Tangent Lie algebra of derived Artin stacks

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December 12, 2013

Abstract
Since the work of Mikhail Kapranov in [Kap], it is known that the shifted tangent complex $T_X[-1]$ of a smooth algebraic variety $X$ is endowed with a weak Lie structure. Moreover any complex of quasi-coherent sheaves on $X$ is endowed with a weak Lie action of this tangent Lie algebra. We will generalize this result to (finite enough) derived Artin stacks, without any smoothness assumption. This in particular applies to (finite enough) singular schemes. This work uses tools of both derived algebraic geometry and $\infty$-category theory.

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Introduction
It is a common knowledge that the shifted tangent complex $T_X[-1]$ of a nice enough geometric stack $X$ in characteristic zero should be endowed with a Lie structure. Moreover any quasi-coherent sheaf on $X$ should admits a somehow natural structure of representation of the Lie algebra $T_X[-1]$. In the article [Kap], Mikhail Kapranov proves the existence of such a structure in (the homotopy category of) the derived category of quasi-coherent sheaves over $X$, when $X$ is a complex manifold. The Lie bracket is there given by the Atiyah class of the tangent complex. The Lie-algebra $T_X[-1]$ should encode the geometric structure of the formal neighbourhood of the diagonal $X \to X \times X$. Moreover, given a $k$-point of $X$, the pullback of the tangent Lie algebra corresponds to a formal moduli problem as Vladimir
Hinich and later Jacob Lurie studied in [Hin] and in [DAG-X]. This formal moduli problem is the formal neighbourhood of the point at hand.

In this article we use the tools of derived algebraic geometry (see [TV] for a review) and the machinery of ∞-categories (as in [HTT]) to define formal stacks over a derived stack and to study the link with Lie algebras over X. This leads to our main theorem.

**Theorem.** Let X be an algebraic derived stack locally of finite presentation over a field k of characteristic zero.

- There is an ∞-category dSt_X^f of formal stacks over X and an adjunction
  \[ F_X : \text{dgLie}_X \rightleftarrows \text{dSt}_X^f : \mathcal{L}_X \]

- Let \( \ell_X \) denote the image of the formal completion of the diagonal of X by \( \mathcal{L}_X \). The underlying module of \( \ell_X \) is quasi-isomorphic to the tangent complex of X shifted by \(-1\) (Theorem 2.0.1).

- The forgetful functor
  \[ \text{dgRep}_X(\ell_X) \to \text{L}_{\text{qcoh}}(X) \]
  from representations of \( \ell_X \) to quasi-coherent sheaves over X admits a colimit-preserving section
  \[ \text{L}_{\text{qcoh}}(X) \to \text{dgRep}_X(\ell_X) \]
  (Theorem 2.3.8).

This result specializes to the case of a smooth algebraic variety X. It ensures \( T_X[-1] \) have a weak Lie structure in the derived category of complexes of quasi-coherent sheaves over X. Another consequence is a weak action of \( T_X[-1] \) over any complex of quasi-coherent sheaves over X, in the sense of [Kap]. Let us also emphasize that the adjunction of the first item is usually not an equivalence. It is when the base X is affine and noetherian.

In the first part of this article, we build an adjunction between formal stacks and dg-Lie algebras when the base is a commutative differential graded algebra A over a field of characteristic zero. In the second part, we define some notion of formal stacks over any derived stack X. Gluing the adjunction of part 1, we obtain an adjunction between formal stacks over X and quasi-coherent Lie algebras over X. We then prove the existence of the Lie structure \( \ell_X \) on \( T_X[-1] \). The last pages deal with the action of \( \ell_X \) on any quasi-coherent sheaf over X.

The author is grateful to Bertrand Toën and to Marco Robalo for the many interesting discussions we had about this article. He would also like to thank Dimitri Ara, Mathieu Anel and Damien Calaque.

**A bit of conventions.** Throughout this article k will be a field of characteristic zero. We will use the toolbox about ∞-categories from [HTT]. Given A a commutative dg-algebra over k concentrated in non positive degree, we will denote:

- by \( \text{dgMod}_A \) the ∞-category of (unbounded) dg-modules over A.
- by \( \text{cdga}_A \) the ∞-category of (unbounded) commutative dg-algebras over A.
- by \( \text{cdga}_A^{\leq 0} \) the ∞-category of A-dg-algebras concentrated in non positive degree.
- by \( \text{dgLie}_A \) the ∞-category of (unbounded) dg-Lie algebras over A.

Each one of those four ∞-categories appears as the underlying ∞-category of a model category, which we will denote the same way. When \( \mathcal{C} \) is an ∞-category, we will denote by \( \mathcal{P}(\mathcal{C}) \) the ∞-category of simplicial presheaves over \( \mathcal{C} \).
1 Lie algebras and formal stacks over a cdga

In this part we will mimic a construction found in Lurie’s [DAG-X]

**Theorem 1.0.1** (Lurie). Let \( k \) be a field of characteristic zero. There is an adjunction of \( \infty \)-categories:

\[
C_k : \text{dgLie}_k \rightleftarrows \left( \text{cdga}_{k/k} \right)^{\text{op}} : D_k
\]

Whenever \( L \) is a dg-Lie algebra:

(i) If \( L \) is freely generated by a dg-module \( V \) then the algebra \( C_k(L) \) is equivalent to the square zero extension \( k \otimes V \langle -1 \rangle \).

(ii) If \( L \) is concentrated in positive degree and every vector space \( L^n \) is finite dimensional, then the adjunction morphism \( L \to D_k C_k L \) is an equivalence.

The goal is to extend this result to more general basis, namely a commutative dg-algebra over \( k \) concentrated in non positive degree. The existence of the adjunction and the point (i) will be proved over any basis, the analog of point (ii) will need the base dg-algebra to be noetherian.

Throughout this section, \( A \) will be a commutative dg-algebra concentrated in non-positive degree over a field \( k \) of characteristic zero.

### 1.1 Algebraic theory of dg-Lie algebras

Let us consider the adjunction \( \text{Free} : \text{dgLie}_A \rightleftarrows \text{dgMod}_A : \text{Forget} \). Given an \( \infty \)-category \( \mathcal{C} \) with all finite coproducts, we will denote by \( \mathcal{P}_\Sigma(\mathcal{C}) \) its completion under sifted colimits.

Recall (see [HTT], part 5.5.8) that this is the \( \infty \)-category \( \text{Fct}^\times(\mathcal{C}^{\text{op}}, \text{sSets}) \) of \( \infty \)-functors preserving finite products.

**Definition 1.1.1.** Let \( \text{dgMod}_A^{f, n \geq 1} \) denote the full-\( \infty \)-category of \( \text{dgMod}_A \) spanned by the free dg-modules of finite type whose generators are in positive degree. An object of \( \text{dgMod}_A^{f, n \geq 1} \) is thus (equivalent to) the dg-module

\[
\bigoplus_{i=1}^n A^p[-i]
\]

for some \( n \geq 1 \) and some family \( (p_1, \ldots, p_n) \) of non negative integers.

Let \( \text{dgLie}_A^{f, n \geq 1} \) denote the essential image of \( \text{dgMod}_A^{f, n \geq 1} \) in \( \text{dgLie}_A \) by the \( \infty \)-functor Free.

**Proposition 1.1.2.** The Yoneda functors

\[
\text{dgMod}_A \to \mathcal{P}_\Sigma(\text{dgMod}_A^{f, n \geq 1}) = \text{Fct}^\times \left( \left( \text{dgMod}_A^{f, n \geq 1} \right)^{\text{op}}, \text{sSets} \right)
\]

\[
\text{dgLie}_A \to \mathcal{P}_\Sigma(\text{dgLie}_A^{f, n \geq 1}) = \text{Fct}^\times \left( \left( \text{dgLie}_A^{f, n \geq 1} \right)^{\text{op}}, \text{sSets} \right)
\]

are equivalences of \( \infty \)-categories.

**Remark 1.1.3.** The above proposition implies that every dg-Lie algebra is colimit of a sifted diagram of objects in \( \text{dgLie}_A^{f, n \geq 1} \).
Proof. Every dg-module can be obtained as the colimit of a diagram in $\text{dgMod}_A^{f,\geq 1}$ and objects of $\text{dgMod}_A^{f,\geq 1}$ are compact projective in $\text{dgMod}_A$ (an object is compact projective if the functor it corepresents preserves sifted colimits). The proposition 5.5.8.25 from [HTT] makes the first functor an equivalence.

The forgetful functor $\text{Forget}$ is monadic. Every dg-Lie algebra can thus be obtained as a colimit of a simplicial diagram with values in the $\infty$-category of free dg-Lie algebras (see [HAlg], prop.6.2.2.12). From those two facts we deduce that every dg-Lie algebra is a colimit of objects in $\text{dgLie}_A^{f,\geq 1}$. The forgetful functor $\text{Forget}$ preserves sifted colimits and objects in $\text{dgLie}_A^{f,\geq 1}$ are thus compact projective in $\text{dgLie}_A$. □

Remark 1.1.4. The above proposition implies that when $A \to B$ is a morphism in $\text{cdga}_k^{\leq 0}$, the following square of $\infty$-categories commutes:

\[
\begin{array}{ccc}
\text{dgLie}_A & \sim & \mathcal{P}_\Sigma(\text{dgLie}_A^{f,\geq 1}) \\
B \otimes_A - & \downarrow & (B \otimes_A -) \downarrow \\
\text{dgLie}_B & \sim & \mathcal{P}_\Sigma(\text{dgLie}_B^{f,\geq 1})
\end{array}
\]

The following proposition actually proves that this comes from a natural transformation between functors $\text{cdga}_k^{\leq 0} \to \text{Cat}_\infty$.

Proposition 1.1.5. There are cofibered $\infty$-categories $\int \text{dgLie}$ and $\int \mathcal{P}_\Sigma(\text{dgLie}_A^{f,\geq 1})$ over $\text{cdga}_k^{\leq 0}$ and an equivalence over $\text{cdga}_k^{\leq 0}$:

\[
\begin{array}{ccc}
\int \mathcal{P}_\Sigma(\text{dgLie}_A^{f,\geq 1}) & \sim & \int \text{dgLie} \\
\downarrow & & \downarrow \\
\text{cdga}_k^{\leq 0} & \to & \text{cdga}_k^{\leq 0}
\end{array}
\]

Proof. Let us define $C$ as the following category.

- An object is couple $(A, L)$ where $A \in \text{cdga}_k^{\leq 0}$ and $L \in \text{dgLie}_A$.
- A morphism $(A, L) \to (B, L')$ is a morphism $A \to B$ and a morphism of $A$-dg-Lie algebras $L \to L'$.

We define $\int \text{dgLie}$ to be the localization of $C$ along quasi-isomorphisms. Using proposition 2.4.19 of [DAG-X], there is a cocartesian fibration of $\infty$-categories $p: \int \text{dgLie} \to \text{cdga}_k^{\leq 0}$.

Let us define $D$ as the following category.

- An object is couple $(A, f)$ where $A \in \text{cdga}_k^{\leq 0}$ and $f$ is a simplicial presheaf over $\text{dgLie}_A^{f,\geq 1}$.
- A morphism $(A, f) \to (B, g)$ is a morphism $A \to B$ and a morphism of presheaves $f \to g(B \otimes_A -)$.

A morphism in $D$ is an equivalence if the map of dg-algebras is a quasi-isomorphism and if the map of presheaves is a local equivalence (regarding to the localization $\mathcal{P}(\text{dgLie}_A^{f,\geq 1}) \to \mathcal{P}_\Sigma(\text{dgLie}_A^{f,\geq 1})$). We define $\int \mathcal{P}_\Sigma(\text{dgLie}_A^{f,\geq 1})$ to be the localization of $D$ along equivalences. Using once again proposition 2.4.19 of [DAG-X], there is a cocartesian fibration of $\infty$-categories $p: \int \mathcal{P}_\Sigma(\text{dgLie}_A^{f,\geq 1}) \to \text{cdga}_k^{\leq 0}$.
We can define the same way a cocartesian fibration \( \mathcal{P}(\mathbf{dgLie}^{f,r,\geq 1}) \rightarrow \mathbf{cdga}_{\leq 0} \) endowed with fully faithful functors to \( \mathbf{dgLie} \) and to \( \mathbf{P} \Sigma_{p} \mathbf{dgLie}^{f,r,\geq 1} \).

The functor \( F_{0} \) has a relative left Kan extension \( F \) (see [HTT] 4.3.2.14). Because of proposition 1.1.2, it now suffices to prove that \( F \) preserves cocartesian morphisms. This is a consequence of [HTT] prop. 4.3.1.12.

1.2 Poincaré-Birkhoff-Witt over a cdga in characteristic zero

**Theorem 1.2.1.** Let \( A \) be a commutative dg-algebra over a field \( k \) of characteristic zero. For any dg-Lie algebra over \( A \), there is a natural isomorphism of \( A \)-dg-modules

\[
\text{Sym}_{A} L \cong u_{A} L
\]

**Proof.** Recall that \( u_{A} L \) can be endowed with a bialgebra structure such that an element of \( L \) is primitive in \( u_{A} L \). The morphism \( L \rightarrow u_{A} L \) therefore induces a morphism of dg-bialgebras

\[
T_{A} L \rightarrow u_{A} L
\]

can be composed with the symmetrization map

\[
\text{Sym}_{A} L \rightarrow T_{A} L
\]

given by

\[
x_{1} \otimes \ldots \otimes x_{n} \mapsto \frac{1}{n!} \sum_{\sigma} \varepsilon(\sigma,x) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}
\]

where \( \sigma \) varies in the permutation group \( \mathfrak{S}_{n} \) and where \( \varepsilon(\sigma,x) \) is a group morphism \( \mathfrak{S}_{n} \rightarrow \{-1,+1\} \) determined by the value on the permutations \((i,j)\)

\[
\varepsilon((i,j),x) = (-1)^{|x_{i}| |x_{j}|}
\]

We finally get a morphism of \( A \)-dg-coalgebras \( \phi : \text{Sym}_{A} L \rightarrow u_{A} L \). Let us take \( n \geq 1 \) and let us assume that the image of \( \phi \) contains \( u_{A}^{n-1} L \). The image of a symmetric tensor

\[
x_{1} \otimes \ldots \otimes x_{n}
\]

by \( \phi \) is the class

\[
\frac{1}{n!} \sum_{\sigma} \varepsilon(\sigma,x) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}
\]

which can be rewritten

\[
x_{1} \otimes \ldots \otimes x_{n} + \sum_{\alpha} \pm \frac{1}{n!} y_{\alpha}^{n} \otimes \ldots \otimes y_{\alpha}^{n-1}
\]

where \( y_{\alpha}^{n} \) is either some of the \( x_{i} \)'s or some bracket \([x_{j},x_{k}]\). This implies that \( u_{A}^{n} L \) is in the image of \( \phi \) and we therefore show recursively that \( \phi \) is surjective (the filtration of \( u_{A} L \) is exhaustive).

There is moreover a section

\[
u_{A} L \rightarrow \text{Sym}_{A} L
\]

for which a formula is given in [Coh] and which concludes the proof. \( \square \)
1.3 Almost finite cellular objects

Let $A$ be a commutative dg-algebra over $k$.

Definition 1.3.1. Let $M$ be an $A$-dg-module.

- We will denote by $MC(M)$ the mapping cone of the identity of $M$.
- We will say that $M$ is an almost finite cellular object if there is a diagram

$$
0 \to A^{P_0} = M_0 \to M_1 \to \ldots
$$

whose colimit is $M$ and such that for any $n$, the morphism $M_n \to M_{n+1}$ fits into a cocartesian diagram

$$
\begin{array}{ccc}
A^{P_n}[n] & \longrightarrow & M_n \\
\downarrow & & \downarrow \\
MC(A^{P_n}[n]) & \longrightarrow & M_{n+1}
\end{array}
$$

Remark 1.3.2. The definition above states that a dg-module $M$ is an almost finite cellular object if it is obtained from 0 by gluing a finite number of cells in each degree (although the total number of cells is not finite).

Lemma 1.3.3. Let $\phi: M \to N$ be a morphism of $A$-dg-modules.

- If $M$ is an almost finite cellular object then it is cofibrant.
- Assume both $M$ and $N$ are almost finite cellular objects. The morphism $\phi$ is a quasi-isomorphism if and only if for any field $l$ and any morphism $A \to l$ the induced map $\phi_k: M \otimes_A l \to N \otimes_A l$ is a quasi-isomorphism.

Proof. Assume $M$ is an almost finite cellular object. Let us consider a diagram $M \to Q \leftarrow P$ where the map $P \to Q$ is a trivial fibration. Each morphism $M_n \to M_{n+1}$ is a cofibration and there thus is a compatible family of lifts $(M_n \to P)$. This gives us a lift $M \to P$. The $A$-dg-module $M$ is cofibrant.

Let now $\phi$ be a morphism $M \to N$ between almost finite cellular objects and that the morphism $\phi_l$ is a quasi-isomorphism for any field $l$ under $A$. Replacing $M$ with the cone of $\phi$ (which is also an almost finite cellular object) we may assume that $N$ is trivial. Notice first that an almost finite cellular object is concentrated in non positive degree. Notice also that for any $n$ the truncated morphism $\phi_{\geq -n}: M_{n+1}^{\geq -n} \to M^{\geq -n}$ is a quasi-isomorphism. We then have

$$
0 \simeq H^j \left( M \otimes_A l \right) \simeq H^j \left( M_n \otimes_A l \right)
$$

whenever $-n < j \leq 0$ and for any $A \to k$. Since $H^j (M_n \otimes_A l) \simeq 0$ if $j \leq -n - 2$ the $A$-dg-module $M_n$ is perfect and of amplitude $[-n-1,-n]$. This implies the existence of two projective modules $P$ and $Q$ (ie retracts of some power of $A$) fitting in a cofibre sequence (see [TV])

$$
P[n] \to M_n \to Q[n+1]
$$

The dg-module $M_n$ is then cohomologically concentrated in degree $]-\infty, -n]$, and so is $M$. This being true for any $n$ we deduce that $M$ is contractible. □
The next lemma requires the base $A$ to be noetherian. Recall that $A$ is noetherian if $H^0(A)$ is noetherian and if $H^n(A)$ is trivial when $n$ is big enough and of finite type over $H^0(A)$ for any $n$.

Lemma 1.3.4. Assume $A$ is noetherian. If $B$ is an object of $\text{cdga}_{/A}^{\leq 0}$ such that:

- The $H^0(A)$-algebra $H^0(B)$ is finitely presented,
- For any $n \geq 1$ the $H^0(B)$-module $H^{-n}(B)$ is of finite type,

then the $A$-dg-module $L_{B/A} \otimes_B A$ is an almost finite cellular object.

**Proof.** Because the functor $(A \to B \to A) \mapsto L_{B/A} \otimes_B A$ preserves colimits, it suffices to prove that $B$ is an almost finite cellular object in $\text{cdga}_{/A}^{\leq 0}$. This means we have to build a diagram

$$B_0 \to B_1 \to \ldots$$

whose colimit is equivalent to $B$ and such that for any $n \geq 1$ the morphism $B_{n-1} \to B_n$ fits into a cocartesian diagram

\[
\begin{array}{ccc}
A[R_1^{n-1}, \ldots, R_q^{n-1}]dR_1^{n-1} = 0 & \to & B_{n-1} \\
\downarrow r_i \mapsto dU_i & & \\
A[U_1^n, \ldots, U_q^n, X_1^n, \ldots, X_p^n]dX_p^n = 0 & \to & B_n
\end{array}
\]

where $R_i^{n-1}$ is a variable in degree $-(n-1)$ and $X_1^n$ and $U_i^n$ are variables in degree $-n$.

We build such a diagram recursively. Let

$$H^0(B) \cong H^0(A)[X_1^n, \ldots, X_p^n]/(R_1^{0}, \ldots, R_q^{0})$$

be a presentation of $H^0(B)$ as a $H^0(A)$-algebra. Let $B_0$ be $A[X_1^n, \ldots, X_p^n]$ equipped with a morphism $\phi_0: B_0 \to B$ given by a choice of coset representatives of $X_1^n, \ldots, X_p^n$ in $B$.

The induced morphism $H^0(B_0) \to H^0(B)$ is surjective and its kernel is of finite type (as a $H^0(A)$-module).

Let $n \geq 1$. Assume $\phi_{n-1}: B_{n-1} \to B$ has been defined and satisfies the properties:

- If $n = 1$ then the induced morphism of $H^0(A)$-modules $H^0(B_0) \to H^0(B)$ is surjective and its kernel $K_0$ is a $H^0(A)$-module of finite type.
- If $n \geq 2$, then the morphism $\phi_{n-1}$ induces isomorphisms $H^{-i}(B_{n-1}) \to H^{-i}(B)$ for $H^0(A)$-modules if $i = 0$ and of $H^0(B)$-modules for $1 \leq i \leq n - 2$.
- If $n \geq 2$ then the induced morphism of $H^0(B)$-modules $H^{-n+1}(B_{n-1}) \to H^{-n+1}(B)$ is surjective and its kernel $K_{n-1}$ is a $H^0(B)$-module of finite type.

Let $n \geq 1$. Let $X_1^n, \ldots, X_p^n$ be generators of $H^{-n}(B)$ as a $H^0(B)$-module and $R_1^{n-1}, \ldots, R_q^{n-1}$ be generators of $K_{n-1}$. Let $B_n$ be the pushout:

\[
\begin{array}{ccc}
A[R_1^{n-1}, \ldots, R_q^{n-1}]dR_1^{n-1} = 0 & \to & B_{n-1} \\
\downarrow r_i \mapsto dU_i & & \\
A[U_1^n, \ldots, U_q^n, X_1^n, \ldots, X_p^n]dX_p^n = 0 & \to & B_n
\end{array}
\]
Let \( r_n^1, \ldots, r_q^1 \) be the images of \( R_1^n, \ldots, R_q^n \) (respectively) by the composite morphism

\[
A[R_1^n, \ldots, R_q^n]^{dR_1^n-1} \to B_{n-1} \to B
\]

There exist \( u^n_1, \ldots, u^n_q \in B \) such that \( du^n_i = r^n_i \) for all \( i \). Those \( u^n_1, \ldots, u^n_q \) together with a choice of coset representatives of \( X^n_1, \ldots, X^n_p \) in \( B \) induce a morphism

\[
A[U^n_1, \ldots, U^n_q, X^n_1, \ldots, X^n_p]^{dX^n_k-1} \to B
\]

which induces a morphism \( \phi_n : B_n \to B \).

If \( n = 1 \) then a quick computation proves the isomorphism of \( H^0(A) \)-modules

\[
H^0(B_1) \cong H^0(B_0)/(R_1^0, \ldots, R_q^0) \cong H^0(B)
\]

If \( n \geq 2 \) then the truncated morphism \( B^{2-n}_n \to B_{n-1}^{2-n} \) is a quasi-isomorphism and the induced morphisms \( H^{-i}(B_n) \cong H^{-i}(B) \) are thus isomorphisms of \( H^0(B) \)-modules for \( i \leq n - 2 \). We then get the isomorphism of \( H^0(B) \)-modules

\[
H^{-n+1}(B_n) \cong H^{-n+1}(B_{n-1})/(R_1^{n-1}, \ldots, R_q^{n-1}) \cong H^{-n+1}(B)
\]

The natural morphism \( \theta : H^{-n}(B_n) \to H^{-n}(B) \) is surjective. The \( H^0(B) \)-module \( H^{-n}(B_n) \) is of finite type and because \( H^0(B) \) is noetherian, the kernel \( K_n \) of \( \theta \) is also of finite type. The recursivity is proven and it now follows that the morphism \( \text{colim}_n B_n \to B \) is a quasi-isomorphism.

**Definition 1.3.5.** Let \( L \) be a dg-Lie algebra over \( A \).

- We will say that \( L \) is very good if there exist a finite sequence
  
  \[
  0 = L_0 \to L_1 \to \ldots \to L_n = L
  \]
  
  such that each morphism \( L_i \to L_{i+1} \) fits into a cocartesian square

\[
\begin{array}{ccc}
\text{Free}(A[-p_i]) & \to & L_i \\
\downarrow & & \downarrow \\
\text{Free}(M^c(A[-p_i])) & \to & L_{i+1}
\end{array}
\]

where \( p_i \geq 2 \).

- We will say that \( L \) is good if it is quasi-isomorphic to a very good dg-Lie algebra.

- We will say that \( L \) is almost finite if it is cofibrant and if its underlying graded module is isomorphic to

\[
\bigoplus_{i \geq 1} A^n_{\alpha_i}[-i]
\]

**Remark 1.3.6.** Any dg-Lie algebra in \( \text{dgLie}_A^{f,0,\geq 1} \) is very good.

**Lemma 1.3.7.**

- Any very good dg-Lie algebra is almost finite.

- The underlying dg-module of a cofibrant dg-Lie algebra is cofibrant.
**Proof.** Any free dg-Lie algebra generated by some \( A[-p] \) with \( p \geq 2 \) is almost finite. Considering a pushout diagram

\[
\begin{array}{c}
\text{Free}(A[-p]) \\
\downarrow \\
\text{Free}(\text{Sym}(A[-p])) \\
\downarrow \\
L
\end{array}
\quad \begin{array}{c}
\text{Free}(M^c(A[-p])) \\
\downarrow \\
L'
\end{array}
\]

Whenever \( L \) is almost finite, so is \( L' \). This proves the first item.

Let now \( L \) be a dg-Lie algebra over \( A \). There is a morphism of dg-modules \( L \to U_A L \). The Poincaré-Birkhoff-Witt theorem states that the dg-module \( U_A L \) is isomorphic to \( \text{Sym}_A L \). There is therefore a retract \( U_A L \to L \) of the universal morphism \( L \to U_A L \). The functor \( U_A: \text{dgLie}_A \to \text{dgAlg}_A \) preserves cofibrant objects and using a result of [SS], so does the forgetful functor \( \text{dgAlg}_A \to \text{dgMod}_A \). We therefore deduce that if \( L \) is cofibrant in \( \text{dgLie}_A \), it is also cofibrant in \( \text{dgMod}_A \).

**Definition 1.3.8.** Let \( \text{dgLie}^\text{good}_A \) denote the sub-\( \infty \)-category of \( \text{dgLie}_A \) spanned by good dg-Lie algebras.

### 1.4 Homology and cohomology of dg-Lie algebras

The content of this section can be found in [DAG-X] when the base is a field. Proofs are simple avatars of Lurie’s on a more general base \( A \). Let then \( A \) be a commutative dg-algebra concentrated in non-positive degree over a field \( k \) of characteristic zero.

**Definition 1.4.1.** Let \( A[\eta] \) denote the (contractible) commutative \( A \)-dg-algebra generated by one element \( \eta \) of degree -1 such that \( \eta^2 = 0 \) and \( d\eta = 1 \). For any \( A \)-dg-Lie algebra \( L \), the tensor product \( A[\eta] \otimes_A L \) is still an \( A \)-dg-Lie algebra and we can thus define the homology of \( L \):

\[
H_A(L) = U_A \left( A[\eta] \otimes_A L \right) \otimes_{U_A L} A
\]

where \( U_A: \text{dgLie}_A \to \text{dgAlg}_A \) is the functor sending a Lie algebra to its enveloping algebra. This construction defines a strict functor:

\[
H_A: \text{dgLie}_A \to A/\text{dgMod}_A
\]

**Remark 1.4.2.** The homology \( H_A(L) \) of \( L \) is isomorphic as a graded module to \( \text{Sym}_A(L[1]) \), the symmetric algebra built on \( L[1] \). The differentials do not coincide though. The one on \( H_A(L) \) is given on homogenous objects by the following formula:

\[
d(\eta.x_1 \otimes \cdots \otimes \eta.x_n) = \sum_{i < j} (-1)^{T_{ij}} \eta.[x_i, x_j] \otimes \eta.x_1 \otimes \cdots \otimes \eta.x_i \otimes \cdots \otimes \eta.x_j \otimes \cdots \otimes \eta.x_n
\]

\[
- \sum_i (-1)^{S_i} \eta.x_1 \otimes \cdots \otimes \eta.d(x_i) \otimes \cdots \otimes \eta.x_n
\]

where \( \eta.x \) denotes the point in \( L[1] \) corresponding to \( x \in L \).

\[
S_i = i - 1 + |x_1| + \cdots + |x_{i-1}|
\]

\[
T_{ij} = (|x_i| - 1)S_i + (|x_j| - 1)S_j + (|x_i| - 1)(|x_j| - 1) + |x_i|
\]

The coalgebra structure on \( \text{Sym}_A(L[1]) \) is compatible with this differential and the isomorphism above induces a coalgebra structure on \( H_A(L) \) given for \( x \in L \) homogenous by:

\[
\Delta(\eta.x) = \eta.x \otimes 1 + 1 \otimes \eta.x
\]
Proposition 1.4.3. The functor $H_A$ preserves quasi-isomorphisms. It induces an $\infty$-functor between the corresponding $\infty$-categories, which we will denote the same way:

$$H_A : \text{dgLie}_A \to A/\text{dgMod}_A$$

Proof. Let $L \to L'$ be a quasi-isomorphism of $A$-dg-Lie algebras. Both $H_A(L)$ and $H_A(L')$ are endowed with a natural filtration denoted $H^\leq_n A$ (resp $L'$) induced by the canonical filtration of $\text{Sym}_A(L[1])$. Because quasi-isomorphisms are stable by filtered colimits, it is enough to prove that each morphism $H^\leq_n A(L) \to H^\leq_n A(L')$ is a quasi-isomorphism. The case $n = 0$ is trivial. Let us assume $H^\leq_{n-1} A(L) \to H^\leq_{n-1} A(L')$ to be a quasi-isomorphism. There are short exact sequences:

$$0 \to H^\leq_{n-1} A(L) \to H^\leq_n A(L) \to \text{Sym}_A^n(L[1]) \to 0$$

$$0 \to H^\leq_{n-1} A(L') \to H^\leq_n A(L') \to \text{Sym}_A^n(L'[1]) \to 0$$

The base dg-algebra $A$ is of characteristic zero and the morphism $\theta$ is thus a retract of the quasi-isomorphism $L[1]^{\otimes n} \to L'[1]^{\otimes n}$ (where the tensor product is taken over $A$).

Proposition 1.4.4. Let $A \to B$ be a morphism in $\text{cdga}^{<0}_{k}$. The following square is commutative:

$$\begin{array}{ccc}
\text{dgLie}_A & \xrightarrow{H_A} & A/\text{dgMod}_A \\
\downarrow B \otimes_A - & & \downarrow B \otimes_A - \\
\text{dgLie}_B & \xrightarrow{H_B} & B/\text{dgMod}_B
\end{array}$$

Proof. This follows directly from the definition.

Corollary 1.4.5. Let $L$ be in $\text{dgLie}^{fr, > 1}_A$ generated by some dg-module $M$. The homology of $L$ quasi-isomorphic to the trivial square zero extension $A \to A \oplus M[1]$.

Proof. This a consequence of the previous proposition and the corresponding result over a field in Lurie’s theorem 1.0.1.

Definition 1.4.6. Let $L$ be an object of $\text{dgLie}_A$. We define cohomology of $L$ as the dual of its homology:

$$C_A(L) = H_A(L)^\vee = \text{Hom}_A(H_A(L), A)$$

It is equipped with a commutative algebra structure (see remark 1.4.2). This defines an $\infty$-functor:

$$C_A : \text{dgLie}_A \to \left(\text{cdga}_{A/A}\right)^{op}$$

Remark 1.4.7. The cohomology of an object of $\text{dgLie}^{fr, > 1}_A$ is concentrated in non positive degree as it is, as a graded module, isomorphic to $\text{Sym}_A(L[1]([-1])]$. The following proposition proves the cohomology of a good dg-Lie algebra is also concentrated in non-positive degree.

Proposition 1.4.8. The $\infty$-functor $C_A$ maps colimits of $\text{dgLie}_A$ to limits of $\text{cdga}_{A/A}$.
Proof (sketch of a). For a complete proof, the author refers to the proof of proposition 2.2.12 in [DAG-X]. We will only transcript here the main arguments.

A commutative $A$-dg-algebra $B$ is the limit of a diagram $B_n$ if and only if the underlying dg-module is the limit of the underlying diagram of dg-modules. It is thus enough to consider the composite $\infty$-functor $dgLie_A \to (cdga_A/A)^{op} \to (dgMod_A/A)^{op}$. This $\infty$-functor is equivalent to $(H_A(-)^{op})$. It is then enough to prove the $\infty$-functor $H_A: dgLie_A \to A/dgMod_A$ to preserve colimits.

To do so, we will first focus on the case of sifted colimits, which need only to be preserved by the composite $\infty$-functor $dgLie_A \to A/dgMod_A \to dgMod_A$. This last $\infty$-functor is the (filtered) colimits of the $\infty$-functors $H_A$ as introduced in the proof of 1.4.3. We now have to prove that the $\infty$-functors $H_A: dgLie_A \to dgMod_A$ preserve sifted colimits. There is a fiber sequence of $\infty$-functors

$$H_A^{n-1} \to H_A^n \to \text{Sym}_A((-)[1])$$

The $\infty$-functors $\text{Sym}_A((-)[1])$ preserve sifted colimits in characteristic zero and an inductive process proves that $H_A$ preserves sifted colimits too.

We now have to treat the case of finite coproducts. The initial object is obviously preserved. Let $L = L' \sqcup L''$ be a coproduct of $dgLie$ algebras. We proved in remark 1.1.3 that $L'$ an $L''$ can be written as sifted colimits of objects of $dgLie_A^{fr,\geq 1}$. We will now prove that $H_A$ preserves the coproduct $L = L' \sqcup L''$ when $L'$ and $L''$ (and thus $L$ too) are in $dgLie_A^{fr,\geq 1}$. This corresponds to the following cocartesian diagram

$$
\begin{array}{c}
A \\
\downarrow \\
A \oplus M'[1] \\
\downarrow \\
A \oplus M''[1] \longrightarrow A \oplus (M' \oplus M'')[1]
\end{array}
$$

where $M'$ and $M''$ are objects of $dgMod_A^{fr,\geq 1}$ generating $L'$ and $L''$ respectively. \hfill $\square$

Definition 1.4.9. The $\infty$-functor $C_A$ is a colimit-preserving functor between presentable $\infty$-categories. It therefore admits a right adjoint which we will denote by $D_A$.

Lemma 1.4.10. Let $B \to A$ be a morphism of $cdga_k^{\geq 0}$. The following diagram of $\infty$-categories commutes:

$$
\begin{array}{ccc}
dgLie_B^{\text{good}} & \longrightarrow & (cdga_B^{\leq 0}/B)^{op} \\
A \otimes_B - \downarrow & & A \otimes_B - \downarrow \\
dgLie_A^{\text{good}} & \longrightarrow & (cdga_A^{\leq 0}/A)^{op}
\end{array}
$$

Proof. Let us first remark that proposition 1.4.4 gives birth to a natural transformation $A \otimes_B C_B(-) \to C_A(A \otimes_B -)$. Let $L \in dgLie_B^{\text{good}}$. The $B$-dg-module $C_A(A \otimes_B L)$ is equivalent to $\text{Hom}_B(H_B(L), A)$. We thus study the natural morphism

$$\phi_L: A \otimes_B C_B(L) \to \text{Hom}_B(H_B(L), A)$$

Let us consider the case of the free dg-Lie algebra $L = \text{Free}(B[-p])$ with $p \geq 1$. If $B$ is the base field $k$ then $H_k(L)$ is perfect (corollary 1.4.5) and the morphism $\phi_L$ is an equivalence. If $B$ is any $k$-dg-algebra then $L$ is equivalent to $B \otimes_k \text{Free}(k[-p])$ and we conclude using proposition 1.4.4 that $\phi_L$ is an equivalence.
To prove the general case of any good dg-Lie algebra it is now enough to ensure that if $L_1 \to L_0 \to L_2$ is a diagram of good dg-Lie algebras such that $\phi_{L_1}$, $\phi_{L_0}$ and $\phi_{L_2}$ are equivalences then so is $\phi_L$, with $L = L_1 \cup_{L_0} L_2$. This can be tested in $\text{dgMod}_A$ in which tensor product and fiber product commute.

**Corollary 1.4.11.** The composite $\infty$-functor $\text{dgMod}_A \to \text{dgLie}_A \to (\text{cdga}_{/A})^{\text{op}}$ is equivalent to the $\infty$-functor $M \to A \otimes M^\vee [-1]$.

**Proof.** The $\infty$-category $\text{dgMod}_A$ is generated under (sifted) colimits by $\text{dgMod}_{A}^{f.t. \geq 1}$. The functors at hand coincide on $\text{dgMod}_{A}^{f.t. \geq 1}$ and both preserve colimits.

**Lemma 1.4.12.** Assume $A$ is noetherian. Let $L$ be a good dg-Lie algebra over $A$. The adjunction morphism $L \to D_A C_A L$ is a quasi-isomorphism.

**Proof.** Let us first prove that the morphism at hand is equivalent, as a morphism of dg-modules, to the natural morphism $L \to L^\vee \vee$. Because of corollary 1.4.11, the composite $\infty$-functor $(\text{cdga}_{/A})^{\text{op}} \to \text{dgLie}_A \to \text{dgMod}_A$ is equivalent to the functor:

$$\left( A \otimes B \to A \right) \mapsto \left( A \otimes \mathbb{L}_B[A][1] \right)^\vee$$

It now suffices to prove that the following morphism is an equivalence:

$$f: \mathbb{L}_{C_A L/A} \otimes_{C_A L} A \to L^\vee [-1]$$

As soon as $L$ is good and $A$ noetherian, the cohomology $C_A L$ of $L$ satisfies the finiteness conditions of lemma 1.3.4. Because $L$ is almost finite (as a dg-Lie algebra), the dg-module $L^\vee [-1]$ is an almost finite cell object. Both domain and codomain of the morphism $f$ are thus almost finite cellular $A$-dg-modules. It is then enough to consider $f \otimes_A k$ for any field $k$ and any morphism $A \to k$

$$f \otimes_A k: \left( \mathbb{L}_{C_A (L/A)} \otimes_A k \right) \to \left( L \otimes k \right)^\vee [-1]$$

The lemma 1.4.10 gives us the equivalence $C_A (L) \otimes_A k \simeq C_A (L \otimes_A k)$ and the morphism $f \otimes_A k$ is thus equivalent to the morphism

$$\mathbb{L}_{C_A (L \otimes_A k) / k} \otimes k \to \left( L \otimes k \right)^\vee [-1]$$

This case is exactly Lurie’s result 1.0.1 (ii).

We now prove that $L \to L^\vee \vee$ is an equivalence. We can assume that $L$ is very good. The underlying $A$-dg-module of $L$ is cofibrant and the dual can be computed naively. Now $L$ is almost finite. There is therefore a family $(n_i)$ of integers and an isomorphism of graded modules

$$L = \bigoplus_{i \geq 1} A^{n_i}[-i]$$

The dual of $L$ is isomorphic to $\prod_{i \geq 1} A^{n_i}[i]$ with an extra differential. Because $A$ in concentrated on non positive degree, the dual $L^\vee$ is equivalent to $\bigoplus_{i \geq 1} A^{n_i}[i]$ (with the extra differential). The natural morphism $L \to L^\vee \vee$ therefore corresponds to the morphism

$$\bigoplus_{i \geq 1} A^{n_i}[-i] \to \prod_{i \geq 1} A^{n_i}[-i]$$

which is a quasi-isomorphism as soon as $A$ is cohomologically bounded.
Remark 1.4.13. The base dg-algebra $A$ needs to be cohomologically bounded for that lemma to be true. Taking $L = \text{Free}(A^2[-1])$ the adjunction morphism is equivalent to

$$L \twoheadrightarrow L^\vee$$

which is not a quasi-isomorphism.

1.5 Formal stack over a dg-algebra

Throughout this section $A$ will denote an object of $\text{cdga}^{\leq 0}_k$.

Definition 1.5.1. Let $\text{dgExt}_A$ denote the full sub-category of $\text{cdga}^{\leq 0}_A/A$ spanned be the trivial square zero extensions $A \oplus M$, where $M$ is a free $A$-dg-module of finite type concentrated in non positive degree.

Definition 1.5.2. A formal stack over $A$ is a functor $\text{dgExt} \rightarrow \mathcal{S}\mathcal{E}t\mathcal{s}$ preserving finite products. We will denote by $\mathcal{dSt}_A^f$ the $\infty$-category of such formal stacks.

$$\mathcal{dSt}_A^f = \mathcal{P}_\Sigma(\text{dgExt}^{\text{op}}_A)$$

Proposition 1.5.3. Let $A$ be in $\text{cdga}^{\leq 0}_k$ and let $S = \text{Spec} A$. There is an adjunction

$$\mathcal{F}_A: \text{dgLie}_A \rightleftarrows \mathcal{dSt}_A^f: \mathcal{L}_A$$

such that

- The composite functor $\mathcal{dSt}_A^f \rightarrow \text{dgLie}_A \rightarrow \text{dgMod}_A$ is equivalent to the functor $X \mapsto \mathcal{T}_{X/S,x}[-1]$ where $\mathcal{T}_{X/S,x}$ is the tangent complex of $X$ over $S$ at the point natural point $x$ of $X$: it is the dg-module representing the product-preserving functor

$$\left(\text{dgMod}_A^{f,n,\geq 1}\right)^{\text{op}} \rightarrow \mathcal{S}\mathcal{E}t\mathcal{s}$$

$$M \mapsto X(A \oplus M^\vee [-1])$$

- The functor $\mathcal{L}_A$ is conservative and preserves sifted colimits.

- If moreover $A$ is noetherian then the functors $\mathcal{L}_A$ and $\mathcal{F}_A$ are equivalences of $\infty$-categories.

Definition 1.5.4. Let $X$ be a formal stack. The Lie algebra $\mathcal{L}_A X$ will be called the tangent Lie algebra of $X$ (over $A$).

Proof (of the proposition). Let us prove the first item. The $\infty$-functor $C_A$ induces a functor

$$C_A: \text{dgLie}_A^{f,n,\geq 1} \rightarrow \text{dgExt}_A^{\text{op}}$$

which composed with the Yoneda embedding defines a functor $\phi: \text{dgLie}_A^{f,n,\geq 1} \rightarrow \mathcal{dSt}_A^f$. This last functor extends by sifted colimits to

$$\mathcal{F}_A: \text{dgLie}_A \simeq \mathcal{P}_\Sigma(\text{dgLie}_A^{f,n,\geq 1}) \rightarrow \mathcal{dSt}_A^f$$

Because $C_A$ preserves coproducts, the functor $\mathcal{F}_A$ admits a right adjoint $\mathcal{L}_A$ given by right-composing by $C_A$. The composed functor

$$\mathcal{dSt}_A^f \rightarrow \mathcal{P}_\Sigma(\text{dgLie}_A^{f,n,\geq 1}) \rightarrow \mathcal{P}_\Sigma(\text{dgMod}_A^{f,n,\geq 1}) \simeq \text{dgMod}_A$$

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then corresponds to the functor
\[ X \mapsto X(C_A(Free(-))) \simeq X(A \oplus (-)^\vee[-1]) \]
This proves the first item. The functor
\[ C_A : (\text{dgLie}_A^{f, \geq 1})^{op} \to \text{dgExt}_A \]
is essentially surjective. This implies that \( \mathcal{L}_X \) is conservative. Let us consider the commutative diagram
\[
\begin{array}{ccc}
P \Sigma (\text{dgExt}_A^{op}) & \xrightarrow{\mathcal{L}_X} & P \Sigma (\text{dgLie}_A^{f, \geq 1}) \\
\downarrow & & \downarrow \\
P (\text{dgExt}_A^{op}) & \xrightarrow{C^*_A} & P (\text{dgLie}_A^{f, \geq 1})
\end{array}
\]
The functors \( i, j \) and \( C^*_A \) preserve sifted colimits and therefore so does \( \mathcal{L}_A \). The third item is a corollary of its fully faithfulness (lemma 1.4.12) under the added assumption.

Until the end of this section, we will focus on proving that the definition we give of a formal stack is equivalent to Lurie’s definition of a formal moduli problem in [DAG-X], as soon as the base dg-algebra \( A \) is noetherian.

**Definition 1.5.5.** An augmented \( A \)-dg-algebra \( B \in \text{cdga}_A^{\leq 0} / A \) is called artinian if there is sequence
\[ B = B_0 \to \ldots \to B_n = A \]
and for \( 0 \leq i < n \) an integer \( p_i \geq 1 \) such that
\[ B_i \simeq B_{i+1} \times_{A[p_i]} A \]
where \( A[\varepsilon_{p_i}] \) denote the square zero extension \( A \oplus A[p_i] \).

We denote by \( \text{dgArt}_A \) the full sub-\( \infty \)-category of \( \text{cdga}_A^{\leq 0} / A \) spanned by the artinian dg-algebras.

**Definition 1.5.6.** Assume \( A \) is noetherian. A formal moduli problem over \( A \) is a functor \( X : \text{dgArt}_A \to \text{sSets} \) satisfying the conditions:
\[(F1) \text{ For } n \geq 1 \text{ and } B \in \text{dgArt}_A / A[\varepsilon_n] \text{ the following natural morphism is an equivalence:} \]
\[ X \left( B \times_{X(A[\varepsilon_n])} A \right) \simeq X(B) \times_{X(A)} X(A) \]
\[(F2) \text{ The simplicial set } X(A) \text{ is contractible.} \]

Let \( \text{dSt}^f_A \) denote the full sub-\( \infty \)-category of the category of functors \( \text{dgArt}_A \to \text{sSets} \) spanned by the formal moduli problems. This is an accessible localization of the presentable \( \infty \)-category \( \mathcal{P}(\text{dgArt}_A) \) of simplicial presheaves over \( \text{dgArt}_A \).

**Proposition 1.5.7.** Let \( A \in \text{cdga}_A^{\leq 0} \) be noetherian. The left Kan extension of the functor \( i : \text{dgExt}_A \to \text{dgArt}_A \) induces an equivalence of \( \infty \)-categories
\[ j : \text{dSt}^f_A \to \text{dSt}^f_A \]
Proof. We will actually prove that the composed functor

\[ f : \text{dgLie}_A \rightarrow \text{dSt}^i_A \rightarrow \overset{\sim}{\text{dSt}^i_A} \]

is an equivalence. The functor \( f \) admits a right adjoint \( g = \mathcal{L}_{Ai}^* \).

Given \( n \geq 1 \) and a diagram \( B \rightarrow A[\varepsilon_n] \leftarrow A \) in \( \text{dgArt}_A \), lemma 1.4.12 implies that the natural morphism

\[ D_A(p)Bq > D_A(pA)\varepsilon_n sq \overset{\sim}{\rightarrow} D_A(pA) \]

is an equivalence. For any \( B \in \text{dgArt}_A \) the adjunction morphism \( B \rightarrow C_A \text{D}A_B \) is then an equivalence. Given \( L \in \text{dgLie}_A \) the functor \( \text{D}^*L : \text{dgArt}_A \rightarrow \text{sSets} \) defined by \( \text{D}^*(B) = \text{Map}(\text{D}^*_A(B), L) \) is a formal moduli problem. The natural morphism \( \text{id} \rightarrow \text{D}^*g \) of \( \infty \)-functors from \( \text{dSt}^i_A \) to itself is therefore an equivalence. The same goes for the morphism \( g \text{D}^* \rightarrow \text{id} \) of \( \infty \)-functors from \( \text{dgLie}_A \) to itself. The functor \( g \) is an equivalence, so is \( f \) and so is \( j \).

2 Tangent Lie algebra

We now focus on gluing the functors built in the previous section. The base will be a derived Artin stack \( X \). One should think of a derived Artin stack as a representable moduli problem over cdga’s (see [TV] for a definition). Such an object admits a cotangent complex \( L_X \) representing the functor of derivations over \( X \). It is a quasi-coherent sheaf over \( X \). When \( X \) is locally of finite presentation, its cotangent complex is perfect and we can thus consider its dual, the tangent complex \( T^*_X \). We will prove the following statement.

**Theorem 2.0.1.** Let \( X \) be a derived Artin stack locally of finite presentation. Then there is a Lie algebra \( \ell_X \) over \( X \) whose underlying module is equivalent to the shifted tangent complex \( T^*_X[-1] \) of \( X \).

Moreover if \( f : X \rightarrow Y \) is a morphism between algebraic stacks locally of finite presentation then there is a tangent Lie morphism \( \ell_X \rightarrow f^*\ell_Y \). More precisely, there is an \( \infty \)-functor

\[ X/\text{dSt}^{\text{alg,lp}} \rightarrow \ell_X/\text{dgLie}_X \]

sending a map \( f : X \rightarrow Y \) to a morphism \( \ell_X \rightarrow f^*\ell_Y \).

2.1 Formal stacks and Lie algebras over a derived Artin stack

Let \( A \rightarrow B \) be a morphism in \( \text{cdga}_k^{\leq 0} \) (where \( k \) is once again a field of characteristic zero). There is an exact scalar extension \( \infty \)-functor \( B \otimes_A - : \text{dgExt}_A \rightarrow \text{dgExt}_B \) and therefore an adjunction

\[ \left( B \otimes_A - \right) : \overset{\sim}{\text{dSt}^i_A} \rightleftarrows \overset{\sim}{\text{dSt}^i_B} : \left( B \otimes_A - \right)^* \]

**Proposition 2.1.1.** There is a diagram of \( \infty \)-categories

\[
\begin{array}{ccc}
\text{dgLie} & \xrightarrow{\mathcal{F}} & \overset{\sim}{\text{dSt}^i} \\
\downarrow p & & \downarrow q \\
\text{cdga}_k^{\leq 0} & & \\
\end{array}
\]

where \( p \) and \( q \) are cocartesian fibrations and where the functor \( \mathcal{F} \) preserves cocartesian morphisms.
Proof. We define \( C \) to be the following category.

- An object is a pair \((A, X)\) where \( A \in \text{cdga}_{k}^{\leq 0}\) and \( X \) is a simplicial presheaf over \( \text{dgExt}_{A}^{op} \).
- A morphism \((A, X) \to (B, Y)\) is a map \( A \to B \) together with a map \( X \to Y(B \otimes A) \).

We will say that a map \((A, X) \to (B, Y)\) is an equivalence if the map \( A \to B \) is a quasi-isomorphism and if the map \( X \to Y(B \otimes A) \) is a local equivalence (regarding the localization \( \text{dgExt}_{A}^{op} \to \text{dSt}_{A}^{f} \)). We define the \( \infty \)-category \( \text{dSt}^{f} \) as the localization of \( C \) along equivalences. The \( \infty \)-category \( \int \text{dSt}^{f} \) is cofibered over \( \text{cdga}_{k}^{\leq 0} \).

Recall from proposition 1.1.5 that the cofibered category \( \int \Sigma_{\text{dgLie}}^{f, \geq 1} \) is equivalent to \( \int \text{dgLie} \) and comes from localizing some category \( D \). There is a functor \( f: C \to D \) mapping an object \((A, X)\) to the couple \((p_{C}A, X)\).

The functor \( f \) preserves equivalences and it induces an \( \infty \)-functor

\[
\mathcal{L}: \int \text{dSt}^{f} \to \int \Sigma_{\text{dgLie}}^{f, \geq 1}
\]

This last functor commutes with the projection to \( \text{cdga}_{k}^{\leq 0} \). Let \( A \in \text{cdga}_{k}^{\leq 0} \). The functor induced by \( \mathcal{L} \) between the fiber categories \( \text{dSt}^{f}_{A} \to \int \Sigma_{\text{dgLie}}^{f, \geq 1} \) is nothing but \( \mathcal{L}_{A} \) and thus admits a left adjoint \( \mathcal{F}_{A} \). Using corollary 1.4.10 we obtain that for any \( A \to B \) the diagram

\[
\begin{array}{ccc}
\int \Sigma_{\text{dgLie}}^{f, \geq 1}_{A} & \xrightarrow{\mathcal{F}_{A}} & \int \text{dSt}^{f}_{A} \\
\downarrow & & \downarrow \\
\int \Sigma_{\text{dgLie}}^{f, \geq 1}_{B} & \xrightarrow{\mathcal{F}_{B}} & \int \text{dSt}^{f}_{B}
\end{array}
\]

is commutative. The proposition 8.3.2.11 of [HAlg] gives the functor \( \mathcal{L} \) a left adjoint \( \mathcal{F} \) relative to \( \text{cdga}_{k}^{\leq 0} \) and proposition 8.3.2.6 of [HAlg] finish the proof.

Remark 2.1.2. We can see that the functor \( \mathcal{F}: \int \Sigma_{\text{dgLie}}^{f, \geq 1} \to \int \text{dSt}^{f} \) is the relative left Kan extension to the functor

\[
\text{Spec}(C(-)): \int \text{dgLie}^{f, \geq 1} \to \int \text{dgExt}^{op} \to \int \text{dSt}^{f}
\]

Proposition 2.1.3. The induced functor

\[
\text{dgLie}: \text{cdga}^{\leq 0}_{k} \to \text{Pr}_{\leq}^{L}
\]

is a stack for the fqc topology.

Proof. The functor \( \text{dgLie} \) is endowed with a forgetful natural transformation to \( \text{dgMod} \), the stack of dg-modules. This forgetful transformation is pointwise conservative and preserves limits. This implies that \( \text{dgLie} \) is also a stack.
**Definition 2.1.4.** Let $X$ be an algebraic derived stack. The $\infty$-category of formal stacks over $X$ is

$$\mathcal{dSt}_X^f = \lim_{\Spec A \to X} \mathcal{dSt}_A^f$$

The $\infty$-category of dg-Lie algebras over $X$ is

$$\mathcal{dgLie}_X = \lim_{\Spec A \to X} \mathcal{dgLie}_A$$

where both limits are taken in $\Pr^L_Z$. There is a colimit preserving functor

$$\mathcal{F}_X : \mathcal{dgLie}_X \to \mathcal{dSt}_X$$

It admits a right adjoint denoted by $\mathcal{L}_X$.

**Remark 2.1.5.** The functor $\mathcal{L}_X$ may not be local on $X$. It does not need to commute with base change.

### 2.2 Tangent Lie algebra of a derived Artin stack locally of finite presentation

**Definition 2.2.1.** Let us consider $\mathcal{C}$ the following category.

- An object is a pair $(A, F \to G)$ where $A$ is a cdga over $k$ and where $F \to G$ is a morphism in the category of presheaves over $\mathfrak{cdga}_A^{\geq 0}$.

- A morphism $(A, F \to G) \to (B, F' \to G')$ is the datum of a morphism $A \to B$ together with a commutative square

\[
\begin{array}{ccc}
F & \longrightarrow & F' \\
\downarrow & & \downarrow \\
G & \longrightarrow & G'
\end{array}
\]

of presheaves over $\mathfrak{cdga}_B^{\geq 0}$.

We set $\int \mathcal{P}(\mathfrak{dAff})^\Delta_1$ to be the $\infty$-localization of $\mathcal{C}$ along quasi-isomorphisms and weak equivalences of presheaves. The $\infty$-category $\int \mathcal{P}(\mathfrak{dAff})^\Delta_1$ is cofibered over $\mathfrak{cdga}_k^{\geq 0}$.

**Definition 2.2.2.** Let $\mathcal{D}$ denote the following category.

- An object is a pair $(A, F)$ where $F$ is a simplicial presheaf over the opposite category of morphisms in $\mathfrak{cdga}_A^{\leq 0}$.

- A morphism $(A, F) \to (B, G)$ is a morphism $A \to B$ and a map $F \to G$ as simplicial presheaves over $\mathfrak{dAff}_B^{\Delta_1}$.

We will denote by $\int \mathcal{P}(\mathfrak{dAff}^\Delta_1)$ the $\infty$-category obtained by $\mathcal{D}$ by localizing along weak equivalences.

**Lemma 2.2.3.** There is a relative adjunction

$$f : \int \mathcal{P}(\mathfrak{dAff}^\Delta_1) \to \int \mathcal{P}(\mathfrak{dAff})^\Delta_1 : g$$
over \( \text{cdga}_k^{<0} \). They can be described on the fibers as follows. Let \( A \in \text{cdga}_k^{<0} \). The left adjoint \( f_A \) is given on morphisms between affine schemes to the corresponding morphism of representable functors. The right adjoint \( g_A \) maps a morphism \( F \to G \) to the representable simplicial presheaf

\[
\text{Map}(-, F \to G)
\]

**Proof.** Let us define a functor \( C \to D \) mapping \( (A, F \to G) \) to the functor

\[
(S \to T) \mapsto \text{Map}(S \to T, F \to G)
\]

We can now derive this functor (replacing therefore \( F \to G \) by a fibrant replacement). We get an \( \infty \)-functor

\[
g : \int \mathcal{P}(\text{dAff})^{\Delta^1} \to \int \mathcal{P}(\text{dAff}^{\Delta^1})
\]

which commutes with the projections to \( \text{cdga}_k^{<0} \). Let \( A \) be in \( \text{cdga}_k^{<0} \) and let \( g_A \) be the induced functor

\[
\mathcal{P}(\text{dAff})^{\Delta^1}_A \to \mathcal{P}(\text{dAff}^{\Delta^1}_A)
\]

It naturally admits a left adjoint. Namely the left Kan extension \( f_A \) to the Yoneda embedding

\[
\text{dAff}^{\Delta^1}_A \to \mathcal{P}(\text{dAff}^{\Delta^1}_A)
\]

For any morphism \( A \to B \) in \( \text{cdga}_k^{<0} \), there is a canonical morphism

\[
f_B \left( \left( B \otimes_A X \right) \right) \to \left( B \otimes_A X \right)
\]

which is an equivalence. [When \( X = \text{Spec} A' \to \text{Spec} A'' \) is representable then both left and right hand sides are equivalent to \( \text{Spec} B' \to \text{Spec} B'' \) where \( B' = B \otimes_A A' \) and \( B'' = B \otimes_A A'' \).]

Propositions 8.2.3.11 of [HAlg] finishes the proof. \( \square \)

**Proposition 2.2.4.** The cofibered category \( \int \mathcal{P}(\text{dAff})^{\Delta^1} \) admits a full sub-category

\[
\int * / \mathcal{P}(\text{dAff})
\]

classified by the functor

\[
A \mapsto \text{Spec} A / \mathcal{P}(\text{dAff})_A
\]

There is a cofibered category over \( \text{cdga}_k^{<0} \)

\[
\int \mathcal{P}(\text{dAff}^*)
\]

classifying an \( \infty \)-functor \( \text{cdga}_k^{<0} \to \text{Pr}_L^{\infty} \)

\[
A \mapsto \mathcal{P}(\text{Spec} A / \text{dAff}_A)
\]

Moreover, the adjunction of lemma 2.2.3 induces a relative adjunction

\[
\int \mathcal{P}(\text{dAff}^*) \rightleftarrows \int * / \mathcal{P}(\text{dAff})
\]

over \( \text{cdga}_k^{<0} \).
Proof. The $\infty$-category $\int^*/\mathcal{P}(\mathsf{dAff})$ is spanned by the pairs $(A, \text{Spec} A \to T)$. Let us now focus on $\int \mathcal{P}(\mathsf{dAff}^*)$. Its definition is analogue to that of $\int \mathcal{P}(\mathsf{dAff}^\Delta)$. We define the restriction functor

$$\int \mathcal{P}(\mathsf{dAff}^\Delta) \to \int \mathcal{P}(\mathsf{dAff}^*)$$

It admits a fiberwise left adjoint, namely the left Kan extension, which commutes with base change. This defines a relative left adjoint

$$\int \mathcal{P}(\mathsf{dAff}^*) \to \int \mathcal{P}(\mathsf{dAff}^\Delta)$$

Composing with the relative adjunction of lemma 2.2.3, we get a relative adjunction

$$\int \mathcal{P}(\mathsf{dAff}^*) \rightleftarrows \int \mathcal{P}(\mathsf{dAff}^\Delta)$$

The left adjoint factors through $\int^*/\mathcal{P}(\mathsf{dAff})$ and the composed functor

$$\int^*/\mathcal{P}(\mathsf{dAff}) \to \int \mathcal{P}(\mathsf{dAff}^\Delta) \to \int \mathcal{P}(\mathsf{dAff}^*)$$

is its relative right adjoint.

Proposition 2.2.5. Let $X$ be an algebraic derived stack. There are functors

$$\phi: X/\mathcal{P}(\mathsf{dAff}^\Delta)/X \to \lim_{\text{Spec} A \to X} \text{Spec} A/\mathcal{P}(\mathsf{dAff}_A)$$

and

$$\theta: \lim_{\text{Spec} A \to X} \text{Spec} A/\mathcal{P}(\mathsf{dAff}_A) \to \lim_{\text{Spec} A \to X} \mathcal{P}(\mathsf{dAff}_A^*)$$

Proof. The functor $\phi$ is given by the following construction:

$$X/\mathcal{P}(\mathsf{dAff}^\Delta)/X \to \lim_{\text{Spec} A \to X} \text{Spec} A/\mathcal{P}(\mathsf{dAff}_A)/\text{Spec} A \simeq \lim_{\text{Spec} A \to X} \text{Spec} A/\mathcal{P}(\mathsf{dAff}_A)$$

The second functor is constructed as follows. The proposition 2.2.4 implies that there is a functor

$$\int \mathcal{P}(\mathsf{dAff}^*) \to \int^*/\mathcal{P}(\mathsf{dAff})$$

preserving cocartesian morphisms (over $\mathsf{cdga}^{\geq 0}_k$). We can therefore build the functor

$$\lim_{\text{Spec} A \to X} \mathcal{P}(\mathsf{dAff}_A^*) \to \lim_{\text{Spec} A \to X} \text{Spec} A/\mathcal{P}(\mathsf{dAff}_A)$$

This functor still preserves colimits and thus admits a right adjoint $\theta$.

Remark 2.2.6. Note that the functor $\theta$ naturally commutes with base change. We can indeed draw the commutative diagram (where $S \to T$ is a morphism between affine derived schemes)

$$T/\mathsf{dSt}/T \longrightarrow \mathcal{P}(T/\mathsf{dSt}/T) \longrightarrow \mathcal{P}(T/\mathsf{dAff}/T)$$

$$S/\mathsf{dSt}/S \longrightarrow \mathcal{P}(S/\mathsf{dSt}/S) \longrightarrow \mathcal{P}(S/\mathsf{dAff}/S)$$

The left hand side square commutes by definition of base change. The right hand side square also commutes as the restriction along a fully faithful functor preserves base change.
Definition 2.2.7. Let $X$ be an algebraic derived stack. Let us define the formal completion functor
\[ (-)^f : X / \mathcal{P}(\text{dAff}_k)/X \to \lim_{\text{Spec } A \to X} \mathcal{P}(\text{dgExt}_A)^{\text{op}} \]
as the composed functor
\[
\begin{align*}
X / \mathcal{P}(\text{dAff}_k)/X &\to \lim_{\text{Spec } A \to X} \text{Spec } A / \mathcal{P}(\text{dAff}_A) \\
&\to \lim_{\text{Spec } A \to X} \mathcal{P}(\text{dAff}_A^a) \\
&\to \lim_{\text{Spec } A \to X} \mathcal{P}(\text{dgExt}_A)^{\text{op}}
\end{align*}
\]

Remark 2.2.8. Let $u : S = \text{Spec } A \to X$ be a point. The functor $u^*(-)^f$ maps a pointed stack $Y$ over $X$ to the functor $\text{dgExt}_A \to \text{sSets}$
\[ B \mapsto \text{Map}_{S/\to X}(\text{Spec } B, Y) \]

Definition 2.2.9. Let $X$ be an algebraic derived stack. Let $\text{dSt}_X^{\text{alg}, \text{lfp}}$ denote the full subcategory of $X / \mathcal{P}(\text{dAff})/X$ spanned by those $X \to Y \to X$ such that $Y$ is a derived Artin stack.

Lemma 2.2.10. The restriction of $(-)^f$ to $\text{dSt}_X^{\text{alg}, \text{lfp}}$ has image in $\text{dSt}_X^f$.

Proof. We have to prove that whenever $Y$ is a pointed algebraic stack over $X$ then $Y^f$ is formal over $X$. Because of remark 2.2.6, it suffices to treat the case of an affine base. The result then follows from the existence of an obstruction theory for $Y \to X$.

Definition 2.2.11. Let $X$ be an algebraic stack locally of finite presentation. We define its tangent Lie algebra as the $X$-dg-Lie algebra
\[ \ell_X = \mathcal{L}_X \left( (X \times X)^f \right) \]
where the product $X \times X$ is a pointed stack over $X$ through the diagonal and the first projection.

Proof (of theorem 2.0.1). Let us denote $Y = X \times X$ and let $u : S = \text{Spec } A \to X$ be a point. The derived stack $X$ has a global tangent complex and the natural morphism $u^*\ell_X = u^*\mathcal{L}_X Y^f \to \mathcal{L}_A(u^*Y^f)$ is an equivalence. The functor $u^*Y^f$ maps $B$ in $\text{dgExt}_A$ to
\[ \text{Map}_{S/\to X}(\text{Spec } B, X \times X) \simeq \text{Map}_S(\text{Spec } B, X) \]
We deduce using corollary 1.4.11 that the underlying $A$-dg-module of $\mathcal{L}_A(u^*Y^f)$ therefore represents the functor
\[
\begin{align*}
\text{dgMod}^{\text{fr,} \geq 1}_A &\to \text{sSets} \\
M &\mapsto \text{Map}_S(\text{Spec } (A \oplus M^\vee [-1]), X)
\end{align*}
\]
Using once again that $X$ has a global tangent complex we conclude that the underlying module of $\ell_X$ is indeed $T_X[-1]$.

Let us now consider the functor
\[ X / \text{dSt}_X^{\text{alg}, \text{lp}} \to \text{dSt}_X^{\text{alg}} \]
mapping a morphism $X \to Z$ to the stack $X \times Z$ pointed by the graph map $X \to X \times Z$ and endowed with the projection morphism to $X$. Composing this functor with $(-)^f$ and $\mathcal{L}_X$ we finally get the wanted functor

$$X/\text{dSt}^{\text{alg}, \text{lp}} \to \ell_X/\text{dgLie}_X$$

Observing that $X \times Z$ is equivalent to $X \times \hat{Z}$ and because $Z$ has a global tangent complex, we deduce

$$\mathcal{L}_X((X \times Z)^f) \simeq u^*\ell_Z$$

$\Box$

2.3 Derived categories of formal stacks

**Definition 2.3.1.** Let $A$ be any cdga $\mathbb{K}^{\leq 0}$ and $L \in \text{dgLie}_A$. The category $\text{dgRep}_A(L)$ of representations of $L$ is endowed with a combinatorial model structure for which equivalences are exactly the $L$-equivariant quasi-isomorphisms and for which the fibrations are those maps sent onto fibrations by the forgetful functor to $\text{dgMod}_A$.

**Lemma 2.3.2.** Let $L$ be an $A$-dg-Lie algebra. There is a Quillen adjunction

$$f_L : \text{dgMod}_{C_A L} \rightleftarrows \text{dgRep}_A(L) : g_L$$

Given by

$$f_L : M \mapsto \mathcal{U}_A \left( A[\eta] \otimes_A L \right) \otimes_{C_A L} M$$

$$g_L : N \mapsto \text{Hom}_L \left( \mathcal{U}_A \left( A[\eta] \otimes_A L \right), N \right)$$

where $A[\eta] \otimes_A L$ is as in section 1.4. Moreover, if $L$ is good then the $\infty$-functor associated to $f_L$ is fully faithful.

**Remark 2.3.3.** The image $g_L(N)$ is a model for the cohomology $R\text{Hom}_L(A, N)$ of $L$ with values in $N$.

**Proof.** The fact that $f$ and $g$ are adjoint functors is immediate. The functor $f$ preserves quasi-isomorphisms (see the proof of 1.4.3) and fibrations. This is therefore a Quillen adjunction.

Let us prove now that $f_{\text{perf}_B}$ is fully faithful. Let $M$ and $N$ be two $B$-dg-modules. There is a map $\text{Map}(M, N) \to \text{Map}(fM, fN)$. Fixing $M$ (resp. $N$), the set of $N$’s (resp. $M$’s) such that this map is an equivalence is stable under extensions, shifts and retracts. It is therefore sufficient to prove that the map $\text{Map}(B, B) \to \text{Map}(fB, fB)$ is an equivalence, which follows from the definition (if we look at the dg-modules of morphisms, then both domain and codomain are quasi-isomorphic to $B = C_A L$).

To prove that $f : \text{dgMod}_B \to \text{dgRep}_A(L)$ is fully faithful, we only need to prove that $f$ preserves perfect objects. It suffices to prove that $f_L B \simeq A$ is perfect in $\text{dgRep}_A(L)$. The (non commutative) $A$-dg-algebra $\mathcal{U}_A(A[\eta] \otimes_A L)$ is a finite cellular object (because $L$ is good) and is endowed with a morphism to $A$. The forgetful functor $\text{dgMod}_A \to \text{dgMod}_{A[\eta] \otimes A L}$ therefore preserves perfect objects (see [TV]).

Let us consider the following category
• An object is a pair \( (A, L) \) with \( A \in \text{cdga}_k \leq 0 \) and with \( L \in \text{dgLie}_A \).

• A morphism \( (A, L) \to (B, L') \) is a map \( A \to B \) together with a map \( L' \to L \otimes_A B \) in \( \text{dgLie}_B \).

Let us denote by \( \int \text{dgLie}^{op} \) the \( \infty \)-category obtained from it by localizing along quasi-isomorphisms.

Let \( \mathcal{C} \) be the following category

• An object is a triple \( (A, L, M) \) with \( A \in \text{cdga}_k \leq 0 \), with \( L \in \text{dgLie}_A \) and with \( M \in \text{dgRep}_A(L) \).

• A morphism \( (A, L, M) \to (B, L', N) \) is a map \( A \to B \) together with a map \( L' \to L \otimes_A B \) in \( \text{dgLie}_B \) and a morphism \( M \otimes_A B \to N \) of \( L' \)-modules.

Let us denote by \( \int \text{dgRep} \) the localization of \( \mathcal{C} \) along quasi-isomorphisms. There is a natural cocartesian fibration \( p: \int \text{dgRep} \to \int \text{dgLie}^{op} \) (see \([\text{DAG-X}]\) prop. 2.4.29).

Let \( \mathcal{D} \) be the following category

• An object is a triple \( (A, L, V) \) with \( A \in \text{cdga}_k \leq 0 \), with \( L \in \text{dgLie}_A \) and with \( V \in \text{dgMod}_{C, A} L \).

• A morphism \( (A, L, V) \to (B, L', W) \) is a map \( A \to B \) together with a map \( L' \to L \otimes_A B \) in \( \text{dgLie}_B \) and a morphism of \( C, A \)-dg-modules \( V \otimes_{C, A} L \to W \).

Let us denote by \( \int \text{dgMod}_{C(\cdot)} \) the \( \infty \)-category obtained from \( \mathcal{D} \) by localizing along quasi-isomorphisms. There is a natural cocartesian fibration \( q: \int \text{dgMod}_{C(\cdot)} \to \int \text{dgLie}^{op} \).

Let us build a functor \( g: \mathcal{C} \to \mathcal{D} \)

• The image of an object \( (A, L, M) \) is the triple \( (A, L, V) \) where \( V \) is the \( C, A \)-dg-module \( \text{Hom}_L(\mathcal{U}_A(\{\eta\} \otimes_A L), M) \).

• The image of an arrow \( M \otimes_A B \to N \) is the composition

\[
\text{Hom}_L(\mathcal{U}_A(\{\eta\} \otimes_A L), M) \otimes_{C, A} L' \to \text{Hom}_L(\mathcal{U}_B(\{\eta\} \otimes_B L'), M \otimes_B L')
\]

\[
\to \text{Hom}_L(\mathcal{U}_B(\{\eta\} \otimes_B L'), N)
\]

where the first map sends a tensor \( \lambda \otimes \mu \) to \( (\lambda \otimes \text{id}) \cdot \mu \) with

\[
\lambda \otimes \text{id}: \mathcal{U}_B(\{\eta\} \otimes_B L') \to \mathcal{U}_B(\{\eta\} \otimes_B L) = \mathcal{U}_A(\{\eta\} \otimes_A L) \otimes_B M \otimes_A B
\]

The functor \( g \) induces an \( \infty \)-functor

\[
g: \int \text{dgRep} \to \int \text{dgMod}_{C(\cdot)}
\]

which commutes with the natural cocartesian fibrations to \( \int \text{dgLie}^{op} \).
Proposition 2.3.4. The functor $g$ admits a left adjoint $f$ relative to $\int \text{dgLie}^{op}$. There is therefore a commutative diagram of $\infty$-categories

\[
\begin{array}{ccc}
\int \text{dgMod}_{C(-)} & \xrightarrow{f} & \int \text{dgRep} \\
\downarrow{p} & & \downarrow{q} \\
\int \text{dgLie}^{op}
\end{array}
\]

where $f$ preserves cocartesian morphisms.

Proof. Whenever we fix $(A, L)$ in $\int \text{dgLie}^{op}$, the functor $g$ restricted to the fiber categories admits a left adjoint (see lemma 2.3.2). Moreover when $(A, L) \to (B, L')$ is a morphism in $\int \text{dgLie}^{op}$, the following squares weakly commute:

\[
\begin{array}{ccc}
\text{dgMod}_{C_A L} & \xrightarrow{f^*_L} & \text{dgRep}_A(L) \\
\downarrow{-\otimes_{C_A L} C_B(L \otimes_A B)} & & \downarrow{-\otimes_{L \otimes_B (L \otimes_A B)}} \\
\text{dgMod}_{C_B(L \otimes_A B)} & \xrightarrow{f^*_L \otimes_A B} & \text{dgRep}_B(L \otimes_A B) \\
\downarrow{-\otimes_{C_B(L \otimes_A B)} C_B(L')} & & \downarrow{\text{Forget}} \\
\text{dgMod}_{C_B L'} & \xrightarrow{f^*_L \otimes B} & \text{dgRep}_B(L')
\end{array}
\]

This proves that $f$ satisfies the requirements of prop. 8.3.2.11 of [HAlg].

There are two $\infty$-functor $\text{cdga}^{\leq 0}_k \to \text{Cat}_{\infty}$ given by

\[
A \mapsto \text{cdga}^{\leq 0}_A \quad \text{and} \quad A \mapsto \text{dgCat}_A
\]

and base change functors (see [Toë]). They are linked by a natural transformation given informally by

\[
\text{cdga}^{\leq 0}_A \to \text{dgCat}_A \\
B \mapsto \text{dgMod}^{\text{cod}}_B
\]

(where $\text{dgMod}^{\text{cod}}_B$ is the dg-category over $A$ of cofibrant $B$-dg-modules). They correspond to a diagram of categories

\[
\begin{array}{ccc}
\int \text{cdga}^{\leq 0} & \xrightarrow{} & \int \text{dgCat} \\
\downarrow{\text{cdga}^{\leq 0}_k} & & \downarrow{\text{cdga}^{\leq 0}_k} \\
\text{cdga}^{\leq 0}_k
\end{array}
\]

We can restrict this functor to the cofibered category $\int \text{dgExt}$ of trivial square root extension by free modules of finite type. Let us denote by $\int \mathcal{P}_{\Sigma}(\text{dgExt}^{op})$ the cofibered category over $\text{cdga}^{\leq 0}_k$ corresponding to the functor $A \mapsto \mathcal{P}_{\Sigma}(\text{dgExt}^{op}_A)$. We get a diagram of cofibered
categories over $\text{cdga}_{\leq 0}^k$

\[
\begin{align*}
\text{dgCat}^{\text{op}} & \xleftarrow{F_0} \text{dgExt}^{\text{op}} \\
\text{cdga}_{\leq 0}^k & \xleftarrow{\text{L}_{\text{qcoh}}} \{ \text{dgExt}^{\text{op}} \}
\end{align*}
\]

The functor $F_0$ admits a relative left Kan extension $\text{L}_{\text{qcoh}}$ as drawn above. The functor $\text{L}_{\text{qcoh}}$ preserves cocartesian morphisms. Composing with the dg-nerve functor allows us the following definition.

**Definition 2.3.5.** Let $X$ be an algebraic stack. We define the derived category of a formal stack $Y$ over $X$ as the well-defined limit of $\infty$-categories

\[
\mathbb{L}^X_{\text{qcoh}}(Y) = \lim_{\text{Spec } A \to X} \lim_{\text{Spec } B \to Y} \text{dgMod}_B
\]

where $Y_A$ is the pullback of $Y$ along the morphism $\text{Spec } A \to X$.

**Definition 2.3.6.** Let $X$ be an algebraic stack and let $L$ be an $X$-dg-Lie algebra. Let us define the $\infty$-category of representations of $L$ as the limit

\[
\text{dgRep}_X(L) = \lim_{\text{Spec } A \to X} \text{dgRep}_A(L_A)
\]

It is naturally endowed with a conservative forgetful functor to $\mathbb{L}_{\text{qcoh}}(X)$.

**Proposition 2.3.7.** Let $X$ be an algebraic stack locally of finite presentation. Let $L$ be an $X$-dg-Lie algebra. There is a fully faithful functor

\[
\mathbb{L}^X_{\text{qcoh}}(\mathcal{F}_X L) \to \text{dgRep}_X(L)
\]

**Proof.** It suffices to consider the affine case. Let us then assume $X = \text{Spec } A$ and let us write $L$ as a sifted colimit of dg-Lie algebras

\[
L_{\alpha} \in \text{dgLie}_{A^{\text{fr}, \geq 1}}
\]

The functor is then

\[
\mathbb{L}^A_{\text{qcoh}}(\mathcal{F}_A L) \cong \lim_{\alpha} \mathbb{L}^A_{\text{qcoh}}(\mathcal{F}_A L_{\alpha}) \cong \lim_{\alpha} \text{dgMod}_{A^{\text{fr}, \geq 1}}(L_{\alpha}) \to \lim_{\alpha} \text{dgRep}_A(L_{\alpha}) \leftarrow \text{dgRep}_A(L)
\]

where the last equivalence comes from [DAG-X], lemma 2.4.32.

**Theorem 2.3.8.** Let $X$ be an algebraic stack locally of finite presentation. There is an $\infty$-functor

\[
\text{Rep}_X : \mathbb{L}_{\text{qcoh}}(X) \to \text{dgRep}_X(\ell_X)
\]

It is a retract of the forgetful functor.
**Proof.** Let us first remark that $L_{\text{qcoh}}(X)$ can be obtained as the derived category $L^X_{\text{qcoh}}(X)$ of the formal stack $X$ over $X$. This trivial formal stack is final in the category $dSt^f_X$ and there is therefore a morphism of formal stacks over $X$

$\mathcal{F}_X \ell_X \to X$

We get

$L_{\text{qcoh}}(X) \simeq L^X_{\text{qcoh}}(X) \to L^X_{\text{qcoh}}(\mathcal{F}_X \ell_X) \to \mathbf{dgRep}_X(\ell_X)$

which locally around an $A$-point and decomposing $\ell_{X,A}$ into a sifted colimit of $\ell_{X,A}^\alpha \in \mathbf{dgLie}_A^{fr,>1}$ is the limit of the functors

$\phi_\alpha : \mathbf{dgMod}_A \to \mathbf{dgMod}_{C,A}(\ell_{X,A}^\alpha) \to \mathbf{dgRep}_A(\ell_{X,A}^\alpha)$

The functor $\phi_\alpha$ is given on a dg-module $M$ by

$\phi_\alpha(M) \simeq U_A \left( A[n] \otimes \ell_{X,A}^\alpha \right) \otimes M$

There is also a free functor $F_\alpha : \mathbf{dgMod}_A \to \mathbf{dgRep}_A(\ell_{X,A}^\alpha)$ given by

$F_\alpha(M) = U_A(\ell_{X,A}^\alpha) \otimes M$

We therefore get a natural transformation $F_\alpha \to \phi_\alpha$ which glues into a natural transformation

$\text{Free} \to \text{Rep}_X$

Both $\text{Rep}_X$ and $\text{Free}$ admit right adjoints, a functor $g_X$ and the forgetful functor $\text{Forget}$ and there is thus a natural transformation of endofunctors of $L_{\text{qcoh}}(X)$

$id \to g_X \text{Rep}_X \to \text{Forget Rep}_X$

This natural transformation is locally

$M \to \left( M \otimes B \right) \otimes U_A \left( A[n] \otimes \ell_{X,A}^\alpha \right)$

which is an equivalence.

\[\square\]

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