ABSOLUTELY GRADED FLOER HOMOLOGIES AND INTERSECTION FORMS FOR FOUR-MANIFOLDS WITH BOUNDARY

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Abstract. In [22], we introduced absolute gradings on the three-manifold invariants developed in [21] and [20]. Coupled with the surgery long exact sequences, we obtain a number of three- and four-dimensional applications of this absolute grading including strengthenings of the “complexity bounds” derived in [20], restrictions on knots whose surgeries give rise to lens spaces, and calculations of $\text{HF}^+$ for a variety of three-manifolds. Moreover, we show how the structure of $\text{HF}^+$ constrains the exoticness of definite intersection forms for smooth four-manifolds which bound a given three-manifold. In addition to these new applications, the techniques also provide alternate proofs of Donaldson’s diagonalizability theorem and the Thom conjecture for $\mathbb{CP}^2$.

1. Introduction

In [21] and [20], we introduced Floer-homology invariants ($\widehat{HF}$, $HF^\pm$, and $HF^\infty$) for oriented three-manifolds equipped with Spin$^c$ structures. In [20], we derived a number of exact sequences relating $HF^+$ (and also $\widehat{HF}$) for three-manifolds which differ by surgeries. In [22], we introduced the corresponding four-dimensional theory, where cobordisms $W$ between three-manifolds induce maps $\widehat{F}$, $F^\pm$, $F^\infty$ between the corresponding Floer homology theories on the boundaries. One by-product of this four-dimensional theory is an absolute rational lift to the relative $\mathbb{Z}$ grading on the homology groups for a three-manifold $Y$ endowed with a torsion Spin$^c$ structure (i.e. one whose first Chern class is a torsion element of $H^2(Y; \mathbb{Z})$).

The present article presents a number of three- and four-dimensional applications of the interplay between the absolute gradings and the surgery exact sequences.

1.1. The correction term $d(Y)$. After two introductory sections (Section 2 where we review some of the terminology from the earlier articles, and Section 3, where we pin down the relationship between the absolute gradings and the exact sequence), we give the first application of the absolute gradings: a “correction term” for three-manifolds. For simplicity, we presently assume that $Y$ is a rational homology three-sphere, equipped with a Spin$^c$ structure $t$. With the help of the absolute grading, we define a numerical invariant $d(Y, t)$ for $Y$, which is the minimal degree of any non-torsion class in $HF^+(Y, t)$.
coming from $HF^\infty(Y, t)$. This invariant is analogous to a gauge-theoretic invariant of Frøyshov (see [12]).

The basic properties of $d$ are conveniently stated with the help of a result which we state after introducing some terminology.

**Definition 1.1.** The three-dimensional Spin$^c$ homology bordism group $\theta^c$ is the set of equivalence classes of pairs $(Y, t)$ where $Y$ is a rational homology three-sphere, and $t$ is a Spin$^c$ structure over $Y$, and the equivalence relation identifies $(Y_1, t_1) \sim (Y_2, t_2)$ if there is a (connected, oriented, smooth) cobordism $W$ from $Y_1$ to $Y_2$ with $H_i(W, \mathbb{Q}) = 0$ for $i = 1$ and $2$, which can be endowed with a Spin$^c$ structure $s$ whose restrictions to $Y_1$ and $Y_2$ are $t_1$ and $t_2$ respectively. The connected sum operation endows this set with the structure of an Abelian group (whose unit is $S^3$ endowed with its unique Spin$^c$ structure).

There is a classical homomorphism

$$\rho: \theta^c \rightarrow \mathbb{Q}/\mathbb{Z}$$

(see for instance [1]), defined as follows. Consider a rational homology three-sphere $(Y, t)$, and let $X$ be any four-manifold equipped with a Spin$^c$ structure $s$ with $\partial X \cong Y$ and $s|\partial X \cong t$. Then

$$\rho(Y, t) \equiv \frac{c_1(s)^2 - \text{sgn}(X)}{4} \pmod{2\mathbb{Z}}$$

where $\text{sgn}(X)$ denotes the signature of the intersection form of $X$.

**Theorem 1.2.** The numerical invariant $d(Y, t)$ descends to give a group homomorphism

$$d: \theta^c \rightarrow \mathbb{Q}$$

which is a lift of $\rho$. Moreover, $d$ is invariant under conjugation; i.e. $d(Y, t) = d(Y, \overline{t})$.

Additivity of the correction term under connected sums, and its behaviour under orientation reversal, is established Section 4. Its rational homology cobordism invariance is established only after more of the four-dimensional theory is developed in Section 9. When $b_1(Y) > 0$, torsion Spin$^c$ structures can be endowed with a collection of correction terms. In Subsection 4.2, we discuss this construction in the case where $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$.

### 1.2. Regularized Euler characteristics

When $Y$ is an integral homology three-sphere, $HF^+(Y)$ is infinitely generated. (We drop the Spin$^c$ structure from the notation, since it is unique.) To obtain a finitely generated group, we must pass to the quotient

$$HF^+_{\text{red}}(Y) = HF^+(Y)/HF^\infty(Y).$$

We have the following relationship between the Euler characteristic of $HF^+_{\text{red}}(Y)$, the correction term for $Y$, and Casson’s invariant $\lambda(Y)$,
Theorem 1.3. Let $Y$ be an integer homology three-sphere. Then, the following relationship holds between Casson’s invariant, the Euler characteristic of $HF^+$ (thought of as a $\mathbb{Z}$-module), and the correction term:

$$\lambda(Y) = \chi(HF^+_{\text{red}}(Y)) - \frac{1}{2}d(Y),$$

where here Casson’s invariant is normalized so that, if $\Sigma(2, 3, 5)$ denotes the Poincaré homology sphere, oriented as the boundary of the negative-definite $E_8$ plumbing, then $\lambda(\Sigma(2, 3, 5)) = -1$.

1.3. Invisible three-manifolds. When $Y$ is an integer homology three-sphere, we define its complexity $N(Y)$ to be the rank of $HF_{\text{red}}(Y)$. Let $K \subset Y$ be a knot in $Y$, $r$ be a rational number, and let $Y_r(K)$ be the integer homology three-sphere obtained by $r$ surgery on $Y$ along $K$. In Theorem 1.8 of [20], we gave lower bounds on the sum of complexities of $Y$ and $Y_{1/n}(K)$ (where $n$ is a whole number), in terms of the Alexander polynomial of $K$. Using $d(Y)$ in conjunction with $N(Y)$, these bounds can be strengthened, according to the following:

Theorem 1.4. Let $Y$ be an integral homology three-sphere and $K \subset Y$ be a knot, whose symmetrized Alexander polynomial is

$$\Delta_K(T) = a_0(K) + \sum_{i=1}^{d} a_i(K) \cdot (T^i + T^{-i}),$$

then for each positive integer $n$, there is a bound

$$n \left( |t_0(K)| + 2 \sum_{i=1}^{d} |t_i(K)| \right) \leq N(Y) + \frac{d(Y)}{2} + N(Y_{1/n}(K)) - \frac{d(Y_{1/n}(K))}{2},$$

where

$$t_i(K) = \sum_{j=1}^{d} j a_{|i|+j}(K).$$

An integral homology three-sphere $Y$ for which both $N(Y) = 0 = d(Y)$ is called invisible, since these invariants do not distinguish it from $S^3$. Of course (in view of Theorem 1.3), all invisible homology three-spheres have $\lambda(Y) = 0$. We prove in Section 6 that the set of invisible three-manifolds $Y$ is closed under connected sum and orientation reversal.

Theorem 1.4 has the following immediate consequence:

Corollary 1.5. If $K \subset Y$ is a knot with non-trivial Alexander polynomial in an invisible three-manifold, then any non-trivial surgery on $K$ gives a three-manifold which is not invisible.
1.4. **Surgeries giving lens spaces.** The graded Floer homologies can also be used to place restrictions on knots $K \subset S^3$ whose surgeries give lens spaces. For example, we have the following:

**Theorem 1.6.** If $K \subset S^3$ is a knot with the property that some integral surgery along $K$ with integral coefficient $p$ with $|p| \leq 4$ gives a lens space, then $HF^+(Y_0(K)) \cong HF^+(S^2 \times S^1)$. In particular, the Alexander polynomial of $K$ is trivial.

**Remark 1.7.** The “cyclic surgery theorem” of Culler-Gordon-Luecke-Shalen [5] ensures that if $K$ is not a torus knot, and some surgery along $K$ gives a lens space, then that surgery is integral. It is conjectured in [14] that the trivial knot is the only knot $K$ with the property that some surgery on $K$ gives a lens space $L(p, 1)$ with $|p| \leq 4$.

A $+5$-surgery on the right-handed trefoil gives the lens space $-L(5, 1)$, so care must be taken in generalizing Theorem 1.6. The following result holds for general $p$:

**Theorem 1.8.** Suppose that $K \subset S^3$ is a knot with the property that $+p$ surgery on $K$ gives the lens space $L(p, 1)$ (with its canonical orientation). Then, $HF^\infty(S^3_0(K)) \cong HF^\infty(S^2 \times S^1)$ as absolutely graded groups; in particular, the Alexander polynomial of $K$ is trivial.

In fact, Gordon has conjectured (see [13]) that if $+p$ surgery on a knot $K \subset S^3$ gives $L(p, 1)$, then $K$ is the unknot. Moreover, Berge [3] has conjectured a complete list of knots which give rise to lens spaces.

Theorems 1.6 and 1.8 are proved in Section 7, where they are given as corollaries to a theorem, Theorem 7.2, which, for any fixed $p$ and $q$, gives strong restrictions on knots $K \subset S^3$ for which $+p$ surgery gives the lens space $L(p, q)$. Specifically, if $K$ is such a knot, the theorem shows how $HF^+$ of the zero-surgery (and hence the Alexander polynomial of the knot) is determined, up to a finite indeterminacy, by the correction terms for $L(p, q)$. The indeterminacy comes from certain allowed permutations of these correction terms. The terms themselves are calculated in Proposition 4.8 (see [24] for a more conceptual interpretation of these numbers). For another illustration of these constraints, we tabulate at the end of this paper (c.f Subsection 10.3) the possible Alexander polynomials of knots in $S^3$ for which some positive integral surgery $p \leq 26$ gives a lens space.

Turned around, Theorem 7.2 also gives obstructions to realizing lens spaces as integral surgeries on a knot in the three-sphere. For example, these techniques show that the lens space $L(22, 3)$ – and indeed, any lens spaces in the family $L(2k(3 + 8k), 2k + 1)$ where $k$ is any positive integer not divisible by 4 – is not obtained by integral surgery on a single knot $K \subset S^3$. This obstruction is “$S^3$-specific”. An argument of Fintushel and Stern gives a necessary and sufficient condition for $L(p, q)$ to be realized as integral surgery on a knot in a homology sphere: the condition is that $\pm q$ be a square modulo $p$, a condition which all the lens spaces in this family satisfy. (This application is described in Proposition 7.9.)
1.5. **Some calculations.** We give several calculations of $N(Y)$ and $d(Y)$, including calculations of both invariants for integer homology spheres obtained as surgeries on the $(p, q)$ torus knot. Indeed, we calculate all of $HF^+$ for the Brieskorn spheres $\Sigma(2, 3, 6n \pm 1)$. Moreover, we calculate $HF^+(E_n)$ where $E_n$ is the three-manifold obtained by $1/n$ surgery with $n > 0$ on the figure eight knot in $S^3$ (c.f. Proposition 8.3):

$$HF^+_k(E_n) \cong \begin{cases} 
\mathbb{Z} & \text{if } k \equiv 0 \pmod{2} \text{ and } k \geq 0 \\
\mathbb{Z}^n & \text{if } k = -1 \\
0 & \text{otherwise}
\end{cases}.$$

We also give the following simple calculation for the invariants of $T^3$.

**Proposition 1.9.** Let $T^3$ denote the three-dimensional torus. Then, we have $H_1(T^3; \mathbb{Z})$-module isomorphisms:

$$\widehat{HF}(T^3) \cong H^2(T^3; \mathbb{Z}) \oplus H^1(T^3; \mathbb{Z}),$$

$$HF^+(T^3) \cong \left( H^2(T^3; \mathbb{Z}) \oplus H^1(T^3; \mathbb{Z}) \right) \otimes \mathbb{Z}[U^{-1}],$$

$$HF^\infty(T^3) \cong \left( H^2(T^3; \mathbb{Z}) \oplus H^1(T^3; \mathbb{Z}) \right) \otimes \mathbb{Z}[U, U^{-1}].$$

The absolute grading is symmetric, in the sense that $\widehat{gr}(H^2(T^3; \mathbb{Z}) \subset \widehat{HF}(T^3)) = 1/2$, $\widehat{gr}(H^1(T^3; \mathbb{Z}) \subset \widehat{HF}(T^3)) = -1/2$.

Further calculations are given for surgeries on certain pretzel knots, via surgery long exact sequences which mirror the skein relations for calculating Alexander polynomials.

1.6. **Intersection forms of definite four-manifolds.** By analyzing the maps $F^\infty_{W,s}$ for negative-definite cobordisms, we give an alternative proof of Donaldson’s diagonalizability theorem:

**Theorem 1.10.** (Donaldson) If $X$ is a smooth, closed, oriented four-manifold with definite intersection form, then the form of $X$ is diagonalizable over $\mathbb{Z}$.

Indeed, applying the same idea to four-manifolds which bound a given integer homology three-sphere $Y$, we obtain the following analogue of a theorem of Frøyshov.

Let $Y$ be an integral homology three-sphere, and $X$ be a four-manifold which bounds $Y$. There is an intersection form

$$Q_X : (H_2(X; \mathbb{Z})/\text{Tors}) \otimes (H_2(X; \mathbb{Z})/\text{Tors}) \longrightarrow \mathbb{Z}.$$  

A characteristic vector for this intersection form is an element $\xi \in H_2(X; \mathbb{Z})/\text{Tors}$ with

$$\xi \cdot v \equiv v \cdot v \pmod{2}$$

for each $v \in H_2(X; \mathbb{Z})/\text{Tors}$. (We write $\xi \cdot \eta$ for $Q_X(\xi, \eta)$, and $\xi^2$ for $Q_X(\xi, \xi)$.)
Theorem 1.11. Let $Y$ be an integral homology three-sphere, then for each negative-definite four-manifold $X$ which bounds $Y$, we have the inequality:

$$\xi^2 + \text{rk}(H^2(X; \mathbb{Z})) \leq 4d(Y),$$

for each characteristic vector $\xi$.

This theorem has an obvious generalization to rational homology three-spheres, and, indeed, generalizations for three-manifolds with $b_1(Y) > 0$ discussed in Section 9.3. When $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}$, the inequality can be used as obstruction to realizing $Y$ as the boundary of an integral homology $S^2 \times D^2$. Moreover, when applied to certain circle bundles over two-manifolds, the intersection form bounds provide another proof of the Thom conjecture for $\mathbb{C}P^2$ first proved by Kronheimer-Mrowka [16] and Morgan-Szabó-Taubes [19], c.f. Subsection 9.4. The proof given here is analogous to a Seiberg-Witten proof given recently by Strle, see [25].

In Section 10, we close with some other applications of the intersection form bounds, as combined with the calculations from Section 8. Specifically, we show how the results constrain the intersection forms of four-manifolds which bound certain Seifert fibered spaces. In particular, in Subsection 10.2, we exhibit a three-manifold whose first homology is isomorphic to $\mathbb{Z}$ and which can be expressed as surgery on a certain two-component link in $S^3$, but which cannot be expressed as surgery on a single knot. Finally, we close with a table of allowed Alexander polynomials of knots giving rise to (small) lens spaces.

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2. Floer homologies and maps between them

In [22], we showed how the constructions from [21] and [20] can give rise to a naturally associated assignment which associates to a three-manifold \( Y \) equipped with a Spin\( c \) structure \( t \), four relatively \( \mathbb{Z}/c_1(t) \) \( \cup H^1(Y; \mathbb{Z}) \mathbb{Z} \)-graded Abelian groups \( \widehat{HF}, HF^+, HF^-, \) and \( HF^\infty \), the latter three of which are already endowed with the action by a polynomial algebra \( \mathbb{Z}[U] \), where multiplication by \( U \) lowers relative degree by 2. All four groups are acted upon by the exterior algebra \( \Lambda^*H_1(Y; \mathbb{Z})/\text{Tors} \).

In fact, these groups are related by functorially associated assignment which associates to a three-manifold \( Y \) equipped with a Spin\( c \) structure \( t \), four relatively \( \mathbb{Z}/c_1(t) \) \( \cup H^1(Y; \mathbb{Z}) \mathbb{Z} \)-graded Abelian groups \( \widehat{HF}, HF^+, HF^-, \) and \( HF^\infty \), the latter three of which are already endowed with the action by a polynomial algebra \( \mathbb{Z}[U] \), where multiplication by \( U \) lowers relative degree by 2. All four groups are acted upon by the exterior algebra \( \Lambda^*H_1(Y; \mathbb{Z})/\text{Tors} \).

When \( W \) is a (smooth) cobordism from a three-manifold \( Y_1 \) to \( Y_2 \), equipped with a Spin\( c \) structure \( s \) whose restrictions to the two boundary components are \( t_1 \) and \( t_2 \) respectively, then there are induced maps between the long exact sequences:

\[
\begin{align*}
... & \longrightarrow HF^-(Y_1, t_1) \underset{F_{W,s}}{\longrightarrow} HF^\infty(Y_1, t_1) \underset{F_{W,s}^\infty}{\longrightarrow} HF^+(Y_1, t_1) \longrightarrow ...
\end{align*}
\]

and

\[
\begin{align*}
... & \longrightarrow \widehat{HF}(Y_1, t_1) \underset{\hat{F}_{W,s}}{\longrightarrow} HF^+(Y_1, t_1) \underset{\hat{F}_{W,s}^+}{\longrightarrow} HF^+(Y_1, t_1) \longrightarrow ...
\end{align*}
\]

The maps \( \widehat{HF}, F_{W,s}, F_{W,s}^\infty, F_{W,s}^+ \) are uniquely defined up to multiplication by an overall sign \( \pm 1 \).

These maps also admit an action by \( \Lambda^*H_1(W)/\text{Tors} \). This action is related to the analogous actions over the three-manifolds, according to the following formula. Fix arbitrary elements \( \gamma_i \in H_1(Y_i)/\text{Tors} \) for \( i = 1, 2 \), and let \( \gamma \in H_1(W)/\text{Tors} \) be the homology class obtained as \( \gamma = j_1(\gamma_1) - j_2(\gamma_2) \), where for \( i = 1, 2 \), \( j_i \) denotes the natural inclusion \( j_i : H_1(Y_i) \longrightarrow H_1(X) \). Then,

\[
\pm(\gamma \otimes F_{W,s})(\xi) = F_{W,s}(\gamma_1 \cdot \xi) - \gamma_2 \cdot F_{W,s}(\xi).
\]
In particular, if $\gamma_1$ and $\gamma_2$ are homologous in $W$, then
\[ F_{W,s}(\gamma_1 \cdot \xi) = \gamma_2 \cdot F_{W,s}(\xi). \]

Suppose now that $Y$ is equipped with a torsion Spin$^c$ structure. In Theorem 7.1 of [22], we showed that $HF^c(Y, t)$, and also $\hat{HF}(Y, t)$ can be given an absolute $\mathbb{Q}$ grading $\hat{gr}$ which lifts the relative $\mathbb{Z}$ grading. It is uniquely characterized by the following properties:

- $\hat{\iota}$, $\iota$, and $\pi$ above preserve the absolute grading
- $\hat{HF}(S^3)$ is supported in absolute grading zero
- if $W$ is a cobordism from $Y_1$ to $Y_2$, and $\xi \in HF^\infty(Y_1, t_1)$, then

\[ \hat{gr}(F_{W,s}(\xi)) - \hat{gr}(\xi) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}, \]

where $t_i = s|Y_i$ for $i = 1, 2$.

The existence and characterization of this absolute grading is the key result we use here from [22].
3. Exact sequences and absolute gradings

Recall that in [20], surgery long exact sequences for \(HF^+\) and \(\widehat{HF}\) were established. We give now a graded refinement, focusing mainly on the case of \(HF^+\). The key point is the relationship between the maps in the long exact sequences and the maps induced by the cobordisms obtained by surgeries on knots.

More concretely, recall that when \(K\) is a knot in an integral homology three-sphere, Theorem 10.1 of [20] gives a long exact sequence

\[
\cdots \to HF^+(Y) \xrightarrow{F_1} HF^+(Y_0) \xrightarrow{F_2} HF^+(Y_1) \xrightarrow{F_3} HF^+(Y) \to \cdots,
\]

where \(Y_0 = Y_0(K)\) and \(Y_1 = Y_1(K)\) are the three-manifolds obtained by 0-surgery and +1-surgery on \(Y\) along \(K\) (using its canonical framing). Here,

\[
HF^+(Y_0) = \bigoplus_{t \in \text{Spin}^c(Y_0)} HF^+(Y_0, t).
\]

The maps \(F_1\) and \(F_2\) were defined by counting pseudo-holomorphic triangles in a Heegaard triple. An easy comparison between the definition of the maps in [20] and the maps associated to two-handles (Subsection 4.1 of [21]) shows that \(F_1\) and \(F_2\) are sums of maps associated to cobordisms; i.e.

\[
F_1 = \sum_{s \in W_0} \pm F_{W_0,s}, \quad \text{and} \quad F_2 = \sum_{s \in W_1} \pm F_{W_1,s},
\]

where \(W_0\) is the cobordism from \(Y\) to \(Y_0\) defined by attaching a single two-handle to \(Y\) (with 0-framing), and \(W_1\) is the cobordism from \(Y_0\) to \(Y_1\) defined by attaching a single two-handle to \(Y_0\). (The signs are chosen as in the proof of Theorem 10.1 of [20] to make the sequence exact.) Observe that \(\text{Spin}^c\) structures on both \(W_0\) and \(W_1\) are uniquely determined by their restrictions to \(Y_0\).

Note that the map \(F_3\) arises as the coboundary map of an associated long exact sequence, though one could alternatively realize it, too, as a sum of maps belonging to cobordisms (compare [4]). However, for our present purposes, it suffices to understand properties of the maps \(F_1\) and \(F_2\).

**Lemma 3.1.** Let \(K \subset Y\) be a knot in an integral homology three-sphere, and let \(t_0\) denote the \(\text{Spin}^c\) structure over \(Y_0\) with trivial first Chern class. In the exact sequence

\[
\cdots \to HF^+(Y) \xrightarrow{F_1} HF^+(Y_0) \xrightarrow{F_2} HF^+(Y_1) \xrightarrow{F_3} HF^+(Y) \to \cdots,
\]

the component of \(F_1\) mapping into \(HF^+(Y_0, t_0)\) (now thought of as absolutely \(\mathbb{Q}\)-graded) has degree \(-1/2\), the restriction of \(F_2\) to \(HF^+(Y_0, t_0)\) has degree \(-1/2\).

**Proof.** In view of the above remarks, the component of \(F_1\) landing in the \(HF^+(Y_0, t_0)\) summand is the map induced by the cobordism \(W_0\), endowed with the \(\text{Spin}^c\) structure \(s_0\) with trivial first Chern class. This cobordism is obtained by a single one-handle.
addition (so its $\chi = 0$), and its signature $\sigma = 0$. Thus, by Equation (4), the result follows. The map $F_2$ is also obtained by a single two-handle addition, with signature zero.

**Remark 3.2.** Note that the above discussion could have been made using $\widehat{HF}$ in place of $HF^+$, with only notational changes.

When $Y$ is an integral homology three-sphere, we can compare the absolute grading of $HF^\circ(Y)$ as defined in Section 7 of [22] with the absolute $\mathbb{Z}/2\mathbb{Z}$ grading as defined in [20].

Observe first that for an integral homology three-sphere, the absolute grading of Section 7 of [22] is actually a $\mathbb{Z}$ lift (rather than a $\mathbb{Q}$ lift) of the usual relative grading. To see this, let $W$ be a a cobordism containing only two-handles from $S^3$ to $Y$ (used in the definition of the absolute grading), and observe that

$$\frac{c_1(s)^2 - 2\chi(W) - 3\text{sgn}(W)}{4} \equiv \frac{c_1(s)^2 - \text{sgn}(W)}{4}, \quad (\text{mod } \mathbb{Z})$$

(where $\text{sgn}(W)$ is the signature of the intersection form of $W$) which is integral since $c_1(s)$ is a characteristic vector for the definite, unimodular form $H^2(W; \mathbb{Z})$.

Next, recall that in general $HF^\circ(Y, t)$ comes equipped with an absolute $\mathbb{Z}/2\mathbb{Z}$ grading, defined in Subsection 11.4 of [20]. When $Y$ is an integral homology three-sphere, this is the $\mathbb{Z}/2\mathbb{Z}$ grading characterized by the property that

$$\chi(\widehat{HF}(Y)) = 1.$$  

(Since there is only one Spin$^c$ structure in this case, we drop it from the notation.)

An equivalent (but technically more useful) formulation of this grading follows from the calculation of $HF^\infty$ of an integer homology three-sphere, Theorem 11.1 of [20], where it is shown that $HF^\infty(Y) \cong \mathbb{Z}[U, U^{-1}]$. We then define the $\mathbb{Z}/2\mathbb{Z}$ degree so that $HF^\infty(Y)$ is supported in even degree.

**Proposition 3.3.** If $Y$ is an integral homology three-sphere, then the parity of the absolute $\mathbb{Z}$ grading defined above is the absolute $\mathbb{Z}/2\mathbb{Z}$ grading defined in [20].

**Proof.** Both gradings (mod 2) agree on $Y$, if and only if they agree on $Y_1$. This is easily seen from the long exact sequence connecting $HF^+(Y)$, $HF^+(Y_0)$, and $HF^+(Y_1)$, according to which the composite map from $Y$ to $Y_1$ shifts the absolute $\mathbb{Z}$-degree by $-1$ (modulo two); it also shifts the absolute $\mathbb{Z}/2\mathbb{Z}$ degree. The result then follows from the model case $S^3$, together with fact that every integer homology can be constructed from $S^3$ by a sequence of $\pm 1$ surgeries. 

Another case which will interest us is that of three-manifolds $Y_0$ with $H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}$. For such a manifold, the absolute $\mathbb{Z}/2\mathbb{Z}$ grading from [20] is defined so that the image of the action by $H_1(Y_0; \mathbb{Z})$ on $HF^\infty(Y_0)$ has degree zero modulo 2.
Proposition 3.4. Let $Y_0$ be a three-manifold with $H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}$, and let $t_0$ be the Spin$^c$ structure with $c_1(t_0) = 0$. Then the parity of $\tilde{gr} + \frac{1}{2}$ is the absolute $\mathbb{Z}/2\mathbb{Z}$ grading given above.

Proof. By surgering out the one-dimensional homology, we can fit $Y_0$ into a long exact sequence for $Y$, $Y_0$, and $Y_1$, where $Y$ and $Y_1$ are integer homology three-spheres. The result then follows from the grading in the long exact sequence (Lemma 3.1) together with Proposition 3.3 above.

We now turn to the graded refinement of the long exact sequence, after introducing some notation. Let $Y$ be a closed, oriented three-manifold and $a \in \mathbb{R}$, and let $s_0$ be a torsion Spin$^c$ structure. Then, we let

$$HF^+_\leq a(Y,s_0) \subset HF^+(Y,s_0)$$

denote the subgroup generated by homogeneous elements $\xi$ with degree $\tilde{gr}(\xi) \leq a$. Let $\mathcal{T} \subset \text{Spin}^c(Y)$ denote the subset of torsion Spin$^c$ structures, then

$$HF^+_\leq a(Y) = \left( \bigoplus_{s_0 \in \mathcal{T}} HF^+_\leq a(Y,s_0) \right) \oplus \left( \bigoplus_{t \in \text{Spin}^c(Y) \setminus \mathcal{T}} HF^+(Y,t) \right).$$

We have the following:

Theorem 3.5. Let $K \subset Y$ be a knot in an integral homology three-sphere. Then, for all sufficiently large $n$, we have an exact sequence

$$\cdots \longrightarrow HF^+_{\leq 2n+1}(Y) \xrightarrow{F_1} HF^+_{\leq 2n+1}(Y_0) \xrightarrow{F_2} HF^+_{\leq 2n+1}(Y_1) \xrightarrow{F_3} \cdots$$

Proof. Observe that $\text{Ker} F_1 \subset HF^+(Y)$ is a finitely generated $\mathbb{Z}$-module, since for all sufficiently large $k$, the $s_0$ component of $F_1$ carries $HF^+_{2k}(Y)$ isomorphically to $HF^+_{2k-1/2}(Y_0,s_0)$, and $HF^+_{2k+1}(Y) = 0$ (we are using here the $HF^\infty$ characterization of the $\mathbb{Z}/2\mathbb{Z}$ grading and Proposition 3.3 to compare it with the absolute $\mathbb{Z}$ grading). Moreover, this kernel is the image of $F_3(HF^+(Y_1))$. It follows that we can find an $n$ large enough that

$$F_3(HF^+(Y_1)) = F_3(HF^+_{\leq 2n+1}(Y_1)).$$

Moreover, since $\bigoplus_{t \neq s_0} HF^+(Y_0,t)$ is finitely generated, we have that $F_2$ maps $HF^+_{\leq 2n+1}(Y_0)$ into $HF^+_{\leq 2n+1}(Y_1)$, for all sufficiently large $n$.

The theorem then follows from the Lemma 3.1.
3.1. More exact sequences: fractional surgeries. There are analogues of the refined exact sequence (Theorem 3.5) which holds for fractional surgeries, as well.

Recall that in Theorem 10.14 of [20], we established an exact sequence for fractional surgeries of the form

\[ \ldots \longrightarrow \text{HF}^+(Y) \xrightarrow{\bar{F}_1} \text{HF}^+(Y_0; \mathbb{Z}/q\mathbb{Z}) \xrightarrow{\bar{F}_2} \text{HF}^+(Y_{1/q}) \xrightarrow{\bar{F}_3} \text{HF}^+(Y) \longrightarrow \ldots, \]

where \( Y \) is an integral homology sphere, and \( q \) is any real number. In the above sequence,

\[ \text{HF}^+(Y_0; \mathbb{Z}/q\mathbb{Z}) = \bigoplus_{t \in \text{Spin}^c(Y_0)} \text{HF}^+(Y_0, t; \mathbb{Z}/q\mathbb{Z}) \]

is a sum of \( \text{HF}^+ \)-groups twisted by some representation

\[ H_1(Y_0; \mathbb{Z}) \longrightarrow \mathbb{Z}/q\mathbb{Z}. \]

Observe that the \( t_0 \)-component of \( \text{HF}^+(Y_0; \mathbb{Z}/q\mathbb{Z}) \) (where, as usual, \( c_1(t_0) = 0 \)) can be given an absolute \( \mathbb{Q} \)-grading (inherited from the absolute grading of the untwisted group).

Defining \( \text{HF}^+_{\leq 2n+1}(Y_0; \mathbb{Z}/q\mathbb{Z}) \) with respect to this grading on the torsion component, we have the following truncated exact sequence (analogous to Theorem 3.6):

**Theorem 3.6.** Let \( K \subset Y \) be a knot in an integral homology three-sphere. Then, for all sufficiently large \( n \), there is an exact sequence

\[ \ldots \longrightarrow \text{HF}^+_{\leq 2n+1}(Y) \xrightarrow{\bar{F}_1} \text{HF}^+_{\leq 2n+1}(Y_0, \mathbb{Z}/q\mathbb{Z}) \xrightarrow{\bar{F}_2} \text{HF}^+_{\leq 2n+1}(Y_{1/q}) \xrightarrow{\bar{F}_3} \text{HF}^+(Y) \longrightarrow \ldots, \]

The proof of the above theorem is complicated by the fact that in the proof of the fractional surgeries exact sequence (Theorem 10.14 of [20]), the maps \( \bar{F}_2 \) and \( \bar{F}_3 \) were constructed by counting pseudo-holomorphic triangles (and thus they correspond to maps of cobordisms), while the map \( \bar{F}_1 \) is an induced coboundary map. Thus, our aim is to give a construction of \( \bar{F}_1 \) by counting pseudo-holomorphic triangles. This construction diverges slightly from the context set up in Subsection 4.1 of [22], but the following analogue of Lemma 3.1 still holds (and from this, Theorem 3.6 follows easily):

**Proposition 3.7.** Let \( K \subset Y \) be a knot in an integral homology three-sphere. There is an exact sequence

\[ \ldots \longrightarrow \text{HF}^+(Y) \xrightarrow{\bar{F}_1} \text{HF}^+(Y_0; \mathbb{Z}/q\mathbb{Z}) \xrightarrow{\bar{F}_2} \text{HF}^+(Y_{1/q}) \xrightarrow{\bar{F}_3} \text{HF}^+(Y) \longrightarrow \ldots, \]

with the property that the component of \( \bar{F}_1 \) mapping into \( \text{HF}^+(Y_0, t_0; \mathbb{Z}/q\mathbb{Z}) \) has degree \(-1/2\), the restriction of \( \bar{F}_2 \) to \( \text{HF}^+_{\leq 2n+1}(Y_0, t_0; \mathbb{Z}/q\mathbb{Z}) \) has degree \(-1/2\).

To prove the above proposition, we find it convenient to introduce a construction closely related to twisted coefficients, and to verify that the associated maps satisfy the naturality properties needed to establish exactness for the surgery sequence. We return
to the proof of Proposition 3.7 and Theorem 3.6 at the end of the present section, after this lengthy digression.

3.2. Covering spaces. We introduce Floer homology groups associated to covering spaces of the symmetric product, and establish enough naturality properties for the above applications (Theorem 3.6).

Fix a two-manifold $\Sigma$. A one-dimensional cohomology class $\xi \in H^1(\Sigma, \mathbb{Z}/n\mathbb{Z}) \cong H^1(\text{Sym}^g(\Sigma), \mathbb{Z}/n\mathbb{Z})$ describes an $n$-fold cyclic covering space of $\text{Sym}^g(\Sigma)$, which we denote by

$$\Pi_\xi : \widetilde{\text{Sym}}^g(\Sigma)_\xi \rightarrow \text{Sym}^g(\Sigma).$$

We assume that the covering space is connected, which is equivalent to the condition that there is a homology class in $\Sigma$ whose pairing with $\xi$ is one. Observe that $\xi$ also gives an $n$-fold covering of $\Sigma$, which we denote by $\widetilde{\Sigma}$, and there is a branched covering space

$$\text{Sym}^g(\widetilde{\Sigma}) \rightarrow \widetilde{\text{Sym}}^g(\Sigma)_\xi.$$

Next, fix a Heegaard triple $(\Sigma, \alpha, \beta, z)$, and let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be $g$-tuples of simple, closed curves in $\widetilde{\Sigma}$, with the property that the $\Pi_\xi \widetilde{\alpha}_i$ is homotopic to some multiple of $\alpha_i$ and $\Pi_\xi \widetilde{\beta}_i$ is homotopic to some multiple of $\beta_i$. There are induced tori $\widetilde{T}_\alpha$ and $\widetilde{T}_\beta$ inside $\widetilde{\text{Sym}}^g(\Sigma)_\xi$.

We can define chain complexes analogous to the $CF^-, CF^\infty, CF^+$, and $\widetilde{CF}$. For instance, we can define $\widetilde{CF}^\infty(\widetilde{\alpha}, \widetilde{\beta}, z)$ generated by pairs $[\widetilde{x}, i]$ where $i \in \mathbb{Z}$ and $\widetilde{x} \in \widetilde{T}_\alpha \cap \widetilde{T}_\beta$ so that $s_z(\Pi(\widetilde{x}))$, with

$$\partial[\widetilde{x}, i] = \sum_{\widetilde{y} \in \widetilde{T}_\alpha \cap \widetilde{T}_\beta} \sum_{\phi \in \pi_2(\widetilde{x}, \widetilde{y})} \#(\mathcal{M}(\phi)) \cdot [\widetilde{y}, i - n_z(\Pi \circ \phi)].$$

Here, of course, $\pi_2(\widetilde{x}, \widetilde{y})$ denotes the space of homotopy classes of Whitney disks in $\widetilde{\text{Sym}}^g(\Sigma)_\xi$ which connect $\widetilde{x}$ to $\widetilde{y}$. We can also define its quotient complex $\widetilde{CF}^+(\widetilde{\alpha}, \widetilde{\beta})$, where the integer is required to be non-negative. In fact, for notational simplicity, we will focus on this case.

The verification that these are, in fact, complexes follows by modifying the discussion in of [21]. We outline the main steps.

First, of course, we must use families of almost-complex structures on the lift $\widetilde{\text{Sym}}^g(\Sigma)_\xi$ to obtain genericity.

Next, observe that a Whitney disk $\phi$ in $\widetilde{\text{Sym}}^g(\Sigma)_\xi$ has a corresponding domain $\mathcal{D}(\phi)$ in $\Sigma$, whose boundary lies in the images of the $\widetilde{\alpha}$ and $\widetilde{\beta}$. Holomorphic curves in $\widetilde{\text{Sym}}^g(\Sigma)_\xi$ give rise to branched covers of the disk, which map holomorphically into $\Sigma$, as before. The energy bounds of [21] follow directly.
Next, since the map from $\tilde{\text{Sym}}^g(\Sigma)_{\xi}$ to $\text{Sym}^g(\Sigma)$ is a covering space, we see that $\pi_2(\text{Sym}^g(\Sigma)_{\xi}) \cong \pi_2(\text{Sym}^g(\Sigma))$, and also that $T\tilde{\text{Sym}}^g(\Sigma)_{\xi} \cong \Pi_{\xi} T\text{Sym}^g(\Sigma)$. Thus, the dimension counts from [21] can be used to show that generic choices of $\tilde{\alpha}$ and $\tilde{\beta}$, two-dimensional moduli spaces miss the holomorphic spheres (which could otherwise spoil $\bar{\partial}^2 = 0$).

For boundary degenerations, observe that our homological hypotheses show that for any $\phi \in \pi_2(\tilde{x}, \tilde{x})$, $D(\phi)$ consists of some number of copies of $\Sigma$. It then as before that the only types of boundary degenerations which can occur (in the proof that $\partial^2 = 0$) are those in the homotopy class $O_{\tilde{x}} + S \in \pi_2(\tilde{x}, \tilde{x})$. As before, a $j$-holomorphic representative for this homotopy class is generically injective, and hence, spaces of boundary degenerations are smooth for generic perturbations of the almost-complex structure induced over $\tilde{\text{Sym}}^g(\Sigma)_{\xi}$. The total number of these can be calculated by degenerating the base $\Sigma$, to see that the signed count is zero.

Finally, when there are relations between the homology classes in the span of $\tilde{\alpha}$ and those of $\tilde{\beta}$, then we need corresponding admissible hypotheses to show that the sums are a priori finite. Weak admissibility in this context, for instance, (which is sufficient for the purposes of $HF^+$) is the requirement that all homologies between the spans of $\tilde{\alpha}$ and $\tilde{\beta}$, when projected down to $\Sigma$, always have both positive and negative multiplicities.

With these remarks in place, then, we have “lifted” a chain complex $\tilde{CF}^+(\Sigma, \tilde{\alpha}, \tilde{\beta}, \tilde{z})$, and also its corresponding homology group.

We relate this construction directly to the three-manifolds invariants in two cases. Fix a Heegaard triple $(\Sigma, \alpha, \beta, z)$ for an oriented three-manifold $Y$, and also a one-dimensional cohomology class $\xi \in H^1(\Sigma; \mathbb{Z})$. Fix lifts $\tilde{\alpha}$ and $\tilde{\beta}$ for $\alpha$ and $\beta$ under the map $\Pi_{\xi}$. In this case, $\tilde{CF}^+(\tilde{\alpha}, \tilde{\beta})$ splits into summands indexed by $t \in \text{Spin}^c(Y)$, where $\tilde{CF}^+(\tilde{\alpha}, \tilde{\beta}, t)$ is generated by $\tilde{x}$ with $s_{\xi}(\Pi_{\xi}\tilde{x}) = t$.

There will be two subcases of particular importance to us. First, suppose that $\xi \in H^1(\Sigma, \mathbb{Z}/p\mathbb{Z})$ satisfies the following two conditions

- there is an element in the span of the $\alpha$ whose evaluation on $\xi$ is 1
- the restriction of $\xi$ to the span of the $\beta$ is trivial.

In this case, $\tilde{T}_\alpha = \Pi_{\xi}^{-1}(T_\alpha)$, while $\tilde{T}_\beta$ is one of the $n$ tori which project to $T_\beta$.

**Proposition 3.8.** Fix a one-dimensional cohomology class $\xi$ with the property that the induced covering space of $T_\alpha$ is connected, while the restriction of $\xi$ to the span of the $\beta$ is trivial. Then, for any choice of $\tilde{\beta}$ as above, we have a natural identification

$$\tilde{HF}^+(\tilde{\alpha}, \tilde{\beta}, t) \cong HF^+(\alpha, \beta, t).$$

**Proof.** According to the assumption, $\Pi_{\xi}^{-1}(T_\beta)$ consists of $n$ disjoint tori. Thus, each intersection point $x \in T_\alpha \cap T_\beta$ corresponds to a unique intersection point of $\tilde{T}_\alpha$ with
Moreover, if \( \tilde{x}, \tilde{y} \in \tilde{T}_\alpha \cap \tilde{T}_\beta \) lie over \( x, y \in T_\alpha \cap T_\beta \), the usual covering space theory gives an identification between Whitney disks connected \( x \) to \( y \) with Whitney disks connecting \( \tilde{x} \) to \( \tilde{y} \). By using the lift of a family of almost-complex structures over \( \text{Sym}^g(\Sigma) \), it is easy to see that the boundary maps are identified, as well.

Thus, the identification takes place actually on the chain complex level.

For the second case, we start again with a pointed Heegaard diagram \((\Sigma, \alpha, \beta, z)\) for \( Y \), and a class \( \xi \in H^1(\Sigma; \mathbb{Z}/n\mathbb{Z}) \), only now we suppose that both covering spaces \( \tilde{T}_\alpha = \Pi^{-1}_\xi(T_\alpha) \) and \( \tilde{T}_\beta = \Pi^{-1}_\xi(T_\beta) \) are connected (i.e. there are homology classes in the spans of both \([\alpha_1], \ldots, [\alpha_g]\) and \([\beta_1], \ldots, [\beta_g]\) whose pairing with \( \xi \) is one).

Observe that the covering space \( \text{Sym}^g(\Sigma)_\xi \) gives rise to an additive map

\[ A: \pi_2(x, y) \longrightarrow \mathbb{Z}/n\mathbb{Z}, \]

in the following way. For each \( x \in T_\alpha \cap T_\beta \), choose some lift \( \tilde{x}_0 \in \tilde{T}_\alpha \cap \tilde{T}_\beta \). Next, each \( \phi \in \pi_2(x, y) \), has a unique lift starting at \( \tilde{x}_0 \). Its other endpoint \( \tilde{y} \) (which depends only on the homotopy class of \( \phi \)) lies over \( y \); thus we can define the element \( A(\phi) \in \mathbb{Z}/n\mathbb{Z} \) to be the element with the property that \( \tilde{y} = A(\phi) \cdot \tilde{y}_0 \). It is easy to see that this \( A \) is additive under juxtaposition, hence it is an additive assignment of the kind required to define homology with twisted coefficients \( HF^+(Y, t; \mathbb{Z}/n\mathbb{Z}) \). (c.f. Subsection 4.10 of [20]).

**Proposition 3.9.** Let \((\Sigma, \alpha, \beta, z)\) be a Heegaard diagram for \( Y \), and fix a class \( \xi \) as above, with the property that the induced covering spaces for both \( T_\alpha \) and \( T_\beta \) are connected. Then, for the additive assignment defined above, there is a natural identification

\[ HF^+(Y, t; \mathbb{Z}/n\mathbb{Z}) \cong \tilde{HF}^+(\tilde{T}_\alpha, \tilde{T}_\beta; t). \]

**Proof.** We have an identification of \( \mathbb{Z}/n\mathbb{Z} \)-sets:

\[ \tilde{T}_\alpha \cap \tilde{T}_\beta \cong (T_\alpha \cap T_\beta) \times \mathbb{Z}/n\mathbb{Z}. \]

Thus, the generators of \( CF^+(Y, t) \) and \( \tilde{CF}^+(\tilde{T}_\alpha, \tilde{T}_\beta; t) \) are identified. The boundary maps are easily seen to coincide.

Of course, if we move the \( \beta \) by an isotopy which does not cross the basepoint, there is an induced isotopy amongst the \( \tilde{\beta} \), which does not project to the basepoint – hence leaving the homology groups unchanged. But we have more freedom to move the \( \tilde{\beta} \).

**Proposition 3.10.** Suppose that \( \{\zeta_1, \ldots, \zeta_g\} \) are curves in \( \tilde{\Sigma} \) which are simultaneously isotopic to \( \tilde{\beta} \) via an isotopy which never projects to the basepoint \( z \in \Sigma \), then the isotopy induces an identification

\[ \tilde{CF}^+(\tilde{T}_\alpha, \tilde{T}_\beta) \cong \tilde{CF}^+(\tilde{T}_\alpha, \tilde{T}_\zeta), \]
where the second chain complex is defined using the torus $\tilde{T}_\zeta = \zeta_1 \times \ldots \times \zeta_g$.

**Proof.** We adapt the usual proof of isotopy invariance, noting that the branched covering
\[
\tilde{\Sigma}^g \rightarrow \text{Sym}^g(\tilde{\Sigma}) \rightarrow \text{Sym}^g(\Sigma)_\zeta
\]
gives us a bound of the energy of holomorphic disks in $\text{Sym}^g(\Sigma)_\zeta$ in terms of the induced homology class in $\tilde{\Sigma}^g$. This energy remains bounded under dynamical boundary conditions, provided that the $\tilde{\beta}$ are moved by an exact Hamiltonian in $\tilde{\Sigma}$.

### 3.3. The fractional surgeries long exactness

We can now modify the proof of the long exact sequence for fractional surgeries (Theorem 10.14 of [20]) to have a better understanding of the maps involved. We assume for simplicity that $Y$ is an integer homology three-sphere, and $K \subset Y$ is a knot with the canonical framing, so that in particular $b_1(Y_0) = 1$.

Consider the Heegaard quadruple $(\Sigma, \alpha, \beta, \gamma, \delta, z)$ as in Section 10.3 of [20]. In particular, $(\Sigma, \alpha, \beta, \gamma, z)$ is a Heegaard triple subordinate to the knot $K \subset Y$ (with its 0-framing), $(\Sigma, \alpha, \gamma, \delta, z)$ is a Heegaard triple subordinate to a knot in $Y_0$, which in turn represents the a canonical cobordism from $Y_0$ to $Y_{1/q}$, while $(\Sigma, \alpha, \delta, \beta)$ represents a cobordism from the union of $Y_1/q$ and a lens space to $Y$.

Let $\xi \in H^1(\Sigma; Z/qZ)$ be the Poincaré dual of $\beta_g$. Clearly, $\xi$ evaluates trivially on the span of the $\beta$ and $\delta$. However, $\langle \xi, \gamma_g \rangle = 1$. It follows that there is some element in the span of $\alpha$ whose pairing with $\xi$ is one.

Since in $H_1(\Sigma; Z)$ we have the relation
\[
\beta_g + q\gamma_g = \delta_g,
\]
we have the corresponding relation in $H_1(\tilde{\Sigma}; Z)$:
\[
\tilde{\beta}_g + \tilde{\gamma}_g = \tilde{\delta}_g.
\]

Observe that we have isomorphisms
\[
HF^{\leq 0}(\tilde{\beta}, \tilde{\gamma}) \cong HF^{\leq 0}(\tilde{\delta}, \tilde{\beta}) \cong HF^{\leq 0}(\#^g(S^1 \times S^2)),
\]
in view of Proposition 3.8, so these first two groups have (up to sign) canonical top-dimensional generators which we denote by $\tilde{\Theta}_{\beta, \gamma}$ and $\tilde{\Theta}_{\delta, \beta}$ respectively. Similarly, we have (up to sign) a canonical top dimensional generator $\tilde{\Theta}_{\gamma, \delta}$ for
\[
HF^{\leq 0}(\tilde{\gamma}, \tilde{\delta}) \cong HF^{\leq 0}(\#^g(S^2 \times S^1), Z/qZ) \cong HF^{\leq 0}(\#^g(S^2 \times S^1)).
\]
(The first isomorphism is an application of Proposition 3.9, and the second is an easy calculation.)
Define
\[ \tilde{F}_1: \text{HF}^+(Y) \cong \text{HF}^+(\tilde{T}_\alpha, \tilde{T}_\beta) \xrightarrow{\otimes \tilde{\Theta}_{\beta, \gamma}} \text{HF}^+(\tilde{T}_\alpha, \tilde{T}_\gamma) \cong \text{HF}^+(Y_0; \mathbb{Z}/n\mathbb{Z}), \]

where the first isomorphism is induced from Proposition 3.8, the second isomorphism is induced from Proposition 3.9, and the tensor product map is defined by counting (zero-dimensional spaces of) pseudo-holomorphic triangles in \( \text{Sym}^g(\Sigma) \). Similarly, we define
\[ \tilde{F}_2: \text{HF}^+(Y_0; \mathbb{Z}/n\mathbb{Z}) \cong \text{HF}^+(\tilde{T}_\alpha, \tilde{T}_\gamma) \xrightarrow{\otimes \tilde{\Theta}_{\gamma, \delta}} \text{HF}^+(\tilde{T}_\alpha, \tilde{T}_\gamma) \cong \text{HF}^+(Y_1/q), \]

Finally, \( \tilde{F}_3 \) is defined as it is in [20] (where it is simply denoted \( F_3 \)) by
\[ \tilde{F}_3: \text{HF}^+(Y_1/q) \cong \text{HF}^+(T_\alpha, T_\beta) \xrightarrow{\otimes \tilde{\Theta}_{\delta, \beta}} \text{HF}^+(T_\alpha, T_\beta) \cong \text{HF}^+(Y). \]

We now have the precise statement of the fractional long exact sequence:

**Proposition 3.11.** The maps \( \tilde{F}_1, \tilde{F}_2, \) and \( \tilde{F}_3 \) defined above fit into a long exact sequence:
\[ \ldots \longrightarrow \text{HF}^+(Y) \xrightarrow{\tilde{F}_1} \text{HF}^+(Y_0; \mathbb{Z}/q\mathbb{Z}) \xrightarrow{\tilde{F}_2} \text{HF}^+(Y_1/q) \xrightarrow{\tilde{F}_3} \ldots \]

**Proof.** Use a pointed Heegaard multi-diagram
\[ (\Sigma, \alpha, \beta, \gamma, \delta, z) \]
as above, so that \((\alpha, \beta)\) describes \( Y \), \((\beta, \gamma)\) describes \( Y_0 \), \((\alpha, \delta)\) describes \( Y_1 \). Also, fix the covering space of \( \text{Sym}^g(\Sigma) \) described above, and lifts \( \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \) and \( \tilde{\delta} \) of the corresponding curves.

Observe that there is a smooth isotopy taking \( \tilde{\gamma}_g \) arbitrarily close to the juxtaposition of \( \tilde{\beta}_g \) with \( \tilde{\delta}_g \), which does not project to the basepoint \( z \in \Sigma \) (see Figure 1). We denote the induced isotopy of \( \tilde{T}_\gamma \) by \( \Psi_t \) and new isotopic \( \gamma \)-torus \((\Psi_1(T_\gamma))\) by \( \tilde{T}'_\gamma \). The proof of Theorem 10.1 of [20] (using the filtrations when \( \tilde{\gamma}_g \) is a sufficiently close approximation to the juxtaposition of \( \tilde{\beta}_g \) and \( \tilde{\delta}_g \)) then gives a exactness (in the middle) for the maps
\[ \tilde{\text{HF}}^+(\tilde{\alpha}, \tilde{\beta}) \xrightarrow{\otimes \tilde{\Theta}_{\beta, \gamma}} \tilde{\text{HF}}^+(\tilde{\alpha}, \tilde{\gamma'}) \xrightarrow{\otimes \tilde{\Theta}_{\gamma', \delta}} \tilde{\text{HF}}^+(\tilde{\alpha}, \tilde{\delta}). \]

Here, \( \otimes \tilde{\Theta}_{\beta, \gamma'} \) is shorthand for the map
\[ \eta \mapsto \sum_{s \in X_{\alpha, \beta, \gamma'}} \pm \tilde{f}_{\alpha, \beta, \gamma'}(\eta \otimes \tilde{\Theta}_{\beta, \gamma'}, s), \]
(with an appropriate choice of signs) where, of course, \( \tilde{f}_{\alpha, \beta, \gamma'} \) counts all holomorphic triangles in \( \text{Sym}^g(\Sigma) \). Also, \( \tilde{\Theta}_{\beta, \gamma'} \in \text{HF}^{\leq 0}(\tilde{\beta}, \tilde{\gamma}, s_0) \) is a generator which was explicitly written down in [20]. It has an alternative, more algebraic characterization (up to a
sign) as the generator of $HF^{\leq 0}(\tilde{\beta}, \tilde{\gamma}, s_0) \cong HF^{\leq 0}(\#^{g-1}(S^2 \times S^1), s_0)$ with maximal (relative) degree.

We claim that functoriality of the triangle construction in $\tilde{\Sym}_g^q(\Sigma)_\xi$, together with exactness in the middle for Exact Sequence (6), gives exactness in the middle for:

\begin{equation}
\begin{array}{c}
HF^+ (\tilde{\alpha}, \tilde{\beta}) \otimes \tilde{\Theta}_{\beta, \gamma} \to HF^+ (\tilde{\alpha}, \tilde{\gamma}) \otimes \tilde{\Theta}_{\gamma, \delta} \to HF^+ (\tilde{\alpha}, \tilde{\delta}).
\end{array}
\end{equation}

Observe that the maps here correspond to the maps $\tilde{F}_1$ and $\tilde{F}_2$ from the statement of the proposition.

**Figure 1. Isotopy of $\tilde{\gamma}$** We illustrate the isotopy of $\tilde{\gamma}$ for the $1/q$ surgery exact sequence, when $q = 3$. The lower picture takes place in a genus one surface, the upper picture takes place in its triple cover (which unwinds $\gamma$ to give $\tilde{\gamma}$). The curve $\tilde{\gamma}'$ approximates the juxtaposition of the lifts $\tilde{\beta}$ and $\tilde{\delta}$ and is isotopic to $\tilde{\gamma}$ through an isotopy which does not project to the basepoint.
To see this, note that the isotopy of $\tilde{\gamma}_g$ to $\tilde{\gamma}_g'$ gives rises to isomorphisms

$$
\tilde{\Gamma}_{\alpha;\gamma,\gamma'} : \tilde{HF}^+ (\tilde{\alpha}, \tilde{\gamma}) \to \tilde{HF}^+ (\tilde{\alpha}, \tilde{\gamma}'),
$$

$$
\tilde{\Gamma}_{\beta;\gamma,\gamma'} : \tilde{HF}^+ (\tilde{\beta}, \tilde{\gamma}) \to \tilde{HF}^+ (\tilde{\beta}, \tilde{\gamma}'),
$$

$$
\tilde{\Gamma}_{\gamma,\gamma';\delta} : \tilde{HF}^+ (\tilde{\gamma}, \tilde{\delta}) \to \tilde{HF}^+ (\tilde{\gamma}', \tilde{\delta})
$$
defined analogously to the maps of isotopies downstairs. It is straightforward to see that these are functorial for triangles (compare Theorem 2.3 of [22]). In particular, we have that

$$
\tilde{F}_{\alpha;\beta,\gamma} (\xi \otimes \tilde{\Gamma}_{\beta;\gamma,\gamma'} (\eta)) = \tilde{\Gamma}_{\alpha;\gamma,\gamma'} \circ \tilde{F}_{\alpha,\beta,\gamma} (\xi \otimes \eta)
$$

$$
\tilde{F}_{\alpha,\beta,\gamma} (\tilde{\Gamma}_{\alpha;\beta,\gamma} (\xi) \otimes \tilde{\Gamma}_{\beta;\gamma,\gamma'} (\eta)) = \tilde{\Gamma}_{\alpha;\beta,\gamma} (\xi \otimes \eta).
$$

It follows that the following diagram is commutative:

$$
\begin{array}{ccc}
\tilde{HF}^+ (\tilde{\alpha}, \tilde{\beta}) & \xrightarrow{\otimes \tilde{\Theta}_{\beta,\gamma}} & \tilde{HF}^+ (\tilde{\alpha}, \tilde{\gamma}) \\
\downarrow{\text{Id}} & & \downarrow{\tilde{\Gamma}_{\alpha;\gamma,\gamma'}} \\
\tilde{HF}^+ (\tilde{\alpha}, \tilde{\beta}) & \xrightarrow{\otimes \tilde{\Theta}_{\gamma,\delta}} & \tilde{HF}^+ (\tilde{\alpha}, \tilde{\delta})
\end{array}
$$

Again, the generators $\tilde{\Theta}_{\beta,\gamma}$ and $\tilde{\Theta}_{\gamma,\delta}$ are uniquely characterized (up to sign) by the property that they are top-dimensional dimensional generators. It follows (since $\tilde{\Gamma}_{\beta;\gamma,\gamma'}$ and $\tilde{\Gamma}_{\gamma,\gamma';\delta}$ are isomorphisms on the level of homology) that

$$
\tilde{\Gamma}_{\beta;\gamma,\gamma'} (\tilde{\Theta}_{\beta,\gamma}) = \pm \tilde{\Theta}_{\beta,\gamma'} \quad \text{and} \quad \tilde{\Gamma}_{\gamma,\gamma';\delta} (\tilde{\Theta}_{\gamma,\delta}) = \pm \tilde{\Theta}_{\gamma,\delta}
$$

Thus, Theorem 10.1 establishes exactness along the bottom row of Diagram (8), which in turn implies exactness along the top row of the same diagram.

Of course, by repeating these arguments only now isotoping the curve $\delta_g$ to $\gamma_g + \beta_g$ (which we can do downstairs in $\Sigma$), we establish exactness for the maps:

$$
\tilde{HF}^+ (\tilde{\alpha}, \tilde{\gamma}) \xrightarrow{\otimes \tilde{\Theta}_{\gamma,\gamma'}} \tilde{HF}^+ (\tilde{\alpha}, \tilde{\gamma'}) \xrightarrow{\otimes \tilde{\Theta}_{\gamma',\beta}} \tilde{HF}^+ (\tilde{\alpha}, \tilde{\beta}).
$$

(Indeed, this was the isotopy used for the case of fractional surgeries in Theorem 10.14 of [20]); and hence, by the same naturality arguments, we get exactness for

$$
\tilde{HF}^+ (\tilde{\alpha}, \tilde{\gamma}) \xrightarrow{\otimes \tilde{\Theta}_{\gamma,\delta}} \tilde{HF}^+ (\tilde{\alpha}, \tilde{\delta}) \xrightarrow{\otimes \tilde{\Theta}_{\delta,\beta}} \tilde{HF}^+ (\tilde{\alpha}, \tilde{\beta}).
$$

$\square$
3.4. **Proof of the truncated exact sequence for fractional surgeries.** Proposition 3.7 follows easily from Proposition 3.11.

**Proof of Proposition 3.7.** We use the maps $\tilde{F}_1$ and $\tilde{F}_2$ as defined in the proof of Proposition 3.11 above. Clearly, the holomorphic triangles which comprise $\tilde{F}_1$ (before we move $T'$ by an isotopy) project down to holomorphic triangles for the Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$. Moreover, it is easy to see that the Maslov indices of these triangles agree with the Maslov indices of their projections. Thus, the degree shift of the $t_0$-component of $\tilde{F}_1$ must agree with the degree shift of its projection. But the projection represents the natural cobordism from $Y$ to $Y_0$, so this degree shift is $\frac{1}{2}$. The same remarks apply to the map $\tilde{F}_2$. 

**Proof of Theorem 3.6** This follows immediately from Proposition 3.7 in the same way that Theorem 3.5 followed from Lemma 3.1.

$\square$
4. THE CORRECTION TERM

With the help of the absolute grading, we can define the following numerical invariant for integer homology three-spheres or, more generally, rational homology three-spheres and Spin$^c$ structures:

**Definition 4.1.** Let $Y$ be a rational homology three-sphere. The correction term $d(Y,t)$ is the minimal grading ($\tilde{\gr}$) of any non-torsion element in the image of $HF^\infty(Y,t)$ in $HF^+(Y,t)$.

This invariant is an analogue of a gauge-theoretic invariant defined by Frøyshov, see [12]. As we shall see (like its gauge-theoretic analogue), the invariant contains information about the intersection forms of four-manifolds which bound $Y$.

**Proposition 4.2.** Let $(Y,t)$ be a rational homology three-sphere. Then, we have that

$$d(Y,t) = d(Y,\overline{t})$$

and

$$d(Y,t) = -d(-Y,t).$$

**Proof.** The conjugation invariance of the correction term follows from the conjugation invariance of the maps associated to cobordisms (c.f Theorem 3.5 of [22]).

We verify Equation (9).

Consider the natural long exact sequence connecting $HF^-(Y,t)$, $HF^\infty(Y,t)$ and $HF^+(Y,t)$ (Exact Sequence (3)). From this sequence, together with the fact that $HF^\infty(Y,t) \cong \mathbb{Z}[U,U^{-1}]$ (c.f. Theorem 11.1 of [20]), it follows that if we define $d^-(Y,t)$ to be the maximal $k$ for which the map $\iota_k : HF^-(Y,t) \to HF^\infty_k(Y,t)$ is non-trivial, then

$$d^-(Y,t) = d(Y,t) - 2.$$  \hfill (10)

Now there is a duality map $D$ which gives a map from $HF^\circ$ homology of $Y$ to $HF^\circ$ cohomology of $-Y$. In its precise graded version, (c.f. Proposition 7.11 of [22]), we obtain a commutative diagram

$$
\begin{array}{ccc}
HF^\infty_k(Y,t) & \xrightarrow{\pi_k} & HF^+_k(Y,t) \\
\downarrow D^\infty & & \downarrow D^+ \\
\mathbb{Z} \cong HF^{\infty-k-2}(-Y,t) & \xrightarrow{i^{-k-2}} & HF^{-k-2}(-Y,t)
\end{array}
$$

(Here, $D[x,i] = [x,-i-1]^*$ is the map from the chain complexes for $Y$, $(\Sigma,\alpha,\beta,z)$, to the cochain complex for $-Y$, $(-\Sigma,\alpha,\beta,z)$. The bottom row is the map on cohomology, and the vertical maps are all isomorphisms. Now, by the universal coefficient theorem in
cohomology and the fact that $HF^\infty(Y, t)$ is a free module in each dimension, it follows that the image of $\sigma_k$ has rank one if and only if the map

$$\sigma_k : HF^-_{k-2}(-Y, t) \to HF^\infty_{-k+2}(-Y, t)$$

is non-trivial. In view of this, and Equation (10), Equation (9) follows. $\blacksquare$

We conclude the subsection with the proof of the additivity of the correction term under the connected sum operation:

**Theorem 4.3.** If $(Y, t)$ and $(Z, u)$ are rational homology three-spheres equipped with Spin$^c$ structures, then

$$d(Y#Z, t#u) = d(Y, t) + d(Z, u).$$

The proof occupies the rest of the present subsection.

We define first a natural transformation

$$F^o_{Y#Z, t#u} : HF^o(Y, t) \otimes HF^{\leq 0}(Z, u) \to HF^o(Y#Z, t#u)$$

as follows. Observe that for some $g_1$ and $g_2$ there is a pair-of-pants cobordism from $Y#(\#^{g_2}(S^2 \times S^1)) \coprod (\#^{g_1}(S^2 \times S^1))#Z$ to $Y#Z$. To define this, consider pointed Heegaard diagrams $(\Sigma_1, \alpha_1, \beta_1, z_1)$ and $(\Sigma_2, \alpha_2, \beta_2, z_2)$ for $Y$ and $Z$ respectively. Then the Heegaard triple

$$(\Sigma_1#\Sigma_2, \alpha_1\alpha_2, \beta_1\beta_2, \alpha_2, \beta_2)$$

describes such a cobordism. In the above notation, $\alpha_1\alpha_2$ denotes the union of $\alpha_1$ and $\alpha_2$, thought of now as circles in $\Sigma_1#\Sigma_2$. Now, we define $F^o_{Y#Z, t#u}$ to be the composite of the map

$$HF^o(Y, t) \otimes HF^{\leq 0}(Z, u) \to HF^o(Y#(\#^{g_2}(S^2 \times S^1)) \coprod (\#^{g_1}(S^2 \times S^1))#Z)$$

induced by the cobordism of one-handles with the map

$$HF^o(Y#(\#^{g_2}(S^2 \times S^1)) \coprod (\#^{g_1}(S^2 \times S^1))#Z) \to HF^o(Y#Z, t#u)$$

defined by counting holomorphic triangles in the Heegaard triple. (Of course, we perturb the circles to achieve admissibility.)

**Proposition 4.4.** The map $F^o_{Y#Z, t#u}$ defined above is independent of the Heegaard diagrams used for $Y$ and $Z$. Moreover, if $W$ is a cobordism from $Y$ to $Y'$ equipped with Spin$^c$ structure $s$, then the following diagram is commutative:

$$\begin{array}{ccc}
HF^o(Y, t) \otimes HF^{\leq 0}(Z, u) & \xrightarrow{F^o_{Y#Z, t#u}} & HF^o(Y#Z, t#u) \\
FW, s \otimes \text{Id} & & \downarrow F^o_W((Z \times [0,1]), s#u) \\
HF^o(Y', t') \otimes HF^{\leq 0}(Z, u) & \xrightarrow{F^o_{Y'#Z, t'#u}} & HF^o(Y'#Z, t'#u). \\
\end{array}$$
Proof. For simplicity, we consider the case where \( W \) is composed entirely of two-handles. Then, the commutative diagram follows from associativity of the triangle construction. More specifically, suppose that \( W \) is represented by the Heegaard triple \( (\Sigma_1, \alpha_1, \beta_1, \gamma, z_1) \). Then, by associativity, the following diagram commutes:

\[
\begin{array}{c}
HF^0(\alpha_1, \beta_1') \otimes HF^0(\alpha_2', \beta_2) \\
\downarrow \\
HF^0(\alpha_1 \alpha_2, \beta_1' \alpha_2') \otimes HF^0(\beta_1' \alpha_2', \beta_1 \beta_2) \\
\end{array}
\]

id \otimes (\otimes \Theta_{\beta_1, \beta_2, \gamma_1, \beta_1'}) \downarrow \otimes \Theta_{\beta_1, \beta_2, \gamma_1, \beta_1'}

The first map corresponds to the one-handles. Observe that \( \Theta_{\beta_1, \beta_2, \gamma_1, \beta_1'} \) is the canonical generator for \( HF^0(\beta_1 \beta_2, \gamma_1 \beta_1') \), which describes a connected sum of \( S^1 \times S^2 \). Thus, going around the above diagram the two ways corresponds to the two possible compositions.

The case of one- and three-handles follows easily (compare Section 4 of [22]). Independence of the Heegaard diagrams follows in the same manner.

Lemma 4.5. Let \( Z = S^3 \), and fix the canonical element \( \Theta \in HF^0(S^3, t_0) \). Then

\[
F_{Y \# S^3, t \# t_0}^0(\cdot \otimes \Theta): HF^0(Y, t) \longrightarrow HF^0(Y \# S^3, t \# t_0) \cong HF^0(Y, t)
\]

is the identity map.

Proof. This follows from the usual stabilization invariance of the triangle construction.

Lemma 4.6. Let \((Z, u)\) be a rational homology sphere equipped with a \( \text{Spin}^c \) structure. Let \( \Theta_Z \in HF^0(Z, u) \) be an element whose image in \( HF^\infty(Z, u) \) under the natural map is a generator. Then, the map

\[
F_{S^3, t_0 \# u}^\infty(\cdot \otimes \Theta_Z): HF^\infty(S^3, t_0) \longrightarrow HF^\infty(Z, u)
\]

is an isomorphism which carries the canonical element of \( HF^\infty(S^3, t_0) \) to the image of \( \Theta_Z \) in \( HF^\infty(Z, u) \).

Proof. By naturality, we have a commutative diagram:

\[
\begin{array}{c}
HF^0(S^3) \xrightarrow{\otimes \Theta_Z} HF^0(Z) \\
\downarrow \iota_{S^3} \downarrow \iota_Z \\
HF^\infty(S^3) \xrightarrow{\otimes \Theta_Z} HF^\infty(Z)
\end{array}
\]
Proposition 4.7. Let \( \Theta_Z \in HF^{\leq 0}(Z, u) \) be an element whose image in \( HF^\infty(Z, u) \) under the natural map is a generator. Then, 
\[
F^\infty_{Y \# Z, t \# u}(\cdot \otimes \Theta_Z) : HF^\infty(Y, t) \to HF^\infty(Y \# Z, t \# u)
\]
is an isomorphism of relatively graded groups. Moreover, 
\[
\tilde{\gamma}(F^\infty_{Y \# Z, t \# u}(\xi \otimes \Theta_Z)) = \tilde{\gamma}(\xi) + \tilde{\gamma}(\Theta_Z).
\]

**Proof.** Each integer homology three-sphere can be obtained from \( S^3 \) by a sequence of \( \pm 1 \) surgeries. When \( Y \) is an integer homology sphere, the result follows from induction on the length of such a sequence, with Lemma 4.6 as the base case (when \( Y \cong S^3 \)).

For the inductive step, suppose that the result holds for an integer homology sphere \( Y \). We claim that for each knot \( K \subset Y \), the result also holds for \( Y_1 = Y_{\pm 1}(K) \). We concentrate on the case where the sign is \(+1\). Then, there is a map of long exact sequences:

\[
\begin{array}{ccccccc}
\ldots & \to & HF^+(Y) & \to & HF^+(Y_0) & \to & HF^+(Y_1) & \to & \ldots \\
\downarrow F^\bot_{Y \# Z}(\cdot \otimes \Theta_Z) & & \downarrow F^\bot_{Y_0 \# Z}(\cdot \otimes \Theta_Z) & & \downarrow F^\bot_{Y_1 \# Z}(\cdot \otimes \Theta_Z) & & \\
\ldots & \to & HF^+(Y \# Z, [u]) & \to & HF^+(Y_0 \# Z, [u]) & \to & HF^+(Y_1 \# Z, [u]) & \to & \ldots
\end{array}
\]

We use the convention that if \( M \) is a three-manifold, then 
\[
HF^\infty(M \# Z, [u]) = \bigoplus_{\{\Theta \in Spin^c(M \# Z) \mid \Theta \equiv \Theta_Z\}} HF^\infty(M \# Z, t),
\]
and the vertical maps are obtained by summing the maps defined in Proposition 4.4. Indeed, the above squares commute by Proposition 4.4. It follows immediately that \( F^\infty_{Y_0 \# Z}(\cdot \otimes \Theta_Z) \) sends the image of the \( \gamma \)-action \( (\gamma \in H_1(Y_0; Z)) \) in \( HF^\infty(Y_0, t_0) \) to the image of the \( \gamma \)-action in \( HF^\infty(Y_0 \# Z, t_0 \# u) \). It follows from naturality that 
\[
F^\infty_{Y_0 \# Z, t_0 \# u}(\cdot \otimes \Theta_Z) : HF^\infty(Y_0, t_0) \to HF^\infty(Y_0, t_0 \# u)
\]
is an isomorphism in all degrees. Commutative Diagram (11) then forces \( F^\infty_{Y_1 \# Z}(\cdot \otimes \Theta_Z) \) to be an isomorphism, as well. (The case of \((-1)\)-surgery follows by repeating the above discussion using the \((-1)\)-surgery exact sequence.)

The case of rational homology spheres follows by induction on the rank of the first homology group, and a comparison of long exact sequences, parallel to the proof of Theorem 11.1 of [20] in the case of rational homology spheres. \( \square \)
Proof of Theorem 4.3. Fix $\Theta \in HF^{\leq 0}(Z,u)$ of degree $d(Z,u)$. Then, we have the following commutative diagram

$$
\begin{array}{ccc}
HF_k(Y,t) & \xrightarrow{F^\infty_{Y\#Z,t\#u}(\cdot \otimes \Theta Z)} & HF_k(Z,t) \\
\downarrow & & \downarrow \\
HF^+(Y,t) & \xrightarrow{F^+_{Y\#Z,t\#u}(\cdot \otimes \Theta Z)} & HF^+(Z,t).
\end{array}
$$

From this it follows easily that

$$
d(Y,t) + d(Z,u) \leq d(Y\#Z,t\#u).
$$

By the same reasoning, we have the inequality

$$
d(-Y,t) + d(-Z,u) \leq d(-Y\#Z,t\#u).
$$

In view of Equation (9), we can conclude that

$$
d(Y,t) + d(Z,u) = d(Y\#Z,t\#u),
$$

as claimed.

4.1. Correction terms for lens spaces. We give an inductive formula for the correction terms of lens spaces.

Let $p$ and $q$ be a pair of relatively prime, positive integers. The lens space $-L(p,q)$ can be given a Heegaard diagram $(E,\alpha,\gamma,z)$, where $E$ is an oriented two-manifold with genus $g = 1$. If we let $\alpha$ be the “horizontal” circle $S^1 \times \{0\}$ and $\beta$ be the vertical one $\{0\} \times S^1$, then $\gamma$ is a smoothly embedded curve homologous to $-q\alpha + p\beta$ (which we can take to be a “straight” circle

$$
\theta \mapsto (e^{\frac{2\pi i (\theta + \theta_0)}{q}}, e^{\frac{2\pi i (\theta + \theta_1)}{p}}).
$$

There is a canonical circular ordering of the Spin$^c$ structures over $-L(p,q)$ (i.e. a labeling of the Spin$^c$ structures by elements $i \in \mathbb{Z}/p\mathbb{Z}$), which we can describe as follows. Consider the pointed Heegaard triple $(E,\alpha,\beta,\gamma,z)$, where the baspoint is placed so that all the coefficients of the triply-periodic domain connecting $\alpha$, $\beta$, and $\gamma$ are negative, and order the intersection points of $\alpha$ with $\gamma$ circularly (about $\alpha$), so that the $(p-1)^{st}$ one modulo $p$ is the one adjacent to the basepoint. (See Figure 2.)

Proposition 4.8. Fix positive, relatively prime integers $p > q$, and also choose an integer with $0 \leq i < p + q$. Then, with respect to the above ordering of the Spin$^c$ structures over $-L(p,q)$, we have the following inductive formula:

$$
d(-L(p,q),i) = \left(\frac{pq - (2i + 1 - p - q)^2}{4pq}\right) - d(-L(q,r),j),
$$

where $r$ and $j$ are the reductions modulo $q$ of $p$ and $i$ respectively.
Figure 2. Ordering of Spin\textsuperscript{c} structures over \(-L(3,2)\). This is an illustration of a Heegaard triple \((E, \alpha, \beta, \gamma, z)\), representing a cobordism between \(-L(3,2), -L(2,1), \) and \(S^3\). The non-positive integers represent multiplicities of the triply-periodic domain used in the proof of Proposition 4.8, while the integers labelling the intersection points between \(\beta \cap \gamma\) and \(\alpha \cap \gamma\) represent the canonical orderings of the Spin\textsuperscript{c} structures over \(-L(2,1)\) and \(-L(3,2)\) respectively. The darkly-shaded triangle (when taken with multiplicity one) is the domain associated to the triangle \(\psi_1\) in the proof of Proposition 4.8.

**Proof.** Observe that the Heegaard triple \((E, \alpha, \beta, \gamma, z)\) represents a cobordism \(X_{\alpha,\beta,\gamma}\) with

\[
\partial X_{\alpha,\beta,\gamma} = -Y_{\alpha,\beta} - Y_{\beta,\gamma} + Y_{\alpha,\gamma} = S^3 - Y_{\beta,\gamma} - L(p,q).
\]

Indeed, it is easy to see that \(Y_{\beta,\gamma} \cong -L(q,r)\) (and in fact the cobordism obtained from \(X_{\alpha,\beta,\gamma}\) after filling in the \(S^3\) is the usual cobordism between \(L(q,r)\) and \(L(p,q)\) given by a single two-handle addition).

Observe that there are \(p + q\) triangles \(\{\psi_0, ..., \psi_{p+q-1}\}\) with non-negative domains but for which \(n_z(\psi_0) = 0\). We order these so that \(D(\psi_0) < D(\psi_1) < ... < D(\psi_{p+q-1})\).

Clearly, with the above conventions, \(\psi_i\) connects the \(i^{th}\) Spin\textsuperscript{c} structure (where \(i\) is thought of as an element of \(\mathbb{Z}/q\mathbb{Z}\)) on \(-L(q,r)\) to the \(i^{th}\) Spin\textsuperscript{c} structure (now regarding \(i\) as an element of \(\mathbb{Z}/p\mathbb{Z}\)) on \(-L(p,q)\). Clearly, now, the induced map \(F^{\leq 0}_{\alpha,\beta,\gamma}\), determined by \(\psi_i\) takes the tensor product of generators in \(HF^{\leq 0}(S^3) \otimes HF^{\leq 0}(-L(q,r),j)\) isomorphically to the generator of \(HF^{\leq 0}(-L(p,q),i)\) (since there is a unique holomorphic triangle in the torus representing the Spin\textsuperscript{c} structure). Thus, as in Equation (4), it follows that

\[
d(-L(p,q),i) + d(-L(q,r),j) = \frac{c_1(g_z(\psi_i))^2 + 1}{4}.
\]

(13)
Strictly speaking, the holomorphic triangle is not the one corresponding to the two-handle addition: the diagram \((E, \{\beta\}, \{\gamma\}, z)\) represents \(L(q, r)\), rather than a connected sum of \(S^2 \times S^1\). Thus, we use the variant of this dimension shift formula as in Proposition 4.7.

Calculating the right-hand-side of the above equation is an application of Proposition 6.3 of [22], a formula which gives the evaluation of the first Chern class of the \(\text{Spin}^c\) structure underlying a triangle in terms of combinatorial data on the Heegaard triple. Let \(\mathcal{P}\) be the generator of the group of triply-periodic domains, which is nowhere positive. The quantities appearing in Equation (8) of [22] are easily seen to be:

\[
\begin{align*}
\hat{\chi}(\mathcal{P}) &= 0 \\
\#(\partial \mathcal{P}) &= p + q + 1 \\
n_z(\mathcal{P}) &= 0 \\
\sigma(\psi_i, \mathcal{P}) &= -p - q + i + 1.
\end{align*}
\]

It now follows from Equation (8) of [22] that

\[
\langle c_1(s_z(\psi_i)), H(\mathcal{P}) \rangle = 2i + 1 - p - q.
\]

Clearly, \(H(\mathcal{P})\) is the generator of the compactly-supported cohomology, and \(H(\mathcal{P})^2 = -pq\), we can combine the above with Equation (13) to obtain Equation (12).

\[
(14)
\]

**4.2. Correction terms for three-manifolds with \(b_1 > 0\).** There are versions of the correction term for three-manifolds whose first Betti number is positive, as well. For instance, one could use totally twisted coefficients to define a rational invariant for three-manifolds equipped with torsion \(\text{Spin}^c\) structures. However, more structure can be obtained by using untwisted coefficients (and still a torsion \(\text{Spin}^c\) structure). For simplicity, we restrict ourselves presently to the case where \(H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}\). In this case, there is a unique \(\text{Spin}^c\) structure \(t_0\) with \(c_1(t_0) = 0\).

**Definition 4.9.** Suppose that \(H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}\). Then, there are two correction terms \(d_{\pm 1/2}(Y_0)\), where \(d_{\pm 1/2}(Y_0)\) is the minimal grading of any non-torsion element in the image of \(HF^{\infty}(Y_0, t_0)\) in \(HF^+(Y, t_0)\) with grading \(\pm 1/2\) modulo 2.

**Proposition 4.10.** Let \(H_1(Y_0; \mathbb{Z}) \cong \mathbb{Z}\). Then,

\[
d_{1/2}(Y_0) - 1 \leq d_{-1/2}(Y_0).
\]

Moreover,

\[
d_{\pm 1/2}(Y_0, t) = d_{\pm 1/2}(Y_0, \bar{t}),
\]

and

\[
d_{\pm 1/2}(Y_0, t) = -d_{\mp 1/2}(-Y_0, t).
\]

(15)
Proof. The first inequality follows from the algebra structure of $HF^\infty(Y_0, t_0)$, together with the grading as given in Proposition 3.4: if $\xi_0$ is any non-zero element with $\tilde{\text{gr}}(\xi_0) \equiv -\frac{1}{2}$ (mod $\mathbb{Z}$) in $HF^+(Y_0, t_0)$ coming from $HF^\infty(Y_0, t_0)$, then there must be an element $\xi_1$ of degree one greater (also coming from $HF^\infty(Y_0, t_0)$), with $\theta \cdot \xi_1 = \xi_0$, where $\theta$ is some element of $H_1(Y_0, t_0)$.

The other two equations follow exactly as in the case where $b_1(Y) = 0$ (Proposition 4.2).

The following proposition gives a relationship between correction terms for integral homology spheres and the correction terms for zero-surgeries on knots in them. This result will be generalized in Section 9 (see especially Theorem 9.11 and Corollary 9.14).

Proposition 4.11. Let $K \subset Y$ be a knot in an integral homology three-sphere. Then,

$$d(Y) - \frac{1}{2} \leq d_{-1/2}(Y_0),$$

and

$$d_{+1/2}(Y_0) - \frac{1}{2} \leq d(Y_1).$$

Proof. The first inequality follows from the fact that for the induced map $F_1$ on the cobordism from $Y$ to $Y_0$,

$$F_1^\infty : HF_k^\infty(Y) \longrightarrow HF_{k-1/2}^\infty(Y_0, t_0)$$

is an isomorphism for all even integers $k$, together with commutativity of the diagram

$$
\begin{array}{ccc}
HF_k^\infty(Y) & \xrightarrow{F_1^\infty} & HF_{k-1/2}^\infty(Y_0, t_0) \\
\downarrow & & \downarrow \\
HF_k^+(Y) & \xrightarrow{F_1^+} & HF_{k-1/2}^+(Y_0, t_0)
\end{array}
$$

The second follows similarly, using the fact that

$$F_2^\infty : HF_{k+1/2}^\infty(Y_0) \longrightarrow HF_k^\infty(Y_1)$$

is an isomorphism for all even integers $k$. \qed

4.3. Knots in $S^3$. In light of Proposition 4.11, the correction terms for homology $S^1 \times S^2$ can be used to give obstructions to obtaining a given three-manifold as zero-surgery on a knot in the three-sphere. Specifically, since $d(S^3) = 0$, we see that if $Y_0$ is obtained as zero-surgery on a knot in $S^3$, then

$$-\frac{1}{2} \leq d_{-1/2}(Y_0).$$
Moreover, by reflecting the knot and using Proposition 4.10, we also obtain the bound
\[ d_{1/2}(Y_0) \leq \frac{1}{2}. \]
We will give a generalization of this observation in Theorem 9.11 (see especially Corollary 9.13).

In another direction, we can think of the correction terms as giving rise to knot invariants (by considering the two correction terms associated to the zero-surgery). In fact, we find convenient to package the information as follows:

\[ \sigma_+(K) = \frac{d_{-1/2}(S^3_0(K)) - d_{1/2}(S^3_0(K)) + 1}{2} \]
and
\[ \sigma_-(K) = \frac{d_{-1/2}(S^3_0(K)) + d_{1/2}(S^3_0(K))}{2} \]
(where \( S^3_0(K) \) denotes zero-surgery along \( K \), given the orientation induced from \( K \), and the correction term is calculated in the unique torsion \( \text{Spin}^c \) structure).

In view of the above results, \( \sigma_+(K) \) is a non-negative integer, while \( \sigma_-(K) \) is an integer. Moreover, if \( r: S^3 \to S^3 \) is an orientation-reversing diffeomorphism of \( S^3 \) to itself, then \( \sigma_+(r(K)) = \sigma_+(K) \), while \( \sigma_-(r(K)) = -\sigma_-(K) \).

Indeed, the \( \sigma_{\pm}(K) \) could alternatively be defined using only correction terms for integral homology three-spheres, in view of the following result:

**Proposition 4.12.** Let \( K \subset S^3 \) be an oriented knot in the three-sphere. Then,

\[ d_{1/2}(S^3_0(K)) - \frac{1}{2} = d(S^3_1(K)) \]
\[ d(S^3_{-1}(K)) - \frac{1}{2} = d_{-1/2}(S^3_0(K)) \]

**Proof.** This is a direct consequence of the surgery long exact sequence, in view of the structure of \( HF^+(S^3) \). \( \square \)
5. The renormalized Euler characteristic and the Casson invariant

Let $Y$ be an integer homology three-sphere (so that it has a unique Spin$^c$ structure, which we drop from the notation). We define a renormalized Euler characteristic by

$$\hat{\chi}(Y) = \chi(HF^+_{\text{red}}(Y)) - \frac{1}{2} d(Y).$$

Let $\Sigma(2, 3, 5)$ denote the Poincaré homology sphere, oriented as the boundary of the negative-definite $E_8$ plumbing.

**Theorem 5.1.** Let $Y$ be an integer homology three-sphere. Then, the renormalized Euler characteristic agrees with Casson's invariant:

$$\hat{\chi}(Y) = \lambda(Y),$$

where here Casson's invariant is normalized so that $\lambda(\Sigma(2, 3, 5)) = -1$.

We will use the surgery long exact sequence for $HF^+$, and the following observation.

**Lemma 5.2.** Let $Y$ be an integral homology three-sphere. Then, for all sufficiently large $n$, we have that

$$\hat{\chi}(Y) = \chi(HF^+_{\leq 2n-1}(Y)) - n.$$

**Proof.** This follows easily from the structure of $HF^\infty(Y)$.

Recall that when $Y_0$ is an integer homology $S^1 \times S^2$, the Euler characteristic $\chi(HF^+_{\leq 2n+1}(Y_0, s_0))$ is independent of $n$, provided that $n$ is sufficiently large. We let $\chi^{\text{trunc}}(HF^+(Y_0, s_0))$ denote this integer.

**Proposition 5.3.** If $K \subset Y$ is a knot in an integral homology three-sphere, and let $s_0$ denote the Spin$^c$ structure on $Y_0$ with trivial $c_1$. Then we have the following:

$$\hat{\chi}(Y) - \hat{\chi}(Y_1) = \chi^{\text{trunc}}(Y_0, s_0) + \sum_{s \neq s_0} \chi(HF^+(Y_0, s)).$$

**Proof.** This follows by taking the Euler characteristic of the exact sequence in the form stated in Theorem 3.5, and applying the observation of Lemma 5.2.

**Proof of Theorem 5.1.** The right-hand-side of Equation (16) can be calculated from the results of [20]. Specifically if we write the Alexander polynomial of $Y - K$ as

$$\Delta_K = a_0 + \sum_{i=1}^d a_i(T^i + T^{-i}),$$

and

$$t_i = \sum_{j=1}^d j a_{|i|+j},$$

then

$$\hat{\chi}(Y) - \hat{\chi}(Y_1) = \chi^{\text{trunc}}(Y_0, s_0) + \sum_{s \neq s_0} \chi(HF^+(Y_0, s)).$$
then according to Theorem 9.1 of [20] (with Proposition 11.12 of [20] to pin down the sign, bearing in mind that for an integer homology $S^1 \times S^2$, the mod 2 reduction of $\tilde{\gamma} + \frac{1}{2}$ gives the absolute $\mathbb{Z}/2\mathbb{Z}$ grading on $Y_0$ which is used to determine the sign of the Euler characteristic, according to Proposition 3.4), for each $i \neq 0$, we have that

$$\chi(HF^+(Y_0, s)) = -t_i,$$

where $c_1(s)$ is $2i$ times a generator of $H^2(Y_0; \mathbb{Z})$, while

$$\chi^{\text{trunc}}(HF^+(Y_0, s_0)) = -t_0,$$

for the torsion $\text{Spin}^c$ structure (this case is handled in Theorem 11.15). Plugging these values into Equation (16), we see that $\hat{\chi}(Y, s)$ satisfies the same surgery formula as Casson’s invariant. Moreover, we know that $\hat{\chi}(S^3) = 0$. Since Casson’s invariant is characterized by its $+1$ surgery formula and this normalization, the theorem follows. $\square$
6. The renormalized complexity and surgeries

In [20], we defined a numerical invariant for integer homology three-spheres

\[ N(Y) = \text{rk} \text{HF}_{\text{red}}(Y). \]

Let \( K \subset Y \) be a knot, and let \( Y_1 \) be the manifold obtained by +1 surgery along \( K \), then Theorem 1.8 of [20] gives a bound:

\[ \text{max}(-t_0, 0) + 2 \sum_{i=1}^{d} |t_i(K)| \leq N(Y) + N(Y_1). \]

Taken with the correction term, however, \( N(Y) \) becomes more effective at distinguishing \( Y \) and \( Y_1 \), as follows:

**Theorem 6.1.** Let \( Y \) be an integral homology three-sphere and \( K \subset Y \) be a knot, then there is a bound:

\[ |t_0(Y)| + 2 \sum_{i=1}^{d} |t_i(K)| \leq N(Y) + \frac{d(Y)}{2} + N(Y_1) - \frac{d(Y_1)}{2}. \]

**Proof.** By surgery long exact sequence, we have exactness in the middle for

\[ \text{HF}_{\text{red}}(Y) \xrightarrow{F_1^{\text{red}}} \text{HF}_{\text{red}}(Y_0) \xrightarrow{F_2^{\text{red}}} \text{HF}_{\text{red}}(Y_1). \]

Indeed, the image of \( F_2^{\text{red}} \) lies in the kernel of the map \( F_3^{\text{red}} : \text{HF}_{\text{red}}(Y_1) \rightarrow \text{HF}_{\text{red}}(Y) \), while the kernel of \( F_1^{\text{red}} \) contains the image of \( F_3^{\text{red}} \). Thus,

\[ \text{rk} \text{HF}_{\text{red}}(Y_0) = \text{rk} \text{Im} F_1^{\text{red}} + \text{rk} \text{Im} F_2^{\text{red}} \]

We claim that

\[ \text{rk} \text{Im} F_2^{\text{red}} \leq \text{rk} \text{HF}_{\text{red}}(Y_1) - \left( \frac{d_{-1/2}(Y_0) + \frac{1}{2} - d(Y)}{2} \right). \]

This is true because (thanks to the absolute gradings in the exact sequence, Lemma 3.1) there is a module \( V_1 \) of rank \( D_Y \) in the image of \( HF^\infty(Y) \) which maps to zero in \( HF^+(Y_0) \). It follows (from the surgery long exact sequence on \( HF^+ \)) that \( HF^+(Y_1) \) surjects onto \( V_1 \subset HF^+(Y) \). In fact, the map

\[ F_3^+ : HF^+(Y_1) \rightarrow HF^+(Y) \]

factors as

\[ HF^+(Y_1) \longrightarrow HF_{\text{red}}(Y_1) \longrightarrow HF^+(Y), \]

since the map \( F_3^\infty \) is trivial. Moreover, the image of \( F_3^{\text{red}} \) is clearly contained in the kernel of this map. But there is a module \( W_1 \) of rank at least \( D_Y \) in \( HF_{\text{red}}(Y_1) \) which maps to \( V_1 \). This establishes Inequality (19).
Similarly, we claim that

\[ \text{rkIm} F_1^{\text{red}} \leq \text{rk} HF_{\text{red}}(Y) - \left( \frac{d(Y_1)}{2} - \frac{d_{1/2}(Y_0)}{2} \right). \tag{20} \]

There is a module \( V_2 \) or rank \( D_{Y_1} = \frac{d(Y_1)}{2} - \frac{d_{1/2}(Y_0)}{2} \) in the image of \( HF^\infty(Y_0) \) in \( HF^{\pm}(Y_0) \) which maps to zero in \( Y_1 \). It follows (from the surgery long exact sequence on \( HF^{\pm} \)) that there is a submodule of \( HF^{\pm}(Y) \) which surjects onto \( V_2 \). The grading of \( V_2 \) is congruent to \( 1/2 \) modulo 2 (see Proposition 3.4, so clearly, \( W_2 \) consists of elements with grading \( 1 \) modulo 2 in \( HF^{\pm}(Y_0) \), so \( W_2 \) injects into \( HF_{\text{red}}(Y) \). On the other hand, \( W_2 \) maps to zero under \( F_1^{\text{red}} \). This establishes Inequality (20).

We claim that for all sufficiently large \( n \),:

\[ \text{rk} HF_{\text{red}}(Y_0, s_0) \leq \text{rk} HF^{\pm}_{\leq 2n+1}(Y_0, s_0) - \left( \frac{d_{-1/2}(Y_0) - d_{1/2}(Y_0) + 1}{2} \right). \]

Combining the Euler characteristic calculations on \( Y_0 \) (Theorem 9.1 of [20] when the Spin\(^c\) structure is non-torsion and Theorem 11.15 of [20] when it is), Equation (18) and Inequalities (19), and (20), we obtain the result claimed.

It is natural to consider the following class of three-manifolds:

**Definition 6.2.** An integer homology three-sphere \( Y \) is said to be invisible if \( d(Y) = 0 \) and \( N(Y) = 0 \).

Invisibility is independent of the orientation of \( Y \), according to the following:

**Proposition 6.3.** Let \( Y \) be an integral homology three-sphere. Then, \( N(-Y) = N(Y) \) and \( d(-Y) = -d(Y) \).

**Proof.** The fact that \( N(Y) = N(-Y) \) follows from duality between \( HF^{\pm}(Y) \) and \( HF_{-}(Y) \) (the latter being cohomology) (see Proposition 7.3 of [21] and also Section 5 of [22]). The claim for \( d \) was established in Proposition 4.2.

Of course, \( S^3 \) is invisible. By the additivity of \( d \) under connected sum (Theorem 4.3) and the following lemma, the set of invisible three-manifolds is closed under connected sum:

**Lemma 6.4.** If \( N(Y_1) = 0 \) and \( N(Y_2) = 0 \), then \( N(Y_1 \# Y_2) = 0 \), as well.

**Proof.** From the long exact sequence connecting \( \hat{HF} \) with \( HF^{\pm} \), it follows that \( N(Y) = 0 \) if and only if \( \hat{HF}(Y) \) has rank one. The connected sum claim then follows from the Künneth formula for connected sums on \( \hat{HF} \) (Proposition 7.2 of [21]).
As we shall see in Section 8, \( N(\Sigma(2, 3, 5)) = 0 \). So, it follows from Lemma 6.4 that \( \Sigma(2, 3, 5) \neq -\Sigma(2, 3, 5) \) is an invisible three-manifold. Theorem 6.1 has the following consequence for invisible three-manifolds:

**Corollary 6.5.** Let \( Y \) be an invisible three-manifold, and \( K \subset Y \) be a knot in \( Y \) with non-trivial Alexander polynomial. Then \( Y_1 \) is not invisible.

**Proof.** The Alexander polynomial of a knot is non-trivial if and only if the left-hand-side of Inequality (17) is positive; so the result follows from the inequality.

Using the long exact sequence for \( 1/n \) surgeries, we have the following generalization of Theorem 6.1:

**Theorem 6.6.** Let \( Y \) be an integral homology three-sphere and \( K \subset Y \) be a knot, then there is a bound

\[
n \left( |t_0(Y)| + 2 \sum_{i=1}^d |t_i(K)| \right) \leq N(Y) + \frac{d(Y)}{2} + N(Y_1/n) - \frac{d(Y_1/n)}{2}.
\]

**Proof.** The proof proceeds exactly as before, substituting the long exact sequence for \( 1/n \) surgeries, with the additional observation that in each degree \( k \), \( HF^\infty_k(Y_0, s_0; \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z} \). Note that we are using Proposition 3.7 in place of Lemma 3.1.

This has the following immediate consequence:

**Corollary 6.7.** Let \( Y \) be an invisible three-manifold, and \( K \subset Y \) be a knot in \( Y \) with non-trivial Alexander polynomial. Then any non-trivial surgery on \( K \) gives a three-manifold which is not invisible.
7. INTEGER SURGERIES IN THE THREE-SPHERE

We will now consider consequences of the graded exact sequences to the situation where \( K \subset S^3 \) is a knot with the property that \(+p\) surgery on \( K \) gives a lens space.

Indeed, as we shall show, when \( K \subset S^3 \) is a knot for which \( S^3_0(K) \cong L(p,q) \), then, the absolute gradings together with the long exact sequence for integer surgeries (Theorem 10.19 of [20]) determine the structure of \( HF^+(S^3_0(K)) \). The methods here can be thought of as elaborations on the proof of Theorem 6.1. Note that one need consider only integer surgeries on the knot \( K \), since if a non-integral surgery on a knot in \( S^3 \) gives rise to a lens space, then the knot must be a torus knot, according to the “cyclic surgery theorem” of Culler-Gordon-Luecke-Shalen, [5].

In fact, most of the results we discuss here for integer surgeries readily generalize to the case where \( S^3 \) is replaced by an arbitrary invisible three-manifold (in the sense of Definition 6.2), and the lens space is replaced by an arbitrary three-manifold \( L \) with \( HF^+_\text{red}(L) = 0 \) and \( H_1(L,\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \). However, we state most of our results for \( S^3 \) and lens spaces, in the interest of exposition.

We recall now the integral surgeries long exact sequence. Let \( K \subset Y \) be a knot in an integral homology three-sphere and \( p \) is a positive integer, then Theorem 10.19 of [20] gives a map

\[ Q: \text{Spin}^c(Y_0) \longrightarrow \text{Spin}^c(Y_p) \]

and a long exact sequence of the form:

\[
\ldots \xrightarrow{F_1} HF^+(Y_0, [t]) \xrightarrow{F_2} HF^+(Y_p, t) \xrightarrow{F_3} HF^+(Y) \longrightarrow \ldots, \tag{21}
\]

where

\[ HF^+(Y_0, [t]) = \bigoplus_{t' \in Q^{-1}(t)} HF^+(Y_0, t'). \]

To describe \( Q \) topologically, recall that the integer surgery naturally gives rise to a cobordism \( W \) between \( Y_0, Y_p \), and the lens space \( L(p,1) \). The proof of the long exact sequence (see especially Proposition 10.15 of [20]), gives a preferred Spin\(^c\) structure \( s \) over \( L(p,1) \). If \( s \) is any Spin\(^c\) structure over \( Y_0 \), then \( Q(t) \) is the Spin\(^c\) structure over \( Y_p \) for which the triple of Spin\(^c\) structures \( s, Q(t) \) and \( u \) extend over \( W \). Indeed, the preferred Spin\(^c\) structure over \( L(p,1) \) is characterized by the following:

**Proposition 7.1.** Let \( N \) be a neighborhood of a two-sphere \( S \) with self-intersection number \(-p\). Then, \( u \) is the Spin\(^c\) structure which extends to a Spin\(^c\) structure \( s \) over \( N \) with

\[ \langle c_1(s), [S] \rangle = p. \]

**Proof.** The intersection point representing \( u \) is adjacent to the basepoint (compare Figure 7 of [20]). The rest is an application of Proposition 6.3 of [22] (as in Proposition 4.8).
There are purely algebraic constraints on realizing a given map from $\text{Spin}^c(Y_0)$ to $\text{Spin}^c(Y)$ as a map of the type $Q$ above. Of course, the two spaces of $\text{Spin}^c$ structures are principal homogeneous spaces for $H^2(Y_0) \cong \mathbb{Z}$ and $H^2(Y_p) \cong \mathbb{Z}/p\mathbb{Z}$ respectively, and the map $Q$ must be equivariant under this action (given a surjective group homomorphism from $H^2(Y_0) \longrightarrow H^2(Y_p)$). In addition, both spaces admit actions by $\mathbb{Z}/2\mathbb{Z}$, given by conjugating the $\text{Spin}^c$ structures, and the map $Q$ must also be equivariant under these $\mathbb{Z}/2\mathbb{Z}$-actions, as well.

In the following statement, recall (c.f. Proposition 4.8) that for each positive integer $p$ and each congruence class $i \in \mathbb{Z}/p\mathbb{Z}$

$$d(L(p,1),i) = \frac{(2j - p)^2 - p}{4p},$$

where we take $j$ to be the integer in the equivalence class $j \equiv i \pmod{p}$ with $0 \leq j < p$. Note that in that proposition we gave an explicit identification $\text{Spin}^c(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$.

When describing $\text{Spin}^c$ structures over the zero-surgery $Y_0$, we will find it convenient to use an identification $\text{Spin}^c(Y_0) \cong \mathbb{Z}$ induced from a choice of generator $H$ for $H_2(Y_0,\mathbb{Z})$. In particular, we write $HF^+(Y_0,i)$ to denote the group for $Y_0$ associated to the $\text{Spin}^c$ structure $t_i \in \text{Spin}^c(Y_0)$ with the property that

$$\langle c_1(s_i), [H] \rangle = 2i$$

(note that the group $HF^+(Y_0,i)$ is actually independent of the choice of generator $H$, since the groups are invariant under conjugation).

**Theorem 7.2.** Let $K \subset S^3$ be a knot in $S^3$ with the property that $p > 0$ surgery on $K$ gives the lens space $L(p,q)$, and let $Y_0 = S^3_0(K)$. Then, $HF^+(Y_0)$ has the following structure:

- The group $HF^\infty(Y_0,0)$ surjects onto $HF^+(Y_0,0)$, and $HF^+(Y_0,0)$ contains no torsion. Thus, $HF^+(Y_0,0)$ is determined by $d_{\pm 1/2}(Y_0,0)$. In fact,

$$d_{-1/2}(Y_0,0) = -\frac{1}{2},$$

and

$$d_{1/2}(Y_0,0) = d(L(p,q),Q(0)) - d(L(p,1),0) + \frac{1}{2},$$

- For each $i$ with $|i| \leq p/2$, all non-zero homogeneous elements in $HF^+(Y_0,i)$ have odd grading, and in fact we have an isomorphism of $\mathbb{Z}[U]$-modules

$$HF^+(Y_0,i) \cong \mathbb{Z}[U]/U^\ell,$$
where the integer \( \ell = \ell(p, q, Q, i) \) is given by the formula:

\[
2\ell = -d(L(p, q), Q(i)) + d(L(p, 1), i) \geq 0,
\]

For each \( i \) with \( |i| > p/2 \), \( HF^+(Y_0, i) = 0 \).

**Remark 7.3.** Of course, no generality is lost by focusing on the case of \( +p \) surgeries. If \( -p \) surgery on a knot \( K \) gives the lens space \( L(p, q) \), then we can apply the above theorem to the reflection of \( K \), \( r(K) \), bearing in mind that \( +p \) surgery on \( r(K) \) gives the lens space \( -L(p, q) = L(p, p-q) \), and also that \( S^3_0(r(K)) = -S^3_0(K) \).

**Remark 7.4.** The methods of this section actually prove a stronger statement: if \( L \) is any three-manifold with \( HF^+_{\text{red}}(L) = 0 \) and \( K \subset Y \) is a knot in an invisible three-manifold with the property that \( Y_p(K) \cong L \), then \( HF^+(Y_0) \) is uniquely specified by formulas similar to those appearing in the statement of Theorem 7.2, depending on the correction terms for \( Y \) and \( L \) (and the correspondence \( Q \)). We do not spell these out at present, since the case of lens space surgeries seems to be the most natural.

Before turning to the proof, we give some consequences of the above theorem.

**Corollary 7.5.** For each pair of relatively prime integers \((p, q)\), there is a finite set of symmetric Laurent polynomials in a variable \( T \) (explicitly determined by \( p \) and \( q \)) which can arise as the Alexander polynomial of a knot \( K \subset S^3 \) with the property that \( S^3_p(K) \cong L(p, q) \).

More precisely, if \( K \subset S^3 \) is a knot with \( S^3_p(K) \cong L(p, q) \), then there is a one-to-one correspondence

\[
\sigma: \mathbb{Z}/p\mathbb{Z} \rightarrow \text{Spin}^c(L(p, q))
\]

with the property for each integer \( i \), we have that

\[
0 \leq 2t_i(K) = \begin{cases} 
-d(L(p, q), \sigma(i)) + d(L(p, 1), i) & \text{if } 2|i| \leq p \\
0 & \text{otherwise,}
\end{cases}
\]

and the correspondence \( \sigma \) satisfies the following symmetries:

- \( \sigma(-i) = \overline{\sigma(i)} \)
- there is an isomorphism \( \phi: \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \) with the property that

\[
\sigma(i) - \sigma(j) = \phi(i - j).
\]

**Proof.** If \( K \) is as above, the equation for the torsions \( t_i(K) \) is an immediate consequence of Theorem 7.2, together with the relationship between the Euler characteristic of \( HF^+ \) for \( Y_0 \) and the torsion invariants for \( Y_0 \). The symmetry properties of \( \sigma \) follow immediately from the corresponding symmetries of \( Q \). Since there are only finitely many different possible choices for \( Q \) (corresponding to the various choices of \( \sigma \)), and the Alexander polynomial of \( K \) is uniquely determined by the torsion coefficients (c.f. Equation (1)), the first statement in the corollary follows.
This has the following special cases (stated in the introduction as Theorems 1.6 and 1.8):

**Corollary 7.6.** Suppose that $K \subset S^3$ is a knot with the property that some integer surgery along $K$ with coefficient $p$ with $|p| \leq 4$ gives a lens space, then $HF^+(S^3_K(0)) \cong HF^+(S^2 \times S^1)$ as absolutely graded groups; in particular, the Alexander polynomial of $K$ is trivial.

**Proof.** This follows from a case-by-case analysis. For each $p$ with $|p| \leq 4$, and each $q \neq 1$, it is easy to see that there is no one-to-one correspondence $\sigma$ between Spin$^c(L(p, q))$ and Spin$^c(L(p, 1))$ for which all differences $-d(L(p, q), t)) + d(L(p, 1), \sigma(t))$ are all non-negative, even integers. Indeed, when $q = 1$, the only possible correspondence is the trivial one (for which all the differences are zero), forcing $HF^+(S^3_K(0)) \cong HF^+(S^2 \times S^1)$.

The proof is straightforward, given that the correction terms for $L(2, 1)$ are given by $(-\frac{3}{4}, \frac{1}{4})$; the correction terms for $L(3, 1)$ are $(\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6})$, and those for $L(4, 1)$ are $(\frac{3}{4}, 0, -\frac{1}{4}, 0)$. \qed

The above result is special to the case where $|p| < 5$. For instance, the lens space $L(5, 4) = -L(5, 1)$ can be realized as $+5$-surgery on the right-handed trefoil knot, whose Alexander polynomial is non-trivial. Indeed, the constraints given above show that any knot $K$ with the property that $S^3_p(K) = L(5, 4)$, the Alexander polynomial is given by $\Delta_K = T - 1 + T^{-1}$. However, we do not have the following statement for general $p$ (see also Section 10.3 for a table of possible Alexander polynomials for knots giving $L(p, q)$, for small values of $p$):

**Corollary 7.7.** Suppose that $K \subset S^3$ is a knot with the property that $+p$ surgery on $K$ gives the lens space $L(p, 1)$. Then, $HF^\infty(S^3_p(K)) \cong HF^\infty(S^2 \times S^1)$ as absolutely graded groups; in particular, the Alexander polynomial of $K$ is trivial.

**Proof.** Suppose that $+p$ surgery on $K$ gives the lens space $L(p, 1)$. If $HF^+(S^3_p(K)) \not\cong HF^\infty(S^2 \times S^1)$, then there must be some non-zero $t_i(K)$, and hence there must be some $i \in \mathbb{Z}/p\mathbb{Z}$ with the property that $-d(L(p, 1), \sigma(i)) + d(L(p, 1), i) \neq 0$ (according to Theorem 7.2). Thus, there must be some (possibly different) $j \in \mathbb{Z}/p\mathbb{Z}$ so that $-d(L(p, 1), \sigma(j)) + d(L(p, 1), j)$ is negative. But this contradicts the non-negativity of the $t_i$ from Theorem 7.2. \qed

In a different direction, Theorem 7.2 gives obstructions to realizing a given lens space as integral surgery on a knot in $S^3$. For example, the condition of integrality of $\ell$ (as given in Equation (22)) could be viewed as an obstruction to obtaining $L(p, q)$ in this way. But it is not particularly strong: since it uses the correction term only modulo $2\mathbb{Z}$, it gives an obstruction only to obtaining $L(p, q)$ on a knot in any homology three-sphere. Now, there is a complete characterization of such lens spaces, due to Fintushel-Stern [8]:

\[\text{Corollary 7.8.} \]
Proposition 7.8. The lens space $L(p,q)$ can be obtained as integral surgery on a knot in an integral homology three-sphere if and only if $\pm q$ is a square modulo $p$.

Proof. We consider the link obtained by a single unknot with framing $p/q$, and another knot which links this with linking number $x$, and which is given the framing $n$. Let $Y$ be the three-manifold obtained as surgery on this link. Then $|H_1(Y; \mathbb{Z})| = |np - qx^2|$. Thus, if $\pm q$ is a square mod $p$, we can find an $n$ and $x$ such that $Y$ is a homology three-sphere.

Conversely, if $Y$ is a homology three-sphere with a knot on which an integral surgery gives $L(p,q)$, we can find instead a knot in $L(p,q)$, on which an integral surgery gives $Y$. Thus, the above argument shows that $\pm q$ is a square modulo $p$.

However, the non-negativity of the $t_i$ (coming from the the absolute $\mathbb{Z}/2\mathbb{Z}$ grading of the $\mathbb{Z}[U]/U^4$ as above) gives a more refined obstruction to realizing a fixed lens space $L(p,q)$ as surgery on some knot in $S^3$. We content ourselves here with one infinite family of lens spaces ruled out by this obstruction.

Proposition 7.9. Consider the family of lens spaces $L(p,q)$ parameterized by positive integers $k$ not divisible by four, with $p = 2k(3 + 8k)$, and $q = 2k + 1$. These spaces cannot be obtained as integral surgeries on any knot in $S^3$, though they all arise as integral surgeries on knots in homology spheres. Moreover, these lens spaces can be obtained by integral surgeries on two-component links in $S^3$.

Proof. We consider the Spin$^c$ structure on $L(2k(3 + 8k), 2k + 1)$ labelled by the integer $k$, according to the ordering given in Proposition 4.8. Since 
$$2k(3 + 8k) \equiv 1 \pmod{1 + 2k},$$
the other lens space appearing in the inductive formula is $-L(1 + 2k, 1)$. Thus after two iterations of Equation (12), we get that
$$d(-L(2k(3 + 8k), 2k + 1), k) = \frac{1 - 8k}{4}.$$In the same manner,
$$d(-L(2k(3 + 8k), 2k + 1), 4k(1 + 2k)) = \frac{1 + 2k}{4}.$$We claim also that both Spin$^c$ structures in question are spin structures. To see this, observe that the Spin$^c$ structure labelled by $i = 4k(1 + 2k)$ extends over the cobordism between $L(p,q)$ and $L(q,1)$ whose first Chern class is trivial (according to Equation (14)). Now the Spin$^c$ structure labeled $k$ differs from $i$ by $p/2$ times a generator of $H^2(L(p,q); \mathbb{Z})$ thus it, too, must come from a Spin structure.

Thus, the correspondence $Q$ must pair the spin correction term $\frac{8k - 1}{4}$ for $L(p,q)$ with one of the two possible correction terms for $L(q,1)$ which come from Spin structures,
namely, $-1/4$ and $\frac{1+2k}{4}$. Pairing with the first is ruled out by the positivity criterion of Theorem 7.2, while pairing with the second is ruled out by the integrality condition of that same theorem, in light of our hypothesis that $k$ is not divisible by 4. It follows that $+p$ surgery on a knot cannot give $L(p,q)$.

To rule out $-L(p,q)$, we observe that the other spin correction term for this manifold, $\frac{1+2k}{4}$, cannot pair with $\frac{1-p}{4}$ by the integrality criterion (and the hypothesis that $k$ is not divisible by 4). On the other hand, pairing it with $-1/4$ is once again ruled out by the positivity criterion.

On the other hand, these lens spaces all arise as surgeries in homology spheres. Specifically, consider the plumbing diagram consisting of a tree with a central node and three chains of spheres. The central node has a sphere of square $-1$, the first chain consists of spheres of square $-2$ and $x$ (to be revealed later), the second consists of a single sphere labelled with $-8k-1$, and the third consists of one node with square $-3$, and then a chain of $2k-1$ spheres with self-intersection number $-2$ (see Figure 3). When the sphere labelled with $x$ is left off, the three-manifold described is simply $+1$-surgery on the (right-handed) torus knot of type $(2,4k+1)$ (as can be seen by successively blowing down $-1$-spheres in the plumbing diagram). When $x = 0$, the three-manifold described is the lens space $L(p,q)$: the $x = 0$ sphere cancels the $-2$ sphere in the first chain, and we can then blow down the central $-1$ sphere, to obtain a single chain $2k+1$ of two-spheres, the first of which is labelled with $-8k$, and the rest with self-intersection number $-2$.

Indeed, the above procedure applied to the entire plumbing diagram, i.e. keeping the first unknot (labelled now with $x = 0$) allows us to express the lens spaces in the given family as integral surgeries on two-component links (one of whose components is the $(2,4k+1)$ torus knot, with framing $+1$).

Having seen consequences of Theorem 7.2, we turn to its proof, after some lemmas.

For the statement of Theorem 7.2, it is useful to have an alternate characterization of $Q$ (at least, up to conjugation).

Fix a knot $K \subset S^3$, and let $W(Y,K,p)$ denote the cobordism from $S^3$ to $Y_p$. This has $b_2(W) = 1$, and indeed, it has a compactly supported class $\Sigma$ with $\Sigma \cdot \Sigma = p$.

**Lemma 7.10.** Let $t$ be a Spin$^c$ structure over $Y_0$, with $\langle c_1(t), H \rangle = 2i$, where $H \in H_2(Y; \mathbb{Z})$ is a generator. Then, the Spin$^c$ structure $Q(t)$ extends over $W(Y,K,p)$ as a Spin$^c$ structure $s$ with

$$\pm \langle c_1(s), [\Sigma] \rangle \equiv p + 2i \pmod{2p}.$$

**Proof.** We can juxtapose the cobordisms $W_0$ from $Y$ to $Y_0$ and the cobordism $W$ from $Y_0 \coprod L(p,1)$ to $Y_p$ to obtain a composite cobordism $X$. By the definition of $Q$, we can find a Spin$^c$ structure over $W$ whose restriction to $Y_p$ is $Q(t)$ and whose restriction to
Figure 3. Plumbing pictures for $L(2k(3+8k), 2k+1)$. This plumbing picture exhibits $L(2k(3+8k), 2k+1)$ as integral surgery on the homology sphere $-\Sigma(2, 4k+1, 8k+1)$ (setting $x = 0$). Blowing down all $(-1)$-spheres, we obtain an description of the lens space as integral surgery on a two-component link.

$L(p, 1)$ is the canonical $\mathfrak{u}$. We can then extend this Spin$^c$ structure from $W$ to obtain a Spin$^c$ structure $s$ over all of $X$.

Now, the composite cobordism $X$ admits a different decomposition as the internal connected sum of the canonical cobordism $W(Y, K, p)$ from $Y$ to $Y_p$ with the null-cobordism of $-L(p, 1)$ as a neighborhood of a sphere $S$ of square $-p$. (This decomposition is easily seen by decomposing the Heegaard quadruple $(\Sigma, \alpha, \beta, \gamma, \delta, z)$ as a juxtaposition of two Heegaard triples in two ways.) Correspondingly, the homology $H_2(X; \mathbb{Z})$ is generated by $[\Sigma]$ and $[S]$. The image of the generator of $H$ in $X$ is represented by $\pm PD(\Sigma) - PD(S)$. It follows that

$$\langle c_1(s)|Y_0, H\rangle = \pm \langle c_1(s), [\Sigma]\rangle - \langle c_1(s), [S]\rangle \equiv \pm \langle c_1(s), [\Sigma]\rangle - p \pmod{2p}.$$ 

We also have the following result concerning degree shifts.

**Lemma 7.11.** The component of $F_1$ in Exact Sequence (21) above which carries $HF^+(Y)$ into the $t_0$-component of $HF^+(Y_0, [Q(t_0)])$ has degree $-1/2$, while the restriction of $F_2$ to the $HF^+(Y_0, t_0)$-summand of $HF^+(Y_0, [Q(t_0)])$ has degree $\left(\frac{p-3}{4}\right)$.

**Proof.** Consider the proof of the integral surgeries long exact sequence from [20], only now isotope the $\gamma$- rather than the $\delta$-curves. In so doing, we can realize $F_1$ and $F_2$ as maps defined by counting holomorphic triangles (where now $F_2$ belongs to a cobordism between $Y_0$, $Y_p$ and $L(p, 1)#(\#^{p-1}(S^2 \times S^1))$). In particular, $F_1$ is a sum of maps induced by a single two-handle addition, so it decreases grading by $1/2$. 

\[\]
The map $F_2$ counts pseudo-holomorphic triangles, induced from
\[ HF^+(Y_0, t_0) \otimes HF^{\leq 0}(-L(p, 1)\#^g-1(S^1 \times S^1), u) \longrightarrow HF^+(Y_p, t), \]
by substituting in the canonical generator of $HF^{\leq 0}(-L(p, 1)\#^g-1(S^1 \times S^1), u)$. To calculate the degree shift of this map, one follows the usual routine. We let $(\Sigma, \alpha, \beta, \gamma, \delta, z)$ be the Heegaard quadruple representing the integer surgery (so that $Y_{\alpha,\beta} = Y$, $Y_{\alpha,\gamma} = Y_0$, $Y_{\alpha,\delta} = Y_p$) fix intersection points $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $y \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ (representing $t$), and $w \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ (representing $Q(t)$), and let $\Theta_{\beta,\gamma}$, $\Theta_{\beta,\delta}$, and $\Theta_{\gamma,\delta}$ be intersection points with the canonical grading (the top non-zero grading in $HF^{\leq 0}$ of the corresponding three-manifolds). Indeed, we select the intersection points $x$, $y$, and $w$ so that there are triangles $\varphi_1 \in \pi_2(x, \Theta_{\beta,\gamma}, y)$, $\varphi_2 \in \pi_2(y, \Theta_{\gamma,\delta}, w)$ with $\mu(\varphi_1) = \mu(\varphi_2) = 0$. (As usual, in the case where $g = 1$, we need to stabilize once to achieve this.)

Now we can find alternative triangles $\psi_1 \in \pi_2(x, \Theta_{\beta,\delta}, y)$ and $\psi_2 \in \pi_2(\Theta_{\beta,\gamma}, \Theta_{\beta,\delta}, \Theta_{\gamma,\delta})$ with $\mu(\psi_2) = 0$, so that the square obtained by juxtaposing $\psi_1$ and $\psi_2$ is homotopic to the square obtained by juxtaposing $\varphi_1$ and $\varphi_2$. Since the the Spin$^c$ structure determined by $\varphi_1 + \varphi_2$ factors through the torsion Spin$^c$ structure on $Y_0$, we can use the previous lemma to conclude that if $\Sigma$ is a generator for $H_2$ of the cobordism from $Y$ to $Y_p$ (corresponding to the triple $(\Sigma, \alpha, \beta, \delta, z)$), then
\[ \langle c_1(g_1(\psi_1)), [\Sigma] \rangle = \pm p. \]
This, together with the additivity of the Maslov index, which forces $\mu(\psi_1) = 0$, gives that
\[ \tilde{\gr}(w) - \tilde{\gr}(w) = \frac{p - 5}{4}. \]
Thus, the degree shift of $F_2$ is calculated by
\[ \tilde{\gr}(w) - \tilde{\gr}(w) = \tilde{\gr}(w) - \tilde{\gr}(w) + \tilde{\gr}(x) - \tilde{\gr}(y) = \frac{p - 5}{4} + \frac{1}{2}, \]
(bearing in mind that the degree shift of $F_1$ is $\frac{1}{2}$).

It will be useful also to have the following result for three-manifolds with trivial $HF^+_{\text{red}}$.

**Lemma 7.12.** Let $K \subset Y$ be a knot in an integer homology three-sphere with $HF^+(Y) = 0$, let $p$ be a positive integer, and suppose that $HF^+_{\text{red}}(Y_p) = 0$. Then, for all integers $n \geq p$, $HF^+_{\text{red}}(Y_n) = 0$.

**Proof.** Recall that for a rational homology sphere $L$, it is always the case that
\[ |H^2(L; \mathbb{Z})| \leq \text{rk} H\hat{F}(L). \]
Moreover, the condition that $HF^+_{\text{red}}(L) = 0$ is equivalent to the condition that $\hat{H}F(L)$ is a free $\mathbb{Z}$-module with
\[ |H^2(L; \mathbb{Z})| = \text{rk} H\hat{F}(L). \]
Thus, to establish the lemma, it suffices to show that if \( \widehat{HF}(Y) \) and \( \widehat{HF}(Y_n) \) are free \( \mathbb{Z} \)-modules of rank 1 and \( n \) respectively, then \( \widehat{HF}(Y_{n+1}) \) is a free \( \mathbb{Z} \)-module of rank \( n+1 \). This, in turn, follows readily from the general form of the surgery long exact sequence (Theorem 10.12 of [20]), which specializes to give exactness in:

\[
\ldots \longrightarrow \widehat{HF}(Y) \longrightarrow \widehat{HF}(Y_n) \longrightarrow \widehat{HF}(Y_{n+1}) \longrightarrow \widehat{HF}(Y) \longrightarrow \ldots
\]

It follows immediately that \( \widehat{HF}(Y_{n+1}) \) is a free \( \mathbb{Z} \)-module whose rank satisfies:

\[
|H_2(Y_{n+1}; \mathbb{Z})| = n + 1 \leq \text{rk} \widehat{HF}(Y_{n+1}) \leq \text{rk} \widehat{HF}(Y) + \text{rk} \widehat{HF}(Y_n) = 1 + n.
\]

\[\square\]

**Proof of Theorem 7.2.** The case of \( Q(0) \) is analogous to the case of \( +1 \) surgeries considered earlier (in the proof of Theorem 6.1). Since in this case, the map \( HF^\infty(Y_0, [Q(0)]) \to HF^\infty(Y_p, Q(t_0)) \) is surjective, so we still have exactness in the middle for

\[
HF^+_\text{red}(Y) \longrightarrow HF^+_\text{red}(Y_0, [Q(0)]) \longrightarrow HF^+_\text{red}(Y_p, Q(0)),
\]

where, under the identification of Spin\(^c\)(\( Y_0 \)) \( \cong \mathbb{Z} \) given earlier, we have that

\[
HF^+_\text{red}(Y_0, [Q(0)]) \cong \bigoplus_{k \in \mathbb{Z}} HF^+_\text{red}(Y_0, kp).
\]

Since we assume that \( Y = S^3 \) and \( Y_p = L(p, q) \), it follows that \( HF^+_\text{red}(Y_0, [Q(0)]) = 0 \). This forces \( HF^+(Y_0, kp) = 0 \) for all \( k \neq 0 \). Also, \( HF^+(Y_0, 0) \) is determined by \( d_{\pm 1/2}(Y_0) \):

\[
HF^+_k(Y_0, 0) = \begin{cases} 
\mathbb{Z} & \text{if } k \equiv -\frac{1}{2} \pmod{2\mathbb{Z}} \text{ and } k \geq d_{-1/2}(Y_0) \\
\mathbb{Z} & \text{if } k \equiv \frac{1}{2} \pmod{2\mathbb{Z}} \text{ and } k \geq d_{1/2}(Y_0) \\
0 & \text{otherwise}
\end{cases}
\]

Moreover, the exact sequence guarantees that \( F_1 \) maps the image of \( HF^\infty(S^3) \) injectively into \( HF^\infty(Y_0, 0) \), while \( F_3 \) maps the elements coming from \( HF^\infty(Y_p, Q(0)) \) (which in our present case is all of \( HF^+(L(p, q), Q(t_0)) \) trivially into \( HF^+(S^3) \)). Thus, it follows that actually \( F_1 \) maps \( S^3 \) injectively into \( HF^+(Y_0, t_0) \). Since \( F_2 \) lowers degree by \( 1/2 \) (Lemma 7.11), it follows that \( d_{-1/2}(Y_0, t_0) = -1/2 \). Again, by exactness, \( F_2 \) maps the elements of grading \( \equiv 1/2 \pmod{2} \) injectively to \( L(p, q) \). By Lemma 7.11, it then follows that \( d_{1/2}(Y_0, t_0) \) is calculated by the formula claimed.

Having determined \( HF^+(Y_0, i) \) for all \( i \equiv 0 \pmod{p} \), we turn to the case of integers \( i \in \mathbb{Z} \) with the property that \( t_0 \not\in Q(i) \). In this case, \( HF^+(Y_0, [Q(i)]) \) is a finitely \( \mathbb{Z} \)-module generated, so clearly \( U^d HF^+(Y_0, [Q(i)]) = 0 \) for sufficiently large powers of \( d \). It follows (since \( U^d : HF^+(S^3) \to HF^+(S^3) \) is surjective for all \( d \)) that the image of \( HF^+(S^3) \) under \( F_1 \) is trivial, so that \( HF^+(Y_0, [Q(i)]) \) is a \( \mathbb{Z}[U] \)-submodule of
\( \text{HF}^+(L(p, q), t) \cong \mathbb{Z}[U^{-1}] \), which is a finitely generated \( \mathbb{Z} \)-module. Thus, it follows that there is some integer \( \ell \) with the property that

\[
\bigoplus_{j \equiv i \pmod{p}} \text{HF}^+(Y_0, j) \cong \mathbb{Z}[U]/U^\ell.
\]  

(23)

Indeed, since the module on the right is a cyclic \( \mathbb{Z}[U] \)-module, it follows immediately that for each integer \( i \) with \( |i| < p \), there is at most one \( j \equiv i \pmod{p} \) with \( \text{HF}^+(Y_0, j) \neq 0 \). To prove the complete proof, it remains to show that the integer in this equivalence class is the one with minimal absolute value, and then to see that \( \ell \) is determined as in Equation (22).

To show the minimality of \( j \), we proceed as follows. Observe first that the above arguments apply to a more general setting: we have shown that if \( K \subset S^3 \) is a knot with the property that for some positive integer \( n \), \( S^3_n(K) \sim L \), where \( L \) is a three-manifold with \( \text{HF}^+_{red}(L) = 0 \), then for each \( i \in \mathbb{Z}/n\mathbb{Z} \) there is at most one integer \( j \equiv i \pmod{n} \) with \( \text{HF}^+(Y_0, j) \neq 0 \). Now, returning to the lens case, note that if \( \text{HF}^+(Y_0, j) \neq 0 \), then it is also the case that \( \text{HF}^+(Y_0, -j) \neq 0 \) (by the conjugation invariance of the invariants) – so we can assume without loss of generality that \( j > 0 \). But both \( \text{HF}^+(Y_0, j) \) and \( \text{HF}^+(Y_0, -j) \) lie in the same \( 2j \)-orbit \( 2j \cdot H^2(Y_0; \mathbb{Z}) \); so it follows immediately that \( \text{HF}^+_{red}(Y_{2j}) \neq 0 \). But then Lemma 7.12 forces \( 2j < p \).

Now, it remains to express the integer \( \ell \) from Equation (23) in terms of correction terms. To this end, observe that the map \( F_3 \) is realized as a sum of maps belonging to the canonical cobordism \( W_p \) from \( Y_p \) to \( Y \):

\[
F_3 = \sum_{\{s \in \text{Spin}^c(W_p) \mid \phi_{Y_p} = Q(i)\}} \pm F^+_{W_p, s},
\]

where here \( W_p = -W(Y, K, p) \) in the notation of Lemma 7.10. Moreover, each of these component maps \( F^+_{W_p, s} \) is induced from the corresponding map on \( \text{HF}^\infty, F^\infty_{W_p, s} \). Thus, these various maps differ only by a dimension shift. Moreover, they must all be isomorphisms, since \( \text{HF}^+(Y_0, [Q(i)]) \) is finitely generated.

To calculate the dimension shift, observe that, the cobordism \( W_p \) from \( Y_p \) to \( Y \) has \( b_2 = 1 \), containing a surface \( \Sigma \) with \( \Sigma \cdot \Sigma = -p \). Now, if \( s \) is any Spin\(^c \) structure over \( W_p \), then we have that

\[
\langle c_1(s), S \rangle = -p + 2j,
\]

and indeed the integer \( j \) uniquely characterizes \( s \). Fixing the restriction of the Spin\(^c \) structure to \( Y_p \), fixes the congruence class of \( j \pmod{p} \). By the dimension formula, the map \( F^\infty_{W_p, s} \) shifts degree by

\[
\frac{p - (2j - p)^2}{4p}.
\]
When \( j \neq 0 \pmod{\mathbb{Z}} \) (and this is equivalent to the assumption that \( t \neq Q(t_0) \)), there is a unique maximal such dimension shift for all Spin\(^c\) structures with given restriction to \( Y_p \), which is found by letting \( i \) be the representative of its congruence class with \( 0 \leq i < p \). Indeed, according to Lemma 7.10, writing \( s|Y_p = Q(t) \), we have that
\[
2i \equiv \langle c_1(t), H \rangle \pmod{2p},
\]
so the maximal dimension shift is given by \(-d(L(p, 1), i)\) (compare Equation (14)).

It is now easy to see (using the fact that the \( F_{W_p,s}^{+} \) are all isomorphisms, and since the \( F_{W_p,s}^{+} \) are all induced from the maps on \( HF^{\infty} \)) that that the dimension \( \ell \) of the kernel of \( F_3 \) is given by
\[
2\ell = d(Y) - d(Y_p, Q(t)) + d(L(p, 1), i),
\]
where \( \langle c_1(t), H \rangle \equiv 2i \pmod{2p} \). In the present case, since \( Y = S^3 \), \( d(Y) = 0 \), we see that \( \ell \) satisfies Equation (22).
8. Calculations

We have seen several general results obtained by combining the absolute gradings with the surgery long exact sequences. In the present section, we calculate the Floer homologies for a number of three-manifolds using these techniques. More calculations will be given in [23].

8.1. Surgeries on torus knots, revisited. We begin with the trefoil.

The Alexander polynomial for the trefoil is $T - 1 + T^{-1}$, so $t_0 = 1$, and all other $t_i = 0$. Let $Y_0$ denote the manifold obtained by 0-surgery on the right-handed trefoil. Recall that $+5$ surgery on the right-handed trefoil gives rise to the lens space $L(5, 4)$. Thus, it follows from the long exact sequence (as applied in Theorem 7.2) that

$$HF_k^+(Y_0, s_0) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv 1/2 \pmod{2} \text{ and } k \geq -3/2 \\ \mathbb{Z} & \text{if } k \equiv -1/2 \pmod{2} \text{ and } k \geq -1/2 \\ 0 & \text{otherwise} \end{cases}$$

(24) and $HF_k^+(Y_0, s) = 0$ if $s \neq s_0$. Letting $\gamma \in H_1(Y_0, \mathbb{Z})$ be a generator, the action by $\gamma$ is an isomorphism $HF_k^+(Y_0, s_0) \rightarrow HF_{k-1}^+(Y_0, s_0)$ if $k \equiv 1/2 \pmod{2}$ and $k \geq 1/2$, and the action is trivial otherwise.

Recall that if $p, q,$ and $r$ are a triple of relatively prime integers, then the Briskorn variety $V(p, q, r)$ is the locus

$$V(p, q, r) = \{(x, y, z) \in \mathbb{C}^3|x^p + y^q + z^r = 0, |x|^2 + |y|^2 + |z|^2 = 1\}.$$ 

The Brieskorn sphere $\Sigma(p, q, r)$ is the homology sphere obtained by $V(p, q, r) \cap S^5$ (where $S^5 \subset \mathbb{C}^3$ is a standard three-sphere). This three-manifold inherits a natural orientation, as the boundary of $V(p, q, r) \cap B^6$ (which in turn is a manifold away from the origin). Now, with these orientation conventions, the three-manifold obtained as $+1$ surgery on the right-handed trefoil knot is $-\Sigma(2, 3, 5)$. Another application of the long exact sequence, this time with $+1$ surgery, shows that $HF_k^+(-\Sigma(2, 3, 5)) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } k \geq -2 \\ 0 & \text{otherwise} \end{cases}.$

Moreover,

$$U: HF_k^+(-\Sigma(2, 3, 5)) \rightarrow HF_{k-2}^+(-\Sigma(2, 3, 5))$$

is trivial when $k = 0$, otherwise it is an isomorphism: i.e. $d(-\Sigma(2, 3, 5)) = -2$ and $HF_{\text{red}}^+(-\Sigma(2, 3, 5)) = 0$.

Now, the Brieskorn sphere $\Sigma(2, 3, 7)$ is obtained as $-1$ surgery on the right-handed trefoil. The exact sequence for $-1$ surgery now reads:

$$\cdots \rightarrow HF^+(\Sigma(2, 3, 7)) \rightarrow HF^+(Y_0) \rightarrow HF^+(S^3) \rightarrow \cdots$$
Since $HF^+_k(S^3) = 0$ for all $k < 0$, the generator of $HF^+_{-3/2}(Y_0, s)$ must come from a generator of $HF^+_1(\Sigma(2, 3, 7))$. Thus, we get that

$$HF^+_k(\Sigma(2, 3, 7)) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } k \geq 0 \\ \mathbb{Z} & \text{if } k = -1 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

Moreover, $HF_{\text{red}}(\Sigma(2, 3, 7))$ has rank one and $d(\Sigma(2, 3, 7)) = 0$.

Indeed, in a similar vein, we can consider the manifold $Z_{-n}$ obtained by $-n$-surgery on the right-handed trefoil for any negative integer $-n$. Applying the long exact sequence for surgeries with negative integer coefficients, we get again that $HF^+_{\text{red}}(Z_{-n}, s) = 0$ for each $s \neq Q(s_0)$, while

$$HF^+_k(\Sigma(Z_{-n}, Q(s_0))) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } k \geq 0 \\ \mathbb{Z} & \text{if } k = -1 \\ 0 & \text{otherwise} \end{cases},$$

with $d(Z_{-n}, Q(s_0)) = 0$. This gives an alternate calculation of Proposition 8.2 from [21], in view of the fact that $Z_{-n} = -Y_n$, and that in [21], the surgery was performed on the left-handed trefoil. Indeed, even when $n$ is even, we get an explicit characterization of the Spin$^c$ structure $Q(s_0)$, justifying Remark 8.7 of [21].

To calculate fractional surgeries on the right-handed trefoil, we must first understand $HF^+(Y_0; \mathbb{Z}/n\mathbb{Z})$, for a surjective representation $\mathbb{Z} \cong H^1(Y_0; \mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$. For this, we apply the long exact sequence for positive integer surgeries with twisted coefficients, c.f. Theorem 10.23 of [20]. (Actually, there we considered the universal twisting, Larent polynomials in $T$, whereas here we specialize to $\mathbb{Z}/n\mathbb{Z}$ twisting, but the proof there works for any specialization.) The long exact sequence in this context reads as follows:

$$... \to HF^+(Y_0, s_0) \to HF^+(L(5, 4), Q(s_0))[\mathbb{Z}/n\mathbb{Z}] \to HF^+(S^3)[\mathbb{Z}/n\mathbb{Z}] \to ...$$

In the above notation, if $A$ is a $\mathbb{Z}$-module $A[\mathbb{Z}/n\mathbb{Z}]$ denotes the induced $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ module $A \otimes _{\mathbb{Z}} [\mathbb{Z}/n\mathbb{Z}]$. Of course, as a $\mathbb{Z}$ module, this is simply a direct sum of $n$ copies of $A$. Recall also that $HF^\infty(Y_0, s_0) \cong \mathbb{Z}$ for all $k \equiv \frac{1}{2} \pmod{1}$. It follows then that

$$HF^+_k(Y_0, s_0) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv 1/2 \pmod{2} \text{ and } k \geq 1/2 \\ \mathbb{Z} & \text{if } k \equiv -1/2 \pmod{2} \text{ and } k \geq -1/2 \\ \mathbb{Z}^n \cong [\mathbb{Z}/n\mathbb{Z}] & \text{if } k = -3/2 \\ 0 & \text{otherwise} \end{cases}$$

Now applying the exact sequence for $1/n$ surgeries for positive integers $n$, we get that

$$HF^+_k(Z_{1/n}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } k \geq 0 \\ \mathbb{Z}^n & \text{if } k = -2 \\ 0 & \text{otherwise} \end{cases}$$
and \( d(Z_{1/n}) = -2 \). Observe that \( Z_{1/n} \) is the Brieskorn sphere \( -\Sigma(2, 3, 6n - 1) \). Similarly, using the sequence for \( -1/n \) surgeries, we get that
\[
HF_k^+(Z_{-1/n}) \cong \begin{cases} 
\mathbb{Z} & \text{if } k \text{ is even and } k \geq 0 \\
\mathbb{Z}^n & \text{if } k = -1 \\
0 & \text{otherwise}
\end{cases},
\]
with \( d(Z_{-1/n}) = 0 \). Note also that \( Z_{-1/n} \cong \Sigma(2, 3, 6n + 1) \).

The key point which facilitated the above calculation was that some positive integral surgery on the trefoil gives rise to a lens space. More generally, we have the following:

**Proposition 8.1.** Let \( K \subset S^3 = Y \) be a knot with the property that some \( +p \) surgery on \( S^3 \) gives a lens space, and let \( Y_0 \) denote the three-manifold obtained by \( 0 \)-surgery along \( K \). Then, for each \( i \neq 0 \),
\[
HF^+(Y_0(K), i) \cong \mathbb{Z}[U]/U^{t_i}
\]
as a \( \mathbb{Z}[U] \) module, which is annihilated by the action of \( H_1(Y_0; \mathbb{Z}) \), where \( t_i = t_i(K) \).

For \( i = 0 \), \( HF^+(Y_0, 0) \) is a quotient of \( HF^\infty(Y_0, 0) \), and
\[
d_{-1/2}(Y_0) = -\frac{1}{2} \quad \text{and} \quad d_{1/2}(Y_0) = \frac{1}{2} - 2t_0.
\]

Moreover, we have that
\[
d(Y_{1/n}) = 2t_0 \quad \text{and} \quad N(Y_{1/n}) = (n - 1) \cdot t_0, +2n \sum_{i=1}^\infty t_i
\]
while
\[
d(Y_{-1/n}) = 0 \quad \text{and} \quad N(Y_{-1/n}) = n \cdot t_0 + 2n \sum_{i=1}^\infty t_i.
\]

**Proof.** The statement about \( Y_0 \) follows from the integral surgeries long exact sequence, as applied in Theorem 7.2. The statements about \( Y_{\pm 1/n} \) then follow easily from the fractional surgeries long exact sequences, as above. \(\square\)

Note that the above proposition applies to arbitrary torus knots: fix relatively prime positive integers \( p \) and \( q \), and let \( K_{p,q} \) denote the right-handed \((p, q)\) torus knot. It follows from Kirby calculus that some positive surgery of \( S^3 \) along \( K_{p,q} \) always gives a lens space. Recall also that the Alexander polynomial of \( K_{p,q} \) is given by
\[
\Delta_{K_{p,q}}(T) = T^{p+q-2} \frac{(1 - T)(1 - T^{pq})}{(1 - T^p)(1 - T^q)}
\]
(26)
The \( t_i(K_{p,q}) \) can be calculated from this in the obvious way.

Indeed, there are other knots satisfying the hypothesis of Proposition 8.1, including, for example, the \((-2, 3, 7)\) pretzel knot (see [8] for this and more examples).
8.2. **Surgeries on the figure-eight knot.** Using the Borromean rings as a stepping-stone (compare [10]), we can calculate $HF^+$ for fractional surgeries on the figure eight knot.

Following notation from [10], let $M\{p, q, r\}$ denote the three-manifold obtained from $S^3$ by surgeries on the Borromean rings with coefficients $p, q,$ and $r$. It is an exercise in Kirby calculus to see that $M\{-1, 0, 1\}$ is the manifold obtained by $p$-surgery on the figure eight knot in $S^3$, while $M\{p, 1, 1\}$ is $p$-surgery on the right-handed trefoil. In particular, $M\{-1, 1, 1\} \cong \Sigma(2, 3, 7)$.

**Proposition 8.2.** The manifold $M\{-1, 0, 1\}$, which is zero-surgery on the figure eight knot has

$$HF^+_k(M\{-1, 0, 1\}) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv \frac{1}{2} \pmod{\mathbb{Z}} \text{ and } k \geq \frac{1}{2} \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } k = -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. $$

Moreover, $d_{-1/2} = -1/2$ and $d_{1/2} = 1/2$.

**Proof.** Use the long exact sequence for the three-manifolds

$S^3 \cong M\{-1, \infty, 1\}, M\{-1, 0, 1\}, M\{-1, 1, 1\} \cong \Sigma(2, 3, 7),$ and Equation (25).

**Proposition 8.3.** Let $E_n$ denote the three-manifold obtained by $1/n$-surgery on the figure eight knot in $S^3$ (with integral $n > 0$). Then,

$$HF^+_k(E_n) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv 0 \pmod{2} \text{ and } k \geq 0 \\ \mathbb{Z}^n & \text{if } k = -1 \\ 0 & \text{otherwise} \end{cases}. $$

Moreover, $d(E_n) = 0$.

**Proof.** Using the surgery exact sequence for the three-manifolds

$M\{-1, 0, 1\}, M\{-1, \infty, 1\}, M\{-1, 1, 1\}$

with twisted coefficients (in $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$), we see that

$$HF^+_k(M\{-1, 0, 1\}; \mathbb{Z}/n\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \equiv \frac{1}{2} \pmod{\mathbb{Z}} \text{ and } k \geq \frac{1}{2} \\ \mathbb{Z} \oplus \mathbb{Z}[\mathbb{Z}/n\mathbb{Z}] & \text{if } k = -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}. $$

The result then follows from the $1/n$ surgery exact sequence. □
8.3. Connected sums. We remark that more examples can also be constructed using the connected sum theorem for $HF^-$ (see Theorem 13.1 of [20]).

For example, it follows easily from Equation (25) (together with the usual long exact sequence relating $HF^-$ and $HF^+$, Equation (3)) that $HF^-(\Sigma(2, 3, 7))$ is generated as a $\mathbb{Z}[U]$-module by a generator $\alpha \in HF^-_2(\Sigma(2, 3, 7))$ (with $U \cdot \alpha = 0$), and a free summand generated by an element $\theta \in HF^-_2(\Sigma(2, 3, 7))$. It follows from the connected sum theorem that for

$$Y = \Sigma(2, 3, 7) \# \Sigma(2, 3, 7),$$

$HF^-(Y)$ is generated as a $\mathbb{Z}[U]$-algebra by elements

$$(\theta \otimes \theta), (\alpha \otimes \alpha), (\alpha \otimes \theta), (\theta \otimes \alpha) \in HF^-_2(Y), \quad (\alpha \ast \alpha) \in HF^-_3(Y);$$

and, with the exception of $(\theta \otimes \theta)$, all of the other generators are annihilated by $U$. Dualizing again, we see that $HF^+(Y)$ is generated by three elements in $HF^+_1(Y)$, and one in $HF^+_3(Y)$, in addition to the chain of generators coming from $HF^\infty(Y)$ (bearing in mind that $d(Y) = 0$).

8.4. The three-torus.

Proposition 8.4. Let $T^3$ denote the three-dimensional torus. Then, we have $H_1(T^3; \mathbb{Z})$-module isomorphisms:

$$\widetilde{HF}(T^3) \cong H^2(T^3; \mathbb{Z}) \oplus H^1(T^3; \mathbb{Z}),$$

$$HF^+(T^3) \cong \left( H^2(T^3; \mathbb{Z}) \oplus H^1(T^3; \mathbb{Z}) \right) \otimes_{\mathbb{Z}} \mathbb{Z}[U^{-1}],$$

$$HF^\infty(T^3) \cong \left( H^2(T^3; \mathbb{Z}) \oplus H^1(T^3; \mathbb{Z}) \right) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}].$$

The absolute grading is symmetric, in the sense that $\text{gr}(H^2(T^3; \mathbb{Z}) \subset \widetilde{HF}(T^3)) = 1/2$, $\text{gr}(H^1(T^3; \mathbb{Z}) \subset \widetilde{HF}(T^3)) = -1/2$.

Observe that $HF^\infty(T^3)$ is smaller than $HF^\infty(\#^3(S^1 \times S^2))$. By analogy with Seiberg-Witten theory, this corresponds to the singular reducible in the character variety, giving rise to a center manifold picture, compare [17] and [18].

Proof. This, too, is proved by considering surgeries on the Borromean rings, continuing notation from the previous section. We find it most convenient to calculate $\widetilde{HF}$, first. Since $\widetilde{HF}(M\{1, 1, 1\}) \cong \mathbb{Z}$ which is supported in dimension $-2$, and $\widetilde{HF}(M\{1, 1, \infty\}) = \widetilde{HF}(S^3) \cong \mathbb{Z}$ (supported in dimension zero), it follows from the surgery exact sequence that $\widetilde{HF}(M\{0, 1, 1\}) \cong \mathbb{Z} \oplus \mathbb{Z}$, where the generators have degree $-1/2$ and $-3/2$. Since $b_1(M\{0, 0, 1\}) = 2$, it follows from Theorem 11.1 of [20], that for each $k$, $HF^\infty_k(M\{0, 0, 1\}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, another application of the surgery exact sequence, for the triple $M\{0, \infty, 1\} \cong S^1 \times S^2$, and $M\{0, 0, 1\}$ and $M\{0, 1, 1\}$, gives us
that $\hat{HF}(0,0,1) \cong \mathbb{Z}^2 \oplus \mathbb{Z}^2$, with two generators in dimension 0 and two in dimension $-1$.

Our final surgery exact sequence is applied to the triple $M\{0,0,\infty\} \cong \#^2(S^1 \times S^2)$, $M\{0,0,0\} \cong T^3$, and $M\{0,0,1\}$. Recall that $\hat{HF}(\#^2(S^1 \times S^2)) = \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}$ in gradings 1, 0, and $-1$ respectively. In this case, the surgery long exact sequence does not uniquely determine the groups $\hat{HF}(T^3)$, so we proceed as follows.

For simplicity, we will work over a field $\mathbb{F}$. Observe first that for each degree $i$, $\hat{HF}(T^3, \mathbb{F}) \cong \hat{HF}_{-i}(T^3, \mathbb{F})$, since $T^3$ admits an orientation-reversing diffeomorphism.

The long exact sequence then implies that $\hat{HF}(T^3) = A \oplus A$ (in degrees $1/2$ and $-1/2$), where $A$ is a $\mathbb{F}$-vector space of dimension $\leq 3$. Now, from the long exact sequence associated to the triple $(\hat{HF}, HF^+, HF^+)$, and since $\hat{HF}$ is supported in only two consecutive dimensions, it follows easily that $HF^+(T^3, \mathbb{F}) \cong (A \oplus A) \otimes \mathbb{F}[U^{-1}]$, and also that $HF^\infty(T^3, \mathbb{F}) \cong (A \oplus A) \otimes \mathbb{F}[U, U^{-1}]$.

We argue that the dimension of $A$ can be no smaller than 3, with the help of the calculation of $HF^\infty$ in the completely twisted case (Theorem 11.3 of [20]). This gives

$$HF^\infty(T^3, \mathbb{F}[H^1(T^3; \mathbb{Z})]) \cong \mathbb{F}[U, U^{-1}]$$

as a module over the ring of Laurent polynomials $\mathbb{F}[H^1(T^3; \mathbb{Z})]$. Now, we have an identification

$$CF^\infty(T^3, \mathbb{F}) \cong CF^\infty(T^3, \mathbb{F}[H^1(T^3; \mathbb{Z})]) \otimes_{\mathbb{F}[H^1(T^3; \mathbb{Z})]} \mathbb{F},$$

giving rise to a universal coefficients spectral sequence

$$\text{Tor}^i_{\mathbb{F}[H^1(T^3; \mathbb{Z})]}(HF^\infty(T^3; \mathbb{F}[H^1(T^3; \mathbb{Z})])) \Rightarrow HF_{i+j}(T^3, \mathbb{F}).$$

Clearly, $\text{Tor}^i_{\mathbb{F}[H^1(T^3; \mathbb{Z})]}(\mathbb{F}) \cong H_*(T^3; \mathbb{F})$. Thus, the $E_2$ term in the spectral sequence has the form:

$$\begin{array}{cccccc}
: & : & : & : & : & : \\
\mathbb{F} & \mathbb{F}^3 & \mathbb{F}^3 & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
0 & 0 & 0 & 0 & 0 & 0 \\
\mathbb{F} & \mathbb{F}^3 & \mathbb{F}^3 & \mathbb{F} & \mathbb{F} & \mathbb{F} \\
0 & 0 & 0 & 0 & 0 & 0 \\
: & : & : & : & : & : 
\end{array}$$

In particular, the only possible non-trivial differential is $d_3$ from the leftmost to the rightmost columns, giving a lower bound of 3 on the dimension of $A$.

Together with our previous upper bound, it follows that the dimension of $A$ is three. The proposition with $\mathbb{Z}$ coefficients then follows easily from the above statement, using fields $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$ for all primes $p$. □
We can calculate $\HF^+ (T^3)$ for completely twisted coefficients by modifying the above techniques. To state the answer, observe that there is a canonical map
\[ \epsilon : \mathbb{F}[H^1(T^3; \mathbb{Z})] \rightarrow \mathbb{Z}, \]
which sends all of $H^1(T^3; \mathbb{Z})$ to 1.

**Proposition 8.5.** There is an identification of $\mathbb{Z}[H^1(T^3; \mathbb{Z})]$-modules:
\[
\HF^+_k (T^3, s_0) \cong \begin{cases} 
0 & \text{if } k \equiv 3/2 \pmod{2} \text{ and } k \geq 3/2 \\
\mathbb{Z} & \text{if } k \equiv 1/2 \pmod{2} \text{ and } k \geq 1/2 \\
\ker \epsilon & \text{if } k = -1/2 \\
0 & \text{otherwise.}
\end{cases}
\]

Let $\mathbb{L}(t_1, \ldots, t_b)$ denote Laurent polynomials in $b$ variables, so that if $Y$ has first Betti number $b$, then $\mathbb{Z}[H^1(Y; \mathbb{Z})] \cong \mathbb{L}(t_1, \ldots, t_b)$. In this notation, then, $\ker \epsilon$ for $T^3$ consists of Laurent polynomials $f(t_1, t_2, t_3)$, with $f(1, 1, 1) = 0$.

**Lemma 8.6.** Identifying $\mathbb{Z}[H^1(M\{0, 1, 1\})] \cong \mathbb{L}(t)$, we have an identification of $\mathbb{L}(t)$-modules:
\[
\HF^+_k (M\{0, 1, 1\}) \cong \begin{cases} 
\mathbb{Z} & \text{if } k \equiv -1/2 \pmod{2} \text{ and } k \geq -\frac{1}{2} \\
\mathbb{L}(t) & \text{if } k = -3/2 \\
0 & \text{otherwise}
\end{cases}.
\]

**Proof.** Consider the twisted surgery sequence for $\HF$ connecting $M\{\infty, 1, 1\} \cong S^3$, $M\{0, 1, 1\}$, and $M\{1, 1, 1\} \cong -\Sigma(2, 3, 5)$. It follows that $\HF(M\{0, 1, 1\}) \cong \mathbb{L}(T) \oplus \mathbb{L}(T)$, generated in dimensions $-1/2$ and $-3/2$.

In general, it follows from the long exact sequence relating $\HF$ and $\HF^+$ that if $\HF_k (Y) = 0$ for all $k \geq m$, then the map $U : \HF^+_{i+1} (Y) \rightarrow \HF^+_{i-1} (Y)$ is an isomorphism for all $i \geq m$; i.e. $\HF^+_j (Y) \cong \HF^+_{\infty} (Y)$ for all $j \geq m-1$. Applying this principle to the above calculation of $\HF(M\{0, 1, 1\})$, and the general calculation of $\HF^+ (Y)$ from Theorem 11.3 of [20], the lemma follows.

**Lemma 8.7.**
\[
\HF^+_k (M\{0, 0, 1\}) \cong \begin{cases} 
\mathbb{Z} & \text{if } k \equiv 1 \pmod{2} \text{ and } k \geq 1 \\
0 & \text{if } k \equiv 0 \pmod{2} \text{ or } k \leq -2 \\
\mathbb{Z} \oplus \mathbb{L}(t_1, t_2) & \text{if } k = -1.
\end{cases}
\]
Moreover, the reduced homology group consists of only $\mathbb{L}(t_1, t_2)$ in dimension $-1$.

**Proof.** Using the long exact sequence for $\HF$ connecting $M\{0, 1, \infty\} \cong S^1 \times S^2$, $M\{0, 1, 0\}$, and $M\{0, 1, 1\}$, it follows that $\HF_k (M\{0, 1, 0\}) = 0$ for all $k \neq -1, 0$. 
It then follows that $HF^+_{-3/2}(M\{0,1,1\})$ has the claimed form in all dimensions except possibly $k = -1$. Let $f_1$ denote the map

$$f_1: HF^+(M\{0,1,1\})[t_2, t_2^{-1}] \rightarrow HF^+(M\{0,1,\infty\})[t_2, t_2^{-1}].$$

Around $k = -1$, we have the exact sequence reads:

$$0 \rightarrow (\text{Im} f_1) \cap \mathbb{L}(t_2) \rightarrow \mathbb{L}(t_2) \rightarrow HF^+_{-1/2}(M\{0,0,1\}) \rightarrow \mathbb{L}(t_1, t_2) \rightarrow 0,$$

where we use the identification $HF^+_{-1/2}(M\{0,1,\infty\})[t_2, t_2^{-1}] \cong \mathbb{L}(t_2)$. Observe here that the variable $t_1$ corresponds to a generator for the cohomologies of $S^1 \times S^2$ and $M\{0,1,1\}$, while $t_2$ corresponds to a new generator in $H^1(M\{0,0,1\})$.

We claim that the cokernel of the first map in the above exact sequence is $\mathbb{Z}$. To see this, observe that, $U$ induces an isomorphism

$$HF^+_{-3/2}(M\{0,1,\infty\})[t_2, t_2^{-1}] \cong HF^+_{-1/2}(M\{0,1,\infty\})[t_2, t_2^{-1}]$$

which gives an identification of submodules

$$(\text{Im} f_1) \cap HF^+_{3/2}(M\{0,1,\infty\})[t_2, t_2^{-1}] \cong (\text{Im} f_1) \cap HF^+_{-1/2}(M\{0,1,\infty\})[t_2, t_2^{-1}],$$

in view of the fact that multiplication by $U$ also induces an isomorphism

$$HF^+_{-3/2}(M\{0,1,1\})[t_2, t_2^{-1}] \cong HF^+_{-1/2}(M\{0,1,1\})[t_2, t_2^{-1}].$$

Thus, we get an identification between quotients modules

$$\frac{HF^+_{3/2}(M\{0,1,\infty\})[t_2, t_2^{-1}]}{(\text{Im} f_1) \cap HF^+_{3/2}(M\{0,1,\infty\})[t_2, t_2^{-1}]} \cong \frac{HF^+_{-1/2}(M\{0,1,\infty\})[t_2, t_2^{-1}]}{HF^+_{-1/2}(M\{0,0,1\})} \cong \mathbb{Z}.$$

Thus, the exact sequence in Equation (27) shows that

$$HF_{-1}(M\{0,0,1\}) \cong \mathbb{Z} \oplus \mathbb{L}(t_1, t_2),$$

as claimed.

\[ \square \]

**Proof of Proposition 8.5.** From the surgery long exact sequence for $HF$ applied to the triple $M\{0,0,\infty\} \cong \#^3(S^1 \times S^2)$, $M\{0,0,0\} \cong T^3$, and $M\{0,0,1\}$, it follows that $HF_k(T^3) = 0$ for all $k > 1/2$, so $HF^+_{k}(T^3) \cong HF^\infty_k(T^3)$ for all $k \geq 1/2$. Again, the orientation-reversing diffeomorphism of $T^3$ shows that $HF^-_{k}(T^3) \cong HF^\infty_k(T^3) = 0$, and thus that $HF^+_{k}(T^3) = 0$ for all $k \leq -3/2$.

Thus, it remains to identify $HF^+_{-1/2}(T^3)$. 
The long exact sequence gives
\begin{equation}
0 \longrightarrow HF_{-1/2}(T^3) \longrightarrow \mathbb{L}(t_3) \oplus \mathbb{L}(t_1, t_2, t_3) \xrightarrow{f^+_1} \mathbb{L}(t_3) \longrightarrow 0,
\end{equation}
where the last map is the restriction of
\[ f^+: HF^+(M\{0, 0, 1\})[t_3, t_3^{-1}] \longrightarrow HF^+(M\{0, 0, \infty\})[t_3, t_3^{-1}] \]
to the part in degree $-1$ (c.f. Lemma 8.7).

We claim that if we further restrict $f^+$ to the summand $\mathbb{L}(t_3) \subset \mathbb{L}(t_3) \oplus \mathbb{L}(t_1, t_2, t_3)$, i.e.
\[ HF^\infty(M\{0, 0, \infty\}) \subset HF^+_{-1}(M\{0, 0, \infty\}), \]
then that restriction is an injection with cokernel $\mathbb{Z}$. To see this, observe that the map induced on $HF^\infty$, $f^\infty$ has some component $g^\infty$ which preserves $\mathbb{Z}$-degree, and all the other components are translates of $g^\infty$ by various powers of the $U$-action.

In view of this, the kernel of $f^+_1$ is identified with the kernel of the induced surjection
\[ \mathbb{L}(t_1, t_2, t_3) \longrightarrow \mathbb{Z} \cong \text{Coker}(f^-_1|_{\mathbb{L}(t_3)}). \]
Any such surjection must carry $t_1$, $t_2$, and $t_3$ to units in $\mathbb{Z}$, and hence its kernel must be identified, as a $\mathbb{L}(t_1, t_2, t_3)$-module, with $\text{ker} \epsilon$. □

8.5. The skein exact sequence and some pretzel knots. Let $Y$ be a three-manifold which is obtained by surgery on a knot $K_+ \subset S^3$. Suppose that $D \subset S^3$ is an embedded disk in $S^3$ which meets $K_+$ in a pair of intersection points, but with opposite sign. There is a projection of $K_+$ for which the two strands passing through $D$ project to a crossing which is “positive” in the usual sense of knot theory (see Figure 4). Now, let $K_-$ be a new knot obtained from $K_+$ by changing the over-crossing to an undercrossing. Let $Y_+ = S^3_1(K_+)$, $Y_- = S^3_1(K_-)$, and let $Y_0$ denote the three-manifold obtained as a $+1$ surgery on $K_+$, followed by $0$-surgery on the curve $\gamma = \partial D$. By handlesliding $K$ over $\gamma$, we see that $Y_0$ could alternatively be thought of as $+1$ surgery on $K_-$ followed by $0$-surgery on $\gamma$. Indeed, if $K'$ is obtained from $K_+$ by twisting an arbitrary number of times about $D$, the manifold obtained as $+1$ surgery on $K'$ followed by $0$-surgery on $\gamma$ is diffeomorphic to $Y_0$. (Note that the manifold $Y_0$ has an alternate description as a sewn-up link complement: let $L_1$ and $L_2$ be the two components of the link obtained as the self-connected sum of $K$ using some arc in $D$, then $Y_0$ is an identification space for $S^3 - L_1 - L_2$, as observed in [15]. However, this alternate description will not be used in the present discussion.)

There is a long exact sequence of the form
\[ \ldots \longrightarrow HF^+(Y_-) \longrightarrow HF^+(Y_0) \longrightarrow HF^+(Y_+) \longrightarrow \ldots \]
Indeed, this is a special case of the general surgery long exact sequence (Theorem 10.12 of [20]), in view of the fact that $Y_+$ is obtained from $Y_-$ by a $+1$ surgery on the curve $\gamma$ (as can be seen by handle-slides over $\gamma$). Long exact sequences of this kind were introduced by Floer for his instanton homology (see [11] and [4]). The analogy with
Conway’s “skein relations” for the Alexander polynomial (and its various quantum generalizations), should be evident.

Observe also that we have stated a simplified form: the sequence holds with arbitrary integral surgery coefficient on $K_+$ (provided that we perform the same surgery over $K_-$, and the $K_+$ component to obtain $Y_0$); it also holds if the knot we choose is a single component of a Kirby calculus link.

Under favorable circumstances, we can use this “skein exact sequence” to calculate $HF^+$ of three-manifolds. We illustrate this for three-stranded pretzel knots, with odd, positive multiplicities.

Let $a$, $b$ and $c$ be any three integers, and let $P(a, b, c)$ denote the pretzel knot with with three tassels, with $a$, $b$ and $c$ crossings, counted with the sign conventions of Figure 4, respectively. For example, $P(1,1,1)$ is the right-handed trefoil, $P(-1,-1,-1)$ is the left-handed trefoil, $P(-1,1,c)$ is the unknot, and $P(-1, -1, 3)$ is the figure eight knot. (See Figure 5.)

**Proposition 8.8.** Let $\ell$, $m$, and $n$ be three non-negative integers, and let $Y(\ell, m, n)$ denote the three-manifold obtained as $0$-surgery on the pretzel knot $P(2\ell+1, 2m+1, 2n+1)$. Then,

$$HF^+_k(Y) \cong \begin{cases} 
\mathbb{Z}^{A+1} & \text{if } k = -3/2 \\
\mathbb{Z} & \text{if } k \geq -1/2 \text{ and } k \equiv \frac{1}{2} \pmod{\mathbb{Z}} \\
0 & \text{otherwise},
\end{cases}$$

where $A = mn + \ell n + \ell m + \ell + m + n$. In fact, $d_{-1/2}(Y) = -1/2$ and $d_{1/2}(Y) = -3/2$.

We find it convenient to pass through $\hat{HF}$. We calculate $\hat{HF}$ for an auxiliary three-manifold before proceeding to the proof of Proposition 8.8. Let $Y(\ell, m, *)$ denote the three-manifold which is obtained by $0$ surgery on $P(2\ell + 1, 2m + 1, 1)$, and then $0$-surgery along an unknot $\gamma_3$ which circles the third tassel. In this notation, the skein exact sequence reads

$$\ldots \rightarrow \hat{HF}(Y(0, 0, -1)) \rightarrow \hat{HF}(Y(0, 0, *)) \rightarrow \hat{HF}(Y(0, 0, 0)) \rightarrow \ldots$$
Figure 5. The (3,3,1) pretzel knot. This is the pretzel knot (3, 3, 1) with the above notation.

Observe that if we choose \( \ell, m \) and \( \ell', m' \) so that \( \ell + m = \ell' + m' \), then there is an identification \( Y(\ell, m, *) \cong Y(\ell', m', *) \). To see this, observe that each twist along the second tassel can be thought of as \(-1\)-surgery on a standard curve \( \gamma_2 \) which goes around that tassel. Handlesliding \( \gamma_2 \) over \( \gamma_3 \), we obtain a curve \( \gamma_1 \), with framing \(-1\), which circles the first tassel, which we can then remove, hence trading twists along the second tassel for twists along the first.

Lemma 8.9. Fix non-negative integers \( \ell \) and \( m \). Then,

\[
\widehat{HF}_k(Y(\ell, m, *)) \cong \begin{cases} 
\mathbb{Z}^{\ell+m+2} & \text{if } k = 0, -1 \\
0 & \text{otherwise}
\end{cases}
\]

Proof. We set \( \ell = 0 \), and fix an arbitrary field \( \mathbb{F} \). We prove that \( \widehat{HF}_k(Y(0, m, *)) \) (with coefficients in the field \( \mathbb{F} \), which we suppress from the notation) has rank \( \ell + m + 2 \) in dimensions \( k = 0 \) and \(-1\), and has rank 0 in all other degrees.

To this end, we claim that there is a skein exact sequence, which reads:

\[
\ldots \longrightarrow \widehat{HF}(Y(0, m, *)) \xrightarrow{F} \widehat{HF}(T^3) \xrightarrow{G} \widehat{HF}(Y(0, m + 1, *)) \longrightarrow \ldots,
\]

where the middle term \( \widehat{HF}(T^3) \) was calculated in Proposition 8.4. (Though we remind the reader that we are using coefficients in \( \mathbb{F} \).)

To see this, observe that for any \( m \), we can use handleslides (over \( \gamma_2 \)) to pass from the link \( P(1, 2m + 1, 1) \cup \gamma_2 \cup \gamma_3 \) (all with framing zero) to the link \( P(1, -1, 1) \cup \gamma_2 \cup \gamma_3 \), which is the Borromean rings. This identifies the middle term with \( HF^+(T^3) \).
Moreover, it also follows that $Y(0,0,*)$ is the three-manifold $M\{0,0,1\}$ in the notation of Section 8.2. Thus, when $m = 0$, the lemma was established during the proof of Proposition 8.4.

For the inductive step, the inductive hypothesis and the skein exact sequence clearly give that $\widehat{HF}_k(Y(0,m+1,*))$ for all $k \neq -1,0$. Moreover, substituting $H^2(T^3)$ and $H^1(T^3)$ for $\widehat{HF}_{1/2}(T^3)$ and $\widehat{HF}_{-1/2}(T^3)$ respectively into the surgery long exact sequence, we get exactness for:

$$0 \to H^2(T^3) \overset{G\frac{1}{2}}{\to} \widehat{HF}_0(Y(0,m+1,*)) \overset{F_0}{\to} H^1(T^3) \overset{G\frac{1}{2}}{\to} \widehat{HF}_{-1}(Y(0,m+1,*))$$

Observe that if $\delta$ is a curve which links $\gamma$ once (i.e. a copy of the curve along which we perform the surgery to go from $T^3$ to $Y(0,m+1,*))$, then the map $G$ annihilates the image of $\delta \cdot \widehat{HF}(T^3)$ (this follows from the fact that $G$ is induced by cobordisms, together with naturality of the $H^1$-action); but this image is two-dimensional inside $H^1(T^3)$. Thus, the rank of $F_0$ is either two or three.

On the other hand, the following diagram commutes

$$\begin{array}{ccc}
\widehat{HF}_0(Y(0,m+1,*)) & \overset{F_0}{\to} & \widehat{HF}_{-1}(Y(0,m+1,*)) \\
\downarrow & & \downarrow \\
\widehat{HF}_{1/2}(T^3) & \overset{\cong}{\to} & \widehat{HF}_{-1/2}(T^3),
\end{array}$$

and since $\widehat{HF}_k(Y) = 0$ for all $k > 0$, it follows that

$$HF_0^+(Y(0,m+1,*)) \cong HF^\infty_0(Y(0,m+1,*)) \cong \mathbb{F}^2.$$

This rules out the possibility that $F_0$ surjects.

It follows then that the rank of $\widehat{HF}_0(Y(0,m+1,*))$ is $m + 3$. This in turn forces $\widehat{HF}_{-1}(Y(0,m+1,*))$ as well (since all other groups are zero, and the Euler characteristic of $\widehat{HF}$ is trivial). Thus, we have established the lemma for all $m$. Since the field $\mathbb{F}$ was arbitrary, the case of integer coefficients follows immediately from the universal coefficients theorem.

Since $Y(\ell,m,*) \cong Y(0,\ell + m,*)$, the lemma follows for all non-negative $\ell$ and $m$. $\square$

**Proof of Proposition 8.8.**

As usual, we fix an arbitrary field $\mathbb{F}$. We prove for all non-negative $\ell$, $m$, and $n$ that

$$\widehat{HF}_k(Y(\ell,m,n)) \cong \begin{cases} 
\mathbb{F}^{A+1} & \text{if } k = -1/2 \text{ or } -3/2 \\
0 & \text{otherwise,}
\end{cases}$$

where, as before, $A = mn + \ell n + \ell m + \ell + m + n$.

To establish Equation (29) for three-manifolds with $\ell = m = 0$, we proceed as follows. Observe that $Y(-1,0,0) \cong S^1 \times S^2$ (since $P(-1,1,2n+1)$ is the unknot; indeed, the
curve $\gamma_3$ is not linked with $P(-1, 1, 2n + 1))$, giving a skein exact sequence
\[\ldots \longrightarrow \hat{HF}(S^1 \times S^2) \xrightarrow{F} \hat{HF}(Y(\ast, 0, n)) \xrightarrow{G} \hat{HF}(Y(0, 0, n)) \longrightarrow \ldots\]

In view of Lemma 8.9, we get that
\[\hat{HF}_{-1/2}(S^1 \times S^2) \cong \mathbb{F} \xrightarrow{F_{-1/2}} \mathbb{F}^{n+2} \xrightarrow{G_{-1}} \hat{HF}_{-3/2}(Y(0, 0, n)) \longrightarrow 0\]

Letting $\delta$ be a curve linking the surgery curve $\gamma$, we have once again that $G_{-1}$ annihilates the image of the $\delta$-action on $\hat{HF}(Y(\ast, 0, n))$. Observe that image of that action is non-trivial: this follows easily from the fact that the $H_1$ action on $HF_{\infty}(Y(\ast, 0, n))$ is non-trivial. Thus, $HF_{-3/2}(Y(0, 0, n)) \cong \mathbb{F}^{n+1}$. Next, let $\epsilon$ be a curve linking the pretzel knot, so that it represents non-trivial homology classes in both $Y(0, 0, n)$ and $Y(\ast, 0, n)$. The map $F$ is equivariant with respect to action by $\epsilon$, and since the generator of $\hat{HF}_{-1/2}(S^1 \times S^2)$ lies in the image of this action, it follows that the map $F_{1/2}$ is injective. It then follows immediately that $\hat{HF}_k(Y(0, 0, n))$ has the stated form for all $k$.

Having established Equation (29) for $Y(0, m, n)$ with $m = 0$, we prove the equation with $\ell = 0$ and arbitrary $m, n$ by induction on $m$. In this case, the skein exact sequence reads:
\[\ldots \longrightarrow \hat{HF}(Y(0, m, n)) \xrightarrow{F} \hat{HF}(Y(0, \ast, n)) \xrightarrow{G} \hat{HF}(Y(0, m + 1, n)) \longrightarrow \ldots\]

showing that $\hat{HF}(Y(0, m + 1, n))$ is supported in dimensions $k = -3/2$ and $k = -1/2$. Thus
\[\hat{HF}_{-1/2}(Y(0, m, n)) \xrightarrow{F_{-1/2}} \mathbb{F}^{n+2} \longrightarrow \hat{HF}_{-3/2}(Y(0, m + 1, n)) \longrightarrow \mathbb{F}^{mn+m+n} \longrightarrow 0.\]

We claim that the image of $F_{-1/2}$ is no more than one dimensional, by the commutativity of the following diagram:
\[\begin{array}{ccc}
\hat{HF}_{-1/2}(Y(0, m, n)) & \longrightarrow & HF_{-1/2}^+(Y(0, m, n)) \\
F_{-1/2} & & \downarrow \\
\hat{HF}_{-1}(Y(0, \ast, n)) & \longrightarrow & HF_{-1}^+(Y(0, \ast, n))
\end{array}\]

bearing in mind that, since $\hat{HF}_k(Y(0, m, n)) = 0$ for all $k \geq 1/2$, we have that $HF_{-1/2}^+(Y(0, m, n)) \cong HF_{-1/2}^+(Y(0, m, n)) \cong \mathbb{F}$; and also, since $HF_{k}(Y(0, \ast, n)) = 0$ for all $k < -1$, the natural map from $\hat{HF}_{-1}(Y(0, \ast, n))$ to $HF_{-1}^+(Y(0, \ast, n))$ is an isomorphism.

On the other hand, the image of $F_{-1/2}$ is no less than one dimensional. Letting $\delta$ be a curve linking $\gamma_2$, it is easy to see that $HF_{-1}(Y(0, \ast n))$ contains an element in the image $\delta \cdot HF_0(Y(0, \ast n))$.

Thus, we have shown that the rank of $HF_{-1/2}^+(Y(0, m+1, n))$ is $(m+1)n+(m+1)+n$. Since the ranks of $HF_k^+(Y(0, m+1, n))$ agree for $k = -1/2$ and $k = -3/2$, we have
established Equation (29) for three-manifolds of the form $Y(0, m, n)$ for arbitrary $m$ and $n$.

This same argument is easily modified to give the inductive step (on $\ell$), establishing Equation (29) for all non-negative $\ell$, $m$, and $n$.

Going from Equation (29) to the corresponding statement with coefficients in $\mathbb{Z}$ follows, as usual, from the universal coefficients theorem, while going from Equation (29) to the proposition (using $HF^+$) is now a straightforward application of the long exact sequence relating $\widehat{HF}(Y)$ with $HF^+(Y)$. \qed
9. Negative-definite intersection forms of smooth four-manifolds

The aim of the present section is to use the maps associated to cobordisms to give restrictions on intersection forms of smooth four-manifolds.

We begin by giving another proof of the celebrated diagonalizability theorem of Donaldson [6]. The proof parallels the Seiberg-Witten proof, though the discussion of reducibles is replaced with the behaviour of the maps on $HF^\infty$. In Subsection 9.2, we give generalizations for four-manifolds-with-boundary, which use the correction term $d(Y)$ for the boundary, in the spirit of Frøyshov [12]. In Subsection 9.3, we give further generalizations for four-manifolds which bound three-manifolds with larger $b_1(Y)$. As an application of these inequalities, we give another proof of the “Thom conjecture” for $\mathbb{CP}^2$ in Subsection 9.4.

9.1. Intersection forms of closed, smooth four-manifolds. In the present subsection, we give another proof of the following result:

**Theorem 9.1.** (Donaldson) If $X$ is a smooth, closed, oriented four-manifold with definite intersection form, then the form is diagonalizable over $\mathbb{Z}$.

The theorem follows from two propositions, regarding the map on $HF^\infty$ induced by the addition of a two-handle.

We say that $HF^\infty(Y)$ is standard if for each torsion Spin$^c$ structure $t_0$,

$$HF^\infty(Y, t_0) \cong (\Lambda^b H^1(Y; \mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}],$$

where $b = b_1(Y)$.

**Lemma 9.2.** Let $Y$ be a three-manifold with $b_1(Y) = b > 0$, equipped with a torsion Spin$^c$ structure $t_0$. Then in each dimension $i$, we have that

$$\operatorname{rk} HF^\infty_i(Y, t_0) \leq 2^{b-1}.$$

**Proof.** Consider the universal coefficients spectral sequence whose $E_2$ term is

$$\operatorname{Tor}^\mathbb{Z}_{[H^1(Y)]}(\mathbb{Z}, \mathbb{Z}) \otimes HF^\infty_0(Y, t_0),$$

and which converges to $HF^\infty_{i+j}(Y, t_0)$. By the calculation of $HF^\infty_0(Y, s_0)$ (Theorem 11.3 of [20]), in each degree, the rank of the the $E_2$ term is $2^{b-1}$. Now, the ranks of the $E_r$ are non-increasing in $r$, the inequality follows. \hfill $\Box$

Actually, the rank bound above holds with coefficients in any field $\mathbb{F}$, in view of the identification $\operatorname{Tor}^\mathbb{F}_{[H^1(Y; \mathbb{Z})]}(\mathbb{F}, \mathbb{F}) \cong \Lambda^b H_1(Y; \mathbb{F})$.

**Proposition 9.3.** Let $Y$ be closed oriented three-manifold, and $\mathbb{K} \subset Y$ a framed knot, framed so that the cobordism $W(\mathbb{K})$ has $b_2^-(W(\mathbb{K})) = 0 = b_2^+(W(\mathbb{K}))$. Let $s$ be a Spin$^c$ structure on $W(\mathbb{K})$ whose restrictions to the boundary components $Y$ and $Y(\mathbb{K})$, $t$ and
\( \xi \), are torsion. When \( \mathbb{K} \) represents a non-torsion class in \( H_1(Y) \), then if \( HF^\infty(Y, t) \) is standard, then the induced map
\[
F_{W(\mathbb{K}), s}^\infty \colon HF^\infty(Y, t) \longrightarrow HF^\infty(Y(\mathbb{K}), t)
\]
vanishes on the kernel of the action by \([K]\), inducing an isomorphism on
\[
HF^\infty(Y, t)/\text{Ker}[K] \cong HF^\infty(Y(\mathbb{K}), t).
\]
If \( \mathbb{K} \) represents a torsion class in \( H_1(Y) \) and \( HF^\infty(Y(\mathbb{K}), t) \) is standard, then the map
\[
HF^\infty(Y, t) \longrightarrow HF^\infty(Y(\mathbb{K}), t)
\]
induces an isomorphism between \( HF^\infty(Y, t) \) and the kernel of the action by \([L]\) on \( HF^\infty(Y(\mathbb{K}), t) \), where \([L] \in H_1(Y(\mathbb{K})) \) is the core of the glued-in solid torus.

**Proof.** Suppose that \( HF^\infty(Y) \) is standard, and \( \mathbb{K} \subset Y \) is a framed knot whose underlying knot \( K \) represents a non-torsion homology class. Assume first that we are working with \( HF^+ \) and \( HF^\infty \) with coefficients in a field, which we drop from the notation for simplicity.

We find another framing on \( K \), denoted \( \mathbb{K}' \) with the property that \( Y \) and \( Y(\mathbb{K}) \) fit into a surgery long exact sequence

\[
\begin{align*}
\cdots & \longrightarrow HF^+(Y(\mathbb{K}')) \\
& \xrightarrow{F_1} HF^+(Y) \\
& \xrightarrow{F_2} HF^+(Y(\mathbb{K})) \\
& \longrightarrow \cdots,
\end{align*}
\]
where we use the general form of the exact sequence, as stated in Theorem 10.12 of [20]. (The framing \( \mathbb{K}' \) is one bigger than the framing \( \mathbb{K} \).) Here, as usual, \( F_1 \) and \( F_2 \) are sums (taken with appropriate signs) of the maps induced by the two-handle additions.

We have a splitting for each torsion \( \text{Spin}^c \) structure \( t \) on \( Y \) of \( HF^\infty(Y, t) \cong \text{Ker}[K] \oplus \text{ker}[K]^+ \), where \( \text{Ker}[K] = \text{Im}[K] \) is the kernel of the action by \([K] \in H_1(Y) \) on \( HF^\infty(Y, t) \) and \( \text{ker}[K]^+ \) is a complementary subspace taken isomorphically to \( \text{Ker}[K] \) by multiplication by \([K]\).

It follows from the fact that the knot \( K \) is torsion in \( W(\mathbb{K}) \) that \( F_{W(\mathbb{K}), s}^+ \) vanishes on the image of the action of \([K]\) on \( HF^\infty(Y, s|Y) \), for each \( \text{Spin}^c \) structure \( s \) over \( W(\mathbb{K}) \); i.e. \( F_{W(\mathbb{K}), s}^\infty \) is trivial on the kernel of \([K]\). In fact, the induced map
\[
F_{W(\mathbb{K}), s}^\infty \colon HF^\infty(Y, t)/\text{Ker}[K] \longrightarrow HF^\infty(Y(\mathbb{K}), t)
\]
is injective, because if it had a kernel element, that would give \( \xi \in HF^\infty(Y, t) \) with \( K \cdot \xi \neq 0 \) so that \( F_{W(\mathbb{K}), s}^\infty(\xi) = 0 \). Moreover, that element \( \xi \) would be in the kernel of \( F_{W(\mathbb{K}), s'}^\infty \) for any \( s' \in \text{Spin}^c(W(\mathbb{K})) \) whose restriction \( s'|Y \cong t \). (This is clear, by moving the basepoint \( z \) in the Heegaard triple representing the cobordism from \( Y \) to \( Y(\mathbb{K}) \).) Thus, \( F_2(\xi) = 0 \) in the surgery long exact sequence. By taking preimage of \( \xi \) under a sufficiently large power of the \( U \) map, this would give rise to a non-zero element in the image of \( HF^+(Y(\mathbb{K}')) \) inside \( HF^+(Y) \) which does not lie in the kernel of the action by
[K]. But such an element cannot exist, since [K] annihilates the image of $HF^{+}(Y(\mathbb{K}'))$ (as $K$ is torsion in the cobordism from $Y(\mathbb{K}')$ to $Y$).

Since, in each dimension $k$, the rank of $HF^{\infty}_{k}(Y, t)/\text{Ker}[K]$ agrees with the upper bound of the rank of $HF^{\infty}_{k}(Y(\mathbb{K}), t)$ given by Lemma 9.2 (here we are using the hypothesis that $Y$ has standard $HF^{\infty}$), it follows that $F^{\infty}_{W(\mathbb{K}), s}$ is an isomorphism, and that $HF^{\infty}(Y(\mathbb{K}))$ is standard.

To pass from field to $\mathbb{Z}$ coefficients in this case, we observe that the above proof actually shows that $F^{\infty}_{W(\mathbb{K}), s}$ always gives an injection. Applying Lemma 9.2 with coefficients in $\mathbb{Z}/p\mathbb{Z}$ for each prime, it follows that $HF^{\infty}(Y(\mathbb{K}), t)$ is a free module. Now, the fact that $F^{\infty}_{W(\mathbb{K}), s}$ is an isomorphism over $\mathbb{Z}$ follows easily from the universal coefficients theorem, together with the fact that $F^{\infty}_{W(\mathbb{K}), s}$ is an isomorphism with coefficients in each $\mathbb{Z}/p\mathbb{Z}$.

When $\mathbb{K}$ represents a torsion class in $Y$, from the previous arguments (letting $Y(\mathbb{K})$ play the role of $Y$ earlier and $Y(\mathbb{K}')$ play the role of $Y(\mathbb{K})$), we see that $\text{Ker}[L]$ maps to zero in $Y(\mathbb{K}')$. In particular, it follows that there is an element of $HF^{\infty}(Y, t)$ which maps to an element of $\text{Ker}[L]$, which generates the submodule $\text{Ker}[L]$ over the ring $\mathbb{Z}[U, U^{-1}] \otimes_{\mathbb{Z}} \Lambda^{*}H_{1}(Y(\mathbb{K}); \mathbb{Z})$. From naturality of the maps of cobordism, it follows that $F^{\infty}_{W(\mathbb{K}), s}$ surjects onto $\text{Ker}[L]$. Another appeal to Lemma 9.2 and the hypothesis on the standardness of $HF^{\infty}(Y(\mathbb{K}))$ gives that the map is an isomorphism (with coefficients in any field, and hence with coefficients in $\mathbb{Z}$ as above). \hfill $\Box$

**Proposition 9.4.** Let $Y$ be a closed oriented three-manifold, and $\mathbb{K} \subset Y$ a framed knot, framed so that the cobordism $W(\mathbb{K})$ has $b_{2}^{+}(W(\mathbb{K})) = 1$. If $HF^{\infty}(Y)$ is standard, then for each $\text{Spin}^{c}$ structure $s$ over $W(\mathbb{K})$ whose restriction to the boundary components $Y$ and $Y(\mathbb{K}')$ is torsion, the induced map $F^{\infty}_{W(\mathbb{K}), s}$ is an isomorphism.

**Proof.** Let $\mathbb{K}'$ denote the same knot, endowed with a framing one greater than $\mathbb{K}$, i.e. so that Exact Sequence (30) holds. Now, there are two cases. Either $b_{1}(Y(\mathbb{K}')) = b_{1}(Y)$, or $b_{1}(Y(\mathbb{K}')) = b_{1}(Y) + 1$. In both cases, we consider the long exact sequence between $Y(\mathbb{K}), Y$, and $Y(\mathbb{K}')$ (observing that all three maps are induced by cobordisms).

When $b_{1}(Y(\mathbb{K}')) = b_{1}(Y)$, we claim that the cobordism from $Y(\mathbb{K}')$ to $Y$ has $b_{2}^{+} = 1$. Since the associated map shifts the absolute $\mathbb{Z}/2\mathbb{Z}$ grading by one, and $HF^{\infty}$ is supported in even degrees, it follows that the corresponding map on $HF^{\infty}$ must vanish (c.f. Lemma 8.2 of [22]). Thus, the image of $HF^{+}(Y(\mathbb{K}'))$ inside $HF^{+}(Y)$ is finitely generated. Since $HF^{+}(Y)$ is infinitely generated, it follows that the map on $HF^{\infty}$ from $Y$ to $Y(\mathbb{K})$ must be injective. Restricting attention to $\mathbb{Z}/p\mathbb{Z}$ coefficients where $p$ is any prime, and counting ranks as in Lemma 9.2, it follows that the map from $Y$ to $Y(\mathbb{K})$ is an isomorphism. We can then pass to $\mathbb{Z}$ coefficients as before.

Assume that $b_{1}(Y(\mathbb{K}')) = b_{1}(Y) + 1$, and again work with $\mathbb{Z}/p\mathbb{Z}$ coefficients. The knot complement gives a natural representation $H^{1}(Y(\mathbb{K}')) \rightarrow \mathbb{Z}$, which in turn gives
us a possible twisting of $HF^\infty(Y(\mathbb{K}', \mathbb{Z}/p\mathbb{Z}[T, T^{-1}])$. The twisted long exact sequence reads as follows:

$$... \rightarrow HF^+(Y(\mathbb{K}))[T, T^{-1}] \rightarrow \frac{HF^+(Y(\mathbb{K}', \mathbb{Z}[T, T^{-1}]})}{HF^+(Y)(T, T^{-1})} \rightarrow ..., $$

which we can further specialize to $\mathbb{Z}/p\mathbb{Z}$ coefficients. If the map $F^\infty_{W(\mathbb{K}), s}: HF^\infty(Y, \mathbb{Z}/p\mathbb{Z}) \rightarrow HF^\infty(Y(\mathbb{K}), \mathbb{Z}/p\mathbb{Z})$ had kernel, it would follow that $HF^\infty(Y(\mathbb{K}', \mathbb{Z}/p\mathbb{Z}[T, T^{-1}])$ would have (infinitely many) submodules with non-trivial $T$-action. But this contradicts the fact that the group $HF^\infty(Y(\mathbb{K}', \mathbb{Z}/p\mathbb{Z})$ with totally twisted coefficients has a trivial action by the group-ring $\mathbb{Z}/p\mathbb{Z}[H^1(Y; \mathbb{Z})]$. It follows that the map $F^\infty_{W(\mathbb{K}), s}$ is injective and hence, by Lemma 9.2 (in view of the fact that $HF^\infty(Y, \mathbb{Z}/p\mathbb{Z}, t)$ is standard), it is an isomorphism. Again, since this argument works for all primes $p$, the statement holds for integral coefficients as well.

The proof of Theorem 9.1 relies on the following result of Elkies, see [7]. Recall that if

$$Q: V \otimes V \rightarrow \mathbb{Z}$$

is bilinear form over $\mathbb{Z}$, then $\xi \in V$ is called a characteristic vector if for all $v \in V$,

$$Q(\xi, v) \equiv Q(v, v) \pmod{2}.$$ 

We denote the set of characteristic vectors for $Q$ by $\Xi(Q)$.

**Theorem 9.5.** (Elkies) Let $Q$ be a negative-definite unimodular bilinear form over $\mathbb{Z}$. Then,

$$0 \leq \max_{\xi \in \Xi(Q)} Q(\xi, \xi) + n,$$

with equality if and only if the bilinear form $Q$ is diagonalizable over $\mathbb{Z}$.

**Proof of Theorem 9.1.** Without loss of generality, we can assume that $b_1(X) = 0$ (by surgering out the one-dimensional homology). We give $X$ a handle decomposition with a unique zero- and four-handle, and let $W$ be the associated cobordism from $S^3$ to $S^3$. Decompose $W = W_1 \cup W_2 \cup W_3$ into its one-, two-, and three-handles.

We claim that for any $\text{Spin}^c$ structure $\mathfrak{s}$ over $X$,

$$F^\infty_{W, \mathfrak{s}}: HF^\infty(S^3, t_0) \rightarrow HF^\infty(S^3, t_0)$$

is an isomorphism. To see this, we think of $W_2$ as given by a framed link $L = \mathbb{K}_1 \cup ... \cup \mathbb{K}_m$ in $\#^{n_1}(S^2 \times S^1)$ (where $n_1$ is the number of one-handles in $X$ for our handle-decomposition), and let

$$Y_0 = \#^{n_1}(S^2 \times S^1), Y_1 = Y_0(\mathbb{K}_1), Y_2 = Y_1(\mathbb{K}_2), ..., Y_m = \#^{n_3}(S^2 \times S^1)$$

(where $n_3$ is the number of three-handles in the handle-decomposition). We claim that since $b_2^+ (X) = 0$, the restriction of $\text{Spin}^c$ to $Y_i$ is always torsion, and also $\delta H^1(Y_i; \mathbb{Z}) \subset X$
is disjoint from \( K \) isomorphically; moreover, by Proposition 9.4 the further composite then we can reorder the knots so that \( K \) in the first sequence, it follows that both \( K \) to be in a standard ordering.

Moreover, we claim that we can order the knots so that for \( i = 1, \ldots, a \), \( b_1(Y_i) \) is decreasing, for \( i = a+1, \ldots, b \), \( b_1(Y_i) \) is constant (and hence zero), and for \( i = a+1, \ldots, m \), \( b_1(Y_i) \) is increasing. We call this a standard ordering. To achieve a standard ordering, we use two moves.

If
\[
b_1(Y) < b_1(Y(\mathbb{K}_1)) > b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)),
\]
then we can reorder the knots so that
\[
b_1(Y) > b_1(Y(\mathbb{K}_2)) < b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)).
\]
To see this, observe that the inequality \( b_1(Y) < b_1(Y(\mathbb{K}_1)) \) implies that \( K_1 \) is a torsion class in \( Y \). It is our goal now to rule out the possibility that \( b_1(Y) \leq b_1(Y(\mathbb{K}_2)) \). If this inequality were satisfied, then it would follow that \( K_2 \) is torsion in \( Y \). Now, if \( K_1 \) and \( K_2 \) were unlinked, then \( K_2 \) would be torsion in \( Y(\mathbb{K}_1) \) as well, contradicting the assumption that \( b_1(Y(\mathbb{K}_1)) < b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)) \). If \( K_1 \) and \( K_2 \) had linking number \( \ell \neq 0 \), then we would be able to find a pair of surfaces \( F_1, F_2 \in \mathcal{W}(\mathbb{K}_1 \cup \mathbb{K}_2) \) (by capping off the null-homologies of \( n_1K_1 \) and \( n_2K_2 \) in \( Y - \mathbb{K}_1 \cup \mathbb{K}_2 \)) with \( F_1 \cdot F_1 = 0, F_2 \cdot F_2 = 0, \) and \( F_1 \cdot F_2 = \ell \), which contradicts \( b_1^+(\mathcal{W}(\mathbb{K}_1 \cup \mathbb{K}_2)) = 0 \).

For the second move, observe that if
\[
b_1(Y) < b_1(Y(\mathbb{K}_1)) = b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)),
\]
then we can reorder the knots so that
\[
b_1(Y) = b_1(Y(\mathbb{K}_2)) < b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)).
\]
To see this, observe that since the first Betti numbers of the three-manifolds do not drop in the first sequence, it follows that both \( K_1 \) and \( K_2 \) represent torsion classes in \( H_1(Y) \); moreover it also follows that \( K_1 \) and \( K_2 \) are unlinked. Thus, \( K_2 \) bounds a Seifert surface which is disjoint from \( K_1 \). Since \( b_1(Y(\mathbb{K}_1)) = b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)) \), the Seifert framing of \( K_2 \) does not agree with the prescribed framing \( \mathbb{K}_2 \). It follows that \( b_1(Y) = b_1(Y(\mathbb{K}_2)) \), and hence also that \( b_1(Y(\mathbb{K}_2)) < b_1(Y(\mathbb{K}_1 \cup \mathbb{K}_2)) \).

Clearly, by applying the above two moves as necessary, we can arrange for the knots to be in a standard ordering.

It follows easily from the definition of the maps induced by one-handles that
\[
F_{W_1,s_1}(HF^\infty(S^3, t_0)) \subset HF^\infty(Y_0, s_0)
\]
is the subgroup
\[
\ker[K_1] \cap \ldots \cap \ker[K_n] \subset HF^\infty(Y_0, s_0).
\]
Now, since \( HF^\infty(Y_0, s|Y_0) \) is standard \( (Y_0 \cong \#^n(S^2 \times S^1)) \) successively applying Proposition 9.3 (in the case where the \( b_1(Y(\mathbb{K})) < b_1(Y) \)), we see that \( HF^\infty(S^3) \) is mapped isomorphically; moreover, by Proposition 9.4 the further composite \( \mathbb{K}_{a+1} \cup \ldots \mathbb{K}_b \) maps
$HF^\infty(S^3, t_0)$ isomorphically to $HF^\infty(Y_\text{hor}, t|Y_\text{hor})$. Successively applying Proposition 9.3 (this time, in the case where the knots are homologically trivial), we see that the composite cobordism carries $HF^\infty(S^3, t_0)$ isomorphically to the subgroup of $HF^\infty(Y_m, t|Y_m)$ which is annihilated by the action of $H_1(Y_m)$. But it follows easily from the definition of the maps induced by three-handles that this group is carried isomorphically to $HF^\infty(S^3, t_0)$ under $F_{W_3, s|W_3}$.

Since $F_{W_3, s|W}$ is an isomorphism, and $HF^\infty(S^3, t_0) \rightarrow HF^+(S^3, t_0)$ is surjective, it follows that

$$F_{W_3, s|W}^+: HF^+(S^3, t_0) \rightarrow HF^+(S^3, t_0)$$

is surjective; in particular, we can find some $\xi \in HF^+(S^3)$ so that $F_{W, s}(\xi) \neq 0$ and $\tilde{\text{gr}}(F_{W, s}(\xi)) = 0$. Thus, the dimension formula (Equation (4)) gives that

$$\tilde{\text{gr}}(F_{W, s}(\xi)) - \tilde{\text{gr}}(\xi) = \frac{c_1(s)^2 - 2\chi(W) - 3\text{sgn}(W)}{4} = \frac{c_1(s)^2 + b_2(X)}{4} \leq 0.$$

This shows that for any characteristic vector for the intersection form of $H^2(X)$, we have that

$$\xi^2 + n \leq 0.$$

It follows then from Elkies’ theorem cited above that the intersection form $H^2(X; Z)$ is diagonalizable.

9.2. Intersection forms of definite four-manifolds bounding homology three-spheres. We give now the generalization of Theorem 9.1 to four-manifolds bounding rational homology three-spheres.

Let $Y$ be a rational homology three-sphere and $X$ be a four-manifold which bounds $Y$. The intersection form of $X$ determines a non-degenerate bilinear form

$$Q_X : (H_2(X; Z)/\text{Tors}) \otimes (H_2(X; Z)/\text{Tors}) \rightarrow \mathbb{Q}.$$

More precisely, the image lies in the subgroup $\frac{1}{|H_1(Y; Z)|} \mathbb{Z}$.

**Theorem 9.6.** Let $Y$ be a rational homology three-sphere, and fix a Spin$^c$ structure $t$ over $Y$. Then, for each smooth, negative-definite four-manifold $X$ which bounds $Y$, and for each Spin$^c$ structure $s \in \text{Spin}^c(X)$ with $s|Y \equiv t$, we have that

$$c_1(s)^2 + \text{rk}(H^2(X; Z)) \leq 4d(Y, t).$$

**Proof.** We view $X$ minus a point as a cobordism $W$ from $S^3$ to $Y$, and proceed as in the proof of Theorem 9.1, to prove that for each Spin$^c$ structure $s$ over $X$,

$$F_{W, s|W}^\infty : HF^\infty(S^3, t_0) \rightarrow HF^\infty(Y, t)$$

is an isomorphism. Note that now the two-handles give rise to a cobordism to $Y \# (#^n_3(S^2 \times S^1))$ which, again, has standard $HF^\infty$. The map

$$F_{W, s|W}^\infty : HF^\infty(Y \# (#^n_3(S^2 \times S^1)), t#t_0) \rightarrow HF^\infty(Y, t)$$
induces an isomorphism from \( \text{Ker} H_1(\#^n_3(S^2 \times S^1); \mathbb{Z}) \subset HF^\infty(Y \#(\#^n_3(S^2 \times S^1)), t \# t_0) \) onto \( HF^\infty(Y, t) \), proving the claimed isomorphism.

From this isomorphism, together with the commutative square

\[
\begin{array}{ccc}
HF^\infty_i(S^3, t_0) & \xrightarrow{F_{W,s}^-} & HF^\infty_{d(Y)}(Y, t) \\
\downarrow & & \downarrow \\
HF^+_i(S^3, t_0) & \xrightarrow{F_{W,s}^+} & HF^+_d(Y, t),
\end{array}
\]

it follows that we can find an element \( \xi \in HF^+(S^3, t_0) \) with the property that \( \tilde{g}(F_{W,s}^+(\xi)) = d(Y, t) \). Thus, we conclude that

\[
\tilde{g}(F_{W,s}(\xi)) - \tilde{g}(\xi) = \frac{c_1(s)^2 - 2\chi(W) - 3\text{sgn}(W)}{4} = \frac{c_1(s)^2 + b_2(X)}{4} \leq d(Y, t).
\]

As a special case, when \( Y \) is an integral homology sphere, the induced bilinear form takes values in \( \mathbb{Z} \), and it is unimodular.

**Corollary 9.7.** Let \( Y \) be an integral homology three-sphere, then for each negative-definite four-manifold \( X \) which bounds \( Y \), we have the inequality:

\[
Q_X(\xi, \xi) + \text{rk}(H^2(X; \mathbb{Z})) \leq 4d(Y),
\]

for each characteristic vector \( \xi \).

**Proof.** This is an immediate consequence of Theorem 9.6, with the observation that each characteristic vector for \( Q_X \) is the first Chern class of some Spin\(^c\) structure over \( X \).

**Corollary 9.8.** If \( Y \) is an integer homology three-sphere with \( d(Y) < 0 \). Then, there is no negative-definite four-manifold \( X \) with \( \partial X = Y \).

**Proof.** This is an immediate consequence of Theorem 9.6 and Elkies’ theorem.

Another consequence of Theorem 9.6 is the rational homology bordism invariance of \( d(Y, t) \). Let \( (Y_1, t_1) \) and \( (Y_2, t_2) \) be a pair of rational homology three-spheres equipped with Spin\(^c\) structures. We say that \( (Y_1, t_1) \) and \( (Y_2, t_2) \) are *rational homology cobordant* if there is a cobordism \( W \) from \( Y_1 \) to \( Y_2 \) with \( H_*(W; \mathbb{Q}) \cong H_*(S^3 \times [0, 1]; \mathbb{Q}) \), which can be equipped with a Spin\(^c\) structure \( s \) with \( s|Y_1 = t_1 \), and \( s|Y_2 = t_2 \).
Proposition 9.9. If \((Y_1, t_1)\) and \((Y_2, t_2)\) are rational homology cobordant (rational homology three-spheres equipped with Spin\(^c\) structures), then \(d(Y_1, t_1) = d(Y_2, t_2)\). In particular, if \((Y, t)\) is a rational homology three-sphere which bounds a rational homology four-ball \(W\), so that \(t\) can be extended over \(W\), then \(d(Y, t) = 0\).

Proof. The proof of Theorem 9.1 shows that if \(W\) is a cobordism from \(Y_1\) to \(Y_2\) with \(b_2^+(W) = 0\), then the map

\[
F_{W, s}^\infty : HF^\infty(Y_1, t_1) \longrightarrow HF^\infty(Y_2, t_2)
\]

is an isomorphism. As before, it follows that

\[
d(Y_2, t_1) - d(Y_1, t_2) \geq \frac{c_1(s)^2 - 2\chi(W) - 3\text{sgn}(W)}{4}
\]

for any Spin\(^c\) structure \(s\) over \(W\). When \(W\) is a rational homology bordism, then the above formula gives \(d(Y_2, t_2) \geq d(Y_1, t_1)\). By reversing the orientation of \(W\) (and using Equation (9)), we see that \(d(Y_1, t_1) \geq d(Y_2, t_2)\), as well. \(\Box\)

Thus, \(d\) can be viewed as an obstruction to finding a homology ball bounding \(Y\). Indeed, we have now all the ingredients for Theorem 1.2 stated in the introduction:

Proof of Theorem 1.2. First, we observe that \(d\) depends only on the Spin\(^c\) cobordism class of a rational homology sphere \(Y\) and Spin\(^c\) structure \(t\); but this was established in Proposition 9.9 above. The fact that \(d\) is a homomorphism follows from this, together with the additivity of \(d\) under the connected sum operation, which was established in Theorem 4.3. The fact that \(d\) lifts the homomorphism \(\rho\) (defined in the introduction) follows immediately from the dimension shift formula for the absolute grading (Equation (4)). Finally, conjugation invariance was established in Proposition 4.2. \(\Box\)

9.3. Intersection forms for definite four-manifolds bounding other three-manifolds. Constraints can be given on (semi-definite) intersection forms for four-manifolds which bound three-manifolds with \(b_1(Y) > 0\); we will focus our attention primarily to the case where \(H_1(Y; \mathbb{Z}) \cong \mathbb{Z}\). But first, we set up some terminology. We have the following easy consequence of Poincaré-Lefschetz duality:

Lemma 9.10. Let \(X\) be an oriented four-manifold with boundary \(Y\), and let \(V\) denote the image of \(H^2(X, Y; \mathbb{Z})\) in \(H^2(X; \mathbb{Z})/\text{Tors}\). Then, the cup product descends to give a non-degenerate bilinear form

\[
Q_X : V \otimes V \longrightarrow \mathbb{Z}.
\]

When \(H_1(Y; \mathbb{Z})\) has no torsion, the associated bilinear form is unimodular.

Proof. Non-degeneracy is an immediate consequence of the fact that the cup-product pairing

\[
\cup : H^2(X; \mathbb{Z})/\text{Tors} \otimes H^2(X, Y; \mathbb{Z})/\text{Tors} \longrightarrow \mathbb{Z}
\]
is nondegenerate.

To prove the second claim (assuming \( H_1(Y; \mathbb{Z}) \) has no torsion), we let \( K \) denote the kernel of the natural map \( H_1(Y; \mathbb{Z}) \cong H^2(Y; \mathbb{Z}) \to H^3(X, Y; \mathbb{Z}) \). Now, we get the following (split) short exact sequence

\[
0 \longrightarrow \left( \frac{H^2(X,Y; \mathbb{Z})}{\delta H^1(Y; \mathbb{Z})} \right) / \text{Tors} \longrightarrow H^2(X; \mathbb{Z}) / \text{Tors} \longrightarrow K \longrightarrow 0,
\]

by considering the Mayer-Vietoris sequence, and using the fact that \( K \) has no torsion. Thus, we can choose bases for \( \left( \frac{H^2(X,Y; \mathbb{Z})}{\delta H^1(Y; \mathbb{Z})} \right) / \text{Tors} \), a Poincaré dual basis for its image under \( \iota \), which we can then extend to a basis for \( H^2(X; \mathbb{Z}) / \text{Tors} \) by basis vectors which project to a basis for \( K \). With respect to these bases, it is easy to see that \( \iota \) is represented by the intersection matrix \( Q_X \), augmented by a zero matrix. Now, since the cokernel of \( \iota \) has no torsion, \( Q_X \) must be unimodular.

\[\text{Theorem 9.11.}\] Let \( X \) be a smooth, oriented four-manifold which bounds a three-manifold \( Y \) with \( H_1(Y; \mathbb{Z}) \cong \mathbb{Z} \). Let \( Q_X \) denote the induced pairing on

\[ V = \text{Im} \left( \frac{H^2(X,Y; \mathbb{Z})}{\delta H^1(Y; \mathbb{Z})} \right) / \text{Tors}, \]

and suppose that \( Q \) is negative definite. Then, if the restriction map \( H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \) is the trivial map, then for each characteristic vector \( \xi \) for \( Q_X \), we have that

\[
Q_X(\xi, \xi) + \text{rk}(V) \leq 4d_{-1/2}(Y) + 2; \tag{31}
\]

while if the restriction map \( H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \) is non-trivial, then for each characteristic vector \( \xi \), we have that

\[
Q_X(\xi, \xi) + \text{rk}(V) \leq 4d_{1/2}(Y) - 2. \tag{32}
\]

\[\text{Remark 9.12.}\] Of course, in the first case, \( \text{rk}(V) = \text{rk}H^2(X) - 1 \), while in the second case, \( \text{rk}(V) = \text{rk}H^2(X) \). Indeed, in the second case, \( V \cong H^2(X,Y)/\text{Tors} \cong H^2(X)/\text{Tors} \).

Before turning to the proof, we state the above inequalities in the case where \( X \) has no two-dimensional homology.

\[\text{Corollary 9.13.}\] Suppose that \( Y \) is a three-manifold with \( H_1(Y; \mathbb{Z}) \cong \mathbb{Z} \). Then, if \( Y \) bounds an integral homology \( S^2 \times D^2 \), then \( d_{-1/2}(Y) \geq -1/2 \), while if \( Y \) bounds an integral homology \( D^3 \times S^1 \), then \( d_{1/2}(Y) \geq 1/2 \).

\[\text{Proof.}\] Apply Inequalities (31) and (32), observing that in both applications, the right-hand-side is zero. \[\square\]
Proof of Theorem 9.11. Assume first that the restriction map $H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$ is trivial. We proceed exactly as in the proofs of Theorem 9.1 and Theorem 9.6. First, we surger out all of $b_1(X)$ without affecting its intersection form. Then, we puncture $X$ in a point and view the resulting space as a cobordism $W$ from $S^3$ to $Y$. Order the two-handles of $W$ as in the proof of Theorem 9.1, and observe that since $b_1(Y) < 3$, its $HF^\infty$ is standard (c.f. Theorem 11.1 of [20]). Thus, it follows from this that all the three-manifolds encountered in the sequence of two-handle additions have standard $HF^\infty$, so Propositions 9.3 and 9.4 apply.

In this manner, we show that

$$F_{W,s|W}^\infty : H^\infty(S^3, t_0) \to H^\infty(Y, t)$$

is injective, mapping onto the image of the action by $\gamma \in H_1(Y; \mathbb{Z})$. Moreover, we have that $\chi(W) = \chi(X) - 1$, $\text{sgn}(W) = -\text{rk}(V)$, $b_0(X) = 1$, $b_1(X) = 0$, $b_2(X) = \text{rk}V + 1$, $b_3(X) = \text{rk}H^1(X, Y; \mathbb{Z}) = 0$, and $b_4(X) = 0$; thus, the dimension formula implies that this degree is

$$\frac{c_1(s)^2 + \text{rk}(V) - 2}{4}.$$

In particular, since the generator of $HF^\infty(Y, t_0)$ of degree $d_{1/2}$ lies in the image of the $\gamma$ action, Inequality (31) follows.

In the case where the map $H^1(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z})$ is non-trivial, we proceed as above, except that now we surger out one-dimensional homology in $X$ until $b_1(X) = 1$ and the map in $H^1$ remains non-trivial. In this case, the corresponding map $F_{W,s|W}^\infty$ remains injective, only its image is complementary to the image of the action by $\gamma \in H_1(Y; \mathbb{Z})$. Inequality (32) follows (bearing in mind that the first Betti number of the four-manifold is one).

This proof also gives the following generalization of Proposition 4.11:

**Corollary 9.14.** Let $K \subset Y$ be a knot in an integral homology three-sphere. Then, we have the following inequalities (where here $n$ is any positive integer):

$$d_{1/2}(Y_0) - \frac{1}{2} \leq d(Y_{1/(n+1)}) \leq d(Y_{1/n}) \leq d(Y)$$

$$d(Y) \leq d(Y_{-1/n}) \leq d(Y_{-1/(n+1)}) \leq d_{-1/2}(Y_0) + \frac{1}{2}.$$

Furthermore, when $Y \cong S^3$, for all positive integers $n$, we have that

$$d_{1/2}(Y_0) - \frac{1}{2} = d(Y_{1/n}) \leq 0 \leq d(Y_{-1/n}) = d_{-1/2}(Y_0) + \frac{1}{2}.$$

**Proof.** Observe first that there are cobordism with $b_2^+ = 0$ connecting $Y_0$ to $Y_{1/(n+1)}$ to $Y_{1/n}$ to $Y$ to $Y_{-1/(n+1)}$ to $Y_{-1/n}$ and back to $Y_0$. 


To construct the cobordism from $Y_0$ to $Y_{1/(n+1)}$, we first take the knot $K \subset Y$ with 0-surgery, and then perform surgery along an additional unknot $L_0 \subset Y$ which links $K$ once, given with framing $-(n+1)$. To go from there on to $Y_{1/n}$, we perform another surgery along a knot $L_1$ which links $L_0$ once, with framing $-1$.

To go from $Y_{-1}$ to $Y_{-1/n}$, we start with $K$ with framing $-1$, and we surger along linear plumbing diagram (of length $N$), with each link given with coefficient $-2$. Surgering along one more linking circle (with coefficient $-2$) gives the cobordism to $Y_{-1/(n+1)}$, while framing the linking circle with coefficient $-1$ gives the required cobordism to $Y_0$.

Moreover, in the cobordisms connecting $Y_0$ to $Y_{1/n}$, the image of $H_1(Y_0)$ is non-trivial, while for the cobordism from $Y_{-1/n}$ to $Y_0$, the one-dimensional homology bounds. The correction terms are always non-increasing under these cobordisms: for the first and last cobordisms, we use the proof of Theorem 9.11, while for the intermediate steps, we use the version given in Theorem 9.6 (the intersection forms for these intermediate forms are obviously diagonalizable, so an appeal to Elkies' result is unnecessary).

The case where $Y = S^3$ then follows from Proposition 4.12.

The most important ingredient in the proof of Theorem 9.11 is that $HF^\infty$ of any three-manifold with $b_1(Y) = 1$ is standard. Thus, Theorem 9.11 has obvious generalizations to any three-manifold $Y$ with $b_1(Y) = 1$; and there is also a version with $b_1(Y) = 2$, which splits into cases according to the possible images of $H^1(X;\mathbb{Z})$ in $H^1(Y;\mathbb{Z})$. When $b_1(Y) > 2$, however, the arguments run into difficulties, since $HF^\infty$ need not be standard (e.g. when $Y = T^3$), and hence the maps on $HF^\infty$ induced by the cobordisms could be trivial. Indeed, it is also the case that the image of $H^1(X;\mathbb{Z})$ in $H^1(Y;\mathbb{Z})$ cannot be arbitrary: for instance, the cup product rules out the possibility of a four-manifold $X$ bounding $T^3$ so that the map $H^1(X,\mathbb{Z}) \rightarrow H^1(T^3;\mathbb{Z})$ has finite cokernel.

However, Theorem 9.11 generalizes readily to arbitrary three-manifolds $Y$ with standard $HF^\infty$. To keep the notation simple, we state this only in the case where the restriction map on $H^1$ is trivial.

**Theorem 9.15.** Let $Y$ be a three-manifold with standard $HF^\infty$, equipped with a torsion Spin$^c$ structure $t$, and let $d_b(Y, t)$ denote its “bottom-most” correction term, i.e. the one corresponding to the generator of $HF^\infty(Y, t)$ which is in the kernel of the action by $H_1(Y)$. Then, for each negative semi-definite four-manifold $W$ which bounds $Y$ so that the restriction map $H^1(W;\mathbb{Z}) \rightarrow H^1(Y;\mathbb{Z})$ is trivial, we have the inequality:

$$c_1(s)^2 + b_2^-(W) \leq 4d_b(Y, t) + 2b_1(Y)$$

for all Spin$^c$ structures $s$ over $W$ whose restriction to $Y$ is $t$.

**Proof.** Follow the proof of Theorem 9.11.
9.4. The minimal genus problem in $\mathbb{C}P^2$. We give a proof of the Thom conjecture for $\mathbb{C}P^2$, based on the theory developed thus far. This result was first proved by Kronheimer-Mrowka [16] and Morgan-Szabó-Taubes [19]. The proof we give here is analogous to a Seiberg-Witten proof given recently by Strle, see [25].

**Theorem 9.16.** (Kronheimer-Mrowka, Morgan-Szabó-Taubes) Let $\Sigma \subset \mathbb{C}P^2$ be a smoothly embedded two-manifold, which represents $m > 0$ times a generator $H \in H_2(\mathbb{C}P^2; \mathbb{Z})$. Then,

$$m^2 - 3m \leq 2g(\Sigma) - 2;$$

i.e. the holomorphic curves in $\mathbb{C}P^2$ minimize genus in their homology class.

The proof is based on the results from Section 9 on intersection forms (specifically, Theorem 9.15), together with the following calculation for circle bundles over two-manifolds.

**Lemma 9.17.** Let $Y$ be a circle bundle over a two-manifold, oriented as the boundary of a tubular neighborhood $N$ of a two-manifold $\Sigma$ with self-intersection number $\Sigma \cdot \Sigma = -n < 0$; and indeed, suppose that $n \geq 2g$, where $g$ denotes the genus of $\Sigma$. Let $u$ be the Spin$^c$ structure over $N$ with

$$\langle c_1(u), [\Sigma] \rangle = -n + 2g,$$

and $t$ be its restriction to $Y$. Then we have an isomorphism of relatively graded groups

$$HF^+(Y, t) \cong HF^+(\#^{2g}(S^2 \times S^1), t_0),$$

with the bottom-most generator of $HF^+(Y, t)$ in degree

$$\frac{1}{4} \left( \frac{g^2}{n} - \frac{n}{4} \right).$$

**Proof.** Consider the integral surgeries long exact sequence (Theorem 10.19 of [20])

$$\cdots \longrightarrow HF^+(S^1 \times \Sigma_g) \longrightarrow HF^+(\#^{2g}(S^2 \times S^1)) \longrightarrow HF^+(Y) \longrightarrow \cdots$$

Recall that the above sequence decomposes, according to Spin$^c$ structures over $Y$, where we use a sum over all Spin$^c$ structures over $S^1 \times \Sigma_g$ in the fiber of $Q$: Spin$^c(S^1 \times \Sigma) \longrightarrow$ Spin$^c(Y)$. If $s$ is a Spin$^c$ structure with $c_1(s) = \ell[PD(\Sigma_g)]$, then it is easy to see that $Q(s)$ is the restriction to $Y$ of a Spin$^c$ structure $u$ over $N$ with $\langle c_1(u), [\Sigma] \rangle = \ell - n$ (compare Lemma 7.10).

Now, by the adjunction inequality for the three-manifold $S^1 \times \Sigma_g$ (c.f. Theorem 8.1 of [20]), $HF^+(S^1 \times \Sigma_g, s)$ is trivial for all Spin$^c$ structures with $Q(s) = t$: it is non-trivial only for those Spin$^c$ structures $s$ for which $c_1(s) = \ell[PD(\Sigma_g)]$ with $|\ell| \leq 2g - 2$ (and it is trivial for all Spin$^c$ structures whose first Chern class is not a multiple PD[\Sigma_g]). Thus, the map in the long exact sequence

$$HF^+(\#^{2g}(S^1 \times S^2), t_0) \longrightarrow HF^+(Y, t)$$
is an isomorphism (of relatively graded groups). In fact, the map can be interpreted as a sum of maps
\[
\sum_{\{s \in \text{Spin}^c(W) \mid s|_Y \simeq t\}} \pm F^+_{W,s},
\]
where \(W\) is a single two-handle addition of \(\#^{2g}(S^1 \times S^2)\) giving rise to \(Y\). The term in this sum which shifts degree down the least corresponds to the Spin\(^c\) structure \(s\) with
\[
\frac{c_1(s)^2 + 1}{4} = \frac{1}{4} \left(1 - \frac{(2g - n)^2}{n}\right).
\]
Since the bottom-most generator of \(HF^+(\#^{2g}(S^1 \times S^2), t_0)\) has degree \(-g\), the result follows.

Proof of Theorem 9.16. Suppose that \(\Sigma_0 \subset \mathbb{C}P^2\) violates Inequality (33). By adding handles locally if necessary, we can find another embedded surface \(\Sigma \subset \mathbb{C}P^2\) (representing the same homology class) with
\[
m^2 - 3m = 2g(\Sigma).
\]
Let \(\mathfrak{f}\) be the Spin\(^c\) structure over \(\mathbb{C}P^2\) whose first Chern class is represented by \(-3H\) (this is the canonical class of \(\mathbb{C}P^2\)). Then the restriction of \(\mathfrak{f}\) to a tubular neighborhood of \(\Sigma\) satisfies the hypotheses of Lemma 9.17, so that if \(Y\) denotes the boundary of this tubular neighborhood, then
\[
(34) \quad d_b(Y, \mathfrak{f}|Y) = -2 + \left(\frac{3m - m^2}{2}\right).
\]
Let \(W\) be the four-manifold with boundary obtained by deleting a tubular neighborhood of \(\Sigma\) from \(\mathbb{C}P^2\). Indeed, according to the above lemma, \(HF^\infty\) of \(Y\) is standard. Moreover, \(H_2(W; \mathbb{Q})\) is clearly trivial, as is the map \(H^1(W; \mathbb{Q}) \rightarrow H^1(Y; \mathbb{Q})\). Thus, Theorem 9.15 applies, and gives the inequality
\[
-g = \frac{3m - m^2}{2} \leq d_b(Y, \mathfrak{f}|Y),
\]
which contradicts Equation (34).\(\square\)
10. Examples

In this final section, we illustrate the intersection form results from Section 9 by combining them with the calculations from Section 8. In Section 10.1, we study four-manifolds which bound surgeries on torus knots, in Section 10.2 we exhibit a homology $S^1 \times S^2$ which is not surgery on a single knot. Although some of the results contained in the present section have alternate proofs using more classical gauge-theoretic techniques (especially the first subsection, which contains some results which can be found in the work of [9], [2], [12]), we include the present discussion to give the reader a better feel for the theorems in the present paper. In the final subsection, we illustrate the results of Section 7, and specifically Corollary 7.5 of that section, by including a table containing all possible symmetric Laurent polynomials in $T$ which can arise as Alexander polynomials of knots in $S^3$ whose $+p$ surgery, for positive integral $p \leq 26$, gives a lens space.

10.1. Intersection form bounds. Continuing notation from Section 8, we let $Y_{p,q}(0)$ denote the three-manifold obtained by zero-surgery on the right-handed $(p,q)$ torus knot.

Proposition 10.1. Let $X_1$ be a four-manifold with $b_2^+ (X_1) = 0$ and $\partial X_1 = Y_{2,3}(0)$. Then, the map from $H^1(\mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z})$ is trivial and the intersection form of $X_1$ is diagonalizable. Similarly, let $X_2$ be a four-manifold with $b_2^+ (X_2) = 0$ and $\partial X_2 = -Y_{2,3}(0)$. Then, if $H^1(X_2; \mathbb{Z}) \rightarrow H^1(Y; \mathbb{Z})$ is non-trivial, then the intersection form of $X_2$ is diagonalizable. Moreover, if the map on $H^1$ is trivial, then if $Q_{X_2}$ denotes the intersection form of $X_2$ (on $V = \text{Im}(H^2(X_2, Y; \mathbb{Z}) \rightarrow H^2(X_2))$, we have that

$$Q_{X_2}(\xi, \xi) + \text{rk}(V) \leq 8.$$ 

Proof. We have seen in Section 8 that

$$d_{-1/2}(Y_{2,3}(0)) = -1/2 \quad \text{and} \quad d_{1/2}(Y_{2,3}(0)) = -3/2.$$ 

Thus, the case where $H^1(X_1; \mathbb{Z}) \rightarrow H^1(Y_{2,3}(0))$ is non-trivial is ruled out by Inequality (32) (together with Elkies’ theorem), while the diagonalizability of $Q$ in the other case is forced by Inequality (31) (and another application of Elkies’ theorem).

The correction terms for $-Y_{2,3}(0)$ (which we could alternatively think of as zero-surgery on the left-handed trefoil knot) are determined by Equation (15). The rest of the proposition then follows from another application of Theorem 9.11.

Indeed, the inequalities obtained in the above proposition are all sharp. Let $X_1$ denote four-manifold obtained by attaching a zero-framed two-handle to the four-ball along a right-handed trefoil. Let $X_2$ denote the four-manifold whose Kirby calculus description is given in Figure 6: there is a single one-handle, and a pair of two-handles added with framing $-1$ each (along unlinked circles), so that the three circles form...
the Borromean rings. Let $X_3$ denote the four-manifold obtained by plumbing nine $-2$-spheres as pictured in Figure 7.

Now, clearly, $b_2^+(X_1) = 0$ and $\partial X_1 = Y_{2,3}(0)$ (and its intersection form is trivially diagonalizable). Moreover, $\partial X_2 = -Y_{2,3}(0)$, and its intersection form is $(-1) \oplus (-1)$ (i.e. it is negative-definite and diagonal), and the map $H^1(X_2; \mathbb{Z}) \to H^1(Y_{2,3}(0); \mathbb{Z})$ is an isomorphism. Finally, the intersection form of $X_3$ induced on $V$ is easily seen to be the negative-definite form $E_8$. According to the inequality in the above proposition, this is the largest rank of any even intersection form which bounds $-Y_{2,3}(0)$.

In fact, Proposition 10.1 admits the following generalization:

**Proposition 10.2.** In the following statements, $p$ and $q$ can be any pair of positive, relatively prime integers (both greater than 1).

![Diagram](image)

**Figure 6.** **Kirby calculus description for $X_2$.** This is the Kirby calculus description of the four-manifold $X_2$ described above, with $\partial X_2 = Y_{2,3}(0)$.

![Diagram](image)

**Figure 7.** **A four-manifold which bounds zero-surgery on the trefoil.** The manifold here ($X_3$) is obtained by plumbing $-2$ spheres: each vertex represents a sphere of self-intersection $-2$. Each edge corresponds to an intersection between spheres. This configuration is simply-connected, and its boundary is $-Y_{2,3}(0)$. 
• For all natural numbers $n$, the manifolds $-\Sigma(p,q,pqn-1)$ cannot bound a four-manifold $X$ with $b^+_2(X) = 0$.

• The intersection form of any four-manifold $X$ with $b^+_2(X) = 0$ which bounds $\Sigma(p,q,pqn+1)$ is diagonalizable.

• If $X$ is a four-manifold with $b^+_2(X) = 0$ and $\partial X = Y_{p,q}(0)$, then the map $H^1(X;\mathbb{Z}) \to H^1(Y_{p,q}(0))$ is trivial and the intersection form of $X$ is diagonalizable.

**Proof.** First observe that if $K_{p,q}$ is the $(p,q)$ torus knot, then $t_0(K) > 0$. This follows from the fact that all the non-zero coefficients of the Alexander polynomial (Equation (26)) are $\pm 1$, and they come in alternating signs, with the top coefficient +1.

All the above results are direct consequences of this sign, the calculations relating the correction terms with $t_0$ (Proposition 8.1), Elkies’ result, and Theorem 9.6 or 9.11 as appropriate.

10.2. **On manifolds which are not surgery on a knot.** We give a simple illustration of Corollary 9.13. Consider the three-manifold $Y_{-2}$ given by the Kirby calculus description pictured in Figure 8.

This three-manifold can alternately be given as a plumbing as in Figure 9, substituting in $k = -2$.

From the plumbing description, it is clear that $Y_{-2}$ is obtained from $L(49, 40)$ (which is the manifold obtained by leaving off the vertex labelled by $k$) by attaching a single two-handle with framing $-2$. Since $w_2$ of the plumbing diagram can be represented by the sum of the Poincaré duals of the $(-7)$-sphere and the $(-5)$-sphere, it follows easily that $W$ is a spin cobordism. Let $s_0$ denote the Spin$^c$ structure on $W$ with $c_1(s_0) = 0$. We claim that the map $F^\infty_{W,s_0}$ is non-trivial. This follows easily from the fact that the map $F^+_{W,s_0}$ is a summand in the map $F_2$ belonging to a surgery long exact

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\[ \begin{aligned} & \text{Figure 8. Kirby calculus description of $Y_{-2}$.} \end{aligned} \]
sequence:

\[ \ldots \longrightarrow HF^+(L(49, 40)) \longrightarrow HF^+(Y_{-2}) \longrightarrow HF^+(L(49, 44)) \longrightarrow \ldots, \]

where \( L(49, 44) \) is the manifold which is boundary of the plumbing pictured in Figure 9 with \( k = -1 \). Using Proposition 4.8, we see that the correction term for \( L(49, 40) \) in its spin structure is \(-2\), and as usual one can see that the \( F^\infty_{W, s_0} \) decreases degree by \( 1/2 \). It follows that \( d_{-1/2}(Y_{-2}) \leq -5/2 \). (In fact, it follows from another glance at the above exact sequence that \( F^\infty_{W, s_0} \) can have no kernel, and hence that \( d_{-1/2}(Y_{-2}) = -5/2 \).)

Thus, according to Corollary 9.13, it follows that \( Y_{-2} \) is not obtained as the zero-surgery on a one-component knot in \( S^3 \) (and, indeed, that \( Y_{-2} \) does not bound any homology \( S^2 \times D^2 \)).

10.3. Alexander polynomials of knots giving lens spaces. Results from Section 7 (c.f. Corollary 7.5) give, for each pair of relatively prime integers \((p, q)\), an explicitly calculable, finite list of symmetric polynomials in \( T \) which contains the Alexander polynomials of all knots \( K \subset S^3 \) with the property that \( S^3_p(K) \cong L(p, q) \). It is interesting to compare the with the conjecture of Berge [3].

This list is determined as follows. Given \((p, q)\), we list the correction terms \( d(L(p, q), i) \) and \( d(L(p, 1), i) \) for \( i = 0, \ldots, p - 1 \), as determined in Proposition 4.8. For each multiplicative generator generator \( h \in (\mathbb{Z}/p\mathbb{Z})^* \), and each \( c \in \mathbb{Z}/p\mathbb{Z} \) we define a corresponding sequence \( t_i \) (indexed by integers \( i \)) by the formula

\[ 0 \leq 2t_i = \begin{cases} -d(L(p, q), c + h \cdot i) + d(L(p, 1), i) & \text{if } 2|i| \leq p \\ 0 & \text{otherwise}. \end{cases} \]

If the numbers \( t_i \) are all non-negative integers, then there is a Laurent polynomial in \( T \)

\[ \Delta(T) = a_0 + \sum_{i=1}^{\infty} a_i(T^i + T^{-i}) \]

whose coefficients are given by

\[ a_i = \begin{cases} t_{i-1} - 2t_i + t_{i+1} & \text{if } i \neq 0 \\ 1 + t_{-1} - 2t_0 + t_1 & \text{if } i = 0 \end{cases} \]
Let $F(p, q)$ denote the set of all symmetric Laurent polynomials obtained in this manner. In particular, observe that the number of polynomials in $F(p, q)$ is bounded by $p$ times the number of integers in $\{1, \ldots, p-1\}$ which are relatively prime to $p$ (note that this is a crude estimate, which can easily be improved, using the conjugation symmetry).

It follows from Corollary 7.5 that if $K \subset S^3$ is any knot with the property that $S^3_p(K) \cong L(p, q)$, then its Alexander polynomial $\Delta_K$ appears in the list $F(p, q)$.

We include here a table of $F(p, q)$ for all integers $p \leq 26$. To conserve space, we do not display $F(p, q)$ when either:

- $q = 1$ (for then $F(p, 1) = \{1\}$, according to Theorem 1.8),
- $F(p, q) = \emptyset$,
- we have already displayed $F(p, q')$, where $q$ and $q'$ are related by $q \cdot q' \equiv 1 \pmod{p}$, because in this case $L(p, q') \cong L(p, q)$, so that $F(p, q) = F(p, q')$. 

(compare Equation (1)).
Note that in this list, $\mathcal{F}(21,16)$ consists of two polynomials (while all other sets displayed consist of a single polynomial).

$\mathcal{F}(5,4) = \{ -1 + T^{-1} + T \}$

$\mathcal{F}(7,4) = \{ -1 + T^{-1} + T \}$

$\mathcal{F}(9,7) = \{ 1 + T^{-2} - T^{-1} - T + T^2 \}$

$\mathcal{F}(10,9) = \{ 1 + T^{-6} - 2T^{-5} + T^{-4} + T^{-3} - T^{-2} - T^2 + T^3 + 2T^4 - 2T^5 + T^6 \}$

$\mathcal{F}(11,9) = \{ 1 + T^{-3} - T^{-2} - T^2 + T^3 \}$

$\mathcal{F}(11,4) = \{ 1 + T^{-2} - T^{-1} + T + T^2 \}$

$\mathcal{F}(13,12) = \{ -3 + T^{-6} - 2T^{-5} + 2T^{-4} - T^{-3} + 2T^{-1} + 2T - T^3 + 2T^4 - 2T^5 + T^6 \}$

$\mathcal{F}(13,10) = \{ -1 + T^{-3} - T^{-2} - T^{-1} + T - T^2 + T^3 \}$

$\mathcal{F}(13,9) = \{ 1 + T^{-3} - T^{-2} - T^2 + T^3 \}$

$\mathcal{F}(14,11) = \{ -1 + T^{-4} - T^{-3} + T^{-1} + T - T^3 + T^4 \}$

$\mathcal{F}(15,4) = \{ -1 + T^{-3} - T^{-2} + T^{-1} + T - T^2 + T^3 \}$

$\mathcal{F}(16,9) = \{ -1 + T^{-4} - T^{-3} + T^{-1} + T - T^3 + T^4 \}$

$\mathcal{F}(17,16) = \{ -3 + T^{-9} - 2T^{-8} + T^{-7} + T^{-6} - 2T^{-4} + T^{-3} + T^{-2} + T^{-1} + T + T^2 + T^3 - 2T^4 + T^5 + T^6 + T^7 - 2T^8 + T^9 \}$

$\mathcal{F}(17,15) = \{ 1 + T^{-7} - 2T^{-6} + 2T^{-5} - T^{-4} + T^{-2} - T^{-1} - T + T^2 - T^3 + 2T^4 - 2T^5 - 2T^6 + T^7 \}$

$\mathcal{F}(17,13) = \{ 1 + T^{-4} - T^{-3} + T^{-2} + T^{-1} - T + T^2 - T^3 + T^4 \}$

$\mathcal{F}(18,13) = \{ 1 + T^{-5} - T^{-4} + T^{-2} - T^{-1} - T + T^2 - T^3 + T^4 \}$

$\mathcal{F}(19,17) = \{ 1 + T^{-9} - 2T^{-8} + T^{-7} + T^{-6} - T^{-5} + T^{-3} - T^{-2} - T^2 + T^3 - 5 + T^6 + T^7 - 2T^8 + T^9 \}$

$\mathcal{F}(19,19) = \{ -1 + T^{-6} - T^{-5} + T^{-4} + T^{-2} - T^{-1} - T + T^2 + T^3 + T^4 \}$

$\mathcal{F}(19,16) = \{ 1 + T^{-5} - T^{-4} + T^{-2} - T^{-1} - T + T^2 - T^3 + T^4 \}$

$\mathcal{F}(20,9) = \{ 1 + T^{-6} - T^{-5} + T^{-3} - T^{-2} - T^2 + T^3 - T^5 + T^6 \}$

$\mathcal{F}(21,16) = \{ -1 + T^{-5} - T^{-4} - T^{-3} - T^{-2} + T^{-1} + T - T^2 + T^3 - T^4 + T^5, -1 + T^{-6} - T^{-5} + T^{-2} + T^3 - T^5 + T^6 \}$

$\mathcal{F}(22,9) = \{ 1 + T^{-6} - T^{-5} + T^{-3} - T^{-2} - T^2 + T^3 - T^5 + T^6 \}$

$\mathcal{F}(23,18) = \{ -1 + T^{-7} - T^{-6} + T^{-4} - T^{-3} + T^{-1} + T - T^3 + T^4 - T^6 + T^7 \}$

$\mathcal{F}(23,16) = \{ -1 + T^{-7} - T^{-6} + T^{-5} - T^{-2} + T^{-1} + T - T^2 + T^3 - T^6 + T^7 \}$

$\mathcal{F}(23,6) = \{ -1 + T^{-5} - T^{-4} + T^{-3} - T^{-2} + T^{-1} + T - T^2 + T^3 - T^4 + T^5 \}$

$\mathcal{F}(25,24) = \{ -5 + 2T^{-12} - 3T^{-11} + 2T^{-9} + T^{-8} - 4T^{-7} + 2T^{-6} + 2T^{-5} - 2T^{-4} - T^{-3} + 2T^{-2} + 2T^{-1} + 2T + 2T^2 - T^3 - 2T^4 + 2T^5 + 2T^6 - 4T^7 + T^8 + 2T^9 - 3T^{11} + 2T^{12} \}$

$\mathcal{F}(25,21) = \{ 1 + T^{-9} - 2T^{-8} + 2T^{-7} - T^{-6} + T^{-4} - T^{-3} + T^{-2} - T^{-1} - T + T^2 - T^3 + T^4 - T^6 + 2T^7 - 2T^8 + T^9 \}$

$\mathcal{F}(25,19) = \{ 1 + T^{-6} - T^{-5} + T^{-4} - T^{-3} + T^{-2} - T^{-1} + T + T^2 + T^3 - T^4 - T^5 + T^6 \}$

$\mathcal{F}(25,16) = \{ 1 + T^{-8} - T^{-7} + T^{-4} - T^{-3} + T^{-2} - T^{-1} - T + T^2 + T^3 + T^4 - T^7 + T^8 \}$

$\mathcal{F}(25,14) = \{ -1 + T^{-7} - T^{-6} + T^{-4} - T^{-3} + T^{-2} - T^{-1} + T - T^3 + T^4 - T^6 + T^7 \}$

$\mathcal{F}(26,25) = \{ 1 + 3T^{-14} - 5T^{-13} + T^{-12} + 2T^{-11} + T^{-10} - 3T^{-8} + 2T^{-6} + 2T^{-5} + 2T^{-4} - 2T^{-3} - 5 + T^{-2} - 2T^3 + T^4 + T^5 + T^6 + 3T^8 + T^9 + T^{10} + T^{11} + T^{12} - 5T^{13} + 3T^{14} \}$

$\mathcal{F}(26,23) = \{ -1 + T^{-12} - 2T^{-11} + T^{-10} + T^{-9} - T^{-8} + T^{-5} - T^{-4} + T^{-3} + T^{-2} + T^2 - T^4 + T^5 - T^8 + T^{10} - 2T^{11} + T^{12} \}$
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