Norm Varieties and Algebraic Cobordism

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Abstract

We outline briefly results and examples related with the bijectivity of the norm residue homomorphism. We define norm varieties and describe some constructions. We discuss degree formulas which form a major tool to handle norm varieties. Finally we formulate Hilbert’s 90 for symbols which is the hard part of the bijectivity of the norm residue homomorphism, modulo a theorem of Voevodsky.

2000 Mathematics Subject Classification: 12G05.
Keywords and Phrases: Milnor’s $K$-ring, Galois cohomology, Cobordism.

Introduction

This text is a brief outline of results and examples related with the bijectivity of the norm residue homomorphism—also called “Bloch-Kato conjecture” and, for the mod 2 case, “Milnor conjecture”.

The starting point was a result of Voevodsky which he communicated in 1996. Voevodsky’s theorem basically reduces the Bloch-Kato conjecture to the existence of norm varieties and to what I call Hilbert’s 90 for symbols. Unfortunately there is no text available on Voevodsky’s theorem.

In this exposition $p$ is a prime, $k$ is a field with char $k \neq p$ and $K^M_n k$ denotes Milnor’s $n$-th $K$-group of $k$ [15], [19].

Elements in $K^M_n k/p$ of the form

$$u = \{a_1, \ldots, a_n\} \mod p$$

are called symbols (mod $p$, of weight $n$).

A field extension $F$ of $k$ is called a splitting field of $u$ if $u_F = 0$ in $K^M_n F/p$.

Let

$$h_{(n,p)} : K^M_n k/p \rightarrow H^0_{\acute{e}t}(k, \mu^n_p),$$

$$\{a_1, \ldots, a_n\} \mapsto (a_1, \ldots, a_n)$$

be the norm residue homomorphism.

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1. Norm varieties

All successful approaches to the Bloch-Kato conjecture consist of an investigation of appropriate generic splitting varieties of symbols. This goes back to the work of Merkurjev and Suslin on the case $n = 2$ who studied the $K$-cohomology of Severi-Brauer varieties \[12\]. Similarly, for the case $p = 2$ (for $n = 3$ by Merkurjev, Suslin \[14\] and the author \[18\], for all $n$ by Voevodsky \[23\]) one considers certain quadrics associated with Pfister forms. For a long time it was not clear which sort of varieties one should consider for arbitrary $n, p$. In some cases one knew candidates, but these were non-smooth varieties and desingularizations appeared to be difficult to handle. Finally Voevodsky proposed a surprising characterization of the necessary varieties. It involves characteristic numbers and yields a beautiful relation between symbols and cobordism theory.

**Definition.** Let $u = \{a_1, \ldots, a_n\} \mod p$ be a symbol. Assume that $u \neq 0$. A norm variety for $u$ is a smooth proper irreducible variety $X$ over $k$ such that

1. The function field $k(X)$ of $X$ splits $u$.
2. $\dim X = d := p^{n-1} - 1$.
3. $s_d(X) \not\equiv 0 \mod p$.

Here $s_d(X) \in \mathbb{Z}$ denotes the characteristic number of $X$ given by the $d$-th Newton polynomial in the Chern classes of $TX$. It is known (by Milnor) that in dimensions $d = p^n - 1$ the number $s_d(X)$ is $p$-divisible for any $X$. If $k \subset \mathbb{C}$ one may rephrase condition 3 by saying that $X(\mathbb{C})$ is indecomposable in the complex cobordism ring mod $p$.

We will observe in section 2.4 that the conditions for a norm variety are birational invariant.

The name “norm variety” originates from some constructions of norm varieties, see section 3.

We conclude this section with the “classical” examples of norm varieties.

**Example.** The case $n = 2$. Assume that $k$ contains a primitive $p$-th root $\zeta$ of unity. For $a, b \in k^*$ let $A_\zeta(a, b)$ be the central simple $k$-algebra with presentation

\[ A_\zeta(a, b) = \langle u, v \mid u^p = a, v^p = b, vu = \zeta uv \rangle. \]

The Severi-Brauer variety $X(a, b)$ of $A_\zeta(a, b)$ is a norm variety for the symbol $\{a, b\} \mod 2$.

**Example.** The case $p = 2$. For $a_1, \ldots, a_n \in k^*$ one denotes by

\[ \langle a_1, \ldots, a_n \rangle = \bigotimes_{i=1}^n \langle 1, -a_i \rangle, \]

the associated $n$-fold Pfister form \[9, 21\]. The quadratic form

\[ \varphi = \langle a_1, \ldots, a_{n-1} \rangle \perp \langle -a_n \rangle \]

is called a Pfister neighbor. The projective quadric $Q(\varphi)$ defined by $\varphi = 0$ is a norm variety for the symbol $\{a_1, \ldots, a_n\} \mod 2$. 
2. Degree formulas

The theme of “degree formulas” goes back to Voevodsky’s first text on the Milnor conjecture (although he never formulated explicitly a “formula”) [22]. In this section we formulate the degree formula for the characteristic numbers $s_d$. It shows the birational invariance of the notion of norm varieties.

The first proof of this formula relied on Voevodsky’s stable homotopy theory of algebraic varieties. Later we found a rather elementary approach [11], which is in spirit very close to “elementary” approaches to the complex cobordism ring [16], [4].

For our approach to Hilbert’s 90 for symbols we use also “higher degree formulas” which again were first settled using Voevodsky’s stable homotopy theory [3]. These follow meanwhile also from the “general degree formula” proved by Morel and Levine [10] in characteristic 0 using factorization theorems for birational maps [1].

We fix a prime $p$ and a number $d$ of the form $d = p^n - 1$.

For a proper variety $X$ over $k$ let

$$I(X) = \deg(\text{CH}_0(X)) \subseteq \mathbb{Z}$$

be the image of the degree map on the group of 0-cycles. One has $I(X) = i(X)\mathbb{Z}$ where $i(X)$ is the “index” of $X$, i.e., the gcd of the degrees $[k(x):k]$ of the residue class field extensions of the closed points $x$ of $X$. If $X$ has a $k$-point (in particular if $k$ is algebraically closed), then $I(X) = \mathbb{Z}$. The group $I(X)$ is a birational invariant of $X$. We put

$$J(X) = I(X) + p\mathbb{Z}.$$

Let $X$, $Y$ be irreducible smooth proper varieties over $k$ with dim $Y = \text{dim } X = d$ and let $f: Y \to X$ be a morphism. Define $\deg f$ as follows: If $\text{dim } f(Y) < \text{dim } X$, then $\deg f = 0$. Otherwise $\deg f \in \mathbb{N}$ is the degree of the extension $k(Y)/k(X)$ of the function fields.

**Theorem (Degree formula for $s_d$).**

$$\frac{s_d(Y)}{p} = (\deg f) \frac{s_d(X)}{p} \mod J(X).$$

**Corollary.** The class

$$\frac{s_d(X)}{p} \mod J(X) \in \mathbb{Z}/J(X)$$

is a birational invariant.

**Remark.** If $X$ has a $k$-rational point, then $J(X) = \mathbb{Z}$ and the degree formula is empty. The degree formula and the birational invariants $s_d(X)/p \mod J(X)$ are phenomena which are interesting only over non-algebraically closed fields. Over the complex numbers the only characteristic numbers which are birational invariant are the Todd numbers.
We apply the degree formula to norm varieties. Let $u$ be a nontrivial symbol mod $p$ and let $X$ be a norm variety for $u$. Since $k(X)$ splits $u$, so does any residue class field $k(x)$ for $x \in X$. As $u$ is of exponent $p$, it follows that $J(X) = p\mathbb{Z}$.

**Corollary (Voevodsky).** Let $u$ be a nontrivial symbol and let $X$ be a norm variety for $u$. Since $k(X)$ splits $u$, so does any residue class field $k(x)$ for $x \in X$. As $u$ is of exponent $p$, it follows that $J(X) = p\mathbb{Z}$.

It follows in particular that the notion of norm variety is birational invariant. Therefore we may call any irreducible variety $U$ (not necessarily smooth or proper) a norm variety of a symbol $u$ if $U$ is birational isomorphic to a smooth and proper norm variety of $u$.

### 3. Existence of norm varieties

**Theorem.** Norm varieties exists for every symbol $u \in K^M_n k/p$ for every $p$ and every $n$.

As we have noted, for the case $n = 2$ one can take appropriate Severi-Brauer varieties (if $k$ contains the $p$-th roots of unity) and for the case $p = 2$ one can take appropriate quadrics.

In this exposition we describe a proof for the case $n = 3$ using fix-point theorems of Conner and Floyd in order to compute the non-triviality of the characteristic numbers. Our first proof for the general case used also Conner-Floyd fix-point theory. Later we found two further methods which are comparatively simpler. However the Conner-Floyd fix-point theorem is still used in our approach to Hilbert’s 90 for symbols.

Let $u = \{a, b, c\} \mod p$ with $a, b, c \in k^*$. Assume that $k$ contains a primitive $p$-th root $\zeta$ of unity, let $A = A\zeta(a, b)$ and let

$$MS(A, c) = \{ x \in A \mid \text{Nrd}(x) = c \}.$$ 

We call $MS(A, c)$ the Merkurjev-Suslin variety associated with $A$ and $c$. The symbol $u$ is trivial if and only if $MS(A, c)$ has a rational point. The variety $MS(A, c)$ is a twisted form of $SL(p)$.

**Theorem.** Suppose $u \neq 0$. Then $MS(A, c)$ is a norm variety for $u$.

Let us indicate a proof for a subfield $k \subset \mathbb{C}$ (and for $p > 2$). Let $U = MS(A, c)$. It is easy to see that $k(U)$ splits $u$. Moreover one has $\dim U = \dim A - 1 = p^2 - 1$. It remains to show that there exists a proper smooth completion $X$ of $U$ with nontrivial characteristic number.

Let

$$\bar{U} = \{ [x, t] \in \mathbb{P}(A \oplus k) \mid \text{Nrd}(x) = ct^p \}$$

be the naive completion of $U$. We let the group $G = \mathbb{Z}/p \times \mathbb{Z}/p$ act on the algebra $A$ via

$$(r, s) \cdot u = \zeta^r u, \quad (r, s) \cdot v = \zeta^s v.$$ 

This action extends to an action on $\mathbb{P}(A \oplus k)$ (with the trivial action on $k$) which induces a $G$-action on $\bar{U}$. Let $\text{Fix}(\bar{U})$ be the fixed point scheme of this action. One
finds that $\text{Fix}(\bar{U})$ consists just of the $p$ isolated points $[1, \zeta^i \sqrt[p]{c}], i = 1, \ldots, p$, which are all contained in $U$.

The variety $U$ is smooth, but $\bar{U}$ is not. However, by equivariant resolution of singularities \[2\], there exists a smooth proper $G$-variety $X$ together with a $G$-morphism $X \to \bar{U}$ which is a birational isomorphism and an isomorphism over $U$. It remains to show that

$$\frac{s_d(X)}{p} \not\equiv 0 \mod p.$$  

For this we may pass to topology and try to compute $s_d(X(C))$. We note that for odd $p$, the Chern number $s_d$ is also a Pontryagin number and depends only on the differentiable structure of the given variety. Note further that $X$ has the same $G$-fixed points as $\bar{U}$ since the desingularization took place only outside $U$.

Consider the variety

$$Z = \left\{ \left[ \sum_{i,j=1}^p x_{ij} u^i v^j, t \right] \in \mathbb{P}(A \oplus k) \mid \sum_{i,j=1}^p x_{ij}^p = ct^p \right\}.$$  

This variety is a smooth hypersurface and it is easy to check

$$\frac{s_d(Z)}{p} \not\equiv 0 \mod p.$$  

As a $G$-variety, the variety $Z$ has the same fixed points as $X$ (“same” means that the collections of fix-points together with the $G$-structure on the tangent spaces are isomorphic). Let $M$ be the differentiable manifold obtained from $X(C)$ and $-Z(C)$ by a multi-fold connected sum along corresponding fixed points. Then $M$ is a $G$-manifold without fixed points. By the theory of Conner and Floyd \[5, 7\] applied to $(\mathbb{Z}/p)^2$-manifolds of dimension $d = p^2 - 1$ one has

$$\frac{s_d(M)}{p} \equiv 0 \mod p.$$  

Thus

$$\frac{s_d(X)}{p} \equiv \frac{s_d(Z)}{p} \mod p$$  

and the desired non-triviality is established.

**The functions $\Phi_n$.** We conclude this section with examples of norm varieties for the general case.

Let $a_1, a_2, \ldots$ be a sequence of elements in $k^*$. We define functions $\Phi_n = \Phi_{a_1, \ldots, a_n}$ in $p^n$ variables inductively as follows.

$$\Phi_0(t) = t^p,$$

$$\Phi_n(T_0, \ldots, T_{p-1}) = \Phi_{n-1}(T_0) \prod_{i=1}^{p-1} (1 - a_n \Phi_{n-1}(T_i)).$$

Here the $T_i$ stand for tuples of $p^{n-1}$ variables. Let $U(a_1, \ldots, a_n)$ be the variety defined by

$$\Phi_{a_1, \ldots, a_{n-1}}(T) = a_n.$$  

**Theorem.** Suppose that the symbol $u = \{a_1, \ldots, a_n\}$ mod $p$ is nontrivial. Then $U(a_1, \ldots, a_n)$ is a norm variety of $u$.  

4. Hilbert’s 90 for symbols

The bijectivity of the norm residue homomorphisms has always been considered as a sort of higher version of the classical Hilbert’s Theorem 90 (which establishes the bijectivity for \( n = 1 \)). In fact, there are various variants of the Bloch-Kato conjecture which are obvious generalizations of Hilbert’s Theorem 90: The Hilbert’s Theorem 90 for \( K^M_n \) of cyclic extensions or the vanishing of the motivic cohomology group \( H^{n+1}(k, \mathbf{Z}(n)) \). In this section we describe a variant which on one hand is very elementary to formulate and on the other hand is the really hard part of the Bloch-Kato conjecture (modulo Voevodsky’s theorem).

Let \( u = \{a_1, \ldots, a_n\} \in K^M_n k/p \) be a symbol. Consider the norm map

\[
N_u = \sum_F N_{F/k} : \bigoplus_F K_1 F \to K_1 k
\]

where \( F \) runs through the finite field extensions of \( k \) (contained in some algebraic closure of \( k \)) which split \( u \). Hilbert’s Theorem 90 for \( u \) states that \( \ker N_u \) is generated by the “obvious” elements.

To make this precise, we consider two types of basic relations between the norm maps \( N_{F/k} \).

Let \( F_1, F_2 \) be finite field extensions of \( k \). Then the sequence

\[
K_1(F_1 \otimes F_2) \xrightarrow{(N_{F_1} \otimes k, -N_{F_1, F_2}/F_2)} K_1 F_1 \oplus K_1 F_2 \xrightarrow{N_{F_1/k + N_{F_2}/k}} K_1 k
\]

is a complex.

Further, if \( K/k \) is of transcendence degree 1, then the sequence

\[
K_2 K \xrightarrow{d_K} \bigoplus_v K_1 \kappa(v) \xrightarrow{N} K_1 k
\]

is a complex. Here \( v \) runs through the valuations of \( K/k \), \( d_K \) is given by the tame symbols at each \( v \) and \( N \) is the sum of the norm maps \( N_{\kappa(v)/k} \). The sum formula \( N \circ d_K = 0 \) is also known as Weil’s formula.

We now restrict again to splitting fields of \( u \). The maps in (1) yield a map

\[
\mathcal{R}_u = \sum_{F_1, F_2} (N_{F_1 \otimes F_2/k, -N_{F_1 \otimes F_2/F_2}}) : \bigoplus_{F_1, F_2} K_1(F_1 \otimes F_2) \to \bigoplus_F K_1 F
\]

with \( N_u \circ \mathcal{R}_u = 0 \). Let \( C \) be the cokernel of \( \mathcal{R}_u \) and let \( N'_u : C \to K_1 k \) be the map induced by \( N_u \). Then the maps in (2) yield a map

\[
\mathcal{S}_u = \sum_K d_K : \bigoplus_K K_2 K \to C
\]

with \( N'_u \circ \mathcal{S}_u = 0 \) where \( K \) runs through the splitting fields of \( u \) of transcendence degree 1 over \( k \) (contained in some universal field). Let \( H_0(u, K_1) \) be the cokernel of \( \mathcal{S}_u \) and let \( N_u : H_0(u, K_1) \to K_1 k \) be the map induced by \( N'_u \).
Hilbert's 90 for symbols. For every symbol $u$ the norm map

$$N_u : H_0(u, K_1) \to K_1 k$$

is injective.

**Example.** If $u = 0$, then it is easy to see that $N_u$ is injective. In fact, it is a trivial exercise to check that $N_u'$ is injective.

**Example.** The case $n = 1$. The splitting fields $F$ of $u = \{a\}$ mod $p$ are exactly the field extensions of $k$ containing a $p$-th root of $a$. It is an easy exercise to reduce the injectivity of $N_u$ (in fact of $N_u'$) to the classical Hilbert's Theorem 90, i.e., the exactness of

$$K_1 L \xrightarrow{1-\sigma} K_1 L \xrightarrow{N_{L/k}} K_1 k$$

for a cyclic extension $L/k$ of degree $p$ with $\sigma$ a generator of Gal($L/k$).

**Example.** The case $n = 2$. Assume that $k$ contains a primitive $p$-th root $\zeta$ of unity. The splitting fields $F$ of $u = \{a, b\}$ mod $p$ are exactly the splitting fields of the algebra $A_{\zeta}(a, b)$. One can show that

$$H_0(u, K_1) = K_1 A_{\zeta}(a, b)$$

with $N_u$ corresponding to the reduced norm map $\text{Nrd}$. Hence in this case Hilbert's 90 for $u$ reduces to the classical fact $SK_1 A = 0$ for central simple algebras of prime degree.

**Example.** The case $p = 2$. The splitting fields $F$ of $u = \{a_1, \ldots, a_n\}$ mod 2 are exactly the field extensions of $k$ which split the Pfister form $\langle\langle a_1, \ldots, a_n\rangle\rangle$ or, equivalently, over which the Pfister neighbor $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle \perp -a_n$ becomes isotropic. Hilbert's 90 for symbols mod 2 had been first established in [17]. This text considered similar norm maps associated with any quadratic form (which are not injective in general). A treatment of the special case of Pfister forms is contained in [8].

**Remark.** One can show that the group $H_0(u, K_1)$ as defined above is also the quotient of $\oplus_F K_1 F$ by the $R$-trivial elements in ker $N_u$. This is quite analogous to the description of $K_1 A$ of a central simple algebra $A$: the group $K_1 A$ is the quotient of $A^*$ by the subgroup of $R$-trivial elements in the kernel of $\text{Nrd}$: $A^* \to F^*$. Similarly for the case $p = 2$: In this case the injectivity of $N_u$ is related with the fact that for Pfister neighbors $\varphi$ the kernel of the spinor norm $\text{SO}(\varphi) \to k^*/(k^*)^2$ is $R$-trivial.

In our approach to Hilbert's 90 for symbols one needs a parameterization of the splitting fields of symbols.

**Definition.** Let $u = \{a_1, \ldots, a_n\}$ mod $p$ be a symbol. A $p$-generic splitting variety for $u$ is a smooth variety $X$ over $k$ such that for every splitting field $F$ of $u$ there exists a finite extension $F'/F$ of degree prime to $p$ and a morphism $\text{Spec} F' \to X$.

**Theorem.** Suppose $\text{char} k = 0$. Let $m \geq 3$ and suppose for $n \leq m$ and every symbol $u = \{a_1, \ldots, a_n\}$ mod $p$ over all fields over $k$ there exists a $p$-generic splitting variety for $u$ of dimension $p^{n-1} - 1$. Then Hilbert's 90 holds for such symbols.

The proof of this theorem is outlined in [20].
For $n = 2$ one can take here the Severi-Brauer varieties and for $n = 3$ the Merkurjev-Suslin varieties. Hence we have:

**Corollary.** Suppose $\text{char } k = 0$. Then Hilbert’s 90 holds for symbols of weight $\leq 3$.

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