Some results on the Sombor indices of graphs

Igor Milovanović, Emina Milovanović, Akbar Ali, Marjan Matejić

1 Faculty of Electronic Engineering, University of Niš, Niš, Serbia
2 Department of Mathematics, Faculty of Science, University of Hail, Hail, Saudi Arabia

(Received: 16 April 2021. Received in revised form: 1 May 2021. Accepted: 1 May 2021. Published online: 4 May 2021.)

Abstract

This paper is concerned with three recently introduced degree-based graph invariants; namely, the Sombor index, the reduced Sombor index and the average Sombor index. The Sombor index, being the simplest one among the aforementioned three invariants, has attracted a significant attention from researchers within a very short time [3, 7–9, 14, 15, 19, 23, 26–31, 34, 35, 39, 41, 42].

The first aim of this paper is to give some results that may be helpful in proving a recently proposed conjecture concerning the Sombor index. Establishing inequalities related to the aforementioned three graph invariants is the second aim of this paper.

Keywords: graph invariant; topological index; Sombor indices; bounds; chemical graph theory.

2020 Mathematics Subject Classification: 05C05, 05C09, 05C90.

1. Introduction

The study of the mathematical aspects of the degree-based graph invariants (also known as topological indices) is considered to be one of the very active research areas within the field of chemical graph theory [17]. Recently, the mathematical chemist Ivan Gutman [18], one of the pioneers of chemical graph theory, proposed a geometric approach to interpret degree-based graph invariants and based on this approach, he devised three new graph invariants; namely the Sombor index, the reduced Sombor index and the average Sombor index. The Sombor index, being the simplest one among the aforementioned three invariants, has attracted a significant attention from researchers within a very short time [3, 7–9, 14, 15, 19, 23, 26–31, 34, 35, 39, 41, 42].

The first aim of this paper to give some results that may be helpful in proving a conjecture concerning the Sombor index posed in the reference [35]. In order to state this conjecture, we need some definitions first. An acyclic graph is the graph containing no cycle. For a graph $G$, its cyclomatic number $\nu(G)$ (or simply $\nu$) is the least number of edges whose deletion makes the graph $G$ as acyclic. A $\nu$-cyclic graph is the one having the cyclomatic number $\nu$. A pendant vertex of a graph is a vertex of degree 1. For $\nu \geq 1$, denote by $H_{n,\nu}$ the graph deduced from the star graph of order $n$ by adding $\nu$ edge(s) between a fixed pendent vertex and $\nu$ other pendent vertices.

Conjecture 1.1. [35] For the fixed integers $n$ and $\nu$ with $6 \leq \nu \leq n - 2$, $H_{n,\nu}$ is the only graph attaining the maximum Sombor index in the class of all $\nu$-cyclic connected graphs of order $n$.

Establishing inequalities related to the Sombor index, the reduced Sombor index and the average Sombor index is another aim of this paper.

2. Preliminaries

Let $G$ be a graph. Denote by $E(G)$ and $V(G)$ the edge set and vertex set, respectively, of $G$. Denote by $i \sim j$ the edge connecting the vertices $v_i, v_j \in V(G)$. For a vertex $v_i \in V(G)$, its degree is denoted by $d_i(G)$ (or simply by $d_i$). A regular graph is the one in which all of its vertices have the same degree. For an edge $e \in E(G)$, its degree is the number of edges adjacent to $e$. By an edge-regular graph, we mean a graph in which all of its edges have the same degree. A graph of order $n$ is also known as an $n$-vertex graph. Denote by $G - v_i$ and $G - v_i v_j$ the graphs obtained from $G$ by removing the vertex $v_i$ and the edge $v_i v_j$, respectively. The $n$-vertex complete graph and the $n$-vertex star graph are denoted as $K_n$ and $K_{1,n-1}$, respectively. From the notations $E(G)$, $V(G)$, $\nu(G)$ and $d_i(G)$, we remove "($G$)" whenever the graph under consideration is clear. The graph-theoretical notation and terminology used in this paper but not defined here, may be found in some standard graph-theoretical books, like [4, 6, 10].

*Corresponding author (Igor.Milovanovic@elfak.ni.ac.rs).
If $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $|E(G)| = m$ then the Sombor index, the average Sombor index and the reduced Sombor index of the graph $G$ are defined as

$$SO(G) = \sum_{i<j} \sqrt{d_i^2 + d_j^2}, \quad SO_{av}(G) = \sum_{i<j} \sqrt{\left( d_i - \frac{2m}{n} \right)^2 + \left( d_j - \frac{2m}{n} \right)^2} \quad \text{and} \quad SO_{red}(G) = \sum_{i<j} \sqrt{(d_i - 1)^2 + (d_j - 1)^2},$$

respectively.

Most of the degree-based graph invariants can be written [22,38] as:

$$BID(G) = \sum_{i<j} f(d_i, d_j),$$

where $f$ is a symmetric non-negative real-valued function of $d_i$ and $d_j$. The graph invariants having the form (1) are known as the bond incident degree indices [36], BID indices in short [2]. Those choices of the function $f$ are given in Table 1 that correspond to the graph invariants used in the next sections.

| Function $f(d_i, d_j)$ | Equation (1) corresponds to | Symbol |
|------------------------|-----------------------------|--------|
| $d_i + d_j$            | first Zagreb index [20,21]   | $M_1$  |
| $d_i d_j$              | second Zagreb index [20]     | $M_2$  |
| $2(d_i + d_j)^{-1}$    | harmonic index [13]         | $H$    |
| $d_i^{-2} + d_j^{-2}$  | inverse degree [13]         | $ID$   |
| $|d_i - d_j|$            | Albertson index [1]         | $Ab$   |
| $2\sqrt{d_i d_j}(d_i + d_j)^{-1}$ | geometric-arithmetic index [37] | $GA$ |
| $d_i d_j(d_i + d_j)^{-1}$ | inverse sum indeg index [38] | $ISI$ |
| $d_i(d_j)^{-1} + d_j(d_i)^{-1}$ | symmetric division deg [38] | $SDD$ |
| $(d_i + d_j)(4d_i d_j)^{-1/2}$ | arithmetic-geometric index [11] | $AG$ |
| $d_i^2 + d_j^2$        | forgotten topological index [16] | $F$  |

3. Towards the proof of Conjecture 1.1

The $p$-Sombor index of a graph $G$ is denoted by $SO_p(G)$ and is defined [35] as the sum of the quantities $(d_i^p + d_j^p)^{1/p}$ over all edges $i \sim j$ of $G$, where $p$ is not equal to 0. The first lemma (Lemma 3.1) of this section gives an upper bound on a generalized variant $SO_{p,q}$ of the $p$-Sombor index:

$$SO_{p,q}(G) = \sum_{i<j} [(d_i + q)^p + (d_j + q)^p]^{1/p} = \sum_{i<j} \varphi_{p,q}(i \sim j),$$

where $q$ is a real number provided that $\varphi_{p,q}(i \sim j)$ is a real number for every edge $i \sim j$ of $G$. The name $(p,q)$-Sombor index may be associated with the graph invariant $SO_{p,q}$.

**Lemma 3.1.** If $G$ is a graph of size $m \geq 1$ then for any real number $q$, it holds that

$$SO_{2,q}(G) \leq \sqrt{m[F(G) + 2q \cdot M_1(G)] + 2q^2 m}$$

with equality if and only if there exist a fixed real number $t$ such that $(d_i + q)^2 + (d_j + q)^2 = t$ for every edge $i \sim j$ of $G$, where $F(G)$ and $M_1(G)$ are the forgotten topological index and first Zagreb index of $G$, respectively; see Table 1.

**Proof.** From Cauchy-Bunyakovsky-Schwarz’s inequality, it follows that

$$\left( \sum_{i<j} \sqrt{(d_i + q)^2 + (d_j + q)^2} \right)^2 \leq \left( \sum_{i<j} (1) \right) \left( \sum_{i<j} [(d_i + q)^2 + (d_j + q)^2] \right)$$

(2)
\[ = m[F(G) + 2q \cdot M_1(G) + 2q^2m]. \]

Note that the equality sign in (2) holds if and only if there exist a fixed real number \( t' \) such that \( \sqrt{(d_i + q)^2 + (d_j + q)^2} = t' \) for every edge \( i \sim j \) of \( G \).

Next, the bound on the invariant \( SO_{2,q} \) given in Lemma 3.1 is used to derive another bound on \( SO_{2,q} \) in terms of the parameters \( m \) and \( q \) only (see Lemma 3.4); however, to proceed, bounds on the forgotten topological index \( F \) and first Zagreb index \( M_1 \) in terms of \( m \) are required first.

**Lemma 3.2.** For any \( n \)-vertex graph \( G \) of size \( m \) with \( 0 \leq m \leq n - 1 \), it holds that
\[ F(G) \leq m(m^2 + 1) \]
with equality if and only if the star \( S_{m+1} \) is a component of \( G \).

**Proof.** We fix \( n \) and use induction on \( m \). For \( m = 0, 1 \), the lemma is obviously true; thus, the induction starts. Suppose that \( G \) is an \( n \)-vertex graph of size \( k \) such that \( 0 \leq k \leq n - 1 \) and \( k \geq 2 \). Take an edge \( i \sim j \). Without loss of generality, assume that \( d_j \leq d_i \). Note that \( d_i + d_j \leq k + 1 \), which gives \( d_i^2 + d_j^2 - (d_i + d_j) \leq (k + 1 - d_j)^2 + d_j^2 - (k + 1) \) and hence the equation \( F(G) - F(G - v_i v_j) = 3(d_i^2 + d_j^2 - d_i - d_j) + 2 \) gives
\[ F(G) - F(G - v_i v_j) \leq 3[(k + 1 - d_j)^2 + d_j^2 - (k + 1)] + 2, \tag{3} \]
where the equality sign in (3) holds if and only if \( d_i + d_j = k + 1 \). (It needs to be mentioned here that, throughout this proof, \( d_r \) denotes the degree of a vertex \( v_r \) in \( G \) not in \( G - v_i v_j \).) The inequalities \( d_j \leq d_i \) and \( d_i + d_j \leq k + 1 \) confirm that \( 2d_j \leq k + 1 \), which forces that the right hand side of (3) is maximum if and only if \( d_j = 1 \). Thus, (3) gives
\[ F(G) - F(G - v_i v_j) \leq 3(k^2 - k) + 2, \tag{4} \]
with equality if and only if \( d_i = k \) and \( d_j = 1 \). Also, by inductive hypothesis, it holds that
\[ F(G - v_i v_j) \leq (k - 1)((k - 1)^2 + 1], \tag{5} \]
with equality if and only if the star \( S_k \) is a component of \( G - v_i v_j \). Thus, from (4) and (5), it follows that \( F(G) \leq k(k^2 + 1) \) with equality if and only if the star \( S_{k+1} \) is a component of \( G \). This completes the induction and hence the proof.

**Lemma 3.3.** [40] For any \( n \)-vertex graph \( G \) of size \( m \) with \( 0 \leq m \leq n - 1 \), it holds that
\[ M_1(G) \leq m(m + 1) \]
with equality if and only if
\[ \begin{cases} \text{the star } S_{m+1} \text{ is a component of } G, & \text{ if } m \neq 3, \\ \text{either the star } S_1 \text{ is a component of } G \text{ or the cycle } C_3 \text{ is a component of } G, & \text{ if } m = 3. \end{cases} \]

The next result follows directly from Lemmas 3.1, 3.2 and 3.3.

**Lemma 3.4.** For any \( n \)-vertex graph \( G \) of size \( m \) with \( 0 \leq m \leq n - 1 \) and for any non-negative real number \( q \), it holds that
\[ SO_{2,q}(G) \leq m\sqrt{m^2 + 2q(m + q + 1) + 1} \]
with equality if and only if the star \( S_{m+1} \) is a component of \( G \).

The next two results were proven in [35].

**Lemma 3.5.** [35] If \( \nu \) and \( n \) are fixed integers such that \( 0 \leq \nu \leq n - 2 \) then the graph attaining the maximum Sombor index in the class of all connected \( \nu \)-cyclic graphs of order \( n \) has the maximum degree \( n - 1 \).

**Lemma 3.6.** [35] Let \( \nu \) and \( n \) be fixed integers such that \( 2 \leq \nu \leq n - 2 \). Let \( G \) be a graph with the maximum value of the Sombor index in the class of all connected \( \nu \)-cyclic graphs of order \( n \). If \( (d_1, d_2, \ldots, d_n) \) is the vertex-degree sequence of \( G \) such that \( d_1 \geq d_2 \geq \cdots \geq d_n \), then the vertex \( v_2 \) is adjacent to all non-pendent vertices of \( G \), where \( d_i = d_{v_i} \) for \( v_i \in V(G) \).
For a real number $\alpha$, define the graph invariant $SO_{V, \alpha}$ as follows

$$SO_{V, \alpha}(G) = \sum_{v_i \in V(G)} \sqrt{(d_i + 1)^2 + \alpha^2}.$$

Towards the Proof of Conjecture 1.1. Let $G$ be a $\nu$-cyclic connected graph of order $n$ with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $6 \leq \nu \leq n - 2$. Let $(d_1, d_2, \ldots, d_n)$ be the vertex-degree sequence of $G$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$, where $d_i = d_{v_i}$ for $v_i \in V(G)$. If either $d_1 \leq n - 2$ or $v_2$ is not adjacent to any non-pendent vertex of $G$ then from either Lemma 3.5 or Lemma 3.6, respectively, it follows that $G$ does not have the maximum value of $SO$ in the class of all $\nu$-cyclic connected graphs of order $n$. In what follows, assume that $d_1 = n - 1$ and that $v_2$ is adjacent to all non-pendent vertices of $G$. Note that the graph $G - v_1$ has exactly one connected non-trivial component $C$ and the subgraph induced by $V(C)$ (the vertex set of $C$) has a vertex of degree $|V(C)| - 1$, and that $G - v_1$ has the size $m' = \nu$, where $6 \leq m' \leq n - 2 = |V(G - v_1)| - 1$. Thus, from Lemma 3.4, it follows that

$$SO(G) = SO_{V,n-1}(G - v_1) + SO_{2,1}(G - v_1)$$

$$\leq SO_{V,n-1}(G - v_1) + \nu \sqrt{(\nu + 1)^2 + 4},$$

where the equality sign in (7) holds if and only if the star $S_{\nu + 1}$ is a component of $G - v_1$.

We believe that the next result concerning the invariant $SO_{V, \alpha}$ is true. However, at the present moment, we do not have its proof; thus, we state it as a conjecture (if one proves this conjecture then from (7), the proof of Conjecture 1.1 follows directly).

Conjecture 3.1. For any $n$-vertex graph $G$ of size $m$ with $6 \leq m \leq n - 1$, it holds that

$$SO_{V,n-1}(G) \leq m\sqrt{(n - 1)^2 + 4 + \sqrt{(n - 1)^2 + (m + 1)^2 + (n - m - 1)\sqrt{(n - 1)^2 + 1}}}$$

with equality if and only if the star $S_{m+1}$ is a component of $G$.

4. Some relations between Sombor indices and other degree-based graph invariants

Before establishing the main results of this section, we first recall an inequality for the real number sequences reported in [33].

Lemma 4.1. [33] Let $x = (x_i), i = 1, 2, \ldots, n$, be a sequence of non-negative real numbers and $a = (a_i), i = 1, 2, \ldots, n$, a sequence of positive real numbers. Then, for any $r \geq 0$ holds

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^{r}} \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r}.$$  \hspace{1cm} (8)

Equality holds if and only if $r = 0$ or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

In the next theorem we determine a relationship between $SO(G)$ and $M_1(G)$ and $ISI(G)$.

Theorem 4.1. Let $G$ be a connected graph. Then

$$SO(G) \leq \sqrt{M_1(G)(M_1(G) - 2ISI(G))}.$$  \hspace{1cm} (9)

Equality holds if and only if $G$ is an edge-regular graph.

Proof. From the definitions of $M_1(G)$ and $ISI(G)$ we have that

$$M_1(G) - 2ISI(G) = \sum_{i \neq j} (d_i + d_j) - \sum_{i \neq j} \frac{2d_id_j}{d_i + d_j} = \sum_{i \neq j} \frac{d_i^2 + d_j^2}{d_i + d_j}.$$  \hspace{1cm} (10)

On the other hand, for $r = 1, x_i := \sqrt{d_i^2 + d_j^2}, a_i := d_i + d_j$, with summation performed over all adjacent vertices $v_i$ and $v_j$ in $G$, the inequality (8) becomes

$$\sum_{i \neq j} \frac{d_i^2 + d_j^2}{d_i + d_j} \geq \frac{\left(\sum_{i \neq j} \sqrt{d_i^2 + d_j^2}\right)^2}{\left(\sum_{i \neq j} (d_i + d_j)\right)}.$$
that is
\[ \sum_{i \sim j} d_i^2 + d_j^2 \geq \frac{SO(G)^2}{M_1(G)}. \]  
(11)

From the above and inequality (10) we arrive at (9).

Equality in (11) holds if and only if \( \frac{d_i^2 + d_j^2}{d_i + d_j} \) is constant for any pair of adjacent vertices \( v_i \) and \( v_j \) in \( G \). Suppose that \( v_j \) and \( v_k \) are adjacent to vertex \( v_i \). Then
\[ \sqrt{d_i^2 + d_j^2} = \sqrt{d_i^2 + d_k^2} \]
that is
\[ 2d_i(d_j^2 - d_jd_k)(d_j - d_k) = 0. \]

From the above identity it follows that equality in (9) holds if and only if \( G \) is an edge-regular graph.

\[ \square \]

**Corollary 4.1.** Let \( G \) be a connected graph with \( n \geq 2 \) vertices and \( m \) edges. Then
\[ SO(G) \leq \sqrt{M_1(G)} \left( M_1(G) - \frac{2m^2}{n} \right). \]  
(12)

Equality holds if and only if \( G \) is an edge-regular graph.

**Proof:** The inequality (12) is obtained from (9) and
\[ ISI(G) \geq \frac{m^2}{n}, \]
which was proven in [12].

\[ \square \]

**Corollary 4.2.** Let \( G \) be a connected graph with \( n \geq 2 \) vertices and \( m \) edges. Then
\[ SO(G) \leq \frac{m}{n-1} \sqrt{\left( 2m + (n-1)(n-2) \right) \left( \frac{2m}{n} + (n-1)(n-2) \right)}. \]  
(13)

Equality holds if and only if \( G \cong K_n \) or \( G \cong K_{1,n-1} \).

**Proof:** The inequality (13) is obtained from (12) and
\[ M_1(G) \leq m \left( \frac{2m}{n-1} + n - 2 \right), \]
which was proven in [5].

\[ \square \]

**Corollary 4.3.** Let \( T \) be a tree with \( n \geq 2 \) vertices. Then
\[ SO(T) \leq (n-1)\sqrt{n^2 - 2n + 2}. \]  
(14)

Equality holds if and only if \( T \cong K_{1,n-1} \).

The inequality (14) was proven in [18].

Proofs of the following theorems are analogous to that of Theorem 4.1, thus omitted.

**Theorem 4.2.** Let \( G \) be a connected graph with \( m \geq 1 \) edges. Then
\[ SO_{red}(G) \leq \sqrt{M_1(G)}(M_1(G) - 2ISI(G) + H(G) - 2m). \]

Equality holds if and only if \( G \) is an edge-regular graph.

**Theorem 4.3.** Let \( G \) be a connected graph with \( n \geq 2 \) vertices and \( m \) edges. Then
\[ SO_{avr}(G) \leq \sqrt{M_1(G)} \left( M_1(G) - 2ISI(G) + \frac{4m^2}{n^2}H(G) - \frac{4m^2}{n} \right). \]

Equality holds if and only if \( G \) is an edge-regular graph.
Theorem 4.4. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$SO(G) \leq \sqrt{\sum_{i,j} \left( \frac{d_i + d_j}{\sqrt{d_id_j}} - 2 \sqrt{d_i d_j} \right)^2} = \sum_{i,j} \frac{d_i^2 + d_j^2}{d_i + d_j}.$$  

Equality holds if and only if $G$ is an edge-regular graph.

The next theorem reveals a connection between Sombor index and indices $F(G)$, $M_2(G)$, $AG(G)$ and $GA(G)$.

Theorem 4.5. Let $G$ be a connected graph. Then

$$SO(G) \leq \sqrt{\frac{1}{2}(F(G) + 2M_2(G))(2AG(G) - GA(G))}. \quad (15)$$

Equality holds if and only if $G$ is regular.

Proof. The following identity holds

$$2AG(G) - GA(G) = \sum_{i,j} \left( \frac{d_i + d_j}{\sqrt{d_id_j}} - 2 \sqrt{d_id_j} \right) = \sum_{i,j} \frac{d_i^2 + d_j^2}{\sqrt{d_id_j}(d_i + d_j)}. \quad (16)$$

By the arithmetic–geometric mean inequality (see e.g. [32]) we have that

$$\sqrt{d_id_j} \leq \frac{1}{2}(d_i + d_j). \quad (17)$$

Combining (16) and (17) gives

$$2AG(G) - GA(G) \geq 2 \sum_{i,j} \frac{d_i^2 + d_j^2}{(d_i + d_j)^2}. \quad (18)$$

On the other hand, for $r = 1$, $x_i := \sqrt{d_i^2 + d_j^2}$, $a_i := (d_i + d_j)^2$, with summation performed over all adjacent vertices $v_i$ and $v_j$ in $G$, the inequality (8) transforms into

$$\sum_{i,j} \frac{d_i^2 + d_j^2}{(d_i + d_j)^2} = \sum_{i,j} \left( \frac{\sqrt{d_i^2 + d_j^2}}{(d_i + d_j)^2} \right)^2 \geq \frac{\left( \sum_{i,j} \sqrt{d_i^2 + d_j^2} \right)^2}{\sum_{i,j} (d_i + d_j)^2},$$

that is

$$\sum_{i,j} \frac{d_i^2 + d_j^2}{(d_i + d_j)^2} \geq \frac{SO(G)^2}{F(G) + 2M_2(G)}. \quad (19)$$

Now, from the above and (18) we arrive at (15).

Equality in (17) holds if and only if $d_i = d_j$ for any pair of adjacent vertices $v_i$ and $v_j$ in $G$, which implies that equality in (15) holds if and only if $G$ is regular.

Corollary 4.4. Let $G$ be a connected graph. Then

$$SO(G) \leq \sqrt{F(G)(2AG(G) - GA(G))}. \quad (20)$$

Equality holds if and only if $G$ is regular.

Proof. By the AM–GM inequality we have that

$$F(G) = \sum_{i,j} (d_i^2 + d_j^2) \geq \sum_{i,j} 2d_id_j = 2M_2(G).$$

The inequality (20) is obtained from the above and (15).

Corollary 4.5. Let $G$ be a connected graph and $\Delta$ be its maximum vertex degree. Then

$$SO(G) \leq \sqrt{\Delta M_1(G)(2AG(G) - GA(G))}. \quad (21)$$

Equality holds if and only if $G$ is regular.

Proof. The following is valid

$$F(G) = \sum_{i=1}^{n} d_i^2 \leq \Delta \sum_{i=1}^{n} d_i^2 = \Delta M_1(G).$$

From the above and inequality (20) we obtain the required result.
The following inequality was proven in [24] for the real number sequences.

**Lemma 4.2.** [24] Let \( p = (p_i) \), \( i = 1, 2, \ldots, n \) be a sequence of non-negative real numbers and \( a = (a_i) \), \( i = 1, 2, \ldots, n \), positive real number sequence. Then, for any real \( r, r \leq 0 \) or \( r \geq 1 \), holds

\[
\left( \sum_{i=1}^{n} p_i \right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left( \sum_{i=1}^{n} p_i a_i^r \right)^r. \tag{21}
\]

When \( 0 \leq r \leq 1 \) the opposite inequality is valid.

Equality holds if and only if \( r = 0 \), or \( r = 1 \), or \( a_1 = a_2 = \cdots = a_n \), or \( p_1 = p_2 = \cdots = p_t = 0 \) and \( a_{t+1} = \cdots = a_n \), for some \( t \), \( 1 \leq t \leq n - 1 \).

In the next theorem we determine a relationship between \( SO(G) \) and \( ID(G) \), \( F(G) \) and \( M_2(G) \).

**Theorem 4.6.** Let \( G \) be a connected graph. Then

\[
SO(G) \leq \sqrt{ID(G)F(G)M_2(G)} . \tag{22}
\]

Equality holds if and only if \( G \) is an edge-regular graph.

**Proof.** For \( r = 2 \), \( p_i := d_i^2 + d_j^2 \), \( a_i := \frac{1}{d_i d_j} \), with summation performed over all pairs of adjacent vertices \( v_i \) and \( v_j \) in \( G \), the inequality (21) becomes

\[
\sum_{i<j} (d_i^2 + d_j^2) \sum_{i<j} \frac{d_i^2 + d_j^2}{d_i d_j^2} \geq \left( \sum_{i<j} \frac{d_i^2 + d_j^2}{d_i d_j} \right)^2 ,
\]

that is

\[
ID(G)F(G) \geq \left( \sum_{i<j} \frac{d_i^2 + d_j^2}{d_i d_j} \right)^2 . \tag{23}
\]

On the other hand, for \( r = 1 \), \( x_i := \sqrt{d_i^2 + d_j^2} \), \( a_i := d_i d_j \), with summation performed over all pairs of adjacent vertices \( v_i \) and \( v_j \) in \( G \), the inequality (8) becomes

\[
\sum_{i<j} \frac{d_i^2 + d_j^2}{d_i d_j} = \sum_{i<j} \left( \frac{\sqrt{d_i^2 + d_j^2}}{d_i d_j} \right)^2 \geq \left( \sum_{i<j} \frac{\sqrt{d_i^2 + d_j^2}}{d_i d_j} \right)^2 ,
\]

that is

\[
\sum_{i<j} \frac{d_i^2 + d_j^2}{d_i d_j} \geq \frac{SO(G)^2}{M_2(G)} . \tag{24}
\]

Now, from (23) and (24) we arrive at (22).

Equality in (23) holds if and only if \( d_id_j \) is constant for any pairs of adjacent vertices \( v_i \) and \( v_j \) in \( G \). Suppose that vertices \( v_j \) and \( v_k \) are adjacent to \( v_i \). In that case, we have that \( d_id_j = d_id_k \), that is \( d_j = d_k \). This means that equality (23) holds if and only if \( G \) is an edge-regular graph. Equality in (24) holds if and only if \( \sqrt{\frac{d_i^2 + d_j^2}{d_id_j}} \) is constant for any pair of adjacent vertices \( v_i \) and \( v_j \) in \( G \). Suppose that vertices \( v_j \) and \( v_k \) are adjacent to \( v_i \). In that case holds \( \sqrt{\frac{d_i^2 + d_j^2}{d_id_j}} = \sqrt{\frac{d_i^2 + d_k^2}{d_id_k}} \), that is \( d_j = d_k \). This means that equality in (24) holds if and only if \( G \) is an edge-regular graph.

One can easily verify that from (24) the inequality

\[
SO(G) \leq \sqrt{M_2(G)SDD(G)} ,
\]

(which was proven in [31]) follows.

**Corollary 4.6.** Let \( G \) be a connected graph. Then

\[
SO(G) \leq \sqrt[4]{ID(G)F(G)^3} .
\]

Equality holds if and only if \( G \) is regular.
Theorem 4.7. Let $G$ be a connected graph. Then
\[
SO(G) \geq \sqrt{\frac{M_1(G)^2 + Alb(G)^2}{2}}.
\] (25)
Equality holds if and only if $G$ is an edge-regular graph.

Proof. The following identities are valid
\[
SO(G) - \sum_{i,j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} = \sum_{i,j} \frac{(d_i - d_j)^2}{\sqrt{d_i^2 + d_j^2}},
\]
and
\[
SO(G) + \sum_{i,j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} = \sum_{i,j} \frac{(d_i + d_j)^2}{\sqrt{d_i^2 + d_j^2}}.
\]
Taking $r = 1$, $x_i := |d_i - d_j|$, and $a_i := \sqrt{d_i^2 + d_j^2}$ in inequality (8) with summation performed over all pairs of adjacent vertices $v_i$ and $v_j$ in $G$, we obtain
\[
SO(G) - \sum_{i,j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} \geq \frac{Alb(G)^2}{SO(G)}.
\]
Similarly, taking $r = 1$, $x_i := d_i + d_j$, and $a_i := \sqrt{d_i^2 + d_j^2}$ in inequality (8) with summation performed over all pairs of adjacent vertices $v_i$ and $v_j$ in $G$, we obtain
\[
SO(G) + \sum_{i,j} \frac{2d_i d_j}{\sqrt{d_i^2 + d_j^2}} \geq \frac{M_1(G)^2}{SO(G)}.
\]
From the above inequalities we obtain the assertion of the Theorem 4.7.

Corollary 4.7. Let $G$ be a connected graph. Then
\[
SO(G) \geq \frac{\sqrt{3}}{2} M_1(G).
\] (26)
Equality holds if and only if $G$ is regular.

Proof. Since $Alb(G)^2 \geq 0$, the inequality (26) is obtained from (25).

The inequality (26) was proven in [15, 31] (see also [19]). By a similar arguments, the following results can be proven.

Theorem 4.8. Let $G$ be a graph with $m \geq 1$ edges. Then
\[
SO_{red}(G) \geq \sqrt{\frac{(M_1(G) - 2m)^2 + Alb(G)^2}{2}}.
\]
Equality holds if and only if $G$ is an edge-regular graph.

Theorem 4.9. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Then
\[
SO_{avr}(G) \geq \sqrt{\frac{(M_1(G) - \frac{4m^2}{n})^2 + Alb(G)^2}{2}}.
\]
Equality holds if and only if $G$ is an edge-regular graph.

From Theorems 4.8 and 4.9 we have the following corollaries.

Corollary 4.8. Let $G$ be a graph with $m \geq 1$ edges. Then
\[
SO_{red}(G) \geq \frac{\sqrt{2}}{2} (M_1(G) - 2m).
\] (27)
Equality holds if and only if $G$ is regular or each of its components is regular.

Corollary 4.9. Let $G$ be a connected graph with $n \geq 2$ vertices and $m$ edges. Then
\[
SO_{avr}(G) \geq \frac{\sqrt{2}}{2} \left(M_1(G) - \frac{4m^2}{n}\right).
\] (28)
Equality holds if and only if $G$ is regular.

Inequalities (27) and (28) were proven in [31] (see also [19]).
Acknowledgment

This research has been funded by Scientific Research Deanship at University of Ha’il, Saudi Arabia, through project number RG-20 031.

References

[1] M. O. Albertson, The irregularity of a graph, *Ars Combin.* 46 (1997) 219–225.
[2] A. Ali, Z. Raza, A. A. Bhatti, Bond incident degree (BID) indices of polyomino chains: a unified approach, *Appl. Math. Comput.* 287–288 (2016) 28–37.
[3] S. Alikhani, N. Ghanbari, Sombor index of polymers, *MATCH Commun. Math. Comput. Chem.* 86 (2021) 715–728.
[4] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, London, 2008.
[5] D. Caen, An upper bound on the sum of squares of degrees in a graph, *Discrete Math.* 155 (1996) 245–248.
[6] G. Chartrand, L. Lesniak, P. Zhang, *Graphs & Digraphs*, Sixth Edition, CRC Press, Boca Raton, 2016.
[7] R. Cruz, I. Gutman, J. Rada, Sombor index of chemical graphs *Appl. Math. Comput.* 399 (2021) Art# 126018.
[8] R. Cruz, J. Rada, Extremal values of the Sombor index in unicyclic and bicyclic graphs, *J. Math. Chem.* 59 (2021) 1098–1116.
[9] K. C. Das, A. Š. Cevik, I. N. Cangul, Y. Shang, On Sombor Index, *Symmetry* 13 (2021) Art# 140.
[10] R. Diestel, *Graph Theory*, Third Edition, Springer, New York, 2005.
[11] M. Erlasi, A. Irmananeš, On ordinary generalized geometric–Uarithmetic index, *Appl. Math. Lett.* 24 (2011) 582–587.
[12] F. Falahati–Nezhad, M. Azari, T. Došlić, Sharp bounds on the inverse sum indeg index, *Discrete Appl. Math.* 217 (2017) 185–195.
[13] S. Fabijowicz, On conjectures of Graffiti–II, *Congr. Numer.* 60 (1987) 187–197.
[14] X. Fang, L. You, H. Liu, The expected values of Sombor indices in random hexagonal chains, phenylene chains and Sombor indices of some chemical graphs, arXiv:2103.07172 [math.CO], (2021).
[15] S. Filipovski, Relations between Somber index and some degree–based topological indices, *Iranian J. Math. Chem.* 12 (2021) 19–26.
[16] B. Furtula, I. Gutman, A forgotten topological index, *J. Math. Chem.* 53 (2015) 1184–1190.
[17] I. Gutman, Degree-based topological indices, * Croat. Chem. Acta* 86 (2013) 351–361.
[18] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.* 86 (2021) 11–16.
[19] I. Gutman, Some basic properties of Sombor indices, *Open J. Discrete Appl. Math.* 4 (2021) 1–3.
[20] I. Gutman, B. Rušič, N. Trinajstić, C. F. Wilcox, Graph theory and molecular orbitals. XII. Acyclic polyenes, *J. Chem. Phys.* 62 (1975) 3399–3405.
[21] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* 17 (1972) 535–538.
[22] B. Hollas, The covariance of topological indices that depend on the degree of a vertex, *MATCH Commun. Math. Comput. Chem.* 54 (2005) 177–187.
[23] B. Horoldagva, C. Xu, On Sombor index of graphs, *MATCH Commun. Math. Comput. Chem.* 86 (2021) 703–713.
[24] J. L. W. V. Jensen, Sur les functions convexes et les inequalites entre les valeurs moyennes, *Acta Math.* 30 (1906) 175–193.
[25] X. Li, H. Zhao, Trees with the first three smallest and largest generalized topological indices, *MATCH Commun. Math. Comput. Chem.* 50 (2004) 57–62.
[26] Z. Lin, On the spectral radius and energy of the Sombor matrix of graphs, arXiv:2102.03960 [math.CO], (2021).
[27] H. Liu, Ordering chemical graphs by their Sombor indices, arXiv:2103.05995 [math.CO], (2021).
[28] H. Liu, Maximum Sombor index among cacti, arXiv:2103.07924 [math.CO], (2021).
[29] H. Liu, L. You, Y. Huang: Ordering chemical graphs by Sombor indices and its applications, *MATCH Commun. Math. Comput. Chem.* 87 (2022), In press.
[30] H. Liu, L. You, Z. Tang, J. B. Liu, On the reduced Sombor index and its applications, *MATCH Commun. Math. Comput. Chem.* 86 (2021) 729–753.
[31] I. Milovanović, E. Milovanović, M. Matejić, On some mathematical properties of Sombor indices, *Bull. Int. Math. Virtual Inst.* 11 (2021) 341–353.
[32] D. S. Mitrinović, P. M. Vasić, *Analytic Inequalities*, Springer, Berlin, 1970.
[33] J. Radon, Uber Die Absolut Additiven Mengenfunktionen, *Wien. Sitzungsber.* 122 (1913) 1295–1438.
[34] I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.* 86 (2021) 53–53.
[35] I. Redžepović, Chemical applicability of Sombor indices, *J. Serb. Chem. Soc.*, DOI: 10.2298/JSC201215006R, In press.
[36] T. Réti, T. Došlić, A. Ali, On the Sombor index of graphs, *Contrib. Math.* 3 (2021) 11–18.
[37] D. Vukičević, J. Duščević, Bond additive modeling 10. Upper and lower bounds of bond incident degree indices of catacondensed fluoranthenes, *Chem. Phys. Lett.* 515 (2011) 186–189.
[38] D. Vukičević, B. Furtula, Topological index based on the ratios of geometrical and arithmetical means of end-vertex degrees of edges, *J. Math. Chem.* 46 (2009) 1389–1376.
[39] D. Vukičević, M. Gašperov, Bond additive modeling 1. Adriatic indices, *Croat. Chem. Acta* 83 (2010) 243–260.
[40] Z. Wang, Y. Mao, Y. Li, B. Furtula, On relations between Sombor and other degree-based indices, *J. Appl. Math. Comput.*, DOI: 10.1007/s12190-021-01516-x, In press.
[41] K. Xu, K. C. Das, S. Balachandran, Maximizing the Zagreb indices of \((m,n)\)-graphs, *MATCH Commun. Math. Comput. Chem.* 72 (2014) 641–654.
[42] T. Zhou, Z. Lin, L. Miao, The Sombor index of trees and unicyclic graphs with given matching number, arXiv:2103.04645 [math.CO], (2021).
[43] T. Zhou, Z. Lin, L. Miao, The Sombor index of trees and unicyclic graphs with given maximum degree, arXiv:2103.07947 [math.CO], (2021).