Abstract: In this paper, we obtain pointwise convergence of solutions to the Schrödinger equation along a class of curves in $\mathbb{R}^2$ by the polynomial partitioning.

Keywords: Schrödinger equation; Pointwise convergence; Polynomial partitioning; Broadness

Mathematics Subject Classification: 35Q41

1 Introduction

The solution to the Schrödinger equation

$$iu_t - \Delta u = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

with initial datum $u(x, 0) = f$, is formally written as

$$e^{it\Delta}f(x) := \int_{\mathbb{R}^n} e^{i(x \cdot \xi + t|\xi|^2)} \hat{f}(\xi) \, d\xi.$$

The problem about finding optimal $s$ for which

$$\lim_{t \to 0} e^{it\Delta}f(x) = f(x), \text{ a.e.}$$

whenever $f \in H^s(\mathbb{R}^n)$, was first considered by Carleson [1], and extensively studied by Sjölin [9] and Vega [10], who proved independently the convergence for $s > 1/2$ in all dimensions. Dahlberg and Kenig [3] showed that the convergence does not hold for $s < 1/4$ in any dimension. When $n = 2$, Du and Li [5] proved the convergence result for $s > 3/8$ by the polynomial partitioning; Du, Guth and Li [6] obtained the sharp result $s > 1/3$ by the polynomial partitioning and $l^2$ decoupling method.

By Cho, Lee and Vargas [2], a general generalization of the pointwise convergence problem is to ask a.e. convergence along a wider approach region instead of vertical lines. One of such problems is to

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consider non-tangential convergence to the initial data, it was shown by Sjölin and Sjögren [8] that non-tangential convergence fails for $s \leq n/2$. Another problem is to consider the relation between the degree of the tangency and regularity when $(x, t)$ approaches to $(x, 0)$ tangentially. One of the model problems raised by [2] is

$$\lim_{t \to 0} e^{it\Delta} f(\gamma(x, t)) = f(x) \ a.e. \quad (1.3)$$

when $n = 1$, here the curves $\gamma$ approach $(x, 0)$ tangentially to the hyperplane $\{(x, t) : t = 0\}$. Ding and Niu [4] improved the result of [2], but this problem is still open for $n \geq 2$.

In this paper, we consider this problem when $n = 2$ and

$$\gamma(x, t) = x - \sqrt{t} \mu, \quad (1.4)$$

where $\mu$ is a unit vector in $\mathbb{R}^2$. The convergence result (1.3) follows from

**Theorem 1.1.** For $2 \leq p \leq 3.2$, if $f \in H^s(\mathbb{R}^2)$, $s > 3/8$, then there exists a constant $C > 0$ such that

$$\sup_{t \in (0, 1)} \left\| e^{it\Delta} f(\gamma(x, t)) \right\|_{L^p(B(0, 1))} \leq C \| f \|_{H^s}. \quad (1.5)$$

**Remark 1.2.** When $\mu = (1, 0)$, $s \geq 11/32$ is showed to be necessary if (1.5) holds. In fact, take

$$\hat{f}(\xi) := \psi \left( \frac{\xi - \lambda \mu}{\lambda^{1/2}} \right),$$

$\psi$ is a non-negative Schwartz function. By rescaling, it follows that

$$\left| e^{it\Delta} f(\gamma(x, t)) \right| \sim \lambda$$

when $|t| \leq \lambda^{-1}$ and $|\lambda^{1/2} (x_1 + \sqrt{t} \lambda + 2t \lambda, x_2)| \leq C$, therefore, (1.5) implies that

$$\lambda \lambda^{-1/2p} \leq \lambda^{1/2+s}.$$  

The desired condition follows from the fact that $\lambda$ can be sufficiently large.

By Littlewood-Paley Theorem and parabolic rescaling, Theorem 1.1 can be reduced to

**Theorem 1.3.** For $2 \leq p \leq 3.2$, $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\sup_{t \in (0, R]} \left\| e^{it\Delta} f(\gamma(x, t)) \right\|_{L^p(B(0, R))} \leq C_\varepsilon R^{\frac{7}{8} + \frac{5}{8} + \varepsilon} \| f \|_{L^2} \quad (1.6)$$

for all $R \geq 1$, all $f$ with $\text{supp} \hat{f} \subset A(1) = \{\xi : |\xi| = 1\}$.

## 2 Proof of Theorem 1.1

For convenience of the proof, we define a new operator

$$e^{it\Pi} f(x) := e^{it\Delta} f(\gamma(x, t)) = \int_{\mathbb{R}^2} e^{i(x \cdot \xi - \sqrt{t} \mu \cdot \xi + t|\xi|^2)} \hat{f}(\xi) \, d\xi. \quad (2.1)$$
Proof of Theorem 1.1: For any $f \in H^s(\mathbb{R}^2)$, we use Littlewood-Paley decomposition,

$$f = \sum_{k \geq 0} f_k,$$

(2.2)

where $supp\hat{f}_0 \subset B(0,1), supp\hat{f}_k \subset A(2^k), k \geq 1$. If Theorem 1.3 holds, when $p = 3.2$, for any $R \geq 1$, $\hat{g} \in C^\infty_c(\mathbb{R}^2)$ with $supp\hat{g} \subset A(1)$, it holds

$$\left\| \sup_{t \in (0,R]} |e^{it\mathcal{H}} g| \right\|_{L^p(B(0,R))} \leq C_\varepsilon R^\varepsilon \|g\|_{L^2},$$

(2.3)

this implies

$$\left\| \sup_{t \in (0,R]} |e^{it\mathcal{H}} g| \right\|_{L^p(B(0,R))} \leq \left\| \sup_{t \in (0,R']} |e^{it\mathcal{H}} g| \right\|_{L^p(B(0,R'))} \leq C_\varepsilon R^{2\varepsilon} \|g\|_{L^2}. $$

By parabolic rescaling,

$$\begin{cases} x = Ry, \\ t = R^2 s, \end{cases}$$

we have

$$e^{it\mathcal{H}} g(x) = \int_{\mathbb{R}^2} e^{i(x \cdot \xi - \sqrt{t} |\xi|^2)} \hat{g}(\xi) d\xi = \int_{\mathbb{R}^2} e^{i(y \cdot R\xi - \sqrt{t} |R\xi|^2 + |\xi|^2)} \hat{g}(\xi) d\xi$$

$$= R^{-2} \int_{\mathbb{R}^2} e^{i(y \cdot \eta - \sqrt{t} \mu \cdot \eta + |\eta|^2)} \hat{g}(\eta) d\eta$$

$$= R^{-2} \int_{\mathbb{R}^2} e^{i(y \cdot \eta - \sqrt{t} \mu \cdot \eta + |\eta|^2)} \hat{g}_1(\eta) d\eta$$

$$= R^{-2} e^{is\mathcal{H}} g_1(y),$$

where $\hat{g}_1(\eta) := \hat{g}(\frac{\eta}{R})$, so that $supp\hat{g}_1 \subset A(R)$. It follows that

$$\left\| \sup_{t \in (0,R]} |e^{it\mathcal{H}} g| \right\|_{L^p(B(0,R))} = R^{2/p-2} \left\| \sup_{s \in (0,1]} |e^{is\mathcal{H}} g_1| \right\|_{L^p(B(0,1))}, \|g\|_{L^2} = R^{-1} \|g_1\|_{L^2},$$

combining it with (2.3), we have

$$\left\| \sup_{t \in (0,1]} |e^{it\mathcal{H}} g_1| \right\|_{L^p(B(0,1))} \leq C_\varepsilon R^{1-2/p+2\varepsilon} \|g_1\|_{L^2}$$

(2.4)

for $supp\hat{g}_1 \subset A(R)$. Apply (2.4) to each $f_k, k \geq 1$,

$$\left\| \sup_{t \in (0,1]} |e^{it\mathcal{H}} f_k| \right\|_{L^p(B(0,1))} \leq C_\varepsilon 2^{k(1-2/p+2\varepsilon)} \|f_k\|_{L^2},$$

(2.5)

And for $f_0$, by (2.8) below, it holds

$$\left\| \sup_{t \in (0,1]} |e^{it\mathcal{H}} f_0| \right\|_{L^p(B(0,1))} \leq \|f_0\|_{L^2},$$

(2.6)
Combining (2.2), (2.5) and (2.6),

\[
\left\| \sup_{t \in (0, R)} |e^{it\mathcal{H}} f| \right\|_{L^p(B(0,1))} \leq \sum_{k \geq 0} \left\| \sup_{t \in (0, R)} |e^{it\mathcal{H}} f_k| \right\|_{L^p(B(0,1))} \leq \sum_{k \geq 0} C_\epsilon 2^{k(1-2/p+2\varepsilon)} \|f_k\|_{L^2} \\
\leq \sum_{k \geq 0} C_\epsilon 2^{k(1-2/p+2\varepsilon)} 2^{-ks} \|f\|_{L^2} \\
\leq C \|f\|_{L^2},
\]

the last inequality follows from the fact that \( s > 3/8 \) and \( \varepsilon \) can be sufficiently small. Notice that the case \( 2 \leq p < 3.2 \) can be easily obtained from the case \( p = 3.2 \) by Hölder’s inequality.

**Proof of Theorem 1.3.** In order to prove (1.6), it suffices to prove that

\[
\left\| \sup_{t \in (0, R)} |e^{it\mathcal{H}} f| \right\|_{L^p(B(0, R))} \leq C_\epsilon M^{-\varepsilon^2} R^{\frac{2}{p} - \frac{2}{p} + 

\varepsilon} \|f\|_{L^2} \tag{2.7}
\]

for all \( R \geq 1, \xi_0 \in B(0,1), M \geq 1, \) and any \( f \) with \( \supp \hat{f} \subset B(\xi_0, M^{-1}) \).

We will prove (2.7) by induction on the physical radius \( R \) and frequency radius \( 1/M \). So we need to check the base of the induction.

**Base of the induction.** From

\[
|e^{it\mathcal{H}} f| \leq M^{-1} \|f\|_{L^2},
\]

it is easy to see that

\[
\left\| \sup_{t \in (0, R)} |e^{it\mathcal{H}} f| \right\|_{L^p(B(0,R))} \leq M^{-1} R^{\frac{2}{p} - \frac{2}{p} + \varepsilon} \|f\|_{L^2} \tag{2.8}
\]

for all \( R \geq 1, M \geq 1, \) so (2.7) is trivial when \( M \geq R^{10} \).

When \( \sqrt{R} \leq M \leq R^{10} \), we adopt wave packets decomposition for \( f \). Let \( \varphi \) be a Schwartz function from \( \mathbb{R} \) to \( \mathbb{R} \), \( \hat{\varphi} \) is non-negative and supported in a small neighborhood of the origin, and identically 1 in another smaller interval. Let \( \theta = \prod_{j=1}^2 \theta_j \) denote the rectangle in the frequency space with center \((c(\theta_1), c(\theta_2))\) and

\[
\hat{\varphi}_\theta(\xi_1, \xi_2) = \prod_{j=1}^2 \frac{1}{|\theta_j|^{1/2}} \hat{\varphi} \left( \frac{\xi_j - c(\theta_j)}{|\theta_j|} \right).
\]

A rectangle \( \nu \) in the physical space is said to be dual to \( \theta \) if \( |\theta_j| \nu_j = 1, j = 1, 2, \) and \((\theta, \nu)\) is said to be a tile. Let \( T \) be a collection of all tiles with fixed dimensions and coordinate axes. Define

\[
\hat{\varphi}_{\theta, \nu}(\xi) = e^{-ic(\nu) \cdot \xi} \hat{\varphi}_\theta(\xi),
\]

we have the following representation

\[
f = \sum_{(\theta, \nu) \in T} f_{\theta, \nu} = \sum_{(\theta, \nu) \in T} \langle f, \varphi_{\theta, \nu} \rangle \varphi_{\theta, \nu}. \tag{2.9}
\]
We will only use $(\theta, \nu)$ where $\theta$ is an $R^{-1/2}$ cube in frequency space and $\nu$ is an $R^{1/2}$ cube in physical space. It is clear that

$$\sum_{(\theta, \nu) \in T} |(f, \varphi_{\theta, \nu})|^2 = \|f\|_{L^2}^2.$$  

For any Schwartz function $f$ with $\text{supp} \hat{f} \subset B(0,1)$, we only need to consider all $\theta$’s that range over $\text{supp} \hat{f}$.

Set

$$\psi_{\theta, \nu} = e^{itH} \varphi_{\theta, \nu},$$

by the representation (2.9), we have

$$e^{itH} f = \sum_{(\theta, \nu) \in T} e^{itH} f_{\theta, \nu} = \sum_{(\theta, \nu) \in T} (f, \varphi_{\theta, \nu}) \psi_{\theta, \nu}. \tag{2.10}$$

Next, we consider the localization of $\psi_{\theta, \nu}$ in $B(0,R) \times [0,R]$. In fact,

$$\psi_{\theta, \nu}(x,t) \chi_{[0,R]}(t)$$

$$= \int_{\mathbb{R}^2} e^{i(x - \sqrt{\mu} \xi + t|\xi|^2)} e^{-ic(\nu) \xi} \varphi_{\theta}(\xi) d\xi \times \chi_{[0,R]}(t)$$

$$= \sqrt{R} \int_{\mathbb{R}^2} e^{i(x - \sqrt{\mu} \xi + t|\xi|^2)} e^{-ic(\nu) \xi} \prod_{j=1}^2 \varphi\left(\frac{\xi_j - \epsilon(\theta_j)}{R^{-1/2}}\right) d\xi \times \chi_{[0,R]}(t)$$

$$= \frac{1}{\sqrt{R}} \int_{\mathbb{R}^2} e^{i((x-c(\nu)) - R^{-1/2} \eta + c(\theta)) - \sqrt{\mu} (R^{-1/2} \eta + c(\theta) + t|R^{-1/2} \eta + c(\theta)|^2)} \prod_{j=1}^2 \varphi(\eta_j) d\eta \times \chi_{[0,R]}(t),$$

the phase function

$$\phi(x,t,\eta) = (x-c(\nu) - R^{-1/2} \eta + c(\theta) - \sqrt{\mu} (R^{-1/2} \eta + c(\theta) + t|R^{-1/2} \eta + c(\theta)|^2).$$

By simple calculation,

$$\nabla_{\eta} \phi(x,t,\eta) = R^{-1/2} (x-c(\nu) + 2tc(\theta)) + R^{-1/2} \sqrt{\mu} + 2R^{-1} t\eta.$$

It is obvious that in $B(0,R) \times [0,R]$, 

$$|\psi_{\theta, \nu}(x,t)| \leq \frac{1}{\sqrt{R}} \chi_{T_{\theta, \nu}}(x,t), \tag{2.11}$$

where $T_{\theta, \nu} := \{(x,t) : 0 \leq t \leq R, |x-c(\nu) + 2tc(\theta)| \leq R^{\frac{1}{2} + \delta}\}, \delta \ll \varepsilon$, is a tube with direction

$$G(\theta) = (-2c(\theta), 1).$$

When $M \geq \sqrt{R}$, there is only one possible $\theta$, therefore all tubes are in the same direction. By the definition of $\nu$, $|c(\nu_1) - c(\nu_2)| \geq R^{1/2}, \nu_1 \neq \nu_2$, these tubes are also essentially disjoint. What’s more, the projection of $T_{\theta, \nu}$ on $x$-plane is contained in an $R^{1/2} \times R$ rectangle, denoted by $S_{\theta, \nu}$, by (2.11),

$$|\psi_{\theta, \nu}(x,t)| \leq \frac{1}{\sqrt{R}} \chi_{T_{\theta, \nu}}(x,t) \leq \frac{1}{\sqrt{R}} \chi_{S_{\theta, \nu}}(x) \chi_{[0,R]}(t). \tag{2.12}$$
Combining (2.10) and (2.12), we have
\[
\left\| \sup_{t \in [0, R]} |e^{it\mathbf{H}} f| \right\|_{L^p(B(0, R))}^p = \int_{B(0, R)} \sup_{t \in [0, R]} |e^{it\mathbf{H}} f|^p \, dx \\
\leq \int_{B(0, R)} \sup_{t \in [0, R]} \left| \sum_{(\theta, \nu) \in T} \langle f, \varphi_{\theta, \nu} \rangle \psi_{\theta, \nu} \right|^p \, dx \\
\leq \int_{B(0, R)} \sup_{t \in [0, R]} \sum_{(\theta, \nu) \in T} |\langle f, \varphi_{\theta, \nu} \rangle|^p |\psi_{\theta, \nu}|^p \, dx \\
\leq R^{-p/2} \int_{B(0, R)} \sup_{t \in [0, R]} \sum_{(\theta, \nu) \in T} |\langle f, \varphi_{\theta, \nu} \rangle|^p \chi_{\mathcal{S}_{a, \nu}}(x) \chi_{[0, R]}(t) \, dx \\
\leq R^{\frac{3-2p}{2}} + O(\delta) \| f \|^p_{L^2},
\]
from which (2.7) follows.

Therefore, we only need to consider the case \( M \leq \sqrt{R} \). On the other hand, when \( R \leq C \) for some constant \( C > 0 \), the result is true by (2.8). So we can assume that \( R \) is sufficiently large. This completes the base of our induction. Now we are ready to prove Theorem 1.3.

Choose non-negative Schwartz functions \( \psi_1(t) \) and \( \psi_2(t) \), such that \( \psi_1(t) \) is supported in a sufficiently small neighborhood of \( [0, R^{-1}] \), and identically 1 on \( [0, R^{-1}] \), \( \psi_2(t) \) is supported in a sufficiently small neighborhood of \( [R^{-1}, 1] \), and identically 1 on \( [R^{-1}, 1] \). We have
\[
\left\| \sup_{t \in [0, R]} |e^{it\mathbf{H}} f| \right\|_{L^p(B(0, R))} \leq \left\| \sup_{t \in [0, R]} |e^{it\mathbf{H}} f| \psi_1 \left( \frac{t}{R} \right) \right\|_{L^p(B(0, R))} + \left\| \sup_{t \in [0, R]} |e^{it\mathbf{H}} f| \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B(0, R))} =: I_1 + I_2.
\]

If \( I_1 \) dominates, \( t \) is localized in a sufficiently small neighborhood of \( [0, R^2] \), the oscillatory integral
\[
e^{it\mathbf{H}} = \int_{\mathbb{R}^2} e^{i(t \xi - \sqrt{1-\xi^2})(\xi^2 + t) \frac{d \xi}{2}}
\]
shows that \( |e^{it\mathbf{H}} f| \psi_1 \left( \frac{t}{R} \right) \) is essentially supported in \( B(0, R^{1-\varepsilon}) \times [0, R^2] \). Therefore,
\[
I_1 \leq \left\| \sup_{t \in [0, R^{1-\varepsilon}]} |e^{it\mathbf{H}} f| \psi_1 \left( \frac{t}{R} \right) \right\|_{L^p(B(0, R^{1-\varepsilon}))} \leq 2C_\varepsilon M^{-\varepsilon} R^{(1-\varepsilon)\left(\frac{3}{2} - \frac{\varepsilon}{2} + \varepsilon\right)} \| f \|^p_{L^2}
\]
\[
\leq R^{-\varepsilon^2} C_\varepsilon M^{-\varepsilon} R^{\frac{3}{2} - \frac{\varepsilon}{2} + \varepsilon} \| f \|^p_{L^2},
\]
since \( R \) is sufficiently large, then (2.13) and (2.14) finished the induction.

We consider the case when \( I_2 \) dominates. Let \( K \) be a large parameter such that \( K \ll R^3 \), we decompose \( B(0, R) \) into balls \( B_K \) of radius \( K \), and interval \([0, R]\) into intervals \( I_K \) of length \( K \). We write
\[
\left\| \sup_{t \in [0, R]} |e^{it\mathbf{H}} f| \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B(0, R))}^p = \sum_{B_K \subset B(0, R)} \int_{B_K} \left( \sup_{I_K \subset [0, R]} \int_{I_K} \right) \left\| e^{it\mathbf{H}} f \right\|^p_{L^p(B(0, R))} \, dx.
\]
We divide $B\left(\xi, M^{-1}\right)$ into balls $\tau$ of radius $(KM)^{-1}$, $f = \sum f_\tau$, $\hat{f}_\tau = \hat{f}\big|_\tau$. For each $B_K \times I_K^1$ and a parameter $A \in \mathbb{Z}^+$, we choose 1-dimensional sub-spaces $V_1^0, V_2^0, \ldots, V_A^0$ such that

$$
\mu_{e^{it\varphi}}\left(B_K \times I_K^1\right) := \min_{v_1, v_2, \ldots, v_A} \left(\max_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} \int_{B_K \times I_K^1} |e^{itH}f|_2^p \psi_2\left(\frac{t}{R}\right) dx dt\right)
$$

achieves the minimum. We say that $\tau \in V_0$ if

$$
\inf_{\xi \in \tau} \text{Angle} \left(\frac{(-2\xi, 1)}{(-2\xi, 1)^t}, V_0\right) \leq (KM)^{-1}.
$$

Then from (2.15),

$$
\left\| \sup_{t \in (0, R)} \left| e^{itH}f \right|_2^p \psi_2\left(\frac{t}{R}\right) \right\|_{L^p(B(0, R))}^p
=$n \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \left| \sum_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p dx 
\leq \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \left| \sum_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p dx 
+ \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \left| \sum_{\tau \in \text{some } V_0, \alpha = 1, 2, \ldots, A} e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p dx
=$$n I_3 + I_4.

If $I_3$ dominates, we have

$$
I_3 \leq \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \left| \sum_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p dx
\leq K^{O(1)} \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \left| \max_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p dx
= K^{O(1)} \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \left| \max_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p dx
= K^{O(1)} \sum_{B_K \subset B(0, R)} \int_{B_K} \sup_{I_K^1 \subset [0, R)} \sup_{t \in I_K^1} \min_{V_1, V_2, \ldots, V_A} \left(\max_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} \int_{B_K \times I_K^1} |e^{itH}f_\tau|_2^p \psi_2\left(\frac{t}{R}\right) d\tau dt\right),
$$

where we used the fact that $|e^{itH}f_\tau|_2^p \psi_2\left(\frac{t}{R}\right)$ is essentially constant $c_K^p$ on $B_K \times I_K^1$. Denote

$$
\left\| e^{itH}f_\tau \left| \psi_2\left(\frac{t}{R}\right) \right|^p \right\|_{L^p(B_K \times I_K^1 \times [0, R))}^p
:= \sum_{B_K \subset B(0, R)} \sup_{I_K^1 \subset [0, R)} \min_{V_1, V_2, \ldots, V_A} \left(\max_{\tau \notin V_0, \alpha = 1, 2, \ldots, A} \int_{B_K \times I_K^1} |e^{itH}f_\tau|_2^p \psi_2\left(\frac{t}{R}\right) d\tau dt\right),
$$

by Theorem 2.1 below, we have

$$
I_3 \leq K^{O(1)} R^{2\alpha^2} \left(\frac{\xi}{\xi} \right)^p M^{-\alpha} R^{2\alpha^2 - 2\alpha + \varepsilon} \|f\|_L^p.
$$
which can be approximated by Theorem 2.1. For \( 2 \leq p \leq 3.2 \) and \( k = 2 \), for any \( \varepsilon > 0 \), there exist positive constants \( A = A(\varepsilon) \) and \( C(K, \varepsilon) \) such that
\[
\left\| e^{itH} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{k,A}L^\infty(B(0,R) \times [0,R])} \leq C(K, \varepsilon) R^{\frac{2}{p} - \frac{1}{2} + \varepsilon} \| f \|_{L^2}, \tag{2.18}
\]
for all \( R \geq 1 \), \( \xi_0 \in B(0,1) \), \( M \geq 1 \), all \( f \) with \( \text{supp} \psi \subset B(\xi_0, M^{-1}) \).

We will prove Theorem 2.1 from Section 3 to Section 7.

3 Preliminaries for the proof of Theorem 2.1

For any subset \( U \subset B(0,R) \times [0,R] \), we define
\[
\left\| e^{itH} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{k,A}L^\infty(U)} := 
\left( \sum_{B_K \subset B(0,R)} \sup_{I_K' \subset [0,R]} \left| \frac{U \cap (B_K \times I_K')}{|B_K \times I_K'|} \right|^{1/p} \right)^{1/p},
\]
which can be approximated by
\[
\left\| e^{itH} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{k,A}L^s(U)} := 
\left( \sum_{B_K \subset B(0,R)} \left( \sum_{I_K' \subset [0,R]} \left| \frac{U \cap (B_K \times I_K')}{|B_K \times I_K'|} \right|^{1/p} \right)^{1/q} \right)^{1/p},
\]
i.e.,
\[
\left\| e^{itH} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{k,A}L^\infty(U)} = \lim_{q \to +\infty} \left\| e^{itH} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{k,A}L^q(U)},
\]
which implies that Theorem 2.1 can be turned to prove Theorem 3.1.
**Theorem 3.1.** For $2 \leq p \leq 3.2$ and $k = 2$, for any $\varepsilon > 0$, $1 \leq q < +\infty$, there exist positive constants $A = A(\varepsilon)$ and $C(\varepsilon)$ such that
\[
\left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L_2}(B(0, R) \times [0, R])} \leq C(\varepsilon) R^{\frac{2}{pq} - \frac{3}{2} + \varepsilon} \| f \|_{L^2},
\]
for all $R \geq 1$, $\xi_0 \in B(0, 1), M \geq 1$, all $f$ with $\text{supp} \hat{f} \subset B(\xi_0, M^{-1})$.

Instead of Theorem 3.1, we will prove Theorem 3.2 below.

**Theorem 3.2.** For $2 \leq p \leq 3.2$ and $k = 2$, for any $\varepsilon > 0$, $1 \leq q < +\infty$, there exist positive constants $\overline{A} = \overline{A}(\varepsilon)$ and $\overline{C}(\varepsilon)$ such that
\[
\left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L_2}(B(0, R') \times [0, R])} \leq C(\varepsilon) R^{\frac{2}{pq} - \frac{3}{2} + \varepsilon} \| f \|_{L^2},
\]
for any fixed $R \geq 1$, all $1 \leq R' \leq R$, $1 \leq A \leq \overline{A}$, $\xi_0 \in B(0, 1), M \geq 1$, all $f$ with $\text{supp} \hat{f} \subset B(\xi_0, M^{-1})$.

We will prove Theorem 3.2 by induction on $R'$ and $A$, we will check the base of the induction.

**The base of the induction.** Given $R > 1$, for any $1 \leq R' \leq R$, it is easy to see
\[
\left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L_2}(B(0, R') \times [0, R])} \leq C(\varepsilon) R^{1/pq} R^\frac{2}{pq} \| f \|_{L^2}.
\]
(1) When $R'$ is controlled by some constant $C$, then (3.1) holds.

(2) When $A = 1$, then (3.1) holds even though $A$ does not appear in the right side of (3.2). In fact, we choose $\overline{A}$ such that $\delta \log \overline{A} = 100$, therefore
\[
\left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L_2}(B(0, R') \times [0, R])} \leq C(\varepsilon) R^{1/pq} R^\frac{2}{pq} \| f \|_{L^2} \\
\leq C(\varepsilon) R^{1/pq} R^{100} \left( R' \right)^{\frac{2}{pq} - \frac{3}{2} + \varepsilon} \| f \|_{L^2} \\
= C(\varepsilon) R^{1/pq} R^{\delta(\log \overline{A} + \log A)} \left( R' \right)^{\frac{2}{pq} - \frac{3}{2} + \varepsilon} \| f \|_{L^2},
\]
this completes the base of the induction. What’s more, by the analysis in Section 2, we only need to consider the case $KM \leq R^{1/2}$.

In order to prove Theorem 3.2, we need some basic inequalities:

**Lemma 3.3.** (1) If $U_1$ and $U_2$ are two subsets of $B(0, R) \times [0, R]$, then for $1 \leq q < +\infty$,
\[
\left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L^q}(U_1 \cup U_2)} \leq \left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L^q}(U_1)} + \left\| e^{t\mathbf{H}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L^q}(U_2)}.
\]
(2) Given non-negative integers $A, A_1, A_2$, $A = A_1 + A_2$, then for $1 \leq q < +\infty$,
\[
\left\| e^{t\mathbf{H}} (f + g) \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L^q}(U)} \leq C_p \left( \left\| e^{t\mathbf{H}} f \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L^q}(U)} + \left\| e^{t\mathbf{H}} g \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^{p}_{L^q}(U)} \right).
\]
(3) If $1 \leq p \leq r$, $U \subset S_U \times I_U \subset B(0, R) \times [0, R]$, where $S_U$ and $I_U$ are subsets paralleled to the $x$-plane and $t$-axe respectively, then for $1 \leq q < +\infty$,

$$\left\| e^{itf}f_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U)} \leq C_K \left( |S_U| |I_U|^{1/q} \right)^{\frac{1}{q} - \frac{1}{p}} \left\| e^{itf}f_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U)}.$$

The proof of Lemma 3.3 is very similar to Lemma 3.1, [5], So we omit the proof here.

4 Polynomial partitioning

The main tool we will use is polynomial partitioning.

**Lemma 4.1.** Suppose $f_1, f_2, ..., f_N$ are functions defined on $\mathbb{R}^n$ with $\text{supp} \tilde{f}_j \subset B^n (0, 1), U_1, U_2, ..., U_N$ are subsets of $B^n (0, R) \times [0, R]$, and $1 \leq p, q < +\infty$, $\Pi$ is a linear sub-space in $\mathbb{R}^{n+1}$ with dimension $m$, $1 \leq m \leq n + 1$, $\pi$ is the orthogonal projection from $\mathbb{R}^{n+1}$ to $\Pi$, then there exists a non-zero polynomial $P_\Pi$ defined on $\Pi$ of degree no more than $C_m N^{1/m}$, such that $P (z) = P_\Pi (\pi (z)), z \in \mathbb{R}^{n+1}$, satisfies

$$\left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P > 0\})}^{p} = \left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P < 0\})}^{p}, \quad j = 1, 2, ..., N. \quad (4.1)$$

**Proof:** Let $V = \{ P (z) = P_\Pi (\pi (z)) : \text{Deg} P_\Pi \leq D \}$, note that $V$ is a vector space of dimension $D^m$, choose $D$ such that $D^m \sim N + 1$, i.e., $D \leq C_m N^{1/m}$, without less of generality, we may assume $\text{Dim} V = N + 1$ and identify $V$ with $\mathbb{R}^{N+1}$. We define a function $G : S^N \subset V \setminus \{0\} \rightarrow \mathbb{R}^N$ as

$$G (P) := \{ G_j (P) \}_{j=1}^{N},$$

where

$$G_j (P) := \left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P > 0\})}^{p} - \left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P < 0\})}^{p}.$$

It is obvious that $G (-P) = -G (P)$. If the function $G$ is continuous, then Lemma 4.1 follows from the Borsuk - Ulam Theorem. So we only need to check the continuity of $G_j$.

Suppose $P_l \rightarrow P$ in $V \setminus \{0\}$, note that

$$|G_j (P_l) - G_j (P)| \leq 2 \left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P_l \leq 0\})}^{p},$$

so we have

$$\lim_{l \rightarrow +\infty} \left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P_l \leq 0\})}^{p} \leq \left\| e^{itf}f_j \psi_2 \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^p L^q(U \cap \{P^{-1} (0) \})}^{p} = 0.$$

This implies that $G$ is continuous on $V \setminus \{0\}$.

We use this Lemma to prove the following partitioning result:
**Theorem 4.2.** Suppose that \( f \) is a function defined on \( \mathbb{R}^n \) with \( \text{supp} \hat{f} \subset B^n (0,1) \), \( U \) is a subset of \( B^n (0,R) \times [0, R'] \), and \( 1 \leq p, q < +\infty \), \( \Pi \) is a linear sub-space in \( \mathbb{R}^{n+1} \) with dimension \( m \), \( 1 \leq m \leq n+1 \), \( \pi \) is the orthogonal projection from \( \mathbb{R}^{n+1} \) to \( \Pi \), then there exists a non-zero polynomial \( P_\Pi \) defined on \( \Pi \) of degree no more than \( D \), and \( P(z) = P_\Pi (\pi(z)) \) such that \( \Pi \) is a union of \( \sim_m D^m \) disjoint open sets \( O_{\Pi,i} \), \( \mathbb{R}^{n+1} \setminus Z(P) \) is a union of \( \sim_m D^m \) disjoint open sets \( O_i = \pi^{-1}(O_{\Pi,i}) \), and for each \( i \), we have

\[
\left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U)}^p \leq C_m D^m \left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap O_i)}^p.
\]  

(4.2)

**Proof:** By Lemma 4.1, we obtain a polynomial \( Q_1 \) of degree \( \leq C \),

\[
Q_1(z) = Q_{\Pi,1}(\pi(z)),
\]

such that

\[
\left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap \{Q_1 > 0\})}^p = \left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap \{Q_1 < 0\})}^p.
\]

Next by Lemma 4.1 again, we have a polynomial \( Q_2 \) of degree \( \leq C_m 2^{1/m} \), such that

\[
\left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap \{Q_1 > 0, Q_2 > 0\})}^p = \left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap \{Q_1 > 0\} \setminus \{Q_2 < 0\})}^p,
\]

\[
\left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap \{Q_1 < 0, Q_2 < 0\})}^p = \left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap \{Q_1 < 0\} \setminus \{Q_2 > 0\})}^p.
\]

Continuing inductively, we construct polynomials \( Q_1, Q_2, \ldots, Q_s \),

\[
Q_l(z) = Q_{\Pi,l}(\pi(z)), l = 1, 2, \ldots, s.
\]

Set \( P := \bigcap_{l=1}^s Q_s \), where \( \deg Q_l \leq C_m 2^{(l-1)/m} \), therefore \( \deg P(z) \leq C_m 2^{s/m} \), and the sign conditions of polynomials cut \( \mathbb{R}^{n+1} \setminus Z(P) \) into \( 2^s \) cells \( O_i \) such that

\[
\left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U)}^p \leq C_m 2^s \left\| e^{it\Pi} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{BL_{k,A}^s L^q(U \cap O_i)}^p.
\]

Choose \( s \) such that \( 2^{s/m} \in [D/2, D] \), then we have \( \deg P \leq D \) and the number of cells \( O_i \) is \( C_m D^m \). It is obvious that \( \Pi \) is divided by \( C_m D^m \) cells \( O_{\Pi,i} \) determined by the sign conditions of \( Q_{\Pi,l}, l = 1, 2, \ldots, s \). This completes the proof of Theorem 4.2.

Same as the analysis in [5], by a slight modification in Theorem 4.2, we assume that all the varieties appear in our argument are transverse complete intersections. For any \( 1 \leq m \leq n \), we say that a variety \( Z(P_1, P_2, ..., P_{n+1-m}) \subset \mathbb{R}^n \times \mathbb{R} \) is a transverse complete intersection if for each \( z \in Z(P_1, P_2, ..., P_{n+1-m}) \),

\[
\nabla P_1(z) \wedge \nabla P_2(z) \wedge \ldots \wedge \nabla P_{n+1-m}(z) \neq 0.
\]

## 5 Proof of Theorem 3.2

**Proof of Theorem 3.2.** By Lemma 3.3 (3), it is sufficient to prove Theorem 3.2 for \( p = 3.2 \). We assume that (3.1) holds for \( A \leq \frac{\pi}{2} \) and \( R' \leq \frac{R}{2} \), next prove it for \( A = A' \) and \( R' = R \).
We say that we are in the algebraic case if there is a transverse complete intersection \( Z(P) \) of dimension 2, where \( \deg P(z) \leq D = D(\varepsilon) \), so that
\[
\left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty(B(0,R) \times [0,R])} \leq C \left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty((B(0,R) \times [0,R]) \cap N_{R^{1/2+\delta}}(Z(P)))}, \tag{5.1}
\]
for each \( \theta, \nu \). By the fundamental theorem of Algebra, see [5], for each \( (\theta, \nu) \), we have
\[
\left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty(B(0,R) \times [0,R])} \leq C \left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty((B(0,R) \times [0,R]) \cap N_{R^{1/2+\delta}}(Z(P)))}
\]
here \( N_{R^{1/2+\delta}}(Z(P)) \) denotes the \( R^{1/2+\delta} \) neighborhood of \( Z(P) \). Otherwise we are in the cellular case.

**Cellular case.** We will use polynomial partitioning. By Theorem 4.2, there exists a non-zero polynomial \( P(z) = \prod Q_i(z) \) of degree at most \( D \) such that \( (\mathbb{R}^2 \times \mathbb{R}) \setminus Z(P) \) is a union of \( \sim D^3 \) disjoint cells \( O_i \) such that for each \( i \), we have
\[
\left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty(B(0,R) \times [0,R])} \leq CD^3 \left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty((B(0,R) \times [0,R]) \cap O_i)}.
\]
Moreover, \( Z(P) \) is a transverse complete intersection of dimension 2.

Put
\[
W := N_{R^{1/2+\delta}}(Z(P)), O_i^* := O_i \setminus W.
\]
Since we are in the cellular case and \( W \subset \bigcup O_i N_{R^{1/2+\delta}}(Z(Q_i)) \), the contribution from \( W \) is negligible. Hence for each \( i \),
\[
\left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty(B(0,R) \times [0,R])} \leq CD^3 \left\| e^{itH} f_2 \left( \frac{t}{R} \right) \right\|_{BL^p L^\infty((B(0,R) \times [0,R]) \cap O_i^*)}. \tag{5.2}
\]
For each cell \( O_i^* \), we set
\[
T_i := \{ (\theta, v) \in T : T_{\theta,v} \cap O_i^* \neq \emptyset \}.
\]
For the function \( f \), we define
\[
f_i := \sum_{(\theta, v) \in T_i} f_{\theta,v}.
\]
It follows that on \( O_i^* \),
\[
e^{itH} f \sim e^{itH} f_i.
\]
By the fundamental theorem of Algebra, see [5], for each \( (\theta, v) \in T \), we have
\[
\text{Card} \{ i : (\theta, v) \in T_i \} \leq D + 1.
\]
Hence
\[
\sum_i \| f_i \|_{L^2}^2 \leq CD \| f \|_{L^2}^2,
\]
by pigeonhole principle, there exists \( O_i^* \) such that
\[
\| f_i \|_{L^2}^2 \leq CD^{-2} \| f \|_{L^2}^2. \tag{5.3}
\]
So for such $i$, by (5.2), the induction on $R'$ and (5.3), we have
\[
\left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \leq CD^3 \left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \\
\leq \sum_{B_{R/2} cover B(0,R)} \left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \leq CD^3 \left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \\
\leq CD^3 \sum_{B_{R/2} cover B(0,R)} \left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \leq CD^3 \left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \\
\leq CD^3 \left( C(K, \varepsilon) R^3 \right)^p.
\]
choose $D$ sufficiently large such that $CD^{3-p} \ll 1$, this completes the induction.

**Algebraic case.** We decompose $B(0,R) \times \{0, R\}$ into balls $B_j$ of radius $\rho$, $\rho^{1/2+\delta_2} = R^{1/2+\delta}$. Choose $\delta \ll \delta_2$, so that $\rho \sim R^{1-O(\delta_2)}$. For each $j$ we define
\[
T_j := \{(\theta, v) \in T : T_{\theta,v} \cap N_{R^{1/2+\delta}}(Z(P)) \cap B_j \neq \emptyset\},
\]
and
\[
f_j := \sum_{(\theta,v) \in T_j} f_{\theta,v}.
\]
On each $B_j \cap N_{R^{1/2+\delta}}(Z(P))$, we have
\[
e^{i\theta f} \sim e^{i\theta f_j}.
\]
Therefore,
\[
\left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B(0,R)\times \{0, R\})} \leq \sum_{j} \left\| e^{i\theta f_{\psi_2}\left(\frac{t}{R}\right)} \right\|_{BL_{P/2,\infty}^{p}(B_j \cap N_{R^{1/2+\delta}}(Z(P)))}.
\]
We further divide $T_j$ into tubes that are tangential to $Z$ and tubes that are transverse to $Z$. We say that $T_{\theta,v}$ is tangential to $Z$ in $B_j$ if the following two conditions hold:

**Distance condition:**

\[
T_{\theta,v} \cap 2B_j \subset N_{R^{1/2+\delta}}(Z(P)) \cap 2B_j = N_{\rho^{1/2+\delta}}(Z(P)) \cap 2B_j.
\]

**Angle condition:** If $z \in Z \cap N_{O(R^{1/2+\delta})}(T_{\theta,v}) \cap 2B_j = Z \cap N_{O(\rho^{1/2+\delta})}(T_{\theta,v}) \cap 2B_j$, then

\[
\text{Angle}(G(\theta), T_{z}Z) \leq C \rho^{-1/2+\delta_2}.
\]

The tangential wave packets are defined by
\[
T_{j,\text{tang}} := \{(\theta,v) \in T_j : T_{\theta,v} \text{ is tangent to } Z \text{ in } B_j\},
\]
and the transverse wave packets
\[
T_{j,\text{trans}} := T_j \setminus T_{j,\text{tang}}.
\]
Set

$$f_{j,\text{tang}} := \sum_{(\theta, \nu) \in T_{j, \text{tang}}} f_{\theta, \nu}, \quad f_{j, \text{trans}} := \sum_{(\theta, \nu) \in T_{j, \text{trans}}} f_{\theta, \nu},$$

so

$$f_j = f_{j, \text{tang}} + f_{j, \text{trans}}.$$

Therefore, we have

$$\left\| e^{itH} f_{j, \text{tang}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B(0, R) \times [0, R])}^p \leq \sum_j \left\| e^{itH} f_{j, \text{tang}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B_j)}^p \leq \sum_j \left\| e^{itH} f_{j, \text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B_j)}^p + \sum_j \left\| e^{itH} f_{j, \text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B_j)}^p.$$

We will treat the tangential term and the transverse term respectively.

**Algebraic transverse case.** In this case, the transverse term dominates, by induction on the radius $R'$,

$$\left\| e^{itH} f_{j, \text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B_j)}^p \leq \left\| e^{itH} f_{j, \text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B(0, R) \times [0, R])}^p \leq C (K, \varepsilon) R^2 \left( \log \frac{R}{\rho} \right) R^R \| f_{j, \text{trans}} \|_{L^2} \leq R^{O(\delta) - 2O(\delta_2)} C (K, \varepsilon) R^{R^2} \| f_j \|_{L^2},$$

where $B_{\rho}$ denotes the projection of $B_j$ on the $x$-plane. By [5] we have

$$\sum_j \| f_{j, \text{trans}} \|^2_{L^2} \leq C (D) \| f \|^2_{L^2}. \quad (5.4)$$

Then

$$\sum_j \left\| e^{itH} f_{j, \text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{BL^p_{\xi \eta} L^q(B_j)}^p \leq R^{O(\delta) - 2O(\delta_2)} \left[ C (K, \varepsilon) R^{R^2 \| f \|_{L^2}} \right]^p \sum_j \| f_{j, \text{trans}} \|^p_{L^2} \leq R^{O(\delta) - 2O(\delta_2)} C (D) \left[ C (K, \varepsilon) R^{R^2 \| f \|_{L^2}} \right]^p.$$

The induction follows by choosing $\delta \ll \varepsilon \delta_2$ and the fact that $R$ is sufficiently large.

**Algebraic tangential case.** in this case, the tangential term dominates, we need to do wave packets decomposition in $B_j$ at scale $\rho$.

**Wave packet decomposition in $B_j$.** Choose $(\overline{B}, \overline{\nu})$ as before where $\overline{B}$ is a $\rho^{-1/2}$-cube in frequency space and $\overline{\nu}$ is a $\rho^{1/2}$-cube in physical space. We can decompose $f$ as

$$f = \sum_{(\overline{B}, \overline{\nu}) \in \Gamma} f_{\overline{B}, \overline{\nu}} = \sum_{(\overline{B}, \overline{\nu}) \in \Gamma} \left( f, \phi_{\overline{B}, \overline{\nu}} \right) \psi_{\overline{B}, \overline{\nu}},$$

where
where
\[
\hat{\varphi}_\varpi (\xi) = e^{-ic(\varpi) \cdot \xi} \hat{\varphi}_\varpi (\xi),
\]
\[
\hat{\varphi}_\varpi (\xi_1, \xi_2) = \frac{1}{\rho^{1/2}} \prod_{j=1}^{2} \hat{\varphi} \left( \frac{\xi_j - c(\theta_j)}{\rho^{1/2}} \right).
\]

Set \((x_0, t_0)\) as the center of \(B_j\). In order to decompose wave packets in \(B_j\), we need to modify the base such that
\[
\hat{\tilde{\varphi}}_\varpi \varpi (x, t) = e^{-ix_0 \cdot \xi + i\sqrt{\rho \mu} \cdot \xi - it_0 |\xi|^2} \hat{\varphi}_\varpi (\xi),
\]
so we set
\[
\hat{\tilde{\varphi}}_\varpi \varpi = e^{-ix_0 \cdot \xi + i\sqrt{\rho \mu} \cdot \xi - it_0 |\xi|^2} \hat{\varphi}_\varpi (\xi),
\]
then
\[
f = \sum_{(\varpi, \varpi) \in T} \langle f, \hat{\tilde{\varphi}}_\varpi \varpi \rangle \hat{\tilde{\varphi}}_\varpi \varpi
\]
Therefore,
\[
e^{itH}f = \sum_{(\varpi, \varpi) \in T} \langle f, \hat{\tilde{\varphi}}_\varpi \varpi \rangle \hat{\tilde{\psi}}_\varpi \varpi,
\]
where
\[
\hat{\tilde{\psi}}_\varpi \varpi = e^{itH} \hat{\tilde{\varphi}}_\varpi \varpi
\]
As the previous analysis, we restrict \(\hat{\tilde{\psi}}_\varpi \varpi\) in \(B_j\), then we have
\[
|\hat{\tilde{\psi}}_\varpi \varpi (x, t)| \leq \rho^{-1/2} \chi_{T_{\varpi \varpi}} (x, t),
\]
the tube \(T_{\varpi \varpi}\) is defined by
\[
T_{\varpi \varpi} := \{(x, t) \in B_j : |x - x_0 - c(\varpi) + 2c(\varpi)(t - t_0)| \leq \rho^{1/2} \delta, |t - t_0| \leq \rho \}.
\]
For each \((\theta, \nu) \in T_{j, \text{tang}}\), we consider the decomposition of \(f_{\theta, \nu}\),
\[
f_{\theta, \nu} = \sum_{(\varpi, \varpi) \in T} \langle f_{\theta, \nu}, \hat{\tilde{\varphi}}_\varpi \varpi \rangle \hat{\tilde{\varphi}}_\varpi \varpi
\]
\((\varpi, \varpi)\) which contribute to \(f_{\theta, \nu}\) satisfy
\[
|c(\theta) - c(\varpi)| \leq 2\rho^{-1/2},
\]
and
\[
|c(\nu) - c(\varpi) - x_0 - 2t_0c(\theta)| \leq R^{1/2} \delta.
\]
From (5.7) we know that
\[
\text{Angle} \left( G(\theta), G(\varpi) \right) \leq 2\rho^{-1/2},
\]
and (5.8) implies that if \((x, t) \in T_{v, \nu} \cap B_j\), then

\[
|x - c(\nu) + 2c(\theta) t| \leq CR^{1/\delta_m}, \tag{5.10}
\]

i.e., \(T_{v, \nu} \subset N_{R^{1/\delta_m}}(T_{v, \nu} \cap B_j)\).

We introduce the definition of \(R^{-1/2+\delta_m}\)-tangent to \(Z\) in \(B\) with radius \(R\). Suppose that \(Z = Z(P_1, ..., P_{3-m})\) is a transverse complete intersection in \(\mathbb{R}^2 \times \mathbb{R}\). We say that \(T_{v, \nu}\) (with scale \(R\)) is \(\left(\frac{R}{2}\right)^{-1/2+\delta_m}\)-tangent to \(Z\) if the following two conditions hold:

1. **Distance condition:**
   
   \[
   T_{v, \nu} \subset N_{(\frac{R}{2})^{1/2+\delta_m}}(Z) \cap B.
   \]

2. **Angle condition:** If \(Z \cap N_{\theta}(\left(\frac{R}{2}\right)^{1/2+\delta_m}) (T_{v, \nu}) \cap B\), then
   
   \[
   \angle \left(\frac{R}{2}\right)^{-1/2+\delta_m} \leq C(R^{1/\delta_m}).
   \]

Moreover, set

\[
T_Z := \left\{ (\theta, v) : T_{v, \nu} \text{ is } \left(\frac{R}{2}\right)^{-1/2+\delta_m} \text{-tangent to } Z \text{ in } B \right\},
\]

we say that \(f\) is concentrated in wave packets from \(T_Z\) in \(B\) if

\[
\sum_{(\theta, v) \in T_Z} \|f_{\theta, v}\|_{L^2} \leq \text{RapDec} \left(\frac{R}{2}\right) \|f\|_{L^2}.
\]

We claim that new wave packets of \(f_{j, \nu} \) are \(\rho^{-1/2+\delta_2}\)-tangent to \(Z\) (\(P\)) in \(B_j\) (note that we do not make a separate notation for convenience). In fact, if \(Z \cap N_{\theta}(\rho^{1/\delta_2}) (T_{v, \nu}) \cap B_j\), then \(z \in Z \cap N_{\theta}(\rho^{1/\delta_2}) (T_{v, \nu}) \cap B_j\), therefore

\[
\angle \frac{\rho}{2}\left(\frac{R}{2}\right)^{-1/2+\delta_2} \leq \angle \left(\frac{R}{2}\right)^{-1/2+\delta_2} + \angle \left(\frac{R}{2}\right)^{1/\delta_2} \leq C\rho^{-1/2+\delta_2}.
\]

Also,

\[
T_{v, \nu} \subset N_{R^{1/\delta_2}}(T_{v, \nu} \cap B_j) \cap B_j = N_{\rho^{1/\delta_2}}(T_{v, \nu} \cap B_j) \cap B_j \subset N_{\rho^{1/\delta_2}}(Z(P)) \cap B_j.
\]

Note that \(B_j \subset B_\rho \times [0, R]\), whenever Theorem 5.1 below holds true, we have

\[
\left\| e^{it\mathcal{H}} f_{j, \nu} \psi_2 \left(\frac{t}{\rho}\right) \right\|_{L^p(B_j)} \leq \left[ \rho(2+1/q)(1-p^{-1}/(4+\delta)) \right] \left\| e^{it\mathcal{H}} f_{j, \nu} \psi_2 \left(\frac{t}{\rho}\right) \right\|_{L^p(B_j)} \leq \left[ \rho(2+1/q)(1-p^{-1}/(4+\delta)) \right] \left\| e^{it\mathcal{H}} f_{j, \nu} \psi_2 \left(\frac{t}{\rho}\right) \right\|_{L^p(B_\rho \times [0, R])} \leq \left[ \rho(2+1/q)(1-p^{-1}/(4+\delta)) \right] C \left( \rho^{1/2+\delta_2} \right) \left( \log \frac{\rho}{\log R} \right) \left( \frac{R}{\rho} \right)^{\delta_2} \left\| f_{j, \nu} \psi_2 \right\|_{L^2(B_j)}\]
where we choose $C(K, \varepsilon) \geq C(K, D, \frac{\varepsilon}{2})$, therefore,

$$\sum_j \left\| e^{itH} j_{\text{tang}} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^j(0, R))}^p \leq R^{O(\delta)} R^{O(\delta) - \varepsilon / 2} \left[ C(K, \varepsilon) R^d R^e \| f \|_{L^2} \right]^p,$$

the induction closes for the fact that $\delta \ll \delta_2 \ll \varepsilon$ and $R$ is sufficiently large.

**Theorem 5.1.** Suppose that $Z(P) \subset \mathbb{R}^2 \times \mathbb{R}$ is a transverse complete intersection determined by some $P(z)$ with $\deg P(z) \leq D_Z$. For all $f$ with $\text{supp} \tilde{f} \subset B(0, 1)$, and fixed $R \geq 1$, if $B(0, R') \times [0, R]$ contains a ball (tube) $B$ of radius $R'$ such that $f$ is concentrated in wave packets from $T_Z$ in $B$, here $1 \leq R' \leq R$, then for any $\varepsilon > 0$ and $p > 4$, there exist positive constants $\overline{A} = \overline{A}(\varepsilon)$ and $C(K, D_Z, \varepsilon)$ such that

$$\left\| e^{itH} f \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^0(0, R') \times [0, R])} \leq C(K, D_Z, \varepsilon) R^d \log(\overline{A}) R^e \left( \frac{R}{R'} \right)^{\frac{d}{2} + \varepsilon} \| f \|_{L^2}$$

(5.11)

holds for all $1 \leq A \leq \overline{A}$. 

### 6 Proof of Theorem 5.1

We will again use the induction on $R'$ and $A$ to prove Theorem 5.1, the base of the induction is done as in Section 3. And we only consider the case $KM < R^{1/2 - O(\delta_1)}$. We assume that the result holds for $A \leq \overline{A}$ and $R' \leq \frac{2}{3}$, next prove it for $A = \overline{A}$ and $R' = R$, this completes the induction.

Set $D = D(\varepsilon, D_Z)$, we say we are in algebraic case if there is transverse complete intersection $Y \subset Z$ of dimension 1 defined using polynomials of degree no more than $D$, such that

$$\left\| e^{itH} f \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^0(0, R') \times [0, R])} \leq C \left\| e^{itH} f \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^0(0, R') \times [0, R]) \cap N_{R^{1/2 + \delta_2}}(Y)}.$$

Otherwise we are in the cellular case.

**Cellular case.** We first identify a significant piece $N_1$ of $(B(0, R) \times [0, R]) \cap N_{R^{1/2 + \delta_2}}(Z(P))$, where locally $Z(P)$ behaves like a 2-plane $V$, such that

$$\left\| e^{itH} f \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^0(0, R) \times [0, R])} \leq C \left\| e^{itH} f \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^0(0, R) \times [0, R]) \cap N_{R^{1/2 + \delta_2}}(Z(P))} \leq C \left\| e^{itH} f \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(B^0(0, R) \times [0, R]) \cap N_{R^{1/2 + \delta_2}}(N_1)}.$$

(6.1)

By Theorem 4.2, there exists a polynomial $Q(z) := \prod_{l=1}^s Q_l$ with $\deg Q(z) \leq D$, where polynomials $Q_1, Q_2, ..., Q_s$

$$Q_l(z) = Q_{V,l}(\pi(z)), \, l = 1, 2, ..., s,$$
π is the orthogonal projection from \( \mathbb{R}^2 \times \mathbb{R} \) to \( \mathbb{R}^2 \times \mathbb{R}/Z(Q) \) is divided into \( D^2 \) cells \( O_i \) such that
\[
\left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(N_1)}^p \leq CD^2 \left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(N_1 \cap O_i)}^p .
\] (6.2)

For each \( l \), the variety \( Y_l = Z(P, Q_l) \) is a transverse complete intersection of dimension 1. Define \( W := N_{R^{1/2}+\varepsilon} (Z(Q)) \), \( O'_i := O_i \setminus W \). By the analysis in [7], we have
\[
W \cap N_1 \subset \cup_i N_{O_i(R^{1/2}+\varepsilon)}(Y_l),
\]
since we are in the cellular case, the contribution from \( W \) is negligible. So we have
\[
\left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(N_1)}^p \leq CD^2 \left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(N_1 \cap O'_i)}^p .
\] (6.3)

Therefore, from (6.1)-(6.3) we actually obtain
\[
\left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(B(0,R) \times [0,R])}^p \leq CD^2 \left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(B(0,R) \times [0,R])}^p .
\] (6.4)

For each cell \( O'_i \), we set
\[
T_i := \left\{ (\theta, v) \in T : T_{\theta, v} \cap O'_i \neq \emptyset \right\} .
\]

For the function \( f \), we define
\[
f_i := \sum_{(\theta, v) \in T_i} f_{\theta, v}.
\]

It follows that on \( O'_i \),
\[
e^{itH} f \sim e^{itH} f_i .
\] (6.5)

By the fundamental theorem of Algebra, for each \( (\theta, v) \in T \), we have
\[
\text{Card} \{ i : (\theta, v) \in T_i \} \leq D + 1.
\]

Hence
\[
\sum_i \| f_i \|_{L^2}^2 \leq CD \| f \|_{L^2}^2 ,
\]

by pigeonhole principle, there exists \( O'_i \) such that
\[
\| f_i \|_{L^2}^2 \leq CD^{-1} \| f \|_{L^2}^2 .
\] (6.6)

So by (6.4), (6.5), the induction on \( R' \), and (6.6), we have
\[
\left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(B(0,R) \times [0,R])}^p \leq CD^2 \left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(B(0,R) \times [0,R])}^p \leq CD^2 \sum_{B_{R/2} \text{ cover } B(0,R)} \left\| e^{itH} f \phi_2 \left( \frac{t}{R} \right) \right\|_{BL_{\infty, A}^p L^q(B_{R/2} \times [0,R])}^p \leq CD^2 \left( C(\kappa, D_{Z, \varepsilon}) R^{\frac{1}{2\varepsilon}} \frac{R}{2} \left\| f \right\|_{L^2} + \varepsilon \right)^p \leq CD^2 \left( C(\kappa, D_{Z, \varepsilon}) R^{\frac{1}{2\varepsilon}} R^{\frac{1}{2\varepsilon} - \frac{1}{2} + \varepsilon} \| f \|_{L^2} \right)^p ,
\]
choose $D$ sufficiently large such that $CD^{2-\frac{2}{p}} \ll 1$, this completes the induction.

**Algebraic case.** In the algebraic case, there exists a transverse complete intersection $Y \subset Z(P)$ of dimension 1, determined by polynomial with degree no more than $D = D(\varepsilon, D_Z)$, so that

$$\|e^{itH}f\psi_2 \left( \frac{t}{R} \right) \|_{BL^p_{x,A}L^q(B(0,R) \times [0,R])} \leq C \|e^{itH}f\psi_2 \left( \frac{t}{R} \right) \|_{BL^p_{x,A}L^q((B(0,R) \times [0,R])\cap N_{R^{1/2+\delta_2}}(Y))}.$$ 

We decompose $B(0,R) \times [0,R]$ into balls $B_j$ of radius $\rho$, $\rho = R^{1/2+\delta_2}$, $\delta_2 \ll \delta_1$, in fact $\rho \sim R^{-O(\delta_1)}$.

For each $j$, we define

$$T_j := \{(\theta, \nu) \in T : T_{\theta,\nu} \cap N_{R^{1/2+\delta_2}}(Y) \cap B_j \neq \emptyset \},$$

and

$$f_j := \sum_{(\theta, \nu) \in T_j} f_{\theta,\nu}.$$

On each $B_j \cap N_{R^{1/2+\delta_2}}(Y)$, we have

$$e^{itH}f \sim e^{itH}f_j.$$

Therefore,

$$\|e^{itH}f\psi_2 \left( \frac{t}{R} \right) \|_{BL^p_{x,A}L^q(B(0,R) \times [0,R])}^p \leq \sum_j \|e^{itH}f\psi_2 \left( \frac{t}{R} \right) \|_{BL^p_{x,A}L^q(B_j)}^p.$$

We further divide $T_j$ into tubes that are tangential to $Y$ and tubes that are transverse to $Y$. We say that $T_{\theta,\nu}$ is tangential to $Y$ in $B_j$ if the following two conditions hold:

**Distance condition:**

$$T_{\theta,\nu} \cap 2B_j \subset N_{R^{1/2+\delta_2}}(Y) \cap 2B_j = N_{R^{1/2+\delta_2}}(Y) \cap 2B_j. \quad (6.7)$$

**Angle condition:** If $z \in Y \cap N_{O(R^{1/2+\delta_2})}(T_{\theta,\nu}) \cap 2B_j = Y \cap N_{O(R^{1/2+\delta_2})}(T_{\theta,\nu}) \cap 2B_j$, then

$$\text{Angle}(G(\theta), T_z Y) \leq C\rho^{-1/2+\delta_1}. \quad (6.8)$$

The tangential wave packets is defined by

$$T_{j,tang} := \{(\theta, \nu) \in T_j : T_{\theta,\nu} \text{ is tangent to } Y \text{ in } B_j \},$$

and the transverse wave packets

$$T_{j,trans} := T_j \setminus T_{j,tang}.$$

Set

$$f_{j,tang} := \sum_{(\theta, \nu) \in T_{j,tang}} f_{\theta,\nu}, \quad f_{j,trans} := \sum_{(\theta, \nu) \in T_{j,trans}} f_{\theta,\nu},$$

so

$$f_j = f_{j,tang} + f_{j,trans}.$$
Therefore, we have
\[
\left\| e^{it\mathcal{H}} f_j \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B(0, R) \times [0, R])} \leq \sum_j \left\| e^{it\mathcal{H}} f_j \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B_j)}
\leq \sum_j \left\| e^{it\mathcal{H}} f_j,\text{tang} \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B_j)}
+ \sum_j \left\| e^{it\mathcal{H}} f_j,\text{trans} \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B_j)}.
\]

We will treat the tangential term and the transverse term respectively. Again, we need to use wave packets decomposition in $B_j$.

**Algebraic tangential case.** In this case, the tangential term dominates. We claim that the new wave packets of $f_j,\text{tang}$ are $\rho^{-1/2+\delta_1}$-tangent to $Y$ in $B_j$. In fact, by (5.9) and (5.10), if $z \in Y \cap N_{O(\rho^{1/2+\delta_1})} \left( T_{\bar{\theta},\bar{\tau}} \right) \cap B_j$, then $z \in Y \cap N_{O(\rho^{1/2+\delta_2})} \left( T_{\bar{\theta},\bar{u}} \right) \cap B_j$, we have
\[
\text{Angle} \left( G(\bar{\theta}), T_\tau Y \right) \leq \text{Angle} \left( G(\bar{\theta}), G(\theta) \right) + \text{Angle} \left( G(\theta), T_\tau Y \right) \leq C\rho^{-1/2+\delta_1}.
\]
Also,
\[
T_{\bar{\theta},\bar{\tau}} \subset N_{\rho^{1/2+\delta_2}} \left( T_{\bar{\theta},\bar{u}} \cap B_j \right) \cap B_j = N_{\rho^{1/2+\delta_1}} \left( T_{\bar{\theta},\bar{u}} \cap B_j \right) \cap B_j \subset N_{O(\rho^{1/2+\delta_1})} \left( Y \right) \cap B_j.
\]
So, we can assume that $f_j,\text{tang}$ is concentrated in wave packets from $T_Y$ in $B_j$. Consider $B_K \times I_K^j$ such that
\[
\left[ N_{O(\rho^{1/2+\delta_1})} \left( Y \right) \cap B_j \right] \cap \left( B_K \times I_K^j \right) \neq \emptyset,
\]
there exists $z_0 \in Y \cap B_j \cap N_{O(\rho^{1/2+\delta_1})} \left( B_K \times I_K^j \right)$, for each $T_{\bar{\theta},\bar{\tau}}$ such that $T_{\bar{\theta},\bar{\tau}} \cap \left( B_K \times I_K^j \right) \neq \emptyset$, we have that $z_0 \in Y \cap B_j \cap N_{O(\rho^{1/2+\delta_1})} \left( T_{\bar{\theta},\bar{\tau}} \right)$, it holds
\[
\text{Angle} \left( G(\bar{\theta}), T_{\tau_0} Y \right) \leq C\rho^{-1/2+\delta_1}.
\]
Then for each $\tau$ with such a $\theta$ in it, it follows
\[
\text{Angle} \left( G(\tau), T_{\tau_0} Y \right) \leq C\rho^{-1/2+\delta_1} \leq (KM)^{-1},
\]
such $\tau$ does not contribute to $\left\| e^{it\mathcal{H}} f_j,\text{tang} \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B_j)}$. Since $f_j,\text{tang}$ is concentrated in wave packets from $T_Y$ in $B_j$,
\[
\left\| e^{it\mathcal{H}} f_j,\text{tang} \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B_j)} \leq \text{RapDec}(\rho) \left\| f \right\|^p_{L^2},
\]
which can be negligible. So we only need to consider the transverse case.

**Algebraic transverse case.** In this case, the transverse term dominates. So we need to estimate
\[
\sum_j \left\| e^{it\mathcal{H}} f_j,\text{trans} \psi_2 \left( \frac{t}{R} \right) \right\|^p_{BL^p_{\ell,\ell} L^p(B_j)}.
\]
Consider the new wave packets decomposition of \( f_{j,\text{trans}} \) in \( B_j \), by (5.9) and (5.10), the new wave packets \( T_{\vec{\theta},\nu} \) satisfy

\[
T_{\vec{\theta},\nu} \subset N_{R^{1/2+\delta}} (T_{\vec{\theta},\nu} \cap B_j) \cap B_j \subset N_{R^{1/2+\delta}} (Z) \cap B_j.
\]  

(6.11)

And if \( z \in Z \cap N_{O(\rho^{1/2+\delta})} \left( T_{\vec{\theta},\nu} \right) \cap B_j \), then

\[
\text{Angle} \left( G(z), T_z Z \right) \leq \text{Angle} \left( G(\vec{\theta}), T_z Z \right) + \text{Angle} \left( G(\vec{\theta}), G(z) \right) \leq C\rho^{-1/2+\delta}.
\]

(6.12)

So \( T_{\vec{\theta},\nu} \) is no longer \( \rho^{-1/2+\delta} \)-tangent to \( Z \) in \( B_j \) because the distance condition is not satisfied.

For each vector \( b \) with \( |b| \leq R^{1/2+\delta} \), define

\[
T_{Z+b} := \left\{ (\vec{\theta}, \nu) : T_{\vec{\theta},\nu} \text{ is } \rho^{-1/2+\delta} \text{-tangent to } Z+b \text{ in } B_j \right\}.
\]

By the angle condition, it turns out that each \( T_{\vec{\theta},\nu} \in T_{Z+b} \) for some \( b \). We set

\[
f_{j,\text{trans},b} := \sum_{(\vec{\theta},\nu) \in T_{Z+b}} f_{\vec{\theta},\nu}
\]

Then on \( B_j \), it holds

\[
\left| e^{itH} f_{j,\text{trans},b} \right|_2 \left( \frac{t}{R} \right) \sim \chi_{N_{\rho^{1/2+\delta}}(Z+b)} (x,t) \left| e^{itH} f_{j,\text{trans}} \right|_2 \left( \frac{t}{R} \right).
\]

(6.13)

Next we choose a set of vectors \( b \in B_{R^{1/2+\delta}} \). We cover \( N_{R^{1/2+\delta}} (Z) \cap B_j \) with disjoint balls of radius \( R^{1/2+\delta} \), and in each ball \( B \) we note the value of \( N_{\rho^{1/2+\delta}} (Z) \cap B \). We will dyadically pigeonhole this volume.

For

\[
B_s := \left\{ B \left( x_0, R^{1/2+\delta} \right) \subset N_{R^{1/2+\delta}} (Z) \cap B_j : B \left( x_0, R^{1/2+\delta} \right) \cap N_{\rho^{1/2+\delta}} (Z) \sim 2^s \right\}.
\]

We select a value of \( s \) so that

\[
\left\| e^{itH} f_{j,\text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(\cup_{s \in B_s} B_j)} \leq (\log R) \left\| e^{itH} f_{j,\text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(\cup_{s \in B_s} B_j)}.
\]

Therefore, we only consider \( \theta, \nu \) such that \( T_{\theta,\nu} \) meets at least one of the balls in \( B_s \). We choose a random set of \( |B_{R^{1/2+\delta}}| / 2^s \) vectors \( b \in B_{R^{1/2+\delta}} \). For a typical ball \( B \left( x_0, R^{1/2+\delta} \right) \in B_s \), the union \( \cup_b N_{\rho^{1/2+\delta}} (Z+b) \cap B_j \) covers a definite fraction of the ball with high probability. It follows

\[
\left\| e^{itH} f_{j,\text{trans}} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(\cup_{s \in B_s} B_j)} \leq (\log R) \sum_b \left\| e^{itH} f_{j,\text{trans},b} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^p(\cup_{s \in B_s} B_j)}.
\]

(6.14)
By the induction on $R'$, we have

$$\left\| e^{LH} f_{j,\text{trans},b} \left( \frac{t}{R} \right) \right\|^p_{BL^p_{k+1} L^q(B_j)} \leq \left\| e^{LH} f_{j,\text{trans},b} \left( \frac{t}{R} \right) \right\|^p_{BL^p_{k+1} L^q(B_j)} \leq \left\| e^{LH} f_{j,\text{trans},b} \left( \frac{t}{R} \right) \right\|^p_{BL^p_{k+1} L^q(B_j)} \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f_{j,\text{trans},b} \right\|_{L^2}^p \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f_{j,\text{trans},b} \right\|_{L^2}^p,$$

therefore, if

$$\sum_j \sum_b \left\| f_{j,\text{trans},b} \right\|^2_{L^2} \sim \sum_j \left\| f_{j,\text{trans}} \right\|^2_{L^2} \leq D \left\| f \right\|^2_{L^2}, \quad (6.15)$$

$$\max_b \left\| f_{j,\text{trans},b} \right\|^2_{L^2} \leq R^{O(\delta_2)} \left( \frac{R}{p} \right)^{-1/2} \left\| f_{j,\text{trans}} \right\|_{L^2}, \quad (6.16)$$

then we have

$$\sum_j \sum_b \left\| e^{LH} f_{j,\text{trans}} \left( \frac{t}{R} \right) \right\|^p_{BL^p_{k+1} L^q(B_j)} \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f_{j,\text{trans}} \right\|_{L^2}^p \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f_{j,\text{trans}} \right\|_{L^2}^p \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f \right\|^p_{L^2} \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f \right\|^p_{L^2} \leq \left[ C \left( K, D, \varepsilon \right) R^4 \left( \log \frac{R}{T} - \log \frac{1}{\varepsilon} \right) R^{1/p} (\rho)^{\frac{1}{p} - \frac{1}{q} + \varepsilon} \right] \left\| f \right\|^p_{L^2},$$

so the induction closes by choosing $\delta_2 \ll \varepsilon \delta_1$ and the fact that $R$ is sufficiently large. This completes the proof of Theorem 5.1.

Next we will prove (6.15) and (6.16). For each $(\theta, \nu) \in T_{j,\text{trans}}$, if $T_{\theta,\nu}$ contributes, then $T_{\theta,\nu}$ intersects some $B \left( x_0, R^{1/2+\delta_2} \right)$ in $B_s$. We have

$$\left\| f_{\theta,\nu} \right\|^2_{L^2} \sim R^{-1/2-\delta_2} \left\| e^{LH} f_{\theta,\nu} \left( \frac{t}{R} \right) \right\|^2_{L^2 \left( B \left( x_0, R^{1/2+\delta_2} \right) \right)}, \quad (6.17)$$
provided Theorem 7.3 below holds true. Set

\[ f_{\theta, \nu, b} := \sum_{(\overline{\theta}, \overline{\nu}) \in T_{x, b} \cap (\theta, \nu)^{\sim}} f_{\overline{\theta}, \overline{\nu}} \]  \tag{6.18}

here (\theta, \nu)^{\sim} denotes the wave packets decomposition of \( f_{\theta, \nu} \) in \( B_j \), it follows

\[ \left\| e^{itH} f_{\theta, \nu, b} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2 \sim \left\| e^{itH} f_{\theta, \nu} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2, \]  \tag{6.19}

using Theorem 7.3 again,

\[ \| f_{\theta, \nu, b} \|_{L^2}^2 \sim R^{-1/2 - \delta_2} \left\| e^{itH} f_{\theta, \nu, b} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2. \]  \tag{6.20}

Notice that the sets \( N_{\rho, 1/2 + \delta_2} (Z + b) \) are essentially disjoint, hence (6.17), (6.18) and (6.20) imply

\[
\sum_b \| f_{\theta, \nu, b} \|_{L^2}^2 \sim R^{-1/2 - \delta_2} \sum_b \left\| e^{itH} f_{\theta, \nu, b} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2 \\
\leq R^{-1/2 - \delta_2} \left\| e^{itH} f_{\theta, \nu} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2 \\
\sim \| f_{\theta, \nu} \|_{L^2}^2.
\]

Therefore

\[
\sum_b \| f_{j, \text{trans}, b} \|_{L^2}^2 = \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \sum_b \| f_{\theta, \nu, b} \|_{L^2}^2 \leq \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \| f_{\theta, \nu} \|_{L^2}^2 = \| f_{j, \text{trans}} \|_{L^2}^2.
\]  \tag{6.21}

Then by (5.4) and (6.21), (6.15) holds.

If Theorem 7.2 below holds true, then for each \( b \),

\[
\| f_{j, \text{trans}, b} \|_{L^2}^2 \leq \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \| f_{\theta, \nu, b} \|_{L^2}^2 \sim R^{-1/2 - \delta_2} \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \left\| e^{itH} f_{\theta, \nu, b} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2 \\
\sim R^{-1/2 - \delta_2} \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \left\| e^{itH} f_{\theta, \nu} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2, \\
\end{align}

and

\[
\left\| e^{itH} f_{\theta, \nu} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2 \leq CR^{Q(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} \left\| e^{itH} f_{\theta, \nu} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2,
\]

therefore

\[
\| f_{j, \text{trans}, b} \|_{L^2}^2 \leq CR^{Q(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} R^{-1/2 - \delta_2} \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \left\| e^{itH} f_{\theta, \nu} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0, R^{1/2 + \delta}))}^2 \\
\leq CR^{Q(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} \sum_{(\theta, \nu) \in T_{j, \text{trans}}} \| f_{\theta, \nu} \|_{L^2}^2 \\
\leq CR^{Q(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} \| f_{j, \text{trans}} \|_{L^2}^2,
\]

in the second inequality above we used Theorem 7.3 again, and (6.16) is obtained.
The following Theorem 7.2 is a generalization of Lemma 6.2 in [7], which is needed in the proof of Theorem 5.1. In order to prove Theorem 7.2, we need a version of the Heisenberg uncertainly principle in [7]:

**Lemma 7.1.** ([7]) Suppose that \( G : \mathbb{R}^n \to \mathbb{C} \) is a function, and that \( \hat{G} \) is supported in a ball \( B(\xi_0, r) \), then for any ball \( B_\rho \) with \( \rho \leq r^{-1} \), we have the inequality

\[
\int_{B_\rho} |G|^2 \leq C \frac{|B_\rho|}{|B_{r^{-1}}|} \int_{B_{r^{-1}}} |G|^2. \tag{7.1}
\]

**Theorem 7.2.** Suppose that \( f \) is concentrated in wave packets from \( T_Z, Z = Z(P) \) is a transverse complete intersection of dimension 2, \( B \) is a ball of radius \( R^{1/2+\delta_2} \) contained in \( B(0, R) \times [0, R] \), \( T_{Z,B} := \{(\theta, \nu) \in T_Z : T_{\theta,\nu} \cap B \neq \emptyset \} \), if

\[
f = \sum_{(\theta, \nu) \in T_{Z,B}} f_{\theta,\nu},
\]

then

\[
\left\| e^{it\hbar} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{L^2(B \cap N_{R^{1/2+\delta_2}}(Z))} \leq C R^{O(\delta_2)} \left( \frac{R^{1/2}}{R^{1/2}} \right)^{-1} \left\| e^{it\hbar} f_{\psi_2} \left( \frac{t}{R} \right) \right\|_{L^2(2B)}. \tag{7.2}
\]

**Proof:** If \( B \cap N_{R^{1/2+\delta_2}}(Z) = \emptyset \), then \( T_{Z,B} = \emptyset \), and there is nothing to prove. So we can assume that \( B \cap N_{R^{1/2+\delta_2}}(Z) \neq \emptyset \), then there exists a point \( z_0 \in Z \) such that \( z_0 \in Z \cap N_{R^{1/2+\delta_2}}(B) \), then for each wave packet \((\theta, \nu) \in T_{Z,B}\), we have

\[
z_0 \in Z \cap N_{R^{1/2+\delta_2}}(T_{\theta,\nu}).
\]

By the definition of \( T_Z \), we have

\[
\text{Angle} \left( G(\theta), T_{\theta,\nu} \right) \leq R^{-1/2+\delta_2}. \tag{7.3}
\]

We can assume \( T_{\theta,\nu} \) is given by

\[
a_1 x_1 + a_2 x_2 + bt = 0, \quad a_1^2 + a_2^2 + b^2 = 1, \quad |(a_1, a_2)| \geq 1,
\]

(7.3) and (7.4) imply

\[
| -2c(\theta) \cdot a + b | \leq CR^{-1/2+\delta_2},
\]

this restricts all \( \theta \) to a strip of width \( R^{-1/2+\delta_2} \) paralleled to \((a_2, -a_1)\), we denote it by \( S \). The Fourier transform of \( e^{it\hbar} f_{\psi_2}(\hbar) \) is supported in

\[
\left\{ (\xi_1, \xi_2, \xi_3) : (\xi_1, \xi_2) \in \theta, \left| \xi_3 - \xi_1^2 - \xi_2^2 \right| \leq R^{-\varepsilon/2} \right\},
\]

therefore, the Fourier transform of \( e^{it\hbar} f_{\psi_2}(\hbar) \) is supported in

\[
\left\{ (\xi_1, \xi_2, \xi_3) : (\xi_1, \xi_2) \in S, \left| \xi_3 - \xi_1^2 - \xi_2^2 \right| \leq R^{-\varepsilon/2} \right\}.
\]
Suppose that $\Pi$ is a 1-dimension linear sub-space of $\mathbb{R}^3$ parallel to $(a_1,a_2,0)$, then the projection of the Fourier transform of $e^{itH}\psi_2(\frac{t}{R})$ on $\Pi$ is supported in a ball of radius $R^{-1/2+\delta_2}$. If we view $e^{itH}\psi_2(\frac{t}{R})$ as a function defined on $\Pi$, then for each $x \in B \cap \Pi$, Lemma 7.1 implies

$$
\int_{\Pi \cap B(x,\rho^{1/2+\delta_2})} \left| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right|^2 \leq \sum_{B_{\rho^{1/2+\delta_2}} \text{ cover } B(x,\rho^{1/2+\delta_2})} \int_{\Pi \cap B(x,\rho^{1/2+\delta_2})} \left| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right|^2 \leq R^{O(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} \int_{\Pi \cap B(x,\rho^{1/2+\delta_2})} \left| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right|^2 \leq R^{O(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} \int_{\Pi \cap B(x,\rho^{1/2+\delta_2})} \left| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right|^2,
$$

here we used the fact that on $\Pi$ which passing through $B$, $e^{itH}\psi_2(\frac{t}{R})$ is essentially supported in $\Pi \cap 2B$, see also [7]. By [7], $\Pi \cap B \cap N_{\rho^{1/2+\delta_2}}(Z) \subset N_{\rho^{1/2+\delta_2}}(\Pi \cap Z) \cap B(2\Pi \cap 2B)$, and $N_{\rho^{1/2+\delta_2}}(\Pi \cap Z) \cap B(2\Pi \cap 2B)$ can be covered by $R^{O(\delta_2)}$ balls $\Pi \cap B(x,\rho^{1/2+\delta_2})$, $x \in B \cap \Pi$, so we get the bound

$$
\left\| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(\Pi \cap B \cap 2B)}^2 \leq R^{O(\delta_2)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-1} \left\| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(\Pi \cap 2B)}^2,
$$

(7.2) is obtained by integrating over all $\Pi$ paralleled to $(a_1,a_2,0)$ and this completes the proof of Theorem 7.2.

In the proof of Theorem 5.1, we also used the following generalization of Lemma 3.4 in [7]:

**Theorem 7.3.** Suppose that $f$ is concentrated in a set of wave packets $T$ and that for every $(\theta, \nu) \in T$, $T_{\theta,\nu} \cap B(\theta,R) \neq \emptyset$, $z = (x_0,t_0)$, $t_0 \leq R$, for some radius $r \sim R^{1/2+\delta_2}$. Then

$$
\left\| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(z,10r))}^2 \sim r \|f\|_{L^2}^2.
$$

**Proof:** Suppose $z = (x_0,t_0)$, for each $t$ in the range $t_0 - r \leq t \leq t_0 + r$, each $(\theta, \nu) \in T$, $T_{\theta,\nu} \cap (\mathbb{R}^2 \times \{t\}) \subset B(x_0,5r)$, therefore, (7.5) follows from the facts that

$$
\left\| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(z,10r))}^2 \geq \left\| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(x_0,5r) \times (t_0-r,t_0+r))}^2 = \int_{t_0-r}^{t_0+r} \int_{B(x_0,5r)} |e^{itH}f|^2 \psi_2 \left( \frac{t}{R} \right) dx dt = \int_{t_0-r}^{t_0+r} \int_{B(x_0,5r)} |f|^2 \psi_2 \left( \frac{t}{R} \right) dx dt = \|f\|_{L^2}^2 \int_{t_0-r}^{t_0+r} \left\| e^{itH}\psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(z,10r))}^2 dt \geq \left( \frac{r}{2} \right) \|f\|_{L^2}^2,
$$


and
\[
\left\| e^{itH} \psi_2 \left( \frac{t}{R} \right) \right\|_{L^2(B(z,10r))}^2 \leq \int_{t_0-10r}^{t_0+10r} \int_{\mathbb{R}^2} \left| e^{itH} f \right|^2 \, dx \left| \psi_2 \left( \frac{t}{R} \right) \right|^2 \, dt = \| f \|_{L^2}^2 \int_{t_0-10r}^{t_0+10r} \left| \psi_2 \left( \frac{t}{R} \right) \right|^2 \, dt \\
\leq \| f \|_{L^2}^2 \int_{t_0-10r}^{t_0+10r} 1 \, dt \leq 20r \| f \|_{L^2}^2 .
\]

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