On absorbing set for 3D Maxwell–Schrödinger
damped driven equations in bounded region

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Abstract

We consider the 3D damped driven Maxwell–Schrödinger equations in a bounded region under suitable boundary conditions. We establish new a priori estimates, which provide the existence of global finite energy weak solutions and bounded absorbing set. The proofs rely on the Sobolev type estimates for magnetic Schrödinger operator.

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1 Introduction

The Maxwell–Schrödinger, coupled equations form a fundamental dynamical system of Quantum Theory. These equations describe crucial phenomena of the matter-radiation interaction which is in the center of Quantum Theory and its applications. In particular, in the applications to the design and optimal control of quantum high-frequency electronic devices: laser, maser, klystron, magnetron, traveling wave tube, synchrotron, electron microscope, and others. The importance of these questions was pointed out in early paper by Kapitza [8]. Thus, a rigorous investigation of the long-time asymptotics for solutions of these equations is indispensable for physical applications.

However, the mathematical theory of these nonlinear evolutionary equations is currently in an initial stage. Respectively, applications of these equations rely on the perturbation theory which cannot provide the long-time behaviour of solutions of these equations. On the other hand, almost all applications require to know the long-time behaviour.

Till now the design and control of quantum devices uses mainly the quasiclassical approximation, which treats electrons as classical particles. For example, in the fundamental monograph [9], the words ‘quantum’
and ‘Schrödinger equation’ were not even mentioned. The most important exception is the study of the laser and maser action, based on 1D coupled Maxwell–Schrödinger equations [6, 7]. However, these studies are not rigorous, which was one of our motivations for the present investigation.

2 Damped driven Maxwell–Schrödinger equations

We consider the coupled damped driven Maxwell–Schrödinger equations (MS) in a bounded domain $V \subset \mathbb{R}^3$ with a smooth boundary $\Gamma := \partial V$. We choose the units where $e = -1$ and $m = e = \hbar = 1$. Then in the Coulomb gauge $\text{div} A(x,t) \equiv 0$ the equations read (cf.[1, 2, 10, 11])

$$
\begin{align*}
\dot{A}(x,t) &= \Delta A(x,t) - \sigma A(x,t) + \mathcal{P}^j(x,t), \\
i \psi \psi = (1 - i \varepsilon)H(t) \psi(t) - i E(t) \psi(t)
\end{align*}
$$

Here $\sigma > 0$ is the electrical conductance of the medium, $\varepsilon, \gamma > 0$ are the absorption coefficients, and $\mathcal{P}$ denotes the orthogonal projection onto free-divergent vector fields from the Hilbert space $L^2(V) \otimes \mathbb{R}^3$. Further, $E(t) := \langle \psi(t), H(t) \psi(t) \rangle$, where $H(t)$ is the Schrödinger operator

$$
H(t) := \frac{1}{2} D^2(t) + \phi(x) + A^0(x,t), \\
D(t) := -i \nabla + A(x,t) + A_p(x,t).
$$

Here $A_p(x,t)$ is an external ‘pumping potential’, and $\phi(x)$ stands for a static external potential (in the case of an atom, $\phi(x)$ is the nucleus potential). Finally, the charge and current densities are expressed in the wave function and the Maxwell potentials as

$$
\rho(x,t) = |\psi(x,t)|^2, \\
\mathbf{j}(x,t) = \text{Re} \left[ \bar{\psi}(x,t) D(t) \psi(x,t) \right].
$$

**Remark 2.1.** We introduce in the Schrödinger equation of the system (2.1) the novel specific nonlinear damping term $-i \gamma E(t) \psi(t)$ which plays the key role in our approach.

3 Boundary conditions

We choose the boundary conditions modelling ideally conducting diamagnetic materials (like cooper, silver, gold, etc.). In such materials the electric and magnetic field should vanish as well as the charge and current surface densities. Hence, the tangential component of the electric field

$$
E(x,t) = -\mathcal{A}(x,t) - \nabla A^0(x,t), \\
\mathcal{A}(x,t) := A(x,t) + A_p(x,t)
$$

vanishes on the boundary $\Gamma = \partial V$ as well as the normal component of the magnetic field $B(x,t) = \text{rot} A(x,t)$. More precisely, we assume that

$$
A^0(x,t) = 0, \\
\mathbf{n}(x) \times \mathcal{A}(x,t) = 0, \\
\mathbf{n}(x) \cdot \text{rot} \mathcal{A}(x,t) = 0, \\
x \in \Gamma, t > 0,
$$

(3.4)

where $\mathbf{n}(x)$ is the outward normal to the resonator boundary at the point $x \in \Gamma$. We slightly reinforce the middle boundary conditions assuming

$$
\mathbf{n}(x) \times \mathcal{A}(x,t) = 0, \\
x \in \Gamma, t > 0.
$$

(3.5)

Then the second conditions of (3.4) hold by differentiation. Moreover, then the third condition follows from (3.5) in local orthogonal coordinates. Indeed, let a point $y \in \partial V$ and $x_3 = 0$ on the tangent plane $T_y \Gamma$. Then (3.5) means that

$$
\mathcal{A}_1(y,t) = \mathcal{A}_2(y,t) = 0 \iff \mathcal{A}(y,t) = C(y) \mathbf{n}(y), \\
y \in \Gamma.
$$

(3.6)

These identities imply that

$$
\partial_k \mathcal{A}_j(y,t) = 0, \\
y \in \Gamma, \quad k, j = 1, 2
$$

(3.7)

if $\mathcal{A} \in C^1(\Gamma) \otimes \mathbb{R}^3$. In particular, $\partial_1 \mathcal{A}_2(y,t) - \partial_2 \mathcal{A}_1(y,t) = 0$ which implies the last condition of (3.4).

Finally, for the electronic wave function we assume the Dirichlet boundary condition

$$
\psi(x,t) = 0, \\
x \in \partial V, t > 0,
$$

(3.8)

which ensures the absence of electronic current on the boundary: $\mathbf{j}(x,t) = 0$ for $x \in \partial V$. 

4 Hamiltonian structure

The Hamiltonian functional is defined by

$$\mathcal{H}(A, \Pi, \psi, t) = \frac{1}{2} |c^2||\Pi|^2 + ||\text{rot} A||^2 + \frac{1}{2} (\psi, H_0(t) \psi), \quad H_0(t) := \frac{1}{2} D_o^2(t) + \phi(x) + \frac{1}{2} A^0(x). \quad (4.9)$$

Here $D_o(t) := -i \nabla + A(x) + A_p(x, t)$ and $A^0(x) := (-\Delta)^{-1} \rho(\cdot) = (-\Delta)^{-1} |\psi(\cdot)|^2$, where $(-\Delta)^{-1}$ is specified with the Dirichlet boundary conditions for $A^0(x, t)$ from (3.4). The system (2.1) under the boundary conditions (3.4), (3.8) can be formally written as

$$\dot{A}(t) = \mathcal{H}_\Pi, \quad \dot{\Pi}(t) = -\mathcal{H}_A - \sigma \Pi(x, t), \quad i \dot{\psi}(t) = (1 - i \epsilon) \mathcal{H}_\psi - i \gamma E(t) \psi(t). \quad (4.10)$$

5 Comments on previous results

- The Maxwell–Schrödinger system of type (2.1) in a bounded region was not considered previously. Such system was considered in [10, 11] for the case of the infinite space $V = \mathbb{R}^d$ with $d = 1, 2, 3$ for $\sigma = \epsilon = \gamma = 0$, $c = 1$ and zero pumping $A_p(x, t) = 0$:

$$\begin{cases}
\dot{A}(x, t) = \Delta A(x, t) + Pj(x, t), & \Delta A^0(x, t) = -\rho(x, t) \\
i \psi(x)(t) = H(t) \psi(t)
\end{cases}, \quad x \in \mathbb{R}^d. \quad (5.11)$$

For this system the existence of global solutions for all finite energy initial states was proved for the first time by Guo, Nakamitsu and Strauss [10]. Their approach relies on application of the Gagliardo–Nirenberg interpolation inequality. The uniqueness of the solution was not proved.

The complete result on the well-posedness in the energy space was established by Bejenaru and Tataru [11] providing strong a priori estimates. The methods [11] rely on microlocal analysis of pseudodifferential operators with “rough symbols”. In particular, these methods provide Lemma 11 of [11]: For each $0 \leq s \leq 2$ the operator $1 - [V - i \lambda]^2$ is a diffeomorphism $H^s(\mathbb{R}^3) \rightarrow H^{2-s}(\mathbb{R}^3)$ which depends continuously on $A \in H^1(\mathbb{R}^3)$. This lemma is a refinement of Proposition A.1 of [12].

- **Dissipative autonomous evolutionary PDEs.** The theory of attractors and long-time behaviour of solutions of such equations originated in the works of Ball, Foias, Hale, Henry, Temam, and was developed further by Babin and Vishik, Chepyzhov, Haraux, Ilyin, Miranville, Pata, Zelik, and others for the Navier-Stokes, reaction-diffusion, Ginzburg–Landau, damped wave and nonlinear Schrödinger, and sine-Gordon equations [22]–[40].

- **Hamiltonian autonomous evolutionary PDEs.** My team has a long time experience working with the theory of global attractors for nonlinear Hamiltonian evolutionary PDEs. I initiated this theory in 1990. The theory was inspired by Böhm’s transitions between quantum stationary states and resulted in more than 50 papers including joint papers in collaboration with H. Spohn, V. Buslaev and others. The global attraction to a compact attractor was established for a list of nonlinear Hamiltonian PDEs, see the surveys [3, 4, 5]. The proofs rely on a novel application of subtle tools of Harmonic Analysis: the Wiener Tauberian theorem, the Titchmarsh convolution theorem, the theory of quasimeasures, and others.

Tao established in [38] the existence of the global attractor for radial solutions to nonlinear defocusing Schrödinger equation without damping in $\mathbb{R}^n$ with $n \geq 11$.

- **Damped driven nonlinear wave and Ginzburg-Landau equations.** Absorbing sets and global attractors were constructed i) for damped driven nonlinear wave equations by Haraux [28, 29] for almost periodic external force, see also Ghidaglia and Temam [27], Mora and Solà-Morales [31], Babin, Chepyzhov and Vishik [22]–[24], and others, and ii) for the damped driven Ginzburg-Landau equations by Ghidaglia and Heron [35] (see also [22, 32]) in the case of a bounded region $V \subset \mathbb{R}^d$ with $n = 1, 2$.

- **Damped driven nonlinear Schrödinger equations.** The theory of global attractors was developed by Ghidaglia [34] on a bounded interval $V \subset \mathbb{R}$, by Wang [39] on the circle $V = \mathbb{R}/\mathbb{Z}$, by Abounouh [33] on a bounded region $V \subset \mathbb{R}^2$, and by Laurençot [37] on $V = \mathbb{R}^N$ with $N \leq 3$. These results on attraction were developed in [36, 39]. In these papers the pumping term does not depend on time, and in [34] the pumping term is time-periodic. The main achievement in these papers is the construction of a compact global attractor in the energy space $H^1$ relying on the Ball ideas [23].
A compact global attractor in the energy space for 2D weakly damped driven nonlinear Schrödinger equation with general nonlinearity was constructed in joint paper of the PI with E. Kopylova [40] for general bounded region \( V \subset \mathbb{R}^2 \) and almost periodic driving.

**The Maxwell–Klein–Gordon and other coupled equations.** For various coupled equations the results on well-posedness, existence of solitary waves and their effective dynamics were obtained by P. D’Ancona, M. Esteban, S. Klainerman, M. Machedon, S. Selberg, E. Séré, D. Stuart, and others, [12]–[21].

### 6 Sobolev type estimates for magnetic Schrödinger operator

We denote the spaces \( L^p = L^p(V), H^s = H^s(V), H^s_o = H^s_o(V) \), and \( \| \cdot \| \) is the norm in \( L^2 \). Denote \( X = [H^1 \otimes \mathbb{R}^3] \oplus [L^2(\mathbb{R}^3) \otimes \mathbb{R}^3] \oplus H^1 \) the Hilbert space of states \( X = (A, \Pi, \psi) \) satisfying the boundary conditions (3.4) and \( \text{div}A(x) = \text{div} \Pi(x) = 0 \) for \( x \in V \).

We will prove below in Lemma A.1 the bound for magnetic potential

\[
\|A\|_{L^2}^2 \leq C\|\nabla A\|^2, \quad A \in \mathbb{A}. \tag{6.12}
\]

This bound holds since the Laplacian \( \Delta \) under the boundary conditions (3.5) is nonnegative and symmetric on a dense domain \( D \subset \mathbb{A} \), and hence, it admits the selfadjoint extension. Finally, the spectrum is discrete and zero is not an eigenvalue. In Lemma A.2 we prove the equivalence of norms for magnetic Schrödinger operator

\[
b_1(\|A(t)\|^2_{H^1})|\psi|^2_{H^1} \leq \|D(t)\psi\| + \|\psi\|^2 \leq b_2(\|A(t)\|^2_{H^1})|\psi|^2_{H^1}, \quad \psi \in H^1, \tag{6.13}
\]

where \( D(t) := i\mathcal{N} - \Delta(x, t) \), \( b_1(r) > 0 \) (respectively \( b_2(r) > 0 \)) is a decreasing (respectively an increasing) function of \( r \geq 0 \). Similarly,

\[
b_1(\|A(t)\|^2_{H^1})|\psi|^2_{H^1} \leq \|H(t)\psi\| + \|\psi\|^2 \leq b_2(\|A(t)\|^2_{H^1})|\psi|^2_{H^1}, \quad \psi \in H^2 \cap H^1. \tag{6.14}
\]

The bounds (6.13) and (6.14) extend Lemma 11 of [11] to the case of bounded region. For the proof of (6.13) we will show that the difference of \( D^2(t) \) with \(-\Delta\) is a relatively compact operator.

We will assume that the potential \( \phi(x) \) is bounded,

\[
\sup_{x \in V} |\phi(x)| < \infty. \tag{6.15}
\]

Hence, we can assume that it is positive,

\[
\phi(x) \geq \varkappa > 0, \quad x \in V \tag{6.16}
\]

since the potential is defined up to an additive constant. Hence,

\[
E(t) = \langle \psi, H(t)\psi \rangle \geq \|D(t)\psi\|^2 + \varkappa\|\psi\|^2 + \langle \rho, (-\Delta)^{-1}\rho \rangle. \tag{6.17}
\]

Now (6.13) implies

\[
E(t) \geq \varkappa_1(\|A(t)\|^2_{H^1})|\psi|^2_{H^1}, \tag{6.18}
\]

where \( \varkappa_1 > 0 \) is a decreasing function. Hence, the standard Sobolev estimates together with (6.17) imply from (6.12) and (6.13) that

\[
\begin{align}
\|A\|_{L^p}^2 & \leq C\sum_k \|\nabla_k A\|^2, \quad A \in \mathbb{A}, \\
\|\psi\|_{L^p}^2 & \leq b(\|A(t)\|_{H^1})E(t), \quad \psi \in H^1.
\end{align} \tag{6.19}
\]

Here the last bound extends Lemma 11 of [11] to the case of bounded region.
7 A priori estimates

First, we obtain a priori estimates for sufficiently smooth solutions \((A(x,t), \Pi(x,t), \psi(x,t))\) of the Maxwell–Schrödinger system \((2.1)\), where all functions are \(C^0(\mathbb{R}^4)\). We plan to get rid of this smoothness assumption and establish the same estimates for all finite energy solutions. We will assume for the pumping field \(A_p(x,t)\) that it is \textit{almost periodic} and

\[
\sup_{x \in \mathbb{R}} \|A_p(x,t)\| + \|\nabla A_p(x,t)\| + |\dot{A}_p(x,t)| < \infty. \tag{7.20}
\]

Differentiating the charge \(Q(t) := \|\psi(t)\|^2\) we have

\[
Q(t) = \langle \psi(t), \psi(t) \rangle + \langle \psi(t), \Psi(t) \rangle = \langle (-i - \varepsilon)H(t)\psi(t), \psi(t) \rangle + \langle \psi(t), (-i - \varepsilon)H(t)\psi(t) \rangle - 2\gamma E(t)Q(t)
\]

\[
= -2\varepsilon E(t) - 2\gamma E(t)Q(t) \leq -2\varepsilon \varepsilon Q(t) - 2\gamma \varepsilon Q^2(t), \tag{7.21}
\]

where we used (6.17). Hence,

\[
Q(t) \leq Q(0). \tag{7.22}
\]

Now differentiating the energy \(\delta(t) := \mathcal{H}(\Pi(t), A(t), \psi(t), t)\) and using (2.1) and (6.17), (7.20), we get

\[
\delta(t) = \langle \mathcal{H}_a, \dot{A} \rangle + \langle \mathcal{H}_\Pi, \dot{\Pi} \rangle + \langle \mathcal{H}_\psi, \dot{\psi} \rangle + \mathcal{H}_t
\]

\[
= \langle \mathcal{H}_a, \dot{A} \rangle + \langle \mathcal{H}_\Pi, -\sigma \Pi + \mathcal{H}_\psi, (-i - \varepsilon)\mathcal{H}_\psi - \gamma E(t)\psi(t) - \langle D(t)\psi, \dot{A}_p \psi \rangle
\]

\[
\leq -\sigma \|\Pi(t)\|^2 - \varepsilon \langle H(t)\psi(t), H(t)\psi(t) \rangle - \gamma E^2(t) + C_p E(t) + C_p \|D(t)\psi(t)\|\|\psi(t)\| \leq C_1 \leq \infty, \tag{7.23}
\]

since \(\mathcal{H}_\psi = H(t)\psi(t)\). Hence, (2.2) and (6.13), and (6.14) imply a priori estimate

\[
\|\nabla A(t)\|^2 + \|\Pi(t)\|^2 + \|\psi(t)\|_{\Omega^1}^2 + \varepsilon \int_0^t b_1(\|A(s)\|_{\Omega^1})\|\psi(s)\|_{\Omega^1}^2 ds \leq C(t + 1), \|\psi(t)\| \leq C < \infty, \quad t > 0. \tag{7.24}
\]

8 Absorbing set

The estimates (7.24) are insufficient to prove the existence of a bounded absorbing set. We will follow the ideas of Haraux [28] (see also [22]) introducing the functional

\[
\Phi(A, \Pi, \psi, t) = \mathcal{H}(A, \Pi, \psi, t) + \eta \langle \Pi, A \rangle \tag{8.25}
\]

with a small \(\eta > 0\). Differentiating \(\Phi(t) := \Phi(A(t), \Pi, \psi(t), t)\) and using (7.23), we obtain

\[
\Phi(t) = \delta(t) + \eta \langle \Pi(t), \Pi(t) \rangle + \eta \langle \Pi(t), A(t) \rangle
\]

\[
\leq -\sigma \|\Pi(t)\|^2 - \varepsilon \langle H(t)\psi(t), H(t)\psi(t) \rangle - \gamma E^2(t) + C_p E(t) + \eta \langle \Pi(t), \Pi(t) \rangle
\]

\[
+ \eta \langle \Delta A(t) - \sigma \Pi(t) + j(t), A(t) \rangle \tag{8.26}
\]

The most problematic term

\[
\langle j(t), A(t) \rangle = -\operatorname{Re} \langle \overline{\psi(t)}D(t)\psi(t), A(t) \rangle \tag{8.27}
\]

can be estimated using the Sobolev-type estimates (6.19):

\[
\|j(t), A(t)\| \leq C\|A(t)\|_{L^5} \|\psi(t)\|_{L^3} \cdot \|D(t)\psi(t)\| \leq C_1 \|\nabla A(t)\|_{L^{1/2}} \|D(t)\psi(t)\|
\]

\[
\leq \delta \|\nabla A(t)\|^2 + \frac{C_2}{\delta} E^2(t), \tag{8.28}
\]
where the last inequality holds by (6.17). For the remaining terms similar estimates follow from the first estimate (6.12) and from (7.20):

\[ |(\Pi(t), A(t))| \leq \delta \| \nabla A(t) \|^2 + \frac{1}{\delta} \| \Pi(t) \|^2. \]  

(8.29)

Now (8.26) implies that for any \( \delta > 0 \)

\[ \Phi(t) \leq -\eta(1-3\delta) \| \nabla A(t) \|^2 - (\sigma - \eta \frac{n}{\delta}) \| \Pi(t) \|^2 - (\gamma - C_2 \frac{\eta}{\delta}) E^2(t) - \epsilon \| H(t) \psi(t) \|^2 + C_p. \]  

(8.30)

It remains to choose \( \delta, \eta > 0 \) such that

\[ \min(\eta(1-3\delta), \sigma - \eta \frac{n}{\delta}, \gamma - C_2 \frac{\eta}{\delta}) > 0. \]  

(8.31)

Then (8.30) and (4.9) imply that

\[ \Phi(t) \leq -\alpha \mathcal{H}(t) - \epsilon \| H(t) \psi(t) \|^2 + C_p, \quad t > 0, \]  

(8.32)

where \( \alpha > 0 \) and \( C_p \in \mathbb{R} \) do not depend on the solution. However, (6.12) implies that for sufficiently small \( \eta > 0 \) we have

\[ c \mathcal{H}(A, \Pi, \psi, t) \leq \Phi(A, \Pi, \psi, t) \leq C \mathcal{H}(A, \Pi, \psi, t) \]  

(8.33)

with \( c, C > 0 \). Hence, (8.32) and (6.14) imply that for small \( \eta > 0 \)

\[ \Phi(t) \leq -\beta \Phi(t) - \epsilon \| H(t) \psi(t) \|^2 + C_p, \quad t > 0, \]  

(8.34)

where \( \beta > 0 \) and \( C_p \in \mathbb{R} \) do not depend on the solution. Now the integration yields

\[ \Phi(t) + \epsilon \int_0^t e^{-\beta(t-s)} \| H(s) \psi(s) \|^2 ds \leq \Phi(0) e^{-\beta t} + \frac{C_p}{\beta}, \quad t > 0. \]  

(8.35)

Hence, (4.9) and (6.17), (8.33) imply that for sufficiently small \( \eta > 0 \) a priori estimate (7.24) refines to

\[ \| \nabla A(t) \|^2 + \| \Pi(t) \|^2 + \| D(t) \psi \|^2 + \| \psi \|^2 + \epsilon \int_0^t e^{-\beta(t-s)} \| H(s) \psi(s) \|^2 ds \leq C \Phi(0) e^{-\beta t} + \frac{C_p}{\beta}, \quad t > 0. \]  

(8.36)

Now (6.13) and (6.14) imply that

\[ \| \nabla A(t) \|^2 + \| \Pi(t) \|^2 + b_1(M) \| \psi \|^2_{H^1} + \epsilon b_1(M) \int_0^t e^{-\beta(t-s)} \| \psi(s) \|^2_{H^1} ds \leq C \Phi(0) e^{-\beta t} + \frac{C_p}{\beta}, \quad t > 0. \]  

(8.37)

where

\[ M := \sup_{i \geq 0} \| A(t) \|_{H^1} \leq C \sup_{i \geq 0} \| \nabla A(t) \|_{H^1} < \infty \]

by (8.36). Let us write the system (2.1) as

\[ \dot{X}(t) = F(X(t), t), \quad X(t) = (A(t), \Pi(t), \psi(t)). \]  

(8.38)

**Corollary 8.1.** The bounds (8.37) imply for solutions \( X(t) \) of (8.38)

\[ \| X(t) \|^2_{\mathcal{X}} + \epsilon_1 \int_0^t e^{-\beta(t-s)} \| \psi(s) \|^2_{H^1} ds \leq C \Phi(0) e^{-\beta t} + \frac{C_p}{\beta}, \quad t > 0, \]  

(8.39)

where \( \epsilon_1, \beta > 0 \). Hence, for any \( R > 0 \) the set \( \mathcal{B} := \{ Y \in \mathbb{X} : \| Y \|^2_{\mathcal{X}} \leq C(1 + \frac{C_p}{\beta}) \} \) absorbs the ball \( \{ Y \in \mathbb{X} : \| Y \|_{\mathcal{X}} \leq R \} \) for large times \( t > t_R \).
A Proof of the Sobolev type estimates for magnetic Schrödinger operator

Let a point \( y \in \Gamma := \partial \Omega \) and \( x_3 = 0 \) on the tangent plane \( T_y \Gamma \). Then (3.7) together with \( \text{div} \, A(y, t) \equiv 0 \) for \( y \in \Omega \) implies that

\[
\partial_3 A_3(y, t) = 0 \tag{A.40}
\]

if \( A \in C^1(\overline{\Omega}) \otimes \mathbb{R}^3 \). Hence, the boundary conditions (3.6), (A.40) can be written as

\[
A_3(x) = 0, \quad \nabla_n A_n(x) = 0, \quad x \in \Gamma, \tag{A.41}
\]

where \( A_3(x) \) is the tangential to the boundary projection, while \( A_n(x) \) is the normal to the boundary projection of \( A(x) \). These boundary conditions and the Stokes formula imply that for \( A \in C^2(\overline{\Omega}) \otimes \mathbb{R}^3 \)

\[
\langle A(t), \Delta A(t) \rangle = \int_\Gamma A(x, t) \cdot \nabla A(x, t) dx - \sum_k \| \nabla_k A(t) \|^2 = - \sum_k \| \nabla_k A(t) \|^2, \tag{A.42}
\]

since the field \( A(x) \) is orthogonal to the boundary \( \Gamma \) while \( \nabla_n A(x) \) is parallel to the boundary at any point \( x \in \Gamma \) due to the boundary conditions (A.41). Hence, the variational derivative

\[
D_\Delta \sum_k \| \nabla_k A \|^2 = -2 \Delta A \tag{A.43}
\]

if \( A \in C^2(\overline{\Omega}) \otimes \mathbb{R}^3 \) and satisfies the boundary conditions (A.41).

**Lemma A.1.** The Laplacian \( \Lambda = -\Delta \) is symmetric and nonnegative on the dense domain \( D_0 = A'(\Omega) \cap C^\infty(\overline{\Omega}) \otimes \mathbb{R}^3 \) in the Hilbert space of vector fields \( \mathcal{X} := L^2(\Omega) \otimes \mathbb{R}^3 \), and the identity holds

\[
\langle A, \Lambda A \rangle = \sum_k \| \nabla_k A \|^2, \quad A \in D_0. \tag{A.44}
\]

The operator \( \Lambda \) admits a selfadjoint extension with a domain \( D \subset A' \), and

\[
\langle A, \Lambda A \rangle \geq \delta \| A \|^2, \quad A \in D, \tag{A.45}
\]

where \( \delta > 0 \).

**Proof.** The Green formula implies that for any vector fields \( A_1, A_2 \in D_0 \)

\[
\langle \Lambda A_1, A_2 \rangle - \langle A_1, \Lambda A_2 \rangle = - \int_\Gamma [\nabla_n A_1(x) \cdot A_2(x) - A_1(x) \cdot \nabla_n A_2(x)] dx = 0 \tag{A.46}
\]

which follows similarly to (A.42). Moreover, the operator \( \Lambda \) is nonnegative on the domain \( D_0 \) by (A.42), and hence, it admits a selfadjoint extension by the Friedrichs theorem. The identity (A.44) follows from (A.42).

Finally, \( \Lambda \) is an elliptic operator in the bounded region \( \Omega \), and the boundary conditions (A.41) satisfy the Shapiro–Lozinskii condition. Hence, the spectrum of \( \Lambda \) is discrete. Namely, \( \Lambda - z \) is invertible for \( z < 0 \) and the resolvent \( R(z) = (\Lambda - z)^{-1} \) is a compact selfadjoint operator in \( \mathcal{X} \) by the elliptic theory and the Sobolev embedding theorem. Finally, the spectra of \( \Lambda - z \) and of \( \Lambda \) differ by the shift.

Now to prove (A.45) it suffices to check that \( \lambda = 0 \) is not an eigenvalue. Indeed, \( \Lambda A = 0 \) implies \( A \in D \) by the elliptic theory, and hence (A.44) implies

\[
\langle A, \Lambda A \rangle = \sum_k \| \nabla_k A \|^2 = 0. \tag{A.47}
\]

Hence, \( A(x) \equiv a \in \mathbb{R}^3 \) for \( x \in \Omega \). Finally, the boundary conditions (A.41) imply that \( a = 0 \).

To prove (6.13) let us rewrite it equivalently as

\[
\langle \psi, \Lambda \psi \rangle \leq C_2(\| A \|_{H^1}) \langle \psi, L \psi \rangle, \quad \psi \in H^1, \tag{A.48}
\]

where \( \Lambda := -\Delta + 1 \) and \( L := (i \nabla + A(x))^2 + 1 \). Now (A.48) follows from the next lemma with \( \delta < 1 \).
Lemma A.2. Let $A \in \mathcal{A}$. Then the difference $T := L - \Lambda$ admits the estimate

$$
|\langle \psi, T\psi \rangle| \leq \delta \langle \psi, \Lambda \psi \rangle + C_\delta(\|A\|_{H^1}) \|\psi\|, \quad \psi \in H^1
$$

(A.49)

for any $\delta > 0$, where $C_\delta(\cdot)$ is a continuous increasing function on $[0, \infty)$.

Proof. The difference $T$ reads

$$
T = 2iA(x)\nabla + \Lambda^2(x),
$$

(A.50)

where we have used that $\text{div} A(x) \equiv 0$. Further, it suffices to prove (A.49) for $\psi \in D_0 := C_0^\infty(\Omega)$ which is dense in the space $H^1$. Hence, the operators $\Lambda$ and $T$ can be considered as the operators on entire space $\mathbb{R}^3$. In particular, let us extend $A(x)$ by zero outside $\Omega$ and define the powers $\Lambda^s$ for $s \in \mathbb{R}$ by the Fourier transform. Then (A.49) can be reduced by the substitution $\psi = \Lambda^{-1/2}\phi$ to the estimate

$$
|\langle \phi, \Lambda^{-1/2}T\Lambda^{-1/2}\phi \rangle| \leq \delta \|\phi\|^2 + C_\delta(\|A\|_{H^1}) \|\Lambda^{-1/2}\phi\|^2, \quad \phi \in L^2(\mathbb{R}^3).
$$

(A.51)

This reduction is not equivalent, but (A.51) implies (A.49) since every function $\psi \in D_0$ admits the representation $\psi = \Lambda^{-1/2}\phi$ with $\phi \in L^2(\mathbb{R}^3)$.

In the case $A(x) \in C_0^\infty(\mathbb{R}^3)$ the operator $\Lambda^{-1/2}T\Lambda^{-1/2}$ is the PDO of order $-1$, and estimates of type (A.51) in this case follows from the interpolation inequality for the Sobolev norms. However, the constant $C_\delta$ is known to depend on some derivatives of the symbol of the composition. So we should find another arguments to prove that this constant depends only on the norm $\|A\|_{H^1}(\Omega)$.

For this purpose we note that the operator $\Lambda^{-1/2}$ is the multiplication by $\langle \xi \rangle^{-1/2}$ in the Fourier transform. Hence, it is the convolution with a distribution $S(x) \in L^1_{\text{loc}}(\mathbb{R}^3)$ which is a radial smooth function for $x \neq 0$ and asymptotically homogeneous at the origin

$$
S(x) \sim C|x|^{-2}, \quad |x| \to 0.
$$

(A.52)

Hence,

$$
S \in L_q^s(\mathbb{R}^3), \quad q \in [1, \frac{3}{2}).
$$

(A.53)

Thus, $\Lambda^{-1/2} = S*$, and hence,

$$
\Lambda^{-1/2}T\Lambda^{-1/2} = 2iS* A(x)\nabla S* + S* A^2(x)S*,
$$

(A.54)

Let us estimate the first term on the right hand side. The second can be bounded similarly.

We denote by $\nabla S*$ the composition of $\nabla$ with $S*$. This composition is the bounded operator in $L^2(\mathbb{R}^3)$ since it is the multiplication by $-i\langle \xi \rangle^{-1/2}$ in the Fourier transform. Thus,

$$
\|\nabla S* \phi\| \leq C\|\phi\|.
$$

(A.55)

Further, the multiplication by $A \in L^6$ maps continuously $L^2(\mathbb{R}^3)$ into $L^{3/2}(\mathbb{R}^3)$ by the Hölder inequality.

$$
\|A(x)\nabla S* \phi\|_{L^{3/2}(\mathbb{R}^3)} \leq C\|\phi\|.
$$

(A.56)

Moreover, supp $A \subset \overline{\Omega}$, and hence, the convolution $S* [A(x)\nabla S* \phi] \in L^{3-\alpha'}(\Omega)$ with sufficiently small $\alpha' > 0$ by (A.53) and the Young theorem on the convolution. Similarly,

$$
\|A^\alpha S* [A(x)\nabla S* \phi]\|_{L^1(\Omega)} \leq C\|\phi\|
$$

(A.57)

for small $\alpha > 0$. This follows by the same Young theorem since $\Lambda^\alpha S*$ for $\alpha < 1/2$ is the operator of convolution with the distribution $S^\alpha \in L^1_{\text{loc}}(\mathbb{R}^3)$ which admits the asymptotics

$$
S^\alpha(x) \sim C|x|^{-2-2\alpha}, \quad |x| \to 0.
$$

(A.58)

This asymptotics holds since the Fourier transform of $S^\alpha$ equals to $\langle \xi \rangle^{-1/2 + \alpha}$.

Finally, (A.57) means that the operator $\Lambda^\alpha S* [A(x)\nabla S* \phi]$ is bounded in $L^2(\Omega)$ since the region $\Omega$ is bounded. In other words, the operator $S* [A(x)\nabla S* \phi]$ is continuous from $L^2(\Omega)$ to the Sobolev space $H^{2\alpha}(\Omega)$ with sufficiently small $\alpha > 0$. Hence, the bound of type (A.51) for the first term on the right hand side of (A.54) follows from the interpolation inequality for the Sobolev norms. The second term can be bounded similarly. □
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