TRANSFER OPERATORS, ATOMIC DECOMPOSITION AND
THE BESTIARY

DANIEL SMANIA

Abstract. Arbieto and S. recently used atomic decomposition to study transfer operators. We give a long list of old and new expanding dynamical systems for which those results can be applied, obtaining the quasi-compactness of transfer operator acting on Besov spaces of measure spaces with a good grid.

Contents

I. INTRODUCTION. 2

II. THE MAIN INGREDIENTS 4

1. Regular Branches 4
   1.1. Bilipschitz maps on Ahlfors-regular quasi-metric spaces. 4
   1.2. $C^1$-diffeomorphisms on $\mathbb{R}^D$. 6

2. Regular Potentials 10
   2.1. Hölder jacobian 10
   2.2. Non-flat critical points on an interval 11
   2.3. $1/\beta$-bounded variation potentials 13

3. Some strongly regular domains in $\mathbb{R}^D$ 15

III. THE TOY MODEL. 16

4. Linear expanding map acting on the circle 16

IV. THE BESTIARY. 17

5. Markovian expanding maps 17

6. Conformal expanding repellers 22

7. Piecewise Bi-Lipchitz maps on the interval with $1/\beta+\epsilon$-bounded variation potentials 24

8. Continuous $C^{1+\beta+\epsilon}$-piecewise expanding maps on the interval 27

9. Piecewise expanding maps on the interval with jacobian in $B^{1/p}_{p,\infty}$ 28

10. Infinitely many branches with small images 31

11. Lorenz maps with non-flat singularities 34

12. Generic piecewise expanding maps on $\mathbb{R}^D$ 38

2010 Mathematics Subject Classification. 37C30, 37D20, 37C40, 30H25, 37D35, 37A05, 37A25, 37A30, 37A50, 37E05, 47A35, 47B65, 60F05, 60F17, 42B35, 42B35, 42C15.

Key words and phrases. transfer operator, atomic decomposition, Besov space, Ruelle, Perron-Frobenious, quasi-compact, Lasota-Yorke, decay of correlations, expanding map, decay of correlations, ergodic theory, central limit theorem, almost sure invariance principle.

D.S. was partially supported by CNPq 307617/2016-5, CNPq 430351/2018-6 and FAPESP Projeto Temático 2017/06463-3.
In S. [33] we defined Besov spaces on measure spaces endowed with a "good grid". This allowed Arbieto and S. [2] to give sufficient conditions for the transfer operator of maps acting on these measure spaces to be quasi-compact and to satisfy the Lasota-Yorke inequality. Many nice statistical properties of Besov observables follow.

Bestiaries were popular in the Middle Ages in Europe. Those were a compendium of wonderful animals. We offer a compendium of exquisite piecewise expanding maps, and we prove (sometimes conditioned to a priori estimate) the quasi-compactness and Lasota-Yorke inequalities for their transfer operator acting on Besov spaces. We list most of the examples in Table 1.

We order our presentation in such way we go from the simplest one, linear expanding maps on the circle, to the most complex example, piecewise expanding maps on $\mathbb{R}^D$.

Our first examples are Markovian expanding maps and conformal expanding maps. They do not have discontinuities and their branches have large images. This allows us to give precise estimates to the essential spectral radius for the transfer operator. These class of examples includes subshifts of finite type and hyperbolic rational maps acting on its Julia sets.

Intervals maps are our next class of examples. Those include piecewise $C^{1+}$-diffeomorphism expanding maps, piecewise Bi-Lipchitz maps with $p$-bounded variation jacobian and Lorenz maps. We also obtain results for piecewise Bi-Lipchitz maps with certain Besov jacobians, a very general class of potentials, but we need a priori estimate in this case.

The last example is given by generic piecewise $C^{1+}$-diffeomorphisms on $\mathbb{R}^D$. This is a more complex situation because large iterations deforms shapes in a more extreme way.

We focus ourselves to obtain the quasi-compactness and the Lasota-Yorke inequality for the transfer operator acting on Besov spaces. With the exception of the a recent result by Nakano and Sakamoto [26] (see also Baladi and Holschneider [4]) for smooth expanding maps on manifolds (no discontinuities) and the the work on Thomine [35] on transfer operators of piecewise $C^{1+}$ expanding maps on manifolds acting on Sobolev spaces, all our results are new, in particular the results for
Besov spaces on phases spaces with very low regularity, as either symbolic spaces or hyperbolic Julia sets.

Moreover often these results imply that Besov observables have nice statistical properties, as the almost sure invariance principle and exponential decay of correlations. We refer to Arbieto and S. [2] for these consequences.

Another consequence for most of the examples here is that the support of every absolutely continuous invariant measure is an open subset of phase space (up to a set of zero measure). This, as far as we know, it is also a new result in some cases, as for generic piecewise $C^{1+}$ expanding maps in $\mathbb{R}^D$.

One may wonder if the atomic decomposition methods in [2] could be applied to more classes of maps, as measure expanding solenoidal attractors studied by Tsujii [38], Avila, Gouëzel, Tsujii [3], Bamón, Kiwi, Rivera-Letelier, Urzúa [5], and maps with critical points. In the latter class there is a previous study of the Besov regularity of the density of invariant measures by Chazottes, Collet and Schmitt [13].

| Class                        | Example                              | $p$       | It needs a priori estimate? | Map has discontinuities? |
|------------------------------|--------------------------------------|-----------|----------------------------|--------------------------|
| Hölder Jacobian              | Markovian Maps                       | $[1, \infty)$ | No                         | “No”                     |
| Complex analytic map         | Conformal expanding repellors        | $[1, \infty)$ | No                         | No                       |
| Interval maps                | Bounded variation Jacobian           | $\sim 1$  | No                         | Yes                      |
|                              | Piecewise $C^{1+}$-smooth maps       | $\sim 1$  | No                         | Yes                      |
|                              | Jacobian in $B^{1/p}_{p,\infty}$     | $[1, \infty)$ | Yes                     | Yes                      |
| Lorenz maps                  |                                      | $1$       | No                         | Yes                      |
| Tent family                  |                                      | $[1, \infty)$ | No                     | No                       |
| Cowieson-type maps           | $C^{1+}$ Piecewise Smooth Maps       | $\sim 1$  | Generic                   | Yes                      |

Table 1. The Bestiary. The first column describes for each values of $p$ the transfer operator is quasi-compact on $B^{p,q}_{p,q}$. The second column tells us if we need an a priori estimate as an assumption. The last column says if we allow discontinuities on the maps under consideration.
II. THE MAIN INGREDIENTS

1. Regular Branches

1.1. Bilipschitz maps on Ahlfors-regular quasi-metric spaces. Let $I$ be a metric space with a quasi-metric $d$ and a finite measure $m$. Then $(I,d,M)$ is an Ahlfors-regular quasi-metric space if there is $D, C_1$ and $r_0$ with the following property. For every $x \in I$ and $r \in (0,r_0)$

\[ \frac{1}{C_1} r^D \leq m(B_d(x,r)) \leq C_1 r^D. \]

There is a good grid $\mathcal{P}$ in $I$ and there is $\eta, C_2, C_3, C_4, C_5 \geq 0$ and $\lambda_1 \in (0,1)$ with the following property (see Proposition 2.1 in [34]). For every $Q \in \mathcal{P}^k$ there is $z_Q \in Q$ satisfying

\[ B_d(z_Q, C_2 \lambda_1^k) \subset Q, \]

\[ \text{diam}_d Q \leq C_3 \lambda_1^k \]

and

\[ m(\{x \in Q: d(x, I \setminus Q) \leq C_4 t \lambda_1^k\} \leq C_5 t^\eta m(Q). \]

We can consider the space $B_{s,p,q}$ associated with $(I,m,\mathcal{P})$. If $\eta > sp$ then $B_{s,p,q}$ indeed does not depend on the particular good grid we choose and it is called the Besov space of $(I,m)$. A classical example to keep in mind is $[0,1]$ endowed with the euclidean metric and the Lebesgue measure. In this case we can take the usual dyadic good grid and $\eta = 1$.

**Proposition 1.1.** Suppose $D_{s,p} < \hat{\eta}$. There is $C_6$ with the following property. Let $\Omega, \Omega' \subset I$ be open sets and $h: \Omega \to \Omega'$ be a bilipschitz map. In particular there is $C_7, C_8 > 0$ such that for every $x, y \in \Omega$

\[ C_7 d(x,y) \leq d(h(x), h(y)) \leq C_8 d(x,y) \]

and there is $C_9, C_{10} > 0$ such that for every measurable set $A \subset \Omega$

\[ C_9 \leq \frac{|h(A)|}{|A|} \leq C_{10}. \]

Let $Q \subset I$ be an open subset such that there is $z_Q \in Q$ satisfying

\[ B_d(z_Q, C_{11} \text{diam}_d Q) \subset Q, \]

and

\[ m(\{x \in Q: d(x, I \setminus h(Q)) \leq C_{12} t \text{diam}_d h(Q)\} \leq C_{13} t^\hat{\eta} m(h(Q)) \]

for some $C_{11}, C_{12}, C_{13}, \text{and } \hat{\eta} > 0$. Then there is $z_{h(Q)}$ such that

\[ B_d(z_{h(Q)}, C_{11} \frac{C_7}{C_8} \text{diam}_d h(Q)) \subset h(Q). \]

and

\[ m(\{x \in h(Q): d(x, I \setminus h(Q)) \leq C_{12} \frac{C_7}{C_8} t \text{diam}_d h(Q)\} \leq C_{13} \frac{C_{10}}{C_9} t^\hat{\eta} m(h(Q)). \]
Moreover \( h(Q) \) is a \( (1 - sp, C_{14}, \lambda_1^{\eta - Dsp}) \)-regular domain, with
\[
C_{14} = C_6 \frac{C_{13} C_{10}}{(C_2 C_8)^\eta (C_3 C_5)} D.
\]

**Proof.** Note that \( z_{h(Q)} = h(x_Q) \) satisfies
\[
B_d(z_{h(Q)}, C_7 \frac{C_r}{C_8} \text{diam}_d h(Q)) \subset h(Q).
\]
Suppose that \( x \in h(Q) \) and it satisfies
\[
d(x, I \setminus h(Q)) \leq C_{12} \frac{C_7}{C_8} t \text{diam}_d h(Q).
\]
Then
\[
d(h^{-1}(x), I \setminus Q) \leq C_{12} t \text{diam}_d Q.
\]
Consequently
\[
h^{-1}\{x \in h(Q) : d(x, I \setminus h(Q)) \leq C_{12} \frac{C_7}{C_8} t \text{diam}_d h(Q)\}
\]
is contained in
\[
\{x \in Q : d(x, I \setminus Q) \leq C_{12} t \text{diam}_d Q\},
\]
so
\[
m(h^{-1}\{x \in h(Q) : d(x, I \setminus h(Q)) \leq C_{12} \frac{C_7}{C_8} t \text{diam}_d h(Q)\})
\]
\[
\leq m(\{x \in Q : d(x, I \setminus Q) \leq C_{12} t \text{diam}_d Q\})
\]
\[
\leq C_{13} t^\eta m(Q)
\]
\[
\leq C_{13} t^\eta m(h(Q)).
\]
and finally
\[
m(\{x \in h(Q) : d(x, I \setminus h(Q)) \leq C_{12} \frac{C_7}{C_8} t \text{diam}_d h(Q)\})
\]
\[
\leq C_{10} m(h^{-1}\{x \in h(Q) : d(x, I \setminus h(Q)) \leq C_{12} \frac{C_7}{C_8} t \text{diam}_d h(Q)\})
\]
\[
\leq C_{13} \frac{C_6 C_{10}}{C_9} t^\eta m(h(Q)).
\]
By (1.9), (1.10) and Proposition 2.2 in [34] it follows that \( h(Q) \) is a \( (1 - sp, C_{14}, \lambda_1^{\eta - Dsp}) \)-regular domain.

**Corollary 1.1.** There is \( C_{15} \) such that for every \( Q \in \mathcal{P} \) the set \( h(Q) \) is a \( (1 - sp, C_{16}, \lambda_1^{\eta - Dsp}) \)-regular domain, with
\[
C_{16} = C_6 \frac{C_{13} C_{10}}{(C_2 C_8)^\eta (C_3 C_5)} D.
\]

**Proof.** Note that
\[
B_d(z_Q, C_2 \frac{C_r}{C_3} \text{diam}_d Q) \subset Q,
\]
and
\[
m(\{x \in Q : d(x, I \setminus Q) \leq C_4 \frac{C_7}{C_8} t \text{diam}_d Q\}) \leq C_5 t^\eta m(Q).
\]
1.2. \(C^1\)-diffeomorphisms on \(\mathbb{R}^D\). A \(D\)-cube in \(\mathbb{R}^D\) is a set \(K \subset \mathbb{R}^D\) defined as

\[ K = \{ x_0 + \sum_{i=1}^{D} \alpha_i v_i, \ \alpha_i \in [0,1] \}, \]

where \(x_0 \in \mathbb{R}^D\) and \(B = \{ v_1, \ldots, v_n \} \) is a basis of \(\mathbb{R}^D\). We can consider the Lebesgue measure \(m_K\) on \(K\) normalized such that \(m_K(K) = 1\) and the dyadic grid \(\mathcal{D}_K\) defined as \(Q \in \mathcal{D}_K^m\) if there are integers \(0 \leq j_i < 2^m\) such that

\[ Q = \{ x_0 + \sum_{i=1}^{D} \alpha_i v_i, \ \alpha_i \in \left[ \frac{j_i}{2^m}, \frac{j_i + 1}{2^m} \right] \}. \]

Of course \(\mathcal{D}_K\) is good grid in \((K, m_K)\). If we consider the metric \(d_K\) on \(K\) such that \(d_K(x, y) = |x - y|_{K}\), where \(|\cdot|_{K}\) comes from an inner product that turns \(B\) into an orthonormal basis then \((K, d_K, m_k)\) is an \(D\)-dimensional Ahlfors regular metric space and \(\mathcal{P} = \mathcal{D}_K\) is good grid on it that satisfies (1.2), (1.3) and (1.4), with \(\eta = 1\) and the constants that appears there may be chosen such that they do not depend on the the chosen orthonormal basis and \(D\)-cube \(K\).

The corresponding Besov space \(\mathcal{B}^s_{p,q}(K, m, \mathcal{D}_K)\), with \(0 < s < 1/p, p \in [1, \infty)\) and \(q \in [1, \infty]\), coincides with the classical Besov space in the homogeneous space \((K, m_K)\) (see [34]).

**Proposition 1.2.** Let \((K, m_K, \mathcal{P})\) be a measure space with a good grid where \(K\) is a compact subset of \(\mathbb{R}^D\), \(m_K\) is the Lebesgue measure up to a scaling factor and \(\mathcal{P}\) is a good grid satisfying (1.1), (1.2), (1.3) and (1.4) taking \(d\) as the euclidean distance multiplied by a factor. Then there is \(C_{17}\), that depends only on the constants that appears in these conditions, such that the following holds. Let

\[ 0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_D \]

and

\[ Q = F[\Pi_{i=1}^{D}[0, \beta_i]], \]

where \(F\) is an isometry of \(\mathbb{R}^D\). Suppose that \(F(Q) \subset K\). Then \(Q\) is a \((1 - sp, C_{17}\Pi_{i \neq 1} \beta_i/\beta_1)^{sp}, \lambda_1^{1 - Dsp})\)-regular domain on \((K, m_K, \mathcal{P})\).

**Proof.** Note that if \(C_{18}^{-1}d\) is the euclidean distance then (1.1) implies

\[ \frac{C_{19}}{C_{18}^D} \frac{\lambda_1^{k_0(Q)}}{C_{18}^{D}} \leq \frac{m(A)}{m_K(A)} \leq \frac{C_{19}}{C_{18}^D} \frac{\lambda_1^{k_0(Q)}}{C_{18}^{D}} \]

for every measurable set \(A\), where the constant \(C_{19}\) is universal and \(m\) is the Lebesgue measure (without any normalization). Moreover

\[ \frac{C_{2}}{C_{18}} \lambda_1^{k_0(Q)} \leq \beta_1 \leq \frac{2C_{3}}{\lambda_1 C_{18}} \lambda_1^{k_0(Q)}. \]

We claim that

\[ m(x \in \mathbb{R}^D: d(x, \mathbb{R}^D \setminus Q) < tC_{18}\beta_1) \leq 2Dt m(Q). \]

It is enough to prove the claim for the case \(F = Id\). We have that

\[ \{ x \in \mathbb{R}^D: d(x, \mathbb{R}^D \setminus Q) \leq \lambda C_{18}\beta_1 \} \]

is contained in

\[ \cup_j \{ (x_1, \ldots, x_D): x_i \in [0, \beta_1] \text{ for } i \neq j, \text{ and } x_j \in [0, \beta_1] \cup [\beta_j - t\beta_1, \beta_j]\}, \]
so
\[ m(x \in \mathbb{R}^D): \ d(x, \mathbb{R}^D \setminus Q) \leq tC_{18}\beta_1) \]
\[ \leq \sum_j 2t\beta_i \Pi_{i\neq j} \beta_i \]
\[ \leq 2t(\sum_j \frac{\beta_i}{\beta_j}) \Pi_{i\neq j} \beta_i \]
\[ \leq 2Dt m(Q). \]

This proves the claim. Define \( F^k(Q) \subset P^k \) recursively as
\[ F^{k+1}(Q) = \{ P \in P^{k+1}: P \subset Q \} \]
and
\[ F^k(Q) = \{ P \in P^k: P \subset Q \} \] \( \cup \) \( \bigcup_{k \leq k} F^j(Q) W \).

Note that if \( d(x, \mathbb{R}^n \setminus Q) > C_3\lambda_1^{-1} \) then there is \( W \in \bigcup_{k \leq k} F^j(Q) \) such that \( x \in W \), so
\[ \sum_{P \in F^k(Q)} m(P) \leq m(x \in \mathbb{R}^D): \ d(x, \mathbb{R}^D \setminus Q) < C_3\lambda_1^{-1} \]
\[ \leq 2DC_2 \lambda_1^{k-1} m(Q) \leq 2DC_3 \lambda_1^{k-\beta_0} m(Q). \]
Since
\[ \frac{C_{19}C_1^{1+D}}{C_{18}C_2^{1+D}} \lambda_1^D \leq m(P) \leq \frac{C_{19}C_1^{1+D}}{C_{18}C_2^{1+D}} \lambda_1^D \]
for every \( P \in F^k(Q) \), we have that
\[ \# F^k(Q) \leq \frac{2DC_3 C_1^{1+D}}{C_{19}C_2^{1+D}} \lambda_1^{k-\beta_0} C_1^{2+D} \lambda_1^{Dk} m(Q). \]

Denote
\[ C_{20} = \frac{2DC_3 C_1^{2+D}}{C_2^{1+D}}. \]

We conclude that
\[ \sum_{P \in F^k(Q)} m_K(P)^{1-sp} \]
\[ \leq \left( \frac{C_{18}C_1}{C_{19}} \right)^{1-sp} \sum_{P \in F^k(Q)} m(P)^{1-sp} \]
\[ \leq \left( \frac{C_{18}C_1}{C_{19}} \right)^{1-sp} C_2 \lambda_1^{k-\beta_0} C_1^{2+D} \lambda_1^{Dk} \left( \frac{C_{19}}{C_{18}} \right)^{1-sp} \lambda_1^{D(1-sp)} m(Q) \]
\[ \leq C_{20} C_3^{D(1-sp)} C_1^{3(1-sp)} \lambda_1^{k-\beta_0} C_1^{2+D} \lambda_1^{D(1-sp)} \lambda_1^{Dk} \left( \frac{C_{19}}{C_{18}} \right)^{1-sp} m(Q)^{1-sp} \]
\[ \leq C_{20} C_3^{D(1-sp)} C_1^{3(1-sp)} \lambda_1^{k-\beta_0} C_1^{2+D} \lambda_1^{D(1-sp)} \left( \frac{C_{19}}{C_{18}} \right)^{1-sp} \lambda_1^{Dk} \left( \frac{C_{19}}{C_{18}} \right)^{1-sp} m(Q)^{1-sp} \]
\[ \leq C_{17} \lambda_1^{(1-Dsp)(k-\beta_0)} \left( \frac{C_{19}}{C_{18}} \right)^{1-sp} m_K(Q)^{1-sp}, \]
where it is worth noting that \( C_{17} \) does not depend on the normalising factor \( C_{18} \).
Proposition 1.3. Let \((K, m, \mathcal{P})\) be a measure space with a good grid as in Proposition 1.2. For every small \(\delta\) there is \(C_{21}\) such that the following holds. Let \(A: \mathbb{R}^D \to \mathbb{R}^D\) be an invertible linear transformation. Let
\[
0 < \alpha_1 \leq \cdots \leq \alpha_D
\]
be such that \(\{\alpha_i^2\}\) are the eigenvalues of \(AA^*\), repeating the eigenvalues the number of times corresponding to its multiplicities. Let \(W\) be a bounded open set satisfying (1.7) and (1.8), with \(Dsp < \min\{\hat{\eta}, \eta\}\). If \(A(W) \subseteq K\) then \(A(W)\) is a regular domain in \((K, m, \mathcal{P})\). The constant \(C_{21}\) depends only on \(\delta\) and the constants in (1.2), (1.3) and (1.4) for the good grid \(\mathcal{P}\) considering \(d\) as either the euclidean distance or a multiply of it.

Proof. Let \(\hat{B} = \{v_1, \ldots, v_n\}\) be a orthonormal basis of \(\mathbb{R}^D\) such that \(AA^*v_i = \alpha_i^2v_i\). Then \(\hat{B} = \{A(v_1)/\alpha_1, \ldots, A(v_D)/\alpha_D\}\) is also an orthonormal basis. Consider a \(D\)-cube \(\hat{K}\) with sides parallels to the basis \(\hat{B}\) such that \(Q \subseteq \hat{K}\). Then \((\hat{K}, d_{\hat{K}}, \mathcal{D}_{\hat{K}})\) is measure space with a good grid satisfying (1.2), (1.3) and (1.4) with \(\eta = 1\), \(\lambda_1 = 1/2\) and the other constants that appears there may be chosen such that they do not depend on the chosen orthonormal basis and \(D\)-cube \(\hat{K}\). By Proposition 1.1 (take \(h = Id\)) we have that \(W\) is a \((1 - sp, C_{14}, (1/2)\hat{\eta} - Dsp)\)-regular domain in \((\hat{K}, d_{\hat{K}}, \mathcal{D}_{\hat{K}})\), with
\[
C_{14} = C_6 \frac{C_{13}}{C_{12}^\delta C_{11}^D}.
\]
and \(C_6\) does not depend on the chosen orthonormal basis and \(D\)-cube \(\hat{K}\), that is, we can find families \(\hat{\mathcal{F}}^j(W) \subseteq \mathcal{D}_{\hat{K}}^j\) such that
\[
\sum_j \sum_{P \in \hat{\mathcal{F}}^j(W)} 1_P = 1_W
\]
and
\[
\sum_{P \in \hat{\mathcal{F}}^j(W)} m_{\hat{K}}(P)^{1 - sp} \leq C_{14}(1/2)(j - k_0(W, \mathcal{D}_{\hat{K}}))(\hat{\eta} - Dsp)m_{\hat{K}}(W)^{1 - sp}.
\]
Note that for every \(P \in \cup_j \hat{\mathcal{F}}^j(W)\) we have that \(A(P)\) is a set as in Proposition 1.2, where \(\beta_i = \alpha_i c\), for some \(c > 0\). So \(A(P)\) is \((1 - sp, C_{17}\Pi_{i \neq 1} \alpha_i/\alpha_1, \lambda_1^{1 - Dsp})\)-regular domain on \((K, m_K, \mathcal{P})\), so there are families \(\mathcal{F}^j(A(P)) \subseteq \mathcal{P}^j\) such that
\[
\sum_j \sum_{R \in \mathcal{P}^j(A(P))} 1_W = 1_{A(R)}
\]
and
\[
\sum_{R \in \mathcal{P}^j(A(P))} m_K(R)^{1 - sp} \leq C_{17}(\Pi_{i \neq 1} \alpha_i/\alpha_1)^{sp}\lambda_1^{(j - k_0(A(P), \mathcal{P}))(1 - Dsp)}m_K(A(P))^{1 - sp}.
\]
We have \(m(A(P)) = \alpha_1 \cdots \alpha_D m(P)\). Let \(C_{22}^{-1}d\) and \(C_{23}^{-1}d\) be the euclidean metric. Note that (replacing the constants if necessary) we may assume that both
that satisfies (1.7) we have
\[
\frac{1}{C_{24}} (1/2)^{k_0(Q,D_K)} \frac{C_{22}}{C_{23}} \alpha \leq \lambda_1^{k_0(A(Q),\mathcal{P})} \leq \alpha \frac{C_{22}}{C_{23}} (1/2)^{k_0(Q,D_K)} C_{24},
\]
in particular (1.11) gives
\[
(1.12) \quad \sum_{P \in \mathcal{F}(W)} \sum_{k_0(A(P),\mathcal{P})=j} \sum_{\alpha} m(R)^{1-sp} \leq C_{25} \lambda_1^{j-k_0(A(W),\mathcal{P})} m(A(W))^{1-sp},
\]
so
\[
\sum_{j \leq \ell} \sum_{P \in \mathcal{F}(W)} \sum_{k_0(A(P),\mathcal{P})=j} \sum_{\alpha} m(R)^{1-sp} \leq C_{17} C_{25} (\Pi j \neq 1) \frac{\alpha_1}{\alpha_1} \lambda_1^{(1-j)(\delta-Dsp)} \sum_{j \leq \ell} m(A(P))^{1-sp} \leq C_{21} (\Pi j \neq 1) \frac{\alpha_1}{\alpha_1} \lambda_1^{(1-j)(\delta-Dsp)} m(A(W))^{1-sp}.
\]
and the same inequality holds replacing \( m \) by \( m_K \).

**Proposition 1.4.** Let \((K, m, \mathcal{P})\) be a measure space with a good grid as in Proposition 1.2. For every small \( \delta \) there is \( C_{26} > 0 \) such that the following holds. Let \( O \) be an open set and \( h: O \to \mathbb{R}^d \) be a \( C^1 \)-diffeomorphism and \( O_1 \subset O \) be an open subset such that \( \overline{O_1} \) is compact. There is \( \delta_1 > 0 \) such that if

A. \( W \) is an open set satisfying (1.7) and (1.8), with \( Dsp < \min\{\tilde{\eta}, \eta\} \),

B. \( W \subset O_1 \) and diam \( W \leq \delta_1 \),

C. \( h(W) \subset K \),

then \( h(W) \) is a

\[
(1 - sp, C_{26}(\Pi j \neq 1) \frac{\alpha_1}{\alpha_1} \lambda_1^{(1-j)(\delta-Dsp)})
\]
regular domain in \((K, m, \mathcal{P})\). Here \( 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \) are such that \( \{\alpha_i^2\} \) are the eigenvalues of \( AA^* \), with \( A = D_{x_0} h \) and \( x_0 \) is an arbitrary element of \( W \). The constant \( C_{26} \) depends only on \( \delta \) and the constants in (1.2), (1.3) and (1.4) for the good grid \( \mathcal{P} \) considering \( d \) as either the euclidean distance or a multiply of it.

**Proof.** Since \( \overline{O_1} \) is compact we can write

\[
h(y) - h(x) = D_x h(y - x) + R(x, y)
\]
where

\[
\lim_{|x - y| \to 0} \frac{|R(x, y)|}{|x - y|} = 0.
\]
Since \( x \mapsto D_x h \) is continuous in \( O_1 \) and \( D_x h \) is invertible for every \( x \in O_1 \), it is easy to see that there is \( \delta_1 > 0 \) such that

\[
\frac{1}{2} |x - y| \leq |(D_{x_0} h)^{-1}h(y) - (D_{x_0} h)^{-1}h(x)| \leq 2|x - y|
\]

for every \( x, y \in B(x_0, \delta_1) \), \( x_0 \in O_1 \). Apply Proposition 1.1 for \( g(x) = (D_{x_0} h)^{-1} \circ h(x) \) and \( W \), then apply Proposition 1.3 for \( A = D_{x_0} h \) and \( g(W) \).

\[ \square \]

2. Regular Potentials

Proposition 2.1. There is \( C_{27} \), that depends only on the good grid \( \mathcal{P} \), with the following property.

A. Suppose that \( \Omega \subset I \) is an interval and the function

\[ g: \Omega \to \mathbb{R} \]

satisfies

\[ |g(x) - g(y)| \leq C_{28}d(x, y)^{D(\beta + \epsilon)} \]

for every \( x, y \in \Omega \). Then if \( W \in \mathcal{P} \) is such that \( W \subset \Omega \) and

\[ C_{28}|W|^{\beta + \epsilon} \leq \sup_W g, \]

then

\[ |g_{1W}|_{B_{r, s}^p(W, P_W, A_t^s)} \leq 2C_{27}(\sup_W g)|W|^{1/p - \beta}. \]

B. If additionally the function \( g \) is the jacobian of \( h \), that is, there

\[ m(h(A)) = \int_A g \, dm, \]

for every measurable set \( A \). If \( W, Q \in \mathcal{P} \) be such that \( W \subset J \), \( Q \subset I \) and \( h(W) \subset Q \) and

\[ C_{28}(\text{diam } h^{-1}(Q))^{D(\beta + \epsilon)} + C_{28}|W|^{\beta + \epsilon} \leq \frac{|Q|}{|h^{-1}(Q)|}, \]

then

\[ |g_{1W}|_{B_{r, s}^p(W, P_W, A_t^s)} \leq 2C_{27} \frac{|Q|}{|h^{-1}(Q)|}|W|^{1/p - \beta}. \]

Proof. Let \( W, Q \in \mathcal{P} \) be such that \( W \subset J \), \( Q \subset I \) and \( h(W) \subset Q \). The function \( \phi: I \to \mathbb{R} \) given by \( \phi(x) = 0 \) if \( x \notin W \), and

\[ \phi(x) = \frac{g(x)}{C_{28}|W|^{\beta + \epsilon} + \sup_W g}|W|^{\beta - 1/p} 1_W \]

otherwise, is a \((\beta, \beta + \epsilon, p)\)-Hölder atom supported on \( W \). Indeed

\[ |\phi|_{\infty} \leq |W|^{\beta - 1/p} \]

and for every \( x, y \in W \)

\[
|\phi(x) - \phi(y)| \leq \frac{C_{28}}{C_{28}|W|^{\beta + \epsilon} + \sup_W g}|W|^{\beta - 1/p}d(x, y)^{D(\beta + \epsilon)}
\]

\[
\leq \frac{C_{28}|W|^{\beta + \epsilon}}{C_{28}|W|^{\beta + \epsilon} + \sup_W g}|W|^{\beta - 1/p - (\beta + \epsilon)}d(x, y)^{D(\beta + \epsilon)}
\]

\[ \leq |W|^{\beta - 1/p - (\beta + \epsilon)}d(x, y)^{D(\beta + \epsilon)} \]
This implies that
\[ g_{1W} = (C_{28}|W|^{\beta+\varepsilon} + \sup_W g)|W|^{1/p-\beta}\phi, \]
so
\[ |g_{1W}|_{L^p,P,W,A_{\mu,q}^{\beta,q}} \leq C_{27}(C_{28}|W|^{\beta+\varepsilon} + \sup_W g)|W|^{1/p-\beta} \]
where \( C_{27} \) depends only on the good grid \( P \).

**Proof of A.** We have
\[ |g_{1W}|_{L^p,P,W,A_{\mu,q}^{\beta,q}} \leq C_{27}(C_{28}|W|^{\beta+\varepsilon} + \sup_W g)|W|^{1/p-\beta} \]
\[ \leq 2C_{27}(\sup_W g)|W|^{1/p-\beta}. \]

**Proof of B.** Note that for every \( x \in h^{-1}(Q) \)
\[ g(x) = \frac{1}{|h^{-1}(Q)|} \int_{h^{-1}(Q)} g(x) \, dm(y) \]
\[ = \frac{1}{|h^{-1}(Q)|} \int_{h^{-1}(Q)} g(y) \, dm(y) + \frac{1}{|h^{-1}(Q)|} \int_{h^{-1}(Q)} g(x) - g(y) \, dm(y) \]
\[ \leq \frac{|Q|}{|h^{-1}(Q)|} + C_{28}(\text{diam } h^{-1}(Q))^{D(\beta+\varepsilon)}. \]
so in particular
\[ \sup_W g \leq \frac{|Q|}{|h^{-1}(Q)|} + C_{28}(\text{diam } h^{-1}(Q))^{D(\beta+\varepsilon)}. \]

So if
\[ C_{28}(\text{diam } h^{-1}(Q))^{D(\beta+\varepsilon)} + C_{28}|W|^{\beta+\varepsilon} \leq \frac{|Q|}{|h^{-1}(Q)|}. \]
Then
\[ |g_{1W}|_{L^p,P,W,A_{\mu,q}^{\beta,q}} \leq C_{27}(C_{28}|W|^{\beta+\varepsilon} + \sup_W g)|W|^{1/p-\beta} \]
\[ \leq C_{27}\left(\frac{|Q|}{|h^{-1}(Q)|} + C_{28}(\text{diam } h^{-1}(Q))^{D(\beta+\varepsilon)} + C_{28}|W|^{\beta+\varepsilon}|W|^{1/p-\beta} \right) \]
\[ \leq 2C_{27}\left(\frac{|Q|}{|h^{-1}(Q)|} \right)|W|^{1/p-\beta}. \]

\[ \square \]

2.2. Non-flat critical points on an interval. Here we consider the interval \([0,1]\) with the dyadic grid and the Lebesgue measure. Define \( h: [0,1] \rightarrow [0,1] \) as \( h(x) = x^\alpha \), with \( \alpha > 1 \) and
\[ g(x) = h'(x) = \alpha x^{\alpha-1}. \]

To simplify the notation we denote \( \gamma = \alpha - 1 \in (0, \infty) \).

**Lemma 2.2.** If \( \gamma \in (0,1) \) then \( g \) is \( \gamma \)-Hölder continuous, that is, there is \( K_\gamma \) such that
\[ |g(x) - g(y)| \leq K_\gamma |x - y|^\gamma. \]
Proof. Consider $0 \leq y < x$ and
\[ T(y, x) = \frac{x^\gamma - y^\gamma}{(x - y)^\gamma}. \]
Of course $T(0, x) = 1$ for every $x > 0$. Note that $T(\lambda x, \lambda y) = T(x, y)$ for every $\lambda > 0$, so $\sup_{0 < y < x} T(x, y) = \sup_{1 < x} T(1, x)$. It is easy to see that
\[ \lim_{x \to 1^+} T(1, x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} T(1, x) = 1, \]
so we can take $K_\gamma = (\gamma + 1) \sup_{0 \leq y < x} T(x, y) < \infty$. \qed

Lemma 2.3. Suppose $\gamma > 0$ and $b \in (c, d) \subset [0, \infty)$. Then
\[ \sup_{c < b \leq d} \frac{b^\gamma(d - c)}{d^{1+\gamma} - c^{1+\gamma}} \leq 1. \]

Proof. For every $b, c, d$ satisfying $0 < c < b < d$ define
\[ T(c, b, d) = \frac{b^\gamma(d - c)}{d^{1+\gamma} - c^{1+\gamma}}. \]
If $c = 0$ then
\[ T(0, b, d) = \frac{b^\gamma}{d^\gamma} \leq 1. \]
Note that for every $\lambda > 0$ and $c > 0$ we have
\[ T(\lambda c, \lambda b, \lambda d) = T(c, b, d), \]
so we have
\[ \sup_{0 < c < b \leq d} \frac{b^\gamma(d - c)}{d^{1+\gamma} - c^{1+\gamma}} = \sup_{1 < b \leq d} \frac{b^\gamma(d - 1)}{d^{1+\gamma} - 1} \leq \sup_{1 < b \leq d} \frac{d^{1+\gamma} - d^\gamma}{d^{1+\gamma} - 1} \leq 1. \] \qed

Proposition 2.4. Suppose that $\beta < \min\{1, \gamma\}$. Then there is $C_{29}$ such that the following holds. Let $W$ and $Q$ be intervals such that $W \subset h^{-1}(Q)$. Then
\[ |g1_W|_{B^\beta_{p,q}(W, P, A_{1/q})} \leq C_{29}|W|^{1/p - \beta} \frac{|Q|}{h^{-1}Q}. \]

Proof. We consider two cases:

Case A. Suppose $\gamma < 1$. Choose $\delta > 0$ such that $\beta + 2\delta = \gamma = \alpha - 1$. Then by Lemma 2.2
\[ |g(x) - g(y)| \leq K_\gamma |x - y|^{\gamma + 2\delta}. \]
Let $W = [a, b]$ and $h^{-1}Q = [c, d]$. Define
\[ \phi(x) = \frac{|W|^{1/p}}{(K_\gamma + \gamma + 1)b^\gamma} g \cdot 1_W. \]
We claim that $\phi$ is a $(\beta, \beta + \delta, p)$-Hölder atom supported on $W$. Indeed
\[ |\phi|_{\infty} \leq \frac{|W|^{\beta - 1/p}(\gamma + 1)b^\gamma}{(K_\gamma + \gamma + 1)b^{\beta + 2\delta}} \leq |W|^{\beta - 1/p}. \]
and for every \( x, y \in W \) we have

\[
|\phi(x) - \phi(y)| \leq \frac{K_\gamma |W|^{\beta-1/p}}{(K_\gamma + \gamma + 1)b^{\beta+\delta}}|x - y|^{\beta+2\delta} \\
\leq \frac{|W|^{\beta-1/p}b^\delta}{b^{\beta+\delta}}|x - y|^{\beta+\delta} \\
\leq \frac{|W|^{\beta-1/p}}{b^{\beta+\delta}}|x - y|^{\beta+\delta} \\
\leq |W|^{\beta-1/p-(\beta+\delta)}|x - y|^{\beta+\delta}.
\]

Consequently by Lemma 2.3

\[
|g^1_W|_{B_{p,q}^\beta(W\!,\! P_W\!,\! A_{szp,q})} \leq C_29|W|^{1/p-\beta}\frac{\gamma y^{\gamma^2-1}d - c}{Q} \\
\leq C_29|W|^{1/p-\beta}\frac{|Q|}{|h^{-1}Q|}.
\]

Case B. Suppose \( \gamma \geq 1 \). Choose \( \delta > 0 \) such that \( \beta + \delta < 1 \). If \( 0 \leq x < y \) we have

\[
|g(x) - g(y)| \leq (1 + \gamma)\gamma y^{\gamma^2-1}|x - y| \leq (1 + \gamma)\gamma y^{-(\beta+\delta)}|x - y|^{\beta+\delta}.
\]

Define

\[
\phi(x) = \frac{|W|^{\beta-1/p}}{(\gamma + 1)^{2b^\gamma}}g \cdot 1_W.
\]

We claim that \( \phi \) is a \((\beta, \beta + \delta, p)\)-Hölder atom supported on \( W \). Indeed

\[
|\phi|_\infty \leq \frac{|W|^{\beta-1/p}(\gamma + 1)b^\gamma}{(\gamma + 1)^{2b^\gamma}} \leq |W|^{\beta-1/p}.
\]

and for every \( x, y \in W = [a, b] \), with \( 0 \leq x < y \), we have

\[
|\phi(x) - \phi(y)| \leq \frac{(1 + \gamma)\gamma y^{-(\beta+\delta)}}{(\gamma + 1)^2b^\gamma}|W|^{\beta-1/p}|x - y|^{\beta+\delta} \\
\leq \frac{|W|^{\beta-1/p}}{b^{\beta+\delta}}|x - y|^{\beta+\delta} \\
\leq \frac{|W|^{\beta-1/p}}{b^{\beta+\delta}}|x - y|^{\beta+\delta} \\
\leq |W|^{\beta-1/p-(\beta+\delta)}|x - y|^{\beta+\delta}.
\]

and now we can complete the proof exactly as in Case A. \( \Box \)

2.3. \( 1/\beta \)-bounded variation potentials. Here we consider the interval \([0, 1]\) with the dyadic grid and the Lebesgue measure.

**Proposition 2.5.** There is \( C_{30} \), that does not depend on \( h \), with the following property.

A. Suppose that the function

\[ g: \Omega \to \mathbb{R} \]
has finite $1/(\beta + \epsilon)$-bounded variation. Then there is a finite partition by intervals \( \{\Omega'_1, \ldots, \Omega'_n\} \) of \( \Omega' \) such that if \( W \in P \) and \( W \subset h^{-1}(\Omega'_i) \), for some \( i \), then

\[
|g1_W|_{B^\beta_{p,q}(W;P_{W,A_{p,q}^{x}})} \leq C_{30}(\sup_{W} |W|)^{1/p - \beta}.
\]

**B.** Suppose additionally that the function \( g \) is the jacobian of \( h \), that is,

\[
m(h(A)) = \int_A g \ dm.
\]

for every measurable set \( A \). Then if \( W, Q \in P \) be such that \( W \subset \Omega, Q \subset \Omega'_i \) for some \( i \), and \( h(W) \subset Q \) then

\[
|g1_W|_{B^\beta_{p,q}(W;P_{W,A_{p,q}^{x}})} \leq 3C_{30} |Q| |W|^{1/p - \beta}.
\]

**Proof.** Let \( W \in P \), with \( W \subset \Omega \). Define the function

\[
\phi = \frac{|W|^{s-1/p}}{\text{var}_{1/\beta}(g,W) + \sup_{W} g} g1_W.
\]

Of course \( |\phi|_\infty \leq |W|^{s-1/p} \) and \( \text{var}_{1/\beta}(\phi,W) \leq |W|^{s-1/p} \). So \( \phi \) is a \( A^{h^W}_{s,p,\beta}(W) \)-atom, consequently

\[
|g1_W|_{B^\beta_{p,q}(W;P_{W,A_{p,q}^{x}})} \leq C_{30}(\text{var}_{1/\beta}(g,W) + \sup_{W} g)|W|^{1/p - s}.
\]

Since the $1/(\beta + \epsilon)$-bounded variation of \( g \) is finite, we can find a finite partition by intervals \( \{\Omega'_1, \ldots, \Omega'_n\} \) of \( \Omega' \) such that for each \( i \)

\[
\text{var}_{1/\beta}(g,h^{-1}(\Omega'_i)) = \sup \left\{ \sum_{k=0}^{n} |g(x_{k+1}) - g(x_k)|^{1/\beta} \right\} < \min \{ \sup_{\Omega} g, \frac{1}{C_{10}} \}.
\]

where the sup runs over all possible finite sequences \( x_k < x_{k+1} \) and \( x_k \in \text{int} h^{-1}(\Omega'_i) \).

Let \( W \subset h^{-1}(Q) \), with \( Q \subset \Omega'_i \) for some \( i \).

**Proof of A.** We have

\[
|g1_W|_{B^\beta_{p,q}(W;P_{W,A_{p,q}^{x}})} \leq 2C_{30}(\sup_{\Omega} g)|W|^{1/p - s}.
\]

**Proof of B.** Note that for every \( x \in h^{-1}(Q) \)

\[
g(x) = \frac{1}{|h^{-1}(Q)|} \int_{h^{-1}(Q)} g(y) \ dm(y)
= \frac{1}{|h^{-1}(Q)|} \int_{h^{-1}(Q)} g(y) \ dm(y) + \frac{1}{|h^{-1}(Q)|} \int_{h^{-1}(Q)} g(x) - g(y) \ dm(y)
\leq \frac{|Q|}{|h^{-1}(Q)|} + \frac{1}{C_{10}} \leq 2\frac{|Q|}{|h^{-1}(Q)|},
\]
so

\[ |g1_W|_{L^p_{\alpha}(W,P_{\alpha}W,\mathcal{A}_p^\alpha)} \leq C_{30} \left( \text{var}_{1/\beta}(g,W) + \sup_W g \right) |W|^{1/p-s} \leq C_{30} \left( \frac{1}{C_{10}} + \frac{|Q|}{|h^{-1}(Q)|} \right) |W|^{1/p-s} \leq 3C_{30} \frac{|Q|}{|h^{-1}(Q)|} |W|^{1/p-s}. \]

\[quare\]

3. Some strongly regular domains in \( \mathbb{R}^D \)

Let \((I,m,\mathcal{P})\) be a measure space with a good grid. Recall that a subset \( \Omega \subset I \) is a \((\alpha,C_{31},t)\)-strongly regular domain (see [33]) if for every \( Q \in \mathcal{P} \), \( i \geq t \) and \( k \geq k_0(Q \cap \Omega) \) there are families \( \mathcal{F}^k(Q \cap \Omega) \subset \mathcal{P}^k \) satisfying

A. We have \( Q \cap \Omega = \bigcup_{k \geq k_0(Q \cap \Omega)} \left( Q \cap \Omega \right) \bigcup_{P \in \mathcal{F}^k(Q \cap \Omega)} P \).

B. If \( P,W \in \bigcup_{k} \mathcal{F}^k(Q \cap \Omega) \) and \( P \neq W \) then \( P \cap W = \emptyset \).

C. We have

\[ \sum_{P \in \mathcal{F}^k(Q \cap \Omega)} |P|^\alpha \leq C_{31} |Q|^\alpha. \]

(3.14)

It is easy to prove that

**Proposition 3.1.** There is \( C_{32} > 0 \) such that the following holds. Let \( K = \bigcup_i \tilde{M}_i \), where \( M_i \) is a compact \((D-1)\)-dimensional \( C^1 \)-manifold with boundary embedded in \( \mathbb{R}^D \). Moreover assume that for every \( x \in \partial K \) there is \( r_x > 0 \) such that

\[ \#\{i : B(x,r) \cap M_i \neq \emptyset\} \leq N \]

for every \( r < r_x \). Then there is \( r_0 \) such that for every \( x \in K \) and \( r \in (0,r_0) \) we have

\[ \frac{1}{C_{32}} r^{D-1-1} \leq m_{D-1}(B(x,r) \cap K) \leq C_{32} N r^{D-1}. \]

(3.15)

Here \( m_{D-1} \) denotes the \((D-1)\)-dimensional Hausdorff measure. We emphasize that \( C_{32} \) does not depend on \( K \).

A \( N \)-good \( C^r \) domain \( P \) in \( \mathbb{R}^D \) is an open subset of \( \mathbb{R}^D \) for which \( \partial P \) is compact and there a finite number of \((D-1)\)-dimensional \( C^m \) manifolds with boundary \( M_i \) embedded in \( \mathbb{R}^D \), with \( m \geq 1 \), such that

\[ \partial P \subset \bigcup_{i \leq k} M_i \]

and such that for every \( x \in \partial P \) there is \( r_x > 0 \) satisfying

\[ \#\{i : B(x,r) \cap M_i \neq \emptyset\} \leq N \]

for every \( r < r_x \). A simple example of a \( N \)-good \( C^r \) domain is a convex set defined as the intersection of a finite number of half-spaces in \( \mathbb{R}^D \).

We say that \( N \)-good \( C^1 \) domain has a regular Whitney stratification if we can choose the the manifolds with boundary \( M_i \) such that \( \bigcup_i M_i \) has a Whitney regular \( C^1 \) stratification. We will not need this property in this section, however it will be useful to study generic piecewise expanding maps on \( \mathbb{R}^D \) in Section 12 following an argument similar to Cowieson [16].
Corollary 3.1. For every $N$ there exists $C_{33}$ such that the following holds. For every $N$-good $C^r$ domain $P$ in $\mathbb{R}^D$ there exists $t$ such that $P$ is a $(1 - \frac{1}{D}, C_{33}, t)$-strongly regular domain.

Proof. This follows from Proposition 3.1 and Proposition 4.1 in [34]. □

Remark 3.2. The class of strongly regular domains in $\mathbb{R}^D$ is much wider than the class of $N$-good $C^1$ domains. For instance a domain whose boundary $K$ is a cone with circular base is a strongly regular domain since it satisfies (3.15) if we replace $C_{32}$ for some appropriated constant. Indeed certain domains with fractal boundary are also strongly regular domains (see Remark 4.2 in [34]). However the advantage of $N$-good $C^1$ domains is that $C_{32}$ does not depend on the particular domain we are considering, that is very handy for estimate the essential spectral radius of transfer operators.

III. THE TOY MODEL.

4. Linear expanding map acting on the circle

Let $\ell \in \mathbb{N} \setminus \{0, 1\}$ and define $f_\ell : [0, 1] \to [0, 1]$ as $f_\ell(x) = \ell x \mod 1$. Let $D^k_\ell$ be the partition of $[0, 1]$ in $\ell^k$ intervals with same size. Then $D_\ell = \langle D^k_\ell \rangle_{k \in \mathbb{N}}$ is a good grid. The map $f_\ell$ is a classic example of expanding map. The Lebesgue measure $m$ on $[0, 1]$ is an invariant probability for $f_\ell$. Our goal is to prove the Lasota-Yorke inequality for $f_\ell$ in the space $B^{s, 1}_{[0, 1], m, D_\ell}$, with $s \in (0, 1)$. That is, we will find $j > 0$, $C > 0$ and $\lambda \in (0, 1)$ such that

$$|L^{\ell^j}_\ell(\phi)|_{B^{s, 1}_{[0, 1], m, D_\ell}} \leq C|\phi|_1 + \lambda |\phi|_{B^{s, 1}_{[0, 1], m, D_\ell}}$$

for every $\phi \in B^{s, 1}_{[0, 1], m, D_\ell}$.

This is not a new result, however the we with this example since its simplicity allows us to give a very detailed and yet short proof of the Lasota-Yorke inequality that illustrate the methods of this paper.

Consider the Ruelle-Perron-Frobenious operator of $f_\ell$

$$(L_\ell \psi)(x) = \sum_{i=0}^{\ell^j - 1} \frac{1}{\ell^j} \psi\left(\frac{x - i}{\ell^j}\right).$$

We have that $D^j_\ell$ is a markovian partition for $f_\ell$. Moreover if $P \in D^k_\ell$ and $k \geq j$, then $Q = f_\ell(P) \in D^{k-j}_\ell$ and

$$L_\ell(a_P) = L_\ell(|P|^{s-1} 1_P) = \frac{1}{\ell^j} |P|^{s-1} 1_Q = \frac{1}{\ell^j} |\ell^j|^{s-1} |P|^{s-1} 1_Q = \frac{1}{\ell^j} a_Q.$$

And if $P \in D^k_\ell$ and $k < j$ then $P$ is an union of $\ell^{-k}$ elements of $D^j_\ell$, so

$$L_\ell(a_P) = L_\ell\left(|P|^{s-1} \sum_{P \in D^j_\ell} 1_Q\right) = \frac{\ell^{-k}}{\ell^j} |P|^{s-1} \ell^{-ks} a_{[0, 1]}.$$
By Arbieto and S. [2, Proposition 5.5] there is $C_{GC}$ such that for every $P \in \mathcal{D}_k$ there are linear functionals $\phi \mapsto k^\phi_P$ in $(L^1)^*$ such that every $\phi \in \mathcal{B}_{1,1}^1([0,1], m, \mathcal{D}_k)$ has a $\mathcal{B}_{1,1}^1$-representation

$$\phi = \sum_k \sum_{P \in \mathcal{D}_k^k} k^\phi_P a_P$$

satisfying

$$\sum_k \sum_{P \in \mathcal{D}_k^k} |k^\phi_P| \leq C_{GC} |\phi|_{\mathcal{B}_{1,1}^1}.$$ 

In particular

$$\mathcal{L}_j(\phi) = \left( \sum_{k<j} \sum_{P \in \mathcal{D}_k^k} k^\phi_P a_{[0,1]} + \sum_{k \geq j} \sum_{P \in \mathcal{D}_k^k} k^\phi_P a_{[0,1]} \right)$$

and

$$|\mathcal{L}_j(\phi)|_{\mathcal{B}_{1,1}^1} = |\mathcal{L}_j(\phi)|_{\mathcal{B}_{1,1}^1} \leq \left( \sum_{k<j} \sum_{P \in \mathcal{D}_k^k} |k^\phi_P| + \sum_{k \geq j} \sum_{P \in \mathcal{D}_k^k} |k^\phi_P| \right) \leq \left( \sum_{k<j} \sum_{P \in \mathcal{D}_k^k} |k^\phi_P| \right) |\phi|_{(L^1)^*} + C_{GC} \ell_{s,1} |\phi|_{\mathcal{B}_{1,1}^1},$$

We can choose $j$ large enough to have

$$\lambda = \frac{C_{GC}}{\ell_{s,1}} < 1$$

so we obtain the Lasota-Yorke inequality.

IV. THE BESTIARY.

5. Markovian expanding maps

Markovian maps arise in the very beginning of the study of the metric theory of expanding maps, as the Gauss map and the linear expanding maps on the circle. The work of Ruelle [29] deals with the one-sided shift, another example of such maps.

Sinai [32] constructed Markov partitions for expanding maps on manifolds [30] and hyperbolic diffeomorphisms, See Bowen [7] and Parry and Pollicott [27] for more details.

Let $(I, m)$ be a probability space. Suppose that there is collection of subsets \{I_1, \ldots, I_n\} of $I$ with $n \geq 2$, a transformation

$$f: \cup_i I_i \to I$$

satisfying

A. We have $I_i \cap I_j = \emptyset$ for every $i \neq j$.

B. $m(I_i) > 0$ for every $i$ and $m(I \setminus \cup_i I_i) = 0.$
C. The set $f(I_i)$ is measurable, and $f: I_i \to f(I_i)$ is a bijection with measurable inverse $h_i: f(I_i) \to I_i$ and Jacobian $w$, that is, for every measurable set $A \subset I_i$

$$m(f(A)) = \int_A w \, dm.$$ 

Moreover $\inf w > 1$.

D. For every $i$ the set $f(I_i)$ is a union of at least two elements of $\{I_1, \ldots, I_n\}$. Then we can define a sequence of partitions of $I$ recursively as $P^0 = \{I\}$, $P^1 = \{I_1, \ldots, I_n\}$ and for $k \geq 1$

$$P^{k+1} = \{h_i(Q), \text{ where } Q \in P^k \text{ and } Q \subset f(I_i)\}.$$ 

Note that $P = (P^k)_k$ is a nested sequence of partitions and every element of $P^k$ has at least two children. Of course the $j$-th iteration $f^j$ of $f$ has similar properties. Indeed for every $P \in P^j$ we have that

$$f^j: P \to f^j(P)$$

is a bijection with Jacobian

$$w_P(x) = \prod_{i=0}^{j-1} w(f^i(x))$$

and a measurable inverse, denoted $h_P$. Moreover

$$f^j(P) \in \{f(I_1), \ldots, f(I_n)\}.$$ 

Now assume additionally

E. (Bowen Condition) We have

$$\sup_k \sup_{P \in P^k} \sup_{x,y \in P} \left| \sum_{i=0}^{k-1} \ln w(f^i x) - \sum_{i=0}^{k-1} \ln w(f^i y) \right| < \infty$$

and there is $C_{34} > 0$ such that

$$\frac{1}{C_{34}} \leq w(x) \leq C_{34}.$$ 

for every $x$.

Define for each $P \in P^k$ the function $w_P : P \to \mathbb{R}_+$ as

$$w_P(x) = \prod_{i=0}^{k-1} w(f^i(x)).$$

It easily follows that there is $C_{35} > 0$ such that for every $x, P, k$ satisfying $x \in P \in P^k$

$$\frac{1}{C_{35}} w_P(x) \leq \frac{1}{|P|} \leq C_{35} w_P(x).$$

and $P$ is a good grid. We can define a metric in $I$ as

$$d(x, y) = \inf \{|Q| : x, y \in Q \in \cup_k P^k\}.$$ 

Then $(I, m, d)$ is an Ahlfors-regular metric space ($D = 1$). Note that every inverse branch $h_P$, with $P \in P^j$ is bi-Lipchitz and satisfies (1.6) for some $C_{10}(j)$ that may depend on $j$. Now assume additionally

F. There is some $C$ such that $w$ satisfies

$$|w(x) - w(y)| \leq C d(x, y)^{\beta + \epsilon}$$

for every $x, y \in I_i, i \leq n$. 
\textbf{Theorem 5.1.} Let $f$ be a Markovian map as above. Then
\[ r_{\text{ess}}(\Phi, \mathcal{B}_{p,q}^1(I, m, \mathcal{P})) \leq \left( \liminf_{j \to \infty} \left( \sum_{P \in \mathcal{P}_j} |P|^{1+sp'} \right)^{1/p'} \right). \]

In particular $r_{\text{ess}}(\Phi) \leq (\inf w)^{-s} < 1$.

\textit{Proof.} We have that
\[ g_P(x) = \frac{1}{w_P(h_P(x))} \]
is the Jacobian of $h_P$. Using the usual bounded distortion argument one can show that there is $C_{36}$ such that
\[ |\ln g_P(x) - \ln g_P(y)| \leq C_{36}d(x, y)^{\beta + \epsilon} \]
for every $x, y \in f^k(P)$, $P \in \mathcal{P}^k$, $k \in \mathbb{N}$. In particular there is $\delta > 0$ such that if $d(x, y) < \delta$ then
\[ \frac{1}{2} \leq \frac{g_P(x)}{g_P(y)} \leq 2. \]
In particular if $Q \subset P$, $x \in h_P^{-1}(Q)$ and $\text{diam } h_P^{-1}(Q) < \delta$ then
\[ \frac{1}{2} g_P(x) \leq \frac{|Q|}{|h_P^{-1}(Q)|} \leq 2g_P(x). \]
Moreover the Mean Value Theorem gives
\[ (5.16) \quad |g_P(x) - g_P(y)| \leq 2g(x)C_{36}d(x, y)^{\beta + \epsilon}. \]

Denote $C_{28}(P, x) = 2C_{36}g(x)$. Reduce $\delta$ if necessary such that $4C_{36}\delta^{\beta + \epsilon} < 1/2$. Then if $W \subset h_P^{-1}(Q)$ we have
\[ C_{28}(P, x)(\text{diam } h_P^{-1}(Q))^{\beta + \epsilon} + C_{28}(P, x)|W|^{\beta + \epsilon} \leq 4C_{36}g(x)\delta^{\beta + \epsilon} \leq \frac{1}{2} g_P(x) \leq \frac{|Q|}{|h_P^{-1}(Q)|}. \]

Choose $i_0 \geq 2$ such that $\text{diam } R < \delta$ for every $R \in \mathcal{P}^{i_0}$. Define $\Lambda_j = \mathcal{P}^{j+i_0}$. Given $R \in \Lambda_j$, let $\tilde{h}_R$ be the restriction of $h_P$ to $f^j(R)$, where $P \in \mathcal{P}^j$ satisfies $R \subset P$. Note that $\tilde{h}_R^{-1}(R)$ is a union of elements of $\mathcal{P}^{i_0}$. In an analogous way, define $\tilde{g}_R$ as the restriction of $g_P$ to $f^j(R)$. Then the Ruelle-Perron-Frobenious of $f^j$ can be written as
\[ (\Phi^j\psi)(x) = \sum_{\tilde{h} \in \Lambda_j} \tilde{g}_R(x)\psi(\tilde{h}_R(x)). \]

By Proposition 2.1.B there is $C_{27}$, that depends only on the good grid $\mathcal{P}$ (in particular it does not depend on $j$) with the following property. If $W, Q \in \mathcal{P}$ are such that $W \subset f^j(R)$, $Q \subset R \in \Lambda_j$ and $\tilde{h}_i(W) \subset Q$ then
\[ |\tilde{g}_R 1_W|_{\mathcal{B}_{p,q}^1(W, \mathcal{P}_{W, \Lambda_j})} \leq 2C_{27} \frac{|Q|}{|h_R^{-1}(Q)|}|1/p-\beta. \]

Due the Bowen condition we have that
\[ |\tilde{g}_R 1_W|_{\mathcal{B}_{p,q}^1(W, \mathcal{P}_{W, \Lambda_j})} \leq C_{D RP}(R) \left( \frac{|Q|}{|h_R^{-1}(Q)|} \right)^{1/p-s+\epsilon} |W|^{1/p-\beta}, \]
where
\[ C_{D RP}(R) = C_{37} \left( \frac{|R|}{|h_R^{-1}(R)|} \right)^{1-(1/p-s+\epsilon)}, \]
where $C_{37}$ does not depend on $R \in \Lambda_j$ and $j$.

Of course $R \in \Lambda_j$ is a $(1-\beta p, 1, j)$-strongly regular domain. Moreover for every $Q \in \mathcal{P}^k$, with $k \geq j$ we have that
\begin{equation}
\# \{ R \in \Lambda_j : R \cap Q \neq \emptyset \} \leq 1.
\end{equation}
Furthermore for every $Q \in \mathcal{P}^k$ satisfying $Q \subset R \in \Lambda_j$ we have that $\tilde{h}_R^{-1}(Q) = f^j(Q)$ is an element of $\mathcal{P}^{k-j}$. In particular we have that $\tilde{h}_R^{-1}(Q)$ is a $(1-s p, C_{DGD1}, \lambda_{DGD2})$-regular domain, where $C_{DGD1} = 1$ and $\lambda_{DGD2} \in (0, 1)$ can be chosen so close to 0 as we want to.

Finally note that if $a_R = j$ for $R \in \Lambda_j$ then
\[ |k_0(Q) - k_0(\tilde{h}_R^{-1}(Q))| = a_R. \]
and (again due Bowen condition) there is $C_{DC1}$ such that taking
\[ \lambda_{DC2}(R) = \left( \frac{|R|}{|\tilde{h}_R^{-1}(R)|} \right)^{1/j} \]
we have
\begin{equation}
\frac{|Q|}{|\tilde{h}_R^{-1}(Q)|} \leq C_{DC1}(\lambda_{DC2}(R))|k_0(Q) - k_0(\tilde{h}_R^{-1}(Q))|.
\end{equation}
for every $Q \in \mathcal{P}$ satisfying $Q \subset R$. If we choose $\lambda_{DGD2}$ small enough we obtain
\begin{equation}
\lambda_{DRS2}(R) = \max\{ (\lambda_{DC2}(R))^{1/p}, \lambda_{DGD2} \} = (\lambda_{DC2}(R))^{1/p} < 1.
\end{equation}
Denote $\lambda_{DRS2} = \sup_{R \in \mathcal{U} \cup \Lambda_j} \lambda_{DRS2}(R)$. Note that $\lambda_{DRS2} \leq (\inf w)^{-\epsilon} < 1$. Choose $\gamma_{DRS3} \in (0, 1)$. So
\begin{align*}
\Theta_R &= C_{DC1}^\epsilon C_{DRP}(R) C_{1/p}^{1/p} (\lambda_{DRS2}(R))^{a_R(1-\gamma_{DRS3})} \\
&= C_{DC1}^\epsilon C_{49} \left( \frac{|R|}{|\tilde{h}_R^{-1}(R)|} \right)^{1-(1/p-s+\epsilon)} C_{DGD1}^{1/p} \left( \frac{|R|}{|\tilde{h}_R^{-1}(R)|} \right)^{\epsilon(1-\gamma_{DRS3})} \\
&= C_{DC1}^\epsilon C_{38} \left( \frac{|R|}{|\tilde{h}_R^{-1}(R)|} \right)^{1-(1/p-s)-\epsilon \gamma_{DRS3}}.
\end{align*}
By Proposition 11.2, the Ruelle-Perron-Frobenious $\Phi^j$ has a $(C_{GSR} C_{39}(j), C_{GSR} C_{40}(j), \gamma_{DRS3})$-essential slicing, for some constant $C_{39}(j)$ and
\[ C_{40}(j) = \left( \sum_{R \in \Lambda_j} \Theta_R^{p'} \right)^{1/p'} \]
Since there is $C > 0$ such that $|\tilde{h}_R^{-1}(R)| \geq C$ for every $R \in \Lambda_j$ we have that
\[ C_{40}(j) \leq C_{DC1}^\epsilon C_{38} C_{41} \left( \sum_{R \in \mathcal{P}^k \cap \Lambda_j} |R|^{1+sp'-\epsilon \gamma_{DRS3}p'} \right)^{1/p'} \]
Corollary 10.1 in [2] tell us that we can write $\Phi^j = K_j + R_j$, where $K_j$ is a finite-rank operator and
\[ |R_j| \leq \frac{2}{1 - \lambda_{DRS3}^{1/p}} C_{GSR} C_{40}(j) C_{GC} \]
\[ \leq \frac{2}{1 - (\inf w)^{-\epsilon} \gamma_{DRS3}} C_{GSR} C_{40}(j) C_{GC} \]
where \( C_{GBS}, C_{GC} \) depends only on the good grid. It follows that the essential spectral radius of \( \Phi \) is at most

\[
\left( \lim_{j \to \infty} \left( \sum_{P \in P_j} |P|^{1 + sp'} \right)^{1/j} \right)^{1/p'}.
\]

and since \( \gamma_{DRS} \) can be taken arbitrarily small we obtained the upper bound

\[
r_{ess}(\Phi) \leq \left( \lim_{j \to \infty} \left( \sum_{P \in P_j} |P|^{1 + sp'} \right)^{1/j} \right)^{1/p'}.
\]

for the essential spectral radius of \( \Phi \). It is easy to see that

\[
r_{ess}(\Phi) \leq (\inf w)^{-s} < 1.
\]

\[\square\]

**Corollary 5.1.** Suppose that every branch of \( f \) is onto, that is, \( f(P) = I \) for every \( P \in \mathcal{P} \). Consider the transfer operator defined by

\[
(\tilde{\Phi} \psi)(x) = \sum_{P \in \mathcal{P}_1} g_P^{1 + sp'} \psi(h_P(x)).
\]

Then \( \tilde{\Phi} : L^\infty \to L^\infty \) is a bounded operator and

\[
r_{ess}(\Phi, \mathcal{B}_{p,q}) \leq (r(\tilde{\Phi}, L^\infty))^{1/p'}.
\]

in the case that \( f \) is continuous in a topological space with a borelian measure \( m \) we have

\[
r(\tilde{\Phi}, L^\infty) = e^{P_{top}((1 + sp') \log w)},
\]

where \( P_{top} \) denotes the topological pressure with respect to \( f \).

**Proof.** The Bowen condition implies that

\[
\sum_{P \in \mathcal{P}_j} |P|^{1 + sp'} \leq C_{42} |\tilde{\Phi}^j 1_j|_{\infty} = C_{42} |\tilde{\Phi}^j|_{\infty}.
\]

for some \( C_{42} \) that does not depend on \( j \), so we obtain

\[
r_{ess}(\Phi, \mathcal{B}_{p,q}) \leq (r(\tilde{\Phi}, L^\infty))^{1/p'}.
\]

\[\square\]

A problem with the above approach for Markovian maps is that \( \mathcal{B}_{p,q} \) a priori depends on the Markov partition \( \mathcal{P} \) under consideration (and consequently depends on \( f \)). So the space \( \mathcal{B}_{p,q} \) is a ad hoc space in this approach. On the other hand, we can use this approach with many situations we have a symbolic dynamics acting on subshift of finite type, as in full shifts, expanding maps on compact sets (in particular, expanding maps on compact manifolds) without discontinuities and Markov expanding interval maps.
6. Conformal expanding repellers

Let $I$ be a compact set in Riemann sphere $\overline{\mathbb{C}}$ endowed with the spherical metric and suppose that $f: I \to I$ is an (open) expanding repeller such that $f$ has a conformal extension to a neighbourhood of $I$ in $\overline{\mathbb{C}}$. We call $f: I \to I$ a conformal expanding repeller (as in Przytycki and Urbański [28]). An important example is obtained taking $f$ as a hyperbolic rational map and $I$ as its Julia set. Let $D$ be the Hausdorff dimension of $I$. Let $m$ the $D$-dimensional Hausdorff measure restrict to $I$ and normalized such that $m(I) = 1$. Then $|f'|^D$ is the Jacobian of $f$ with respect to $m$, that is, if $A \subset I$ is Borellian set with small diameter then

$$m(f(A)) = \int_A |f'|^D \, dm.$$ 

Moreover $m$ is geometric measure, that is, it satisfies (1.1) if we take $d$ as the spherical metric. We could now consider a Markov partition for $(f, I)$ and use the methods there using a $B_{s,p,q}$. We will use a new method here. Since $(I, d, m)$ is an Ahlfors-regular space (so in particular a homogeneous space), there is a Besov space $B_{s,p,q}$ that does not depend one the particular choice of a Markov partition. Indeed, using Christ [14] one can construct a good grid $P$ for $I$ with the following properties. There are constants $\eta, C_{43}, C_{44}, C_{45}, C_{46} \geq 0$ and $\lambda \in (0, 1)$ such that for every $Q \in P_k$ with $k \geq 1$, there is $z_Q \in Q$ satisfying

$$B_d(z_Q, C_{43} \lambda^k_2) \subset Q,$$

$$\text{diam}_d Q \leq C_{44} \lambda^k_2$$

and

$$m\{x \in Q: d(x, I \setminus Q) \leq C_{45} t \lambda^k_2\} \leq C_{46} t^{\eta} m(Q).$$

See Proposition 2.1 in S. [34] for details. If $sp < \eta$ then $B_{s,p,q}(I, m, P)$ coincides with the corresponding Besov space $B_{s,p,q}(I, m, d)$ of the homogeneous space $(I, m, d)$ as defined by Han, Lu and Yang [18]. From now one we assume $s \in (0, 1)$, $p \geq 1$ and $sp < \min\{1, \eta\}$.

**Theorem 6.1.** Let $f: I \to I$ be a conformal repeller with dimension $D$. Then

$$r_{ess}(\Phi, B_{s,p,q}) \leq \min |f'|^{-D_s} < 1.$$ 

Here $B_{s,p,q}$ is the Besov space of the homogenous space $(I, m, d)$.

**Proof.** Since $f$ is expanding and conformal, there is $\delta_0 > 0$ and $\alpha > 1$ with the following property. For every $x \in I$ we can find a domain $V_x$ such that

$$f: V_x \to B_{\infty}(f(x), 2\delta_0)$$

is conformal and its inverse $h_x$ is a contraction, that is

$$d_{\infty}(h_x(z), h_x(y)) \leq \frac{1}{\alpha} d_{\infty}(z, y)$$

for every $x, y \in B_{\infty}(f(x), 2\delta_0)$. In particular

$$V_x \subset B_{\infty}(x, 2\delta_0)$$

and we can define the inverse branches of $f^j$

$$h_{j,x}: B_{\infty}(f^j(x), 2\delta_0) \to \mathbb{C}.$$
as
\[ h_{j,x}(y) = h_x \circ \cdots \circ h_{f^{j-2}(x)} \circ h_{f^{j-1}(x)}(y). \]

Define
\[ g_{j,x}(y) = |h'_{j,x}(y)|^D. \]

Using Koebe Lemma one can prove there is \( C_{47} \) such that for every \( x \in I \) and \( j \)
\[ |g_{j,x}(z) - g_{j,x}(y)| \leq 2g(y)C_{47}d(z,y)^{D(\beta + \epsilon)} \]
for every \( y, z \in B_{\delta}(f^j(x), \delta_0) \). Using arguments quite similar to those in Section 5 we can show that there is \( \delta_1 < \delta_0 \) such that If \( W, Q \in P \) satisfy \( Q \subset h_{x,j}B_{\delta}(f^j(x), \delta_0) \), \( \text{diam } h_{x,i}^{-1}(Q) < \delta_1 \) and \( W \subset h_{x,j}^{-1}Q \) then
\[ |g_{x,j}W|_{\beta,\varrho}(W,P,W,A_{p,q}) \leq 2C_{48} \frac{|Q|}{|h_{x,j}^{-1}(Q)|} |W|^{1/p - \beta}. \]

Note that (use Koebe again) there is \( \tilde{k} \) with the following property. For every \( j \) and \( m \)-almost every \( x \in I \) there is \( P_{x,j} \in \mathcal{Q}_k \) such that \( P_{x,j} \subset h_{x,j}(B_{\delta}(f^j(x), \delta_1/2), x \in P_{x,j} \) and \( f^j(P_{x,j}) \) contains at least one element of \( P^k \). In particular there is \( \delta_2 > 0 \) such that \( m(f^j(P_{x,j})) \geq \delta_2 \) for every \( x, j \). Since \( P \) is a nested sequence of partitions of \( I \) one can find a finite family \( \Lambda_j \subset \{ P_{x,j} \}_{x \in I} \) that is a partition of \( I \). For every \( R \in \Lambda_j \) denote by \( h_R \) and \( g_R \) the restrictions of \( h_{x,i} \) and \( g_{x,i} \) to \( R \). So if \( Q \subset R \) then
\[ |gR1W|_{\beta,\varrho}(W,P,W,A_{p,q}) \leq C_{DP}(R) \left( \frac{|Q|}{|h_R^{-1}(Q)|} \right)^{\frac{1}{p} - s + \epsilon} |W|^{1/p - \beta}, \]
where
\[ C_{DP}(R) = C_{49} \left( \frac{|R|}{|h_R^{-1}(R)|} \right)^{1/(1 - s + \epsilon)} \]
Note that this map \( h_R \) has bounded distortion in the sense that for every \( Q \subset P \), with \( Q \in \mathcal{P} \) we have that \( f^j(Q) \) is an \((1-\epsilon, C_{DGD1}, \lambda_{DGD2})\)-regular domain, where \( C_{DGD1}, \lambda_{DGD2} \) does not depend on \( j \). Indeed since \( f^j \) is conformal, so using Koebe Lemma one can show that in every small open sets close to \( I \) the map \( f^j \) can be written as \( f^j(x) = h(\alpha e^{i\varphi} x) \), where \( h \) is a bilipchitz function that satisfies (1.5), and \( C_\varphi \) does not depend on \( j \). Then we can use an argument similar to Proposition 1.1, noticing that the \( D \)-dimensional Haussdorff measure behaves quite well under the action of scalings and rotations.

We can see the Ruelle-Perron-Frobenious operator of \( f^j \) as
\[ (\Phi^j \psi)(x) = \sum_{R \in \Lambda_j} g_R(x) \psi(h_R(x)). \]

Let \( k_j = \max\{k_0(R) : R \in \Lambda_j \} \). Then for every \( Q \in \mathcal{P} \), with \( k \geq k_j \) we have that
\[ \# \{ R \in \Lambda_j : R \cap Q \neq \emptyset \} \leq 1. \]
Moreover
\[ C_{50} \lambda_2^D|k_0(Q) - k_0(h_R^{-1}(Q))| \leq \frac{|Q|}{|h_R^{-1}(Q)|} \leq C_{DC1} \lambda_2^D|k_0(Q) - k_0(h_R^{-1}(Q))| \]
for appropriated constants \( C_{50}, C_{DC1} \). Indeed due the bounded distortion of \( h_R \) we have
\[ C_{51} \lambda_2^D|k_0(Q) - k_0(h_R^{-1}(Q))| \leq \frac{|R|}{|h_R^{-1}(R)|} \leq C_{52} \lambda_2^D|k_0(Q) - k_0(h_R^{-1}(Q))| \]
and consequently
\[ |k_0(Q) - k_0(h_R^{-1}(Q))| \geq \frac{1}{D \ln \lambda_2} \ln \frac{|R|}{|h_R^{-1}(R)|} - \frac{\ln C_51}{D \ln \lambda_2}, \]
so define
\[ a_R = \frac{1}{D \ln \lambda_2} \ln \frac{|R|}{|h_R^{-1}(R)|} - \frac{\ln C_51}{D \ln \lambda_2}. \]

Take \( \lambda_{DC2}(R) = \lambda_2^D \). Now we cannot choose \( \lambda_{DG2}^D \) to be close to zero anymore. But we can choose \( \epsilon \) small enough such that
\[ \lambda_{DG2}^D(R) = \max\{\lambda_2^{D\epsilon}, \lambda_{DG2}^{1/p}\} = \lambda_2^{D\epsilon} < 1 \]
Then the \( \Theta_R \) has expression identical to the expression in Section 5, so we can use the same arguments to conclude that
\[ r_{\text{ess}}(\Phi) \leq (\liminf_{j \to \infty} \left( \sum_{P \in \Lambda_j} |P|^{1+sp'/j} \right)^{1/j})^{1/p'}. \]
The nature of \( \Lambda_j \) is more mysterious here, however \( \Lambda_j \) is a partition of \( I \), so we can yet obtain the estimate
\[
\left( \sum_{P \in \Lambda_j} |P|^{1+sp'/j} \right)^{1/j} \\
\leq (\sum_{P \in \Lambda_j} |P|^{1/j} (\max_{P \in \Lambda_j} |P|)^{sp'/j})^{1/j} \\
\leq (\max_{P \in \Lambda_j} |P|)^{sp'/j} \\
\leq \delta^{sp'/j} (\max_{P \in \Lambda_j} |P|)^{sp'/j} \\
\leq \delta^{sp'/j} \min |f'|^{-Ds}.
\]

so \( r_{\text{ess}}(\Phi, B_{p,q}^s) \leq \min |f'|^{-Ds} < 1. \)

7. PIECEWISE BIO-LIPCHITZ MAPS ON THE INTERVAL WITH 1/(\( \beta + \epsilon \))-BOUNDED VARIATION POTENTIALS

That is our first class examples of maps that do not have a Markov partition. The study of ergodic theory of piecewise monotone maps on the interval is quite long. Lasota and Yorke [23] studied \( C^2 \) expanding maps on the interval. Keller and Hofbauer [20][19] considered maps with bounded variation jacobian. Keller [21] studied of transfer operators with \( p \)-bounded variation Jacobians, which includes piecewise \( C^{1+} \)-diffeomorphisms. In particular he obtained statistical properties for certain bounded observables.

Consider a map
\[ f: \cup_{i \in \Lambda} I_i \to I \]
where \( I = [0, 1] \) and \( I_i \) are intervals and \( \Lambda \) is finite, where \( \Lambda_1 = \{I_i\} \) is a partition of \( I \). We assume that \( f: I_i \to I \) is continuous and there is \( \alpha, \beta > 0 \) satisfying for every \( i \in \Lambda \) and \( x, y \in I_i \)
\[ \beta|x - y| \geq |f(x) - f(y)| \geq \alpha|x - y|. \]
Suppose that \( w: \cup_{i} \to I \) is a \( 1/(\beta + \epsilon) \)-bounded variation function such that \( \inf w > 0 \). Note that \( w \) also have finite \( 1/(\beta + \epsilon') \)-bounded variation for every \( \epsilon' \), and
sometimes it will be useful to reduce \( \epsilon \). Of course \( f^j \) is also a piecewise expanding map with a corresponding dynamical partition of \( I \) by intervals \( \Lambda_j \).

As usual denote by \( h_J \) the inverse of \( f^j : J \to f^j(J) \) and the induced potential by \( g_J \), that has \( 1/(\beta + \epsilon) \)-bounded variation on \( J \). Let \( D \) be the dyadic good grid of \( I \). Denote

To simplify the notation, denote

\[
m(J) = \inf_{Q \subset J} \frac{|Q|}{|h_J^{-1}(Q)|}, \quad M(J) = \sup_{Q \subset J} \frac{|Q|}{|h_J^{-1}(Q)|}.
\]

Note that \( m(J) \) and \( M(J) \) are finite since the branches of \( f^j \) are bi-Lipschitzian maps.

By Proposition 2.5.A, we can replace the partition \( \Lambda_j \) by a finer finite partition \( \Lambda_{j'} \) such that for every \( W \in D \) and \( W \subset f^j(J) \), with \( J \in \Lambda_j \), we have

\[
|g_J 1_W|_{GP_{p,q}(W,P_{W,A_{\rho_{p,q}}})} \leq C_{30} (\sup_W g_J)|W|^{1/p-\beta}.
\]

so

\[
|g_J 1_W|_{GP_{p,q}(W,P_{W,A_{\rho_{p,q}}})} \leq C_{DRP}(J) \left( \frac{|Q|}{|h_J^{-1}(Q)|} \right)^{1/p-s+\epsilon} |W|^{1/p-\beta},
\]

where

\[
C_{DRP}(J) = C_{30} (\sup_W g_J)m(J)^{-(1/p-s+\epsilon)},
\]

Note that branches of \( f^j \) takes intervals to intervals, and every interval in \( I \) is a \((1-sp,C_{DG1},\lambda_{DG2})\)-regular domain and \((1-\beta p, C_{DG3},0)\)-strongly regular domain, where the constants does not depend on the interval. For every interval \( A \subset I \) we have

\[2^{-k_0(A)} \leq |A| \leq 2^{-(k_0(A)-2)\cdot}.\]

Let \( J \in \hat{\Lambda}_j \) and consider an interval \( Q \subset J \). If \( j \) is large enough we have

\[k_0(h_J^{-1}(Q)) < k_0(Q)\]

and consequently

\[
\frac{1}{4} \left( \frac{1}{2} \right)^{k_0(Q)-k_0(h_J^{-1}(Q))} \leq \left( \frac{|Q|}{|h_J^{-1}(Q)|} \right)^{2^{-(k_0(Q)-2)} \cdot 2^{-k_0(h_J^{-1}(Q))}} \leq \frac{4}{2} \left( \frac{1}{2} \right)^{k_0(Q)-k_0(h_J^{-1}(Q))}.
\]

so we can take \( C_{DC1} = 4 \) and \( \lambda_{DC2}(P) = 1/2 \). Moreover we can choose

\[
|k_0(Q) - k_0(h_J^{-1}(Q))| \geq a_J = \frac{\ln M(J)}{\ln 2} - 2 \geq \frac{\ln \alpha}{\ln 2} - 2.
\]

We can take \( \epsilon \) small enough such that

\[
\lambda_{DRS3}(J) = \max\{(1/2)^\epsilon, \lambda_{DG2}^{1/p}\} = (1/2)^\epsilon < 1
\]

So in this case

\[
\Theta J = C_{DC1}^{\epsilon} C_{DRP}(J) C_{DG1}^{1/p} (1/2)^{a_J(1-\gamma_{DRS3})} = C_{DC1}^{\epsilon} C_{DG3}(\sup_W g_J) m(J)^{-(1/p-s+\epsilon)} M(J)^{\epsilon(1-\gamma_{DRS3})}
\]

Moreover there is \( k_j \) such that for every \( Q \in P^k \), with \( k \geq k_j \) we have that

\[
\#\{J \in \Lambda_j : J \cap Q \neq \emptyset\} \leq 2.
\]
By Proposition 11.2 in [2], the Ruelle-Perron-Frobenious $\Phi^j$ of $f^j$ has a

$$(C_{GSR} C_{55}(j), C_{GSR} C_{56}(j), \gamma_{DRS3})$$

essential slicing, for some constant $C_{55}(j)$ and

$$C_{56}(j) = 2C_{53}^{1/p} \left( \sum_{J \in A_j} \Theta^p_{J} \right)^{1/p'}$$

Corollary 10.1 in [2] tell us that we can write $\Phi^j = K_j + F_j$, where $F_j$ is a finite-rank bounded operator in $B_{p,q}^s$ and

$$|K_j| \leq \frac{2}{1 - (1/2)^{\gamma_{DRS3} C_{GBS} C_{40}(j) C_{GC}}}.$$
so to get quasicompactness of $\Phi$ on $B_{1,q}^*$ we need
\[ |K_j| \leq 4C_{53}C_{58} \sup_{J \in \Lambda_j} \left( \frac{|J|}{|h_{\Lambda_j}(J)|} \right)^{s-\epsilon \gamma_{DRS}} < 1. \]
Note that the constants $C_{53}, C_{58}$ may depend on $\epsilon$. Since $f$ is expanding (inf $|f'| > 1$) then
\[ |K_j| \leq 4C_{53}C_{58} (\inf |f'|)^{\beta(s-\epsilon \gamma_{DRS})} < 1. \]
and we get
\[ r_{ess}(\Phi, B_{1,q}^*) \leq \lim_{j \to \infty} \inf (4C_{53}C_{58} (\inf |f'|)^{s-\epsilon \gamma_{DRS}})^{1/j} = (\inf |f'|)^{s-\epsilon \gamma_{DRS}}. \]
Taking $\epsilon \to 0$ we obtain $r_{ess}(\Phi, B_{1,q}^*) \leq (\inf |f'|)^s < 1$. We can also obtain the quasi-compactness of $\Phi$ on $B_{p,q}^*$ for $p$ close to 1. \(\square\)

8. CONTINUOUS $C^{1+\beta+\epsilon}$-PIECEWISE EXPANDING MAPS ON THE INTERVAL

Here we show that

**Theorem 8.1.** In the setting of the Section 7 we may consider the case when $f$ is continuous and every branch of $f$ is a $C^{1+\beta+\epsilon}$-diffeomorphism that is expanding (\(\alpha > 1\)) and with $w = |f'|$. Then
\[ r_{ess}(\Phi, B_{p,q}^*) \leq e^{h_{top}(f)/p'} (\inf |f'|)^{(1/p'+s)}, \]
where $h_{top}(f)$ denotes the topological entropy of $f$. In particular $\Phi$ acts as a quasi-compact operator on $B_{p,q}^*$ for $p \sim 1$.

**Proof.** In Section 7 we started defining $\Lambda_j$ as the partition given by the intervals of monotonicity of $f^j$. It is well known that the rate of the growth of the cardinality of $\Lambda_j$ is related with the topological entropy of $f$. Indeed
\[
(8.29) \quad \lim_{j \to \infty} \frac{1}{j} \ln \# \Lambda_j = h_{top}(f).
\]

But in Section 7 we needed to replace $\Lambda_j$ by a refinement of it. This makes harder to understand the growth of $\# \Lambda_j$. But in the case of continuous $C^{1+\beta+\epsilon}$-piecewise expanding case, we may also need to do a similar procedure, but in a more orderly fashion. Indeed, using the same bounded distortion arguments we used in Section 5 one can prove that there is $\delta > 0$ such that for every $j$ and every maximal monotone inverse branch $h_J$ with $J \in \Lambda_j$ we have that (7.24) holds for every $W \in D$ such that $W \subset f^j(J)$ and $diam W < \delta$ and (7.28) holds for every $x, y \in f^j(J)$ such that $|x - y| < \delta$. So to obtain the appropriated refinement $\Lambda_j$ such that (7.24) holds $W \subset f^j(J)$ we just need to cut every interval of $\Lambda_j$ in subintervals in such way that the image of each subinterval by $f^j$ has diameter smaller than $\delta$. This can be made in such way that the number of elements in the new partition has at most the number of elements in the original partition times $1 + 1/\delta$. In particular (8.29) remains true. Using the results of Section 7 one can show that
\[
r_{ess}(\Phi, B_{p,q}^*) \leq \left( \lim_{j \to \infty} \inf \left( \sum_{J \in \Lambda_j} (\inf |f^j(J)|)^{(1+p')/(1+p')} \right)^{1/p'} \right)^{1/j} \leq e^{h_{top}(f)/p'} (\inf |f'|)^{(1/p'+s)}. \]
Corollary 8.1. Consider the tent family $f_t: [-1,1] \to [-1,1]$, with $1/2 < t \leq 1$, given by $f_t(x) = -2t|x| + 2t - 1$. We have $\exp(h_{top}(f_t)) = 2t = |f'_t|$. So

$$
\rho_{\text{ess}}(\Phi_{f_t}, B_{p,q}^*) \leq (2t)^{-s} < 1.
$$

for every $s, p, q$ satisfying $0 < s < 1/p$, $p \in [1, \infty)$ and $q \in [1, \infty]$.

9. Piecewise expanding maps on the interval with jacobian in $B_{p,\infty}^{1/p}$

The class of jacobians for which the quasi-compactness of the transfer operator is obtained in this section is, as far as we know, of the lowest regularity in the literature. The setback is that we need that the dynamics satisfies an a priori estimate.

We consider here $I = [0,1]$ with the good dyadic grid $D$.

Theorem 9.1. There is $C_{59} > 0$ such that the following holds. Let $\Lambda$ be a finite set, $\theta_r \in (0,1)$ for every $r \in \Lambda$. Let $\{I_r\}_{r \in \Lambda}$ be a finite partition of $I = [0,1]$ by intervals satisfying $|I_r|^{\theta_r^{-1}} < 1$. If $p > 1$ and $\epsilon_0 > 0$ satisfies

$$
(\#\Lambda)^{1/p'} (\max \{ \theta_r \})^{1/p' + s - \epsilon_0} \leq C_{59}
$$

then there is $\delta > 0$ with the following property.

Choose a collection of intervals $\{J_r\}_{r \in \Lambda}$ in $I$ such that $\theta_r = |I_r|/|J_r| < 1$ for every $r$. Let $I_r = [a_r, b_r]$, $J_r = [c_r, d_r]$. Suppose that $\alpha_r: I \to \mathbb{R}$ belongs to $B_{p,\infty}^{1/p}(I) \cap L^\infty$,

$$
\int \alpha_r \ dm = 0,
$$

$|\alpha_r|_\infty < \min \{1-\theta_r, \theta_r \}$ and $\alpha_r(x) = 0$ for every $x \notin J_r$. Define

$$
h_r: J_r \to I_r
$$
as the bi-Lipchitz map

$$h_r(x) = a_r + \int_{[r, x]} \alpha_r + \theta_r \, dm.$$  

Consider the piecewise expanding map

$$F: \cup_r I_r \to I$$

defined by $F(x) = h_r^{-1}(x)$ for $x \in I_r$. If

$$\max_r \{ |\alpha_r|_{L^p} + |\alpha_r|_{\infty} \} < \delta$$

then

A. The transfer operator $\Phi$ is quasi-compact on $B^a_{p,q}$.
B. $\Phi$ satisfies the Lasota-Yorke inequality for the pair $(B^a_{p,q}, L^1)$,
C. $F$ has an absolutely continuous invariant probability $\mu$,
D. Every absolutely continuous invariant probability $\mu = \rho \, \nu$ satisfies $\rho \in B^a_{p,q}$.

Proof. We can assume that $\max \{ \theta_r \} < 1/10$. The Jacobian of $h_r$ is $g_r(x) = \alpha_r(x) + \theta_r$. Recall that by Proposition 18.10 in [33] there is $C_{\delta_0}$ such that for every $f \in B^1_{p,\infty}(I) \cap L^\infty$ we have

$$|f1_W|_{B^\beta_{p,q}(W, D)} \leq (C_{\delta_0} |f|_{B^1_{p,q}} + |f|_{\infty}) |W|^{1/p - \beta}$$

for every interval $W \in I$. In particular

$$|g_r1_W|_{B^\beta_{p,q}(W, D)} \leq (\theta_r + C_{\delta_0} |\alpha_r|_{B^1_{p,q}} + |\alpha_r|_{\infty}) |W|^{1/p - \beta}$$

for every $W \subset J_r$, $W \in D$. If $\delta$ is small enough then

$$(1 - 1/4)\theta_r \leq \frac{|Q|}{|h^{-1}_r(Q)|} \leq (1 + 1/4)\theta_r < \frac{1}{8}$$

for every interval $Q \subset I_r$, $r \in \Lambda$. Consequently $|k_0(h^{-1}_r(Q)) - k_0(Q)| \geq 1$, so we can take $C_{DC1} = 1$ and $\lambda_{DC2} = 1/2$ and $a_r = 1$. Recall that every interval in is a $(1 - sp, C_{DGD1}, \lambda_{DGD2})$-regular domain, provided we choose $C_{DGD1} > 0$, $\lambda_{DGD2} \in (0, 1)$ properly, and in particular this holds for intervals such as $F(Q)$, where $Q$ is an interval inside $I_r$ for some $r$. Fix $\epsilon \in (0, \epsilon_0)$ such that $1/p - s + \epsilon < 1$ and

$$\lambda_{DRS2} = \max \{ (\lambda_{DC2})^\epsilon, \lambda_{DGD2}^{1/p} \} = (1/2)^\epsilon.$$  

For every $W \subset h^{-1}_r(Q)$ we have

$$|g_r1_W|_{B^\beta_{p,q}(W, D)} \leq \frac{5}{4} \theta_r |W|^{1/p - \beta}$$

$$\leq \frac{5}{4} \left(1 + \frac{1}{\theta_r} \right)^{1/p - s + \epsilon} \left(\frac{|Q|}{|h^{-1}_r(Q)|}\right)^{1/p - s + \epsilon} |W|^{1/p - \beta},$$

so we take

$$C_{DRP}(r) = 2\theta_r^{1 - (1/p - s + \epsilon)}.$$  

So

$$\Theta_r = C_{DC1}^\varepsilon(r)C_{DRP}(r)C_{DGD1}^\varepsilon \lambda_{DC2}^{1 - \gamma_{DRS3}}$$

$$\leq 2C_{DGD1}^\varepsilon \theta_r^{1 - (1/p - s + \epsilon)} (1/2)^{1 - \gamma}.$$
Let \( t \) be such that
\[
M = \sup_{P \in D^k} \# \{ r \in \Lambda : I_r \cap P \neq \emptyset \} \leq 2.
\]

Of course
\[(9.31) \quad N = \sup_{P \in \mathcal{H}} \# \{ r \in \Lambda \text{ s.t. } P \subset J_r \} \leq \# \Lambda \]

Note that every interval, and in particular the intervals in the partition \( \{ I_r \} \), are \((1 - \beta p, C_{33}, t)\)-strongly regular domain, for some universal \( C_{33} \).

We can apply Theorem 12.4 in [2] with \( p = 1 \) to conclude that \( \Phi \) has a \((C_{GSR} C_{61}, C_{GSR} C_{62}, t)\)
essential slicing on \( B_{1,q}^s \), for some \( C_{61} \) and \( C_{62} = C_{1}^{1/p} C_{2}^{1/p} \)
and Corollary 13.1 therein tell us that \( \Phi \) has a similar upper bound for its essential spectral radius bounded and consequently if
\[
\# \Lambda^{1/p'} \left( \max \{ \theta_r \} \right)^{1/p' - s} < C_{1}^{(1 - p)/(1 - p')}
\]
is small enough we can apply Theorem 14.1 and Corollaries 14.1 and 14.2 in [2] to conclude that
\[
\sigma_{ess}(\Phi, B_{p,q}) < 1,
\]
and we can obtain the Lasota-Yorke inequality applying Theorem 14.1 and Corollaries 14.1 and 14.2, and Theorem 15.1 in [2].

\[\square\]

Remark 9.2. Let \( p > 1 \). Potentials in \( B_{p,\infty}^{1/p} \) can be very irregular. Consider for instance
\[
\alpha : [0, 1/2] \rightarrow [-1, 1]
\]
given by \( \alpha(x) = \sin(2\pi \log x) \). Then \( \alpha \in B_{p,\infty}^{1/p}(I) \). Indeed if \( \alpha_+(x) = \max\{\alpha(x), 0\} \)
\[
\alpha_+ = \sum_k \sum_{Q \in D^k} c_Q a_Q
\]
where \( a_Q(x) = 1_{F}(x) \) is the \((1/p, p)\)-Souza’s atom supported on \( Q \), \( c_I = 0 \) and
\[
c_Q = \inf_Q \alpha_+ - \inf_Q \alpha_+,
\]
if \( Q \in D^k \) with \( k \geq 1 \) and \( Q' \) is such that \( Q \subset Q' \) and \( Q' \in D^{k-1} \). If \( Q \subset [2^{-(i+1)}, 2^{-i}] \) we have
\[
|c_Q| \leq C 2^i |Q|,
\]
and \( c_Q = 0 \) for every \( Q \) such that \( Q \notin [2^{-(i+1)}, 2^{-i}] \) for every \( i \), so
\[
\sum_{Q \in D^k} |c_Q|^p = \sum_{i < k} \sum_{Q \in D^k} \sum_{Q \subset [2^{-(i+1)}, 2^{-i}]} |c_Q|^p \leq C \sum_{i < k} 2^{(i-k)p+(k-i)} \leq C \sum_{i < k} 2^{(k-i)(1-p)} \leq \frac{C}{1 - 2^{1-p}}.
\]
and consequently \( \alpha_+ \in B^{1/p}_{p,\infty} \). In the analogous way \( \alpha_-(x) = \max\{-\alpha(x),0\} \in B^{1/p}_{p,\infty} \). So \( \alpha = \alpha_+ - \alpha_- \in B^{1/p}_{p,\infty} \). In particular we can construct \( \alpha_r \) using \( \alpha \) in such way to satisfy Theorem 9.1 (See Figure 1).

10. Infinitely many branches with small images

There are three main motivations for the family of expanding maps we study in this section. First, we provided examples of maps with jacobian in \( B^{1/p}_{p,\infty} \) but without absolutely continuous invariant probabilities and whose transfer operator is not quasi-compact on \( B^{1/p}_{p,\infty} \). Secondly it also provides examples of expanding maps with infinitely many branches and whose images of most of the branches are very small, and yet the transfer operator is quasi-compact. Moreover the maps in this family have a dynamical behaviour quite similar to induced maps of unimodal maps that appears in the study of the existence of wild attractors. See Bruin, Keller, Nowicki and van Strien [8], Keller and Nowicki [22], Bruin, Keller and St. Pierre [9], and Moreira and S. [25].

For every \( r \geq 0 \) define \( L_r = [2^{-(r+1)}, 2^{-r}] \). Note that \( L_r \in D^{r+1} \).

**Lemma 10.1.** There is \( C_{63} \) such that for every \( \Lambda \subset \mathbb{N} \) the set

\[
\Omega = \bigcup_{i \in \Lambda} L_r
\]

is a \( (1 - \beta p, C_{63}, 0) \)-strongly regular domain.

**Proof.** Given \( Q \in D^k \), for some \( k \), there are three cases.

**Case i.** Suppose \( Q \neq [0,2^{-k}] \) and \( Q \subset L_r \) for some \( r \notin \Lambda \). In this case \( Q \cap \Omega = \emptyset \), so pick \( \mathcal{F}^i(\Omega \cap Q) = \emptyset \) for every \( i \).

**Case ii.** Suppose \( Q \neq [0,2^{-k}] \) and \( Q \subset L_r \) for some \( r \in \Lambda \), so \( Q \cap \Omega = Q \). In this case pick \( \mathcal{F}^k(\Omega \cap Q) = \{Q\} \) and \( \mathcal{F}^i(\Omega \cap Q) = \emptyset \) for every \( i \neq k \).
Case iii. If $Q = [0, 2^{-k}]$ then pick either
$$F^i(\Omega \cap Q) = \{ L_{i+1} \}$$
if $i + 1 \in \Lambda$ and $i + 1 \geq k$, or $F^i(\Omega \cap Q) = \emptyset$ otherwise.
In all cases
$$Q \cap \Omega = \bigcup_i \bigcup_{P \in F^i(\Omega \cap Q)} P$$
and
$$\sum_{P \in F^i(\Omega \cap Q)} |P|^{1-\beta} \leq \sum_{i \geq k} 2^{-i(1-\beta)} \leq C_632^{-k(1-\beta)} = C_63|Q|^{1-\betap}.$$ 
This proves the Lemma. □

Let $i_0 \in \mathbb{N}$. For each $j = 0, 1, 2, 3$ let
$$I_{i_0}^j = \bigcup_{i \geq i_0 \mod 3} L_i.$$ 
For $\alpha > 0$ define
$$G_{\alpha, \zeta, i}: L_i \to I$$
as
$$G_{\alpha, \zeta, i}(x) = \begin{cases} 
\alpha(7/2 - 13\zeta/6)(x - 1/2^{i+1}) + \alpha/2^{i+2}, & \text{for } 1/2 \leq 2^ix < 11/16, \\
\alpha(7/2 - 5\zeta/6)(x - 11/2^{i+4}) + \alpha(1/2^{i+1} - 13\zeta/2^{i+5}), & \text{for } 11/16 \leq 2^ix < 7/8, \\
\alpha(7/2 + 9\zeta/2)(x - 1/2^i) + \alpha/2^{i-1}, & \text{for } 7/8 \leq 2^ix < 1.
\end{cases}$$ 
Note that $G_{\alpha, \zeta, i}$ is continuous, monotone and $|G'_{\alpha, \zeta, i}| \geq 4\alpha/3$ for every $\alpha > 0$ and $\zeta \in [0, 1]$. Moreover $G_{1, \zeta, i} L_i = L_{i+1} \cup L_i \cup L_{i-1}$. Define
$$F_{\alpha, \zeta, j, i_0}: I_j^{i_0} \to I$$
as $F_{\alpha, \zeta, j, i_0}(x) = G_{\alpha, \zeta, i}(x)$ for $x \in L_i \subset I_j^{i_0}$. For every $\alpha > 0$ we have that $F_{\alpha, \zeta, j, i_0}$ is injective.

Fix $\alpha > 0$ and choose
$$i_0 = \min\{i \geq 0: \alpha 2^{-i+1} \leq 1\}.$$ 
Divide the interval $[2^{-i_0}, 1]$ in $k_0$ intervals $I_{-1}, \ldots, I_{-k_0}$ of same size,
$$\Lambda = \{-k_0, \ldots, -1, 0, 1, 2, 3\}$$
and $I_j = I_j^{i_0}$ for $j \geq 0$. Consider the piecewise linear map
$$F_{\alpha, \zeta, k_0}: \bigcup_{r \in \Lambda} I_r \to I$$
where $F_{\alpha, \zeta, k_0}(x) = G_{\alpha, \zeta, j, i_0}(x)$ for $x \in I_j$, if $j \geq 0$, and
$$F_{\alpha, \zeta, k_0}: I_j \to I$$
is an onto affine map if $j < 0$. In particular $|G'_{\alpha, \zeta, k_0}| \geq k_0$ on $I_j$ for $j < 0$. Denote the inverse of $F_{\alpha, \zeta, k_0}$ on $I_j$ by $h_j$ and $J_j = F_{\alpha, \zeta, k_0}(I_j)$.

**Theorem 10.2.** We have
A. The Dirac mass supported in \{0\} is the unique physical measure of \(G_{1,1,k_0}\) and its basin of attraction is the whole \(I\) (up to a set of zero Lebesgue measure), so in particular for every \(n\)

\[
\Phi^n: B^a_{p,q} \to B^a_{p,q}
\]

is not a quasi-compact operator and it does not satisfy the Lasota-Yorke inequality for the pair \((B^a_{p,q}, L^1)\).

B. For every \(\alpha > 0\), \(\zeta \in (0,1], p \in [1,\infty), q \in [1,\infty)\) we have that \(\Phi\) is a bounded operator acting on \(B^a_{p,q}\).

C. Let \(p \in (1,\infty)\) and \(q \in [1,\infty)\). If \(k_0\) and \(\alpha\) are large enough and \(\zeta \in [0,1]\) then \(\Phi\) is a quasi-compact operator on \(B^a_{p,q}\) and it satisfies the Lasota-Yorke inequality for the pair \((B^a_{p,q}, L^1)\).

**Proof of A.** Note that \(G_{1,1,k_0}\) is an expanding markovian map. It is not difficult to show that \(G_{1,1,k_0}\) has the following property: if \(S\) satisfies \(G_{1,1,k_0}^{-1}(S) = S\) then either \(m(S) = 0\), \(m(S) = 1\) or

\[
S \subset \{x \in I: \lim_{n \to \infty} G^n_{1,1,k_0}(x) = 0\}.
\]

So if the basin of attraction of the Dirac mass supported in \(\{0\}\) has positive Lebesgue measure then it has full Lebesgue measure since its complement is backward invariant. Note that \(G_{1,1,k_0}\) is conjugated with a skew product close to 0. Indeed

Define \(g_{1,1}: [1/2, 1] \to [1/2, 1]\) as

\[
g_{1,1}(x) = \begin{cases} 8x/3 - 5/6, & 1/2 \leq x < 11/16, \\ 8x/3 - 4/3, & 11/16 \leq x < 7/8, \\ 4x - 3, & 7/8 \leq x < 1. \end{cases}
\]

and \(\psi: [1/2, 1] \to \{-1, 0, 1\}\) as

\[
\psi(x) = \begin{cases} -1, & 1/2 \leq x < 11/16, \\ 0, & 11/16 \leq x < 7/8, \\ 1, & 7/8 \leq x < 1. \end{cases}
\]

Consider the skew product

\[
T: [1/2, 1] \times \mathbb{Z} \to [1/2, 1] \times \mathbb{Z}
\]

given by \(T(x, i) = (g_{1,1}(x), i + \psi(x))\). Define

\[
H: [1/2, 1] \times \mathbb{Z} \to (0, +\infty)
\]

by \(H(x, i) = 2^{-i}x\). Then \(H \circ T(x, i) = G_{1,1,k_0} \circ H(x, i)\), provided \(i\) is large enough. Note that the (normalised) Lebesgue measure \(m\) on \([1/2, 1]\) is the unique absolutely continuous invariant probability of \(g_{1,1}\) and \(m\) is ergodic. Since

\[
\int \psi \, dm > 0
\]

it follows that for \(x \in [0, 1]\) in a subset \(S\) of positive Lebesgue measure we have

\[
\lim_{n \to \infty} G^n_{1,1,k_0}(x) = 0,
\]

which implies that \(S\) has full Lebesgue measure. So the Dirac mass at 0 is the unique physical measure of \(G_{1,1,k_0}\) and its basin of attraction has full Lebesgue measure. It follows that \(\Phi^n\) is not quasi-compact on \(B^a_{p,q}\), otherwise \(G_{1,1,k_0}\) would have an absolutely continuous invariant probability. \(\square\)
Proof of B. Choose \( k_0 \) large enough such for every \( \alpha > 0 \) and \( \zeta \in [0,1] \) we have that the transfer operator \( \Phi_1 \) of the restriction

\[
F_{\alpha,\zeta,k_0} : \cup_{i<0} I_i \to I
\]

can be written as

\[
\Phi_1 = T_1 + K_1,
\]

where \( T_1 \) is a finite-rank operator, \( |K_1|_{B^{p,q}} < 1/4 \) and it also satisfies the Lasota-Yorke inequality

\[
|\Phi_1 f|_{B^{p,q}} \leq 4^{-1}|f|_{B^{p,q}} + C_{64}|f|_1.
\]

Consider the transfer operator \( \Phi_2 \) of the restriction

\[
F_{\alpha,\zeta,k_0} : \cup_{i\geq0} I_i \to I
\]

By Lemma 10.1 every domain \( I_j \), with \( j = 0,1,2,3 \), is a \((1-\beta p, C_{63},0)\)-strongly regular domain. For every \( P = [a,b] \in D^k \) such that \( P \subset J \) there is \( [c,d] \subset [a,b] \) such that

\[
h_i \cdot 1_P = \frac{1}{\alpha}((7/2 - 13\zeta/6)^{-1}1_{[a,c]} + (7/2 - 5\zeta/6)^{-1}1_{[c,d]} + (7/2 + 9\zeta/2)^{-1}1_{[d,b]})
\]

so if \( \zeta \in [0,1] \) then

\[
|h_i \cdot 1_W|_{B^{p,q}} \leq \frac{C_{65}}{\alpha}|W|^{1/p-\beta} \leq \frac{C_{66}}{\alpha^{1-1/p+s-\epsilon}}\left(\frac{|W|}{|h_i^{-1}(W)|}\right)^{1/p-s+\epsilon}|W|^{1/p-\beta}.
\]

Let \( C_{DRP} = C_{66}/\alpha^{1-1/p+s-\epsilon} \). We can take as usual \( C_{DC1} = 1 \) and \( \lambda_{DC2} = 1/2 \) and

\[
a_r = -\frac{\log \alpha}{\log 2} - C_{67},
\]

Fix \( \gamma = 1/2 \). Choosing \( \epsilon \) small enough we have

\[
\Theta_i = C_{68}1/\alpha^{1-1/p+s-\epsilon}/2
\]

for every \( i = 0,1,2,3 \). By Theorem 12.5 in [2] we have that \( \Phi_2 \) is a bounded operator in \( B^{p,q}_s \) with

\[
|\Phi_2|_{B^{p,q}_s} \leq C_{69}/\alpha^{1-1/p+s-\epsilon}/2,
\]

so if \( \alpha \) is large enough then \( |\Phi_2|_{B^{p,q}_s} \leq 1/4 \) and consequently \( \Phi = \Phi_1 + \Phi_2 \) is a quasi-compact operator and it satisfies the Lasota-Yorke inequality for the pair \((B^{p,q}_s,L^1)\).

\[\Box\]

11. Lorenz maps with non-flat singularities

One of the motivations to the results Keller [21] is to study Lorenz maps, an important class of examples that appears in the study of singular hyperbolic flows. Here we obtain the quasi-compactness in a spaces of functions with more general class of observables, that includes unbounded ones.

**Proposition 11.1.** Let \( \Lambda_1 \) be a collection of pairwise disjoint intervals of \( \hat{I} = [a,b] \) and

\[
F : \cup_{J \in \Lambda_1} J \to \hat{I}
\]

be a map with following property. For every \( J \in \Lambda_1 \) we have that the restriction of \( F \) to \( J \) is

- either a \( C^{1+\beta+\epsilon} \)-diffeomorphism,
or
\[ F(x) = (D_J x - a_J)^{1/(1+\gamma)}. \]
for \( x \in J = [a_J, b_J], \ D_J > 0 \) and \( \beta < \min\{1, \gamma\}. \) In this case we say that \( F : J \to F(J) \) is a Lorenz branch.

Then the Ruelle-Perron-Frobenious operator \( \Phi \) of \( F \) with \( g_J = |h'_J| \) is a bounded operator in \( \mathcal{B}_{1,q}^r = \mathcal{B}_{1,q}^r(\bar{I}, \mathcal{D}, m) \) that can be written as
\[ \Phi = Z_F + K_F \]
where \( Z_F \) is a bounded finite rank operator on \( \mathcal{B}_{1,q}^r \) and
\[
(11.32) \quad |K_F|_{\mathcal{B}_{1,q}^r} \leq \frac{2}{1 - (1/2)\gamma} C_{GBS}^4 \gamma + 1 (2C_{27} + C_{29}) \alpha^{-(s + \gamma)} C_{GBS}. \]

Here \( \alpha = \inf |F'|. \) In particular if \( \alpha \) is large enough we have that \( \Phi \) is quasi-compact in \( \mathcal{B}_{1,q}^r. \)

**Proof.** To deal with the non-Lorenz branches, we will use methods similar to those in Section 7. Proposition 2.1.B we can refine \( \Lambda \) (that is, replace intervals in \( \Lambda \) by finite collections of pairwise disjoint intervals that covers the original intervals) in such way that for every non-Lorenz branch \( F : J \to f(J) \) we have
\[
|gJ^p|_{\mathcal{B}_{p,q}^r(W, pW, \mathcal{A})} \leq 2C_{27} \frac{|Q|}{|h^{-1}(Q)|} |W|^{1/p - \beta} \\
\leq 2C_{27} \alpha^{-(1 - 1/p + s - \epsilon)} \left( \frac{|Q|}{|h^{-1}(Q)|} \right)^{1/p - s + \epsilon} |W|^{1/p - \beta},
\]
for every \( Q, J \in \mathcal{D} \) such that \( Q \subset J \) and \( W \subset f(Q). \) Here \( g_J = |h'_J| \) and \( h_J \) is the corresponding inverse branch and \( C_{27} \) depends only on the good grid \( \mathcal{D}. \)

On the other hand if \( J \in \Lambda_1 \) is a Lorenz branch then by Proposition 2.4 we have
\[
|gJ^p|_{\mathcal{B}_{p,q}^r(W, pW, \mathcal{A})} \leq C_{29} \frac{|Q|}{|h^{-1}(Q)|} |W|^{1/p - \beta} \\
\leq C_{29} \alpha^{-(1 - 1/p + s - \epsilon)} \left( \frac{|Q|}{|h^{-1}(Q)|} \right)^{1/p - s + \epsilon} |W|^{1/p - \beta},
\]
where \( Q \) and \( W \) satisfy the same conditions as before.

Note that (7.25) and (7.26) also holds (with \( j = 1 \)) for every \( J \in \Lambda_1. \) Taking \( \epsilon \) small enough we obtain
\[
\Theta_J = C_{DC1}^\epsilon C_{DRP}(J)C_{DGD1}(1/2)^{\gamma_j(1 - \gamma_{DRS3})} \\
\leq 4^{\epsilon + 1}(2C_{27} + C_{29}) \alpha^{-(1 - 1/p + s - \epsilon)} \alpha^{-(1 - \gamma_{DRS3})}
\]
and consequently (11.32) holds.

**Corollary 11.1.** Let
\[ f : \cup_{J \in \Lambda_1} J \to I \]
be a map with following property. For every \( J \in \Lambda_1 \) we have that the restriction of \( f \) to \( J \) is
i. (Branch Type I) either a \( C^{1 + \beta + \epsilon} \) diffeomorphism,
Proof. Let $F$ be the Ruelle-Perron-Frobenius of $Z$ where the constant $I$ given by $M$ is the Ruelle-Perron-Frobenius of $f$ for every $\psi$ that implies $F$ is a quasi-compact operator acting on $B^s_{1,q}$ and satisfying (11.33)

\[ r_{\text{ess}}(\Phi,B^s_{1,q}) \leq \alpha^{-s}. \]

If $\alpha = \inf |f'| > 1$ then $\Phi$ is a quasi-compact operator acting on $B^s_{1,q}$ and satisfying

\[ r_{\text{ess}}(\Phi,B^s_{1,q}) \leq \alpha^{-s}. \]

**Proof.** Let $I = [c,d]$. We claim that there are two functions $F_1: A \to I$ and $F_2: B \to \hat{I}$, where $\hat{I} = [c,2d-c]$ such that

- We have that $I \subset A \cup B \subset \hat{I} = [a,\hat{b}]$. Moreover $A$ and $B$ are disjoint and $|\hat{I}| = 2^t|I|$ for some $t \in \mathbb{N}$.
- Both $F_1$ and $F_2$ satisfy the assumptions of Proposition 11.1, $\inf |F'_i| > \alpha^j$ and $\inf |F'_i| > \alpha$.
- We have

\[ f_j(x) = \begin{cases} 
F_1(x), & \text{if } x \in A, \\
F_2^{j+1}(x), & \text{if } x \in B \cap I. 
\end{cases} \]

In particular if

\[ \Phi_{F_i}: B^s_{1,q}(\hat{I}, D_1, m) \to B^s_{1,q}(\hat{I}, D_1, m) \]

is the Ruelle-Perron-Frobenious of $F_i$, with $i = 1, 2$ then for every small $\epsilon, \gamma_{\text{DRS}}$ we can write

\[ \Phi_{F_1} + \Phi_{F_2} = Z_j + K_j, \]

where $Z_j$ has finite rank and

\[ |K_j|_{B^s_{1,q}(I,D_1,m)} \leq C\alpha^{-j(s+\epsilon\gamma_{\text{DRS}})}, \]

where the constant $C$ may depends on $\epsilon$ but does not depend on $j$. It is easy to see that the inclusion

\[ \mathcal{I}: B^s_{1,q}(I,D_1,m) \to B^s_{1,q}(\hat{I}, D_1, m) \]

given by $\mathcal{I}(\psi) = \psi$ is continuous, as well the multiplier

\[ \mathcal{M}: B^s_{1,q}(\hat{I}, D_1, m) \to B^s_{1,q}(I,D_1,m) \]

given by $\mathcal{M}(\psi) = 1_t \psi$.

If

\[ \Phi_f: B^s_{1,q}(I,D_1,m) \to B^s_{1,q}(I,D_1,m) \]

is the Ruelle-Perron-Frobenious of $f$ we have

\[ \Phi_f^j\psi = \Phi_{F_1}^j\psi + \Phi_{F_2}^j\psi \]

for every $\psi \in L^1(m)$ with support on $I$. So

\[ \Phi_f^j = \mathcal{M}Z_j\mathcal{I} + \mathcal{M}K_j\mathcal{I}, \]

Since $\mathcal{M}Z_j\mathcal{I}$ is a finite rank operator and $\mathcal{M}K_j\mathcal{I}$ and

\[ |\mathcal{M}K_j\mathcal{I}|_{B^s_{1,q}(I,D_1,m)} \leq |\mathcal{M}C\alpha^{-j(s+\epsilon\gamma_{\text{DRS}})}|\mathcal{I} \]

that implies

\[ r_{\text{ess}}(\Phi_f,B^s_{1,q}(I,D_1,m)) \leq \alpha^{-s}. \]
It remains to prove the claim. Let $J_1, \ldots, J_k$ be an enumeration of the elements $J \in \Lambda$ such that $f^j : J \to f^j(J)$ is a branch of Type II. We are going to define recursively a family of intervals $\{U_1, \ldots, U_k\}$, intervals

$$I_0 \subset I_1 \subset \cdots \subset I_k$$

and respective partitions $\Lambda^i$ of $I_i$ by intervals and maps

$$f_i : \cup_{J \in \Lambda^i} J_i \to I_i$$

such that $J_\ell \in \Lambda_i$ and $f_i = f^j$ on $J_\ell$ for every $\ell > i$. Define $f_0 = f^j$, $I_0 = I$ and $\Lambda^0 = \Lambda_j$. Assume we have defined $f_i, I_i, \text{ and } \Lambda^i$. Let $J_{i+1} = [a_i, b_i]$. We have that

$$f^j = f_i : J_{i+1} \to f^j(J_{i+1})$$

can be written as

$$f^j = \psi_n \circ E_n \circ \phi_n \circ \psi_{n-1} \circ E_{n-1} \circ \phi_{n-1} \circ \cdots \psi_1 \circ E_1 \circ \phi_1,$$

where $1 \leq n \leq j$. Here

$$\phi_i : [a_i, b_i] \to [0, c_i]$$

and

$$\phi_j : [0, E_\ell(c_i)] \to [a_{\ell+1}, b_{\ell+1}]$$

are $C^{1+\gamma}$ diffeomorphisms, the map $E_\ell$ is defined by $E_\ell(x) = (D_\ell x)^{1/(1+\gamma)}$, with $D_\ell \in \{D_J\}_{J \in \Lambda_i}$, and $a_\ell = 0$ for $1 \leq \ell < n$, and moreover there is $C_70$ such that

$$\frac{1}{C_70} \leq \phi_\ell(x), \psi_\ell(x) \leq C_70,$$

for every $j$, and every branch $J \in \Lambda_j$ of $f^j$ that is not a diffeomorphism, and every $\ell < n$.

Choose $A$ such that $A/C_70 = a^{2j}$. Define $\omega_1(x) = A\phi_1(x)$, $\omega_2(x) = A\phi_2 \circ \psi_1(Ax)$, $\omega_3(x) = A\phi_2 \circ \psi_2(Ax)$, and $\omega_{n+1}(x) = \psi_n(Ax)$. Also set $\tilde{E}_\ell(x) = (D_\ell A^{-2-\gamma})^{1/(1+\gamma)}$. Of course

$$f^j(x) = \omega_{n+1} \circ \tilde{E}_n \circ \omega_n \circ \cdots \circ \tilde{E}_2 \circ \omega_2 \circ \tilde{E}_1 \circ \omega_1(x).$$

Assume that $\phi_0(a_1) = 0$ (the case $\phi_0(b_1) = 0$ is similar). Now define $U_i = [a_i + \delta, b_i]$, $\tilde{R}_0 = [a_1 + \delta, b_1]$, $\tilde{R}_0 = \omega_1([a_1 + \delta])$, and recursively $\tilde{R}_\ell = \tilde{E}_\ell(R_\ell)$ and $R_{\ell+1} = \omega_{\ell+1}(R_\ell)$. If $\delta > 0$ is small enough we have

$$\min_{\ell} \min_{x \in \tilde{R}_{\ell-1}} \omega_\ell(x), \min_{x \in R_\ell} \tilde{E}_\ell(x) > a^{2j}$$

and

$$2j a^{2j} |\tilde{R}_0| + \sum_{\ell} |\tilde{R}_\ell| + \sum_{\ell} |R_\ell| < 2^{-i} |I|$$

(11.34)

Let $\pi_\ell$, with $1 \leq \ell \leq n + 1$, and $\tilde{\pi}_\ell$, with $0 \leq \ell \leq n$, be affine isometries, where $\pi_{n+1}(x) = \tilde{\pi}_0(x) = x$, and such that

$$\{\pi_\ell(R_\ell)\}_{1 \leq \ell \leq n} \cup \{\tilde{\pi}_\ell(\tilde{R}_\ell)\}_{1 \leq \ell \leq n}$$

is a family of pairwise disjoint intervals outside $I_\ell$ such that

$$\tilde{\Lambda} = (\Lambda^i \setminus \{J\}) \cup \{[a_i + \delta, b_i], \tilde{R}_0\} \cup \{\pi_\ell(R_\ell)\}_{1 \leq \ell \leq n} \cup \{\tilde{\pi}_\ell(\tilde{R}_\ell)\}_{1 \leq \ell \leq n}$$

is a partition of an interval $I_\ell = [c_i, d_i] \supset I_0$. Note that $f^j$ is a diffeomorphism on the interval $[a_1 + \delta, b_1]$. Define $f^j$ as $f_j$ on every element of $\Lambda^i \setminus \{J\}$ and on $[a_1 + \delta, b_1]$, as $\tilde{\pi}_\ell \circ \tilde{E}_\ell \circ \pi^{-1}$ on $\pi_\ell(\tilde{R}_\ell)$, for every $1 \leq \ell \leq n$, and as $\pi_{\ell+1} \circ \omega_\ell \circ \pi^{-1}$ on $\pi_\ell(\tilde{R}_\ell)$ for every $0 \leq \ell \leq n$. We have $|f^j| \geq a^{2j}$ everywhere.
Let $Y_0 = \tilde{R}_0, Y_1, \ldots, Y_{2(j-n)}$ be pairwise disjoint intervals that are outside $\tilde{I}$, and such that

$$\Lambda_{i+1} = \tilde{\Lambda}_i \cup \{Y_1, \ldots, Y_{2(j-n)}\}$$

is a partition of an interval $I_{i+1}$ and $|Y_i| = \alpha^i|\tilde{R}_0|$ for $i \leq 2(j-n)$. Define $f_{i+1}$ as $\tilde{f}_i$ in $\tilde{I} \setminus \tilde{R}_0$, as the orientation preserving affine map on $Y_i$ satisfying $f_{i+1}(Y_i) = Y_{i+1}$, for each $i < 2(j-n)$ and as $f_{i+1}(x) = \tilde{f}_i(\theta(x))$ on $Y_{2(j-n)}$, where $\theta$ is the orientation preserving affine map such that $\theta(Y_{2(j-n)}) = Y_0 = \tilde{R}_0$, in particular $|\theta'| = \alpha^{-2(j-n)}$ and consequently $|f_{i+1}(x)| = |\tilde{f}_i(\theta(x))|\alpha^{-2(j-n)} \geq \alpha^{2n} \geq \alpha^2$.

We conclude that $|f_{i+1}'| \geq \alpha$ everywhere. This completes the recursive construction of $f_k$.

Due (11.34) we have $|I_{i+1} \setminus I_i| \leq 2^{-i}|I|$, so $|I_k| \leq 2^{|I|}$ and $I_k \subset \tilde{I}$. To conclude the proof of the claim, take

$$A = (A_i \setminus \{J_1, \ldots, J_k\}) \cup \{U_1, \ldots, U_k\},$$

$$B = I_k \setminus A, F_1 \text{ equal to } f^i \text{ on } A \text{ and } F_2 \text{ equals to } f_k^{2j+1} \text{ on } B.$$

\[\square\]

12. Generic piecewise expanding maps on $\mathbb{R}^D$

Piecewise smooth expanding maps on $\mathbb{R}^D$ received a lot of attention. See Góra and Boyarsky [17], Adl-Zarabi [1] and Saussol [31]. The goal of this section is to obtain generic results for piecewise expanding maps on $\mathbb{R}^D$ as in Cowieson [16] [15].

Let $I_r$ be a finite piecewise partition made of open sets in $\mathbb{R}^D$, and $m \geq 1$. Consider the set $D^m_{exp}(\{I_r\})$ of maps $F: I \to I$ so that

A. for each $r$ the map $F: I_r \to I$ extends as a $C^m$ diffeomorphism on an open neighborhood of $\overline{I_r}$,

B. there is $\lambda > 1$ such that $|f^i(x) \cdot w| \geq \lambda |w|$ for every $x \in I, w \in \mathbb{R}^D$.

One can give a topology on $D^m_{exp}(\{I_r\})$ considering the product topology of the $C^\infty$ topologies on each branch $F: I_r \to I$.

The remarkable result by Cowieson [16] [15] shows that in an open and dense set of maps $F \in D^m_{exp}(\{I_r\})$, with $m \geq 2$ and the partition $\{I_r\}$ is a $C^m$ partition (see Cowieson [16] for details) then the map $F$ has an absolutely continuous invariant probability and whose density has bounded variation. We improve this result with

**Theorem 12.1.** Let $\{I_r\}$ be a finite partition of a good $C^1$ domain in $\mathbb{R}^D$ such that every $I_r$ is a $N$-good $C^1$ domain with a regular Whitney stratification. For a map $F$ in an open and dense subset of $D^1_{exp}+\epsilon(\{I_r\})$, with $\beta, \epsilon > 0$, $0 < s < \beta < 1/D$, the Perron-Frobenious operator $\Phi: B^s_{1,q} \to B^s_{1,q}$ is quasi-compact with

\[
\sigma_{ess}(\Phi, B^s_{1,q}) \leq (\inf_{x} \min_{|v| = 1}|D_x F \cdot v|)^{-D s}.
\]

and it satisfies the Lasota-Yorke inequality for the pair $(L^1, B^s_{1,q})$. Moreover every absolutely continuous invariant probability of $F$ has a density that is $B^s_{1,q}$-positive, so in particular its support is an open set of $I$ (up to a subset of zero Lebesgue measure).

**Proof.** Let $p \in [1, \infty)$. By a transversality argument as in Cowieson [16] [15] one can prove that for an open and dense set of maps $F$ in $D^1_{exp}+\epsilon(\{I_r\})$ the map $F^n$ is a piecewise expanding $C^{1+\beta+\epsilon}$ map defined in a partition $\{I^n_r\} \subset \Lambda_n$ that consists of $B$-good $C^1$ domains with a regular Whitney stratification, for some $B$ that does
not depend on n. Denote by $h_r$ the corresponding inverse branches. By Proposition 1.4 there is $\delta > 0$ such that if $Q \in \mathcal{D}$, $\text{diam } Q < \delta$ and $Q \subset I^n_r$ then $h_r^{-1}(Q)$ is a regular domain in $(K, m, \mathcal{D})$. Here $0 < \alpha_1(Q) \leq \alpha_2(Q) \leq \cdots \leq \alpha_n(Q)$ are such that $\{\alpha_i^2(Q)\}$ are the eigenvalues of $A_QA_Q^*$, with $A_Q = D_{x_Q}h_r^{-1}$ and $x_Q$ is an arbitrary element of $Q$. Here we must choose $x \in Q$. Replacing $\{I^n_r\}_{r \in \Lambda_n}$ by an appropriately finer partition and increasing $B$ if necessary we can additionally assume that $\text{diam } I^n_r < \delta$ for every $r \in \Lambda_n$ and that for every $Q \in \mathcal{D}$, satisfying $Q \subset I^n_r$, $r \in \Lambda_n$ and $n$ we have that $h_r^{-1}(Q)$ is a regular domain in $(K, m, \mathcal{D})$. Here $0 < \alpha_1(r) \leq \alpha_2(r) \leq \cdots \leq \alpha_n(r)$ are such that $\{\alpha_i^2(r)\}$ are the eigenvalues of $A_rA_r^*$, with $A_r = D_{x_r}h_r^{-1}$ and $x_r$ is an arbitrary element of $I_r$. So we can take $C_{DG1}(r) = C_{71}(\Pi_{i \neq 1} \frac{\alpha_i(r)}{\alpha_1(r)})^{sp}$ and $\lambda_{DG2}(r) = \lambda_1^{(1-\delta)(1-Dsp)}.$ Notice that (refining $\{I^n_r\}_r$ once again)

\begin{equation}
|Q| \leq C_{72} \Pi_i \frac{1}{\alpha_i(r)}.
\end{equation}

(12.36)

for some $C_{72}$ and every $Q \in I^n_r$, $r \in \Lambda_n$, with $Q \in \mathcal{D}$. On the other hand we have

\begin{equation}
\frac{1}{C_{73}} \left( \frac{1}{2} \right)^{D(k_0(Q) - k_0(h_r^{-1}(Q)))} \leq \frac{1}{\alpha_1(r)^{sp}} \leq C_{73} \left( \frac{1}{2} \right)^{D(k_0(Q) - k_0(h_r^{-1}(Q)))}
\end{equation}

for some $C_{73}$. In particular

\begin{equation}
|k_0(h_r^{-1}(Q)) - k_0(Q)| \geq a_r = \frac{|\ln \alpha_1(r)|}{\ln 2} - \frac{\ln C_{73}}{\mathcal{D} \ln 2}.
\end{equation}

and if $\alpha_1 > 1$ then

\begin{equation}
\frac{|Q|}{h_r^{-1}(Q)} \leq C_{74} \Pi_{i \neq 1} \alpha_i(r)^{-sp} \frac{1}{\alpha_1(r)^{sp}} \left( \frac{1}{2} \right)^{D(k_0(h_r^{-1}(Q)) - k_0(Q))}.
\end{equation}

Take $\lambda_{DC2} = 1/2^D$ and

\begin{equation}
C_{DC1} = C_{74} \Pi_{i \neq 1} \frac{\alpha_i(r)}{\alpha_1(r)}.
\end{equation}

Since the Jacobian $g_r(x) = |\text{Det } Dh_r|$ is $(\beta + \epsilon)$-Hölder and $F$ is piecewise expanding, one can use the same argument as in the proof of Theorem 5.1 to conclude that

\begin{equation}
|g_r W|_{B^p_{\beta, \sigma}(W, P_w, A_w^*)} \leq C_{DRP}(r) \left( \frac{|Q|}{h_r^{-1}(Q)} \right)^{1/p-s+\epsilon} |W|^{1/p-\beta},
\end{equation}

provided $W \in h_r^{-1}Q$, with $Q \subset I^n_r$, where

\begin{equation}
C_{DRP}(r) = C_{18} \left( \Pi_i \frac{1}{\alpha_i(r)} \right)^{1-(1/p-s+\epsilon)}.
\end{equation}
and $C_{18}$ does not depend on $r \in \Lambda_n$ and $n$. Here we may need to refine the partition \( \{ I^n_r \}_{r \in \Lambda_n} \) again. Finally we obtain

\[
\Theta_r = C^*_{DGC} (r) C_{DGP} (r) C_{DGD}^{1/p} (1/2)^{D_{\alpha,r} (1 - \gamma_{D_{RS\epsilon}})} \\
\leq C_{75} \left( \Pi_{i \neq 1} \frac{\alpha_1 (r)}{\alpha_i (r)} \right)^{r - s} \left( \Pi_i \frac{1}{\alpha_i (r)} \right)^{1 - \left( 1/p - s + \epsilon \right)} \frac{1}{\alpha_1 (r)^{D_{\epsilon} (1 - \gamma_{D_{RS\epsilon}})}} \\
\leq C_{75} \frac{1}{\alpha_1 (r)^{D (s - \epsilon)}} \left( \Pi_i \frac{1}{\alpha_i (r)} \right)^{1 - 1/p} \frac{1}{\alpha_1 (r)^{D_{\epsilon} (1 - \gamma_{D_{RS\epsilon}})}} \\
\leq C_{75} \left( \Pi_i \frac{1}{\alpha_i (r)} \right)^{1 - 1/p} \frac{1}{\alpha_1 (r)^{D (s - \epsilon)_{D_{RS\epsilon}}}}.
\]

(12.39)

In particular for $p = 1$ we have

\[
\Theta_r \leq C_{75} \alpha_1 (r)^{- D (s - \epsilon)_{D_{RS\epsilon}}} \leq \left( \inf \min_x |D_x F \cdot v| \right)^{- D_n (s - \epsilon)_{D_{RS\epsilon}}}.
\]

Let $t_n$ be such that

\[
M_n = \sup_{P \in P^k, k \geq t_n} \# \left\{ r \in \Lambda_n : I^n_r \cap P \neq \emptyset \right\} \leq B < \infty
\]

for every $n$. Due Corollary 3.1 we can increase $t_n$ such that every $I^n_r$, with $r \in \Lambda_n$, is a $(1 - \frac{1}{7}, C_{33}, t_n)$-strongly regular domain.

We can apply Theorem 12.4 in [2] with $p = 1$ and

\[
T \leq B \left( \inf \min_x |D_x F \cdot v| \right)^{- D_n (s - \epsilon)_{D_{RS\epsilon}}}
\]

and Corollary 13.1 therein we conclude that $\Phi^n$ has a $(C_{DSFR}, C_{DSES})$-essential slicing with

\[
C_{DSES} \leq C_{76} \left( \inf \min_x |D_x F \cdot v| \right)^{- D_n (s - \epsilon)_{D_{RS\epsilon}}}
\]

so it has a similar upper bound for its essential spectral radius bounded and consequently (since $\epsilon$ can be taken arbitrarily small)

\[
\sigma_{ess} (\Phi^n, B^n_{1,q}) \leq \left( \inf \min_x |D_x F \cdot v| \right)^{- D_s}.
\]

Moreover by Theorem 14.1 in [2] shows that the operator $\Phi^n$ satisfies the Lasota-Yorke inequality for the pair $(L^1, B^n_{1,q})$ and Theorem 15.1 proves that the support of the invariant measure is an open set up to a set of zero Lebesgue measure. \( \square \)

**Remark 12.2.** It is easy to construct examples of piecewise smooth metric-expanding maps for which that transfer operator is quasi-compact. For instance, consider the square $[0,1]^2$ with the dyadic good grid $\mathcal{D}$. Choose $k_0$, $\{ I_r \}_r = \mathcal{D}^{k_0}$ and consider

\[
F: \cup_r I_r \to [0,1]^2
\]

such that $F: I_r \to Q_r$ is an affine bijection and $Q_r$ is choose to be one of the rectangles in the l.h.s. of Figure 2 (the largest square in the picture is $[0,1]^2$). If $k_0$ is large enough then the "Winky Face" map $F$ is piecewise metric-expanding map that satisfies the conclusions of Theorem 12.1.

Nakano and Sakamoto [26] proved the quasi-compactness of the transfer operator of smooth expanding maps on manifolds with no discontinuities and they gave an estimate to the essential spectral radius. The estimate in (12.35) is quite similar to
their estimate in that case. One of main features of our methods it that it allows us to give a very good description of the support of the invariant measure. This is quite rare in high-dimensional settings, except if some additional transitivity assumption holds.

One may wonder if the results for all piecewise expanding affine maps by Buzzi [12] in the plane and Tsujii [37] (in $\mathbb{R}^D$), as well results for all piecewise expanding real analytic maps by Buzzi [11] and Tsujii [36] can be generalised for Besov spaces.

Consider solenoidal attractors as studied in Tsujii [38] and Avila, Gouézel and Tsujii [3]. This is an interesting case of study since these maps are measure-expanding but not metric-expanding maps.

Avila, Gouézel and Tsujii [3] proved that for generic solenoidal attractors the support of the absolutely continuous invariant measure of a generic solenoidal attractor has non empty interior and Bamón, Kiwi, Rivera-Letelier and Urzúa [6] proved that on certain conditions the support is an open set. It is an interesting question if one can use atomic decomposition methods to study transfer operators in this setting.

V. COMPARING $B^s_{p,q}$ WITH OTHER FUNCTION SPACES IN LITERATURE.

Bellow we show to the reader that the class of observables for which we obtain results for the quasi-compactness of the transfer operator and also good statistical properties is quite wide, and often include previous function spaces that appears in the literature on transfers operators. Note that we do not claim that functions of Besov spaces on measure spaces with good grid always have good statistical properties for all the dynamical systems the cited authors took under consideration.

13. Keller’s spaces

The most influential result in the study of transfer operators for potentials with low regularity ($p$-bounded variation potentials) in one-dimensional dynamics was obtained by Keller [21]. Following Keller’s notation, given an interval $I$ and $y \in I$ define $S_\epsilon(y) = \{x \in I : |x - y| < \epsilon\}$, and

$$osc(h, \epsilon, y) = \text{ess sup}\{|h(x_1) - h(x_2)|, (x_1, x_2) \in S_\epsilon(y) \times S_\epsilon(y)\},$$

where the essential sup is taken with respect to the Lebesgue measure on $I \times I$. Define

$$OSC_1(h, \epsilon) = \int osc(h, \epsilon, y) \, dm(y),$$

and define

$$var_{1,1/p}(h) = \sup_{0<\epsilon\leq 1} \frac{OSC_1(h, \epsilon)}{\epsilon^{1/p}}.$$

The Keller’s space of functions of generalized bounded variation $BV_{1,1/p}$ is the space of functions in $L^1(m)$ with the norm

$$||f||_{BV_{1,1/p}} = ||f||_{L^1(m)} + var_{1,1/p}(f).$$

Keller considered a piecewise expanding map of the interval $I$ with a finite partition and such that $1/f'$ has $p$-bounded variation. He proved the such transfer operator
acts as a quasi-compact operator on $BV_{p,1/p}$. The relation with our Banach spaces is given by

**Proposition 13.1.** Let $\mathcal{D}$ be the dyadic partition of the interval $I = [0,1]$ and $0 < s < 1$. We have $BV_{1,s} \subset B_{1,\infty}^s(\mathcal{D})$. Moreover the corresponding inclusion is continuous.

**Proof.** Indeed, let $f \in BV_{1,s}$. If $x_Q \in Q$, where $Q$ is an interval, then

$$osc_1(f, Q) = \inf_c \int_Q |f(x) - c| \, dm(x) \leq \frac{1}{|Q|} \int_Q \int_Q |f(x) - f(y)| \, dm(x) \, dm(y)$$

$$\leq \frac{1}{|Q|} \int_Q \int_Q osc(f, |Q|, x) \, dm(x) \, dm(y) = \int_Q osc(f, |Q|, x) \, dm(x)$$

so we have

$$\sum_{Q \in \mathcal{D}} |Q|^{-s} osc_1(f, Q) \leq 2^{is} \sum_{Q \in \mathcal{D}} \int_Q osc(f, |Q|, x) \, dm(x)$$

$$\leq 2^{is} OSC_1(f, 2^{-i})$$

$$\leq 2^s ||f||_{BV_{1,s}}.$$

We complete the proof applying Theorem 15.1. $\square$

Note that $BV_{1,s} \neq B_{1,\infty}^s(\mathcal{D})$ since we know (Keller [21]) that $BV_{1,s} \subset L^\infty(m)$ and there are unbounded functions in the Besov space $B_{1,\infty}^s(\mathcal{D})$. Saussol [31] used Keller’s approach to study transfer operator in higher dimensions.

**14. Liverani’s spaces**

Liverani [24] presented a new approach to deal with one-dimensional transfer operators with low regularity. His methods were extended successfully to some higher dimensional settings. For every $s \in (0,1)$ he consider the Banach space $B_s$ of all complex-valued borelian functions in $L^{1/s}(I,m)$, with $I = [-\pi, \pi]$, for which the norm

$$(14.40) \quad ||f||_s = \sup \{| \int g'f \, dm : g \in C^1, |g|_{C^s(I)} \leq 1 \}$$

is finite. Liverani consider certain classes of piecewise Holder continuous potentials and piecewise expanding maps on $I$ with an infinite partition and gave estimate to the essential spectral radius of the associated transfer operator acting on $B_s$.

**Proposition 14.1.** Let $\mathcal{D}$ be the good grid of dyadic partitions of the interval $I = [0,1]$ and $0 < s < 1$. We have $B_{1,1}^s \subset B_{1-s} \subset B_{1,\infty}^s$. Moreover these inclusions are continuous.

**Proof.** Let $f \in B_{1,1}^s(\mathcal{D})$ and

$$f = \sum_k \sum_{Q \in p_k} c_Q a_Q$$
be a $B^1_{1,1}(D)$-representation of $f$. Let $g \in C^1, |g|_{C^{1-s}(I)} \leq 1$. Then
\[
\left| \int g' f \, dm \right| = \left| \sum_k \sum_{Q \in \mathcal{P}^k} c_Q \int g' a_Q \, dm \right|
\leq \sum_k \sum_{Q \in \mathcal{P}^k} |c_Q| |Q|^{s-1} \left| \int_Q g' \, dm \right|
\leq \sum_k \sum_{Q \in \mathcal{P}^k} |c_Q|.
\] (14.41)

So $|f|_{B_{1-s}} \leq |f|_{B_{1,1}}$. Note that for the dyadic partition $D$ and Lebesgue measure $m$, the Haar basis indexed by $\hat{\mathcal{H}}(I) = \{I\} \cup_{Q \subset I} \mathcal{H}_Q$ constructed in Section 14 in [33] is just the classical Haar basis, since for every $Q \in \mathcal{D}^k$ there are exactly two intervals $Q_1, Q_2 \in \mathcal{D}^{k+1}$ such that $Q \subset Q_i$ satisfying $\mathcal{H}_Q = \{S^Q\}$, where $S = (S_1^Q, S_2^Q)$, with $S_i^Q = \{Q_i\}$. Then
\[
\phi_{SQ} = \frac{1}{|Q|^{1/2}} \left(1_{Q_1} - 1_{Q_2}\right).
\]
Let $f \in B_{1-s}$. Then $f \in L^{1/(1-s)}(m)$, so
\[
f = \sum_{S \in \hat{\mathcal{H}}(I)} d_S \phi_S,
\]
where
\[
d_S = \int f \phi_S \, dm.
\]
if $d_S \geq 0$ set $\hat{\phi}_S = \phi_S$, otherwise let $\hat{\phi}_S = -\phi_S$. Define
\[
\psi_k = \sum_{Q \in \mathcal{D}^k} \hat{\phi}_{SQ}.
\]
Then
\[
\int f \psi_k \, dm = \sum_{Q \in \mathcal{D}^k} |d_{SQ}|.
\]
Note that $\psi_k$ is the derivative of the Lipschitz function
\[
\Psi_k(x) = \int_0^x \psi_k \, dm,
\]
and satisfies $|\psi_k|_\infty \leq 2^{-k/2}$. Let $x, y \in I$, with $x \leq y$ and
\[
[a, b] = \bigcup_{Q \in \mathcal{D}^k} \{Q\}
\]
Since
\[
\int_Q \psi_k \, dm = \int_Q \hat{\phi}_{SQ} \, dm = 0
\]
and $|Q| = 2^{-k}$ for every $Q \in \mathcal{D}^k$ we have that
\[
|\Psi_k(y) - \Psi_k(x)| \leq |\Psi_k(y) - \Psi_k(b)| + |\Psi_k(a) - \Psi_k(x)|
\leq 2^{k/2}(|y - b| + |z - x|)^s(|y - b| + |z - x|)^{1-s}
\leq 2^{k/2+s-k} |y - x|^{1-s},
\]
The last inequality follows from $\max\{|y - b|, |z - x|\} \leq 2^{-k}$. So

$$|\psi_k|_{C^1(I)} \leq C \, 2^{(1/2-s)k}$$

and by Liverani [24]

$$\sum_{Q \in D^k} |dS_Q| = \int f \psi_k \, dm \leq C 2^{(1/2-s)k} |f|_{B_{1-s}}.$$ 

So

$$\sum_{Q \in D^k} |dS_Q||Q|^{1/2-s} \leq C |f|_{B_{1-s}}.$$

So by Theorem 15.1 in [33] we have that $f \in B^{s}_{1,\infty}$.

15. Thomine’s result for Sobolev spaces

Thomine [35] studied the action of the transfer operator of certain piecewise expanding maps on $\mathbb{R}^n$ on the classical Sobolev spaces

$$\mathcal{H}_p^s(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}((1 + |\eta|^2)^{s/2} |\mathcal{F}u|)(\eta) \in L^p(\mathbb{R}^n) \}.$$ 

Here $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$. It is well known that $\mathcal{H}_p^s(\mathbb{R})$ coincides with the Besov space $B^{s}_{p,p}(\mathbb{R})$, whose restriction to the interval $I = [0, 1]$ is exactly $B^s_{p,p}(\mathcal{D})$, where $\mathcal{D}$ is the sequence of dyadic partitions of $[0, 1]$.

16. Butterley’s spaces

Let $I$ be an interval. Butterley [10] proved the quasi-compactness of certain transfer operators with piecewise Hölder potentials acting on a Banach space $\mathcal{B}_s \subset L^1(m)$, where $m$ is the Lebesgue measure on $I$, and

$$||f||_{\mathcal{B}_s} := \inf \left\{ \sup_{0 < k \leq 1} \left( k^{-s} |f_k|_{L^1(m)} + k^{1-s} ||f_k||_{BV(I)} \right) \right\}$$

where the infimum runs over all possible families $\{f_k\}_{0 < k \leq 1} \subset L^1(m)$.

**Proposition 16.1.** We have $\mathcal{B}_s \subset B^{s}_{1,\infty}(\mathcal{D})$. Moreover the corresponding inclusion is continuous.

**Proof.** Indeed, let $f \in \mathcal{B}_s$. So choosing $k = 2^{-i}$ we can find a family $\{g_i\}_{i \in \mathbb{N}} \subset L^1(m)$ such that

$$|g_i - f|_{L^1(I)} \leq 2^{-i(1-s+1)} ||f||_{\mathcal{B}_s} \text{ and } ||g_i||_{BV(I)} \leq 2^{i(1-s+1)} ||f||_{\mathcal{B}_s}.$$ 

for every $i \in \mathbb{N}$. So

$$\sum_{Q \in D^i} |Q|^{-s} \text{osc}_1(f, Q) \leq 2 ||f||_{\mathcal{B}_s} + 2^{is} \sum_{Q \in D^i} \text{osc}_1(g_i, Q)$$

$$\leq 2 ||f||_{\mathcal{B}_s} + 2^{(s-1)} ||g_i||_{BV(I)}$$

$$\leq 4 ||f||_{\mathcal{B}_s}.$$ 

We complete the proof applying Theorem 15.1 in [33]. □
References

[1] K. Adl-Zarabi. Absolutely continuous invariant measures for piecewise expanding $C^2$ transformations in $\mathbb{R}^n$ on domains with cusps on the boundaries. Ergodic Theory Dynam. Systems, 16(1):1–18, 1996.

[2] A. Arbieto and D. Smania. Transfer operators and atomic decomposition, 2019.

[3] A. Avila, S. Gouëzel, and M. Tsujii. Smoothness of solenoidal attractors. Discrete Contin. Dyn. Syst., 15(1):21–35, 2006.

[4] V. Baladi and M. Holschneider. Approximation of nonessential spectrum of transfer operators. Nonlinearity, 12(3):525–538, 1999.

[5] R. Bamón, J. Kiwi, J. Rivera-Letelier, and R. Urzúa. On the topology of solenoidal attractors of the cylinder. Ann. Inst. H. Poincaré Anal. Non Linéaire, 23(2):209–236, 2006.

[6] R. Bamón, J. Kiwi, J. Rivera-Letelier, and R. Urzúa. On the topology of solenoidal attractors of the cylinder. Ann. Inst. H. Poincaré Anal. Non Linéaire, 23(2):209–236, 1996.

[7] R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, Vol. 470. Springer-Verlag, Berlin-New York, 1975.

[8] H. Bruin, G. Keller, T. Nowicki, and S. van Strien. Wild Cantor attractors exist. Ann. of Math. (2), 143(1):97–130, 1996.

[9] H. Bruin, G. Keller, and M. St. Pierre. Adding machines and wild attractors. Ergodic Theory Dynam. Systems, 17(6):1267–1287, 1997.

[10] O. Butterley. An alternative approach to generalised BV and the application to expanding interval maps. Discrete Contin. Dyn. Syst., 33(8):3355–3363, 2013.

[11] J. Buzzi. Absolutely continuous invariant probability measures for arbitrary expanding piecewise $\mathbb{R}$-analytic mappings of the plane. Ergodic Theory Dynam. Systems, 20(3):697–708, 2000.

[12] J. Buzzi. Thermodynamical formalism for piecewise invertible maps: absolutely continuous invariant measures as equilibrium states. In Smooth ergodic theory and its applications (Seattle, WA, 1999), volume 69 of Proc. Sympos. Pure Math., pages 749–783. Amer. Math. Soc., Providence, RI, 2001.

[13] J.-R. Chazottes, P. Collet, and B. Schmitt. Statistical consequences of the Devroye inequality for processes. Applications to a class of non-uniformly hyperbolic dynamical systems. Nonlinearity, 18(5):2341–2364, 2005.

[14] M. Christ. $A(T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math., 60/61(2):601–628, 1990.

[15] W. J. Cowieson. Stochastic stability for piecewise expanding maps in $\mathbb{R}^d$. Nonlinearity, 13(5):1745–1760, 2000.

[16] W. J. Cowieson. Absolutely continuous invariant measures for most piecewise smooth expanding maps. Ergodic Theory Dynam. Systems, 22(4):1061–1078, 2002.

[17] P. Góra and A. Boyarsky. Absolutely continuous invariant measures for piecewise expanding $C^2$ transformation in $\mathbb{R}^N$. Israel J. Math., 67(3):272–286, 1989.

[18] Y. Han, S. Lu, and D. Yang. Inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. Approx. Theory Appl. (N.S.), 15(3):37–65, 1999.

[19] F. Hofbauer and G. Keller. Equilibrium states for piecewise monotonic transformations. Ergodic Theory Dynam. Systems, 2(1):23–43, 1982.

[20] F. Hofbauer and G. Keller. Ergodic properties of invariant measures for piecewise monotonic transformations. Math. Z., 180(1):119–140, 1982.

[21] G. Keller. Generalized bounded variation and applications to piecewise monotonic transformations. Z. Wahrsch. Verw. Gebiete, 69(3):461–478, 1985.

[22] G. Keller and T. Nowicki. Fibonacci maps re(al)visited. Ergodic Theory Dynam. Systems, 15(1):99–120, 1995.

[23] A. Lasota and J. A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. Trans. Amer. Math. Soc., 186:481–488 (1974), 1973.

[24] C. Liverani. A footnote on expanding maps. Discrete Contin. Dyn. Syst., 33(8):3741–3751, 2013.

[25] C. G. Moreira and D. Smania. Metric stability for random walks (with applications in renormalization theory). In Frontiers in complex dynamics, volume 51 of Princeton Math. Ser., pages 261–322. Princeton Univ. Press, Princeton, NJ, 2014.
[26] Y. Nakano and S. Sakamoto. Spectra of expanding maps on besov spaces. *Discrete and Continuous Dynamical Systems*, 39(4):1779–1797, 2019.
[27] W. Parry and M. Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, (187-188):268, 1990.
[28] F. Przytycki and M. Urbański. *Conformal fractals: ergodic theory methods*, volume 371 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2010.
[29] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. *Comm. Math. Phys.*, 9:267–278, 1968.
[30] D. Ruelle. The thermodynamic formalism for expanding maps. *Comm. Math. Phys.*, 125(2):239–262, 1989.
[31] B. Saussol. Absolutely continuous invariant measures for multidimensional expanding maps. *Israel J. Math.*, 116:223–248, 2000.
[32] J. G. Sinaǐ. Markov partitions and y-diffeomorphisms. *Funkcional. Anal. i Priložen*, 2(1):64–89, 1968.
[33] D. Smania. Besov-ish spaces through atomic decomposition, 2019.
[34] D. Smania. Classic and exotic Besov spaces induced by good grids, 2019.
[35] D. Thomine. A spectral gap for transfer operators of piecewise expanding maps. *Discrete Contin. Dyn. Syst.*, 30(3):917–944, 2011.
[36] M. Tsujii. Absolutely continuous invariant measures for piecewise real-analytic expanding maps on the plane. *Comm. Math. Phys.*, 208(3):605–622, 2000.
[37] M. Tsujii. Absolutely continuous invariant measures for expanding piecewise linear maps. *Invent. Math.*, 143(2):349–373, 2001.
[38] M. Tsujii. Fat solenoidal attractors. *Nonlinearity*, 14(5):1011–1027, 2001.

Departamento de Matemática, Instituto de Ciências Matemáticas e da Computação-Universidade de São Paulo (ICMC/USP), Caixa Postal 668, São Carlos-SP, Brazil.

Email address: smania@icmc.usp.br
URL: http://conteudo.icmc.usp.br/pessoas/smania/