Integrals Over Polytopes, Multiple Zeta Values and Polylogarithms, and Euler’s Constant

Jonathan Sondow
209 West 97th Street
New York City
New York 10025 USA
jsondow(at)alumni.princeton.edu

and

Sergey Zlobin
Faculty of Mechanics and Mathematics
Moscow State University
Leninskie Gory, Moscow 119899 RUSSIA
sirg.zlobin(at)mail.ru

Abstract

Let $T$ be the triangle with vertices (1,0), (0,1), (1,1). We study certain integrals over $T$, one of which was computed by Euler. We give expressions for them both as a linear combination of multiple zeta values, and as a polynomial in single zeta values. We obtain asymptotic expansions of the integrals, and of sums of certain multiple zeta values with constant weight. We also give related expressions for Euler’s constant. In the final section, we evaluate more general integrals – one is a Chen (Drinfeld-Kontsevich) iterated integral – over some polytopes that are higher-dimensional analogs of $T$. This leads to a relation between certain multiple polylogarithm values and multiple zeta values.

Contents

1 Introduction

2 The integral $I_n$ for $n \geq 0$

3 Asymptotic expansion of $I_n$

4 Applications to multiple zeta values

5 The integral $I_{-1}$ and Euler’s constant

6 Integrals over higher-dimensional analogs of $T$

7 References
1 Introduction

Let $T$ be the triangle defined by

$$T := \{(x, y) \in [0, 1]^2 | x + y \geq 1\},$$

with vertices (1,0), (0,1), (1,1). In this paper we study the integral over $T$

$$I_n := \iint_T \frac{(-\ln xy)^n}{xy} dxdy$$

for $n = -1, 0, 1, 2, \ldots$. We also consider integrals over several polytopes that are higher-dimensional analogs of $T$.

Euler computed an iterated integral equivalent to $I_0$, and found that

$$I_0 = \int_0^1 \int_0^{1-x} \frac{dx}{xy} = \int_0^1 \frac{\ln(1-x)}{x} dx = \int_0^1 \sum_{r=1}^{\infty} \frac{x^{r-1}}{r} dx = \sum_{r=1}^{\infty} \frac{1}{r^2} = \zeta(2).$$

Using integration by parts he derived formula (22), and used it to calculate $\zeta(2)$ correctly to six decimals—see [5, Section 1.2], [7, pp. 43-45].

We generalize Euler’s result to $n = 0, 1, 2, \ldots$ by showing that $I_n$ is equal to an integer linear combination of multiple zeta values

$$\zeta(s_1, \ldots, s_l) := \sum_{n_1 > n_2 > \cdots > n_l > 0} \frac{1}{n_1^{s_1} \cdots n_l^{s_l}}$$

of weight $s_1 + \cdots + s_l = n + 2$. We also express $I_n$ as a polynomial in single zeta values.

Theorem 1 Let $n \geq 0$ be an integer.

(i) Then

$$I_n = n! \sum_{k=0}^{n} \zeta(n-k+2, \{1\}_k),$$

where $\{1\}_k$ denotes $1, 1, \ldots, 1$ ($k$ times).

(ii) Moreover, $I_n$ is equal to an explicit polynomial of several variables with rational coefficients in the values of the Riemann zeta function $\zeta(2), \zeta(3), \ldots, \zeta(n+2)$.

(Theorem 1, Corollary 1, and Lemma 1 were obtained by the second author in [17]..) The proof is given in Section 2, along with the explicit formula. Examples are

$$I_0 = \zeta(2), \quad I_1 = \zeta(3) + \zeta(2, 1) = 2\zeta(3), \quad I_2 = 2 (\zeta(4) + \zeta(3, 1) + \zeta(2, 1, 1)) = \frac{9}{2} \zeta(4),$$

$$I_3 = 6 (\zeta(5) + \zeta(4, 1) + \zeta(3, 1, 1) + \zeta(2, 1, 1, 1)) = 36\zeta(5) - 12\zeta(2)\zeta(3).$$

The cases $n = 0, 1, \text{and} 2$ are particularly simple.

Corollary 1 For $n = 0, 1, \text{and} 2$, the integral $I_n$ is a rational multiple of $\zeta(n+2)$.

For $n = 0 \text{ and } 1$, this also follows from Beukers’ [2] formulas for $\zeta(2)$ and $\zeta(3)$ as integrals over the unit square

$$S := [0, 1]^2.$$
Namely, the change of variables $x = X, y = 1 - XY$ transforms both $I_0$ into

$$I_0 = \int \int_T \frac{dxdy}{xy} = \int \int_S \frac{dXdY}{1 - XY} = \zeta(2)$$

and $\frac{1}{2}I_1$ into

$$\frac{1}{2}I_1 = \frac{1}{2} \int \int_T -\frac{\ln xy}{xy} dxdy = \int \int_T -\frac{\ln x}{xy} dxdy = \int \int_S -\frac{\ln X}{1 - XY} dXdY = \frac{1}{2} \int \int_S -\frac{\ln XY}{1 - XY} dXdY = \zeta(3).$$

Here is an outline of the proof of Theorem 1. We first prove

**Lemma 1** If $k \geq 0$ and $l \geq 0$ are integers, then

$$I_{k,l} := \int \int_T (\frac{\ln x}{x})^k (\frac{\ln y}{y})^l dxdy = k!!l!\zeta(l + 2, \{1\}_k). \quad (4)$$

If in addition $l \geq 1$, then

$$J_{k,l} := \int_0^1 (\frac{\ln(1-x)}{1-x})^k (\ln x)^l dx = k!!l!\zeta(l + 1, \{1\}_k). \quad (5)$$

Expanding $(-\ln x - \ln y)^n$, part (i) follows immediately. To prove the lemma, we show that $(l + 1)I_{k,l} = J_{k,l+1}$, and then evaluate the integral $J_{k,l}$. Part (ii) of Theorem 1 follows, using a formula in [9] for $J_{k,l}$ in terms of single zeta values.

As an application, we obtain an explicit version of a result in [3].

**Corollary 2** If $n \geq 2$ and $k \geq 0$, then the multiple zeta value $\zeta(n, \{1\}_k)$ can be explicitly represented as a polynomial of several variables with rational coefficients in the single zeta values $\zeta(2), \zeta(3), \ldots, \zeta(n + k)$.

Lemma 1 also affords a simple proof of a special case of the duality theorem for multiple zeta values (see, for example, [5, Section 2.8]).

**Corollary 3** If $k \geq 0$ and $l \geq 0$, then $\zeta(k + 2, \{1\}_l) = \zeta(l + 2, \{1\}_k)$.

For instance, $\zeta(2, 1) = \zeta(3)$ and $\zeta(2, 1, 1) = \zeta(4)$. Using these equalities, we give a second proof of Corollary 1. However, unlike the cases $n = 0$ and 1, we do not have a proof of the case $n = 2$ of Corollary 1 that does not use Theorem 1.

On the basis of numerical evidence and examples such as (3), we make the

**Conjecture 1** The integral $I_n$ is not a rational multiple of $\zeta(n + 2)$ when $n > 2$.

This has not been proved for a single value of $n$. However, using Theorem 1 (ii), we give a conditional proof for all $n = 3, 4, \ldots$, assuming a standard conjecture (see, for example, [16, Introduction]).

**Theorem 2** If the numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$ are algebraically independent over the rationals, then Conjecture 1 is true.
Using a lemma we prove which gives an asymptotic expansion for the coefficients of the Taylor series of certain meromorphic functions (Lemma 2), we estimate $I_n$ for $n$ large.

**Theorem 3** The asymptotic equivalence

$$I_n \sim 2n! \quad (n \to \infty)$$

(6)

holds. More precisely, the following asymptotic expansion is valid:

$$\frac{I_n}{n!} \approx 2 + \frac{6}{2^{n+2}} + \frac{20}{3^{n+2}} + \frac{70}{4^{n+2}} + \cdots \quad (n \to \infty),$$

where the numerator of the $k$-th term is $\binom{2k}{k}$, for $k = 1, 2, \ldots$.

This in turn gives an estimate for the sum of the multiple zeta values $\zeta(m-k, \{1\}_k)$ of constant weight $m$.

**Corollary 4** The average of the multiple zeta values $\zeta(m), \zeta(m-1, 1), \ldots, \zeta(2, \{1\}_{m-2})$ is asymptotic to $2/m$ as $m$ tends to infinity. In fact, the following asymptotic expansion holds:

$$\sum_{k=0}^{m-2} \zeta(m-k, \{1\}_k) \approx 2 + \frac{6}{2^m} + \frac{20}{3^m} + \frac{70}{4^m} + \cdots \quad (m \to \infty).$$

Another application of Theorem 3 is a curious result.

**Corollary 5** The series

$$\sum_{n=0}^{\infty} (-1)^n \frac{I_n}{n!}$$

diverges, but is Abel summable to $1/2$.

Let us now go "down" from $I_0$ to $I_{-1}$.

**Question** Can one evaluate the integral

$$I_{-1} = \iint_T \frac{dxdy}{x y (-\ln xy)} = 1.7330025 \ldots$$

(7)

in terms of more familiar constants?

Surprisingly, it turns out that $I_{-1}$ involves all the integrals $I_0, I_1, I_2, \ldots$ (hence all multiple zeta values $\zeta(m, \{1\}_k)$ for $m \geq 2$ and $k \geq 0$).

**Theorem 4** If $\text{li}$ is the logarithmic integral function, then

$$I_{-1} = \sum_{n=0}^{\infty} (-1)^n \frac{I_n}{(n+1)!} + \int_0^1 \frac{\text{li}(x - x^2)}{x} dx + 1.$$

(Compare the convergent series here with the divergent series in Corollary 5.)
We now transform the double integral $I_{-1}$ into single integrals, one involving the generalized binomial coefficient

$$\binom{s}{t} := \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)}.$$

**Proposition 1** The following integral formulas for $I_{-1}$ are valid:

$$I_{-1} = \int_0^\infty \left(1 - \frac{1}{(2t)^t}\right) \frac{dt}{t^2} = \int_0^1 \ln \left(1 + \frac{\ln(1-x)}{\ln x}\right) \frac{dx}{x}. \quad (8)$$

Expanding the first integrand in a power series, we find that the $n$th coefficient involves the integral $I_n$.

**Theorem 5** If $0 < |t| < 1$, then

$$\left(1 - \frac{1}{(2t)^t}\right) \frac{1}{t^2} = \sum_{n=0}^\infty (-1)^n \frac{I_n}{n!} t^n. \quad (9)$$

An application is Corollary 5.

We now relate $I_{-1}$ to Euler’s constant $\gamma$, which is defined as the limit

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right).$$

If one thinks of $\gamma$ as "\(\zeta(1)\)" then from the formulas $I_2 = 9\zeta(4)$, $I_1 = 2\zeta(3)$, and $I_0 = \zeta(2)$ one might expect that $I_{-1}$ involves $\gamma$. This is also suggested by the similarity between the double integral (7) for $I_{-1}$ and the double integral for Euler’s constant [12], [14]

$$\gamma = \iint_S \frac{1-X}{(1-X)(-\ln XY)} dXdY. \quad (10)$$

Formula (8) leads to another, related connection between $I_{-1}$ and $\gamma$. Namely, when $t = n$ is a positive integer, $\binom{2n}{n}$ is the central binomial coefficient $\binom{2n}{n}$, which figures in the formulas for Euler’s constant

$$\binom{2n}{n} \gamma = A_n - L_n + \iint_S \frac{(X(1-X)Y(1-Y))^n}{(1-X)(-\ln XY)} dXdY \quad (n \geq 1)$$

and

$$\gamma = \frac{A_n - L_n}{\binom{2n}{n}} + O \left(\frac{1}{2^{6n}\sqrt{n}}\right) \quad (n \to \infty),$$

where $A_n$ is a certain rational number and $L_n$ is a particular linear form in logarithms [12].

If in (10) we perform the change of variables $X = x, Y = (1-y)/x$, we obtain an integral over the triangle $T$ for Euler’s constant,

$$\gamma = \iint_T \frac{1-x}{xy(-\ln(1-y))} dxdy, \quad (11)$$

analogous to the triangle integral (7) for $I_{-1}$. 
We find an analog for $\gamma$ of the first integral for $I_{-1}$ in (8), which involves the generalized binomial coefficient $\binom{2n}{t}$. (There exist classical analogs for $\gamma$ of the second integral, which involves logarithms.)

**Proposition 2** The following formula for Euler’s constant is valid:

$$\gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2(t+k)} dt.$$  

As an application, if we integrate termwise, and exponentiate the resulting series, we recover Ser’s infinite product for $e^\gamma$ [11] (rediscovered in [13], [15]):

$$e^\gamma = \prod_{k=2}^{\infty} \left( \prod_{j=1}^{k} j^{-1} j^{-1} \right)^{1/k} = \left( \frac{2}{1} \right)^{1/2} \left( \frac{2^2}{1 \cdot 3^3} \right)^{1/3} \left( \frac{2^3 \cdot 4^4}{1 \cdot 3^6 \cdot 5} \right)^{1/4} \cdots.$$ 

The rest of the paper is organized as follows. In Sections 2 and 3 we establish the non-asymptotic and asymptotic results, respectively, on $I_n$ for $n \geq 0$. The applications to multiple zeta values are proved in Section 4, and in Section 5 we prove the formulas for $I_{-1}$ and $\gamma$. The final section is devoted to generalizing $I_n$ to integrals over higher-dimensional analogs of the triangle $T$; one is a Chen (Drinfeld-Kontsevich) iterated integral (see Remark 3). An application is a relation between certain multiple polylogarithm values and multiple zeta values (Corollary 7).

## 2 The integral $I_n$ for $n \geq 0$

We prove the non-asymptotic results on $I_0, I_1, I_2, \ldots$ stated in the Introduction.

**Lemma 1** If $k \geq 0$ and $l \geq 0$ are integers, then

$$I_{k,l} := \int_{T} \int \frac{(-\ln x)^k (-\ln y)^l}{xy} dxdy = k!l! \zeta(l+2, \{1\}_k).$$

If in addition $l \geq 1$, then

$$J_{k,l} := \int_0^1 (\frac{-\ln(1-x))^k}{1-x} (-\ln x)^l dx = k!l! \zeta(l+1, \{1\}_k).$$

**Proof.** We have

$$I_{k,l} = \int_0^1 \frac{(-\ln x)^k}{x} \int_{1-x}^1 \frac{(-\ln y)^l}{y} dy dx = \int_0^1 \frac{(-\ln x)^k}{x} \cdot \frac{(-\ln(1-x))^{l+1}}{l+1} dx.$$ 

Replacing $x$ with $1-x$, we see that

$$I_{k,l} = \frac{J_{k,l+1}}{l+1}.$$ 

Thus (4) follows from (5). To prove (6), we multiply the formula [16, Section 1]
\((-\ln(1-x))^k = k! \sum_{n_1>n_2>\cdots>n_k>0} \frac{x^{n_1}}{n_1\cdots n_k}\)

by \((1-x)^{-1} = 1 + x + x^2 + \cdots\), and substitute the resulting series
\[
\frac{(-\ln(1-x))^k}{1-x} = k! \sum_{m\geq n_1>n_2>\cdots>n_k>0} \frac{x^m}{n_1\cdots n_k}
\]
into the integral (5) for \(J_{k,l}\). We then integrate termwise, using the fact that
\[
\int_0^1 x^m (-\ln x)^l dx = \frac{l!}{(m+1)^{l+1}}.
\]
The result is
\[
J_{k,l} = k! l! \sum_{m>n_1>n_2>\cdots>n_k>0} \frac{1}{m!^{l+1} n_1\cdots n_k} = k! l! \zeta(l+1, \{1\}_k),
\]
and the lemma follows. \(\square\)

**Theorem 1** If \(n \geq 0\), then \(I_n\) can be expressed both

(i) in terms of multiple zeta values as
\[
I_n = n! \sum_{k=0}^n \zeta(n-k+2, \{1\}_k)
\]

(ii) and in terms of single zeta values as
\[
I_n = \sum_{k=0}^n \binom{n}{k} \frac{J_{k,n-k+1}}{n-k+1}
\]

where the integral \(J_{k,n-k+1}\), defined in (5), is given by the formula [9]
\[
J_{k,l} = k! l! \sum_{p=1}^l \frac{(-1)^{p+1}}{p!} \sum_{t_1} t_1^{1} \cdots \sum_{t_p} t_p \sum_{l_1}^{t_1} \cdots \sum_{l_p}^{t_p} \frac{\zeta(t_1) \cdots \zeta(t_p)}{l_1! \cdots l_p!},
\]
the sum on \(t_i\) being taken over all sets of integers \(\{t_1, \ldots, t_p\}\) with
\[
t_i > 1, \quad \sum_{i=1}^p t_i = k + l + 1,
\]
and the sum on \(l_i\) over all sets of integers \(\{l_1, \ldots, l_p\}\) with
\[
0 < l_i < t_i, \quad \sum_{i=1}^p l_i = l.
\]

**Proof.** Expanding \((-\ln xy)^n = (-\ln x - \ln y)^n\) in the definition (1) of \(I_n\), and applying (4), gives (12). Hence, using (5), formula (13) holds. Finally, the evaluation (14) of the integral (5) for \(J_{k,l}\) is proved in [9]. \(\square\)

**Corollary 1** For \(n = 0, 1, \text{ and } 2\), the integral \(I_n\) is a rational multiple of \(\zeta(n+2)\).
We give two proofs. The first is short, but uses Theorem 1 (ii), whose proof depends on [9]. The second is longer, but is self-contained (except for a formula due to Euler): it uses Theorem 1 (i) and Corollary 3, whose proofs do not rely on other papers.

Proof 1. For $n = 0, 1, 2$, formulas (13) and (14) yield $I_0 = J_{0,1} = \zeta(2)$ and $I_1 = \frac{1}{2}J_{0,2} + J_{1,1} = 2\zeta(3)$ and $I_2 = \frac{1}{3}J_{0,3} + \frac{1}{2}J_{1,2} + J_{2,1} = \frac{3}{4}\zeta(4)$.

Proof 2. In the Introduction, we showed that $I_0 = \zeta(2)$. Using the same method, together with the formula $\int_0^1 x^{k-1}(-\ln x)\,dx = k^{-2}$, we obtain

\[
I_1 = 2 \int_T \frac{\ln x}{xy}\,dxdy = 2 \int_0^1 \ln(1-x)x\,dx = 2\sum_{k=1}^\infty \frac{1}{k} \int_0^1 x^{k-1}(-\ln x)\,dx = 2\zeta(3).
\]

Alternatively, $I_0 = \zeta(2)$ and $I_1 = \zeta(3) + \zeta(2,1)$ by Theorem 1 (i), and $\zeta(2,1) = \zeta(3)$ by Corollary 3.

In order to prove that $I_2 = \frac{9}{2}\zeta(4)$, it suffices, by Theorem 1 (i) and Corollary 3, to apply Euler’s formula $\zeta(3,1) = \frac{1}{4}\zeta(4)$. (For the latter, take $n = 3$ in his equation (9.5) of [1, p. 252].)

**Theorem 2** If the numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \zeta(9), \ldots$ are algebraically independent over the rationals, then $I_n$ is not a rational multiple of $\zeta(n+2)$ when $n > 2$.

**Proof.** First take the case $n = 3m - 2$, with $m > 1$. The integral $I_n$ is equal to a linear combination (13) of integrals $J_{k,l}$ with positive coefficients. Each $J_{k,l}$ is equal to a polynomial (14) of several variables in single zeta values. Now in (14) the monomial $\zeta(3)^m$ appears only when $p = m$, and then its coefficient is nonzero and has sign $(-1)^{m+1}$. Hence the expression for $I_n$ the coefficient of $\zeta(3)^m$ is nonzero. It follows, using the hypothesis, that $I_n$ cannot be a rational multiple of $\zeta(n+2) = \zeta(3m)$.

The cases $n = 3m + 3$ and $n = 3m + 5$, with $m > 0$, are similar: consider the monomials $\zeta(3)^m\zeta(5)$ and $\zeta(3)^m\zeta(7)$, respectively. The remaining cases $n = 3$ and $n = 5$ can be handled by direct calculation, completing the proof.

**Theorem 5** If $0 < |t| < 1$, then

\[
\frac{1}{t^2} \left(1 - \frac{1}{(2t)^{n}}\right) = \sum_{n=0}^\infty (-1)^n I_n t^n.
\]

**Proof.** The generating function

\[
\sum_{k,l \geq 0} x^{k+1} y^{l+1} (l+2, \{1\}_k) = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}
\]

(compare (19)) is derived in [3]. If $x = y = -t$, then the series converges when $|t| < 1$. Setting $k + l = n$, we obtain

\[
t^2 \sum_{n=0}^\infty (-1)^n t^n \sum_{k=0}^n (n-k+2, \{1\}_k) = 1 - \frac{1}{(2t)^{n}}.
\]

Applying (2), the theorem follows.
3 Asymptotic expansion of $I_n$

Using Theorem 5 and the next lemma, we estimate the integral $I_n$ when $n$ is large.

**Lemma 2** Suppose that the function $f(z)$ is meromorphic in the complex plane and has only simple poles $z_1, z_2, \ldots$, with residues $r_1, r_2, \ldots$, respectively. If $0 < |z_1| \leq |z_2| \leq \cdots$, then the coefficients of the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

have the asymptotic expansion

$$a_n \approx -\frac{r_1}{z_1^{n+1}} - \frac{r_2}{z_2^{n+1}} - \cdots \quad (n \to \infty).$$

**Remark 1** Recall [6, Section 1.5] that the last formula means that, for every fixed positive integer $k$,

$$a_n = -\frac{r_1}{z_1^{n+1}} - \cdots - \frac{r_k}{z_k^{n+1}} + O\left(\frac{r_{k+1}}{z_{k+1}^{n+1}}\right) \quad (n \to \infty). \quad (17)$$

**Proof of Lemma 2.** A meromorphic function has only finitely many poles in any bounded region, so for each $k \geq 1$ there exists $l > k$ with $|z_l| < |z_{l+1}|$. Note that the only singularities of the function

$$f(z) = \sum_{j=1}^{l} \frac{r_j}{z - z_j} = \sum_{n=0}^{\infty} \left( a_n + \frac{r_1}{z_1^{n+1}} + \cdots + \frac{r_l}{z_l^{n+1}} \right) z^n \quad (18)$$

are $z_{l+1}, z_{l+2}, \ldots$. It follows, using the inequalities $0 < |z_1| \leq |z_2| \leq \cdots$, that the radius of convergence of the series (18) is equal to $|z_{l+1}|$. As $|z_1| < |z_{l+1}|$, we may substitute $z = z_l$ into the series. Therefore,

$$\lim_{n \to \infty} \left( a_n + \frac{r_1}{z_1^{n+1}} + \cdots + \frac{r_k}{z_k^{n+1}} + \frac{r_{k+1}}{z_{k+1}^{n+1}} + \cdots + \frac{r_l}{z_l^{n+1}} \right) z_l^n = 0.$$ 

Since $|z_{k+1}| \leq |z_{k+2}| \leq \cdots \leq |z_l|$, the limit implies the asymptotic formula (17). This proves the lemma. \hfill \Box

**Theorem 3** We have $I_n \sim 2n!$ as $n$ tends to infinity. More precisely, the following asymptotic expansion is valid:

$$\frac{I_n}{n!} \approx 2 + \frac{6}{2^{n+2}} + \frac{20}{3^{n+2}} + \frac{70}{4^{n+2}} + \cdots \quad (n \to \infty),$$

where the numerator of the $k$-th term is ${2k \choose k}$ for $k = 1, 2, \ldots$.

**Proof.** Denote the function on the left side of (9) by $f(t)$. Aside from a removable singularity at $t = 0$, the singularities of $f(t)$ are simple poles at $t = -1, -2, \ldots$. A calculation shows that the residue at $t = -k$ is equal to
Res\((f; -k) = \text{Res}\left(\frac{1}{t^2} \left(1 - \frac{\Gamma(t + 1)^2}{\Gamma(2t + 1)}\right); -k\right) = -\frac{1}{k^2} \lim_{t \to -k} (t + k) \frac{\Gamma(t + 1)^2}{\Gamma(2t + 1)} = \frac{1}{k} \binom{2k}{k}
\]
for \(k = 1, 2, \ldots\). Using Theorem 5 and Lemma 2, the second statement (which implies the first) follows.

□

**Corollary 5** The series

\[
\sum_{n=0}^{\infty} (-1)^n \frac{I_n}{n!}
\]
diverges, but is Abel summable to \(1/2\).

**Proof.** The divergence follows from (6). Letting \(t \to 1^-\) in (15), we obtain the desired Abel summation. □

## 4 Applications to multiple zeta values

Using the results obtained on \(I_n\), we study multiple zeta values of the form \(\zeta(m, \{1\}_k)\).

**Corollary 2** If \(m \geq 2\) and \(k \geq 0\), then the multiple zeta value \(\zeta(m, \{1\}_k)\) can be explicitly represented as a polynomial of several variables with rational coefficients in the single zeta values \(\zeta(2), \zeta(3), \ldots, \zeta(m + k)\).

**Proof.** Set \(l = m - 1\) in (5) and (14). □

**Remark 2** This result, including the polynomial formula (at least implicitly), was first obtained in [3], using the generating function (see [3] for the equivalence with (16))

\[
\sum_{k,l \geq 0} x^{k+1}y^{l+1} \zeta(l + 2, \{1\}_k) = 1 - \exp\left(\sum_{n=2}^{\infty} \frac{x^n + y^n - (x+y)^n}{n} \zeta(n)\right).
\]  \hspace{1cm} (19)

**Corollary 3** If \(k \geq 0\) and \(l \geq 0\), then \(\zeta(k + 2, \{1\}_l) = \zeta(l + 2, \{1\}_k)\).

**Proof.** Making the change of variables \(x, y \to y, x\) in the integral (4), the symmetry of the triangle \(T\) yields \(I_{k,l} = I_{l,k}\). Using Lemma 1, the result follows. □

**Corollary 4** The average of the multiple zeta values \(\zeta(m), \zeta(m - 1, 1), \ldots, \zeta(2, \{1\}_{m-2})\) is asymptotic to \(2/m\) as \(m\) tends to infinity. In fact, the following asymptotic expansion holds:

\[
\sum_{k=0}^{m-2} \zeta(m - k, \{1\}_k) \approx 2 + \frac{6}{2m} + \frac{20}{3m} + \frac{70}{4m} + \cdots \ (m \to \infty).
\]

**Proof.** Setting \(m = n + 2\) in Theorems 1 and 3 gives the desired expansion. It follows that the average in question is asymptotic to \(\frac{2}{m-1} \sim \frac{2}{m}\) as \(m\) tends to infinity. □
5 The integral $I_{-1}$ and Euler’s constant

We prove the results on $I_{-1}$ and $\gamma$ stated in the Introduction.

**Proposition 1** The following single integral formulas for the double integral $I_{-1}$ are valid:

$$I_{-1} = \int_0^\infty \left( 1 - \frac{1}{t^2} \right) dt = \int_0^1 \ln \left( 1 + \frac{\ln(1 - x)}{\ln x} \right) dx.$$

*Proof.* In (7) make the substitution

$$-\frac{1}{\ln xy} = \int_0^\infty (xy)^t dt$$

and change the order of integration, obtaining

$$I_{-1} = \int_0^\infty \int_0^1 \int_{1-x}^1 (xy)^{t-1} dy dx dt = \int_0^\infty \int_0^1 (x^{t-1} - x^{t-1}(1 - x)^t) dx \frac{dt}{t}$$

$$= \int_0^\infty \left( \frac{1}{t} - \frac{\Gamma(t)\Gamma(t+1)}{\Gamma(2t+1)} \right) \frac{dt}{t},$$

using Euler’s integral for the beta function. Replacing $\Gamma(t)$ with $t^{-1}\Gamma(t+1)$, the first equality follows. To see that the second integral is also equal to $I_{-1}$, integrate with respect to $y$ in (7).

**Theorem 4** If $\li$ is the logarithmic integral, then

$$I_{-1} = \sum_{n=0}^\infty (-1)^n \frac{I_n}{(n+1)!} + \int_0^1 \frac{\li(x - x^2)}{x} dx + 1. $$

*Proof.* Make the substitution

$$-\frac{1}{\ln xy} = \int_0^1 (xy)^t dt - \frac{xy}{\ln xy}$$

in (7). Using the proof of Proposition 1, we get

$$I_{-1} = \int_0^1 \left( 1 - \frac{1}{(2t)^2} \right) \frac{dt}{t^2} - \int_T \frac{dx dy}{\ln xy}. $$

Substituting the series [15] into the first integral, we integrate termwise and obtain the series in the desired formula. Letting $y = u/x$ in the second integral gives

$$\int_T \frac{dx dy}{\ln xy} = \int_0^1 x \int_{x-x^2}^x \frac{du}{\ln u} dx = \int_0^1 \frac{\li(x) - \li(x - x^2)}{x} dx,$$

and the following calculation (see [8, Section 6.212]) completes the proof:

$$\int_0^1 \frac{\li(x)}{x} dx = \lim_{q \to 0} \int_0^1 \frac{\li(x)}{x^{q+1}} dx = \lim_{q \to 0} \frac{\ln(1 - q)}{q} = -1. \qed$$
Proposition 2 The following formula for Euler’s constant is valid:

\[ \gamma = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k^2} \frac{1}{t+k} dt. \]

Proof. In (20) we replace \( xy \) with \( 1 - y \). Substituting the result into (11), we change the order of integration and get

\[ \gamma = \int_0^\infty \int_0^1 \int_0^1 \frac{1-x}{xy} (1-y)^t dxdydt = \int_0^\infty \int_0^1 \frac{-\ln(1-y) - y(1-y)^t} y dydt \]

\[ = \int_0^\infty \sum_{k=2}^\infty \frac{1}{k} \int_0^1 y^{k-1}(1-y)^t dydt. \]

Using the proof of Proposition 1, we obtain the desired formula. \( \square \)

6 Integrals over higher-dimensional analogs of \( T \)

There are several ways to generalize the triangle \( T \) and the integral \( I_n \). The simplest generalization of \( T \) is the polytope

\[ V_m := \{(x_1,x_2,\ldots,x_m) \in [0,1]^m | x_1 + x_j \geq 1, j = 2,\ldots,m \}. \]

Theorem 6 For \( m \geq 2 \) and \( n \geq 0 \), the integral

\[ K_{m,n} := \int \cdots \int_{V_m} \frac{(-\ln(x_1x_2\cdots x_m))^n}{x_1x_2\cdots x_m} dx_1dx_2\cdots dx_m \]

is equal to an integer linear combination of multiple zeta values of weight \( m + n \), namely,

\[ K_{m,n} = n! \sum_{k_2,\ldots,k_m \geq 0, k_1 = 1, k_1 + \cdots + k_m = n} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2 + 1)! \cdots (k_m + 1)!} \zeta(k_2 + \cdots + k_m + m, \{1\} k_1). \quad (21) \]

It is also equal to a polynomial of several variables with rational coefficients in values of the Riemann zeta function at integers.

Proof. Expanding \((-\ln(x_1x_2\cdots x_m))^n\) gives

\[ K_{m,n} = \sum_{k_1 \geq 0, \ldots, k_m \geq 0, k_1 + \cdots + k_m = n} \frac{n!}{k_1!k_2!\cdots k_m!} \times \int_0^1 \left( \int_{1-x_1}^1 \frac{(-\ln x_1)^{k_1}}{x_1} dx_1 \right) \left( \int_{1-x_2}^1 \frac{(-\ln x_2)^{k_2}}{x_2} dx_2 \right) \cdots \left( \int_{1-x_m}^1 \frac{(-\ln x_m)^{k_m}}{x_m} dx_m \right) dx_1. \]

Since

\[ \int_{1-x_1}^1 \frac{(-\ln x)^k}{x} dx = \frac{(-\ln(1-x_1))^{k+1}}{k+1}, \]

12
we get

$$K_{m,n} = \sum_{k_1 \geq 0, \ldots, k_m \geq 0} \frac{n!}{k_1!(k_2+1)! \cdots (k_m+1)!} \int_0^1 \frac{(-\ln x_1)^{k_1}}{x_1} \frac{(-\ln(1-x_1))^{k_2+\cdots+k_m+m-1}}{x_1} \, dx_1.$$  

Evaluating the last integral using formulas (3) and (14), the theorem follows. □

Taking $m = 2$, the polytope $V_2$ is the triangle $T$, the integral $K_{2,n}$ is the triangle integral $I_n$, and Theorem 6 reduces to Theorem 1. In particular, formula (21) for $K_{m,n}$ is a weighted version of formula (2) for $I_n$.

**Corollary 6** If $m \geq 2$, then $K_{m,0} = (m-1)!\zeta(m)$.

**Proof.** Taking $n = 0$ in Theorem 6 forces $k_1 = k_2 = \cdots = k_m = 0$ in (21). □

There is another, more natural proof of Corollary 6, one that does not use Theorem 6.

**Second proof of Corollary 6.** We use the representation

$$\zeta(m) = \int \cdots \int_{[0,1]^m} \frac{dx_1 \cdots dx_m}{1-x_1 \cdots x_m}.$$  

(To prove this formula, expand the integrand in a geometric series and integrate termwise.) We perform the change of variables

$$x_1 = y_m, \quad x_2 = \frac{y_{m-1}}{y_m}, \quad x_3 = \frac{y_{m-2}}{y_{m-1}}, \ldots, \quad x_{m-1} = \frac{y_2}{y_3}, \quad x_m = \frac{1-y_1}{y_2}.$$  

(For $m = 2$, compare this with the transformation of $I_0$ into Beukers’ integral for $\zeta(2)$ in the Introduction.) We get

$$\zeta(m) = K'_m := \int \cdots \int y_1 \cdots y_m \, dy_1 \cdots dy_m,$$

where the integral is over the polytope defined by $1 \geq y_m \geq y_{m-1} \geq \cdots \geq y_2 \geq 0$, $y_2+y_1 \geq 1$, $y_1 \leq 1$. By symmetry, we may interchange the variables $y_i$ and $y_j$ if $i > j > 1$. Using all permutations of $y_2, \ldots, y_m$, we arrive at

$$(m-1)!\zeta(m) = \int \cdots \int_{V'_m} y_1 \cdots y_m \, dy_1 \cdots dy_m,$$

where the integral is over

$$V'_m := \{(y_1, y_2, \ldots, y_m) \in [0,1]^m | y_1 + y_j \geq 1, \quad j = 2, \ldots, m\}.$$  

This is the integral $K_{m,0}$, and the second proof is complete. □

**Remark 3** The integral $K'_m$ is equivalent to the Chen (Drinfeld-Kontsevich) integral [16, Section 1]

$$\int \cdots \int \frac{dY_1 dY_2 \cdots dY_m}{(1-Y_1)Y_2 \cdots Y_m}$$

13
over the polytope $1 \geq Y_m \geq \cdots \geq Y_1 \geq 0$: set $y_1 = 1 - Y_1$ and $y_j = Y_j$ for $j = 2, \ldots, m$.

If we rewrite (21) as

$$K_{m,n} = n! \sum_{p=0}^{n} a_{m,p} \zeta(m + p, \{1\}_{n-p}),$$

where $a_{m,p}$ denotes the sum of the multinomial coefficients

$$a_{m,p} := \sum_{k_2 \geq 0, \ldots, k_m \geq 0} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2 + 1)! \cdots (k_m + 1)!},$$

then for any fixed $m \geq 2$ one can derive a closed expression for $a_{m,p}$. For example,

$$a_{2,p} = 1, \quad a_{3,p} = 4 \cdot 2^p - 2, \quad a_{4,p} = 27 \cdot 3^p - 24 \cdot 2^p + 3.$$ 

In general, we have

**Proposition 3** Fix $m \geq 2$ and $p \geq 0$. Then the integers $a_{m,p}$, $a_{m-1,p+1}$, $\ldots$, $a_{2,p+m-2}$ satisfy the recurrence

$$\sum_{t=0}^{m-2} \binom{m-1}{t} a_{m-t,p+t} = (m-1)^{p+m-1}.$$ 

**Proof.** First note that if we denote

$$S_{m,p} := \{(k_2, \ldots, k_m) \in \mathbb{Z}^{m-1} | k_j \geq -1, j = 2, \ldots, m; k_2 + \cdots + k_m = p\},$$

then

$$\sum_{S_{m,p}} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2 + 1)! \cdots (k_m + 1)!} = \sum_{l_2 \geq 0, \ldots, l_m \geq 0} \frac{(l_2 + \cdots + l_m)!}{l_2! \cdots l_m!} = (m-1)^{p+m-1}.$$ 

Now note that if $S_{m,p,t}$ is the subset of $S_{m,p}$ consisting of those $(m-1)$-tuples $(k_2, \ldots, k_m)$ with exactly $t$ numbers among the $k_j$ equal to $-1$, then

$$\sum_{S_{m,p,t}} \frac{(k_2 + \cdots + k_m + m - 1)!}{(k_2 + 1)! \cdots (k_m + 1)!} = \binom{m-1}{t} \sum_{l_2 \geq 0, \ldots, l_m \geq 0} \frac{(l_2 + \cdots + l_m + m - 1 - t)!}{(l_2 + 1)! \cdots (l_m + 1)!}$$

$$= \binom{m-1}{t} a_{m-t,p+t}.$$ 

Finally, since $S_{m,p}$ is the disjoint union

$$S_{m,p} = \bigcup_{t=0}^{m-2} S_{m,p,t},$$

the proposition follows. \qed
Another generalization of the triangle $T$ is the polytope

$$W_m := \{ (x_1, \ldots, x_m) \in [0, 1]^m | x_i + x_j \geq 1, 1 \leq i < j \leq m \}.$$ 

Note that it is symmetric in all variables, unlike $V_m$.

We first generalize the triangle integral $I_0$ to an integral over $W_m$. Then we extend to a generalization of $I_n$ over $W_m$ for all $n \geq 0$.

Recall that, for all complex $s$ and all $z$ with $|z| < 1$, the polylogarithm $Li_s(z)$ is defined by the convergent series

$$Li_s(z) := \sum_{r=1}^{\infty} \frac{z^r}{r^s}.$$ 

**Theorem 7** If $m \geq 2$, then the integral

$$L_m := \int \cdots \int_{W_m} \frac{dx_1 \cdots dx_m}{x_1 \cdots x_m}$$

is equal to a polynomial of several variables with integer coefficients in the values $\ln 2$, $\zeta(m)$, and $Li_s(1/2)$ for $s = 2, 3, \ldots, m$, namely,

$$L_m = m!\zeta(m) - (m - 1) \ln^m 2 - m! \sum_{p=0}^{m-2} \frac{\ln^p 2}{p!} Li_{m-p} \left( \frac{1}{2} \right).$$

**Proof.** The symmetry of $W_m$ yields

$$L_m = m! \int \cdots \int_{W'_m} \frac{dx_1 \cdots dx_m}{x_1 \cdots x_m},$$

where $W'_m$ is the polytope defined by $0 \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq 1$ and $x_1 + x_2 \geq 1$.

Integrating consecutively with respect to $x_m, x_{m-1}, \ldots, x_2$, and setting $x = x_1, y = x_2$, we obtain

$$L_m = m(m-1) \int_H \frac{(-\ln y)^{m-2}}{xy} dx dy,$$

where $H$ is the triangle defined by $0 \leq x \leq y \leq 1$ and $x + y \geq 1$. (Thus $H$ is the upper half of the triangle $T$ when bisected by the line $y = x$.) Since $H$ is also defined by $1/2 \leq y \leq 1$ and $1 - y \leq x \leq y$, we see that

$$L_m = m(m - 1) \int_{1/2}^1 \frac{(-\ln y)^{m-2}(\ln y - \ln(1 - y))}{y} dy$$

$$= (m - 1) \left( -\ln 2 + m \int_{1/2}^1 \frac{(-\ln y)^{m-2}(-\ln(1 - y))}{y} dy \right).$$

The series expansion

$$\frac{-\ln(1 - y)}{y} = \sum_{r=1}^{\infty} \frac{y^{r-1}}{r}$$

15
yields
\[
\int_{1/2}^{1} \frac{(-\ln y)^{m-2}(-\ln(1-y))}{y} dy = \sum_{r=1}^{\infty} \frac{1}{r} \int_{1/2}^{1} y^{r-1}(-\ln y)^{m-2} dy.
\]

Now consider the identity
\[
\int_{1/2}^{1} y^{t+r-1} dy = \frac{1}{t+r} - \frac{1}{(t+r)2^{t+r}}.
\]
If we differentiate \(l\) times with respect to \(t\), and then set \(t = 0\), we obtain
\[
\int_{1/2}^{1} y^{r-1}(-\ln y)^{t} dy = \frac{1}{r^{t+1}} - \frac{1}{2^r} \sum_{p=0}^{l} \ln p \frac{\ln^{p} 2}{r^{l-p+1}p!}.
\]
Putting \(l = m - 2\), the theorem follows. \(\square\)

**Example 1** Taking \(m = 2\) gives
\[
L_{2} = \iint_{W_{2}} \frac{1}{xy} = 2\zeta(2) - \ln^2 2 - 2 \text{Li}_{2} \left( \frac{1}{2} \right).
\]
On the other hand, \(L_{2} = I_{0} = \zeta(2)\). This proves Euler’s formula for the dilogarithm at 1/2 [5, Section 1.2], [7, pp. 43-45], [10, Section 1.4]:
\[
\text{Li}_{2} \left( \frac{1}{2} \right) = \sum_{r=1}^{\infty} \frac{1}{r^{2}2^{r}} = \frac{\zeta(2)}{2} - \frac{\ln^2 2}{2}. \tag{22}
\]

Now take \(m = 3\). Using Landen’s formula for the trilogarithm at 1/2 [10, Equation 6.12],
\[
\text{Li}_{3} \left( \frac{1}{2} \right) = \frac{7\zeta(3)}{8} - \frac{\pi^2 \ln 2}{12} + \frac{\ln^3 2}{6}. \tag{23}
\]
we get
\[
L_{3} = \iiint_{W_{3}} \frac{dx_{1}dx_{2}dx_{3}}{x_{1}x_{2}x_{3}} = 6\zeta(3) - 2\ln^3 2 - 6 \text{Li}_{3} \left( \frac{1}{2} \right) - 6 \ln 2 \text{Li}_{2} \left( \frac{1}{2} \right) = \frac{3}{4}\zeta(3).
\]
Thus, surprisingly, \(L_{3}\) and \(I_{1}\) are both rational multiples of \(\zeta(3)\).

Finally, setting \(m = 4\) and using the formulas for \(\text{Li}_{2}(1/2)\) and \(\text{Li}_{3}(1/2)\), we obtain
\[
L_{4} = \frac{4}{15}\pi^4 - \ln^4 2 + \pi^2 \ln^2 2 - 21\zeta(3) \ln 2 - 24 \text{Li}_{4} \left( \frac{1}{2} \right).
\]

We now generalize \(I_{n}\) to an integral over the polytope \(W_{m}\). First, we extend the definition of the polylogarithm \(\text{Li}_{s}(z)\) by defining the *multiple polylogarithm*
\[
\text{Li}_{s_{1},...,s_{l} \ (z)} := \sum_{n_{1}>n_{2}>...>n_{l}>0} \frac{z^{n_{1}}}{n_{1}^{s_{1}} \cdots n_{l}^{s_{l}}},
\]
...
Theorem 8 If \( m \geq 2 \) and \( n \geq 0 \), then the integral
\[
M_{m,n} := \int \cdots \int_{W_m} \frac{(-\ln(x_1 \cdots x_m))^n}{x_1 \cdots x_m} dx_1 \cdots dx_m
\]
is equal to a polynomial of several variables with rational coefficients in the values \( \ln 2 \), \( \zeta(a, \{1\}_{m+n-a}) \) with \( m \leq a \leq m+n \), and \( \text{Li}_{b,\{1\}_c}(1/2) \) with \( b+c \leq m+n \), \( b \geq 2 \), \( 0 \leq c \leq n \). Explicitly, if \( A(k_2) := \frac{1}{k_2!} \) and if
\[
A(k_2, \ldots, k_m) := \frac{1}{k_2! \cdots k_m!} \cdot \frac{1}{(k_m+1)(k_{m-1}+k_m+2) \cdots (k_3 + \cdots + k_m + m - 2)}
\]
for \( m \geq 3 \), then
\[
M_{m,n} = m! n! \sum_{k_1 \geq 0, \ldots, k_m \geq 0 \atop k_1 + \cdots + k_m = n} A(k_2, \ldots, k_m) \left[ (k_2 + \cdots + k_m + m - 2)! \zeta(k_2 + \cdots + k_m + m, \{1\}_{k_1}) \right. \\
- \frac{\ln^{m+n+2} 2}{(k_1 + 1)!(m + n)} - (k_2 + \cdots + k_m + m - 2)! \sum_{p=0}^{k_2 + \cdots + k_m + m - 2} \frac{\ln^p 2}{p!} \text{Li}_{k_2 + \cdots + k_m + m - p, \{1\}_{k_1}}(1/2) \bigg].
\]

Proof. The proof is similar to that of Theorem 7 (the case \( n = 0 \)). \( \square \)

Notice the equality of the integrals \( M_{2,n} = I_n \).

As an application of Theorem 8, we obtain the following relation between certain multiple polylogarithm values and multiple zeta values. (The relation can also be deduced from the Hölder convolution formula in [4, Equation (7.2)].)

Corollary 7 If \( n \geq 0 \), then
\[
\sum_{k=0}^{n} \sum_{p=0}^{n-k} \frac{\ln^p 2}{p!} \text{Li}_{n-k+2-p,\{1\}_k} \left( \frac{1}{2} \right) = \frac{1}{(n+2)!} \ln^{n+2} 2 + \frac{1}{2} \sum_{k=0}^{n} \zeta(n-k+2, \{1\}_k).
\]

Proof. In Theorem 8, take \( m = 2 \) and set \( k_1 = k \), so that \( k_2 = n - k \). Then \( A(k_2) = \frac{1}{(n-k)!} \) and
\[
M_{2,n} = 2n! \sum_{k=0}^{n} \left[ \zeta(n-k+2, \{1\}_k) - \frac{\ln^{n+2} 2}{(k+1)!(n-k)!(n+2)} - \sum_{p=0}^{n-k} \frac{\ln^p 2}{p!} \text{Li}_{n-k+2-p,\{1\}_k} \left( \frac{1}{2} \right) \right].
\]

Now substitute \( M_{2,n} = I_n \), and apply Theorem 1 (i) and the identity
\[
\sum_{k=0}^{n} \frac{1}{(k+1)!(n-k)!} = \frac{2^{n+1} - 1}{(n+1)!}.
\]

Example 2 The case \( n = 0 \) is Euler’s formula \([22]\) for \( \text{Li}_2(1/2) \). Taking \( n = 1 \) and substituting \( \zeta(2,1) = \zeta(3) \) gives the relation
\[
\text{Li}_3 \left( \frac{1}{2} \right) + \text{Li}_2 \left( \frac{1}{2} \right) \ln 2 + \text{Li}_{2,1} \left( \frac{1}{2} \right) = -\frac{\ln^3 2}{2} + \zeta(3),
\]
which is a special case of [4, Equation (7.3)]. Using the values of \( \text{Li}_2(1/2) \) and \( \text{Li}_3(1/2) \) in \((22)\) and \((23)\), we get the formula

\[
\text{Li}_{2,1}(1/2) = \sum_{r=2}^{\infty} \frac{1}{r^2 2^r} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r-1}\right) = \frac{\zeta(3)}{8} - \frac{\ln^3 2}{6}.
\]

Finally, adding \( \text{Li}_3(1/2) \) recovers Ramanujan’s summation [1, p. 258]

\[
\sum_{r=1}^{\infty} \frac{1}{r^2 2^r} \left(1 + \frac{1}{2} + \cdots + \frac{1}{r}\right) = \frac{\zeta(3)}{12} - \pi^2 \ln 2.
\]

7 References

1. B. C. Berndt, *Ramanujan’s Notebooks*, Part I, Springer-Verlag, New York, 1985.
2. F. Beukers, A note on the irrationality of \( \zeta(2) \) and \( \zeta(3) \), *Bull. London Math. Soc.* **11** (1979) 268-272.
3. J. M. Borwein, D. M. Bradley, and D. J. Broadhurst, Evaluations of \( k \)-fold Euler/Zagier sums: a compendium of results for arbitrary \( k \), *Electron. J. Combin.* **4** (1997) no. 2, R5.
4. J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, Special values of multiple polylogarithms, *Trans. Amer. Math. Soc.* **353** (2001) 907-941.
5. P. Cartier, Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents, *Astérisque* **282** (2002) 137-173.
6. N. G. de Bruijn, *Asymptotic Methods in Analysis*, Dover Publications, New York, 1981.
7. W. Dunham, *Euler: The Master of Us All*, Dolciani Mathematical Expositions No. 22, Mathematical Association of America, Washington, D.C., 1999.
8. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th edition, A. Jeffrey and D. Zwillinger, editors, Academic Press, San Diego, 2000.
9. K. S. Köhlig, J. A. Mignaco, and E. Remiddi, On Nielsen’s generalized polylogarithms and their numerical calculation, *BIT* **10** (1970) 38-74.
10. L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, New York, 1981.
11. J. Ser, Sur une expression de la fonction \( \zeta(s) \) de Riemann, *C. R. Acad. Sci. Paris Sér. I Math.* **182** (1926) 1075-1077.
12. J. Sondow, Criteria for irrationality of Euler’s constant, *Proc. Amer. Math. Soc.* **131** (2003) 3335-3344.
13. J. Sondow, An infinite product for \( e^\gamma \) via hypergeometric formulas for Euler’s constant, \( \gamma \) (2003, preprint); available at [http://arXiv.org/abs/math/0306008](http://arXiv.org/abs/math/0306008).
14. J. Sondow, Double integrals for Euler’s constant and \( \ln \frac{4}{\pi} \) and an analog of Hadjicostas’s formula, *Amer. Math. Monthly* **112** (2005) 61-65.
15. J. Sondow, A faster product for \( \pi \) and a new integral for \( \ln \frac{4}{\pi} \), *Amer. Math. Monthly* **112** (2005) 729-734.
16. M. Waldschmidt, Multiple polylogarithms: an introduction, in *Number Theory and Discrete Mathematics*, Proceedings of the International Conference in Honour of Srinivasa Ramanujan, Chandigarh, 2000, A. K. Agarwal et al., editors, Birkhäuser, Basel, 2002, pp. 1-12.
17. S. Zlobin, On a certain integral over a triangle (2005, preprint); available at http://arXiv.org/abs/math/0511239.