Abstract. Researchers have studied the *first-passage time* of financial time series and observed that the smallest time interval needed for a stock index to move a given distance is typically shorter for negative than for positive price movements. The same is not observed for the index constituents, the individual stocks. We use the discrete wavelet transform to show that this is a long, rather than short, timescale phenomenon—if enough low frequency content of the price process is removed, the asymmetry disappears. We also propose a model which explains the asymmetry in terms of prolonged, correlated downward movements of individual stocks.

**Keywords:** models of financial markets, stochastic processes
1. Introduction

Modeling the statistical properties of financial time series has long been an active area of research, in the fields of both economics and physics. The traditional object of study has been the returns of various assets, i.e. the size of price movements over fixed time intervals. Inspired by research in the field of turbulence, Simonsen et al [8] asked the ‘inverse’ question: what is the smallest time interval needed for an asset to cross a fixed return level $\rho$? Figure 1 shows the distribution of this random variable, the first-passage time, for the Dow Jones Industrial Average index, for $\rho = \pm 5\%$. As noted by Jensen et al [5], the most likely first-passage time is shorter for $\rho = -5\%$ than for $\rho = +5\%$, which they refer to as the gain/loss asymmetry. Intriguingly, the same asymmetry is not observed in the constituents of the index, the individual stocks [6].

One explanation for the gain/loss asymmetry in the index is that, occasionally, the stocks move in a highly correlated manner, and that this tends to happen for downward movements rather than upward movements [3]. To support this view, Donangelo et al [3] proposed the asymmetric synchronous market model, where the log stock prices move like independent random walks for most days, but sometimes move downwards together. This model exhibits gain/loss asymmetry for the index, but not for the individual stocks.

In this paper we investigate on what timescale the gain/loss asymmetry emerges. To this end, we use the discrete wavelet transform and estimate the first-passage time for high-pass filtrations of the time series. Apparently, if enough low frequency content is removed, the gain/loss asymmetry disappears—the asymmetry is due to effects on quite long timescales, 64–128 days and longer. Together with the fact that individual stocks do not exhibit gain/loss asymmetry, this indicates that the asymmetry is due to prolonged, correlated downward movements of the stock prices. This is contrary to the model from Donangelo et al [3], where the correlated losses are ‘local’ in time. Indeed, we illustrate that the gain/loss asymmetry in that model emerges on a rather short timescale, which is inconsistent with our empirical findings. Finally, we construct a generalization of the asymmetric synchronous market model, where the gain/loss asymmetry emerges on longer timescales, consistently with the empirical findings.
Figure 1. Estimated distribution of the first-passage time $\tau_\rho$ for the log price of the Dow Jones industrial average index. The graphs correspond to $\rho = +5\%$ (stars) and $\rho = -5\%$ (rings). The solid lines are the fitted generalized gamma density functions.

2. First-passage time on multiple timescales

Throughout this section we let $\{X_t\}_{t \geq 0}$ denote the logarithm of a financial time series, for instance the price of a stock index. The first-passage time $\tau_\rho$ of the level $\rho$ is defined as

$$\tau_\rho = \begin{cases} \inf \{s > 0; X_{t+s} - X_t \geq \rho\} & \text{if } \rho > 0, \\ \inf \{s > 0; X_{t+s} - X_t \leq \rho\} & \text{if } \rho < 0, \end{cases}$$

and is assumed to be independent of $t$. The distribution of $\tau_\rho$ is estimated in a straightforward manner from a time series $X = (X_{t_1}, \ldots, X_{t_N})$. Consider $\rho > 0$, and let $t_{n+m}$ be the smallest time point such that $X_{t_{n+k}} - X_{t_n} \geq \rho$, if such a time point exists. In that case, $t_{n+m} - t_n$ is viewed as an observation of $\tau_\rho$. (If $\rho < 0$, take instead $t_{n+m}$ such that $X_{t_{n+k}} - X_{t_n} \leq \rho$.) Running $n$ from 1 to $N - 1$ gives a set of observations from which the distribution of $\tau_\rho$ is estimated as the empirical distribution. Given the empirical distribution, we follow Jensen et al [5] and compute a fit of the density function for the generalized gamma distribution. This density is plotted as a solid line together with the empirical distribution in all figures, to guide the eye—we do not discuss the fitted parameters, nor claim that $\tau_\rho$ truly follows a generalized gamma distribution.

Given a time series $X = (X_{t_1}, \ldots, X_{t_N})$, the level-$J$ discrete wavelet transform\(^1\) yields an additive decomposition

$$X = \sum_{j=1}^{J} D_j + S_j,$$

\(^1\) See, e.g., Gencay et al [4] or Percival and Walden [7].
where $D_j$ is the $j$th-level detail and $S_j$ is the $J$th-level smooth. Essentially, $D_j$ is a band-pass filtration and $S_j$ is a low pass filtration of $X$. If $X$ consists of daily observations, then $D_j$ contains changes on a timescale of between $2^{j-1}$ and $2^j$ days, and $S_J$ contains changes on timescales longer than $2^J$ days. The signal $R_j := X - S_j$ can thus be seen as a ‘detrended’ version of $X$, where the time horizon of the removed trends increases exponentially with $J$.

We considered the time series $X$ of daily observations of the Dow Jones Industrial Average index, henceforth DJIA, and computed\(^2\) the filtered signals $R_6, R_8$ and $R_{10}$ (see figure 2). We then estimated the distribution of the first-passage time for $R_6, R_8, R_{10}$ and $X$, for $\rho = \pm 5\%$ (the standard deviation of daily log returns of DJIA is roughly 1%).

\(^2\) In all our experiments we use LA(8), the least asymmetric wavelet filter of length 8 (see Daubechies [2]), but our results are robust in the sense that other choices yield very similar results.
Figure 3 shows the result: the gain/loss asymmetry is absent for $R_6$, and gradually emerges for $R_8$ and $R_{10}$. This indicates that if enough low frequency components of the signal are removed, the gain/loss asymmetry disappears. Since the level $J = 6$ corresponds to 32–64 days, the asymmetry appears mainly due to effects on rather long timescales: 64–128 days and longer. The same pattern was observed for other stock indices, like the S&P500 and Nasdaq (not reported). The effect is not influenced by the fact that the increments in $R_6$, $R_8$ and $R_{10}$ have standard deviations slightly different to $X$—re-scaling the high-pass filtrations does not change the result.

Figure 4 shows the results from the same analysis for the IBM stock price. We see that there is virtually no asymmetry for the IBM stock, at any timescale, which is consistent with the observations from Johansen et al [6]. We have also confirmed this for several other individual stocks.

In the light of the finding that the gain/loss asymmetries in stock indices are due to effects on long rather than short timescales, it is natural to question the structure of the asymmetric synchronous market model (ASMM) given by Donangelo et al [3]. In that model, the features giving rise to the gain/loss asymmetry by construction take place on
Figure 4. Estimated distribution of the first-passage time \( \tau \rho \) for \( X \), the log price of the IBM stock, and its high-pass filtrations \( R_6, R_8 \), and \( R_{10} \). The graphs correspond to \( \rho = +5\% \) (stars) and \( \rho = -5\% \) (rings). The solid lines are the fitted generalized gamma density functions.

the shortest possible timescale. To see to what extent the gain/loss asymmetry in ASMM resembles that of DJIA when considered on multiple timescales, we performed the analysis for a realization of the model. That is, we let \( X \) be a time series generated by ASMM, and estimated the first-passage time densities\(^3\) for \( R_6, R_8, R_{10} \), and \( X \). Figure 5 shows the result—the gain/loss symmetry is evidently present already on shorter timescales, for \( R_6 \) and \( R_8 \), which is in contrast with the empirical findings (cf figure 3).

3. A generalized asymmetric synchronous market model

In this section we propose a new model to remedy the failure of ASMM in reproducing the gain/loss asymmetry when considered on multiple timescales. The model can be seen as a

\(^3\) As in Donangelo et al [3], we take \( \rho = \pm 5\sigma \) where \( \sigma \) equals the daily volatility (the standard deviation of the log price) of the model index. This corresponds to \( \rho = \pm 5\% \) in the case of the Dow Jones index, since the daily volatility there is roughly 1%.
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Figure 5. Estimated distribution of the first-passage time $\tau_\rho$ for $X$, the log price in a realization of ASMM with $N = 30$ stocks, and its high-pass filtrations $R_6$, $R_8$ and $R_{10}$. The graphs correspond to $\rho = 5\sigma$ (stars) and $\rho = -5\sigma$ (rings). The solid lines are the fitted generalized gamma density functions.

A generalization of ASMM—it has one ‘regular’ state where the stocks move independently, and one ‘distressed’ state where their moves are highly correlated. The most important difference between our model and ASMM is that the market may remain in the distressed state for several days. Conceptually, this is in correspondence with the empirical findings from section 2, that the gain/loss asymmetry is due to effects on long rather than short timescales.

In the regular state, all stocks follow independent geometric Brownian motions with drift $\mu_r$ and standard deviation $\xi$:

$$dS_i(t) = \mu_r S_i(t) \, dt + \xi S_i(t) \, dW_i(t),$$

where $S_i(t)$ denotes the price at time $t$ of the $i$th stock and $W_i$ are independent Brownian motions, for $i = 1, \ldots, N$. In the distressed state, the stocks have a lower drift $\mu_d < \mu_r$ and all the stocks move together:

$$dS_i(t) = \mu_d S_i(t) \, dt + \xi S_i(t) \, dW(t),$$

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Figure 6. Estimated distribution of the first-passage time $\tau_p$ for $X$, the log price in a realization of our proposed model with $N = 30$ stocks, and its high-pass filtrations $R_6$, $R_8$ and $R_{10}$. The graphs correspond to $\rho = 5\sigma$ (stars) and $\rho = -5\sigma$ (rings).

where a single Brownian motion $W$ drives the price changes in all the stocks. Solving the stochastic differential equations and sampling with $\Delta t = 1/250$ gives the following discrete time dynamics. If the market is in the regular state,

$$S_i(t + \Delta t) = S_i(t)e^{(\mu_r - \xi^2/2)\Delta t + \xi \sqrt{\Delta t} Z_i},$$

where $Z_i$, $i = 1, \ldots, N$, are independent standard normals. If the market is in the distressed state,

$$S_i(t + \Delta t) = S_i(t)e^{(\mu_d - \xi^2/2)\Delta t + \xi \sqrt{\Delta t} Z},$$

where the price changes in all the stocks are driven by a single standard normal $Z$.

For any given day, we consider the possibility of changing states: we let $p_{rd}$ denote the probability of changing from the regular to the distressed state, and let $p_{dr}$ denote the probability of the converse transition. It is natural to take $p_{rd}$ smaller than $p_{dr}$—the
Figure 7. Estimated distribution of the first-passage time $\tau_\rho$ for $X$, the log price in a realization of our proposed model with $N=1$ stock, and its high-pass filtrations $R_6$, $R_8$, and $R_{10}$. The graphs correspond to $\rho = 5\sigma$ (stars) and $\rho = -5\sigma$ (rings). The solid lines are the fitted generalized gamma density functions.

market will then be in the regular state more often than not. The index is defined by

$$I(t) := \frac{1}{N} \sum_{i=1}^{N} S_i(t).$$

Note that ASMM can be seen as a particular case of this model, with $p_{dr} = 1$—the transition from the distressed to the normal state always happens in one day. Just as in ASMM, the individual stocks are less affected than the index by a transition to the distressed state. For the individual stocks, only the drift changes, but for the index, the volatility is markedly higher, since all the stocks suddenly move in the same direction.

We fix the following parameters: $\Delta t = 1/250$, $\xi = 0.3$, $\mu_d = -0.10$, $p_{rd} = 1/200$, $p_{dr} = 1/25$, and choose the regular drift $\mu_r$ in order to make $\mathbb{E}[(I(t + \Delta t) - I(t))/I(t)] = 2.58 \times 10^{-4}$ (this is the historical daily mean return of DJIA). As above, let $\rho = \pm 5\sigma$, where $\sigma$ denotes the standard deviation of the daily log returns of the index. Figure 6 shows the estimated first-passage density for the index and its high-pass filtrations, for
$N = 30$ stocks. We see that the multiresolution analysis resembles that of DJIA: the gain/loss asymmetry is absent for $R_6$ but emerges for $R_8$ and $R_{10}$. Figure 7 shows the result from the same analysis but with $N = 1$ stock: there is no gain/loss asymmetry in this case. Again, this is consistent with the empirical findings, that individual stocks do not exhibit gain/loss asymmetry.

It is important to remark that although our model qualitatively reproduces the gain/loss asymmetry in stock indices, it does not perfectly describe the height of the peaks in the empirical first-passage time densities—much like ASMM. In particular, for DIJA, the magnitude of the peak of the density is lower for $\rho = +5\%$ than for $\rho = -5\%$, reflecting a slightly heavier tail (see figure 3). More fundamentally, in the real market there is probably no such thing as a state where all the stocks move in the same direction. However, our results suggest that one should consider regimes in the market where a significant proportion of the stocks tend to move downwards in a highly correlated manner.

Finally, we note that if $\mu_d > \mu_r$, the model gives the opposite gain/loss asymmetry—the most likely first-passage time is shorter for positive than for negative $\rho$ (not reported).

4. Conclusion

Using wavelet multiresolution analysis, we have shown that the so-called gain/loss asymmetry observed in stock indices is due to effects on long rather than short timescales—the asymmetry disappears if enough low frequency content is removed from the signal. Moreover, we generalize the asymmetric synchronous market model to include prolonged periods of correlated downward movements of the index constituents. This model qualitatively reproduces the asymmetry.

According to our generalized asymmetric synchronous market model, the stock market occasionally enters a distressed state of correlated downward movements and stays there for a prolonged period of time, expected to last $1/p_d$ days. In the future, we would like to investigate whether it is possible to estimate this quantity from data by comparing empirical first-passage time distributions to the distributions implied by the model, in effect ‘calibrating’ the model. We are intrigued by the question of how such information may relate to the so-called heterogeneous market hypothesis [1]—that is, that various participants in the market have different time horizons and dealing frequencies. The wavelet multiscale decomposition of the signal could potentially be used be to investigate the emergence of the gain/loss asymmetry even further. For instance, what happens if the signal is band-pass filtered instead of high-pass filtered—is it possible to pinpoint the timescale of the origin of the asymmetry?

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