A GENERAL REALIZATION THEOREM FOR MATRIX-VALUED HEGLOTZ-NEVANLINNA FUNCTIONS

SERGEY BELYI, SEppo HASSI, HENK De SNOO, AND EDUARD TSEKANOVSKII

Dedicated to Damir Arov on the occasion of his 70th birthday
and to Yury Berezanski˘ı on the occasion of his 80th birthday

Abstract. New special types of stationary conservative impedance and scattering systems, the so-called non-canonical systems, involving triplets of Hilbert spaces and projection operators, are considered. It is established that every matrix-valued Herglotz-Nevanlinna function of the form

\[ V(z) = Q + Lz + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma(t), \]

\( z \in \mathbb{C} \setminus \mathbb{R}, \)

can be realized as a transfer function of such a new type of conservative impedance system. In this case it is shown that the realization can be chosen such that the main and the projection operators of the realizing system satisfy a certain commutativity condition if and only if \( L = 0. \) It is also shown that \( V(z) \) with an additional condition (namely, \( L \) is invertible or \( L = 0 \)), can be realized as a linear fractional transformation of the transfer function of a non-canonical scattering \( F_+ \)-system. In particular, this means that every scalar Herglotz-Nevanlinna function can be realized in the above sense.

Moreover, the classical Livšic systems (Brodski˘ı-Livšic operator colligations) can be derived from \( F_+ \)-systems as a special case when \( F_+ = I \) and the spectral measure \( d\Sigma(t) \) is compactly supported. The realization theorems proved in this paper are strongly connected with, and complement the recent results by Ball and Staffans.

1. Introduction

An operator-valued function \( V(z) \) acting on a finite-dimensional Hilbert space \( \mathcal{E} \) belongs to the class of matrix-valued Herglotz-Nevanlinna functions if it is holomorphic on \( \mathbb{C} \setminus \mathbb{R} \), if it is symmetric with respect to the real axis, i.e., \( V(z)^* = V(\bar{z}) \), \( z \in \mathbb{C} \setminus \mathbb{R} \), and if it satisfies the positivity condition

\[ \text{Im} \ V(z) \geq 0, \quad z \in \mathbb{C}_+. \]

It is well known (see e.g. [30]) that matrix-valued Herglotz-Nevanlinna functions admit the following integral representation:

\[ V(z) = Q + Lz + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \]
where \( Q = Q^* , \) \( L \geq 0 , \) and \( \Sigma(t) \) is a nondecreasing matrix-valued function on \( \mathbb{R} \) with values in the class of nonnegative matrices in \( \mathcal{E} \) such that
\[
\int_{\mathbb{R}} \frac{(d\Sigma(t)x,x)}{1+t^2} < \infty , \quad x \in \mathcal{E} .
\]

The problem considered in this paper is the general operator representation of these functions with an interpretation in system theory, i.e., in terms of linear stationary conservative dynamical systems. This involves new types of stationary conservative impedance and scattering systems (non-canonical systems) involving triplets of Hilbert spaces and projection operators. The exact definition of both types of non-canonical systems is given below. It turns out that every matrix-valued Herglotz-Nevanlinna function can be realized as a matrix-valued transfer function of this new type of conservative impedance system. Moreover, assuming an additional condition on the matrix \( L \) in (1.1) (\( L \) is invertible or \( L = 0 \)), it is shown that such a function is realizable as a linear fractional transformation of the transfer matrix-valued function of a conservative stationary scattering \( F_+ \)-system. In this case the main operator of the impedance system is the “real part” of the main operator of the scattering \( F_+ \)-system. In particular, it follows that every scalar Herglotz-Nevanlinna function can be realized in the above mentioned sense.

This gives a complete solution of the realization problems announced in “Unsolved problems in mathematical systems and control theory” [34] in the framework of modified Brodski˘ı-Livˇsic operator colligations (in the scalar case via impedance and scattering systems, in the matrix-valued case via impedance systems). Furthermore, the classical canonical systems of the M.S. Livˇsic type (Brodski˘ı-Livˇsic operator colligations) can be derived from \( F_+ \)-systems as a special case when \( F_+ = I \) and the spectral measure \( d\Sigma(t) \) is compactly supported.

Realizations of different classes of holomorphic matrix-valued functions in the open right half-plane, unit circle, and upper half-plane play an important role in the spectral analysis of non-self-adjoint operators, interpolation problems, and system theory; see [2]–[21], [24]–[40], [42]–[52]. For special classes of Herglotz-Nevanlinna functions such operator realizations are known.

Consider, for instance, a matrix-valued Herglotz-Nevanlinna function of the form
\[
V(z) = \int_a^b \frac{d\Sigma(t)}{t-z}, \quad z \in \mathbb{C} \setminus \mathbb{R} ,
\]
with \( \Sigma(t) \) a nondecreasing matrix-valued function on the finite interval \((a,b) \subset \mathbb{R} \).

Then \( V(z) \) has an operator realization of the form
\[
V(z) = K^\ast (A - zI)^{-1}K, \quad z \in \mathbb{C} \setminus \mathbb{R} ,
\]
where \( A \) is a bounded self-adjoint operator acting on a Hilbert space \( \mathcal{H} \) and \( K \) is a bounded invertible operator from the Hilbert space \( \mathcal{E} \) into \( \mathcal{H} \). Such realizations are due to M.S. Brodski˘ı and M.S. Livˇsic; they have been used in the theory of characteristic operator-valued functions as well as in system theory in the following sense (cf. [37]–[59], [24], [25], [40]). Let \( J \) be a bounded, self-adjoint, and unitary operator in \( \mathcal{E} \) which satisfies \( \text{Im} A = JKJ^\ast \). Then the aggregate
\[
\Theta = \begin{pmatrix} A & K & J \\ \mathcal{H} & \mathcal{E} \end{pmatrix}
\]
or

\[
\begin{align*}
(A - zI)x &= KJ\varphi_- , \\
\varphi_+ &= \varphi_- - 2iK^*x,
\end{align*}
\]

is the corresponding, so-called, canonical system or Brodskii-Livšic operator colligation, where \(\varphi_- \in \mathcal{E}\) is an input vector, \(\varphi_+ \in \mathcal{E}\) is an output vector, and \(x\) is a state space vector in \(\mathcal{H}\). The function \(W_\Theta(z)\), defined by

\[
W_\Theta(z) = I - 2iK^*(A - zI)^{-1}KJ, \tag{1.7}
\]

such that \(\varphi_+ = W_\Theta(z)\varphi_-\), is the transfer function of the system \(\Theta\) or the characteristic function of operator colligation. Such type of systems appear in the theory of electrical circuits and have been introduced by M.S. Livšic [38]. The relation between \(V(z)\) in (1.4) and \(W(z)\) in (1.7) is given by

\[
V(z) = i[W(z) + I]^{-1}|W(z) - I|J.
\]

For an extension of the class of (compactly supported) Herglotz-Nevanlinna functions in (1.3) involving a linear term as in (1.1), see [31]–[33], [43]–[44]. Obviously, general matrix-valued Herglotz-Nevanlinna functions \(V(z)\) cannot be realized in the form (1.10) even by means of a canonical system (a Brodskii-Livšić operator colligation, cf., e.g. [19]-[21]. The system described by (1.8) is called a canonical Livšić system or Brodskii-Livšić rigged operator colligation, cf., e.g. [19]-[21]. The operator-valued function

\[
W_\Theta(z) = I - 2iK^*(\mathcal{A} - zI)^{-1}KJ \tag{1.9}
\]

is a transfer function (or characteristic function) of the system \(\Theta\). It was shown in [19] that a matrix-valued function \(V(z)\) acting on a Hilbert space \(\mathcal{E}\) of the form (1.1) can be represented and realized in the form

\[
V(z) = i[W_\Theta(z) + I]^{-1}|W_\Theta(z) - I| = K^*(A_R - zI)^{-1}K, \tag{1.10}
\]

where \(W_\Theta(z)\) is a transfer function of some canonical scattering \((J = I)\) system \(\Theta\), and where the “real part” \(A_R = \text{Re}(\mathcal{A} + \mathcal{A}^*)\) of \(\mathcal{A}\) satisfies \(A_R \supset A\) if and only if the function \(V(z)\) in (1.1) satisfies the following two conditions:

\[
\begin{align*}
L &= 0 ,
Qx &= \int_R \frac{1}{1 + t^2}d\Sigma(t)x \quad \text{when} \quad \int_R (d\Sigma(t)x,x)_{\mathcal{E}} < \infty.
\end{align*}
\]

This shows that general matrix-valued Herglotz-Nevanlinna functions \(V(z)\) acting on \(\mathcal{E}\) cannot be realized in the form (1.10) even by means of a canonical system (a Brodskii-Livšić rigged operator colligation) \(\Theta\) of the form (1.8).

The main purpose of the present paper is to solve the general realization problem for matrix-valued Herglotz-Nevanlinna functions. The case of Herglotz-Nevanlinna functions of the form (1.1) with a bounded measure was considered in [31]–[33]. In
the general case, an appropriate realization for these functions will be established by introducing new types systems: so-called non-canonical \(\Delta_+\)-systems and \(F_+\)-systems. A \(\Delta_+\)-system or impedance system can be written as

\[
\begin{cases}
(D - zF_+)x = K\varphi_-, \\
\varphi_+ = K^*x,
\end{cases}
\]

where \(D\) and \(F_+\) are self-adjoint operators acting from \(H_+\) into \(H_-\) and in addition \(F_+\) is an orthogonal projector in \(H_+\). In this case the associated transfer function is given by

\[
V(z) = K^*(D - zF_+)^{-1}K.
\]

It will be shown that every matrix-valued Herglotz-Nevanlinna function can be represented in the form (1.13).

Another type of realization problem deals with so-called non-canonical \(\Delta_+\), \(\Delta_-\) systems, (1.14)

\[
\begin{cases}
(A - zF_+)x = KJ\varphi_-, \\
\varphi_+ = \varphi_- - 2iK^*x,
\end{cases}
\]

also called rigged \(F_+\)-colligations. This colligation can be expressed via an array similar to the Brodskii-Livšic rigged operator colligation (1.8):

\[
\Theta_{F_+} = \begin{pmatrix} A & F_+ \\ K J & E \end{pmatrix}.
\]

The additional ingredient in (1.15) is the operator \(F_+\) which is an orthogonal projection in \(H_+\) and \(H_\). The corresponding transfer function (or \(F_+\)-characteristic function) is

\[
W_{\Theta_{F_+}}(z) = I - 2iK^*(A - zF_+)^{-1}KJ.
\]

It will be shown that every matrix-valued Herglotz-Nevanlinna function with an invertible matrix \(L\) (or \(L = 0\)) in (1.1) can be represented in the form

\[
V(z) = K^*(A_R - zF_+)^{-1}KJ,
\]

where \(A_R = \frac{1}{2}(A + A^*)\) is the “real part” of the main operator \(A\) in the corresponding \(F_+\)-colligation. The corresponding \(F_+\)-characteristic function \(W_{\Theta_{F_+}}(z)\) is related to the Herglotz-Nevanlinna function \(V(z)\) via

\[
V(z) = i[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I].
\]

Moreover, it will also be shown that the operators \(D\) and \(F_+\) in (1.13) can be selected so that they satisfy a certain commutativity condition precisely when the linear term in (1.1) is absent, i.e., if \(L = 0\). When \(F_+ = I\) the constructed realization reduces to the Brodskii-Livšic rigged operator colligation (canonical system) (1.8) as well as to the classical Brodskii-Livšic operator colligation (canonical system) when the measure \(d\Sigma(t)\) in (1.1) is compactly supported; this includes all the previous results in the realization problem for matrix-valued Herglotz-Nevanlinna functions. The results in this paper depend in an essential way on the theory of extensions in rigged Hilbert spaces [51], [50]; a concise exposition of this theory is provided in [51].

A different approach to realization problems is due to J.A. Ball and O.J. Staffans [16], [17], [46], [47], [48]. In particular, they consider canonical input-state-output
systems of the type

\[
\begin{aligned}
\dot{x} &= Ax(t) + Bu(t), \\
y(t) &=Cx(t) + Du(t),
\end{aligned}
\]

with the transfer mapping

\[
T(s) = D + C(sI - A)^{-1}B.
\]

It follows directly from [16], [17], and [48] that for an arbitrary Herglotz-Nevanlinna function \(V(z)\) of the type (1.1) with \(L = 0\) the function \(-iV(iz)\) can be realized in the form (1.19) by a canonical impedance conservative system (1.18) considered in [16], [17], and [48]. However, this does not contradict the criteria for the canonical realizations (1.9), (1.10), (1.11) established by two of the authors in [19] due to the special type \((F_+ = I)\) of the Livšic systems (Brodskiĭ-Livšic rigged operator colligations) under consideration. Theorem 4.1 of the present paper provides a general result for non-canonical realizations of such functions. The general realization case involving a non-zero linear term in (1.1) is also implicitly treated by Ball and Staffans in [16], [17].

The authors would like to thank Joe Ball and Olof Staffans for valuable discussions and important remarks.

2. Some preliminaries

Let \(\mathcal{H}\) be a Hilbert space with inner product \((x, y)\) and let \(A\) be a closed linear operator in \(\mathcal{H}\) which is Hermitian, i.e., \((Ax, y) = (x, Ay)\), for all \(x, y \in \text{dom} A\). In general, \(A\) need not be densely defined. The closure of its domain in \(\mathcal{H}\) is denoted by \(\text{dom} A = \mathcal{H}_0\). In the sequel \(A\) is often considered as an operator from \(\mathcal{H}_0\) into \(\mathcal{H}\). Then the adjoint \(A^*\) of \(A\) is a densely defined operator from \(\mathcal{H}\) into \(\mathcal{H}_0\). Associated to \(A\) are two Hilbert spaces \(\mathcal{H}_+\) and \(\mathcal{H}_-\), the spaces with a positive and a negative norm. The space \(\mathcal{H}_+\) is \(\text{dom} A^*\) equipped with the graph inner product:

\[
(f, g)_+ = (f, g) + (A^* x, A^* y), \quad f, g \in \text{dom} A^*,
\]

while \(\mathcal{H}_-\) is the corresponding dual space consisting of all linear functionals on \(\mathcal{H}_+\), which are continuous with respect to \(\|\cdot\|_+\). This gives rise to a triplet \(\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-\) of Hilbert spaces, which is often called a rigged Hilbert space associated to \(A\). The norms of these spaces satisfy the inequalities

\[
\|x\| \leq \|x\|_+, \quad x \in \mathcal{H}_+, \quad \text{and} \quad \|x\|_- \leq \|x\|, \quad x \in \mathcal{H}.
\]

In what follows the prefixes (\(+\)-), (\(-\)-), and (\(-\)-) will be used to refer to corresponding metrics, norms, or inner products of rigged Hilbert spaces. Recall that there is an isometric operator \(R\), the so-called Riesz-Berezanskiĭ operator, which maps \(\mathcal{H}_-\) onto \(\mathcal{H}_+\) such that

\[
\begin{aligned}
(x, y)_- &= (x, Ry) = (Rx, y) = (Rx, R^* y)_+ , \quad x, y \in \mathcal{H}_- , \\
(u, v)_+ &= (u, R^{-1} v) = (R^{-1} u, v) = (R^{-1} u, R^{-1} v)_-, \quad u, v \in \mathcal{H}_+ ,
\end{aligned}
\]

see [22]. A closed densely defined linear operator \(T\) in \(\mathcal{H}\) is said to belong to the class \(\Omega_A\) if:

(i) \(A = T \cap T^*\) (i.e. \(A\) is the maximal common symmetric part of \(T\) and \(T^*\));
(ii) \(-i\) is a regular point of \(T\).
An operator \( A \in [\mathcal{H}_+, \mathcal{H}_-] \) is called a \((*)\)-extension of \( T \in \Omega_A \) if the inclusions
\[
T \subset A \quad \text{and} \quad T^* \subset A^*
\]
are satisfied. Here the adjoints are taken with respect to the underlying inner products and \([\mathcal{H}_1, \mathcal{H}_2]\) stands for the class of all linear bounded operators between the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). An operator \( A \in [\mathcal{H}_+, \mathcal{H}_-] \) is a bi-extension of \( A \) if \( A \supset A \) and \( A^* \supset A \). Clearly, every \((*)\)-extension \( A \) of \( T \in \Omega_A \) is a bi-extension of \( A \). A bi-extension \( A \) of \( A \) is called a self-adjoint bi-extension if \( A = A^* \) and the operator \( \tilde{A} \) defined by
\[
\tilde{A} = \left\{ (x, Ax) : x \in \mathcal{H}_+, Ax \in \mathcal{H} \right\}.
\]
is a self-adjoint extension of \( A \) in the original Hilbert space \( \mathcal{H} \). A \((*)\)-extension \( A \) of \( T \) is called correct if its “real part” \( A_R := \frac{1}{2}(A + A^*) \) is a self-adjoint bi-extension of \( A \).

For two operators \( A \) and \( B \) in a Hilbert space \( \mathcal{H} \) the set of all points \( z \in \mathbb{C} \) such that the operator \( (A - zB)^{-1} \) exists on \( \mathcal{H} \) and is bounded will be denoted by \( \rho(A, B) \) and \( \rho(A) = \rho(A, I) \). For some basic facts concerning resolvent operators of the form \( (A - zB)^{-1} \), see [38, 39, 41].

Now proper definitions for both \( \Delta_+ \)-systems and \( F_+ \)-systems can be given.

**Definition 2.1.** Let \( A \) be a closed symmetric operator in a Hilbert space \( \mathcal{H} \) and let \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) be the rigged Hilbert space associated with \( A \). The system of equations
\[
\begin{align}
(D - zF_+)x &= K\phi_-, \\
\phi_+ &= K^*x,
\end{align}
\]
where \( \mathcal{E} \) is a finite-dimensional Hilbert space is called a \( \Delta_+ \)-system or impedance system if:

1. \( D \in [\mathcal{H}_+, \mathcal{H}_-] \) is a self-adjoint bi-extension of \( A \);
2. \( K \in [\mathcal{E}, \mathcal{H}_-] \) with \( \ker K = \{0\} \) (i.e. \( K \) is invertible);
3. \( F_+ \) is an orthogonal projection in \( \mathcal{H}_+ \) and \( \mathcal{H}_- \);
4. \( \rho(D, F_+, K) \) of all points \( z \in \mathbb{C} \) where \( (D - zF_+)^{-1} \) exists on \( \mathcal{H} \) and \( \ker K \cap \rho(\mathcal{H}) \) is open.

**Definition 2.2.** Let \( A \) be a closed symmetric operator in a Hilbert space \( \mathcal{H} \) and let \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) be the rigged Hilbert space associated with \( A \). The array
\[
\Theta = \Theta_{F_+} = \left( \begin{array}{ccc}
\mathcal{H}_+ & \mathcal{H} & \mathcal{H}_- \\
\alpha & F_+ & K \\
\mathcal{E} & J & \mathcal{E}
\end{array} \right),
\]
where \( \mathcal{E} \) is a finite-dimensional Hilbert space is called an \( F_+ \)-colligation or an \( F_+ \)-system if:

1. \( \alpha \in [\mathcal{H}_+, \mathcal{H}_-] \) is a correct \((*)\)-extension of \( T \in \Omega_A \);
2. \( J = J^* = J^{-1} : \mathcal{E} \to \mathcal{E} \);
3. \( \alpha - A^* = 2iKJK^* \), where \( K \in [\mathcal{E}, \mathcal{H}_-] \) and \( \ker K = \{0\} \) (\( K \) is invertible);
4. \( F_+ \) is an orthogonal projection in \( \mathcal{H}_+ \) and \( \mathcal{H}_- \);
5. The set \( \rho(\alpha, F_+, K) \) of all points \( z \in \mathbb{C} \), where \( (\alpha - zF_+)^{-1} \) exists on \( \mathcal{H} \) and \( \ker K \cap \rho(\mathcal{H}) \) is open;
6. The set \( \rho(\alpha_R, F_+, K) \) of all points \( z \in \mathbb{C} \), where \( (\alpha_R - zF_+)^{-1} \) exists on \( \mathcal{H} \) and \( \ker K \cap \rho(\mathcal{H}) \) are both open;
To each $F_+$-system ($F_+$-colligation) in Definition 2.2 one can associate a transfer function, or a characteristic function, via

\[(2.5) \quad W_{\Theta}(z) = I - 2iK^*(\hat{A} - zF_+)^{-1}KJ.\]

**Proposition 2.3.** Let $\Theta_{F_+}$ be an $F_+$-colligation of the form (2.4). Then for all $z,w \in \rho(\hat{A}, F_+, K)$,

\[W_{\Theta_{F_+}}(z)JW_{\Theta_{F_+}}^*(w) - J = 2i(\bar{w} - z)K^*(\hat{A} - zF_+)^{-1}F_+(\hat{A}^* - \bar{w}F_+)^{-1}K,\]

\[W_{\Theta_{F_+}}^*(w)JW_{\Theta_{F_+}}(z) - J = 2i(\bar{w} - z)JK^*(\hat{A}^* - \bar{w}F_+)^{-1}F_+(\hat{A} - zF_+)^{-1}KJ.\]

**Proof.** By the properties (iii) and (vi) in Definition 2.2, one has for all $z,w \in \rho(\hat{A}, F_+, K)$

\[(\hat{A} - zF_+)^{-1} - (\hat{A}^* - \bar{w}F_+)^{-1} = (\hat{A} - zF_+)^{-1}[(\hat{A}^* - \bar{w}F_+) - (\hat{A} - zF_+)](\hat{A}^* - \bar{w}F_+)^{-1},\]

\[= (z - \bar{w})(\hat{A} - zF_+)^{-1}F_+(\hat{A}^* - \bar{w}F_+)^{-1} - 2i(\hat{A} - zF_+)^{-1}JK^*(\hat{A}^* - \bar{w}F_+)^{-1}.\]

This identity together with (2.5) implies that

\[W_{\Theta_{F_+}}(z)JW_{\Theta_{F_+}}^*(w) - J = [I - 2iK^*(\hat{A} - zF_+)^{-1}KJ][J + 2iJK^*(\hat{A}^* - \bar{w}F_+)^{-1}K] - J\]

\[= 2i(\bar{w} - z)K^*(\hat{A} - zF_+)^{-1}F_+(\hat{A}^* - \bar{w}F_+)^{-1}K.\]

This proves the first equality. Likewise one proves the second identity by using

\[(\hat{A} - zF_+)^{-1} - (\hat{A}^* - \bar{w}F_+)^{-1} = (z - \bar{w})(\hat{A}^* - \bar{w}F_+)^{-1}F_+(\hat{A} - zF_+)^{-1}\]

\[- 2i(\hat{A}^* - \bar{w}F_+)^{-1}JK^*(\hat{A} - zF_+)^{-1}.\]

This completes the proof. \qed

Proposition 2.3 shows that the transfer function $W_{\Theta_{F_+}}(z)$ in (2.5) associated to an $F_+$-system of the form (2.4) is $J$-unitary on the real axis, $J$-expansive in the upper halfplane, and $J$-contractive in the lower halfplane with $z \in \rho(\hat{A}, F_+, K)$.

There is another function that one can associate to each $F_+$-system $\Theta_{F_+}$ of the form (2.4). It is defined via

\[(2.6) \quad V_{\Theta_{F_+}}(z) = K^*(\hat{A}_R - zF_+)^{-1}K, \quad z \in \rho(\hat{A}_R, F_+, K),\]

where $\rho(\hat{A}_R, F_+, K)$ is defined above. Clearly, $\rho(\hat{A}_R, F_+, K)$ is symmetric with respect to the real axis.
Theorem 2.4. Let $\Theta_{F_+}$ be an $F_+$-system of the form (2.4) and let $W_{\Theta_{F_+}}(z)$ and $V_{\Theta_{F_+}}(z)$ be defined by (2.6) and (2.10), respectively. Then for all $z, w \in \rho(A_R, F_+, K),$

$$V_{\Theta_{F_+}}(z) - V_{\Theta_{F_+}}(w)^* = (z - \bar{w})K^*(A_R - z F_+)^{-1}F_+(A_R - \bar{w}F_+)^{-1}K,$$

(2.7) $V_{\Theta_{F_+}}(z)$ is a matrix-valued Herglotz-Nevanlinna function, and for each $z \in \rho(A_R, F_+, K) \cap \rho(A, F_+, K)$ the operators $I + iV_{\Theta_{F_+}}(z)J$ and $I + W_{\Theta_{F_+}}(z)$ are invertible. Moreover,

$$V_{\Theta_{F_+}}(z) = i[W_{\Theta_{F_+}}(z) + I]^{-1}[W_{\Theta_{F_+}}(z) - I]J$$

and

$$W_{\Theta_{F_+}}(z) = [I + iV_{\Theta_{F_+}}(z)J]^{-1}[I - iV_{\Theta_{F_+}}(z)J].$$

Proof. For each $z, w \in \rho(A_R, F_+, K)$ one has

$$(A_R - z F_+)^{-1} - (A_R - \bar{w}F_+)^{-1} = (z - \bar{w})(A_R - z F_+)^{-1}F_+(A_R - \bar{w}F_+)^{-1},$$

(2.10) In view of (2.6) this implies (2.7).

Clearly,

$$V_{\Theta_{F_+}}(z)^* = V_{\Theta_{F_+}}(z).$$

Moreover, it follows from (2.4) and the definition (2.6) that $V_{\Theta_{F_+}}(z)$ is a matrix-valued Herglotz-Nevanlinna function.

The following identity with $z \in \rho(A, F_+, K) \cap \rho(A_R, F_+, K)$

$$(A_R - z F_+)^{-1} - (A_R - \bar{z}F_+)^{-1} = i(A_R - \bar{z}F_+)^{-1}\text{Im}(A_R - z F_+)^{-1}$$

leads to

$$K^*(A_R - z F_+)^{-1}K - K^*(A_R - \bar{z}F_+)^{-1}K = iK^*(A_R - \bar{z}F_+)^{-1}KJK^*(A_R - z F_+)^{-1}K.$$
3. Impedance Realizations of Herglotz-Nevanlinna Functions

The realization of Herglotz-Nevanlinna functions has been obtained for various subclasses. In this section earlier realizations are combined to present a general realization of an arbitrary Herglotz-Nevanlinna function by an impedance system. The following lemma is essentially contained in [32]; for completeness a full proof is presented here.

**Lemma 3.1.** Let $Q$ be a self-adjoint operator in a finite-dimensional Hilbert space $E$. Then $V(z) = Q$ admits a representation of the form

\begin{equation}
V(z) = K^*(D - zF_+)^{-1}K, \quad z \in \rho(D, F),
\end{equation}

where $K$ is an invertible mapping from $E$ into a Hilbert space $H$, $D$ is a bounded self-adjoint operator in $H$, and $F_+$ is an orthogonal projection in $H$ whose kernel $\ker F_+$ is finite-dimensional.

**Proof.** First assume that $Q$ is invertible. Let $H = E$, let $K$ be any invertible mapping from $E$ onto $H$, and let $D = KQ^{-1}K^*$. Then $D$ is a bounded self-adjoint operator in $H$. Clearly, $V(z) = K^*(D - zF)^{-1}K$ with $F = 0$, an orthogonal projection in $H$. In the general case, $Q$ can be written as the sum of two invertible self-adjoint operators $Q = Q^{(1)} + Q^{(2)}$ (for example, $Q^{(1)} = Q - \varepsilon I$ and $Q^{(2)} = \varepsilon I$, where $\varepsilon$ is a real number), so that

\begin{equation}
Q^{(1)} = K^{(1)*}(D^{(1)} - zF^{(1)})^{-1}K^{(1)}, \quad Q^{(2)} = K^{(2)*}(D^{(2)} - zF^{(2)})^{-1}K^{(2)},
\end{equation}

where $K^{(i)}$ is an invertible operator from $E$ into a Hilbert space $H^{(i)} = E$, $D^{(i)}$ is a bounded self-adjoint operator in $H^{(i)}$, and $F^{(i)} = 0$ is an orthogonal projection in $H^{(i)}$, $i = 1, 2$. (Note that since $K^{(i)}$ is an arbitrary invertible operator from $E$ into $H^{(i)}$ it may as well be chosen as $K^{(i)} = I_E$). Define

$$
\mathcal{H} = \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)}, \quad K = \begin{pmatrix} K^{(1)} \\ K^{(2)} \end{pmatrix}, \quad D = \begin{pmatrix} D^{(1)} & 0 \\ 0 & D^{(2)} \end{pmatrix}, \quad F_+ = \begin{pmatrix} F^{(1)} & 0 \\ 0 & F^{(2)} \end{pmatrix}.
$$

Then $K$ is an invertible operator from $E$ into the Hilbert space $\mathcal{H}$, $D$ is a bounded self-adjoint operator, and $F_+ = 0$ is an orthogonal projection in $\mathcal{H}$. Moreover,

\begin{align*}
Q &= Q^{(1)} + Q^{(2)} \\
&= K^{(1)*}(D^{(1)} - zF^{(1)})^{-1}K^{(1)} + K^{(2)*}(D^{(2)} - zF^{(2)})^{-1}K^{(2)} \\
&= K^*(D - zF_+)^{-1}K,
\end{align*}

which proves the lemma. \qed

Herglotz-Nevanlinna functions of the form \[11\] which satisfy the conditions in \[19\], \[20\] can be realized by means of the theory of regularized generalized resolvents, \[19\], \[20\]. By means of Lemma 3.1 these realizations can be extended to Herglotz-Nevanlinna functions of the form \[11\] with $L = 0$.

**Theorem 3.2.** Let $V(z)$ be a Herglotz-Nevanlinna function, acting on a finite-dimensional Hilbert space $\mathcal{E}$, with the integral representation

\begin{equation}
V(z) = Q + \int_R \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma(t),
\end{equation}

where \( Q = Q^* \) and \( \Sigma(t) \) is a nondecreasing matrix-valued function on \( \mathbb{R} \) satisfying (3.1). Then \( V(z) \) admits a realization of the form
\[
V(z) = K^* (\mathcal{D} - zF_+)^{-1} K, \quad z \in \mathbb{C} \setminus \mathbb{R} \subset \rho(\mathcal{D}, F_+, K),
\]
where \( \mathcal{D} \in [\mathcal{H}_+, \mathcal{H}_-] \) is a self-adjoint bi-extension, \( \mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_- \) is a rigged Hilbert space, \( F_+ \) is an orthogonal projection in \( \mathcal{H}_+ \) and \( \mathcal{H} \) is an injective (invertible) operator from \( \mathcal{E} \) into \( \mathcal{H}_+ \), \( K^* \in [\mathcal{H}_+, \mathcal{E}] \). Moreover, the operators \( \mathcal{D} \) and \( F_+ \) can be selected such that the following commutativity condition holds:
\[
(3.4) \quad F_-, \mathcal{D} = \mathcal{D} F_+, \quad \mathcal{D} = R^{-1} F_+ R \in [\mathcal{H}_-, \mathcal{H}_-],
\]
where \( R \) is the Riesz-Berezanskiǐ operator defined in (2.1).

**Proof.** According to [19, Theorem 9] each matrix-valued Herglotz-Nevanlinna function of the form (3.2) admits a realization of the form (3.3) for Herglotz-Nevanlinna functions \( V(z) \) which do not satisfy the condition (3.6), the realization result in Lemma 3.1 will be used. Denote by \( \mathcal{E}_1 \) the linear subspace of vectors \( x \in \mathcal{E} \) with the property (3.7) and let \( \mathcal{E}_2 = \mathcal{E} \oplus \mathcal{E}_1 \), so that \( \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \). Rewrite \( Q \) in the block matrix form
\[
Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad Q_{ij} = P_{E_i} Q |_{E_j}, \quad j = 1, 2,
\]
and let \( \Sigma(t) = (\Sigma_{ij}(t))^2_{i,j=1} \) be decomposed accordingly. Observe, that by (3.1), (3.2), (3.4) the integrals
\[
(3.8) \quad G_{11} := \int_{\mathbb{R}} \frac{t}{1 + t^2} d\Sigma_{11}(t), \quad G_{12} := \int_{\mathbb{R}} \frac{t}{1 + t^2} d\Sigma_{12}(t)
\]
are convergent. Let the self-adjoint matrix \( G \) be defined by
\[
(3.9) \quad G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},
\]
where \( C = C^* \) is arbitrary. Now rewrite \( V(z) = V_1(z) + V_2(z) \) with
\[
(3.10) \quad V_1(z) = Q - G, \quad V_2(z) = G + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma(t).
\]
Clearly, for every \( x \in \mathcal{E}_1 \) the equality
\[
Gx = \int_{\mathbb{R}} \frac{t}{1 + t^2} d\Sigma(t)x
\]
holds. Consequently, \( V_2(z) \) admits the following representation
\[
(3.11) \quad V_2(z) = K^*_2 (\mathcal{A}^{(2)}_R - zI)^{-1} K_2
\]
where $K_2 : \mathcal{E} \to \mathcal{H}_2$, $K_2^* : \mathcal{H}_2 \to \mathcal{E}$ with $\mathcal{H}_2 \subset \mathcal{H}_2 \subset \mathcal{H}_2$ a rigged Hilbert space, and where $A_R^{(2)} = \frac{1}{2}(A^{(2)} + (A^{(2)})^*)$ is a self-adjoint bi-extension of a Hermitian operator $A_2$. The operator $K_2$ is invertible and has the properties
\begin{equation}
\text{ran} \ K_2 \subset \text{ran} (A^{(2)} - zI), \quad \text{ran} \ K_2 \subset \text{ran} (A^{(2)} - zI),
\end{equation}
where $\mathcal{H}_2 \ni (A^{(2)} - zI)^{-1}K_2 \in [\mathcal{E}, \mathcal{H}_2^+], \quad (A^{(2)} - zI)^{-1}K_2 \in [\mathcal{E}, \mathcal{H}_2^+]$,
for further details, see [10]. Now, by Lemma 3.1 the function $V_1(z)$ admits the representation
\begin{equation}
V_1(z) = K_1^*(D_1 - zF_{+1})^{-1}K_1,
\end{equation}
where $D_1 = D_1^*$ and $F_{+1} = 0$ are acting on a finite-dimensional Hilbert space $\mathcal{H}_1 = \mathcal{E} \oplus \mathcal{E}$ and where $K_1 : \mathcal{E} \to \mathcal{H}_1$ is invertible. Recall from Lemma 3.1 that
\begin{equation}
D_1 = \begin{pmatrix} D_1^{(1)} & 0 \\ 0 & D_2^{(2)} \end{pmatrix}, \quad K_1 = \begin{pmatrix} K_1^{(1)} \\ K_1^{(2)} \end{pmatrix},
\end{equation}
where $K_1^{(i)} : \mathcal{E} \to \mathcal{E}, i = 1, 2$, and $D_1^{(1)}, D_2^{(2)}$ are defined by means of the decomposition of $Q - G$ into the sum of two invertible self-adjoint operators
\begin{equation}
Q - G = (Q^{(1)} - G^{(1)}) + (Q^{(2)} - G^{(2)}).
\end{equation}
Then
\begin{equation}
D_1^{(i)} = K_1^{(i)*}(Q^{(i)} - G^{(i)})^{-1}K_1^{(i)}, \quad i = 1, 2.
\end{equation}
To obtain the realization 3.3 for $V(z)$ in 3.2, introduce the following triplet of Hilbert spaces
\begin{equation}
\mathcal{H}^{(1)}_+ := \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{H}_2 \subset \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{H}_2 \subset \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{H}_2 := \mathcal{H}^{(1)},
\end{equation}
i.e., a rigged Hilbert space corresponding to the block representation of symmetric operator $D_1 \oplus A_2$ in $\mathcal{H}^{(1)} := \mathcal{H}_1 \oplus \mathcal{H}_2$ (where $\mathcal{H}_1 = \mathcal{E} \oplus \mathcal{E}$). Also introduce the following operators
\begin{equation}
\mathbb{D} = \begin{pmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_1^{(2)} & 0 \\ 0 & 0 & A_R^{(2)} \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} K_1^{(1)} \\ K_1^{(2)} \\ K_2 \end{pmatrix}.
\end{equation}
It is straightforward to check that
\begin{equation}
V(z) = V_1(z) + V_2(z)
= K_1^{(1)*}(D_1^{(1)} - zF_{+1})^{-1}K_1^{(1)} + K_1^{(2)*}(D_2^{(2)} - zF_{+1})^{-1}K_1^{(2)}
+ K_2^{*}(A_R^{(2)} - zI)^{-1}K_2
= K^*(\mathbb{D} - zF_+)^{-1}K.
\end{equation}
By the construction, $A_2 \subset \mathcal{A}_R^{(2)} = (\mathcal{A}_R^{(2)})^* \subset \mathcal{A}_R^{(2)}$, where
\begin{equation}
\mathcal{A}_R^{(2)} = \{ f, g \in \mathcal{A}_R^{(2)} : g \in \mathcal{F} \}
\end{equation}
and $A_2$ is a symmetric operator in $\mathcal{F}_2$, cf. [12]. Moreover, $\mathbb{D}$ as an operator in $[\mathcal{H}^{(1)}_+, \mathcal{H}^{(1)}_-]$ is self-adjoint, i.e. $\mathbb{D} = \mathbb{D}^*$, and since
\begin{equation}
\widehat{\mathbb{D}} = \begin{pmatrix} D_1 & 0 \\ 0 & A_R^{(2)} \end{pmatrix} \subset \begin{pmatrix} D_1 & 0 \\ 0 & A_R^{(2)} \end{pmatrix} = \mathbb{D},
\end{equation}
and $A = D_1 \oplus A_2 \subset \hat{D}$, the operator $\mathbb{D}$ is a self-adjoint bi-extension of the Hermitian operator $A$ in $\mathcal{H}_1 \oplus \mathcal{H}_2$. It is easy to see that with operators in (3.16) one obtains the representation (3.2) for $V(z)$ in (3.2) and the system constructed with these operators satisfy the definition 2.1 of a $\Delta_+^-$-system.

Finally, from (3.16) one obtains $F^-\mathbb{D} = \mathbb{D}F^+$, where $F^+$ and $F^-$ are connected as in (3.4). This completes the proof of the theorem. $\square$

Remark 3.3. According to the recent results by Staffans [48] an operator-function $(-i)V(iz)$, where $V(z)$ is defined by (3.2) can be realized by an impedance system of the form (1.18)–(1.19) (see also [16], [17], [46], [47]). This realization is carried out by using a different approach and does not possess some of the properties contained in Theorem 3.2.

The general impedance realization result for Herglotz-Nevanlinna functions of the form (1.1) is now built on Theorem 3.2 and a representation for linear functions.

Lemma 3.4. Let $L$ be a nonnegative matrix in a finite-dimensional Hilbert space $\mathcal{E}$. Then it admits a realization of the form

$$zL = z\hat{K}^*P\hat{K} = K_3^*(D_3 - zF_3)^{-1}K_3,$$

where $D_3$ is a self-adjoint matrix in a Hilbert space $\mathcal{H}_3$, $P$ is the orthogonal projection onto $\overline{\mathcal{R}an} L$, and $K_3$ is an invertible operator from $\mathcal{E}$ into $\mathcal{H}_3$.

Proof. Since $L \geq 0$, there is a unique nonnegative square root $L^{1/2} \geq 0$ of $L$ with $\ker L^{1/2} = \ker L$, $\overline{\mathcal{R}an} L^{1/2} = \overline{\mathcal{R}an} L$.

Define the operator $\hat{K}$ in $\mathcal{E}$ by

$$\hat{K}u = \begin{cases} u, & u \in \ker L; \\ L^{1/2}u, & u \in \overline{\mathcal{R}an} L. \end{cases}$$

Then $\hat{K}$ is invertible and $L^{1/2} = P\hat{K}$, where $P$ denotes the orthogonal projection onto $\overline{\mathcal{R}an} L$. Define

$$\mathcal{H}_3 = \mathcal{E} \oplus \mathcal{E}, \quad K_3 = \begin{pmatrix} P\hat{K} \\ \hat{K} \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad F^+,3 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Then $K_3$ is an invertible operator from $\mathcal{E}$ into $\mathcal{H}_3$, $D_3$ is a bounded self-adjoint operator, and $F^+,3$ is an orthogonal projection in $\mathcal{H}_3$. Moreover,

$$V_3(z) = zL = z\hat{K}^*P\hat{K} = K_3^*(D_3 - zF^+,3)^{-1}K_3.$$

This completes the proof. $\square$

The general realization result for Herglotz-Nevanlinna functions of the form (1.1) is now obtained by combining the earlier realizations.

Theorem 3.5. Let $V(z)$ be a matrix-valued Herglotz-Nevanlinna function in a finite-dimensional Hilbert space $\mathcal{E}$ with the integral representation

$$V(z) = Q + zL + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\Sigma(t),$$

where $Q = Q^*$, $L \geq 0$, and $\Sigma(t)$ is a nondecreasing nonnegative matrix-valued function on $\mathbb{R}$ satisfying (1.2). Then $V(z)$ admits a realization of the form

$$V(z) = K^*(\mathbb{D} - zF^+)^{-1}K.$$
where $\mathbb{D} \in [\mathcal{H}_+ + \mathcal{H}_-]$ is a self-adjoint bi-extension in a rigged Hilbert space $\mathcal{H}_+ \subset \mathcal{H}$, $F_+$ is an orthogonal projection in $\mathcal{H}_+$ and $K \in [\mathcal{E}, \mathcal{H}_+]$ is an invertible mapping from $\mathcal{E}$ into $\mathcal{H}_-$. 

**Proof.** Define the following matrix functions

$$V_1(z) = Q + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\Sigma(t), \quad V_2(z) = zL.$$ 

According to Theorem 3.2 the function $V_1(z)$ has a representation

$$V_1(z) = K_1^*(\mathbb{D}_1 - zF_{+1})^{-1}K_1,$$

where $\mathbb{D}_1$, $K_1$ and $F_{+1}$ are given by the formula (3.16). We recall that $\mathbb{D}_1$ is a self-adjoint bi-extension in a rigged Hilbert space $\mathcal{H}_+(1) \subset \mathcal{H}_{(1)} \subset \mathcal{H}_{(1)}^+$ given by (3.15), $F_{+1}$ is an orthogonal projection in $\mathcal{H}_+(1)$, and $K_1$ is an invertible mapping from $\mathcal{E}$ into $\mathcal{H}_{(1)}$. According to Lemma 3.3 the functions $V_2(z)$ has a realization of the form (3.14) with components $\mathcal{H}_3$, $D_3$, $K_3$ and $F_{+3}$ described by (3.21).

Now the final result follows by introducing the rigged Hilbert space $\mathcal{H}_3 \oplus \mathcal{H}_3(1) \subset \mathcal{H}_3 \oplus \mathcal{H}_3(1) \subset \mathcal{H}_3 \oplus \mathcal{H}_3(1)$ and the operators

$$\mathbb{D} = \begin{pmatrix} D_3 & 0 \\ 0 & \mathbb{D}_1 \end{pmatrix} \in [\mathcal{H}_3 \oplus \mathcal{H}_3(1), \mathcal{H}_3 \oplus \mathcal{H}_3(1)]^+, \quad F_+ = \begin{pmatrix} F_{+3} & 0 \\ 0 & F_{+1} \end{pmatrix}, \quad K = \begin{pmatrix} K_3 \\ K_1 \end{pmatrix}.$$ 

It is straightforward to check that with these operators one obtains the representation (3.21) for $V(z)$ in (3.23) and the system constructed with these operators satisfy the definition (2.4) of a $\mathcal{D}_+$-system. \qed

For the sake of clarity an extended version for the impedance realization in the proof of Theorem 3.3 is provided. The rigged Hilbert space used is

$$E \oplus E \oplus E \oplus E \oplus H_{+2} \subset E \oplus E \oplus E \oplus E \oplus H_{+2} \subset E \oplus E \oplus E \oplus E \oplus H_{-2},$$

and the operators are given by

$$\mathbb{D} = \begin{pmatrix} 0 & iI & 0 & 0 & 0 \\ -iI & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1^{(1)} & 0 & 0 \\ 0 & 0 & D_1^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \hat{K}_R^{(2)} \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} P\hat{K} \\ \hat{K} \\ K_1^{(1)} \\ K_1^{(2)} \\ K_2 \end{pmatrix}.$$ 

All the operators in (3.26) are defined above.

In conclusion of this section it is observed that the general impedance realization case involving a non-zero linear term in (3.23) is also implicitly treated by Ball and Staffans in 16, 17.

4. **$F_+$-system realization results**

In the general impedance realization results in Theorem 3.2 and Theorem 3.3 the realizations are in terms of the operators in (3.3) and (3.24), respectively. It remains to identify the Herglotz-Nevanlinna functions as transforms of transfer functions of appropriate conservative systems.
Theorem 4.1. Let $V(z)$ be a Herglotz-Nevanlinna function acting on a finite-dimensional Hilbert space $\mathcal{E}$ with the integral representation
\begin{equation}
V(z) = Q + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma(t),
\end{equation}
where $Q = Q^*$ and $\Sigma(t)$ is a nondecreasing matrix-valued function on $\mathbb{R}$ satisfying (1.2). Then the function $V(z)$ can be realized in the form
\begin{equation}
V(z) = i[W_{\Theta F_+}(z) + I]^{-1}[W_{\Theta F_+}(z) - I],
\end{equation}
where $W_{\Theta F_+}(z)$ is the transfer function given by (2.5) of an $F_+$-system defined in (2.4). The $F_+$-system in (2.4) can be taken to be a scattering system.

Proof. By Theorem 3.2 the function $V(z)$ can be represented in the form $V(z) = K^*(\mathbb{D} - zF_+)^{-1}K$, where $K$, $\mathbb{D}$, and $F_+$ are as in (3.10) corresponding to the decomposition
\begin{equation}
V(z) = V_1(z) + V_2(z),
\end{equation}
where $V_1(z) = Q - G$, $V_2(z) = G + \int_{\mathbb{R}} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\Sigma(t),$ with a self-adjoint operator $G$ of the form (3.4). With the notations used in the proof of Theorem 3.2 one may rewrite $V_1(z)$ and $V_2(z)$ as in (3.11) with
\begin{equation}
D_1^{(1)} = (Q - G - \varepsilon I)^{-1}, \quad D_1^{(2)} = (\varepsilon I)^{-1}, \quad K_1^{(1)} = \lambda I_{\mathcal{E}}, \quad K_1^{(2)} = I_{\mathcal{E}}, \quad K_2^{(2)} = I_{\mathcal{E}},
\end{equation}
where $K_2^{(2)}$ is associated to a ($\ast$)-extension $A^{(2)}$ of an operator $T_2 \in \Omega_{A_2}$ for which $-i \in \rho(T_2)$, cf. (19). The remaining operators are defined in (3.10).

Recall that $K_2$ and the resolvents $(A^{(2)} - zI)^{-1}, (A^{(2)}_R - zI)^{-1}$ satisfy the properties (3.12). To construct an $F_+$-system of the form (3.11) introduce the operator $\tilde{A}$ by
\begin{equation}
\tilde{A} = \mathbb{D} + iKK^* \in [\mathcal{H}_+, \mathcal{H}_-],
\end{equation}
where $K$, $\mathbb{D}$, and $F_+$ are defined in (3.10). Then the block-matrix form of $\tilde{A}$ is
\begin{equation}
\tilde{A} = \begin{pmatrix}
D_1^{(1)} + i\lambda^2 I & i\lambda I & i\lambda K_2^{(2)} \\
i\lambda I & D_1^{(2)} + iI & iK_2^{(2)} \\
i\lambda K_2 & iK_2 & A^{(2)}
\end{pmatrix}.
\end{equation}
Let
\begin{equation}
\Theta_{F_+} = \begin{pmatrix}
\tilde{A} & F_+ & K \\
\mathcal{D}_+ & \mathcal{H}_+ & \mathcal{H}_- & \mathcal{E}
\end{pmatrix},
\end{equation}
where the rigged Hilbert triplet $\mathcal{D}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is defined in (3.11), i.e.,
\begin{equation}
\mathcal{E} \oplus \mathcal{E} \oplus \mathcal{D}_+ \subset \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{H}_2 \subset \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{H}_-.
\end{equation}
It remains to show that all the properties in Definition 2.2 are satisfied. For this purpose, consider the equation
\begin{equation}
(\tilde{A} - zF_+)x = (\mathbb{D} + iKK^*)x - zF_+x = Ke,
\end{equation}
or
\begin{equation}
\begin{pmatrix}
D_1^{(1)} + i\lambda^2 I & i\lambda I & i\lambda K_2^{(2)} \\
i\lambda I & D_1^{(2)} + iI & iK_2^{(2)} \\
i\lambda K_2 & iK_2 & A^{(2)} - zI
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
\lambda e \\
e \\
K_2 e
\end{pmatrix}.
\end{equation}
Using the decomposition of the operators and taking into account that
\[ \mathcal{A}^{(2)} = \mathcal{A}^{(2)}_R + iK_2K_2^* \]
this equation can be rewritten in form of the following system
\[
\begin{align*}
D_1^{(1)} x_1 + i\lambda^2 Ix_1 + i\lambda Ix_2 + i\lambda K_2^* x_3 = \lambda e, \\
D_1^{(2)} x_2 + i\lambda Ix_1 + i\lambda x_2 + iK_2^* x_3 = e, \\
(A^{(2)} - zI)x_3 + i\lambda K_2 x_1 + iK_2 x_2 = K_2 e.
\end{align*}
\]
(4.6)

or
\[
\begin{align*}
\frac{1}{\lambda} D_1^{(1)} x_1 + i\lambda Ix_1 + i\lambda x_2 + iK_2^* x_3 = e, \\
D_1^{(2)} x_2 + i\lambda Ix_1 + i\lambda x_2 + iK_2^* x_3 = e, \\
(A^{(2)} - zI)x_3 + i\lambda K_2 x_1 + iK_2 x_2 = K_2 e.
\end{align*}
\]
(4.7)

In a neighborhood of \((-i)\) the resolvent \((A^{(2)} - zI)^{-1}\) is well defined so that by
(3.12) the third equation in (4.6) can be solved for \(x_3:\)

\[
x_3 = (A^{(2)} - zI)^{-1} K_2 e - i(A^{(2)} - zI)^{-1} K_2 (\lambda x_1 + x_2).
\]
Substitute \(1\) into the first line of the system yields

\[
\frac{1}{\lambda} D_1^{(1)} x_1 + iI(\lambda x_1 + x_2) + K_2^* (A^{(2)} - zI)^{-1} K_2 (\lambda x_1 + x_2) = e - iK_2^* (A^{(2)} - zI)^{-1} K_2 e,
\]
Denoting the right hand side by \(C\) and using \(2\), we get

\[
C = e - iK_2^* (A^{(2)} - zI)^{-1} K_2 e = \frac{1}{2} [I + W_{\theta_2}(z)] e.
\]
Then

\[
\frac{1}{\lambda} D_1^{(1)} x_1 + iI(\lambda x_1 + x_2) + K_2^* (A^{(2)} - zI)^{-1} K_2 (\lambda x_1 + x_2) = C,
\]
Multiply both sides by \(2i\) and using \(2\), one more time yields

\[
\frac{2i}{\lambda} D_1^{(1)} x_1 - [I + W_{\theta_2}(z)] (\lambda x_1 + x_2) = 2iC.
\]
Denoting for further convenience \(B = [I + W_{\theta_2}(z)]\) we obtain

\[
\frac{2i}{\lambda} D_1^{(1)} x_1 - \lambda Bx_1 - Bx_2 = 2iC.
\]
or
\[
(4.8)
\frac{2i}{\lambda} D_1^{(1)} x_1 - \lambda Bx_1 - 2iC = Bx_2.
\]
Now we subtract the second equation of the system from the first and obtain

\[
D_1^{(1)} x_1 = \lambda D_1^{(2)} x_2,
\]
or
\[
(4.9)
\lambda(D_1^{(1)})^{-1} D_1^{(2)} x_2 = x_1.
\]
Applying \(1\) to \(1\), we get

\[
2i D_1^{(2)} x_2 - B \lambda^2 (D_1^{(1)})^{-1} D_1^{(2)} x_2 - Bx_2 = 2iC,
\]
and using \(1\),

\[
\frac{2i}{\varepsilon} I x_2 - B(\lambda^2 (Q - G - \varepsilon I) \frac{1}{\varepsilon} + I) x_2 = 2iC,
\]
or
\[
(4.10)
\left(2i I - [I + W_{\theta_2}(z)] [\lambda^2 (Q - G) + \varepsilon(1 - \lambda^2 I)] \right) x_2 = 2i\varepsilon C.
\]
Choosing $\lambda$ and $\varepsilon$ sufficiently small the matrix on the left hand side of (4.10) can be made invertible for $z = -i$. Using an invertibility criteria from [24] we deduce that (4.10) is also invertible in a neighborhood of $(-i)$. Consequently, the system (4.6) has a unique solution and $(A - zF_+)^{-1}K$ is well defined in a neighborhood of $(-i)$.

In order to show that the remaining properties in Definition 2.2 are satisfied we need to present an operator $T \in \Omega_A$ such that $A$ is a correct $(\cdot)^*$-extension of $T$. To construct $T$ we note first that $(A - zF_+)\mathcal{H}_+ \supset \mathcal{H}$ for some $z$ in a neighborhood of $(-i)$. This can be confirmed by considering the equation

$$
(A - zF_+)x = g, \quad x \in \mathcal{H}_+,
$$

and showing that it has a unique solution for every $g \in \mathcal{H}$. The procedure then is reduced to solving the system (4.6) with an arbitrary right hand side $g \in \mathcal{H}$. Following the steps for solving (4.6) we conclude that the system (4.11) has a unique solution. Similarly one shows that $(A^* - zF_+)\mathcal{H}_+ \supset \mathcal{H}$. Using the technique developed in [21] we can conclude that operators $(A + iF_+)^{-1}$ and $(A^* - iF_+)^{-1}$ are $(-\cdot,\cdot)$-continuous. Define

$$
T = A, \quad \text{dom}\, T = (A + iF_+)\mathcal{H}_+,
$$

$$
T_1 = A^*, \quad \text{dom}\, T_1 = (A^* - iF_+)\mathcal{H}_+.
$$

One can see that both $\text{dom}\, T$ and $\text{dom}\, T_1$ are dense in $\mathcal{H}$ while operator $T$ is closed in $\mathcal{H}$. Indeed, assuming that there is a vector $\phi \in \mathcal{H}$ that is $(\cdot)$-orthogonal to $\text{dom}\, T$ and representing $\phi = (A^* - iF_+)\psi$ we can immediately get $\phi = 0$. It is also easy to see that $T_1 = T^*$. Thus, operator $T$ defined by (4.12) fits the definition of correct $(\cdot)^*$-extension for operator $A$. Property (vi) of Definition 2.2 follows from Theorem 4.1 and the fact that $A_R = \mathbb{D}$.

Consequently all the properties for an $F_+$-system $\Theta$ in Definition 2.2 are fulfilled with the operators and spaces defined above. \hfill \Box

Now the principal result of the paper will be presented.

**Theorem 4.2.** Let $V(z)$ be a matrix-valued Herjolfs-Nevanlinna function in a finite-dimensional Hilbert space $\mathcal{E}$ with the integral representation

$$
V(z) = Q + zL + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\Sigma(t),
$$

where $Q = Q^*$, $L \geq 0$ is an invertible matrix, and $\Sigma(t)$ is a nondecreasing nonnegative matrix-valued function on $\mathbb{R}$ satisfying (1.2). Then $V(z)$ can be realized in the form

$$
V(z) = i|W_{\Theta F_+}(z) + I|^{-1}|W_{\Theta F_+}(z) - I|
$$

where $W_{\Theta F_+}(z)$ is a matrix-valued transfer function of some scattering $F_+$-system of the form (2.1).

**Proof.** Decompose the function $V(z)$ as follows:

$$
V_1(z) = Q + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\Sigma(t) \quad \text{and} \quad V_2(z) = zL,
$$

and use the earlier realizations for each of these functions.

By Theorem 4.1 the function $V_1(z)$ can be represented by

$$
V_1(z) = i|W_{\Theta F_1+}(z) + I|^{-1}|W_{\Theta F_1+}(z) - I|,
$$

where $W_{\Theta F_1+}(z)$ is a matrix-valued transfer function of some scattering $F_+$-system of the form (2.1).
where $W_{\mathcal{F}_1^+}(z)$ is a matrix-valued transfer function of some scattering $F_1^+$

system,

$$W_{\mathcal{F}_1^+}(z) = I - 2izK_1^*(A_1 - zF_1^+)^{-1}K_1,$$

(4.15)

$A_1 = \mathbb{D}_1 + iK_1K_1^*$ maps $\mathcal{H}_1$ continuously into $\mathcal{H}_2$. $\mathbb{D}_1$ is a self-adjoint bi-extension, and $\mathbb{D}_1 \in [\mathcal{H}_1, \mathcal{H}_2], K_1 \in [\mathfrak{E}, \mathcal{H}_1]$, $K_2 \in [\mathfrak{E}, \mathcal{H}_2]$

Following the proof of Theorem 3.5, the function $V_2(z)$ can be represented in the form

$$V_2(z) = K_2^*(D_2 - zF_2^+)^{-1}K_2,$$

where

$$D_2 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad F_2^+ = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad K_2 = \begin{pmatrix} P\hat{K} \\ \hat{K} \end{pmatrix},$$

and $P$ and $\hat{K}$ are as in (3.20), so that $K_2$ is an invertible operator from $\mathfrak{E}$ into $\mathcal{H}_2 = \mathfrak{E} \oplus \mathfrak{E}$. Introduce the triplet $\mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}_1 \oplus \mathcal{H}_2$, and consider the operator

$$(4.17) \quad \mathcal{A} = \mathbb{D} + iKK^*$$

from $\mathcal{H}_1 \oplus \mathcal{H}_2$ into $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by the block form

$$\mathcal{A} = \begin{pmatrix} \mathbb{D}_1 & 0 \\ 0 & D_2 \end{pmatrix} + i\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix} = \begin{pmatrix} \mathcal{A}_1 & iK_1K_2^* \\ iK_2K_1^* & \mathcal{A}_2 \end{pmatrix}.$$ (4.18)

Here $\mathcal{A}_2 = D_2 + iK_2K_2^*$. It will be shown that the equation

$$(4.19) \quad (\mathcal{A} - zF_+)x = Ke, \quad e \in \mathfrak{E},$$

with

$$(4.20) \quad F_+ = \begin{pmatrix} F_1^+ & 0 \\ 0 & F_2^+ \end{pmatrix}, \quad K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix},$$

has always a unique solution $x \in \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$(\mathcal{A} - zF_+)^{-1}K \in [\mathfrak{E}, \mathcal{H}_1 \oplus \mathcal{H}_2].$$

Taking into account (4.18), the equation (4.19) can be written as the following system

$$(4.21) \quad \begin{cases} (\mathcal{A}_1 - zF_1^+)x_1 + iK_1K_2^*x_2 = K_1e, \\ (\mathcal{A}_2 - zF_2^+)x_2 + iK_2K_1^*x_1 = K_2e, \end{cases}$$

where

$\mathcal{A}_1 = \mathbb{D}_1 + iK_1K_1^*, \quad \mathcal{A}_2 = D_2 + iK_2K_2^*.$

By Theorem 4.1, it follows that

$$(\mathcal{A}_1 - zF_1^+)^{-1}K_1 \in [\mathfrak{E}, \mathcal{H}_1].$$

Therefore, the first equation in (4.21) gives

$$(4.22) \quad x_1 = (\mathcal{A}_1 - zF_1^+)^{-1}K_1e - i(\mathcal{A}_1 - zF_1^+)^{-1}K_1K_2^*x_2.$$ 

Now substituting $x_1$ in the second equation in (4.21) yields

$$(4.23) \quad (\mathcal{A}_2 - zF_2^+)x_2 + K_2K_1^*(\mathcal{A}_1 - zF_1^+)^{-1}K_1K_2^*x_2$$

$$= K_2e - iK_2K_1^*(\mathcal{A}_1 - zF_1^+)^{-1}K_1e.$$
Taking into account (4.15), (4.16), and (4.18) the identity (4.23) leads to
\[
\left(2iI - (D_2 - zF_+^{(2)})^{-1}K_2[I + W_{\Theta_{\Delta_1}}(z)]K_2^*\right)x_2
\]
\[
= 2i(D_2 - zF_+^{(2)})^{-1}\left(K_2e - iK_2K_1^*(A_1 - zF_+^{(1)})^{-1}K_1e\right).
\]
It will be shown that the matrix-function on the lefthand side, in front of \(x_2\), is invertible. First by straightforward calculations one obtains
\[
(D_2 - zF_+^{(2)})^{-1} = \begin{pmatrix} zI & iI \\ -iI & 0 \end{pmatrix} \in [\mathcal{E} \oplus \mathcal{E}, \mathcal{E} \oplus \mathcal{E}].
\]
The matrix function \(M(z)\) defined by
\[
M(z) = I + W_{\Theta_{\Delta_1}}(z) \in [\mathcal{E}, \mathcal{E}]
\]
is invertible by Theorem 3.1. It follows from (4.16) that
\[
K_2M(z) = \begin{pmatrix} \hat{P}\hat{K} \\ \hat{K} \end{pmatrix} M(z) = \begin{pmatrix} L^{1/2}M(z) \\ \hat{K}M(z) \end{pmatrix} \in [\mathcal{E} \oplus \mathcal{E}, \mathcal{E} \oplus \mathcal{E}],
\]
and that
\[
K_2M(z)K_2^* = \begin{pmatrix} L^{1/2}M(z)L^{1/2} \\ \hat{K}M(z)\hat{K} \end{pmatrix} \in [\mathcal{E} \oplus \mathcal{E}, \mathcal{E} \oplus \mathcal{E}].
\]
For any \(2 \times 2\) block-matrix
\[
Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
with entries in \([\mathcal{E}]\) define the matrix-function
\[
N(z) = 2iI - (D_2 - zF_+^{(2)})^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = i \begin{pmatrix} 2 + zai - c & zbi - d \\ a & b + 2 \end{pmatrix}.
\]
Since the matrix \(L > 0\) is invertible it follows that \(\ker L = \{0\}\) and \(\hat{K} = L^{1/2}\). Now choose
\[
Z = \begin{pmatrix} L^{1/2}M(z)L^{1/2} \\ L^{1/2}M(z)L^{1/2} \end{pmatrix} = \begin{pmatrix} A_0 & A_0 \\ A_0 & A_0 \end{pmatrix},
\]
where \(A_0 = A_0(z) = L^{1/2}M(z)L^{1/2}\). Note that the matrix-function \(A_0\) is invertible and that \(A_0^{-1} = L^{-1/2}M(z)^{-1}L^{-1/2}\). With this choice of \(Z\) one obtains
\[
N = N(z) = i \begin{pmatrix} 2I + zA_0 - A_0 & zA_0 - A_0 \\ A_0 & A_0 + 2I \end{pmatrix}.
\]
To investigate the invertibility of \(N\) consider the system
\[
\begin{pmatrix} 2I + zA_0 - A_0 & zA_0 - A_0 \\ A_0 & A_0 + 2I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]
or
\[
\begin{cases}
(2I + zA_0 - A_0)x_1 + (zA_0 - A_0)x_2 = 0, \\
A_0x_1 + (A_0 + 2)x_2 = 0.
\end{cases}
\]
Solving the second equation for \(x_1\) yields
\[
\begin{cases}
2x_1 + zA_0x_1 - A_0x_1 + zA_0x_2 - A_0x_2 = 0, \\
x_1 = -x_2 - 2A_0^{-1}x_2.
\end{cases}
\]
Substituting \(x_1\) into the first equation gives
\[
(2A_0^{-1} + z)A_0^{-1}x_2 = 0.
\]
or equivalently,
\begin{equation}
A_0 x_2 = \frac{2i}{z} x_2.
\end{equation}
Recall that
\[ A_0 = A_0(z) = L^{1/2} M(z) L^{1/2} = L^{1/2} [I + W_{\Theta_1}(z)] L^{1/2}. \]
For every \( z \) in the lower half-plane \( W_{\Theta_1}(z) \) is a contraction (see \( 19 \)) and thus \( \| A_0(z) \| \leq 2 \| L \|. \) This means that for every \( z \) \( (\text{Im}\ z < 0) \) the norm of the left hand side of \( 4.24 \) is bounded while the norm of the right side can be made unboundedly large by letting \( z \to 0 \) along the imaginary axis. This leads to a conclusion that \( x_2 = 0 \) and then also \( x_1 = 0. \) Hence, \( N = N(z) \) is invertible.

Consequently,
\begin{equation}
2i I - (D_2 - z F^{(2)}_+) K_2 [I + W_{\Theta_1}(z)] K_2^* = 0
\end{equation}
is invertible and \( x_2 \) depends continuously on \( e \in \mathcal{E} \) in \( 123, \) while \( 122 \) shows that \( x_1 \) depends continuously on \( e \in \mathcal{E}. \)

Now we will follow the steps taken in the proof of the Theorem 4.1 to show that the remaining properties in Definition 2.2 are satisfied. We introduce an operator \( T \in \Omega_A \) such that \( A \) is a correct \((*)\)-extension of \( T. \) To construct \( T \) we note first that \((A - z F^*_-) \mathcal{F} \supset \mathcal{F} \) for some \( z \) in a neighborhood of \((-i)\). This can be confirmed by considering the equation
\begin{equation}
(A - z F^*_-) x = g, \quad x \in \mathcal{F},
\end{equation}
and showing that it has a unique solution for every \( g \in \mathcal{F}. \) The procedure then is reduced to solving the system \( 4.21 \) with an arbitrary right hand side \( g \in \mathcal{F}. \) Inspecting the steps of solving \( 4.21 \) we conclude that the system \( 4.26 \) has a unique solution. Similarly one shows that \((A^* - z F^*_+) \mathcal{S} \supset \mathcal{S} \).

Once again relying on \( 51 \) we can conclude that operators \( (A^* + i F^*_+) \) and \( (A^* - i F^*_+) \) are \((-\cdot,\cdot)\)-continuous and define
\begin{equation}
\begin{aligned}
T &= A, \quad \text{dom } T = (A + i F^*_+) \mathcal{S}, \\
T_1 &= A^*, \quad \text{dom } T_1 = (A^* - i F^*_+) \mathcal{S}.
\end{aligned}
\end{equation}
Using similar to the proof of Theorem 4.1 arguments we note that both \( \text{dom } T \) and \( \text{dom } T_1 \) are dense in \( \mathcal{S} \) while operator \( T \) is closed in \( \mathcal{S}. \) It is also easy to see that \( T_1 = T^*. \) Thus, operator \( T \) defined by \( 4.27 \) fits the definition of correct \((*)\)-extension for operator \( A. \) Property (vi) of Definition 2.2 follows from Theorem 4.2 and the fact that \( A_R = \mathbb{D}. \)

Therefore, the array
\begin{equation}
\Theta_{F_+} = \begin{pmatrix}
\mathcal{S}_{\Theta_1} \oplus \mathcal{S}_2 & \mathcal{S}_1 \subset \mathcal{S}_2 & \mathcal{S}_{\Theta_1} \oplus \mathcal{S}_2 \\
K & F^*_+ & I
\end{pmatrix} \mathcal{E}
\end{equation}
is an \( F_+ \)-system and \( V(z) \) admits the realizations
\[ V(z) = K^*(\mathbb{D} - z F_+)^{-1} K = i [W_{\Theta_{F_+}}(z) - I]^{-1} [W_{\Theta_{F_+}}(z) - I]. \]
This completes the proof. \( \square \)

It was shown in \( 33 \) that for the case of compactly supported measure in \( 4.13 \) the function \( V(z) \) can be realized without the restriction on the invertibility of the linear term \( L. \)
5. Minimal Realization

Recall that a symmetric operator $A$ in a Hilbert space $\mathcal{H}$ is called a prime operator \cite{51, 19} if there exists no reducing invariant subspace on which it induces a self-adjoint operator. A notion of a minimal realization is now defined along the lines of the concept of prime operators. An $F_+$-system of the form \cite{24} is called $F_+$-minimal if there are no nontrivial reducing invariant subspaces $\mathcal{H}_1 = \overline{\mathcal{H}_1}^+$, $(\mathcal{H}_1^+)$ is a (+)-subspace of ran $F_+$ of $\mathcal{H}$ where the symmetric operator $A$ induces a self-adjoint operator. Here the closure is taken with respect to (+)-metric. In the case that $F_0 = I$ this definition coincides with the one used for rigged operator colligations in \cite{24, 19}.

**Theorem 5.1.** Let the matrix-valued Herglotz-Nevanlinna function $V(z)$ be realized in the form

\begin{equation}
V(z) = i[W_{\Theta F_+}(z) + I]^{-1}[W_{\Theta F_+}(z) - I],
\end{equation}

where $W_{\Theta F_+}(z)$ is the transfer function of some $F_+$-system \cite{24}. Then this $F_+$-system can be reduced to an $F_+$-minimal system of the form \cite{24} and its transfer function gives rise to an $F_+$-minimal realization of $V(z)$ via \cite{51}.

**Proof.** Let the matrix-valued Herglotz-Nevanlinna function $V(z)$ be realized in the form \cite{51} with an $F_+$-system of the type \cite{24}. Assume that its symmetric operator $A$ has a reducing invariant subspace $\mathcal{H}_1 = \overline{\mathcal{H}_1}^+$, $(\mathcal{H}_1^+)$ is a (+)-subspace of ran $F_+$ on which it generates a self-adjoint operator $A_1$. Then there is the following (·, ·)-orthogonal decomposition

\begin{equation}
\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1, \quad A = A_0 \oplus A_1,
\end{equation}

where $A_0$ is an operator induced by $A$ on $\mathcal{H}^0$.

The identity \cite{52} shows that the adjoint of $A$ in $\mathcal{H}$ admits the orthogonal decomposition $A^* = A_0^* \oplus A_1$. Now consider operators $T \supset A$ and $T^* \supset A$ as in the definition of the system $\Theta_{F_+}$. It is easy to see that both $T$ and $T^*$ admit the (·, ·)-orthogonal decompositions

\begin{equation}
T = T_0 \oplus A_1,
\end{equation}

and

\begin{equation}
T^* = T_0^* \oplus A_1,
\end{equation}

where $T_0 \supset A_0$ and $T_0^* \supset A_0$. Since $T \in \Omega_A$, the identity $A_0 \oplus A_1 = T \cap T^* = (T_0 \cap T_0^*) \oplus A_1$ holds and $-i$ is a regular point of $T = T_0 \oplus A_1$ or, equivalently, $-i$ is a regular point of $T_0$. This shows that $T_0 \in \Omega_{A_0}$. Clearly,

\begin{equation}
\mathcal{H}_+ = \mathcal{H}_0^+ \oplus \mathcal{H}_1^+ = \text{dom } A_0^* \oplus \text{dom } A_1.
\end{equation}

This decomposition remains valid in the sense of (+)-orthogonality. Indeed, if $f_0 \in \mathcal{H}_0^+$ and $f_1 \in \mathcal{H}_1^+ = \text{dom } A_1$, then by considering the adjoint of $A : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ one obtains

\begin{align*}
(f_0, f_1)_+ &= (f_0, f_1) + (A^* f_0, A^* f_1) \\
&= (f_0, f_1) + (A_0^* f_0, A_1 f_1) \\
&= 0 + 0 = 0.
\end{align*}
Consequently, the inclusions $\mathcal{H}_+ \subset \mathcal{H}_0 \subset \mathcal{H}_-$ can be rewritten in the following decomposed forms

$$
\mathcal{H}_+^0 \oplus \mathcal{H}_+^1 \subset \mathcal{H}_0^0 \oplus \mathcal{H}_0^1 \subset \mathcal{H}_-^0 \oplus \mathcal{H}_-^1
$$

Since $A_1$ is selfadjoint in $\mathcal{H}_1$, $A_0$ is a correct ($\star$)-extension of $T_0$, cf. (2.2). Moreover,

$$
\frac{\mathcal{A} - \mathcal{A}^*}{2i} = \left( (\mathcal{A}_0 \oplus A_1) - (\mathcal{A}_0^* \oplus A_1) \right)
= \frac{\mathcal{A}_0 - \mathcal{A}_0^*}{2i} + \frac{A_1 - A_1^*}{2i}
= \frac{\mathcal{A}_0 - \mathcal{A}_0^*}{2i} \oplus \frac{A_1 - A_1^*}{2i} \oplus O.
$$

where $O$ stands for the zero operator. Decompose $K \in [\mathcal{E}, \mathcal{H}_-]$ according to $\mathcal{H}_- = \mathcal{H}_0^0 \oplus \mathcal{H}_0^1$ as follows $K = K_0 \oplus K_1$. Then (5.4) implies that

$$
KJK^* = K_0JK_0^* \oplus O.
$$

Since $\dim \mathcal{E} < \infty$ and $\ker K = \{0\}$, one has $K^* = \mathcal{E}$ and therefore also $\ker JK^* = \mathcal{E}$. According to (5.4), $K_1(\ker JK^*) = \{0\}$ and therefore $K_1 = 0$, or equivalently, $K = K_0 \oplus O$. Let $P_0^1$ be the orthogonal projection operator of $\mathcal{H}_+$ onto $\mathcal{H}_0^0$ and let $P_+^1 = I - P_0^1$. Then $K^* = K_0^*P_+^1$, since for all $f \in \mathcal{E}$, $g \in \mathcal{H}_+$ one has

$$(Kf, g) = (K_0f, g) = (K_0f, g_0 + g_1) = (K_0f, g_0) + (K_0f, g_1) = (f, K_0^*g_0) = (f, K_0^*P_0^1g).$$

Since $\mathcal{H}_0^1$ is a closed subspace of $\ker F_+$, $P_0^1 = I - P_+^1$ commutes with $F_+$ and therefore $F_+^1 := F_0^1P_+^1$ defines an orthogonal projection in $\mathcal{H}_0^1$.

Now, let $e \in \mathcal{E}$, let $z \in \rho(\mathcal{A}, F_+, K)$, and let $x = x^0 + x^1 \in \mathcal{H}_+$ be such that

$$(\mathcal{A} - zF_+)x = Ke.$$

Since $K = K_0 \oplus O$ the previous identity is equivalent to

$$(\mathcal{A}_0 \oplus A_1 - zF_+)(x^0 + x^1) = (K_0 \oplus O)e.$$

Since $F_+x^1 = x^1$ and $P_0^1$ commutes with $F_+$, this yields

$$(\mathcal{A}_0 - zF_0^1)x^0 = K_0e, \quad (A_1 - zI)x^1 = 0.$$

It follows from the previous equations that $z \in \rho(A_1)$ because $z \in \rho(\mathcal{A}, F_+, K)$. Thus, $\rho(\mathcal{A}, F_+, K) \subset \rho(\mathcal{A}_0, F_0^1, K_0)$ and hence $x^0 = (\mathcal{A}_0 - zF_0^1)^{-1}K_0e$. On the other hand, $x^0 = x = (\mathcal{A} - zF_+)^{-1}Ke$ and therefore for all $e \in \mathcal{E}$ one obtains

$$(\mathcal{A} - zF_+)^{-1}Ke = (\mathcal{A}_0 - zF_0^1)^{-1}K_0e$$

and

$$K^*(\mathcal{A} - zF_+)^{-1}Ke = \mathcal{K}_0^*(\mathcal{A}_0 - zF_0^1)^{-1}K_0e.$$
This means that the transfer functions of the system $\Theta_{F_+}$ in (2.4) and of the system 

\[
\Theta_{F_+} = \begin{pmatrix} \mathcal{A}_0 & \mathcal{F}_0^0 & \mathcal{F}_0^- \\ \mathcal{J}_0^+ \subset \mathcal{Y}_0^+ \subset \mathcal{Y}_0^- & \mathcal{F}_0^+ \\ \mathcal{E} \end{pmatrix}
\]

coincide. Therefore, the system $\Theta_{F_+}$ in (2.4) can be reduced to an $F_+$-minimal system of the same form such that the corresponding transfer functions coincide. This completes the proof of the theorem. □

The definition of minimality can be extended to $\Delta_+$-systems in the same manner. Moreover, an $F_+$-system of the form (1.14)

\[
\begin{align*}
(A - z F_+) x &= KJ \varphi, \\
\varphi_+ &= \varphi_- - 2iK^* x,
\end{align*}
\]

and a $\Delta_+$-system of the form (1.12)

\[
\begin{align*}
(A_R - z F_+) x &= K \varphi, \\
\varphi_+ &= K^* x,
\end{align*}
\]

where $A_R$ is the real part of $A$, are minimal (or non-minimal) simultaneously.

For the $\Delta_+$-systems constructed in Section 3 the minimality can be characterized as follows.

**Theorem 5.2.** The realization of the matrix-valued Herglotz-Nevanlinna function $V(z)$ constructed in Theorem 3.5 is minimal if and only if the symmetric part $A_2$ of $\mathcal{A}_R^{(2)}$ defined by (3.26) is prime.

**Proof.** Assume that the system constructed in Theorem 3.5 is not minimal. Let $\mathcal{Y}_1$ (with $\mathcal{Y}_1 \subset \mathcal{Y}_2$) be a reducing invariant subspace from Theorem 3.1 on which $A$ generates a self-adjoint operator $A_1$. Then $\mathcal{D} = \mathcal{D}_0 \oplus A_1$ and it follows from the block representations of $\mathcal{D}$ and $F_+$ in (3.26) that $\mathcal{Y}_1$ is necessarily a subspace of $\mathcal{Y}_2$ in (3.26) while $\mathcal{Y}_1^+$ is a subspace of $\mathcal{Y}_2^+$. To see this let us describe $\mathcal{D}_+$ first. According to (3.26), $\mathcal{Y}_1^+ \subset \mathcal{Y}_1 \subset \mathcal{Y}_- = \mathcal{E}^i \oplus \mathcal{Y}_{1+2} \subset \mathcal{E}^i \oplus \mathcal{Y}_2 \subset \mathcal{E}^i \oplus \mathcal{Y}_{1-2}$ and hence every vector $x \in \mathcal{Y}_1^+$ can be written as

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \text{ where } x_1, x_2, x_3, x_4 \in \mathcal{E}, x_5 \in \mathcal{Y}_{1+2}.
\]

By (3.26),

\[
F_+ x = \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ x_5 \end{pmatrix}, \text{ and } \mathcal{D}(F_+ x) = \begin{pmatrix} i x_2 \\ 0 \\ 0 \\ 0 \\ \mathcal{A}_R^{(2)} x_5 \end{pmatrix}.
\]

This means that $x \in \mathcal{Y}_1^+ \subset \mathcal{D}_+$ only if $x_2 = 0$. Therefore the only possibility for a reducing invariant subspace $\mathcal{Y}_1$ is to be a subspace of $\mathcal{Y}_2$ while $\mathcal{Y}_1^+$ is a subspace of $\mathcal{Y}_{1+2}$. This proves the claim $\mathcal{Y}_1^+ \subset \mathcal{Y}_{1+2}$. Consequently, $\mathcal{Y}_1$ is a reducing invariant subspace for the symmetric operator $A_2$, in which case the operator $A_2$ is not prime.

Conversely, if the symmetric operator $A_2$ is not prime, then a reducing invariant subspace on which $A_2$ generates a self-adjoint operator is automatically a reducing
invariant subspace for the operator $A$ which belongs to $\text{ran } F_+$. This completes the proof.

Finally, Theorem 5.2 implies that a realization of an arbitrary matrix-valued Herglotz-Nevanlinna function in Theorem 5.5 can be provided by a minimal $\Delta_+$-system.

6. Examples

The paper will be concluded with some simple illustrations of the main realization result.

Example 1. Consider the following Herglotz-Nevanlinna function

\begin{equation}
V(z) = 1 + z - i \tanh \left( \frac{i}{2} z l \right), \quad z \in \mathbb{C} \setminus \mathbb{R},
\end{equation}

where $l > 0$. An explicit $F_+$-system $\Theta_{F_+}$ will be constructed so that $V(z) \equiv i\{W_{\Theta_{F_+}}(z) + I\}^{-1}\{W_{\Theta_{F_+}}(z) - I\} = V_{\Theta_{F_+}}(z)$. Let the differential operator $T_2$ in $\mathcal{H}_2 = L^2_{[0,l]}$ be given by

$$T_2x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom } T_2 = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2, x(0) = 0 \},$$

with adjoint

$$T_2^* x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom } T_2^* = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2, x(l) = 0 \}.$$

Let $A_2$ be the symmetric operator defined by

\begin{equation}
A_2x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom } A_2 = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2, x(0) = x(l) = 0 \},
\end{equation}

with adjoint

$$A_2^* x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom } A_2^* = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2 \}.$$ 

Then $\mathcal{H}_+ = \text{dom } A_2^* = W^1_2$ is a Sobolev space with the scalar product

$$(x, y)_+ = \int_0^l x(t)\overline{y(t)} \, dt + \int_0^l x'(t)\overline{y'(t)} \, dt.$$

Now consider the rigged Hilbert space

$$W^1_2 \subset L^2_{[0,l]} \subset (W^1_2)_-,$$

and the operators

$$A_2^* x = \frac{1}{i} \frac{dx}{dt} + ix(0) [\delta(x - l) - \delta(x)],$$

$$A_2^* x = \frac{1}{i} \frac{dx}{dt} + ix(l) [\delta(x - l) - \delta(x)],$$

where $x(t) \in W^1_2$ and $\delta(x)$, $\delta(x - l)$ are delta-functions in $(W^1_2)_-$. Define the operator $K_2$ by

$$K_2 c = c \cdot \frac{1}{\sqrt{2}} [\delta(x - l) - \delta(x)], \quad c \in \mathbb{C}^1,$$

so that

$$K_2^* x = \left( x, \frac{1}{\sqrt{2}} [\delta(x - l) - \delta(x)] \right) = \frac{1}{\sqrt{2}} [x(l) - x(0)],$$
for \( x(t) \in W_2^1 \).

Let \( D_1 = K_1 Q_1^{-1} K_1^* = 1 \), where \( Q = 1 \), and \( K_1 = 1, K_1 : \mathbb{C} \rightarrow \mathbb{C} \). Following [3,25] define

\[
\mathcal{H}_3 = \mathbb{C} \oplus \mathbb{C}, \quad K_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad F_{+,3} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.
\]

Now the corresponding \( F_+ \)-system can be constructed. According to (3.20) one has

\[
\mathbb{D} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbb{K}_{2,R} \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} 1 \\ 1 \\ 1 \\ K_2 \end{pmatrix},
\]

and it follows from [4.17 - 4.18] that

\[
\mathbb{A} = \mathbb{D} + iKK^* = \begin{pmatrix} i & 2i & iK_2^* \\ 0 & i & iK_2^* \\ i & i & 1+i \\ iK_2 & iK_2 & iK_2 \end{pmatrix}.
\]

Consequently, the corresponding \( F_+ \)-system is given by

\[
\Theta_{F_+} = \begin{pmatrix} \mathbb{A} & K \\ F_+ & I \end{pmatrix},
\]

where \( \mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \) and all the operators are described above. It is well known (see for example [24]) that the symmetric operator \( A_2 \) defined in (6.2) does not have nontrivial invariant subspaces on which it induces self-adjoint operators. Thus, the \( F_+ \)-system in (6.3) is an \( F_+ \)-minimal realization of the function \( V(z) \) in (6.1), cf. Section 5. The transfer function of this system is

\[
W_{\Theta_{F_+}}(z) = \frac{2 - i(1 + e^{iz})(z + 1)}{2e^{iz} + i(1 + e^{iz})(z + 1)} = \frac{1 - iz - \tanh \left( \frac{iz}{2} \right)}{1 + iz + \tanh \left( \frac{iz}{2} \right)}.
\]

**Example 2.** Consider the following Herglotz-Nevanlinna function

\[
V(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left( -i \tanh (\pi iz) \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{1-i} \end{pmatrix} .
\right.
\]

An explicit \( F_+ \)-system \( \Theta_{F_+} \) will be constructed so that \( V(z) = i[\Theta_{F_+}(z) + I]^{-1}[\Theta_{F_+}(z) - I] = V_{\Theta_{F_+}}(z) \). Let \( T_{21} \) be a differential operator \( \mathcal{H}_2 = L^2_{[0,2\pi]} \) given by

\[
T_{21}x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom} T_{21} = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2, x(0) = 0 \},
\]

with adjoint

\[
T_{21}^*x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom} T_{21}^* = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2, x(2\pi) = 0 \}.
\]

Let \( A_{21} \) be the symmetric operator defined by

\[
A_{21}x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom} A_{21} = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2, x(0) = x(2\pi) = 0 \},
\]

with adjoint

\[
A_{21}^*x = \frac{1}{i} \frac{dx}{dt}, \quad \text{dom} A_{21}^* = \{ x(t) \in \mathcal{H}_2 : x'(t) \in \mathcal{H}_2 \}.
\]
Then $\delta_+ = \text{dom} A_{21}^* = W_2^1$ is a Sobolev space with the scalar product

$$(x, y)_+ = \int_0^{2\pi} x(t)\overline{y(t)} \, dt + \int_0^{2\pi} x'(t)\overline{y'(t)} \, dt.$$  

Consider the rigged Hilbert space

$$W_2^1 \subset L^2_{[0,2\pi]} \subset (W_2^1)_-,$$

and the operators

$$\mathcal{A}_{21} x = \frac{1}{i} \frac{dx}{dt} + i x(0) [\delta(x - 2\pi) - \delta(x)],$$

$$\mathcal{A}_{21}^* x = \frac{1}{i} \frac{dx}{dt} + i x(2\pi) [\delta(x - 2\pi) - \delta(x)],$$

where $x(t) \in W_2^1$ and $\delta(x), \delta(x - 2\pi)$ are delta-functions in $(W_2^1)_-$. Define the operator $K_{21}$ by

$$K_{21} c = c \cdot \frac{1}{\sqrt{2}} [\delta(x - 2\pi) - \delta(x)], \quad c \in \mathbb{C}^1,$$

so that

$$K_{21}^* x = \left( x, \frac{1}{\sqrt{2}} [\delta(x - 2\pi) - \delta(x)] \right) = \frac{1}{\sqrt{2}} [x(2\pi) - x(0)],$$

for $x(t) \in W_2^1$. Define

$$(6.6) \quad T_{22} = \begin{pmatrix} i & i \\ -i & 1 \end{pmatrix} \quad \text{and} \quad K_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and set

$$(6.7) \quad \mathcal{A}_2 = \begin{pmatrix} \mathcal{A}_{21} & 0 \\ 0 & T_{22} \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} K_{21} & 0 \\ 0 & K_{22} \end{pmatrix}. $$

Now let $D_1 = K_1 Q^{-1} K_1^* = I_2$, where $Q = I_2$, and $K_1 = I_2$, $K_1 : \mathbb{C}^2 \to \mathbb{C}^2$. Following (3.21) define

$$\delta_3 = \mathbb{C}^4, \quad K_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad F_{+,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Now the corresponding $F_+$-system will be constructed. According to (3.20) one has

$$(6.8) \quad \mathbb{D} = \begin{pmatrix} D_3 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & D_1 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \mathcal{A}_{2,R} \\ 0 & \vdots & 0 & I \end{pmatrix}, \quad \mathbb{F}_+ = \begin{pmatrix} F_{+,3} & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \vdots & 0 & I \end{pmatrix}, \quad K = \begin{pmatrix} K_3 \\ \vdots \\ K_1 \\ \vdots \end{pmatrix}$$

and it follows from (3.17) - (3.18) that

$$(6.9) \quad \mathbb{A} = \mathbb{D} + i KK^*.$$  

Consequently, the corresponding $F_+$-system is given by

$$(6.10) \quad \Theta_{F_+} = \begin{pmatrix} \mathbb{C}^6 \oplus W_2^1 \subset \mathbb{C}^6 \oplus L^2_{[0,2\pi]} \subset \mathbb{C}^6 \oplus (W_2^1)_- & K & F_+ & I \end{pmatrix},$$
where all the operators are described above. The transfer function of this system is given by
\[
W_{\Theta F}(z) = \begin{pmatrix}
\frac{1-i-zi-\text{tanh}(\pi iz)}{1+i+zi+\text{tanh}(\pi iz)} & 0 \\
0 & \frac{z^3+i-3+i}{z^3+i+3-i}
\end{pmatrix}.
\]

It is easy to see that the maximal symmetric part of the operator $T_{22}$ in (6.6) is a non-densely defined operator
\[
A_{22} = \begin{pmatrix}
0 & i \\
-i & 1
\end{pmatrix}, \quad \text{dom } A_{22} = \left\{\begin{pmatrix} 0 \\ c \end{pmatrix} : c \in \mathbb{C} \right\}.
\]

Consequently, the symmetric operator $A_2$ defined by $A_2$ in (6.7), $D$ in (6.8), and $A$ in (6.9) is given by
\[
A_2 = \begin{pmatrix}
\frac{1}{\sqrt{i}} & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & -i & 0 & 1
\end{pmatrix},
\]
\[
\text{dom } A_2 = \left\{\begin{pmatrix} x(t) \\ 0 \\ c \end{pmatrix} : x(t), x'(t) \in \mathcal{H}_2, x(0) = x(2\pi) = 0, c \in \mathbb{C} \right\}.
\]

Hence, this operator $A_2$ does not have nontrivial invariant subspaces on which it induces self-adjoint operators. Thus, $F_+$-system in (6.10) is an $F_+$-minimal realization of the function $V(z)$ in (6.11), cf. Section 5.

References

[1] D. Alpay, A. Dijksma, J. Rovnyak, and H.S.V. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Oper. Theory Adv. Appl., 96, Birkhäuser Verlag, Basel, 1997.
[2] D. Alpay and E.R. Tsekanovskii, "Interpolation theory in sectorial Stieltjes classes and explicit system solutions", Lin. Alg. Appl., 314 (2000), 91–136.
[3] D. Alpay and E.R. Tsekanovskii, "Subclasses of Herglotz-Nevanlinna matrix-valued functions and linear systems", *Dynamical systems and differential equations*, (ed. J. Du and S. Hu), An added volume to *Discrete and continuous dynamical systems*, (2001), 1–14.
[4] Yu.M. Arlinski˘ı, "On inverse problem of the theory of characteristic functions of unbounded operator colligations", Dopovidi Akad. Nauk Ukrain. RSR 2, Ser. A, 105–109.
[5] Yu.M. Arlinski˘ı and E.R. Tsekanovskii, "Linear systems with Schrödinger operators and their transfer functions", Oper. Theory Adv. Appl., 149, 2004, 47–77.
[6] Yu.M. Arlinski˘ı and E.R. Tsekanovskii, "Constant J-unitary factor and operator-valued transfer functions", *Dynamical Systems and Differential Equations*, (ed. W.Feng, S.Hu and X.Lu), A supplemental volume to *Discrete and continuous dynamical systems*, (2003), 48–56.
[7] D. Arov, "Passive linear systems and scattering theory", in *Dynamical Systems, Control Coding, Computer Vision*, vol.25 of *Progress in Systems and Control Theory*, (1999), Birkhäuser Verlag, 27–44.
[8] D. Arov, "Darlington realization of matrix-valued functions", Math. USSR Izvestija, 7 (1973), 1295–1326.
[9] D. Arov and H. Dym, "J-inner matrix-functions, interpolation and inverse problems for canonical systems III.More on the inverse monodromy problem", Integr. Equat. Oper. Th., 36 (2000), 127–181.
[10] D. Arov and L.Z. Grossman, "Scattering matrices in the theory of unitary extensions of isometric operators", Math. Nachr., 157, (1992), 105–123.
[11] D. Arov and M.A. Nudelman, "Passive linear stationary dynamical scattering systems with continuous time", Integral Equat. Oper. Th., 24 (1996), 1–45.
[12] J.A. Ball, "Linear systems, operator model theory and scattering multivariable generalizations", *Operator theory: Advances and Applications*, (Winnipeg, MB, 1998), pp. 151–178, Fields Inst. Commun. 25, Amer. Math. Soc. Providence, RI, 2000.
[13] J.A. Ball and N. Cohen, “de Branges-Rovnyak operator models and systems theory: a survey”, Oper. Theory Adv. Appl., 50 (1991), 93–136.
[14] J.A. Ball, I. Gohberg, and L. Rodman, “Realization and interpolation of rational matrix functions”, Oper. Theory Adv. Appl., 33 (1988), 1–72.
[15] J.A. Ball, I. Gohberg, and L. Rodman, Interpolation of rational matrix functions, Vol. 45, Oper. Theory Adv. Appl., Birkhäuser, 1990.
[16] J.A. Ball and O.J. Staffans, “Conservative state-space realizations of dissipative system behaviors”, Report No. 37, Institute Mittag-Leffler, (2002/2003), 55 pp.
[17] J.A. Ball and O.J. Staffans, “Conservative state-space realizations of dissipative system behaviors”, Integr. Equ. Oper. Theory (Online), Birkhäuser, 2005, DOI 10.1007/s00020-003-1356-3.
[18] H. Bart, I. Gohberg, and M.A. Kaashoek, Minimal factorization of matrix and operator functions, Oper. Theory Adv. Appl., 1, Birkhäuser Verlag, Basel, 1979.
[19] S.V. Belyi and E.R. Tsekanovski˘ı, "Realization theorems for operator-valued R-functions", Oper. Theory Adv. Appl., 98 (1997), 55–91.
[20] S.V. Belyi and E.R. Tsekanovski˘ı, "On classes of realizable operator-valued R-functions", Oper. Theory Adv. Appl., 115 (2000), 85–112.
[21] S.V. Belyi, S. Hassi, H.S.V. de Snoo, and E.R. Tsekanovski˘ı, “On the realization of inverse of Stieltjes functions”, Proceedings of MTNS-2002, University of Notre Dame, CD-ROM, 11p., 2002.
[22] Yu.M. Berezanski˘ı, Expansion in eigenfunctions of self-adjoint operators, vol. 17, Transl. Math. Monographs, AMS, Providence, 1968.
[23] L. de Branges and J. Rovnyak, "Canonical models in quanum scattering theory", in Perturbation theory and its applications in quantum mechanics, Wiley & Sons, New York-London-Sydney, 1966.
[24] M.S. Brodski˘ı, Triangular and Jordan representations of linear operators, Moscow, Nauka, 1969 (Russian).
[25] M.S. Brodski˘ı and M.S. Livˇ sic, "Spectral analysis of non-selfadjoint operators and intermediate systems", Uspekhi Matem. Nauk, 23, no. 1, 79, (1958), 3–84 (Russian).
[26] R.F. Curtain and H. Zwart, An introduction to infinite-dimensional linear systems theory, Springer-Verlag, New York, 1995.
[27] I. Dovzenko and E.R. Tsekanovski˘ı, "Classes of Stieltjes operator-functions and their conservative realizations", Dokl. Akad. Nauk SSSR, 311 no. 1 (1990), 18–22.
[28] P.A. Fuhrmann, Linear systems and operators in Hilbert space, McGraw-Hill, New York, 1981.
[29] F. Gesztesy, N.J. Kalton, K.A. Makarov, and E.R. Tsekanovski˘ı, "Some applications of operator-valued Herglotz functions", Oper. Theory Adv. Appl., 123 (2001), 271–321.
[30] F. Gesztesy and E.R. Tsekanovski˘ı, "On matrix-valued Herglotz functions", Math. Nachr., 218 (2000), 61–138.
[31] S. Hassi, H.S.V. de Snoo, and E.R. Tsekanovski˘ı, "An addendum to the multiplication and factorization theorems of Brodskii-Livšic-Potapov", Applicable Analysis, 77 (2001), 125–133.
[32] S. Hassi, H.S.V. de Snoo, and E.R. Tsekanovski˘ı, "Commutative and noncommutative representations of matrix-valued Herglotz-Nevanlinna functions", Applicable Analysis, 77 (2001), 135–147.
[33] S. Hassi, H.S.V. de Snoo, and E.R. Tsekanovski˘ı, "Realizations of Herglotz-Nevanlinna functions via F-colligations", Oper. Theory Adv. Appl., 132 (2002), 183–198.
[34] S. Hassi, H.S.V. de Snoo, and E.R. Tsekanovski˘ı, "The realization problem for Herglotz-Nevanlinna functions", in Unsolved problems in mathematical systems and control theory, (ed. V. Blondel and A. Megretski), Princeton University Press, 2004, 8–13.
[35] J.W. Helton, "Systems with infinite-dimensional state space: the Hilbert space approach", Proc. IEEE, 64 (1976), no. 1, 145–160.
[36] M. Kuijper, First-order representations of linear systems, Birkhäuser-Verlag, Basel-Boston, 1994.
[37] M.S. Livšic, "On spectral decomposition of linear non-selfadjoint operators", Math. Sbornik., 30, no.76 (1954), 145–198 (Russian).
[38] M.S. Livšic, Operators, oscillations, waves, Moscow, Nauka, 1966 (Russian).
[39] M.S. Livšic and V.P. Potapov, "A theorem on the multiplication of characteristic matrix-functions", Dokl. Akad. Nauk SSSR, 72 (1950), 625–628 (Russian).
[40] M.S. Livšic and A.A. Yantsevich, *Operator colligations in Hilbert spaces*, Kharkov University Press, 1971 (Russian) [English translation: V.H. Winston & Sons, 1979]

[41] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space*, North-Holland, New York, 1970.

[42] J.W. Polderman and J.C. Willems, *Introduction to mathematical system theory: a behavioural approach*, Springer, 1998.

[43] A. Rutkas and N. Radbel, "Linear operator pencils and noncanonical systems", Teor. Funk. Anal. i Prilozhen., 17 (1973), 3–14, (Russian).

[44] A. Rutkas, “Characteristic function and a model of a linear operator pencil”, Teor. Funk. Anal. i Prilozhen., 45 (1986), 98–111 (Russian) [English Transl., J.Soviet Math., 48 (1990), 451–464].

[45] D. Salamon, "Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach", Trans. Amer. Math. Soc., 300 (1987), 383–431.

[46] O.J. Staffans, *Well-posed linear systems: Part I*, Book manuscript, available at [http://www.abo.fi/~staffans/](http://www.abo.fi/~staffans/) 2002

[47] O.J. Staffans, “Passive and conservative continuous time impedance and scattering systems, Part I: Well posed systems”, Math. Control Signals Systems, 15, (2002), 291–315.

[48] O.J. Staffans, Passive and conservative infinite-dimensional impedance and scattering systems (from a personal point of view), in Mathematical Systems Theory in Biology, Communication, Computation, and Finance, J. Rosenthal and D. S. Gilliam, eds, IMA Volumes in Mathematics and its Applications 134, Springer-Verlag, New York, 2002, pp. 375-413.

[49] O.J. Staffans and G. Weiss, “Transfer functions of regular linear systems, Part II: the system operator and the Lax-Phillips semigroup”, Trans. Amer. Math. Soc., 354, (2002), 3229–3262.

[50] E.R. Tsekanovski˘ı, "Accretive extensions and problems on Stieltjes operator-valued functions realizations", Oper. Theory Adv. Appl., 59 (1992), 328–347.

[51] E.R. Tsekanovski˘ı and Yu.L. Shmul’yan, "The theory of bi-extensions of operators on rigged Hilbert spaces. Unbounded operator colligations and characteristic functions", Russ. Math. Surv., 32 (1977), 73–131.

[52] G. Weiss, "Transfer functions of regular linear systems. Part I: characterizations of regularity", Trans. Amer. Math. Soc., 342 (1994), 827–854.