FORMAL AND RIGID GEOMETRY: AN INTUITIVE INTRODUCTION, AND SOME APPLICATIONS

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Abstract. We give an informal introduction to formal and rigid geometry, and we discuss some applications in algebraic and arithmetic geometry and singularity theory, with special emphasis on recent applications to the Milnor fibration and the motivic zeta function by J. Sebag and the author.

1. Introduction

Let $R$ be a complete discrete valuation ring, with quotient field $K$, and residue field $k$. We choose a uniformizing parameter $\pi$, i.e. $\pi$ generates the unique maximal ideal of $R$. Geometers may take $R = \mathbb{C}[[t]]$, the ring of formal power series over the complex numbers, with $K = \mathbb{C}((t))$, $k = \mathbb{C}$, $\pi = t$, while number theorists might prefer to think of $R = \mathbb{Z}_p$, the ring of $p$-adic integers, with $K = \mathbb{Q}_p$, $k = \mathbb{F}_p$, $\pi = p$.

Very roughly, a formal scheme over $R$ consists of an algebraic variety over $k$, together with algebraic information on an infinitesimal neighbourhood of this variety. If $X$ is a variety over $R$, we can associate to $X$ its formal completion $\hat{X}$ in a natural way. It is a formal scheme over $R$, and can be seen as an infinitesimal tubular neighbourhood of the special fiber $X_0$ in $X$. Its underlying topological space coincides with the space underlying $X_0$, but additional infinitesimal information is contained in the sheaf of regular functions on $\hat{X}$.

An important aspect of the formal scheme $\hat{X}$ is the following phenomenon. A closed point $x$ on the scheme-theoretic generic fiber of $X$ over $K$ has coordinates in some finite extension of the field $K$, and (unless $X$ is proper over $R$) there is no natural way to associate to the point $x$ a point of the special fiber $X_0/k$ by reduction modulo $\pi$. However, by inverting $\pi$ in the structure sheaf of $\hat{X}$, we can associate a generic fiber $X_\eta$ to the formal scheme $\hat{X}$, which is a rigid variety over $K$. Rigid geometry provides a satisfactory theory of analytic geometry over non-archimedean fields. A point on $X_\eta$ has coordinates in the ring of integers of some finite extension $K'$ of $K$: if we denote by $R'$ the normalization of $R$ in $K'$, we can canonically identify $X_\eta(K')$ with $\hat{X}(R')$. By reduction modulo $\pi$, we obtain a canonical “contraction” of the generic fiber $X_\eta$ to the special fiber $X_0$. Roughly speaking, the formal scheme $\hat{X}$ has the advantage that its generic and its special fiber are tightly connected, and what glues them together are the $R'$-sections on $X$, where $R'$ runs over the finite extensions of $R$.

A main disadvantage of rigid geometry is the artificial nature of the topology on rigid varieties: it is not a classical topology, but a Grothendieck topology. In the nineties, Berkovich developed his spectral theory of non-archimedean spaces. His spaces carry a true topology, which allows to apply classical techniques from
algebraic topology. In particular, the unit disc \( R \) becomes connected by arcs, while it is totally disconnected w.r.t. its \( \pi \)-adic topology.

In Section 3, we give a brief survey of the basic theory of formal schemes, and Section 4 is a crash course on rigid geometry. Section 5 contains the basic definitions of Berkovich’ approach to non-archimedean geometry. In the final Section 6, we briefly discuss some applications of the theory, with special emphasis on the relation with arc spaces of algebraic varieties, and the Milnor fibration.

This intuitive introduction merely aims to provide some insight into the theory of formal schemes and rigid varieties. We do not provide proofs; instead, we chose to give a list of more thorough introductions to the different topics dealt with in this note.

2. Conventions and notation

- For any field \( F \), we denote by \( F^{\text{alg}} \) an algebraic closure, and by \( F^{\text{s}} \) the separable closure of \( F \) in \( F^{\text{alg}} \).
- If \( S \) is any scheme, a \( S \)-variety is a separated reduced scheme of finite type over \( S \).
- For any locally ringed space (site) \( X \), we denote the underlying topological space (site) by \( |X| \).
- Throughout this note, \( R \) denotes a complete discrete valuation ring, with residue field \( k \), and quotient field \( K \). We fix a uniformizing parameter \( \pi \), i.e. a generator of the maximal ideal of \( R \). For any integer \( n \geq 0 \), we denote by \( R_n \) the quotient ring \( R/(\pi^{n+1}) \). A finite extension \( R' \) of \( R \) is by definition the normalization of \( R \) in some finite field extension \( K' \) of \( K \); \( R' \) is again a complete discrete valuation ring.
- Once we fix a value \( |\pi| \in [0,1[ \), the discrete valuation \( v \) on \( K \) defines a non-archimedean absolute value \( |.| \) on \( K \), with \( |z| = |\pi|^v(z) \) for \( z \in K^* \). This absolute value induces a topology on \( K \), called the \( \pi \)-adic topology. The ideals \( \pi^n R \), \( n \geq 0 \), form a fundamental system of open neighbourhoods of the zero element in \( K \). The \( \pi \)-adic topology is totally disconnected. The absolute value on \( K \) extends uniquely to an absolute value on \( K^{\text{alg}} \). For any integer \( m > 0 \), we endow \( (K^{\text{alg}})^m \) with the norm \( \|z\| := \max_{i=1,...,m} |z_i| \).

3. Formal geometry

In this note, we will only consider formal schemes topologically of finite type over the complete discrete valuation ring \( R \). This case is in many respects simpler than the general one, but it serves our purposes. For a more thorough introduction to the theory of formal schemes, we refer to [23 §10], [22, no. 182], [28], or [9].

Intuitively, a formal scheme \( X_\infty \) over \( R \) consists of its special fiber \( X_0 \), which is a scheme of finite type over \( k \), endowed with a structure sheaf containing additional algebraic information on an infinitesimal neighbourhood of \( X_0 \).

3.1. Affine formal schemes. For any tuple of variables \( x = (x_1,\ldots,x_m) \), we define a \( R \)-algebra \( R\{x\} \) as the projective limit

\[
R\{x\} := \lim_{\rightarrow} R_n[\bar{x}]
\]
The $R$-algebra $R\{x\}$ is canonically isomorphic to the algebra of converging power series over $R$, i.e. the subalgebra of $R[[x]]$ consisting of the elements

$$c(x) = \sum_{i=(i_1,\ldots,i_m) \in \mathbb{N}^m} \left( c_i \prod_{j=1}^m x_j^{i_j} \right) \in R[[x]]$$

such that $c_i \to 0$ (w.r.t. the $\pi$-adic topology on $K$) as $|i| = i_1 + \ldots + i_m$ tends to $\infty$. This means that for each $n \in \mathbb{N}$, there exists a value $i_0 \in \mathbb{N}$ such that $c_i$ is divisible by $\pi^n$ in $R$ if $|i| > i_0$. Note that this is exactly the condition which guarantees that the images of $c(x)$ in the quotient rings $R_n[[x]]$ are actually polynomials, i.e. belong to $R_n[x]$. The algebra $R\{x\}$ can also be characterized as the sub-algebra of $R[[x]]$ consisting of those power series which converge on the closed unit disc $R^m = \{ z \in K^m \mid \| z \| \leq 1 \}$, since an infinite sum converges in a non-archimedean field iff its terms tend to zero. One can show that $R\{x\}$ is Noetherian [23, 0.(7.5.4)].

An $R$-algebra $A$ is called topologically of finite type (tft) over $R$, if it is isomorphic to an algebra of the form $R\{x_1,\ldots,x_m\}/I$, for some integer $m > 0$ and some ideal $I$. For any integer $n \geq 0$, we denote by $A_n$ the quotient ring $A/(\pi^{n+1})$. It is a $R_n$-algebra of finite type. Then $A$ is the limit of the projective system $(A_n)_{n \in \mathbb{N}}$, and if we endow each ring $A_n$ with the discrete topology, then $A$ becomes a topological ring w.r.t. the limit topology (the $\pi$-adic topology on $A$). By definition, the ideals $\pi^n A, n > 0$, form a fundamental system of open neighbourhoods of the zero element of $A$.

To any tft $R$-algebra $A$, we can associate a ringed space $\text{Spf } A$. It is defined as the direct limit

$$\text{Spf } A := \lim_{\longrightarrow} \text{Spec } A_n$$

in the category of topologically ringed spaces (where the topology on $\mathcal{O}_{\text{Spec } A_n}$ is discrete for every $n$), so the structure sheaf $\mathcal{O}_{\text{Spf } A}$ is a sheaf of topological $R$-algebras in a natural way. Moreover, one can show that the stalks of this structure sheaf are local rings. A tft affine formal $R$-scheme is a locally topologically ringed space in $R$-algebras which is isomorphic to a space of the form $\text{Spf } A$.

Note that the transition morphisms $\text{Spec } A_m \to \text{Spec } A_n, m \leq n$, are nilpotent immersions and therefore homeomorphisms. Hence, the underlying topological space $|\text{Spf } A|$ of $\text{Spf } A$ is the set of open prime ideals $J$ of $A$ (i.e. prime ideals containing $\pi$), endowed with the Zariski topology, and it is canonically homeomorphic to $|\text{Spec } A_0|$.

So we see that $\text{Spf } A$ is the locally topologically ringed space in $R$-algebras

$$\left( |\text{Spec } A_0|, \lim_{\longrightarrow} \mathcal{O}_{\text{Spec } A_n} \right)$$

In particular, we have $\mathcal{O}_{\text{Spf } A}(\text{Spf } A) = A$. Whenever $f$ is an element of $A$, we denote by $D(f)$ the set of open prime ideals of $A$ which do not contain $f$. This is an open subset of $|\text{Spf } A|$, and the ring of sections $\mathcal{O}_{\text{Spf } A}(D(f))$ is the $\pi$-adic completion $A_f$ of the localization $A_f$. 
A morphism between \( tft \) affine formal \( R \)-schemes is by definition a morphism of locally ringed spaces in \( R \)-algebras. If \( h : A \to B \) is a morphism of \( tft \) \( R \)-algebras, then \( h \) induces a direct system of morphisms of \( R \)-schemes \( \Spec B_n \to \Spec A_n \) and by passage to the limit a morphism of \( tft \) affine formal \( R \)-schemes \( \Spec f(h) : \Spec B \to \Spec A \). The resulting functor \( \Spec \) induces an equivalence between the opposite category of \( tft \) \( R \)-algebras, and the category of \( tft \) affine formal \( R \)-schemes, just like in the algebraic scheme case.

The special fiber \( X_0 \) of the affine formal \( R \)-scheme \( X_∞ = \Spec A \) is the \( k \)-scheme \( X_0 = \Spec A_0 \). As we’ve seen, the natural morphism of topologically locally ringed spaces \( X_0 \to X_∞ \) is a homeomorphism.

**Example 1.** Any finite extension \( R' \) of \( R \) is a \( tft \) \( R \)-algebra. The affine formal scheme \( \Spec R' \) consists of a single point, corresponding to the maximal ideal of \( R' \), but the ring of sections on this point is the entire ring \( R' \). So in some sense the infinitesimal information in the topology of \( \Spec R' \) (the generic point) is transferred to the structure sheaf of \( \Spec R' \).

If \( A = R[x, y]/(\pi - xy) \) and \( X_∞ = \Spec A \), then as a topological space \( X_∞ \) coincides with its special fiber \( X_0 = \Spec k[x, y]/(xy) \), but the structure sheaf of \( X_∞ \) is much “thicker” than the one of \( X_0 \). The formal \( R \)-scheme \( X_∞ \) should be seen as an infinitesimal tubular neighborhood around \( X_0 \).

### 3.2. Formal schemes

A formal scheme \( X_∞ \) topologically of finite type (\( tft \)) over \( R \) is a locally topologically ringed space in \( R \)-algebras, which has a finite open cover by \( tft \) affine formal \( R \)-schemes. A morphism between \( tft \) formal \( R \)-schemes is a morphism of locally ringed spaces in \( R \)-algebras.

It is often convenient to describe \( X_∞ \) in terms of the direct system \((X_n := X_∞ \times_R R_n)_{n≥0}\). The locally ringed space \( X_n \) is a scheme of finite type over \( R_n \), for any \( n \); if \( X_∞ = \Spec A \), then \( X_n = \Spec A_n \). For any pair of integers \( 0 ≤ m ≤ n \), the natural map of \( R \)-schemes \( u_{m,n} : X_m \to X_n \) induces an isomorphism of \( R_m \)-schemes \( X_m \cong X_n \times_{R_n} R_m \). The scheme \( X_0 \) is called the special fiber of \( X_∞ \), and \( X_n \) is the \( n \)-th infinitesimal neighborhood of \( X_0 \) in \( X_∞ \) (or “thickening”). The natural morphism of locally topologically ringed spaces \( X_n \to X_∞ \) is a homeomorphism for each \( n ≥ 0 \).

Conversely, if \((X_n)_{n≥0}\) is a direct system of \( R \)-schemes of finite type such that \( π^{n+1} = 0 \) on \( X_n \) and such that the transition morphism \( u_{m,n} : X_m \to X_n \) induces an isomorphism of \( R_m \)-schemes \( X_m \cong X_n \times_{R_n} R_m \) for each \( 0 ≤ m ≤ n \), then this direct system determines a \( tft \) formal \( R \)-scheme \( X_∞ \) by putting

\[
X_∞ := \lim_{\longrightarrow \atop n} X_n
\]

as a locally topologically ringed space in \( R \)-algebras.

In the same way, giving a morphism \( f : X_∞ \to Y_∞ \) between \( tft \) formal \( R \)-schemes amounts to giving a series of morphisms \((f_n : X_n \to Y_n)_{n≥0}\), where \( f_n \) is a morphism of \( R_n \)-schemes, and all the squares

1Such a morphism is automatically continuous w.r.t. the topology on the structure sheaves, since it maps \( π \) to itself; because the topology is the \( π \)-adic one, it is determined by the \( R \)-algebra structure. This is specific to so-called \( R \)-adic formal schemes and does not hold for more general formal \( R \)-schemes.
commute. In other words, a morphism of \textit{stft} formal $R$-schemes consists of a compatible system of morphisms between all the infinitesimal neighbourhoods of the special fibers.

The formal scheme $X_\infty$ is called separated if the scheme $X_n$ is separated for each $n$. In fact, this will be the case as soon as the special fiber $X_0$ is separated. We will work in the category of separated formal schemes, topologically of finite type over $R$; we’ll call these objects \textit{stft} formal $R$-schemes.

A \textit{stft} formal scheme $X_\infty$ over $R$ is flat if its structure sheaf has no $\pi$-torsion. A typical example of a non-flat \textit{stft} formal $R$-scheme is one with an irreducible component concentrated in the special fiber. A flat \textit{stft} formal $R$-scheme can be thought of as a continuous family of schemes over the infintesimal disc $\text{Spf } R$. Any \textit{stft} formal $R$-scheme has a maximal flat closed formal subscheme, obtained by killing $\pi$-torsion.

3.3. Coherent modules. Let $A$ be a \textit{stft} $R$-algebra. An $A$-module $N$ is coherent iff it is finitely generated. Any such module $N$ defines a sheaf of modules on $\text{Spf } A$ in the usual way. A coherent sheaf of modules $\mathcal{N}$ on a \textit{stft} formal $R$-scheme $X_\infty$ is obtained by gluing coherent modules on affine open formal subschemes.

A more convenient description is the following: the category of coherent sheaves $\mathcal{N}$ on $X_\infty$ is equivalent to the category of direct systems $(\mathcal{N}_n)_{n \geq 0}$, where $\mathcal{N}_n$ is a coherent sheaf on the scheme $X_n$, and the $\mathcal{O}_{X_n}$-linear transition map $v_{m,n} : \mathcal{N}_m \to \mathcal{N}_n$ induces an isomorphism of coherent $\mathcal{O}_{X_n}$-modules $\mathcal{N}_m \cong u_{n,m}^* \mathcal{N}_n$ for any pair $m \leq n$. Morphisms between such systems are defined in the obvious way.

3.4. The completion functor. Let $X$ be any Noetherian scheme and $\mathcal{J}$ a coherent ideal sheaf on $X$, and denote by $V(\mathcal{J})$ the closed subscheme of $X$ defined by $\mathcal{J}$. The $\mathcal{J}$-adic completion $\widehat{X}/\mathcal{J}$ of $X$ is the limit of the direct system of schemes $(V(\mathcal{J}^n))_{n > 0}$ in the category of topologically ringed spaces (where $\mathcal{O}_{V(\mathcal{J}^n)}$ carries the discrete topology). This is, in fact, a formal scheme, but in general not of the kind we have defined before; we include the construction here for later use. If $h : Y \to X$ is a morphism of Noetherian schemes, and if we denote by $\mathcal{K}$ the inverse image $\mathcal{J}\mathcal{O}_Y$ of $\mathcal{J}$ on $Y$, then $h$ defines a direct system of morphisms of schemes $V(\mathcal{K}^n) \to V(\mathcal{J}^n)$ and by passage to the limit a morphism of topologically locally ringed spaces $Y/\mathcal{K} \to \widehat{X}/\mathcal{J}$, called the $\mathcal{J}$-adic completion of $h$.

If $X$ is a separated $R$-scheme of finite type and $\mathcal{J}$ is the ideal generated by $\pi$, then the $\mathcal{J}$-adic completion of $X$ is the limit of the direct system $(X_n = X \times_R R_n)_{n \geq 0}$, and this is a \textit{stft} formal $R$-scheme which we denote simply by $X$. It is called the formal ($\pi$-adic) completion of the $R$-scheme $X$. Its special fiber $X_0$ is canonically isomorphic to the fiber of $X$ over the closed point of $\text{Spec } R$. The formal scheme $\widehat{X}$ is flat iff $X$ is flat over $R$. Intuitively, $\widehat{X}$ should be seen as the infinitesimal tubular neighbourhood of $X_0$ in $X$. As a topological space, it coincides with $X_0$, but additional infinitesimal information is contained in the structure sheaf.
Example 2. If $X = \text{Spec } R[x_1, \ldots, x_n]/(f_1, \ldots, f_\ell)$, then its formal completion $\hat{X}$ is simply $\text{Spf } R[x_1, \ldots, x_n]/(f_1, \ldots, f_\ell)$.

By the above construction, a morphism of separated $R$-schemes of finite type $f : X \to Y$ induces a morphism of formal $R$-schemes $\hat{f} : \hat{X} \to \hat{Y}$ between the formal $\pi$-adic completions of $X$ and $Y$. We get a completion functor

$$\hat{\cdot} : (\text{sft} - \text{Sch}/R) \to (\text{sft} - \text{For}/R) : X \mapsto \hat{X}$$

where $(\text{sft} - \text{Sch}/R)$ denotes the category of separated $R$-schemes of finite type, and $(\text{sft} - \text{For}/R)$ denotes the category of separated formal schemes, topologically of finite type over $R$.

For a general pair of separated $R$-schemes of finite type $X, Y$, the completion map

$$C_{X,Y} : \text{Hom}_{(\text{sft} - \text{Sch}/R)}(X,Y) \to \text{Hom}_{(\text{sft} - \text{For}/R)}(\hat{X}, \hat{Y}) : f \mapsto \hat{f}$$

is injective, but not bijective. It is a bijection, however, if $X$ is proper over $R$: this is a corollary of Grothendieck’s Existence Theorem; see [24, 5.4.1]. In particular, the completion map induces a bijection between $R'$-sections of $X$, and $R'$-sections of $\hat{X}$ (i.e. morphisms of formal $R$-schemes $\text{Spf } R' \to \hat{X}$), for any finite extension $R'$ of the complete discrete valuation ring $R$. Indeed: $\text{Spec } R'$ is a finite, hence proper $R$-scheme, and its formal $\pi$-adic completion is $\text{Spf } R'$.

Example 3. If $X = \text{Spec } B$, with $B$ an $R$-algebra of finite type, and $Y = \text{Spec } R[z]$, then

$$\text{Hom}_{(\text{sft} - \text{Sch}/R)}(X,Y) = B$$

on the other hand, if we denote by $\hat{B}$ the $\pi$-adic completion of $B$, then $\hat{X} = \text{Spf } \hat{B}$ and $\hat{Y} = \text{Spf } R[z]$, and we find

$$\text{Hom}_{(\text{sft} - \text{For}/R)}(\hat{X}, \hat{Y}) = \hat{B}$$

The completion map $C_{X,Y}$ is given by the natural injection $B \to \hat{B}$; it is not surjective in general, but it is surjective if $B$ is finite over $R$.

If $X$ is a separated $R$-scheme of finite type, and $N$ is a coherent sheaf of $\mathcal{O}_X$-modules, then $N$ induces a direct system $(N_n)_{n \geq 0}$, where $N_n$ is the pull-back of $N$ to $X_n$. This system defines a coherent sheaf of modules $\hat{N}$ on $\hat{X}$. If $X$ is proper over $R$, it follows from Grothendieck’s Existence Theorem that the functor $\mathcal{N} \to \hat{N}$ is an equivalence between the category of coherent $\mathcal{O}_X$-modules and the category of coherent $\mathcal{O}_{\hat{X}}$-modules [24, 5.1.6]. Moreover, there is a canonical isomorphism $H^q(\hat{X}, \hat{N}) \cong H^q(X, N)$ for each coherent $\mathcal{O}_X$-module $N$ and each $q \geq 0$.

If a $\text{sft}$ formal $R$-scheme $Y_\infty$ is isomorphic to the $\pi$-adic completion $\hat{Y}$ of a separated $R$-scheme $Y$ of finite type, we call the formal scheme $Y_\infty$ *algebrizable*, with algebraic model $Y$. The following theorem is the main criterion to recognize algebrizable formal schemes [24, 5.4.5]: if $Y_\infty$ is proper over $R$, and $\mathcal{L}$ is an invertible $\mathcal{O}_{Y_\infty}$-bundle such that the pull-back $\mathcal{L}_0$ of $\mathcal{L}$ to $Y_0$ is ample, then $Y_\infty$ is algebrizable. Moreover, the algebraic model $Y$ for $Y_\infty$ is unique up to canonical isomorphism, there exists a unique line bundle $\mathcal{M}$ on $Y$ with $\mathcal{L} = \mathcal{M}$, and $\mathcal{M}$ is ample. For an example of a proper formal $\mathbb{C}[[t]]$-scheme which is not algebrizable, see [28, 5.24(b)].
3.5. **Formal blow-ups.** Let $X_\infty$ be a flat stft formal $R$-scheme, and let $\mathcal{I}$ be a coherent ideal sheaf on $X_\infty$ such that $\mathcal{I}$ contains a power of the uniformizing parameter $\pi$. We can define the formal blow-up of $X_\infty$ at the center $\mathcal{I}$ as follows [13 §2]: if $X_\infty = \text{Spf } A$ is affine, and $I$ is the ideal of global sections of $\mathcal{I}$ on $X_\infty$, then the formal blow-up of $X_\infty$ at $\mathcal{I}$ is the $\pi\mathcal{O}_{\text{Spec } A}$-adic completion of the blow-up of Spec $A$ at $I$. The general case is obtained by gluing.

The formal blow-up of $X_\infty$ at $\mathcal{I}$ is again a flat stft formal $R$-scheme, and the composition of two formal blow-ups is again a formal blow-up [13 2.1+2.5]. If $X$ is a separated $R$-scheme of finite type and $\mathcal{I}$ is a coherent ideal sheaf on $X$ containing a power of $\pi$, then the formal blow-up of $\hat{X}$ at $\hat{\mathcal{I}}$ is canonically isomorphic to the $\pi$-adic completion of the blow-up of $X$ at $\mathcal{I}$.

4. **Rigid geometry**

In this note, we’ll only be able to cover the basics of rigid geometry. We refer the reader to the books [10] [20] and the research papers [8] [13] [37] [40] for a more thorough introduction. A nice survey on Tate’s approach to rigid geometry can be found in [39].

4.1. **Analytic geometry over non-Archimedean fields.** Let $L$ be a non-archimedean field (i.e. a field which is complete w.r.t. an absolute value which satisfies the ultrametric property); we assume that the absolute value on $L$ is non-trivial. For instance, if $K$ is our complete discretely valued field, then we can turn $K$ into a non-archimedean field by fixing a value $|\pi| \in ]0,1]$ and putting $|x| = |\pi|^{v(x)}$ for $x \in K^*$, where $v$ denotes the discrete valuation on $K$ (by convention, $v(0) = \infty$ and $|0| = 0$).

The absolute value on $L$ extends uniquely to any finite extension of $L$, and hence to $L^s$ and $L^{alg}$. We denote by $\hat{L}^{alg}$ the completion of $L^{alg}$, and by $\hat{L}^s$ the closure of $L^s$ in $\hat{L}^{alg}$; these are again non-archimedean fields. We denote by $L^o$ the valuation ring $\{x \in L \mid |x| \leq 1\}$, by $L^{oo}$ its maximal ideal $\{x \in L \mid |x| < 1\}$, and by $\hat{L}$ the residue field $L^o/L^{oo}$. For $L = K$ we have $L^o = R$, $L^{oo} = (\pi)$ and $\hat{L} = k$.

Since $L$ is endowed with an absolute value, one can use this structure to develop a theory of analytic varieties over $L$ by mimicking the construction over $C$. Na"ively, we can define analytic functions on open subsets of $L^n$ as $L$-valued functions which are locally defined by a converging power series with coefficients in $L$. However, we are immediately confronted with some pathological phenomena. Consider, for instance, the $p$-adic unit disc

$$Z_p = \{x \in \mathbb{Q}_p \mid |x| \leq 1\}$$

The partition

$$\{pZ_p, 1 + pZ_p, \ldots, (p - 1) + pZ_p\}$$

is an open cover of $Z_p$ w.r.t. the $p$-adic topology. Hence, the characteristic function of $pZ_p$ is analytic, according to our na"ive definition. This contradicts some elementary properties that one expects an analytic function to have. The cause of this and similar pathologies, is the fact that the unit disc $Z_p$ is totally disconnected with respect to the $p$-adic topology. In this approach, there are “too many” analytic functions, and “too few” analytic varieties (for instance, with this definition, any compact $p$-adic manifold is isomorphic to a disjoint union of $i$ unit discs, with $i \in \{0, \ldots, p - 1\}$ its Serre invariant [39]).
Rigid geometry is as a more refined approach to non-Archimedean analytic geometry, turning the unit disc into a connected space. Rigid spaces are endowed with a certain Grothendieck topology, only allowing a special type of covers.

We’ll indicate two possible approaches to the theory of rigid varieties over $L$. The first one is due to Tate [40], the second one to Raynaud [37]. If we return to our example of the $p$-adic unit disc $\mathbb{Z}_p$, Tate’s construction can be understood as follows. In fact, we already know what the “correct” algebra of analytic functions on $\mathbb{Z}_p$ should be: the power series with coefficients in $K$ which converge globally on $\mathbb{Z}_p$. Tate’s idea is to start from this algebra, and then to construct a space on which these functions naturally live. This is similar to the construction of the spectrum of a ring in algebraic geometry. Raynaud observed that a certain class of Tate’s rigid varieties can be characterized in terms of formal schemes.

4.2. Tate algebras. The basic objects in Tate’s theory are the algebras of converging power series over $L$

$$T_m = L\{x_1, \ldots, x_m\} = \{\alpha = \sum_{i \in \mathbb{N}^m} (\alpha_i \prod_{j=1}^m x_j^i) \in L[[x_1, \ldots, x_m]] | |\alpha_i| \to 0 \text{ as } |i| \to \infty\}$$

where $|i| = \sum_{j=1}^m i_j$. The convergence condition implies in particular that for each $\alpha$ there exists $i_0 \in \mathbb{N}$ such that for $|i| > i_0$ the coefficient $\alpha_i$ belongs to $L^{o}$. The algebra $T_m$ is the algebra of power series over $L$ which converge on the closed unit polydisc $(L^{o})^m$ in $L^m$ (since an infinite sum converges in a non-archimedean field iff its terms tend to zero). Note that, for $L = K$, $T_m \cong R\{x_1, \ldots, x_m\} \otimes_R K$.

Analogously, we can define an algebra of converging power series $B\{x_1, \ldots, x_m\}$ for any Banach algebra $B$. The algebra $T_m$ is a Banach algebra for the sup-norm $\|f\|_{\sup} = \max |\alpha_i|$. It is Noetherian, and any ideal $I$ is closed, so that the quotient $T_m/I$ is again a Banach algebra w.r.t. the residue norm.

A Tate algebra, or $L$-affinoid algebra, is a $L$-algebra $A$ isomorphic to such a quotient $T_m/I$. The residue norm on $A$ depends on the presentation $A \cong T_m/I$. However, any morphism of $L$-algebras $T_m/I \to T_m/J$ is automatically continuous, so in particular, the residue norm on $A$ is well-defined up to equivalence, and the induced topology on $A$ is independent of the chosen presentation. For any maximal ideal $y$ of $A$, the residue field $A/y$ is a finite extension of $L$. For proofs of all these facts, we refer to [20] 3.2.1.

By Proposition 1 of [10] 7.1.1, the maximal ideals $y$ of $T_m$ correspond bijectively to $G(L^{alg}/L)$-orbits of tuples $(z_1, \ldots, z_m)$, with $z_i \in (L^{alg})^o$, via the map

$$y \mapsto \{(\varphi(x_1), \ldots, \varphi(x_m)) | \varphi : T_m/y \hookrightarrow L^{alg}\}$$

where $\varphi$ runs through the $L$-embeddings of $T_m/y$ in $L^{alg}$. In particular, for any morphism of $L$-algebras $\psi : T_m \to L^{alg}$ and any index $i$, the element $\psi(x_i)$ belongs to $(L^{alg})^o$. It follows that $\psi$ is contractive, in the sense that $|\psi(a)| \leq \|a\|_{\sup}$ for any $a$ in $T_m$.

The fact that we obtain tuples of elements in $(L^{alg})^o$, rather than $L^{alg}$, might look strange at first; it is one of the most characteristic properties of Tate’s rigid varieties. Let us consider an elementary example. If $z$ is an element of $L$, then $x - z$ is invertible in $L\{x\}$ iff $z$ does not belong to $L^o$. Indeed, for $z \neq 0$ the coefficients of the formal power series $1/(x - z) = -(1/z) \sum_{i \geq 0} (x/z)^i$ tend to zero iff $|z| > 1$, i.e. iff $z \notin L^o$. So $(x - z)$ only defines a maximal ideal in $L\{x\}$ if $z \in L^o$. 
4.3. Affinoid spaces. The category of $L$-affinoid spaces is by definition the opposite category of the category of Tate algebras over $L$. For any $L$-affinoid space $X$, we’ll denote the corresponding Tate algebra by $A(X)$, and we call it the algebra of analytic functions on $X$. Conversely, for any Tate algebra $A$, we denote the corresponding affinoid space by $\text{Sp} A$ (some authors use the notation $\text{Spm}$ instead). For any $m \geq 0$, the affinoid space $\text{Sp} T_m$ is called the closed unit disc of dimension $m$ over $L$.

To any $L$-affinoid space $X = \text{Sp} A$, we associate the set $X^\flat$ of maximal ideals of the Tate algebra $A = A(X)$. If we present $A$ as a quotient $T_m/(f_1, \ldots, f_n)$, then elements of $(\text{Sp} A)^\flat$ correspond bijectively to $G(L^\text{alg}/L)$-orbits of tuples $z = (z_1, \ldots, z_m)$, with $z_i \in (L^\text{alg})^\flat$, and $f_j(z) = 0$ for each $j$. In particular, if $L$ is algebraically closed and $X$ is the closed unit disc $\text{Sp} T_1$, then

$$X^\flat = L^\flat = \{ x \in L \mid |x| \leq 1 \}$$

We’ve seen above that, for any maximal ideal $x$ of $A$, the quotient $A/ x$ is a finite extension of $L$, so it carries a unique prolongation of the absolute value $|\cdot|$ on $L$. Hence, for any $f \in A$ and any $x \in (\text{Sp} A)^\flat$, we can speak of the value $f(x)$ of $f$ at $x$ (the image of $f$ in $A/ x$), and its absolute value $|f(x)|$. In this way, elements of $A$ are viewed as functions on $(\text{Sp} A)^\flat$. Note that, if $x$ is a prime ideal of $A$, there is in general no canonical way to extend the absolute value on $L$ to the extension $A/ x$. This is one of the reasons for working with the maximal spectrum $(\text{Sp} A)^\flat$, rather than the prime spectrum $\text{Spec} A$. In Berkovich’ theory (Section 5), the notion of point is generalized by admitting any prime ideal $x$ and specifying an extension of the absolute value on $L$ to $A/ x$.

The spectral semi-norm on $A$ is defined by

$$\|f\|_{\text{sup}} := \sup_{x \in X^\flat} |f(x)|$$

It is a norm iff $A$ is reduced. By the maximum modulus principle [10, 6.2.1.4], this supremum is, in fact, a maximum, i.e. there is a point $x$ in $X^\flat$ with $|f(x)| = \|f\|_{\text{sup}}$. Moreover, for $A = T_m$, this definition coincides with the one in the previous section, by [10, 5.1.4.6].

If $\varphi : A \to B$ is a morphism of $L$-affinoid algebras, then for any maximal ideal $x$ in $B$, $\varphi^{-1}(x)$ is a maximal ideal in $A$, since $B/ x$ is a finite extension of $L$. Hence, any morphism of $L$-affinoid spaces $h : X \to Y$ induces a map $h^\flat : X^\flat \to Y^\flat$ on the associated sets. A morphism of $L$-affinoid spaces $h : X \to Y$ is called a closed immersion if the corresponding morphism of $L$-affinoid algebras $A(Y) \to A(X)$ is surjective.

We could try to endow $X^\flat$ with the initial topology w.r.t. the functions $x \mapsto |f(x)|$, where $f$ varies in $A$. If $L$ is algebraically closed and if we identify $(\text{Sp} L \{x\})^\flat$ with $L^\flat$, then this topology is simply the topology on $L^\flat$ defined by the absolute value. It is totally disconnected, so it does not have the nice properties we are looking for.

4.4. Open covers. A morphism $h : Y \to X$ of $L$-affinoid spaces is called an open immersion, if it satisfies the following universal property: for any morphism $g : Z \to X$ of $L$-affinoid spaces such that the image of $g^\flat$ is contained in the image of $h^\flat$ in $X^\flat$, there is a unique morphism $g' : Z \to Y$ such that $g = h \circ g'$. If $h$ is an open immersion, the image $D$ of $h^\flat$ in $X^\flat$ is called an affinoid domain. One can show that the map $h^\flat$ is always injective [10, 7.2.2.1], so it identifies the set $D$
with \( Y^\circ \). The \( L \)-affinoid space \( Y \) and the open immersion \( h : Y \to X \) are uniquely determined by the affinoid domain \( D \), up to canonical isomorphism. With slight abuse of notation, we will identify the affinoid domain \( D \) with the \( L \)-affinoid space \( Y \), so that we can think of an affinoid domain as an affinoid space sitting inside \( X \), and we can speak of the Tate algebra \( A(D) \) of analytic functions on \( D \). If \( E \) is a subset of \( D \), then \( E \) is an affinoid domain in \( D \) if and only if it is an affinoid domain in \( X \). In this case, the universal property yields a restriction map \( A(D) \to A(E) \). The intersection of two affinoid domains is again an affinoid domain, but this does not always hold for their union. If \( h : Y \to X \) is a morphism of \( L \)-affinoid spaces, then the inverse image of an affinoid domain in \( X \) is an affinoid domain in \( Y \).

**Example 4.** Consider the closed unit disc \( X = \text{Sp} L\{x\} \). For \( a \in L^o \) and \( r \) in the value group \( |L^*| \), we denote by \( D(a, r) \) the “closed disc” \( \{ z \in X^o \mid |x(z) - a| \leq r \} \), and by \( D^{-}(a, r) \) the “open disc” \( \{ z \in X^o \mid |x(z) - a| < r \} \). We will see below that the disc \( D(a, r) \) is an affinoid domain in \( X \), with \( A(D(a, r)) = L(x, T)/r - \rho T \), where \( \rho \) is any element of \( L \) with \( |\rho| = r \). On the other hand, the disc \( D^{\circ}(a, r) \) can not be an affinoid domain in \( X \), since \( \rho(r) = a \) does not reach its maximum on \( D^\circ(a, r) \).

Assume now that \( L \) is algebraically closed. By Theorem 2 in [10, 9.7.2] the affinoid domains in \( X \) are the finite disjoint unions of subsets of the form

\[
D(a_0, r_0) \setminus \bigcup_{i=1}^q D^{-}(a_i, r_i)
\]

with \( a_i \in L^o \) and \( r_i \in [L^o\cap]0, 1[ \) for \( i = 0, \ldots, q \).

An affinoid cover of \( X \) is a finite set of open immersions \( u_i : U_i \to X \) such that the images of the maps \( u_i \) cover \( X^\circ \). A special kind of affinoid covers is constructed as follows: take analytic functions \( f_1, \ldots, f_n \) in \( A(X) \), and suppose that these elements generate the unit ideal \( A(X) \). Consider, for each \( i = 1, \ldots, n \), the \( L \)-affinoid space \( U_i \) given by

\[
A(U_i) = A(X)\{T_1, \ldots, T_n\}/(f_j - T_jf_i)_{j=1, \ldots, n}
\]

The obvious morphism \( u_i : U_i \to X \) is an open immersion, and \( U_i \) is called a rational subspace of \( X \). The image of \( (u_i)^\circ \) is the set of points \( x \) of \( X^\circ \) such that \( |f_j(x)| \geq |f_j(x)| \) for \( j = 1, \ldots, n \). Indeed: using the fact that a morphism of \( L \)-algebras \( T_n \to L^{\text{alg}} \) is contractive (Section 4.2), and the assumption that \( f_1, \ldots, f_n \) and \( f_j - T_jf_i \) generate \( A(X) \), one shows that a morphism of \( L \)-algebras \( \phi : A(X) \to L^{\text{alg}} \) factors through a morphism of \( L \)-algebras \( \psi_i : A(U_i) \to L^{\text{alg}} \) if and only if \( \psi(f_j) = 0 \) and \( \psi(T_j) = \psi(f_j)/\psi(f_i) \) belongs to \( (L^{\text{alg}})^o \), i.e. \( |\psi(f_j)| \leq |\psi(f_i)| \). In this case, \( \psi_i \) is unique.

The set of morphisms \( \{u_1, \ldots, u_n\} \) is an affinoid cover, and is called a standard cover. It is a deep result that any affinoid domain of \( X \) is a finite union of rational subsets of \( X \), and any affinoid cover of \( X \) can be refined by a standard cover [10, 7.3.5.3+8.2.2.2].

One of the cornerstones in the theory of rigid varieties is Tate’s Acyclicity Theorem [10, 8.2.1.1]. It states that analytic functions on any affinoid cover \( \{u_i : U_i \to X\}_{i \in I} \) satisfy the gluing property: the sequence

\[
A(X) \to \prod_{i \in I} A(U_i) \Rightarrow \prod_{(i, j) \in I^2} A(U_i \cap U_j)
\]

is exact.
Now we can define, for each $L$-affinoid space $X$, a topology on the associated set $X^\flat$. It will not be a topology in the classical sense, but a Grothendieck topology, a generalization of the topological concept in the framework of categories. A Grothendieck topology specifies a class of opens (admissible opens) and, for each admissible open, a class of covers (admissible covers). These have to satisfy certain axioms which allow to develop a theory of sheaves and cohomology in this setting. A space with a Grothendieck topology is called a site. Any topological space (in the classical sense) can be viewed as a site in a canonical way: the admissible opens and admissible covers are the open subsets and the open covers. For our purposes, we do not need the notion of Grothendieck topology in its most abstract and general form: a sufficient treatment is given in [10, 9.1.1].

The weak $G$-topology on an $L$-affinoid space $X$ is defined as follows: the admissible open sets of $X^\flat$ are the affinoid domains, and the admissible covers are the affinoid covers [10, 9.1.4]. Any morphism $h$ of $L$-affinoid spaces is continuous w.r.t. the weak $G$-topology (meaning that the inverse image under $h^\flat$ of an admissible open is again an admissible open, and the inverse image of an admissible cover is again an admissible cover). We can define a presheaf of $L$-algebras $O_X$ on $X^\flat$ with respect to this topology, by putting $O_X(D) = A(D)$ for any affinoid domain $D$ of $X$ (with the natural restriction maps). By Tate’s Acyclicity Theorem, $O_X$ is a sheaf. Note that the exact definition of the weak $G$-topology varies in literature: sometimes the admissible opens are taken to be the finite unions of rational subsets in $X$, and the admissible covers are the covers by admissible opens with a finite subcover (e.g. in [20, §4.2]).

In the theory of Grothendieck topologies, there is a canonical way to refine the topology without changing the associated category of sheaves [10, 9.1.2]. This refinement is important to get good gluing properties for affinoid spaces, and to obtain continuity of the analytification map (Section 4.6). This leads to the following definition of the strong $G$-topology on a $L$-affinoid space $X$.

- The admissible open sets are (possibly infinite) unions $\cup_{i \in I} D_i$ of affinoid domains $D_i$ in $X$, such that, for any morphism of $L$-affinoid spaces $h : Y \to X$, the image of $h^\flat$ in $X^\flat$ is covered by a finite number of $D_i$.
- An admissible cover of an admissible open subset $V \subset X^\flat$ is a (possibly infinite) set of admissible opens $\{V_j \mid j \in J\}$ in $X^\flat$ such that $V = \cup_j V_j$, and such that, for any morphism of $L$-affinoid spaces $\varphi : Y \to X$ with $\text{Im}(\varphi^\flat) \subset V$, the cover $\{\varphi^\flat)^{-1}(V_j) \mid j \in J\}$ of $Y$ can be refined by an affinoid cover.

Any morphism of $L$-affinoid spaces is continuous w.r.t. the strong $G$-topology. The strong $G$-topology on $X = \text{Sp} A$ is finer than the Zariski topology on the maximal spectrum of $A$ (this does not hold for the weak $G$-topology). From now on, we’ll endow all $L$-affinoid spaces $X$ with the strong $G$-topology. The structure sheaf $O_X$ of $X$ extends uniquely to a sheaf of $L$-algebras w.r.t. the strong $G$-topology, which is called the sheaf of analytic functions on $X$. One can show that its stalks are local rings. In this way, we associate to any $L$-affinoid space $X$ a locally ringed site in $L$-algebras $(X^\flat, O_X)$.

For any morphism of $L$-affinoid spaces $h : Y \to X$, there is a morphism of sheaves of $L$-algebras $O_X \to (h^\flat)_* O_Y$ which defines a morphism of locally ringed spaces $(Y^\flat, O_Y) \to (X^\flat, O_X)$ (if $D = \text{Sp} A$ is an affinoid domain in $X$, then $(h^\flat)^{-1}(D)$ is an affinoid domain $\text{Sp} B$ in $Y$ and there is a natural morphism of $L$-algebras
A \to B$ by the universal property defining affinoid domains). This construction defines a functor from the category of $L$-affinoid spaces to the category of locally ringed spaces in $L$-algebras, and this functor is fully faithful \[10\] 9.3.1.1, i.e. every morphism of locally ringed sites in $L$-algebras $((\text{Sp} \, B), \mathcal{O}_{\text{Sp} \, B}) \to ((\text{Sp} \, A), \mathcal{O}_{\text{Sp} \, A})$ is induced by a morphism of $L$-algebras $A \to B$. With slight abuse of notation, we will call the objects in its essential image also $L$-affinoid spaces, and we’ll identify an $L$-affinoid space $X$ with its associated locally ringed site in $L$-algebras $(X^{\mathbb{D}}, \mathcal{O}_X)$.

If $D$ is an affinoid domain in $X$, then the strong $G$-topology on $X$ restricts to the strong $G$-topology on $D$, and the restriction of $\mathcal{O}_X$ to $D$ is the sheaf of analytic functions $\mathcal{O}_D$.

One can check that the affinoid space $\text{Sp} \, T_m$ is connected with respect to the strong and the weak $G$-topology, for any $m \geq 0$. More generally, connectedness of an $L$-affinoid space $X = \text{Sp} \, A$ is equivalent for the weak $G$-topology, the strong $G$-topology, and the Zariski topology \[10\] 9.1.4, Prop. 8, and it is also equivalent to the property that the ring $A$ has non-trivial idempotents; so the $G$-topologies nicely reflect the algebraic structure of $A$.

**Example 5.** Let $X$ be the closed unit disc $\text{Sp} \, L\{x\}$. The set

$$U_1 = \{z \in X \mid |x(z)| = 1\} = \{z \in X \mid |x(z)| \geq 1\}$$

is a rational domain in $X$, so it is an admissible open already for the weak $G$-topology. The algebra of analytic functions on $U_1$ is given by

$$\mathcal{O}_X(U_1) = L\{x,T\}/(xT - 1)$$

The set

$$U_2 = \{z \in X \mid |x(z)| < 1\}$$

is not an admissible open for the weak $G$-topology (it cannot be affinoid since the function $|x(.)|$ does not reach a maximum on $U_2$), but it is an admissible open for the strong $G$-topology: we can write it as an infinite union of rational domains

$$U_2^{(n)} = \{z \in X \mid |x(z)|^n \leq |a|\}$$

where $n$ runs through $\mathbb{N}^*$ and $a$ is any non-zero element of $L^{\mathbb{D}}$.

This family satisfies the finiteness condition in the definition of the strong $G$-topology: if $Y \to X$ is any morphism of $L$-affinoid spaces whose image is contained in $U_2$, then by the maximum principle (Section 4.3) the pull-back of the function $|x(.)|$ to $Y$ reaches its maximum on $Y$, so the image of $Y$ is contained in $U_2^{(n)}$ for $n$ sufficiently large.

The algebra $\mathcal{O}_X(U_2)$ of analytic functions on $U_2$ consists of the elements $\sum_{i \geq 0} a_i x^i$ of $L[[x]]$ such that $|a_i| r^i$ tends to zero as $i \to \infty$, for any $r \in ]0,1[$.

Hence, we can write $X$ as a disjoint union $U_1 \sqcup U_2$ of admissible opens. This does not contradict the fact that $X$ is connected, because $\{U_1, U_2\}$ is not an admissible cover, since it cannot be refined by a (finite!) affinoid cover.

**4.5. Rigid varieties.** Now, we can give the definition of a general rigid variety over $L$. It is a set $X$, endowed with a Grothendieck topology and a sheaf of $L$-algebras $\mathcal{O}_X$, such that $X$ has an admissible cover $\{U_i\}_{i \in I}$ with the property that each locally ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an $L$-affinoid space. An admissible open

---

1 To be precise, this Grothendieck topology should satisfy certain additional axioms; see \[10\] 9.3.1.4.
U in X is called an affinoid domain in X if (U, O_X|U) is isomorphic to an L-affinoid space. If X is affinoid, this definition is compatible with the previous one. A morphism Y → X of rigid varieties over L is a morphism of locally ringed spaces in L-algebras.

A rigid variety over L is called quasi-compact if it is a finite union of affinoid domains. It is called quasi-separated if the intersection of any pair of affinoid domains is quasi-compact, and separated if the diagonal morphism is a closed immersion.

4.6. Analytification of a L-variety. For any L-scheme X of finite type, we can endow the set X^o of closed points of X with the structure of a rigid L-variety.

More precisely, by [8, 0.3.3] and [30, 5.3] there exists a functor ( . )^an : (ft−Sch/L) → (Rig/L) from the category of L-schemes of finite type, to the category of rigid L-varieties, such that

(1) For any L-scheme of finite type X, there exists a natural morphism of locally ringed sites

\[ i : X^an \rightarrow X \]

which induces a bijection between the underlying set of X^an and the set X^o of closed points of X. The couple (X^an, i) satisfies the following universal property: for any rigid variety Z over L and any morphism of locally ringed sites j : Z → X, there exists a unique morphism of rigid varieties j' : Z → X^an such that j = i ∘ j'.

(2) If f : X' → X is a morphism of L-schemes of finite type, the square

\[
\begin{array}{ccc}
(X')^an & \xrightarrow{f^an} & X^an \\
\downarrow i & & \downarrow i' \\
X' & \xrightarrow{f} & X
\end{array}
\]

commutes.

(3) The functor ( . )^an commutes with fibered products, and takes open (resp. closed) immersions of L-schemes to open (resp. closed) immersion of rigid L-varieties. In particular, X^an is separated if X is separated.

We call X^an the analytification of X. It is quasi-compact if X is proper over L, but not in general. The analytification functor has the classical GAGA properties: if X is proper over L, then analytification induces an equivalence between coherent O_X-modules and coherent O_X^an-modules, and the cohomology groups agree; a closed rigid subvariety of X^an is the analytification of an algebraic subvariety of X; and for any L-variety Y, all morphisms X^an → Y^an are algebraic. These results can be deduced from Grothendieck’s Existence Theorem; see [31, 2.8].

Example 6. Let D be the closed unit disc Sp L{x}, and consider the endomorphism σ of D mapping x to a · x, for some non-zero a ∈ L^an. Then σ is an isomorphism from D onto the affinoid domain D(0, |a|) in D (notation as in Example 4). The rigid affine line (A^1_L)^an is the limit of the direct system

\[ D \xrightarrow{\sigma} D \xrightarrow{\sigma} \ldots \]

in the category of locally ringed sites in L-algebras. Intuitively, it is obtained as the union of an infinite number of concentric closed discs whose radii tend to ∞.
4.7. Rigid spaces and formal schemes. Finally, we come to a second approach to the theory of rigid spaces, due to Raynaud [37]. We will only deal with the case where \( L = K \) is a complete discretely valued field, but the theory is valid in greater generality (see [13]).

We've seen before that the underlying topological space of a stft formal \( R \)-scheme \( X_\infty \) coincides with the underlying space of its special fiber \( X_0 \). Nevertheless, the structure sheaf of \( X_\infty \) contains information on an infinitesimal neighbourhood of \( X_0 \), so one might try to construct the generic fiber \( X_\eta \) of \( X_\infty \). As it turns out, this is indeed possible, but we have to leave the category of (formal) schemes: this generic fiber \( X_\eta \) is a rigid variety over \( K \).

4.8. The affine case. Let \( A \) be an algebra topologically of finite type over \( R \), and consider the affine formal scheme \( X_\infty = \text{Spf} A \). The tensor product \( A \otimes_R K \) is a \( K \)-affinoid algebra, and the generic fiber \( X_\eta \) of \( X_\infty \) is simply the \( K \)-affinoid space \( \text{Sp} A \otimes_R K \).

Let \( K' \) be any finite extension of \( K \), and denote by \( R' \) the normalization of \( R \) in \( K' \). There exists a canonical bijection between the set of morphisms of formal \( R \)-schemes \( \text{Spf} R' \rightarrow X_\infty \), and the set of morphisms of rigid \( K \)-varieties \( \text{Sp} K' \rightarrow X_\eta \). Consider a morphism of formal \( R \)-schemes \( \text{Spf} R' \rightarrow X_\infty \), or, equivalently, a morphism of \( R \)-algebras \( A \rightarrow R' \). Tensoring with \( K \) yields a morphism of \( K \)-algebras \( A \otimes_R K \rightarrow R' \otimes_R K \cong K' \), and hence a \( K' \)-point of \( X_\eta \). Conversely, for any morphism of \( K \)-algebras \( A \otimes_R K \rightarrow K' \), the image of \( A \) will be contained in \( R' \), since we've already seen in Section [12] that the image of \( R\{x_1, \ldots, x_m\} \) under any morphism of \( K \)-algebras \( T_m \rightarrow K^{\text{alg}} \) is contained in the normalization \( R^{\text{alg}} \) of \( R \) in \( K^{\text{alg}} \).

To any \( R' \)-section on \( X_\infty \), we can associate a point of \( X_\infty \), namely the image of the singleton \( |\text{Spf} R'| \). In this way, we obtain a specialization map of sets

\[
sp : |X_\eta| \rightarrow |X_\infty| = |X_0|
\]

4.9. The general case. The construction of the generic fiber for general stft formal \( R \)-schemes \( X_\infty \) is obtained by gluing the constructions on affine charts. The important point here is that the specialization map \( sp \) is continuous: if \( X_\infty = \text{Spf} A \) is affine, then for any open formal subscheme \( U_\infty \) of \( X_\infty \), the inverse image \( sp^{-1}(U_\infty) \) is an admissible open in \( X_\eta \); in fact, if \( U_\infty = \text{Spf} B \) is affine, then \( sp^{-1}(U_\infty) \) is an affinoid domain in \( X_\eta \), canonically isomorphic to \( U_\eta = \text{Sp} B \otimes_R K \).

Hence, the generic fibers of the members \( U_\infty^{(j)} \) of an affine open cover of a stft formal \( R \)-scheme \( X_\infty \) can be glued along the generic fibers of the intersections \( U_\infty^{(j)} \cap U_\infty^{(j')} \) to obtain a rigid \( K \)-variety \( X_\infty \), and the specialization maps glue to a continuous map

\[
sp : |X_\eta| \rightarrow |X_\infty| = |X_0|
\]

This map can be enhanced to a morphism of ringed sites by considering the unique morphism of sheaves

\[
sp^* : \mathcal{O}_{X_\infty} \rightarrow sp_* \mathcal{O}_{X_\eta}
\]

which is given by the natural map

\[
\mathcal{O}_{X_\infty}(U_\infty) = A \rightarrow A \otimes_R K = sp_*(\mathcal{O}_{X_\eta}(U_\infty))
\]

on any affine open formal subscheme \( U_\infty = \text{Spf} A \) of \( X_\infty \).

The generic fiber of a stft formal \( R \)-scheme is a separated, quasi-compact rigid \( K \)-variety. The formal scheme \( X_\infty \) is called a formal \( R \)-model for the rigid \( K \)-variety.
$X_\eta$. Since the generic fiber is obtained by inverting $\pi$, it is clear that the generic fiber does not change if we replace $X_\infty$ by its maximal flat closed formal subscheme (by killing $\pi$-torsion). If $K'$ is a finite extension of $K$ and $R'$ the normalization of $R$ in $K'$, then we still have a canonical bijection $X_\infty(R') = X_\eta(K')$.

The construction of the generic fiber is functorial: a morphism of stft formal $R$-schemes $h : Y_\infty \to X_\infty$ induces a morphism of rigid $K$-varieties $h_\eta : Y_\eta \to X_\eta$, and the square

$$
\begin{array}{ccc}
Y_\eta & \xrightarrow{h_\eta} & X_\eta \\
\downarrow & & \downarrow \\
Y_\infty & \xrightarrow{h} & X_\infty
\end{array}
$$

commutes. We get a functor

$$(\cdot)_\eta : (\text{stft} - \text{For}/R) \to (\text{sqc} - \text{Rig}/K) : X_\infty \mapsto X_\eta$$

from the category of stft formal $R$-schemes, to the category of separated, quasi-compact rigid $K$-varieties.

For any locally closed subset $Z$ of $X_0$, the inverse image $sp^{-1}(Z)$ is an admissible open in $X_\infty$, called the tube of $Z$ in $X_\infty$, and denoted by $|Z|$. If $Z$ is open in $X_0$, then $|Z|$ is canonically isomorphic to the generic fiber of the open formal subscheme $Z_\infty = (|Z|, \mathcal{O}_{X_\infty}|_Z)$ of $X_\infty$. The tube $|Z|$ is quasi-compact if $Z$ is open, but not in general.

Berthelot showed in [8, 0.2.6] how to construct the generic fiber of a broader class of formal $R$-schemes, not necessarily tft. If $Z$ is closed in $X_0$, then $|Z|$ is canonically isomorphic to the generic fiber of the formal completion of $X_\infty$ along $Z$ (this formal completion is the locally topologically ringed space with underlying topological space $|Z|$ and structure sheaf $\varprojlim_{\infty} \mathcal{O}_{X_\infty}/\mathcal{I}_Z^n$ where $\mathcal{I}_Z$ is the defining ideal sheaf of $Z$ in $X_\infty$). In particular, if $Z$ is a closed point $x$ of $X_0$, then $|x|$ is the generic fiber of the formal spectrum of the completed local ring $\hat{\mathcal{O}}_{X_\infty,x}$ with its adic topology (we did not define this notion; see [23, 10.1]).

**Example 7.** Let $X_\infty$ be an affine stft formal $R$-scheme, say $X_\infty = \text{Spf} A$. Consider the residue classes $f_1, \ldots, f_r$ in $A_0$ and denote by $Z$ the closed subscheme of $X_0$ defined by the residue classes $\overline{f}_1, \ldots, \overline{f}_r$ in $A_0$. The tube $|Z|$ of $Z$ in $X_\infty$ consists of the points $x$ of $X_\eta$ with $|f_i(x)| < 1$ for $i = 1, \ldots, r$ (since this condition is equivalent to $f_i(x) \equiv 0$ mod $(K^{alg})^{\infty}$).

If $X_\infty = \text{Spf} R\{x\}$, then $X_\eta$ is the closed unit disc $\text{Sp} K\{x\}$, and the special fiber $X_0$ is the affine line $\mathbb{A}_K^1$. If we denote by $O$ the origin in $X_0$ and by $V$ its complement, then $|V|$ is the affinoid domain $U_1 = \text{Sp} K\{x, T\}/(xT - 1)$

from Example 4 (the “boundary” of the closed unit disc), and $|O|$ is the open unit disc $U_2$ from the same example. The first one is quasi-compact, the second is not.
4.10. **Localization by formal blow-ups.** The functor \( (\_ )_\eta \) is not an equivalence. One can show that formal blow-ups are turned into isomorphisms [13 4.1]. Intuitively, this is clear: the center \( I \) of a formal blow-up contains a power of \( \pi \), so it becomes the unit ideal after inverting \( \pi \).

In some sense, this is the only obstruction. Denote by \( \mathcal{C} \) the category of flat \( stft \) formal \( R \)-schemes, localized with respect to the formal blow-ups. This means that we artificially add inverse morphisms for formal blow-ups, thus turning them into isomorphisms. The objects of \( \mathcal{C} \) are simply the flat \( stft \) formal \( R \)-schemes, but a morphism in \( \mathcal{C} \) from \( Y_\eta \) to \( X_\eta \) is given by a triple \((Y'_\eta, \varphi_1, \varphi_2)\) where \( \varphi_1 : Y'_\eta \to Y_\eta \) is a formal blow-up, and \( \varphi_2 : Y'_\eta \to X_\eta \) a morphism of \( stft \) formal \( R \)-schemes.

We identify this triple with another triple \((Y''_\eta, \psi_1, \psi_2)\) if there exist a third triple \((Z_\eta, \chi_1, \chi_2)\) and morphisms of \( stft \) formal \( R \)-schemes \( Z_\eta \to Y'_\eta \) and \( Z_\eta \to Y''_\eta \) such that the obvious triangles commute.

Since admissible blow-ups are turned into isomorphisms by the functor \( (\_ )_\eta \), it factors though a functor \( \mathcal{C} \to (sqc - \text{Rig}/K) \). Raynaud [37] showed that this is an equivalence of categories (a detailed proof is given in [13]). This means that the category of separated, quasi-compact rigid \( K \)-varieties, can be described entirely in terms of formal schemes. To give an idea of this dictionary between formal schemes and rigid varieties, we list some results. Let \( X \) be a separated, quasi-compact rigid variety over \( K \).

- **[13 4.1(e),4.7]** There exists a flat \( stft \) formal \( R \)-scheme \( X_\infty \) such that \( X \) is isomorphic to \( X_\eta \).
- **[13 4.1(c+d)]** If \( X_\eta \) and \( Y_\eta \) are \( stft \) formal \( R \)-schemes and \( \varphi : Y_\eta \to X_\eta \) is a morphism of rigid \( K \)-varieties, then in general, \( \varphi \) will not extend to a morphism \( Y_\infty \to X_\infty \) on the \( R \)-models. However, by Raynaud’s result, there exist a formal blow-up \( f : Y'_\infty \to Y_\infty \) and a morphism of \( stft \) formal \( R \)-schemes \( g : Y'_\infty \to X_\infty \), such that \( \varphi = g_\eta \circ (f_\eta)^{-1} \). If \( \varphi \) is an isomorphism, we can find \((Y'_\infty, f, g)\) with both \( f \) and \( g \) formal blow-ups.
- **[13 4.4]** For any affinoid cover \( \mathcal{U} \) of \( X \), there exist a formal model \( X_\infty \) of \( X \) and a Zariski cover \( \{U_1, \ldots, U_s\} \) of \( X_\eta \) such that \( \mathcal{U} = \{U_1[\ldots]\} \).

See [13 14 15 16] for many other results.

**Example 8.** Consider the \( stft \) formal \( R \)-schemes

\[
X_\infty = \text{Spf} \ R\{x\}/(x^2 - 1) \\
Y_\infty = \text{Spf} \ R\{x\}/(x^2 - \pi^2)
\]

The generic fibers \( Y_\eta \) and \( X_\eta \) are isomorphic (both consist of two points \( \text{Sp} \ K \)) but it is clear that there is no morphism of \( stft \) formal \( R \)-schemes \( Y_\infty \to X_\infty \) which induces an isomorphism between the generic fibers. The problem is that the section \( x/\pi \) is not defined on \( Y_\infty \); however, blowing up the ideal \( (x, \pi) \) adds this section to the ring of regular functions, and the formal blow-up scheme is isomorphic to \( X_\infty \).

Next, consider the \( stft \) formal \( R \)-scheme \( Z_\infty = \text{Spf} \ R\{x\} \), and the standard cover of \( Z_\eta \) defined by the couple \((x, \pi)\). The cover consists of the closed disc \( D(0, |\pi|) \) and the closed annulus \( Z_\eta \setminus D^{-}(0, |\pi|) \) (notation as in Example 4). These sets are not tubes in \( Z_\infty \), since by Example 4 both sets have non-empty intersection with the tube \( |O| \) (but do not coincide with it). But if we take the formal blow-up \( Z'_\infty \to Z_\infty \) at the ideal \( (x, \pi) \), then the rational subsets in our standard cover are precisely the generic fibers of the blow-up charts \( \text{Spf} \ R\{x, T\}/(xT - \pi) \) and \( \text{Spf} \ R\{x, T\}/(x - \pi T) \).
4.11. **Proper $R$-varieties.** Now let $X$ be a separated scheme of finite type over $R$, and denote by $X_K$ its generic fiber. We denote by $(X_K)^o$ the set of closed points of $X_K$. By [8, 0.3.5], there exists a canonical open immersion $\alpha : (\hat{X})_{\eta} \to (X_K)^{an}$. If $X$ is proper over $R$, then $\alpha$ is an isomorphism.

For a proper $R$-scheme of finite type $R$, we can describe the specialization map 

$$sp : (X_K)^o = |(X_K)^{an}| = |\hat{X}_{\eta}| \to |\hat{X}| = |X_0|$$

as follows: let $x$ be a closed point of $X_K$, denote by $K'$ its residue field, and by $R'$ the normalization of $R$ in $K'$. The point $x$ defines a morphism $x : Spec K' \to X$. The valuative criterion for properness guarantees that the morphism $Spec R' \to Spec R$ lifts to a unique morphism $h : Spec R' \to X$ with $h_K = x$. If we denote by 0 the closed point of $Spec R'$, then $sp(x) = h(0) \in |X_0|$.

In general, the open immersion $\alpha : \hat{X}_{\eta} \to (X_K)^{an}$ is strict. Consider, for instance, a proper $R$-variety $X$, and let $X'$ be the variety obtained by removing a closed point $x$ from the special fiber $X_0$. Then $X'_K = X_K$; however, by taking the formal completion $\hat{X'}$, we lose all the points in $\hat{X}_{\eta}$ that map to $x$ under $sp$, i.e. $\hat{X}_{\eta} = \hat{X}_{\eta}\backslash\{x\}$. We'll see an explicit example in the following section. This is another instance of the fact that the rigid generic fiber $\hat{X}_{\eta}$ is “closer” to the special fiber than the scheme-wise generic fiber $X'_K$.

4.12. **Example: the projective line.** Let $X$ be the affine line $Spec R[x]$ over $R$; then $X_K = Spec K[x]$, and $(X_K)^{an}$ is the rigid affine line $(h_K)^{an}$ from Example 4.6 On the other hand, $\hat{X} = Spf R\{x\}$ and $\hat{X}_{\eta}$ is the closed unit disc $Sp K\{x\}$.

The canonical open immersion $\hat{X}_{\eta} \to (X_K)^{an}$ is an isomorphism onto the affinoid domain in $(X_K)^{an}$ consisting of the points $z$ with $|x(z)| \leq 1$.

If we remove the origin $O$ from $X$, we get a scheme $X'$ with $X'_K = X_K$. However, the formal completion of $X'$ is

$$\hat{X'} = Spf R\{x,T\}/(xT - 1)$$

and its generic fiber is the complement of $|O|$ in $\hat{X}_{\eta}$ (see Example 4.7).

Now let us turn to the projective line $\mathbb{P}^1_R = Proj R[x,y]$. The analytic projective line $(\mathbb{P}^1_K)^{an}$ can be realized in different ways. First, consider the usual affine cover of $\mathbb{P}^1_K$ by the charts $U_1 = Spec K[x/y]$ and $U_2 = Spec K[y/x]$. Their analytifications $(U_1)^{an}$ and $(U_2)^{an}$ are infinite unions of closed discs (see Example 4.6) centered at 0, resp. $\infty$. Gluing along the admissible opens $(U_1)^{an} - \{0\}$ and $(U_2)^{an} - \{\infty\}$ in the obvious way, we obtain $(\mathbb{P}^1_K)^{an}$.

On the other hand, we can look at the formal completion $\hat{\mathbb{P}}^1_R$ by the results in Section 4.11 we know that its generic fiber is canonically isomorphic to $(\mathbb{P}^1_K)^{an}$. The stft formal $R$-scheme $\hat{\mathbb{P}}^1_R$ is covered by the affine charts $V_1 = Spf R\{x/y\}$ and $V_2 = Spf R\{y/x\}$ whose intersection is given by

$$V_0 = Spf R\{x/y, y/x\}/((x/y)(y/x) - 1)$$

We’ve seen in Example 4.7 that the generic fibers of $V_1$ and $V_2$ are closed unit discs around $x/y = 0$, resp. $y/x = 0$, and that $(V_0)^{an}$ coincides with their boundaries. So in this way, $(\mathbb{P}^1_K)^{an}$ is realized as the Riemann sphere obtained by gluing two closed unit discs along their boundaries.
5. Berkovich spaces

We recall some definitions from Berkovich’ theory of analytic spaces over non-archimedean fields. We refer to [2], or to [6] for a short introduction. A very nice survey of the theory and some of its applications is given in [19].

For a commutative Banach ring with unity \((\mathcal{A}, \| \cdot \|)\), the spectrum \(\mathcal{H}(\mathcal{A})\) is the set of all bounded multiplicative semi-norms \(x : \mathcal{A} \to \mathbb{R}_+\) (where “bounded” means that there exists a number \(C > 0\) such that \(x(a) \leq C\|a\|\) for all \(a \in \mathcal{A}\)). If \(x\) is a point of \(\mathcal{H}(\mathcal{A})\), then \(x^{-1}(0)\) is a prime ideal of \(\mathcal{A}\), and \(x\) descends to an absolute value \(|\cdot|\) on the quotient field of \(\mathcal{A}/x^{-1}(0)\). The completion of this field is called the residue field of \(x\), and denoted by \(\mathcal{H}(x)\). Hence, any point \(x\) of \(\mathcal{H}(\mathcal{A})\) gives rise to a bounded ring morphism \(\chi_x\) from \(\mathcal{A}\) to the complete valued field \(\mathcal{H}(x)\), and \(x\) is completely determined by \(\chi_x\). In this way, one can characterize the points of \(\mathcal{H}(\mathcal{A})\) as equivalence classes of bounded ring morphisms from \(\mathcal{A}\) to a complete valued field \([2, 1.2.2(ii)]\), just as one can view elements of the spectrum Spec \(B\) of a commutative ring \(B\) either as prime ideals in \(B\) or as equivalence classes of ring morphisms from \(B\) to a field.

If we denote the image of \(f \in \mathcal{A}\) under \(\chi_x\) by \(f(x)\), then \(x(f) = |f(x)|\). We endow \(\mathcal{H}(\mathcal{A})\) with the weakest topology such that \(\mathcal{H}(\mathcal{A}) \to \mathbb{R} : x \mapsto |f(x)|\) is continuous for each \(f\) in \(\mathcal{A}\). This topology is called the spectral topology on \(\mathcal{H}(\mathcal{A})\). If \(\mathcal{A}\) is not the zero ring, it makes \(\mathcal{H}(\mathcal{A})\) into a non-empty compact Hausdorff topological space \([2, 1.2.1]\). A bounded morphism of Banach algebras \(\mathcal{A} \to \mathcal{B}\) induces a continuous map \(\mathcal{H}(\mathcal{B}) \to \mathcal{H}(\mathcal{A})\) between their spectra. In particular, the spectrum of \(\mathcal{A}\) only depends on the equivalence class of \(\|\cdot\|\).

If \(L\) is a non-archimedean field with non-trivial absolute value and \(A\) is an \(L\)-affinoid algebra (these are called strictly \(L\)-affinoid in Berkovich’ theory) then \(A\) carries a Banach norm, well-defined up to equivalence (Section \[1.2\]). The spectrum \(\mathcal{H}(A)\) of \(A\) is called a strictly \(L\)-affinoid analytic space; Berkovich endows these topological spaces with a structure sheaf of analytic functions. General strictly \(L\)-analytic spaces are obtained by gluing strictly \(L\)-affinoid spaces.

Any maximal ideal \(x\) of \(A\) defines a point of \(\mathcal{H}(A)\): the bounded multiplicative semi-norm sending \(f \in A\) to \(|f(x)|\). This defines a natural injection \(\text{Sp } A \to \mathcal{H}(A)\), whose image consists of the points \(y\) of \(\mathcal{H}(A)\) with \(\mathcal{H}(y) : L < \infty\). So \(\mathcal{H}(A)\) contains the “classical” rigid points of \(\text{Sp } A\), but in general also additional points \(z\) with \(z^{-1}(0)\) not a maximal ideal. Beware that the natural map

\[
\mathcal{H}(A) \to \text{Spec } A : z \mapsto z^{-1}(0)
\]

is not injective, in general: if \(P \in \text{Spec } A\) is not a maximal ideal, there may be several bounded absolute values on \(A/P\) extending the absolute value on \(L\).

For a Hausdorff strictly \(L\)-analytic space \(X\), the set of rigid points

\[
X_{\text{rig}} := \{ x \in X \mid |\mathcal{H}(x) : L| < \infty \}
\]

can be endowed with the structure of a quasi-separated rigid variety over \(L\) in a natural way. Moreover, the functor \(X \mapsto X_{\text{rig}}\) induces an equivalence between the category of paracompact strictly \(L\)-analytic spaces, and the category of quasi-separated rigid varieties over \(L\) which have an admissible affinoid covering of finite type \([3, 1.6.1]\). The space \(X_{\text{rig}}\) is quasi-compact if and only if \(X\) is compact.

The big advantage of Berkovich spaces is that they carry a “true” topology instead of a Grothendieck topology, with very nice features (Hausdorff, locally
connected by arcs, . . .). As we’ve seen, Berkovich obtains his spaces by adding points to the points of a rigid variety (not unlike the generic points in algebraic geometry) which have an interpretation in terms of valuations. We refer to [2, 1.4.4] for a description of the points and the topology of the closed unit disc \( D = \mathbb{A}(L\{x\}) \).

To give a taste of these Berkovich spaces, let us explain how two points of \( D_{\text{rig}} \) can be joined by a path in \( D \). We assume, for simplicity, that \( L \) is algebraically closed. For each point \( a \) of \( D_{\text{rig}} \) and each \( \rho \in [0, 1] \) we define \( D(a, \rho) \) as the set of points \( z \) in \( D_{\text{rig}} \) with \( |x(z) - x(a)| \leq \rho \). This is not an affinoid domain if \( \rho \notin [L^\ast] \).

Any such disc \( E = D(a, \rho) \) defines a bounded multiplicative semi-norm \( ||f||_E \) on the Banach algebra \( L\{x\} \), by mapping \( f = \sum_{n=0}^{\infty} a_n(T-a)^n \) to

\[ |f|_E = \sup_{z \in E} |f(z)| = \max_n |a_n| \rho^n \]

and hence, \( E \) defines a Berkovich-point of \( D \). Now a path between two points \( a, b \) of \( D_{\text{rig}} = L^\circ \) can be constructed as follows: put \( \delta = |x(a) - x(b)| \) and consider the path

\[ \gamma : [0, 1] \to D : t \mapsto \begin{cases} D(a, 2t\delta), & \text{if } 0 \leq t \leq 1/2, \\ D(b, 2(1-t)\delta), & \text{if } 1/2 \leq t \leq 1. \end{cases} \]

Geometrically, this path can be seen as a closed disc around \( a \), growing continuously in time \( t \) until it contains \( b \), and then shrinking to \( b \).

A remarkable feature of Berkovich’ theory is that it can also be applied to the case where \( L \) carries the trivial absolute value. If \( k \) is any field, and \( X \) is an algebraic variety over \( k \), then we can endow \( k \) with the trivial absolute value and consider the Berkovich analytic space \( X^{an} \) associated to \( X \) over \( k \) [2, 3.5]. Surprisingly, the topology of \( X^{an} \) contains non-trivial information on \( X \). For instance, if \( k = \mathbb{C} \), then the rational singular cohomology \( H_{\text{sing}}(X^{an}, \mathbb{Q}) \) of \( X^{an} \) is canonically isomorphic to the weight zero part of the rational singular cohomology of the complex analytic space \( X(\mathbb{C}) \) [7, 1.1(c)]. We refer to [39] and [41] for other applications of analytic spaces w.r.t. trivial absolute values.

Let us mention that there are still alternative approaches to non-archimedean geometry, such as Fujiwara and Kato’s Zariski-Riemann spaces [21], or Huber’s adic spaces [27]. See [43] for a (partial) comparison.

6. Some applications

6.1. Relation to arc schemes and the Milnor fibration.

6.1.1. Arc spaces. Let \( k \) be any field, and let \( X \) be a separated scheme of finite type over \( k \). Put \( R = k[[t]] \). For each \( n \geq 1 \), we define a functor

\[ F_n : (k - \text{alg}) \to (\text{Sets}) : A \to X(A \otimes_k R_n) \]

from the category of \( k \)-algebras to the category of sets. It is representable by a separated \( k \)-scheme of finite type \( \mathcal{L}_n(X) \) (this is nothing but the Weil restriction of \( X \times_k R_n \) to \( k \)). For any pair of integers \( m \geq n \geq 0 \), the truncation map \( R_m \to R_n \) induces by Yoneda’s Lemma a morphism of \( k \)-schemes

\[ \pi^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X) \]

It is easily seen that these morphisms are affine, and hence, we can consider the projective limit

\[ \mathcal{L}(X) := \varprojlim_n \mathcal{L}_n(X) \]
in the category of $k$-schemes. This scheme is called the arc scheme of $X$. It satisfies $\mathcal{L}(X)(k') = X(k'[t])$ for any field $k'$ over $k$ (these points are called arcs on $X$), and comes with natural projections

$$\pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X)$$

In particular, we have a morphism $\pi_0 : \mathcal{L}(X) \to \mathcal{L}_0(X) = X$. For any subscheme $Z$ of $X$, we put $\mathcal{L}(X)_Z = \mathcal{L}(X) \times_X Z$. By Yoneda’s lemma, a morphism of separated $k$-schemes of finite type $h : Y \to X$ induces $k$-morphisms $h : \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$, and by passage to the limit, a $k$-morphism $h : \mathcal{L}(Y) \to \mathcal{L}(X)$.

If $X$ is smooth over $k$, the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are fairly well understood: if $X$ has pure dimension $d$, then, for each pair of integers $m \geq n \geq 0$, $\pi_m^n$ is a locally trivial fibration with fiber $A^d_k$ (w.r.t. the Zariski topology). If $x$ is a singular point of $X$, however, the scheme $\mathcal{L}(X)_x$ is still quite mysterious. It contains a lot of information on the singular germ $(X, x)$; interesting invariants can be extracted by the theory of motivic integration (see [17, 18, 44]).

The schemes $\mathcal{L}(X)_x$ and $\mathcal{L}(X)$ are not Noetherian, in general, which complicates the study of their geometric properties. Already the fact that they have only finitely many irreducible components if $k$ has characteristic zero, is a non-trivial result. We will show how rigid geometry allows to translate questions concerning the arc space into arithmetic problems on rigid varieties.

6.1.2. The relative case. Let $k$ be any algebraically closed field of characteristic zero\(^3\) and put $R = k[[t]]$. For each integer $d > 0$, $K = k((t))$ has a unique extension $K(d)$ of degree $d$ in a fixed algebraic closure $K^\text{alg}$ of $K$, obtained by joining a $d$-th root of $t$ to $K$. We denote by $R(d)$ the normalization of $R$ in $K(d)$. For each $d > 0$, we choose of a $d$-th root of $t$ in $K^\text{alg}$, denoted by $\sqrt[d]t$, such that $(\sqrt[d]t)^e = \sqrt[e]t$ for each $d, e > 0$. This choice defines an isomorphism of $k$-algebras $R(d) \cong k[[\sqrt[d]t]]$. It also induces an isomorphism of $R$-algebras

$$\varphi_d : R(d) \to R(d)' : \sum_{i \geq 0} a_i (\sqrt[d]t)^i \mapsto \sum_{i \geq 0} a_i t^i$$

where $R(d)'$ is the ring $R$ with $R$-algebra structure given by

$$R \to R : \sum_{i \geq 0} b_i t^i \mapsto \sum_{i \geq 0} b_i t^{id}$$

Let $X$ be a smooth irreducible variety over $k$, endowed with a dominant morphism $f : X \to \mathbb{A}^1_k = \text{Spec} k[t]$. We denote by $\hat{X}$ the formal completion of the $R$-scheme $X_R = X \times_{k[t]} k[[t]]$; we will also call this the $t$-adic completion of $f$. Its special fiber $X_0$ is simply the fiber of $f$ over the origin.

There exists a tight connection between the points on the generic fiber $\hat{X}_0$ of $\hat{X}$, and the arcs on $X$. For any integer $d > 0$, we denote by $\mathcal{X}(d)$ the closed subscheme of $\mathcal{L}(X)$ defined by

$$\mathcal{X}(d) = \{ \psi \in \mathcal{L}(X) \mid f(\psi) = t^d \}$$

We will construct a canonical bijection

$$\varphi : \hat{X}_0(\mathcal{X}(d)) \to \mathcal{X}(d)(k)$$

such that the square

\(^3\)This condition is only imposed to simplify the arguments.
commutes.

As we’ve seen in Section 4.3, the specialization morphism of ringed sites \( sp: \hat{X}_\eta \to \hat{X} \) induces a bijection \( \hat{X}_\eta(K(d)) \to \hat{X}(R(d)) \), and the morphism \( sp \) maps a point of \( \hat{X}_\eta(K(d)) \) to the reduction modulo \( \sqrt[d]{\mathfrak{f}} \) of the corresponding point of \( \hat{X}(R(d)) \). By Grothendieck’s Existence Theorem (Section 3.4), the completion functor induces a bijection \( (X_R)(R(d)) \to \hat{X}(R(d)) \). Finally, the \( R \)-isomorphism \( \varphi_d: R(d) \to R(d)' \) induces a bijection

\[
(X_R)(R(d)) \to (X_R)(R(d)') = X(d)(k)
\]

In other words, if we take an arc \( \psi: \text{Spec } R \to X \) with \( f(\psi) = t^d \), then the morphism \( \hat{\psi}_\eta \) yields a \( K(d) \)-point on \( X_\eta \), and this correspondence defines a bijection between \( X(d)(k) \) and \( \hat{X}_\eta(K(d)) \). Moreover, the image of \( \psi \) under the projection \( \pi_0: \mathcal{L}(X) \to X \) is nothing but the image of the corresponding element of \( \hat{X}_\eta(K(d)) \) under the specialization map \( sp: |\hat{X}_\eta| \to |\hat{X}| = |X_\eta| \).

The Galois group \( G(K(d)/K) = \mu_d(k) \) acts on \( \hat{X}_\eta(K(d)) \), and its action on the level of arcs is easy to describe: if \( \psi \) is an arc \( \text{Spec } R \to X \) with \( f(\psi) = t^d \), and \( \xi \) is an element of \( \mu_d(k) \), then \( \xi \cdot \psi(t) = \psi(\xi \cdot t) \).

The spaces \( X(d) \), with their \( \mu_d(k) \)-action, are quite close to the arc spaces appearing in the definition of the motivic zeta function associated to \( f \) [18, 3.2]. In fact, the motivic zeta function can be realized in terms of the motivic integral of a Gelfand-Leray form on \( \hat{X}_\eta \), and the relationship between arc schemes and rigid varieties can be used in the study of motivic zeta functions and the monodromy conjecture, as is explained in [35, 34].

6.1.3. The absolute case. This case is easily reduced to the previous one. Let \( X \) be any separated \( k \)-scheme of finite type, and consider its base change \( X_R = X \times_k R \). We denote by \( \hat{X} \) the formal completion of \( X_R \).

There exists a canonical bijection between the sets \( \mathcal{L}(X)(k) \) and \( X_R(R) \). Hence, by the results in the previous section, \( k \)-rational arcs on \( X \) correspond to \( K \)-points on the generic fiber \( \hat{X}_\eta \) of \( \hat{X} \), by a canonical bijection

\[
\varphi: \mathcal{L}(X)(k) \to \hat{X}_\eta(K)
\]

and the square

\[
\begin{array}{ccc}
\mathcal{L}(X)(k) & \xrightarrow{\varphi} & \hat{X}_\eta(K) \\
\pi_0 \downarrow & & \downarrow sp \\
X(k) & \xrightarrow{=} & X(k)
\end{array}
\]

commutes. So the rigid counterpart of the space \( \mathcal{L}(X)_Z := \mathcal{L}(X) \times_X Z \) of arcs with origin in some closed subscheme \( Z \) of \( X \), is the tube \( |Z| \) of \( Z \) in \( \hat{X}_\eta \) (or rather, its set of \( K \)-rational points).
Of course, the scheme structure on $\mathcal{L}(X)$ is very different from the analytic structure on $\tilde{X}_\eta$. Nevertheless, the structure on $\tilde{X}_\eta$ seems to be much richer than the one on $\mathcal{L}(X)$, and one might hope that some essential properties of the non-Noetherian scheme $\mathcal{L}(X)$ are captured by the more “geometric” object $\tilde{X}_\eta$. Moreover, there exists a satisfactory theory of étale cohomology for rigid $K$-varieties (see for instance [3] or [27]), making it possible to apply cohomological techniques to the study of the arc space.

6.1.4. The analytic Milnor fiber. Let $g : \mathbb{C}^m \to \mathbb{C}$ be an analytic map, and denote by $Y_0$ the analytic space defined by $g = 0$. Let $x$ be a point of $Y_0$. Consider an open disc $D := B(0, \eta)$ of radius $\eta$ around the origin in $\mathbb{C}$, and an open disc $B := B(x, \epsilon)$ in $\mathbb{C}^m$. We denote by $D^\times$ the punctured disc $D - \{0\}$, and we put

$$X' := B \cap g^{-1}(D^\times)$$

Then, for $0 < \eta \ll \epsilon \ll 1$, the induced map

$$g' : X' \to D^\times$$

is a $C^\infty$ locally trivial fibration, called the Milnor fibration of $g$ at $x$. It is trivial if $g$ is smooth at $x$. Its fiber at a point $t$ of $D^\times$ is denoted by $F_x(t)$, and it is called the (topological) Milnor fiber of $g$ at $x$ (w.r.t. $t$). To remove the dependency on the base point, one constructs the canonical Milnor fiber $F_x$ by considering the fiber product

$$F_x := X' \times_{D^\times} \overline{D^\times}$$

where $\overline{D^\times}$ is the universal covering space

$$\overline{D^\times} = \{z \in \mathbb{C} \mid \Im(z) > -\log \eta\} \to D^\times : z \mapsto \exp(iz)$$

Since this covering space is contractible, $F_x$ is homotopically equivalent to $F_x(t)$. The group of covering transformations $\pi_1(D^\times)$ acts on the singular cohomology of $F_x$: the action of the canonical generator $z \mapsto z + 2\pi i \pi_1(D^\times)$ is called the monodromy transformation of $g$ at $x$. The Milnor fibration $g'$ was devised in [32] as a tool to gather information on the topology of $Y_0$ near $x$.

We return to the algebraic setting: let $k$ be an algebraically closed field of characteristic zero, put $R = k[[t]]$, let $X$ be a smooth irreducible variety over $k$, and let $f : X \to \mathbb{A}^1_k = \text{Spec} k[t]$ be a dominant morphism. As before, we denote by $X$ the formal $t$-adic completion of $f$, with generic fiber $\tilde{X}_\eta$. For any closed point $x$ on $X_0$, we put $\mathcal{F}_x := |x|$, and we call this rigid $K$-variety the analytic Milnor fiber of $f$ at $x$. This object was introduced and studied in [33, 34]. We consider it as a bridge between the topological Milnor fibration and arc spaces; a tight connection between these data is predicted by the motivic monodromy conjecture. See [35] for more on this point of view.

The topological intuition behind the construction is the following: the formal neighbourhood $\text{Spf} R$ of the origin in $\mathbb{A}^1_k = \text{Spec} k[t]$, corresponds to an infinitesimally small disc around the origin in $\mathbb{C}$. Its inverse image under $f$ is realized as the $t$-adic completion of the morphism $f$: the formal scheme $\tilde{X}$ should be seen as a tubular neighborhood of the special fiber $X_0$ defined by $f$ on $X$. The inverse image of the punctured disc becomes the “complement” of $X_0$ in $\tilde{X}$, i.e. the generic fiber $\tilde{X}_\eta$ of $\tilde{X}$. The specialization map $sp$ can be seen as a canonical “contraction” of $\tilde{X}_\eta$ on $X_0$, such that $\mathcal{F}_x$ corresponds to the topological space $X'$ considered above.
Note that this is not really the Milnor fiber yet: we had to base-change to a universal cover of $D^{\times}$, which corresponds to considering $\mathcal{F}_x \hat{\otimes}_K \tilde{K}^{alg}$ instead of $\mathcal{F}_x$, by the dictionary between finite covers of $D^{\times}$ and finite extensions of $K$. The monodromy action is translated into the Galois action of $G(K^{alg}/K) \cong \hat{\mathbb{Z}}(1)(k)$ on $\mathcal{F}_x \hat{\otimes}_K \tilde{K}^{alg}$.

It follows from the results in Section 6.1 that, for any integer $d > 0$, the points in $\mathcal{F}_x(K(d))$ correspond canonically to the arcs

$$\psi : \text{Spec } k[[t]] \to X$$

satisfying $f(\psi) = t^d$ and $\pi_0(\psi) = x$. Moreover, by Berkovich’ comparison result in [5, 3.5] (see also Section 6.3), there are canonical isomorphisms

$$H^i_{\text{ét}}(\mathcal{F}_x \hat{\otimes}_K \tilde{K}^{alg}, \mathbb{Q}_\ell) \cong R^i\psi_!(\mathbb{Q}_\ell)_x$$

such that the Galois action of $G(K^{alg}/K)$ on the left hand side corresponds to the monodromy action of $G(K^{alg}/K)$ on the right. Here $H^i_{\text{ét}}$ is étale $\ell$-adic cohomology, and $R\psi_!$ denotes the étale $\ell$-adic nearby cycle functor associated to $f$. In particular, if $k = \mathbb{C}$, this implies that $H^i_{\text{ét}}(\mathcal{F}_x \hat{\otimes}_K \tilde{K}^{alg}, \mathbb{Q}_\ell)$ is canonically isomorphic to the singular cohomology $H^i_{\text{sing}}(F_x, \mathbb{Q}_\ell)$ of the canonical Milnor fiber $F_x$ at $x$, and that the action of the canonical topological generator of $G(K^{alg}/K) = \hat{\mathbb{Z}}(1)(\mathbb{C})$ corresponds to the monodromy transformation, by Deligne’s classical comparison theorem for étale and analytic nearby cycles [1, XIV]. In view of the motivic monodromy conjecture, it is quite intriguing that $\mathcal{F}_x$ relates certain arc spaces to monodromy action; see [35] for more background on this perspective.

6.2. Deformation theory and lifting problems. Suppose that $R$ has mixed characteristic, and let $X_0$ be a scheme of finite type over the residue field $k$. Illusie sketches in [28, 5.1] the following problem: is there a flat scheme $X$ of finite type over $R$ such that $X_0 = X \times_R k$? Grothendieck suggested the following approach: first, try to construct an inductive system $X_n$ of flat $R_n$-schemes of finite type such that $X_n \cong X_m \times_{R_m} R_n$ for $m \geq n \geq 0$. In many situations, the obstructions to lifting $X_n$ to $X_{n+1}$ live in a certain cohomology group of $X_0$, and when these obstructions vanish, the isomorphism classes of possible $X_{n+1}$ correspond to elements in another appropriate cohomology group of $X_0$. Once we found such an inductive system, its direct limit is a flat formal $R$-scheme $X_{\infty}$, topologically of finite type. Next, we need to know if this formal scheme is algebrizable, i.e. if there exists an $R$-scheme $X$ whose formal completion $\hat{X}$ is isomorphic to $X_{\infty}$. This scheme $X$ would be a solution to our lifting problem. A useful criterion to prove the existence of $X$ is the one quoted in Section 6.4 if $X_0$ is proper and carries an ample line bundle that lifts to a line bundle on $X_{\infty}$, then $X_{\infty}$ is algebrizable. Moreover, the algebraic model $X$ is unique up to isomorphism by Grothendieck’s existence theorem (Section 6.4). For more concrete applications of this approach, we refer to Section 5 of [28].

6.3. Nearby cycles for formal schemes. Berkovich used his étale cohomology theory for non-archimedean analytic spaces, developed in [3], to construct nearby and vanishing cycles functors for formal schemes [4, 5]. His formalism applies, in particular, to stft formal $R$-schemes $X_{\infty}$, and to formal completions of such formal schemes along closed subschemes of the special fiber $X_0$. Let us denote by $R\psi_!$ the functor of nearby cycles, both in the algebraic and in the formal setting. Suppose that $k$ is algebraically closed. Let $X$ be a variety over $R$, and denote by $\hat{X}$ its formal completion, with generic fiber $\hat{X}_n$. Let $Y$ be a closed subscheme of $X_0$, and
let $\mathcal{F}$ be an étale constructible sheaf of abelian groups on $X \times_R K$, with torsion orders prime to the characteristic exponent of $k$. Then Berkovich associates to $\mathcal{F}$ in a canonical way an étale sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}_\eta$, and an étale sheaf $\mathcal{F}/\mathcal{Y}$ on the tube $|\mathcal{Y}|$. His comparison theorem [5, 3.1] states that there are canonical quasi-isomorphisms

$$R\psi_\eta(\mathcal{F}) \cong R\psi_\eta(\tilde{\mathcal{F}})$$

Moreover, by [5, 3.5] there is a canonical quasi-isomorphism

$$R\Gamma(Y, R\psi_\eta(\mathcal{F})|_Y) \cong R\Gamma(|Y|, \tilde{\mathcal{F}}/\mathcal{Y})$$

In particular, if $x$ is a closed point of $X_0$, then $R^i\psi_\eta(\mathbb{Q}_\ell)_x$ is canonically isomorphic to the $i$-th $\ell$-adic cohomology space of the tube $|x[\tilde{\mathcal{K}}^s]$. Similar results hold for tame nearby cycles and vanishing cycles.

This proves a conjecture of Deligne’s, stating that $R\psi_\eta(\mathcal{F})|_Y$ only depends on the formal completion of $X$ along $Y$. In particular, the stalk of $R\psi_\eta(\mathcal{F})$ at a closed point $x$ of $X_0$ only depends on the completed local ring $\tilde{\mathcal{O}}_{X,x}$.

6.4. Semi-stable reduction for curves. Bosch and Lütkebohmert show in [12, 11] how rigid geometry can be used to construct stable models for smooth projective curves over a non-archimedean field $L$, and uniformizations for Abelian varieties. Let us briefly sketch their approach to stable reduction of curves.

If $A$ is a reduced Tate algebra over $L$, then we define

$$A^o = \{ f \in A | ||f||_{\sup} \leq 1 \}$$

$$A^{oo} = \{ f \in A | ||f||_{\sup} < 1 \}$$

Note that $A^o$ is a subring of $A$, and that $A^{oo}$ is an ideal in $A^o$. The quotient $\tilde{A} := A^o/A^{oo}$ is a reduced algebra of finite type over $\tilde{L}$, by [10, 1.2.5.7+6.3.4.3], and $\tilde{X} := \text{Spec } \tilde{A}$ is called the canonical reduction of the affinoid space $X := \text{Sp } \tilde{A}$. There is a natural reduction map $X \to \tilde{X}$ mapping points of $X$ to closed points of $\tilde{X}$. The inverse image of a closed point $x$ of $\tilde{X}$ is called the formal fiber of $X$ at $x$; it is an open rigid subspace of $X$.

Let $C$ be a projective, smooth, geometrically connected curve over $L$, and consider its analytification $C^{an}$. By a technical descent argument we may assume that $L$ is algebraically closed. The idea is to construct a finite admissible cover $\mathcal{U}$ of $C^{an}$ by affinoid domains $\tilde{U}$ whose canonical reductions $\tilde{U}$ are semi-stable. If the cover $\mathcal{U}$ satisfies a certain compatibility property, the canonical reductions $\tilde{U}$ can be glued to a semi-stable $\tilde{L}$-variety. From this cover $\mathcal{U}$ one constructs a semi-stable model for $C$. The advantage of passing to the rigid world is that the Grothendieck topology on $C^{an}$ is much finer then the Zariski topology on $C$, allowing finer patching techniques.

To construct the cover $\mathcal{U}$, it is proved that smooth points and ordinary double points on $\tilde{U}$ can be recognized by looking at their formal fiber in $\tilde{U}$. For instance, a closed point $x$ of $\tilde{U}$ is smooth if its formal fiber is isomorphic to an open disc of radius 1. An alternative proof based on rigid geometry is given in [20, 5.6].

6.5. Constructing étale covers, and Abhyankar’s Conjecture. Formal and rigid patching techniques can also be used in the construction of Galois covers; see [26] for an introduction to this subject. This approach generalizes the classical Riemann Existence Theorem for complex curves to a broader class of base fields. Riemann’s Existence Theorem states that, for any smooth connected complex curve
there is an equivalence between the category of finite étale covers of $X$, the category of finite analytic covering spaces of the complex analytic space $X^{an}$, and the category of finite topological covering spaces of $X(\mathbb{C})$ (w.r.t. the complex topology). So the problem of constructing an étale cover is reduced to the problem of constructing a topological covering space, where we can proceed locally w.r.t. the complex topology and glue the resulting local covers. In particular, it can be shown in this way that any finite group is the Galois group of a finite Galois extension of $\mathbb{C}(x)$, by studying the ramified Galois covers of the complex projective line.

The strategy in rigid geometry is quite similar: given a smooth curve $X$ over a non-archimedean field $L$, we consider its analytification $X^{an}$. We construct an étale cover $Y'$ of $X^{an}$ by constructing covers locally, and gluing them to a rigid variety. Then we use a GAGA-theorem to show that $Y'$ is algebraic, i.e. $Y' = Y^{an}$ for some curve $Y$ over $L$; $Y$ is an étale cover of $X$. Of course, several technical complications have to be overcome to carry out this strategy.

We list some results that can be obtained by means of these techniques, and references to their proofs.

- (Harbater) For any finite group $G$, there exists a ramified Galois cover $f : X \to \mathbb{P}_L^1$ with Galois group $G$, such that $X$ is absolutely irreducible, smooth, and projective, and such that there exists a point $x$ in $X(L)$ at which $f$ is unramified. An accessible proof by Q. Liu is given in [29]; see also [42] § 3.

- (Abhyankar’s Conjecture for the projective line) Let $k$ be an algebraically closed field of characteristic $p > 0$. A finite group $G$ is the Galois group of a covering of $\mathbb{P}_k^1$, only ramified over $\infty$, iff $G$ is generated by its elements having order $p^n$ with $n \geq 1$. This conjecture was proven by Raynaud in [38]. This article also contains an introduction to rigid geometry and étale covers.

- (Abhyankar’s Conjecture) Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $X$ be a smooth connected projective curve over $k$ of genus $g$, let $\xi_0, \ldots, \xi_r$ (r $\geq 0$) be distinct closed points on $X$, and let $\Gamma_{g,r}$ be the topological fundamental group of a complex Riemann surface of genus $g$ minus $r + 1$ points (it is the free group on $2g + r$ generators). Put $U = X \setminus \{\xi_0, \ldots, \xi_r\}$. A finite group $G$ is the Galois group of an unramified Galois cover of $U$, iff every prime-to-$p$ quotient of $G$ is a quotient of $\Gamma_{g,r}$. This conjecture was proven by Harbater in [25].

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