Soliton Induced Singularities
in 2 d Gravity and their Evaporation

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ABSTRACT

Positive energy singularities induced by Sine-Gordon solitons in 1+1 dimensional dilaton gravity with positive and negative cosmological constant are considered. When the cosmological constant is positive, the singularities combine a white hole, a timelike singularity and a black hole joined smoothly near the soliton center. When the cosmological constant is negative, the solutions describe two timelike singularities joined smoothly near the soliton center. We describe these spacetimes and examine their evaporation in the one loop approximation.

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Since the pioneering work of Hawking\textsuperscript{1} and others\textsuperscript{2} in the mid-seventies, the classical and quantum analysis of singularities in gravity has led to a lively discussion\textsuperscript{3,4} on some fundamental physical problems that are expected to arise in an eventual theory of quantum gravity. Hawking’s discovery that black holes evaporate thermally raised a deep puzzle in physics, the possibility of information loss in quantum gravity. Attempts to solve this problem have characteristically been hampered by technical difficulties, particularly the perturbative non-renormalizability of quantum gravity.

Not long ago, Witten and others\textsuperscript{5} discovered a 1+1 dimensional model of gravity with non-trivial dynamics which contains many of the key features of the four dimensional theory, in particular the formation and evaporation of black holes and naked singularities. As the most interesting features of the four dimensional theory are retained in this model and the dynamics is considerably simpler than in four dimensions, one has a potentially useful device to use in examining some of the interesting physics underlying the problems posed by Hawking’s early work. With this motivation, a lot of effort has been directed toward understanding 1+1 dimensional dilaton gravity.

An early attempt at understanding the dynamical formation and evaporation of black holes in 1+1 dimensions appeared in the work of Callan, Giddings, Harvey and Strominger (CGHS)\textsuperscript{6} in which the authors coupled matter degrees of freedom to the original Witten model by way of conformally invariant scalar fields whose solution was taken to be a shock-wave travelling at constant advanced time. Neglecting the back reaction, the incoming shock-wave was seen to radiate away before the black hole forms but, because the dilaton coupling is not small at the turn around point, the back reaction becomes important and the one loop approximation is an unreliable indicator of what is actually happening. Various improvements have since been made to the original CGHS model\textsuperscript{7,8}, but in some form or other they have all had their failures and a reliable understanding even of two dimensional black hole dynamics is yet to come (for a review on the status of the CGHS and other related models, see [9]). Naked singularities were also analyzed within the context of the CGHS model with a negative cosmological constant.\textsuperscript{10} In the one loop approximation, neglecting the back reaction, the naked singularity evaporates catastrophically emitting all its energy at early retarded times. It was argued that for energetic shock waves the dilaton coupling is weak in the evaporation region (at the “explosion” point) making the one loop approximation a good indicator of the underlying physics.
Recently a model of 1+1 dimensional dilaton gravity with Sine-Gordon solitons\textsuperscript{11} was proposed as being of interest from the point of view of non-linear integrable systems. The model may also be interesting from the point of view of soliton solutions in four dimensional general relativity for spacetimes admitting an Abelian two parameter group of isometries.\textsuperscript{12} It lacks conformal invariance in the matter sector, but one hopes that the model’s integrability will eventually facilitate its full canonical quantization. The singularities induced by the incoming solitons are qualitatively quite different from those induced by the shock wave of the CGHS model. They are made up of spacelike and timelike pieces joined smoothly by lightlike singularities at the soliton center. Thus the model combines black holes and naked singularities and one might ask what quantum effects do to these objects and what is the ultimate fate of the incoming soliton, taking into account its Hawking radiation. (As the classical stress energy falls off exponentially on $\mathcal{I}^+$, the dominant effect there is its Hawking evaporation.) As a step towards answering this question, we briefly describe the classical solutions, showing that they do indeed describe the dynamical formation of positive energy singularities, and then examine their evaporation in the one loop approximation. It turns out remarkably that the Hawking radiation does not differ significantly from its shock wave counterpart if natural boundary conditions are imposed.

We consider the CGHS action modified to include the Sine-Gordon potential

\begin{equation}
S = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[ e^{-2\phi} \left( -R + 4(\nabla \phi)^2 + \Lambda \right) - \frac{1}{2}(\nabla f)^2 + 4\mu^2 e^{-2\phi}(\cos f - 1) \right]
\end{equation}

where $\phi$ is the dilaton, $f$ is a matter field and $\Lambda$ is the cosmological constant which we will take to be either positive or negative in what follows. $R$ is the two dimensional scalar curvature, and $g_{\mu\nu}$ is the two metric. Our conventions are those of Weinberg.\textsuperscript{13} This action without the $f$–matter fields arises in two dimensional string theory.

The classical equations of motion follow by variation with respect to the metric, dilaton and matter fields. The metric equations

\begin{equation}
0 = T_{\mu\nu} = e^{-2\phi} \left[ 2\nabla_\mu \nabla_\nu \phi - \frac{1}{2} e^{2\phi} \nabla_\mu f \nabla_\nu f \\
+ g_{\mu\nu} \left( 2(\nabla \phi)^2 - 2\nabla^2 \phi - \Lambda + \frac{1}{4} e^{2\phi} (\nabla f)^2 - 2\mu^2 (\cos f - 1) \right) \right]
\end{equation}

form a set of constraints on the allowable field configurations. The dilaton and matter
The equations are
\[
4 \nabla^2 \phi - 4(\nabla \phi)^2 - R + 4 \left( \frac{\Lambda}{4} + \mu^2 (\cos f - 1) \right) = 0 \quad (3a)
\]
and
\[
-\nabla^2 f + 4 \mu^2 e^{-2\phi} \sin f = 0 \quad (3b)
\]
It is simplest to analyze the above equations in the conformal gauge, where the metric has the form
\[
g_{\mu\nu} = e^{2\rho} \eta_{\mu\nu}, \quad (4)
\]
and in lightcone coordinates \( x^\pm = x^0 \pm x^1 \) which we use hereafter. The constraints and equations of motion reduce to
\[
0 = T_{++} = e^{-2\rho} \left[ -4 \partial_+ \rho \partial_+ \phi + 2 \partial_+^2 \phi \right] - \frac{1}{2} (\partial_+ f)^2
\]
\[
0 = T_{--} = e^{-2\rho} \left[ -4 \partial_- \rho \partial_- \phi + 2 \partial_-^2 \phi \right] - \frac{1}{2} (\partial_- f)^2 \quad (5)
\]
\[
0 = T_{+-} = e^{-2\rho} \left[ -2 \partial_+ \partial_- \phi + 4 \partial_+ \phi \partial_- \phi + \frac{\Lambda}{4} e^{2\rho} + \mu^2 e^{2\rho} (\cos f - 1) \right]
\]
and
\[
-4 \partial_+ \partial_- \phi + 4 \partial_+ \phi \partial_- \phi + 2 \partial_+ \partial_- \rho + e^{2\rho} \left[ \frac{\Lambda}{4} + \mu^2 (\cos f - 1) \right] = 0
\]
\[
+ \partial_+ \partial_- f + \mu^2 e^{2(\rho - \phi)} \sin f = 0 \quad (6)
\]
When the dilaton equation in (6) is combined with the last of the three constraints in (5), one sees that the conformal factor, \( \rho(x) \), is equal to the dilaton, \( \phi(x) \) up to a harmonic function, \( h(x) \). However, a choice of \( h(x) \) is essentially a choice of coordinate system because the choice of conformal gauge does not fix the conformal subgroup of diffeomorphisms. Choosing a coordinate system such that \( h(x) = 0 \), the general solution, satisfying the constraints has the form \( (e^{-2\rho} = e^{-2\phi} = \sigma) \)
\[
f_{\text{kink}} = 4 \tan^{-1} e^{(\Delta - \Delta_0)}
\]
\[
\sigma = a + bx^+ + cx^- - \frac{\Lambda}{4} x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0) \quad (7)
\]
in terms of \( \Delta = \gamma_+ x^+ + \gamma_- x^- \), where
\[
\gamma_\pm = \pm \mu \sqrt{\frac{1 \pm v}{1 \mp v}}, \quad (8)
\]
v is the velocity of the soliton, \( f(x, t) = f(x + vt), \Delta = \Delta_0 \) is its center which we take
without loss of generality to be greater than or equal to zero, and $a$, $b$ and $c$ are arbitrary constants. The classical energy momentum tensor of the incoming soliton is

$$
T_{++}^f = \frac{1}{2}(\partial_+ f)^2 = \frac{4\gamma_+^2}{\cosh^2(\Delta - \Delta_0)}
$$

$$
T_{--}^f = \frac{1}{2}(\partial_- f)^2 = \frac{4\gamma_-^2}{\cosh^2(\Delta - \Delta_0)}
$$

$$
T_{+-}^f = \mu^2(\cos f - 1) = -\frac{2\mu^2}{\cosh^2(\Delta - \Delta_0)}.
$$

(9)

To fix the constants of integration one needs to impose some reasonable physical conditions. Thus, we require that the metric reduces to the linear dilaton vacuum in the absence of the incoming soliton, which we define as the limit $T_{\mu\nu}^f \to 0$, $\Delta_0 = 0$. This gives

$$
\sigma = -\frac{\Lambda}{4}x^+x^- - 2\ln \cosh(\Delta - \Delta_0)
$$

(10)

There is also the antikink solution

$$
f_{antikink} = 4\arctan e^{-(\Delta - \Delta_0)}
$$

$$
\sigma = -\frac{\Lambda}{4}x^+x^- - 2\ln \cosh(\Delta - \Delta_0)
$$

(11)

which may be analyzed analogously. We will work only with the kink solution in (7) and (10). When $\Lambda > 0$ equation (10) represents a spacetime that admits a positive energy singularity combining a white hole, a timelike singularity and a black hole, all smoothly joined along the soliton center (by a white hole we mean a spacelike naked singularity). When the cosmological constant is negative, the spacetime admits two naked singularities joined smoothly at the soliton center. In every case, the curvature singularity is at $\sigma = 0$ as can be seen by inserting (10) into the expression

$$
R = +2e^{-3\rho}\nabla^2 e^\rho - 2e^{-4\rho}(\nabla e^\rho)^2
$$

$$
= +4\sigma \partial_+ \partial_- \ln \sigma
$$

(12)

for the curvature scalar. The Bondi energy\textsuperscript{14} of the singularity on $I^+$ is straightforward to calculate in each case, as the spacetime admits a Killing vector near null infinity. For
example, near $\mathcal{I}_R^+$ and for positive cosmological constant, the metric behaves as

$$\sigma \to -\lambda^2 x^+ x^- - 2\Delta_0 - \frac{4\mu^2}{\lambda^2} + 2 \ln 2$$  \hspace{1cm} (13)$$

where

$$x^+ = x^+ + \frac{2\gamma_-}{\lambda^2}$$
$$x^- = x^- + \frac{2\gamma_+}{\lambda^2}$$ \hspace{1cm} (14)$$

The (timelike) Killing vector is $\xi^\mu = (x^+, x^-)$. Let $t_{\mu\nu}$ be a linearization of $\mathcal{T}_{\mu\nu}$ about the (dilaton) vacuum so that $j_\mu = t_{\mu\nu} \xi^\nu$ is a conserved current in the asymptotic region. Consider a solution that is asymptotic to the vacuum with $\phi = \phi^{(0)} + \delta \phi$, where $\phi^{(0)} = -\ln(-\lambda^2 x^+ x^-)/2$. The current density takes the form

$$j_+ = 2\lambda \partial_{x^+} \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_{x^+} \delta \phi + x^- \partial_{x^-} \delta \phi \right] \right)$$
$$j_- = 2\lambda \partial_{x^-} \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_{x^+} \delta \phi + x^- \partial_{x^-} \delta \phi \right] \right).$$ \hspace{1cm} (15)$$

The conservation of $j_\mu$ implies the existence of two charges, $Q^+ = \int_{\mathcal{I}_R^-} dx^- j_-$ and $Q^- = \int_{\mathcal{I}_R^+} dx^+ j_+$ which evolve the system in the direction of increasing $x^+$ and $x^-$ respectively and close to infinity. The current density is a total derivative, so its integral can be measured as a surface term. Thus, for example, one obtains the conserved charge (the Bondi energy)

$$M = Q^- = 2\lambda \left( e^{-2\phi^{(0)}} \left[ \delta \phi + x^+ \partial_{x^+} \delta \phi + x^- \partial_{x^-} \delta \phi \right] \right)_{\mathcal{I}_R^-}$$ \hspace{1cm} (16)$$

on $\mathcal{I}_R^+$. If $\Lambda = 4\lambda^2 > 0$ the spacetime described by

$$\sigma = -\lambda^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0)$$ \hspace{1cm} (17)$$

is a combination of a white hole, a black hole, and a naked singularity. The Kruskal diagram displayed in figure I shows the singularity along with the trajectory of the soliton.
center. Directing our attention to the region on right, the observer on $\mathcal{I}_R^+$ measures the (Bondi) mass

$$M_R = 4\lambda \left( \ln 2 - \frac{2\mu^2}{\lambda^2} + \Delta_0 \right).$$ (18)

and the soliton center is seen to emerge from the merging of a white hole and a timelike singularity at $(x^+ = 0, x^- = \Delta_0/\gamma_-)$. The white hole extends from $(x^+ = 2\gamma_-/\lambda^2, x^- = -\infty)$ on $\mathcal{I}_R^-$ to $(x^+ = 0, x^- = \Delta_0/\gamma_-)$. Here it smoothly turns into a timelike line proceeding to $(x^- = 0, x^+ = \Delta_0/\gamma_+)$, where the soliton is reabsorbed. At this point the singularity once again turns spacelike smoothly and reaches $\mathcal{I}_R^+$ at $(x^+ = \infty, x^- = -2\gamma_+/\lambda^2)$. All singularities are asymptotically spacelike as they approach $\mathcal{I}^\pm$. Although the figure was drawn for a particular choice of parameters, the qualitative behavior of the singularities is independent of the choice as long as the Bondi mass is positive.

The soliton never enters the left region (we have taken $\Delta_0 \geq 0$) and the observer on $\mathcal{I}_L^+$ measures the mass

$$M_L = 4\lambda \left( \ln 2 - \frac{2\mu^2}{\lambda^2} - \Delta_0 \right)$$ (19)

If $M_R$ and $M_L$ are both positive, the singularities on the left and on the right have the same features. As $\Delta_0 \to 0$ (figure II) the left and right singularities merge, in the limit forming a white hole that extends from $x^+ = -2\gamma_-/\lambda^2$ on $\mathcal{I}_R^-$ to $x^- = 2\gamma_+/\lambda^2$ on $\mathcal{I}_L^-$ and a black hole that stretches from $x^- = -2\gamma_+/\lambda^2$ on $\mathcal{I}_R^+$ to $x^+ = 2\gamma_-/\lambda^2$ on $\mathcal{I}_L^+$, intersecting at the origin. The length of the timelike naked singularity has shrunk to zero and the masses measured on both null infinities are now the same. Even if they are zero ($2\mu^2 = \lambda^2 \ln 2$) soliton energy and momentum is present throughout the spacetime.

The spacetime with negative cosmological constant ($\Lambda = -4\lambda^2$)

$$\sigma = \lambda^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0)$$ (20)

is shown in figure III. All singularities are timelike as they approach $\mathcal{I}^\pm$. In the top region the timelike singularity approaches $\mathcal{I}_R^+$ at $(x^+ = \infty, x^- = 2\gamma_+/\lambda^2)$ and approaches $\mathcal{I}_L^+$ at $(x^+ = -2\gamma_-/\lambda^2, x^- = \infty)$. The two timelike sections merge at a white hole in the region where the soliton center enters the spacetime. The latter emerges at $(x^+ = \Delta_0/\gamma_+, x^- = \infty)$. The spacetime with negative cosmological constant ($\Lambda = -4\lambda^2$)

$$\sigma = \lambda^2 x^+ x^- - 2 \ln \cosh(\Delta - \Delta_0)$$ (20)
0) and travels to \( t^0 \). The Bondi mass depends on whether the asymptotic observer is located on \( \mathcal{I}_R^+ \) or \( \mathcal{I}_L^+ \),

\[
M_R = 4\lambda \left( \frac{2\mu^2}{\lambda^2} + \Delta_0 + \ln 2 \right), \quad \text{and} \quad M_L = 4\lambda \left( \frac{2\mu^2}{\lambda^2} - \Delta_0 + \ln 2 \right),
\]

(21)

the qualitative behavior again being independent of the parameter values chosen if both masses are positive. The masses are equal when \( \Delta_0 = 0 \). In this limit one has two naked singularities, the first extending from \( x^- = -2\gamma_+/\lambda^2 \) on \( \mathcal{I}_L^- \) to \( x^- = 2\gamma_+/\lambda^2 \) on \( \mathcal{I}_R^+ \) and the other from \( x^+ = 2\gamma_-/\lambda^2 \) on \( \mathcal{I}_R^- \) to \( x^+ = -2\gamma_-/\lambda^2 \) on \( \mathcal{I}_L^+ \), intersecting at the origin (figure IV).

We would now like to include quantum effects in this model. As seen from (9), the classical stress energy tensor is exponentially vanishing on \( \mathcal{I}_R^+ \) (\( x^+ = \infty \)) so that the Hawking evaporation is dominant here. To include quantum effects in two dimensions one needs only the trace of the stress tensor, the other components being determined by the conservation equations in keeping with Wald’s axioms.\(^{15}\) The geometric contribution to the trace can depend only on the scalar curvature, \( R \), as it is the only available geometric invariant. Thus the quantum correction to the trace of the stress tensor must be given by

\[
T^{(q)}_{\mu} = -4\sigma T^q_{++} = -\alpha R = -4\alpha \sigma \partial_+ \partial_- \ln \sigma \tag{22}
\]

where \( \alpha \) is some positive dimensionless constant. The two conservation equations now can be integrated to yield the components of the stress tensor in terms of its trace,

\[
\langle T_{++} \rangle = T^f_{++} + T^q_{++} = T^f_{++} - \int \frac{dx^-}{\sigma} \partial_+(\sigma T^q_{+-}) + A(x^+),
\]

\[
\langle T_{--} \rangle = T^f_{--} + T^q_{--} = T^f_{--} - \int \frac{dx^+}{\sigma} \partial_- (\sigma T^q_{++}) + B(x^-),
\]

(23)

where \( A(x^+) \) and \( B(x^-) \) are boundary condition dependent functions of \( x^+ \) and \( x^- \) respectively. Consider the case of positive cosmological constant and the observer on the right. A consistent solution should admit no incoming radiation on \( \mathcal{I}_R^- \) other than any matter fields that might be present and vanish in the absence of the soliton, that is, in
the linear dilaton vacuum. The stress tensor satisfying these conditions is

\[
\begin{align*}
\langle T_{++} \rangle &= T_{++}^f + T_{++}^q = T_{++}^f - \alpha \left( \frac{\partial_+^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_+ \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^++^2} \\
\langle T_{--} \rangle &= T_{--}^f + T_{--}^q = T_{--}^f - \alpha \left( \frac{\partial_-^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_- \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^-2} \\
\langle T_{+-} \rangle &= T_{++}^f + \alpha \partial_+ \partial_- \ln \sigma
\end{align*}
\]

where \( x^+ = x^+ + 2\gamma_+ / \lambda^2 \). It is most convenient to analyze the above expressions in the coordinate system in which the metric is manifestly asymptotically flat. Define, therefore the coordinates \( \sigma^\pm = t \pm x \) by

\[
\begin{align*}
x^+ &= \frac{1}{\lambda} e^{\lambda \sigma^+} - \frac{2\gamma_-}{\lambda^2} \\
x^- &= -\frac{1}{\lambda} e^{-\lambda \sigma^-} - \frac{2\gamma_+}{\lambda^2}
\end{align*}
\]

Thus \( \sigma^+ \to \infty \) corresponds to the lightlike surface \( x^+ = \infty \) and \( \sigma^- \to \infty \) to the lightlike surface \( x^- = -\infty \), while \( \sigma^- \to \infty \) corresponds to the lightlike surface \( x^- = -2\gamma_- / \lambda^2 \). Transforming the expressions in (23) to the new system, one finds that

\[
\begin{align*}
\langle T_{++}^\sigma \rangle &\to 0, \quad \langle T_{+-}^\sigma \rangle \to 0 \\
\langle T_{--}^\sigma \rangle &\to \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 + \frac{2\gamma_+}{\lambda} e^{\lambda \sigma^-} \right)^2} \right].
\end{align*}
\]

\( \langle T_{--}^\sigma \rangle \) is the outgoing flux at \( I_R^+ \). It grows smoothly from zero at \( x^- = -\infty \) to a maximum of \( \alpha \lambda^2 / 2 \) at \( x^- = 2\gamma_+ / \lambda^2 \) on \( I_R^+ \). As expressed in (26) and (27) the quantum stress tensor represents the Hawking flux from the singularity. It depends on the soliton mass parameter but not on the mass of the singularity itself. This would seem to be a general feature of the Hawking evaporation in this model, having been shown to be true for the radiation from a black hole formed by an incoming shock wave in the CGHS model. The integrated flux along \( I_R^+ \) is the total energy lost by the incoming soliton. As the flux rapidly approaches its maximum value of \( \alpha \lambda^2 / 2 \), the integrated flux shows an infinite loss of energy if the integral is performed up to the future horizon at \( x^- = 2\gamma_+ / \lambda^2 \). However, as CGHS pointed
out, this is a consequence of having neglected the back reaction and must not be taken seriously. Instead, one can try to estimate the retarded time, \( x^- \) at which the integrated Hawking radiation is equal to the mass of the singularity, \( M = 4\lambda(\Delta_0 - 2\mu^2/\lambda^2 + \ln 2) \).

One finds

\[
\int_{-\infty}^{\sigma^-} d\sigma^- \langle T_{\sigma^-} \rangle = \frac{\alpha \lambda}{2} \left[ 1 - \frac{1}{1 + \frac{2\gamma_+}{\lambda}} + \ln \left( 1 + \frac{2\gamma_+}{\lambda} e^{\lambda \sigma^-} \right) \right] = M \tag{28}
\]

For a small mass singularity, the retarded time is given by

\[
x^- = -\frac{2\gamma_+}{\lambda^2} \left( 1 + \frac{\alpha \lambda}{M} \right) \tag{29}
\]

which, when na"ively traced backwards, corresponds to the point

\[
(x^+_\tau, x^-_\tau) = \left( \frac{\Delta_0}{\gamma_+} + \frac{2\gamma_-}{\lambda^2} \left[ 1 + \frac{\alpha \lambda}{M} \right], -\frac{2\gamma_+}{\lambda^2} \left[ 1 + \frac{\alpha \lambda}{M} \right] \right) \tag{30}
\]

on the soliton trajectory. If \( x^+_\tau < 0 \), the soliton energy has evaporated earlier than the appearance of its center in the spacetime. An observer on \( \mathcal{I}_{R}^- \) sees a white hole which rapidly radiates away all its energy. This of course is true only if the mass of the singularity is small. On the other hand, if \( x^+_\tau \) is greater than zero, the soliton does enter the spacetime evaporating eventually by \( x^-_\tau \) in (29). How reliable is the estimate of its lifetime above?

Assuming that the soliton center does enter the spacetime before evaporating completely, the dilaton coupling constant at the turn around point,

\[
ed^\phi = \frac{1}{\sqrt{2 \left( 1 + \frac{\alpha \lambda}{M} \right) \left[ \frac{M}{4\lambda} - \ln 2 - \frac{2\mu^2\alpha}{\lambda M} \right]}} \tag{31}
\]

is large for a small mass singularity and signals the breakdown of the one loop approximation.

On the other hand, if \( M \) is large,

\[
\int_{-\infty}^{\sigma^-} d\sigma^- \langle T_{\sigma^-} \rangle \sim \frac{\alpha \lambda}{2} \left[ \ln \frac{2\gamma_+}{\lambda} + \lambda \sigma^- \right] = M \tag{32}
\]
or

\[ x^- = - \frac{2\gamma_+}{\lambda^2} \left[ 1 + e^{-2M/\alpha \lambda} \right] \tag{33} \]

which, when traced back corresponds to the point

\[ (x^+ = \frac{\Delta_0}{\gamma_+} + \frac{2\gamma_+}{\lambda^2} \left[ 1 + e^{-2M/\alpha \lambda} \right], -\frac{2\gamma_+}{\lambda^2} \left[ 1 + e^{-2M/\alpha \lambda} \right]) \tag{34} \]

on the soliton trajectory. The dilaton coupling at this point has the value

\[ e^\phi = \frac{1}{\sqrt{2 \left( 1 + e^{-2M/\alpha \lambda} \right) \left( \frac{M}{4\lambda} - \ln 2 - \frac{2\mu^2}{\lambda} e^{-2M/\alpha \lambda} \right)}} \tag{35} \]

and is small in the limit of large \( M \). The soliton evaporates completely by the time the observer has reached the event horizon and the black hole never forms. Moreover this is the limit in which the one loop approximation is a satisfactory indication of what may actually be happening. Similar conclusions can be drawn in the CGHS (shock wave) model. In the low mass limit that the dilaton coupling is large at the turn around point, being proportional only to \( \sqrt{1/\alpha} \) and signalling a breakdown in the one loop approximation. However, in the large mass limit the dilaton coupling behaves as the inverse square root of the mass. Thus, there seems to be no essentially new feature in the evaporation of the soliton.

Next consider an observer in the left quadrant. As we have mentioned, because of our choice of \( \Delta_0 > 0 \) the soliton center never enters this region and this observer lives in a universe inhabited only by the tail of the soliton energy and the singularity described earlier. The appropriate boundary conditions on the Hawking stress tensor are (a) its vanishing in the absence of the soliton and (b) no flux across \( \mathcal{I}_L^+ \). It follows that the quantum contribution to the stress tensor is given by

\[ T^q_{++} = -\alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_+ \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^+^2} \]

\[ T^q_{--} = -\alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left[ \frac{\partial_- \sigma}{\sigma} \right]^2 \right) - \frac{\alpha}{2x^-^2} \tag{36} \]

\[ T^q_{+-} = \alpha \partial_+ \partial_- \ln \sigma \]
where \( x^- = x^- - 2\gamma_+ / \lambda^2 \). Going to the system \( \sigma^\pm = t \pm x \) given by

\[
\begin{align*}
x^+ &= - \frac{1}{\lambda} e^{-\lambda \sigma^+} + \frac{2\gamma^+}{\lambda^2} \\
x^- &= \frac{1}{\lambda} e^{\lambda \sigma^-} + \frac{2\gamma^-}{\lambda^2}
\end{align*}
\]

(37)

in which the metric is manifestly flat at null infinity, \( \sigma^+ \to \infty \) corresponds to the lightlike line \( x^+ = 2\gamma_- / \lambda^2 \) and \( \sigma^- \to -\infty \) to the lightlike line \( x^+ = 2\gamma_+ / \lambda^2 \) while \( \sigma^+ \to -\infty \) and \( \sigma^- \to \infty \) correspond to the respective lightlike infinities) one finds on \( I^+_L \)

\[
\langle T_{--}^\sigma \rangle \to 0, \quad \langle T_{++}^\sigma \rangle \to 0
\]

(38)

and

\[
\langle T_{++}^\sigma \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 - \frac{2\gamma^-}{\lambda} e^{-\lambda \sigma^+} \right)^2} \right]
\]

(39)

In this part of the world, the radiation is at its maximum value (of \( \alpha \lambda^2 / 2 \)) at early advanced times and decreases to zero in the far future of \( I^+_L \). The integrated flux over any interval is infinite, that is the singularity “explodes”, giving up all its energy in a burst and wiping itself out.

We turn now to solitons in 2d gravity with a negative cosmological constant. We consider the observer in the top quadrangle. In the absence of \( I^- \) the only reasonable boundary condition one may impose upon the stress tensor is that it vanish in the absence of the soliton, a limit we have defined earlier. The quantum corrections to the stress tensor then take the form

\[
\begin{align*}
T^q_{++} &= - \alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left( \frac{\partial_+ \sigma}{\sigma} \right)^2 \right) - \frac{\alpha}{2x^+^2} \\
T^q_{--} &= - \alpha \left( \frac{\partial^2 \sigma}{\sigma} - \frac{1}{2} \left( \frac{\partial_- \sigma}{\sigma} \right)^2 \right) - \frac{\alpha}{2x^-^2} \\
T^q_{+-} &= \alpha \partial_+ \partial_- \ln \sigma
\end{align*}
\]

(39)

Again, to analyze the tensors it is convenient to go to a system in which the metric is
asymptotically flat: define the system \( \sigma^\pm = t \pm x \) by

\[
x^+ = \frac{1}{\lambda} e^{\lambda \sigma^+} - \frac{2\gamma_-}{\lambda^2} \\
x^- = \frac{1}{\lambda} e^{\lambda \sigma^-} + \frac{2\gamma_+}{\lambda^2}
\]  

(40)

Thus, \( \sigma^- \to -\infty \) corresponds to the lightlike line \( x^- = 2\gamma_+ / \lambda^2 \) and \( \sigma^+ \to -\infty \) to the lightlike line \( x^+ = -2\gamma_- / \lambda^2 \) while \( \sigma^\pm \to \infty \) correspond to the respective lightlike infinities.

The fluxes across both \( \mathcal{I}^+_L \) and \( \mathcal{I}^+_R \) are now non-vanishing, each approaching a maximum of \( \alpha \lambda^2 / 2 \) at early times and decreasing steadily in the far future, as \( t^0 \) is approached. Thus, on \( \mathcal{I}^+_R \), for instance, one finds

\[
\langle T^{\sigma}_{++} \rangle \to 0, \quad \langle T^{\sigma}_{+-} \rangle \to 0
\]  

(41)

and

\[
\langle T^{\sigma}_{--} \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 + \frac{2\gamma_+}{\lambda} e^{-\lambda \sigma^-} \right)^2} \right], \quad (42)
\]

and on \( \mathcal{I}^+_L \)

\[
\langle T^{\sigma}_{--} \rangle \to 0, \quad \langle T^{\sigma}_{+-} \rangle \to 0
\]  

(43)

and

\[
\langle T^{\sigma}_{++} \rangle = \frac{\alpha \lambda^2}{2} \left[ 1 - \frac{1}{\left( 1 - \frac{2\gamma_-}{\lambda} e^{-\lambda \sigma^-} \right)^2} \right]. \quad (44)
\]

The tensors are again independent of the mass of the singularities but depend on the soliton mass parameter. The integrated flux over any interval is again infinite because the flux itself approaches a steady state at early times. Of course this is a consequence of having neglected the back reaction of the radiation on the spacetime geometry.

This picture is also similar to that developed in the shock wave model. Even if quantum gravity does permit the formation of naked singularities, they will evaporate catastrophically ("explode") due to the Hawking radiation. To justify this statement, one must check the validity of the one loop picture by considering the strength of the dilaton
coupling constant at the point on the soliton center at which the singularities are expected to detonate. On $I_R^+$ this is at the retarded time $x^- = 2\gamma_+ / \lambda^2$ which, when traced back corresponds to the point $(x^+_+ , x^-_+ ) = (1/\gamma_+ (\Delta_0 + 2\mu^2/\lambda), 2\gamma_+ / \lambda^2)$ on the soliton center and gives for the dilaton coupling

$$e^\phi = \frac{1}{\sqrt{2 \left( \frac{2\mu^2}{\lambda^2} + \Delta_0 \right)}}.$$  

This is indeed small when the Bondi mass, $M_R$ is large, i.e., the soliton mass parameter is large. On the other hand, on $I_L^+$ this is at the advanced time $x^+ = -2\gamma_- / \lambda^2$ which, when traced back, corresponds to the point $(x^+_-, x^-_-) = (-2\gamma_- / \lambda^2, 1/\gamma_- (\Delta_0 - 2\mu^2 / \lambda))$ on the soliton center and gives for the dilaton coupling

$$e^\phi = \frac{1}{\sqrt{2 \left( \frac{2\mu^2}{\lambda^2} - \Delta_0 \right)}}.$$  

which is again small when the Bondi mass, $M_L$, is large (or the soliton mass parameter is large).

We shall not discuss here the remaining case, the lower quadrant in 1+1 dilaton gravity with a negative cosmological constant. In this case an observer is obstructed from reaching future null infinity by the naked singularity. Unlike the case of the upper quadrant where the naked singularity forms simultaneously with the emergence of the soliton center, in the lower quadrant the soliton emerges in the past. This case may therefore be thought of as a dynamical model for the “formation” of a naked singularity. Because of the inaccessibility of null infinity to an observer the analysis of the Hawking radiation requires somewhat different considerations than the cases we have already discussed. The analysis of this very interesting case will be reported elsewhere.

In this article we have examined the singularities induced by an incoming soliton in two dimensional dilaton gravity both with a positive and negative cosmological constant. The singularities are neither purely spacelike nor purely timelike but a combination of the two joined smoothly by lightlike singularities along the soliton center. The classical stress energy tensor of the Sine- Gordon field is exponentially vanishing at infinity and the Hawking radiation is dominant there. We have therefore examined the Hawking evaporation of these singularities. To do so it was necessary to impose reasonable boundary
conditions on the Hawking tensor. Arguing that the most sensible conditions are (a) the vanishing of the tensor in the absence of the soliton energy-momentum and (b) the absence of incoming radiation on $I^-$, we showed that the essential character of the radiation does not differ significantly from that produced by an incoming shock wave (the CGHS model) in the two cases.

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Figure Captions:

Figure I: The Kruskal diagram for $\Lambda > 0$ and $\Delta_0 > 0$. Regions I & III are physical ($\sigma > 0$). A black hole, a white hole, and a timelike singularity are joined smoothly along the soliton center.

Figure II: The Kruskal diagram for $\Lambda > 0$ and $\Delta_0 = 0$. Regions I & III are physical ($\sigma > 0$). A black hole and a white hole intersect at the origin.

Figure III: The Kruskal diagram for $\Lambda < 0$ and $\Delta_0 > 0$. Regions II & IV are physical ($\sigma > 0$). Two timelike singularities are joined smoothly at the soliton center.

Figure IV: The Kruskal diagram for $\Lambda < 0$ and $\Delta_0 = 0$. Regions II & IV are physical ($\sigma > 0$). Two timelike singularities intersect at the origin.
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FIGURE I
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FIGURE II
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FIGURE III
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FIGURE IV