Nonlocal Control in the Conduction Coefficients: Well-Posedness and Convergence to the Local Limit

Evgrafov, Anton ; Bellido, José C.

Published in:
SIAM Journal on Control and Optimization

Link to article, DOI:
10.1137/19M126181X

Publication date:
2020

Document Version
Peer reviewed version

Citation (APA):
Evgrafov, A., & Bellido, J. C. (2020). Nonlocal Control in the Conduction Coefficients: Well-Posedness and Convergence to the Local Limit. SIAM Journal on Control and Optimization, 58(4), 1769–1794. https://doi.org/10.1137/19M126181X

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
NON-LOCAL CONTROL IN THE CONDUCTION COEFFICIENTS:
WELL POSEDNESS AND CONVERGENCE TO THE LOCAL LIMIT

ANTON EVGRAFOV† AND JOSÉ C. BELLIDO‡

Abstract. We consider a problem of optimal distribution of conductivities in a system governed
by a non-local diffusion law. The problem stems from applications in optimal design and more specif-
ically topology optimization. We propose a novel parametrization of non-local material properties.
With this parametrization the non-local diffusion law in the limit of vanishing non-local interaction
horizons converges to the famous and ubiquitously used generalized Laplacian with SIMP (Solid
Isotropic Material with Penalization) material model. The optimal control problem for the limiting
local model is typically ill-posed and does not attain its infimum without additional regularization.
Surprisingly, its non-local counterpart attains its global minima in many practical situations, as we
demonstrate in this work. In spite of this qualitatively different behaviour, we are able to partially
characterize the relationship between the non-local and the local optimal control problems. We also
complement our theoretical findings with numerical examples, which illustrate the viability of our
approach to optimal design practitioners.

Key words. Nonlocal optimal design, nonlocal optimal control in the coefficients, convergence
to local problems, numerical approximation of nonlocal problems

AMS subject classifications. 49J21, 49J45, 49J35, 80M50

1. Introduction. Nonlocal problems receive a lot of attention nowadays owing
to a wide range of applications they have in a variety of contexts. In particular, we
mention micromechanics [45], image processing [34], phase transitions [2], pattern for-
mation [32], population dispersal [20], optimal design [4] and shape optimization [31],
onimal control [22] and inverse problems [23]; see also a very recent monograph on
the subject [25]. Among nonlocal problems, fractional or nonlocal diffusion plays a
central role. It has attracted enormous interest and a great deal of work has been
done over the past twenty years. The number of references on the subject and connec-
tions of nonlocal diffusion with remarkable applications is really overwhelming. We
refer the interested readers to the monographs [7, 17] and the references therein. In
the context of continuum mechanics the non-local modelling goes back at least to the
Eringen’s model [27, 28], and more recently has been focused on peridynamical mod-
elling [35, 49, 36, 53, 26, 37]. These latter models refrain from using the gradients
of the state fields with the goal of unified description of singular phenomena, such
as fracture or dislocation. When considering scalar equations, such as for example
the steady state heat equation, peridynamics equations may essentially be seen as a
nonlocal diffusion equations on bounded domains [1, 5].

In this paper we consider a prototypical optimal design problem for diffusion
phenomena, in which one has to determine the best way of distributing conduct-
ing materials inside a given computational domain. For models governed by the
local diffusion phenomena such problems have been studied for a long time and are
quite well understood, see for example [3, 11, 41] and references therein. As a very

*Submitted to the editors 14.05.2019.
Funding: AE’s research is financially supported by the Villum Fonden through the Villum
Investigator Project InnoTop. The work of JCB is funded by FEDER EU and Ministerio de Economía
y Competitividad (Spain) through grant MTM2017-83740-P.
†Department of Mechanical Engineering, Technical University of Denmark, DK–2800 Kgs.
Lyngby, Denmark (aaev@mek.dtu.dk).
‡Department of Mathematics, University of Castilla-La Mancha, 13071–Ciudad Real, Spain (Jose-
Carlos.Bellido@uclm.es).
rough summary we can say that these problems are typically ill-posed and do not attain their infimum. One possible way of dealing with this issue is to bring the limits of minimizing sequences into consideration and interpret them as composite materials obtained from mixing the original materials in the sense described by the theory of homogenization. Another option is to restrict the set of considered material distributions by introducing constraints or penalty functions with regularizing effect. An extremely successful example of the latter approach, which is nowadays widely utilized in engineering practice, is the SIMP (Solid Isotropic Material with Penalization) material parametrization model combined with additional regularization techniques [11, 12, 13, 48]. An interesting recent study, which can be interpreted in the light of comparing the two outlined approaches in the context of the steady state heat conduction, is [54].

Research in optimal design of systems described by the non-local governing equations is in its infancy. Our present model is inspired by the recent studies [4, 6], where a very similar optimal design problem for a nonlocal diffusion state law was analyzed. The main novelty of our work is the way in which the material properties (conductivities) enter into the model. We chose a nonlocal interpolation of material properties that allows us to establish a natural link between the nonlocal optimal design problem and a local one in which material properties are interpolated by the SIMP scheme. Our main objective is to analyze this nonlocal optimal design problem and to characterize its relationship with the local optimal design problem as the nonlocal parameter (the interaction horizon) goes to zero. We have documented our preliminary findings in a brief note [29], where we have been primarily concerned with the qualitative relation between our non-local model and a particular heuristic regularization method (“sensitivity filtering”, see [48]) for the local model with SIMP. This work includes the technical results, proofs, and numerical experiments, which have been omitted from [29] owing to the space requirements.

The outline of the paper is as follows. In Section 2 we formulate the non-local state equations and discuss their well-posedness. We also state the nonlocal optimal design problem, which is going to be the main subject of our study, as well as its local counterpart. Section 2 also includes Theorem 2.3, an interesting result illustrating the continuity of the conductivity-to-state map for the non-local problem. In Section 3 we include several existence results for the nonlocal optimal design problem depending on the SIMP penalization parameter. In Section 4 we address the convergence of the nonlocal problems to the local one and the connection of our proposed model with SIMP in the local case. Finally, in Section 5, a numerical approach to the nonlocal optimal design problems is described and several numerical examples are presented.

2. Problem formulation and preliminaries.

2.1. Local control in the conduction coefficients. We begin our discussion with a well understood optimal control problem in the conduction coefficients of generalized Laplace equation, or topology optimization through material distribution, see for example [18, 3, 11] and references therein. This problem will serve us both as a reference physical model and also as a limiting problem later on.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded, and connected domain. In this domain we consider the generalized Laplace equation with a volumetric source $f \in L^2(\Omega)$, homogeneous Dirichlet boundary conditions, and a spatially heterogeneous diffusion (conduction) coefficient $\kappa^{loc}$. Its weak solution $u \in U_0 = W_0^{1,2}(\Omega)$ satisfies
the variational statement
\[ a_{\kappa_{\text{loc}}}(u, v) = \ell(v), \quad \forall \, v \in U_0, \]
where the parametric bilinear form \( a_{\kappa_{\text{loc}}} \) and the linear functional \( \ell \) are given by
\[ \begin{align*}
  a_{\kappa_{\text{loc}}}(u, v) &= \int_{\Omega} \kappa_{\text{loc}}(x) \nabla u(x) \cdot \nabla v(x) \, dx, \\
  \ell(v) &= \int_{\Omega} f(x) v(x) \, dx.
\end{align*} \]

We recall that
\[ u \text{ solves } (2.1) \iff u = \arg \min_{v \in U_0} I_{\kappa_{\text{loc}}}(v), \]
where \( I_{\kappa_{\text{loc}}}(v) = a_{\kappa_{\text{loc}}}(v, v) - 2\ell(v) \) is the associated quadratic energy functional. We will utilize the shorthand notation \( S_{\text{loc}}(\kappa_{\text{loc}}) := \arg \min_{v \in U_0} I_{\kappa_{\text{loc}}}(v) \) for the control coefficient-to-state operator for this system.

In connection with this governing equation we consider a problem of optimal distribution of conductive material in \( \Omega \) under simple constraints. Specifically, we will define the following convex set of admissible material distributions:
\[ \mathcal{A} = \left\{ \rho \in L^\infty(\Omega) \mid \rho \leq \rho(\cdot) \leq \overline{\rho}, \text{ a.e. in } \Omega, \int_{\Omega} \rho(x) \, dx \leq \gamma |\Omega| \right\}, \]
where \( \gamma \in ]\underline{\rho}, \overline{\rho}[ \) is a given volume fraction, and \( 0 < \underline{\rho} < \overline{\rho} < \infty \) are given bounds.\(^1\) We assume that the local material conductivity \( \kappa_{\text{loc}} \) is related to the control parameter \( \rho \in \mathcal{A}^\delta \) through the so-called SIMP (Simple Interpolated Material with Penalization) model, see [11, 3]: \( \kappa_{\text{loc}}(x) = \rho^p(x) \), where \( p \geq 1 \) plays a role of a penalty parameter in certain optimal design problems. Each \( \rho \in \mathcal{A}^\delta \) is therefore mapped into \( \kappa_{\text{loc}} \) satisfying the bounds \( \underline{\kappa} = \rho^\delta \leq \kappa_{\text{loc}} \leq \overline{\kappa} = \overline{\rho}^\delta \).

For a given performance functional \( J : \mathcal{A} \times U_0 \to \mathbb{R} \) we consider the following optimization problem:
\[ \min_{\rho \in \mathcal{A}} J(\rho, S_{\text{loc}}(\rho^\delta)). \]

Note that the use of “minimize” instead of “inf” does not automatically mean that the infimum is attained in (2.5). In fact, in most interesting situations this is not the case without additional regularization of the problem (2.5). We will return to this issue in Section 3.

Of particular interest to us will be the case of \( J(\rho, u) = \ell(u) \), corresponding to compliance minimization. Note that in this case the reduced compliance can be expressed in a variety of ways:
\[ \ell(\rho) := \ell(S_{\text{loc}}(\rho^\delta)) = a_{\rho^\delta}(S_{\text{loc}}(\rho^\delta), S_{\text{loc}}(\rho^\delta)) = -2I_{\rho^\delta}(S_{\text{loc}}(\rho^\delta)). \]

The last expression in conjunction with (2.3) allows us to state the compliance minimization problem as a saddle point problem, which will be used later on.

\(^1\)Our setup can be easily generalized to the situation when \( \rho, \overline{\rho} \in L^\infty(\Omega) \).
2.2. Non-local state equation. We will now introduce a non-local analogue of
the local governing equations (2.1), (2.2). We will use $\delta > 0$ to denote the non-local
interaction horizon, and $B(x, \delta) = \{ y \in \mathbb{R}^n : |x - y| < \delta \}$ to denote an open ball
of radius $\delta$ centered at $x \in \mathbb{R}^n$. Let $\Omega_\delta = \bigcup_{x \in \Omega} B(x, \delta)$ be the set of points located
within distance $\delta$ from points in $\Omega$, and $\Gamma_{\text{nl}} = \Omega_\delta \setminus \bar{\Omega}$ be the “non-local boundary” of
$\Omega$. We consider nonlocal linear diffusion equations, that in this scalar framework can
also be seen as peridynamics equations. Similarly to (2.1), they will be formulated
with the help of the following parametric bilinear form $a_{\delta, k}(\cdot, \cdot)$:

\begin{equation}
(2.7) \quad a_{\delta, k}(u, v) = \int_{\Omega_\delta} \int_{\Omega_\delta} \kappa(x, x')A_\delta(|x - x'|) \frac{u(x) - u(x')}{|x - x'|} \frac{v(x) - v(x')}{|x - x'|} \, dx \, dx',
\end{equation}

while we use the same linear functional $\ell(\cdot)$ defined in (2.2). In the equations above,
$\kappa \in L^\infty(\Omega_\delta \times \Omega_\delta)$ is a “nonlocal conductivity”, and $A_\delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a radial kernel
satisfying certain conditions. More precisely, we assume that

\begin{equation}
(2.8) \quad \kappa \in K^\delta = \{ \tilde{\kappa} \in L^\infty(\Omega_\delta \times \Omega_\delta) \mid \tilde{\kappa} \leq \tilde{\kappa}(x, x') = \tilde{\kappa}(x', x) \leq \pi, \forall x, x' \in \Omega_\delta \},
\end{equation}

for some $0 < \kappa < \pi < +\infty^2$, and that $A_\delta$ satisfies the following conditions:

\begin{align}
(2.9) & \quad \frac{1}{n} \int_{\mathbb{R}^n} A_\delta(|x|) \, dx = 1, \\
(2.10) & \quad \supp A_\delta(|\cdot|) \subset B(0, \delta), \quad \forall \delta > 0,
\end{align}

and there exists $s \in (0, 1)$ and $c_3 > 0$ such that

\begin{equation}
(2.11) \quad A_\delta(|x|) \geq \frac{c_3}{|x|^{n+2s-2}}, \quad \forall x \in B(0, \delta/2) \setminus \{0\}, \delta > 0.
\end{equation}

Let $U^\delta = \{ u \in L^2(\Omega_\delta) \mid a_{\delta, 1}(u, u) < +\infty \}$, and $U^\delta_{0}$ be the topological closure of
$C^\infty_c(\Omega)$ (where we extend the functions in $C^\infty_c(\Omega)$ by 0 outside of $\Omega$) in $U^\delta$ with respect
to the semi-inner product $a_{\delta, 1}(\cdot, \cdot)$. Since the nonlocal conductivities are uniformly
bounded from above and away from zero, for any $\kappa \in K^\delta$ and $u \in U^\delta$ we have the estimates

$$\kappa a_{\delta, 1}(u, u) \leq a_{\delta, \kappa}(u, u) \leq \pi a_{\delta, 1}(u, u).$$

Consequently, $u \in U^\delta$ if and only if $\|u\|_{L^2(\Omega_\delta)} < +\infty$ and $a_{\delta, \kappa}(u, u) < +\infty$. In this
notation, the non-local governing equations we will study can be stated as follows:
find $u \in U^\delta_{0}$ such that

\begin{equation}
(2.12) \quad a_{\delta, \kappa}(u, v) = \ell(v), \quad \forall v \in U^\delta_{0}.
\end{equation}

2.3. Non-local control in the conduction coefficients. We will now intro-
duce a non-local analogue of (2.5), where the local governing equations (2.1), (2.2) are
replaced with their non-local analogue introduced in the previous section. Similarly
to (2.4), we define the following convex set of admissible material distributions:

\begin{equation}
(2.13) \quad A^\delta = \left\{ \rho \in L^\infty(\Omega_\delta) \mid \bar{\rho} \leq \rho(\cdot) \leq \overline{\rho}, \text{a.e. in } \Omega_\delta, \int_{\Omega} \rho(x) \, dx \leq \gamma|\Omega| \right\}.
\end{equation}

\footnote{Dependence on $\delta$ in the definition of the set $K^\delta$ could be omitted by considering nonlocal conductivities defined on the whole space $\mathbb{R}^n \times \mathbb{R}^n$.}
where the parameters are exactly as in (2.4). We still assume that the local material conductivity \( \kappa_{\text{loc}} \) satisfies the SIMP model \( \kappa_{\text{loc}}(x) = \rho^p(x), \quad x \in \Omega_\delta \). In addition, we assume that the non-local conductivity \( \kappa(x,x') \) entering (2.12) is simply a geometric mean of \( \kappa_{\text{loc}}(x) \) and \( \kappa_{\text{loc}}(x') \), that is,

\[
(2.14) \quad \kappa(x,x') = \sqrt{\kappa_{\text{loc}}(x) \kappa_{\text{loc}}(x')} = \rho^{p/2}(x)\rho^{p/2}(x').
\]

For a given performance functional \( J : \mathcal{A}^\delta \times \mathcal{U}^\delta \rightarrow \mathbb{R} \) we consider the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad J(\rho, u), \\
\text{subject to} & \quad u \text{ solving (2.12) with } \kappa(x,x') = \rho^{p/2}(x)\rho^{p/2}(x').
\end{align*}
\]

We will still refer to the case \( J(\rho, u) = \ell(u) \) as the compliance minimization problem.

**Remark 2.1.** Essentially, the only difference between the non-local optimal design problem considered here and the one in [4] is the way the control variables enter the bilinear form. In order to preserve the symmetry of the form with respect to \((x,x')\) one takes an average between the local conductivities at \(x\) and \(x'\) to be the non-local conductivity in the bilinear form defining the non-local governing equations. The authors of [4] take an arithmetic mean, whereas we use a geometric mean for the same purpose. The main advantage of our choice is that it allows us to establish existence of optimal solutions to a non-local version of the classical compliance minimization problem discussed in Section 2.1 without regularizing it even in the non-convex case \( 1 < p \leq 2 \) as we will prove in Proposition 3.3.

**2.4. Well-posedness and continuity of the conductivity-to-state operator.** In this subsection we recall a couple of known results about the non-local equation (2.12).

Under the assumptions (2.9)–(2.11), \( a_{\delta,1}(\cdot,\cdot) \) defines an inner product on \( \mathcal{U}^\delta_0 \) thereby making it a Hilbert space [5]. We denote by \( \| \cdot \|_{\mathcal{U}^\delta_0} \) the norm induced by this inner product. Furthermore, owing to (2.11), there are constants \( \tilde{c}_\delta > 0, \tilde{c}_\delta > 0 \) independent of \( u \) such that for any \( \delta > 0, \kappa \in \mathcal{K}^\delta \) and \( u \in \mathcal{U}^\delta_0 \) we have the inequalities

\[
(2.16) \quad a_{\delta,\kappa}(u,u) \geq c_{\delta, \kappa} \int_{\Omega_\delta} \int_{\Omega_\delta \cap B(x,\delta/2)} \frac{(u(x) - u(x'))^2}{|x-x'|^{n+2s}} \, dx' \, dx \\
\quad \geq \tilde{c}_{\delta, \kappa} \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{(u(x) - u(x'))^2}{|x-x'|^{n+2s}} \, dx' \, dx \geq \tilde{c}_{\delta, \kappa} \| u \|_{L^2(\Omega)}^2,
\]

where the second inequality is a consequence of [9, Proposition 6.1] and the third inequality is established in [9, Lemma 6.2]. When combined with Cauchy–Bunyakovsky–Schwarz inequality, (2.16) implies that \( \ell : \mathcal{U}^\delta_0 \rightarrow \mathbb{R} \) is a continuous linear functional with respect to \( \| \cdot \|_{\mathcal{U}^\delta_0} \). Consequently we have verified the necessary assumptions for the application of Lax–Milgram Lemma (see, for example, [16]) allowing us to conclude the following.

**Theorem 2.2.** For each \( \delta \in [0,\delta_0], \kappa \in \mathcal{K}^\delta \) and \( f \in L^2(\Omega) \) there is a unique solution \( u \in \mathcal{U}^\delta_0 \) of (2.12). Furthermore, this solution minimizes the associated quadratic energy functional:

\[
(2.17) \quad I_{\delta,\kappa}(v) := \frac{1}{2} a_{\delta,\kappa}(v,v) - \ell(v)
\]
that is, \( u = \arg \min_{v \in \mathcal{U}_0^\delta} I_{\delta, \kappa}(v) \).

This well-posedness result allows us to univocally define the coefficient-to-state operator \( S_{\delta}(\kappa) := \arg \min_{v \in \mathcal{U}_0^\delta} I_{\delta, \kappa}(v) \), exactly as in the local case. Furthermore, we put the reduced non-local compliance to be

\[
(2.18) \quad c_\delta(\rho) := \ell(S_{\delta}(\kappa)) = a_\kappa(S_{\delta}(\kappa), S_{\delta}(\kappa)) = -2I_{\delta, \kappa}(S_{\delta}(\kappa)),
\]

where \( \kappa(x, x') = \rho^{p/2}(x)\rho^{p/2}(x') \).

Nonlocal diffusion and peridynamics equations have been studied extensively. Existence results for a much more general class of nonlinear and nonlocal variational principles have been obtained in \([9]\). Particularly, well-posedness of equation (2.12) has been shown in \([5, \text{Theorem 1.2}]\) for slightly more general conditions on the kernel \( A_\delta \). More specifically, hypothesis (2.11) could be further relaxed without sacrificing the conclusions of Theorem 2.2. Even though our assumption (2.11) is not optimal if one is only concerned with solvability of the non-local governing equations, it is convenient for us since it implies the compact embedding of \( \mathcal{U}_0^\delta \) into \( L^2(\Omega_\delta) \), and ultimately a certain continuity of the coefficient-to-state operator for the non-local equation (2.12).

Indeed, let us revisit the string of inequalities (2.16). Note that the third term in (2.16) is nothing else but the Gagliardo seminorm \( \|u\|_{W^{s,2}(\Omega_\delta)} \). Therefore, in addition to the continuous embedding of \( \mathcal{U}_0^\delta \) into \( L^2(\Omega_\delta) \), we also have a continuous embedding of \( \mathcal{U}_0^\delta \) into the fractional Sobolev space \( W^{s,2}(\Omega_\delta) \). Since the latter space is compactly embedded into \( L^2(\Omega_\delta) \) (see \([24, \text{Theorem 7.1}]\)), we also have a compact embedding of \( \mathcal{U}_0^\delta \) into \( L^2(\Omega_\delta) \).

Let us now discuss the conductivity-to-state operator for the non-local equation (2.12). In the case of the classical, local diffusion (generalized Laplace) equation, the conductivity-to-state operator \( S^{\text{loc}} \) is famously not continuous with respect to weak* convergence of conductivities in \( K^\delta \). In fact a stronger \( H \)-convergence, or \( G \)-convergence in our self-adjoint case, has been specifically defined to obtain such a result \([38, 39, 50, 51, 52]\). The notion of \( H \)-convergence has been recently extended to the nonlocal \( p \)-Laplacian in \([30]\), proving its sequential compactness for bounded coefficients. In stark contrast, the conductivity-to-state operator \( S_{\delta} \) for the non-local equation (2.12) is continuous with respect to weak* convergence of conductivities in \( K^\delta \). This fact has been established in \([4, \text{Theorem 6}]\). For the sake of completeness and clarity we include a simpler proof of this result, which is in the same spirit as the simple characterization of \( H \)-convergence for fractional \( p \)-Laplacian equations given by the authors in \([8]\).

\[ \text{Theorem 2.3. Let us fix } \delta \in [0, \delta_0], \text{ and } f \in L^2(\Omega). \text{ Consider a sequence of conductivities } \kappa_j \in K^\delta, \text{ together with the corresponding sequence } u_j = S_{\delta}(\kappa_j) \in \mathcal{U}_0^\delta \text{ of solutions to (2.12). Assume that } \kappa_j \to \hat{\kappa}, \text{ weak* in } L^\infty(\Omega_\delta \times \Omega_\delta), \text{ and let } \hat{u} = S(\hat{\kappa}) \in \mathcal{U}_0^\delta \text{ be the corresponding solution to (2.12). Then, we have } \lim_{j \to \infty} \|u_j - \hat{u}\|_{L^2(\Omega_\delta)} = \lim_{j \to \infty} \|u_j - \hat{u}\|_{\mathcal{U}_0^\delta} = 0. \]

\[ \text{Proof. Recall that } \hat{u} \text{ is the unique minimizer of } I_{\delta, \hat{\kappa}}, \text{ and } u_j \text{ is that of } I_{\delta, \kappa_j}. \]

Keeping in mind that \( \kappa_j \to \hat{\kappa}, \text{ weak* in } L^\infty(\Omega_\delta \times \Omega_\delta), \) and the inclusion

\[
A_{\delta}(|x - x'|) \frac{(\hat{u}(x) - \hat{u}(x'))^2}{|x - x'|^2} \in L^1(\Omega_\delta \times \Omega_\delta)
\]

we have the inequality

\[
(2.19) \quad \limsup_{j \to \infty} I_{\delta, \kappa_j}(u_j) \leq \lim_{j \to \infty} I_{\delta, \hat{\kappa}}(\hat{u}) = I_{\delta, \hat{\kappa}}(\hat{u}).
\]
Let us now extract a subsequence $\{(\kappa_{j'}, u_{j'})\}$, $j' = 1, 2, \ldots$ from the original sequence such that $\liminf_{j' \to \infty} I_{\delta, \kappa_{j'}}(u_{j'}) = \lim_{j' \to \infty} I_{\delta, \kappa_{j'}}(u_{j'})$. Note that as a direct consequence of (2.16) we get an uniform estimate $\|u_{j}\|_I^{2} \leq a_{\delta, \kappa_{j}}(u_{j}, u_{j}) = \ell(u_{j}) \leq c_{\delta}^{-1/2}(\pi/\ell)^{1/2}\|f\|_{\mathcal{L}^{2}(\Omega)}\|u_{j}\|_{\mathcal{U}_{0}^{d}}$, $j = 1, 2, \ldots$ Therefore there exists $u \in \mathcal{U}_{0}^{d}$ and a further subsequence, labelled by $\{u_{j''}\}$, $j'' = 1, 2, \ldots$, such that $u_{j''} \to u$, weakly in $\mathcal{U}_{0}^{d}$, $u_{j''} \to u$, strongly in $L^{2}(\Omega_{\delta})$, and $u_{j''}(x) \to u(x)$, a.e. in $\Omega_{\delta}$. Let us define the finite measures

$$
\mu_{j}(E) = \int_{E} \kappa_{j}(x, x') \, dx \, dx', \quad j = 1, 2, \ldots
$$

and

$$
\tilde{\mu}(E) = \int_{E} \hat{k}(x, x') \, dx \, dx', \quad j = 1, 2, \ldots
$$

where $E \subset \Omega_{\delta} \times \Omega_{\delta}$ is an arbitrary Lebesgue measurable set and $\chi_{E} \in L^{1}(\Omega_{\delta} \times \Omega_{\delta})$ is its characteristic function. Weak* convergence of $\kappa_{j}$ to $\hat{k}$ implies the strong convergence of measures $\lim_{j \to \infty} \mu_{j} = \tilde{\mu}$ (i.e. $\lim_{j \to \infty} \mu_{j}(E) = \tilde{\mu}(E)$ for any measurable set $E \subset (\Omega_{\delta} \times \Omega_{\delta})$). Furthermore, in view of (2.19) and the continuity of $\ell$ the non-negative sequence $a_{\delta, \kappa_{j}}(u_{j}, u_{j}) = 2[I_{\delta, \kappa_{j}}(u_{j}) + \ell(u_{j})]$ is bounded from above. Therefore we can apply the generalized Fatou’s lemma [46, Prop. 17, pg. 269]:

$$
\liminf_{j \to \infty} I_{\delta, \kappa_{j}}(u_{j}) = \lim_{j'' \to \infty} \left[ \frac{1}{2} \int_{\Omega_{\delta} \times \Omega_{\delta}} A_{\delta}(|x - x'|) \left(\frac{(u_{j''}(x) - u_{j''}(x'))^{2}}{|x - x'|^{2}}\right) \, dx \, dx' \right] \] - \lim_{j'' \to \infty} \ell(u_{j''}) \geq \frac{1}{2} \int_{\Omega_{\delta} \times \Omega_{\delta}} A_{\delta}(|x - x'|) \left(\frac{(u(x) - u'(x'))^{2}}{|x - x'|^{2}}\right) \, dx \, dx' = I_{\delta, \hat{k}}(\hat{u}) - I_{\delta, \kappa_{j}}(u_{j}) = 0.
$$

In particular, $\lim_{j \to \infty} I_{\delta, \kappa_{j}}(u_{j}) = I_{\delta, \hat{k}}(\hat{u})$. The strong convergence of $u_{j}$ towards $\hat{u}$ in $\mathcal{U}$, and owing to the continuous embedding also in $L^{2}(\Omega_{\delta})$, follows from the already established facts as follows:

$$
0 \leq \limsup_{j \to \infty} \|u_{j} - \hat{u}\|_{\mathcal{U}_{0}^{d}}^{2} \leq \lim_{j \to \infty} \left[ a_{\delta, \kappa_{j}}(u_{j}, u_{j}) - 2a_{\delta, \kappa_{j}}(u_{j}, \hat{u}) + a_{\delta, \kappa_{j}}(\hat{u}, \hat{u}) \right] = -2I_{\delta, \kappa_{j}}(u_{j}) + 2I_{\delta, \kappa_{j}}(\hat{u}) = 0.
$$

3. Existence of optimal conductivity distributions. Existence of solutions for the optimization problem (2.15) in the special case of compliance minimization, which corresponds to $J(\rho, u) = \ell(u)$, has been briefly outlined in [29]. For the sake of keeping this manuscript self-contained, we include the short proofs of these results here. We also establish existence of solutions to (2.15) for more general objective functions, but for a specific penalization value $p = 2$.

3.1. Compliance minimization, convex case: $p = 1$. Let us first consider the convex case of compliance minimization, which is obtained by setting $p = 1$, $J(\rho, u) = \ell(u)$ in (2.15). The argument appeals to the convexity of the problem\(^3\), see [18].

**Proposition 3.1.** The compliance minimization problem (2.15) (that is, $J(\rho, u) = \ell(u)$), admits an optimal solution $(\rho^{*}, u^{*}) \in A^{1} \times \mathcal{U}_{0}^{d}$ for $p = 1$.\(^3\)

\footnote{Note that the same arguments apply to the practically uninteresting case $0 < p < 1$ as well.}
Proof. In view of the existence of states for every conductivity distribution \( \rho \in \mathcal{A}^4 \), and also since these states satisfy the energy minimization principle \((2.17)\), our optimal design problem can be equivalently stated as the following saddle point problem:

\[
\max_{\rho \in \mathcal{A}^4} \min_{u \in \mathcal{U}_0^6} I_{\delta,\kappa}(u),
\]

where \( \kappa(x,x') = \rho^{1/2}(x)\rho^{1/2}(x') \). Note that the map \( \mathbb{R}^2_+ \ni (\xi, \eta) \mapsto \xi^{1/2}\eta^{1/2} \in \mathbb{R} \) is concave. Consequently, the map \( \mathcal{A}^4 \ni \rho \mapsto I_{\delta,\kappa}(u) \in \mathbb{R} \) is concave. Consequently, the map

\[
\begin{align*}
&\text{(with respect to } \mathbb{L}^\infty(\Omega_\delta)\text{-norm) for each } u \in \mathcal{U}_0^6, \text{ where continuity is owing to the dominated Lebesgue convergence theorem. Therefore, the map } \\
&\mathcal{A}^4 \ni \rho \mapsto I_{\delta,\kappa}(u) \in \mathbb{R} \\
&\text{is weak* sequentially upper semicontinuous in } \mathcal{L}^\infty(\Omega_\delta). \text{ This property is preserved under taking the infimum, therefore } \mathcal{A}^4 \ni \rho \mapsto \min_{u \in \mathcal{U}_0^6} I_{\delta,\kappa}(u) \in \mathbb{R} \text{ is weak* sequentially upper semicontinuous in } \mathcal{L}^\infty(\Omega_\delta). \\
&\text{Finally, the set } \mathcal{A}^4 \text{ is non-empty, closed and convex in } \mathcal{L}^\infty(\Omega_\delta), \text{ thereby also weak* sequentially compact. It only remains to apply the Weierstrass' existence theorem to conclude the proof.}
\end{align*}
\]

3.2. Compliance minimization, nonconvex case: \( 1 < p \leq 2 \). Contrary to the case \( p = 1 \), where lower semicontinuity in the appropriate weak* topology holds in the local case thereby ensuring the existence of optimal solutions \([18]\), for \( p > 1 \) the nonlinear dependence on the control variable in the state equation destroys these properties. Consequently, additional regularization is required to guarantee the existence of optimal solutions for the local compliance minimization problem with SIMP, which is also confirmed and reflected in numerous numerical algorithms based on SIMP method \([3, 11]\). Surprisingly, the non-local compliance minimization problem still attains its infimum even in the non-convex case \( 1 < p \leq 2 \). We begin with the following simple statement.

**Lemma 3.2.** The map \( \mathcal{A}^4 \ni \rho \mapsto \rho(x)\rho(x') \in \mathcal{L}^\infty(\Omega_\delta \times \Omega_\delta) \) is sequentially continuous in the weak* topology of \( \mathcal{L}^\infty \).

**Proof.** Consider a sequence \( \rho_k \in \mathcal{A}^4, k = 1, 2, \ldots \) with \( \rho_k \rightharpoonup \hat{\rho} \), weak* in \( \mathcal{L}^\infty(\Omega_\delta) \). Let us take an arbitrary \( \psi \in \mathcal{L}^1(\Omega_\delta \times \Omega_\delta) \). Owing to Fubini’s theorem, the sequence \( \phi_k(x) = \int_{\Omega_\delta} \psi(x,x')\rho_k(x') \, dx' \), \( k = 1, 2, \ldots \), converges towards \( \hat{\phi}(x) = \int_{\Omega_\delta} \psi(x,x')\hat{\rho}(x') \, dx' \), for almost all \( x \in \Omega_\delta \). As the elements of this sequence are dominated by an \( \mathcal{L}^1(\Omega_\delta) \) function \( \mathcal{P} \int_{\Omega_\delta} |\psi(x,x')| \, dx' \), Lebesgue’s dominated convergence theorem applies implying that \( \lim_{k \to \infty} \| \phi_k - \hat{\phi} \|_{\mathcal{L}^1(\Omega_\delta)} = 0 \). Finally,

\[
0 \leq \lim_{k \to \infty} \left| \int_{\Omega_\delta} \int_{\Omega_\delta} \psi(x,x')|\rho_k(x')\rho_k(x) - \hat{\rho}(x')\hat{\rho}(x)| \, dx' \, dx \right| \leq \lim_{k \to \infty} \left| \int_{\Omega_\delta} \hat{\phi}(x)|\rho_k(x) - \hat{\rho}(x)| \, dx \right| + \lim_{k \to \infty} \mathcal{P}\| \phi_k - \hat{\phi} \|_{\mathcal{L}^1(\Omega_\delta)} = 0.
\]

With this in mind, we can extend Proposition 3.1 to the non-convex case.

**Proposition 3.3.** The compliance minimization problem \((2.15)\) (that is, \( J(\rho, u) = \ell(u) \)), admits an optimal solution \( (\rho^*, u^*) \in \mathcal{A}^4 \times \mathcal{U}_0^6 \) for \( 1 < p \leq 2 \).

**Proof.** As in the proof of Proposition 3.1 it is sufficient to establish the weak* sequential upper semicontinuity of the map \( \mathcal{A}^4 \ni \rho \mapsto I_{\delta,\kappa}(u) \in \mathbb{R} \) with \( \kappa(x,x') = \rho^{1/2}(x)\rho^{1/2}(x') \).
$\rho^p/2(x)\rho^p/2(x')$, since upper semicontinuity is preserved under taking minimum over $u \in \mathcal{U}_\delta^0$. The required property follows easily from the norm-continuity and concavity of the map $K^\delta : \tilde{k} \mapsto I_{\delta, p^2/2}(u) \in \mathbb{R}$ for $0 < p \leq 2$ and Lemma 3.2.

3.3. More general objective functions: $p = 2$. SIMP has been successfully utilized within other contexts than compliance minimization, see [11]. The non-local optimal design problem we consider admits optimal solutions without the need for further regularization in the special case $p = 2$ for a wide class of objective functions.

**Proposition 3.4.** Let $p = 2$, and assume that the objective function $(\rho, u) \mapsto J(\rho, u)$ is sequentially lower semicontinuous with respect to weak* topology of $L^\infty(\Omega_\delta)$ $\times$ norm topology of $\mathcal{U}_\delta^0$. Then the optimal design problem (2.15) admits an optimal solution $(\rho^*, u^*) \in \mathcal{A}_\delta \times \mathcal{U}_\delta^0$.

**Proof.** In view of weak* compactness of $\mathcal{A}_\delta$ in $L^\infty(\Omega_\delta)$, in order to apply the direct method of calculus of variations and conclude the existence of optimal solutions it is sufficient to establish that for an arbitrary minimizing sequence $(\rho_k, u_k) \in \mathcal{A}_\delta \times \mathcal{U}_\delta^0$ with $\rho_k \to \tilde{\rho}$, weak* in $L^\infty(\Omega_\delta)$, we have the corresponding convergence $u_k \to \tilde{u}$ in $\mathcal{U}_\delta^0$, where $\tilde{u} = S_\delta(\tilde{\rho})$ with $\kappa(x, x') = \tilde{\rho}(x)\tilde{\rho}(x')$. However, this follows immediately from Theorem 2.3 in view of Lemma 3.2.

4. Convergence to the local problem as $\delta \to 0$: connection to SIMP.

We will now turn our attention to the relationship between the non-local compliance minimization problem and the local one, which arises as a natural candidate for the limiting object for vanishing non-local interaction horizons $\delta \to 0$. More specifically, we would like to understand whether infimum values of the nonlocal problems converge to the infimum of the local problem, and/or whether sequences of minimizers of the nonlocal problems converge towards minimizers of the local problem. The standard framework for studying variational convergence of functionals, which is equipped with the precise vocabulary for formulating and answering such questions, is that of $\Gamma$-convergence [15]. Unfortunately, in our situation it is impossible to expect the local compliance minimization problem to be the $\Gamma$-limit of the nonlocal compliance minimization problems for any $p \in [1, 2]$. Indeed, the $\Gamma$-limit is always a lower semicontinuous functional in the topology in which the $\Gamma$-convergence is set [15]. However, the local compliance functional in the presence of SIMP penalization with $p > 1$ is not lower semicontinuous in a relevant topology. This is precisely the fundamental reason for the lack of optimal solutions to the local compliance minimization problem in this situation, the fact which is well documented and understood in the literature [11, 3].

In spite of this unfortunate insurmountable obstacle, in this section we would like to investigate what kind of relationship between the two problems can be salvaged for any $p \in [1, 2]$.

In order to succinctly discuss convergence of minimizers it would be convenient to put them into the same function space, which is not a priori the case given the fact that $\Omega_\delta$ decreases to $\Omega$ as $\delta \to 0$. Since we are only concerned with small $\delta > 0$, we fix an arbitrary $\delta_0 > 0$ and will only consider $\delta \in [0, \delta_0]$. This allows us to consider material distributions to be elements of the “largest” space $L^\infty(\Omega_{\delta_0})$, extending them by $\rho$ outside of their domain of definition $\Omega_\delta$, $0 < \delta < \delta_0$. (The same applies to the limiting local model, if we “by continuity” put $\Omega_0 = \Omega$.) In a similar fashion we will extend the state functions by 0 outside of their domain of definition $\Omega_\delta$, $0 \leq \delta < \delta_0$.

With such an extension we have that both $\xi_\delta$ and $c$ are defined on a subset of the same function space, $L^\infty(\Omega_{\delta_0})$, which we equip with weak* topology. Convergence $\delta \to 0$ will be understood as convergence for any sequence $\delta_j \to 0$ as $j \to \infty$. 

This manuscript is for review purposes only.
4.1. “\(T\)-lower semi-continuity”. The first result is in the spirit of the \(\liminf\)
inequality of \(T\)-convergence, but with the unfortunate exception that the functional
arguments do not converge in the natural topology of the function space we work
with, expect for \(p = 1\).

Proposition 4.1. Let \(\rho_\delta \in L^\infty(\Omega_{\delta_0})\) be such that \(\rho \leq \rho_\delta(x), \rho(x) \leq \overline{\rho}\) for almost
all \(x \in \Omega_{\delta_0}\). Assume that \(\rho_\delta^{p/2} \to \rho^p\), weak* in \(L^\infty(\Omega_{\delta_0})\) as \(\delta \to 0\). Let \(u_\delta = S_\delta(\kappa_\delta)\)
be the solution of (2.12), where \(\kappa_\delta(x, x') = \rho_\delta^{p/2}(x)\rho_\delta^{p/2}(x')\), and let \(u = S(\rho^p)\) be the
solution to (2.1). Then

\[
\lim inf_{\delta \to 0} \kappa_\delta(x, x') \geq \kappa(x, x').
\]

Proof. For convenience we put \(\kappa(x, x') = \rho^{p/2}(x)\rho^{p/2}(x')\); then \(\lim_{\delta \to 0} \kappa_\delta(x, x') = \kappa(x, x')\), for almost all \((x, x') \in \Omega^2_{\delta_0}\). Note that owing to [14, Theorem 1] we have the
inclusion \(u \in \mathcal{U}_{\delta_0}^0\), for all \(\delta \in [0, \delta_0]\). Therefore, we can test (2.12) with \(v = u\) to get the
equality

\[
a_{\delta, \kappa_\delta}(u_\delta, u) = \ell(u) = a_{p\rho}(u, u).
\]

Consequently, the difference \(L_\delta := \kappa_\delta(u_\delta) - \ell(\rho_\delta) = \ell(u_\delta - u)\) can be written as

\[
L_\delta = a_{\delta, \kappa_\delta}(u_\delta, u_\delta) - 2a_{\delta, \kappa_\delta}(u_\delta, u) + a_{p\rho}(u, u)
\]

\[
= a_{\delta, \kappa_\delta}(u_\delta, u_\delta) - 2a_{\delta, \kappa_\delta}(u_\delta, u) + a_{\delta, \kappa_\delta}(u, u) + [a_{p\rho}(u, u) - a_{\delta, \kappa_\delta}(u, u)].
\]

Now we have \(L_\delta^{(1)} = a_{\delta, \kappa_\delta}(u_\delta - u, u_\delta - u) \geq 0\) because \(a_{\delta, \kappa_\delta}(\cdot, \cdot)\) is an inner product
on \(\mathcal{U}_{\delta_0}^0\), and therefore

\[
L_\delta \geq L_\delta^{(2)},
\]

and hence in order to prove the result it is enough to show that

\[
\lim inf_{\delta \to 0} L_\delta^{(2)} \geq 0.
\]

But, applying Young’s inequality to the nonlocal conductivity

\[
\rho_\delta^{p/2}(x)\rho_\delta^{p/2}(x') \leq \frac{1}{2} (\rho_\delta^p(x) + \rho_\delta^p(x'))
\]

we have that

\[
-a_{\delta, \kappa_\delta}(u, u) \geq - \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} \frac{\rho_\delta^p(x) + \rho_\delta^p(x')}{2} A_\delta(|x - x'|)(u(x) - u(x'))^2 \frac{dx}{|x - x'|^2} dx' \]

\[
= - \int_{\Omega_{\delta_0}} \rho_\delta^p(x') \int_{\Omega_{\delta_0}} A_\delta(|x - x'|)(u(x) - u(x'))^2 \frac{dx}{|x - x'|^2} dx',
\]

and the last term converges to \(-a_{p\rho}(u, u)\) owing to [14, Corollary 1] and the weak
convergence of \(\rho_\delta^p\). Consequently

\[
\lim inf_{\delta \to 0} [a_{p\rho}(u, u) - a_{\delta, \kappa_\delta}(u, u)] \geq 0
\]

and the proof is finished. \(\Box\)
4.2. Pointwise convergence: \( \lim_{\delta \to 0} \epsilon_{\delta}(\rho) = \epsilon(\rho) \). The following result establishes the pointwise convergence of \( \epsilon_{\delta} \) to \( \epsilon \) in \( L^\infty(\Omega_{\delta}) \). It should be understood as a lim sup-inequality in \( \Gamma \)-convergence, where the recovering sequence is the constant sequence.

In order to prove it we need the following lemma, which establishes that the estimate (2.16) can be made uniform with respect to small \( \delta > 0 \).

**Lemma 4.2.** There exists a constant \( \delta \in ]0, \delta_0[ \), and a constant \( C_{\delta} > 0 \), independent from \( \delta \) and \( u \), such that we have the inequality 

\[
C_{\delta} \| u \|_{L^2(\Omega)} \leq \| u \|_{U_\delta}^\delta
\]

for any \( \delta \in ]0, \delta_0[ \) and \( u \in U_\delta^\delta \).

**Proof.** For the sake of contradiction, we assume that for each \( j = 1, 2, ... \) there is \( \delta_j \in ]0, \delta_0[ \), and \( u_j \in U_\delta^\delta \) such that

\[
\lim_{j \to \infty} \delta_j = 0, \quad \lim_{j \to \infty} \| u_j \|_{U_\delta^\delta} = 0, \quad \text{but} \quad \| u_j \|_{L^2(\Omega)} = 1.
\]

Note that since \( u_j \) is extended by zero outside of \( \Omega_{\delta_j} \), \( u_j \equiv 0 \) in \( \Omega_{\delta_0} \setminus \Omega \). When combined with the smallness of support of \( A_{\delta_j}(\cdot | \cdot) \), see (2.10), this implies the equality

\[
\int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} A_{\delta_j}(|x - x'|) \frac{u_j(x) - u_j(x')}{|x - x'|^2} \, dx \, dx' = a_{\delta_j}(u_j, u_j) = \| u_j \|_{U_\delta^\delta}^2.
\]

We can therefore apply [42, Theorem 1.2], which asserts that the sequence \( u_j \) is relatively compact in \( L^2(\Omega_{\delta_0}) \) with all its accumulation points being in \( W^{1,2}(\Omega_{\delta_0}) \).

Let \( u_0 \in W^{1,2}(\Omega_{\delta_0}) \) be such an \( L^2(\Omega_{\delta_0}) \)-accumulation point of \( u_j \); in particular

\[
\| u_0 \|_{L^2(\Omega)} = 1.
\]

Since the accumulation point does not depend on any finite number of terms in the sequence \( u_j \), we can utilize the estimate in [42, Theorem 1.2] as follows:

\[
\int_{\Omega_{\delta_0}} \| \nabla u_0(x) \|^2 \, dx \leq \limsup_{j \to \infty} \{ a_{\delta_j}(u_j, u_j) \} = 0.
\]

Therefore, \( u_0 \) must be a constant on \( \Omega_{\delta_0} \). On the other hand we have the pointwise (in fact, finite) convergence \( \lim_{j \to \infty} u_j(x) = 0 \), \( x \in \Omega_{\delta_0} \) \( \setminus \) closure \( \Omega \), and consequently \( u_0 \equiv 0 \) on \( \Omega_{\delta_0} \) \( \setminus \) closure \( \Omega \). Therefore \( u_0 \equiv 0 \) on \( \Omega_{\delta_0 \setminus \Omega} \), which contradicts the previously established fact \( \| u_0 \|_{L^2(\Omega)} = 1 \). \( \square \)

*Remark 4.3.* Note that Lemma 4.2 implies the coercivity of the nonlocal equation (2.12) even in the absence of assumption (2.11), as [42, Theorem 1.2] does not require such a condition. However, as we mentioned previously, for our purposes the assumption (2.11) is a natural hypothesis, as it implies the continuous embedding of \( U_\delta^\delta \) into \( W^{1,2}(\Omega_{\delta}) \) and therefore also the compact embedding into \( L^2(\Omega_{\delta}) \).

Before staying the main result of this section we need to define the following class of designs:

\[
\tilde{\mathcal{A}}^\delta = \left\{ \rho \in \mathcal{A}^\delta : \rho(x) = \sum_{k=1}^K \alpha_k \chi_{B_k}(x), \ K \in \mathbb{N}, \ \alpha_k > 0, \right\}
\]

\( B_k \) open, pairwise disjoint, and such that \( \bigcup_{k=1}^\delta B_k \supseteq \Omega_{\delta} \).

Note that \( \tilde{\mathcal{A}}^\delta \) is not just the class of simple functions in \( \mathcal{A}^\delta \), but the class of simple functions *supported* on open sets. This subtle but important restriction is going to be needed to apply \( \Gamma \)-convergence results of [43, 10] in the Step 2 of the proof below.
Proposition 4.4. Consider an arbitrary $\rho \in \hat{A}$. Let $u_\delta = S_\delta(\kappa) \in U_0^\delta$ be the sequence of solutions to (2.12) corresponding to a fixed $f \in L^2(\Omega)$ and $\kappa(x, x') = \rho^{p/2}(x)\rho^{p/2}(x')$, but varying $\delta \to 0$. Let further $u = S^{\text{loc}}(\rho) \in W_0^{1,2}(\Omega)$ be the weak solution to the local generalized Laplace problem (2.1).

Then
\[
\lim_{\delta \to 0} \|u_\delta - u\|_{L^2(\Omega)} = 0, \quad \text{and} \quad \lim_{\delta \to 0} a_{\delta,\kappa}(u_\delta, u_\delta) = a_{\rho^p}(u, u).
\]

Consequently,
\[
\lim_{\delta \to 0} \epsilon_\delta(\rho) = \epsilon(\rho),
\]
for any $\rho \in \hat{A}$.

Proof. Let $\hat{\delta} \in [0, \delta_0]$ and $C_\delta > 0$ be those established in Lemma 4.2. Since $u_\delta = S_\delta(\kappa)$ solves (2.12), for all $\delta \in [0, \hat{\delta}]$ we get the estimate
\[
C_\delta^2 \|u_\delta\|_{L^2(\Omega)}^2 \leq \epsilon_\delta a_{\delta,1}(u_\delta, u_\delta) \leq a_{\delta,\kappa}(u_\delta, u_\delta) \leq \|f\|_{L^2(\Omega)} \|u_\delta\|_{L^2(\Omega)},
\]
and consequently the uniform stability estimates
\[
\|u_\delta\|_{L^2(\Omega)} = \|u_\delta\|_{L^2(\Omega)} \leq \frac{1}{C_\delta^2 \|f\|_{L^2(\Omega)}}, \quad \text{and} \quad \|u_\delta\|_{U_0^\delta} \leq \frac{1}{C_\delta^2 \|f\|_{L^2(\Omega)}}.
\]

Utilizing [42, Theorem 1.2] as in Lemma 4.2, we establish the existence of an $L^2(\Omega_\delta)$-accumulation point $u_0 \in W_0^{1,2}(\Omega)$ of $u_\delta$. Let $\delta_j$, $j = 1, 2, \ldots$ be a sequence realizing convergence towards this accumulation point, that is $\lim_{j \to \infty} \delta_j = 0$ and $\lim_{j \to \infty} \|u_{\delta_j} - u_0\|_{L^2(\Omega)} = 0$. As mentioned previously, in the following discussion both $u_0$ and $u$, the solution of the limiting local problem, are extended by 0 outside of $\Omega$.

Step 1. We claim that
\[
(4.1) \quad \limsup_{j \to \infty} I_{\delta_j,\kappa}(u_{\delta_j}) \leq I_{\rho^p}(u).
\]

Indeed, owing to (2.9) and [14, Theorem 1] we have the bound $\|u\|_{U_0^\delta_j}$ \leq $C\|u\|_{W^{1,2}(\Omega)}$ and therefore also the inclusion $u \in U_0^\delta_j$. Consequently, owing to the variational characterization (2.17) we have the inequalities $I_{\delta_j,\kappa}(u_{\delta_j}) \leq I_{\delta_j,\kappa}(u)$, for each $\delta_j$, $j = 1, 2, \ldots$. Therefore, in order to establish (4.1) it is sufficient to show the inequality $\limsup_{j \to \infty} a_{\delta_j,\kappa}(u, u) \leq a_{\rho^p}(u, u)$. But this inequality follows from the direct application of Proposition 4.1 to the constant sequence $\rho$:
\[
\liminf_{j \to \infty} \left(-2I_{\delta_j,\kappa}(u_{\delta_j})\right) \geq -2I_{\rho^p}(u)
\]
and therefore (4.1) holds.

Step 2. We claim that
\[
I_{\rho^p}(u_0) \leq \liminf_{j \to \infty} I_{\delta_j,\kappa}(u_{\delta_j}).
\]

Note that owing to the strong $L^2$ convergence $\lim_{j \to \infty} \|u_{\delta_j} - u_0\|_{L^2(\Omega)} = 0$ it is sufficient to prove the inequality
\[
a_{\rho^p}(u_0, u_0) \leq \liminf_{j \to \infty} a_{\delta_j,\kappa}(u_{\delta_j}, u_{\delta_j}).
\]
Let us recall that $\rho \in \mathcal{A}^{\delta_0}$ is a simple function supported on open sets, that is

$$\rho(x) = \sum_{i=1}^{I} \alpha_i \chi_{B_i}(x),$$

where $I \in \mathbb{N}$, $\alpha_i > 0$ and the $B_i$ open and pairwise disjoint such that $\mathrm{closure}(\cup_{i=1}^{I} B_i) \supset \Omega_{\delta_0}$. Then

$$a_{\delta_j,\kappa}(u_{\delta_j}, u_{\delta_j}) = \int_{\Omega_{\delta_0}} \int_{\Omega_{\delta_0}} \rho^{p/2}(x)\rho^{p/2}(x') A_{\delta_j}(|x-x'|) \frac{(u_{\delta_j}(x) - u_{\delta_j}(x'))^2}{|x-x'|^2} \, dx' \, dx

= \sum_{i,j=1}^{I} \alpha_i \alpha_j \int_{B_i} \int_{B_j} A_{\delta_j}(|x-x'|) \frac{(u_{\delta_j}(x) - u_{\delta_j}(x'))^2}{|x-x'|^2} \, dx' \, dx

\geq \sum_{i=1}^{I} \alpha_i^{p/2} \int_{B_i} \int_{B_i} A_{\delta_j}(|x-x'|) \frac{(u_{\delta_j}(x) - u_{\delta_j}(x'))^2}{|x-x'|^2} \, dx' \, dx.$$  

Applying the $\Gamma$-convergence results in [43, 10] we conclude that

$$\liminf_{j \to \infty} \int_{B_i} \int_{B_j} A_{\delta_j}(|x-x'|) \frac{(u_{\delta_j}(x) - u_{\delta_j}(x'))^2}{|x-x'|^2} \, dx' \, dx \geq \int_{B_i} |\nabla u_0(x)|^2 \, dx.$$  

Summing up these inequalities and recalling that $u_0$ vanishes outside $\Omega$ we can conclude that

$$\liminf_{j \to \infty} a_{\delta_j,\kappa}(u_{\delta_j}, u_{\delta_j}) \geq a_{\rho^\ast}(u_0, u_0).$$

Since $\lim_{j \to \infty} \|u_{\delta_j} - u_0\|_{L^2(\Omega)} = 0$ this is sufficient to conclude the proof of step 2 for simple functions $\rho$. Notice that the requirement of $B_i$ to be open is necessary in order to apply the results in [43, 10].

**Step 3: Conclusion.** Combining the inequalities obtained in steps 1 and 2 we obtain the following string of inequalities:

$$\limsup_{j \to \infty} I_{\delta_j,\kappa}(u_{\delta_j}) \leq I_{\rho^\ast}(u) \leq I_{\rho^\ast}(u_0) \leq \liminf_{j \to \infty} I_{\delta_j,\kappa}(u_{\delta_j}).$$

The variational characterization of the local problem (2.3) and the uniqueness of solutions to (2.1) implies that $u = u_0$. Therefore the family of solutions $\{u_{\delta_j}\}_{0 < \delta < 1}$, which is relatively compact in $L^2(\Omega_{\delta_j})$, has only one accumulation point, and the sequence $\delta_j$, $j = 1, 2, \ldots$ selected in the beginning of the proof is in fact arbitrary. This finishes the proof of the proposition.

The conclusions of Proposition 4.4 hold in fact for an even larger class of material distributions than $\mathcal{A}^{\delta_0}$.

**Corollary 4.5.** Suppose that $\rho \in \mathcal{A}^{\delta_0}$ be such that there exists a sequence $\{\rho_i\}_i \subset \mathcal{A}^{\delta_0}$ such that $\rho_i(x) \leq \rho(x)$ and $\lim_{i \to \infty} \rho_i(x) = \rho(x)$ a.e. in $x \in \Omega_{\delta_0}$. Then the conclusion of Proposition 4.4 holds for this $\rho$.

**Proof.** Note that Steps 1 and 3 of the proof above do not utilize the simple structure of $\rho$. Thus, it is only necessary to amend Step 2 of the proof, which we do here.

Owing to our assumptions, there exists a sequence of non-negative simple functions $\rho_i$, which approximates $\rho$ almost everywhere in $\Omega_{\delta_0}$ from below. Let $\kappa_i(x, x') = \ldots$
\[ \rho_i^{p/2}(x) \rho_j^{p/2}(x'). \] Then \( \kappa_i(x,x') \leq \kappa(x,x') \) for almost all \( (x,x') \in \Omega_{\delta_0}^{2}. \) Therefore, in view of Proposition 4.4, for each \( i = 1, 2, \ldots \) we can write:

\[
\lim_{j \to \infty} \inf \alpha_{\delta_j, \kappa}(u_{\delta_j}, u_{\delta_j}) \geq \lim_{j \to \infty} \inf \alpha_{\delta_j, \kappa}(u_{\delta_j}, u_{\delta_j}) \geq a_{\rho_0}^{p}(u_0, u_0).
\]

It remains to take the limit with respect to \( i \to \infty \) and utilize the dominated Lebesgue convergence theorem to reach the inequality claimed in Step 2 of the proof of Proposition 4.4.

**Remark 4.6.** Note that one can approximate an arbitrary bounded and measurable \( \rho \) pointwise with a non-decreasing sequence of simple functions, see [47, Theorem 1.17]. Unfortunately, in Corollary 4.5 we need that these simple functions are additionally supported on open, and not just measurable, sets \( B_i \). Therefore we have not succeeded in showing the result for a general \( \rho \in \mathcal{A}^{b_{0}} \). We actually believe, and conjecture, that Corollary 4.5 is true without this additional restriction.

### 4.3. Discussion of \( \Gamma \)-convergence

General \( \Gamma \)-convergence theory establishes that \( \Gamma \)-convergence together with equi-coercivity implies both convergence of infima to the minimum of the limit problem, and that any cluster point of any sequence of minimizers is a minimizer of the limit problem [15]. As we have pointed out above, the unfortunate fact that convergence of functional arguments for liminf result, Proposition 4.1, is not weak* convergence in \( L^{\infty}(\Omega_{\delta_0}) \) spoils the possibility of a general \( \Gamma \)-convergence result for any \( p \in [1, 2] \). Furthermore, local compliance for \( p \in [1, 2] \) is not lower semicontinuous and consequently cannot be a \( \Gamma \)-limit. In addition, the restricting hypothesis on the admissible material distributions in the statement of Corollary 4.5 rules out the possibility of obtaining a general \( \Gamma \)-convergence result even for \( p = 1 \). Still, we can point out at a few inequalities characterizing the relationship between the non-local and the local optimal control problems in the convex case \( p = 1 \).

On the one hand, for each \( \delta > 0 \) we can take a minimizer \( \rho_{\delta} \) for the non-local compliance minimization. Clearly the sequence \( \rho_{\delta} \) is bounded in \( L^{\infty}(\Omega_{\delta_0}) \). Therefore, there exists a sequence \( \delta_j \), with \( \delta_j \to 0 \), and \( \rho \in \mathcal{A} \), such that \( \rho_{\delta_j} \rightharpoonup \rho \) weak* in \( L^{\infty}(\Omega_{\delta_0}) \). Taking into account that \( p = 1 \), Proposition 4.1 implies

\[
m \leq c(\rho) \leq \lim_{j \to \infty} \inf \epsilon_{\delta_j}(\rho_{\delta_j}) = \lim_{j \to \infty} \inf m_{\delta_j},
\]

where

\[
m = \inf_{\rho \in \mathcal{A}} c(\rho), \quad m_{\delta} = \min_{\rho \in \mathcal{A}_{\delta}}(\rho).
\]

As this argument can be made for any sequence \( \delta_j \) converging to zero, we have that

\[
m_j \leq \lim_{\delta \to 0} \inf m_{\delta}.
\]

Furthermore, if \( \inf_{\rho \in \mathcal{A}} c(\rho) \) is attained at \( \hat{\rho} \in \mathcal{A} \) satisfying the assumptions of Corollary 4.5, then the inequality (4.3) becomes equality:

\[
m = c(\hat{\rho}) = \lim_{j \to \infty} \epsilon_{\delta_j}(\hat{\rho}) \geq \lim_{j \to \infty} \sup m_{\delta_j},
\]

for any \( \delta_j \to 0 \).

Finally, if Corollary 4.5 were true for any \( \rho \in \mathcal{A}^{b_{0}} \), then we would have that \( \epsilon_{\delta} \Gamma \)-converges to \( c \) as \( \delta \) goes to 0. Indeed, weak* topology is metrizable on bounded sets of \( L^{\infty}(\Omega_{\delta_0}) \) (cf. [16, Theorem 3.28]). Therefore \( \mathcal{A}^{b_{0}} \) equipped with the weak* topology is a metric space, and therefore \( \Gamma \)-convergence requires only two facts to hold, namely, limsup and liminf inequalities. These would then be direct consequences of Proposition 4.1 and Corollary 4.5, respectively.
5. Numerical experiments. The objective of this section is to numerically illustrate the behaviour of the proposed optimization model, with emphasis on the results established in the previous sections. All our numerical experiments are performed with \( n = 2 \), \( \Omega = ]0,1[^2 \), \( A_\delta(|x|) = c_{\text{nrm}} |x|^{-(n+2\nu-2)} \max\{0,\delta^2 - |x|^2\}^\beta \), with \( \beta = 3.0 \) and \( c_{\text{nrm}} \) determined from (2.9). Additionally, we use \( \rho = 10^{-5} \), \( p = 1.0 \), \( s \in \{1/3, 2/3\} \), \( \delta \in \{0.05, 0.1, 0.2\} \), and \( p \in \{1, 2\} \).

5.1. Galerkin FEM discretization of the state equations. The variational formulation (2.12) with a symmetric and coercive bilinear form \( a_{\delta,h}(|\cdot|,\cdot) \) naturally lends itself for an application of Galerkin method. In our numerical experiments we only consider polyhedral sets \( \Omega \), and therefore we proceed in the standard fashion by decomposing \( \Omega_\delta \) into a union of shape-regular simplices \( \Omega_\delta^0 \), where \( h > 0 \) will denote a characteristic size (diameter) of the elements in our mesh. We make sure that \( \Omega_\delta^0 \) conforms with the subdivision of \( \Omega_\delta \) into \( \Omega \) and \( \Gamma^{nl} \), see Fig. 1. Functions in \( U \)

![Fig. 1. Three types of meshes used in the present work and their corresponding labels. In this figure, we use \( \delta = 0.2 \), and decomposition of \( \Omega_\delta \) into \( \Omega \) and \( \Gamma^{nl} \) is illustrated with color. Gmsh is utilized for generating unstructured meshes [33].](image)

![Fig. 2. Left: reference elements \( T_1 \) (red), and elements \( T_2 \) (orange), participating in the pre-computation of the integrals. Note that only half of the elements in the proximity of the reference cell need to be considered owing to the symmetry of the bilinear form with respect to the change of the integration variables \( x \mapsto x' \). Right: contour plot of \( A_\delta(|\cdot-x'|) \) for \( x \) located at the barycenter of one of the reference elements. \( \delta = 0.2 \) and \( s = 1/3 \) is used.](image)

will be approximated with continuous piecewise-linear polynomials \( u^h \in U^h = \{v^h \in U \cap C^0(\Omega_\delta) : v^h|_T \text{ is a linear polynomial, } vT \in \Omega_\delta^0\} \). Naturally we put \( U_0^h = U^h \cap U_0 \), which leads us to the following discrete variational principle (system of linear algebraic
Assembly process for the right hand side of this system is completely standard, whereas in order to assemble the left hand side of this system we need to loop over all pairs \((T_1, T_2) \in \Omega^3_\delta \times \Omega^3_\delta\) of elements in the mesh, which are not further than the distance of 2\(\delta\) from each other, and compute the local integral contribution to \(a_{\delta,\kappa}(u_h, v_h)\), that is, the integral over \(T_1 \times T_2\). Note that when \(T_1 \cap T_2 \neq \emptyset\), the integrand is unbounded; even when \(T_1 \cap T_2 = \emptyset\) the integrand is not a polynomial function. In our implementation we utilize the quadratures described in [19], which are tailored for a nearly identical situation.\(^4\) In order to avoid committing a variational crime by not integrating the bilinear form precisely, we first estimate how many quadrature points we need for the accurate integration; the results are reported in Fig. 3. Despite the fact that the assumptions imposed in [19] are not always satisfied we observe exponential convergence of the quadratures. However, note the unusual scaling of the x-axis; in the most singular case \(k = 2\) corresponding to \(T_1 = T_2\) we need approximately 15\(^5\) = 759375 quadrature points (when using Gauss–Jacobi quadrature in the singular direction, see [19] for details) to achieve nearly full IEEE double precision accuracy before the round off errors start to play a role!

Because of such a high cost of elemental integration, and because the number of integrals in a quasi-uniform grid grows as \(O(\delta^n h^{-2n})\), we focus on regular grids (see Fig. 1). In this setting we only need to evaluate integrals for a fixed “reference” \(T_1\) and varying \(T_2\), thereby bringing the number of integrals down to \(O(\delta^n h^{-n})\) as shown in Fig. 2. Even with this preprocessing, both the work and memory requirements for the global matrix assembly scale as \(O(\delta^n h^{-2n})\). Putting this into perspective, for Grid2 with \(h = 2^{1/2}/320\) (i.e., each side of the unit square \(\Omega\) is discretized with 320 elements) we need approximately 4.7Mb to store the precomputed integrals and approximately 3.87 \cdot 10^3Mb to store the assembled matrix. Direct solver such as UMFPACK [21] quickly run out of memory for problems with \(\delta = 0.1\), and we switch to CG-accelerated Ruge–Stuben AMG solver PyAMG [40] (even smoothed aggregation is too much memory and computationally demanding).

### 5.1.1. \(h\)-convergence test

In order to test the code, we use the method of manufactured solutions, see e.g. [44]. We put \(\delta = 0.1\), \(s \in \{1/3, 2/3\}\), \(\kappa \equiv 1\), and let the analytical solution to be \(u_{\text{ana}}(x, y) = [x(1-x)y(1-y)]^s \sin(2\pi(x+y^2))\), when \(x, y \in \Omega\), and zero otherwise. The corresponding right hand side can be (numerically) computed as
\[
f(x, y) = -2 \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \int_0^{2\pi} A_\delta(r) \frac{u_{\text{ana}}(x + r \cos(\theta), y + r \sin(\theta)) - u_{\text{ana}}(x, y)}{r} \, d\theta \, dr,
\]
which is evaluated using the standard adaptive quadrature package in SciPy. The results of this test are shown in Fig. 4. In both cases we do observe convergence, although it is difficult to say whether we reach the asymptotic convergence rate (we run the simulations until we run out of memory), or indeed whether the adaptive quadratures evaluate (5.2) accurately enough.

---

\(^4\)It should be noted that our integrands do not always satisfy the assumptions imposed in [19] as the terms \((u(x) - u(x'))/|x - x'|\) and \((v(x) - v(x'))/|x - x'|\) are only bounded and not continuous across the boundaries \(T_1 \cap T_2\).
5.1.2. "Γ-convergence" test. Another test we perform is that of “numerical Γ-convergence”, that is, we try to illustrate Proposition 4.4. Namely we put $s = 1/3$, $\rho(x, y) = \rho + (\rho - \rho) \exp\{-[(x - m_x)^2 + (y - m_y)^2]/\sigma\}$ with $(m_x, m_y) = (1/2, 2/3)$ and $\sigma = 0.1$, and $\kappa$ computed from $\rho$ using the SIMP model with $p = 2$, $u_{\text{ana}}(x, y) = \sin(2\pi x)\sin(\pi y)$, for $x, y \in \Omega$, and zero otherwise. We compute...
\( f = -\text{div}[\rho^p \nabla u_{\text{ana}}] \), and solve a variety of non-local problems with varying \( \delta \) and \( h \) on Grid2. The results are summarized in Fig. 5. The main observation is that the quantity \( e(\delta) = \| u_{\text{ana}} - \lim_{h \to 0} u_{\delta,h} \|_{L^2(\Omega)} \), where \( u_{\delta,h} \) is the numerical solution to the discretized non-local problem (5.1), appears to be decreasing with \( \delta \). Unsurprisingly, one needs finer resolution meshes to resolve non-local problems with smaller values of \( \delta \).

\[ \begin{array}{c|c|c|c|c|c|c} \hline h & \text{relative error} \\ \hline h = 7.071e-02 & 10^{-3} & 10^{-2} \\ h = 3.536e-02 & 10^{-3} & 10^{-2} \\ h = 1.768e-02 & 10^{-3} & 10^{-2} \\ h = 8.839e-03 & 10^{-3} & 10^{-2} \\ h = 4.419e-03 & 10^{-3} & 10^{-2} \\ \hline \end{array} \]

Fig. 5. Verification of convergence of the discretization (and the numerical implementation) with respect to \( h \)-refinement. Left: \( s = 1/3 \), right: \( s = 2/3 \). The relative error is measured with respect to \( L^2(\Omega) \) norm.

5.2. Solving the optimization problem. In our implementation we solve a discretized version of the optimization problem (2.15) with a tiny discrepancy: instead of the constraint \( \int_{\Omega} \rho(x) \, dx \leq \gamma |\Omega| \) in (2.13) we have implemented the constraint \( \int_{\Omega} \rho(x) \, dx \leq \gamma |\Omega| \). We focus exclusively on compliance minimization, that is, \( J(\rho, u) = \ell(u) \), and we employ the so-called optimality criterion (OC) scheme for solving these problems. This choice is primarily owing to the popularity of OC in the topology optimization community, and any other gradient-based non-linear constrained optimization algorithm could be utilized in its place. Within the OC scheme, given a current material distribution \( \rho_k \) we first compute the corresponding state \( u_{\rho_k} \) by solving the discretized system (5.1), and then the derivatives of non-local compli-
The new material distribution is defined by a simple pointwise update scheme

\[ \rho_{k+1} = \pi_{B_k}[\rho_k(-\nabla \epsilon_\delta(\rho_k)/\lambda_{k+1})^\xi], \]

where \( \pi_{B_k}[\cdot] \) is a projection operator onto a closed, convex, and non-empty set \( B_k = \{ \rho \in L^2(\Omega_\delta) \mid \max[\rho, (1 - \eta)\rho_k] \leq \rho \leq \min[\rho, (1 + \eta)\rho_k] \} \), and \( \eta > 0 \) and \( \xi \in (0, 1) \) are trust-region like and damping parameters, respectively, and \( \nabla \epsilon_\delta(\rho_k) \) is \( L^2(\Omega_\delta) \) representation of the directional derivatives \( \epsilon_\delta'(\rho_k; \xi) \). Finally, \( \lambda_{k+1} \) is computed by finding the root of the equation \( \int_{\Omega_\delta} \rho_{k+1}(x)\,dx = \gamma|\Omega| \) using, for example, the bisection algorithm. We put \( \eta = 0.2, \xi = 0.5 \), and stop the algorithm when \( \|\rho_{k+1} - \rho_k\|_{L^2(\Omega_\delta)} < 10^{-4} \). For more details see [11].

### 5.2.1. Convex case: \( p = 1 \)

In the the “easy” convex case corresponding to \( p = 1 \) we can expect that the optimization algorithm computes approximations to (discretized approximations of) globally optimal solutions. The computed local conductivities \( \kappa_{loc} = \rho \) and the corresponding states for several values of \( \delta \) are shown in Figure 6. Note that in this case the local compliance minimization problem (2.5) admits globally optimal solutions, which is also shown in Figure 6 with the corresponding state. The qualitative resemblance between the solutions to the non-local and local problems is clear from these pictures. Additionally, we seem to have a quantitative connection between the two problems, illustrated in Table 1.

Note that the number of optimization iterations needed to solve the problem is virtually independent from the value of the non-local horizon \( \delta \) or size of the mesh \( h \). Still, each iteration of the optimization algorithm, which requires solving the discretized state equations, becomes significantly more costly for larger \( \delta > 0 \).

### Table 1

Summary of the results for the convex case \( p = 1 \). \( J^* \): optimal value; \( N \): number of OC iterations. \( \delta = 0 \) corresponds to the limiting local model (solved using a separate code) which admits optimal solutions in the convex case \( p = 1 \), see [18, 11, 3].

| \( \delta \)   | \( h \)       | \( J^* \)       | \( N \) |
|--------------|---------------|-----------------|-------|
| 0.2          | 3.54 \cdot 10^{-2} | 9.0727 \cdot 10^{-2} | 26    |
| 0.2          | 1.77 \cdot 10^{-2} | 9.1778 \cdot 10^{-2} | 27    |
| 0.2          | 8.84 \cdot 10^{-3} | 9.2408 \cdot 10^{-2} | 27    |
| 0.2          | 4.42 \cdot 10^{-3} | 9.2798 \cdot 10^{-2} | 27    |
| 0.1          | 3.54 \cdot 10^{-2} | 7.8627 \cdot 10^{-2} | 28    |
| 0.1          | 1.77 \cdot 10^{-2} | 7.9479 \cdot 10^{-2} | 32    |
| 0.1          | 8.84 \cdot 10^{-3} | 7.9930 \cdot 10^{-2} | 32    |
| 0.1          | 4.42 \cdot 10^{-3} | 8.0218 \cdot 10^{-2} | 32    |
| 0.05         | 3.54 \cdot 10^{-2} | 7.3251 \cdot 10^{-2} | 29    |
| 0.05         | 1.77 \cdot 10^{-2} | 7.3896 \cdot 10^{-2} | 34    |
| 0.05         | 8.84 \cdot 10^{-3} | 7.4276 \cdot 10^{-2} | 36    |
| 0.05         | 4.42 \cdot 10^{-3} | 7.4484 \cdot 10^{-2} | 35    |
| 0           | 3.13 \cdot 10^{-3} | 6.9460 \cdot 10^{-2} | 38    |

This manuscript is for review purposes only.
5.2.2. Nonconvex case: $p = 2$. A significantly more interesting case, in view of the very different behaviour of the non-local and local problems, corresponds to the non-convex optimization problem (2.15) with $p = 2$. In this case, for the local problem the intermediate values of the conductivity are effectively penalized by the underlying physics and one can expect the computed optimized conductivity distributions to be of “bang-bang” structure, assuming either the lowest or the highest possible values of conductivity everywhere. Note also that in this case the local optimization problem (2.5) does not admit optimal solutions, and therefore we have no local solution to compare with.

The computed local conductivities $\kappa^{loc} = \rho^p$ and the corresponding states for several values of $\delta$ are shown in Figure 7. They agree with our expectations. One can also note that for smaller $\delta$ the computed conductivity distributions display smaller features and progressively more oscillatory character. Indeed, in the case of the local problem, minimizing sequences consist of conductivities, which are locally highly oscillatory (periodic) and can be mathematically understood as converging to composite materials, see [11, 3]. Quantitative information about these solutions is shown in Table 2. Note that for smaller $\delta$ the optimization algorithm requires significantly more iterations to converge (which does not mean that it is more difficult to solve, as each iteration is less computationally expensive in this case).

The non-existence of optimal solutions for the local compliance minimization problem manifests itself numerically as “mesh-dependence” of optimal designs, where progressively more oscillatory conductivity distributions are encountered as one refines the computational mesh. This is often used as an “engineering” test of existence of optimal solutions for a given problem. The optimal conductivity distributions corresponding to $\delta = 0.2$ and a range of discretization levels is shown in Figure 8. Indeed, one can recognize the sequence of convergent shapes as the discretization gets finer and finer. Again, this example works by pure coincidence: indeed as the problem
is non-convex the optimization algorithm may end up in different local minima at different discretization levels, and indeed we have observed such behavior for other values of $\delta$.

Finally, we perform a “cross-check” of the computed non-local designs by evaluating all computed designs for all values of $\delta$. We expect (but of course cannot guarantee this, as we can only hope to find locally optimal solutions and not global ones) that the distribution optimized for a specific value of $\delta$ would outperform other designs computed for different values of $\delta$. The results of this test are shown in Table 3. Indeed, our expectations are confirmed: within each row the diagonal element is the smallest one.

![Conductivity distribution and Temperature distribution](image)

**Fig. 7.** Pictures of the computed optimal designs for $p = 2$. Top line: optimal design (conductivity distribution $\kappa_{\text{loc}} = \rho^2$), bottom line: corresponding optimal states. From left to right: $\delta = 0.2$, $\delta = 0.1$, $\delta = 0.05$. Note the slightly different color range and the presence of $\Gamma_{\text{nl}}$ for the nonlocal problems.

| $\delta$   | $h$    | $J^*$   | $N$  |
|------------|--------|---------|------|
| 0.2        | $3.54 \cdot 10^{-2}$ | 1.6902 $\cdot 10^{-1}$ | 1080 |
| 0.2        | $1.77 \cdot 10^{-2}$  | 1.7327 $\cdot 10^{-1}$  | 1062 |
| 0.2        | $8.84 \cdot 10^{-3}$  | 1.7516 $\cdot 10^{-1}$  | 1300 |
| 0.2        | $4.42 \cdot 10^{-3}$  | 1.7613 $\cdot 10^{-1}$  | 1139 |
| 0.1        | $3.54 \cdot 10^{-2}$  | 1.2857 $\cdot 10^{-1}$  | 514  |
| 0.1        | $1.77 \cdot 10^{-2}$  | 1.3242 $\cdot 10^{-1}$  | 1116 |
| 0.1        | $8.84 \cdot 10^{-3}$  | 1.3480 $\cdot 10^{-1}$  | 1395 |
| 0.1        | $4.42 \cdot 10^{-3}$  | 1.3560 $\cdot 10^{-1}$  | 1521 |
| 0.05       | $3.54 \cdot 10^{-4}$  | 1.0721 $\cdot 10^{-1}$  | 291  |
| 0.05       | $1.77 \cdot 10^{-2}$  | 1.1066 $\cdot 10^{-1}$  | 790  |
| 0.05       | $8.84 \cdot 10^{-3}$  | 1.1139 $\cdot 10^{-1}$  | 1358 |
| 0.05       | $4.42 \cdot 10^{-3}$  | 1.1185 $\cdot 10^{-1}$  | 2394 |
A. EVGRAFOV AND J.C. BELLIDO

Fig. 8. “Mesh-independence” test (owing to the non-convexity of the problem we cannot guarantee that the optimization algorithm converges to the “same” local minimum on different discretizations). Conductivity distributions $\kappa^{\text{loc}} = \rho^2$ are shown. We put $p = 2$, $\delta = 0.2$, and $h \in \{3.54 \cdot 10^{-2}, 1.77 \cdot 10^{-2}, 8.84 \cdot 10^{-3}, 4.42 \cdot 10^{-3}\}$.

Table 3
“Cross-checking” of the computed designs, $h = 4.42 \cdot 10^{-3}$ in all cases.

| $\delta$ | $J$ for $\delta = 0.05$ | $J$ for $\delta = 0.1$ | $J$ for $\delta = 0.2$ |
|----------|------------------------|------------------------|------------------------|
| 0.05     | $1.1185 \cdot 10^{-1}$ | $1.1724 \cdot 10^{-1}$ | $1.3225 \cdot 10^{-1}$ |
| 0.1      | $1.3954 \cdot 10^{-1}$ | $1.3560 \cdot 10^{-1}$ | $1.4172 \cdot 10^{-1}$ |
| 0.2      | $1.9678 \cdot 10^{-1}$ | $1.8213 \cdot 10^{-1}$ | $1.7613 \cdot 10^{-1}$ |

5.2.3. Nonconvex case: $p > 2$. Note that the sheer fact that the results of Subsection 3.2 do not apply in the case $p > 2$ does not mean that the non-local compliance minimization problem (2.15) does not admit solutions. In order to find the answer to this question a different type of analysis is needed. Intuitively, in view of the loss of convexity in the proof of Proposition 3.3, we believe the answer to this question is negative. The following computations, which mirror the “mesh-independence” experiment of the previous subsection, provide some support (but by no means conclusive evidence) for our pessimism: see Figure 9. Note that unlike in Figure 8 the oscillatory behaviour of $\kappa^{\text{loc}}$ does not seem to disappear with mesh refinement, indicating that Proposition 3.3 may be “sharp” for the considered model.

5.2.4. Arithmetic instead of geometric averaging of local conductivities. The proofs in Subsection 3.2 illustrate the value of utilizing geometric averaging in defining the non-local conductivity in (2.14). Nevertheless, one may wonder what happens in the case of arithmetic averaging. Figure 10 provides an example of such a computation, where we use $p \in \{1, 2\}$, $\delta = 0.2$, and $h \in$
Fig. 9. “Mesh-independence” test for $p = 3$ (top row) and $p = 4$ (bottom row). As before, conductivity distributions $\kappa^{loc} = \rho^p$ are shown. We use $\delta = 0.2$, and $\rho \in \{3.54 \cdot 10^{-2}, 1.77 \cdot 10^{-2}, 8.84 \cdot 10^{-3}\}$, from left to right, as in the previous subsection.

Even though our analysis (or analysis in [4, 6]) does not cover the non-convex case $p = 2$, it appears that there is convergence of local conductivities even in this situation. Note however that both cases $p = 1$ and $p = 2$ produce similar optimal objective values ($J^* \approx 9.19 \cdot 10^{-2}$ and $J^* \approx 8.58 \cdot 10^{-2}$). The first number compares very well with the one we have previously reported in Table 1 for $\delta = 0.2$, whereas the second one gives 50% smaller objective than in the geometric averaging case, see Table 2. This is consistent with the fact that geometric averaging results in a significantly smaller interaction between points in the nearly non-conductive and conductive domains than in the case of arithmetic averaging, which intuitively seems to be more “physical” to us. Indeed the “holes” in the computed material distributions, see Figure 10, bottom row, do not make much physical sense at least for the usual, local diffusion problem. Consequently, we believe that the compliance functional is underestimated for the case of the arithmetic mean parametrization of non-local conductivities and $p > 1$.

REFERENCES

[1] B. AKSOYLU AND T. MENGESHA, Results on nonlocal boundary value problems, Numer. Funct. Anal. Optim., 31 (2010), pp. 1301–1317, https://doi.org/10.1080/01630563.2010.519136.

[2] G. ALBERTI AND G. BELLETTINI, A nonlocal anisotropic model for phase transitions. I. The optimal profile problem, Math. Ann., 310 (1998), pp. 527–560, https://doi.org/10.1007/s002080050159.

[3] G. ALLAIRE, Shape optimization by the homogenization method, vol. 146, Springer Science & Business Media, 2012.

[4] F. ANDRÉS AND J. MUÑOZ, Nonlocal optimal design: a new perspective about the approximation of solutions in optimal design, Journal of Mathematical Analysis and Applications, 429 (2015), pp. 288–310.

[5] F. ANDRÉS AND J. MUÑOZ, A type of nonlocal elliptic problem: Existence and approximation through a Galerkin–Fourier method, SIAM Journal on Mathematical Analysis, 47 (2015), pp. 498–525.

[6] F. ANDRÉS AND J. MUÑOZ, On the convergence of a class of nonlocal elliptic equations and
Fig. 10. “Mesh-independence” test for arithmetic averaging of local conductivities with \( p = 1 \) (top row) and \( p = 2 \) (bottom row). As before, conductivity distributions \( \kappa^{\text{loc}} = \rho^p \) are shown. We use \( \delta = 0.2 \), and \( h \in \{3.54 \cdot 10^{-2}, 1.77 \cdot 10^{-2}, 8.84 \cdot 10^{-3}\} \), from left to right.

Related optimal design problems, Journal of Optimization Theory and Applications, 172 (2017), pp. 33–55.

[7] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, and J. J. Toledo-Melero, Nonlocal diffusion problems, vol. 165 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI; Real Sociedad Matemática Española, Madrid, 2010, https://doi.org/10.1090/surv/165.

[8] J. C. Bellido and A. Evgrafov, A simple characterization of H-convergence for a class of nonlocal problems, arXiv e-prints, (2019), arXiv:1903.11585, https://arxiv.org/abs/1903.11585.

[9] J. C. Bellido and C. Mora-Corral, Existence for nonlocal variational problems in peridynamics, SIAM J. Math. Anal., 46 (2014), pp. 890–916, https://doi.org/10.1137/130911548.

[10] J. C. Bellido, C. Mora-Corral, and P. Pedregal, Hyperelasticity as a Γ-limit of peridynamics when the horizon goes to zero, Calc. Var. Partial Differential Equations, 54 (2015), pp. 1643–1670, https://doi.org/10.1007/s00526-015-0839-9.

[11] M. Bendsoe and O. Sigmund, Topology Optimization: Theory, Methods and Applications. ISBN: 3-540-42992-1, Springer, 2003.

[12] M. P. Bendsoe and N. Kikuchi, Generating optimal topologies in structural design using a homogenization method, Computer methods in applied mechanics and engineering, 71 (1988), pp. 197–224.

[13] B. Bourdin, Filters in topology optimization, International journal for numerical methods in engineering, 50 (2001), pp. 2143–2158.

[14] J. Bourgain, H. Brezis, and P. Mironescu, Another look at Sobolev spaces, in Optimal control and partial differential equations, IOS, Amsterdam, 2001, pp. 439–455.

[15] A. Braides, Γ-convergence for beginners, vol. 22 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2002, https://doi.org/10.1093/acprof:oso/9780198507840.001.0001.

[16] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer Science & Business Media, 2010.

[17] C. Bucur and E. Valdinoci, Nonlocal diffusion and applications, vol. 20 of Lecture Notes of the Unione Matematica Italiana, Springer, Cham; Unione Matematica Italiana, Bologna, 2016, https://doi.org/10.1007/978-3-319-28739-3.

[18] J. Céa and K. Malanowski, An example of a max-min problem in partial differential equations, SIAM Journal on Control, 8 (1970), pp. 305–316.

[19] A. Chernov, T. von Petersdorff, and C. Schwab, Quadrature algorithms for high dimensional singular integrands on simplices, Numerical algorithms, 70 (2015), pp. 847–874.
This manuscript is for review purposes only.
[47] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, third ed., 1987.

[48] O. Sigmund, *On the design of compliant mechanisms using topology optimization*, Journal of Structural Mechanics, 25 (1997), pp. 493–524.

[49] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, Journal of the Mechanics and Physics of Solids, 48 (2000), pp. 175–209.

[50] O. Sigmund, *On the design of compliant mechanisms using topology optimization*, Journal of Structural Mechanics, 25 (1997), pp. 493–524.

[51] S. Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all’equazione del calore*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 21 (1967), pp. 657–699.

[52] L. Tartar, *Compensated compactness and applications to partial differential equations*, in *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, vol. 39 of Res. Notes in Math., Pitman, Boston, Mass.-London, 1979, pp. 136–212.

[53] L. Tartar, *Estimation de coefficients homogénisés*, in Computing methods in applied sciences and engineering (Proc. Third Internat. Sympos., Versailles, 1977), I, vol. 704 of Lecture Notes in Math., Springer, Berlin, 1979, pp. 364–373.

[54] G. Voyiadjis, ed., *Peridynamic Theory and Its Applications*, Springer, New York, 2014.

[55] F. W. Y. Suna and O. Sigmund, *On the non-optimality of tree structures for heat conduction*, International Journal of Heat and Mass Transfer, 122 (2018), pp. 660–680.