COHOMOLOGICALLY INDUCED DISTINGUISHED REPRESENTATIONS AND A NON-VANISHING HYPOTHESIS FOR ALGEBRAICITY OF CRITICAL L-VALUES

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Abstract. Let $G$ be a real reductive group with an involution $\sigma$ on it. Let $H$ be an open subgroup of $G^\sigma$ with a character $\chi$ on it. Associated to a “theta stable”, “$\sigma$-split” parabolic subalgebra, and using the Zuckerman functor, we construct representations of $G$ together with $\chi$-equivariant linear functionals on them. We apply this construction to prove a non-vanishing hypothesis of H. Grobner and A. Raghuram in the study of algebraicity of critical $L$-values.

1. Introduction

For any positive integer $n$, write $\Pi_{2n}^{\text{coh}}(\mathbb{R})$ for the set of isomorphism classes of irreducible Casselman-Wallach representations $\pi$ of $\text{GL}_{2n}(\mathbb{R})$ such that

- $\pi|_{\text{SL}_{\pm}^{2n}(\mathbb{R})}$ is unitarizable and tempered, and
- there is a representation $F$ in $\hat{\text{GL}}_{2n}(\mathbb{C})$ such that the total relative Lie algebra cohomology
  \[ H^*(g_{2n}, K_{2n}; \pi \otimes F) \neq 0. \]

Here $g_{2n} := \mathfrak{gl}_{2n}(\mathbb{C})$, $K_{2n} := \text{SO}(2n) \mathbb{R}_+ \subset \text{GL}_{2n}(\mathbb{R})$, and $\hat{\text{GL}}_{2n}(\mathbb{C})$ is the set of isomorphism classes of irreducible algebraic finite-dimensional representations of the complex group $\text{GL}_{2n}(\mathbb{C})$. Recall that a representation of a real reductive group is said to be Casselman-Wallach if it is Fréchet, smooth, of moderate growth, and its underlying Harish-Chandra module is admissible and finitely generated. The reader may consult [Cas], [Wal, Chapter 11] or [BK] for more details about Casselman-Wallach representations. The usual notion of real reductive groups is explained in Section 3.1. To ease notation, we do not distinguish a representation with its underlying vector space, or an irreducible representation with its isomorphism class, or a one-dimensional representation with its corresponding character.

The representations in $\Pi_{2n}^{\text{coh}}(\mathbb{R})$ are real components of cuspidal automorphic representations of $\text{GL}_{2n}$ which are regular algebraic in the sense of Clozel ([Clo]). The

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reader is referred to [Clo, Section 3] as a general reference for the following discussion concerning representations in $\Pi_{2n}^{coh}(\mathbb{R})$ and their cohomology. See also [Mah, Section 3.1], [RS1, Section 5], [GR1, Section 5.5] or [GR2, Section 3.4].

It follows from Vogan-Zuckerman’s theory of cohomological representations ([VZ]) that

$$\Pi_{2n}^{coh}(\mathbb{R}) = \{ \pi_{l,w} \mid w \in \mathbb{Z}, 1 = (l_1 > l_2 > \cdots > l_n > 0) \in (w + 1 + 2\mathbb{Z})^n \},$$

where

$$\pi_{l,w} = \pi_l \otimes |\det|^w,$$

and

$$\pi_l := \text{Ind}_{P_{2,2,\cdots,2}}^{\text{GL}_{2n}(\mathbb{R})} D_{l_1} \hat{\otimes} D_{l_2} \hat{\otimes} \cdots \hat{\otimes} D_{l_n} \quad \text{(normalized smooth induction)}.$$ 

Here $P_{2,2,\cdots,2} \subset \text{GL}_{2n}(\mathbb{R})$ is the block-wise up triangular group corresponding to the partition $2n = 2 + 2 + \cdots + 2$, and $D_{l_j}$ is the unique irreducible Casselman-Wallach representation of $\text{GL}_{2}(\mathbb{R})$ which is unitarizable, tempered, and has infinitesimal character $(l_j^2, -l_j^2)$.

Denote by $F_{l,-w}$ the representation in $\hat{\text{GL}}_{2n}(\mathbb{C})$ of highest weight

$$\lambda_1 - \rho_{2n} - (\frac{w}{2}, \frac{w}{2}, \cdots, \frac{w}{2}, \frac{w}{2}) \in \mathbb{Z}^{2n},$$

where

$$\lambda_1 := \left( \frac{l_1}{2}, \frac{l_2}{2}, \cdots, \frac{l_n}{2}, -\frac{l_n}{2}, \cdots, -\frac{l_2}{2}, -\frac{l_1}{2} \right)$$

is the infinitesimal character of $\pi_l$, and

$$\rho_{2n} := \left( \frac{2n-1}{2}, \frac{2n-3}{2}, \cdots, \frac{1}{2}, -\frac{1}{2}, \cdots, \frac{3-2n}{2}, \frac{1-2n}{2} \right)$$

is a half sum of positive roots. Then $F_{l,-w}$ is the only representation in $\hat{\text{GL}}_{2n}(\mathbb{C})$ so that

$$H^*(g_{2n}, K_{2n}; \pi_{l,w} \otimes F_{l,-w}) \neq 0.$$ 

The highest degree of non-vanishing of (3) is

$$p_0 := n^2 + n - 1.$$ 

Recall that the cohomology (3) is computed by the complex

$$\{ \text{Hom}_{K_{2n}}(\wedge^j(g_{2n}/t_{2n}), \pi_{l,w} \otimes F_{l,-w}) \}_{j \in \mathbb{Z}},$$

with certain coboundary maps. Here and henceforth, we use the corresponding lower case Gothic letter to indicate the complexified Lie algebra of a Lie group.
(For example, $\mathfrak{t}_{2n}$ is the complexified Lie algebra of $K_{2n}$.) At the highest degree of non-vanishing, it turns out that

\[
H^\mathbf{p}_\mathbf{0}(\mathfrak{g}_{2n}, K_{2n}; \pi_{1, w} \otimes F_{1, -w}) = \text{Hom}_{K_{2n}}(\wedge^{\mathbf{p}_\mathbf{0}}(\mathfrak{g}_{2n}/\mathfrak{t}_{2n}), \pi_{1, w} \otimes F_{1, -w})
\]

\[
= \bigoplus_{\epsilon \in \mathbb{Z}/2\mathbb{Z}} \text{Hom}_{\text{det}_L}(\wedge^{\mathbf{p}_\mathbf{0}}(\mathfrak{g}_{2n}/\mathfrak{t}_{2n}), \pi_{1, w} \otimes F_{1, -w}),
\]

with each summand of (4) has dimension one. Here “Hom$_{\text{det}_L}$” indicates the space of linear maps $f$ such that

\[
f(k.v) = \det(k)^{\epsilon} k.(f(v)), \quad k \in \text{O}(2n), \ v \in \wedge^{\mathbf{p}_\mathbf{0}}(\mathfrak{g}_{2n}/\mathfrak{t}_{2n}).
\]

Similar notation will be used later on without further explanation.

Put

\[
I^+ := (l_1 + 1, l_2 + 1, \ldots, l_n + 1),
\]

and let $\tau_+$ be the corresponding irreducible representation of $\text{O}(2n)$, as in Section 2.1. It is easy to see that $\tau_+$ occurs with multiplicity one in $\pi_{1, w}$ as the unique minimal $\text{O}(2n)$-type in the sense of Vogan (cf. Remark 2.5). We fix an $\text{O}(2n)$-equivariant embedding and view $\tau_+$ as a subspace of $\pi_{1, w}$.

The key concrete result of this paper is the following

**Theorem A.** Let $\pi_{1, w}$ be a representation in $\Pi_{2n}^{\text{coh}}(\mathbb{R})$ as in (1). Let $\chi = \chi_1 \otimes \chi_2$ be a character of $\text{GL}_n(\mathbb{R}) \times \text{GL}_n(\mathbb{R})$ so that $\chi_1 \chi_2 = \det^{w}$. Then the restriction to $\tau_+$ induces a linear isomorphism

\[
\text{Hom}_{\text{GL}_n(\mathbb{R}) \times \text{GL}_n(\mathbb{R})}(\pi_{1, w}, \chi) \cong \text{Hom}_{\text{O}(n) \times \text{O}(n)}(\tau_+, \chi),
\]

and both spaces in (6) have dimension one.

Now we consider the finite-dimensional representation $F_{1, -w}$. Put

\[
I^- := \begin{cases} 
1 - (2n - 1, 2n - 3, \ldots, 3, 1 + 1/2), & \text{if } l_n = 1 \text{ and } w/2 \text{ is odd;} \\
1 - (2n - 1, 2n - 3, \ldots, 3, 1), & \text{otherwise},
\end{cases}
\]

and write $\tau_-$ for the corresponding irreducible representation of $\text{O}_{2n}(\mathbb{C})$, again as in Section 2.1. Then $\tau_-$ occurs with multiplicity one in $F_{1, -w}$ as the unique maximal $\text{O}_{2n}(\mathbb{C})$-type (see Lemma 2.4 and Remark 2.5). We fix an $\text{O}_{2n}(\mathbb{C})$-equivariant embedding and view $\tau_-$ as a subspace of $F_{1, -w}$.

The finite-dimensional counterpart of Theorem A is

**Theorem B.** Let $F_{1, -w}$ be a finite-dimensional representation of $\text{GL}_{2n}(\mathbb{C})$ as in (2). Let $w_1, w_2 \in \mathbb{Z} \cap [1 - \frac{l_n - w}{2}, \frac{l_n - 1 - w}{2}]$ be two integers so that $w_1 + w_2 = -w$. Then the restriction to $\tau_-$ induces a linear isomorphism

\[
\text{Hom}_{\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})}(F_{1, -w}, \det^{w_1, w_2}) \cong \text{Hom}_{\text{O}_n(\mathbb{C}) \times \text{O}_n(\mathbb{C})}(\tau_-, \det^{w_1, w_2}),
\]

and both spaces in (7) have dimension one.
Here \( \det^{w_1,w_2} \) denotes the character \( \det^{w_1} \otimes \det^{w_2} \) on \( \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \), and similar notation will be used later on without further explanation.

**Remark.** The set \( 1/2 + (\mathbb{Z} \cap [\frac{1-l_n-w}{2}, \frac{l_n-1-w}{2}]) \) consists precisely all critical points of the standard local \( L \)-function \( L(s, \pi_{l_n,w}) \) (cf. [GR2, Proposition 6.1.1]).

To formulate the non-vanishing hypothesis of the title, put

\[
h_{2n} := \text{gl}_n(\mathbb{C}) \times \text{gl}_n(\mathbb{C}),
\]

and

\[
c_{2n} := h_{2n} \cap \ell_{2n} = (\mathfrak{o}_n(\mathbb{C}) \times \mathfrak{o}_n(\mathbb{C})) \oplus \mathbb{C}.
\]

It is an important coincidence that

\[
\text{dim}(h_{2n}/c_{2n}) = p_0.
\]

Write

\[
j_{2n} : h_{2n}/c_{2n} \to g_{2n}/\ell_{2n}
\]

for the inclusion map.

Let \( \chi = \chi_1 \otimes \chi_2 \) be as in Theorem A, and let \( w_1, w_2 \) be as in Theorem B. Define \( \epsilon_{x_i} \in \mathbb{Z}/2\mathbb{Z} \) so that

\[
\chi_i|_{\mathfrak{o}(n)} = \det^\epsilon_{x_i}, \quad i = 1, 2,
\]

and put

\[
\epsilon_0 := n - 1 + \epsilon_{x_1} + w_1 = n - 1 + \epsilon_{x_2} + w_2 \in \mathbb{Z}/2\mathbb{Z}.
\]

Using Theorem A and B, we prove

**Theorem C.** Let \( \varphi_\chi \) be a nonzero element of the left hand side of (6), and let \( \varphi_{w_1,w_2} \) be a nonzero element of the left hand side of (7). Then the linear functional

\[
\text{Hom}(\bigwedge^{p_0}(g_{2n}/\ell_{2n}), \pi_{1,w} \otimes F_{1,-w}) \to \text{Hom}(\bigwedge^{p_0}(h_{2n}/c_{2n}), \chi \otimes \det^{w_1,w_2}),
\]

\[
f \mapsto (\varphi_\chi \otimes \varphi_{w_1,w_2}) \circ f \circ \bigwedge^{p_0} j_{2n}
\]

do not vanish on the one-dimensional space

\[
\text{Hom}_{\det^{\epsilon_0}}(\bigwedge^{p_0}(g_{2n}/\ell_{2n}), \pi_{1,w} \otimes F_{1,-w}),
\]

and does vanish on the one-dimensional space

\[
\text{Hom}_{\det^{\epsilon_{0}+1}}(\bigwedge^{p_0}(g_{2n}/\ell_{2n}), \pi_{1,w} \otimes F_{1,-w}).
\]

Here “\( \text{Hom} \)” in (9) indicates the space of all linear maps. Theorem C is a slight generalization of the non-vanishing hypothesis of H. Grobner and A. Raghuram in their study of algebraicity of critical \( L \)-values using Shalika models, see [GR2, Hypothesis 6.6.2]. As pointed out by them, up to minor variations, such a hypothesis appears in many previous articles, see, for instance, Ash-Ginzburg [AG], Harris [Har], Kasten-Schmidt [KS], Kazhdan-Mazur-Schmidt [KMS], Mahnkopf [Mah], Raghuram [Rag]
and Schmidt [Schm]. We expect that the method of this paper also works for these variations. The reader is referred to the afore mentioned articles for the importance of these hypotheses in the study of algebraicity of critical L-values.

In Section 2, we deal with finite-dimensional representation theory problems encountered in this paper. In particular, we prove Theorem B, and prove Theorem C by assuming Theorem A. Section 3 is devoted to a general construction of cohomologically induced distinguished representations. This lies at the heart of the paper and is interesting in itself. As an application of this construction, we prove Theorem A in Section 4.

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2. Finite-dimensional distinguished representations

2.1. Irreducible representations of $O_{2n}(\mathbb{C})$. We start with a review of Cartan-Weyl’s highest weight theory in the case of even orthogonal groups (cf. [GW, Section 5.5.5]). Let $V$ be a complex vector space of even dimension $2n \geq 2$, equipped with a non-degenerate symmetric bilinear form on it. Denote by $\hat{O}(V)$ the set of isomorphism classes of irreducible algebraic finite-dimensional representations of $O(V)$. We intend to parameterize it by the set

$$P_{O_{2n}} := \{ l = (l_1 \geq l_2 \geq \cdots \geq l_n) \in \mathbb{Z}^{n-1} \times \{-1/2, 0, 1, 2, \cdots \} \}.$$

That is, for every $l \in P_{O_{2n}}$, we shall define a representation $\tau_l \in \hat{O}(V)$, and every representation in $\hat{O}(V)$ is uniquely of the form $\tau_l$.

When $n = 1$, $\tau_{-1/2}$ and $\tau_0$ are defined to be the determinant character and the trivial character, respectively. Fix one of the two embeddings of $\mathbb{C}^\times$ in $O(V)$, for $l \geq 1$, we define $\tau_l$ to be the two-dimensional representation of $O(V)$ which has weights $\pm l$ when restricted to $\mathbb{C}^\times$.

In general, fix a flag

$$F = (X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_{n-1})$$
of totally isotropic spaces in $V$ so that $\dim X_j = j$, $j = 0, 1, \ldots, n - 1$. Denote by $B_{\mathcal{F}}$ the stabilizer of $\mathcal{F}$ in $O(V)$. Then we have an exact sequence

$$1 \to U_{\mathcal{F}} \to B_{\mathcal{F}} \to (\mathbb{C}^\times)^{n-1} \times O(X_{n-1}^\perp/X_{n-1}) \to 1,$$

where $U_{\mathcal{F}}$ is the unipotent radical of $B_{\mathcal{F}}$, and the identification

$$\text{GL}(X_1/X_0) \times \text{GL}(X_2/X_1) \times \cdots \times \text{GL}(X_{n-1}/X_{n-2}) = (\mathbb{C}^\times)^{n-1}$$

is used. Now for every $l = (l_1, l_2, \cdots, l_n) \in \mathcal{P}_{O(2n)}$, we define $\tau_l$ to be the unique representation in $\hat{O}(V)$ so that $\tau_l^{U_{\mathcal{F}}}$ descends to the irreducible representation

$$(\cdot)^{l_1} \otimes (\cdot)^{l_2} \otimes \cdots \otimes (\cdot)^{l_{n-1}} \otimes \tau_{l_n}$$

of $(\mathbb{C}^\times)^{n-1} \times O(X_{n-1}^\perp/X_{n-1})$. Here and henceforth, a superscript group (or Lie algebra) indicates the invariants of the group action (or the Lie algebra action, respectively).

As usual, we identify irreducible representations of the compact group $O(2n)$ with representations in $\hat{O}_{2n}(\mathbb{C})$.

### 2.2. Distinguished representations of $O_{2n}(\mathbb{C})$

For simplicity of notation, put $O_n^2 := O_n(\mathbb{C}) \times O_n(\mathbb{C})$. A slight modification of Helgason’s proof of Cartan-Helgason Theorem show the following lemma (cf. [Hel, Chapter V, Theorem 4.1]). We should not go to the details.

**Lemma 2.1.** Let $\tau_l \in \hat{O}_{2n}(\mathbb{C})$ with $l = (l_1, l_2, \cdots, l_n) \in \mathcal{P}_{O(2n)}$, and let $\epsilon_1, \epsilon_2 \in \mathbb{Z}/2\mathbb{Z}$. If one of the following three conditions

(a) $l_n > 0$ and $l_1, l_2, \cdots, l_n \in \epsilon_1 + \epsilon_2$,
(b) $l_n = 0$, and $l_1, l_2, \cdots, l_{n-1} \in \epsilon_1 = \epsilon_2 = 2\mathbb{Z}$,
(c) $l_n = -1/2$, and $l_1, l_2, \cdots, l_{n-1} \in \epsilon_1 - 1 = \epsilon_2 - 1 = 2\mathbb{Z}$,

is satisfied, then

$$\dim \text{Hom}_{O_n^2}(\tau_l, \det^{\epsilon_1, \epsilon_2}) = 1.$$ 

Recall that every representation in $\hat{O}_{2n}(\mathbb{C})$ is self dual. We should not provide a proof of the following elementary lemma.

**Lemma 2.2.** Let $\tau \in \hat{O}_{2n}(\mathbb{C})$ and let $\epsilon_1, \epsilon_2 \in \mathbb{Z}/2\mathbb{Z}$. If $\phi$ is a nonzero element of $\text{Hom}_{O_n^2}(\tau, \det^{\epsilon_1, \epsilon_2})$, then $\phi \otimes \phi$ does not vanish on the one-dimensional space $(\tau \otimes \tau)_{O_{2n}(\mathbb{C})}$. 
2.3. Distinguished representations of $\text{GL}_{2n}(\mathbb{C})$. Denote by $F_{\mu}$ the irreducible algebraic finite-dimensional representation of $\text{GL}_{2n}(\mathbb{C})$ with highest weight

$$
\mu := (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{2n-1} \geq \mu_{2n}) \in \mathbb{Z}^{2n}.
$$

The following lemma is an instance of H. Schlichtkrull’s generalization of Cartan-Helgason Theorem ([Sch2, Theorem 7.2]), see also [Kna, Theorem 2.1]).

**Lemma 2.3.** Assume that

$$
\mu_1 + \mu_{2n} = \mu_2 + \mu_{2n-1} = \cdots = \mu_n + \mu_{n+1} = w_{\mu}
$$

for some integer $w_{\mu}$. If $w_1, w_2 \in \mathbb{Z} \cap [\mu_{n+1}, \mu_n]$ and $w_1 + w_2 = w_{\mu}$, then

$$
\dim \text{Hom}_{\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})}(F_{\mu}; \text{det}^{w_1, w_2}) = 1.
$$

2.4. The maximal $\text{O}_{2n}(\mathbb{C})$-type. As usual, for any integer $m \geq 0$, we identify the group of algebraic characters on $(\mathbb{C}^\times)^m$ with $\mathbb{Z}^m$, and write $e_1, e_2, \cdots, e_m$ for the standard basis of $\mathbb{Z}^m$.

Fix an embedding

$$
\gamma_{2n} : (\mathbb{C}^\times)^{2n} \hookrightarrow \text{GL}_{2n}(\mathbb{C})
$$

of algebraic groups which sends $(a_1, a_2, \cdots, a_{2n})$ to the matrix

$$
\begin{bmatrix}
\frac{a_1+a_2}{2} & 0 & \cdots & 0 & \cdots & 0 & \frac{a_1-a_2}{2} \\
0 & \frac{a_2+a_{2n-1}}{2} & \cdots & 0 & \cdots & 0 & \frac{a_2-a_{2n-1}}{2} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{a_n+a_{n+1}}{2} & \cdots & \frac{a_n-a_{n+1}}{2} & 0 \\
0 & 0 & \cdots & \frac{a_{n+1}+a_{2n}}{2} & \cdots & \frac{a_{n+1}-a_{2n}}{2} & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{a_{2n-1}-a_2}{2} & \cdots & 0 & \cdots & \frac{a_{2n-1}+a_2}{2} & 0 \\
\frac{a_{2n-1}-a_2}{2} & 0 & \cdots & 0 & \cdots & \frac{a_{2n-1}+a_2}{2} & \frac{a_{2n-1}+a_2}{2}
\end{bmatrix},
$$

where $i = \sqrt{-1} \in \mathbb{C}$ is the fixed square root of $-1$. View $(\mathbb{C}^\times)^{2n}$ as a Cartan subgroup of $\text{GL}_{2n}(\mathbb{C})$ via the embedding (13). Then the corresponding root system is

$$
\{ \pm(e_i - e_j) \mid 1 \leq i < j \leq 2n \} \subset \mathbb{Z}^{2n}.
$$

Fix a Borel subalgebra $\mathfrak{b}_{2n}$ of $\mathfrak{g}_{2n}$, which corresponds to the positive system

$$
\{ e_i - e_j \mid 1 \leq i < j \leq 2n \} \subset \mathbb{Z}^{2n}
$$

of (14). Put

$$
T_0 := \gamma_{2n}^{-1}(\text{O}_{2n}(\mathbb{C})) = \{(a_1, a_2, \cdots, a_n, a_n^{-1}, \cdots, a_2^{-1}, a_1^{-1}) \in (\mathbb{C}^\times)^{2n} \} = (\mathbb{C}^\times)^n,
$$

and view it as a Cartan subgroup of $\text{O}_{2n}(\mathbb{C})$. Put

$$
\mathfrak{b}_0 := \mathfrak{b}_{2n} \cap \mathfrak{o}_{2n}(\mathbb{C}).
$$
This is the Borel subalgebra of \( \mathfrak{so}_{2n}({\mathbb C}) \) corresponding to the positive system
\[
\{ e_i \pm e_j \mid 1 \leq i < j \leq n \}
\]
of the root system of \( O_{2n}({\mathbb C}) \).

Let \( \mu \) be as in (12) and let \( v^+_\mu \in F_\mu \) be a nonzero highest weigh vector with respect to \( b_{2n} \). Then it is also a highest weight vector with respect to \( b_2 \). Therefore it generates an irreducible representation of \( O_{2n}({\mathbb C}) \). It is easy to see that this representation is isomorphic to \( \tau_{l_\mu} \), where

\[
l_\mu := \begin{cases} 
(\mu_1 - \mu_{2n}, \mu_2 - \mu_{2n-1}, \cdots, \mu_n - \mu_{n+1} - 1/2), & \text{if } \mu_n = \mu_{n+1} \text{ is odd;} \\
(\mu_1 - \mu_{2n}, \mu_2 - \mu_{2n-1}, \cdots, \mu_n - \mu_{n+1}), & \text{otherwise.}
\end{cases}
\]

**Lemma 2.4.** The irreducible representation \( \tau_\mu \) of \( O_{2n}({\mathbb C}) \) occurs with multiplicity one in \( F_\mu \).

**Proof.** The lemma follows by noting that \( v^+_\mu \) has weight
\[
(\mu_1 - \mu_{2n}, \mu_2 - \mu_{2n-1}, \cdots, \mu_n - \mu_{n+1})
\]
with respect to \( T_0 \), and this weight has multiplicity one in \( F_\mu \). \( \square \)

**Remark 2.5.** As a slight modification of Vogan’s definition ([Vog1, Definition 5.1]), we define the size of a representation \( \tau_\mu \) of \( O_{2n}({\mathbb C}) \) to be
\[
|\tau_\mu| := \sqrt{\sum_{j=1}^n (l_j + 2n - 2j)^2}, \quad l = (l_1, l_2, \cdots, l_n) \in \mathcal{P}_{O_{2n}}.
\]

Then \( \tau_\mu \) is the unique maximal \( O_{2n}({\mathbb C}) \)-type of \( F_\mu \), namely, all other representations in \( O_{2n}({\mathbb C}) \) which occur in \( F_\mu \) has strictly smaller size than \( \tau_\mu \).

### 2.5. Highest weight vectors in distinguished representations.

Let \( K \) be a reductive linear algebraic group defined over \( \mathbb{C} \), with an algebraic involution \( \sigma \) on it. Let \( C \) be an open subgroup of \( K^\sigma \). Let \( q \) be a \( \sigma \)-split parabolic subalgebra of the Lie algebra \( \mathfrak{k} \) of \( K \), namely, \( q \cap \sigma(q) \) is a Levi factor for both parabolic subalgebras \( q \) and \( \sigma(q) \). Denote by \( U \) the unipotent radical of the normalizer of \( q \) in \( K \).

As usual, we use “\( U \)” to indicate the universal enveloping algebra of a complex Lie algebra. The following lemma is useful to us.

**Lemma 2.6.** Let \( F \) be an irreducible algebraic finite-dimensional representation of \( K \) and let \( \chi : C \to \mathbb{C}^\times \) be an algebraic character. If \( C \) meets every connected component of \( K \), then every nonzero element in \( \text{Hom}_C(F, \chi) \) does not vanish on \( F^U \).

**Proof.** Note that \( \mathfrak{c} + q = \mathfrak{k} \) since \( q \) is \( \sigma \)-split, where \( \mathfrak{c} \) denotes the Lie algebra of \( C \). Therefore
\[
\mathbb{C}.F^U \supset U(\mathfrak{c}).F^U = U(\mathfrak{c})U(q).F^U = U(\mathfrak{k}).F^U = K_0.F^U,
\]
where $K_0$ is the identity connected component of $K$. Therefore
\[ C.F^U \supset C.(K_0.F^U) = K.F^U = F, \]
and the lemma easily follows. \hfill \Box

2.6. **Proof of Theorem B.** We use the notation of the Introduction in the remaining part of this section. It follows from Lemma 2.3 and 2.1 that both
\[ (17) \quad \text{Hom}_{GL_n(C) \times GL_n(C)}(F_{1,-w}, \det^{w_1,w_2}) \]
and
\[ (18) \quad \text{Hom}_{O_n(C) \times O_n(C)}(\tau_{1-}, \det^{w_1,w_2}) \]
have dimension one.

Write
\[ (19) \quad \sigma_{2n} : GL_{2n}(C) \to GL_{2n}(C) \quad \text{and} \quad \sigma_{2n} : g_{2n} \to g_{2n} \]
for the conjugations by the diagonal matrix with the first $n$ diagonal entries 1 and the last $n$ diagonal entries $-1$. Note that the Borel subalgebra $b_{2n}$ of Section 2.4 is $\sigma_{2n}$-split. Let $v_{1,-w}^+ \in F_{1,-w}$ be a nonzero highest weight vector with respect to $b_{2n}$. By Lemma 2.6, a nonzero functional in (17) does not vanish on $v_{1,-w}^+$. It does not vanish on $\tau_{1-}$ since $v_{1,-w}^+ \in \tau_{1-}$ by the argument of Section 2.4. This proves Theorem B.

2.7. **The space** $\wedge^{p_0}(g_{2n}/k_{2n})$. Put
\[ b_t := b_{2n} \cap \ell_{2n} = b_o \oplus \mathbb{C}. \]
The space $b_{2n}/b_t$ has dimension $p_0$ and $\wedge^{p_0}(b_{2n}/b_t)$ has weight
\[ l_O := (2n, 2n - 2, \ldots, 2) \]
with respect to the Cartan subgroup $T_O$ of (16). Furthermore, $\wedge^{p_0}(b_{2n}/b_t) \subset \wedge^{p_0}(g_{2n}/\ell_{2n})$ consists of highest weight vectors with respect to $b_t$. Therefore it generates an irreducible representation of $O_{2n}(C)$ which is isomorphic to $\tau_{1O}$.

**Lemma 2.7.** The irreducible representation $\tau_{1O}$ of $O_{2n}(C)$ occurs with multiplicity one in $\wedge^{p_0}(g_{2n}/\ell_{2n})$.

**Proof.** Similar to the proof of Lemma 2.4, the lemma holds because that the weight $l_O$ occurs with multiplicity one in $\wedge^{p_0}(g_{2n}/\ell_{2n})$. \hfill \Box

Fix a nonzero $O_{2n}(C)$-equivariant linear map
\[ \eta_O : \wedge^{p_0}(g_{2n}/\ell_{2n}) \to \tau_{1O}. \]
Recall the inclusion map $j_{2n} : b_{2n}/\ell_{2n} \to g_{2n}/\ell_{2n}$ from (8).
Proposition 2.8. The map
\[ \eta_O \circ (\wedge^p \eta_{2n}) : \wedge^p (\mathfrak{h}_{2n}/\mathfrak{c}_{2n}) \to \tau_O \]
is nonzero.

Proof. Identify \( \mathfrak{g}_{2n}/\mathfrak{t}_{2n} \) with the space \( \mathfrak{s}_{2n} \) of symmetric trace free matrices in \( \mathfrak{g}_{2n} \), and define on it an \( O_{2n}(\mathbb{C}) \)-invariant non-degenerate symmetric bilinear form
\[ \langle x, y \rangle := \text{tr}(xy), \quad x, y \in \mathfrak{g}_{2n}. \]
This induces a symmetric bilinear form \( \langle , \rangle_{\wedge} \) on \( \wedge^p (\mathfrak{g}_{2n}/\mathfrak{t}_{2n}) \). In order the prove the proposition, it suffices to show that the one-dimensional spaces \( \wedge^p (\mathfrak{h}_{2n}/\mathfrak{c}_{2n}) \) and \( \wedge^p (\mathfrak{b}_{2n}/\mathfrak{b}_t) \) are not perpendicular to each other under the form \( \langle , \rangle_{\wedge} \), or equivalently, the paring
\[ \langle , \rangle : \mathfrak{h}_{2n}/\mathfrak{c}_{2n} \times \mathfrak{b}_{2n}/\mathfrak{b}_t \to \mathbb{C} \]
is non-degenerate.

Denote by \( \sigma^0_{2n} : \mathfrak{g}_{2n}/\mathfrak{t}_{2n} \to \mathfrak{g}_{2n}/\mathfrak{t}_{2n} \) the map induced by \( \sigma_{2n} \). Let \( x \in \mathfrak{h}_{2n}/\mathfrak{c}_{2n} \) so that
\[ \langle x, \mathfrak{b}_{2n}/\mathfrak{b}_t \rangle = \{0\}. \tag{20} \]
Then
\[ \langle x, \sigma^0_{2n}(\mathfrak{b}_{2n}/\mathfrak{b}_t) \rangle = \langle \sigma^0_{2n}(x), \sigma^0_{2n}(\mathfrak{b}_{2n}/\mathfrak{b}_t) \rangle = \langle x, \mathfrak{b}_{2n}/\mathfrak{b}_t \rangle = \{0\}. \tag{21} \]
Since \( \sigma_{2n}(\mathfrak{b}_{2n}) \) is opposite to \( \mathfrak{b}_{2n} \), we have
\[ \mathfrak{b}_{2n}/\mathfrak{b}_t + \sigma^0_{2n}(\mathfrak{b}_{2n}/\mathfrak{b}_t) = \mathfrak{g}_{2n}/\mathfrak{t}_{2n}. \tag{22} \]
We conclude that \( x = 0 \) by combining (20), (21) and (22). This proves the proposition. \( \square \)

Lemma 2.9. The one dimension representation \( \wedge^p (\mathfrak{h}_{2n}/\mathfrak{c}_{2n}) \) of \( O_n(\mathbb{C}) \times O_n(\mathbb{C}) \) corresponds to the character \( \det^{n-1} \otimes \det^{n-1} \).

This is easy and we omit the proof.

2.8. Proof of Theorem C.

Lemma 2.10. Let \( \epsilon_i \in \mathbb{Z}/2\mathbb{Z}, \ i = 1, 2, \cdots, 7 \), and assume that
\[ \epsilon_1 + \epsilon_3 + \epsilon_5 = \epsilon_2 + \epsilon_4 + \epsilon_6 = \epsilon_7. \]
If \( \varphi_O, \varphi_+ \) and \( \varphi_- \) are respectively nonzero elements in
\[ \text{Hom}_{O_n}(\det^{\epsilon_1,\epsilon_2}, \tau_O), \text{ Hom}_{O_n}(\tau_{1^+}, \det^{\epsilon_3,\epsilon_4}) \text{ and } \text{Hom}_{O_n}(\tau_{1^-}, \det^{\epsilon_5,\epsilon_6}), \]
then the map
\[ \text{Hom}_{\det^{\epsilon_7}}(\tau_O, \tau_{1^+} \otimes \tau_{1^-}) \to \text{Hom}(\det^{\epsilon_1,\epsilon_2}, \det^{\epsilon_3,\epsilon_4} \otimes \det^{\epsilon_5,\epsilon_6}), \]
\[ f \mapsto (\varphi_+ \otimes \varphi_-) \circ f \circ \varphi_O \]
is nonzero.

Proof. Without loss of generality assume that \( \varepsilon_T = 0 \). By taking the transpose of \( \varphi_O \), the lemma is equivalent to saying that the map

\[
\varphi'_O \otimes \varphi_- \otimes \varphi_+ : \mathfrak{t}_O \otimes \mathfrak{t}_- \otimes \mathfrak{t}_+ \rightarrow \det^{\varepsilon_1,\varepsilon_2} \otimes \det^{\varepsilon_5,\varepsilon_6} \otimes \det^{\varepsilon_3,\varepsilon_4}
\]

does not vanish on \((\mathfrak{t}_O \otimes \mathfrak{t}_- \otimes \mathfrak{t}_+)_{O_{2n}(\mathbb{C})}\), where \( \varphi'_O \) is a nonzero element of \( \text{Hom}_{O_{2n}(\mathbb{C})}(\mathfrak{t}_O, \det^{\varepsilon_1,\varepsilon_2}) \).

Denote by \( v_l \in \mathfrak{t}_O \) and \( v_l^- \in \mathfrak{t}_- \) two nonzero highest weight vectors of weights \( l_1 = (2n, 2n - 2, \ldots, 2) \) and \( (l_1 - (2n - 1), l_2 - (2n - 3), \ldots, l_n - 1) \), respectively, with respective to \( \mathfrak{b}_o \). Then \( v_l \otimes v_l^- \in \mathfrak{t}_O \otimes \mathfrak{t}_- \) generates an irreducible representation \( \tau'_l \) of \( O_{2n}(\mathbb{C}) \) which is isomorphic to \( \tau^l \). The functional \( \varphi'_O \otimes \varphi_- \) does not vanish on \( \tau'_l \) since it does not vanish on \( v_l \otimes v_l^- \) by Lemma 2.6. Now apply Lemma 2.2 and we finish the proof. \( \square \)

Assuming Theorem A, we are now ready to prove Theorem C. For the first assertion, it suffices to show that the map

\[
\text{Hom}_{\det^{0}}(\mathfrak{t}_O, \mathfrak{t}_+ \otimes \mathfrak{t}_-) \rightarrow \text{Hom}(\Lambda^{p_0}(\mathfrak{h}_{2n}/\mathfrak{c}_{2n}), \chi \otimes \det^{w_1.w_2}),
\]

\[ f \mapsto (\varphi_\chi \otimes \varphi_{w_1,w_2}) \circ f \circ (\eta_\chi \otimes \Lambda^{p_0}j_{2n}) \]

is nonzero. In view of Lemma 2.9 and 2.10, this is a consequence of Theorem A, Theorem B and Proposition 2.8.

The second assertion holds because the image of (11) under the map (9) is contained in

\[
\text{Hom}_{\det^{0+1}}(\mathfrak{t}_O, \mathfrak{t}_+ \otimes \mathfrak{t}_-) \rightarrow \text{Hom}(\Lambda^{p_0}(\mathfrak{h}_{2n}/\mathfrak{c}_{2n}), \chi \otimes \det^{w_1.w_2}),
\]

and the later space clearly vanishes.

3. Cohomologically induced distinguished representations

This section may be read independently.

3.1. Generalities on distinguished representations. Let \( G \) be a real reductive group, namely, it is a Lie group with the following properties:

- the complexified Lie algebra \( \mathfrak{g} \) of \( G \) is reductive,
- it has only finitely many connected components,
- there is a connected closed subgroup of \( G \) with finite center whose complexified Lie algebra equals to \([\mathfrak{g}, \mathfrak{g}]\).

Let \( \sigma \) be a continuous involution on \( G \) and let \( H \) be an open subgroup of the \( \sigma \)-fixed point group \( G^\sigma \).

Fix a Cartan involution \( \theta \) of \( G \) which commutes \( \sigma \) (for its existence, cf. [Ber]). Let \( \mathfrak{q} \) be a parabolic subalgebra of \( \mathfrak{g} \). We say that \( \mathfrak{q} \) is real \( \sigma \)-split if

\[
\tilde{\mathfrak{q}} = \mathfrak{q} \quad \text{and} \quad \theta(\mathfrak{q}) = \sigma(\mathfrak{q}),
\]
and say that \( q \) is \( \theta \)-stable \( \sigma \)-split if
\[
\theta(q) = q \quad \text{and} \quad \bar{q} = \sigma(q).
\]
Here \( \sigma, \theta : g \to g \) are respectively complexified differentials of the involutions \( \sigma, \theta : G \to G \), and \( \bar{\cdot} : g \to g \) is the complex conjugation with respect to the real Lie algebra of \( G \).

Fix a character \( \chi : H \to \mathbb{C}^\times \). It is of general interest to construct Casselman-Wallach representations which map to the induced representation
\[
\text{Ind}^G_H \chi := \{ f \in C^\infty(G) \mid f(hx) = \chi(h)f(x), \ h \in H, \ x \in G \}.
\]
This is the same as constructing Casselman-Wallach representations \( \pi \) of \( G \), together with \( H \)-equivariant continuous linear functionals \( \varphi : \pi \to \chi \). We call such a pair \((\pi, \varphi)\) a \( \chi \)-distinguished representation of \( G \).

One general way to get distinguished representations is the ordinary parabolic induction associated to real \( \sigma \)-split parabolic subalgebras of \( g \). This is studied by many authors for trivial \( \chi \), see, for instance, Oshima-Sekiguchi [OS], Ólafsson [Ola], Delorme [Del], van den Ban [Ban] and Brylinski-Delorme [BrD]. On the other hand, Flensted-Jensen [FJ] and Oshima-Matsuki [OM] construct all discrete series representations on \( H \setminus G \), when they exist. Schlichtkrull [Sch1] and Vogan [Vog2] prove that these are isomorphic to cohomologically induced representations associated to certain \( \theta \)-stable \( \sigma \)-split parabolic subalgebras, without describing the corresponding \( H \)-invariant functionals.

Note that \( H \) is a real reductive group and is stable under \( \theta \). Put \( K := G^\theta \) and \( C := H^\theta \), which are maximal compact subgroups of \( G \) and \( H \), respectively. The following automatic continuity theorem is due to van den Ban-Delorme [BaD, Theorem 1] and Brylinski-Delorme [BrD, Theorem 1] for trivial \( \chi \), and Delorme confirms to the author its validity in general.

**Theorem 3.1.** Let \( E \) be a finitely generated admissible \((g, K)\)-module. Then the restriction induces a linear isomorphism
\[
\text{Hom}_H(E^\infty, \chi) \cong \text{Hom}_hC(E, \chi),
\]
where \( E^\infty \) denotes the Casselman-Wallach globalization of \( E \).

In view of Theorem 3.1, one may study distinguished representations in the setting of Harish-Chandra modules. We note that associated to a parabolic subalgebra which is either real \( \sigma \)-split or \( \theta \)-stable \( \sigma \)-split, one may construct distinguished representations (in the setting of Harish-Chandra modules) by purely algebraic method. We carry out the construction in the \( \theta \)-stable \( \sigma \)-split case in the remaining part of this section. The real \( \sigma \)-split case will be studied in more detail by the author and Chen-Bo Zhu in a paper in preparation.
3.2. Some subgroups and Lie subalgebras. Assume that \( q \) is a \( \theta \)-stable \( \sigma \)-split parabolic subalgebra of \( g \). We introduce the following diamond of groups

\[
\begin{array}{c}
G' \\
H' \quad \subseteq \quad K' \\
C'
\end{array}
\quad \begin{array}{c}
G \\
H \quad \subseteq \quad K
\end{array}
\]

where

\[
G' := N_G(q) = N_G(\bar{q}) \quad \text{(the normalizer)},
\]

and

\[
H' := G' \cap H, \quad K' := G' \cap K, \quad C' := H' \cap K'.
\]

Denote by \( n \) the nilpotent radical of \( q \cap [g, g] \). The parabolic subalgebras \( q \) and \( \bar{q} \) decompose as

\[
q = g' \oplus n \quad \text{and} \quad \bar{q} = g' \oplus \bar{n},
\]

and they are opposite to each other. Put

\[
q_c := q \cap k \quad \text{and} \quad n_c := n \cap k.
\]

Lemma 3.2. One has that

\[
g = h + q = h + \bar{q}
\]

and

\[
\mathfrak{k} = \mathfrak{c} + q_c = \mathfrak{c} + \bar{q}_c.
\]

Proof. This is known to experts. We sketch a proof for completeness. Consider the real form

\[
g_0 := \{x \in g \mid (\sigma \circ \theta)(\bar{x}) = x\}
\]

of \( g \). Then \( q \) is real with respect to \( g_0 \), and \( \sigma|_{g_0} \) is a Cartan involution. Therefore

\[
g = h + q
\]

by the infinitesimal version of Langlands decomposition. Apply “\( - \)” to (26), we get

\[
g = h + \bar{q}.
\]

The equalities (25) is proved in the same way.

Lemma 3.3. One has that

\[
\dim \mathfrak{c}/\mathfrak{c}' = \dim n_c = \dim \bar{n}_c = \frac{1}{2} \dim \mathfrak{k}/\mathfrak{k}'.
\]
Proof. We have
\[ c \cap \bar{q}_c = \sigma (c \cap \bar{q}_c) = c \cap q_c = c \cap (q_c \cap \bar{q}_c) = c \cap t' = c'. \]
Therefore by (25),
\[ c/c' = c/(c \cap \bar{q}_c) \cong t/\bar{q}_c \cong n_c \]
as vector spaces, which implies the first equality of the lemma. Other equalities are obvious. \qed

Write $S$ for the value of the equalities of Lemma 3.3.

3.3. Cohomological induction. Let $E'$ be a finitely generated admissible $(\mathfrak{g}', K')$-module. View it as a $(\bar{q}, K')$-module through the trivial $\bar{n}$-action. Then
\[ E^\circ := U(\mathfrak{g}) \otimes_{U(\bar{q})} E' \]
is a $(\mathfrak{g}, K')$-module, where $\mathfrak{g}$ acts by left multiplication, and $K'$ acts by the tensor product of its adjoint action on $U(\mathfrak{g})$ and its given action on $E'$.

Denote by $\Gamma_{K',K}$ the Zuckerman functor from the category of $(\mathfrak{g}, K')$-modules to the category of $(\mathfrak{g}, K)$-modules, and by $\Gamma_{K',K}^j$ its derived functors, $j = 0, 1, 2, \cdots$. We are concerned with the $(\mathfrak{g}, K)$-module
\[ E := \Gamma_{\bar{q}}^S(E') := \Gamma_{K',K}^S(E^\circ). \]

As usual, we use “Ad” to indicate the adjoint action in variant contexts. Denote by $\mathbb{C}[K]$ the space of left $K$-finite (or equivalently, right $K$-finite) smooth functions on $K$. (Similar notation is used for other compact Lie groups.) In order to describe the module $E$ more explicitly, we introduce an action of the quadruple
\[ (\mathfrak{t}, K') \times (\mathfrak{g}, K) \]
on the space
\[ \mathbb{C}[K] \otimes E^\circ, \]
as follows:

- the pair $(\mathfrak{t}, K')$ acts by the tensor product of the left translation on $\mathbb{C}[K]$ and the restriction of the $(\mathfrak{g}, K')$-action on $E^\circ$;
- the group $K$ acts on $\mathbb{C}[K] \otimes E^\circ$ through the right translation on $\mathbb{C}[K]$;
- the Lie algebra $\mathfrak{g}$ acts by
\[ (X.f)(k) := (\text{Ad}_k X).f(k), \quad k \in K, \ f \in \mathbb{C}[K] \otimes E^\circ. \]
In (31) and as usual, we identify $\mathbb{C}[K] \otimes E^\circ$ with a space of $E^\circ$-valued functions on $K$. 

Under these actions, $\mathbb{C}[K] \otimes E^o$ becomes a $(\mathfrak{t}, K')$-module as well as a weak $(\mathfrak{g}, K)$-module (cf. [KV, Chapter I, Section 5] for the notion of weak $(\mathfrak{g}, K)$-modules). Furthermore, the $(\mathfrak{t}, K')$-action and the $(\mathfrak{g}, K)$-action commute with each other. Therefore the relative Lie algebra cohomology group

$$H^j(\mathfrak{t}, K'; \mathbb{C}[K] \otimes E^o), \quad j = 0, 1, 2, \ldots,$$

carries a $(\mathfrak{g}, K)$-action. It turns out that (32) is actually a $(\mathfrak{g}, K)$-module (not only a weak $(\mathfrak{g}, K)$-module), and is canonically isomorphic to the derived functor module $\Gamma^j_{K', K}(E^o)$ (cf. [DV], see also [MP, Theorem 1.6]).

### 3.4. $\chi$-equivariant functionals.

Let $C'$ act on $\wedge S(c/c')$ by the adjoint action, and let $\mathfrak{h}$ act on it trivially. Then $\wedge S(c/c')$ is an $(\mathfrak{h}, C')$-module. Write $$\chi^o := \wedge S(c/c') \otimes \chi,$$ to be viewed as a tensor product $(\mathfrak{h}, C')$-module. Now assume that we are given an $(\mathfrak{h}', C')$-equivariant linear map $$\varphi' : E' \to \chi^o,$$ and we shall construct an $(\mathfrak{h}, C)$-equivariant functional $\varphi : E \to \chi$ in what follows.

**Lemma 3.4.** There is a unique $(\mathfrak{h}, C')$-equivariant linear map $\varphi^o : E^o \to \chi^o$ which extends $\varphi'$.

**Proof.** We have that $$\mathfrak{h} \cap \bar{\mathfrak{q}} = \sigma(\mathfrak{h} \cap \bar{\mathfrak{q}}) = \mathfrak{h} \cap \mathfrak{q} = \mathfrak{h} \cap (\mathfrak{q} \cap \bar{\mathfrak{q}}) = \mathfrak{h} \cap \mathfrak{g}' = \mathfrak{h}'.$$

Combining with (24), we get $$E^o = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} E' = U(\mathfrak{h}) \otimes_{U(\mathfrak{h}')} E'$$
as an $(\mathfrak{h}, C')$-module. Therefore the lemma is a form of Frobenious reciprocity. □

Similar to the action of the quadruple (29) on $\mathbb{C}[K] \otimes E^o$, based on the $(\mathfrak{h}, C')$-action on $\chi^o$, we define an action of the quadruple

$$\mathfrak{c} \times (\mathfrak{h}, C)$$
on the space $$\mathbb{C}[C] \otimes \chi^o.$$ Note that the quadruple (33) is component-wise contained in the quadruple (29), and the map

$$r_{K,C} \otimes \varphi^o : \mathbb{C}[K] \otimes E^o \to \mathbb{C}[C] \otimes \chi^o$$is $(\mathfrak{c}, C') \times (\mathfrak{h}, C)$-equivariant. Here $r_{K,C}$ is the restriction map, and $\varphi^o$ is as in Lemma 3.4.

The following lemma is routine to check.
Lemma 3.5. Let the pairs \((c, C')\) and \((h, C)\) act on \(\chi^o = \wedge^S(c/c') \otimes \chi\) through their actions on \(\wedge^S(c/c')\) and \(\chi\), respectively. Then the linear map
\[
C[C] \otimes \chi^o \rightarrow \chi^o,
\]
\[
f \mapsto \int_C \chi(c)^{-1} f(c) \, dc,
\]
is \((c, C') \times (h, C)\)-equivariant, where \(\chi^o\) is the normalized Haar measure on \(C\), and as usual, \(C[C] \otimes \chi^o\) is viewed as a space of \(\chi^o\)-valued functions on \(C\).

View \(\chi^o\) as a \((c, C') \times (h, C)\)-module as in Lemma 3.5, then we have

Lemma 3.6. One has an identification
\[
\widetilde{H}^S(c, C'; \chi^o) = \chi
\]
of \((h, C)\)-modules.

Proof. Note that
\[
\widetilde{H}^S(c, C'; \wedge^S(c/c')) = \text{Hom}_{C'}(\wedge^S(c/c'), \wedge^S(c/c')) = C
\]
as a vector space. Therefore
\[
\widetilde{H}^S(c, C'; \chi^o) = \widetilde{H}^S(c, C'; \wedge^S(c/c')) \otimes \chi = C \otimes \chi = \chi.
\]
\[ \square \]

Restriction of cohomology yields an \((h, C)\)-equivariant linear map
\[
E = \widetilde{H}^S(f, K'; C[K] \otimes \chi^o) \rightarrow \widetilde{H}^S(c, C'; C[K] \otimes \chi^o),
\]
and the composition of (34) and (35) yields an \((h, C)\)-equivariant linear map
\[
\widetilde{H}^S(c, C'; C[K] \otimes \chi^o) \rightarrow \widetilde{H}^S(c, C'; \chi^o) = \chi.
\]
Finally we obtain the desired \((h, C)\)-equivariant linear map
\[
\varphi := \Gamma ^S_q(\varphi') : E \rightarrow \chi
\]
by composing (37) and (38).

Remark. It seems that meromorphic continuation is needed in order to construction general \(\chi\)-equivariant functionals on \(\Gamma ^S_{K', K}(\text{Hom}_{U(g)}(U(g), E')_{K'})\). The later is another form of cohomologically induction which is isomorphic to \(E\) when \(E'\) lies in a “general position”. This is similar to the construction of \(H\)-invariant linear functionals on the usual parabolically induced representations, where meromorphic continuation is used in the literature.
3.5. **Bottom layers.** Let $F$ be an irreducible representation of $K$. The matrix coefficient map
\[ m_F : F \otimes F^* \rightarrow \mathbb{C}[K], \quad u \otimes v \mapsto (k \mapsto v(ku)) \]
is $K \times K$-equivariant, with the first factor of $K \times K$ acts on $\mathbb{C}[K]$ by right translations, and the second one acts by left translations. We have a $K$-space containment
\[ (40) \quad E = H^S(\mathfrak{t}, K'; \mathbb{C}[K] \otimes E^o) \supset H^S(\mathfrak{t}, K'; F \otimes F^* \otimes E^o) = F \otimes H^S(\mathfrak{t}, K'; F^* \otimes E^o). \]

Let $F'$ be an irreducible representation of $K'$ which is $K'$-equivariantly embedded in $E'$. Assume that we are given a $K'$-equivariant linear map $\beta_0 : \wedge^S n_c \rightarrow (F^*)^{\bar{n}_c} \otimes F'$. It induces a map $\beta_0 : \wedge^S n_c \rightarrow F^* \otimes E^o$ since $(F^*)^{\bar{n}_c} \subset F^*$ and $F' \subset E' \subset E^o = U(\mathfrak{g}) \otimes U(\bar{\mathfrak{g}}) E'$.

The following lemma is essentially known and routine to check.

**Lemma 3.7.** The composition map
\[ \tilde{\beta}_0 : \wedge^S (\mathfrak{t}/\mathfrak{t}') \rightarrow \wedge^S n_c \rightarrow F^* \otimes E^o \]
is a cocycle of degree $S$ in the complex $\{ \text{Hom}_{K'}(\wedge^j (\mathfrak{t}/\mathfrak{t}'), F^* \otimes E^o) \}_{j \in \mathbb{Z}}$ which computes the cohomology group $H^S(\mathfrak{t}, K'; F^* \otimes E^o)$. Here
\[ p_{nc} : \mathfrak{t}/\mathfrak{t}' \rightarrow \mathfrak{t}/\bar{\mathfrak{t}}_c = n_c \]
is the quotient map.

The map $\tilde{\beta}_0$ of the above lemma represents an element $[\tilde{\beta}_0]$ of $H^S(\mathfrak{t}, K'; F^* \otimes E^o)$. Combining this with (40), we get a $(K$-equivariant) bottom layer map
\[ \beta : F \rightarrow E, \quad v \mapsto v \otimes [\tilde{\beta}_0]. \]

3.6. **Non-vanishing of $\varphi$ on Bottom layers.** Recall that we are given an $(\mathfrak{h}', C')$-equivariant linear map $\varphi' : E' \rightarrow \chi^o = \wedge^S (\mathfrak{c}/\mathfrak{c}') \otimes \chi$.

By tensoring with $(F^*)^{\bar{n}_c}$, its restriction to $F'$ yields a linear map $\varphi_F : (F^*)^{\bar{n}_c} \otimes F' \rightarrow (F^*)^{\bar{n}_c} \otimes \chi^o$.

The image of the composition map
\[ (41) \quad \varphi_F \circ \beta_0 : \wedge^S n_c \rightarrow (F^*)^{\bar{n}_c} \otimes \chi^o \]
is at most one-dimensional and is therefore of the form $v_{F^*} \otimes \chi^o$ for some $v_{F^*} \in (F^*)^{\bar{n}_c}$.

We have the following criteria of non-vanishing of $\varphi$ on bottom layers:
Proposition 3.8. The functional $\varphi \circ \beta$ on $F$ is nonzero if and only if
\[ \int_C \chi(c) c.v_F^* \, dc \neq 0. \]

Proof. Let $v \in F$. Then $\beta(v) \in E$ is represented by the cocycle
\[ v \otimes \tilde{\beta}_0 \in \text{Hom}_{K'}(\wedge^S(c/t'), F \otimes F^* \otimes E^o) \subset \text{Hom}_{K'}(\wedge^S(c/t'), \mathbb{C}[K] \otimes E^o). \]
Recall the map
\[ r_{K,C} \otimes \varphi^o : \mathbb{C}[K] \otimes E^o \to \mathbb{C}[C] \otimes \chi^o \]
form (34), and write
\[ j_c : c/c' \to \mathfrak{t}/\mathfrak{t}' \]
for the inclusion map. Denote by $b_v$ a generator of the image of the composition of
\[ \wedge^S(c/c') \xrightarrow{\wedge^S j_c} \wedge^S(c/t') \xrightarrow{v \otimes \tilde{\beta}_0} \mathbb{C}[K] \otimes E^o \xrightarrow{r_{K,C} \otimes \varphi^o} \mathbb{C}[C] \otimes \chi^o, \]
to be viewed as a $\chi^o$-valued function on $C$. Then by the construction of $\varphi$, we know that $\varphi(\beta(v)) = 0$ if and only if
\[ (42) \quad \int_C \chi(c)^{-1} b_v(c) \, dc = 0. \]

Note that the map $v \otimes \tilde{\beta}_0$ is the composition of
\[ \wedge^S(c/c') \xrightarrow{\wedge^S j_c} \wedge^S(c/t') \xrightarrow{v \otimes \beta_0} \mathbb{C}[K] \otimes E^o. \]
Since the composition of
\[ \wedge^S(c/c') \xrightarrow{\wedge^S j_c} \wedge^S(c/t') \xrightarrow{v \otimes \beta_0} \mathbb{C}[K] \otimes E^o \]
is a linear isomorphism, $b_v$ a generator of the image of the composition of
\[ \wedge^S(n_c) \xrightarrow{v \otimes \beta_0} \mathbb{C}[K] \otimes E^o \xrightarrow{r_{K,C} \otimes \varphi^o} \mathbb{C}[C] \otimes \chi^o. \]
This composition map is the same as the composition of
\[ (43) \quad \wedge^S(n_c) \xrightarrow{v \otimes \beta_0} F \otimes F^* \otimes E^o \xrightarrow{1_F \otimes 1_{F^*} \otimes \varphi^o} F \otimes F^* \otimes \chi^o \xrightarrow{(r_{K,C} \circ m_F) \otimes 1_{\chi^o}} \mathbb{C}[C] \otimes \chi^o. \]
By the definition of $v_{F^*}$, the image of the composition of the first two maps in (43) is $v \otimes v_{F^*} \otimes \chi^o$. Hence (42) is equivalent to
\[ \int_C \chi(c)^{-1} v_{F^*}(c,v) \, dc = 0, \]
and the proposition follows. \qed
**Remark.** The argument of this section goes through if we replace \( q \) by an arbitrary \( \theta \)-stable parabolic subalgebra of \( g \) (with the assumption that \( q \cap \bar{q} \) is a common Levi factor of \( q \) and \( \bar{q} \), as usual), and replace \( H \) by an arbitrary \( \theta \)-stable closed subgroup of \( G \) so that \( q + h = g \). This should provide a construction of discrete series for general spherical reductive homogeneous spaces. We hope to study this in a future work.

4. **Proof of Theorem A**

We continue with the notation of last section and specify the discussion to the following case:

- the group \( G = \text{GL}_{2n}(\mathbb{R}) \),
- the involution \( \sigma = \sigma_{2n} \) (see (19)),
- the Cartan involution \( \theta \) is the inverse transpose of matrices,
- the parabolic subalgebra \( q = b_{2n} \) (see (15)), which is \( \theta \)-stable \( \sigma \)-split.

Identify \( G' \) with the group

\[
(\mathbb{C}^\times)^n = \left\{ (a_1, a_2, \ldots, a_n, \bar{a}_n, \ldots, \bar{a}_2, \bar{a}_1) \in (\mathbb{C}^\times)^{2n} \right\}
\]

through the embedding \( \gamma_{2n} \) (see (13)). It is a fundamental Cartan subgroup of \( \text{GL}_{2n}(\mathbb{R}) \). The groups in (23) become

\[
\begin{array}{ccc}
(\mathbb{C}^\times)^n & \subset & \text{GL}_{2n}(\mathbb{R}) \\
(\mathbb{R}^\times)^n & \subset & (\mathbb{S}^1)^n \\
\{\pm1\}^n & \subset & (\text{GL}_n(\mathbb{R}))^2 \\
& & (\text{O}(n))^2 \\
\end{array}
\]

Here \( \mathbb{S}^1 \) is the group of complex numbers of modulus one.

We also use the notation of the introduction. Let \( E' \) be the one-dimensional algebraic representation of \( \gamma_{2n}((\mathbb{C}^\times)^{2n}) \) corresponding to the character

\[
\lambda_1 + \rho_{2n} + \left( \frac{w}{2}, \frac{w}{2}, \ldots, \frac{w}{2}, \frac{w}{2} \right) \in \mathbb{Z}^{2n}.
\]

View it as a representation of \( G' \) through restriction, and we form the \((\mathfrak{g}, K)\)-module

\[
E := \Gamma_\mathfrak{g}^S(E')
\]

as in last section. Here \( S := \frac{1}{2} \dim \mathfrak{k}_c/\mathfrak{t}_c = n^2 - n \).

Write \( F' := E' \), viewed as a representation of \( K' = (\mathbb{S}^1)^n \). It has weight

\[
(l_1 + 2n - 1, l_2 + 2n - 3, \ldots, l_n + 1).
\]
Recall the representation $\tau_1$ of $O(2n)$ from the introduction. Put $F := \tau_1$. Then $F^* \cong F$ and $(F^*)^{n_c}$ has dimension two, with weights
\[-l_1 - 1, -l_2 - 1, \cdots, -l_n - 1\] and \[-l_1 - 1, -l_2 - 1, \cdots, l_n + 1\]. Note the one-dimensional representation $\wedge S_{n_c}$ of $K'$ has weight
\[(2n - 2, 2n - 4, \cdots, 0)\].

Fix a nonzero $K'$-equivariant linear map $\beta_0 : \wedge S_{n_c} \to (F^*)^{n_c} \otimes F'$, which exists and is unique up to scalar multiplications. We form the bottom layer map
\[\beta : F \to E\]
as in last section.

Recall from Theorem A that we are given a character
\[\chi = \chi_1 \otimes \chi_2 : H = GL_n(\mathbb{R}) \times GL_n(\mathbb{R}) \to \mathbb{C}^\times\]
such that
\[(44) \quad \chi_1 \chi_2 = \det^w.\]
Note that $\wedge^S(c/c')$ is trivial as an $(\mathfrak{h}, C')$-module, and (44) ensures that
\[\chi|_{H'} \cong E'|_{H'},\]
both correspond to the character
\[H' = (\mathbb{R}^\times)^n \subset \mathbb{C}^\times, \quad (a_1, a_2, \cdots, a_n) \mapsto (a_1 a_2 \cdots a_n)^w.\]
Therefore $E'$ is isomorphic to $\chi^\circ = \wedge^S(c/c') \otimes \chi$ as an $H'$-representation. Fix an $H'$-equivariant linear isomorphism
\[\varphi' : E' \to \chi^\circ.\]
As in last section, we construct an $(\mathfrak{h}, C')$-equivariant linear map
\[\varphi : E \to \chi.\]

Note that $v_{F^*}$ (see (41)) is a nonzero element in $(F^*)^{n_c}$ of weight
\[-l_1 - 1, -l_2 - 1, \cdots, -l_n - 1.\]

**Lemma 4.1.** One has that
\[\int_C \chi(c) \cdot v_{F^*} \, dc \neq 0.\]

**Proof.** View $F^*$ as an irreducible representation of the real form $O(n, n)$ of $O_{2n}(\mathbb{C})$. Then a slight modification of Helgason’s original proof of Cartan-Helgason Theorem shows the Lemma (cf. [Hel, Chapter V, Theorem 4.1]).
Now Proposition 3.8 implies that

(45) \[ \varphi \circ \beta \neq 0. \]

The following proposition is essentially an instance of [VZ, Proposition 6.1]. We should not go to the details.

**Proposition 4.2.** The underlying \((\mathfrak{g}, K)\)-module of \(\pi_{l,w}\) is isomorphic to \(E\).

In view of Proposition 4.2, and apply the automatic continuity theorem (Theorem 3.1), we obtain a \(\text{GL}_n(\mathbb{R}) \times \text{GL}_n(\mathbb{R})\)-equivariant continuous linear map

\[ \varphi : \pi_{l,w} \rightarrow \chi \]

which does not vanish on the minimal \(O(2n)\)-type \(\tau^+_l\).

We finish the proof of Theorem A by applying the following multiplicity one theorem. It is proved by Aizenbud and Gourevitch when \(\chi_{p,q}\) is trivial ([AGS, Theorem 8.2.4]), and their proof actually works in the general case.

**Theorem 4.3.** Let \(p, q \geq 0\). Let \(\pi_{p,q}\) be an irreducible Casselman-Wallach representation of \(\text{GL}_{p+q}(\mathbb{R})\) and let \(\chi_{p,q}\) be a character on \(\text{GL}_p(\mathbb{R}) \times \text{GL}_q(\mathbb{R})\), then

\[ \dim \text{Hom}_{\text{GL}_p(\mathbb{R}) \times \text{GL}_q(\mathbb{R})}(\pi_{p,q}, \chi_{p,q}) \leq 1. \]

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