CONVEXITY OF TROPICAL POLYTOPES

MARIANNE JOHNSON and MARK KAMBITES

School of Mathematics, University of Manchester, Manchester M13 9PL, England.

ABSTRACT. We study the relationship between min-plus, max-plus and Euclidean convexity for subsets of $\mathbb{R}^n$. We introduce a construction which associates to any max-plus convex set with compact projectivisation a canonical matrix called its dominator. The dominator is a Kleene star whose max-plus column space is the min-plus convex hull of the original set. We apply this to show that a set which is any two of (i) a max-plus polytope, (ii) a min-plus polytope and (iii) a Euclidean polytope must also be the third. In particular, these results answer a question of Sergeev, Schneider and Butkovič [15] and show that row spaces of tropical Kleene star matrices are exactly the “polytopes” studied by Joswig and Kulas [12].

The notion of tropical convexity (also known as max-plus or min-plus convexity) has long played an important role in max-plus algebra and its numerous application areas (see for example [3, 4]). More recently, applications have emerged in areas of pure mathematics as diverse as algebraic geometry [7] and semigroup theory [9, 10].

Recall that a subset of $\mathbb{R}^n$ is called max-plus convex if it is closed under the operations of componentwise maximum (“max-plus sum”) and of adding a fixed value to each component (“tropical scaling”). A max-plus polytope is a non-empty max-plus convex set which is generated under these operations by finitely many of its elements; max-plus polytopes are exactly the row spaces (or column spaces) of matrices over the max-plus semiring. There are obvious dual notions of min-plus convexity and min-plus polytopes. Min- and max-plus polytopes are sometimes called tropical polytopes [11].

The projectivisation of a max-plus polytope is the set of orbits of its points under the action of tropical scaling. Since any point can be scaled to put 0 in the first coordinate (say), restricting the polytope to points with first coordinate 0 gives a cross-section of the scaling orbits, and hence a natural identification of the projectivisation with a subset of $\mathbb{R}^{n-1}$. A subset which arises from a max-plus polytope in this way we term a projective max-plus polytope. In general, a projective max-plus polytope is a compact Euclidean polyhedral complex in $\mathbb{R}^{n-1}$; it may or may not be a convex set in the ordinary Euclidean metric. Joswig and Kulas [12] studied the class

1Email Marianne.Johnson@maths.manchester.ac.uk.
2Email Mark.Kambites@manchester.ac.uk.
3Typically one fixes upon either the “min convention” or the “max convention” and uses terms such as “tropically convex” and “tropical polytope” to refer to min-plus or max-plus as appropriate. A key feature of this paper is that we study the relationship between min-plus and max-plus convexity, so for the avoidance of ambiguity we will tend not to use the word “tropical”.

1
of projective max-plus polytopes\footnote{Formally speaking they studied projective min-plus polytopes, but the difference is immaterial because negation of vectors forms a trivial duality between min-plus and max-plus convex sets while preserving Euclidean convexity.} which are Euclidean convex, and hence also Euclidean polytopes. These sets, which they term \textit{polytropes}, turn out to have numerous interesting properties and applications (see for example\footnote{In fact \cite{15} is written in the language of max-times algebra and geometry; this is exactly equivalent to max-plus via the logarithm map, save for the fact that they work with a $0$ element, which corresponds to a $-\infty$ element in the max-plus case. For simplicity we work here without $-\infty$, but the results of \cite{15} all specialise to apply in this case. The concept corresponding to Euclidean convexity in the max-times setting is log-convexity.} \cite{6} \cite{15} \cite{17}).

At around the same time, Sergeev, Schneider and Butkovič\footnote{Formally speaking they studied projective min-plus polytopes, but the difference is immaterial because negation of vectors forms a trivial duality between min-plus and max-plus convex sets while preserving Euclidean convexity.} studied the tropical polytopes arising as row spaces of \textit{Kleene stars}. Kleene stars are a class of particularly well-behaved idempotent max-plus matrices which play a vital role in almost all aspects of max-plus algebra (see Section 1 for a definition and \cite{6} for a comprehensive introduction). They prove, among many other interesting things, that the column space of a max-plus\footnote{Formally speaking they studied projective min-plus polytopes, but the difference is immaterial because negation of vectors forms a trivial duality between min-plus and max-plus convex sets while preserving Euclidean convexity.} Kleene star is always Euclidean convex \cite{15} Propositions 2.6 and 3.1]; since such column spaces are max-plus polytopes by definition, this means they are all examples of Joswig-Kulas \textit{“polytropes”}. The question of whether the converse holds was posed (with a slightly differently phrasing) in \cite{15, p. 2400}: is a Euclidean-convex max-plus polytope necessarily the column space of a Kleene star? One aim of the present paper is to answer this question in the affirmative, establishing that \textit{“polytropes”} are exactly column (and row) spaces of Kleene stars. This unifies the research in \cite{12} and \cite{15}, and opens up the possibility of using methods from each to address the kind of questions considered in the other.

(Note that in the abstract and introduction of \cite{6} it is stated that a matrix with $0$s on the diagonal has Euclidean-convex max-plus column space if and only if it is a max-plus Kleene star. Some of our results would follow from this claim, but in fact the claim is false: for example the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has $0$s on the diagonal and Euclidean-convex max-plus column space but is not a max-plus Kleene star. The main result \cite{6} Theorem 2.1] is correct, but does not suffice to establish either the more general claim in the abstract, or our own results.)

The key to our approach is to study min-plus closures of max-plus polytopes (or more generally, max-plus convex sets with compact projectivisation). We introduce (in Section 2) an elementary, but surprisingly powerful, construction which canonically associates a matrix to each such set. This matrix, which we term the \textit{dominator} of the set, is always a max-plus Kleene star, and we show that its max-plus column space is exactly the min-plus convex hull of the original set. In particular, if the chosen set was already min-plus convex then it is exactly the column space of the dominator. Combining this result with its min/max dual and with results of \cite{15} yields two elegant and useful characterisations of min-plus convexity for max-plus convex sets with compact projectivisation:
Theorem A. Let $P \subseteq \mathbb{R}^n$ be a max-plus convex set with compact projectivisation. Then the following are equivalent:

(i) $P$ is min-plus convex;
(ii) $P$ is the max-plus column space of a max-plus Kleene star;
(iii) $P$ is the min-plus column space of a min-plus Kleene star.

Of course Theorem A also has a natural dual statement, obtained by exchanging “min” and “max” throughout.

A non-trivial corollary of Theorem A is that the min-plus convex hull of a max-plus polytope is always a max-plus polytope. (The fact that such a set is max-plus convex is an easy consequence of the distributivity of maximum over minimum, but the fact that it must be max-plus finitely generated is less obvious.)

We then turn our attention to Euclidean convexity, showing that for a max-plus polytope, being min-plus convex and being Euclidean convex are in fact equivalent. Indeed, combining this statement with its max-min dual we see that (modulo issues of finite generation), any two of the three notions of convexity under consideration together imply the third:

Theorem B. Let $P \subseteq \mathbb{R}^n$. If any two of the following statements hold, then so does the remaining one:

(i) $P$ is a max-plus polytope;
(ii) $P$ is a min-plus polytope;
(iii) $P$ is Euclidean convex.

Notice that the hypothesis and condition (iii) are invariant under exchanging min and max, while conditions (i) and (ii) are dual to each other. As a consequence of this, only two implications need to be proved to establish the theorem. One of these implications is proved via Kleene stars, by combining Theorem A with a result of Sergeev, Schneider and Butkovič [15]; the other is established by an elementary, but quite technical, direct argument.

In contrast to Theorem A, the finite generation hypotheses implied by the word “polytope” in conditions (i) and (ii) of Theorem B are essential, and cannot be relaxed to assume only compactness of the projectivisation. Indeed, we shall see below (Section 3) an example of a Euclidean convex, max-plus convex set with compact projectivisation which is nevertheless not min-plus convex.

Theorem A and Theorem B can be combined in various ways to show the equivalence of various other combinations of conditions. Probably the most interesting example is the promised proof that the “polytopes” of Joswig and Kulas [12] are exactly the column spaces of Kleene stars as studied by Sergeev, Schneider and Butkovič [15].

Corollary C. A max-plus [min-plus] polytope is a Euclidean polytope if and only if it is the column space of a max-plus [respectively, min-plus] Kleene star.

1. Residuation and domination

We write $a \oplus b := \max(a, b)$, $a \ominus b := \min(a, b)$ and $a \otimes b := a + b$ for all $a, b \in \mathbb{R}$. The operations $\oplus$ and $\otimes$ (respectively $\ominus$ and $\otimes$) give $\mathbb{R}$
the structure of a semiring, called the max-plus (min-plus) semiring. The two semiring structures each induce natural multiplication operations for suitably-sized matrices over \( \mathbb{R} \), which we denote by \( \otimes \) and \( \boxplus \) respectively.

For \( x \in \mathbb{R}^n \) we write \( x_i \) to denote the \( i \)th entry of \( x \). There is a natural partial order on \( \mathbb{R}^n \) defined by \( x \leq y \) if and only if \( x_i \leq y_i \) for all \( i \). The operations \( \oplus \) and \( \boxplus \) extend entrywise to elements of \( \mathbb{R}^n \) in the obvious way, and indeed are the least upper bound and greatest lower bound operations respectively, with regard to the order. There is also a tropical scaling action of \( \mathbb{R} \) on \( \mathbb{R}^n \) given by

\[
\lambda \otimes (x_1, \ldots, x_n) = (\lambda \otimes x_1, \ldots, \lambda \otimes x_n) = (x_1 + \lambda, \ldots, x_n + \lambda).
\]

These operations make \( \mathbb{R}^n \) into a module over both the max-plus and the min-plus semiring.

A subset of \( \mathbb{R}^n \) is called max-plus convex\(^6\) (respectively, min-plus convex) if it is closed under \( \oplus \) (respectively \( \boxplus \)) and tropical scaling. The max-plus convex hull of any subset \( P \subseteq \mathbb{R}^n \) is the smallest max-plus convex set containing \( P \), or equivalently the set of all finite max-plus linear combinations of elements in \( P \), or the submodule of \( \mathbb{R}^n \) (viewed as a module over the max-plus semiring) generated by \( P \); we denote it \( \text{Span}_{\oplus}(P) \). The min-plus convex hull \( \text{Span}_{\boxplus}(P) \) is defined dually. If \( M \) is a matrix over \( \mathbb{R} \) then its max-plus column space \( \text{Col}_{\oplus}(M) \) is the max-plus convex hull of its columns; the min-plus column space \( \text{Col}_{\boxplus}(M) \), max-plus row space \( \text{Row}_{\oplus}(M) \) and min-plus row space \( \text{Row}_{\boxplus}(M) \) are defined similarly in the obvious way.

For all \( x, y \in \mathbb{R}^n \) we define

\[
\langle x \mid y \rangle = \max\{\lambda \in \mathbb{R} : \lambda \otimes x \leq y\} = \min_i \{y_i - x_i\}.
\]

This operation is a residuation operator in the sense of residuation theory\(^2\), and has been extensively applied in max-plus algebra and geometry (see for example \([1, 4, 5, 6]\)). We record a number of useful properties, which the reader can easily verify using the definition above:

\[
\langle x \mid y \rangle = -\lambda \otimes \langle x \mid y \rangle = \langle x \mid -\lambda \otimes y \rangle \tag{1.1}
\]

\[
\langle x \mid y \rangle \boxplus \langle x' \mid y \rangle = \langle x \oplus x' \mid y \rangle \leq \langle x \mid y \rangle \leq \langle x \mid y \oplus y' \rangle \tag{1.2}
\]

\[
\langle x \mid y \rangle \boxplus \langle x' \mid y' \rangle = \langle x \mid y \boxplus y' \rangle \leq \langle x \mid y \rangle \leq \langle x \boxplus x' \mid y \rangle \tag{1.3}
\]

We say that \( x \) dominates \( y \) in position \( i \) if \( \langle x \mid y \rangle = y_i - x_i \). It follows from (1.1) that domination is scale invariant, that is, if \( x \) dominates \( y \) in position \( i \), then \( \lambda \otimes x \) dominates \( \mu \otimes y \) in position \( i \) for all \( \lambda, \mu \in \mathbb{R} \). We denote the set of all elements of \( \mathbb{R}^n \) dominated by \( x \) in position \( i \) by \( \text{Dom}_i(x) \). If \( P \) is a subset of \( \mathbb{R}^n \), we say that \( x \) dominates \( P \) in position \( i \) if \( P \subseteq \text{Dom}_i(x) \).

**Lemma 1.1.** For each \( x \in \mathbb{R}^n \) and each coordinate \( i \), the set \( \text{Dom}_i(x) \) is max-plus convex, min-plus convex and Euclidean convex.

**Proof.** Since domination is scale invariant we see that \( \text{Dom}_i(x) \) is closed under tropical scaling. Thus, in order to show that \( \text{Dom}_i(x) \) is max-plus and min-plus convex, it suffices to show that \( u \oplus v, u \boxplus v \in \text{Dom}_i(x) \) for all \( u, v \in \text{Dom}_i(x) \).

---

\(^6\)The term convex comes from an alternative characterisation in terms of tropical line segments; see \([7]\).
It follows from (1.2) that
\[ \langle x \mid u \oplus v \rangle \geq \langle x \mid u \rangle \oplus \langle x \mid v \rangle. \]
By assumption the latter is equal to \((u_i - x_i) \oplus (v_i - x_i)\), or in other words, \((u \oplus v)_i - x_i\). Conversely, it is immediate from the definition of the bracket that
\[ (u \oplus v)_i - x_i \geq \min_k ((u \oplus v)_k - x_k) = \langle x \mid u \oplus v \rangle \]
so we have
\[ (u \oplus v)_i - x_i = \langle x \mid u \oplus v \rangle \]
as required to show that \(u \oplus v\) is dominated by \(x\) in position \(i\).

On the other hand by (1.3) we have
\[ \langle x \mid u \boxplus v \rangle = \langle x \mid u \rangle \boxplus \langle x \mid v \rangle, \]
giving
\[ \langle x \mid u \boxplus v \rangle = (u_i - x_i) \boxplus (v_i - x_i) = (u \boxplus v)_i - x_i, \]
showing that \(u \boxplus v\) is dominated by \(x\) in position \(i\).

Finally, let \(u, v \in \text{Dom}_i(x)\), \(t \in [0, 1]\) and let \(z = tu + (1 - t)v\) be a point on the line segment between \(u\) and \(v\). It follows from the fact that \(u, v \in \text{Dom}_i(x)\) that
\[ u_i \leq u_j + (x_i - x_j), \quad v_i \leq v_j + (x_i - x_j), \quad \text{for all } j. \]
Since \(0 \leq t, (1 - t)\) for all \(j\) we have
\[ z_i = tu_i + (1 - t)v_i \leq t(u_j + (x_i - x_j)) + (1 - t)(v_j + (x_i - x_j)) \leq tu_j + (1 - t)v_j + (x_i - x_j) \leq z_j + (x_i - x_j). \]
In other words, \(x\) dominates \(z\) in position \(i\).

The fact that \(\text{Dom}_i(x)\) is Euclidean convex can also be deduced from the work of Develin and Sturmfels [7, Lemma 10].

**Lemma 1.2.** Let \(S\) be a subset of \(\mathbb{R}^n\) and \(P\) its max-plus (or min-plus or Euclidean) convex hull. Then \(x\) dominates \(P\) in position \(i\) if and only if \(x\) dominates \(S\) in position \(i\).

**Proof.** One implication is immediate from the definition of domination, while the other follows straight from Lemma 1.1.

**Lemma 1.3.** Suppose that \(c_1, \ldots, c_n \in \mathbb{R}^n\) are such that for all \(i\) and \(j\), \(c_i\) dominates \(c_j\) in position \(i\). Then the intersection \(\bigcap_i \text{Dom}_i(c_i)\) is a max-plus polytope (with generating set \(c_1, \ldots, c_n\)) that is min-plus and Euclidean convex.

**Proof.** Let \(D = \bigcap_i \text{Dom}_i(c_i)\). By Lemma 1.1, \(D\) is an intersection of sets which are max-plus, min-plus and Euclidean convex, and so is itself max-plus, min-plus and Euclidean convex. It follows straight from the hypothesis that \(c_1, \ldots, c_n \in D\) and, since \(D\) is max-plus convex, also that
\[ \text{Span}_\oplus(c_1, \ldots, c_n) \subseteq D. \]
Moreover, if \( y \in D \), that is, \( y \in \text{Dom}_i(c_i) \) for all \( i \), then it is easy to see that for all \( i \) the inequality \( \langle c_i \mid y \rangle \otimes c_i \leq y \) holds, with equality in position \( i \). Thus we have
\[
y = \bigoplus_{i=1}^{n} \langle c_i \mid y \rangle \otimes c_i \in \text{Span}_{\oplus}(c_1, \ldots, c_n)
\]
so that \( D \subseteq \text{Span}_{\oplus}(c_1, \ldots, c_n) \subseteq D \). \( \square \)

2. Kleene stars and dominator matrices

Recall that a matrix \( A \in M_n(\mathbb{R}) \) is called a (max-plus) Kleene star\(^7\) if \( A \otimes A = A \) (that is, \( A \) is idempotent) and all diagonal entries of \( A \) are 0. Kleene stars have numerous fascinating properties and important applications, and have long played a central role in max-plus algebra; see for example [3] for a full introduction.

The following lemma provides a connection between Kleene stars and the concept of domination introduced in Section 1.

**Lemma 2.1.** Let \( K \in M_n(\mathbb{R}) \) be a max-plus Kleene star, with columns \( c_1, \ldots, c_n \). Then
\[
\text{Col}_{\oplus}(K) = \bigcap_i \text{Dom}_i(c_i).
\]
In particular, \( \text{Col}_{\oplus}(K) \) is min-plus and Euclidean convex.

**Proof.** Since \( K \) is a Kleene star, it follows from [11, Lemma 5.3(ii)] that \( \langle c_j \rangle_i = \langle c_i \mid c_j \rangle \) for all \( i \) and \( j \). Thus we have,
\[
\langle c_i \mid c_j \rangle = (c_j)_i - (c_i)_i,
\]
hence showing that each \( c_i \) dominates each \( c_j \) in position \( i \). It then follows from Lemma [1,3] that \( \bigcap_i \text{Dom}_i(c_i) \) is the max-plus polytope generated by the columns \( c_i \), (that is, the column space of \( K \)) and that this polytope is min-plus and Euclidean convex. \( \square \)

The fact that the column space of a max-plus Kleene star is Euclidean convex was first proved (in the language of max-times algebra, and working with a zero element) by Sergeev, Schneider and Butkovič [15, Propositions 2.6 and 3.1].

For our next lemma, we shall need some facts about the duality between the row and column spaces of a tropical matrix. Given a matrix \( A \in M_n(\mathbb{R}) \), the max-plus duality maps of \( A \) are defined as follows
\[
\rho_A : \text{Row}_{\oplus}(A) \rightarrow \text{Col}_{\oplus}(A), \quad r \mapsto A \otimes (-r)^T \text{ for all } r \in \text{Row}_{\oplus}(A),
\]
\[
\chi_A : \text{Col}_{\oplus}(A) \rightarrow \text{Row}_{\oplus}(A), \quad c \mapsto (-c)^T \otimes A \text{ for all } c \in \text{Col}_{\oplus}(A).
\]
The maps \( \rho_A \) and \( \chi_A \) are mutually inverse bijections between the max-plus row space and the max-plus column space of the matrix \( A \); they have many interesting properties — see for example [3,7,9]. When the matrix \( A \) is

---

\(^7\)This is a slightly non-standard definition, which we prefer here since it avoids introducing additional terminology for which we have no further use. For the standard definition see for example [3]; for equivalence of the two definitions see for example [15, Proposition 2.1].
a Kleene star, it turns out that the duality maps have a particularly nice form:

**Lemma 2.2.** Let $K$ be a max-plus Kleene star. Then

(i) The negated matrix $-K$ is a min-plus Kleene star.
(ii) $\chi_K(x) = -x^T$ for all $x \in \text{Col}_\oplus(K)$.
(iii) $\rho_K(y) = -y^T$ for all $y \in \text{Row}_\oplus(K)$.
(iv) $\text{Col}_\oplus(K) = \text{Col}_\oplus(-K^T)$.
(v) $\text{Row}_\oplus(K) = \text{Row}_\oplus(-K^T)$.

**Proof.** (i) This is immediate from the fact that negation is an isomorphism between the max-plus and min-plus semirings.

(ii) Let $x \in \text{Col}_\oplus(K)$. By Lemma 2.4, for every $i$ we have that the $i$th column of $K$ dominates $x$ in position $i$. By definition this means $x_i - K_{i,i} \leq x_j - K_{j,i}$ for all $i$ and $j$, or by negating and rewriting in tropical notation,

$$(-x)_i \oplus K_{i,i} \geq (-x)_j \oplus K_{j,i}$$

for all $i$ and $j$. But $K$ is a Kleene star so $K_{i,i} = 0$ and hence

$$(\chi_K(x))_i = ((-x)^T \oplus K)_i = \bigoplus_{j=1}^n (-x)_j \oplus K_{j,i} = (-x)_i \oplus K_{i,i} = (-x)_i$$

for all $i$, that is, $\chi_K(x) = -x^T$.

(iii) The proof is dual to (ii).

(iv) It follows from (ii) and (iii) that the max-plus column space $\text{Col}_\oplus(K)$ is exactly the negation of the max-plus row space. Because negation is an isomorphism between the max-plus and min-plus semirings, this means the column space is the min-plus convex hull of the negated rows of $K$, in other words, of the columns of $-K^T$. But this is by definition $\text{Col}_\oplus(-K^T)$.

(v) The proof is dual to (iv). \qed

**Lemma 2.3.** The min-plus convex hull of a max-plus convex set is max-plus convex.

**Proof.** Let $P \subseteq \mathbb{R}^n$ be a max-plus convex set and let $u, v$ be two elements of the min-plus closure $\text{Span}_\oplus(P)$. It suffices to show that the max-plus sum $u \oplus v$ is contained in $\text{Span}_\oplus(P)$. Since $u, v \in \text{Span}_\oplus(P)$ and $P$ is already closed under tropical scaling, we can write $u = u_1 \oplus \cdots \oplus u_k$ and $v = v_1 \oplus \cdots \oplus v_m$ for some $u_1, \ldots, u_k, v_1, \ldots, v_m \in P$. Then it is easy to check (using distributivity of max over min) that

$$u \oplus v = ((u_1 \oplus v_1) \oplus \cdots \oplus (u_k \oplus v_1)) \oplus \cdots \oplus ((u_1 \oplus v_m) \oplus \cdots \oplus (u_k \oplus v_m))$$

Since each max-plus sum $u_i \oplus v_j$ is an element of $P$, it follows that $u \oplus v$ is in $\text{Span}_\oplus(P)$. \qed

**Lemma 2.4.** Let $P \subseteq \mathbb{R}^n$ be a non-empty max-plus convex set with compact projectivisation.

(i) For each coordinate $i$, the set $P_i = \{ u \in P \mid u_i \geq 0 \}$ has a unique greatest lower bound $d_i$. The element $d_i$ lies in the min-plus convex hull of $P$ and the $i$th component of $d_i$ is 0.
(ii) If \( x \) is in the min-plus convex hull of \( P \), then \( d_i \) dominates \( x \) in position \( i \), that is, \( \langle d_i \mid x \rangle = x_i \) and hence
\[
x = \bigoplus_i \langle d_i \mid x \rangle \otimes d_i.
\]

Proof. (i) Fix \( i \). For each \( j \neq i \), it follows from compactness of the projectivisation that the set \( \{u_j \mid u \in P_i\} \) is topologically closed and bounded below, and hence contains a minimum element. Thus, we may choose \( w_j \in P_i \) satisfying
\[
(w_j)_i \geq 0 \quad \text{and} \quad (w_j)_j = \min\{u_j \mid u \in P_i\}.
\]
Note that \((w_j)_i = 0\); indeed, if not then \((- (w_j)_i) \otimes w_j\) would be in \( P_i \) with \( j \)th coordinate strictly smaller than that of \( w_j \), giving a contradiction.

Now let \( d_i \) be the min-plus sum of the elements \( w_j \). Then the \( i \)th component of \( d_i \) is 0. We claim that \( d_i \) is the greatest lower bound of \( P_i \). By construction, \( d_i \leq w_j \) for each \( j \neq i \), so by the definition of \( w_j \) we have
\[
(d_i)_j \leq (w_j)_j \leq u_j \quad \text{for all} \; u \in P_i \; \text{and} \; j \neq i.
\]
Moreover, using the definition of \( P_i \) we have
\[
(d_i)_i = 0 \leq u_i \quad \text{for all} \; u \in P_i.
\]
Thus, \( d_i \) is a lower bound for \( P_i \). If \( y \) is any other lower bound for \( P_i \), then in particular \( y \leq w_j \) for all \( j \neq i \) and hence \( y \leq d_i \).

(ii) Suppose first that \( x \in P \). Consider the element \( y = (- x_i) \otimes x \). Then \( y_i = 0 \) and \( y \in P \) so \( y \in P_i \) and hence \( d_i \leq y \). Moreover, \( (d_i)_i = 0 = y_i \), so \( \langle d_i \mid y \rangle = 0 \) and using (1.1) we have
\[
\langle d_i \mid x \rangle = \langle d_i \mid x_i \otimes y \rangle = x_i \otimes 0 = x_i
\]
as required to show that \( d_i \) dominates \( x \) in position \( i \). Thus, \( d_i \) dominates \( P \) in position \( i \), and hence by Lemma 1.2 \( d_i \) dominates the min-plus convex hull of \( P \).

Now if \( x \) is in the min-plus convex hull of \( P \) then since \( \langle d_i \mid x \rangle \otimes d_i \leq x \) and \( \langle d_i \mid x \rangle \otimes (d_i)_i = \langle d_i \mid x \rangle = x_i \), it follows that
\[
x = \bigoplus_i \langle d_i \mid x \rangle \otimes d_i.
\]

\[\square\]

The previous lemma motivates a key definition. Let \( P \) be a max-plus convex subset of \( \mathbb{R}^n \) with compact projectivisation. We define the (min-plus) dominator matrix of \( P \), denoted \( D_P \), to be the matrix whose \( i \)th column is the greatest lower bound of the set \( \{u \in P : u_i \geq 0\} \). There is of course a natural dual concept of the max-plus dominator of a closed min-plus convex set in \( \mathbb{R}^n \).

**Proposition 2.5.** Let \( P \subseteq \mathbb{R}^n \) be max-plus convex with compact projectivisation, and let \( D_P \) be the min-plus dominator matrix of \( P \). Then \( D_P \) is a max-plus Kleene star, with max-plus column space exactly the min-plus convex hull of \( P \). (In particular, the min-plus convex hull of \( P \) is a max-plus polytope.)
Proof. Let $P'$ denote the min-plus convex hull $\text{Span}_\oplus(P)$. By Lemma 2.4(i), each column $d_i$ is contained in $P'$. Thus by Lemma 2.3 we see that $\text{Col}_\oplus(D_P) \subseteq P'$. On the other hand, it follows from Lemma 2.4(ii) that $P' \subseteq \text{Col}_\ominus(D_P)$. Thus $P' = \text{Col}_\ominus(D_P)$.

Now since each column $d_i$ lies in $P'$, applying Lemma 2.4 to these elements shows that $d_i$ dominates $d_j$ in position $i$ for all $i$ and $j$. Thus by Lemma 1.3 we see that $\text{Col}_\ominus(D_P) = \bigcap_i \text{Dom}_i(d_i)$.

In particular $P'$, being the max-plus column space of a matrix, is a max-plus polytope, and hence has compact projectivisation, so Lemma 2.4 also applies with $P$ replaced by $P'$. By Lemma 2.4(i) the set

$$P'_i = \{ u \in P' \mid u_i \geq 0 \}$$

has a greatest lower bound $f_i$, where $(f_i)_i = 0$ and $f_i$ is contained in the min-plus convex hull of $P'$, that is, in $P'$ itself. Hence by Lemma 2.4(ii), $(d_i)_{f_i} = (f_i)_i = 0$ and $(f_i | d_i) = (d_i)_i = 0$, giving $f_i = d_i$. In other words, $d_i$ is the greatest lower bound of the set $P'_i$.

Now fix $j$ and for each $k$ let $y_k = -(d_{j})_k \otimes d_j$. Then $y_k \in \text{Col}_\ominus(D_P) = P'$ and $(y_k)_k = 0$, so $y_k \in P'_i$ and hence $d_k \leq y_k$. In particular this gives

$$(d_k)_i \leq -(d_{j})_k \otimes (d_j)_i = (y_k)_i$$

for all $i, k$, and hence

$$(d_k)_i + (d_{j})_k \leq (d_j)_i$$

for all $i, k$.

In other words,

$$(D_P \otimes D_P)_{i,j} = \bigoplus_k (D_P)_{i,k} \otimes (D_P)_{k,j} = \bigoplus_k ((d_k)_i + (d_{j})_k) \leq (d_j)_i = (D_P)_{i,j}.$$  

On the other hand, since the diagonal entries of $D_P$ are all 0, it is clear that $(D_P \otimes D_P)_{i,j} \geq (D_P)_{i,j}$, so that $D_P$ is idempotent. Thus we have shown that $D_P$ is a max-plus idempotent matrix with 0’s on the diagonal; in other words, $D_P$ is a max-plus Kleene star.

We consider next the important special case where $P$ is both max-plus and min-plus convex, and therefore has both a min-plus and a max-plus dominator matrix.

**Theorem 2.6.** Let $P \subseteq \mathbb{R}^n$ be both max-plus and min-plus convex with compact projectivisation. Then $P$ is both a max-plus polytope and a min-plus polytope, and its max-plus dominator is the negated transpose of its min-plus dominator.

**Proof.** That $P$ is both a max-plus polytope and a min-plus polytope follows from Proposition 2.5 and its dual.

Let $D_P$ be the min-plus dominator of $P$ considered as a max-plus convex set. We aim first to show that $P$ is the min-plus column space of $-D_P^T$. It follows from Proposition 2.4 that $D_P$ is a max-plus Kleene star with max-plus column space $P$. By Lemma 2.2(i), $-D_P$ is a min-plus Kleene star and it follows easily that $-D_P^T$ is also a min-plus Kleene star. Moreover, by Lemma 2.2(iv), the min-plus column space of $-D_P^T$ is exactly $P$.

Now let $D_P$ be the max-plus dominator of $P$. Using the fact $P$ is both min-plus and max-plus convex, it follows from the dual to Proposition 2.5 that $D_P$ is a min-plus Kleene star with min-plus column space $P$. 


However, it follows from a result of Sergeev [13, Proposition 6] that there is at most one Kleene star with a given column space, so we conclude that $D'_P = -(D_P)^T$ as required. □

**Corollary 2.7.** Let $P \subseteq \mathbb{R}^n$ be max-plus and min-plus convex with compact projectivisation, and let $D_P$ be the min-plus dominator matrix of $P$. Then

*(i)* $P$ is a max-plus polytope generated by the columns of $D_P$.

*(ii)* $P$ is a min-plus polytope generated by the rows of $-D_P$.

### 3. Proofs of the main theorems

In this section we apply the technical results so far to establish our main theorems, as described in the introduction.

**Theorem A.** Let $P \subseteq \mathbb{R}^n$ be max-plus convex with compact projectivisation. Then the following are equivalent:

*(i)* $P$ is min-plus convex;

*(ii)* $P$ is the max-plus column space of a max-plus Kleene star;

*(iii)* $P$ is the min-plus column space of a min-plus Kleene star.

**Proof of Theorem A.** Suppose first that $P$ is min-plus convex. Then by Proposition 2.5 we see that $P = \text{Col}_\oplus(D_P)$, where $D_P$ is a max-plus Kleene star, establishing that (i) implies (ii).

It follows from Lemma 2.2 that conditions (ii) and (iii) are equivalent. (Indeed, if $P$ is the max-plus column space of a max-plus Kleene star $K$, then it is the min-plus column space of the min-plus Kleene star formed by taking the negated transpose of $K$ and vice versa.)

Finally, that (ii) implies (i) is immediate from Lemma 2.1. □

**Theorem B.** Let $P \subseteq \mathbb{R}^n$. If any two of the following statements hold, then so does the remaining one:

*(i)* $P$ is a max-plus polytope;

*(ii)* $P$ is a min-plus polytope;

*(iii)* $P$ is Euclidean convex.

**Proof.** We shall show that (i) and (ii) together imply (iii), and that (i) and (iii) together imply (ii). The remaining required implication — that (ii) and (iii) together imply (i) — is dual to the latter case by exchanging min and max.

Suppose first that (i) and (ii) hold. Then by Theorem A, $P$ is the max-plus column space of a max-plus Kleene star and hence by Lemma 2.1 (or [13, Propositions 2.6 and 3.1]) $P$ is Euclidean convex.

Now suppose (i) and (iii) hold, and let $V = \{v_1, \ldots, v_m\}$ be a weak max-plus basis for $P$. Using the type decomposition (see [7]) of $P$ with respect to $V$ we note that $P$ is the union of finitely many bounded cells $X_S$, each of which is compact, max-plus convex and min-plus convex. We shall show that given any finite collection of compact min-plus convex subsets of $\mathbb{R}^n$, if the union is Euclidean convex then it is also min-plus convex. It therefore follows from this result that $P$ is min-plus convex, and hence by application of Theorem A we see that $P$ is indeed a min-plus polytope.
Let $X_1, \ldots, X_k$ be a finite collection of compact, min-plus convex subsets of $\mathbb{R}^n$ whose union (call it $P$) is Euclidean convex. Suppose for a contradiction that $P$ is not min-plus convex. Clearly $P$ is closed under tropical scaling, so we must be able to choose $x, y \in P$ with $x \oplus y \not\in P$. Write $q = x \oplus y$. Since $P$ is compact we may assume without loss of generality that $x$ is minimal in $P$ with $x \oplus y = q$, in the weak sense that there is no $x' \in P$ with $q < x' < x$. (Indeed, having fixed $y$ and $q$, choose $x \in P$ with $x \oplus y = q$ such that the first coordinate is minimum possible, the second coordinate minimum possible subject to the first and so forth.)

Now consider the Euclidean line segment from $x$ to $y$. Note that this is a proper line segment, since $x = y$ would imply $q \in P$, giving a contradiction. Since $P$ is Euclidean-convex, the line segment is contained in $P$. Since $P$ is the union of the (finitely many) $X_i$s and are all closed, it is easy to see that there must be some $i$ such that $X_i$ contains both $x$ and some non-end-point (call it $z$) of the line segment. (Indeed, if we consider a sequence of non-endpoints on the line segment converging to $x$, there must be some $X_i$ which contains infinitely many terms, hence which contains a sequence converging to $x$, hence which contains $x$.) Notice that $z$ is a weighted average of $x$ and $y$, both of which are larger than $q$, so it must itself be larger than $q$.

Now consider the point $x \oplus z$. This lies in $X_i$, since the latter is min-plus convex, and hence in $P$. Since $x$ and $z$ are both larger than $q$, we have $x \oplus z \geq q$, and by definition $x \oplus z \leq x$. By the minimality assumption on $x$, this means that we must have $x \oplus z = x$, in other words, $x \leq z$.

But $z$ is a weighted average of $x$ and $y$, so this implies $x \leq y$, whereupon $q = x \oplus y = x \in P$, giving a contradiction. □

Notice that if $P \subseteq \mathbb{R}^n$ is max-plus convex with compact projectivisation and satisfies any of the equivalent conditions of Theorem A, then $P$ must be Euclidean convex (this can be seen using Lemma 2.1 for instance). We remark that there are however max-plus convex sets in $\mathbb{R}^n$ which have compact projectivisation and are Euclidean convex, yet not min-plus convex. For example, consider the max-plus convex hull $P \subseteq \mathbb{R}^3$ of the points
\[ \{(1,0,0), (a,-a,0) : 0 \leq a \leq 1\}. \]

It is easy to see that the projectivisation of $P$ is a Euclidean triangle in $\mathbb{R}^2$. However, the min-plus sum of any two distinct elements of the form $(a,-a,0)$ lies outside $P$. Note that this means $P$ cannot be a max-plus polytope: indeed, if it were the convex hull of finitely many of its elements then by Theorem B it would have to be also a min-plus polytope.

References

[1] F. L. Baccelli, G. Cohen, G. J. Olshder, and J.-P. Quadrat. Synchronization and linearity. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Ltd., Chichester, 1992.
[2] T. S. Blyth and M. F. Janowitz. Residuation theory. Pergamon Press, Oxford, 1972. International Series of Monographs in Pure and Applied Mathematics, Vol. 102.
[3] P. Butkovič. Max-linear systems: theory and algorithms. Springer, 2010.
[4] P. Butkovič, H. Schneider, and S. Sergeev. Generators, extremals and bases of max cones. Linear Algebra Appl., 421(2-3):394–406, 2007.
[5] G. Cohen, S. Gaubert, and J.-P. Quadrat. Duality and separation theorems in idempotent semimodules. Linear Algebra Appl., 379:395–422, 2004.
[6] M. J. de la Puente. On tropical Kleene star matrices and alcoved polytopes. *Kybernetika*, 49(6):897–910, 2013.
[7] M. Develin and B. Sturmfels. Tropical convexity. *Doc. Math.*, 9:1–27 (electronic), 2004.
[8] S. Gaubert and S. Sergeev. Cyclic projectors and separation theorems in idempotent convex geometry. *J. Mathematical Sciences*, 155:815–829, 2007.
[9] C. Hollings and M. Kambites. Tropical matrix duality and Green’s $D$ relation. *J. London Math. Soc.*, 86:520–538, 2012.
[10] M. Johnson and M. Kambites. Green’s $J$-order and the rank of tropical matrices. *J. Pure Appl. Algebra*, 217:280–292, 2013.
[11] M. Johnson and M. Kambites. Idempotent tropical matrices and finite metric spaces. *Adv. Geom.*, 14:253–276, 2014.
[12] M. Joswig and K. Kulas. Tropical and ordinary convexity combined. *Adv. Geom.*, 10(2):333–352, 2010.
[13] S. Sergeev. Max-plus definite matrix closures and their eigenspaces. *Linear Algebra Appl.*, 421(2-3):182–201, 2007.
[14] S. Sergeev. Multiorder, Kleene stars and cyclic projectors in the geometry of max cones. In *Tropical and idempotent mathematics*, volume 495 of *Contemp. Math.*, pages 317–342. Amer. Math. Soc., Providence, RI, 2009.
[15] S. Sergeev, H. Schneider, and P. Butkovič. On visualization scaling, subeigenvectors and Kleene stars in max algebra. *Linear Algebra Appl.*, 431(12):2395–2406, 2009.
[16] N. M. Tran. Polytropes and tropical eigenspaces: cones of linearity. *Discrete Comput. Geom.*, 51(3):539–558, 2014.
[17] A. Werner and J. Yu. Symmetric alcoved polytopes. *Electron. J. Combin.*, 21(1):Paper 1.20, 14, 2014.