Superconformal Symmetry and Correlation Functions

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Abstract
Four-dimensional $\mathcal{N}$-extended superconformal symmetry and correlation functions of quasi-primary superfields are studied within the superspace formalism. A superconformal Killing equation is derived and its solutions are classified in terms of supertranslations, dilations, Lorentz transformations, $R$-symmetry transformations and special superconformal transformations. In general, due to the invariance under supertranslations and special superconformal transformations, superconformally invariant $n$-point functions reduce to one unspecified $(n-2)$-point function which must transform homogeneously under the remaining rigid transformations, i.e. dilations, Lorentz transformations and $R$-symmetry transformations. Based on this result, we are able to identify all the superconformal invariants and obtain the general form of $n$-point functions for scalar superfields. In particular, as a byproduct, a selection rule for correlation functions is derived, the existence of which in $\mathcal{N} = 4$ super Yang-Mills theory was previously predicted in the context of AdS/CFT correspondence [1]. Superconformally covariant differential operators are also discussed.

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1 Introduction & Summary

Superconformal field theories have been of renewed attention after the Maldacena conjecture that the string/M theory on $AdS_{d+1}$ backgrounds is dual to a conformal field theory in a spacetime of dimension, $d$, which is interpreted as the boundary of $AdS_{d+1}$ [2–4]. As all the known nontrivial conformal field theories in higher than two dimensions are supersymmetric theories [3–14], it is natural to consider a group which combines supersymmetry and conformal symmetry together, i.e. the superconformal group. In fact, the pioneering work on supersymmetry in four-dimensions [17] introduced the $\mathcal{N} = 1$ superconformal symmetry, though it is broken at quantum level.

Contrary to the ordinary conformal symmetry, not all spacetime dimensions allow superconformal symmetry. The standard supersymmetry algebra admits an extension to a superconformal algebra only if $d \leq 6$ [18] (for a review see [19]). In particular in four-dimensions, which is of our interest in this paper, the bosonic part of the superconformal algebra is

$$o(2, 4) \oplus u(N) .$$

(1.1)

Hence the four-dimensional superconformal group is identified with a supermatrix group, $SU(2, 2|N)$ [21,22] or its complexification, $SL(4|N; \mathbb{C})$ [22,23]. Normally $N \geq 5$ cases are excluded from the renormalization point of view, as theories with more than four supercharges must have spins higher than one such as graviton/gravitino and it is unlikely that supergravity theories are renormalizable. However, such a restriction on the value of $\mathcal{N}$ is not relevant to our work.

According to the conjecture [2–4], four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ is dual to type IIB string theory on $AdS_5 \times S^5$ in the limit of small $gYM$ and large but fixed ’t Hooft coupling, $g_{YM}^2 N$. In this limit, the string theory can be effectively described by tree level type IIB supergravity, while the field theory dual is strongly coupled. As the perturbative approach breaks down in the strongly coupled CFT side, to check the conjectured duality it is desirable to have non-perturbative understanding on super Yang-Mills theory. This motivates us to explore four-dimensional $\mathcal{N}$-extended superconformal symmetry and correlation functions subject to the symmetry as done in the present paper.

In our previous work [21], six-dimensional ($\mathcal{N}, 0$) superconformal symmetry was analyzed in terms of coordinate transformations on superspace and through dimensional reduction basic features of four-dimensional $\mathcal{N}$-extended superconformal symmetry were obtained. In the present paper, in a similar fashion to [21,24] but in a self-contained manner, we analyze four-dimensional $\mathcal{N}$-extended superconformal symmetry on superspace. Our main results concern the general forms of superconformally invariant $n$-point functions for
quasi-primary superfields. In particular, as a byproduct, we obtain a selection rule for correlation functions of the component fields, \( \psi^I(x) \), appearing in the power series expansions of quasi-primary superfields in Grassmann coordinates, \( \theta \) and \( \bar{\theta} \). The selection rule states that if the sum of the \( R \)-symmetry charge, \( \kappa_i \), is not zero then the correlation function of the component fields vanishes

\[
\langle \psi^{I_1}_{x_1} \cdots \psi^{I_n}_{x_n} \rangle = 0 \quad \text{if} \quad \sum_{i=1}^n \kappa_i \neq 0.
\]

The existence of this kind of selection rule in \( \mathcal{N} = 4 \) super Yang-Mills theory was previously predicted by Intriligator within the context of \( AdS/CFT \) correspondence, as the dual IIB supergravity contains a corresponding \( U(1) \) symmetry [1]. Therefore our results provide a supporting evidence for the Maldacena conjecture, as the selection rule here is derived by purely considering the symmetry on CFT side without referring to the string side.

The contents of the present paper are as follows. In section 2, we first define the four-dimensional \( \mathcal{N} \)-extended superconformal group in terms of coordinate transformations on superspace as a generalization of the definition of ordinary conformal transformations. We then derive a superconformal Killing equation, which is a necessary and sufficient condition for a supercoordinate transformation to be superconformal. The general solutions are identified in terms of supertranslations, dilations, Lorentz transformations, \( R \)-symmetry transformations and special superconformal transformations, where \( R \)-symmetry is given by \( U(N) \) as in eq.(1.1) and supertranslations and special superconformal transformations are dual to each other through superinversion map. The four-dimensional \( \mathcal{N} \)-extended superconformal group is then identified with a supermatrix group, \( SU(2,2|\mathcal{N}) \), having dimensions \((15+\mathcal{N}^2)|8\mathcal{N}\) as known. However, we point out that for \( \mathcal{N} = 4 \) case an equivalence relation must be imposed on the supermatrix group and so the four-dimensional \( \mathcal{N} = 4 \) superconformal group is isomorphic to a quotient group of the supermatrix group.

In section 3, we obtain an explicit formula for the finite nonlinear superconformal transformations of the supercoordinates, \( z \), parameterizing superspace and discuss several representations of the superconformal group. We also construct matrix or vector valued functions depending on two or three points in superspace which transform covariantly under superconformal transformations. For two points, \( z_1 \) and \( z_2 \), we find a matrix, \( \hat{I}(z_1,z_2) \), which transforms covariantly like a product of two tensors at \( z_1 \) and \( z_2 \). For three points, \( z_1, z_2, z_3 \), we find ‘tangent’ vectors, \( Z_i \), which transform homogeneously at \( z_i, i = 1,2,3 \). These are crucial variables for obtaining two-point, three-point and general \( n \)-point correlation functions later.

In section 4, we discuss the superconformal invariance of correlation functions for quasi-primary superfields and exhibit general forms of two-point, three-point and \( n \)-point func-
tions. Explicit formulae for two-point functions of superfields in various cases are given. In general, due to the invariance under supertranslations and special superconformal transformations, $n$-point functions reduce to one unspecified $(n-2)$-point function which must transform homogeneously under the rigid transformations - dilations, Lorentz transformations and $R$-symmetry transformations. We then identify all the superconformal invariants and obtain the general form of $n$-point functions of scalar superfields. As a byproduct, we derive the selection rule for correlation functions (1.2).

In section 3, superconformally covariant differential operators are discussed. The conditions for superfields, which are formed by the action of spinor derivatives on quasi-primary superfields, to remain quasi-primary are obtained. In general, the action of differential operator on quasi-primary fields generates an anomalous term under superconformal transformations. However, with a suitable choice of scale dimension and $R$-symmetry charge, we show that the anomalous term may be cancelled. We regard this analysis as a necessary step to write superconformally invariant actions on superspace, as the kinetic terms in such theories may consist of superfields formed by the action of spinor derivatives on quasi-primary superfields.

In the appendix, the explicit form of superconformal algebra and a method of solving the superconformal Killing equation are exhibited.

Recent review on the implications of $\mathcal{N} = 1$ superconformal symmetry for four-dimensional quantum field theories is contained in [25] and some related works on superconformally invariant correlation functions can be found in [26–33]. $\mathcal{N} = 1$ superconformal symmetry on curved superspace is studied in [34–36] and conformally covariant differential operators in non-supersymmetric theories are discussed in [37,38].

2 Superconformal Symmetry in Four-dimensions

In this section we first define the four-dimensional $\mathcal{N}$-extended superconformal group on superspace and then discuss its superconformal Killing equation along with the solutions.
2.1 Four-dimensional Superspace

The four-dimensional supersymmetry algebra has the standard form with $P_\mu = (H, -P)$

\[
\{Q_{aa}, Q^b_\dot{\alpha}\} = 2\delta^b_a \sigma^\mu_{a\dot{\alpha}} P_\mu, \quad (2.1)
\]

\[
[P_\mu, P_\nu] = [P_\mu, Q_{aa}] = [P_\mu, \bar{Q}^a_\dot{\alpha}] = \{Q_{aa}, Q_{bb}\} = \{\bar{Q}^a_\dot{\alpha}, \bar{Q}^b_\dot{\beta}\} = 0,
\]

where $1 \leq \alpha, \dot{\alpha} \leq 2$, $1 \leq a \leq N$ and $Q_{aa}$ satisfies

\[
Q_{aa}^\dagger = \bar{Q}^a_\dot{\alpha}. \quad (2.2)
\]

$P_\mu, Q_{aa}$ and $\bar{Q}^a_\dot{\alpha}$ generate a supergroup, $G_T$, with parameters, $z^M = (x^\mu, \theta^{a\alpha}, \bar{\theta}^{\dot{\alpha}})$, which are coordinates on superspace. The general element of $G_T$ is written in terms of these coordinates as

\[
g(z) = e^{(z \cdot P + \theta^{a\alpha} Q_{aa} + \bar{\theta}^{\dot{\alpha}})} . \quad (2.3)
\]

Corresponding to eq.(2.2) we may assume $\theta^{a\alpha}$ to satisfy

\[
\theta^{a\alpha\ast} = \bar{\theta}^{\dot{\alpha}} , \quad (2.4)
\]

so that

\[
g(z)^\dagger = g(z)^{-1} = g(-z). \quad (2.5)
\]

The Baker-Campbell-Haussdorff formula with the supersymmetry algebra (2.1) gives

\[
g(z_1)g(z_2) = g(z_3) , \quad (2.6)
\]

where

\[
x^\mu_3 = x^\mu_1 + x^\mu_2 + i\theta_1^{a\sigma} \sigma^\mu \bar{\theta}_{2a} - i\theta_2^{a\sigma} \sigma^\mu \bar{\theta}_{1a} , \quad \theta^a_3 = \theta^a_1 + \theta^a_2 , \quad \bar{\theta}_{3a} = \bar{\theta}_{1a} + \bar{\theta}_{2a}. \quad (2.7)
\]

Letting $z_1 \rightarrow -z_2$ we may get the supertranslation invariant one forms, $e^M = (e^\mu, d\theta^{a\alpha}, d\bar{\theta}^{\dot{\alpha}})$, where

\[
e^\mu(z) = dx^\mu + i d\theta^a \sigma^\mu \bar{\theta}_a - i \theta^a \sigma^\mu d\bar{\theta}_a . \quad (2.8)
\]

The exterior derivative, $d$, on superspace is defined as

\[
d \equiv dz^M \frac{\partial}{\partial z^M} = e^M D_M = e^\mu \partial_\mu + d\theta^{a\alpha} D_{a\alpha} - d\bar{\theta}^{\dot{\alpha}} D_{\dot{\alpha}} . \quad (2.9)
\]

\(^1\text{See Appendix A for our notations and some useful equations.}\)
where \( D_M = (\partial_\mu, D_{a\alpha}, -\bar{D}^{\dot{a}}_\dot{\alpha}) \) are covariant derivatives
\[
\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad D_{a\alpha} = \frac{\partial}{\partial \theta^{a\alpha}} - i(\sigma^\mu \bar{\theta})_\alpha \frac{\partial}{\partial x^\mu}, \quad \bar{D}^{\dot{a}}_\dot{\alpha} = - \frac{\partial}{\partial \bar{\theta}^{\dot{a}\dot{\alpha}}} + i(\bar{\theta}^\alpha \sigma^\mu)_{\dot{\alpha}} \frac{\partial}{\partial x^\mu},
\]
satisfying the anti-commutator relations
\[
\{ D_{a\alpha}, \bar{D}^{\dot{a}}_{\dot{\alpha}} \} = 2i \delta_a^{\dot{a}} \sigma_{a\dot{a}}^\mu \partial_\mu.
\]
Under an arbitrary superspace coordinate transformation, \( z \rightarrow z' \), \( e^M \) and \( D_M \) transform as
\[
e^M(z') = e^N(z) R^M_N(z), \quad D'_M = R^{-1}_M e^N(z) D_N,
\]
so that the exterior derivative is left invariant
\[
e^M(z) D_M = e^M(z') D'_M,
\]
where \( R^N_M(z) \) is a \((4 + 4\mathcal{N}) \times (4 + 4\mathcal{N})\) supermatrix of the form
\[
R^N_M(z) = \begin{pmatrix}
R^\nu_\mu(z) & \partial_\mu \bar{\theta}^{\dot{a}\beta} & \partial_\mu \bar{\theta}^{\dot{a}\dot{\beta}} \\
B_{a\alpha}^\mu(z) & D_{a\alpha} \theta^b \theta^{b\beta} & D_{a\alpha} \bar{\theta}^{b\beta} \\
-B_{\dot{a}\dot{\alpha}}^\mu(z) & -\bar{D}_{\dot{a}\dot{\alpha}} \theta^a \theta^{a\beta} & -\bar{D}_{\dot{a}\dot{\alpha}} \bar{\theta}^{a\dot{\beta}}
\end{pmatrix},
\]
with
\[
R^\nu_\mu(z) = \frac{\partial x'^\nu}{\partial x^\mu} + i \frac{\partial \theta^a}{\partial x^\mu} \sigma^\nu \bar{\theta}^a - i \theta^{a\alpha} \sigma^\nu \partial_\mu \bar{\theta}^a, \tag{2.15}
\]
\[
B_{a\alpha}^\mu(z) = D_{a\alpha} x'^\mu + i D_{a\alpha} \theta^b \sigma^\mu \bar{\theta}^b + i \theta^{a\alpha} \sigma^\mu D_{a\alpha} \bar{\theta}^b, \tag{2.16}
\]
\[
\bar{D}_{\dot{a}\dot{\alpha}}^\mu(z) = \bar{D}_{\dot{a}\dot{\alpha}} x'^\mu + i \bar{D}_{\dot{a}\dot{\alpha}} \theta^a \sigma^\mu \bar{\theta}^a + i \theta^{a\alpha} \sigma^\mu \bar{D}_{\dot{a}\dot{\alpha}} \bar{\theta}^a = (B_{a\alpha}^\mu(z))^\dagger. \tag{2.17}
\]

### 2.2 Superconformal Group & Killing Equation

The superconformal group is defined here as a group of superspace coordinate transformations, \( z \rightarrow z' \), that preserve the infinitesimal supersymmetric interval length, \( e^2 = \eta_{\mu\nu} e^\mu e^\nu \) up to a local scale factor, so that
\[
e^2(z) \rightarrow e^2(z') = \Omega^2(z; g) e^2(z), \tag{2.18}
\]
where $\Omega(z;g)$ is a local scale factor. This requires $B_{\alpha a}^{\mu}(z) = B_{\tilde{\alpha} a}^{\mu}(z) = 0$

$$D_{\alpha a} x^{\mu} + i D_{\alpha a} \theta^b \sigma^\mu \tilde{\theta}'_b + i \theta^b \sigma^\mu D_{\alpha a} \tilde{\theta}'_b = 0,$$  

$$\tilde{D}_{\tilde{\alpha} a} x^{\mu} + i \tilde{D}_{\tilde{\alpha} a} \theta^b \sigma^\mu \tilde{\theta}'_b + i \theta^b \sigma^\mu \tilde{D}_{\tilde{\alpha} a} \tilde{\theta}'_b = 0,$$  

and

$$e^{\mu}(z') = e^{\nu}(z) R_{\nu}^{\mu}(z; g), \quad (2.20)$$

$$R_{\mu}^{\lambda}(z; g) R_{\nu}^{\sigma}(z; g) \eta_{\lambda \nu} = \Omega^2(z; g) \eta_{\mu \sigma}, \quad \det R(z; g) = \Omega^{4}(z; g). \quad (2.21)$$

Hence $R_{MN}^N$ in eq.(2.14) is of the form

$$R_{MN}^N(z; g) = \begin{pmatrix} R_{\mu}^{\nu}(z; g) & \partial_{\mu} \theta^{b \beta} & \partial_{\mu} \tilde{\theta}^{\beta} \\ 0 & D_{\alpha a} \theta^{b \beta} & D_{\alpha a} \tilde{\theta}^{\beta} \\ 0 & -\tilde{D}_{\tilde{\alpha} a} \theta^{b \beta} & -\tilde{D}_{\tilde{\alpha} a} \tilde{\theta}^{\beta} \end{pmatrix}. \quad (2.22)$$

$R_{\mu}^{\nu}(z; g)$ is a representation of the superconformal group. Under successive superconformal transformations, $z \to g \to z'$ giving $z \to g'' \to z''$, we have

$$R(z; g) R(z'; g') = R(z; g''). \quad (2.23)$$

Infinitesimally $z' \approx z + \delta z$, eq.(2.15) gives

$$D_{\alpha a} h^{\mu} = -2i (\sigma^\mu \lambda^a)_{\alpha}, \quad \tilde{D}_{\tilde{\alpha} a} h^{\mu} = 2i (\tilde{\lambda}^a \sigma^\mu)_{\tilde{\alpha}}, \quad (2.24)$$

where we define

$$\lambda^a = \delta \theta^a, \quad \tilde{\lambda}_{\tilde{a}} = \delta \tilde{\theta}_{\tilde{a}}, \quad (2.25)$$

$$h^{\mu} = \delta x^{\mu} + i \delta \theta^a \sigma^\mu \tilde{\theta}_{\tilde{a}} - i \theta^a \sigma^\mu \delta \tilde{\theta}_{\tilde{a}}. \quad (2.26)$$

Infinitesimally $R_{\mu}^{\nu}$ from eq.(2.15) is of the form

$$R_{\mu}^{\nu} \approx \delta_{\mu}^{\nu} + \partial_{\mu} h^{\nu}, \quad (2.26)$$

so that the condition (2.21) reduces to the ordinary conformal Killing equation

$$\partial_{\mu} h^{\nu} + \partial_{\nu} h^{\mu} \propto \eta_{\mu \nu}. \quad (2.27)$$

\textsuperscript{2}More explicit form of $R_{MN}^N$ is obtained later in eq.(3.49).
Eq. (2.27) follows from eq. (2.24). Using the anti-commutator relation for $D_a\alpha$ and $\bar{D}_b\check{\alpha}$, we get from eq. (2.24)

$$\delta_a^b h^\mu = \frac{1}{2} \left( D_a\lambda^\beta \sigma^\mu \tilde{\sigma}^\nu - (\tilde{\sigma}_\nu \sigma^\mu \bar{D}^b_a \check{\lambda}_a)^\alpha \right),$$  

(2.28)

and hence

$$\delta_a^b (\partial_\mu h_\nu + \partial_\nu h_\mu) = (D_a\lambda^{ba} - \bar{D}^b_a \check{\lambda}_a) \eta_{\mu\nu},$$  

(2.29)

which implies eq. (2.27). Thus eq. (2.24) is a necessary and sufficient condition for a super-coordinate transformation to be superconformal.

With the notation as in eq. (A.15) we write

$$h_\alpha \tilde{\alpha} = h^\mu \sigma_\mu \tilde{\alpha},$$  

$$\tilde{h}^\alpha = h^\mu \tilde{\sigma}_\mu \tilde{\alpha},$$

(2.30)

and using eq. (A.7), eq. (2.24) is equivalent to

$$D_a(\alpha h_\beta) \tilde{\beta} = 0,$$  

$$\bar{D}_a(\bar{\beta} \tilde{h}_{\bar{\beta}}) = 0,$$

(2.31)

or

$$D_{\alpha a} \tilde{h}^{\beta \gamma} = \frac{1}{2} \delta^{\beta \gamma} D_a \tilde{h}^\alpha \tilde{\gamma},$$  

$$\bar{D}_a \bar{\tilde{h}}^{\bar{\alpha} \bar{\beta}} = \frac{1}{2} \delta^{\bar{\alpha} \bar{\beta}} \bar{D}_a \bar{\tilde{h}}^\beta \tilde{\gamma},$$

(2.32)

while $\lambda^{a\alpha}, \bar{\lambda}_a^\check{\alpha}$ are given by

$$\lambda^{a\alpha} = -i \frac{1}{8} \bar{D}_a \tilde{h}^{\alpha \alpha},$$  

$$\bar{\lambda}_a^\check{\alpha} = i \frac{1}{8} D_{a a} \tilde{h}^{\check{\alpha} \alpha}.$$  

(2.33)

Eq. (2.31) may therefore be regarded as the fundamental superconformal Killing equation and its solutions give the generators of extended superconformal transformations in four-dimensions. The general solution is \footnote{A method of obtaining the solution (2.34) is demonstrated in Appendix B}:

$$\tilde{h}(z) = \tilde{x}_- b \tilde{x}_+ - \tilde{x}_-(\tilde{w} - \frac{1}{2} \lambda + 4 \rho_a \theta^a) + (w + \frac{1}{2} \lambda - 4 \bar{\theta}_a \bar{\rho}^a) \tilde{x}_+$$  

$$-4i \bar{\theta}_a t^a b^b \bar{\theta}^b + 2 \Omega \bar{\theta}_a \theta^a + 4 i (\bar{\varepsilon}_a \theta^a - \bar{\theta}_a \varepsilon^a) + \bar{\alpha},$$

(2.34)

where $a^\mu, b^\mu, \lambda \in \mathbb{R}, \Omega \in S^1, t \in su(N)$ and for $w_{\mu\nu} = -w_{\nu\mu}$ we define

$$w = \frac{1}{4} w_{\mu\nu} \tilde{\sigma}^\mu \sigma^\nu,$$  

$$\bar{w} = \frac{1}{4} w_{\mu\nu} \sigma^\mu \tilde{\sigma}^\nu.$$  

(2.35)

For later use it is worth to note

$$\epsilon \bar{w}^t \bar{\epsilon}^{-1} = -\bar{w},$$  

$$\epsilon^{-1} w^t \bar{\epsilon} = -w,$$  

$$\bar{w}^t = -w.$$  

(2.36)
Eq. (2.34) also gives
\[ \lambda^a = \varepsilon^a + \frac{1}{2}(\lambda + i\Omega)\theta^a - \theta^a \bar{w} + t^a_b \theta^b + \theta^a b \bar{x}_+ - i \bar{\rho}^a \bar{x}_+ - 4(\theta^a \rho_b) \theta^b, \quad (2.37) \]
and
\[ \delta \bar{x}_+ = \bar{x}_+ b \bar{x}_+ - 4 \bar{x}_+ \rho_a \theta^a + \lambda \bar{x}_+ + w \bar{x}_+ - \bar{x}_+ \bar{w} + 4i \bar{\varepsilon}_a \theta^a + \bar{\alpha}. \quad (2.38) \]
Note that \( \delta \bar{x}_+, \lambda^a \) are functions of \( \bar{x}_+, \theta^a \) only, which can be also directly shown from eq.(2.24). In fact, the superconformal group can be obtained alternatively by imposing the super-diffeomorphisms to leave the chiral subspaces of superspace invariant. The chiral structures are given by
\[ z^M_+ = (x^\mu_+, \theta^{\alpha a}), \quad z^M_- = (x^\mu_-, \bar{\theta}^a). \quad (2.39) \]
In this approach, one needs to solve a reality condition \[39, 40\]
\[ \delta \bar{x}_+(\bar{x}_+, \theta) - \delta \bar{x}_-(\bar{x}_-, \bar{\theta}) = 4i \bar{\lambda}_a(\bar{x}_-, \bar{\theta})\theta^a + 4i \bar{\theta}_a \lambda^a(\bar{x}_+, \theta). \quad (2.40) \]

2.3 Extended Superconformal Transformations

In summary, the generators of extended superconformal transformations in four-dimensions acting on the four-dimensional superspace, \( \mathbb{R}^4|\mathcal{N} \), with coordinates, \( z^M = (x^\mu, \theta^a, \bar{\theta}^a) \), can be classified as

1. Supertranslations, \( a, \varepsilon, \bar{\varepsilon} \)
\[ \delta x^\mu = a^\mu + i\varepsilon^a \sigma^\mu \bar{\theta}_a - i\theta^a \sigma^\mu \varepsilon_a, \quad \delta \theta^a = \varepsilon^a, \quad \delta \bar{\theta}_a = \bar{\varepsilon}_a. \quad (2.41) \]
This is consistent with eq.(2.7).

2. Dilations, \( \lambda \)
\[ \delta x^\mu = \lambda x^\mu, \quad \delta \theta^a = \frac{1}{2} \lambda \theta^a, \quad \delta \bar{\theta}_a = \frac{1}{2} \lambda \bar{\theta}_a. \quad (2.42) \]

3. Lorentz transformations, with \( w, \bar{w} \) defined in eq.(2.36)
\[ \delta x^\mu = w^\mu_{\nu} x^\nu, \quad \delta \theta^a = -\theta^a \bar{w}, \quad \delta \bar{\theta}_a = w \bar{\theta}_a. \quad (2.43) \]

4. \( R \)-symmetry transformations, \( U(\mathcal{N}) \), of dimension \( \mathcal{N}^2, t, \Omega \)
\[ \delta x^\mu = 0, \quad \delta \theta^a = t^a_b \theta^b + i \frac{1}{2} \Omega \theta^a, \quad \delta \bar{\theta}_a = -\bar{\theta}_b t^b_a - i \frac{1}{2} \Omega \bar{\theta}_a, \quad (2.44) \]
where the \( \mathcal{N} \times \mathcal{N} \) matrix, \( t \), is a SU(\( \mathcal{N} \)) generator, i.e. \( t^1 = -t, \ t^a_a = 0 \) and \( \Omega \in S^1 \).
5. Special superconformal transformations, $b, \rho, \bar{\rho}$

$$\delta x^\mu = 2x \cdot b x^\mu - x^2 b^\mu + \theta^a \sigma^\mu \bar{x}_+ \rho_a + \bar{\rho}^a \bar{x}_- \sigma^\mu \bar{\theta}_a + 2\theta^a b \bar{\theta}_b \sigma^\mu \bar{\theta}_a,$$

$$\delta \theta^a = \theta^a b \bar{x}_+ - i \bar{\rho}^a \bar{x}_+ - 4(\theta^a \rho_b) \theta^b,$$

$$\delta \bar{\theta}_a = \bar{x}_- b \bar{\theta}_a + i \bar{x}_- \rho_a - 4 \bar{\theta}_b (\bar{\rho}^b \bar{\theta}_a).$$

(2.45)

2.4 Superinversion

In four-dimensions we define superinversion, $z^M \xrightarrow{i_{s}} z'^M = (x'^\mu, \theta'^a, \bar{\theta}'_a) \in \mathbb{R}^{4|4N}$, by

$$x'^\mu = -\frac{x^\mu}{x^2}, \quad \theta'^a = -i \frac{1}{x^2} \varepsilon^{-1} \bar{x}_- \theta_a \bar{\zeta}^{ba}, \quad \bar{\theta}'_a = i \frac{1}{x^2} \varepsilon^{-1} x_+ \theta^b \bar{\zeta}_{ba},$$

(2.46)

where $\mathcal{N} \times \mathcal{N}$ matrices, $\zeta^{ab}, \bar{\zeta}_{ab}$ satisfy

$$\zeta^{ab} \bar{\zeta}_{bc} = \delta^a_c, \quad \zeta^{ab} = \bar{\zeta}^{ba}, \quad \bar{\zeta}_{ab} = (\zeta^{ab})^*.$$  

(2.47)

Eq. (2.46) may be rewritten as

$$\theta'^a = i \frac{1}{x^2} \bar{\theta}^a \bar{x}_-, \quad \bar{\theta}'_a = -\zeta^{ba} \theta^b \bar{\zeta};$$

$$\bar{\theta}'_a = -i \frac{1}{x^2} \bar{x}_+ \bar{\theta}_a, \quad \bar{\theta}_a = \epsilon \theta^b \bar{\zeta}_{ba}.$$  

(2.48)

It is easy to verify that superinversion is idempotent

$$i_{s}^2 = 1.$$  

(2.49)

Using

$$\bar{e} = e^\mu \bar{\sigma}_\mu = d\bar{x}_+ - 4i \bar{\theta}_a d\theta^a, \quad e = e^\mu \sigma^\mu = dx_+ + 4i d\theta_a \bar{\theta}^a,$$

we get under superinversion

$$\bar{e}(z') = x_+^{-1} e(z) x_+^{-1}, \quad e(z') = \bar{x}_-^{-1} \bar{e}(z) \bar{x}_+^{-1},$$

(2.51)

and hence

$$e^2(z') = \Omega^2(z; i_s) e^2(z), \quad \Omega(z; i_s) = \frac{1}{\sqrt{x_+^2 x_-^2}}.$$  

(2.52)
Eq. (2.51) can be rewritten as

\[ e^\mu(z') = e^\nu(z) R_\nu^\mu(z; i_s), \]  
\[ R_\nu^\mu(z; i_s) = \frac{1}{2} \text{tr} (\tilde{x}_-^{-1} \tilde{\sigma}_\nu \tilde{x}_+^{-1} \tilde{\sigma}_\mu) = \frac{1}{2} \text{tr} (x_+^{-1} \sigma_\nu x_-^{-1} \sigma_\mu), \]  
\[ (2.53) \]

or using eq. (A.6b)

\[ R_\nu^\mu(z; i_s) = \frac{1}{2} \text{tr} (\tilde{x}^{-1} \tilde{\sigma}_\nu \tilde{x}^{-1} \tilde{\sigma}_\mu), \]
\[ (2.54) \]

If we consider a transformation, \( z \xrightarrow{i_s \circ g \circ i_s} z' \), where \( g \) is a four-dimensional superconformal transformation, then we get from eqs. (2.37, 2.38)

\[ \delta \tilde{x}_+ = \tilde{x}_+ a \tilde{x}_+ - 4 \tilde{x}_+ \tilde{\varepsilon}_a \theta^a - \lambda \tilde{x}_+ + w \tilde{x}_+ - \tilde{x}_+ \tilde{w} + 4i \tilde{\rho}_a \theta^a + \tilde{b}, \]
\[ \delta \theta^a = \tilde{\rho}^a - \frac{1}{2} (\lambda + i \Omega) \theta^a - \theta^a \tilde{w} - (\zeta t^i \tilde{\zeta})^a_b \theta^b + \theta^a a \tilde{x}_+ - i \tilde{\varepsilon}_a \tilde{x}_+ - 4(\theta^a \tilde{\varepsilon}_a) \theta^b, \]
\[ (2.55) \]

where

\[ \tilde{\rho}^a = -\zeta^{ba} \rho_b \varepsilon^{-1}; \quad \tilde{\rho}_a = \varepsilon^{-1} \tilde{\rho}^{bt} \tilde{\zeta}_{ba}; \]
\[ \tilde{\varepsilon}_a = \varepsilon \varepsilon^{bt} \tilde{\zeta}_{ba}; \quad \tilde{\varepsilon}_a = -\zeta^{ba} \varepsilon^{bt} \tilde{\zeta}. \]
\[ (2.56) \]

Hence, under superinversion, the superconformal transformations are related by

\[ K \equiv \begin{pmatrix} a^\mu \\ b^\mu \\ \tilde{\varepsilon}^a \\ \rho_a \\ \lambda \\ \Omega \\ w^\mu_b \\ t^\mu_b \end{pmatrix} \xrightarrow{i_s \circ (b, \tilde{\rho}) \circ i_s} \begin{pmatrix} b^\mu \\ a^\mu \\ \tilde{\rho}_a \\ \tilde{\varepsilon}_a \\ -\lambda \\ -\Omega \\ w^\mu_b \\ -(\zeta t^i \tilde{\zeta})^a_b \end{pmatrix}. \]
\[ (2.57) \]

In particular, special superconformal transformations (2.45) can be obtained by

\[ z \xrightarrow{i_s \circ (b, \tilde{\rho}) \circ i_s} z' \equiv z_s(z; u), \]
\[ (2.58) \]

where \( (b, \tilde{\rho}) \) is a supertranslation and \( u^M = (b^\mu, \tilde{\rho}^a, \tilde{\rho}^a_\tilde{\zeta}). \)
2.5 Superconformal Algebra

The generator of infinitesimal superconformal transformations, $\mathcal{L}$, is given by

$$\mathcal{L} = h^\mu \partial_\mu + \lambda^{a\alpha} D_{a\alpha} - \bar{\lambda}_{\dot{a}}^{\dot{\alpha}} \bar{D}_{\dot{a} \dot{\alpha}}. \quad (2.59)$$

If we write the commutator of two generators, $\mathcal{L}_1, \mathcal{L}_2$, as

$$[\mathcal{L}_2, \mathcal{L}_1] = \mathcal{L}_3 = h^\mu_3 \partial_\mu + \lambda^{a\alpha}_3 D_{a\alpha} - \bar{\lambda}_{\dot{a}}^{\dot{\alpha}}_3 \bar{D}_{\dot{a} \dot{\alpha}}_3, \quad (2.60)$$

then $h^\mu_3, \lambda^{a\alpha}_3, \bar{\lambda}_{\dot{a}}^{\dot{\alpha}}_3$ are given by

$$h^\mu_3 = h^\mu_2 \partial_\mu h^\mu_1 + 2i \lambda^{a\alpha}_2 \sigma^\rho \bar{\lambda}^{2a}_a - (1 \leftrightarrow 2),$$

$$\lambda^{a\alpha}_3 = h^\mu_2 \partial_\mu \lambda^{a\alpha}_1 + \lambda^{b\beta}_2 D_{b\beta} \lambda^{a\alpha}_1 - (1 \leftrightarrow 2),$$

$$\bar{\lambda}_{\dot{a}}^{\dot{\alpha}}_3 = h^\mu_2 \partial_\mu \bar{\lambda}_{1\dot{a}}^{\dot{\alpha}} - \bar{\lambda}_{\dot{a}}^{\dot{\alpha}}_2 \bar{D}_{\dot{a}} \bar{\lambda}_{1\dot{a}}^{\dot{\alpha}} - (1 \leftrightarrow 2),$$

and $h^\mu_3, \lambda^{a\alpha}_3, \bar{\lambda}_{\dot{a}}^{\dot{\alpha}}_3$ satisfy eq. (2.24) verifying the closure of the Lie algebra. Explicitly with eqs. (2.34, 2.37) we get

$$a^a_3 = w^\mu_1 \nu a^\nu_2 + \lambda_1 a^\mu_2 + 2i \epsilon^a_1 \sigma^\mu \bar{\epsilon}^{2a}_2 - (1 \leftrightarrow 2),$$

$$\epsilon^a_3 = -\epsilon^a_2 \bar{w}_1 + \frac{1}{2} \lambda_1 \epsilon^a_2 - i \bar{\rho}^a_1 \bar{\delta}^2_2 + \tau^a_1 \epsilon^b_2 + i \frac{1}{2} \Omega_{1a} \epsilon_2^a - (1 \leftrightarrow 2),$$

$$\bar{\epsilon}^a_3 = \bar{w}_1 \bar{\epsilon}^{2a}_1 + \frac{1}{2} \lambda_1 \bar{\epsilon}^{2a}_2 + i \bar{\delta}^{2a}_1 \rho_1 - \bar{\epsilon}^{2b}_2 \tau^b_1 - i \frac{1}{2} \Omega_{1a} \bar{\epsilon}^{2a}_2 - (1 \leftrightarrow 2),$$

$$\lambda_3 = 2a_2 b_1 + 2(\rho^a_1 \bar{\epsilon}^{2a}_2 + \epsilon^a_2 \rho_1) - (1 \leftrightarrow 2),$$

$$w^{a\nu}_3 = w^\mu_1 \nu w^{\lambda}_2 \nu + 2(a^a_2 b^\nu_1 - a^a_2 b^\nu_1) + 2(\epsilon^a_2 \sigma^{\mu \nu} \bar{\rho}^{a\nu}_1 \rho_1 - \bar{\rho}^a_1 \bar{\sigma}^{\mu \nu} \bar{\epsilon}^{2a}_2) - (1 \leftrightarrow 2),$$

$$b^a_3 = w^\mu_1 \nu b^\nu_2 - \lambda_1 b^\mu_2 + 2i \bar{\rho}^a_1 \bar{\sigma}^\mu \rho_2 - (1 \leftrightarrow 2),$$

$$\rho_{3a} = \bar{w}_1 \rho_{2a} - \frac{1}{2} \lambda_1 \rho_{2a} + i b_2 \bar{\epsilon}^{1a}_1 - \rho_{2b} \tau^b_1 - i \frac{1}{2} \Omega \rho_{2a} - (1 \leftrightarrow 2),$$

$$\bar{\rho}^a_3 = -\bar{\rho}^a_2 w_1 - \frac{1}{2} \lambda_1 \bar{\rho}^a_2 + i \epsilon^a_1 b_2 + \tau^a_1 \bar{\rho}^b_2 + i \frac{1}{2} \Omega_{1a} \bar{\rho}^a_2 - (1 \leftrightarrow 2),$$

$$t^a_{3b} = (t^a_1 t^b_1) + 4(\epsilon^a_2 \rho_{2b} - \bar{\rho}^a_2 \bar{\epsilon}^{1b}_1) - \frac{4}{\mathcal{N}}(\epsilon^a_1 \rho_{2c} - \bar{\rho}^a_1 \bar{\epsilon}^{1c}_1) \delta^a_b - (1 \leftrightarrow 2),$$

$$\Omega_3 = 2i(\frac{1}{\mathcal{N}} - 1)(\epsilon^a_2 \rho_{1a} - \bar{\rho}^a_1 \bar{\epsilon}^{2a}_1) - (1 \leftrightarrow 2).$$
From eq.(2.62) we can read off the explicit forms of four-dimensional superconformal algebra as exhibited in Appendix C.

Now, we consider $\mathcal{N} \neq 4$ case and $\mathcal{N} = 4$ case separately. For $\mathcal{N} \neq 4$ case, if we define a $(4 + \mathcal{N}) \times (4 + \mathcal{N})$ supermatrix, $M$, as

\[
M = \begin{pmatrix}
  w + \frac{1}{2} \lambda + i \frac{1}{2} \psi & -i \hat{a} & 2 \tilde{\epsilon}_b \\
  -i b & 2 \rho^a & 2 \tilde{\rho}_b \\
  2 \tilde{\rho}^a & 2 \rho_b & t^a + i 2 \frac{\psi}{N} \delta^a_b
\end{pmatrix}, \tag{2.63}
\]

where

\[
\psi = \frac{\Omega}{\frac{4}{N} - 1}, \tag{2.64}
\]

then the relation above (2.62) agrees with the matrix commutator

\[
[M_1, M_2] = M_3. \tag{2.65}
\]

This can be verified using eqs.(A.7, A.11, A.12).

In general, for $\mathcal{N} \neq 4$, $M$ can be defined as a $(4, \mathcal{N})$ supermatrix subject to

\[
\text{str } M = 0, \tag{2.66}
\]

and a reality condition

\[
B M B^{-1} = -M^\dagger, \quad B = \begin{pmatrix}
  0 & 1 & 0 \\
  1 & 0 & 0 \\
  0 & 0 & -1
\end{pmatrix}. \tag{2.67}
\]

Supermatrix of the form (2.63) is the general solution of these two equations.

The $4 \times 4$ matrix appearing in $M$,

\[
\begin{pmatrix}
  w + \frac{1}{2} \lambda & -i \hat{a} \\
  -i b & \bar{w} - \frac{1}{2} \lambda
\end{pmatrix}, \tag{2.68}
\]

corresponds to a generator of $O(2, 4) \cong SU(2, 2)$ as demonstrated in Appendix D. Thus, the $\mathcal{N} \neq 4$ superconformal group in four-dimensions may be identified with the supermatrix group generated by supermatrices of the form $M$ (2.63), which is $SU(2, 2|\mathcal{N}) \equiv G_s$ having dimensions $(15 + \mathcal{N}^2|8\mathcal{N})$. 

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When $\mathcal{N} = 4$, similar analysis is also possible with a subtle modification. In this case, $M$ is a $(4, 4)$ supermatrix satisfying the reality condition (2.67) and, instead of eq.(2.66),

$$\text{str } M = -2i\Omega.$$  \hfill (2.69)

Such a supermatrix, $M$, is of the general form

$$M = \mathcal{M} + i\frac{1}{2}\Omega \begin{pmatrix} 0 & 0 \\ 0 & \delta_a \end{pmatrix} + i\frac{1}{2}\psi \begin{pmatrix} 1 & 0 \\ 0 & \delta_a \end{pmatrix},$$  \hfill (2.70)

where $\mathcal{M}$ is of the form (2.63) with $\psi = 0$, and $\psi$ in eq.(2.70) is just an arbitrary real bosonic variable. Essentially we add the $\Omega$ term to eq.(2.63) to get eq.(2.70). Same as $\mathcal{N} \neq 4$ case, the matrix commutator of $M_1$ and $M_2$ reproduces eq.(2.62) as in eq.(2.65), though $\psi$ is arbitrary.

As the variable, $\psi$, is auxiliary, one might be tempted to fix its value, or more generally let it be a function of the parameters of superconformal transformations, $a^\mu$, $b^\mu$, $\Omega$, · · · and try to determine the function. However, this is not possible. The commutator of $\mathcal{M}_1$ and $\mathcal{M}_2$ includes $\psi$ type term

$$[\mathcal{M}_1, \mathcal{M}_2] = \mathcal{M}_3 + i\frac{1}{2}\psi_3 \mathbf{1},$$ \hfill (2.71)

$$\psi_3 = 2i(\varepsilon^a_1 \rho_1 a - \varepsilon^a_1 \rho_2 a + \bar{\rho}_2^a \bar{\varepsilon}_1 a - \bar{\rho}_1^a \bar{\varepsilon}_2 a),$$

and $\psi_3$ can not be expressed in terms of the parameters, $a^\mu_3$, $b^\mu_3$, $\Omega_3$, · · ·, appearing in eq.(2.62). Hence it is not possible to put $\psi$ as a function of the superconformal transformation parameters. Therefore four-dimensional $\mathcal{N} = 4$ superconformal algebra is represented by $(4, 4)$ supermatrices, $M$, satisfying the reality condition (2.67) with an equivalence relation, $\sim$, imposed

$$M_1 \sim M_2 \quad \text{if } M_1 - M_2 = i\psi \mathbf{1} \quad \text{for some } \psi \in \mathbb{R}.$$  \hfill (2.72)

We note that an extra condition, $\text{str } M = 0$, defines an invariant subalgebra of the whole four-dimensional $\mathcal{N} = 4$ superconformal algebra. This invariant subalgebra forms a simple Lie superalgebra. In the literature the four-dimensional $\mathcal{N} = 4$ superconformal algebra is often identified with this simple Lie superalgebra, the $R$-symmetry of which is $\text{su}(4)$ rather than $\text{u}(4)$ \cite{18}, as the $\Omega$ term in eq.(2.70) is neglected. However, we emphasize here that the whole $\mathcal{N} = 4$ superconformal algebra may contain a $\text{u}(1)$ factor which has non-trivial

\footnote{An alternative approach may be taken as in \cite{41}, where a modified supermatrix commutator is introduced for $\text{SL}(m|m)$.}
commutator relations with other generators as seen in eq. (2.62) or eq. (C.9).

The $\mathcal{N} = 4$ superconformal group in four-dimensions, $G_s$, is now identified with a quotient group of the supermatrix group, as it is isomorphic to the supermatrix group generated by supermatrices of the form $M$ \(\text{(2.70)}\) with an equivalence relation imposed on the supermatrix group element, $G$, from eq. (2.72)

\[ G_1 \sim G_2 \quad \text{if} \quad G_1^{-1}G_2 = e^{i\psi} \quad \text{for some} \quad \psi \in \mathbb{R}. \quad (2.73) \]

We also note that the four-dimensional $\mathcal{N} = 4$ superconformal group has dimensions (31|32) and is isomorphic to a semi-direct product of U(1) and a simple Lie supergroup. Therefore, by breaking the U(1) symmetry, the four-dimensional $\mathcal{N} = 4$ superconformal group can be reduced to the simple Lie subgroup having dimensions (30|32).

## 3 Coset Realization of Transformations

In this section, we first obtain an explicit formula for the finite nonlinear superconformal transformations of the supercoordinates and discuss several representations of the superconformal group. We then construct matrix (vector) valued functions depending on two (three) points in superspace which transform covariantly under superconformal transformations. These are crucial variables for obtaining two-point, three-point and general $n$-point correlation functions later.

### 3.1 Superspace as a Coset

To obtain an explicit formula for the finite nonlinear superconformal transformations, we first identify the superspace, $\mathbb{R}^{4|4\mathcal{N}}$, as a coset, $G_s/G_0$, where $G_0 \subset G_s$ is the subgroup generated by matrices, $M_0$, of the form \(\text{(2.63)}\) with $a^\mu = 0$, $\varepsilon^a = 0$ and depending on parameters $b^\mu$, $\rho_a$, $\bar{\rho}_a$, $\lambda$, $\Omega$, $w_{\mu\nu}$, $t^a$. The group of supertranslations, $G_T$, parameterized by coordinates, $z^M \in \mathbb{R}^{4|4\mathcal{N}}$, has been defined by general elements as in eq. (2.3), with the group property given by eqs. (2.6, 2.7). Now we may represent it by supermatrices\footnote{The subscript, $T$, denotes supertranslations.}.

\[
G_T(z) = \exp \begin{pmatrix}
0 & -i\bar{x} & 2\bar{\theta}_b \\
0 & 0 & 0 \\
0 & 2\theta^a & 0
\end{pmatrix} = \begin{pmatrix}
1 & -i\bar{x}_+ & 2\bar{\theta}_b \\
0 & 1 & 0 \\
0 & 2\theta^a & \delta^a_b
\end{pmatrix}.
\]  

(3.1)

Note that $G_T(z)^{-1} = G_T(-z)$.

In general an element of $G_s$ can be uniquely decomposed as $G_TG_0$. Thus for any element
$G(g) \in G_s$ we may define a superconformal transformation, $z \rightarrow z'$, and an associated element $G_0(z; g) \in G_0$ by

$$G(g)^{-1}G_T(z)G_0(z; g) = G_T(z').$$

(3.2)

If $G(g) \in G_T$ then clearly $G_0(z; g) = 1$.

Infinitesimally eq. (3.2) becomes

$$\delta G_T(z) = MG_T(z) - G_T(z)\dot{M}_0(z),$$

(3.3)

where $M$ is given by eq. (2.63) or eq. (2.70) and $\dot{M}_0(z)$, the generator of $G_0$, has the form

$$\dot{M}_0(z)$$

$$= \left( \begin{array}{ccc}
\hat{w}(z) + \frac{1}{2}\lambda(z) + i\frac{1}{2}\psi(z) & 0 & 0 \\
-i\dot{\theta} & \tilde{\hat{w}}(z) - \frac{1}{2}\lambda(z) + i\frac{1}{2}\psi(z) & 2\hat{\rho}_b(z) \\
2\tilde{\phi}^a(z) & 0 & \hat{t}^a_b(z) + i\frac{2}{N}(\psi(z) + \delta^a_N\Omega)\delta^a_b \end{array} \right).$$

(3.4)

The components depending on $z$ are given by

$$\hat{w}(z) = w - 4\bar{\theta}_a\rho^a + \bar{x}_b + \frac{1}{2}\text{tr}(4\bar{\theta}_a\rho^a - \bar{x}_b)1,$$

$$\tilde{\hat{w}}(z) = \tilde{w} + 4\rho_a\theta^a - b\bar{x}_b - \frac{1}{2}\text{tr}(4\rho_a\theta^a - b\bar{x}_b)1 = -\hat{w}(z)^\dagger,$$

$$\lambda(z) = \lambda + 2b\cdot x + 2(\theta^a\rho_a + \bar{\rho}^a\bar{\theta}_a) = \frac{1}{4}\partial_\mu h^\mu(z),$$

$$\psi(z) = \psi + 2\theta^a b\bar{\theta}_a + 2i(\theta^a\rho_a - \bar{\rho}^a\bar{\theta}_a),$$

$$\hat{t}^a_b(z) = t^a_b + 4i\theta^a b\bar{\theta}_b + 4(\bar{\rho}^a\bar{\theta}_b - \theta^a\rho_b) - \frac{1}{N}(4i\theta^b\bar{\theta}_c + 4\bar{\rho}^b\bar{\theta}_c - 4\theta^b\rho_c)\delta^a_b,$$

$$\hat{\rho}_a(z) = \rho_a - ib\bar{\theta}_a = -i\frac{1}{4}\sigma^a\partial_\mu \lambda_a(z),$$

$$\tilde{\phi}^a(z) = \bar{\phi}^a + i\theta^a b = \hat{\rho}_a(z)^\dagger.$$  

Writing $\delta G_T(z) = \mathcal{L}G_T(z)$ we may verify that $\mathcal{L}$ is identical with eq. (2.59).

$$\hat{w}(z), \tilde{\hat{w}}(z)$$

are also written as

$$\hat{w}(z) = \frac{1}{4}\hat{w}_{\mu\nu}(z)\bar{\sigma}^\mu\sigma^\nu, \tilde{\hat{w}}(z) = \frac{1}{4}\tilde{w}_{\mu\nu}(z)\sigma^\mu\bar{\sigma}^\nu,$$

with

$$\hat{w}_{\mu\nu}(z) = w_{\mu\nu} + 4x_{[\mu}b_{\nu]} + \theta^a\sigma_{[\mu}\bar{\sigma}_{\nu]}(2\rho_a - ib\bar{\theta}_a) - (2\bar{\rho}_a + i\theta^a b)\bar{\sigma}_{[\mu}\sigma_{\nu]}\bar{\theta}_a = -\partial_{[\mu}h_{\nu]}(z).$$

(3.6)
The definitions (3.5) can be summarized by
\[ D_{b\beta} \lambda^{a\alpha} = \frac{1}{2} \delta^a_b \delta^\alpha_\beta (\lambda(z) + i\hat{\Omega}(z)) - \delta^a_b \tilde{w}^\alpha_\beta(z) + \delta^a_\beta \hat{t}^a_b(z), \tag{3.7} \]
and they give
\[ [D_{a\alpha}, \mathcal{L}] = \frac{1}{2} (\lambda(z) + i\hat{\Omega}(z)) D_{a\alpha} - \tilde{w}^\alpha_{\alpha\beta}(z) D_{a\beta} + \hat{t}^b_{a\alpha}(z) D_{b\alpha}, \tag{3.8} \]
where
\[ \hat{\Omega}(z) = \left( \frac{4}{N} - 1 \right) \hat{\psi}(z) + \delta^4_{\mathcal{N}} \Omega. \tag{3.9} \]
For later use we note
\[ D_{a\alpha} \tilde{w}_{\mu\nu}(z) = 2(\sigma_{[\mu} \tilde{\sigma}_{\nu]})_{\alpha}^\beta \hat{\rho}_{a\beta}(z), \]
\[ D_{a\alpha} \hat{\lambda}(z) = 2\hat{\rho}_{a\alpha}(z), \]
\[ D_{a\alpha} \hat{\psi}(z) = 2i\hat{\rho}_{a\alpha}(z), \]
\[ D_{a\alpha} \hat{t}^b_{c\alpha}(z) = -4\delta^b_{a\alpha} \hat{\rho}_{b\alpha}(z) + \frac{4}{N} \delta^b_{c} \hat{\rho}_{a\alpha}(z). \tag{3.10} \]
The above analysis can be simplified by reducing $G_0(z; g)$. To achieve this we let
\[ Z_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.11} \]
and then
\[ M_0 Z_0 = Z_0 H_0, \quad H_0 = \begin{pmatrix} \tilde{w} - \frac{1}{2} \lambda + i\frac{1}{2} \hat{\psi} & 2\hat{\rho}_b \\ 0 & \hat{t}^b_a + i\frac{2}{N} (\hat{\psi} + \delta^4_{\mathcal{N}} \Omega) \delta^a_b \end{pmatrix}. \tag{3.12} \]
Now if we define
\[ Z(z) \equiv G_T(z) Z_0 = \begin{pmatrix} -i\hat{\xi}_+ & 2\hat{\theta}_b \\ 1 & 0 \\ 2\hat{\theta}^a & \delta^a_b \end{pmatrix}, \tag{3.13} \]
then $Z(z)$ transforms under infinitesimal superconformal transformations as
\[ \delta Z(z) = \mathcal{L} Z(z) = M Z(z) - Z(z) H(z), \tag{3.14} \]
where $H(z)$ is given by

$$H(z) = \left( \begin{array}{cc} \hat{w}(z) - \frac{1}{2}\hat{\lambda}(z) + i\frac{1}{4}\hat{\psi}(z) & 2\hat{\rho}_b(z) \\ 0 & \hat{t}_b^a(z) + i\frac{2}{N}(\hat{\psi}(z) + \delta_{A'}\Omega)\delta^a_b \end{array} \right).$$

(3.15)

From eqs. (2.60, 2.65) considering $[L_2, L_1]Z(z) = L_3Z(z)$,

(3.16)

we get

$$H_3(z) = L_2H_1(z) - L_1H_2(z) + [H_1(z), H_2(z)],$$

(3.17)

which gives separate equations for $\tilde{w}$, $\hat{\lambda}$, $\hat{\psi}$, $\hat{\rho}_a$ and $\hat{t}_{ab}^a$, thus $\hat{\lambda}_3 = L_2\hat{\lambda}_1 - L_1\hat{\lambda}_2$, and so on.

As a conjugate of $Z(z)$ we define $\bar{Z}(z)$ by

$$\bar{Z}(z) = \bar{Z}(0)G_T(z)^{-1},$$

(3.19)

and corresponding to eq. (3.14) we have

$$\delta\bar{Z}(z) = \bar{D}Z(z) = \bar{H}(z)\bar{Z}(z) - \bar{Z}(z)M,$$

(3.20)

where

$$\bar{H}(z) = \left( \begin{array}{cc} \hat{w}(z) + \frac{1}{2}\hat{\lambda}(z) + i\frac{1}{4}\hat{\psi}(z) & 0 \\ 2\hat{\rho}_b(z) & \hat{t}_b^a(z) + i\frac{2}{N}(\hat{\psi}(z) + \delta_{A'}\Omega)\delta^a_b \end{array} \right).$$

(3.21)

### 3.2 Finite Transformations

Finite superconformal transformations can be obtained by exponentiation of infinitesimal transformations. To obtain a superconformal transformation, $z \xrightarrow{g} z'$, we therefore solve the differential equation

$$\frac{d}{dt}z_t^M = L^M(z_t), \quad z_0 = z, \quad z_1 = z',$$

(3.22)

where, with $L$ given in eq. (2.59), $L^M(z)$ is defined by

$$L = L^M(z)\partial_M.$$

(3.23)
From eq.(3.14) we get
\[
\frac{d}{dt}Z(z_t) = MZ(z_t) - Z(z_t)H(z_t),
\]
which integrates to
\[
Z(z_t) = e^{tM}Z(z)K(z,t),
\]
where \(K(z,t)\) satisfies
\[
\frac{d}{dt}K(z,t) = -K(z,t)H(z_t), \quad K(z,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Hence for \(t = 1\) with \(K(z,1) \equiv K(z;g)\) the superconformal transformation, \(z \xrightarrow{g} z'\), from eq.(3.25) becomes
\[
Z(z') = G(g)^{-1}Z(z)K(z;g), \quad G(g)^{-1} = e^{M}.
\]

\(G_0(z;g)\) in eq.(3.2) is related to \(K(z;g)\) from eq.(3.27) by
\[
G_0(z;g)Z_0 = Z_0K(z;g).
\]

In general \(K(z;g)\) is of the form
\[
K(z;g) = \begin{pmatrix} L(z^+;g) & 2 \Sigma_b(z;g) \\ 0 & u^a_b(z;g) \end{pmatrix}.
\]

From
\[
\tilde{w}(z) - \frac{i}{2} \tilde{\lambda}(z) + i \frac{1}{2} \tilde{\psi}(z) = \tilde{w} - \frac{i}{2} \lambda + i \frac{1}{2} \psi + 4 \rho_\alpha \theta^a - b \tilde{x}_+,
\]
\(L(z^+;g)\) is defined on chiral superspace, and since \(L(z^+;g)\) is a 2 \(\times\) 2 matrix, we have
\[
L(z^+;g)\epsilon L(z^+;g)^t = \det L(z^+;g) \epsilon.
\]

Infinitesimally this is consistent with eq.(2.36).

We decompose \(L(z^+;g)\) as
\[
\hat{L}(z^+;g) = \Omega_+(z^+;g)^{-\frac{1}{2}}L(z^+;g),
\]
\[
\Omega_+(z^+;g) = \det L(z^+;g),
\]
where \(\hat{L} \in \text{SL}(2,\mathbb{C})\), the 2 \(\times\) 2 matrices with determinant one.

Since \(\frac{d}{dt}u(z,t) = -u(z,t)\hat{t}(z_t) + i \frac{2}{\lambda} \hat{\psi}(z_t) + i \frac{1}{2} \delta^4 \lambda \Omega\), \(u^t = u^{-1}\) and hence \(u \in U(\mathcal{N})\).

If we write
\[
\hat{u}(z;g) = \frac{u(z;g)}{(\det u(z;g))^\frac{1}{N}} ,
\]
then \( \hat{u} \in SU(\mathcal{N}) \).

From eq. (3.27) \( \bar{Z} \) transforms as

\[
\bar{Z}(z') = \bar{K}(z; g)\bar{Z}(z)G(g),
\]

where

\[
\bar{K}(z; g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} K(z; g)^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \bar{L}(z_-; g) & 0 \\ -2\Sigma(z; g) & u^{-1}(z; g) \end{pmatrix},
\]

\[
\bar{L}(z_-; g) = L(z_+; g)^\dagger,
\]

\[
\bar{\Sigma}(z; g) = \Sigma(z; g)^\dagger.
\]

In a similar fashion to eq. (3.32a) we write

\[
\hat{\bar{L}}(z_-; g) = \Omega^{-1}(z_-; g)\bar{L}(z_-; g) = \hat{\bar{L}}(z_+; g)^\dagger \in SL(2, \mathbb{C}),
\]

\[
\Omega^{-1}(z_-; g) = \det L(z_-; g) = \Omega_+(z_+; g)^*.
\]

If we define for superinversion, \( z \xrightarrow{i_s} z' \), (2.48)

\[
G(i_s)^{-1} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & \epsilon^{-1} \\ 0 & 0 \end{pmatrix},
\]

\[
K(z; i_s) = \begin{pmatrix} 1 & 0 \\ i\bar{\xi}_- & -2\theta a \end{pmatrix} = \bar{Z}(z')^t.
\]

Similarly we have

\[
\bar{K}(z; i_s)\bar{Z}(z)G(i_s) = Z(z')^t,
\]

where

\[
\bar{K}(z; i_s) = \begin{pmatrix} -i(\bar{\xi}_-)^{-1} & 0 \\ 2i\bar{\xi}_a\theta\bar{\xi}_-^{-1} & \delta_a \end{pmatrix}.
\]

For later use, we also define with eq. (3.9 1.22)

\[
\frac{d}{dt} \Upsilon(z, t) = i\hat{\Omega}(z_t)\Upsilon(z, t), \quad \Upsilon(z, 0) = 1,
\]

\[
\Upsilon(z; g) \equiv \Upsilon(z, 1),
\]
and
\[ \Omega(z; g) \equiv \sqrt{\Omega_+(z_+; g)\Omega_-(z_-; g)}. \] (3.42)

Note that
\[ \Omega(z; g)^* = \Omega(z; g), \quad \Upsilon(z; g)^* = \Upsilon(z; g)^{-1}. \] (3.43)

Since
\[ \text{sdet } G = \exp(\text{str } \ln G), \] (3.44)
when \( \mathcal{N} = 4 \), \( \Upsilon(z; g) \) is related to the superdeterminant of \( G(g) \in G_s \)
\[ \Upsilon(z; g) = e^{\Omega} = \sqrt{\text{sdet } G(g)}. \] (3.45)

If
\[ \sigma^\nu R^\mu_{\nu}(z; g) = L(z_+; g)\sigma^\mu \hat{L}(z_-; g), \] (3.46)
then \( R^\mu_{\nu}(z; g) \) is identical to the definition (2.20), since infinitesimally
\[ \hat{\lambda}(z)\sigma^\mu - \hat{\omega}(z)\sigma^\nu + \sigma^\nu \hat{\omega}_\nu^\mu(z) = \hat{\lambda}(z)\sigma^\mu - \sigma^\nu \hat{\omega}_\nu^\mu(z) = \sigma^\nu \partial_\nu h^\mu(z), \] (3.47)
which agrees with eq.(2.26). Furthermore eq.(3.46) shows that the definition (2.21) of \( \Omega(z; g) \) is consistent with eq.(3.42). We may normalize \( R^\mu_{\nu}(z; g) \) as well
\[ \hat{R}^\mu_{\nu}(z; g) = \Omega(z; g)^{-1}R^\mu_{\nu}(z; g) \in \text{SO}(1, 3) \cong \text{SL}(2, \mathbb{C})/\mathbb{Z}_2. \] (3.48)

### 3.3 Representations

Based on the results in the previous subsection, it is easy to show that the matrix, \( \mathcal{R}_M^N(z; g) \), given in eq.(2.23) is of the form
\[
\mathcal{R}_M^N(z; g) = \begin{pmatrix}
R^\nu_{\mu}(z; g) & i(S^b(z; g)\bar{\sigma}_\mu L(z_+; g))^\beta \nu & -i(L(z_+; g)\bar{\sigma}_\mu \Sigma^b(z; g))^\beta \nu \\
0 & (\Omega(z; g)\Upsilon(z; g))^-\hat{L}_\beta^\alpha(z_+; g)\hat{u}^{-1}a(z; g) & 0 \\
0 & 0 & \left(\frac{\Omega(z; g)}{\Upsilon(z; g)}\right)^{\frac{1}{2}}\hat{L}_\beta^\alpha(z_-; g)\hat{u}^{a}b(z; g)
\end{pmatrix}.
\] (3.49)

Since \( \mathcal{R}_M^N(z; g) \) is a representation of the four dimensional \( \mathcal{N} \)-extended superconformal group, each of the following also forms a representation of the group, though it is not a faithful representation
\[
\Omega(z; g) \in \text{D}, \quad \Upsilon(z; g) \in \text{U}(1), \quad \hat{R}(z; g) \in \text{SO}(1, 3),
\]
\[
\hat{L}(z_+; g) \in \text{SL}(2, \mathbb{C}), \quad \hat{L}(z_-; g) \in \text{SL}(2, \mathbb{C}), \quad \hat{u}(z; g) \in \text{SU}(\mathcal{N}),
\] (3.50)
where D is the one dimensional group of dilations.

Under the successive superconformal transformations, $g'' : z \rightarrow z' \rightarrow z''$, they satisfy

$$\Omega(z; g)\Omega(z'; g') = \Omega(z; g''), \quad \text{and so on.} \quad (3.51)$$

We note that when $N \neq 4$, $\Omega(z_+; g)$ and $\Omega(z_-; g)$ can be written as

$$\Omega(z_+; g) = \Omega(z; g)\Upsilon(z; g)^{N\over N-1},$$

$$\Omega(z_-; g) = \Omega(z; g)\Upsilon(z; g)^{-N\over N-1}. \quad (3.52)$$

Hence, they also form representations of the $N \neq 4$ superconformal group. On the other hand, in the case of $N = 4$, due to the arbitrariness of $\psi$ in eq.(2.70) $L(z_+; g)$ and $L(z_-; g)$ do not form representations. They do so only if the equivalence relation (2.73) is imposed, but this will give just $\hat{L}(z_+; g), \hat{L}(z_-; g)$ and $\Omega(z; g)$.

### 3.4 Functions of Two Points

In this subsection, we construct matrix valued functions depending on two points, $z_1$ and $z_2$, in superspace which transform covariantly like a product of two tensors at $z_1$ and $z_2$ under superconformal transformations.

If $F(z)$ is defined for $z \in \mathbb{R}^{4|4N}$ by

$$F(z) = \bar{Z}(0)G_T(z)Z(0) = \begin{pmatrix} -i\bar{x}_+ & 2\bar{\theta}_b \\ 2\theta^a & \delta^a_b \end{pmatrix}, \quad (3.53)$$

then $F(z)$ satisfies

$$F(-z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} i\bar{x}_- & -2\bar{\theta}_b \\ -2\theta^a & \delta^a_b \end{pmatrix}, \quad (3.54)$$

and the superdeterminant of $F(z)$ is given by

$$\text{sdet} \ F(z) = -x_-^2. \quad (3.55)$$

If we consider

$$\begin{pmatrix} 1 & 0 \\ -2i\theta^a\bar{x}_+^{-1} & 1 \end{pmatrix} F(z) \begin{pmatrix} 1 & -2i\bar{x}_+^{-1}\bar{\theta}_b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -i\bar{x}_+ & 0 \\ 0 & \nu^a_b(-z) \end{pmatrix}, \quad (3.56)$$
then this defines $v^a_b(z)$ as
\[ v^a_b(z) = \delta^a_b + 4i \frac{1}{x_{-}^2} \theta^a x_{-} \bar{\theta}_b. \] (3.57)

From eqs. (3.55, 3.56) it is evident that
\[ \det v(z) = \frac{x_+^2}{x_-^2}. \] (3.58)

It is useful to note
\[ v^a_b(-z) = v^{-1}_{-a} b(z) = v^{\dagger}_{-a} b(z) = \delta^a_b - 4i \frac{1}{x_{+}^2} \theta^a x_{+} \bar{\theta}_b. \] (3.59)

Now with the supersymmetric interval for $\mathbb{R}^{4|4N}$ superspace defined by
\[ G_T(z_2)^{-1} G_T(z_1) = G_T(z_{12}), \quad z_{+}^M = (x_{12}^a, \theta_{12}^a, \bar{\theta}_{12}^a) = -z_{-}^M, \]
\[ x_{12}^\mu = x_{1}^\mu - x_{2}^\mu + i \theta_{12}^a \sigma^a \bar{\theta}_{2a} - i \theta_{21}^a \sigma^a \bar{\theta}_{12}^a, \quad \theta_{12}^a = \theta_{12}^a - \theta_{2}^a, \quad \bar{\theta}_{12}^a = \bar{\theta}_{12}^a - \bar{\theta}_{2}^a, \] (3.60)
we may write
\[ \bar{Z}(z_2) Z(z_1) = F(z_{12}) = \begin{pmatrix} i \bar{x}_{21} & -2\bar{\theta}_{21b} \\ -2\theta_{21}^a & \delta^a_b \end{pmatrix}, \] (3.61)
and
\[ \text{sdet} F(z_{12}) = -x_{12}^2, \quad \det v(z_{12}) = \frac{x_{21}^2}{x_{12}^2}, \] (3.62)
where
\[ x_{21}^\mu = x_{2-}^\mu - x_{1+}^\mu - 2i \theta_{21}^a \sigma^a \bar{\theta}_{2a} = x_{21}^\mu + i \theta_{21}^a \sigma^a \bar{\theta}_{12}^a = (x_{21})^\mu, \] (3.63)
\[ x_{12}^\mu = x_{1-}^\mu - x_{2+}^\mu + 2i \theta_{12}^a \sigma^a \bar{\theta}_{1a} = x_{12}^\mu + i \theta_{12}^a \sigma^a \bar{\theta}_{12}^a = (x_{12})^\mu. \]

From eqs. (3.27, 3.34) $F(z_{12})$ transforms as
\[ F(z'_{12}) = K(z_2; g) F(z_{12}) K(z_1; g). \] (3.64)

In particular, with eqs. (3.29, 3.35), this gives transformation rules for $\bar{x}_{12}'$ and $\bar{x}_{21}'$
\[ \bar{x}_{12}' = \bar{L}(z_{1-}; g) \bar{x}_{12} L(z_{2+}; g), \] (3.65a)
\[ \bar{x}_{21}' = \bar{L}(z_{2-}; g) \bar{x}_{21} L(z_{1+}; g), \] (3.65b)
so that

\[ x_{12}^2 = \Omega_-(z_1-; g)\Omega_+(z_2+; g)x_{12}^2, \quad (3.66a) \]

\[ x_{21}^2 = \Omega_-(z_2-; g)\Omega_+(z_1+; g)x_{21}^2, \quad (3.66b) \]

and in particular

\[ x_{12}^2 x_{21}^2 = \Omega(z_1; g)^2\Omega(z_2; g)^2 x_{12}^2 x_{21}^2, \quad (3.67a) \]

\[ \left( \frac{x_{12}^2}{x_{21}^2} \right)^{\frac{1}{2}} = \frac{\Upsilon(z_1; g)^2}{\Upsilon(z_2; g)^2} \left( \frac{x_{12}^2}{x_{21}^2} \right)^{\frac{1}{2}}. \quad (3.67b) \]

From eqs. (3.46, 3.65a) \( \text{tr}(\sigma^\mu \tilde{x}_{12} \sigma^\nu \tilde{x}_{21}) \) transforms covariantly as

\[ \text{tr}(\sigma^\mu \tilde{x}_{12} \sigma^\nu \tilde{x}_{21}') = \text{tr}(\sigma^\lambda \tilde{x}_{12} \sigma^\rho \tilde{x}_{21}) R_{\lambda\mu}(z_1; g) R_{\rho\nu}(z_2; g). \quad (3.68) \]

Since \( v^a(z_{21}) \) transforms infinitesimally as

\[ \delta v(z_{21}) = \hat{t}(z_2)v(z_{21}) - v(z_{21})\hat{t}(z_1) + i \frac{2}{N} (\hat{\psi}(z_2) - \hat{\psi}(z_1))v(z_{21}), \quad (3.69) \]

finitely it transforms as

\[ v(z_{21}') = u^{-1}(z_2; g)v(z_{21})u(z_1; g). \quad (3.70) \]

From eqs. (3.38, 3.39) \( F(z_{12}) \) transforms under superinversion as

\[ \bar{K}(z_2; i_s)F(z_{12})K(z_1; i_s) = F(-z_{12}')^t, \quad (3.71) \]

which gives

\[ \tilde{x}_{2-1} \tilde{x}_{21} \tilde{x}_{1+} = -x_{12}', \quad x_{12}^2 = \frac{x_{21}^2}{x_{2-1} x_{1+}^2}. \quad (3.72) \]

and

\[ \check{z} v^{-1}(z_2)v(z_{21})v(z_1)\zeta = v(z_{21}')^t. \quad (3.73) \]

Eq. (3.72) shows that eq. (3.68) holds for superinversion as well

\[ \text{tr}(\sigma^\mu \tilde{x}'_{12} \sigma^\nu \tilde{x}'_{21}) = \text{tr}(\sigma^\lambda \tilde{x}_{12} \sigma^\rho \tilde{x}_{21}) R_{\lambda\mu}(z_1; i_s) R_{\rho\nu}(z_2; i_s). \quad (3.74) \]
3.5 Functions of Three Points

In this subsection, for three points, \(z_1, z_2, z_3\) in superspace, we construct ‘tangent’ vectors, \(Z_i\), which transform homogeneously at \(z_i\), \(i = 1, 2, 3\).

With \(z_{21} \mapsto (z_{21})',\) \(z_{31} \mapsto (z_{31})'\), we define \(Z_i^M = (X_i^\mu, \Theta_{1a}, \bar{\Theta}_{1a}) \in \mathbb{R}^{4|4N}\) by

\[
G_T((z_{31})')^{-1}G_T((z_{21})') = G_T(z_1).
\]

Explicit expressions for \(Z_i^M\) can be obtained by calculating

\[
\bar{Z}((z_{31})')Z((z_{21})') = F(z_1) = \begin{pmatrix} -i \tilde{X}_1 + & 2\tilde{\Theta}_{1b} \\ 2\tilde{\Theta}_1^a & \delta^a_b \end{pmatrix}.
\] (3.76)

We get

\[
\tilde{X}_{1+} = -x_{13}^{-1}x_{23}x_{21}^{-1},
\]

\[
\Theta_a^1 = i(\tilde{\theta}_{21}^a x_{21}^{-1} - \tilde{\theta}_{31}^a x_{31}^{-1}), \quad \tilde{\Theta}_{1a} = i(x_{13}^{-1}\tilde{\theta}_{13a} - x_{12}^{-1}\tilde{\theta}_{12a}).
\] (3.77)

Using

\[
x_{13} + x_{21} + 4i\tilde{\theta}_{13a}\tilde{\theta}_{21}^a = x_{23},
\] (3.78)

one can assure

\[
\tilde{X}_{1-} = \tilde{X}_{1+} - 4i\tilde{\Theta}_{1a}\Theta_a^1 = x_{12}^{-1}x_{32}x_{31}^{-1} = \tilde{X}_{1+}^t,
\]

\[
\tilde{X}_1 = \frac{1}{2}(\tilde{X}_{1+} + \tilde{X}_{1-}) = X_1^\mu \tilde{\sigma}_\mu.
\] (3.79)

It is evident from eq.(3.75) that under \(z_2 \leftrightarrow z_3,\) \(z_1 \rightarrow -z_1\).

Associated with \(F(z)\) given in eq.(3.53) we define \(\bar{F}(z)\) by

\[
\bar{F}(z) = \begin{pmatrix} \epsilon & 0 \\ 0 & \zeta \end{pmatrix} F(z)^t \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & \bar{\zeta} \end{pmatrix} = \begin{pmatrix} i\tilde{X}_{1+} & 2\tilde{\Theta}_b \\ 2\tilde{\Theta}_1^a & \delta^a_b \end{pmatrix}.
\] (3.80)

With this definition we may write

\[
\bar{F}(z_1) = \begin{pmatrix} iX_{1+} & 2\tilde{\Theta}_{1b} \\ 2\tilde{\Theta}_1^a & \delta^a_b \end{pmatrix},
\] (3.81)

where

\[
X_{1+} = -\bar{x}_{31}^{-1}\bar{x}_{23}\bar{x}_{13}^{-1},
\]

\[
\tilde{\Theta}_{1a} = i(\bar{x}_{31}^{-1}\tilde{\theta}_{31a} - \bar{x}_{21}^{-1}\tilde{\theta}_{21a}) , \quad \tilde{\Theta}_1^a = i(\theta_{12}^a \bar{x}_{12}^{-1} - \theta_{13}^a \bar{x}_{13}^{-1}).
\] (3.82)
\( \tilde{F}(z_1) \) transforms infinitesimally as
\[
\delta \tilde{F}(z_1) = \left( \begin{array}{cc}
\tilde{w}(z_1) - \frac{1}{2} \lambda(z_1) + i \frac{1}{2} \tilde{\psi}(z_1) & 0 \\
0 & \hat{t}(z_1) + i \frac{2}{N} \hat{\psi}(z_1) + \delta^4 \lambda \Omega
\end{array} \right) \tilde{F}(z_1)
\]
\[
- \tilde{F}(z_1) \left( \begin{array}{cc}
\tilde{w}(z_1) + \frac{1}{2} \lambda(z_1) + i \frac{1}{2} \tilde{\psi}(z_1) & 0 \\
0 & \hat{t}(z_1) + i \frac{2}{N} \hat{\psi}(z_1) + \delta^4 \lambda \Omega
\end{array} \right),
\]
and hence for finite transformations
\[
\tilde{F}(z'_1) = \left( \begin{array}{cc}
L(z_{1+}; g)^{-1} & 0 \\
0 & u(z_1; g)^{-1}
\end{array} \right) \tilde{F}(z_1) \left( \begin{array}{cc}
\bar{L}(z_{1-}; g)^{-1} & 0 \\
0 & u(z_1; g)
\end{array} \right).
\]
Thus \( z_1 \) transforms homogeneously at \( z_1 \), as ‘tangent’ vectors do.
Explicitly we have from eq.(3.84)
\[
X'_{1+} = \Omega(z_1; g)^{-1} \hat{L}(z_{1+}; g)^{-1} X_{1+} \hat{L}(z_{1-}; g)^{-1},
\]
\[
\tilde{\Theta}'_{1a} = \Omega(z_1; g)^{-1} \frac{1}{2} \Upsilon(z_1; g)^{-1} \hat{L}(z_{1+}; g)^{-1} \tilde{\Theta}_{1b} \hat{u}^b_a(z_1; g),
\]
\[
\tilde{\Theta}'_{1a} = \Omega(z_1; g)^{-1} \frac{1}{2} \Upsilon(z_1; g)^{-1} \hat{u}^{-1}_{1b}(z_1; g) \tilde{\Theta}_{1b} \hat{L}(z_{1-}; g)^{-1}.
\]
\( X_{1-} \) also transforms in the same way as \( X_{1+} \) in eq.(3.85) and hence
\[
X'_{1-} = \Omega(z_1; g)^{-1} X_{1+} \hat{R}_{\nu}^\mu(z_1; g),
\]
\[
\Theta'_{1a} \sigma^\mu \tilde{\Theta}_{1a} = \Omega(z_1; g)^{-1} \Theta_{1\nu} \sigma^\mu \tilde{\Theta}_{1a} \hat{R}_{\nu}^\mu(z_1; g).
\]
From eq.(3.84) we get
\[
\text{sdet} \tilde{F}(z_1) = \text{sdet} F(z_1) = -X'^2_{1-} = -\frac{x_{32}^2}{x_{12}^2 x_{31}^2}. \tag{3.87}
\]
If we define a function \( \tilde{v}(z) \in U(\mathcal{N}) \) by
\[
\tilde{v}_{\alpha}^a(z) = (\zeta v(z)^T \zeta)^a_{\alpha} = \delta_{\alpha}^a - 4i \frac{1}{x_1^2} \tilde{\theta}^a \tilde{x}_{-} \tilde{\theta}_b,
\]
then a direct calculation leads
\[
\tilde{v}(z_1) = v(z_{13}) v(z_{32}) v(z_{21}). \tag{3.89}
\]
Similarly for $R_{\mu \nu}(z; i_s)$ given in eq.(2.53) we have

$$R(z_1; i_s) = x_{12}^2 x_{21}^2 x_{31}^2 x_{13}^2 R(z_{12}; i_s) R(z_{23}; i_s) R(z_{31}; i_s).$$

(3.90)

From eqs.(3.68, 3.70) $\tilde{v}(z_1), R(z_1; i_s)$ transform homogeneously at $z_1$ under superconformal transformation, $z \rightarrow z'$,

$$\tilde{v}(z_1') = u^{-1}(z_1; g) \tilde{v}(z_1) u(z_1; g),$$

(3.91a)

$$R(z_1'; i_s) = \Omega(z_1; g) R^{-1}(z_1; g) R(z_1; i_s) R(z_1; g).$$

(3.91b)

Under superinversion, $z_j \rightarrow z_j'$, $j = 1, 2, 3$, $z_1$ transforms to $z_1'$, from eq.(3.72), as

$$\tilde{X}_1' = \tilde{x}_1 + x_{1-} \tilde{x}_{1-},$$

(3.92)

$$\Theta_1^a = iv^{-1} b(z_1) \tilde{\Theta}_1^b \tilde{x}_{1-},$$

and hence

$$X_1^\mu = \Omega(z_1; i_s)^{-2} X_1^\nu R_{\nu \mu}(z_1; i_s),$$

(3.93a)

$$\Theta_1^a \sigma^\nu \tilde{\Theta}_1^b = -\Omega(z_1; i_s)^{-2} \Theta_1^a \sigma^\nu \tilde{\Theta}_1^b R_{\nu \mu}(z_1; i_s).$$

(3.93b)

Note the minus sign in eq.(3.93b).

By taking cyclic permutations of $z_1, z_2, z_3$ in eq.(3.77) we may define $z_2, z_3$. We find $z_2, z_3$ are related to $z_1$ in a simple form

$$F((z_2)'), \quad \tilde{F}(z_1'), \quad (z_3)'.$$

(3.94)

where $(z)' = ((X)', (\Theta)'^a, (\tilde{\Theta})'_a)$ is defined by superinversion, $z \rightarrow \tilde{z}$ (Z)'.

Explicitly we have

$$X_2' + \tilde{\sigma} = \tilde{x}_{21} X_{1+} x_{12}, \quad (\Theta_2)'^a = -iv_{b}(z_{21}) \tilde{\Theta}_1^b \tilde{x}_{12},$$

(3.95a)

$$\tilde{X}_{3+} = \tilde{x}_{31}^{-1} X_{1+}' \sigma x_{13}^{-1}, \quad \Theta_3^a = -iv_{b}(z_{13}) (\tilde{\Theta}_1^b) x_{13}^{-1}.$$
From eqs. (3.77, 3.79) we get

\[
\frac{X^2_{1+}}{X^2_{1-}} = \frac{x^2_{12}x^2_{23}x^2_{31}}{x^2_{21}x^2_{13}x^2_{32}},
\]

(3.96a)

\[
\frac{X_{1+} \cdot X_{1-}}{\sqrt{X^2_{1+} X^2_{1-}}} = \frac{\text{tr}(x_{21}\bar{x}_{23}x_{13}\bar{x}_{12}x_{32}\bar{x}_{31})}{2\sqrt{x^2_{12}x^2_{21}x^2_{23}x^2_{32}x^2_{31}x^2_{13}}}. \tag{3.96b}
\]

These expressions are invariant under cyclic permutations of \(z_1, z_2, z_3\) and hence

\[
\frac{X^2_{2+}}{X^2_{2-}} = \frac{X^2_{3+}}{X^2_{3-}}, \tag{3.97a}
\]

\[
\frac{X_{2+} \cdot X_{2-}}{\sqrt{X^2_{2+} X^2_{2-}}} = \frac{X_{3+} \cdot X_{3-}}{\sqrt{X^2_{3+} X^2_{3-}}}. \tag{3.97b}
\]

From eq. (3.86a) these are invariants for any continuous superconformal transformation and furthermore from eq. (3.93a) the latter is invariant under superinversion along with

\[
\frac{X^2_{1+} + X^2_{1-}}{X^2_{1+} X^2_{1-}}. \tag{3.98}
\]

Note that such invariants, depending on three points, do not exist in ordinary conformal theories and that in the case of \(\mathcal{N} = 1\) due to the identity (A.19a) those two variables are not independent [24, 25].

4 Superconformal Invariance of Correlation Functions

In this section we discuss the superconformal invariance of correlation functions for quasi-primary superfields and exhibit general forms of two-point, three-point and \(n\)-point functions.

4.1 Quasi-primary Superfields

We first assume that there exist quasi-primary superfields, \(\Psi^I(z)\), which under the superconformal transformation, \(z \rightarrow z'\), transform as

\[
\Psi^I \rightarrow \Psi'^I, \quad \Psi'^I(z') = \Psi^I(z)D^I_f(z; g). \tag{4.1}
\]

\(^6\)In [24] it was explicitly shown that the chiral/anti-chiral superfields and supercurrents in some \(\mathcal{N} = 1\) theories are quasi-primary.
\( D(z; g) \) obeys the group property so that under the successive superconformal transformations, \( g'' : z \xrightarrow{g} z' \xrightarrow{g'} z'' \), it satisfies

\[
D(z; g)D(z'; g') = D(z; g''), \tag{4.2}
\]

and hence also

\[
D(z; g)^{-1} = D(z'; g^{-1}). \tag{4.3}
\]

We choose here \( D(z; g) \) to be a representation of \( \text{SL}(2, \mathbb{C}) \times \text{SU}(\mathcal{N}) \times \text{U}(1) \times D \), which is a subgroup of the stability group at \( z = 0 \), and so we decompose the spin index, \( \iota \), of superfields into \( \text{SL}(2, \mathbb{C}) \) index, \( \rho \), and \( \text{SU}(\mathcal{N}) \) index, \( r \), as \( \Psi^\iota \equiv \Psi^\rho_r \). Now \( D^\iota_I_J(z; g) \) is factorized as

\[
D^\iota_I_J(z; g) = D^\rho_\sigma(\hat{L}(z_+; g))D^r_s(\hat{u}(z; g))\Omega(z; g)^{-\eta}\Upsilon(z; g)^{-\kappa}, \tag{4.4}
\]

where \( D^\rho_\sigma(\hat{L}) \), \( D^r_s(\hat{u}) \) are representations of \( \text{SL}(2, \mathbb{C}) \), \( \text{SU}(\mathcal{N}) \) respectively, while \( \eta \) and \( \kappa \) are the scale dimension and \( R \)-symmetry charge of \( \Psi^\rho_r \) respectively.

Infinitesimally

\[
\delta \Psi^\rho_r(z) = -(\mathcal{L} + \eta\hat{\lambda}(z) + i\kappa\hat{\Omega}(z))\Psi^\rho_r(z) - \Psi^\sigma_r(z)\frac{i}{2}(s_{\mu\nu})_{\rho s}^{\sigma}\hat{w}^{\mu\nu}(z) - \Psi^s_r(z)\frac{1}{2}(s^a_{\phantom{a}b})_{\rho s}^{\phantom{\rho}r}t^b_a(z), \tag{4.5}
\]

where \( s_{\mu\nu}, s^a_{\phantom{a}b} \) are matrix generators of \( \text{SO}(1, 3), \text{SU}(\mathcal{N}) \) satisfying

\[
[s_{\mu\nu}, s_{\lambda\rho}] = -\eta_{\mu\lambda}s_{\nu\rho} + \eta_{\mu\rho}s_{\nu\lambda} + \eta_{\nu\lambda}s_{\mu\rho} - \eta_{\nu\rho}s_{\mu\lambda}, \tag{4.6}
\]

\[
[s^a_{\phantom{a}b}, s^c_{\phantom{c}d}] = 2(\delta^a_{\phantom{a}d}s^c_{\phantom{c}b} - \delta^c_{\phantom{c}d}s^a_{\phantom{a}b}),
\]

and hence

\[
\left[\frac{1}{2} s^a_{\phantom{a}b}t_a^1, \frac{1}{2} s^c_{\phantom{c}d}t_b^c \right] = \frac{1}{2} s^a_{\phantom{a}b}[t_1, t_2]_a. \tag{4.7}
\]

From eqs.\((3.13, 3.17)\) using eq.\((4.4)\) we have

\[
\delta_3 \Psi^\rho_r = [\delta_2, \delta_1] \Psi^\rho_r. \tag{4.8}
\]

It is useful to consider the complex conjugate superfield of \( \Psi^\rho_r \)

\[
\bar{\Psi}^{\rho r}(z) = \Psi^\rho_r(z)\dagger. \tag{4.9}
\]

\( \bar{\Psi}^{\rho r}(z) \) transforms as

\[
\bar{\Psi}^{\rho r}(z') = \Omega(z; g)^{-\eta}\Upsilon(z; g)^{\kappa}\bar{D}^\rho_\sigma(\hat{L}(z_-; g))D^r_s(\hat{u}(z; g)^{-1})\bar{\Psi}^s(z). \tag{4.10}
\]
Superconformal invariance for a general \( n \)-point function requires
\[
\langle \Psi^{I_1}_1(z_1)\Psi^{I_2}_2(z_2)\cdots\Psi^{I_n}_n(z_n) \rangle = \langle \Psi^{I_1}_1(z_1)\Psi^{I_2}_2(z_2)\cdots\Psi^{I_n}_n(z_n) \rangle .
\]  
(4.11)

In superconformal field theories on chiral superspace, the representation of \( U(1) \times D \) is given by
\[
\Omega_+(z_+; g) - \eta ,
\]  
(4.12)
so that for \( \mathcal{N} \neq 4 \), \( \eta \) and \( \kappa \) are related by, from eq.(3.52),
\[
\eta + \left( \frac{4}{N} - 1 \right) \kappa = 0 .
\]  
(4.13)
On the other hand when \( \mathcal{N} = 4 \), as shown in subsection 3.3, \( \Omega_+(z_+; g) \) does not form a representation of the \( \mathcal{N} = 4 \) superconformal group, and hence there is no conventional way of defining quasi-primary chiral/anti-chiral superfields in \( \mathcal{N} = 4 \) superconformal theories. We speculate that this fact makes it difficult to construct four-dimensional \( \mathcal{N} = 4 \) superconformal theories on chiral superspace.

### 4.2 Two-point Correlation Functions

The solution for the two-point function of the quasi-primary superfields, \( \Psi^r, \bar{\Psi}^r \), has the general form\(^7\)
\[
\langle \bar{\Psi}^{\rho r}(z_1)\Psi^{\sigma s}(z_2) \rangle = C_{\Psi} \frac{I^{\rho\sigma}(\hat{x}_{12}) I^r_s(\hat{\nu}(z_{12}))}{(x^2_{12})^{\frac{1}{2}(\eta-(\frac{4}{N}-1)\kappa)}(x^2_{21})^{\frac{1}{2}(\eta+(\frac{4}{N}-1)\kappa)}} ,
\]  
(4.14)
where we define
\[
\hat{x}_{12} = \frac{x_{12}}{(x^2_{12})^{\frac{1}{2}}} \in SL(2, \mathbb{C}) ,
\]  
(4.15)
\[
\hat{\nu}^a_b(z_{12}) = \left( \frac{x^2_{21}}{x^2_{12}} \right)^{\frac{1}{2}} \left( \delta^a_b + 4i\theta^a_{12}\tilde{x}^{-1}_{12}\tilde{\theta}^b_{12b} \right) \in SU(\mathcal{N}) ,
\]
and \( I^{\rho\sigma}(\hat{x}_{12}), I^r_s(\hat{\nu}(z_{12})) \) are tensors transforming covariantly according to the appropriate representations of \( SL(2, \mathbb{C}), SU(\mathcal{N}) \) which are formed by decomposition of tensor products of \( \hat{x}_{12}, \hat{\nu}(z_{12}) \).

---

\(^7\)See subsection 4.4 for a proof.
Under superconformal transformations, $I^\rho\sigma(\hat{\chi}_{12})$ and $I^r_s(v(z_{12}))$ satisfy from eqs. (3.65a, 3.70)

\[ D(\hat{L}(z_{1-}; g))I(\hat{\chi}_{12})D(\hat{L}(z_{2+}; g)) = I(\hat{\chi}'_{12}) , \]
\[ D(\hat{u}(z_1; g))^{-1}I(\hat{v}(z_{12}))D(\hat{u}(z_2; g)) = I(\hat{v}(z'_{12})) . \]

As examples, we first consider the chiral/anti-chiral scalar and spinorial fields, $S(z_+), \bar{S}(z_-), \phi^\alpha(z_+), \bar{\phi}^\alpha(z_-)$ in $\mathcal{N} \neq 4$ theories which transform as

\[ S'(z'_+) = \Omega_+(z_+; g)^{-\eta}S(z_+), \]
\[ \bar{S}'(z'_-) = \Omega_-(z_--; g)^{-\eta}\bar{S}(z_-), \]
\[ \phi'^\alpha(z'_+) = \Omega_+(z_+; g)^{-\eta}\phi^\beta(z_+; g)\hat{L}^\alpha_\beta(z_+; g), \]
\[ \bar{\phi}'^\alpha(z'_-) = \Omega_-(z_--; g)^{-\eta}\hat{L}^\dot{\alpha}_{\dot{\beta}}(z_-; g)\bar{\phi}^\dot{\beta}(z_-), \]

so that from eq. (4.13) $\eta + (4N - 1)\kappa = 0$ and $s_{\mu\nu} \to \frac{1}{2}\sigma_{[\mu}\bar{\sigma}_{\nu]}$ for the spinorial fields. The two-point functions of them are

\[ \langle \bar{S}(z_1)S(z_2) \rangle = C_S \frac{1}{(x_{12}^2)^{\eta}} , \]
\[ \langle \bar{\phi}^\dot{\alpha}(z_1)\phi^\alpha(z_2) \rangle = C_{\phi^\alpha} \frac{(\hat{\chi}_{12})_{\dot{\alpha}\alpha}}{(x_{12}^2)^{\eta}} . \]

For a real vector field, $V^\mu(z)$, where the representation of $SL(2, \mathbb{C})$ is given by $\hat{R}_{\mu}^\nu(z; g)$ and the $R$-symmetry charge is zero, $\kappa = 0$, we have

\[ \langle V^\mu(z_1)V_\nu(z_2) \rangle = C_V \frac{I^{\mu\nu}(z_{12})}{(x_{12}^2)^{\frac{3}{2}}}, \quad I^{\mu\nu}(z_{12}) = \frac{1}{2}\text{tr}(\sigma^\mu\hat{x}_{12}\sigma^\nu\hat{x}_{21}) . \]

From eq. (A.10) one can show

\[ I_{\mu \nu}(z_{12}) = \frac{1}{2}\text{tr}(\sigma_{\mu}\hat{x}_{12}\sigma_{\nu}\hat{x}_{21}) = \frac{1}{2}\text{tr}(\bar{\sigma}_{\mu}\hat{x}_{21}\bar{\sigma}_{\nu}\hat{x}_{12}) , \]

where

\[ \hat{x} = \frac{x}{(x^2)^{\frac{3}{2}}} = \hat{x}^{-1} . \]
Hence $I_{\mu\nu}(z_{12})$ satisfies

$$I_{\mu\nu}(z_{12})I_{\lambda\nu}(z_{12}) = \delta_{\mu}^\lambda. \quad (4.23)$$

Note that $I(z_{12}) \propto R(z_{12}; \bar{i}_a)$, where $R(z; \bar{i}_a)$ is given by eq. (2.53).

For gauge fields, $\psi_a(z)$, $\bar{\psi}^a(z)$, which transform as

$$\psi'_a(z') = \Omega(z; g)^{-\kappa} \psi_b(z) \hat{u}^b_a(z; g), \quad (4.24a)$$

$$\bar{\psi}'^a(z') = \Omega(z; g)^{-\kappa} \bar{\psi}^b(z) \hat{\bar{u}}^b_a(z; g), \quad (4.24b)$$

the two-point function of them is

$$\langle \bar{\psi}^a(z_1)\psi_b(z_2) \rangle = C_{\psi} \hat{v}^{a\beta}(z_{12}) \frac{(x_{12}^2)^{\eta} - (x_{12}^2)^{(\eta+1)}}{(x_{12}^2)^{\eta} - (x_{12}^2)^{(\eta-1)}}. \quad (4.25)$$

Note that to have non-vanishing two-point correlation functions, the scale dimensions, $\eta$, of the two fields must be equal and the $R$-symmetry charges must have the same absolute value with opposite signs, $\kappa$, $-\kappa$, as shown in subsection 4.4 later.

For a real vector superfield, $V^\mu(z)$, if we define

$$V_{\alpha\dot{\alpha}}(z) = \sigma_{\mu\alpha} V^\mu(z), \quad (4.26)$$

from eq. (4.20) we get

$$\langle V_{\alpha\dot{\alpha}}(z_1)V_{\beta\dot{\beta}}(z_2) \rangle = 2C_{V} \frac{(\tilde{x}_{12})_{\alpha\dot{\alpha}}(\tilde{x}_{12})_{\beta\dot{\beta}}}{(x_{12}^2)^{\eta}}. \quad (4.27)$$

From

$$D_{\alpha\dot{\alpha}}(z_1)x_{12}^\mu = 2i(\sigma^\mu \bar{\theta}_{12a})_\alpha, \quad D_{\alpha\dot{\alpha}}(z_1)x_{12}^\mu = 0, \quad (4.28)$$

with $\tilde{D}^\alpha\dot{\alpha} = \zeta^{a\beta} \bar{e}^{-1\alpha\beta} D_{\beta\dot{\beta}}$, we get

$$\tilde{D}^\alpha\dot{\alpha}(z_1)\langle V_{\alpha\dot{\alpha}}(z_1)V_{\beta\dot{\beta}}(z_2) \rangle = 4iC_{V}(\eta - 3) \frac{\bar{\theta}_{12\beta}(x)_{\beta\dot{\beta}}}{(x_{12}^2)^{\eta-1}}, \quad (4.29)$$

and hence $\langle V_{\alpha\dot{\alpha}}(z_1)V_{\beta\dot{\beta}}(z_2) \rangle$ is conserved if $\eta = 3$

$$\tilde{D}^\alpha\dot{\alpha}(z_1)\langle V_{\alpha\dot{\alpha}}(z_1)V_{\beta\dot{\beta}}(z_2) \rangle = 0 \quad \text{if} \quad \eta = 3. \quad (4.30)$$

The anti-commutator relation for $D_{\alpha\dot{\alpha}}$, $\tilde{D}^\alpha\dot{\alpha}$ (2.11) implies also

$$\frac{\partial}{\partial x_1^\mu}\langle V^\mu(z_1)V^\nu(z_2) \rangle = 0 \quad \text{if} \quad \eta = 3. \quad (4.31)$$
4.3 Three-point Correlation Functions

The solution for the three-point correlation function of the quasi-primary superfields, $\Psi^\rho_r$, has the general form

$$\langle \Psi^\rho_{1r}(z_1)\Psi^\sigma_{2s}(z_2)\Psi^\tau_{3t}(z_3) \rangle = H^\rho_{\sigma\tau}(Z) I^\rho_{\sigma\tau}(\hat{L}) ,$$

where $Z^M = (X^\mu, \Theta^a, \bar{\Theta}^a) \in \mathbb{R}^{4|4N}$ is given by eq.(3.75).

Superconformal invariance (4.11) is now equivalent to

$$H^\rho_{\sigma\tau}(Z) D^\rho_{\sigma\tau}(\hat{L}) = H^\rho_{\sigma\tau}(Z') D^\rho_{\sigma\tau}(\hat{L}) ,$$

where

$$Z'^M = (X^\nu \hat{R}_{\nu\mu}(\hat{L}), \Theta^a \hat{L}, \bar{\Theta}^a \hat{L}) .$$

In general there are a finite number of linearly independent solutions of eq.(4.33a), and this number may be reduced by imposing extra restrictions on the correlation function.

---

8See subsection 4.4 for a proof.
As an example, we consider the three-point correlation function of a real vector superfield, $V^\mu(z)$, where $\kappa = 0$. From eq.(4.32) we may write

$$
\langle V^\mu(z_1)V^\nu(z_2)V^\lambda(z_3) \rangle = \frac{H^{\mu\nu\lambda}(z_1)I_{\nu'}(z_{12})I_{\lambda'}(z_{13})}{(x_{12}^2x_{23}^2x_{31}^2x_{13}^2)^{\frac{1}{2}n}}.
$$

(4.35)

Since eq.(4.33a) is obtained by considering invariance under continuous superconformal transformations, invariance under superinversion which is a discrete map may give an extra restriction. Besides the superconformal invariance, the three-point function has additional symmetry under permutations of the superfields. Furthermore, for supercurrents we may require the correlation function to satisfy the conservation equations like eqs.(4.30, 4.31). More explicitly, under superinversion we may require $V^\mu(z)$ transforms to

$$
V^\mu(z') = -V^\nu(z)\hat{R}_\nu^\mu(z; i_s)\Omega(z; i_s)^{-n}.
$$

(4.36)

The occurrence of the minus sign in $\mathcal{N} = 1$ Wess-Zumino model and vector superfield theory was verified in [24]. Invariance under superinversion, $z_j \xrightarrow{i_s} z_j'$, $j = 1, 2, 3$, implies using eqs.(2.52, 3.72, 3.74)

$$
H^{\mu\nu\lambda}(z_1)\hat{R}_\nu^\mu(z_1; i_s)\hat{R}_\nu^\nu(z_1; i_s)\hat{R}_\lambda^\lambda(z_1; i_s) = -\Omega^{-n}(z_1; i_s)H^{\mu\nu\lambda}(z_1'),
$$

(4.37)

which also implies using eqs. [3.92, 3.95a]

$$
\Omega(z_{12}; i_s)^nH^{\mu\nu\lambda}(z_1)\hat{R}_\nu^\mu(z_{12}; i_s)\hat{R}_\nu^\nu(z_{12}; i_s)\hat{R}_\lambda^\lambda(z_{12}; i_s) = -H^{\mu\nu\lambda}(-z_2'),
$$

(4.38)

where $(z_2)'$ is given by superinversion, $z_2 \xrightarrow{i_s} (z_2)'$.

From $\langle V^\mu(z_1)V^\nu(z_2)V^\lambda(z_3) \rangle = \langle V^\mu(z_1)V^\lambda(z_3)V^\nu(z_2) \rangle$ we have

$$
H^{\mu\nu\lambda}(Z) = H^{\mu\lambda\nu}(Z),
$$

(4.39)

and from $\langle V^\mu(z_1)V^\nu(z_2)V^\lambda(z_3) \rangle = \langle V^\nu(z_2)V^\lambda(z_3)V^\mu(z_1) \rangle$ we have using eqs.(3.90, 4.38) with $Z \xrightarrow{i_s} (Z)'$

$$
H^{\mu\lambda\nu}(-(Z)') = -\Omega(Z; i_s)^{-n}H^{\nu\lambda\mu}(Z)\hat{R}_\nu^\lambda(Z; i_s).
$$

(4.40)

Imposing these extra conditions it was shown that the three-point correlation functions of supercurrents in $\mathcal{N} = 1$ theories have two linearly independent forms [24]. Similarly, the three-point functions of real scalar superfields in $\mathcal{N} = 1$ theories have also two linearly independent solutions [24].

Invariance under $R$-symmetry transformations (4.33a) implies that $H^{\mu\nu\lambda}(Z)$ is a function of $X^\mu$, $\Theta^{aa}\hat{\Theta}_a^\alpha$ or equivalently $X^\mu$, $\Theta^{a}\sigma^\mu\hat{\Theta}_a$, as demonstrated in subsection 4.3, and hence we may put

$$
H^{\mu\nu\lambda}(Z) = H^{\mu\nu\lambda}(X^\lambda, \Theta^{a}\sigma^\lambda\hat{\Theta}_a).
$$

(4.41)
4.4  \( n \)-point Correlation Functions - in general

In this subsection we show that the solution for \( n \)-point correlation functions of the quasi-primary superfields, \( \Psi^\rho_r \), has the general form

\[
\langle \Psi^\rho_1 r_1(z_1) \cdots \Psi^\rho_n r_n(z_n) \rangle 
= H^{\rho_1 r_1 \rho_2 r_2 \cdots \rho_n r_n}(Z_1, \cdots, Z_{n-2}) \prod_{k=2}^n \frac{I^{\rho_k r_k}(\hat{x}_{1k}) I^{r_k \rho_k}(\hat{z}_{1k})}{(x_{1k}^2)^{\frac{1}{2}(\eta_k - (\frac{1}{2} - 1)\kappa_k)} (x_{1k}^2)^{\frac{1}{2}(\eta_k + (\frac{1}{2} - 1)\kappa_k)}},
\]

where in a similar fashion to eq.(3.75) \( Z_1, \cdots, Z_{n-2} \) are given, with \( z_{k1} \rightarrow \tilde{z}_{k1} \), \( k \geq 2 \), by

\[
G_T(\tilde{z}_{j1})^{-1}G_T(z_{1j-1}) = G_T(z_{1(j-1)}), \quad j = 2, 3, \cdots, n-1.
\]

We note that all of them are ‘tangent’ vectors at \( z_1 \).

Superconformal invariance \([111]\) is equivalent to

\[
H^{\rho_1 r_1 \rho_2 r_2 \cdots \rho_n r_n}(Z_1, \cdots, Z_{n-2}) \prod_{k=2}^n \frac{I^{\rho_k r_k}(\hat{x}_{1k}) I^{r_k \rho_k}(\hat{z}_{1k})}{(x_{1k}^2)^{\frac{1}{2}(\eta_k - (\frac{1}{2} - 1)\kappa_k)} (x_{1k}^2)^{\frac{1}{2}(\eta_k + (\frac{1}{2} - 1)\kappa_k)}},
\]

as well as

\[
Z^{M}_{(j)} = (X_{(j)}^\nu, \hat{R}_{\nu}(\hat{L}), \Theta^a_{(j)} \hat{L}, \hat{\Theta}_{(j)a}),
\]

\[
Z^{M}_{(j)} = (X_{(j)}^\mu, \hat{R}_{\mu}(\hat{L}), \Theta_{(j)} \hat{L}, \hat{\Theta}_{(j)a}),
\]

\[
Z^{M}_{(j)} = (X_{(j)}^\nu, \hat{K}_{\nu}(\hat{L}), \Theta_{(j)} \hat{K}_{\nu}, \hat{\Theta}_{(j)a}),
\]

\[
Z^{M}_{(j)} = (X_{(j)}^\nu, \hat{K}_{\nu}(\hat{L}), \Theta_{(j)} \hat{K}_{\nu}, \hat{\Theta}_{(j)a}),(4.42)
\]

Thus, in general \( n \)-point functions reduce to one unspecified \((n - 2)\)-point function which must transform homogeneously under the rigid transformations,
implies

The superconformal invariance of the correlation function (4.11), using eqs. (3.67a, 3.85, 4.16a),

Now we consider a superconformal transformation, \( z \rightarrow z' \), defined by

\[
G_T(z') = G_T(z'_1) G_T(z; u),
\]

\[
G_T(z) = G_T(z_1)^{-1} G_T(z),
\]

where \( z_s(z'; u) \) is a special superconformal transformation given in eq. (2.58) and \( z'_1 \) can be arbitrary. Since

\[
H^\rho_{r_1 r_2 \cdots r_n} (\tilde{z}_{21}, \tilde{z}_{31}, \cdots, \tilde{z}_{n1}) \text{ possesses a supertranslational invariance}
\]

\[
H^\rho_{r_1 r_2 \cdots r_n} (\tilde{z}_{21}, \tilde{z}_{31}, \cdots, \tilde{z}_{n1}) = H^\rho_{r_1 r_2 \cdots r_n} (\tilde{z}'_{21}, \tilde{z}'_{31}, \cdots, \tilde{z}'_{n1}).
\]

\( ^9 \) The key idea in this proof first appeared in [24].

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Thus we can write
\[ H^{\rho_1 \rho_2 \cdots \rho_n r_n}(\tilde{z}_{21}, \tilde{z}_{31}, \ldots, \tilde{z}_{n1}) = H^{\rho_1 \rho_2 \cdots \rho_n r_n}(Z_{1(1)}, \cdots, Z_{1(n-2)}) , \]
\[ G_T(Z_{1(j-1)}) = G_T(\tilde{z}_{n1})^{-1}G_T(\tilde{z}_{j1}) . \]

With the transformation rule for \( Z_1 \) (3.85), eq.(4.50) completes our proof. \( Q.E.D. \)

We note that, in the case of \( n = 2 \), \( H^{\rho_1 \rho_2} \) is independent of \( z_1, z_2 \) and eqs.(4.44c,d) show that two-point functions vanish if \( \eta_1 \neq \eta_2 \) or \( \kappa_1 \neq -\kappa_2 \). Furthermore, if the representation is irreducible then \( H = 1 \) by Schur’s Lemma.

### 4.5 Selection Rule & Superconformal Invariants

We begin with fields, \( \psi^I(x) \), depending on \( x \in \mathbb{R}^4 \) which are obtained by letting the Grassmann coordinates inside quasi-primary superfields, \( \Psi^I(x, \theta^a, \bar{\theta}^a) \), be zero
\[ \psi^I(x) \equiv \Psi^I(x, 0, 0) . \]

They are the lowest order term appearing in the power series expansions of superfields in Grassmann coordinates. The superconformal invariance under \( U(1) \) transformations (4.44d) implies for arbitrary \( \Omega \in S^1 \)
\[ \langle \psi^I_1(x_1) \cdots \psi^I_n(x_n) \rangle = e^{i(\kappa_1 + \cdots + \kappa_n)}\Omega \langle \psi^I_1(x_1) \cdots \psi^I_n(x_n) \rangle , \]

hence, if the sum of the \( R \)-symmetry charge, \( \kappa_i \), is not zero then the correlation function must vanish as exhibited in eq.(1.2)
\[ \langle \psi^I_1(x_1) \cdots \psi^I_n(x_n) \rangle = 0 \quad \text{if} \quad \sum_{i=1}^{n} \kappa_i \neq 0 . \]

This selection rule can be generalized further to all the other component fields in the power series expansions of superfields
\[ \Psi^I(x, \theta^a, \bar{\theta}^a) = \psi^I(x) + \psi^I_{a\alpha}(x)\theta^{a\alpha} + \bar{\theta}^{\dot{a}\dot{\alpha}}\bar{\psi}^{I\dot{a}\dot{\alpha}}(x) + \cdots . \]

If we define the \( R \)-symmetry charge of the component fields, \( \psi^I(x), \psi^I_{a\alpha}(x), \bar{\psi}^{I\dot{a}\dot{\alpha}}(x) \), etc. as \( \kappa, \kappa + \frac{1}{2}, \kappa - \frac{1}{2} \), etc. respectively, then the invariance under \( U(1) \) transformation,
\[ \langle \Psi^I_1(x_1, \theta^{a_1}, \bar{\theta}^{\dot{a}_1}) \cdots \rangle = e^{i(\kappa_1 + \cdots + \kappa_n)}\Omega \langle \Psi^I_1(x_1, e^{i\frac{\Omega}{2}}\theta^{a_1}, e^{-i\frac{\Omega}{2}}\bar{\theta}^{\dot{a}_1}) \cdots \rangle , \]
implies that the selection rule (4.53) holds for all the component fields. The existence of this kind of selection rule in $\mathcal{N} = 4$ super Yang-Mills theory was previously predicted by Intriligator within the context of $AdS$/CFT correspondence, as the dual IIB supergravity contains a corresponding U(1) symmetry [1]. Therefore our results provide a supporting evidence for the Maldacena conjecture, as the selection rule here is derived by purely considering the symmetry on CFT side without referring to the string side.

Essentially, for $\mathcal{N} \neq 4$ case, the selection rule exists since the four-dimensional $\mathcal{N} \neq 4$ superconformal group includes the U(1) factor inevitably. However in $\mathcal{N} = 4$ case, as verified in subsection 2.5, the corresponding superconformal group is isomorphic to a semi-direct product of U(1) and a simple Lie supergroup so that it can be reduced to the simple Lie subgroup by breaking the U(1) symmetry. In this case, the selection rule will not be applicable to the corresponding $\mathcal{N} = 4$ superconformal theory, since the U(1) representation becomes trivial, $\Upsilon(z; g) = 1$, and the $R$-symmetry charge is not defined.

Now, we consider correlation functions of quasi-primary scalar superfields, $\Psi(z)$. From eqs. (4.44a, b) $H(Z_{1(1)}, \cdots, Z_{1(n-2)})$ must be SL(2, C) × SU($\mathcal{N}$) invariant and hence it is a function of SL(2, C) × SU($\mathcal{N}$) invariants. According to [12], these invariants can be obtained by contracting in all possible ways the spinorial indices of $\epsilon_{\alpha\beta}, \epsilon^{-1\alpha\beta}, \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}, \bar{\epsilon}^{\dot{1}\dot{\alpha}\dot{\beta}}, (X_{1(i)}{a\dot{a}}), (X_1^{(i)}){a\dot{a}}$, (4.56)

On the other hand if we write

$X_{1(1)} = (X_1^{(1)}){a\dot{a}}$,

where $\hat{z}_{1(j)}^M = (\hat{X}_1^{(j)}{a\dot{a}})$ are normalized $z_{1(j)}^M$

$\hat{X}_1^{(j)}{a\dot{a}} = \frac{X_1^{(j)}{a\dot{a}}}{(X_{1(1)}{a\dot{a}})^2}$,

then using eq. (A.20a) one can show that $X_{1(1)}, X_{1(J)}$ are all the invariants for SL(2, C) × SU($\mathcal{N}$) × U(1) × D and hence invariants for the whole $\mathcal{N}$-extended superconformal group. Note that from eq. (3.93b), some of them are pseudo invariants under superinversion. Explicitly, we may reproduce the invariants depending on three points (3.97) as

$$(\hat{X}_1^{(1)+})^2 = \frac{X_{1(1)}^{(1)+} \cdot X_{1(1)}^{(1)-}}{X_{1(1)}^{2(1)+} \cdot X_{1(1)}^{2(1)-}}, \quad \hat{X}_{1(1)+} \cdot \hat{X}_{1(1)-} = \frac{X_{1(1)}^{(1)+} \cdot X_{1(1)}^{(1)-}}{\sqrt{X_{1(1)}^{2(1)+} \cdot X_{1(1)}^{2(1)-}}}, \quad (4.59)$$
and using

$$X^2_{1(l-1)+} = \frac{x^2_{1\ell n}}{x^2_{1l} x^2_{1n}}; \quad (4.60)$$

$$X^2_{(l,m)+} = \frac{x^2_{lm}}{x^2_{l1} x^2_{1m}}; \quad (4.61)$$

$$X^\mu_{(l,m)+} = X^\mu_{1(l-1)+} - X^\mu_{1(m-1)-} + 2i\Theta^a_1(a_1)\Theta^\mu_1(a_{m-1}) \sigma^1 \Theta^1(a_{m-1}), \quad (4.62)$$

we may also obtain cross ratio type invariants depending on four points, \(z_r, z_s, z_t, z_u\)

$$\frac{x^2_{rs} x^2_{tu}}{x^2_{rt} x^2_{ts}}. \quad (4.63)$$

From \(\hat{X}^2_{1(l+1)} \hat{X}^2_{1(l)-} = 1\), the number of different \(X_{1(I)} \cdot X_{1(J)}, 1 \leq I, J \leq (n-1)(n-2)\) is

$$\frac{1}{2}(n^2 - 3n + 4)(n^2 - 3n + 1). \quad \text{However, a vector, } \alpha, \text{ in } d\text{-dimensions may be specified by } d\text{ equations, } \alpha \cdot \beta = c_i; 1 \leq i \leq d. \quad \text{Thus the number of independent superconformal invariants or } X_{1(I)} \cdot X_{1(J)} \text{ in four-dimensions, } \#_n, \text{ is}$$

$$\#_n = \begin{cases} 2 & \text{for } n = 3, \\ 4n^2 - 12n + 1 & \text{for } n \geq 4. \end{cases} \quad (4.63)$$

This result holds for \(\mathcal{N} \geq 2\), and for \(\mathcal{N} = 1\) due to the identity \((\mathcal{A}19a)\) the number of independent \(X_{1(I)} \cdot X_{1(J)}\) reduces further to

$$\#_n = \begin{cases} 1 & \text{for } n = 3, \\ 4n^2 - 12n & \text{for } n \geq 4. \end{cases} \quad (4.64)$$

In the case of \(\sum_i \kappa_i = 0\), \(H(\hat{z}_{1(1)}, \cdots, \hat{z}_{1(n-2)})\) must be \(\text{SL}(2, \mathbb{C}) \times \text{SU}(\mathcal{N}) \times \text{U}(1) \times \text{D} \) invariant. Hence it is a function of \(X_{1(I)} \cdot X_{1(J)}\) and the \(n\)-point correlation function of scalar superfields reduces to an arbitrary \(\#_n\) variable function. The \(n\)-point function in this case may have the following general form

$$\langle \Psi_1(z_1) \cdots \Psi_n(z_n) \rangle = \frac{F(X_{1(I)} \cdot X_{1(J)})}{\prod_{l \neq m} (x^2_{1m})^{\Delta_{lm}}}, \quad (4.65)$$

\(^{10}\)Similar analysis for \(\mathcal{N} = 1\) was done by Osborn. \cite{23}.
\[ \Delta_{lm} = -\frac{1}{2(n-1)(n-2)} \sum_{i=1}^{n} \eta_{i} + \frac{1}{2(n-2)}(\eta_{l} + \eta_{m}) + \frac{1}{2n}(\frac{4}{N} - 1)(\kappa_{l} - \kappa_{m}). \]  

(4.66)

\[ F(X_{1(I)} \cdot X_{1(J)}) \text{ is related to } H(Z_{1(1)}, \cdots, Z_{1(n-2)}) \text{ by} \]

\[ F(X_{1(I)} \cdot X_{1(J)}) = H(\hat{Z}_{1(1)}, \cdots, \hat{Z}_{1(n-2)}) \prod_{2 \leq l \neq m} (\hat{X}_{1(l,m)}^2)^{\Delta_{lm}}. \]  

(4.67)

This relation can be derived using eq.(4.44c) and the following identity which holds when \((\frac{4}{N} - 1) \sum_{i=1}^{n} \kappa_{i} = 0\)

\[
\left( \prod_{l \neq m} (x_{lm}^2)^{\Delta_{lm}} \right) \left( \prod_{k=2}^{n} \left( x_{1k}^2 \right)^{-\frac{1}{2}(\eta_{k} - (\frac{4}{N} - 1)\kappa_{k})} \right) \left( x_{1k1}^2 \right)^{-\frac{1}{2}(\eta_{k} + (\frac{4}{N} - 1)\kappa_{k})} \prod_{2 \leq l \neq m} \left( \frac{\lambda x_{lm}^2}{x_{1l}^2 x_{1m}^2} \right)^{\Delta_{lm}},
\]  

(4.68)

where \(\lambda \in \mathbb{R}\).

### 4.6 Non-supersymmetric Case

In particular, here, we consider the non-supersymmetric case, i.e. \(\mathcal{N} = 0\). Quasi-primary fields, \(\Psi^{\rho}(x)\), transform under the conformal transformations, \(x \xrightarrow{g} x'\), as

\[
\Psi^{\rho}(x) \rightarrow \Psi'^{\rho}(x') = \Psi^{\rho}(x)D_{\sigma}^{\rho}(\hat{L}(x; g))\Omega(x; g)^{-\eta}. \]

(4.69)

\(n\)-point function has the general form

\[
\langle \Psi^{\rho_{1}}_{1}(x_{1}) \cdots \Psi^{\rho_{n}}_{n}(x_{n}) \rangle = \frac{H_{\rho_{1} \cdots \rho_{n}}(X_{1(1)}, \cdots, X_{1(n-2)})I_{\rho_{12}}^{\rho_{2}}(\hat{X}_{12}) \cdots I_{\rho_{1n}}^{\rho_{n}}(\hat{X}_{1n})}{(x_{12}^2)^{n_2} \cdots (x_{1n}^2)^{n_n}},
\]  

(4.70)

where \(H_{\rho_{1} \cdots \rho_{n}}(X_{1(1)}, \cdots, X_{1(n-2)})\) is a function depending on \(n-2\) points, \(X_{1(j)}, 1 \leq j \leq n-2\),

\[
(X_{1(j)})^{\mu} = \frac{x_{1(j+1)1}^\mu}{x_{11}^\mu} - \frac{x_{1(j+1)1}^\mu}{x_{1(j+1)1}^\mu}, \quad X_{1(j)}^2 = \frac{x_{1(j+1)n}^2}{x_{1n}^2 x_{1(j+1)1}^2}. \]

(4.71)
Under conformal transformations, $X_{1(j)}$ transforms homogeneously at $x_1$ as in eq.(3.86a). Conformal invariance is equivalent to

$$H_{\rho_1\rho_2\cdots\rho_n}(X_{(1)}, \cdots, X_{(n-2)}) D_{\rho'_1}^\rho_1(\hat{L}) = H_{\rho_1\rho_2'\cdots\rho_n'}(X'_{(1)}, \cdots, X'_{(n-2)}) \bar{D}^\rho_2 \rho_2(\hat{L}) \cdots \bar{D}^\rho_n \rho_n(\hat{L}),$$

(4.72a)

$$X_{(j)}^\mu = X'_{(j)} R_\mu(\hat{L}),$$

$$H_{\rho_1\rho_2\cdots\rho_n}(X_{(1)}, \cdots, X_{(n-2)}) = \lambda^{-\eta_1+\eta_2+\cdots+\eta_n} H_{\rho_1\rho_2\cdots\rho_n}(X''_{(1)}, \cdots, X''_{(n-2)}),$$

(4.72b)

$$X''_{(j)}^\mu = \lambda X_{(j)}^\mu.$$  

We note that this result holds in arbitrary dimension, $d$.

As an example, we consider scalar fields, $\Psi(x)$. The $n$-point function of them is from eq.(4.65)

$$\langle \Psi_1(x_1) \cdots \Psi_n(x_n) \rangle = \frac{F(\hat{X}_{1(i)} \cdot \hat{X}_{1(j)})}{\prod_{l<m} (x_{lm}^2)^{(\eta_1+\cdots+\eta_m)/2(n-2)-(\eta_1+\cdots+\eta_m)/2(n-2)}},$$

(4.73)

where

$$\hat{X}_{1(i)} = \frac{X_{1(i)}}{\sqrt{X_{1(1)}}}.$$  

(4.74)

Alternatively, from eq.(4.63), we may regard $F(\hat{X}_{1(i)} \cdot \hat{X}_{1(j)})$ as an arbitrary function of cross ratios $(x_{ij}^2 x_{kl}^2) / (x_{ik}^2 x_{jl}^2)$.

The number of independent $\hat{X}_{1(i)} \cdot \hat{X}_{1(j)}$, $1 \leq i, j \leq n-2$ or cross ratios in $d$-dimensions, $\#_{n|d}$, is

$$\#_{n|d} = \left\{ \begin{array}{ll} \frac{1}{2} n(n-3) & \text{for } n \leq d+2, \\ dn - \frac{1}{2}(d+1)(d+2) & \text{for } n > d+2, \end{array} \right.$$  

(4.75)

and the $n$-point function of scalar fields reduces to an arbitrary $\#_{n|d}$ variable function.
5 Superconformally Covariant Operators

In general acting on a quasi-primary superfield, $\Psi^\rho_r(z)$, with the spinor derivative, $D_{aa}$, does not lead to a quasi-primary field. For a superfield, $\Psi^\rho_r$, from eqs.(3.8, 3.10, 4.3) we have

$$D_{aa}\delta \Psi^\rho_r = -(\mathcal{L} + (\eta + \frac{1}{2})\lambda + i(\kappa + \frac{1}{2})\hat{\Omega})D_{aa}\Psi^\rho_r$$

$$+ \tilde{w}_{\alpha}^{}\beta D_{a\beta}\Psi^\rho_r - D_{aa}\Psi^\sigma_r \frac{1}{2}(s_{\mu\nu}\tilde{w}^{\mu\nu})_{\sigma}^{}$$

$$- \tilde{t}_{a}^{} D_{ba}\Psi^\rho_r - D_{aa}\Psi^\rho_{s} \frac{1}{2}(s_{bc}\tilde{\epsilon}_{b}^{}c_{b}^{})_{s}^{}$$

$$+ 2\hat{\rho}_{b\beta}(\Psi Y^{b\beta}_{aa})_{r}^{}.$$  \hspace{1cm} (5.1)

We may connect the generator of $\text{SL}(2, \mathbb{C})$ to $\text{SO}(1, 3)$ by

$$s_{\alpha}^{}\beta \equiv -\frac{1}{2}s_{\mu\nu}(\sigma[^{\mu\nu}]_{\alpha}^{\beta}),$$  \hspace{1cm} (5.2)

where

$$[s_{\alpha}{}_{\beta}, s_{\gamma}{}_{\delta}] = \epsilon_{\alpha\delta}s_{\beta\gamma} + \epsilon_{\beta\delta}s_{\alpha\gamma} + \epsilon_{\alpha\gamma}s_{\beta\delta} + \epsilon_{\beta\gamma}s_{\alpha\delta},$$

$$s_{\alpha}{}_{\beta} = s_{\alpha}{}_{\gamma}\epsilon_{\gamma}{}_{\beta} = s_{\beta}{}_{\alpha}. $$ \hspace{1cm} (5.3)

$Y^{b\beta}_{aa}$ is given by

$$Y^{b\beta}_{aa} = (((4N - 1)\kappa - \eta)\delta^{b}_{\alpha}\delta^{\beta}_{\alpha} + \delta^{b}_{\alpha}s_{\alpha}{}_{\beta} + 2s_{b}^{a}\delta^{\beta}_{\alpha}.$$  \hspace{1cm} (5.4)

To ensure that $D_{aa}\Psi^\rho_r$ is quasi-primary it is necessary that the terms proportional to $\hat{\rho}$ vanish and this can be achieved by restricting $D_{aa}\Psi^\rho_r$ to an irreducible representation of $\text{SL}(2, \mathbb{C}), \text{SU}(N)$ and choosing a particular value of $\eta$ and $\kappa$ so that $\Psi Y = 0$. The changes of the scale dimension and the $R$-symmetry charge, $\eta \rightarrow \eta + \frac{1}{2}$, $\kappa \rightarrow \kappa + \frac{1}{2}$, in eq.(5.1) are also apparent from eq.(2.12)

$$D_{aa} = \Omega(z; g)^{1/2}Y(z; g)^{1/2}L_{\alpha}^{\beta}(z; g)\tilde{u}^{-1b}_{a}(z; g)D_{b\beta}.$$  \hspace{1cm} (5.5)
As an illustration we consider tensorial fields, \( \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \), which transform as

\[
\delta \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} = - (\mathcal{L} + \eta \hat{\lambda} + i \kappa \hat{\Omega}) \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \\
+ \sum_{p=1}^{k} \tilde{w}_{\alpha_p}^\beta \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} - \sum_{q=1}^{l} \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \hat{w}_{\beta}^\alpha \hat{a}_q \\
- \sum_{i=1}^{m} \Psi_{a_1 \ldots a_1 \ldots a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \hat{c}_{\alpha_i} + \sum_{j=1}^{n} \tilde{a}_{b_j} \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l}.
\]

(5.6)

Note that spinorial indices, \( \alpha, \hat{\alpha} \) may be raised by \( \epsilon^{-1}, \tilde{\epsilon}^{-1} \).

For \( \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \) we have

\[
(\Psi Y_{\alpha a}^{b \beta})_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} = -2 \sum_{p=1}^{k} \delta_a^b \delta_{\alpha_p}^\beta \Psi_{a_1 \ldots a_m a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \\
+ 2 \sum_{i=1}^{m} \delta_{a_i}^b \delta_{\alpha}^\beta \Psi_{a_1 \ldots a_1 \ldots a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \\
- 2 \sum_{j=1}^{n} \delta_a^{b_j} \delta_{\alpha}^\beta \Psi_{a_1 \ldots a_1 \ldots a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l} \\
+ \left( (\frac{4}{N} - 1) \kappa - \eta + k + 2 \frac{1}{N} (n - m) \right) \delta_a^b \delta_{\alpha}^\beta \Psi_{a_1 \ldots a_1 \ldots a_1 \ldots a_k \hat{a}_1 \ldots \hat{a}_l}.
\]

(5.7)

In particular, eqs. (5.7) shows that the following are quasi-primary

\[
\begin{align*}
D_{(b \beta \Psi_{a_1 \ldots a_m a_1 \ldots a_k})_{a_1 \ldots \hat{a}_l}} & \quad \text{if } \eta - \left( \frac{4}{N} - 1 \right) \kappa = -k + 2(1 - \frac{1}{N})m, \quad (5.8a) \\
\bar{\zeta}_{(a \hat{D}^{b \beta} \Psi_{a_1 \ldots a_m a_1 \ldots a_k})_{a_1 \ldots \hat{a}_l}} & \quad \text{if } \eta - \left( \frac{4}{N} - 1 \right) \kappa = 3 + 2(1 - \frac{1}{N})m, \quad (5.8b) \\
\bar{\zeta}_{(b \beta \Psi_{a_1 \ldots a_m a_1 \ldots a_k})_{a_1 \ldots \hat{a}_l}} & \quad \text{if } \eta - \left( \frac{4}{N} - 1 \right) \kappa = -k - 2(1 + \frac{1}{N})m, \quad (5.8c) \\
\bar{\zeta}_{(b \beta \Psi_{a_1 \ldots a_m a_1 \ldots a_k})_{a_1 \ldots \hat{a}_l}} & \quad \text{if } \eta - \left( \frac{4}{N} - 1 \right) \kappa = 3 - 2(1 + \frac{1}{N})m, \quad (5.8d)
\end{align*}
\]

where () , [ ] denote the usual symmetrization, anti-symmetrization respectively and obviously eqs. (5.8a)(d) are nontrivial if \( m + 1 \leq N \).

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Now we consider the case where more than one spinor derivative, $D_{a\alpha}$, act on a quasi-primary superfield. In this case, it is useful to note
\begin{equation}
D_{(a(\alpha D_{b)\beta}) = 0, \quad D_{[a[\alpha D_{b]\beta]} = 0, \quad (5.9)
\end{equation}
and
\begin{equation}
D_{a\alpha} = 0 \quad (5.10)
\end{equation}
From eq.(5.7) one can show that the following are quasi-primary
\begin{equation}
D_{[b_1(\beta_1 \cdots D_{b_n\beta_n}\Psi_{a_1 \cdots a_m}]_{\alpha_1 \cdots \alpha_l} \quad \text{if} \quad \eta - \left(\frac{4}{N} - 1\right)\kappa = 2 - k - 2n - 2(1 + \frac{1}{N})m, \quad (5.11a)
\end{equation}
\begin{equation}
D_{(bc}\Psi_{a_1 \cdots a_m})_{\alpha_1 \cdots \alpha_l} \quad \text{if} \quad \eta - \left(\frac{4}{N} - 1\right)\kappa = 2 + 2(1 - \frac{1}{N})m, \quad (5.11b)
\end{equation}
where
\begin{equation}
D_{bc} = \epsilon^{-\gamma}\beta D_{b\beta}D_{c\gamma}. \quad (5.12)
\end{equation}
Eq.(5.11a) is nontrivial if $m + n \leq N$.
Similar analysis for $\bar{D}_{a\dot{\alpha}}$ is ready to be done by taking complex conjugates of the results (5.8a, 5.11a). We also note that in $N \neq 4$ chiral superfield theories, $\eta$ and $\kappa$ are related by $\eta + (\frac{1}{N} - 1)\kappa = 0$ as demonstrated in eq.(4.13).

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Appendix

A Notations & Useful Equations

With the four-dimensional Minkowskian metric, \( \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1) \), the \( 4 \times 4 \) gamma matrices, \( \gamma^\mu, \mu = 0, 1, \cdots, 3 \), satisfy the Clifford algebra

\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}.
\]  
(A.1)

The gamma matrices for even dimensions can be chosen in general to have the form

\[
\gamma^\mu = \left( \begin{array}{cc} 0 & \sigma^\mu \\
\tilde{\sigma}^\mu & 0 \end{array} \right),
\]  
(A.2)

and to satisfy

\[
\gamma_5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \left( \begin{array}{cc} 1 & 0 \\
0 & -1 \end{array} \right).
\]  
(A.3)

We also assume the hermiticity condition

\[
\gamma^0 \gamma_\mu \gamma^0 = \gamma^\mu.
\]  
(A.4)

The \( 2 \times 2 \) matrices, \( \sigma^\mu, \tilde{\sigma}^\mu \) satisfy from eqs. (A.1, A.3)

\[
\sigma^\mu \tilde{\sigma}^\nu + \tilde{\sigma}^\nu \sigma^\mu = 2 \eta^{\mu\nu},
\]  
(A.5)

and

\[
\frac{1}{2} \text{tr}(\sigma^\mu \tilde{\sigma}^\nu) = \eta^{\mu\nu},
\]  
(A.6a)

\[
\frac{1}{2} \text{tr}(\sigma^\mu \tilde{\sigma}^\nu \sigma^\lambda \tilde{\sigma}^\rho) = \eta^{\mu\nu} \eta^{\lambda\rho} + \eta^{\mu\rho} \eta^{\nu\lambda} - \eta^{\mu\lambda} \eta^{\nu\rho} - i \epsilon^{\mu\nu\lambda\rho},
\]  
(A.6b)

where we put \( \epsilon_{0123} = -\epsilon^{0123} = 1 \).

\( \sigma^\mu \) and \( \tilde{\sigma}^\mu \) separately form bases of \( 2 \times 2 \) matrices with the completeness relation

\[
\sigma^\mu_{\alpha\dot{\alpha}} \bar{\sigma}^\mu_{\dot{\alpha}\alpha} = 2 \delta^\mu_{\alpha} \delta^{\dot{\alpha}}_{\dot{\alpha}}.
\]  
(A.7)

The coefficient on the right hand side may be determined by eq.(A.6a).

Charge conjugation matrix \( C \) satisfies

\[
C \gamma^\mu C^{-1} = -\gamma^\mu,
\]  
(A.8)
and has the form
\[ C = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} , \]  
(A.9)
where \( \epsilon_{\alpha\beta} \), \( \bar{\epsilon}_{\dot{\alpha}\dot{\beta}} \) are the \( 2 \times 2 \) anti-symmetric matrices, \( \epsilon_{12} = \bar{\epsilon}_{12} = 1 \) with inverses, \( (\epsilon^{-1})^{\alpha\beta} \), \( (\bar{\epsilon}^{-1})^{\dot{\alpha}\dot{\beta}} \).

Eq. (A.8) implies
\[ \epsilon \tilde{\sigma}^{\mu} \epsilon = -\sigma^{\mu} . \]  
(A.10)

From eqs. (A.7, A.10) we get
\[ \sigma^{\mu}_{\alpha\dot{\alpha}} \sigma_{\beta\dot{\beta}} = 2 \epsilon_{\alpha\dot{\alpha}} \epsilon_{\beta\dot{\beta}} , \]  
(A.11)
\[ (\sigma^{\mu} \tilde{\sigma}^{\nu})_{\alpha}^{\beta} (\sigma_{\mu} \tilde{\sigma}_{\nu})_{\gamma}^{\delta} = 4 (\delta_{\alpha}^{\beta} \delta_{\gamma}^{\delta} - 2 \delta_{\alpha}^{\delta} \delta_{\gamma}^{\beta}) , \]  
(A.11)
\[ (\sigma^{\mu} \tilde{\sigma}^{\nu})_{\alpha}^{\beta} (\tilde{\sigma}_{\mu} \sigma_{\nu})_{\dot{\alpha}}^{\dot{\beta}} = 0 . \]  
(A.11)

It is useful to note
\[ \delta_{\alpha}^{\beta} \delta_{\gamma}^{\delta} - \delta_{\alpha}^{\delta} \delta_{\gamma}^{\beta} = \epsilon_{\alpha\gamma} \epsilon^{-1,\beta} . \]  
(A.12)

We may choose
\[ \sigma^{0} = \bar{\sigma}^{0} = 1 , \]  
(A.13)
then from eqs. (A.4, A.3)
\[ \sigma^{i} = -\bar{\sigma}^{i} , \quad \sigma^{\mu}_{\dot{\mu}} = \sigma^{\mu} , \quad i = 1, 2, 3 . \]  
(A.14)

In four-dimensions there is a unique correspondence between a general four vector, \( v^{\mu} \), and a \( 2 \times 2 \) matrix, \( \nu_{\alpha\dot{\alpha}} \) or \( \tilde{\nu}_{\alpha\dot{\alpha}} \), through
\[ v_{\alpha\dot{\alpha}} = v^{\mu} \sigma_{\mu\alpha\dot{\alpha}} , \quad v^{\mu} = \frac{1}{2} \text{tr}(\tilde{\sigma}^{\mu} v) , \]  
(A.15)
\[ \tilde{\nu}_{\alpha\dot{\alpha}} = \nu^{\mu} \tilde{\sigma}_{\mu}^{\dot{\alpha}} \alpha , \quad \nu^{\mu} = \frac{1}{2} \text{tr}(\sigma^{\mu} \tilde{v}) . \]  
(A.15)

With this notation it is convenient to introduce the variables
\[ \tilde{x}_{\pm} = \tilde{x} \pm 2i \bar{\theta}_{a} \theta^{a} . \]  
(A.16)

Note that
\[ \tilde{x}_{\pm} = x_{\pm}^{\mu} \tilde{\sigma}_{\mu} , \quad x_{\pm}^{\mu} = x^{\mu} \mp i \theta^{a} \sigma^{\mu}_{\dot{\alpha}} \theta^{a} , \]  
(A.17)
\[ \tilde{x}_{\pm}^{\dagger} = \tilde{x}_{\mp} , \]  
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and

\[ x_\pm = x_\mu^\pm \sigma_\mu = x \mp 2i \bar{\theta}_a \tilde{\theta}^a. \] (A.18)

Some useful identities relevant to the present paper are

\[ \theta^a \bar{\theta}^\beta = -\frac{1}{2} \epsilon_{\dot{a} \dot{b}} \theta^a \bar{\theta}^\dot{b}, \quad \bar{\theta}^a \theta^\beta = -\frac{1}{2} \epsilon_{\dot{a} \dot{b}} \bar{\theta}^\dot{a} \theta^b, \] (A.19a)

\[ \theta^a \bar{\theta}^\sigma \theta^\mu \bar{\theta}^\nu = \frac{1}{2} \theta^a \bar{\theta}^\sigma \theta^\mu \bar{\theta}^\nu, \] (A.19b)

\[ \theta^a \bar{\theta}^\mu \sigma^\nu \bar{\theta}^\dot{a} = \frac{4}{x^2} (\theta x \bar{\theta})^2, \] (A.19c)

\[ \theta^a \bar{\theta}^\beta = \frac{1}{2} \theta_1 \sigma^\mu \bar{\theta}_2 \sigma_\mu \theta^a, \] (A.20a)

\[ \epsilon_{\mu_1 \cdots \mu_d} \epsilon_{\nu_1 \cdots \nu_d} = \sum_{p=1}^{d!} \text{sign}(p) \delta_{\mu_1 \nu_1} \cdots \delta_{\mu_d \nu_d} \quad p: \text{permutations}, \] (A.20b)

\[ \epsilon_{\mu_1 \cdots \mu_d} x^{\mu_1} \cdots x^{\mu_d} = \pm \sqrt{\epsilon^{\mu_1 \cdots \mu_d} x(1) \cdot x(2) \cdots x(d)}, \] (A.20c)

\[ \det(\theta^a \bar{\theta}_b) = (\frac{1}{2})^N (N + 1)! \prod_{a=1}^{N} \theta^a \bar{\theta}_a. \] (A.21)

### B Solution of Superconformal Killing Equation

To solve the superconformal Killing equation (2.31) we first note that from eq.(2.32)

\[ D_{b \beta} D_{a \alpha} \tilde{h}^{\gamma \gamma} = \frac{1}{4} \delta_\gamma^\alpha D_{b \beta} D_{a \alpha} \tilde{h}^{\gamma \delta}. \] (B.1)

Contracting \( \beta \) and \( \gamma \) indices gives

\[ D_{b \delta} D_{a \alpha} \tilde{h}^{\gamma \delta} = 0, \] (B.2)

and hence

\[ D_{b \beta} D_{a \alpha} \tilde{h}^{\gamma \gamma} = 0. \] (B.3)

Now we write a general solution of eq.(2.27) as

\[ h^\mu(z) = a^\mu(\theta, \bar{\theta}) + \lambda(\theta, \bar{\theta}) x^\mu + w^\mu_\nu(\theta, \bar{\theta}) x^\nu + 2x \cdot b(\theta, \bar{\theta}) x^\mu - x^2 b^\mu(\theta, \bar{\theta}), \] (B.4)
where \( w_{\mu\nu}(\theta, \bar{\theta}) + w_{\nu\mu}(\theta, \bar{\theta}) = 0 \).

Then eq.(B.4) can be written in terms of \( \tilde{h} = h^\mu \tilde{\sigma}_\mu \) as

\[
\tilde{h}(z) = \tilde{x}_- b(\theta, \bar{\theta}) \tilde{x}_+ + \tilde{x}_- W(\theta, \bar{\theta}) + \tilde{W}(\theta, \bar{\theta}) \tilde{x}_+ + \tilde{A}(\theta, \bar{\theta}) ,
\]

(B.5)

where

\[
W(\theta, \bar{\theta}) = -\frac{1}{4} w_{\mu\nu}(\theta, \bar{\theta}) \sigma^\mu \tilde{\sigma}^\nu + \frac{i}{2} \lambda(\theta, \bar{\theta}) - 2ib(\theta, \bar{\theta}) \tilde{\theta}_a \theta^a = \tilde{W}(\theta, \bar{\theta})^\dagger ,
\]

\[
\tilde{A}(\theta, \bar{\theta}) = \tilde{a}(\theta, \bar{\theta}) - i \frac{1}{2} w_{\mu\nu}(\theta, \bar{\theta}) (\tilde{\theta}_a \theta^a \sigma^{[\mu} \tilde{\sigma}^{\nu]} + \tilde{\sigma}^{[\mu} \sigma^{\nu]} \tilde{\theta}_a \theta^a) + 4\theta^a b(\theta, \bar{\theta}) \tilde{\theta}_a \theta^{b} = \tilde{\Lambda}(\theta, \bar{\theta})^\dagger .
\]

(B.6)

Essentially we may regard \( W(\theta, \bar{\theta}) \) as an arbitrary \( 2 \times 2 \) matrix and \( \tilde{A}(\theta, \bar{\theta}) \) as an arbitrary \( 2 \times 2 \) hermitian matrix.

The variables, \( \tilde{x}_\pm \) defined in eq.((A.16)), satisfy

\[
D_{aa} \tilde{x}_+^{\hat{\beta}\hat{\alpha}} = -4i\delta^\beta_\alpha \tilde{\theta}_a^{\hat{\beta}}, \quad D_{aa} \tilde{x}_-^{\hat{\beta}\hat{\alpha}} = 0 ,
\]

(B.7)

\[
\tilde{D}_{\alpha}^{\hat{\beta}\hat{\alpha}} \tilde{x}_+^{\hat{\beta}} = 0, \quad \tilde{D}_{\alpha}^{\hat{\beta}\hat{\alpha}} \tilde{x}_-^{\hat{\beta}} = 4i\delta_\beta^\alpha \theta^{\alpha \beta} ,
\]

which ensure that substituting eq.(B.5) into eq.(2.32) leads independent equations for \( b(\theta, \bar{\theta}), W(\theta, \bar{\theta}), \tilde{A}(\theta, \bar{\theta}) \).

After substituting eq.(B.5) into eq.(2.32), we get from the \( x^2 \)-terms

\[
D_{aa} (b(\theta, \bar{\theta}) \tilde{\sigma}^\mu)_{\hat{\beta}}^{\hat{\alpha}} = \frac{1}{2} \delta^\beta_{\alpha} D_{a\gamma} (b(\theta, \bar{\theta}) \tilde{\sigma}^{\mu})_{\hat{\beta}}^{\hat{\gamma}} .
\]

(B.8)

From eq.(A.7)

\[
\delta^\gamma_{\beta} D_{aa} b_{\hat{\delta}\hat{\gamma}}(\theta, \bar{\theta}) = \frac{1}{2} \delta^\gamma_{\beta} D_{a\gamma} b_{\hat{\delta}\hat{\gamma}}(\theta, \bar{\theta}) ,
\]

(B.9)

and hence

\[
D_{aa} b(\theta, \bar{\theta}) = 0 .
\]

(B.10)

In a similar fashion, or taking complex conjugate, one can show \( D_{a}^{\hat{\beta}} b(\theta, \bar{\theta}) = 0 \). Therefore \( b^\mu(\theta, \bar{\theta}) \) is independent of \( \theta, \bar{\theta} \) and eq.(B.7) shows that \( \tilde{x}_- b \tilde{x}_+ \) is a solution of eq.(2.32).

The remaining terms lead

\[
(D_{aa} \tilde{W}(\theta, \bar{\theta}) \tilde{x}_-)_{\hat{\beta}\hat{\gamma}} + (\tilde{x}_- D_{aa} \tilde{W}(\theta, \bar{\theta}))_{\hat{\beta}\hat{\gamma}} + 4i(D_{aa} \tilde{W}(\theta, \bar{\theta}) \tilde{\theta}_b)_{\hat{\beta}} \theta^{b\hat{\gamma}} + D_{aa} \tilde{A}^{\hat{\beta}\hat{\gamma}}(\theta, \bar{\theta})
\]

\[
= \frac{1}{2} \delta^\beta_{\alpha} \{(D_{a\gamma} \tilde{W}(\theta, \bar{\theta}) \tilde{x}_-)_{\hat{\beta}\hat{\gamma}} + (\tilde{x}_- D_{a\gamma} \tilde{W}(\theta, \bar{\theta}))_{\hat{\beta}\hat{\gamma}} + 4i(D_{a\gamma} \tilde{W}(\theta, \bar{\theta}) \tilde{\theta}_b)_{\hat{\beta}} \theta^{b\hat{\gamma}} + D_{a\gamma} \tilde{A}^{\hat{\beta}\hat{\gamma}}(\theta, \bar{\theta}) \}.
\]

(B.11)
This gives two separate equations

\[ D_{aa}(\tilde{W}(\theta, \bar{\theta})\tilde{\sigma}^\mu + \bar{\sigma}^\mu W(\theta, \bar{\theta}))^{\beta \gamma} = \frac{1}{2}\delta_\alpha^\beta D_{a\gamma}(\tilde{W}(\theta, \bar{\theta})\tilde{\sigma}^\mu + \bar{\sigma}^\mu W(\theta, \bar{\theta}))^{\beta \gamma}, \quad (B.12) \]

\[ D_{aa}(\tilde{A}(\theta, \bar{\theta}) + 4i\tilde{W}(\theta, \bar{\theta})\bar{\theta}_b\theta^b)^{\beta \gamma} = \frac{1}{2}\delta_\alpha^\beta D_{a\gamma}(\tilde{A}(\theta, \bar{\theta}) + 4i\tilde{W}(\theta, \bar{\theta})\bar{\theta}_b\theta^b)^{\beta \gamma}. \quad (B.13) \]

Eq. (B.12) is equivalent, from eq. (A.7), to

\[ \delta_\gamma^\beta D_{aa}W^\beta_\gamma(\theta, \bar{\theta}) + \delta_\gamma^\beta D_{aa}W^\gamma_\beta(\theta, \bar{\theta}) = \frac{1}{2}\delta_\alpha^\beta(D_{a\alpha}W^\beta_\gamma(\theta, \bar{\theta}) + \delta_\beta^\gamma D_{a\beta}W^\gamma_\beta(\theta, \bar{\theta})). \quad (B.14) \]

By contracting \( \beta \) and \( \gamma \) indices one can solve \( D_{aa}\tilde{W}^\alpha_\beta(\theta, \bar{\theta}) \) in terms of \( D_{aa}W^\gamma_\beta(\theta, \bar{\theta}) \), and hence eq. (B.14) is equivalent to

\[ D_{aa}\tilde{W}^\alpha_\beta(\theta, \bar{\theta}) = \frac{1}{3}\delta_\alpha^\beta(D_{\alpha\beta}W^\gamma_\beta(\theta, \bar{\theta}) + 2D_{\alpha\beta}W^\beta_\gamma(\theta, \bar{\theta})) = \frac{1}{2}\delta_\alpha^\beta D_{aa}W^\gamma_\beta(\theta, \bar{\theta}). \quad (B.15a) \]

\[ D_{aa}W^\gamma_\beta(\theta, \bar{\theta}) = \frac{1}{3}\delta_\alpha^\gamma(D_{\alpha\beta}W^\gamma_\beta(\theta, \bar{\theta}) + 2D_{\alpha\beta}W^\beta_\gamma(\theta, \bar{\theta})) = \frac{1}{2}\delta_\beta^\gamma D_{aa}W^\beta_\beta(\theta, \bar{\theta}). \quad (B.15b) \]

Eq. (B.13) gives from eq. (B.3)

\[ 0 = D_{\alpha\beta}D_{aa}\tilde{A}^{\gamma\gamma}(\theta, \bar{\theta}) + 4i(D_{\alpha\beta}D_{aa}\tilde{W}(\theta, \bar{\theta})\bar{\theta}_c\theta^c)^{\gamma\gamma} - 4i(D_{aa}\tilde{W}(\theta, \bar{\theta})\bar{\theta}_b \gamma_{\beta})^\gamma, \quad (B.16) \]

and \( \{D_{aa}, D_{\alpha\beta}\} = 0 \) implies

\[ 0 = \delta_\alpha^\gamma(D_{aa}\tilde{W}(\theta, \bar{\theta})\bar{\theta}_b)^{\gamma} + \delta_\alpha^\gamma(D_{\alpha\beta}\tilde{W}(\theta, \bar{\theta})\bar{\theta}_a)^{\gamma}, \quad (B.17) \]

and hence from eq. (B.15a)

\[ 0 = D_{aa}\tilde{W}^\alpha_\alpha(\theta, \bar{\theta})\bar{\theta}_b. \quad (B.18) \]

Therefore \( \tilde{W}^\alpha_\alpha(\theta, \bar{\theta}) \) is of the form

\[ \tilde{W}^\alpha_\alpha(\theta, \bar{\theta}) = \tilde{\omega}(\bar{\theta}) + w'(\theta)\bar{\theta}^{2N}, \quad \bar{\theta}^{2N} = \bar{\theta}_1^2 \bar{\theta}_2^2 \cdots \bar{\theta}_N^2. \quad (B.19) \]

We may require \( w'(0) = 0 \).

Substituting this expression into eq. (B.15a) gives

\[ D_{aa}\tilde{W}^\alpha_\alpha(\theta, \bar{\theta}) = \frac{1}{2}\delta_\alpha^\alpha D_{aa}w'(\theta)\bar{\theta}^{2N}, \quad (B.20) \]

and hence we can put

\[ \tilde{W}^\alpha_\alpha(\theta, \bar{\theta}) = \frac{1}{2}\delta_\alpha^\alpha \tilde{\omega}'(\bar{\theta})\bar{\theta}^{2N} + \tilde{\omega}_\alpha(\theta), \quad (B.21) \]
Substituting this expression again into eq. \((B.15a)\) gives
\[ D_{a\alpha} w'(\theta) \bar{\theta}^{2N} = -\bar{w}'(\bar{\theta}) D_{a\alpha} \theta^{2N}, \]  
(B.22a)

\[ D_{a\beta} w_\alpha^{\beta}(\theta) = 2 D_{a\alpha} w_\beta^{\beta}(\theta). \]  
(B.22b)

Thus we get
\[ w'(\theta) = 2ic \theta^{2N}, \quad \bar{\theta}^{2N} = \theta_1^2 \theta_2^2 \cdots \theta_N^2 = \bar{\theta}^{2N}, \]  
(B.23)

and from eq. \((B.15b)\)
\[ D_{a\alpha} w_\gamma^{\beta}(\theta) = \frac{1}{2} \delta_\gamma^{\beta} D_{a\delta} w_\delta^{\gamma}(\theta), \]  
(B.24)

where \(c \in \mathbb{R}\).

From eq. \((B.3)\)
\[ 0 = D_{a\alpha} D_{b\beta} \tilde{A}(\theta, \bar{\theta}). \]  
(B.25)

Hence we can put
\[ D_{a\alpha} w_\beta^{\beta}(\theta) = 4 \rho_{a\alpha}. \]  
(B.26)

Therefore we get the solution for \(\bar{\theta}_a^{\beta}(\theta, \bar{\theta})\)
\[ \bar{\theta}(\theta, \bar{\theta}) = -4c \theta^{2N} \bar{\theta}^{2N} - 4\rho_a \theta^a - \frac{1}{4} w_{\mu\nu} \sigma^{\mu\nu} + \frac{1}{2} \lambda. \]  
(B.27)

However, \(\tilde{h}(\tilde{z})\) is independent of \(c\) and hence we can put \(c = 0\).

Now, eq. \((B.13)\) reads
\[ D_{a\alpha} \tilde{A}_\beta^{\alpha}(\theta, \bar{\theta}) = \frac{1}{2} \delta_\alpha^{\beta} D_{a\gamma} \tilde{A}_\gamma^{\alpha}(\theta, \bar{\theta}), \]  
\[ D_{a\alpha} \tilde{A}_\gamma^{\beta}(\theta, \bar{\theta}) = \frac{1}{2} \delta_\beta^{\alpha} D_{a\gamma} \tilde{A}_\gamma^{\beta}(\theta, \bar{\theta}), \]  
(B.28)

and hence
\[ \tilde{D}_a^b D_{a\alpha} \tilde{A}_\beta^{\alpha}(\theta, \bar{\theta}) = \frac{1}{4} \delta_\beta^{\alpha} \delta_\alpha^{\beta} \tilde{D}_a^b D_{a\gamma} \tilde{A}_\gamma^{\gamma}(\theta, \bar{\theta}). \]  
(B.29)

Since \(D_{a\alpha} D_{b\beta} \tilde{A}(\theta, \bar{\theta}) = 0\) we can put
\[ \tilde{D}_a^a D_{a\gamma} \tilde{A}_\gamma^{\gamma}(\theta, \bar{\theta}) = -16it^a_b + 8\Omega \delta^a_b, \]  
(B.30a)

and hence
\[ D_{a\alpha} \tilde{A}_\alpha^{\alpha}(\theta, \bar{\theta}) = -8i \bar{\varepsilon}_a^{\alpha}, \]  
(B.30b)

where \(\Omega \in \mathbb{R}\) and \(t \in \text{su}(N)\) i.e. \(t^\dagger = -t, \text{tr} t = 0\).

Eq. \((B.30a)\) gives the general solution for \(\tilde{A}(\theta, \bar{\theta})\)
\[ \tilde{A}(\theta, \bar{\theta}) = -4i \bar{\theta}_a t^a b \theta^b + 2\Omega \bar{\theta}_a \theta^a + 4i (\bar{\varepsilon}_a \theta^a - \bar{\theta}_a \varepsilon^a) + \tilde{a}. \]  
(B.31)

All together, we get the general solution of the superconformal Killing equation \((2.34)\).
C Basis for Superconformal Algebra

We write the superconformal generators in general as
\[ K \cdot P = a^\mu P_\mu + \varepsilon^a Q_a + \bar{Q}^a \bar{\varepsilon}_a + \lambda D + i \Omega R + \frac{1}{2} w^{\mu \nu} M_{\mu \nu} + b^\mu K_\mu + S^a \rho_a + \bar{\rho}^a \bar{S}_a + t^a A_b^a , \] (C.1)
for
\[ K = (a^\mu, b^\mu, \varepsilon^a, \rho_a, \bar{\rho}^a, \lambda, \Omega, w^{\mu \nu}, t^a ) , \] (C.2a)
\[ P = (P_\mu, K_\mu, Q_a, \bar{Q}^a, S^a, \bar{S}_a, D, R, M^{\mu \nu}, A^a_b) , \] (C.2b)
where the SU(\(N\)) generators, \(A^a_b\), satisfy \(A^\dagger = -A, \ \text{tr}A = 0\). The superconformal algebra can now be obtained by imposing
\[ [K_1 \cdot P, K_2 \cdot P] = -i K_3 \cdot P , \] (C.3)
where \(K_3\) is given by eq.(2.62). From this expression, we can read off the following superconformal algebra.

- Poincaré algebra
\[ [P_\mu, P_\nu] = 0 , \quad [M_{\mu \nu}, P_\lambda] = i(\eta_{\mu \lambda} P_\nu - \eta_{\nu \lambda} P_\mu) , \] (C.4)
\[ [M_{\mu \nu}, M_{\lambda \rho}] = i(\eta_{\mu \lambda} M_{\nu \rho} - \eta_{\mu \rho} M_{\nu \lambda} - \eta_{\nu \lambda} M_{\mu \rho} + \eta_{\nu \rho} M_{\mu \lambda}) . \]

- Supersymmetry algebra
\[ \{Q_a, \bar{Q}^b\} = 2 \delta_a^b \sigma_{\dot{a} \dot{a}}^\mu P_\mu , \]
\[ [M_{\mu \nu}, Q_a] = i \frac{1}{2} \sigma_{[\mu} \bar{\sigma}_{\nu]} Q_a , \]
\[ [M_{\mu \nu}, \bar{Q}^a] = -i \frac{1}{2} \bar{Q}^a \sigma_{[\mu} \bar{\sigma}_{\nu]} , \]
\[ [P_\mu, Q_a] = [P_\mu, \bar{Q}^a] = [Q_a, Q_b] = [\bar{Q}^a, \bar{Q}^b] = 0 . \]
• Special superconformal algebra

\[
[K_\mu, K_\nu] = 0 ~, \quad [M_{\mu\nu}, K_\lambda] = i(\eta_{\mu\lambda} K_\nu - \eta_{\nu\lambda} K_\mu) ~,
\]

\[
\{\bar{S}_a^\dot{\alpha}, S^{b\dot{\alpha}}\} = 2\delta_a^b \bar{\sigma}^{\mu\dot{\alpha}} K_\mu ~,
\]

\[
[M_{\mu\nu}, S^a] = -i\frac{1}{2} S^a \sigma_{[\mu \sigma_{\nu]}},
\]

\[
[M_{\mu\nu}, \bar{S}_a] = i\frac{1}{2} \bar{\sigma}_{[\mu \sigma_{\nu}]} \bar{S}_a ~,
\]

\[
[K_\mu, S^{a\alpha}] = [K_\mu, \bar{S}_a^\dot{\alpha}] = \{S^{a\alpha}, S^{b\beta}\} = \{\bar{S}_a^\dot{\alpha}, \bar{S}_b^\dot{\beta}\} = 0 .
\]

• Cross terms between \((P, Q, \bar{Q})\) and \((K, S, \bar{S})\)

\[
[P_\mu, K_\nu] = 2i(M_{\mu\nu} + \eta_{\mu\nu}D) ~,
\]

\[
[P_\mu, S^a] = -\bar{Q}^a \bar{\sigma}_\mu ~, \quad [P_\mu, \bar{S}_a] = \bar{\sigma}_\mu Q_a ~,
\]

\[
[K_\mu, Q_a] = \sigma_\mu \bar{S}_a ~, \quad [K_\mu, \bar{Q}^a] = -S^a \sigma_\mu ~,
\]

\[
\{Q_{a\alpha}, S^{b\beta}\} = i\delta_a^b (2\delta_\alpha^\beta D + (\sigma_{[\mu} \bar{\sigma}_{\nu]})^\beta \sigma_{\mu}) - 2i\delta_\alpha^\beta (A^b_a + (\frac{4}{N} - 1)\delta_a^b R) ~,
\]

\[
\{\bar{Q}_a^\dot{\alpha}, \bar{S}_b^\dot{\beta}\} = -i\delta_a^b (2\delta_\dot{\alpha}^\dot{\beta} D - (\bar{\sigma}_{[\mu} \sigma_{\nu]})^\dot{\beta} \bar{\sigma}_{\mu}) - 2i\delta_\dot{\alpha}^\dot{\beta} (\bar{A}^a_b + (\frac{4}{N} - 1)\delta_a^b R) ~,
\]

\[
\{Q_{a\alpha}, \bar{S}_b^\dot{\beta}\} = \{\bar{Q}_a^\dot{\alpha}, S^{b\beta}\} = 0 .
\]

• Dilations

\[
[D, P_\mu] = -iP_\mu ~, \quad [D, K_\mu] = iK_\mu ~,
\]

\[
[D, Q_a] = -i\frac{1}{2} Q_a ~, \quad [D, \bar{Q}^a] = -i\frac{1}{2} \bar{Q}^a ~,
\]

\[
[D, S^a] = i\frac{1}{2} S^a ~, \quad [D, \bar{S}_a] = i\frac{1}{2} \bar{S}_a ~,
\]

\[
[D, D] = [D, R] = [D, M_{\mu\nu}] = [D, A^a_b] = 0 .
\]

\[51\]
• R-symmetry, $U(1) \times SU(N)$

\[
[R, Q_a] = -i \frac{1}{2} Q_a, \quad [R, \bar{Q}^a] = i \frac{1}{2} \bar{Q}^a,
\]
\[
[R, S^a] = i \frac{1}{2} S^a, \quad [R, \bar{S}^a] = -i \frac{1}{2} \bar{S}^a,
\]
\[
[R, R] = [R, P_\mu] = [R, K_\mu] = [R, M_{\mu\nu}] = [R, A^a_{\mu}] = 0,
\]
\[
[A^a_{\mu}, A^c_{\nu}] = 2i(\delta^c_\mu A^a_{\nu} - \delta^a_\nu A^c_{\mu}), \quad (C.9)
\]
\[
[A^a_{\mu}, Q_c] = -2i\delta^a_c Q_b, \quad [A^a_{\mu}, \bar{Q}^c] = 2i\delta^c_\mu \bar{Q}^a,
\]
\[
[A^a_{\mu}, S^c] = 2i\delta^c_\mu S^a, \quad [A^a_{\mu}, \bar{S}^c] = -2i\delta^a_\mu \bar{S}^b,
\]
\[
[A^a_{\mu}, P_\mu] = [A^a_{\mu}, K_\mu] = [A^a_{\mu}, M_{\mu\nu}] = 0.
\]

D Realization of $O(2, 4) \cong SU(2, 2)$ structure in $M$

We exhibit explicitly the relation of the four-dimensional conformal group to $O(2, 4) \cong SU(2, 2)$ by introducing six-dimensional gamma matrices with $A = 0, 1, \cdots, 5$

\[
\begin{pmatrix}
0 & \Sigma^A \\
\bar{\Sigma}^A & 0
\end{pmatrix}.
\]

$\Sigma^A, \bar{\Sigma}^A$ satisfy

\[
\Sigma^A \bar{\Sigma}^B + \Sigma^B \bar{\Sigma}^A = 2G^{AB}, \quad (D.2)
\]

where $G^{AB} = \text{diag}(+1, -1, -1, -1, -1, +1)$. In particular, here we choose $\Sigma^A, \bar{\Sigma}^A$ as

\[
\Sigma^\mu = \begin{pmatrix}
\bar{\sigma}^\mu & 0 \\
0 & \sigma^\mu
\end{pmatrix}, \quad \Sigma^4 = \begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}, \quad \Sigma^5 = \begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix},
\]

\[
\bar{\Sigma}^\mu = \begin{pmatrix}
\sigma^\mu & 0 \\
0 & \bar{\sigma}^\mu
\end{pmatrix}, \quad \bar{\Sigma}^4 = -\Sigma^4, \quad \bar{\Sigma}^5 = -\Sigma^5.
\]

$\Sigma_A, \bar{\Sigma}_A$ satisfy

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \Sigma_A \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} = \bar{\Sigma}_A^\dagger = \Sigma^A.
\]

(D.4)
For the supermatrix, $M$, given in eq.(2.63), we may now express the $4 \times 4$ part in terms of $\Sigma^{AB} \equiv \frac{1}{2} \Sigma^{[A \Sigma_B]}$ as

$$m \equiv \begin{pmatrix} w + \frac{i}{2} \lambda & -i \tilde{a} \\ -i \tilde{b} & \tilde{w} - \frac{i}{2} \lambda \end{pmatrix} = \frac{1}{2} w_{AB} \Sigma^{AB},$$  \hspace{1cm} (D.5)

where $w_{45}$, $w_{\mu4}$, $w_{\mu5}$ are given by

$$\omega_{45} = \lambda, \quad \omega_{\mu4} = a_\mu - b_\mu, \quad \omega_{\mu5} = a_\mu + b_\mu.$$  \hspace{1cm} (D.6)

$\Sigma^{AB}$ generates the Lie algebra of $O(2,4)$

$$[\Sigma^{AB}, \Sigma^{CD}] = -G^{AC} \Sigma^{BD} + G^{AD} \Sigma^{BC} + G^{BC} \Sigma^{AD} - G^{BD} \Sigma^{AC}.$$  \hspace{1cm} (D.7)

Eq.(2.67), the condition on $M$, is satisfied partially by eq.(D.4).

In general, $m$ can be defined as a $4 \times 4$ matrix subject to $\text{tr} m = 0$ and a reality condition

$$bm + m^\dagger b = 0, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (D.8)

Now, if we write

$$\tilde{m} = p^{-1} m p, \quad b = pj p^{-1}, \quad p = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  \hspace{1cm} (D.9)

then $\text{tr} \tilde{m} = 0$ and eq.(D.8) is equivalent to

$$j \tilde{m} + \tilde{m}^\dagger j = 0.$$  \hspace{1cm} (D.10)

Hence $\tilde{m} \in \text{su}(2,2)$.

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