Size sensitive packing number for Hamming cube and its consequences

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Abstract

We prove a size-sensitive version of Haussler’s Packing lemma [7] for set-systems with bounded primal shatter dimension, which have an additional size-sensitive property. This answers a question asked by Ezra [9]. We also partially address another point raised by Ezra regarding overcounting of sets in her chaining procedure. As a consequence of these improvements, we get an improvement on the size-sensitive discrepancy bounds for set systems with the above property. Improved bounds on the discrepancy for these special set systems also imply an improvement in the sizes of relative $(\varepsilon, \delta)$-approximations and $(\nu, \alpha)$-samples.

1 Introduction

A set system or range space $(X, \mathcal{S})$ is a ground set $X$ and a collection $\mathcal{S} \subseteq 2^X$ of subsets of $X$, called ranges.

In this paper we are interested in set systems that have bounded primal shatter dimension. So, let’s begin by recalling the definition of primal shatter function which plays an important role in this paper:

**Definition 1** (Primal shatter function; see [15]). The primal shatter function of a set system $(X, \mathcal{S})$ is defined as

$$\pi_{\mathcal{S}}(m) = \max_{Y \subseteq X, |Y| = m} |\mathcal{S}|_{Y}$$

where $\mathcal{S}|_{Y} = \{S \cap Y : S \in \mathcal{S}\}$.

A set system $(X, \mathcal{S})$ with $|X| = n$ has a primal shatter dimension $d$ if for all $m \leq n$

$$\pi_{\mathcal{S}}(m) = O(m^d).$$

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1 At the time of submission, we have become aware of a similar packing result proven simultaneously by Ezra. However, we note that our proof of the main packing lemma is quite different from hers. Also, the focus of our paper is on discrepancy bounds and sampling complexity.

2 Note that for the rest of this paper we will call $\mathcal{S}|_{Y}$ the projection of $Y$ onto $\mathcal{S}$.

3 Strictly speaking, the primal shatter dimension is defined over a family $\mathcal{F} = \{\mathcal{F}_i\}_{i=1}^{\infty}$ of set systems, where for each $i$, $\mathcal{F}_i$ is a sub-family of set systems whose ground set has exactly $i$ elements. The constant of proportionality is common for all members of $\mathcal{F}$. 
Even though it is natural to consider VC-dimension of set systems that arise in geometric settings, but a set system \((X, S)\) with VC-dimension \(d\) also implies that the primal shatter dimension of \((X, S)\) is \(d\), see \([15]\). From this point onward we will be looking at set systems that have bounded primal shatter dimension.

For a set system \((X, S)\), a subset \(P \subseteq S\) is \(\delta\)-separated if for all \(S_1, S_2 (\neq S_1) \in \mathcal{P}\) we have more than \(\delta\) elements in the set \(S_1 \Delta S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)\), i.e., symmetric difference distance \(|S_1 \Delta S_2|\) between \(S_1\) and \(S_2\) is strictly greater than \(\delta\). A \(\delta\)-packing for \((X, S)\) is inclusion-maximal \(\delta\)-separated subsets of \(S\).

Let \(X = [n] = \{1, \ldots, n\}\) be the ground set and \(S\) be a subset of \(2^X\). Then it is natural to associate sets \(S \in \mathcal{S}\) with the \(n\)-dimensional Hamming cube, where \(S\) will be mapped to the vertex \(v_S\) whose \(i\)-th coordinate is “one” if \(i \in S\), otherwise it is “zero”, i.e., \(v_S\) is the indicator vector for the set \(S\). In this setting symmetric difference distance \(|S_1 \Delta S_2|\) between two set \(S_1\) and \(S_2\) becomes equal to the Hamming distance between the two vertices \(v_{S_1}\) and \(v_{S_2}\) in the Hamming cube. Therefore the problem of finding \(\delta\)-packing boils down to finding inclusion-maximal set of vertices such that for any two vertices the Hamming distance is greater than \(\delta\).

The problem of bounding the size of a \(\delta\)-packing of a set system has been an important question. In a breakthrough paper \([7]\), Haussler proved an asymptotically tight bound on the size of the largest \(\delta\)-packing of a set systems with bounded primal shatter dimension:

**Theorem 2** (Haussler’s packing lemma \([7, 15]\)). Let \(d > 1\) and \(C\) be constants, and let \((X, S)\) be a set system with \(|X| = n\) and whose primal shatter function satisfies \(\pi_S(m) \leq Cm^d\) for all \(1 \leq m \leq n\), i.e., primal shatter dimension \(d\). If \(\delta\) be an integer, \(1 \leq \delta \leq n\), and let \(P \subseteq S\) be \(\delta\)-packed then

\[
|P| = O((n/\delta)^d).
\]

Note that the constant in big-O depends only on \(d\) and \(C\).

Matoušek \([15]\) remarked that Haussler’s proof of the packing lemma uses a “probabilistic argument which looks like a magician’s trick”. Haussler actually proved the result for set systems with bounded VC-dimension, but it was verified by Wernisch \([19]\) to also work for set systems with bounded primal shatter dimension. For a proof of Theorem 2 refer to Matoušek’s book on Geometric Discrepancy \([15]\), where Matoušek followed Chazelle’s \([6]\) simplified proof of the above theorem.

Ezra \([9]\) refined the definition of primal shatter dimension to make it size sensitive. Specifically, for any \(Y \subseteq X\) with \(|Y| = m\), where \(1 \leq m \leq n\), and for any parameter \(1 \leq k \leq m\), the number of sets of size at most \(k\) in the projection \(S|_Y\) of \(Y\) onto \(S\) is \(Cm^{d_1}k^{d-d_1}\), where \(C\) is a constant, \(d\) is the primal shatter dimension and \(1 \leq d_1 \leq d\). This is a generalisation of primal shatter function, and for the rest of this paper we will call \(d_1\) and \(d_2 = d - d_1\) size-sensitive shattering dimensions (or constants). Ezra \([9]\) gave a bound of \(O\left(\frac{j^d}{2^{d_2-j^d}}\right)\), for packings of sets of size \(O(n/2^{d-1})\) having separation \(n/2^j\). Ezra further conjectured that the factor of \(j^d\) was not essential and could be removed. This would have made the bound optimal up to some constants.\(^4\)

\(^4\)Similar to the primal shatter dimension, we mention the caveat that the size-sensitive shattering constants are defined for a family of set systems, where \(n\) and \(k\) both go to infinity, independently of each other.
The main contribution of this paper is to get the following a size sensitive analog of the Haussler’s packing result:

**Theorem 3** (Size sensitive packing lemma). Let \( \delta \in [n] \), and let \( \mathcal{P} \) be a \( \delta \)-separated set system having primal shatter dimension \( d \), and size-sensitive shattering constants \( d_1 \) and \( d_2 \). Let \( \mathcal{P}_l \) be the sets of size \( l \) in \( \mathcal{P} \). Let \( M = M(l) = |\mathcal{P}_l| \). Then

\[
M \leq c^* \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2},
\]

where \( c^* \) is independent of \( n, l, \delta \).

Applying the above bound to Ezra’s scenario, we get \( O(2^{jd/2^{(i-1)d_2}}) \). Thus, we prove that the extra polylog factors in Ezra’s bound can be removed and answer her question in the affirmative.

**Combinatorial discrepancy**  Given a set system \((X, S)\) where \( X = [n] \), in combinatorial discrepancy we are interested in finding a bi-coloring \( \chi : X \to \{-1, +1\} \) such that worst imbalance \( \max_{S_i \in S} |\chi(S_i)| \), where \( \chi(S_i) = \sum_{j \in S_i} \chi(j) \), in the set system is minimised. The discrepancy of \((X, S)\) is defined as

\[
\text{disc}(S) = \min_{\chi} \max_{S \in S} |\chi(S)|.
\]

Using partial coloring or entropy method of Beck \[3\] and an innovative chaining method (originally due to Kolmogorov) to get a decomposition of sets in the range space, Matoušek \[14, 15\] proved an important result for the case of set system with bounded primal shatter dimension:

**Theorem 4** \((14, 15)\). Let \( d > 1 \) be a constant, and let \((X, S)\) be a set system with \( \pi_m(S) \leq Cm^d \), where \( C > 0 \) and \( d > 1 \) are constants. Then \( \text{disc}(S) = O(n^{1/2 - 1/2d}) \), where the constant is big-\( O \) depends only on \( d \) and \( C \).

Ezra \[9\] generalised the above result to be case of set systems with size sensitive primal shatter dimensions \( d_1 \) and \( d_2 \) and also make the discrepancy dependent on the size of the sets:

**Theorem 5** \((9)\). Let \((X, S)\) be a finite set system of primal shatter dimension \( d \) with the additional property that in any set system restricted to \( Y \subseteq X \), the number of sets of size \( k \leq |Y| \) is \( O(|Y|^{d_1} k^{d - d_1}) \), where \( 1 \leq d_1 \leq d \). Then

\[
\text{disc}(S) = \begin{cases} 
O \left( |S|^{d_2/(2d)} n^{(d_1 - 1)/(2d)} \log^{1/2 + 1/2d} n \right), & \text{if } d_1 > 1 \\
O \left( |S|^{d_2/(2d)} \log^{3/2 + 1/2d} n \right), & \text{if } d_1 = 1 
\end{cases}
\]

where \( d_2 = d - d_1 \).

This bound is slightly suboptimal for the case for points and halfspaces in the plane. Har-Peled and Sharir. \[10\] proved that for the case of points and halfspaces in the plane
the discrepancy bound for a set \( S \) is \( O(|S|^{1/4} \log n) \), but the bound in Theorem 5 is a considerable improvement for the case of points and halfspaces in three dimensional space obtained by Sharir and Zaban \[17\], which extended the construction of Har-Peled and Sharir \[10\].

Using our new size sensitive packing bound and Ezra’s \[9\] refinement of Matoušek’s chaining trick \[14, 15\], we get the following improvement to Theorem 5.

**Theorem 6.** Let \((X, S)\) be a (finite) set system of primal shatter dimension \( d \) and size-sensitive shatter constants \( d_1 \leq d \) and \( d_2 = d - d_1 \). Then

\[
disc(S) = \begin{cases} 
O\left(|S|^{d_2/(2d)}n^{(d_1-1)/(2d)} f(|S|, n)\right), & \text{if } d_1 > 1 \\
O\left(|S|^{d_2/(2d)} f(|S|, n) \log n\right), & \text{if } d_1 = 1
\end{cases}
\]

where \( f(|S|, n) = \sqrt{1 + 2\log \left(1 + \log \min\{|S|, \frac{n}{|S|}\}\right)} \). A coloring with the above discrepancy bounds can be computed in expected polynomial time.

Note that the proof of both Theorems 5 and 6 fundamentally uses the recent improvement to Beck’s entropy method \[3\] by Lovett and Meka \[13\].

A set system \((X, S)\) is called low degree if for all \( j \in X \), \( j \) appears in at most \( t \leq n \) sets in \( S \). Beck and Fiala has been conjectured that discrepancy of low degree set system is \( O(\sqrt{t}) \) \[2\]. This is known as the Beck-Fiala conjecture. Beck and Fiala \[5\], using a linear programming approach showed that discrepancy of low degree set systems is bounded by \( 2t - 1 \). Using entropy method, one can obtain a constructive bound of \( O(\sqrt{t \log n}) \) \[2, 13\]. See also \[18\]. Currently the best bound (non-constructive) is by Banaszczyk \[1\], who proved that the discrepancy is bounded by \( O(\sqrt{t \log n}) \). We prove in Section 5 that for low degree set systems with \( d_1 = 1 \), the discrepancy is bounded by \( O(t^{1/2-1/2d} \sqrt{\log \log t \log n}) \). Specifically, for the case of points and halfspaces in 2-dimensional setting, we get \( O(t^{1/4} \sqrt{\log \log t \log n}) \). Also note that our result is constructive, i.e., in expected polynomial time we can find a coloring that matches the above discrepancy bound.

**Relative \((\varepsilon, \delta)\)-approximation and \((\nu, \alpha)\)-sample** Using the improved size sensitive discrepancy bounds for sets we will be able to improve on the previous bounds for relative \((\varepsilon, \delta)\)-approximation and \((\nu, \alpha)\)-sample.

For a finite set system \((X, S)\), we define for \( S \in S \) \( \overline{X}(S) = \frac{|S \cap X|}{X} \). For a given \( 0 < \varepsilon < 1 \) and \( 0 < \delta < 1 \), a subset \( Z \subseteq X \) is a relative \((\varepsilon, \delta)\)-approximation if \( \forall S \in S \)

\[ \overline{X}(S)(1 - \delta) \leq \overline{Z}(S) \leq \overline{X}(S)(1 + \delta), \text{ if } \overline{X}(S) \geq \varepsilon, \text{ and} \]

\[ \overline{X}(S) - \delta \varepsilon \leq \overline{Z}(S) \leq \overline{X}(S) + \delta \varepsilon, \text{ otherwise} \]

Har-Peled and Sharir \[10\] showed that the notion of relative \((\varepsilon, \delta)\)-approximation and \((\nu, \alpha)\)-sample are equivalent if \( \nu \) is proportional to \( \varepsilon \) and \( \alpha \) is proportional to \( \delta \). A \((\nu, \alpha)\)-sample of a set system \((X, S)\) is a subset \( Z \subseteq X \) satisfying the following inequality \( \forall S \in S \):

\[
d_{\nu}(\overline{X}(S), \overline{Z}(S)) := \frac{|\overline{X}(S) - \overline{Z}(S)|}{\overline{X}(S) + \overline{Z}(S) + \nu} < \alpha.
\]
Relative $(\epsilon, \delta)$-approximation is an important tool to tackle problems in approximate range counting [10].

In Section 6, we prove the following bound on the size of relative $(\epsilon, \delta)$-approximation which is an improvement over the previous bounds [10, 8, 9], the most recent one being [9], who gave a bound of

$$\max \left\{ O \left( \log n \right), O \left( \frac{\log 1/(\epsilon \delta)}{\epsilon^d \sigma^{d+1}} \right) \right\},$$

for $d_1 > 1$, and

$$\max \left\{ O \left( \log \frac{3d+1}{d+1} n \right), O \left( \frac{\log 1/(\epsilon \delta)}{\epsilon^d \sigma^{d+1}} \right) \right\},$$

for $d_1 = 1$.

**Theorem 7.** Let $(X, S)$ be a set system with $|X| = n$, primal shatter dimension $d$ and size sensitive shattering constants $d_1$ and $d_2 = d - d_1$. Then for $0 < \epsilon < 1$ and $0 < \delta < 1$, $(X, S)$ has a relative $(\epsilon, \delta)$-approximation of size

$$O \left( \frac{\log \log \frac{2d+1}{d+1} 1 / \epsilon \delta}{\epsilon^d \sigma^{d+1}} \right).$$

for $d_1 > 1$, and

$$\max \left\{ O \left( \log \frac{3d+1}{d+1} n \right), O \left( \frac{\log 2d+1 1 / \epsilon \delta \log \log \frac{2d+1}{d+1} \epsilon \delta}{\epsilon^d \sigma^{d+1}} \right) \right\}$$

for $d_1 = 1$. The constant in big-$O$ depends only on $d$, and a relative $(\epsilon, \delta)$-approximation with above bounds can be computed in expected polynomial time.

# 2 Preliminaries

In this section we cover some basic concepts of discrepancy theory, especially in a geometric setting, which will be needed in the following sections.

## 2.1 The Beck-Spencer method, and the Lovett-Meka algorithm

In the discrepancy upper bound problem, given a universe of elements $X$, and a subset $S$ of its power set, $S \subset 2^X$, we wish to find a coloring $\chi : X \to [-1, 1]$ which minimizes the imbalance in every set $S \in S$. In [3], Beck introduced a technique to obtain such colorings - the method of partial coloring. The idea is to color a substantial fraction of elements, while leaving others uncolored. This allows for low discrepancy in the partially colored universe. The remaining elements are then colored recursively. The technique was then further developed by Spencer [4], and is one of the major techniques used extensively in discrepancy theory. In the proofs of Beck and Spencer, the partial coloring was an existential result, and did not yield polynomial time algorithms to give low-discrepancy colorings. Recently however, a breakthrough result of Bansal [2] provided the first polynomial time
algorithm to obtain low-discrepancy colorings whose existence was implied by the Beck-Spencer technique. Subsequently, Lovett and Meka\[13\] gave a constructive version of the Beck-Spencer partial coloring lemma. We describe their lemma below:

Given a parameter $\delta \geq 0$, a partial coloring of $X$ is a function $\chi : X \to [-1, 1]$, where if for some $x$, $|\chi(x)| \geq 1 - \delta$, then we say that $x$ is colored, otherwise $x$ is uncolored.

**Lemma 8** (Lovett-Meka\[13\]). Let $(X, \mathcal{S})$ be a set system with $|X| = n$. Let $\Delta : \mathcal{S} \to \mathbb{R}_+$ be such that

$$
\sum_{S \in \mathcal{S}} \exp(-\Delta^2_S/(16|S|)) \leq n/16.
$$

Then, there exists $\chi : X \to [-1, 1]^n$ with $|\{i : |\chi_i| = 1\}| \geq n/2$, such that $|\text{sum}_{i \in S} \chi_i| \leq \Delta_S + 1/\text{poly}(n)$ for every $S \in \mathcal{S}$. Further, there exists a randomized $\text{poly}(|S|, n)$-time algorithm to find $\chi$.

The above lemma can be recursively applied on the remaining uncolored elements of $X$, to obtain a full coloring function $\chi : X \to [-1, -1+\delta] \cup [1-\delta, 1]$. This can be rounded to a coloring in $[-1, 1]^n$ by choosing $\delta$ sufficiently small. We shall refer to each such application of Lemma\[S\] as one round of the Lovett-Meka algorithm; a complete coloring, then, requires $O(\log n)$ such rounds. Denoting the bound for the set $S$ in the $j$-th round by $\Delta_{S,j}$, the discrepancy $|\sum_{i \in S} \chi_i|$ of the set $S$ in the final coloring is bounded from above by $\Delta_S = \sum_j \Delta_{S,j}$.

### 2.2 Chaining, and size-sensitive shattering constants

Now we describe the chaining decomposition, as used by Matoušek\[14, 15\], and further refined by Ezra\[9\]. We are given a set system $(X, \mathcal{S})$. For each $j = 0, \ldots, \log n = k$, we first form a maximal $\delta = n/2^j$-separated family, $\mathcal{F}_j \subseteq 2^X$. Clearly, $\mathcal{F}_k = \mathcal{S}$, and $\mathcal{F}_0 = \emptyset$. Since each family $\mathcal{F}_i$ is maximal, for each $F_i \in \mathcal{F}_i$, there exists a $F_{i-1} \in \mathcal{F}_{i-1}$, such that $|F_i \Delta F_{i-1}| \leq n/2^{i-1}$. If $F_i \in \mathcal{F}_{i-1}$, then we are done. Otherwise, suppose the statement were not true, then $F_i$ would have symmetric difference at least $n/2^{i-1}$ from every member of $\mathcal{F}_{i-1}$, and so would have to be a member of $\mathcal{F}_{i-1}$, which contradicts the maximality of $\mathcal{F}_{i-1}$.

The first decomposition Notice that since every $S \in \mathcal{S}$ lies in $\mathcal{F}_k$, applying the above property, we can find for each such $S$, $F_{k-1} \in \mathcal{F}_{k-1}$ such that the Hamming distance of $S$, $F_{k-1}$ is at most $n/2^{k-1} = 2$. Let $A_k = S \setminus F_{k-1}$, and $B_k = F_{k-1} \setminus S$, and let $F_k$ denote $S$. Clearly, $S = F_k = (F_{k-1} \cup A_{k-1}) \setminus B_{k-1}$. Extending this argument further, we get that

$$
S = F_k = (\ldots (((F_0 = \emptyset \cup A_1) \setminus B_1) \cup A_2) \setminus B_2) \cup \ldots \cup A_k) \setminus B_k
$$

where for each $j = 1, \ldots, k$, $F_{j-1} \in \mathcal{F}_{j-1}$ is the closest neighbour of $F_j$ in $\mathcal{F}_{j-1}$, and $A_i := F_i \setminus F_{j-1}$, and $B_j := F_{j-1} \setminus F_i$. We call the sequence $S = F_k \rightarrow F_{k-1} \rightarrow \ldots F_1$ as the closest-neighbour chain of $S$. 
This decomposition is now made size-sensitive by the following refinement: partition the sets in $S$ into $S_1, \ldots, S_k$, where for $S \in S$, $S \in S_i$ if and only if

$$\frac{n}{2^i} \leq S \leq \frac{n}{2^{i-1}}.$$  

For a fixed $S_i \in S_i$, consider the truncated closest-neighbour chain and the corresponding decomposition:

$$S_i = F_i^k \rightarrow F_i^{k-1} \rightarrow \ldots F_i^1,$$

$$S = (\ldots ((F_{i-1} \cup A_i) \setminus B_i) \cup \ldots \cup A_k) \setminus B_k.$$  

We now construct the size-sensitive families $F_i^j$, by following, for each $S \in S_i$, the truncated closest-neighbour chain of $S_i$, and assigning each $F_i^j$ in this chain to the family $F_i^j$. The following properties can be easily proven using the triangle-inequality on the closest-neighbour chain of $S_i$:

**Proposition 9.** For each for the sets $F_i^j \in F_i^j$, $j = i - 1, \ldots, k$, we have:

$$|S \Delta F_i^j| < O\left(\frac{n}{2^{j-1}}\right).$$

**Proposition 10.** For each for the sets $F_i^j \in F_i^j$, $j = i - 1, \ldots, k$, we have:

$$|F_i^j| < O\left(\frac{n}{2^{j-1}}\right).$$

Similar to the definitions of $A_j, B_j$, we construct the size-sensitive families $A_i^j = F_j^1 \setminus F_{j-1}^1$, and $B_i^j = F_{j-1}^1 \setminus F_j^1$, for each $F_j^1 \in F_j^1$. Finally, for each fixed $i = 1, \ldots, k$, let $\mathcal{M}_i^j$ denote the collection of $A_i^j, B_i^j$.

Observe that for each $i, j$, we have $|\mathcal{M}_i^j| = 2|F_j^1|$. Further, the size of each $A_i^j, B_i^j \in \mathcal{M}_i^j$, is $O\left(n/2^{i-1}\right)$. We shall later apply the Beck-Spencer partial coloring technique, on the set-system $\left(X, \bigcup_{i,j} \mathcal{M}_i^j\right)$.

### 3 Size-sensitive packing bound

In this section, we shall prove a size-sensitive version of Haussler’s upper bound for $\delta$-separated systems in set-systems of bounded primal shatter dimension. By Haussler’s result [7], we know that $M = O(n/\delta)^d = (n/\delta)^{d_1}(l/\delta)^{d_2}.g(n, l, \delta)^d$, where $g(n, l, d) = O((n/l)^{d_2})$. We want to show the optimum upper bound for $g$ is independent of $n, l$. We shall build on Chazelle’s presentation of Haussler’s proof, (which has been described by Matoušek as “a magician’s trick”) as explained in [13]. We shall show that the optimal bound (up to constants) is in fact, $g = c^*$, where $c^*$ is the fixed point of $f(x) = c' \log x$, with $c' > 0$ independent of $n, l, \delta$. 
Intuition We provide some intuition for our extension of Haussler’s proof below (at least to the reader familiar with it). A naïve attempt to extend Haussler’s proof to size-sensitive shattering constants fails because the proof essentially uses a random sampling set $A$, which for our purposes, has to behave somewhat like a $(\varepsilon, \delta/n)$-approximation, (at least with respect to upper-bounds on the intersection sizes), with at least a constant probability. This is unlikely to be true for a random set $A$. We shall therefore, not require that $A$ behave like an $(\varepsilon, \delta/n)$- approximation. Instead, we shall allow some sets in $P_l$ to have larger than expected intersections with $A$, and control the expected number of such ‘bad’ sets by our choice of the size of the sample set $A$.

Details Let $A \subset X$ be a random set, constructed by choosing each element $u \in X$ randomly with probability $p = \frac{3dK}{\delta}$, where $K \geq 1$ is a parameter to be fixed later. Let $s := |A|$. Define $H = P_l|_A$. Consider the unit distance graph $UD(H)$. For each set $Q \in H$, define the weight of $Q$ as:

$$w(Q) = \#\{S \in P_l : S \cap H = Q\}.$$ 

Observe that $\sum_{Q \in H} w(Q) = M$.

Let $E = E(UD(H))$, the edge set of $UD(H)$. Now define the weight of an edge $e = (Q, Q') \in E$ as

$$w(e) = \min(w(Q), w(Q')).$$

Let $W := \sum_{e \in E} w(e)$. We claim that

Claim 11. For any $A \subset X$,

$$W \leq 2d \sum_{Q \in H} w(Q) = 2dM.$$ 

Proof. The proof is based on the following lemma, proved by Haussler [7] for set systems with bounded VC dimension, and later verified by Wernisch [19] to also work for set systems with bounded primal shatter dimension. The following version appears in Matoušek’s book on Geometric Discrepancy [15]:

Lemma 12 ([7], [15]). Let $S$ be a set-system of primal shatter dimension $d$ on a finite set $X$. Then the unit-distance graph $UD(S)$ has at most $d|V(S)|$ edges.

Let $S$ be $H$. Since $H$ has primal shatter dimension $d$, the lemma implies that there exists a vertex $v \in V(H)$, whose degree is at most $2d$. Removing $v$, the total vertex weight drops by $w(v)$, and the total edge weight drops by at most $2dw(v)$. Continuing the argument until all vertices are removed, we get the claim. □

Next, we shall prove a lower bound on the expectation $E[W]$. Choose a random element $a \in A$. Let $A' := A \setminus \{a\}$. Note that $A'$ is a random subset of $X$, chosen with probability $p' = p - 1/n$. Crucially, one can consider the above process equivalent to first choosing $A'$ by selecting each element of $X$ with probability $p'$, and then selecting a uniformly random element $a \in X \setminus A'$ with probability $1/n$.

Let $E_1 \subset E$ be those edges $(Q, Q')$ of $E$ for which $Q \Delta Q' = \{a\}$, and let $W_1 = \sum_{e \in E_1} w(e)$. We need to lower bound $E[W_1]$. Given $A' \subset X$, let $Y = Y(A') := \#\{S \in P_l : S \cap H = A'\}$.
\(|S \cap A'| > c(l/\delta)\) i.e. the number of sets in \(P_1\), each of whose intersection with \(A'\) has more than \(c(l/\delta)\) elements, (where \(c\) shall be chosen appropriately). Let \(Nice\) denote the event \((Y \leq 8E[Y]) \wedge (np/2 \leq s \leq 3np/2) = N_Y \wedge N_s\). Conditioning \(W\) on \(Nice\), we get:

\[
E[W] = \Pr[Nice] E[W|Nice] + \Pr[\bar{Nice}] E[W|\bar{Nice}]
\]

By Markov’s inequality:

\[
Pr[N_Y] = \Pr[Y \leq 8E[Y]] \leq 1/8,
\]

and using Chernoff bounds,

\[
Pr[\bar{N}_s] = \Pr[|s - np| > np/2] \leq 2e^{-36dKn/(3.2^2\delta)} << 1/4,
\]

since \(n/\delta \geq 1\). We get that \(Pr[Nice] \geq 7/8 - e^{-4dK} \geq 3/4\) for \(dK \geq 1\). Hence,

\[
E[W] \geq (3/4)E[W|Nice] \geq \frac{3(np/2)}{4}E[W_1|Nice],
\]

where the last inequality follows by symmetry of the choice of \(a\) from \(A\), and the lower bound on \(s\) when the event \(Nice\) holds.

Hence, \(E[W] \geq (3np/8)E[W_1|Nice]\). So to lower bound \(E[W_1]\) up to constants, it suffices just to lower bound \(E[W_1|Nice]\). Let \(W_2\) denote \(W_1|Nice\). Consider now \(E[W_2|A']\).

That is, consider a fixed set \(A'\) whose size is between \(np/2\) and \(3np/2\), and which is such that the number of sets \(S \in P_1\) which intersect \(A'\) in more than \(cl/\delta\) vertices, is at most \(8E[Y]\). We shall lower bound \(W_1\) for this choice of \(A'\).

By definition, \(W_1 = \sum_{e \in E_1} w(e)\). Consider the equivalence classes of \(P_1\) formed by their intersections with \(A'\):

\[
P_i = P_1 \cup P_2 \cup \ldots \cup P_r.
\]

Define \(Bad \subset [r]\) to be those indices \(j\) for which \(P_j\) is such that

\[
\forall S \in P_j : |S \cap A'| > 8c(l/\delta).
\]

Further, let \(Good\) be \([r] \setminus Bad\). Since \(Nice\) holds, we have:

\[
\sum_{j \in Bad} |P_j| \leq 8E[Y].
\]

Consider a class \(P_i\) such that \(i \in Good\). Let \(P_1 \subset P_i\) be those sets in \(P_i\) which contain \(a\), and let \(P_2 = P_i \setminus P_1\). Let \(b = |P_i|, b_1 = |P_1|\) and \(b_2 = |P_2|\). Then the edge \(e \in E_1\) formed by the projection of \(P_i\) in \(A\), has weight

\[
w(e) = \min(b_1, b_2) \geq \frac{b_1 b_2}{b}.
\]

For a given ordered pair of sets \(S, S' \in P_i\), the probability that \(a \in S \Delta S'\) is \(\frac{\delta}{n - |A'|}\), which is at least \(\frac{\delta}{n}\). Therefore, the expected weight of \(e\) (conditioned on \(Nice\) and \(A'\)) is at least:

\[
E[w(e)|Nice \cap A'] \geq \frac{b(b - 1)}{b} \frac{\delta}{n} - (b - 1)\frac{\delta}{n} = (|P_i| - 1)\frac{\delta}{n}.
\]
Hence, the expected weight of $E_1$ is:

$$E[W_2|A'] \geq \sum_{e \in E_1} w(e) \geq \sum_{i \in \text{Good}} (|P_i| - 1) \frac{\delta}{n}$$

But by the size-sensitive shattering property, we have that

$$\forall j \in \text{Good}, \ |(P_j|A')| \leq Cs^{d_1} (clp)^{d_2}.$$ Substituting in the lower bound for $E[W_2]$, we get:

$$E[W_2|A'] \geq \left( \left( \sum_{i \in \text{Good}} |P_i| \right) - C(2np)^{d_1} (clp)^{d_2} \right) \frac{\delta}{n} \geq \left( |P_i| - 8E[Y] - C(6dK)^{d_1} c^{d_2} \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} \right) \frac{\delta}{n} \geq \left( M - 8E[Y] - C_1 K^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} \right) \frac{\delta}{n},$$

where $C_1 = C(6d)^{d_1} c^{d_2}$. Since the above holds for each $A'$ which satisfies Nice, we get that

$$E[W_2] \geq \left( M - 8E[Y] - C_1 K^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} \right) \frac{\delta}{n}.$$ Comparing with the upper bound on $W$,

$$(3np/8)E[W_1|\text{Nice}] \leq E[W] \leq 2dM,$$

and substituting the lower bound $E[W_1|\text{Nice}]$, and solving for $M$, we get

$$M \leq \frac{(27K/4) \left( 8E[Y] + C_1 K^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} \right)}{(27K/4 - 1)}.$$ The following claim therefore, completes the proof:

**Claim 13.** For $K = \max\{1, (\log g)/36\}$,

$$E[Y] \leq C_2 \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2}.$$ Indeed, substituting the choice of $K$ and the value of $E[Y]$ from Claim 13, we get that

$$g^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} = M \leq \frac{C_1 K^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} + 8C_2 \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2}}{1 - 4/27K} \leq \frac{C_3 K^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2}}{1 - 4/27K} \leq C_4 (\max\{1, \log g\})^d \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2}.$$
This implies that \( g \leq C_4 \max \{1, \log g\} \). Since for any non-negative \( g \), we have \( g \geq C_4 \log g \), therefore, it suffices to take \( g \leq C_4 \), i.e. \( g = c^* \), where \( c^* \) is a constant independent of \( n, l, \delta \).

It only remains to prove Claim 13.

**Proof of Claim 13.** The proof follows easily from Chernoff bounds. Fix \( S \in \mathcal{P}_i \). Let \( Z = |A' \cap S| \). Then \( E[Z] = |S|p' = lp' \). Since \( A' \) is a random set chosen with probability \( p' = p - 1/n \), the probability that \( Z \geq clp' = 36cdKl/\delta - cl/n \) is upper bounded using Chernoff bounds, as:

\[
\Pr[Z - E[Z] > (c - 1)E[Z]] \leq e^{-E[Z]} \leq e^{-36dKl/\delta},
\]

for \( c = 1.01e \) and \( n \geq 100 \), say. Hence the expected number \( E[Y] \) of sets, each of which intersect \( A' \) in more than \( 36cdKl/\delta \) elements, is at most:

\[
E[Y] \leq M.e^{-36dKl/\delta} \leq Me^{-36dK},
\]

since \( l \geq \delta \). Substituting the value of \( M \) and also \( K \) in terms of \( f \), we have

\[
E[Y] \leq g^d (n/\delta)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} e^{-36dK} \leq \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2} e^{d(\ln g - 36K)} \leq \left( \frac{n}{\delta} \right)^{d_1} \left( \frac{l}{\delta} \right)^{d_2}
\]

for \( K \geq (\ln g)/36 \). \( \square \)

This completes the proof of Theorem 3.

## 4 Size-sensitive discrepancy bounds

Now we shall use the framework of Ezra [9], together with the result proved in the previous section, to obtain improved bounds on the discrepancy of set systems having bounded primal shatter dimension and size-sensitive shattering constants \( d_1 \) and \( d_2 \). Such set systems are often encountered in geometric settings, e.g. points and half-spaces in \( d \) dimensions. In addition, we shall also derive an easy corollary for the Beck-Fiala setting, i.e. where each element of the universe \( X \) has bounded degree. In order to keep the exposition as self-contained as possible, we briefly describe Ezra’s framework first.

The basic idea, as in [13], and [9], will be to consider the set system formed by \( A_j^i \) and \( B_j^i \) for each \( i, j \in \{1, \ldots, k\} \), and bound the discrepancy for this system. We shall then sum the discrepancies of the components of each set in the original system, to obtain the total discrepancy.

For each \( i \) and \( j \), define \( \mathcal{M}_j^i \) as the collection of the sets \( A_j^i, B_j^i \). In each iteration of the Lovett-Meka algorithm, we set a common discrepancy bound \( \Delta_j^i \) for all the sets in \( \mathcal{M}_j^i \). Note that the construction of \( A_j^i \) and \( B_j^i \) implies that for each \( i, j \), \( |A_j^i| = |B_j^i| = O(n/2^{j-1}) \).

By Theorem 3 we have that for each \( i \) and \( j \), \( |\mathcal{M}_j^i| = O \left( \frac{2^{jd}}{n^{2^{j-1}}2^j} \right) \). Further, note that for a fixed \( i \), by the construction, the size of each original set \( S \in \mathcal{S} \) is \( O(n/2^{j-1}) \). Grouping the sets having the same discrepancy parameter \( \Delta_j^i \), together, we see that we need:

\[
\sum_{i=1}^k \sum_{j=1}^k C \cdot \frac{2^{jd}}{2^{2d(j-1)}}, exp \left( -\frac{(\Delta_j^i)^2}{16s_j} \right) \leq \frac{n}{16},
\]
where \( k := \log n \), and \( s_j = n/2^{j-1} \). Define
\[
j_0 := \frac{\log n + d_2(i-1)}{d} - B,
\]
where \( B \) is a suitable constant to be set later. Proceeding as in \[9\], we shall split the sum into two parts: \( j > j_0 \) and \( j \leq j_0 \). Set
\[
\Delta_j := A \cdot \frac{1}{(1 + |j - j_0|)^2} \left( \frac{n^{1/2 - 1/(2d)}}{2(i-1)(d_2/(2d))} \right) \cdot \sqrt{1 + 2 \log h},
\]
where \( h = (k/2 - |i - k/2|) \).

For the case when \( j > j_0 \), let \( j = j_0 + r \).

\[
\sum_{i=1}^{k} \sum_{j>j_0} C \cdot \frac{2^{jd}}{2d_2(i-1)} \cdot \exp \left( -\frac{(\Delta_j)^2}{16s_j} \right) \leq \sum_{i=1}^{k} \sum_{r=1}^{k-j_0} C \cdot \frac{n^{2^r d}}{2d_2} \cdot \exp \left( -\frac{A^2 2^{r-(B+1)}(1 + 2 \log h)}{16(1 + r)^4} \right)
\]
\[
\leq \sum_{i=1}^{k} C \cdot \frac{n^{2^r d}}{2d_2 h^2} \cdot \sum_{r=1}^{k-j_0} 2^{r} \cdot \exp \left( -\frac{A^2 2^{r-(B+1)}}{16(1 + r)^4} \right)
\]
\[
\leq \sum_{i=1}^{k} C \cdot \frac{n^{2^r d}}{2d_2 h^2} \cdot \sum_{r=1}^{k-j_0} \exp \left( r d \ln 2 - \frac{A^2 2^{r-(B+1)}}{16(1 + r)^4} \right)
\]
The inner summation over \( r \) can be easily seen to converge to a constant, since the exponent can be made negative for suitably large \( A \), and almost doubles with increase in \( r \). The summation over \( i \), converges to \( \Theta(n) \), and can be made much smaller than \( n/32 \), by suitably adjusting the constant \( B \).

For the second part, the calculations proceed as below:
For \( j \leq j_0 \), \( i = 1..k \), just upper bound the exponent by 1. Now we have:
\[
\sum_{i=1}^{k} \sum_{j=1}^{j_0} C \cdot \frac{2^{jd}}{2d_2(i-1)}
\]
Reverse the order of summation:
\[
\sum_{j=0}^{\log n/d} \sum_{i=j+1}^{i_0} \frac{2^{jd}}{2d_2(i-1)}
\]
where \( i_0 = 1 + \frac{\log n - jd}{d_2} \). This further simplifies to:
\[
\sum_{j=0}^{\log n/d} \sum_{i=j+1}^{i_0} C \cdot \frac{2^{jd}}{2d_2(i-1)} \leq \sum_{j=0}^{\log n/d} 2C \cdot \frac{2^{jd}}{2d_2(j)2d_2}
\]
\[
= \sum_{j=0}^{\log n/d} 2C \cdot 2^{jd-2d_2(d_2-1)dB}
\]
\[
\leq 2C \cdot 2^{\log n/(d_1/d) - dB} = n^{d_1/d}/2dB
\]
which is much less than \( n \), for suitable value of \( B \).

To get the discrepancy bound for the original set \( S \), which had size in \([n/2^i, n/2^i]\), we need to sum up the discrepancies over the \( F_j^i \) in the chain corresponding to \( S \): \( \Delta_S = \sum_j \Delta_j^i \).

Here, the factor \( \frac{A}{(1+i-j_0)^2} \) in our choice of \( \Delta_j^i \) ensures that this sum is essentially of the order of the maximal term, which occurs when \( j = j_0 \). Further, when \( d_1 = 1 \), the log \( n \) rounds of the Lovett-Meka algorithm induce an extra logarithmic factor. In the case \( d_1 > 1 \) this does not happen, because the factor of \( n^{(d_1-1)/(2d)} \) present in the discrepancy bound sets up a geometrically decreasing series.

Therefore, in terms of the size of the original set \( S \), which was in \([n/2^i, n/2^i]\), we get:

\[
\Delta_S \leq \sum_{j=1}^{k} \Delta_j^i = \sum_{i=1}^{k} \frac{A}{2^{i-1}(d_2/(2d))} n^{1/2-1/(2d)} \sqrt{1 + 2 \log(k/2 - |i - k/2|)}
\]

\[
= \begin{cases} 
O\left(|S|^{d_2/(2d)} n^{(d_1-1)/(2d)} \sqrt{\log f}\right), & \text{if } d_1 > 1 \\
O\left(|S|^{d_2/(2d)} \log n \sqrt{\log f}\right), & \text{if } d_1 = 1 
\end{cases}
\]

where

\[ f = \begin{cases} 
1 + 2 \log |S|, & \text{if } |S| \leq n^{1/2} \\
1 + 2 \log(n/|S|), & \text{if } |S| \geq n^{1/2} 
\end{cases} \]

This completes the proof of Theorem 6.

5 ‘Well-behaved’ Beck-Fiala systems

In many geometric settings, the range space is such that for any subset of elements in the universe, the number of projections of the range space on this subset is linear in the size of the subset. For example, points and axis-parallel rectangles in the plane, or points and axis-parallel boxes in three dimensions, points and half-spaces in two and three dimensions, etc. In general, following Ezra [9], we call a set-system ‘well-behaved’ if it has bounded primal shatter dimension, and \( d_1 = 1 \) for this system. In this section, we shall prove a general result for such systems, under the Beck-Fiala setting.

Details

In the Beck-Fiala setting, each element \( x \in X \) has degree bounded by \( t \), i.e. belongs to at most \( t \) many ranges or sets. Suppose, in addition, the range space is also planar, i.e. the ranges appear as polygons on a plane, then we obtain the following result:

Theorem 14. Let \((X,S)\) be a (finite) set system with bounded primal shatter dimension \( d \), and size-sensitive constants \( d_1 = 1 \), and \( d_2 = d - d_1 \). Further, each element belongs to at most \( t \) sets. Then the discrepancy of this set system is given by:

\[
\text{disc}(S) = O\left(t^{1/2-1/2d} \sqrt{\log \log t \log n}\right).
\]

Note that this discrepancy bound is constructive, i.e., in expected polynomial time we can find a coloring that matches the above discrepancy bound.
Proof Sketch: The proof follows quite simply. First, we observe that in a Beck-Fiala-type system with maximum degree $t$, the number of sets having size more than $32t$ is less than $n/32$. If we ensure that $\sum_{S: |S| \leq 32t} \exp(-\Delta_S^2/16|S|)$ is at most $n/32$, then each of the remaining sets $S$ can be assigned zero discrepancy (i.e. $\forall S: |S| \geq 32t : \Delta_S = 0$) throughout the $O(\log n)$ rounds of the algorithm. We apply Theorem 6 for $d_1 = 1$ only for the sets whose size is at most $32t$, and set $\Delta_S = 0$ for each set that has more than $32t$ vertices. Thus we get that the maximum discrepancy is $O(t^{1/2}/2^d \sqrt{\log \log t \log n})$.

Example. Observe that Theorem 14 implies that for the case of points and halfspaces in the 2-dimensional case, in the Beck-Fiala setting, the discrepancy is bounded by $O(t^{1/4} \sqrt{\log \log t \log n})$.

6 Improving relative $(\varepsilon, \delta)$-approximation bound via discrepancy

Har-Peled and Sharir [10] showed that the notion of relative $(\varepsilon, \delta)$-approximation and $(\nu, \alpha)$-sample are equivalent if $\nu$ is proportional $\varepsilon$ and $\alpha$ is proportional to $\delta$. In this section we will be working with $(\nu, \alpha)$-sample, and the improvements in the bounds we get in $(\nu, \alpha)$-sample size will directly imply improvement in the size of relative $(\varepsilon, \delta)$-approximation.

We will use the halving technique [16, 10, 9] repeatedly to get a $(\nu, \alpha)$-sample. In the analysis of this procedure the size sensitive discrepancy bound for the set system $(X, \mathcal{S})$ plays an important role. The reason we could improve on the previous bounds of $(\nu, \alpha)$-sample because we improved on the size sensitive discrepancy bounds in Theorem 6.

In this construction the set $X$ is repeatedly halved in each iterations until one obtains a $(\nu, \alpha)$-sample of appropriate size. W.l.o.g we will assume for the rest of this section that $X \in \mathcal{S}$. Let $X_0 = X$. In each iteration $i \geq 1$, the set $X_{i-1}$ is partitioned into sets $X_i$ and $X_{i-1}'$ where $X_i$ is colored $+1$ and $X_{i-1}'$ is colored $-1$. Note that the coloring corresponds to the discrepancy bounds obtained in Theorem 6. Assume, w.l.o.g., $|X_{i-1}| \geq |X_{i-1}'|$. We continue this process until a $(\nu, \alpha)$-sample of desired size is obtained.

Case of $d_1 > 1$ Since we assume $X_0 = X \in \mathcal{S}$, we would get for each iteration $i$, $X_{i-1} \in \mathcal{S} |_{X_{i-1}}$. From Theorem 6, we get, for all $S \in \mathcal{S}$, that

$$||X_i \cap S| - |X_i' \cap S|| \leq K_d|X_{i-1} \cap S|^{d-1-d_1} \frac{d_1+1}{2d} f(|X_{i-1} \cap S|, |X_{i-1}|).$$

(2)

The constant $K_d$ depends only on $d$ and the function $f(\cdot)$ is defined in Theorem 6.

Taking $S = X_{i-1}$ and using the fact $|X_i| + |X_i'| = |X_{i-1}|$ we get

$$|X_i| - (|X_{i-1}| - |X_i|) \leq K_d|X_{i-1}|^{d-1-d_1} \frac{d_1+1}{2d} \text{ as } f(|X_{i-1}|, |X_{i-1}|) = 1$$

$$= K_d|X_{i-1}|^{d-1} \frac{d_1+1}{2d}$$

(3)

Therefore

$$|X_i| \leq \frac{|X_{i-1}|}{2} \leq K_d \frac{d_1+1}{2d}.$$

(4)

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This implies

\[ |X_i| \leq \frac{|X_{i-1}|}{2} \left( 1 + \frac{K_d |X_{i-1}|}{|X_{i-1}|} \right) \]

\[ = \frac{|X_{i-1}|}{2} \left( 1 + \frac{K_d}{|X_{i-1}|} \right) \]

Write \( |X_i| \) as

\[ |X_i| = \frac{|X_{i-1}|}{2} (1 + \delta_{i-1}) \text{ where } 0 \leq \delta_{i-1} \leq \frac{K_d}{|X_{i-1}|^{\frac{d+1}{2d}}} \]

Above inductive formula gives the following recursive formula:

\[ |X_i| = \frac{|X_0|}{2^i} \prod_{j=0}^{i-1} (1 + \delta_j) \]

\[ \leq \frac{|X_0|}{2^i} \exp \left\{ \sum_{j=0}^{i-1} \delta_j \right\} \]

\[ \leq \frac{n}{2^i} \exp \left\{ K_d \sum_{j=0}^{i-1} \left( \frac{j}{n} \right)^{1/2+1/2d} \right\} \]

The last inequality follows from the fact \( |X_i| \geq |X_{i-1}|/2 \) (by construction) and \( \delta_j \leq \frac{K_d}{|X_{i-1}|^{\frac{d+1}{2d}}} \). The exponential term in Eq. (5) to \( O(1) \) if

\[ i \leq \log n - \frac{2d}{d+1} \log K_d. \]

Stopping the procedure for the above mentioned bound we get \( n_i = \Omega(K_d^{\frac{2d}{d+1}}) \).

Observe that

\[ |\bar{X}_{i-1}(S) - \bar{X}_i(S)| = \left| \frac{|S \cap X_{i-1}|}{|X_{i-1}|} - \frac{|S \cap X_i|}{|X_i|} \right| \]

\[ = \left| \frac{|S \cap X_i| + |S \cap X_{i-1}|}{X_{i-1}} - \frac{|S \cap X_i|}{|X_i|} \right| \]

\[ = \left| \frac{|S \cap X_i|}{X_{i-1}} + \delta_{i-1} \right| \left| \frac{|S \cap X_{i-1}|}{X_{i-1}} \right| \]

\[ \leq \frac{2|S \cap X_i|}{|X_i|} \] as \( X_{i-1} = X_i \cup X'_i \)

\[ \text{add. & sub. \ } \frac{|S \cap X_i|}{|X_{i-1}|} \]

\[ = \delta_{i-1} \left| \frac{|S \cap X_{i-1}|}{|X_i|} \right| \]

\[ \text{as } |X_{i-1}| = \frac{|X_{i-1}|(1+\delta_{i-1})}{2} \]

\[ = \frac{|S \cap X'_i|}{|X_i|} + \left( \frac{2\delta_{i-1}}{1+\delta_{i-1}} \right) \left| \frac{|S \cap X_{i-1}|}{|X_{i-1}|} \right| \]

\[ \text{add. & sub. \ } \frac{|S \cap X'_i|}{|X_i|} \]

\[ = \delta_{i-1} \left| \frac{|S \cap X_{i-1}|}{|X_i|} \right| \]

\[ \text{as } |X_{i-1}| = \frac{2|X_i|}{(1+\delta_{i-1})} \]

\[ \text{as } \left| X_{i-1} \right| = \frac{2|X_i|}{(1+\delta_{i-1})} \]
Note that the second term in the above equation is bounded by
\[
\delta_{i-1} \overline{X}_i(S) \leq \delta_{i-1} \left( \overline{X}_i(S) + \overline{X}_{i-1}(S) + \nu \right)
\]

There now we would try to bound the first term \( \left| \frac{|S \cap X'_i| - |S \cap X_i|}{|X_{i-1}|} \right| \).

From Theorem 6, the fact that \( f(|X_{i-1} \cap S|, |X_{i-1}|) = O(\log \log |X_{i-1}|) \), and \( x^t \leq \frac{(x+y)}{y^{1-t}} \), \( \forall x \geq 0, y > 0, & t \in [0, 1] \), we get that there exists \( K_d' \) such that
\[
\frac{|S \cap X'_i| - |S \cap X_i|}{|X_{i-1}|} \leq K_d' (\overline{X}_{i-1}(S) + \nu) \log \log |X_{i-1}|
\]

\[
\leq K_d' \log |X_{i-1}| \overline{X}_{i-1}(S) + \overline{X}_{i-1}(S) + \nu \tag{7}
\]

From Eqs (6) and (7), we get
\[
d_\nu(\overline{X}_{i-1}(S), \overline{X}_i(S)) \leq \frac{K_d'}{|X_{i-1}|^{\frac{d+1}{2d}}} \left(1 + \frac{\log \log |X_i|}{\nu^{\frac{d+1}{2d}}} \right)
\]

\[
\leq \frac{2K_d' \log |X_{i-1}|}{|X_{i-1}|^{\frac{d+1}{2d}}} \frac{|X_{i-1}|^{\frac{d+1}{2d}}}{\nu^{\frac{d+1}{2d}}}
\]

Using the fact that \( d_\nu(\cdot, \cdot) \) satisfies triangle inequality [11, 12], we get
\[
d_\nu(\overline{X}_0(S), \overline{X}_i(S)) \leq \sum_{j=1}^{i} d_\nu(\overline{X}_{j-1}(S), \overline{X}_j(S))
\]

\[
\leq O \left( \frac{\log \log n_i}{\nu^{\frac{d+1}{2d}} n_i^{-\frac{1}{2d}}} \right), \tag{8}
\]

the constant in big-O depends only on \( d \).

This implies to get \( d_\nu(\overline{X}_0(S), \overline{X}_i(S)) < \alpha \), we need
\[
n_{i-1} = \Omega \left( \frac{\log \log \frac{2d}{\nu \alpha}}{\nu^{\frac{d+1}{2d}} \alpha^{\frac{2d}{d+1}}} \right)
\]

Therefore there exists a \((\nu, \alpha)\)-sample of size
\[
O \left( \frac{\log \log \frac{2d}{\nu \alpha}}{\nu^{\frac{d+1}{2d}} \alpha^{\frac{2d}{d+1}}} \right).
\]
Case of $d_1 = 1$ Using the same technique as for the case of $d_1 > 1$, we will get the following bound for $(\nu, \alpha)$-sample size

$$
\max \left\{ O \left( \log \frac{2d}{d+1} n \right), O \left( \frac{\log \frac{2d}{d+1} n \log \log \frac{2d}{d+1} n}{\nu \alpha^{d+1}} \right) \right\}.
$$

Note that the constant in big-$O$ depends only on $d$.

This completes the proof of Theorem 7.

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