In this paper we study the boundedness of global pseudo-differential operators on smooth manifolds. By using the notion of global symbol we extend a classical condition of Hörmander type to guarantee the $L^p$-$L^q$-boundedness of global operators. First we investigate $L^p$-boundedness of pseudo-differential operators in view of the Hörmander-Mihlin condition. We also prove $L^\infty$-$BMO$ estimates for pseudo-differential operators. Later, we concentrate our investigation to settle $L^p$-$L^q$ boundedness of the Fourier multipliers and pseudo-differential operators for the range $1 < p \leq 2 \leq q < \infty$. On the way to achieve our goal of $L^p$-$L^q$ boundedness we prove two classical inequalities, namely, Paley inequality and Hausdorff-Young-Paley inequality for smooth manifolds. Finally, we present the applications of our boundedness theorems to the well-posedness properties of different types of the nonlinear partial differential equations.

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2010 Mathematics Subject Classification. 58J40; Secondary 47B10, 47G30, 35S30.

Key words and phrases. Pseudo-differential operator, nonharmonic analysis, manifold, Hausdorff-Young-Paley inequality, multiplier, boundedness, non-linear partial differential equation.

The authors were supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations. MR was also supported in parts by the EPSRC Grant EP/R003025/1, by the Leverhulme Research Grant RPG-2017-151.
1. Introduction

In this paper we investigate classical conditions for the boundedness of multipliers and more generally, pseudo-differential operators in the context of the Fourier analysis arising from the spectral decomposition of a model operator \( L \) on a smooth manifold \( \mathcal{M} \) (which can be closed or with smooth boundary). To explain the results in this paper, let us recall the following classical results of Fourier analysis. If \( \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) is the Fourier transform on \( \mathbb{R}^n \),

\[
(\mathcal{F} f)(\xi) := \int_{\mathbb{R}^n} e^{-i2\pi x \cdot \xi} f(x) \, dx, \quad \xi \in \mathbb{R}^n, \quad f \in C_0^\infty(\mathbb{R}^n),
\]

the function \( m \) is measurable on \( \mathbb{R}^n \), and the function \( \psi \in C_0(\mathbb{R}^n) \) is a test function, under the following conditions,

1. (Hörmander Mihlin Condition)

\[
\|m\|_{L^s(\mathcal{M})} = \sup_{r > 0} \| r^{s-\frac{2}{p}} \mathcal{F}[m(\cdot)\psi(|\cdot|^r)] \|_{L^2(\mathbb{R}^n)} < \infty, \quad s > n/2,
\]

2. (Paley-type inequality)

\[
M_\psi := \sup_{t > 0} \{ \xi \in \mathbb{R}^n : \psi(\xi) \geq t \} < \infty,
\]

the operators \( T_m \) and \( T \) defined by

1'.

\[
T_m f(x) := \int_{\mathbb{R}^n} e^{i2\pi x \cdot \xi} m(\xi)(\mathcal{F} f)(\xi) d\xi, \quad f \in C_0^\infty(\mathbb{R}^n),
\]

2'.

\[
T f(\xi) := (\mathcal{F} u)(\xi)\phi(\xi)^{2\left(1 - \frac{1}{p}\right)}, \quad f \in C_0^\infty(\mathbb{R}^n),
\]

admit bounded extensions \( T_m : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \), for \( 1 < p < \infty \), and \( T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \), when \( 1 < p \leq 2 \). These two classical results are due to Hörmander (see [25, pages 105 and 120]). So, the Hörmander Mihlin Condition assures the \( L^p \)-boundedness of multipliers of the Fourier transform, while, the Paley-type inequality describes the growth of the Fourier transform of a function in terms of its \( L^p \)-norm. Interpolating the Paley-inequality with the Hausdoff-Young inequality one can obtain the following Hörmander’s version of the Hausdorff-Young-Paley inequality,

\[
\left( \int_{\mathbb{R}^n} |(\mathcal{F} f)(\xi)\phi(\xi)^{\frac{1}{r} - \frac{1}{\rho'}}|^r d\xi \right)^{\frac{1}{r}} \leq \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq r \leq p' < \infty, \quad 1 < p < 2. \tag{1.6}
\]

Also, as a consequence of the Hausdorff-Young-Paley inequality, Hörmander [25, page 106] proves that the condition

\[
\sup_{t > 0} t^b \{ \xi \in \mathbb{R}^n : m(\xi) \geq t \} < \infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{b}, \tag{1.7}
\]

where \( 1 < p \leq 2 \leq q < \infty \), implies the existence of a bounded extension of \( T_m : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \). The aim of this paper is to extend these results to the case of smooth-manifolds, by using the Fourier analysis associated to a model operator \( L \) on \( \mathcal{M} \). To formulate our results more precisely, let \( L \) be a pseudo-differential operator of
order $m$ on the interior $M$ of $\overline{M}$ in the sense of Hörmander. This means that in every coordinate chart on the interior $M$, $L$ agrees with a pseudo-differential operator of order $m$ in some open subset of $\mathbb{R}^{\dim(M)}$.

We assume that some boundary conditions called (BC) are fixed and lead to a discrete spectrum with a family of eigenfunctions yielding a Riesz basis in $L^2(\overline{M})$. However, it is important to point out that the operator $L$ does not have to be self-adjoint or an elliptic differential operator. For a discussion on general bi-orthogonal systems we refer the reader to Bari [3] and Gelfand [18]. Now we formulate our assumptions precisely. The discrete set of eigenvalues and eigenfunctions will be indexed by a countable set $I$. We consider the spectrum $\{\lambda_\xi \in \mathbb{C} : \xi \in I\}$ of $L$ with corresponding eigenfunctions in $L^2(M)$ denoted by $u_\xi$, i.e.,

$$Lu_\xi = \lambda_\xi u_\xi \text{ in } M, \quad \text{for all } \xi \in I, \quad (1.8)$$

and the eigenfunctions $u_\xi$ satisfy the boundary conditions (BC). We can think of (BC) as defining the domain of the operator $L$. The conjugate spectral problem is

$$L^*v_\xi = \lambda_\xi v_\xi \text{ in } M, \quad \text{for all } \xi \in I,$$

which we equip with the conjugate boundary conditions $(BC)^\ast$. We assume that the functions $u_\xi, v_\xi$ are normalised, i.e. $\|u_\xi\|_{L^2} = \|v_\xi\|_{L^2} = 1$ for all $\xi \in I$. Moreover, we can take biorthogonal systems $\{u_\xi\}_{\xi \in I}$ and $\{v_\xi\}_{\xi \in I}$, i.e. $(u_\xi, v_\eta)_{L^2} = 0$ for $\xi \neq \eta$, and $(u_\xi, v_\eta)_{L^2} = 1$ for $\xi = \eta$.

$$\langle f, g \rangle_{L^2} = \int_M f(x)\overline{g(x)}\,dx$$

is the usual inner product of the Hilbert space $L^2(M)$. We also assume that the system $\{u_\xi\}$ is a Riesz basis of $L^2(\overline{M})$, i.e. for every $f \in L^2(\overline{M})$ there exists a unique series $\sum_{\xi \in I} a_\xi u_\xi$ that converges to $f$ in $L^2(\overline{M})$. It is well known that (cf. [3]) the system $\{u_\xi\}$ is a basis of $L^2(\overline{M})$ if and only if the system $\{v_\xi\}$ is a basis of $L^2(\overline{M})$. Our analysis will be based on the quantization process carried by the non-harmonic analysis developed in [30, 31]. So, if $C_L^\infty(\overline{M}) := \cap_{k=1}^\infty \text{Dom}(L^k)$, an $L$-pseudo-differential operator is a continuous linear operator $A : C_L^\infty(\overline{M}) \to C_L^\infty(\overline{M})$, defined by

$$Af(x) \equiv T_m f(x) := \sum_{\xi \in I} u_\xi(x)m(x, \xi)(\mathcal{F}_L f)(\xi), \quad f \in C_L^\infty(\overline{M}). \quad (1.9)$$

The $L$-symbol of $A$ is the function $m : \overline{M} \times I \to \mathbb{C}$, and $\mathcal{F}_L f$ is the $L$-Fourier transform of $f$ at $\xi \in I$, which is defined via,

$$\widehat{f}(\xi) \equiv (\mathcal{F}_L f)(\xi) := \int_M f(x)\overline{v_\xi(x)}\,dx.$$
• With the notation of Definition 3.2, every $L$-pseudo-differential operator $A$, can be realised as a pseudo-multiplier of $L$ via (3.4) associating to $A$ a continuous function $\tau_m : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{C}$ interpolating the values of the symbol $m$ of $A$ in the variable $\xi \in \mathcal{I}$, in terms of the spectrum of $|L|$, in such a way that $m(x, \xi) = \tau_m(x, \lambda \xi)$. In Theorem 3.6, we prove that the Hörmander-Mihlin condition,

$$
\|\tau_m\|_{L^1, \mathcal{H}^s} = \sup_{r>0, x \in \mathcal{M}} r^{(s+\tfrac{Q_m}{2})} \langle \cdot \rangle^s \mathcal{F} [\tau_m(x, \cdot) \psi(r^{-1})] \|L^2(\mathbb{R}) < \infty,
$$

with $s$ large enough, and $\psi$ implies that $A \equiv T_m$ defined by (1.9) admits a bounded extension on $L^p(\mathcal{M})$, for all $1 < p < \infty$. This in particular implies that, if $m$ satisfies the Marcinkiewicz type condition

$$
\sup_{x \in \mathcal{M}} |\partial_{\alpha,\beta} \tau_m(x, \omega)| \leq C_{\alpha,\beta}(1 + |\omega|)^{-|\alpha|}, \quad \omega \in \mathbb{R},
$$

the operator $A \equiv T_m$ in (1.9) admits a bounded extension on $L^p(\mathcal{M})$, for all $1 < p < \infty$. Similar conditions are studied in Theorem 3.9 in the $L^\infty(\mathcal{M})$-$BMO(\mathcal{M})$ setting.

• We prove the following Paley-Inequality (see Theorem 4.2): Let $1 < p \leq 2$, and let us assume that

$$
\sup_{\xi \in \mathcal{I}} \left( \sup_{\xi \in \mathcal{I}} \left( \frac{\|v_{\xi}\|_{L^\infty(\mathcal{M})}}{\|u_{\xi}\|_{L^\infty(\mathcal{M})}} \right) \right) < \infty. \quad (1.11)
$$

If $\varphi(\xi)$ is a positive sequence in $\mathcal{I}$ such that

$$
M_{\varphi} := \sup_{t>0} t \sum_{\xi \in \mathcal{I}} \|u_{\xi}\|_{L^\infty(\mathcal{M})}^2
$$

is finite, then for every $f \in L^p(\mathcal{M})$ we have

$$
\left( \sum_{\xi \in \mathcal{I}} |\mathcal{F}_L(f)(\xi)|^p \|u_{\xi}\|_{L^\infty(\mathcal{M})}^{2-p} \varphi(\xi)^{2-p} \right)^{\frac{1}{p}} \lesssim M_{\varphi} \|f\|_{L^p(\mathcal{M})}. \quad (1.12)
$$

• Assuming (1.11), the Hausdorff-Young-Paley inequality (see Theorem 4.6) takes the form,

$$
\left( \sum_{\xi \in \mathcal{I}} \left( |\mathcal{F}_L f(\xi)| \varphi(\xi)^{\frac{1}{p} - \frac{1}{p'}} \right)^{b} \|u_{\xi}\|_{L^\infty(\mathcal{M})}^{1 - \frac{b}{p}} \|v_{\xi}\|_{L^\infty(\mathcal{M})}^{1 - \frac{b}{p'}} \right)^{\frac{1}{p}} \lesssim_{p'} M_{\varphi}^{\frac{1}{p} - \frac{1}{p'}} \|f\|_{L^p(\mathcal{M})}, \quad (1.13)
$$

provided that

$$
M_{\varphi} := \sup_{t>0} t \sum_{\xi \in \mathcal{I}} \|u_{\xi}\|_{L^\infty(\mathcal{M})}^2 < \infty.
$$

• Assuming

$$
\sup_{\xi \in \mathcal{I}} \left( \frac{\|v_{\xi}\|_{L^\infty(\mathcal{M})}}{\|u_{\xi}\|_{L^\infty(\mathcal{M})}} \right) < \infty \quad \text{and} \quad \sup_{\xi \in \mathcal{I}} \left( \frac{\|u_{\xi}\|_{L^\infty(\mathcal{M})}}{\|v_{\xi}\|_{L^\infty(\mathcal{M})}} \right) < \infty, \quad (1.14)
$$
in Theorem 4.10, for $1 < p \leq 2 \leq q < \infty$, we prove that under the weak-$\ell^b$, condition with $\frac{1}{b} = \frac{1}{p} - \frac{1}{q}$,

$$\sup_{s > 0, x \in M} s \left( \sum_{\xi \in I} \max\{ \| u_\xi \|_{L^2(M)}^2, \| v_\xi \|_{L^2(M)}^2 \} \right)^{\frac{1}{p}} < \infty, \quad (1.15)$$

for $|\beta| \leq \rho$, with $\rho$ large enough, the operator $A \equiv T_m : L^p(M) \to L^q(M)$, extends to a bounded linear operator.

Finally, we apply the above $L^p - L^q$ results to the non-linear partial differential equations (PDEs):

- Let us denote by $L^2(M)$ the Hilbert space $L^2$ on $M$. In the nonlinear stationary problem case, we consider the following equation in $L^2(M)$

$$Au = |Bu|^p + f,$$

where $A, B : L^2(M) \to L^2(M)$ and $1 \leq p < \infty$.

- As an example of the application to the nonlinear heat equation, we study the Cauchy problem in the space $L^\infty(0, T; L^2(M))$

$$u_t(t) - |Bu(t)|^p = 0, \quad u(0) = u_0,$$

where $B$ is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.

- In the non-linear wave equation case, we study the following initial value problem (IVP)

$$u_{tt}(t) - b(t)|Bu(t)|^p = 0,$$

$$u(0) = u_0, \quad u_t(0) = u_1,$$

where $b$ is a positive bounded function depending only on time, $B$ is a linear operator in $L^2(M)$ and $1 \leq p < \infty$.

In all of these cases, we establish well-posedness properties of the solutions in the space $L^\infty(0, T; L^2(M))$. We also note that the operators $B$ in our examples have a nature of integro-differential operators.

**Remark 1.1.** Let us observe that for the $n$-torus, $\overline{M} = T^n \equiv [0, 1)^n$, we have that $\partial M = \emptyset$, and if we choose $L = \Delta_{T^n}$ being the Laplacian on the torus, then $\nu_\xi = u_\xi = e_\xi$, where $e_\xi(x) := e^{2\pi i x \cdot \xi}$, $x \in T^n$, $\xi \in \mathcal{I} = \mathbb{Z}^n$. In this case, our main results recover the classical periodic Paley-Inequality,

$$\left( \sum_{\xi \in \mathbb{Z}^n} |\hat{f}(\xi)|^p \varphi(\xi)^{2-p} \right)^{\frac{1}{p}} \lesssim M_{\varphi}^{\frac{2-p}{p}} \| f \|_{L^p(T^n)}, \quad (1.16)$$

and the periodic Hausdorff-Young-Paley inequality

$$\left( \sum_{\xi \in \mathbb{Z}^n} \left( |\hat{f}(\xi)| \varphi(\xi)^{\frac{1}{b} - \frac{1}{p}} \right)^b \right)^{\frac{1}{b}} \lesssim_p M_{\varphi}^{\frac{b}{b-p}} \| f \|_{L^p(T^n)}.$$

$$\left( \sum_{\xi \in \mathbb{Z}^n} \left( |\hat{f}(\xi)| \varphi(\xi)^{\frac{1}{b} - \frac{1}{p}} \right)^b \right)^{\frac{1}{b}} \lesssim_p M_{\varphi}^{\frac{b}{b-p}} \| f \|_{L^p(T^n)}.$$

(1.17)
Observe that the condition (1.15), takes the form

\[
\sup_{x \in \mathbb{T}^n} \sup_{s > 0} s^b \# \{ \xi \in \mathbb{Z}^n : m(x, \xi) \geq s \}^{\frac{1}{b}} := \sup_{s > 0, x \in \mathbb{T}^n} s \left( \sum_{\xi \in \mathbb{Z}^n} \left| \partial^{|\beta|} m(x, \xi) \right| > s \right)^{\frac{1}{b}} < \infty,
\]

for \(|\beta| \leq \lceil n/p \rceil + 1\), which implies that the periodic operator

\[
Af(x) = \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi x \cdot \xi} m(x, \xi) (\mathcal{F}_{\Delta^m} f)(\xi), \quad f \in C^\infty(\mathbb{T}^n),
\]

admits a bounded extension from \(L^p(\mathbb{T}^n)\) into \(L^q(\mathbb{T}^n)\), for \(1 < p \leq 2 \leq q < \infty\), and \(\frac{1}{b} = \frac{1}{p} - \frac{1}{q}\).

**Remark 1.2.** The Hörmander condition for pseudo-multipliers, in particular, associated with the harmonic oscillator on \(M = \mathbb{R}^n\), has been studied in [8] and [7] and references therein. In this work we will generalise such analysis to the case of arbitrary smooth manifolds.

**Remark 1.3.** The periodic Paley-inequality, Hausdorff-Young-Paley inequality, and the \(L^p(\mathbb{T}^n)-L^q(\mathbb{T}^n)\), estimate are known to be sharp. We refer the reader to Littlewood and Paley [22, 23], and Zygmund [38] for details.

**Remark 1.4.** The classical periodic inequalities in Remark 1.1, together with the the \(L^p(\mathbb{T}^n)-L^q(\mathbb{T}^n)\) estimate above, were first extended to the case of compact homogeneous manifolds in the work of Akylzhanov, the third author and Nursultanov [1], by taking \(\overline{M} = M = G/K, \partial M = \emptyset\), with \(G\) being a compact Lie group and \(K\) one of its closed subgroups. The Paley-inequality, the Hausdorff-Young-Paley inequality, and the \(L^p(M)-L^q(M)\) estimates obtained in [1], used the notion of matrix-valued symbol and also a matrix-valued Fourier transform. If we consider the model operator \(L\) being \(L \equiv \mathcal{L}_{G/K}\), that is the lifting of the Laplacian \(\mathcal{L}_G\) on \(G\), to \(M\), the inequalities obtained here are different of the obtained in [1], because we use scalar-valued symbols and a scalar-valued Fourier transform. However they are related in some sense. Recently, in [11] the \(L^p-L^q\) boundedness of spectral multipliers of the anharmonic oscillator has been investigated by Chatzakou and the second author. The anharmonic oscillator can be thought as a self-adjoint prototype for model operator \(L\) when \(M = \mathbb{R}^n\).

**Remark 1.5.** The sharpness of the Paley-inequality on compact homogeneous manifolds was discussed in [1, page 1529], and in particular in the case of \(M = \text{SU}(2)\), with the notion of monotone matrices (see Definition 1.8 of [1]).

**Remark 1.6.** Some results on \(L^p\)-Fourier multipliers in the spirit of the Hörmander-Mihlin theorem, are also known on locally compact groups (see the paper of Akylzhanov and the third author [2]). The classical work of Coifman and Weiss [13] includes the case of the group \(\text{SU}(2)\), the reference [34] for general compact Lie groups, and [17] for graded Lie groups. The case of pseudo-differential operators on compact Lie groups (and also in graded groups) can be found in [9] and [14].
Remark 1.7. If $L$ admits a self-adjoint extension $L^*$ on $L^2(M)$, then we have that $u_\xi = v_\xi$ for every $\xi \in \mathcal{I}$, and the condition (1.11) holds true. In this case, $L \subset L^{**}$, which means that Dom($L$) $\subset$ Dom($L^*$), and for every $f \in$ Dom($L$), $L f = L^* f$. In this privileged situation we have,

$$\sup_{\xi \in \mathcal{I}} \left( \frac{\|u_\xi\|_{L_\infty(M)}}{\|u_\xi\|_{L_\infty(M)}} \right) = \sup_{\xi \in \mathcal{I}} \left( \frac{\|a_\xi\|_{L_\infty(M)}}{\|a_\xi\|_{L_\infty(M)}} \right) = 1. \quad (1.20)$$

Remark 1.8. If $\overline{M}$ is a geodesically complete Riemannian manifold, the $L^\infty$-$\text{BMO}$ boundedness of pseudo-differential operators will be considered in Theorem 3.9.

This work is organised as follows. In Section 2 we present some basics about the non-harmonic analysis developed in [30, 31]. In Section 3, we prove our Hörmander-Mihlin condition and also our Marcinkiewicz type condition. The Paley-inequality, Hausdorff-Young-Paley inequality, and the $L^p$-$L^q$ boundedness of pseudo-differential operators will be investigated in Section 4. Finally, in Section 5, we obtain some applications of our main results. Indeed, we obtain some applications to non-linear PDEs.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \leq cB$ for a suitable constant $c > 0$, where as $A \simeq B$ if $A \leq cB$ and $B \leq dA$, for suitable $c, d > 0$.

2. Preliminaries

Let $\overline{M}$ be a manifold with boundary. This means that the interior of $\overline{M}$, denoted by $M$, is the set of points in $\overline{M}$ which have neighbourhoods homeomorphic to an open subset of $\mathbb{R}^n$. The boundary of $\overline{M}$, denoted $\partial M$, is the complement of $M$ in $\overline{M}$. The boundary points can be characterised as those points which are mapped on the boundary hyperplane of $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ under some coordinate chart. If $M$ is a manifold with boundary of dimension $n$ then $\partial M \neq \emptyset$ is a manifold (without boundary) of dimension $n - 1$. We will assume that $M$ is orientable. This implies the orientability of $\partial M$. So, we assume that $\overline{M}$ is endowed with a density $\overline{dx}$. In practice, we can assume that $\overline{dx}$ is defined by a non-trivial volume form $\overline{dx} = \omega dx_1 \wedge \cdots \wedge dx_n$ on $\overline{M}$. A function $f : \overline{M} \to \mathbb{C}$ is smooth at $x \in M$, if there exists a chart $(\phi, V)$ on $M$, where $V$ is a neighbourhood of $x$, $V \subset M$, and $\phi : V \to W = \phi(V) \subset \mathbb{R}^n$ is a coordinate path, such that the mapping $f \circ \phi^{-1} : \phi(V) \to \mathbb{C}$ is smooth. If $x \in \overline{M} \setminus M = \partial M$, we say that $f : \overline{M} \to \mathbb{C}$ is smooth at $x$, if there exists a chart $(\phi, V)$ on $M$, where $V$ is neighbourhood of $x \in \partial M$, and $\phi : V \to \phi(V) = W \cap (\mathbb{R}^{n-1} \times [0, \infty))$, with $W$ being an open subset of $\mathbb{R}^n$, such that the mapping $f \circ \phi^{-1} : W \cap (\mathbb{R}^{n-1} \times [0, \infty)) \to \mathbb{C}$, is the restriction to $W \cap (\mathbb{R}^{n-1} \times [0, \infty))$ of a smooth map $g : W \to \mathbb{C}$, i.e. $g|_{W \cap (\mathbb{R}^{n-1} \times [0, \infty))} = f$.

We will denote by $C^{\infty}(\overline{M})$ the set of smooth functions $f$ over $\overline{M}$. We will denote by $\partial^\alpha f := \partial_\alpha^\beta g|_W \circ \phi$, the partial derivatives of $f$, defined in local coordinates on $\overline{M}$. We will denote by $L^p(\overline{M})$, $1 \leq p < \infty$, the Lebesgue spaces associated to $\overline{dx}$. For $p = \infty$, $L^\infty(\overline{M})$ denotes the set of essentially $\overline{dx}$-bounded functions.

We will describe some elements involved in the quantization of pseudo-differential operators on manifolds as developed by the third and last author in [30] and [31]. The space

$$C^\infty_{L^k}(\overline{M}) := \cap_{k=1}^\infty \text{Dom}(L^k) \quad (2.1)$$
where Dom($L^k$) := \{ $f \in L^2(M) \mid L^j f \in \text{Dom}(L), j = 0, 1, \cdots, k - 1$\}, so that the boundary condition (BC) are satisfied by the operators $L^j$. The Fréchet topology of $C^\infty_L(M)$ is given by the family of norms

$$\|f\|_{C^k_L} := \max_{j \leq k} \|L^j f\|_{L^2(M)}, \quad k \in \mathbb{N}_0,$$

which consists of all rapidly decreasing functions. Similarly, we define $C^\infty_{L^*}(\overline{M})$ corresponding to the adjoint $L^*$ by

$$C^\infty_{L^*}(\overline{M}) := \bigcap_{k=1}^\infty \text{Dom}((L^*)^k)$$

where Dom$((L^*)^k)$ := \{ $f \in L^2(M) \mid (L^*)^j f \in \text{Dom}(L), j = 0, 1, \cdots, k - 1$\}, which satisfy the adjoint boundary conditions corresponding to the operator $L^*$. The Fréchet topology of $C^\infty_{L^*}(\overline{M})$ is given by the family of norms

$$\|f\|_{C^k_{L^*}} := \max_{j \leq k} \|(L^*)^j f\|_{L^2(M)}, \quad k \in \mathbb{N}_0,$$

Since $\{u_\xi\}$ and $\{v_\xi\}$ are dense in $L^2(M)$ we have that $C^\infty_L(M)$ and $C^\infty_{L^*}(M)$ are dense in $L^2(M)$.

In order to introduce a global definition of the Fourier transform let us introduce the space $\mathcal{S}(\mathcal{I})$, which consists of all rapidly decreasing functions $\phi : \mathcal{I} \to \mathbb{C}$. This means that for any $N \in \mathbb{N}$, there exists a constant $C_{\phi,N}$ such that $|\phi(\xi)| \leq C_{\phi,N} |\xi|^{-N}$ for all $\xi \in \mathcal{I}$. The space $\mathcal{S}(\mathcal{I})$ forms a Fréchet space with the family of semi-norms $p_k(\phi) := \sup_{\xi \in \mathcal{I}} |\xi|^k |\phi(\xi)|$. The $L$-Fourier transform is a bijective homeomorphism $\mathcal{F}_L : C^\infty_L(M) \to \mathcal{S}(\mathcal{I})$ defined by

$$(\mathcal{F}_L f)(\xi) := \hat{f}(\xi) := \int_M f(x) \overline{v_\xi(x)} \, dx. \quad (2.2)$$

The inverse operator $\mathcal{F}_L^{-1} : \mathcal{S}(\mathcal{I}) \to C^\infty_L(M)$ is given by

$$(\mathcal{F}_L^{-1} h)(x) := \sum_{\xi \in \mathcal{I}} h(\xi) u_\xi(x)$$

so that the Fourier inversion formula is given by

$$f(x) = \sum_{\xi \in \mathcal{I}} \hat{f}(\xi) u_\xi(x), \quad f \in C^\infty_L(M). \quad (2.3)$$

Similarly, the $L^*$-Fourier transform is a bijective homeomorphism $\mathcal{F}_{L^*} : C^\infty_{L^*}(\overline{M}) \to \mathcal{S}(\mathcal{I})$ defined by

$$(\mathcal{F}_{L^*} f)(\xi) := \hat{f}_*(\xi) := \int_M f(x) \overline{u_\xi(x)} \, dx. \quad (\text{where } \hat{f}_* := \mathcal{F}_{L^*} f)$$

Its inverse $\mathcal{F}_{L^*}^{-1} : \mathcal{S}(\mathcal{I}) \to C^\infty_{L^*}(M)$ is given by $$(\mathcal{F}_{L^*}^{-1} h)(x) := \sum_{\xi \in \mathcal{I}} h(\xi) v_\xi(x)$$ so that the conjugate Fourier inversion formula is given by

$$f(x) = \sum_{\xi \in \mathcal{I}} \hat{f}_*(\xi) v_\xi(x), \quad f \in C^\infty_{L^*}(M). \quad (2.4)$$
The space $\mathcal{D}'(M) := \mathcal{L}(C^\infty_c(\overline{M}, \mathbb{C}))$ of linear continuous functionals on $C^\infty_c(\overline{M})$ is called the space of $L$-distributions. By dualising the inverse $L$-Fourier transform $\mathcal{F}_L^{-1} : \mathcal{S}(\mathcal{I}) \to C^\infty_c(\overline{M})$, the $L$-Fourier transform extends uniquely to the mapping

$$\mathcal{F}_L : \mathcal{D}'(M) \to \mathcal{S}'(\mathcal{I})$$

by the formula $\langle \mathcal{F}_L w, \phi \rangle := \langle w, \mathcal{F}_L^{-1} \phi \rangle$ with $w \in \mathcal{D}'(M)$, $\phi \in \mathcal{S}(\mathcal{I})$. The space $l^2_L := \mathcal{F}_L(L^2(\overline{M}))$ is defined as the image of $L^2(\overline{M})$ under the $L$-Fourier transform. Then the space of $l^2_L$ consists of the sequences of the Fourier coefficients of function in $L^2(\overline{M})$, in which Plancherel identity holds, for $a, b \in l^2_L$,

$$(a, b)_{l^2_L} := \sum_{\xi \in \mathcal{I}} a(\xi)(\mathcal{F}_L \circ \mathcal{F}_L^{-1}b(\xi)).$$  \hspace{1cm} (2.5)

For $f \in \mathcal{D}'(M) \cap \mathcal{D}'_c(\mathcal{I})$ and $s \in \mathbb{R}$, we say that $f \in \mathcal{H}^s_L(M)$ if and only if $\langle \xi \rangle^s \hat{f}(\xi) \in l^2_L$,

provided with the norm

$$\|f\|_{\mathcal{H}^s_L} := \left( \sum_{\xi \in \mathcal{I}} \langle \xi \rangle^{2s} \hat{f}(\xi) \overline{\hat{f}}(\xi) \right)^{1/2}.$$

Now, we will present the definition of global pseudo-differential operator as developed in \cite{30}. If $m : \overline{M} \times \mathcal{I} \to \mathbb{C}$ is a smooth function, which means that $m(\cdot, \xi) \in C^\infty_c(\overline{M})$, for every $\xi \in \mathcal{I}$, the pseudo-differential operator associated to $m$, is defined by

$$Af(x) = \sum_{\xi \in \mathcal{I}} u_\xi(x)m(x, \xi)\hat{f}(\xi), \ f \in \text{Dom}(A).$$  \hspace{1cm} (2.6)

In those cases where $A : C^\infty_c(\overline{M}) \to C^\infty_c(\overline{M})$ is a continuous linear operator with symbol $\sigma : \mathcal{I} \to \mathbb{C}$, that does not depends on $x \in \overline{M}$, we say that $A$ is a $L$-Fourier multiplier. Indeed, such operators satisfy the identity

$$\mathcal{F}_L(Af)(\xi) = \sigma(\xi)\mathcal{F}_L(f)(\xi)$$

for every $f \in C^\infty_c(\overline{M})$ and for every $\xi \in \mathcal{I}$.

### 3. $L^p-L^p$ BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS

#### 3.1. Hörmander-Mihlin condition for pseudo-differential operators.

In this section we investigate the $L^p$-boundedness of global pseudo-differential operators on a manifold $\overline{M} = M \cup \partial M$, where $M$ is the interior of $\overline{M}$ and $\partial M$ is its boundary. We will denote by $L^\circ$ the densely defined operator given by

$$L^\circ u_\xi = \lambda_\xi u_\xi, \quad \xi \in \mathcal{I}.$$  

The results presented here also allow the case $\partial M = \emptyset$. We will assume the following facts,
HMII: there exist $-\infty < \gamma^{(1)}_p, \gamma^{(2)}_p < \infty$, satisfying
\[
\|u_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma^{(1)}_p}, \quad \|v_\xi\|_{L^p(M)} \lesssim |\lambda_\xi|^{\gamma^{(2)}_p}, \quad 1 \leq p \leq \infty. \tag{3.1}
\]
HMII: The operator $\sqrt{L^\circ L}$ satisfies the Weyl-eigenvalue counting formula
\[
N(\lambda) := \sum_{\xi \in I: |\lambda_\xi| \leq \lambda} = O(\lambda^{Q'}), \quad \lambda \to \infty,
\]
where $Q > 0$. If $Q' > Q$, then $N(\lambda) = O(\lambda^{Q'})$, $\lambda \to \infty$, so that we assume that $Q$ is the smallest real number satisfying (3.2).

Remark 3.1. The first assumption (HMII) means that the $L^p$-norms of the biorthonormal system $\{u_\xi\}_{\xi \in I}$ and $\{v_\xi\}_{\xi \in I}$ growth polynomially, while (HMIII) assures that we have a suitable control on the spectrum of $L$. If $M$ is a closed manifold and $L$ is an elliptic self-adjoint and positive pseudo-differential operator, is known that in (3.2), $Q = \dim(M)$. Other kind of operators appear for example when $L$ is the positive sub-Laplacian on a closed manifold $M$, in this case (3.2) holds with $Q$ being the Hausdorff dimension associated to the Carnot-Carathéodory distance associated with $L$.

We observe that $\gamma^{(1)}_2 = \gamma^{(2)}_2 = 0$ in view that the functions $u_\xi$ are considered with $L^2(M)$-norm normalised. We will denote
\[
\gamma_p := \gamma^{(1)}_p + \gamma^{(2)}_p. \quad \tag{3.3}
\]

Now, we will precise the kind of pseudo-differential that we will analyse in this section. We will refer to them as pseudo-multipliers. We will define it as follows.

Definition 3.2. Let $A : C^\infty(L^\circ L) \to C^\infty(L^\circ L)$ be a continuous linear operator defined as in (2.6). We say that the pseudo-differential operator $A$ is a pseudo-multiplier associated with $L$ (pseudo-multiplier for short), if there exists a continuous function $\tau_m : M \times \mathbb{R} \to \mathbb{C}$, such that for every $\xi \in I$, and $x \in M$, we have $m(x, \xi) = \tau_m(x, |\lambda_\xi|)$.

In this case, we say that $A$ is the pseudo-multiplier associated with $\tau_m$. Clearly,
\[
Af(x) \equiv \tau_m(x, \sqrt{L^\circ L})f(x) := \sum_{\xi \in I} u_\xi(x)\tau_m(x, |\lambda_\xi|)\tilde{f}(\xi), \tag{3.4}
\]
for all $f \in C^\infty(L^\circ L)$.

Remark 3.3. There is a one to one correspondence between pseudo-differential operators mapping $C^\infty(L^\circ L)$ into itself and pseudo-multipliers. Indeed, starting with a pseudo-multiplier defined by (3.4), we can associate to it a symbol via $m(x, \xi) := \tau_m(x, |\lambda_\xi|)$, and viceversa, starting with a pseudo-differential operator defined by (2.6), we can define for every $\lambda_\xi$, $\tau'_m(x, |\lambda_\xi|) := m(x, \xi)$, and after that we can interpolate $\{\tau'_m(x, |\lambda_\xi|)\}_{x \in M, \xi \in I}$, with a continuous function $\tau_m : M \times \mathbb{R} \to \mathbb{C}$, in such a way that
\[
\tau_m|_{\mathbb{R} \times \{\lambda_\xi\}_{\xi \in I}} = \{\tau'_m(x, |\lambda_\xi|)\}_{x \in M, \xi \in I} = \{m(x, \xi)\}_{x \in M, \xi \in I}.
\]

Remark 3.4. The approach in proving the $L^p$-estimates for this section comes from starting with a function $\tau_m : M \times \mathbb{R} \to \mathbb{C}$, satisfying the Hörmander condition
\[
\|\tau_m\|_{L^\infty(M)} = \sup_{r > 0, x \in M} r^{(s - \frac{d}{2p})}\|\langle \cdot \rangle^s \mathcal{F}[\tau_m(x, \cdot)\psi(r^{-1} \cdot)]\|_{L^2(\mathbb{R})} < \infty, \tag{3.5}
\]
where $Q_m \in \mathbb{R}$, and later we consider for such a function $\tau_m$, the pseudo-differential operator $T_m$, with symbol $\tau_m|_{\mathbb{Z}^d \times \{ |\lambda| \} \in \xi \in \mathbb{I}} = \{ m(x, \xi) \}|_{(x, \xi) \in \mathbb{Z}^d \times \mathbb{I}}$ obtained from the restriction of $\tau_m : \mathbb{M} \times \mathbb{R} \rightarrow \mathbb{C}$, to the set $\mathbb{M} \times \{ |\lambda| \} \in \mathbb{I}$. Because there are infinite continuous extensions $\tau_m$ for $m$, the Hörmander Mihlin condition depends on the extension $\tau_m$ under consideration. In practice, however, we can start with a function $\tau : \mathbb{M} \times \mathbb{R} \rightarrow \mathbb{C}$ satisfying (3.5) (with $\tau$ instead of $\tau_m$) and we can consider the pseudo-multiplier associated to $\tau$ which defines a pseudo-differential operator bounded on $L^p(\mathbb{M})$, (for $s$ large enough). Important examples of pseudo-multipliers, are the spectral multipliers of $\sqrt{L^pL}$ which are defined by

$$
\tau(\sqrt{L^pL})f(x) := \sum_{\xi \in \mathbb{I}} u_\xi(x)\tau(|\lambda\xi|)\hat{f}(\xi),
$$

for all $f \in C^\infty_c(\mathbb{M})$. Of particular interest are the functions of positive elliptic operators $E$, $\tau(E)$ on a closed manifold, satisfying estimates of the type $|\partial^j_t \tau(t)| \lesssim (1 + t)^{-\rho|j|}$, $\rho > 0$, (see e.g. [12] and references therein). The prototype in this situation is the positive Laplacian $\Delta_{(M,g)}$ on a closed Riemannian manifold $(M, g)$.

**Remark 3.5.** We summarise the assumptions of this section keeping in mind that if we know how the spectrum of $\sqrt{L^pL}$ behaves (in the form of (HMII)), if we can estimate polynomially the $L^p$-norms of the eigenfunctions, and we encode the symbol of a pseudo-differential operator $A$, $m$ in terms of the function $\tau_m$, we expect to provide information on the boundedness of $A$, on $L^p(\mathbb{M})$, (or from $L^\infty(\mathbb{M})$ to $\text{BMO}(\mathbb{M})$), by using conditions of Hörmander Mihlin type on $\tau_m$. One reason for this is that $\mathbb{M} \times \text{Spectrum}(\sqrt{L^pL})$ is contained in the domain of $\tau_m$.

### 3.2. $L^p$-boundedness of pseudo-multipliers of $L$

In this section we prove the Hörmander-Mihlin theorem for operators on a manifold $\mathbb{M}$, possibly with $\partial M \neq \emptyset$, allowing also the case $\partial M = \emptyset$.

**Theorem 3.6.** Let $\mathbb{M}$ be a smooth manifold with boundary and let $A : C^\infty_c(\mathbb{M}) \rightarrow C^\infty_c(\mathbb{M})$ be the pseudo-multiplier defined in (3.4). Let us assume that $\tau_m$ satisfies the following Hörmander condition,

$$
\|\tau_m\|_{l.u., H^s} = \sup_{r > 0, x \in \mathbb{M}} r^{(s - Q_m)}\|(|\cdot|)^s\mathcal{F}[\tau_m(x, \cdot)|_{\tau_m(r^{-1}, \cdot)}]\|_{L^2(\mathbb{R})} < \infty,
$$

for $s > \max\{1/2, \gamma_p + Q + (Q_m/2)\}$. Then $A \equiv T_m : L^p(\mathbb{M}) \rightarrow L^p(\mathbb{M})$ extends to a bounded linear operator for all $1 < p < \infty$.

**Proof.** We choose a function $\psi_0 \in C^\infty_c(\mathbb{R})$, $\psi_0(\lambda) = 1$, if $|\lambda| \leq 1$, and $\psi(\lambda) = 0$, for $|\lambda| \geq 2$. For every $j \geq 1$, let us define $\psi_j(\lambda) = \psi_0(2^{-j+1}\lambda) - \psi_0(2^{-j-1}\lambda)$. Then we have

$$
\sum_{l \in \mathbb{N}_0} \psi_l(\lambda) = 1, \text{ for every } \lambda > 0.
$$

Let us consider $f \in C^\infty_c(\mathbb{M})$. We will decompose the function $m$ as

$$
\tau_m(x, |\lambda\xi|) = \tau_m(x, |\lambda\xi|)(\psi_0(|\lambda\xi|) + \psi_1(|\lambda\xi|)) \sum_{k=2}^\infty m_k(x, \xi),
$$

(3.9)
where
\[ m_k(x, \xi) := \tau_m(x, |\lambda \xi|) \cdot \psi_k(|\lambda \xi|). \]

Let us define the sequence of pseudo-differential operators \( T_{m_j}, j \in \mathbb{N} \), associated to every symbol \( m_j \), for \( j \geq 2 \), and by \( T_0 \) the operator with symbol

\[ \sigma \equiv \tau_m(x, |\lambda \xi|)(\psi_0(|\lambda \xi|) + \psi_1(|\lambda \xi|)). \]

Then we want to show that the operator series

\[ T_0 + S_m, \quad S_m := \sum_k T_{m_k}, \quad (3.10) \]

satisfies,

\[ \|T_m\|_{\mathcal{B}(L^p(M))} \leq \|T_0\|_{\mathcal{B}(L^p(M))} + \sum_k \|T_{m_k}\|_{\mathcal{B}(L^p(M))}, \quad (3.11) \]

where the series in the right hand side converges. So, we want to estimate every norm \( \|T_{m_j}\|_{\mathcal{B}(L^p(M))} \). For this, we will use the fact that for \( f \in C_0^\infty(M) \),

\[ \|T_{m_j}f\|_{L^p(M)} = \sup\{|(T_{m_j}f, g)|_{L^2(M)} : \|g\|_{L^{p'}}(M) = 1\}. \quad (3.12) \]

In fact, for \( f \) and \( g \) as above we have

\[
(T_{m_k}f, g)_{L^2(M)} = \int_M T_{m_k}f(x)g(x)dx
\]

\[
= \int_M \sum_{2^k \leq |\lambda \xi| < 2^{k+1}} m(x, \xi) \hat{f}(\xi)u_\xi(x)g(x)dx
\]

\[
= \int_M \int_M \sum_{2^k \leq |\lambda \xi| < 2^{k+1}} m(x, \xi) f(y)u_\xi(x)\bar{\psi}(y)g(x)dydx.
\]

Now, in order use that \( \tau_m \) satisfies the Hörmander condition, we will use the Euclidean Fourier transform. Indeed, for every \( x \in M \) let us denote the inverse Euclidean Fourier transform of the function

\[ \tau_m(x, \cdot)\psi(2^{-k} \cdot) : \omega \mapsto \tau_m(x, \omega)\psi(2^{-k} \omega), \]

by \( \mathcal{F}^{-1}[\tau_m(x, \cdot)\psi(2^{-k} \cdot)]. \) So, for every \( \xi \in \mathbb{Z}, \omega = |\lambda \xi| \in \mathbb{R}, \) and we have

\[ m_k(x, \xi) := \tau_m(x, |\lambda \xi|)\psi(2^{-k}|\lambda \xi|) \]

\[ = \mathcal{F}^{-1}(\mathcal{F}[\tau_m(x, \cdot)\psi(2^{-k} \cdot)])(\xi) = \int_{\mathbb{R}} \mathcal{F}[\tau_m(x, \cdot)\psi(2^{-k} \cdot)](z)e^{2\pi i|\lambda \xi|z}dz. \]

Consequently,

\[
|\langle T_{m_k}f, g \rangle_{L^2(M)}| \\
\leq \sum_{2^k \leq |\lambda \xi| < 2^{k+1}} \sup_{x \in M} \int_{\mathbb{R}} |\mathcal{F}[\tau_m(x, \cdot)\psi(2^{-k} \cdot)](z)|dz \\
\times \|f\|_{L^p} \|g\|_{L^{p'}} \|u_\xi\|_{L^p} \|\psi\|_{L^{p'}} \\
\lesssim \sum_{2^k \leq |\lambda \xi| < 2^{k+1}} \sup_{x \in M} \int_{\mathbb{R}} |\mathcal{F}[\tau_m(x, \cdot)\psi(2^{-k} \cdot)](z)|dz
\]
Because following Marcinkiewicz type condition pseudo-differential operators (defined in (3.7)): Let \( L \) be a smooth manifold with boundary and let \( \tau_m \) satisfies the following Marcinkiewicz type condition

\[
\sup_{x \in \overline{M}} | \partial^\alpha_\tau \tau_m(x, \omega) | \leq C_{\alpha, \beta} (1 + |\omega|)^{-|\alpha|}, \quad (x, \omega) \in \overline{M} \times \mathbb{R},
\]

we get the inequality

\[
\sum_{2^k \leq |\lambda| < 2^{k+1}} \| \tau_m \|_{L^p(\mathbb{R})} \leq \sum_{2^k \leq |\lambda| < 2^{k+1}} \| \tau_m \|_{L^p(\mathbb{R})} \cdot 2^{-k(s-Q)} |\lambda|^{-Q}.
\]

So, we can estimate the operator norm of \( T_{m_k} \) by

\[
\| T_{m_k} \|_{\mathcal{B}(L^p)} \leq \sum_{2^k \leq |\lambda| < 2^{k+1}} \| \tau_m \|_{L^p(\mathbb{R})} \cdot 2^{-k(s-Q)} |\lambda|^{-Q}.
\]

Since

\[
\| T_0 f \|_{L^p(\overline{M})} \leq \| m(\cdot, 0) \|_{L^\infty(\overline{M})} \| f \|_{L^p(\overline{M})},
\]

we have the boundedness of \( T_0 \) on \( L^p \). It is clear that if we want to end the proof, we need to estimate \( I := \sum_{k \geq 0} \| T_{m_k} \|_{\mathcal{B}(L^p(\overline{M}))} \). Consequently, we obtain

\[
0 < I \leq \| T_0 \|_{\mathcal{B}(L^p)} + \sum_{k=1}^{\infty} 2^{-k(s-Q-\frac{Qm}{2})} \| \tau_m \|_{L^p(\mathbb{R})} < \infty,
\]

for \( s > Q + \frac{Qm}{2} + \gamma_p \). So, we have

\[
\| T_m \|_{\mathcal{B}(L^p)} \leq C(\| \tau_m \|_{L^p(\mathbb{R})} + \| m \|_{L^\infty}).
\]

The proof is complete. \( \square \)

As an application of the Hörmander-Mihlin theorem proved above, we will prove that the following Marcinkiewicz type condition also implies the \( L^p \) boundedness of pseudo-differential operators (defined in (1.9)).

**Theorem 3.7.** Let \( \overline{M} \) be a smooth manifold with boundary and let \( A : C_0^\infty(\overline{M}) \rightarrow C_0^\infty(\overline{M}) \) be the pseudo-multiplier defined in (3.4). Let us assume that \( \tau_m \) satisfies the following Marcinkiewicz type condition

\[
\sup_{x \in \overline{M}} | \partial^\alpha_\tau \tau_m(x, \omega) | \leq C_{\alpha, \beta} (1 + |\omega|)^{-|\alpha|}, \quad (x, \omega) \in \overline{M} \times \mathbb{R},
\]
for $|\alpha| \leq \rho$, where $\rho \in \mathbb{N}$, and $\rho > \max\{1/2, \gamma_p + Q + (1/2)\}$. Then $A \equiv T_m : L^p(M) \to L^p(M)$ extends to a bounded linear operator for all $1 < p < \infty$.

**Proof.** For the proof, we will use that the Sobolev space $H^s(\mathbb{R})$ defined by those functions $g$ satisfying $\|g\|_{H^s(\mathbb{R})} := \|(z)^s(\mathcal{F} g)\|_{L^2(\mathbb{R})} < \infty$, has the equivalent norm

$$\|g\|_{H^s(\mathbb{R})} := \sum_{|\beta| \leq s} \|\partial^\beta g\|_{L^2(\mathbb{R})}, \quad (3.15)$$

when $s$ is an integer (see, e.g. [16], p. 163). We will show that

$$\sup_{k > 0, x \in M} 2^{k(\rho - \frac{1}{2})} \|\tau_m(x, \cdot)\psi(2^{-k} \cdot)\|_{H^\rho} = \sup_{k > 0, x \in M} \|\tau_m(x, 2^k \cdot)\psi(\cdot)\|_{H^\rho} < \infty, \quad (3.16)$$

provided that $\rho$ is an integer. From the estimate

$$\|\tau_m(x, 2^k \cdot)\psi(\cdot)\|_{H^\rho} \asymp \|\tau_m(x, 2^k \cdot)\psi(\cdot)\|_{H^\rho} = \sum_{|\beta| \leq \rho} \|\partial^\beta (\tau_m(x, 2^k \cdot)\psi(\cdot))\|_{L^2(\mathbb{R})}, \quad (3.17)$$

we will estimate the $L^2$-norms of the derivatives $\partial^\beta (\tau_m(x, 2^k \cdot)\psi(\cdot))(\xi)$. By the Leibniz rule we have

$$\partial^\beta (\tau_m(x, 2^k \xi)\psi(\xi)) = \sum_{|\alpha| \leq |\beta|} 2^{k|\alpha|}(\partial^\alpha \tau_m)(x, 2^k \xi)\partial^{\beta - \alpha} \psi(\xi).$$

So, we obtain

$$\|\partial^\beta (\tau_m(x, 2^k \cdot)\psi(\cdot))\|_{L^2} \leq \sum_{|\alpha| \leq |\beta|} C_\alpha \|\partial^{\beta - \alpha} \psi(\cdot)\|_{L^2}, \quad (3.18)$$

where we have used that $(3.14)$ implies the estimate $|2^{k|\alpha|}(\partial^\alpha \tau_m)(x, 2^k \cdot)| \leq C_\alpha$, for $k$ large enough. Now, $(3.16)$ follows by summing both sides of $(3.18)$ over $|\beta| \leq \rho$. Thus, if we use Theorem 3.6 with $Q_m/2 = 1/2$ and $s = \rho$, we finish the proof because the condition $(3.14)$ implies that $(3.7)$ holds true and consequently we obtain the boundedness of $A$ on $L^p(M)$. \hfill \square

3.3. $L^\infty$-BMO boundedness for pseudo-differential operators. Next, we will study the $L^\infty(\overline{M})$-BMO($\overline{M}$) boundedness for pseudo-differential operators on compact manifolds with boundary. In this subsection assume that $(M, g)$ is a geodesically complete Riemannian manifold. So, let us fix the geodesical geodesic distance $d(\cdot, \cdot)$ on $M$. Under the condition that $(M, g)$ is geodesically complete we can assure that every point in the boundary $\partial M$ can be connected with other points in $M$ using a geodesic path. This allows us to define balls on the boundary using the geodesic distance $d(\cdot, \cdot)$ defined by the Riemannian metric $g$ (see e.g. Pigola and Veronelli [29]).

The ball of radius $r > 0$, is defined as

$$B(x, r) = \{y \in \overline{M} : d(x, y) < r\}.$$

Then the BMO space on $\overline{M}$, $BMO(\overline{M})$, is the space of locally integrable functions $f$ satisfying

$$\|f\|_{BMO(\overline{M})} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where $f_B := \frac{1}{|B|} \int_B f(x) dx$. 
and $B$ ranges over all balls $B(x_0, r)$, with $(x_0, r) \in \overline{M} \times (0, \infty)$.

**Remark 3.8.** If $(\overline{M}, g)$ is a Riemannian metric and with the geodesic distance $d(\cdot, \cdot)$, $(\overline{M}, d)$ is a complete metric space, then $(\overline{M}, g)$ is geodesically complete (see e.g. Theorem A and Corollary B of Pigola and Veronelli [29]).

The Hardy space $H^1(\overline{M})$ will be defined via the atomic decomposition. Thus, $f \in H^1(\overline{M})$ if and only if $f$ can be expressed as $f = \sum_{j=1}^{\infty} c_j a_j$, where $\{c_j\}_{j=1}^{\infty}$ is a sequence in $l^1(\mathbb{N})$, and every function $a_j$ is an atom, i.e., $a_j$ is supported in some ball $B = B_j$, $\int_{B_j} a_j(x) dx = 0$, and

$$
\|a_j\|_{L^\infty(G)} \leq \frac{1}{|B_j|}.
$$

The norm $\|f\|_{H^1(\overline{M})}$ is the infimum over all possible series $\sum_{j=1}^{\infty} |c_j|$. Furthermore, if $dx$ satisfies the doubling property, the space $BMO(\overline{M})$ is the dual of $H^1(\overline{M})$, which can be deduced from the general work on complete metric spaces with the doubling property due to Carbonaro, Mauceri, and Meda [6].

(a). If $\phi \in BMO(\overline{M})$, then $\Phi : f \mapsto \int_{\overline{M}} f(x) \phi(x) dx$, admits a bounded extension on $H^1(\overline{M})$.

(b). Conversely, every continuous linear functional $\Phi$ on $H^1(\overline{M})$ arises as in (a) with a unique element $\phi \in BMO(\overline{M})$.

$$
\|f\|_{BMO(\overline{M})} = \sup_{\|g\|_{H^1} = 1} \left| \int_{\overline{M}} f(x) g(x) dx \right|, \quad \|g\|_{H^1} = \sup_{\|f\|_{BMO} = 1} \left| \int_{\overline{M}} f(x) g(x) dx \right|.
$$

So, the $L^\infty$-$BMO$ boundedness for pseudo-differential operators is considered as follows.

**Theorem 3.9.** Let $\overline{M}$ be a geodesically complete Riemannian manifold with (possibly empty) boundary $\partial M$, and let $A : C^\infty_0(\overline{M}) \to C^\infty_0(\overline{M})$ be the pseudo-multiplier defined in (3.4). Let us assume that one of the following two conditions hold.

1. $\|\tau_m\|_{L^\infty(\overline{M})} = \sup_{r > 0, x \in \overline{M}} r^{(s-\frac{Q_m}{2})} \| \mathcal{F}[\tau_m(x, \cdot) \psi(\cdot r^{-1})] \|_{L^2(\mathbb{R})} < \infty$,

   for $s > \max\{1/2, (Q_m/2) + Q + \gamma_\infty\}$.

2. $\sup_{x \in \overline{M}} |\partial_{\omega}^\alpha \tau_m(x, \omega)| \leq C_{\alpha, \beta} (1 + |\omega|)^{-|\alpha|}$, $(x, \omega) \in \overline{M} \times \mathbb{R}$,

   for all $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \leq \rho$, where $\rho \in \mathbb{N}$, and $\rho > \max\{1/2, (Q_m/2) + Q + \gamma_\infty\}$.

Then, $A \equiv T_m : L^\infty(\overline{M}) \to BMO(\overline{M})$ extends to a bounded operator.

**Proof.** Let us assume that $\tau_m$ satisfies (3.20). This is the relevant assumption, because in Theorem 3.7, we have proved that a function satisfying (3.21) also satisfies (3.20). Let us consider $f \in L^\infty(\overline{M})$. Similar as in Theorem 3.6, we choose a function $\psi_0 \in$
Consequently, for some $\Omega$ where we have denoted $C_{0}^{\infty}(\mathbb{R})$, $\psi_{0}(\lambda) = 1$, if $|\lambda| \leq 1$, and $\psi(\lambda) = 0$, for $|\lambda| \geq 2$. For every $j \geq 1$, let us define $\psi_{j}(\lambda) = \psi_{0}(2^{-j}\lambda) - \psi_{0}(2^{-j+1}\lambda)$. Then we have
\[
\sum_{l \in \mathbb{N}_{0}} \psi_{l}(\lambda) = 1, \quad \text{for every } \lambda > 0. \tag{3.22}
\]
We will decompose the symbol $\tau_{m}$ as
\[
\tau_{m}(x, |\lambda\xi|) = \tau_{m}(x, |\lambda\xi|)(\psi_{0}(|\lambda\xi|) + \psi_{1}(|\lambda\xi|)) + \sum_{k=2}^{\infty} m_{k}(x, \xi) \tag{3.23}
\]
where we have denoted
\[
m_{k}(x, \xi) := \tau_{m}(x, |\lambda\xi|) \cdot \psi_{k}(|\lambda\xi|).
\]
Let us define the sequence of pseudo-differential operators $T_{m_{j}}$, $j \in \mathbb{N}$, associated to every symbol $m_{j}$, for $j \geq 2$, and by $T_{0}$ the operator with symbol
\[
\sigma \equiv \tau_{m}(x, |\lambda\xi|)(\psi_{0}(|\lambda\xi|) + \psi_{1}(|\lambda\xi|)).
\]
Because, $f \in L^{\infty}(\mathbb{M})$ and for every $j$, $T_{m_{j}}$ has symbol with compact support in the $\xi$-variable, $T_{m_{j}} : L^{\infty}(\mathbb{M}) \to L^{\infty}(\mathbb{M})$ is bounded, and consequently $T_{m_{j}}f \in L^{\infty}(\mathbb{M}) \subset BMO(\mathbb{M})$. Now, because $T_{m_{j}}f \in BMO(\mathbb{M})$, we will estimate its $BMO$-norm $\|T_{m_{j}}f\|_{BMO(\mathbb{M})}$. By using that every symbol $m_{k}$ has variable $\xi$ supported in $\{\xi \in \mathcal{I} : 2^{k-1} \leq |\lambda\xi| \leq 2^{k+1}\}$, we have
\[
T_{m_{k}}f(x) = \sum_{2^{k-1} \leq |\lambda\xi| \leq 2^{k+1}} m_{k}(x, \xi)u_{\xi}(x)\hat{f}(\xi), \quad x \in \mathbb{M}.
\]
Consequently,
\[
\|T_{m_{k}}f\|_{BMO(\mathbb{M})} \leq \sum_{2^{k-1} \leq |\lambda\xi| \leq 2^{k+1}} \|m_{k}(\cdot, \xi)u_{\xi}(\cdot)\|_{BMO(\mathbb{M})}\hat{f}(\xi). \tag{3.24}
\]
From (3.19) and by using the Euclidean Fourier inversion formula applied to $\tau_{m_{k}}(x, \cdot) := \tau_{m}(x, \cdot) \cdot \psi_{k}(\cdot)$ we have,
\[
\|m_{k}(\cdot, \xi)u_{\xi}(\cdot)\|_{BMO(\mathbb{M})} =\sup_{\Omega_{0} \in H^{1}(\mathbb{M})} \left| \int_{\mathbb{M}} m_{k}(x, \xi)u_{\xi}(x)\Omega_{0}(x)dx \right|
\]
\[
= \left| \int_{\mathbb{M}} \int_{\mathbb{R}} e^{i2\pi|\lambda\xi|z} \bar{\tau}_{m_{k}}(x, z)dz u_{\xi}(x)\Omega(x)dx \right|
\]
\[
\leq \sup_{x \in \mathbb{M}} \int_{\mathbb{R}} |\bar{\tau}_{m_{k}}(x, z)||dz \times \int_{\mathbb{M}} |u_{\xi}(x)||\Omega(x)|dx,
\]
for some $\Omega \in H^{1}(\mathbb{M})$, such that $\|\Omega\|_{H^{1}} = 1$. Let us note that, for every $\varepsilon > 0$, there exists a decomposition of $\Omega$ given by
\[
\Omega = \sum_{j=1}^{\infty} c_{j}a_{j},
\]
where \( \{c_j\}_{j=1}^{\infty} \) is a sequence in \( \ell^1(\mathbb{N}) \), and every function \( a_j \) is an atom, i.e., \( a_j \) is supported in some ball \( B = B_j \), satisfying the cancellation property: \( \int_{B_j} a_j(x) \, dx = 0 \), with

\[
\|a_j\|_{L^\infty(B)} \leq \frac{1}{|B_j|},
\]
and

\[
\|\Omega\|_{H^1(\mathcal{M})} = 1 \leq \sum_{j=1}^{\infty} |c_j| < 1 + \varepsilon.
\]

Observe that

\[
\int_{\mathcal{M}} |u_{\xi}(x)| |\Omega(x)| \, dx \leq \sum_{j=1}^{\infty} |c_j| \|u_{\xi}\|_{L^\infty(\mathcal{M})} \int_{\mathcal{M}} |a_j(x)| \, dx = \sum_{j=1}^{\infty} |c_j| \|u_{\xi}\|_{L^\infty(\mathcal{M})} \int_{B_j} |a_j(x)| \, dx
\]

\[
\leq \sum_{j=1}^{\infty} |c_j| \|u_{\xi}\|_{L^\infty(\mathcal{M})} \|a_j\|_{L^\infty(\mathcal{M})} |B_j|
\]

\[
\leq (1 + \varepsilon) \|u_{\xi}\|_{L^\infty(\mathcal{M})}.
\]

By the Cauchy-Schwarz inequality, and the condition \( s > 1/2 \), we have

\[
\int_{\mathbb{R}} |\hat{\tau}_{mk}(x, z)| \, dz \leq \left( \int_{\mathbb{R}} (z)^{2s} |\hat{\tau}_{mk}(x, z)|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}} (z)^{-2s} \, dz \right)^{\frac{1}{2}}. \tag{3.25}
\]

Consequently, we claim that

\[
\int_{\mathbb{R}} |\hat{\tau}_{mk}(x, z)| \, dz \lesssim \|\tau_m\|_{l.u., \mathcal{H}^s} \times 2^{-k(s - \frac{Q_m}{2})}. \tag{3.26}
\]

Indeed,

\[
\int_{\mathbb{R}} |\hat{\tau}_{mk}(x, z)| \, dz \lesssim \|\tau_{mk}(x, \cdot)\|_{\mathcal{H}^s(\mathbb{R})} = \|\tau_{m}(\cdot) \psi(2^{-k} \cdot)\|_{\mathcal{H}^s(\mathbb{R})}
\]

\[
\lesssim \|\tau_m\|_{l.u., \mathcal{H}^s} \times 2^{-k(s - \frac{Q_m}{2})}.
\]

So, we obtain

\[
\|m_k(\cdot, \xi) u_{\xi}(\cdot)\|_{BMO(\mathcal{M})} \leq \|\tau_{m}\|_{l.u., \mathcal{H}^s} \times 2^{-k(s - \frac{Q_m}{2})} \times \int_{\mathcal{M}} |u_{\xi}(x)| |\Omega(x)| \, dx
\]

\[
\leq \|\tau_{m}\|_{l.u., \mathcal{H}^s} \times 2^{-k(s - \frac{Q_m}{2})} (1 + \varepsilon) \|u_{\xi}\|_{L^\infty(\mathcal{M})}.
\]

Thus, we can write

\[
\|T_{mk} f\|_{BMO(\mathcal{M})} \leq \sum_{2^{k-1} \leq |\lambda| \leq 2^{k+1}} \|\tau_{m}\|_{l.u., \mathcal{H}^s} 2^{-k(s - \frac{Q_m}{2})} \|u_{\xi}\|_{L^\infty(\mathcal{M})} |\hat{f}(\xi)|
\]

\[
\leq \sum_{2^{k-1} \leq |\lambda| \leq 2^{k+1}} \|\tau_{m}\|_{l.u., \mathcal{H}^s} 2^{-k(s - \frac{Q_m}{2})} \|u_{\xi}\|_{L^\infty(\mathcal{M})} \|u_{\xi}\|_{L^1} \|f\|_{L^\infty}.
\]
Thus, the analysis above implies the following estimate for the operator norm of $T_{m_k}$, for all $k \geq 2$,

$$
\|T_{m_k}\|_{\mathcal{B}(L^\infty(M), \text{BMO}(M))} \lesssim \sum_{2k-1 \leq |\lambda| \leq 2k+1} \|\tau_m\|_{l.u. H^s} 2^{-k(s-\frac{Q_m}{2})} \|u_\xi\|_{L^\infty(M)} \|u_\xi\|_{L^1(M)}
$$

$$
\lesssim \sum_{2k-1 \leq |\lambda| \leq 2k+1} 2^{k\gamma_\infty} \times \|\tau_m\|_{l.u. H^s} \times 2^{-k(s-\frac{Q_m}{2})}
$$

$$
\approx 2^{kQ} \times 2^{k\gamma_\infty} \times \|\tau_m\|_{l.u. H^s} \times 2^{-k(s-\frac{Q_m}{2})}.
$$

Now, by using that $T_0$ is an operator whose symbol has compact support in the $\xi$-variables, we conclude that $T_0$ is bounded from $L^\infty(M)$ to $\text{BMO}(M)$ and

$$
\|T_0\|_{\mathcal{B}(L^\infty(M), \text{BMO}(M))} \leq C \|m\|_{L^\infty}.
$$

This analysis, allows us to estimate, the operator norm of $T_m$ as follows,

$$
\|T_m\|_{\mathcal{B}(L^\infty(M), \text{BMO}(M))} \leq \|T_0\|_{\mathcal{B}(L^\infty(M), \text{BMO}(M))} + \sum_{k} \|T_{m_k}\|_{\mathcal{B}(L^\infty(M), \text{BMO}(M))}
$$

$$
\lesssim \|m\|_{L^\infty} + \sum_{k=1}^{\infty} 2^{-k(s-Q-\frac{Q_m}{2})-\gamma_\infty} \|\tau_m\|_{l.u. H^s}
$$

$$
\leq C(\|m\|_{L^\infty} + \|\tau_m\|_{l.u. H^s}) < \infty,
$$

provided that $s > (Q_m/2) + Q + \gamma_\infty$. So, we have proved the $L^\infty$-$\text{BMO}$ boundedness of $T_m$. \qed

Now, observe that in view of the duality $(H^1)' = \text{BMO}$, we can use the duality argument to deduce the following estimate for $L$-Fourier multipliers.

**Corollary 3.10.** Let $\overline{M}$ be a geodesically complete Riemannian manifold with (possibly empty) boundary $\partial M$, and let $A : C^\infty_0(\overline{M}) \to C^\infty_0(\overline{M})$ be an $L$-Fourier multiplier. Let us assume that one of the following two conditions hold.

1. $\|\tau_m\|_{l.u. H^s} = \sup_{r > 0} r^{s-\frac{Q_m}{2}} \|\langle \cdot \rangle^s \mathcal{F}[\tau_m(\cdot)\psi(r^{-1} \cdot)]\|_{L^2(\mathbb{R})} < \infty$, \hspace{1cm} (3.27)

   for $s > \max\{1/2, \gamma_\infty + Q + (Q_m/2)\}$.

2. $|\partial_\omega^\alpha \tau_m(\omega)| \leq C_{\alpha, \beta}(1 + |\omega|)^{-|\alpha|}$, $\omega \in \mathbb{R}$, \hspace{1cm} (3.28)

   for all $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \leq \rho$, where $\rho \in \mathbb{N}$, and $\rho > \max\{1/2, (Q_m/2) + Q + \gamma_\infty\}$.

Then, $A$ admits a bounded extension from $L^\infty(\overline{M})$ into $\text{BMO}(\overline{M})$ and from the Hardy space $H^1(\overline{M})$ to $L^1(\overline{M})$. 

4. $L^p$-$L^q$ BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS FOR $1 < p \leq 2 \leq q < \infty$

This section is devoted to the study of $L^p$-$L^q$ boundedness of the pseudo-differential operators and Fourier multipliers on manifolds $\mathcal{M}$. To accomplish this aim we will first prove some inequalities, namely, Paley inequality and Hausdorff-Young-Paley inequality in our setting which eventually yield us the boundedness results. Before stating our main results of this section we recall the definition of relevant $L^p$-spaces on the discrete set $\mathcal{I}$ from [30].

We describe the $p$-Lebesgue versions of the spaces of Fourier coefficients. These spaces can be considered as the extension of the usual $\ell^p$ spaces on the discrete set $\mathcal{I}$ adapted to the fact that we are dealing with biorthogonal systems.

Thus, we introduce the spaces $l^p_L = l^p(L)$ as the spaces of all $a \in S'(\mathcal{I})$ such that

$$
\|a\|_{l^p(L)} := \left( \sum_{\xi \in \mathcal{I}} |a(\xi)|^p |u_\xi|^{2-p} \right)^{1/p} < \infty, \quad \text{for } 1 \leq p \leq 2,
$$

(4.1)

and

$$
\|a\|_{l^p(L)} := \left( \sum_{\xi \in \mathcal{I}} |a(\xi)|^p \|v_\xi\|^{2-p}_{L^\infty(\mathcal{M})} \right)^{1/p} < \infty, \quad \text{for } 2 \leq p < \infty,
$$

(4.2)

and, for $p = \infty$,

$$
\|a\|_{l^\infty(L)} := \sup_{\xi \in \mathcal{I}} \left( |a(\xi)| \cdot \|v_\xi\|^{-1}_{L^\infty(\mathcal{M})} \right) < \infty.
$$

We note that in the case of $p = 2$, we have already defined the space $l^2(L)$ by the norm (2.5). There is no problem with this since the norms (4.1)-(4.2) with $p = 2$ are equivalent to that in (2.5).

Analogously, we also introduce spaces $l^p_\ast = l^p(L^\ast)$ as the spaces of all $b \in S'(\mathcal{I})$ such that the following norms are finite:

$$
\|b\|_{l^p(L^\ast)} = \left( \sum_{\xi \in \mathcal{I}} |b(\xi)|^p \|v_\xi\|^{2-p}_{L^\infty(\Omega)} \right)^{1/p}, \quad \text{for } 1 \leq p \leq 2,
$$

$$
\|b\|_{l^p(L^\ast)} = \left( \sum_{\xi \in \mathcal{I}} |b(\xi)|^p \|u_\xi\|^{2-p}_{L^\infty(\Omega)} \right)^{1/p}, \quad \text{for } 2 \leq p < \infty,
$$

$$
\|b\|_{l^\infty(L^\ast)} = \sup_{\xi \in \mathcal{I}} \left( |b(\xi)| \cdot \|u_\xi\|^{-1}_{L^\infty(\Omega)} \right).
$$

For more discussion on this we refer to [30]. The following Hausdorff-Young inequality is proved by the last two authors in [30].

**Theorem 4.1** (Hausdorff-Young inequality). Let $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There is a constant $C_p \geq 1$ such that for all $f \in L^p(\mathcal{M})$ we have

$$
\left( \sum_{\xi \in \mathcal{I}} |\mathcal{F}_L(f)(\xi)|^{p'} \|v_\xi\|^{2-p'}_{L^\infty(\mathcal{M})} \right)^{\frac{1}{p'}} = \|\hat{f}\|_{l^{p'}(L)} \leq C_p \|f\|_{L^p(\mathcal{M})}.
$$
Similarly, we also have

\[
\left( \sum_{\xi \in I} |\mathcal{F}_{L^p}(f)(\xi)| |u_\xi|^{2-p'} L_\infty(M) \right)^{\frac{1}{p'}} = \|\hat{f}\|_{L^p(L^p)} \leq C_p \|f\|_{L^p(M)}.
\]

In this direction, we present the following Paley-type inequality.

4.1. Hausdorff-Young-Paley inequality. In [25], Lars Hörmander established a Paley-type inequality for the Fourier transform on \( \mathbb{R}^n \). The following inequality is an analogue of this inequality for the \( L^p \)-Fourier transform on manifolds. This inequality was established by the third author and his collaborators for compact homogeneous spaces and for locally compact unimodular groups [1, 2].

**Theorem 4.2** (\( L^p \)-Paley-type inequality). Let \( 1 < p \leq 2 \) and

\[
\sup_{\xi \in I} \left( \frac{\|u_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}} \right) < \infty.
\]

If \( \varphi(\xi) \) is a positive sequence in \( I \) such that

\[
M_\varphi := \sup_{t > 0} \sum_{\xi \in I} t \|u_\xi\|_{L^\infty(M)}^2 < \infty,
\]

then for every \( f \in L^p(M) \) we have

\[
\left( \sum_{\xi \in I} |\mathcal{F}_{L}(f)(\xi)| |u_\xi|^{2-p'} L_\infty(M) \varphi(\xi)^{2-p} \right)^{\frac{1}{p'}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(M)}.
\]

**Proof.** Let \( \nu \) be the measure on \( I \) defined by \( \nu(\xi) := \varphi^2(\xi) \|u_\xi\|_{L^\infty(M)}^2 \) for \( \xi \in I \). Now, we define weighted spaces \( L^p(I, \nu) \), \( 1 \leq p \leq 2 \), as the spaces of complex (or real) sequences \( a = \{a_\xi\}_{\xi \in I} \) such that

\[
\|a\|_{L^p(I, \nu)} := \left( \sum_{\xi \in I} |a_\xi|^p \varphi^2(\xi) \|u_\xi\|_{L^\infty(M)}^2 \right)^{\frac{1}{p}} < \infty.
\]

We show that the sublinear operator \( A : L^p(M) \to L^p(I, \nu) \) defined by

\[
Af := \left\{ \frac{|\mathcal{F}_{L}(f)(\xi)|}{\|u_\xi\|_{L^\infty(M)} \varphi(\xi)} \right\}_{\xi \in I}
\]

is well-defined and bounded from \( L^p(M) \) to \( L^p(I, \nu) \) for \( 1 < p \leq 2 \). In other words, we claim that we have the estimate

\[
\|Af\|_{L^p(I, \nu)} = \left( \sum_{\xi \in I} \left( \frac{|\mathcal{F}_{L}(f)(\xi)|}{\|u_\xi\|_{L^\infty(M)} \varphi(\xi)} \right)^p \varphi^2(\xi) \|u_\xi\|_{L^\infty(M)}^2 \right)^{\frac{1}{p}} \lesssim M_\varphi^{\frac{2-p}{p}} \|f\|_{L^p(M)},
\]

(4.5)
which would give us (4.3) and where we set

\[ M_\varphi := \sup_{t > 0} t \sum_{\xi \in \mathcal{I}, \varphi(\xi) \geq t} \| u_\xi \|_{L^2(M)}^2 . \]

To prove this we will show that \( A \) is of weak-type \((2, 2)\) and of weak-type \((1, 1)\). More precisely, with the distribution function,

\[ \nu_I(y; Af) = \sum_{\xi \in \mathcal{I}, |Af(\xi)| \geq y} \| u_\xi \|_{L^2(M)}^2 \varphi^2(\xi) \]

we show that

\[ \nu_I(y; Af) \leq \left( \frac{M_2 \| f \|_{L^2(M)}}{y} \right)^2 \]

with norm \( M_2 = 1 \), \hspace{1cm} (4.6)

\[ \nu_I(y; Af) \leq \frac{M_1 \| f \|_{L^1(M)}}{y} \]

with norm \( M_1 = M_\varphi \). \hspace{1cm} (4.7)

Then (4.5) will follow by the Marcinkiewicz interpolation theorem. Now, to show (4.6), using Plancherel identity we get

\[ y^2 \nu_I(y; Af) \leq \sup_{y > 0} y^2 \nu_I(y; Af) =: \| Af \|_{L^2(I, \nu)}^2 \leq \| Af \|_{L^2(M)}^2 \]

\[ = \sum_{\xi \in \mathcal{I}} \left( \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)}} \right)^2 \varphi^2(\xi) \| u_\xi \|_{L^2(M)}^2 \]

\[ = \sum_{\xi \in \mathcal{I}} |F_L(f)(\xi)|^2 = \| F_L(f) \|_{L^2(M)} = \| f \|_{L^2(M)}^2 . \]

Thus, \( A \) is type \((2, 2)\) with norm \( M_2 \leq 1 \). Further, we show that \( A \) is of weak type \((1, 1)\) with norm \( M_1 = M_\varphi \); more precisely, we show that

\[ \nu_I \{ \xi \in I : \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)}} > y \} \lesssim M_\varphi \frac{\| f \|_{L^1(M)}}{y} . \] \hspace{1cm} (4.8)

Here, the left hand side is the weighted sum \( \sum \varphi^2(\xi) \| u_\xi \|_{L^\infty(M)}^2 \) taken over those \( \xi \in I \) such that \( \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)}} > y \). From the definition of the Fourier transform it follows that

\[ |F_L(f)(\xi)| \leq \| v_\xi \|_{L^\infty(M)} \| f \|_{L^1(M)} . \]

Therefore, we get

\[ y \leq \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)}} \leq \frac{\| f \|_{L^1(M)}}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1}} . \]

Using this, we get

\[ \left\{ \xi \in I : \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)}} > y \right\} \subset \left\{ \xi \in I : \frac{\| f \|_{L^1(M)}}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1}} > y \right\} . \]
for any \( y > 0 \). Consequently,

\[
\nu \left\{ \xi \in I : \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1}} > y \right\} \leq \nu \left\{ \xi \in I : \frac{\| f \|_{L^1(M)}}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1}} > y \right\}.
\]

By setting \( w := \frac{\| f \|_{L^1(M)}}{y} \), we get

\[
\nu \left\{ \xi \in I : \frac{|F_L(f)(\xi)|}{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1}} > y \right\} \leq \sum_{\xi \in I} \varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1} \leq w \tag{4.9}
\]

We claim that

\[
\sum_{\xi \in I} \varphi(\xi) \| u_\xi \|_{L^\infty(M)}^2 \leq M \varphi w. \tag{4.10}
\]

In fact, we have

\[
\sum_{\xi \in I} \varphi(\xi) \| u_\xi \|_{L^\infty(M)}^2 \leq w
\]

\[
= \sum_{\xi \in I} \| u_\xi \|_{L^\infty(M)}^2 \int_0^{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1} \leq w} d\tau.
\]

We can interchange sum and integration with the fact that \( c := \sup_{\xi \in I} \left( \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)} \right) < \infty \) to get

\[
\sum_{\xi \in I} \| u_\xi \|_{L^\infty(M)}^2 \int_0^{\varphi(\xi) \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1} \leq w} d\tau \leq \int_0^{w^2 c^2} d\tau \sum_{\xi \in I} \| u_\xi \|_{L^\infty(M)}^2 \| v_\xi \|_{L^\infty(M)}^{-1} \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1} \leq w
\]

Further, we make a substitution \( \tau = t^2 \), yielding

\[
\int_0^{w^2 c^2} d\tau \sum_{\xi \in I} \| u_\xi \|_{L^\infty(M)}^2 \| v_\xi \|_{L^\infty(M)}^{-1} \| u_\xi \|_{L^\infty(M)} \| v_\xi \|_{L^\infty(M)}^{-1} \leq w
\]
\[
2 \int_0^{\infty} t dt \sum_{t \leq \varphi(\xi) \leq w} \|u_\xi\|_{L^\infty(M)}^2 \sum_{t \leq \varphi(\xi)} \|u_\xi\|_{L^\infty(M)}^2 \leq 2 \int_0^{\infty} t dt \sum_{t \leq \varphi(\xi)} \|u_\xi\|_{L^\infty(M)}^2.
\]

Since
\[
t \sum_{t \leq \varphi(\xi)} \|u_\xi\|_{L^\infty(M)}^2 \leq \sup_{t > 0} t \sum_{t \leq \varphi(\xi)} \|u_\xi\|_{L^\infty(M)}^2 = M_\varphi
\]
is finite by assumption, we have
\[
2 \int_0^{\infty} t dt \sum_{t \leq \varphi(\xi)} \|u_\xi\|_{L^\infty(M)}^2 \lesssim M_\varphi w = \frac{M_\varphi \|f\|_{L^1(M)}}{y}.
\]
This proves (4.10). Therefore, we have proved inequalities (4.6) and (4.7). Then by using the Marcinkiewicz interpolation theorem with \( p_1 = 1, p_2 = 2 \) and \( \frac{1}{p} = 1 - \theta + \frac{\theta}{2} \), we now obtain
\[
\left( \sum_{\xi \in \mathcal{I}} \left( \frac{\|\mathcal{F}_{L}(f)(\xi)\|_{L^\infty(M)} \varphi(\xi)}{\|u_\xi\|_{L^\infty(M)} \varphi(\xi)} \right)^p \|u_\xi\|_{L^\infty(M)} \varphi(\xi)^2 \right)^{\frac{1}{p}}
= \|A_{f}\|_{L^{p}(\mathcal{I}, \nu)} \lesssim M_\varphi^{2-p} \|f\|_{L^p(M)},
\]
yielding (4.3).

Now, we state the Paley inequality associated with the \( L^* \)-Fourier transform. The proof is verbatim to the Paley inequality for \( L \)-Fourier transform above with the use \( L^* \)-Fourier transform and \( \nu \)-spaces.

**Theorem 4.3 (\( L^* \)-Paley-type inequality).** Let \( 1 < p \leq 2 \) and
\[
\sup_{\xi \in \mathcal{I}} \left( \frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}} \right) < \infty.
\]
If \( \varphi(\xi) \) is a positive sequence in \( \mathcal{I} \) such that
\[
M_\varphi := \sup_{t > 0} t \sum_{t \leq \varphi(\xi)} \|v_\xi\|_{L^\infty(M)}^2 < \infty,
\]
then for every \( f \in L^p(M) \) we have
\[
\left( \sum_{\xi \in \mathcal{I}} |\mathcal{F}_{L^*}(f)(\xi)|^p \|v_\xi\|_{L^\infty(M)}^{2-p} \varphi(\xi)^{2-p} \right)^{\frac{1}{p}} \lesssim M_\varphi^{2-p} \|f\|_{L^p(M)}. \tag{4.11}
\]

The following theorem [5] is useful to obtain one of our crucial results.
**Theorem 4.4.** Let \( d\mu_0(x) = \omega_0(x) d\mu(x) \), \( d\mu_1(x) = \omega_1(x) d\mu(x) \), and write \( L^p(\omega) = L^p(\omega_0) \) for the weight \( \omega \). Suppose that \( 0 < p_0, p_1 < \infty \). Then

\[
(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta,p} = L^p(\omega),
\]

where \( 0 < \theta < 1, \frac{1}{\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) and \( \omega = \omega_0^{\frac{p(1-\theta)}{\theta p_0}} \omega_1^{\frac{\theta}{p_1}} \).

The following corollary is immediate.

**Corollary 4.5.** Let \( d\mu_0(x) = \omega_0(x) d\mu(x) \), \( d\mu_1(x) = \omega_1(x) d\mu(x) \). Suppose that \( 0 < p_0, p_1 < \infty \). If a continuous linear operator \( A \) admits bounded extensions, \( A : L^p(Y, \mu) \to L^{p_0}(\omega_0) \) and \( A : L^p(Y, \mu) \to L^{p_1}(\omega_1) \), then there exists a bounded extension \( A : L^p(Y, \mu) \to L^b(\omega) \) of \( A \), where \( 0 < \theta < 1, \frac{1}{\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) and \( \omega = \omega_0^{\frac{p(1-\theta)}{\theta p_0}} \omega_1^{\frac{\theta}{p_1}} \).

Using the above corollary we now present Hausdorff-Young-Paley inequality.

**Theorem 4.6** (\( L \)-Hausdorff-Young-Paley inequality). Let \( 1 < p \leq 2 \), and let \( 1 < p \leq b \leq p' \leq \infty \), where \( p' = \frac{p}{p-1} \) and

\[
\sup_{\xi \in \mathcal{I}} \left( \frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}} \right) < \infty.
\]

If \( \varphi(\xi) \) is a positive sequence in \( \mathcal{I} \) such that

\[
M_\varphi := \sup_{t > 0} t \sum_{\xi \in \mathcal{I}} \|u_\xi\|_{L^\infty(M)}^2
\]

is finite, then for every \( f \in L^p(M) \) we have

\[
\left( \sum_{\xi \in \mathcal{I}} \left( |F_L f(\xi)| \varphi(\xi)^{\frac{1}{b'}} \right)^b \|u_\xi\|_{L^\infty(M)}^\frac{1}{b} \|v_\xi\|_{L^\infty(M)}^\frac{1}{b} \right)^\frac{1}{b} \leq_p M_\varphi \frac{1}{p} \|f\|_{L^p(M)}. \tag{4.12}
\]

**Proof.** From Theorem 4.2, the operator defined by

\[
A f(\xi) := \left\{ \frac{F_L f(\xi)}{\|u_\xi\|_{L^\infty(M)}} \right\}_{\xi \in \mathcal{I}},
\]

is bounded from \( L^p(M) \) to \( L^p(\mathcal{I}, \omega_0) \), where \( \omega_0 = \|u_\xi\|_{L^\infty(M)}^2 \varphi(\xi)^{2-p} \). From Theorem 4.1, we deduce that \( A : L^p(M) \to L^p(\mathcal{I}, \omega_1) \) with \( \omega_1(\xi) = \|u_\xi\|_{L^\infty(M)}^p \|v_\xi\|_{L^\infty(M)}^{2-p} \), admits a bounded extension. By using the real interpolation we will prove that \( A : L^p(M) \to L^b(\mathcal{I}, \omega) \), \( p \leq b \leq p' \), is bounded, where the space \( L^p(\mathcal{I}, \omega) \) is defined by the norm

\[
\|\sigma\|_{L^p(\mathcal{I}, \omega)} := \left( \sum_{\xi \in \mathcal{I}} |\sigma(\xi)|^p w(\xi) \right)^\frac{1}{p}
\]

and \( \omega(\xi) \) is positive sequence over \( \mathcal{I} \) to be determined. To compute \( \omega \), we can use Corollary 4.5, by fixing \( \theta \in (0,1) \) such that \( \frac{1}{\theta} = \frac{1-\theta}{p} + \frac{\theta}{p'} \). In this case \( \theta = \frac{p-b}{b(p-2)} \), and

\[
\omega = \omega_0^{\frac{p(1-\theta)}{\theta p_0}} \omega_1^{\frac{\theta}{p_1}} = \varphi(\xi)^{1-\frac{1}{p}} \|u_\xi\|_{L^\infty(M)}^{(1-\frac{1}{b})} \|v_\xi\|_{L^\infty(M)}^{(1-\frac{1}{b})}. \tag{4.13}
\]
Thus we finish the proof. □

Analogously, by interpolating the Hausdorff-Young inequality for $L^r$-Fourier transform and $L^r$-Paley type inequality (Theorem 4.3) we obtain the following $L^r$-version of Hausdorff-Young-Paley inequality.

**Theorem 4.7** ($L^r$-Hausdorff-Young-Paley inequality). Let $1 < p \leq 2$, and let $1 < p \leq b \leq p' \leq \infty$, where $p' = \frac{p}{p-1}$ and

$$\sup_{\xi \in \mathcal{I}} \left( \frac{\| u_\xi \|_{L^\infty(M)}}{\| v_\xi \|_{L^\infty(M)}} \right) < \infty.$$ 

If $\varphi(\xi)$ is a positive sequence in $\mathcal{I}$ such that

$$M_\varphi := \sup_{t > 0} \sum_{t \leq \varphi(\xi)} \| v_\xi \|_{L^\infty(M)}$$

is finite, then for every $f \in L^p(M)$ we have

$$\left( \sum_{\xi \in \mathcal{I}} \left( |F_{L^r} f(\xi)| \varphi(\xi)^{\frac{1}{b} - \frac{1}{p'}} \right)^b \| v_\xi \|_{L^\infty(M)} \| u_\xi \|_{L^\infty(M)} \right)^{\frac{1}{b}} \lesssim_p M_\varphi^{\frac{1}{b} - \frac{1}{p'}} \| f \|_{L^p(M)}. \quad (4.14)$$

### 4.2. $L^p$-boundedness

In this subsection we will prove the $L^p$-boundedness of Fourier multipliers related of model operator $L$ on manifold $M$. This was proved for the torus in [28] using a different method.

**Theorem 4.8.** Let $1 < p \leq 2 \leq q < \infty$ and assume that

$$\sup_{\xi \in \mathcal{I}} \left( \frac{\| v_\xi \|_{L^\infty(M)}}{\| u_\xi \|_{L^\infty(M)}} \right) < \infty \quad \text{and} \quad \sup_{\xi \in \mathcal{I}} \left( \frac{\| u_\xi \|_{L^\infty(M)}}{\| v_\xi \|_{L^\infty(M)}} \right) < \infty. \quad (4.15)$$

Suppose that $A : C^\infty_c(M) \to C^\infty_c(M)$ is a $L$-Fourier multiplier with $L$-symbol $\sigma_{A,L}$ on $M$, that is, $A$ satisfies

$$F_L(Af)(\xi) = \sigma_{A,L}(\xi) F_L f(\xi), \quad \text{for all } \xi \in \mathcal{I},$$

where $\sigma_{A,L} : \mathcal{I} \to \mathbb{C}$ is a function. Then we have

$$\| A \|_{\mathcal{B}(L^p(M), L^q(M))} \lesssim \sup_{s > 0} \left( \sum_{\xi \in \mathcal{I}} \max \{ \| u_\xi \|_{L^\infty(M)}^2, \| v_\xi \|_{L^\infty(M)}^2 \} \right)^{\frac{1}{2}} \left( \sum_{|\sigma_{A,L}(\xi)| \geq s} \max \{ \| u_\xi \|_{L^\infty(M)}^2, \| v_\xi \|_{L^\infty(M)}^2 \} \right)^{\frac{1}{2}}.$$ 

Before starting the proof we would like to notice here that for $p \leq q'$ we only need the first inequality in (4.15) above.

**Proof.** Let us first assume that $p \leq q'$, where $\frac{1}{q} + \frac{1}{q'} = 1$. Since $q' \leq 2$, the Hausdorff-Young inequality gives that

$$\| Af \|_{L^q(M)} \lesssim \| F_L(Af) \|_{L^{q'}(L)} = \| \sigma_{A,L} F_L(f) \|_{L^{q'}(L)}$$

$$= \left( \sum_{\xi \in \mathcal{I}} |\sigma_{A,L}(\xi)|^{q'} |F_L(f)(\xi)|^{q'} \| u_\xi \|_{L^\infty(M)}^{2-q'q} \right)^{\frac{1}{q'}}. \quad (4.16)$$
Now, we are in a position to apply Theorem 4.6. Set \( \frac{1}{p} - \frac{1}{q} = \frac{1}{r} \). By applying Theorem 4.6 in (4.16) by taking \( \varphi(\xi) := \left( |\sigma_{A,L}(\xi)| \| u_\xi \|_{L^\infty(\mathcal{M})}^{\frac{1}{q}} \| v_\xi \|_{L^\infty(\mathcal{M})}^{\frac{1}{q'}} \right) \) with \( b = q' \), we get

\[
\| A f \|_{L^r(\mathcal{M})} \lesssim \| \sigma_{A,L} F_L(f) \|_{L^{r'}(\mathcal{M})} \]

\[
= \left( \sum_{\xi \in \mathcal{I}} |\sigma_{A,L}(\xi)|^{q'} |F_L(f)(\xi)|^{q'} \| u_\xi \|_{L^\infty(\mathcal{M})}^{2-q'} \right)^{\frac{1}{q'}}
\]

\[
= \left( \sum_{\xi \in \mathcal{I}} \left( |F_L f(\xi)| \varphi(\xi) \right)^{q'} \| u_\xi \|_{L^\infty(\mathcal{M})}^{1-\frac{q'}{r'}} \| v_\xi \|_{L^\infty(\mathcal{M})}^{\frac{1}{r'}} \right)^{\frac{1}{q'}}
\]

\[
\lesssim \left( \sup_{s > 0} s \sum_{\xi \in \mathcal{I}} \| u_\xi \|_{L^\infty(\mathcal{M})}^{\frac{2}{r'}} \right)^{\frac{1}{q'}} \| f \|_{L^p(\mathcal{M})},
\]

(4.12)

for all \( f \in L^p(\mathcal{M}) \), in view of \( \frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'} = \frac{1}{r'} \). Thus, we obtain

\[
\| A \|_{\mathcal{B}(L^p(\mathcal{M}), L^q(\mathcal{M}))} \lesssim \left( \sup_{s > 0} s \sum_{\xi \in \mathcal{I}} \| u_\xi \|_{L^\infty(\mathcal{M})}^{\frac{2}{r'}} \right)^{\frac{1}{q'}}
\]

Now, using the condition (4.15) we deduce that \( \| u_\xi \|_{L^\infty(\mathcal{M})} \| v_\xi \|_{L^\infty(\mathcal{M})}^{-1} \| v_\xi \|_{L^\infty(\mathcal{M})}^{\frac{1}{q'}} \lesssim 1 \) and so

\[
\| u_\xi \|_{L^\infty(\mathcal{M})} \| v_\xi \|_{L^\infty(\mathcal{M})}^{\frac{1}{q'}} \| v_\xi \|_{L^\infty(\mathcal{M})}^{\frac{1}{q'}} \lesssim 1.
\]

Therefore, we get

\[
\| A \|_{\mathcal{B}(L^p(\mathcal{M}), L^q(\mathcal{M}))} \lesssim \left( \sup_{s > 0} s \sum_{\xi \in \mathcal{I}} \| u_\xi \|_{L^\infty(\mathcal{M})}^{\frac{2}{r'}} \right)^{\frac{1}{q'}} = \left( \sup_{s > 0} s^{\frac{1}{q'}} \sum_{\xi \in \mathcal{I}} \| u_\xi \|_{L^\infty(\mathcal{M})}^{\frac{2}{r'}} \right)^{\frac{1}{q'}}
\]

\[
= \sup_{s > 0} s \left( \sum_{\xi \in \mathcal{I}} \| u_\xi \|_{L^\infty(\mathcal{M})}^{\frac{2}{r'}} \right)^{\frac{1}{q'}}
\]
\[ \|A\|_{\mathcal{B}(L^p(M),L^q(M))} \leq \sup_{s>0} s \left( \sum_{\xi \in \mathcal{I}} \max\{\|u_\xi\|_{L^\infty(M)}^2, \|v_\xi\|_{L^\infty(M)}^2\} \right)^{\frac{1}{p'}} < \infty. \]

Now we consider the case \( q' \leq p \) so that \( p' \leq q = (q')' \). Using the duality of \( L^p \)-spaces we have \( \|A\|_{\mathcal{B}(L^p(M),L^q(M))} = \|A^*\|_{\mathcal{B}(L^{q'}(M),L^{p'}(M))} \). The \( L^* \)-symbol of \( \sigma_{A^*,L^*}(\xi) \) of the adjoint operator \( A^* \), which is an \( L^* \)-Fourier multiplier, is equal to \( \sigma_{A,L}(\xi) \) and obviously we have \( |\sigma_{A,L}(\xi)| = |\sigma_{A^*,L^*}(\xi)| \) (see Proposition 3.6 in [15]). Now, the idea is to proceed as the case \( p \leq q' \) but this time for \( L^* \)-Fourier multiplier \( A^* \) we get, as an application of Hausdorff-Young inequality for \( L^* \)-Fourier transform and Theorem 4.7, that

\[ \|A\|_{\mathcal{B}(L^p(M),L^q(M))} \leq \sup_{s>0} s \left( \sum_{\xi \in \mathcal{I}} \max\{\|u_\xi\|_{L^\infty(M)}^2, \|v_\xi\|_{L^\infty(M)}^2\} \right)^{\frac{1}{p'}} \]

Therefore, in the view of \( \frac{1}{p} - \frac{1}{q} = \frac{1}{r} = \frac{1}{q'} - \frac{1}{p'} \) we have

\[ \|A\|_{\mathcal{B}(L^p(M),L^q(M))} \leq \sup_{s>0} s \left( \sum_{\xi \in \mathcal{I}} \max\{\|u_\xi\|_{L^\infty(M)}^2, \|v_\xi\|_{L^\infty(M)}^2\} \right)^{\frac{1}{p} - \frac{1}{q}}, \]

proving the Theorem 4.8.

\[ \square \]

In case when \( M \) is a compact manifold the condition \( 1 < p \leq 2 < q < \infty \) can be replaced by \( 1 < p, q < \infty \).

**Corollary 4.9.** Let \( 1 < p, q < \infty \) and assume that

\[ \sup_{\xi \in \mathcal{I}} \left( \frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}} \right) < \infty \quad \text{and} \quad \sup_{\xi \in \mathcal{I}} \left( \frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}} \right) < \infty. \]

Suppose that \( A : C^\infty_L(M) \to C^\infty_L(M) \) is a \( L \)-Fourier multiplier with \( L \)-symbol \( \sigma_{A,L} \) on a compact manifold \( M \). If \( 1 < p, q \leq 2 \), then

\[ \|A\|_{\mathcal{B}(L^p(M),L^q(M))} \leq \sup_{s>0} s \left( \sum_{\xi \in \mathcal{I}} \max\{\|u_\xi\|_{L^\infty(M)}^2, \|v_\xi\|_{L^\infty(M)}^2\} \right)^{\frac{1}{p} - \frac{1}{2}}, \]

while for \( 2 \leq p, q < \infty \) we have

\[ \|A\|_{\mathcal{B}(L^p(M),L^q(M))} \leq \sup_{s>0} s \left( \sum_{\xi \in \mathcal{I}} \max\{\|u_\xi\|_{L^\infty(M)}^2, \|v_\xi\|_{L^\infty(M)}^2\} \right)^{\frac{1}{p'} - \frac{1}{2}}. \]
Proof. Let us assume that $1 < p, q \leq 2$. Using the compactness of $\overline{M}$, we have $\|A\|_{\mathcal{B}(L^p(\overline{M}), L^q(\overline{M}))} \lesssim \|A\|_{\mathcal{B}(L^p(\overline{M}), L^2(\overline{M}))}$ and therefore, Theorem 4.8 gives

$$\|A\|_{\mathcal{B}(L^p(\overline{M}), L^q(\overline{M}))} \lesssim \sup_{s > 0} \left( \sum_{\xi \in \mathcal{I}} \max_{|\sigma_{A,L}(\xi)| \geq s} \{\|u_{\xi}\|_{L^\infty(\overline{M})}^2, \|v_{\xi}\|_{L^\infty(\overline{M})}^2\} \right)^{\frac{1}{2} \cdot \frac{1}{p}} \cdot \frac{1}{q} \cdot \frac{1}{2} .$$

Now, let us assume that $2 \leq p, q < \infty$. Then $1 < p', q' \leq 2$, and using the first part of the proof we deduce

$$\|A\|_{\mathcal{B}(L^p(\overline{M}), L^q(\overline{M}))} = \|A^*\|_{\mathcal{B}(L^{p'}(\overline{M}), L^{q'}(\overline{M}))} \lesssim \sup_{s > 0} \left( \sum_{\xi \in \mathcal{I}} \max_{|\sigma_{A,L}(\xi)| \geq s} \{\|u_{\xi}\|_{L^\infty(\overline{M})}^2, \|v_{\xi}\|_{L^\infty(\overline{M})}^2\} \right)^{\frac{1}{2} \cdot \frac{1}{p'}} \cdot \frac{1}{q'} \cdot \frac{1}{2} .$$

Thus, we finish the proof. \(\square\)

The following theorem presents our main result of this section on $L^p$-$L^q$ boundedness of pseudo-differential operators.

**Theorem 4.10.** Let $1 < p \leq 2 \leq q < \infty$ and assume that

$$\sup_{\xi \in \mathcal{I}} \left( \frac{\|v_{\xi}\|_{L^\infty(\overline{M})}}{\|u_{\xi}\|_{L^\infty(\overline{M})}} \right) < \infty \quad \text{and} \quad \sup_{\xi \in \mathcal{I}} \left( \frac{\|u_{\xi}\|_{L^\infty(\overline{M})}}{\|v_{\xi}\|_{L^\infty(\overline{M})}} \right) < \infty.$$

Suppose that $A : C^\infty_L(\overline{M}) \to C^\infty_L(\overline{M})$ is a continuous linear operator with $L$-symbol $\sigma_{A,L} : \overline{M} \times \mathcal{I} \to \mathbb{C}$, where $\overline{M}$ is a compact manifold, satisfying

$$\|\sigma_{A,L}(\beta)\|_{(\beta)} := \sup_{s > 0, y \in \overline{M}} s \left( \sum_{\xi \in \mathcal{I}} \max_{|\partial_y \sigma_{A,L}(y,\xi)| \geq s} \{\|u_{\xi}\|_{L^\infty(\overline{M})}^2, \|v_{\xi}\|_{L^\infty(\overline{M})}^2\} \right)^{\frac{1}{2} \cdot \frac{1}{q}} \cdot \frac{1}{p} \cdot \frac{1}{2} < \infty, \quad (4.17)$$

for all $|\beta| \leq \left[ \frac{\dim(M)}{q} \right] + 1$, where $\partial_y$ denotes the local partial derivative (see Section 2).

If $\partial M \neq \emptyset$, let us assume additionally that $\operatorname{supp}(\sigma_{A,L}) \subset \{(y, \xi) \in \overline{M} \times \mathcal{I} : y \in \overline{M} \setminus V\}$ where $V \subset \overline{M}$ is an open neighbourhood of the boundary $\partial M$. Then $A$ admits a bounded extension from $L^p(\overline{M})$ into $L^q(\overline{M})$.

**Proof.** Let us assume that $f \in C^\infty_0(\overline{M})$. First, assume that $\partial M \neq \emptyset$. For every $y \in \overline{M}$, we define

$$A_y f(x) := \sum_{\xi \in \mathcal{I}} u_{\xi}(x) \sigma_{A,L}(y, \xi) \hat{f}(\xi), \quad (4.18)$$
where for every \( \beta = \text{int}(M) \) get Therefore, by using the change of the order of integration and Fubini Theorem we get

\[
\|A f\|_{L^q(M)}^q = \int_M |A f(x)|^q dx = \int_M |A f(x)|^q dx
\]

\[
\leq \int_{M \setminus V} \sup_{y \in M \setminus V} |A_y f(x)|^q dx.
\]

Now, the compactness of \( M \), and the local Sobolev embedding theorem on \( M \setminus V \subset M = \text{int}(M) \), implies

\[
\sup_{y \in M \setminus V} |A_y f(x)|^q \leq C \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \int_M |\partial_y^\beta A_y f(x)|^q dy dx
\]

\[
\leq \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \int_M |\partial_y^\beta A_y f(x)|^q dy dx
\]

where for every \( \beta \in \mathbb{N}^{\text{dim} M} \), the operator \( \partial_y^\beta A \) is defined by

\[
(\partial_y^\beta A_y) f(x) := \sum_{\xi \in I} u_\xi(x)(\partial_y^\beta \sigma_{A,L})(y, \xi) \hat{f}(\xi). \tag{4.19}
\]

Therefore, by using the change of the order of integration and Fubini Theorem we get

\[
\|A f\|_{L^q(M)}^q \leq \int_M \sup_{y \in M \setminus V} |A_y f(x)|^q dx \leq C \int_M \sup_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \int_M |\partial_y^\beta A_y f(x)|^q dy dx
\]

\[
\leq C \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \int_M |\partial_y^\beta A_y f(x)|^q dx = \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \sup_{y \in M} \|\partial_y^\beta A_y f\|_{L^q(M)}^q
\]

\[
\leq C \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \sup_{y \in M} \|f\| \text{Op}(\partial_y^\beta \sigma_{A,L}\gamma) f\|_{L^q(M), L^q(M)} \|f\|_{L^q(M)}^q
\]

\[
\leq \left[ \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \sup_{s > 0, y \in M} s \left( \sum_{\xi \in I} \max\{\|u_\xi\|_{L^\infty(M)}, \|v_\xi\|_{L^\infty(M)}\} \right)^{\frac{1}{p} - \frac{1}{q}} \right] \|f\|_{L^q(M)}^q,
\]

where the last inequality follows from Theorem 4.8. Hence,

\[
\|A \|_{\mathcal{B}(L^p(M), L^q(M))} \leq \sup_{s > 0, y \in M} s \left( \sum_{|\beta| \leq \frac{\text{dim}(M)}{q}} + 1 \sup_{\xi \in I} \max\{\|u_\xi\|_{L^\infty(M)}, \|v_\xi\|_{L^\infty(M)}\} \right)^{\frac{1}{p} - \frac{1}{q}} < \infty.
\]
Now, if $\partial M = \emptyset$, we can take $V = \emptyset$ above and the proof above works in this case. Thus, we finish the proof of Theorem 4.10.

The following corollary is an analogue of Corollary 4.9 for pseudo-differential operators. The proof of this corollary follows similar to Corollary 4.9 by using Theorem 4.10.

**Corollary 4.11.** Let $1 < p, q < \infty$ and assume that

$$
\sup_{\xi \in \mathcal{I}} \frac{\|v_\xi\|_{L^\infty(M)}}{\|u_\xi\|_{L^\infty(M)}} < \infty \quad \text{and} \quad \sup_{\xi \in \mathcal{I}} \frac{\|u_\xi\|_{L^\infty(M)}}{\|v_\xi\|_{L^\infty(M)}} < \infty.
$$

Suppose that $A : C^\infty_0(M) \to C^\infty_0(M)$ is a continuous linear operators with $L$-symbol $\sigma_{A,L} : \overline{M} \times \mathcal{I} \to \mathbb{C}$, where $\overline{M}$ is a compact manifold, satisfying

- for $1 < p, q \leq 2$,

$$
\sup_{s > 0, y \in \overline{M}} s \left( \sum_{\xi \in \mathcal{I}} \max_{|\partial^0 \sigma_{A,L}(y,\xi)| \geq s} \{\|u_\xi\|_{L^\infty(M)}, \|v_\xi\|_{L^\infty(M)}\} \right)^{\frac{1}{p} - \frac{1}{2}} < \infty,
$$

for all $|\beta| \leq \left[ \frac{\dim(M)}{q} \right] + 1$, and

- for $2 \leq p, q < \infty$,

$$
\sup_{s > 0, y \in \overline{M}} s \left( \sum_{\xi \in \mathcal{I}} \max_{|\partial^0 \sigma_{A,L}(y,\xi)| \geq s} \{\|u_\xi\|_{L^\infty(M)}, \|v_\xi\|_{L^\infty(M)}\} \right)^{\frac{1}{q} - \frac{1}{2}} < \infty,
$$

for all $|\beta| \leq \left[ \frac{\dim(M)}{p} \right] + 1$.

If $\partial M \neq \emptyset$, let us assume additionally that $\text{supp}(\sigma_{A,L}) \subset \{(y, \xi) \in \overline{M} \times \mathcal{I} : y \in \overline{M} \setminus V \}$ where $V \subset \overline{M}$ is an open neighbourhood of the boundary $\partial M$. Then $A$ admits a bounded extension from $L^p(\overline{M})$ into $L^q(\overline{M})$.

5. Applications to Non-Linear PDEs

In this section we illustrate some applications of our main results. In particular, we discuss applications of the above $L^p$–$L^q$ boundedness theorems to some nonlinear PDEs. Especially, the main interest is to establish the well-posedness properties of nonlinear equations.

5.1. **Nonlinear Stationary Equation.** Let us consider nonlinear stationary equation in the Hilbert space $L^2_0(M)$

$$
Au = |Bu|^p + f,
$$

where $A, B : L^2_0(M) \to L^2_0(M)$ and $1 < p < \infty$. For any $s \in \mathbb{R}$ let us denote by $\mathcal{H}_s^p(M)$ the subspace of $L^2_0(M)$ such that

$$
\mathcal{H}_s^p(M) := \{u \in L^2_0(M) : L^s u \in L^2(M)\}.
$$
By $\mathcal{H}_L^\infty(\overline{M})$ we denote
\[ \mathcal{H}_L^\infty(\overline{M}) := \bigcap_{s=1}^{\infty} \mathcal{H}_L^s(\overline{M}). \]

**Lemma 5.1** ([30]). Let $A$ be an $L$-elliptic pseudo-differential operator with $L$-symbol $\sigma_A \in S^\mu(\overline{M} \times \mathbb{L})$, $\mu \in \mathbb{R}$, and let $Au = f$ in $\overline{M}$, $u \in \mathcal{H}_L^\infty(\overline{M})$. Then we have the estimate
\[ \|u\|_{\mathcal{H}_L^{s+r}(\overline{M})} \leq C_{sN}(\|f\|_{\mathcal{H}_L^2(\overline{M})} + \|u\|_{\mathcal{H}_L^{-N}(\overline{M})}), \]
for any $s, N \in \mathbb{R}$.

By using Lemma 5.1, we conclude estimates for the solution of the equation (5.1) as the following statement.

**Corollary 5.2.** Let $1 \leq p < \infty$. Suppose that the conditions of Lemma 5.1 holds. In addition, we assume that $B$ is a Fourier multiplier as in Theorem 4.10 bounded from $L^2(\overline{M})$ to $L^{2p}(\overline{M})$. Then any solution of the equation (5.1) satisfies the inequality
\[ \|u\|_{L^2(\overline{M})} \leq C_N(\|u\|_{L^{2p}(\overline{M})} + \|f\|_{L^2(\overline{M})} + \|u\|_{\mathcal{H}_L^{-N}(\overline{M})}), \]
for any $N \in \mathbb{R}$.

**Proof.** By using Lemma 5.1, we have
\[ \|u\|_{L^2(\overline{M})} \leq C_N(\|(Bu)^p\|_{L^{2p}(\overline{M})} + \|f\|_{L^2(\overline{M})} + \|u\|_{\mathcal{H}_L^{-N}(\overline{M})}). \]
Finally, from Theorem 4.10, we obtain
\[ \|u\|_{L^2(\overline{M})} \leq C_N(\|u\|_{L^{2p}(\overline{M})}^p + \|f\|_{L^2(\overline{M})} + \|u\|_{\mathcal{H}_L^{-N}(\overline{M})}). \]

\[ \Box \]

### 5.2. Nonlinear Heat Equation

Let us consider the Cauchy problem for the nonlinear evolutionary equation in the space $L^\infty(0, T; L^2(\overline{M}))$
\[ u_t(t) - |Bu(t)|^p = 0, u(0) = u_0, \tag{5.2} \]
where $B$ is a linear operator in $L^2(\overline{M})$ and $1 < p < \infty$.

**Definition 5.3.** We say that the Cauchy problem (5.2) admits a solution $u$ if it satisfies
\[ u(t) = u_0 + \int_0^t |Bu(\tau)|^p d\tau \tag{5.3} \]
in the space $L^\infty(0, T; L^2(\overline{M}))$ for every $T < \infty$.

We say that the Cauchy problem (5.2) admits a local solution $u$ if it satisfies the equation (5.3) in the space $L^\infty(0, T^*; L^2(\overline{M}))$ for some $T^* > 0$.

**Theorem 5.4.** Let $1 < p < \infty$. Suppose that $B$ is a Fourier multiplier as in Theorem 4.8 bounded from $L^2(\overline{M})$ to $L^{2p}(\overline{M})$. Then the Cauchy problem (5.2) has a local solution in $L^\infty(0, T; L^2(\overline{M}))$, that is, there exists $T^* > 0$ such that the Cauchy problem (5.2) has a solution in $L^\infty(0, T^*; L^2(\overline{M}))$. 

Proof. We start by integrating in $t$ the equation (5.2),

$$u(t) = u_0 + \int_0^t |B(u(\tau))|^p \, d\tau.$$  

By taking the $L^2$-norm on both sides, one obtains

$$\|u(t)\|_{L^2(\mathcal{M})}^2 \leq C(\|u_0\|_{L^2(\mathcal{M})}^2 + t \int_0^t \| (B(u(\tau))) \|_{L^2(\mathcal{M})}^{2p} \, d\tau),$$

since

$$\left( \int_0^t |B(u(\tau))|^p \, d\tau \right)^2 \leq t \int_0^t |B(u(\tau))|^p \, d\tau$$

and

$$\int_M \int_0^t |B(u(\tau))|^{2p} \, dx \, d\tau = \int_M \int_0^t |B(u(\tau))|^{2p} \, dx \, d\tau = \int_0^t \| (B(u(\tau))) \|_{L^{2p}(\mathcal{M})}^{2p} \, d\tau.$$  

Now, using Theorem 4.8, we get

$$\|u(t)\|_{L^2(\mathcal{M})}^2 \leq C(\|u_0\|_{L^2(\mathcal{M})}^2 + t \int_0^t \| u(\tau) \|_{L^2(\mathcal{M})}^{2p} \, d\tau),$$

(5.4)

for some constant $C$ independent from $u_0$ and $t$.

Finally, by taking $L^\infty$-norm in time on both sides of the estimate (5.4), one obtains

$$\|u(t)\|_{L^\infty(0,T;L^2(\mathcal{M}))} \leq C(\|u_0\|_{L^2(\mathcal{M})}^2 + T^2 \| u \|_{L^\infty(0,T;L^2(\mathcal{M}))}^{2p}).$$

(5.5)

Let us introduce the following set

$$S_c := \{ u \in L^\infty(0,T;L^2(\mathcal{M})) : \|u\|_{L^\infty(0,T;L^2(\mathcal{M}))} \leq c \|u_0\|_{L^2(\mathcal{M})} \},$$

(5.6)

for some constant $c \geq 1$. Then we have

$$\|u_0\|_{L^2(\mathcal{M})}^2 + T^2 \| u \|_{L^\infty(0,T;L^2(\mathcal{M}))}^{2p} \leq \|u_0\|_{L^2(\mathcal{M})}^2 + T^2 c^{2p} \|u_0\|_{L^2(\mathcal{M})}^{2p}.$$  

Finally, to be $u$ from the set $S_c$ it is enough to have

$$\|u_0\|_{L^2(\mathcal{M})}^2 + T^2 c^{2p} \|u_0\|_{L^2(\mathcal{M})}^{2p} \leq c^2 \|u_0\|_{L^2(\mathcal{M})}^2.$$  

It can be obtained by requiring the following,

$$T \leq T^* := \sqrt{c^2 - 1} \frac{1}{c^p \|u_0\|_{L^2(\mathcal{M})}}.$$  

Thus, by applying the fixed point theorem, there exists a unique local solution $u \in L^\infty(0,T^*;L^2(\mathcal{M}))$ of the Cauchy problem (5.2).  

By using Theorem 4.10, one obtains:
**Theorem 5.5.** Let $1 < p < \infty$. Suppose that $B$ is a continuous linear operators with $L$-symbol $\sigma_{B,L} : \overline{M} \times I \rightarrow \mathbb{C}$, satisfying the condition in Theorem 4.10. Then the Cauchy problem (5.2) admits a local solution in $L^\infty(0, T; L^2(\overline{M}))$, that is, there exists $T^* > 0$ such that the Cauchy problem (5.2) has a solution in $L^\infty(0, T^*; L^2(\overline{M}))$.

### 5.3. Nonlinear Wave Equation

Now we study well-posedness properties of the initial value problem (IVP) for the nonlinear wave equation (NLWE) in $L^\infty(0, T; L^2(\overline{M}))$

\[
\begin{align*}
\text{u}_{tt}(t) - b(t)|Bu(t)|^p = 0, \\
u(0) = u_0, \quad u_t(0) = u_1,
\end{align*}
\]

where $b$ is a positive bounded function depending only on time, $B$ is a linear operator in $L^2(\overline{M})$ and $1 < p < \infty$.

**Definition 5.6.** We say that IVP (5.7) admits a global solution $u$ if it satisfies

\[
u(t) = u_0 + tu_1 + \int_0^t (t - \tau)b(\tau)|Bu(\tau)|^p d\tau
\]

in the space $L^\infty(0, T; L^2(\overline{M}))$ for every $T < \infty$.

We say that the Cauchy problem (5.7) admits a local solution $u$ if it satisfies the equation (5.8) in the space $L^\infty(0, T^*; L^2(\overline{M}))$ for some $T^* > 0$.

**Theorem 5.7.** Let $1 \leq p < \infty$. Assume that $u_0, u_1 \in L^2(\overline{M})$. Suppose that $B$ is a Fourier multiplier as in Theorem 4.8 bounded from $L^2(\overline{M})$ to $L^{2p}(\overline{M})$.

(i) Assume that $\|b\|_{L^2(0,T)} < \infty$ for some $T > 0$. Then the Cauchy problem (5.7) has a local solution in $L^\infty(0, T; L^2(\overline{M}))$, that is, there exists $T^* > 0$ such that the Cauchy problem (5.7) has a solution in $L^\infty(0, T^*; L^2(\overline{M}))$.

(ii) Suppose that $u_1$ is identically equal to zero. Let $\gamma > 3/2$. Moreover, assume that $\|b\|_{L^2(0,T)} \leq c T^{-\gamma}$ for every $T > 0$, where $c$ does not depend on $T$. Then, for every $T > 0$, the Cauchy problem (5.7) has a global solution in the space $L^\infty(0, T; L^2(\overline{M}))$ for sufficiently small $u_0$ in $L^2$-norm.

**Proof.** (i) We start by two times integrating in $t$ the equation (5.7)

\[
u(t) = u_0 + tu_1 + \int_0^t (t - \tau)b(\tau)|Bu(\tau)|^p d\tau.
\]

By taking the $L^2$-norm on both sides, for $t < T$ one obtains

\[
\|u(t)\|_{L^2(\overline{M})} \leq C(\|u_0\|_{L^2(\overline{M})}^2 + t\|u_1\|_{L^2(\overline{M})}^2 + t^2\|b\|_{L^2(0,T)}^2 \int_0^t \|(Bu(\tau))\|_{L^{2p}(\overline{M})}^{2p} d\tau),
\]
since
\[
\left| \int_0^t (t - \tau)b(\tau)(Bu(\tau))^p d\tau \right|^2 \leq \left( \int_0^t \left| b(\tau)(Bu(\tau))^p \right| d\tau \right)^2 \leq t^2 \int_0^t \left| b(\tau) \right|^2 d\tau \int_0^t \left| Bu(\tau) \right|^{2p} d\tau
\]
and
\[
\int_{\mathcal{M}} \int_0^t \left| Bu(\tau) \right|^{2p} d\tau dx = \int_{\mathcal{M}} \int_0^t \left| Bu(\tau) \right|^{2p} dx d\tau = \int_0^t \left| Bu(\tau) \right|^{2p} d\tau d\mathcal{M}.
\]
Now, using conditions on the operator $B$, we get
\[
\|u(t)\|^2_{L^2(M)} \leq C(\|u\|^2_{L^2(M)} + \|u\|^2_{L^2(M)} + t^2\|b\|^2_{L^2(M)} \int_0^t \|u(\tau)\|^2_{L^2(M)} d\tau), \tag{5.9}
\]
for some constant $C$ not depending on $u_0$, $u_1$ and $t$. Finally, by taking the $L^\infty$-norm in time on both sides of the estimate (5.9), one obtains
\[
\|u\|^2_{L^\infty(0,T;L^2(M))} \leq C(\|u\|^2_{L^2(M)} + T\|u\|^2_{L^2(M)} + T^3\|b\|^2_{L^2(0,T)}\|u\|^2_{L^\infty(0,T;L^2(M))}). \tag{5.10}
\]
Let us introduce the set
\[
S_c := \{u \in L^\infty(0,T;L^2(M)) : \|u\|^2_{L^\infty(0,T;L^2(M))} \leq c(\|u\|^2_{L^2(M)} + T\|u\|^2_{L^2(M)}) \} \tag{5.11}
\]
for some constant $c \geq 1$. Then we have
\[
\|u_0\|^2_{L^2(M)} + T\|u_1\|^2_{L^2(M)} + T^3\|b\|^2_{L^2(M)}\|u_0\|^2_{L^\infty(0,T;L^2(M))} \leq \|u_0\|^2_{L^2(M)} + T\|u_1\|^2_{L^2(M)} + T^3\|b\|^2_{L^2(0,T)}\|u_0\|^2_{L^2(M)} \tag{5.12}
\]
Observe that, to be $u$ from the set $S_c$ it is enough to have
\[
\|u_0\|^2_{L^2(M)} + T\|u_1\|^2_{L^2(M)} + T^3\|b\|^2_{L^2(0,T)}\|u_0\|^2_{L^2(M)} + T\|u_1\|^2_{L^2(M)} \leq c(\|u_0\|^2_{L^2(M)} + T\|u_1\|^2_{L^2(M)}),
\]
It can be obtained by requiring the following
\[
T \leq T^* := \min \left[ \left( \frac{c - 1}{\|b\|^2_{L^2(0,T)} C^p\|u_0\|^2_{L^2(M)}} \right)^{1/3}, \left( \frac{c - 1}{\|b\|^2_{L^2(0,T)} C^p\|u_1\|^2_{L^2(M)}} \right)^{1/3} \right].
\]
Thus, by applying the fixed point theorem, there exists a unique local solution $u \in L^\infty(0,T^*;L^2(M))$ of the Cauchy problem (5.2).

Now we prove Part (ii). By repeating the arguments of the proof of Part (i), we start from (5.10). By taking into account assumptions, we have
\[
\|u\|^2_{L^\infty(0,T;L^2(M))} \leq C(\|u\|^2_{L^2(M)} + T^{3-2\gamma}\|u\|^2_{L^\infty(0,T;L^2(M))}). \tag{5.13}
\]
Fix $c \geq 1$. Introduce the set
\[
S_c := \{u \in L^\infty(0,T;L^2(M)) : \|u\|^2_{L^\infty(0,T;L^2(M))} \leq cT^{8\gamma}\|u\|^2_{L^2(M)} \},
\]
with $\gamma_0 > 0$ is to be defined later. Now, we have
\[
\|u_0\|_{L^2(M)}^2 + T^3\|b\|_{L^2(0,T;L^2(M))}^{2p} \leq \|u_0\|_{L^2(M)}^2 + T^{3-2\gamma + \gamma_0 p}\|u_0\|_{L^2(M)}^{2p},
\]
where $\gamma_0$ to be chosen later.

To guarantee $u \in S_c$, we require that
\[
\|u_0\|_{L^2(M)}^2 + T^3 - 2\gamma + \gamma_0 p \leq \|u_0\|_{L^2(M)}^2 c T^{-\tilde{\gamma} + \gamma_0}.
\]
Now by choosing $0 < \gamma_0 < \frac{2\gamma - 3}{p}$ such that
\[
\tilde{\gamma} := 3 - 2\gamma + \gamma_0 p < 0,
\]
we obtain
\[
c T^{-\tilde{\gamma} + \gamma_0} \leq \|u_0\|_{L^2(M)}^{2p - 2}.
\]
From the last estimate, we conclude that for any $T > 0$ there exists sufficiently small $\|u_0\|_{L^2(M)}$ such that IVP (5.7) has a solution. It proves Part (ii) of Theorem 5.7. □

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