Ramanujan’s Harmonic Number Expansion

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Abstract
An algebraic transformation of the DeTemple-Wang half-integer approximation to the harmonic series produces the general formula and error estimate for the Ramanujan expansion for the nth harmonic number.

1 Introduction
Entry 9 of Chapter 38 of B. Berndt’s edition of Ramanujan’s Notebooks, Volume 5 [1, p. 521] reads:

“Let \( m := \frac{n(n+1)}{2} \), where \( n \) is a positive integer. Then, as \( n \) approaches infinity,

\[
\sum_{k=1}^{n} \frac{1}{k} \sim \frac{1}{2} \ln(2m) + \gamma + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \frac{1}{1680m^4} + \frac{1}{2310m^5}
\]

\[- \frac{191}{360360m^6} + \frac{29}{30030m^7} - \frac{2833}{1166880m^8} + \frac{140051}{17459442m^9} - \cdots.\]

Berndt’s proof simply verifies (as he himself explicitly notes) that Ramanujan’s expansion coincides with the standard Euler expansion

\[
H_n := \sum_{k=1}^{n} \frac{1}{k} \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots
\]

\[= \ln n + \gamma - \sum_{k=1}^{\infty} \frac{B_k}{n^k}\]

where \( B_k \) denotes the \( k^{\text{th}} \) Bernoulli number and \( \gamma := 0.57721 \cdots \) is Euler’s constant.
However, Berndt does not give the general formula for the coefficient of $\frac{1}{m^r}$ in Ramanujan’s expansion, nor does he prove that it is an asymptotic series in the sense that the error in the value obtained by stopping at any particular stage in Ramanujan’s series is less than the next term in the series. Indeed we have been unable to find any error analysis of Ramanujan’s series.

Although it is true that the asymptotics of the harmonic numbers were already determined by the Euler expansion, later mathematicians have offered alternative approximative formulas (see [2, 3, 4, 5, 6, 8]). Ramanujan’s formula is one of the most accurate (see [8]).

However, the Euler expansion apparently does not easily lend itself to an error analysis of Ramanujan’s expansion, nor to a general formula for the coefficient of $\frac{1}{m^r}$.

In an earlier paper (see [7]) we proved that the first five terms of the Ramanujan expansion indeed form an asymptotic expansion in the sense above, but we did not prove the general case.

The general case is the subject of the following:

Theorem 1. For any integer $p \geq 1$ define:

$$R_p := \frac{(-1)^{p-1}}{2p \cdot 8^p} \left\{ 1 + \sum_{k=1}^{p} \binom{p}{k} (-4)^k B_{2k} \left( \frac{1}{2} \right) \right\},$$

where $B_{2k}(x)$ is the Bernoulli polynomial of order $2k$. Put

$$m := \frac{n(n+1)}{2}$$

where $n$ is a positive integer. Then, for every integer $r \geq 1$, there exists a $\Theta_r$, $0 < \Theta_r < 1$, for which the following equation is true:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{1}{2} \ln(2m) + \gamma + \sum_{p=1}^{r} \frac{R_p}{m^p} + \Theta_r \cdot \frac{R_{r+1}}{m^{r+1}}$$

We observe that the formula for $R_p$ can be written symbolically as follows:

$$R_p = -\frac{1}{2p} \left( \frac{4B^2 - 1}{8} \right)^p$$

where we write $B_{2m} \left( \frac{1}{2} \right)$ in place of $B^{2m}$ after carrying out the above expansion.

2 Proof of Ramanujan’s Expansion

Proof. We begin with the half-integer approximation to $H_n$ due to DeTemple and Wang (see [4]): For any positive integer $r$ there exists a $\theta_r$, $0 < \theta_r < 1$, for which the following
The equation is true:

\[ H_n = \ln \left( n + \frac{1}{2} \right) + \gamma + \sum_{p=1}^{r} \frac{D_p}{(n + \frac{1}{2})^{2p}} + \theta_r \cdot \frac{D_{r+1}}{(n + \frac{1}{2})^{2r+2}} \]  \hspace{1cm} (5)

where

\[ D_p := -\frac{B_{2p} (\frac{1}{2})}{2p}. \]  \hspace{1cm} (6)

Since

\[ \left( n + \frac{1}{2} \right)^2 = 2m + \frac{1}{4} \]

we obtain:

\[ \sum_{p=1}^{r} \frac{D_p}{(n + \frac{1}{2})^{2p}} = \sum_{p=1}^{r} \frac{D_p}{(2m)^p (1 + \frac{1}{8m})^p} = \sum_{p=1}^{r} \frac{D_p}{(2m)^p} \left( 1 + \frac{1}{8m} \right)^{-p} = \sum_{p=1}^{r} \frac{D_p}{(2m)^p} \sum_{k=0}^{\infty} \binom{-p}{k} \frac{1}{8^k m^k} = \sum_{p=1}^{r} \frac{D_p}{2^p} \sum_{k=0}^{\infty} (-1)^k \binom{k+p-1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{p+k}} = \sum_{p=1}^{r} \left\{ \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \binom{p-1}{p-s} \frac{1}{8^{p-s}} \right\} \cdot \frac{1}{m^p} + E_r \]

where

\[ E_r := \frac{D_1}{2} \sum_{k=r}^{\infty} (-1)^k \binom{k}{k} \frac{1}{8^k} \cdot \frac{1}{m^{1+k}} + \frac{D_2}{2^2} \sum_{k=r-1}^{\infty} (-1)^k \binom{k+1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{2+k}} + \cdots + \frac{D_r}{2^r} \sum_{k=1}^{\infty} (-1)^k \binom{k+r-1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{r+k}} \]

Substituting the right hand side of the last equation into the right hand side of (5) we obtain:

\[ H_n = \ln \left( n + \frac{1}{2} \right) + \gamma + \sum_{p=1}^{r} \left\{ \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \binom{p-1}{p-s} \frac{1}{8^{p-s}} \right\} \cdot \frac{1}{m^p} + E_r + \theta_r \cdot \frac{D_{r+1}}{(n + \frac{1}{2})^{2r+2}}. \]  \hspace{1cm} (7)
Moreover,

\[
\ln \left( n + \frac{1}{2} \right) = \frac{\ln \left( n + \frac{1}{2} \right)^2}{2} \\
= \frac{1}{2} \ln \left( 2n + 1 \right) \\
= \frac{1}{2} \ln(2m) + \frac{1}{2} \ln \left( 1 + \frac{1}{8m} \right) \\
= \frac{1}{2} \ln(2m) + \frac{1}{2} \sum_{l=1}^{\infty} (-1)^{l-1} \frac{1}{l 8^l m^l} \\
= \frac{1}{2} \ln(2m) + \frac{1}{2} \sum_{l=1}^{r} (-1)^{l-1} \frac{1}{l 8^l m^l} + \epsilon_r
\]

where

\[
\epsilon_r := \sum_{l=r+1}^{\infty} (-1)^{l-1} \frac{1}{2l 8^l m^l}.
\]

Substituting the right-hand side of this last equation into (7) we obtain

\[
H_n = \frac{1}{2} \ln(2m) + \frac{1}{2} \sum_{l=1}^{r} (-1)^{l-1} \frac{1}{l 8^l m^l} + \gamma + \sum_{p=1}^{r} \left\{ \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \frac{1}{p-1} \right\} \cdot \frac{1}{m^p} \\
+ \epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}} \\
= \frac{1}{2} \ln(2m) + \gamma + \sum_{p=1}^{r} \left\{ (-1)^{p-1} \frac{1}{2p 8^p} + \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \frac{1}{p-1} \right\} \cdot \frac{1}{m^p} \\
+ \epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n+\frac{1}{2})^{2r+2}}
\]

Therefore, we have obtained Ramanujan's expansion into powers of \( \frac{1}{m} \), and the coefficient of \( \frac{1}{m^p} \) is

\[
R_p = \left\{ (-1)^{p-1} \frac{1}{2p 8^p} + \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \frac{1}{p-1} \right\} \cdot \frac{1}{8^{p-s}}
\]  
(8)
But,
\[
\frac{D_s}{2^s} (-1)^{p-s} \left( \frac{p-1}{p-s} \right) \frac{1}{8^{p-s}} = -\frac{B_{2s} \left( \frac{1}{2} \right)}{2^s} (-1)^{p-s} \left( \frac{p-1}{p-s} \right) \frac{1}{8^{p-s}}
\]
\[
= (-1)^{p-s-1}\frac{B_{2s} \left( \frac{1}{2} \right)}{2^{2s}} \left( \frac{p-1}{p-s} \right) \frac{1}{8^{p-s}}
\]

and therefore
\[
R_p = (-1)^{p-1} \frac{1}{2p8^p} + \sum_{s=0}^{p-1} \frac{D_s}{2^s} (-1)^{p-s} \left( \frac{p-1}{p-s} \right) \frac{1}{8^{p-s}}
\]
\[
= (-1)^{p-1} \frac{1}{2p8^p} + \sum_{s=0}^{p-1} (-1)^{p-s-1}\frac{B_{2s} \left( \frac{1}{2} \right)}{2^{2s}} \left( \frac{p-1}{p-s} \right) \frac{1}{8^{p-s}}
\]
\[
= (-1)^{p-1} \left\{ \frac{1}{2p8^p} + \sum_{s=1}^{p} (-1)^r \frac{B_{2s} \left( \frac{1}{2} \right)}{2^{2s}} \left( \frac{p-1}{p-s} \right) \frac{1}{8^{p-s}} \right\}
\]
\[
= (-1)^{p-1} \left\{ \frac{1}{2p8^p} + \sum_{s=1}^{p} (-1)^r \frac{B_{2s} \left( \frac{1}{2} \right)}{2 \cdot 2^s} \frac{1}{p} \frac{1}{8^{p-s}} \right\}
\]
\[
= (-1)^{p-1} \left\{ 1 + \sum_{s=1}^{p} \left( \frac{p}{s} \right) (-4)^s B_{2s} \left( \frac{1}{2} \right) \right\}
\]

Thus, the formula for \( H_n \) takes the form:
\[
H_n = \frac{1}{2} \ln(2m) + \gamma + \sum_{p=1}^{\infty} \frac{(-1)^{p-1}}{2p8^p} \left\{ 1 + \sum_{s=1}^{p} \left( \frac{p}{s} \right) (-4)^s B_{2s} \left( \frac{1}{2} \right) \right\} \cdot \frac{1}{m^p}
\]
(9)
\[
+ \epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n + \frac{1}{2})^{2r+2}}
\]
(10)

We see that (9) is the Ramanujan expansion with the general formula for the coefficient, \( R_p \), of \( \frac{1}{m^p} \), as given in the statement of the theorem, while (10) is (an undeveloped form of) the error term.

We will now estimate the error.

To do so we will use the fact that the sum of a convergent alternating series, whose terms (taken with positive sign) decrease monotonically to zero, is equal to any partial sum plus a positive proper fraction of the first neglected term (with sign).

Thus,
\[
\epsilon_r := \sum_{l=r+1}^{\infty} (-1)^{l-1} \frac{1}{2l8^l m^l} = \alpha_r (-1)^r \frac{1}{2(r+1)8^{(r+1)} m^{r+1}}
\]
where 0 < \alpha_r < 1.

Moreover,

\[
E_r = \frac{D_1}{2^1} \sum_{k=r}^{\infty} (-1)^k \binom{k}{k} \frac{1}{8^k} \cdot \frac{1}{m^{1+k}} + \frac{D_2}{2^2} \sum_{k=r-1}^{\infty} (-1)^k \binom{k+1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{2+k}} + \cdots \\
+ \frac{D_r}{2^r} \sum_{k=1}^{\infty} (-1)^k \binom{k+r-1}{k} \frac{1}{8^k} \cdot \frac{1}{m^{r+k}}
\]

\[
= \left\{ \delta_1 \frac{D_1}{2^1} (-1)^r \frac{r}{8^r} + \delta_2 \frac{D_1}{2^2} (-1)^{r-1} \frac{r}{8^{r-1}} + \cdots + \delta_r \frac{D_r}{2^r} (-1)^1 \frac{r}{1} \right\} \frac{1}{m^{r+1}}
\]

where 0 < \delta_k < 1 \text{ for } k = 1, 2, \cdots, r \text{ and } 0 < \Delta_r < 1.

Finally

\[
\theta_r \cdot \frac{D_{r+1}}{(n + \frac{1}{2})^{2r+2}} = \theta_r \cdot \frac{D_{r+1}}{(2m)^{r+1} (1 + \frac{1}{8m})^{r+1}} = \delta_{r+1} \cdot \frac{D_{r+1}}{2^{r+1}} \cdot \frac{1}{m^{r+1}}
\]

where 0 < \delta_{r+1} < 1.

Thus, the total error is equal to

\[
\epsilon_r + E_r + \theta_r \cdot \frac{D_{r+1}}{(n + \frac{1}{2})^{2r+2}} = \Theta_r \cdot \left\{ (-1)^r \frac{1}{2(r+1)8^{r+1}} + \sum_{q=1}^{r+1} \frac{D_{2q}}{2^q} (-1)^{r-q+1} \frac{r}{8^{r-q+1}} \right\} \frac{1}{m^{r+1}}
\]

\[
= \Theta_r \cdot R_{r+1}
\]

by (8), where 0 < \Theta_r < 1, which is of the form as claimed in the theorem. This completes the proof.

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