FINITE CYCLES OF INDECOMPOSABLE MODULES

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Dedicated to Raymundo Bautista on the occasion of his 70th birthday

Abstract. We solve a long standing open problem concerning the structure of finite cycles in the category $\mathsf{mod} \ A$ of finitely generated modules over an arbitrary artin algebra $A$, that is, the chains of homomorphisms $M_0 \xrightarrow{f_1} M_1 \xrightarrow{} \cdots \xrightarrow{f_r} M_r = M_0$ between indecomposable modules in $\mathsf{mod} \ A$ which do not belong to the infinite radical of $\mathsf{mod} \ A$. In particular, we describe completely the structure of an arbitrary module category $\mathsf{mod} \ A$ whose all cycles are finite. The main structural results of the paper allow to derive several interesting combinatorial and homological properties of indecomposable modules lying on finite cycles. For example, we prove that for all but finitely many isomorphism classes of indecomposable modules $M$ lying on finite cycles of a module category $\mathsf{mod} \ A$ the Euler characteristic of $M$ is well defined and nonnegative. As another application of these results we obtain a characterization of all cycle-finite module categories $\mathsf{mod} \ A$ having only a finite number of functorially finite torsion classes. Moreover, new types of examples illustrating the main results of the paper are presented.

0. Introduction

Throughout the paper, by an algebra is meant an artin algebra over a fixed commutative artin ring $K$, which we shall assume (without loss of generality) to be basic and indecomposable. For an algebra $A$, we denote by $\mathsf{mod} \ A$ the category of finitely generated right $A$-modules and by $\mathsf{ind} \ A$ the full subcategory of $\mathsf{mod} \ A$ formed by the indecomposable modules. The Jacobson radical $\mathsf{rad} \ A$ of $\mathsf{mod} \ A$ is the ideal generated by all nonisomorphisms between modules in $\mathsf{ind} \ A$, and the infinite radical $\mathsf{rad}^{\infty} \ A$ of $\mathsf{mod} \ A$ is the intersection of all powers $\mathsf{rad}^{i} \ A$, $i \geq 1$, of $\mathsf{rad} \ A$. By a result of Auslander [7], $\mathsf{rad}^{\infty} \ A = 0$ if and only if $A$ is of finite representation type, that is, $\mathsf{ind} \ A$ admits only a finite number of pairwise nonisomorphic modules (see also [33] for an alternative proof of this result). On the other hand, if $A$ is of infinite representation type then $(\mathsf{rad}^{\infty} \ A)^2 \neq 0$, by a result proved in [20].

An important combinatorial and homological invariant of the module category $\mathsf{mod} \ A$ of an algebra $A$ is its Auslander-Reiten quiver $\Gamma_A$. Recall that $\Gamma_A$ is a valued translation quiver whose vertices are the isomorphism classes $\{X\}$ of modules $X$ in $\mathsf{ind} \ A$, the arrows correspond to irreducible homomorphisms between modules in $\mathsf{ind} \ A$, and the translation is the Auslander-Reiten translation $\tau_A = D\mathsf{Tr}$. We shall not distinguish between a module in $\mathsf{X}$ in $\mathsf{ind} \ A$ and the corresponding vertex $\{X\}$ of $\Gamma_A$. If $A$ is an algebra of finite representation type, then every nonzero nonisomorphism in $\mathsf{ind} \ A$ is a finite sum of composition of reducible homomorphisms between modules in $\mathsf{ind} \ A$, and hence we may recover $\mathsf{mod} \ A$ from the translation quiver $\Gamma_A$. In general, $\Gamma_A$ describes only the quotient category $\mathsf{mod} \ A/\mathsf{rad}^{\infty} \ A$.

Let $A$ be an algebra and $M$ a module in $\mathsf{ind} \ A$. An important information concerning the structure of $M$ is coded in the structure and properties of its support algebra $\mathsf{Supp}(M)$ defined as follows. Consider a decomposition $A = P_M \oplus Q_M$ of $A$ in $\mathsf{mod} \ A$ such that the simple summands of the semisimple module $P_M/\mathsf{rad}P_M$ are exactly the simple composition factors of $M$. Then $\mathsf{Supp}(M) = A/t_A(M)$, where $t_A(M)$ is the ideal in $A$ generated by the images of all homomorphisms from $Q_M$ to $A$ in $\mathsf{mod} \ A$. We note that $M$ is an indecomposable module over $\mathsf{Supp}(M)$. Clearly, we may realistically hope to describe the structure of $\mathsf{Supp}(M)$ only for modules $M$ having some distinguished properties.

1991 Mathematics Subject Classification. 16G10, 16G60, 16G70.

Key words and phrases. Cycles of modules, Generalized multicoil algebras, Generalized double tilted algebras, Auslander-Reiten quiver.

This work was completed with the support of the research grant DEC-2011/02/A/ST1/00216 of the Polish National Science Center and the CIMAT Guanajuato, México.
A prominent role in the representation theory of algebras is played by cycles of indecomposable modules (see [13], [48], [62], [71]). Recall that a cycle in ind $A$ is a sequence

$$M_0 \xrightarrow{f_1} M_1 \rightarrow \cdots \rightarrow M_{r-1} \xrightarrow{f_r} M_r = M_0$$

of nonzero nonisomorphisms in ind $A$ [62], and such a cycle is said to be finite if the homomorphisms $f_1, \ldots, f_r$ do not belong to $\text{rad}_A^\infty$ (see [4], [5]). Following Ringel [62], a module $M$ in ind $A$ which does not lie on a cycle in ind $A$ is called directing. The following two important results on directing modules were established by Ringel in [62]. Firstly, if $A$ is an algebra with all modules in ind $A$ being directing, then $A$ is of finite representation type. Secondly, the support algebra $\text{Supp}(M)$ of a directing module $M$ over an algebra $A$ is a tilted algebra $\text{End}_H(T)$, for a hereditary algebra $H$ and a tilting module $T$ in mod $H$, and $M$ is isomorphic to the image $\text{Hom}_H(T, I)$ of an indecomposable injective module $I$ in mod $H$ via the functor $\text{Hom}_H(T, -) : \text{mod } H \rightarrow \text{mod } \text{End}_H(T)$. In particular, it follows that, if $A$ is an algebra of infinite representation type, then ind $A$ always contains a cycle. Moreover, it has been proved independently by Peng and Xiao [51] and Skowroński [69] that the Auslander-Reiten quiver $\Gamma_A$ of an algebra $A$ admits at most finitely many $\tau_A$-orbits containing directing modules. Hence, in order to obtain information on the support algebras $\text{Supp}(M)$ of nondirecting modules in ind $A$, it is natural to study properties of cycles in ind $A$ containing $M$. A module $M$ in ind $A$ is said to be cycle-finite if $M$ is nondirecting and every cycle in ind $A$ passing through $M$ is finite. Obviously, every indecomposable module over an algebra of finite representation type is cycle-finite. Examples of cycle-finite indecomposable modules over algebras of infinite representation type are provided by all indecomposable modules in the stable tubes of tame hereditary algebras [25], canonical algebras [62], or more generally concealed canonical algebras [36]. Following Assem and Skowroński [13], [43], an algebra $A$ is said to be cycle-finite if all cycles in ind $A$ are finite. The class of cycle-finite algebras is wide and contains the following distinguished classes of algebras: the algebras of finite representation type, the tame tilted algebras [28], [32], [62], the tame double tilted algebras [59], the tame generalized double tilted algebras [60], the tubular algebras [62], [63], the iterated tubular algebras [57], the tame quasi-tilted algebras [37], [76], the tame generalized multicoil algebras [46], the algebras with cycle-finite derived categories [4], and the strongly simply connected algebras of polynomial growth [74]. We also mention that a selfinjective algebra $A$ is cycle-finite if and only if $A$ is of finite representation type [31]. On the other hand, frequently an algebra $A$ admits a Galois covering $R \rightarrow R/G = A$, where $R$ is a cycle-finite locally bounded category and $G$ is an admissible group of automorphisms of $R$, which allows to reduce the representation theory of $A$ to the representation theory of cycle-finite algebras being finite convex subcategories of $R$ (see [55] and [74] for some general results). For example, every finite dimensional selfinjective algebra $A$ of polynomial growth over an algebraically closed field $K$ admits a canonical standard form $\overline{A}$ (geometric socle deformation of $A$) such that $\overline{A}$ has a Galois covering $R \rightarrow R/G = \overline{A}$, where $R$ is a cycle-finite selfinjective locally bounded category and $G$ is an admissible infinite cyclic group of automorphisms of $R$, the Auslander-Reiten quiver $\Gamma_{\overline{A}}$ of $\overline{A}$ is the orbit quiver $\Gamma_R/G$ of $\Gamma_R$, and the stable Auslander-Reiten quivers of $A$ and $\overline{A}$ are isomorphic (see [66], [80]). We refer to [13], [43], [73] for some general results on the structure of cycle-finite algebras and their module categories.

In the paper we are concerned with the problem of describing the support algebras of cycle-finite modules over arbitrary (artin) algebras. We note that this may be considered as a natural extension of the problem concerning the structure of support algebras of directing modules, solved by Ringel in [62]. Namely, the directing modules in ind $A$ may be viewed as modules $M$ in ind $A$ for which every oriented cycle of nonzero homomorphisms in ind $A$ containing $M$ consists entirely of isomorphisms. The considered problem, initiated more than 25 years ago in [4], turned out to be very difficult, and many researchers involved to its solution resigned. The main obstacle for solution of this problem was the large complexity of finite cycles of indecomposable modules and the fact that all cycles of
Let $A$ be an algebra and $M$ be a cycle-finite module in $\text{ind} \ A$. Then every cycle in $\text{ind} \ A$ passing through $M$ has a refinement to a cycle of irreducible homomorphisms in $\text{ind} \ A$ containing $M$ and consequently $M$ lies on an oriented cycle in the Auslander-Reiten quiver $\Gamma_A$ of $A$. Following Malicki and Skowroński [45], we denote by $\mathcal{C}_A$ the cyclic quiver of $A$ obtained from $\Gamma_A$ by removing all acyclic vertices (vertices not lying on oriented cycles in $\Gamma_A$) and the arrows attached to them. Then the connected components of the translation quiver $\mathcal{C}_A$ are said to be cyclic components of $\Gamma_A$. It has been proved in [45] that two modules $X$ and $Y$ in $\text{ind} \ A$ belong to the same cyclic component of $\Gamma_A$ if and only if there is an oriented cycle in $\Gamma_A$ passing through $X$ and $Y$. For a cyclic component $\Gamma$ of $\mathcal{C}_A$, we consider a decomposition $A = P_T \oplus Q_T$ of $A$ in $\text{mod} \ A$ such that the simple summands of the semisimple module $P_T/\text{rad} P_T$ are exactly the simple composition factors of indecomposable modules in $\Gamma$, the ideal $t_A(\Gamma)$ in $A$ generated by the images of all homomorphisms from $Q_T$ to $A$ in $\text{mod} \ A$, and call the quotient algebra $\text{Supp}(\Gamma) = A/t_A(\Gamma)$ the support algebra of $\Gamma$. Observe now that $M$ belongs to a unique cyclic component $\Gamma(M)$ of $\Gamma_A$ consisting entirely of cycle-finite indecomposable modules, and the support algebra $\text{Supp}(M)$ of $M$ is a quotient algebra of the support algebra $\text{Supp}(\Gamma(M))$ of $\Gamma(M)$. A cyclic component $\Gamma$ of $\Gamma_A$ containing a cycle-finite module is said to be a cycle-finite cyclic component of $\Gamma_A$. We will prove that the support algebra $\text{Supp}(\Gamma)$ of a cycle-finite cyclic component $\Gamma$ of $\Gamma_A$ is isomorphic to an algebra of the form $e_T A e_T$ for an idempotent $e_T$ of $A$ whose primitive summands correspond to the vertices of a convex subquiver of the valued quiver $Q_A$ of $A$. On the other hand, the support algebra $\text{Supp}(M)$ of a cycle-finite module $M$ in $\text{ind} \ A$ is not necessarily an algebra of the form $e A e$ for an idempotent $e$ of $A$ (see Section 7).

The main results of the paper provide a conceptual description of the support algebras of cycle-finite cyclic components of $\Gamma_A$. The description splits into two cases. In the case when a cycle-finite cyclic component $\Gamma$ of $\Gamma_A$ is infinite, we prove that $\text{Supp}(\Gamma)$ is a suitable gluing of finitely many generalized multicoil algebras (introduced by Malicki and Skowroński in [46]) and algebras of finite representation type, and $\Gamma$ is the corresponding gluing of the associated cyclic generalized multicoils via finite translation quivers. In the second case when a cycle-finite cyclic component $\Gamma$ is finite, we prove that $\text{Supp}(\Gamma)$ is a generalized double tilted algebra (in the sense of Reiten and Skowroński [60]) and $\Gamma$ is the core of the connecting component of this algebra.

We would like to mention that the generalized multicoil algebras form a prominent class of algebras of global dimension at most 3, containing the class of quasitilted algebras of canonical type, and are obtained by sophisticated gluings of concealed canonical algebras using admissible algebra operations, generalizing the coil operations proposed by Assem and Skowroński in [5]. The generalized double tilted algebras form a distinguished class of algebras, containing all tilted algebras and all algebras of finite representation type, and can be viewed as two-sided gluings of tilted algebras. The tilted algebras and quasitilted algebras of canonical type were under intensive investigation over the last two decades by many representation theory algebraists. Hence, the main results of the paper give a good understanding of the support algebras of cycle-finite cyclic components. On the other hand, the results and examples presented in the paper create new interesting open problems and research directions (see Section 1).

The paper is organized as follows. In Section 1 we present the main results of the paper and related background. In Section 2 we describe properties of cyclic components of the Auslander-Reiten quivers of algebras, applied in the proofs of the main theorems. Sections 3, 4 and 5 are devoted to the proofs of Theorems 1.1, 1.2 and 1.10 respectively. In Sections 6 and 7 we present new types of examples, illustrating the main results of the paper.
For basic background on the representation theory applied here we refer to [8, 10, 62, 64, 65, 82].

The main results of the paper have been proved during the visit of P. Malicki and A. Skowroński at the Centro de Investigación en Mathemáticas (CIMAT) in Guanajuato (November 2012), who would like to thank J. A. de la Peña and CIMAT for the warm hospitality and wonderful conditions for the successful realization of this joint research project. The results were presented by the first named author during the conferences ”Advances in Representation Theory of Algebras” (Guanajuato, December 2012) and ”Perspectives of Representation Theory of Algebras” (Nagoya, November 2013).

1. Main results and related background

In order to formulate the main results of the paper we need special types of components of the Auslander-Reiten quivers of algebras and distinguished classes of algebras with separating families of Auslander-Reiten components.

Let $A$ be an algebra. For a subquiver $\Gamma$ of $\Gamma_A$, we denote by $\text{ann}_A(\Gamma)$ the intersection of the annihilators $\text{ann}_A(X) = \{a \in A \mid Xa = 0\}$ of all indecomposable modules $X$ in $\Gamma$, and call the quotient algebra $B(\Gamma) = A/\text{ann}_A(\Gamma)$ the faithful algebra of $\Gamma$. By a component of $\Gamma_A$ we mean a connected component of the translation quiver $\Gamma_A$. A component $C$ of $\Gamma_A$ is called regular if $C$ contains neither a projective module nor an injective module, and semiregular if $C$ does not contain both a projective and an injective module. It has been shown in [58] and [59] that a regular component $C$ of $\Gamma_A$ contains an oriented cycle if and only if $C$ is a stable tube (is of the form $\mathbb{Z}A_\infty/\langle \tau^r \rangle$, for a positive integer $r$). Moreover, Liu proved in [29] that a semiregular component $C$ of $\Gamma_A$ contains an oriented cycle if and only if $C$ is a ray tube (obtained from a stable tube by a finite number (possibly zero) of ray insertions) or a coray tube (obtained from a stable tube by a finite number (possibly zero) of coray insertions). A component $C$ of $\Gamma_A$ is said to be coherent [44] (see also [21]) if the following two conditions are satisfied:

(C1) For each projective module $P$ in $C$ there is an infinite sectional path

$$P = X_1 \to X_2 \to \cdots \to X_i \to X_{i+1} \to X_{i+2} \to \cdots.$$ 

(C2) For each injective module $I$ in $C$ there is an infinite sectional path

$$\cdots \to Y_{j+2} \to Y_{j+1} \to Y_j \to \cdots \to Y_2 \to Y_1 = I.$$ 

Further, a component $C$ of $\Gamma_A$ is said to be almost cyclic if its cyclic part $\mathcal{C}$ is a cofinite subquiver of $\mathcal{C}$. We note that the stable tubes, ray tubes and coray tubes of $\Gamma_A$ are special types of almost cyclic coherent components. In general, it has been proved by Malicki and Skowroński in [45] that a component $C$ of $\Gamma_A$ is almost cyclic and coherent if and only if $C$ is a generalized multicoil, obtained from a finite family of stable tubes by a sequence of admissible operations (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*). On the other hand, a component $C$ of $\Gamma_A$ is said to be almost acyclic if all but finitely many modules of $C$ are acyclic. It has been proved by Reiten and Skowroński in [60] that a component $C$ of $\Gamma_A$ is almost acyclic if and only if $C$ admits a multisection $\Delta$. Moreover, for an almost acyclic component $C$ of $\Gamma_A$, there exists a finite convex subquiver $c(C)$ of $C$ (possibly empty), called the core of $C$, containing all modules lying on oriented cycles in $C$ (see [60] for details). A family $\mathcal{C} = \{C_i\}_{i \in I}$ of components of $\Gamma_A$ is said to be generalized standard if $\text{rad}^\infty_A(X,Y) = 0$ for all modules $X$ and $Y$ in $\mathcal{C}$ [58], and sincere if every simple module in mod $A$ occurs as a composition factor of a module in $\mathcal{C}$. Finally, following Assem, Skowroński and Tomé [3], a family $\mathcal{C} = \{C_i\}_{i \in I}$ of components of $\Gamma_A$ is said to be separating if the components in $\Gamma_A$ split into three disjoint families $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that:

(S1) $\mathcal{C}^A$ is a sincere generalized standard family of components;

(S2) $\text{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\text{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\text{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;
(S3) any morphism from $P^A$ to $Q^A$ in mod $A$ factors through the additive category $\text{add}(C^A)$ of $C^A$. We then say that $C^A$ separates $P^A$ from $Q^A$ and write

$$\Gamma_A = P^A \cup C^A \cup Q^A.$$  

We mention that then the families $P^A$ and $Q^A$ are uniquely determined by the separating family $C^A$, and $C^A$ is a faithful family of components in $\Gamma_A$, that is, $\text{ann}_A(C^A) = 0$.

In the representation theory of algebras an important role is played by the canonical algebras introduced by Ringel in \[62\] and \[63\]. Every canonical algebra $\Lambda$ is of global dimension at most 2 and its Auslander-Reiten quiver $\Gamma_{\Lambda}$ admits a canonical separating family $T^\Lambda$ of stable tubes, so $\Gamma_{\Lambda}$ admits a disjoint union decomposition $\Gamma_{\Lambda} = P^\Lambda \cup T^\Lambda \cup Q^\Lambda$. Then an algebra $C$ of the form $\text{End}_{\Lambda}(T)$, with $T$ a tilting module in the additive category $\text{add}(P^A)$ of $P^A$ is called a concealed canonical algebra of type $\Lambda$, and $T^C = \text{Hom}_{\Lambda}(T, T^\Lambda)$ is a separating family of stable tubes in $\Gamma_{C}$, so we have a disjoint union decomposition $\Gamma_{C} = P^C \cup T^C \cup Q^C$. It has been proved by Lenzing and de la Peña in \[36\] that an algebra $A$ is a concealed canonical algebra if and only if $\Gamma_A$ admits a separating family $T^A$ of stable tubes. The concealed canonical algebras form a distinguished class of quasitilted algebras, which are the endomorphism algebras $\text{End}_{\mathcal{H}}(T)$ of tilting objects $T$ in abelian hereditary $K$-categories $\mathcal{H}$ \[27\]. By a result due to Happel, Reiten and Smalø proved in \[26\], an algebra $A$ is a quasitilted algebra if and only if $\text{gl. dim } A \leq 2$ and every module $X$ in $\text{ind } A$ satisfies $\text{pd}_A X \leq 1$ or $\text{id}_A X \leq 1$. Further, it has been proved by Happel and Reiten in \[26\] that the class of quasitilted algebras consists of the tilted algebras (the endomorphism algebras $\text{End}_{\mathcal{H}}(T)$ of tilting modules $T$ over hereditary algebras $H$) and the quasitilted algebras of canonical type (the endomorphism algebras $\text{End}_{\mathcal{H}}(T)$ of tilting objects $T$ in abelian hereditary categories $\mathcal{H}$ whose derived category $D^b(\mathcal{H})$ is equivalent to the derived category $D^b(\text{mod } A)$ of the module category mod $A$ of a canonical algebra $\Lambda$). Moreover, it has been proved by Lenzing and Skowroński in \[37\] (see also \[76\]) that an algebra $A$ is a quasitilted algebra of canonical type if and only if $\Gamma_A$ admits a separating family $T^A$ of semiregular tubes (ray and coray tubes), and if and only if $A$ is a semiregular branch enlargement of a concealed canonical algebra $C$. We are now in position to introduce the class of generalized multicoll algebras \[46\], being sophisticated gluings of quasitilted algebras of canonical type, playing the fundamental role in first main result of the paper. It has been proved by Malicki and Skowroński in \[46\] that the Auslander-Reiten quiver $\Gamma_{A}$ of an algebra $A$ admits a separating family of almost cyclic coherent components if and only if $A$ is a generalized multicoll algebra, that is, a generalized multicoll enlargement of a product $C = C_1 \times \ldots \times C_m$ of concealed canonical algebras $C_1, \ldots, C_m$ using modules from the separating families $T^{C_1}, \ldots, T^{C_m}$ of stable tubes of $\Gamma_{C_1}, \ldots, \Gamma_{C_m}$ and a sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1')-(ad 5'). For a generalized multicoll algebra $A$, there is a unique quotient algebra $A^{(l)}$ of $A$ which is a product of quasitilted algebras of canonical type having separating families of ray tubes (the left quasitilted algebra of $A$) and a unique quotient algebra $A^{(r)}$ of $A$ which is a product of quasitilted algebras of canonical type having separating families of ray tubes (the right quasitilted algebra of $A$) such that $\Gamma_A$ has a disjoint union decomposition (see \[46\] Theorems C and E))

$$\Gamma_A = P^A \cup C^A \cup Q^A,$$

where

- $P^A$ is the left part $P^{A(l)}$ in a decomposition $\Gamma_{A(l)} = P^{A(l)} \cup T^{A(l)} \cup Q^{A(l)}$ of the Auslander-Reiten quiver $\Gamma_{A(l)}$ of the left quasitilted algebra $A^{(l)}$ of $A$, with $T^{A(l)}$ a family of coray tubes separating $P^{A(l)}$ from $Q^{A(l)}$;
- $Q^A$ is the right part $Q^{A(r)}$ in a decomposition $\Gamma_{A(r)} = P^{A(r)} \cup T^{A(r)} \cup Q^{A(r)}$ of the Auslander-Reiten quiver $\Gamma_{A(r)}$ of the right quasitilted algebra $A^{(r)}$ of $A$, with $T^{A(r)}$ a family of ray tubes separating $P^{A(r)}$ from $Q^{A(r)}$;
- $C^A$ is a family of generalized multicolls separating $P^A$ from $Q^A$, obtained from stable tubes in the separating families $T^{C_1}, \ldots, T^{C_m}$ of stable tubes of the Auslander-Reiten quivers

\[5\]
\(\Gamma_{C_1, \ldots, C_m}\) of the concealed canonical algebras \(C_1, \ldots, C_m\) by a sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1')-(ad 5'), corresponding to the admissible operations leading from \(C = C_1 \times \ldots \times C_m\) to \(A\):

- \(C^A\) consists of cycle-finite modules and contains all indecomposable modules of \(T^{A(i)}\) and \(T^{A(r)}\);
- \(P^A\) contains all indecomposable modules of \(P^{A(r)}\);
- \(Q^A\) contains all indecomposable modules of \(Q^{A(i)}\).

Moreover, in the above notation, we have

- \(\text{gl. dim } A \leq 3\);
- \(\text{pd}_A X \leq 1\) for any indecomposable module \(X\) in \(P^A\);
- \(\text{id}_A Y \leq 1\) for any indecomposable module \(Y\) in \(Q^A\);
- \(\text{pd}_A M \leq 2\) and \(\text{id}_A M \leq 2\) for any indecomposable module \(M\) in \(C^A\).

A generalized multicoil algebra \(A\) is said to be \textit{tame} if \(A^{(l)}\) and \(A^{(r)}\) are products of tilted algebras of Euclidean types or tubular algebras. We also note that every tame generalized multicoil algebra is a cycle-finite algebra.

The following theorem is the first main result of the paper.

\textbf{Theorem 1.1.} Let \(A\) be an algebra and \(\Gamma\) be a cycle-finite infinite component of \(\Lambda\). Then there exist infinite full translation subquivers \(\Gamma_1, \ldots, \Gamma_r\) of \(\Gamma\) such that the following statements hold.

(i) For each \(i \in \{1, \ldots, r\}\), \(\Gamma_i\) is a cyclic coherent full translation subquiver of \(\Gamma\).
(ii) For each \(i \in \{1, \ldots, r\}\), \(\text{Supp}(\Gamma_i) = B(\Gamma_i)\) and is a generalized multicoil algebra.
(iii) \(\Gamma_1, \ldots, \Gamma_r\) are pairwise disjoint full translation subquivers of \(\Gamma\) and \(\Gamma^{cc} = \Gamma_1 \cup \ldots \cup \Gamma_r\) is a maximal cyclic coherent and cofinite full translation subquiver of \(\Gamma\).
(iv) \(B(\Gamma \setminus \Gamma^{cc})\) is of finite representation type.
(v) \(\text{Supp}(\Gamma) = B(\Gamma)\).

It follows from the above theorem that all but finitely many modules lying in an infinite cycle-finite component \(\Gamma\) of \(\Lambda\) can be obtained from indecomposable modules in stable tubes of concealed canonical algebras by a finite sequence of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1')-(ad 5') (see [60] Section 3) for details). We refer also to [34] and [70] for some results on the composition factors of indecomposable modules lying in stable tubes of the Auslander-Reiten quivers of concealed canonical algebras, and to [49] for the structure of indecomposable modules lying in coils. We would like to stress that the cycle-finiteness assumption imposed on the infinite component \(\Gamma\) of \(\Lambda\) is essential for the validity of the above theorem. Namely, it has been proved in [77], [79] that, for an arbitrary finite dimensional algebra \(B\) over a field \(K\), a module \(M\) in \(\text{mod } B\), and a positive integer \(r\), there exists a finite dimensional algebra \(A\) over \(K\) such that \(B\) is a quotient algebra of \(A\), \(\Gamma_A\) admits a faithful generalized standard stable tube \(T\) of rank \(r\), \(T\) is not cycle-finite, and \(M\) is a subfactor of all but finitely many indecomposable modules in \(T\). This shows that in general the problem of describing the support algebras of infinite cyclic components (even stable tubes) of Auslander-Reiten quivers is difficult.

In order to present the second main result of the paper, we need the class of generalized double tilted algebras introduced by Reiten and Skowroński in [50] (see also [2], [19] and [59]). A \textit{generalized double tilted algebra} is an algebra \(B\) for which \(\Gamma_B\) admits a separating almost acyclic component \(C\).

For a generalized double tilted algebra \(B\), the Auslander-Reiten quiver \(\Gamma_B\) has a disjoint union decomposition (see [60] Section 3)]

\[
\Gamma_B = P^B \cup C^B \cup Q^B,
\]

where

- \(C^B\) is an almost acyclic component separating \(P^B\) from \(Q^B\), called a \textit{connecting component} of \(\Gamma_B\);
There exist hereditary algebras $H_1^{(l)}, \ldots, H_n^{(l)}$ and tilting modules $T_1^{(l)} \in \text{mod } H_1^{(l)}, \ldots, T_m^{(l)} \in \text{mod } H_m^{(l)}$ such that the tilted algebras $B_1^{(l)} = \text{End}_{H_1^{(l)}}(T_1^{(l)}), \ldots, B_m^{(l)} = \text{End}_{H_m^{(l)}}(T_m^{(l)})$ are quotient algebras of $B$ and $\mathcal{P}_B$ is the disjoint union of all components of $\Gamma_{B_1^{(l)}}, \ldots, \Gamma_{B_m^{(l)}}$ contained entirely in the torsion-free parts $\mathcal{Y}(T_1^{(l)}), \ldots, \mathcal{Y}(T_m^{(l)})$ of mod $B_1^{(l)}, \ldots, \text{mod } B_m^{(l)}$ determined by $T_1^{(l)}, \ldots, T_m^{(l)}$.

There exist hereditary algebras $H_1^{(r)}, \ldots, H_n^{(r)}$ and tilting modules $T_1^{(r)} \in \text{mod } H_1^{(r)}, \ldots, T_n^{(r)} \in \text{mod } H_n^{(r)}$ such that the tilted algebras $B_1^{(r)} = \text{End}_{H_1^{(r)}}(T_1^{(r)}), \ldots, B_n^{(r)} = \text{End}_{H_n^{(r)}}(T_n^{(r)})$ are quotient algebras of $B$ and $\mathcal{Q}_B$ is the disjoint union of all components of $\Gamma_{B_1^{(r)}}, \ldots, \Gamma_{B_n^{(r)}}$ contained entirely in the torsion parts $\mathcal{X}(T_1^{(r)}), \ldots, \mathcal{X}(T_n^{(r)})$ of mod $B_1^{(r)}, \ldots, \text{mod } B_n^{(r)}$ determined by $T_1^{(r)}, \ldots, T_n^{(r)}$.

Every indecomposable module in $C^B$ not lying in the core $c(C^B)$ of $C^B$ is an indecomposable module over one of the tilted algebras $B_1^{(l)}, \ldots, B_m^{(l)}$, $B_1^{(r)}, \ldots, B_n^{(r)}$.

Every nondirecting indecomposable module in $C^B$ is cycle-finite and lies in $c(C^B)$.

$\text{id}_{\mathcal{P}_B} X \leq 1$ for all indecomposable modules $X$ in $\mathcal{P}_B$.

$\text{id}_{\mathcal{Q}_B} Y \leq 1$ for all indecomposable modules $Y$ in $\mathcal{Q}_B$.

For all but finitely many indecomposable modules $M$ in $C^B$, we have $\text{pd}_B M \leq 1$ or $\text{id}_B M \leq 1$.

Then $B^{(l)} = B_1^{(l)} \times \ldots \times B_m^{(l)}$ is called the left tilted algebra of $B$ and $B^{(r)} = B_1^{(r)} \times \ldots \times B_n^{(r)}$ is called the right tilted algebra of $B$. We note that the class of algebras of finite representation type coincides with the class of generalized double tilted algebras $B$ with $\Gamma_B$ being the connecting component $C^B$ (equivalently, with the tilted algebras $B^{(l)}$ and $B^{(r)}$ being of finite representation type (possibly empty)). Finally, a generalized double tilted algebra is said to be tame if the tilted algebras $B^{(l)}$ and $B^{(r)}$ are generically tame in the sense of Crawley-Boevey [22], [23]. We note that every tame generalized double tilted algebra is a cycle-finite algebra. We would like to mention that there exist generalized double tilted algebras of infinite representation type of arbitrary global dimension $d \in \mathbb{N} \cup \{\infty\}$. We refer also to [29], [40], [67] for useful characterizations of tilted algebras.

The following theorem is the second main result of the paper.

**Theorem 1.2.** Let $A$ be an algebra and $\Gamma$ be a cycle-finite finite component of $\Gamma_A$. Then the following statements hold.

(i) $\text{Supp}(\Gamma)$ is a generalized double tilted algebra.

(ii) $\Gamma$ is the core $c(C^{B(\Gamma)})$ of a unique almost acyclic connecting component $C^{B(\Gamma)}$ of $\Gamma_{B(\Gamma)}$.

(iii) $\text{Supp}(\Gamma) = B(\Gamma)$.

We would like to point that every finite cyclic component $\Gamma$ of an Auslander-Reiten quiver $\Gamma_A$ contains both a projective module and an injective module (see Corollary 2.6), and hence $\Gamma_A$ admits at most finitely many finite cyclic components. We refer also to [35], [83], [84] for some results concerning double tilted algebras with connecting components containing nondirecting indecomposable modules.

An idempotent $e$ of an algebra $A$ is said to be convex provided $e$ is a sum of pairwise orthogonal primitive idempotents of $A$ corresponding to the vertices of a convex valued subquiver of the quiver $Q_A$ of $A$ (see Section 2 for definition). The following direct consequence of Theorems 1.1 and 2.3 provides a handy description of the faithful algebra of a cycle-finite component of $\Gamma_A$.

**Corollary 1.3.** Let $A$ be an algebra and $\Gamma$ be a cycle-finite component of $\Gamma_A$. Then there exists a convex idempotent $e_\Gamma$ of $A$ such that $\text{Supp}(\Gamma)$ is isomorphic to the algebra $e_\Gamma A e_\Gamma$.

The third main result of the paper is a consequence of Theorems 1.1 and 1.2 and the results established in [47], Theorem 1.3.

**Theorem 1.4.** Let $A$ be an algebra. Then, for all but finitely many isomorphism classes of cycle-finite modules $M$ in $\text{ind } A$, the following statements hold.
The following fact is a direct consequence of the part (iii) of Theorem 1.5.

\[ \text{Corollary 1.6.} \]
Let \( A \) be an algebra. Then the number of isomorphism classes of modules \( X \) in \( \text{ind} A \) with \( [X] = [M] \) is bounded by \( m \).

(iii) \( \text{The number of isomorphism classes of cycle-finite modules } M \text{ in } \text{ind} A \text{ with } \text{Ext}^1_A(M, M) = 0 \text{ is finite.} \)

A module \( M \) in \( \text{mod} A \) with \( \text{Ext}^1_A(M, M) = 0 \) is frequently called \textit{rigid}. Following Adachi, Iyama and Reiten \cite{AdachiIyamaReiten}, a module \( M \) in a module category \( \text{mod} A \) is said to be \( \tau_A \)-rigid if \( \text{Hom}_A(M, \tau_A M) = 0 \). We note that every \( \tau_A \)-rigid module in \( \text{mod} A \) is rigid and all directing modules in \( \text{ind} A \) are \( \tau_A \)-rigid. The following fact is a direct consequence of the part (iii) of Theorem 1.5.

\[ \text{Corollary 1.6.} \]
Let \( A \) be an algebra. Then the number of isomorphism classes of cycle-finite \( \tau_A \)-rigid modules in \( \text{ind} A \) is finite.

Following Auslander and Reiten \cite{AuslanderReiten}, one associates with each nonprojective module \( X \) in a module category \( \text{ind} A \) the number \( \alpha(X) \) of indecomposable direct summands in the middle term
\[ 0 \to \tau_A X \to Y \to X \to 0 \]
of the almost split sequence with the right term \( X \). It has been proved by Bautista and Brenner \cite{BautistaBrenner} that, if \( A \) is an algebra of finite representation type and \( X \) a nonprojective module in \( \text{ind} A \), then \( \alpha(X) \leq 4 \), and if \( \alpha(X) = 4 \) then \( Y \) admits a projective-injective indecomposable direct summand \( P \), and hence \( X = P/\text{soc}(P) \). In \cite{BautistaBrenner} Liu proved that the same is true for any indecomposable nonprojective module \( X \) lying on an oriented cycle of the Auslander-Reiten quiver \( \Gamma_A \) of any algebra \( A \), and consequently for any nonprojective cycle-finite module in \( \text{ind} A \).
The following theorem is a direct consequence of Theorems [1.1] and [1.2] and [45, Corollary B], and provides more information on almost split sequences of cycle-finite modules.

**Theorem 1.7.** Let $A$ be an algebra. Then, for all but finitely many isomorphism classes of nonprojective cycle-finite modules $M$ in $\text{ind} A$, we have $\alpha(M) \leq 2$.

In connection to Theorem 1.7, we would like to mention that, for a cycle-finite algebra $A$ and a nonprojective module $M$ in $\text{ind} A$, we have $\alpha(M) \leq 5$, and if $\alpha(M) = 5$ then the middle term of the almost split sequence in $\text{mod} A$ with the right term $M$ admits a projective-injective indecomposable direct summand $P$, and hence $M = P/\text{soc}(P)$ (see [14, Conjecture 1], [44] and [56]).

The next theorem describe the structure of the module category $\text{ind} A$ of an arbitrary cycle-finite algebra $A$, and is a direct consequence of Theorems 1.1 and 1.2 as well as [43, Theorem 2.2] and its dual.

**Theorem 1.8.** Let $A$ be a cycle-finite algebra. Then there exist tame generalized multicoil algebras $B_1, \ldots, B_p$ and tame generalized double tilted algebras $B_{p+1}, \ldots, B_q$ which are quotient algebras of $A$ and the following statements hold.

1. $\text{ind} A = \bigcup_{i=1}^{p} \text{ind} B_i$.
2. All but finitely many isomorphism classes of modules in $\text{ind} A$ belong to $\bigcup_{i=1}^{p} \text{ind} B_i$.
3. All but finitely many isomorphism classes of nonprojective modules in $\text{ind} A$ belong to generalized multicoils of $\Gamma_{B_1}, \ldots, \Gamma_{B_p}$.

The next theorem extends the homological characterization of strongly simply connected algebras of polynomial growth established in [53] to arbitrary cycle-finite algebras, and is a direct consequence of Theorem [4.3] and the properties of directing modules described in [62, 2.4(8)].

**Theorem 1.9.** Let $A$ be a cycle-finite algebra. Then, for all but finitely many isomorphism classes of modules $M$ in $\text{ind} A$, we have $|\text{Ext}_{A}^{1}(M, M)| \leq |\text{End}_{A}(M)|$ and $\text{Ext}_{A}^{r}(M, M) = 0$ for $r \geq 2$.

Recently, the problem of describing all algebras $A$ having only finitely many isomorphism classes of $\tau_A$-rigid modules in $\text{ind} A$ was raised to be important. Namely, it has been proved by Iyama, Reiten, Thomas and Todorov (talk by Reiten at the conference in Nagoya) that the functorially finite torsion classes in a module category $\text{mod} A$ form a complete lattice if and only if there is only a finite number of pairwise nonisomorphic $\tau_A$-rigid modules in $\text{ind} A$. Recall that a torsion class in $\text{mod} A$ is a full subcategory $\mathcal{T}$ of $\text{mod} A$ which is closed under factor modules and extensions, and $\mathcal{T}$ is functorially finite if $\mathcal{T} = \text{Fac}(M)$ for a module $M$ in $\text{mod} A$, where $\text{Fac}(M)$ is the category of all factor modules of $M$. Furthermore, by a recent result of Iyama and Jasso (talk by Jasso at the conference in Nagoya), the finiteness of the isomorphism classes of $\tau_A$-rigid modules in a module category $\text{ind} A$ implies that the geometric realization of the simplicial complex of support $\tau_A$-tilting pairs is homeomorphic to the $(n - 1)$-dimensional sphere, where $n$ is the rank of the $K_0(A)$.

The following theorem provides a complete characterization of cycle-finite algebras $A$ having only finitely many isomorphism classes of $\tau_A$-rigid indecomposable modules.

**Theorem 1.10.** Let $A$ be a cycle-finite algebra. The following statements are equivalent.

1. The number of isomorphism classes of $\tau_A$-rigid modules in $\text{ind} A$ is finite.
2. The number of isomorphism classes of rigid modules in $\text{ind} A$ is finite.
3. The number of isomorphism classes of directing modules in $\text{ind} A$ is finite.
4. $A$ is of finite representation type.

Then we have the following consequence of Theorem 1.10 [1, Theorem 2.7] and a recent result by Iyama and Jasso.

**Corollary 1.11.** Let $A$ be a cycle-finite algebra. The following statements are equivalent.
Moreover, a set of pairwise orthogonal primitive idempotents of \( \tau_P(\Delta) \) is of infinite representation type, except \( \Delta = A_n \) with \( n \leq 4 \). Therefore, there are algebras of infinite representation type having only finitely many isomorphism classes of \( \tau \)-rigid indecomposable modules.

We end this section with some questions related to the results described above.

In [38, 39] Liu introduced the notions of left and right degrees of irreducible homomorphisms of modules and showed their importance for describing the shapes of the components of the Auslander-Reiten quivers of algebras. In particular, Liu pointed out in [38] that every cycle of irreducible homomorphisms between indecomposable modules in a module category \( \text{mod} A \) contains an irreducible homomorphism of finite left degree and an irreducible homomorphism of finite right degree. It would be interesting to describe the degrees of irreducible homomorphisms occurring in cycles of cycle-finite modules (see [15, 16, 17, 18] for some results in this direction).

In [52] de la Peña proved that the support algebra of a directing module over a tame algebra over an algebraically closed field is a tilted algebra being a gluing of at most two representation-infinite tilted algebras of Euclidean type. It would be interesting to know if the support algebra \( \text{Supp}(\Gamma) \) of a cycle-finite finite component \( \Gamma \) in the cyclic quiver \( \gamma \Gamma_A \) of a cycle-finite algebra is a gluing of at most two representation-infinite tilted algebras of Euclidean type. In general, it is not clear how many tilted algebras may occur in the decompositions of the left tilted algebra and the right tilted algebra of the support algebra \( \text{Supp}(\Gamma) \) of a cycle-finite component \( \Gamma \) of the cyclic quiver \( \gamma \Gamma_A \) of an algebra \( A \) (see Examples [71] and [72]).

2. Cyclic components

In this section we recall some concepts and describe some properties of cyclic components of the Auslander-Reiten quivers of algebras. Let \( A \) be an algebra (basic, indecomposable) and \( e_1, \ldots, e_n \) be a set of pairwise orthogonal primitive idempotents of \( A \) with \( 1_A = e_1 + \cdots + e_n \). Then

- \( P_i = e_i A, i \in \{1, \ldots, n\} \), is a complete set of pairwise nonisomorphic indecomposable projective modules in \( \text{mod} A \);
- \( I_i = D(Ae_i), i \in \{1, \ldots, n\} \), is a complete set of pairwise nonisomorphic indecomposable injective modules in \( \text{mod} A \);
- \( S_i = \text{top}(P_i) = e_i A/e_i \text{rad} A, i \in \{1, \ldots, n\} \), is a complete set of pairwise nonisomorphic simple modules in \( \text{mod} A \);
- \( S_i = \text{soc}(I_i) \), for any \( i \in \{1, \ldots, n\} \).

Moreover, \( F_i = \text{End}_A(S_i) \cong e_i A e_i / e_i(\text{rad} A) e_i \), for \( i \in \{1, \ldots, n\} \), are division algebras. The quiver \( Q_A \) of \( A \) is the valued quiver defined as follows:

- the vertices of \( Q_A \) are the indices \( 1, \ldots, n \) of the chosen set \( e_1, \ldots, e_n \) of primitive idempotents of \( A \);
- for two vertices \( i \) and \( j \) in \( Q_A \), there is an arrow \( i \to j \) from \( i \) to \( j \) in \( Q_A \) if and only if \( e_i(\text{rad} A) e_j / e_i(\text{rad} A)^2 e_j \neq 0 \). Moreover, one associates to an arrow \( i \to j \) in \( Q_A \) the valuation \((d_{ij}, d'_{ij})\), so we have in \( Q_A \) the valued arrow
  \[
  i \xrightarrow{d_{ij}, d'_{ij}} j,
  \]
  with the valuation numbers \( d_{ij} = \dim_{F_i} e_i(\text{rad} A) e_j / e_i(\text{rad} A)^2 e_j \) and \( d'_{ij} = \dim_{F_i} e_i(\text{rad} A) e_j / e_i(\text{rad} A)^2 e_j \).
It is known that $Q_A$ coincides with the Ext-quiver of $A$. Namely, $Q_A$ contains a valued arrow $i \xrightarrow{(d_{ij}, d'_{ij})} j$ if $\text{Ext}^1_A(S_i, S_j) \neq 0$ and $d_{ij} = \text{dim}_F \text{Ext}^1_A(S_i, S_j)$, $d'_{ij} = \text{dim}_F \text{Ext}^1_A(S_i, S_j)$. An algebra $A$ is called \textit{triangular} provided its quiver $Q_A$ is acyclic (there is no oriented cycle in $Q_A$). We shall identify an algebra $A$ with the associated category $A^*$ whose objects are the vertices $1, \ldots, n$ of $Q_A$, $\text{Hom}_A(i, j) = e_j A e_i$ for any objects $i$ and $j$ of $A^*$, and the composition of morphisms in $A^*$ is given by the multiplication in $A$. For a module $M$ in $\text{mod} A$, we denote by $\text{supp}(M)$ the full subcategory of $A = A^*$ given by all objects $i$ such that $Me_i \neq 0$, and call the \textit{support} of $M$. More generally, for a translation subquiver $C$ of $\Gamma_A$, we denote by $\text{supp}(C)$ the full subcategory of $A$ given by all objects $i$ such that $Xe_i \neq 0$ for some indecomposable module $X$ in $C$, and call it the \textit{support} of $C$. Then a module $M$ in $\text{mod} A$ (respectively, a family of components $C$ in $\Gamma_A$) is said to be \textit{sincere} if $\text{supp}(M) = A$ (respectively, if $\text{supp}(C) = A$). Finally, a full subcategory $B$ of $A$ is said to be a \textit{convex subcategory} of $A$ if every path in $Q_A$ with source and target in $B$ has all vertices in $B$. Observe that, for a convex subcategory $B$ of $A$, there is a fully faithful embedding of mod $B$ into mod $A$ such that mod $B$ is the full subcategory of mod $A$ consisting of the modules $M$ with $Me_i = 0$ for all objects $i$ of $A$ which are not objects of $B$.

An essential role in further considerations will be played by the following result proved in [15, Proposition 5.1].

\textbf{Proposition 2.1.} Let $A$ be an algebra and $X, Y$ be modules in $\text{ind} A$. Then $X$ and $Y$ belong to the same component of $\epsilon \Gamma_A$ if and only if there is an oriented cycle in $\Gamma_A$ passing through $X$ and $Y$.

We prove now the following property of cycle-finite cyclic components.

\textbf{Proposition 2.2.} Let $A$ be an algebra and $\Gamma$ be a cycle-finite component of $\epsilon \Gamma_A$. Then $\text{supp}(\Gamma)$ is a convex subcategory of $A$.

\textbf{Proof.} Let $C = \text{supp}(\Gamma)$. Assume to the contrary that $C$ is not a convex subcategory of $A$. Then $Q_A$ contains a path

$$i = i_0 \xrightarrow{(d_{i_0}, d'_{i_0})} i_1 \xrightarrow{(d_{i_1}, d'_{i_1})} i_2 \cdots \xrightarrow{(d_{i_{s-1}}, d'_{i_{s-1}})} i_s = j,$$

with $s \geq 2$, $i, j$ in $C$ and $i_1, \ldots, i_{s-1}$ not in $C$. Since $Q_A$ coincides with the Ext-quiver of $A$, we have $\text{Ext}^1_A(S_{i_{t-1}}, S_i) \neq 0$ for $t \in \{1, \ldots, s\}$. Then there exist in $\text{mod} A$ nonsplitable exact sequences

$$0 \rightarrow S_{i_{t-1}} \rightarrow L_t \rightarrow S_{i_t} \rightarrow 0,$$

for $t \in \{1, \ldots, s\}$. Clearly, $L_1, \ldots, L_s$ are indecomposable modules in $\text{mod} A$ of length 2. In particular, we obtain nonzero nonisomorphisms $f_r : L_r \rightarrow L_{r-1}$ with $\text{Im} f_r = S_{i_r-1}$, for $r \in \{2, \ldots, s\}$. Consider now the ideal $J$ in $A$ of the form

$$J = Ae_i (\text{rad} A)e_i (\text{rad} A) + (\text{rad} A)e_{i_{s-1}} (\text{rad} A)e_j A$$

and the quotient algebra $B = A/J$. Since $i_1$ and $i_{s-1}$ do not belong to $C = \text{supp}(\Gamma)$, for any module $M$ in $\Gamma$, we have $Me_{i_1} = 0$ and $Me_{i_{s-1}} = 0$, and consequently $MJ = 0$. This shows that $\Gamma$ is a cyclic component of $\Gamma_B$. Moreover, it follows from the definition of $J$ that $S_{i_1}$ is a direct summand of the radical $\text{rad} P^*_{i} e_i B$ of $S_{i_1}$ in $\text{mod} B$ and $S_{i_{s-1}}$ is a direct summand of the socle factor $I^*_j/S_j$ of the injective envelope $I^*_j = D(Be_j)$ of $S_j$ in $\text{mod} B$. Further, since $i$ and $j$ are in $C$, there exist indecomposable modules $X$ and $Y$ in $\Gamma$ such that $S_i$ is a composition factor of $X$ and $S_j$ is a composition factor of $Y$. Then we infer that $\text{Hom}_B(P^*_{i}, X) \neq 0$ and $\text{Hom}_B(Y, I^*_j) \neq 0$, because $\Gamma$ consists of $C$-modules, and hence $B$-modules. It follows from Proposition 2.1 that we have in $\Gamma$ a path from $X$ to $Y$. Therefore, we obtain in $\text{ind} A$ a cycle of the form

$$X \rightarrow \cdots \rightarrow Y \rightarrow I^*_j \rightarrow S_{i_{s-1}} \rightarrow L_{s-1} \rightarrow \cdots \rightarrow L_2 \rightarrow S_{i_1} \rightarrow P^*_i \rightarrow X,$$

which is an infinite cycle, because $X$ and $Y$ belong to $\Gamma$ but $S_{i_1}$ and $S_{i_{s-1}}$ are not in $\Gamma$. This contradicts the cycle-finiteness of $\Gamma$. Hence $C = \text{supp}(\Gamma)$ is indeed a convex subcategory of $A$. \qed
Proposition 2.3. Let $A$ be an algebra and $\Gamma$ be a cycle-finite component of $\mathcal{I} \Gamma_A$. Consider a decomposition $A = P_\Gamma \oplus Q_\Gamma$ of $A$ in $\text{mod} \ A$ such that the simple summands of $P_\Gamma/\text{rad}P_\Gamma$ are exactly the simple composition factors of the indecomposable modules in $\Gamma$. Then there exists an idempotent $e_\Gamma$ of $A$ such that $P_\Gamma = e_\Gamma A$, $Q_\Gamma = (1-e_\Gamma)A$, $t_A(\Gamma) = A(1-e_\Gamma)A$, and $e_\Gamma A e_\Gamma$ is isomorphic to the endomorphism algebra $\text{End}_A(P_\Gamma)$. In follows from Proposition 2.2 that $e_\Gamma$ is a convex idempotent of $A$. Observe also that $\text{End}_A(P_\Gamma)$ is the algebra of the support category $\text{supp}(\Gamma)$ of $\Gamma$. The next result gives another description of $\text{End}_A(P_\Gamma)$ in case the component $\Gamma$ of $\mathcal{I} \Gamma_A$ is cycle-finite.

**Proposition 2.3.** Let $A$ be an algebra and $\Gamma$ be a cycle-finite component of $\mathcal{I} \Gamma_A$. Consider a decomposition $A = P_\Gamma \oplus Q_\Gamma$ of $A$ in $\text{mod} \ A$ such that the simple summands of $P_\Gamma/\text{rad}P_\Gamma$ are exactly the simple composition factors of the indecomposable modules in $\Gamma$. Then the algebras $\text{Supp}(\Gamma)$ and $\text{End}_A(P_\Gamma)$ are isomorphic.

**Proof.** Observe that the support algebra $\text{Supp}(\Gamma) = A/t_A(\Gamma)$ is isomorphic to the endomorphism algebra $\text{End}_A(P_\Gamma/t_A(\Gamma))$. Moreover, $P_\Gamma t_A(\Gamma)$ is the right $A$-submodule of $P_\Gamma$ generated by the images of all homomorphisms from $Q_\Gamma$ to $P_\Gamma$ in $\text{mod} \ A$. For any homomorphism $f \in \text{End}_A(P_\Gamma)$ we have the canonical commutative diagram in $\text{mod} \ A$ of the form

\[
\begin{array}{cccccc}
0 & \rightarrow & P_\Gamma t_A(\Gamma) & \rightarrow & P_\Gamma & \rightarrow & P_\Gamma/t_A(\Gamma) & \rightarrow & 0 \\
& f' & \downarrow f & & & \downarrow f & & \downarrow f & \\
0 & \rightarrow & P_\Gamma t_A(\Gamma) & \rightarrow & P_\Gamma & \rightarrow & P_\Gamma/t_A(\Gamma) & \rightarrow & 0,
\end{array}
\]

where $f'$ is the restriction of $f$ to $P_\Gamma t_A(\Gamma)$ and $f$ is induced by $f$. Clearly, by the projectivity of $P_\Gamma$ in $\text{mod} \ A$, every homomorphism $g \in \text{End}_A(P_\Gamma/t_A(\Gamma))$ is of the form $f$ for some homomorphism $f \in \text{End}_A(P_\Gamma)$. This shows that the assignment $f \rightarrow f$ induces an epimorphism $\text{End}_A(P_\Gamma) \rightarrow \text{End}_A(P_\Gamma/t_A(\Gamma))$ of algebras. Assume now that $f = 0$ for a homomorphism $f \in \text{End}_A(P_\Gamma)$. Then $\text{Im} f \subseteq P_\Gamma t_A(\Gamma)$. On the other hand, it follows from the definition of $t_A(\Gamma)$ that there is an epimorphism $v : Q_\Gamma^m \rightarrow P_\Gamma t_A(\Gamma)$ in $\text{mod} \ A$ for some positive integer $m$. Using the projectivity of $P_\Gamma$ in $\text{mod} \ A$, we conclude that there is a homomorphism $u : P_\Gamma \rightarrow Q_\Gamma^m$ such that $f = vu$. But $f \neq 0$ implies that $u \neq 0$ and $v \neq 0$, and then a contradiction with the convexity of $\text{Supp}(\Gamma)$ in $A = A^*$ established in Proposition 2.2. Hence $f = 0$. Therefore, the canonical epimorphism of algebras $\text{End}_A(P_\Gamma) \rightarrow \text{End}_A(P_\Gamma/t_A(\Gamma))$ is an isomorphism, and so the algebras $\text{End}_A(P_\Gamma)$ and $\text{Supp}(\Gamma)$ are isomorphic.

The following fact proved by Bautista and Smalø in [12] (see also [82, Corollary III. 11.3]) will be essential for our considerations.

**Proposition 2.4.** Let $A$ be an algebra and

\[ X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{r-1} \rightarrow X_r = X \]

a cycle in $\Gamma_A$. Then there exists $i \in \{2, \ldots, r\}$ such that $\tau_A X_i \cong X_{i-2}$.

**Lemma 2.5.** Let $A$ be an algebra and $\Gamma$ be a cyclic component of $\Gamma_A$. Assume that

\[ X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{r-1} \rightarrow X_r = X \]

is a cycle in $\Gamma$. Then the following statements hold.

(i) If all modules $X_i$, $i \in \{1, \ldots, r\}$, are nonprojective, then $\Gamma$ contains a cycle of the form

\[ \tau_A X = \tau_A X_0 \rightarrow \tau_A X_1 \rightarrow \cdots \rightarrow \tau_A X_{r-1} \rightarrow \tau_A X_r = \tau_A X. \]

(ii) If all modules $X_i$, $i \in \{1, \ldots, r\}$, are noninjective, then $\Gamma$ contains a cycle of the form

\[ \tau_A^{-1} X = \tau_A^{-1} X_0 \rightarrow \tau_A^{-1} X_1 \rightarrow \cdots \rightarrow \tau_A^{-1} X_{r-1} \rightarrow \tau_A^{-1} X_r = \tau_A^{-1} X. \]
Proof. It follows from Proposition 2.4 that there exists \( i \in \{2, \ldots, r\} \) such that \( \tau_A X_i = X_{i-2} \), or equivalently, \( X_i = \tau_A^{-1} X_{i-2} \). Hence, if all modules \( X_i, i \in \{1, \ldots, r\} \), are nonprojective, then we have in \( \Gamma_A \) a cycle
\[
\tau_A X_i = \tau_A X_0 \to \tau_A X_1 \to \cdots \to \tau_A X_{i-1} \to \tau_A X_i \to \cdots \to \tau_A X_{r-1} \to \tau_A X_r = \tau_A X_i
\]
with \( \tau_A X_i = X_{i-2} \), and hence all modules of this cycle belong to the cyclic component \( \Gamma \) containing \( X_{i-2} \). Similarly, if all modules \( X_i, i \in \{1, \ldots, r\} \), are noninjective, then we have in \( \Gamma_A \) a cycle
\[
\tau_A^{-1} X_i = \tau_A^{-1} X_0 \to \tau_A^{-1} X_1 \to \cdots \to \tau_A^{-1} X_{i-1} \to \tau_A^{-1} X_i \to \cdots \to \tau_A^{-1} X_{r-1} \to \tau_A^{-1} X_r = \tau_A^{-1} X_i
\]
with \( X_i = \tau_A^{-1} X_{i-2} \), and hence all modules of this cycle belong to the cyclic component \( \Gamma \) containing \( X_i \).
\hfill \Box

Corollary 2.6. Let \( A \) be an algebra and \( \Gamma \) a finite cyclic component of \( \Gamma_A \). Then \( \Gamma \) contains a projective and an injective module.

Proof. Assume \( \Gamma \) does not contain a projective module. Then it follows from Lemma 2.5 that, for any indecomposable module \( X \) in \( \Gamma \), \( \tau_A X \) is also a module in \( \Gamma \). Since \( \Gamma \) is a finite translation quiver, this implies that \( \Gamma = \tau_A \Gamma \), and hence \( \Gamma \) is a component of \( \Gamma_A \). Then there exists an indecomposable algebra \( B \) (a block of \( A \)) such that \( \Gamma \) is a component of \( \Gamma_B \), and consequently \( \Gamma = \Gamma_B \), by the well known theorem of Auslander (see [82, Theorem III. 10.2]). But this is a contradiction, because \( \Gamma_B \) contains projective modules. Therefore, \( \Gamma \) contains a projective module. The proof that \( \Gamma \) contains an injective module is similar.
\hfill \Box

Let \( A \) be an algebra and \( C \) a component of \( \Gamma_A \). We denote by \( \mathcal{C} \) the left stable part of \( C \) obtained by removing from \( C \) the \( \tau_A \)-orbits of projective modules and the arrows attached to them, and by \( \mathcal{C} \) the right stable part of \( C \) obtained by removing in \( C \) the \( \tau_A \)-orbits of injective modules and the arrows attached to them. We note that, if \( C \) is infinite, then \( \mathcal{C} \) or \( \mathcal{C} \) is nonempty.

The following proposition will be applied in the proofs of our main theorems.

Proposition 2.7. Let \( A \) be an algebra, \( C \) a component of \( \Gamma_A \), and \( \Sigma \) an infinite family of cycle-finite modules in \( C \). Then one of the following statements hold.

(i) The stable part \( \mathcal{C} \) of \( C \) contains a stable tube \( D \) having infinitely many modules from \( \Sigma \).

(ii) The left stable part \( \mathcal{C} \) of \( C \) contains a component \( D \) with an oriented cycle and an injective module such that the cyclic part \( \mathcal{D} \) of \( D \) contains infinitely many modules from \( \Sigma \).

(iii) The right stable part \( \mathcal{C} \) of \( C \) contains a component \( D \) with an oriented cycle and a projective module such that the cyclic part \( \mathcal{D} \) of \( D \) contains infinitely many modules from \( \Sigma \).

Proof. (1) Assume first that there is a \( \tau_A \)-orbit \( O \) in \( C \) containing infinitely many modules from \( \Sigma \). Consider the case when \( O \) contains infinitely many left stable modules from \( \Sigma \). Then there exist a module \( M \) in \( O \cap \Sigma \) and an infinite sequence \( 0 = r_0 < r_1 < r_2 < \ldots \) of integers such that the modules \( \tau_A^{r_i} M, i \in \mathbb{N} \), belong to \( O \cap \Sigma \). Let \( D \) be the component of \( \mathcal{C} \) containing the modules \( \tau_A^{r_i} M, i \in \mathbb{N} \). We have two cases to consider.

Assume \( D \) contains an oriented cycle. Observe that \( D \) is not a stable tube, and hence does not contain a \( \tau_A \)-periodic module, because \( D \) contains infinitely many modules from the \( \tau_A \)-orbit \( O \). Hence, applying [39, Lemma 2.2 and Theorem 2.3], we conclude that \( D \) contains an infinite sectional path
\[
\cdots \to \tau_A^{t} X_s \to \cdots \to \tau_A^{t} X_2 \to \tau_A^{t} X_1 \to X_s \to \cdots \to X_2 \to X_1,
\]
where \( t > s \geq 1 \), \( X_i \) is an injective module for some \( i \in \{1, \ldots, s\} \), and each module in \( D \) belongs to the \( \tau_A \)-orbit of one of the modules \( X_i \). Clearly, then there is a nonnegative integer \( m \) such that all modules \( \tau_A^{r_i} M, r \geq m \), belong to the cyclic part \( \mathcal{D} \) of \( D \). Therefore, the statement (ii) holds.

Assume \( D \) is acyclic. Then it follows from [39, Theorem 3.4] that there is an acyclic locally finite valued quiver \( \Delta \) such that \( D \) is isomorphic to a full translation subquiver of \( \mathbb{Z}\Delta \), which is closed
under predecessors. But then there exists a positive integer $i$ such that $\tau_{A}^i M$ is not a successor of a projective module in $C$, and consequently does not lie on an oriented cycle in $C$. On the other hand, $\tau_{A}^i M$ belongs to $\Sigma$, and then is a cycle-finite indecomposable module, so lying on a cycle in $C$, a contradiction.

Similarly, if $O$ contains infinitely many right stable modules from $\Sigma$, then the statement (iii) holds.

(2) Assume now that every $\tau_A$-orbit in $C$ contains at most finitely many modules from $\Sigma$. Since $\Sigma$ is an infinite family of modules, we infer that there is an infinite component $D$ of the stable part $sC$ of $C$ containing infinitely many modules from $\Sigma$. We have two cases to consider.

Assume $D$ contains an oriented cycle. Then it follows from [35 Corollary] (see also [38 Theorems 2.5 and 2.7]) that $D$ is a stable tube. Thus the statement (i) holds.

Assume $D$ is acyclic. Applying [35 Corollary] again, we conclude that there exists an infinite locally finite acyclic valued quiver $\Delta$ such that $D$ is isomorphic to the translation quiver $\mathbb{Z}\Delta$. Let $n$ be the rank of the Grothendieck group $K_0(A)$ of $A$. Then there is a module $M$ in $D \cap \Sigma$ such that the length of any walk in $C$ from a nonstable module in $C$ to a module in the $\tau_A$-orbit $O(M)$ of $M$ is at least $2n$. Then it follows from [21, Lemma 1.5] (see also [69, Lemma 4]) that, for each positive integer $s$, there exists a path

$$M = X_0 \to X_1 \to \cdots \to X_t = \tau_A^s M$$

in $\text{ind } A$ with all modules $X_i$ in $C$, and consequently a cycle in $\text{ind } A$ passing through $M$ and $\tau_A^s M$, because there is a path

$$\tau_A^s M = Y_0 \to Y_1 \to \cdots \to Y_r = M$$

of irreducible homomorphisms in $\text{ind } A$. Moreover, $M$ is a cycle-finite module, as a module from $\Sigma$. This shows that $C$ contains oriented cycles passing through $M$ and any module $\tau_A^s M$, $s \geq 1$. We also note that there is a component $D'$ of the left stable part $lC$ of $C$ containing all $\tau_A$-orbits of $D$. Then there is an infinite locally finite acyclic valued subquiver $\Delta'$ containing $\Delta$ as a full valued subquiver, such that $D'$ is isomorphic to a full translation subquiver of $\mathbb{Z}\Delta'$, which is closed under predecessors. Then there exists a positive integer $m$ such that the module $\tau_A^m M$ is not a successor of a projective module in $C$, and then $\tau_A^m M$ does not lie on an oriented cycle in $C$, a contradiction. □

**Corollary 2.8.** Let $A$ be an algebra and $\Gamma$ be a cycle-finite infinite component of $\tau_G A$. Then $\tau G$ or $r \Gamma$ admits a component $D$ containing an oriented cycle and infinitely many modules of $\Gamma$.

### 3. Proof of Theorem 1.1

Let $A$ be an algebra and $\Gamma$ be a cycle-finite infinite component of $\tau_G A$. Consider the component $C$ of $\tau_G A$ containing the translation quiver $\Gamma$. Since $\Gamma$ is infinite and cyclic, we conclude from Corollary 2.8 that $lC$ or $rC$ contains a connected component $\Sigma$ containing an oriented cycle and infinitely many modules of $\Gamma$. We claim that there exists a cyclic coherent full translation subquiver $\Omega$ of $\Gamma$ containing all modules of the cyclic part $\tau_C$ of $\Sigma$. We have three cases to consider:

1. Assume $\Sigma$ is contained in the stable part $sC = lC \cap rC$ of $C$. Then $\Sigma$ is an infinite stable translation quiver containing an oriented cycle, and hence $\Sigma$ is a stable tube, by the main result of [35]. Clearly, the stable tube $\Sigma$ is a cyclic and coherent translation quiver. Since $\Sigma$ is a component of $lC$ and a component of $rC$, we conclude that $\Gamma$ contains a cyclic coherent full translation subquiver $\Omega$ such that $\Sigma$ is obtained from $\Omega$ by removing all finite $\tau_A$-orbits without $\tau_A$-periodic modules.

2. Assume $\Sigma$ is a component of $lC$ containing at least one injective module. Then it follows from [39 Lemma 2.2 and Theorem 2.3] that $\Sigma$ contains an infinite sectional path

$$\cdots \to \tau_A^r X_s \to \cdots \to \tau_A^s X_2 \to \tau_A^s X_1 \to X_s \to \cdots \to X_2 \to X_1,$$

where $r > s \geq 1$, $X_i$ is an injective module for some $i \in \{1, \ldots, s\}$, and each module in $\Sigma$ belongs to the $\tau_A$-orbit of one of the modules $X_i$. Observe that there exists an infinite
sectional path in \( \Sigma \)
\[
X_s \to \tau_A^{-r-1}X_1 \to \cdots
\]
starting from \( X_s \). Let \( p \) be the minimal element in \( \{1, \ldots, s\} \) such that there exists an infinite sectional path in \( \Sigma \) starting from \( X_p \). Then \( \Gamma \) contains a cyclic coherent full translation subquiver \( \Omega \) such that \( \Sigma \) is obtained from \( \Omega \) by removing the \( \tau_A \)-orbits of projective modules \( \mathcal{P} \) lying on infinite sectional paths in \( \Sigma \) of the forms
\[
P \to \cdots \to X_j \to \cdots \to \tau_A^{r-j+p-1}X_1 \to \cdots
\]
for some \( j \in \{p, \ldots, s\} \), or
\[
P \to \cdots \to \tau_A^{mr}X_1 \to \tau_A^{mr-1}X_{i+1} \to \cdots
\]
for some \( m \geq 1 \) and \( i \in \{1, \ldots, s\} \).

(3) Assume \( \Sigma \) is a component of \( \mathcal{C} \) containing at least one projective module. Then it follows from \cite{duals} duals of Lemma 2.2 and Theorem 2.3 that \( \Sigma \) contains an infinite sectional path
\[
X_1 \to X_2 \to \cdots \to X_i \to \tau_A^{-m}X_1 \to \tau_A^{-m}X_2 \to \cdots \to \tau_A^{-m}X_i \to \cdots,
\]
where \( m > t \geq 1 \), \( X_j \) is an injective module for some \( j \in \{1, \ldots, t\} \), and each module in \( \Sigma \) belongs to the \( \tau_A \)-orbit of one of the modules \( X_j \). Observe that there exists an infinite sectional path in \( \Sigma \)
\[
\cdots \to \tau_A^{-m+1}X_1 \to X_t
\]
ending in \( X_t \). Let \( q \) be the minimal element in \( \{1, \ldots, t\} \) such that there exists an infinite sectional path in \( \Sigma \) ending in \( X_q \). Then \( \Gamma \) contains a cyclic coherent full translation subquiver \( \Omega \) such that \( \Sigma \) is obtained from \( \Omega \) by removing the \( \tau_A \)-orbits of injective modules \( \mathcal{I} \) lying on infinite sectional paths in \( \Sigma \) of the forms
\[
\cdots \to \tau_A^{-m+t-1}X_1 \to \cdots \to X_i \to \cdots \to I
\]
for some \( i \in \{q, \ldots, t\} \), or
\[
\cdots \to \tau_A^{-ms+1}X_{j+1} \to \tau_A^{-ms}X_j \to \cdots \to I
\]
for some \( s \geq 1 \) and \( j \in \{1, \ldots, t\} \).

Let \( \Gamma_1, \ldots, \Gamma_t \) be all maximal cyclic coherent pairwise different full translation subquivers of \( \Gamma \). Clearly, \( \Gamma_1, \ldots, \Gamma_t \) are pairwise disjoint. For each \( i \in \{1, \ldots, t\} \), consider the support algebra \( B^{(i)} = \text{Supp}(\Gamma_i) \) of \( \Gamma_i \).

Fix \( i \in \{1, \ldots, t\} \). We shall prove that \( B^{(i)} \) is a generalized multicoil algebra and \( \Gamma_i \) is the cyclic part of a generalized multicoil \( \Gamma_i^{\bullet} \) of \( B^{(i)} \), and consequently \( \Gamma_i \) is a cyclic generalized multicoil full translation subquiver of \( \Gamma_{B^{(i)}} \). Since \( \Gamma_i \) is a cyclic coherent full translation subquiver of the component \( C \) of \( \Gamma_A \) and of \( \Gamma \), it follows from the proofs of Theorems A and F in \cite{duals} that \( \Gamma_i \), considered as a translation quiver, is a generalized multicoil, and consequently can be obtained from a finite family \( T_1^{(i)}, \ldots, T_{p_i}^{(i)} \) of stable tubes by an iterated application of admissible operations of types (ad 1)-(ad 5) and their duals (ad 1*)-(ad 5*). We note that all vertices of the stable tubes \( T_1^{(i)}, \ldots, T_{p_i}^{(i)} \) are indecomposable modules of \( \Gamma \), and the stable tubes \( T_1^{(i)}, \ldots, T_{p_i}^{(i)} \) can be obtained from \( \Gamma \) by removing the modules of \( \Gamma \setminus (T_1^{(i)} \cup \cdots \cup T_{p_i}^{(i)}) \) and shrinking the corresponding sectional paths in \( \Gamma \) with the ends at the modules in \( T_1^{(i)} \cup \cdots \cup T_{p_i}^{(i)} \) into the arrows. We claim now that \( \Gamma_i \) is a generalized standard full translation subquiver of \( \Gamma_A \). Suppose that \( \text{rad}^\infty(X, Y) \neq 0 \) for some indecomposable \( A \)-modules \( X \) and \( Y \) lying in \( \Gamma_i \). Then, applying Proposition \ref{finitecycles}, we conclude that there is in \( \text{ind} A \) an infinite cycle
\[
X \xrightarrow{f} Y \xrightarrow{f_1} Z_1 \xrightarrow{f_2} Z_2 \cdots \xrightarrow{f_{t-1}} Z_t \xrightarrow{f_t} X
\]
where $Z_1, \ldots, Z_t = X, Y$ are modules in $\Gamma_i$, $f_1, \ldots, f_t$ are irreducible homomorphisms and $0 \neq f \in \text{rad}^\infty(X, Y)$, a contradiction with the cycle-finiteness of $\Gamma$. Similarly, there is no path in $\text{ind } B^{(i)}$ of the form

$$X \xrightarrow{g} Z \xrightarrow{h} Y$$

with $X$ and $Y$ in $\Gamma_i$ and $Z$ not in $\Gamma_i$ (external short path of $\Gamma_i$ in $\text{ind } B$ in the sense of [58]). Since $\Gamma_i$ is a sincere cyclic coherent full translation subquiver of $\Gamma_{B^{(i)}}$, applying [46, Theorem A] (and its proof), we conclude that $B^{(i)}$ is a generalized multicoil algebra, $\Gamma_i$ is the cyclic part of a generalized multicoil $\Gamma_i^*$ of $\Gamma_{B^{(i)}}$, and $\text{ann } B^{(i)}(\Gamma_i) = \text{ann } B^{(i)}(\Gamma_i^*) = 0$, and hence $B^{(i)} = B(\Gamma_i) = B(\Gamma_i^*)$. For each $j \in \{1, \ldots, p_l\}$, consider the quotient algebra $C_j = A/\text{ann}_A(T_j^{(i)})$ of $A$ by the annihilator $\text{ann}_A(T_j^{(i)})$ of the family of indecomposable $A$-modules forming $T_j^{(i)}$. Then $C_j^{(i)}$ is a concealed canonical algebra and $T_j^{(i)}$ is a stable tube of $\Gamma_{C_j^{(i)}}$. We note that we may have $C_j = C_k^{(i)}$ for $j \neq k$ in $\{1, \ldots, p_l\}$. Then denoting by $C^{(i)}$ the product of pairwise different algebras in the family $C^{(i)}_1, \ldots, C^{(i)}_{p_l}$, with respect to the annihilators $\text{ann}_A(T_1^{(i)}), \ldots, \text{ann}_A(T_{p_l}^{(i)})$ of $T_1^{(i)}, \ldots, T_{p_l}^{(i)}$, we obtain that $B^{(i)}$ is a generalized multicoil enlargement of $C^{(i)}$ involving the stable tubes $T_1^{(i)}, \ldots, T_{p_l}^{(i)}$ and admissible operations of types $(a)$-$\text{ad } 5)$ corresponding to the translation quiver operations leading from the stable tubes $T_1^{(i)}, \ldots, T_{p_l}^{(i)}$ to the generalized multicoil $\Gamma_i^*$. Further, by [46, Theorem C], we have the following additional properties of $B^{(i)}$:

1. There is a unique factor algebra (not necessarily connected) $B_l^{(i)}$ of $B^{(i)}$ (the left part of $B^{(i)}$) obtained from $C^{(i)}$ by an iteration of admissible operations of type $(\text{ad } 1^*)$ and a family $\tilde{T}_1^{(i)}, \ldots, \tilde{T}_{p_l}^{(i)}$ of coray tubes in $\Gamma_{B_l^{(i)}}$, obtained from the stable tubes $T_1^{(i)}, \ldots, T_{p_l}^{(i)}$ by the corresponding coray insertions, such that $B^{(i)}$ is obtained from $B_l^{(i)}$ by an iteration of admissible operations of types $(\text{ad } 1^*)$-$\text{ad } 5)$ and $\Gamma_i^*$ is obtained from the family $\tilde{T}_1^{(i)}, \ldots, \tilde{T}_{p_l}^{(i)}$ by an iteration of admissible operations of types $(\text{ad } 1^*)$-$\text{ad } 5)$ corresponding to those leading from $B_l^{(i)}$ to $B^{(i)}$.

2. There is a unique factor algebra (not necessarily connected) $B_r^{(i)}$ of $B^{(i)}$ (the right part of $B^{(i)}$) obtained from $C^{(i)}$ by an iteration of admissible operations of type $(\text{ad } 1)$ and a family $\tilde{T}_1^{(i)}, \ldots, \tilde{T}_{p_l}^{(i)}$ of ray tubes in $\Gamma_{B_r^{(i)}}$, obtained from the stable tubes $T_1^{(i)}, \ldots, T_{p_l}^{(i)}$ by the corresponding ray insertions, such that $B^{(i)}$ is obtained from $B_r^{(i)}$ by an iteration of admissible operations of types $(\text{ad } 1^*)$-$\text{ad } 5)$ and $\Gamma_i^*$ is obtained from the family $\tilde{T}_1^{(i)}, \ldots, \tilde{T}_{p_l}^{(i)}$ by an iteration of admissible operations of types $(\text{ad } 1^*)$-$\text{ad } 5)$ corresponding to those leading from $B_r^{(i)}$ to $B^{(i)}$.

As a consequence, the generalized multicoil $\Gamma_i^*$ of $\Gamma_{B^{(i)}}$ admits a left border $\Delta_l^{(i)}$ and a right border $\Delta_r^{(i)}$ having the following properties:

(a) $\Delta_l^{(i)}$ and $\Delta_r^{(i)}$ are disjoint and unions of finite sectional paths of $\Gamma_i$;

(b) $\Gamma_i$ is the full translation subquiver of $\Gamma_i^*$ consisting of all modules which are both successors of modules lying in $\Delta_l^{(i)}$ and predecessors of modules lying in $\Delta_r^{(i)}$;

(c) $\Gamma_i^* \setminus \Gamma_i$ consists of a finite number of directing $B^{(i)}$-modules;

(d) Every module in $\Gamma \setminus \Gamma_i$ which is a predecessor of a module in $\Gamma_i$ is a predecessor of a module in $\Delta_l^{(i)}$;

(e) Every module in $\Gamma \setminus \Gamma_i$ which is a successor of a module in $\Gamma_i$ is a successor of a module in $\Delta_r^{(i)}$;

(f) $B(\Delta_l^{(i)}) = \text{Supp}(\Delta_l^{(i)})$ and is a product of tilted algebras of equioriented Dynkin types $\Delta_n$ and $\Delta_l^{(i)}$ is the union of sections of the connecting components of the indecomposable parts of $B(\Delta_l^{(i)})$;

(g) $B(\Delta_r^{(i)}) = \text{Supp}(\Delta_r^{(i)})$ and is a product of tilted algebras of equioriented Dynkin types $\Delta_n$ and $\Delta_r^{(i)}$ is the union of sections of the connecting components of the indecomposable parts of $B(\Delta_r^{(i)})$. 
We denote by $\Gamma^{cc}$ the union of the translation subquivers $\Gamma_1, \ldots, \Gamma_t$. We claim that $\Gamma \setminus \Gamma^{cc}$ consists of finitely modules and $\Gamma^{cc}$ is a maximal cyclic coherent full translation subquiver of $\Gamma$. Suppose that infinitely many modules of $\Gamma$ are not contained in $\Gamma^{cc}$. We have the following properties of modules in $\Gamma \setminus \Gamma^{cc}$. Since $\Gamma$ is a connected component of $c\Gamma_A$, by Proposition 2.1 for any modules $M$ in $\Gamma \setminus \Gamma^{cc}$ and $N$ in $\Gamma^{cc}$, there is an oriented cycle in $\Gamma$ passing through $M$ and $N$. Moreover, if $N$ belongs to $\Gamma_t$, then every such a cycle is of the form

$$M \to \cdots \to X \to \cdots \to N \to \cdots \to Y \to \cdots \to M$$

with $X$ in $\Delta^{(i)}$ and $Y$ in $\Delta^{(i)}$. Applying Proposition 2.7 to the infinite family $\Sigma = \Gamma \setminus \Gamma^{cc}$ of cycle-finite modules, we obtain that the left stable part $c\mathcal{C}$ or the right stable part $\mathcal{C}$ of $\Gamma$ admits an infinite component $\Sigma'$ containing an oriented cycle and infinitely many modules from $\Gamma \setminus \Gamma^{cc}$. Then, as in the first part of the proof, we infer that there exists a cyclic coherent full translation subquiver $\Omega'$ of $\Gamma$ containing all modules of $\Sigma'$. Obviously, $\Omega'$ is disjoint with $\Gamma_1, \ldots, \Gamma_t$, and this contradicts to our choice of $\Gamma_1, \ldots, \Gamma_t$. Therefore, indeed, $\Gamma \setminus \Gamma^{cc}$ consists of finitely many modules.

Our next aim is to show that the algebra $B(\Gamma \setminus \Gamma^{cc}) = A/\text{ann}_A(\Gamma \setminus \Gamma^{cc})$ is of finite representation type. We abbreviate $D = B(\Gamma \setminus \Gamma^{cc})$. Observe that, if every indecomposable module from mod $D$ lies in $\Gamma \setminus \Gamma^{cc}$, then $D$ is of finite representation type. Therefore, assume that mod $D$ admits an indecomposable module $Z$ which is not in $\Gamma \setminus \Gamma^{cc}$. Let $M$ be the direct sum of all indecomposable $A$-modules lying in $\Gamma \setminus \Gamma^{cc}$. Moreover, let $D = P' \oplus P''$ be a decomposition of $D$ in mod $D$, where $P'$ is the direct sum of all indecomposable projective $D$-modules lying in $\Gamma \setminus \Gamma^{cc}$ and $P''$ is the direct sum of the remaining indecomposable projective $D$-modules. Observe that $M$ is a faithful module in mod $D$ and hence we have a monomorphism of right $D$-modules $P'' \to M$, which then factors through a direct sum of modules lying on the sum $\Delta^{(i)} \cup \cdots \cup \Delta^{(i)}$ of the right parts $\Delta^{(i)} \cup \cdots \cup \Delta^{(i)}$ of $\Gamma_1, \ldots, \Gamma_t$, and consequently $P''$ is a module over the algebra $B(\Lambda^{(1)} \times \cdots \times B(\Lambda^{(t)}))$. Consider also a projective cover $\pi : P_D(Z) \to Z$ of $Z$ in mod $D$. Let $P_D(Z) = P_D^l(Z) \oplus P_D^r(Z)$, where $P_D^l(Z)$ is a direct sum of direct summands of $P'$ and $P_D^r(Z)$ is a direct sum of direct summands of $P''$, and denote by $\pi' : P_D^l(Z) \to Z$ and $\pi'' : P_D^r(Z) \to Z$ the restrictions of $\pi$ to $P_D^l(Z)$ and $P_D^r(Z)$, respectively. Then $\pi' : P_D^l(Z) \to Z$ factors through a direct sum of modules lying on the sum $\Delta^{(i)} \cup \cdots \cup \Delta^{(i)}$ of the left parts $\Delta^{(i)} \cup \cdots \Delta^{(i)}$ of $\Gamma_1, \ldots, \Gamma_t$, because $Z$ does not belong to $\Gamma \setminus \Gamma^{cc}$. In particular, we obtain that $\pi'(P_D^l(Z))$ is a module over the algebra $B(\Lambda^{(1)} \times \cdots \times B(\Lambda^{(t)}))$. Summing up, we conclude that $Z = \pi'(P_D^l(Z)) + \pi''(P_D^r(Z))$ is a module over the quotient algebra

$$A = B(\Lambda^{(1)} \times \cdots \times B(\Lambda^{(t)})) \times B(\Lambda^{(1)} \times \cdots \times B(\Lambda^{(t)}),$$

of $A$, which is an algebra of finite representation type as a product of tilted algebras of Dynkin types $\Lambda_n$. Therefore, we obtain that every module from ind $D$ which is not in $\Gamma \setminus \Gamma^{cc}$ is an indecomposable module in ind $\Lambda$. Since $\Gamma \setminus \Gamma^{cc}$ is finite, we conclude that $D$ is of finite representation type.

Finally, let $B = \text{Supp}(\Gamma) = A/t_A(\Gamma)$. Then $\Gamma$ is a sincere cycle-finite component of $c\Gamma_B$ and $\text{ann}_B(\Gamma) = \text{ann}_A(\Gamma)/t_A(\Gamma)$. Hence, in order to show that $\text{Supp}(\Gamma) = B(\Gamma)$, it is enough to prove that $\Gamma$ is a faithful translation subquiver of $\Gamma_B$. Let $B = P \oplus Q$ be a decomposition in mod $B$ such that $Q$ is the direct sum of all indecomposable projective modules lying in $\Gamma$ and $P$ the direct sum of the remaining indecomposable projective right $B$-modules. Then $P$ is a direct sum of indecomposable projective modules over the product $B^{(1)} \times \cdots \times B^{(t)}$ of generalized multicoil algebras $B^{(1)}, \ldots, B^{(t)}$. Since $B^{(i)} = \text{Supp}(\Gamma_i) = B(\Gamma_i)$ for any $i \in \{1, \ldots, t\}$, we conclude that there is a monomorphism $P \to N^m$ for a module $N$ in mod $B$ being a direct sum of indecomposable modules lying in $\Gamma^{cc} = \Gamma_1 \cup \cdots \cup \Gamma_t$ and a positive integer $m$. Clearly, then there is a monomorphism in mod $B$ of the form $B = P \oplus Q \to (N \oplus Q)^m$, and consequently $\Gamma$ is a faithful component of $\Gamma_B$. Therefore, we obtain the equality $\text{Supp}(\Gamma) = B(\Gamma)$. 


4. Proof of Theorem 1.2

Let $A$ be an algebra and $\Gamma$ be a cycle-finite finite component of $\Gamma_A$. Moreover, let $B = A/t_A(\Gamma)$ be the support algebra of $\Gamma$. Observe that $\Gamma$ is a sincere cycle-finite component of $\Gamma_B$. We will show that $B$ is a generalized double tilted algebra, applying [78, Theorem]. Since $\Gamma$ is a finite component of $\Gamma_B$, it follows from Corollary 2.6 that $\Gamma$ contains a projective module and an injective module. Hence, applying Proposition 2.1 we conclude that there exists in $\Gamma$ a path from an injective module to a projective module. Let

$$I = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m = P$$

be an arbitrary path in ind $B$ from an indecomposable injective module $I$ to an indecomposable projective module $P$. Since $\Gamma$ is a sincere translation subquiver of $\Gamma_B$, there exist indecomposable modules $M$ and $N$ in $\Gamma$ such that $\text{Hom}_B(P,M) \neq 0$ and $\text{Hom}_B(N,I) \neq 0$. Further, it follows from Proposition 2.11 that there exists a path in ind $B$ from $M$ to $N$. Therefore, we obtain in ind $B$ a cycle of the form

$$M \to \cdots \to N \to X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{m-1}} X_{m-1} \xrightarrow{f_m} X_m \to M,$$

and this is a finite cycle, because $M$ and $N$ belong to the cycle-finite component $\Gamma$ of $\Gamma_B$. This shows that all the modules $X_0, X_1, \ldots, X_{m-1}, X_m$ belong to the finite translation quiver $\Gamma$ of $\Gamma_B$. Then it follows from [78, Theorem] that $B$ is a quasitilted algebra or a generalized double tilted algebra. Furthermore, by [21, Corollary (E)], the Auslander-Reiten quiver of a quasitilted algebra which is not a tilted algebra consists of semiregular components. Clearly, every tilted algebra is a generalized double tilted algebra [60]. Since the cyclic component $\Gamma$ of $\Gamma_B$ contains a path from an injective module to a projective module, we obtain that $B$ is a generalized double tilted algebra. Hence, it follows from [60, Section 3] that $\Gamma_B$ admits an almost acyclic component $C$ with a faithful multisection $\Delta$. Recall that, following [60, Section 2], a full connected subquiver $\Delta$ of $C$ is called a multisection if the following conditions are satisfied:

(i) $\Delta$ is almost acyclic.
(ii) $\Delta$ is convex in $C$.
(iii) For each $\tau_B$-orbit $O$ in $C$, we have $1 \leq |\Delta \cap O| < \infty$.
(iv) $|\Delta \cap O| = 1$ for all but finitely many $\tau_B$-orbits $O$ in $C$.
(v) No proper full convex subquiver of $\Delta$ satisfies (i)–(iv).

Moreover, for a multisection $\Delta$ of a component $C$, the following full subquivers of $C$ were defined in [60]:

$$\Delta'_l = \{ X \in \Delta; \text{there is a nonsectional path in } C \text{ from } X \text{ to a projective module } P \},$$

$$\Delta'_r = \{ X \in \Delta; \text{there is a nonsectional path in } C \text{ from an injective module } I \text{ to } X \},$$

$$\Delta''_l = \{ X \in \Delta; \tau_A^{-1}X \notin \Delta'_l \}, \quad \Delta''_r = \{ X \in \Delta; \tau_AX \notin \Delta'_r \},$$

$$\Delta_l = (\Delta \setminus \Delta'_l) \cup \tau_A\Delta''_r, \quad \Delta_c = \Delta'_l \cap \Delta'_r, \quad \Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1}\Delta''_l.$$

Then $\Delta_l$ is called the left part of $\Delta$, $\Delta_r$ the right part of $\Delta$, and $\Delta_c$ the core of $\Delta$. The following basic properties of $\Delta$ have been established in [60, Proposition 2.4]:

(a) Every cycle of $C$ lies in $\Delta_c$.
(b) $\Delta_c$ is finite.
(c) Every indecomposable module $X$ in $C$ is in $\Delta_c$, or a predecessor of $\Delta_l$ or a successor of $\Delta_r$ in $C$.

It follows also from [60, Theorem 3.4, Corollary 3.5] and the known structure of the Auslander-Reiten quivers of tilted algebras (see [28, 32, 62, 65]) that every component of $\Gamma_B$ different from $C$ is a semiregular component. Hence the cyclic component $\Gamma$ is a translation subquiver of $C$, and consequently is contained in the core $\Delta_c$ of $\Delta$. We also know from [60, Proposition 2.11] that, for
another multisection \( \Sigma \) of \( C \), we have \( \Sigma_c = \Delta_c \). Thus \( \Delta_c \) is a uniquely defined core \( c(C) \) of the connecting component \( C \) of \( \Gamma_B \). We claim that \( \Gamma = c(C) \). Let \( X \) be a module in \( \Delta_c = \Delta'_l \cap \Delta'_r \). Then there are nonsectional paths in \( C \) from \( X \) to an indecomposable projective module \( P \) and from an indecomposable injective module \( I \) to \( X \). Moreover, there exist indecomposable modules \( Y \) and \( Z \) in \( \Gamma \) such that \( \text{Hom}_B(P,Y) \neq 0 \) and \( \text{Hom}_B(Z,I) \neq 0 \), because \( \Gamma \) is a sincere translation subquiver of \( \Gamma_B \). Further, by Proposition 2.1, we have in \( \Gamma \) a path from \( Y \) to \( Z \). Hence we obtain in ind \( B \) a cycle of the form

\[
X \to \cdots \to P \to Y \to \cdots \to Z \to I \to \cdots X,
\]

which is a finite cycle because \( Y \) and \( Z \) belong to the cycle-finite component \( \Gamma \) of \( \Gamma_B \). Therefore, there is in \( C \) a cycle passing through the modules \( X \), \( Y \) and \( Z \), and so \( X \) belongs to \( \Gamma \). This shows that \( \Gamma = \Delta_c = c(C) \).

Let \( B^{(l)} = \text{Supp}(\Delta_l) \) be the support algebra of the left part \( \Delta_l \) of \( \Delta \) (if \( \Delta_l \) is nonempty) and \( B^{(r)} = \text{Supp}(\Delta_r) \) be the support algebra of the right part \( \Delta_r \) of \( \Delta \) (if \( \Delta_r \) is nonempty). Then the following description of ind \( B \) follows from the results established in [60, Section 3]:

1. \( B^{(l)} \) is a tilted algebra (not necessarily indecomposable) such that \( \Delta_l \) is a disjoint union of sections of the connecting components of the indecomposable parts of \( B^{(l)} \) and the category of all predecessors of \( \Delta_l \) in ind \( B \) coincides with the category of all predecessors of \( \Delta_l \) in ind \( B^{(l)} \), or \( B^{(l)} \) is empty in case \( \Delta_l \) is empty.

2. \( B^{(r)} \) is a tilted algebra (not necessarily indecomposable) such that \( \Delta_r \) is a disjoint union of sections of the connecting components of the indecomposable parts of \( B^{(r)} \) and the category of all successors of \( \Delta_r \) in ind \( B \) coincides with the category of all successors of \( \Delta_r \) in ind \( B^{(r)} \), or \( B^{(r)} \) is empty in case \( \Delta_r \) is empty.

3. Every indecomposable module in ind \( B \) is either in \( \Gamma = c(C) \), a predecessor of \( \Delta_l \) in ind \( B \), or a successor of \( \Delta_r \) in ind \( B \).

4. If \( \Delta_l \) is nonempty, then \( \Delta_l \) is a faithful subquiver of \( \Gamma_B^{(l)} \), and hence \( B^{(l)} \) is the faithful algebra \( B(\Delta_l) = A/\text{ann}_A(\Delta_l) \) of \( \Delta_l \).

5. If \( \Delta_r \) is nonempty, then \( \Delta_r \) is a faithful subquiver of \( \Gamma_B^{(r)} \), and hence \( B^{(r)} \) is the faithful algebra \( B(\Delta_r) = A/\text{ann}_A(\Delta_r) \) of \( \Delta_r \).

We will prove now that \( B \) coincides with the faithful algebra \( B(\Gamma) = A/\text{ann}_A(\Gamma) \) of \( \Gamma \). Observe that \( t_A(\Gamma) \subseteq \text{ann}_A(\Gamma) \) and \( \text{ann}_B(\Gamma) = \text{ann}_A(\Gamma)/t_A(\Gamma) \). Therefore, it is sufficient to show that \( \Gamma \) is a faithful subquiver of \( \Gamma_B \). Let \( M_\Gamma \) be the direct sum of all indecomposable \( B \)-modules lying in \( \Gamma \). Then \( M_\Gamma \) is a sincere module in mod \( B \), by definition of \( B \). In order to show that \( M_\Gamma \) is a faithful \( B \)-module, it is enough to prove that there is a monomorphism \( B \to (M_\Gamma)^n \) for some positive integer \( n \). Let \( B = P^{(l)} \oplus P^{(c)} \) be a decomposition of \( B \) in mod \( B \) such that the indecomposable direct summands of \( P^{(c)} \) are exactly the indecomposable projective \( B \)-modules lying in the core \( \Gamma = c(C) \) of \( C \). Clearly, \( P^{(c)} \) is a direct summand of \( M_\Gamma \), and hence there is a monomorphism \( P^{(c)} \to M_\Gamma \) in mod \( B \). On the other hand, the indecomposable direct summands of \( P^{(l)} \) form a complete family of pairwise nonisomorphic indecomposable projective right modules over the left tilted algebra \( B^{(l)} \) of \( B \). Hence, if \( P^{(l)} = 0 \), or equivalently the left part \( \Delta_l \) of \( \Delta \) is empty, then \( B = P^{(l)} \) and \( M_\Gamma \) is a faithful module in mod \( B \), as required. Therefore, assume that \( \Delta_l \) is nonempty.

Let \( \Gamma^{(l)} \) be the family of all indecomposable modules \( X \) in \( \Gamma \) such that there is an arrow \( Y \to X \) in \( C \) with \( Y \) from \( \Delta_l \). We claim that, for any module \( X \) in \( \Gamma^{(l)} \), there exists an indecomposable projective module \( P \) in \( \Gamma \) such that \( \text{Hom}_B(P,X) \neq 0 \). We may assume that \( X \) is not projective. Then \( \tau_B X \) is an indecomposable module not lying in \( \Gamma \), because we have a path \( \tau_B X \to Y \to X \) in \( C \), with \( Y \) in the cyclic component \( \Gamma \) of \( \Gamma_B \) and \( Y \) not in \( \Gamma \). Observe that then \( \tau_B X \in \Delta_l \) because \( X \) is in \( \Gamma = \Delta_c = \Delta'_l \cap \Delta'_r \). Consider now an oriented cycle in \( \Gamma \)

\[
X = X_0 \to X_1 \to \cdots \to X_{r-1} \to X_r = X
\]
passing through $X$. It follows from Proposition 2.4 that there exists $i \in \{2, \ldots, r\}$ such that $\tau_B X_i \cong X_{i-2}$. Since $\tau_B X$ does not belong to $\Gamma$, we then conclude that there is in $\Gamma$ a sectional path

$$X_s \to X_{s+1} \to \cdots \to X_{r-1} \to X_r = X$$

with $X_s = P$ an indecomposable projective module. Hence we obtain that $\text{Hom}_B(P, X) \neq 0$, because the composition of irreducible homomorphisms in $\text{mod} \ B$ corresponding to arrows of a sectional path in $\Gamma_B$ is nonzero, by a theorem of Bautista and Smalø [12] (see also [S2] Theorem III.11.2). Observe also that, for any module $Y$ lying on $\Delta_l$, we have $\text{Hom}_B(P_{(c)}, Y) = 0$, because $Y$ is a module in $\text{mod} \ B^{(l)}$. This leads to the following property of modules in $\Gamma^{(l)}$: any irreducible homomorphism $f : Y \to X$ with $X$ in $\Gamma^{(l)}$ and $Y$ in $\Delta_l$ is a monomorphism.

Consider now the family $\Omega^{(l)}$ of all indecomposable modules $Y$ in $\Delta_l$ such that there is an arrow in $\mathcal{C}$ from $Y$ to a module $X$ in $\Gamma$, and hence in $\Gamma^{(l)}$. Moreover, let $M^{(l)}$ be the direct sum of all indecomposable modules in $\Omega^{(l)}$. Observe that $M^{(l)}$ is a right $B^{(l)}$-module. Moreover, for any module $Y$ in $\Omega^{(l)}$, there is an irreducible monomorphism $Y \to X$ in $\text{mod} \ B$ with $X$ lying in $\Gamma^{(l)}$. This implies that there is a monomorphism in $\text{mod} \ B$ of the form $M^{(l)} \to (M_l)^m$ for some positive integer $m$. We will prove that $M^{(l)}$ is a faithful right $B^{(l)}$-module. Let $P$ be an indecomposable projective module in $\text{mod} \ B^{(l)}$, or equivalently, an indecomposable direct summand of $P^{(l)}$. Since $M_l$ is a sincere module in $\text{mod} \ B$, we conclude that there is an indecomposable module $Z$ in $\mathcal{G}$ such that $\text{Hom}_B(P, Z) \neq 0$. Further, the radical $\text{radEnd}_B(M_l)$ of the endomorphism algebra $\text{End}_B(M_l)$ in nilpotent. Then there exist a path of irreducible homomorphisms

$$Z_{t+1} \xrightarrow{g_{t+1}} Z_t \xrightarrow{g_t} Z_{t-1} \to \cdots \to Z_2 \xrightarrow{g_2} Z_1 \xrightarrow{g_1} Z_0 = Z$$

and a homomorphism $v_{t+1} : P \to Z_{t+1}$ in $\text{mod} \ B$ with $g_1 g_2 \cdots g_t g_{t+1} v_{t+1} \neq 0$, $Z_0, Z_1, \ldots, Z_t$ indecomposable modules in $\Gamma$ and $Z_{t+1}$ an indecomposable module in $\Delta_l$ (see [S2] Proposition III.10.1). This implies that $\text{Hom}_B(P, M^{(l)}) \neq 0$ because $Z_{t+1}$ is a direct summand of $M^{(l)}$. Therefore, $M^{(l)}$ is a sincere right $B^{(l)}$-module. We know also that $B^{(l)}$ is a tilted algebra and $\Delta_l$ is a disjoint union of sections of the connecting components of the indecomposable parts of $B^{(l)}$ and the category of all predecessors of $\Delta_l$ in $\text{ind} \ B$ coincides with the category of all predecessors of $\Delta_l$ in $\text{ind} \ B^{(l)}$. Then we conclude that, for any indecomposable module $L$ in $\text{ind} \ B^{(l)}$, we have

$$\text{Hom}_{B^{(l)}}(L, M^{(l)}) = 0 \text{ or } \text{Hom}_{B^{(l)}}(M^{(l)}, \tau_{B^{(l)}} L) = 0.$$ 

Summing up, we proved that $M^{(l)}$ is a sincere module in $\text{mod} \ B^{(l)}$ which is not the middle of a short chain in the sense of [61] (see also [9]). Then it follows from [61] Corollary 3.2 that $M^{(l)}$ is a faithful module in $\text{mod} \ B^{(l)}$. Hence, there exists a monomorphism $B^{(l)} \to (M^{(l)})^s$ in $\text{mod} \ B^{(l)}$ for some positive integer $s$.

Finally, since there exist monomorphisms $M^{(l)} \to (M_l)^m$ and $P^{(l)} \to (M_l)$, and $B = P^{(l)} \oplus P_{(c)}$ with $P^{(l)} = B^{(l)}$ in $\text{mod} \ B$, we obtain that there is a monomorphism in $\text{mod} \ B$ of the form $B \to (M_l)^n$ for some positive integer $n$. Therefore, $M_l$ is a faithful module in $\text{mod} \ B$, and consequently $B = B(\Gamma)$. This finishes the proof of the theorem.

In connection with the final part of the above proof, we mention that, by a recent result proved by Jaworska, Malicki and Skowroński in [29], an algebra $A$ is a tilted algebra if and only if there exists a sincere module $M$ in $\text{mod} \ A$ such that for any module $X$ in $\text{ind} \ A$, we have $\text{Hom}_A(X, M) = 0$ or $\text{Hom}_A(M, \tau_A X) = 0$. Moreover, all modules $M$ in a module category $\text{mod} \ A$ not being the middle of short chains have been described completely in [50].

5. Proof of Theorem 1.10

Let $A$ be a cycle-finite algebra. Clearly, the statement (iv) implies the statements (i), (ii), (iii). Assume $A$ is of infinite representation type. Then it follows from [72] Corollary 4.3 that there is an idempotent $e$ in $A$ such that $B = A/AeA$ is a tame concealed algebra. Then $\text{ind} \ B$ admits infinitely
many pairwise nonisomorphic directing postprojective (respectively, preinjective) modules. Hence, ind $B$ has infinitely many pairwise nonisomorphic rigid modules, and consequently ind $A$ has infinitely many pairwise nonisomorphic rigid modules. Applying Theorem 1.5 we conclude that ind $A$ contains infinitely many pairwise nonisomorphic directing modules, because $A$ is a cycle-finite algebra. Then $\Gamma_A$ contains a $\tau_A$-orbit containing infinitely many directing modules, by [51, Theorem 2.7] or [69, Corollary 2]. Hence $\Gamma_A$ contains either an acyclic left stable full translation subquiver $D$ which is closed under predecessors or an acyclic right stable full translation subquiver $E$ which is closed under successors. Since $A$ is a cycle-finite algebra, applying [44, Theorem 2.2] and its dual, we conclude that $D$ (respectively, $E$) consists entirely of directing modules, which are obviously also $\tau_A$-rigid and rigid indecomposable modules. Therefore, any of the statements (i), (ii), (iii) implies the statement (iv).

6. Examples: infinite cyclic components

In this section we present examples illustrating Theorem 1.1.

Example 6.1. Let $K$ be a field and $A = KQ/I$ the bound quiver algebra given by the quiver $Q$ of the form

and $I$ the ideal in the path algebra $KQ$ of $Q$ over $K$ generated by the elements $\alpha\beta - \sigma\gamma$, $\xi\eta - \mu\nu$, $\pi\kappa - \xi\rho\alpha\beta$, $\rho\varphi$, $\psi\rho$, $jl$, $dc$, $ed$, $gd$, $hg$, $hf$, $ih$, $av$, $rs$, $st$, $r^2$. Then $A$ is a cycle-finite algebra and $\Gamma_A$
admits a component \( C \) of the form

\[
\begin{array}{c}
\text{modules} \\
S_1 \rightarrow S_2 \rightarrow S_3 \\
\vdots \\
S_{12} \rightarrow S_{13} \\
S_{14} \rightarrow S_{15} \rightarrow S_{16} \\
\vdots \\
S_{24} \rightarrow S_{25} \\
\end{array}
\]

The cyclic part \( \mathcal{C} \) of \( C \) consists of one infinite component \( \Gamma \) and one finite component \( \Gamma' \) described as follows. The infinite cyclic component \( \Gamma \) is obtained by removing from \( \mathcal{C} \) the modules \( S_{12}, S_{17}, S_{18}, P_{17}, S_{24}, M, S_{26}, P_{26}, N, I_{26}, W, I_{24}, S_{25} \), and the arrows attached to them. The finite cyclic component \( \Gamma' \) is the full translation subquiver of \( \mathcal{C} \) given by the vertices \( S_{26}, P_{26}, N, I_{26}, W, I_{24} \). The maximal cyclic coherent part \( \Gamma^{cc} \) of \( \Gamma \) is the full translation subquiver of \( \mathcal{C} \) obtained by removing from \( \mathcal{C} \) the modules \( S_{12}, I_{13}, T, S_{14}, P_{15} = I_{14}, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, P_{20} = I_{21}, S_{20}, S_{17}, P_{17}, S_{18}, S_{24}, M, S_{26}, P_{26}, N, I_{26}, W, I_{24}, S_{25} \), and the arrows attached to them. Further, \( \Gamma^{cc} \) is the cyclic part of the maximal almost cyclic coherent full translation subquiver \( \Gamma^* \) of \( \mathcal{C} \) obtained by removing from \( \mathcal{C} \) the modules \( P_{15} = I_{14}, S_{15}, P_{21}, S_{22}, L, P_{22}, R, I_{15}, I_{22}, S_{21}, P_{20} = I_{21}, S_{26}, P_{26}, I_{26}, W, N, I_{24} \) and the arrows attached to them, and shrinking the sectional path \( M \rightarrow N \rightarrow I_{24} \rightarrow S_{25} \) to the arrow \( M \rightarrow S_{25} \).

Let \( B = A/\text{ann}_A(\Gamma) \). Then \( B = A/\text{ann}_A(\Gamma^*) \), because \( \text{ann}_A(\Gamma) = \text{ann}_A(\Gamma^*) \). Observe that \( B = KQ_B/I_B \), where \( Q_B \) is the full subquiver of \( Q \) given by all vertices of \( Q \) except \( 15, 21, 22, 26 \), and \( I_B = I \cap KQ_B \). We claim that \( B \) is a tame generalized multicoil algebra. Consider the path algebra \( C = K\Sigma \) of the full subquiver \( \Sigma \) of \( Q \) given by the vertices \( 4, 5, 6, 7, 8, 9 \). Then \( C \) is a hereditary algebra of Euclidean type \( \tilde{D}_5 \), and hence a tame concealed algebra. It is known that \( \Gamma_C \) admits an infinite family \( \mathcal{T}^C_\lambda \), \( \lambda \in \Lambda(C) \), of pairwise orthogonal generalized standard stable tubes, having a unique stable tube, say \( \tau_C \), of rank \( 3 \) with the mouth formed by the modules \( S_6 = \tau_C S_7, S_7 = \tau_C E, E = \tau_C S_6 \), where \( E \) is the unique indecomposable \( C \)-module with the dimension vector

\[
\dim E = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad (\text{see } \cite{25} \text{ Section 6} \text{ and } \cite{64} \text{ Theorem XIII 2.9}).
\]

Then \( B \) is the generalized multicoil enlargement of \( C \), obtained by applications of the following admissible operations:

- two admissible operations of types (ad 1*), with the pivots \( S_6 \) and \( S_{12} \), creating the vertices \( 11, 12, 13, 14 \) and the arrows \( \varphi, a, b, c \);
- two admissible operations of types (ad 1*), with the pivots \( E \) and \( S_{2} \), creating the vertices \( 2, 1, 0 \) and the arrows \( \beta, \gamma, \kappa, \omega, \theta \);
- two admissible operations of types (ad 1), with the pivots \( S_{7} \) and \( S_{16} \), creating the vertices \( 16, 17, 18, 19, 20 \) and the arrows \( \psi, i, m, j, i \);
- one admissible operation of type (ad 3) with the pivot the radical of \( P_{10} \), creating the vertex 10 and the arrows \( \xi, \mu, \pi \);
one admissible operation of type \((ad\, 1^*)\) with the pivot \(V\) being the unique indecomposable module of dimension 2 having \(S_{11}\) as the socle and \(S_6\) as the top, creating the vertices 23, 24, 25 and the arrows \(v, t, u\).

Then the left part \(B^{(l)}\) of \(B\) is the convex subcategory of \(B\) (and of \(A\)) given by the vertices 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 23, 24, 25, and is a tilted algebra of Euclidean type \(\mathbb{E}_{14}\) with the connecting postprojective component \(P_{B^{(l)}}\) containing all indecomposable projective \(B^{(l)}\)-modules. The right part \(B^{(r)}\) of \(B\) is the convex subcategory of \(B\) (and of \(A\)) given by the vertices 0, 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 17, 18, 19, 20, and is a tilted algebra of Euclidean type \(\mathbb{E}_{14}\) with the connecting preinjective component \(Q_{B^{(r)}}\) containing all indecomposable injective \(B^{(r)}\)-modules. We also note that the left border \(\Delta_l\) of the generalized multicoil \(\Gamma^*\) of \(B\) is given by the quivers \(P_{17} \rightarrow S_{17}\) and \(S_{29}\), and the right border \(\Delta_r\) of \(\Gamma^*\) is given by the quivers \(T \rightarrow I_{13} \rightarrow S_{12}\) and \(S_{24} \rightarrow M\). Further, the algebra \(B(\Gamma \setminus \Gamma^{cc}) = A/\text{ann}_A(\Gamma \setminus \Gamma^{cc})\) is the disjoint union of three representation-finite convex subcategories of \(A\): \(D_l\) given by the vertices 12, 13, 14, 15, 20, 21, 22, \(D_2\) given by the vertices 17, 18, and \(D_3\) given by the vertices 24, 25, 26. We note that \(D_3\) is the faithful algebra \(B(\Gamma')\) of the finite cyclic component \(\Gamma'\). It follows from [46 Theorems C and F] that the Auslander-Reiten quiver \(\Gamma_B\) of the generalized multicoil enlargement \(B\) of \(C\) is of the form

\[
\Gamma_B = P^B \cup C^B \cup Q^B,
\]

where \(P^B = P_{B^{(l)}}\), \(Q^B = Q_{B^{(r)}}\), and \(C^B\) is the family \(C^B_\lambda, \lambda \in \Lambda(C)\), of pairwise orthogonal generalized multicoils such that \(C^B_\lambda = \Gamma^*\) and \(C^B_\lambda = \mathcal{T}_\lambda^C\) for all \(\lambda \in \Lambda(C) \setminus \{1\}\). Hence \(\Gamma_A\) is of the form

\[
\Gamma_A = P^A \cup C^A \cup Q^A,
\]

where \(P^A = P_{B^{(l)}}\), \(Q^A = Q_{B^{(r)}}\), and \(C^A\) is the family \(C^A_\lambda, \lambda \in \Lambda(C)\), of pairwise orthogonal generalized standard components such that \(C^A_\lambda = C\), \(C^A_\lambda = \mathcal{T}_\lambda^C\) for all \(\lambda \in \Lambda(C) \setminus \{1\}\). Moreover, we have

\[
\text{Hom}_A(C^A, P^A) = 0, \text{Hom}_A(Q^A, C^A) = 0, \text{Hom}_A(Q^A, P^A) = 0.
\]

In particular, \(A\) is a cycle-finite algebra with \((\text{rad}_A^\infty)^3 = 0\).

**Example 6.2.** Let \(K\) be a field and \(B = KQ/J\) the bound quiver algebra given by the quiver \(Q\) of the form

![Quiver Diagram](http://example.com/qr.jpg)
and $J$ the ideal in the path algebra $KQ$ of $Q$ over $K$ generated by the elements $\varphi_1\psi_1, \eta_1\varphi_2, \omega_1\sigma_3\sigma_2\sigma_1, \pi_3\psi_3\psi_2, \lambda_2\mu_2, \eta_2\kappa_2, \gamma_1\gamma_2, \pi_1\theta_1, \nu_1\theta_2, \alpha_3\rho_1, \lambda_1\rho_2, \xi_1\rho_1, \alpha_1\beta_3 + \gamma_1\gamma_2\gamma_3, \alpha_1\delta_1, \alpha_2\sigma_1, \epsilon_1\delta_1, \epsilon_1\alpha_2, \epsilon_3\delta_2, \gamma_1\mu_1, \nu_1\gamma_3$. Denote by $P_k, I_k, S_k$ the indecomposable projective module, the indecomposable injective module, and the simple module in $\text{mod } B$ at the vertex $k$ of $Q$. Then $\Gamma_B$ admits a cyclic component $C$ obtained by identification the sectional paths $H_1 \to H_2 \to H_3, L_1 \to L_2, N_1 \to N_2$ and the module $S_{21}$ occurring in the following three translation quivers: $C_1$ of the form

$C_2$ of the form

and $C_3$ of the form
where \( X = I_{38} = P_{32}, Y = I_{39} = P_{22}, Z = I_{40} = P_{16}, N_2 = I_9 \) and the vertical dashed lines have to be identified in order to obtain the translation quivers \( C_1, C_2 \) and \( C_3 \). We claim that \( B \) is a generalized multicoil algebra. Denote by \( Q_C \) the full subquiver of \( Q \) given by the vertices \( 0, 1, 2, 3, 4, 5, 6, 7, 8 \). Consider the bound quiver algebra \( C = KQ_C/J_C \) with \( J_C \) the ideal in \( KQ_C \) generated by \( \alpha_1 \alpha_2 \alpha_3 \alpha_4 + \beta_1 \beta_2 \beta_3 + \gamma_1 \gamma_2 \gamma_3 \) and the path algebra \( D = KQ_D \) of the full subquiver \( Q_D \) of \( Q \) given by the vertices \( 25, 26, 27, 28 \). Then \( C \) is a canonical algebra of wild type and \( D \) is a canonical algebra of Euclidean type \( \tilde{A}_3 \). It is known that \( \Gamma_C \) admits an infinite family \( \mathcal{T}_r^C, \lambda \in \Lambda(C), \) of pairwise orthogonal stable tubes, having a unique stable tube, say \( T^C_1 \), of rank 4 with the mouth formed by the modules \( S_1 = \tau_C S_2, S_2 = \tau_C S_3, S_3 = \tau_C E, E = \tau_C S_1 \), where \( E \) is the unique indecomposable \( C \)-module with the dimension vector \( \dim E = 1 \quad 0 \quad 0 \quad 1 \), and a stable tube, say \( T^C_2 \), of rank 3 with the mouth formed by the modules \( S_5 = \tau_C S_7, S_7 = \tau_C F, F = \tau_C S_5 \), where \( F \) is the unique indecomposable \( C \)-module with the dimension vector \( \dim F = 1 \quad 1 \quad 1 \quad 1 \) (see [22] (3.7)). Moreover, \( \Gamma_D \) admits an infinite family \( \mathcal{T}_r^D, \mu \in \Lambda(D), \) of pairwise orthogonal stable tubes, having a stable tube, say \( T^D_1 \), of rank 2 with the mouth formed by the modules \( S_{26} = \tau_D G, G = \tau_D S_{26} \), where \( G \) is the unique indecomposable \( D \)-module with the dimension vector \( \dim G = 1 \quad 1 \). Denote by \( C_i = KQ_{C_i}/J_{C_i} \), the bound quiver algebra, where \( Q_{C_i} \) is the full subquiver of \( Q \) given by the vertices \( 0, 1, 2, \ldots, i \), \( i \geq 8 \) \((C_8 = C)\), \( J_{C_i} = J \cap KQ_{C_i} \), and by \( D_j = KQ_{D_j}/J_{D_j} \), the bound quiver algebra, where \( Q_{D_j} \) is the full subquiver of \( Q \) given by the vertices \( 25, 26, 27, \ldots, j \), \( j \geq 28 \) \((D_{28} = D)\), \( J_{D_j} = J \cap KQ_{D_j} \). Moreover, for each \( k \in \{8, 9, \ldots, 39\} \) (respectively, \( k \in \{28, 29, \ldots, 33\} \) and \( l \in \{0, 1, \ldots, 39\} \) (respectively, \( l \in \{28, 29, \ldots, 33\} \)), we denote by \( I^D_C, I^C_D, S^C_I \) (respectively, \( I^D_C, I^C_D, S^C_I \)) the indecomposable projective module, the indecomposable injective module, and the simple module in \( \text{mod } C_k \) (respectively, \( \text{mod } D_k \)) at the vertex \( l \) of \( Q_{C_k} \) (respectively, \( \text{of } Q_{D_k} \)). Then \( B \) is the generalized multicoil enlargement of \( C \times D \), obtained by applications of the following admissible operations:

- one admissible operation of type (ad 1*) with the pivot \( S^C_3 \), creating the vertices 9, 10, 11 and the arrows \( \delta_1, \delta_2, \delta_3 \);
- one admissible operation of type (ad 1*) with the pivot \( S^C_5 \), creating the vertices 12, 13, 14, 15 and the arrows \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \);
- one admissible operation of type (ad 1) with the pivot \( S^C_7 \), creating the vertices 16, 17 and the arrows \( \xi_1, \xi_2 \);
- one admissible operation of type (ad 4) with the pivot \( S^C_9 \), and the finite sectional path \( S^C_{10} \rightarrow I^C_{10} \), creating the vertices 18, 19 and the arrows \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \);
- one admissible operation of type (ad 1*) with the pivot \( S^C_{13} \), creating the vertices 20, 21 and the arrows \( \mu_1, \mu_2 \);
- one admissible operation of type (ad 1) with the pivot \( S^C_{21} \), creating the vertices 22, 23, 24 and the arrows \( \nu_1, \nu_2, \nu_3 \);
- one admissible operation of type (ad 1*) with the pivot \( S^D_{26} \), creating the vertices 29, 30, 31 and the arrows \( \psi_1, \psi_2, \psi_3 \);
- one admissible operation of type (ad 1) with the pivot \( W \) being the unique indecomposable module of dimension 2 having \( S^D_{29} \) as the socle and \( S^D_{26} \) as the top, creating the vertices 32, 33 and the arrows \( \eta_1, \eta_2 \);
- one admissible operation of type (ad 4) with the pivot \( S^D_{33} \), and the finite sectional path \( I^D_{13} \rightarrow I^D_{14} \rightarrow I^D_{15} \), creating the vertex 34 and the arrows \( \omega_1, \omega_2 \);
- one admissible operation of type (ad 4) with the pivot \( S^D_{24} \) and the finite sectional path \( I^D_{30} \rightarrow S^D_{31} \), creating the vertices 35, 36 and the arrows \( \pi_1, \pi_2, \pi_3 \);
- one admissible operation of type (ad 4) with the pivot \( S^C_{20} \) and the module \( S^C_{21} \), creating the vertex 37 and the arrows \( \lambda_1, \lambda_2 \);
- one admissible operation of type (ad 2*) with the pivot \( P^C_{32} \), creating the vertex 38 and the arrows \( \kappa_1, \kappa_2 \).
• one admissible operation of type $(ad 2^*)$ with the pivot $P_{22}^{C_{18}}$, creating the vertex 39 and the arrows $\theta_1$, $\theta_2$.

• one admissible operation of type $(ad 2^*)$ with the pivot $P_{16}^{C_{39}}$, creating the vertex 40 and the arrows $\rho_1$, $\rho_2$.

Then the left part $B^{(l)}$ of $B$ is the convex subcategory of $B$ being the product $B^{(l)} = B_1^{(l)} \times B_2^{(l)}$, where $B_1^{(l)} = KQ_1^{(l)} / J_1^{(l)}$ is the branch coextension of the canonical algebra $C$ and $B_2^{(l)} = KQ_2^{(l)} / J_2^{(l)}$ is the branch coextension of the canonical algebra $D$ given by the quivers

![Diagram of $Q_1^{(l)}$ and $Q_2^{(l)}$ quivers]

and the ideals $J_1^{(l)} = KQ_1^{(l)} \cap J$ in $KQ_1^{(l)}$ and $J_2^{(l)} = KQ_2^{(l)} \cap J$ in $KQ_2^{(l)}$. The right part $B^{(r)}$ of $B$ is the convex subcategory of $B$ being the product $B^{(r)} = B_1^{(r)} \times B_2^{(r)}$, where $B_1^{(r)} = KQ_1^{(r)} / J_1^{(r)}$ is the branch extension of the canonical algebra $C$ and $B_2^{(r)} = KQ_2^{(r)} / J_2^{(r)}$ is the branch extension of the canonical algebra $D$ given by the quivers

![Diagram of $Q_1^{(r)}$ and $Q_2^{(r)}$ quivers]
and the ideals $J_1^{(r)} = KQ_1^{(r)} \cap J$ in $KQ_1^{(r)}$ and $J_2^{(r)} = KQ_2^{(r)} \cap J$ in $KQ_2^{(r)}$. It follows from [46, Theorems C and F] that the Auslander-Reiten quiver $\Gamma_B$ of the generalized multicoil enlargement $B$ of $C \times D$ is of the form

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{C}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B, \mathcal{C}^B, \mathcal{Q}^B$ are of the following families of components:

- $\mathcal{C}^B$ is a family of pairwise orthogonal generalized multicoils consisting of the faithful cyclic component $\mathcal{C}$ (described above), the family $\mathcal{T}_C^\lambda$, $\lambda \in \Lambda(C) \setminus \{1, 2\}$, of stable tubes of $\Gamma_C$, and the family $\mathcal{T}_D^\mu$, $\mu \in \Lambda(D) \setminus \{1\}$, of stable tubes of $\Gamma_D$;
- $\mathcal{P}^B = \mathcal{P}^{B^{(l)}}$ and consists of the unique postprojective component $\mathcal{P}(A_1^{(l)})$ of the wild concealed algebra $A_1^{(l)}$ being the convex subcategory of $B_1^{(l)}$ given by all object of $B_1^{(l)}$ except 8, the unique postprojective component $\mathcal{P}(B_2^{(l)}) = \mathcal{P}^{B_{2}^{(l)}}$ of the tilted algebra $B_2^{(l)}$ of Euclidean type $\tilde{A}_7$, one component with the stable part $\mathbb{Z}\tilde{A}_\infty$ containing the indecomposable projective $B_1^{(l)}$-module at the vertex 8, and infinitely many regular components of the form $\mathbb{Z}\tilde{A}_\infty$;
- $\mathcal{Q}^B = \mathcal{Q}^{B^{(r)}}$ and consists of the unique preinjective component $\mathcal{Q}(A_1^{(r)})$ of the wild concealed algebra $A_1^{(r)}$ being the convex subcategory of $B_1^{(r)}$ given by all object of $B_1^{(r)}$ except 0, the unique preinjective component $\mathcal{Q}(B_2^{(r)}) = \mathcal{Q}^{B_{2}^{(r)}}$ of the tilted algebra $B_2^{(r)}$ of Euclidean type $\tilde{A}_9$, one component with the stable part $\mathbb{Z}\tilde{A}_\infty$ containing the indecomposable injective $B_1^{(r)}$-module at the vertex 0, and infinitely many regular components of the form $\mathbb{Z}\tilde{A}_\infty$.

Moreover, we have

$$\text{Hom}_B(\mathcal{C}^B, \mathcal{P}^B) = 0, \text{Hom}_B(\mathcal{Q}^B, \mathcal{C}^B) = 0, \text{Hom}_B(\mathcal{Q}^B, \mathcal{P}^B) = 0.$$

We also note that $B$ is not a cycle-finite algebra, because $\Gamma_B$ contains regular components of the form $\mathbb{Z}\tilde{A}_\infty$ (see [46, Lemma 3]).

Finally, we mention that the cyclic component $\mathcal{C}$ of $\Gamma_B$ is the cyclic generalized multicoil obtained from the stable tubes $\mathcal{T}_C^1$, $\mathcal{T}_C^2$ of $\Gamma_C$ and the stable tube $\mathcal{T}_D^1$ of $\Gamma_D$ by the 14 translation quiver admissible operations [45, Section 2] corresponding to the 14 admissible algebra operations leading from $C \times D$ to $B$, described above. We also point that the cyclic component $\mathcal{C}$ has a Möbius strip configuration obtained by identifying in $C_3$ two sectional paths $N_1 \rightarrow N_2$.

### 7. Examples: finite cyclic components

In this section we present examples illustrating Theorem 1.2 and showing faithful almost acyclic Auslander-Reiten components of new types.

**Example 7.1.** Let $K$ be a field, $n \geq 7$ a natural number, and $A_n = KQ_n/I_n$ the bound quiver algebra given by the quiver $Q_n$ of the form

![Diagram](https://example.com/diagram)

and $I_n$ the ideal in the path algebra $KQ_n$ of $Q_n$ over $K$ generated by the elements $\varepsilon^2, \eta \varepsilon$ and $\beta \gamma - \rho \delta \omega$. Then the category $\text{mod } A_n$ is equivalent to the category $\text{rep}_K(Q_n, I_n)$ of the $K$-linear representations of the bound quiver $(Q_n, I_n)$. Consider the indecomposable module $M_n$ in $\text{mod } A_n$ corresponding to
the indecomposable representation in $\text{rep}_K(Q_n, I_n)$ of the form

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

We note that $M_n$ is a faithful $A_n$-module, and hence $B(M_n) = A_n$. Let $\Omega_n$ be the full subquiver of $Q_n$ given by the vertices $2, 3, 4, 5, 6, \ldots, n-1, n$ and the arrows $\gamma, \delta, \omega, \sigma_6, \ldots, \sigma_{n-1}, \sigma_n$, and $H_n = K\Omega_n$ the associated path algebra. Then $H_n$ is a hereditary algebra. Observe that $H_7, H_8, H_9$ are hereditary algebras of Dynkin types $E_6, E_7, E_8$ (respectively), $H_{10}$ is a hereditary algebra of Euclidean type $E_8$, and, for $n \geq 11$, $H_n$ is a hereditary algebra of wild type. For each $i \in \{0,1,\ldots,n-1,n\}$, we denote by $P_i, I_i, S_i$ the indecomposable projective module, the indecomposable injective module, the simple module in mod $A_n$ at the vertex $i$ of $Q_n$. Moreover, for each $j \in \{2,3,4,\ldots,n-1,n\}$, we denote by $I_j^*$ the indecomposable injective module in mod $H_n$ at the vertex $j$ of $\Omega_n$. Further, let $\Lambda = K[\varepsilon]/(\varepsilon^2)$. Then $P_0$ is the indecomposable projective module in mod $\Lambda$ and $S_0$ is its top. Finally, observe that $A_n$ is the one-point extension algebra

\[
\begin{bmatrix}
\Lambda \times H_n & 0 \\
S_0 \oplus I_n^* & K
\end{bmatrix}
\]

of $\Lambda \times H_n$ by the module $S_0 \oplus I_n^*$, with the extension vertex 1. Since $I_n^*$ is the indecomposable injective module in mod $H_n$ and $H_n$ is a hereditary algebra, we conclude that

\[
\text{Hom}_{\Lambda \times H_n}(S_0 \oplus I_n^*, \tau_{H_n} X) = \text{Hom}_{H_n}(I_n^*, \tau_{H_n} X) = 0
\]

for any module $X$ in ind $H_n$. Then, applying [65, Corollary XV.1.7] (see also [81, Lemma 5.6]), we conclude that every almost split sequence in mod $H_n$ is an almost split sequence in mod $A_n$. This implies that the Auslander-Reiten quiver $\Gamma_{H_n}$ of $H_n$ is a full translation subquiver of the Auslander-Reiten quiver $\Gamma_{A_n}$ of $A_n$. In particular, we obtain that the preinjective component $Q(H_n)$ of $\Gamma_{H_n}$, containing the indecomposable injective modules $I_j^*, j \in \{2,3,4,\ldots,n-1,n\}$, is a full translation subquiver of a component $\mathcal{C}_n$ of $\Gamma_{A_n}$ which is closed under predecessors. Then the direct calculation shows that $\mathcal{C}_n$ is a component of the form

\[
\begin{bmatrix}
\Lambda \times H_n & 0 \\
S_0 \oplus I_n^* & K
\end{bmatrix}
\]

We also note that $\mathcal{C}_n$ admits a unique multisection $\Delta = \Delta_n$ consisting of all indecomposable modules in $\mathcal{C}_n$ which lie on oriented cycles passing through the simple module

\[
\begin{bmatrix}
\Lambda \times H_n & 0 \\
S_0 \oplus I_n^* & K
\end{bmatrix}
\]
Moreover, we have $\Delta'_c = \Delta = \Delta'_r$, and hence $\Delta = \Delta_c$. Further, the left part $\Delta_l$ of $\Delta$ coincides with $\tau_{A_n} \Delta''_r$ and consists of the indecomposable modules $I_j^n$, for $j \in \{2, 3, 4, \ldots, n - 1, n\}$. Similarly, the right part $\Delta_r$ of $\Delta$ coincides with $\tau_{A_n}^{-1} \Delta''_l$ and consists of the indecomposable injective modules $I_1, I_2, I_3, I_4$. Therefore, the left tilted part $A_n^{(l)}$ of $A_n$ is the hereditary algebra $H_n$ and the right tilted part $A_n^{(r)}$ of $A_n$ is the path algebra $K\Sigma$ of the quiver $\Sigma$ of the form

$$2 \xrightarrow{\beta} 1 \xrightarrow{\varrho} 3 \xrightarrow{\delta} 4.$$  

Observe now that $\Delta = \Delta_c$ is a cycle-finite finite component of $\Gamma_{A_n}$ containing the faithful indecomposable module $M_n$, because $C_n$ is a generalized standard component of $\Gamma_A$ closed under successors in $\text{ind} \ A_n$. In particular, we conclude that $\Delta$ is the cyclic component $\Gamma(M_n)$ of the module $M_n$ and $B(\Gamma(M_n)) = B(M_n) = A_n$ is a generalized double tilted algebra. We also mention that $A_7, A_8, A_9$ are of finite representation type with $\Gamma_{A_7} = C_7, \Gamma_{A_8} = C_8, \Gamma_{A_9} = C_9$, and hence are cycle-finite algebras. Further, $A_{10}$ is a cycle-finite algebra of infinite representation type whose Auslander-Reiten quiver has the disjoint union decomposition

$$\Gamma_{A_{10}} = \mathcal{P}(H_{10}) \cup \mathcal{T}^{H_{10}} \cup \mathcal{C}_{10},$$

where $\mathcal{P}(H_{10})$ is the postprojective component and $\mathcal{T}^{H_{10}}$ an infinite family of pairwise orthogonal generalized standard stable tubes of $\Gamma_{H_{10}}$. On the other hand, the algebras $A_n$, for $n \geq 11$, are not cycle-finite because their Auslander-Reiten quivers admit regular components of $\Gamma_{H_n}$ being of the form $\mathbb{Z} \mathbb{A}_\infty$, and hence consisting of indecomposable modules lying on infinite cycles (see [69, Theorem]). More precisely, for $n \geq 11$, the Auslander-Reiten quiver of $A_n$ has the disjoint union decomposition

$$\Gamma_{A_n} = \mathcal{P}(H_n) \cup \mathcal{R}(H_n) \cup \mathcal{C}_n,$$

where $\mathcal{P}(H_n)$ is the postprojective component and $\mathcal{R}(H_n)$ is an infinite family of regular components of the form $\mathbb{Z} \mathbb{A}_\infty$ in $\Gamma_{H_n}$. We also mention that the algebras $A_n$, for $n \geq 7$, are of infinite global dimension, because the simple module $S_0$ is of infinite projective dimension.

**Example 7.2.** Let $K$ be a field, $m, n \geq 8$ natural numbers, and $B_{m,n} = KQ_{m,n}/I_{m,n}$ the bound quiver algebra given by the quiver $Q_{m,n}$ of the form

![Quiver Diagram](attachment:quiver.png)

and $I_{m,n}$ the ideal in the path algebra $KQ_{m,n}$ of $Q_{m,n}$ over $K$ generated by the elements $\alpha \beta, \sigma \alpha, \beta \delta, \sigma \gamma \delta, \xi \varrho - \eta \omega \mu, \varphi \psi - \lambda \theta \nu$. Then the category $\text{mod} B_{m,n}$ is equivalent to the category $\text{rep}_K(Q_{m,n}, I_{m,n})$ of the $K$-linear representations of the bound quiver $(Q_{m,n}, I_{m,n})$. Consider the indecomposable module
$M_m$ in mod $B_{m,n}$ corresponding to the indecomposable representation in $\text{rep}_K(Q_{m,n}, I_{m,n})$ of the form

\[
\begin{array}{cccccc}
1 & 0 & 1 & 0 & 1 & 0 \\
K & K & K & K & K & K \\
1 & 1 & 1 & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & 1 & 1 & 1 \\
K & K & K & K & K & K \\
\end{array}
\]

and the indecomposable module $N_n$ in mod $B_{m,n}$ corresponding to the indecomposable representation in $\text{rep}_K(Q_{m,n}, I_{m,n})$ of the form

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & K & K & K & K & K \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & K & K & K & K & K \\
\end{array}
\]

We note that $M_m \oplus N_n$ is a faithful $B_{m,n}$-module.

Let $\Omega_m$ be the subquiver of $Q_{m,n}$ given by the vertices 3, 4, 5, 6, 7, \ldots, $m-1$, $m$ and the arrows $\varphi$, $\omega$, $\mu$, $\alpha_7$, $\alpha_8$, \ldots, $\alpha_{m-1}$, $\alpha_m$, and $H_m = K\Omega_m$ the path algebra of $\Omega_m$ over $K$. Similarly, let $\Omega'_n$ be the subquiver of $Q_{m,n}$ given by the vertices $3', 4', 5', 6', 7', \ldots, (n-1)'$, $n'$ and the arrows $\varphi$, $\theta$, $\lambda$, $\beta_7$, $\beta_8$, \ldots, $\beta_{n-1}$, $\beta_n$, and $H'_n = K\Omega'_n$ the path algebra of $\Omega'_n$ over $K$. Then $H_m$ and $H'_n$ are hereditary algebras. Moreover, $H_6$ and $H'_6$ are of Dynkin type $E_6$, $H_9$ and $H'_9$ are of Dynkin type $E_7$, $H_{10}$ and $H'_{10}$ are of Dynkin type $E_8$, $H_11$ and $H'_{11}$ are of Euclidean type $\tilde{E}_8$, and $H_m$ and $H'_n$, for $m, n \geq 12$, are of wild type. For each $i \in \{3, 4, \ldots, m-1, m\}$, we denote by $I_i^*$ the indecomposable injective $H_m$-module at the vertex $i$. Similarly, for each $j' \in \{3', 4', \ldots, (n-1)', n'\}$, we denote by $P_{j'}^*$ the indecomposable projective $H'_{n'}$-module at the vertex $j'$. Furthermore, for each vertex $i$ of $Q_{m,n}$, we denote by $P_i$, $I_i$, $S_i$ the indecomposable projective module, the indecomposable injective module, and the simple module in mod $B_{m,n}$ at the vertex $i$. Finally, we denote by $\Sigma$ the subquiver of $Q_{m,n}$ given by the vertices 0, 1, 1' and the arrows $\alpha$, $\beta$, $\gamma$, and $\Lambda = K\Sigma/J$ the bound quiver algebra with $J$ the ideal in the path algebra $K\Sigma$ of $\Sigma$ over $K$ generated by $\alpha\beta$. We denote by $R$ and $T$ the indecomposable modules in mod $\Lambda$ corresponding to the representations

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
K & K & K & K & K & K \\
\end{array}
\]

and

\[
\begin{array}{cccccc}
\begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \end{bmatrix} \\
K & K & K & K \\
\end{array}
\]

in $\text{rep}_K(\Sigma, J)$, respectively. Moreover, denote by $P_1^*$ the indecomposable projective $\Lambda$-module at the vertex 1 and by $I_1'$ the indecomposable injective $\Lambda$-module at the vertex 1', and observe that $P_0$ is the indecomposable projective $\Lambda$-module at the vertex 0 and $I_0$ is the indecomposable injective $\Lambda$-module at the vertex 0.

We claim that $B_{m,n}$ is a generalized double tilted algebra and the indecomposable modules $M_m$ and $N_n$ belong to a cycle-finite cyclic component $\Gamma_{m,n}$, and hence $B(\Gamma_{m,n}) = B_{m,n}$. More precisely,
we will show that $\Gamma_{m,n}$ is the cyclic part of the almost acyclic generalized standard component $C_{m,n}$ of $\Gamma_{B_{m,n}}$ obtained by identification the modules $R, T$ and $S_0$ occurring in the following two translation quivers: $C_m$ of the form

![Translation Quiver Diagram]

and $C_n^+$ of the form

![Translation Quiver Diagram]

Let $Q_m$ be the subquiver of $Q_{m,n}$ given by the vertices $1', 0, 1, 2, 3, 4, 5, 6, 7, \ldots, m-1, m$ and the arrows $\alpha, \beta, \gamma, \xi, \eta, \omega, \mu, \alpha_7, \alpha_8, \ldots, \alpha_{m-1}, \alpha_m, J_m$ the ideal in the path algebra $KQ_m$ of $Q_m$ over $K$ generated by $\alpha \beta, \sigma \alpha, \xi \eta - \eta \omega \mu$, and $C_m = KQ_m/J_m$ the associated bound quiver algebra. Then $C_m$ is the one-point extension algebra

$$
\begin{bmatrix}
\Lambda \times H_m & 0 \\
R \oplus I_m^* & K
\end{bmatrix}
$$

of $\Lambda \times H_m$ by the module $R \oplus I_m^*$, with the extension vertex 2. Since $I_m^*$ is the indecomposable injective module over the hereditary algebra $H_m$, we conclude that

$$\text{Hom}_{\Lambda \times H_m}(R \oplus I_m^*, \tau H_m X) = \text{Hom}_{H_m}(I_m^*, \tau H_m X) = 0$$

for any indecomposable module $X$ in $\text{mod } H_m$. Then, applying [65 Corollary XV.1.7] (or [81 Lemma 5.6]), we conclude that every almost split sequence in $\text{mod } H_m$ is an almost split sequence in $\text{mod } C_m$. This implies that the Auslander-Reiten quiver $\Gamma_{H_m}$ of $H_m$ is a full translation subquiver of the Auslander-Reiten quiver $\Gamma_{C_m}$ of $C_m$. Moreover, a direct calculation shows that the component $C_m$ of $\Gamma_{C_m}$, containing the indecomposable injective $H_m$-modules $I^*_i$, $i \in \{3, 4, 5, 6, 7, \ldots, m-1, m\}$, is the
translation quiver obtained from the translation quiver $C^-_m$ and the translation quiver below

\[ \begin{array}{ccc}
S_0 & \overset{R}{\rightarrow} & T_1' \\
S_1' & \overset{T}{\rightarrow} & S_1 \\
I_0 & \overset{P_0}{\rightarrow} & \text{T} \\
\end{array} \]

by identifying the common modules $R, T_1', S_1, T, I_0$ and $S_0$. We observe that the Auslander-Reiten quiver $\Gamma_{C_m}$ consists of the component $C_m$ and the components of $\Gamma_{H_m}$ different from the preinjective component. We also note that the indecomposable module $M_m$ is a unique sincere module in $\text{ind} C_m$, and $M_m$ is not a faithful module in $\text{mod} C_m$.

Dually, let $Q'_n$ be the subquiver of $Q_{m,n}$ given by the vertices $1, 0, 1', 2', 3', 4', 5', 6', 7', \ldots, (n-1)'$, $n$ and the arrows $\alpha, \beta, \gamma, \delta, \varphi, \varphi, \theta, \lambda, \beta_7, \beta_8, \ldots, \beta_{n-1}, \beta_n$, $J'_n$ the ideal in the path algebra $KQ'_{n}$ of $Q'_{n}$ over $K$ generated by $\alpha \beta, \beta \delta, \varphi \psi - \lambda \theta \nu$, and $C'_n = KQ'_{n}/J'_n$ the associated bound quiver algebra. Then $C'_n$ is the one-point extension algebra

\[
\begin{bmatrix}
K & 0 \\
\text{Hom}_K(R \oplus P^*_n, K) & \Lambda \times H^*_n
\end{bmatrix}
\]

of $\Lambda \times H^*_n$ by the module $R \oplus P^*_n$, with the coextension vertex $2'$. Since $P^*_n$ is the indecomposable projective module over the hereditary algebra $H^*_n$, we conclude that

\[
\text{Hom}_{\Lambda \times H^*_n}(\tau^{-1}_n Y, R \oplus P^*_n) = \text{Hom}_{H^*_n}(\tau^{-1}_n Y, P^*_n) = 0
\]

for any indecomposable module $Y$ in $\text{mod} H^*_n$. Then, applying the dual of [65 Corollary XV.1.7] (or [81 Lemma 5.6]), we conclude that every almost split sequence in $\text{mod} H^*_n$ is an almost split sequence in $\text{mod} C'_n$. This implies that the Auslander-Reiten quiver $\Gamma_{H^*_n}$ of $H^*_n$ is a full translation subquiver of the Auslander-Reiten quiver $\Gamma_{C'_n}$ of $C'_n$. Moreover, a direct calculation shows that the component $C'_n$ of $\Gamma_{C'_n}$, containing the indecomposable projective $H^*_n$-modules $P^*_n, j' \in \{3', 4', 5', 6', 7', \ldots, (n-1)', n'\}$, is the translation quiver obtained from the translation quiver $C^-_m$ and the translation quiver below

\[ \begin{array}{ccc}
S_0 & \overset{R}{\rightarrow} & T_1' \\
S_1' & \overset{T}{\rightarrow} & S_1 \\
I_0 & \overset{P_0}{\rightarrow} & \text{T} \\
\end{array} \]

by identifying the common modules $S_1', P_0, T_1, R, T$ and $S_0$. We observe that the Auslander-Reiten quiver $\Gamma_{C'_n}$ consists of the component $C'_n$ and the components of $\Gamma_{H'_n}$ different from the postprojective component. We also note that the indecomposable module $N_n$ is a unique sincere module in $\text{ind} C'_n$, and $N_n$ is not a faithful module in $\text{mod} C'_n$.

Further, we observe that the algebra $B_{m,n} = KQ_{m,n}/I_{m,n}$ is the one-point extension algebra

\[
\begin{bmatrix}
C'_n \times H_m & 0 \\
R \oplus I_m & K
\end{bmatrix}
\]

of $C'_n \times H_m$ by the module $R \oplus I_m$, with the extension vertex $2$. It follows from the structure of the Auslander-Reiten quiver $\Gamma_{C'_n}$ of $C'_n$ that, for any indecomposable module $Z$ in $\text{mod} C'_n$ nonisomorphic to the simple module $S_0$, we have

\[
\text{Hom}_{C'_n \times H_m}(R \oplus I_m, \tau_{C'_n} Z) = \text{Hom}_{C'_n}(R, \tau_{C'_n} Z) = 0.
\]

Then, applying [65 Corollary XV.1.7] (or [81 Lemma 5.6]) again, we conclude that every almost split sequence in $\text{mod} C'_n$ with the right term nonisomorphic to $S_0$ is an almost split sequence in $\text{mod} B_{m,n}$. This shows that the translation quiver obtained from $\Gamma_{C'_n}$ by removing the module $S_0$ and two arrows
attached to it is a full translation subquiver of $\Gamma_{B_{m,n}}$. In particular, we conclude that the almost split sequence in $\text{mod} \ C'_n$ with the left term $S_0$ is an almost split sequence in $\text{mod} \ B_{m,n}$.

Finally, we observe that the algebra $B_{m,n} = KQ_{m,n}/I_{m,n}$ is also the one-point coextension algebra

$$\begin{bmatrix} K & 0 \\ \text{Hom}_K(R \oplus P^*_n, K) & C_m \times H'_n \end{bmatrix}$$

of $C_m \times H'_n$ by the module $R \oplus P^*_n$, with the coextension vertex $2'$. It follows also from the structure of the Auslander-Reiten quiver $\Gamma_{C_m}$ of $C_m$ that, for any indecomposable module $Z$ in $\text{mod} \ C_m$ nonisomorphic to the simple module $S_0$, we have

$$\text{Hom}_{C_m \times H'_n}(\tau_{C_m}^{-1}Z, R \oplus P^*_n) = \text{Hom}_{C_m}(\tau_{C_m}^{-1}Z, R) = 0.$$

Then, applying the dual of [65, Corollary XV.1.7] (or [81, Lemma 5.6]) again, we conclude that every almost split sequence in $\text{mod} \ C_m$ with the left term nonisomorphic to $S_0$ is an almost split sequence in $\text{mod} \ B_{m,n}$. This shows that the translation quiver obtained from $\Gamma_{C_m}$ by removing the module $S_0$ and two arrows attached to it is a full translation subquiver of $\Gamma_{B_{m,n}}$. In particular, we obtain that the almost split sequence in $\text{mod} \ C_m$ with the right term $S_0$ is also an almost split sequence in $\text{mod} \ B_{m,n}$.

Summing up, we proved that $\Gamma_{B_{m,n}}$ contains the component $C_{m,n}$ of the required form, containing the preinjective component $Q(H_m)$ of $\Gamma_{H_m}$ as a full translation subquiver closed under predecessors and the postprojective component $P(H'_n)$ of $\Gamma_{H'_n}$ as a full translation subquiver closed under successors. Moreover, the Auslander-Reiten quiver $\Gamma_{B_{m,n}}$ of $B_{m,n}$ has a disjoint union decomposition

$$\Gamma_{B_{m,n}} = P_{m,n} \cup C_{m,n} \cup Q_{m,n}$$

such that

- $P_{m,n}$ is empty for $m \in \{8, 9, 10\}$;
- $P_{11,n}$ consists of the postprojective component $P(H_{11})$ of Euclidean type $\tilde{E}_8$ and an infinite family $\mathcal{T}^{H_{11}}$ of pairwise orthogonal generalized standard stable tubes in $\Gamma_{H_{11}}$;
- $P_{m,n}$, for $m \geq 12$, consists of the postprojective component $P(H_m)$ of wild type and an infinite family of regular components of the form $ZA_{\infty}$ in $\Gamma_{H_m}$;
- $Q_{m,n}$ is empty for $n \in \{8, 9, 10\}$;
- $Q_{m,11}$ consists of the preinjective component $Q(H'_{11})$ of Euclidean type $\tilde{E}_8$ and an infinite family $\mathcal{T}^{H'_{11}}$ of pairwise orthogonal generalized standard stable tubes in $\Gamma_{H'_{11}}$;
- $Q_{m,n}$, for $n \geq 12$, consists of the preinjective component $Q(H'_n)$ of wild type and an infinite family of regular components of the form $ZA_{\infty}$ in $\Gamma_{H'_n}$.

Finally, observe that $C_{m,n}$ is an almost acyclic component of $\Gamma_{B_{m,n}}$ whose cyclic part $\Gamma_{m,n}$ is connected and consists of all indecomposable modules in $C_{m,n}$ which lie on oriented cycles passing through the simple module $S_0$. In fact, $\Gamma_{m,n}$ is the unique multisection $\Delta$ of $C_{m,n}$, and so $\Delta = \Gamma_{m,n}$.

Further, $\Gamma_{m,n}$ contains the indecomposable modules $M_m$ and $N_n$. Since $M_m \oplus N_n$ is a faithful module in $\text{mod} \ B_{m,n}$, we conclude that $\Gamma_{m,n}$ is a faithful cyclic component of $\Gamma_{B_{m,n}}$, and hence $B_{m,n} = B(\Gamma_{m,n}) = B_{m,n}/\text{ann}B_{m,n}(\Gamma_{m,n})$. Observe also that $B_{m,n} = \text{Supp}(\Gamma_{m,n})$. In particular, $C_{m,n}$ is a faithful component of $\Gamma_{B_{m,n}}$. Moreover, the left part $\Delta_l$ of $\Delta$ coincides with $\tau_{B_{m,n}} \Delta'_l$ and consists of the indecomposable modules $I_{i}^l$, $i \in \{3, 4, \ldots, m - 1\}$, and the indecomposable modules $P_1$, $\tau_{B_{m,n}}^{-1}P_2$, $\tau_{B_{m,n}}^{-1}P_3$, $\tau_{B_{m,n}}^{-1}P_4$, $\tau_{B_{m,n}}^{-1}P_5$. Similarly, the right part $\Delta_r$ of $\Delta$ coincides with $\tau_{B_{m,n}}^{-1} \Delta'_r$ and consists of the indecomposable modules $P_{j'}$, $j' \in \{3', 4', \ldots, (n - 1)'\}$, and the indecomposable modules $I_{3}$, $\tau_{B_{m,n}}I_{2}$, $\tau_{B_{m,n}}I_{3}$, $\tau_{B_{m,n}}I_{4}$, $\tau_{B_{m,n}}I_{5}$. Observe also that $Q(H_m)$ is a generalized standard component of $\Gamma_{H_m}$, $P(H'_n)$ is a generalized standard component of $\Gamma_{H'_n}$, and $\text{Hom}_{B_{m,n}}(P, Q) = 0$ for any indecomposable modules $P \in P(H'_n)$ and $Q \in Q(H_m)$. This shows that $C_{m,n}$ is a generalized standard component of $\Gamma_{B_{m,n}}$. Then it follows from [63, Theorem 3.1] that $B_{m,n}$ is a generalized double tilted algebra. Moreover, the left tilted part $B_{m,n}^{(l)}$ is the product $H_m \times H'$ of $H_m$ and the path
algebra $H' = K\Omega'$ of the quiver $\Omega'$ of the form

```
1' \psi / \nu \rightarrow 2' \psi \\
\downarrow 3' \phi \downarrow \\
4' \theta \downarrow \\
\downarrow 5'
```

and Dynkin type $D_5$, while the right tilted part $B_{m,n}^{(r)}$ is the product $H_n' \times H$ of $H_n'$ and the path algebra $H = K\Omega$ of the quiver $\Omega$ of the form

```
3 \xi \rightarrow 2 \sigma \\
\downarrow \eta \\
\downarrow 1 \\
\downarrow \omega \\
\downarrow 4
```

and Dynkin type $D_5$. In particular, we obtain that $B_{m,n}$ is a tame generalized double tilted algebra (equivalently, cycle-finite algebra) if and only if $m, n \in \{8, 9, 10, 11\}$. Clearly, $B_{m,n}$ is of finite representation type if and only if $m, n \in \{8, 9, 10\}$. Finally, we note that the algebras $B_{m,n}$, for all $m, n \geq 8$, are of global dimension three, with the simple module $S_1$ having the projective dimension three.

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