Quantum information reclaiming after amplitude damping

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Abstract
We design an effective method to investigate the quantum information reclaim from the environment after amplitude damping has occurred. In particular, we address the question of optimal measurement on the environment to perform the best possible correction on two- and three-dimensional quantum systems. While for qubits we show that the entanglement fidelity is the same for all possible measurements, for qudits we find that different measurements give rise to different values of the entanglement fidelity. By searching over all possible measurements on the environment we uncover the optimal one leading to the maximum entanglement fidelity.

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1. Introduction
Decoherence is a fundamental problem for the realization of quantum information tasks. The unavoidable interaction of a system with the surrounding environment disturbs the state and causes decoherence [1]. Usually decoherence implies an irreversible flow of information from the system to the environment with a consequent wash out of coherence features of the system. A useful active strategy to defend quantum coherence against environmental noise is that of quantum error correcting codes [2]. These codes, the first of which was discovered by Shor [3] and Steane [4], redundantly encode information in a manner that allows one to correct errors while preserving coherence, and thus the encoded information. Loosely speaking, this way of controlling coherence can be thought as a closed loop (feedback) control because it involves measurements on the system (code) to extract the error syndrome and recovery operations back to the system depending on the measurement outcomes.

Having access to the environment, one could think of reclaiming the information lost into the environment and restore the (quantum) coherence features of the system without resorting to any redundancy in the system space. This would be a clear example of feedback
control [5]. This method is practical for systems interacting with sufficiently controllable environments, that is, environments on which the necessary measurements can be performed (e.g. in experiments inside optical cavities [6]). Besides that, the problem is of importance from both theoretical and foundational perspectives because it answers the question of whether or not classical retrieved information from the environment can be useful to restore quantum information features.

As is shown in Figure 1, in this scheme one can make a measurement on the final state of the environment and considering its classical result recognize what kind of error has occurred on the system due to the interaction with the environment. Then, a proper correction should be performed on the system to reduce the effect of decoherence. Recently, a lot of attention has been devoted to this scheme from different aspects. In [7, 8], the capacity for this scenario has been studied and in [9] it has been shown that in certain cases repeated application of this scheme allows the effects of decoherence to be completely removed. In [5], it has been shown that there is a correspondence between measurement on the environment and Kraus representation of the map [10]. To be more explicit, any classical outcome of the measurement \( \alpha \) can be interpreted as the error \( t_\alpha \) (Kraus operator) occurred on the system. The recovery \( R_\alpha \) should then be performed to reduce the effect of the error \( t_\alpha \). For a given measurement the optimal recovery scheme (the recovery necessary to restore the maximum value of information) has been derived [5]. However, it is clear that to be able to reclaim more information from the environment, one should search for a proper measurement on the environment among all possible measurements. In fact, the success of this technique resides not only on the optimal recovery but also on the optimality of measurement.

Here our aim is to find the optimal measurement, the measurement which can provide the highest achievable coherence if the optimal recovery is performed for the correction. As an example of optimal measurement, consider a map which admits a decomposition in terms of unitary operators. If one performs a measurement on the environment which corresponds to such a decomposition of the map, the error operators will be proportional to unitary operators and hence it is possible to reverse their effect. This implies that this measurement is optimal. However such a decomposition does not exist for all the channels; hence, finding the optimal measurement is not trivial in general. It is well known that finding the optimal algorithm for quantum feedback control is challenging because it must involve an optimization over the manner of measurement besides that of the actuation process [11].

The optimization is usually done according to a cost function (figure of merit) which quantifies the performance of the correction scheme. We will see later (discussion after
equation (21)) that this is not possible unless we know the properties of the map. This fact makes it impossible to present a general form of the optimal measurement for a general map. In addition to that, even if we restrict our attention to a specific map, overcoming the obstacle of optimizing the cost function over all possible measurements is not straightforward. Despite all these difficulties we can shed light on this problem by analyzing amplitude damping channels which successfully describe a processes with gradual loss of information from the system to the environment [12]. Furthermore, this channel is not a random unitary channel, so finding the optimal measurement for this channel is highly non-trivial. In addressing this problem one can naively guess that the optimal measurement might be the one corresponding to the canonical Kraus representation, i.e. the one defined in terms of orthogonal Kraus operators (orthogonal in the sense of operators trace scalar product). Actually, we will show that for qubits all possible measurements on the environment are equivalent to the canonical one in the sense of recovering the errors by feedback control. However, for amplitude damping channels acting on qutrits the situation is different and we will find the optimal measurement which does not correspond to the canonical Kraus representation. In such a way we provide the optimal algorithm for feedback control of quantum coherence in the presence of amplitude damping.

The layout of this paper is as follows. In section 2, we briefly present the main conceptual and computational tools needed to restore quantum coherence by means of a quantum feedback control scheme (figure 1). In section 3, we discuss the significance of optimal measurement within the specific context of our work and design a method to find the optimal measurement on the environment. The amplitude damping channel is introduced in section 4. For $d = 2$, we show that all decompositions or all measurements on the environment are equivalent for restoring quantum coherence via feedback control. Furthermore, we study the amplitude damping channel for qutrits. We find that for $d = 3$ the canonical Kraus decomposition is sub-optimal. Indeed, by means of analytical methods, we construct a class of (non-canonical) Kraus decompositions leading to a higher performance than the one obtained via the canonical decomposition. The paper concludes in section 5 with a discussion.

2. Quantum feedback control scheme

In this section, we describe the general scheme for error correction using the information we gain by measuring the state of the environment. The evolution of the state in interaction with an environment or the action of any quantum channel on the input state can be described by considering the unitary evolution of the system together with the environment. Considering Hilbert spaces of initial and final states $\mathcal{H}_1$ and $\mathcal{H}_2$ and also the Hilbert spaces for initial and final environments $K_1$ and $K_2$, the general evolution can be described by a unitary operator $U : \mathcal{H}_1 \otimes K_1 \rightarrow \mathcal{H}_2 \otimes K_2$. The final state of the system is given by tracing over the environment after the interaction:

$$T(\rho) = \text{Tr}_{K_2}[U(\rho \otimes \sigma)U^\dagger]$$

where $\rho$ and $\sigma$ are positive trace-class operators belonging to $\mathcal{L}(\mathcal{H})$, the space of all linear operators on Hilbert space $\mathcal{H}$. The map $T : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}_2)$ is a CPT map (completely positive trace preserving map) which describes the channel or evolution of the system after the interaction with the environment. Our aim is to gather information about the errors occurred on the system by measuring the environment after interaction with the system. A general POVM measurement on $K_2$ is described by a family of operators $M_\alpha \in \mathcal{L}(K_2)$ where

$$\sum_\alpha M_\alpha = I, \quad M_\alpha > 0.$$
The index \( \alpha \) labels the classical result of the measurement. To understand how this classical information can be used to gain information about the final state of the system, let us consider an arbitrary observable \( A \in \mathcal{L}(\mathcal{H}_2) \) and its expectation value:

\[
\langle A \rangle = \text{Tr}_{\mathcal{H}_2}(T(\rho)A) = \text{Tr}_{\mathcal{H}_2}[U(\rho \otimes \sigma)U^\dagger(A \otimes I)].
\]  

(3)

Replacing \( I \) from (2) in the above equation, we get

\[
\langle A \rangle = \sum_{\alpha} \text{Tr}_{\mathcal{H}_2}\{\text{Tr}_{\mathcal{H}_2}[U(\rho \otimes \sigma)U^\dagger(A \otimes M_{\alpha})]\}
\]

(4)

where \( T_{\alpha} : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}_2) \) is defined as

\[
T_{\alpha}(\rho) := \text{Tr}_{\mathcal{H}_2}[U(\rho \otimes \sigma)U^\dagger(I \otimes M_{\alpha})].
\]  

(5)

Rewriting the expectation value of \( A \) in the following way:

\[
\langle A \rangle = \sum_{\alpha} p_{\alpha} \text{Tr}[T_{\alpha}(\rho)A],
\]  

(6)

we can conclude that \( p_{\alpha} = \text{Tr}(T_{\alpha}(\rho)) \) is the probability of getting \( \alpha \) as the classical result of measurement and the density matrix \( \frac{1}{p_{\alpha}}T_{\alpha}(\rho) \) as the selected state of the system after the measurement. Therefore, by performing the measurement on the environment the channel is decomposed as

\[
T = \sum_{\alpha} T_{\alpha}.
\]  

(7)

Let us recall from [5] that the most informative measurements on the environment are those for which the selected output of the channel after the measurement can be described by a single Kraus operator \( t_{\alpha} : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \):

\[
T_{\alpha}(\rho) = t_{\alpha}\rho t_{\alpha}^\dagger.
\]  

(8)

From now on we assume that the initial state of the environment is pure. It has been proved in [5] that when the initial state of the environment is pure, every decomposition of the channel in form (7) can be realized by a measurement on the environment. We also assume that we can perform the most informative measurement on the environment. Therefore, designing different POVM measurements on the environment is equivalent to considering different Kraus representations of the original channel:

\[
T(\rho) = \sum_{\alpha} t_{\alpha}\rho t_{\alpha}^\dagger, \quad \sum_{\alpha} t_{\alpha}^\dagger t_{\alpha} = I.
\]  

(9)

To perform the correction we design a recovery channel \( R_{\alpha} : \mathcal{L}(\mathcal{H}_2) \rightarrow \mathcal{L}(\mathcal{H}_1) \) that depends on the measurement outcomes. The state of the system after performing the correction becomes \( \frac{1}{p_{\alpha}}R_{\alpha}(T_{\alpha}(\rho)) \) with probability \( p_{\alpha} \) and the overall channel \( T_{\text{corr}} : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}_1) \) is given by

\[
T_{\text{corr}} = \sum_{\alpha} R_{\alpha} \circ T_{\alpha},
\]  

(10)

where \( \circ \) denotes the composition of the maps. The closer the \( T_{\text{corr}} \) is to the \( \text{id} \) map, the more successful is the scheme to recover quantum information. To quantify the performance of the correction scheme, we use entanglement fidelity as measure. For a general map \( \Phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}) \) the entanglement fidelity is defined as follows [13, 14]:

\[
F(\Phi) = \langle \Psi|\Phi \otimes I (|\Psi\rangle\langle\Psi|)|\Psi\rangle,
\]  

(11)
where $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ is the maximally entangled state and $I$ is the identity map. For a map $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ with Kraus operators $A_i$, the entanglement fidelity is given by [12]

$$F(\Phi) = \frac{1}{d^2} \sum_i |\text{tr} A_i|^2,$$

(12)

where $d = \dim \mathcal{H}$. We are interested in $F(T_{\text{corr}})$, the entanglement fidelity of the corrected map. Given $T(\rho)$ in (9) and using the Kraus representation of the recovery channel $R_\alpha : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1)$:

$$R_\alpha (\rho') = \sum_\beta r^{a\alpha}_\beta \rho r^{a\dagger}_\beta, \quad \sum_\beta r^{a\dagger}_\beta r^{a}_\beta = I,$$

(13)

the entanglement fidelity of the corrected channel becomes

$$F(T_{\text{corr}}) = \frac{1}{d^2} \sum_{a,\beta} \left| \text{tr} \left( r^{(a)}_\beta t_\alpha \right) \right|^2.$$

(14)

Entanglement fidelity reaches identity and quantum information can receive complete correction if and only if for all $\alpha$, $t_\alpha t^{\dagger}_\alpha = \tau_\alpha I$ with $\tau_\alpha \geq 0$ and $\sum_\alpha \tau_\alpha = 1$ [5]. In particular, when $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$, channels with Kraus operators proportional to unitary are completely correctable. In this case the optimal measurement on the environment is the one corresponding to the unitary decomposition of the map and the correction scheme is very clear because the evolution of the system is reversible.

For those cases where quantum information cannot be restored completely, the correction schemes which give the maximum entanglement fidelity for a given measurement are known. More precisely, using the Cauchy–Schwarz inequality, it has been shown [5] that for every family of recovery channels $R_\alpha : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1)$ and a given measurement corresponding to the Kraus decomposition $T(\rho) = \sum_\alpha t_\alpha \rho t^{\dagger}_\alpha$, the entanglement fidelity $F(T_{\text{corr}})$ is such that

$$F(T_{\text{corr}}) \leq \frac{1}{d^2} \sum_\alpha \left( \text{tr} |t_\alpha|^2 \right)^2,$$

(15)

where $|t| = \sqrt{\text{tr} t}$ is the modulus of $t$. By introducing the polar decomposition of the Kraus operators of $T$

$$t_\alpha = u_\alpha |t_\alpha|,$$

where $u_\alpha$ is a unitary operator, the optimal recovery channel in equation (13) can be applied by a single Kraus operator $r^{a\alpha}_{\mu\beta} = u^{\dagger}_\mu r^{a\dagger}_\beta$:

$$R_\alpha (\rho') = u^{\dagger}_\mu \rho' u_\mu.$$

Performing this recovery, the equality in equation (15) is obtained. Therefore, even if quantum information cannot be restored completely, the optimal recovery can be applied to attain the maximum in equation (15). In the rest of the paper we denote this quantity by

$$\tilde{F}(T_{\text{corr}}) = \frac{1}{d^2} \sum_\alpha \left( \text{tr} |t_\alpha| \right)^2.$$

(16)

It is important to note that for a given measurement on the environment which corresponds to the Kraus representation of the map with $t_\alpha$s as Kraus operators, $\tilde{F}(T_{\text{corr}})$ can be obtained by performing the optimal recovery. However, this result leaves the question of the optimal measurement on the environment unanswered. It is clear that by changing the measurement on the environment (choosing another Kraus representation of the map) $\tilde{F}(T_{\text{corr}})$ changes. In the next section we discuss about the optimal measurement on the environment which gives the maximum of $\tilde{F}(T_{\text{corr}})$ over all possible measurements.
3. Optimal measurement

To preserve the highest value of information, we should optimize the entanglement fidelity not only over all possible recovery schemes but also over all possible measurements on the environment. To find the optimal measurement on the environment, we need to maximize the entanglement fidelity over all possible Kraus decompositions of the channel, knowing that the optimal recovery can be applied. To be more specific, we should maximize $\tilde{F}(T_{\text{corr}})$ in equation (16) over all possible Kraus representations. To make the problem tractable, we restrict our attention to Kraus representations with the same number $N$ of Kraus operators as the canonical one. We start working from the canonical representation of a given map:

$$T(\rho) = \sum_{k=0}^{N-1} C_k \rho C_k^\dagger,$$

which is a decomposition characterized by orthogonal Kraus operators, i.e. the Kraus operators satisfying $\text{tr} (C_k C_k^\dagger) = c_k \delta_{k,k'}$, $c_k \in \mathbb{R}^+$. To search over all the Kraus decompositions of the map for the one corresponding to the optimal measurement we consider a general $N$-dimensional unitary transformation $V$ with $[V_{k,l}]$ elements and construct a new set of Kraus operators $\{B_k\}$ for the channel:

$$B_k = \sum_{l=0}^{N-1} V_{k,l} C_l.$$

Afterwards we maximize $\tilde{F}(T_{\text{corr}}) = \frac{1}{d^2} \sum_k (\text{tr} |B_k|^2)^2$ over the parameters of the unitary matrix $V$. Using the unitary transformation which maximizes $\tilde{F}(T_{\text{corr}})$, we can obtain the Kraus decompositions from the canonical decomposition which corresponds to the optimal measurement on the environment.

It is important to note that instead of considering a general unitary operator in $U(N)$ it is sufficient to perform the maximization over the parameters in $SU(N)$. This is because any operator $V \in U(N)$ can be written as $V = e^{i\phi} U$ with $U \in SU(N)$ and $\phi \in \mathbb{R}$. Constructing the new set of Kraus operators by means of $V$, we get

$$B_k = \sum_{l=0}^{N-1} V_{k,l} C_l = e^{i\phi} \sum_{l=0}^{N-1} U_{k,l} C_l.$$

However, only the absolute values of $B_k$s play a role in $\tilde{F}(T_{\text{corr}})$

$$|B_k| = \sqrt{B_k^\dagger B_k} = \left( \sum_{l,l'} U_{k,l}^* U_{k,l'} C_l^\dagger C_{l'} \right)^{\frac{1}{2}}.$$

Thus, $\tilde{F}(T_{\text{corr}})$ does not depend on the phase $e^{i\phi}$ and therefore we can restrict our attention to Kraus representations obtained from the canonical decomposition by transformations in $SU(N)$ without loss of generality. Therefore,

$$\tilde{F}(T_{\text{corr}}) = \frac{1}{d^2} \sum_k \left( \text{tr} |B_k|^2 \right)^2 = \frac{1}{d^2} \sum_k \left( \text{tr} \left( \sum_{l,l'} U_{k,l}^* U_{k,l'} C_l^\dagger C_{l'} \right)^{\frac{1}{2}} \right)^2$$

should be maximized over the element of $U \in SU(N)$. To compute $\tilde{F}(T_{\text{corr}})$ in terms of the parameters of the special unitary matrix $U$, we should find $|B_k|$. As it has been made explicit in equation (20) this cannot be done unless one studies a specific map.
In the following section we find the optimal measurement to perform correction on the amplitude damping channel. This is a non-trivial case because this map is not a random unitary map. Therefore, it is not included in the category of completely correctable maps discussed after equation (14), with trivial optimal measurement and the correction scheme.

4. Amplitude damping channel

The general behavior of processes with energy dissipation to the environment is well characterized by the amplitude damping channel. The gradual dissipation of energy or amplitude damping channel is described by the interaction between the system and the environment, modeled by harmonic oscillators, through the unitary operator [12]:

\[ U = e^{-i\chi(ab^\dagger + a^\dagger b)} , \]

where \( \chi \) is proportional to the coupling constant between the system and the environment. The operators \( a \) and \( a^\dagger \) (resp. \( b \) and \( b^\dagger \)) denote annihilation and creation operators of the system (resp. environment). By truncating the Fock basis of the Bosonic mode of the system to length \( d \), we describe a qudit system. The Kraus operators, \( C_m = \langle me | U | 0 \rangle \) (the subscript \( e \) indicate the environment state), for this \( d \)-dimensional amplitude damping channel are given by

\[ C_m = \sum_{n=m}^{d-1} \sqrt{C(n,m)(1-p)^{n-m} p^m} |n-m\rangle \langle n| , \quad m = 0, \ldots , d-1 , \]  

(22)

where \( p = \sin^2 \chi \) is the probability of loosing a single quantum of energy, \( |n\rangle \) are the number states and \( C(n,m) = \binom{n}{m} \) is the binomial coefficient. It is straightforward checking that this is the canonical representation

\[ \text{tr} (C_m^\dagger C_m) = c_m \delta_{m,m'}, \quad c_m = \sum_n C(n,m)(1-p)^{n-m} p^m . \]

Using the definition of the Kraus operators in (22), it is clear that \( C_m^\dagger C_m \) is diagonal:

\[ C_m^\dagger C_m = \sum_{n=m}^{d-1} C(n,m)(1-p)^{n-m} p^m |n\rangle \langle n| , \]  

(23)

which makes finding the modulus of \( C_m \) easy. From equations (23) and (16), it is straightforward to find that the highest value of the entanglement fidelity for the canonical representation of the amplitude damping channel is

\[ \tilde{F}_c(T_{\text{corr}}) = \frac{1}{d^2} \sum_{m=0}^{d-1} (\text{tr} |C_m|)^2 \]

\[ = \frac{1}{d^2} \sum_{m=0}^{d-1} \left( \sum_{n=0}^{d-1} \sqrt{C(n,m)(1-p)^{n-m} p^m} \right)^2 , \]  

(24)

where the subscript \( c \) stands for canonical. In the following, we study the problem of the optimal measurement in two- and three-dimensional systems where analytical analysis can be performed.


4.1. Optimal measurement in dimension two

For systems with two-dimensional Hilbert space, to find the Kraus decomposition corresponding to the optimal measurement on the environment, we start from the canonical representation of the map:

\[
C_0 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|, \\
C_1 = \sqrt{p}|0\rangle\langle 1|.
\] (25)

The entanglement fidelity for canonical Kraus representation \(\tilde{F}_c(T_{\text{corr}})\) is given by

\[
\tilde{F}_c(T_{\text{corr}}) = \frac{1 + \sqrt{1-p}}{2}.
\] (26)

Considering the general two-dimensional special unitary operator

\[
U = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1,
\] (27)

we introduce a general set of Kraus operators \(\{B_k\}\) as follows:

\[
B_0 = \begin{pmatrix} \alpha & \beta \sqrt{p} \\ 0 & \alpha \sqrt{1-p} \end{pmatrix}, \quad B_1 = \begin{pmatrix} -\beta & \alpha \sqrt{p} \\ 0 & -\beta \sqrt{1-p} \end{pmatrix}.
\] (28)

For this set of Kraus operators, the entanglement fidelity is given by

\[
\tilde{F}_B(T_{\text{corr}}) = \frac{1}{4} \sum_{m=0}^{1} \text{tr}^2(|B_m|) = \frac{1 + \sqrt{1-p}}{2}.
\] (29)

Comparing equations (26) and (29), we conclude that all representations with two Kraus operators give the same entanglement fidelity as the canonical one. So the performance of this scheme is independent of the kind of measurement that should be performed on the environment.

To answer the question of whether or not the same result is valid in higher dimensions, we face the problem of diagonalizing \(d\)-dimensional matrices \(B_k B_l\) to compute the entanglement fidelity. Therefore, in the next subsection we study the amplitude damping channel in three dimensions. This will give us some insights on possible advantages (or disadvantages) of performing measurements corresponding to the canonical representation.

4.2. Optimal measurement in dimension three

For three-dimensional Hilbert space the canonical decomposition of the channel is described by the following Kraus operators:

\[
C_0 = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1| + (1-p)|2\rangle\langle 2|, \\
C_1 = \sqrt{p}|0\rangle\langle 1| + \sqrt{2}p(1-p)|1\rangle\langle 2|, \\
C_2 = p|0\rangle\langle 2|.
\] (30)

If we do not apply the feedback control scheme to make any correction on the final state of the system, by using equation (12) we find that the entanglement fidelity of this map is

\[
F(T) = \frac{1}{9} \sum_{k=1}^{3} |\text{tr} C_k|^2 = \frac{1}{9} (2 - p + \sqrt{1-p})^2.
\] (31)
To reclaim the information lost in the environment, we perform the measurement corresponding to the canonical Kraus decomposition of the map and apply the optimal recovery scheme using the outcomes of the measurement. The entanglement fidelity of the corrected channel becomes

$$\tilde{F}_c(T_{\text{corr}}) = \frac{1}{9} \sum_{k=1}^{3} (\text{tr} |C_k|)^2 = \frac{1}{9} [(2 - p + \sqrt{1 - p})^2 + (\sqrt{p} + \sqrt{2p(1 - p)})^2 + p^2].$$  \hspace{1cm} (32)

Figure 2 shows the entanglement fidelity of the map with (dash line) and without (dot line) correction versus the error probability $p$. It is clear from this figure that performing optimal recovery after the measurement on the environment corresponding to the canonical representation of the map is completely helpful. However, this success does not guarantee that the measurement corresponding to the canonical representation is the optimal measurement. As explained in section 3, in order to find the optimal measurement we maximize $\tilde{F}(T_{\text{corr}})$ over all possible Kraus representations with $N = 3$ number of Kraus operators. To construct the general Kraus representation from the canonical one, we can restrict ourselves to the unitary operators in $SU(3)$ without loss of generality. As starting working hypothesis, we consider the following two subgroups of $SU(3)$, $G_1$ and $G_2$, leading to equal or higher entanglement fidelity than the one given by the canonical Kraus representation. We show that the decomposition giving the highest value of entanglement fidelity can be constructed by means of unitary transformations in $G_2$.

4.2.1. Equi-canonical class. The first subgroup $G_1$ that we study is defined as

$$G_1 := \left\{ U_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 + |\beta|^2 = 1 \right\}. \hspace{1cm} (33)$$
The general Kraus operators constructed in terms of the elements of this subgroup are given by
\[ B_0 = C_0, \]
\[ B_1 = \alpha C_1 + \beta C_2, \tag{34} \]
\[ B_2 = -\bar{\beta} C_1 + \bar{\alpha} C_2. \]

To calculate the entanglement fidelity \( \tilde{F}_B(T_{\text{corr}}) \) for this class of Kraus operators,
\[ \tilde{F}_B(T_{\text{corr}}) = \frac{1}{9} \sum_{k=0}^{2} (\text{tr} |B_k|)^2, \]

at first we need to compute \( |B_k| \)'s and therefore the eigenvalues of the \( B_1^\dagger B_1 \)\( \times \)s. It turns out that the non-vanishing eigenvalues of \( B_1^\dagger B_1 \) are given by
\[ \lambda_1, \lambda_1' = \frac{a \pm \sqrt{a^2 - b^2}}{2}, \tag{35} \]
with
\[ a = p^2 + 3p(1 - p)|\alpha|^2, \quad b = 2p|\alpha|^2 \sqrt{2(1 - p)}. \tag{36} \]
The non-vanishing eigenvalues of \( B_2^\dagger B_2 \) are
\[ \lambda_2, \lambda_2' = \frac{a' \pm \sqrt{a'^2 - b'^2}}{2}, \tag{37} \]
with
\[ a' = p^2 + 3p(1 - p)|\beta|^2, \quad b' = 2p|\beta|^2 \sqrt{2(1 - p)}. \tag{38} \]

Therefore, the entanglement fidelity is given by
\[ \tilde{F}_B(T_{\text{corr}}) = \frac{1}{9} [(2 - p + \sqrt{1 - p})^2 + a + a' + b + b'] \]
\[ = \frac{1}{9} [(2 - p + \sqrt{1 - p})^2 + (\sqrt{p} + \sqrt{2p(1 - p)})^2 + p^2]. \tag{39} \]

Comparing (39) with (32), we see that for all the new set of Kraus operators \( B_k \) in (34), the entanglement fidelity \( \tilde{F}_B(T_{\text{corr}}) \) in equation (39) equals the entanglement fidelity for the canonical Kraus representation:
\[ \tilde{F}_B(T_{\text{corr}}) = \tilde{F}_c(T_{\text{corr}}). \tag{40} \]

This means there is no advantage in performing measurements corresponding to the canonical representation over the class of Kraus decomposition in (34).

As a side remark, we note that other sets of Kraus operators \( \{B'_{k}\} \) leading to the entanglement fidelity \( \tilde{F}_B(T_{\text{corr}}) \) in (39) can be constructed. Introduce the two \( SU(3) \)-operators \( g_1 \) and \( g_2 \) as follows:
\[ g_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{41} \]

and consider the new set of Kraus operators \( \{B'_k\} \) given by
\[ B'_k = \sum_l (g_j)_{k,l} B_l \quad j = 1, 2. \tag{42} \]

From (42), we note that the only effect of the \( g \)s on the vector \( B \equiv (B_0, B_1, B_2)' \) is that of cyclically permuting the vector components. Therefore,
\[ \tilde{F}_B(T_{\text{corr}}) = \tilde{F}_B(T_{\text{corr}}). \]
where the subscript $B'$ stands for the Kraus operators in (42). The class of unitary operators giving the Kraus representations $\{B'_k\}$ from the canonical ones are simply the following left cosets of $G_1$:

\[
\begin{align*}
   g_1 G_1 & := \left\{ \begin{pmatrix} 0 & -\bar{\beta} & \bar{\alpha} \\ 1 & 0 & 0 \\ 0 & \alpha & \beta \end{pmatrix} \right| \alpha^2 + |\beta|^2 = 1 \right\}, \\
   g_2 G_1 & := \left\{ \begin{pmatrix} 0 & \alpha & \bar{\beta} \\ 1 & 0 & 0 \\ 0 & \bar{\alpha} & \bar{\beta} \end{pmatrix} \right| \alpha^2 + |\beta|^2 = 1 \right\}.
\end{align*}
\]

(43)

Therefore, any measurement corresponding to the Kraus representation obtained from the canonical one via transformations in $G_1 \cup g_1 G_1 \cup g_2 G_1$ leads to the same entanglement fidelity $\tilde{F}_c(T_{corr})$ in (32).

4.2.2. Super-canonical class. In the previous subsection we showed that there is a large class of Kraus representations leading to the same entanglement fidelity that can be achieved by the canonical representation. A more interesting question is whether or not we can design a measurement on the environment that gives entanglement fidelity values higher than the canonical one. To answer this question, we consider a new subgroup $G_2$ of $SU(3)$:

\[
G_2 := \left\{ U_2 = \begin{pmatrix} \gamma & 0 & \delta \\ 0 & 1 & 0 \\ -\bar{\delta} & 0 & \bar{\gamma} \end{pmatrix} \right| \gamma^2 + |\delta|^2 = 1 \right\}.
\]

(44)

In terms of $U_2$, the new set of Kraus operators becomes

\[
\begin{align*}
   D_0 &= \gamma C_0 + \delta C_2, \\
   D_1 &= C_1, \\
   D_2 &= -\bar{\delta} C_0 + \bar{\gamma} C_2.
\end{align*}
\]

(45)

Following the line of reasoning presented in the previous subsection, we obtain

\[
\begin{align*}
   \text{tr} |D_0| &= \left( \frac{g + \sqrt{g^2 - h^2}}{2} \right)^{\frac{1}{2}} + \left( \frac{g - \sqrt{g^2 - h^2}}{2} \right)^{\frac{1}{2}} + \sqrt{1 - p} |\gamma|, \\
   g &= p^2 + 2(1 - p)|\gamma|^2, \\
   h &= 2(1 - p)|\gamma|^2.
\end{align*}
\]

(46)

Similarly

\[
\begin{align*}
   \text{tr} |D_2| &= \left( \frac{k + \sqrt{k^2 - l^2}}{2} \right)^{\frac{1}{2}} + \left( \frac{k - \sqrt{k^2 - l^2}}{2} \right)^{\frac{1}{2}} + \sqrt{1 - p} |\delta|, \\
   k &= p^2 + 2(1 - p)|\delta|^2, \\
   l &= 2(1 - p)|\delta|^2.
\end{align*}
\]

(47)

(48)

(49)
Therefore, the entanglement fidelity, $\tilde{F}_D(T_{\text{corr}})$ obtained from the new set of Kraus representation in equation (45), becomes (appendix)

$$\tilde{F}_D(T_{\text{corr}}) = \tilde{F}_c(T_{\text{corr}}) + \frac{2\sqrt{1-p}}{9}\Omega,$$

(50)

with

$$\Omega = |\gamma|\sqrt{g + h} + |\delta|\sqrt{k + i} - (2 - p).$$

(51)

Since $\Omega$ is strictly positive (appendix), it follows that

$$\tilde{F}_D(T_{\text{corr}}) > \tilde{F}_c(T_{\text{corr}}).$$

(52)

We conclude that all the measurements corresponding to the Kraus decompositions constructed using the elements of group $G_2$ lead to entanglement fidelity values higher than the one obtained by means of the canonical representation. Using the same arguments of the previous subsection, we can find a larger class of unitary transformations giving rise to Kraus representations with higher entanglement fidelity than the canonical one. Such transformations belong to the following left cosets of $G_2$:

$$g_1G_2 := \left\{ \begin{pmatrix} -\bar{\delta} & 0 & \bar{\gamma} \\ \gamma & 0 & \delta \\ 0 & 1 & 0 \end{pmatrix} \ | \ |\gamma|^2 + |\delta|^2 = 1 \right\},$$

(53)

$$g_2G_2 := \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -\bar{\delta} & 0 & \bar{\gamma} \\ \gamma & 0 & \delta \end{pmatrix} \ | \ |\gamma|^2 + |\delta|^2 = 1 \right\},$$

where $g_1$ and $g_2$ are given in equation (41). Therefore, the operators in the set

$$G_2 \cup g_1G_2 \cup g_2G_2$$

(54)

give rise to decompositions that work better than the canonical one. We can find the best decomposition in this class by maximizing $\Omega$ in equation (51) over the parameters defining the transformations in class (54). It follows that the maximum is achieved for $|\gamma| = |\delta| = \frac{1}{\sqrt{2}}$:

$$\Omega_{\text{max}} = \sqrt{2 + 2(1-p)^2} - (2 - p).$$

(55)

Thus, the maximum entanglement fidelity in this class becomes

$$[\tilde{F}_D(T_{\text{corr}})]_{\text{max}} = \frac{1}{9}[5 - 2 + 2p\sqrt{2(1-p)} + 2\sqrt{2(1-p)}(2 - 2p + p^2)],$$

(56)

and can be achieved by performing measurements corresponding to the decompositions arising from the canonical one by means of the following special unitary transformations:

$$U_{G_2} = \begin{pmatrix} e^{\frac{i\theta}{\sqrt{2}}} & 0 & e^{\frac{i\phi}{\sqrt{2}}} \\ 0 & 1 & 0 \\ -e^{\frac{i\phi}{\sqrt{2}}} & 0 & e^{\frac{i\theta}{\sqrt{2}}} \end{pmatrix}, \quad U_{g_1G_2} = \begin{pmatrix} -e^{\frac{i\phi}{\sqrt{2}}} & 0 & e^{\frac{i\theta}{\sqrt{2}}} \\ e^{\frac{i\theta}{\sqrt{2}}} & 0 & e^{\frac{i\phi}{\sqrt{2}}} \\ 0 & 1 & 0 \end{pmatrix}, \quad U_{g_2G_2} = \begin{pmatrix} 0 & 1 & 0 \\ e^{\frac{i\phi}{\sqrt{2}}} & 0 & e^{\frac{i\theta}{\sqrt{2}}} \\ 0 & 1 & 0 \end{pmatrix}.$$
4.2.3. Maximum entanglement fidelity. It is impractical to analytically prove that the entanglement fidelity in equation (56) is the global maximum not just the local maximum in the super-canonical class. However, we are able to show this numerically. We have generated $n = 10^5$ random unitary operators by sampling from $U(3)$ according to the Haar measure. Specifically it has been done by using the Ginibre ensemble [15] consisting of matrices whose entries are independent and identically distributed standard normal complex random variables and then orthonormalizing such matrices [16]. The chosen cardinality $n$ of the randomly generated distinct Kraus representations is the smallest positive integer number necessary to obtain convergence to the numerically found global maximum. Any other randomly generated set of Kraus representations with higher cardinality $m > n$ converges to the same global maximum. Our numerical analysis implies that such global maximum coincides with the analytical expression in equation (56). Thus, we conclude that $[F_D(T_{corr})]_{\text{max}}$ in (56) (red line in figure 2) is indeed the global maximum and the optimal measurements on the environment correspond to the Kraus decompositions arising from the canonical decomposition by the special unitary transformation in equation (57).

5. Discussions

Different strategies can be used to defeat the effect of decoherence. In this paper, we focus on restoring quantum coherence by reclaiming quantum information lost to the environment surrounding the system. The main idea is to perform a measurement on the environment and then, relying on the classical result of the measurement, perform an appropriate correction on the system to decrease or remove the effect of decoherence (see figure 1). While for a given measurement, the optimal recovery operation is known [5], here we address the important question of what kind of measurement yields the highest performance of this strategy. To answer this question, the cost function (entanglement fidelity) quantifying the performance of this scheme should be maximized over all possible measurements on the environment. Although the method we use here is applicable in principle to any desirable map, the maximization of entanglement fidelity should be done for every specific channel. Actually it is impossible to give a general optimal feedback algorithm for arbitrary channels.

In view of such difficulties, we choose to study here the amplitude damping channel which successfully models the gradual loss of information from the system to the environment. After having described the amplitude damping channel in an arbitrary dimension, we focused on two-dimensional systems. We showed that all measurements corresponding to Kraus representations of the map with two Kraus operators (measurements with two outcomes) give the same value for the entanglement fidelity. It implies that for the amplitude damping channel in $d = 2$, the reduction of decoherence using the quantum feedback control scheme does not depend on the details of the measurement made on the environment. Our numeric studies show that the same statement is valid even if we consider measurements with three or four outcomes.

Although in two-dimensional Hilbert space the performance of quantum feedback control does not depend on measurement details, the situation becomes different when we increase the dimension to three. By studying the three-dimensional amplitude damping channel, we have showed there is a class of measurements or class of Kraus representations of the amplitude damping channel which perform as good as canonical representation. We named it equi-canonical class and analytically found the entanglement fidelity that can be attained by performing such measurements. Interestingly, we introduced another class of Kraus representations, the super-canonical class, which leads to an entanglement fidelity higher than the one obtained in the equi-canonical class. We analytically found the maximum
entanglement fidelity in the super-canonical class and the Kraus representations by which this maximum can be attained. By means of numeric techniques, we discovered that the maximum entanglement fidelity we have found is not only the maximum in this class but also the global maximum over all possibilities when considering the most general Kraus representations with three Kraus operators. Furthermore, our numeric studies show that the same statement is true even if we consider measurement with four outcomes. Indeed, motivated by numerical evidences, we feel like conjecturing that this might be true for any number of Kraus operators (measurement outcomes). The extension of our study to \(d\)-dimensional systems faces the difficulty of evaluating the cost function characterizing the performance of our method. The cost function is formally written in terms of \(tr^2(|B_k|)\), where \(B_k\)s are \(d\)-dimensional matrices. So to evaluate the cost function we need to diagonalize the \(d\)-dimensional matrices \(B_k^*B_k\). However, in general, the diagonalization of an arbitrary \(d\)-dimensional matrix is beyond the currently known analytical techniques.

As a final remark, we point out that although our findings are limited to amplitude-damping channels (very common sources of errors in quantum information processing tasks) for qubits and qutrits, we are confident that our analysis contains a sufficient amount of technical details to pave the way for further investigations along many directions. One could be the application of the technique to different quantum noisy channels. Another could be the study of multi-steps feedback control up to continuous time control where some results are already known [17].

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Appendix

In what follows, we justify the positivity of \(\Omega_1\) in (51). Recall that \(\tilde{F}_{D}(T_{\text{corr}})\) is given by

\[
\tilde{F}_{D}(T_{\text{corr}}) = \frac{1}{9} \sum_{k=0}^{2} (\text{tr}|D_k|)^2,
\]

with \(D_1 = C_1\). Therefore, \(\tilde{F}_{D}(T_{\text{corr}})\) becomes

\[
\tilde{F}_{D}(T_{\text{corr}}) = \frac{1}{9}[\text{tr}|C_1|^2 + (\text{tr}|D_0|)^2 + (\text{tr}|D_2|)^2]. \tag{A.1}
\]

Replacing \(\text{tr}|D_0|\) and \(\text{tr}|D_2|\) from equations (46) and (48) into the above equation and using the following identity:

\[
|\alpha|\sqrt{a} + |\beta|\sqrt{b} = \sqrt{|\alpha|^2a + |\beta|^2b + 2|\alpha||\beta|\sqrt{ab}}, \tag{A.2}
\]

we obtain

\[
\tilde{F}_{D}(T_{\text{corr}}) = \frac{1}{9} [\text{tr}^2(|C_1|) + 2p^2 + 5(1 - p) + 2\sqrt{1 - p}(\gamma|\sqrt{g + h + |\delta|\sqrt{k + l}})]. \tag{A.3}
\]

Using equation (32), we can rewrite the above equation to get equation (50):

\[
\tilde{F}_{D}(T_{\text{corr}}) = \tilde{F}_{c}(T_{\text{corr}}) + \frac{2\sqrt{1 - p}}{9} \Omega
\]
with
\[ \Omega = |\gamma|\sqrt{g + h + |\delta|\sqrt{k + l}} - (2 - p). \]

To prove that \( \Omega > 0 \), we first use (A.2) to rewrite \( \Omega \) as follows:
\[ \Omega = \sqrt{|\gamma|^2(g + h) + |\delta|^2(k + l) + 2|\gamma||\delta|\sqrt{(g + h)(k + l)}} - (2 - p) \]
\[ = \sqrt{(2 - p)^2 - 8(1 - p)|\gamma|^2|\delta|^2 + 2|\gamma||\delta|\sqrt{p^4 + 4p^2(1 - p) + 16(1 - p)^2|\gamma|^2|\delta|^2}} - (2 - p). \]
\[ \text{(A.4)} \]

Since the following inequality holds:
\[ 2|\gamma||\delta|\sqrt{p^4 + 4p^2(1 - p) + 16(1 - p)^2|\gamma|^2|\delta|^2} > 8(1 - p)|\gamma|^2|\delta|^2, \]
\[ \text{(A.5)} \]
we conclude that \( \Omega > 0 \).

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