COHOMOLOGICAL HALL ALGEBRAS AND THEIR REPRESENTATIONS VIA TORSION PAIRS

DUILIU-EMANUEL DIACONESCU, MAURO PORTA, AND FRANCESCO SALA

ABSTRACT. In this paper, starting from a torsion pair $\langle T, F \rangle$ on the heart of a $t$-structure $\tau$ on a stable $\infty$-category $C$, satisfying certain natural conditions, we construct cohomological, K-theoretical and categorified Hall algebras associated to the (derived) moduli stack $\text{Coh}_T(C, \tau)$ of objects of $T$ and corresponding left and right actions on the Borel-Moore, K-theory, bounded derived category of the (derived) moduli stack $\text{Coh}_F(C, \tau)$ of objects of $F$, respectively. In particular, this allows us to attach to $\langle T, F \rangle$ canonical subalgebras of the endomorphism algebras of the Borel-Moore homology and K-theory of $\text{Coh}_T(C, \tau)$, whose “positive parts” are the cohomological and K-theoretical Hall algebras of $T$, respectively. We call them the Yangian and quantum loop algebra of the torsion pair. The theory of cohomological and K-theoretical Hall algebras suffers from its long-standing limitation to only be able to produce “positive parts” of whole algebras: our construction suggests a new direction that might lead to overcome this difficulty. We also provide a geometric sufficient criterion ensuring that the commutator of two different operators vanish.

In the quiver case, we can obtain the action of the two-dimensional cohomological Hall algebra of a quiver on the cohomology of Nakajima quiver varieties within our framework. Besides the quiver case, we also apply our framework to two explicit torsion pairs on a smooth projective complex surface, and we investigate the corresponding Hall algebras and their representations associated to them. In these cases, our formalism provides Hall algebra constructions of Neguţ’s action of the Ding-Iohara-Miki of $S$ on the K-theory of moduli spaces of Gieseker-stable sheaves on $S$ and DeHority’s approach to the realization of representations of affinization of Lorentzian-Kac-Moody algebras via moduli spaces of rank one torsion-free sheaves on $S$.

Finally, we slightly modify our method to construct representations of the cohomological (resp. K-theoretical, categorified) Hall algebra of zero-dimensional sheaves on $S$ on the Borel-Moore homology (resp. K-theory, bounded derived category) of the (derived) moduli stack of Pandharipande-Thomas stable pairs on surfaces and “dually” on (derived) relative Hilbert schemes of points. Contrary to the three previous examples, the action comes from extensions of flags rather than extensions of sheaves (as it is more commonly encountered in the theory of cohomological Hall algebras).

CONTENTS

1. Introduction
2. Representations of COHAs arising from $m$-flags
3. CoHAs and representations associated to a torsion pair
4. COHA of a surface and its representation via torsion-free sheaves
5. Perverse COHA of a surface and its perverse torsion-free representation
6. Representations of the COHA of a surface via stable pairs
7. COHA of a quiver and its representation via quiver varieties
Appendix A. Complements on the Segal conditions
References

2020 Mathematics Subject Classification. Primary: 14A20; Secondary: 14A30, 14F08, 17B37.
Key words and phrases. Hall algebras, Yangians, categorification, stable $\infty$-categories, torsion pairs, tilting theory.
The work of D.-E. D was partially supported by NSF grant DMS-180241, while the work of F. S. was partially supported by JSPS KAKENHI Grant Number JP21K03197.
1. INTRODUCTION

Cohomological and K-theoretical Hall algebras are associative algebra structures on the Borel-Moore homology and K-theory, respectively, of certain moduli stacks of objects of a category, which are flat with respect to a fixed $t$-structure. Their categorified version (in the sense of [PS19]) is a $E_{1}$-monoidal structure on the stable $\infty$-category of complexes with coherent cohomology on suitable derived enhancements of these moduli stacks. The most common examples of these moduli stacks are those parametrizing coherent sheaves on smooth projective curves or surfaces, and their non-commutative counterparts, i.e., those parametrizing representations of a quiver or of the preprojective algebra of a quiver.

The cohomological Hall algebra of the preprojective algebra of a quiver is a subalgebra of a positive nilpotent part of the Maulik-Okounkov Yangian of the same quiver [SV17] (and conjecturally the former coincides with the latter). In [VV22], a similar relation has been established for K-theoretical Hall algebras and quantum loop algebras. On the other hand, only few results about the ‘quantum nature’ of COHAs of coherent sheaves are known. For example, the cohomological Hall algebra of the zero-dimensional coherent sheaves on a smooth quasi-projective complex surface $S$ has been characterized in [Zha21, KV19].

Another current limitation of the theory of cohomological Hall algebras is that they are supposed to geometrically realize only positive nilpotent parts of the full Yangians. An open problem of the theory is how to realize the full Yangian starting from the cohomological Hall algebra. When one deals with classical Hall algebras of hereditary categories instead of two-dimensional cohomological Hall algebras, this problem has been solved algebraically by defining a bialgebra structure on the Hall algebra and taking the (reduced) Drinfeld double (see e.g. [Sch12] and references therein). In the cohomological setting, it is not clear at the moment how to define a coproduct which would make the algebra a bialgebra.

The present paper presents a direction to solve both current limitations of the theory of cohomological Hall algebras.

From torsion pairs to cohomological Hall algebras and their representations. Let us introduce the general framework we work on.

Let $\mathcal{C}$ be a $\mathbb{C}$-linear finite type stable $\infty$-category endowed with a $t$-structure $\tau$ which satisfies openness of flatness. This condition ensures that the derived moduli stack $\text{Coh}_{\text{ps}}(\mathcal{C}, \tau)$ of $\tau$-flat pseudo-perfect objects of $\mathcal{C}$ is a geometric derived stack locally of finite presentation over $\mathbb{C}$. Let $\text{Coh}_{\text{ps}}^{\text{ext}}(\mathcal{C}, \tau)$ be the derived moduli stack of extensions, which is a geometric derived stack locally of finite presentation over $\mathbb{C}$, and denote by

$$\partial_{0} \times \partial_{2} : \text{Coh}_{\text{ps}}^{\text{ext}}(\mathcal{C}, \tau) \to \text{Coh}_{\text{ps}}(\mathcal{C}, \tau) \times \text{Coh}_{\text{ps}}(\mathcal{C}, \tau)$$

and

$$\partial_{1} : \text{Coh}_{\text{ps}}^{\text{ext}}(\mathcal{C}, \tau) \to \text{Coh}_{\text{ps}}(\mathcal{C}, \tau)$$

the maps that send an extension to the extreme objects and to the middle object, respectively. Let $\nu = (\mathcal{T}, \mathcal{F})$ be a torsion pair on the heart $\mathcal{C}^{c}$ of the $t$-structure $\tau$. We shall assume that it is open in the sense of [AB13, Definition 1.2.1], so that also the tilted $t$-structure $\tau_{\nu}$ satisfies the openness of flatness as well, so the corresponding moduli stacks of flat objects and extensions of flat objects.

---

1Here, we mean the Grothendieck group of coherent sheaves.

2The conjecture is true in the case of the one-loop quiver and finite and affine ADE quivers.
are geometric and locally of finite presentation over $\mathbb{C}$. Moreover, we denote by $\text{Coh}_\tau(\mathcal{E}, \tau)$ and $\text{Coh}_\tau(\mathcal{E}, \tau)$ the open substacks of $\text{Coh}_{ps}(\mathcal{E}, \tau)$ parametrizing flat objects belonging to $\mathcal{T}$ and $\mathcal{F}$, respectively.

The first main result of the paper is the following structural result.

**Theorem A** (Theorem 3.21). Let $\mathcal{E}$ be a finite type compactly generated stable $\infty$-category equipped with a $t$-structure $\tau = (\mathcal{E}^{<0}, \mathcal{E}^{>0})$ which is left and right complete, compatible with filtered colimits, and satisfies openness of flatness. Let $\nu = (\mathcal{T}, \mathcal{F})$ be an open torsion pair on $\mathcal{E}^{\heartsuit}$. Assume that:

1. the tilted $t$-structure $\tau_\nu$ satisfies openness of flatness;
2. both maps
   
   $$\partial_0 \times \partial_2 : \text{Coh}_{ps}^{\text{ext}}(\mathcal{E}, \tau) \to \text{Coh}_{ps}(\mathcal{E}, \tau) \times \text{Coh}_{ps}(\mathcal{E}, \tau)$$

   and
   
   $$\partial_0 \times \partial_2 : \text{Coh}_{ps}^{\text{ext}}(\mathcal{E}, \tau_\nu) \to \text{Coh}_{ps}(\mathcal{E}, \tau_\nu) \times \text{Coh}_{ps}(\mathcal{E}, \tau_\nu)$$

   are derived lci, while both maps
   
   $$\partial_1 : \text{Coh}_{ps}^{\text{ext}}(\mathcal{E}, \tau) \to \text{Coh}_{ps}(\mathcal{E}, \tau)$$

   and
   
   $$\partial_1 : \text{Coh}_{ps}^{\text{ext}}(\mathcal{E}, \tau_\nu) \to \text{Coh}_{ps}(\mathcal{E}, \tau_\nu)$$

   are representable by proper algebraic spaces.
3. the abelian category $\mathcal{T}$ is a Serre subcategory of $\mathcal{E}^{\heartsuit}$;
4. the maps
   
   $$\text{Coh}_\tau(\mathcal{E}, \tau) \to \text{Coh}_{ps}(\mathcal{E}, \tau)$$

   $$\text{Coh}_\tau(\mathcal{E}, \tau) \to \text{Coh}_{ps}(\mathcal{E}, \tau_\nu)$$

   are representable by open and closed embeddings;

Then

- the pro-$\infty$-category\(^3\) $\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$ has the structure of a $\boxtimes_1$-monoidal category. In particular, the K-theory $G_0(\text{Coh}_{\tau}(\mathcal{E}, \tau))$ and the Borel-Moore homology $H_*^{BM}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$ of $\text{Coh}_{\tau}(\mathcal{E}, \tau)$ have the structure of associative algebras, and

- the pro-$\infty$-category $\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$ has both the structure of a categorical left and of a categorical right module over $\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$. In particular, the K-theory $G_0(\text{Coh}_{\tau}(\mathcal{E}, \tau))$ (resp. the Borel-Moore homology $H_*^{BM}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$) of $\text{Coh}_{\tau}(\mathcal{E}, \tau)$ has both the structure of a left and a right $G_0(\text{Coh}_{\tau}(\mathcal{E}, \tau))$-module (resp. $H_*^{BM}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$-module).

Thanks to the left and right module structures, we are able to give the following definition.

**Definition** (Definition 3.23). The categorified quantum loop algebra $\mathcal{H}(\mathcal{T}, \mathcal{F})$ of the pair $(\mathcal{T}, \mathcal{F})$ is the monoidal subcategory of the monoidal $\infty$-category of endofunctors $\text{End}(\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau)))$ generated by the images of the two monoidal functors

$$a_\ell : \text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau)) \to \text{End}(\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau))),$$

$$a_r : \text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau)) \to \text{End}(\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau)))$$

corresponding to the two module structures of $\text{Coh}_{ps}^{\text{b}}(\text{Coh}_{\tau}(\mathcal{E}, \tau))$.

The quantum loop algebra $\mathcal{U}(\mathcal{T}, \mathcal{F})$ of the pair $(\mathcal{T}, \mathcal{F})$ is the subalgebra of $\text{End}(G_0(\text{Coh}_{\tau}(\mathcal{E}, \tau)))$ generated by the images of the two maps of associative algebras

$$a_\ell : G_0(\text{Coh}_{\tau}(\mathcal{E}, \tau)) \to \text{End}(G_0(\text{Coh}_{\tau}(\mathcal{E}, \tau))),$$

\(^3\)The reader can safely ignore the “pro” attribute in first reading. It is a way to encode and categorify the natural topology on G-theory and Borel-Moore homology induced by quasi-compact open exhaustion of the stack $\text{Coh}_{\tau}(\mathcal{E}, \tau)$.
\[ a_r : \mathcal{G}_0(\mathbf{Coh}_f(\mathcal{E}, \tau)) \longrightarrow \text{End}(\mathcal{G}_0(\mathbf{Coh}_f(\mathcal{E}, \tau))) \]
corresponding to the two module structures of \( \mathcal{G}_0(\mathbf{Coh}_f(\mathcal{E}, \tau)) \). Similarly, we define the Yangian \( \mathcal{Y}_{(\mathcal{F}, \mathcal{F})} \) of the pair \( (\mathcal{F}, \mathcal{F}) \).

Let \( x \) be a class in the Borel-Moore homology (resp. K-theory, or \( \mathbf{Coh}_b^\text{pro} \)) of \( \mathbf{Coh}_f(\mathcal{E}, \tau) \) and denote by \( a_l(x) \) and \( a_r(x) \) the corresponding (categorical) operators induced by the left and right actions, respectively. It is important to stress that in general the commutator \( [a_l(x), a_r(x)] \) does not vanish\(^4\). Nevertheless, we address from this abstract point of view the question of when two classes \( x, y \) induce commuting operators. We find a criterion of geometric origin (see Corollary 3.27), that is applied in the geometric situation described in the next section.

**Cohomological Hall algebras of coherent sheaves on a surface and their representations.** Let \( S \) be a smooth projective irreducible complex surface. We apply our framework in two examples. The first concerns the torsion pair \( (\mathbf{Coh}_\text{tor}(S), \mathbf{Coh}_{\text{ff}}(S)) \) of the standard heart of \( \mathbf{Coh}(S) \) formed by torsion and torsion-free sheaves, respectively. The standard \( t \)-structure of \( \text{Perf}(S) \) and this torsion pair satisfy the assumptions of the above theorem. Moreover, the action of the cohomological/K-theoretical/categorified Hall algebra of \( \mathbf{Coh}_\text{tor}(S) \) preserves the rank of the torsion-free sheaves. Thus, the second main result is the following.

**Theorem B** (cf. Theorem 4.3 and Corollary 4.4).

- The stable pro-\( \infty \)-category \( \mathbf{Coh}_\text{tor}^\text{pro}(\mathbf{Coh}_\text{tor}(S)) \) has a \( \mathcal{E}_1 \)-monoidal structure. In particular, \( \mathcal{G}_0(\mathbf{Coh}_\text{tor}(S)) \) and \( H^\text{BM}_*(\mathbf{Coh}_\text{tor}(S)) \) have the structure of associative algebras.

- The stable pro-\( \infty \)-category \( \mathbf{Coh}_\text{tor}^\text{pro}(\mathbf{Coh}_{\text{ff}}(S,r)) \) has the structure of a left and a right categorical module over \( \mathbf{Coh}_\text{tor}^\text{pro}(\mathbf{Coh}_\text{tor}(S)) \). In particular, \( \mathcal{G}_0(\mathbf{Coh}_{\text{ff}}(S,r)) \) (resp. \( H^\text{BM}_*(\mathbf{Coh}_r(\mathbf{Coh}_\text{tor}(S))) \) has the structure of a left and a right module over \( \mathcal{G}_0(\mathbf{Coh}_\text{tor}(S)) \) (resp. over \( H^\text{BM}_*(\mathbf{Coh}_\text{tor}(S)) \)).

We show in Corollary 4.5 the vanishing of the commutators between (categorical operators) induced by classes in Borel-Moore homologies (resp. K-theory, \( \mathbf{Coh}_b^\text{pro} \)) of the moduli stacks \( \mathbf{Coh}(Z_1) \) and \( \mathbf{Coh}(Z_2) \) corresponding to two disjoint closed subschemes \( Z_1 \) and \( Z_2 \) of \( S \).

The first part of the Theorem B has been already proved by [Zha21] in the K-theoretical case for zero-dimensional sheaves and in general by [KV19] in the cohomological and K-theoretical case, while in [PS19] in the categorified case. Moreover, the left action has been already constructed in [KV19] in the cohomological and K-theoretical case. The two main advances of the theorem is the categorification of the left action and above all the definition of the right action. From a technical point of view, the existence of the action is guaranteed by a general restriction mechanism established in Appendix A. Furthermore, a version of Theorem B holds also by replacing \( \mathbf{Coh}_\text{tor}(S) \) with the derived stack \( \mathbf{Coh}_{\text{dim}}(S) \) of zero-dimensional sheaves on \( S \).

We recover – from the viewpoint of cohomological and K-theoretical Hall algebras – both Neguț’s construction [Neg19] of the action of the Ding-Iohara-Miki algebra of \( S \) on the K-theory of moduli spaces of Gieseker-stable sheaves on \( S \) and DeHority’s construction [DeH20] of the action of affinizations of Lorentzian Kac–Moody algebras on the cohomology of moduli spaces of rank one Gieseker-stable sheaves on a K3 surface. Both Neguț’s and DeHority’s approach are based on the use of explicit operators given by Hecke correspondences: from one side this allows them to compute explicitly the relations between them, but on the other side this forces them to consider only certain operators since it is crucial for them to understand the geometry of Hecke correspondences. This limitation does not appear if one uses directly cohomological Hall algebras. Thus, our result provides a new approach to extend their results by considering bigger algebras than those obtained by them.

\(^4\)This is something already known in the literature if one uses certain families of Nakajima types operators as Neguț [Neg19] in K-theory or DeHority [DeH20] in cohomology: we recover their results thanks to Theorem B below.
For the second example we consider, we first tilt the standard \( t \)-structure of \( \text{Coh}(S) \) by the torsion pair whose torsion part is given by zero-dimensional sheaves. Let \( \tau_A \) be the corresponding tilted \( t \)-structure and let \( A \) be its heart. We call its objects \textit{perversely coherent sheaves} on \( S \). We consider the torsion pair \((A\text{tor}, A\text{tf})\) of \( A \) whose torsion part \( A\text{tor} \) consists of rank zero complexes of \( A \). This torsion pair is “dual” to the first one via the equivalences \((-)\)\text{\textup{op}}[2] : \( A\text{tor} \xrightarrow{\sim} \text{Coh}_{\text{tor}}(S) \) and \((-)\)\text{\textup{op}}[1] : \( A\text{tf} \xrightarrow{\sim} \text{Coh}_{\text{tf}}(S) \). Thus, we obtain:

\textbf{Theorem C} (Theorem 5.13).

(1) The equivalence

\[ \text{(-)}\text{\textup{op}}[2] : \text{Coh}_{\text{tor}}(S, \tau_A^{\text{op}}) \xrightarrow{\sim} \text{Coh}_{\text{tor}}(S) \]

induces an equivalence of \( \mathbb{E}_1 \)-monoidal stable pro-\( \infty \)-categories:

\[ \Gamma : \text{Co}\mathbb{H}_{\text{pro}}^b(\text{Coh}_{\text{tor}}(S, \tau_A^{\text{op}})) \xrightarrow{\sim} \text{Co}\mathbb{H}_{\text{pro}}^b(\text{Coh}_{\text{tor}}(S)) . \]

It induces isomorphisms of associative algebras:

\[ \Gamma : \text{G}_0(\text{Coh}_{\text{tor}}(S, \tau_A))^{\text{op}} \xrightarrow{\sim} \text{G}_0(\text{Coh}_{\text{tor}}(S)) , \]

\[ \Gamma : \text{H}_s^{BM}(\text{Coh}_{\text{tor}}(S, \tau_A))^{\text{op}} \xrightarrow{\sim} \text{H}_s^{BM}(\text{Coh}_{\text{tor}}(S)) . \]

(2) The equivalences

\[ \text{(-)}\text{\textup{op}}[1] : \text{Coh}_{\text{tf}}(S, \tau_A^{\text{op}}, r) \xrightarrow{\sim} \text{Coh}_{\text{tf}}(S;r) \]

and \[ \text{(-)}\text{\textup{op}}[2] : \text{Coh}_{\text{tor}}(S, \tau_A^{\text{op}}) \xrightarrow{\sim} \text{Coh}_{\text{tor}}(S) \]

induce an equivalence of left and right categorical modules over \( \text{Co}\mathbb{H}_{\text{pro}}^b(\text{Coh}_{\text{tor}}(S)) \):

\[ \Psi : \text{Co}\mathbb{H}_{\text{pro}}^b(\text{Coh}_{\text{tf}}(S, \tau_A^{\text{op}}, r)) \xrightarrow{\sim} \text{Co}\mathbb{H}_{\text{pro}}^b(\text{Coh}_{\text{tf}}(S;r)) . \]

At the K-theory and Borel-Moore homology levels we have isomorphisms

\[ \text{G}_0(\text{Coh}_{\text{tf}}(S, \tau_A^{\text{op}}, r)) \xrightarrow{\sim} \text{G}_0(\text{Coh}_{\text{tf}}(S;r)) , \]

\[ \text{H}_s^{BM}(\text{Coh}_{\text{tf}}(S, \tau_A^{\text{op}}, r)) \xrightarrow{\sim} \text{H}_s^{BM}(\text{Coh}_{\text{tf}}(S;r)) \]

of left and right modules over \( \text{G}_0(\text{Coh}_{\text{tor}}(S)) \) and of \( \text{H}_s^{BM}(\text{Coh}_{\text{tor}}(S)) \), respectively.

We use the \( t \)-structure \( \tau_A \) to construct left and right representations of the categorified and cohomological Hall algebras of \( \text{Coh}_{0\text{-dim}}(S) \) via Pandharipande-Thomas stable pairs. Recall that a stable pair is a pair consisting of a pure one-dimensional sheaf \( \mathcal{F} \) on \( S \) and a section \( s : \mathcal{O}_S \to \mathcal{F} \) with zero-dimensional cokernel. It is easy to see that this datum is equivalent to that of an exact triangle \( 0_S \to \mathcal{F} \to \mathcal{E} \) where we ask \( \mathcal{F}[1] \in A\text{tor} \) and \( E \in A\text{tf} \) (see Proposition 6.1). While the action of the previous theorem is essentially a simultaneous restriction of the action induced by the multiplication of \( \text{Coh}(S, \tau_A) \) and of its tilting, in the case of stable pairs the fundamental mechanism underlying the action is slightly more involved: we obtain it as a restriction of an abstract action in correspondence of the derived stack of perfect complexes \( \text{Perf}(S) \) on the derived stack \( \text{FlagPerf}^{(2)}(S) \) of flags of length 2 (see §2 for the precise definition and construction of this action). Our third main result reads:

\textbf{Theorem D} (cf. Corollaries 6.28 and 6.31). Let \( \mathcal{P}(S) \) be the derived moduli stack of Pandharipande-Thomas stable pairs and by \( \mathcal{P}(S) \) its classical truncation. Then, the pro-\( \infty \)-category \( \text{Co}\mathbb{H}_{\text{pro}}^b(\mathcal{P}(S)) \) has the structure of a left and right categorical module over the \( \mathbb{E}_1 \)-monoidal \( \infty \)-category \( \text{Co}\mathbb{H}_{\text{pro}}^b(\text{Coh}_{0\text{-dim}}(S)) \).

In particular,

\[ \text{G}_0(\mathcal{P}(S)) \quad \text{and} \quad \text{H}_s^{BM}(\mathcal{P}(S)) \]

are left and right modules of \( \text{G}_0(\text{Coh}_{0\text{-dim}}(S)) \) and \( \text{H}_s^{BM}(\text{Coh}_{0\text{-dim}}(S)) \), respectively.
In the homology case, we expect an induced action of a shifted Yangian\(^5\). We leave this aspect to a future investigation.

Note that in the local surface case, Toda constructed a right categorical module structure on \(\text{Coh}^b_{\text{ps}}\) of Pandharipande-Thomas moduli spaces of stable pairs over the categorified Hall algebra of zero-dimensional sheaves (cf. \[Tod20, \S 4\]). In this case, there is no left categorical module structure because of a wall-crossing phenomenon which does not appear in our two-dimensional case.

As shown in [PT10, Appendix B], moduli spaces of stable pairs are “dual” to relative Hilbert schemes of points. More precisely, let \(H_{1,\text{pure}}(S)\) be the Hilbert scheme parametrizing pure one-dimensional subschemes \(C \subseteq S\). Let \(\mathcal{C} \subseteq S \times H_{1,\text{pure}}(S)\) be the universal curve and consider the (underived) relative Hilbert scheme \(\text{Hilb}(\mathcal{C}/H_{1,\text{pure}}(S))\) of \(H_{1,\text{pure}}(S)\)-flat families of zero-dimensional quotients of \(\mathcal{O}_C\). Let \(\mathcal{P}_S(B)\) be the derived enhancement of \(\text{Hilb}(\mathcal{C}/H_{1,\text{pure}}(S))\) introduced in \(\S 5.3\) (to be more precise, see Remark 6.40). By using the duality, we are able to prove:

**Theorem E** (cf. Theorem 6.41 and Corollary 6.43). The \(\text{pro}-\infty\)-category \(\text{Coh}^b_{\text{ps}}(\mathcal{P}_S(B))\) has the structure of a left and right categorical module over the \(\mathbb{E}_1\)-monoidal \(\infty\)-category \(\text{Coh}^b_{\text{ps}}(\text{Coh}^0_{\text{dim}}(S))\). In particular, 

\[
G_0(\text{Hilb}(\mathcal{C}/H_{1,\text{pure}}(S))) \quad \text{and} \quad H^\text{BM}_s(\text{Hilb}(\mathcal{C}/H_{1,\text{pure}}(S)))
\]

are left and right modules of \(G_0(\text{Coh}^0_{\text{dim}}(S))\) and \(H^\text{BM}_s(\text{Coh}^0_{\text{dim}}(S))\), respectively.

One can wonder if the above result holds also for \(\text{Coh}_{\text{ps}}(S)\). We are able to lift only the left action at the level of stable pairs and only the right action at the level of relative Hilbert schemes, while for the other action, we found a no-go result coming from a geometric constraint (cf. Corollary 6.13).

Finally, the results above can be extended to more general stable pairs, for which \(\mathcal{O}_S\) is replaced by any locally free sheaf \(\mathcal{V}\) of finite rank.

**Cohomological Hall algebras of quivers and Yangians.** Our framework applies also to the quiver case: we recover the known construction of the action of the two-dimensional cohomological Hall algebra of a quiver on the cohomology of Nakajima quiver varieties associated to the same quiver, but also we construct new actions on the cohomology of other quiver varieties.

Let \(Q\) be a quiver, let \(w \in \mathbb{N}^Q\), and let \(Q^w\) be its corresponding Crawley-Boevey quiver (cf. Definition 7.1). Denote by \(\Pi_Q^w\) the preprojective algebra of \(Q\) and by \(\mathcal{G}_w(Q^w)\) the derived preprojective algebra of \(Q\) (see Definition 7.1).

Set \(\mathcal{E}_{Q^w} := \mathcal{G}_w(Q^w)-\text{Mod}\). Then, the heart of the standard \(t\)-structure of \(\mathcal{E}_{Q^w}\) is the abelian category \(\text{Mod}(\Pi_Q^w)\) of representations of \(\Pi_Q^w\). Let \(\mathcal{I} := \text{Mod}(\Pi_Q)\) be the category of representations of \(\Pi_Q\): it can be canonically realized as a full subcategory of \(\text{Mod}(\Pi_Q^w)\), which is a torsion part of a torsion pair \((\mathcal{I}, \mathcal{F}) := (\mathcal{I}^\perp)\) of \(\text{Mod}(\Pi_Q^w)\), which is open. The finite-dimensional representations belonging to \(\mathcal{I}\) are exactly those who are \(\infty\)-co-generated in the sense of [CB01, Page 261]. The corresponding moduli stack \(\text{Coh}_{\mathcal{I}}^b(\mathcal{E}_{Q^w}, \tau_{\text{std}})\) is an open substack of \(\text{Coh}_{\mathcal{F}}(\mathcal{E}_{Q^w}, \tau_{\text{std}})\).

The moduli stack \(\text{Coh}_{\mathcal{I}}^b(\mathcal{E}_{Q^w}, \tau_{\text{std}})\) decomposes with respect to the dimension of the finite-dimensional representations of \(\Pi_Q^w\) into open and closed substacks. In particular, \(\text{Coh}_{\mathcal{I}}^b(\mathcal{E}_{Q^w}, \tau_{\text{std}})\) is the open and closed substack of \(\text{Coh}_{\mathcal{I}}^b(\mathcal{E}_{Q^w}, \tau_{\text{std}})\) defined by the condition that the vector space at the vertex \(\infty\) is zero. Denote by \(\text{Coh}_{\mathcal{I}}^b(\Pi_Q^w; w_\infty)\) the moduli stack of finite-dimensional representations of \(\Pi_Q^w\) belonging to \(\mathcal{I}\), for which the dimension of the vector space at the vertex \(\infty\) is \(w_\infty\). Note that the classical truncation of \(\text{Coh}_{\mathcal{I}}^b(\Pi_Q^w; 1)\) admits a fine moduli space, which is the Nakajima quiver variety \(M_{Q, \theta}(w)\) of \(\theta\)-stable framed representations with framed dimension vector \(w\). Here, \(\theta := (1, \ldots, 1)\).

By applying the framework described in the previous section, we get the following.

---

\(^5\)In three noncompact toric three-dimensional case, some conjectures in this direction have been made in [RSY+20].

\(^6\)We warmly thank Olivier Schiffmann for suggesting us to consider this torsion pair in the quiver setting.
Theorem F (Theorem 7.6). The stable pro-infty-category $\text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{ps}}(\Pi_Q))$ has a $E_1$-monoidal structure. In particular, $G_0(\text{Coh}_{\text{ps}}(\Pi_Q))$ and $H_\text{BM}^*(\text{Coh}_{\text{ps}}(\Pi_Q))$ have the structure of associative algebras.

For each dimension vector $w_\omega \in \mathbb{N}$, $\text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{ps}}(\Pi_{Q_\omega}; w_\omega))$ has the structure of a left categorical module over $\text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{ps}}(\Pi_Q))$. In particular, $G_0(\text{Coh}_{\text{ps}}(\Pi_{Q_\omega}; w_\omega))$ and $H_\text{BM}^*(\text{Coh}_{\text{ps}}(\Pi_{Q_\omega}; w_\omega))$ have the structure of left modules over $G_0(\text{Coh}_{\text{ps}}(\Pi_Q))$ and over $H_\text{BM}^*(\text{Coh}_{\text{ps}}(\Pi_Q))$, respectively.

Fix $w_\omega = 1$. Then, $G_0(\text{M}_{Q,\omega}(w))_C$ and $H_\text{BM}^*(\text{M}_{Q,\omega}(w))$ have the structure of left modules over $G_0(\text{Coh}_{\text{ps}}(\Pi_Q))_C$ and $H_\text{BM}^*(\text{Coh}_{\text{ps}}(\Pi_Q))$, respectively. Similar statements hold equivariantly with respect to the torus $T$ introduced in [SV20, §3.3].

The theorem recovers the constructions of the two-dimensional cohomological Hall algebra of a quiver and its categorification (cf. [SV13a, SV13b, SV20, YZ18, VV22, DPSa]) and the left action of the two-dimensional cohomological Hall algebra of a quiver on the cohomology of Nakajima quiver varieties (cf. [SV13a, SV20]).

The right action is not obtained since the second condition of assumption (2) of Theorem A is not satisfied by the tilted heart. In a future version of the present paper we extend this construction to the cohomological Hall algebra of nilpotent representations of the preprojective algebra of $Q$, obtaining a left and right actions.

1.1. Outline. In §2, we introduce the general framework which allows us to construct representations of cohomological Hall algebras and their categorifications via moduli stacks of flags of perfect complexes. Given a torsion pair, §3 deals with an application of this framework to construct a cohomological Hall algebra associated to the torsion part of the torsion pair and a representation of it associated to the torsion-free part of the torsion pair. We apply this construction to two torsion pairs naturally associated to a smooth projective complex surface $S$: the first torsion pair is the one having as torsion part the usual torsion sheaves on $S$ and the study of the corresponding cohomological Hall algebra and its representation is the subject of §4. The second torsion pair is the one having as torsion part only zero-dimensional sheaves on $S$: the study of the corresponding cohomological Hall algebra and its representation is given in §5. Moreover, the two torsion pairs are “dual” to each other and the relation with the Hall algebras is investigated in the same section. §6 investigates the construction of representations of cohomological Hall algebras via stable pairs. Finally, in §7 we apply this construction to the quiver setting and we recover the left action of the two-dimensional cohomological Hall algebra of a quiver on the cohomology of Nakajima quiver varieties of [SV20]. Appendix §A deals with the technical machinery of 2-Segal representations, which is heavily used to obtain the results in §2.

1.2. Notation. For a smooth projective complex surface $S$, let $K_0(S)$ be the Grothendieck group of $S$ and let $N(S)$ be the numerical Grothendieck group of $S$, where the latter is defined by:

$$N(S) := K_0(S) \equiv .$$

Here, $F_1, F_2 \in K_0(S)$ satisfy $F_1 \equiv F_2$ if $\text{ch}(F_1) = \text{ch}(F_2)$. Then $N(S)$ is a finitely generated free abelian group.

We denote by $NS(S)$ the Neron-Severi group of $S$. For a coherent sheaf $E$ on $S$ whose support has dimension less than or equal to one, we denote by $\ell(E) \in NS(S)$ the fundamental one cycle of $E$.

Let $\text{Coh}_{\text{ps}}(S) \subset \text{Coh}(S)$ be the subcategory of sheaves $E$ with $\dim \text{Supp}(E) \leq 1$. We define the subgroup $N_{\leq 1}(S) \subset N(S)$ to be

$$N_{\leq 1}(S) := \text{Im}(K_0(\text{Coh}_{\leq 1}(S)) \rightarrow N(S)).$$

Note that we have an isomorphism $N_{\leq 1}(S) \simeq NS(S) \oplus \mathbb{Z}$ sending $E$ to the pair $(\ell(E), \chi(E))$. We shall identify an element $v \in N_{\leq 1}(S)$ with $(b, n) \in NS(S) \oplus \mathbb{Z}$ by the above isomorphism.

For any $E \in \text{Perf}(S)$, we set $E^\vee := \text{R}Hom_{\text{ps}}(S)(E, \mathcal{O}_S)$; while for any coherent sheaf $E$ on $S$, we set $E^* := \text{Hom}_S(E, \mathcal{O}_S) \simeq \mathcal{H}^0(E^\vee)$, which is the usual dual sheaf.
We use the attribute “geometric” instead of “algebraic” or “Artin” for a (derived) stack, following the convention in [PS19].

Acknowledgments. The paper was developed during the Research in Pair “2227p: Representation theory of two-dimensional categorified Hall Algebras of curves” at the Mathematisches Forschungsinstitut Oberwolfach by the last two-named authors and during a research visit of the second-named author at Department of Mathematics of the University of Pisa under the 2021 Visiting Fellow program of the University of Pisa. We thank both institutions for providing us an exceptional research environment. We would also like to thank Olivier Schiffmann for various stimulating discussions and Matthew B. Young for explaining his paper [You18] to us.

2. REPRESENTATIONS OF COHAS ARISING FROM m-FLAGS

In this section, we construct representations of cohomological Hall algebras via stacks of \( m \)-flags. The relevant combinatorial notion here is that of relative 2-Segal spaces [Wal, You18], which we briefly review in Appendix A.

In this section, we shall fix a compactly supported stable \( \infty \)-category \( C \).

2.1. Categorical representations on \( m \)-flags. Recall from [PS19, DPSa] that we have a simplicial derived stack

\[
S_* \text{Perf}_{ps}(\mathcal{C}) : \Delta^{op} \rightarrow dSt.
\]

The derived stack \( S_1 \text{Perf}_{ps}(\mathcal{C}) \) of 1-simplexes is simply denoted by \( \text{Perf}_{ps}(\mathcal{C}) \), and it is the derived stack sending \( A \in \text{CAlg} \) to the maximal \( \infty \)-groupoid contained inside

\[
\text{Fun}((\mathcal{C}^{\omega})^{op}, \text{Perf}(A)).
\]

We refer to such objects as \( A \)-families of pseudo-perfect objects of \( \mathcal{C} \).

Fix an integer \( m \geq 1 \). We set

\[
\text{FlagPerf}_{ps}(m)(\mathcal{C}) := S_m \text{Perf}_{ps}(\mathcal{C}),
\]

and we refer to this as the derived stack of \( m \)-flags of pseudo-perfect objects in \( \mathcal{C} \).

Remark 2.1. Unraveling the definitions, we see that the derived stack \( \text{FlagPerf}_{ps}(m)(\mathcal{C}) \) can informally be described as the functor sending \( A \in \text{CAlg} \) to diagrams of the form

\[
\begin{array}{cccccccccccccccccc}
0 & \rightarrow & F_{0,1} & \rightarrow & F_{0,2} & \rightarrow & \cdots & \rightarrow & F_{0,m-1} & \rightarrow & F_{0,m} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
0 & \rightarrow & F_{1,2} & \rightarrow & \cdots & \rightarrow & F_{1,m-1} & \rightarrow & F_{1,m} & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
& & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
F_{m-2,m-1} & \rightarrow & F_{m-2,m} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & F_{m-1,m} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & & & 0 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\end{array}
\]

\[\text{(2.1)}\]

\(\text{Perf}_{ps}(\mathcal{C})\) is Toen-Vaqué moduli stack of objects [TV07].
where every square is a pullback in \( \text{Fun}((\mathcal{C}^\omega)^\text{op}, \text{Perf}(A)) \). We refer to such a diagram as an \( m \)-flag of pseudo-perfect objects of \( \mathcal{C} \).

When \( m = 2 \), further unraveling the definitions shows that \( \text{FlagPerf}_\text{ps}^2(\mathcal{C}) \) parametrizes diagrams \( \mathcal{F} \) of the form

\[
\begin{array}{ccc}
0 & \rightarrow & F_{0,1} \\
\downarrow & & \downarrow \\
0 & \rightarrow & F_{1,2} \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0
\end{array}
\]

where the central square is asked to be a pullback. In other words, we can identify \( \text{FlagPerf}_\text{ps}^2(\mathcal{C}) \) with the stack \( \text{Perf}^\text{ext}_{\text{ps}}(\mathcal{C}) \) parametrizing extensions of pseudo-perfect objects in \( \mathcal{C} \). With respect to this representation and to the canonical identification \( S_1 \text{Perf}_{\text{ps}}(\mathcal{C}) \simeq \text{Perf}_{\text{ps}}(\mathcal{C}) \), we see that

\[
\partial_0(\mathcal{F}) = F_{0,1}, \quad \partial_1(\mathcal{F}) = F_{0,2}, \quad \partial_2(\mathcal{F}) = F_{1,2}.
\]

We further fix an \( (m-1) \)-flag \( V \in \text{FlagPerf}_{\text{ps}}^{(m-1)}(\mathcal{C}) \), which we represent as the following diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & V_{0,1} & \rightarrow & V_{0,2} & \rightarrow & \cdots & \rightarrow & V_{0,m-1} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
0 & \rightarrow & V_{1,2} & \rightarrow & \cdots & \rightarrow & V_{1,m-1} \\
\downarrow & & \cdots & & \downarrow & & \cdots & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
& & \vdots & & \vdots & & \vdots & & \vdots \\
& & V_{m-2,m-1} & & \cdots & & \cdots & & \cdots \\
& & \downarrow & & \downarrow & & \cdots & & \downarrow \\
& & 0 & & & & & & 0
\end{array}
\]

The boundary map \( \partial_m : [m] \rightarrow [m-1] \) in \( \Delta^\text{op} \) induces a forgetful functor

\[
\partial_m : \text{FlagPerf}_{\text{ps}}^{(m)}(\mathcal{C}) \rightarrow \text{FlagPerf}_{\text{ps}}^{(m-1)}(\mathcal{C}).
\]

**Definition 2.2.** Let \( V \in \text{FlagPerf}_{\text{ps}}^{(m-1)}(\mathcal{C}) \). We define the derived stack \( \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; V) \) of \( V \)-flags of length \( m \) as the fiber product

\[
\text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; V) \rightarrow \text{FlagPerf}_{\text{ps}}^{(m)}(\mathcal{C})
\]

\[
\text{Spec}(k) \xrightarrow{\text{xy}} \text{FlagPerf}_{\text{ps}}^{(m-1)}(\mathcal{C})
\]

where \( \text{xy} : \text{Spec}(k) \rightarrow \text{FlagPerf}_{\text{ps}}^{(m-1)}(\mathcal{C}) \) is the map corresponding to the flag \( V \).

\[\text{Here and in what follows, } \partial_\ast \text{ denotes the } \ast\text{-th face map.}\]
Applying the right version of Construction A.6 to $S_\bullet \text{Perf}_{\text{ps}}(\mathcal{C})$ and the $(m-1)$-flag $\mathcal{V}$, we obtain a relative 2-Segal simplicial derived stack that we denote as

$$u^\ell_\bullet : S_\bullet \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; \mathcal{V}) \longrightarrow S_\bullet \text{Perf}_{\text{ps}}(\mathcal{C}).$$

(2.2)

When $m = 1$, the choice of $\mathcal{V}$ is empty. In this case, we therefore simply denote the above simplicial object by

$$u^\ell_\bullet : S_\bullet \text{FlagPerf}_{\text{ps}}^{(1),\dagger}(\mathcal{C}) \longrightarrow S_\bullet \text{Perf}_{\text{ps}}(\mathcal{C}).$$

Remark 2.3. In light of Theorem A.1, we think of $\text{Perf}_{\text{ps}}(\mathcal{C})$ as an algebra in the $\infty$-category of correspondences $\text{Corr}(\text{dSt})$ acting on the derived stack $\text{FlagPerf}_{\text{ps}}^{m}(\mathcal{C}; \mathcal{V})$. The action is implemented by the correspondence

$$S_\bullet \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; \mathcal{V}) \longrightarrow \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; \mathcal{V}) \times \text{Perf}_{\text{ps}}(\mathcal{C}).$$

Unraveling the definition, we see that $S_\bullet \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; \mathcal{V})$ fits in the following fiber product:

$$S_\bullet \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; \mathcal{V}) \longrightarrow S_{m+1} \text{Perf}_{\text{ps}}(\mathcal{C})$$

$$\text{Spec}(k) \longrightarrow S_{m-1} \text{Perf}_{\text{ps}}(\mathcal{C}),$$

(2.3)

where $\partial_{m,m+1}$ is induced by the map $[m+1] \to [m-1]$ in $\Delta^{op}$ that avoids $m$ and $m+1$ inside $[m+1]$, while the morphism $x_V$ is the morphism classifying the $(m-1)$-flag $\mathcal{V}$. In other words, $S_\bullet \text{FlagPerf}_{\text{ps}}^{(m),\dagger}(\mathcal{C}; \mathcal{V})$ can be informally described as the derived stack parametrizing diagrams of the form

$$
\begin{array}{cccccccc}
0 & \longrightarrow & V_{0,1} & \longrightarrow & V_{0,2} & \longrightarrow & \cdots & \longrightarrow & V_{0,m-1} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
0 & \longrightarrow & V_{1,2} & \longrightarrow & \cdots & \longrightarrow & V_{1,m-1} & \longrightarrow & F_{1,m} \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \downarrow \\
V_{m-2,m-1} & \longrightarrow & F_{m-2,m} & \longrightarrow & F_{m-2,m+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{m-1,m} & \longrightarrow & F_{m-1,m+1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_{m,m+1} \\
\end{array}
$$

(2.4)

\footnote{The discrepancy between the choice of either left or right version in Construction A.2 and the superscript $\ast$ in $S_\bullet(\mathcal{C})$ is justified from the fact that, in this way, the notation is compatible with the classical literature on (classical, cohomological, K-theoretical) Hall algebras and their representations. See [PS19].}
The morphism \( u_1 \) sends the diagram (2.4) to \( F_{m,m+1} \), the morphism \( \partial_0 \) sends the diagram (2.4) to the full V-flag determined by the chain
\[
F_{0,m+1} \longrightarrow F_{1,m+1} \longrightarrow \cdots \longrightarrow F_{m-1,m+1},
\]
while the morphism \( \partial_1 \) sends the diagram (2.4) to the full V-flag determined by the chain
\[
F_{0,m} \longrightarrow F_{1,m} \longrightarrow \cdots \longrightarrow F_{m-1,m}.
\]

\[\triangle\]

**Construction 2.4.** Let \( T \) and \( F \) be derived stacks and let
\[
j : T \longrightarrow \text{Perf}_{ps}(\mathcal{C}) \quad \text{and} \quad i : F \longrightarrow \text{FlagPerf}_{ps}^{(m),t}(\mathcal{C}; V)
\]
be two morphisms. Choose as well a morphism \( 0_T : \text{Spec}(k) \rightarrow T \) lifting the zero morphism \( \text{Spec}(k) \rightarrow \text{Perf}(S) \) classifying the zero perfect complex. We can arrange these data into the following diagram
\[
\rho : \begin{array}{ccc}
F & \longrightarrow & \text{Spec}(k) \\
\downarrow i & & \downarrow j \\
\text{FlagPerf}_{ps}^{(m),t}(\mathcal{C}; V) & \longrightarrow & \text{Spec}(k) \\
\downarrow 0_T & & \downarrow 0 \\
& & \text{Perf}_{ps}(\mathcal{C})
\end{array}
\]
We obtain in this way a boundary datum for the relative simplicial derived stack (2.2), in the sense of Definition A.25. Thus, applying Construction A.26, we obtain a new relative simplicial derived stack
\[
u_{\ell} : \mathcal{S}_1^{\ell} \text{FlagPerf}_{F,T}^{(m),t}(\mathcal{C}; V) \longrightarrow \mathcal{S}_1^{\ell} \text{Perf}_{T}(\mathcal{C}).
\]

\[\odot\]

**Remark 2.5.** Unwinding the definitions, we see that \( \mathcal{S}_1^{\ell} \text{Perf}_{T}(\mathcal{C}) \simeq T \) and that \( \mathcal{S}_2^{\ell} \text{Perf}_{T}(\mathcal{C}) \) fits in the following pullback square
\[
\begin{array}{ccc}
\mathcal{S}_2^{\ell} \text{Perf}_{T}(\mathcal{C}) & \longrightarrow & \mathcal{S}_2^{\ell} \text{Perf}_{ps}(\mathcal{C}) \\
\downarrow & & \downarrow \partial_0 \times \partial_1 \times \partial_2 \\
T \times T \times T & \longrightarrow & \text{Perf}_{ps}(\mathcal{C}) \times \text{Perf}_{ps}(\mathcal{C}) \times \text{Perf}_{ps}(\mathcal{C})
\end{array}
\]
Similarly, \( \mathcal{S}_1^{\ell} \text{FlagPerf}_{F,T}^{(m),t}(\mathcal{C}; V) \simeq F \) and \( \mathcal{S}_1^{\ell} \text{FlagPerf}_{F,T}^{(m),t}(\mathcal{C}; V) \) fits in the following pullback square:
\[
\begin{array}{ccc}
\mathcal{S}_1^{\ell} \text{FlagPerf}_{F,T}^{(m),t}(\mathcal{C}; V) & \longrightarrow & \mathcal{S}_1^{\ell} \text{FlagPerf}_{ps}^{(m),t}(\mathcal{C}; V) \\
\downarrow & & \downarrow \partial_0 \times \partial_1 \times u_1 \\
F \times F \times T & \longrightarrow & \text{FlagPerf}_{ps}^{(m),t}(\mathcal{C}; V) \times \text{FlagPerf}_{ps}^{(m),t}(\mathcal{C}; V) \times \text{Perf}_{ps}(\mathcal{C})
\end{array}
\]

\[\triangle\]

At this point, Corollary A.28 immediately implies:

**Proposition 2.6.**

(1) Assume that the square
\[
\begin{array}{ccc}
\mathcal{S}_2^{\ell} \text{Perf}_{T}(\mathcal{C}) & \longrightarrow & \mathcal{S}_2^{\ell} \text{Perf}_{ps}(\mathcal{C}) \\
\downarrow & & \downarrow \partial_0 \times \partial_2 \\
T \times T & \longrightarrow & \text{Perf}_{ps}(\mathcal{C}) \times \text{Perf}_{ps}(\mathcal{C})
\end{array}
\]
is a pullback. Then \( \mathcal{S}_1^{\ell} \text{Perf}_{T}(\mathcal{C}) \) is a 2-Segal derived stack.
(2) Assume furthermore that at least one between
\[ S_\ell \text{FlagPerf}^{(m),\dagger}_F \rightarrow S_\ell \text{FlagPerf}^{(m),\dagger}_F \]
and
\[ \partial_0 \times \partial_2 : S_2 \text{Perf}_T(\mathcal{C}) \rightarrow T \times T \]
is a pullback. Then \( S_\ell \text{FlagPerf}^{(m),\dagger}_F \rightarrow S_\ell \text{Perf}_T(\mathcal{C}) \) is a relative 2-Segal derived stack.

**Corollary 2.7.** Assume that both \( T \) and \( F \) are derived geometric stacks.

(1) Assume that condition (1) of Proposition 2.6 is satisfied by \( T \). In addition, assume that

(i) the map
\[ \partial_0 \times \partial_2 : S_2 \text{Perf}_T(\mathcal{C}) \rightarrow T \times T \]
is derived lci, and

(ii) the map
\[ \partial_1 : S_2 \text{Perf}_T(\mathcal{C}) \rightarrow T \]
is representable by proper algebraic spaces.

Then, \( \text{Coh}^\text{b}_{\text{pro}}(T) \) has the structure of an \( E_1 \)-monoidal stable pro-\( \infty \)-category, whose underlying tensor product is given by the composition
\[ \text{Coh}^\text{b}_{\text{pro}}(T) \times \text{Coh}^\text{b}_{\text{pro}}(T) \xrightarrow{(\partial_1) \circ (\partial_2 \times \partial_0)^*} \text{Coh}^\text{b}_{\text{pro}}(T). \]

In particular, \( G_0(T) \) and \( H^\text{BM}_s(T) \) have the structures of associative algebras.

(2) Assume that conditions (1) and (2) of Proposition 2.6 are satisfied by \( T, F \), and conditions (1)-(i) and (1)-(ii) are satisfied by \( T \). In addition, assume that

(i) the map
\[ \partial_0 \times u_{1}^i : S_\ell \text{FlagPerf}^{(m),\dagger}_F \rightarrow F \times T \]
is derived lci, and

(ii) the map
\[ \partial_1 : S_\ell \text{FlagPerf}^{(m),\dagger}_F(\mathcal{C}; \mathbb{V}) \rightarrow F \]
is representable by proper algebraic spaces.

Then, \( \text{Coh}^\text{b}_{\text{pro}}(F) \) has the structure of a right categorical module over the \( E_1 \)-monoidal stable pro-\( \infty \)-category \( \text{Coh}^\text{b}_{\text{pro}}(T) \), whose underlying action is given by the composition
\[ \text{Coh}^\text{b}_{\text{pro}}(T) \times \text{Coh}^\text{b}_{\text{pro}}(F) \xrightarrow{(\partial_1) \circ (u_{1}^i \times \partial_0)^*} \text{Coh}^\text{b}_{\text{pro}}(F). \]

In particular, \( G_0(F) \) and \( H^\text{BM}_s(F) \)
are right modules of $G_0(T)$ and $H^R_{BM}(T)$, respectively.

(3) Assume that conditions (1) and (2) of Proposition 2.6 are satisfied by $T, F$, and conditions (1)-(i) and (1)-(ii) are satisfied by $T$. In addition, assume that

(i) the map

$$\partial_1 \times u_1^I : S^I_{\mathfrak{Flag}_{P}^*(m), \tau}^T (\mathfrak{C}, V) \to F \times T$$

is derived lci, and

(ii) the map

$$\partial_0 : S^I_{\mathfrak{Flag}_{P}^*(m), \tau} (\mathfrak{C}, V) \to F$$

is representable by proper algebraic spaces.

Then, $Coh^b_{\mathfrak{pro}}(F)$ has the structure of a left categorical module over the $\mathcal{E}_1$-monoidal $\infty$-category $Coh^b_{\mathfrak{pro}}(T)$, whose underlying action is given by the composition

$$Coh^b_{\mathfrak{pro}}(F) \times Coh^b_{\mathfrak{pro}}(T) \overset{\otimes}{\to} Coh^b_{\mathfrak{pro}}(F \times T) \overset{(\partial_0)_{\tau \circ (\partial_1 \times u_1^I)^*}}{\to} Coh^b_{\mathfrak{pro}}(F).$$

In particular,

$$G_0(F) \quad \text{and} \quad H^R_{BM}(F)$$

are left modules of $G_0(T)$ and $H^R_{BM}(T)$, respectively.

**Proof.** Reasoning as in the proof of [PS19, Theorem 4.9], the result follows directly combining Proposition 2.6 and Theorem A.1. \qed

**Example 2.8.** Let $k$ be a field and let $\mathfrak{C}$ be a $k$-linear finite type compactly generated stable $\infty$-category. Fix a $t$-structure $\tau = (\mathfrak{C}^{\leq 0}, \mathfrak{C}^{> 0})$ on $\mathfrak{C}$. Assume that $\tau$ is compatible with filtered colimits, left and right complete and that it satisfies openness of flatness;\footnote{This notion has been introduced in the literature in the case of smooth and proper categories: see e.g. [BLM+21, §10.1] and the references therein.} this guarantees the existence of the derived moduli stack $\text{Coh}^b_{\mathfrak{ps}}(\mathfrak{C}, \tau)$ of $\tau$-flat pseudo-perfect objects of $\mathfrak{C}$, which is a geometric derived stack locally of finite presentation over $k$ (see [DPSa] for details). Set

$$T := \text{Coh}^b_{\mathfrak{ps}}(\mathfrak{C}, \tau).$$

Fix furthermore an $(m - 1)$-flag $V$ and let $F$ be the open substack of $\text{Perf}^b_{\mathfrak{ps}}(\mathfrak{C}, V)$ parametrizing $V$-flags of the form (2.1) where we ask $F_{i,n}$ to be $\tau$-flat for $i = 0, 1, \ldots, m$. Then the assumptions of Proposition 2.6 are satisfied and we write

$$S^I_{\mathfrak{Flag}_{\mathfrak{Coh}}^*(m), \tau} (\mathfrak{C}, \tau; V) \to S^*_{\mathfrak{Coh}_{\mathfrak{ps}}}(\mathfrak{C}, \tau)$$

for the resulting 2-Segal stack. \triangle

For later applications, we conclude this subsection by the following observation. In the previous setting, choose maps $\alpha : T' \to T$ and $\mu : F' \to F$ of derived stacks. Assume that the morphism $0_T : \text{Spec}(k) \to T$ lifts to a morphism $0_{T'} : \text{Spec}(k) \to T'$. Then:

**Corollary 2.9.** Assume that $T, T', F$ and $F'$ are geometric derived stacks and that the following conditions are met:

1. the pair $(T,F)$ satisfies assumption (3) of Corollary 2.7;
2. the stack $T'$ satisfies assumption (1) of Proposition 2.6 and the map

$$\partial_1 : S^I_{\mathfrak{Perf}^b_{\mathfrak{T}}}(\mathfrak{C}) \to T'$$

is representable by proper algebraic spaces;
3. the map $\alpha : T' \to T$ is representable by proper algebraic spaces;
(4) the map
\[ \partial_1 \times u'_1 : S_1^{\text{flag} \mathcal{F}_{F',T'}(\emptyset; V)} \to F' \times T' \]

is derived lci;

(5) the square
\[
\begin{array}{ccc}
S_1^{\text{flag} \mathcal{F}_{F',T'}(\emptyset; V)} & \to & S_1^{\text{flag} \mathcal{F}_{F,T}(\emptyset; V)} \\
\partial_0 \times u'_1 & \downarrow & \partial_0 \times u'_1 \\
F' \times T' & \to & F \times T
\end{array}
\]
is a pullback (this happens for instance if both pairs \((T, F)\) and \((T', F')\) make the square (2.5) into a pullback).

Then the pair \((T', F')\) also satisfies assumption (3) of Corollary 2.7.

Proof. To begin with, consider the following ladder of commutative squares:
\[
\begin{array}{ccc}
S_2 \mathcal{F}_{T}(\emptyset) & \to & S_2 \mathcal{F}_{T}(\emptyset) & \to & S_2 \mathcal{F}_{ps}(\emptyset) \\
\partial_2 \times h_0 & \downarrow & \partial_2 \times h_0 & \downarrow & \partial_2 \times h_0 \\
T' \times T' & \to & T \times T & \to & \mathcal{F}_{ps}(\emptyset) \times \mathcal{F}_{ps}(\emptyset)
\end{array}
\]

Since the pair \((T, F)\) satisfies assumption (3) of Corollary 2.7, the middle vertical map is derived lci and the right square is a pullback. Since \(T'\) satisfies assumption (1) of Proposition 2.6, the outer square is a pullback. Thus, the left square is a pullback as well, and hence the leftmost vertical arrow is derived lci. As the map \(\partial_3 : \mathcal{F}_{ps}(\emptyset) \to T'\) is representable by proper algebraic spaces by assumption, we see that \(T'\) satisfies conditions (1)-(i) and (1)-(ii) in Corollary 2.7.

We are now left to prove that the map
\[ \partial_1 \times u'_1 : S_1^{\text{flag} \mathcal{F}_{F',T'}(\emptyset; V)} \to F' \times T' \]
is derived lci and that the map
\[ \partial_0 : S_1^{\text{flag} \mathcal{F}_{F',T'}(\emptyset; V)} \to F' \]
is representable by proper algebraic spaces. The former holds by assumption. For the latter, consider the following commutative diagram:
\[
\begin{array}{ccc}
S_1^{\text{flag} \mathcal{F}_{F',T'}(\emptyset; V)} & \to & X_{F,F',T'} \\
\rho_0 & \downarrow & \rho_1 \\
X_{F,F',T'} & \to & X_{F,F',T'} \\
F' \times T' & \mu \times \text{id}_{T'} & \to & F \times T
\end{array}
\]

where the objects \(X_{s,s,s}\) are defined by asking all the squares to be pullbacks. Then the map \(\partial_0\) is canonically identified with the composite \(q_2 \circ q_1 \circ q_0\). Although the upper left square may not be a pullback, condition (5) implies that \(q_0\) is an equivalence. Since the pair \((T, F)\) satisfies assumption (2) of Corollary 2.7, we see that \(q_2\) is representable by proper schemes. Since \(\alpha\) is representable by proper schemes, the same goes for \(q_1\). The conclusion follows. \qed
Notation 2.10. In practice, we will often apply Corollary 2.9 with $T = \text{Coh}_{ps}(\mathcal{C}, \tau)$ and $F = \text{FlagCoh}_{ps}^{(m),+}(\mathcal{C}, \tau; V)$, where $\tau$ is a $t$-structure as in Example 2.8. In this case, we will rather denote the relative 2-Segal derived stack

$$S^\bullet_{\text{FlagPerf}_{F,T}^{(m),+}}(\mathcal{C}; V) \longrightarrow S^\bullet_{\text{Perf}_{T}}(\mathcal{C})$$

by the notation

$$S^\bullet_{\text{FlagCoh}_{F,T}^{(m),+}}(\mathcal{C}, \tau; V) \longrightarrow S^\bullet_{\text{Coh}_{T}}(\mathcal{C}, \tau) \, .$$

$\triangle$

Remark 2.11.

1. Instead of the right version, we can apply the left version of Construction A.6 to $S^\bullet_{\text{Perf}_{ps}}(\mathcal{C})$. Then, all the statements above hold also after replacing “left” with “right” and vice versa.

2. Running Construction A.2 instead of Construction A.6, we would get a relative 2-Segal derived stack

$$u^\xi _\ell : S^\bullet _{\text{FlagPerf}_{ps}^{(m)}}(\mathcal{C}) \longrightarrow S^\bullet_{\text{Perf}_{ps}}(\mathcal{C}) ,$$

and all the results of this section equally apply to this setup. For $m = 1$ there is no difference between the $V$-marked version and the unmarked one.

$\triangle$

2.2. The relative (co)tangent complex for $m$-flags. We keep the same notations of the previous subsection. In Corollaries 2.7 and 2.9, we assumed that the map

$$\partial _1 \times u^\xi _1 : S^\bullet _{\text{FlagPerf}_{F,T}^{(m),+}}(\mathcal{C}; V) \longrightarrow F \times T$$

is derived lci.

We now provide a method to compute the cotangent complex of this morphism, starting with the simplest case, i.e., $F = \text{FlagPerf}_{ps}^{(m),+}(\mathcal{C}; V)$ and $T = \text{Perf}_{ps}(\mathcal{C})$. Consider the morphism in $\Delta^{op}$

$$\text{ev}_{m-1} : [m-1] \longrightarrow [0]$$

that selects $m - 1$. It induces a natural transformation $\text{ev}_{m-1} \ast \text{id}_{[n]} : [m-1] \ast [n] \rightarrow [0] \ast [n]$ which yields, via Construction A.6, a natural morphism

$$\lambda^0 _\bullet : S^\bullet _{\text{FlagPerf}_{ps}^{(m),+}}(\mathcal{C}; V) \longrightarrow S^\bullet _{\text{FlagPerf}_{ps}^{(1)}}(\mathcal{C}) ,$$

compatible with the structural maps to $S^\bullet_{\text{Perf}_{ps}}(\mathcal{C})$. With respect to the notation of Remarks 2.1 and 2.3, the morphism $\lambda^0 _0$ corresponds to the morphism

$$\lambda^0 _1 : \text{FlagPerf}_{ps}^{(m),+}(\mathcal{C}; V) \longrightarrow \text{Perf}_{ps}(\mathcal{C})$$

which sends a diagram of the form (2.1) to $F_{0,m}$. On the other hand, thanks to the description of $S^\bullet _{\text{FlagPerf}_{ps}^{(m),+}}(\mathcal{C}; V)$ as a pullback (cf. Equation (2.3)), $\lambda^0 _1$ is explicitly described as the morphism

$$\lambda^0 _1 : S^\bullet _{\text{FlagPerf}_{ps}^{(m),+}}(\mathcal{C}; V) \longrightarrow S^\bullet _{\text{FlagPerf}_{ps}^{(2)}}(\mathcal{C}) \simeq \text{FlagPerf}_{ps}^{(1)}(\mathcal{C}) \simeq \text{Perf}^{ext}_{ps}(\mathcal{C}) ,$$
which sends a diagram of the form (2.4) to the sub-diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & F_{0,m} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & F_{m,m+1} \\
\end{array}
\]

Proposition 2.12. The square

\[
\begin{array}{ccc}
S'_1\FlagPerf_{ps}^{(m),+}(\mathcal{C}; \mathcal{V}) & \longrightarrow & S'_1\FlagPerf_{ps}^{(1)}(\mathcal{C}) \\
\bigtriangleup_1 \times u_1' & \Downarrow & \bigtriangleup_1 \times \text{id} \\
\FlagPerf_{ps}^{(m),+}(\mathcal{C}; \mathcal{V}) \times \Perf_{ps}(\mathcal{C}) & \longrightarrow & \FlagPerf_{ps}^{(1)}(\mathcal{C}) \times \Perf_{ps}(\mathcal{C})
\end{array}
\]

is a pullback. Thus, the relative tangent complex $T_x$ of the left vertical map at a point $x: \text{Spec}(A) \rightarrow S'_1\FlagPerf_{ps}^{(m),+}(\mathcal{C}; \mathcal{V})$ classifying a diagram of the form (2.4) fits into the following natural fiber sequence

\[
T_x \longrightarrow \text{RHom}_{\mathcal{C}}(F_{m,m+1}, F_{0,m+1})[1] \oplus \text{RHom}_{\mathcal{C}}(F_{0,m+1}, F_{0,m})[1] \longrightarrow \text{RHom}_{\mathcal{C}}(F_{0,m+1}, F_{0,m+1})[1].
\]

Proof. Unraveling the definitions, the first half follows from the fact that the Waldhausen construction $S_1\Perf_{ps}(\mathcal{C})$ is a 2-Segal derived stack (applying [DK19, Proposition 2.3.2-(3)] with $n = m + 1, i = 0$ and $j = n - 1 = m$). For the second half, we apply [PS19, Proposition 3.2] to the map $\bigtriangleup_1 \times u_1': S'_1\FlagPerf_{ps}^{(1)}(\mathcal{C}) \rightarrow \FlagPerf_{ps}^{(1)}(\mathcal{C}) \times \Perf_{ps}(\mathcal{C})$, that implies that $T_x$ is identified with the limit of the following diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{RHom}_{\mathcal{C}}(F_{0,m+1}, F_{0,m+1})[1] \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{RHom}_{\mathcal{C}}(F_{0,m}, F_{0,m+1})[1] \\
\end{array}
\]

Thus, we see that $T_x$ fits in the following pullback square

\[
\begin{array}{ccc}
T_x & \longrightarrow & \text{RHom}_{\mathcal{C}}(F_{0,m+1}, F_{0,m})[1] \\
\downarrow & & \downarrow \\
\text{RHom}_{\mathcal{C}}(F_{m,m+1}, F_{0,m+1})[1] & \longrightarrow & \text{RHom}_{\mathcal{C}}(F_{0,m+1}, F_{0,m+1})[1]
\end{array}
\]

whence the conclusion. 

3. COHAS AND REPRESENTATIONS ASSOCIATED TO A TORSION PAIR

In this section we fix a field $k$ and a $k$-linear compactly generated stable $\infty$-category $\mathcal{C}$ equipped with a $t$-structure $\tau = (\mathcal{C}_{\geq 0}, \mathcal{C}_{> 0})$\textsuperscript{11} which is left and right complete and compatible with filtered colimits. We further assume that $\mathcal{C}$ is of finite type and that $\tau$ satisfies the openness of flatness, so that there exists the derived moduli stack $\text{Coh}_{ps}(\mathcal{C}, \tau)$ of $\tau$-flat pseudo-perfect objects of $\mathcal{C}$, which is a geometric derived stack, locally of finite presentation over $k$ (cf. [DPSa]).

\textsuperscript{11}We will use the cohomological notation for the $t$-structures.
In [DPSa] it was shown that under certain conditions on the pair \((\mathcal{E}, \tau)\), it is possible to attach to it various Hall algebras (cohomological, \(K\)-theoretical, categorified). Our main goal is to prove that a torsion pair on \(\mathcal{E}^\triangleright\) gives rise to several Hall algebras and their representations.

3.1. Families of torsion pairs, their CoHAs, and their representations. We start by recalling the notion of torsion pair.

**Definition 3.1.** Let \(\mathcal{A}\) be an abelian category. A torsion pair in \(\mathcal{A}\) is a pair \(\nu = (\mathcal{T}, \mathcal{F})\) of full subcategories such that

- for any \(T \in \mathcal{T}\) and \(F \in \mathcal{F}\) one has \(\text{Hom}_{\mathcal{A}}(T, F) = 0\);
- every \(X \in \mathcal{A}\) fits into an exact sequence
  \[0 \to T \to E \to F \to 0\]

with \(T \in \mathcal{T}\) and \(F \in \mathcal{F}\).

In this case, we refer to \(\mathcal{T}\) as the torsion part of \(\nu\) and to \(\mathcal{F}\) as the torsion-free part of \(\nu\).

The proof of next lemma consists of standard category theory and we leave it to the reader.

**Lemma 3.2.** Let \(\mathcal{A}\) be an abelian category and let \(\nu = (\mathcal{T}, \mathcal{F})\) be a torsion pair on \(\mathcal{A}\). Then:

1. both \(\mathcal{T}\) and \(\mathcal{F}\) are closed under extensions;
2. if \(T \to T'\) is an epimorphism in \(\mathcal{A}\) and \(T \in \mathcal{T}\), then \(T' \in \mathcal{T}\);
3. if \(F' \to F\) is a monomorphism in \(\mathcal{A}\) and \(F \in \mathcal{F}\), then \(F' \in \mathcal{F}\).

**Definition 3.3.** A full subcategory \(\mathcal{A}'\) of \(\mathcal{A}\) is a Serre subcategory, if for any short exact sequence

\[0 \to E' \to E \to E'' \to 0\]

in \(\mathcal{A}\), we have \(E \in \mathcal{A}'\) if and only if \(E', E'' \in \mathcal{A}'\).

In the following, we will also consider torsion pairs \(\nu = (\mathcal{T}, \mathcal{F})\) for which \(\mathcal{T}\) is a Serre subcategory of \(\mathcal{A}\).

**Lemma 3.4.** Let \(f : F \to G\) be a morphism of derived stacks. If \(f\) is representable by open immersions, then for every commutative algebra \(A \in \text{CAlg}\), the induced map \(f_A : F(A) \to G(A)\) is \((-1)\)-truncated in \(\mathcal{S}\), and hence it is fully faithful. Furthermore, \(f\) is an equivalence if and only if for every field \(k\), every morphism \(\text{Spec}(k) \to G\) factors through \(f\).

**Proof.** For the first half, it is enough to observe that the diagonal morphism \(F \to F \times_G F\) is an equivalence, which is obvious from the fact that \(f\) is representable by open immersions. For the second half, it is enough to prove that for every \(A \in \text{CAlg}\) the map \(F_A := \text{Spec}(A) \times_G F \to \text{Spec}(A)\) is an equivalence. By assumption this is an open subscheme of \(\text{Spec}(A)\), and hence it is an isomorphism if and only if for every field \(k\), every morphism \(\text{Spec}(k) \to \text{Spec}(A)\) has image contained in \(F_A\). The conclusion follows. \(\square\)

**Remark 3.5.** Let \(A \in \text{CAlg}\) and let \(\mathcal{E}^\triangleright_A\) for the heart of the induced \(t\)-structure on \(\mathcal{E}_A\). Write \(\text{Coh}^\triangleright_{ps}(\mathcal{E}_A, \tau_A)\) for the full subcategory of \(\mathcal{E}^\triangleright_A\) spanned by those objects \(F \in \mathcal{E}^\triangleright_A\) such that for every compact object \(G \in \mathcal{E}_A\), \(\text{Hom}_{\mathcal{E}^\triangleright_A}(G, F) \in \text{Perf}(A)\). Since short exact sequences in \(\mathcal{E}^\triangleright_A\) are fiber sequences in \(\mathcal{E}_A\), we see that \(\text{Coh}^\triangleright_{ps}(\mathcal{E}_A, \tau_A)\) is an abelian subcategory of \(\mathcal{E}^\triangleright_A\). \(\triangle\)

**Notation 3.6.** Let \(U : \text{Coh}(\mathcal{E}, \tau)\) be a morphism representable by open immersions. Given \(A \in \text{CAlg}\), Lemma 3.4 implies that \(U(A)\) is a full subgroupoid of \(\text{Coh}^\triangleright_{ps}(\mathcal{E}, \tau)(A) = \text{Coh}_{ps}(\mathcal{E}_A, \tau_A)^\triangleright\). We let \(\text{Coh}_U(\mathcal{E}_A, \tau_A)\) be the full subcategory of \(\text{Coh}^\triangleright_{ps}(\mathcal{E}_A, \tau_A)\) spanned by the objects that belong to the image of \(U(A)\). \(\triangle\)
Assumption 3.7. Let $T$ and $F$ be two open substacks of $\text{Coh}(\mathcal{C}, \tau)$ such that for every field $k$, the subcategories $(\text{Coh}_T(\mathcal{C}_k, \tau_k), \text{Coh}_F(\mathcal{C}_k, \tau_k))$ form a torsion pair on the abelian category $\mathcal{C}_k^{\perp}$. Thus, we think about $T$ and $F$ as parametrizing together families of torsion pairs on $\mathcal{C}$.

Lemma 3.2 immediately implies the following result.

Lemma 3.8. The squares

\[
\begin{array}{ccc}
\mathcal{S}_2 \text{Coh}_T(\mathcal{C}, \tau) & \longrightarrow & \mathcal{S}_2 \text{Coh}_{ps}(\mathcal{C}, \tau) \\
\downarrow & & \downarrow \\
T \times T & \longrightarrow & \text{Coh}_{ps}(\mathcal{C}, \tau) \times \text{Coh}_{ps}(\mathcal{C}, \tau)
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{S}_2 \text{Coh}_F(\mathcal{C}, \tau) & \longrightarrow & \mathcal{S}_2 \text{Coh}_{ps}(\mathcal{C}, \tau) \\
\downarrow & & \downarrow \\
F \times F & \longrightarrow & \text{Coh}_{ps}(\mathcal{C}, \tau) \times \text{Coh}_{ps}(\mathcal{C}, \tau)
\end{array}
\]

are pullback.

Similarly, the squares

\[
\begin{array}{ccc}
\mathcal{S}_1^T \text{FlagCoh}^{(1)}_{F,T}(\mathcal{C}, \tau) & \longrightarrow & \mathcal{S}_1^T \text{FlagCoh}_{ps}^{(1)}_{F,T}(\mathcal{C}, \tau) \\
\downarrow & & \downarrow \\
F \times T & \longrightarrow & \text{Coh}_{ps}(\mathcal{C}, \tau) \times \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau)
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{S}_1^T \text{FlagCoh}^{(1)}_{F,T}(\mathcal{C}, \tau) & \longrightarrow & \mathcal{S}_1^T \text{FlagCoh}_{ps}^{(1)}_{F,T}(\mathcal{C}, \tau) \\
\downarrow & & \downarrow \\
F \times T & \longrightarrow & \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau) \times \text{Coh}_{ps}(\mathcal{C}, \tau)
\end{array}
\]

are pullback.

Remark 3.9. Unraveling the definitions, we see that all the three derived stacks $\mathcal{S}_2 \text{Coh}(\mathcal{C}, \tau)$, $\mathcal{S}_1^T \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau)$ and $\mathcal{S}_1^T \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau)$ can be identified with the derived stack $\text{Coh}_{ps}^{ext}(\mathcal{C}, \tau)$ parametrizing extensions of $\tau$-flat pseudo-perfect objects in $\mathcal{C}$. However, we use the notation $\mathcal{S}_2 \text{Coh}(\mathcal{C}, \tau)$ to indicate the algebra structure in correspondences of $\text{Coh}_{ps}(\mathcal{C}, \tau)$, while we write $\mathcal{S}_1^T \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau)$ to denote the left action in correspondences of $\text{Coh}_{ps}(\mathcal{C}, \tau)$ on itself. Similarly, $\mathcal{S}_1^T \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau)$ corresponds to the right action of $\text{Coh}_{ps}(\mathcal{C}, \tau)$ on itself.

Proof of Lemma 3.8. We give the argument for the first square, as the others follow in an analogous way. Since $T \to \text{Coh}_{ps}(\mathcal{C}, \tau)$ is representable by open immersions, Lemma 3.4 shows that it is enough to prove that for every field $k$ the square obtained by taking $k$-points is a pullback. Lemma 3.4 also guarantees that the bottom horizontal map is fully faithful, so it follows that the top horizontal one is fully faithful as well. To check essential surjectivity, recall that a $k$-point of $\text{Coh}_{ps}^{ext}(\mathcal{C}, \tau)$ can be represented as a fiber sequence

\[
E := E_{01} \longrightarrow E_{02} \longrightarrow E_{12},
\]

in $\mathcal{C}_k$, where in addition the $E_{ij}$ are pseudo-perfect and $\tau$-flat. Since $k$ is a field, we have that $E_{ij} \in \mathcal{C}_k^{\perp}$, and therefore the $E_{ij}$ belong to the abelian category $\text{Coh}_k^{\perp}(\mathcal{C}_k, \tau_k)$. This immediately implies that the map on the left is a monomorphism, and that the one on the right is an epimorphism in $\text{Coh}_k^{\perp}(\mathcal{C}_k, \tau_k)$. Thus, Lemma 3.2(1) shows that if $\delta_2(E) = E_{01}$ and $\delta_0(E) = E_{12}$ are in $\text{Coh}_k(\mathcal{C}_k, \tau_k)$, the same goes for $\delta_1(E) = E_{02}$. The conclusion follows.

With the same proof, we also obtain the following.
Lemma 3.10. Assume that for every field $k$ the abelian category $\text{Coh}_T(\mathcal{E}_k, \tau_k)$ is a Serre subcategory of $\text{Coh}^{\vee}_{ps}(\mathcal{E}_k, \tau_k)$. Then the square
\[
\begin{array}{ccc}
S_2\text{Coh}_T(\mathcal{E}, \tau) & \longrightarrow & S_2\text{Coh}_{ps}(\mathcal{E}, \tau) \\
\downarrow \alpha_1 & & \downarrow \alpha_1 \\
T & \longrightarrow & \text{Coh}_{ps}(\mathcal{E}, \tau)
\end{array}
\]
is a pullback. The same assertion holds replacing $T$ by $F$.

Combining Proposition 2.6 and Lemma 3.8 we deduce that the morphisms
\[
S_2^{\Phi}\text{FlagCoh}_{T,F}(\mathcal{E}, \tau) \longrightarrow S_2\text{Coh}_F(\mathcal{E}, \tau) \quad \text{and} \quad S_2^{\Phi}\text{FlagCoh}_{F,T}(\mathcal{E}, \tau) \longrightarrow S_2\text{Coh}_T(\mathcal{E}, \tau)
\]
are relative 2-Segal derived stacks. Then, Lemma 3.8 and Corollary 2.9 immediately imply the following result.

Corollary 3.11. Keep Assumption 3.7. In addition, assume that the morphism
\[
\partial_0 \times \partial_2 : S_2\text{Coh}_{ps}(\mathcal{E}, \tau) \longrightarrow \text{Coh}_{ps}(\mathcal{E}, \tau) \times \text{Coh}_{ps}(\mathcal{E}, \tau)
\]
is derived lci and the morphism
\[
\partial_1 : S_2\text{Coh}_{ps}(\mathcal{E}, \tau) \longrightarrow \text{Coh}_{ps}(\mathcal{E}, \tau)
\]
is representable by proper algebraic spaces. Then:

1. If the morphism
\[
\partial_1 : S_2\text{Coh}_F(\mathcal{E}, \tau) \longrightarrow T
\]
is representable by proper algebraic spaces, then $\text{ Coh}^b_{\text{pro}}(T)$ has an induced $\mathbb{E}_1$-monoidal structure. In particular, $G_0(T)$ and $H^*_{BM}(T)$ have the structure of associative algebras. This is the case if for every field $k$, the abelian category $\mathcal{E}_k$ is a Serre subcategory of $\mathcal{E}_k$.

2. If the morphism
\[
\partial_1 : S_2\text{Coh}_F(\mathcal{E}, \tau) \longrightarrow F
\]
is representable by proper algebraic spaces, then $\text{ Coh}^b_{\text{pro}}(F)$ has an induced $\mathbb{E}_1$-monoidal structure. In particular, $G_0(F)$ and $H^*_{BM}(F)$ have the structure of associative algebras. This is the case if for every field $k$, the abelian category $\mathcal{E}_k$ is a Serre subcategory of $\mathcal{E}_k$.

3. In addition to point (1), assume that the morphism
\[
T \longrightarrow \text{Coh}_{ps}(\mathcal{E}, \tau)
\]
is representable by open and closed embeddings. Then $\text{ Coh}^b_{\text{pro}}(F)$ has an induced structure of a left categorical module over $\text{ Coh}^b_{\text{pro}}(T)$. In particular, $G_0(F)$ (resp. $H^*_{BM}(F)$) has the structure of a left module over $G_0(T)$ (resp. over $H^*_{BM}(T)$).

4. In addition to point (1), assume that the morphism
\[
F \longrightarrow \text{Coh}_{ps}(\mathcal{E}, \tau)
\]
is representable by open and closed embeddings. Then $\text{ Coh}^b_{\text{pro}}(T)$ has an induced structure of a right categorical module over $\text{ Coh}^b_{\text{pro}}(F)$. In particular, $G_0(T)$ (resp. $H^*_{BM}(T)$) has the structure of a right module over $G_0(F)$ (resp. over $H^*_{BM}(F)$).

Proof. First of all, Lemma 3.10 shows that indeed if $T$ or $F$ define Serre subcategories of $\mathcal{E}_k$ for every field $k$, then the assumptions of (1) and (2) are satisfied.

We first show that the theses of (1) and (2) hold. For this, it is enough to show that, setting
\[
T = F := \text{Coh}_{ps}(\mathcal{E}, \tau), \quad T' := T \quad \text{and} \quad F' := F,
\]
the assumptions of Corollary 2.9 are satisfied. Now, condition (3) in Corollary 2.7 follows from the assumptions on \( \text{Coh}_{ps}(\mathcal{C}, \tau) \). Thus, condition (1) in Corollary 2.9 is satisfied. The first half of Lemma 3.8 implies the first half of condition (2) in Corollary 2.9, while second half holds true thanks to assumption (1). Condition (3) in Corollary 2.9 is implied by assumption (2).

The second part of Lemma 3.8 implies condition (5) in Corollary 2.9 is satisfied. Finally, condition (4) in Corollary 2.9 is automatically satisfied: indeed, under the identification

\[
S^1_s \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau) \simeq S^2 \text{Coh}_{ps}(\mathcal{C}, \tau),
\]

the map

\[
\partial_{1} \times u^L : S^1_s \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau) \to \text{Coh}_{ps}(\mathcal{C}, \tau) \times \text{Coh}_{ps}(\mathcal{C}, \tau)
\]
corresponds to the map (3.1), which is derived lci by assumption. We therefore obtain a commutative square

\[
\begin{array}{ccc}
S^1_s \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau) & \longrightarrow & S^2 \text{FlagCoh}_{ps}^{(1)}(\mathcal{C}, \tau) \\
\downarrow & & \downarrow \\
F \times T & \longrightarrow & \text{Coh}_{ps}(\mathcal{C}, \tau) \times \text{Coh}_{ps}(\mathcal{C}, \tau)
\end{array}
\]

where both horizontal arrows are open immersions and the right vertical map is derived lci. It follows that the left vertical map is derived lci as well.

Now, the theses of (1) and (2) hold similarly, by using a “right action” version of Corollary 2.9 for

\[
T = F := \text{Coh}_{ps}(\mathcal{C}, \tau), \quad F' := T \quad \text{and} \quad T' := F,
\]
in which we replace \( S^1_s \) by \( S^2_s \) and “left” by “right” in condition (3) of Corollary 2.7 and in Corollary 2.9.

3.2. Torsion COHAs and their torsion-free representations. In this section, we shall choose open substacks \( T \) and \( F \) which can be canonically defined from a torsion pair \( \nu = (T, F) \) on \( \mathcal{C}^\diamond \). To perform this construction, we first recall the tilting procedure.

**Construction 3.12.** Fix a torsion pair \( \nu = (T, F) \) on \( \mathcal{C}^\diamond \). Define

\[
\mathcal{C}_\nu^{\leq 0} := \left\{ F \in \mathcal{C}^{\leq 0} \mid \mathcal{H}_t^{\geq 0}(F) \in \mathcal{T} \right\}.
\]

Since the \( t \)-structure \( \tau \) is compatible with filtered colimits and the inclusion \( \mathcal{T} \subseteq \mathcal{C}^\diamond \) commutes with colimits, it follows that \( \mathcal{C}_\nu^{\leq 0} \) is closed under colimits. Consider now a fiber sequence

\[
F' \longrightarrow F \longrightarrow F''
\]
in \( \mathcal{C} \). Passing to the long exact sequence of homotopy groups we find

\[
\cdots \longrightarrow \mathcal{H}_t^{-1}(F'') \longrightarrow \mathcal{H}_t^{0}(F') \longrightarrow \mathcal{H}_t^{1}(F) \longrightarrow \mathcal{H}_t^{0}(F''/F') \longrightarrow \mathcal{H}_t^{1}(F') \longrightarrow \cdots.
\]

It immediately follows that if \( F, F' \in \mathcal{C}_\nu^{\leq 0} \) (resp. \( F', F'' \in \mathcal{C}_\nu^{\leq 0} \)) then \( F'' \in \mathcal{C}_\nu^{\leq 0} \) (resp. \( F \in \mathcal{C}_\nu^{\leq 0} \)). In particular, \( \mathcal{C}_\nu^{\leq 0} \) is closed under filtered colimits and cofibers, and hence under arbitrary colimits, and it is furthermore closed under extensions as well. Applying [Lur17, Proposition 1.4.4.11], we deduce the existence of a unique \( t \)-structure

\[
\tau_\nu := (\mathcal{C}_\nu^{\leq 0}, \mathcal{C}_\nu^{\geq 0})
\]

whose connective part is \( \mathcal{C}_\nu^{\leq 0} \).

**Definition 3.13.** We call \( \tau_\nu := (\mathcal{C}_\nu^{\leq 0}, \mathcal{C}_\nu^{\geq 0}) \) the \( (\nu) \)-tilted \( t \)-structure obtained from \( \nu \). We denote by \( \mathcal{C}_\nu^{\diamond} \) the corresponding heart.

**Remark 3.14.**
(1) In the setting of the previous construction, it is easy to characterize $\mathcal{E}_v^{\geq 0}$. Indeed,

$$\mathcal{E}_v^{\geq 0} = \left\{ F \in \mathcal{C} \mid \text{Map}(G, F) = 0 \text{ for every } G \in \mathcal{E}_v^{< -1} \right\}.$$

Since we obviously have $\mathcal{E}_v^{< -2} \subseteq \mathcal{E}_v^{< -1}$, it follows that $\mathcal{E}_v^{> 0} \subseteq \mathcal{E}_v^{>-1}$. Furthermore, if $T \in \mathcal{F}$, then $T[1] \in \mathcal{E}_v^{<-1}$, and therefore if $F \in \mathcal{E}_v^{> 0}$ then

$$0 \simeq \text{Map}_C(T[1], F) \simeq \text{Map}_C(T, \tau^{-1}_C(F)),$$

so that $\mathcal{H}_T^{-1}(F)$ must belong to the right orthogonal to $\mathcal{F}$, i.e. $\mathcal{H}_T^{-1}(F) \in \mathcal{F}$. The vice-versa being obvious, we obtain

$$\mathcal{E}_v^{> 0} = \left\{ F \in \mathcal{E}_v^{>-1} \mid \mathcal{H}_T^{-1}(F) \in \mathcal{F} \right\}.$$

(2) It follows from the discussion above that $\mathcal{E}_v^{\triangleright}$ is characterized as follows:

$$\mathcal{E}_v^{\triangleright} = \left\{ E \in \mathcal{C} \mid \mathcal{H}_T^0(E) \in \mathcal{F}, \mathcal{H}_T^{-1}(E) \in \mathcal{F}, \text{ and } \mathcal{H}_T^i(E) = 0 \text{ for } i \neq -1, 0 \right\}.$$

(3) Assume that both $\mathcal{F}$ and $\mathcal{E}_v^{\triangleright}$ are compactly generated and that the inclusion $\mathcal{F} \hookrightarrow \mathcal{E}_v^{\triangleright}$ preserves compact objects. Then the inclusion $\mathcal{F} \hookrightarrow \mathcal{E}_v^{\triangleright}$ commutes with filtered colimits, and therefore it follows from the explicit descriptions of $\mathcal{E}_v^{< 0}$ and of $\mathcal{E}_v^{> 0}$ that the tilted $t$-structure $\tau_v$ is again compatible with filtered colimits.

The following lemma follows from the above remarks.

**Lemma 3.15** ([Pol07, Lemma 1.1.2]). Let $\tau_{(1)} = (\mathcal{E}_v^{< 0}, \mathcal{E}_v^{> 0})$ and $\tau_{(2)} = (\mathcal{E}_v^{< 0}, \mathcal{E}_v^{> 0})$ be two $t$-structures on $\mathcal{C}$. Denote by $\mathcal{E}_v^{< 0}$ and $\mathcal{E}_v^{> 0}$ the corresponding hearts. The $t$-structures $(\tau_{(1)}, \tau_{(2)})$ satisfy

$$\mathcal{E}_v^{< -1} \subseteq \mathcal{E}_v^{< 0}_{(1)} \subseteq \mathcal{E}_v^{< 0}_{(2)}$$

if and only if $\tau_{(2)}$ is the tilting of $\tau_{(1)}$ with respect to the torsion pair $(\mathcal{E}_v^{< 0}, \mathcal{E}_v^{> 0}, \mathcal{E}_v^{< 0}, \mathcal{E}_v^{> 0})$ in $\mathcal{E}_v^{\triangleright}$.

We collect the following result which will be useful later on.

**Lemma 3.16.** Let $\nu = (\mathcal{F}, \mathcal{F})$ be a torsion pair on $\mathcal{E}_v^{\triangleright}$. If $\mathcal{F}$ is a Serre subcategory of $\mathcal{E}_v^{\triangleright}$, then it is a Serre subcategory of $\mathcal{E}_v^{\triangleright}$ as well.

**Proof.** Since $\mathcal{F}$ is the torsion-free part of a torsion pair on $\mathcal{E}_v^{\triangleright}$ by Lemma 3.15, Lemma 3.2 shows that it is closed under extensions and subobjects. It is then enough to prove that it is closed under quotients. To prove this, let

$$0 \longrightarrow T_1 \longrightarrow T \longrightarrow T_2 \longrightarrow 0$$

be a short exact sequence in $\mathcal{E}_v^{\triangleright}$ and assume that $T \in \mathcal{F}$. Then $T_1 \in \mathcal{F}$ as well, and passing to the associated long exact sequence with respect to the $t$-structure $\tau$ we find

$$0 \longrightarrow \mathcal{H}_T^{-1}(T_2) \longrightarrow \mathcal{H}_T^0(T_1) \longrightarrow \mathcal{H}_T^0(T) \longrightarrow \mathcal{H}_T^0(T_2) \longrightarrow 0.$$

Since $T_2 \in \mathcal{E}_v^{\triangleright}$ by assumption, $\mathcal{H}_T^{-1}(T_2) \in \mathcal{F}$. On the other hand, $T_1 \simeq \mathcal{H}_T^0(T_1)$ is torsion. Therefore the injectivity of the map $\mathcal{H}_T^{-1}(T_2) \rightarrow T_1$ together with the assumption that $\mathcal{F}$ is a Serre subcategory of $\mathcal{E}_v^{\triangleright}$ implies that $\mathcal{H}_T^{-1}(T_2)$ belongs to $\mathcal{F}$ as well, which implies $\mathcal{H}^{-1}(T_2) = 0$. Thus, $T_2 \in \mathcal{F}$. \qed
Construction 3.17. Given \((\mathcal{C}, \tau)\) and a torsion pair \(v = (\mathcal{I}, \mathcal{F})\) on \(\mathcal{C}^\triangledown\), we define the derived stack \(\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau)\) as the fiber product
\[
\begin{array}{ccc}
\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau) & \longrightarrow & \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau^v) \\
\downarrow & & \downarrow \\
\operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) & \longrightarrow & \operatorname{Perf}_{\mathcal{F}}(\mathcal{C})
\end{array}
\]

Let \([1]: \operatorname{Perf}_{\mathcal{F}}(\mathcal{C}) \to \operatorname{Perf}_{\mathcal{F}}(\mathcal{C})\) be the morphism corresponding to the shift by 1 in \(\mathcal{C}\). Then we define \(\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau)\) as the fiber product
\[
\begin{array}{ccc}
\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau) & \longrightarrow & \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau^v) \\
\downarrow & & \downarrow \\
\operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) & \longrightarrow & \operatorname{Perf}_{\mathcal{F}}(\mathcal{C}) [1] \to \operatorname{Perf}_{\mathcal{F}}(\mathcal{C})
\end{array}
\]

Definition 3.18 (cf. [AB13, Definition A.2]). We say that a torsion pair \(v = (\mathcal{I}, \mathcal{F})\) on \(\mathcal{C}^\triangledown\) is open if the morphisms
\[
\begin{array}{ccc}
\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau) & \longrightarrow & \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \\
\end{array}
\]

are representable by open immersions of schemes.

Warning 3.19. Let \(v = (\mathcal{I}, \mathcal{F})\) be an open torsion pair on \(\mathcal{C}^\triangledown\). Let \(A\) be a discrete commutative algebra. Since we have the obvious inclusions \((\mathcal{C}_A)^{\leq -1} \subseteq (\mathcal{C}_A)^{\leq 0} \subseteq (\mathcal{C}_A)^{\leq 0}\), so Lemma 3.15 shows that the induced t-structure \((\tau_A)_v\) is obtained as a tilting of \(\tau_A\) by a torsion pair \(v_A\), whose torsion part is given by \(\mathcal{E}_A^\triangledown \cap \mathcal{E}_A^\triangledown\). If \(F \in \operatorname{Perf}(\mathcal{C}_A)\) classifies an \(A\)-point of \(\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau)\) then, since \(A\) is discrete, we obviously have that \(F\) belongs to the torsion part of \(v_A\). However, the reader should be aware that the converse is typically false: in order for an object in the torsion part of \(v_A\) to classify an \(A\)-point of \(\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau)\) it has to be flat with respect to both \(\tau_A\) and \(\tau_A\). This is always the case when \(A\) is a field, but it is typically false otherwise.

Notation 3.20. The maps \(\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau) \to \operatorname{Coh}(\mathcal{C}, \tau)\) and \(\operatorname{Coh}_\mathcal{I}(\mathcal{C}, \tau) \to \operatorname{Coh}(\mathcal{C}, \tau)\) define 2-Segal substacks of \(\mathcal{S}_\mathcal{C}(\mathcal{C}, \tau)\) and \(\mathcal{S}_\mathcal{C}(\mathcal{C}, \tau)\) via Proposition 2.6 and Lemma 3.8. To keep the notation under control, we simply denote them by \(\mathcal{S}_\mathcal{C}(\mathcal{C}, \tau)\) and \(\mathcal{S}_\mathcal{C}(\mathcal{C}, \tau)\). We apply a similar convention to all the other possible combinations of \(\mathcal{I}, \mathcal{F}\), left and right.

Theorem 3.21. Let \(\mathcal{C}\) be a finite type compactly generated stable \(\infty\)-category equipped with a t-structure \(\tau = (\mathcal{C}^{\leq 0}, \mathcal{C}^{> 0})\) which is left and right complete, compatible with filtered colimits, and satisfies openness of flatness in the sense of [DPSa]. Let \(v = (\mathcal{I}, \mathcal{F})\) be an open torsion pair on \(\mathcal{C}^\triangledown\). Assume that:

1. the tilted t-structure \(\tau_v\) satisfies openness of flatness;
2. both maps
   \[
   \partial_0 \times \partial_2: \mathcal{S}_2 \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \longrightarrow \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \times \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau)
   \]
   
   and
   \[
   \partial_0 \times \partial_2: \mathcal{S}_2 \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \longrightarrow \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \times \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau)
   \]
   
   are derived lci, while both maps
   \[
   \partial_1: \mathcal{S}_2 \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \longrightarrow \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau)\]
   
   and
   \[
   \partial_1: \mathcal{S}_2 \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau) \longrightarrow \operatorname{Coh}_{\mathcal{F}}(\mathcal{C}, \tau)
   \]
   
   are representable by proper algebraic spaces.
3. the abelian category \(\mathcal{I}\) is a Serre subcategory of \(\mathcal{C}^\triangledown\).
(4) the maps

\[
\text{Coh}_\tau (\mathcal{E}, \tau) \longrightarrow \text{Coh}_{ps}(\mathcal{E}, \tau) \quad (3.2)
\]
\[
\text{Coh}_\tau (\mathcal{E}, \tau) \longrightarrow \text{Coh}_{ps}(\mathcal{E}, \tau_\ell) \quad (3.3)
\]

are representable by open and closed embeddings;

Then \( \text{Coh}_\tau (\mathcal{E}, \tau) \) seen as a substack of \( \text{Coh}_{ps}(\mathcal{E}, \tau) \) satisfies assumption \((2_\tau)\) of Corollary 3.11, while \( \text{Coh}_\tau (\mathcal{E}, \tau) \) seen as a substack of \( \text{Coh}_{ps}(\mathcal{E}, \tau_\ell) \) satisfies assumption \((2_\tau)\) of Corollary 3.11. As a consequence:

- \( \text{Coh}_{ps}^b(\text{Coh}_\tau (\mathcal{E}, \tau)) \) inherits the structure of a \( \mathbb{E}_1 \)-monoidal pro-\( \infty \)-category. In particular, \( \mathcal{G}_0(\text{Coh}_\tau (\mathcal{E}, \tau)) \) and \( H_{BM}^\mathcal{BM}(\text{Coh}_\tau (\mathcal{E}, \tau)) \) have the structure of associative algebras, and

- \( \text{Coh}_{ps}^b(\text{Coh}_\tau (\mathcal{E}, \tau)) \) has both the structure of a categorical left and of a categorical right module over \( \text{Coh}_{pro}^b(\text{Coh}_\tau (\mathcal{E}, \tau)) \). In particular, \( \mathcal{G}_0(\text{Coh}_\tau (\mathcal{E}, \tau)) \) (resp. \( H_{BM}^\mathcal{BM}(\text{Coh}_\tau (\mathcal{E}, \tau)) \)) has both the structure of a left and a right \( \mathcal{G}_0(\text{Coh}_\tau (\mathcal{E}, \tau)) \)-module (resp. \( H_{BM}^\mathcal{BM}(\text{Coh}_\tau (\mathcal{E}, \tau)) \)-module).

**Proof.** Observe that the morphisms \( \text{Coh}_\tau (\mathcal{E}, \tau) \to \text{Coh}_{ps}(\mathcal{E}, \tau) \) and \( \text{Coh}_\tau (\mathcal{E}, \tau) \to \text{Coh}_{ps}(\mathcal{E}, \tau_\ell) \) extend to pushout squares, fitting in the following pullback square:

\[
\begin{array}{ccc}
\text{S}_*\text{Coh}_\tau (\mathcal{E}, \tau) & \longrightarrow & \text{S}_*\text{Coh}_{ps}(\mathcal{E}, \tau) \\
\downarrow & & \downarrow \\
\text{S}_*\text{Coh}_{ps}(\mathcal{E}, \tau_\ell) & \longrightarrow & \text{S}_*\text{Perf}_{ps}(\mathcal{E})
\end{array}
\]

In other words, we obtain a canonical equivalence of 2-Segal derived stacks

\[
\text{S}_*\text{Coh}_\tau (\mathcal{E}, \tau) \approx \text{S}_*\text{Coh}_\tau (\mathcal{E}, \tau_\ell).
\]

Now, Assumption 3.7 is satisfied since the torsion pair is open. Assumptions (1) and (2) guarantee that the \( \text{Coh}_\tau (\mathcal{E}, \tau) \) satisfies condition \((1_\ell)\) of Corollary 3.11 is satisfied for (and, dually, thanks to Lemma 3.16 that \( \text{Coh}_{ps}(\mathcal{E}, \tau_\ell) \) satisfies condition \((1_\ell)\) of loc. cit.) Equation (3.2) in assumption (4) shows that \( \text{Coh}_\tau (\mathcal{E}, \tau) \) also satisfies condition \((2_\ell)\) of Corollary 3.11. Therefore, the existence of the algebra structure and of the left action follows.

On the other hand, Equation (3.3) of assumption (4) shows that \( \text{Coh}_\tau (\mathcal{E}, \tau_\ell) \) satisfies condition \((2_\ell)\) of Corollary 3.11. Thus, we obtain that \( \text{Coh}_{ps}^b(\text{Coh}_\tau (\mathcal{E}, \tau_\ell)) \) carries a right categorical module structure over \( \text{Coh}_{pro}^b(\text{Coh}_\tau (\mathcal{E}, \tau_\ell)) \) (and similarly for \( \mathcal{G}_0 \) and \( H_{BM}^\mathcal{BM} \)). At this point, the conclusion follows by observing that the shift \([1]: \text{Perf}_{ps}(\mathcal{E}) \to \text{Perf}_{ps}(\mathcal{E}) \) induces an equivalence

\[
[1]: \text{Coh}_\tau (\mathcal{E}, \tau) \longrightarrow \text{Coh}_{\tau}[1](\mathcal{T}, \tau^\nu).
\]

\[\Box\]

**Corollary 3.22.** Keep the assumptions of Theorem 3.21. In addition, let \( T \to \text{Coh}_\tau (\mathcal{E}, \tau) \) be a morphism representable by open and closed immersions and let \( F \to \text{Coh}_\tau (\mathcal{E}, \tau) \) be a morphism representable by open immersions for which the squares

\[
\begin{array}{ccc}
\text{S}_2\text{Coh}_\tau (\mathcal{E}, \tau) & \longrightarrow & \text{S}_2\text{Coh}_{ps}(\mathcal{E}, \tau) \\
\downarrow & & \downarrow \mathcal{D}_0 \times \mathcal{D}_2 \quad ,
\end{array}
\]

\[
\begin{array}{ccc}
\text{S}_2\text{Coh}_\tau (\mathcal{E}, \tau) & \longrightarrow & \text{S}_2\text{Coh}_{ps}(\mathcal{E}, \tau) \\
\downarrow & & \downarrow \mathcal{D}_1 \quad ,
\end{array}
\]

\[
\begin{array}{ccc}
T \times T & \longrightarrow & \text{Coh}_{ps}(\mathcal{E}, \tau) \times \text{Coh}_{ps}(\mathcal{E}, \tau) \\
\downarrow & & \downarrow \mathcal{D}_1 
\end{array}
\]

(3.4)
For this, we first look at $S_\theta$ goal is to provide a geometric interpretation of the commutator of the left and the right actions.

\[ \text{Definition 3.23.} \quad \text{The categorified quantum loop algebra} \ \mathcal{H}_{(T,F)} \text{ of the pair} \ (T,F) \ \text{is the monoidal subcategory of the monoidal} \ \infty \text{-category of endofunctors} \ \text{End}(\mathcal{C}_0(F)) \ \text{generated by the images of the two monoidal functors} \]

\[ a_\ell : \mathcal{C}_0(T) \rightarrow \text{End}(\mathcal{C}_0(F)), \]

\[ a_\tau : \mathcal{C}_0(T) \rightarrow \text{End}(\mathcal{C}_0(F)), \]

corresponding to the two module structures of $\mathcal{C}_0(F)$.

The quantum loop algebra $U\mathcal{H}_{(T,F)}$ of the pair $(T,F)$ is the subalgebra of $\text{End}(\mathcal{C}_0(F))$ generated by the images of the two maps of associative algebras

\[ a_\ell : \mathcal{C}_0(T) \rightarrow \text{End}(\mathcal{C}_0(F)), \]

\[ a_\tau : \mathcal{C}_0(T) \rightarrow \text{End}(\mathcal{C}_0(F)), \]

corresponding to the two module structures of $\mathcal{C}_0(F)$. Similarly, we define the Yangian $\mathcal{Y}_{(T,F)}$ of the pair $(T,F)$.

3.3. Categorified commutators. Let us place ourselves again in the context of Theorem 3.21. Our goal is to provide a geometric interpretation of the commutator of the left and the right actions. For this, we first look at $S_3\text{Perf}_{ps}(\mathcal{C})$, and observe that the simplicial identities guarantee that the diagram

\[ \begin{array}{ccc}
\text{Perf}_{ps}(\mathcal{C}) \times S_2\text{Perf}_{ps}(\mathcal{C}) \times \text{Perf}_{ps}(\mathcal{C}) & \xrightarrow{\text{id} \times \partial_0 \times \partial_2} & \text{Perf}_{ps}(\mathcal{C}) \\
\text{Perf}_{ps}(\mathcal{C}) \times \text{Perf}_{ps}(\mathcal{C}) & \xrightarrow{\partial_0 \times \partial_2} & S_2\text{Perf}_{ps}(\mathcal{C}) \\
\text{Perf}_{ps}(\mathcal{C}) \times S_2\text{Perf}_{ps}(\mathcal{C}) & \xrightarrow{\partial_0 \times \partial_2} & S_2\text{Perf}_{ps}(\mathcal{C}) \\
\end{array} \]
commutes. Recall that $S_3 \text{Perf}(\mathcal{C})$ parametrizes flags of the form
\[
\begin{array}{ccc}
0 & \rightarrow & F_{01} \\
\downarrow & & \downarrow \\
0 & \rightarrow & F_{12} \\
\downarrow & & \downarrow \\
0 & \rightarrow & F_{23} \\
\downarrow & & \\
0 & \rightarrow 
\end{array}
\]
(3.7)

Define now an open substack $S_3 \text{FlagCoh}^{(1)}_{((\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$ of $S_3 \text{Perf}(\mathcal{C})$ parametrizing flags $\mathcal{F}$ of the form (3.7) satisfying

- $\partial_3(\mathcal{F}) \in S'_1 \text{FlagCoh}^{(1)}_{(\mathcal{F}, (\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau_{-\omega})$ and $\partial_{0,1}(\mathcal{F}) = F_{23} \in \text{Coh}_\mathcal{T}(\mathcal{C}, \tau)$;

- $\partial_1(\mathcal{F}) \in S'_1 \text{FlagCoh}^{(1)}_{(\mathcal{F}, (\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$,

where $\tau_{-\omega} := \tau_{\mathcal{T}[1]}^{-1}$ denotes the anti-tilted t-structure.\(^{12}\)

Remark 3.24. Note that for the flags of the form (3.7) parametrized by $S_3 \text{FlagCoh}^{(1)}_{((\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$ one has that $F_{13}$ is only a $\tau$-flat pseudo-perfect complex. \(\triangle\)

$S_3 \text{FlagCoh}^{(1)}_{((\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$ fits into the convolution diagram encoding the composition of the right action with the left action:\(^{13}\)

\[
\begin{array}{ccc}
S_3 \text{FlagCoh}^{(1)}_{((\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau) & \xrightarrow{\partial_1} & S'_1 \text{FlagCoh}^{(1)}_{(\mathcal{F}, (\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau) \\
\downarrow & & \downarrow \\
S'_1 \text{FlagCoh}^{(1)}_{(\mathcal{T}, [1], \mathcal{T})}(\mathcal{C}, \tau_{-\omega}) \times \text{Coh}_\mathcal{T}(\mathcal{C}, \tau) & \xrightarrow{\partial_0 \times \partial_2 \times \text{id}} & (\text{Coh}_\mathcal{T}_{[1]}(\mathcal{C}, \tau_{-\omega}) \times \text{Coh}_\mathcal{T}(\mathcal{C}, \tau)) \times \text{Coh}_\mathcal{T}(\mathcal{C}, \tau)
\end{array}
\]

We set
\[
p_{r, t} := (\partial_0 \times \partial_2 \times \text{id}) \circ (\partial_3 \times \partial_{0,1}) \quad \text{and} \quad q^{r, t} := \partial_1 \circ \partial_1.
\]

Observe that $p_{r, t}$ is derived lci, while $q^{r, t}$ is representable by proper algebraic spaces.

Define similarly an open substack $S_3 \text{FlagCoh}^{(1)}_{((\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$ of $S_3 \text{Perf}(\mathcal{C})$ parametrizing flags $\mathcal{F}$ of the form (3.7) satisfying

- $\partial_0(\mathcal{F}) \in S'_1 \text{FlagCoh}^{(1)}_{(\mathcal{F}, (\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$ and $\partial_{2,3}(\mathcal{F}) = F_{01} \in \text{Coh}_{\mathcal{T}[1]}(\mathcal{C}, \tau_{-\omega})$;

- $\partial_2(\mathcal{F}) \in S'_1 \text{FlagCoh}^{(1)}_{(\mathcal{T}, [1], \mathcal{T})}(\mathcal{C}, \tau_{-\omega})$.

Remark 3.25. Note that for the flags of the form (3.7) parametrized by $S_3 \text{FlagCoh}^{(1)}_{((\mathcal{T}, \mathcal{T}), \mathcal{T})}(\mathcal{C}, \tau)$ one has that $F_{02}$ is only a $\tau_{-\omega}$ pseudo-perfect complex. \(\triangle\)

\(^{12}\)In this subsection, we prefer to work with the torsion pair $(\mathcal{T}, \mathcal{T}[1])$ rather than $(\mathcal{T}[1], \mathcal{T})$, since it makes our computations clearer to the reader.

\(^{13}\)As we pointed out before, when we consider the left action (resp. right action) the stack which gives rise to the algebra is on the “right” (resp. “left”) with respect to the stack which gives rise to the module.
$S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau)$ fits into the convolution diagram encoding the composition of the left action with the right action:

$$
\begin{array}{c}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau) \\
\downarrow^{\delta_2 \times \delta_0}
\end{array}
\begin{array}{c}
S_1^1\text{FlagCoh}^{(1)}_{(\mathcal{J}_{[-1]},\mathcal{J})}(\mathcal{E}, \tau_{-\nu}) \\
\downarrow^{\delta_1}
\end{array}
\begin{array}{c}
\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)
\end{array}
$$

$\text{Coh}_{\mathcal{J}_{[-1]}}(\mathcal{E}, \tau_{-\nu}) \times (\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau) \times \text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau))$

We set

$$p_{\ell, r} := (id \times \delta_0 \times \delta_2) \circ (\delta_{2,3} \times \delta_0) \quad \text{and} \quad q^{\ell, r} := \partial_1 \circ \partial_2.$$

Observe that $p_{\ell, r}$ is derived lci, while $q^{\ell, r}$ is representable by proper algebraic spaces.

Summarizing the discussion so far, we have diagrams

$$
\begin{array}{ccc}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau) \\
\downarrow^{p_{\ell, r}} \\
\text{Coh}_{\mathcal{J}_{[-1]}}(\mathcal{E}, \tau_{-\nu}) \times \text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau) \\
\downarrow^{q^{\ell, r}} \\
\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)
\end{array}
\begin{array}{ccc}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau) \\
\downarrow^{p_{\ell, r}} \\
\text{Coh}_{\mathcal{J}_{[-1]}}(\mathcal{E}, \tau_{-\nu}) \times \text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau) \\
\downarrow^{q^{\ell, r}} \\
\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)
\end{array}
$$

The maps $p_{\ell, r}$ and $q^{\ell, r}$ induce after passing to $\text{Coh}_{\text{pro}}^b$ the functor

$$\text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}_{[-1]}}(\mathcal{E}, \tau_{-\nu})) \otimes \text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)) \otimes \text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)) \rightarrow \text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau))$$

that can be written

$$G' \otimes \mathcal{E} \otimes \mathcal{G} \rightarrow q^{\ell, r}_*(p_{\ell, r}^*(G' \boxtimes \mathcal{E} \boxtimes \mathcal{G})) \simeq (G' \otimes \mathcal{E}) \otimes \mathcal{G}.$$

On the other hand, the morphism $p_{\ell, r}$ and $q^{\ell, r}$ induce in the same way the functor

$$\text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}_{[-1]}}(\mathcal{E}, \tau_{-\nu})) \otimes \text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)) \otimes \text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)) \rightarrow \text{Coh}_{\text{pro}}^b(\text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau))$$

that can be written

$$G' \otimes \mathcal{E} \otimes \mathcal{G} \rightarrow q^{\ell, r}_*(p_{\ell, r}^*(G' \boxtimes \mathcal{E} \boxtimes \mathcal{G})) \simeq G' \otimes (\mathcal{E} \otimes \mathcal{G}).$$

Now, we discuss when the two compositions coincide. First, we define $S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau)$ as the open substack of $S_3\text{Perf}(\mathcal{E})$ parametrizing flags $\mathcal{F}$ of the form for which $F_0[-1], F_{23} \in \text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)$ and $F_{12}, F_{03} \in \text{Coh}_{\mathcal{J}}(\mathcal{E}, \tau)$. Finally, we define $S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau)$ as the fiber pullback

$$
\begin{array}{c}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau) \\
\downarrow
\end{array}
\begin{array}{c}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau)
\end{array}
\begin{array}{c}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau) \\
\downarrow
\end{array}
\begin{array}{c}
S_3\text{FlagCoh}^{(1)}_{(\mathcal{J},\mathcal{J},\mathcal{I})}(\mathcal{E}, \tau)
\end{array}.$$
Here, all the maps are open immersions.

**Proposition 3.26.**

1. Assume that $\mathcal{T}$ is a Serre subcategory of $\mathcal{C}^\circ$. For every geometric point $y: \text{Spec}(k) \to S$, the natural morphisms $$(F_{13})_{\text{tor}} \to F_{23} \quad \text{and} \quad (F_{13})_{\text{tor}} \to H^1_{\tau}(F_{01})$$ are monomorphisms in the heart $\mathcal{C}^\circ$ of $\mathcal{T}$.

2. For every geometric point $y: \text{Spec}(k) \to S$, the natural morphisms $H^1_{\tau}(F_{01}) \to H^1_{\tau}(F_{02})$ and $H^0_{\tau}(F_{23}) \to H^1_{\tau}(F_{02})$ are surjective in the heart $\mathcal{C}^\circ$ of $\mathcal{T}$.

**Proof.** Let us prove the first assertion. Consider the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & (F_{13})_{\text{tor}} & \longrightarrow & F_{13} & \longrightarrow & (F_{13})_{\text{t.f.}} & \longrightarrow & 0 \\
& & \downarrow \alpha & & \downarrow & & \downarrow & \\
0 & \longrightarrow & (F_{23})_{\text{tor}} & \sim & F_{23} & \longrightarrow & 0
\end{array}
$$

whose rows are exact. Applying the snake lemma we obtain that the exact sequence

$$0 \longrightarrow \ker(\alpha) \longrightarrow F_{12} \longrightarrow (F_{13})_{\text{t.f.}}.
$$

Since $F_{12}$ is torsion free by assumption, it follows that $\ker(\alpha)$ is torsion free as well. On the other hand, $\ker(\alpha)$ is a subobject of $(F_{13})_{\text{tor}}$. Since $\mathcal{T}$ is a Serre subcategory of $\mathcal{C}^\circ$, it follows that $\ker(\alpha)$ is torsion as well, and hence $\ker(\alpha) = 0$. This proves the injectivity of the first map. For the second map, we consider instead the following diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & (F_{13})_{\text{tor}} & \longrightarrow & F_{13} & \longrightarrow & (F_{13})_{\text{t.f.}} & \longrightarrow & 0 \\
& & \downarrow \alpha' & & \downarrow & & \downarrow & \\
0 & \longrightarrow & H^1_{\tau}(F_{01})_{\text{tor}} & \sim & H^1_{\tau}(F_{01}) & \longrightarrow & 0
\end{array}
$$

Reasoning as above, we see that $\ker(\alpha')$ is a subobject of $F_{03}$, which is torsion-free by assumption. At the same time, $\ker(\alpha')$ is also a subobject of $(F_{13})_{\text{tor}}$, and hence (since $\mathcal{T}$ is a Serre subcategory), we deduce that $\ker(\alpha') = 0$.

We now prove the second assertion. Starting from the fiber sequence $F_{01} \to F_{02} \to F_{12}$, we obtain the long exact sequence

$$H^1_{\tau}(F_{01}) \longrightarrow H^1_{\tau}(F_{02}) \longrightarrow H^1_{\tau}(F_{12}) = 0,$$

which implies the first surjectivity. On the other hand, starting from $F_{02} \to F_{03} \to F_{23}$, we obtain

$$H^0_{\tau}(F_{23}) \longrightarrow H^1_{\tau}(F_{02}) \longrightarrow H^1_{\tau}(F_{03}) = 0,$$

whence the conclusion. 

**Corollary 3.27.** Let $i: \mathcal{T} \to \text{Coh}_{\mathcal{T}_{\geq 1}}(\mathcal{C}, \tau_{\geq 0})$ and $i': \mathcal{T}' \to \text{Coh}_{\mathcal{T}}(\mathcal{C}, \tau)$ be two closed immersions. Assume that the following condition holds:

- $\mathcal{T}$ is a Serre subcategory of $\mathcal{C}^\circ$. 


• for every geometric point $x$: $\text{Spec}(k) \to \text{Coh}(\mathcal{E}, \tau)$ classifying an object $M \in \mathcal{C}_k$, if there exist two surjections
  
  $$\mathcal{H}_1^\tau(F) \longrightarrow M \quad \text{and} \quad \mathcal{H}_0^\tau(F') \longrightarrow M$$

  in $\mathcal{C}_k$, such that $F$ is induced by a point $y \in T(k)$ via the embedding $i$, and $F'$ is classified by a point $y' \in T'(k)$ via the embedding $i'$, then $M = 0$.

• for every geometric point $x$: $\text{Spec}(k) \to \text{Coh}(\mathcal{E}, \tau)$ classifying an object $M \in \mathcal{C}_k$, if there exist two monomorphisms

  $$M \longrightarrow \mathcal{H}_1^\tau(F) \quad \text{and} \quad M \longrightarrow \mathcal{H}_0^\tau(F')$$

  in $\mathcal{C}_k$, such that $F$ is induced by a point $y \in T(k)$ via the embedding $i$, and $F'$ is classified by a point $y' \in T'(k)$ via the embedding $i'$, then $M = 0$.

In this case, the fiber product

$$(T \times \text{Coh}_\mathcal{F}(\mathcal{E}, \tau) \times T') \times _{\mathcal{Coh}_\mathcal{F}[1]}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{S}_3 \text{FlagCoh}_1^{(1)}}(\mathcal{E}, \tau)$$

is equivalent to

$$(T \times \text{Coh}_\mathcal{F}(\mathcal{E}, \tau) \times T') \times _{\mathcal{Coh}_\mathcal{F}[1]}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{S}_3 \text{FlagCoh}_1^{(1)}}(\mathcal{E}, \tau).$$

In particular, for any $\mathcal{G} \in \text{Coh}_{b, \text{pro}}^b(T)$, $\mathcal{G}' \in \text{Coh}_{b, \text{pro}}^b(T')$, and $\mathcal{E} \in \text{Coh}_{b, \text{pro}}^b(\text{Coh}_\mathcal{F}(\mathcal{E}, \tau))$ we get

$$(i'_* \mathcal{G}' \otimes \mathcal{E}) \otimes i_* \mathcal{G} \simeq i'_* \mathcal{G}' \otimes (\mathcal{E} \otimes i_* \mathcal{G}).$$

**Proof.** Both stacks are both equivalent to

$$\left(T \times \text{Coh}_\mathcal{F}(\mathcal{E}, \tau) \times T'\right) \times _{\mathcal{Coh}_\mathcal{F}[1]}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{Coh}_\mathcal{F}}(\mathcal{E}, \tau) \times _{\mathcal{S}_3 \text{FlagCoh}_1^{(1)}}(\mathcal{E}, \tau),$$

thanks to Proposition 3.26. \(\Box\)

4. COHA OF A SURFACE AND ITS REPRESENTATION VIA TORSION-FREE SHEAVES

4.1. Algebra and representations. Let $S$ be a smooth projective irreducible complex surface. Consider the torsion pair on $\text{Coh}(S)$:

$$\mathcal{T} = \text{Coh}_{\text{tor}}(S) := \{ \mathcal{E} \in \text{Coh}(S) \mid \text{dim}(\mathcal{E}) \leq 1 \},$$

$$\mathcal{F} = \text{Coh}_{\text{lf}}(S) := \{ \mathcal{E} \in \text{Coh}(S) \mid \mathcal{E} \text{ is torsion-free} \}.$$  

We write $\mathcal{T}_\mathcal{B}$ for the t-structure obtained tilting the standard t-structure of $\text{Perf}(S)$ with respect to the above torsion pair. We denote by $\mathcal{B}$ the heart of $\mathcal{T}_\mathcal{B}$.

**Remark 4.1.** The t-structure $\mathcal{T}_\mathcal{B}$ can be realized as the perverse t-structure (in the sense of [Bay09, §3.1]) associated with the perversity function

$$p: \{0, 1, 2\} \to \mathbb{Z}, \quad p(n) := -\left\lfloor \frac{n}{2} \right\rfloor,$$

i.e.,

$$p(0) = p(1) = 0 \quad \text{and} \quad p(2) = -1.$$  

The function $p$ is called the large volume perversity function. \(\triangle\)

**Remark 4.2.** Note that $\text{Coh}_{\text{tor}}(S)$ is uniquely characterized as the subcategory of $\text{Coh}(S)$ consisting by rank zero sheaves. Thus, it is straightforward to see that $\text{Coh}_{\text{tor}}(S)$ is Serre subcategory of $\text{Coh}(S)$, hence it is also a Serre subcategory of $\mathcal{B}$ by Lemma 3.16. \(\triangle\)
Recall that $\mathcal{C}_S := \text{QCoh}(S) = \text{Ind}((\text{Perf}(S)))$. Denote by $\text{Coh}(S) = \text{Coh}(\mathcal{C}_S, \tau_{\text{std}})$ the derived moduli stack of coherent sheaves on $S$, where $\tau_{\text{std}}$ denotes the standard $t$-structure. Construction 3.17 yields two substacks

\[
\text{Coh}_{\text{tor}}(S) := \text{Coh}_\tau(\mathcal{C}_S, \tau_{\text{std}}) \quad \text{and} \quad \text{Coh}_{\text{eft}}(S) := \text{Coh}_\tau(\mathcal{C}_S, \tau_{\text{std}})
\]

of $\text{Coh}(S)$ respectively parametrizing torsion and torsion-free sheaves on $S$.

These substacks have been shown to be open in [AB13, Example A.4], hence $(\mathcal{T}, \mathcal{F})$ is an open torsion pair. Moreover, this yields that the classical stack of flat objects in the tilted heart $\mathcal{B}$ is geometric and locally of finite presentation over $C$ by [AB13, Corollary A.9]. Thus, it follows from [DPSa] that the tilted $t$-structure $\tau_B$ satisfies openness of flatness and hence that $\text{Coh}(S, \tau_B) := \text{Coh}(\mathcal{C}_S, \tau_B)$ is a geometric derived stack locally of finite presentation over $C$, which is open inside $\text{Perf}(S)$.

Furthermore, Construction 3.17 immediately implies that the natural morphisms

\[
\text{Coh}_{\text{tor}}(S) \to \text{Coh}(S, \tau_B) \quad \text{and} \quad [1] : \text{Coh}_{\text{eft}}(S) \to \text{Coh}(S, \tau_B)
\]

are representable by open immersions.

Recall that one has a decomposition of $\text{Coh}(S)$ into open and closed substacks

\[
\text{Coh}(S) = \bigsqcup_{(r, \beta, n) \in \mathbb{Z} \times \text{NS}(S) \times \mathbb{Z}} \text{Coh}(S; r, \beta, n),
\]

where each component corresponds to coherent sheaves $E$ on $S$ with rank $r$, first Chern class $\beta$, and Euler characteristic $n$. The stack $\text{Coh}(S, \tau_B)$ admits a similar decomposition. Thus, $\text{Coh}_{\text{tor}}(S)$ is equivalent to the connected component corresponding to zero rank objects of both $\text{Coh}(S)$ and $\text{Coh}(S, \tau_B)$, and hence it is open and closed inside both $\text{Coh}(S)$ and $\text{Coh}(S, \tau_B)$.

The main result of this section is the following:

**Theorem 4.3.** The stable pro-$\infty$-category $\text{Coh}^b_{\text{pro}}(\text{Coh}_{\text{tor}}(S))$ has a $E_1$-monoidal structure. In particular, $G_0(\text{Coh}_{\text{tor}}(S))$ and $H^B_*(\text{Coh}_{\text{tor}}(S))$ have the structure of associative algebras.

The stable pro-$\infty$-category $\text{Coh}^b_{\text{pro}}(\text{Coh}_{\text{eft}}(S))$ has the structure of a left and a right categorical module over $\text{Coh}^b_{\text{pro}}(\text{Coh}_{\text{tor}}(S))$. In particular, $G_0(\text{Coh}_{\text{eft}}(S))$ (resp. $H^B_*(\text{Coh}_{\text{eft}}(S))$) has the structure of a left and a right module over $G_0(\text{Coh}_{\text{tor}}(S))$ (resp. over $H^B_*(\text{Coh}_{\text{tor}}(S))$).

**Proof.** It is enough to apply Theorem 3.21. We already argued that both $\tau_{\text{std}}$ and $\tau_B$ satisfy openness of flatness. Assumption (1) has been verified in [PS19] for $\tau_{\text{std}}$ and in [DPSa] for $\tau_B$. Assumption (2) follows from Remark 4.2. Finally we argued above that $\text{Coh}_{\text{tor}}(S)$ is closed in both $\text{Coh}(S, \tau_{\text{std}})$ and in $\text{Coh}(S, \tau_B)$, hence also assumption (3) is verified. Thus, the conclusion follows.

We have an decomposition of the stack $\text{Coh}_{\text{eft}}(S)$ of torsion-free sheaves on $S$:

\[
\text{Coh}_{\text{eft}}(S) = \bigsqcup_{(r, \beta, n) \in \mathbb{Z} \times \text{NS}(S)} \text{Coh}_{\text{eft}}(S; r, \beta, n).
\]

Set

\[
\text{Coh}_{\text{eft}}(S; r) := \bigsqcup_{(\beta, n) \in \text{NS}(S)} \text{Coh}_{\text{eft}}(S; r, \beta, n).
\]

By applying Corollary 3.22 to $T := \text{Coh}_{\text{tor}}(S)$ and $F = \text{Coh}_{\text{eft}}(S; r)$, we get the following:

**Corollary 4.4.** The stable pro-$\infty$-category $\text{Coh}^b_{\text{pro}}(\text{Coh}_{\text{eft}}(S; r))$ has the structure of a left and a right categorical module over $\text{Coh}^b_{\text{pro}}(\text{Coh}_{\text{tor}}(S))$. In particular, $G_0(\text{Coh}_{\text{eft}}(S; r))$ (resp. $H^B_*(\text{Coh}_{\text{eft}}(S; r))$) has the structure of a left and a right module over $G_0(\text{Coh}_{\text{tor}}(S))$ (resp. over $H^B_*(\text{Coh}_{\text{tor}}(S))$).

\[14\]Recall that any $t$-structure on $\text{Perf}(S)$ canonically lifts to a $t$-structure on $\text{QCoh}(S)$: we will not distinguish between the two.
Let $\mathcal{H}_{\text{tor}, t, f}$, $U_{\text{tor}, t, f}$, and $Y_{\text{tor}, t, f}$ be the algebras of Definition 3.23 associated to the pair $(\text{Coh}_{\text{tor}}(S), \text{Coh}_f(S))$. Similarly, let $Y_{\text{tor}, t, f}(r), U_{\text{tor}, t, f}(r)$, and $Y_{\text{tor}, t, f}(r)$ be the algebras associated to $(\text{Coh}_{\text{tor}}(S), \text{Coh}_f(S)(r))$. The decomposition of $\text{Coh}_{t, f}(S)$ with respect to the rank induces the decompositions:

$$\mathcal{H}_{t, f} \cong \bigoplus_r \mathcal{H}_{t, f}(r), \quad U_{t, f} \cong \bigoplus_r U_{t, f}(r) \quad \text{and} \quad Y_{t, f} \cong \bigoplus_r Y_{t, f}(r).$$

Let $\text{Coh}_{0, \text{dim}}(S)$ be the derived moduli stack of zero-dimensional torsion sheaves on $S$. Note that the square (3.4) is a pullback with $T := \text{Coh}_{0, \text{dim}}(S)$, since the subcategory $\text{Coh}_{0, \text{dim}}(S)$ is Serre and $\text{Coh}_{0, \text{dim}}(S)$ is open and closed in $\text{Coh}_{t, f}(S)$. Thus, we can obtain analogous to Theorem 4.3 and Corollary 4.4 for the pairs $(T := \text{Coh}_{0, \text{dim}}(S), F := \text{Coh}_{t, f}(S))$ and $(T := \text{Coh}_{0, \text{dim}}(S), F := \text{Coh}_{t, f}(S(1)))$, by applying Corollary 3.22. Correspondingly, we can introduce the algebras of Definition 3.23 associated to these pairs and have the following decompositions:

$$\mathcal{H}_{0, \text{dim}, t, f} \cong \bigoplus_{r, c_1} \mathcal{H}_{0, \text{dim}, t, f}(r, c_1), \quad U_{0, \text{dim}, t, f} \cong \bigoplus_{r, c_1} U_{0, \text{dim}, t, f}(r, c_1)$$

and

$$Y_{0, \text{dim}, t, f} \cong \bigoplus_{r, c_1} Y_{0, \text{dim}, t, f}(r, c_1),$$

where the sums are over all possible ranks $r \in \mathbb{Z}_{\geq 1}$ and first Chern classes $c_1 \in \text{NS}(S)$ of torsion-free coherent sheaves on $S$.

4.2. Vanishing of categorified commutators. We now apply the study of the categorified commutators of §3.3 to provide a geometric criterion guaranteeing that two operator commute. Fix two closed subschemes $Z_1$ and $Z_2$ of $S$ and let

$$i_1: Z_1 \rightarrow S \quad \text{and} \quad i_2: Z_2 \rightarrow S$$

be the natural inclusions. There are induced morphisms

$$j_1: \text{Coh}(Z_1) \rightarrow \text{Coh}_{t, f}(S) \quad \text{and} \quad j_2: \text{Coh}(Z_2) \rightarrow \text{Coh}_{t, f}(S),$$

given by the pushforward along $i_1$ and $i_2$. It follows from that the morphisms $j_1$ and $j_2$ are closed immersions. In particular, for $i = 1, 2$ we obtain functors

$$j_i^*: \text{Coh}_{\text{pro}}(\text{Coh}(Z_i)) \rightarrow \text{Coh}_{\text{pro}}(\text{Coh}_{t, f}(S)).$$

Assume now that $Z_1 \cap Z_2 = \emptyset$. This guarantees that, if $F, F'$ are coherent sheaves on $Z_1$ and $Z_2$ respectively, and we are given surjections (resp. monomorphisms)

$$i_{1, *}(F) \twoheadrightarrow M \quad \text{and} \quad i_{2, *}(F') \twoheadrightarrow M$$

(resp. $M \hookrightarrow i_{1, *}(F)$ and $M \hookrightarrow i_{2, *}(F')$)

for some coherent sheaf $M$ on $S$, then $\text{supp}(M) \subseteq Z_1 \cap Z_2$, and hence necessarily $M = 0$. Thus, the assumptions of Corollary 3.27 are satisfied. Writing $\tau\tau_B := \tau_B[-1]$ and setting $\text{Coh}(S, \tau\tau_B) := \text{Coh}(0, \tau\tau_B)$, we obtain:

**Corollary 4.5.** Let $Z_1$ and $Z_2$ of $S$ be two disjoint closed subschemes. Denote by $i_i$ the corresponding inclusions of subschemes and by $j_i$ the inclusions of the corresponding moduli stacks of sheaves, for $i = 1, 2$. For $G_1 \in \text{Coh}_{\text{pro}}(\text{Coh}(Z_1))$ and $G_2 \in \text{Coh}_{\text{pro}}(\text{Coh}(Z_2))$, we have

$$(j_2, G_2 \otimes \mathcal{E}) \otimes \Phi(j_1, G_1) \simeq j_2, G_2 \otimes (\mathcal{E} \otimes \Phi(j_1, G_1))$$

for any $\mathcal{E} \in \text{Coh}_{\text{pro}}(\text{Coh}_{t, f}(S))$. Therefore the operators induced by $G_1$ and $G_2$ on $\text{Coh}_{\text{pro}}(\text{Coh}_{t, f}(S))$ commute. Here, we denote by $\Phi: \text{Coh}_{\text{pro}}(\text{Coh}_{t, f}(S)) \rightarrow \text{Coh}_{\text{pro}}(\text{Coh}_{\tau\tau_B}(S))$ induced by the equivalence $[-1]: \text{Coh}_{t, f}(S) \rightarrow \text{Coh}_{\tau\tau_B}(S)$. Similar statements hold in K-theory and Borel-Moore homology.
4.3. COHAs of zero-dimensional sheaves and moduli spaces of Gieseker-stable sheaves. In this section we compare our approach to that of Neguţ [Neg19]. In loc. cit., Neguţ introduced operators acting on the K-theory of moduli spaces of Gieseker-stable sheaves on S arising from certain Hecke correspondences associated to zero-dimensional sheaves and proved that they generate an algebra satisfying relations resembling those of the Ding-Iohara-Miki algebra\textsuperscript{15}. We shall call Neguţ’s algebra the Ding-Iohara-Miki algebra of S.

Let $H$ be an ample divisor and let $\text{Coh}^{H,s\text{g}}_{\text{lf}}(S)$ be the derived moduli stack of $H$-Gieseker-(semi)-stable torsion-free sheaves on S, which is an open substack of $\text{Coh}_{\text{lf}}(S)$. We have a decomposition into open and closed substacks

$$\text{Coh}^{H,s\text{g}}_{\text{lf}}(S) := \bigsqcup_{r,c_1,c_2} \text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1,c_2)$$

with respect to the rank $r$ and the Chern classes $c_1, c_2$ of stable torsion free sheaves. Fix $r, c_1$ and define

$$\text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1) := \bigsqcup_{c_2} \text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1,c_2).$$

Let $\mathcal{M}_S(r,c_1)$ be the (coarse) moduli space of the classical truncation $^c\text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1)$ of $\text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1)$.

Now, let us fix $r$ and $c_1 \in \text{NS}(S)$ such that $\text{g.c.d.}(r,c_1 \cdot H) = 1$ (cf. [Neg19, Assumption A]). This implies that all semistable sheaves are automatically stable. We can apply Corollary 3.22 to $T := \text{Coh}_{0\text{-dim}}(S)$ and $F := \text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1)$. Therefore, we obtain the following result.

**Theorem 4.6.** The stable pro-$\infty$-category $^b\text{Coh}^\text{pro}(\text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1))$ has the structure of a left and a right categorical module over $^b\text{Coh}^\text{pro}(\text{Coh}_{0\text{-dim}}(S))$. In particular, $G_0(\mathcal{M}_S(r,c_1))_C$ (resp. $H_0^{BM}(\mathcal{M}_S(r,c_1))$) has the structure of a left and a right module over $G_0(\text{Coh}_{0\text{-dim}}(S))_C$ (resp. over $H_0^{BM}(\text{Coh}_{0\text{-dim}}(S))$).

**Proof.** We have already discussed that the square (3.4) is a pullback with $T := \text{Coh}_{0\text{-dim}}(S)$, since the subcategory $\text{Coh}_{0\text{-dim}}(S)$ is Serre, and $\text{Coh}_{0\text{-dim}}(S)$ is open and closed in $\text{Coh}_{\text{tor}}(S)$. Moreover, thanks to [Neg19, Proposition 5.5], the squares (3.6) and (3.5) are pullback for $F := \text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1)$ and $T := \text{Coh}_{0\text{-dim}}(S)$, and $\mathcal{M}_S(r,c_1)$ is open in $\text{Coh}_{\text{lf}}(S)$. Thus, Corollary 3.22 applies and the assertion follows.

Thanks to the Theorem, we can define the corresponding algebras following Definition 3.23: we denote them by

$$\mathcal{H}_{0\text{-dim,at}}(r,c_1), \mathcal{U}_{0\text{-dim,at}}(r,c_1) \text{ and } y_{0\text{-dim,at}}(r,c_1).$$

**Remark 4.7.** Since $\text{Coh}^{H,s\text{g}}_{\text{lf}}(S;r,c_1)$ is open in $\text{Coh}_{\text{lf}}(S;r,c_1)$, we have corresponding pullbacks in Borel-Moore homology, K-theory, and bounded derived category. They induce a functor of monoidal categories and homomorphisms of associative algebras, respectively:

$$\mathcal{H}_{0\text{-dim,lf}}(r,c_1) \rightarrow \mathcal{H}_{0\text{-dim,at}}(r,c_1), \mathcal{U}_{0\text{-dim,lf}}(r,c_1) \rightarrow \mathcal{U}_{0\text{-dim,at}}(r,c_1),$$

and

$$y_{0\text{-dim,lf}}(r,c_1) \rightarrow y_{0\text{-dim,at}}(r,c_1).$$

\[ \triangle \]

Now, let us introduce the algebra acting on the K-theory of $\mathcal{M}_S(r,c_1)$, which is related to the Ding-Iohara-Miki algebra of $S$: let $\mathcal{U}_{\text{pt,at}}(r,c_1)$ be the subalgebra of $\mathcal{U}_{0\text{-dim,at}}(r,c_1)$ generated by $G_0(\text{Coh}_{\text{pt}}(S;1))_C$, where $\text{Coh}_{\text{pt}}(S;1)$ is the closed substack of $\text{Coh}_{0\text{-dim}}(S)$ parametrizing zero-dimensional sheaves on $S$ scheme-theoretically supported at a single point and of length one. Let us recall [Neg19, Assumption S]: the canonical bundle $K_S$ of $S$ is either trivial or satisfies $c_1(K_S) \cdot H < 0$. Under this assumption, $\mathcal{M}_S(r,c_1,c_2)$ is a smooth projective variety. [Neg19, Theorem 1.2] yields the following result.

\textsuperscript{15}Note that this does not coincide with the elliptic Hall algebra, rather the latter is a quotient of the former.
Proposition 4.8. Under Assumptions A and S, the algebra $U_{\text{pt, at}}(r, c_1)$ is independent of $r, c_1$ and it coincides with the Ding-Iohara-Miki algebra of $S$ introduced in [Neg19].

Remark 4.9. We can define the monoidal subcategory $\mathcal{H}_{\text{pt, at}}(r, c_1)$ of $\mathcal{H}_{\dim, \text{at}}(r, c_1)$ generated by $\text{Coh}^b_{\text{pro}}(\text{Coh}_{\text{at}}(S; 1))$. This contains the endofunctors defined by Neguț in [Neg22], by upgrading from K-theory to bounded derived category the construction in [Neg19].

4.4. COHAs of one-dimensional sheaves and moduli spaces of rank one stable sheaves. A rank one torsion-free sheaf is automatically Gieseker-stable, with respect to any ample line bundle. Let $\mathcal{M}_S(1)$ be the (coarse) moduli space of rank one Gieseker-stable sheaves on $S$, which is a smooth projective variety. We have the following:

Corollary 4.10. $G_0(\mathcal{M}_S(1))_{\text{C}}$ (resp. $H^\text{BM}_1(\mathcal{M}_S(1))$) has the structure of a left and a right module over $G_0(\text{Coh}_{\text{tor}}(S))_{\text{C}}$ (resp. over $H^\text{BM}_1(\text{Coh}_{\text{tor}}(S))$).

Thanks to the above corollary, we can define the algebras of Definition 3.23 associated to the pair $(\text{Coh}_{\text{tor}}(S), \mathcal{M}_S(1))$. In particular, we denote by $\mathcal{Y}_{\text{tor}, r, k=1}$ the corresponding Yangian.

From now on, let $S$ be a K3 surface such that $\text{NS}(S)$ is generated by irreducible $-2$ curves and any pair of irreducible $-2$ curves on $S$ are either disjoint or intersect transversally at a single point. Let $\mathcal{Y}_{\text{NS}(S), r, k=1}$ be the subalgebra of $\mathcal{Y}_{\text{tor}, r, k=1}$ generated by the fundamental classes of the substacks $\text{Coh}_{\text{tor}}(S; 0, c_1, c_2)$ parametrizing torsion sheaves with first Chern class given by a $-2$ curve and with arbitrary second Chern class. The main result of [DeH20] yields the following.

Proposition 4.11. The algebra $\mathcal{Y}_{\text{NS}(S), r, k=1}$ is isomorphic to the modified universal enveloping algebra $\mathcal{U}({\mathfrak{g}}(\text{NS}(S)))$ of the affine Lie algebra $\mathfrak{g}(\text{NS}(S))$, introduced in [DeH20].

5. Perverse COHA of a surface and its perverse torsion-free representation

5.1. Reminders on perverse coherent sheaves. Let $S$ be a smooth projective irreducible complex surface. By analogy with [Tod09, §2.2], we define the following full subcategories of $\text{Coh}(S)$:

$\text{Coh}_0(S) := \{ \mathcal{E} \in \text{Coh}(S) \mid \dim \text{Supp}(\mathcal{E}) = 0 \}$,
$\text{Coh}_{\geq 1}(S) := \{ \mathcal{E} \in \text{Coh}(S) \mid \text{Hom}_S(T, \mathcal{E}) = 0 \text{ for any } T \in \text{Coh}_0(S) \}$.

Essentially the same proof as in [Tod09, Lemma 2.10] yields:

Lemma 5.1. The pair $(\text{Coh}_0(S), \text{Coh}_{\geq 1}(S))$ is a torsion pair of $\text{Coh}(S)$.

We denote by $\tau_A$ the $t$-structure on $\text{Perf}(S)$ obtained by tilting the standard $t$-structure with respect to the torsion pair $(\text{Coh}_0(S), \text{Coh}_{\geq 1}(S))$, and we denote by $A$ the heart of $\tau_A$.

In [AB13, Example A.4-(1)] it is shown that the torsion pair $(\text{Coh}_0(S), \text{Coh}_{\geq 1}(S))$ is open. Thus, the classical stack of flat objects belonging to $A$ is geometric and locally of finite presentation over $\mathbb{C}$ by [AB13, Corollary A.9]. Therefore, the $t$-structure $\tau_A$ satisfies openness of flatness by [DPSa] and there exists a derived moduli stack $\mathcal{S}_S(\text{Coh}(S, \tau_A)) := \text{Coh}(\mathcal{C}_S, \tau_A)$ parametrizing $\tau_A$-flat families of objects in $\text{Perf}(S)$ (cf. [DPSa]). We obtain in this way a 2-Segal derived stack $\mathcal{S}_S(\text{Coh}(S, \tau_A))$, and it follows from [DPSa] that the map

$\partial_0 \times \partial_1 : \mathcal{S}_S(\text{Coh}(S, \tau_A)) \rightarrow \text{Coh}(S, \tau_A) \times \text{Coh}(S, \tau_A)$

is derived lci, while the map

$\partial_1 : \mathcal{S}_S(\text{Coh}(S, \tau_A)) \rightarrow \text{Coh}(S, \tau_A)$

is representable by proper algebraic spaces.

As usual, this gives rise to Hall algebra structures on $\text{Coh}^b_{\text{pro}}(\text{Coh}(S, \tau_A))$, $G_0(\text{Coh}(S, \tau_A))$ and $H^1_1(\text{Coh}(S, \tau_A))$. 

Remark 5.2. Notice that the abelian category $\mathcal{A}$ is the heart of the perverse $t$-structure given by the perversity function $p: \{0,1,2\} \to \mathbb{Z}$ given by
\[ p(0) = 0 \quad \text{and} \quad p(1) = -1 = p(2). \]
Notice that this is nothing but the dual perversity (in the sense of [Bay09, Definition 3.1.1]) of the large volume function $p$ of Remark 4.1. \hfill \triangle

Following [Tod09, §2.3], let $\mathcal{A}_{\text{tor}}$ be the smallest full subcategory of $\mathcal{A}$ closed under extensions and containing both $\mathcal{F}[1]$ for pure one-dimensional sheaves $\mathcal{F}$, and $\mathcal{O}_x$ for $x \in S$; we also set:
\[ \mathcal{A}_{\text{tor}} := \{ E \in \mathcal{A} \mid \text{Hom}_\mathcal{A}(F, E) = 0 \text{ for any } F \in \mathcal{A}_{\text{tor}} \} \subset \mathcal{A}. \]

As before, the following lemma can be proven by an argument parallel to the one given in [Tod09, Lemma 2.16].

Lemma 5.3. The pair $(\mathcal{A}_{\text{tor}}, \mathcal{A}_{\text{tor}})$ is a torsion pair of $\mathcal{A}$.

Definition 5.4. We call the objects of $\mathcal{A}_{\text{tor}}$ torsion perverse sheaves on $S$, while the objects of $\mathcal{A}_{\text{tor}}$ torsion-free perverse sheaves on $S$.

Applying Construction 3.17 to the torsion pair $(\mathcal{A}_{\text{tor}}, \mathcal{A}_{\text{tor}})$, we obtain two substacks
\[ \text{Coh}_{\text{tor}}(S, \tau_\mathcal{A}) := \text{Coh}_{\mathcal{A}_{\text{tor}}}(\mathcal{E}_S, \tau_\mathcal{A}) \quad \text{and} \quad \text{Coh}_{\text{tor}}(S, \tau_\mathcal{A}) := \text{Coh}_{\mathcal{A}_{\text{tor}}}(\mathcal{E}_S, \tau_\mathcal{A}) \]
of $\text{Coh}(S, \tau_\mathcal{A})$. Our goal in this section is to show that Theorem 3.21 can be applied to the current setup. In order to do this, the first thing to do is to check that $(\mathcal{A}_{\text{tor}}, \mathcal{A}_{\text{tor}})$ is an open torsion pair. For the torsion part, this is not difficult to see:

Lemma 5.5. An object $E \in \mathcal{A}$ belongs to $\mathcal{A}_{\text{tor}}$ if and only if $\text{rk}(E) = 0$. In particular,
\begin{itemize}
  \item $\mathcal{H}^0(E)$ is zero-dimensional and $\mathcal{H}^{-1}(E)$ is pure one-dimensional for any $E \in \mathcal{A}_{\text{tor}}$,
  \item $\mathcal{A}_{\text{tor}}$ is a Serre subcategory of $\mathcal{A}$, and
  \item $\text{Coh}_{\text{tor}}(S, \tau_\mathcal{A})$ is both open and closed inside $\text{Coh}(S, \tau_\mathcal{A})$.
\end{itemize}

Proof. Since the rank is additive, it immediately follows from the definition that every object $E$ in $\mathcal{A}_{\text{tor}}$ satisfies $\text{rk}(E) = 0$. Vice-versa, if $E \in \mathcal{A}$, then $\mathcal{H}^0(E)$ is zero-dimensional by definition. If $\text{rk}(E) = 0$, then we also have $\text{rk}(\mathcal{H}^{-1}(E)) = 0$, and therefore $\mathcal{H}^{-1}(E) \in \text{Coh}_{\text{tor}}(S)$. On the other hand, since $E \in \mathcal{A}$, we also have $\mathcal{H}^{-1}(E) \in \text{Coh}_{\text{tor}}(S)$, and therefore $\mathcal{H}^{-1}(E)$ is pure one-dimensional.

For statement (2), since $\mathcal{A}_{\text{tor}}$ is the torsion part of a torsion pair, it is automatically closed under quotients and extensions. If
\[ 0 \to T_1 \to T \to T_2 \to 0 \]
is a short exact sequence in $\mathcal{A}$ and $T \in \mathcal{A}_{\text{tor}}$, then $T_2 \in \mathcal{A}_{\text{tor}}$, whence $\text{rk}(T_1) = \text{rk}(T) - \text{rk}(T_2) = 0$. Thus, $T_1 \in \mathcal{A}_{\text{tor}}$.

For the final statement, consider the group homomorphism
\[ \text{cl}: \mathcal{A} \to \mathbb{Z} \oplus \text{NS}(S) \oplus \mathbb{Z} \]
sending an object $E$ to the pair $(\text{rk}(E), \text{ch}_1(E), \chi(E))$; it induces a decomposition of $\text{Coh}(S, \tau_\mathcal{A})$ into open and closed substacks
\begin{equation}
\text{Coh}(S, \tau_\mathcal{A}) = \bigcup_{v \in \mathbb{Z} \oplus \text{NS}(S) \oplus \mathbb{Z}} \text{Coh}(S, \tau_\mathcal{A}; v),
\end{equation}
where each component corresponds to objects $E \in \mathcal{A}$ such that $\text{cl}(E) = v$. Then the first half of the lemma shows that
\[ \text{Coh}_{\text{tor}}(S, \tau_\mathcal{A}) = \bigcup_{\text{rk}(v) = 0} \text{Coh}(S, \tau_\mathcal{A}; v). \]
Therefore $\text{Coh}_{\text{tor}}(S, \tau_\mathcal{A})$ is both open and closed inside $\text{Coh}(S, \tau_\mathcal{A})$. \hfill \square
On the other hand, to prove that $\text{Coh}_{t,f}(S, \tau_A)$ is open inside $\text{Coh}(S, \tau_A)$ requires some extra work. Rather than doing it directly, we provide a duality result that yields a friendlier understanding of $A_{t,f}$.

5.2. Duality. In order to provide a more insightful description of $A_{t,f}$, we start by relating the heart $\mathcal{A}$ with the heart $\mathcal{B}$ considered in the previous section. The precise relation between these two abelian categories is given by the following result:

**Lemma 5.6.** Let $\tau_A^\text{op}$ be the opposite t-structure on $\text{Perf}(S)^\text{op}$ induced by $\tau_A$. Then the self-equivalence $(-)^\vee[2]: \text{Perf}(S)^\text{op} \to \text{Perf}(S)$ is t-exact with respect to the t-structure $\tau_A^\text{op}$ on $\text{Perf}(S)^\text{op}$ and the t-structure $\tau_B$ on $\text{Perf}(S)$. In particular, it restricts to an equivalence $(-)^\vee[2]: A^\text{op} \to \mathcal{B}^\text{op}$.

**Proof.** The second half of the statement is an obvious consequence of the first half. To prove the t-exactness, let $\omega_S \in \text{Pic}(S)$ be the canonical line bundle of $S$. Then the self-equivalence $- \otimes_{O_S} \omega_S: \text{Perf}(S) \to \text{Perf}(S)$ is t-exact with respect to the standard t-structure. Furthermore, this functor sends both $\text{Coh}_{t,\text{tor}}(S)$ and $\text{Coh}_{t,f}(S)$ to themselves. It follows that this functor is $\tau_B$-exact, and therefore to prove the claim it is enough to prove that the composite $\mathcal{D} := (-)^\vee[2] \otimes_{O_S} \omega_S: \text{Perf}(S)^\text{op} \to \text{Perf}(S)^\text{op}$ is t-exact with respect to the t-structure $\tau_A^\text{op}$ on the source and $\tau_B$ on the target. Since $\tau_A$ and $\tau_B$ are the perverse t-structures associated with dual perversity functions (see Remark 5.2), the conclusion follows directly from [Bay09, Proposition 7.2.1].

The main result of this section can be formulated as follows:

**Theorem 5.7.** The self-equivalence $(-)^\vee[2]: A^\text{op} \to \mathcal{B}$ identifies the torsion pair $(A^\text{op}_{t,f}, A^\text{op}_{t,\text{tor}})$ on $A^\text{op}$ with the torsion pair $(\text{Coh}_{t,f}(S)[1], \text{Coh}_{t,\text{tor}}(S))$ on $\mathcal{B}$. In other words, $(-)^\vee[2]$ also induces equivalences

$$(-)^\vee[2]: A^\text{op}_{t,\text{tor}} \to \text{Coh}_{t,\text{tor}}(S),$$

$$(-)^\vee[2]: A^\text{op}_{t,f} \to \text{Coh}_{t,f}(S)[1].$$

**Proof.** Since $(-)^\vee[2]$ induces an equivalence between $A^\text{op}$ and $\mathcal{B}$, it is enough to prove the following two statements:

1. an object $E \in \text{Perf}(S)$ belongs to $A_{t,\text{tor}}$ if and only if $E^\vee[2]$ belongs to $\text{Coh}_{t,\text{tor}}(S)$;

2. an object $E \in \text{Perf}(S)$ belongs to $A_{t,f}$ if and only if $E^\vee[2]$ belongs to $\text{Coh}_{t,f}(S)[1]$.

Moreover, since $A^\text{op}_{t,f}$ is the left orthogonal to $A^\text{op}_{t,\text{tor}}$ inside $A^\text{op}$, and similarly $\text{Coh}_{t,f}(S)[1]$ is the left orthogonal of $\text{Coh}_{t,\text{tor}}(S)$ inside $\mathcal{B}$, it is actually enough to prove the first statement.

We start by proving the “only if” part of statement (1). Attached to any object $E$ of $A_{t,\text{tor}}$ there is a canonical fiber sequence

$$\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E),$$

Applying $(-)^\vee[2]$, we obtain

$$\mathcal{H}^0(E)^\vee[2] \to E^\vee[2] \to \mathcal{H}^{-1}(E)^\vee[1].$$
Lemma 5.5 guarantees that $\mathcal{H}^{-1}(E)$ is a pure one-dimensional sheaf and $\mathcal{H}^0(E)$ is zero-dimensional. Hence [HL10, Proposition 1.1.6-(i)] provides us with canonical identifications

$$
\mathcal{H}^0(E)^\vee[2] \simeq \mathcal{E}xt_2^2(\mathcal{H}^0(E), \mathcal{O}_S) \quad \text{and} \quad \mathcal{H}^{-1}(E)^\vee[1] \simeq \mathcal{E}xt_1^2(\mathcal{H}^{-1}(E), \mathcal{O}_S).
$$

This shows at the same time that $\mathcal{H}^0(E)^\vee[2]$ is zero-dimensional and that $\mathcal{H}^{-1}(E)^\vee[1]$ is purely 1-dimensional. Thus, $\mathcal{H}^i(E^\vee[2]) = 0$ unless $i = 0$, and $\mathcal{H}^0(E^\vee[2])$ is a torsion sheaf.

For the “if” of statement (1), it is enough to prove that if $E \in \text{Coh}_{\text{tor}}(S)$, then $E^\vee[2]$ belongs to $A_{\text{tor}}$. Observe that Lemma 5.1 attaches to $E$ a canonical fiber sequence

$$
E_0 \rightarrow E \rightarrow E_{\geq 1},
$$

where $E_0$ is the maximal zero-dimensional subsheaf of $E$, and $E_{\geq 1} \in \text{Coh}_{\geq 1}(S)$. Since $E \in \text{Coh}_{\text{tor}}(S)$, $E_{\geq 1}$ is purely 1-dimensional. Applying $(-)^\vee[2]$ we obtain

$$
E_{\geq 1}^\vee[2] \rightarrow E^\vee[2] \rightarrow E_0^\vee[2].
$$

Then $E_0^\vee[2] \simeq \mathcal{E}xt_2^2(E_0, \mathcal{O}_S)$ is zero-dimensional, and $E_{\geq 1}^\vee[2] \simeq \mathcal{E}xt_1^2(E_{\geq 1}, \mathcal{O}_S)[1]$ is purely 1-dimensional. Thus, $E^\vee[2] \in A_{\text{tor}}$.  

**Corollary 5.8.** The morphism of derived stacks

$$
\text{Coh}_{\text{lf}}(S, \tau_A) \rightarrow \text{Coh}(S, \tau_A)
$$

is representable by open immersions.

**Proof.** Since we already argued that the $t$-structure $\tau_A$ satisfies openness of flatness, we see that $\text{Coh}(S, \tau_A)$ is open in $\text{Perf}(S)$. It is therefore enough to prove that the same goes for $\text{Coh}_{\text{lf}}(A)$. Let $j: \text{Coh}_{\text{lf}}(A) \rightarrow \text{Perf}(S)$ be the canonical morphism. Since $(\quad)^\vee[1]: \text{Perf}(S)^{\text{op}} \simeq \text{Perf}(S)$ induces a self equivalence of the derived stack $\text{Perf}(S)$ (which we still denote in the same way), we see that it is enough to prove that the map

$$
(\quad)^\vee[1] \circ j: \text{Coh}_{\text{lf}}(S, \tau_A) \rightarrow \text{Perf}(S)
$$

is an equivalence. However, Theorem 5.7 canonically identifies this map with the inclusion of $\text{Coh}_{\text{lf}}(S)$, which is indeed open in $\text{Coh}(S)$, and hence, a posteriori, in $\text{Perf}(S)$.  

We finish this discussion on duality with the following two structural results on $A_{\text{lf}}$ and $A_{\text{tor}}$:

**Corollary 5.9.** Let $E \in A_{\text{lf}}$. Then:

1. the sheaf $\mathcal{H}^{-1}(E)$ is locally free of rank $-\text{rk}(E)$;
2. one has

$$
\text{rk}(E) = -\text{rk}(E^\vee[1]);
$$

3. the sequence

$$
0 \rightarrow \mathcal{H}^0(E^\vee[1]) \rightarrow \mathcal{H}^{-1}(E) \rightarrow \mathcal{E}xt^2(\mathcal{H}^0(E), \mathcal{O}_S) \rightarrow 0
$$

is short exact and corresponds to the one associated to the inclusion of $\mathcal{H}^0(E^\vee[1])$ into its undervived double dual.

**Proof.** To begin with, Theorem 5.7 allows to write $E \simeq E^\vee[1]$, where $E \in \text{Coh}_{\text{lf}}(S)$. Let $W$ be the undervived double dual of $E$. Since $E$ is torsion free, $W$ is locally free, the canonical map $E \rightarrow W$ is injective and its cokernel $T$ is zero-dimensional. They fit into the short exact sequence

$$
0 \rightarrow E \rightarrow W \rightarrow T \rightarrow 0. \tag{5.2}
$$

Rotating, we obtain a fiber sequence

$$
T \rightarrow E[1] \rightarrow W[1].
$$

Applying $(-)^\vee[2]$, we obtain

$$
W[1] \rightarrow E^\vee[1] \rightarrow T^\vee[2].
$$
Passing to the long exact sequence and using the fact that $\mathcal{T}^\vee[2]$ is concentrated in degree zero, we obtain

$$\mathcal{H}^{-1}(E) \simeq \mathcal{H}^{-1}(\mathcal{E}^\vee[1]) \simeq \mathcal{W}^\vee,$$

and thus we deduce that $\mathcal{H}^{-1}(E)$ is locally free.

Consider now the canonical fiber sequence

$$\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E).$$

Applying $(-)^\vee[1]$ and rotating, we obtain the fiber sequence

$$\mathcal{E} \to \mathcal{H}^{-1}(E)^\vee \to \mathcal{H}^0(E)^\vee[2].$$

Since $\mathcal{H}^{-1}(E)^\vee \simeq \mathcal{W}$, and $\mathcal{H}^0(E)^\vee[2]$ has amplitude $[0, 0]$, the above fiber sequence corresponds to the short exact sequence (5.2). Hence

$$\mathcal{T} \simeq \mathcal{H}^0(E)^\vee[2] \simeq \mathcal{E}_{\mathcal{H}^0(E), \mathcal{O}_S}.$$

Finally, since $\mathcal{T}$ is zero-dimensional, it follows that

$$\text{rk}(\mathcal{E}) = \text{rk}(\mathcal{H}^{-1}(E)^\vee) = \text{rk}(\mathcal{H}^{-1}(E)) = -\text{rk}(E).$$

The proof is therefore achieved. \qed

**Corollary 5.10.**

1. Under the equivalence $(-)^\vee[2]$: $\text{Coh}_\text{tor}(S) \simeq A_{\text{tor}}^{\text{op}}$, pure 1-dimensional sheaves correspond to objects $F \in A_{\text{tor}}$ satisfying $\mathcal{H}^0(F) = 0$.

2. Under the equivalence $(-)^\vee[2]$: $\text{Coh}_{\text{lf}}(S)[1] \simeq A_{\text{lf}}^{\text{op}}$, $\text{Vect}(S)[1]$ corresponds to the full subcategory of $A_{\text{lf}}^{\text{op}}$ spanned by objects $E$ satisfying $\mathcal{H}^0(E) = 0$. Here, $\text{Vect}(S) \subset \text{Coh}(S)$ denotes the subcategory of locally free sheaves on $S$.

**Proof.** Using [HL10, Proposition 1.1.6], the first statement follows from Lemma 5.5, while the second one follows from Corollary 5.9-(1). \qed

### 5.3. Algebra and representations

As a consequence of Lemmas 5.5 and Corollary 5.8, assumptions (1) and (2) of Corollary 3.11 are satisfied. Let now $\tau^A_v$ be the tilting of $\tau_A$ with respect to the torsion pair $v = (A_{\text{tor}}, A_{\text{lf}})$, and let us denote by $A^v$ the tilted heart. Since the tilting of $T_A$ with respect to the torsion pair $(\text{Coh}_\text{tor}(S)[1], \text{Coh}_\text{tor}(S))$ coincides with the shift by 1 of the standard t-structure, which satisfies again openness of flatness we immediately deduce from Theorem 5.7 that $\tau^A_v$ also satisfies openness of flatness. Moreover, it follows that $(A_{\text{lf}}[1], A_{\text{tor}})$ is an open torsion pair on $A^v$ and that $A_{\text{tor}}$ is a Serre subcategory of $A^v$. Finally, we also deduce that the map of derived stacks

$$\text{Coh}_\text{tor}(S, \tau_A) \to \text{Coh}(S, \tau^A_v)$$

is representable by open and closed embeddings. Thus, all the assumptions of Theorem 3.21 are satisfied, and therefore we obtain:

**Theorem 5.11.** The stable pro-$\infty$-category $\text{Coh}^\text{pro}_{\text{Coh}_{\text{tor}}(S, \tau_A)}$ has a $\mathbb{E}_1$-monoidal structure. In particular, $G_0(\text{Coh}_{\text{tor}}(S, \tau_A))$ and $H_*^{\text{BM}}(\text{Coh}_{\text{tor}}(S, \tau_A))$ have the structure of associative algebras.

The stable pro-$\infty$-category $\text{Coh}^\text{pro}_{\text{Coh}_{\text{lf}}(S, \tau_A)}$ has the structure of a left and a right categorical module over $\text{Coh}^\text{pro}_{\text{Coh}_{\text{tor}}(S, \tau_A)}$. In particular, $G_0(\text{Coh}_{\text{lf}}(S, \tau_A))$ (resp. $H_*^{\text{BM}}(\text{Coh}_{\text{lf}}(S, \tau_A))$) has the structure of a left and a right module over $G_0(\text{Coh}_{\text{tor}}(S, \tau_A))$ (resp. over $H_*^{\text{BM}}(\text{Coh}_{\text{tor}}(S, \tau_A))$).

As in Subsection 4.1, we can refine this action keeping track of the decomposition (5.1). More precisely, setting

$$\text{Coh}(S, \tau_A; r) := \bigsqcup_{\text{rk}(v) = -r} \text{Coh}(S, \tau_A; v)$$

and

$$\text{Coh}_{\text{lf}}(S, \tau_A; r) := \bigsqcup_{\text{rk}(v) = -r} \text{Coh}_{\text{lf}}(A_{\text{lf}} S, \tau_A; v),$$
we obtain:

**Corollary 5.12.** The stable pro-$\infty$-category $\text{Coh}_{\text{tor}}^b(S, \tau_A; r)$ has the structure of a left and a right categorical module over $\text{Coh}_{\text{tor}}^b(S, \tau_A)$. 

In particular, $G_0(\text{Coh}_{\text{tor}}^b(S, \tau_A; r))$ (resp. $H_*^{BM}(\text{Coh}_{\text{tor}}^b(S, \tau_A; r))$) has the structure of a left and a right module over $G_0(\text{Coh}_{\text{tor}}^b(S, \tau_A))$ (resp. over $H_*^{BM}(\text{Coh}_{\text{tor}}^b(S, \tau_A))$).

As it is natural to expect, Theorem 5.7 implies that the algebras and their representations constructed in Theorems 4.3 and 5.11 coincide. Indeed, Lemma 5.6 provides an identification of 2-Segal derived stacks

$$\mathcal{S}_s^{\mathbb{C}}\text{Coh}(S, \tau_A^\text{op}) \simeq \mathcal{S}_s\text{Coh}(S, \tau_B),$$

and compatibly an identification of relative 2-Segal derived stacks

$$\mathcal{S}_s^{\mathbb{C}}\text{Coh}(S, \tau_A^\text{op}) \simeq \mathcal{S}_s\text{Coh}(S, \tau_B) \quad \text{and} \quad \mathcal{S}_s\text{Coh}(S, \tau_A^\text{op}) \simeq \mathcal{S}_s\text{Coh}(S, \tau_B).$$

Paired with Theorem 5.7 and with the results of §4, this yields:

**Theorem 5.13.**

(1) The equivalence

$$(-)^V[2]: \text{Coh}_{\text{tor}}^b(S, \tau_A^\text{op}) \xrightarrow{\simeq} \text{Coh}_{\text{tor}}^b(S)$$

induces an equivalence of $\mathcal{E}_1$-monoidal stable pro-$\infty$-categories:

$$\Gamma: \text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{tor}}^b(S, \tau_A^\text{op})) \xrightarrow{\simeq} \text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{tor}}^b(S)).$$

It induces isomorphisms of associative algebras:

$$\Gamma: G_0(\text{Coh}_{\text{tor}}^b(S, \tau_A))^\text{op} \xrightarrow{\simeq} G_0(\text{Coh}_{\text{tor}}^b(S)), \quad \Gamma: H_*^{BM}(\text{Coh}_{\text{tor}}^b(S, \tau_A))^\text{op} \xrightarrow{\simeq} H_*^{BM}(\text{Coh}_{\text{tor}}^b(S)).$$

(2) The equivalences

$$(-)^V[1]: \text{Coh}_t(S, \tau_A^\text{op}; r) \rightarrow \text{Coh}_t(S; r)$$

and

$$(-)^V[2]: \text{Coh}_{\text{tor}}^b(S, \tau_A^\text{op}) \xrightarrow{\simeq} \text{Coh}_{\text{tor}}^b(S)$$

induce an equivalence of left and right categorical modules over $\text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{tor}}^b(S))$:

$$\Psi: \text{Coh}_{\text{pro}}^b(\text{Coh}_t(S, \tau_A^\text{op}; r)) \xrightarrow{\simeq} \text{Coh}_{\text{pro}}^b(\text{Coh}_t(S; r)).$$

At the $K$-theory and Borel-Moore homology levels we have isomorphisms

$$G_0(\text{Coh}_t(S, \tau_A^\text{op}; r)) \xrightarrow{\simeq} G_0(\text{Coh}_t(S; r)), \quad H_*^{BM}(\text{Coh}_t(S, \tau_A^\text{op}; r)) \xrightarrow{\simeq} H_*^{BM}(\text{Coh}_t(S; r))$$

of left and right modules over $G_0(\text{Coh}_{\text{tor}}^b(S))$ and of $H_*^{BM}(\text{Coh}_{\text{tor}}^b(S))$, respectively.

**Remark 5.14.** For every $\infty$-groupoid $K \in \mathcal{S}$, there is a functorial self-equivalence $\text{inv}_K: K_{\text{op}} \simeq K$. This induces an equivalence of derived stacks

$$\text{Coh}(S, \tau_A) \simeq \text{Coh}(S, \tau_A^\text{op}),$$

which nevertheless does not propagate through the natural 2-Segal structure on both sides. Concretely, this means that $\mathcal{E}_1$-monoidal pro-$\infty$-category $\text{Coh}_{\text{pro}}^b(\text{Coh}(S, \tau_A^\text{op}))$ has $\text{Coh}_{\text{pro}}^b(\text{Coh}(S, \tau_A))$ as underlying pro-$\infty$-category, and it has the opposite tensor structure.
6. Representations of the COHA of a Surface via Stable Pairs

6.1. Preliminaries on stable pairs. Following [PT09], a Pandharipande-Thomas stable pair is a pair \((\mathcal{F}, s)\) consisting of a pure one-dimensional sheaf \(\mathcal{F}\) on \(S\) and a global section \(s\) of \(\mathcal{F}\), which is generically surjective (i.e., the cokernel of \(s\) is zero-dimensional). In this section, we provide an alternative definition, which would be more suitable for us\(^{16}\).

**Proposition 6.1.** Let \(\mathcal{V}\) be a locally free sheaf on \(S\) of finite rank. For a fiber sequence

\[
\mathcal{V} \xrightarrow{s} \mathcal{F} \longrightarrow E
\]

in \(\text{Perf}(S)\), the following statements are equivalent:

1. \(\mathcal{F}\) is purely one-dimensional coherent sheaf and the morphism \(s\): \(\mathcal{V} \to \mathcal{F}\) has zero-dimensional cokernel;
2. \(E\) belongs to \(\mathcal{A}_{\text{lf}}\), and \(\mathcal{F}[1]\) belongs to \(\mathcal{A}_{\text{tor}}\);
3. \(E\) belongs to \(\mathcal{A}_{\text{lf}}\), \(\mathcal{F}[1]\) belongs to \(\mathcal{A}\) and \(\text{rk}(E) = -\text{rk}(\mathcal{V})\).

In this case, we further have \(\text{ch}_1(E) = \text{ch}_1(\mathcal{F})\) if and only if \(\text{ch}_1(\mathcal{V}) = 0\).

**Proof.** We first prove that (1) \(\Rightarrow\) (2).

Since \(\mathcal{F}\) is purely one-dimensional, we have \(\mathcal{F}[1] \in \mathcal{A}_{\text{tor}} \subseteq \mathcal{A}\) by definition. So we only have to show that \(E \in \mathcal{A}_{\text{lf}}\). Taking the long exact sequence of cohomology sheaves associated to (6.1) we obtain

\[
0 = \mathcal{H}^{-1}(\mathcal{F}) \longrightarrow \mathcal{H}^{-1}(\mathcal{E}) \longrightarrow \mathcal{H}^0(\mathcal{V}) \longrightarrow \mathcal{H}^0(\mathcal{F}) \longrightarrow \mathcal{H}^0(\mathcal{E}) \longrightarrow 0.
\]

The central terms are canonically identified with \(\mathcal{V}\) and \(\mathcal{F}\), respectively. Thus \(\mathcal{H}^{-1}(\mathcal{E})\) is torsion-free, and our assumption guarantees that \(\mathcal{H}^0(\mathcal{E})\) is zero-dimensional. In other words, \(E \in \mathcal{A}\). Rotating the sequence (6.1) we obtain the fiber sequence

\[
E \longrightarrow \mathcal{V}[1] \xrightarrow{s[1]} \mathcal{F}[1],
\]

where the three terms belong to \(\mathcal{A}\). It follows that this is in fact a short exact sequence in \(\mathcal{A}\) and therefore that the map \(E \to \mathcal{V}[1]\) is injective. Since \(\mathcal{V}\) is locally free, we have that \(\mathcal{V}'\) is again locally free, and in particular \((\mathcal{V}[1])' \simeq \mathcal{V}'[1]\) belongs to \(\text{Coh}_{\text{lf}}(S)[1]\). Hence, Theorem 5.7 guarantees that \(\mathcal{V}[1]\) belongs to \(\mathcal{A}_{\text{lf}}\). Thus, Lemma 3.2-(3) guarantees that \(E \in \mathcal{A}_{\text{lf}}\). Finally, Lemma 5.5 guarantees that \(\text{rk}(\mathcal{F}[1]) = 0\), and hence that

\[
\text{rk}(E) = \text{rk}(\mathcal{V}[1]) = -\text{rk}(\mathcal{V}).
\]

We now prove that (2) \(\Rightarrow\) (1).

Since \(\mathcal{F}[1] \in \mathcal{A}_{\text{tor}}\), Lemma 5.5 guarantees that \(\text{rk}(\mathcal{F}[1]) = 0\), that \(\mathcal{H}^0(\mathcal{F}) \simeq \mathcal{H}^{-1}(\mathcal{F}[1])\) is purely one-dimensional and that \(\mathcal{H}^1(\mathcal{F}) \simeq \mathcal{H}^0(\mathcal{F}[1])\) is zero-dimensional. Passing to the long exact sequence of cohomology sheaves associated to (6.1), we obtain

\[
0 = \mathcal{H}^1(\mathcal{V}) \longrightarrow \mathcal{H}^1(\mathcal{F}) \longrightarrow \mathcal{H}^1(\mathcal{E}) = 0.
\]

Thus, \(\mathcal{F} \simeq \mathcal{H}^{-1}(\mathcal{F}[1])\) is purely one-dimensional. Finally, the sequence (6.2) canonically identifies the cokernel of \(\mathcal{V} \to \mathcal{F}\) with \(\mathcal{H}^0(E)\), which is zero-dimensional because \(E \in \mathcal{A}\).

Finally, the equivalence (3) \(\Leftrightarrow\) (2) follows directly from Lemma 5.5: indeed, if \(\mathcal{F}[1]\) belongs to \(\mathcal{A}\), then \(\mathcal{F}[1] \in \mathcal{A}_{\text{tor}}\) if and only if \(\text{rk}(\mathcal{F}[1]) = 0\), and this is equivalent to ask \(\text{rk}(E) = -\text{rk}(\mathcal{V})\).

We now reformulate the notion of Pandharipande-Thomas stable pairs.

\(^{16}\)Our definition makes evident a possible generalization of a notion of stable pairs canonically associated to a torsion pair of \(\text{Perf}(S)\). See § 6.5.1 for more on this topic.
Definition 6.2. Let $V$ be a locally free sheaf of finite rank on $S$. A $V$-stable pair on $S$ is a fiber sequence $V \rightarrow F \rightarrow E$ satisfying the equivalent conditions of Proposition 6.1. When $V = \mathcal{O}_S^r$, we refer to $V$-stable pairs as rank $r$ stable pairs. When $r = 1$, we simply refer to them as stable pairs. We denote by $P(S; V)$ the $\infty$-category of $V$-stable pairs. When $V = \mathcal{O}_S^1$, we write $P(S; 1)$ instead of $P(S; \mathcal{O}_S^1)$. 

Remark 6.3. Our viewpoint on stable pairs resembles that in [Bri11] for Pandharipande-Thomas stable pairs on Calabi-Yau threefolds.

Remark 6.4. Fix a locally free sheaf $V$ of finite rank. Under the equivalence $\text{Perf}^\Delta(S) \simeq \text{Perf}^\mathrm{ext}(S)$, the $\infty$-category of $V$-stable pairs is equivalent to the $\infty$-category of pairs $(F, s)$, where $F[1] \in \mathcal{A}_{\tau \text{f}}$ and $s : V \rightarrow F$ is a morphism in $\text{Perf}(S)$ with the property that $\text{cofib}(s) \in \mathcal{A}_{\tau \text{f}}$. Alternatively, we can ask $F$ to be a pure one-dimensional sheaf and the cokernel of $s$ to be zero-dimensional. 

Remark 6.5. Let $X$ be a Calabi-Yau threefold. In [Tod09, Lemma 4.5] it is shown that the forgetful functor

$$\partial_0 : P(X; 1) \rightarrow \mathcal{A}_{\tau \text{f}}$$

sending a stable pair $V \xrightarrow{\phi} F \rightarrow E$ to $E = \text{cofib}(s)$ is an equivalence. This is not the case for surfaces, and indeed the proof given in loc. cit. uses as essential ingredient the fact that for any $E \in \mathcal{A}_{\tau \text{f}}$, there is a canonical isomorphism $\text{Hom}_\mathcal{A}(E, \mathcal{O}_X[1]) \simeq \mathbb{C}$. This is obtained as follows: start with the fiber sequence

$$\mathcal{H}^{-1}(E)[1] \rightarrow E \rightarrow \mathcal{H}^0(E) ;$$

applying $(-)^V[1]$ and passing to the long exact sequence of cohomology groups we obtain

$$\text{Ext}^1_X(\mathcal{H}^0(E), \mathcal{O}_X) \rightarrow \text{Ext}^1_X(E, \mathcal{O}_X) \rightarrow \text{Ext}^0_X(\mathcal{H}^{-1}(E), \mathcal{O}_X) \rightarrow \text{Ext}^2_X(\mathcal{H}^0(E), \mathcal{O}_X) .$$

Since $X$ is a threefold, the extremes vanish, and therefore the central morphism is an isomorphism. Finally, since $\mathcal{H}^{-1}(E)$ is the ideal sheaf of a curve, we have

$$\text{Ext}^0_X(\mathcal{H}^{-1}(E), \mathcal{O}_X) \simeq \mathbb{C} .$$

On the other hand, in the surface case, we have by Serre duality

$$\text{Ext}^2_X(\mathcal{H}^0(E), \mathcal{O}_X) \simeq \mathcal{H}^0(\mathcal{H}^0(E) \otimes \omega_S)^\vee ,$$

which is typically non-zero. Moreover, in the rank one case, $\mathcal{H}^{-1}(E)$ is a line bundle, and thus we see that $\text{Hom}_\mathcal{S}(\mathcal{H}^{-1}(E), \mathcal{O}_S)$ is in general not isomorphic to $\mathbb{C}$. Thus, one cannot prove the existence of a canonical morphism $E \rightarrow \mathcal{O}_S[1]$. 

6.2. Extension of stable pairs by torsion perverse sheaves. The goal of this section is to show that the various Hall algebras attached to $\text{Coh}_{\tau \text{f}}(S, \tau_A)$ naturally act on stable pairs. As usual, the fundamental mechanism is given by extensions. In the case at hand, it takes the following form:

Definition 6.6. Let $E := (V \rightarrow F \rightarrow E)$ be a $V$-stable pair and let $G \in \mathcal{A}_{\tau \text{f}}$. An extension of $E$ by $G$ is a commutative diagram $E$ in $\text{Perf}(S)$

$$\begin{array}{ccc}
0 & \rightarrow & V \\
\downarrow & & \downarrow \\
0 & \rightarrow & E \\
\downarrow & & \downarrow \\
0 & \rightarrow & G \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

(6.3)
all of whose squares are pullback, and where we further ask that \( V \to \mathcal{F}' \to E' \) is a \( \mathcal{V} \)-stable pair.

**Notation 6.7.** Compatible with the simplicial notation of Appendix A, in the situation of the above definition we set

\[
\partial_1(E) := (V \to \mathcal{F} \to E), \quad \partial_0(E) := (V \to \mathcal{F}' \to E'), \quad u_1^\ell(E) := G .
\]

**Proposition 6.8.** Let \( A \) be a diagram in \( \text{Perf}(S) \) of the form (6.3), where \( G \in \mathcal{A}_{\text{tor}} \). Assume that \( \partial_0(E) = (V \to \mathcal{F}' \to E') \) is a \( \mathcal{V} \)-stable pair. Then \( \partial_1(E) = (V \to \mathcal{F} \to E) \) is a \( \mathcal{V} \)-stable pair if and only if \( E \in \mathcal{A} \).

**Proof.** If \( \partial_1(E) \) is a \( \mathcal{V} \)-stable pair, then \( E \in \mathcal{A}_{\text{tor}} \), by definition, and in particular it belongs to \( \mathcal{A} \). On the other hand, if \( E \in \mathcal{A} \), then \( E \to \mathcal{F}' \to G \) is a fiber sequence in \( \text{Perf}(S) \) and all its terms belong to \( \mathcal{A} \). Thus, it is a short exact sequence in \( \mathcal{A} \). In particular, \( E \to \mathcal{F}' \) is a monomorphism and therefore Lemma 3.2-(3) guarantees that \( E \in \mathcal{A}_{\text{tor}} \). Moreover, since \( \text{rk}(G) = 0 \), we also obtain \( \text{rk}(E) = \text{rk}(E') = -\text{rk}(V) \). Finally, consider the fiber sequence

\[
G \longrightarrow \mathcal{F}[1] \longrightarrow \mathcal{F}'[1] .
\]

Since its extremes belong to \( \mathcal{A} \), we see that \( \mathcal{F}[1] \notin \mathcal{A} \) as well. Thus, the conclusion follows from Proposition 6.1. \( \square \)

### 6.3. Left representations via stable pairs

Fix a locally free sheaf \( \mathcal{V} \) of finite rank on \( S \). To keep the notation under control, we let

\[
\text{Perf}^\dagger_{\text{ps}}(S; \mathcal{V}) := \text{Flag}\text{Perf}^{(2),\dagger}_{\text{ps}}(\mathcal{O}_S; \mathcal{V}) ,
\]

where the latter the stack of \( \mathcal{V} \)-flags of length 2 (see Definition 2.2). Intuitively, it is the derived stack parametrizing fiber sequences of the form

\[
E := (V \longrightarrow \mathcal{F} \longrightarrow E) .
\]

Compatibly with the simplicial notation of the Appendix, we set

\[
\partial_0(E) := E \quad \text{and} \quad \partial_1(E) := \mathcal{F} .
\]

We set

\[
\mathcal{S}^\dagger_{\text{ps}}\text{Perf}_{\text{ps}}(S; \mathcal{V}) := \mathcal{S}^\dagger_{\text{ps}}\text{Flag}\text{Perf}^{(2),\dagger}_{\text{ps}}(\mathcal{O}_S; \mathcal{V}) ,
\]

where the latter is the simplicial derived stack of (2.2), obtained via Construction A.6. This simplicial stack comes equipped with a natural morphism

\[
u^\dagger_{\text{ps}}: \mathcal{S}^\dagger_{\text{ps}}\text{Perf}_{\text{ps}}(S; \mathcal{V}) \longrightarrow \mathcal{S}_{\text{ps}}\text{Perf}_{\text{ps}}(S) ,
\]

which is a relative 2-Segal stack.

We are going to show that this action induces an action at the level of \( \mathcal{V} \)-stable pairs. To begin with, following Definition 6.2, we introduce the moduli stack of \( \mathcal{V} \)-stable pairs as the fiber product

\[
P(S; \mathcal{V}) \longrightarrow \text{Perf}_{\text{ps}}(S; \mathcal{V}) \quad \downarrow \partial_1[1] \times \partial_0 \quad \text{Coh}_{\text{tor}}(S, \tau_A) \times \text{Coh}_{\text{tor}}(S, \tau_A) \longrightarrow \text{Perf}_{\text{ps}}(S) \times \text{Perf}_{\text{ps}}(S)
\]

When \( \mathcal{V} = \mathcal{O}^{\otimes r}_S \), we write \( P(S; r) \) instead of \( P(S; \mathcal{V}) \).

Specializing Construction 2.4 with \( T := \text{Coh}_{\text{tor}}(S, \tau_A) \) and \( F := P(S; \mathcal{V}) \), we obtain a new relative simplicial derived stack that we denote

\[
u^\dagger_{\text{ps}}: \mathcal{S}^\dagger_{\text{ps}}P(S; \mathcal{V}) \longrightarrow \mathcal{S}_{\text{ps}}\text{Coh}_{\text{tor}}(S, \tau_A) .
\]

In order to prove that this is a relative 2-Segal stack and prove that it gives rise to actual representations, it is convenient to provide a different description.
Introduce the auxiliary derived stack $\text{Perf}^l_{0,A}(S; V)$ defined via the fiber product

$$
\begin{array}{c}
\text{Perf}^l_{0,A}(S; V) \\
\downarrow \\
\text{Coh}_{l.f.}(S, \tau_A) \\
\end{array} \longrightarrow \begin{array}{c}
\text{Perf}^l_{ps}(S) \\
\end{array},
$$

which can informally be described as the derived stack parametrizing $V$-extensions of the form $V \to F \to E$, where $E \in A$, but where no condition is put on $F$.

Specializing Construction 2.4 with $T := \text{Coh}_{\text{tor}}(S, \tau_A)$ and $F := \text{Perf}^l_{0,A}(S; V)$ we obtain the relative simplicial derived stack

$$u^l_* : S^l_* \text{Perf}^l_{0,A}(S; V) \longrightarrow S^l_* \text{Coh}_{\text{tor}}(S, \tau_A). \quad (6.5)$$

**Lemma 6.9.** The square

$$
\begin{array}{c}
S^l_1 \text{Perf}^l_{0,A}(S; V) \\
\downarrow \partial_1 \times u^l_1 \\
\text{Perf}^l_{0,A}(S; V) \times \text{Coh}_{\text{tor}}(S, \tau_A) \\
\end{array} \longrightarrow \begin{array}{c}
S^l_1 \text{Perf}^l_{ps}(S; V) \\
\downarrow \partial_1 \times u^l_1 \\
\text{Perf}^l_{ps}(S; V) \times \text{Perf}^l_{ps}(S) \\
\end{array}
$$

is a pullback. In particular, (6.5) is a relative 2-Segal derived stack.

**Proof.** The second half follows directly from the first one and Proposition 2.6. It is thus enough to prove the first statement. Unwinding the definitions, we first see that both horizontal morphisms are representable by open immersions. Therefore, it is enough to prove that we indeed have a pullback square after evaluating at geometric points. In this case, we are called to prove that given a diagram of the form (6.3), if $E \in A$ and $G \in A_{\text{tor}}$, then $E' \in A$ as well. However, since $E \to E' \to G$ is a fiber sequence, the conclusion follows directly from the long exact sequence of homotopy groups associated to the $t$-structure $\tau_A$. \hfill $\square$

We now observe that the derived moduli stack of stable pairs $P(S; V)$ maps canonically to $\text{Perf}^l_{0,A}(S; V)$. We therefore obtain a commutative diagram $\rho$

$$
\begin{array}{c}
P(S; V) \\
\downarrow \\
\text{Perf}^l_{0,A}(S; V) \\
\end{array} \longrightarrow \begin{array}{c}
\text{Spec}(C) \\
\downarrow \\
\text{Spec}(C) \\
\end{array} \longrightarrow \begin{array}{c}
\text{Coh}_{\text{tor}}(S, \tau_A) \\
\downarrow \\
\text{Coh}_{\text{tor}}(S, \tau_A) \\
\end{array},
$$

which defines a boundary datum for the relative 2-Segal derived stack (6.5) in the sense of Definition A.25.

Applying Construction A.26, we obtain once again the stable pair action (6.4). The usefulness of this description of (6.4) is explained by the following:

**Proposition 6.10.** The square

$$
\begin{array}{c}
S^l_1 P(S; V) \\
\downarrow \partial_0 \\
P(S; V) \\
\end{array} \longrightarrow \begin{array}{c}
S^l_1 \text{Perf}^l_{0,A}(S; V) \\
\downarrow \partial_0 \\
\text{Perf}^l_{0,A}(S; V) \\
\end{array}
$$

is a pullback. In particular, (6.4) is a relative 2-Segal derived stack.

**Proof.** Observe first that the horizontal maps are representable by open immersions. Therefore, it is enough to check that the statement is true after evaluating on geometric points. In this case, we are called to show that given a diagram $E$ of the form (6.3), if $V \to F' \to E'$ is a $V$-stable pair, $G \in A_{\text{tor}}$ and $E \in A$, then $V \to F \to E$ is also a $V$-stable pair. This is guaranteed to be true by
Proposition 6.8. For the second part of the statement, observe that what we just showed implies immediately that the square

\[
\begin{array}{ccc}
S_1^! P(S; V) & \longrightarrow & S_1^! \text{Perf}_{\partial_A}^+(S; V) \\
\downarrow \partial_0 \times u_1^! & & \downarrow \partial_0 \times u_1^! \\
P(S; V) \times \text{Coh}_{\text{tor}}(S, \tau_A) & \longrightarrow & \text{Perf}_{\partial_A}^+(S; V) \times \text{Coh}_{\text{tor}}(S, \tau_A)
\end{array}
\]

is a pullback, so that the conclusion follows from Lemma 6.9 and Corollary A.28. □

Having constructed a 2-Segal action of \(\text{Coh}_{\text{tor}}(S, \tau_A)\) on \(P(S; V)\), we proceed to check the conditions of Corollary 2.9. We begin with properness, and we need to fix the following notation.

Notation 6.11. Let \(\pi_0^\text{tf}: \partial_0: P(S; V) \longrightarrow \text{Coh}_{\text{tf}, \tau}(S, \tau_A)\) be the morphism that sends a \(V\)-stable pair \(E = (V \to F \to E)\) to its \(A_{\text{tf}}\) component \(E = \partial_0(E)\). It can be promoted to a morphism of simplicial derived stacks \(\pi_0^\text{tf}: S_1^! P(S; V) \longrightarrow S_1^! \text{FlagCoh}_{\text{tf, tor}}^{(1)}(S, \tau_A)\), and the level 1 component \(\pi_1^\text{tf}\) can be explicitly described as the map sending an extension of stable pairs of the form (6.3) to the sub-extension \(E \to E' \to G\). △

Lemma 6.12. The commutative square

\[
\begin{array}{ccc}
S_1^! P(S; V) & \longrightarrow & S_1^! \text{FlagCoh}_{\text{tf, tor}}^{(1)}(S, \tau_A) \\
\downarrow \partial_0 & & \downarrow \partial_0 \\
P(S; V) & \longrightarrow & \text{Coh}_{\text{tf}, \tau}(S, \tau_A)
\end{array}
\]

is a pullback square. In particular, the left vertical morphism is representable by proper algebraic spaces.

Proof. Since \(A_{\text{tor}}\) is a Serre subcategory of \(A\), Corollary 3.11 implies that the right vertical map is representable by proper algebraic spaces. Thus all we have to do is to prove that the square of the statement is a pullback. Unwinding the definitions we see that we have to show that given any solid diagram of the form

\[
\begin{array}{cccc}
0 & \longrightarrow & V & \twoheadrightarrow & F & \twoheadrightarrow & F' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E & \longrightarrow & E' & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & & & & \\
\end{array}
\]

where \(G \in A_{\text{tor}}\), \(E \in A\) and \(V \to F' \to E'\) is a \(V\)-stable pair, then there exists a unique way (up to a contractible space of choices) of finding an object \(F\) together with the dashed arrows making the above diagram into an extension of stable pairs in the sense of Definition 6.6. Since the Waldhausen construction \(S_1^! \text{Perf}_{\text{ps}}(S)\) is a 2-Segal object, we see that setting

\(F := \text{fib}(F' \to G)\)

provides a unique way of filling the above diagram inside \(S_1^! \text{Perf}_{\text{ps}}(S)\), and therefore inside \(S_1^! \text{FlagPerf}_{\text{ps}}^{(2)}(S; V)\). Since \(S_1^! P(S; V)\) is open inside \(S_1^! \text{FlagPerf}_{\text{ps}}^{(2)}(S; V)\), all we are left to
We now look at the map
\[ \delta_1 \times u_1 : S_1^1 \mathbb{P}(S; V) \to \mathbb{P}(S; V) \times \text{Coh}_{\text{der}}(S, \tau_A). \] (6.6)

As consequence of Proposition 2.12, we obtain the following.

**Corollary 6.13.** The tangent complex \( T \) of the map (6.6) at a point \( x : \text{Spec}(A) \to S_1^1 \mathbb{P}(S; V) \) classifying an extension of stable pairs of the form (6.3) fits in the following natural fiber sequence:
\[ T \to R\text{Hom}_{S_A}(G, F')[1] \oplus R\text{Hom}_{S_A}(F', F)[1] \to R\text{Hom}_{S_A}(F', F')[1]. \]

In particular, letting \( T := G^\vee[2] \) and \( P' := (F')^\vee[1] \), we obtain an isomorphism
\[ H^2(T) \cong \text{Ext}^2_{S_A}(P', T). \]

Thus, if \( T \) is a zero-dimensional coherent sheaf and \( A \) is a field, we have \( H^2(T) = 0 \).

**Proof.** The existence of the given fiber sequence is the exact consequence of Proposition 2.12. Passing to the long exact sequence of cohomology groups, we obtain
\[ \text{Ext}_{S_A}^2(G, F') \oplus \text{Ext}_{S_A}^2(F', F) \to \text{Ext}_{S_A}^2(F', F') \to H^2(T) \to \]
\[ \to \text{Ext}_{S_A}^3(G, F') \oplus \text{Ext}_{S_A}^3(F', F) \to \text{Ext}_{S_A}^3(F', F'). \]

Setting \( P := F^\vee[1] \) and applying Theorem 5.7, we obtain
\[ \text{Ext}_{S_A}^3(F', F') \cong \text{Ext}_{S_A}^3(P', P') = 0 \quad \text{and} \quad \text{Ext}_{S_A}^3(F', F) \cong \text{Ext}_{S_A}^3(P, P') = 0 \cdot \]

Similarly,
\[ \text{Ext}_{S_A}^3(G, F') \cong \text{Ext}^2(P', T). \]

The conclusion therefore follows if we can prove that the first map is surjective.

After applying Theorem 5.7, it takes the following form:
\[ \text{Ext}_{S_A}^1(P', T) \oplus \text{Ext}_{S_A}^2(P, P') \to \text{Ext}_{S_A}^2(P', P'). \]

We claim that the second component is surjective. To see this, start from the fiber sequence \( F \to F' \to G \). Applying \((-)^{\vee}[2] \) and rotating it, it induces the fiber sequence \( P' \to P \to T \). Applying \( R\text{Hom}_{S_A}(-, P') \) and passing to the long exact sequence of cohomology groups, we find
\[ \text{Ext}_{S_A}^2(T, P') \to \text{Ext}_{S_A}^2(P, P') \to \text{Ext}_{S_A}^3(P', P') \to \text{Ext}_{S_A}^3(T, P') \cong 0, \]
whence the conclusion.

Finally, the last statement is an obvious consequence of Serre duality: since
\[ \text{Ext}_{S_A}^3(P', T)^{\vee} \cong \text{Hom}_{S_A}(T, P' \otimes \omega_{S_A/A}), \]
we see that if \( T \) is 0-dimensional the purity of \( P' \) implies the vanishing of the above group. \( \square \)

**Theorem 6.14.** Let \( \text{Coh}_{\text{pd}}(S) \) be the moduli stack of zero-dimensional coherent sheaves on \( S \). Then, the pro-\( \infty \)-category \( \text{Coh}_{\text{pd}}^b(\mathbb{P}(S; V)) \) carries a right categorical module structure over \( \text{Coh}_{\text{pd}}^b(\text{Coh}_{\text{pd}}(S)) \).

In particular, \( \text{Coh}_{\text{pd}}^b(\mathbb{P}(S; V)) \) (resp. \( \text{Coh}_{\text{pd}}^b(\mathbb{P}(S; V)) \)) is a right module over \( \text{Coh}_{\text{pd}}^b(\text{Coh}_{\text{pd}}(S)) \) (resp. over \( \text{Coh}_{\text{pd}}^b(\text{Coh}_{\text{pd}}(S)) \)).

**Proof.** Consider
\[ T := \text{Coh}_{\text{pd}}(S) = \text{Coh}_{\text{pd}}(S, \tau_{\text{std}}^\text{op}) \quad \text{and} \quad F := \mathbb{P}(S; V). \]

First, note that the category of zero-dimensional sheaves on \( S \) is a Serre subcategory of both \( \text{Coh}(S) \) and \( A \) (the latter via duality - Theorem 5.7). Moreover, we will interpret \( \text{Coh}_{\text{pd}}(S) \) as
the open and closed substack of both \( \text{Coh}_{\text{tor}}(S) \) and \( \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}}) \) defined by the condition that the torsion objects have zero first Chern class.

The property of being a Serre subcategory implies that the condition (1) of Proposition 2.6 is satisfied by \( T \). Furthermore, thanks to Proposition 6.8, also condition (2) of Proposition 2.6 is satisfied by \( T, F \).

Now note that \( \text{Coh}_{0\text{-dim}}(S) \) fits into the following pullback squares:

\[
\begin{array}{ccc}
\mathcal{S}_2 \text{Perf}_T(\mathcal{E}_S) & \to & \mathcal{S}_2 \text{Perf}_{\text{tor}}(\mathcal{E}_S) \\
\downarrow \partial_0 \times \partial_2 & & \downarrow \partial_0 \times \partial_2 \\
\text{Coh}_{0\text{-dim}}(S, \tau_{\text{std}}) \times \text{Coh}_{0\text{-dim}}(S, \tau_{\text{std}}) & \to & \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}}) \times \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}})
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{S}_2 \text{Perf}_T(\mathcal{E}_S) & \to & \mathcal{S}_2 \text{Perf}_{\text{tor}}(\mathcal{E}_S) \\
\downarrow \partial_1 & & \downarrow \partial_1 \\
\text{Coh}_{0\text{-dim}}(S, \tau_{\text{std}}) & \to & \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}})
\end{array}
\]

The right vertical maps are derived lci and representable by proper algebraic spaces, respectively, as explained in §5.3. Then, the left vertical maps satisfy the same properties, hence conditions (1)-(i) and - (ii) of Corollary 2.7 are satisfied by \( \text{Coh}_{0\text{-dim}}(S) \).

Finally, the map

\[
\partial_1 \times u^f_1 : \mathcal{S}'_1 \text{FlagPerf}^\text{tor}_{F,T}(\mathcal{E}_S; \mathcal{V}) \to \mathcal{P}(S; \mathcal{V}) \times \text{Coh}_{0\text{-dim}}(S)
\]

fits into the pullback square

\[
\begin{array}{ccc}
\mathcal{S}'_1 \text{FlagPerf}^\text{tor}_{F,T}(\mathcal{E}_S; \mathcal{V}) & \to & \mathcal{S}'_1 \mathcal{P}(S; \mathcal{V}) \\
\downarrow \partial_1 \times u^f_1 & & \downarrow \partial_1 \times u^f_1 \\
\mathcal{P}(S; \mathcal{V}) \times \text{Coh}_{0\text{-dim}}(S, \tau_{\text{std}}) & \to & \mathcal{P}(S; \mathcal{V}) \times \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}})
\end{array}
\]

Thanks to the computation of the relative cotangent complex of the right vertical map in Corollary 6.13, we see that the left vertical map is derived lci. Using a similar argument and Lemma 6.12, one shows that

\[
\partial_0 : \mathcal{S}'_1 \text{FlagPerf}^\text{tor}_{F,T}(\mathcal{E}_S; \mathcal{V}) \to \mathcal{P}(S; \mathcal{V})
\]

is representable by proper algebraic spaces.

Thus, all the assumptions in conditions (1) and (3) of Corollary 2.7 are satisfied. The assertion follows noticing that the duality (Theorem 5.7) exchanges left and right representations (a phenomenon seen already in Theorem 5.13). \( \square \)

6.4. Right representations via stable co-pairs. Recall that a \( \mathcal{V} \)-stable pair is a fiber sequence \( \mathcal{V} \to \mathcal{F} \to \mathcal{E} \), where \( \mathcal{F}[1] \in \mathcal{A}_{\text{tor}} \) and \( \mathcal{E} \in \mathcal{A}_{\text{f}} \). This gives rise to the projection map

\[
\pi^\text{tor}_{\mathcal{V}f} : \mathcal{P}(S; \mathcal{V}) \to \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}})
\]

of Notation 6.11. Recall also that \( \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}}) \) acts both from the left and from the right on \( \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}}) \).

In the previous section we saw that the left action restricted to the substack \( \text{Coh}_{0\text{-dim}}(S, \tau_{\text{std}}) \subseteq \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}}) \) of zero-dimensional coherent sheaves, can be lifted to an action at the level of stable pairs. We are now going to analyze the right action; the main result of the section is that the same lifting is possible, and this time it will take place at the level of the whole \( \text{Coh}_{\text{tor}}(S, \tau_{\mathcal{A}}) \).
Mimicking the idea of Theorem 3.21, we start looking for an analogue of stable pairs in the tilted heart \( A^v \), where \( v = (A_{\text{tor}}, A_{\text{t.f.}}) \) is the torsion pair of Lemma 5.3. Recall that \((A_{\text{t.f.}}[1], A_{\text{tor}})\) is again a torsion pair on \( A^v \).

We write as usual
\[
A_{\text{tor}}^v := A_{\text{t.f.}}[1] \quad \text{and} \quad A_{\text{t.f.}}^v := A_{\text{tor}}.
\]

We start with the following definition.

**Definition 6.15.** Let \( V \) be a locally free sheaf of finite rank on \( S \).

1. A \( V \)-stable co-pair is a fiber sequence
   \[
   F \rightarrow E \rightarrow V[2],
   \]
   where \( F \in A_{\text{t.f.}}^v \) and \( E \in A_{\text{tor}}^v \).

2. Let \( E = (F \rightarrow E \rightarrow V[2]) \) and let \( G \in A_{\text{t.f.}}^v \). An extension of \( E \) by \( G \) is a commutative diagram \( \mathbb{E} \) in \( \text{Perf}(S) \)
   \[
   \begin{array}{ccc}
   0 & \rightarrow & G & \rightarrow & F' & \rightarrow & E' & \rightarrow & 0 \\
   0 & \rightarrow & F & \rightarrow & E & \rightarrow & 0 \\
   0 & \rightarrow & V[2] & \rightarrow & 0
   \end{array}
   \]
   (6.7)
   where every square is asked to be a pullback and where we further ask \( F' \rightarrow E' \rightarrow V[2] \) to be a \( V \)-stable co-pair.

\(\square\)

**Notation 6.16.** Let \( \mathbb{E} \) be an extension of \( V \)-stable co-pairs of the form (6.7).Compatibly with the simplicial notation of the Appendix A, we set
\[
\partial_0(\mathbb{E}) := (F \rightarrow E \rightarrow V[2]) \quad \text{and} \quad \partial_1(\mathbb{E}) := (F' \rightarrow E' \rightarrow V[2]).
\]

We equally set
\[
u_1(\mathbb{E}) := G.
\]

\(\triangle\)

The following lemma is tautological.

**Lemma 6.17.** Let \( V \) be a locally free sheaf of finite rank on \( S \). A fiber sequence
\[
F \rightarrow E \rightarrow V[2]
\]
is a \( V \)-stable co-pair if and only if the shifted-rotated sequence
\[
V \rightarrow F[-1] \rightarrow E[-1]
\]
is a \( V \)-stable pair.

**Corollary 6.18.** Let \( V \) be a locally free sheaf of finite rank on \( S \). Let
\[
E = (F \rightarrow E \rightarrow V[2])
\]
be a fiber sequence in \( \text{Perf}(S) \), where \( F \) and \( E \) belong to \( A^v \). Then the following statements are equivalent:

1. \( E \) is a \( V \)-stable co-pair;
2. \( E \in A^v_{\text{tor}} \) and \( \text{rk}(E) = -\text{rk}(V) \);
\( E \in \mathcal{A}_\text{tor}^\heartsuit \) and \( \text{rk}(\mathcal{F}) = 0. \)

**Proof.** This follows at once combining Proposition 6.1 and Lemma 6.17. \( \square \)

Using Corollary 6.18 instead of Proposition 6.1, the same proof of Proposition 6.8 yields:

**Proposition 6.19.** Let \( \mathcal{E} \) be an extension of \( \mathcal{V} \)-stable co-pairs of the form (6.7). Assume that \( G \in \mathcal{A}_\text{tor} \) and that \( \partial_1(\mathcal{E}) = (\mathcal{F}' \to \mathcal{E}' \to \mathcal{V}[2]) \) is a \( \mathcal{V} \)-stable co-pair. Then \( \partial_0(\mathcal{E}) = (\mathcal{F} \to \mathcal{E} \to \mathcal{V}[2]) \) is a \( \mathcal{V} \)-stable co-pair if and only if \( E \in \mathcal{A}_\heartsuit \). 

We now introduce just as in the previous subsection the derived stack \( \mathcal{P}_c(S; \mathcal{V}) \) of stable co-pairs. As the procedure is formally identical, we limit ourselves to explain the broad steps. To begin with, applying the left version of Construction A.6 to the 2-Segal derived stack \( S\mathcal{P}_\text{ps}(S) \) with \( \mathcal{V} := \mathcal{V}[2] \), we obtain a relative 2-Segal derived stack

\[
u_c^* : S\mathcal{P}_\text{ps}^\dagger(S; \mathcal{V}) \colonequals S\mathcal{P}_\text{ps}^\dagger(\mathcal{C}_S; \mathcal{V}[2]) \longrightarrow S\mathcal{P}_\text{ps}(S).
\]

Unraveling the definitions, we see that

\[
\mathcal{P}_\text{ps}^\dagger(S; \mathcal{V}) = S\mathcal{P}_\text{ps}^\dagger(S; \mathcal{V})
\]

parametrizes fiber sequences of the form

\[
\mathcal{F} \longrightarrow E \longrightarrow \mathcal{V}[2].
\]

On the other hand, \( S\mathcal{P}_\text{ps}^\dagger(S; \mathcal{V}) \) parametrizes extensions of the form (6.7), where there is still no restriction whatsoever on the terms \( G, \mathcal{F}, \mathcal{F}', E \) and \( E' \). We now define the derived moduli stack \( \mathcal{P}_c(S; \mathcal{V}) \) of stable co-pairs as the fiber product

\[
\begin{array}{ccc}
\mathcal{P}_c(S; \mathcal{V}) & \longrightarrow & \mathcal{P}_\text{ps}^\dagger(S; \mathcal{V}) \\
\downarrow & & \downarrow \\
\text{Coh}_\text{lf}(S, \tau^\heartsuit_\Lambda) \times \text{Coh}_\text{tor}(S, \tau^\heartsuit_\Lambda) & \longrightarrow & \mathcal{P}_\text{ps}(S) \times \mathcal{P}_\text{ps}(S)
\end{array}
\]

Then, Lemma 6.17 immediately implies the following:

**Lemma 6.20.** The self-equivalence

\[
\rho : S\mathcal{P}_\text{ps}(S) \longrightarrow S\mathcal{P}_\text{ps}(S)
\]

that sends a fiber sequence \( E_1 \to E_2 \to E_3 \) to the shifted-rotated sequence \( E_2[1] \to E_3[1] \to E_1[2] \) induces an equivalence

\[
\rho : \mathcal{P}(S; \mathcal{V}) \simeq \mathcal{P}_c(S; \mathcal{V}).
\]

On the other hand, applying Construction A.26 to the relative 2-Segal derived stack (6.8) with the choice of the boundary datum given by the diagram

\[
\begin{array}{ccc}
\mathcal{P}_c(S; \mathcal{V}) & \longrightarrow & \text{Spec}(\mathbb{C}) \\
\downarrow & & \downarrow \\
\mathcal{P}_\text{ps}^\dagger(S; \mathcal{V}) & \longrightarrow & \mathcal{P}_\text{ps}(S)
\end{array}
\]

we obtain a relative simplicial derived stack

\[
u_c^* : S\mathcal{P}_c(S; \mathcal{V}) \longrightarrow S\mathcal{Coh}_\text{lf}(S, \tau^\heartsuit_\Lambda)
\]

The same procedure that leads to Proposition 6.10 yields:

**Proposition 6.21.** The morphism (6.9) is a relative 2-Segal derived stack.

In order to convey the information contained in the 2-Segal property into an actual right representation, we need to verify the assumptions of Corollary 2.9. We start with properness.
Notation 6.22. Let
\[ \pi_0^{\text{lef}} : \mathcal{P}(S; V) \to \text{Coh}_{\text{lef}}(S, \tau_A^u) \]
be the morphism that sends a stable co-pair \( E = (\mathcal{F} \to \mathcal{E} \to \mathcal{V}[2]) \) to \( \partial_2(E) = \mathcal{F} \). It can be promoted to a morphism of simplicial derived stacks
\[ \pi_*^{\text{lef}} : S^i \mathcal{P}(S; V) \to S_* \text{Coh}_{\text{lef}}(S, \tau_A^u), \]
and the level 1 component \( \pi_1^{\text{lef}} \) can be explicitly described as the map sending an extension of \( \mathcal{V} \)-stable co-pairs of the form (6.7) to the sub-extension \( \mathcal{G} \to \mathcal{F}' \to \mathcal{F} \). \( \square \)

Lemma 6.23. The square
\[
\begin{array}{ccc}
S^i \mathcal{P}(S; V) & \xrightarrow{\pi_i^{\text{lef}}} & S^i \text{FlagCoh}_{\text{lef,tor}}^1(S, \tau_A^u) \\
\downarrow \delta_1 & & \downarrow \delta_1 \\
\mathcal{P}(S; V) & \xrightarrow{\pi_1^{\text{lef}}} & \text{Coh}_{\text{lef}}(S, \tau_A^u)
\end{array}
\]
is a pullback. In particular, the left vertical map is representable by proper algebraic spaces.

Proof. As already observed in §5.3, the map
\[ \text{Coh}_{\text{tor}}(S, \tau_A) = \text{Coh}_{\text{lef}}(S, \tau_A^u) \to \text{Coh}(S, \tau_A^u) \]
is representable by open and closed immersions. Therefore, Corollary 3.11-(2) guarantees that the right vertical map is representable by proper algebraic spaces. It is then enough to prove the first half of the statement, i.e., that the given square is a pullback. Unwinding the definitions, we have to prove that given a solid diagram of the form
\[
\begin{array}{c}
0 \to G \to \mathcal{F}' \to E' \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to \mathcal{F} \to \cdots \to E \\
\downarrow \quad \downarrow \\
0 \to \mathcal{V}[2] \\
\end{array}
\]
where \( G \in \mathcal{A}_\text{tor}^u, \mathcal{F}' \to E' \to \mathcal{V}[2] \) is a \( \mathcal{V} \)-stable co-pair and \( \mathcal{F} \in \mathcal{A}^u \), there is a unique way (up to a contractible space of choices) of defining \( E \) and the dashed arrows making the above diagram into an extension of \( \mathcal{V} \)-stable copairs in the sense of Definition 6.15-(1). Since the Waldhausen construction \( S_* \text{Perf}_{\text{ps}}(S) \) satisfies the 2-Segal property, we see that setting
\[ E := \text{cofib}(G \to E') \]
provides the unique filling inside \( S_3 \text{Perf}_{\text{ps}}(S) \) (and hence in \( S^i \text{FlagPerf}_{\text{ps}}^1(S; V) \)) we were looking for. To complete the proof, we only have to check that in this situation \( \mathcal{F} \to E \to \mathcal{V}[2] \) is automatically a \( \mathcal{V} \)-stable co-pair. To see this, observe first that, since both \( \mathcal{F} \) and \( \mathcal{V}[2] \) belong to \( \mathcal{A}^u \), we automatically have that \( E \in \mathcal{A}^u \) as well. The fiber sequence \( G \to E' \to E \) is therefore a short exact sequence in \( \mathcal{A}^u \), and since \( E' \in \mathcal{A}^u_{\text{tor}} \), it follows that \( E \in \mathcal{A}^u_{\text{tor}} \) as well. Finally, since \( \text{rk}(G) = 0 \), we deduce \( \text{rk}(E) = \text{rk}(E') = -\text{rk}(V) \), so the conclusion follows from Proposition 6.8. \( \square \)

We now turn to the map
\[ \iota_1' \times \partial_0 : S^i \mathcal{P}(S; V) \to \text{Coh}_{\text{lef}}(S, \tau_A^u) \times \mathcal{P}(S; V). \]
Contrary to what happens in the previous subsection, we have the following.
Proposition 6.24. The square
\[
\begin{array}{ccc}
S_1^Pc(S; V) & \longrightarrow & S_1^\text{FlagCoh}_{t, f, \text{tor}}^{(1)}(S, \tau_A^v) \\
\downarrow_{\alpha_1 \times \alpha_0} & & \downarrow_{\alpha_0} \\
\text{Coh}_{t, f}(S, \tau_A^v) \times Pc(S; V) & \longrightarrow & \text{Coh}_{t, f}(S, \tau_A^v) \times \text{Coh}_{\text{tor}}(S, \tau_A^v)
\end{array}
\]
is a pullback. In particular, the left vertical map is derived lci.

Proof. It has already been verified in Theorem 3.21 that the right vertical map is derived lci. We are therefore left to check that the above square is a pullback. Equivalently, we have to check that the square
\[
\begin{array}{ccc}
S_1^Pc(S; V) & \longrightarrow & S_1^\text{FlagCoh}_{t, f, \text{tor}}^{(1)}(S, \tau_A^v) \\
\downarrow_{\alpha_0} & & \downarrow_{\alpha_0} \\
Pc(S; V) & \longrightarrow & \text{Coh}_{\text{tor}}(S, \tau_A^v)
\end{array}
\]
is a pullback. Unraveling the definitions, we are called to show that given a solid diagram of the form
\[
\begin{array}{ccc}
0 & \longrightarrow & G \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & E'
\end{array}
\]
where \(G \in \mathcal{A}_v\), \(\mathcal{F} \to E \to \mathcal{V}[2]\) is a \(\mathcal{V}\)-stable co-pair and \(E' \in \mathcal{A}_v^\text{tor}\), there is a unique way (up to a contractible space of choices) to define \(\mathcal{F}'\) together with the dashed morphisms making the above diagram into an extension of \(\mathcal{V}\)-stable co-pairs in the sense of Definition 6.15-(2). As usual, the 2-Segal property satisfied by the Waldhausen construction \(\mathcal{S}_* \text{Perf} \mathcal{S}\) shows that the only possible choice is to take
\[
\mathcal{F}' := \text{fib}(E' \to \mathcal{V}[2]),
\]
and we are left to check that in this situation the fiber sequence \(\mathcal{F}' \to E' \to \mathcal{V}[2]\) is a \(\mathcal{V}\)-stable co-pair. Since both \(G\) and \(\mathcal{F}\) belong to \(\mathcal{A}_v^t = \mathcal{A}_v^\text{tor}\), the same goes for \(\mathcal{F}'\). Moreover, \(E'\) belongs to \(\mathcal{A}_v^\text{tor}\) by assumption and since \(\text{rk}(G) = 0\), we have
\[
\text{rk}(E') = \text{rk}(E) = -\text{rk}(\mathcal{V}).
\]
Thus, the conclusion follows from Proposition 6.19.

The following is the main result of this section.

Theorem 6.25. The pro-\(\infty\)-category \(\text{Coh}_{\text{pro}}(\mathcal{P}(S; V))\) has the structure of a right categorical module over the \(E_1\)-monoidal \(\infty\)-category \(\text{Coh}_{\text{pro}}(\text{Coh}_{t, f}(A^v))\). In particular,
\[
\text{G}_0(\mathcal{P}(S; V)) \quad \text{and} \quad H_{BM}^\infty(\mathcal{P}(S; V))
\]
are right modules of \(\text{G}_0(\text{Coh}_{t, f}(A^v))\) and \(H_{BM}^\infty(\text{Coh}_{t, f}(A^v))\), respectively.
Proof. We need to check the assumptions of Corollary 2.7-(2) for
\[ T := \text{Coh}_{\text{tor}}(S, \tau_{A}^v) \quad \text{and} \quad F := \text{P}^c(S; V). \]
First, note that condition (1) of Proposition 2.6 and conditions (1)-(i) and (1)-(ii) of Corollary 2.7 are satisfied by \( T \), since in Theorem 3.21 we observed that there is a canonical equivalence
\[ S_* \text{Coh}_{\text{tor}}(A) \cong S_* \text{Coh}_{\text{ff}}(A^v), \]
and the above conditions are satisfied by \( \text{Coh}_{\text{tor}}(A) \), as already shown in Theorem 5.11. Moreover, the map
\[ u'_1 \times \partial_0: S'_1 \text{P}^c(S; V) \to \text{Coh}_{\text{ff}}(S, \tau_{A}^v) \times \text{P}^c(S; V). \]
is derived lci by Proposition 6.24, while the map
\[ \partial_1: S'_1 \text{P}^c(S; V) \to \text{P}^c(S; V) \]
is representable by proper algebraic spaces, as shown in Lemma 6.23. Thus, all the assumptions of Corollary 2.7-(2) are verified and the assertion follows. \( \square \)

Corollary 6.26. The pro-\( \infty \)-category \( \text{Coh}_{\text{pro}}^b(\text{P}(S; V)) \) carries a right categorical module structure over \( \text{Coh}_{\text{tor}}^b(\text{Coh}_{\text{tor}}(A)) \). Thus, \( G_0(\text{P}(S; V)) \) (resp. \( H^\bullet_{\text{BM}}(\text{P}(S; V)) \)) is a right module over \( G_0(\text{Coh}_{\text{tor}}(A)) \) (resp. over \( H^\bullet_{\text{BM}}(\text{Coh}_{\text{tor}}(A)) \)).

Proof. First, Theorem 3.21 yields a canonical equivalence
\[ S_* \text{Coh}_{\text{tor}}(A) \cong S_* \text{Coh}_{\text{ff}}(A^v), \]
while Lemma 6.20 yields an equivalence of derived stacks
\[ \rho: \text{P}(S; V) \to \text{P}^c(S; V). \]
Via this equivalence, the right 2-Segal action of \( \text{Coh}_{\text{tor}}(A) \) on \( \text{P}^c(S; V) \) is transferred to a right 2-Segal action of \( \text{Coh}_{\text{tor}}(A) \) on \( \text{P}(S; V) \). The statements for the categorical, \( G \)-theoretical and Borel-Moore homology actions follow automatically. \( \square \)

Remark 6.27. Since \( \text{Coh}_{\text{tor}}(S)^{\text{op}} \cong \text{Coh}_{\text{tor}}(A) \), we obtain that \( \text{Coh}_{\text{pro}}^b(\text{P}(S; V)) \) carries a left categorical module structure over \( \text{Coh}_{\text{tor}}^b(\text{Coh}_{\text{tor}}(S)) \). Thus, \( G_0(\text{P}(S; V)) \) (resp. \( H^\bullet_{\text{BM}}(\text{P}(S; V)) \)) is a left module over \( G_0(\text{Coh}_{\text{tor}}(S)) \) (resp. over \( H^\bullet_{\text{BM}}(\text{Coh}_{\text{tor}}(S)) \)). \( \triangle \)

Corollary 6.28. The pro-\( \infty \)-category \( \text{Coh}_{\text{pro}}^b(\text{P}(S; V)) \) has the structure of a left and right categorical module over the \( E_1 \)-monoidal \( \infty \)-category \( \text{Coh}_{\text{pro}}^b(\text{Coh}_{\text{tor}}^0(\text{std}(S))). \) In particular,
\[ G_0(\text{P}(S; V)) \quad \text{and} \quad H^\bullet_{\text{BM}}(\text{P}(S; V)) \]
are left and right modules of \( G_0(\text{Coh}_{\text{tor}}^0(\text{std}(S))) \) and \( H^\bullet_{\text{BM}}(\text{Coh}_{\text{tor}}^0(\text{std}(S))) \), respectively.

Proof. Using Corollary 6.26 we obtain a left categorical module structure of the pro-\( \infty \)-category \( \text{Coh}_{\text{pro}}^b(\text{P}(S; V)) \) over \( \text{Coh}_{\text{tor}}^b(\text{Coh}_{\text{tor}}^0(\text{std}(S))) \) via the inclusion \( \text{Coh}_{\text{tor}}^0(\text{std}(S), \tau_{\text{std}}^0) \subset \text{Coh}_{\text{tor}}(S, \tau_A) \), while the right (categorical) structure is constructed in Theorem 6.14. \( \square \)

Remark 6.29. In the local surface case, Toda constructed a right categorical module structure of \( \text{Coh}_{\text{tor}}^b \) of Pandharipande-Thomas moduli spaces of stable pairs over the categorified Hall algebra of zero-dimensional sheaves (cf. [Tod20, §4]). In this case, there is no left categorical module structure because of a wall-crossing phenomenon which does not appear in our two-dimensional case. \( \triangle \)

Remark 6.30. Following [Tod20, §6], we can study the left and right action given by the K-theory of the moduli stack of zero-dimensional sheaves on \( S \) of length one: one obtains the same relations as loc.cit. \( \triangle \)
Consider \( \mathcal{V} = \mathcal{O}_S \). Then \( \mathcal{P}(S; \mathcal{O}_S) \) is equivalent to the moduli space \( \mathcal{P}(S) \) of Pandharipande-Thomas stable pairs on \( S \), which is a projective scheme.\(^\text{17}\) We have the following:

**Corollary 6.31.** The stable pro-\( \infty \)-category \( \text{Coh}_{\text{pro}}^\infty (\mathcal{P}(S)) \) has the structure of a left and a right categorical module over \( \text{Coh}_{\text{pro}}^\infty (\text{Coh}_{\text{dim}}^\infty (S)) \). In particular, \( \mathcal{G}_0(\mathcal{P}(S)) \) (resp. \( H_{BM}^t(\mathcal{P}(S)) \)) has the structure of a left and a right module over \( \mathcal{G}_0(\text{Coh}_{\text{dim}}^\infty (S)) \) (resp. over \( H_{BM}^t(\text{Coh}_{\text{dim}}^\infty (S)) \)).

### 6.5. Duality

In what follows, we want to understand what category we obtain by applying the duality functor \((-)\vee[2] \) of Theorem 5.7 to the category of stable pairs and derive an analog of Theorem 6.25 for the corresponding derived moduli stack. It is straightforward to see that the dual to a stable pair

\[
\mathcal{V} \to \mathcal{F} \to \mathcal{E}
\]

is the fiber sequence

\[
\mathcal{V}^\vee \to \mathcal{E}^\vee[1] \to \mathcal{F}^\vee[1],
\]

which belongs to \( \text{Coh}^\vee(\mathcal{S}) \), and \( \mathcal{E}^\vee[1] \) is torsion free, while \( \mathcal{F}^\vee[1] \) is purely 1-dimensional.

In order to promote this bijection to an equivalence of moduli spaces, it is more convenient to introduce a further abstraction of the \( \infty \)-category of \( \mathcal{V} \)-stable pairs, that depends functorially on an \( \infty \)-category \( \mathcal{C} \), a \( t \)-structure \( \tau \) on \( \mathcal{C} \), a torsion pair \( \nu \) on \( \mathcal{C}^\vee \) and an object \( \mathcal{V} \in \mathcal{C} \). In this way, the above bijection immediately becomes an equivalence of \( \infty \)-categories, as a straightforward application of the functoriality and of Theorem 5.7.

#### 6.5.1. \( \mathcal{V} \)-stable (co-)pairs in a stable \( \infty \)-category

We start by the following definition:

**Definition 6.32.** A framed tilting datum is a 4-tuple \( \mathcal{C} = (\mathcal{C}, \tau, \nu, \mathcal{V}) \), where \( \mathcal{C} \) is a stable \( \infty \)-category, \( \tau \) is a \( t \)-structure on \( \mathcal{C} \), \( \nu \) is a torsion pair on \( \mathcal{C}^\vee \) and \( \mathcal{V} \in \mathcal{C} \) is an object.

Given a framed tilting datum \( \mathcal{C} \) as above, we refer to \( \mathcal{V} \) as the framing and to the triple \( (\mathcal{C}, \tau, \nu) \) as the tilting datum underlying \( \mathcal{C} \). We leave as an exercise to verify that framed tilting data can be naturally organized into an \( \infty \)-category \( \mathcal{T}^\mathcal{fr} \). Concretely, given two framed tilting data \( \mathcal{C} = (\mathcal{C}, \tau, \nu, \mathcal{V}) \) and \( \mathcal{D} = (\mathcal{D}, \tau, \nu, \mathcal{W}) \) a morphism from \( \mathcal{C} \) to \( \mathcal{D} \) is a pair \( (F, \alpha) \), where

\[
F: \mathcal{C} \to \mathcal{D}
\]

is an \( t \)-exact stable functor with the property that \( F(\mathcal{C}^\vee_{\text{tor}}) \subseteq \mathcal{D}^\vee_{\text{tor}} \) and \( F(\mathcal{C}^\vee_{\text{tf}}) \subseteq \mathcal{D}^\vee_{\text{tf}} \), and

\[
\alpha: F(\mathcal{V}) \tilde{\to} \mathcal{W}
\]

is an equivalence in \( \mathcal{D} \).

**Notation 6.33.** Let \( \mathcal{C} \) be a stable \( \infty \)-category. Given an object \( \mathcal{V} \in \mathcal{C} \), we let \( \mathcal{C}(\mathcal{V}) \) be the fiber product

\[
\mathcal{C}(\mathcal{V}) \to S_2\mathcal{C} \quad \downarrow a_2 \quad \mathcal{V} \to \mathcal{C}
\]

Informally speaking, objects in \( \mathcal{C}(\mathcal{V}) \) are fiber sequences of the form \( \mathcal{V} \to \mathcal{F} \to \mathcal{E} \), and morphisms are morphisms of fiber sequences that restrict to the identity on \( \mathcal{V} \). We refer to \( \mathcal{C}(\mathcal{V}) \) as the \( \infty \)-category of \( \mathcal{V} \)-pairs.

Replacing \( a_2 \) with \( a_1 \) in the above diagram, we denote by \( \mathcal{C}(\mathcal{V}) \) the resulting category, and we refer to it as the \( \infty \)-category of \( \mathcal{V} \)-co-pairs.

\(^\text{17}\)PT10, Proposition 5.2 shows that, for a K3 surface \( S \), the moduli space of stable pairs of fixed Chern class \( \beta \) and Euler characteristic \( n \) is smooth for \( \beta \in \text{NS}(S) \) irreducible.
Definition 6.34. Let $C = (\mathcal{C}, \tau, \nu, \mathcal{V}) \in T_{fr}$, and write $\mathcal{C}^\tau$ for the heart of $\tau$ and $v = (\mathcal{C}_{tor}^\tau, \mathcal{C}_{tf}^\tau)$ for the torsion pair.

1. A $C$-pair is a fiber sequence in $\mathcal{C}$ of the form
   $$\mathcal{V} \to \mathcal{F} \to \mathcal{E}$$
   where $\mathcal{F}[1] \in \mathcal{C}_{tor}^\tau$ and $\mathcal{E} \in \mathcal{C}_{tf}^\tau$. We denote by $P(C)$ the full subcategory of $\mathcal{C}^\tau(\mathcal{V})$ spanned by $C$-pairs.

2. A $C$-co-pair is a fiber sequence in $\mathcal{C}$ of the form
   $$\mathcal{F} \to \mathcal{E} \to \mathcal{V}[1]$$
   where $\mathcal{F} \in \mathcal{C}_{tor}^\tau$ and $\mathcal{E} \in \mathcal{C}_{tf}^\tau$. We denote by $P^c(C)$ the full subcategory of $\mathcal{C}^\tau(\mathcal{V}[1])$ spanned by $C$-co-pairs.

It is straightforward to check that the constructions $P(C)$ and $P^c(C)$ give rise to functors
   $$P, P^c : T_{fr} \to \text{Cat}_\infty.$$ 

In order to combine this construction with Theorem 5.7, we need to understand certain natural symmetries of $T_{fr}$ and how they interact with the previous two functors.

Construction 6.35.

1. Let $\mu_2 := \{1, \zeta_2\} \subseteq \mathbb{C}^\times$, where we write $\zeta_2$ instead of $-1$ to avoid confusion below. Recall that there is a canonical action of $\mu_2$ on $\text{Cat}_\infty$ that takes an infinite category $\mathcal{C}$ to its opposite $\mathcal{C}^{op}$. This action preserves stable infinite-categories, and therefore it lifts to an action on $\text{Cat}^{st}_\infty$. Let $l\text{-Cat}^{st}_\infty$ be the infinite-category of stable infinite-categories equipped with $l$-structures, and $l$-exact functors between them. The $\mathbb{Z}/2\mathbb{Z}$-action can be lifted to $l\text{-Cat}^{st}_\infty$, observing that if $\tau = (\mathcal{C}^{\geq 0}, \mathcal{C}^{\geq 0})$ is a $l$-structure on $\mathcal{C}$, then $\tau^{op} := ((\mathcal{C}^{\geq 0})^{op}, (\mathcal{C}^{\leq 0})^{op})$ is a $l$-structure on $\mathcal{C}^{op}$.

2. On the other hand, $l\text{-Cat}^{st}_\infty$ also carries a canonical action of $\mathbb{Z}$, which is uniquely determined by the fact that $1$ acts on $(\mathcal{C}, \tau)$ by shifting the $l$-structure:
   $$1 \cdot (\mathcal{C}, \tau) := (\mathcal{C}, \tau[1]).$$
   The two actions do not commute: indeed the endo-functor $(-)^{op} : \text{Cat}^{st}_\infty \to \text{Cat}^{st}_\infty$ takes the self-equivalence $[1] : \mathcal{C} \simeq \mathcal{C}$ to the self-equivalence $[-1] : \mathcal{C}^{op} \simeq \mathcal{C}^{op}$. Thus,
   $$1 \cdot (\zeta_2 \cdot \tau) = \tau^{op}[1] = ((\mathcal{C}^{\geq 0})^{op}[1], (\mathcal{C}^{\leq 0})^{op}[1])$$
   $$= ((\mathcal{C}^{\geq 0}[-1])^{op}, (\mathcal{C}^{\leq 0}[-1])^{op})$$
   $$= ((\mathcal{C}^{\geq 1})^{op}, (\mathcal{C}^{\leq 1})^{op}),$$
   while
   $$\zeta_2 \cdot (1 \cdot \tau) = (\tau[1])^{op} = ((\mathcal{C}^{\geq 0}[1])^{op}, (\mathcal{C}^{\leq 0}[1])^{op}) = ((\mathcal{C}^{\geq -1})^{op}, (\mathcal{C}^{\leq -1})^{op}).$$
   The same computation immediately shows that
   $$\tau^{op}[-1] = (\tau[1])^{op},$$
   which in turn implies that $l\text{-Cat}^{st}_\infty$ carries a natural $\mu_2 \ltimes \mathbb{Z}$-action, where $\mu_2$ acts on $\mathbb{Z}$ by multiplication by $-1$.

3. Let now $T$ the infinite-category of stable infinite-categories $\mathcal{C}$ equipped with a $l$-structure $\tau = (\mathcal{C}^{\geq 0}, \mathcal{C}^{\geq 0})$ and a torsion pair $v = (A_{tor}, A_{tf})$ on the heart $A$ of $\tau$. The $\mu_2 \ltimes \mathbb{Z}$-action on $l\text{-Cat}^{st}_\infty$ canonically lifts to $T$ by setting
   $$1 \cdot v := (A_{tor}[1], A_{tf}[1]) \quad \text{and} \quad \zeta_2 \cdot v := v^{op} := ((A_{tf})^{op}, (A_{tor})^{op}).$$
The tilting operation provides an extension of this action to the bigger group $\mu_2 \ltimes \mathbb{Z}[\frac{1}{2}]$, where $\frac{1}{2}$ acts by

$$\frac{1}{2} \cdot (\tau, \nu) := (\tau_0, \nu_0).$$

Here, $\tau_0$ is the $t$-structure obtained by tilting $\tau$ with respect to $\nu$, and

$$\nu_0 := (A_{t,f}[1], A_{tor})$$

is the induced tilted torsion pair.

(4) Finally, let us consider $\mathbb{T}^{fr}$. The action of $\mu_2 \ltimes \mathbb{Z}[\frac{1}{2}]$ on $\mathbb{T}$ extends to $\mathbb{T}^{fr}$, by setting

$$\xi_2 \cdot \mathcal{V} := \mathcal{V}^{\text{op}}[-2] \quad \text{ and } \quad \frac{1}{2} \cdot \mathcal{V} := \mathcal{V}[1].$$

Here $\mathcal{V}^{\text{op}}$ denote the object $\mathcal{V} \in \mathcal{C}$ seen as an object in $\mathcal{C}^{\text{op}}$, and the shift $[2]$ is understood computed in $\mathcal{C}^{\text{op}}$. In other words:

$$\mathcal{V}^{\text{op}}[-2] \simeq (\mathcal{V}[2])^{\text{op}}.$$

Finally, observe that since $1 = \frac{1}{2} + \frac{1}{2}$, the above definition implies $1 \cdot \mathcal{V} := \mathcal{V}[2]$.

In what follows, we will exclusively need the notation introduced in this construction. For this reason, we allowed ourselves to ignore the technical details that would be needed to properly construct an $\infty$-categorical action of $\mu_2 \ltimes \mathbb{Z}[\frac{1}{2}]$ on $\mathbb{T}^{fr}$.

Lemma 6.36. Let $\mathcal{C} = (\mathcal{C}, \tau, \nu, \mathcal{V}) \in \mathbb{T}^{fr}$. Then:

1. The shift-rotation self-equivalence $\rho: \mathcal{S}_2 \mathcal{C} \to \mathcal{S}_2 \mathcal{C}$ sending a fiber sequence $F_1 \to F_2 \to F_3$ to $F_2[1] \to F_3[1] \to F_1[2]$ induces a natural equivalence

$$\rho: P^\mathcal{C}(\frac{1}{2} \cdot \mathcal{C}) \simeq P(\mathcal{C}).$$

2. The canonical equivalence $\omega: (\mathcal{S}_2 \mathcal{C})^{\text{op}} \to \mathcal{S}_1(\mathcal{C}^{\text{op}})$ sending a fiber sequence $F_1 \to F_2 \to F_3$ to $F_3^{\text{op}} \to F_2^{\text{op}} \to F_1^{\text{op}}$ induces a natural equivalence

$$\rho^{-1} \circ \omega: P(\xi_2 \cdot \mathcal{C}) \simeq P(\mathcal{C})^{\text{op}},$$

where $\rho^{-1}$ is the inverse to the shift-rotation functor $\rho$ considered in the previous point.

Proof. First observe that the shift-rotation functor induces a canonical equivalence

$$\rho: \mathcal{C}^{\text{fr}}(\mathcal{V}) \to \mathcal{C}^{\text{fr}}(\mathcal{V}[2]).$$

Now, a $\mathcal{V}$-pair $\mathcal{V} \to \mathcal{F} \to E$ belongs to $P(\mathcal{C}) = P_{\tau, \nu}(\mathcal{C}; \mathcal{V})$ if and only if $\mathcal{F}[1] \in A_{tor}$ and $E \in A_{t,f}$. After applying $\rho$, we obtain the fiber sequence

$$\mathcal{F}[1] \to E[1] \to \mathcal{V}[2],$$

and $\mathcal{F}[1] \in A_{tor} = (A_{t,f})_\mathcal{C}$, while $E[1] \in (A_{t,f})_{\mathcal{C}} = (A_{\nu})_{tor}$. Therefore, the above sequence is a co-pair for $\frac{1}{2} \cdot \mathcal{C} = (\mathcal{C}, \tau_0, \nu_0, \mathcal{V}[1])$. Thus, $\rho$ induces a well defined functor

$$\rho: P(\mathcal{C}) \to P^\mathcal{C}(\frac{1}{2} \cdot \mathcal{C}),$$

and it is straightforward to check that it is an equivalence. This proves statement (1).

We now prove statement (2). To begin with, the natural equivalence $\omega$ tautologically induces an equivalence

$$\omega: (\mathcal{C}^{\text{fr}}(\mathcal{V}))^{\text{op}} \simeq (\mathcal{C}^{\text{op}})^{\text{fr}}(\mathcal{V}^{\text{op}}),$$

which sends a $\mathcal{V}$-pair $\mathcal{V} \to \mathcal{F} \to E$ to $E^{\text{op}} \to \mathcal{F}^{\text{op}} \to \mathcal{V}^{\text{op}}$. Further applying

$$\rho^{-1}: (\mathcal{C}^{\text{op}})^{\text{fr}}(\mathcal{V}^{\text{op}}) \to (\mathcal{C}^{\text{op}})^{\text{fr}}(\mathcal{V}^{\text{op}}[-2]),$$

we obtain

$$\mathcal{V}^{\text{op}}[-2] \to E^{\text{op}}[-1] \to \mathcal{F}^{\text{op}}[-1].$$
We now observe that $E^{\text{op}}[-1][1] = E^{\text{op}} \in (A_{\text{tf}})^{\text{op}} = (E^{\text{op}})^{\text{op \ast}}_{\text{tor}}$, while $F^{\text{op}}[-1] \simeq (F[1])^{\text{op}}$ belongs to $(A_{\text{tor}})^{\text{op}} = (E^{\text{op}})^{\text{op \ast}}_{\text{tor}}$. Thus, $\omega$ induces a well defined functor
\[
\rho^{-1} \circ \omega : \mathbf{P}(C)^{\text{op}} \to \mathbf{P}(\xi_{2} \cdot C),
\]
which is readily checked to be an equivalence. □

Now, we are ready to state the main result of this section, which follows from Theorem 5.7.

**Theorem 6.37.** Let $S$ be a smooth projective irreducible complex surface and let $\mathcal{V}$ be a locally free sheaf on $S$. Let $C := (\text{Perf}(S), \tau_{\text{std}}, v_{\text{fs}}, \mathcal{V}^{\vee}[-1])$ and $A := (\text{Perf}(S), \tau_{A}, v_{A}, \mathcal{V})$. Then there are canonical equivalences
\[
\mathbf{P}(A)^{\text{op}} \simeq \mathbf{P}(1/2 \cdot C) \quad \text{and} \quad \mathbf{P}^{\text{c}}(1/2 \cdot A)^{\text{op}} \simeq \mathbf{P}^{\text{c}}(1 \cdot C).
\]

The former can informally be described as sending a fiber sequence $\mathcal{V} \to F \to E$ to $\mathcal{V}^{\vee} \to E^{\vee}[1] \to F^{\vee}[1]$, and the latter can be described as sending a fiber sequence $F \to E \to \mathcal{V}[2]$ to $E^{\vee}[3] \to F^{\vee}[3] \to \mathcal{V}^{\vee}[2]$.

**Proof.** To begin with, we observe that the functor
\[
(\cdot)^{\vee}[2] : \text{Perf}(S)^{\text{op}} \to \text{Perf}(S)
\]
takes $\mathcal{V}^{\vee}[2] = (\mathcal{V}[2])^{\text{op}}$ to $\mathcal{V}^{\vee}[-2][2] \simeq \mathcal{V}^{\vee}$. Thus, Theorem 5.7 can be restated saying that the anti-equivalence $(-)^{\vee}[2]$ induces an equivalence
\[
(-)^{\vee}[2] : \xi_{2} \cdot A \simeq \frac{1}{2} \cdot C
\]
in $\mathcal{T}^{\text{fr}}$. At this point, applying repeatedly Lemma 6.36, we find:
\[
\mathbf{P}(A)^{\text{op}} \simeq \mathbf{P}(\xi_{2} \cdot A) \simeq \mathbf{P}(\frac{1}{2} \cdot C).
\]

Similarly,
\[
\mathbf{P}^{\text{c}}(\frac{1}{2} \cdot A)^{\text{op}} \simeq \mathbf{P}(A)^{\text{op}} \simeq \mathbf{P}(\frac{1}{2} \cdot C) \simeq \mathbf{P}^{\text{c}}(1 \cdot C).
\]

The explicit formulas are obtained by unwinding the definitions of these equivalences. □

**Remark 6.38.** In other words, the above theorem can be stated saying that for a fiber sequence $\mathcal{V} \to F \to E$ the following statements are equivalent:

1. The fiber sequence is a $\mathcal{V}$-stable pair, i.e. $F$ is purely 1-dimensional and $E \in A_{\text{tf}}$;
2. All the terms of the associated fiber sequence $\mathcal{V}^{\vee} \to E^{\vee}[1] \to F^{\vee}[1]$ belongs to $\text{Coh}^{\text{op}}(S)$, and $E^{\vee}[1]$ is torsion free, while $F^{\vee}[1]$ is purely 1-dimensional.

### 6.5.2. Duality and representations

**Theorem 6.37** has an immediate counterpart at the level of moduli stacks. Indeed, mimicking the construction of $\mathbf{P}(S; \mathcal{V})$ performed in §6.3, we introduce the derived stack $\mathbf{P}_{B}(S; \mathcal{V})$ as the fiber product
\[
\begin{array}{ccc}
\mathbf{P}_{B}(S; \mathcal{V}) & \xrightarrow{\mathbf{P}_{B}(S; \mathcal{V})} & \text{Perf}_{\text{ps}}(S; \mathcal{V}) \\
\downarrow & & \downarrow \quad \partial_{1} \times_{0} \\
\text{Coh}_{\text{tf}}(S) \times \text{Coh}_{\text{tor}}(S) & \xrightarrow{\mathbf{P}_{B}(S; \mathcal{V})} & \text{Perf}_{\text{ps}}(S) \times \text{Perf}_{\text{ps}}(S)
\end{array}
\]

**Remark 6.39.** Let $\text{Coh}_{1\text{-pure}}(S)$ be the open substack of $\text{Coh}_{\text{tor}}(S)$ parametrizing pure 1-dimensional coherent sheaves. Combining Proposition 6.1, Theorem 5.7 and Remark 6.38 we deduce that the
The natural projection $P_B(S; V^\vee) \to \text{Coh}_{t^\text{or}}(S)$ factors through $\text{Coh}_{1\text{-pure}}(S)$. This leads to the following alternative description of $P_B(S; V)$: let $P_{1\text{-pure}}(S)$ be the fiber product

$$P_{1\text{-pure}}(S; V^\vee) \longrightarrow \text{Perf}_{\text{ps}}(S; V^\vee)$$

$$\downarrow$$

$$\text{Coh}_{1\text{-pure}}(S) \longrightarrow \text{Perf}_{\text{ps}}(S)$$

Concretely, $P_{1\text{-pure}}(S; V^\vee)$ parametrizes extensions of the form $V^\vee \to E \to F$, where $F$ is purely 1-dimensional. Observe that this automatically implies that $E \in \text{Coh}(S)$. At this point, it follows that

$$P_B(S; V^\vee) \longrightarrow P_{1\text{-pure}}(S; V^\vee)$$

$$\downarrow$$

$$\text{Coh}_{t^\text{f.}}(S) \longrightarrow \text{Perf}_{\text{ps}}(S)$$

is a fiber product, thus realizing $P_B(S; V^\vee)$ as an open substack inside $P_{1\text{-pure}}(S; V^\vee)$. △

**Remark 6.40.** When $V = O_S$, we can provide a more explicit description of the (underived truncation of) $P_B(S; V^\vee)$. Let $H_{1\text{pure}}(S)$ be the Hilbert scheme parametrizing pure one-dimensional subschemes $C \subset S$. Let $C \subset S \subset H_{1\text{pure}}(S)$ be the universal curve and consider the (underived) relative Hilbert scheme $\text{Hilb}(C/H_{1\text{pure}}(S))$ of $H_{1\text{pure}}(S)$-flat families of zero-dimensional quotients of $O_C$. Then [PT10, Proposition B.8] yields a canonical identification

$$\mathcal{P}_B(S; O_S) \simeq \text{Hilb}(C/H_{1\text{pure}}(S)).$$

△

At this point, Theorem 6.37 immediately implies:

**Theorem 6.41.** There is a canonical equivalence

$$P(S; V) \simeq P_B(S; V^\vee).$$

Thus, $\text{Coh}^B_{\text{pro}}(P_B(S; V^\vee))$ is a right categorical module over $\text{Coh}^B_{\text{pro}}(\text{Coh}_{t^\text{or}}(S))$ and a left categorical module over $\text{Coh}^B_{\text{pro}}(\text{Coh}_{t^\text{dim}}(S))$. Similar statements hold in $K$-theory and in Borel-Moore homology.

**Proof.** For every affine derived scheme $T = \text{Spec}(A)$, let $p_S: T \times S \to S$ be the canonical projection. There is a canonical equivalence

$$\text{Perf}^f(T \times S; p_S^*(V))^\text{op} \simeq \text{Perf}^f(T \times S; p_S^*(V^\vee))^\text{op},$$

that sends a $p_S^*(V)$-extension $p_S^*(V) \to F \to E$ to the $p_S^*(V^\vee)$-extension $V^\vee \to E^\vee[1] \to F^\vee[1]$. Passing to the maximal $\infty$-groupoids and using the canonical equivalence

$$\text{inv}: \text{Perf}^f(T \times S; p_S^*(V))^\sim \simeq (\text{Perf}^f(T \times S; p_S^*(V))^\text{op})^\sim,$$

this yields a canonical equivalence of derived stacks

$$\text{Perf}^f(S; V) \simeq \text{Perf}^f(S; V^\vee).$$

Theorem 6.37 and Remark 6.38 imply that this equivalence restricts to an equivalence $P(S; V) \simeq P_B(S; V^\vee)$. The existence of the actions at the categorified (resp. $K$-theory, Borel-Moore homology) level is then a direct consequence of Theorems 6.14 and 6.25. □

**Remark 6.42.** Rather than transferring the left and the right action via the equivalence $P(S; V) \simeq P_B(S; V^\vee)$, it would be possible to independently construct, mimicking what was done in §6.3 and §6.4 for the $t$-structure $\tau_B$ and the torsion pair $(\text{Coh}_{t^\text{f.}}(S)[1], \text{Coh}_{t^\text{tor}}(S))$. In this case, the former theorem would be upgraded, with no additional cost, to state that the equivalence $P(S; V) \simeq P_B(S; V^\vee)$ is compatible with the natural actions on both sides. △
Corollary 6.43. With respect to the notations of Remark 6.40, $G_0(\text{Hilb}(C/H_{1,\text{pure}}(S)))$ is a right module over $G_0(\text{Coh}_0(S))$ and a left module over $G_0(\text{Coh}_{0,\text{dim}}(S))$. Similar statements hold for Borel-Moore homology.

Proof. Since $G_0$ and Borel-Moore homology are insensitive to the derived structure, this immediately follows combining Theorem 6.41 with Remark 6.40. □

7. COHA OF A QUIVER AND ITS REPRESENTATION VIA QUIVER VARIETIES

7.1. Preprojective algebras. Let $Q$ be a quiver, i.e., an oriented graph, with a finite vertex set $Q_0$ and a finite arrow set $Q_1$. For any arrow $a \in Q_1$, we denote by $s(a)$ the source of $a$ and by $t(a)$ the target of $a$. The path algebra $CQ$ of the quiver $Q$ is the associative algebra with basis all possible paths of length $\ell \geq 0$ of $Q$, endowed with the multiplication given by concatenation of paths, whenever possible, otherwise zero.

The double $Q^{\text{db}}$ of $Q$ is the quiver that has the same vertex set as $Q$ and whose set of arrows $Q_1^{\text{db}}$ is a disjoint union of the set $Q_1$ of arrows of $Q$ and of the set $$Q_1^{\text{opp}} := \{ a^* | a \in Q_1 \}$$ consisting of an arrow $a^*$ for any arrow $a \in Q_1$, with the reverse orientation (i.e., $s(a^*) = t(a)$ and $t(a^*) = s(a)$).

Definition 7.1 (cf. [BCS20, §4.1.4]). The derived preprojective algebra $\mathcal{G}_2(Q)$ is the derived push-out

$$\mathcal{G}_2(Q) := CQ^{\text{db}} \bigcup_{C[x]} C$$

where the morphism $C[x] \to CQ^{\text{db}}$ sends $x$ to

$$\sum_{a \in Q_1} (aa^* - a^*a) \in CQ^{\text{db}}.$$ (7.1)

The preprojective algebra $\Pi_Q$ of $Q$ is the 0-th cohomology of $\mathcal{G}_2(Q)$, i.e., the quotient of the path algebra $CQ^{\text{db}}$ by the two-sided ideal generated by the element (7.1).

Remark 7.2. The notion of derived preprojective algebra was originally introduced by Ginzburg [Gin06]. It is equivalent to the notion of 2-Calabi-Yau completion of $CQ$ by Keller [Kel11]. As pointed out for example in loc.cit., when $Q$ is not a finite Dynkin quiver, $\mathcal{G}_2(Q)$ is quasi-isomorphic to its preprojective algebra $\Pi_Q$. △

7.2. COHA of the preprojective algebra of a quiver and its representation via quiver varieties.

Definition 7.3 ([CB01]). Let $w \in \mathbb{N}^{Q_0}$. The Crawley-Boevey quiver associated to $Q$ is the quiver $Q^w$ whose set of vertices is given by $Q_0 \sqcup \{ \infty \}$, where $\infty$ is a new additional vertex, and whose set of arrows is the disjoint union of $Q_1$ and the set of $w_i$ additional edges of the form $\beta_i : i \to \infty$ for each vertex $i \in Q_0$. □

Fix a quiver $Q$. Consider the abelian category $\text{Mod}(\Pi_Q^w)$ of representations of $\Pi_Q^w$. Following e.g. [Gin12, §5], a representation $M$ corresponds to a pair of vector spaces over $C$

$$M := \bigoplus_{i \in Q_0} M_i \text{ and } M_\infty,$$

and a quadruple $(x, y, \alpha, \beta)$ of collections of $C$-linear maps

$$x_a : M_{s(a)} \to M_{t(a)} \text{, } y_a : M_{t(a)} \to M_{s(a)} \text{, } \alpha_i : M_\infty \to M_i \text{ and } \beta_i : M_i \to M_\infty$$

satisfying the preprojective relations. Consider the subcategory $\mathcal{T}$ of $\text{Mod}(\Pi_Q^w)$ consisting of those representations $M$ for which $M_\infty = 0$. This is a Serre subcategory. Moreover, it is equivalent to the category $\text{Mod}(\Pi_Q)$ of representations of $\Pi_Q$. Set $\mathcal{F} := \mathcal{T}^\perp$. Then, $v = (\mathcal{T}, \mathcal{F})$ is a torsion pair of $\text{Mod}(\Pi_Q^w)$.
Remark 7.4. Fix $\theta_\infty \in \mathbb{Q}$, set

$$\theta := (1, \ldots, 1) \in \mathbb{Q}^\delta_0 \quad \text{and} \quad \overline{\theta} := (\theta, \theta_\infty) \in \mathbb{Q}^\delta_0 \oplus \mathbb{Q}.$$ 

Define the $\overline{\theta}$-slope of a finite-dimensional representation $M = (M, M_\infty, x, y, \delta)$ of $\Pi_{\mathbb{Q}^\omega}$ as

$$\mu_{\overline{\theta}}(M) := \frac{\sum_{i \in \mathbb{Q}_0} \dim M_i + \theta_\infty \dim M_\infty}{\sum_{i \in \mathbb{Q}_0} \dim M_i + \dim M_\infty}.$$ 

Fix $\theta_\infty < 1$. Then, for any finite-dimensional representation $M$ we get $\mu_{\overline{\theta}}(M) \leq 1$ and $\mu_{\overline{\theta}}(M) = 1$ if and only if $M_\infty = 0$. Thus, the finite-dimensional representations $M$ belonging to $\mathcal{F}$ are all $\overline{\theta}$-semistable of slope one. On the other hand, the finite-dimensional representations belonging to $\mathcal{F}$ are those $M$ for which $\mu_{\overline{\theta}}(M) < 1$. This condition is equivalent to the requirement that all sub-representations of $M$ have a nonzero vector space associated to the vertex $\infty$. Following [CB01, Page 261], if $M$ satisfies this condition, we say that is $\infty$-co-generated. \hfill \square

Set $\mathcal{C}_{\mathbb{Q}^\omega} := \mathbb{Q}_2(\mathbb{Q}^\omega)_{-\text{mod}}$. Since $\mathbb{Q}_2(\mathbb{Q}^\omega)$ is concentrated in homologically nonnegative degrees, the heart of the standard $t$-structure of $\mathcal{C}_{\mathbb{Q}^\omega}$ is the abelian category Mod($\Pi_{\mathbb{Q}^\omega}$) of modules over the corresponding preprojective algebra $\Pi_{\mathbb{Q}^\omega}$. Note that $\tau_{\text{std}}$ satisfies openness of flatness.

The moduli stack $\mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$ of $\tau_{\text{std}}$-flat pseudo-perfect objects of $\mathcal{C}_{\mathbb{Q}^\omega}$ is a geometric derived stack locally of finite presentation over $\mathcal{C}$ (see [DPSa]). Its truncation is the usual moduli stack of finite-dimensional representations of the $\Pi_{\mathbb{Q}^\omega}$. The stack $\mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$ admits a decomposition into open and closed substacks depending on the dimension vectors $v$ and $w_\infty$ of $M$ and $M_\infty$, respectively:

$$\mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}}) = \bigsqcup_{v \in \mathbb{N}^{\delta_0}, w_\infty \in \mathbb{N}} \mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}}; v, w_\infty).$$

Now, we note that

$$\mathcal{Coh}_{\tau}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}}) \simeq \bigsqcup_{v \in \mathbb{N}^{\delta_0}} \mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}}; v, 0).$$

Hence, $\mathcal{Coh}_{\tau}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$ is an open and closed substack of $\mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$. Moreover,

$$\mathcal{Coh}_{\tau}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}}) \simeq \mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}}) \simeq \mathcal{Coh}_{ps}(\Pi_{\mathbb{Q}}),$$

where $\mathcal{C}_{\mathbb{Q}} := \mathbb{Q}_2(\mathbb{Q})_{-\text{mod}}$. Furthermore, $\mathcal{Coh}_{\tau}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$ is an open substack of $\mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$. We denote it by $\mathcal{Coh}_{ps}^s(\Pi_{\mathbb{Q}})$, since it is a moduli stack of stable objects, as justified by the following remark.

Remark 7.5. As shown in [CB01, Page 261], the coarse moduli space of the classical truncation of $\mathcal{Coh}_{ps}^s(\Pi_{\mathbb{Q}}; v, 1)$ is the Nakajima quiver variety $M_{\mathbb{Q}}(v, w)$ of $\theta$-stable representations, in the sense of Nakajima, of the preprojective algebra of the framed quiver of $\mathbb{Q}$ with dimension vectors $v$ and $w_\infty$. \hfill \square

Set

$$\mathcal{Coh}_{ps}^s(\Pi_{\mathbb{Q}}; w_\infty) := \bigsqcup_{v \in \mathbb{N}^{\delta_0}} \mathcal{Coh}_{ps}^s(\Pi_{\mathbb{Q}}; v, w_\infty).$$

It is straightforward to verify that $\mathcal{Coh}_{ps}(\Pi_{\mathbb{Q}})$, seen as a substack of $\mathcal{Coh}_{ps}(\mathcal{C}_{\mathbb{Q}^\omega}, \tau_{\text{std}})$, satisfies assumption (2.) of Corollary 3.11. Thus, we get the following result.

Theorem 7.6. The stable pro-$\infty$-category $\mathcal{Coh}_{pro}(\mathcal{Coh}_{ps}(\Pi_{\mathbb{Q}}))$ has a $E_1$-monoidal structure. In particular, $G_0(\mathcal{Coh}_{ps}(\Pi_{\mathbb{Q}}))$ and $\mathcal{H}^1_{BM}(\mathcal{Coh}_{ps}(\Pi_{\mathbb{Q}}))$ have the structure of associative algebras.

\hfill 18\text{Here, $\mu_{\overline{\theta}}(M)$ is the $\overline{\theta}$-slope of the maximal destabilizing sub-representation of $M$.}
For each dimension vector \( w_\infty \), \( \text{Coh}^b_{\text{pro}}(\text{Coh}^s_{\text{ps}}(\Pi_{Q^w}; w_\infty)) \) has the structure of a left categorical module over \( \text{Coh}^b_{\text{pro}}(\text{Coh}^s_{\text{ps}}(\Pi_Q)) \). In particular, \( G_0(\text{Coh}^s_{\text{ps}}(\Pi_{Q^w}; w_\infty)) \) (resp. \( H^{BM}_s(\text{Coh}^s_{\text{ps}}(\Pi_{Q^w}; w_\infty)) \)) has the structure of a left module over \( G_0(\text{Coh}^s_{\text{ps}}(\Pi_Q)) \) (resp. over \( H^{BM}_s(\text{Coh}^s_{\text{ps}}(\Pi_Q)) \)).

Fix \( w_\infty = 1 \). Then, \( G_0(M_{Q^w}(w))_C \) and \( H^{BM}_s(M_{Q^w}(w)) \) have the structure of a left modules over \( G_0(\text{Coh}^s_{\text{ps}}(\Pi_Q))_C \) and \( H^{BM}_s(\text{Coh}^s_{\text{ps}}(\Pi_Q)) \), respectively. Similar statements hold equivariantly with respect to the torus \( T \) introduced in [SV20, §3.3].

Note that the first part of the above theorem, i.e., the constructions of the two-dimensional cohomological Hall algebra of a quiver and its categorification have been already given in [SV13a, SV13b, SV20, YZ18, VV22] (see also [DPSa] for the latter). The left action of the two-dimensional cohomological Hall algebra of a quiver on the cohomology of Nakajima quiver varieties has been constructed in [SV13a, SV20].

**Remark 7.7.** Note that we cannot apply directly Theorem 3.21 since the second condition in assumption (2) of it is not verified by the tilted heart, while the rest of the assumptions of the theorem are verified. \( \triangle \)

### Appendix A. Complements on the Segal conditions

There are two conceptual ingredients behind the associativity of the Hall multiplication (be it in the classical, cohomological, \( K \)-theoretical or categorified context): the Waldhausen construction and the 2-Segal property.

A similar paradigm can be developed for representations of Hall algebras, as it was independently shown by Young [You18] and Walde [Wal]. In this appendix we briefly review this theory, and lay out a couple of useful foundational results that are needed throughout the main body of this paper.

#### A.1. 2-Segal objects and their modules

Let \( \mathcal{T} \) be a presentable \( \infty \)-category. We refer to [DK19, §2] for background material on (pointed) 1- and 2-Segal objects, and to [Wal, §3.5.2] and [You18, §1.3] for background material on relative 2-Segal objects.

For \( k \in \{1, 2\} \) we denote by \( k \text{-Seg}_k(\mathcal{T}) \) the full subcategory of \( \text{Fun}(\Delta^{op}, \mathcal{T}) \) spanned by pointed and unital \( k \)-Segal objects. We denote by \( 1\text{-Seg}_{1, \text{rel}}(\mathcal{T}) \) (resp. \( 1\text{-Seg}_{1, \text{rel}}(\mathcal{T}) \)) the full subcategory of \( \text{Fun}(\Delta^{op} \times \Delta^1, \mathcal{T}) \) spanned by pointed and unital relative left (resp. right) \( k \)-Segal objects. Finally, we denote by \( 2\text{-Seg}_{2, \text{rel}}(\mathcal{T}) \) the full subcategory of \( \text{Fun}(\Delta^{op} \times \Delta^1, \mathcal{T}) \) spanned by relative 2-Segal objects.

Let us recall the following fundamental result about these objects:

**Theorem A.1** (Lurie, Dyckerhoff-Kapranov, Young).

1. There exists an equivalence of \( \infty \)-categories

\[
\text{1-Seg}_{1, \text{rel}}(\mathcal{T}) \simeq \text{LMod}(\mathcal{T}^\times) \quad \text{and} \quad \text{1-Seg}_{1, \text{rel}}(\mathcal{T}) \simeq \text{RMod}(\mathcal{T}^\times),
\]

which restricts to an equivalence

\[
\text{1-Seg}_1(\mathcal{T}) \simeq \text{Mon}_{\mathcal{E}_1}(\mathcal{T}^\times).
\]

2. There exists a canonical functor of \( \infty \)-categories

\[
\text{2-Seg}_1(\mathcal{T}) \rightarrow \text{Mon}_{\mathcal{E}_1}(\text{Corr}^\times(\mathcal{T})),
\]

which informally sends a pointed 2-Segal object \( A \), to the object \( A_1 \) equipped with the multiplication given by the span

\[
A_1 \times A_1 \xleftarrow{\delta_1 \times \delta_2} A_2 \xrightarrow{\delta_1} A_1,
\]

\[\text{In a future version of the paper, we will extend this result by proving that after passing to the cohomological Hall algebra of nilpotent representations of } \Pi_Q, \text{ we obtain a left and right action.}\]
and whose higher coherences are controlled by the higher simplices of \(A_\bullet\).

(3) The functor of the previous point lifts to a functor of \(\infty\)-categories

\[
2\text{-Seg}_c^\text{rel}(\mathcal{T}) \longrightarrow \text{LMod}(\text{Corr}^\times(\mathcal{T})) .
\]

which informally sends a pointed relative 2-Segal object \(f_\bullet : M_\bullet \rightarrow A_\bullet\),

where \(A_1\) is equipped with the above multiplication and \(M_0\) is equipped with the action

\[
A_1 \times M_0 \quad \xrightarrow{f_1 \times \partial_0} \quad M_1 \quad \xrightarrow{\partial_1} \quad M_0,
\]

and whose higher coherences are controlled by the higher simplices of \(M_\bullet\).

\[\text{Proof.}\] The first statement is proven in [Lur17, Propositions 4.1.2.10 and 4.2.2.9]. Statement (2) is proven in [DK19, Theorem 11.1.6]. Finally, statement (3) is proven in [You18, Theorem 4.2]. \(\square\)

Any associative (unital) monoid \(M\) acts on himself by multiplication on the left and on the right. It is easy to give a purely simplicial description of this phenomenon. The following construction generalizes [Lur17, Example 4.2.2.4] and [You18, Proposition 2.5]:

\[\text{Construction A.2.}\] Let \(m \geq 1\) be an integer and define the function \(a_m : \{0, 1\} \rightarrow \mathbb{Z}\) by setting

\[
a_m(i) = m - 1 - im = \begin{cases} m - 1 & \text{if } i = 0, \\ -1 & \text{if } i = 1. \end{cases}
\]

Consider now the functors

\[\ell^{act}(m), r^{act}(m) : \Delta^{op} \times \Delta^1 \longrightarrow \Delta^{op}\]

defined by

\[\ell^{act}(m)([n], i) := [n] \star [a_m(i)]\quad\text{ and }\quad r^{act}(m)([n], i) := [a_m(i)] \star [n],\]

where \(\star\) denotes the join operation. Given an \(\infty\)-category \(\mathcal{T}\) with products and a simplicial object \(A : \Delta^{op} \rightarrow \mathcal{T}\), we write

\[\ell A(m) := A \circ \ell^{act}(m)\quad\text{ and }\quad r A(m) := A \circ r^{act}(m),\]

and we refer to them as the \(\text{left}\) and \(\text{right} m\text{-flags action objects}.\) Observe that by definition we have

\[\ell A(m)([0], 0) = A([m]) = r A(m)([0], 0).\]

\[\text{Proposition A.3.}\] Let \(m \geq 1\) be an integer and let \(A : \Delta^{op} \rightarrow \mathcal{T}\) be a simplicial object. If \(A\) is (unital) 1-Segal, then \(\ell A(m)\) and \(r A(m)\) are (unital) relative left and right 1-Segal objects, respectively. If \(A\) is (unital) 2-Segal, then both \(\ell A(m)\) and \(r A(m)\) are (unital) relative 2-Segal objects.

\[\text{Proof.}\] When \(m = 1\) this coincides exactly with [Lur17, Example 4.2.2.4] (in the 1-Segal case) and with [You18, Proposition 2.5] (in the 2-Segal case). The proof of the general case is identical, and we leave it to the interested reader. \(\square\)

When \(m \geq 2\) the above construction can be slightly refined. We begin fixing the following notation:

\[\text{Notation A.4.}\] We let \(\Delta_+\) be the augmented simplicial category. Its objects are possibly empty finite linearly ordered sets. Conventionally, we denote by \([-1]\) the empty poset, which becomes the initial object of \(\Delta_+\). Moreover, for every \([n] \in \Delta_+\) we set

\[\[n] \star [-2] := [-1] \quad\text{ and }\quad [-2] \star [n] := [-1].\]

Given an integer \(m \geq 1\) we let

\[\sigma_m : \Delta^{op} \times \Delta^1 \longrightarrow \Delta^{op}_+\]

by setting

\[\sigma_m([n], i) := [a_m(i) - 1].\]
Notice that when \( i = 1 \), \( a_m(i) - 1 = -2 \), and therefore by convention we have \( σ^*_m([n], 1) = σ^*_m([n], 1) = [-1] \) for every \([n] \in Δ^op\).

\[\square\]

**Remark A.5.** Let \( T \) be an \( \infty \)-category with products and let \( A: Δ^op \to T \) be a simplicial object. We extend \( A \) to an augmented simplicial object \( A_+: Δ^op \to T \) by setting \( A_+([-1]) := 1_T \), where \( 1 \) denotes the final object of \( T \). Then the composite \( A_+ \circ σ_m \) is canonically identified with the morphism \( A([-m+1]) \to 1 \), where both the source and the target are seen as constant simplicial objects. Since \( 1 \) is the final object of \( T \), it is immediate to see that \( A_+ \circ σ_m \) is a relative (left 1-Segal and hence) 2-Segal object. \[\triangle\]

**Construction A.6.** Let \( m \geq 1 \) be an integer. The boundary maps

\[\partial_{m-1}: [m-1] \to [m-2] \quad \text{and} \quad \partial_1: [m-1] \to [m-2]\]

induce natural transformations

\[σ^*_m: \operatorname{act}^m \to σ_{m-1} \quad \text{and} \quad δ^*_m: \operatorname{act}^m \to σ_{m-1} \].

Let \( T \) be an \( \infty \)-category with products and let \( A: Δ^op \to T \) be a simplicial object. Denoting by \( 1 \) the final object of \( T \), let \( V: 1 \to A([-m+2]) \) be a morphism in \( T \). Reviewing \( id_1: 1 \to 1 \) as a constant simplicial object in \( T \), we review \( V \) as a morphism

\[
\begin{array}{ccc}
1 & \xrightarrow{V} & A([-m+2]) \\
\downarrow{id_1} & & \downarrow{1} \\
1 & & 1
\end{array}
\]

where the columns are seen as constant simplicial objects in \( T \). Furthermore, Remark A.5 canonically identifies the right column with \( A_+ \circ σ_{m-1} \). We therefore review \( V \) as a morphism \( id_1 \to A_+ \circ σ_{m-1} \). At this point, we define \( ℓ_A^{(m),+} \) and \( r_A^{(m),+} \) as the fiber products

\[
\begin{array}{ccc}
ℓ_A^{(m),+} & \xrightarrow{id_1} & \operatorname{id} \\
\downarrow{ℓ_A^{(m)}} & \downarrow{V} & \downarrow{V} \\
A_+ \circ σ_m & \xrightarrow{r_A^{(m),+}} & A_+ \circ σ_m
\end{array}
\]

taken in \( \operatorname{Fun}(Δ^op \times Δ^1, T) \).

\[\ominus\]

**Remark A.7.** The previous construction makes equally sense for \( m = 1 \), but since \( A_+([-1]) = A_+([-1]) = 1 \), there is only one possible choice for the morphism \( V \), and the resulting objects \( ℓ_A^{(1),+} \) and \( r_A^{(1),+} \) coincide with \( ℓ_A^{(1)} \) and \( r_A^{(1)} \), respectively. \[\triangle\]

In virtue of the previous remark, the following result subsumes Proposition A.3:

**Proposition A.8.** Let \( m \geq 2 \) be an integer. Let \( T \) be an \( \infty \)-category with products and a final object \( 1 \). Let \( A: Δ^op \to T \) be a simplicial object and let \( V: 1 \to A([-m+2]) \) be any morphism. If \( A \) is (unital) 1-Segal, then both \( ℓ_A^{(m),+} \) and \( r_A^{(m),+} \) are (unital) relative left and right 1-Segal objects, respectively. If \( A \) is (unital) 2-Segal, then both \( ℓ_A^{(m),+} \) and \( r_A^{(m),+} \) are (unital) 2-Segal objects.

**Proof.** Observe that both \( id_1: 1 \to 1 \) and \( A_+ \circ σ_m \) are both unital relative left and right 1-Segal objects. Recall moreover that the collection of (unital) relative left and right 1-Segal and (unital) relative 2-Segal objects are closed under limits in \( \operatorname{Fun}(Δ^op \times Δ^1, T) \). Thus, the conclusion follows from Proposition A.3 and the very definition of \( ℓ_A^{(m),+} \) and \( r_A^{(m),+} \). \[\square\]
A.2. 1-Coskeletality. Fix a presentable ∞-category $\mathcal{T}$. Checking that a simplicial object $X_\bullet \in \text{Fun}(\Delta^{\text{op}}, \mathcal{T})$ is 1- or 2-Segal involves a priori to check infinitely many conditions. However, when the simplicial structure on $X_\bullet$ is built out of finitely many simplexes, it is desirable to be able to decrease the number of needed checks. In this subsection we work out this idea, which will turn out to be useful later on in this appendix.

Notation A.9. Let $j_1 : \Delta^{\text{op}}_{\leq 1} \to \Delta$ be the canonical inclusion. We write

$$\cosk_1 := j_{1,*} \circ j_1^* : \text{Fun}(\Delta^{\text{op}}, \mathcal{T}) \longrightarrow \text{Fun}(\Delta^{\text{op}}_{\leq 1}, \mathcal{T}),$$

and we refer to it as the 1-coskeleton functor. Here $j_1^*$ denotes the restriction along $j_1$ and $j_{1,*}$ denotes the left Kan extension along $j_1$.

Observe that there is a canonical natural transformation $\text{id} \to \cosk_1$, corresponding to the unit of the adjunction $j_1^* \dashv j_{1,*}$. Similarly, we define the 1-skeleton functor as $\sk_1 := j_{1,1} \circ j_{1,1}^!$, where $j_{1,1}$ denotes the left Kan extension along $j_1$.

Definition A.10. We say that a morphism $f_\bullet : X_\bullet \to Y_\bullet$ in $\text{Fun}(\Delta^{\text{op}}, \mathcal{T})$ is 1-coskeletal if the square

$$\begin{array}{ccc}
X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\
\downarrow & & \downarrow \\
\cosk_1(X_\bullet) & \xrightarrow{\cosk_1(f_\bullet)} & \cosk_1(Y_\bullet)
\end{array}$$

is a pullback. △

Notation A.11. The Yoneda embedding provides a canonical equivalence

$$\text{Fun}(\Delta^{\text{op}}, \mathcal{T}) \simeq \text{Fun}^R(\text{PSh}(\Delta)^{\text{op}}, \mathcal{T}).$$

In particular, given a simplicial space $K \in \text{PSh}(\Delta)$ we obtain a limit-preserving functor

$$\text{ev}_K : \text{Fun}(\Delta^{\text{op}}, \mathcal{T}) \longrightarrow \mathcal{T},$$

which evaluates a simplicial object $X_\bullet$ on the simplicial space $K$. We write $X_K := \text{ev}_K(X_\bullet)$. △

Remark A.12. If $X_\bullet \in \text{Fun}(\Delta^{\text{op}}, \mathcal{T})$, then with the previous notation we have $X_{\Delta^n} \simeq X_n$. Set

$$\beta^n := \Delta^n \cdot \Pi \Delta \cdot \Pi \Delta \cdot \cdots,$$

where the pushouts are computed via the maps induced by the spans $\{i-1, i\} \leftarrow \{i\} \to \{i, i+1\}$. Then the canonical map $\beta^n : \Delta^n \to \Delta^1$ induces a morphism

$$X_n \longrightarrow X_{\beta^n} \simeq X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1.$$

Saying that $X_\bullet$ is pointed 1-Segal is equivalent (by definition) to say that these maps are equivalence for every $n \geq 0$. A similar description holds for 2-Segal objects, see [DK19, §2.3]. △

Lemma A.13. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a 1-coskeletal morphism. Then for every morphism $K \to H$ in $\text{PSh}(\Delta)$, the square

$$\begin{array}{ccc}
X_H & \longrightarrow & X_K \times_Y H \\
\downarrow & & \downarrow \\
X_{\sk_1(H)} & \longrightarrow & X_{\sk_1(K)} \times_{\sk_1(Y)} \sk_1(H)
\end{array}$$

is a pullback.

Proof. Unraveling the definition of 1-coskeletal morphism and using the fact that the evaluations $\text{ev}_K$ and $\text{ev}_H$ commute with limits, we readily reduce to the case where $Y_\bullet \to \cosk_1(Y_\bullet)$ is an equivalence. In this case, the map $X_\bullet \to \cosk_1(X_\bullet)$ is an equivalence as well, and therefore we are reduced to check that for every $Z_\bullet \in \text{Fun}(\Delta^{\text{op}}, \mathcal{T})$ and every $L \in \text{PSh}(\Delta)$ the map

$$\text{ev}_L(\cosk_1(Z_\bullet)) \longrightarrow \text{ev}_{\sk_1(L)}(\cosk_1(Z_\bullet))$$
is an equivalence. Composing with the functors Hom_{\mathcal{T}}(T, -) as T varies on a set of generators for \mathcal{T}, we are immediately reduced to the case \mathcal{T} = \mathcal{S}, where the latter is the \infty\text{-category of spaces. In this case, the conclusion follows since the adjunction sk_1 \dashv \cosk_1 holds. □

**Notation A.14.** When dealing with relative Segal objects, we write \Delta^1 = \{m \to a\} and represent objects in Fun(\Delta^{op} \times \Delta^1, \mathcal{T}) by the notation \mathcal{X}_\bullet. We refer to \mathcal{X}_\bullet^m := \mathcal{X}_\bullet|_{\Delta^{op} \times \{m\}} as the underlying module object and to \mathcal{X}_\bullet^n := \mathcal{X}_\bullet|_{\Delta^{op} \times \{a\}} as the underlying algebra object. △

**Definition A.15.**

1. We say that a morphism \mathcal{f}_\bullet : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet in Fun(\Delta^{op}, \mathcal{T}) is 1-Segal if for every \(n \geq 1\) the map

\[
X_n \to X_{3^n} \times_{Y_{3^n}} Y_n
\]

is an equivalence. We say that \(\mathcal{f}_\bullet\) is pointed 1-Segal if it is 1-Segal and the map \(f_0 : X_0 \to Y_0\) is an equivalence.

2. We say that a morphism \(\mathcal{f}_\bullet^\ast : \mathcal{X}_\bullet^\ast \to \mathcal{Y}_\bullet^\ast\) in Fun(\Delta^{op} \times \Delta^1, \mathcal{T}) is relative left (resp. right) 1-Segal if for every \(n \geq 1\) the morphism \(\mathcal{f}_\bullet^\ast : \mathcal{X}_\bullet^n \to \mathcal{Y}_\bullet^n\) is 1-Segal and for every \(n \geq 0\) the square

\[
\begin{array}{ccc}
X_n & \to & X_0^n \\
\downarrow & & \downarrow \\
Y_n & \to & Y_0^n
\end{array}
\]

is a pullback, where the m-component of the horizontal maps are induced by \(\Delta^0 \to \Delta^n\) (resp. by \(\Delta^0 \to \Delta^n\)). □

**Warning A.16.** Let \(\mathcal{f}_\bullet : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet\) be a morphism in Fun(\Delta^{op}, \mathcal{T}). Reviewing \(\mathcal{f}_\bullet\) as an object in Fun(\Delta^{op} \times \Delta^1, \mathcal{T}), we see that for \(\mathcal{f}_\bullet\) it makes sense at the same time to be 1-Segal (as a morphism) or relative left (or right) 1-Segal (as an object). These two notions do not coincide. However, if \(\mathcal{Y}_\bullet\) itself is 1-Segal, then if \(\mathcal{f}_\bullet\) is relative left (or right) 1-Segal, it follows that \(\mathcal{X}_\bullet\) is 1-Segal (see [You18, Proposition 2.1]), and therefore that the morphism \(\mathcal{f}_\bullet\) is 1-Segal as well (since any morphism between 1-Segal objects is automatically 1-Segal). △

**Proposition A.17.** Let \(\mathcal{f} : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet\) be a morphism in Fun(\Delta^{op}, \mathcal{T}). If \(\mathcal{f}\) is 0-coskeletal, then it is 1-Segal. Moreover, \(\mathcal{f}\) is pointed 1-Segal if and only if \(\mathcal{f}\) is an equivalence.

**Proof.** Since \(\mathcal{f}\) is 0-coskeletal, the square

\[
\begin{array}{ccc}
\cosk_0(\mathcal{X}_\bullet) & \to & \cosk_0(\mathcal{Y}_\bullet) \\
\downarrow^{\cosk_0(\mathcal{f})} & & \downarrow^{\cosk_0(\mathcal{f})} \\
\cosk_0(\mathcal{X}_\bullet) & \to & \cosk_0(\mathcal{Y}_\bullet)
\end{array}
\]

is a pullback. Now, we can canonically identify \(\cosk_0(\mathcal{X}_\bullet)\) with the \v{C}ech nerve of \(X_0 \to 1_\mathcal{T}\), where \(1_\mathcal{T}\) is the final object in \(\mathcal{T}\). Thus, the proof of [Lur09, Proposition 6.1.2.11] shows that \(\cosk_0(\mathcal{f})\) is 1-Segal. For the same reason, \(\cosk_0(\mathcal{Y}_\bullet)\) is 1-Segal, and therefore map \(\cosk_0(\mathcal{f})\) is 1-Segal as well. Since the above square is a pullback, we deduce that \(\mathcal{f}\) is 1-Segal as well, proving the first half of the statement. For the second half, saying that \(\mathcal{f}\) is pointed is equivalent to say that the induced map \(f_0 : X_0 \to Y_0\) is an equivalence. Since \(\cosk_0(\mathcal{f})\) is the right Kan extension along \(\Delta_0^{op} \to \Delta^{op}\) of the morphism \(f_0\), the conclusion follows. □
Proposition A.18. Let $f : X_\bullet \to Y_\bullet$ be a morphism in $\text{Fun}(\Delta^{\text{op}}, T)$. If $f$ is $1$-coskeletal, then it is $1$-Segal if and only if the diagram

$$
\begin{array}{ccc}
X_2 & \longrightarrow & X_1 \times_{X_0} X_1 \\
\downarrow & & \downarrow \\
Y_2 & \longrightarrow & Y_1 \times_{Y_0} Y_1
\end{array}
$$

is a pullback square. Furthermore, it is pointed $1$-Segal if and only if in addition to this condition the map $X_0 \to Y_0$ is an equivalence.

Proof. The pointed case is an obvious consequence of the unpointed one, so we only deal with the latter. Let $j : \Delta^{\text{op}}_{\leq 1} \to \Delta^{\text{op}}$ be the natural inclusion. Fix an integer $n \geq 0$. Since $f$ is $1$-coskeletal, Lemma A.13 provides us with the following pullback square:

$$
\begin{array}{ccc}
X_n & \longrightarrow & X_{\text{sk}_1(\Delta^n)} \times_{Y_{\text{sk}_1(\Delta^n)}} Y_n \\
\downarrow & & \downarrow \\
X_{\text{sk}_1(\Delta^n)} & \longrightarrow & X_{\text{sk}_1(\Delta^n)} \times_{Y_{\text{sk}_1(\Delta^n)}} Y_{\text{sk}_1(\Delta^n)}
\end{array}
$$

It is therefore enough to prove that the condition of the statement implies that the bottom row is an equivalence. We proceed by induction on $n$. When $n = 1$, $\text{sk}_1(\Delta^1) = \Delta^1 = \beta^1$, and therefore the statement is trivial. When $n = 2$, the assertion is true by assumption. Let therefore $n \geq 3$. For $j = n + 1, n, n - 1, \ldots, 0$ we define inductively simplicial sets $\beta^n(j)$ and $\text{sk}_1(\Delta^n)(j)$ together with morphisms

$$
\alpha_j : \beta^n(j) \to \text{sk}_1(\Delta^n)(j) \quad \text{and} \quad \beta_j : \text{sk}_1(\Delta^n)(j) \to \text{sk}_1(\Delta^n)(j - 1),
$$

as follows:

1. for $j = n + 1$ we set $\text{sk}_1(\Delta^n)(n + 1) := \text{sk}_1(\Delta^n)$, $\beta^n(n + 1) := \beta^n$ and we let $\alpha_{n+1}$ be the canonical map; furthermore we conventionally set $\text{sk}_1(\Delta^n)(n + 2) = \varnothing$ so that $\beta_{n+1}$ is uniquely determined;

2. for $j = n$ we set $\beta^n(n) := \beta^n$, and we define $\text{sk}_1(\Delta^n)(n)$, $\alpha_n$ and $\beta_n$ via the following diagram:

$$
\begin{array}{ccc}
\Delta^0 & \xrightarrow{n - 1} & \beta^n(n) \\
\downarrow & & \downarrow \\
\Delta^1 & \xrightarrow{n} & \text{sk}_1(\Delta^n)(n)
\end{array}
$$

where we require the square on the right to be a pushout;

3. for $j \leq n$ we define $\beta^n(j)$, $\text{sk}_1(\Delta^n)(j)$, $\alpha_j$ and $\beta_{j+1}$ for $j \leq n$, we define the same data at level $j - 1$ by forming the following diagram:

$$
\begin{array}{ccc}
\Delta^0 \amalg \Delta^0 & \xrightarrow{(j - 1, n)} & \beta^n(j) \\
\downarrow & & \downarrow \\
\Delta^1 & \xrightarrow{\alpha_{j-1}} & \text{sk}_1(\Delta^n)(j - 1)
\end{array}
$$

where both squares are required to be pushouts.

Observe that by construction $\text{sk}_1(\Delta^n)(0) = \text{sk}_1(\Delta^n)$ and that the canonical map $\beta^n \to \text{sk}_1(\Delta^n)$ can be factored as the composition

$$
\beta^n = \beta^n(n) \xrightarrow{\alpha_n} \text{sk}_1(\Delta^n)(n) \xrightarrow{\beta_{n-1}} \text{sk}_1(\Delta^n)(n - 1) \xrightarrow{\beta_{n-2}} \cdots \xrightarrow{\beta_1} \text{sk}_1(\Delta^n)(0) = \text{sk}_1(\Delta^n).
$$
This reduces us to check that the map
\[ X_{sk_1(\Delta^n)}(n) \rightarrow X_{\beta^n(n)} \times_{Y_{sk_1(\Delta^n)(n)}} Y_{sk_1(\Delta^n)(n)} \]
induced by \( a_n \) and the maps
\[ X_{sk_1(\Delta^n)(j-1)} \rightarrow X_{sk_1(\Delta^n)(j)} \times_{Y_{sk_1(\Delta^n)(j)}} Y_{sk_1(\Delta^n)(j-1)} \]
induced respectively by the \( \beta_j \)'s are equivalences.

For \( a_n \), it is enough to observe that the relevant map fits in the following pullback square
\[
\begin{array}{ccc}
X_{sk_1(\Delta^n)}(n) & \rightarrow & X_{\beta^n(n)} \times_{Y_{sk_1(\Delta^n)(n)}} Y_{sk_1(\Delta^n)(n)} \\
\downarrow & & \downarrow \\
X_{sk_1(\Delta^{n-1})} & \rightarrow & X_{\beta^{n-1}} \times_{Y_{sk_1(\Delta^{n-1})}} Y_{sk_1(\Delta^{n-1})}
\end{array}
\]
and that the inductive hypothesis of level \( n - 1 \) guarantees that the bottom row is an equivalence.

We now deal with the morphisms \( \beta_j \) for \( j \leq n - 1 \). Observe that the outer square in the diagram (A.1) can be alternatively be decomposed in the following ladder:
\[
\begin{array}{cccc}
\Delta^0 \coprod \Delta^0 & \rightarrow & \Lambda^2_1 & \rightarrow & sk_1(\Delta^n)(j) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^1 & \rightarrow & sk_1(\Delta^2) & \rightarrow & sk_1(\Delta^n)(j-1)
\end{array}
\]
Since the left and the outer squares are pushout, the same goes for the right one. Thus, the induced square
\[
\begin{array}{ccc}
X_{sk_1(\Delta^n)(j-1)} & \rightarrow & X_{sk_1(\Delta^n)(j)} \times_{Y_{sk_1(\Delta^n)(j)}} Y_{sk_1(\Delta^n)(j-1)} \\
\downarrow & & \downarrow \\
X_{sk_1(\Delta^2)} & \rightarrow & X_{\Lambda^2_1} \times_{Y_{\Lambda^2_1}} Y_{sk_1(\Delta^2)}
\end{array}
\]
is a pullback, and the bottom row is an equivalence. The conclusion follows. \( \square \)

In order to keep the notations under control, it is useful to introduce the following convention, that will be applied through the remainder of this appendix:

**Notation A.19.** We represent \( \Delta^1 \) as \( \{ m \rightarrow a \} \). Given a functor \( F : \Delta^{op} \times \Delta^1 \rightarrow \mathcal{T} \), we write
\[ F^a := F|_{\Delta^{op} \times \{ a \}} \quad \text{and} \quad F^m := F|_{\Delta^{op} \times \{ m \}} , \]
and we refer to these functors as the \( a \)- and \( m \)-components of \( F \). Often (but not always), we represent functors \( \Delta^{op} \times \Delta^1 \rightarrow \mathcal{T} \) by notations analogous to \( X_{\bullet} \). In this case, we mean that \( \ast \in \{ a, m \} \) and that we are given a natural transformation \( X_{\ast}^m \rightarrow X_{\ast}^a \). \( \triangle \)

The following two propositions generalize [You18, Propositions 2.1 and 2.4] to morphisms. The proofs are straightforward generalizations of the proofs in loc. cit., and they are left to the interested reader:

**Proposition A.20.** A morphism \( f_{\ast} : X_{\ast}^a \rightarrow Y_{\ast}^a \) in \( \text{Fun}(\Delta^{op} \times \Delta^1, \mathcal{T}) \). Assume that the morphism \( f_{\ast}^{0} : X_{\ast}^{0} \rightarrow Y_{\ast}^{0} \) is 1-Segal. Then \( f_{\ast} \) is relative left 1-Segal if and only if the morphism \( f_{\ast}^{m} : X_{\ast}^{m} \rightarrow Y_{\ast}^{m} \) is 1-Segal and the square
\[
\begin{array}{ccc}
X_{1}^{m} & \rightarrow & X_{1}^{a} \times_{Y_{1}^{a}} X_{0}^{m} \\
\downarrow & & \downarrow \\
Y_{1}^{m} & \rightarrow & Y_{1}^{a} \times_{Y_{0}^{a}} Y_{0}^{m}
\end{array}
\]

is a pullback.

**Proposition A.21.** Let \( f^*: X^*_A \to Y^*_A \) be a morphism in \( \text{Fun}(\Delta^{op} \times \Delta^1, \mathcal{T}) \). If \( f^*_A \) is relative left or right 1-Segal, then it is relative 2-Segal.

### A.3. Induced algebras and modules.

We now address the following question, which plays a central role in this paper.

Let \( s_*: X^m_* \to X^a_* \) be a relative 2-Segal object with values in \( \mathcal{T} \). Given morphisms \( A \to X^a_* \) and \( M \to X^m_* \), we are going to spell out some sufficient conditions that guarantee that the simplicial structure of \( X^a_* \) induces a relative 2-Segal object \( X^a_*(A, M) \) satisfying

\[
X^a_1(A, M) = A \quad \text{and} \quad X^m_1(A, M) = M.
\]

**Notation A.22.** Let \( \mathbb{I} \) be the (non full) subcategory of \( \Delta^{op} \times \Delta^1 \) depicted as follows:

\[
\begin{align*}
\{ ([0], m) \} & \quad \downarrow \quad \{ ([1], a) \} \\
\uparrow^i \quad \downarrow^\phi & \quad \uparrow^j \quad \downarrow^i
\end{align*}
\]

Observe that \( \mathbb{I} \) fits in between \( \Delta^{op}_{\leq 0} \times \Delta^1 \) and \( \Delta^{op}_{\leq 1} \times \Delta^1 \).

We set

\[
\Delta_{\leq 0}^{op} \times \Delta^1 \xrightarrow{i_0} \mathbb{I} \xrightarrow{i_1} \Delta_{\leq 1}^{op} \times \Delta^1 \xrightarrow{i} \Delta^{op} \times \Delta^1
\]

be the natural inclusions.

We set

\[
ev_0 := j_1 \circ i_1 \quad \text{and} \quad j_0 := ev_0 \circ i_0 = j_1 \circ i_1 \circ i_0.
\]

Compatibly with Notation A.19, we let \( \mathbb{I}^a \) and \( \mathbb{I}^m \) be the full subcategories of \( \mathbb{I} \) spanned, respectively, by the objects \( \{ ([1], a), ([0], a) \} \) and \( \{ [0], m \} \). For \( \star \in \{ a, m \} \), we write \( i_0^\star, j_0^\star, i_1^\star, j_1^\star \) and \( ev_\star \) for the restriction of the above functors to \( \Delta_{\leq 0}^{op} \times \{ \star \} \), \( \mathbb{I}^\star, \Delta_{\leq 1}^{op} \times \{ \star \} \) and \( \Delta^{op} \times \{ \star \} \), respectively.

**Notation A.23.** Given an object \( \mathcal{A} = (\pi: M \to A_0, s: A_0 \to A_1) \in \text{Fun}(\mathbb{I}, \mathcal{T}) \), we set

\[
\mathcal{A}^a := (s: A_0 \to A_1) \quad \text{and} \quad \mathcal{A}_0 := (\pi: M \to A_0).
\]

**Lemma A.24.** Let \( \mathcal{A} = (\pi: M \to A_0, s: A_0 \to A_1) \in \text{Fun}(\mathbb{I}, \mathcal{T}) \).

1. There is a canonical equivalence

\[
ev_{\mathbb{I}^a_*}(\mathcal{A}^a) \simeq ev_{1,\mathcal{A}^a}(\mathcal{A}) := ev_{1,\mathcal{A}^a}(\mathcal{A})|_{\Delta^{op} \times \{ a \}^a}.
\]

In particular, the \( a \)-component of \( ev_{1,\mathcal{A}^a}(\mathcal{A}) \) is 1-coskeletal.

2. The square

\[
\begin{array}{c}
ev_{\mathbb{I}^a_*}(\mathcal{A})^m \xrightarrow{ev_{\mathbb{I}^a_*}(\mathcal{A})^m} \quad j_0^m(M) \\
\downarrow \quad \downarrow^{j_0^m(\pi)}
\end{array}
\]

is a pullback in \( \text{Fun}(\Delta^{op}, \mathcal{T}) \). In particular, the structural morphism \( ev_{\mathbb{I}^a_*}(\mathcal{A})^m \to ev_{\mathbb{I}^a_*}(\mathcal{A})^a \) between the \( m \)- and the \( a \)-components of \( ev_{\mathbb{I}^a_*}(\mathcal{A}) \) is 0-coskeletal, and hence 1-Segal.
Proof. The first part of statement (1) follows directly from the fact that there are no morphisms from objects of $\mathbb{I}^a$ to objects of $\mathbb{I}^m$. The second half follows immediately from the functoriality of right Kan extensions, as $ev_i^a$ factors through $j_i^1: \Delta^0_{\leq 1} \times \{a\} \to \Delta^0 \times \{a\}$.

We now prove statement (2). By definition, $ev_i^m$ is just the inclusion $\Delta^0_{\leq 1} \times \{m\} \to \Delta^0 \times \{m\}$, while $j_0^m$ is the inclusion $\Delta^0_{\leq 0} \times \{m\} \to \Delta^0 \times \{a\}$. It follows that both $ev_{i^m}(M)$ and $j_0^{i^m}(A_0)$ are $0$-coskeletal, and so the same goes for the morphism between them. Thus, given that the square (A.2) is a pullback, it follows that the structural morphism $ev_{i^m}(A)^m \to ev_{i^m}(A)^a$ is $0$-coskeletal, and Proposition A.17 guarantees that it is $1$-Segal. So, we only have to check that (A.2) is a pullback square.

We start with the following observation. Let $k \in \{0,1\}$ and consider the functor
\[
j_{k,s}: \text{Fun}(\Delta^0_{\leq k} \times \Delta^1, \mathcal{T}) \to \text{Fun}(\Delta^0 \times \Delta^1, \mathcal{T})
\]
given by right Kan extension along $j_k$.

Thanks to the canonical equivalences
\[
\text{Fun}(\Delta^0_{\leq k} \times \Delta^1, \mathcal{T}) \simeq \text{Fun}(\Delta^0_{\leq k}, \text{Fun}(\Delta^1, \mathcal{T})) \quad \text{and} \quad \text{Fun}(\Delta^0 \times \Delta^1, \mathcal{T}) \simeq \text{Fun}(\Delta^0, \text{Fun}(\Delta^1, \mathcal{T}))
\]
and to the fact that limits in $\text{Fun}(\Delta^1, \mathcal{T})$ are computed level-wise, we see that given a functor $X_{\bullet}^s: \Delta^0_{\leq k} \times \Delta^1 \to \mathcal{T}$, one has canonical equivalences
\[
j_{k,s}(X_{\bullet}^s)^a \simeq j_{k,s}(X_{\bullet}^s)^a \quad \text{and} \quad j_{k,s}(X_{\bullet}^s)^m \simeq j_{k,s}(X_{\bullet}^m).
\]
It follows that the square (A.2) is obtained by applying $j_{1,s}$ to the square
\[
i_{1,s}(A)^m \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
are pullbacks. Since the horizontal arrows of the right square are the canonical projections, both statements are obvious, and the conclusion follows. □

**Definition A.25.** Let \( X \) be a relative simplicial object. A boundary datum \( \rho \) for \( X \) is the given of an object \( A = (s: A_0 \to A_1, \pi: M \to A_0) \) in \( \text{Fun}(I, T) \) and a morphism \( f := f^s: A \to ev^1_*(X^s) \). We can represent a boundary datum \( \rho = (A, f) \) as the datum of the following commutative diagram:

\[
\begin{array}{ccc}
M & \longrightarrow & A_0 \\
\downarrow f^m & & \downarrow f^s \\
X_0 & \longrightarrow & X_0^s
\end{array}
\]

**Construction A.26.** Let \( X \) be a relative simplicial object and let \( \rho = (A, f) \) be a boundary datum for \( X \). We define the relative simplicial object \( X_{(\rho)} \) as the fiber product

\[
X_{(\rho)} \longrightarrow X^s \\
\downarrow ev_*(A) & \quad & \downarrow ev_*(f) \\
A_1 \times_{A_0} A_1 & \longrightarrow & X_1^s \times_{X_0^s} X_1^s
\]

**Proposition A.27.** In the setting of Construction A.26, assume that the square

\[
\begin{array}{ccc}
X_2^s(\rho) & \longrightarrow & X_2^s \\
\downarrow & & \downarrow \\
A_1 \times_{A_0} A_1 & \longrightarrow & X_1^s \times_{X_0^s} X_1^s
\end{array}
\]

is a pullback. Then the morphism between the \( a \)-components

\[
X_2^a(\rho) \longrightarrow X_2^a
\]

is 1-Segal. If in addition the square

\[
\begin{array}{ccc}
X_1^m(\rho) & \longrightarrow & A_1 \times_{A_0} M \\
\downarrow & & \downarrow \\
X_1 & \longrightarrow & X_1^a \times_{X_0^a} X_1^a
\end{array}
\]

is a pullback, then the morphism \( X^m_{(\rho)} \to X^m \) is relative left 1-Segal. Furthermore, replacing \( \partial_0 \) by \( \partial_1 \) in the above square and asking it to be a pullback implies that \( X^m_{(\rho)} \to X^m \) is relative right 1-Segal.

**Proof.** Applying Lemma A.24-(1), we see that the morphism \( X^m_{(\rho)} \to X^m \) is 1-coskeletal. Using Proposition A.18 we reduce the check that the square (A.4) is a pullback, which holds by assumption. This proves the first half of the proposition. For the second half, observe our assumption together with Proposition A.20 shows that it is enough to prove that the map \( X^m_{(\rho)} \to X^m \) is 1-Segal. Set

\[
Y^m := X^a(\rho) \times_{X^s} X^m 
\]

so that the former map factors as

\[
X^m_{(\rho)} \longrightarrow Y^m \longrightarrow X^m
\]

The map \( \psi: Y^m \to X^m \) is by definition the base change of the map \( X^m_{(\rho)} \to X^m \) we just proved to be 1-Segal. Thus, \( \psi \) is 1-Segal itself. To complete the proof, it is therefore enough to prove that
\( \varphi \) is 1-Segal. For this, define \( Z_m^* \) by asking the right bottom square in the following commutative diagram

\[
\begin{array}{ccc}
X_m^*(\rho) & \xrightarrow{\varphi} & Y_m^* \\
\downarrow & & \downarrow \\
\text{ev}_{L,s}(A)^m & \rightarrow & Z_m^* \\
\downarrow & & \downarrow \\
\text{ev}_{L,s}(\text{ev}_I^*(X_m^*))^m & \rightarrow & \text{ev}_{L,s}(\text{ev}_I^*(A))^n
\end{array}
\]

to be a pullback. Lemma A.24-(2) implies that the bottom diagonal map and the rightmost lower vertical map are 1-Segal. Thus, the map \( Z_m^* \rightarrow \text{ev}_{L,s}(\text{ev}_I^*(A))^n \) is 1-Segal. It automatically follows that \( \text{ev}_{L,s}(A)^m \rightarrow Z_m^* \) is 1-Segal. To complete the proof, it is therefore enough to check that the upper left square is a pullback. Observe that the upper outer rectangle and the bottom lower right square are pullbacks by definition. So, the transitivity property of pullback squares reduces us to check that the right outer vertical rectangle is a pullback. However, it can alternatively be decomposed as

\[
\begin{array}{ccc}
Y_m^* & \rightarrow & X_m^* \\
\downarrow & & \downarrow \\
X_s^*(\rho) & \rightarrow & X_s^* \\
\downarrow & & \downarrow \\
\text{ev}_{L,s}(\text{ev}_I^*(A))^n & \rightarrow & \text{ev}_{L,s}(\text{ev}_I^*(X_s^*))^n
\end{array}
\]

and now both squares are pullback by definition. The conclusion follows. \( \square \)

**Corollary A.28.** In the setting of Proposition A.27, if in addition \( X_s^* \) is a relative 2-Segal object, the same goes for \( X_s^*(\rho) \).

**Proof.** This is an immediate consequence of Propositions A.21 and A.27. \( \square \)

**References**

[AB13] D. Arcara and A. Bertram, Bridgeland-stable moduli spaces for K-trivial surfaces, J. Eur. Math. Soc. (EMS) 15 (2013), no. 1, 1–38, With an appendix by Max Lieblich.

[Bay09] A. Bayer, Polynomial Bridgeland stability conditions and the large volume limit, Geom. Topol. 13 (2009), no. 4, 2389–2425.

[BLM+21] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, and P. Stellari, Stability conditions in families, Publ. Math. Inst. Hautes Études Sci. 133 (2021), 157–325.

[BCS20] T. Bozec, D. Calaque, and S. Scherotzke, Relative critical loci and quiver moduli, J. Amer. Math. Soc. 24 (2011), no. 4, 969–998.

[CB01] W. Crawley-Boevey, Geometry of the moment map for representations of quivers, Compositio Math. 126 (2001), no. 3, 257–293.

[DeH20] S. DeHority, Affinizations of Lorentzian Kac-Moody algebras and Hilbert schemes of points on K3 surfaces, arXiv:2007.04953, 2020.

[DPS20] D.-E. Diaconescu, M. Porta, and F. Sala, McKay correspondence, cohomological Hall algebras and categorification, arXiv:2004.13685, 2020.

[DPSa] D.-E. Diaconescu, Derived moduli of flat objects, in preparation. 7, 8, 13, 16, 17, 22, 29, 32, 56, 57

[DPS+ b] D.-E. Diaconescu, M. Porta, F. Sala, O. Schiffmann, and É. Vasserot, Cohomological Hall algebra of a surface and affine Yangians, in preparation.

[Gin06] V. Ginzburg, Calabi-Yau algebras, arXiv:math/0612139, 2006.

[Gin12] V. Ginzburg, Lectures on Nakajima’s quiver varieties, Geometric methods in representation theory: I, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 145–219.

[HL10] D. Huybrechts and M. Lehn, The geometry of moduli spaces of sheaves, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010.
[Kel11] B. Keller, Deformed Calabi-Yau completions, J. Reine Angew. Math. 654 (2011), 125–180, With an appendix by Michel Van den Bergh. 55

[KV19] M. Kapranov and E. Vasserot, The cohomological Hall algebra of a surface and factorization cohomology, arXiv:1901.07641, 2019. 2, 4

[Lur09] J. Lurie, Higher topos theory, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. 61

[Lur17] , Higher algebra, available at his webpage, 2017. 20, 58

[Neg19] A. Neguţ, Shuffle algebras associated to surfaces, Selecta Math. (N.S.) 25 (2019), no. 3, Paper No. 36, 57, 4, 31, 32

[Neg22] , Hecke correspondences for smooth moduli spaces of sheaves, Publ. Math. Inst. Hautes Études Sci. 135 (2022), 337–418. 32

[Poli07] A. Polishchuk, Constant families of 1-structures on derived categories of coherent sheaves, Mosc. Math. J. 7 (2007), no. 1, 109–134, 167. 21

[PT09] R. Pandharipande and R. P. Thomas, Curve counting via stable pairs in the derived category, Invent. Math. 178 (2009), no. 2, 407–447. 38

[PS19] M. Porta and F. Sala, Two-dimensional categorified Hall algebras, arXiv:1903.07253, 2019. 2, 4, 6, 8, 10, 13, 16, 29

[RSV17] M. Rapcak, Y. Soibelman, Y. Yang, and G. Zhao, Cohomological Hall algebras and perverse coherent sheaves on toric Calabi-Yau 3-folds, arXiv:2007.13365, 2020. 6

[Sch12] O. Schiffmann, Lectures on Hall algebras, Geometric methods in representation theory. II, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 1–141. 2

[SV13a] O. Schiffmann and E. Vasserot, Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}^2$, Publ. Math. Inst. Hautes Études Sci. 118 (2013), 213–342. 7, 57

[SV13b] , The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$, Duke Math. J. 162 (2013), no. 2, 279–366. 7, 57

[SV17] , On cohomological Hall algebras of quivers: Yangians, arXiv:1705.07491, 2017. 2

[SV20] , On cohomological Hall algebras of quivers: generators, J. Reine Angew. Math. 760 (2020), 59–132. 7, 57

[Tod09] Y. Toda, Limit stable objects on Calabi-Yau 3-folds, Duke Math. J. 149 (2009), no. 1, 157–208. 32, 33, 39

[Tod20] , Hall-type algebras for categorical Donaldson-Thomas theories on local surfaces, Selecta Math. (N.S.) 26 (2020), no. 4, Paper No. 62, 72. 6, 49

[TV07] B. Toën and M. Vaquié, Moduli of objects in dg-categories, Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 3, 387–444. 8

[VV22] M. Varagnolo and E. Vasserot, K-theoretic Hall algebras, quantum groups and super quantum groups, Selecta Math. (N.S.) 28 (2022), no. 1, Paper No. 7, 56. 2, 7, 57

[Wal] T. Walde, Hall monoidal categories and categorical modules, arXiv:1611.08241. 8, 57

[You18] M. B. Young, Relative 2-Segal spaces, Algebr. Geom. Topol. 18 (2018), no. 2, 975–1039. 8, 57, 58, 61, 63

[YZ18] Y. Yang and G. Zhao, The cohomological Hall algebras of a preprojective algebra, Proc. Lond. Math. Soc. (3) 116 (2018), no. 5, 1029–1074. 7, 57

[Zha21] Y. Zhao, On the K-theoretic Hall algebra of a surface, Int. Math. Res. Not. IMRN (2021), no. 6, 4445–4486. 2, 4