A NOTE ON MOTIVIC INTEGRATION IN MIXED CHARACTERISTIC

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Abstract. We introduce a quotient of the Grothendieck ring of varieties by identifying classes of universally homeomorphic varieties. We show that the standard realization morphisms factor through this quotient, and we argue that it is the correct value ring for the theory of motivic integration on formal schemes and rigid varieties in mixed characteristic.

The present note is an excerpt of a detailed survey paper which will be published in the proceedings of the conference “Motivic integration and its interactions with model theory and non-archimedean geometry” (ICMS, 2008).

1. Introduction

The Grothendieck ring $K_0(Var_F)$ of varieties over a field $F$ arises naturally as the universal ring of additive and multiplicative invariants of such varieties. Taking the class of a variety in the Grothendieck ring is the most general way to “measure the size” of the variety. In recent years, the Grothendieck ring of varieties has received much attention, because of its role as value ring in several theories of motivic integration.

In spite of this renewed interest, many basic questions on the structure of the Grothendieck ring remain unanswered. The main difficulty is that it may be very hard to decide whether two given varieties have distinct classes in the Grothendieck ring. The central question in this context is the one raised by Larsen and Lunts in [3, 1.2].

Question 1.1 (Larsen-Lunts). Let $F$ be a field, and let $X$ and $Y$ be $F$-varieties such that $[X] = [Y]$ in $K_0(Var_F)$. Is it true that $X$ and $Y$ are piecewise isomorphic, i.e., that we can find an integer $n > 0$, subvarieties $X_1, \ldots, X_n$ of $X$ and subvarieties $Y_1, \ldots, Y_n$ of $Y$ such that $X_i$ is $F$-isomorphic to $Y_i$ for every $i$ in $\{1, \ldots, n\}$?

This question has been answered affirmatively in certain cases [4][12], but it remains open in general. In the present note, we raise a different question.

Question 1.2. Let $F$ be a field of characteristic $p > 0$. If $X$ and $Y$ are $F$-varieties such that there exists a universal homeomorphism of $F$-schemes $Y \to X$, is it true that $[X] = [Y]$ in $K_0(Var_F)$?

It seems reasonable to expect that Question 1.2 has a negative answer, in general. In characteristic zero, universally homeomorphic varieties are piecewise isomorphic (Proposition 3.1). In positive characteristic, Question 1.2 is in some sense orthogonal to Question 1.1, since there are examples of universally homeomorphic $F$-varieties that are not piecewise isomorphic. The most basic example is the following: assume that $F$ is imperfect, and let $F'$ be a non-trivial finite purely...
inseparable extension of $F$. Then the morphism $\text{Spec } F' \to \text{Spec } F$ is a universal homeomorphism. We do not know if $[\text{Spec } F'] = [\text{Spec } F]$ in $K_0(\text{Var}_F)$.

For every field $F$, we introduce a quotient $K_0(\text{Var}_F)$ of the Grothendieck ring of $F$-varieties by identifying classes of universally homeomorphic varieties (Definition 3.2). We call this quotient the modified Grothendieck ring of $F$-varieties. If $F$ has characteristic zero, then the projection morphism

$$K_0(\text{Var}_F) \to K_0(\text{Var}_F)$$

is an isomorphism (Proposition 3.3). In any characteristic, there exists a canonical isomorphism

$$K_0(\text{Var}_F) \to K_0(\text{ACF}_F)$$

to the Grothendieck ring of the theory $\text{ACF}_F$ of algebraically closed fields over $F$ (Proposition 3.7). We show that the standard realization morphisms of the Grothendieck ring of varieties (étale realization, Poincaré polynomial, ...) factor through the modified Grothendieck ring $K_0(\text{Var}_F)$ (Proposition 4.1). In fact, this is what makes Question 1.2 hard to answer: we cannot use the standard realization morphisms to distinguish classes of universally homeomorphic varieties in the Grothendieck ring.

In Section 5 we fill a gap in the proof of the change of variables theorem for motivic integrals on formal schemes in mixed characteristic [11, 8.0.5]. To this aim, it is necessary to replace the Grothendieck ring of varieties by the modified version introduced in Definition 3.2. We emphasize that this correction only affects the theory of motivic integration in mixed characteristic; in equal characteristic $p \geq 0$, the results in [11, 8.0.5] are valid as stated. The modification is harmless for the applications of the theory, because the standard realization morphisms factor through the modified Grothendieck ring. In Sections 5.2 and 5.3 we give a list of changes that should be made to the literature.

The present note is an excerpt of a detailed survey paper, which will be published in the proceedings of the conference “Motivic integration and its interactions with model theory and non-archimedean geometry” (ICMS, 2008).

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Notations. We denote by $(\cdot)_{\text{red}}$ the functor from the category of schemes to the category of reduced schemes that maps a scheme $X$ to its maximal reduced closed subscheme $X_{\text{red}}$. If $S$ is a Noetherian scheme, then an $S$-variety is a reduced separated $S$-scheme of finite type.

2. The Grothendieck ring of varieties

Let $S$ be a Noetherian scheme. For the definition of the Grothendieck ring of $S$-varieties $K_0(\text{Var}_S)$ and its localization $\mathcal{M}_S$, we refer to [7, 2.1]. We recall some of the main realization morphisms. For details, the reader may consult [7, § 2.1].

Point counting. If $F$ is a finite field, then there exists a unique ring morphism

$$\sharp : K_0(\text{Var}_F) \to \mathbb{Z}$$

that maps $[X]$ to the cardinality of the set $X(F)$ for every $F$-variety $X$. 
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Euler characteristic. If $F$ is a field, and $\ell$ a prime invertible in $F$, then there exists a unique ring morphism

$$\chi_{\text{top}} : K_0(\text{Var}_F) \to \mathbb{Z}$$

that maps $[X]$ to the $\ell$-adic Euler characteristic of $X$ for every $F$-variety $X$. It localizes to a ring morphism

$$\chi_{\text{top}} : \mathcal{M}_F \to \mathbb{Z}$$

These morphisms are independent of $\ell$.

Galois realization. Let $F$ be a field, and $\ell$ a prime invertible in $F$. We fix a separable closure $F^s$ of $F$. We denote by $G_F$ the absolute Galois group of $F$, and by $K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell)$ the Grothendieck ring of $\ell$-adic Galois representations of $F$. There exists a unique ring morphism

$$\text{Gal} : K_0(\text{Var}_F) \to K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell)$$

that maps $[X]$ to

$$\sum_{i=0}^{2 \dim(X)} (-1)^i [H^i_{\text{et}}(X \times_F F^s, \mathbb{Q}_\ell)] \in K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell)$$

for every $F$-variety $X$. It localizes to a ring morphism

$$\text{Gal} : \mathcal{M}_F \to K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell)$$

Étale realization. Let $\ell$ be a prime, and $S$ a Noetherian $\mathbb{Z}[1/\ell]$-scheme. There exists a unique ring morphism

$$\text{ét} : K_0(\text{Var}_S) \to K_0(D^b_c(S, \mathbb{Q}_\ell))$$

that maps $[X]$ to the class of $R(g_X)_! \mathbb{Q}_\ell$ for every $S$-variety $X$, where we denote by $g_X : X \to S$ the structural morphism. It localizes to a ring morphism

$$\text{ét} : \mathcal{M}_S \to K_0(D^b_c(S, \mathbb{Q}_\ell))$$

Poincaré realization. Let $S$ be a Noetherian scheme. The Poincaré realization

$$P_S : K_0(\text{Var}_S) \to \mathcal{C}(S, \mathbb{Z}[T])$$

is defined in [7, 8.12]. The target $\mathcal{C}(S, \mathbb{Z}[T])$ is the ring of constructible functions on $S$ with values in $\mathbb{Z}[T]$ [7, §8.3].

3. The modified Grothendieck ring

3.1. Trivializing universal homeomorphisms. Recall that a morphism of schemes

$$f : X \to Y$$

is called a universal homeomorphism if for every morphism of schemes $Y' \to Y$, the morphism

$$f_{Y'} : X \times_Y Y' \to Y'$$

obtained from $f$ by base change is a homeomorphism [1 2.4.2]. This property is obviously stable under base change. If $f$ is of finite presentation, then $f$ is a universal homeomorphism if and only if $f$ is finite, surjective, and purely inseparable [2 8.11.6]. We call two schemes $X$ and $Y$ universally homeomorphic if there exists a universal homeomorphism from $X$ to $Y$ or from $Y$ to $X$. If $S$ is a scheme and $X$ and $Y$ are $S$-schemes, then we call $X$ and $Y$ universally $S$-homeomorphic if there exists a universal homeomorphism of $S$-schemes $X \to Y$ or $Y \to X$. 
Proposition 3.1. If \( f : X \to Y \) is a universal homeomorphism of finite type between Noetherian \( \mathbb{Q} \)-schemes, then there exists a finite partition \( \{Y_1, \ldots, Y_r\} \) of \( Y \) into locally closed subsets, such that, if we endow \( Y_i \) with its reduced induces structure, the morphism \( (X \times_Y Y_i)_\text{red} \to Y_i \) is an isomorphism for each \( i \in \{1, \ldots, r\} \).

Proof. Since \( f_\text{red} : X_\text{red} \to Y_\text{red} \) is still a universal homeomorphism \([1. 2.4.3(vi)]\), we may assume that \( X \) and \( Y \) are reduced. By Noetherian induction, it is enough to find a non-empty open subscheme \( U \) of \( Y \) such that \( X \times_Y U \to U \) is an isomorphism. In particular, we may assume that \( Y \) is irreducible. Then \( X \) is irreducible, because it is homeomorphic to \( Y \). If we denote by \( \eta_Y \) the generic point of \( Y \), then its inverse image in \( X \) consists of a unique point \( \eta_X \), which is the generic point of \( X \). The residue field \( \kappa(\eta_X) \) is a purely inseparable extension of the residue field \( \kappa(\eta_Y) \) of \( \eta_Y \). Since these fields have characteristic zero, we see that \( f \) induces an isomorphism \( \kappa(\eta_X) \cong \kappa(\eta_Y) \), so that the restriction of \( f \) to some dense open subset of \( X \) is an open immersion. This concludes the proof. \( \square \)

Definition 3.2. Let \( S \) be a Noetherian scheme. We denote by \( I_S^{\text{sh}} \) the ideal in \( K_0(\text{Var}_S) \) generated by elements of the form \([X] - [Y]\), where \( X \) and \( Y \) are universally \( S \)-homeomorphic separated \( S \)-schemes of finite type. We put

\[
K_0(\text{Var}_S) = K_0(\text{Var}_S)/I_S^{\text{sh}}
\]

and we call this quotient the modified Grothendieck ring of \( S \)-varieties.

For every separated \( S \)-scheme of finite type \( Z \), we denote by \( \langle Z \rangle \) the image of \([Z] \) in \( K_0(\text{Var}_S) \). We put \( \mathbb{L}_S = \langle \mathbb{A}^1_S \rangle \), and we denote by \( \mathcal{M}_S^{\text{mod}} \) the localization of \( K_0(\text{Var}_S) \) with respect to \( \mathbb{L}_S \).

The dimensional completion \( \hat{\mathcal{M}}_S^{\text{mod}} \) is the separated completion of \( \mathcal{M}_S^{\text{mod}} \) with respect to the descending filtration \( F^* \mathcal{M}_S^{\text{mod}} \), where for every \( i \in \mathbb{Z} \), \( F^i \mathcal{M}_S^{\text{mod}} \) is the subgroup of \( \mathcal{M}_S^{\text{mod}} \) generated by the elements of the form \( (X) \mathbb{L}_S^j \) with \( X \) a separated \( S \)-scheme of finite type and \( j \) an element of \( \mathbb{Z} \) such that

\[
\dim(X/S) + j \leq -i
\]

Here \( \dim(X/S) \) denotes the relative dimension of \( X \) over \( S \).

If \( S = \text{Spec} A \) for some Noetherian ring \( A \), then we also write \( I_A^{\text{sh}}, K_0(\text{Var}_A), \mathbb{L}_A, \mathcal{M}_A^{\text{mod}} \) and \( \hat{\mathcal{M}}_A^{\text{mod}} \) instead of \( I_S^{\text{sh}}, K_0(\text{Var}_S), \mathbb{L}_S, \mathcal{M}_S^{\text{mod}} \) and \( \hat{\mathcal{M}}_S^{\text{mod}} \).

Let \( \tilde{S} \) be a Noetherian scheme, \( X \) a separated \( S \)-scheme of finite type, and \( C \) constructible subset of \( X \). We can write \( C \) as a disjoint union of locally closed subsets \( C_1, \ldots, C_r \) of \( X \). If we endow \( C_i \) with its reduced induced structure, for every \( i \), then the class \([C]\) of \( C \) in \( K_0(\text{Var}_S) \) is defined by

\[
[C] = [C_1] + \ldots + [C_r]
\]

This definition does not depend on the choice of \( C_1, \ldots, C_r \). We denote by \( \langle C \rangle \) the image of \([C]\) in \( K_0(\text{Var}_S) \).

Proposition 3.3. If \( S \) is a Noetherian scheme over \( \mathbb{Q} \), then \( I_S^{\text{sh}} \) is the zero ideal, and the projection

\[
K_0(\text{Var}_S) \to K_0(\text{Var}_S)
\]

is an isomorphism.
Proof. This follows immediately from Proposition 3.1 and the scissor relations in the Grothendieck ring $K_0(Var_S)$. □

If $S$ is not a $\mathbb{Q}$-scheme, we do not know if $I_{S}^h$ is different from zero. If $S$ is the spectrum of a field $F$ of positive characteristic, then $I_{S}^h$ is trivial if and only if Question 1.2 in the introduction has a positive answer.

3.2. **Base change and direct image.** The definitions of the modified Grothendieck ring $K_{0}(\text{Var}_{S})$ and its localization $\mathcal{M}_{S}^{\text{mod}}$ are compatible with base change and direct image. If $f: T \to S$ is a morphism of Noetherian schemes, then there exists a unique ring morphism

$$f^{*} : K_{0}(Var_{S}) \to K_{0}(Var_{T})$$

such that $f^{*}(X) = \langle X \times_{S} T \rangle$ for every separated $S$-scheme $X$ of finite type. It localizes to a ring morphism

$$f^{*} : \mathcal{M}_{S}^{\text{mod}} \to \mathcal{M}_{T}^{\text{mod}}$$

If $g: S \to U$ is a separated morphism of finite type between Noetherian schemes, then there exists a unique morphism of abelian groups

$$g_{!} : K_{0}(Var_{S}) \to K_{0}(Var_{U})$$

such that for every separated $S$-scheme $X$ of finite type, we have $g_{!}(X) = \langle X|_{U} \rangle$ (here $X|_{U}$ denotes the $U$-scheme obtained by composing the structural morphism $X \to S$ with the morphism $g$). Moreover, there exists a unique morphism of abelian groups

$$g_{!} : \mathcal{M}_{S}^{\text{mod}} \to \mathcal{M}_{U}^{\text{mod}}$$

such that

$$g_{!}(\langle X \rangle) = \langle X|_{U} \rangle$$

for every separated $S$-scheme of finite type $X$ and every integer $i$.

3.3. **Fibrations.**

**Lemma 3.4.** Let $S$ be a Noetherian scheme, and let $X$, $Y$ and $Z$ be separated $S$-schemes of finite type. Let $f : X \to Y$ be a morphism of $S$-schemes, and assume that for every perfect field $F$ and every morphism of schemes $\text{Spec} F \to Y$, there exists a universal homeomorphism of $F$-schemes

$$X \times_{Y} \text{Spec} F \to Z \times_{S} \text{Spec} F$$

Then

$$\langle X \rangle = \langle Y \rangle \cdot \langle Z \rangle$$

in $K_{0}(Var_{S})$.

**Proof.** By Noetherian induction, it is enough to find a non-empty open subscheme $U$ of $Y$ such that

$$\langle X \times_{Y} U \rangle = \langle U \rangle \cdot \langle Z \rangle$$

in $K_{0}(Var_{S})$. We may assume that $Y$ is affine and integral. We denote by $B$ the ring of regular functions on $Y$. Let $F$ be the perfect closure of the function field $\text{Frac}(B)$ of $Y$. We know that there exists a universal homeomorphism of $F$-schemes

$$g : X \times_{Y} \text{Spec} F \to Z \times_{S} \text{Spec} F$$
The \( B \)-algebra \( F \) is the direct limit of its finitely generated sub-\( B \)-algebras. Hence, by [2, 8.8.2 and 8.10.5], there exist a finitely generated sub-\( B \)-algebra \( B' \) of \( F \), and a universal homeomorphism of \( B' \)-schemes

\[
g' : X_Y \text{Spec} B' \to Z_S \text{Spec} B'
\]
such that \( g \) is obtained from \( g' \) by base change from \( \text{Spec} B' \) to \( \text{Spec} F \).

Since \( \text{Spec} B' \to Y \) is purely inseparable over the generic point of \( Y \), and the generic point of \( Y \) is the projective limit of the dense open subschemes of \( Y \), it follows from [2, 8.10.5] that there exists a dense open subscheme \( U \) of \( Y \) such that

\[
\text{Spec} B' \times_Y U \to U
\]
is a universal homeomorphism.

Looking at the diagram of universal homeomorphisms of separated \( S \)-schemes of finite type

\[
\begin{array}{ccc}
X_Y (\text{Spec} B' \times_Y U) & \xrightarrow{g' \times_Y \text{id}_U} & Z_S (\text{Spec} B' \times_Y U) \\
\downarrow & & \downarrow \\
X_Y U & & Z_S U
\end{array}
\]
we find that

\[
\langle X_Y U \rangle = \langle U \rangle \cdot \langle Z \rangle
\]
in \( K_0(\text{Var}_S) \).

3.4. The Grothendieck ring of the theory \( ACF_F \). The modified Grothendieck ring arises naturally in the setting of model theory. Let \( F \) be a field, and denote by \( ACF_F \) the theory of algebraically closed fields over \( F \) in the language \( \mathcal{L}_F \) of \( F \)-algebras.

Recall that formulas in \( \mathcal{L}_F \) consist of quantifiers and Boolean combinations of polynomial equations with coefficients in \( F \). For every formula \( \varphi(x_1, \ldots, x_n) \) in \( \mathcal{L}_F \) and every \( F \)-algebra \( A \), we denote by \( S_\varphi(A) \) the subset of \( A^n \) defined by \( \varphi \).

We say that two formulas \( \varphi(x_1, \ldots, x_m) \) and \( \psi(y_1, \ldots, y_n) \) in \( \mathcal{L}_F \) are \( ACF_F \)-equivalent, if there exists a third formula \( \eta(x_1, \ldots, x_m, y_1, \ldots, y_n) \) such that, for every algebraically closed field \( L \) that contains \( F \), the set \( S_\varphi(L) \subset L^m \) and \( S_\psi(L) \subset L^n \). We say that \( \eta \) defines an \( ACF_F \)-equivalence between \( \varphi \) and \( \psi \).

**Definition 3.5.** Let \( F \) be a field. The Grothendieck group \( K_0(ACF_F) \) of the theory \( ACF_F \) is the quotient of the free abelian group on \( ACF_F \)-equivalence classes \([\varphi]\) of formulas \( \varphi \) in \( \mathcal{L}_F \), by the subgroup generated by elements of the form

\[
[\varphi \land \psi] + [\varphi \lor \psi] - [\varphi] - [\psi]
\]
where \( \varphi \) and \( \psi \) are formulas in \( \mathcal{L}_F \) with the same sets of free variables.

We endow \( K_0(ACF_F) \) with the unique ring structure such that, for all formulas \( \varphi \) and \( \psi \) in \( \mathcal{L}_F \) in disjoint sets of free variables, we have

\[
[\varphi] \cdot [\psi] = [\varphi \land \psi]
\]
in \( K_0(ACF_F) \).
The unit for the multiplication in $K_0(ACF_F)$ is the class of the formula $\psi = (0 = 0)$. For every $F$-algebra $A$, this formula defines the set $S_0(A) = A^0 = \{pt\}$.

If $n$ is an element of $\mathbb{N}$, and $i_X : X \rightarrow A^n_F$ is an immersion of $F$-schemes, then there exists a formula $\varphi(x_1, \ldots, x_n)$ such that

$$S_\varphi(L) = X(L) \subset L^n$$

for every field $L$ that contains $F$. We call such a formula $\varphi$ an $i_X$-formula. It is not unique. If $Y$ is a quasi-affine $F$-variety, then we say that a formula $\psi$ in $L_F$ is a $Y$-formula if it is an $i_Y$-formula for some immersion $i_Y : Y \rightarrow A^n_F$, with $m \in \mathbb{N}$. Again, such a $Y$-formula is not unique, but its class $[\psi]$ in $K_0(ACF_F)$ only depends on $Y$, and not on the choice of the immersion $i_Y$ or the formula $\psi$.

**Lemma 3.6.** Let $F$ be a field. Let $m$ and $n$ be elements of $\mathbb{N}$, and let $C$ and $D$ be constructible subsets of $A^n_F$, resp. $A^m_F$. Assume that there exists a formula $\eta(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in $L_F$ such that, for every algebraically closed field $L$ containing $F$, the set

$$S_\eta(L) \subset L^{m+n}$$

is the graph of a bijection between $C(L) \subset L^m$ and $D(L) \subset L^n$. Then we have

$$\langle C \rangle = \langle D \rangle$$

in $K_0(Var_F)$.

**Proof.** By quantifier elimination relative to $ACF_F$, there exists a unique constructible subset $E$ of $A^{m+n}_F$ such that

$$S_\eta(L) = E(L) \subset L^{m+n}$$

for every algebraically closed field $L$ that contains $F$. It suffices to show that $\langle C \rangle = \langle E \rangle$ in $K_0(Var_F)$. We denote by

$$\pi : A^{m+n}_F \rightarrow A^m_F$$

the projection onto the first $m$ coordinates. By Noetherian induction on $C$, it is enough to prove that there exists a non-empty open subset $U$ of $C$ such that $U$ is locally closed in $A^n_F$, $V = \pi^{-1}(U) \cap E$ is locally closed in $A^{m+n}_F$, and the morphism $V \rightarrow U$ induced by $\pi$ is a universal homeomorphism if we endow $U$ and $V$ with the reduced induced structure.

We may assume that $C$ is irreducible and locally closed in $A^n_F$, and we endow $C$ with its reduced induced structure. Let $E'$ be an integral subscheme of $A^{m+n}_F$ whose support is contained in $E$ and such that the morphism $\pi' : E' \rightarrow C$ induced by $\pi$ is dominant. Then $\pi'$ is a quasi-finite morphism of $F$-varieties. Hence, there exists a non-empty open subset $U$ of $C$ such that, putting $V = E' \times_C U$, the morphism $V \rightarrow U$ is finite and surjective. Surjectivity implies, in particular, that $V = E \cap \pi^{-1}(U)$. It follows from our assumptions that $V \rightarrow U$ is purely inseparable, so that it is a universal homeomorphism. \hfill $\square$

**Proposition 3.7.** Let $F$ be a field. There exists a unique ring morphism

$$\alpha : K_0(Var_F) \rightarrow K_0(ACF_F)$$

such that, for every quasi-affine $F$-variety $X$, the image of $[X]$ under $\alpha$ is the class in $K_0(ACF_F)$ of an $X$-formula $\varphi_X$. 
The morphism \( \alpha \) is surjective, and its kernel equals \( \Gamma_F^{uh} \), so that \( \alpha \) factors through an isomorphism

\[ K_0(\text{Var}_F) \to K_0(\text{ACF}_F) \]

**Proof.** Uniqueness if clear, since the classes of affine \( F \)-varieties generate the Grothendieck ring \( K_0(\text{Var}_F) \). So let us prove the existence of \( \alpha \).

We define the Grothendieck ring \( K_0(\text{QAff}_F) \) of quasi-affine \( F \)-varieties as follows. As an abelian group, \( K_0(\text{QAff}_F) \) is the quotient of the free abelian group on isomorphism classes \([X]'\) of quasi-affine \( F \)-varieties \( X \) by the subgroup generated by elements of the form

\[ [X]' - [Y]' - [X \setminus Y]' \]

with \( X \) a quasi-affine \( F \)-variety and \( Y \) a closed subvariety of \( X \). We endow \( K_0(\text{QAff}_F) \) with the unique ring structure such that for all quasi-affine \( F \)-varieties \( X \) and \( Y \), we have

\[ [X]' \cdot [Y]' = [(X \times_F Y)_{\text{red}}]' \]

It follows easily from the scissor relations in the Grothendieck ring that there is a unique morphism of abelian groups

\[ \beta : K_0(\text{QAff}_F) \to K_0(\text{Var}_F) \]

that maps \([X]'\) to \([X]\) for every quasi-affine \( F \)-variety \( X \), and that \( \beta \) is an isomorphism of rings. It is also straightforward to check that there exists a unique ring morphism

\[ \alpha' : K_0(\text{QAff}_F) \to K_0(\text{ACF}_F) \]

that maps \([X]'\) to \([\varphi_X]\) for every quasi-affine \( F \)-variety \( X \), where \( \varphi_X \) is an \( X \)-formula. We can define \( \alpha \) as \( \alpha' \circ \beta^{-1} \).

Now, we show that \( \ker(\alpha) \) contains the ideal \( \Gamma_F^{uh} \). Let \( f : X \to Y \) be a universal homeomorphism of \( F \)-varieties. We choose a partition \( \{Y_1, \ldots, Y_n\} \) of \( Y \) into locally closed subsets, and we endow \( Y_i \) with its reduced induced structure, for each \( i \). We may assume that \( Y_i \) is affine for every \( i \). If we put \( X_i = (X \times_Y Y_i)_{\text{red}} \), then the morphism \( X_i \to Y_i \) is a universal homeomorphism, \( X_i \) is affine, and we have

\[ [X] = \sum_{i=1}^r [X_i] \]

\[ [Y] = \sum_{i=1}^r [Y_i] \]

in \( K_0(\text{Var}_F) \). So it is enough to prove that \( \alpha([X] - [Y]) = 0 \) if \( f : X \to Y \) is a universal homeomorphism of affine \( F \)-varieties. We denote by \( \Gamma_f \) the graph of \( f \) in \( X \times_F Y \), and we choose closed immersions \( i_X : X \to \mathbb{A}_F^n \) and \( i_Y : Y \to \mathbb{A}_F^n \), with \( m, n \in \mathbb{N} \). These induce a closed immersion \( i_f : \Gamma_f \to \mathbb{A}_F^{m+n} \). If we choose an \( i_X \)-formula \( \varphi_X \), an \( i_Y \)-formula \( \varphi_Y \) and an \( i_f \)-formula \( \varphi_f \), then \( \varphi_f \) defines an \( \text{ACF}_F \)-equivalence between \( \varphi_X \) and \( \varphi_Y \), so that \( \alpha([X]) = \alpha([Y]) \).

Hence, the morphism \( \alpha \) factors through a ring morphism

\[ \gamma : K_0(\text{Var}_F) \to K_0(\text{ACF}_F) \]

We'll show that it is an isomorphism by constructing its inverse. If \( \varphi(x_1, \ldots, x_m) \) is a formula in \( L_F \), then by quantifier elimination relative to \( \text{ACF}_F \), there exists a unique constructible subset \( C_\varphi \) of \( \mathbb{A}^m_L \) such that, for every algebraically closed field \( L \) that contains \( F \), the subsets \( S_\varphi(L) \) and \( C_\varphi(L) \) of \( L^m \) coincide. It follows from
Lemma 3.6 that the class \( \langle C_\varphi \rangle \) of \( C_\varphi \) in \( K_0(\text{Var}_F) \) only depends on the \( ACF_F \)-equivalence class of \( \varphi \). It is easily seen that there exists a unique ring morphism

\[ \delta : K_0(ACF_F) \to K_0(\text{Var}_F) \]

that maps \( [\varphi] \) to \( [C_\varphi] \) for every formula \( \varphi \) in \( L_F \). This ring morphism is inverse to \( \gamma \). \( \square \)

4. Compatibility with realization morphisms

In this section, we’ll prove that the realization morphisms from Section 2 factor through the modified Grothendieck ring.

**Proposition 4.1.** Let \( S \) be a Noetherian scheme, and let \( X \) and \( Y \) be separated \( S \)-schemes of finite type such that there exists a universal homeomorphism of \( S \)-schemes

\[ f : X \to Y \]

We denote by \( g_X \) and \( g_Y \) the structural morphisms from \( X \), resp. \( Y \), to \( S \).

1. If \( S \) is the spectrum of a finite field \( F \), then \( X(F) \) and \( Y(F) \) have the same cardinality. In particular, the point counting realization

\[ \sharp : K_0(\text{Var}_F) \to \mathbb{Z} \]

factors through a ring morphism

\[ \sharp : K_0(\text{Var}_F) \to \mathbb{Z} \]

2. Let \( \ell \) be a prime invertible on \( S \). The natural morphism

\[ R(g_Y)_!(\mathbb{Q}_\ell) \to R(g_X)_!(\mathbb{Q}_\ell) \]

in \( D^b_c(S, \mathbb{Q}_\ell) \) induced by \( f \) is an isomorphism. In particular, the étale realization

\[ \text{ét} : K_0(\text{Var}_S) \to K_0(D^b_c(S, \mathbb{Q}_\ell)) \]

factors through a ring morphism

\[ \text{ét} : K_0(\text{Var}_S) \to K_0(D^b_c(S, \mathbb{Q}_\ell)) \]

which localizes to a ring morphism

\[ \text{ét} : \mathcal{M}_{S}^{mod} \to K_0(D^b_c(S, \mathbb{Q}_\ell)) \]

3. If \( F \) is a field, and \( \ell \) a prime number invertible in \( F \), then the Galois realization

\[ \text{Gal} : K_0(\text{Var}_F) \to K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell) \]

factors through a ring morphism

\[ \text{Gal} : K_0(\text{Var}_F) \to K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell) \]

which localizes to a ring morphism

\[ \text{Gal} : \mathcal{M}_{F}^{mod} \to K_0(\text{Rep}_{G_F} \mathbb{Q}_\ell) \]

4. The Poincaré polynomials \( P(g_X) : S \to \mathbb{Z}[T] \) and \( P(g_Y) : S \to \mathbb{Z}[T] \) are equal. In particular, the Poincaré realization

\[ P_S : K_0(\text{Var}_S) \to \mathcal{C}(S, \mathbb{Z}[T]) \]

factors through a ring morphism

\[ P_S : K_0(\text{Var}_S) \to \mathcal{C}(S, \mathbb{Z}[T]) \]
which localizes to a ring morphism

\[ P_S : \mathcal{M}_S^{\text{mod}} \to C(S, \mathbb{Z}[T, T^{-1}]) \]

**Proof.** (1) Since \( f \) is a universal homeomorphism and \( F \) is perfect, the map

\[ f(F) : X(F) \to Y(F) \]

is a bijection.

(2) By Grothendieck’s spectral sequence for the composition of the functors \( f_! = f_* \) and \((g_Y)_!\), it suffices to show that

\[
\begin{align*}
(f_\ast \mathbb{Q}_\ell) & \cong \mathbb{Q}_\ell, \\
R^j f_\ast (\mathbb{Q}_\ell) & \equiv 0 \text{ for } j > 0.
\end{align*}
\]

The isomorphism (4.1) follows immediately from the fact that \( f \) is a universal homeomorphism. The equality (4.2) follows from finiteness of \( f \).

(3) This is a special case of point (2).

(4) By definition of the Poincaré polynomial \[7, 8.12\], it is enough to consider the case where \( S \) is the spectrum of a field \( F \) of characteristic \( p > 0 \). By \[2, 8.8.2 \text{ and } 8.10.5\], there exist a finitely generated sub-\( \mathbb{F}_p \)-algebra \( A \) of \( F \), and a universal homeomorphism \( f' : X' \to Y' \) of separated \( A \)-schemes of finite type, such that \( f : X \to Y \) is obtained from \( f' \) by base change from Spec \( A \) to \( S = \text{Spec} F \). We denote by \( g_{X'} \) and \( g_{Y'} \) the structural morphisms from \( X' \) and \( Y' \) to Spec \( A \).

If we denote by \( \eta \) the generic point of Spec \( A \) and by \( \kappa(\eta) \) its residue field, then the Poincaré polynomial \( P(g_{X'}) \), resp. \( P(g_{Y'}) \), is equal to the Poincaré polynomial of the separated \( \kappa(\eta) \)-scheme of finite type \( X' \times_A \kappa(\eta) \), resp. \( Y' \times_A \kappa(\eta) \) \[7, 8.12\]. Since \( P(g_{X'}) \) and \( P(g_{Y'}) \) are constructible functions on Spec \( A \) \[7, 8.12\], it suffices to show that their values coincide at all closed points of Spec \( A \). Hence, we may assume that \( F \) is a finite field. In this case, the result follows immediately from point (2), by definition of the Poincaré polynomial of a variety over a finite field \[7, 8.1\]. \( \square \)

**Corollary 4.2.** If \( F \) is a field, then the étale Euler characteristic

\[ \chi_{\text{top}} : K_0(\text{Var}_F) \to \mathbb{Z} \]

factors through a ring morphism

\[ \chi_{\text{top}} : K_0(\text{Var}_F) \to \mathbb{Z} \]

which localizes to a ring morphism

\[ \chi_{\text{top}} : \mathcal{M}_F^{\text{mod}} \to \mathbb{Z} \]

5. Motivic integration in mixed characteristic

In this section, we will fill a gap in the proof of the change of variables theorem for motivic integrals on formal schemes in mixed characteristic \[11, 8.0.5\]. To this aim, it is necessary to replace the localized Grothendieck ring of varieties by the modified version introduced in Definition \[3.2\]. We emphasize that this correction only affects the theory of motivic integration in mixed characteristic; in equal characteristic \( p \geq 0 \), the results in \[11, 7.1.3 \text{ and } 8.0.5\] are valid as stated.
schemes. Indeed, only when $F$ is a perfect field containing $k$ and $\text{Spec } F \rightarrow Gr_n(X)$ a morphism of $k$-schemes. We only compute the fibers $Gr_n(Y) \times_{Gr_n(X)} \text{Spec } F$

when $F$ is a perfect field containing $k$ and $\text{Spec } F \rightarrow Gr_n(X)$ a morphism of $k$-schemes. Indeed, only when $F$ is perfect, we can identify the set $Gr(X)(F)$ with the set $X(R_F)$, where

$$R_F = R \otimes_{W(k)} W(F)$$

is a complete discrete valuation ring that is an unramified extension of $R$. Then the proof of \cite{11} 7.1.3 shows that there exists an isomorphism of $F$-schemes

$$Gr_n(Y) \times_{Gr_n(X)} \text{Spec } F \cong k_F^e$$

so that we can deduce from Lemma 5.1 that

$$\langle \pi_{n,Y}(B) \rangle = \langle \pi_{n,X}(A) \rangle \bar{L}_k^e$$

in $K_0(Var_k)$. However, we cannot conclude that the equality in point (2) of Lemma 5.1 holds. Therefore, we have to replace Lemma 5.1 by the following statement.

**Lemma 5.2 (Corrected form).** Assume that the structural morphism $Y \rightarrow \mathbb{D}$ is smooth. Let $B \subset Gr(Y)$ be an $m$-cylinder, for some $m \in \mathbb{N}$, and put $A = h(B)$. Assume that $e$ and $e'$ are elements of $\mathbb{N}$ such that $\text{ord}_n(\text{Jac})(\varphi) = e$ for all $\varphi \in B$, and $A \subset Gr^{(e')}_{X}(X)$. Then $A$ is a cylinder.

Moreover, if the restriction of $h$ to $B$ is injective, then, for all integers $n \geq \max(2e + c_X, m + e)$, the following properties hold.

(1) If $\varphi$ and $\varphi'$ belong to $B$ and $\pi_{n,X}(h(\varphi)) = \pi_{n,X}(h(\varphi'))$, then $\pi_{n-e,Y}(\varphi) = \pi_{n-e,Y}(\varphi')$,

(2) We have $\langle \pi_{n,Y}(B) \rangle = \langle \pi_{n,X}(A) \rangle \bar{L}_k^e$ in $K_0(Var_k)$.
The statements in [11, 8.0.3] and [11, 8.0.5] (change of variables theorem) have to be modified accordingly, replacing the completed Grothendieck ring $\hat{\mathcal{M}}_k$ by the completed modified Grothendieck ring $\hat{\mathcal{M}}^\text{mod}_k$.

We emphasize that we do not know a counter-example to the original statement in Lemma 5.1 since we do not know if the projection morphism

$$K_0(\text{Var}_k) \to K_0(\text{Var}_k)$$

is an isomorphism.

5.2. Integration on rigid varieties and the motivic Serre invariant. If $R$ has mixed characteristic, the modification of the change of variables theorem in [11, 8.0.5] also affects the theory of motivic integration on rigid $K$-varieties developed in [5]. Theorem-Definition 4.1.2 in [5] should be replaced by the following statement (only in the mixed characteristic case; in equal characteristic $p \geq 0$, all the results in [5] are valid as stated). For terminology and notations, we refer to [5].

Theorem-Definition 5.3. Let $X$ be a smooth separated quasi-compact rigid variety over $K$, of pure dimension $d$. Let $\omega$ be a differential form in $\Omega^d_X(\mathcal{X})$.

(1) Let $\mathcal{X}$ be a stft formal $R$-model of $X$. Then the function $\text{ord}_{\mathcal{X}}(\omega)$ is exponentially integrable on $\text{Gr}(\mathcal{X})$ and the image of the motivic integral

$$\int_{\text{Gr}(\mathcal{X})} L^{-\text{ord}_{\mathcal{X}}(\omega)} d\mu$$

in $\hat{\mathcal{M}}^\text{mod}_k$ does not depend on the model $\mathcal{X}$. We denote it by $\int_X \omega d\mu$.

(2) Assume moreover that $\omega$ is a gauge form, i.e., that it generates $\Omega^d_{\mathcal{X}}$ at every point of $X$, and assume that some open formal subscheme $\mathcal{U}$ of $X$ is a weak Néron model of $X$. Then the function $\text{ord}_{\mathcal{X}}(\omega)$ takes only a finite number of values on $\text{Gr}(\mathcal{X})$, and its fibres are stable cylinders. The image of the motivic integral

$$\int_{\text{Gr}(\mathcal{X})} L^{-\text{ord}_{\mathcal{X}}(\omega)} d\tilde{\mu}$$

in $\hat{\mathcal{M}}^\text{mod}_k$ does not depend on the model $\mathcal{X}$. We denote it by $\int_X \omega d\tilde{\mu}$.

If $R$ has mixed characteristic, the definition of the motivic Serre invariant [5, 4.5.1 and 4.5.3] has to be modified accordingly.

Theorem-Definition 5.4. Let $X$ be a smooth separated quasi-compact rigid $K$-variety, and let $\mathcal{U}$ be a weak Néron model of $X$. The class

$$\langle \mathcal{U}_s \rangle \in K_0(\text{Var}_k)/\langle L_k - 1 \rangle$$

of the special fiber $\mathcal{U}_s$ of $\mathcal{U}$ only depends on $X$, and not on the choice of weak Néron model. We denote it by $S(X)$, and we call it the motivic Serre invariant of $X$.

5.3. Further corrections to the literature. The theory of motivic integration on formal schemes and rigid varieties in mixed characteristic has been applied in several other articles. All of them can easily be corrected, replacing the Grothendieck ring of varieties by the modified Grothendieck ring of varieties. This is harmless for the applications of the theory, since all the realization morphisms that are used factor through the modified Grothendieck ring (see Section 4). Let us
indicate some of the changes that should be made (only in the mixed characteristic case; in equal characteristic \( p \geq 0 \), all the results are valid as stated).

In \([8, 4.18]\), the last two lines of the statement should be replaced by:

\[ \ldots \langle \pi_n(B) \rangle = \langle \pi_n(A) \rangle \tilde{L}_X \text{ in } K_0(\text{Var}_{X_s}). \]

In \([8, 4.19]\), the ring \( \mathcal{M}_{X_s} \) should be replaced by \( \mathcal{M}_{X_s}^{\text{mod}} \), and in \([8, 4.20(2)]\), the ring \( \mathcal{M}_{X_s} \) should be replaced by \( \mathcal{M}_{X_s}^{\text{mod}} \). Likewise, in \([8, 6.2]\), the Grothendieck rings have to be replaced by their modified analogues. In particular, the motivic Serre invariant of a generically smooth \( \text{stft} \) formal \( R \)-scheme \( X_\infty \) of pure relative dimension \([8, 6.2]\) is well-defined in \( K_0(\text{Var}_{X_s})/(\tilde{L}_X - \langle X_s \rangle) \).

In \([9, 3.2]\), the various Grothendieck rings should be replaced by the modified Grothendieck rings. The trace formula in \([9, 5.4]\) remains valid, because the \( \ell \)-adic Euler characteristic factors through the modified Grothendieck ring (Corollary 4.2).

In \([10, 5]\), the various Grothendieck rings should be replaced by the modified Grothendieck rings.

In Sections 4 and 5.3 of \([6]\), the various Grothendieck rings should be replaced by the modified Grothendieck rings. The trace formula in \([6, 6.4]\) remains valid, because the \( \ell \)-adic Euler characteristic factors through the modified Grothendieck ring (Corollary 4.2).

In \([7, 5]\), in particular in Theorem 5.4 and Definition 5.5, the motivic Serre invariant should take its values in \( K_0(\text{Var}_{k})/(\tilde{L}_k - 1) \). The results in Sections 6 and 7 of \([7]\) remain valid, because the Poincaré polynomial and the \( \ell \)-adic Euler characteristic factor through the modified Grothendieck ring (Proposition 4.1(4) and Corollary 4.2).

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