TIME-FREQUENCY LOCALIZATION OPERATORS AND A BEREZIN TRANSFORM

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ABSTRACT. Time-frequency localization operators are a quantization procedure that maps symbols on \( \mathbb{R}^{2d} \) to operators and depends on two window functions. We study the range of this quantization and its dependence on the window functions. If the short-time Fourier transform of the windows does not have any zero, then the range is dense in the Schatten \( p \)-classes. The main tool is new version of the Berezin transform associated to operators on \( L^2(\mathbb{R}^d) \). Although some results are analogous to results about Toeplitz operators on spaces of holomorphic functions, the absence of a complex structure requires the development of new methods that are based on time-frequency analysis.

1. INTRODUCTION

We study the problem whether an arbitrary operator on \( L^2(\mathbb{R}^d) \) can be approximated by a time-frequency localization operator. The formalism of time-frequency localization operators is a quantization procedure and maps functions on phase space \( \mathbb{R}^{2d} \) to operators acting on \( L^2(\mathbb{R}^d) \). Time-frequency localization operators are widely used in physics and engineering. In physics, a special case of localization operators (with Gaussian window) has been around for a long time in connection with quantization under the name (Anti-) Wick operators in the work of Berezin [1]. In analysis, they are used to approximate pseudodifferential operators, see [11,26,32]. In the form considered here localization operators were introduced and studied by Daubechies [12] and Ramanathan and Topiwala [29]. In signal processing localization operators are used for time-frequency masking and feature extraction of signals from a time-frequency representation.

For a more formal discussion of our results, let us introduce the time-frequency shifts \( \pi(z)f(t) = e^{2\pi i \xi \cdot t} f(t-x) \) for \( z = (x,\xi) \in \mathbb{R}^{2d} \) and \( t \in \mathbb{R}^d \). If \( a \) is a function on \( \mathbb{R}^{2d} \) and \( \varphi_1, \varphi_2 \) are two functions in \( L^2(\mathbb{R}^d) \), so-called window functions, then the localization operator \( A_{a}^{\varphi_1,\varphi_2} \) is defined formally as

\[
A_{a}^{\varphi_1,\varphi_2}f = \int_{\mathbb{R}^{2d}} a(z) \langle f, \pi(z)\varphi_1 \rangle \pi(z)\varphi_2 \, dz.
\]

Roughly speaking, the action of \( A_{a}^{\varphi_1,\varphi_2} \) consists of multiplying the short-time Fourier transform \( z \mapsto \langle f, \pi(z)\varphi_1 \rangle \) by a symbol \( a \) (this yields a function on \( \mathbb{R}^{2d} \)) and then projecting back to \( L^2(\mathbb{R}^d) \). This procedure resembles the definition of Toeplitz...
operators in complex analysis, and indeed, localization operators are also called Toeplitz operators by some authors [34, 35]. The basic properties of localization operators are well understood. There are numerous results about the boundedness properties between function spaces, the Schatten class properties, and the symbolic calculus for localization operators. The reader should consult the surveys [10, 26, 36] for the basic properties and [5, 7, 9, 21, 24] for further discussion.

In this paper we examine the question how large the class of localization operators (with symbols from some prescribed class, e.g., from Schatten class) is compared to the standard spaces of operators (e.g., the Hilbert-Schmidt class). Can an arbitrary operator be approximated by a localization operator, and, if yes, in which topology? In terms of physics, what is the range of the quantization procedure given by localization operators?

Motivated by the analogy to Toeplitz operators, we will answer this question in terms of a related dual mapping, the so-called Berezin transform. Fix \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \), and let \( T \in B(L^2(\mathbb{R}^d)) \) be a bounded operator on \( L^2(\mathbb{R}^d) \). The Berezin transform maps operators to functions and is defined to be

\[
\mathcal{B}T(z) := \langle T\pi(z)\varphi_2, \pi(z)\varphi_1 \rangle, \quad z \in \mathbb{R}^{2d}.
\]

In the following \( S^p \) denotes the Schatten \( p \)-class on \( L^2(\mathbb{R}^d) \) for \( 1 \leq p < \infty \). For \( p = \infty \), \( S^\infty \) consists of all compact operators on \( L^2(\mathbb{R}^d) \). The conjugate index is \( p' = p/(p-1) \). Time-frequency localization operators and the Berezin transform are related as follows.

**Theorem 1.1.** Let \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d), \ 1 \leq p' < \infty \). Then the range \( \{ A^{\varphi_1,\varphi_2}_a : a \in L^{p'}(\mathbb{R}^{2d}) \} \) is norm-dense in \( S^{p'}(L^2(\mathbb{R}^d)) \), if and only if the Berezin transform \( \mathcal{B} \) is one-to-one on \( S^p \).

In other words, the quantization procedure \( a \mapsto A^{\varphi_1,\varphi_2}_a \) has dense range in the Schatten class \( S^{p'} \), if and only if the Berezin transform is one-to-one on \( S^p \). This insight leads us to investigate the properties of the Berezin transform.

Here is a simple statement about the injectivity of the Berezin transform without any additional technical definitions.

**Proposition 1.2.** Let \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \).

(i) If \( 1 \leq p \leq 2 \) and \( V_{\varphi_1}\varphi_2(z) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \), then \( \mathcal{B} \) is one-to-one from \( S^p \) to \( L^p(\mathbb{R}^{2d}) \). In this case, \( \{ A^{\varphi_1,\varphi_2}_a : a \in L^{p'}(\mathbb{R}^{2d}) \} \) is norm-dense in \( S^{p'} \).

(ii) If \( p \geq 2 \) and \( \mathcal{B} \) is one-to-one from \( S^p \) to \( L^p(\mathbb{R}^{2d}) \), then \( V_{\varphi_1}\varphi_2(z) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \).

The main point of this proposition is that the injectivity of the Berezin transform depends on the properties of the windows \( \varphi_1 \) and \( \varphi_2 \).

By combining Theorem 1.1 and Proposition 1.2 we obtain a complete characterization when the quantization \( a \mapsto A^{\varphi_1,\varphi_2}_a \) has dense range in the class of Hilbert-Schmidt operators \( S^2 \).

**Corollary 1.3.** The range \( \{ A^{\varphi_1,\varphi_2}_a : a \in L^{p'}(\mathbb{R}^{2d}) \} \) is norm-dense in \( S^2 \), if and only if \( \langle \varphi_2, \pi(z)\varphi_1 \rangle \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \).
For $p = \infty$, we will see that the range of $a \mapsto A_{a}^{\varphi_{1}, \varphi_{2}}$ from $L^{\infty}(\mathbb{R}^{d})$ to $B(L^{2}(\mathbb{R}^{d}))$ is dense in the weak operator topology, but fails to be norm-dense. In fact, for the Fourier transform $\mathcal{F}$ we will show (Theorem 5.6) that

$$\|\mathcal{F} - A_{a}^{\varphi_{1}, \varphi_{2}}\|_{op} \geq 1$$

for all $a \in L^{\infty}(\mathbb{R}^{d})$ and all $\varphi_{1}, \varphi_{2} \in L^{2}(\mathbb{R}^{d})$. Therefore the class of localization operators cannot be norm-dense in $B(L^{2}(\mathbb{R}^{d}))$.

Theorem 1.1 and Proposition 1.2 are formulated for $L^{p}$-symbols. For such symbols the corresponding mapping properties between symbols and operators are easy to understand \cite{4,36}. However, in the context of time-frequency analysis a less restrictive calculus is available that also includes distributional symbols. In Sections 4 and 5 we will formulate more technical versions of Theorem 1.1 and Proposition 1.2 for distributional symbol classes.

Theorem 1.1 and Proposition 1.2, at least for the cases $p = 1$ and $p = \infty$, have many predecessors in complex analysis, see, e.g., \cite{2,3,14–16}. Perhaps the closest results are due to Berger and Coburn who show that Toeplitz operators on Bargmann-Fock space with symbols of compact support are dense in the trace class and in the compact operators \cite[Thm. 9]{3}.

Finally, Theorem 1.1 can also be formulated in a more abstract context, where one needs only a resolution of the identity $\text{Id} = \int_{X} P(x) d\mu(x)$ with rank-one operators $P(x)$. This formalism was introduced by B. Simon \cite{33} and some special cases of Theorem 1.1 could be attributed to him. We emphasize that in the context of time-frequency analysis we prove a much sharper result for distributional symbol classes that cannot be formulated in the axiomatic setting.

Perhaps the main surprise of Theorem 1.1 is the explicit dependance of the density property on the properties of the window functions. This is a new aspect in our approach, because window functions do not even occur in the definition of classical Toeplitz operators. The analysis of the interaction between window functions and the properties of time-frequency localization operators is the main novelty of this paper and requires tools that are specific for time-frequency analysis.

The paper is organized as follows: In Section 2 we collect the necessary prerequisites from time-frequency analysis, in particular, we recall the definition of modulation spaces and some facts from harmonic analysis, and we summarize the mapping properties of localization operators. In Section 3 we introduce the Berezin transform and study its mapping properties. In Section 4 we investigate the kernel of the Berezin transform. Finally, in Section 5 we study the approximation of arbitrary operators by time-frequency localization operators and prove the theorems discussed in the introduction.

## 2. Prerequisites

### 2.1. Short-Time Fourier Transform, Wigner Distribution and Modulation Spaces.

Let $x, \omega \in \mathbb{R}^{d}$ and $z = (x, \omega) \in \mathbb{R}^{2d}$. We define the translation operator on $L^{2}(\mathbb{R}^{d})$ as $T_{x}f(t) := f(t - x)$ and the modulation operator as $M_{\omega}f(t) := e^{2\pi i \omega \cdot t} f(t)$, for $f \in L^{2}(\mathbb{R}^{d})$. The time-frequency shift $\pi(z)$ is then given
as
\[ \pi(z)f(t) := \pi(x, \omega)f(t) := M_\omega T_x f(t) = e^{2\pi i \omega \cdot t} f(t - x). \]
The time-frequency shifts are unitary operators on the Hilbert space \( L^2(\mathbb{R}^d) \) and isometries on \( L^p(\mathbb{R}^d) \).

We frequently use the following bilinear time-frequency distributions.

The short-time Fourier transform of \( f \) with respect to a window \( g \) (STFT) is defined as
\[ V_g f(x, \omega) := \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t)g(t-x)e^{-2\pi i t \cdot \omega} \, dt \]
for \( x, \omega \in \mathbb{R}^d \).

We list some standard properties of the short-time Fourier transform.

**Lemma 2.1.** Let \( f, g \in L^2(\mathbb{R}^d) \). Then the following holds:

(i) Covariance property: \( |V_g(\pi(w)f)(z)| = |V_g f(z - w)| \)

(ii) Isometry property: \( V_g f \) is in \( L^2(\mathbb{R}^{2d}) \) and \( \|V_g f\|_2 = \|f\|_2 \|g\|_2 \).

(iii) Let \( J(z_1, z_2) = (z_2, -z_1) \) be the standard symplectic form on \( \mathbb{R}^{2d} \), then
\[ V_{\pi(z)g}(\pi(z)f)(w) = e^{2\pi i J_z \cdot w} V_g f(w) \quad w, z \in \mathbb{R}^{2d}. \]

For the representation of operators we will also need the Wigner distribution. The (cross) Wigner distribution of \( f \in L^2(\mathbb{R}^d) \) and \( g \in L^2(\mathbb{R}^d) \) is defined to be
\[ W(f, g)(x, \omega) := \int_{\mathbb{R}^d} f(x + \frac{t}{2})g(x - \frac{t}{2})e^{-2\pi i t \cdot \omega} \, dt \]
for \( x, \omega \in \mathbb{R}^d \).

The following formula connects the Wigner distribution with the STFT via the Fourier transform. Let \( f, g \in L^2(\mathbb{R}^d) \). Then
\[ \widehat{W(f, g)}(x, \omega) = e^{-\pi i x \cdot \omega} V_g f(-\omega, x), \quad \text{for all } x, \omega \in \mathbb{R}^d. \]

**Definition 2.2.** (Modulation Spaces). Let \( g \in \mathcal{S}(\mathbb{R}^d) \) be a fixed non-zero Schwartz function, and \( 1 \leq p, q \leq \infty \). We define the modulation space
\[ M^{p,q}(\mathbb{R}^d) := \{ f \in S'(\mathbb{R}^d) : V_g f \in L^{p,q}(\mathbb{R}^{2d}) \}. \]
The norm on \( M^{p,q} \) is given by
\[ \|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} := \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^{2d}} |V_g f(x, \omega)|^p \, dx \, d\omega \right)^{1/p} \]
with modifications if \( p = \infty \) or \( q = \infty \). We write \( M^p = M^{p,p} \).

The theory of modulation spaces is explained in detail in the monographs [23] and [20]. We mention only the following properties: Every modulation space \( M^{p,q} \) is a Banach space. Its definition does not depend on the chosen window function \( g \); different windows \( g \in \mathcal{S}(\mathbb{R}^d) \) yield equivalent norms [23]. The duality of modulation spaces is analogous to that of Lebesgue spaces: If \( 1 \leq p, q < \infty \), then the dual space of \( M^{p,q} \) is given by \( M^{p',q'} \), where \( p' = \frac{p}{p-1} \) is the conjugate index of \( p \). Unlike
the Lebesgue spaces $L^p(\mathbb{R}^d)$, modulation spaces are embedded into each other: $S \subseteq M^1 \subseteq M^p \subseteq M^q \subseteq M^\infty \subseteq S'$ for $1 \leq p \leq q \leq \infty$. Furthermore, 

$$M^{p,1} \subseteq L^p \subseteq M^{p,\infty}$$

for all $1 \leq p \leq \infty$.

For further reference we note the following smoothness properties of STFTs [7, 23]. If $f, g \in S(\mathbb{R}^d)$, then $V_q f \in S(\mathbb{R}^{2d})$; if $f, g \in M^1(\mathbb{R}^d)$, then $V_q f \in M^1(\mathbb{R}^{2d})$. Finally, we will use the following convolution [7,34]: $M^\infty \ast M^1 \subseteq M^{\infty,1}$ with a norm estimate

$$\|f \ast g\|_{M^{\infty,1}} \leq C\|f\|_{M^\infty} \|g\|_{M^1}.$$

### 2.2. Compact Operators and Schatten Classes

Let $T : H \to H$ be a compact and self-adjoint operator on a Hilbert space $H$. Then there exists a sequence of eigenvalues $(\lambda_j)_{j \in \mathbb{N}} \subseteq \mathbb{R}$ and an orthonormal set of eigenvectors $(\phi_j)_{j \in \mathbb{N}} \subseteq H$ such that $\lim_{j \to \infty} \lambda_j = 0$, $T \phi_j = \lambda_j \phi_j$ for all $j \in \mathbb{N}$, and

$$T f = \sum_{j=1}^\infty \lambda_j \langle f, \phi_j \rangle \phi_j$$

for all $f \in H$, with convergence of the series in the norm of $H$. The singular values $s_j(T)$ of a compact operator $T : H \to H$ are the square-roots of the eigenvalues of the (compact self-adjoint positive) operator $T^* T$. We write

$$s_j(T) := \lambda_j (T^* T)^{1/2},$$

and assume that $s_j(T) \geq s_{j+1}(T)$. If $(s_j(T))_{j \in \mathbb{N}} \in \ell^p(\mathbb{N})$ for some $1 \leq p < \infty$, then $T : H \to H$ belongs to the Schatten $p$-class $S^p$. Equipped with the norm $\|T\|_{S^p} = (\sum_j s_j(T)^p)^{1/p}$, $S^p$ is a Banach space for $1 \leq p < \infty$. The space $S^1$ consists of the trace class operators, the space $S^2$ consists of all Hilbert-Schmidt operators. For $p = \infty$, we define $S^\infty$ to be the set of all compact operators. Then we have the duality relation $(S^p)^* = S^{p'}$, where $p' = \frac{p}{p-1}$ is the conjugate index of $p$, $1 \leq p < \infty$.

### 2.3. Localization Operators

A time-frequency localization operator can be interpreted as a multiplier for the short-time Fourier transform. First we analyze a given function $f$ on $\mathbb{R}^d$, a “signal” in engineering language, by taking its short-time Fourier transform with window $\varphi_1$. This yields a time-frequency representation of $f$ on $\mathbb{R}^d \times \mathbb{R}^d$. Then we extract the interesting parts of this representation by multiplying the STFT with a suitable “mask”, the so-called symbol, a function $a$ on $\mathbb{R}^d \times \mathbb{R}^d$. Finally, we synthesize a new signal by applying the adjoint of the short-time Fourier transform (possibly with some other window $\varphi_2$). This algorithm of time-frequency filtering leads to the formal definition of the localization operator:

$$A^\varphi_{a,\varphi_2} f = V^*_\varphi_2 (a \cdot V_{\varphi_1} f) = \int_{\mathbb{R}^{2d}} a(z) \cdot V_{\varphi_1} f(z) \pi(z) \varphi_2 \ dz.$$
In proofs it will be convenient to use the weak definition of a localization operator. If \( \varphi_1, \varphi_2, f, g \in L^2(\mathbb{R}^d) \), then
\[
\langle A_a^{\varphi_1, \varphi_2} f, g \rangle = \langle a, V_{\varphi_1} f V_{\varphi_2} g \rangle.
\]
Here the first bracket is the inner product on \( L^2(\mathbb{R}^d) \), whereas the second bracket is on \( L^2(\mathbb{R}^{2d}) \). By extending the inner product to a duality bracket, one can define a localization operator conveniently on many other function spaces. For example, (3) makes sense, when \( a \in S'(\mathbb{R}^{2d}) \) and \( \varphi_1, \varphi_2, f, g \in S(\mathbb{R}^d) \), consequently, the localization operator \( A_a^{\varphi_1, \varphi_2} \) defines a continuous operator \( A_a^{\varphi_1, \varphi_2} \) from \( S(\mathbb{R}^d) \) to \( S'(\mathbb{R}^d) \). As a special case we mention that for \( a \in M^\infty(\mathbb{R}^{2d}) \) and \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) the operator \( A_a^{\varphi_1, \varphi_2} \) is bounded on all modulation spaces \( M^{p,q}(\mathbb{R}^d), 1 \leq p, q \leq \infty \).

The following table summarizes the basic mapping properties of the correspondence \( A : a \mapsto A_a^{\varphi_1, \varphi_2} \) from symbols to localization operators. In the result for compact operators \( L^0(\mathbb{R}^{2d}) \) denotes the subspace of compactly supported functions in \( L^\infty(\mathbb{R}^{2d}) \). For proofs see the given references.

| Symbol | Windows | Localization Operator | Ref. |
|--------|---------|-----------------------|-----|
| \( S'(\mathbb{R}^{2d}) \) | \( S(\mathbb{R}^d) \) | \( \Psi DO S(\mathbb{R}^d) \to S'(\mathbb{R}^d) \) | [12] |
| \( L^\infty(\mathbb{R}^{2d}) \) | \( L^2(\mathbb{R}^d) \) | \( B(\mathbb{L}^2(\mathbb{R}^d)) \) | [30] |
| \( L^p(\mathbb{R}^{2d}), 1 \leq p < \infty \) | \( L^2(\mathbb{R}^d) \) | \( S^\infty(\mathbb{L}^2(\mathbb{R}^d)) \) | [21] |
| \( M^{\infty, \infty}(\mathbb{R}^{2d}) \) | \( M^1(\mathbb{R}^d) \) | \( B(M^{p,q}(\mathbb{R}^d)), 1 \leq p, q \leq \infty \) | [7] |
| \( M^{p,\infty}(\mathbb{R}^{2d}), 1 \leq p < \infty \) | \( M^1(\mathbb{R}^d) \) | \( S^p(\mathbb{L}^2(\mathbb{R}^d)) \) | [7] |

Table 1. Localization operators with different symbols and windows

We mention that the symbol class \( M^{p,\infty} \) is significantly larger than just \( L^p \). For example, the modulation space \( M^{p,\infty} \) contains discrete measures of the form \( a = \sum_j c_j \delta_{z_j} \), where the coefficient sequence \( c \in \ell^p \) and \( \{z_j\} \) is a uniformly discrete subset of \( \mathbb{R}^{2d} \). In particular, with the symbol class \( M^{p,\infty} \) one can treat many aspects of the theory of so-called Gabor multipliers that was initiated in [19].

2.4. Kernel Theorems. For some technical arguments we will use the pseudodifferential operator representation of an operator. We state explicitly two representation theorems that use the Wigner distribution \( W(f,g) \) of \( f \) and \( g \).

**Proposition 2.3.** (i) Pool’s Theorem [23]: If \( T \in S^2(L^2(\mathbb{R}^d)) \), then there exists a unique symbol \( \sigma \in L^2(\mathbb{R}^{2d}) \) such that
\[
\langle Tf, g \rangle = \langle \sigma, W(g, f) \rangle
\]
for all \( f, g \in L^2(\mathbb{R}^d) \).

(ii) Kernel theorem for modulation spaces [18, 23]: Let \( T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d) \) be a bounded linear operator. Then there exists a unique \( \sigma \in M^\infty(\mathbb{R}^{2d}) \) such that
\[
\langle Tf, g \rangle = \langle \sigma, W(g, f) \rangle
\]
for all \( f, g \in M^1(\mathbb{R}^d) \).
2.5. Density theorems. Tauberian theorems address the question when the span of all translates of a given function $f$ is dense in a space. It is therefore not surprising that they can be applied with profit to other density questions. We need three versions.

**Proposition 2.4.** (i) $L^2$-version: Let $f \in L^2(\mathbb{R}^d)$. Then $\overline{\text{span}\{T_z f \mid z \in \mathbb{R}^d\}} = L^2(\mathbb{R}^d)$ if and only if $\hat{f}(\omega) \neq 0$ for almost all $\omega \in \mathbb{R}^d$.

(ii) $L^1$-version: Let $f \in L^1(\mathbb{R}^d)$. Then $\overline{\text{span}\{T_z f \mid z \in \mathbb{R}^d\}} = L^1(\mathbb{R}^d)$, if and only if $\hat{f}(\omega) \neq 0$ for all $\omega \in \mathbb{R}^d$.

(iii) $M^1$-version: Let $f \in M^1(\mathbb{R}^d)$. Then $\overline{\text{span}\{T_z f \mid z \in \mathbb{R}^d\}} = M^1(\mathbb{R}^d)$ (where the closure is taken with respect to the $M^1$-norm), if and only if $\hat{f}(\omega) \neq 0$ for all $\omega \in \mathbb{R}^d$.

The $L^1$ and $L^2$-versions are standard, the $L^1$-version is one of many famous theorems of Wiener [31], the $M^1$-version is contained in [17,30,31] (the ideal theory in the Banach algebras $L^1$ and $M^1$ is the same).

Let us emphasize that the obvious generalizations to $L^p(\mathbb{R}^d)$ for $p \neq 1, 2, \infty$ are false. By a very deep result of Lev and Olevski [27] the density of $\text{span}\{T_z f \mid z \in \mathbb{R}^d\}$ in $L^p$ depends on subtle properties of $f$, and not only on the measure of the zero set of $f$. For this reason the formulation of Proposition 1.2 is divided into a sufficient condition and a slightly weaker necessary condition (rather than a complete characterization).

3. The Berezin Transform

In this section we study a general Berezin transform for operators on $L^2(\mathbb{R}^d)$. The Berezin transform maps operators to functions, whereas quantization rules map functions to operators.

**Definition 3.1** (Berezin Transform). Let $T \in B(L^2(\mathbb{R}^d))$. Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. The Berezin transform $B$ maps $T$ to the function on $\mathbb{R}^{2d}$ given by $B_T(z) := \langle T \pi(z) \varphi_2, \pi(z) \varphi_1 \rangle$, $z \in \mathbb{R}^{2d}$.

**REMARKS:** 1. The Berezin transform and its properties depend on the window functions $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, but we will simply write $B$ instead of $B_{\varphi_1, \varphi_2}$. This practice should not lead to any confusion.

2. If $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then the Berezin transform can be defined for all operators from $T : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$. In the following we will derive several continuity properties of the Berezin transform for different spaces of operators and windows.

For technical arguments we will use two alternative representations of the Berezin transform.

**Lemma 3.2.** If $T$ is bounded from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$ and $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then there exists a unique symbol $\sigma \in M^\infty(\mathbb{R}^{2d})$, such that

$$B_T(z) = \langle \sigma, T W(\varphi_1, \varphi_2) \rangle.$$

If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ and $T \in \mathcal{S}'$, then (4) holds with $\sigma \in L^2(\mathbb{R}^{2d})$. 
Proof. Let $\sigma$ be the kernel of $T$ in the pseudodifferential operator representation of $T$ given in Theorem 2.3. If $T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$, then $\sigma \in M^\infty(\mathbb{R}^{2d})$; if $T \in \mathcal{S}$, then $\sigma \in L^2(\mathbb{R}^{2d})$. The identity (4) now follows from the identity

$$BT(z) = \langle T\pi(z)\varphi_2, \pi(z)\varphi_1 \rangle = \langle \sigma, W(\pi(z)\varphi_1, \pi(z)\varphi_2) \rangle = \langle \sigma, T_2W(\varphi_1, \varphi_2) \rangle,$$

where in the last equality we have used the covariance property of the Wigner distribution. $\blacksquare$

Lemma 3.3. Assume that $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ are two orthonormal systems in $L^2(\mathbb{R}^d)$ and that $(s_n)_{n \in \mathbb{N}} \in \ell^\infty$ is a bounded sequence of complex numbers. If

$$Tf = \sum_{n \in \mathbb{N}} s_n \langle f, g_n \rangle h_n,$$

then

$$BT(z) = \sum_{n \in \mathbb{N}} s_n V_{\varphi_1} h_n(z) V_{\varphi_2} g_n(z)$$

for every $z \in \mathbb{R}^{2d}$.

Proof. Since $(g_n)$ and $(h_n)$ are orthonormal systems in $L^2(\mathbb{R}^d)$, the series defining $T$ converges in $L^2(\mathbb{R}^d)$ for every $f$ and $T$ is bounded on $L^2(\mathbb{R}^d)$. We compute

$$BT(z) = \langle T\pi(z)\varphi_2, \pi(z)\varphi_1 \rangle$$

$$= \left\langle \sum_{n \in \mathbb{N}} s_n \langle \pi(z)\varphi_2, g_n \rangle h_n, \pi(z)\varphi_1 \right\rangle$$

$$= \sum_{n \in \mathbb{N}} s_n \langle \pi(z)\varphi_2, g_n \rangle \langle h_n, \pi(z)\varphi_1 \rangle$$

$$= \sum_{n \in \mathbb{N}} s_n V_{\varphi_2} g_n(z) V_{\varphi_1} h_n(z).$$

$\blacksquare$

In particular, the assumptions of Lemma 3.3 are satisfied for compact operators and for operators belonging to some Schatten $p$-class $\mathcal{S}^p$, $1 \leq p \leq \infty$, since every $T \in \mathcal{S}^p$ possesses a singular value decomposition

$$T = \sum_{n \in \mathbb{N}} s_n \langle \cdot, g_n \rangle h_n$$

that satisfies the assumptions of Lemma 3.3.

3.1. Boundedness of the Berezin Transform. In the following we study the mapping properties of the Berezin transform on the Schatten $p$-classes.
**Theorem 3.4.** Let $1 \leq p \leq \infty$ and $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$.

The Berezin transform is bounded from $S^p$ to $L^p(\mathbb{R}^{2d})$ with operator norm

$$
\|B\|_{S^p \to L^p} \leq \|\varphi_1\|_2 \|\varphi_2\|_2.
$$

If $T \in S^\infty$, then $BT$ is in $C_0(\mathbb{R}^{2d})$, the continuous functions vanishing at infinity.

**Proof.** We prove the boundedness for $p = 1$ and $p = \infty$ and then use interpolation.

(i) Case $p = \infty$: If $T \in B(L^2(\mathbb{R}^d))$, then

$$
|BT(z)| = \langle T \pi(z) \varphi_2, \pi(z) \varphi_1 \rangle
$$

$$
\leq \|T\|_{B(L^2)} \|\pi(z) \varphi_2\|_2 \|\pi(z) \varphi_1\|_2
$$

$$
= \|T\|_{B(L^2)} \|\varphi_2\|_2 \|\varphi_1\|_2
$$

for all $z \in \mathbb{R}^d$, hence $B$ is bounded from $B(L^2)$ to $L^\infty$ with

$$
\|B\|_{B(L^2) \to L^\infty} \leq \|\varphi_1\|_2 \|\varphi_2\|_2.
$$

The function $BT$ is continuous since the mapping $z \mapsto \pi(z) \varphi$ is continuous from $\mathbb{R}^d$ to $L^2(\mathbb{R}^d)$ for arbitrary $\varphi \in L^2(\mathbb{R}^d)$.

Next assume that $T$ is a compact operator and $\lim_{n \to \infty} |z_n| = \infty$ for a sequence $(z_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$. Then, for arbitrary $\phi, \psi \in L^2(\mathbb{R}^d)$,

$$
\langle \psi, \pi(z_n) \phi \rangle = V_\phi \psi(z_n) \longrightarrow 0 = \langle \psi, 0 \rangle
$$

by the version of the Riemann-Lebesgue Lemma for the STFT. Thus $\pi(z_n) \phi$ converges weakly to 0. Since $T$ is compact, the sequence $T(\pi(z_n) \phi_1)$ converges to 0 in the $L^2$-norm. But then

$$
BT(z_n) = \langle T \pi(z_n) \phi_1, \pi(z_n) \phi_2 \rangle \longrightarrow 0
$$

for $n \to \infty$. Consequently, $BT \in C_0(\mathbb{R}^{2d})$.

(ii) Case $p = 1$: If $T \in S^1$ is trace class, then $T$ possesses a spectral representation

$$
Tf = \sum_k s_k \langle f, g_k \rangle h_k, \quad f \in L^2(\mathbb{R}^d),
$$

with two orthonormal systems $(g_k)_{k \in \mathbb{N}}, (h_k)_{k \in \mathbb{N}}$, and singular values $(s_k)_{k \in \mathbb{N}} \in \ell^1$, such that $\sum_{k \in \mathbb{N}} s_k = \|T\|_{S^1} < \infty$. Then by Lemma 3.3

$$
BT(z) = \sum_k s_k V_{\varphi_1} h_k(z) V_{\varphi_2} g_k(z).
$$

Since $V_{\varphi_1} h_k$ and $V_{\varphi_2} g_k$ are in $L^2(\mathbb{R}^{2d})$ by Lemma 2.1(ii), the product $V_{\varphi_1} h_k V_{\varphi_2} g_k$ is in $L^1(\mathbb{R}^{2d})$. Therefore the above series converges absolutely in $L^1(\mathbb{R}^{2d})$ with

$$
\|BT\|_1 = \|\sum_k s_k V_{\varphi_1} h_k V_{\varphi_2} g_k\|_1 \leq \sum_k s_k \|V_{\varphi_1} h_k V_{\varphi_2} g_k\|_1
$$

$$
\leq \sum_k s_k \|V_{\varphi_1} h_k\|_2 \|V_{\varphi_2} g_k\|_2 = \|\varphi_1\|_2 \|\varphi_2\|_2 \sum_k s_k
$$

$$
= \|\varphi_1\|_2 \|\varphi_2\|_2 \|T\|_{S^1} < \infty.
$$

Hence $BT \in L^1(\mathbb{R}^{2d})$, and $B : S^1 \to L^1(\mathbb{R}^{2d})$ is bounded.
(iii) Interpolation: We use the preceding steps as the endpoints for complex interpolation with \( L^p = [L^1, L^\infty]_\theta \) and \( S^p = [S^1, S^\infty]_\theta = [S^1, B(L^2)]_\theta \). The norm estimate follows from

\[
\|B\|_{S^p \to L^p} \leq \|B\|_{S^1 \to L^1}^{\theta} \|B\|_{S^\infty \to L^\infty}^{1-\theta} \\
\leq (\|\varphi_1\|_2 \|\varphi_2\|_2)^{\theta} (\|\varphi_1\|_2 \|\varphi_2\|_2)^{1-\theta} \\
= \|\varphi_1\|_2 \|\varphi_2\|_2.
\]

\[\square\]

Next we consider the Berezin transform with windows in the modulation space \( M^1(\mathbb{R}^d) \) and strengthen the preceding results. We first recall an important local property of short-time Fourier transforms from [7] (estimate (23) in Cor. 4.2).

**Lemma 3.5.** If \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) and \( g, h \in L^2(\mathbb{R}^d) \), then the product of STFTs \( V_{\varphi_1}g \overline{V_{\varphi_2}h} \) is in \( M^1(\mathbb{R}^{2d}) \) and satisfies the estimate

\[
\|V_{\varphi_1}g \overline{V_{\varphi_2}h}\|_{M^1} \leq C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|g\|_2 \|h\|_2.
\]

**Theorem 3.6.** Let \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) and \( 1 \leq p \leq \infty \). Then \( B \) is bounded from \( S^p \) to \( M^p,1(\mathbb{R}^{2d}) \) and satisfies

\[
\|BT\|_{M^p,1} \leq C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|T\|_{S^p}
\]

for all \( T \in S^1 \), with some fixed constant \( C > 0 \).

**Proof.** Again we prove the boundedness for \( p = 1 \) and \( p = \infty \) and then use interpolation.

(i) Case \( p = 1 \): Assume \( T \in S^1 \). Using the spectral representation of \( T \), Lemma 3.3 implies that

\[
BT(z) = \sum_{n \in \mathbb{N}} s_n V_{\varphi_1}h_k(z) \overline{V_{\varphi_2}g_k(z)},
\]

where \((s_n)_{n \in \mathbb{N}} \in \ell^1 \), \( s_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( \|T\|_{S^1} = \sum_{n \in \mathbb{N}} s_n \).

By Lemma 3.5, each product of STFTs \( V_{\varphi_1}h_k(z) \overline{V_{\varphi_2}g_k(z)} \) is in \( M^1(\mathbb{R}^{2d}) \) and satisfies

\[
\|V_{\varphi_1}h_k \overline{V_{\varphi_2}g_k}\|_{M^1} \leq C \|h_k\|_2 \|\varphi_1\|_{M^1} \|g_k\|_2 \|\varphi_2\|_{M^1}.
\]

Consequently the series for \( BT \) converges absolutely in \( M^1(\mathbb{R}^{2d}) \), and

\[
\|BT\|_{M^1} \leq \sum_{n \in \mathbb{N}} s_n \|V_{\varphi_1}h_k \overline{V_{\varphi_2}g_k}\|_{M^1}
\]

\[
\leq C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \sum_{n \in \mathbb{N}} s_n \\
= C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|T\|_{S^1}.
\]
Assume that of the right-hand side denote the explained in the following statement. Again between the Berezin transform and time-frequency localization operators is explained in the following statement. Since \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \), the Wigner distribution \( W(\varphi_1, \varphi_2) \) is in \( M^1(\mathbb{R}^d) \) by \[23\] Thm. 12.1.2, and

\[
\|W(\varphi_1, \varphi_2)\|_{M^1} \leq C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}.
\]

Finally, we use the convolution relation for modulation spaces \( M^\infty \ast M^1 \subseteq M^\infty,1 \) from \[7\] and thus obtain the norm estimate

\[
\|\mathcal{B}T\|_{M^\infty,1} \leq C_0 \|\sigma\|_{M^\infty} \|W(\varphi_1, \varphi_2)\|_{M^1} \leq C \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} \|T\|_{B(L^2)}.
\]

(iii) For \( 1 < p < \infty \) we use complex interpolation with \( S^p = [S^1, B(L^2)]_\theta \) and

\[
M^{p,1}(\mathbb{R}^d) = [M^1(\mathbb{R}^d), M^\infty,1(\mathbb{R}^d)]_\theta = [M^{1,1}(\mathbb{R}^d), M^\infty,1(\mathbb{R}^d)]_\theta.
\]

3.2. The Berezin Transform and Localization Operators. The connection between the Berezin transform and time-frequency localization operators is explained in the following statement. Again \( p' = \frac{p}{p-1} \) denotes the conjugate exponent of \( p \).

**Theorem 3.7.** Assume that \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d), T \in S^p, 1 \leq p \leq \infty, \) and \( a \in L^{p'}(\mathbb{R}^d) \). Then

\[
\langle \mathcal{A}a, T \rangle = \langle a, \mathcal{B}T \rangle.
\]

Here the brackets on the left-hand side denote the \( S^p \)-\( S^{p'} \)-duality, and the brackets on the right-hand side denote the \( L^p \)-\( L^{p'} \)-duality.

Likewise, if \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d), 1 \leq p < \infty, T \in S^p, \) and \( a \in M^{p',\infty}(\mathbb{R}^d) \), then \[9\] holds with the \( M^{p',\infty} \)-\( M^{p,1} \)-duality on the right hand side.

**Proof.** Let \( T \in S^p \) and \( a \in L^{p'}(\mathbb{R}^d) \) or \( a \in M^{p',\infty}(\mathbb{R}^d) \), with appropriate windows. Choose an arbitrary orthonormal basis \( (e_n)_{n \in \mathbb{N}} \) of \( L^2(\mathbb{R}^d) \), then we have

\[
\langle \mathcal{A}a, T \rangle = \langle A^{\varphi_1, \varphi_2}_a, T \rangle
\]

\[
= \text{trace}(T^* A^{\varphi_1, \varphi_2}_a),
\]

\[
= \sum_{n \in \mathbb{N}} \langle T^* A^{\varphi_1, \varphi_2}_a e_n, e_n \rangle
\]

\[
= \sum_{n \in \mathbb{N}} \langle A^{\varphi_1, \varphi_2}_a e_n, Te_n \rangle
\]

Since \( A^{\varphi_1, \varphi_2}_a \in S^{p'} \) (see Table 1) and \( T \in S^p \), all orthonormal bases yield the same sum.
Now consider the singular value decomposition of $T$,

$$T = \sum_k s_k \langle \cdot, g_k \rangle h_k,$$

with $(s_k) \in \ell^p$. The orthonormal system $(g_k)_k$ can be completed to an orthonormal basis $(g_n)_{n \in \mathbb{N}}$ of $L^2(\mathbb{R}^d)$. We obviously have $T g_k = s_k h_k$ for $g_k$ a member of the original collection $(g_k)_k$, whereas $T g_n = 0$ for all $g_n$ that were joined to the original system $(g_k)_k$ to form a complete basis. Thus, choosing the orthonormal basis $(g_n)_{n \in \mathbb{N}}$ for $(e_n)_{n \in \mathbb{N}}$ in (10), we obtain

$$\langle Aa, T \rangle = \sum_{n \in \mathbb{N}} \langle A_{\alpha}^{\varphi_1, \varphi_2} g_n, T g_n \rangle$$

$$= \sum_k \langle A_{\alpha}^{\varphi_1, \varphi_2} g_k, T g_k \rangle$$

$$= \sum_k s_k \langle A_{\alpha}^{\varphi_1, \varphi_2} g_k, h_k \rangle.$$

Using the weak definition of $A_{\alpha}^{\varphi_1, \varphi_2}$ from (3), i.e., $\langle A_{\alpha}^{\varphi_1, \varphi_2} g_k, h_k \rangle = \langle a, V_{\varphi_2} h_k \overline{V_{\varphi_1} g_k} \rangle$ for all $k$, we find that

$$\langle Aa, T \rangle = \sum_k s_k \langle a, V_{\varphi_2} h_k \overline{V_{\varphi_1} g_k} \rangle.$$

(11)

Now let

$$T_N = \sum_{k=1}^N s_k \langle \cdot, g_k \rangle h_k.$$

be the finite-rank approximation of $T$, so that $T_N \to T$ in $S^p$ for $1 \leq p \leq \infty$.

Lemma 3.3 yields the representation

$$\mathcal{B} T_N = \sum_{k=1}^N s_k V_{\varphi_2} g_k V_{\varphi_1} h_k.$$

Hence

$$\langle a, \mathcal{B} T_N \rangle = \langle a, \sum_{k=1}^N s_k V_{\varphi_2} g_k V_{\varphi_1} h_k \rangle$$

$$= \sum_{k=1}^N s_k \langle a, V_{\varphi_2} g_k V_{\varphi_1} h_k \rangle$$

$$= \langle Aa, T_N \rangle$$

by (11). Now let $N \to \infty$. Since $\mathcal{B}$ is bounded and the duality bracket $\langle \cdot, \cdot \rangle$ is continuous on $S^p \times S^p$, $L^p \times L^p$, and on $M^p, \infty \times M^{p,1}$, we conclude

$$\langle a, \mathcal{B} T \rangle = \langle Aa, T \rangle$$

for every $a \in L^p(\mathbb{R}^{2d})$ (or $a \in M^{p,\infty}(\mathbb{R}^{2d})$) and $T \in S^p$.  

The following statement is a simple corollary of Theorem 3.7.
Theorem 3.8. Let $1 \leq p < \infty$.

(i) Assume $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Then the operator $A : L^{p'}(\mathbb{R}^{2d}) \to S^p$ is the Banach space adjoint of the operator $B : S^p \to L^p(\mathbb{R}^{2d})$.

(ii) Assume $\varphi_2, \varphi_2 \in M^1(\mathbb{R}^d)$. Then the operator $A : M^{p',\infty}(\mathbb{R}^{2d}) \to S^p$ is the Banach space adjoint of the operator $B : S^p \to M^{p,1}(\mathbb{R}^{2d})$.

Thus in both cases we have $B^* = A$.

Proof. The adjoint of $B$ is the unique operator $B^* : (L^p(\mathbb{R}^{2d}))^* \to (S^p)^*$ and $B^* : (M^{p,1}(\mathbb{R}^{2d}))^* \to (S^p)^*$, respectively, such that

$$\langle BT, a \rangle = \langle T, B^* a \rangle$$

for all $T \in S^p$ and all $a \in L^{p'}(\mathbb{R}^{2d})$ (or $a \in M^{p',\infty}(\mathbb{R}^{2d})$). Since $(L^p)^* = L^{p'}$ and $(M^{p,1})^* = M^{p',\infty}$ and since by Theorem 3.7

$$\langle BT, a \rangle = \langle T, A a \rangle$$

for all $T \in S^p$ and all $a \in L^{p'}(\mathbb{R}^{2d})$ (or $a \in M^{p',\infty}(\mathbb{R}^{2d})$), we conclude that $B^* = A$.

\[ \blacksquare \]

For the $L^p$-case we obtain a symmetric statement.

Corollary 3.9. Let $1 < p < \infty$ and $p'$ be the conjugate exponent. The Berezin transform $B : S^p \to L^p(\mathbb{R}^{2d})$ is the Banach space adjoint of the operator $A : L^{p'}(\mathbb{R}^{2d}) \to S^p$, i.e. $A^* = B$.

For $p = 1$, the operator $B : S^1 \to L^1(\mathbb{R}^{2d})$ is the Banach space adjoint of $A : L^0(\mathbb{R}^{2d}) \to S^\infty$.

Proof. The statement is clear for $1 < p < \infty$, because $L^p$ and $S^p$ are reflexive Banach spaces. For $p = 1$ we observe that $A$ maps $L^0(\mathbb{R}^{2d})$ (the compactly supported functions in $L^\infty(\mathbb{R}^{2d})$) into $S^\infty$ (see Table 1), consequently, $A^*$ maps $S^1 = (S^\infty)^*$ into $L^1(\mathbb{R}^{2d}) = L^0(\mathbb{R}^{2d})^*$. So (9) is still well-defined, and thus $A^* = B$.

\[ \blacksquare \]

Note that Corollary 3.9 does not hold for the case of modulation spaces, since $M^{p,\infty}$ is not reflexive.

4. THE KERNEL OF THE BEREZIN TRANSFORM

In this section we study the kernel of the Berezin transform and its dependence of the windows $\varphi_1$ and $\varphi_2$. In particular, we derive conditions when $B$ is one-to-one.

In the following every operator-valued integral is to be understood in the weak sense. Thus $T = \int s(z)\pi(z) dz$ means that

$$\langle T f, g \rangle_{\mathbb{R}^d} = \int_{\mathbb{R}^{2d}} s(z)\pi(z)f, g) dz = \langle s, Vf g \rangle_{\mathbb{R}^{2d}}.$$

for suitable test functions $f$ and $g$. For $s \in M^\infty(\mathbb{R}^{2d})$ and $f, g \in M^1(\mathbb{R}^d)$, the integral defines a bounded operator from $M^1(\mathbb{R}^d)$ to $M^\infty(\mathbb{R}^d)$. The support of a
distribution $s \in M^\infty(\mathbb{R}^d)$ is the smallest closed set $E$ such that $\langle s, f \rangle = 0$ for every function $f \in M^1(\mathbb{R}^d)$ with compact support in the complement $E^c$.

**Proposition 4.1.** Assume that $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$ and let $E = \{ z \in \mathbb{R}^d : V_{\varphi_1, \varphi_2}(z) = 0 \}$. Let $s \in M^\infty(\mathbb{R}^d)$ and $T : M^1(\mathbb{R}^d) \to M^\infty(\mathbb{R}^d)$ be given by

$$T = \int_{\mathbb{R}^d} s(z) \pi(z) \, dz.$$ 

Then the following holds: If $T$ is in the kernel of $B$, then supp $s \subseteq E$.

In particular, if $V_{\varphi_1, \varphi_2}(z) \neq 0$ for all $z \in \mathbb{R}^d$, then the Berezin transform $B$ is one-to-one.

**Proof.** The Berezin transform of $T$ is

$$BT(z) = \langle T \pi(z) \varphi_2, \pi(z) \varphi_1 \rangle = \langle s, \pi(z) \varphi_1 \rangle = \langle s, M_{\mathcal{J}z} \varphi_1 \varphi_2 \rangle,$$

where we have used Lemma 2.1(iii) in the last equality.

If $BT(z) = 0$ for all $z \in \mathbb{R}^d$, then

$$BT(z) = \langle s, M_{\mathcal{J}z} \varphi_1 \varphi_2 \rangle = \langle s \overline{\varphi_1 \varphi_2}^\dagger (-\mathcal{J} z) = 0$$

for all $z \in \mathbb{R}^d$. Thus the distribution $s \overline{\varphi_1 \varphi_2}$ is zero in $M^\infty(\mathbb{R}^d)$. Choose a test function $\Psi \in M^1(\mathbb{R}^d)$ with compact support $K \subseteq E^c$. Since $V_{\varphi_1, \varphi_2}(z) \neq 0$ for $z \notin E$, there exists a function $\Psi_1 \in M^1(\mathbb{R}^d)$, such that $\Psi_1(z) = \frac{1}{\overline{\varphi_1 \varphi_2}(z)}$ for $z \in K$ (see [31] or [25, VIII.6.1]). Thus we can write $\Psi$ as

$$\Psi = \Psi_1 (V_{\varphi_1, \varphi_2}) \in M^1(\mathbb{R}^d)$$

on all of $\mathbb{R}^d$. Consequently

$$\langle \sigma, \Psi \rangle = \langle \sigma, \Psi_1 (V_{\varphi_1, \varphi_2}) \rangle = \langle \sigma \overline{V_{\varphi_1, \varphi_2}}, \Psi_1 \Psi \rangle = 0.$$

This is true for all $\Psi$ with compact support in $E^c$, and therefore supp $s \subseteq E$, as claimed.  

We next investigate the injectivity of the Berezin transform on the class of Hilbert-Schmidt operators.

**Theorem 4.2.** Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. Then the Berezin transform $B : S^2 \to L^2(\mathbb{R}^d)$ is one-to-one, if and only if $V_{\varphi_1, \varphi_2}(z) \neq 0$ for almost all $z \in \mathbb{R}^d$.

**Proof.** Note that, by formula (1), $V_{\varphi_1, \varphi_2}(z) \neq 0$ for almost all $z \in \mathbb{R}^d$ if and only if $W(\varphi_1, \varphi_2)^\dagger(z) \neq 0$ for almost all $z \in \mathbb{R}^d$. If $T \in S^2$, there is a unique symbol $\sigma \in L^2(\mathbb{R}^d)$ such that

$$BT(z) = \langle \sigma, T \rangle W(\varphi_1, \varphi_2)$$

$$= \langle \sigma, M_{\mathcal{J}z} W(\varphi_1, \varphi_2) \rangle$$

$$= \langle \sigma, W(\varphi_1, \varphi_2)^\dagger(-z) \rangle.$$
Since both \( \sigma \in L^2(\mathbb{R}^d) \) and \( W(\varphi_1, \varphi_2) \in L^2(\mathbb{R}^{2d}) \), the product \( \hat{\sigma} \cdot W(\varphi_1, \varphi_2) \) is in \( L^1(\mathbb{R}^{2d}) \) and its Fourier transform is well-defined.

First assume that \( W(\varphi_1, \varphi_2) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \). If \( \mathcal{B}T \equiv 0 \), then \( \hat{\sigma} \cdot W(\varphi_1, \varphi_2) = 0 \) almost everywhere everywhere.

Conversely, assume that there exists a set \( E \subseteq \mathbb{R}^{2d} \) of positive measure such that \( W(\varphi_1, \varphi_2) = 0 \) for \( z \in E \). Without loss of generality we may assume that \( 0 < |E| < \infty \). Choose \( \sigma \in L^2(\mathbb{R}^{2d}) \) as the characteristic function \( \hat{\sigma} = \chi_E \) of \( E \). Then \( \hat{\sigma} \cdot W(\varphi_1, \varphi_2) = 0 \) on \( \mathbb{R}^{2d} \), hence \( \mathcal{B}T \equiv 0 \) for the Hilbert-Schmidt operator \( T \in S^2 \) associated with symbol \( \sigma \). Since \( \sigma \neq 0 \) and thus \( T \neq 0 \), \( \mathcal{B} \) has a non-trivial kernel and is not one-to-one.

**Corollary 4.3.** Let \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \).

(i) If \( 1 \leq p < 2 \) and \( V_{\varphi_1} \varphi_2 \neq 0 \) almost everywhere, then \( \mathcal{B} \) is one-to-one from \( S^p \) to \( L^p(\mathbb{R}^{2d}) \).

(ii) If \( p > 2 \) and \( \mathcal{B} \) is one-to-one from \( S^p \) to \( L^p(\mathbb{R}^{2d}) \), then \( V_{\varphi_1} \varphi_2 \neq 0 \) almost everywhere.

**Proof.** (i) The assumption \( V_{\varphi_1} \varphi_2 \neq 0 \) almost everywhere implies that \( \mathcal{B} \) is one-to-one on \( S^2 \). In particular \( \mathcal{B} \) is one-to-one on the smaller space \( S^p \subseteq S^2 \) for \( p < 2 \).

(ii) If \( \mathcal{B} \) is one-to-one on \( S^p \) for \( p > 2 \), the \( \mathcal{B} \) is one-to-one on \( S^2 \subseteq S^p \) and by Theorem 4.2 \( V_{\varphi_1} \varphi_2 \neq 0 \) almost everywhere.

The following example illustrates the differences between Proposition 4.1 and Theorem 4.2. Assume that \( V_{\varphi_1} \varphi_1(w) = 0 \) for some \( w \in \mathbb{R}^{2d} \), and consider the time-frequency shift \( T = \pi(w) \in B(L^2(\mathbb{R}^d)) \). Using Lemma 2.1(iii), the Berezin transform of \( \pi(w) \) is given as

\[
\mathcal{B}T(z) = \langle T\pi(z)\varphi_2, \pi(z)\varphi_1 \rangle = \langle \pi(w)\pi(z)\varphi_2, \pi(z)\varphi_1 \rangle = e^{-2\pi i z \cdot w} \hat{V}_{\varphi_2} \varphi_1(w) = 0.
\]

Thus \( \mathcal{B}T = 0 \in L^\infty(\mathbb{R}^{2d}) \), but \( T \neq 0 \in B(L^2(\mathbb{R}^d)) \), hence \( \mathcal{B} \) is not one-to-one.

This example can be summarized as follows.

**Corollary 4.4.** Let \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \). If \( \mathcal{B} : B(L^2(\mathbb{R}^d)) \to L^\infty(\mathbb{R}^{2d}) \) is one-to-one, then \( V_{\varphi_1} \varphi_2(z) \neq 0 \) for all \( z \in \mathbb{R}^{2d} \).

As a consequence of Proposition 4.1 we will derive the following fact. For \( T \in B(L^2(\mathbb{R}^d)) \) let

\[
K_T(z, w) = \langle T\pi(z)\varphi_1, \pi(w)\varphi_2 \rangle
\]

be the essential kernel associated to \( T \). We note that the Berezin transform \( \mathcal{B}T \) is the diagonal of \( K_T \). The essential kernel appears in the description of an operator \( T \) on the level of STFTs. It is well known (see, e.g., [9]) that every operator \( T \) can...
be written as
\[ Tf = (\|\varphi_1\|_2 \|\varphi_2\|_2)^{-2} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \langle f, \pi(z) \varphi_1 \rangle \langle T \pi(z) \varphi_1, \pi(w) \varphi_2 \rangle \pi(w) \varphi_2 \, dz \, dw \]
\[ = (\|\varphi_1\|_2 \|\varphi_2\|_2)^{-2} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} V_{\varphi_1} f(z) K_T(z, w) \pi(w) \varphi_2 \, dz \, dw \]
with a suitable weak interpretation of the integrals. For example, if \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) and \( T \) is bounded from \( M^1(\mathbb{R}^d) \) to \( M^\infty(\mathbb{R}^d) \), then the essential kernel is continuous and bounded, and the integral is well defined in the weak sense.

**Corollary 4.5.** Assume that \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) and \( \langle \varphi_2, \pi(z) \varphi_1 \rangle \neq 0 \) for all \( z \in \mathbb{R}^{2d} \). Then \( K_T \) is uniquely determined by its diagonal \( BT \).

**Proof.** Since \( V_{\varphi_1} \varphi_2 \) does not have any zero, the Berezin transform is one-to-one on the space of all bounded operators from \( M^1 \) to \( M^\infty \) (Proposition 4.1). Since \( T \) is uniquely determined by \( BT \), so is its kernel \( K_T \).

Statements of this type can be found in the theory of Toeplitz operators on complex domains, see for instance [3,22] for Toeplitz operators on Bargmann-Fock space. In these cases, the statement of Corollary 4.5 is immediate because the kernel is an analytic function. It is quite surprising that such a statement can be proved without explicit analytic structure. Englis [16] uses the above property as an explicit assumption in his work on Berezin quantization for general function spaces.

### 5. Density Results

In this section we investigate whether a given operator on \( L^2(\mathbb{R}^d) \) can be approximated by a localization operator with respect to various norms. In particular, we would like to understand when the set of localization operators is dense in \( B(L^2(\mathbb{R}^d)) \) or in \( S^p \).

For this we will combine Theorem 3.8 and Corollary 3.9 with the following facts (cf. [9]):

Let \( X, Y \) be Banach spaces and \( T : X \to Y \) be a bounded operator and let \( T^* : Y^* \to X^* \) be the (Banach space) adjoint operator.

- \( T^* \) is one-to-one on \( Y^* \), if and only if the range of \( T \) is norm-dense in \( Y \).
- \( T \) is one-to-one on \( X \), if and only if the range of \( T^* \) is \( w^* \)-dense in \( X^* \).

Thus, in view of Theorem 3.8 the quantization \( a \to Aa \) has dense range, if and only if the Berezin transform \( B \) is one-to-one. This property was studied in Section 4. We now apply these results and derive sufficient conditions for the density of localization operators in a space of operators.

First we treat the density in \( B(L^2(\mathbb{R}^d)) \).

**Theorem 5.1.** Let \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \) and \( A : L^\infty(\mathbb{R}^{2d}) \to B(L^2(\mathbb{R}^d)) \).

If \( V_{\varphi_1} \varphi_2(z) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \), then \( \text{ran}(A) \) is \( w^* \)-dense in \( B(L^2(\mathbb{R}^d)) \).
Proof. By Corollary 4.3, the condition \( V_{\varphi_1,\varphi_2}(z) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \) implies that the Berezin transform \( \mathcal{B} : S^1 \to L^1(\mathbb{R}^{2d}) \) is one-to-one. Since \( \mathcal{A} = \mathcal{B}^* \) by Theorem 3.8, the injectivity of the Berezin transform \( \mathcal{B} : S^1 \to L^1(\mathbb{R}^{2d}) \) is equivalent to the weak*-density of the range of \( \mathcal{A} \).

For the density of localization operators in the Schatten \( p \)-classes we have the following results. As the precise statements depend on the window class and on the range of \( p \), we provide separate formulations.

**Theorem 5.2.** Assume that \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \) and \( 2 \leq p' < \infty \), and \( \mathcal{A} : L^{p'}(\mathbb{R}^{2d}) \to S^{p'} \). If \( V_{\varphi_1,\varphi_2}(z) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \), then \( \text{ran}(\mathcal{A}) = \{ A_{\mathbf{a}}^{\varphi_1,\varphi_2} : a \in L^{p'} \} \) is norm-dense in \( S^{p'} \).

**Proof.** The proof is the same as for Theorem 5.1.

**Theorem 5.3.** Assume that \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) and \( 1 \leq p' < 2 \). Let \( \mathcal{A} : L^{p'}(\mathbb{R}^{2d}) \to S^{p'} \). If \( V_{\varphi_1,\varphi_2}(z) \neq 0 \) for all \( z \in \mathbb{R}^{2d} \), then \( \text{ran}(\mathcal{A}) = \{ A_{\mathbf{a}}^{\varphi_1,\varphi_2} : a \in L^{p'} \} \) is norm-dense in \( S^{p'} \).

**Proof.** The proof is again identical to the proof of Theorem 5.1, we use Proposition 4.3 instead of Theorem 5.2.

Analogous results hold for the symbol class \( M^{p,\infty} \), which is significantly larger than \( L^p \).

**Theorem 5.4.** Let \( \varphi_1, \varphi_2 \in M^1(\mathbb{R}^d) \) and \( \mathcal{A} : M^{p',\infty}(\mathbb{R}^{2d}) \to S^p \), \( 1 < p' < \infty \). Assume that either (a) \( 1 < p' < 2 \) and \( V_{\varphi_1,\varphi_2}(z) \neq 0 \) for all \( z \in \mathbb{R}^{2d} \), or (b) \( 2 \leq p' < \infty \) and \( V_{\varphi_1,\varphi_2}(z) \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \).

Then \( \text{ran}(\mathcal{A}) = \{ A_{\mathbf{a}}^{\varphi_1,\varphi_2} : a \in L^{p'} \} \) is norm-dense in \( S^{p'} \).

**Proof.** By Theorems 5.2 and 5.3 the range of \( \mathcal{A} \) from \( L^{p'}(\mathbb{R}^d) \) to \( S^{p'} \) is norm-dense. Since \( L^{p'}(\mathbb{R}^{2d}) \subseteq M^{p',\infty}(\mathbb{R}^{2d}) \) for \( 1 \leq p' \leq \infty \), we deduce that
\[
\{ A_{\mathbf{a}}^{\varphi_1,\varphi_2} : a \in L^{p'}(\mathbb{R}^{2d}) \} \subseteq \{ A_{\mathbf{a}}^{\varphi_1,\varphi_2} : a \in M^{p',\infty}(\mathbb{R}^{2d}) \} \subseteq S^{p'}
\]
is also norm-dense in \( S^{p'} \).

In the special case of \( p' = 2 \), we obtain a complete characterization for the density in terms of the windows (as highlighted in the introduction).

**Theorem 5.5.** Let \( \varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \). Then the following conditions are equivalent:

(i) The range \( \{ A_{\mathbf{a}}^{\varphi_1,\varphi_2} : a \in L^2(\mathbb{R}^{2d}) \} \) is norm-dense in \( S^2(L^2(\mathbb{R}^d)) \).

(ii) The Berezin transform \( \mathcal{B} \) is one-to-one on \( S^2 \).

(iii) \( \langle \varphi_2, \pi(z) \varphi_1 \rangle \neq 0 \) for almost all \( z \in \mathbb{R}^{2d} \).

**Proof.** The equivalence of conditions (i) and (ii) follows from Theorem 3.8, the equivalence of conditions (ii) and (iii) is stated in Theorem 4.2.
This result says that every Hilbert-Schmidt operator can be approximated by localization operators. Constructive approximation procedures and error estimates were studied in [13].

**Remark:** The equivalence (i) $\Leftrightarrow$ (iii) in Theorem 5.5 can be proved easily and directly with the Weyl calculus. The Weyl symbol of the localization operator $A_a^\varphi_1,\varphi_2$ is given by $a \ast W(\varphi_2, \varphi_1)$ (see, e.g., [30]). By Theorem 2.3 the mapping $A : L^2(\mathbb{R}^d) \to S^2$ has dense range, if and only if the subspace $L^2 \ast W(\varphi_2, \varphi_1)$ is dense in $L^2(\mathbb{R}^{2d})$. According to Theorem 2.4(i), this is the case, if and only if $W(\varphi_2, \varphi_1)(z) \neq 0$ for almost all $z \in \mathbb{R}^{2d}$.

This simple argument cannot be extended to arbitrary Schatten classes for two reasons: first, we do not have an explicit description of the Weyl symbols of operators $A$ in terms of $\varphi_1, \varphi_2$. Second, except for the spaces $M^1, L^1, L^2$ we do not know how to characterize the translation-invariant subspace generated by a function $g$ (in our case $g = W(\varphi_2, \varphi_1)$). This is a very subtle point: already for $L^p, p \neq 1, 2, \infty$, the density of $\text{span}\{T_z g : z \in \mathbb{R}^{2d}\}$ does not only depend on the zero set of $\hat{g}$ [27]. Therefore the characterization of the injectivity of the Berezin transform on Schatten $p$-classes seem to be a difficult problem and lies currently beyond our techniques.

Finally we address the failure of density of $\text{ran}(A)$ in $B(L^2(\mathbb{R}^d))$ in the operator norm. For an intuitive understanding, we remark that localization operators modify the phase space content locally, but they do not move energy in phase space. Therefore one may expect that localization operators cannot approximate operators that describe phase space transformations, such as the Fourier transform $\mathcal{F}$ or Fourier integral operators. Indeed, we have the following negative result. For a comparable result in the theory of Toeplitz operators we refer to [2].

**Theorem 5.6.** Let $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d) \setminus \{0\}$ be arbitrary and $a \in L^\infty(\mathbb{R}^{2d})$. Then

$$\|\mathcal{F} - A_a^{\varphi_1,\varphi_2}\|_{op} \geq 1.$$ 

Consequently, the set of all localization operators with symbols in $L^\infty(\mathbb{R}^{2d})$ is not dense in $B(L^2(\mathbb{R}^d))$ with respect to the operator norm.

**Proof.** For $f \in L^2(\mathbb{R}^d)$ we have

$$\|\mathcal{F}f - A_a^{\varphi_1,\varphi_2}f\|^2 = \|\hat{f}\|^2 + \|A_a^{\varphi_1,\varphi_2}f\|^2 - 2 \Re \langle \hat{f}, A_a^{\varphi_1,\varphi_2}f \rangle$$

$$\geq \|\hat{f}\|^2 + \|A_a^{\varphi_1,\varphi_2}f\|^2 - 2 |\langle \hat{f}, A_a^{\varphi_1,\varphi_2}f \rangle|$$

$$= \|\hat{f}\|^2 + \|A_a^{\varphi_1,\varphi_2}f\|^2 - 2 |\langle a, V_{\varphi_2}f \rangle|.$$

Now fix an arbitrary non-zero $g \in L^2(\mathbb{R}^d)$. We will show that

$$\lim_{w \to \infty} |\langle a, V_{\varphi_2}(\pi(w))V_{\varphi_1}(\pi(w))g \rangle| = 0.$$ 

Let $\varepsilon > 0$ be given and choose $R > 0$ such that

$$\int_{[R,R]} \int_{[-R,R]^{2d}} |V_{\varphi_1} g(z)|^2 dz \geq (1 - \varepsilon) \|g\|^2 \|\varphi_1\|^2.$$
and
\[\iint_{[-R,R]^2d} |V \varphi_2 \hat{g}(z)|^2 \, dz \geq (1 - \varepsilon) \|g\|^2_2 \|\varphi_2\|^2_2.\]

This is always possible because \(V \varphi g \in L^2(\mathbb{R}^d)\) by the isometry property of the STFT (Lemma 2.1).

For \(z = (x, \omega) \in \mathbb{R}^d\) let \(\mathcal{J} z = (\omega, -x)\). Then \(\hat{\pi(z) g} = e^{2 \pi i x \cdot \omega} \pi(\mathcal{J} z) \hat{g}\).

Now define \(U_\varepsilon := \mathcal{J} w + [-R, R]^2d\) and \(V_\varepsilon := w + [-R, R]^2d\). By the covariance property (2.1)(i) and a change of variables we obtain
\[\iint_{V_\varepsilon} |V \varphi_1(\pi(w)g)(z)|^2 \, dz = \int \int_{w_{[-R,R]^{2d}}} |V \varphi_1 g(z - w)|^2 \, dz\]
\[= \int \int_{[-R,R]^{2d}} |V \varphi_1 g(z)|^2 \, dz \geq (1 - \varepsilon) \|g\|^2_2 \|\varphi_2\|^2_2\]
\[= (1 - \varepsilon) \|\pi(w)g\|^2_2 \|\varphi_2\|^2_2 ,\]

and similarly
\[\iint_{U_\varepsilon} |V \varphi_2(\pi(w)g)(z)|^2 \, dz = \int \int_{[-R,R]^{2d}} |V \varphi_2 \hat{g}(z)|^2 \, dz \geq (1 - \varepsilon) \|\pi(w)g\|^2_2 \|\varphi_1\|^2_2 .\]

By choosing \(|w|\) large enough, we can achieve \(U_\varepsilon \cap V_\varepsilon = (\mathcal{J} w + [-R, R]^{2d}) \cap (w + [-R, R]^{2d}) = \emptyset\), so that \(U_\varepsilon \subseteq \mathbb{R}^d \setminus V_\varepsilon\) and \(V_\varepsilon \subseteq \mathbb{R}^d \setminus U_\varepsilon\). Then
\[\iint_{U_\varepsilon} |V \varphi_1(\pi(w)g)(z)|^2 \, dz \leq \iint_{\mathbb{R}^d \setminus V_\varepsilon} |V \varphi_1(\pi(w)g)(z)|^2 \, dz \leq \varepsilon \|\pi(w)g\|^2_2 \|\varphi_1\|^2_2 \]
and
\[\iint_{V_\varepsilon} |V \varphi_2(\pi(w)g)(z)|^2 \, dz \leq \iint_{\mathbb{R}^d \setminus U_\varepsilon} |V \varphi_2(\pi(w)g)(z)|^2 \, dz \leq \varepsilon \|\pi(w)g\|^2_2 \|\varphi_2\|^2_2 .\]

Writing \(f = \pi(w)g\), we conclude that
\[|\langle a, V \varphi_1 f V \varphi_2 \hat{f} \rangle| = |\iint_{\mathbb{R}^d} a(z) V \varphi_1 f(z) V \varphi_2 \hat{f}(z) \, dz|\]
\[\leq \|a\|_{\infty} \iint_{\mathbb{R}^d} |V \varphi_1 f(z)| \, |V \varphi_2 \hat{f}(z)| \, dz \]
\[= \|a\|_{\infty} \left( \iint_{U_\varepsilon} |V \varphi_1 f(z)| \, |V \varphi_2 \hat{f}(z)| \, dz + \iint_{\mathbb{R}^d \setminus U_\varepsilon} |V \varphi_1 f(z)| \, |V \varphi_2 \hat{f}(z)| \, dz \right) \]
\[\leq \|a\|_{\infty} \left( \left( \iint_{U_\varepsilon} |V \varphi_1 f(z)|^2 \, dz \right)^{1/2} \left( \iint_{U_\varepsilon} |V \varphi_2 \hat{f}(z)|^2 \, dz \right)^{1/2} \right. \]
\[+ \left. \left( \iint_{\mathbb{R}^d \setminus U_\varepsilon} |V \varphi_1 f(z)|^2 \, dz \right)^{1/2} \left( \iint_{\mathbb{R}^d \setminus U_\varepsilon} |V \varphi_2 \hat{f}(z)|^2 \, dz \right)^{1/2} \right) \]
\[\leq \|a\|_{\infty} \left( \left( \iint_{U_\varepsilon} |V \varphi_1 f(z)|^2 \, dz \right)^{1/2} \right) \|V \varphi_2 \hat{f}\|_2 .\]
+ \|V_{\varphi_1} f\|_2 \left( \int_{\mathbb{R}^{2d}\setminus U_\varepsilon} |V_{\varphi_2} \hat{f}(z)|^2 \, dz \right)^{1/2} \right) \\
\leq \|a\|_\infty \left[ \|f\|_2 \|\varphi_1\|_2 \cdot \sqrt{\varepsilon} \|f\|_2 \|\varphi_2\|_2 + \sqrt{\varepsilon} \|f\|_2 \|\varphi_1\|_2 \cdot \|f\|_2 \|\varphi_2\|_2 \right] \\
= 2 \|a\|_\infty \|\varphi_1\|_2 \|\varphi_2\|_2 \|f\|_2^2 \cdot \sqrt{\varepsilon} \\
= C \|g\|_2^2 \cdot \sqrt{\varepsilon}

This estimate holds for all sufficiently large \( w \in \mathbb{R}^d \), and the constant is \( C = 2 \|a\|_\infty \|\varphi_1\|_2 \|\varphi_2\|_2 \) independently of \( w \in \mathbb{R}^d \). Thus the claim

\[
\lim_{w \to \infty} \left| \langle a, V_{\varphi_1}(\pi(w)g) V_{\varphi_2}(\pi(w)\hat{g}) \rangle \right| = 0
\]

is proved.

For nonzero \( g \in L^2(\mathbb{R}^d) \) and \( \varepsilon > 0 \) choose \( w \in \mathbb{R}^d \) such that

\[
\left| \langle a V_{\varphi_1}(\pi(w)g), V_{\varphi_2}(\pi(w)\hat{g}) \rangle \right| \leq \frac{\varepsilon}{2} \|g\|_2^2.
\]

Then with \( f := \pi(w)g \) we have

\[
\|\mathcal{F} f - A^\varphi_a f\|_2^2 \geq \|\hat{f}\|_2^2 + \|A^\varphi_a f\|_2^2 - 2 \left| \langle V_{\varphi_2} \hat{f}, aV_{\varphi_1} f \rangle \right| \\
\geq (1 - \varepsilon)\|g\|_2^2 = (1 - \varepsilon)\|f\|_2^2.
\]

since \( \|f\|_2 = \|\hat{f}\|_2 = \|g\|_2 = \|\hat{g}\|_2 \). This shows

\[
\|\mathcal{F} - A^\varphi_a\|_{B(L^2)} \geq 1
\]

for all \( a \in L^\infty(\mathbb{R}^d) \), which was to be proved.

A similar argument applies to arbitrary operators of the metaplectic representation, since these operators correspond to linear transformations of phase space. By a more sophisticated argument one can also show that the Fourier transform cannot be approximated by a localization operator \( A^\varphi_a \) with a distributional symbol \( a \in M^\infty \) instead of \( a \in L^\infty(\mathbb{R}^d) \).

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