Light-cone formulation and spin spectrum of non-critical fermionic string

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Abstract

A free fermionic string quantum model is constructed directly in the light-cone variables in the range of dimensions $1 < d < 10$. It is shown that after the GSO projection this model is equivalent to the fermionic massive string and to the non-critical Ramond-Neveu-Schwarz string. The spin spectrum of the model is analysed. For $d = 4$ the character generating functions is obtained and the particle content of first few levels is numerically calculated.

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1 Introduction

It was recently shown that the covariant quantization of the Rammond-Neveu-Schwarz string modified by adding the supersymmetric Liouville sector with vanishing cosmological constants leads in the dimensions $1 < d < 10$ to a family of tachyon free unitary models \[1\]. For every member of this family the Neveu-Schwarz sector does not contain any massless states which justifies the name – fermionic massive string – introduced in \[1\]. One of the quantum models characterised by the largest subspace of null states is equivalent to the non-critical RNS spinning string truncated in the Neveu-Schwarz sector to a tachyon free subspace of the fermion parity operator. It is called the critical fermionic massive string. The aim of the present paper is to develop the light-cone formulation of this model. Our motivation is twofold. First of all all the light-cone formulation seems indispensable for analysing the splitting-joining interaction. Secondly, it can be used to calculate the particle content of the model.

The paper is organised as follows. In Section 2 we define a quantum free string model directly in the light-cone variables. We shall call it the fermionic non-critical light-cone string or simply the light-cone string. In this model the longitudinal degrees of freedom are described by a background charge Fock space realisation \[2\] of the superconformal Verma module with the central charge $\hat{c} = 10 - d$, and the highest weight $h = \frac{1}{2}$. A similar construction motivated by the Liouville theory was first discussed by Marnelius in the context of the non-critical Polyakov fermionic string \[3\]. The GSO projection is introduced as a projection on a suitably chosen eigenspace of the (world-sheet) fermion parity operator.

In Section 3 we show that after the GSO projection the fermionic non-critical light-cone string is equivalent to the critical massive string and therefore to the suitably projected Rammond-Neveu-Schwarz non-critical string.

In Section 4 the spectrum of the light-cone string is analysed. For $d = 4$ the expansion of the character generating function in terms of irreducible characters is derived. It is illustrated by numerical calculations of spin content of first few levels. The spectrum of the GSO projected tachyon free model is also calculated. The corresponding results for the closed non-critical light-cone string are presented in Section 5. An interesting feature of the model is that the closed string spectrum does not contain space-time fermions.

The spectra of the open and the closed critical massive strings derived in this paper exclude the fundamental string interpretation of the model. It might however a good candidate for an effective low energy description of strong interactions. It shares two important features of the critical fermionic string - the light-cone formulation and the absence of tachyons. Whether it is enough for a consistent interaction is an interesting open problem.

2 Fermionic non-critical light-cone string

Let us fix a light-cone frame $\{e_\pm, e_1, \ldots, e_{d-2}\}$ in $d$-dimensional Minkowski space normalised by $e_\pm^2 = 0$, $e_+ \cdot e_- = -1$, and $e_i \cdot e_j = \delta_{ij}$ for $i, j = 1, \ldots, d - 2$. We shall use the following notation for the light-cone components of a vector $V$

$$V^\pm = e_\pm \cdot V , \quad V^i = e_i \cdot V , \quad \nabla = V^i e_i .$$
The fermionic non-critical light-cone string is defined as a representation of the algebra

\[ [a^i_0, q^j_0] = -i \delta^{ij}, \quad [a^+_0, q^-_0] = i, \quad [c_0, q^L_0] = -i, \]

\[ [a^i_m, a^j_n] = m \delta^{ij} \delta_{m,-n}, \quad [c_m, c_n] = m \delta_{m,-n}, \]

\[ \{b^r_r, b^s_s\} = \delta^{rs} \delta_{r,-s}, \quad \{d_r, d_s\} = \delta_{r,-s}, \tag{1} \]

supplemented by the conjugation properties

\[
(a^i_0)\dagger = a^i_0, \quad (q^i_0)\dagger = q^i_0, \quad (c_0)\dagger = c_0, \quad (q^L_0)\dagger = q^L_0, \\
(a^+_0)\dagger = a^+_0, \quad (q^-_0)\dagger = q^-_0, \\
(a^i_m)\dagger = a^{-i}_{-m}, \quad (b^r_r)\dagger = b^{-r}_{-r}, \quad (c_m)\dagger = c_{-m}, \quad (d_r)\dagger = d_{-r},
\]

where \( m, n \in \mathbb{Z}; r, s \in \mathbb{Z} + \frac{1}{2} \). The operators \( P^+ = \sqrt{\alpha} a^+_0, P^i = \sqrt{\alpha} a^i_0 \), and \( x^- = \frac{1}{\sqrt{\alpha}} q^-_0, x^i = \frac{1}{\sqrt{\alpha}} q^i_0 \) are interpreted as components of the total momentum of the string, and the barycentric coordinates, respectively.

Let us denote by \( F_i(p^+, \overline{p}) \) the Fock space generated by the algebra of non-zero modes (with negative labels) out of the unique vacuum state \( \Omega_\epsilon \) satisfying

\[ P^i \Omega_\epsilon = p^i \Omega_\epsilon, \quad P^+ \Omega_\epsilon = p^+ \Omega_\epsilon, \quad c_0 \Omega_\epsilon = \lambda \Omega_\epsilon. \]

The space of states is a direct integral of Hilbert spaces over the spectrum of momentum operators

\[ H_\epsilon = \int \frac{dp^+}{|p^+|} d^{d-2}p \ H_\epsilon(p^+, \overline{p}) . \]

In the Neveu-Schwarz sector (\( \epsilon = 1 \))

\[ H_1(p^+, \overline{p}) = F_1(p^+, \overline{p}) . \]

In the Ramond sector (\( \epsilon = 0 \)) the fermionic zero modes \( b^0_0, d_0 \) form the real Euclidean Clifford algebra \( \mathcal{C}(d - 1, 0) \). If one requires a well defined fermion parity operator the zero mode sector of \( H_0(p^+, \overline{p}) \) must carry a representation of the Clifford algebra \( \mathcal{C}(d, 0) \). We assume that this sector is described by an irreducible representation \( D(d) \) of the complexified Clifford algebra \( \mathcal{C}^C(d) = \mathcal{C}(d, 0) \otimes \mathcal{C}_i \), and

\[ H_0(p^+, \overline{p}) = F_0(p^+, \overline{p}) \otimes D(d) . \]

The representation of the algebra (\( \mathbb{F} \)) on \( H_0(p^+, \overline{p}) \) is given by

\[ a^i_m = \tilde{a}^i_m \otimes 1, \quad c_m = \tilde{c}_m \otimes 1, \quad m \neq 0, \]

\[ b^i_r = \tilde{b}^i_r \otimes \Gamma^F, \quad d_r = \tilde{d}_r \otimes \Gamma^F, \quad r \neq 0, \]

\[ b^0_0 = 1 \otimes \frac{1}{\sqrt{2}} \Gamma^i, \quad d_0 = 1 \otimes \frac{1}{\sqrt{2}} \Gamma^L, \]

where \( \tilde{a}^i_m, \tilde{c}_m, \tilde{b}^i_r, \tilde{d}_r \) denote the operators on \( F_0(p^+, \overline{p}) \) representing the non-zero bosonic and fermionic modes, and \( \Gamma^1, ..., \Gamma^{d-2}, \Gamma^L, \Gamma^F \) are the gamma matrices of the \( D(d) \) representation.
In order to construct generators of a unitary representation of the Poincare group we introduce the operators

\[
L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} : \bar{a}_{-n} \cdot \bar{a}_{m+n} : + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r : \bar{b}_{-r} \cdot \bar{b}_{r+m} : + (1 - \epsilon) \frac{d+1}{16} \delta_{m,0}
\]

\[
+ \frac{1}{2} \sum_{n \in \mathbb{Z}} : c_{-n} c_{m+n} : + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} r : d_{-r} d_{r+m} : + 2i \sqrt{\beta} m c_m + 2 \beta \delta_{m,0} ,
\]

\[
G_r = \sum_{n \in \mathbb{Z}} \bar{a}_{-n} \cdot \bar{b}_{n+r} + \sum_{n \in \mathbb{Z}} c_{-n} d_{n+r} + 4i \sqrt{\beta} r d_r ,
\]

forming an \( N = 1 \) superconformal algebra with the central charge \( \hat{c} = d - 1 + 32 \beta \)

\[
[L_m, L_n] = (m - n) L_{m+n} + \frac{1}{8} (d - 1 + 32 \beta) (m^3 - m) \delta_{m,-n} ,
\]

\[
[L_m, G_r] = (\frac{1}{2} m - r) G_{m+r} ,
\]

\[
\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2} (d - 1 + 32 \beta) (r^2 - \frac{1}{4}) \delta_{r,-s} .
\]

The generators of the translations in the longitudinal and the transverse directions are given by the operators \( P^+ \) and \( P^i \), respectively. The generator of the translation in the \( x^+\)-direction is defined by

\[
P^- = \frac{\alpha}{P^+} (L_0 - a_0) .
\]

The \( x^+ \) coordinate is regarded as an evolution parameter and \( P^- \) plays the role of the string Hamiltonian. The generators of the Lorentz group are defined by

\[
M^{ij} = x^i P^j - x^j P^i - i \sum_{n > 0} \frac{1}{n} (a^i_{-n} a^j_n - a^j_{-n} a^i_n)
\]

\[
+ (1 - \epsilon) i b^i_0 b^j_0 - i \sum_{r > 0} (b^i_{-r} b^j_r - b^j_{-r} b^i_r) ,
\]

\[
M^{i+} = x^i P^+ ,
\]

\[
M^{+-} = \frac{1}{2} (P^+ x^- + x^- P^+) ,
\]

\[
M^{i-} = \frac{1}{2} (P^- x^i + x^i P^-) - x^- P^i - \frac{i}{a_0} \sum_{n > 0} \frac{1}{n} (a^i_{-n} L_n - L_n a^i_n)
\]

\[
+ (1 - \epsilon) \frac{i}{a_0} b^i_0 G_0 - \frac{i}{a_0} \sum_{r > 0} (b^i_{-r} G_r - G_r b^i_r) ,
\]

The algebra of the generators \( P^+, P^-, P^i, M^{+-}, M^{i+}, M^{i-}, M^{ij} \) closes to the Lie algebra of the Poincare group up to some anomalous terms. They appear only in the commutators

\[
[M^{i-}, M^{j-}] = \frac{1}{8a_0^2} \sum_{n > 0} (\Delta n - \bar{\Delta} \frac{1}{n}) (a^i_{-n} a^j_n - a^j_{-n} a^i_n)
\]

\[
+ \frac{1}{2a_0^2} \sum_{r > 0} (\Delta - \bar{\Delta} \frac{1}{r^2}) (b^i_{-r} b^j_r - b^j_{-r} b^i_r) ,
\]

where \( \Delta = d - 9 + 32 \beta, \bar{\Delta} = 16\alpha_0 - d + 1 - 32 \beta \), and vanish if and only if \( \beta = \frac{1}{32} (9 - d) \), and \( \alpha_0 = \frac{1}{2} \). The first condition implies that the operators \( P^-, M^{i-} \) are self-adjoint only
in the range $2 \leq d \leq 9$. The second leads to the following expression for the mass square operator
\[ M_2^2 = 2\alpha (R_\epsilon + \lambda^2 \frac{d-1}{16}) \]
where $R_\epsilon = \sum_{m>0} (\bar{a}_m \cdot a_m + c_{-m} c_m) + \sum_{r>0} r (\bar{b}_r \cdot b_{r+m} + d_{-r} d_{r+m})$ is the level operator. Note that in the covariant massive string model the eigenvalue $\lambda$ of the bosonic Liouville zero mode $c_0$ is restricted by the constraint $c_0 = 0$. In the present construction it is regarded as a free real parameter.

It follows from (2) that for $\lambda^2$ small enough the ground states in the Neveu-Schwarz sector are tachyonic. One can try to solve this problem by introducing the GSO projection \[ \tilde{F}_\epsilon = \sum_{r>0} \tilde{b}_r \cdot \tilde{b}_r + \sum_{r>0} \tilde{d}_r \tilde{d}_r \]
be the fermion number operator on $F_\epsilon(p^+, \bar{p})$. We introduce the fermion parity operators on the total Hilbert space $H = H_0 \oplus H_1$:
\[ (-1)^F = (-1)^{\tilde{F}_0} \otimes \Gamma^F \oplus (-1)^{\tilde{F}_1+1} \]
The GSO projection is defined as the projection on the +1 eigenspace of $(-1)^F$. In the case of even dimensions there exists another operator $\Theta$ with all the properties of the fermion parity operator, and anticommuting with $(-1)^F$. One can show that the GSO projections with respect to $\Theta$, and $(-1)^F$ lead to equivalent models.

### 3 Equivalence to other models

In this section we shall show that the light-cone string is equivalent to the critical fermionic massive string recently introduced in \[1\]. In the covariantly quantized fermionic massive string the conditions for physical states can be solved in terms of the transversal $A^i_m, B^i_r$, the super-Liouville $C_m, D_r$, and the "shifted" longitudinal $A^L_m, B^L_r$. For details concerning the DDF construction and the notation used in this section we refer to \[1\].

The critical fermionic massive string corresponds to a special choice of the parameters $\beta = \frac{9-d}{22}, m_0^2 = 0$. In this case all states containing the "shifted" longitudinal excitations are null. The space of physical states can be identified with all states generated by the transverse and the super-Liouville DDF operators $A^i_m, B^i_r, C_m, D_r$. They have the same (anti)commutation relations and the conjugation properties, as the light-cone excitations $a^i_m, b^i_r, c_m, d_r$. Also the continuous spectra of the bosonic zero modes in both models are identical. In the critical massive string the representation of the transverse, the Liouville, and the fermion parity gamma matrices on the on-mass-shell physical states coincides with the representation $D(d)$. The only difference is that in the covariant model one gets a neutral, while in the light-cone string a positive definite scalar product. This discrepancy is not essential - the subspaces with definite products in the covariant model are eigenspaces of the the fermion parity operator \[1\]. Note that the neutral product of the covariant model is a consequence of the assumption that the zero and the non-zero fermionic modes have the same conjugation properties.

One way to show the equivalence of the Poincare group representations is to calculate the commutators of the Poincare generators with the DDF operators. This calculations can be facilitated by the technique of the leading terms \[8\]. It is based on the observation that the DDF operators expressed in terms of elementary excitations are uniquely determined by their leading terms i.e. parts of such expressions which do not contain any
The problem of the spin spectrum is to decompose the unitary representation of the Poincare group on the Hilbert space of string into irreducible representation. It follows from formula (2) that the decomposition of $H_\epsilon$ into representations of a fixed mass coincides with the level structure

$$H_\epsilon = \bigoplus_{N \geq 0} \int \frac{dp^+}{|p^+|} d^{d-2}\mathbf{p} H^N_\epsilon(p^+, \mathbf{p}) \ , \ R_\epsilon H^N_\epsilon(p^+, \mathbf{p}) = NH^N_\epsilon(p^+, \mathbf{p}) .$$

For $\lambda^2$ in the range $0 \leq \lambda^2 < \frac{d-1}{8}$ the lowest level subspace $H^0_\epsilon$ in the Neveu-Schwarz sector carries an irreducible tachyonic representation. For $\lambda^2 = \frac{d-1}{8}$, $H^0_\epsilon$ is a massless, and for $\lambda^2 > \frac{d-1}{8}$, a massive scalar representation.

### 4 Spin spectrum

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In the Ramond sector the 0-level subspaces $H^0_\ell(p^+, \mathcal{P})$ are by construction isomorphic with the irreducible representation $D(d)$ of the complex Clifford algebra $\mathcal{C}^C(d)$. For a massive momentum ($\lambda^2 > 0$) the representation of the little group $\text{Spin}(d-1)$ on $D(d)$ is a direct sum of two isomorphic fundamental irreducible representations $S(d-1)$ of $\text{Spin}(d-1)$.

In the massless case ($\lambda^2 = 0$) the maximal compact subgroup of the little group is $\text{Spin}(d-2)$. In the odd dimensions $D(d)$ is a direct sum of two fundamental irreducible representations of $\text{Spin}(d-2)$ while in the even dimensions it is a direct sum of four such representations. In particular for $d = 4$ and $\lambda^2 = 0$ the zero level in the Ramond sector contains two pairs of the left, and the right Weyl spinors.

Since all higher levels are massive, the spaces $H^N_\ell(p^+, \mathcal{P})$ should be decomposed into irreducible representations of the little group $\text{Spin}(d-1)$. For every momentum $p$ with $m^2 = \alpha (2N + \lambda^2 - \varepsilon \frac{d-1}{2})$, $N > 0$, one can choose a light-cone frame such that $p^+ = \sqrt{\alpha}, \mathcal{P} = 0$, and the little group is generated by

$$G^j = M^{j-} - \frac{m^2}{2\alpha}M^{j+} \ , \ M^{ji} .$$

We shall use the method developed in the case of the bosonic light-cone string [5]. It relies on the observation that, as far as the character generating function is concerned, the vector representation of $\text{Spin}(d-2)$ formed by the transverse excitation can be extended to a vector representation of $\text{Spin}(d-1)$ by means of the Liouville excitations. The only novelty is that in the present case we have two vector representations $V^B_m$ and $V^F_r$, spanned by the creation operators

$$a^a_{-m} = \left\{ \begin{array}{l} \kappa_m c_{-m} \hspace{1cm} a = 0 \hspace{1cm} 1 \leq a \leq d-2 \hspace{1cm} , \hspace{1cm} b^a_{-r} = \left\{ \begin{array}{l} \kappa_r d_{-r} \hspace{1cm} a = 0 \hspace{1cm} 1 \leq a \leq d-2 \hspace{1cm} . \end{array} \right. \right.$$ 

The normalisation constants $\kappa_m = \frac{\lambda - 2i \sqrt{\beta k}}{\sqrt{\lambda^2 + 4m^2 k}}$, $\kappa_r = \frac{\lambda - 4i \sqrt{\beta r}}{\sqrt{\lambda^2 + 16r^2}}$ are chosen in order to obtain the canonical antisymmetric matrix generators $D^{(i)}$ of $\text{Spin}(d-1)$ vector representation:

$$\left[ G^j , A^a_{-m} \right] = i \sqrt{\lambda^2 + 4m^2 \beta} \ D^{(i)}_{ab} A^b_{-m} + \ldots ,$$

$$\left[ G^j , B^a_{-r} \right] = i \sqrt{\lambda^2 + 16r^2 \beta} \ D^{(i)}_{ab} B^b_{-r} + \ldots .$$

The dots in the formulae above denote all terms of higher order in the excitation operators. Such terms do not contribute to the character functions.

The subspace $H^N_\ell(\sqrt{\alpha}, 0)$ decomposes into a direct sum of tensor products of the symmetric tensor powers of $V^B_m$, the antisymmetric tensor powers of $V^F_r$, and $D(d)$. Then using the method of [5] one can write the character of the Spin$(d - 1)$ representation on $H^N_\ell(\sqrt{\alpha}, 0)$ as

$$\chi^N_\ell = \sum_{N_B + N_F = N} \sum_{p_B \in P(N_B)} \sum_{p_F \in P(N_F)} \prod_{m_k \in p_B} \prod_{m_r \in p_F} \chi^{m_k}_{S} \chi^{m_r}_{A} \chi^{0}_{\ell} ,$$

where the sum runs over all partitions $p_B = \{m_k\}$, $p_F = \{m_r\}$ of the bosonic $N_B$, and the fermionic $N_F$ level number. The symbols $\chi^{m_k}_{S}$, and $\chi^{m_r}_{A}$ stand for the characters of the $m_k$-th symmetric, and the $m_r$-th antisymmetric tensor power of the vector representation of Spin$(d - 1)$, respectively. Finally, $\chi^{0}_{\ell}$ is given by

$$\chi^{0}_{\ell} = 2\chi_{S(d-1)} , \hspace{1cm} \chi^{0}_{1} = 1 ,$$

(3)
where $\chi_{S(d-1)}$ is the character of the fundamental irreducible representation of Spin$(d-1)$. Using the formulae for characters of tensor products [3] one gets the character generating function

$$\chi_\epsilon(t, g) = \sum_{N \geq 0} \epsilon^N \chi^N(g) = \prod_{k \in \mathbb{I}} \frac{1}{\det(1 - t^k \mathcal{D}_\nu(g))} \prod_{(1+\epsilon)r \in \mathbb{I}} \det(1 + t^r \mathcal{D}_\nu(g)) \chi_\epsilon^0(g)$$

where $\mathcal{D}_\nu$ denotes the vector representation of Spin$(d-1)$, and $\mathbb{I}$ is the set of all positive integers. The expansions of $\chi_\epsilon(t, g)$ in terms of irreducible characters can be found using the techniques developed by Curtright and Thorn [7] for strings with only transverse excitations. In the case of $d = 4$ one gets:

$$\chi_\epsilon(t, \varphi) = 2^{1-\epsilon} t^{\frac{1}{8}(1-\epsilon)} p^4(t) \pi_\epsilon(t) \sum_{l \in \mathbb{I} \cup \frac{d-1}{2}} \chi_l(\varphi)$$

$$\sum_{k \in \mathbb{I}} (-1)^{k-1} (1 - t^k) \sum_{m \in \mathbb{I} \cup \frac{d-1}{2}} t^{(k(k-1)+m^2)} (1 - t^{m+\frac{1}{2}}(t^{k|m|} - t^{k(l+m+1)})$$

where

$$p(t) = \prod_{n \in \mathbb{I}} (1 - t^n)^{-1} , \quad \pi_\epsilon(t) = \prod_{(1+\epsilon)r \in \mathbb{I}} (1 + t^r)$$

and $\chi_l(\varphi)$ is the character of the spin $l$ irreducible representation of Spin$(d-1)$. For $\lambda^2 = 0$ the spin spectrum up to 6-th level is presented on Fig.1.

The doubling of the spectrum in the Ramond sector [3] is related to the presence of the fermionic zero mode in the Liouville sector. For all dimensions in the range $1 < d < 10$ the GSO projection removes this doubling without any extra conditions for the parameters of the model. In the Neveu-Schwarz sector it simply removes the integer levels. The GSO truncated spectrum is presented on Fig.2.

## 5 Closed string

The closed string Hilbert space $H_c$ can be constructed as the subspace of the tensor product of two copies of the open string Hilbert spaces $(H_0 \oplus H_1) \otimes (\bar{H}_0 \oplus \bar{H}_1)$ determined by the conditions

$$a_i^i = \bar{a}_0^i = \frac{P_i^i}{2\sqrt{\alpha}} , \quad a_0^+ = \bar{a}_0^+ = \frac{P_0^+}{2\sqrt{\alpha}} , \quad c_0 = \bar{c}_0 = \lambda$$

and annihilated by the twist operator $T = (R_0 \oplus R_1) \otimes 1 - 1 \otimes (\bar{R}_0 \oplus \bar{R}_1)$.

Since the mass levels of different sectors of the open non-critical string never coincide [2] the mixed sectors $H_0 \otimes \bar{H}_1 , H_1 \otimes \bar{H}_0$ are excluded. In consequence the spectrum of the closed fermionic string does not contain space-time fermions. This is in fact a common feature of all the covariant closed string models corresponding to the family of non-critical open strings considered in [1].

The representation of the Poincare generators are constructed in a standard manner. In particular, the Hamiltonian $P^-$ generating the $x^+$-evolution, and the mass square operator $M^2\epsilon$ are given by

$$P^- = \frac{\alpha}{P^+} (L_0 + \bar{L}_0 - 1) , \quad M^2\epsilon = 4\alpha(R + \bar{R} + \lambda^2 - \epsilon^d)$$
The character generating function can be calculated as the "diagonal" part (i.e. all terms of the form $t^N t'^N$) of the product of two open string generating functions

$$\chi^{(\text{closed})}_\epsilon(t, g) = \text{Diag}(\chi_{\epsilon}(t, g)\chi_{\epsilon}(t', g))|_{t=t'}.$$ 

For $d = 4$, and $\lambda^2 = 0$ the results of the numerical calculations of first few levels, before, and after GSO projection are presented on Fig.3, and on Fig.4, respectively.

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Fig. 1

$D = 4$

- $\alpha(0)_{\text{NS}} = -\frac{3}{16}$
- $\alpha(0)_{\text{R}} = 0$
$$D = 4$$

- $\alpha(0)_{NS} = -\frac{3}{16}$
- $\alpha(0)_{R} = 0$

Fig. 2
Fig. 3

\[ D = 4 \]
\[ \bullet \alpha(0)_{\text{NS–NS}} = \frac{-6}{16} \]
\[ \circ \alpha(0)_{\text{R–R}} = 0 \]
$D = 4$

- $\alpha(0)_{\text{NS–NS}} = -\frac{6}{16}$
- $\alpha(0)_{\text{R–R}} = 0$

Fig. 4