The modified Power function distribution

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Abstract: Recently, a lot of new, improved, flexible and robust probability distributions have been developed from the existing distributions to encourage their applications in diverse fields. This paper proposes a new lifetime distribution called the Modified Power function (MPF) distribution, the distribution belongs to the Marshall-Olkin-G family of distribution and it’s an extension of the one parameter Power function distribution. The MPF distribution enjoys a close form distributional expression. Some of its statistical properties including possible transformations are presented. The paper suggests the use of maximum likelihood method of parameter estimation for estimating the parameters of the new distribution. The applicability of the distribution was illustrated with two real data-sets and its goodness-of-fit was compared with that of the Exponential, Weibull, Lindley Exponential, Exponentiated Exponential, Kumaraswamy, Power function and Beta distributions by using the AIC, AICc, CAIC, BIC, HQC, W∗ and A∗ goodness-of-fit measures and the results shows that the MPF distribution is the best candidate for the data-sets.

Keywords: Power function distribution; Marshall-Olkin-G family; reliability; failure rate

1. Introduction
Over many decades ago the survival/reliability and failure characteristics of such devices, items, or equipments that are liable to falling out of use have hitherto been studied with so many probability

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PUBLIC INTEREST STATEMENT
The accuracy and dependability of results of any statistical modelling rely heavily on the choice of model used. Right data with wrong model often yield vague and misleading estimates and predictions; and consequently, to erroneous and complicated interpretations and conclusions. This paper presents a new model called the Modified Power function (MPF) distribution. The distribution is an important generalization of the Power function distribution whose beauty stems from its very simplicity and handiness in modelling variables that takes values between 0 and 1. For example data arising from meteorological measurements and several other scientific experiments often take such values and can be modelled by the MPF distribution. Additionally, in real application situation the distribution have a better fitting capability in comparison with some other well-known distributions.
distributions by reliability Engineers, Statisticians, Actuarial scientist and others. However, the intricacy of data-sets that regularly evolve in practice have rendered standard probability distributions like the: Exponential, Weibull, Kumaraswamy, Pareto, Lindley, Log-logistic, Power function, Gamma, Dagum, Raleigh, Log-normal, Fréchet, Gumbel, etc; unfit for modelling such complicated data-sets. Interestingly, a lot of methods have been introduced in the literature for improving the flexibility of these distributions in order to provide better fits to the ever increasing complex data-sets in their many guises. For instance, we present a list of some of the available methods in the literature that is used for constructing new family of probability distributions with their corresponding pioneer authors as follows: Exponentiated-G family by Gupta and Kundu (2001), Transmuted-G family by Shaw and Buckley (2009), Marshall–Olkin-G family by Marshall and Olkin (1997), Beta-G family by Eugene, Lee, and Famoye (2002), Zografos-Balakrishnan-G family by Nadarajah, Cordeiro, and Ortega (2015), Kumaraswamy-G family by Cordeiro and de Castro (2011), Lomax-G family by Cordeiro, Ortega, Popović, and Pescim (2014), Symmetric-G skew family by Ferreira and Steel (2012), McDonald-G family by Alexander, Cordeiro, Ortega, and Sarabia (2012), Lindley-G family by Cakmakyapan and Ozel (2016), Gamma-G family by Zografos and Balakrishnan (2009), Generalized beta-G family by Alexander et al. (2012), Gamma uniform-G family by Torabi and Hedesh (2012), Beta exponential-G family by Alzaatreh, Lee, and Famoye (2013), Modified beta-G family by Nadarajah, Teimouri, and Shih (2014), Exponentiated exponential Poisson-G family by Ristić and Nadarajah (2014), Truncated-exponential skew-symmetric-G family by Nadarajah, Nassiri, and Mohammadpour (2014); etc. All these extended families are robust and have additional flexibility compared to the original distribution in which they were derived from due to the presence of additional shape parameter(s).

The idea in this paper is to extend the one parameter Power function distribution to a more flexible distribution by adopting the Marshall and Olkin’s scheme, in order to make it more versatile in analysing a variety of complex data-sets. For a collection of different versions of the Power function distribution readers are referred to Table 1 of Tahir, Alizadeh, Mansoor, Cordeiro, and Zubair (2014). The cumulative density function (cdf) of the particular parameterization of the Power function distribution under study in this paper is of the form

\[ G(x) = 1 - (1 - x)^\delta; \quad 0 < x < 1; \quad \delta > 0, \quad (1) \]

while the corresponding probability density function (pdf) is given by:

\[ g(x) = \delta (1 - x)^{\delta - 1}; \quad 0 < x < 1; \quad \delta > 0; \quad (2) \]

where \( \delta \) is a shape parameter. This distribution is a competitor and a special case of the beta distribution. The distribution is useful in describing the characteristics of random variables that are confined in the open interval of 0 and 1 in real life. For instance, the distribution could be used in meteorology to model sunshine data as in Table 2 of Sulaiman, Oo, Wahab, and Zakaria (1999).

The formulation of the Marshall–Olkin family of distributions is as follows

\[ \bar{F}(x) = \frac{\gamma G(x)}{1 - (1 - \gamma)G(x)}; \quad -\infty < x < \infty; \quad 0 < \gamma < \infty, \quad (3) \]

while the cdf is \( F(x) = 1 - \bar{F}(x) \) and the pdf is given by:
where $\bar{F}(x)$ is the complementary cumulative density function (reliability/survival function) of the Marshall-Olkin-G family, $\bar{G}(x)$ and $g(x)$ are the reliability and pdf corresponding to the baseline distribution (original distribution), respectively, and $\gamma$ is the additional shape parameter. Notably, the Marshall-Olkin distribution reduces to the baseline distribution when $\gamma = 1$. To see some of the distributions that have been modified according to Marshall and Olkin (1997) readers are referred to Barreto-Souza, Lemonte, and Cordeiro (2013) and Cordeiro, Lemonte, and Ortega (2014).

Different parameterizations of the Power function distribution have been extensively studied in the literature. For example; Meniconi and Barry (1996) proposed the two parameter Power function distribution as a simple alternative to the Exponential distribution when it comes to modelling failure data; particularly, those that are related to electrical components. Tahir et al. (2014) extended the two-parameter Power function distribution to a more general and flexible four-parameter Weibull-Power function distribution as an adequate distribution for modelling survival data. They studied some of its statistical properties and the bivariate extension was also proposed. Naveed Shahzad and Asghar (2016) proposed the Transmuted Power function distribution generalizing the Power function distribution according to Shaw and Buckley (2009). Hanif, Al-Ghamdi, Khan, and Shahba (2015) estimated the parameter of the one-parameter Power function distribution using the Bayesian method (Gibbs sampler). They compared the performance of the estimates obtained with two priors (Weibull and Generalized gamma distribution) with those obtained by the method of maximum likelihood. Saleem, Aslam, and Economou (2010) modelled a heterogeneous population using the two-component mixture of one-parameter Power function distribution. Zarrin, Saxena, and Kamal (2013) provides some analytical results for reliability computation and Bayesian estimation for a system reliability whose applied stress and strength behaves like the two-parameter Power function distribution. Naveed-Shahzad, Asghar, Shehzad, and Shahzadi (2015) showed by Monte-Carlo simulation study that, the method of L-moments provides better estimates of the parameters of the two-parameter Power function distribution than the one that is based on the methods of moments and maximum likelihood. Zaka and Akhter (2013) estimated the parameters of the two-parameter Power function distribution through the methods of least squares, relative least squares and ridge regression.

Kumar and Khan (2014) presented concise expressions for single and product moments of the generalized order statistics (gos) of the three-parameter Power function distribution and discussed their characterization based on the conditional moments of the gos. Saran and Pandey (2004) derived and discussed the linear unbiased estimates of the parameters of a three-parameter Power function distribution based on the kth record values. Chang (2007) provides the characterizations of the two-parameter Power function distribution using the independence record values. Lim and Lee (2013) gave some proof of a characterization of the two-parameter Power function distribution using the lower record values. And Ahsanullah, Shakil, and Golam Kibria (2013) presented a new characterization of the two-parameter Power function distribution based on the lower records.

The rest of this paper contains the following sections: Section 2 is the introduction of the Modified Power function (MPF) distribution and its properties; Section 3 is the parameter estimation; Section 4 is the Monte-Carlo simulation study; Section 5 is the application of the new distribution and Section 6 is the conclusion.

2. Model definition

The $cdf$ of the MPF distribution is given by;

$$F(x) = 1 - \frac{\gamma(1-x)^\delta}{1 - (1 - \gamma)(1-x)^\delta}; \quad 0 < x < 1; \quad \delta; \quad \gamma > 0,$$

(5)

with pdf

$$f(x) = \frac{\gamma g(x)}{(1 - (1 - \gamma)\bar{G}(x))^2}; \quad -\infty < x < \infty; \quad 0 < \gamma < \infty,$$

(4)
\( f(x) = \gamma \delta (1 - x)^{\delta - 1}[1 - (1 - \gamma)(1 - x)^\delta]^{-2}; \ 0 < x < 1; \ \delta, \ \gamma > 0, \) \hfill (6)

while, the reliability function \( F(x) \), which gives the probability that a system will survive beyond a specified time say \( x \) is defined by;

\[ F(x) = \frac{\gamma (1 - x)^\delta}{1 - (1 - \gamma)(1 - x)^\delta}, \] \hfill (7)

and the reversed hazard rate function \( rhf \) which is defined by;

\[ H(x) = \lim_{\Delta x \to 0} \frac{P[x - \Delta x < X \leq |X \leq x]}{\Delta x} = \frac{f(x)}{1 - F(x)} = \frac{\gamma \delta (1 - x)^{\delta - 1}[1 - (1 - \gamma)(1 - x)^\delta]}{(1 - (1 - \gamma)(1 - x)^\delta)[(1 - (1 - \gamma)(1 - x)^\delta) - \gamma(1 - x)^\delta]}, \]

and it gives the instantaneous failure rate of a system at time \( x \) given that if failed before time \( x \).

### 2.1. Asymptotics and shapes

The asymptotic and shape characteristics of the \( pdf \) in Equation (6) and the \( rhf \) in Equation (8) are outlined in this section.

The asymptotic behaviour of the \( pdf \) is

\[ f(0) = \frac{\delta}{\gamma}; \ \delta, \ \gamma > 0 \]

and

\[ f(1) = \begin{cases} \infty, & \text{if } \delta < 1, \ \gamma > 0, \\ 0, & \text{if } \delta \geq 1, \ \gamma > 0. \end{cases} \]

While the asymptotic behaviour of the reversed hazard rate function is given by:

\[ H(0) = H(1) = \infty; \ \forall \ \delta, \ \gamma > 0. \]

To characterize the shape of the MPF distribution, we start by obtaining the first derivative of its \( pdf \) which we have as

\[ f'(x) = -\frac{\gamma \delta (1 - x)^{\delta - 1}(\delta - 1)}{(1 - x)[1 - (1 - \gamma)(1 - x)^\delta]^2} = \frac{2\gamma \delta^2(1 - x)^{\delta - 1}(1 - \gamma)(1 - x)^\delta}{[1 - (1 - \gamma)(1 - x)^\delta]^3(1 - x)}. \]

The \( f'(x) < 0 \) indicate that the MPF distribution could be monotone decreasing. It has a critical point \( x_0 \) at which it is maximum, the critical point of the function is given by;

\[ x_0 = 1 - \exp \left( -\frac{\log \left( \frac{\delta - 1}{\delta \gamma - \delta + 1} \right)}{\delta} \right); \ \gamma, \ \delta > 1. \]

Thus, \( \exists \ x < x_0 \) such that \( f(x) \) is increasing and \( x > x_0 \) such that \( f(x) \) is decreasing, then we say that \( f(x) \) has a single mode at \( x_0 \). For different parameter values, the \( pdf \) of the MPF distribution could take any of the following shapes:
\[
\lim_{x \to 0} f(x) = \begin{cases} 
\text{constant}, & \text{if } \delta, \gamma = 1, \\
\text{increasing}, & \text{if } \delta \leq 1 \text{ and } \gamma > 1 \text{ or if } \delta < 1 \text{ and } \gamma \geq 1, \\
\text{decreasing}, & \text{if } \delta > 1 \text{ and } \gamma \leq 1 \text{ or if } \delta \geq 1 \text{ and } \gamma < 1, \\
\text{unimodal}, & \text{if } \delta, \gamma > 1. 
\end{cases}
\]

Another important shape characteristic of the MPF distribution is bathtub shape. This shape can be verified by showing that the density function in Equation (6) is convex.

\[
f''(x) = \frac{\gamma \delta (1-x)^{\delta-1} \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]} - \frac{\gamma \delta (1-x)^{\delta-1} \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]^2} + \frac{4\gamma \delta^2 (1-x)^{\delta-1} \delta - 1 \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]^2} \frac{6\gamma \delta^3 (1-x)^{\delta-1} \delta - 1 \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]^2} \frac{2\gamma \delta^3 (1-x)^{\delta-1} \delta - 1 \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]^2} - \frac{2\gamma \delta^3 (1-x)^{\delta-1} \delta - 1 \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]^2}
\]

Clearly, \(f''(x) > 0\), and the convexity of \(f(x)\) is confirmed. Figure 1 illustrate all the possible shapes of the pdf of the MPF distribution.

To characterize the shape of the reversed hazard rate function \(H(x)\) of the MPF distribution, we start by obtaining its first derivative which we obtained as

\[
H'(x) = \frac{\gamma \delta (1-x)^{\delta-1} \delta - 1}{(1-x)(1-(1-\gamma)(1-x)\delta)} - \frac{\gamma \delta^2 (1-x)^{\delta-1} \delta - 1 \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]} \frac{\gamma \delta^3 (1-x)^{\delta-1} \delta - 1 \delta - 1}{(1-x)^2[1-(1-\gamma)(1-x)\delta]^2}
\]

\(H'(x) < 0\) implies that the reversed hazard rate function is decreasing (reverse-J) and \(H'(x)\) has two real roots at

\[
X_0^{(1,2)} = \begin{cases} 
1 - \left[\frac{\gamma \delta^2 + \gamma \delta + \gamma}{2(\delta \gamma - \delta \gamma - 1)}\right]^\frac{1}{2}, & \delta \gamma - \delta + \gamma - 1 > 0, \\
1 - \left[\frac{\gamma \delta^2 + \gamma \delta + \gamma}{2(\delta \gamma - \delta \gamma - 1)}\right]^\frac{1}{2}, & \delta \gamma - \delta + \gamma - 1 > 0.
\end{cases}
\]

Hence, for some \(\delta\) and \(\gamma\), \(X < X_0^{(1,2)}\) such that \(H(x)\) is decreasing, \(X_0^{(1)} < X < X_0^{(2)}\) such that \(H(x)\) is constant and \(X > X_0^{(2)}\) such that \(H(x)\) is increasing. Thus, the rhf of the MPF distribution is bathtub shaped.
See Figure 2 for possible shapes of the reversed hazard rate function of the MPF distribution.
2.2. Transformations
If a random variable $X$ is continuous and differentiable, then it can be transformed into another random variable $Y$ by $f(y) = f(x)/|dy/dx|$ and using a suitable transformation relation such as $y = f(x)$, where $|dy/dx|$ is called the Jacobian of the transformation. We have presented some of the possible transformations of the MPF distribution in Appendix 1.

2.3. Quantile function and random number generation
By inverting Equation (5) we obtain the quantile function of the MPF distribution as

$$X_Q = 1 - \left[ \frac{1 - Q}{1 - Q + \gamma} \right]^\frac{1}{\delta}; 0 < Q < 1; \delta; \gamma > 0. \quad (9)$$

When $Q = 0.50$ in Equation (9) we obtain the median of the MPF distribution as

$$X_{0.50} = 1 - \left( \frac{1}{1 + \gamma} \right)^\frac{1}{\delta}; \delta; \gamma > 0.$$

If $Q \sim U(0, 1)$ then we can simulate random variables from the MPF distribution through the inversion of $cdf$ method with Equation (9). Also, using Equation (9) we can obtain the Bowley skewness (B) due to Bowley (1901/1920) and Moors kurtosis (M) due to Moors (1986). Remarkably, these measures does not depend on the moments of the distribution and are almost insensitive to outliers, unlike the classical skewness and kurtosis statistics.

The Bowley skewness statistic is given by:

$$B = \frac{X_{3/4} + X_{1/4} - 2X_{2/4}}{X_{3/4} - X_{1/4}},$$

while, the Moors kurtosis statistic is given by:

$$M = \frac{X_{1/8} - X_{1/8} + X_{7/8} - X_{5/8}}{X_{6/8} - X_{2/8}}.$$
Figure 3 illustrates the Bowley skewness (right panel) and Moors kurtosis (left panel) for the MPF distribution.

The plots of the Bowley skewness of the MPF distribution indicate that the distribution could be asymmetric (positive or negative) or symmetric, while the plots of the Moors kurtosis indicate that the distribution is heavy tailed. Notably, the variability of the two measures depends on the values of the two shape parameters ($\gamma$ and $\delta$).

2.4. Useful expansion

To present a straightforward analytical derivation of some important properties of the new distribution, the following expansion of the pdf of the MPF distribution in Equation (6) is handy. Given that $0 \leq q < 1$ we have that

$$f(x) = y\delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} [1 - y]^{i} [1 - x]^{\delta(i+1)-1}; 0 < x < 1; \delta, y > 0.$$  \hspace{1cm} (10)

2.5. Moments

**Lemma 2.1** If $X$ follows the MPF distribution then, its $k$th crude moment is given by:

$$\mu'_k = y\delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} [1 - y]^{i} \cdot B(k+1; \delta[i+1]).$$

where $B(\cdot; \cdot)$ is the Beta function which is defined as $B(\alpha; \beta) = \int_{0}^{1} x^{\alpha-1} (1 - x)^{\beta-1} \, dx$.

**Proof** Using Equation (10) we have

$$\mu'_k = \int_{0}^{1} x^k f(x) \, dx$$

$$= \int_{0}^{1} x^{k} y\delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} [1 - y]^{i} [1 - x]^{\delta(i+1)-1} \, dx$$

$$= y\delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} [1 - y]^{i} \cdot B(k+1; \delta[i+1]).$$ \hspace{1cm} (11)
Corollary 2.1.1 The first and second moment are given by:

\[ \mu'_1 = \gamma \delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} (1 - \gamma)^i \cdot B(2; \delta(i+1)), \]

and

\[ \mu'_2 = \gamma \delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} (1 - \gamma)^i \cdot B(3; \delta(i+1)), \]

respectively, while the variance \( \text{V}(X) \) is given by:

\[ \text{V}(X) = \gamma \delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} (1 - \gamma)^i \cdot \{B(3; \delta(i+1))\}^2 \]

2.6. Generating function

Apart from producing moments, the moment generating function (mgf) could be used to describe and characterize the distribution of a random variable say \( X \).

Lemma 2.2 If \( X \) is distributed according to the MPF distribution, then its mgf \( M_X(t) \) could be obtained through the general definition of the mgf of a continuous random variable, which is defined as

\[ M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx}f(x)dx = \sum_{k=0}^{\infty} \frac{t^k \mu'_k}{\Gamma(k+1)}. \]

Thus the mgf of the MPF distribution is

\[ M_X(t) = \gamma \delta \sum_{k=0}^{\infty} \frac{t^k \Gamma(i+2)}{\Gamma(i+1)\Gamma(k+1)\Gamma(2)} (1 - \gamma)^i \cdot B(k+1; \delta(i+1)). \]

Proof The proof is straightforward, hence we omit it.

2.7. Incomplete moment

In Economics, the incomplete moment forms a basic tool for constructing measures of inequality such as the Lorenz and Bonferroni curves. The kth incomplete moment of a continuous random variable is defined as

\[ \mathcal{I}(X, k) = \int_{-\infty}^{\infty} \hat{f}(x)dx. \]

Lemma 2.3 If \( X \) is distributed according to the MPF distribution then, its kth incomplete moment is given by:

\[ \mathcal{I}(X, k) = \gamma \delta \sum_{i=0}^{\infty} \frac{\Gamma(i+2)}{\Gamma(i+1)\Gamma(2)} (1 - \gamma)^i \cdot B_x(k+1; \delta(i+1)) \]

where \( B_x(\cdot; \cdot) \) is the incomplete Beta function which is defined as \( B_x(\alpha; \beta) = \int_{0}^{x} x^{\alpha-1}(1 - x)^{\beta-1}dx. \)

Proof The proof is analogous to that of Lemma 2.1.

2.8. Entropy

Entropies are used to quantify the variation, likelihood, or randomness of a random variable. In this section, we present the Rényi entropy measure due to Rényi (1961) of the MPF distribution. The Rényi
entropy generalizes the following entropies: Hartley, Shannon, collision and min-entropy and it is
given by:

\[ I_\varphi(\varphi) = \frac{1}{1 - \varphi} \log \left( \int f^\varphi(x)dx \right), \quad \text{for } \varphi > 0 \setminus \{1\}. \]  

\[ (12) \]

**Lemma 2.4** If \( X \) follows the MPF distribution then, its Rényi entropy measure is given by:

\[ I_\varphi(\varphi) = \frac{1}{1 - \varphi} \log \left( \gamma \delta \sum_{i=0}^\infty \frac{\Gamma(i + 2)}{\Gamma(i + 1)\Gamma(2)} \right)^\varphi \cdot \frac{(1 - \gamma)^\psi}{\psi \delta(i + 1) - 1 + 1}. \]

\[ (13) \]

**Proof** Using Equation (12) we have

\[ X^\varphi = \left[ \gamma \delta \sum_{i=0}^\infty \frac{\Gamma(i + 2)}{\Gamma(i + 1)\Gamma(2)} (1 - \gamma)\Gamma(i + 1) \right]^\varphi \cdot \frac{(1 - \gamma)^\psi}{\psi \delta(i + 1) - 1 + 1}. \]

\[ (14) \]

substituting \( y = 1 - x \) into Equation (13) we have

\[ X^\varphi = \left[ \gamma \delta \sum_{i=0}^\infty \frac{\Gamma(i + 2)}{\Gamma(i + 1)\Gamma(2)} (1 - \gamma)\Gamma(i + 1) \right]^\varphi \cdot \frac{(1 - \gamma)^\psi}{\psi \delta(i + 1) - 1 + 1}. \]

thus, the Rényi entropy measure

\[ I_\varphi(\varphi) = \frac{1}{1 - \varphi} \log \left( \gamma \delta \sum_{i=0}^\infty \frac{\Gamma(i + 2)}{\Gamma(i + 1)\Gamma(2)} \right)^\varphi \cdot \frac{(1 - \gamma)^\psi}{\psi \delta(i + 1) - 1 + 1}. \]

\[ (15) \]

2.9. Lorenz curves

Lorenz curve is a popular tool in Economics which gives a graphical representation of the distribution
of income or wealth. It was originally introduced by Lorenz (1905) for characterizing the inequality
of wealth distribution. Ever since the introduction, Lorenz curve have increasingly been applied in
other fields including reliability. Lorenz curve of a continuous random variable is defined as

\[ L(x) = \frac{1}{\mu} \int_{-\infty}^x tf(t)dt. \]

**Lemma 2.5** If \( X \) is MPF distributed then, its Lorenz curve is given by:

\[ L(x) = \sum_{i=1}^\infty \frac{B_i(2, \delta(i + 1))}{B(2, \delta(i + 1))}, \]

where \( B(\cdot; \cdot) \) is the Beta function and \( B_x(\cdot; \cdot) \) is the incomplete Beta function.

**Proof** The proof is similar to that of Lemma 2.1.
2.10. Order statistics
The distribution of the kth order statistics denoted by \( f_{x_{(k)}}(x) \) of an n sized random sample \( X_1, X_2, X_3, \ldots, X_n \) is generally given by:

\[
f_{x_{(k)}}(x) = \frac{n!}{(k-1)! (n-k)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} f(x)
\]

\[
= \frac{n!}{j!(k-1)! (n-k-j)!} f(x) \sum_{j=0}^{n-k} (-1)^j [F(x)]^{j+k-1}
\]

The density of the kth order statistics of the MPF distribution could be obtained by substituting

Equations (5) and (10) into Equation (16) then, we have the kth order statistics of the MPF distribution as

\[
f_{x_{(k)}}(x) = \delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{r=0}^\infty \frac{(-1)^j x^r \Gamma(n+1) \Gamma(j+k+1) [1-\gamma]^j [1-x]^m}{\Gamma(j+1) \Gamma(k) \Gamma(n-k-j+1) \Gamma(\ell+1) \Gamma(\ell+k-\ell)}
\]

The rth crude moment of the kth order statistics of the MPF distribution is obtained as follows

\[
E(X_{(k)}^r) = \int_0^1 x^r [1-x]^\ell [1-\gamma] [1-x]^m \, dx
\]

\[
= \frac{\ell!}{\ell!(\ell+r+m+1)} \Gamma(\ell+1) \Gamma(\ell+r+m+1)
\]

where

\[
J_{i,j,\ell}(x) = \delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{r=0}^\infty \frac{(-1)^j x^r \Gamma(n+1) \Gamma(j+k+1) [1-\gamma]^j [1-x]^m}{\Gamma(j+1) \Gamma(k) \Gamma(n-k-j+1) \Gamma(\ell+1) \Gamma(\ell+k-\ell)}
\]

and

\[
J'_{i,j,\ell,m}(x) = \delta \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{r=0}^\infty \frac{(-1)^j x^r m \Gamma(n+1) \Gamma(j+k+1) [1-\gamma]^j [1-x]^m}{\Gamma(j+1) \Gamma(k) \Gamma(n-k-j+1) \Gamma(\ell+1) \Gamma(\ell+k-\ell)}
\]

3. Parameter estimation
Here, we estimate the parameters of the MPF distribution through the method of maximum likelihood. Suppose X is distributed according to the distribution with pdf \( f(x) \) then; the probability of obtaining the estimates of the parameters \( \theta \) that could generate the observed sample of size n, say \( x_1, x_2, \ldots, x_n \), is referred to as the maximum likelihood while, the estimates are called the maximum likelihood estimates.

The \textit{mle} of any random variable X is defined by;

\[
L(\theta) = \prod_{i=1}^n f(x_i; \theta) ; \theta \in \Theta
\]
where $\theta$ is the vector of parameters belonging to $f(x)$ and $\Theta$ is the parameter space.

The mle of the MPF distribution is derived using Equation (6) as follows

\[
\mathcal{L}(\theta) = \prod_{i=1}^{n} \{\gamma \delta (1 - x_i)^{\delta-1} [1 - (1 - \gamma)(1 - x_i)^{\delta}]^{-2}\} = (\gamma \delta)^n \prod_{i=1}^{n} \{(1 - x_i)^{\delta-1} [1 - (1 - \gamma)(1 - x_i)^{\delta}]^{-2}\}
\]

where $\theta = (\delta, \gamma)'$.

The log-likelihood function $\mathcal{L}(\theta)$ is given by:

\[
\mathcal{L}(\theta) = n \log(\gamma \delta) + (\delta - 1) \sum_{i=1}^{n} \log(1 - x_i) - 2 \sum_{i=1}^{n} \log(1 - [1 - \gamma][1 - x_i]^\delta)
\]

and the partial derivatives of Equation (21) w. r. t $\gamma$ and $\delta$ are

\[
\frac{\partial \mathcal{L}(\theta)}{\partial \gamma} = \frac{n}{\gamma} - 2 \sum_{i=1}^{n} \frac{[1 - x_i]^{\delta}}{1 - [1 - \gamma][1 - x_i]^\delta},
\]

and

\[
\frac{\partial \mathcal{L}(\theta)}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n} \log(1 - x_i) + 2 \sum_{i=1}^{n} \frac{[1 - \gamma][1 - x_i]^\delta \log(1 - x_i)}{1 - [1 - \gamma][1 - x_i]^\delta},
\]

respectively.

The mle estimates of $\gamma$ and $\delta$ could be obtained by setting Equations (22) and (23) to zero and solving them simultaneously. But, these equations cannot be solved analytically because of their nonlinear structure. Indeed, there are a lot of in-built optimization functions in most of the available mathematical and statistical softwares that one could use, instead. For instance, the mle and mle2 functions that are under the bbmle package and the nlm, nlminb and optim functions which are under the stats package, etc. could offer numerical solution to such problems in R, R-language (R Core Team, 2013).

Denote $\Omega = (\hat{\gamma}, \hat{\delta})'$, we have that under some standard regularity conditions, $\sqrt{n}(\hat{\Theta} - \Theta)$ have a multivariate normal distribution $N_2(0, J_{-1}(\Theta))$, where $J_{-1}(\Theta)$ is the expected information matrix defined by: $-E(\partial^2 \mathcal{L}(\Theta)/\partial \Theta \partial \Theta')$. The asymptotic behaviour of the expected information matrix can be approximated by the observed information matrix, denoted by $I_{-1}(\Theta)$. Where the diagonal entries of $I_{-1}(\Theta)$ are the variance of ($\hat{\Theta}$) while the off-diagonal entries are the covariances. Given that $\sqrt{n}(\hat{\Theta} - \Theta) \sim N_2(0, I_{-1}(\hat{\Theta}))$ is available, we can perform statistical inference for functions of $\Theta$. For example, the approximate $100(1 - \epsilon)$% two-sided confidence interval of the model parameters $\Theta$ could be calculated as:

\[
\hat{\Theta} \pm Z_{\frac{\epsilon}{2}} \sqrt{I_{-1}(\hat{\Theta})},
\]

where $I_{-1}(\hat{\Theta})$ are the diagonal entries of the observed information matrix, and $Z_{\frac{\epsilon}{2}}$ is the upper $\epsilon/2$th percentile of the standard normal distribution.

The observed information matrix of the MPF distribution is given by:

\[
I_{-1}(\hat{\Theta}) = -\left(\begin{array}{cc}
\frac{\partial^2 \mathcal{L}(\Theta)}{\partial \gamma^2} & \frac{\partial^2 \mathcal{L}(\Theta)}{\partial \gamma \partial \delta} \\
\frac{\partial^2 \mathcal{L}(\Theta)}{\partial \delta \partial \gamma} & \frac{\partial^2 \mathcal{L}(\Theta)}{\partial \delta^2}
\end{array}\right).
\]
where

\[
\frac{\partial^2 \hat{L}(\Omega)}{\partial \gamma^2} = -\frac{n}{\gamma^2} + 2 \sum_{i=1}^{n} \frac{(1-x_i)^{2\delta}}{(1-\gamma)(1-x_i)^\delta}^2,
\]

\[
\frac{\partial^2 \hat{L}(\Omega)}{\partial \delta^2} = -\frac{n}{\delta^2} + 2 \sum_{i=1}^{n} \left[ \frac{(1-\gamma)(1-x_i)^\delta \log^2(1-x_i)}{1-(1-\gamma)(1-x_i)^\delta} + \frac{(1-\gamma)^2(1-x_i)^{2\delta} \log^2(1-x_i)}{(1-\gamma)(1-x_i)^\delta}^2 \right],
\]

and

\[
\frac{\partial^2 \hat{L}(\Omega)}{\partial \gamma \partial \delta} = -2 \sum_{i=1}^{n} \left[ \frac{(1-x_i)^{\delta} \log(1-x_i)}{1-(1-\gamma)(1-x_i)^\delta} + \frac{(1-\gamma)(1-x_i)^{2\delta} \log(1-x_i)}{1-(1-\gamma)(1-x_i)^\delta}^2 \right].
\]

4. Monte-Carlo simulation

To investigate the performance of the mle estimates in estimating the parameters \(\Omega = (\delta, \gamma)\) of the MPF distribution we conduct a Monte-Carlo simulation study. First, different sample sizes 20, 30, 40, \ldots, 400 were drawn from the MPF distribution with parameters \(\delta\) (10.00 and 7.00) and \(\gamma\) (0.25 and 0.80) using the inverse transform method; secondly, the parameters of the distribution were estimated and; finally, the algorithm switches between drawing a specific sample size \((n)\) and estimating the parameters of the MPF distribution in 5,000 \((N)\) iterations. For each sample size the parameter estimates, standard errors, bias and mean square errors MSE were computed and plotted in Figures 4 and 5. Where

- \(\bar{\Omega} = 1/N \sum_{i=1}^{n} \hat{\Omega}_i\)
- \(SE_\delta = \sqrt{\sum_{i=1}^{N} (\hat{\Omega}_i - \bar{\Omega})^2 / (N - 1)}\)
- \(Bias = \bar{\hat{\Omega}} - \Omega; i = 1, \ldots, n\)
- \(MSE = 1/N \sum_{i=1}^{n} (\hat{\Omega}_i - \Omega)^2\)

The simulation results as shown in Figures 4 and 5 indicates that the mle method provides good estimates of the parameters of the MPF distribution because the parameter estimates stabilizes and approximates to the true value as \(n\) increases while the standard errors, bias and MSE decreases with increasing \(n\).
5. Application

In this section we demonstrate the usefulness of the MPF distribution with two data-sets. The first data whose basic statistics are presented in Table 1 are on the Anxiety performance of a group of 166 normal women that were reported in Bourguignon, Ghosh, and Cordeiro (2016). The second illustration is based on the Evaporation data in Table 2 that was extracted from the monthly publication of climatological data of the National Oceanic and Atmospheric Administration (NOAA). The data is on the daily pan evaporation in hundredths of inches that was recorded in September 2016 in San Joaquin Drainage 05 Friant Government Camp, California, USA. The data is freely available at http://www.ncdc.noaa.gov/oa/ncdc.html. Some of the basic statistics of the Evaporation data are listed in Table 3. The values of the two data-sets correspond to the support of the MPF distribution and this characteristic is one of the main motivation for these illustrations.

The MPF distribution with seven other competing distributions: exponential (Exp), Weibull, Lindley Exponential (LE) due to Bhati, Malik, and Vaman (2015), Exponentiated exponential (EE) due to Gupta and Kundu (2001), Kumaraswamy, Power function (PF) and Beta distribution would be fitted to the data-sets and their performance would be compared by the following goodness-of-fit measures: AIC, AICc, CAIC, BIC, HQC, $W^*$ statistics due to Cramér (1928) and Von Mises (1928) and $A^*$ statistics due to Anderson and Darling (1952).

### Table 1. Some descriptive statistics of the Anxiety data

| n         | Min | Max  | $Q_1$  | $Q_3$  | Mean  | Median |
|-----------|-----|------|--------|--------|-------|--------|
| 166.000000 | 0.010000 | 0.690000 | 0.010000 | 0.130000 | 0.091205 | 0.030000 |
| NAs       | Sum | SE Mean | Variance | St dev | Skewness | Kurtosis |
| 0.000000  | 15.140000 | 0.010285 | 0.017560 | 0.132513 | 2.211515 | 4.862916 |

### Table 2. Evaporation data

|       | 0.28 | 0.29 | 0.27 | 0.17 | 0.33 | 0.26 | 0.32 | 0.24 |
|-------|------|------|------|------|------|------|------|------|
| 0.28  | 0.26 | 0.30 | 0.29 | 0.21 | 0.23 | 0.25 | 0.27 | 0.26 |
| 0.32  | 0.28 | 0.18 | 0.18 | 0.26 | 0.42 | 0.11 | 0.34 | 0.32 |
Akaike information criterion (AIC),

\[
\text{AIC} = -2 \hat{\mathcal{L}} + 2k
\]

AIC with a correction (AICc),

\[
\text{AICc} = \text{AIC} + \frac{2k(k+1)}{n-k-1}
\]

Consistent Akaike information criterion (CAIC),

\[
\text{CAIC} = -2 \hat{\mathcal{L}} + k(\log(n) + 1)
\]

Bayesian information criterion (BIC),

\[
\text{BIC} = -2 \hat{\mathcal{L}} + k \log(n)
\]

Hannan-Quinn information criterion (HQC),

\[
\text{HQC} = -2 \hat{\mathcal{L}} + 2k \log \log(n)
\]

Cramér-von Mises \(W^*\) criterion,

\[
W^* = \frac{1}{12n} + \sum_{i=1}^{n} \left[ \frac{2i-1}{2n} - \hat{F}(x_i) \right]^2
\]

Anderson-Darling \(A^*\) criterion,

\[
A^* = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1)[\log \hat{F}(x_i) + \log(1 - \hat{F}(x_{n+1-i}))]
\]

where \(\hat{\mathcal{L}}\), \(k\), \(n\) and \(\hat{F}(\cdot)\) correspond to the estimate of the model maximized log-likelihood function, number of parameters in the distribution, sample size of the fitted data and the estimated distribution function under the ordered data, respectively.

The distribution with the smallest goodness-of-fit measures is considered as the best candidate for the given data. Results from the model fittings are listed in Tables 4 and 5, while Figures 6 and 7 shows the pdf’s and cdf’s of the estimated distributions superimposed on the empirical ones.

The estimated Fisher information matrix of the MPF distribution under the Anxiety data shows that the mle of \(\gamma\) and \(\delta\) converges to the global maximum. The inverse of this matrix gives the variance-covariance matrix of the distribution parameters. The Fisher information matrix is given by:
The estimated Fisher information matrix of the MPF distribution under the Evaporation data shows that the mle of $\gamma$ and $\delta$ converges to the global maximum. The inverse of this matrix gives the variance-covariance matrix of the distribution parameters. The Fisher information matrix is given by:

$$I_n(\hat{\Omega}) = \begin{pmatrix} 8618.0894 & -338.49807 \\ -338.4981 & 14.48771 \end{pmatrix}.$$  

The estimated Fisher information matrix of the MPF distribution under the Evaporation data shows that the mle of $\gamma$ and $\delta$ converges to the global maximum. The inverse of this matrix gives the variance-covariance matrix of the distribution parameters. The Fisher information matrix is given by:

$$I_n(\hat{\Omega}) = \begin{pmatrix} 5.177010 \times 10^{-6} & -0.002349911 \\ -2.349911 \times 10^{-3} & 1.143606048 \end{pmatrix}.$$  

Comparing the results in Tables 4 and 5, one can quickly see that the MPF distribution with the smallest goodness-of-fit statistics appears the best candidate for the data under consideration and the pdf (left panel) and cdf (right panel) plots of the fitted distributions in Figures 6 and 7 does not suggest otherwise regarding the appropriacy of the MPF distribution in modelling the two data-sets.
Table 5. Results from modelling the Evaporation data

| Models  | Estimates   | SE  | AIC       | AICc       | CAIC       | HQC       | BIC       | W*        | A*        |
|---------|-------------|-----|-----------|------------|------------|-----------|-----------|------------|-----------|
| MPF     | 1412.42721  | 1694.2972 | -81.69939 | -81.25494 | -76.89699 | -80.80287 | -78.89699 | 0.05896    | 0.38224   |
| γ       | 22.71681    | 3.60488 |           |            |            |           |           |            |           |
| δ       | 3.699135    | 0.675434 | -16.48597 | -16.34311  | -14.08477 | -16.03772 | -15.08477 | 14.15077   | 8.69495   |
| Exp     | 79.49685    | -79.05241 | -74.69446 | -78.60034  | -76.69446 |           |           | 0.17703    | 0.57561   |
| 𝑎       | 5.0164320   | 0.68282010 |          |            |            |           |           |            |           |
| 𝑏       | 0.2935308   | 0.01125956 |          |            |            |           |           |            |           |
| LE      | 36.38278    | 16.203963 | -70.44084 | -69.9964   | -65.63845 | -69.54433 | -67.63845 | 0.75064    | 1.52985   |
| 𝑎       | 15.03919    | 1.995754 |           |            |            |           |           |            |           |
| 𝛾       | 35.42952    | 16.159197 | -70.45895 | -70.01451  | -65.65656 | -69.56244 | -67.65656 | 0.74429    | 1.52852   |
| 𝛿       | 15.03847    | 1.992487 |           |            |            |           |           |            |           |
| Kumaras- | -79.47926   | -79.03482 | -74.67687 | -78.58275  | -76.67687 | -76.67687 |           | 0.17211    | 0.57551   |
| swalmy  | 5.011266    | 0.688616 |           |            |            |           |           |            |           |
| 𝛾       | 464.365774  | 374.21693 | -25.75337 | -25.61051  | -23.35217 | -25.30511 | -24.35217 | 13.62780   | 8.00053   |
| 𝛿       | 3.139363    | 0.5732239 |           |            |            |           |           |            |           |
| Beta    | -77.61904   | -77.17459 | -72.81664 | -76.72253  | -74.81664 |           |           | 0.77112    | 0.84949   |
| 𝛾       | 13.32467    | 3.407187 |           |            |            |           |           |            |           |
| 𝛿       | 36.01972    | 9.320629 |           |            |            |           |           |            |           |

Figure 6. The estimated density plots of the fitted distributions superimposed on the empirical density plot of the Anxiety data.
6. Concluding remarks

This paper have contributed a new distribution to the Marshall-Olkin-G family of distributions. The new distribution is a generalization of the one parameter Power function distribution called the MPF distribution. Some of the statistical properties of the MPF distribution have been derived, such as the quantile, skewness, kurtosis, moments, variance, generating functions, Rényi’s entropy, Lorenz curve, order statistics and some useful transformations are given. The method of maximum likelihood estimation (mle) was used to estimate the parameters of the new distribution and results based on Monte-Carlo simulation study, supports the use of the mle method for estimating the parameters of the MPF distribution. The goodness-of-fit of the MPF distribution was demonstrated with two real data-sets and the results from the data modelling shows that the MPF distribution offers a better fit to the data-sets than the other competing distributions. We hope that the new proposed distribution will be highly utilized across all relevant fields.

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Erratum

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References

Ahmad, M., Shakil, M., & Golam Kibria, B. M. G. (2013). A characterization of the power function distribution based on lower records. ProbStat Forum, 6, 68–72.
Alexander, C., Cordeiro, G. M., Ortega, E. M., & Sarabia, J. M. (2012). Generalized beta-generated distributions. Computational Statistics & Data Analysis, 56, 1880–1897.

Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71, 63–79.
Anderson, T. W., & Darling, D. A. (1952). Asymptotic theory of certain “goodness of fit” criteria based on stochastic processes. The Annals of Mathematical Statistics, 23, 193–212.
Barreto-Souza, W., Lemonte, A. J., & Cordeiro, G. M. (2013). General results for the Marshall-Olkin’s family of distributions. Anais da Academia Brasileira de Ciências, 85, 3–21.
Bhati, D., Malik, M. A., & Vaman, H. J. (2015). Lindley-Exponential distribution: properties and applications. Metron, 73, 335–357.
Bourguignon, M., Ghosh, J., & Cordeiro, G. M. (2016). General results for the transmuted family of distributions and new models. Journal of Probability and Statistics, 2016, 12 pages.
Bowley, A. L. (1901/1920). Elements of Statistics. London: P.S. King & Son. Or in a later edition: Bowley, A. L. Elements of Statistics, 4th Edn (New York, Charles Scribner).
Cakmakyapan, S., & Ozel, G. (2016). The Lindley family of distributions: Properties and applications. Hacettepe Journal of Mathematics and Statistics, 46, 1–27.
Castellares, F., & Lemonte, A. J. (2016). On the Marshall-Olkin extended distributions. Communications in Statistics-Theory and Methods, 45, 4537–4555.
Chang, S. K. (2007). Characterizations of the power function distribution by the independence of record values. Journal of the Chungcheong Mathematical Society, 20, 139–146.

Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. Journal of Statistical Computation and Simulation, 81, 883–898.

Cordeiro, G. M., Lemonte, A. J., & Ortega, E. M. (2014). The Marshall-Olkin family of distributions: Mathematical properties and new models. Journal of Statistical Theory and Practice, 8, 343–366.

Cordeiro, G. M., Ortega, E. M., Popovic, B. V., & Pescim, R. R. (2014). The Lomax generator of distributions: Properties, minification process and regression model. Applied Mathematics and Computation, 247, 465–486.

Cramer, H. (1928). On the composition of elementary errors. Almqvist & Wiksells.

Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics—Theory and methods, 31, 497–512.

Ferreira, J. T. S., & Steel, M. F. (2012). A constructive representation of univariate skewed distributions. Journal of the American Statistical Association, 101, 823–835.

Ghitany, M. E., Al-Hussaini, E. K., & Al-Jarallah, R. A. (2005). Marshall-Olkin extended Weibull distribution and its application to censored data. Journal of Applied Statistics, 32, 1025–1034.

Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: An alternative to gamma and Weibull distributions. Biometrical Journal, 43, 117–130.

Hanif, S., Al-Ghamdi, S. D., Khan, K., & Shahbaz, M. Q. (2015). Bayesian estimation for parameters of power function distribution under various priors. Mathematical Theory and Modeling, 5.

Kumar, D., & Khan, R. U. (2014). Moments of power function distribution based on ordered random variables and characterization. Sri Lankan Journal of Applied Statistics, 15, 91–105.

Lim, E. H., & Lee, M. Y. (2013). A characterization of the power function distribution by independent property of lower record values. Journal of the Chungcheong Mathematical Society, 26, 269–273.

Lorenz, M. O. (1905). Methods of measuring the concentration of wealth. Publications of the American Statistical Association, 9, 209–219.

Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika, 84, 641–652.

Meniconi, M., & Barry, D. M. (1998). The power function distribution: A useful and simple distribution to assess electrical component reliability. Microelectronics Reliability, 38, 1207–1212.

Moors, J. J. A. (1986). The meaning of kurtosis: Darlington reexamined. The American Statistician, 40, 283–284.

Nadarajah, S., Cordeiro, G. M., & Ortega, E. M. (2015). The Zografos-Balakrishnan-G family of distributions: Mathematical properties and applications. Communications in Statistics—Theory and Methods, 44, 186–215.

Nadarajah, S., Nossir, V., & Momhamadpour, A. (2014). Truncated-exponential skew-symmetric distributions. Statistics, 48, 872–895.

Nadarajah, S., Teimouri, M., & Shih, S. H. (2014). Modified beta distributions. Sankhya B, 76, 19–48.

Naveed Shahzad, M., & Asghar, Z. (2014). Transmuted power function distribution: A more flexible distribution. Journal of Statistics and Management Systems, 17, 519–539.

Naveed-Shahzad, M., Asghar, Z., Shehzad, F., & Shahzadi, M. (2015). Parameter estimation of power function distribution with TL-moments. Revista Colombiana de Estadistica, 38, 321–334.

Pal, M., Ali, M. M., & Woo, J. (2006). Exponentiated Weibull distribution. Statistics, 66, 139–147.

R Core Team. (2013). R: A language and environment for statistical computing. Vienna: R Foundation for Statistical Computing. Retrieved from http://www.Rproject.org/

Rényi, A. L. F. R. E. D. (1961). On measures of entropy and information. In Fourth Berkeley symposium on mathematical statistics and probability (Vol. 1, pp. 547–561). Berkeley: University of California Press.

Ristić, M. M., & Kundu, D. (2015). Marshall-Olkin generalized exponential distribution. Metron, 73, 317–333.

Ristić, M. M., & Nadarajah, S. (2014). A new lifetime distribution. Journal of Statistical Computation and Simulation, 84, 135–150.

Saleem, M., Ashlam, M., & Economou, P. (2010). On the Bayesian analysis of the mixture of power function distribution using the complete and the censored sample. Journal of Applied Statistics, 37, 25–40.

Saran, J., & Pandey, A. (2004). Estimation of parameters of a power function distribution and its characterization by k-th record values. Statistica, 64, 523–536.

Shaw, W. T., & Buckley, I. R. (2009). The alchemy of probability distributions: Beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.

Sulaiman, M. Y., Oo, W. H., Wahab, M. A., & Zakaria, A. (1999). Application of beta distribution model to Malaysian sunshine data. Renewable Energy, 18, 573–579.

Tahir, M., Alizadeh, M., Mansoor, M., Cordeiro, G. M., & Zubair, M. (2015). The Weibull-Power function distribution with applications. Hacetettepe University Bulletin of Natural Sciences and Engineering Series B: Mathematics and Statistics.

Torabi, H., & Hedesh, N. M. (2012). The gamma-uniform distribution and its applications. Kybernetika, 48, 16–30.

Von Mises, R. (1928). The alchemy of probability. Statistik und Wahrheit. Almqvist & Wiksells.

Zografos, K., & Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inferences. Statistical Methodology, 6, 344–362.
Appendix 1

**Lemma 1.1** If $X \sim$ MPF distribution with $\gamma; \delta > 0$ and $y = \left[-1/\lambda \log(x)\right]^{1/\delta}$ then $Y \sim$ MOW (Marshall-Olkin Weibull) distribution with parameters $\lambda; \beta; \gamma; \delta > 0$ due to Ghitany, Al-Hussaini, and Al-Jarallah (2005).

**Corollary 1.1.1** The MOW distribution reduces to the MOE (Marshall-Olkin exponential) due to Ristić and Kundu (2015), EW (exponentiated Weibull) due to Pal, Ali, and Woo (2006), EE (exponentiated exponential) due to Gupta and Kundu (2001), Weibull and exponential distribution when $\beta = 1; \gamma = 1; \beta; \gamma; \delta = 1$ and $\gamma; \beta; \delta = 1$, respectively.

**Lemma 1.2** If $X \sim$ MPF distribution with $\gamma; \delta > 0$ and $y = 1 - x$ then $Y \sim$ MOPF (Marshall-Olkin Power function) distribution ($f(y) = \gamma \delta y^{\delta - 1}[1 - (1 - \gamma) y^{\delta}]^{-2}; 0 < y < 1; \gamma; \delta > 0$)-New.

**Corollary 1.2.1** The MOPF distribution reduces to the baseline distribution ($f(y) = \delta y^{\delta - 1}; 0 < y < 1; \delta > 0$) when $\gamma = 1$.

**Lemma 1.3** If $X \sim$ MPF distribution with $\gamma; \delta > 0$ and $y = x^{1/\delta}$ then $Y \sim$ MOKw (Marshall-Olkin Kumaraswamy) distribution due to Castellares and Lemonte (2016) ($f(y) = \gamma \delta y^{\alpha - 1}[1 - y^{\alpha}]^{\delta - 1}[1 - \gamma][1 - y^{\alpha}]^{\delta - 1}[1 - (1 - \gamma)[1 - y^{\alpha}]^{\delta - 1}]; 0 < y < 1; \alpha; \gamma; \delta > 0$).

**Corollary 1.3.1** The MOKw distribution reduces to the Kumaraswamy distribution when $\gamma = 1$.

**Proof** The proofs of Lemmas 1.1, 1.2 and 1.3 are very trivial; hence, have been omitted. □