Error Correction and Symmetrization in Quantum Computers

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Abstract

Errors in quantum computers are of two kinds: sudden perturbations to isolated qubits, and slow random drifts of all the qubits. The latter may be reduced, but not eliminated, by means of symmetrization, namely by using many replicas of the computer, and forcing their joint quantum state to be completely symmetric. On the other hand, isolated errors can be corrected by quantum codewords that represent a logical qubit in a redundant way, by several physical qubits. If one of the physical qubits is perturbed, for example if it gets entangled with an unknown environment, there still is enough information encoded in the other physical qubits to restore the logical qubit, and disentangle it from the environment. The recovery procedure may consist of unitary operations, without the need of actually identifying the error.

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1. Introduction

A computer is a physical system, subject to the ordinary laws of nature. No error ever occurs in the application of these laws. What we call an error is a mismatch between what the computer does and what we wanted it to do. This may be caused by incorrect programming (software errors, that I shall not consider here), or by imperfect hardware. The computer engineer’s problem is to design the hardware in such a way that common flaws, which are unavoidable, will almost never cause errors in the final output (namely, in the relevant parts of the final state of the computer).

In classical computers, logical bits, having values 0 or 1, are implemented in a highly redundant way by bistable elements, such as magnetic domains. The bistability is enforced by coupling the physical bits to a dissipative environment. Errors may then occur, because of thermal fluctuations and other hardware imperfections. To take care of these errors, various correction methods have been developed [1], involving the use of redundant bits (that are implemented by additional bistable elements).

In quantum computers, the situation is more complicated: in spite of their name, the logical “qubits” (quantum binary digits) are not restricted to the discrete values 0 and 1. Their value can be represented by any point on the surface of a Poincaré sphere. Moreover, any set of qubits can be in an entangled state: none of the individual qubits has a pure quantum state, it is only the state of all the qubits together that is pure [2]. The continuous nature of qubit states implies that there can be no intrinsic stabilizing mechanism, and error control becomes critical.

Here, a distinction must be made between quantum computers of the Benioff type [3, 4], where quantum hardware is used for implementing classical logic, and computers that are fundamentally quantal [5], and can do more than just mimicking classical computation. In the former case, there are instants of time at which all the qubits ought to represent definite values, 0 or 1. They are not then in a quantum superposition, and error correction can be done as for a classical computer [6]. On the other hand, in a computer of the Deutsch type [5], the quantum state of the computer typically is an entangled state of all the qubits, and classical methods of error correction are not applicable. What can be done then depends on the nature of the anticipated errors.

In general, we may write the Hamiltonian of the computer as $H = H_0 + H_1$, where $H_0$ is the Hamiltonian of an ideal error free computer, and $H_1$ represents the influence of the
environment. The latter is unknown to the computer designer, except statistically. That Hamiltonian acts on a Hilbert space which is the tensor product of those representing the computer and the environment. The designer’s problem is to distill, from the computer’s variables, a subset giving with probability close to 1 the correct result of the computation, irrespective of the unknown form of $H_1$ and of the initial state state of the environment.

Two different types of errors ought to be considered: accidental large disturbances to isolated qubits (e.g., a residual gas molecule may hit one of them), and small, random, uncorrelated drifts of all the qubits.

The first type of error can be corrected by using codewords, as first shown by Shor [7]. A codeword is a representation of a logical qubit by means of several physical qubits. There were 9 qubits in Shor’s codewords. It is now known that the minimal number is 5. In particular, Bennett et al. have constructed 5-qubit codewords that have the remarkable property of being invariant under a cyclic permutation of the qubits [8]. In all these quantum codewords, the physical qubits are in a highly entangled state, chosen in such a way that, if any one of the qubits gets entangled with an unknown environment, there still is enough information stored in the other qubits to restore the codeword and to unitarily disentangle it from the environment, irrespective of the unknown state of the latter.

The second type of error, continuous random drifts of all the qubits, cannot be eliminated by using codewords, but can be reduced by symmetrizing the joint quantum state of several identical computers [9, 10]. This symmetrization method and some of its variants are discussed in Sect. 2. The encoding of logical qubits into codewords consisting of several physical qubits is explained in Sect. 3. How to actually restore the initial state of a corrupted codeword is shown in Sect. 4. Finally, Sect. 5 is devoted to a fundamental issue: error correction without detection of the error syndromes.

2. Symmetrization

In its original version [9, 10], the symmetrization method involved the use of $R$ identical replicas of the entire computer. At preset times, the joint quantum state of the $R$ computers is projected onto the symmetric subspace of their common Hilbert space (for example, by measuring whether or not the state is symmetric, and aborting the com-
putation if the answer is negative). As shown below, if small errors randomly affect all the qubits, this projection reduces the average error by a factor $R$. On the other hand, symmetrization gives poor results if a single qubit goes completely astray: we then have a non-symmetric state that is almost orthogonal to the symmetric subspace, and the computation is almost always aborted. Indeed, if one of the computers has a state orthogonal to that of all the others, the probability is only $1/R$ that the joint state will be projected onto the symmetric subspace, and in that case, the error is not eliminated, but rather uniformly spread over all the $R$ computers! Obviously, large isolated errors cannot be handled in this way. Their correction requires the use of specially designed codewords.

There is however a more efficient protocol for error correction by symmetrization. The $R$ computers can be arranged in pairs, and each one of the $R/2$ pairs is symmetrized separately. The process can then be repeated with different pairing arrangements, if we wish to further improve the symmetry. With such a pairwise symmetrization, if a computer accidentally gets into a state orthogonal to that of all the other ones, there is a 50% chance that the pair containing the bad computer will be eliminated, and a 50% chance that the error will be equally shared by the two computers. Repeating this process many times, so that each computer has many partners, ultimately leads to the elimination of a bad computer, together with one good one. There still are $R-2$ good computers available for continuing the work.

A more complicated (and probably more realistic) model would be to assume that any computer may occasionally fail when one of the logical steps is executed. This event must be rare enough so that the total probability of failure of any given computer during the entire computation is less than $\frac{1}{2}$. Pairwise symmetrizations are performed between any two logical steps (the pairs are chosen in such a way that each computer is compared with many other ones during the complete computation). Most errors are then eliminated, and the surviving computers contain, on the average, less than one defective result. In this theoretical model, an “error” means a state that is orthogonal to the correct one. This has to be generalized to the case of less radical errors. It is plausible that repeated pairwise symmetrizations are in general preferable to a single overall symmetrization, but a formal proof is still needed.

Clearly, the poor efficiency of the symmetrization method, in the case of large isolated errors, is due to possible failures of the symmetry tests (also known as “quantum measure-
ments”). When a test fails, we must discard a pair of computers, if not the entire process. However, there is no need of measuring anything in order to force a quantum state to stay in a symmetric subspace. A measurement is not a supernatural event. It is an ordinary dynamical process, and any error correction that may result from it should also be obtainable as a consequence of a unitary evolution, governed by ordinary dynamical laws. Indeed, a much simpler method for enforcing symmetry of the quantum state is to impose on the $R$ computers an extra static potential that vanishes in the symmetric subspace, and has a very large value in all the orthogonal (asymmetric) states. Effectively, in the $R$ computers, any $R$ homologous physical qubits behave as if they were $R$ bosons. Likewise, if the qubits of a codeword have an internal symmetry, such as the cyclic symmetry of the codewords in ref. [8], we may protect their cyclic subspace by erecting around it a high potential barrier.

The result of such a symmetrizing potential is analogous to a continuous Zeno effect (ref. [2], pp. 392–400). To test its effectiveness, consider the simple example of two computer memories, each one consisting of a single qubit, initially in the state $\left(\frac{\alpha}{\beta}\right)$, which is unknown. We want these computer memories to be stable: there should be no evolution of the two qubits. The problem is to protect them against random fluctuations of the environment. Let us use for this discussion the terminology and notations appropriate to spin-$\frac{1}{2}$ particles. A symmetric state of the pair belongs to the triplet ($J = 1$) representation, while the singlet ($J = 0$) is antisymmetric.

Consider the Hamiltonian

$$H_0 = (1 - J^2/2) \Omega,$$  

where $\Omega$ is a large positive constant. Since $J^2 = J(J + 1)$, this potential vanishes in the triplet state, and is equal to $\Omega$ for a singlet. As a simple model of perturbation, let a phase error be generated by

$$H_1 = \mu \sigma_{A_z} + \nu \sigma_{B_z},$$

where $\mu$ and $\nu$ are constant coefficients much smaller than $\Omega$, and the subscripts $A$ and $B$ refer to the two qubits. This can also be written as

$$H_1 = \epsilon (\sigma_{A_z} + \sigma_{B_z}) + \eta (\sigma_{A_z} - \sigma_{B_z}),$$
where \( \epsilon = (\mu + \nu)/2 \) and \( \eta = (\mu - \nu)/2 \). The \( \epsilon \) term in \( H_1 \) is symmetric, it commutes with \( H_0 \), and therefore this kind of perturbation cannot be eliminated by symmetrization. Indeed, the evolution of the qubit state \( (\alpha \beta) \) is given (if we ignore the \( \eta \) term, for simplicity) by \( \alpha(t) = \alpha(0) e^{-i \epsilon t} \) and \( \beta(t) = \beta(0) e^{i \epsilon t} \). If there were \( R \) qubits, instead of just two, the symmetric part of the perturbation (which cannot be eliminated by symmetrization) would have as its coefficient the arithmetic average of the individual perturbations. If the latter are random and independent, that average is expected to be smaller than the r.m.s. perturbation by a factor \( \sqrt{R} \), and therefore the error probability is reduced by a factor \( R \). No further improvement can be expected.

On the other hand, the error due to the antisymmetric part of \( H_1 \) can be considerably reduced. Written with the Bell basis [11], the initial state of the pair is

\[
\left( \frac{\alpha}{\beta} \right) \otimes \left( \frac{\alpha}{\beta} \right) = \frac{\alpha^2 + \beta^2}{\sqrt{2}} \Phi^+ + \frac{\alpha^2 - \beta^2}{\sqrt{2}} \Phi^- + \sqrt{2} \alpha \beta \Psi^+,
\]

where \( \Phi^+ \), \( \Phi^- \), and \( \Psi^+ \) are the triplet states corresponding to \( J_x = 0 \), \( J_y = 0 \), and \( J_z = 0 \), respectively. The antisymmetric part of the perturbation has matrix elements given by

\[
(\sigma_{Az} - \sigma_{Bz}) \Phi^\pm = 0,
\]

and

\[
(\sigma_{Az} - \sigma_{Bz}) \Psi^\pm = \Psi^\mp,
\]

where \( \Psi^- \) is the singlet state. The nontrivial part of the Hamiltonian thus involves only the \( \Psi^\pm \) subspace. We can write (ignoring for simplicity the \( \epsilon \) contribution, which is symmetric)

\[
H = H_0 + H_1 = \begin{pmatrix}
0 & \eta \\
\eta & \Omega
\end{pmatrix}.
\]

It is easy to find the eigenvalues and eigenvectors of this Hamiltonian. The initial state (4) can be written as a linear combination of these two eigenstates, and its time evolution obtained explicitly: the \( \Phi^\pm \) terms have constant amplitudes, and, for \( \eta \ll \Omega \), the \( \Psi^+ \) term in (4) evolves as

\[
\Psi^+ \to e^{i \eta^2 t / \Omega} \Psi^+ + (\eta / \Omega) (e^{-i \Omega t} - 1) \Psi^-,
\]

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where terms of order \((\eta / \Omega)^2\) have been neglected. If we could make the potential energy \(\Omega\) arbitrarily large (as we do in an ideal “quantum measurement” context, where the interaction with the measuring apparatus is assumed arbitrarily strong), then the \(\Psi^+\) term in the state vector would be perfectly stabilized, and the \(\Psi^-\) term (which is antisymmetric) would never appear. For large but finite \(\Omega\), the amplitude of the \(\Psi^-\) term, initially zero, always remains small. On the other hand, the \(\Psi^+\) term undergoes a slow secular drift, which definitely is an error, but is nevertheless compatible with the symmetry constraint. The same kind of drift also occurs for repeated discrete symmetrization \([9]\), because the symmetric state obtained at each step may contain a small residual error, and these errors gradually accumulate.

These considerations can now be generalized from 2 to \(R\) computers, each one having many qubits. It may seem that a global potential is required, involving all of them at once, a proposal that would be a technological nightmare. Fortunately, this is not necessary: it is enough to take \(R(R - 1)/2\) identical potentials, one for each pair of computers. If any two computers are in a symmetric state, then all \(R\) computers are in a symmetric state, by definition. However, the comparison of two computers cannot be done bitwise: the states of the complete computers have to be compared. How to actually do that by means of a coherent sequence of two-qubit interactions requires a complicated protocol \([10]\), beyond the scope of this review.

3. **Encoding and decoding**

Let us now turn our attention to the case of large errors, occurring in a few, isolated qubits. The latter are materialized by single quanta, such as trapped ions \([12]\). Their coupling to a dissipative environment (which was the standard stabilizing mechanism for classical bits) is to be avoided as much as possible, because it readily leads to decoherence, namely to the loss of phase relationships. Yet, disturbances due to the environment cannot be completely eliminated: e.g., even if there are no residual gas molecules in the vacuum of an ion trap, there still are the vacuum fluctuations of the quantized electromagnetic field, which induce spontaneous transitions between the energy levels of the ions. Therefore, error control is an essential part of any quantum communication or computing system.

This goal is much more difficult to achieve than classical error correction, because
Qubits cannot be read, or copied, or duplicated, without altering their quantum state in an unpredictable way [13]. The feasibility of quantum error correction, which for some time had been in doubt, was first demonstrated by Shor [7]. As in the classical case, redundancy is an essential element, but this cannot be a simple repetitive redundancy, where each bit has several identical replicas and a majority vote is taken to establish the truth. This is because qubits, contrary to ordinary classical bits, can be entangled, and usually they are. As a trivial example, in the singlet state of two spin-$\frac{1}{2}$ particles, each particle, taken separately, is in a completely random state. Therefore, comparing the states of spin-$\frac{1}{2}$ particles that belong to different (redundant) singlets would give no information whatsoever.

All quantum error correction methods [7, 8, 14–17] use several physical qubits for representing a smaller number of logical qubits (usually a single one). These physical qubits are prepared in a carefully chosen, highly entangled state. None of these qubits, taken alone, carries any information. However, a large enough subset of them may contain a sufficient amount of information, encoded in relative phases, for determining and exactly restoring the state of the logical qubits, including their entanglement with the other logical qubits in the quantum computer.

I shall now review the quantum mechanical principles that make error correction possible. (I shall not discuss how to actually design new codewords; the most efficient techniques involve a combination of classical coding theory and of the theory of finite groups [17].) Note that, since quantum codewords span only a restricted subspace of the complete physical Hilbert space, the unitary operations that generate the quantum dynamical evolution (that is, the computational process) are subject to considerable arbitrariness. The latter is similar to the gauge freedom in quantum field theory. Quantum codewords can thus serve as a simple toy model for investigating the quantization of constrained dynamical systems, such as field theories with gauge groups [18].

In the following, I shall usually consider codewords that represent a single logical qubit. It is also possible, and it is more efficient, to encode several qubits into larger codewords. Ideally, it would be best to encode the entire computer into one super-codeword (but then, how to program the evolution of that super-codeword would be a very difficult problem, that might be solvable only by another quantum computer!). However, no new physical principles are involved in the simultaneous encoding of several qubits, and the simple case
of a single qubit is sufficient for illustrating these principles.

The quantum state of a single logical qubit will be denoted as

$$\psi = \alpha |0\rangle + \beta |1\rangle,$$

where the coefficients $\alpha$ and $\beta$ are complex numbers. The symbols $|0\rangle$ and $|1\rangle$ represent any two orthogonal quantum states, such as “up” and “down” for a spin, or the ground state and an excited state of a trapped ion. In a quantum computer, there are many logical qubits, typically in a collective, highly entangled state, and any particular qubit has no definite state. I shall still use the same symbol $\psi$ for representing the state of the entire computer, and Eq. (9) could now be written as

$$\psi = |\alpha\rangle \otimes |0\rangle + |\beta\rangle \otimes |1\rangle,$$

where one particular qubit has been singled out for the discussion, and the symbols $|\alpha\rangle$ and $|\beta\rangle$ represent the collective states of all the other qubits, that are correlated with $|0\rangle$ and $|1\rangle$, respectively. However, to simplify the notation and improve readability, I shall still write the computer state as in Eq. (9). In the following, Dirac’s ket notation will in general not be used for generic state vectors (such as $\psi$, $\alpha$, $\beta$) and the $\otimes$ sign will sometimes be omitted, when the meaning is clear. Kets will be used only for denoting basis vectors such as $|0\rangle$ and $|1\rangle$, and their direct products. The latter will be labelled by binary numbers, such as

$$|9\rangle \equiv |01001\rangle \equiv |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle.$$

In order to encode the qubit $\psi$ in Eq. (9), we introduce an auxiliary system, called ancilla (this is the Latin word for housemaid). The ancilla is made of $n$ qubits, initially in a state $|000\ldots\rangle$. We shall use $2^n$ mutually orthogonal vectors $|a\rangle$, with $a = 0, 1, \ldots$ (written in binary notation), as a basis for the quantum states of the ancilla. The labels $a$ are called syndromes, because, as we shall see, the presence of an ancilla with $a \neq 0$ may serve to identify an error in the encoded system that represents $\psi$.

Encoding is a unitary transformation, $E$, performed on a physical qubit and its ancilla together:

$$|z\rangle \otimes |a = 0\rangle \rightarrow E \left(|z\rangle \otimes |a = 0\rangle\right) \equiv |Z_0\rangle,$$
where $z$ and $Z$ are either 0 or 1 (the index 0 in $|Z_0\rangle$ means that there is no error at this stage). The unitary transformation $E$ is executed by a quantum circuit (an array of quantum gates). However, from the theorist’s point of view, it is also convenient to consider $|z\rangle \otimes |a = 0\rangle$ and $|Z_0\rangle$ as two different representations of the same qubit $|z\rangle$: its logical representation, and its physical representation. The first one is convenient for discussing matters of principle, such as quantum algorithms, while the physical representation shows how qubits are actually materialized by distinct physical systems (which may be subject to independent errors). These two different representations are analogous to the use of normal modes vs. local coordinates for describing the small oscillations of a mechanical system [19]. One description is mathematically simple, the other one refers to directly accessible quantities.

4. Error correction

Since there are $2^n$ syndromes (including the null syndrome for no error), it is possible to identify and correct up to $2^n - 1$ different errors that affect the physical qubits, with the help of a suitable decoding method, as explained below. Let $|Z_a\rangle$, with $a = 0, \ldots, 2^n - 1$, be a complete set of orthonormal vectors describing the physical qubits of which the codewords are made: $|0_0\rangle$ and $|1_0\rangle$ are the two error free states that represent $|0\rangle$ and $|1\rangle$, and all the other $|0_{a}\rangle$ and $|1_{a}\rangle$ are the results of errors (affecting one physical qubit in the codeword, or several ones, this does not matter at this stage). These $|Z_a\rangle$ are defined in such a way that $|0_{a}\rangle$ and $|1_{a}\rangle$ result from definite errors in the same physical qubits of $|0_{0}\rangle$ and $|1_{0}\rangle$: for example, the third qubit is flipped, $\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \rightarrow \left(\begin{array}{c} \beta \\ \alpha \end{array}\right)$, and the seventh one has a phase error, $\left(\begin{array}{c} \alpha \\ \beta \end{array}\right) \rightarrow \left(\begin{array}{c} \alpha \\ -\beta \end{array}\right)$.

We thus have two complete orthonormal bases, $|z\rangle \otimes |a\rangle$ and $|Z_a\rangle$. These two bases uniquely define a unitary transformation $E$, such that

\[ E (|z\rangle \otimes |a\rangle) = |Z_a\rangle, \]  

and

\[ E^\dagger |Z_a\rangle = |z\rangle \otimes |a\rangle, \]  

\[ 10 \]
where \(a\) runs from 0 to \(2^n - 1\). Here, \(E\) is the encoding matrix, and \(E^\dagger\) is the decoding matrix. If the original and corrupted codewords are chosen in such a way that \(E\) is a real orthogonal matrix (not a complex unitary one), then \(E^\dagger\) is the transposed matrix, and therefore \(E\) and \(E^\dagger\) are implemented by the same quantum circuit, executed in two opposite directions. (If \(E\) is complex, the encoding and decoding circuits must also have opposite phase shifts.)

The \(2^n - 1\) standard errors \(|Z_0\rangle \rightarrow |Z_a\rangle\) are not the only ones that can be corrected by the \(E^\dagger\) decoding. Any error of type

\[
|Z_0\rangle \rightarrow U |Z_0\rangle = \sum_a c_a |Z_a\rangle,
\]

is also corrected, since

\[
E^\dagger \sum_a c_a |Z_a\rangle = |z\rangle \otimes \sum_a c_a |a\rangle,
\]

is a direct product of \(|z\rangle\) with the ancilla in some irrelevant corrupted state. Note that no knowledge of the syndrome is needed in order to correct the error \([6]\). Error correction is a logical operation that can be performed automatically, without having to execute quantum measurements. We definitely know that the error is corrected, even if we don’t know the nature of that error. I shall return to this issue in Sect. 5.

It is essential that the result on the right hand side of (16) be a direct product. Only if the new ancilla state is the same for \(|z\rangle = |0\rangle\) and \(|z\rangle = |1\rangle\), and therefore also for the complete computer state in Eq. (10), is it possible to coherently detach the ancilla from the rest of the computer, and replace it by a fresh ancilla (or restore it to its original state \(|a = 0\rangle\) by a dissipative process involving still another, extraneous, physical system). This means, in the graphical formalism of quantum circuits, that the “wires” corresponding to the old ancilla stop, and new “wires” enter into the circuit, with a standard quantum state for the new ancilla. There is some irony in this introduction of a dissipative process for stabilizing a quantum computer. The latter was originally conceived as an analog device with a continuous evolution, and it is now brought one step closer to a conventional digital computer!

There are many plausible scenarios for the emergence of coherent superpositions of corrupted states, as in (16). For example, in an ion trap, a residual gas molecule, whose wave function is spread over a domain much larger than the inter-ion spacing, can be
scattered by all the ions together, as by a diffraction grating, and then all the ions are left in a collective recoil state (namely, a coherent superposition of states where one of the ions recoiled and the other ones did not). Furthermore, mixtures of errors of type (16) are also corrigible. Indeed, if

$$\rho = \sum_j p_j \sum_{ab} c_{ja} |Z_a\rangle \langle Z_b| c_{jb}^*,$$

with $p_j > 0$ and $\sum p_j = 1$, then

$$E^\dagger \rho E = |z\rangle \langle z| \otimes \sum_j p_j \sum_{ab} c_{ja} |a\rangle \langle a| c_{jb}^*,$$

again is a direct product of the logical qubit and the corrupted ancilla in a mixed state.

In particular, these mixtures include the case where a physical qubit in the codeword gets entangled with an unknown environment, which is the typical source of error. Let $\eta$ be the initial, unknown state of the environment, and let its interaction with a physical qubit cause the following unitary evolution:

$$|0\rangle \otimes \eta \rightarrow |0\rangle \otimes \mu + |1\rangle \otimes \nu,$$

$$|1\rangle \otimes \eta \rightarrow |0\rangle \otimes \sigma + |1\rangle \otimes \tau,$$

where the new environment states $\mu$, $\nu$, $\sigma$, and $\tau$, are also unknown, except for unitarity constraints. Now assume that the physical qubit, which has become entangled with the environment in such a way, was originally part of a codeword,

$$|Z_0\rangle = |X_{Z0}\rangle \otimes |0\rangle + |X_{Z1}\rangle \otimes |1\rangle,$$

where the index $Z$ means 0 or 1. (The index 0 may also refer to the error free state of a codeword. The interpretation of a subscript 0 should be obvious from the context.) The codeword $|Z_0\rangle$, together with its environment, thus evolves as

$$Z_0 \otimes \eta \rightarrow Z' = X_{Z0} \otimes \left( |0\rangle \otimes \mu + |1\rangle \otimes \nu \right) + X_{Z1} \otimes \left( |0\rangle \otimes \sigma + |1\rangle \otimes \tau \right),$$

where I have omitted most of the ket signs, for brevity. This can also be written as

$$Z' = \left[ X_{Z0} \otimes |0\rangle + X_{Z1} \otimes |1\rangle \right] \frac{\mu + \tau}{2} + \left[ X_{Z0} \otimes |0\rangle - X_{Z1} \otimes |1\rangle \right] \frac{\mu - \tau}{2} + \left[ X_{Z0} \otimes |1\rangle + X_{Z1} \otimes |0\rangle \right] \frac{\nu + \sigma}{2} + \left[ X_{Z0} \otimes |1\rangle - X_{Z1} \otimes |0\rangle \right] \frac{\nu - \sigma}{2}.$$
On the right hand side, the vectors

\begin{align}
Z_0 & := XZ_0 \otimes |0\rangle + XZ_1 \otimes |1\rangle, \\
Z_r & := XZ_0 \otimes |0\rangle - XZ_1 \otimes |1\rangle, \\
Z_s & := XZ_0 \otimes |1\rangle + XZ_1 \otimes |0\rangle, \\
Z_t & := XZ_0 \otimes |1\rangle - XZ_1 \otimes |0\rangle,
\end{align}

(23)
correspond, respectively, to a correct codeword, to a phase error (|1\rangle \rightarrow -|1\rangle), a bit error (|0\rangle \leftrightarrow |1\rangle), which is the only classical type of error, and to a combined phase and bit error. If these three types of errors can be corrected, we can also correct any type of entanglement with the environment, as we shall soon see.

For this to be possible, it is sufficient that the eight vectors in Eq. (23) be mutually orthogonal. The simplest way of achieving this orthogonality is to construct the codewords |0_0\rangle and |1_0\rangle in such a way that the following scalar products hold:

\begin{align}
\langle X_{Zy}, X_{Z'y'} \rangle = \frac{1}{2} \delta_{ZZ'} \delta_{yy'}.
\end{align}

(24)
(There are 10 such scalar products, since each index in this equation may take the values 0 and 1.) If these conditions are satisfied, the decoding of \( Z' \) by \( E^\dagger \) gives, by virtue of Eq. (14),

\begin{align}
E^\dagger Z' = |z\rangle \otimes \left(|a = 0\rangle \otimes \frac{\mu + \tau}{2} + |r\rangle \otimes \frac{\mu - \tau}{2} + |s\rangle \otimes \frac{\nu + \sigma}{2} + |t\rangle \otimes \frac{\nu - \sigma}{2}\right),
\end{align}

(25)
where \(|r\rangle\), \(|s\rangle\), and \(|t\rangle\) are various corrupted states of the ancilla. The expression in parentheses is an entangled state of the ancilla and the unknown environment. We cannot know it explicitly, but this is not necessary: it is sufficient to know that it is the same state for \(|z\rangle = |0\rangle\) and \(|z\rangle = |1\rangle\), and any linear combination thereof, as in Eq. (14). We merely have to discard the old ancilla and bring in a new one.

This is how codewords fight entanglement with entanglement. How to actually construct codewords that satisfy Eq. (24), when any one of their physical qubits is singled out, is a difficult problem, best handled by a combination of classical codeword theory and finite group theory [17]. I shall not enter into this subject here. I only mention that in order to correct an arbitrary error in any one of its qubits, a codeword must have at least five qubits: each one contributes three distinct vectors, like \( Z_r, Z_s, \) and \( Z_t \) in Eq. (23), and these, together with the error free vector \( Z_0 \), make 16 vectors for each logical qubit.
value, and therefore $32 = 2^5$ in the total. Longer codewords can correct more than one erroneous qubit. For example, Steane’s linear code [15], with 7 qubits, can correct not only any error in a single physical qubit, but also a phase error, $|1\rangle \rightarrow -|1\rangle$, in one of them, and a bit error, $|0\rangle \leftrightarrow |1\rangle$, in another one (check! $1 + 7 \times 3 + 7 \times 6 = 2^7 - 1$). A well designed codeword is one where the orthogonal basis $|Z_a\rangle$ corresponds to the most plausible physical sources of errors, e.g., single bit errors, rather than complicated types of errors involving several qubits in a coherent way.

The error correction method proposed above, in Eq. (14), is conceptually simple, but it has the disadvantage of leaving the logical qubit $|z\rangle$ in a “bare” state, vulnerable to new errors that would be not be detected. It is therefore necessary to re-encode that qubit immediately, with another ancilla (or with the same ancilla, reset to $|a = 0\rangle$ by interaction with still another system). A more complicated but safer method is to bring in a second ancilla, in a standard state $|b = 0\rangle$, and have it interact with the complete codeword in such a way that

$$|Z_a\rangle \otimes |b = 0\rangle \rightarrow |Z_0\rangle \otimes |b = a\rangle. \quad (26)$$

This again is a unitary transformation, which can be implemented by a quantum circuit. Note that now the unitary matrix that performs that error recovery is of order $2^{2n+1}$, instead of $2^{n+1}$.

Naturally, errors can also occur in the encoding and decoding processes themselves. More sophisticated methods must therefore be designed, that allow fault tolerant computation. An adaptive strategy is indicated, with several alternative paths for error correction. Most paths fail, because new errors are created; however, these errors can be detected, and there is a high probability that one of the paths will eventually lead to the correct result. As a consequence, the error correction circuits are able to correct old errors faster than they introduce new ones. There is then a high probability for keeping the number of errors small enough, so that the correction machinery can successfully deal with them [20, 21].

Finally, let us note that the symmetrization method, that was discussed in Sect. 2, can also be applied to individual codewords, if the latter have an internal symmetry. For example the codewords of ref. [8] are invariant under cyclic permutations of their 5 qubits. These codewords have the property that if any four qubits are correct, it is
always possible to restore the remaining defective qubit. However, the codeword error correction procedure definitely requires four qubits to be correct, and it cannot cope with small drifts of all five qubits, or even two of them. Therefore, it is helpful to test once in a while the cyclic symmetry of the codeword: successful tests will reduce the amplitude of small errors. Unfortunately, just as in the case of inter-computer symmetrization, an unsuccessful test leads to an asymmetric state, and forces us to completely discard the incorrect codeword. One of the logical qubits is then missing, and the computation can proceed only if there is enough redundancy among the logical qubits themselves (not only in their representation by physical qubits), for example, if they are parts of higher order codewords.

Instead of continually testing the symmetry of a codeword, it is also possible to force its physical qubits to respect that symmetry by introducing a high potential barrier that prevents access to asymmetric states, as in Eq. (27):

$$H_0 \rightarrow H_0 + \Omega \left(1 - |0_0\rangle\langle 0_0| - |1_0\rangle\langle 1_0|\right).$$

(27)

In this way, an error that turns a codeword state into a new quantum state lying in the orthocomplement of the legal subspace, can be produced only by investing a large amount of energy, $\Omega$. All corrigible errors are of this type, and are therefore prevented. However, an incorrigible error creates a state that is not orthogonal to both codewords, and therefore it would not be prevented by the additional potential in Eq. (27). Incorrigible errors remain uncorrected, of course. At most, their probability of occurrence can be reduced.

5. To know or not to know

It was already noted that no knowledge of the syndrome is needed in order to correct a corrigible error [6]. Error correction is a logical operation. It is part of the software, and can be performed automatically, without involving irreversible quantum measurements. We can be sure that the error is corrected, even if we don’t know the nature of that error.

This situation is reminiscent of the teleportation of an unknown quantum state [22]: the classical information sent by the emitter is not correlated to the quantum state that is teleported, and the emitter does not know which state she sends. The receiver also does not know, and cannot know, which state he receives. However, he can be sure
that this state is identical to the one that was in the emitter’s hands, before the teleportation process began. The teleportation process can also be achieved by an ordinary quantum circuit [23], without performing any measurement. Alternatively, that circuit can be interrupted by a quantum measurement, and classical information transferred in a conventional way to another point, where the circuit restarts. There, the classical information is used for performing a unitary transformation that brings the teleportation process to successful completion.

Likewise, there is no fundamental difference between a “conscious” error correction and an “unconscious” one. Figure 1 describes how a codeword is measured, in order to determine the syndrome and correct the error. On the other hand, Fig. 2(a) shows how the same error correction can be performed automatically, thanks to Eq. (26). In that case, the syndrome need not be known, but it still is encoded in an ancilla. It may therefore be measured, if we wish so, in order to restore that ancilla to its standard state, $|b = 0\rangle$. This is shown in Fig. 2(b). From a comparison of these figures, it is clear that automatic correction is conceptually simpler, and makes use of fewer hardware resources than those needed for measuring and then correcting an error.

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Captions of figures

Fig. 1. The corrupted codeword $|Z'\rangle$ interacts with a measuring apparatus, initially in state $|A_0\rangle$. The codeword and the apparatus get entangled (their states $|Z_a\rangle$ and $|A_a\rangle$ are correlated). Amplification brings the apparatus into the classical domain, and gives a definite value to $a$. The apparatus then selects the unitary transformation $U_a$ that restores the correct codeword state, $|Z_0\rangle$.

Fig. 2. (a) The corrupted codeword $|Z'\rangle$ interact with a fresh ancilla $|b_0\rangle$, and returns to the legal state $|Z_0\rangle$, while the used ancilla in state $|b'\rangle$ may be discarded. (b) Instead of discarding the ancilla, it is possible to measure it by means of an apparatus, initially in state $|A_0\rangle$. The ancilla and the apparatus then get entangled, and the restoration process continues as in Fig. 1.
Figure 1

Figure 2