Self-dual Maxwell field in 3D gravity with torsion and dynamical role of central charges

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Abstract. We show that general structure of the field equations for the system of a self-dual Maxwell field coupled to 3D gravity with torsion implies that classical central charges have a new dynamical role. By using simple correspondence between self-dual solutions with torsion and their Riemannian counterparts we construct two exact solutions, corresponding to the sectors with a massless and massive Maxwell field, and calculate their conserved charges.

1. Introduction
The three-dimensional (3D) gravity has been actively studied for nearly three decades as an interesting model which shares the same conceptual features with general relativity (GR). Among many outstanding results [1], the discovery of the BTZ black hole [2] was particularly important, since it improved our basic understanding of the classical and quantum black hole dynamics.

Mielke and Baekler (MB) [3] proposed a new geometric framework for 3D gravity by replacing the traditional Riemannian geometry of GR by the Riemann-Cartan geometry, in which the gravitational dynamics is characterized by both the curvature and the torsion. Such an approach led to a number of interesting results [4, 5, 6, 7], which confirm that the topological MB model has a rich dynamical structure. The most recent results are related to the electrically charged sector of the theory [8, 9]. By analyzing static configurations of the MB model, we found an exact solution with azimuthal electric field [8]. Here, we continue the investigation of the charged sector by studying the self-dual Maxwell field as a source.

The layout of the paper is as follows. In section 2 we give a brief account of 3D gravity with torsion, with Maxwell field modified by a topological mass term ($\mu$) [10] as a source. In section 3, we impose a restriction on the Maxwell field by requesting it to be self-dual. As a consequence, we find that two possible values of the classical central charge of 3D gravity with torsion [5] are directly related to the two self-duality sectors of the Maxwell field. We establish a simple correspondence between self-dual solutions with torsion, and their Riemannian counterparts. In section 4, we construct two exact, stationary and spherically symmetric solutions, characterized by $\mu = 0$ and $\mu \neq 0$ and calculate their conserved charges—energy, angular momentum and electric charge. Section 5 is devoted to concluding remarks.

We use the same conventions as in [9]: the Latin indices refer to the local Lorentz frame, the Greek indices refer to the coordinate frame; the middle alphabet letters ($i, j, k, ..., \mu, \nu, \lambda, ...$) run over 0,1,2, the first letters of the Greek alphabet ($\alpha, \beta, \gamma, ...$) run over 1,2; the metric components in the local Lorentz frame are $\eta_{ij} = (+, -, -)$; totally antisymmetric tensor $\varepsilon^{ijk}$ and the related tensor density $\varepsilon^{\mu\nu\rho}$ are both normalized so that $\varepsilon^{012} = 1$; the Hodge-star operation is $\ast$. 

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2. Lagrangian and field equations

Theory of gravity with torsion can be naturally described as a Poincaré gauge theory (PGT). Basic gravitational variables in PGT are the triad $b^i$ and the Lorentz connection $A^i = -A^i_j (1\text{-forms})$, and the corresponding field strengths are the torsion $T^i$ and the curvature $R^i_j$. In 3D, one can introduce the notation $A^i (1\text{-forms})$, and the corresponding field strengths are the torsion $T^i$ expressed in terms of the energy-momentum tensor $T_{ij}$, which yields:

$$ T^i = db^i + \varepsilon^i_{jk} \omega^j \wedge b^k, \quad R^i_j = d\omega^i_j + \frac{1}{2} \varepsilon^i_{jk} \omega^j \wedge \omega^k. \quad (2.1) $$

PGT is characterized by a useful identity:

$$ \omega^i \equiv \tilde{\omega}^i + K^i, \quad (2.2a) $$

where $\tilde{\omega}^i$ is the Levi-Civita (Riemannian) connection, and $K^i$ is the contortion 1-form, defined implicitly by $T^i =: \varepsilon^i_{mn} K^m \wedge b^n$. Using this identity, one can express the curvature $R_i = R_i(\omega)$ in terms of its Riemannian piece $\tilde{R}_i = R_i(\tilde{\omega})$ and the contortion $K_i$:

$$ 2R_i \equiv 2\tilde{R}_i + 2\tilde{\nabla} K_i + \varepsilon_{imn} K^m \wedge K^n. \quad (2.2b) $$

The covariant derivative $\nabla = \tilde{\nabla}(\omega)$ acts on a tangent-frame spinor/tensor in accordance with its spinorial/tensorial structure.

The antisymmetry of the Lorentz connection $A^i_j$ implies that the geometric structure of PGT corresponds to Riemann-Cartan geometry.

The topological MB model for 3D gravity [3] represents a natural generalization of GR with a cosmological constant (GR$_\Lambda$):

$$ L_0 = 2ab^i R_i - \frac{A}{3} \varepsilon_{ijb} b^i b^j + \alpha_3 \left( \omega^i d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^j \omega^k \right) + \alpha_4 b^i T_i, \quad (2.3a) $$

where $a = 1/16\pi G$ and the wedge product sign $\wedge$ is omitted for simplicity. The complete dynamics of the topologically massive Maxwell field coupled to 3D gravity with torsion is described by:

$$ L = L_0 + L_M, \quad L_M := -\frac{1}{2} F'^* F - \frac{b}{2} A F, \quad (2.3b) $$

where $F = dA$.

The gravitational field equations are obtained by varying $L$ with respect to $b^i$ and $\omega^i$. In the nondegenerate sector with $\Delta := \alpha_3 \alpha_4 - \alpha^2 \neq 0$, they take the following simple form [8, 9]:

$$ T_i = \varepsilon_{imn} K^m b^n, \quad K^m = \frac{1}{2} (pb^m + ut^m), \quad (2.4a) $$

$$ 2R_i = q \varepsilon_{imn} b^m b^n - v \varepsilon_{imn} t^m b^n, \quad (2.4b) $$

where $p := (\alpha_3 \alpha_4 + \alpha_4 a)/\Delta$, $u := \alpha_3/\Delta$, $q := -((\alpha_2^2 + aA)/\Delta$, $v := a/\Delta$, and the 1-form $t^i$ is expressed in terms of the energy-momentum tensor $T^i_k$:

$$ t^i := -\left( T^i_k - \frac{1}{2} \delta^i_k T \right) b^k, \quad T^i_k := -F'^{im} F_{km} + \frac{1}{4} \delta^i_k F^2, \quad (2.4c) $$

with $T = T^k_k$ and $F^2 = F'^{mn} F_{mn}$. These equations, together with the modified Maxwell equations

$$ d'^* F + \mu F = 0, \quad (2.4c) $$

define the complete dynamics of our system. The Cartan curvature $R_i$ is calculated using the identity (2.2b),

$$ 2R_i = 2\tilde{R}_i + u \tilde{\nabla} t_i + \varepsilon_{imn} \left( \frac{p^2}{4} b^m b^n + \frac{up}{2} t^m b^n + \frac{u^2}{4} t^m t^n \right), \quad (2.5) $$

where the Maxwell field contribution is compactly represented by the 1-form $t^i$. 

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3. General dynamical characteristics of self-dual solutions

General structure of the field equations (2.4) depends essentially on the form of the Maxwell field. In this section, we discuss some important dynamical characteristics of exact solutions corresponding to the self-dual Maxwell field as a source.

We are looking for a spherically symmetric and stationary solution. Choosing the local coordinates \( x^\mu = (t, r, \varphi) \), we make the following ansatz for the triad and for the Maxwell field:

\[
\begin{align*}
    b^0 &= N \, dt, \\
    b^1 &= B^{-1} \, dr, \\
    b^2 &= K (d\varphi + C dt), \\
    F &= E b^0 b^1 - H b^1 b^2.
\end{align*}
\]

Here, \( N, B, C, K \) and \( E, H \) are six unknown functions of the radial coordinate \( r \).

We assume a generalized self-duality of the Maxwell field:

\[
E = \epsilon H, \quad \epsilon^2 = 1.
\]

After introducing a new radial coordinate \( \rho = \rho(r) \), defined by \( d\rho = dr/B \), we find that the first integrals of the modified Maxwell equations are given as

\[
\begin{align*}
    EK &= -Q_e \exp(-\epsilon \mu \rho), \\
    -HN &= (Q_m - Q_e C) \exp(-\epsilon \mu \rho),
\end{align*}
\]

where \( Q_e, Q_m \) are integration constants, and \( \epsilon Q_e N = K(Q_m - Q_e C) \), as a consequence of (3.3). The self-duality condition implies \( F^2 = 0 \) and simplifies the form of the electromagnetic energy-momentum tensor,

\[
T^i_{\ j} = \begin{pmatrix}
    E^2 & 0 & EH \\
    0 & 0 & 0 \\
    -EH & 0 & -H^2
\end{pmatrix},
\]

as well as the form of \( t^i \):

\[
t^0 = -E (Eb^0 + H t^2), \quad t^1 = 0, \quad t^2 = -\epsilon t^0.
\]

This defines the contortion as in (2.4a), and exhausts the content of the first field equation.

Going now to the second field equation (2.4b), we calculate the Cartan curvature (2.5). Then, by substituting this result into (2.4b), we obtain:

\[
2 \tilde{R}_i = \Lambda_{\text{eff}} \, \epsilon_{imn} b^m b^n - V \epsilon_{imn} t^m b^n, \\
V := v + \frac{pu}{2} + \epsilon \frac{u}{\ell}.
\]

Note that the factor \( V = V(\epsilon) \) is proportional to the classical central charge \( c(\epsilon) \), characterizing the asymptotic conformal structure of 3D gravity with torsion [5]:

\[
V(\epsilon) = \frac{1}{\Delta} \frac{1}{24\pi \ell} c(\epsilon), \quad c(\mp 1) = 24\pi \left[ a\ell + \alpha_3 \left( \frac{p\ell}{2} \mp 1 \right) \right].
\]

- Equation (3.5) reveals a new feature of the central charges \( c(\mp 1) \), showing that they have a direct dynamical influence on the self-duality modes \( \epsilon = \mp 1 \) of the system.

The set of equations (3.5) is overdetermined. However, there is a consistent solution of this system, which can obtained either directly [9], or by noting that in the limit \( u \to 0 \), or equivalently, \( V \to v = -1/a \), equation (3.5) becomes equivalent to the Einstein equation for the self-dual Maxwell field in Riemannian 3D gravity. This property can be formulated as the following constructive statement:

- Starting with any self-dual solution in Riemannian 3D gravity, one can generate the related self-dual solution with torsion by making the replacement \( v \to V \).

In what follows, we shall illustrate the power of this theorem by constructing two different solutions of (3.5), belonging to the sectors with \( \mu = 0 \) and \( \mu \neq 0 \), respectively.
4. Exact self-dual solutions

4.1. Solution with $\mu = 0$

The condition $\mu = 0$ refers to the standard, massless Maxwell field. The field equations define the form of $K^2$. To realize the construction of the solution, we first conveniently fix $g_2$ by demanding $g_2 = r_0^2$. After choosing the radial coordinate $r$ by $B = (r^2 - r_0^2)/\ell r$ and imposing the boundary condition $C \rightarrow 0$ for $r \rightarrow \infty$, we find the general solution for $\mu = 0$ [9]:

\[
B = \frac{r^2 - r_0^2}{\ell r}, \quad N = \frac{r^2 - r_0^2}{\ell K}, \quad K^2 = r^2 + r_0^2 \ln \left| \frac{r^2 - r_0^2}{r_0^2} \right| + h_1, \quad C = \frac{\epsilon}{\ell} \left( 1 - \frac{r^2 - r_0^2}{K^2} \right), \quad E = \epsilon H = -\frac{Q_\epsilon}{K},
\]

where $h_1 = g_1 - r_0^2$. To complete the solution, we display also the electromagnetic potential:

\[
A = \left( \frac{Q_\epsilon}{2} \ln \left| \frac{r^2 - r_0^2}{r_0^2} \right| + h_2 \right) \left( dt + \epsilon \ell d\varphi \right).
\]

In the limit $u \rightarrow 0$, or $V \rightarrow -1/a$, the solution (4.1) coincides with the Kamata-Koikawa self-dual solution, found in Riemannian GR$_A$ [11] (see also [12]). Thus, the field configuration (4.1) represent a generalization of the Kamata-Koikawa solution to the self-dual solution with torsion. In the limit $Q_\epsilon \rightarrow 0$, the solution (4.1) reduces asymptotically to the vacuum state of the BTZ-like black hole with torsion [5].

The scalar Cartan curvature of the self-dual solution (4.1) is $R = -6q$, while $\tilde{R} = -6A_{\text{eff}}$. However, the form of $R_q$ implies that the metric of the solution is not maximally symmetric. To gain a deeper insight into the nature of the self-dual solution (4.1), we now turn our attention to its conserved charges.

The family of solutions (4.1) is parametrized by $(Q_\epsilon, h_1, h_2)$. Considering the neutral limit $Q_\epsilon \rightarrow 0$ in Riemannian GR$_A$, Kamata and Koikawa [11] concluded that one should fix the parameter $h_1$ to zero, as it leads to $K^2 \rightarrow r^2$ in this limit. However, the choice $h_1 = 0$ leads to divergent values of energy and angular momentum [11, 9].

An interesting attempt to handle these divergences by a suitable regularization procedure was proposed in [15]. In this procedure, we enclose the system in a circle $C_{\text{as}}$ having a large, but finite radius $r_{\text{as}}$. This circle represents a regularized spatial boundary at infinity, and the asymptotic region is defined by $r \rightarrow r_{\text{as}}$. Then, we make a choice of the boundary conditions by fixing the values of dynamical variables at $C_{\text{as}}$. Finally, we take the limit $r_{\text{as}} \rightarrow \infty$.

In our example, after introducing $C_{\text{as}}$ we choose the boundary conditions $K^2 = r_{\text{as}}^2$ and $A = 0$ [9]; in this way, we effectively eliminate the logarithmic terms from the boundary $C_{\text{as}}$. The standard canonical procedure yields the following expressions for the energy $E$, the angular momentum $M$ and the electric charge $Q$ of our self-dual solution [9]:

\[
\ell E = \epsilon Q_\epsilon \frac{u}{12\ell} c(-\epsilon) = \epsilon M, \quad Q = 2\pi Q_\epsilon.
\]

Although the regularized self-dual solution has the same leading asymptotic terms as the black hole with torsion, its conserved charges $E$ and $M$ are quite different. The reason for this lies in the fact that $E$ and $M$ are basically determined by the sub-leading asymptotic terms. This is clearly seen in the canonical formalism, where the analysis of the relevant surface terms shows that the non-vanishing $E$ and $M$ are generated by a combined effect of the Maxwell field ($\ell^2$) and the gravitational Chern-Simons term ($u \sim \alpha_3$).

The regularized solution and the one with $h_1 = h_2 = 0$, are two different members of the same family (4.1), which have different conserved charges. Another difference is found in their geometric properties, since the regularized solution is valid only in the asymptotic region [9].
4.2. Solution with \( \mu \neq 0 \)

The topological mass \( \mu \) is introduced to regularize the asymptotic behavior of the Maxwell field. As shown in [9], the solution for \( K^2 \) in the case \( \mu \neq 0 \) (with \( \epsilon \mu \ell \neq -1 \)), takes the form

\[
K^2 = g_1 + g_2 \exp(2\rho/\ell) - \frac{r_0^2}{\mu^2 \ell^2 + \epsilon \mu \ell} \exp(-2\epsilon \mu \rho),
\]

while the general self-dual solution with torsion (for \( \mu \neq 0, \epsilon \mu \ell \neq -1 \)) is given by:

\[
N = \frac{g_3 \exp(2\rho/\ell)}{K}, \quad C = \frac{Q_m}{Q_e} - \epsilon \frac{N}{K}, \quad E = \epsilon H = -\frac{Q_e}{K} \exp(-\epsilon \mu \rho),
\]

where \( g_1, g_2 \) and \( g_3 \) are constants of integration.

For a fixed \( \epsilon \), the solution is characterized by eight parameters. This number can be significantly reduced by imposing various physical/geometric requirements. A convenient normalization of the time coordinate leads to \( g_3 = \ell \). Demanding that \( C \) vanishes at spatial infinity, we obtain \( Q_m/Q_e = \epsilon/\ell \). The choice \( g_2 = \ell^2 \) ensures that in the limit \( Q_e \to 0 \), \( g_1 \to 0 \), the solution (4.3) reduces to the black hole vacuum. Finally, we note that solutions with vanishing Maxwell field at spatial infinity are characterized by \( \epsilon \mu > 0 \). Hence, we impose the requirement \( \epsilon \mu = 1/\ell \) to simplify the calculations [13].

Since the solution (4.3) is defined only up to a choice of the radial coordinate, we are free to choose \( K = r \). Expressed in terms of \( r \), the solution (4.3) reads:

\[
K = r, \quad B = \frac{\sqrt{(r^2 - g_1)^2 + 2\ell^2 r_0^2}}{\ell r}, \quad N = \frac{(r^2 - g_1 + \ell^2 B)}{2\ell r},
\]

\[
C = \epsilon \left(\frac{1}{\ell} - \frac{N}{r}\right), \quad E = -\frac{Q_e}{r} \sqrt{\frac{r}{N r}}, \quad A = -Q_e \ell \sqrt{\frac{r}{N r}} (dt + \epsilon \ell d\phi).
\]

This result represents a natural generalization of the self-dual solution found by Fernando and Mansouri [13], in the context of Riemannian theory. In the limit \( Q_e \to 0 \), the solution reduces to the extreme black hole with torsion [5].

From the above expressions, we conclude that (4.4) is a perfectly regular solution, since \( B > 0 \) and \( N > 0 \) over the whole range of \( r \), while the Cartan curvature is constant, \( R = -6q \).

To calculate the conserved charges we first fix the asymptotic form of the fields in (4.4):

\[
N \sim \frac{r}{\ell} - \frac{g_1}{\ell r}, \quad B \sim \frac{r}{\ell} - \frac{g_1}{\ell r}, \quad C \sim \frac{g_1}{\ell r^2}, \quad E \sim \frac{Q_e}{r^2}, \quad A_\mu \sim \frac{Q_e}{r}.
\]

Note that the electromagnetic potential has the same asymptotic behavior as in 4D, due to the presence of the topological mass \( \mu \), which acts as an infrared regulator and modifies the asymptotics of all the fields.

Because of the fast asymptotic decrease of the Maxwell potential, the electromagnetic contribution to the energy and angular momentum vanishes. Moreover, the asymptotics defined by (4.5) coincides with that of the extreme BTZ black hole with torsion [5], so that the energy and angular momentum of our self-dual solution (4.4) are given by:

\[
\ell E = \frac{g_1}{6\ell} c(-\epsilon) = \epsilon M.
\]

In the Riemannian limit \( u \to 0 \), these expressions agree with those obtained in the quasilocal formalism by Fernando and Mansouri [13], and by Dereli and Obukhov [14].

The electric charge of the solution vanishes:

\[
Q = 0.
\]

Thus, despite the existence of the radial electric field, its asymptotic fall off is too fast to produce a nonvanishing electric charge. The constant \( Q_e \) cannot be interpreted as the electric charge, and our Fernando-Mansouri-like solution should be called a neutral self-dual solution.
5. Concluding remarks

(1) Studying dynamical properties of the self-dual Maxwell field modified by a topological mass term and coupled to 3D gravity with torsion, we found that the dynamical evolution is directly influenced by the classical central charges.

(2) We constructed two exact, stationary and self-dual solutions with torsion, characterized by \( \mu = 0 \) and \( \mu \neq 0 \). They represent respective generalizations of the Kamata-Koikawa and Fernando-Mansouri solutions, found earlier in Riemannian GR\( \Lambda \). For each of these solutions, we calculated its conserved charges.

Acknowledgments

This work was supported by the Serbian Science Foundation.

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