POWERS OF DEHN TWISTS GENERATING RIGHT-ANGLED ARTIN GROUPS

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Abstract. We give a bound for the exponents of powers of Dehn twists to generate a right-angled Artin group. Precisely, if $F$ is a finite collection of pairwise distinct simple closed curves on a finite type surface and if $N$ denotes the maximum of the intersection numbers of all pairs of curves in $F$, then we prove that $\{T_\gamma^n \mid \gamma \in F\}$ generates a right-angled Artin group for all $n \geq N^2 + N + 3$. This extends a previous result of Koberda, who proved the existence of a bound possibly depending on the underlying hyperbolic structure of the surface. In the course of the proof, we obtain a universal bound depending only on the topological type of the surface in certain cases, which partially answers a question due to Koberda.

1. Introduction

1.1. Background and Main Theorem. Throughout this paper, we fix an orientable surface $\Sigma$ of genus $n_g$ with $n_p$ punctures. An essential simple closed curve is a circle embedded in $\Sigma$, which is not homotopic to a point and a puncture. We will always assume that $3n_g + n_p - 3 > 0$, so that $\Sigma$ admits a hyperbolic structure and contains at least one essential simple closed curve. The (geometric) intersection number between two essential simple closed curves $\alpha$ and $\beta$ is the infimum of $|\alpha' \cap \beta'|$ among all simple closed curves $\alpha'$ and $\beta'$ homotopic to $\alpha$ and $\beta$, respectively.

If $\text{Homeo}^+(\Sigma)$ is the group of orientation-preserving self-homeomorphisms fixing punctures pointwise and $\text{Homeo}_0(\Sigma)$ is the group of self-homeomorphisms homotopic to the identity, then the mapping class group of $\Sigma$ is defined as

$$\text{Mod}(\Sigma) = \text{Homeo}^+(\Sigma) / \text{Homeo}_0(\Sigma).$$

For an essential simple closed curve $\gamma$, a (right-handed) Dehn twist along $\gamma$ is a self-homeomorphism which is constructed by twisting a regular neighborhood of $\gamma$ once. A more precise definition, especially the meaning of the term right-handed, is given in Remark 3.4; see also Farb–Margalit’s book [FM12]. We write $T_\gamma$ for the mapping class of a (right-handed) Dehn twist along $\gamma$.

On the other hand, if $\Gamma = (V(\Gamma), E(\Gamma))$ is a simplicial graph, the right-angled Artin group of $\Gamma$ is the group with the presentation

$$A(\Gamma) = \langle V(\Gamma) \mid [a, b] = 1 \text{ if } \{a, b\} \in E(\Gamma) \rangle.$$

Let $\text{lcm}\{a, b\}$ be the least common multiple of integers $a$ and $b$. We set $\text{lcm}\{a, 0\} = 0$ for all integer $a$. If $F$ is a set of essential simple closed curves, we write $\Delta(F) := \max\{\text{lcm}\{i(\alpha, \beta), i(\beta, \gamma)\} \mid \alpha, \beta, \gamma \in F\}$. Note that $\Delta(F) \geq \max_{\alpha, \beta \in F} i(\alpha, \beta)$. A multicurve is a set of pairwise disjoint essential simple closed curves. Then our main theorem is the following.

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Theorem 1.1 (Main theorem). Let $\mathcal{F}$ be a nonempty finite set of essential simple closed curves on $\Sigma$ which are not homotopic to each other. Let $\Gamma$ be the simplicial graph whose vertex set is $\mathcal{F}$ satisfying that two vertices are joined by an edge if and only if their intersection number is zero. If $\Delta(\mathcal{F}) \geq 2$ and $m$ is an integer satisfying that $|m| \geq \max_{\alpha, \beta, \gamma \in \mathcal{F}, i(\beta, \gamma) > 0} \frac{\Delta(\mathcal{F}) + 2 \cdot i(\alpha, \gamma) + 1}{i(\beta, \gamma)}$, then the surjective homomorphism $\varphi : A(\Gamma) \to \langle \{T_\gamma^m | \gamma \in \mathcal{F}\} \rangle$ sending each alphabet $\gamma$ to $T_\gamma^m$ is an isomorphism. Meanwhile, if $\Delta(\mathcal{F}) = 1$ and $|m| \geq 7$, then $\varphi$ is an isomorphism.

Remark 1.2. In Theorem 1.1, because two Dehn twists $T_\alpha^m$ and $T_\beta^m$ commute with each other if and only if $i(\alpha, \beta) = 0$, there is a surjective homomorphism $\varphi : A(\Gamma) \to \langle \{T_\gamma^m | \gamma \in \mathcal{F}\} \rangle$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
F(\mathcal{F}) & \xrightarrow{\varphi} & \langle \{T_\gamma^m | \gamma \in \mathcal{F}\} \rangle \\
A(\Gamma) \xrightarrow{\varphi^{-1}} & & \\
\end{array}
\]

where $F(\mathcal{F})$ is the free group generated by $\mathcal{F}$.

Corollary 1.3. Let $\mathcal{F}$ be a finite set of essential simple closed curves on $\Sigma$ which are not homotopic to each other. Assume that there is an integer $N \geq 2$ such that for every $\alpha, \beta \in \mathcal{F}$, either $i(\alpha, \beta) = 0$ or $i(\alpha, \beta) = N$. Then the group generated by $\{T_\gamma^6 | \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group.

Proof. Assume that $\mathcal{F}$ is not a multicurve. By the assumption, $\Delta(\mathcal{F}) = N$ and $\max_{\alpha, \beta, \gamma \in \mathcal{F}, i(\beta, \gamma) > 0} \frac{\Delta(\mathcal{F}) + 2 \cdot i(\alpha, \gamma) + 1}{i(\beta, \gamma)} + 2 \leq 5 + \frac{1}{N}$. By Theorem 1.1, the group generated by $\{T_\gamma^6 | \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group. $\square$

Corollary 1.3 is a partial answer to Koberda’s question [Kob12b, Question 1.16].

Corollary 1.4. Let $\mathcal{F}$ be a finite set of essential simple closed curves on $\Sigma$ which are not homotopic to each other. If there is an integer $N \geq 2$ such that $i(\alpha, \beta) \leq N$ for all $\alpha, \beta \in \mathcal{F}$, then the group generated by $\{T_\gamma^m | \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group for all integers $m \geq N^2 + N + 3$.

Proof. Assume that $\mathcal{F}$ is not a multicurve. Since $\Delta(\mathcal{F}) \leq N(N - 1)$, we have $\max_{\alpha, \beta, \gamma \in \mathcal{F}, i(\beta, \gamma) > 0} \frac{\Delta(\mathcal{F}) + 2 \cdot i(\alpha, \gamma) + 1}{i(\beta, \gamma)} + 2 \leq \frac{N(N-1)+2N+1}{1} + 2 = N^2 + N + 3$. Therefore, by Theorem 1.1, the group generated by $\{T_\gamma^m | \gamma \in \mathcal{F}\}$ is isomorphic to a right-angled Artin group whenever $m \geq N^2 + N + 3$. $\square$

For two Dehn twists, there is a complete classification for the isomorphism type of $\langle T_\alpha^m, T_\beta^n \rangle$. (cf. [FM12, Table in Section 3.5.2]) One can also verify from the table that the second powers of two Dehn twists generate a free abelian or free group of rank 2. Clay–Leininger–Mangahas [CLM12] showed that every right-angled Artin group can be quasi-isometrically embedded into some mapping class group of a surface of finite type. Kim–Koberda [KK16] proved that every embedding $A(\Gamma) \hookrightarrow \Mod(\Sigma)$ implies a graph embedding $\Gamma \hookrightarrow \mathcal{C}(\Sigma)$ into the curve graph of a surface $\Sigma$ if $\Sigma$ has low complexity.
Koberda [Kob12b] proved that sufficiently large powers of pseudo-Anosov mapping classes on subsurfaces and Dehn twists generate a right-angled Artin group. Kuno [Kun17] showed that handlebody groups contains right-angled Artin groups as subgroups. Funar [Fun14] proved that, if the set of simple closed curves is sparse, then the second powers of their Dehn twists generate a right-angled Artin group. Runnels [Run] found other bounds of exponents for powers of Dehn twists in a different setting. Crisp–Paris [CP01] solved a similar problem for Artin groups.

1.2. Hyperbolic Structure. Let us now fix a hyperbolic structure of $\Sigma$ given by a covering map

$$\xi : \mathbb{H}^2 \to \Sigma.$$  

Unless stated otherwise, all essential simple closed curves we consider are assumed to be geodesics. In particular, if we have two distinct essential simple closed geodesics $\alpha$ and $\beta$, we have

$$|\alpha \cap \beta| = i(\alpha, \beta).$$

1.3. The Proof of Theorem 1.1. Two geodesics $\tilde{\alpha}$ and $\tilde{\beta}$ on $\mathbb{H}^2$ are said to cross each other, written by $\tilde{\alpha} \Leftarrow \tilde{\beta}$, if they intersect each other transversally. The following is a technical definition to prove Theorem 1.1.

**Definition 1.5** (First definition of $PP_n$). Let $A$ be a multicurve on $\Sigma$. Let $\alpha$ be a simple closed geodesic in $A$. Let $\beta$ be a simple closed geodesic crossing $\alpha$. For an integer $n > 2$, a simple closed geodesic $\gamma$ is said to be contained in $PP_n(A, \alpha, \beta)$ if for some lift $\tilde{\gamma}$ of $\gamma$, for some lift $\tilde{\alpha}$ of $\alpha$ and for some (at least) $n$ lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\beta$, the following hold. (See Figure 1.)

(i) For each $i \in \{1, \ldots, n\}$, the lifts $\tilde{\alpha}$, $\tilde{\beta}_i$ and $\tilde{\gamma}$ cross each other.

(ii) If a lift $\tilde{\alpha}'$ of $A$ crosses both $\tilde{\gamma}$ and some $\tilde{\beta}_i$, then $\tilde{\alpha}' = \tilde{\alpha}$.

The general definition of $PP_n$ can be found in Definition 4.1. Roughly speaking, the indicator $n$ of $PP_n(A, \alpha, \beta)$ is a “combinatorial tangent” of a simple closed geodesic on the grid of $\alpha$ (vertical) and $\beta$ (horizontal). That is, if a simple closed geodesic $\gamma$ belongs to $PP_n(A, \alpha, \beta)$ with large $n$, it means that $\gamma$ follows $\alpha$ for a long time with respect to $A$ and $\beta$. It is a key notion for the proof of Theorem 1.1 to overcome difficulties. (Compare with [Kob12b, Section 5.7].)

**Theorem 1.6.** Let $A$ and $B$ be multicurves on $\Sigma$ satisfying that $i(\alpha, \beta) > 0$ for some $\alpha \in A$ and $\beta \in B$. Then the following hold.

(1) If a simple closed geodesic $\gamma$ belongs to $PP_n(A, \alpha, \beta)$, then

$$n \leq \text{lcm}\{i(\alpha, \beta), i(\beta, \gamma)\}.$$
(2) Let $\gamma$ be a simple closed geodesic crossing $\beta$. If $n$ is an integer more than two and $m$ is another integer satisfying that

$$|m| \geq \frac{n + 2 \cdot i(\alpha, \gamma)}{i(\beta, \gamma)} + 2,$$

then

$$T^m_\beta \left( \{ \alpha \} \cup \text{PP}_n(\mathcal{A}, \alpha, \beta) \right) \subseteq \text{PP}_n(\mathcal{B}, \beta, \gamma).$$

(3) For every $\gamma \in A - \{ \alpha \}$ and $m \in \mathbb{Z}$, we have

$$T^m_\gamma \left( \text{PP}_n(\mathcal{A}, \alpha, \beta) \right) \subseteq \text{PP}_n(\mathcal{A}, \alpha, \beta).$$

The proof of Theorem 1.6 is on the page 15. This theorem gives a setting for ping-pong technique. The original ping-pong lemma can be found in Koberda [Kob12a, Kob12b].

**Definition 1.7.** A reduced word $W = \gamma_n^{k_n} \ldots \gamma_1^{k_1}$ of $A(\Gamma)$ is said to be central if $\gamma_i \neq \gamma_j$ and $[\gamma_i, \gamma_j] = 1$ for all distinct $i, j \in \{1, \ldots, n\}$. And we say a reduced word $W$ of $A(\Gamma)$ is of central form if there exist central words $W_1, \ldots, W_n$ such that $W = W_n \ldots W_1$ and the last alphabet of $W_i$ does not commute with the last alphabet of $W_{i+1}$ for every $i \in \{1, \ldots, n - 1\}$.

**Proof of Theorem 1.1.** If $\Gamma$ is the join of some nonempty subgraphs $\Gamma_1$ and $\Gamma_2$, then $A(\Gamma) \cong A(\Gamma_1) \times A(\Gamma_2)$. In this case, we can divide the proof of Theorem 1.1 into the cases for $\Gamma_1$ and $\Gamma_2$. So we reduce this proof to the case that $\Gamma$ is not the join of two nonempty subgraphs. In other words, we assume that each simple closed geodesic in $\mathcal{F}$ crosses another simple closed geodesic of $\mathcal{F}$.

Note that every nonidentity element of $A(\Gamma)$ admits a word of central form. Choose $W = W_n \ldots W_1 \in A(\Gamma)$ be a nonidentity reduced word of central form. We will show that $\varphi(W)$ acts nontrivially on the set of simple closed geodesics. For each $i = 1, \ldots, n$, let $\text{supp}(W_i)$ be the subset of $\mathcal{F}$ containing the alphabets of $W_i$. Then $\text{supp}(W_i)$ is a multicurve. For each $i$, let $\gamma_i$ be the last alphabet of $W_i$. And let $\gamma_{n+1}$ be an arbitrary simple closed geodesic in $\mathcal{F}$ crossing $\gamma_n$. Set $\alpha := \gamma_2$.

Write $W_1 = \gamma_{1,j_1}^{k_{1,j_1}} \ldots \gamma_{1,j_1-1}^{k_{1,j_1-1}} \gamma_{1,1}^{k_{1,1}}$. Let

$$N := \begin{cases} \Delta(\mathcal{F}) + 1 & \text{if } \Delta(\mathcal{F}) \geq 2, \\ 3 & \text{if } \Delta(\mathcal{F}) = 1. \end{cases}$$

Then we have

$$\varphi(\gamma_1^{k_1})\alpha = T^{mk_1}_{\gamma_1}(\alpha) \in \text{PP}_N(\text{supp}(W_1), \gamma_1, \gamma_2)$$

by Theorem 1.6(2). Applying Theorem 1.6(3) repeatedly, $\varphi(\gamma_{1,j_1}^{k_{1,j_1}} \ldots \gamma_{1,1}^{k_{1,1}})$ sends $\varphi(\gamma_1^{k_1})\alpha$ into $\text{PP}_N(\text{supp}(W_1), \gamma_1, \gamma_2)$. Consequently, $\text{PP}_N(\text{supp}(W_1), \gamma_1, \gamma_2)$ contains $\varphi(W_1)\alpha$.

Suppose that the simple closed geodesic $\varphi(W_{l-1}W_{l-2} \ldots W_1)\alpha$, denoted by $\alpha_{l-1}$, is contained in $\text{PP}_N(\text{supp}(W_{l-1}), \gamma_{l-1}, \gamma_l)$ for some $l \in \{2, \ldots, n\}$. If we write $W_l = \gamma_{l,j_l}^{k_{l,j_l}} \ldots \gamma_{l,1}^{k_{l,1}} \gamma_l^{k_l}$, then

$$\varphi(\gamma_l^{k_l})\alpha_{l-1} \in \text{PP}_N(\text{supp}(W_l), \gamma_l, \gamma_{l+1})$$

by Theorem 1.6(2). And by Theorem 1.6(3), we have

$$\varphi(W_l)\alpha_{l-1} = \varphi(\gamma_{l,j_l}^{k_{l,j_l}} \ldots \gamma_{l,1}^{k_{l,1}})\varphi(\gamma_l^{k_l})\alpha_{l-1} \in \text{PP}_N(\text{supp}(W_l), \gamma_l, \gamma_{l+1}).$$

In conclusion, it holds that $\varphi(W)\alpha \in \text{PP}_N(\text{supp}(W_n), \gamma_n, \gamma_{n+1})$. 
Since Theorem 1.6(1) implies that $\alpha$ does not belong to $\text{PP}_\mathcal{X}(\text{supp}(W_n), \gamma_n, \gamma_{n+1})$, the action of $W$ on the set of simple closed geodesics is nontrivial. Because $W$ is an arbitrary nonidentity reduced word of $A(\Gamma)$, the action of $A(\Gamma)$ on the set of simple closed geodesics is faithful. Therefore, the surjective homomorphism $\varphi$ is injective, i.e., it is an isomorphism. \[ \square \]

1.4. **Guide to Readers.** Although we closely follow the original approach of Koberda [Kob12b], the material of this paper is essentially self-contained. As we have just shown that Theorem 1.6 implies Theorem 1.1, we will from now only focus on the proof of the former. Dual trees and the set $\text{PP}_n$ are our main tools. Section 2 is an introduction to the actions of fundamental groups of surfaces on dual trees. Proposition 2.8 is the main result of this section. In Section 3, we study the action of lifts of Dehn twists on dual trees. Proposition 3.5 gives how we compute Dehn twists by fundamental groups of surfaces. In Section 4, we investigate the set $\text{PP}_n$ and prove Theorem 1.6. Proposition 4.5 is the technical essence of our paper.

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2. **The Actions of Fundamental Groups of Surfaces on Dual Trees**

For disjoint subspaces $A$ and $B$ of $\mathbb{H}^2$, we say that a geodesic $L$ separates $A$ from $B$ if $A$ and $B$ lie in different connected components of $\mathbb{H}^2 \setminus L$.

**Remark 2.1.** For pairwise disjoint geodesics $L_1, L_2, L_3$ on $\mathbb{H}^2$, we will use the well-known facts.

1. Every geodesic crossing both $L_1$ and $L_3$ also crosses $L_2$ if and only if $L_2$ separates $L_1$ from $L_3$.
2. If some geodesic crosses $L_1, L_2$ and $L_3$, then some $L_i$ separates the others.

**Definition 2.2.** Let $\xi : \mathbb{H}^2 \to \Sigma$ be a covering map. For a simple closed geodesic $\gamma$ on $\Sigma$, the dual tree of $\gamma$ is a triple $(\mathcal{V}_\gamma, d_\gamma, \sigma_\gamma)$ satisfying the following. (We write $\mathcal{V}_\gamma$ simply for the triple $(\mathcal{V}_\gamma, d_\gamma, \sigma_\gamma)$.)

- $\mathcal{V}_\gamma$ is a topological tree embedded into $\mathbb{H}^2$ satisfying that every connected component of $\mathbb{H}^2 \setminus \xi^{-1}(\gamma)$ contains exactly one vertex of $\mathcal{V}_\gamma$ and for every lift $\tilde{\gamma}$ of $\gamma$, there is a unique edge $e$ on $\mathcal{V}_\gamma$ such that $|\mathcal{V}_\gamma \cap \tilde{\gamma}| = |e \cap \tilde{\gamma}| = |e \cap \xi^{-1}(\gamma)| = 1$. (See Figure 2.)
- $d_\gamma$ is a metric on $\mathcal{V}_\gamma$ satisfying that for each edge $e$ of $\mathcal{V}_\gamma$, the length of $e$ is 1 and the intersection between $e$ and a lift of $\gamma$ is the midpoint of $e$.
- $\sigma_\gamma$ is an isometric action of $\pi_1(\Sigma)$ on $(\mathcal{V}_\gamma, d_\gamma)$ satisfying that $\sigma_\gamma(g)(\tilde{\gamma} \cap \mathcal{V}_\gamma) = (g\tilde{\gamma}) \cap \mathcal{V}_\gamma$ for all $g \in \pi_1(\Sigma)$ and a lift $\tilde{\gamma}$ of $\gamma$.

Note that if $p \in \mathcal{V}_\gamma$ is not the midpoint of an edge, then for every $g \in \pi_1(\Sigma)$, two points $\sigma_\gamma(g)p$ (the isometric action on $(\mathcal{V}_\gamma, d_\gamma)$) and $gp$ (the isometric action on $(\mathbb{H}^2, d_{\mathbb{H}^2})$) can be different but they are always contained in the same connected component of $\mathbb{H}^2 \setminus \xi^{-1}(\gamma)$. Note that an element of $\pi_1(\Sigma)$ preserving a lift of $\gamma$ fixes an edge of $\mathcal{V}_\gamma$. 


For a simple closed geodesic $\gamma$ and its lift $\tilde{\gamma}$, we write $N_r(\gamma)$ and $N_r(\tilde{\gamma})$ for the open $r$-neighborhoods of $\gamma$ and $\tilde{\gamma}$, respectively. And we write $\overline{N}_r(\gamma)$ and $\overline{N}_r(\tilde{\gamma})$ for the closure of $N_r(\gamma)$ and $N_r(\tilde{\gamma})$, respectively. The collar lemma (cf. [Bus78]) means that for every simple closed geodesic $\gamma$ on $\Sigma$, there is a positive number $R$, depending only on the length of $\gamma$ such that $\overline{N}_r(\gamma)$ is homeomorphic to an annulus for every $0 < r < R$.

**Proposition 2.3.** Let $\gamma$ be a simple closed geodesic on $\Sigma$. Let $R$ be a positive number such that $\overline{N}_{r}(\gamma)$ is homeomorphic to an annulus for every $0 < r < R$. Then for every $0 < r < R$, there is a $\pi_1(\Sigma)$-equivariant surjective continuous map $\Phi_{\gamma,r} : \mathbb{H}^2 \rightarrow Y_\gamma$ such that the following hold.

1. For each lift $\tilde{\gamma}$ of $\gamma$, if $e(\tilde{\gamma})$ is the edge intersecting $\tilde{\gamma}$ and $u$ is a point in the interior of $e(\tilde{\gamma})$, then $\Phi_{\gamma,r}(u)$ is a one-dimensional subspace of $\overline{N}_r(\tilde{\gamma})$.
2. If $\alpha$ is a simple closed geodesic on $\Sigma$, then there is $0 < R' \leq R$ such that for every $0 < r < R'$ and a lift $\tilde{\alpha}$ of $\alpha$, the image $\Phi_{\gamma,r}(\tilde{\alpha})$ is given as follows:
   (a) the midpoint of an edge if $\alpha = \gamma$;
   (b) a vertex if $\alpha \cap \gamma = \emptyset$;
   (c) a geodesic with respect to $d_\gamma$ if $\alpha \pitchfork \gamma$.

**Proof.** (1) Fix a positive number $r < R$. Let $\rho : S^1 \times [0, 1] \rightarrow \Sigma$ be a topological embedding satisfying that $\text{im } \rho = \overline{N}_r(\gamma)$ and $\rho(S^1 \times \{1/2\}) = \gamma$. Let $\xi : \mathbb{H}^2 \rightarrow \Sigma$ be a covering map. For each lift $\tilde{\gamma}$ of $\gamma$, let $\rho_{\tilde{\gamma}}$ be a lift of $\rho$ whose image contains $\tilde{\gamma}$ as a subset. And let $\psi_\tilde{\gamma} : [0, 1] \rightarrow e(\tilde{\gamma})$ be an isometry satisfying that $\psi_\tilde{\gamma}(0)$ and $\rho_{\tilde{\gamma}}(s, 0)$ are in the same connected component of $\mathbb{H}^2 \setminus \xi^{-1}(\gamma)$ for all $s \in S^1$.

Then for every $g \in \pi_1(\Sigma)$ and $t \in [0, 1]$, the point $\sigma_{\gamma}(g)\psi_\tilde{\gamma}(t)$ is on the edge $e(g\tilde{\gamma})$. So $\sigma_{\gamma}(g)\psi_\tilde{\gamma}$ is an isometry from $[0, 1]$ to $e(g\tilde{\gamma})$. Because $\sigma_{\gamma}(g)\psi_\tilde{\gamma}(0)$ and $g\psi_\tilde{\gamma}(0)$ are in the same connected component of $\mathbb{H}^2 \setminus \xi^{-1}(\gamma)$, we have $\sigma_{\gamma}(g)\psi_\tilde{\gamma} = \psi_\tilde{g}$. With the map $\text{pr} : (s, t) \mapsto t$, we define a map $\Phi_{\gamma,r} : \mathbb{H}^2 \rightarrow Y_\gamma$ by

$$\Phi_{\gamma,r}(p) = \begin{cases} (\psi_\tilde{\gamma} \circ \text{pr} \circ \rho_{\tilde{\gamma}}^{-1})(p) & \text{if } p \in \text{im } \rho_{\tilde{\gamma}} \text{ for some lift } \tilde{\gamma}; \\
\text{the vertex lying in the same component}, & \text{otherwise.} \end{cases}$$
Then \( \Phi_{\gamma,r} \) is well-defined, continuous and surjective, and satisfies the condition (1).

To show that \( \Phi_{\gamma,r} \) is \( \pi_1(\Sigma) \)-equivariant, choose a point \( q \) on \( \mathbb{H}^2 \) and \( g \in \pi_1(\Sigma) \). If \( q \) is a point in \( \mathcal{N}_r(\tilde{\gamma}) = \text{im} \rho_\gamma \) for some lift \( \tilde{\gamma} \), then \( gq \in \mathcal{N}_r(\tilde{g}\tilde{\gamma}) \) = \( \text{im} \rho_{\tilde{g}\tilde{\gamma}} \). Because \( \rho_\gamma \) and \( \rho_{\tilde{g}\tilde{\gamma}} \) are lifts of \( \rho \), we have \( (pr \circ \rho_\gamma^{-1})(q) = (pr \circ \rho_{\tilde{g}\tilde{\gamma}}^{-1})(gq) \). Then \( \sigma_\gamma(g)\Phi_{\gamma,r}(q) = \sigma_{\tilde{g}}(g)(\psi_{\tilde{\gamma}}(pr \circ \rho_{\tilde{g}\tilde{\gamma}}^{-1})(gq)) = \psi_{\tilde{\gamma}}(pr \circ \rho_{\tilde{g}\tilde{\gamma}}^{-1})(gq)) = \Phi_{\gamma,r}(gq) \).

If \( \Phi_{\gamma,r}(q) \) is a vertex \( v \) on \( \mathcal{Y}_\gamma \), then \( q \) and \( v \) are in the same connected component of \( \mathbb{H}^2 \setminus \xi^{-1}(\gamma) \). Since \( gq, gv \) and \( \sigma_\gamma(g)v \) are in the same connected component, we have \( \sigma_\gamma(g)\Phi_{\gamma,r}(q) = \sigma_\gamma(g)v = \Phi_{\gamma,r}(gq) \). Therefore, \( \Phi_{\gamma,r} \) is \( \pi_1(\Sigma) \)-equivariant.

(2) If \( \alpha = \gamma \), then \( \Phi_{\gamma,r}(\tilde{\alpha}) \) is the midpoint of an edge by definition. If \( \alpha \) is disjoint from \( \gamma \), then \( \alpha \) is also disjoint from \( \mathcal{N}_r(\gamma) \) for all \( 0 < r < R(\alpha, \gamma) \) by the collar lemma. So \( \tilde{\alpha} \) is disjoint from the \( r \)-neighborhoods of all lifts of \( \gamma \). Therefore, \( \Phi_{\gamma,r}(\tilde{\alpha}) \) is a vertex.

Assume that \( \alpha \) crosses \( \gamma \). Since \( \alpha \) is compact, there is a positive number \( R(\alpha, \gamma) > 0 \) such that \( \alpha \cap \mathcal{N}_r(\gamma) \) is the disjoint union of line segments intersecting \( \gamma \) for all \( 0 < r < R(\alpha, \gamma) \). So \( \tilde{\alpha} \) crosses \( \tilde{\gamma} \) whenever a lift \( \tilde{\alpha} \) of \( \alpha \) intersects the \( r \)-neighborhood of a lift \( \tilde{\gamma} \) of \( \gamma \). Therefore, \( \Phi_{\gamma,r}(\tilde{\alpha}) \) is the union of edges, which is a geodesic on \( \mathcal{Y}_\gamma \). \( \square \)

The translation length of an isometry \( f \) on the dual tree \( \mathcal{Y}_\gamma \) is

\[
\text{tr}_\gamma f := \inf_{v \in \mathcal{Y}_\gamma} (v, \Phi_{\gamma,r}(v))
\]

For \( g \in \pi_1(\Sigma) \), we write \( \text{tr}_\gamma g \) instead of \( \text{tr}_\gamma \sigma_\gamma(g) \). An element of \( \pi_1(\Sigma) \) is said to be primitive if it cannot be written by a proper power of another element of \( \pi_1(\Sigma) \).

**Lemma 2.4.** If \( \gamma \) is a simple closed geodesic on \( \Sigma \) and \( h \) is a primitive element of \( \pi_1(\Sigma) \) preserving a lift of a simple closed geodesic \( \alpha \), then \( \text{tr}_\gamma h = i(\alpha, \gamma) \).

**Proof.** Let \( \tilde{\alpha} \) be the lift of \( \alpha \) preserved by \( h \). If \( i(\alpha, \gamma) = 0 \), then \( \Phi_{\gamma,r}(\tilde{\alpha}) \) is a point on \( \mathcal{Y}_\gamma \) for some \( r > 0 \). Because \( h \) preserves \( \Phi_{\gamma,r}(\tilde{\alpha}) \), we have \( \text{tr}_\gamma h = 0 = i(\alpha, \gamma) \).

Assume that \( \alpha \) crosses \( \gamma \). By Proposition 2.3(2), it is satisfied that \( h \) preserves the geodesic \( \Phi_{\gamma,r}(\tilde{\alpha}) \) for some \( r > 0 \). If \( p \) is a point on \( \tilde{\alpha} \) such that \( \Phi_{\gamma,r}(p) \) is a vertex, then \( \text{tr}_\gamma h \) is equal to \( d_{\gamma}(\Phi_{\gamma,r}(p), \Phi_{\gamma,r}(\tilde{\alpha})) \) by Bass-Serre theory. Since the geodesic segment joining \( p \) and \( \gamma \) crosses exactly \( i(\alpha, \gamma) \) lifts of \( \gamma \), we have \( d_{\gamma}(\Phi_{\gamma,r}(p), \Phi_{\gamma,r}(\tilde{\alpha})) = i(\alpha, \gamma) \). Therefore, \( \text{tr}_\gamma h = i(\alpha, \gamma) \). \( \square \)

**Lemma 2.5.** Let \( \alpha, \beta \) and \( \gamma \) be simple closed geodesics on \( \Sigma \) crossing each other. If \( x \in \alpha \cap \beta \), \( y \in \beta \cap \gamma \) and \( z \in \gamma \cap \alpha \), then the number of distinct contractible geodesic triangles whose vertices are \( x, y \) and \( z \) is at most 1.
Proof. For contradiction, suppose that there are distinct disks \( D_1 \) and \( D_2 \) whose boundaries are geodesic triangles such that \( x, y \) and \( z \) are vertices of \( \partial D_i \) for all \( i = 1, 2 \). Write \( \partial D_i \) for the boundary of \( D_i \).

If \( \partial D_1 \cap \partial D_2 \) does not contain a side of \( \partial D_1 \) or \( \partial D_2 \) (cf. Figure 3a), then \( D_1 \cup D_2 \) is homotopic to a pair of pants. It implies that \( \alpha, \beta \) and \( \gamma \) are pairwise disjoint, which is a contradiction.

Assume that \( \partial D_1 \) and \( \partial D_2 \) share exactly two sides. (cf. Figure 3b) Then \( D_1 \cup D_2 \) is a disk and its boundary is one of \( \alpha, \beta \) or \( \gamma \). It induces that one of \( \alpha, \beta \) and \( \gamma \) is contractible to a point, which is a contradiction.

If \( \partial D_1 \cap \partial D_2 \) is the disjoint union of a vertex and a side (cf. Figure 3c), then \( D_1 \cup D_2 \) is homotopic to a cylinder. So two of \( \alpha, \beta \) and \( \gamma \) are homotopic to each other, which is a contradiction. Therefore, \( \partial D_1 \) and \( \partial D_2 \) share all sides, which implies that \( D_1 = D_2 \).

Definition 2.6. If \( \gamma \) is a simple closed geodesic on \( \Sigma \) and \( L_1 \) and \( L_2 \) are geodesics on \( \mathbb{H}^2 \), then we write \( \Delta_\gamma(L_1, L_2) \) for the number of lifts of \( \gamma \) crossing both \( L_1 \) and \( L_2 \).

Remark 2.7. Let \( \alpha, \beta \) and \( \gamma \) be simple closed geodesics on \( \Sigma \). For sufficiently small \( r > 0 \), a lift \( \tilde{\alpha} \) of \( \alpha \) crosses a lift \( \tilde{\gamma} \) of \( \gamma \) if and only if the edge containing the midpoint \( \Phi_{\gamma,r}(\tilde{\gamma}) \) belongs to \( \Phi_{\gamma,r}(\tilde{\alpha}) \). For a lift \( \tilde{\beta} \) of \( \beta \), the number \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \) is equal to the length of \( \Phi_{\gamma,r}(\tilde{\alpha}) \cap \Phi_{\gamma,r}(\tilde{\beta}) \).

Proposition 2.8. Let \( \alpha, \beta \) and \( \gamma \) be simple closed geodesics on \( \Sigma \). And let \( \tilde{\alpha} \) and \( \tilde{\beta} \) be lifts of \( \alpha \) and \( \beta \), respectively.

1. If \( \tilde{\alpha} \) is disjoint from \( \tilde{\beta} \), then \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq \min\{i(\alpha, \gamma), i(\beta, \gamma)\} \).
2. If \( \tilde{\alpha} \) crosses \( \tilde{\beta} \), then \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq \lcm\{i(\alpha, \gamma), i(\beta, \gamma)\} \).

Proof. (1) Let \( r > 0 \) be sufficiently small, and let \( \Phi_\gamma = \Phi_{\gamma,r} : \mathbb{H}^2 \to \mathcal{Y}_\gamma \) be a \( \pi_1(\Sigma) \)-equivariant surjective continuous map satisfying Proposition 2.3. Suppose that either \( i(\alpha, \gamma) \) or \( i(\beta, \gamma) \) is zero. Then \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \) has at most one element. So \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) = 0 \).

Assume that both \( i(\alpha, \gamma) \) and \( i(\beta, \gamma) \) are positive. Let \( h \) be a primitive element of \( \pi_1(\Sigma) \) preserving \( \tilde{\alpha} \). Because \( h \) is orientation-preserving, \( \tilde{\alpha} \) does not separate \( \tilde{\beta} \) from \( h\tilde{\beta} \). Since \( h \) is an isometry, \( h\tilde{\beta} \) does not separate \( \tilde{\alpha} \) from \( \tilde{\beta} \). Likewise, \( \tilde{\beta} \) does not separate \( \tilde{\alpha} \) from \( h\tilde{\beta} \). By Remark 2.1, for every lift of \( \gamma \), it is disjoint from some of \( \tilde{\alpha}, \tilde{\beta} \) and \( h\tilde{\beta} \).

Then \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \cap \Phi_\gamma(h\tilde{\beta}) \) is a vertex or is empty. So the intersection between \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \) and \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(h\tilde{\beta}) = \sigma_\gamma(h)(\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})) \) does not contain an edge. Similarly, \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \) does not share an edge with \( \sigma_\gamma(h^{-1})(\Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta})) \). It implies that \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \) is the line segment of length at most \( \text{tr}_\gamma h = i(\alpha, \gamma) \).

In other words, the number of lifts of \( \gamma \) intersecting \( \tilde{\alpha} \) and \( \tilde{\beta} \) simultaneously is at most \( i(\alpha, \gamma) \). Changing the role between \( \alpha \) and \( \beta \), we have \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq i(\beta, \gamma) \). Therefore, \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq \min\{i(\alpha, \gamma), i(\beta, \gamma)\} \).

(2) Write \( N := \lcm\{i(\alpha, \gamma), i(\beta, \gamma)\} \), \( a := N/i(\alpha, \gamma) \) and \( b := N/i(\beta, \gamma) \). For contradiction, assume that \( \Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \geq N + 1 \). Let \( h \) be a primitive element of \( \pi_1(\Sigma) \) preserving \( \tilde{\alpha} \). By Proposition 2.3(2) and the assumption, \( \Phi_\gamma(\tilde{\alpha}) \) and \( \Phi_\gamma(\tilde{\beta}) \) are geodesics on \( \mathcal{Y}_\gamma \) and their intersection is a line segment of length at least \( N + 1 \). So there is an edge \( e \) on \( \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \) such that \( \sigma_\gamma(h^ae) \subset \Phi_\gamma(\tilde{\alpha}) \cap \Phi_\gamma(\tilde{\beta}) \). In other words, there is a lift \( \tilde{\gamma} \) of \( \gamma \) such that both \( \tilde{\gamma} \) and \( h^ae \tilde{\gamma} \) crosses both \( \tilde{\alpha} \) and \( \tilde{\beta} \). See Figure 4.
Because $d_\gamma(\Phi_\gamma(\tilde{\gamma}), \Phi_\gamma(h^a\tilde{\gamma}))$ is a multiple of $i(\beta, \gamma)$, there is a primitive element $g$ of $\pi_1(\Sigma)$ preserving $\tilde{\beta}$ such that $g^b\tilde{\gamma} = h^a\tilde{\gamma}$. Let $A$ and $B$ be the geodesic triangles which are contained in $\tilde{\alpha} \cup \tilde{\beta} \cup \tilde{\gamma}$ and $\tilde{\alpha} \cup \tilde{\beta} \cup g^b\tilde{\gamma}$, respectively. If $\xi : \mathbb{H}^2 \rightarrow \Sigma$ is a covering map, $\xi(A)$ and $\xi(B)$ are distinct contractible geodesic triangles on $\Sigma$ such that they share all vertices. It is a contradiction because of Lemma 2.5. Therefore, $\Delta_\gamma(\tilde{\alpha}, \tilde{\beta}) \leq N = \text{lcm}\{i(\alpha, \gamma), i(\beta, \gamma)\}$. □

Remark 2.9 (Further reading). Morgan–Shalen [MS91] defined the dual trees for measured laminations, which is more general than our definition. Levitt–Paulin [LP97] generally studied geometric actions on trees. Skora [Sko96] showed that nontrivial minimal tree actions of hyperbolic surface groups such that every parabolic isometry fixes a vertex are given from dual trees of measured laminations. Rips and Bestvina–Feigh [BF95, Theorem 9.8] proved that every finitely generated group acting freely on a tree is a free product of surface groups and free abelian groups.

The theory of tree action is originated from Bass–Serre’s work [Ser77, Bas93]. It had been developed by Serre, Bass, Dunwoody (cf. [DD89]) and so on. Sageev’s study [Sag95] provided methods for CAT(0) cube complexes including dual trees of simple closed geodesics. Haglund–Wise [HW08, HW12] developed special cube complexes. It built a foundation for Agol [Ago13] to prove the virtual Haken conjecture.

3. Lifts of Dehn twists

Remark 3.1. In this section, we write $T_\gamma$ as a self-homeomorphism homotopic to a Dehn twist along $\gamma$.

It is well-known that the mapping class group $\text{Mod}(\Sigma)$ is a subgroup of the outer automorphism group of $\pi_1(\Sigma)$. Note that, when $\Sigma$ is closed, $\text{Mod}(\Sigma)$ is an index 2 subgroup of $\text{Out}(\pi_1(\Sigma))$ by the Dehn–Niesen–Baer theorem; see Farb–Margalit [FM12, Theorem 8.1].

Remark 3.2. Let us recall the relationship between mapping classes and outer automorphisms. Let $F$ be a self-homeomorphism of $\Sigma$. If $\xi : \mathbb{H}^2 \rightarrow \Sigma$ is a covering map and $\tilde{F}$ is a self-homeomorphism of $\mathbb{H}^2$ which is a lift of $F$, then $\xi(Fg\tilde{F}^{-1}p) = F\xi(g\tilde{F}^{-1}p) = F\xi(\tilde{F}^{-1}p) = \xi(p)$ for every $g \in \pi_1(\Sigma)$ and $p \in \mathbb{H}^2$. That is, $Fg\tilde{F}^{-1}$ is an element of $\pi_1(\Sigma)$, and the conjugate action of $\tilde{F}$ on $\pi_1(\Sigma)$ is an automorphism of $\pi_1(\Sigma)$. When we consider the collection of lifts of $F$, the set of their conjugate actions is indeed an outer automorphism of $\pi_1(\Sigma)$.

This method is invariant under homotopies of $\Sigma$. If $\Phi_t : \Sigma \rightarrow \Sigma$ is a homotopy from $F$ to another self-homeomorphism $F'$ on $\Sigma$, then for every lift $\tilde{F}$, there is a lift $\tilde{\Phi}_t : \mathbb{H}^2 \rightarrow \mathbb{H}^2$ of $\Phi$ which is a homotopy from a lift of $F$ to $F'$. But varying time $t$ does not change the conjugate action of $\tilde{F}$ because the action of $\pi_1(\Sigma)$ on $\mathbb{H}^2$ is...
discrete. As a result, the mapping class of $F$ determines an outer automorphism of $\pi_1(\Sigma)$.

For a simple closed geodesic $\gamma$, recall that the dual tree $\mathcal{Y}_\gamma$ of $\gamma$ is a metric space with a metric $d_\gamma$ and an isometric action $\sigma_\gamma$. (See Definition 2.2.)

**Proposition 3.3.** If $\gamma$ is a simple closed geodesic on $\Sigma$ and $\tilde{T}_\gamma$ is a lift of a Dehn twist $T_\gamma$ along $\gamma$, then there is an isometry $f$ on the dual tree $\mathcal{Y}_\gamma$ of $\gamma$ such that $\sigma_\gamma(T_\gamma g \tilde{T}_\gamma^{-1})(v) = f \sigma_\gamma(g)f^{-1}(v)$ for all $g \in \pi_1(\Sigma)$ and $v \in \mathcal{Y}_\gamma$.

From now on, we write $\sigma_\gamma(\tilde{T}_\gamma)$ instead of the isometry $f$.

**Proof.** Let $r > 0$ be sufficiently small satisfying that the closed $r$-neighborhood $N(\gamma)$ of $\gamma$ is homeomorphic to an annulus. Let $\xi : \mathbb{H}^2 \to \Sigma$ be a covering map. And let $\Phi_\gamma = \Phi_{\gamma,r} : \mathbb{H}^2 \to \mathcal{Y}_\gamma$ be a $\pi_1(\Sigma)$-equivariant surjective continuous map satisfying Proposition 2.3. Define a relation $\sim$ on the open $r$-neighborhood $\mathcal{N}_r(\gamma)$ such that for all $x, y \in \mathcal{N}_r(\gamma)$,

$$x \sim y \text{ if and only if } \Phi_\gamma(p) = \Phi_\gamma(q) \text{ for some } p \in \xi^{-1}(x) \text{ and } q \in \xi^{-1}(y).$$

Then $\sim$ is an equivalence relation on $\mathcal{N}_r(\gamma)$ because $\Phi_\gamma$ is $\pi_1(\Sigma)$-equivariant. Note that the decomposition of $\mathcal{N}_r(\gamma)$ by $\sim$ is a foliation of circles on $\mathcal{N}_r(\gamma)$ by Proposition 2.3(1).

Passing to homotopy, suppose that $T_\gamma$ supports $\mathcal{N}_r(\gamma)$ and preserves the equivalence relation $\sim$. Then for all points $p, q \in \mathbb{H}^2$ satisfying that $\Phi_\gamma(p) = \Phi_\gamma(q)$, we have $\Phi_\gamma(T_\gamma(p)) = \Phi_\gamma(T_\gamma(q))$. It implies that the map $f := \Phi_\gamma \sigma_\gamma \Phi_\gamma^{-1} : \mathcal{Y}_\gamma \to \mathcal{Y}_\gamma$ is well-defined. Since $\tilde{T}_\gamma$ preserves the separability of lifts of $\gamma$ (i.e., for all lifts $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ of $\gamma$, we have $\tilde{\gamma}_1$ separates $\tilde{\gamma}_2$ from $\tilde{\gamma}_3$ if and only if $\tilde{T}_\gamma(\tilde{\gamma}_1)$ separates $\tilde{T}_\gamma(\tilde{\gamma}_2)$ from $\tilde{T}_\gamma(\tilde{\gamma}_3)$) and the equivalence relation $\sim$, we have $f$ is an isometry on $\mathcal{Y}_\gamma$.

We claim that for every $g \in \pi_1(\Sigma)$ and $v \in \mathcal{Y}_\gamma$, we have $\sigma_\gamma(\tilde{T}_\gamma g \tilde{T}_\gamma^{-1})v = f \sigma_\gamma(g)f^{-1}v.$ Since $gp \in \Phi_{\gamma,r}^{-1}\sigma_\gamma(g)\Phi_{\gamma,r}(p)$ for every point $p \in \mathbb{H}^2$, it holds that $\tilde{T}_\gamma g \tilde{T}_\gamma^{-1}(p)$ is an element of the set $\tilde{T}_\gamma(\Phi_{\gamma,r}^{-1}\sigma_\gamma(g)\Phi_{\gamma,r}(p)).$ The $\Phi_\gamma$-image of the set $\tilde{T}_\gamma(\Phi_{\gamma,r}^{-1}\sigma_\gamma(g)\Phi_{\gamma,r}(p))$ is a point on $\mathcal{Y}_\gamma$ and is equal to $\Phi_\gamma((\tilde{T}_\gamma g \tilde{T}_\gamma^{-1})p).$ Because $\Phi_\gamma$ is $\pi_1(\Sigma)$-equivariant, we have $\sigma_\gamma(\tilde{T}_\gamma g \tilde{T}_\gamma^{-1})\Phi_\gamma(p) = \Phi_\gamma((\tilde{T}_\gamma g \tilde{T}_\gamma^{-1})p) = f \sigma_\gamma(g)f^{-1}\Phi_\gamma(p)$ for all $p \in \mathbb{H}^2$. Because $\Phi_\gamma$ is surjective, we prove the claim. □

**Remark 3.4.** Let us remind the definition of a Dehn twist. Let $\gamma$ be a simple closed geodesic of $\Sigma$. For the annulus $A := (\mathbb{R}/\mathbb{Z}) \times [0, 1]$, let $\iota : A \to \Sigma$ be a topological embedding such that $\iota((\mathbb{R}/\mathbb{Z}) \times \{1/2\}) = \gamma$. Define a map $T : A \to A$ by $T([s], t) = ([s+t], t)$ for every $([s], t) \in (\mathbb{R}/\mathbb{Z}) \times [0, 1]$. Then the map $T_\gamma : \Sigma \to \Sigma$ defined by

$$T_\gamma(x) = \begin{cases} (\iota \circ T \circ \iota^{-1})(x) & \text{if } x \in \iota(A), \\ x & \text{otherwise,} \end{cases}$$

is a Dehn twist. (See Farb-Margalit’s definition [FM12, Section 3.1.1].)

Note that $\pi_1(A) = \langle a \rangle$ acts on $\tilde{A} = \mathbb{R} \times [0, 1]$ by $a^m(s, t) = (s + m, t)$ for all $m \in \mathbb{Z}$ and $(s, t) \in \tilde{A}$. If $\tilde{T} : \tilde{A} \to \tilde{A}$ is the lift of $T$ fixing $\mathbb{R} \times \{0\}$ pointwise, the equation

$$\tilde{T}^m(s, 1) = (s + m, 1) = a^m(s, 1)$$

holds for all $m \in \mathbb{Z}$ and $s \in \mathbb{R}$. We want to apply this equation to lifts of a Dehn twist. Every lift $\tilde{\iota} : \tilde{A} \to \mathbb{H}^2$ of $\iota$ gives an injection $\tilde{\iota}_* : \pi_1(A) \to \pi_1(\Sigma)$. If $\tilde{T}_\gamma$ is
the lift of $T_\gamma$ which fixes a side of $i(\bar{A})$ pointwise, then there is a primitive element $h \in \{i_*(a), i_*(a^{-1})\}$ such that we have
\[
\hat{T}_\gamma^m(p) = h^m p
\]
for every point $p$ on the other side of $i(\bar{A})$ and $m \in \mathbb{Z}$.

**Proposition 3.5.** Let $\gamma$ be a simple closed geodesic on $\Sigma$, and let $v$ be a vertex of the dual tree $\mathcal{Y}_\gamma$. Then there is a lift $\hat{T}_\gamma$ of a Dehn twist $T_\gamma$ such that $\sigma_\gamma(\hat{T}_\gamma)$ fixes the closed 1-neighborhood of $v$ pointwise.

Furthermore, if $w$ is a vertex of $\mathcal{Y}_\gamma$ distinct from $v$ and $\hat{L} : [0, n] \to \mathcal{Y}_\gamma$ is the unit-speed geodesic path from $v$ to $w$, then there are primitive elements $h_1, \ldots, h_n \in \pi_1(\Sigma)$ such that $h_i$ fixes the midpoint $\hat{L}(i - 1/2)$ for each $i = 1, \ldots, n$ and such that $\sigma_\gamma(T^m_\gamma)w = \sigma_\gamma(h_1^m \ldots h_n^m)w$ for every integer $m$; see Figure 5.

**Proof.** Let $r > 0$ be small enough that the closed $r$-neighborhood of $\gamma$ is homeomorphic to an annulus. Let $\Phi_\gamma := \Phi_{\gamma,r} : \mathbb{H}^2 \to \mathcal{Y}_\gamma$ be a $\pi_1(\Sigma)$-equivariant surjective continuous map satisfying Proposition 2.3. Passing to homotopy, suppose that $T_\gamma$ is a Dehn twist along $\gamma$ such that the support of $T_\gamma(\gamma)$ is the $r$-neighborhood of $\gamma$. Let $p$ be a point on $\mathbb{H}^2$ such that $\Phi_\gamma(p) = v$.

Because the projective image of $p$ on $\Sigma$ is not contained in the open support of $T_\gamma$, there is a lift $\hat{T}_\gamma$ of $T_\gamma$ which fixes $p$. For every edge $e$ incident to $v$, if $\hat{\gamma}$ is the lift of $\gamma$ such that $\Phi_\gamma(\hat{\gamma})$ is the midpoint of $e$, then $\hat{T}_\gamma(\hat{\gamma}) = \hat{\gamma}$ because some connected component of the boundary of the $r$-neighborhood of $\hat{\gamma}$ is fixed pointwise by $\hat{T}_\gamma$. So $\sigma_\gamma(T_\gamma)e = e$. Therefore, $\hat{T}_\gamma$ fixes the closed 1-neighborhood of $v$ pointwise.

To prove the second statement in Proposition 3.5, we use an induction on the distance $n$ between $v$ and $w$. If $v$ and $w$ are joined by an edge $e$ and a primitive element $h$ of $\pi_1(\Sigma)$ fixes the edge $e$, then we have $\sigma_\gamma(T^m_\gamma)w = \sigma_\gamma(h^m)w$ for every integer $m$ by the above.

Assume that $n = d_\gamma(v, w) \geq 2$. Let an integer $m$ be given. Let $\hat{T}_\gamma^m$ be the lift of $T_\gamma$ fixing the closed 1-neighborhood of $\hat{L}(1)$. By the induction hypothesis, there are primitive elements $h_2, \ldots, h_n \in \pi_1(\Sigma)$ fixing midpoints $\hat{L}(3/2), \ldots, \hat{L}(n - 1/2)$, respectively, such that $\sigma_\gamma((\hat{T}_\gamma^m)^m)w = \sigma_\gamma(h_2^m \ldots h_n^m)w$.

We claim that $(\hat{T}_\gamma^m)^m = h_1^m \hat{T}_\gamma^m$ for some primitive element $h_1$ fixing $\hat{L}(1/2)$. Let $\hat{\gamma}$ be the lift of $\gamma$ such that $\Phi_\gamma(\hat{\gamma}) = \hat{L}(1/2)$. Then $\hat{T}_\gamma^m$ fixes some boundary component $\partial_+ N_\gamma(\hat{\gamma})$ of the closed $r$-neighborhood of $\hat{\gamma}$ pointwise. By Remark 3.4, there is a primitive element $h_1 \in \pi_1(\Sigma)$ preserving $\hat{\gamma}$ such that $\hat{T}_\gamma^m(p) = h_1^m p$ for every $p \in \partial_+ N_\gamma(\hat{\gamma})$. Because $h_1^{-m} \hat{T}_\gamma^m$ fixes $p$ and is a lift of $T_\gamma^m$, we have $h_1^{-m} \hat{T}_\gamma^m = (\hat{T}_\gamma^m)^m$ by the path lifting property. Therefore, $\sigma_\gamma(\hat{T}_\gamma^m)w = \sigma_\gamma(h_1^m h_2^m \ldots h_n^m)w$. \(\square\)
4. The Proof of Theorem 1.6

Let us prove Theorem 1.6. First, we will introduce the general definition of $\text{PP}_n$. (cf. Definition 1.5) We recall that if $\mathcal{F}$ is a set of simple closed geodesics on $\Sigma$, a lift of $\mathcal{F}$ means a lift of a simple closed geodesic in $\mathcal{F}$. And recall that a multicurve is a set of pairwise disjoint simple closed geodesics on $\Sigma$.

**Definition 4.1** (General definition of $\text{PP}_n$). Let $\mathcal{A}$ and $\mathcal{B}$ be multicurves on $\Sigma$, and let $\alpha$ and $\beta$ be simple closed geodesics in $\mathcal{A}$ and $\mathcal{B}$, respectively. For $n > 2$, a simple closed geodesic $\gamma$ of $\Sigma$ is said to be contained in $\text{PP}_n(\mathcal{A}, \alpha, \beta)$ if for some lift $\tilde{\gamma}$ of $\gamma$, for some lift $\tilde{\alpha}$ of $\alpha$ and (at least) $n$ lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\beta$ such that the following hold.

1. Each $\tilde{\beta}_i$ separates $\tilde{\beta}_{i-1}$ from $\tilde{\beta}_{i+1}$.
2. There is an index $i_0 \in \{1, \ldots, n\}$ such that $\tilde{\beta}_{i_0}$ is a lift of $\beta$.
3. For all $i = 1, \ldots, n$, the triple $\tilde{\gamma}, \tilde{\alpha}$ and $\tilde{\beta}_i$ cross each other.
4. If a lift $\tilde{\alpha}'$ of $\alpha$ intersects $\tilde{\gamma}$ and some $\tilde{\beta}_i$, then $\tilde{\alpha}' = \tilde{\alpha}$.

The set $\text{PP}_n(\mathcal{A}, \alpha, \beta)$ in Definition 1.5 is equal to $\text{PP}_n(\mathcal{A}, \alpha, \{\beta\}, \beta)$ with $\alpha \neq \beta$. We collect basic properties of $\text{PP}_n$ in the next lemma.

**Lemma 4.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be multicurves, and let simple closed geodesics $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ be given. For $n > 2$, the following hold.

1. If $\text{PP}_n(\mathcal{A}, \alpha, \beta, \beta)$ is nonempty, then $i(\alpha, \beta) > 0$.
2. If $2 < m \leq n$, then $\text{PP}_m(\mathcal{A}, \alpha, \beta, \beta) \supseteq \text{PP}_n(\mathcal{A}, \alpha, \beta, \beta)$.
3. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are multicurves satisfying that $\alpha \in \mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\text{PP}_n(\mathcal{A}_1, \alpha, \beta, \beta) \supseteq \text{PP}_n(\mathcal{A}_2, \alpha, \beta, \beta)$.
4. If $\mathcal{B}_1$ and $\mathcal{B}_2$ are multicurves satisfying that $\beta \in \mathcal{B}_1 \subseteq \mathcal{B}_2$, then $\text{PP}_n(\mathcal{A}, \alpha, \beta_1, \beta) \subseteq \text{PP}_n(\mathcal{A}, \alpha, \beta_2, \beta)$.

**Proof.** (1) If some simple closed geodesic is contained in $\text{PP}_n(\mathcal{A}, \alpha, \beta, \beta)$, then $\tilde{\alpha} \neq \tilde{\beta}$ for some lift $\tilde{\alpha}$ of $\alpha$ and lift $\tilde{\beta}$ of $\beta$ by Definition 1.1(ii) and (iii). It implies that $i(\alpha, \beta) > 0$.

(2) Choose a simple closed geodesic $\gamma \in \text{PP}_n(\mathcal{A}, \alpha, \beta, \beta)$. Then there are a lift $\tilde{\gamma}$ of $\gamma$, a lift $\tilde{\alpha}$ of $\alpha$ and $n$ lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\beta$ satisfying (i) to (iv) of Definition 4.1. If $\tilde{\beta}_{i_0}$ (for some $1 < i_0 < n$) is a lift of $\beta$, choose a subsequence $\tilde{\beta}_{j_1}, \ldots, \tilde{\beta}_{j_m}$ of length $m$ from $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$ such that $\tilde{\beta}_{j_1}, \tilde{\beta}_{j_0}$ and $\tilde{\beta}_n$ are contained in the subsequence. Then $\tilde{\gamma}, \tilde{\alpha}$ and the subsequence $\tilde{\beta}_{j_1}, \ldots, \tilde{\beta}_{j_m}$ satisfy (i) to (iv) of Definition 4.1 of $\text{PP}_m$ for $\gamma$. That is, $\gamma \in \text{PP}_m(\mathcal{A}, \alpha, \beta, \beta)$.

(3) Let a simple closed geodesic $\gamma \in \text{PP}_n(\mathcal{A}_2, \alpha, \beta, \beta)$ be given. Then there are a lift $\tilde{\gamma}$ of $\gamma$, a lift $\tilde{\alpha}$ of $\alpha$ and $n$ lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\beta$ satisfying (i) to (iv) of Definition 4.1. If a lift $\tilde{\alpha}'$ of $\mathcal{A}_1$ intersects both $\tilde{\gamma}$ and some $\tilde{\beta}_i$, then $\tilde{\alpha}' = \tilde{\alpha}$ because $\tilde{\alpha}'$ is a lift of $\mathcal{A}_2$. So (iv) of Definition 4.1 holds. Note that the conditions (i) to (iii) of Definition 4.1 are satisfied immediately with $\tilde{\gamma}, \tilde{\alpha}$ and $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_n\}$. Therefore, $\gamma \in \text{PP}_n(\mathcal{A}_1, \alpha, \beta, \beta)$.

We leave (4) as an exercise. □

**Remark 4.3.** The below proposition (Proposition 4.5) induces the converse of Lemma 4.2(1) so the following statement is satisfied. Whenever $\mathcal{A}$ and $\mathcal{B}$ are multicurves and $\alpha$ and $\beta$ are simple closed geodesics in $\mathcal{A}$ and $\mathcal{B}$, respectively, we have $i(\alpha, \beta) > 0$ if and only if $\text{PP}_n(\mathcal{A}, \alpha, \beta, \beta) \neq \emptyset$ for all $n > 2$. 
Lemma 4.4. Let $\alpha$ and $\beta$ be simple closed geodesics crossing each other. If a simple closed geodesic $\gamma$ is contained in $\text{PP}_n(\{\alpha\}, \alpha, \beta) = \text{PP}_n(\{\alpha\}, \alpha, \{\beta\}, \beta)$ for some $n$, then $n \leq \text{lcm}\{i(\alpha, \beta), i(\beta, \gamma)\}$.

Proof. Assume that $\gamma \in \text{PP}_n(\{\alpha\}, \alpha, \beta)$ for some $n > 2$. Then there are a lift $\tilde{\gamma}$ of $\gamma$, a lift $\tilde{\alpha}$ of $\alpha$ and $n$ lifts $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$ of $\beta$ such that $\tilde{\alpha}, \tilde{\beta}_i$, and $\tilde{\gamma}$ cross each other for each $i$ by (iii) of Definition 4.1. Then $n \leq \text{lcm}\{i(\alpha, \beta), i(\beta, \gamma)\}$ by Proposition 2.8(2).

Proposition 4.5. Let $\alpha$, $\beta$, and $\gamma$ be simple closed geodesics such that $i(\alpha, \beta) > 0$ and $i(\beta, \gamma) > 0$. Let $n > 2$ be given. If an integer $m$ satisfies that

$$|m| \geq \frac{n + 2 \cdot i(\alpha, \gamma)}{i(\beta, \gamma)} + 2,$$

then for every multicurve $\mathcal{B}$ containing $\beta$,

$$T^n_{\beta}((\{\alpha\} \cup \text{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)) \subseteq \text{PP}_n(\mathcal{B}, \beta, \{\gamma\}, \gamma).$$

Proof. Fix a multicurve $\mathcal{B}$ containing $\beta$. And choose $\delta \in \{\alpha\} \cup \text{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)$. Let $\tilde{\delta}$ and $\tilde{\alpha}$ be lifts of $\delta$ and $\alpha$, respectively, and let $\tilde{\beta}_1, \tilde{\beta}_0$ and $\tilde{\beta}_1$ be lifts of $\mathcal{B}$ satisfying the following; see Figure 6.

- If $\delta = \alpha$, let $\tilde{\delta} = \tilde{\alpha}$ be an arbitrary lift of $\alpha$. Let $\tilde{\beta}_0$ be a lift of $\beta$ crossing $\tilde{\alpha}$. Let $\tilde{\beta}_1$ be lifts of $\beta$ crossing $\tilde{\alpha}$ such that $\tilde{\beta}_0$ is the unique lift of $\mathcal{B}$ separating $\tilde{\beta}_1$ from $\tilde{\beta}_0$.

- If $\delta \in \text{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)$, then let $\tilde{\delta}$ and $\tilde{\alpha}$ be lifts of $\delta$ and $\alpha$, respectively, and let $\tilde{\beta}_1, \tilde{\beta}_0, \tilde{\beta}_1$ be lifts of $\mathcal{B}$ satisfying (ii) to (iv) of Definition 4.1 and satisfying that $\tilde{\beta}_0$ separates $\tilde{\beta}_1$ from $\tilde{\beta}_0$. If $\tilde{\beta}_0$ and some $\tilde{\beta}_i$, are separated by a lift $\tilde{\beta}'$ of $\mathcal{B}$, then substitute $\tilde{\beta}_i$ to $\tilde{\beta}'$. After then, the lifts $\tilde{\delta}, \tilde{\alpha}, \tilde{\beta}_1, \tilde{\beta}_0, \tilde{\beta}_1$ also satisfy Definition 4.1 for $\delta \in \text{PP}_3(\{\alpha\}, \alpha, \mathcal{B}, \beta)$. Repeating this process, we obtain the fact that $\tilde{\beta}_0$ is the unique lift of $\mathcal{B}$ separating $\tilde{\beta}_1$ from $\tilde{\beta}_0$. Note that $\tilde{\beta}_0$ is a lift of $\beta$ by (ii) of Definition 4.1.

Claim. The number of lifts of $\gamma$ separating $\tilde{\beta}_1$ from $\tilde{\beta}_0$ is at most $2 \cdot i(\alpha, \gamma)$.

Let $h$ be a primitive element of $\pi_1(\Sigma)$ preserving $\tilde{\alpha}$ such that $h\tilde{\beta}_1$ lies in the connected component of $\mathbb{H}^2 \setminus \tilde{\beta}_0$ containing $\tilde{\beta}_0$. Because $h\tilde{\beta}_1$ cannot separate $\tilde{\beta}_0$ from $\tilde{\beta}_1$, either $h\tilde{\beta}_1 = \tilde{\beta}_0$ or $h\tilde{\beta}_0$ separates $\tilde{\beta}_0$ from $h\tilde{\beta}_1$. Since $h$ sends $\tilde{\beta}_0$ into the connected component of $\mathbb{H}^2 \setminus \tilde{\beta}_0$ containing $\tilde{\beta}_1$, either $h\tilde{\beta}_0 = \tilde{\beta}_1$ or $h\tilde{\beta}_0$ separates $\tilde{\beta}_0$ from $h\tilde{\beta}_0$. Combining these facts, we obtain that either $h^2\tilde{\beta}_1 = \tilde{\beta}_1$ or $\tilde{\beta}_1$ separates $\tilde{\beta}_0$ from $h^2\tilde{\beta}_1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Figure 6}
\end{figure}
Let $Y_\gamma$ be the dual tree of $\gamma$. Let $\Phi := \Phi_{\gamma,r} : \mathbb{H}^2 \to Y_\gamma$ be a $\pi_1(\Sigma)$-equivariant surjective continuous map satisfying Proposition 2.3 for some sufficiently small number $r > 0$. For each $i = -1, 0, 1$, it holds that $\Phi_\gamma(\tilde{\alpha})$ interacts $\Phi_\gamma(\tilde{\beta}_i)$ because $\tilde{\alpha} \cap \tilde{\beta}_i$. See Figure 7. Every path joining $\Phi_\gamma(\tilde{\beta}_{-1})$ and $\sigma_\gamma(h^2)\Phi_\gamma(\tilde{\beta}_{-1})$ intersects $\Phi_\gamma(\tilde{\beta}_1)$ by the above. So we have $d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \Phi_\gamma(\tilde{\beta}_1)) \leq d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \sigma_\gamma(h^2)\Phi_\gamma(\tilde{\beta}_{-1}))$.

Because $Y_\gamma$ is a tree, the shortest geodesic between $\Phi_\gamma(\tilde{\beta}_{-1})$ and $\Phi_\gamma(\tilde{\beta}_1)$ is contained in $\Phi_\gamma(\tilde{\alpha})$. Since the translation length of $h$ on $Y_\gamma$ is $i(\alpha, \gamma)$, it is satisfied that

$$d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \Phi_\gamma(\tilde{\beta}_1)) \leq 2 \cdot i(\alpha, \gamma).$$

So the claim holds.

Let $v$ be the vertex on the dual tree $Y_\beta$ of $\beta$, which lies on the geodesic segment connecting $\Phi_\beta(\tilde{\beta}_{-1})$ and $\Phi_\beta(\tilde{\beta}_0)$. Let $T_\beta$ be the lift of a Dehn twist $T_\beta$ which fixes the closed 1-neighborhood of $v$. (cf. Proposition 3.5) Because $\Phi_\beta(\tilde{\beta}_{-1})$ and $\Phi_\beta(\tilde{\beta}_0)$ are contained in the closed 1-neighborhood of $v$, we have $T_\beta^m \tilde{\beta}_{-1} = \tilde{\beta}_{-1}$ and $T_\beta^m \tilde{\beta}_0 = \tilde{\beta}_0$. On the other hand, since $\tilde{\beta}_0$ is the unique lift of $\beta$ separating $\tilde{\beta}_1$ from $\Phi_{\beta^{-1}}(v)$, we have

$$T_\beta^m \tilde{\beta}_1 = h_0^m \tilde{\beta}_1$$

for some primitive element $h_0$ preserving $\tilde{\beta}_0$ by Proposition 3.5.

**Claim. The number of lifts of $\gamma$ separating $\tilde{\beta}_{-1}$ from $T_\beta^m \tilde{\beta}_1$ is at most $n$.**

If $\text{proj} : Y_\gamma \to \Phi_\gamma(\tilde{\beta}_0)$ be the closest-point projection to $\Phi_\gamma(\tilde{\beta}_0)$, let $P_i$ denote $\text{proj}(\Phi_\gamma(\tilde{\beta}_i))$ for each $i = -1, 1$. Then the length $l(P_i)$ of each $P_i$ is at most $i(\beta, \gamma)$ by Proposition 2.8. Let $w'$ and $w''$ be the vertices in $P_1$ satisfying that $d_\gamma(w', \sigma_\gamma(h_0^m)w'') = d_\gamma(P_1, \sigma_\gamma(h_0^m)P_1)$. Since $\Phi_\gamma(\tilde{\beta}_0)$ is preserved by $h_0$, we have $d_\gamma(P_1, \sigma_\gamma(h_0^m)P_1) = d_\gamma(w', \sigma_\gamma(h_0^m)w'')$

$$\geq d_\gamma(w', \sigma_\gamma(h_0^m)w') - d_\gamma(\sigma_\gamma(h_0^m)w', \sigma_\gamma(h_0^m)w'')$$

$$\geq |m| \cdot \text{tr}_c h_0 - l(P_1)$$

$$\geq (|m| - 1) \cdot i(\beta, \gamma).$$

By the hypothesis and the previous claim,

$$d_\gamma(\Phi_\gamma(\tilde{\beta}_{-1}), \sigma_\gamma(h_0^m)\Phi_\gamma(\tilde{\beta}_1)) \geq d_\gamma(P_{-1}, \sigma_\gamma(h_0^m)P_1)$$

$$\geq d_\gamma(P_1, \sigma_\gamma(h_0^m)P_1) - d_\gamma(P_{-1}, P_1) - l(P_{-1})$$

$$\geq (|m| - 1) \cdot i(\beta, \gamma) - 2 \cdot i(\alpha, \gamma) - i(\beta, \gamma)$$

$$\geq n.$$
So there are at least \( n \) edges of \( \gamma \) between \( \Phi_\gamma(\tilde{\beta}_{-1}) \) and \( \sigma_\gamma(h_0^m)\Phi_\gamma(\tilde{\beta}_1) \). In other words, there are \( n \) lifts \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \) of \( \gamma \) separating \( \tilde{\beta}_{-1} \) from \( h_0^m\tilde{\beta}_1 \). We proved the claim.

Since \( \tilde{T}_m^m\tilde{\delta} \) crosses both \( \tilde{T}_m^m\tilde{\beta}_{-1} = \tilde{\beta}_{-1} \) and \( \tilde{T}_m^m\tilde{\beta}_1 \), it also crosses all \( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \). See Figure 8. Note that \( \tilde{T}_m^m\tilde{\beta}_0 \) is the unique lift of \( B \) separating \( \tilde{\beta}_{-1} \) from \( \tilde{T}_m^m\tilde{\beta}_1 \). If a lift \( \tilde{\beta'} \) of \( B \) intersects \( \tilde{T}_m^m\tilde{\delta} \) and some \( \tilde{\gamma}_i \), then it must separates \( \tilde{\beta}_{-1} \) from \( h_0^m\tilde{\beta}_1 \) so that \( \tilde{\beta'} = \tilde{\beta}_0 \). Hence, \( T_m^m\delta \) is contained in \( \text{PP}_n(B, \beta, \{0\}, \gamma) \).

**Proposition 4.6.** Let \( A \) and \( B \) be multicurves, and let \( \alpha \in A \) and \( \beta \in B \) be simple closed geodesics such that \( i(\alpha, \beta) > 0 \). Then

\[
T_\gamma^m(\text{PP}_n(A, \alpha, B, \beta)) \subseteq \text{PP}_n(A, \alpha, B, \beta)
\]

for all \( \gamma \in A - \{\alpha\} \), \( n > 2 \) and \( m \in \mathbb{Z} \).

**Proof.** Choose \( \delta \in \text{PP}_n(A, \alpha, B, \beta) \). Let \( \tilde{\delta} \) and \( \tilde{\alpha} \) be lifts of \( \delta \) and \( \alpha \), respectively, and let \( \tilde{\beta}_1, \ldots, \tilde{\beta}_n \) be lifts of \( B \) satisfying (i) to (iv) of Definition 4.1 for \( \tilde{\delta} \in \text{PP}_n(A, \alpha, B, \beta) \).

Let \( \tilde{\alpha}_{-1} \) and \( \tilde{\alpha}_1 \) be lifts of \( A \) such that \( \tilde{\alpha} \) is the unique lift of \( A \) separating \( \tilde{\alpha}_{-1} \) from \( \tilde{\alpha}_1 \). Because \( \gamma \) is disjoint from \( \alpha \), there is a lift \( \tilde{T}_\gamma \) of a Dehn twist \( T_\gamma \) whose support is disjoint from \( \tilde{\alpha} \).

In this case, we have \( \tilde{T}_\gamma^m\tilde{\alpha}_i = \tilde{\alpha}_i \) for all \( i = -1, 1 \). So \( \tilde{T}_\gamma^m\tilde{\delta} \) still crosses both \( \tilde{\alpha}_{-1} \) and \( \tilde{\alpha}_1 \). It implies that all \( \tilde{\beta}_1, \ldots, \tilde{\beta}_n \) cross \( \tilde{T}_\gamma^m\tilde{\delta} \). Because \( \tilde{\alpha} \) is the unique lift of \( A \) separating \( \tilde{\alpha}_{-1} \) from \( \tilde{\alpha}_1 \), (iv) of Definition 4.1 also holds for \( \tilde{T}_\gamma\tilde{\beta} \). Therefore, \( T_\gamma^m\delta \in \text{PP}_n(A, \alpha, B, \beta) \).

**Proof of Theorem 1.6.** (1) and (3) are direct consequences of Lemma 4.4 and Proposition 4.6.

(2) By Lemma 4.2(2), (3) and (4), we have

\[
\text{PP}_n(A, \alpha, \beta) \subseteq \text{PP}_3(\{\alpha\}, \alpha, \beta) \subseteq \text{PP}_3(\{\alpha\}, \alpha, B, \beta).
\]

Therefore, \( T_\beta^m(\{\alpha\} \cup \text{PP}_n(A, \alpha, \beta)) \subseteq \text{PP}_n(B, \beta, \gamma) \) by Proposition 4.5.

**References**

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