REGION OF V ARIABILITY FOR SPIRALLIKE FUNCTIONS
WITH RESPECT TO A BOUNDARY POINT

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Abstract. Let \( F_\mu \) denote the class of all non-vanishing analytic functions \( f \) in the unit disk \( \mathbb{D} \) with \( f(0) = 1 \), and for \( \mu \in \mathbb{C} \), such that \( \text{Re} \mu > 0 \) satisfying
\[
\text{Re} \left( \frac{2\pi z f'(z)}{\mu f(z)} + \frac{1 + z}{1 - z} \right) > 0 \quad \text{in} \ \mathbb{D}.
\]
For any fixed \( z_0 \) in the unit disk and \( \lambda \in \overline{\mathbb{D}} \), we shall determine the region of variability \( V(z_0, \lambda) \) for \( \log f(z_0) \) when \( f \) ranges over the class
\[
\mathcal{F}_\mu(\lambda) = \left\{ f \in \mathcal{F}_\mu : f'(0) = \frac{\mu}{\pi}(\lambda - 1) \right\}.
\]
In the final section we graphically illustrate the region of variability for several sets of parameters.

1. Introduction

We denote the class of analytic functions in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) by \( \mathcal{H}(\mathbb{D}) \), and we think of \( \mathcal{H}(\mathbb{D}) \) as a topological vector space endowed with the topology of uniform convergence over compact subsets of \( \mathbb{D} \). Denote by \( \mathcal{S}^* \) the subclass of functions \( \phi \in \mathcal{H}(\mathbb{D}) \) with \( \phi(0) = 0 \) such that \( \phi \) maps \( \mathbb{D} \) univalently onto a domain \( \Omega = \phi(\mathbb{D}) \) that is starlike with respect to the origin. That is, \( t \phi(z) \in \phi(\mathbb{D}) \) for each \( t \in [0,1] \). It is well known that for \( \phi \in \mathcal{H}(\mathbb{D}) \) with \( \phi(0) = 0 = \phi'(0) - 1 \), \( \phi \in \mathcal{S}^* \) if and only if
\[
\text{Re} \left( \frac{z \phi'(z)}{\phi(z)} \right) > 0, \quad z \in \mathbb{D}.
\]
Functions in \( \mathcal{S}^* \) are referred to as starlike functions. Denote by \( \mathcal{C} \) the subclass of functions \( \phi \in \mathcal{H}(\mathbb{D}) \) with \( \phi(0) = 0 \) such that \( \phi \) maps \( \mathbb{D} \) univalently onto a convex domain. It is well known that for \( \phi \in \mathcal{H}(\mathbb{D}) \) with \( \phi(0) = 0 = \phi'(0) - 1 \), \( \phi \in \mathcal{C} \) if and only if \( z \phi' \in \mathcal{S}^* \). Functions in \( \mathcal{C} \) are referred to as convex functions. We refer to the books [2, 5] for a detailed discussion on these two classes. Although, the class of starlike functions (with respect to an interior point) has been studied extensively among many other subclasses, little was known about starlike functions with respect to a boundary point until the work of Robertson [15]. Motivated by the work in [15] and characterizations of this class of functions, some advancement in this direction has taken place (see [17, 19, 4, 6]). On the other hand, there does not seem to be any development on spiral-like functions with respect to a boundary

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point until the recent work of Elin et al. [3] (see also [4]). More recently, Aharonov et al. [1] provide with a natural geometric approach to discuss spiral-like functions with respect to a boundary point and the conditions described in [1] cover the results studied by others. On the other hand, several authors have studied region of variability problems for various subclasses of univalent functions in $\mathcal{H}(\mathbb{D})$, see [8, 9, 12, 13, 14, 18, 19]. For example, it is well-known that for each fixed $z_0 \in \mathbb{D}$, the region of variability

$$V(z_0) = \{ \log \phi'(z_0) : \phi \in \mathcal{C}, \phi'(0) = 1 \}$$

is the set $\{ \log(1-z)^{-2} : |z| \leq |z_0| \}$.

Let $F_\mu$ denote the class of functions $f \in \mathcal{H}(\mathbb{D})$, and non-vanishing in $\mathbb{D}$ with $f(0) = 1$, and for $\mu \in \mathbb{C}$, such that $\text{Re} \mu > 0$ satisfying

$$\text{Re} P_f(z) > 0, \quad z \in \mathbb{D},$$

where

$$P_f(z) = \frac{2\pi z f''(z)}{\mu f(z)} + \frac{1+z}{1-z}.$$  \hspace{1cm} \text{(1.1)}$$

Clearly $P_f(0) = 1$. Basic properties and a number of equivalent characterizations of the class $F_\mu$ are formulated in [1]. The case $\mu = \pi$ coincides with the class introduced by Robertson [15] who has generated interest on this class, and its associated classes. It is also known that functions in $F_\pi$ are either close-to-convex or just the constant 1.

For $f \in F_\mu$, we denote by $\log f$ the single-valued branch of the logarithm of $f$ with $\log f(0) = 0$. Herglotz representation for analytic function with positive real part in $\mathbb{D}$ shows that if $f \in F_\mu$, then there exists a unique positive unit measure $\nu$ on $(-\pi, \pi]$ such that

$$\frac{2\pi z f''(z)}{\mu f(z)} + \frac{1+z}{1-z} = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}} d\nu(t),$$

and hence, a computation gives that

$$\log f(z) = \frac{\mu}{\pi} \int_{-\pi}^{\pi} \log \left( \frac{1-z}{1-ze^{-it}} \right) d\nu(t);$$

or equivalently

$$f(z) = (1-z)^{\mu/\pi} \exp \left\{ \frac{\mu}{\pi} \int_{-\pi}^{\pi} \log \left( \frac{1-z}{1-ze^{-it}} \right) d\nu(t) \right\}.$$  \hspace{1cm} \text{(1.2)}$$

Let $\mathcal{B}_0$ be the class of analytic functions $\omega$ in $\mathbb{D}$ such that $|\omega(z)| \leq 1$ in $\mathbb{D}$ and $\omega(0) = 0$. Consequently, for each $f \in F_\mu$ there exists an $\omega_f \in \mathcal{B}_0$ of the form

$$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) + 1}, \quad z \in \mathbb{D},$$

and conversely. It is a simple exercise to see that

$$P'_f(0) = 2\omega'_f(0) = 2 \left( \frac{\pi}{\mu} f'(0) + 1 \right)$$

or equivalently

$$|P'_f(0)| = 2|\pi/\mu f'(0) + 1| \leq 2.$$  \hspace{1cm} \text{(1.3)}$$

Suppose that $f \in F_\mu$. Then, a simple application of the classical Schwarz lemma (see for example [2, 10, 11]) shows that

$$|P'_f(0)| = 2|\pi/\mu f'(0) + 1| \leq 2,$$
because $|\omega_f'(0)| \leq 1$. Using (1.2), one can obtain by a computation that
\[
\frac{\omega''(0)}{2} = \frac{P''(0)}{4} - \lambda^2, \quad \text{and} \quad P''(0) = \frac{4\pi}{\mu} f''(0) - \frac{4\mu}{\pi} (\lambda - 1)^2 + 4
\]
so that
\[
\frac{\omega''(0)}{2} = \frac{\pi}{\mu} f''(0) - \frac{\mu}{\pi} (\lambda - 1)^2 + 1 - \lambda^2.
\]
Also if we let
\[
g(z) = \frac{\omega_f(z) - \lambda}{1 - \lambda \omega_f(z)}, \quad \text{for} \ |\lambda| < 1,
\]
and $g(z) = 0$ for $|\lambda| = 1$, then we see that
\[
g'(0) = \begin{cases} \frac{1}{1 - |\lambda|^2} \left( \frac{\omega_f(z)}{z} \right)' \bigg|_{z=0} = \frac{1}{1 - |\lambda|^2} \left( \frac{\omega''(0)}{2} \right) & \text{for} \ |\lambda| < 1 \\ 0 & \text{for} \ |\lambda| = 1. \end{cases}
\]
We note that for $|\lambda| < 1$,
\[
|g'(0)| \leq 1 \iff \frac{|\omega''(0)|}{2(1 - |\lambda|^2)} \leq 1 
\implies \frac{1}{1 - |\lambda|^2} \left| \frac{\pi}{\mu} f''(0) - \frac{\mu}{\pi} (\lambda - 1)^2 + 1 - \lambda^2 \right| \leq 1 
\iff f''(0) = \frac{\mu}{\pi} \left( a(1 - |\lambda|^2) + \frac{\mu}{\pi} (\lambda - 1)^2 - (1 - \lambda^2) \right)
\]
for some $a \in \mathbb{C}$ with $|a| \leq 1$. Consequently, for $\lambda \in \overline{D} = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and for $z_0 \in \mathbb{D}$ fixed, it is natural to introduce
\[
\mathcal{F}_\mu(\lambda) = \left\{ f \in \mathcal{F}_\mu : f'(0) = \frac{\mu}{\pi} (\lambda - 1) \right\} \\
V(z_0, \lambda) = \left\{ \log f(z_0) : f \in \mathcal{F}_\mu(\lambda) \right\}
\]
From (1.3) and the normalization condition introduced in the class $\mathcal{F}_\mu(\lambda)$, we observe that $\omega_f'(0) = \lambda$. The main aim of this paper is to determine the region of variability $V(z_0, \lambda)$ for $\log f(z_0)$ when $f$ ranges over the class $\mathcal{F}_\mu(\lambda)$. The precise geometric description of the set $V(z_0, \lambda)$ is established in Theorem 2.6.

2. Basic properties of $V(z_0, \lambda)$ and the Main result

To state our main theorem, we need some preparation. For a positive integer $p$,
\[
(S^*)^p = \{ f = f_0^p : f_0 \in S^* \}
\]
and recall the following result from [18].

**Lemma 2.1.** Let $f$ be an analytic function in $\mathbb{D}$ with $f(z) = z^p + \cdots$. If
\[
\text{Re} \left( 1 + z \frac{f''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D},
\]
then $f \in (S^*)^p$.

Now, we list down some basic properties of $V(z_0, \lambda)$.

**Proposition 2.2.** We have
(1) $V(z_0, \lambda)$ is compact.
(2) $V(z_0, \lambda)$ is convex.
(3) for $|\lambda| = 1$ or $z_0 = 0$,

\[(2.3) \quad V(z_0, \lambda) = \left\{ \frac{\mu}{\pi} \log \left( \frac{1 - z_0}{1 - \lambda z_0} \right) \right\}.
\]

(4) for $|\lambda| < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, $V(z_0, \lambda)$ has $(\mu/\pi) \log \left( \frac{1 - z_0}{1 - \lambda z_0} \right)$ as an interior point.

**Proof.** (1) Since $\mathcal{F}_\mu(\lambda)$ is a compact subset of $\mathcal{H}(\mathbb{D})$, it follows that $V(z_0, \lambda)$ is also compact.

(2) If $f_0, f_1 \in \mathcal{F}_\mu(\lambda)$ and $0 \leq t \leq 1$, then the function

\[f_t(z) = \exp \{ (1 - t) \log f_0(z) + t \log f_1(z) \}\]

is evidently in $\mathcal{F}_\mu(\lambda)$. Also, because of the representation of $f_t$, we see easily that the set $V(z_0, \lambda)$ is convex.

(3) If $z_0 = 0$, (2.3) trivially holds. If $|\lambda| = |\omega f'(0)| = 1$, then it follows from the classical Schwarz lemma that $\omega f(z) = \lambda z$, which implies

\[P_f(z) = 1 + \frac{\lambda z}{1 - \lambda z} \quad \text{and} \quad f(z) = \left( \frac{1 - z}{1 - \lambda z} \right)^{\frac{\mu}{\pi}}.
\]

Consequently,

\[V(z_0, \lambda) = \left\{ \frac{\mu}{\pi} \log \left( \frac{1 - z_0}{1 - \lambda z_0} \right) \right\}.
\]

(4) For $|\lambda| < 1$, and $a \in \mathbb{D}$, we define

\[\delta(z, \lambda) = \frac{z + \lambda}{1 + \lambda z},
\]

and

\[(2.4) \quad H_{a,\lambda}(z) = \exp \left( \frac{\mu}{\pi} \int_0^z \frac{\delta(a \zeta, \lambda) - 1}{(1 - \delta(a \zeta, \lambda) \zeta)(1 - \zeta)} d\zeta \right), \quad z \in \mathbb{D}.
\]

First we claim that $H_{a,\lambda} \in \mathcal{F}_\mu(\lambda)$. For this, we compute

\[
\frac{2\pi z H'_{a,\lambda}(z)}{\mu H_{a,\lambda}(z)} = \frac{2z(\delta(az, \lambda) - 1)}{(1 - \delta(az, \lambda) z)(1 - z)} = \frac{2z \delta(az, \lambda)}{1 - \delta(az, \lambda) z} - \frac{2z}{1 - z}
\]

and so, we see easily that

\[
\frac{2\pi z H'_{a,\lambda}(z)}{\mu H_{a,\lambda}(z)} + \frac{1 + z}{1 - z} = \frac{1 + \delta(az, \lambda) z}{1 - \delta(az, \lambda) z}.
\]

As $\delta(az, \lambda)$ lies in the unit disk $\mathbb{D}$, $H_{a,\lambda} \in \mathcal{F}_\mu(\lambda)$ and the claim follows. Also we observe that

\[(2.5) \quad \omega_{H_{a,\lambda}}(z) = z \delta(az, \lambda).
\]
Next we claim that the mapping $\mathbb{D} \ni a \mapsto \log H_{a,\lambda}(z_0)$ is a non-constant analytic function of $a$ for each fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{D}$. To do this, we put

$$h(z) = \frac{2\pi}{\mu(1 - |\lambda|^2)} \frac{\partial}{\partial a} \left\{ \log H_{a,\lambda}(z) \right\} \bigg|_{a=0}.$$ 

A computation gives

$$h(z) = 2 \int_0^z \frac{\zeta}{(1 - \lambda \zeta)^2} \, d\zeta = z^2 + \cdots$$

from which it is easy to see that

$$\Re \left\{ \frac{zh''(z)}{h'(z)} \right\} = \Re \left\{ \frac{1 + \lambda z}{1 - \lambda z} \right\} > 0, \quad z \in \mathbb{D}.$$

By Lemma 2.1 there exists a function $h_0 \in \mathcal{S}^*$ with $h = h_0^2$. The univalence of $h_0$ together with the condition $h_0(0) = 0$ implies that $h(z_0) \neq 0$ for $z_0 \in \mathbb{D} \setminus \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto \log H_{a,\lambda}(z_0)$ is a non-constant analytic function of $a$ and hence, it is an open mapping. Thus, $V(z_0, \lambda)$ contains the open set $\{ \log H_{a,\lambda}(z_0) : |a| < 1 \}$. In particular,

$$\log H_{0,\lambda}(z_0) = (\mu/\pi) \log \left( \frac{1 - z_0}{1 - \lambda z_0} \right)$$

is an interior point of $\{ \log H_{a,\lambda}(z_0) : a \in \mathbb{D} \} \subset V(z_0, \lambda). \quad \square$

We remark that, since $V(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, the boundary $\partial V(z_0, \lambda)$ is a Jordan curve and $V(z_0, \lambda)$ is the union of $\partial V(z_0, \lambda)$ and its inner domain.

Now we state our main result and the proof will be presented in Section 3.

**Theorem 2.6.** For $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V(z_0, \lambda)$ is the Jordan curve given by

$$(-\pi, \pi) \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0) = \frac{\mu}{\pi} \int_0^{z_0} \frac{\delta(e^{i\theta} \zeta, \lambda) - 1}{(1 - \delta(e^{i\theta} \zeta, \lambda)\zeta)(1 - \zeta)} \, d\zeta.$$ 

If $\log f(z_0) = \log H_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{F}_\mu(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f(z) = H_{e^{i\theta},\lambda}(z)$.

3. **Proof of Theorem 2.6**

**Proposition 3.1.** For $f \in \mathcal{F}_\mu(\lambda)$ we have

$$(3.2) \quad \left| \frac{f'(z)}{f(z)} - \frac{\mu}{\pi} c(z, \lambda) \right| \leq \frac{\mu}{\pi} r(z, \lambda), \quad z \in \mathbb{D},$$

where

$$c(z, \lambda) = \frac{|z|^2(z - \lambda)(1 - \bar{\lambda}) - (1 - \lambda)(1 - \bar{\lambda}z)}{(1 - z)(1 - |z|^2)(1 + |z|^2 - 2\Re(\lambda z))}, \quad \text{and}$$

$$r(z, \lambda) = \frac{(1 - |\lambda|^2)|z|}{(1 - |z|^2)(1 + |z|^2 - 2\Re(\lambda z))}.$$ 

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$. 
Proof. Let \( f \in \mathcal{F}_\mu(\lambda) \). Then there exists \( \omega_f \in \mathcal{B}_0 \) satisfying (1.2). As noticed in the introduction through (1.3) and the normalization of \( f \), we have \( \omega_f(0) = \lambda \). It follows from the Schwarz lemma (see for example [2, 10, 11]) that

\[
\frac{|\omega_f(z) - \lambda|}{1 - \lambda \omega_f(z)} \leq |z|, \quad z \in \mathbb{D}.
\]  

(3.3)

From (1.1) and (1.2) this is equivalent to

\[
\frac{f'(z)}{f(z)} - \frac{\mu}{\pi} A(z, \lambda) \leq |z| |\tau(z, \lambda)|,
\]  

(3.4)

where

\[
\begin{aligned}
A(z, \lambda) &= \frac{\lambda - 1}{(1 - \lambda z)(1 - z)} \\
B(z, \lambda) &= \frac{1 - \lambda}{(1 - z)(z - \lambda)} \\
\tau(z, \lambda) &= \frac{z - \lambda}{1 - \lambda z}.
\end{aligned}
\]  

(3.5)

A simple calculation shows that the inequality (3.4) is equivalent to

\[
\left| \frac{f'(z)}{f(z)} - \frac{\mu}{\pi} A(z, \lambda) + |z|^2 |\tau(z, \lambda)|^2 B(z, \lambda) \right| \leq \frac{|\mu| |z| |\tau(z, \lambda)| |A(z, \lambda) + B(z, \lambda)|}{1 - |z|^2 |\tau(z, \lambda)|^2}.
\]  

(3.6)

Using (3.5) we can easily see that

\[
1 - |z|^2 |\tau(z, \lambda)|^2 = \frac{(1 - |z|^2)(1 + |z|^2 - 2\Re(\lambda z))}{1 - \lambda z^2},
\]

\[
A(z, \lambda) + B(z, \lambda) = \frac{1 - |\lambda|^2}{(1 - \lambda z)(z - \lambda)}
\]

and

\[
A(z, \lambda) + |z|^2 |\tau(z, \lambda)|^2 B(z, \lambda) = \frac{(\lambda - 1)(1 - \lambda z^2) + |z|^2 (z - \lambda)(1 - \lambda)}{(1 - z)|1 - \lambda z|^2}.
\]

Thus, by a simple computation, we see that

\[
\frac{A(z, \lambda) + |z|^2 |\tau(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2 |\tau(z, \lambda)|^2} = c(z, \lambda)
\]

and

\[
\frac{|z| |\tau(z, \lambda)| |A(z, \lambda) + B(z, \lambda)|}{1 - |z|^2 |\tau(z, \lambda)|^2} = r(z, \lambda).
\]

Now the inequality (3.2) follows from these equalities and (3.6).

It is easy to see that the equality occurs for a \( z \in \mathbb{D} \) in (3.2), when \( f = H_{e^{i\theta}, \lambda} \) for some \( \theta \in \mathbb{R} \). Conversely if the equality occurs for some \( z \in \mathbb{D} \setminus \{0\} \) in (3.2), then the equality must hold in (3.3). Thus from the Schwarz lemma there exists a \( \theta \in \mathbb{R} \) such that \( \omega_f(z) = z\delta(e^{i\theta} z, \lambda) \) for all \( z \in \mathbb{D} \). This implies \( f = H_{e^{i\theta}, \lambda} \).

The choice of \( \lambda = 0 \) gives the following result which may need a special mention.
Corollary 3.7. For \( f \in \mathcal{F}_\mu(0) \) we have
\[
\left| \frac{f'(z)}{f(z)} - \frac{\mu(z^2 - 1)}{\pi(1 - z)(1 - |z|^4)} \right| \leq \frac{\mu}{\pi} |z|, \quad z \in \mathbb{D}.
\]

For each \( z \in \mathbb{D} \setminus \{0\} \), equality holds if and only if \( f = H_{e^{i\omega}, \theta} \) for some \( \theta \in \mathbb{R} \).

Corollary 3.8. Let \( \gamma : z(t), 0 \leq t \leq 1, \) be a \( C^1 \)-curve in \( \mathbb{D} \) with \( z(0) = 0 \) and \( z(1) = z_0 \). Then we have
\[
V(z_0, \lambda) \subset \left\{ w \in \mathbb{C} : |w - \frac{\mu}{\pi} C(\lambda, \gamma)| \leq \frac{|\mu|}{\pi} R(\lambda, \gamma) \right\},
\]
where
\[
C(\lambda, \gamma) = \int_0^1 c(z(t), \lambda) z'(t) \, dt \quad \text{and} \quad R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda)|z'(t)| \, dt.
\]

Proof. For \( f \in \mathcal{F}_\mu(\lambda) \), it follows from Proposition 3.1 that
\[
\left| \log f(z_0) - \frac{\mu}{\pi} C(\lambda, \gamma) \right| = \left| \int_0^1 \left\{ \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t) \, dt \right|
\leq \int_0^1 \left| \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right| |z'(t)| \, dt
\leq \frac{|\mu|}{\pi} \int_0^1 r(z(t), \lambda)|z'(t)| \, dt = \frac{|\mu|}{\pi} R(\lambda, \gamma).
\]
Since \( \log f(z_0) \in V(z_0, \lambda) \) was arbitrary, the conclusion follows. \( \square \)

For the proof of our next result, we need the following lemma.

Lemma 3.9. For \( \theta \in \mathbb{R} \) and \( \lambda \in \mathbb{D} \), the function
\[
G(z) = \frac{\mu}{\pi} \int_0^z \frac{e^{i\theta} \zeta}{\{1 + (\lambda e^{i\theta} - \lambda) \zeta - e^{i\theta} \zeta^2\}^2} \, d\zeta, \quad z \in \mathbb{D},
\]
has a double zero at the origin and no zeros elsewhere in \( \mathbb{D} \). Furthermore there exists a starlike univalent function \( G_0 \) in \( \mathbb{D} \) such that \( G = (\mu/(2\pi))e^{i\theta}G_0^2 \) and \( G_0(0) = G_0'(0) = 1 = 0 \).

Proof. Let \( b = \text{Im}(\lambda e^{i\theta/2}) \in \mathbb{R} \). Then a computation shows that
\[
1 + (\lambda e^{i\theta} - \lambda) z - e^{i\theta} z^2 = (1 - z/z_1)(1 - z/z_2),
\]
where
\[
z_1 = e^{-i\theta/2}(ib + \sqrt{1 - b^2}) \quad \text{and} \quad z_2 = e^{-i\theta/2}(ib - \sqrt{1 - b^2}).
\]
From this we have
\[
\frac{G''(z)}{G'(z)} - \frac{1}{z} = \frac{d}{dz} \left\{ \log \frac{G'(z)}{z} \right\} = \frac{2/z_1}{1 - z/z_1} + \frac{2/z_2}{1 - z/z_2}.
\]
Since \( |z_1| = |z_2| = 1 \), we have for \( z \in \mathbb{D} \)
\[
\text{Re} \left\{ 1 + \frac{zG''(z)}{G'(z)} \right\} = \text{Re} \left\{ \frac{1 + z/z_1}{1 - z/z_1} \right\} + \text{Re} \left\{ \frac{1 + z/z_2}{1 - z/z_2} \right\} > 0.
\]
Applying Lemma 2.4 to $(2\pi/\mu)e^{i\theta}G(z)$ with $p = 2$ there exists a $G_0 \in S^*$ such that $G = (\mu/(2\pi))e^{i\theta}G_0^2$.

**Proposition 3.10.** Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have $\log H_{e^{i\theta}}(z_0) \in \partial V(z_0, \lambda)$. Furthermore if $\log f(z) = \log H_{e^{i\theta}}(z_0)$ for some $f \in F_{\mu}(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f = H_{e^{i\theta}}$.

**Proof.** From (2.4) we have

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} = \frac{\mu}{\pi} \frac{\delta(az, \lambda) - 1}{1 - \delta(az, \lambda)} = \frac{\mu}{\pi} \frac{(\lambda - 1) + (1 - \overline{\lambda})az}{(1 - z)(1 + (\lambda - \lambda)z - az^2)}.$$  

Using (3.5) we compute

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} = \frac{\mu}{\pi} A(z, \lambda) = \frac{\mu(1 - |\lambda|^2)az}{\pi(1 - \lambda z)(1 + (\overline{\lambda}a - \lambda)z - az^2)},$$

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} + \frac{\mu}{\pi} B(z, \lambda) = \frac{\mu(1 - \lambda^2)}{\pi(z - \lambda)(1 + (\overline{\lambda}a - \lambda)z - az^2)}$$

and hence we obtain that

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} - \frac{\mu}{\pi} c(z, \lambda) = \frac{1}{1 - |z|^2} \left\{ \left( \frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} - \frac{\mu}{\pi} A(z, \lambda) \right) - |z|^2 \tau(z, \lambda) \left( \frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} + \frac{\mu}{\pi} B(z, \lambda) \right) \right\}$$

$$= \frac{\mu(1 - |\lambda|^2)[a(1 - \overline{\lambda}z) - \overline{z}(\overline{\lambda} - \lambda)]}{\pi(1 - |z|^2)(1 + |z|^2 - 2Re(\lambda z))(1 + (\overline{\lambda}a - \lambda)z - az^2)} = \frac{r(z, \lambda) \frac{\mu az}{\pi |z|} \left( \frac{|1 + (\overline{\lambda}a - \lambda)z - az^2|^2}{(1 + (\overline{\lambda}a - \lambda)z - az^2)^2} \right).}$$

Now by substituting $a = e^{i\theta}$ we easily see that

$$\frac{H'_{e^{i\theta}}(z)}{H_{e^{i\theta}}(z)} - \frac{\mu}{\pi} c(z, \lambda) = r(z, \lambda) \frac{\mu e^{i\theta}z}{\pi |z|} \left( \frac{|1 + (\overline{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2|^2}{(1 + (\overline{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2)^2} \right).$$

Putting $G(z)$ as in Lemma 3.9 we get that

$$\frac{H'_{e^{i\theta}}(z)}{H_{e^{i\theta}}(z)} - \frac{\mu}{\pi} c(z, \lambda) = \frac{|\mu|}{\pi} r(z, \lambda) \left( \frac{G'(z)}{|G'(z)|} \right)$$

and there exists a starlike univalent function $G_0 \in \mathbb{D}$ such that $G = (\mu/(2\pi))e^{i\theta}G_0^2$ and $G_0(0) = G'_0(0) - 1 = 0$. As the function $G_0$ is starlike, for any $z_0 \in \mathbb{D} \setminus \{0\}$ the linear segment joining $0$ and $G_0(z_0)$ entirely lies in $G_0(\mathbb{D})$. Now, we define $\gamma_0$ by

$$\gamma_0 : z(t) = G_0^{-1}(tG_0(z_0)), \quad 0 \leq t \leq 1.$$ 

Since $G(z(t)) = (\mu/(2\pi))e^{i\theta}(G_0(z(t)))^2 = (\mu/(2\pi))e^{i\theta}(tG_0(z_0))^2 = t^2G(z_0)$, we have

$$G'(z(t))z'(t) = 2tG(z_0), \quad t \in [0, 1].$$
Using (3.13) and (3.11) we have

\[(3.14) \log H_{e^{i\theta}, \lambda}(z_0) - \frac{\mu}{\pi} C(\lambda, \gamma_0) = \int_0^1 \left\{ \frac{H'_{e^{i\theta}, \lambda}(z(t))}{H_{e^{i\theta}, \lambda}(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t) \, dt \]

\[= \frac{|\mu|}{\pi} \int_0^1 \frac{r(z(t), \lambda) G'(z(t)) z'(t)}{G(z(t)) z'(t)} |z'(t)| \, dt \]

\[= \frac{G(z_0)}{G(z_0)} \frac{|\mu|}{\pi} \int_0^1 r(z(t), \lambda) \, dt \]

\[= \frac{G(z_0)}{G(z_0)} \frac{|\mu|}{\pi} R(\lambda, \gamma_0), \]

where \( C(\lambda, \gamma_0) \) and \( R(\lambda, \gamma_0) \) are defined as in Corollary 3.8. Thus, we have

\[\log H_{e^{i\theta}, \lambda}(z_0) \in \partial \mathbb{D} \left( \frac{\mu}{\pi} C(\lambda, \gamma_0), \frac{|\mu|}{\pi} R(\lambda, \gamma_0) \right).\]

Also, from Corollary 3.8 we have

\[\log H_{e^{i\theta}, \lambda}(z_0) \in V(z_0, \lambda) \subset \overline{\mathbb{D}} \left( \frac{\mu}{\pi} C(\lambda, \gamma_0), \frac{|\mu|}{\pi} R(\lambda, \gamma_0) \right).\]

Hence, we conclude that \( \log H_{e^{i\theta}, \lambda}(z_0) \in \partial V(z_0, \lambda). \)

Finally, we prove the uniqueness of the curve. Suppose that

\[\log f(z_0) = \log H_{e^{i\theta}, \lambda}(z_0)\]

for some \( f \in \mathcal{F}_\mu(\lambda) \) and \( \theta \in (-\pi, \pi]. \) We introduce

\[h(t) = \frac{G(z_0)}{G(z_0)} \left\{ \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t),\]

where \( \gamma_0 : z(t), 0 \leq t \leq 1, \) is given by (3.12). Then, \( h(t) \) is continuous function in \([0, 1]\) and satisfies

\[|h(t)| \leq \frac{|\mu|}{\pi} r(z(t), \lambda) |z'(t)|.\]

Furthermore, we have from (3.14)

\[\int_0^1 \text{Re} \, h(t) \, dt = \int_0^1 \text{Re} \left\{ \frac{G(z_0)}{G(z_0)} \left\{ \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t) \right\} dt \]

\[= \text{Re} \left\{ \frac{G(z_0)}{G(z_0)} \left\{ \log f(z_0) - \frac{\mu}{\pi} C(\lambda, \gamma_0) \right\} \right\} \]

\[= \text{Re} \left\{ \frac{G(z_0)}{G(z_0)} \left\{ \log H_{e^{i\theta}, \lambda}(z_0) - \frac{\mu}{\pi} C(\lambda, \gamma_0) \right\} \right\} \]

\[= \frac{|\mu|}{\pi} \int_0^1 r(z(t), \lambda) |z'(t)| \, dt.\]

Thus, we have

\[h(t) = \frac{|\mu|}{\pi} r(z(t), \lambda) |z'(t)| \quad \text{for all} \ t \in [0, 1].\]
From (3.11) and (3.13), it follows that
\[
\frac{f'}{f} = \frac{H'_{e^{i\theta},\lambda}}{H_{e^{i\theta},\lambda}} \text{ on } \gamma_0.
\]
By applying the identity theorem for analytic functions, we get
\[
\frac{f'}{f} = \frac{H'_{e^{i\theta},\lambda}}{H_{e^{i\theta},\lambda}} \text{ in } \mathbb{D}
\]
and hence, by normalization, \( f = H_{e^{i\theta},\lambda} \) in \( \mathbb{D} \).

**Proof of Theorem 2.6.** We need to prove that the closed curve
\[
(-\pi, \pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0)
\]
is simple. Suppose that
\[
\log H_{e^{i\theta_1},\lambda}(z_0) = \log H_{e^{i\theta_2},\lambda}(z_0)
\]
for some \( \theta_1, \theta_2 \in (-\pi, \pi] \) with \( \theta_1 \neq \theta_2 \). Then, from Proposition 3.10, we have
\[
H_{e^{i\theta_1},\lambda} = H_{e^{i\theta_2},\lambda}.
\]
From (2.5) this gives a contradiction that
\[
e^{i\theta_1}z = \tau \left( \frac{\omega H_{e^{i\theta_1},\lambda}}{z}, \lambda \right) = \tau \left( \frac{\omega H_{e^{i\theta_2},\lambda}}{z}, \lambda \right) = e^{i\theta_2}z.
\]
Thus, the curve must be simple.

Since \( V(z_0, \lambda) \) is a compact convex subset of \( \mathbb{C} \) and has nonempty interior, the boundary \( \partial V(z_0, \lambda) \) is a simple closed curve. From Proposition 3.1, the curve \( \partial V(z_0, \lambda) \) contains the curve \( (-\pi, \pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0) \). Recall the fact that a simple closed curve cannot contain any simple closed curve other than itself. Thus, \( \partial V(z_0, \lambda) \) is given by \( (-\pi, \pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0) \).

### 4. Geometric view of Theorem 2.6

Using Mathematica 4.1, we describe the boundary of the set \( V(z_0, \lambda) \). Here we give the Mathematica program which is used to plot the boundary of the set \( V(z_0, \lambda) \).

We refer [16] for Mathematica program. The short notations in this program are of the form: “z0 for \( z_0 \), “lam for \( \lambda \)” and “mu for \( \mu \”).

```mathematica
Remove["Global`*"];

z0 = Random[] Exp[I Random[Real, {-Pi, Pi}]]
lam = Random[] Exp[I Random[Real, {-Pi, Pi}]]
mu = Random[Real, {0, 10^-3}] + I Random[Real, {-10^-3, 10^-3}]

Q[lam_, the_] := ((lam - 1) + (1 - Conjugate[lam]) Exp[I*the]*z)/
    ((1 - z)((1 + Conjugate[lam]* Exp[I*the] - lam)*z )
    - Exp[I*the]*z*z));

myf2[lam_, the_, z0_] := mu/Pi NIntegrate[Q[lam, the], {z, 0, z0}];
image = ParametricPlot[{Re[myf2[lam, the, z0]],
    Im[myf2[lam, the, z0]]}, {the, -Pi, Pi},
    AspectRatio -> Automatic];
(*Clear[z0, lam, mu];*)
The following pictures give the geometric view of the boundary of the set $V(z_0, \lambda)$. Each of the following figures contain two pictures which describe the boundary of the set $V(z_0, \lambda)$ for fixed value of $z_0 \in \mathbb{D} \setminus \{0\}$, $\lambda \in \mathbb{D}$ and $\mu \in \mathbb{C}$ such that Re $\mu > 0$. The corresponding values for each picture are given in a column at the bottom of the picture. Note that according to Proposition 2.2 the region bounded by the curve $\partial V(z_0, \lambda)$ is compact and convex.

**Figure 1.** Region of variability for $\log f(z_0)$

$z_0 = -0.173777 + 0.0869191 i$
$\lambda = -0.196029 + 0.480913 i$
$\mu = 32796 + 64560.2 i$

$z_0 = -0.713811 - 0.0997298 i$
$\lambda = -0.225338 + 0.323073 i$
$\mu = 69097.4 + 83886.6 i$

**Figure 2.** Region of variability for $\log f(z_0)$

$z_0 = -0.734426 + 0.61942 i$
$\lambda = -0.0564481 - 0.00656122 i$
$\mu = 54025 - 5108.28 i$

$z_0 = -0.69693 - 0.601351 i$
$\lambda = -0.0416728 - 0.683999 i$
$\mu = 23944.2 + 50613.5 i$
Figure 3. Region of variability for $\log f(z_0)$

$z_0 = 0.0150249 + 0.994594i$
$\lambda = -0.219752 - 0.256693i$
$\mu = 16828.1 - 35690.8i$

$z_0 = 0.378332 - 0.90135i$
$\lambda = 0.366791 - 0.600223i$
$\mu = 5006.59 - 46769.8i$

Figure 4. Region of variability for $\log f(z_0)$

$z_0 = 0.80351 + 0.549035i$
$\lambda = -0.55886 + 0.0419296i$
$\mu = 83278.8 - 90464.3i$

$z_0 = 0.691568 + 0.644823i$
$\lambda = 0.126172 + 0.137643i$
$\mu = 47178.4 + 83497.8i$
Figure 5. Region of variability for log $f(z_0)$

$z_0 = 0.737135 + 0.496542i$

$\lambda = -0.00646307 - 0.0167039i$

$\mu = 14038.5 + 9544.66i$

Figure 6. Region of variability for log $f(z_0)$

$z_0 = 0.556307 - 0.814404i$

$\lambda = 0.226895 - 0.384635i$

$\mu = 13589.3 - 25797.8i$

$z_0 = 0.880992 - 0.328223i$

$\lambda = -0.0326596 + 0.656304i$

$\mu = 39935.5 + 11412i$

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