Morita equivalence classes of blocks with elementary abelian
defect groups of order 32

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Abstract

We classify the Morita equivalence classes of blocks with elementary abelian defect
groups of order 32 with respect to a complete discrete valuation ring with an algebraically
closed residue field of characteristic two. As a consequence we prove Harada’s conjecture
for all these blocks, and we prove Broué’s abelian defect group conjecture for 30 of the
34 determined equivalence classes of blocks.

Keywords: Donovan’s conjecture; Finite groups; Morita equivalence; Block theory;
Modular representation theory.

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1 Introduction

In the following let $\mathcal{O}$ be a complete discrete valuation ring, with field of fractions $K$ of
characteristic 0 and residue field $k$, an algebraically closed field of characteristic two. Note
that $K$ and $k$ cannot both be algebraically closed, but we can assume $K$ to be large enough
for all finite groups considered in this paper. The triple $(K, \mathcal{O}, k)$ is usually called
\textit{p-modular system}. We say that two algebras $A$ and $B$ are Morita equivalent if there is an equivalence of
categories of $A$-modules and $B$-modules. Since in characteristic 2 the representation theory
of most group algebras where the group has a non-cyclic Sylow 2-subgroup is wild (with
a few tame exceptions), classifying blocks or modules up to isomorphism is not a realistic
expectation, which is why we use a weaker equivalence.

Given a finite group $G$, we consider blocks of the group algebras $\mathcal{O}G$ and $kG$. Given a
block $B$ of $\mathcal{O}G$, we can obtain a block $\overline{B}$ of $kG$ via the canonical map $B \mapsto \overline{B} = B \otimes_\mathcal{O} k$. Moreover, a Morita equivalence between blocks of group algebras over $\mathcal{O}$ implies a Morita
equivalence over the same group algebras over $k$, while the converse is not known to be true.
Hence, classifying blocks over $\mathcal{O}$ is, in general, harder than classifying blocks over $k$.

For a $p$-block $B$ of $\mathcal{O}G$ we can consider the defect group $D$, a $p$-subgroup of $G$ defined up
to conjugation as the unique maximal vertex of the $B$-modules. A $B$-subpair is a pair $(Q, B_Q)$
where $Q$ is a subgroup of $D$ and $B_Q$ is a block of $\mathcal{O}QC_G(Q)$ with Brauer correspondent $B$.
When $Q = D$, the Brauer subpairs $(D, B_D)$ are all $G$-conjugate. The inertial quotient of $B$ is defined as $E = N_G(D, B_D)/C_G(D)$, where $N_G(D, B_D)$ denotes the stabilizer of $B_D$ in
$N_G(D)$. In general $E$ is a $p'$-group and, when $D$ is abelian, $B$ is nilpotent if and only if $E = 1$.

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We denote the number of irreducible characters of \( KG \) in the block \( B \) as \( k(B) \), and the number of irreducible Brauer characters of \( kG \) in the block \( B \) as \( l(B) \). Moreover, we denote as \( F \) the fusion system of \( D \) given by the block \( B \) (see [33, 8.1]). For a group algebra \( OG \), we denote as \( B_0(OG) \) the principal block, the one that contains the trivial character.

Donovan’s conjecture states that for each isomorphism class of a \( p \)-group \( D \) there is a finite number of Morita equivalence classes of blocks of finite groups with defect group \( D \). However, note that defect groups and inertial quotients are not known in general to be invariant under Morita equivalence, but there is no known counterexample. Donovan’s conjecture has been proved, in particular, over \( k \) for elementary abelian 2-groups in [12], and later generalized to abelian 2-groups in [16] over \( k \) and in [17] over \( O \). However, in both cases the proof does not produce an explicit list of all the classes for each fixed defect group.

Our purpose is to describe the Morita equivalence classes of blocks with defect group \((C_2)^5\).

**Theorem.** Let \( G \) be a finite group, and let \( B \) be a block of \( O\!G \) with elementary abelian defect group \( D \) of order 32. Then \( B \) is Morita equivalent to precisely one of 34 Morita equivalence classes (explicitly listed in Theorem 5.2). Moreover, if a block \( C \) of \( O\!H \) for a finite group \( H \) is Morita equivalent to \( B \), then the defect group of \( C \) is isomorphic to \( D \).

In Section 2, we list preliminary results and some reductions that we use in the proof of the main theorem. In Section 3 we look at perfect isometries between certain blocks, that we use to extend the main theorem of [26] over \( O \), which allows us to examine blocks of groups that cover a block of a normal subgroup with index a power of two. In Section 4 we give background on crossed products and Picard groups, which allow us to examine blocks of groups that cover a block of a normal subgroup with odd index, and apply this method to study some cases that arise when proving our main result. In Section 5 we prove our main theorem, list all the classes and investigate whether Broué’s abelian defect group conjecture holds for these blocks. In Section 6 we prove a conjecture of Harada for all the blocks determined in Section 5.

2 Reductions and technical lemmas

Given a normal subgroup \( N \triangleleft G \), \( B \) a block of \( O\!G \) and \( b \) a block of \( O\!N \), we say that \( B \) covers \( b \) when \( Bb \neq 0 \). The structures of \( B \) and \( b \) are closely related in this case. For instance, as shown in [15, 15.1], a defect group of \( b \) is the intersection of a defect group of \( B \) with \( N \). This relation is the main tool that we use to obtain our classification.

In some cases, we will require equivalences stronger than Morita equivalence to be able to study certain blocks. We give a brief recap here (better, more detailed descriptions can be found, among many others, in [33]).

As we mentioned in the introduction, two blocks \( B \) and \( C \) of finite groups are Morita equivalent if their module categories are equivalent. An alternative, more explicit way to define this equivalence is to say that there is a \((B,C)\)-bimodule \( M \) and a \((C,B)\)-bimodule \( N \) such that \( M \otimes_C N \simeq B \) as \((B,B)\)-bimodules and \( N \otimes_B M \simeq C \) as \((C,C)\)-bimodules. We say that the Morita equivalence is *realized* by \( M \) and \( N \).
Two blocks are \emph{basic Morita equivalent} if they are Morita equivalent via an equivalence realized by bimodules with endopermutation source. Two blocks are \emph{source algebra equivalent} (or \emph{Puig equivalent}) if they are Morita equivalent via an equivalence realized by bimodules with trivial source. Note that each equivalence is stronger than the ones above it.

Given a finite group \( G \), a block \( B \) of \( OG \) is said to be quasiprimitive if for any normal subgroup \( N \triangleleft G \) there is a unique block \( b \) of \( ON \) covered by \( B \). This is equivalent, by [11, 15.1], to the requirement that each \( b \) is \( G \)-stable under the action of \( G \) by conjugation on \( ON \). We can reduce to quasiprimitive blocks in many situations, including our main theorem, using a classic reduction usually denoted as Fong I. We report it here for the reader’s convenience.

\begin{theorem}[6.8.3, \cite{33}]\label{thm:fongI}
Let \( G \) be a finite group, and let \( N \) be a normal subgroup of \( G \). Let \( b \) be a block of \( ON \) and \( B \) be a block of \( OG \) that covers \( b \) (i.e. \( Bb \neq 0 \)). Let \( H \) be the stabiliser of \( b \) in \( G \), acting by conjugation. Then there is a unique block \( C \) of \( OH \) covered by \( B \) and covering \( b \), and \( C \) is Morita equivalent to \( B \) and shares a defect group with \( B \).
\end{theorem}

We also use the following version of the Fong II reduction, used whenever \( G \) has a normal \( p' \)-subgroup.

\begin{theorem}[\cite{30}]\label{thm:fongII}
Let \( G \) be a finite group and \( N \triangleleft G \). Let \( B \) be a block of \( OG \) with defect group \( D \) that covers a \( G \)-stable nilpotent block \( b \) of \( ON \) with defect group \( D \cap N \). Then there are finite groups \( M \triangleleft L \) such that \( M \simeq D \cap N \), \( L/M \simeq G/N \), there is a subgroup \( DL \subseteq L \) with \( DL \simeq D \) and \( DL \cap M = D \cap N \), and there is a central extension \( \tilde{L} \) of \( L \) by a \( p' \)-group, and a block \( \tilde{B} \) of \( O\tilde{L} \) which is Morita equivalent to \( B \) and has defect group \( \tilde{D} \simeq DL \simeq D \). If \( B \) is the principal block, then so is \( \tilde{B} \).
\end{theorem}

\begin{corollary}[\cite{14}]\label{cor:quasiprimitive}
Let \( G \) be a finite group, let \( N \triangleleft G \) with \( N \not\leq Z(OG)O_{p'}(G) \). Let \( B \) be a quasiprimitive \( p \)-block of \( OG \) covering a nilpotent block \( b \) of \( ON \). Then there is a finite group \( H \) with \( [H : O_{p'}(Z(H))] < [G : O_{p'}(Z(G))] \) and a block \( B_H \) with isomorphic defect group to the one of \( B \), such that \( B_H \) is Morita equivalent to \( B \).
\end{corollary}

Given a quasiprimitive block \( B \) of \( OG \), a normal subgroup \( N \triangleleft G \) and a block \( b \) of \( ON \) covered by \( B \), and given a chain of subnormal subgroups \( N = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_t = G \) we define a \emph{block chain} to be any sequence of blocks \( b_i \) of \( ON_i \) such that \( b_{i+1} \) covers \( b_i \), \( b_0 = b \) and \( b_t = B \). Note that whenever \( N_i \triangleleft G \), there is a unique block \( b_i \) covered by \( B \), so in particular if the chain consists of normal subgroups of \( G \), then the block chain between \( b \) and \( B \) is uniquely determined.

We prove a result about series of normal subgroups of solvable groups that we will need later.

\begin{lemma}\label{lem:series}
Let \( G \) be a finite group and \( B \) a quasiprimitive block of \( OG \) with a defect group \( D \) of order \( p^n \). Let \( N \) be a normal subgroup of \( G \) and let \( b \) be a block of \( ON \) covered by \( N \). If \( G/N \) is solvable, then \( DN/N \) is a Sylow \( p \)-subgroup of \( G/N \).
\end{lemma}

\begin{proof}
Since \( G/N \) is solvable, it is in particular \( \ell \)-solvable for each prime \( \ell \) that divides \( |G| \). Then we can consider the upper \( p \)-series of \( G/N \) (see [22, Chapter 6.3] )
\[
1 \triangleleft O_p(G/N) \triangleleft O_{p.p'}(G/N) \triangleleft O_{p,p',p}(G/N) \triangleleft \cdots \triangleleft G/N
\]
Every subgroup in the chain is a characteristic subgroup of \( G/N \), and note that each index is either a power of \( p \) or prime to \( p \). We can now take a preimage of this series under
\[ \pi : G \to G/N \] to obtain

\[ N_0 = N \triangleleft N_1 \triangleleft N_2 \triangleleft \cdots \triangleleft N_t = G \]

and consider the corresponding block chain given by the unique blocks \( b_i \) of \( \mathcal{O}N_i \) covered by \( B \). Let \( D_i \) be the defect group of \( b_i \). We distinguish two cases:

- \([N_{i+1} : N_i]\) is prime to \( p \). Then \( b_i \) and \( b_{i+1} \) share a defect group, so \( D_i = D_{i+1} \).
- \([N_{i+1} : N_i]\) is a power of \( p \). Then \( b_i \) is \( N_{i+1} \)-stable because it is \( G \)-stable, and \( b_{i+1} \) is the unique block of \( \mathcal{O}N_{i+1} \) covering \( b_i \) (see [20, 5.3.5]). Then by [11 15.1] we have that \([D_{i+1} : D_i] = [N_{i+1} : N_i]_p\).

Then

\[ [D : D_0] = [D : D_{t-1}] \cdots [D_2 : D_1][D_1 : D_0] = [G : N_{t-1}]_p \cdots [N_2 : N_1]_p[N_1 : N_0]_p = |G/N|_p \]

Since from [11 15.1] \( D_0 = D \cap N \), we are done. \( \square \)

We can relate inertial quotients of a block \( B \) and a block \( b \) covered by \( B \) with the following lemma.

**Lemma 2.5.** Let \( G \) be a finite group and \( N \triangleleft G \) with \([G : N] = \ell\), a prime such that \( \ell \neq p \). Let \( B \) be a block of \( \mathcal{O}G \) with an abelian defect group \( D \) that covers a \( G \)-stable block \( b \) of \( \mathcal{O}N \). Let \( E_B \) and \( E_b \) be the inertial quotients of \( B \) and \( b \) respectively. Then either \( |E_B| = \ell |E_b| \) or \(|E_B| \) divides \(|E_b|\).

**Proof.** This argument is extracted from the proof of [14 2.11]. First, note that \( B \) and \( b \) share a defect group \( D \). Let \( B_D \) and \( b_D \) be blocks of \( CG(D) \) and \( C_N(D) \) respectively, such that \((D, B_D)\) is a \( B \)-Brauer subpair and \((D, b_D)\) is a \( b \)-Brauer subpair. With a little abuse of notation, in the following we denote the block idempotents corresponding to each block using the same letters.

Now we use an argument extracted from [24 2.1]. By definition, \( Br_N^G(B)B_D \neq 0 \). Note that, since \( B \) and \( b \) share a defect group, \( Br_N^G(B) \) is equal to the restriction of \( Br_N^G(B) \) to \( (\mathcal{O}N)_D \). The block \( B_D \) covers at least one block \( \tilde{b}_D \) of \( C_N(D) \). Let \( \tilde{b} = (\tilde{b}_D)^N \). Then \( Br_N^G(b)\tilde{b}_D \neq 0 \). Since \( B_D \in Z(\mathcal{O}CG(D)) \), we can write

\[ Br_N^G(B\tilde{b})B_D\tilde{b}_D = Br_N^G(B)B_D Br_N^G(b)\tilde{b}_D \neq 0 \]

so \( B\tilde{b} \neq 0 \), which implies that \( B \) covers \( \tilde{b} \), and so \( b = \tilde{b} \) since, being \( G \)-stable, \( b \) is the unique block of \( \mathcal{O}N \) covered by \( B \) ([11 15.1]). Hence, we can choose \( b_D = \tilde{b}_D \) and have that \( B_D \) covers \( b_D \).

If \( CG(D) \neq C_N(D) \), we distinguish two cases:

- If \( N_G(D, b_D) \neq N_N(D, b_D) \) then \( b_D \) is the unique block of \( C_N(D) \) covered by \( B_D \). Then \( N_G(D, B_D) \leq N_G(D, b_D) \), so \( E_B \leq N_G(D, b_D)/CG(D) \).

Since \( |N_G(D, b_D)| = \ell \cdot |N_N(D, b_D)| \) and also \( |CG(D)| = \ell |C_N(D)| \), it follows that \(|E_B| \) divides \(|E_b|\).
– If \( N_G(D,b_D) = N_N(D,b_D) \) then \( B_D \) covers all \( \ell \) conjugates of \( b_D \). In particular, it is the unique block that covers \( b_D \), hence \( N_G(D,b_D) \leq N_G(D,B_D) \), and actually \( N_G(D,B_D) = N_G(D,b_D)C_G(D) = N_N(D,b_D)C_G(D) \) Therefore

\[
E_B = \frac{N_G(D,B_D)}{C_G(D)} = \frac{C_G(D)N_N(D,b_D)}{C_G(D)} \simeq \frac{N_N(D,b_D)}{N_N(D,b_D) \cap C_G(D)} = \frac{N_N(D,b_D)}{C_N(D)} = E_b
\]

If \( C_G(D) = C_N(D) \) then \( b_D = B_D \) and therefore \( N_N(D,b_D) \leq N_G(D,b_D) = N_G(D,B_D) \). Then \( E_b \leq E_B \), so in particular \( |E_B| = |E_b| \) or \( |E_B| = |E_b|/\ell \).

The following lemma relates the defect groups and the inertial quotients of blocks over \( \mathcal{O} \) and over \( k \). It seems to be common knowledge that the inertial quotient is the same in both situations, but we were unable to find a precise reference, so we prove a weaker result that is enough for our purposes.

**Lemma 2.6.** Let \( G \) be a finite group, and let \( B \) be a block of \( \mathcal{O}G \). Then we can consider \( \overline{B} = B \otimes_\mathcal{O} k \), the corresponding block of \( kG \). Let \( D \) and \( E \) be an abelian defect group and an inertial quotient of \( B \), and \( \overline{D} \), \( \overline{E} \) of \( \overline{B} \). Then \( D = \overline{D} \) and \( E \leq \overline{E} \).

**Proof.** The claim about the defect group is Proposition 6.1.6 in [33]. Let \( Z(\mathcal{O}G) \to Z(kG), x \mapsto \overline{x} \) be the canonical map. If \( b \) is the Brauer correspondent of \( B \) in \( \mathcal{O}N_G(D) \), then \( \overline{b} \in kN_G(D) \) is the Brauer correspondent of \( \overline{B} \) since the Brauer correspondent is unique. Let \( c \) be a block of \( \mathcal{O}C_G(D) \) covered by \( b \). The inertial quotient is defined as \( E = N_G(D,c)/C_G(D) \). Then \( \overline{c} \) is also covered by \( \overline{b} \) since the kernel of the canonical map does not contain nonzero idempotents (as seen in the proof of Theorem 6.1.6 in [33], and hence \( bc \neq 0 \) implies that \( \overline{bc} \neq 0 \). Since the inertial quotient is independent of the choice of the block of \( kC_G(D) \), we have that \( \overline{E} = N_G(D,\overline{c})/C_G(D) \).

If \( g \in N_G(D) \) is such that \( cg = c \), then clearly \( \overline{c}g = \overline{c} \), so \( E \leq \overline{E} \). □

A block \( B \) of \( \mathcal{O}G \) is nilpotent covered if there exists a group \( \tilde{G} \triangleright G \) and a nilpotent block \( \tilde{B} \) of \( \mathcal{O}\tilde{G} \) such that \( \tilde{B} \) covers \( B \). We say that \( B \) is inertial if it is basic Morita equivalent to its Brauer correspondent. The following lemma relates these two concepts.

**Lemma 2.7 ([44], [57]).** Let \( G \) be a finite group and let \( N \triangleleft G \). Let \( b \) be a \( p \)-block of \( \mathcal{O}N \) covered by a block \( B \) of \( \mathcal{O}G \). Then

1. If \( B \) is inertial, then \( b \) is inertial.
2. If \( b \) is nilpotent covered, then \( b \) is inertial.
3. If \( p \) does not divide \( [G : N] \) and \( b \) is inertial, then \( B \) is inertial.
4. If \( b \) is nilpotent covered, then it has abelian inertial quotient.

**Proof.** (i), (ii) and (iv) are respectively Theorem 3.13 and Corollary 4.3 in [44]. (iii) is the main theorem of [57]. □
Given two finite groups $N \triangleleft G$ and a block $b$ of $\mathcal{O}N$, we define $G[b]$ as the group of elements of $G$ acting as inner automorphisms on $b \otimes \mathcal{O} k$. We will use the following result, extracted from [23], when dealing with automorphisms of products of quasisimple groups, similarly to what has been done in [13].

**Lemma 2.8.** Let $G$ be a finite group and $B$ a block of $\mathcal{O}G$ with defect group $D$. Let $N$ be a normal subgroup of $G$ that contains $D$, and suppose that $B$ covers a $G$-stable block $b$ of $\mathcal{O}N$. Let $\hat{b}$ be a block of $\mathcal{O}G[b]$ covered by $B$. Then

(i) $b$ is source algebra equivalent to $B'$. In particular, it has isomorphic inertial quotient.
(ii) $B$ is the unique block of $\mathcal{O}G$ that covers $B'$.

**Proof.** Proposition 2.2 in [23] gives the source algebra equivalence between $\hat{b} \otimes \mathcal{O} k$ and $b \otimes \mathcal{O} k$. From Proposition 7.8 in [40], this equivalence lifts to $\hat{b}$ and $b$. Part (ii) follows from Theorem 3.5 in [36].

In particular, we will use the following corollary, which is proved applying Lemma 2.8 to the cases $G = G[b]$ and $N = G[b]$.

**Corollary 2.9 ([13]).** Let $G$ be a finite group and let $N \triangleleft G$ be a normal subgroup with prime index $\ell \neq p$. Let $b$ be a $G$-stable $p$-block of $\mathcal{O}N$, and let $B$ be a block of $\mathcal{O}G$ that covers $b$. Then either $B$ is the unique block of $\mathcal{O}G$ that covers $b$, or $B$ is source algebra equivalent to $b$.

Given two groups $G_1$ and $G_2$, every block of $\mathcal{O}(G_1 \times G_2)$ is of the form $b_1 \otimes b_2$, where $b_i$ is a block of $\mathcal{O}G_i$. It is well known that ordinary representation theory of central products is very similar to the one of direct products, which still holds when looking at $p$-blocks as long as the shared center is a $p'$-group.

**Lemma 2.10 (7.5 [46], 1.5 [11]).** Let $G = G_1 * G_2$ be a central product of finite groups $G_1$ and $G_2$, and let $B$ be a $p$-block of $\mathcal{O}G$ with defect group $D$, and $B_i$ the unique block of $\mathcal{O}G_i$ covered by $B$, with defect group $D_i$. Then

1. $D = D_1 \ast D_2 = D_1 * D_2$ is a defect group of $B$.
2. If $G_1 \cap G_2$ has order prime to $p$, then $B$ is isomorphic to $B_1 \otimes B_2$.
3. $B$ is nilpotent if and only if $B_i$ is nilpotent for each $i$.

Note that this lemma can be generalized to a central product of any number of groups.

Now we focus on the case when $p = 2$, and $D = (C_2)^n$. For the convenience of the reader, we start by writing the classification of 2-blocks with a smaller elementary abelian defect group.

**Proposition 2.11 ([11], [19], [32], [7], [13], [14]).** Let $G$ be a finite group and let $B$ be a block of $\mathcal{O}G$ with defect group $D = (C_2)^n$ where $n \leq 4$. Then:

1. If $n = 1$ then $B$ is Morita equivalent to $\mathcal{O}C_2$. In particular, $B$ is nilpotent.
2. If $n = 2$ then $B$ is source algebra equivalent to $\mathcal{O}(C_2)^2$, $\mathcal{O}A_4$ or $B_0(\mathcal{O}A_5)$. 

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3. If \( n = 3 \) then \( B \) is Morita equivalent to \( \mathcal{O}(C_2)^3 \), \( \mathcal{O}(A_4 \times C_2) \), \( B_0(\mathcal{O}(A_5 \times C_2)) \), \( \mathcal{O}((C_2)^3 \times C_7) \), \( B_0(\mathcal{O}SL_2(8)) \), \( \mathcal{O}((C_2)^3 \times (C_7 \times C_3)) \), \( B_0(\mathcal{O}A_1) \), \( B_0(\mathcal{O}Aut(SL_2(8))) \).

4. If \( n = 4 \) then \( B \) is Morita equivalent to: \( \mathcal{O}(C_2)^4 \), \( \mathcal{O}(A_4 \times (C_2)^2) \), \( B_0(\mathcal{O}(A_5 \times (C_2)^2)) \), \( \mathcal{O}((C_2)^4 \times (C_3)^2) \), \( \mathcal{O}(((C_2)^4 \times C_3) \times C_3) \), \( \mathcal{O}(((C_2)^3 \times C_7) \times C_2) \), \( B_0(\mathcal{O}SL_2(8) \times C_2) \), \( \mathcal{O}(A_4 \times A_1) \), \( B_0(\mathcal{O}(A_5 \times A_5)) \), \( \mathcal{O}((C_2)^4 \times C_{15}) \), \( \mathcal{O}(((C_2)^3 \times (C_7 \times C_3)) \times C_2) \), \( B_0(\mathcal{O}SL_2(16)) \), \( B_0(\mathcal{O}(J_1 \times C_2)) \), \( B_0(\mathcal{O}Aut(SL_2(8) \times C_2)) \), or a nonprincipal block of \( \mathcal{O}((C_2)^4 \times 3^+)^2 \).

Moreover, in each of these situations Morita equivalent blocks have isomorphic inertial quotients, and these blocks cannot be Morita equivalent to a block with a non-isomorphic defect group.

Proof. Blocks with cyclic defect groups have been classified in [1]. The result for Klein four defect group appears, for instance, in [19], [32], [7]. The results for \( n = 3 \) and \( n = 4 \) are the main theorems of [13] and [14] respectively.

Numerical invariants for blocks with defect group \( (C_2)^5 \) have not been completely determined. Nevertheless, using Brauer’s second main theorem and a result from [16] we can obtain a list of possibilities for these values, that we use to deal with certain situations in the proof of the main theorem.

At the end of this paper we will have classified all Morita equivalence classes of blocks with defect group \( (C_2)^5 \), and since \( k(B) \) and \( l(B) \) are invariant under Morita equivalence we will know exactly what cases occur.

**Proposition 2.12.** Let \( B \) be a block of \( \mathcal{O}G \) where \( G \) is a finite group, with defect group \( D = (C_2)^5 \) and inertial quotient \( E \) of order \( e(B) \). Then one of the following holds:

(i) \( e(B) = 1 \) and \( k(B) = 32 \), \( l(B) = 1 \).

(ii) \( e(B) = 3 \), \( |C_D(E)| = 8 \) and \( k(B) = 32 \), \( l(B) = 3 \).

(iii) \( e(B) = 3 \), \( |C_D(E)| = 2 \) and \( k(B) = 16 \), \( l(B) = 3 \).

(iv) \( e(B) = 5 \), and \( k(B) = 16 \), \( l(B) = 5 \).

(v) \( e(B) = 7 \), and \( k(B) = 32 \), \( l(B) = 7 \).

(vi) \( e(B) = 9 \), and \( k(B) = 32 \), \( l(B) = 9 \) or \( k(B) = 16 \), \( l(B) = 1 \).

(vii) \( e(B) = 15 \), and \( k(B) = 32 \), \( l(B) = 15 \).

(viii) \( e(B) = 21 \), \( |C_D(E)| = 4 \), and \( k(B) = 32 \), \( l(B) = 5 \).

(ix) \( e(B) = 21 \), \( |C_D(E)| = 1 \), and \( k(B) = 32 \), \( l(B) = 21 \) or \( k(B) = 24 \), \( l(B) = 13 \) or \( k(B) = 16 \), \( l(B) = 5 \).

(x) \( e(B) = 31 \), and \( k(B) = 32 \), \( l(B) = 31 \) or \( k(B) = 24 \), \( l(B) = 23 \) or \( k(B) = 16 \), \( l(B) = 15 \) or \( k(B) = 8 \), \( l(B) = 7 \).

(xi) \( e(B) = 63 \), and \( k(B) \leq 32 \) and \( k(B) - l(B) = 17 \) or \( k(B) - l(B) = 9 \).

(xii) \( e(B) = 155 \), and \( k(B) \leq 32 \) and \( k(B) - l(B) = 5 \).
Proof. Since $D$ is abelian, the inertial quotient $E = N_G(D, B_D)/C_G(D)$ is a subgroup of $\text{Aut}(D) = \text{Out}(D) = \text{GL}_5(2)$. First, we give a classification of the subgroups with odd order of $\text{GL}_5(2)$, which gives us all the possibilities for the isomorphism class of $E$ and its action on $D$.

An explicit computation (using Magma [9]) gives the following diagram (where $A \to B$ if there is a subgroup $C \leq B$ such that $A \simeq C$, the action of $C$ on $D$ is the same as the one of $A$, and $|B|/|C|$ is a prime).

![Diagram](attachment:image.png)

where we distinguish the two different actions of $C_3$ and $C_7 \times C_3$ (corresponding to two pairs of distinct conjugacy classes in $\text{GL}_5(2)$) as follows: we denote by $(C_3)_1$ the action on $D$ such that $C_D(C_3) \simeq (C_2)^3$, which happens for example in the group $A_4 \times (C_2)^3$, and we denote by $(C_3)_2$ the action such that $C_D(C_3) = C_2$: the generator of this subgroup is the 5th power of the Singer cycle $C_{15}$ of $(C_2)^4$. Similarly, we denote by $(C_7 \times C_3)_1$ the action on $D$ such that $C_D(C_7 \times C_3) = (C_2)^2$, or equivalently the one such that the subgroup $C_3 \leq C_7 \times C_3$ acts as $(C_3)_1$. We denote by $(C_7 \times C_3)_2$ the other one, where $C_D(E) = 1$ and $C_3 \leq C_7 \times C_3$ acts as $(C_3)_2$. We have proved that $e(B)$ can only take the values specified in the statement of the theorem.

Whenever $C_D(E) \neq 1$, we can use Proposition 16 in [17] to immediately obtain our claims. This proves cases (i)-(viii).

From Proposition 21 in [17], we have that $k(B) \leq 32$. Now we use the same argument as in [31, 2.1]. Let a subsection be a pair $(u, b_u)$ where $u \in D$ and $b_u$ is a block of $C_G(u)$. Whenever there is a nontrivial subsection $(u, b_u)$ such that $l(b_u) = 1$ then $|D| = 32$ is a sum of $k(B)$ odd squares of integers, which implies that $k(B) \in \{8, 16, 24, 32\}$. In particular, such subsection always exists when $E$ is abelian (case (x)). Since $D$ is abelian then $B$ is a controlled block, meaning that the fusion system $\mathcal{F}(B) = \mathcal{F}_D(D \times E)$. So to compute subsections it is enough to consider a set of representatives $R$ of the orbits of $D$ under the action of $E$. Recall that from Brauer’s second main theorem $k(B) = \sum_{(u, b_u) \in R} l(b_u)$, so in particular $k(b) - l(b) = \sum_{(u, b_u) \in R, u \neq 1} l(b_u)$.

(ix) If $E$ is abelian, then there are four subsections $(1, B)$, $(u_1, b_1)$, $(u_2, b_2)$, $(u_3, b_3)$ with $l(b_1) = 3, l(b_2) = 7, l(b_3) = 1$. So $k(B) - l(B) = 11$ and our claim is proved.

If $E$ is not abelian, there are four subsections $(1, B)$, $(u_1, b_1)$, $(u_2, b_2)$, $(u_3, b_3)$ with $l(b_1) = 3, l(b_2) = 7, l(b_3) = 1$. In particular, there is a subsection of length 1 so $k(B) \in \{8, 16, 24, 32\}$. Now $k(B) - l(B) = 11$ and we are done.

(x) In this case $E$ is abelian and there are only two subsections, $(1, B)$ and $(u, b_u)$, with $l(b_u) = 1$. So $k(B) - l(B) = 1$ and we are done.
Proposition 2.13. Let \( G \) be a quasisimple group, and let \( B \) be a block of \( OG \) with defect group \( D \) contained in \((C_2)^5\). Then one or more of the following occurs:

(i) \( B \) is the principal block, \( G \) is simple and \( G \simeq SL_2(8), SL_2(16), SL_2(32), J_1 \) or \( 2G_2(q) \) with \( q = 3^{2m+1}, m \geq 1 \)

(ii) \( B \) is the unique nonprincipal block of \( G \simeq Co_3 \) with defect group \((C_2)^5\)

(iii) \( B \) is Morita equivalent to a block \( C \) with an isomorphic defect group \( D \) of \( OM \) where \( M = M_0 \times M_1 \) is a subgroup of \( G \) such that \( M_0 \) is abelian and the block of \( M_1 \) covered by \( C \) has defect group \( C_2 \times C_2 \). In this case, \( G \) is of type \( D_n(q) \) or \( E_7(q) \), where \( n = 2t \) for \( t \) odd and \( q \) is a power of an odd prime.

(iv) \( O_2(G) \leq (C_2)^3 \) and \( D/O_2(G) \) has defect group \( C_2 \times C_2 \).

(v) \( B \) is nilpotent covered. In this case, \( G/Z(G) \) is of type \( A_n(q) \) or \( E_6(q) \) where \( q \) is a power of an odd prime.

Proof. (i)-(iv) and the first claim of (v) follow immediately from Theorem 6.1 and Proposition 5.3 of [12] (see also 4.1 in [16]), that tell us the isomorphism class of \( G/Z(G) \). Since all the Schur multipliers of the groups listed in (i) and (ii) are trivial, in those cases \( G \) is actually simple. The last claim of (v) is implied by Lemma 4.2 and Proposition 5.4 in [12] by noting that in the proof of Theorem 6.1 the only case in which \( B \) is nilpotent covered but not nilpotent is when the hypothesis of Proposition 5.4 are satisfied.

We also need to examine outer automorphisms of quasisimple groups, and in particular we need the following fact.

Proposition 2.14. Let \( N \) be a finite quasisimple group of type \( A_n \) or \( E_6 \), and let \( G \) be a finite group such that \( N \triangleleft G \leq \text{Aut}(N), C_G(N) \leq N \) and the 2-rank of \( G/N \) is 2 or more. Then \( G \) has a normal subgroup of index 2.

Proof. Recall that since \( N \) is quasisimple \( \text{Aut}(N) \leq \text{Aut}(N/Z(N)) \) ([16, 7.6]). Since \( C_G(N) \leq N \), there is an injective map \( G/N \to \text{Out}(N) \).

From [6] (see also [49, 1.5]), if \( N/Z(N) = A_n(q) \) then \( \text{Out}(N) \leq C_{(n+1,q-1)} \times (C_f \times C_2) \). If the latter \( C_2 \) is contained in \( \text{Out}(N) \), we are done. Otherwise our hypothesis implies that
n, q are odd and f is even. But then we can decompose \( C_f = C_g \times C_{2f} \) for some odd g, and we are done.

If \( N/Z(N) = 2A_6(q) \) then \( \text{Out}(N) \leq C_{(n+1,q-1)} \times C_{2f} \), and we are done.

If \( N/Z(N) = E_6(q) \) then \( \text{Out}(N) \leq C_3 \times (C_f \times C_2) \), and we are done.

Finally, if \( N/Z(N) = 2E_6(q) \) then \( \text{Out}(N) \leq C_3 \times C_{2f} \), a contradiction. \( \square \)

### 3 Perfect isometries

When classifying Morita equivalence classes of blocks over \( O \) instead of over \( k \), one of the main obstacles is that the main result of [26] does not extend immediately to \( O \). However, using a result from Watanabe [56] and a method developed by Puig and Usami in [42], we are able to extend this result in the specific cases that we need to prove our theorem. In particular, when dealing with blocks with a cyclic inertial quotient again a result by Watanabe [56] gives the required extension. We also need a similar result when the inertial quotient is \( C_3 \times C_3 \) or \( (C_7 \times C_3)_1 \), which is what we obtain in this section. For the convenience of the reader, we introduce some notation and quickly summarize the method introduced in [42].

Let \( B \) be a block of \( OG \) for a finite group \( G \), with defect group \( D \) and inertial quotient \( E \), and let \( e \) be a fixed block of \( kCG(D) \) with Brauer correspondent \( B \), that is, a maximal Brauer subpair \( (D,e) \). Let \( NG(D,e) = NG(D,e)/D \) and \( CG(D) = CG(D)/D \), and let \( \pi \) be the image of \( e \) in \( kCG(D) \). Then a consequence of Brauer’s first main theorem is that \( kCG(D) \pi \) is a simple \( k \)-algebra. Therefore, from the Skolem-Noether theorem, we have the following exact sequence

\[
1 \rightarrow k^\times \rightarrow (kCG(D)\pi)^\times \rightarrow \text{Aut}(kCG(D)\pi) \rightarrow 1
\]

There is an injective map \( \iota : NG(D,e) \rightarrow \text{Aut}(kCG(D)\pi) \) given by the action of the group, and we can define the central extension of \( NG(D,e) \) by \( k^\times \) given by its action on \( kCG(D)\pi \) explicitly. We have the following commutative and exact diagram

\[
\begin{array}{ccc}
1 & \rightarrow & k^\times \\
| & & \downarrow \iota \\
1 & \rightarrow & NG(D,e) \\
\end{array}
\]

where we define \( \overline{NG(D,e)} \) as the group \( \{ (\alpha, \beta) \in kCG(D)\pi \times NG(D,e) : \pi(\alpha) = \iota(\beta) \} \).

There is an injective group homomorphism \( \phi : CG(D) \rightarrow \overline{NG(D,e)} \), \( z \mapsto (\overline{z}, z) \). Since the image of \( CG(D) \) intersects trivially the image of \( k^\times \), the quotient group \( \overline{NG(D,e)}/\phi(CG(D)) \) is still a central extension of \( E \) by \( k^\times \). If we denote by \( \overline{NG(D,e)}^\phi \) the opposite group, there is an exact sequence

\[
1 \rightarrow CG(D) \rightarrow \overline{NG(D,e)}^\phi \rightarrow \hat{E} \rightarrow 1
\]

where we denote by \( \hat{E} \) the opposite central extension of \( E \) by \( k^\times \). Let \( \hat{L} = P \rtimes \hat{E} \).

To understand this better we can look at [41, 5], where the concept of \( k^\times \)-group is studied; the concept is equivalent to central extensions of a group \( H \) by \( k^\times \). For a \( k^\times \)-group \( \hat{H} \), if there exists a finite group \( H \) such that \( H = \hat{H}/k^\times \) then there is a finite subgroup \( H' \) of \( H \) such that \( \hat{H} = ZH' \) and \( Z \cap H' \subset [H', H'] \), where \( Z \) is the image of \( k^\times \) in \( H \).
Let $Z'$ be the set of $z \in H'$ that are the image of an element $\lambda_z$ of $k^\times$, and let $e'$ be the central idempotent $\frac{1}{|Z'|} \sum_{z \in Z'} \lambda_z z^{-1} \in \mathcal{O}H'$. Then there is an interior $G$-algebra isomorphism

$$\mathcal{O}H' e' \simeq \mathcal{O}_s \hat{G}$$

induced by the inclusion $H' \subset \hat{H}$. Here the twisted algebra is, of course, defined over $\mathcal{O}$ (and hence over $K$) in the same way as over $k$ (explicitly in 5.12 of [41]).

Let $\mathcal{CF}_K(\hat{L}), \mathcal{CF}_K(G, B)$ denote the set of central functions over $K_s \hat{L}$ and $K GB$ respectively (note the asterisk), and let $\mathcal{L}_K(\hat{L})$ and $\mathcal{L}_K(G, B)$ denote the Grothendieck groups of the categories of, respectively, the twisted group algebra $K_s \hat{L}$ and $K GB$. Moreover, let $\mathcal{L}_K^0(\hat{L})$ denote the kernel of the restriction map $\mathcal{L}_K(\hat{L}) \to \mathcal{L}_k(\hat{L})$, and $\mathcal{L}_K^0(G, B)$ the kernel of $\mathcal{L}_K(G, B) \to \mathcal{L}_k(G, B)$. Finally, let $\mathcal{BCF}_K(\hat{L}), \mathcal{BCF}_K(G, B)$ denote the set of Brauer central functions (that is, central functions over the $p$-regular elements of $G$) over $K_s \hat{L}$ and $K GB$ respectively. From now on, whenever considering central functions or characters over subgroups or quotients of subgroups of $\hat{L}$, we implicitly do so in the twisted group algebras $K_s \hat{L}$ or $\mathcal{O}_s \hat{L}$.

Let $\lambda$ be a generalized character of a defect group $D$ of $B$ such that whenever $(x, b_x) \in (D, e)$ and $z \in G$ such that $(x, b_x) z^2 \in (D, e)$ we have $\lambda(x) = \lambda(x^z)$. Given a character $\chi \in \mathcal{L}_K(G, B)$, we can define $\lambda \ast \chi$ as in [3], which is another generalised character of $B$. In the following, whenever $\lambda$ is a generalised character of a subgroup of $D$, we consider its inflation to $D$ implicitly.

Let $X$ be an upwardly closed $G$-stable set of subgroups of $D$. Puig and Usami define the notion of a $(G, B)$-local system on $X$, that is, a collection of perfect isometries

$$\{\Gamma_Y : \mathcal{L}_K(C_L(Y)) \to \mathcal{L}_K(C_G(Y), e^{C_G(Y)})\}_{Y \in X}$$

satisfying the following conditions:

1. $\forall Y \in X, \forall \eta \in \mathcal{CF}_K(C_L(Y)), \forall s \in E$ we have $\Gamma_Y(\eta)^s = \Gamma_Y(\eta^s)$
2. $\forall Y \in X, \forall \eta \in \mathcal{L}_K(C_L(Y)), \Delta_Y(\eta) := \sum_{u \in U_Y} e_{C_G(Y)}(\Gamma_Y(u)(d^u_{C_L(Y)}(\eta)))$, where $U_Y$ is a set of representatives for the orbits of $C_E(Y)$ in $D$, and $d^u_H$ and $e^u_H$ are the twisted restriction maps as defined in [42 2.10].

In particular, $\Delta_Y(\mathcal{L}_K(C_L(Y))) = \mathcal{L}_K(C_G(Y), e^{C_G(Y)})$ is a perfect isometry, and it satisfies $\Delta_Y(\lambda \ast \eta) = \lambda \ast \Delta_Y(\eta)$ for any $\lambda \in \mathcal{CF}_K(D)^{C_E(Y)}, \eta \in \mathcal{CF}_K(C_L(Q))$. Note that if $\{1\} \in X$, $\Delta_{\{1\}}$ is a perfect isometry between $\mathcal{O}_s \hat{L}$ and $\mathcal{O} GB$.

If we have a $(G, B)$-local system on $X$, we can consider a subgroup $Q$ maximal with respect to the property $Y \not\in X$. We can then consider the sum

$$\sum_{u \in U_Q \setminus Q} e_{C_G(Q)}^u \circ \Gamma_Q(u) \circ d^u_{C_L(Q)}$$

which is well-defined as $Q < Q \cdot \langle u \rangle$, and therefore $Q \cdot \langle u \rangle \in X$, for any $u \in U_Q \setminus Q$. This induces a bijective isometry $\Delta_Q : \mathcal{CF}_K(C_L(Q)) \to \mathcal{CF}_K(C_G(Q), e^{C_G(Q)})$ such that $\Delta_Q(\mathcal{L}_K(C_L(Q))) = \mathcal{L}_K(C_G(Q), e^{C_G(Q)})$ (see [43 3.7]). In order to extend $X$ to a $(G, B)$-local system on a bigger set of subgroups, we use this lemma.
Lemma 3.1 (3.11, [42]). With the notations above, let $X$ be an upwardly closed $G$-stable set of subgroups of $D$, and let $Q$ be a subgroup of $D$ maximal with respect to the property $Q \not\subseteq X$. Let $\Gamma$ be a $(G, B)$-local system on $X$. Then $\Gamma$ can be extended to a $(G, B)$-local system $\hat{\Gamma}$ on $\hat{X} = X \cup \{Q^e\}_{e \in E}$ if and only if the bijective isometry $\Delta_Q^0$ can be extended to an $N_E(Q)$-stable bijective isometry
\[
\Delta_Q : \mathcal{F}_K(C_L(Q)) \to \mathcal{F}_K(C_G(Q), \pi^G(Q))
\]
such that $\Delta_Q(L_K(C_L(Q))) = L_K(C_G(Q), \pi^G(Q))$.

We are also going to use the following fact:

Lemma 3.2. Let $G$ be a finite group, let $B$ be a block of $OG$ with nontrivial defect. Let $\chi \in \text{Irr}(B)$. Then there exists $\theta \in \mathcal{L}_K^0(G, B)$ such that $(\theta, \chi) \neq 0$.

Proof. Suppose otherwise, so there exists a $\chi \in \text{Irr}(B) \cap \mathcal{L}_K^0(G, B)^\perp$. Then as described in [33, 9.2], $\mathcal{L}_K^0(G, B)^\perp = \text{Pr}_G(B)$, hence $\chi$ is the character of a simple projective module. But then $\chi$ forms a block of defect zero of $G$, which is a contradiction.

We are interested in constructing $(G, B)$-local systems on blocks of $OG$ with defect group properly contained in $(C_2)^5$ and non-cyclic inertial quotient (the cyclic case has been done in [50]). As seen in Proposition 2.12, this means that we are interested in blocks with inertial quotient $C_3 \times C_3$ or $(C_7 \times C_3)_1$.

In the following we denote as $\delta_{ij}$ numbers such that $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ otherwise. Given two generalised characters $\chi, \psi$ of $G$ we write $(\chi, \psi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g)$ for the usual inner product between characters.

Proposition 3.3. Let $G$ be a finite group and $B$ be a block of $OG$ with defect group $D \cong (C_2)^5$ and inertial quotient $E = C_3 \times C_3$. Then there is a $(G, B)$-local system on the set of all subgroups of $D$.

Proof. As usual, let $k(B) = |\text{Irr}(B)|$ and $l(B) = |\text{IBr}(B)|$. We will repeatedly use Proposition 16 in [41] and Proposition 2.12 to determine $k(b)$ and $l(b)$ for various blocks $b$, without making further reference to it.

We proceed with an inductive argument on $X$. As a base case, when $X = \{D\}$, the existence is known by [42, 3.4.2]. Now suppose that there is a $(G, B)$-local system on $X$, and let $Q$ be a subgroup of $D$ maximal with respect to the property $Q \not\subseteq X$. We consider $\hat{X} = X \cup \{Q^e\}_{e \in E}$ and prove that the isometry $\Delta_Q^0$ can always be extended to an $N_E(Q)$-stable isometry. Then Lemma 3.1 proves the result. Let $\pi_Q = \pi^{G(Q)}$ as above.

From Proposition 2.12, either $k(B) = 32$, $l(B) = 9$ or $k(B) = 16$, $l(B) = 1$.

(a) First we investigate the situation with $k(B) = 32$. In this case, $O_4L \cong OL$ (see [40, 10.4]) where $L = D \times E$. Let $D = [D, E] \times C_D(E) = P \times R \cong (A_4 \times A_4) \times C_2$.

(a_1) Suppose that $C_E(Q) = 1$. Then $\pi_Q$ is nilpotent, and $C_L(Q)/Q = D/Q$. Let $\rho$ be the trivial character of $D/Q$. Then $\mathcal{L}_K^0(C_L(Q)/Q) = \sum_{\sigma \in \text{Irr}(D/Q)} (\sigma - \rho)\mathbb{Z}$. Note that $(\sigma_1 - \rho, \sigma_2 - \rho) = 2$.

- If $D/Q = C_2$ there is a unique $\sigma \neq \rho$ in $\text{Irr}(C_L(Q)/Q)$. We can label $\text{Irr}(\pi_Q) = \{\chi_1, \chi_2\}$ such that $\Delta_Q^0(\sigma - \rho) = \varepsilon_1\chi_1 + \varepsilon_2\chi_2$
for some \( \varepsilon_i \in \{ \pm 1 \} \). Then defining \( \Delta_Q(\sigma) = \varepsilon_1\chi_1, \Delta_Q(\rho) = \varepsilon_2\chi_2 \) extends the isometry, and the extension is clearly \( N_E(Q) \)-stable since \( |\text{Irr}(b_Q)| = 2 \) and \( [N_E(Q) : C_E(Q)] \) is odd.

- If \( D/Q > C_2 \), then there are at least two different characters \( \sigma_1 \neq \sigma_2 \) different from \( \rho \) in \( \text{Irr}(C_L(Q)/Q) \). Hence we can label characters of \( \text{Irr}(b_Q) \) such that

\[
\Delta_Q^0(\sigma_1 - \rho) = \varepsilon_2\chi_2 - \varepsilon_1\chi_1, \quad \Delta_Q^0(\sigma_2 - \rho) = \varepsilon_3\chi_3 - \varepsilon_1\chi_1
\]

since \( (\sigma_i - \rho, \sigma_j - \rho) = 1 + \delta_{ij} \). Now if we consider any other character \( \sigma_i \in \text{Irr}(C_L(Q)/Q) \) different from \( \sigma_1, \sigma_2 \) we claim that there is a unique \( \chi \in \text{Irr}(b_Q) \) such that \( \Delta_Q^0(\sigma_i - \rho) = \varepsilon\chi - \varepsilon_1\chi_1 \). In fact, \( (\sigma_i - \rho, \sigma_j - \rho) = 1 \) for \( j = 1, 2 \), but \( \Delta_Q^0(\sigma_1 - \rho) = \varepsilon_2\chi_2 + \varepsilon_3\chi_3 \) leads to a contradiction, as \( 0 = \varepsilon_2\chi_2(1) + \varepsilon_3\chi_3(1) = 2\varepsilon_1\chi_1(1) \neq 0 \). So \( \chi_1 \) is in the support of \( \Delta_Q^0(\sigma_i - \rho) = \varepsilon\chi - \varepsilon_1\chi_1 \), which implies that \( \chi \neq \chi_2, \chi_3 \): in fact, suppose that \( \chi = \chi_2 \). Then

\[
1 = (\sigma_1 - \rho, \sigma_1 - \rho) = (\Delta_Q^0(\sigma_1 - \rho), \Delta_Q^0(\sigma_1 - \rho)) = 2
\]

which is a contradiction, and the same holds for \( \chi = \chi_3 \). Hence we can associate to each \( \sigma_i \) a uniquely determined character \( \chi_{\sigma_i} \), which we denote as \( \chi_{i+1} \).

We can now define \( \Delta_Q(\sigma_i) = \varepsilon_{i+1}\chi_{i+1} \). This extends the isometry, as required. Since every character was uniquely determined by \( \Delta_Q^0 \), the \( N_E(Q) \)-stability is guaranteed.

(a2) Suppose that \( C_E(Q) = E \). Then \( N_E(Q) = E \), and \( Q \subseteq R \), which implies that either \( Q = C_2 \) or \( Q = 1 \).

- If \( Q = C_2 \), then \( C_L(Q)/Q = T_1 \times T_2 \), where \( T_i \simeq A_4 \). Let \( \text{Irr}(T_1) = \{ \theta_1, \theta_2, \theta_3, \theta_4 \} \) and \( \text{Irr}(T_2) = \{ \phi_1, \phi_2, \phi_3, \phi_4 \} \), where \( \theta_4 \) and \( \phi_4 \) are the non-linear characters. Let \( \theta = \theta_1 + \theta_2 + \theta_3 - \theta_4 \), and \( \phi = \phi_1 + \phi_2 + \phi_3 - \phi_4 \). Then a basis for \( C_L(K)(Q)/Q \) is given by

\[
B_0^0 = \{ B_i = \hat{\theta}\phi_i \}_{i=1,2,3,4} \cup \{ B_{j+4} = \theta_j \hat{\phi} \}_{j=1,2,3}
\]

Note that \( (\theta\phi_i, \theta\phi_j) = 4\delta_{ij} = (\theta\hat{i}, \theta\hat{j}) \) and that this inner product is preserved by \( \Delta_Q^0 \). Hence we know that, for every element \( b \) of the basis, \( \Delta_Q^0(b) \) is the sum of four distinct characters in \( \text{Irr}(\overline{e_Q}) \) with coefficients \( \pm 1 \), since the only other possibility to obtain 4 would be \( \Delta_Q^0(b) = 2\chi_i \), but then \( 0 = 2\chi_i(1) \) which is a contradiction.

The block \( \overline{e_Q} \) of \( C_G(Q)/Q \) has defect group \( (C_2)^4 \) and inertial quotient \( C_3 \times C_3 \). From Theorem 1 in [54], \( k(e_Q) = 32 \), \( l(e_Q) = 9 \), therefore \( k(\overline{e_Q}) = 16 \) and \( l(\overline{e_Q}) = 9 \).

From now on, we say that a character \( \chi \) is in the support of a generalised character \( \Phi \) if \( (\Phi, \chi) \neq 0 \).

Consider a \( 7 \times 16 \) matrix \( M = (m_{xy}) \) defined as follows: every row represents an element of \( B_K^0 \), every column a character of \( \text{Irr}(\overline{e_Q}) \), and \( m_{xy} = \langle \Delta_Q^0(B_x), \chi_y \rangle \). Then very row contains exactly 4 nonzero elements and every entry is either \(-1, 0 \) or \( 1 \): hence, there are exactly 28 nonzero entries in the matrix, and Lemma 3.2 implies that every column contains at least one nonzero element. In particular, a nonzero \( m_{xy} \) means that \( \chi_y \) is in the support of \( \Delta_Q^0(B_x) \).

Consider \( B_1 + B_2 + B_3 + B_4 - B_5 - B_6 - B_7 = \theta_4 \hat{\phi} \). We can label characters without loss of generality such that \( \Delta_Q^0(\theta_4 \hat{\phi}) = \chi_4 + \chi_8 + \chi_{12} + \chi_{16} \). For brevity from now on we call these four characters special.
Now we use a counting argument: every column corresponding to a non-special character contains at least two nonzero entries, since it contains at least one and in the linear combination that is equal to $\theta_i \phi$ the associated character does not appear, which means that it appears in the support of some other element of $B^0_K$ with an opposite sign: this gives at least 24 nonzero entries in the non-special columns. Since the total is 28, it follows that the columns corresponding to the special characters contain exactly one nonzero entry, and that every other column contains exactly two nonzero entries.

The fact that $\Sigma^0_Q$ is an isometry tells us that whenever $i < j$

$$(B_i, B_j) = (\Sigma^0_Q(B_i), \Sigma^0_Q(B_j)) = \begin{cases} 0 & \text{if } i \leq 3, j \leq 4 \\ 1 & \text{if } i \leq 4, j \geq 5 \\ 0 & \text{if } i \geq 5, j \geq 6 \end{cases}$$

Note that, for any $i$, the support of $\Sigma^0_Q(B_i)$ cannot contain two special characters. In fact for each given $i$ there are at least three more characters $B_j$ such that $(B_i, B_j) = 1$, which means that the supports of the image of each pair under $\Sigma^0_Q$ share at least one character: but the special characters can only appear in the support of only one such support (since their column contains a single nonzero entry), and any character can appear in at most two, which means that the support of $\Sigma^0_Q(B_i)$ contains at least three characters that appear in the support of another image of an element of $B^0_K$ (hence, non-special).

With a similar argument, the supports of the images of $B_5, B_6, B_7$ cannot contain any special character, since there are four characters $(B_1, B_2, B_3, B_4)$ with which they share an element in the supports of their images under $\Sigma^0_Q(B_i)$, so every character in their supports also appears in the support of the image of another element of $B^0_K$. We can therefore label elements in $\text{Irr}(\Sigma_Q)$ such that $\Sigma^0_Q(B_i)$ contains $\chi_{4i}$ for $i \leq 4$. We can also label characters such that

$$\Sigma^0_Q(B_1) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4$$

for some $\varepsilon_i \in \{\pm 1\}$. We know that the supports of the images of $B_5, B_6$ and $B_7$ share exactly one character with $\Sigma^0_Q(B_1)$, and that these are all distinct. Hence, we can label elements (switching labels of $\chi_1, \chi_2, \chi_3$ if needed) such that $\chi_i$ is in the support of $\Sigma^0_Q(B_{i+4})$ for $i = 1, 2, 3$, which in particular implies that these three characters do not appear in any other support. Repeating the same argument for $B_2, B_3$ and $B_4$, this means that the supports of the images of $B_1, B_2, B_3, B_4$ are disjoint. We can therefore label characters in $\text{Irr}(\Sigma_Q)$ such that

$$\Sigma^0_Q((\theta_1 + \theta_2 + \theta_3 - \theta_4) \phi_1) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4$$

$$\Sigma^0_Q((\theta_1 + \theta_2 + \theta_3 - \theta_4) \phi_2) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8$$

$$\Sigma^0_Q((\theta_1 + \theta_2 + \theta_3 - \theta_4) \phi_3) = \varepsilon_9 \chi_9 + \varepsilon_{10} \chi_{10} + \varepsilon_{11} \chi_{11} + \varepsilon_{12} \chi_{12}$$

$$\Sigma^0_Q((\theta_1 + \theta_2 + \theta_3 - \theta_4) \phi_4) = \varepsilon_{13} \chi_{13} + \varepsilon_{14} \chi_{14} + \varepsilon_{15} \chi_{15} + \varepsilon_{16} \chi_{16}$$

Then if we define $\Sigma_Q : \mathcal{L}_K(C_L(Q)/Q) \rightarrow \mathcal{L}_K(C_G(Q)/Q, \Sigma_Q)$ as

$$\Sigma_Q(\theta_i \phi_j) = \varepsilon_i + 4(j-1) \chi_{i+4(j-1)}$$
is an isometry that extends $\Sigma^0_Q$ as required. Since $N_E(Q) = C_E(Q)$, stability is automatic.

- If $Q = 1$, then $C_L(Q) = T_1 \times T_2 \times V$, where $T_1 \simeq A_4$ and $V \simeq C_2$. Let $\text{Irr}(T_1)$ be as in case (a1), and let $\text{Irr}(V) = \{\eta_1, \eta_2\}$. A basis for $L^0_k(C_L(Q))$ is given by

\[
\{B_{i+7(h-1)} = \hat{\theta}_i \phi_j \eta_k \}_{i=1,2,3,4} \cup \{B_{j+4+7(h-1)} = \theta_j \hat{\phi} \eta_h \}_{j=1,2,3,4} \cup \{\theta_i \phi_j (\eta_1 - \eta_2)\}_{i=1,2,3,4}
\]

Note that $(\hat{\theta}_i \phi_j \eta_k, \hat{\phi}_j \eta_h) = 4 \delta_{ij} \delta_{kh}$, and that $(\theta_i \phi_j (\eta_1 - \eta_2), \theta_h \phi_k (\eta_1 - \eta_2)) = 2 \delta_{ij} \delta_{kh}$.

As $C_G(Q) = G$, $k(\Sigma_Q) = k(B) = 32$ and $l(B) = 9$. We can consider the $14 \times 32$ matrix $M_{xy}$, where the rows are indexed by the elements $B_i$, and repeat the argument of the previous case: every row has four nonzero entries, and there are eight special characters (those in the supports of the images under $\Delta^0_1$ of the linear combinations $B_1 + B_2 + B_3 + B_4 - B_5 - B_6 - B_7$ and $B_8 + B_9 + B_{10} + B_{11} - B_{12} - B_{13} - B_{14}$). We label them $\{\chi_{i4}\}_{i=1,\ldots,8}$. Now there are 56 nonzero entries in $M_{xy}$, and again from Lemma 3.2 every column has at least one nonzero entry, so every special column has exactly one nonzero element, and every other column has exactly two. We can prove with an identical argument that the images of $B_1, B_2, B_3, B_4, B_8, B_9, B_{10}, B_{11}$ have pairwise disjoint supports, and that hence we can label $\text{Irr}(B) = \{\chi_1, \ldots, \chi_{32}\}$ such that

\[
\Delta^0_1(\hat{\theta}_i \phi_j \eta_h) = \varepsilon_1 + 4m \chi_1 + \varepsilon_2 + 4m \chi_2 + \varepsilon_3 + 4m \chi_3 + \varepsilon_4 + 4m \chi_4
\]

where $m = i + 4(h-1) - 1$. Then defining $\Delta_1(\theta_i \phi_j \eta_h) = \hat{\varepsilon}_1 + 4m \eta_1 + \varepsilon_2 + 4m \eta_2 + \varepsilon_3 + 4m \eta_3 + \varepsilon_4 + 4m \eta_4$ gives an isometry that extends $\Delta^0_1$ as required. Again, since $N_E(Q) = C_E(Q)$, stability is automatic.

(a3) Suppose that $C_E(Q) = C_3$. Then $C_D(C_E(Q)) = (C_2)^3$, since $C_E(Q)$ cannot act as $(C_3)_2$ (in the notation of Proposition 2.2) without having $Q = R$, but in that case $C_E(R) = E$. Therefore either $Q = (C_2)^3$, or $Q = C_2 \times C_2$, or $Q = C_2 \neq R$.

- If $Q = C_2$, then $C_L(Q) = C_2 \times A_4 \times C_2 = T \times W$ where $W = C_2 \times C_2$ and $T = A_4$. Let $\text{Irr}(W) = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and let $\text{Irr}(T) = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where $\phi_4$ is the nonlinear character. As before, let $\hat{\phi} = \phi_1 + \phi_2 + \phi_3 - \phi_4$. Then a basis for $L^0_k(C_L(Q)/Q)$ is given by $\{\hat{\phi} \beta_i\}_{i=1,2,3,4} \cup \{\beta_{i+1} - \beta_i \phi_j\}_{i=1,2,3,j=1,2,3}$. Note that $(\beta_i \hat{\phi}, \beta_j \hat{\phi}) = 4 \delta_{ij}

The block $\Sigma_Q$ of $C_G(Q)/Q$ has defect group $D = (C_2)^4$ and inertial quotient $E = C_3$, with $C_{\Sigma_Q}(E) = C_2 \times C_2$. Therefore, $k(\Sigma_Q) = 16$ and $l(\Sigma_Q) = 3$. We can label characters in $\text{Irr}(\Sigma_Q) = \{\chi_1, \ldots, \chi_{16}\}$ such that

\[
\Sigma_Q(\hat{\phi} \beta_1) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4
\]

Then since $(\hat{\phi} \beta_1, (\beta_2 - \beta_1) \phi_i) = \pm 1$ for any $i$, there is a unique character, that we label as $\chi_{4+i}$, such that

\[
\Sigma_Q((\beta_2 - \beta_1) \phi_i) = \varepsilon_{4+i} \chi_{4+i} - \varepsilon_i \chi_i
\]

Which then, since $(\hat{\phi} \beta_2, (\beta_2 - \beta_1) \phi_i) = \pm 1$, implies

\[
\Sigma_Q(\hat{\phi} \beta_2) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8
\]
Iterating this process, we can label characters such that
\[
\Sigma_0^0(\hat{\phi}b_j) = \varepsilon_4i-3\chi_{4i-3} + \varepsilon_{4i-2}\chi_{4i-2} + \varepsilon_{4i-1}\chi_{4i-1} + \varepsilon_{4i}\chi_{4i}
\]
\[
\Sigma_0^0(\hat{\phi}\beta_i - \beta_i)\phi_j = \varepsilon_{4i+j}\chi_{4i+j} - \varepsilon_{4(i-1)+j}\chi_{4(i-1)+j}
\]
Then defining \(\Sigma_0^0(\hat{\phi}\beta_i) = \varepsilon_{j+4(i-1)}\chi_{j+4(i-1)}\) gives an isometry that extends \(\Sigma_0^0\), and since \(N_E(Q) = C_E(Q) = C_3\), \(N_E(Q)\)-stability is automatic.

- If \(Q = C_2 \times C_2\), then either \(R \subseteq Q\), or \(R \cap Q = \{1\}\). In both cases, \(C_L(Q)/Q = T \times V\) where \(T = A_4\), \(V = C_2\). Let \(\text{Irr}(V) = \{\beta_1, \beta_2\}\) and \(\text{Irr}(T) = \{\phi_1, \phi_2, \phi_3, \phi_4\}\), where \(\phi_4\) is the non-linear character. As before, let \(\hat{\phi} = \phi_1 + \phi_2 + \phi_3 - \phi_4\). Then a basis for \(C^*_k(C_L(Q)/Q)\) is given by \(\{\hat{\phi}\beta_i\}_{i=1,2} \cup \{(\beta_2 - \beta_1)\phi_i\}_{i=1,2,3}\). Note that \((\beta_i\hat{\phi}, \hat{\beta}_j\hat{\phi}) = 4\delta_{ij}\).

The block \(\overline{\Sigma_0^0}\) of \(G_Q(Q)/Q\) has defect group \(D = (C_2)^3\) and inertial quotient \(E = C_3\). Therefore, \(k(B) = 8\). We can repeat an identical argument to that in the previous case, to obtain that we can label characters in \(\text{Irr}(\overline{\Sigma_0^0})\) as \(\{\chi_1, \ldots, \chi_8\}\) such that
\[
\Sigma_0^0(\hat{\phi}\beta_1) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4
\]
\[
\Sigma_0^0(\hat{\phi}\beta_2) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8
\]
Then defining \(\Sigma_0^0(\hat{\phi}\beta_3) = \varepsilon_{1+4(j-1)}\chi_{1+4(j-1)}\) gives an isometry that extends \(\Sigma_0^0\).

If \(R \subseteq Q\), \(N_E(Q) = C_E(Q)\) so the stability condition is automatic. Otherwise, \(N_E(Q) = E\). To prove the stability of the extension \(\Sigma_0^0\) defined above it is enough to observe that the basis of \(C_L(Q)/Q\) is pointwise fixed by the action of \(E\), and that every label on \(\text{Irr}(\overline{\Sigma_0^0})\) is uniquely determined by our process: in fact, \(\chi_k\) is the unique character in the intersection of the supports of the images under \(\Sigma_0^0\) of \(\hat{\beta}_{[k/4]}\) and \(\theta_{E}(\beta_2 - \beta_1)\) where \(\overline{k} = k \mod 4\), so \(E\)-stability follows from the one of \(\Sigma_0^0\). Hence, we are done.

- If \(Q = (C_2)^3\), then \(R \subseteq Q\), and \(C_L(Q)/Q = A_4\). Let \(\text{Irr}(A_4) = \{\theta_1, \theta_2, \theta_3, \theta_4\}\), then \(L_0^0(C_L(Q)/Q) = \mathbb{Z}(\theta_1 + \theta_2 + \theta_3 - \theta_4)\). Note that \((\theta_1 + \theta_2 + \theta_3 - \theta_4), (\theta_1 + \theta_2 + \theta_3 - \theta_4)) = 4\).

The block \(\overline{\Sigma_0^0}\) of \(G_Q(Q)/Q\) has defect group \(C_2 \times C_2\) and inertial quotient \(C_3\), which implies that \(k(B) = 4\), \(l(B) = 3\). Label \(\text{Irr}(\overline{\Sigma_0^0})\) such that
\[
\Sigma_0^0(\hat{\phi}_1 + \hat{\phi}_2 + \hat{\phi}_3 - \hat{\phi}_4) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4
\]
then defining \(\Sigma_0^0(\hat{\phi}_i) = \varepsilon_i \chi_i\) gives an isometry that extends \(\Sigma_0^0\). Here \(N_E(Q) = E\), and \(N_E(Q)/C_E(Q)\) acts trivially on \(D/Q \subseteq C_L(Q)/Q\). In particular, it fixes every \(\theta_i\). We want to show that \(E\) also fixes every \(\chi_i\); equivalently, since \(\Sigma_0^0\) is already \(N_E(Q)\)-stable, we need to show that \(E\) acts trivially on \(\text{Irr}_k(C_G(Q)/Q, \overline{\Sigma_0^0})\). Suppose not: then, following the same type of argument in [12 4.9] for any \(\phi \in \text{Irr}(C_G(Q)/Q, \overline{\Sigma_0})\) the induced character \(\text{Ind}_{C_G(Q)/Q}^{N_G(Q, e_Q)/Q} \phi\) is an irreducible Brauer character, hence since \((N_G(Q, e_Q)/Q)/(C_G(Q)/Q) \approx E/C_E(Q)\) there are at most three isomorphism classes of simple \(k(N_G(Q, b_Q)/Q)\)-modules (note that \(\overline{b_Q}\) is not a block of \(kN_G(Q, e_Q)/Q\)). However, from Lemma 3.14 in [12] there is a bijection that preserves defect groups and inertial quotients between blocks of \(k(N_L(Q)/Q)\) and blocks of \(k(N_G(Q, b_Q)/Q)\) that cover \(\overline{b_Q}\). In particular, \(k(N_L(Q)/Q) = k(A_4 \times C_3)\), which has three blocks with 3 simple modules each, so \(k(N_G(Q, b_Q)/Q)\overline{b_Q}\) has at least 9 simple modules, which is a contradiction. Then \(E\) acts trivially on \(\text{IBr}(C_G(Q)/Q, \overline{\Sigma_0^0})\), so \(\Sigma_0^0\) is \(N_E(Q)\)-stable.
(b) Now suppose that \( k(B) = 16, l(B) = 1 \). Then the central extension \( \hat{E} \) does not split, and we have \( \mathcal{O}_L \hat{L} = \mathcal{O}_L \hat{b}' \) where \( L' = ((C_2)^4 \times 3_1^{1+2}) \times C_2 \), and \( b' \) is a nonprincipal block of \( L' \). Note that it does not matter what central extension (\( 3_1^{1+2} \) or \( 3_1^{1+2} \)) or nonprincipal block we choose, since these are all Morita equivalent by [14].

(b_1) Suppose that \( |C_E(Q)| = 1 \). Then \( C_L(Q) = k^x \times D \), and we can apply the same argument as in case (a_1) to determine the unique extension of \( \overline{Q} \).

(b_2) Suppose that \( C_E(Q) = E \). Then \( N_E(Q) = E \) as well, and either \( Q = R = C_2 \), or \( Q = 1 \).

- If \( Q = C_2 \), then \( \mathcal{O}_C(C_L(Q)/Q) \simeq \mathcal{O}((C_2)^4 \times 3_1^{1+2})\overline{b}' \), where \( \overline{b}' \) is the unique block dominated by \( \overline{b}' \) (see [46, 1.22]), and \( k(\overline{b}') = 8, l(\overline{b}') = 1 \), so \( \dim(K) = 7 \).

The character table of this block is (omitting \( p \)-singular conjugacy classes)

| order(\( g \)) | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
|--------------|---|---|---|---|---|---|---|---|---|---|
| \( \theta_{1,1} \leq \theta \leq \theta_8 \) | 3 | 3J | \( -3J - 3 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

where \( J^3 = 1 \) is a primitive root of unity. A basis of \( \mathcal{L}_K^0(C_L(Q)/Q) \) is

\[
\{(\theta_i - \theta_{i-1})\}_{i=1,...,6} \cup \{\hat{\theta} = \theta_5 + \theta_6 + \theta_7 - \theta_8\}
\]

The block \( \overline{Q} \) of \( C_G(Q)/Q \) has defect group \( (C_2)^4 \) and inertial quotient \( C_3 \times C_3 \). From Theorem 1 in [54], \( k(\overline{Q}) = 16, l(\overline{Q}) = 1 \), and therefore \( k(\overline{Q}) = 8 \) and \( l(\overline{Q}) = 1 \).

Noting that \( (\hat{\theta}, \hat{\theta}) = 4 \), we can label characters in \( \text{Irr}(\overline{Q}) \) such that

\[
\overline{\Delta}_Q(\hat{\theta}) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8
\]

and then, since \( (\hat{\theta}, \theta_1 - \theta_3) = -1 \) and \( (\theta_1 - \theta_5, \theta_4 - \theta_3) = 2 \), so the supports have one character in common (without loss of generality assumed to be \( \chi_3 \)), label

\[
\overline{\Delta}_Q(\theta_4 - \theta_5) = \varepsilon_4 \chi_4 - \varepsilon_5 \chi_5
\]

and repeating the argument (considering that \( (\theta_i - \theta_{i+1}, \theta_{i+1} - \theta_{i+2}) = -1 \) and that \( (\theta_i - \theta_{i+1}, \theta_i - \theta_{i+1}) = 2 \))

\[
\overline{\Delta}_Q(\theta_i - \theta_{i+1}) = \varepsilon_i \chi_i - \varepsilon_{i+1} \chi_{i+1}
\]

Then defining \( \overline{\Delta}_Q(\theta_i) = \varepsilon_i \chi_i \) gives an isometry that extends \( \overline{\Delta}_Q \) as required. Since \( N_E(Q) = C_E(Q) \), stability is automatic.

- If \( Q = 1 \), then we need to look at \( \mathcal{O}_L \hat{L} \simeq \mathcal{O}_L((C_2)^4 \times 3_1^{1+2}) \times C_2)\overline{b}' \). With the same notation for characters of \( (C_2)^4 \times 3_1^{1+2} \) as in the previous case, let \( \text{Irr}(C_2) = \{\beta_1, \beta_2\} \). Then a basis of \( \mathcal{L}_K^0(\mathcal{O}_L \hat{L}) \) (which has dimension 15) is given by \( \{\theta(\beta_i - \beta_{i+1})\}_{i=1,...,6} \cup \{\theta \beta_1\} \cup \{\theta_i(\beta_2 - \beta_1)\}_{i=1,...,8} \). We can repeat the same argument as in the previous case to show that we can label the characters of \( \text{Irr}(\overline{Q}) \) such that

\[
\Delta_{\overline{Q}}(\hat{\theta} \beta_1) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8
\]

\[
\Delta_{\overline{Q}}(\theta_i - \theta_{i+1}) = \varepsilon_i \chi_i - \varepsilon_{i+1} \chi_{i+1}
\]
and then, considering that \((\theta_i - \theta_{i+1})\beta_1, \theta_i(\beta_2 - \beta_1)) = -1\) and \((\theta_i(\beta_2 - \beta_1), \theta_i(\beta_2 - \beta_1)) = 2\), we can label the remaining characters such that
\[
\Delta^0_i(\theta_i(\beta_2 - \beta_1)) = \varepsilon_{i+8}x_{i+8} - \varepsilon_ix_i
\]
Then defining \(\Delta_1(\theta_i\beta_j) = \varepsilon_{i+8(j-1)}x_{i+8(j-1)}\) gives an isometry that extends \(\Delta^0_i\) as required. Since \(N_E(Q) = C_E(Q)\), stability is automatic.

(b3) Suppose that \(C_E(Q) = C_3\). Note that the only possibility for this to happen is for \(C_D(C_E(Q)) = (C_2)^3\), as otherwise \(Q = R\) and so \(C_E(Q) = E\). In particular, this implies that \(Q = (C_2)^3\), or \(Q = C_2 \times C_2\), or \(Q = C_2 \neq R\).

- If \(Q = C_2\), then \(O_s(C_L^E)/Q \simeq O(A_4 \times (C_2)^2 \times C_3)b\), where \(b\) is any of the three blocks. In particular \(k(b) = 16\), \(l(b) = 3\). Moreover, the block \(\overline{e}Q\) of \(C_E(Q)/Q\) has defect group \((C_2)^4\) and inertial quotient \(C_3\), and since the action of the inertial quotient \(C_E(Q) = C_3\) centralises a \((C_2)^2\) in the defect group, \(k(\overline{e}Q) = 16\) and \(l(\overline{e}Q) = 3\).

- If \(Q = (C_2)^3\), then \(O_s(C_L^E)/Q \simeq O(A_4 \times C_2 \times C_3)b\), where \(b\) is any of the three blocks. In particular, \(k(b) = 8\), \(l(b) = 3\). Moreover, the block \(\overline{e}Q\) of \(C_E(Q)/Q\) has defect group \((C_2)^3\) and inertial quotient \(C_3\), which implies that \(k(\overline{e}Q) = 8\) and \(l(\overline{e}Q) = 3\).

- If \(Q = (C_2)^3\), then \(O_s(C_L^E)/Q \simeq O(A_4 \times C_3)b\), where \(b\) is any of the three blocks. In particular, \(k(b) = 4\), \(l(b) = 3\). Moreover, the block \(\overline{e}Q\) of \(C_E(Q)/Q\) has defect group \((C_2)^2\) and inertial quotient \(C_3\), which implies that \(k(\overline{e}Q) = 4\) and \(l(\overline{e}Q) = 3\).

We have considered all three problems of extending isometries between blocks with these characteristics in case (a3), so just repeating those arguments proves that \(\overline{\Sigma}_Q\) can be extended to an \(N_E(Q)\)-stable isometry \(\Sigma_Q\).

We have shown the existence of a \((G, B)\)-local system on the set of all subgroups of \(D\).

**Proposition 3.4.** Let \(G\) be a finite group and \(B\) be a block of \(OG\) with defect group \(D \cong (C_2)^5\) and inertial quotient \(E = (C_7 \times C_3)_1\) (meaning that \(C_D(E) = C_2 \times C_2\)). Then there is a \((G, B)\)-local system on the set of all subgroups of \(D\).

**Proof.** As usual, let \(k(B) = |\text{Irr}(B)|\) and \(l(B) = |\text{IBr}(B)|\). We will repeatedly use Proposition 16 in [47] and Proposition 2,12 to determine \(k(b)\) and \(l(b)\) for various blocks \(b\), without making further reference to it.

Just as in Proposition 3.3, we proceed with an inductive argument on \(X\). As a base case, when \(X = \{D\}\), the existence is known by [42, 3.4.2]. Now suppose that there is a \((G, B)\)-local system on \(X\), and let \(Q\) be a subgroup of \(D\) maximal with respect to the property \(Q \not\subseteq X\).

We consider \(\tilde{X} = X \cup \{Q^e\}_{e \in E}\) and prove that the isometry \(\overline{\Sigma}_Q\) can always be extended to an \(N_E(Q)\)-stable isometry. Then Lemma 3.1 proves the result. Let \(\overline{e}Q = \overline{\text{E}}_C(Q)\) as above.

From Proposition 2,12, \(k(B) = 32\), \(l(B) = 5\).

We have \(O_\tilde{L} \simeq OL\) where \(L = D \times E\) (see [40, 10.4]). Let \(D = [D, E] \times C_D(E) = P \times R\) where \(R = (C_2)^2\) and \(P \simeq ((C_2)^5 \times (C_7 \times C_3))\).

First, note that while a priori \(C_E(Q) \in \{1, C_3, C_7, C_7 \times C_3\}\), if \(C_7 \subseteq C_E(Q)\) then \(C_E(Q) = C_7 \times C_3\). So we only have to consider three possibilities, one for each possibility for \(C_E(Q)\).
(c₁) Suppose that \( C_E(Q) = 1 \). Then \( \sigma \) is nilpotent, and \( C_L(Q)/Q = D/Q \). Let \( \sigma_1 \) be the trivial character of \( D/Q \). Then \( \mathcal{L}_K^0(C_L(Q)/Q) = \sum_{\sigma \in \text{Irr}(D/Q)}(\sigma_1 - \rho)\mathbb{Z} \). Note that \( (\sigma_1 - \rho, \sigma_1 - \rho) = 2 \). We can use the same argument as in case (a₁) of Proposition 3.3 to show that \( \mathcal{X}_Q^0 \) can be extended to a unique \( \mathcal{X}_Q^0 \).

(c₂) Suppose that \( C_E(Q) = E \). Then \( N_E(Q) = E \) as well, so the \( N_E(Q) \)-stability of any extension of \( \mathcal{X}_Q^0 \) is automatic. Moreover, \( Q \subseteq R \) so \( Q \in \{1, C_2, (C_2)^2\} \).

- If \( Q = (C_2)^2 \). Then \( C_L(Q)/Q = (C_2)^3 \times (C_7 \times C_3) \), so \( \dim(\mathcal{L}_K^0(C_L(Q)/Q)) = 3 \). The character table of \( C_L(Q)/Q \) is

| order(g) | 1    | 2    | 3    | 3    | 6    | 7    | 7    |
|---------|------|------|------|------|------|------|------|
| ρ₁      | 1    | 1    | 1    | 1    | 1    | 1    | 1    |
| ρ₂      | 1    | 1    | 2    | 1    | 2    | 1    | 1    |
| ρ₃      | 3    | 3    | 0    | 0    | 0    | J    | J    |
| ρ₄      | 3    | 3    | 0    | 0    | 0    | 0    | J    |
| ρ₅      | 7    | 1    | 1    | -1   | -1   | 0    | 0    |
| ρ₆      | 7    | -1   | 1    | 0    | 0    | 0    | J    |
| ρ₇      | 7    | -1   | 2    | 0    | 2    | 0    | 0    |
| ρ₈      | 7    | -1   | 2    | 0    | 0    | 0    | 0    |

where \( \omega \) and \( \gamma \) are primitive roots of unity such that \( \omega^3 = 1, \gamma^6 = 1 \) and \( J = \frac{-1 + \sqrt{7}i}{2} \).

Hence, a basis of \( \mathcal{L}_K^0(C_L(Q)/Q) \) is given by

\[
\begin{align*}
\theta_1 &= \rho_1 + \rho_5 + \rho_1 - \rho_6 \\
\theta_2 &= \rho_4 + \rho_3 + \rho_2 - \rho_7 \\
\theta_3 &= \rho_4 + \rho_5 + \rho_3 - \rho_8
\end{align*}
\]

For every generalised character \( \theta \) we denote \( S_\theta = \text{supp}(\mathcal{X}_Q^0(\theta)) \), the set of all ordinary irreducible characters of the ambient group that are not orthogonal to \( \mathcal{X}_Q^0(\theta) \).

Note that \( (\theta_1, \theta_2) = 2 + 2\delta_{ij} \). Let \( \text{Irr}(b_Q) = \{\chi_1, \ldots, \chi_8\} \), then \( (\mathcal{X}_Q^0(\theta_1), \mathcal{X}_Q^0(\theta_1)) = 4 \).

Since \( \mathcal{X}_Q^0(\theta_1) = 2\varepsilon_i \chi_i \) leads to a contradiction since \( 2\varepsilon_i \chi_i(1) \neq 0 \), we can label characters in \( \text{Irr}(b_Q) \) such that

\[
\mathcal{X}_Q^0(\theta_1) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_1 \chi_1 + \varepsilon_6 \chi_6
\]

where \( \varepsilon_i = \pm 1 \). Since \( (\theta_1, \theta_2) = 2 \) then \( |S_{\theta_1} \cap S_{\theta_2}| = 2 \) or 4. Suppose that it is 4. Then since \( |S_{\theta_1}| = 4 \) the supports are the same, and exactly one character appears with an opposite sign to that in \( \mathcal{X}_Q^0(\theta_1) \): without loss of generality we can suppose it is \( \chi_6 \).

Then \( \mathcal{X}_Q^0(\theta_2) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_1 \chi_1 - \varepsilon_6 \chi_6 \). However, in this case \( \mathcal{X}_Q^0(\theta_1 - \theta_2) = 2\varepsilon_6 \chi_6 \), which is a contradiction since \( 2\varepsilon_6 \chi_6(1) \neq 0 \). Hence \( |S_{\theta_1} \cap S_{\theta_2}| = 2 \) and we can suppose without loss of generality that the two characters in the intersection are \( \chi_4 \) and \( \chi_5 \).

Then

\[
\mathcal{X}_Q^0(\theta_2) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_2 \chi_2 + \varepsilon_7 \chi_7
\]

Now, from Lemma 3.2 \( \chi_3, \chi_8 \in S_{\theta_3} \), and with the same logic as before \( |S_{\theta_1} \cap S_{\theta_2} \cap S_{\theta_3}| = 2 \) for \( i = 1, 2 \). This implies that necessarily \( |S_{\theta_1} \cap S_{\theta_2} \cap S_{\theta_3}| = 2 \), so

\[
\mathcal{X}_Q^0(\theta_3) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_3 \chi_3 + \varepsilon_8 \chi_8
\]
We can then define the extension of $\Sigma^0_Q$ explicitly, as

$$\overline{\Sigma}_Q(\rho_i) = \varepsilon_i \chi_i$$

- If $Q = C_2$, then $C_L(Q)/Q = ((C_2)^3 \times (C_7 \times C_3)) \times C_2$, so $\dim(C^0_L(C_L(Q)/Q)) = 11$. The irreducible characters of $C_L(Q)/Q$ are $\rho_i\sigma_j$, where $\rho_i$ is a character of $(C_2)^3 \times (C_7 \times C_3)$ labeled as in the previous case and $\sigma_j$ is an irreducible character of $C_2$. In particular a basis of $L^0_K(C_L(Q)/Q)$ is given by $\{\theta_i\sigma_j\}_{i=1,...,3} \cup \{\rho_i(\sigma_1 - \sigma_2)\}_{i=1,...,8}$. Note that $(\theta_i\sigma_1, \theta_j\sigma_1) = 2 + 2\delta_{ij}$. We can apply the same argument as before to the image under $\overline{\Sigma}_Q^0$ of $\{\theta_i\sigma_j\}$ for each fixed $j$ to obtain that $|\Sigma_j| = |S_{\theta_i\sigma_j} \cup S_{\theta_j\sigma_j} \cup S_{\theta_i\sigma_j}| = 6$ or 8, as now we do not know if the intersection of these three supports is empty or not. Suppose it is empty, so $|\Sigma_j| = 6$. Then we have

$$\overline{\Sigma}_Q^0(\theta_1\sigma_1) = \varepsilon_1 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_1 \chi_4 + \varepsilon_6 \chi_6$$
$$\overline{\Sigma}_Q^0(\theta_2\sigma_1) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_2 \chi_2 + \varepsilon_7 \chi_7$$
$$\overline{\Sigma}_Q^0(\theta_3\sigma_1) = \varepsilon_1 \chi_4 + \varepsilon_6 \chi_6 + \varepsilon_2 \chi_2 + \varepsilon_7 \chi_7$$

Since $(\rho_i(\sigma_1 - \sigma_2), \rho_i(\sigma_1 - \sigma_2)) = 2$, and $(\rho_i(\sigma_1 - \sigma_2), \theta_i\sigma_1) = 1$ for any $i = 1, 2, 3$, then $S_{\rho_i(\sigma_1 - \sigma_2)}$ intersects each $S_{\theta_i\sigma_1}$ in one element. However, $|S_{\rho_i(\sigma_1 - \sigma_2)}| = 2$, which is a contradiction (there is no choice of the pair of characters that makes this possible, as each pair appears in one of the $S_{\theta_i\sigma_1}$). Therefore, the intersection of the three supports $S_{\theta_1\sigma_1}$, $S_{\theta_2\sigma_1}$, $S_{\theta_3\sigma_1}$ is not empty, so we can label the characters as

$$\overline{\Sigma}_Q^0(\theta_1\sigma_1) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_1 \chi_1 + \varepsilon_6 \chi_6$$
$$\overline{\Sigma}_Q^0(\theta_2\sigma_1) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_2 \chi_2 + \varepsilon_7 \chi_7$$
$$\overline{\Sigma}_Q^0(\theta_3\sigma_1) = \varepsilon_4 \chi_4 + \varepsilon_5 \chi_5 + \varepsilon_3 \chi_3 + \varepsilon_8 \chi_8$$

Then since $(\rho_i(\sigma_1 - \sigma_2), \theta_i\sigma_1) = 1$ for any $i = 1, 2, 3$, in particular considering $i = 4, 5$ it is clear that we can also label characters such that

$$\overline{\Sigma}_Q^0(\rho_4(\sigma_1 - \sigma_2)) = \varepsilon_4 \chi_4 - \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2$$
$$\overline{\Sigma}_Q^0(\rho_5(\sigma_1 - \sigma_2)) = \varepsilon_5 \chi_5 - \varepsilon_1 \chi_1$$

and, therefore, using that $(\rho_i(\sigma_1 - \sigma_2), \rho_j(\sigma_1 - \sigma_2)) = 2\delta_{ij}$,

$$\overline{\Sigma}_Q^0(\rho_i(\sigma_1 - \sigma_2)) = \varepsilon_i \chi_i - \varepsilon_i \chi_i$$

We can then define the extension of $\overline{\Sigma}_Q^0$ explicitly as

$$\overline{\Sigma}_Q(\rho_i\sigma_j) = \varepsilon_{i+(8j-8)}\chi_{i+(8j-8)}$$

- If $Q = 1$, then $C_L(Q)/Q = L$, and $\dim(C^0_L(L)) = 27$. The characters of $L$ are of the form $\rho_i\sigma_j$, where $\rho_i$ is a character of $(C_2)^3 \times (C_7 \times C_3)$ and $\sigma_j$ is a character of $C_2 \times C_2$. In particular, keeping the notation for the $\rho_i$ characters, a basis of $L^0_K(L)$ is given by
\[\{\theta_i\}_{i=1,\ldots,3} \cup \{\rho_i(\sigma_j - \sigma_{j+1})\}_{i=1,\ldots,8, j=1,2,3}\]. With the same argument as in the previous case, we can say that there is a labeling of \(\text{Irr}(B)\) as \(\{\chi_1, \ldots, \chi_{32}\}\) such that

\[
\Delta_0'\left(\theta_1\sigma_1\right) = \varepsilon_1\chi_4 + \varepsilon_5\chi_5 + \varepsilon_1\chi_1 + \varepsilon_6\chi_6
\]
\[
\Delta_0'\left(\theta_2\sigma_1\right) = \varepsilon_1\chi_4 + \varepsilon_5\chi_5 + \varepsilon_2\chi_2 + \varepsilon_7\chi_7
\]
\[
\Delta_0'\left(\theta_3\sigma_1\right) = \varepsilon_1\chi_4 + \varepsilon_5\chi_5 + \varepsilon_3\chi_3 + \varepsilon_8\chi_8
\]
\[
\Delta_0'\left(\rho_i(\sigma_1 - \sigma_2)\right) = \varepsilon_1\chi_1 - \varepsilon_{i+8}\chi_{i+8}
\]

Now we want to determine \(\Delta_0'\left(\rho_i(\sigma_2 - \sigma_3)\right)\) and \(\Delta_0'\left(\rho_i(\sigma_3 - \sigma_4)\right)\). We can compute

\[
\theta_1\sigma_2 = (\rho_4 + \rho_5 + \rho_1 - \rho_6)\sigma_2
\]
\[
= (\rho_4 + \rho_5 + \rho_1 - \rho_6)(\sigma_1 - \sigma_2)
\]
\[
= \theta_1\sigma_1 - \rho_4(\sigma_1 - \sigma_2) - \rho_5(\sigma_1 - \sigma_2) - \rho_1(\sigma_1 - \sigma_2) + \rho_6(\sigma_1 - \sigma_2)
\]
Hence

\[
\Delta_0'\left(\theta_1\sigma_2\right) = \varepsilon_4\chi_4 + \varepsilon_5\chi_5 + \varepsilon_1\chi_1 + \varepsilon_6\chi_6 - \varepsilon_4\chi_4 + \varepsilon_{12}\chi_{12} - \varepsilon_5\chi_5 + \varepsilon_{13}\chi_{13} - \varepsilon_1\chi_1 + \varepsilon_9\chi_9 - \varepsilon_6\chi_6 + \varepsilon_{14}\chi_{14}
\]
\[
= \varepsilon_{12}\chi_{12} + \varepsilon_{13}\chi_{13} + \varepsilon_9\chi_9 + \varepsilon_{14}\chi_{14}
\]
and similarly we can obtain

\[
\Delta_0'\left(\theta_2\sigma_2\right) = \varepsilon_{12}\chi_{12} + \varepsilon_{13}\chi_{13} + \varepsilon_{10}\chi_{10} + \varepsilon_{15}\chi_{15}
\]
\[
\Delta_0'\left(\theta_3\sigma_2\right) = \varepsilon_{12}\chi_{12} + \varepsilon_{13}\chi_{13} + \varepsilon_{11}\chi_{11} + \varepsilon_{16}\chi_{16}
\]

Now we consider \(\rho_i(\sigma_2 - \sigma_3)\). Again we can say, with the same argument used in the previous case to determine the image of \(\rho_i(\sigma_1 - \sigma_2)\), that we can label the irreducible characters in \(\text{Irr}(B)\) such that

\[
\Delta_0'\left(\rho_4(\sigma_2 - \sigma_3)\right) = \varepsilon_{12}\chi_{12} - \varepsilon_20\chi_{20}
\]
\[
\Delta_0'\left(\rho_5(\sigma_2 - \sigma_3)\right) = \varepsilon_{13}\chi_{13} - \varepsilon_{21}\chi_{21}
\]
which are the only possibilities since \((\rho_i(\sigma_1 - \sigma_2), \rho_j(\sigma_2 - \sigma_3)) = -\delta_{ij}\). Now, again using that \((\rho_i(\sigma_2 - \sigma_3), \rho_j(\sigma_2 - \sigma_3)) = 2\delta_{ij}\), we can label characters such that

\[
\Delta_0'\left(\rho_i(\sigma_2 - \sigma_3)\right) = \varepsilon_{i+8}\chi_{i+8} - \varepsilon_{i+16}\chi_{i+16}
\]

We can now compute

\[
\Delta_0'\left(\theta_1\sigma_3\right) = \varepsilon_20\chi_{20} + \varepsilon_{21}\chi_{21} + \varepsilon_{17}\chi_{17} + \varepsilon_{22}\chi_{22}
\]
\[
\Delta_0'\left(\theta_2\sigma_3\right) = \varepsilon_20\chi_{20} + \varepsilon_{21}\chi_{21} + \varepsilon_{18}\chi_{18} + \varepsilon_{23}\chi_{23}
\]
\[
\Delta_0'\left(\theta_3\sigma_3\right) = \varepsilon_20\chi_{20} + \varepsilon_{21}\chi_{21} + \varepsilon_{19}\chi_{19} + \varepsilon_{24}\chi_{24}
\]

Finally, we consider \(\rho_i(\sigma_3 - \sigma_4)\). We can repeat the argument above, together with the fact that \((\rho_i(\sigma_2 - \sigma_3), \rho_j(\sigma_3 - \sigma_4)) = -\delta_{ij}\) and \((\rho_i(\sigma_1 - \sigma_2), \rho_j(\sigma_3 - \sigma_4)) = 0\) to obtain that

\[
\Delta_0'\left(\rho_i(\sigma_3 - \sigma_4)\right) = \varepsilon_{i+16}\chi_{i+16} - \varepsilon_{i+24}\chi_{i+24}
\]

We can then define the extension of \(\Delta_0'\) explicitly as

\[
\Delta_0'(\rho_i\sigma_j) = \varepsilon_{i+(8j-8)}\chi_{i+(8j-8)}
\]
(c₃) Suppose that $C_E(Q) = C_3$. Then $Q \cap P = C_2$, and $Q \in \{(C_2)^3, (C_2)^2, C_2\}$. Hence, since $C_7 < E$ does not fix $Q \cap P$, $N_E(Q) = C_E(Q)$ again and so the $N_E(Q)$-stability of any extension of $\widehat{\Delta}^0_Q$ is automatic.

- If $Q = (C_2)^3$, then $C_L(Q)/Q = A_4$, so $\dim(L^0_K(C_L(Q)/Q)) = 1$. Let $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ be the irreducible characters of $A_4$, where $\theta_4$ is the non-linear one. Let $\bar{\theta} = \theta_1 + \theta_2 + \theta_3 - \theta_4$. Then $L^0_K(C_L(Q)/Q) = \mathbb{Z}\bar{\theta}$, and $(\bar{\theta}, \bar{\theta}) = 4$. Note that $\widehat{\Delta}^0_Q(\theta) \neq \pm 2\chi_i$ since $\pm 2\chi_i(1) \neq 0$. Then we can label $\text{Irr}(\bar{b}_Q) = \chi_1, \ldots, \chi_8$ such that

$$\widehat{\Delta}^0_Q(\theta_1 + \theta_2 + \theta_3 - \theta_4) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4$$

and then it is immediate that $\widehat{\Delta}^0_Q(\theta_i) = \varepsilon_i \chi_i$ extends the isometry.

- If $Q = (C_2)^2$, then $C_L(Q)/Q = A_4 \times C_2$, so $\dim(L^0_K(C_L(Q)/Q)) = 5$. The characters of $C_L(Q)/Q$ are $\theta_i \sigma_j$, where $\theta_i$ is a character of $A_4$ (where $\theta_4$ is the nonlinear character and we let $\theta = \theta_1 + \theta_2 + \theta_3 - \theta_4$) and $\sigma_j$ is a character of $C_2$. Then a basis of $L^0_K(C_L(Q)/Q)$ is given by $\{\theta \sigma_i\}_{i=1,2} \cup \{\theta_i(\sigma_1 - \sigma_2)\}_{i=1,\ldots,3}$. We can label $\text{Irr}(\bar{b}_Q) = \chi_1, \ldots, \chi_8$ such that

$$\widehat{\Delta}^0_Q(\bar{\theta} \sigma_1) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4$$

Since $(\theta \sigma_i, \theta_j(\sigma_1 - \sigma_2)) = (-1)^{i+j}$ for $i, j = 1, 2, 3$, then we can label characters to get

$$\widehat{\Delta}^0_Q(\theta_1(\sigma_1 - \sigma_2)) = \varepsilon_1 \chi_1 - \varepsilon_5 \chi_5$$

Since $(\theta_1(\sigma_1 - \sigma_2), \theta_2(\sigma_1 - \sigma_2)) = 0$ and $(\bar{\theta} \sigma_1, \bar{\theta} \sigma_2(\sigma_1 - \sigma_2)) = 1$, then we can label characters such that $\widehat{\Delta}^0_Q(\theta_2(\sigma_1 - \sigma_2)) = \varepsilon_2 \chi_2 - \varepsilon_6 \chi_6$. With the same logic, we also obtain that we can choose the labels such that

$$\widehat{\Delta}^0_Q(\bar{\theta}_3(\sigma_1 - \sigma_2)) = \varepsilon_3 \chi_3 - \varepsilon_7 \chi_7$$

We can now compute

$$\widehat{\Delta}^0_Q(\bar{\theta} \sigma_2) = \widehat{\Delta}^0_Q\left(\bar{\theta} \sigma_1 - \sum_{i=1}^{3} \theta_i(\sigma_1 - \sigma_2)\right) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8$$

We can then define the extension of $\widehat{\Delta}^0_Q$ explicitly as

$$\widehat{\Delta}^0_Q(\theta_i \sigma_j) = \varepsilon_{i+(4j-4)} \chi_{i+(4j-4)}$$

- If $Q = C_2$, then $C_L(Q)/Q = A_4 \times (C_2)^2$, so $\dim(L^0_K(C_L(Q)/Q)) = 13$. The characters of $C_L(Q)/Q$ are $\theta_i \sigma_j$, where $\theta_i$ is a character of $A_4$ with the same notations as in the previous case and $\sigma_j$ is a character of $C_2 \times C_2$. Then a basis of $L^0_K(C_L(Q)/Q)$ is given by $\{\theta \sigma_i\}_{i=1,\ldots,4} \cup \{\theta_i(\sigma_1 - \sigma_2)\}_{i=1,2,3,j=1,2,3}$. Repeating the argument of the previous case, we can label the characters of $\text{Irr}(\bar{b}_Q)$ such that

$$\widehat{\Delta}^0_Q(\bar{\theta} \sigma_1) = \varepsilon_1 \chi_1 + \varepsilon_2 \chi_2 + \varepsilon_3 \chi_3 + \varepsilon_4 \chi_4$$

$$\widehat{\Delta}^0_Q(\bar{\theta} \sigma_2) = \varepsilon_5 \chi_5 + \varepsilon_6 \chi_6 + \varepsilon_7 \chi_7 + \varepsilon_8 \chi_8$$

$$\widehat{\Delta}^0_Q(\theta_i(\sigma_1 - \sigma_2)) = \varepsilon_i \chi_i - \varepsilon_{i+4} \chi_{i+4} \quad \text{(for } i = 1, 2, 3)$$

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Now we can repeat the argument by considering \( \hat{\theta}\sigma_2 \) and the characters \( \theta_i(\sigma_2 - \sigma_3) \), to get that

\[
\overline{\Sigma}_Q^0(\theta_i(\sigma_2 - \sigma_3)) = \varepsilon_{i+4}\chi_{i+4} - \varepsilon_{k_i}\chi_{k_i}
\]

for certain indices \( k_i \). We claim that \( k_i \notin \{1, \ldots, 8\} \) (so that these characters do not appear in the support of \( \hat{\theta}\sigma_1 \), since we already know that \( k_i \) is not 5, 6, 7 or 8). Consider

\[
\overline{\Sigma}_Q^0(\theta_i(\sigma_1 - \sigma_3)) = \overline{\Sigma}_Q^0(\theta_i(\sigma_1 - \sigma_2) + \theta_i(\sigma_2 - \sigma_3)) = \varepsilon_i\chi_i - \varepsilon_{k_i}\chi_{k_i}
\]

Since \( (\hat{\theta}\sigma_1, \theta_i(\sigma_1 - \sigma_3)) = 1 \) and \( i = 1, 2, 3 \), our claim is proved. Hence, we can label characters such that

\[
\overline{\Sigma}_Q^0(\hat{\theta}\sigma_3) = \varepsilon_9\chi_9 + \varepsilon_{10}\chi_{10} + \varepsilon_{11}\chi_{11} + \varepsilon_{12}\chi_{12}
\]

Finally, we can repeat the same argument, this time considering \( \hat{\theta}\sigma_3 \) and the characters \( \theta_i(\sigma_3 - \sigma_4) \), and again with a similar argument as above (or just using Lemma \ref{lem:extension}) we obtain

\[
\overline{\Sigma}_Q^0(\hat{\theta}\sigma_3) = \varepsilon_{13}\chi_{13} + \varepsilon_{14}\chi_{14} + \varepsilon_{15}\chi_{15} + \varepsilon_{16}\chi_{16}
\]

We can then define the extension of \( \overline{\Sigma}_Q^0 \) explicitly as

\[
\overline{\Sigma}_Q(\theta_i\sigma_j) = \varepsilon_{i+(4j-4)}\chi_{i+(4j-4)}
\]

We have shown the existence of a \((G, B)\)-local system on the set of all subgroups of \( D \). \( \square \)

The existence of a \((G, B)\)-local system can be combined with the following Lemma to obtain Morita equivalences between certain types of blocks.

**Lemma 3.5** (\cite{55}). Let \( N \) be a normal subgroup of \( G \) whose index is a power of \( p \). Let \( b \) be a block of \( ON \) that is covered by a block \( B \) of \( OG \) and also \( G \)-stable. Suppose that \( D \), a defect group of \( B \), decomposes as \( D = Q \times (D \cap N) \), and let \( C = CG(Q) \). Let \( b_D \) be a root of \( B \) in \( OC_G(D) \), and let \( b_Q = b_D^{CG(Q)} \). If there exists a perfect isometry \( I \) between \( L_K(C, b_Q) \) and \( L_K(G, B) \), with the following property

\[
I(\lambda \ast \chi') = \lambda \ast I(\chi') \quad (\forall \lambda \in \text{Irr}(R), \forall \chi \in \text{Irr}(B'))
\]

then \( B \cong OQ \otimes_O \Omega b \) (as \( O \)-algebras).

Combining our results with the main theorem of \cite{56} about blocks with a cyclic inertial quotient, we obtain the following extensions of the main theorem of \cite{26} to blocks over \( O \).

**Proposition 3.6.** Let \( G \) be a finite group and \( B \) be a block of \( OG \) with defect group \( D \cong (C_2)^5 \) and inertial quotient which is either \( C_3, C_5, C_7, C_15, C_3 \times C_3 \) or \( C_7 \times C_3 \) of type 1 (as defined in \cite{212}). Suppose that there is \( N \triangleleft G \) with \( [G : N] = 2^k \) and that \( B \) covers a \( G \)-stable block \( b \) of \( ON \). Let \( Q \leq D \) with \( G = NQ \). Then \( B \) is Morita equivalent to the block \( b \otimes OQ \) of \( O(N \times Q) \).

**Proof.** Whenever \( E \) is cyclic of order 3, 5, 7 or 15, Proposition \ref{lem:extension} implies that \( l(B) = |E| \). Then there is a \((G, B)\)-local system on the set \( X \) of all subgroups of \( D \), as detailed in the proof of the main theorem of \cite{56}. When \( E = C_3 \times C_3 \) or \( (C_7 \times C_3)_1 \), then there is a \((G, B)\)
In particular in each case we have a perfect isometry

$$\Delta_1 : \mathcal{L}_K(C_L(Q)) \rightarrow \mathcal{L}_K(G,B)$$

such that $\Delta_1(\lambda \ast \eta) = \lambda \ast \Delta_1(\eta)$ for any $\lambda \in \text{Irr}(C_D(E))$, $\eta \in \mathcal{L}_K(C_L(Q))$. The block $e_{N_G(D,e)}$ of $O_N(G(D,e))$ is Morita equivalent to the block $b'$ of $OL'$, since the cocycle that defines the twist of the twisted group algebra determined by $N_G(D,e)e$ (in the sense specified in the main theorem of [28]) is precisely the one that determines the central extension of $L$ by $k^*$ (see also [51, 2.5]). We can compose $\Delta_1$ with this Morita equivalence to get a perfect isometry

$$I : \mathcal{L}_K(N_G(D,e), e_{N_G(D,e)}) \rightarrow \mathcal{L}_K(G,B)$$

such that $I(\lambda \ast \eta) = \lambda \ast I(\eta)$ for any $\lambda \in \text{Irr}(C_D(E))$, $\eta \in \mathcal{L}_K(N_G(D,e), e_{N_G(D,e)})$. We can now apply the same argument to $C_G(Q), e_{C_G(Q)}$ since this block also has defect group $D$ and inertial quotient $E$. Note that $N_G(D,e) \subseteq C_G(Q)$, hence $N_{C_G(Q)}(D,e) = N_G(D,e)$, which means that we get a perfect isometry

$$J : \mathcal{L}_K(N_G(D,e), e_{N_G(D,e)}) \rightarrow \mathcal{L}_K(C_G(Q), e_{C_G(Q)})$$

such that $J(\lambda \ast \eta) = \lambda \ast J(\eta)$ for any $\lambda \in \text{Irr}(C_D(E))$, $\eta \in \mathcal{L}_K(N_G(D,e), e_{N_G(D,e)})$. Now $K = I \circ J^{-1}$ is a perfect isometry such that $K(\lambda \ast \eta) = \lambda \ast K(\eta)$ for any $\lambda \in \text{Irr}(C_D(E))$, $\eta \in \mathcal{L}_K(G,B)$, so from Lemma 4.1 we are done.

## 4 Crossed products and Picard groups

We recall the key concepts from [29]. Given a finite group $G$ and a ring with identity $A$, $A$ is a $G$-graded ring if there is a decomposition $A = \bigoplus_{g \in G} A_g$ as additive subgroups such that $A_gA_h \subseteq A_{gh}$, and $A_1$ is a subring of $A$ containing 1.

A $G$-graded ring $A$ is said to be a crossed product of $A_1$ with $G$ if for any $g \in G$, $A_g$ contains at least one unit. We call two $G$-graded rings $A$ and $B$ weakly equivalent if there is an isomorphism of rings $\phi : A \rightarrow B$ such that $\phi(A_g) \subseteq B_g$ for all $g \in G$. Moreover, we say they are equivalent if $\phi$ restricts to the identity map on $A_1 \simeq B_1$.

A key result from Külshammer’s paper is a characterization of all the possible crossed products of a given ring $R$ and a group $G$:

**Theorem 4.1.** The equivalence classes of crossed products of a ring $R$ with a group $G$ are parametrized by pairs $(\omega, \zeta)$, where $\omega : G \rightarrow \text{Out}(R)$ is a homomorphism whose corresponding 3-cocycle in $H^3(G, U(Z(R)))$ is zero, and $\zeta \in H^2(G, U(Z(R)))$ where the action of $G$ on $U(Z(R))$ is induced by $\omega$.

Moreover, weak equivalence classes of crossed products correspond to orbits of $\text{Aut}(R)$ on the set of possible $(\omega, \zeta)$.

We are interested in classifying blocks up to Morita equivalence: given a Morita equivalence class we consider a canonical representative of it, the basic algebra. In fact, it is well known that two Morita equivalent algebras have isomorphic basic algebras, and that moreover any algebra is Morita equivalent to its basic algebra. This is compatible with a crossed product structure, as the following lemma shows.
Lemma 4.2. Let $G$ be a finite group, $N < G$ with $[G : N]$ a prime $\ell \neq p$. Let $X = G/N$. Let $B$ a block of $OG$ that covers a $G$-stable block of $ON$, and let $f$ be a basic idempotent of $b$, i.e. an idempotent such that $fbf$ is a basic algebra of $b$. Then

1. $fBf$ is a crossed product of $fbf$ with $X$.

2. $fBf$ is Morita equivalent to $B$.

Proof. The group algebra $OG$ is a crossed product of $ON$ and $X = G/N$. Since $b$ is $G$-stable, it is the unique block covered by $B$, so the unique block of $ON$ such that $Bb \neq 0$. Hence, $B = OGB$ is also a crossed product of $b = ONb$ with $X$.

The first claim now follows from Proposition 4.15 in [18], noting that for two basic idempotents $e$ and $f$ of $A$ the property $eA \simeq fA$ holds since all the basic idempotents are in the same orbit under conjugation by units of $A$.

Since $b$ is $G$-stable and covered by $B$, $Bb = B$. To prove the second claim, recall that for an algebra $A$ and an idempotent $f$, $A$ and $fAf$ are Morita equivalent if and only if $A \simeq A$ [50] 9.9. Since $b$ and $fbf$ are Morita equivalent, so $bfb = b$, we can write

$$BfB = BbfB = BbB = B$$

which proves the claim. 

Since computing $\text{Out}(fbf)$ is, in general, hard, we study a specific subgroup of the Picard group of $b$ instead. We give the relevant definitions:

Recall that two algebras $A$ and $B$ are Morita equivalent if and only if there is an $A$-$B$-bimodule $M$ and a $B$-$A$-bimodule $N$ such that $M \otimes _B N \simeq A$ and $N \otimes _A M \simeq B$. The Picard group of a block $b$, denoted by $\text{Pic}(b)$, is the group of $b$-$b$-bimodules that induce a self-Morita equivalence of $b$, where the group operation is given by the tensor product. We are also interested in the subgroup $\mathcal{T}(b) \leq \text{Pic}(b)$, where $\mathcal{T}(b)$ is defined as the subgroup of all the bimodules in $\text{Pic}(b)$ with trivial source. The following result, extracted from the main theorem of [4], gives an upper bound for the size of the subgroup $\mathcal{T}(b)$.

Lemma 4.3 ([4]). Let $G$ be a finite group, and let $b$ be a block of $OG$ with abelian defect group $D$ and inertial quotient $E$. Let $\mathcal{F}$ be the fusion system on $D$ determined by $b$. Let $\text{Pic}(b)$ and $\mathcal{T}(b)$ be as defined above. Then there is an exact sequence

$$1 \to \text{Out}_D(A) \to \mathcal{T}(b) \to \text{Out}(D, \mathcal{F}) \to 1$$

where $A = iOGi$ is a source algebra of $b$ and $\text{Out}(D, \mathcal{F})$ is defined as $\text{Aut}(D, \mathcal{F})/\text{Aut}_D(\mathcal{F})$. Moreover, $\text{Out}_D(A)$ is isomorphic to a subgroup of $\text{Hom}(E, k^\times)$.

Following [18] 5.1, we say that $b$ is source algebra equivalent to $c$, if they have a common defect group $D$ and there are source idempotents $i$ of $b$ and $j$ of $c$ such that $iOGi$ is isomorphic to $jOHj$ as interior $D$-algebras. A source algebra equivalence implies a Morita equivalence and it preserves fusion systems, vertices and sources. In particular, $\mathcal{T}(b) \simeq \mathcal{T}(c)$. Moreover, any source algebra equivalence over $k$ implies one over $O$ by (90 7.8).

In the proof of our result we use the following Picard groups, computed in [13].

Proposition 4.4 ([18]). For a block $b$ of a finite group $G$ with defect group $D$, let $\mathcal{T}(b)$ be the subgroup of $\text{Pic}(b)$ of bimodules with trivial source. Then
Moreover, let $Q$ be a finite abelian 2-group. Then

(a) $\text{Pic}(O(C_2^3 \rtimes C_7)) = T(\text{Pic}(C_7^3 \rtimes C_7)) = C_7 \rtimes C_3$.

(b) $\text{Pic}(O(C_2^3 \rtimes (C_7 \rtimes C_3))) = T(\text{Pic}(C_7^3 \rtimes (C_7 \rtimes C_3))) = C_3$.

Moreover, let $Q$ be a finite abelian 2-group. Then

(i) $\text{Pic}(O(A_4 \times Q)) = S_3 \times (Q \rtimes \text{Aut}(Q))$.

(ii) $\text{Pic}(B_0(O(A_5 \times Q))) = C_2 \times (Q \rtimes \text{Aut}(Q))$.

Proof. Case (a) and (b) follow from Lemma 5.2 and Theorem 4.6 in [15] respectively. Case (i) and (ii) are also immediate from Theorem 4.6 in [15].

Method 4.5. We detail our method and the context in which it applies: let $G$ be a finite group and $B$ be a block of $OG$ with defect group $D \cong (C_2)^5$. Suppose that there is $N \triangleleft G$ with $[G : N]$ odd (so $G/N$ is solvable) and that $B$ covers a $G$-stable block $b$ of $ON$. Moreover, suppose that $C_G(N) \leq N$, and that $N = \ker(G \to \text{Out}(b))$ where the map is given by $G$ acting by conjugation on $b$. Note that, since $[G : N]$ is odd, $B$ and $b$ share a defect group.

Let $f$ be a basic idempotent for $b$. From Lemma 4.2, we know that $B$ is Morita equivalent to $fBf$, which is a crossed product of $fbf$ with $G/N$. Let $\omega : G/N \to \text{Out}(fbf)$ be the homomorphism that corresponds to the crossed product weak equivalence class of $fBf$, obtained as the composition of

$$G/N \xrightarrow{\alpha} \text{Out}_*(N) \xrightarrow{\beta} \text{Out}(b) \xrightarrow{\gamma} \text{Pic}(b) = \text{Pic}(fbf) = \text{Out}(fbf)$$

where we define the elements as follows:

(*) We define $\text{Out}_*(N)$ as the subgroup $\{\Phi \in \text{Out}(N) : \forall \phi \in \Phi, \phi(b) = b\}$, where we extend $\phi$ linearly to $ON$. Note that since by definition the block idempotent of $b$ is central in $ON$, each inner automorphism of $N$ fixes $b$, so the action of any automorphism in the coset $\Phi$ does not depend on the choice of the representative. Moreover, note that since we assume that $b$ is $G$-stable we do not need to consider any automorphism of $N$ that does not fix $b$.

(α) For a coset $x \in G/N$, choose a representative $g \in x$. Define $\tau_g \in \text{Aut}(N), n \mapsto s_n = gng^{-1}$, and define $\alpha(x)$ to be its coset in $\text{Out}(N)$. If we choose a different representative $h \in x$ then, since $h = m_g$ for some $m \in N$, $h = mgm^{-1}n$, so the coset in $\text{Out}(N)$ does not depend on the choice of $g$ and $\alpha$ is well defined.

Note that since $b$ is $G$-stable in our situation $\alpha(G/N) \leq \text{Out}_*(N)$.

(β) For a coset $\Phi \in \text{Out}_*(N)$, choose a representative $\phi \in \Phi$, and let $\phi \in \text{Aut}(ON)$ be the automorphism obtained extending $\phi$ linearly. Since $\phi(b) = b$, we define $\beta(\phi)$ to be the coset in $\text{Out}(b)$ of the restriction of $\phi$ to $ONb$. Recall that inner automorphisms of $N$ induce inner automorphisms of $b$: in fact, for $n \in N$, we can consider the decomposition of $ON$ into blocks, and hence the element $nb \in ONb$. Then $\beta$ is well defined.

(γ) For $\phi \in \text{Aut}(b)$, we define the $b$-$b$-bimodule $\phi b$ as: $\phi b = b$ as sets, and $x.m.y = \phi(x)my$ for $x, m, y \in b$. From [10] 55.11, since inner automorphisms give isomorphic bimodules, the map $\gamma$ defined as $\gamma(\phi) = \phi b$ gives an embedding of $\text{Out}(b)$ in $\text{Pic}(b)$.
Note that $\alpha$ and $\gamma$ are always injective maps, and since $N = \ker(G \to \text{Out}(b))$ so is $\beta$.

For any $g \in G$ we have an induced action $\tau_g \in \text{Aut}(b)$ given by conjugation. Since $\omega_b$ is a direct summand of the permutation $O\tau_1$-$\tau_3$-bimodule $\tau_9 \tau_1$, it has trivial source, which means that $\tau_g b \in \mathcal{T}(b)$. Since we are interested in the action of $G$ on $b$, a priori we should examine only the possibilities for $\omega$ corresponding to elements in $\mathcal{T}(b)$. However, depending on the case we are examining, $\mathcal{T}(b)$ is not always well-determined, since this subgroup is not invariant under Morita equivalence in general (and in fact not even for nilpotent blocks, as seen in [11, 7.2]). Hence, when given an arbitrary block $b$ Morita equivalent to a block $c$, a priori we can only state that $\mathcal{T}(b) \leq \text{Pic}(c)$.

The solvability of $G/N$ allows us to consider only crossed products with cyclic groups, as follows: we can consider a chain of subnormal subgroups

$$N = N_0 \triangleleft N_1 \triangleleft \cdots \triangleleft N_t = G$$

such that $\ell_i = [N_i+1 : N_i]$ is prime, and a block chain $b_i$ of $O\tau_1$ such that $b_i$ covers $b_{i-1}$, with $b_t = B$. Note that they all share a defect group since $G/N$ is odd.

Here from Lemma 4.2 $b_1$ is Morita equivalent to a crossed product of $b$ with $N_1/N$, a cyclic subgroup of $G/N$, and the weak crossed product equivalence class is specified by a pair $(\omega_1, \zeta_1)$ as in Theorem 4.1. In this situation, as detailed in [14] (using [29] and [33, 1.2.10]) the group $H^2(N_1/N, \mathcal{U}(Z(fb))) = 1$, so weak equivalence classes of crossed products of $fbf$ and $G/N$ are classified by just orbits of possible $\omega_1$ whose induced 3-cocycle vanishes.

Actually, in the following we consider each possibility for $\omega_1$ without checking the additional requirement of the induced 3-cocycle vanishing, and the existence of examples of blocks of finite groups that induce $\omega_1$ in each case will imply, post-hoc, that the induced 3-cocycle indeed vanishes.

In general, $N_1$ is not guaranteed to be normal in $G$, and this technique would not work immediately on the whole block chain, as while $b = b_0$ is $G$-stable we cannot assume $N_{i+1}$-stability of each $b_i$ to apply Lemma 4.2 and repeat the argument. In every case that we consider, however, we will either be able to assume that each $N_i$ is a normal subgroup of $G$. We can then consider $N_2$ and $b_2$, which is a crossed product of $b_1$ and $N_2/N_1$, noting that possibilities for $\omega_2$ are still controlled by $G/N_1$, and iterate the process to get all possible block chains and, hence, all possible Morita equivalence classes for $B$.

In some cases, different crossed products give representatives of the same Morita equivalence class, because they represent the same action of a group. An example is when $G = N_1$ and $\text{Pic}(b) = C_3$, which a priori gives three different crossed product weak equivalence classes for $fBf$ as a crossed product between $C_3$ and $fbf$, each corresponding to the a different homomorphism $\omega : G/N \to C_3$, explicitly $x \mapsto 1$, $x \mapsto y$, $x \mapsto y^2$ if $C_3 = \langle y \rangle$. However, since the action comes from an element of $G/N$, the two nontrivial possibilities for $\omega$ give two blocks of isomorphic group algebras, where the isomorphism comes from the map $G/N \to G/N$, $x \mapsto x^{-1}$, which implies that the blocks lie in the same Morita equivalence class (and are, in fact, isomorphic as algebras). The same can be said, for example, when $\text{Pic}(b) = C_7$ about the six nontrivial possibilities. We will implicitly use this logic of identifying two objects, not in the same weak equivalence classes as crossed products, through group algebra isomorphisms.
whenever we determine the number of possible Morita equivalence classes for \( B \) in all the rest of the paper.

(†) On the other hand, a priori our analysis of crossed products could produce crossed product algebras of \( b \) and \( G/N \) that are not known to occur as blocks of finite groups. However, in each case we were successful in producing examples of groups that realize the predicted crossed product whenever looking at a block chain of length 2. In some longer chains, however, we were unable to find examples of certain crossed products actually occurring as blocks of finite groups. Explicitly, in some situations with the hypothesis in Method 4.5 above we encounter a block chain \( b_0, b_1, b_2 \) where we can identify all Morita equivalence classes of blocks \( b_1 \) that can cover \( b_0 \), and all classes of blocks \( b_2 \) that can cover \( b_1 \), but we are unable to find a chain of groups that realizes some possible block chains. In fact, the structure of a block \( b_i \) in the chain is determined not only by \( b_{i-1} \), but also by all the blocks below it \( b_0, \ldots, b_{i-2} \). We denote this type of situation by saying that the action is “realized as a crossed product”, meaning that certainly the Morita equivalence class of \( b_2 \) will be among the ones specified, but possibly not all of them actually occur. This does not hinder our classification purpose, as all the Morita equivalence classes determined for \( b_2 \) will, in each case, appear in our list independently of whether they occur in the specific chain of normal subgroups considered in that case. In the following we reference this argument as (†).

Now we look at specific examples that realize certain actions represented in the Picard group with actual examples in finite groups. In [44, 4.4] Puig gives restrictions on the types of blocks that can be nilpotent covered, proving that if a block is nilpotent covered then it is inertial and has an abelian inertial quotient. Moreover, Puig shows how to construct a nilpotent covered inertial block for each given abelian defect group and abelian inertial quotient. We use this construction to give explicit examples of some particular blocks.

**Example 4.6.** Let \( D = (C_2)^2 \), \( E = C_3 \). We can consider the extraspecial group \( S = 3^{1+2}_+ \) of order 27, and the semidirect product \( G = D \rtimes S \) where \( S \) acts as \( S/(C_3 \times C_3) = C_3 \), which can also be seen as \( C_3^2 \rtimes (D \rtimes E) \) acting as \( (D \rtimes E)/D = E \). This group has 3 blocks Morita equivalent to \( O(D \rtimes E) \) and 2 nilpotent blocks, all with defect group \( D \) since \( D = O_3(G) \). Its normal subgroup \( N = C_3 \rtimes (D \rtimes E) \) has 3 blocks Morita equivalent to \( O(D \rtimes E) \). Since every block of \( OG \) covers at least one block of \( ON \), then at least one block of \( ON \) is nilpotent covered (in fact, the two nonprincipal blocks are). This corresponds to the crossed product with \( C_3 \) specified by \( C_3 \leq \text{Pic}(A_4) \).

**Example 4.7.** Let \( D = (C_2)^3 \), \( E = C_7 \). As above, take \( S = 7^{1+2}_+ \) and consider \( G = D \rtimes S \). Then \( G \) has 7 blocks Morita equivalent to \( O(D \rtimes E) \) and 6 nilpotent blocks, all with defect group \( D \). Its normal subgroup \( N = C_7 \times ((C_2)^3 \rtimes C_7) \) has 7 blocks Morita equivalent to \( O(D \rtimes E) \), so at least one of them is nilpotent covered (and in fact all the nonprincipal blocks are). This corresponds to the crossed product with \( C_7 \) specified by \( C_7 \leq \text{Pic}(O((C_2)^3 \rtimes C_7))) \). For brevity, we name this pair of groups \( N_7 \triangleleft G_7 \).

An additional example is given by a similar phenomenon that does not involve nilpotent blocks, but in which the inertial quotient becomes smaller.
Example 4.8. Consider $G = (C_2)^3 \times (C_7 \times 3_+^{1+2})$ and $N = (C_2)^3 \times (C_7 \times C_3) \times C_3$. Then $G$ has five blocks, two of which have 7 simple modules. Then they are Morita equivalent to $O((C_2)^3 \times C_7)$, and since every block of $ON$ is Morita equivalent to $O(C_2)^3 \times (C_7 \times C_3))$ we have an example of a block $B$ of $OG$ covering a block $b$ of $ON$ where the inertial quotient “shrinks” from $C_7 \times C_3$ to just $C_7$, which corresponds to the crossed product with $C_3$ specified by $\text{Pic}(O((C_2)^3 \times (C_7 \times C_3))) = C_3$. For brevity, we name this pair of groups $N_{73} < G_{73}$.

Finally, we examine a case in which by taking a central extension we get a new Morita equivalence class.

Example 4.9. Consider $G = (C_2)^5 \times (C_7 \times 3_+^{1+2})$, where the action is given by the quotient $3_+^{1+2}/Z(3_+^{1+2}) = C_3 \times C_3$ and $N$ is a maximal subgroup of index 3. Then $G$ has five blocks, two of which have 7 simple modules. These blocks are not Morita equivalent to $O((C_2)^3 \times C_7)$, as seen by direct inspection of the Cartan matrix. Moreover, the inertial quotient of these blocks is $(C_7 \times C_3) \times C_3$, and they form a different Morita equivalence class.

In order to prove our main result, we need to examine the possible Morita equivalence classes of blocks that cover specific classes of blocks. We do it in this section, using the methods detailed above.

Proposition 4.10. Let $G$ be a finite group and $B$ be a quasiprimitive block of $OG$ with defect group $D \cong (C_2)^5$. Suppose that there is $N \triangleleft G$ with $[G:N]$ odd, and that $B$ covers a block $b$ of $ON$. Moreover, suppose that $C_G(N) \leq N$ and $N = \ker(G \to \text{Out}(N))$.

(i) If $b$ is Morita equivalent to $O(A_4 \times (C_2)^3)$, then $B$ is Morita equivalent to: $b$, $O_D(O(A_4 \times A_4 \times C_2), O(A_4 \times ((C_2)^3 \times C_7)))$, $O(A_4 \times ((C_2)^3 \times (C_7 \times C_3)))$, $O((C_2)^2 \times ((C_2)^3 \times C_7))$, $O((C_2)^2 \times ((C_2)^3 \times (C_7 \times C_3)))$, $O(((C_2)^5 \times (C_7 \times C_3))$, $O(((C_2)^4 \times (C_3)^2) \times C_2)$, a nonprincipal block of $O((C_2)^3 \times (C_7 \times C_3)\times C_2)$, or a nonprincipal block of $O((C_2)^5 \times (C_7 \times C_3)\times C_2)$.

(ii) If $b$ is Morita equivalent to $B_0(O(A_5 \times (C_2)^3))$, then $B$ is Morita equivalent to: $b$, $B_0(O(A_5 \times A_4 \times C_2))$, $B_0(O(A_5 \times ((C_2)^3 \times C_7)))$ or $B_0(O(A_5 \times ((C_2)^3 \times (C_7 \times C_3))))$.

Proof. First, note that since $\text{Pic}(b)$ is known from Proposition 4.4, and $G/N$ has odd order, in both cases when we apply Method 4.4 we only need to consider possibilities for $\omega$ coming from a maximal odd order subgroup of Pic(b).

We use the groups examined in Examples 4.8 and 4.9 to show that certain crossed product algebras actually occur as blocks of finite groups.

(i) Any block Morita equivalent to $b$ has a Picard group isomorphic to $S_3 \times ((C_2)^3 \times \text{GL}_3(2))$, which contains $C_3 \times (C_7 \times C_3)$ as a maximal subgroup with odd order. Let $\sigma : G \to S_3$ be the homomorphism given by the action of $G$ permuting the three simple modules of $b$. Note that if $G \neq \ker(\sigma)$ then $[G : \ker(\sigma)] = 3$.

Consider as in 4.5 a chain of normal subgroups $\{N_i\}$ of length $t$ where $N_{i-1} = \ker(\sigma)$, and the corresponding block chain $\{b_i\}$. Note that $t \leq 3$ since $G/N$ is isomorphic to a subgroup of Pic(b).

Consider the block $b_1$, which is Morita equivalent to a crossed product of the basic algebra of $b$ with $X_1 = N_1/N$ as detailed before, and let $\omega_1$ be the corresponding
homomorphism. We have four nontrivial possibilities for \((X_1, \omega_1)\) up to equivalence, which give the following Morita equivalence classes for \(b_1\):

(a) \(|X_1| = 7\), and \(b_1\) is Morita equivalent to \(O(A_4 \times ((C_2)^3 \times C_7))\), realized when \(N = A_4 \times (C_2)^3\).

(b) \(|X_1| = 3\), and \(b_1\) is Morita equivalent to \(O(A_4 \times A_4 \times C_2)\) realized again when \(N = A_4 \times (C_2)^3\).

(c) \(|X_1| = 3\), and \(b_1\) is Morita equivalent to \(O(D)\), realized when \(N = PSL_3(7) \times (C_2)^3\) and \(G = PGL_3(7) \times (C_2)^3\).

(d) \(|X_1| = 3\), and \(b_1\) is Morita equivalent to a nonprincipal block of \(O((C_2^4 \times 3_+^{1+2}) \times C_2)\) where the center of \(3_+^{1+2}\) acts trivially and \(N\) is a maximal subgroup of \(G\) with index 3.

Note that in cases (c) and (d) the simple modules of \(b\) are permuted by the action of \(N_1/N\), which means that \(G = N_1\) and \(B = b_1\).

For the other two cases, we consider \(b_2\) and \(N_2/N_1\) and the corresponding \(\omega_2\):

(a) Note that \(C_7 \triangleleft G/N\), so \(N_1 \triangleleft G\). Then \(G/N_1 = (G/N)/(N_1/N) \leq C_7 \times C_3\). The block \(b_2\) is Morita equivalent to a crossed product of the basic algebra of \(b_1\) with \(X_2 = N_2/N_1\), and possibilities for the corresponding \(\omega_2\) are given by conjugacy classes of embeddings of \(X_2\) into \(G/N_1\): therefore \(X_2 = C_3\) and we have three nontrivial possibilities for the Morita equivalence class of \(b_2\):

1. \(b_2\) is Morita equivalent to \(O(A_4 \times ((C_2)^3 \times (C_7 \times C_3)))\), realized when \(N_1 = A_4 \times ((C_2)^3 \times C_7)\).

2. \(b_2\) is Morita equivalent to \(O((C_2)^2 \times ((C_2)^3 \times C_7)))\), realized when \(N_1 = PSL_3(7) \times ((C_2)^3 \times C_7)\).

3. \(b_2\) is Morita equivalent to a nonprincipal block of \(O((C_2)^5 \times (C_7 \times 3_+^{1+2}))\), realized when \(N_1 = N_{734}\).

Note that in cases (2) and (3) the simple modules of \(b\) are permuted transitively by the action of \(N_2/N\), so \(G = N_2\) and \(B = b_2\).

In case (1) note that \(N_2 \triangleleft G\) since \(C_7 \times C_3 \triangleleft G/N\). Consider \(b_3\) and \(N_3\): \(G/N_2 \leq C_3\), and \(b_3\) is Morita equivalent to a crossed product of \(b_2\) with \(X_3 = N_3/N_2\). There are up to two possibilities (see (i)) for the corresponding \(\omega_3\), which gives that \(b_3\) is Morita equivalent to \(O((C_2)^2 \times ((C_2)^3 \times (C_7 \times C_3)))\), realized when \(N_2 = PSL_3(7) \times ((C_2)^3 \times C_7)\), and \(O(((C_2)^3 \times (C_7 \times C_3)_2))\), realized as a crossed product when \(N_2 = N_{73} \times PSL_3(7)\). Now \(G = N_3\) so \(B = b_3\).

(b) Note that, by inspection of all possible chains \(1 < \cdots < G/N\) with cyclic indices, in this case \(G/N \leq C_3 \times C_3\). So, in particular, \(G/N_1 \leq C_3\). It follows that \(G = N_2\), \(b_2\) is Morita equivalent to a crossed product of \(N_2/N_1\) and the basic algebra of \(b_1\) and there are up to two possibilities (see (i)) for the Morita equivalence class of \(b_2\) as \(O((C_2)^3 \times A_4)\), realized when \(N_1 = PSL_3(7) \times C_2 \times A_4\), and \(O(((C_2)^4 \times C_3)_2 \times C_2)\), realized as a crossed product when \(N_1 = PSL_3(7)^2 \times C_2\).

(ii) Any block Morita equivalent to \(b\) has a Picard group isomorphic to \((C_2)^3 \times GL_3(2))\), which contains \(C_7 \times C_3\) as a maximal subgroup with odd order. As in (i), we consider
Proposition 4.11. Let $G$ be a finite group and $B$ be a quasiprimitive block of $\mathcal{O}G$ with defect group $D \cong (C_2)^5$. Suppose that there is $N \triangleleft G$ with $[G : N]$ odd, suppose that $G/N$ is supersolvable and that $B$ covers a $G$-stable block $b$ of $\mathcal{O}N$. Suppose that $C_G(N) \leq N$ and $N = \ker(G \to \text{Out}(b))$. Then if $b$ is Morita equivalent to $B_0(\mathcal{O}(A_5 \times A_5 \times C_2))$ then $B$ is source algebra equivalent to $b$.

Proof. Suppose that $B$ is not Morita equivalent to $b$, and consider as above the action of $G/N$ on $b$ by conjugation. Since $G/N$ is supersolvable, we can consider a chain of normal subgroups $N < N_1 < \cdots < N_t < G$, with prime indices, and a corresponding block chain $b, b_1, \ldots, b_t, B$ where each block covers the ones below it. Note that they all share a defect group.

Let $[N_1 : N] = \ell$, an odd prime. From Corollary 2.14, either $b_1$ is the unique block covering $b$ or $b_1$ is source algebra equivalent to $b$. The block $b$ has $l(b) = 9$, and from the decomposition matrix (preserved under Morita equivalence) we know that, if we consider the character of each projective cover of the simple modules, there is one with 32 irreducible constituents, four with 16 irreducible constituents and four with 8 irreducible constituents, and any automorphism of the block preserves the number of irreducible constituents. Hence, if $b_1$ is the unique block covering $b$ and $\ell \geq 5$ then $N_1/N$ fixes every simple module, which implies that $l(b_1) = 9\ell$. This is a contradiction to Proposition 2.12 since $l(c) \leq k(c) \leq 32$ for any block $c$ with defect group $(C_2)^5$.
If \( \ell = 3 \), suppose that \( b_1 \) is the unique block covering \( b \). Then from Lemma 2.9, \( c(b_1) = 1, 3, 9 \) or 27. First, note that \( c(b_1) = 27 \) is a contradiction to Proposition 2.12. Now either every character of each projective cover of the simple modules is fixed, so \( l(b_1) = 27 \) (a contradiction to Proposition 2.12), or there is one orbit of length 3 and six fixed characters, which gives \( l(b_1) = 1 + 3 \cdot 6 = 19 \) (again a contradiction), or there are two orbits of length 3 and 3 fixed characters, which gives \( l(b_1) = 1 + 1 + 3 \cdot 3 = 11 \) (again a contradiction). Since we know there is at least one fixed character of the projective cover of a simple module, there cannot be three orbits of length 3.

Therefore, the only possibility is that \( b_1 \) is not the unique block that covers \( b \), and hence from Corollary 2.9 that \( b_1 \) is source algebra equivalent to \( b \). Note that, since \( B \) is quasiprimitive and \( N_1 \triangleleft G \), \( b_1 \) is \( G \)-stable. We can now repeat the argument for any other intermediate block \( b_1 \) (replacing \( b_1 = b_{i+1}, b = b_i \)) and compose the equivalences to obtain that \( B \) is source algebra equivalent to \( b \).

**Proposition 4.12.** Let \( G \) be a finite group and \( B \) be a quasiprimitive block of \( OG \) with defect group \( D = (C_2)^5 \). Suppose that there are \( H_1, H_2 \triangleleft G \), \( H = H_1 \times H_2 \) with \( H \triangleleft G \) and \( [G : H] \) odd, and suppose that \( B \) covers \( G \)-stable blocks \( c_i \) of \( OH_1 \), so \( B \) also covers the block \( c = c_1 \otimes c_2 \) of \( OH \). Suppose that \( CG(H) \leq H \) and \( H = \ker(G \to \text{Out}(c)) \). Then either \( B \) is Morita equivalent to \( c \) or one of the following occurs:

1. If \( c_1 \) is Morita equivalent to \( O(A_4 \times Q_1) \), and \( c_2 \) is Morita equivalent to \( O(A_4 \times Q_2) \) where \( Q_1, Q_2 \in \{1, C_2\} \), then \( B \) is Morita equivalent to \( O(A_4 \times (C_2)^3) \), \( O((C_2)^4 \times (C_5)_2 \times C_2) \), \( OD \) or a nonprincipal block of \( O(((C_2)^4 \times 3_{1+2}^{1+2}) \times C_2) \).

2. If \( c_1 \) is Morita equivalent to \( O(A_4 \times Q_1) \) and \( c_2 \) is Morita equivalent to \( B_0(O(A_5 \times Q_2)) \) where \( Q_1, Q_2 \in \{1, C_2\} \), then \( B \) is Morita equivalent to \( b \) or \( B_0(O(A_5 \times (C_2)^3)) \).

3. If \( c_1 \) is Morita equivalent to \( B_0(O(A_5 \times Q_1)) \), and \( c_2 \) is Morita equivalent to \( B_0(O(A_5 \times Q_2)) \) where \( Q_1, Q_2 \in \{1, C_2\} \), then \( B \) is Morita equivalent to \( b \).

4. If \( c_1 \) is Morita equivalent to \( O((C_2)^3 \times C_7) \) and \( c_2 \) is Morita equivalent to \( O(A_4) \), then \( B \) is Morita equivalent to one of the blocks specified in case (i) of Proposition 4.11.

5. If \( c_1 \) is Morita equivalent to \( O((C_2)^3 \times C_7) \) and \( c_2 \) is Morita equivalent to \( B_0(O(A_5)) \), then \( B \) is Morita equivalent to \( B_0(O((C_2)^3 \times A_5)) \), \( B_0(O((C_2)^3 \times (C_7 \times C_3)) \times A_5) \) or \( B_0(O(A_4 \times C_2 \times A_5)) \).

6. If \( c_1 \) is Morita equivalent to \( O((C_2)^3 \times (C_7 \times C_3)) \) and \( c_2 \) is Morita equivalent to \( O(A_4) \), then \( B \) is Morita equivalent to \( O(((C_2)^3 \times C_7) \times A_4) \), \( O(((C_2)^3 \times (C_7 \times C_3)) \times (C_2)^2) \) or a nonprincipal block of \( O((C_2)^5 \times (C_7 \times 3_{1+2}^{1+2})) \).

7. If \( c_1 \) is Morita equivalent to \( O((C_2)^3 \times (C_7 \times C_3)) \) and \( c_2 \) is Morita equivalent to \( B_0(O(A_5)) \), then \( B \) is Morita equivalent to \( B_0(O((C_2)^3 \times C_7) \times A_5)) \).

**Proof.** We use the method detailed before in 4.8 picking a chain of normal subgroups \( \{N_i\} \) of length \( t \) with \( N_0 = N = H \), \( N_t = G \) and the corresponding block chain \( \{b_i\} \), with \( b_0 = c_1 \otimes c_2 \) and \( b_t = B \). Since the action of \( G \) restricts to both \( H_1 \) and \( H_2 \), we only need to consider the subgroup \( T(c_1) \times T(c_2) \) of \( T(c) \), which is contained in Pic(c_1) \times Pic(c_2), as the action of
any other element of Pic(c) cannot come from an action of G. In each case, we denote the maximal subgroup of odd order of this subgroup of Pic(c) as T.

1. In this case Pic(c_1) = S_3 \times Q_4, so T = C_3 \times C_3. As in the previous proposition, let \( \sigma : G \rightarrow S_3 \) be the homomorphism given by the action of G permuting the three simple modules of b_1, and note that if \( G \neq \ker(\sigma) \) then \( |G : \ker(\sigma)| = 3 \). Consider as in 4.5 a chain of normal subgroups \( \{N_i\} \) of length \( t \) with \( N_{t-1} = \ker(\sigma) \), and the corresponding block chain \( \{b_i\} \). Note that \( t \leq 2 \).

Then \( N_1/N = C_3 \) and we have the following possibilities for the Morita equivalence class of \( b_1 \):

(a) \( b_1 \) is Morita equivalent to \( O(A_4 \times (C_2)^3) \) realized when \( H = PSL_3(7)^2 \times C_2 \), and \( N_1 \leq \ker(\sigma) \).

(b) \( b_1 \) is Morita equivalent to \( O(A_4 \times (C_2)^3) \) realized when \( H = PSL_3(7)^2 \times C_2 \), and \( N_1 = G \).

(c) \( b_1 \) is Morita equivalent to \( O(((C_2)^3 \times (C_3)^2) \times C_2) \), realized when \( N = C = PSL_3(7)^2 \times C_2 \) and \( G = (PSL_3(7)^2 \times C_3) \times C_2 \).

Note that in case (b) and (c) the simple modules of c_1 are permuted transitively, so \( G = N_1 \) and \( B = b_1 \). In case (a) \( N_1 < G \), so we can consider \( b_2 \) and \( G/N_1 = C_3 \). There are up to two possibilities (in the sense specified in §) for the Morita equivalence class of \( B : OD \), realized when \( N_1 = PSL_3(7) \times PGL_3(7) \times C_2 \), and a nonprincipal block of \( O(((C_2)^3 \times 3_+^2) \times C_2) \), realized as a crossed product when \( N_1 = A_4 \times (C_2)^3 \).

2. In this case Pic(c_1) = S_3 \times Q_4 and Pic(c_2) = C_2 \times Q_2, so \( T = C_3 \). Then \( t = 1 \), and there is a unique possibility for the Morita equivalence class of \( b_1 = B \), \( B_0/O(A_4 \times (C_2)^3)) \), realized when \( N = A_5 \times PSL_3(7) \).

3. This case is implied by the stronger result in Proposition 4.11. Note that our technique also gives the result, noting that Pic(c_1) = Pic(c_2) = C_2 \times Q_4, a 2-group. So \( T = 1 \), and hence \( t = 0 \) and \( G = H \).

In the following we use the groups examined in Examples 4.7, 4.8 and 4.9 named for brevity \( (G_7, N_7) \), \( (G_{73}, N_{73}) \) and \( (G_{734}, N_{734}) \).

4. In this case Pic(c_1) = C_7 \times C_3 and Pic(c_2) = S_3. Hence \( T = (C_7 \times C_3) \times C_3 \). As before, let \( \sigma : G \rightarrow S_3 \) be the homomorphism given by the action of G permuting the three simple modules of c_2, and note that if \( G \neq \ker(\sigma) \) then \( |G : \ker(\sigma)| = 3 \). Consider as in 4.5 a chain of normal subgroups \( \{N_i\} \) of length \( t \) where \( N_{t-1} = \ker(\sigma) \), and the corresponding block chain \( \{b_i\} \). Note that \( t \leq 3 \).

We have the following possibilities for \( X_1 = N_1/N \) and the Morita equivalence class of \( b_1 \):

(a) \( |X_1| = 7 \), and \( b_1 \) is Morita equivalent to \( O((C_2)^3 \times A_4) \), realized when \( N = N_7 \times A_4 \).

(b) \( |X_1| = 3 \), and \( b_1 \) is Morita equivalent to \( O(((C_2)^3 \times (C_7 \times C_3) \times A_4) \) realized when \( N = ((C_2)^3 \times C_7) \times A_4 \).

(c) \( |X_1| = 3 \), and \( b_1 \) is Morita equivalent to \( O(((C_2)^3 \times C_7) \times (C_2)^2) \), realized when \( N = ((C_2)^3 \times C_7) \times PSL_3(7) \).
6. In this case Pic$(c_1) = C_7 \times C_3$ and Pic$(c_2) = C_2$, so $T = C_7 \times C_3$. We have the following possibilities for $X_1 = N_1/N$ and the Morita equivalence class of $b_1$:

(a) $|X_1| = 7$, and $b_1$ is Morita equivalent to $B_0(\mathcal{O}((C_2)^3 \times A_5))$, which is realized when $N = N_7 \times A_5$.

(b) $|X_1| = 3$, and $b_1$ is Morita equivalent to $B_0(\mathcal{O}((C_2)^3 \times (C_7 \times C_3)) \times A_5)$, realized when $N = ((C_2)^3 \times C_7) \times A_5$.

In case (b) inspection of all possible chains implies $N_1 = G$. In case (a) $N_1 < G$, so we can apply the relevant cases from the proof of Proposition 4.10 to prove our result. Not all cases are actually known to occur as block chains, in the sense specified in (1).

5. In this case Pic$(c_1) = C_7 \times C_3$ and Pic$(c_2) = C_2$, so $T = C_7 \times C_3$. There are up to two possibilities for the Morita equivalence class of $X_1 = N_1/N = C_3$ and the corresponding block chain $b_1$.

(a) $|X_1| = 7$, and $b_1$ is Morita equivalent to $B_0(\mathcal{O}((C_2)^3 \times A_4))$, realized when $N = N_7 \times A_4$.

(b) $|X_1| = 3$, and $b_1$ is Morita equivalent to $B_0(\mathcal{O}((C_2)^3 \times (C_7 \times C_3)) \times (C_2)^2)$ realized when $N = ((C_2)^3 \times (C_7 \times C_3)) \times \text{PSL}_3(7)$.

(c) $b_1$ is Morita equivalent to $\mathcal{O}((C_2)^5 \times (C_7 \times C_3))$, realized when $N = N_{73} \times \text{PSL}_3(7)$.

In cases (b) and (c) the simple modules of $c_2$ are permuted by the action of $X_1$, so $G = N_1$ and $B = b_1$. In case (a) $N_1 < G$ so we consider $b_2$ and $X_2 = N_2/N_1 = C_3$. Then $N_2 = G$ and $b_2 = B$.

There are up to two possibilities for the Morita equivalence class of $B$, in the sense specified in (1): $\mathcal{O}((C_2)^3 \times C_7) \times (C_2)^2)$, realized when $N_1 = ((C_2)^3 \times C_7) \times \text{PSL}_3(7)$, and a nonprincipal block of $\mathcal{O}((C_2)^5 \times (C_7 \times 3_+^2))$, realized as a crossed product when $N = ((C_2)^3 \times C_7) \times \text{PSL}_3(7)$.
7. In this case \( \text{Pic}(c_1) = C_3 \) and \( \text{Pic}(c_2) = C_2 \), so the maximal subgroup of odd order that we need to consider is \( C_3 \). Then \( G = N_1 \), and \( b_1 = B \) is Morita equivalent to \( B_0(\mathcal{O}(((C_2)^3 \times C_7) \times A_5)) \), realized when \( N = N_{73} \times A_5 \).

\[ \square \]

The next lemma deals with situations in which \( b \) is again a block of the direct product of two normal subgroups, but this time one of the groups is fixed up to isomorphism. First, we construct two additional examples, in a way the analogues of Examples 4.8 and 4.9.

**Example 4.13.** If we consider \( G = \text{SL}_2(8) \times S \) where \( S = 3_1^{+2} \) and \( G \) is a central extension of \( (\text{SL}_2(8) \times C_3) \times C_3 \), and its normal subgroup \( N = \text{SL}_2(8) \times C_3 \), then \( G \) has sixteen blocks, eleven of which have defect zero, and five of which have defect group \( (C_2)^3 \). By direct inspection of the Cartan matrices, using the classification in [13], three of the latter are Morita equivalent to \( B_0(\mathcal{O}(\text{Aut}((\text{SL}_2(8)))) \) and the other two are Morita equivalent to \( B_0(\mathcal{O}(\text{SL}_2(8)))) \). All the blocks on \( \mathcal{O}N \) are Morita equivalent to \( B_0(\mathcal{O}(\text{Aut}((\text{SL}_2(8)))) \), so this gives an example of a block Morita equivalent to \( B_0(\mathcal{O}(\text{SL}_2(8)))) \) that covers a block Morita equivalent to \( B_0(\mathcal{O}(\text{Aut}((\text{SL}_2(8)))) \). This corresponds to the crossed product of \( B_0(\mathcal{O}(\text{Aut}((\text{SL}_2(8)))) \) with \( C_3 \) specified by \( \text{Pic}(B_0(\mathcal{O}(\text{Aut}((\text{SL}_2(8)))) \) = C_3 \). For brevity, we name this pair of groups \( N_{73s} \) and \( G_{73s} \).

**Example 4.14.** If we consider \( G = (\text{SL}_2(8) \times (C_2)^2) \times 3_1^{+2} \) and \( G \) is a central extension of \( (\text{SL}_2(8) \times C_3) \times A_4 \), and \( N \) a maximal normal subgroup of index 3, then \( G \) has eight blocks: five have defect group \( (C_2)^2 \), and three have defect group \( (C_2)^3 \). Among the latter, the two nonprincipal blocks have 7 simple modules. These blocks are not Morita equivalent to \( \mathcal{O}((C_2)^3 \times C_7) \) or \( \mathcal{O} \text{SL}_2(8) \), as seen by direct inspection of the Cartan matrix. Moreover, the inertial quotient of these blocks is \( (C_7 \times C_3) \times C_5 \). They form a different Morita equivalence class. For brevity, we name this pair of groups \( N_{73s} \) and \( G_{73s} \).

**Proposition 4.15.** Let \( G \) be a finite group and \( B \) be a quasiprimitive block of \( \mathcal{O}G \) with defect group \( D = (C_2)^3 \). Suppose that there are \( H_1, H_2 \triangleleft G \), \( H = H_1 \times H_2 \) with \( H \triangleleft G \) and \([G : H]\) odd, and suppose that \( B \) covers \( G \)-stable blocks \( c_i \) of \( \mathcal{O}H_i \), so \( B \) also covers the block \( c = c_1 \otimes c_2 \) of \( \mathcal{O} \). Suppose that \( C_G(H) \leq H \) and \( H = \ker(G \to \text{Out}(c)) \).

Suppose that \( H_1 \) is isomorphic to \( \text{SL}_2(8) \) or \( 2\text{G}_2(3^{2m+1}) \) for some \( m \in \mathbb{N} \), and \( c_1 \) is the principal block, or that \( H_1 \) is isomorphic to \( \text{Co}_3 \) and \( c_1 \) is the unique nonprincipal block with defect group \( (C_2)^3 \).

1. If \( H_1 = \text{SL}_2(8) \) and \( c_2 \) is nilpotent then \( B \) is Morita equivalent to \( \text{b} \), or to the principal block of \( \text{SL}_2(8) \times A_4 \), \( \text{Aut}(\text{SL}_2(8)) \times (C_2)^2 \), \( \text{SL}_2(8) \times (C_2)^2 \times C_3 \) or \( \text{Aut}(\text{SL}_2(8)) \times A_4 \) or a nonprincipal block of \( (\text{SL}_2(8) \times (C_2)^2) \times 3_1^{+2} \).

2. If \( H_1 = \text{SL}_2(8) \) and \( c_2 \) is Morita equivalent to \( \mathcal{O}A_4 \) then \( B \) is Morita equivalent to \( \text{b} \), or to the principal block of \( \text{SL}_2(8) \times (C_2)^2 \), \( \text{Aut}(\text{SL}_2(8)) \times A_4 \), \( \text{Aut}(\text{SL}_2(8)) \times (C_2)^2 \) or a nonprincipal block of \( (\text{SL}_2(8) \times (C_2)^2) \times 3_1^{+2} \).

3. If \( H_1 = \text{SL}_2(8) \) and \( c_2 \) is Morita equivalent to \( B_0(\mathcal{O}A_3) \) then \( B \) is Morita equivalent to \( \text{b} \), or to the principal block of \( \text{Aut}(\text{SL}_2(8)) \times A_5 \).
Finally, we reference the groups in Example 4.13 as $N$ actual blocks of finite groups.

4. If $H_1 = 2G_2(3^{2m+1})$ or $H_1 = Co_3$ and $c_2$ is nilpotent, then $B$ is Morita equivalent to $b$ or to the principal block of $\text{Aut}(\text{SL}_2(8)) \times A_4$.

5. If $H_1 = 2G_2(3^{2m+1})$ or $H_1 = Co_3$, and $c_2$ is Morita equivalent to $O_A_4$, then $B$ is Morita equivalent to $b$ or to the principal block of $\text{Aut}(\text{SL}_2(8)) \times (C_2)^2$.

6. If $H_1 = 2G_2(3^{2m+1})$ or $H_1 = Co_3$, and $c_2$ is Morita equivalent to $B_0(\mathcal{O}_A_5)$, then $B$ is Morita equivalent to $b$.

7. If $H_1 = J_1$ and $c_2$ is nilpotent then $B$ is Morita equivalent to $b$ or to the principal block of $L_1 \times A_4$.

8. If $H_1 = J_1$ and $c_2$ is Morita equivalent to $O_A_4$ then $B$ is Morita equivalent to $b$ or to the principal block of $L_1 \times (C_2)^2$.

9. If $H_1 = J_1$ and $c_2$ is Morita equivalent to $B_0(\mathcal{O}_A_5)$ then $B$ is Morita equivalent to $b$.

Proof. We use the same method as in Proposition 4.12, and knowledge of the outer automorphism groups of the various possibilities for $L_1$.

We want to point out that, although not strictly necessary to prove our result, some Picard groups are known from [15]: from Proposition 5.3 and 5.4 in [15], $\text{Pic}(B_0(\mathcal{O}\text{SL}_2(8))) = C_3$, and $\text{Pic}(B_0(\mathcal{O}\text{Aut}(\text{SL}_2(8)))) = \text{Pic}(B_0(\mathcal{O}(2G_2(q)))) = C_3$.

Recall, from [27, 1.5] and [39, 3.3], that the unique nonprincipal block of $\text{Co}_3$ that has defect group $(C_2)^3$ and the principal blocks of $\text{Aut}(\text{SL}_2(8))$ and $2G_2(q)$ are Morita equivalent.

In each case we consider a chain of normal subgroups $\{N_t\}$ of length $t$ with $N_0 = N = H$ and $N_t = G$, and the corresponding block chain $\{b_t\}$. Knowledge of the Picard groups in each case gives a bound to possibilities for $T(c_1)$ and, hence, to nontrivial crossed products. Again as in Proposition 4.12, since the action by $G$ restricts to both $H_1$ and $H_2$, we only need to consider the subgroup $\gamma(\beta(\text{Out}(H_1))) \times T(c_2)$ (see 4.5), which is contained in $\gamma(\beta(\text{Out}(H_1))) \times \text{Pic}(c_2)$, as the action of any other element of $\text{Pic}(c)$ cannot come from an action of $G$. In each case, we denote the maximal subgroup of odd order of this subgroup of $\text{Pic}(c)$ as $T$.

By direct inspection of the decomposition matrix of the principal block of $\mathcal{O}\text{SL}_2(8)$, the block $c_1$ has $l(c_1) = 7$, and if we consider the characters of the projective covers of the simple modules there is one with 7 irreducible constituents, three with 4 irreducible constituents and three with 2 irreducible constituents, and any automorphism of the block preserves the number of irreducible constituents. Hence, there is a homomorphism $\sigma_1 : G \rightarrow S_3$ given by the permutation of the three simple module whose projective covers have 2 constituents. Note that the generator of $\text{Pic}(c_1) = C_3$ does permute these three simple modules.

Finally, we reference the groups in Example 4.13 as $N_{733} < G_{733}$. These groups give examples of certain classes of crossed products as actual blocks of finite groups.

1. If $H_1 = \text{SL}_2(8)$ and $c_2$ is Morita equivalent to $\mathcal{O}(C_2)^2$ then $T = C_3 \times C_3$. Consider as in 4.12 a chain of normal subgroups $\{N_t\}$ of length $t$ where $N_{t-1} = \ker(\sigma_1)$, and the corresponding block chain $\{b_t\}$. Note that $t \leq 2$. Then $X_1 = N_1/N = C_3$, and the Morita equivalence class of $b_1$ is among the following possibilities:

(a) $b_1$ is Morita equivalent to $B_0(\mathcal{O}(\text{SL}_2(8) \times A_4))$, realized when $N = \text{SL}_2(8) \times (C_2)^2$. 

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(b) \( b_1 \) is Morita equivalent to \( B_0(O(\text{Aut}(SL_2(8)) \times (C_2)^2)) \), realized when \( N = SL_2(8) \times (C_2)^2 \).
(c) \( b_1 \) is Morita equivalent to \( B_0(O((SL_2(8) \times (C_2)^2) \times C_3)) \), realized when \( N = SL_2(8) \times (C_2)^2 \).

In cases (b) and (c) the simple modules of \( c_1 \) whose projective covers have 2 irreducible constituents are permuted by the action of \( N_1 \), so \( G = N_1 \) and \( B = b_1 \). In case (a) \( N_1 < G \), so we consider \( b_2 \) as a crossed product of \( b_1 \) with \( G/N_1 = C_3 \). There are up to two possibilities (in the sense specified in (ii)) for the Morita equivalence class of \( B: B_0(O(\text{Aut}(SL_2(8)) \times A_4))) \), realized when \( N = SL_2(8) \times (C_2)^2 \), and a nonprincipal block of \( (SL_2(8) \times (C_2)^2) \times 3^{1+2} \), realized as a crossed product when \( N_1 = N_{733s} \).

2. If \( H_1 = SL_2(8) \) and \( c_2 \) is Morita equivalent to \( O(A_4) \) then \( T = C_3 \times C_3 \). Let \( \sigma_2 : G \to S_3 \) be the homomorphism given by the action of \( G \) permuting the three simple modules of \( c_2 \), and note that if \( G \neq \ker(\sigma_2) \) then \( |G : \ker(\sigma_2)| = 3 \). Consider as in (i) a chain of normal subgroups \( \{N_i\} \) of length \( t \) where \( N_{t-1} = \ker(\sigma_2) \), and the corresponding block chain \( \{b_i\} \). Note that \( t \leq 2 \).

Then \( X_1 = N_1/N = C_3 \), and the Morita equivalence class of \( b_1 \) is among the following possibilities:

(a) \( b_1 \) is Morita equivalent to \( B_0(O(\text{Aut}(SL_2(8)) \times A_4))) \), realized when \( N = SL_2(8) \times A_4 \).
(b) \( b_1 \) is Morita equivalent to \( B_0(O(SL_2(8) \times (C_2)^2))) \), realized when \( N = SL_2(8) \times PSL_3(7) \).
(c) \( b_1 \) is Morita equivalent to a nonprincipal block of \( O(SL_2(8) \times (C_2)^2) \times 3^{1+2} \) where the center acts trivially, realized when \( N = N_{733s} \).

In cases (b) and (c) the simple modules of \( c_2 \) are permuted by the action of \( N_1 \), so \( G = N_1 \) and \( B = b_1 \). In case (a) \( N_1 < G \), so we consider \( b_2 \) as a crossed product of \( b_1 \) with \( N_2/N_1 = C_3 \). There are up to two possibilities (in the sense specified in (i)) for the Morita equivalence class of \( b_2 = B: B_0(O(\text{Aut}(SL_2(8)) \times (C_2)^2))) \), realized when \( N = SL_2(8) \times PSL_3(7) \), and \( B_0(O((SL_2(8) \times (C_2)^2) \times C_3)) \), realized as a crossed product when \( N_1 = N_{73} \times PSL_3(7) \).

3. If \( H_1 = SL_2(8) \) and \( c_2 \) is Morita equivalent to \( B_0(O(A_5)) \) then \( T = C_3 \). Then \( t = 1 \), so \( N_1 = G \) and there is only one nontrivial possibility for \( X_1 = G/N \), that \( B \) is Morita equivalent to \( B_0(O(\text{Aut}(SL_2(8)) \times A_5))) \).

From (i) \( \text{Out}(2G_2(3^{2m+1})) = C_{2m+1} \), and every degree of irreducible characters of the principal block \( b_1 \) occurs with multiplicity 1 or 2, which implies that every automorphism of \( b_1 \) is inner. Hence when \( H_1 = 2G_2(3^{2m+1}) \), with the notation in (i) \( \beta(\text{Out}(H_1)) \) is contained in \( \ker(G \to \text{Out}(c)) \), so in our situation we \( \beta(\alpha(\text{Out}(H_1))) = 1 \) (that is, \( G \) acts trivially on \( H_1 \)). Moreover, \( \text{Out}(C_{2m}) = 1 \). So we can limit our analysis to the subgroup \( T(c_2) \) in the last three cases.

4. If \( H_1 = 2G_2(q) \), for any \( q = 3^{2m+1}, m \in N \) or \( H_1 = C_{2m+1} \), and \( B \) is nilpotent, then \( T = C_3 \). Then \( N_1 = G \) and \( B = b_1 \), and \( B \) is Morita equivalent to \( B_0(O(\text{Aut}(SL_2(8)) \times A_4))) \), realized when \( N = 2G_2(q) \times (C_2)^2 \).
5. If $H_1 = 2G_2(q)$, for any $q = 3^{2m+1}, m \in \mathbb{N}$ or $H_1 = \text{Co}_3$, and $c_2$ is Morita equivalent to $\mathcal{O}A_4$, then $T = C_3$. Then $N_1 = G$ and $B = b_1$, and the Morita equivalence class of $B$ is $B_0(\mathcal{O}(\text{Aut}(SL_2(8)) \times (C_2)^2))$, realized when $N = 2G_2(q) \times \text{PSL}_3(7)$.

6. If $H_1 = 2G_3(q)$, for any $q = 3^{2m+1}, m \in \mathbb{N}$ or $H_1 = \text{Co}_3$, and $c_2$ is Morita equivalent to $B_0(\mathcal{O}A_5)$, then $T = 1$, so $t = 0$ and $G = H$.

The Picard group of the principal block of $J_1$ is not known, but since $\text{Out}(J_1) = 1$ and $N_1$ is isomorphic to $J_1$ we can use this fact instead.

7. If $H_1 = J_1$ and $c_2$ is Morita equivalent to $\mathcal{O}(C_2)^2$ then $\text{Out}(J_1) = 1$ and $\text{Pic}(\mathcal{O}(C_2)^2) = (C_2)^2 \rtimes S_3$, so $T = C_3$. Then $G = N_1, B = b_1$ and there is only one nontrivial possibility for the Morita equivalence class of $B$, $B_0(\mathcal{O}(J_1 \times A_4))$, realized when $N = J_1 \times (C_2)^2$.

8. If $H_1 = J_1$ and $c_2$ is Morita equivalent to $\mathcal{O}A_4$ then $\text{Out}(J_1) = 1$ and $\text{Pic}(\mathcal{O}A_4) = S_3$, so $T = C_3$. Then $G = N_1, B = b_1$ and there is only one nontrivial possibility for the Morita equivalence class of $B$, $B_0(\mathcal{O}(J_1 \times (C_2)^2))$, realized when $N = J_1 \times \text{PSL}_3(7)$.

9. If $H_1 = J_1$ and $c_2$ is Morita equivalent to $B_0(\mathcal{O}A_5)$ then $\text{Out}(J_1) = 1$ and $\text{Pic}(c_2) = C_2$, so $T = \{1\}$, $t = 0$ and $G = N$, so we are done.

In Method 4.5, we assume that $N = \ker(G \to \text{Out}(b))$, and in fact we can always reduce to this situation. Suppose we are in the situation of 4.5, but without the hypothesis $N = \ker(G \to \text{Out}(b))$. Recall the definition of $G[b]_\mathcal{O}$ in Section 2: we define $G[b]_\mathcal{O}$ as the group of elements acting as inner automorphisms on the block $b$ of $\mathcal{O}N$. Then $G[b]_\mathcal{O} < G[b]$ via the canonical map $b \to b \otimes_\mathcal{O} k$. We identify $G[b]_\mathcal{O}$ with $\ker(G \to \text{Out}(b))$.

From Proposition 2.3 there is a unique block $\hat{b}$ of $G[b]_\mathcal{O}$ that is source algebra equivalent to $b$. So in general we can consider $G[b], \hat{b}$ instead of $N, b$ and apply Method 4.5 (since $\text{Pic}(\hat{b}) = \text{Pic}(b)$) to obtain all possible Morita equivalence classes for $B$. However, in Proposition 4.12 and 4.15 we have used the group structure to reduce to particular subgroups of $\text{Pic}(b)$: to generalize these arguments we need to show that when the kernel $G[b]_\mathcal{O}$ is nontrivial we can still reduce to particular subgroups of $\text{Pic}(b)$.

**Corollary 4.16.** Let $G$ be a finite group and $B$ be a quasiprimitive block of $\mathcal{O}G$ with defect group $D = (C_2)^5$. Suppose that are $N_1, N_2 < G$, $N = N_1 \times N_2$ with $N < G$ and $[G : N]$ of odd order, and suppose that $B$ covers $G$-stable blocks $c_i$ of $\mathcal{O}N_i$, so $B$ also covers the block $c = c_1 \otimes c_2$ of $\mathcal{O}N$. Suppose that $C_G(N) \subseteq N$. Then, for each fixed pair of Morita equivalence classes of $c_1, c_2$ listed in cases 1-7 of Proposition 4.12 and cases 1-9 of Proposition 4.15, the Morita equivalence class of $B$ is still among the ones listed in that same case in Proposition 4.12 or Proposition 4.15.

**Proof.** We use the notation of Method 4.5. Let $G[b]_\mathcal{O} = \ker(G \to \text{Out}(N))$. Since each $N_i$ is a normal subgroup of $G$, $\alpha(G/N)$ is contained in $\text{Out}_*(N_1) \times \text{Out}_*(N_2)$. We can consider the map $\beta_i : \text{Out}_*(N_i) \to \text{Out}(b_i)$ defined in the same way as $\beta$ in 4.5 and the map $\beta_1 \beta_2$, obtained by extending each $\beta_i$ to $\text{Out}_*(N_1) \times \text{Out}_*(N_2)$ such that $\beta_i = \text{id}_{N_j}$ when $i \neq j$. Since $b = b_1 \otimes b_2$, it is immediate that $\beta = \beta_1 \beta_2$. In particular then $\beta(G/N)$ is a subgroup of $\text{Pic}(b_1) \times \text{Pic}(b_2)$, through maps $\gamma_i$ again defined as in 4.5.
If we define \( \hat{b} \) to be the unique block of \( G[b^*]\) covered by \( B \) and covering \( b^* \) then from Proposition 2.8, \( \hat{b} \) is source algebra equivalent to \( b \).

Hence, we can apply Proposition 4.12 or Proposition 4.15 (as appropriate), replacing \( N \) and \( b \) with \( G[b^*]\) and \( \hat{b} \), and obtain the same possibilities for the Morita equivalence class of \( B \) since, from the above discussion, \( \gamma(\beta(\alpha(G/G[b^*])))) \leq \text{Pic}(b_1) \times \text{Pic}(b_2) \) and we can replicate the proofs.

Now we want to show that when \( N \) is a perfect group and a central extension of a group \( \overline{N} \) by \( C_2 \) we can use the Picard group of a block dominated by \( b \).

**Method 4.17.** Let \( G \) be a finite group and \( B \) be a quasiprimitive block of \( OG \) with defect group \( D = (C_2)^5 \). Let \( N \triangleleft G \) and let \( b \) be the unique block of \( ON \) covered by \( B \). Suppose that \( N \) is a perfect group, \( C_G(N) \leq N \), and that there is \( Z = C_2 \leq Z(G) \cap N \). Let \( \overline{G} = G/Z \), \( \overline{N} = N/Z \), and let \( \overline{b} \) be the unique block of \( O\overline{N} \) dominated by \( b \). Let \( \mu_Z : OG \to O\overline{G} \): from Theorem 8.11 in [37], in particular, \( \mu_Z(b) = \overline{b} \).

Since \( N \) is perfect, \( \text{Out}(N) = \text{Out}(\overline{N}) \) (see the proof of [36], 7.6). We can define the maps in the same way as in 4.15 and we have the following diagram where \( \omega_Z \) is the map that makes the diagram commutative.

\[
\begin{array}{c}
G/N \xrightarrow{\alpha} \text{Out}(N) \xrightarrow{\beta} \text{Out}(b) \xrightarrow{\gamma} \text{Pic}(b) \\
\downarrow \mu_Z \downarrow \omega_Z \\
G/\overline{N} \xrightarrow{\overline{\alpha}} \text{Out}(\overline{N}) \xrightarrow{\overline{\beta}} \text{Out}(\overline{b}) \xrightarrow{\overline{\gamma}} \text{Pic}(\overline{b})
\end{array}
\]

and note that \( \ker(\overline{\beta}) = G[b^*]\) \( \overline{N}/N \). We claim that \( \omega_Z(\ker(\beta)) \leq \ker(\overline{\beta}) \).

In fact, let \( x \in G/N \) be a coset, and let \( \alpha \in \ker \). If \( \alpha(x) \in \ker \), there exists \( y \in ONb \) such that \( yz = yz \) for any \( z \in ONb \). Then \( \mu_Z(y) = \mu_Z(z) \), and since \( \mu_Z(b) = \overline{b} \) then \( \mu_Z(y) = \omega_Z = \omega_Z = \omega_Z \). Then \( \mu_Z(y) \) induces an inner automorphism of \( O\overline{N} \). Note that a different choice of the representative \( g' = mg \in x \) would give \( y' = mb \in ONb \), so the choice of the representative does not matter. Then \( \alpha(\mu_Z(x)) = \ker(\overline{\beta}) \) and our claim is proved.

Since \( Z \leq N \), a cardinality argument then shows that \( \mu_Z \) induces an isomorphism between the kernels, which can be identified as \( G[b]\) \( \overline{N}/N \simeq G[\overline{b}]\) \( \overline{N}/\overline{N} \). Then in particular the images of the maps \( \beta \) and \( \overline{\beta} \) are isomorphic, so \( \gamma(\beta(\alpha(G/N))) \) is isomorphic to a subgroup of \( \text{Pic}(\overline{b}) \).

We can then apply Method 4.15 to \( G[b], \overline{b} \) but considering all subgroups with odd order of \( \text{Pic}(\overline{b}) \) and obtain all possible Morita equivalence classes for \( B \).

**Corollary 4.18.**

1. The nonprincipal blocks of \( O((C_2)^4 \times 3_+^{1+2}) \times C_2 \) and \( O((C_2)^4 \times 3_+^{1+2}) \times C_2 \) are all Morita equivalent.

2. The nonprincipal blocks of \( O((C_2)^5 \times (C_7 \times 3_+^{1+2})) \) and \( O((C_2)^5 \times (C_7 \times 3_+^{1+2})) \) with 7 simple modules are all Morita equivalent.

3. The nonprincipal blocks of \( O((SL_2(8) \times (C_2)^2) \times 3_+^{1+2}) \) and \( O((SL_2(8) \times (C_2)^2) \times 3_+^{1+2}) \) with 7 simple modules are all Morita equivalent.

**Proof.** This is immediate by considering in each case a maximal subgroup \( N \triangleleft G \) of index 3 and a block \( b \) covered by \( B \) of \( ON \) in each case:
1. The claim immediately follows from \([14, 3.3]\).

2. If we consider \(N = ((C_2)^3 \times C_7) \times A_4 \times C_3\), it is a normal subgroup of both groups \(G_1\) and \(G_2\) listed, and it has only three blocks, all Morita equivalent to \(O((C_2)^3 \times C_7) \times A_4)\). \(N\) and \(G_i\) satisfy the hypothesis of Proposition \([14,12]\), so in particular there is only one possible Morita equivalence class for \(B\) with 7 simple modules.

3. If we consider \(N = (\text{SL}_2(8) \times A_4) \times C_3\), it is a normal subgroup of both groups \(G_1\) and \(G_2\) listed, and it has only three blocks with defect group \((C_2)^5\), all Morita equivalent to \(B_0(\text{O}(\text{SL}_2(8) \times A_4))\). \(N\) and \(G_i\) satisfy the hypothesis of Proposition \([14,15]\), so in particular there is only one possible Morita equivalence class for \(B\) with 7 simple modules.

\[\square\]

5 Blocks with defect group \((C_2)^5\)

First we classify the blocks with a normal defect group.

**Theorem 5.1.** Let \(B\) be a block of \(\text{O}(G)\) where \(G\) is a finite group, and let \(B\) have normal defect group \(D \simeq (C_2)^5\). Then \(B\) is Morita equivalent to the principal block of \(D \rtimes E\) where \(E\) is a subgroup of \(\text{GL}_5(2)\) with odd order, or a nonprincipal block of \((C_2)^4 \times 3^{1+2} \times C_2\), or of \((C_2)^5 \times (C_7 \times 3^{1+2})\).

**Proof.** A result by Külshammer \([28]\) states that every block with a normal defect group \(D\) is Morita equivalent to the twisted group algebra \(\text{O}_\alpha(D \rtimes E)\) where \(E = \text{NG}(D, b_D)/\text{C}_G(D)\) is the inertial quotient. Moreover, each possible \(\alpha\) can be chosen as \(\beta^{-1}\) where \(\beta \in H^2(E, k^\times)\) (see also \([33, 6.14]\)).

To get all possible inertial quotients, since the action on \(D\) has to be faithful, it is enough to consider all the conjugacy classes of odd order subgroups of \(\text{Aut}(D) = \text{GL}_5(2)\). We listed those in Proposition \([2,12]\). Each group algebra \(\text{O}(D \rtimes E)\) is also a block and, therefore, a representative of its class.

It is a standard fact that twisted group algebras \(\text{O}_\alpha H\) can be realized as blocks of the ordinary group algebra of a central extension \(\hat{H}\) of \(H\) by a \(p'\)-group (see, for example, \([30, 10.5]\)), so to produce examples of these Morita equivalence classes it is enough to look at central extensions of \(D \rtimes E\) by an odd subgroup, and hence at the Schur multiplier \(M(E)\) of \(E\). \(M(E)\) is trivial whenever \(E\) is cyclic (see \([33, 1.2.10]\)) and when \(E = C_7 \rtimes C_3\) or \(E = C_3 \rtimes C_5\).

When \(E = C_3 \rtimes C_3\) or \(E = (C_7 \rtimes C_3) \rtimes C_3\), \(M(E) = C_3\), giving two nontrivial possibilities for \(\alpha\) in each case. In the first case, one of them corresponds to the Morita equivalence class whose representative is one of the nonprincipal blocks of \(\text{O}((C_2)^4 \times 3^{1+2}) \times C_2\), where the center of the extraspecial group acts trivially. In the second case, one of them corresponds to the Morita equivalence class whose representative is one of nonprincipal blocks of \(\text{O}((C_2)^5 \times (C_7 \times 3^{1+2}))\), where again the center acts trivially.

From Corollary \([4,18]\) in each case choosing the other possibility for \(\alpha\) (which corresponds to choosing \(3^{1+2}\) instead of \(3^{1+2}\)), gives Morita equivalent blocks, as there is only one possibility for the weak crossed product equivalence class of these blocks in the given situations. \[\square\]
We state and prove our main result.

**Theorem 5.2.** Let \( G \) be a finite group, and let \( B \) be a block of \( OG \) with elementary abelian defect group \( D \) of order 32. Then either \( B \) is Morita equivalent to the principal block of precisely one of the following groups:

\[
\begin{align*}
(i) & \quad (C_2)^5 \quad \text{(inertial quotient 1)} \\
(ii) & \quad A_4 \times (C_2)^3 \quad \text{(i.q. (C_3)_1)} \\
(iii) & \quad A_5 \times (C_2)^3 \quad \text{(i.q. (C_3)_1)} \\
(iv) & \quad ((C_2)^4 \times C_3) \times C_2 \quad \text{(i.q. (C_3)_2)} \\
(v) & \quad ((C_2)^4 \times C_5) \times C_2 \\
(vi) & \quad ((C_2)^5 \times C_7) \times (C_2)^2 \quad \text{(i.q. C_7)} \\
(vii) & \quad SL_2(8) \times (C_2)^2 \quad \text{(i.q. C_7)} \\
(viii) & \quad A_4 \times A_4 \times C_2 \quad \text{(i.q. C_3 \times C_3)} \\
(ix) & \quad A_4 \times A_5 \times C_2 \quad \text{(i.q. C_3 \times C_3)} \\
(x) & \quad A_5 \times A_5 \times C_2 \quad \text{(i.q. C_3 \times C_3)} \\
(xi) & \quad ((C_2)^4 \times C_{15}) \times C_2 \\
(xii) & \quad SL_2(16) \times C_2 \\
(xiii) & \quad ((C_2)^5 \times C_7) \times A_4 \quad \text{(i.q. C_{21})} \\
(xiv) & \quad ((C_2)^5 \times C_7) \times A_5 \quad \text{(i.q. C_{21})} \\
(xv) & \quad SL_2(8) \times A_4 \quad \text{(i.q. C_{21})} \\
(xvi) & \quad SL_2(8) \times A_5 \quad \text{(i.q. C_{21})} \\
(xvii) & \quad ((C_2)^3) \times (C_7 \times C_3)) \times (C_2)^2 \quad \text{(i.q. (C_7 \times C_3)_1)} \\
(xviii) & \quad J_1 \times (C_2)^2 \quad \text{(i.q. (C_7 \times C_3)_1)} \\
(xix) & \quad \text{Aut}(SL_2(8)) \times (C_2)^2 \quad \text{(i.q. (C_7 \times C_3)_1)} \\
(xx) & \quad (C_2)^5 \times (C_7 \times C_3) \quad \text{(i.q. (C_7 \times C_3)_2)} \\
(xxi) & \quad (SL_2(8) \times (C_2)^2) \times C_3 \quad \text{(i.q. (C_7 \times C_3)_2)} \\
(xxii) & \quad (C_2)^5 \times C_{31} \quad \text{(i.q. C_{31})} \\
(xxiii) & \quad SL_2(32) \quad \text{(i.q. C_{31})} \\
(xxiv) & \quad ((C_2)^3) \times (C_7 \times C_3)) \times A_4 \quad \text{(i.q. (C_7 \times C_3 \times C_4)} \\
(xxv) & \quad ((C_2)^3) \times (C_7 \times C_3)) \times A_5 \quad \text{(i.q. (C_7 \times C_3 \times C_4)} \\
(xxvi) & \quad J_1 \times A_4 \quad \text{(i.q. (C_7 \times C_3 \times C_4)} \\
(xxvii) & \quad J_1 \times A_5 \quad \text{(i.q. (C_7 \times C_3 \times C_4)} \\
(xxviii) & \quad \text{Aut}(SL_2(8)) \times A_4 \quad \text{(i.q. (C_7 \times C_3 \times C_4)} \\
(xxix) & \quad \text{Aut}(SL_2(8)) \times A_5 \quad \text{(i.q. (C_7 \times C_3 \times C_4)} \\
(xxx) & \quad (C_2)^5 \times (C_{31} \times C_5) \quad \text{(i.q. C_{31} \times C_5)} \\
(xxxi) & \quad \text{Aut}(SL_2(32)) \quad \text{(i.q. C_{31} \times C_5)}
\end{align*}
\]

or \( B \) is Morita equivalent to a nonprincipal block of one of the following groups (there is one such Morita equivalence class for each group):

\[
\begin{align*}
(a) & \quad ((C_2)^4 \times 3_+^{1+2^2}) \times C_2 \quad \text{(i.q. C_3 \times C_3)} \\
(b) & \quad (C_2)^5 \times (C_7 \times 3_+^{1+2^2}) \quad \text{(i.q. (C_7 \times C_3) \times C_3)} \\
(c) & \quad SL_2(8) \times (C_2)^2 \times 3_+^{1+2^2} \quad \text{(i.q. (C_7 \times C_3) \times C_3)}
\end{align*}
\]

Moreover, if a block \( C \) of \( OH \) for a finite group \( H \) is Morita equivalent to \( B \), then the defect group of \( C \) is isomorphic to \( D \).

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Proof. Let $B$ be a block of $\mathcal{O}G$ for a finite group $G$ with defect $D = (C_2)^5$, such that $B$ is not Morita equivalent to any of the blocks in the statement of the theorem and such that $|G : O_2(Z(G))|, |G|$ is minimised in the lexicographic ordering. First, we show that these hypothesis on $B$ imply three important facts:

(I) $B$ is quasiprimitive, that is, for any normal subgroup $N \triangleleft G$ any block of $\mathcal{O}N$ covered by $B$ is $G$-stable.

In fact, let $N \triangleleft G$, and let $b$ be a block of $\mathcal{O}N$ covered by $B$. We write $I_G(b)$ for the stabiliser of $b$ under conjugation by $G$. Then we can consider the Fong-Reynolds correspondent $B_I$ as in Proposition 2.11 the unique block of $I_G(b)$ covering $b$ and with Brauer correspondent $B$, that is Morita equivalent to $B$ and shares a defect group with it. By minimality, it follows that $I_G(b) = G$, and the same is true for any block of any normal subgroup of $G$.

(II) If there is a normal subgroup $N \triangleleft G$ such that $B$ covers a nilpotent block $b$ of $\mathcal{O}N$, then $N \leq O_2(G)Z(G)$. This follows from minimality and quasiprimity, using Corollary 2.3. In particular, note that this implies $O_2(G) \leq Z(G)$.

(III) $G$ does not have any normal subgroup of index $2$: in fact, suppose by contradiction that there is $N \triangleleft G$ with index $2$. Let $b_N$ be the unique block of $\mathcal{O}N$ covered by $B$. Then from [20, 5.3.5] $B$ is also the unique block covering $b_N$, so from [11, 15.1] $D \cap N$ is a defect group of $b_N$ and $DN/N$ is a Sylow $2$-subgroup of $G/N$. Hence, $b_N$ has defect group $(C_2)^4$ and by Proposition 2.11 we know all the possibilities for its Morita equivalence class and, in particular, its inertial quotient. From the main theorem of [20], $B = B \otimes k$ is Morita equivalent to $(b_N \otimes k) \otimes kC_2$. Moreover, from Maschke’s theorem (see [22, 3.3.2]) there is an $E$-stable decomposition of $D$ as $(D \cap N) \times Q$ where $Q = C_2$. In particular then $Q \leq C_D(E)$. Then $B$ is as in cases (i)-(viii) of Proposition 2.12, so its inertial quotient is cyclic of order $3$, $5$, $7$ or $15$, $(C_3 \times C_3)$ or $(C_7 \times C_3)_1$.

From Proposition 2.6 the inertial quotient $E$ of $B$ is also one of these groups. We can use Proposition 2.12 to conclude that $l(B) = |E|$ whenever $E$ is cyclic, which means that we can apply Proposition 3.1 in every case to obtain that $B$ is Morita equivalent to $b_N \otimes OC_2$, a contradiction since all such blocks already appear in the list. As a corollary of this, we have that $G = O^2(G)$.

Recall the definition of the generalised Fitting subgroup $F^*(G)$ as in [2]; we write $E(G)$ for the layer of $G$, a normal subgroup that is the central product of the components of $G$ (the subnormal quasisimple subgroups). We write $F(G)$ for the Fitting subgroup. The generalised Fitting subgroup is defined as $F^*(G) = F(G)E(G) \triangleleft G$, and it is a central product of $F(G)$ and $E(G)$. A fundamental property is that $C_G(F^*(G)) \leq F^*(G)$, so there is an injective group homomorphism from $G/F^*(G)$ into $Out(F^*(G))$.

In particular, in our situation, (II) implies that $F(G) = Z(G)O_2(G) = O_2'(Z(G))O_2(G)$, and (I) implies that there is a unique block of $F^*(G)$ covered by $B$. We call it $b^*$. By minimality, we can suppose that no subgroup of $O_2'(Z(G))$ is a direct factor of $G$.

The first thing we prove is that we do not need to consider the situation $E(G) = 1$. In fact, if $E(G) = 1$, then since $D$ is a defect group $O_2(G) \leq D$. Since $D$ is abelian, $D \leq C_G(O_2(G)) = C_G(F^*(G)) \leq F^*(G) = O_2(G)Z(G)$ in this case, so $D \leq O_2(G)$ which
implies $D = O_2(G)$. Then $D$ is a normal subgroup of $G$, so $B$ is Morita equivalent to one of the blocks listed in Proposition 6.1, a contradiction.

Write $E(G) = L_1 \ast \cdots \ast L_t$, where each $L_i$ is a component of $G$. We prove that $t \leq 2$. Let $b_E$ be the unique block of $E(G)$ covered by $B$, and let $D_E = D \cap E(G)$ be its defect group. For each $i$, let $b_i$ be the unique block of $L_i$ covered by $b_E$ (note that $E(G) = L_i C_{E(G)}(L_i)$, so $b_i$ is $E(G)$-stable). Let $D_i$ be its defect group.

The components in $E(G)$ are permuted by the action of $G$ by conjugation. Consider a $G$-orbit of components $L = L_{i_1} \ast \cdots \ast L_{i_u}$ for some $u \in \mathbb{N}$. Note that all the components in the same orbit are isomorphic, with isomorphic blocks $b_{i_j}$ and defect groups $D_{i_j}$. Since $L < G$, it has a unique block $b_L$ covered by $B$, that covers each $b_{i_j}$ as determined before. The block $b_L$ cannot be nilpotent because of (II). Hence, from Lemma 2.10 every block $b_{i_j}$ of $L_{i_j}$ is also not nilpotent, so there is a nontrivial inertial quotient $E_{i_j}$ acting on the defect group $D_{i_j}$ of $b_{i_j}$. Then $|D_{i_j}, E_{i_j}| \geq (C_2)^2$, which implies that $D_{i_j}/(D_{i_j} \cap Z(L_{i_j})) \geq (C_2)^2$.

Again from Lemma 2.10 the defect group of $b_{i_j}, D_L = D \cap L$ is a central product of all $D_{i_j}$. Then $D_L \geq ((C_2)^2)^w$. Write the $G$-orbits of components in $E(G)$ as $L^j, j = 1, \ldots, w$, each the central product of $u_j$ isomorphic components. Then

$$|D| = 2^5 \sum_{i=1}^{w} |D_{L^i}| \geq (2^2)^{\sum_{j=1}^{w} u_j}.$$  

It follows that either there are two $G$-stable components $L_1$ and $L_2$, or there is a single orbit of components with $u_1 = 2$. In the latter case $G$ would have a normal subgroup of index 2 given by the kernel of the homomorphism $G \to S_2$ given by the permutation of the components, which contradicts (III). So we have shown that $t \leq 2$ and that if $t = 2$ then both components are normal in $G$.

If $t = 2$, so $E(G) = L_1 \ast L_2$, then the normal subgroup $(L_1 \cap L_2) \leq Z(E(G))$. In particular $O_2(Z(E(G))) \leq O_2(G) \leq D$, so if $O_2(Z(E(G)))$ is not trivial then it has order 2, since for each component $|D_i/(D_i \cap Z(L_i))| \geq 2$.

Consider $O_2(G) \lhd G$. Then $C = C_G(O_2(G)) \lhd G$. Let $c$ be the unique block of $OC$ covered by $B$. Then it has defect group $D \cap C = D$, and $C_G(D) \leq C$ since $O_2(G) \leq D$. From [1, 15.1] then $B$ is the unique block covering $c$, so in particular $[G : C]$ is odd since they share a defect group. Since $F^*(G) \leq C$, then $C/F^*(G) \to Out(E(G))$, since $C$ centralises $F(G) = O_2(G) O_2(Z(G))$, so it is solvable because of Schreier’s conjecture (since $t \leq 2$). Moreover, $G/C$ is solvable since it has odd order. Hence since $(G/F^*(G))/(C/F^*(G)) = G/C$ is solvable and $C/F^*(G)$ is solvable then $G/F^*(G)$ is also solvable.

First, we consider the case $t = 1$. In this case $E(G)$ is a quasisimple group, so from Proposition 2.13 we know all the possibilities that can occur, one or more of the following: $b_E$ can be among blocks of $SL_2(32), SL_2(16), SL_2(8), J_1, Co(3), 2G_2(3^{2n+1})(n > 0), D_n(q), E_7(q)$ for some odd prime power $q$, or it can be nilpotent covered or as in case (iv) of the Proposition.

We examine each case, determining that each possible Morita equivalence class is already in our list. We can show that the information about $b^*$ and $F^*(G)$ determines the structure
of $B$ and $G$ by using the self-centralising property of $F^*(G)$ as follows: for any possible fixed $E(G)$ and its unique block $b_E$ covered by $B$, we look at the possibilities for $O_2(G)$. Then, for each one, we examine the possibilities for the action $G/F^*(G) \to \text{Out}(F^*(G))$ on $F^*(G)$.

First, note that if $L_1$ is as in case (iv) of Proposition 2.13 then $b_E$ is the block of a central extension by an elementary abelian 2-group of a block $b_E$ with defect group $C_2 \times C_2$. In particular, from Proposition 2.11 $b_E$ is source algebra equivalent to $O_A4$ or $B_0(O_A5)$, so from [46, 1.22] and the main theorem of [13] $b_E$ has inertial quotient $(C_3)_1$. Using the language of pointed groups from [43], we note that whenever a block has an abelian defect group there are no essential pointed groups, so all the hypothesis of Corollary 1.14 in [43] are satisfied by the first intermediate central extension by a $C_2$ of $b_E$. Then this block is basic Morita equivalent to the principal block of a central extension by $C_2$ of $A_4$ or $A_5$, and the only central extension with abelian defect groups is the direct product. Now we can consider a central extension by another $C_2$, and repeat the argument. In particular then a repeated application of Corollary 1.14 in [43] yields that in this case $b_E$ is Morita equivalent to $O(A_4 \times (C_2)^y)$ or $B_0(O(A_5 \times (C_2)^y))$ for $y = 0, 1, 2$ or 3. Note that by looking at all the specific possibilities for Schur multipliers of simple groups we could exclude some of the possibilities for $y$, but we do not need to consider this to prove our result.

Note that, from the main theorem of [35] and from [6] the outer automorphism groups of the specific simple groups that we need to consider, $SL_2(2^n)$, $J_1$, $Co_3$ and $2G_2(q)$, only give split extensions of the simple group, so there is a unique extension (the semidirect product). This does not hold in general for every quasisimple group, but we use crossed product arguments (which avoid direct inspection of the extensions by outer automorphisms) in every other situation.

Now we examine each possibility:

If $E(G) = SL_2(32)$, so $b_E$ has defect group $(C_2)^5$, then $D \leq F^*(G)$ so the only possibilities for $G$ are $SL_2(32)$ and $\text{Aut}(SL_2(32))$ since $\text{Out}(F^*(G)) = \text{Out}(SL_2(32)) = C_5$.

If $E(G) = SL_2(16)$, then again $\text{Out}(F^*(G)) = \text{Out}(SL_2(16))$ which has order 4. In particular if $G \neq F^*(G)$ then $G$ has a normal subgroup of index 2, which is a contradiction. So $O_2(G) = C_2$ and $G = F^*(G) = SL_2(16) \times C_2$, again a contradiction since it appears in the list.

If $E(G) = SL_2(8)$, $2G_2(q)$ or $Co_3$, recall from [13] that the relevant blocks of these quasisimple groups are Morita equivalent. Then $|D \cap E(G)| = 8$, and since $\text{Out}(E(G))$ has odd order in each case we can assume that $D \leq F^*(G)$, so $O_2(G) = (C_2)^2$. We can then apply Proposition 4.15 to obtain a contradiction, since the block of $Co_3$ with elementary abelian defect group (of order 8) is Morita equivalent to the principal block of $\text{Aut}(SL_2(8))$ (see [27]), and therefore in each case $B$ is Morita equivalent to a block in our list.

If $E(G) = J_1$, since $\text{Out}(J_1) = 1$ then $O_2(G) = (C_2)^2$, so $D \leq F^*(G)$ and $|G/F^*(G)|$ is odd. Hence $\text{Out}(F^*(G)) \leq C_3$. If $G = F^*(G)$ then $B = b^*$, a contradiction. Otherwise $G = J_1 \times A_4$, which is again a contradiction since the only relevant block already appears in the list.

In every other case, from Lemma 2.4 there is a chain of normal subgroups

$$
F^*(G) \triangleleft b^* \triangleleft N_o \triangleleft N_e \triangleleft G[b_e] \triangleleft G \triangleleft b \triangleleft B
$$

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where \( \ell_\alpha, \ell_\beta, \ell_g \) are odd numbers and \( \ell_e = [D : D \cap F^*(G)] \). Note that \( \ell_e = 2 \) is a contradiction to (III), and \( \ell_e \leq 8 \) otherwise \( b^* \) is nilpotent, contradicting (II).

We consider the corresponding block chain \( b^*, b_0, b_e, b, B \), noting that each block is covered by \( B \) and, hence, is \( G \)-stable. From Lemma 3.1 in [16] we know that \( D \leq G[b^*] \), so \( b_e \) has the same defect group as \( D \), and \( N_b \leq G[b^*] \). In particular, from 2.8 \( b_0 \) is source algebra equivalent to \( b^* \). Therefore, \( b_e \) is Morita equivalent to \( b^* \otimes \mathcal{O}(C_2)^{\log_2(\ell_e)} \), since if \( D \leq F^*(G) \) then \( \ell_e = 1 \) so \( b_0 = b_e \), and when \( \ell_e = 4 \) or 8, \( b_e \) satisfies the hypothesis of Proposition 3.16 with an identical argument as in the proof of (III) above. Again from Lemma 2.8 \( b \) is source algebra equivalent to \( b_e \). Note that the pair \((G[b^*], \hat{b})\) satisfies the hypothesis of Method 1.3.

Suppose that \( \ell_e = 1 \), so \( D \leq F^*(G) \) and \( b^* = b_e \). If \( b_1 \) is nilpotent covered then it is inertial, and hence so is \( b^* \). Hence by Proposition 2.7 \( B \) is inertial, a contradiction by Theorem 5.1. Otherwise \( \hat{b} \) is Morita equivalent to \( \mathcal{O}(A_4 \times \langle C_2 \rangle^3) \) or \( B_0(\mathcal{O}(A_5 \times \langle C_2 \rangle^3)) \), so we can use Proposition 4.10 to obtain a contradiction.

Now suppose that \( \ell_e = 4 \). If \( |D \cap L_1| = 8 \) then if \( b^* = b_1 \) is nilpotent covered Proposition 2.11 gives a contradiction to (III), and if \( b_1 \) is Morita equivalent to \( \mathcal{O}(A_4 \times C_2) \) or \( B_0(\mathcal{O}(A_5 \times C_2)) \), then \( \hat{b} \) is as in the hypothesis of Proposition 4.10 which gives a contradiction. If \( |D \cap L_1| = 4 \) then \( O_2(G) = C_2 \) and \( b_1 \) is Morita equivalent to \( \mathcal{O}(A_4 \times C_2) \) or \( B_0(\mathcal{O}(A_5 \times C_2)) \), so \( \hat{b} \) is again as in the hypothesis of Proposition 4.10 a contradiction.

Finally, suppose that \( \ell_e = 8 \). Then \( |D \cap L_1| = 4 \) and \( |O_2(G)| = 1 \), otherwise \( b^* \) would be nilpotent contradicting (II). So \( b^* \) is Morita equivalent to \( \mathcal{O}(A_4) \), \( B_0(\mathcal{O}(A_5)) \) again a contradiction since \( \hat{b} \) satisfies the hypothesis of Proposition 4.10.

Now suppose that \( t = 2 \), so \( E(G) = L_1 \times L_2 \), where \( L_i \lhd G \). Suppose that \( D \) is not contained in \( F^*(G) \). Then, since \( |D \cap F^*(G)| \leq (2^2)^2 \), \( |G/F^*(G)| \leq 2 \). Then \( G \) has a normal subgroup of index 2, contradicting (III). So we can assume that \( D \leq F^*(G) \). We also assume without loss of generality that \( |D \cap L_1| \geq |D \cap L_2| \).

As we did in the previous case, we consider the chain of normal subgroups and the relative block chain, which now reduces to
\[
F^*(G) \overset{\ell_b}{\underset{\ell_g}{\triangleleft}} G[b_e] \overset{\ell_e}{\triangleleft} G \quad \hat{b} \quad B
\]
since \( D \leq F^*(G) \). Recall that \( \hat{b} \) is source algebra equivalent to \( b^* \) (Proposition 2.8).

Let \( L_\cap = L_1 \cap L_2 \). Note that since \( L_\cap \) is a normal abelian subgroup in \( G \), it is contained in \( F(G) \). As seen above, the Sylow 2-subgroup of \( L_\cap \) can only be trivial or \( C_2 \), and each normal subgroup of odd order is central because of (II). Hence, \( L_\cap \leq Z(G) \). In particular \( G \) acts trivially on its center, so the map \( \alpha \) defined in 4.5 with \( N = F^*(G) \) factors through \( (G/Z(G))/F^*(G)/Z(G) \) \( \rightarrow \) \( \text{Out}(L_1/L_\cap \times L_2/L_\cap) \leq \text{Out}(L_1/Z(L_1)) \times \text{Out}(L_2/Z(L_2)) \) where the last inclusion holds because each \( L_i \) is normal in \( G \) (see also 3 and then 46. 7.6). Whenever \( L_\cap \) has odd order \( b^* = b_1 \otimes b_2 \) from Proposition 2.10 and we can apply the same argument as in the proof of Corollary 4.10 to show that \( \beta = \beta_1 \beta_2 \). Then \( \beta(\alpha(G/N)) \) is a subgroup of \( \text{Pic}(c_1) \times \text{Pic}(c_2) \), and we can apply the method of Corollary 4.10 and Proposition 4.12 to obtain the same possibilities for the Morita equivalence class of \( B \).

If \( |D \cap L_1| = 4 \), then \( O_2(G) = C_2 \) and \( L_\cap, Z(L_1) \) and \( Z(L_2) \) have odd order. In this situation \( b_1 \) is Morita equivalent to \( \mathcal{O}(A_4) \) or \( B_0(\mathcal{O}(A_5)) \), and so is \( b_2 \). Then we can apply
Proposition 4.12 (considering \( H_1 = L_1 \times O_2(G) \)) in light of Corollary 4.16 to \( \tilde{b} \) to obtain a contradiction.

So we can suppose that \(|D \cap L_1| = 8\). If \( L_1 = SL_2(8), \ Z G_2(3^{2m+1}) \), \( Co_3 \) or \( J_1 \) then \( L_1 \) is a simple group, \(|O_2(G)| = 1\) and so \( E(G) = L_1 \times L_2 \). Then Proposition 4.15 in light of Proposition 4.16 applied to \( \tilde{b} \) gives a contradiction.

Suppose that \( L_\infty \) has odd order. Then in particular \(|O_2(G)| = 1\). Then if \( b_1 \) is nilpotent covered, so basic Morita equivalent to a block with a normal defect group \((C_2)^3\), or if \( b_1 \) is as in case (iii) of Proposition 2.13 again Proposition 4.12 in light of Corollary 4.16 applied to \( \tilde{b} \) gives a contradiction.

The last situation we need to deal with is \( E(G) \) being a central product of two components \( L_1, L_2 \) such that \( Z = C_2 \leq L_\infty \). Without loss of generality, up to repeating the arguments above for the odd part of \( L_\infty \), we can suppose that \( L_\infty = Z \). Note that in this situation \( Z \leq O_2(G) \leq F^*(G) \).

Consider \( \overline{b} \), the unique block of \( E(G)/Z = L_1/Z \times L_2/Z \) dominated by \( b^* \).

Since \( E(G)/Z = L_1/Z \times L_2/Z \), the direct product of two simple groups, Lemma 2.10 and Proposition 2.11 imply that \( \overline{b} \) is source algebra equivalent to the principal block of one among \( O(A_4 \times A_4), O(A_4 \times A_3) \) or \( O(A_5 \times A_5) \). In particular, both \( b^* \) and \( \overline{b} \) have inertial quotient isomorphic to \( C_2 \times C_2 \). Recall that source algebra equivalences are realized by trivial source bimodules, and hence are in particular basic Morita equivalences. Therefore the pair \((b^*, \overline{b})\) satisfies the hypothesis of Corollary 1.14 in [43], so \( b^* \) is basic Morita equivalent to the principal block of a central extension by \( C_2 \) of \( X \times Y \) where \( X, Y \in \{ A_4, A_5 \} \): moreover, the only possibility for \( b^* \) to have an abelian defect group is when the central extension is a direct product with \( C_2 \).

If \( b^* \) is basic Morita equivalent to \( O(A_4 \times A_4 \times C_2) \) then it is inertial, so by Proposition 2.4, \( B \) is also inertial, which is a contradiction.

Since any central product of two perfect groups is perfect, we can use the same argument of Lemma 7.6 in [46] to show that \( Aut(E(G)) \leq Aut(L_1/Z \times L_2/Z) \). Then in particular \( Out(E(G)) = Out(E(G)/Z) = Out(L_1) \times Out(L_2) \), since each \( L_i \) is normal in \( G \). Then by direct inspection of all the possibilities (listed in [8]) \( G/F^*(G) \) is a supersolvable group.

If \( b^* \) is basic Morita equivalent to \( B_0(O(A_5 \times A_5 \times C_2)) \), we can apply Proposition 4.11 to show that \( B \) is Morita equivalent to \( b^* \) and hence to (x), a contradiction.

If \( b^* \) is basic Morita equivalent to \( B_0(O(A_4 \times A_5 \times C_2)) \) then we use Method 4.17 from [15, 4.7], \( Pic(B_0(O(A_4 \times A_5))) = S_3 \times C_2 \). Then the maximal subgroup of odd order that we need to consider to bound the possible Morita equivalence classes for \( B \), just as in Method 4.8, is \( C_3 \). In particular if \( B \) is not Morita equivalent to \( \tilde{b} \) there is one possible Morita equivalence class for \( B \) given by the crossed product specified by any nontrivial homomorphism \( \omega : C_3 \to C_3 \), and that is \( B_0(O(A_5 \times (C_2)^3)) \), realized when \( F^*(G) = PSL_3(7) \times A_5 \times C_2 \). Hence \( B \) is Morita equivalent to (iii), a contradiction.

Therefore, in every possible case, we have a contradiction. To see that the classes are distinct it is enough to compute the Cartan matrices for each block, and to note that for the nonprincipal block with only one simple module the number of irreducible characters \( k(O((C_2)^4 \times 3_1^{\omega+2})b) \neq k(OD) \).

The fact that the isomorphism class of an elementary abelian defect group is invariant under Morita equivalences is Corollary 1.6 in [34].
Remark 5.3. A block Morita equivalent to (a) cannot be a principal block, since principal blocks with one simple module are nilpotent from [38, 6.13], but $k((a)) = 16$.
Moreover, from the main theorem of [28] together with Lemma 2.5 in [42], if a block of $OG$ with defect group $D$ and inertial quotient $E$ is principal then its Brauer correspondent is Morita equivalent to a non-twisted group algebra $O(D \rtimes E)$. Then in particular if Broué’s abelian defect group conjecture holds for blocks Morita equivalent to (b) or (c) then they cannot be principal blocks of any finite group, since the Brauer correspondent of (c) is Morita equivalent to (b), which is Morita equivalent to a twisted group algebra with a nontrivial twist, specified by the central extension. We prove the conjecture for (b) in Proposition 5.5.

In general, the inertial quotient of a block is not known to be preserved under Morita equivalences. However, the number of irreducible ordinary characters $k(B)$ and the number of simple modules $l(B)$ are preserved under Morita equivalence, so using the classification above and Proposition 2.12 we can immediately show that in almost every situation, given two Morita equivalent blocks with defect group $(C_2)^5$, the inertial quotient is preserved. The only exceptions are two pairs of inertial quotients in which the numerical invariants coincide, where we give a partial result.

Corollary 5.4. Let $B$ be a block of $OG$ where $G$ is a finite group, with defect group $D = (C_2)^5$ and inertial quotient $E$ of order $e(B)$. Then one of the following holds:

1. If $e(B) = 1$ then $k(B) = 32$, $l(B) = 1$.
2. If $e(B) = 3$ and $|C_D(E)| = 8$ then $k(B) = 32$, $l(B) = 3$.
3. If $e(B) = 3$ and $|C_D(E)| = 2$ then $k(B) = 16$, $l(B) = 3$.
4. If $e(B) = 5$ then $k(B) = 16$, $l(B) = 5$.
5. If $e(B) = 7$ then $k(B) = 32$, $l(B) = 7$.
6. If $e(B) = 9$ then $k(B) = 32$, $l(B) = 9$ or $k(B) = 16$, $l(B) = 1$.
7. If $e(B) = 15$ then $k(B) = 32$, $l(B) = 15$.
8. If $e(B) = 21$ and $|C_D(E)| = 4$ then $k(B) = 32$, $l(B) = 5$.
9. If $e(B) = 21$, $E$ is not abelian and $|C_D(E)| = 1$ then $k(B) = 16$, $l(B) = 5$.
10. If $e(B) = 21$ and $E$ is abelian then $k(B) = 32$, $l(B) = 21$.
11. If $e(B) = 31$ then $k(B) = 32$, $l(B) = 31$.
12. If $e(B) = 63$ then $k(B) = 32$, $l(B) = 15$ or $k(B) = 16$, $l(B) = 7$.
13. If $e(B) = 155$ then $k(B) = 16$, $l(B) = 11$.

In particular, Morita equivalent blocks have isomorphic inertial quotients with the same action on the defect group, except possibly when the Morita equivalence class is (v), (xi) or (xii) in Theorem 5.5.
Proof. Proposition 2.12 implies that $e(B)$ can only assume the values listed above, and that moreover for each fixed pair $(k(B), l(B))$ of blocks appearing in Theorem 5.2 there is a single possible isomorphism class for $E$, with two exceptions: the only pairs of values $(k(B), l(B))$ that occur in two distinct isomorphism classes for $E$ are $(16, 5)$ and $(32, 15)$:

- Let $B$ be such that $k(B) = 16$, $l(B) = 5$. Then $B$ is Morita equivalent to one of (v), (xx), (xxi) in Theorem 5.2. Suppose that $e(B) = 5$. Then from Proposition 2.12 $l(B) = 5$, so from the main theorem of [56] $B$ is perfectly isometric to (v). In particular then $B$ is Morita equivalent to $O(D \rtimes C_5)$, since this block is not perfectly isometric to (xx) or (xxi): in fact, a perfect isometry implies an isomorphism of the centers, but the Loewy lengths of the centers of the three blocks listed above, computed with Magma [9], are respectively 4 for (v) and 3 for (xx) and (xxi). In particular, if $B$ is Morita equivalent to (xx) or (xxi) then $e(B) \neq 5$, so every block in (xx) and (xxi) has $e(B) = 21$.

- Let $B$ be a block of $OG$ with $k(B) = 32$, $l(B) = 15$. Then $B$ is Morita equivalent to one of (xi), (xii), (xxiv), (xxv), (xxvi), (xxvii), (xxviii), (xxix) in Theorem 5.2. Suppose that $e(B) = 15$. Then from Proposition 2.12 $l(B) = 15$, so from the main theorem of [56] $B$ is perfectly isometric to (xi). In particular then $B$ is Morita equivalent to (xi) or (xii), since (xi) is not perfectly isometric to any of the other possibilities for the Morita equivalence class: in fact, as before, a perfect isometry implies an isomorphism of the centers. In this case all the centers have Loewy length 3 but, as computed with Magma [9], the dimension of $J^2(Z((xi)))$ is 15, while it is 21 for each representative between (xxiv)-(xxix). In particular, if $B$ is Morita equivalent to any block in (xxiv)-(xxix) then $e(B) \neq 15$, so every block in (xxiv)-(xxix) has $e(B) = 63$.

At present, we are unable to show that an arbitrary block with defect group $(C_2)^5$ and inertial quotient $(C_7 \rtimes C_3)_2$ is not Morita equivalent to (v), and that a block with defect group $(C_2)^5$, inertial quotient $(C_7 \rtimes C_3) \times C_3$ and 15 simple modules is not Morita equivalent to (xi) or (xii), so these Morita equivalence classes could contain blocks with different inertial quotients. However, we want to point out that an example of such blocks would provide a counterexample to Broué’s abelian defect group conjecture (hence, in particular, it would need to be a nonprincipal block).

With the exception of these cases, we prove Broué’s abelian defect group conjecture for Morita equivalences classes of blocks with defect group $(C_2)^5$. In each case, the equivalences will actually be splendid.

**Proposition 5.5.** Let $B$ be a block of $OG$ with defect group $(C_2)^5$ and inertial quotient $E$. Then $B$ is splendid derived equivalent to its Brauer correspondent $b$ in $NG(D)$, except possibly when $B$ is Morita equivalent to (v), (xi), (xii) or (c) in Theorem 5.2.

If $B$ has inertial quotient $C_5$ or $C_{15}$ then $B$ is splendid derived equivalent to its Brauer correspondent.

Proof. Consider $B$ as a block of $kG$ via the usual map $OG \to kG$, $B \mapsto B \otimes_O k$. A splendid derived equivalence defined over $k$ lifts to one over $O$ by [45] 5.2.]
If $B$ is Morita equivalent to any principal block $B'$ in the list of Theorem 5.2 that is not (v), (xi) or (xii), then $k(B) = k(B')$ and $(l(B) = l(B')$, and in particular by Corollary 5.3 $B$ and $B'$ have the same inertial quotient $E$, acting in the same way on $D$. By Theorem 4.33 in [8] then there is a splendid derived equivalence between $B'$ and its Brauer correspondent $b'$. If $|E|_3 \neq 9$, by [29] $b'$ is Morita equivalent to $k(D \rtimes E)$ since the Schur multiplier of $E$ is trivial. Then by composing these equivalences $B$ is derived equivalent to $k(D \rtimes E)$.

If $|E|_3 = 9$, then when $|E| = 9$ the splendid derived equivalences between (viii), (ix) and (x) can be built from the one between $kA_4$ and $B_0(kA_5)$ (see, for instance, [15]), so $B$ is splendid derived equivalent to (viii), which is its own Brauer correspondent, by composing the equivalences.

When $|E| = 63$, there are splendid derived equivalences between $k((C_2)^3 \rtimes (C_7 \rtimes C_3))$, $k(\text{Aut}(\text{SL}_2(8)))$ and $B_0(kJ_1)$ (see [13]), which together with the equivalence between $kA_4$ and $B_0(kA_5)$ give splendid derived equivalences between all the principal blocks with this inertial quotient. In particular (xxiv) is its own Brauer correspondent, so we are done.

If $B$ is not Morita equivalent to any principal block, then there are two possibilities for the Morita equivalence class of $B$, denoted as (a), (b) in Theorem 5.2. In this situation $B' = b'$, so we can use the same argument as above to obtain the splendid derived equivalence.

When $B$ has $e(B) = 5$ or $15$ we can use Proposition 5.4 to repeat the same argument as above since then the Morita equivalence between $B$ and $B'$ preserves the inertial quotient. In particular then (xi) and (xii) are splendid derived equivalent.

6 Harada’s conjecture

We prove a conjecture of Harada for the Morita equivalence classes we determined, and hence for any block with defect group $(C_2)^5$. The conjecture states that for a block $B$ of a finite group $G$ if a nonempty $J \subseteq \text{Irr}_K(B)$ is such that

$$\sum_{\chi \in J} \chi(1) \chi(g) = 0 \quad \forall g \in G \setminus G^0$$

(†)

where we denote as $G^0$ the set of $p$-regular elements of $G$, then $J = \text{Irr}(B)$. Note that if a subset $J$ of $\text{Irr}(B)$ satisfies Harada’s conjecture, then the complement $\text{Irr}(B) \setminus J$ also does. Lemma 1 in [48] shows that the property (†) above is equivalent to the existence of a vector $v \in \mathbb{Z}^{\text{Irr}(B)}$ such that for every row $d_\chi$ of the decomposition matrix of $B$ it holds that $(d_\chi, v) = \chi(1)$ if $\chi \in J$ and $0$ if $\chi \notin J$. This implies that if a block satisfies Harada’s conjecture, then any other block Morita equivalent to it also does. Then, in particular, it is enough to prove it for each representative determined in Theorem 5.2.

The Basic Set conjecture (21) states that there is always a basic set (i.e. a $\mathbb{Z}$-basis of $\mathbb{Z}^{\text{IBr}(B)}$) consisting of restrictions of ordinary irreducible characters, hence such that the decomposition matrix $Q$ with respect to this basic set contains $\text{Id}^{\text{Irr}(B) \times \text{Irr}(B)}$ as a submatrix.

**Lemma 6.1.** Let $N$ be a finite group and let $b$ be a block of $\mathcal{O}N$ for which Harada’s conjecture and the Basic Set conjecture hold. Then Harada’s conjecture and the Basic Set conjecture hold for $G = N \times (C_2)^2$, or $G = N \times A_4$ for the unique block $B$ of $\mathcal{O}G$ that covers $b$, or $G = N \times A_5$ for the unique block $B$ of $\mathcal{O}G$ that covers $b$ and the principal block of $A_5$. 49
Proof. In the following we use the notation $\mathcal{J} = \text{Irr}(B) \setminus J$.

When $G = N \times (C_2)^2$, the decomposition matrix $Q$ of $B$ is obtained by repeating every row of the decomposition matrix of $b$ twice, so a counterexample $J$ to Harada’s conjecture for $G$ would give a counterexample for $N$ as well, a contradiction. Moreover, the Basic Set conjecture trivially holds for $B$.

When $G = N \times A_4$, the decomposition matrix $Q$ of the block $B = b \times O A_4$ is the Kronecker product of the decomposition matrices of the factors, as seen in [52, 2.5]. In particular, it has the form

$$
\begin{pmatrix}
R & 0 & 0 \\
0 & R & 0 \\
0 & 0 & R \\
R & R & R
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\Delta
\end{pmatrix}
$$

where we denote by $R$ the decomposition matrix of $b$, which can be written as $R = \begin{pmatrix} \text{Id}^{l(b) \times (b)} \\ S \end{pmatrix}$

where $S$ has $k(b) - l(b)$ rows, and we distinguish subsets of rows $\alpha, \beta, \gamma, \Delta$, as above. So the Basic Set conjecture holds for $B$. Since $b$ is not a counterexample to Harada’s conjecture, if $\chi \in \alpha$ is in $J$, then $\alpha \subseteq J$, and the same holds for $\beta$ and $\gamma$. In fact let $\alpha = \{\chi_1, \ldots, \chi_{k(b)}\}$, and let $I = J \cap \alpha$. Then the first $l(b)$ entries of the vector $v$ are such that $v_i = \chi_i(1)$ if $i \in I$ and 0 otherwise. If $J \neq J$ there is a character $\chi_j \in \alpha$ such that $0 = (d_{\chi_j}, v) = \sum x_i v_i = \sum_{i \in I} x_i \chi_i(1)$. Since every character of $\alpha$ is the product of a fixed character $\psi$ of $A_4$ and a character of $N$, then if $I \neq \alpha$ we can obtain a counterexample to Harada’s conjecture for $b$ simply by dividing by the appropriate $\psi(1)$. Then $I = \alpha$. We can repeat an identical argument for $\beta$ and $\gamma$.

If $J = \alpha$ we can consider the first row of the $\Delta$ block, corresponding to a character $\psi$, which gives $0 = \chi_1(1) \neq 0$, a contradiction. Note that even for the rows of $\Delta$ corresponding to the nonidentity rows of $R$ we can use the rows in $\alpha$ to obtain $0 = \chi_j(1) \neq 0$ for some $\chi_j \in \alpha$, where $\chi_j(1)$ is also equal to some linear combination of characters in the first $l(b)$ lines of $\alpha$. We will use this multiple times in the proof of this Proposition. An analogous argument repeated for each row implies that if $\alpha \subset J$ but $\beta, \gamma \subset \mathcal{J}$ then $\Delta \subset J$.

If $J$ contains $\alpha$ and $\beta$, we use the fact that the vector $v$ mentioned above is completely determined by the rows containing just a single nonzero entry, and has the form $v = (v_1, \ldots, v_{3l(b)})$ where $v_i = \chi_j(1)$ (with $j$ such that $Q_{ijh} = \delta_{jh}$) if $\chi_j \in J$ and 0 otherwise. Using this, we can deduce that $J$ contains $\Delta$ as well, since for any row $d_{\chi}$, $\chi \in \Delta$ it holds that

$$(d_{\chi}, v) \geq \chi_\alpha(1) + \chi_\beta(1) > 0$$

for certain $\chi_\alpha \in \alpha, \chi_\beta \in \beta$. But then $J \leq \gamma$, and since by the previous $J = \gamma$ implies that $\Delta \subset \mathcal{J}$, a contradiction, then $\mathcal{J} = \emptyset$ and $J = \text{Irr}(B)$ so Harada’s conjecture holds for $B$.

Since either $J$ or $\mathcal{J}$ contains two among $\alpha, \beta$ and $\gamma$, we have shown that in each case $J = \text{Irr}(B)$ and hence that Harada’s conjecture holds for $B$.

When $G = N \times A_5$, the decomposition matrix $Q$ has the form

$$
\begin{pmatrix}
R & 0 & 0 \\
0 & R & 0 \\
0 & 0 & R \\
R & R & -R
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\Delta
\end{pmatrix}
$$

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so the Basic Set conjecture holds for $B$. Let $J$ be a set such that $(\dagger)$ holds. With identical arguments as in the previous case, we show that if $J$ contains a character from $\alpha$ then $\alpha \leq J$, and that the same holds for $\beta$ and $\gamma$. Moreover, again repeating the argument above, we can show that if $J$ contains exactly one among $\alpha$, $\beta$ and $\gamma$ then $J$ contains $\Delta$ as well. If $J$ contains $\alpha$ and $\beta$, but it has an empty intersection with $\gamma$, then we can deduce that $J$ contains $\Delta$ as well, since for any row $d_{\chi}$, $\chi \in \Delta$ it holds that 

$$(d_{\chi}, v) \geq \chi_{\alpha}(1) + \chi_{\beta}(1) > 0$$

If $J$ contains $\gamma$, by looking at the first row of $\Delta$ we get $0 \neq (v, d_{\chi_{\alpha}}) = v_1 + v_{k(b)+1} - \chi_{2k(b)+1}(1)$, which implies that at least one between $v_1$ and $v_{k(b)+1}$ is nonzero, and hence that $J$ contains $\alpha$ or $\beta$. Hence, up to exchanging $\alpha$ and $\beta$ and considering $\overline{J}$, we need to examine just two cases:

- $J$ contains $\alpha$, but $J$ has an empty intersection with $\beta$ and $\gamma$. In this case the vector $v$ is completely determined, and $\Delta \subseteq J$ by the remark above. But then $\overline{J}$ contains $\beta$ and $\gamma$, so $\Delta \subseteq \overline{J}$ as well, a contradiction.

- $J$ contains $\alpha$, $\beta$ and $\gamma$. In this case the vector $v$ is also completely determined, and it has no nonzero entry. The rows in $\Delta$ then give explicit relations, all of the form $(d_{\chi_{\gamma b}}(v, d_{\chi_{\alpha}}), v) = \chi_{\gamma}(1) + \chi_{k(b)+j}(1) - \chi_{2k(b)+j}(1)$. Each of these numbers is nonzero, since $B$ is the principal block and $D$ is abelian, so all character degrees are odd (see for example [33, 3.15] together with the fact that all characters have height zero as seen in [24]). Then $\Delta \subseteq J$, so $J = \text{Irr}(B)$ and hence Harada’s conjecture holds for $B$.

\[\square\]

**Proposition 6.2.** Let $G$ be a finite group, and let $B$ be a block of $OG$ with defect group $D = (C_2)^5$. Then Harada’s conjecture holds for $B$, that is, if a nonempty $J \subseteq \text{Irr}_K(B)$ is such that

$$\sum_{\chi \in J} \chi(1)\chi(g) = 0 \quad \forall g \in G \setminus G^0 \quad (*)$$

where we denote as $G^0$ the set of $p$-regular elements of $G$, then $J = \text{Irr}(B)$.

**Proof.** The statement is known to hold for 2-solvable groups from Proposition 3 in [18]. For any block with defect group a proper subgroup of $D$ the conjecture holds by Theorem 4 in [38]. For any block Morita equivalent to $b \otimes OC_2$ for a block $b$ with defect group $(C_2)^4$, the conjecture also holds, as the decomposition matrix of $B$ is obtained by repeating every row of the decomposition matrix of $b$ two times, and so a counterexample for $B$ would also give a counterexample for $b$.

Moreover, when $k(B) - l(B) = 1$ the statement is known to hold by [25, 3]. Finally, when there is a representative in the Morita equivalence class of the form $G_1 \times G_2$ where $G_1 \in \{A_4, A_5\}$ and $G_2 \in \{\text{SL}_2(8), \text{Aut}(\text{SL}_2(8)), J_1, (C_2)^3 \times C_7, (C_2)^3 \times (C_7 \times C_3)\}$ we can apply Lemma 6.1 and we are done.

The last three Morita equivalence classes left to examine are the principal blocks (xxi) and (xxxii), and the nonprincipal block (c) in Theorem 5.2. We checked these groups computationally using Magma [9], testing the condition $\sum_{\chi \in J} \chi(1)\chi(g)$ on all 2-singular elements for each subset $J$ with less than $\left\lfloor \frac{k(B)}{2} \right\rfloor$ elements, and Harada’s conjecture holds. \[\square\]
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