On diagonal solutions of the reflection equation

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Abstract
We study solutions of the reflection equation associated with the quantum affine algebra $U_q(\hat{\text{gl}}(N))$ and obtain diagonal K-operators in terms of the Cartan elements of a quotient of $U_q(\text{gl}(N))$. We also consider intertwining relations for these K-operators and find an augmented $q$-Onsager algebra like symmetry behind them.

Keywords: augmented $q$-Onsager algebra, intertwining relation, K-matrix, L-operator, quantum algebra, reflection equation

1. Introduction

The reflection equation [1] is a fundamental object in quantum integrable systems with open boundary conditions [2]. It has the following form

$$ R_{12}(y,x)K_1(x)K_{21}(xy) = K_{21}(y)K_1(x)K_{12}(1/xy), \quad x, y \in \mathbb{C}. $$

(1.1)

Here $R(x)$ is a solution (R-matrix) of the Yang–Baxter equation and $K(x)$ is a K-matrix. The indices 1, 2 denote the space where the operators act non-trivially. In particular, there is a $N \times N$ diagonal matrix solution [3] of the reflection equation associated with the R-matrices [4] for the $N$-dimensional fundamental representation of $U_q(\text{gl}(N))$. These R-matrices are specialization of more general operators called L-operators: $L_{12}(x), \bar{L}_{12}(x) \in U_q(\text{gl}(N)) \otimes \text{End}(\mathbb{C}^N)$. Namely, they are given by $R_{12}(x) = (\pi \otimes 1)L_{12}(x), \bar{R}_{21}(x) = (\pi \otimes 1)\bar{L}_{12}(x)$, where $\pi$ is the fundamental representation of $U_q(\text{gl}(N))$. In this context, a natural problem is to investigate the solutions of the reflection equation associated with the L-operators:

$$ L_{12}(y,x)K_1(x)L_{21}(xy) = K_{21}(y)L_{12}(1/xy)K_1(x)L_{12}(y), \quad (1.2) $$

$$ L_{12}(y,x)K_1(x)L_{21}(xy) = K_{21}(y)L_{12}(1/xy)K_1(x)L_{12}(y), \quad (1.2) $$
where the K-operator $K(x)$ is an operator in $U_q(\hat{gl}(N))$. In this paper, we propose diagonal solutions of (1.2), namely solutions written in terms of only the Cartan elements of (a quotient of) $U_q(\hat{gl}(N))$. Evaluation of the K-operator for the fundamental representation of $U_q(\hat{gl}(N))$ reproduces the $N \times N$ diagonal K-matrix: $K(x) = \pi(K(x))$. In the context of Baxter Q-operators for integrable systems with open boundaries, the essentially same K-operator for $N = 2$ case was previously proposed in [5]. This paper extends this to $N \geq 3$ case in part. We remark that K-operators for Baxter Q-operators appeared first in [6] for the XXX-model ($q = 1$ and $N = 2$ case).

We also rewrite (1.2) as intertwining relations for the K-operator and speculate an explicit form of the underlying symmetry algebra. For $N = 2$ case, it is known that the augmented $q$-Onsager algebra [7, 8], which is a co-ideal subalgebra of $U_q(\hat{sl}(2))$, serves as this. We find an augmented $q$-Onsager algebra like symmetry still exists for an oscillator representation even for $N \geq 3$ case. At a little more abstract level, the relevant algebras will be the reflection equation algebras [2] and some coideal subalgebras of quantum affine algebras [9]. In addition, symmetries of the transfer matrix of open spin chains with diagonal K-matrices are discussed in [10]. Intertwining relations for K-matrices are studied, for example, in [11]. However, an explicit expression for the higher rank analogue of the augmented $q$-Onsager algebra seems to be missing in the literatures. We expect that our results can be a cue for further study on this.

There are several versions of the reflection equation (‘left’, ‘right’, ‘twisted’, ‘untwisted’). However, their solutions, K-matrices, are considered to be related each other by some transformations. Then we concentrate on one of them. Throughout this paper, we use the general gradation\footnote{We emulate [12] and their subsequent papers for this.} of $U_q(\hat{gl}(N))$. This does not produce particularly new results since the difference of the gradation reduces to a simple similarity transformation and a rescaling of the spectral parameter on the level of the R-matrices. However, there is a merit to use it since the parameters in the general gradation play a role as ‘markers’ and make it easier to trace the actions of the generators of $U_q(\hat{gl}(N))$.

2. The quantum affine and finite algebras

In this section, we review the quantum algebras [13–15] of type A and associated L-operators. We also refer the books [16, 17] for review of this subject.

2.1. The quantum affine algebra $U_q(\hat{gl}(N))$

We introduce a $q$-commutator $[X,Y]_q = XY - qXY$, and set $[X,Y]_1 = [X,Y]$. The quantum affine algebra $U_q(\hat{gl}(N))$ is a Hopf algebra generated by the generators\footnote{In this paper, we do not use the degree operator $d$. We will only consider level zero representations. The notation $e_0, f_0$ conventionally used in literatures corresponds to $e_0, f_0$. We assume that the deformation parameter $q = e^h$ ($h \in \mathbb{C}$) is not a root of unity.} $e_i, f_i, k_i$, where $i \in \{1, \ldots, N\}$. For $i, j \in \{1, 2, \ldots, N\}$, the defining relations of the algebra are given by

\begin{align}
[k_i, k_j] &= 0, & [k_i, e_j] &= (\delta_{ij} - \delta_{i+1,j})e_j, & [k_i, f_j] &= -(\delta_{ij} - \delta_{i+1,j})f_j, \\
[e_i, f_j] &= \frac{q^h - q^{-h}}{q - q^{-1}}, \\
[e_i, [e_i, e_j]_q] &= 0 \quad \text{for} \quad |i - j| \geq 2, \\
[e_i, [e_i, [e_i, e_j]_q]_q] &= 0 \quad \text{for} \quad N = 2, \quad i \neq j, \\
[e_i, [e_i, [e_i, e_j]_q]_{q^{-1}}] &= 0 \quad \text{for} \quad N = 3, \quad |i - j| = 1.
\end{align}

(2.1)
where \( h_i = k_i - k_i + 1 \), and \( i, j \) should be interpreted modulo \( N \): \( (N + 1 \equiv 1) \). 
\( \hat{c} = \sum_{i=1}^{N} k_i \) is a central element of the algebra. The algebra has the co-product

\[
\Delta : U_q(\hat{g}(N)) \rightarrow U_q(\hat{g}(N)) \otimes U_q(\hat{g}(N))
\]

defined by

\[
\Delta(e_i) = e_i \otimes 1 + q^{-h_i} \otimes e_i,
\]
\[
\Delta(f_i) = f_i \otimes q^{h_i} + 1 \otimes f_i,
\]
\[
\Delta(k_i) = k_i \otimes 1 + 1 \otimes k_i.
\]

We will also use the opposite co-product defined by

\[
\Delta' = \sigma \circ \Delta,
\]
\[
\sigma \circ (X \otimes Y) = Y \otimes X, \quad X, Y \in U_q(\hat{g}(N)).
\]

In addition to these, there are anti-poid and co-unit, which will not be used here. The Borel sub-algebras \( B_+ \) (resp. \( B_- \)) is generated by the elements \( e_i, k_i \) (resp. \( f_i, k_i \)), where \( i \in \{1, \ldots, N\} \).

There exists a unique element \([13, 18]\) \( \mathcal{R} \in B_+ \otimes B_- \) called the universal R-matrix which satisfies the following relations

\[
\Delta'(a) \mathcal{R} = \mathcal{R} \Delta(a) \quad \text{for} \quad \forall a \in U_q(\hat{g}(N)),
\]
\[
(\Delta \otimes 1) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{23},
\]
\[
(1 \otimes \Delta) \mathcal{R} = \mathcal{R}_{13} \mathcal{R}_{12}
\]

where \( \mathcal{R}_{12} = \mathcal{R} \otimes 1, \mathcal{R}_{23} = 1 \otimes \mathcal{R}, \mathcal{R}_{13} = (\sigma \otimes 1) \mathcal{R}_{23} \). The Yang–Baxter equation

\[
\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12}
\]

is a corollary of these relations (2.4). The universal R-matrix can be written in the form

\[
\mathcal{R} = \mathcal{R} \otimes q^{\sum_{i=1}^{N} k_i \otimes k_i}.
\]

Here \( \mathcal{R} \) is the reduced universal R-matrix, which is a series in \( e_i \otimes 1 \) and \( 1 \otimes f_j \) and does not contain Cartan elements.

2.2. The quantum finite algebra \( U_q(\hat{g}(N)) \)

The finite quantum algebra \( U_q(\hat{g}(N)) \) is generated by the generators \( \{e_{i,j+1}, e_{i+1,j}\}_{i=1}^{N-1} \) and \( \{e_0\}_{i=1}^{N} \) obeying the following defining relations:

\[
[e_i, e_j] = 0, \quad [e_i, e_{j+1}] = (\delta_{ij} - \delta_{i+1,j})e_{j+1}, \quad [e_i, e_{j-1}] = (\delta_{ij} - \delta_{i+1,j})e_{j-1},
\]
\[
[e_{i,j+1}, e_{i+1,j+1}] = \delta_{ij} \frac{q^{s_{i,j} - s_{i+1,j+1}} - q^{-s_{i+1,j+1}}}{q - q^{-1}},
\]
\[
[e_{i,j+1}, e_{i,j+1}] = [e_{i+1,j}, e_{i+1,j}] = 0 \quad \text{for} \quad |i - j| \geq 2,
\]
\[
[e_{i,j+1}, [e_{i+1,j}, e_{i+1,j}]_{q^{-1}}] = [e_{i+1,j}, [e_{i+1,j}, e_{i+1,j}]_{q^{-1}}] = 0 \quad \text{for} \quad |i - j| = 1.
\]

Note that \( c = e_{11} + e_{22} + \cdots + e_{NN} \) is a central element of the algebra. We will also use elements defined recursively by

\[3\] We will use similar notation for the L-operators to indicate the space on which they non-trivially act.
\[ e_{ij} = [e_{ik}, e_{kj}]_q \quad \text{for} \quad i > k > j, \]
\[ e_{ij} = [e_{ik}, e_{kj}]_{q^{-1}} \quad \text{for} \quad i < k < j, \]  \hfill (2.8)
\[ \tau_{ij} = [\tau_{ik}, \tau_{kj}]_{q^{-1}} \quad \text{for} \quad i > k > j, \]
\[ \tau_{ij} = [\tau_{ik}, \tau_{kj}]_q \quad \text{for} \quad i < k < j, \]  \hfill (2.9)

where \( \tau_{i+1,i} = e_{i+1,i}, \) \( \tau_{i+1,i} = e_{i+1,i}. \) We summarize relations among these elements in the appendix.

There is an evaluation map \( \text{ev}_x : U_q(\hat{\mathfrak{g}}(N)) \mapsto U_q(\mathfrak{gl}(N)); \)
\[ e_N \mapsto x^{\delta_{ii}} q^{-c_{ii}} e_N q^{c_{ii}}, \]
\[ f_N \mapsto x^{-\delta_{ii}} q^{c_{ii}} e_N q^{-c_{ii}}, \]
\[ e_i \mapsto x^i e_{i+1}, \quad f_i \mapsto x^{-i} e_{i+1,i} \quad \text{for} \quad 1 \leq i \leq N - 1, \]
\[ k_i \mapsto e_i \quad \text{for} \quad 1 \leq i \leq N, \]  \hfill (2.10)

where \( x \in \mathbb{C} \) is a spectral parameter; and the parameters \( s_i \in \mathbb{Z} \) fix the gradation of the algebra. We will also use another evaluation map \( \text{ev}_x : U_q(\hat{\mathfrak{g}}(N)) \mapsto U_q(\mathfrak{gl}(N)); \)
\[ e_N \mapsto x^{\delta_{ii}} q^{c_{ii}} \tau_{i+1,i} q^{-c_{ii}}, \]
\[ f_N \mapsto x^{-\delta_{ii}} q^{-c_{ii}} \tau_{i+1,i} q^{c_{ii}}, \]
\[ e_i \mapsto x^i e_{i+1}, \quad f_i \mapsto x^{-i} e_{i+1,i} \quad \text{for} \quad 1 \leq i \leq N - 1, \]
\[ k_i \mapsto e_i \quad \text{for} \quad 1 \leq i \leq N. \]  \hfill (2.11)

### 2.3. Representations of quantum algebras

Let \( \pi \) be the fundamental representation of \( U_q(\mathfrak{gl}(N)), \) and \( E_{ij} \) be a \( N \times N \) matrix unit whose \((k,l)\)-element is \( \delta_{kl} \delta_{ij}. \) We chose the basis of the representation space so that \( \pi(e_{ij}) = \pi(\tau_{ij}) = E_{ij} \) holds. The composition \( \pi = \pi \circ \text{ev}_x = \pi \circ \text{ev}_x \) gives an evaluation representation\(^4\) of \( U_q(\mathfrak{gl}(N)). \)

The \( q \)-oscillator algebra is generated by the generators \( c_i, c_i^\dagger, n_i \) \((i \in \{1, 2, \ldots, N\})\) obeying the defining relations:
\[ [c_i, c_i^\dagger]_q = \delta_{ii} q^{-n_i}, \quad [c_i, c_j^\dagger]_q = \delta_{ij} q^{n_j}, \]
\[ [n_i, c_i] = -\delta_{ii} c_i, \quad [n_i, c_i^\dagger] = \delta_{ii} c_i^\dagger, \quad [n_i, n_j] = [c_i, c_j] = [c_i^\dagger, c_j^\dagger] = 0. \]  \hfill (2.12)

A Fock space is given by the actions of the generators on the vacuum vector defined by \( c_i |0\rangle = n_i |0\rangle = 0. \) A highest weight representation of \( U_q(\mathfrak{gl}(N)) \) with the highest weight \( (m, 0, \ldots, 0) \) and the highest weight vector \( |0\rangle \) \((e_{ii}|0\rangle = m \delta_{ii} |0\rangle, \) \( e_{i,i+1}|0\rangle = 0, \) \( 1 \leq i \leq N, 1 \leq j \leq N - 1 \) can be realized in terms of the \( q \)-oscillator algebra (see \( q \)-analogue of the Holstein–Primakoff realization \([19, 20]\)):

\(^4\) Evaluation representations based the map \( \text{ev}_x \) do not necessary coincide with the ones based on the map \( \text{ev}_x, \) for more general representations.
\[ e_{1i} = m - n_2 - \cdots - n_N, \]
\[ e_{ik} = n_k, \quad \text{for} \quad 2 \leq k \leq N, \]
\[ e_{ij} = c_i q^{-\sum_{k=i}^{i-1} n_k}, \quad \tau_{ij} = c_i q^{-\sum_{k=2}^{i-1} n_k} \quad \text{for} \quad 2 \leq j \leq N, \]
\[ e_{il} = c_i^l \left[ m - \sum_{k=2}^{N} n_k \right] q^{-\sum_{k=2}^{l-1} n_k}, \quad \tau_{il} = c_i^l \left[ m - \sum_{k=2}^{N} n_k \right] q^{-\sum_{k=2}^{l-1} n_k} \quad \text{for} \quad 2 \leq i \leq N, \]
\[ e_{jq} = c_i^l q^{-\sum_{k=1}^{j-1} n_k}, \quad \tau_{jq} = c_i^l q^{-\sum_{k=1}^{j-1} n_k} \quad \text{for} \quad 2 \leq i < j \leq N, \]
\[ e_{jq} = c_i^l q^{-\sum_{k=1}^{j-1} n_k}, \quad \tau_{jq} = c_i^l q^{-\sum_{k=1}^{j-1} n_k} \quad \text{for} \quad 2 \leq j < i \leq N, \] (2.13)

where \([x]_q = (q^x - q^{-x})/(q - q^{-1})\). This representation is infinite dimensional and has an invariant subspace if \(m\) is a non-negative integer. Factoring out the invariant subspace, one can obtain the \(m\)th symmetric tensor representation. The \(K\)-operators in section 3 are valid at least for this \(q\)-oscillator realization. We also use this \(q\)-oscillator realization to check the commutation relations for underlying symmetry in section 4.

### 2.4. L-operators

The so-called L-operators are images of the universal R-matrix, which are given by \(L(xy^{-1}) = \phi(xy^{-1})(e_{ij} \otimes \pi_j)R\), \(\Xi(xy^{-1}) = \phi(xy^{-1})(\pi_i \otimes \pi_j)R_{21}\), where \(\phi(xy^{-1})\) and \(\phi(xy^{-1})\) are overall factors. They are solutions of the intertwining relations following from (2.4):

\[ (\pi_i \otimes \pi_j) \Delta'(a) L(xy^{-1}) = L(xy^{-1}) ((\pi_i \otimes \pi_j) \Delta(a)), \] (2.14)
\[ (\pi_i \otimes \pi_j) \Delta(a) \Xi(xy^{-1}) = \Xi(xy^{-1}) ((\pi_i \otimes \pi_j) \Delta'(a)) \quad \forall a \in U_q(gl(N)). \] (2.15)

One can solve\(^5\) these to get (see [15, 21])

\[ L(xy^{-1}) = \sum_{j=1}^{N} \left\{ (xy^{-1})^{-\xi_k + \xi_j} L_{kj}^+ - (xy^{-1})^{-\xi_k + \xi_j} L_{kj}^- \right\} \otimes E_{kj}, \] (2.16)
\[ \Xi(xy^{-1}) = \sum_{j=1}^{N} \left\{ - (xy^{-1})^{\xi_k - \xi_j} T_{kj}^+ + (xy^{-1})^{\xi_k - \xi_j} T_{kj}^- \right\} \otimes E_{kj}, \] (2.17)

where \(\xi_k = s_k + s_{k+1} + \cdots + s_N, \ s = \xi_1\), and the coefficients are related to the generators of \(U_q(gl(N))\) as

\[ L_{kj}^+ = q^{\epsilon_k}, \quad L_{kj}^- = q^{-\epsilon_k}, \]
\[ L_{kj}^+ = (q - q^{-1})e_{jk} q^{\epsilon_j}, \quad L_{kj}^- = 0 \quad \text{for} \quad i > j, \]
\[ L_{kj}^- = -(q - q^{-1})q^{-\epsilon_k} e_{jk}, \quad L_{kj}^+ = 0 \quad \text{for} \quad i < j. \] (2.18)

\(^5\)Taking note on the expression (2.6), we adopt the solutions which satisfy \(L(0) = L(\infty) = \sum_{i=1}^{N} q^{\epsilon_i} \otimes E_i\) for the case \(n_k > 0\) (for all \(k \in \{1, 2, \ldots, N\}\)).
\[ T^+_{ij} = q^{-\epsilon_i}, \quad T^-_{ij} = q^{\epsilon_j}, \]
\[ T^+_{ij} = -(q - q^{-1})q^{-\epsilon_i}e_{ij}, \quad T^-_{ij} = 0 \quad \text{for} \quad i > j, \quad (2.19) \]
\[ T^+_{ij} = (q - q^{-1})e_{ji}q^{\epsilon_j}, \quad T^-_{ij} = 0 \quad \text{for} \quad i < j. \]

We remark that the second L-operator (2.17) follows from the first one (2.16) by the transformation
\[ e_{ij} \mapsto e_{N+1-i,N+1-j}, \quad E_{ij} \mapsto E_{N+1-i,N+1-j} \quad \text{for} \quad 1 \leq i, j \leq N, \]
\[ s_l \mapsto -s_{N-l} \quad \text{for} \quad 1 \leq l \leq N - 1, \quad (2.20) \]
\[ s_N \mapsto -s_N, \]
which is composition of automorphisms of \( U_q(gl(N)) \) and \( U_q(gl(N)) \). In fact, the intertwining relation (2.15) follows from (2.14) by the same transformation (2.20).

Evaluating the L-operators for the fundamental representation\(^6\), we obtain R-matrices [4],
\[
R(x) = (\pi \otimes 1) L(x) = \sum_{i=1}^{N} (q - x^i q^{-1}) E_{ii} \otimes E_{ii} + \sum_{i \neq j} (1 - x^i) E_{ii} \otimes E_{jj}
+ (q - q^{-1}) \sum_{i<j} x^{i-j} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{i>j} x^{i-j} E_{ij} \otimes E_{ji},
(2.21)\]
\[
\overline{R}(x) = (\pi \otimes 1) \overline{L}(x) = \sum_{i=1}^{N} (q - x^{-i} q^{-1}) E_{ii} \otimes E_{ii} + \sum_{i \neq j} (1 - x^{-i}) E_{ii} \otimes E_{jj}
+ (q - q^{-1}) \sum_{i<j} x^{i-j} E_{ij} \otimes E_{ji} + (q - q^{-1}) \sum_{i>j} x^{i-j} E_{ij} \otimes E_{ji}.
(2.22)\]

### 3. The reflection equation and its solutions

In this section, we will derive intertwining relations from the reflection equation associated with the L-operators, and obtain K-operators in terms of the Cartan elements of (a quotient of) \( U_q(gl(N)) \).

We start from the following form of the reflection equation for the R-matrices (2.21) and (2.22):
\[
R_{12} \left( \frac{x}{\lambda} \right) K_1(x) \overline{R}_{12} \left( \lambda y \right) K_2(y) = K_2(y) R_{12} \left( \frac{1}{\lambda y} \right) K_1(x) \overline{R}_{12} \left( \frac{x}{\lambda} \right).
(3.1)\]
It is known that this equation allows a \( N \times N \) diagonal matrix solution (K-matrix) [3]. In our convention, it reads
\[
K(x) = \sum_{k=1}^{a} x^{2(i-\xi_i)} (\epsilon_- + \epsilon_+ x^i) E_{kk} + \sum_{k=a+1}^{N} x^{2(i-\xi_i)} (\epsilon_- + \epsilon_+ x^i) E_{kk},
(3.2)\]

\(^6\) Up to an overall factor, (2.22) coincides with \((\pi_2 \otimes \pi_1)\overline{R}_{23}\) since (3.27) and \(c = 1\) hold true for the fundamental representation.
where \(a \in \{0, 1, \ldots, N\}\), \(e_\pm \in \mathbb{C}\). Now we would like to consider the reflection equation for the L-operators (2.16) and (2.17):

\[
L_{12} \left( \frac{x}{y} \right) K_1(x) L_{12} \left( \frac{1}{xy} \right) K_2(y) = K_2(y) L_{12} \left( \frac{1}{xy} \right) K_1(x) L_{12} \left( \frac{x}{y} \right). \tag{3.3}
\]

The reflection equation (3.1) is the image of (3.3) for \(\pi \otimes 1\). Expanding (3.3) with respect to \(y\), we obtain

\[
\sum_{k=1}^{N} x^{-2\zeta_k} \left( L^+_n K(x) L^-_{ij} - L^-_n K(x) L^+_{ij} \right) = 0 \quad \text{for } i, j \leq a \text{ or } i, j > a, \tag{3.4}
\]

\[
\sum_{k=1}^{N} x^{-2\zeta_k} \left\{ \left( x L^+_n K(x) L^-_{sj} + x^{-s} L^-_n K(x) L^+_{sj} \right) \epsilon_+ + L^+_n K(x) L^-_{sj} \epsilon_- \right\} = 0 \quad \text{for } i \leq a < j, \tag{3.5}
\]

\[
\sum_{k=1}^{N} x^{-2\zeta_k} \left\{ \left( x L^+_n K(x) L^-_{sj} + x^{-s} L^-_n K(x) L^+_{sj} \right) \epsilon_+ + L^+_n K(x) L^-_{sj} \epsilon_- \right\} = 0 \quad \text{for } j \leq a < i. \tag{3.6}
\]

One can rewrite these in terms of \(e_\eta\) and \(\pi_\eta\):

\[
q^{2\epsilon_\eta} K(x) = K(x) q^{2\epsilon_\eta}, \tag{3.7}
\]

\[
x^{-2\zeta} e_{jk} K(x) - (q - q^{-1}) \sum_{k=j+1}^{i-1} x^{-2\zeta} e_{jk} K(x) \pi_{jk} - x^{-2\zeta} K(x) \pi_{jk} = 0 \quad \text{for } j < i \leq a \text{ or } a < j < i, \tag{3.8}
\]

\[
x^{-2\zeta} e_{jk} K(x) + (q - q^{-1}) \sum_{k=i+1}^{j-1} x^{-2\zeta} e_{jk} K(x) \pi_{jk} - x^{-2\zeta} K(x) \pi_{jk} = 0 \quad \text{for } i < j \leq a \text{ or } a < i < j, \tag{3.9}
\]

\[
x^{-2\zeta} e_{jk}(e_+ x^{-s} q^{-2\zeta_0} + e_-) K(x) - (q - q^{-1}) \left( e_+ x^s \sum_{k=1}^{j-1} x^{-2\zeta} q^{\epsilon_\eta} e_{jk} q^{\epsilon_{\eta}} K(x) \pi_{jk} \right.

+ e_+ x^{-s} \sum_{k=j+1}^{N} x^{-2\zeta} e_{jk} K(x) q^{-\epsilon_{\eta}} \pi_{jk} q^{\epsilon_{\eta}} - e_- \sum_{k=j+1}^{j-1} x^{-2\zeta} e_{jk} K(x) \pi_{jk} \left. \right) = 0 \quad \text{for } i \leq a < j. \tag{3.10}
\]

\textsuperscript{7} The commutation relations for the Cartan elements from (3.4) are of the form \(q^{\epsilon_\eta} K(x) q^{-\epsilon_{\eta}} = q^{-\epsilon_{\eta}} K(x) q^{\epsilon_\eta}\), from which \(e_\eta K(x) = K(x) e_\eta\) are deduced (under expansion of \(e_\eta = \log q^{\epsilon_{\eta}} / \log q\)). We have used these to simplify (3.8)–(3.11).
\[ x^{-2\xi} e_{ij} (\epsilon_+ x^q 2^{q^0} + \epsilon_-) K(x) + (q - q^{-1}) \left( \epsilon_+ x^q \sum_{k=1}^{j-1} x^{-2\xi} e_k q^k \bar{K}(x) \bar{q}^k \right) \]
\[ + \epsilon_+ x^{-\xi} \sum_{k=i+1}^{N} x^{-2\xi} q^{-e_k} e_k K(x) q^{-e_k} \bar{q}_k - \epsilon_- \sum_{k=j+1}^{i-1} x^{-2\xi} \epsilon_k K(x) \bar{q}_k \]
\[ - x^{-2\xi} K(x) \bar{q}_{ji} (\epsilon_+ x^{-q} q^{-2e_{ji}} + \epsilon_-) = 0 \quad \text{for} \quad j \leq a < i. \] (3.11)

For \( i \) and \( j \) satisfying \( |i - j| = 1 \), the relations (3.8) and (3.9) reduce to
\[ x^{-2\xi} e_{ji+1} K(x) = x^{-2\xi} e_{i+1} K(x) e_{ji+1}, \quad x^{-2\xi} e_{ji+1} K(x) = x^{-2\xi} K(x) e_{ji+1}, \]
for \( j \neq a. \) (3.12)

Taking note on (2.8) and (2.9) (and (3.7)), one can derive
\[ x^{-2\xi} \bar{q}_{ji} K(x) = x^{-2\xi} K(x) \bar{q}_{ji} \quad \text{for} \quad i, j \leq a \quad \text{or} \quad a < i, j, \] (3.13)
\[ x^{-2\xi} \bar{q}_{ji} K(x) = x^{-2\xi} K(x) e_{ji} \quad \text{for} \quad i, j \leq a \quad \text{or} \quad a < i, j \] (3.14)
from (3.12). Then the relations (3.8) and (3.9) boil down to the relations
\[ e_{ji} - (q - q^{-1}) \sum_{k=i+1}^{j-1} e_k \bar{q}_k - \bar{q}_{ji} = 0 \quad \text{for} \quad j < i. \] (3.15)
\[ e_{ji} + (q - q^{-1}) \sum_{k=i+1}^{j-1} e_k \bar{q}_k - \bar{q}_{ji} = 0 \quad \text{for} \quad i < j. \] (3.16)

which are special cases of (A.2) and (A.3), under (3.13) and (3.14). Thus it suffices to consider (3.12) instead of (3.8) and (3.9). We further rewrite (3.10) and (3.11) under the relations (3.13) and (3.14), in the form of intertwining relations.

For \( i \leq a < j \), (3.10) reduces to
\[ x^{-2\xi} (\epsilon_+ x^q A_{ji} + \epsilon_- B_{ji}) K(x) = x^{-2\xi} K(x) (\epsilon_+ x^q C_{ji} + \epsilon_- D_{ji}), \]
\[ A_{ji} = e_{ji} q^{-2e_{ji}} - (q - q^{-1}) \sum_{k=j+1}^{N} e_k \bar{q}_k q^{-e_k + e_{ji} - 1}, \]
\[ B_{ji} = e_{ji} + (q - q^{-1}) \sum_{k=j+1}^{i-1} e_k \bar{q}_k, \]
\[ C_{ji} = \bar{q}_j q^{2e_{ji}} + (q - q^{-1}) \sum_{k=0}^{j-1} e_k \bar{q}_k q^{e_k + e_{ji} - 2}, \]
\[ D_{ji} = \bar{q}_{ji} - (q - q^{-1}) \sum_{k=i+1}^{a} e_k \bar{q}_k. \] (3.17)
and for $j \leq i < a$, (3.11) reduces to

$$x^{-2\xi}(\epsilon + x^a A_{ji} + \epsilon - B_{ji})K(x) = x^{-2\xi} K(x)(\epsilon + x^a C_{ji} + \epsilon - D_{ji}),$$

$$A_{ji} = e_j q^{2\varepsilon_j} + (q - q^{-1}) \sum_{k=1}^{a} e_k \bar{e}_k q^{\varepsilon_k + \varepsilon_j},$$

$$B_{ji} = e_j - (q - q^{-1}) \sum_{k=j+1}^{a} e_k \bar{e}_k,$$  \hspace{1cm} (3.18)

$$C_{ji} = \bar{e}_j q^{-2\varepsilon_j} - (q - q^{-1}) \sum_{k=j+1}^{N} e_k \bar{e}_k q^{-\varepsilon_k - \varepsilon_j + 2},$$

$$D_{ji} = \bar{e}_j + (q - q^{-1}) \sum_{k=a+1}^{j-1} e_k \bar{e}_k.$$  

We remark that the relation $B_{ji} = D_{ji}$ holds due to (A.2) and (A.3). We find that the following operators\(^8\) satisfy (3.7), (3.12), (3.17) and (3.18) for the $q$-oscillator representation (2.13), and thus solve the reflection equation (3.3).

$$K(x) = q^{2(\varepsilon - 1) \sum_{k=1}^{N} \epsilon_k x^2 \sum_{i=1}^{N} (\theta(1 \leq k \leq a) - \xi_i) \text{cn} \left( \frac{-\epsilon x^{-1} x^a q^{2 \sum_{k=i+1}^{a} \epsilon_k + 2}; q^{-2}}{-\epsilon x^{-1} x^a q^{2 \sum_{k=i}^{a} \epsilon_k}; q^{-2} \biggr) \right)} \right| > 1, \quad \epsilon_+ \neq 0, \hspace{1cm} (3.19)$$

$$K(x) = q^{2(\varepsilon - 1) \sum_{k=1}^{N} \epsilon_k x^2 \sum_{i=1}^{N} (\theta(1 \leq k \leq a) - \xi_i) \text{cn} \left( \frac{-\epsilon x^{-1} x^a q^{2 \sum_{k=i+1}^{a} \epsilon_k + 2}; q^{-2}}{-\epsilon x^{-1} x^a q^{2 \sum_{k=i}^{a} \epsilon_k}; q^{-2} \biggr) \right)} \right| \biggr| < 1, \quad \epsilon_+ \neq 0, \hspace{1cm} (3.20)$$

$$K(x) = x^{-2 \sum_{i=1}^{N} \xi_i e_i} \left( \frac{-\epsilon x^{-1} x^a q^{2 \sum_{k=i+1}^{a} \epsilon_k + 2}; q^{-2}}{-\epsilon x^{-1} x^a q^{2 \sum_{k=i}^{a} \epsilon_k}; q^{-2} \biggr) \right)} \right| \biggr| > 1, \quad \epsilon_- \neq 0, \hspace{1cm} (3.21)$$

$$K(x) = x^{-2 \sum_{i=1}^{N} \xi_i e_i} \left( \frac{-\epsilon x^{-1} x^a q^{2 \sum_{k=i+1}^{a} \epsilon_k + 2}; q^{-2}}{-\epsilon x^{-1} x^a q^{2 \sum_{k=i}^{a} \epsilon_k}; q^{-2} \biggr) \right)} \right| \biggr| < 1, \quad \epsilon_- \neq 0, \hspace{1cm} (3.22)$$

where $c = \sum_{k=1}^{N} e_k$, $\theta$(True) = 1, $\theta$(False) = 0, and $(x; q)_{\infty} = \prod_{j=0}^{\infty} (1 - x q^j)$. One can directly check that (3.19)–(3.22) satisfy (3.7) and (3.12) for the generic generators of $U_q(gl(N))$. One can also show that the operators (3.19)–(3.22) satisfy the following relations

$$K(x) E_{ji} = x^{2(\xi_i - \xi_j)} E_{ji} K(x) \left( \frac{\epsilon_+ x^a q^{2 \sum_{k=i+1}^{a} \epsilon_k + \epsilon_-}}{\epsilon_+ x^{-1} x^a q^{2 \sum_{k=i}^{a} \epsilon_k + 2} + \epsilon_-} \right) \text{ for } j \leq a < i, \hspace{1cm} (3.23)$$

\(^8\)In 2016, we were informed by Belliard that he found a (non-diagonal) solution for the rational case $Y(sl(2))$.  

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\[ K(x)E_{ij} = x^2(\xi-\xi)E_{ij}K(x) \left( \frac{e^+\cdot x^{-j}q^{-2}\sum_{\alpha}e_{\alpha} + e_-}{e^+\cdot x^{-j}q^{-2}\sum_{\alpha}e_{\alpha} + e_-} \right) \text{ for } i \leq a < j, \]

(3.24)

where \( E_{ij} \) is any operator obeying \( q^aE_{ij}q^{-a} = q^\delta_{ij}E_{ij} \). Thus (3.17) and (3.18) reduce to

\[ B_{ij} = A_iq^2\sum_{\alpha}e_{\alpha} = C_iq^{-2}\sum_{\alpha}e_{\alpha} + 2 \text{ for } i \leq a < j, \]

(3.25)

\[ B_{ij} = A_iq^{-2}\sum_{\alpha}e_{\alpha} = C_iq^2\sum_{\alpha}e_{\alpha} - 2 \text{ for } j \leq a < i. \]

(3.26)

These relations (3.25) and (3.26) give a constraint on the class of representations of \( U_q(sl(N)) \). In other words, we have to consider the quotient of \( U_q(sl(N)) \) by the relations (3.25) and (3.26). Moreover, we find that the first evaluation map (2.10) is almost\(^9\) equivalent to the second one (2.11) under these:

\[ \text{ev}_x(e\xi) = q^{-2(c-1)}\text{ev}_x(e\xi), \quad \text{ev}_x(f\xi) = q^{2(c-1)}\text{ev}_x(f\xi) \quad \text{(under (3.25) and (3.26))}. \]

(3.27)

One can check that (2.13) satisfies (3.25) and (3.26). In addition, the fundamental representation of \( U_q(sl(N)) \), which is a quotient of (2.13) at \( m = 1 \), also fulfills (3.25) and (3.26). Then we recover (3.2) from (3.19):

\[ \pi(K(x)) = \frac{x^{-j}\left( -\frac{e^+}{e_+}x^{-j}q^{-2}; q^{-2}\right)_{\infty}}{x^+\left( -\frac{e^+}{e_+}x^{i-j}q^{-2}; q^{-2}\right)_{\infty}} \equiv K(x). \]

(3.28)

It is important to note that the relations (3.25) and (3.26) become trivial for \( N = 2 \) case. This means that the K-operators (3.19)–(3.22) written in terms of the generic Cartan elements solve the reflection equation (3.3) without any constraint on the representations for \( U_q(sl(2)) \) case. In fact, (3.19) for \( N = 2 \) and \( a = 1 \) reproduces\(^10\) the K-operator proposed in [5].

Let us consider a product of the L-operators

\[ L(x)q^{2\sum_{i=1}^{N}\xi_\alpha \overline{\xi}_{\alpha}} \left( \sum_{\alpha}e_{\alpha} \right) q^{-2\sum_{i=1}^{N}\xi_\alpha \overline{\xi}_{\alpha}} = \sum_{i=1}^{N}(-x^i - x^{-i} + G_i) \otimes E_{ii} + (q - q^{-1}) \sum_{i,j} (C_{ij} - A_iq^{2(c-1)})q^{-e_{\alpha} - e_{\alpha} + 1} \otimes E_{ij} \]

\[ + (q - q^{-1}) \sum_{i>j} (A_{ij} - C_iq^{2(c-1)}) \otimes E_{ij}, \]

(3.29)

where \( G_i = G_i^{(+)} + G_i^{(-)} \) is defined by

\[ G_i^{(+)} = q^{2e_{\alpha}} + (q - q^{-1})^2 \sum_{k=1}^{i+1} e_{\alpha} e_{\alpha} q^{-e_{\alpha} + 1}, \]

(3.30)

\[ G_i^{(-)} = q^{2(c-1)} \left( q^{-2e_{\alpha}} + (q - q^{-1})^2 \sum_{k=1}^{i} e_{\alpha} e_{\alpha} q^{-e_{\alpha} - e_{\alpha} + 1} \right). \]

(3.31)

\(^9\)This comes from (3.25) for \( (i,j) = (1,N) \) and (3.26) for \( (i,j) = (N,1) \) under \( a \in \{ 1, 2, \ldots, N - 1 \} \). The factor \( q^{T(2c-1)} \) in the right hand side can be eliminated if one considers composition of the automorphism \( e_i \mapsto q^{-2i}e_i, f_i \mapsto q^{2i}e_i, \) if, \( k \mapsto k \) of \( U_q(sl(N)) \) and the evaluation map \( \text{ev}_x \).

\(^10\) \( e^{s\cdot q^{2(c-1)}}K(s)q^{-1} \) for \( N = 2 \) coincides with equation (4.9) in [5].
Here $A_{ji}$ and $C_{ji}$ for $i < j$ (resp. $i > j$) are the ones in (3.17) (resp. (3.18)) (dependence on the parameter $a$ is irrelevant). In deriving (3.29), we have used (A.2) and (A.3). Note that (3.29) becomes diagonal under the relations
\[ A_{ji}q^{2(c-1)} = C_{ji} \quad \text{for} \quad i < j, \quad \text{and} \quad A_{ji} = C_{ji}q^{2(c-1)} \quad \text{for} \quad j < i. \] (3.32)

We remark that a part of the relations (3.25) and (3.26) automatically follows from (3.32). For $N = 2$, the relation (3.32) becomes trivial, and the diagonal elements $G_1$ and $G_2$ reduce to a Casimir element of $U_q(\mathfrak{gl}(2))$ (see equation (3.5) in [5]). On the other hand, for $N \geq 3$, each diagonal element is not a Casimir element, but rather twisted sums
\[ \sum_{k=1}^{N} G_k^{(+)} q^{2k}, \quad \sum_{k=1}^{N} G_k^{(-)} q^{2k} \quad \text{and} \quad \sum_{k=1}^{N} G_k^q q^{2k} \] (see equation (41) in [25]). However, there are cases where simplifications occur for particular representations. In fact, one can check that the $q$-oscillator representation (2.13) satisfies the condition (3.32), and $G_i = q^{2m_i} + q^{-2}$ holds for any $i$. This is one of the desirable properties (a sort of unitarity condition) for construction of mutually commuting transfer matrices.

4. Augmented $q$-Onsager algebra like symmetry

In this section, we will reconsider the intertwining relations in the previous section and point out their connection to the generators of $U_q(\mathfrak{gl}(N))$. We will also mention our observation on an underlying symmetry behind them.

We find that the intertwining relations (3.13), (3.14), (3.17) and (3.18) can be written in terms of the generators of $U_q(\mathfrak{gl}(N))$:
\[ e^{v_{x-1}(Z_{ji})}K(x) = K(x)e^{v_x(Z_{ji})}, \] (4.1)
where $Z_{ji}$ is defined as follows:

For $1 \leq i = j \leq N$,
\[ Z_{ji} = k_i. \] (4.2)

For $1 \leq j < i \leq a$ or $a + 1 \leq j < i \leq N$,
\[ Z_{ji} = e_{[j,i-1]}. \] (4.3)

For $1 \leq i < j \leq a$ or $a + 1 \leq i < j \leq N$,
\[ Z_{ji} = f_{[i,j-1]}. \] (4.4)

For $j \leq a < i$,
\[ Z_{ji} = \epsilon_+ Z_{ji}^+ + \epsilon_- Z_{ji}^- , \]
\[ Z_{ji}^+ = \begin{cases} [f_{[1,j-1]}, f_{[i,N-1]}]q^{h_i-k_j+1} & \text{for} \quad 2 \leq j \leq a < i \leq N - 1 \\ [f_{[N,i,N-1]}]q^{h_i-k_j+1} & \text{for} \quad 1 = j \leq a < i \leq N - 1 \quad \text{(4.5)} \\ f_{[N,i]}q^{h_i-k_j+1} & \text{for} \quad 2 \leq j \leq a < i = N \\ f_{[i]}q^{h_i-k_j+1} & \text{for} \quad 1 = j \leq a < i = N, \end{cases} \]

\[ \sum_{k=1}^{N} G_k^q q^{2k} \] is a part of the twisted trace of (3.29) (by $1 \otimes \pi(q^{2\sum_{k=1}^{N} k^2})$).
We have observed the following commutation relations among generators:

\[ [e_{[j,a]}, e_{[j+1,b]}]_{q^{-1}} = 0 \quad \text{for} \quad 1 \leq j < a < a + 1 < i \leq N \]

\[ [e_{[j,a]}, e_{[j+1,b]}]_{q} = 0 \quad \text{for} \quad 1 \leq j < a < a + 1 = i \leq N \]

\[ e_{[j,a-1]} = 0 \quad \text{for} \quad 1 \leq j = a < i \leq N. \]

For \( i \leq a < j \):

\[ Z_{ji} = \epsilon_{+} Z_{ji}^{+} + \epsilon_{-} Z_{ji}^{-}, \]

\[ Z_{ji}^{+} = \begin{cases} [\tau_{[j,N-1]}, [e_{N}, e_{[1,j-1]}]_{q^{-1}}]_{q} & \text{for} \quad 2 \leq i \leq a < j \leq N - 1 \\ [\tau_{[j,N-1]}, e_{N}]_{q} & \text{for} \quad 1 = i \leq a < j \leq N - 1 \\ [e_{N}, e_{[1,j-1]}]_{q^{-1}} & \text{for} \quad 2 \leq i \leq a < j = N \\ e_{N} & \text{for} \quad 1 = i \leq a < j = N, \end{cases} \]

\[ Z_{ji}^{-} = \begin{cases} [f_{[a+1,j-1]}, f_{[a]}]_{q^{-1}} q^{k-j+1} & \text{for} \quad 1 \leq i \leq a < a + 1 < j \leq N \\ [f_{[a]}]_{q^{-1}} q^{k-j+1} & \text{for} \quad 1 \leq i \leq a < a + 1 = j \leq N \end{cases} \]

Here we use the following notation for root vectors. For \( i < j \), we define:

\[ e_{[i,a]} = [e_{i}, [e_{i+1}, \ldots, [e_{j-2}, [e_{j-1}, e_{j}]_{q^{-1}}]_{q^{-1}}]_{q^{-1}}]_{q^{-1}}, \]

\[ \tau_{[i,a]} = [e_{i}, [e_{i+1}, \ldots, [e_{j-2}, [e_{j-1}, e_{j}]_{q}]_{q}]_{q}]_{q}, \]

\[ f_{[i,a]} = [f_{i}, [f_{i-1}, \ldots, [f_{j-2}, [f_{j-1}, f_{j}]_{q}]_{q}]_{q}]_{q}, \]

\[ \hat{f}_{[i,a]} = [f_{i}, [f_{i-1}, \ldots, [f_{j-2}, [f_{j-1}, f_{j}]_{q^{-1}}]_{q^{-1}}]_{q^{-1}}]_{q}, \]

and \( e_{ij} = \tau_{ij} = e_{i}, f_{ij} = \hat{f}_{ij} = f_{i} \). We remark that the third case in (4.6) (resp. (4.8)) is a special case of the first or the second ones. In deriving these, we have used (A.2)–(A.10).

We introduce generators composed of Cartan elements

\[ \tau_{ji}^{+} = \epsilon_{+} q^{a_{i} \sum_{k=1}^{j} k - a_{i} \sum_{k=1}^{j-1} k} \quad \tau_{ji}^{-} = \epsilon_{-} q^{\sum_{k=1}^{j} k - \sum_{k=1}^{j-1} k} \quad \text{for} \quad 1 \leq j \leq a < i \leq N, \]

and use notation \( Z_{a i, a} = \epsilon_{a} \), \( Z_{a 1} = \epsilon_{a} \), \( Z_{a+1,a} = f_{a} \), \( Z_{1, N} = f_{N} \), \( \tau_{a}^{+} = \tau_{a+1,a}^{+} \), \( \tau_{a}^{-} = \tau_{a+1,a}^{-} \), \( \hat{\tau}_{a}^{+} = \hat{\tau}_{a+1,a}^{+} \), \( \hat{\tau}_{a}^{-} = \hat{\tau}_{a+1,a}^{-} \). We have observed the following commutation relations among generators for the \( q \)-oscillator representation of \( U_{q}(gl(N)) \) under the evaluation map (2.10) and (2.13), in addition to the relations (2.1) restricted to the generators \( \{e_{i}, f_{i}\}_{i \neq a, N} \) and \( \{k_{i}\}_{i=1}^{N} \):

\[ [\tau_{ji}^{+}, \tau_{jr}^{-}] = [\tau_{ji}^{-}, \tau_{jr}^{+}] = [\tau_{ji}^{+}, k_{i}] = 0 \quad \text{for} \quad 1 \leq j, r \leq a < i \leq s \leq N, \quad 1 \leq l \leq N; \]

\[ [k_{i}, Z_{ji}] = ([\delta_{li} - \delta_{ij}]Z_{ji}) \quad \text{for} \quad 1 \leq l, i, j \leq N; \]
\[ \mathbf{t}_{ij}^{+} Z_{ij} = q^{\theta(1 \leq r \leq 0) - \theta(1 \leq r \leq N) - \theta(0 \leq r \leq 1) + \theta(i \leq r \leq N)} Z_{ij} \mathbf{t}_{ij}^{+}, \]
\[ \mathbf{t}_{ij}^{-} Z_{ij} = q^{-\theta(j \leq r \leq 0) + \theta(a+1 \leq r \leq j) + \theta(j \leq r \leq a) - \theta(a+1 \leq r \leq i)} Z_{ij} \mathbf{t}_{ij}^{-} \]
for \( 1 \leq j \leq a < i \leq N, \quad 1 \leq r, s \leq N; \)
\[ [\epsilon_{a}, [\epsilon_{a}, [\epsilon_{a}, f_{1}]_{q}]_{q}]_{q^{-1}} = \rho_{a}([\epsilon_{a}]^{2} - ([\epsilon_{a}]^{2})^{2}) \epsilon_{a}, \]
\[ [\epsilon_{N}, [\epsilon_{N}, [\epsilon_{N}, f_{1}]_{q}]_{q}]_{q^{-1}} = \rho_{N}([\epsilon_{N}]^{2} - ([\epsilon_{N}]^{2})^{2}) \epsilon_{N}, \]
\[ [f_{1}, [f_{1}, f_{1}]_{q}]_{q^{-1}} = \rho_{1}([f_{1}]^{2} - ([f_{1}]^{2})^{2}) f_{1}, \]
\[ [f_{N}, [f_{N}, [f_{N}, \epsilon_{N}]_{q}]_{q}]_{q^{-1}} = \rho_{N}([f_{N}]^{2} - ([f_{N}]^{2})^{2}) f_{N} \text{ for } 1 \leq a \leq N - 1; \]
\[ [\epsilon_{a}, [\epsilon_{a}, \epsilon_{N}]_{q}]_{q^{-2}} = [f_{a}, [f_{a}, f_{a}]_{q}]_{q^{-2}} = 0, \]
\[ [\epsilon_{N}, [\epsilon_{N}, \epsilon_{N}]_{q}]_{q^{-2}} = [f_{N}, [f_{N}, f_{N}]_{q}]_{q^{-2}} = 0, \]
\[ [f_{N}, \epsilon_{N}]_{q^{-2}} = [\epsilon_{N}, f_{N}]_{q^{-2}} = 0 \text{ for } 2 \leq a \leq N - 2; \]
\[ [\epsilon_{a-1}, [\epsilon_{a-1}, \epsilon_{a}]_{q}]_{q^{-1}} = [\epsilon_{a}, [\epsilon_{a}, \epsilon_{a-1}]_{q}]_{q^{-1}} = [f_{a-1}, [f_{a-1}, f_{a}]_{q}]_{q^{-1}} = [f_{a}, [f_{a}, f_{a-1}]_{q}]_{q^{-1}} = 0, \]
\[ [\epsilon_{1}, [\epsilon_{1}, \epsilon_{N}]_{q}]_{q^{-1}} = [\epsilon_{N}, [\epsilon_{N}, \epsilon_{1}]_{q}]_{q^{-1}} = [f_{1}, [f_{1}, f_{N}]_{q}]_{q^{-1}} = [f_{N}, [f_{N}, f_{1}]_{q}]_{q^{-1}} = 0, \]
\[ [\epsilon_{a-1}, f_{a}]_{q^{-1}} = [\epsilon_{1}, f_{N}]_{q^{-1}} = 0 \text{ for } 2 \leq a \leq N - 1; \]
\[ [\epsilon_{a+1}, [\epsilon_{a+1}, \epsilon_{a}]_{q}]_{q^{-1}} = [\epsilon_{a}, [\epsilon_{a}, \epsilon_{a+1}]_{q}]_{q^{-1}} = [f_{a+1}, [f_{a+1}, f_{a}]_{q}]_{q^{-1}} = [f_{a}, [f_{a}, f_{a+1}]_{q}]_{q^{-1}} = 0, \]
\[ [\epsilon_{N-1}, [\epsilon_{N-1}, \epsilon_{N}]_{q}]_{q^{-1}} = [\epsilon_{N}, [\epsilon_{N}, \epsilon_{N-1}]_{q}]_{q^{-1}} = 0, \]
\[ [f_{N-1}, [f_{N-1}, f_{N}]_{q}]_{q^{-1}} = [f_{N}, [f_{N}, f_{N-1}]_{q}]_{q^{-1}} = 0, \]
\[ [\epsilon_{a+1}, f_{a}]_{q^{-1}} = [\epsilon_{a-1}, f_{a}]_{q^{-1}} = 0 \text{ for } 1 \leq a \leq N - 2; \]
\[ [\epsilon_{i}, \epsilon_{a}] = [\epsilon_{i}, f_{a}] = [f_{i}, f_{a}] = 0 \text{ for } 1 \leq i \leq a - 2 \text{ or } a + 2 \leq i \leq N - 1; \]
\[ [\epsilon_{i}, \epsilon_{N}] = [\epsilon_{i}, f_{N}] = [f_{i}, f_{N}] = 0 \text{ for } 2 \leq i \leq a - 1 \text{ or } a + 1 \leq i \leq N - 2; \]
\[ [f_{i}, \epsilon_{i}] = [f_{i}, \epsilon_{N}] = 0 \text{ for } 1 \leq i \leq a - 1 \text{ or } a + 1 \leq i \leq N - 1; \]
\[ [e_{i-1}, Z_{i}]_{q} = Z_{i+1}, \quad [Z_{i}, f_{i-1}]_{q} = Z_{i+1} q^{-1+k_{-1}q^{-1}} \text{ for } 2 \leq i \leq a < j \leq N; \]
\[ [Z_{i}, e_{j}]_{q^{-1}} = Z_{i+1}, \quad [f_{i}, Z_{j}]_{q} = Z_{i+1} q^{k_{-1}-k_{i+1}} \text{ for } 1 \leq i \leq a < j \leq N - 1; \]
\[ [f_{1}, [f_{1}, [f_{1}, f_{1}]_{q}]_{q}]_{q^{-1}} = q^{-1} \rho_{1} ([f_{1}]^{2})^{2} f_{1} Z_{N}, \]
\[ [f_{N}, [f_{N}, [f_{N}, f_{1}]_{q}]_{q}]_{q^{-1}} = -q^{-1} \rho_{N} ([f_{N}]^{2})^{2} f_{N} Z_{N}, \]
\[ [e_{1}, [e_{1}, [e_{1}, f_{1}]_{q}]_{q}]_{q^{-1}} = -q^{2} \rho_{1} ([e_{1}]^{2})^{2} e_{1} Z_{N} 2 q^{k_{1}-k_{1}}, \]
\[ [e_{N}, [e_{N}, [e_{N}, f_{1}]_{q}]_{q}]_{q^{-1}} = q^{2} \rho_{N} ([e_{N}]^{2})^{2} e_{N} Z_{N} 2 q^{k_{N}-k_{N}} \text{ for } a = 1, \quad N > 2; \]
\[ [f_{N-1}, [f_{N-1}, f_N]]_q]_{q^{-1}} = -q^2 \rho [f_{N-1}]_q^3 f_{N-1} Z_{N,N-1} q^{2k}, \]
\[ [f_N, [f_N, [f_N, f_N]]_q]_{q^{-1}} = q \rho \hat{h}_N [f_N]_q^2 f_N Z_{N,N-1} q^{2k}, \]
\[ [e_{N-1}, [e_{N-1}, [e_{N-1}, e_N]]_q]_{q^{-1}} = \rho \rho N^{-1} [f_{N-1}]_q^3 e_{N-1} Z_{N,N-1} q^{k_1-k_{N-1}}, \]
\[ [e_N, [e_N, [e_N, e_N]]_q]_{q^{-1}} = -q^{-1} \rho \rho N^{-1} [f_{N-1}]_q^3 e_N Z_{N,N-1} q^{k_1-k_{N-1}} \]
for \( a = N - 1, \ N > 2, \) (4.24)

where \( \rho = (q^3 - q^{-3})(q^2 - q^{-2}), \ \hat{\rho} = \rho/(q - q^{-1}). \) We remark that the parameter \( \alpha \) is a fixed parameter in the diagonal K-matrix (3.2). The generators \( e_i \) and \( f_i \) are analogues \(^{14}\) of the generators \( e_i \) and \( f_i \) of the quantum affine algebra. In fact, they satisfy Serre like relations (see, (4.15)–(4.17)), \( Z_{ij} \) are analogue of higher root vectors, which are connected to the generators \( e_i \) and \( f_i \) of the quantum affine algebra through (4.22). The relations (4.14) are characteristic of this system, which look different from the relations of the quantum affine algebra; while their variants (4.23) and (4.24) exist only for \( N \geq 2 \) case. These commutation relations are valid without any restriction of the class of representations for \( N = 2 \) case, and define the so-called augmented q-Onsager algebra \[7, 8\], a realization of which in terms of the generators of \( U_q(sl(2)) \) is known in \[8\]. In contrast, not all of them hold true for generic representations for \( N \geq 3 \) case. Whether the elements \( Z_{ij} \) for \( N \geq 3 \) satisfy closed commutation relations on the level of the algebra remains to be clarified.

5. Rational limit

In this section, we consider the rational limit of some of the formulas in previous sections.

Let \( \epsilon_{\pm} = \mp q^{2p}, \ p = p_+ - p_-, \) and \( u \in \mathbb{C}. \) In order to take the rational limit of the L-operators (2.16) and (2.17), we must renormalize them by the factor \( (q - q^{-1})^{-1}. \) The factor is necessary to get finite non-zero results:

\[ L(u) = \lim_{q \to 1} (q - q^{-1})^{-1} L(q^{-2u}) = \sum_{ij} (su \delta_{ij} + e_{ij}) \otimes E_{ij}, \] (5.1)
\[ \overline{L}(u) = \lim_{q \to 1} (q - q^{-1})^{-1} \overline{L}(q^{-2u}) = \sum_{ij} (-su \delta_{ij} + e_{ij}) \otimes E_{ij}, \] (5.2)

where \( e_{ij} \) are generators of \( gl(N); \)

\[ [e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}. \] (5.3)

The limit of the K-matrix (3.2) can be taken similarly:

\[ K(u) = \lim_{q \to 1} (q - q^{-1})^{-1} K(q^{-2u}) = \sum_{k=1}^{a} (su + p) E_{kk} + \sum_{k=a+1}^{N} (-su + p) E_{kk}. \] (5.4)

As for the K-operators (3.19)–(3.22), we rewrite them in terms of a q-analogue of the gamma function (see for example, [22])

\(^{14}\) This analogy is not perfect since they are mixtures of \( e_i \) and \( f_i \), as can be seen from (4.5)–(4.8).
\[ \Gamma_q(x) = \frac{(q; q)_x}{(q^2; q)_x} (1 - q)^{1-x} \quad \text{for} \quad |q| < 1, \quad (5.5) \]

which reduces to the normal gamma function in the rational limit: \( \lim_{q \to 1} \Gamma_q(x) = \Gamma(x) \).

The limit \( K(u) = \lim_{|q| \to 1+0} K(q^{-2a})(1 - q^{-2})^{1-c+2wu} \) for (3.19) and the limit \( K(u) = \lim_{|q| \to 1-0} K(q^{-2a})(1 - q^{-2})^{1-c+2wu} \) for (3.22) give the same result:

\[ K(u) = \frac{\Gamma(-su + p + \sum_{k=1}^a e_{kk})}{\Gamma(su + p - 1 - \sum_{k=a+1}^N e_{kk})}. \quad (5.6) \]

The limit \( K(u) = \lim_{|q| \to 1-0} K(q^{-2a})(1 - q^{-2})^{1-c+2wu} \) for (3.20) and the limit \( K(u) = \lim_{|q| \to 1+0} K(q^{-2a})(1 - q^{-2})^{1-c+2wu} \) for (3.21) give the same result:

\[ K(u) = \frac{\Gamma(-su + p + \sum_{k=1}^N e_{kk})}{\Gamma(su + p - 1 - \sum_{k=1}^{a+1} e_{kk})}. \quad (5.7) \]

Then, after renormalizing the K-operator as above, the limit of the intertwining relations (3.13) and (3.14) can be taken straightforwardly:

\[ e_{ij} K(u) = K(u) e_{ji} \quad \text{for} \quad i, j \leq a \quad \text{or} \quad i, j \geq a + 1. \quad (5.8) \]

As for the intertwining relations (3.17) and (3.18), in addition to renormalizing the K-operator, one has to divide both sides of them by \((q - q^{-1})\) before taking the limit to get:

\[ \left( (su - p)e_{ij} - \sum_{k=a+1}^N e_{ik}e_{jk} \right) K(u) = K(u) \left( (-su + p)e_{ji} + \sum_{k=1}^a e_{ik}e_{jk} \right) \quad \text{for} \quad i \leq a < j, \quad (5.9) \]

\[ \left( (-su - p)e_{ij} + \sum_{k=1}^a e_{ik}e_{jk} \right) K(u) = K(u) \left( (su - p)e_{ji} - \sum_{k=a+1}^N e_{ik}e_{jk} \right) \quad \text{for} \quad j \leq a < i. \quad (5.10) \]

We find that (5.6) and (5.7) solve the relations (5.8)–(5.10), and thus the reflection equation

\[ L_{12} (v - u) K_1(u) L_{12} (u + v) K_2(v) = K_2(v) L_{12} (-u - v) K_1(u) L_{12} (u - v) \quad \text{for} \quad u, v \in \mathbb{C}; \quad (5.11) \]

which is the rational limit of (3.3), if the following conditions\(^{15}\) are satisfied.

\[ e_{ij} \sum_{k=a+1}^N e_{ik} = e_{ij} \sum_{k=a+1}^N e_{ik} - 1 = \sum_{k=1}^a e_{ik}e_{jk} \quad \text{for} \quad i \leq a < j, \quad (5.12) \]

\[ e_{ij} \sum_{k=1}^a e_{ik} = e_{ij} \sum_{k=1}^a e_{ik} - 1 = \sum_{k=a+1}^N e_{ik}e_{jk} \quad \text{for} \quad j \leq a < i. \quad (5.13) \]

\(^{15}\)These are sufficient conditions. If the parameter \( p \) (in addition to \( u \)) is interpreted as a free one, these are necessary conditions as well. On the other hand, (3.25) and (3.26) are necessary and sufficient conditions independent of \( \epsilon_{\pm} \).
The relations (5.12) and (5.13) correspond to the rational limit of (3.25) and (3.26), respectively. Then the rational limit of (2.13) satisfies (5.12) and (5.13). Note that the above relations reproduce (a part of) a kind of generalized rectangular condition \[16\] [23].

\[ \sum_{k=1}^{N} e_k e_k = \alpha e_k + \beta \delta_{ij}, \quad \alpha, \beta \in \mathbb{C}. \]  

(5.14)

In fact, sum of the first and the second relations in (5.12) or (5.13) gives

\[ e_\beta \left( \sum_{k=1}^{N} e_k e_k - 1 \right) = \sum_{k=1}^{N} e_k e_k \quad \text{for} \quad i \leq a < j \quad \text{or} \quad j \leq a < i. \]  

(5.15)

This corresponds to (5.14) for \( \beta = 0 \) since \( \alpha := \sum_{k=1}^{N} e_k e_k - 1 \) is a central element of \( gl(N) \).

6. Concluding remarks

We have derived intertwining relations (3.13), (3.14), (3.17) and (3.18) from the reflection equation (3.3) for L-operators associated with \( U_q(gl(N)) \) and have obtained the diagonal K-operators (3.19)–(3.22) in terms of the Cartan elements of a quotient of \( U_q(gl(N)) \) by the relations (3.25) and (3.26). The intertwining relations can be expressed in terms of generators of \( U_q(gl(N)) \) as in (4.1). In particular, the elements \( \{ Z_i \} \) (4.2)–(4.8) in the intertwining relations satisfy augmented \( q \)-Onsager algebra like commutation relations for the \( q \)-oscillator representation (2.13) under the evaluation map (2.10). It will be an interesting problem to investigate the algebra generated by \( \{ Z_i \} \) (with some constraint on generators of \( U_q(gl(N)) \), if any), which could be a version of higher rank generalization of the augmented \( q \)-Onsager algebra. The reflection equation (3.1) defines [2] the reflection equation algebra if one adds the third component in the K-matrix:

\[ R_{12} \left( \frac{y}{x} \right) K_{13}(x) \tilde{R}_{12} (xy) K_{23}(y) = K_{23}(y) \tilde{R}_{12} \left( \frac{1}{xy} \right) K_{13}(x) \tilde{R}_{12} \left( \frac{x}{y} \right), \]

\[ K_{13}(x) = \sum_{i,j=1}^{N} E_{ij} \otimes 1 \otimes k_j(x), \quad K_{23}(x) = \sum_{i,j=1}^{N} 1 \otimes E_{ij} \otimes k_i(x). \]  

(6.1)

where the elements \( \{ k_j(x) \} \) generate the algebra. In this context, it will be desirable to clarify the direct connection between \( \{ Z_i \} \) and a certain specialization of \( \{ k_j(x) \} \) for the diagonal K-operators.

In this paper, we have defined the L-operator \( \mathbf{L}_i(x) \) based on the second evaluation map (2.11). It will be worthwhile to define it based on the first one (2.10) as \( \mathbf{L}_i(xy^{-1}) = \phi(xy^{-1})(v_i \otimes \pi_y) \tilde{R}_{21} \) and investigate solutions of the reflection equation. The reason why we avoided \( \mathbf{L}_i(x) \) was that its expression based on the generators \( e_j \) or \( \pi_j \) might be involved for \( N \geq 3 \) case. The conditions (3.25) and (3.26) for the solutions of the reflection equation are the ones that the L-operator \( \mathbf{L}_i(x) \) coincides with \( \mathbf{L}_i(x) \) up to a fine tune by the central element (see the relation (3.27)). The technical difficulty on \( \mathbf{L}_i(x) \) may be resolved if we start from the FRT formulation of the algebra (based on Yang–Baxter relations) [24] and use the matrix elements of the L-operators as the generators of the algebra instead of \( e_j \) or \( \pi_j \).

\[16\] In case the representations are finite dimensional, this condition is satisfied by the representations labeled by rectangular Young diagrams.
Another important problem is construction of Baxter Q-operators for open boundary conditions in the light of [5, 6]. The K-operators obtained in this paper could be building blocks of Q-operators after taking limits on the parameter \(m\) of the \(q\)-oscillator representation, as was already discussed in [5] for \(N = 2\) case.

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Appendix. Relations for \(U_q(gl(N))\)

Here we review relations among the generators of \(U_q(gl(N))\), some of which are used in the main text.

\[
\begin{align*}
  e_{ab} &= [e_a, e_b]_q & \text{for} & & a > c > b, \\
  e_{ab} &= [e_a, e_b]_{q^{-1}} & \text{for} & & a < c < b, \\
  [e_{ab}, e_{ba}] &= q^{e_{ac} - e_{cb}} - q^{-e_{ac} + e_{cb}} & \text{for} & & a < b, \\
  [e_{dc}, e_{ba}] &= -(q - q^{-1})e_{da}e_{bc} & \text{for} & & b < d < a < c \\
 & & & \text{or} & & a < c < b < d, \\
  [e_{dc}, e_{ba}] &= 0 & \text{for} & & d < c < b < a \text{ or } d > c > b > a \text{ or } d < b < a < c \text{ or } d > b > a > c \text{ or } d < c < a < b \text{ or } c < d < b < a \text{ or } d < a < b < c \text{ or } c < b < a < d, \\
  [e_{dc}, e_{ba}] &= -(q - q^{-1})q^{e_{ac} - e_{cb}}e_{da}e_{bc} & \text{for} & & d < a < c < b, \\
  [e_{dc}, e_{ba}] &= (q - q^{-1})e_{dc}e_{bc}q^{e_{la} - e_{ai}} & \text{for} & & a < d < b < c, \\
  [e_{ba}, e_{ab}] &= e_{ba}q^{e_{ac} - e_{cb}} & \text{for} & & a < b < c, \\
  [e_{ba}, e_{ac}] &= q^{e_{ac} - e_{cb}}e_{bc} & \text{for} & & a < c < b, \\
  [e_{db}, e_{cb}] &= e_{db}q^{e_{ac} - e_{cb}} & \text{for} & & a < d < b, \\
  [e_{db}, e_{cb}] &= q^{e_{ac} - e_{cb}}e_{db} & \text{for} & & d < a < b, \\
  [e_{da}, e_{eb}] &= 0 & \text{for} & & a < b < d \text{ or } b < d < a, \\
  [e_{bc}, e_{ba}] &= 0 & \text{for} & & c < a < b \text{ or } b < c < a. \\
\end{align*}
\]

The relations for \(\tau_j\) can be obtained from the above relations through \(q \to q^{-1}\). In addition, one can prove the following relations based on the above relations and induction. We remark that (A.3) for \(j = 1 - 1\) corresponds to equation (10) in [25].

\[
e_{ji} = (q - q^{-1}) \sum_{k=j+1}^{l} e_{ik} \tau_{jk} = \tau_{ji} + (q - q^{-1}) \sum_{k=l+1}^{j-1} e_{ik} \tau_{jk}
\]

\[
\begin{cases}
  \tau_{ji} & \text{for } j \leq l = i - 1, \\
  \tau_{ji} [\tau_{j+1, l+1}, e_{l+1, i}]_{q^{-1}} & \text{for } j \leq l < i - 1, \\
  e_{ji} & \text{for } j = l < i.
\end{cases}
\]
\( \mathcal{e}_{ji} = (q - q^{-1}) \sum_{k=i+1}^{l} e_k \mathcal{e}_{jk} = e_{ji} + (q - q^{-1}) \sum_{k=l+1}^{j-1} e_k \mathcal{e}_{jk} \)

\[
= \begin{cases} 
  e_{ji} & \text{for } i \leq l = j - 1, \\
  [\mathcal{e}_{ji+1}, e_{l+1}]_{q^2}^{-1} & \text{for } i \leq l < j - 1, \\
  \mathcal{e}_{ji} & \text{for } i = l < j,
\end{cases}
\]  
(A.3)

\( e_{ji} + (q - q^{-1}) \sum_{k=l}^{j-1} e_k \mathcal{e}_{jk} q^{e_{ji} - e_{kj}} = [\mathcal{e}_{ji}, e_{l}]_{q^2} q^{e_{ji} - e_{lj}} \) for \( l < j < i \),  
(A.4)

\( e_{ji} + (q - q^{-1}) \sum_{k=l}^{j-1} e_k \mathcal{e}_{jk} q^{e_{ji} - e_{kj}} = [\mathcal{e}_{ji}, e_{l}]_{q^2} q^{e_{ji} - e_{lj}} + 1 \) for \( l < j < i \),  
(A.5)

\( e_{ji} - (q - q^{-1}) \sum_{k=i+1}^{l} e_k \mathcal{e}_{jk} q^{e_{ji} - e_{kj}} = [\mathcal{e}_{ji}, e_{l}]_{q^2}^{-1} q^{e_{ji} - e_{lj}} \) for \( j < i < l \),  
(A.6)

\( e_{ji} - (q - q^{-1}) \sum_{k=i+1}^{l} e_k \mathcal{e}_{jk} q^{e_{ji} - e_{kj}} + 1 = [\mathcal{e}_{ji}, e_{l}]_{q^2} q^{e_{ji} - e_{lj}} \) for \( i < j < l \).  
(A.7)

\( [\mathcal{e}_{ji}, e_{l}]_{q^2} = [\mathcal{e}_{ji+1}, e_{l+1}]_{q^2} q^{e_{ji} - e_{lj} + 1} \) for \( i < j < l \) or \( j < i < l \).  
(A.8)

\( [\mathcal{e}_{ji}, e_{l}]_{q^2} = [\mathcal{e}_{ji-1}, e_{l-1}]_{q^2} q^{e_{ji} - e_{lj} - 1} \) for \( l < j < i \) or \( j < l < i \).  
(A.9)

\( [\mathcal{e}_{ji}, e_{l}]_{q^2}^{-1} = [\mathcal{e}_{ji-1}, e_{l-1}]_{q^2}^{-1} q^{e_{ji} - e_{lj} - 1} \) for \( i < j - 1 < l < j \) or \( j < l - 1 < l < i \).  
(A.10)

We also remark that the third relation in the right hand side of (A.2) (and (A.3)) is a special case of the second one.

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