Production cross section of rotating string

Tsunehide Kuroki\textsuperscript{1} and Toshihiro Matsuo\textsuperscript{2}

\textsuperscript{1}Theory Group, KEK, Tsukuba, Ibaraki 305-0801, Japan
\textsuperscript{2}Department of Physics, National Taiwan Normal University, Taipei City 116, Taiwan

Abstract

We calculate production cross sections of a single rotating string from a collision of two light states in bosonic string theory. We find that the cross sections are written in terms of the modified Bessel function of the first kind with the degree given by the angular momentum in the high energy regime. We also obtain a similar formula from the partial wave expansion of four point amplitudes. Asymptotic behavior of the cross sections is consistent with a well known form factor of a string.
1 Introduction

Ultra-high energy scattering of strings is a genuine probe into their underlying fundamental structure which is invisible in ordinary circumstances. Many interesting features have emerged such as among other things higher genus dominance in hard processes \[1, 2\], saturation on the Cerulus-Martin lower bound in saddle point analysis \[3\], partial wave unitarity \[4, 5\], linear relations \[6, 7, 8\], an appearance of the shock wave metric in the eikonal amplitudes in Regge processes \[9, 10, 11, 12\], and spacetime uncertainty \[14\], etc. Still recently the perturbative high energy scattering amplitudes have been attracted lots of attention and been used to study for various purposes especially in the new light of AdS/CFT \[15, 16, 17, 18\], etc. In the investigations mentioned above basic ingredients are the four point elastic scattering amplitudes of the Veneziano and the Virasoro-Shapiro. On the other hand, three point amplitudes, which are just constants and less interesting when the external states are in lower level tachyons or massless states, become nontrivial when at least one of the external states is set in an extremely highly excited state.

However in general such a highly excited state has a complicated expression of vertex operator. In \[19\] Amati and Russo have studied such a three point process including highly excited states and indeed they have shown that a single highly excited string emits a light state and behaves as a black body with the Hagedorn temperature which is the characteristic temperature of string theory, when the initial states are averaged and the final states are summed over the degeneracy which are in the same mass level. They calculated directly a decay rate instead of an amplitude in such a way that is quite similar to the unitarity relation (optical theorem) by introducing a projection operator to realize highly excited string states as external states which in general are difficult to treat in perturbative calculations. Their idea has been extended and applied to subsequent works in \(e.g.\) \[20, 21, 22, 23\].

Another interesting application of the three point amplitude is to pursue processes of black hole formation in ultra-high energy scatterings and issues of unitarity both of which are closely related. In \[24\] Dimopoulos and Emparan have investigated, among other things, production of a single highly excited string as a black hole progenitor, in view of the correspondence principle \[25, 26, 27\]. They conclude that the total production cross section of a highly excited string is rather large and comparable to the conjectured black hole geometric cross section, though these two do not match at the correspondence point. The cross section of a string grows much faster than that of black hole and thus seems to violate the unitarity bound.

All of these arguments so far have been made without specifying the angular momentum or in other words the impact parameter in the case of scattering events. Namely, these amplitudes are obtained with summing over all contributions from states which have different angular momenta. It is important to calculate more detailed amplitudes or cross sections with the angular momentum specified because the cross sections or S-matrices are basic data that characterize the nature of fundamental strings. Furthermore they are also useful for other practical purposes to reveal various features of a string. For instance, specifying angular momentum or impact parameter provides a

\[1\]See \[13\] for recent progress.
starting point to argue non-locality \cite{28, 29, 30, 31, 32}, and decay rates of a specific state which
has maximal angular momentum have been investigated in \cite{33, 34, 35, 36, 37}.

In this paper we shall calculate a production cross section of a single rotating string from a
collision of two strings, which is described by a three point amplitude via two methods. The first
one is to introduce a projection operator which specifies eigenstates of a definite component of the
angular momentum, say, \( J_{12} \) as done in \cite{19} for the mass level. With a help of this operator we
obtain a closed formula for the production cross section of a single string with a definite angular
momentum. Our formula agrees with \cite{24} when we sum over the angular momentum. A similar
result can be obtained via the optical theorem for a four point amplitude where the total angular
momentum of a single produced string is specified.

This paper is organized as follows. In section two we calculate production cross sections of a
heavy string produced from a collision of two light states without specifying the angular momentum.
We employ both tachyon-tachyon and photon-photon collisions. In section three we calculate
a production cross section of a process in which we specify one of components of the angular
momentum. In section four we derive from the partial wave expansion of a four point amplitude
a production cross section of a rotating string with the total angular momentum specified. In
appendices we present details of formulae used in our calculations.

2 Collision between two light states

In this section we calculate production cross sections of a single heavy string from a collision
between two light states in the bosonic (oriented) open string theory. Here the collision is inclusive
with respect to the angular momentum, that is, we do not specify the angular momentum of the
process. Namely we sum over all the angular momenta.

2.1 tachyon-tachyon collision

As a warming-up exercise, we consider a tachyon-tachyon collision. Let the initial tachyon momenta
be \( k_\mu \) and \( p_\mu \), and the momentum of a final massive string state be \( P_\mu \), which is assumed to be
in a level \( N \) state. The final state is denoted by \( \Phi_N \). The on-shell conditions for the initial state
tachyons are

\[ k^2 = p^2 = 1, \quad (2.1) \]

and for the final state

\[ P^2 = -N + 1, \quad (2.2) \]

where we set \( \alpha' = 1 \). Let \( s = -(p+k)^2 \), then from the energy-momentum conservation \( P_\mu = p_\mu + k_\mu \)
(on-shell condition which will see later) we have

\[ s = N - 1. \quad (2.3) \]
We consider a probability of the collision process which is given as
\[ \mathcal{P}(V(p), V(k) \rightarrow \Phi_N) = \sum_{\Phi|N} |\Phi| \frac{1}{\sqrt{2}} (V(k) + \Theta V(k) \Theta^{-1}) |p| \],
\[ (2.4) \]
where the symbol $\Phi|N$ means that we sum all over the states $\Phi$ in the level $N$. This is because we do not observe detailed information on the final state. They are specified only by the level and thus will have lots of degeneracy as they go to a higher level. Here the tachyon vertex operator\[^{2} \] is
\[ V(k, 1) = g_{\text{o}} : \exp(i k \cdot X(1)) :, \]
\[ (2.5) \]
where $g_{\text{o}}$ is the open string coupling constant. And $\Theta = (-1)^{\hat{N}}$ is the twist operator which takes a vertex operator on one boundary to the other boundary. Here $\hat{N} = \sum_{\alpha_{\mu}} \alpha_{-\mu} \alpha_{\mu}^{\mu}$ is the number operator and $\alpha_{n}$ obeys the commutation relation $[\alpha_{n}, \alpha_{m}] = n \delta_{n-m}$.

We introduce an operator which is a projection to the level $n$ state \[^{[19]} \] :
\[ P_{n} = \oint \frac{dz}{2\pi iz} \hat{N}_{-n}, \]
\[ (2.6) \]
which enables us to write the probability as
\[ \mathcal{P}(V(p), V(k) \rightarrow \Phi_N) = \frac{1}{2} \sum_{i,j=\text{all}} (\langle \langle i | V(-k) P_{N} | j \rangle \rangle V(k) P_{0} | i \rangle \rangle + \langle \langle i | \Theta V(-k) \Theta^{-1} P_{N} | j \rangle \rangle V(k) P_{0} | i \rangle \rangle + \langle \langle i | V(-k) P_{N} | j \rangle \rangle \Theta V(k) \Theta^{-1} P_{0} | i \rangle \rangle + \langle \langle i | \Theta V(-k) \Theta^{-1} P_{N} | j \rangle \rangle \Theta V(k) \Theta^{-1} P_{0} | i \rangle \rangle \]
\[ + \langle \langle i | V(-k, 1) P_{N} V(k, 1) P_{0} + V(-k, 1) P_{N} V(k, -1) P_{0} | i \rangle \rangle, \]
\[ (2.7) \]
where $i, j$ take all the physical states in Hilbert space and we have used $P_{N} = \Theta V_{N} \Theta^{-1}$. Here the trace is taken only on the oscillator parts. As for the zero mode, it is easy to evaluate
\[ |\langle P|e^{ik\cdot\hat{x}}|p\rangle|^{2} = V\delta(P - p - k). \]
\[ (2.8) \]
Hereafter we omit this factor for simplicity and the on-shell condition should be understood. The position of the vertex operator is converted as
\[ z \hat{N} V(k, 1) z^{-\hat{N}} = V(k, z), \]
\[ (2.9) \]
thus we can write the probability as follows
\[ \mathcal{P}(V(p), V(k) \rightarrow \Phi_{s+1}) = \oint_{C_{z_{1}}} \frac{dz_{1}}{2\pi iz_{1}} z_{1}^{-s-1} \oint_{C_{z_{2}}} \frac{dz_{2}}{2\pi iz_{2}} \text{tr}[V(-k, 1)(V(k, z_{1}) + V(k, -z_{1}))(z_{1}z_{2})^{\hat{N}}], \]
\[ (2.10) \]
where the contours $C_{z_{1}}, C_{z_{2}}$ should be taken as $|z_{1}|, |z_{2}| < 1$ so that they will not hit the vertex operators. By change of variables $z_{1} = v, z_{1}z_{2} = w$ (thus $|w| < |v| < 1$) we have
\[ \mathcal{P}(V(p), V(k) \rightarrow \Phi_{s+1}) = \oint \frac{dv}{2\pi iv} v^{-s-1} \oint \frac{dw}{2\pi i w} \text{tr}[V(-k, 1)(V(k, v) + V(k, -v))w^{\hat{N}}]. \]
\[ (2.11) \]
\[^{2} \] Here the argument $z = 1$ in $X(z)$ denotes the origin in the coordinate system $(\tau, \sigma)$ which is related to $z$ as $z = e^{\tau+i\sigma}$. 


Note that there is no momentum integration. This can be seen as a sum of two (oscillator part of) one-loop cylinder amplitudes, one with two vertex operators inserted at the same boundary (planar diagram) and the other with those inserted at both boundaries (non-planar diagram). The trace part is evaluated as (see appendix A for detail)

\[
\text{tr}[V(-k, 1)V(k, v)w^N] = [f(w)]^{2-D} \left[ \hat{\psi}(v, w) \right]^{-2},
\]

where \( D \) is the number of spacetime dimensions and \( f(w) \) and \( \hat{\psi}(v, w) \) are defined in \([A.18]\) and \([A.19]\), respectively.

Now it is straightforward to perform the \( w \) contour integration because it only picks up constant terms in the expansion of the integrand with respect to \( w \). One obtains

\[
\mathcal{P}(V(p), V(k) \rightarrow \Phi_{s+1}) = g_o^2 \int \frac{dv}{2\pi i v} v^{-s-1} \left[ \frac{1}{(1-v)^2} + \frac{1}{(1+v)^2} \right]
\]

\[
= g_o^2 (s+2)(1+(-1)^{s+1}).
\]

Dividing phase space factor we get a cross section:

\[
\sigma_{\text{open}} \simeq g_o^2 (1+(-1)^{s+1}),
\]

where we have taken large \( s \).

In the case of a closed string collision, calculation may be done in a similar way as in the case of the open string. Actually, at the tree level, a squared amplitude of open strings provides that of closed strings \([38]\). Both planar and non-planar diagrams give rise to the same closed string amplitude. Therefore for a collision with two closed string tachyons we have a probability:

\[
\mathcal{P}(V(p), V(k) \rightarrow \Phi_{s+1})_{\text{closed}} \sim g_s^2 s^2,
\]

or cross section:

\[
\sigma_{\text{closed}} \simeq g_s^2 s,
\]

which is precisely the one derived in \([24]\).

### 2.2 photon-photon collision

At high energy a cross section should not depend on the type of initial states. To check this we calculate a cross section of an inclusive process of a collision of two photons. Actually we will obtain a cross section which is essentially the same as that of the tachyon-tachyon case at high energy.

We consider initial photons with polarization and momenta \((\zeta^\mu, k_\mu)\) and \((\xi^\mu, p_\mu)\) and a final massive string state \(\Phi\) of level \(N = s+1\). The initial state polarization is averaged and that of the final state is summed. The probability is

\[
\mathcal{P}(\xi(p), \zeta(k) \rightarrow \Phi_N) = \frac{1}{(D-2)^2} \sum_{\xi, \zeta} \sum_{\Phi|N} |\langle \Phi | V(\zeta, k) | \xi, p \rangle|^2.
\]
We will concentrate on the planar contribution until the last step of the computation. Contribution from the non-planar diagram is easy to include once we obtain the planar part as we saw in the last subsection. Here the photon vertex operator is

\[ V(\zeta, k, 1) = g_0 \exp(ik \cdot X(1) + \zeta \cdot \dot{X}(1))|_{\text{linear in } \zeta}, \tag{2.18} \]

where the normal ordering is irrelevant because \( k^2 = k \cdot \zeta = 0 \) and only the linear term in \( \zeta \) is relevant for us.

With the help of the projection operator we have

\[ \mathcal{P}(\xi(p), \zeta(k) \rightarrow \Phi_{N}) = \frac{1}{(D-2)^2} \sum_{\zeta} \oint \frac{dw}{2\pi iw} w^{-1} \oint \frac{dv}{2\pi iv} v^{1-N} \text{tr}[V(\zeta, -k, 1)V(\zeta, k, v)w^N]. \tag{2.19} \]

The trace part can be calculated as

\[ \text{tr}[V(\zeta, -k, 1)V(\zeta, k, v)w^N] = g_0^2 f(w)^{2-D} \zeta^2 \hat{\Omega}(v, w), \tag{2.20} \]

where

\[ \hat{\Omega}(v, w) = \sum_{n=1}^{\infty} n \left( v^n + \frac{w^n(v^n + v^{-n})}{1 - w^n} \right), \tag{2.21} \]

and we have

\[ \mathcal{P}(\xi(p), \zeta(k) \rightarrow \Phi_{s+1}) = g_0^2 s(1 + (-1)^s), \tag{2.23} \]

where we have added a contribution from the non-planar diagram which is obtained simply by replacing the argument \( v \) to \(-v\) in (2.20). This is the same behavior as that of the previous two tachyons collision at large \( s \).

We close this section by making a remark on the contribution from the non-planar diagram. In (2.13) or (2.23), the non-planar contribution exactly cancels the planar one when \( s \) is even or odd, respectively. Since these probabilities are essentially the pole residues at \( s = N - 1 \) of the corresponding four point amplitudes via the optical theorem as explained in section 4, this observation reflects the fact that when all the four external lines are the same in the four point amplitude, its pole residues in the different channels have the same absolute value with different signs. Therefore, in a general case we do not expect such a cancellation. In fact, in contrast to our case, in [19] the non-planar contribution is shown to be subleading with respect to \( N \) compared to the planar one.
3 Collision with specifying angular momentum

Now we specify the angular momentum of a collision process. We consider tachyons with momentum \( k_\mu \) and \( p_\mu \) to produce a massive string of level \( N \) with an angular momentum. To be precise, we take the angular momentum \( J_{12} \). The final state is denoted as \( \Phi_{(N,J)} \).

The probability in which the final states are summed up over the degeneracy is

\[
P(V(p), V(k) \to \Phi_{(N,J)}) = \sum_{\Phi_{(N,J)}} |\langle \Phi|V(k)|p\rangle|^2.
\]

(3.1)

Again we will concentrate on the planar contribution until the last step of computation.

Now we introduce an operator which is a projection to states with an angular momentum \( J \):

\[
Q_J = \oint \frac{dz}{2\pi iz} z^{j-J},
\]

(3.2)

where

\[
j = \hat{x}^1 \hat{p}^2 - \hat{x}^2 \hat{p}^1 - i \sum_{n=1}^{N} \frac{1}{n} (\alpha_{-n}^1 \alpha_n^2 - \alpha_{-n}^2 \alpha_n^1).
\]

(3.3)

Then the probability is written as

\[
P(V(p), V(k) \to \Phi_{(N,J)}) = \sum_{i,j=\text{all}} \langle i|V(-k)Q_J P_N |j\rangle \langle j|V(k)P_0|i\rangle = \text{tr}[V(-k,1)Q_J P_N V(k,1)P_0].
\]

(3.4)

Note that we do not need to insert \( Q_{J=0} \) to pick up the spin zero tachyon state because this projection is already accomplished by \( P_{N=0} \). Here again the trace is taken only on the oscillator parts. As for the zero mode, let us focus on the 1-2 components (the other components are the same as before):

\[
\langle p|V(-k)q_J|P\rangle \langle P|v(k)|p\rangle = \langle p|e^{-ik-\hat{x}^1 \hat{p}^2 - \hat{x}^2 \hat{p}^1}|P\rangle \langle P|e^{ik-\hat{x}^1 \hat{p}^2 + \hat{x}^2 \hat{p}^1}|P\rangle = \langle p + k|z^{\hat{x}^1 \hat{p}^2 - \hat{x}^2 \hat{p}^1}|P\rangle \delta(p + k - P).
\]

(3.5)

If we set the (1-2 component of) momentum of the final state zero, \( P = 0 \), then

\[
\langle p + k|z^{\hat{x}^1 \hat{p}^2 - \hat{x}^2 \hat{p}^1}|0\rangle = \delta(p + k),
\]

(3.6)

and we obtain

\[
V_2 \delta^{(2)}(p + k).
\]

(3.7)

We will omit this factor in the following. We use the on-shell condition to replace \( N \) by \( s + 1 \).

Then we have

\[
P(V(p), V(k) \to \Phi_{(s+1,J)}) = \oint C_z \frac{dz}{2\pi iz} z^{-J} \oint C_{u_1} \frac{du_1}{2\pi i u_1} u_1^{-s-1} \oint C_{u_2} \frac{du_2}{2\pi i u_2} \times \text{tr}[V(-k,1)z^{\hat{x}^1 \hat{p}^2 - \hat{x}^2 \hat{p}^1} V(k, u_1)(u_1 u_2)^{\hat{x}^1 \hat{p}^2 - \hat{x}^2 \hat{p}^1}].
\]

(3.8)
The contours $C_z, C_{u_1}, C_{u_2}$ satisfy $|z|, |u_1|, |u_2| < 1$ in such a way that they do not encounter the vertex operators. Changing the variables as $u_1 = v, u_1 u_2 = w$ (thus $|w| < |v| < 1$) we get

$$\mathcal{P}(V(p), V(k) \rightarrow \Phi(s+1,J)) = \oint \frac{dz}{2\pi i z} z^{-J} \oint \frac{dv}{2\pi i v} v^{-s-1} \oint \frac{dw}{2\pi i w} \text{tr}[V(-k,1)z^j V(k,v)w^N].$$

(3.9)

The trace part is computed in appendix A. Without loss of generality, one can set the $i = 2, 3, .., d$ components of incoming momenta vanish. Furthermore using the on-shell condition $k^2 = 1$, or $(k_1)^2 = (k_0)^2 + 1$ which yields $s/4 = (k_0)^2 = (k_1)^2 - 1$, we have, including contributions from ghost,

$$\text{tr}[V(-k,1)z^j V(k,v)w^N] = g_0^2[f(w)]^{4-D}[\tilde{\psi}(v,w)]^{s/2}[\tilde{f}(z,w)]^{-1} \left[\tilde{\psi}(z,v,w)\right]^{-\frac{s}{4}+1},$$

(3.10)

where $\tilde{f}(z,w)$ and $\tilde{\psi}(z,v,w)$ are defined in (A.13) and (A.14), respectively. It is evident that (3.10) reproduces (2.12) when $z = 1$, as expected. Again it is straightforward to perform the contour integrations with respect to the variables $w$ and $z$. As for the $w$ contour integration only the $w = 0$ term survives and the $z$ integration provides the modified Bessel function of the first kind,

$$\oint \frac{dz}{2\pi i z} z^{-J} \oint \frac{dw}{2\pi i w} \text{tr}[V(-k,1)z^j V(k,v)w^N] = g_0^2(1 - v)^{s/2} I_J \left(\frac{s}{2} + 2\right) \ln(1 - v).$$

(3.12)

Thus we have

$$\mathcal{P}(s, J) = g_0^2 \oint \frac{dv}{2\pi i v} v^{-s-1}(1 - v)^{\tilde{s}/2} - 2 I_J \left(-\frac{s}{2} \ln(1 - v)\right),$$

(3.13)

where $\tilde{s} := s + 4$. An integral representation of the modified Bessel function enables us to write

$$\mathcal{P}(s, J) = \frac{g_0^2}{\sqrt{\pi} \Gamma(J + 1/2)} \int_{-1}^{1} dz Q_{J-1/2}(z) \oint \frac{dv}{2\pi i v} v^{-s-1}(1 - v)^{\tilde{s}/2} - 2 + 2 \tilde{s}/2 \left(-\frac{s}{2} \ln(1 - v)\right)^J,$$

(3.15)

where

$$Q_J(z) = (1 - z^2)^J.$$

(3.16)

Then the $v$ integration can be easily done and we have

$$\mathcal{P}(s, J) = \frac{g_0^2}{\sqrt{\pi} \Gamma(J + 1/2)^2} \int_{-1}^{1} dz Q_{J-1/2}(z) \left(-\frac{\partial}{\partial z}\right)^J \frac{\Gamma(s + 3 - (z+1)\tilde{s}/2)}{\Gamma(s + 2)\Gamma(2 - (z+1)\tilde{s}/2)}.$$

(3.17)

One can immediately show that the integrand is even or odd function of $z$ according as $s + J + 1$ is even or odd integer, respectively. Hence one can write

$$\mathcal{P}(s, J) = \frac{1 + (-1)^{s+J+1}}{\sqrt{\pi} \Gamma(J + 1/2)^2} \int_{0}^{\tilde{s}/2} dx Q_{J-1/2} \left(1 - \frac{2x}{\tilde{s}}\right) \left(-\frac{\partial}{\partial x}\right)^J \frac{\Gamma(s + 3 - x)}{\Gamma(s + 2)\Gamma(2 - x)}.$$

(3.18)

\[\text{It can be defined by a contour integration:}
I_v(t) = \oint \frac{dz}{2\pi i} z^{-\nu-1} e^{t/2(z+1)/z}.
(11.11)
\]

\[\text{That is}
I_v(t) = \frac{(1/2)^J}{\sqrt{\pi} \Gamma(J + 1/2)} \int_{-1}^{1} dz (1 - z^2)^{J-1/2} e^{-t z} , \text{ Re}(J) > -1/2.
(14.14)
\]
where we have changed the variable as
\[ x := (z + 1)\tilde{s}/2. \] (3.19)

So far the expressions are exact. However, the integration in (3.13) or (3.18) may not be performed rigorously and hence we make an approximation. Let us examine
\[ F(x) := \frac{\Gamma(s + 3 - x)}{\Gamma(s + 2)\Gamma(2 - x)} = \frac{1}{\Gamma(s + 2)} [(s + 2 - x)(s + 1 - x) \cdots (3 - x)(2 - x)], \] (3.20)
in which the bracket part is so called the Pochhammer symbol. It is sufficient to consider \( F(x) \) only in the integration region \( 0 < x < \tilde{s}/2 \). The first zero point is \( x = 2 \) and in between \( 2 < x < \tilde{s}/2 \) it oscillates with taking quite small values of \( \mathcal{O}(1/s^2) \) compared to those in \( 0 < x < 2 \). Thus the integration region can be restricted to \( 0 < x < 2 \), where we can neglect \( \Gamma(2 - x) \) since it gives only an \( \mathcal{O}(1) \) contribution. For large \( s \) we may use the Stirling formula to get
\[ F(x) \sim s^{1-x}. \] (3.21)
This is the only approximation we made in the calculation, which becomes better as \( s \) becomes larger. Therefore we have
\[ \left( -\frac{\partial}{\partial x} \right)^J \! F(x) \sim (\ln s)^J s^{1-x}, \] (3.22)
and (3.18) becomes
\[ \mathcal{P}(s, J) \simeq \frac{(1 + (-1)^{s+J+1})g_0^2(\tilde{s}/2)^{J-1}}{\sqrt{\pi} \Gamma(J + 1/2)2^J} \int_0^{\tilde{s}/2} dx Q_{J-1/2} \left( 1 - \frac{2x}{\tilde{s}} \right) (\ln s)^J s^{1-x}, \] (3.23)
where we recover the original integration region because the integrand is again sufficiently small in \( 2 < x < \tilde{s}/2 \). Thus we finally obtain the probability:
\[ \mathcal{P}(s, J) \simeq g_0^2 s^{1-\tilde{s}/2} I_J \left( \frac{\tilde{s}}{2} \ln s \right) (1 + (-1)^{s+J+1})(1 + (-1)^{s+1}), \] (3.24)
where we have put the contribution from the non-planar diagram which is obtained by replacing the argument \( v \) to \( -v \) in (3.10). It should be checked that summing over all the angular momenta reproduces the previous result (2.13) up to a term suppressed with \( s \) (thus negligible at high energy):
\[ \sum_{J=-\infty}^{\infty} \mathcal{P}(s, J) = \mathcal{P}(s), \] (3.25)
where we have used \( \sum_{J=-\infty}^{\infty} I_J(z) = e^z \) and \( \sum_{J=-\infty}^{\infty}(-1)^J I_J(z) = e^{-z} \).

The cross section \( \sigma(s, J) = \mathcal{P}(s, J)/s \) is given as
\[ \sigma(s, J) \simeq g_0^2 e^{-\xi} I_J(\xi)(1 + (-1)^J)(1 + (-1)^{s+1}), \quad (\xi := \frac{\tilde{s}}{2} \ln s). \] (3.26)

This is one of the main results of this paper. The fact that this cross section vanishes for even \( s \) may have the same reason as in (2.13). On the other hand, it should be noted that it also vanishes
for odd $J$. We regard this result as a reflection of the fact that the system has a symmetry under the $\pi$ rotation of 1-2 plane $\varphi \rightarrow \varphi + \pi$ and that the wave function of the produced rotating string is rotationally symmetric in the 1-2 plane.

Although this expression is obtained by an approximation for large $s$, it is instructive to make a further approximation for the modified Bessel function with large $\xi$ to see the asymptotic behavior of the cross section. In appendix B we give an asymptotic form of the modified Bessel function by a saddle point method. We find (up to the factor $(1 + (-1)^{s})(1 + (-1)^{s+1})$)

$$
\sigma(s, J) \sim g^2_o \frac{\exp\left( -\xi + \xi \sqrt{1 + \frac{J^2}{\xi^2}} - J \ln \left( \frac{J}{\xi} + \sqrt{1 + \frac{J^2}{\xi^2}} \right) \right)}{\sqrt{2\pi \xi \sqrt{1 + \frac{J^2}{\xi^2}}}}. \quad (3.27)
$$

The leading term in the expansion of $J/\xi$ is Gaussian:

$$
\sigma(s, J) \sim \frac{g_s}{\sqrt{2\pi \xi}} \exp \left( -\frac{J^2}{2\xi} \right), \quad (3.28)
$$

which explicitly shows that the cross section with a definite angular momentum decreases with energy, in particular $\sigma(s, J = 0) \sim (s \ln s)^{-1/2}$. The Gaussian factor tells us the form factor of a string. By noting that the angular momentum is related to the impact parameter through $J = b\sqrt{s}/2$, the form factor corresponds to an extended object with size $\sqrt{\ln s}$ as is well known.

### 4 Partial wave expansion

In this section we give another derivation of a cross section with specifying the total angular momentum, or one component $J_{12}$ respectively, based on the optical theorem. In fact, by the optical theorem, we derive the total cross section as the imaginary part of the Veneziano amplitude. Then expanding it in terms of an appropriate base of eigenfunctions of the total angular momentum squared $\hat{L}^2$ or in terms of the one for $\hat{J}_{12}$, we can read off the cross section in question.

We start with the Veneziano amplitude:

$$
V(s, t, u) = A(s, t) + A(t, u) + A(u, s), \quad (4.1)
$$

where

$$
A(s, t) = g^2_o \frac{\Gamma(-1 - s)\Gamma(-1 - t)}{\Gamma(-2 - s - t)}. \quad (4.2)
$$

We make the partial wave expansion of an $s$-channel pole residue by using the Legendre polynomials which are eigenfunctions of the total angular momentum squared:

$$
V(s, t, u) = \sum_L (2L + 1) A_L(s) P_L(1 + 2t/\bar{s}). \quad (4.3)
$$

Note here that in the present case the scattering angle $\theta$ is related to the Mandelstam variables through $\cos \theta = 1 + 2t/\bar{s}$. We are interested in the imaginary part of the Veneziano amplitude.
Noting
\[
\text{Im}_{s \sim (N-1)} \Gamma(-s - 1 + i\epsilon) = \frac{(-1)^{s+1}}{(s + 1)!} \delta(s - N + 1),
\]
then one can easily see
\[
\text{Im}_{s \sim (N-1)} V(s, t, u) = g_0^2 \pi (1 + (-1)^{s+1}) \frac{\Gamma(3 + s + t)}{\Gamma(s + 2)\Gamma(2 + t)} \delta(s - N + 1),
\]
where the term proportional to \((-1)^{s+1}\) stems from \(A(u, s)\). We have other delta functions which represent the on-shell momentum conservation condition for the pole residues. In the following we neglect the \(\delta\)-function factor with the on-shell condition understood. The coefficient function \(A_L(s)\) in the partial wave expansion of this \(s\)-channel pole residue is
\[
A_L(s) = \int_{-1}^{1} dz \text{Im} V(s, t, u) P_L(z), \quad z = 1 + \frac{2t}{s},
\]
\[
= g_0^2 (1 + (-1)^{s+1}) \frac{2\pi}{s} \int_{0}^{\frac{s}{2}} dx Q_L \left(1 - \frac{2x}{s}\right) \frac{\Gamma(s + 3 - x)}{\Gamma(s + 2)\Gamma(2 - x)} P_L \left(1 - \frac{2x}{s}\right), \quad x = -t.
\]
It can be shown by using the Rodrigues formula:
\[
P_n(x) = (-1)^n \frac{d^n}{dx^n} (1 - x^2)^n,
\]
that
\[
P_L \left(1 - \frac{2x}{s}\right) = (1 + \frac{\tilde{s}}{2})^L \left(-\frac{\tilde{s}}{2} \frac{\partial}{\partial x}\right)^L Q_L \left(1 - \frac{2x}{s}\right),
\]
where \(Q_L(x)\) is defined in (3.10). Thus we have
\[
A_L(s) = g_0^2 (1 + (-1)^{s+1}) \frac{2\pi}{s} \frac{\tilde{s}^L (-1)^L}{2^{2L} L!} \int_{0}^{\frac{s}{2}} dx Q_L \left(1 - \frac{2x}{s}\right) \frac{\partial^L}{\partial x^L} \frac{\Gamma(s + 3 - x)}{\Gamma(s + 2)\Gamma(2 - x)}.
\]
The integrand has the same property as we argued to obtain the projection factor in (3.18) which is in this case a consequence of the \(t \leftrightarrow u\) crossing symmetry originating from the one under an interchange of the final states which are identical with each other. It enables us to obtain
\[
A_L(s) = \frac{2\pi (1 + (-1)^{s+1})(1 + (-1)^L)g_0^2 \tilde{s}^{L-1}}{2^{2L} L!}
\]
\[
\times \int_{0}^{\frac{s}{2}} dx Q_L \left(1 - \frac{2x}{s}\right) \left(-\frac{\partial}{\partial x}\right)^L \frac{\Gamma(s + 3 - x)}{\Gamma(s + 2)\Gamma(2 - x)}.
\]
One immediately notices a similarity to (3.18) in the previous section.

As before we take the approximation (3.21) which is valid for large \(s\). The same calculation leads us to the cross section:
\[
\sigma_{open}(s, L) = (2L + 1)A_L/s
\]
\[
\simeq (1 + (-1)^{s+1})(1 + (-1)^L)\frac{g_0^2 \pi \sqrt{\tilde{s} \ln s/2}}{(\tilde{s} \ln s/2)^{1/2}} (2L + 1)I_{L+1/2}(\tilde{s} \ln s/2).
\]
In the argument above we have used the Legendre polynomials as a basis of the expansion, which are the four dimensional (part of) spherical harmonics. Instead one can use $D$-dimensional spherical harmonics which are given by the Gegenbauer polynomials and can repeat the same argument to obtain a result generalized to $D$-dimensions. In particular, we again have the same projection factor as in (4.10) because the integrand in this case also has the same property as in (4.9) which follows from the crossing symmetry. However instead of doing this we provide an alternative derivation of it in the following.

We start with the Regge limit ($s \gg 1$ with $t$ fixed) of the $s$-channel pole residue of the Veneziano amplitude:

$$\text{Im}_{s \sim (N-1)} V(s, t, u) \sim (1 + (-1)^{s+1}) s^{1+t}. \quad (4.15)$$

The limit breaks the $t \leftrightarrow u$ symmetry, however we should keep the crossing symmetry which is a reflection of the fact that the final states are identical as we mentioned before. We may recover it by introducing the term in the other Regge limit ($s \gg 1$ with $u$ fixed) and have

$$\text{Im}_{s \sim (N-1)} V(s, t, u) \sim (1 + (-1)^{s+1})(s^{1+t} + s^{1+u})/2$$

$$= (1 + (-1)^{s+1})se^{-\frac{1}{2} \ln s} (e^{\frac{3}{2} \ln s} + e^{-\frac{3}{2} \ln s})/2. \quad (4.16)$$

It is easy to verify a formula:

$$e^{\xi z} = \sum_{L=0}^{\infty} \frac{2^{3\lambda-1} \Gamma(\lambda + 1/2) \Gamma(\lambda)}{\Gamma(2\lambda)} (L + \lambda)_{\xi}^{-\lambda} I_{L+\lambda}(\xi) C_L^{\lambda}(z), \quad (4.17)$$

where $C_L^{\lambda}(z)$ is the Gegenbauer polynomial and $\lambda = (D - 3)/2$. We apply this formula together with the one obtained by replacing $z$ with $-z$ (thus we have $(-1)^L$ since $C_L^{\lambda}(-z)$ = $(-1)^L C_L^{\lambda}(z)$) to (4.16). Then we get a $D$-dimensional generalization of (4.11) that is

$$\sigma(s, L) \propto g_s^2 (1 + (-1)^{s+1})(1 + (-1)^L) e^{-\xi \ln s} (L + \lambda)_{\xi}^{-\lambda} I_{L+\lambda}(\xi), \quad \xi = \frac{\bar{s}}{2} \ln s. \quad (4.18)$$

However, we notice here that it is not guaranteed that we can always start from an amplitude in the Regge limit. In fact, it should be ensured by the fact that a function we try to expand in terms of spherical harmonics takes a sufficiently small value unless $x$ is small as we saw in (5.22).

---

5 The generating function of the Gegenbauer polynomials is

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{L=0}^{\infty} C_L^{\lambda}(x) t^L, \quad (4.12)$$

and the polynomial $C_L^{\lambda}(x)$ of order $L$ is given explicitly by

$$C_L^{\lambda}(x) = \frac{\Gamma(\lambda + 1/2) \Gamma(L + 2\lambda)}{\Gamma(2\lambda) \Gamma(L + \lambda + 1/2)} \frac{(-1)^L}{2^L L!} \frac{d^L}{dx^L} (1 - x^2)^{-\lambda+1/2} (1 - x^2)^{-\lambda-1/2 + L}, \quad (4.13)$$

which is reduced to the Legendre polynomial for $\lambda = 1/2$. They are normalized by

$$\int_{-1}^{1} dx (1 - x^2)^{-\lambda-1/2} [C_L^{\lambda}(x)]^2 = \frac{2^{1-2\lambda} \pi^2 \Gamma(L + 2\lambda)}{(L + \lambda) \Gamma(L + 1)}, \quad \lambda > -1/2. \quad (4.14)$$

6 See [39] for more detailed treatment and discussions.
One can calculate a partial wave amplitude for closed string case in a similar way. From the Virasoro-Shapiro amplitude:

\[ V(s, t, u) = g_s^2 \prod_{w=s,t,u} \frac{\Gamma(-1-w)}{\Gamma(2+w)}, \]

where \( s + t + u = -4 \) and \( \alpha' = 4 \), we find

\[ \sigma_{\text{closed}}(s, L) \propto g_s^2 (1 + (-1)^L) s e^{-\tilde{s} \ln s} (\tilde{s} \ln s)^{-\lambda} (L + \lambda) I_{L+\lambda}(\tilde{s} \ln s). \]

(4.20)

Note the existence of the projection factor originating from the crossing symmetry, but there is no \((1 + (-1)^{s+1})\) factor for the lack of the nonplanar diagram.

Among all \( L \), the largest contribution is provided by \( L = 0 \) and the cross section asymptotically becomes

\[ \sigma_{\text{open}}(s, L = 0) \sim g_o^2 \left( \frac{s}{2} \ln s \right)^{-\lambda -1/2}, \quad s \gg 1 \]

(4.21)

for open string and

\[ \sigma_{\text{closed}}(s, L = 0) \sim g_s^2 s (s \ln s)^{-\lambda -1/2}, \quad s \gg 1 \]

(4.22)

for closed string. It is evident that the cross section for the open string damps much faster than that of the closed string since the open string has less degrees of freedom than the closed string.

It is also interesting to compare the result in this section with that in the previous one. The cross section with a definite total angular momentum damps more rapidly than the one obtained in the previous section \( \sigma(s, J = 0) > \sigma_{\text{open}}(s, L = 0) \), since the projection to states with a definite total angular momentum is more restricted, and the number of states which contribute to the cross section is smaller than the case of fixed \( J_{12} \). As for the \( L \)-dependence of \( \sigma_{\text{open}}(s, L) \) given in (4.18) we find that it is again basically Gaussian and the same as \( \sigma(s, J) \) given in (3.26) or (3.28) when \( s \) is large enough by using the asymptotic form of the modified Bessel function given in appendix B. However, there is a slight difference in that \( L \) is shifted by \( \lambda \). It may originate from a quantum effect, but it is interesting to clarify its origin.

In the last, we show that we are able to reproduce the formula (3.26) by the partial wave expansion in terms of plane waves which are the eigenfunctions corresponding to \( J \). In other words, the partial wave amplitude is obtained as the Fourier transform of the imaginary part of the Veneziano amplitude. In fact one can explicitly show

\[ \text{Im}_{s \sim (N-1)} V(s, t, u) = \sum_{J=-\infty}^{\infty} P(s, J) e^{iJ\theta}, \]

(4.23)

where \( P(s, J) \) is given by (3.13) and its non-planar pair, which implies that we have

\[ P(s, J) = \int_0^{2\pi} \frac{d\theta}{2\pi} \text{Im}_{s \sim (N-1)} V(s, t, u) e^{-iJ\theta}. \]

(4.24)
5 Summary and discussions

We have calculated the total production cross sections of a single rotating string by both direct and indirect methods. The former is done with the help of the projection operator to eigenstates of the angular momentum, while the latter is by the optical theorem applied to the four point amplitudes. Both result in qualitatively the same formula which is written in terms of the modified Bessel function whose leading behavior has a Gaussian profile with respect to the angular momentum giving the form factor of a string corresponding to an object of size $\sqrt{\ln s}$. The cross sections with a fixed angular momentum damp as center of mass energy $\sqrt{s}$ grows, in contrast to the inclusive case in which the cross section stays constant or raises linearly in $s$ in open and closed string case, respectively. In our case two methods provide similar results as expected and the latter is easier than the former. However, we emphasize here that the former has an advantage that it can be generalized to cases where the optical theorem cannot be applied in a straightforward manner, like specifying other quantum numbers than the angular momentum, or a multi-string production cross section corresponding to a loop diagram which is in fact treated in [21], [22].

It would be intriguing to compare our result with the production cross section of a black hole with an angular momentum $L$ which is evaluated geometrically. It is known that when $L$ is not specified, the production cross section of a string and that of a black hole have different $s$-dependence even at the correspondence point [24]. Now our result enables us to compare $L$-dependence of these two as well. Indeed, they have again quite different behavior as functions of $L$, because the production cross section of a string is essentially Gaussian, while that of a black hole raises with $L$ at least in small $L$ region. Here it should be stressed that we do not take account of string loop effects. In fact, in high energy regime higher genus amplitudes are known to be dominant and hence it is evident that in order to compare the results of string and black hole at the correspondence point, we have to control higher genus amplitudes in a systematic way. Therefore it would be quite interesting to examine how $L$-dependence of the production cross section of a string would change drastically via higher genus amplitudes by using, e.g. the eikonal amplitude developed in [9], [10], assuming that the correspondence works.

Our result may have lots of potential applications. High energy string scattering amplitudes have been basic ingredients in arguments of the connection between string theory and hadron physics through the AdS/CFT correspondence [15], [16], [17], [28]. To apply not only to these hadron physics but also to cosmic string and Polymer physics, etc are also interesting.

Although we do not consider their effects, if scattering energy is high enough to produce some non-perturbative objects like D-branes and NS-branes, then we have to take account of these effects. It would be interesting to investigate these effects on the cross sections. Following [21], [22] it is also interesting to find the form factor or a profile of the wave function of a rotating string. The superstring generalization seems valuable for future use, though essential feature may be captured by bosonic string theory.
Acknowledgements

The authors would like to thank Kin-ya Oda for collaboration in an early stage of the present work. We are also grateful to Hikaru Kawai, Tamiaki Yoneya, and Keiji Igi for valuable discussions. One of the authors T.M would like to thank Takayuki Hirayama, Feng-Li Lin, Dan Tomino for discussions and comments. He also thanks to Hiroshi Itoyama for encouragement. The authors thank the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-07-05 on “String Theory and Quantum Field Theory” were useful to complete this work. The work of T.M is supported by the Taiwan’s National Science Council under grants NSC95-2811-M-003-005.

A Trace calculation

We would like to evaluate (non-zero modes)

\[
\text{tr}[V(k, \rho) z^J V(p, v) w^N] = \prod_{n=1} T_n^{1-2} \prod_{i=0,3,\ldots} T_i^{1-2}, \tag{A.1}
\]

where \( V \) is a vertex operator for a string state.

The normal ordered tachyon vertex operator is

\[
V(k, \rho) = \exp \left( \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n} k \cdot \alpha_n \rho^n \right) \exp \left( -\sqrt{2} \sum_{n=1}^{\infty} \frac{1}{n} k \cdot \alpha_{-n} \rho^{-n} \right). \tag{A.2}
\]

Note here we take \( \alpha' = 1 \) convention as in the main part of the present paper and set \( g_o = 1 \) for simplicity. We first focus on the 1,2 components and use an arrow to represent the (1, 2) component of vectors such as \( \vec{p} = (p_1, p_2) \), etc. The relevant part is

\[
T_n^{1-2}(k, p, \rho, z, v, w) = \text{tr} \left[ \exp \left( \sqrt{2} \frac{\rho_n}{n} \vec{k} \cdot \vec{\alpha}_n \right) \exp \left( -\sqrt{2} \frac{\rho^{-n}}{n} \vec{k} \cdot \vec{\alpha}_{-n} \right) \exp \left( -i \ln \frac{z}{n} \vec{\alpha}_{-n} \times \vec{\alpha}_n \right) \right]. \tag{A.3}
\]

To calculate the trace it is convenient to use the coherent state basis:\textsuperscript{7}

\[
\text{tr}(A) = \int \frac{d^2z_1}{\pi} \frac{d^2z_2}{\pi} e^{-|z_1|^2-|z_2|^2} \langle z_1 z_2 | A | z_1 z_2 \rangle, \tag{A.4}
\]

where

\[
| z_1 z_2 \rangle = \exp \left( \frac{1}{\sqrt{n}} \vec{z} \cdot \vec{\alpha}_{-n} \right) |0\rangle. \tag{A.5}
\]

\textsuperscript{7} In the worldsheet point of view, (A.1) is a finite temperature Green function (two point function on cylinder) between the rotated vertex \( z^{-J} V(k, \rho) z^J \) and \( V(p, v) \) with a worldsheet Hamiltonian \( \beta H = -\ln w \hat{N} - \ln z \hat{J} \). Therefore one can use, as an alternative way, the Wick’s theorem in terms of the finite temperature Green function to find (A.9).
Using the Baker-Campbell-Hausdorff (BCH) formula it yields

\[
T_n^{1-2} = \int \frac{d^2 z_1}{\pi} \int \frac{d^2 z_2}{\pi} e^{-|z_1|^2-|z_2|^2} \exp \left( \sqrt{2} \frac{\rho_n}{\sqrt{n}} \vec{k} \cdot \vec{z}^* - \sqrt{2} \frac{\rho_n w^{n-n}}{\sqrt{n}} \vec{p} \cdot \vec{z} \right) \\
\times \langle 0 | \exp \left( \left( -\sqrt{2} \frac{\rho_n}{\sqrt{n}} \vec{k} + \frac{\vec{z}^*}{\sqrt{n}} \right) \cdot \vec{\alpha}_n \right) \exp \left( -i \frac{\ln z}{n} \vec{\alpha}_- \times \vec{\alpha}_n \right) \\
\times \exp \left( \frac{\rho_n}{\sqrt{n}} \vec{z}^* + \sqrt{2} \frac{\rho_n}{\sqrt{n}} \vec{p} \cdot \vec{\alpha}_- \right) \rangle | 0 \rangle.
\]

(A.6)

To compute this we shall use the formula which is easy to verify

\[
\exp \left( -\frac{iA}{n} \vec{\alpha}_- \times \vec{\alpha}_n \right) \exp \left( \vec{B} \cdot \vec{\alpha}_- \right) | 0 \rangle = \exp \left( (\cos(iA)\alpha^L_1 + \sin(iA)\alpha^R_1) B_1 + (\sin(iA)\alpha^L_2 + \cos(iA)\alpha^R_2) B_2 \right) | 0 \rangle,
\]

(A.7)

and again we use the BCH formula to eliminate all the creation and annihilation operators. Then using the formula:

\[
\int \frac{d^2 z_1}{\pi} \int \frac{d^2 z_2}{\pi} e^{-c_1|z_1|^2 + a_1z_1 + b_1z_1^2 - c_2|z_2|^2 + a_2z_2 + b_2z_2^2 + c_2z_1 + d_2z_2} \\
\times \frac{1}{c_1c_2 - ed} \exp \left( \frac{a_2b_2c_1 + a_1b_1c_2 + a_1b_2e + a_2b_1d}{c_1c_2 - ed} \right),
\]

(A.8)

we arrive at

\[
T_n^{1-2} = \frac{1}{1 - 2cw^n + w^{2n}} \exp \left[ -\frac{2}{n(1 - 2cw^n + w^{2n})} \left\{ s\vec{p} \times \vec{k}(v/\rho)^n - sw^{2n}\vec{p} \times \vec{k}(\rho/v)^n \\
+ (c - w^n)\vec{p} \cdot \vec{k}(v/\rho)^n + w^n(1 - cw^n)\vec{p} \cdot \vec{k}(v/\rho)^n + w^n(c - w^n)(\vec{p} \cdot \vec{k} + \vec{p} \cdot \vec{k}) \right\} \right],
\]

(A.9)

where

\[
c := \cosh(\ln(z)) = \frac{1}{2}(z + z^{-1}), \quad s := i \sinh(\ln(z)) = \frac{i}{2}(z - z^{-1}).
\]

(A.10)

Taking product with respect to \(n\) by using a formula (which is verified by expanding both sides):

\[
\sum_{n=1}^{\infty} \frac{V^n}{n(1 - zw^n)(1 - z^{-1}w^n)} = \sum_{m=0}^{\infty} \frac{z^{m+1} - z^{-m-1}}{z - z^{-1}} \ln(1 - Vw^m),
\]

(A.11)

one obtains

\[
\prod_{n=1}^{\infty} T_n^{1-2}(\vec{k}, \vec{p}, \rho, z, v, w) = \tilde{f}(z, w)^{-1} \left[ \hat{\Psi}(z, v/\rho, w) \right]^{\vec{p} \cdot \vec{k}} \left[ \hat{\Phi}(z, v/\rho, w) \right]^{i\vec{p} \times \vec{k}},
\]

(A.12)

where

\[
\tilde{f}(z, w) := \prod_{m=1}^{\infty} (1 - zw^m)(1 - z^{-1}w^m),
\]

(A.13)
\[ \hat{\Psi}(z, v, w) := (1 - v)^{z+z^{-1}} \prod_{m=1}^{\infty} \frac{(1 - vw^m)^{z+m-1} + z - m - 1}{(1 - w^m)^{2(z+m-z-1)}}, \quad (A.14) \]

and

\[ \hat{\Phi}(z, v, w) := (1 - v)^{z-z^{-1}} \prod_{m=1}^{\infty} \frac{(1 - vw^m)^{z+m-1} - z - m - 1}{(1 - w^m)^{-(z+m-1)}}, \quad (A.15) \]

Here we have used the on-shell condition \( \vec{p} + \vec{k} = 0 \) to convert \( \vec{p}^2 + \vec{k}^2 = -2\vec{p} \cdot \vec{k} \).

It is easy to obtain contributions from other components \( i = 0, 3, \ldots, d \) by setting \( z = 1 \) in the above expressions, then

\[ T_i = \frac{1}{1 - w^n} \exp \left[ -\frac{2}{n(1 - w^n)} \left\{ p^i k^i (v/\rho)^n + w^n p^i k^i (\rho/v)^n + w^n (k^i k^i + p^i p^i) \right\} \right]. \quad (A.16) \]

One may get after taking product with respect to \( n \)

\[ \prod_{n=1}^{\infty} T_n(k, p_i; \rho, 1, v, w) = [f(w)]^{-1} \hat{\psi}(v/\rho, w)^{2p_i k_i}, \quad (A.17) \]

for each \( i = 0, 3, \ldots, d \) and again we have used the on-shell condition \( p_i + k_i = 0 \). Here the Dedekind function \( f(w) \) and \( \hat{\psi}(v, w) \) are given as

\[ f(w) := \prod_{n=1}^{\infty} (1 - w^n), \quad (A.18) \]

and

\[ \hat{\psi}(v, w) := (1 - v)^{z-z^{-1}} \prod_{m=1}^{\infty} \frac{(1 - vw^m)(1 - w^m/v)}{(1 - w^m)^2}. \quad (A.19) \]

Combining all we obtain

\[ \text{tr}[V(k, \rho) z^I V(p, v) w^J] = [f(w)]^{-d+1} \hat{f}(z, w)^{-1} \times \prod_{i=0,3,\ldots,d} \hat{\psi}(v/\rho, w)^{2p_i k_i} \hat{\psi}(v/\rho, w)^{\vec{p} \cdot \vec{k}} \hat{\Phi}(z, v/\rho, w)^{i \vec{p} \times \vec{k}}. \quad (A.20) \]

B  Asymptotic form of the modified Bessel function

The modified Bessel function has a following integral representation for integer \( \nu \):

\[ I_\nu(z) = \text{Re} \left[ \frac{1}{\pi} \int_0^{\pi} d\theta e^{z \cos \theta + i\nu \theta} \right]. \quad (B.1) \]

Let us derive an asymptotic formula for large \( z \) by a saddle point method. Let \( f(\theta) := z \cos \theta + i\nu \theta \), we find a saddle point \( \theta_0 \) which is determined by

\[ \theta_0 = \sin^{-1}(i\nu/z) = i \sinh^{-1}(\nu/z) = i \ln \left( \frac{\nu}{z} + \sqrt{1 + \frac{\nu^2}{z^2}} \right), \quad (B.2) \]
which is pure imaginary for real $\nu, z$. This can be an end point of the integration region by an appropriate modification of the contour. The half of the saddle point thus contributes to the solution. Let $\theta_0$ be a saddle point. The asymptotic form is given as

$$I_\nu(z) \simeq \frac{1}{2} \text{Re} \left[ \frac{e^{f(\theta_0)}}{\pi} \sqrt{\frac{2\pi}{|f''(\theta_0)|}} \right], \quad (B.3)$$

where

$$\text{Re} \exp(f(\theta_0)) = e^{z \sqrt{1 + \nu^2/z^2} - \nu \sinh^{-1}(\nu/z)} = e^{z \sqrt{1 + \nu^2/z^2} - \nu \ln \left( \frac{z}{\nu} + \sqrt{1 + \nu^2/z^2} \right)}, \quad (B.4)$$

and

$$f''(\theta_0) = -z \cos(\theta_0) = -z \sqrt{1 + \nu^2/z^2}, \quad (B.5)$$

thus

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z \sqrt{1 + \nu^2/z^2}}} \exp \left[ z \sqrt{1 + \nu^2/z^2} - \nu \ln \left( \frac{\nu}{z} + \sqrt{1 + \nu^2/z^2} \right) \right], \quad (B.6)$$

References

[1] D. J. Gross and P. F. Mende, “The High-Energy Behavior of String Scattering Amplitudes,” Phys. Lett. B 197 (1987) 129.

[2] D. J. Gross and P. F. Mende, “String Theory Beyond the Planck Scale,” Nucl. Phys. B 303 (1988) 407.

[3] P. F. Mende and H. Ooguri, “BOREL SUMMATION OF STRING THEORY FOR PLANCK SCALE SCATTERING,” Nucl. Phys. B 339 (1990) 641.

[4] M. Soldate, “Partial Wave Unitarity and Closed String Amplitudes,” Phys. Lett. B 186 (1987) 321.

[5] I. J. Muzinich and M. Soldate, “High-Energy Unitarity of Gravitation and Strings,” Phys. Rev. D 37 (1988) 359.

[6] D. J. Gross, “High-Energy Symmetries Of String Theory,” Phys. Rev. Lett. 60 (1988) 1229.

[7] C. T. Chan and J. C. Lee, “Stringy symmetries and their high-energy limits,” Phys. Lett. B 611 (2005) 193 [arXiv:hep-th/0312226].

[8] C. T. Chan, P. M. Ho, J. C. Lee, S. Teraguchi and Y. Yang, “High-energy zero-norm states and symmetries of string theory,” Phys. Rev. Lett. 96 (2006) 171601 [arXiv:hep-th/0505035].
[9] D. Amati, M. Ciafaloni and G. Veneziano, “Superstring Collisions at Planckian Energies,” Phys. Lett. B 197 (1987) 81.

[10] D. Amati, M. Ciafaloni and G. Veneziano, “Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions,” Int. J. Mod. Phys. A 3 (1988) 1615.

[11] D. Amati, M. Ciafaloni and G. Veneziano, “Can Space-Time Be Probed Below The String Size?,” Phys. Lett. B 216 (1989) 41.

[12] G. Veneziano, “String-theoretic unitary S-matrix at the threshold of black-hole production,” JHEP 0411 (2004) 001 [arXiv:hep-th/0410166].

[13] D. Amati, M. Ciafaloni and G. Veneziano, “Towards an S-matrix Description of Gravitational Collapse,” arXiv:0712.1209 [hep-th].

[14] T. Yoneya, “String theory and space-time uncertainty principle,” Prog. Theor. Phys. 103 (2000) 1081 [arXiv:hep-th/0004074].

[15] J. Polchinski and M. J. Strassler, “Hard scattering and gauge/string duality,” Phys. Rev. Lett. 88 (2002) 031601 [arXiv:hep-th/0109174].

[16] J. Polchinski and M. J. Strassler, “Deep inelastic scattering and gauge/string duality,” JHEP 0305 (2003) 012 [arXiv:hep-th/0209211].

[17] R. C. Brower, J. Polchinski, M. J. Strassler and C. I. Tan, “The pomeron and gauge / string duality,” arXiv:hep-th/0603115.

[18] L. F. Alday and J. M. Maldacena, “Gluon scattering amplitudes at strong coupling,” JHEP 0706 (2007) 064 [arXiv:0705.0303 [hep-th]].

[19] D. Amati and J. G. Russo, “Fundamental strings as black bodies,” Phys. Lett. B 454, 207 (1999) [arXiv:hep-th/9901092].

[20] J. L. Manes, “Emission spectrum of fundamental strings: An algebraic approach,” Nucl. Phys. B 621 (2002) 37 [arXiv:hep-th/0109196].

[21] J. L. Manes, “String form factors,” JHEP 0401 (2004) 033 [arXiv:hep-th/0312035].

[22] J. L. Manes, “Portrait of the string as a random walk,” JHEP 0503 (2005) 070 [arXiv:hep-th/0412104].

[23] B. Chen, M. Li and J. H. She, “The fate of massive F-strings,” JHEP 0506 (2005) 009 [arXiv:hep-th/0504040].

[24] S. Dimopoulos and R. Emparan, “String balls at the LHC and beyond,” Phys. Lett. B 526, 393 (2002) [arXiv:hep-ph/0108060].
[25] G. T. Horowitz and J. Polchinski, “A correspondence principle for black holes and strings,” Phys. Rev. D 55, 6189 (1997) [arXiv:hep-th/9612146].

[26] G. T. Horowitz and J. Polchinski, “Self gravitating fundamental strings,” Phys. Rev. D 57, 2557 (1998) [arXiv:hep-th/9707170].

[27] T. Damour and G. Veneziano, “Self-gravitating fundamental strings and black holes,” Nucl. Phys. B 568, 93 (2000) [arXiv:hep-th/9907030].

[28] S. B. Giddings, “High energy QCD scattering, the shape of gravity on an IR brane, and the Froissart bound,” Phys. Rev. D 67 (2003) 126001 [arXiv:hep-th/0203004].

[29] S. B. Giddings, “Locality in quantum gravity and string theory,” Phys. Rev. D 74 (2006) 106006 [arXiv:hep-th/0604072].

[30] S. B. Giddings, “Black hole information, unitarity, and nonlocality,” Phys. Rev. D 74 (2006) 106005 [arXiv:hep-th/0605196].

[31] S. B. Giddings, D. J. Gross and A. Maharana, “Gravitational effects in ultrahigh-energy string scattering,” [arXiv:0705.1816] [hep-th].

[32] S. B. Giddings and M. Srednicki, “High-energy gravitational scattering and black hole resonances,” [arXiv:0711.5012] [hep-th].

[33] R. Iengo and J. G. Russo, “The decay of massive closed superstrings with maximum angular momentum,” JHEP 0211 (2002) 045 [arXiv:hep-th/0210245].

[34] R. Iengo and J. G. Russo, “Semiclassical decay of strings with maximum angular momentum,” JHEP 0303 (2003) 030 [arXiv:hep-th/0301109].

[35] D. Chialva, R. Iengo and J. G. Russo, “Decay of long-lived massive closed superstring states: Exact results,” JHEP 0312 (2003) 014 [arXiv:hep-th/0310283].

[36] D. Chialva, R. Iengo and J. G. Russo, “Search for the most stable massive state in superstring theory,” JHEP 0501 (2005) 001 [arXiv:hep-th/0410152].

[37] R. Iengo and J. G. Russo, “Handbook on string decay,” JHEP 0602 (2006) 041 [arXiv:hep-th/0601072].

[38] H. Kawai, D. C. Lewellen and S. H. H. Tye, “A Relation Between Tree Amplitudes Of Closed And Open Strings,” Nucl. Phys. B 269 (1986) 1.

[39] K. Oda, in preparation.

[40] J. G. Russo and L. Susskind, “Asymptotic level density in heterotic string theory and rotating black holes,” Nucl. Phys. B 437 (1995) 611 [arXiv:hep-th/9405117].