Exact solution for two interacting electrons on artificial atoms and molecules in solids

Amnon Aharony\textsuperscript{1}, Ora Entin-Wohlman\textsuperscript{1} and Yoseph Imry\textsuperscript{2}

\textsuperscript{1} School of Physics and Astronomy, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
\textsuperscript{2} Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

(August 12, 2018)

We present a general scheme for finding the exact eigenstates of two electrons, with on-site repulsive potentials \( \{ U_i \} \), on \( I \) impurities in a macroscopic crystal. The model describes impurities in doped semiconductors and artificial “molecules” in quantum dots. For quantum dots, the energy cost for adding two electrons is bounded by the single-electron spectrum, and does not diverge when \( U_i \to \infty \), implying limitations on the validity of the Coulomb blockade picture. Analytic applications on a one-dimensional chain yield quantum delocalization and magnetic transitions.

31.25.Nj, 71.70.d, 73.20.Hb, 73.23.Ps

There has been much recent interest in the effects of interactions on the localization of electrons in disordered systems \cite{1}. An important question, triggered by recent experiments in two dimensions \cite{2}, concerns the possibility that the interactions help to delocalize the electrons, yielding a metal–insulator transition. For two interacting electrons in a random potential, this possibility was supported by Shepelyansky’s numerical work \cite{3}, by a Thouless type block-scaling argument \cite{4} and by further numerical scaling results \cite{5,6}. The latter took advantage of the fact that, for two electrons on an \( N \)-site lattice with on-site repulsion, it is sufficient to study the two–particle Green’s function only in the \( N \)-dimensional Hilbert space of the doubly occupied sites, instead of the full \( N^2 \)-dimensional space \cite{7}. In the present paper we generalize such considerations for the dilute case, in which only \( I \) out of the \( N \) sites are replaced by impurities. These impurities may have different single-electron (1e) energies \( \{ \epsilon_i \} \) and on-site electron–electron (e–e) interactions \( \{ U_i \} \). As we show, the exact eigenenergies of the two electrons are found from the eigenvalues of a (small) \( I \times I \) matrix, which involves the eigenfunctions and the eigenvalues of the 1e Hamiltonian. One surprising result shows that in general the interacting eigenenergies are bounded between consecutive non–interacting two–electron (2e) energies, so that the energy cost due to the interactions is usually much smaller than the average repulsion \( \langle U \rangle \), assigned to each pair in the Coulomb blockade approach \cite{8}.

Our study is also relevant for quantum dots \cite{9}. A quantum dot coupled to electrodes has been modeled by one impurity \( (I = 1) \) on a one dimensional (1D), \( N \)-site chain \cite{9}. This represents a special case of the Anderson model \cite{10}, with a momentum–dependent hybridization between the impurity and the conductance band. Unlike the case of a momentum–independent hybridization, we find that the behavior of the two electrons on the “dot” has a rich phase diagram, as function of the “dot” site energy \( \epsilon_0 \) and the hybridization, i.e., the hopping energy \( t_0 \) between the “dot” and the leads. These parameters can be tuned experimentally, by varying the voltage on a gate coupled capacitively to the dot and the barriers between the dot and the leads \cite{11}. We also find a delocalization transition at large \( U \), for sufficiently negative \( \epsilon_0 \). Our method allows a similar analytic treatment of the \( I = 1 \) case in higher dimensions.

More complex examples include a cluster of \( I \) impurities coupled to 1D leads, representing one large dot \cite{12}, two separate impurities, representing double quantum dots (or artificial “molecules”) \cite{13}, etc. The latter case may also shed light on the nature of the low–lying states in the impurity band of doped semiconductors, which involve “molecules” of impurity–pairs \cite{14,5}. For \( I = 2 \), the “molecule” adjusts to our bound on the interaction energy by crossing over to the Mott antiferromagnetic (singlet) state, in which the electrons are localized on separate ions \cite{15,13}. Finally, the case \( I = N \) represents the 2e Hubbard model, whose non–random version \( (\epsilon_i = 0, U_i \equiv U) \) can be solved analytically in general dimensions using our method.

We start with the 1e Hamiltonian

\[
\mathcal{H}_0 = \sum_{\langle n,m \rangle} (t_{nm}|n\rangle\langle m| + h.c.) + \sum_{i=1}^{I} \epsilon_i|i\rangle\langle i|,
\]

where \(|i\rangle\) is a (spin–independent) state fully localized on site \( i \), the first sum runs over all the site pairs in the system (including the impurities) and the second sum runs over the \( I \) impurities. We first find the 1e eigenstates \(|\alpha\rangle \equiv \sum_{n} \phi_{\alpha}(n)|n\rangle \) and eigenenergies \( \epsilon_{\alpha} \) of \( \mathcal{H}_0 \). This is usually easy, being a linear problem (in any case, this involves at most the diagonalization of an \( N \times N \) matrix). The on–site interactions take place only on the impurities,

\[
\mathcal{H}_{\text{int}} = \Sigma_{i=1}^{I} U_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow},
\]

where \( \hat{n}_{i\sigma} \equiv |i\sigma\rangle\langle i\sigma| \) is the number operator of electrons with spin \( \sigma = \uparrow, \downarrow \) on site \( i \). It is convenient to construct a basis for the 2e Hilbert space from the eigenstates of \( \mathcal{H}_0 \). These are split into singlet \( (S) \) and triplet \( (T) \) spatial states, both with energies \( \epsilon_{ab} \equiv \epsilon_a + \epsilon_b \):
\begin{equation}
|ab\rangle_S = (|a(1)\rangle b(2)) + |a(2)\rangle b(1)) \zeta_{ab},
\end{equation}

\begin{equation}
|ab\rangle_T = (|a(1)\rangle b(2)) - |a(2)\rangle b(1)) / \sqrt{2},
\end{equation}

where $|a(j)\rangle$ represents the 1e eigenstate of $H_0$ for electron $j$. \zeta_{ab} = 2^{-1/2} \zeta_{ab}^0$ (with the Kronecker $\delta$), and the T states are used only for $a \neq b$. To avoid double counting, we consider only states with $\epsilon_a \leq \epsilon_b$.

Using this basis, the matrix elements of $H_{\text{int}}$ involving T states vanish for the on-site interaction of Eq. (2), and the energies of the T states remain $\epsilon_{ab}$. We hence focus on the S states, and omit the subscript S. The nonzero S–S matrix elements are $\langle ab|H_{\text{int}}|cd\rangle = \Sigma_{i=1}^J U_{iab}(i)^* \eta_{iab}(i) \eta_{icd}(i)$, where $\eta_{ab}(i) = 2\zeta_{ab} \phi_a(i) \phi_b(i)$. We now write the eigenfunctions of $H_0 + H_{\text{int}}$, with energy $E$, as $\Sigma_{\epsilon_a \leq \epsilon_b} \epsilon_{ab} x_{ab}(ab)$. The coefficients must obey

\begin{equation}
(E - \epsilon_{ab}) x_{ab} = \Sigma_i \sqrt{U_{iab}(i)} \eta_{ab}(i)^* A_i,
\end{equation}

where $A_i = \Sigma_{\epsilon_a \leq \epsilon_b} \sqrt{U_{iab}(i)} x_{ab}(i)$. Using (4) we find

\begin{equation}
A_i = \Sigma_{j=1}^J S_{ij} A_j,
\end{equation}

where

\begin{equation}
S_{ij}(E) = \sqrt{U_{iab}(i)} \Sigma_{\epsilon_a \leq \epsilon_b} \eta_{ab}(i) \eta_{ab}(j)^*/(E - \epsilon_{ab}).
\end{equation}

$S_{ij}/\sqrt{U_{iab}(i)} = G(jj; ii; E) \equiv (jj; (E - H_0)\zeta_{ij})^{-1}$ is the non–interacting 2e Green’s function in which the two electrons are on the same site, cf. [14].

In addition to degenerate solutions [14], the new eigenstates are found by requiring a non–zero solution to Eq. (5), namely that the $I \times I$ determinant $D(E) \equiv |S_{ij} - \delta_{ij}|$ vanishes. One way to find the new eigenvalues \{E\} is to find the I eigenvalues $S_i(E)$ of the matrix $S$, and then solve the equations $S_i(E) = 1$. The $A_i$’s are then given by the eigenvectors related to $S_i(E)$, and the \{x_{ab}(E)\}’s are found (up to a normalization constant) from Eq. (4). We emphasize that, unlike a perturbative expansion, this formalism gives exact values for $E$ even for very large $U_{iab}$’s. As we show below, this enables us to find interesting transitions in that limit.

The equation $D(E) = 0$ can be used to find bounds on the energies \{E\}. Since each $S_{ij}$ has poles at every $\epsilon_{ab}$, we generally expect each $S_i$ also to have such poles. Although the residues of some of these poles may vanish in special circumstances (e. g., $S_i = S_{ii} = U_{i}/(E - 2\epsilon_i)$ for isolated impurities, $t_{nm} \equiv 0$, or for special symmetric cases, see below), this is not expected to happen in the general random case, which we discuss now. When $E$ is very close to a non–interacting energy $\epsilon_{ab}$, one has $S_{ij} \approx \sqrt{U_{iab}(i)} \eta_{ab}(j)^*/(E - \epsilon_{ab})$. It is then easy to show that in this approximation, $D(E) \approx (1 - \sqrt{1 - T S}) \approx (1 - \sqrt{1 - (U_{ab}/(E - \epsilon_{ab}))})$, with $U_{ab} = \Sigma_{i=1}^J U_{iab}(i)^2$. Indeed, for very small $U_{iab}$ the equation $D(E) = 0$ reproduces the lowest order perturbation result, $E \approx \epsilon_{ab} + \langle U \rangle_{ab}$. As $E$ crosses through $\epsilon_{ab}$, $(-1)^{D(E)}$ jumps from $-\infty$ to $\infty$. As $E$ increases between two consecutive non–interacting eigenvalues, $(-1)^{D(E)}$ has no singularities, and thus varies smoothly from $\infty$ to $-\infty$. Except for the special cases mentioned above, for which $D(E)$ splits into products involving subspaces of the levels (see below), we thus conclude that $D(E) = 0$ must have at least one (and up to $I$) solution(s) between every pair of such consecutive energies, and the new energies maintain the sequence of the non–interacting ones. Specifically, if the 1e lowest levels are $\epsilon_0$ and $\epsilon_1$ then the non–interacting lowest energies are $\epsilon_{gg} = 2\epsilon_g$ and $\epsilon_{gs} = \epsilon_g + \epsilon_s$, and the interacting ground state energy $\epsilon_{gg}$ obeys $\epsilon_{gg} < \epsilon_{gg} < \epsilon_{gs}$.

Thus, the effective interaction cost for adding the two electrons, $\Delta_{gg} = E_{gg} - \epsilon_{gg}$, is now bounded by $(\epsilon_s - \epsilon_g)$, and is always smaller than $U$. For small $(\epsilon_s - \epsilon_g)$, this cost is negligible. Similar results apply to all the levels except the largest one (see below). This renormalization of the interaction energy has direct consequences for the issue of the Coulomb blockade, where one assumes that $\Delta_{ab} \approx (U)_{ab}$, usually assumed to be independent of $ab$. It is interesting to note that small (and not evenly spaced) values of $\Delta$ were observed in a series of 2D quantum dots [17]. It is tempting to relate these observations to our result.

The simplest example concerns one impurity (or “dot”) on a 1D wire, closed with periodic boundary conditions. We place the impurity at site $i = 0$, with energy $\epsilon_0$ and with real matrix elements $t_{01} = t_{0N-1} \equiv t_0$. The other nearest neighbor matrix elements are set at $t_{n,n+1} \equiv t = 1$, for $n = 1, 2, ..., N - 2$, and all energies are scaled by $t$. The eigenenergies of $H_0$ have the form $\epsilon_k = 2 \cos(k)$. Out of these, $N/2$ belong to odd states, of the form $\phi_k(n) \propto \sin(kn)$, with $k = 2n\pi/N$, $\ell = 1, 2, ..., N/2$ (for convenience, $N$ is even). These states do not “feel” the impurity $\phi_k(0) = 0$, and remain unchanged for all $\epsilon_0$ and $U$. The remaining $N/2$ states are even functions of $n$, with $\phi_k(n) = \phi_k(0)\cos(k(N - n)/2) / \cos(k\pi/2)$ for $n \neq 0$, where $|\phi_k(0)|$ is determined by $\sum_n |\phi_k(n)|^2 = 1$. The allowed values of $k$ are given by

\begin{equation}
2\gamma \tan(k\pi/2) \sin(k) = 2(1 - \gamma) \cos(k) - \epsilon_0,
\end{equation}

where $\gamma \equiv (t_0/t)^2 = t_0^2$. An analysis of this equation yields the surprising phase diagram shown by the full lines in Fig. 1: In region A all the states are delocalized ($k$ is real). In region C (or D+F) there exists one bound state above (or below) the conduction band, with a localization length $1/\kappa_+ + \text{energy} \epsilon_+ = 2 \cosh(\kappa_+) > 2$, (or $1/\kappa_- + \text{energy} \epsilon_- = -2 \cosh(\kappa_-) < -2$), where $\epsilon_\pm = \pm \epsilon_0/2 + \sqrt{\epsilon_0^2/4 - 1 + 2\gamma}$. Finally, both bound states exist in region B. It is interesting to note that when $\epsilon_0$ is inside the original conduction band, $|\epsilon_0| < 2$, the state on the impurity becomes delocalized for any infinitesimal hybridization $\gamma$, and the impurity does not imply localization (as might be anticipated in 1D). Furthermore,
this state becomes localized for large $\gamma$. This surprising result arises since a larger $\gamma$ implies a larger repulsion of the localized state below the band.

Combining the normalization condition with Eq. (3) yields (for $N \gg 1$) $|\phi_b(0)|^2 = 1/[1 + 2\gamma/(\epsilon^2 + 1)]$ for the bound states $b = \pm$, and $|\phi_b(0)|^2 = 2\pi f(k)/N$, with $\pi f(k) = 4\gamma^2 \sin^2(k)/[4\gamma^2 \sin^2(k) + 2(1 - \gamma) \cos(k) - \epsilon_0^2]$. The latter result implies that unless one has a uniform chain, where $\gamma = 1$ and $\epsilon_0 = 0$, the weight of the states with $k$ at the band edges 0 and $\pi$ vanishes on the impurity. This will turn out to be crucial below.

For $I = 1$, Eq. (5) reduces to a single equation, $D(E) = U s(E) - 1$, and the eigenenergies $E$ obey $s(E) = 1/U$. Similar equations are encountered in numerous cases, e. g., in the Cooper 2e problem or in Kohn’s model of the insulating state [3]. Indeed, $s(E)$ jumps from $-\infty$ to $\infty$ as $E$ crosses each non–interacting energy $E_{ab}$, and we find one new eigenvalue between every pair of non–interacting energies, as described above. In regions B, D and F, $\epsilon_f = \epsilon_c = \epsilon$, and $\epsilon_0 = -2$ (the bottom of the band). Thus, $\Delta_{gg}$ is bound by $|\epsilon_c - 2|$, and is always smaller than $U$. In fact, $\Delta_{gg} \to 0$ when the 1e bound state approaches the band, at the line $\epsilon_0 = 2(\gamma - 1)$. For another insight into the smallness of $\Delta_{gg}$, rewrite $s = |\phi_{-}(0)|^2/(E - 2\epsilon_c) - \Gamma(E)$. For $E < \epsilon_{gu} = \epsilon_c - 2$, one has $\Gamma > 0$, and $|\phi_{-}(0)|^2/(E - 2\epsilon_c) = 1/U_{\text{eff}}$, with the smaller renormalized repulsive energy $U_{\text{eff}} = U/(1 + \Gamma U) < U$. It is interesting to note that in regions D and F, $\Gamma$ is proportional to $\gamma$, and therefore the renormalization of $U$ increases with the hybridization. The Coulomb blockade picture is restored for $\gamma \ll 1$.

The details of $s(E)$ depend on the parameters $\epsilon_0$ and $\gamma$. To treat the bands (of states $|kk\rangle$ and $|\pm k\rangle$) in the limit $N \to \infty$, we replace the summations by integrals: $s = \sum_{b,b'} 2|\phi_b(0)|^2 |\phi_{b'}(0)|^2 / (E - \epsilon_b - \epsilon_{b'}) + 2 \sum_b |\phi_b(0)|^2 \int_0^{\pi} dk / (E - 2 \cos(k) - \epsilon_b) + 2 \int_0^{\pi} dk \int_0^{\pi} dk' / (E - 2 \cos(k) - 2 \cos(k'))$.

where the bound states $b, b' = \pm$ are included only when they exist (regions C, D, F and B). For $E$ inside the band we must return to the discrete sum, and the dense energies practically don’t shift.

Our most interesting results arise in region D+F. Here we have only one bound state, $b = -$, so that $s(E)$ is negative for $E < \epsilon_{gg} = 2\epsilon_c$, and decreases from $+\infty$ as $E$ increases above this value, towards the band which starts at $\epsilon_{gu} = \epsilon_c - 2$. For finite $N$, $s$ would diverge towards $-\infty$ as $\epsilon_{gu}$ is approached, implying a persistent bound solution with a discrete energy below the band for all $U$. This would also be the case for the continuum case, $N \to \infty$, if one had a non–zero value of $f(0)$ (from $|\phi_k(0)|^2$ at $k = 0$), due to the divergence of the 1D density of states there. However, as noted above, $f(k)$ vanishes at $k = 0$, yielding a finite value $s(\epsilon_{gu}) \equiv s_c$. We studied $s_c$ as function of $\epsilon_0$ and $\gamma$, and found that in region D of Fig. 1 one has $s_c < 0$, so that the equation $s(E) = 1/U$ still has a discrete bound (“insulating”) state there. However, in region F one has $s_c > 0$, implying a disappearance of this bound state for $U > U_c = 1/s_c$.

This transition is intuitively easy to understand: when $\gamma$ is very small, this transition occurs when the energy of the two electrons in the isolated atom, $2\epsilon_0 + U$, exceeds that of the state in which one electron remains bounded while the other moves to the band, which is equal to $\epsilon_0 - 2$. At finite $\gamma$, a larger negative $\epsilon_0$ implies a smaller localization length $1/\kappa_c$. The electrons are then more localized on the impurity, and $\Delta_{gg}$ is larger, leading to the “insulator to metal” transition from region D to region F. It should be noted that although the bound state is a singlet, with total spin zero, the new ground state in region F has one bound electron and one “free” electron.

Such a state does not feel the e–e repulsion, and is thus practically degenerate with the slightly lower triplet state (for large $N$, the difference is of order $1/N$). Unlike the “insulator” singlet (or “antiferromagnetic”) ground state, which has no net magnetic moment, this “metallic” state in region F is paramagnetic. This difference should be measurable in an external magnetic field.

It is interesting to study the crossover from the “mesoscopic” case, of finite $N$, to the thermodynamic limit discussed above. Firstly, for finite large $N$ the transition from region A to region D is smeared, occurring when $(\epsilon_+ + 2)$ becomes comparable to the spacing between the band states, of order $1/N^2$. Secondly, in the mesoscopic case, $s(E)$ diverges to $-\infty$ as $E$ approaches the lowest band state (slightly above $\epsilon_{gu}$). However, in region F, $s(E)$ first gets very close to $s_c$, and only then drops sharply to $-\infty$. For $U > U_c$, the resulting “bound” state will thus occur very close to $\epsilon_{gu}$, making it almost indistinguishable from the states inside the band. This implies a “smeared” transition from region D to region F for large mesoscopic systems.

Similar interesting effects occur above the band, in regions A, D and F. The highest band energy is $E = 4$, corresponding to the upper limit of the last term in Eq. (6). Since $f(\pi) = 0$, $s(E)$ remains finite as $E$ approaches 4 from above, and $0 < s(E = 4) \equiv s_x < \infty$, $s_x = \infty$ only on the boundary $\epsilon_0 = 2(1 - \gamma)$. As $E$ increases from 4 to $\infty$, $s$ decreases from $s_x$ to zero, implying a discrete solution to the equation $s(E) = 1/U$, with $E > 4$ above the band, if $U > U_x = 1/s_x$. We found (by calculating $x_{ab}$) that the corresponding wave function decays away from the impurity. Since this new state is above the band, it describes an “antibound” 2e excitation, which is reminiscent of the upper Hubbard band at low concentration [15]. Our analysis finds a phase transition of this excited state, from an insulating localized state at $U > U_x$ to...
a conducting state for smaller \( U \). This transition also becomes “ smeared” for finite \( N \).

Now we give a very brief summary of our results for \( I = 2 \), on the 1D ring. Consider the very symmetric case, with the two impurities at sites \( i = i_1, \ i_2 \) having \( \epsilon_i \equiv \epsilon_0, \ U_i \equiv U, \ t_{i_1+i_2} = t_0 \) and \( i_2-i_1 = R \). The 1e real wave functions separate into even and odd ones, with \( \phi_{\alpha}(i_1) = \pm \phi(i_2) \). Depending on \( \epsilon_0, \ t_0 \) and \( R \) one again finds a rich 1e phase diagram. In particular, the low energy 1e states may start with the band, or have a bound even (“bonding”) state or (above a minimal value of \( R, \ R_x \) have also an odd bound (“antibonding”) state between the bonding state and the band. The high symmetry implies that \( S_{11} = S_{22} \) and \( S_{12} = S_{21} \), with eigenvalues \( S_{\pm}(E) = S_{11} \pm S_{12} \). Writing \( S_{\pm}(E) \) as sums over non–interacting singlet states, we find that \( S_{+} \) contains only the even–even and the odd–odd states, while \( S_{-} \) contains only the even–odd states. Thus, \( D(E) = (S_{+} - 1)(S_{-} - 1) \), and the problem splits into two separate spaces. Consequently, \( \Delta_{ab} \) is now bounded by the first non–interacting excited state within the even–even and odd–odd subspace. The analysis of each equation \( S_{\pm} \) is now similar to that of \( I = 1 \), and the resulting phase diagram again contains delocalization and magnetic transitions \([20]\). It is possible to follow the coefficients \( \{x_{ab}\} \), and see exactly how \( U \) causes a gradual crossover from the “molecular” to the “atomic” limits as \( R \) increases \([19]\). In the latter localized states, the effective interaction energy obeys our bound by being of order \( 1/U \). This crossover would be missed for \( R < R_x \), if one were to ignore the hybridization with the band (as commonly assumed \([21]\)).

Our method is easily generalized to higher dimensions and to larger \( I \). It should be particularly helpful in numerical work on random systems. Another simple example concerns the non–random Hubbard model, where \( I = N \). The matrix \( S_i \) is easily diagonalized in Fourier space, yielding the upper Hubbard band. This and other examples will be published elsewhere \([20]\).

We acknowledge many discussions with Yehoshua Levinson, and comments from Peter Wölfe. This research is supported by grants from the Israel Science Foundation and the French Ministry of Research and Technology.

---

[1] B. L. Altshuler and A. G. Aronov, in Electron-Electron Interactions in Disordered Systems, A.L. Efros and M. Pollak, eds. p. 1-154, North Holland, Amsterdam (1985); A. M. Finkelstein, Sov. Phys. JETP 67, 97 (1983).
[2] S. V. Kravchenko et al., Phys. Rev. Lett. 77, 4938 (1996) and references therein.
[3] D. L. Shepelyansky, Phys. Rev. Lett. 73, 2607 (1994).
[4] Y. Imry, Europhys. Lett. 30, 405 (1995); Phys. Stat. Sol. 205, 249 (1998). See also D. Weinnmann, A. Müller-Groeling, J.-L. Pichard and K. Frahm, Phys Rev. Lett. 75, 1598 (1995).
[5] F. von Oppen, T. Wettig and J. Müller, Phys. Rev. Lett. 76, 491 (1996).
[6] M. Ortuño and E. Cuevas. cond-mat/9808104 (1998).
[7] G. Grabert and M.H. Devoret (eds) Single Charge Tunneling, Coulomb Blockade Phenomena in Nanostructures, Nato ASI, Series B: Physics, vol 294. Plenum, NY (1992).
[8] L. P. Kouwenhoven et al., Mesoscopic Electron Transport, Proceedings of the NATO Advanced Study Institute, edited by L. L. Sohn, L. P. Kouwenhoven and G. Schön (Kluwer 1997).
[9] T. K. Ng and P. A. Lee, Phys. Rev. Lett. 61, 1768 (1988).
[10] P. W. Anderson, Phys. Rev. 124, 41 (1961).
[11] A. Yacoby et al., Phys. Rev. Lett., 74, 4047 (1995).
[12] R. Berkowits, Phys. Rev. Lett. 73, 2067 (1994).
[13] e. g. R. H. Blick et al., Phys. Rev. Lett. 80, 4032 (1998); G. Scheldebeck et al., Science 278, 1792 (1997).
[14] e. g. N. F. Mott and E. A. Davis, Electronic Processes in Non-Crystalline Materials (Clarendon, Oxford 1979).
[15] W. A. Harrison, Phys. Rev. B29, 2917 (1984).
[16] If the energy \( \epsilon_{ab} \) is \( M \)-fold degenerate then one can have up to \( (M - 1) \) solutions for which all \( A_i = 0 \), \( E = \epsilon_{ab} \) remains unaffected by the interactions and \( x_{\alpha,\beta} \) are the solutions of \( \sum_{\alpha,\beta} \eta_{\alpha,\beta}(i) x_{\alpha,\beta} = 0 \), where \( \alpha, \beta = 1, ..., M \) denote the degenerate states.
[17] N. B. Zhitenev, R. C. Ashoori, L. N. Pfeiffer and K. W. West, Phys. Rev. Lett. 79, 2308 (1997).
[18] W. Kohn, Phys. Rev. 133, A171 (1964).
[19] P. W. Anderson, Phys. Rev. Lett. 64, 1839 (1990); J. R. Engelbrecht and M. Randeria, Phys. Rev. Lett. 65, 1032 (1990).
[20] A. Aharony, O. Entin-Wohlman and Y. Imry, unpublished.
[21] e. g. Y. Tokura, D. G. Austing and S. Tarucha, Tokyo preprint; B. Partoens, A. Matulis and F. M. Peters, cond-mat/9807174 (1998).
FIG. 1. $\epsilon_0 - \gamma$ phase diagram for the single impurity case.