FUNCTORS OF LIFTINGS OF PROJECTIVE SCHEMES

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ABSTRACT. A classical approach to investigate a closed projective scheme $W$ consists in considering a general hyperplane section of $W$, which inherits many properties of $W$. The inverse problem that consists in finding a scheme $W$ starting from a possible hyperplane section $Y$ is called a lifting problem, and every such scheme $W$ is called a lifting of $Y$. Investigations in this topic can produce methods to obtain schemes with specific properties. For example, any smooth point for $Y$ is smooth also for $W$.

We describe all the liftings of $Y$ with a given Hilbert polynomial by a parameter scheme which is obtained by gluing suitable affine open subschemes in a Hilbert scheme and defines a functor of points. We use constructive methods from Gröbner and marked bases theories. Furthermore, by classical tools we obtain an analogous result for equidimensional liftings. Examples of explicit computations are provided.

INTRODUCTION

Let $K$ be an infinite field, $A$ be a Noetherian $K$-algebra and $\mathbb{P}_A^n$ the $n$-dimensional projective space over $A$. A classical approach to investigate a closed projective scheme $X$ consists in considering a general hyperplane section of $X$, because many properties of $X$ are preserved under general hyperplane sections and can be easier recognized in subschemes of lower dimension.

The inverse problem that consists in finding a scheme $X$ starting from a possible hyperplane section is called a lifting problem and investigations in this topic can produce methods to obtain affine or projective schemes with specific properties.

In this paper, we consider the following lifting problem: given a closed subscheme $Y \subset \mathbb{P}_{K}^{n-1}$, describe all closed subschemes $W \subset \mathbb{P}_A^n$ such that $Y$ is a general hyperplane section of $W$, up to an extension of scalars.

Every such scheme $W$ is called a lifting of $Y$ over $A$ (Definition 3.1) and the saturated defining ideal $I$ of $W$ is called a lifting of the saturated defining ideal $I'$ of $Y$ (see Definition 3.2 and Proposition 3.3). By the definition, we almost suddenly obtain that $W$ is smooth at every point at which $Y$ is smooth (see Proposition 3.5). We always consider ideals that are homogeneous in the polynomial rings $A[x_0, \ldots, x_n]$ and $K[x_0, \ldots, x_{n-1}]$.

The framework of the present paper is functorial and constructive. We describe all the liftings of $Y$ with a given Hilbert polynomial by means of a parameter scheme which is a subscheme of a Hilbert scheme and represents a functor of points. We obtain an analogous result for equidimensional liftings if $Y$ is equidimensional. Both constructive and classical techniques are used. The constructive methods are borrowed from Gröbner and marked bases theories.

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This paper has been motivated by the investigation of the so-called $x_n$-liftings of a homogeneous polynomial ideal that has been faced in [6] from a functorial point of view. We highlight and exploit connections between the lifting problem considered here and the definition of $x_n$-lifting, which can be given in terms of ideals (see [13, 29]) or equivalently in terms of $K$-algebras, like proposed by Grothendieck (see [9, 29] and the references therein) (see Definition 2.4).

We now give a detailed outline of the contents of the paper.

Without loss of generality, we assume that $\mathbb{P}^{n-1}_K$ is defined in $\mathbb{P}^n_A$ by the nullity of the last variable $x_n$ in the polynomial ring $A[x_0, \ldots, x_n]$ (up to an extension of scalars). This assumption allows us to exploit the behavior of Gröbner bases with respect to the degree reverse lexicographic order when we need the saturation of homogeneous ideals and of their initial ideals (see Remark 1.6). The algebraic counterpart of our lifting problem consists in identifying the liftings $W$ of $Y$ with the $x_n$-liftings of ideals having saturation equal to $I'$ (see Proposition 4.3). This characterization implies that all the liftings with a given Hilbert polynomial $p(t)$ can be identified with the points of a disjoint union of locally closed subschemes in the Hilbert scheme $\text{Hilb}^n_{p(t)}$ via suitable Gröbner strata (see Theorem 4.4).

More precisely, we prove that a subscheme $W$ is a lifting of $Y$ only if $I$ belongs to the Gröbner stratum over a monomial ideal which is a lifting of the initial ideal of $I'$, in the further non-restrictive hypothesis that the variable $x_{n-1}$ is generic for $I'$ (see Definition 1.1 and Theorem 4.5). In particular, if the initial ideal of $I'$ is quasi-stable then also the initial ideal of $I$ is quasi-stable (see Theorem 7.2). The proofs of these results are constructive and then produce a method for the computation of the locally closed subschemes in a Hilbert scheme whose disjoint union corresponds to all liftings with a given Hilbert polynomial via Gröbner strata (see Example 7.4).

So, we obtain embeddings of liftings with a given Hilbert polynomial in Gröbner strata and, hence, in a Hilbert scheme. Then, it is natural to look at liftings from a functorial point of view. We are able to define functors related to our liftings as subfunctors of a Hilbert functor (Definitions 5.2 and 5.5). Given a Hilbert polynomial $p(t)$, we prove that the functor $L^{p(t),e}_Y$ of equidimensional liftings and the functor $L_Y^{p(t)}$ of liftings of $Y$ with Hilbert polynomial $p(t)$ are functors of points.

For what concerns the functor $L_Y^{p(t),e}$, we adapt to our situation classical arguments of Algebraic Geometry, like the upper semicontinuity of the dimension of the fibers of a dominant map (see Theorem 6.4).

For what concerns the functor $L_Y^{p(t)}$, the locally closed subfunctors that are represented by the locally closed schemes above introduced in suitable Gröbner strata are not necessarily open subfunctors of $L_Y^{p(t)}$. Hence, we study liftings in marked schemes over truncations of quasi-stable ideals, that are open subschemes in a Hilbert scheme.

Like we observe in Remark 9.5, marked schemes are not sufficient to characterize liftings. Anyway, once we have already obtained a characterization by Gröbner strata, we can exploit the features of marked schemes in order to construct an open covering of the functor $L_Y^{p(t)}$ that provides a scheme representing $L_Y^{p(t)}$ by a gluing procedure (see Lemma 9.4, Theorems 9.6 and 9.8 and Remark 9.9).

The proof of this last result is constructive. In Example 9.10 we exhibit an application of this construction. The computations performed in our examples were made by the symbolic systems Maple [24] and CoCoA [1].
1. Setting

We consider commutative unitary rings and morphisms that preserve the unit.

Let us denote by \( T_\mathbf{x} = \{ x_0^{\alpha_0} \cdots x_n^{\alpha_n} : \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1} \} \) and by \( T_{\mathbf{x},x_n} = \{ x_0^{\alpha_0} \cdots x_n^{\alpha_n} : \alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1} \} \) the sets of terms in the variables \( \mathbf{x} = \{ x_0, \ldots, x_{n-1} \} \) and terms in the variables \( \mathbf{x}, x_n = \{ x_0, \ldots, x_n \} \), respectively. The variables are ordered as \( x_0 > \cdots > x_n \).

For a term \( x^\alpha \) other than 1, we denote by \( \min(x^\alpha) \) the smallest variable appearing in \( x^\alpha \) with a non-zero exponent, by \( \max(x^\alpha) \) the greatest variable appearing in \( x^\alpha \) with a non-zero exponent, and by \( \deg(x^\alpha) = |\alpha| := \sum \alpha_i \) the degree of \( x^\alpha \).

Let \( K \) be an infinite field. We denote the polynomial ring \( K[x_0, \ldots, x_{n-1}] \) by \( K[\mathbf{x}] \) and the polynomial ring \( K[x_0, \ldots, x_n] \) by \( K[\mathbf{x}, x_n] \). For any (Noetherian) \( K \)-algebra \( A \), \( A[\mathbf{x}] \) denotes the polynomial ring \( K[\mathbf{x}] \otimes_K A \) and \( A[\mathbf{x}, x_n] \) denotes \( K[\mathbf{x}, x_n] \otimes_K A \). For every \( a \in A \), we set \( \deg(a) = 0 \). So, \( A[\mathbf{x}] \) and \( A[\mathbf{x}, x_n] \) are standard graded \( A \)-algebras and the polynomials and ideals we consider are homogeneous with respect to the variables \( x_0, \ldots, x_n \). Obviously, \( A[\mathbf{x}] \) is a subring of \( A[\mathbf{x}, x_n] \), hence the following notations and assumption will be stated for \( A[\mathbf{x}, x_n] \) but will hold for \( A[\mathbf{x}] \) too.

For any non-zero homogeneous polynomial \( f \in A[\mathbf{x}, x_n] \), the support \( \text{supp}(f) \) of \( f \) is the set of terms in the variables \( \mathbf{x}, x_n \) that appear in \( f \) with a non-zero coefficient and \( \deg(f) = \max\{\deg(x^\alpha) : x^\alpha \in \text{supp}(f)\} \). We denote by \( \text{coeff}(f) \subset A \) the set of the coefficients in \( f \) of the terms of \( \text{supp}(f) \). For any subset \( \Gamma \subseteq A[\mathbf{x}, x_n], \Gamma_t \) is the set of homogeneous polynomials of \( \Gamma \) of degree \( t \). Furthermore, we denote by \( (\Gamma) \) the \( A \)-module generated by \( \Gamma \). When \( \Gamma \) is a homogeneous ideal, we denote by \( \Gamma_\geq t \) the ideal generated by the homogeneous polynomials of \( \Gamma \) of degree \( \geq t \).

Let \( I \) be a homogeneous ideal of \( A[\mathbf{x}, x_n] \). The saturation of \( I \) is \( I^{\text{sat}} = \{ f \in A[\mathbf{x}, x_n] \mid \forall i = 0, \ldots, n, \exists k_i : x_i^{k_i} f \in I \} \). The ideal \( I \) is saturated if \( I = I^{\text{sat}} \) and is \( m \)-saturated if \( I_t = (I^{\text{sat}})^t, \) for every \( t \geq m \). The satiety of \( I \), denoted by \( \text{sat}(I) \), is the smallest \( m \) for which \( I \) is \( m \)-saturated.

**Definition 1.1.** A linear form \( h \in A[\mathbf{x}, x_n] \) is generic for \( I \) if \( h \) is not a zero-divisor in \( A[\mathbf{x}, x_n]/I^{\text{sat}} \). If \( I \) is an Artinian ideal, then every linear form \( h \) is intended to be generic for \( I \) (for example see [4, Definition (1.5)]).

**Remark 1.2.** If \( h \) is a generic linear form for \( I \), then \( I^{\text{sat}} = (I : h^\infty) := \{ f \in A[\mathbf{x}, x_n] \mid \exists t \geq 0 : h^t f \in I \} \) (see [4]).

A monomial ideal \( J \) of \( A[\mathbf{x}, x_n] \) is an ideal generated by terms. We denote by \( B_J \) the minimal set of terms generating \( J \) and by \( \mathcal{N}(J) \) the sous-escalier of \( J \), i.e. the set of terms outside \( J \). The following definitions apply in every polynomial ring.

**Definition 1.3.** Given a monomial ideal \( J \) and an ideal \( I \), a \( J \)-reduced form modulo \( I \) of a polynomial \( f \) is a polynomial \( \tilde{f} \) such that \( f - \tilde{f} \) belongs to \( I \) and \( \text{supp}(f) \) is contained in \( \mathcal{N}(J) \). If \( \tilde{f} \) is the unique possible \( J \)-reduced form modulo \( I \) of \( f \), then it is called the \( J \)-normal form modulo \( I \) of \( f \) and is denoted by \( \text{Nf}(f) \).

Some types of monomial ideals are particularly interesting due to useful combinatorial properties. In this paper, we will consider the following monomial ideals.

**Definition 1.4.** A monomial ideal \( J \subset A[\mathbf{x}, x_n] \) is quasi-stable if for every \( x^\alpha \in J \) and \( x_j > \min(x^\alpha) \), there is \( t \geq 0 \) such that \( x_j^t x^\alpha \) belongs to \( J \).
Remark 1.5. The smallest variable is generic for any quasi-stable ideal [31, Proposition 4.4(ii)].

With the usual language of Gröbner bases theory, from now, we consider the degree reverse lexicographic term order (degrevlex, for short) and, for every non-zero polynomial \( f \in A[x, x_n] \) and homogeneous ideal \( I \subset A[x, x_n] \), denote by \( \text{in}(f) \) and by \( \text{in}(I) \) the head term of \( f \) and the initial ideal of \( I \) with respect to the degrevlex.

Remark 1.6. Recall that the smallest variable is generic for a homogeneous polynomial ideal \( I \) if and only if it is generic for \( \text{in}(I) \). Indeed, the head term with respect to degrevlex of a homogeneous polynomial \( f \) is divisible by \( x_n^r \) if and only if \( f \) is divisible by \( x_n^r \).

In this paper we consider homogeneous ideals \( I \) generated by either Gröbner bases in \( K[x, x_n] \) or monic Gröbner bases in \( A[x, x_n] \) or marked bases over quasi-stable ideals in \( K[x, x_n] \) and \( A[x, x_n] \) (see Section 8). Hence, the quotients \( A[x, x_n]/I \) are free graded \( A \)-modules and the monomial ideals involved in the considered bases (initial ideals or quasi-stable ideals for marked bases) are always generated by terms. This is a key point for the use of functors we will introduce, because

\( (*) \) both monic Gröbner bases and marked bases over quasi-stable ideals have a good behavior with respect to the extension of scalars, that is if \( \varphi : A \to B \) is a morphism of \( K \)-algebras and \( I \) is an ideal in \( A[x] \) (or \( A[x, x_n] \)) generated by a monic Gröbner basis (resp. a marked basis over a quasi-stable ideal) \( G_I \), then \( I \otimes_A B \) is generated by \( \varphi(G_I) \) which is again a monic Gröbner basis (resp. a marked basis over a quasi-stable ideal), with the same head terms (see [3] and [8, Lemma 6.1]).

Due to previous item \( (*) \), for a homogeneous ideal \( I \) of \( A[x, x_n] \), we can consider the Hilbert function \( h_{A[x,x_n]/I} \) of the free graded \( A \)-module \( A[x, x_n]/I \) and its Hilbert polynomial \( p_{A[x,x_n]/I}(t) \) like the Hilbert function and the Hilbert polynomial of \( K[x, x_n]/\text{in}(I) \), respectively. When we say “ideal \( I \) with Hilbert polynomial \( p(t) \)” we mean \( p_{A[x,x_n]/I}(t) = p(t) \).

Remark 1.7. If \( L \) is a homogeneous ideal of \( K[x, x_n] \) then

\[
LA[x, x_n] = L \otimes_K A, \quad L = LA[x, x_n] \cap K[x, x_n], \quad (LA[x, x_n])^{\text{sat}} = L^{\text{sat}}A[x, x_n],
\]

and if \( I \) is a homogeneous ideal of \( A[x, x_n] \) then \( I^{\text{sat}} \cap K[x, x_n] = (I \cap K[x, x_n])^{\text{sat}} \).

2. **Background I: Gröbner functor and functor of \( x_n \)-liftings**

In this section, referring to [22, 23] and to [6] we collect some known information about the Gröbner functor and the functor of \( x_n \)-liftings.

Both these functors are subfunctors of a Hilbert functor. So, referring to [16, 27], we first recall that the Hilbert functor \( \text{Hilb}^n \) associates to a locally Noetherian \( K \)-scheme \( S \) the following set of flat families of subschemes of \( \mathbb{P}^n_S = \mathbb{P}^n_K \times_{\text{Spec}(K)} S \) parameterized by \( S \),

\[
\text{Hilb}^n(S) = \left\{ W \subset \mathbb{P}^n_S \mid W \text{ is flat over } S \right\},
\]

and to any morphism \( \phi : T \to S \) of locally Noetherian \( K \)-schemes the map

\[
\text{Hilb}^n(\phi) : \text{Hilb}^n(S) \to \text{Hilb}^n(T)
\]

\[
W \mapsto W \times_S T.
\]

Grothendieck [16] shows that the functor \( \text{Hilb}^n \) is representable. More precisely, there exists a locally Noetherian scheme \( \text{Hilb}^n \), called Hilbert scheme, together with a family \( X \subset \mathbb{P}^n_K \times \text{Hilb}^n \)
such that for every scheme \( W \subset \mathbb{P}^n_S \) flat over \( S \), \( W = \mathcal{X} \times_{\text{Hilb}^n_S} S \) for a unique morphism \( S \to \text{Hilb}^n \). We identify every point of a Hilbert scheme with the saturated ideal \( I \subseteq K[x_0, \ldots, x_n] \) that defines the corresponding fiber in \( \mathbb{P}^n_K \).

For every \( W \in \text{Hilb}^n \), if \( I \) is the saturated ideal defining \( W \), there is a numerical polynomial \( p(t) \in \mathbb{Q}[t] \) such that \( \dim_K(K[x_0, \ldots, x_n]/I)_m = p(m) \) for \( m \gg 0 \). This polynomial is the \textit{Hilbert polynomial} of \( W \). If \( p(t) \in \mathbb{Q}[t] \) is the Hilbert polynomial of some scheme in \( \mathbb{P}^n_K \), then \( p(t) \) is called \textit{admissible}, and we define a subfunctor of \( \text{Hilb}^n \):

\[
\text{Hilb}^n_{p(t)}(S) = \left\{ W \subset \mathbb{P}^n_S \mid W \text{ is flat over } S \text{ and has fibers with Hilbert polynomial } p(t) \right\}.
\]

Grothendieck proved the representability of \( \text{Hilb}^n \) using the stratification given by the admissible Hilbert polynomials in \( \mathbb{P}^n_K \). Indeed, we can consider the map \( s \mapsto p_s(t) \) that associates to every point of \( S \) the Hilbert polynomial of the fiber at the point \( s \in S \) of the flat morphism \( W \to S \). Thanks to flatness, this map is locally constant on \( S \).

Hence, \( \text{Hilb}^n \) decomposes as co-product of the above subfunctors:

\[
\text{Hilb}^n = \bigsqcup_{p(t) \text{ admissible for schemes in } \mathbb{P}^n_K} \text{Hilb}^n_{p(t)}.
\]

For every admissible polynomial \( p(t) \) of \( \mathbb{P}^n_K \), \( \text{Hilb}^n_{p(t)} \) is represented by a projective scheme \( \text{Hilb}^n_{p(t)} \).

The fact that \( \text{Hilb}^n \) and \( \text{Hilb}^n_{p(t)} \) are locally Noetherian allows to consider only the restriction of the functors \( \text{Hilb}^n \) and \( \text{Hilb}^n_{p(t)} \) to the category of Noetherian \( K \)-algebras (e.g. [12, Proposition VI-2 and Exercise VI-3]). Hence, \textit{from now} we can consider Noetherian affine schemes over \( K \), hence Noetherian \( K \)-algebras \( A \), instead of locally Noetherian \( K \)-schemes.

Since in this paper we consider only the degrevlex order, we now recall the notion of Gröbner functor in this particular setting.

\textbf{Definition 2.1.} [22, 23] Given a monomial ideal \( J \subset K[x, x_n] \), the \textit{Gröbner functor} \( \text{St}_J : \text{Sets} \to \text{Noeth-K-Alg} \) associates to any Noetherian \( K \)-algebra \( A \) the set \( \text{St}_J(A) := \{ I \subset A[x, x_n] : \text{in}(I) = J \otimes_K A \} \) and to any \( K \)-algebra morphism \( \phi : A \to B \) the function \( \text{St}_J(\phi) : \text{St}_J(A) \to \text{St}_J(B) \) such that the image of an ideal \( I \) is \( I \otimes_A B \). The family \( \text{St}_J(A) \) is called a \textit{Gröbner stratum}.

\textbf{Remark 2.2.} By construction, the ideals belonging to a Gröbner stratum share the same Hilbert function.

The functor \( \text{St}_J \) is the functor of points of an affine scheme [22, Theorem 3.6]. Now, we briefly recall the construction of its representing scheme \( \text{St}_J \), which is called \textit{Gröbner stratum scheme}. We consider a set \( G \) of polynomials of the following shape:

\[
(2.2) \quad G = \left\{ f_\alpha = x^\alpha + \sum_{J(\gamma) = \gamma < x^\alpha} C_{\alpha \gamma} x^\gamma : \text{in}(f_\alpha) = x^\alpha \in B_J \right\} \subset K[C_J][x, x_n]
\]

where \( C_J \) denotes the set of the new variables \( C_{\alpha \gamma} \). Let \( \mathfrak{a}_J \) be the ideal in \( K[C_J] \) generated by the coefficients of the terms of \( T_{x, x_n} \) in \( J \)-reduced forms \( S(f_\alpha, f_\beta) \) of the \( S \)-polynomials \( S(f_\alpha, f_\beta) \) modulo \( G \). Due to [22, Proposition 3.5] the ideal \( \mathfrak{a}_J \) depends only on \( J \) and on the given term.
order, which here is supposed to be the degrevlex. Hence, the ideal \( a_J \) defines the affine scheme \( \text{St}_J = \text{Spec}(K[C_J]/a_J) \).

Next theorem lists some main features of the functor \( \text{St}_J \).

**Theorem 2.3.** ([23, Lemma 5.2], [6, Theorem 2.2] and [22, Theorem 6.3]) Let \( J \subset A[x, x_n] \) be a monomial ideal and \( p(t) \) the Hilbert polynomial of \( A[x, x_n]/J \).

(i) \( \text{St}_J \) is a Zariski sheaf.

(ii) If the terms in \( B_J \) are not divisible by \( x_n \), then

\[
\text{St}_J \cong \text{St}_{J,m}, \text{ for every integer } m
\]

and \( \text{St}_J \) is a locally closed subfunctor of the Hilbert functor \( \text{Hilb}_{p(t)}^n \).

By Theorem 2.3, the scheme \( \text{St}_J \) can be embedded in the Hilbert scheme \( \text{Hilb}_{p(t)}^n \) as a locally closed subscheme. This fact is crucial in order to embed also the \( x_n \)-liftings of a homogeneous ideal in a Hilbert scheme.

**Definition 2.4.** ([13, 29] Let \( H \) be a homogeneous ideal of \( K[x] \). A homogeneous ideal \( I \) of \( A[x, x_n] \) is called a \( x_n \)-lifting of \( H \) if the following conditions are satisfied:

(a) the indeterminate \( x_n \) is a non-zero divisor in \( A[x, x_n]/I \);

(b) \( (I, x_n)/(x_n) \cong HA[x] \) under the canonical projection \( A[x, x_n]/(x_n) \cong A[x] \);

or, equivalently,

(b') \( \{g(x_0, x_1, \ldots, x_{n-1}, 0) : g \in I\} = HA[x] \).

Due to [10, Theorem 2.5] (see also [21, Proposition 6.2.6]), the following constructive characterization of \( x_n \)-liftings is given in [6, Theorem 3.2].

**Theorem 2.5.** ([10, Theorem 2.5], [21, Proposition 6.2.6], [6, Theorem 3.2 and Corollary 3.3]) Let \( H \) be a homogeneous ideal of \( K[x] \). A homogeneous ideal \( I \) of \( A[x, x_n] \) is a \( x_n \)-lifting of \( H \) if and only if the reduced Gröbner basis of \( I \) is of type \( \{f_\alpha + g_\alpha\}_\alpha \), where \( \{f_\alpha\}_\alpha \) is the reduced Gröbner basis of \( H \) and \( g_\alpha \in (x_n)A[x, x_n] \). If \( I \subset A[x, x_n] \) is a \( x_n \)-lifting of \( H \), then \( \text{in}(I) \) is generated by the same terms as \( \text{in}(H) \).

**Definition 2.6.** The functor of \( x_n \)-liftings \( \text{L}_H : \text{Noeth-K-Alg} \to \text{Sets} \) of a homogeneous ideal \( H \) of \( K[x] \) associates to every Noetherian \( K \)-algebra \( A \) the set \( \text{L}_H(A) = \{ I \subseteq A[x, x_n] : \text{I is a } x_n \text{-lifting of } H \} \) and to every morphism of \( K \)-algebras \( \phi : A \to B \) the map

\[
\text{L}_H(\phi) : \text{L}_H(A) \to \text{L}_H(B)
\]

\[
I \mapsto I \otimes_A B.
\]

With the notation of Definition 2.6, denote by \( J \) the ideal \( \text{in}(H)A[x, x_n] \) and by \( p(t) \) the Hilbert polynomial of \( A[x, x_n]/J \). Then, the functor \( \text{L}_H \) is a closed subfunctor of \( \text{St}_J \). More precisely, it is the functor of points of an affine scheme \( \text{L}_H \) which is a closed subscheme of \( \text{St}_J \) and, hence, a locally closed subscheme of the Hilbert scheme \( \text{Hilb}_{p(t)}^n \) [6, Theorem 4.3 and Proposition 6.1]. Moreover, it follows that \( \text{L}_H \) is a Zariski sheaf.

### 3. Liftings of Projective Schemes

In this section, we begin our investigation about the notion of **lifting of a projective scheme**. Recall that we only consider Noetherian affine schemes over \( K \), hence Noetherian \( K \)-algebras.
Definition 3.1. Let $\mathcal{H} = \mathbb{P}_{K}^{n-1}$ be the hyperplane of $\mathbb{P}_{K}^{n}$ defined by the ideal $(x_{n})$, $Y$ be a closed subscheme of $\mathbb{P}_{K}^{n-1}$ with Hilbert polynomial $p_{Y}(t)$ and $A$ be a $K$-algebra. A lifting of $Y$ over $A$ is a closed subscheme $W \in \text{Hilb}^{n}(A)$ of $\mathbb{P}_{A}^{n} = \mathbb{P}_{K}^{n} \times_{\text{Spec}(K)} \text{Spec}(A)$, such that:

(i) $\Delta_{W}(t) = p_{Y}(t)$;
(ii) $W \cap (\mathcal{H} \times_{\text{Spec}(K)} \text{Spec}(A)) = Y \times_{\text{Spec}(K)} \text{Spec}(A)$.

Observe that in Definition 3.1 we assume that the scheme $Y$ is contained in the hyperplane defined by the ideal $(x_{n})$. This assumption is not restrictive because, given a linear form $h \in K[x, x_{n}]$, we can always replace $h$ by the smallest variable $x_{n}$ thanks to a suitable (deterministic) change of coordinates.

Definition 3.2. Let $I'$ be a homogeneous saturated ideal of $K[x]$. A homogeneous saturated ideal $I$ of $A[x, x_{n}]$ is called a lifting of $I'$ if the following conditions are satisfied:

(a) the indeterminate $x_{n}$ is generic for $I$;
(b) $(I, x_{n})/(x_{n}) = I'A[x]$ under the canonical projection $A[x, x_{n}]/(x_{n}) \simeq A[x]$;
or, equivalently,
(b') $(\{g(x_{0}, x_{1}, \ldots, x_{n-1}, 0) : g \in I\})^{\text{sat}} = I'A[x]$.

Proposition 3.3. Let $Y$ be a closed subscheme of $\mathbb{P}_{K}^{n-1}$ defined by the homogeneous saturated ideal $I' \subset K[x]$ and $W \in \text{Hilb}^{n}(A)$ a closed subscheme of $\mathbb{P}_{A}^{n}$ defined by the homogeneous saturated ideal $I \subset A[x, x_{n}]$. Then, $W$ is a lifting of $Y$ if and only if $I$ is a lifting of $I'$.

Proof. By condition (ii) of Definition 3.1, we have $(I, x_{n})^{\text{sat}} = I'A[x] = I' \otimes_{K} A$, in particular the quotient $(A[x, x_{n}]/(I, x_{n}))$ has Hilbert polynomial $p_{Y}(t)$. Thus, by the following short exact sequence

$$0 \to (A[x, x_{n}]/(I : x_{n}))_{t-1} \xrightarrow{x_{n}} (A[x, x_{n}]/I)_{t-1} \to (A[x, x_{n}]/(I, x_{n}))_{t} \to 0$$

condition (i) of Definition 3.1, i.e. $\Delta_{W}(t) = p_{Y}(t)$, implies that the quotient $(A[x, x_{n}]/(I : x_{n})$ has the same Hilbert polynomial $p_{W}(t)$ as $A[x, x_{n}]/I$. Hence, we obtain $(I : x_{n})^{\text{sat}} = I^{\text{sat}} = I$ because $(I : x_{n}) \supseteq I$. In conclusion, we have $I^{\text{sat}} \supseteq (I : x_{n})^{\text{sat}} \supseteq (I : x_{n}) \supseteq I = I^{\text{sat}}$, namely $(I : x_{n}) = I$, which is possible only if $x_{n}$ is generic for $I$.

Conversely, if $I$ is a lifting of $I'$ then it is quite immediate that $W$ is a lifting of $Y$. Indeed, if $x_{n}$ is generic for $I$, then $(I : x_{n}) = I = I^{\text{sat}}$. Hence, $A[x, x_{n}]/(I : x_{n})$ and $A[x, x_{n}]/I$ have the same Hilbert polynomial. From (3.1), we obtain that $\Delta_{p}(t) = p_{Y}(t)$ and condition (ii) of Definition 3.1 also follows.

With the notation we have already introduced in Definition 3.1, we consider a closed subscheme $Y \subset \mathcal{H} \subset \mathbb{P}_{A}^{n}$, where $\mathcal{H}$ is defined by the ideal $(x_{n})$. If $W \subset \mathbb{P}_{A}^{n}$ is a lifting of $Y$, then $\deg W = \deg Y$ and $\dim W = \dim Y + 1$, so there are natural restrictions on the Hilbert polynomial $p(t)$ of $W$ because $\Delta_{p}(t)$ must be the Hilbert polynomial $p_{Y}(t)$ of $Y$. Hence, the non-constant part of the Hilbert polynomial of $W$ is determined by the Hilbert polynomial of $Y$, but in general there are no limits on the constant term of the Hilbert polynomial of $W$, also if we only consider liftings without zero-dimensional components, as shown by the following example.

Example 3.4. For every positive integer $k$, consider the double line $W_{k} \subset \mathbb{P}_{K}^{3}$ defined by the ideal $I = (x_{0}^{2}, x_{0}x_{1}, x_{1}^{2}, x_{0}x_{2}^{k} - x_{1}x_{3}^{k}) \subseteq K[x_{0}, x_{1}, x_{2}, x_{3}]$. The Hilbert polynomial of $W_{k}$ is
$p(t) = 2t + k + 1$ and $(X_k, x_3)$ is a lifting of the double point $Y \subset \mathbb{P}^2_K$ defined by the ideal $I' = (x_0, x_1^2) \subseteq K[x_0, x_1, x_2]$. In conclusion, for every positive integer $k$, we find a lifting of $Y$ with Hilbert polynomial $2t + k + 1$ and without zero-dimensional components.

We conclude this section observing that points that are smooth in $Y$ are also smooth in a lifting $W$ of $Y$.

**Proposition 3.5.** Let $W$ be a lifting of a scheme $Y$. If $Y$ is smooth on a point $P$ then also $W$ is smooth on $P$.

**Proof.** By definition of lifting, $Y$ is a Cartier divisor in $W$. Then the dimension of the Zariski tangent space of a point $y$ in $Y$ is not lower than the dimension of the Zariski tangent space of the point $y$ in $W$ minus 1. Indeed, if $m$ is the local ring at $y$ in $Y$ and $M$ is the local ring at $y$ in $W$, we have $\dim_K \frac{m}{m^2} \geq \dim_K \frac{M}{M^2} - 1$. Hence, if we had $\dim_K \frac{M}{M^2} > \dim(W)$ that we would obtain $\dim_K \frac{m}{m^2} \geq \dim_K \frac{M}{M^2} - 1 > \dim(W) - 1 = \dim(Y)$. \hfill $\Box$

4. **Construction of liftings of projective schemes**

We now investigate relations between the notion of lifting of a saturated homogeneous ideal in $K[x]$ and that of $x_n$-lifting of a homogeneous ideal in $K[x]$, obtaining a constructive characterization of the liftings of projective schemes.

In general a lifting is not a $x_n$-lifting, as shown by the following easy example. Nevertheless, we will show how to recover every lifting of a given saturated ideal by constructing the $x_n$-liftings of suitable families of ideals.

**Example 4.1.** Consider the ideal $I' = (x_0, x_1) \subseteq K[x_0, x_1, x_2]$. The ideal

$$I = (x_0^2 + x_3^2, x_0x_1, x_0x_2, x_1^2, x_1x_2) \subset K[x_0, x_1, x_2, x_3]$$

is a lifting of $I'$ but it is not a $x_3$-lifting of $I'$, as one can easily verify using Theorem 2.5.

**Lemma 4.2.** Let $I' \subseteq K[x]$ be a saturated ideal. If $I \subset A[x, x_n]$ is a lifting of $I'$ then $I' = \left(\frac{(I,x_n)}{(x_n)}\right)_{sat} \cap K[x] = \left(\frac{(I,x_n)}{(x_n)} \cap K[x]\right)_{sat}$.

**Proof.** The definition of lifting immediately implies the thesis by Remark 1.7. \hfill $\Box$

**Proposition 4.3.** Let $I' \subseteq K[x]$ be a homogeneous saturated ideal. A homogeneous ideal $I \subseteq A[x, x_n]$ is a lifting of $I'$ if and only if there exists a homogeneous ideal $H \subseteq K[x]$ such that $H_{sat} = I'$ and $I$ is a $x_n$-lifting of $H$.

**Proof.** First assume that $I$ is a lifting of $I'$ and take $H := \frac{(I,x_n)}{(x_n)} \cap K[x]$. Then, we have $H_{sat} = I'$ by Lemma 4.2. Furthermore, it is immediate that $I$ is a $x_n$-lifting of $H$, because $HA[x] = \frac{(I,x_n)}{(x_n)}$ by Remark 1.7 and $x_n$ is a non-zero divisor in $A[x, x_n]/I$ by definition.

Conversely, let $I \subset A[x, x_n]$ be an ideal which is a $x_n$-lifting of a homogeneous ideal $H \subseteq K[x]$ such that $H_{sat} = I'$. Then, $I$ is a lifting of $I'$, because $x_n$ is generic for $I$, so $I$ is saturated and $x_n$ is generic for $I$. Moreover, $\frac{(I,x_n)}{(x_n)} \cong H \otimes_K A$ implies $((I, x_n)/(x_n))_{sat} = I'A[x, x_n]$, by Remark 1.7. \hfill $\Box$

**Theorem 4.4.** Let $I' \subseteq A[x]$ be a homogeneous saturated ideal and $I \subseteq A[x, x_n]$ a lifting of $I'$. The locus of liftings of $I'$ in $\text{St}_{\text{in}}(I)(A)$ is parameterized by an affine scheme obtained by linear sections of $\text{St}_{\text{in}}(I)$.
Proof. Let $J' := \text{in}(I') \subset K[x]$ and $J := \text{in}(I) \subset A[x, x_n]$. By Proposition 4.3, $I$ is a $x_n$-lifting of a homogeneous ideal $H \subset K[x]$ with $H^{\text{sat}} = I'$. Hence, by Theorem 2.5, $\text{in}(H)$ has the same generators as $J$, so $\langle J_{x_n} \rangle = \text{in}(H) \otimes_K A$ and $\text{in}(H) = J \cap K[x]$.

Let $a_J \subset A[C_J]$ be the defining ideal of the Gröbner stratum scheme $\text{St}_J$ and $a_{J \cap K[x]} \subset K[C_{J \cap K[x]}]$ the defining ideal of the Gröbner stratum scheme $\text{St}_{J \cap K[x]}$, where $C_{J \cap K[x]} \subset C_J$.

We now characterize the ideal $H \subset K[x]$ by the following two conditions. The first condition is that $H$ belongs to the Gröbner stratum $\text{St}_{J \cap K[x]}(K)$, so the reduced Gröbner basis of $H$ consists of polynomials of the following type

\[
f_\beta = x^\beta + \sum_{\mathcal{N}((\text{in}(H))_{|\beta|} \exists x^\gamma < x^\beta} C_{\beta \gamma} x^\gamma, \quad f_\beta \in K[C_{J \cap K[x]}][x],
\]

for every term $x^\beta$ minimal generator of $J$. The second condition is that the polynomials $f_\beta$ belong to $I'$, because $H$ is contained in $I'$. This second condition implies that the saturation of $H$ is $I'$ because $K[x]/I'$ and $K[x]/H$ have the same Hilbert polynomial, by the first condition.

By construction, $\text{in}(H)$ is contained in $I'$. Hence, we have $\mathcal{N}(J') \subseteq \mathcal{N}(\text{in}(H))$ and can obtain a $J$-reduced form $\overline{f_\beta}$ modulo $I'$ of the polynomial $f_\beta$ using the reduced Gröbner basis of $I'$.

Imposing that $\overline{f_\beta}$ is zero we obtain that $H$ is contained in $I'$ and collect some new constraints on the coefficients in the parameters $C_{\beta \gamma}$ in $C_{J \cap K[x]}$. Let $b_{J \cap K[x]} \subset K[C_{J \cap K[x]}]$ be the ideal generated by these constraints, for every $x^\beta \in B_J$. The ideal $a_{J \cap K[x]} + b_{J \cap K[x]} \subset K[C_{J \cap K[x]}]$, hence the affine scheme $\text{Spec} \left( \frac{K[C_{J \cap K[x]}]}{a_{J \cap K[x]} + b_{J \cap K[x]}} \right)$, parameterizes the locus in $\text{St}_{J \cap K[x]}(K)$ of all the ideals $H \subset K[x]$ such that $H^{\text{sat}} = I'$.

Finally, we apply Theorem 2.5 and hence consider

\[
g_\beta := f_\beta + \sum_{x^\delta \in \mathcal{N}(J)_{|\beta|} - 1} C_{\beta \delta} x_n x^\delta, \quad g_\beta \in A[C_J][x, x_n],
\]

for every term $x^\beta$ minimal generator of $J$. The set $\{g_\beta\}_{x^\beta \in B_J}$ is a Gröbner basis with initial ideal $J$ modulo the ideal $a_J \subset A[C_J]$ which defines the Gröbner stratum scheme $\text{St}_J$.

We now observe that if the set of polynomials $g_\beta$ is a Gröbner basis then also the set of polynomials $f_\beta$ is a Gröbner basis, due to the hypothesis on $J$ and $\text{in}(H)$. This fact means that the ideal $a_{J \cap K[x]} A[C_J]$ is contained in $a_J$. Then, the ideal $a_J + b_{J \cap K[x]} A[C_J]$, hence the affine scheme $\text{Spec} \left( \frac{A[C_J]}{a_J + b_{J \cap K[x]} A[C_J]} \right)$, parameterizes the locus of the liftings of $I'$ in the Gröbner stratum $\text{St}_J(A)$.

It remains to show that the constraints we obtain by rewriting the polynomials $f_\beta$ of (4.1) by the reduced Gröbner basis of $I'$ are linear, i.e. the ideal $b_{J \cap K[x]}$ has linear generators. This fact is immediate, because the coefficients of the polynomials of the reduced Gröbner basis of $I'$ belongs to the field $K$. □

Theorem 4.4 describes the locus of liftings of a homogeneous saturated ideal $I' \subset K[x]$ in a Gröbner stratum. In the further hypothesis that the variable $x_{n-1}$ is generic for $I'$, we can recognize what Gröbner strata are candidate to contain these liftings.

**Theorem 4.5.** Let $I' \subset K[x]$ be a homogeneous saturated ideal. If $x_{n-1}$ is generic for $I'$, then the liftings of $I'$ belong to a Gröbner stratum over a monomial lifting $J \subset A[x, x_n]$ of $\text{in}(I')$. 


Proof. It is enough to prove that if \( I \subset A[x, x_n] \) is a lifting of \( I' \) and \( x_{n-1} \) is generic for \( I' \), then \( \left( \frac{(in(I), x_n)}{(x_n)} \right)^{\text{sat}} = in(I') \otimes_K A \). Hence, we will conclude by applying Theorem 4.4 and observing that if \( x_n \) is generic for \( I \) then it is a non-zero divisor for \( \text{in}(I) \).

Let \( J' := \text{in}(I') \) and \( J := \text{in}(I) \). By definition of lifting we have \( \left( \frac{(I, x_n)}{(x_n)} \right)^{\text{sat}} = I' \otimes_K A \). Hence, there exists an integer \( s \geq 0 \) such that \( \left( \frac{(I, x_n)}{(x_n)} \right)_{\geq s} = J'_{\geq s} \otimes_K A \). By [4, Lemma 2.2], we have \( \text{in}(I_{\geq s}, x_n) = (\text{in}(I_{\geq s}), x_n) \) and obtain

\[
\left( \frac{J_{\geq s}, x_n}{(x_n)} \right) = J'_{\geq s} \otimes_K A
\]

because \( \text{in}(I_{\geq s}) = (\text{in}(I))_{\geq s} = J_{\geq s} \) and \( \text{in}(I')_{\geq s} = (\text{in}(I'))_{\geq s} = J'_{\geq s} \). Now, it is enough to recall that \( J' \) is saturated because \( I' \) is saturated and \( x_{n-1} \) is not a zero-divisor in \( K[x]/I' \).

\[\Box\]

Remark 4.6. The condition that \( x_{n-1} \) is generic for \( I' \) is not restrictive because it can always be obtained up to a suitable change of variables, like we will point out in Section 7. On the other hand, the result of Theorem 4.5 does not hold without this hypothesis: for example, for the saturated ideal \( I' = (x_0^2, x_1x_0 + x_2^2) \subset K[x_0, x_1, x_2] \) we obtain \( \text{in}(I') = (x_0^2, x_1x_0, x_2^2x_0, x_2^4) \), that is not saturated.

Due to Theorems 4.4 and 4.5, we know where the liftings of a given saturated polynomial ideal \( I' \subset A[x] \) are located and how they can be constructed. In practice, we proceed in the following way.

Let \( I' \subset K[x] \) be a saturated polynomial ideal such that \( x_{n-1} \) is generic for \( I' \) and let \( p_Y(t) \) be the Hilbert polynomial of the scheme \( Y \) defined by \( I' \). For every admissible polynomial \( p(t) \) for schemes in \( \mathbb{P}_A^n \) such that \( \Delta p(t) = p_Y(t) \), we perform the following instructions:

- we compute the collection \( \mathcal{L} \) of all the monomial liftings \( J \subset A[x, x_n] \) of \( \text{in}(I') \) with Hilbert polynomial \( p(t) \); indeed, the liftings of \( I' \) belong to the Gröbner strata over these monomial liftings by Theorem 4.5;
- for every \( J \in \mathcal{L} \),
  - we construct the polynomials \( f_\beta \) as in (4.1) and impose that they belong to \( I' \) by computing their \( J \)-reduced forms \( \overline{f_\beta} \) modulo \( I' \) using the polynomials of the reduced Gröbner basis of \( I' \); let \( b_{J \cap K[x]} \subset K[C_{J \cap K[x]}] \) be the ideal generated by \( \text{coeff}(\overline{f_\beta}) \), for every \( x^\beta \in B_J \);
  - taking into account the fact that the polynomials \( g_\beta \) as in (4.2) form a reduced Gröbner basis modulo the ideal \( a_J \), the ideal \( a_J + b_{J \cap K[x]}A[C_J] \) defines the locus of the liftings \( I \subset A[x, x_n] \) of \( I' \) with Hilbert polynomial \( p(t) \) and initial ideal \( J \).

5. Functors of liftings

In Section 4 we obtained embeddings of the liftings of a homogeneous saturated ideal \( I' \subset K[x] \) in suitable Gröbner strata by Theorem 4.4. So, it is now natural to study liftings from a functorial perspective because Gröbner strata are described by functors.

The following result is a generalization of [6, Corollary 3.3]. If \( \phi : A \to B \) is a \( K \)-algebra morphism, we denote by \( \phi \) also the natural extension of \( \phi \) to \( A[x, x_n] \) and recall that the image under \( \phi \) of every ideal \( I \) in \( A[x, x_n] \) generates the extension \( I^e = IB[x, x_n] = I \otimes_A B \) (see [3]).
Proposition 5.1. Let $I' \subseteq K[x]$ be a saturated ideal and $\phi : A \to B$ a $K$-algebra morphism. For every lifting $I \subseteq A[x, x_n]$ of $I'$ over $A$ the ideal $I \otimes_A B$ is a lifting of $I'$ over $B$, with the same Hilbert polynomial as $I$.

Proof. Let $G_I$ be the reduced monic Gröbner basis of $I$ and $J = \text{in}(I)$. Then, $\phi(G_I)$ is a monic Gröbner basis of $I \otimes A B$ with the same head terms. Let $H \subseteq K[x]$ be the ideal such that $I$ is a $x_n$-lifting of $H$ and $H^{\text{sat}} = I'$. Let $G_H = \{ f_\beta \} x_\beta \in B_J$ be the reduced (monic) Gröbner basis of $H$. By Theorems 4.4 and 2.5 we have that $G_I$ is of the following type

$$G_I = \{ f_\beta + \sum_{x^i \in N(J)[\beta]-1} c_{\beta \delta} x_n x_\delta \} \beta, \ c_{\beta \delta} \in A.$$

The ideal $I \otimes A B$ is then generated by $\phi(G_I) = \{ f_\beta - \sum_{x^i \in N(J)[\beta]-1} \phi(c_{\beta \delta}) x_n x_\delta \} x_\beta \in B_J$, which is still a reduced Gröbner basis because the polynomials of $G_I$ are monic. Hence, $I \otimes A B$ is a $x_n$-lifting of $H$ by Theorem 2.5 and a lifting of $I'$ by Proposition 4.3.

For what concerns the Hilbert polynomial, it is sufficient to observe that $I$ and $I \otimes_K B$ have the same initial ideal, hence the same Hilbert polynomial. □

Given a scheme $Y \subseteq \mathbb{P}^{n-1}_K$, thanks to Proposition 5.1 we can now easily define some functors concerning the liftings of $Y$. Recall that we only consider Noetherian affine schemes over $K$, hence Noetherian $K$-algebras $A$.

Definition 5.2. Let $Y = \text{Proj} (K[x]/I')$ be a closed subscheme of $\mathbb{P}^{n-1}_K$ with Hilbert polynomial $p_Y(t)$ and $p(t)$ a Hilbert polynomial such that $\Delta p(t) = p_Y(t)$.

(a) The functor of liftings of $Y$, $\text{L}_Y : \text{Noeth}-K-\text{Alg} \to \text{Sets}$, associates to every Noetherian $K$-algebra $A$ the set

$$\text{L}_Y(A) := \{ I \subseteq A[x, x_n] : I \text{ lifting of } I' \}$$

and to every morphism of $K$-algebras $\phi : A \to B$ the map

$$\text{L}_Y(\phi) : \text{L}_Y(A) \to \text{L}_Y(B) \quad I \mapsto I \otimes_A B.$$

(b) The functor of liftings of $Y$ with Hilbert polynomial $p(t)$, $\text{L}_Y^{p(t)} : \text{Noeth}-K-\text{Alg} \to \text{Sets}$, associates to every Noetherian $K$-algebra $A$ the set

$$\text{L}_Y^{p(t)}(A) := \{ I \subseteq A[x, x_n] : I \text{ lifting of } I' \text{ with Hilbert polynomial } p(t) \}$$

and to every morphism of $K$-algebras $\phi : A \to B$ the map

$$\text{L}_Y^{p(t)}(\phi) : \text{L}_Y^{p(t)}(A) \to \text{L}_Y^{p(t)}(B) \quad I \mapsto I \otimes_A B.$$

It is immediate that $\text{L}_Y^{p(t)}$ is a subfunctor of $\text{L}_Y$, and furthermore $\text{L}_Y$ (resp. $\text{L}_Y^{p(t)}$) is a subfunctor of $\text{Hilb}^n$ (resp. $\text{Hilb}^n_{p(t)}$). In fact, recall that the functors $\text{Hilb}^n$ and $\text{Hilb}^n_{p(t)}$ are both representable by locally Noetherian schemes, hence it is enough to consider their restrictions to the category of Noetherian $K$-algebras.
Using the same arguments on Hilbert schemes that give (2.1), we obtain that the functor of liftings of $Y$ decomposes as a co-product of the above subfunctors

\[(5.1) \quad \mathbb{L}_Y = \coprod_{p(t) \text{ admissible for liftings of } Y} \mathbb{L}_{Y}^{p(t)}.\]

**Proposition 5.3.** The functor $\mathbb{L}_{Y}^{p(t)}$ is a Zariski sheaf.

**Proof.** Let $A$ be a Noetherian $K$-algebra and \{\(U_i = \text{Spec}(A_{a_i})\)\}_{i=1,...,r}$ an open covering of $\text{Spec}(A)$. This is equivalent to the fact \((a_1, \ldots, a_r) = A\). Consider a set of ideals $I_i \in \mathbb{L}_{Y}^{p(t)}(A_{a_i})$ such that for any pair of indexes $i \neq j$ we have

\[(5.2) \quad I_{ij} := I_i \otimes_{A_{a_i}} A_{a_i,a_j} = I_j \otimes_{A_{a_j}} A_{a_i,a_j} \in \mathbb{L}_{Y}^{p(t)}(A_{a_i,a_j}).\]

We need to show that there is a unique ideal $I \in \mathbb{L}_{Y}^{p(t)}(A)$ such that $I_i = I \otimes_A A_{a_i}$ for every $i$.

By Proposition 4.3, there are $H_i$ and $H_j$ ideals in $K[x]$ such that $H_i^{\text{sat}} = H_j^{\text{sat}} = I'$ and $I_i$ is a $x_n$-lifting of $H_i$, while $I_j$ is a $x_n$-lifting of $H_j$. By Theorem 2.5 and assumption (5.2), $H_i = H_j \subset K[x]$, hence $I_i$ belongs to $\mathbb{L}_H(A_{a_i})$ and $I_j$ belongs to $\mathbb{L}_H(A_{a_j})$. Since $\mathbb{L}_H$ is a Zariski sheaf, there is a unique $I \subset A[x,x_n]$ such that $I$ is a $x_n$-lifting of $H$, $I \otimes_A A_{a_i} = I_i$ and $I \otimes_A A_{a_j} = I_j$. By Proposition 4.3 we conclude that $I$ belongs to $\mathbb{L}_{Y}^{p(t)}(A)$ and is the unique ideal in $A[x,x_n]$ such that $I_i = I \otimes_A A_{a_i}$ for every $i$. \hfill \Box

Recall that a closed subscheme in $\mathbb{P}_n^K$ is *equidimensional* if all its components have the same dimension, in particular it has no embedded components. Thus, there exists an equidimensional lifting $W$ of a subscheme $Y$ only if $Y$ is equidimensional, i.e. the ideal $I' \subset K[x]$ defines an equidimensional scheme in $\mathbb{P}_n^{K-1}$. We say that a saturated ideal $I \subset A[x,x_n]$ is *equidimensional* if it defines families of equidimensional subschemes.

Next result highlights that base extension preserves the fibers on every $K$-point.

**Proposition 5.4.** Let $W$ be a scheme over a $K$-algebra $A$. If $\phi : A \to B$ is a morphism of $K$-algebras, then the fibers of $W \to \text{Spec}(A)$ are isomorphic to the fibers of $W \times_{\text{Spec}(A)} \text{Spec}(B) \to \text{Spec}(B)$ for every $K$-point.

**Proof.** We show that the fibers in $W$ and in $W \times_{\text{Spec}(A)} \text{Spec}(B)$ can be identified.

Let $\phi : A \to B$ be a morphism of $K$-algebras, $\phi^* : \text{Spec}(B) \to \text{Spec}(A)$ the corresponding morphism and $\text{Spec}(K) \to \text{Spec}(B)$ the morphism associated to a $K$-point of $\text{Spec}(B)$ (e.g. [18, Chapter II, Exercise 2.7]). Moreover, let $\text{Spec}(K) \to \text{Spec}(A)$ be the morphism associated to the $K$-point of $\text{Spec}(A)$ obtained by composition with $\phi^*$ and $W \times_{\text{Spec}(A)} \text{Spec}(K)$ the fiber on this $K$-point. Then, we obtain $\text{Spec}(K) \simeq W \times_{\text{Spec}(A)} \text{Spec}(K)$ due to the transitivity of base extension. \hfill \Box

Due to Proposition 5.4, under the hypothesis that $Y$ is an equidimensional scheme we can now introduce functors of equidimensional liftings.

**Definition 5.5.** Let $Y = \text{Proj}(K[x]/I')$ be an equidimensional closed subscheme of $\mathbb{P}_K^{n-1}$ with Hilbert polynomial $p_Y(t)$ and $p(t)$ a Hilbert polynomial such that $\Delta p(t) = p_Y(t)$.

(a) The functor of *equidimensional liftings of $Y$*, denoted by $\mathbb{L}^e_Y : \text{Noeth-}\text{Alg} \to \text{Sets}$, associates to every Noetherian $K$-algebra $A$ the set

\[\mathbb{L}_Y^e(A) := \{I \subset A[x,x_n] : I \text{ equidimensional lifting of } I'\}\]
and to every morphism of $K$-algebras $\phi : A \to B$ the map
\[
\mathbb{L}_Y^e(\phi) : \mathbb{L}_Y^e(A) \to \mathbb{L}_Y^e(B)
\]
\[
I \mapsto I \otimes_A B.
\]

(b) The functor of equidimensional liftings of $Y$ with Hilbert polynomial $p(t)$, denoted by $\mathbb{L}_Y^{p(t),e}$: Noeth-$K$-Alg $\to$ Sets, associates to every Noetherian $K$-algebra $A$ the set
\[
\mathbb{L}_Y^{p(t),e}(A) := \{ I \subset A[x, x_n] : I \text{ equidimensional lifting of } I' \text{ with Hilbert polynomial } p(t) \}
\]
and to every morphism of $K$-algebras $\phi : A \to B$ the map
\[
\mathbb{L}_Y^{p(t),e}(\phi) : \mathbb{L}_Y^{p(t),e}(A) \to \mathbb{L}_Y^{p(t),e}(B)
\]
\[
I \mapsto I \otimes_A B.
\]

By definition, the functor $\mathbb{L}_Y^e$ (resp. $\mathbb{L}_Y^{p(t),e}$) is a subfunctor of $\mathbb{L}_Y$ (resp. of $\mathbb{L}_Y^{p(t)}$) and we have
\[
\mathbb{L}_Y^e = \prod_{p(t) \text{ admissible for liftings of } Y \text{ in } \mathbb{P}^n_K} \mathbb{L}_Y^{p(t),e}
\]
similarly to formulas (2.1) and (5.1) for $\text{Hilb}^n$ and $\mathbb{L}_Y$.

The following sections are devoted to show that the functors we defined are isomorphic to functors of points. We will begin from $\mathbb{L}_Y^{p(t),e}$, deducing that also $\mathbb{L}_Y^e$ is a functor of points by equality (5.3), because we do not need to introduce other tools. We need some more notions and results (Sections 7 and 8) in order to prove that also $\mathbb{L}_Y^{p(t)}$ is a functor of points (and $\mathbb{L}_Y^e$ too, by equality (5.1)).

6. The functor $\mathbb{L}_Y^{p(t),e}$ is a functor of points

In this section, we prove that for a given Hilbert polynomial $p(t)$ the functor $\mathbb{L}_Y^{p(t),e}$ is a functor of points. We need some preliminary results.

Denote by $\text{Hilb}_p^{n,Y}$ the locus of points in $\text{Hilb}_p^n$ corresponding to schemes that contain $Y$.

**Theorem 6.1.** $\text{Hilb}_p^{n,Y}$ is a closed subscheme of $\text{Hilb}_p^n$ representing a closed subfunctor $\text{Hilb}_p^{n,Y}$ of $\text{Hilb}_p^n$.

**Proof.** If $p_Y(t)$ is the Hilbert polynomial of $Y$, we can consider the Hilbert-flag scheme $\mathcal{F}_p(Y,W)$ (see [20]) and the projections $\pi_1 : \mathcal{F}_p(Y,W) \to \text{Hilb}_p^{n-1}$ and $\pi_2 : \mathcal{F}_p(Y,W) \to \text{Hilb}_p^n$. Thus, $\pi^{-1}(Y)$ is the closed scheme consisting of the couples $(Y, W)$, where $W$ varies among all the closed subschemes of $\mathbb{P}^n_K$ containing $Y$ and having Hilbert polynomial $p(t)$. Thus, $\text{Hilb}_p^{n,Y} = \pi_2(\pi^{-1}(Y)) \simeq \pi^{-1}(Y)$ is the closed subfunctor of $\text{Hilb}_p^n$ consisting of the points in $\text{Hilb}_p^n$ corresponding to schemes containing $Y$. It follows also that $\text{Hilb}_p^{n,Y}$ represents the closed subfunctor of the Hilbert functor that associates to a scheme $S$ the set of subschemes $W$ of $\mathbb{P}^n_K \times S$ containing $Y \times_{\text{Spec}(K)} S$.

From now on we denote by $U_e$ the subset of $\text{Hilb}_p^n$ of points corresponding to families of equidimensional subschemes.

**Proposition 6.2.** [15, Théorème (12.2.1)(iii)] The subset $U_e$ of $\text{Hilb}_p^n$ is open.
Proposition 6.3. [14, Proposition (2.3.4)(iii)] Let $S$ be a locally noetherian $K$-scheme, $W$ be an element of $\text{Hilb}^n_{p(t)}(S)$ and $f : W \to S$ the corresponding flat projection. For every irreducible closed subset $S'$ of $S$, every irreducible component $W'$ of $f^{-1}(S')$ is dominant on $S'$, i.e. $f' = f|_{W'} : W' \to S'$ is dominant.

Theorem 6.4. Let $Y$ be an equidimensional closed subscheme of $\mathbb{P}^{n-1}$ with Hilbert polynomial $p_Y(t)$ and $p(t)$ a Hilbert polynomial such that $\Delta p(t) = p_Y(t)$. Then, $L_{\text{Y}^{p(t)},e}$ is the functor of points of a locally closed subscheme of $\text{Hilb}^n_{p(t)}$.

Proof. Let $S$ be a Noetherian $K$-scheme, $W$ an element of $\text{Hilb}^n_{p(t)}(S)$ and $f : W \to S$ the corresponding flat projection. The fibers of $f$ in $W$ have degree equal to $\deg(Y)$ and dimension equal to $\dim(Y) + 1$ because $\Delta p(t) = p_Y(t)$.

For every irreducible closed subset $S'$ of $S$, let $W'$ be any irreducible component of $f^{-1}(S')$. By Proposition 6.3, $W'$ is dominant on $S'$. Then, also $W' \cap (H \times S')$ is dominant on $S'$, because for every $s' \in S'$ the fiber of $s'$ in $W'$ has dimension at least 1 by construction, and hence $s'$ has a fiber in $W' \cap (H \times S')$ of dimension at least 0. Indeed, the dimension of every fiber in $W' \cap (H \times S')$ is between $\dim(Y) + 1$ and $\dim(Y)$.

Recall that the dimension of the fibers of a dominant morphism is an upper semicontinuous function, namely the subset of $s'$ fibers in $W \cap (H \times S')$ have dimension less than or equal to $\dim(Y)$ is open [26, Chapter I, section 8, Corollary 3]. Since this dimension cannot be strictly lower than $\dim(Y)$ by the previous argument, all the fibers of the above open subset have dimension equal to $\dim(Y)$.

In the above situation, if we assume that $W$ belongs to $\text{Hilb}^n_{p(t)}(U_e)$, where $U_e$ is the open subset of Proposition 6.2, we can also observe that the fibers in $W$ and $H \times \text{Spec}(K)$ intersect properly, implying that the degree of the fibers in $W \cap (H \times \text{Spec}(K) S)$ is less than or equal to $\deg(Y)$ because the fibers and $H \times \text{Spec}(K) S$ are equidimensional (see [17, Corollary 18.5] in case of varieties).

If we also assume that $W$ belongs to $\text{Hilb}^n_{p(t),Y}$, so that $W$ contains $Y$, we obtain that the fibers in $W \cap (H \times \text{Spec}(K) S)$ have degree equal to $\deg(Y)$ because $Y \times \text{Spec}(K) S \subseteq W \cap (H \times \text{Spec}(K) S)$. We can now conclude that there is an open subset in $\text{Hilb}^n_{p(t),Y}$ describing all subschemes $W$ such that $W \cap (H \times \text{Spec}(K) S) = Y \times \text{Spec}(K) S$, namely $W$ is a lifting of $Y$. It is immediate that any equidimensional lifting of $Y$ belongs to this open subset of $\text{Hilb}^n_{p(t),Y}$ and, hence, locally closed subscheme of $\text{Hilb}^n_{p(t)}$.

Remark 6.5. The locally closed subscheme that represents $L_{\text{Y}^{p(t)},e}$, and which has been introduced in the proof of Theorem 6.4, completely describes the locally Cohen-Macaulay liftings of $Y$ when $Y$ is a zero-dimensional scheme.

7. The case of a quasi-stable initial ideal

Let $I' \subset K[x]$ be a saturated homogeneous ideal with $x_{n-1}$ generic. We go back to consider the situation of Theorem 4.5. The number of saturated monomial ideals $J \subset A[x, x_n]$ that are liftings of $\text{in}(I')$ is infinite. Even if we only focus on those monomial liftings of $\text{in}(I')$ having a prescribed Hilbert polynomial, this number is finite, but can still be huge. Hence, we try to reduce the range of such monomial ideals, when it is possible.
In this section, we study liftings of \( I' \) when the initial ideal \( \text{in}(I') \) is quasi-stable. The assumption that \( \text{in}(I') \) is quasi-stable is not restrictive: indeed, this can be obtained by a change of coordinates on \( I' \), and this change does not affect the scheme \( Y = \text{Proj}(K[\mathbf{x}]/I') \) from a geometric point of view. Furthermore, if \( \text{in}(I') \) is quasi-stable, then every monomial lifting \( J \subset A[\mathbf{x}, x_n] \) of \( \text{in}(I') \) is quasi-stable too (Theorem 7.2). Quasi-stability for initial ideals will allow us to use the techniques concerning marked bases over quasi-stable ideals that were developed in [8] (see [2] for the more general case of free modules) and that will be recalled in next section.

**Lemma 7.1.** Let \( J \subset A[\mathbf{x}, x_n] \) be a monomial ideal. If \( J_{\geq s} \) is quasi-stable for some integer \( s \), then \( J \) is quasi-stable.

**Proof.** If \( J_{\geq s} \) is quasi-stable, it is enough to check the condition of Definition 1.4 for every term \( x^\alpha \in J \) with \( |\alpha| < s \). For every \( x_i > \min(x^\alpha) \), take \( x^\alpha x_i^{s-|\alpha|} \in J_{\geq s} \). Then, there is an integer \( t \) such that \( \frac{x^\alpha x_i^{s-|\alpha|}+t}{\min(x^\alpha x_i^{s-|\alpha|})} \) belongs to \( J_{\geq s} \), because \( J_{\geq s} \) is quasi-stable. We conclude by observing that \( \min(x^\alpha) = \min(x^\alpha x_i^{s-|\alpha|}) \) by construction. \( \square \)

**Theorem 7.2.** If \( J' \subset K[\mathbf{x}] \) is a saturated quasi-stable ideal then a monomial lifting \( J \subset A[\mathbf{x}, x_n] \) of \( J' \) is quasi-stable.

**Proof.** Consider the ideal \( L = \frac{(J,x_n)}{(x_n)} \cap K[\mathbf{x}] \). Since quasi-stability is a property concerning the semigroup structure of the ideal generated by the minimal monomial basis of \( J \), regardless of the coefficients of the polynomial ring, it is sufficient to prove that \( L \) is quasi-stable.

By the hypothesis, we have \( L^{\text{sat}} = J' \) and hence, if \( s = \text{sat}(L) \), then \( L_{\geq s} = J'_{\geq s} \) is quasi-stable. By Lemma 7.1(i), \( L \) is quasi-stable too. \( \square \)

**Corollary 7.3.** Let \( I' \subset K[\mathbf{x}] \) be a homogeneous saturated ideal and \( I \subset A[\mathbf{x}, x_n] \) be a lifting of \( I' \). If \( \text{in}(I') \) is quasi-stable, then \( \text{in}(I) \) is a quasi-stable lifting of \( \text{in}(I') \).

**Proof.** This is a consequence of Remarks 1.5 and 1.6 and of Theorems 7.2 and 4.5. \( \square \)

The following example shows an explicit computation of the monomial liftings of \( \text{in}(I') \), when \( \text{in}(I') \) is quasi-stable, and an explicit computation of the ideal \( H \) of Proposition 4.3.

**Example 7.4.** In this example, quasi-stable ideals are computed by the algorithm described in [5]. Let \( \mathbf{x} := \{x_0, \ldots, x_3\} \) and consider the saturated ideal \( I' = (x_0^2, x_0 x_1 + x_1^3, x_0 x_2) \cap (x_2, x_3, x_0 x_1 + x_1^3, x_2^3) \subset K[\mathbf{x}] \) (see [25]). The reduced Gröbner basis of \( I' \) is \( G' = \{x_0 x_2, x_0 x_1 + x_1^3, x_3^2, x_2 x_3^2, x_3^3\} \), hence the initial ideal is the quasi-stable ideal \( J' := \text{in}(I') = (x_0^2, x_0 x_1, x_0 x_2, x_1^2 x_2, x_1^3) \). The Hilbert function of \( \frac{K[\mathbf{x}][I']}{p} \) is

\[
h_{K[\mathbf{x}]/I'}(0) = 1, \quad h_{K[\mathbf{x}]/I'}(1) = 4 \quad \text{and} \quad h_{K[\mathbf{x}]/I'}(t) = 2t + 3 \quad \text{for every } t \geq 2,
\]

hence \( I' \) defines a curve \( Y \) in \( \mathbb{P}^3_K \). A Hilbert polynomial \( p(t) \) such that \( \Delta p(t) = 2t + 3 \) must be of type \( p(t) = t^2 + 4t + c \).

Assume that \( W \subset \mathbb{P}^3_K \) is a surface and is a lifting of \( Y \) over a Noetherian \( K \)-algebra \( A \). We now observe that the Hilbert polynomial of \( W \) is \( p(t) = t^2 + 4t + c \) with \( c \geq 0 \). Indeed, since \( x_4 \) is not a zero-divisor on \( S/I \), we have the short exact sequence

\[
0 \to (A[\mathbf{x}, x_4]/I)_{t-1} \to (A[\mathbf{x}, x_4]/I)_t \to (A[\mathbf{x}, x_4]/(I, x_4))_t \to 0,
\]
that gives \( h_{A[x,x_4]/(I,x_4)}(t) = \Delta h_{A[x,x_4]/I}(t) \), in particular \( p_{A[x,x_4]/(I,x_4)}(t) = \Delta p_{A[x,x_4]/I}(t) \) and \( \Delta h_W(t) \geq h_Y(t) \) for every \( t \). As a consequence, \( h_W(t) \geq \sum_{i=0}^{t} h_Y(i) \), hence

\[
h_{A[x,x_4]/I}(0) = 1, h_{A[x,x_4]/I}(1) = 5 \text{ and } h_{A[x,x_4]/I}(t) \geq t^2 + 4t \text{ for every } t \geq 2,
\]
and \( c = 0 \) is the minimal possible value of the constant term \( c \) for the Hilbert polynomial of \( W \). We now investigate the cases \( c = 0 \) and \( c = 1 \).

If \( c = 0 \), among 56 possible quasi-stable saturated ideals there is a unique quasi-stable ideal \( J \subset K[x, x_4] \) such that \( \left( \frac{J(x_4)}{(x_4)} \right)_{\text{sat}} \cap K[x] = J' \): it is \( J = J' \cdot K[x, x_4] \). Hence, in this case the liftings of \( I' \) belong to the Gröbner stratum \( St_{\gamma}(A) \) and are exactly the \( x_4 \)-liftings of \( I' \) because \( J \) and \( J' \) share the same generators, using Theorem 4.5.

If \( c = 1 \), among 176 possible quasi-stable saturated ideals there are only the following 5 quasi-stable ideals \( J^{(i)} \subset K[x, x_4] \) such that \( \left( \frac{J^{(i)}(x_4)}{(x_4)} \right)_{\text{sat}} \cap K[x] = \text{in}(I') \):

\[
\begin{align*}
J^{(1)} &= \{x_2x_0, x_0x_1 + x_1^2, x_3x_2^2, x_2x_1^2, x_3^3, x_0^3\}, \\
J^{(2)} &= \{x_2x_0, x_0^2 + ax_0x_1 + ax_1^2, x_0x_1x_3 + x_1^2x_3, x_2x_1^2, x_3^3, x_0^3\}, \\
J^{(3)} &= \{x_0x_1 + x_1^2 + bx_0x_2, x_0^2 + cx_0x_2, x_3x_2x_0, x_0x_2^2 + x_2x_1^2, x_3^2\}, \\
J^{(4)} &= \{x_0x_2, x_0x_1 + x_1^2 + x_2^2 + d^2x_0x_2, x_3x_2^2, x_3^2\}, \\
J^{(5)} &= \{x_0x_2, x_0x_1 + x_1^2 + x_2^2 + d^2x_0x_2, x_3x_2^2, x_3^2\}.
\end{align*}
\]

Hence, in this case the liftings of \( I' \) belong to the union of Gröbner strata over the above quasi-stable ideals in the Hilbert scheme \( \text{Hilb}_{1}^{4+4t+1} \). Now, we apply the construction described in the proof of Theorem 4.4 to these monomial ideals and obtain the following families of Gröbner bases for the ideals of type \( H' \):

\[
\begin{align*}
J^{(1)}: \ G^{(1)} &= \{x_2x_0, x_0x_1 + x_1^2, x_3x_2^2, x_2x_1^2, x_3^3, x_0^3\}, \\
J^{(2)}: \ G^{(2)}(a) &= \{x_2x_0, x_0^2 + ax_0x_1 + ax_1^2, x_0x_1x_3 + x_1^2x_3, x_2x_1^2, x_3^3, x_0^3\} \text{ where } a \in A, \\
J^{(3)}: \ G^{(3)}(b, c) &= \{x_0x_1 + x_1^2 + bx_0x_2, x_0^2 + cx_0x_2, x_3x_2x_0, x_0x_2^2 + x_2x_1^2, x_3^2\} \text{ where } b, c \in A, \\
J^{(4)}: \ G^{(4)} &= \{x_2x_0, x_0x_1 + x_1^2 + x_2x_1^2 + x_3x_2^2, x_3^2\}, \\
J^{(5)}: \ G^{(5)}(d) &= \{x_0x_2, x_0x_1 + x_1^2 + x_2x_1^2 + d^2x_0x_2, x_3x_2^2, x_3^2\} \text{ where } d \in A.
\end{align*}
\]

In conclusion, for every \( k = 1, \ldots, 5 \), the liftings of \( I' \) in \( St_{\gamma}(k) \) are the \( x_4 \)-liftings of the ideals generated by the corresponding Gröbner bases \( (G^{(k)}(a, b, c, d)) \) modulo the defining ideal of the Gröbner stratum over \( J^{(k)} \).

8. **Background II: marked functor over a truncation ideal**

In this section, referring to [23, 8] we recall the notion of marked functor over a truncation ideal and its main features. Marked bases and functors over quasi-stable ideals are the main tool we use in Section 9 to prove the representability of \( L_{p}^{(t)} \).

**Definition 8.1.** [30] The **Pommeret cone** of a term \( x^a \) is the set \( C_{P}(x^a) = \{x^a x^b \mid \max(x^b) \leq \min(x^a)\} \). Given a finite set \( M \) of terms, its **Pommeret span** is \( \cup_{x^a \in M} C_{P}(x^a) \). The finite set of terms \( M \) is a **Pommeret basis** of the ideal \( (M) \) if the Pommeret span of \( M \) coincides with the set of terms in \( (M) \) and the Pommeret cones of the terms in \( M \) are pairwise disjoint.

**Theorem 8.2.** [31, Definition 4.3 and Proposition 4.4] A monomial ideal \( J \) is quasi-stable if and only if it has a Pommeret basis.
**Definition 8.3.** [28] A **marked polynomial** is a polynomial $F$ together with a specified term of $\text{supp}(F)$ that will be called **head term** of $F$ and denoted by $\text{Ht}(F)$.

**Definition 8.4.** [11, Definition 5.1] Let $J \subset A[x, x_n]$ be a quasi-stable ideal.

A **$P(J)$-marked set** (or marked set over $P(J)$) $G$ is a set of homogeneous monic marked polynomials $F_\alpha$ in $A[x, x_n]$ such that the head terms $\text{Ht}(F_\alpha) = x^\alpha$ are pairwise different and form the Pommaret basis $P(J)$ of $J$, and $\text{supp}(F_\alpha - x^\alpha) \subset N(J)$.

A **$P(J)$-marked basis** (or marked basis over $P(J)$) $G$ is a $P(J)$-marked set such that $N(J)$ is a basis of $A[x, x_n]/(G)$ as an $A$-module, i.e. $A[x, x_n] = (G) \oplus (N(J))$ as an $A$-module.

Let $J \subset A[x, x_n]$ be a saturated quasi-stable ideal and $m$ a non-negative integer. The ideal $J_{\geq m}$ is called **$m$-truncation** of $J$ and is quasi-stable, because $J$ is. Referring to [11, 23, 8], we now recall the definition of marked functor over the $m$-truncation of a saturated quasi-stable ideal.

**Definition 8.5.** [8, 23] The **marked functor** $\overline{\text{Mf}}_{P(J_{\geq m})} : \text{Noeth-K-Alg} \rightarrow \text{Sets}$ associates to every Noetherian $K$-algebra $A$ the set

$$\overline{\text{Mf}}_{P(J_{\geq m})}(A) := \{I \text{ saturated ideal in } A[x, x_n] : A[x, x_n] = I_{\geq m} \oplus \langle N(J_{\geq m}) \rangle \} =$$

$$= \{I \text{ saturated ideal in } A[x, x_n] : I_{\geq m} \text{ is generated by a } P(J_{\geq m})\text{-marked basis} \}$$

and to every $K$-algebra morphism $\phi : A \rightarrow B$ the function

$$\phi : \overline{\text{Mf}}_{P(J_{\geq m})}(A) \rightarrow \overline{\text{Mf}}_{P(J_{\geq m})}(B) \text{ given by } \phi(I) = I \otimes_A B.$$

**Remark 8.6.**

(i) The fact that $I$ belongs to $\overline{\text{Mf}}_{P(J_{\geq m})}(A)$ does not imply that $I$ belongs to $\overline{\text{Mf}}_{P(J)}(A)$ (e.g. [8, Example 3.8]).

(ii) Observe that if $I$ belongs to $\overline{\text{Mf}}_{P(J_{\geq m})}(A)$, then the Hilbert function of $A[x, x_n]/I_{\geq m}$ is equal to the Hilbert function of $A[x, x_n]/J_{\geq m}$.

**Remark 8.7.** In the original definition of marked functor over a $m$-truncation of a quasi-stable ideal introduced in [23] we exactly read:

$$\overline{\text{Mf}}_{P(J_{\geq m})}(A) := \{M \text{ homogeneous ideal in } A[x, x_n] : A[x, x_n] = M \oplus \langle N(J_{\geq m}) \rangle \}.$$ 

Our presentation is equivalent to the original one because a homogeneous ideal $H \subset A[x, x_n]$ such that $A[x, x_n] = M \oplus \langle N(J_{\geq m}) \rangle$ is exactly of type $I_{\geq m}$, where $I = M_{\text{sat}}$ (see [8, Corollary 3.7]).

Let $\rho$ be the maximal degree of a term in the Pommaret basis $P(J)$ that is divisible by $x_{n-1}$, i.e. $\rho$ is the satiety of the quasi-stable ideal $(J, x_n)/(x_n)$ in $A[x]$ [8, Theorem 2.2].

If $m \geq \rho - 1$, then the marked functor $\overline{\text{Mf}}_{P(J_{\geq m})}$ is a **representable open subfunctor** of the Hilbert functor $\text{Hilb}^n$$_t$, where $p(t)$ is the Hilbert polynomial of $A[x, x_n]/J$, and we denote by $\text{Mf}_P(J_{\geq m})$ its representing scheme (see [23] and [8, Proposition 6.13]).

Moreover, for every $m \geq \rho$, there is a scheme-theoretical isomorphism between $\overline{\text{Mf}}_{P(J_{\geq m-1})}$ and $\overline{\text{Mf}}_{P(J_{\geq m})}$ (see [7, Theorem 5.7] if $J$ is strongly stable and [8, Section 6] in the more general case $J$ is quasi-stable).

For any non-negative integer $m$ we have (see Theorem 2.3(ii) and [23])

$$\text{St}_J \simeq \text{St}_{J_{\geq m}} \subseteq \overline{\text{Mf}}_{P(J_{\geq m})} \quad (8.1)$$
or, equivalently,
\[(8.2) \quad \text{St}_f(A) \simeq \text{St}_{J_{\geq m}}(A) \subseteq \text{Mf}_p(J_{\geq m})(A). \]

9. The functor $L_Y^{p(t)}$ is a functor of points

We can now give a constructive proof of the fact that the functor $L_Y^{p(t)}$ is isomorphic to a functor of points. We use the result described in [12, Theorem VI-14] and [32, Proposition 2.16] and the notion of marked functor together with Theorem 7.2.

**Notation 9.1.** Let $I' \subset K[x]$ be a homogeneous saturated ideal with $J' = \text{in}(I')$ quasi-stable, and $J \subset A[x, x_n]$ a monomial lifting of $J'$. Let $Y$ be the projective subscheme defined by $I'$.

We denote by $L_Y^{p(t)}$ the functor $L_Y^{p(t)} \cap \text{Mf}_p(J_{\geq m})$, considered as a subfunctor of $\text{Hilb}^n_A$. More precisely, the functor $L_Y^{p(t)} : \text{Noeth-K-Alg} \to \text{Sets}$ associates to every Noetherian $K$-algebra $A$ the set $L_Y^{p(t)}(A) := \{L_Y^{p(t)}(A) \cap \text{Mf}_p(J_{\geq m})(A)\}$ and to every $K$-algebra morphism $\phi : A \to B$ the function $\phi : L_Y^{p(t)}(A) \to L_Y^{p(t)}(B)$ given by $\phi(I) = I \otimes_A B$.

**Lemma 9.2.** Let $I' \subset K[x]$ be a homogeneous saturated ideal with $J' = \text{in}(I')$ quasi-stable, $J \subset A[x, x_n]$ a monomial lifting of $J'$ and $\rho$ the satriety of $(J, x_n)/(x_n) \subset A[x]$. If $m \geq \rho$ then:

(i) A saturated ideal $I \subset A[x, x_n]$ belongs to $L_Y^{p(t)}(J_{\geq m-1})(A)$ if and only if $I$ is a lifting of $I'$ and $I_{\geq m-1}$ has a $P(J_{\geq m-1})$-marked basis.

(ii) $L_Y^{p(t)}(J_{\geq m-1})$ is an open subfunctor of $L_Y^{p(t)}$.

**Proof.** The statements are direct consequences of Definition 9.1 and of properties of marked functors. For the openness of $L_Y^{p(t)}(J_{\geq m-1})$ as a subfunctor of $L_Y^{p(t)}$, see for instance [19, Section 1.7, item (2)].

**Lemma 9.3.** Let $J' \subset K[x]$ be a saturated quasi-stable ideal, $J \subset A[x, x_n]$ a monomial lifting of $J'$ and $\rho$ the satriety of $(J, x_n)/(x_n) \subset A[x]$. If $m \geq \rho$ then:

(i) \( \left( \frac{J_{x_n}}{(x_n)} \right)_{\geq m} = J'_{\geq m} \otimes A \), and

(ii) $\mathcal{P}(J_{\geq m}) = \{ x^\alpha \in \mathcal{P}(J_{\geq m}) : x^\alpha \text{ is not divisible by } x_n \}$.

**Proof.** Observe that the terms of the Pommaret basis of $J$ are not divisible by $x_n$ [8, Remark 2.8]. Moreover, recall that in our setting we have $J' \otimes A := \left( \frac{J_{x_n}}{(x_n)} \right)^{\text{sat}}$, where the ideal $\left( \frac{J_{x_n}}{(x_n)} \right) = (\mathcal{P}(J)) \cdot A[x]$ is quasi-stable and $m$-saturated because $m \geq \rho$. Thus, we obtain $J'_{\geq m} \otimes A = \left( \frac{J_{x_n}}{(x_n)} \right)_{\geq m}$ and then use the definition of Pommaret basis.

**Lemma 9.4.** Let $J' \subset K[x]$ be a saturated quasi-stable ideal, $J \subset A[x, x_n]$ a monomial lifting of $J'$ and $\rho$ the satriety of $(J, x_n)/(x_n) \subset A[x]$. For every $m \geq \rho$, if $I \subset A[x, x_n]$ belongs to $\text{Mf}_p(J_{\geq m-1})(A)$, then $\left( \frac{J_{x_n}}{(x_n)} \right)_{\geq m}$ belongs to $\text{Mf}_p(J'_{\geq m})(A)$.

**Proof.** By hypothesis, $I$ has a $\mathcal{P}(J_{\geq m-1})$-marked basis. We want to show that $\left( \frac{J_{x_n}}{(x_n)} \right)_{\geq m}$ has a $\mathcal{P}(J'_{\geq m})$-marked basis. For every $t \geq m$ we have
\[(9.1) \quad h_{A[x, x_n]/(J, x_n)}(t) = h_{A[x, x_n]/(J, x_n)}(J_{x_n})(t) = h_{K[x]/J'}(t). \]
In fact, for every $t \geq m$, we have $h_{A[x,x_n]/I}(t-1)=h_{A[x,x_n]/J}(t-1)$ because $I_{\geq m-1}$ has a $P(J_{\geq m-1})$-marked basis; for every $t \geq 0$, we have $h_{A[x,x_n]/I}(t)=h_{A[x,x_n]/J}(t)$ and $h_{A[x,x_n]/(J,x_n)}(t)=h_{A[x,x_n]/J}(t)$, because $J$ and $I$ are saturated in $A[x,x_n]$ and $x_n$ is generic for $J$ and also for $I$, by [8, Theorem 3.5 and Corollary 3.7]; for every $t \geq m$, we obtain $h_{A[x,x_n]/(J,x_n)}(t)=h_{K[x]/J}(t)$, because $J$ is a lifting of $J'$ and then $(x_n,\cdots,x_n)_{\geq m}=J_{\geq m}\otimes A$ by Lemma 9.3.

The $P(J_{\geq m-1})$-marked basis $G$ of $I_{\geq m-1}$ is of the following type:

$$G = \left\{ f_\alpha = x^\alpha + \sum_{\gamma \in N(J)_{\alpha}} C_{\alpha \gamma} x^\gamma : \text{Ht}(f_\alpha) = x^\alpha \in P(J_{\geq m-1}) \right\} \subset K[C_{J_{\geq m-1}}][x,x_n],$$

where $C_{J_{\geq m-1}}$ is the set of the variables $C_{\alpha \gamma}$. Let $A_{J_{\geq m-1}} \subseteq K[C_{J_{\geq m-1}}]$ be the defining ideal of $Mf_{P(J_{\geq m-1})}$, that is the ideal generated exactly by all the relations which are satisfied by the variables $C_{\alpha \gamma}$ when $G$ is a $P(J_{\geq m-1})$-marked basis. Indeed, the functor $Mf_{P(J_{\geq m-1})}$ is represented by the scheme $\text{Spec} \left( K[C_{J_{\geq m-1}}]/A_{J_{\geq m-1}} \right)$. We now construct a $P(J'_{\geq m})$-marked basis for the ideal $(I,x_n)_{\geq m}$ by taking Lemma 9.3 into account.

For every $x^\alpha \in P(J_{\geq m})$, that is not divisible by $x_n$, we consider its $J_{\geq m-1}$-reduced form $\text{Nm}(x^\alpha)$ modulo $I_{\geq m-1}$ and the homogeneous polynomial $g_\alpha := x^\alpha - \text{Nm}(x^\alpha) \in I_{\geq m-1}$. Observe that $\text{Nm}(x^\alpha)$ exists for properties of marked bases. Then, we take the following set that is a $P(J'_{\geq m})$-marked set by Lemma 9.3 and [8, Proposition 2.10]

$$G' := \{ g_\alpha | x_n=0 : x^\alpha \in P(J_{\geq m}) \text{ and } x^\alpha \text{ not divisible by } x_n \} \cup \{ f_\beta | x_n=0 : f_\beta \in G_{\geq m+1} \}.$$

By construction, the polynomials of $G'$ belong to $(I,x_n)/(x_n)$. We now show that $G'$ is a $P(J'_{\geq m})$-marked basis of $(I,x_n)/(x_n)_{\geq m}$.

By construction, $G'_{\geq m}$ is an independent set of polynomials and hence $h_{A[x,x_n]/(J,x_n)}(m) = h_{A[x,x_n]/G'}(m)$. For every $t \geq m+1$, by induction we suppose that $(I,x_n)/(x_n)_{t-1}$ is equal to $(G')_{t-1}$. By construction, we have $I_t = (x_0,\ldots,x_n)_{t-1} \cdot I_{t-1} + (G_t)$. Hence

$$(I,x_n)/(x_n)_t = A[x]_1 \cdot ((I,x_n)/(x_n))_{t-1} + (G'_t) = A[x]_1 \cdot (G'_{t-1} + (G'_t)) = (G'_{t-1} + (G'_t)).$$

Then, $G'$ is a $P(J'_{\geq m})$-marked set and generates $(I,x_n)/(x_n)_{\geq m}$. Moreover, by [8, Corollary 4.9] $G'$ is a $P(J'_{\geq m})$-marked basis because $h_{A[x,x_n]/I}(t)=h_{K[x]/J}(t)$ for every $t \geq m$, like observed in (9.1). Thus, $(I,x_n)/(x_n)_{\geq m}$ belongs to $Mf_{P(J'_{\geq m})}(A)$. \hfill \square

**Remark 9.5.** Marked schemes do not give us a characterization of liftings because $I'_{\geq m} \in Mf_{P(J'_{\geq m})}(K)$ does not imply $I' \in Mf_{P(J'_{\geq m})}(K)$ [8, Example 3.8]. We can a priori assume that the ideal $I'$ belongs to $Mf_{P(J'_{\geq m})}(K)$ due to the results obtained in Section 4 by Gröbner strata.

We are now ready to prove the main result of the present section, that is $L_{J'_{\geq m-1}}^{\beta(t)}$ is a functor of points.
Theorem 9.6. Let $J' \subset K[x]$ be a saturated quasi-stable ideal, $J \subset A[x, x_n]$ a monomial lifting of $J'$ and $\rho$ the satiety of $(J, x_n)/(x_n) \subset A[x]$. For every $m \geq \rho$, the family of ideals $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}(A)$ is parameterized by an affine subscheme of $\text{Mf}_{\mathcal{P}(J_{\geq m-1})}$ and $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}$ is a functor of points.

Proof. By Lemma 9.2(i), a saturated ideal $I$ belongs to $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}(A)$ if and only if is a lifting of $I'$ and $I_{\geq m-1}$ has a $\mathcal{P}(J_{\geq m-1})$-marked basis.

Assume that $I \subset A[x, x_n]$ is a saturated ideal such that $I_{\geq m-1}$ has a $\mathcal{P}(J_{\geq m-1})$-marked basis $G$ like in (9.2). We now find conditions on the coefficients of the polynomials in $G$ that are equivalent to the fact that $I$ is a lifting of $I'$.

By Lemma 9.4, the ideal $\left(\frac{(I,x_n)}{(x_n)}\right)^{\geq m}$ has a $\mathcal{P}(J'_{\geq m})$-marked basis $G'$, like in (9.3). Hence, by [8, Corollary 3.7], $\left(\frac{(I,x_n)}{(x_n)}\right)^{\geq m}$ is $m$-saturated, i.e. $\left(\frac{(I,x_n)}{(x_n)}\right)^{\geq m} = \left(\frac{(I,x_n)}{(x_n)}\right)^{\text{sat}}_{\geq m}$.

Now we impose that $G'$ is contained in $I'$, hence $((I, x_n)/(x_n))_{\geq m} \subset I'_{\geq m}$.

Let $\mathcal{B}_{J'_{\geq m}} \subseteq K[\mathcal{C}_{J_{\geq m-1}}]$ be the ideal generated by the coefficients of the terms in the $J'_{\geq m}$-normal forms modulo $I'$ of the polynomials in the marked basis $G'$, which can be computed by the reduced Gröbner basis of $I'$ with respect to degrevlex. The ideal $\mathcal{A}_{J_{\geq m-1}} + \mathcal{B}_{J'_{\geq m}}$ exactly gives the conditions that $G'$ is a marked basis and $(G') = ((I, x_n)/(x_n))_{\geq m}$ is contained in $I'$.

Now, we obtain $\left(\frac{(I, x_n)}{(x_n)}\right)^{\text{sat}}_{\geq m} = I'$ by Hilbert polynomial arguments and conclude that $I$ is a lifting of $I'$.

Finally, we can now observe that the functor $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}(A)$ is the functor of points defined by the scheme $\text{Spec} \left( K[\mathcal{C}_{J_{\geq m-1}}]/(\mathcal{A}_{J_{\geq m-1}} + \mathcal{B}_{J'_{\geq m}}) \right)$. \Halmos

Remark 9.7. Differently from the ideal $\mathcal{B}_{J \cap K[x]}$ in the proof of Theorem 4.4, the ideal $\mathcal{B}_{J'_{\geq m}} \subseteq K[\mathcal{C}_{J_{\geq m-1}}]$ that is constructed in the proof of Theorem 9.6 is generated by polynomials which are not necessarily linear. In order to have linear generators, we could repeat the proof of Theorem 4.4 starting from the $\mathcal{P}(J_{\geq m})$-marked basis of $I_{\geq m}$ instead of the $\mathcal{P}(J_{\geq m-1})$-marked basis of $I_{\geq m-1}$. However, the number of variables $\mathcal{C}_{J_{\geq m}}$ is much higher than the number of the variables $\mathcal{C}_{J_{\geq m-1}}$: indeed, the ideal $\mathcal{A}_{J_{\geq m-1}}$ can be obtained from $\mathcal{A}_{J_{\geq m}}$ by an elimination of variables (which is a time-consuming process). This elimination of variables applied on the linear generators of $\mathcal{B}_{J'_{\geq m}} \subseteq K[\mathcal{C}_{J_{\geq m-1}}]$ gives generators of higher degree in a smaller number of variables. If we use the process of the proof of Theorem 9.6, the eliminable variables do not appear from the very beginning, allowing the embedding of $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}$ in a rather small affine space, without an expensive elimination of variables (see [7, Section 5.1] for details).

Theorem 9.8. $\mathcal{L}^{\rho(t)}_{\{Y\}}$ is a functor of points.

Proof. Due to Theorems 9.6 and 7.2, we can consider the finite set of the open subfunctors $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}$ of the functor $\mathcal{L}^{\rho(t)}_{\{Y\}}$, where $J$ varies among all the quasi-stable ideals in $K[x, x_n]$ that are liftings of $J'$ with $\rho(t)$ as Hilbert polynomial of $A[x, x_n]/J$.

By Theorem 9.6, for every of these ideals $J$, the open functor $\mathcal{L}^{\rho(t)}_{\{Y,J_{\geq m-1}\}}$ is exactly the functor of points of the ring $R_J := K[\mathcal{C}_{J_{\geq m}}]/(\mathcal{A}_{J_{\geq m-1}} + \mathcal{B}_{J'_{\geq m}})$. By Theorem 4.5 and by formula (8.2),
for every $K$-algebra $A$, we obtain $L_Y^{p(t)}(A) = \bigcup_J L^{p(t)}_{Y,J \geq m-1}(A)$. Thanks to [12, Theorem VI-14] we can now conclude, because $L_Y^{p(t)}$ is a Zariski sheaf (see Proposition 5.3).

**Remark 9.9.** Concerning the proof of Theorem 9.8, we can observe that the open subfunctors $L_Y^{p(t)}_{J \geq m-1}$ form an open covering of the functor $L_Y^{p(t)}$ (e.g. [32, Definition 2.15]). Thus, [32, Proposition 2.16] clarifies that a scheme defining a functor of points isomorphic to $L_Y^{p(t)}$ can be constructed by means of fiber products of the subfunctors $L^{p(t)}_{Y,J \geq m-1}$.

We conclude this section giving an example of explicit computation of $L_Y^{p(t)}_{J \geq m-1}$ as described in Lemma 9.4 and in Theorem 9.8.

**Example 9.10.** We consider the scheme $Y$ defined by the ideal $I' \subset K[x_0, \ldots, x_3] = K[x]$ of Example 7.4 and the quasi-stable ideal $J^{(2)} = (x_2x_0, x_0^2, x_3x_1x_0, x_2x_1^2, x_1^3, x_1^2x_0)$. We explicitly construct the Noetherian $K$-algebra defining the functor of points isomorphic to $L_Y^{p(t)}$ can be constructed by means of fiber products of the subfunctors $L^{p(t)}_{Y,J \geq m-1}$.

Following the proof of Theorem 9.6, we construct the ideal $A_{J^{(2)} \geq m-1}$. It is sufficient to consider a $\mathcal{P}(J^{(2)} \geq m-1)$-marked set $\mathcal{G}$ as in (9.2): we obtain 7 marked polynomials in the polynomial ring $K[C_{J^{(2)} \geq m-1}][x, x_4]$, with $|C_{J^{(2)} \geq m-1}| = 136$. We impose on the variables $C_{J^{(2)} \geq m-1}$ the conditions ensuring that $\mathcal{G}$ is a $\mathcal{P}(J^{(2)} \geq m-1)$-marked basis (see [8, Theorem 5.1]), obtaining 335 polynomials, which are the generators of $A_{J^{(2)} \geq m-1}$.

We now construct a $\mathcal{P}(J' \geq m)$-marked basis for the ideal $\left(\frac{\mathcal{G}_a(x_4)}{x_4}\right)_{J^{(2)} \geq m}$, following the lines of the proof of Lemma 9.4. We explicitly construct the polynomials $g_a$ and $f_\beta$ of (9.3) computing $\mathcal{P}(J^{(2)} \geq m-1)$-normal forms modulo $(\mathcal{G})_{J^{(2)} \geq m}$. By the polynomial reduction defined in [8, Section 4], we impose that $g_a|_{x_4=0}$ and $f_\beta|_{x_4=0}$ belong to $I'_{J^{(2)} \geq m}$ (see [8, Corollary 4.9]). The conditions that we impose on the normal forms modulo $I'_{J^{(2)} \geq m}$ of the polynomials in $\mathcal{G}'$ are the generators of $B_{J^{(2)} \geq m}$.

These computations also allow us to observe that there are some liftings of $Y$ belonging to $L^{p(t)}_{Y,J^{(2)} \geq m-1}(K)$ which do not belong to $\text{St}_{J^{(2)} \geq m-1}(K)$. For instance, for every $e \in K, e \neq 0$, the following ideal

$$I = (x_2x_0 - ex_1^2 - ex_1x_0, x_0^2, x_3x_1x_0, x_2x_1^2, x_1^3, x_1^2x_0, x_0x_1x_2)$$

is a lifting of $Y'$, it has a marked basis over $J^{(2)}_{\geq 2}$, hence it belongs to $L^{p(t)}_{Y,J^{(2)} \geq m-1}(K)$, but in$(I) = J^{(3)}$: this means that $I$ does not belong to $\text{St}_{J^{(2)} \geq m-1}(K)$. Moreover we observe that the marked basis of $I$ cannot be a Groebner basis with respect to some term order, because the term $x_1x_0$ is higher than $x_2x_0$ with respect to any term order.

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