On the global density slope–anisotropy inequality

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Starting from the central density slope–anisotropy theorem of An & Evans [1], recent investigations have shown that the involved density slope–anisotropy inequality holds not only at the center, but at all radii (i.e. globally) in a very large class of spherical systems with positive phase–space distribution function. Here we present some additional analytical cases that further extend the validity of the global density slope–anisotropy inequality. These new results, several numerical evidences, and the absence of known counter–examples, lead us to conjecture that the global density slope–anisotropy inequality could actually be a universal property of spherical systems with positive distribution function.

INTRODUCTION

In the study of stellar systems based on the “ρ–to–f” approach (where ρ is the material density and f is the associated phase–space distribution function, hereafter DF), ρ is given, and specific assumptions on the internal dynamics of the model are made (e.g. see [2], [3]). In some special cases inversion formulae exist and the DF can be obtained in integral form or as series expansion (see, e.g., [4]–[12]). Once the DF of the system is derived, a non–negativity check should be performed, and in case of failure the model must be discarded as unphysical, even if it provides a satisfactory description of data. Indeed, a minimal but essential requirement to be met by the DF (of each component) of a stellar dynamical model is positivity over the accessible phase–space. This requirement (also known as phase–space consistency) is much weaker than the model stability, but it is stronger than the fact that the Jeans equations have a physically acceptable solution. However, the difficulties inherent in the operation of recovering analytically the DF prevent in general a simple consistency analysis.

Fortunately, in special circumstances phase–space consistency can be investigated without an explicit recovery of the DF. For example, analytical necessary and sufficient conditions for consistency of spherically symmetric multi–component systems with Osipkov–Merritt (hereafter OM) anisotropy ([6], [7]) were derived in [13] (see also [14]) and applied in several investigations (e.g., [15]–[19]). Moreover, in [20] we derived analytical consistency criteria for the family of spherically symmetric, multi–component generalized Cuddeford [10] systems, which contains as very special cases constant anisotropy and OM systems.

Another necessary condition for consistency of spherical systems is given by the “central cusp–anisotropy theorem” by An & Evans [1], an inequality relating the values of the central logarithmic density slope γ and of the anisotropy parameter β of any consistent spherical system:
**Theorem** In every consistent system with constant anisotropy \( \beta(r) = \beta \) necessarily

\[
\gamma(r) \equiv \frac{d \ln \rho(r)}{d \ln r} \geq 2\beta \quad \forall r,
\]  

where

\[
\beta(r) \equiv 1 - \frac{\sigma^2_t(r)}{2\sigma^2(r)}.
\]  

Moreover the same inequality holds asymptotically at the center of every consistent spherical system with generic anisotropy profile.

In the following we call \( \gamma(r) \geq 2\beta(r) \forall r \) the *global* density slope–anisotropy inequality: therefore the An & Evans theorem states that constant anisotropy systems obey to the global density slope–anisotropy inequality. However, constant anisotropy systems are quite special, and so it was a surprise when we found ([18]) that the necessary condition for model consistency derived in [13] for OM anisotropic systems can be rewritten as the global density slope–anisotropy inequality. In other words, the global inequality holds not only for constant anisotropy systems, but also for each component of multi–component OM systems. Prompted by this result, in [20] we introduced the family of multi–component generalized Cuddeford systems, a class of models containing as very special cases both the multi–component OM models and the constant anisotropy systems. We studied their phase–space consistency, obtaining analytical necessary and sufficient conditions for it, and we finally proved that the global density slope–anisotropy inequality is again a necessary condition for model consistency!

The results of [18] and [20], here summarized, revealed the unexpected generality of the global density slope–anisotropy inequality. In absence of counter–examples (see in particular the Discussions in [20]) it is natural to ask whether the global inequality is just a consequence of some special characteristics of the DF of generalized Cuddeford systems, or it is even more general, i.e. it is necessarily obeyed by all spherically symmetric two–integrals systems with positive DF. Here we report on two new interesting analytical cases of models, not belonging to the generalized Cuddeford family, supporting the latter point of view. We also present an alternative formulation of the global density–slope anisotropy inequality. Therefore, even if a proof of the general validity of the global density slope–anisotropy inequality is still missing, some relevant advance has been made, and we now have the proof that entire new families of models do obey the global inequality (see [21] for a full discussion).

**THE DENSITY SLOPE–ANISOTROPY INEQUALITY**

**Multi–component Osipkov–Merritt systems**

The OM prescription to obtain radially anisotropic spherical systems assumes that the associated DF depends on the energy and on the angular momentum modulus of stellar orbits as

\[
f(\mathcal{E}, J) = f(Q), \quad Q = \mathcal{E} - \frac{J^2}{2r_a^2},
\]  

where \( r_a \) is the so–called anisotropy radius (e.g. see [3]). In the formula above \( \mathcal{E} = -\Psi_T - v^2/2 \) is the relative energy per unit mass, \( -\Phi_T = -\Psi_T \) is the relative (total) potential, and \( f(Q) = 0 \) for \( Q \leq 0 \). A multi–component OM system is defined as the superposition
of density components, each of them characterized by a DF of the family (2), but in general with different $r_a$. Therefore, unless all the $r_a$ are identical, a multi–component OM model is not an OM system. It is easy to prove that the radial dependence of the anisotropy parameter associated to such models is

$$\beta(r) = \frac{r^2}{r^2 + r_a^2},$$

i.e. systems are isotropic at the center and increasingly radially anisotropic with radius.

Consistency criteria for multi–component OM models have been derived in [13], while in [18] it was shown that a necessary condition for phase–space consistency of each density component can be rewritten as the global density slope-anisotropy inequality

$$\gamma(r) \geq 2\beta(r) \quad \forall r,$$

i.e. not only constant anisotropy systems but also multi–component OM models follow the global inequality.

**Multi–component generalized Cuddeford systems**

An interesting generalization of OM and constant anisotropy systems was proposed by Cuddeford ([10]; see also [22]), and is obtained by assuming

$$f(\mathcal{E}, J) = J^{2\alpha} h(Q),$$

(5)

where $\alpha > -1$ is a real number and $Q$ is defined as in equation (2). Therefore, both the OM models ($\alpha = 0$), and the constant anisotropy models ($r_a \to \infty$), belong to the family (5). In particular, it is easy to show that from equation (5)

$$\beta(r) = \frac{r^2 - \alpha r_a^2}{r^2 + r_a^2}.$$  

(6)

Remarkably, also for these models a simple inversion formula links the DF to the density profile ([10]). Such inversion formula still holds for multi–component, generalized Cuddeford systems, that we have introduced in [20]. *Each* density component of a generalized Cuddeford model has a DF given by the sum of an arbitrary number of Cuddeford DFs with arbitrary positive weights $w_i$ and possibly different anisotropy radii $r_{ai}$ (but same $h$ function and angular momentum exponent), i.e.

$$f = J^{2\alpha} \sum_i w_i h(Q_i), \quad Q_i = \mathcal{E} - \frac{J^2}{2r_{ai}^2}.$$  

(7)

Of course, the orbital anisotropy distribution characteristic of DF (7) is *not* a Cuddeford one, and quite general anisotropy profiles can be obtained by specific choices of the weights $w_i$, the anisotropy radii $r_{ai}$, and the exponent $\alpha$. However, near the center
$\beta(r) \sim -\alpha$, and $\beta(r) \sim 1$ for $r \to \infty$, independently of the specific values of $w_i$ and $r_{ai}$.

In [20], we have found necessary and sufficient conditions for the consistency of multi–component generalized Cuddeford systems. At variance with the simpler case of OM models, the new models admit a family of necessary conditions, that can be written as simple inequalities involving repeated differentiations of the augmented density expressed as a function of the total potential. Surprisingly, we also showed that the first of the necessary conditions for phase–space consistency can be reformulated as the global density slope–anisotropy inequality (4), which therefore holds at all radii for each density component of multi–component generalized Cuddeford models.

HOW GENERAL IS THE DENSITY SLOPE–ANISOTROPY INEQUALITY?

The natural question posed by the analysis above is whether the global density slope–anisotropy inequality is a peculiarity of multi–component generalized Cuddeford models: after all, only models in this (very large) family have been proved to obey the global inequality. We now continue our study by showing, by direct computation, that two well-known anisotropic models, whose analytical DF is available and not belonging to the generalized Cuddeford family, indeed obey to the global density slope–anisotropy inequality. A full discussion of the following cases, and their place in a broader context, will be presented in [21].

The Dejonghe (1987) anisotropic Plummer model

Dejonghe [9], by using the augmented density approach, studied a family of (one–component) anisotropic Plummer models, with normalized density–potential pair

$$\rho = \frac{3}{4\pi} \frac{\Psi^{5-q}}{(1+r^2)^{q/2}}, \quad \Psi = \frac{1}{\sqrt{1+r^2}}. \quad (8)$$

Both the radial trend of orbital anisotropy and the model DF were recovered analytically:

$$\beta(r) = \frac{q}{2} \frac{r^2}{1+r^2}; \quad f = e^{7/2-q} g \left( \frac{f^2}{2e} \right), \quad (9)$$

where $g$ belongs to the family of hypergeometric functions. In [9] it is shown that the consistency requirement $f \geq 0$ imposes the limitation $q \leq 2$. Well, a direct computation of the logarithmic density slope of the Plummer model (8), together with equation (9), proves that these models obey to the global density slope–anisotropy inequality when $q \leq 2$. 
The Baes & Dejonghe (2002) anisotropic Hernquist model

Baes & Dejonghe [23] considered a family of one–component anisotropic Hernquist models whose normalized density–potential pair is

$$\rho = \frac{(1 + r^2)^{(2(\beta_0 - \beta_{\infty}) - \Psi^{4/2 - 2\beta_0}}}{2\pi r^2\beta_0 (1 - \Psi)^{1 - 2\beta_0}}, \quad \Psi = \frac{1}{1 + r},$$

with $\beta_{\infty} \leq \beta_0$. The corresponding anisotropy parameter and DF are

$$\beta(r) = \frac{\beta_0 + \beta_{\infty}r}{1 + r}; \quad f = e^{5/2 - 2\beta_{\infty} + \beta_0}J^{-2\beta_0} \sum_k \left( \frac{J^2}{2\epsilon} \right)^k g_k(\epsilon),$$

so that $\beta_0$ and $\beta_{\infty}$ are the anisotropy values at the center and at large radii of the system, respectively; note that in this family of models the orbital anisotropy decreases moving away from the center. In equation (11) $g_k$ are hypergeometric functions and, in accordance with the “cusp slope–central anisotropy theorem”, the request of non–negativity imposes $\beta_0 \leq 1/2$ (see [23]). Note that, as in the previous case, the DF is not of the generalized Cuddeford family. Again a comparison of the logarithmic density slope of Hernquist profile (10) with equation (11) shows that, when $\beta_{\infty} \leq \beta_0$ and $\beta_0 \leq 1/2$ also these models obey the global inequality (4)!

Alternative formulation of the density slope–anisotropy inequality

While we refer the reader to [21] for a full discussion of the new results, and for how these find place in a more general context, here we show that the density slope–anisotropy inequality can also be expressed as a condition on the radial velocity dispersion. In fact, the relevant Jeans equation in spherical symmetry reads

$$\frac{d\rho \sigma_r^2}{dr} + \frac{2\beta \rho \sigma_r^2}{r} = \rho \frac{d\Psi_T}{dr}$$

(e.g., [3]). Introducing the logarithmic density slope and rearranging the terms, one finds

$$\gamma(r) - 2\beta(r) = r \left( \frac{d\sigma_r^2}{dr} - \frac{d\Psi_T}{dr} \right) \geq 0$$

as an equivalent, alternative formulation of the density slope–anisotropy inequality. Of course, the proof that a given family of self–consistent models obeys inequality (13) is not easier than the proof that would be obtained by working on phase–space.

CONCLUSIONS

We have shown analytically that two more models, in addition to the whole family of multi–component generalized Cuddeford systems, satisfy the global density slope–anisotropy inequality as a necessary condition for phase–space consistency. This reinforces the conjecture that the global slope–anisotropy relation (4) could be a universal
necessary condition for consistent spherical systems. We recall that additional evidences supporting such idea exist: for example Michele Trenti kindly provided us with a large set of numerically computed \( f_\nu \) models [24], and all of them, without exception, satisfy the inequality \( \gamma(r) \geq 2\beta(r) \) at all radii. Additional numerical findings are mentioned in [20].

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