Algebraic and Polytopic Formulation to Cohomology

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Abstract

The polytopic definition introduced recently describing the topology of manifolds is used to formulate a generating function pertinent to its topological properties. In particular, a polynomial in terms of one variable and a tori underlying this polynomial may be defined that generates an individual cohomological count. This includes the de Rham complex for example, as well as various index theorems by definition such as homotopy. The degree of the polynomials depends on the volume used to define the region parameterizing the manifolds; its potentially complex form and L-series is not presented in this work. However, the polynomials and the relevant torii uniformize the topological properties in various dimensions; in various dimensions this is interesting in view of known topologies.
Introduction

The classification of manifolds has occupied theorists, both physicists and mathematicians of many types, for a long time. In general the construction of invariants such as the dimensions of de Rham complexes or the computation of indices of spectral operators relevant to homotopy or cobordism groups follows from a variety of techniques. The dimension of the space (or manifold) is important for both the technique used to compute the invariants and the end result.

Recently a polytopic construction was introduced so that a set of points in a lattice consisting of $N^d$ entries may be parameterized by a single number $z$. This means that all $q_1$-dimensional manifolds embedded in $q_2$-dimensional space may be labeled by a single number. Each number represents a manifold, and the construction permits possible uniformizations of the properties of manifolds in diverse dimensions, for example a quantity is computable via a function $P(z)$. The polytopes were given in [1]; related work on the L-series is in [2], and on knot invariants and polynomials in [3]. Compact number expressions describing gauge amplitudes might also be relevant.

The invariants are introduced and defined in the subsequent section, and following this, the latticed manifolds are defined and numbered.

Invariants

Consider that a manifold in $d$ dimensions embedded in a lattice of $N_d$ points is labeled by a number from 0 to $N$. Then by definition a polynomial of degree $N$

$$P_I(\tilde{z}) = \sum b_i \tilde{z}^i$$

(1)

generates an invariant $I$ for all manifolds placed in the space. Example would be the individual $(p, q)$ forms in the Dolbeaux cohomology, homotopies, differential structures, etc... Associated to the function (1) is a function,

$$\tilde{z} = \sum c_i z^i ,$$

(2)

which is a map from the basis of the numbers labeling the manifolds to another set of numbers. The latter set of numbers is potentially more convenient for the input into the polynomials in (1).

The space of all the manifolds, each labeled by an integer, generates numbers by the manipulation,
\[
\begin{pmatrix}
P_I(z_1) \\
P_I(z_2) \\
\vdots \\
P(z_{N_d})
\end{pmatrix} =
\begin{pmatrix}
z_1 & z_1^2 & \cdots & z_1^N \\
z_2 & z_2^2 & \cdots & z_2^N \\
\vdots & \vdots & \cdots & \vdots \\
z_{N_d} & z_{N_d}^2 & \cdots & z_{N_d}^N
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_N
\end{pmatrix}.
\]
(3)

An explicit construction of the numbers \(b_i\) via the left inverse show that for integer \(P_I(z_i)\) the numbers may be rationalized into integers by multiplying them via the inverse of the matrix in (3).

The same set of numbers generates an L-series after the numbers \(b_i\) are rationalized to integers and placed into correspondence with the primes via counting them in accordance with the integers \(i\) via \(p_i\), and with the \(b_i\) numbers,

\[
\zeta(s, C') = \prod \left(1 + a_pp^{-s} + p^{1-2s}\right)^{-1},
\]
(4)

and \(-p + b_p = a_p\). The construction in terms of the L-series means that there is an elliptic curve defined for the lattice with \(N\) points with \(b_p\) solutions to

\[
y^2 = x^3 + \alpha_1 x + \alpha_2 \mod p.
\]
(5)

There is a series of curves for the series of \(N\) points (and the dimensionality). Depending on the invariant the curves might have special properties.

A separate modular form is defined by

\[
\sum e^{-P_I(z)} = \sum \Delta_I(w)e^{-w},
\]
(6)

which essentially counts the distributions of the invariants of the embedded manifolds parameterized by the polynomial \(P_I(z)\). The function \(\Delta_I(w)\) counts the number of \(z\)-solutions to \(w = P_I(z)\).

The direct computation of these invariants requires specifying the ordering of the points in the lattice defining the embedding space. The exact ordering of the embedding of the lattice points and their relation to the number labeling the manifold may play an important role in specifying the forms of the polynomials and their associated modular forms, and tori. In other words, an appropriate choice may simplify both the L-series and torus in (4) and (5) (and that of the 'pseudo'-modular form in (6)).
It is of interest to find the best parameterization of the embedded manifolds, via the integers \( z \) and the mapping \( \tilde{z} = f_I(z) \) relevant to the index \( P_I(z) \). This is useful in various dimensions to both relate the \( P_I \) quantities to more symmetric tori (symmetric via their L-series) in \( \mathbb{R}^d \) and also to each other in differing dimensions.

The finite volume specification of the invariants \( P_I \) are expected to have a \( V \to \infty \) limit.

Although the exact form of the invariants are not given here, it would certainly be interesting to find their forms and any coefficient structure in the different dimensions, and at large volume. The L-series formulation could be relevant in the classification of topologies. Also, fractional dimensional manifolds can be characterized by continuation of the \( P_I^{(d)} \) into non-integral dimensions; the continuation could define the properties in this case.

**Polytope construction**

In this section the polytopic, or rather simplicial complex, construction of the manifolds is given. The topology of the manifolds are found essentially by filling in a lattice with a set of points. A basic simplicial complex is defined by ‘connecting the dots’ in \( \mathbb{R}^d \). In the polytopic construction here, space-filling ‘membranes,’ embedded in \( \mathbb{R}^d \), are used. All embeddable manifolds are accessible via this construction (such as a Klein manifold in \( d = 4 \)).

The polytopes (simplicial complexes) considered are constructed via a set of integers that label the points and faces parameterizing the surface. The integers may be given a matrix representation that permits a polynomial interpretation, and hence maps to knot(s) invariant(s).

Take a series of numbers \( a_1a_2 \ldots a_n \) corresponding to the digits of an integer \( p \), with the base of the individual number being \( 2^n \); this number \( a_j \) could be written in base 10 by the usual digits. In this way, upon reduction to base 2 the digits of the base reduced number spans a square with \( n + 1 \) entries. Each number \( a_j \) parameterizes a column with ones and zeros in it. The lift of the numbers could be taken to base 10 with minor modifications, by converting the base of \( p \) to 10 (with possible remainder issues if the number does not ‘fit’ well).

The individual numbers \( a_i \) decompose as \( \sum a_i^m 2^m \) with the components \( a_i^m \) being 0 or 1. Then map the individual number to a point on the plane,

\[
\vec{r}_i^m = a_i^m \times m\hat{e}_1 + a_i^m \times i\hat{e}_2 ,
\]
with the original number mapping to a set of points on the plane via all of the entries in $a_1 a_2 \ldots a_m$. In doing this, a collection of points on the plane is spanned by the original number $p$, which could be a base 10 number. The breakdown of the number to a set of points in the plane is represented in figure 1.

A set of further integers $p_j = a_{1}^{(j)} a_{2}^{(j)} \ldots a_{n}^{(j)}$ are used to label a stack of coplanar lattices with the same procedure to fill in the third dimension. The spacial filling of the disconnected polhedron is assembled through the stacking of the base reduced integers.

Colored polytopes are introduced by taking the integers $p_j$ into the numbers $a_{j}^{(k,m)}$ with base $N$ as opposed to base 2. The individual entries in the lattice are spanned by the vector,

$\vec{r} = \vec{r}^m = a_{i}^{m,k} \times m\hat{e}_1 + a_{i}^{m,k} \times i\hat{e}_2 + a_{i}^{m,k} \times k\hat{e}_3 . \quad (8)$

The base reduced entries of $a_{i}^{m,k}$ may be attributed into 'colors' or a group theory indices labeling a representation.

Next the volume $V$ and the $\partial V$ surface area of the polytope region is deduced from the entries $a_{i}^{m,k}$. The volume is the sum of the individual entries $a_{i}^{m,k}$ over the entire lattice,

$p_j = a_{i,j}^i 2^i \quad V_s = \sum_{i,k,m} a_{i}^{k,m} . \quad (9)$

The surface area of the polytope is a region bounded by the entries of the entries $a_{i}^{m,k}$. The bounded region is found via the differences of the entries $a_i^j$; in two dimensions,

$V_{sf} = \sum_{ij} |a_{i}^j - a_{i-1}^j| - \sum_{ij} |a_{i}^j - a_{i-1}^j| . \quad (10)$

The region bounding the polytope is deduced from the differences in the integers.

The terms in both series, $V_s$ and $V_{sf}$, are defined or computed via the expansions,

$P^i_1 = \sum M^ij_{(1)} p^j \quad P^i_2 = M^ij_{(2)} p^j = \sum |a_i - a_{i-1}|_{pint} , \quad (11)$

$P^i_1 = \sum a|_{pint} , \quad (12)$
defined for the integer $p$ configuration. Even though the individual terms $|a_i - a_i|$ in the summations involved the expansion are absolute value, the entire sum is found via a summation over the individual numbers $p$ parameterizing the lattice and its configuration. (A discussion of the computation of the matrices $M$ is found in [1].)

The 'colored' boundary is given a boundary via the same formalism, but with a generalized difference $|a_i - a_{i-1}|$; group theory or 'color' differences found with a different inner product are possible. The summations for these numbers may also be inverted to obtain the values $a_i$ in terms of $p^j$ and an associated matrix.

An example list of this variables is given in the following table,

\[
\begin{pmatrix}
  p & a_i & p_1 & p_2 \\
  1 & 1 & 1 & 1 \\
  2 & 01 & 1 & 2 \\
  3 & 11 & 2 & 0 \\
  4 & 001 & 1 & 2 \\
  5 & 101 & 2 & 2 \\
  6 & 011 & 2 & 2 \\
  7 & 111 & 3 & 0 \\
\end{pmatrix}
\]

(13)

The number $p$ is listed, followed by the binary format; the integers $p_1$ and $p_2$ are the sums $\sum a_i$ and $\sum |a_i - a_{i+1}|$, in a cyclic fashion around the numbers $p$.

The polyhedron is constructed by the single numbers spanning the multiple layers in 3-d, or by one number with the former grouped as $p_1p_2\ldots p_n$. The generalization to multiple dimensions is straightforward. Also, the sewing and dissection of the manifolds based on the operations on the integer $p$ is straightforward.

**Conclusions**

Manifolds are put in one to one correspondence with the integers via an embedding into a d-dimensional lattice. To each integer $z$ there is a set of latticized points specifying the manifold.

Properties such as cohomology, and other topological ones, are formulated in a uniform sense via mappings

\[
P(\tilde{z}) = \sum b_i \tilde{z}^i, \quad \tilde{z} = \sum c_i z^i.
\]

(14)
The first function specifies the index of the manifold $z$ via its value on $z$. The latter function is a redefinition of the coordinates labeling the lattice configurations; these two functions should enter with each other for each of the topological indices.

The values of the indices, specified on all manifolds within a lattice of size $N^d$, may be put into correspondence with an L-series via,

$$\zeta(s, C) = \prod \left(1 + a_p p^{-s} + p^{1-2s}\right)^{-1}, \quad (15)$$

and $-p + b_p = a_p$; a simple ordering is for the primes to be put into correspondence with the $z$-values and $b_p = P(z)$. The function $\tilde{z} = f(z)$ is chosen to systematize the entries in (15). In this fashion, there is an elliptic curve that specifies the topological index in $d$-dimensions for a lattice of size $V$ (and as $V \to \infty$).
References

[1] G. Chalmers, *Polytopes and Knots*, physics/0503212.

[2] G. Chalmers, *Integer and Rational Solutions to Polynomial Equations*, physics/0503020.

[3] G. Chalmers, *A New Knot Invariant*, physics/0503081.

[4] G. Chalmers, *Very Compact Expressions for Amplitudes*, physics/0502058.