Research Article

Multiattribute Decision-Making Method with Intuitionistic Fuzzy Archimedean Bonferroni Means

Hengzhi Zhao

School of Economics and Management, Hefei University, Hefei 230601, China

Correspondence should be addressed to Hengzhi Zhao; songjt352@163.com

Received 12 February 2021; Revised 1 March 2021; Accepted 13 March 2021; Published 30 April 2021

Academic Editor: Zaoli Yang

Copyright © 2021 Hengzhi Zhao. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The Bonferroni mean (BM) can portray the inter-relationships among the arguments, which is based on algebraic norm. Archimedean norm is the generalization of algebraic norm. In this work, the intuitionistic fuzzy (IF) BMs on the basis of the Archimedean norm are investigated, including the Archimedean norm-based IF arithmetic BM (AN-IFABM) and the Archimedean norm-based IF geometric BM (AN-IFGBM). Then, we discuss their typical properties and several particular cases of the AN-IFABM and AN-IFGBM in detail. Moreover, we design the Archimedean norm-based IF weighted arithmetic BM (AN-IFWABM) and the Archimedean norm-based IF weighted geometric BM (AN-IFWGBM) to consider the importance of each argument and their interconnections. Applying the proposed extensions, an approach is designed to cope with the multiattribute decision-making (MADM) problems. Finally, an efficiency evaluation problem of some public companies in China is analyzed by the proposed approach to demonstrate its applicability and validity.

1. Introduction

Fuzzy sets (FSs) [1] have been widely applied and extended to a variety of fields. However, FS is a set in which each element is denoted by a real number in (0, 1). To compensate for the deficiency, Atanassov [2–4] presented the intuitionistic fuzzy sets (IFSs). In IFSs, considering both membership degree and nonmembership degree, the data information is displayed in 2-tuples [5–12].

The objective of the MADM problem is to select the most acceptable solution from a group of options with the decision makers’ (DMs) preference information. The widely adopted method is the priority ranking method [13–16]. The aggregation methods are utilized to fuse preference information by using the aggregation operators (AOs). Xu [17] defined the operational laws of IFSs and then designed a sequence of AOs with IFSs. By means of geometric mean, Xu and Yager [18] gave some geometric mean AOs with IFSs, and then they provided an application for the MADM problems. Zhao et al. [19] developed the generalized AOs, which are followed by the discussion of their properties. For the new operations in IFSs, Wei [20] constructed some operators and then investigated a novel IF model to cope with MADM problems. Under the probabilistic linguistic information and picture fuzzy environment, Lin et al. [21, 22] designed different types of MADM models to deal with practical problems.

From the abovementioned analyze, we know that these AOs fail to capture the relationship among the aggregation arguments. To offset this flaw, several AOs have been generated. Under the IF information environment, Xu and Xia [23] investigated several new types of AOs on the basis of Choquet integral and Dempster–Shafer theory. Considering the Dempster–Shafer belief structure, Yang and Chen [24] proposed several new operators, which can capture the relationship among the input decision-making information. For the MADM problems with the prioritization ordering among the attributes, Yager [25] utilized the prioritized aggregation (PA) operators to develop a MADM method. In addition, Yu and Xu [26] proposed the prioritized IF aggregation (PIFA) operator to select the information security systems. Based on the power average operator [27], Xu [28] developed a novel IF MADM model. Zhou et al. [29, 30] proposed two generalized IF power aggregation algorithms,
and then they drew up a new MADM method to assign the appropriate weights to the experts.

The BM [31] can be utilized to capture the inter-relationships between input parameters. Some generalizations of BM were further proposed by Yager [32] to enhance their modeling capabilities. Furthermore, Xu and Yager [33] investigated the intuitionistic fuzzy BM (IFBM) and the weighted IFBM (WIFBM). To describe the inter-relationship between arguments, the IF geometric BM (IFGBM) and weighted IFGBM (WIFGBM) were introduced [34]. Under the linguistic Pythagorean fuzzy information environment, Lin et al. [35] developed a new MADM method by using several proposed linguistic Pythagorean fuzzy interaction partitioned Bonferroni mean aggregation operators. Xia et al. [36] defined the generalized IF weighted BM (GIFWBM) and Bonferroni geometric mean (GIFWBG). Lin et al. [37] first introduced the concept of linguistic q-rung orthopair fuzzy set and then designed the linguistic q-rung orthopair fuzzy interactional PGHM operator with the help of Heronian mean.

On the one hand, we can find that abovementioned BMs are all based on algebraic norm, which are special cases of Archimedean norm [38–40]. On the other hand, by using confidence level, Rahman et al. [41] proposed the confidence intuitionistic fuzzy Einstein hybrid averaging (CIFEHA) operator and confidence intuitionistic fuzzy Einstein hybrid geometric (CIFEHG) operator to fuse a group of IFNs into a collection IFN. However, the CIFEHA operator and CIFEHG operator [41] neglect the importance of each argument and their interconnections.

Therefore, in order to cope with these shortcomings, we construct a MADM method with intuitionistic fuzzy Archimedean Bonferroni means. The main contributions of this paper are as follows:

(i) The Archimedean norm-based IF arithmetic BM (AN-IFABM) and the Archimedean norm-based IF geometric BM (AN-IFGBM) are designed
(ii) The typical properties and several particular cases of AN-IFABM and AN-IFGBM are discussed
(iii) The Archimedean \( t \)-norm-based IF weighted arithmetic BM (AN-IFWABM) and the Archimedean \( t \)-norm-based IF weighted geometric BM (AN-IFWGBM) are presented
(iv) A MADM method is developed and applied to evaluate the public companies

The remainder of our work is structured as follows. In Section 2 several concepts relevant to BM and IFSs are introduced. Section 3 investigates two IFBMs based on Archimedean norm, such as AN-IFABM and AN-IFGBM, and develops some specific IFBMs. In Section 4, we present the weighted versions of the AN-IFABM and the AN-IFGBM. Section 5 develops a method for intuitionistic fuzzy MADM with the proposed BMs. Section 6 shows a numerical example to validate our method. The major conclusions of our work are summarized in Section 7.

### 2. Preliminaries

In the following, the BM, geometric BM, IFSs, and Archimedean norm are briefly retrospected. Then, some IF operational laws are presented in terms of Archimedean norm.

#### 2.1. BM and GBM

In order to capture the inter-relationships among attributes, BM [31] and geometric BM [34] are both valuable information aggregation methods, which are extensions of the arithmetic averaging (AA) [40] and the geometric averaging (GA) [42], respectively.

**Definition 1** (see [31]). Assume that \( \eta_i \) for all \( i \in N^* \) are a set of numbers that are not negative and \( b, d \geq 0 \). If

\[
BM^{b,d}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{n(n-1)} \sum_{i<j} \eta_i^b \eta_j^d, \tag{1}
\]

then BM\(^{b,d}\) is called a BM.

**Definition 2** (see [34]). Assume that \( \eta_i \) for all \( i \in N^* \) are a set of numbers that are not negative and \( b, d \geq 0 \). If

\[
GBM^{b,d}(\eta_1, \eta_2, \ldots, \eta_n) = \frac{1}{b+d} \prod_{i<j} (b\eta_i + d\eta_j), \tag{2}
\]

then GBM\(^{b,d}\) is called a GBM.

#### 2.2. Intuitionistic Fuzzy Operational Laws Based on Archimedean Norm

**Definition 3** (see [40]). Let \( a, b, c \in [0, 1] \); a function \( \phi : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a t-norm, if

1. \( \phi(1, a) = a \)
2. \( \phi(a, b) = \phi(b, a) \)
3. \( \phi(a, \phi(b, c)) = \phi(\phi(a, b), c) \)
4. If \( 0 \leq a \leq a' \leq 1 \) and \( 0 \leq b \leq b' \leq 1 \), we have \( \phi(a, b) \leq \phi(a', b') \).

In addition, if a t-norm \( \phi(a, b) \) is continuous and \( \phi(a, a) < a \) for all \( a \in (0, 1) \), then \( \phi(a, b) \) is an Archimedean t-norm.

**Definition 4** (see [40]). Let \( a, b, c \in [0, 1] \); a function \( \psi : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a t-conorm if it satisfies the following requirements:

1. \( \psi(0, a) = a \)
2. \( \psi(a, b) = \psi(b, a) \)
3. \( \psi(a, \psi(b, c)) = \psi(\psi(a, b), c) \)
4. If \( 0 \leq a \leq a' \leq 1 \) and \( 0 \leq b \leq b' \leq 1 \), we have \( \psi(a, b) \geq \psi(a', b') \).

In addition, if a t-conorm \( \psi(a, b) \) is continuous and \( \psi(a, a) > a \) for all \( a \in (0, 1) \), then \( \psi(a, b) \) is an Archimedean t-conorm.

According to [40, 43], we know that a strict Archimedean t-norm \( \phi(a, b) \) is expressed by an additive operator \( \delta \) as...
\( \varphi(a, b) = \delta^{-1}(\delta(a) + \delta(b)) \). Analogously, the dual t-conorm denoted by \( \psi(a, b) = \epsilon^{-1}(\epsilon(a) + \epsilon(b)) \) with \( \epsilon(t) = \delta(1-t) \), where \( \delta: [0, 1] \rightarrow [0, +\infty) \) such that \( \delta(1) = 0 \); then, \( \epsilon: [0, 1] \rightarrow [0, +\infty) \) is a strictly increasing function and \( \epsilon(0) = 0 \).

**Definition 5** (see [2]). An IFS \( I \) in \( X \) is given by

\[
I = \{(x, \mu_I(x), \upsilon_I(x)) | x \in X\},
\]

where the membership degree \( \mu_I: X \rightarrow [0, 1] \), the non-membership degree \( \upsilon_I: X \rightarrow [0, 1] \), and

\[
0 \leq \mu_I(x) + \upsilon_I(x) \leq 1, \quad \forall x \in X.
\]

An intuitionistic fuzzy number (IFN) is denoted by \((\mu_I(x), \upsilon_I(x))\) [18]. Xu and Yager [33] denoted an IFN by \( \theta = (\mu_\theta, \upsilon_\theta) \). Let \( H \) be the set of all IFNs.

**Definition 6** (see [31]). Let \( \theta = (\mu_\theta, \upsilon_\theta) \) be an IFN, the score function of \( \theta \) is \( s(\theta) = \mu_\theta - \upsilon_\theta \) and the accuracy function of \( \theta \) is \( f(\theta) = \mu_\theta + \upsilon_\theta \). Let \( \theta_1 \) and \( \theta_2 \) be two IFNs, then

1. If \( s(\theta_1) > s(\theta_2) \), then \( \theta_1 > \theta_2 \)
2. If \( s(\theta_1) = s(\theta_2) \), then
   a. If \( f(\theta_1) > f(\theta_2) \), then \( \theta_1 > \theta_2 \)
   b. If \( f(\theta_1) = f(\theta_2) \), then \( \theta_1 = \theta_2 \)

**Definition 7** (see [40]). Let \( \theta, \theta_1, \) and \( \theta_2 \) be three IFNs, then

1. \( \theta_1 \oplus \theta_2 = (\psi(\mu_{\theta_1}, \mu_{\theta_2}), \psi(\upsilon_{\theta_1}, \upsilon_{\theta_2})) = (\epsilon^{-1}(\epsilon(\mu_{\theta_1}) + \epsilon(\mu_{\theta_2})), \delta^{-1}(\delta(\upsilon_{\theta_1}) + \delta(\upsilon_{\theta_2}))) \)
2. \( \theta_1 \odot \theta_2 = (\psi(\mu_{\theta_1}, \mu_{\theta_2}), \psi(\upsilon_{\theta_1}, \upsilon_{\theta_2})) = (\delta^{-1}(\delta(\mu_{\theta_1}) + \delta(\mu_{\theta_2})), \epsilon^{-1}(\epsilon(\upsilon_{\theta_1}) + \epsilon(\upsilon_{\theta_2}))) \)
3. \( \lambda \theta = (\epsilon^{-1}(\lambda \epsilon(\mu_\theta)), \delta^{-1}(\lambda \delta(\upsilon_\theta))) \), \( \lambda > 0 \)
4. \( \theta^\lambda = (\delta^{-1}(\lambda \delta(\mu_\theta)), \epsilon^{-1}(\lambda \epsilon(\upsilon_\theta))) \), \( \lambda > 0 \)

**Theorem 1.** Let \( \theta, \theta_1, \) and \( \theta_2 \) be three IFNs, then we have

1. \( \theta_1 \oplus \theta_2 = \theta_2 \oplus \theta_1 \)
2. \( \theta_1 \odot \theta_2 = \theta_2 \odot \theta_1 \)
3. \( \lambda (\theta_1 \oplus \theta_2) = \lambda \theta_1 \oplus \lambda \theta_2, \lambda > 0 \)
4. \( (\theta_1 \odot \theta_2)^\lambda = \theta_1^\lambda \odot \theta_2^\lambda, \lambda > 0 \)
5. \( \lambda_1 \theta \oplus \lambda_2 \theta = (\lambda_1 + \lambda_2) \theta, \lambda_1, \lambda_2 > 0 \)
6. \( \theta_1^\lambda \odot \theta_2^\lambda = \theta_1^\lambda \odot \theta_2^\lambda, \lambda_1, \lambda_2 > 0 \)

### 3. IFBM Based on Archimedean Norm

In MADM, the performance values of an attribute are usually expressed with IFNs, which is a more objective reflection of the DM’s preference. An extension of the IFBM and IFGBM is given for the purpose of aggregating all performance values of the alternatives with respect to all attributes.

#### 3.1. IFABM Based on Archimedean Norm

As defined in Section 2, the operations can be utilized to integrate IF information. In this subsection, we shall investigate the AN-IFABM and then analyze some desirable properties of the AN-IFABM.

**Definition 8.** Let \( \theta_i = (\mu_{\theta_i}, \upsilon_{\theta_i}) \) for all \( i \in N^+ = \{1, 2, \ldots, n\} \) be a group of IFNs and \( b, d \geq 0 \); the AN-IFABM is denoted as

\[
\text{AN-IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j} (\theta_i \odot \theta_j)^{1/(b+d)}.
\]

**Theorem 2.** Let \( \theta_i = (\mu_{\theta_i}, \upsilon_{\theta_i}) \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), then the gathered value by using AN-IFABM is also an IFN, and
\[\text{AN} - \text{IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} (\delta_i \otimes \delta_j)\right)^{1/(b+d)}\]

\[= \left(\delta^{-1} \left(\frac{1}{b+d} \delta \left(\epsilon^{-1} \left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \epsilon \left(\delta^{-1}(b\delta(\mu_\theta) + d\delta(\mu_\theta))\right)\right)\right)\right)\right)^{1/(b+d)}, \quad (6)\]

\[\epsilon^{-1} \left(\frac{1}{b+d} \epsilon \left(\delta^{-1} \left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta \left(\epsilon^{-1}(b\epsilon(\nu_\theta) + d\epsilon(\nu_\theta))\right)\right)\right)\right)^{1/(b+d)}. \quad (7)\]

Proof. With the operations of IFN stated in Section 2, we obtain that

\[\theta_i = (\delta^{-1}(p\delta(\mu_\theta)), \epsilon^{-1}(b\epsilon(\nu_\theta))), \theta_j = (\delta^{-1}(d\delta(\mu_\theta)), \epsilon^{-1}(d\epsilon(\nu_\theta))), \quad \text{for all } i, j, \quad (7)\]

and then

\[\theta_i \otimes \theta_j = (\delta^{-1}(b\delta(\mu_\theta) + d\delta(\mu_\theta)), \epsilon^{-1}(b\epsilon(\nu_\theta) + d\epsilon(\nu_\theta))), \quad (8)\]

According to Definition 7, we can get that

\[\left(\sum_{i,j=1 \atop i \neq j}^{n} \delta \left(\epsilon^{-1}(b\delta(\mu_\theta) + d\delta(\mu_\theta))\right)\right)^{1/(b+d)}, \quad (9)\]
Then, we have

\[
\frac{1}{n(n-1)} \sum_{i,j=1}^{n} \left( \theta_i^k \otimes \theta_j^l \right) = \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \varepsilon \left( \varepsilon^{-1} \left( \sum_{i,j=1}^{n} \varepsilon \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta}) + d \delta(\mu_{\theta}) \right) \right) \right) \right) \right) \right),
\]

\[
\delta^{-1} \left( \frac{1}{n(n-1)} \delta \left( \delta^{-1} \left( \sum_{i,j=1}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta}) + d \delta(\mu_{\theta}) \right) \right) \right) \right) \right),
\]

\[
\delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\nu_{\theta}) + d \delta(\nu_{\theta}) \right) \right) \right),
\]

It follows that

\[
\left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \phi \left( \eta_i^k \otimes \eta_j^l \right) \right)^{1/(b+d)} = \left( \delta^{-1} \left( \frac{1}{n(n-1)} \delta \left( \delta^{-1} \left( \sum_{i,j=1}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\mu_{\theta}) + d \delta(\mu_{\theta}) \right) \right) \right) \right) \right) \right),
\]

\[
\varepsilon^{-1} \left( \frac{1}{b+d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \delta \left( \varepsilon^{-1} \left( b \delta(\nu_{\theta}) + d \delta(\nu_{\theta}) \right) \right) \right) \right) \right),
\]

(11)
i.e., equation (6) holds.

In the following, we prove that the gathered value with AN-IFABM is also an IFN. Since $\delta: [0, 1] \rightarrow [0, +\infty]$ is a strictly decreasing function, and $\varepsilon(t) = \delta(1 - t)$, then $\varepsilon(t)$ and $\varepsilon^{-1}(t)$ are two strictly increasing functions, $\delta^{-1}(t)$ is a strictly decreasing function, and $\varepsilon^{-1}(t) = 1 - \delta^{-1}(t)$. Therefore, we have

$$
\delta^{-1}\left(\frac{1}{b + d} \delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon\left(\delta^{-1}\left(b\delta(\mu_\theta) + d\delta(\mu_\theta)\right)\right)\right)\right) \right) \geq 0,
$$

(12)

$$
\varepsilon^{-1}\left(\frac{1}{b + d} \delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon\left(\delta^{-1}\left(b\delta(\mu_\theta) + d\delta(\mu_\theta)\right)\right)\right)\right) \right) \geq 0.
$$

Afterwards, we will demonstrate that

$$
\delta^{-1}\left(\frac{1}{b + d} \delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon\left(\delta^{-1}\left(b\delta(\mu_\theta) + d\delta(\mu_\theta)\right)\right)\right)\right) \right) + \varepsilon^{-1}\left(\frac{1}{b + d} \delta\left(\varepsilon^{-1}\left(\frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon\left(\delta^{-1}\left(b\delta(\nu_\theta) + d\delta(\nu_\theta)\right)\right)\right)\right) \right) \leq 1.
$$

(13)

Since $\mu_\theta \leq 1 - \nu_\theta, i \in N^*$, then $b \cdot \varepsilon(\nu_\theta) + d \cdot \varepsilon(\nu_\theta) = b \cdot \delta(1 - \nu_\theta) + d \cdot \delta(1 - \nu_\theta) \leq b \cdot \delta(\mu_\theta) + d \cdot \delta(\mu_\theta)$, for all $i, j$, and we have

$$
\varepsilon^{-1}\left(b \cdot \varepsilon(\nu_\theta) + d \cdot \varepsilon(\nu_\theta)\right) \leq \varepsilon^{-1}\left(b \cdot \delta(\mu_\theta) + d \cdot \delta(\mu_\theta)\right) = 1 - \delta^{-1}\left(b \cdot \delta(\mu_\theta) + d \cdot \delta(\mu_\theta)\right).
$$

(14)

As $\delta(t)$ is a strictly decreasing function, one can derive that
Thus, it follows that

\[
\delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \delta \left( \epsilon^{-1} \left( b \cdot \epsilon(\gamma_{ij}) + d \cdot \epsilon(\gamma_{ij}) \right) \right) \right) \leq \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \right) \right) 
\]

\[
= 1 - \epsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \right) \right),
\]

Hence,

\[
\frac{1}{b+d} \epsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \delta \left( \epsilon^{-1} \left( b \cdot \epsilon(\gamma_{ij}) + d \cdot \epsilon(\gamma_{ij}) \right) \right) \right) \right) 
\]

\[
\leq \frac{1}{b+d} \epsilon \left( 1 - \epsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \cdot \delta(\mu_{ij}) + d \cdot \delta(\mu_{ij}) \right) \right) \right) \right) 
\]

\[
= \frac{1}{b+d} \delta \left( \epsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b\delta(\mu_{ij}) + d\delta(\mu_{ij}) \right) \right) \right) \right),
\]
and then

\[
\begin{align*}
\varepsilon^{-1} & \left( \frac{1}{b+d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta^{-1}(b \cdot \varepsilon(\gamma_{\theta}) + d \cdot \varepsilon(\gamma_{\theta})) \right) \right) \right) \\
& \leq \varepsilon^{-1} \left( \frac{1}{b+d} \delta \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon(\delta^{-1}(b \cdot \delta(\mu_{\theta}) + d \cdot \delta(\mu_{\theta}))) \right) \right) \right) \\
& = 1 - \delta^{-1} \left( \frac{1}{b+d} \delta \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon(\delta^{-1}(b \delta(\mu_{\theta}) + d \delta(\mu_{\theta}))) \right) \right) \right),
\end{align*}
\]

i.e.,

\[
\begin{align*}
\varepsilon^{-1} & \left( \frac{1}{b+d} \varepsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta^{-1}(b \cdot \varepsilon(\gamma_{\theta}) + d \cdot \varepsilon(\gamma_{\theta})) \right) \right) \right) \\
& + \delta^{-1} \left( \frac{1}{b+d} \delta \left( \varepsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \varepsilon(\delta^{-1}(b \cdot \delta(\mu_{\theta}) + d \cdot \delta(\mu_{\theta}))) \right) \right) \right) \leq 1,
\end{align*}
\]

which completes the demonstration of Theorem 2. \( \square \)

**Theorem 3** (idempotency). Let \( \theta_i \) for all \( i \in \mathbb{N}^+ \) be a group of IFNs and \( b, d \geq 0 \). If all \( \theta_i = \theta = (\mu_{\theta}, \gamma_{\theta}) \) for all \( i \), then

\[
AN - IFABM_{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = (\mu_{\theta}, \gamma_{\theta}) = \theta.
\]

**Proof.** Since \( \theta_1 = \theta_2 = \cdots = \theta_n = \theta = (\mu_{\theta}, \gamma_{\theta}) \), then we have
AN – IFABM$^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = AN – IFABM^{b,d}(\theta, \theta, \ldots, \theta)$

\[
\begin{align*}
&= \left( \delta^{-1} \left( \frac{1}{b + d} \delta \left( \epsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \epsilon^{-1} \left( \delta^{-1} \left( \delta^{-1} \left( (b + d) \delta(\mu_{\theta}) + d \epsilon(\nu_{0}) \right) \right) \right) \right) \right) \right) , \\
&\quad \epsilon^{-1} \left( \frac{1}{b + d} \delta \left( \epsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \epsilon^{-1} \left( \delta^{-1} \left( \delta^{-1} \left( (b + d) \delta(\mu_{\theta}) + d \epsilon(\nu_{0}) \right) \right) \right) \right) \right) \\
&\quad = \left( \delta^{-1} \left( \frac{1}{b + d} \delta(\epsilon^{-1} \left( \epsilon^{-1} \left( (b + d) \delta(\mu_{\theta}) + d \epsilon(\nu_{0}) \right) \right) \right) \right) , \\
&\quad \epsilon^{-1} \left( \frac{1}{b + d} \delta(\epsilon^{-1} \left( (b + d) \delta(\mu_{\theta}) + d \epsilon(\nu_{0}) \right) \right) \\
&\quad = \left( \delta^{-1} \left( \delta(\mu_{\theta}) \right) , \epsilon^{-1} \left( \epsilon(\nu_{0}) \right) \right) = (\mu_{\theta}, \nu_{0}) = \theta.
\end{align*}
\]

(22)

The proof is completed.

**Remark 1.** If $\theta_i$ for all $i \in N^+$ are a set of the smallest IFNs, i.e., $\theta_1 = \theta_2 = \ldots = \theta_n = \theta^- = (0, 1)$, then

\[
AN – IFABM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = AN – IFABM^{b,d}(\theta^-, \theta^-, \ldots, \theta^-) = (0, 1).
\]

(23)

If $\theta_i$ for all $i \in N^+$ are a group of the largest IFNs, i.e., $\theta_1 = \theta_2 = \ldots = \theta_n = \theta^+ = (1, 0)$, then

\[
AN – IFABM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = AN – IFABM^{b,d}(\theta^+, \theta^+, \ldots, \theta^+) = (1, 0).
\]

(24)

**Theorem 4** (monotonicity). Let $\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})$ and $\delta_i = (\mu_{\delta_i}, \nu_{\delta_i})$ for all $i \in N^+$ be two collections of IFNs, $b, d \geq 0$. If $\mu_{\theta_i} \leq \mu_{\theta_j}$ and $\nu_{\theta_i} \geq \nu_{\theta_j}$, then

\[
AN – IFABM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) \leq AN – IFABM^{b,d}(\delta_1, \delta_2, \ldots, \delta_n).
\]

(25)

\[
\delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta(\epsilon^{-1} \left( b \cdot \epsilon(\nu_{\theta_i}) + d \cdot \epsilon(\nu_{\theta_j}) \right) \right) \right) \geq \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta(\epsilon^{-1} \left( b \cdot \epsilon(\nu_{\delta_i}) + d \cdot \epsilon(\nu_{\delta_j}) \right) \right) \right).
\]

(28)
Hence,

\[
\begin{align*}
\delta^{-1} & \left( \frac{1}{b + d} \right) \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \theta_{j} \right) \right) \\
\leq \delta^{-1} & \left( \frac{1}{b + d} \right) \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \theta_{j} \right) \right) \\
\geq \delta^{-1} & \left( \frac{1}{b + d} \right) \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \theta_{j} \right) \right)
\end{align*}
\]

(29)

(30)

Let \( \theta = \text{AN} - \text{IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) \) and \( \vartheta = \text{AN} - \text{IFABM}^{b,d}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) \), then equations (29) and (30) are equal to \( \mu_{\theta} \leq \mu_{\vartheta} \) and \( \nu_{\vartheta} \geq \nu_{\theta} \). Hence, we have

\[
s(\theta) = \mu_{\theta} - \nu_{\vartheta} \leq \mu_{\theta} - \nu_{\theta} = s(\vartheta).
\]

(31)

Case 1. If \( s(\theta) < s(\vartheta) \), then by using Definition 6, we can get that \( \theta < \vartheta \), i.e.,

\[
\text{AN} - \text{IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) < \text{AN} - \text{IFABM}^{b,d}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n).
\]

(32)

Case 2. If \( s(\theta) = s(\vartheta) \), i.e., \( \mu_{\theta} - \nu_{\theta} = \mu_{\vartheta} - \nu_{\vartheta} \), then \( \mu_{\theta} + \nu_{\vartheta} = \mu_{\vartheta} + \nu_{\vartheta} \). Since \( \mu_{\theta} \leq \mu_{\vartheta} \) and \( \nu_{\vartheta} \geq \nu_{\theta} \), thus \( \mu_{\theta} = \mu_{\vartheta} \) and \( \nu_{\vartheta} = \nu_{\theta} \), then we have

\[
f(\theta) = \mu_{\theta} + \nu_{\theta} = \mu_{\vartheta} + \nu_{\vartheta} = f(\vartheta).
\]

(33)

Then, by Definition 6, we have \( \theta = \vartheta \), i.e.,

\[
\text{AN} - \text{IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \text{AN} - \text{IFABM}^{b,d}(\vartheta_1, \vartheta_2, \ldots, \vartheta_n).
\]

(34)

According to equations (32) and (34), we can obtain that equation (25) holds.

**Theorem 5** (boundedness). Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), and let

\[
\theta^L = \left( \min_i \{\mu_{\theta_i}\}, \max_i \{\nu_{\theta_i}\} \right), \quad \theta^U = \left( \max_i \{\mu_{\theta_i}\}, \min_i \{\nu_{\theta_i}\} \right).
\]

(35)

then we have

\[
\theta^L \leq \text{AN} - \text{IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) \leq \theta^U.
\]

(36)

**Proof.** As \( \min_i \{\mu_{\theta_i}\} \leq \mu_{\theta} \leq \max_i \{\mu_{\theta_i}\} \) and \( \max_i \{\nu_{\theta_i}\} \geq \nu_{\theta} \geq \min_i \{\nu_{\theta_i}\} \), for all \( i \), by Theorem 4, we obtain that

\[
\text{AN} - \text{IFABM}^{b,d}(\theta^L, \theta^L, \ldots, \theta^L) \leq \text{AN} - \text{IFABM}^{b,d}(\theta^L, \theta^L, \ldots, \theta^L) \leq \text{AN} - \text{IFABM}^{b,d}(\theta^U, \theta^U, \ldots, \theta^U).
\]

(37)

Furthermore, according to Theorem 3, we have

\[
\text{AN} - \text{IFABM}^{b,d}(\theta^L, \theta^L, \ldots, \theta^L) = \theta^L,
\]

\[
\text{AN} - \text{IFABM}^{b,d}(\theta^U, \theta^U, \ldots, \theta^U) = \theta^U.
\]

(38)

Hence, from equations (37) and (38), we get that

\[
\theta^L \leq \text{AN} - \text{IFABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) \leq \theta^U.
\]

(39)

\[\Box\]

**Theorem 6** (commutativity). Let \( \theta_i = (\mu_{\theta_i}, \nu_{\theta_i}) \) for all \( i \in N^+ \) be a group of IFNs and \( b, d \geq 0 \), then
AN–IFABM\(^{b,d}(\theta_1,\theta_2,\ldots,\theta_n)\) = AN–IFABM\(^{b,d}(\bar{\theta}_1,\bar{\theta}_2,\ldots,\bar{\theta}_n)\),

where \((\bar{\theta}_1,\bar{\theta}_2,\ldots,\bar{\theta}_n)\) is any permutation of \((\theta_1,\theta_2,\ldots,\theta_n)\).

\begin{equation}
\text{AN–IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1 \atop i \neq j}^n (\theta_i^b \otimes \theta_j^d)\right)^{1/(b+d)} = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1 \atop j \neq i}^n (\theta_i^d \otimes \theta_j^b)\right)^{1/(d+b)}
\end{equation}

\textbf{Theorem 7.} Let \(\theta_i\) for all \(i \in N^+\) be a set of IFNs and \(b, d \geq 0\), then

\begin{equation}
\text{AN–IFABM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \text{AN–IFABM}^{d,b}(\theta_1,\theta_2,\ldots,\theta_n).
\end{equation}

Proof. By using Theorem 1, we get

If we assign the generator \(\delta\) with different forms, then AN–IFABM reduces to some specific intuitionistic fuzzy BMs.

\begin{equation}
\text{IFBM}^{b,d}(\theta_1,\theta_2,\ldots,\theta_n) = \left(1 - \prod_{i,j=1 \atop i \neq j}^n \left(1 - \mu_{\theta_i}^b \mu_{\theta_j}^d\right)^{1/(b+d)}\right)^{1/(b+d)} = \left(1 - \prod_{i,j=1 \atop i \neq j}^n \left(1 - \nu_{\theta_i}^b \nu_{\theta_j}^d\right)^{1/(n(n-1))}\right)^{1/(b+d)}
\end{equation}

Remark 2. If \(\delta(t) = -\log t\), the AN–IFABM converts to the IFBM defined by Xu and Yager [33]:
Remark 3. If $\delta(t) = \log_e (2 - t) - \log_e t$, the AN-IFABM changes to the intuitionistic fuzzy Einstein BM (IFEBM):

$$
\text{IFEBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{2[P(\mu_{\theta_1}, \mu_{\theta_2}) - Q(\mu_{\theta_1}, \mu_{\theta_2})]}{P(\mu_{\theta_1}, \mu_{\theta_2}) + 3Q(\mu_{\theta_1}, \mu_{\theta_2})} \right)^{1/(b+d)}
\left( \frac{M(\gamma_{\theta_1}, \gamma_{\theta_2}) + 3N(\gamma_{\theta_1}, \gamma_{\theta_2})}{M(\gamma_{\theta_1}, \gamma_{\theta_2}) + 3N(\gamma_{\theta_1}, \gamma_{\theta_2})} \right)^{1/(b+d)},
$$

where

$$
P(\mu_{\theta_1}, \mu_{\theta_2}) = \prod_{i,j=1 \atop i \neq j}^n \left( 2 - \mu_{\theta_i} \right)^b \left( 2 - \mu_{\theta_j} \right)^d + 3\mu_{\theta_i}^b \mu_{\theta_j}^d \right)^{1/(n(n-1))},
$$

$$
Q(\mu_{\theta_1}, \mu_{\theta_2}) = \prod_{i,j=1 \atop i \neq j}^n \left( 2 - \mu_{\theta_i} \right)^b \left( 2 - \mu_{\theta_j} \right)^d - \mu_{\theta_i}^b \mu_{\theta_j}^d \right)^{1/(n(n-1))},
$$

$$
M(\gamma_{\theta_1}, \gamma_{\theta_2}) = \prod_{i,j=1 \atop i \neq j}^n \left( 1 + \gamma_{\theta_i} \right)^b \left( 1 + \gamma_{\theta_j} \right)^d + 3(1 - \gamma_{\theta_i})^b (1 - \gamma_{\theta_j})^d \right)^{1/(n(n-1))},
$$

$$
N(\gamma_{\theta_1}, \gamma_{\theta_2}) = \prod_{i,j=1 \atop i \neq j}^n \left( 1 + \gamma_{\theta_i} \right)^b \left( 1 + \gamma_{\theta_j} \right)^d - (1 - \gamma_{\theta_i})^b (1 - \gamma_{\theta_j})^d \right)^{1/(n(n-1))}.
$$

If we take the parameters $b$ and $d$ of the AN-IFABM in various values, then several special conditions can be derived as follows:

Remark 4. If $b \to 0$ or $b \to 0$, then the AN-IFABM is transformed as the Archimedean norm-based generalized IF mean (AN-GIFM) (take $d \to 0$ for example):

$$
\text{AN-GIFM}(\theta_1, \theta_2, \ldots, \theta_n) = \lim_{d \to 0} \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^n (\theta_i^b \otimes \theta_j^d) \right)^{1/(b+d)}
= \left( \frac{1}{n} \sum_{i=1}^n \delta(\theta_i^b) \right)^{1/b}
= \left( \frac{1}{n} \sum_{i=1}^n \delta(\theta_i^b) \right)^{1/b}
= \left( \frac{1}{n} \sum_{i=1}^n \delta(\theta_i^b) \right)^{1/b}
= \left( \frac{1}{n} \sum_{i=1}^n \delta(\theta_i^b) \right)^{1/b}.
$$
Remark 5. If \( b \geq 2 \) and \( d \to 0 \), then the AN-IFABM is transformed as the Archimedean norm-based IF square mean (AN-IFSM):

\[
\text{AN-IFSM}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n} \sum_{i=1}^{n} \theta_i \right)^{1/2} = \left( \delta^{-1} \left( \frac{1}{2} \delta \left( \epsilon^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon(\delta(\mu_{\theta_i})) \right) \right) \right) \right), \epsilon^{-1} \left( \frac{1}{2} \epsilon \left( \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon(\delta(\nu_{\theta_i})) \right) \right) \right) \right). \quad (48)
\]

Remark 6. If \( b = 1 \) and \( d \to 0 \), then the AN-IFABM is transformed as the Archimedean norm-based IF averaging (AN-IFA) [38]:

\[
\text{AN-IFA}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{n} \sum_{i=1}^{n} \theta_i = \left( \epsilon^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon(\mu_{\theta_i}) \right), \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta(\nu_{\theta_i}) \right) \right). \quad (49)
\]

Remark 7. If \( b = d = 1 \), then the AN-IFABM is transformed as the following:

\[
\text{AN-IFISM}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n} \sum_{i=1}^{n} \theta_i \right)^{1/2} = \left( \delta^{-1} \left( \frac{1}{2} \delta \left( \epsilon^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon(\delta(\mu_{\theta_i}) + \delta(\mu_{\theta_j})) \right) \right) \right) \right), \epsilon^{-1} \left( \frac{1}{2} \epsilon \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon(\nu_{\theta_i}) + \epsilon(\nu_{\theta_j}) \right) \right) \right) \right). \quad (50)
\]

which is named the Archimedean norm-based IF interrelated square mean (AN-IFISM).

3.2. IFGBM Based on Archimedean Norm. Motivated by the geometric mean [42], we shall investigate the AN-IFGBM, and we also explore several adequate properties of the AN-IFGBM.

Definition 9. Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \); the AN-IFGBM is expressed as

\[
\text{AN-IFGBM}_{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{b+d} \sum_{i,j=1}^{n} \delta(\theta_i \theta_j) \right) \left( b \theta_i \theta_j d \right)^{1/(n(n-1))}. \quad (51)
\]
Theorem 8. Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), then the gathered value by using the AN-IFGBM is also an IFN, and

\[
\text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b + d} \sum_{i,j=1 \atop i \neq j}^{n} (b \theta_i \theta_d \theta_j)^{1/(n(n-1))}
\]

\[
= \left( \varepsilon^{-1} \left( \frac{1}{b + d} \varepsilon^\delta \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta^{-1} \left( b \varepsilon(\mu_{\theta_i} + d \varepsilon(\mu_{\theta_j})) \right) \right) \right) \right),
\]

\[\delta^{-1} \left( \frac{1}{b + d} \delta \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} \delta^{-1} \left( b \delta(\nu_{\theta_i} + d \delta(\nu_{\theta_j})) \right) \right) \right).
\]

Similarly, the AN-IFGBM also satisfies the properties that the AN-IFAMB has.

Theorem 9 (idempotency). Let \( \theta_i \) for all \( i \in N^+ \) be a group of IFNs and \( b, d \geq 0 \). If all \( \theta_i \) are equal, i.e., \( \theta_i = \theta = (\mu_{\theta}, \nu_{\theta}) \), for all \( i \), then

\[
\text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = (\mu_{\theta}, \nu_{\theta}) = \theta.
\]

Remark 8. If \( \theta_i \) for all \( i \in N^+ \) is a group of the smallest IFNs, i.e., \( \theta_1 = \theta_2 = \cdots = \theta_n = \theta^- = (0, 1) \), then

\[
\text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \text{AN - IFGBM}^{bd}(\theta^-) = (0, 1).
\]

If \( \theta_i \) for all \( i \in N^+ \) are a group of the largest IFNs, i.e., \( \theta_1 = \theta_2 = \cdots = \theta_n = \theta^+ = (1, 0) \), then

\[
\text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \text{AN - IFGBM}^{bd}(\theta^+) = (1, 0).
\]

Theorem 10. (monotonicity). Let \( \theta_1 \) and \( \theta_2 \) for all \( i \in N^+ \) be two collections of IFNs, \( b, d \geq 0 \). If \( \mu_{\theta_0} \leq \mu_{\theta_1} \) and \( \nu_{\theta_0} \geq \nu_{\theta_1} \), for all \( i \), then

\[
\text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) \leq \text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n).
\]

Theorem 11 (boundedness). Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), and let

\[
\theta^l = \left( \min_i \{\mu_{\theta_i}, \max_i \{\nu_{\theta_i}\} \right),
\]

\[
\theta^U = \left( \max_i \{\mu_{\theta_i}, \min_i \{\nu_{\theta_i}\} \right).
\]

then we have

\[
\theta^l \leq \text{AN - IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) \leq \theta^U.
\]

Theorem 12 (commutativity). Let \( \theta_i \) for all \( i \in N^+ \) be a set of IFNs and \( b, d \geq 0 \), then

\[
\text{AN - IFGBM}^{bd}(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_n) = \text{AN - IFGBM}^{bd}(\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_n),
\]

where \( (\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_n) \) is any permutation of \( (\theta_1, \theta_2, \ldots, \theta_n) \).

Moreover, if we assign the additive generator \( \delta \) with different forms, then the AN-IFGBM reduces to some specific intuitionistic fuzzy geometric BMs.

Remark 9. If \( \delta(t) = -\log_2 t \), the AN-IFGBM is transformed as the IFGBM defined by Xia et al. [34]:

\[
\text{IFGBM}^{bd}(\theta_1, \theta_2, \ldots, \theta_n) = \left( 1 - \left( 1 - \prod_{i,j=1 \atop i \neq j}^{n} \left( 1 - (1 - \mu_{\theta_i})^d (1 - \mu_{\theta_j})^d \right)^{1/(n(n-1))} \right) \right) \times \left( 1 - \prod_{i,j=1 \atop i \neq j}^{n} \left( 1 - \nu_{\theta_i} \nu_{\theta_j}^d \right)^{1/(n(n-1))} \right). \]
Remark 10. If $\delta(t) = \log_e(2-t) - \log_e t$, the AN-IFGBM is transformed as the intuitionistic fuzzy Einstein geometric BM (IFGBM):

$$\text{IFGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{M(\mu_{\theta_i}, \mu_{\theta_j}) + 3N(\mu_{\theta_i}, \mu_{\theta_j})}{M(\mu_{\theta_i}, \mu_{\theta_j}) + 3N(\mu_{\theta_i}, \mu_{\theta_j})} \right)^{1/(b+d)} \left( \frac{P(\nu_{\theta_i}, \nu_{\theta_j}) - Q(\nu_{\theta_i}, \nu_{\theta_j})}{P(\nu_{\theta_i}, \nu_{\theta_j}) + 3Q(\nu_{\theta_i}, \nu_{\theta_j})} \right)^{1/(b+d)},$$

(61)

where

$$M(\mu_{\theta_i}, \mu_{\theta_j}) = \prod_{\substack{i,j=1 \atop i \neq j}}^n \left( 1 + \mu_{\theta_i} \right)^b \left( 1 + \mu_{\theta_j} \right)^d + \frac{3}{2} \left( 1 - \mu_{\theta_i} \right)^b \left( 1 - \mu_{\theta_j} \right)^d \left( 1/(n(n-1)) \right)^{1/2},$$

$$N(\mu_{\theta_i}, \mu_{\theta_j}) = \prod_{\substack{i,j=1 \atop i \neq j}}^n \left( 1 + \mu_{\theta_i} \right)^b \left( 1 + \mu_{\theta_j} \right)^d - \left( 1 - \mu_{\theta_i} \right)^b \left( 1 - \mu_{\theta_j} \right)^d \left( 1/(n(n-1)) \right)^{1/2},$$

$$P(\nu_{\theta_i}, \nu_{\theta_j}) = \prod_{\substack{i,j=1 \atop i \neq j}}^n \left( 2 - \nu_{\theta_i} \right)^b \left( 2 - \nu_{\theta_j} \right)^d + \frac{3}{2} \nu_{\theta_i}^b \nu_{\theta_j}^d \left( 1/(n(n-1)) \right)^{1/2},$$

$$Q(\nu_{\theta_i}, \nu_{\theta_j}) = \prod_{\substack{i,j=1 \atop i \neq j}}^n \left( 2 - \nu_{\theta_i} \right)^b \left( 2 - \nu_{\theta_j} \right)^d - \nu_{\theta_i}^b \nu_{\theta_j}^d \left( 1/(n(n-1)) \right)^{1/2}. $$

In the following, we will consider several exceptional cases of the AN-IFGBM with diverse values of the parameters $b$ and $d$.

Remark 11. If $b \to 0$ or $d \to 0$, then the AN-IFGBM is transformed as the Archimedean norm-based generalized IF geometric mean (AN-GIFGM) (take $d \to 0$ for example):

$$\text{AN-GIFGM}(\theta_1, \theta_2, \ldots, \theta_n) = \lim_{d \to 0} \left( \frac{b \delta(b \delta \theta_i)}{b + d \delta(b \delta \theta_i)} \right)^{1/(n(n-1))} = \frac{1}{b} \sum_{i=1}^n \left( b \delta(b \delta \theta_i) \right)$$

(63)

$$(b \theta_i)^{1/10} = \left( \frac{1}{b} \delta^{-1} \left( \delta^{-1} \left( \frac{1}{b} \sum_{i=1}^n \delta(b \delta \theta_i) \right) \right) \right)^{1/10} \delta^{-1} \left( \frac{1}{b} \delta^{-1} \left( \frac{1}{b} \sum_{i=1}^n \delta(b \delta \theta_i) \right) \right)^{1/10}. $$
**Remark 12.** If $b = 2$ and $d \to 0$, then the AN-IFGBM is transformed as

\[
\text{AN-IFSGM}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{2} \bigotimes_{i=1}^{n} (2\theta_i)^{1/n}
\]

which we call the Archimedean norm-based IF square geometric mean (AN-IFSGM).

**Remark 13.** If $b = 1$ and $d \to 0$, the AN-IFGBM is transformed as the Archimedean norm-based intuitionistic fuzzy geometric averaging (AN-IFGA) [40]:

\[
\text{AN-IFGA}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{2} \bigotimes_{i=1}^{n} (\theta_i)^{1/n}
\]

\[
= \left( \varepsilon^{-1} \left( \frac{1}{2} \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta^{-1} \left( \varepsilon^{-1} (\mu_{\theta_i}) \right) \right) \right), \delta^{-1} \left( \frac{1}{2} \delta^{-1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta^{-1} \left( \delta^{-1} (\nu_{\theta_i}) \right) \right) \right) \right).
\]

**Remark 14.** If $b = d = 1$, then the AN-IFGBM is transformed as the Archimedean norm-based IF interrelated geometric square mean (AN-IFIGSM):

\[
\text{AN-IFIGSM}(\theta_1, \theta_2, \ldots, \theta_n)
\]

\[
= \frac{1}{2} \bigotimes_{i \neq j}^{n} (\theta_i \oplus \theta_j)^{1/(mn-1)}
\]

\[
= \varepsilon^{-1} \frac{1}{2}
\]

\[
\cdot \varepsilon^{-1} \frac{1}{mn} \sum_{i \neq j}^{n} \delta^{-1} (\varepsilon^{-1} (\delta^{-1} (\mu_{\theta_i} + \mu_{\theta_j})), \delta^{-1} (\delta^{-1} (\delta^{-1} (\nu_{\theta_i} + \nu_{\theta_j}))).
\]

4. IFWBMs Based on Archimedean Norm

The AN-IFABM and AN-IFGBM mentioned in Section 3 take into consideration the connection between the attributes, in which each element gives the same contribution to AN-IFABM and AN-IFGBM. While, sometimes, different attributes have an unequal importance in real-life problems, thus, each attribute produces different effects on the results. To evaluate the impact of each argument, we develop two IFWBMs with Archimedean norm.
Definition 10. Let \( \theta_i \) for all \( i \in N^* \) be a group of IFNs and \( b, d \geq 0 \); an AN-IFWABM is a mapping AN-IFWABM: \( H^n \rightarrow H \), satisfying

\[
\text{AN - IFWABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \bigoplus_{i,j=1}^{n} (v_i \theta_i)^b \otimes (v_j \theta_j)^d \right)^{1/(b+d)}, \quad (67)
\]

where \( v = (v_1, v_2, \ldots, v_n)^T \) is the weight vector (WV), and \( v_i > 0, \sum_{i=1}^{n} v_i = 1 \).

Theorem 13. Let \( \theta_i \) for all \( i \in N^* \) be a group of IFNs, whose WV is \( v = (v_1, v_2, \ldots, v_n)^T \), and satisfy \( v_i > 0, \sum_{i=1}^{n} v_i = 1 \). Let \( b, d \geq 0 \), then the gathered value using the AN-IFWABM is also an IFN, and

\[
\text{AN - IFWABM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{n(n-1)} \bigoplus_{i,j=1}^{n} (v_i \theta_i)^b \otimes (v_j \theta_j)^d \right)^{1/(b+d)}
\]

\[
= \left( \delta^{-1} \left( \frac{1}{b+d} \right) \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \delta^{-1} \left( \epsilon \left( v_i \theta_i \mu \theta_i \right) \right) \right) + d \delta^{-1} \left( \epsilon \left( v_j \theta_j \nu \theta_j \right) \right) \right) \right) \right) \right) \left( \delta^{-1} \left( \frac{1}{b+d} \right) \left( \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \epsilon \left( \delta^{-1} \left( b \delta^{-1} \left( \epsilon \left( v_i \theta_i \nu \theta_i \right) \right) \right) + d \delta^{-1} \left( \epsilon \left( v_j \theta_j \mu \theta_j \right) \right) \right) \right) \right) \right), \quad (68)
\]

If we assign the generator \( \delta \) with various forms, then one can derive several specific IF weighted BMs.

Remark 15. If \( \delta(t) = -\log_t \), the AN-IFWABM is transformed as the IF weighted BM (IFWBM) [33]:

\[
\text{IFWBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( 1 - \prod_{i,j=1}^{n} \left( 1 - \left( 1 - \mu \theta_i \right)^b \left( 1 - \mu \theta_j \right)^d \right) \right)^{1/(b+d)} \left. 1 - \prod_{i,j=1}^{n} \left( 1 - \left( 1 - \nu \theta_i \right)^b \left( 1 - \nu \theta_j \right)^d \right) \right)^{1/(b+d)}, \quad (69)
\]
\(\text{IFEWBM}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{2^{\left( A\left( \mu_{\theta_j}, \mu_{\theta_j} \right) - B\left( \mu_{\theta_j}, \mu_{\theta_j} \right) \right)}}{A\left( \mu_{\theta_j}, \mu_{\theta_j} \right) + 3B\left( \mu_{\theta_j}, \mu_{\theta_j} \right)} \right)^{1/(b+d)} \)

\[
\begin{align*}
A\left( \mu_{\theta_j}, \mu_{\theta_j} \right) &= \prod_{i,j=1}^{n} \left( \left( \left( 1 + \mu_{\theta_j} \right)^{\nu_i} + 3\left( 1 - \mu_{\theta_j} \right)^{\nu_i} \right)^b \left( \left( 1 + \mu_{\theta_j} \right)^{\nu_i} + 3\left( 1 - \mu_{\theta_j} \right)^{\nu_i} \right)^d \right), \\
B\left( \mu_{\theta_j}, \mu_{\theta_j} \right) &= \prod_{i,j=1}^{n} \left( \left( \left( 1 + \mu_{\theta_j} \right)^{\nu_i} + 3\left( 1 - \mu_{\theta_j} \right)^{\nu_i} \right)^b \left( \left( 1 + \mu_{\theta_j} \right)^{\nu_i} + 3\left( 1 - \mu_{\theta_j} \right)^{\nu_i} \right)^d \right), \\
C\left( \nu_{\theta_j}, \nu_{\theta_j} \right) &= \prod_{i,j=1}^{n} \left( \left( \left( 2 - \nu_{\theta_j} \right)^{\nu_i} + 3\nu_{\theta_j}^{\nu_i} \right)^b \left( \left( 2 - \nu_{\theta_j} \right)^{\nu_i} + 3\nu_{\theta_j}^{\nu_i} \right)^d \right), \\
D\left( \nu_{\theta_j}, \nu_{\theta_j} \right) &= \prod_{i,j=1}^{n} \left( \left( \left( 2 - \nu_{\theta_j} \right)^{\nu_i} + 3\nu_{\theta_j}^{\nu_i} \right)^b \left( \left( 2 - \nu_{\theta_j} \right)^{\nu_i} + 3\nu_{\theta_j}^{\nu_i} \right)^d \right),
\end{align*}
\]

Definition 11. Let \(\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})\) for all \(i \in N^+\) be a group of IFNs and \(b, d \geq 0\); an AN-IFWGBM is defined as

\[
\text{AN-IFWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{1}{b+d} \sum_{i,j=1}^{n} (b \nu_i \nu_{\theta_j} \nu_{\theta_j})^{1/(n(n-1))} \right),
\]

where \(v = (v_1, v_2, \ldots, v_n)^T\) is the WV, and \(v_i > 0, \sum_{i=1}^{n} v_i = 1\).

Theorem 14. Let \(\theta_i = (\mu_{\theta_i}, \nu_{\theta_i})\) for all \(i \in N^+\) be a collection of IFNs, whose WV is \(v = (v_1, v_2, \ldots, v_n)^T\), and satisfy \(v_i > 0, \sum_{i=1}^{n} v_i = 1\). Let \(b, d \geq 0\), then the assembled value using the AN-IFWGBM is also an IFN, and
\[ AN - IFWGBM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{b + d} \prod_{i,j=1}^{n} (b\theta_i^{\varphi_i} + d\theta_j^{\varphi_j})^{1/(n(n-1))} \]

\[
= \left( \varepsilon^{-1} \left( \frac{1}{b + d} \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{j=1}^{n} \delta^{-1} \left( b\varepsilon^{-1}(v_j\delta(\mu_0)) + d\varepsilon^{-1}(v_j\delta(\mu_0))) \right) \right) \right) \right),
\]

\[
\delta^{-1} \left( \frac{1}{b + d} \delta^{-1} \left( \frac{1}{n(n-1)} \sum_{j=1}^{n} \varepsilon^{-1} \left( b\delta^{-1}(v_j\epsilon(\nu_0)) + d\delta^{-1}(v_j\epsilon(\nu_0))) \right) \right) \right).
\]

**Proof.** The demonstration of Theorem 14 is analogous to Theorem 8.
Moreover, given different forms for the additive generator \( \delta \), several generalized IFWGBMs can be obtained as follows.

\[
IFWGBM^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( 1 - \prod_{i,j=1}^{n} \left( 1 - (1 - \mu_0^\varphi_i)(1 - \mu_0^\varphi_j) \right)^{1/(n(n-1))} \right) \left( 1 - \prod_{i,j=1}^{n} \left( 1 - (1 - \nu_0^\varphi_i)(1 - \nu_0^\varphi_j) \right)^{1/(n(n-1))} \right)^{1/(b+d)}
\]

**Remark 17.** If \( \delta(t) = -\log_t(t) \), then the AN-IFWGBM is transformed as the IFWGBM [34]:

\[
\text{IFWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{C(\mu_0, \mu_0) + 3D(\mu_0, \mu_0)}{C(\mu_0, \mu_0) + 3D(\mu_0, \mu_0)} \right)^{1/(b+d)} - \left( \frac{C(\mu_0, \mu_0) - D(\mu_0, \mu_0)}{C(\mu_0, \mu_0) - D(\mu_0, \mu_0)} \right)^{1/(b+d)}
\]

**Remark 18.** If \( \delta(t) = \log_t(2 - t) - \log_t(t) \), then the AN-IFWGBM is transformed as the IF Einstein weighted geometric BM (IFEWGBM):

\[
\text{IFEWGBM}^{b,d}(\theta_1, \theta_2, \ldots, \theta_n) = \left( \frac{C(\mu_0, \mu_0) + 3D(\mu_0, \mu_0)}{C(\mu_0, \mu_0) + 3D(\mu_0, \mu_0)} \right)^{1/(b+d)} \left( \frac{2(A(\nu_0, \nu_0) - B(\nu_0, \nu_0))}{A(\nu_0, \nu_0) + 3B(\nu_0, \nu_0)} \right)^{1/(b+d)} - \left( \frac{C(\mu_0, \mu_0) - D(\mu_0, \mu_0)}{C(\mu_0, \mu_0) - D(\mu_0, \mu_0)} \right)^{1/(b+d)} \left( \frac{2(A(\nu_0, \nu_0) + 3B(\nu_0, \nu_0))}{A(\nu_0, \nu_0) + 3B(\nu_0, \nu_0)} \right)^{1/(b+d)}
\]
where

\[
C(\mu_i, \mu_j) = \prod_{i,j=1 \atop i \neq j}^n \left( \left( 2 - \mu_i \right)^{a_i} + 3 \mu_i^b \right)^\frac{1}{\theta_{ij}^{c_{ij}}} \left( \left( 2 - \mu_j \right)^{a_j} + 3 \mu_j^b \right)^\frac{1}{\theta_{ij}^{d_{ij}}} \times \left( \left( 2 - \mu_i \right)^{a_i} - \mu_i^b \right)^\frac{1}{\theta_{ij}^{e_{ij}}} \left( \left( 2 - \mu_j \right)^{a_j} - \mu_j^b \right)^\frac{1}{\theta_{ij}^{f_{ij}}} \left( \left( 2 - \mu_i \right)^{a_i} - \mu_i^b \right)^\frac{1}{\theta_{ij}^{g_{ij}}} \left( \left( 2 - \mu_j \right)^{a_j} - \mu_j^b \right)^\frac{1}{\theta_{ij}^{h_{ij}}} \right)^{1/(n(n-1))},
\]

\[
D(\mu_i, \mu_j) = \prod_{i,j=1 \atop i \neq j}^n \left( \left( 2 - \mu_i \right)^{a_i} + 3 \mu_i^b \right)^\frac{1}{\theta_{ij}^{c_{ij}}} \left( \left( 2 - \mu_j \right)^{a_j} + 3 \mu_j^b \right)^\frac{1}{\theta_{ij}^{d_{ij}}} \times \left( \left( 2 - \mu_i \right)^{a_i} - \mu_i^b \right)^\frac{1}{\theta_{ij}^{e_{ij}}} \left( \left( 2 - \mu_j \right)^{a_j} - \mu_j^b \right)^\frac{1}{\theta_{ij}^{f_{ij}}} \left( \left( 2 - \mu_i \right)^{a_i} - \mu_i^b \right)^\frac{1}{\theta_{ij}^{g_{ij}}} \left( \left( 2 - \mu_j \right)^{a_j} - \mu_j^b \right)^\frac{1}{\theta_{ij}^{h_{ij}}} \right)^{1/(n(n-1))},
\]

\[
A(\eta_i, \eta_j) = \prod_{i,j=1 \atop i \neq j}^n \left( \left( 1 + \eta_i \right)^{a_i} + 3 \left( 1 - \eta_i \right)^{a_i} \right)^\frac{1}{\theta_{ij}^{c_{ij}}} \left( \left( 1 + \eta_j \right)^{a_j} + 3 \left( 1 - \eta_j \right)^{a_j} \right)^\frac{1}{\theta_{ij}^{d_{ij}}} \times \left( \left( 1 + \eta_i \right)^{a_i} - \eta_i \right)^\frac{1}{\theta_{ij}^{e_{ij}}} \left( \left( 1 + \eta_j \right)^{a_j} - \eta_j \right)^\frac{1}{\theta_{ij}^{f_{ij}}} \left( \left( 1 + \eta_i \right)^{a_i} - \eta_i \right)^\frac{1}{\theta_{ij}^{g_{ij}}} \left( \left( 1 + \eta_j \right)^{a_j} - \eta_j \right)^\frac{1}{\theta_{ij}^{h_{ij}}} \right)^{1/(n(n-1))},
\]

\[
B(\eta_i, \eta_j) = \prod_{i,j=1 \atop i \neq j}^n \left( \left( 1 + \eta_i \right)^{a_i} + 3 \left( 1 - \eta_i \right)^{a_i} \right)^\frac{1}{\theta_{ij}^{c_{ij}}} \left( \left( 1 + \eta_j \right)^{a_j} + 3 \left( 1 - \eta_j \right)^{a_j} \right)^\frac{1}{\theta_{ij}^{d_{ij}}} \times \left( \left( 1 + \eta_i \right)^{a_i} - \eta_i \right)^\frac{1}{\theta_{ij}^{e_{ij}}} \left( \left( 1 + \eta_j \right)^{a_j} - \eta_j \right)^\frac{1}{\theta_{ij}^{f_{ij}}} \left( \left( 1 + \eta_i \right)^{a_i} - \eta_i \right)^\frac{1}{\theta_{ij}^{g_{ij}}} \left( \left( 1 + \eta_j \right)^{a_j} - \eta_j \right)^\frac{1}{\theta_{ij}^{h_{ij}}} \right)^{1/(n(n-1))}.
\]

(76)

5. An Approach for MADM under IF Environment

Here, an IF MADM method is discussed based on AN-
IFWABM and AN-IFWGBM.

For MADM problems with IFNs, let \( Y = \{Y_1, Y_2, \ldots, Y_m\} \) be a discrete set of options and \( R = \{R_1, R_2, \ldots, R_n\} \) be a group of attributes, whose WV is \( v = (v_1, v_2, \ldots, v_n)^\top \), satisfying \( v_j > 0 \) and \( \sum_{j=1}^n v_j = 1 \), where \( v_j \) denotes the impact value of the attribute \( R_j \). Assume that the DMs are demanded to provide the evaluation information of alternative \( Y_i \) under the attribute \( R_j \) with IFFs \( \theta_{ij} = (\mu_{\theta_{ij}}, \nu_{\theta_{ij}}) \) where \( \mu_{\theta_{ij}}, \nu_{\theta_{ij}} \in [0, 1] \), and \( \mu_{\theta_{ij}} + \nu_{\theta_{ij}} \in [0, 1] \).

With the performance of all alternatives available, we can construct an IF decision matrix \( D = (\theta_{ij})_{mn} \).

Next, AN-IFWABM and AN-IFWGBM are availed to handle the IF MADM problems.

**Step 1.** For all attributes \( R_j \) that are benefit type, they need not to be normalized. Hence, we convert the IF decision matrix \( D = (\theta_{ij})_{mn} \) to the normalized IF decision matrix \( B = (\bar{\theta}_{ij})_{mn} \) where

\[
\bar{\theta}_{ij} = \begin{cases} \theta_{ij}, & \text{for benefit attribute } R_j, \\ \theta_{ij}, & \text{for cost attribute } R_j, \end{cases}
\]

\( i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, \)

(77)

and \( \bar{\theta}_{ij} = (\nu_{\theta_{ij}}, \mu_{\theta_{ij}}) \).

**Step 2.** Utilize AN-IFWABM,

\[
\vartheta_i = \text{AN-IFWABM}^{bd}(\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{im})
\]

\[
= \left( \frac{1}{n(n-1)} \sum_{j=1}^n \left( \left( \bar{\theta}_{ij} \right)^b \right)^{1/(b+d)} \right)^{(n(n-1))},
\]

(78)

or AN-IFWGBM,

\[
\vartheta_i = \text{AN-IFWGBM}^{bd}(\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{im})
\]

\[
= \frac{1}{b + d} \sum_{j=1}^n \left( b \bar{\theta}_{ij} d \bar{\theta}_{ik} \right)^{1/(n(n-1))},
\]

(79)

to gather all the individual IFN \( \vartheta_{ij} \) into the overall IFN \( \vartheta_i \), where \( b, d \geq 0 \).

**Step 3.** The values of \( s(\vartheta_i) \) and \( f(\vartheta_i) \) are calculated, respectively. Then, we sort the overall IFVs \( \vartheta_i \) by Definition 6.

**Step 4.** Select the best one \( Y_j \) with the highest priority by the sorting of \( \vartheta_i \).

**Step 5.** End.

6. Application to Efficiency Assessment Problem

As a demonstration of the application of the designed approach, in this section, a test of our method is presented for
the efficiency evaluation problem. Then, a comparison is made between our approach and the methods in Xia et al. [34] and Rahman et al. [41].

The transformation efficiency of scientific and technological achievements for the following 6 companies is evaluated: Chinasoft International (Y1), Geely Automobile Company (Y2), GWT Co., Ltd. (Y3), Tencent Holdings Ltd. (Y4), iFlyteK Co., Ltd. (Y5), and Alibaba (Y6). Assisted by 10 experts from different fields, a DM assessed these companies depending on the following three attributes: technical staff (R1), research spending (R2), and invention patents (R3). After reviewing the documents related to the three attributes in the assessment process, this DM provides the WV of the attributes R1, R2, R3 as \( v = (0.30, 0.50, 0.20)^T \). Suppose that the features of the public companies Y1, Y2, . . . , Y6 for the attributes R1, R2, R3 are expressed by IFNs \( \theta_{ij} = (\mu_{ij}, \nu_{ij}) \) and all \( \theta_{ij} = (\mu_{ij}, \nu_{ij}) \) are included in the IF decision matrix \( D = (\theta_{ij})_{6\times 3} \) (see Table 1).

To choose the most preferred company, we utilize the AN-IFWABM and AN-IFWGBM to solve the IF MADM problems, respectively. The following are the major steps:

(i) **Step 1.** Given that all attributes \( R_j \) are revenue types that do not require normalization, then go to the next step.

(ii) **Step 2.** For the matrix \( D = (\theta_{ij})_{6\times 3} \) in IFSs, we utilize the AN-IFWABM (78) and AN-IFWGBM (79) (we take \( b = d = 1 \) and let the additive generator \( \delta(t) = \log_2(2 - t) - \log_2(t) \) to aggregate all the individual IFNs \( \theta_{ij} \). The overall IFNs \( \theta_i \) of the companies \( Y_i \) are obtained, which are indicated in Table 2.

(iii) **Step 3.** The values of \( s(\theta_i) \) with the overall \( \theta_i \) are calculated, which are indicated in Table 3.

(iv) **Step 4.** Sort all \( Y_i \) by the values of \( s(\theta_i) \); the ordering of the alternatives is presented in Table 4.

One can observe that the ranking of the companies is slightly different as per the BMs deployed. Thus, the results may produce diverse decision-making results by the various BMs used.

Now, if we take \( b = d = 2 \), then by the AN-IFWABM and AN-IFWGBM (let the additive generator \( \delta(t) = \log_2(2 - t) - \log_2(t) \) ), we get the results as shown in Tables 2–4.

In what follows, in view of the same IF decision matrix \( D = (\theta_{ij})_{6\times 3} \), the methods established by Xia et al. [34] and Rahman et al. [41] are applied to get the most desirable company. With the method by Xia et al. [34], the following process steps are obtained.

(i) **Step 1:** see Step 1.

(ii) **Step 2:** applying the IFWBM equation (69) and IFWGBM equation (74) (we take \( b = d = 1 \) and \( b = d = 2 \), respectively) to gather all the individual IFNs \( \theta_{ij} \), respectively, and then get the overall IFNs \( \theta_i \) of the alternatives \( Y_i \), which are shown in Table 5.

(iii) **Step 3:** the values of \( s(\theta_i) \) with the overall IFVs \( \theta_i \) are calculated in Table 6.

(iv) **Step 4:** sort all \( Y_i \) with the scores \( s(\theta_i) \); the ordering of the \( Y_i \) is expressed in Table 7.

By using confidence level, Rahman et al. [41] proposed the CIFHEA operator and CIFHEHG operator to fuse a group of IFNs into a collection IFN and then developed a MADM method. With the help of the method by Rahman et al. [41], the most desirable company can be derived as follows:

(i) **Step A:** see Step 1.

(ii) **Step B:** based on the provided IF decision matrix \( D = (\theta_{ij})_{6\times 3} \), we can determine the confidence level preference matrix \( B = (\beta_{ij})_{6\times 3} \) (where \( \beta_{ij} = nu_{ij} \theta_{ij} \)) in Table 8.

(iii) **Step C:** utilizing the CIFHEA operator and CIFHEHG operator [41] to fuse the confidence level preference values \( \beta_{ij} \) \( (j = 1, 2, 3) \) into the overall confidence level preference values \( \beta_i \) \( (i = 1, 2, \ldots, 6) \) is shown in Table 9.

(iv) **Step D:** by using Definition 6, we can get the score functions of \( \beta_i \) \( (i = 1, 2, \ldots, 6) \), and the ranking order of six companies can be derived, which are shown in Tables 10 and 11.

Obviously, the proposed method in this work yields the slightly different ranking result with the methods by Xia et al. [34] and Rahman et al. [3]. However, the method by Xia et al. [34] is designed to carry combination processes with the algebraic rules of IFNs, which are different from the limit case of general fuzzy sets [36]. On the other hand, the algebraic norm is a special case of Archimedean norm. Therefore, it is understandable that the decision-making process in our approach would be more efficient and common than the model established by Xia et al. [34]. In addition, a slightly different ranking of \( Y_i \) was found which varies in parameters \( b \) and \( d \), which mirrors the risk preferences of DMs. Moreover, with the variation of parameter values, the results of the alternatives may change. Yet, we cannot say which is the perfect order, it relies on the DMs' optimism or pessimism and just reflects the DMs' attitude.

During the decision-making process by Rahman et al. [41], the CIFHEA operator and CIFHEHG operator neglect the importance of each argument and their interconnections. Furthermore, according to the original IF decision matrix \( D = (\theta_{ij})_{6\times 3} \) provided by DM, it is observed that \( \theta_{ij} > \theta_{ij} \); \( j = 1, 2, 3 \), which indicates that Tencent Holdings Ltd. \( (Y_4) \) is preferred to Chinasoft International \( (Y_1) \), i.e., \( Y_4 > Y_1 \). Therefore, the developed MADM method in the paper is more reliable than Rahman et al.'s [41] method.
Table 2: The overall IFNs $\theta_i$.

| $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|------------|------------|------------|------------|------------|------------|
| AN-IFWBM ($b = d = 1$) | (0.2224, 0.7196) | (0.2208, 0.5945) | (0.2284, 0.7511) | (0.2998, 0.6901) | (0.3314, 0.5912) | (0.3012, 0.6551) |
| AN-IFWGBM ($b = d = 1$) | (0.8316, 0.1048) | (0.8270, 0.0439) | (0.8255, 0.1681) | (0.8678, 0.0922) | (0.8575, 0.1012) | (0.8446, 0.0817) |
| AN-IFWBM ($b = d = 2$) | (0.2380, 0.7085) | (0.2216, 0.0439) | (0.2468, 0.7387) | (0.3383, 0.6479) | (0.4229, 0.5345) | (0.3204, 0.6122) |
| AN-IFWGBM ($b = d = 2$) | (0.8270, 0.1076) | (0.8099, 0.0468) | (0.8144, 0.1682) | (0.8533, 0.0984) | (0.8331, 0.1142) | (0.8459, 0.1106) |

Table 3: The scores of the overall IFNs.

| $s(\theta_1)$ | $s(\theta_2)$ | $s(\theta_3)$ | $s(\theta_4)$ | $s(\theta_5)$ | $s(\theta_6)$ |
|---------------|---------------|---------------|---------------|---------------|---------------|
| AN-IFWARM ($b = d = 1$) | −0.4972 | −0.3746 | −0.5227 | −0.3903 | −0.2598 | −0.3359 |
| AN-IFWGBM ($b = d = 1$) | 0.7268 | 0.7831 | 0.6574 | 0.7756 | 0.7563 | 0.7629 |
| AN-IFWARM ($b = d = 2$) | −0.4705 | −0.3626 | −0.4919 | −0.3096 | −0.1116 | −0.2918 |
| AN-IFWGBM ($b = d = 2$) | 0.7164 | 0.7631 | 0.6462 | 0.7549 | 0.7169 | 0.7353 |

Table 4: Ordering of six corporations.

| Ordering | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|----------|------------|------------|------------|------------|------------|------------|
| AN-IFWBM ($b = d = 1$) | $Y_3 > Y_6 > Y_5 > Y_4 > Y_1 > Y_2$ |
| AN-IFWGBM ($b = d = 1$) | $Y_2 > Y_4 > Y_5 > Y_3 > Y_1$ |
| AN-IFWBM ($b = d = 2$) | $Y_2 > Y_4 > Y_5 > Y_3 > Y_1$ |
| AN-IFWGBM ($b = d = 2$) | $Y_2 > Y_4 > Y_5 > Y_3 > Y_1$ |

Table 5: The overall IFNs $\theta_i$.

| $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|------------|------------|------------|------------|------------|------------|
| IFWBM ($b = d = 1$) | (0.1982, 0.6814) | (0.2091, 0.5309) | (0.2023, 0.7229) | (0.2658, 0.6684) | (0.2782, 0.6544) | (0.2768, 0.6907) |
| IFWGBM ($b = d = 1$) | (0.8084, 0.1034) | (0.8101, 0.0438) | (0.8112, 0.1269) | (0.8561, 0.0915) | (0.8381, 0.1006) | (0.8430, 0.1015) |
| IFWBM ($b = d = 2$) | (0.2091, 0.6728) | (0.2089, 0.5248) | (0.2141, 0.7129) | (0.3022, 0.6467) | (0.3279, 0.6248) | (0.2984, 0.6312) |
| IFWGBM ($b = d = 2$) | (0.8039, 0.1056) | (0.7924, 0.0465) | (0.8100, 0.1290) | (0.8406, 0.0971) | (0.8092, 0.1130) | (0.8259, 0.0805) |

Table 6: The scores of the $\theta_i$.

| $s(\theta_1)$ | $s(\theta_2)$ | $s(\theta_3)$ | $s(\theta_4)$ | $s(\theta_5)$ | $s(\theta_6)$ |
|---------------|---------------|---------------|---------------|---------------|---------------|
| IFWBM ($b = d = 1$) | −0.5022 | −0.3218 | −0.5206 | −0.4026 | −0.3762 | −0.4139 |
| IFWGBM ($b = d = 1$) | 0.7050 | 0.7664 | 0.6843 | 0.7647 | 0.7375 | 0.7415 |
| IFWBM ($b = d = 2$) | −0.4637 | −0.3159 | −0.4988 | −0.3445 | −0.2969 | −0.3328 |
| IFWGBM ($b = d = 2$) | 0.6982 | 0.7459 | 0.6810 | 0.7435 | 0.6962 | 0.7454 |

Table 7: Ordering of six corporations.

| Operators | Ordering |
|-----------|----------|
| IFWBM ($b = d = 1$) | $Y_2 > Y_5 > Y_4 > Y_6 > Y_1 > Y_3$ |
| IFWGBM ($b = d = 1$) | $Y_2 > Y_4 > Y_5 > Y_3 > Y_1$ |
| IFWBM ($b = d = 2$) | $Y_2 > Y_4 > Y_5 > Y_3 > Y_1$ |
| IFWGBM ($b = d = 2$) | $Y_2 > Y_4 > Y_5 > Y_3 > Y_1$ |

Table 8: The IF decision matrix $B = (\beta_{ij})_{6 \times 3}$.

| $Y_1$ | $Y_2$ | $Y_3$ | $Y_4$ | $Y_5$ | $Y_6$ |
|-------|-------|-------|-------|-------|-------|
| $R_1$ | (0.4728, 0.4559) | (0.6452, 0.2257) | (0.4515, 0.2013) | (0.7121, 0.2009) | (0.5569, 0.3749) | (0.7855, 0.2004) |
| $R_2$ | (0.6419, 0.3625) | (0.4325, 0.0615) | (0.5596, 0.3749) | (0.3456, 0.4521) | (0.5546, 0.4125) |
| $R_3$ | (0.4256, 0.1526) | (0.4456, 0.1254) | (0.5525, 0.2254) | (0.5546, 0.4125) | (0.5546, 0.4125) |
7. Conclusions
As the Archimedean norm can capture the internal relationship between decision-making information, thus, it is widely used in the design of information fusion and the construction of decision model. This paper focuses on the construction of MADM method with intuitionistic fuzzy Archimedean Bonferroni means. Firstly, we introduced the intuitionistic fuzzy operations based on Archimedean norm. Then, we developed two new kinds of intuitionistic fuzzy BMs with the help of Archimedean norm, such as AN-IFABM and AN-IFGBM. We further discussed their desirable properties, including commutativity, idempotency, monotonicity, and boundedness. Furthermore, an intuitionistic fuzzy MADM method is developed by the proposed intuitionistic fuzzy BM. Finally, the new MADM method is applied to evaluate the efficiency of the conversion of scientific and technological achievements of six public companies in China.

However, the proposed MADM method with intuitionistic fuzzy Archimedean Bonferroni means does not consider the cooperation consensus among DMs. Therefore, in the future, we will investigate the cooperation MADM methods for intuitionistic fuzzy information. The proposed intuitionistic fuzzy Archimedean Bonferroni means can also be expanded to other uncertainty areas, such as Pythagorean fuzzy information and Pythagorean picture fuzzy information.

Data Availability
The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest
The author declares no conflicts of interest.

Table 9: The overall IFNs $\beta_i$ ($i = 1, 2, \ldots, 6$).

| Operators       | $\beta_1$        | $\beta_2$        | $\beta_3$        | $\beta_4$        | $\beta_5$        | $\beta_6$        |
|-----------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| CIFEHA operator | (0.4518, 0.5126) | (0.4859, 0.4015) | (0.4225, 0.5050) | (0.4523, 0.4698) | (0.3956, 0.4125) | (0.4411, 0.4208) |
| CIFEHG operator | (0.6915, 0.1100) | (0.7246, 0.1334) | (0.7364, 0.2634) | (0.7030, 0.1224) | (0.7416, 0.2425) | (0.7154, 0.1643) |

Table 10: The scores $s(\beta_i)$ ($i = 1, 2, \ldots, 6$).

| Operators       | $s(\beta_1)$ | $s(\beta_2)$ | $s(\beta_3)$ | $s(\beta_4)$ | $s(\beta_5)$ | $s(\beta_6)$ |
|-----------------|--------------|--------------|--------------|--------------|--------------|--------------|
| CIFEHA operator | -0.0608      | 0.0844       | -0.0825      | -0.0175      | -0.0169      | -0.0203      |
| CIFEHG operator | 0.5815       | 0.5912       | 0.4730       | 0.5806       | 0.4991       | 0.5511       |

Table 11: Ordering of six corporations.

| Operators       | Ordering |
|-----------------|----------|
| CIFEHA operator | $Y_2 > Y_3 > Y_6 > Y_1 > Y_5$ |
| CIFEHG operator | $Y_2 > Y_1 > Y_4 > Y_6 > Y_5 > Y_3$ |

Acknowledgments
The work was supported by the Major Project of Humanities and Social Sciences of Anhui Provincial Education Department (no. SK2019ZD55) and Analysis of Logistics Status in Anhui Province Based on Big Data (Horizontal Project, 2020).

References
[1] L. A. Zadeh, “Fuzzy sets,” Information and Control, vol. 8, no. 3, pp. 338–353, 1965.
[2] K. T. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 20, no. 1, pp. 87–96, 1986.
[3] K. T. Atanassov, "More on intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 33, no. 1, pp. 37–45, 1989.
[4] Y. Song, X. Wang, L. Lei, and A. Xue, "A novel similarity measure on intuitionistic fuzzy sets with its applications," Applied Intelligence, vol. 42, no. 2, pp. 252–261, 2015.
[5] F. Meng and X. Chen, "Interval-valued intuitionistic fuzzy multi-criteria group decision making based on cross entropy and 2-additive measures," Soft Computing, vol. 19, no. 7, pp. 2071–2082, 2015.
[6] Z. Xu, "Multi-person multi-attribute decision making models under intuitionistic fuzzy environment," Fuzzy Optimization and Decision Making, vol. 6, no. 3, pp. 221–236, 2007.
[7] M. Qiyas, M. A. Khan, S. Khan, and S. Abdullah, "Concept of Yager operators with the picture fuzzy set environment and its application to emergency program selection," International Journal of Intelligent Computing and Cybernetics, vol. 13, no. 4, pp. 455–483, 2020.
[8] F. Jin, L. Pei, H. Chen, and L. Zhou, "Interval-valued intuitionistic fuzzy continuous weighted entropy and its application to multi-criteria fuzzy group decision making," Knowledge-Based Systems, vol. 59, pp. 132–141, 2014.
[9] S. Khan, S. Abdullah, and S. Ashraf, "Decision support technique based on neutrosophic Yager aggregation operators: application in solar power plant locations-case study of Bahawalpur, Pakistan," Mathematical Problems in Engineering, vol. 2020, Article ID 6677676, 21 pages, 2020.
[10] M. Lin, W. Xu, Z. Lin, and R. Chen, "Determine OWA operator weights using kernel density estimation," Economic Research-Ekonomiska Istraživanja, vol. 33, no. 1, pp. 1441–1464, 2020.
[11] J.-J. Peng, J.-Q. Wang, H.-Y. Zhang, and X.-H. Chen, "An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets," Applied Soft Computing, vol. 25, pp. 336–346, 2014.
[12] Y. Y. Chi, G. Q. Bai, and H. Dong, "A new multicriteria decision-making method for the selection of sponge city schemes with Shapley–Choquet aggregation operators," Mathematical Problems in Engineering, vol. 2021, Article ID 6615709, 16 pages, 2021.
[13] Z. Aya and F. Samanioglu, "A hesitant fuzzy linguistic terms set-based AHP-TOPSIS approach to evaluate ERP software packages," International Journal of Intelligent Computing and Cybernetics, vol. 11, pp. 14–25, 2020.
[14] H. Li, L. Lv, F. Li, L. Wang, and Q. Xia, “A novel approach to emergency risk assessment using FMEA with extended MULTIMOORA method under interval-valued Pythagorean fuzzy environment,” International Journal of Intelligent Computing and Cybernetics, vol. 13, no. 1, pp. 41–65, 2020.

[15] X. Q. Xu, J. L. Xie, N. Yue, and H. H. Wang, “Probabilistic uncertain linguistic TODIM method based on the generalized Choquet integral and its application,” International Journal of Intelligent Computing and Cybernetics, 2021.

[16] J. Ye, “Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine,” Soft Computing, vol. 21, no. 3, pp. 817–825, 2017.

[17] Z. S. Xu, “Intuitionistic fuzzy aggregation operators,” IEEE Transactions on Fuzzy Systems, vol. 15, pp. 1179–1187, 2007.

[18] Z. Xu and R. R. Yager, “Some geometric aggregation operators based on intuitionistic fuzzy sets,” International Journal of General Systems, vol. 35, no. 4, pp. 417–433, 2006.

[19] H. Zhao, Z. Xu, M. Ni, and S. Liu, “Generalized aggregation operators for intuitionistic fuzzy sets,” International Journal of Intelligent Systems, vol. 25, no. 1, pp. 1–30, 2010.

[20] G. Wei, “Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making,” Applied Soft Computing, vol. 10, no. 2, pp. 423–431, 2010.

[21] M. Lin, Z. Xu, Y. Zhai, and Z. Yao, “Multi-attribute group decision-making under probabilistic uncertain linguistic environment,” Journal of the Operational Research Society, vol. 69, no. 2, pp. 157–170, 2018.

[22] M. Lin, C. Huang, and Z. Xu, “MULTIMOORA based MCDM model for site selection of car sharing station under picture fuzzy environment,” Sustainable Cities and Society, vol. 53, Article ID 101873, 2020.

[23] Z. Xu and M. Xia, “Induced generalized intuitionistic fuzzy operators,” Knowledge-Based Systems, vol. 24, no. 2, pp. 197–209, 2011.

[24] W. Yang and Z. Chen, “The quasi-arithmetic intuitionistic fuzzy OWA operators,” Knowledge-Based Systems, vol. 27, pp. 219–233, 2012.

[25] R. R. Yager, “Prioritized aggregation operators,” International Journal of Approximate Reasoning, vol. 48, no. 1, pp. 263–274, 2008.

[26] X. Yu and Z. Xu, “Prioritized intuitionistic fuzzy aggregation operators,” Information Fusion, vol. 14, no. 1, pp. 108–116, 2013.

[27] R. R. Yager, “The power average operator,” IEEE Transactions on Systems, Man, and Cybernetics- Part A: Systems and Humans, vol. 31, no. 6, pp. 724–731, 2001.

[28] Z. Xu, “Approaches to multiple attribute group decision making based on intuitionistic fuzzy power aggregation operators,” Knowledge-Based Systems, vol. 24, no. 6, pp. 749–760, 2011.

[29] L. Zhou and H. Chen, “A generalization of the power aggregation operators for linguistic environment and its application in group decision making,” Knowledge-Based Systems, vol. 26, pp. 216–224, 2012.

[30] L. Zhou, H. Chen, and J. Liu, “Generalized power aggregation operators and their applications in group decision making,” Computers & Industrial Engineering, vol. 62, no. 4, pp. 989–999, 2012.

[31] C. Bonferroni, “Sulle medie multiple di potenze,” Bolletino Matematica Italiana, vol. 5, pp. 267–270, 1950.