INITIAL-BOUNDARY VALUE PROBLEM OF THE NAVIER-STOKES EQUATIONS IN THE HALF SPACE WITH NONHOMOGENEOUS DATA

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Abstract. This paper discusses the solvability (global in time) of the initial-boundary value problem of the Navier-Stokes equations in the half space when the initial data $h \in \dot{B}^{\frac{\alpha}{q} - \frac{2}{q}}_{\alpha} (\mathbb{R}^n)$ and the boundary data $g \in \dot{B}^{\frac{\alpha}{q} - \frac{1}{q}}_{\alpha} (\mathbb{R}^n) \cap L^q (\mathbb{R}^n)$, for any $0 < \alpha < 2$ and $q = \frac{n + 2}{\alpha + 1}$. Compatibility condition (1.3) is required for $h$ and $g$.

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1. Introduction

Let $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n | x_n > 0 \}$, $n \geq 2$. In this study, we consider the following nonstationary Navier–Stokes equations

$$
\begin{align*}
  u_t - \Delta u + \nabla p &= -\text{div}(u \otimes u), \\
  \text{div} u &= 0 \text{ in } \mathbb{R}^n_+ \times (0, \infty), \\
  u|_{t=0} &= h, \\
  u|_{x_n=0} &= g,
\end{align*}
$$

where $u = (u_1, \cdots, u_n)$ and $p$ are the unknown velocity and pressure, respectively, $h = (h_1, \cdots, h_n)$ is the given initial data, and $g = (g_1, \cdots, g_n)$ is the given boundary data.

Abundant literature exists on Navier–Stokes equations with homogeneous boundary data ($g = 0$) (See [6, 10, 21, 22, 24, 31, 37] and the references therein).

Further, over the past decade, many mathematicians have focused on studying Navier–Stokes equations with nonhomogeneous boundary data ($g \neq 0$) (See [4, 7, 8, 12, 13, 14, 16, 17, 18, 19, 25, 26, 27, 28, 33, 35, 44] and the references therein).

The study closely relating to our present study is that by G. Grubb [27], who used pseudo-differential operator techniques to realize the local in time existence of solution $u \in B^{\frac{\alpha}{q}}_q (\Omega \times (0, T))$, $\infty > q > \frac{n + 2}{\alpha + 1}$ with $\alpha q > 2$ in the interior or exterior domains, when $h \in \dot{B}^{\frac{\alpha}{q} - \frac{2}{q}}_q (\Omega)$ and $g \in B^{\frac{\alpha}{q} - \frac{1}{q}}_q (\partial \Omega \times (0, T))$ with $g_n = 0$ (When $h = 0$, the result in [27] was given up to the case $\alpha q > 1$(and $\infty > q > \frac{n + 2}{\alpha + 1}$). See also [24, 26, 28, 37]). Here, let $B^s_q (S \subset \mathbb{R}^m)$ be the set of distributions in Besov space $B^s_q (\mathbb{R}^m)$ supported in $\tilde{S}$, and $B^{s \frac{\alpha}{q}}_q (S \times (0, T))$ be the set of distributions in anisotropic Besov space $B^{s \frac{\alpha}{q}}_q (\mathbb{R}^m \times (-\infty, T])$ supported in $\tilde{S} \times [0, T]$.

In Refs. [7, 8, 16, 17, 18, 19], rough initial and boundary data were considered for the local data in the time existence of weak or very weak solutions. In Refs. [4, 33, 44], a mild-type solution was considered in the half space when the rough
initial and boundary data are given. Recently, Chang and Jin [14] studied the 
local in time solvability of Navier–Stokes equations when \( h \in B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n_+) \) and \( g \in B_q^{\lambda-\frac{\lambda}{q}}(\mathbb{R}^{n-1} \times (0, \infty)) \), \( q > n + 2 \). By the same authors [14], this result is extended to 
global time existence with small initial and boundary data.

This study aims to extend the result of [27] to any \( h \in B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n_+) \) and \( g \in B_q^{\lambda-\frac{\lambda}{q}}(\mathbb{R}^{n-1} \times (0, \infty)) \) with \( g_n \in B_q^{\lambda-\frac{\lambda}{q}}(0, \infty; L^q(\mathbb{R}^{n-1})) \) \( \cap L^q(0, \infty; B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^{n-1})) \), where \( q = \frac{n+2}{\alpha+1} \) and \( 0 < \alpha < 2 \). For \( \alpha > \frac{2}{q} \), the following compatibility condition is required:

\[
g|_{t=0} = h|_{x_n=0} \quad \text{on} \quad \mathbb{R}^{n-1}.
\]

(1.2)

The compatibility [12] can be generalized to any \( \alpha > 0 \) and \( q > 1 \) as follows:

\[
g - \Gamma_t \ast \hat{h}|_{x_n=0} \in B_q^{\alpha-\frac{\alpha}{q}, \frac{2}{q}-\frac{\lambda}{q}}(\mathbb{R}^{n-1} \times (0, \infty)),
\]

(1.3)

where \( B_q^{\lambda, \eta}(\mathbb{R}^{n-1} \times (0, \infty)) \) is the completion of \( C_0^\infty(\mathbb{R}^{n-1} \times (0, \infty)) \) in \( B_q^{\alpha, \eta}(\mathbb{R}^{n-1} \times (0, \infty)) \), \( \hat{h} \in B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n) \) is some solenoidal extension of \( h \) to \( \mathbb{R}^n \) with \( \|\hat{h}\|_{B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n)} \approx \|h\|_{B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n)} \) and \( \Gamma_t \ast f|_{x_n=0} := \int_{\mathbb{R}^n} \Gamma(x' - y', y_n, t) f(y) dy \). (According to Lemma 3.7, \( \Gamma_t \ast f|_{x_n=0} \in B_q^{\alpha-\frac{\alpha}{q}, \frac{2}{q}-\frac{\lambda}{q}}(\mathbb{R}^{n-1} \times (0, \infty)) \) for any \( \alpha > 0, q > 1 \) when \( f \in B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n) \).

The following text states our main result.

**Theorem 1.1.** Let \( 0 < \alpha < 2 \) and \( 1 < q = \frac{n+2}{\alpha+1} < \infty \). Further, let \( h \in B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^n_+) \) and \( g \in B_q^{\lambda-\frac{\lambda}{q}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \) with \( g_n \in B_q^{\lambda-\frac{\lambda}{q}}(0, \infty; B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^{n-1})) \) \( \cap L^q(0, \infty; B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^{n-1})) \). In addition, we assume that \( (h, g) \) satisfies the generalized compatibility condition (1.3). Then, there exists \( \varepsilon^* > 0 \) such that if

\[
\|h\|_{B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}_+^n)} + \|g\|_{B_q^{\alpha-\frac{\alpha}{q}, \frac{2}{q}-\frac{\lambda}{q}}(\mathbb{R}^{n-1} \times (0, \infty))} + \|g_n\|_{B_q^{\lambda-\frac{\lambda}{q}}(0, \infty; B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^{n-1}))} + \|g_n\|_{L^q(0, \infty; B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}^{n-1}))} \leq \varepsilon^*,
\]

then the (1.1) has a unique weak solution \( u \in B_q^{\alpha-\frac{\alpha}{q}}(\mathbb{R}_+^n \times (0, \infty)) \).

Section 2 further explains the spaces and notations.

Note that as the nonstationary Navier–Stokes equations is invariant under the scaling,

\[
u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad h_\lambda = \lambda h(\lambda x), \quad g_\lambda(x, t) = \lambda g(\lambda x, \lambda^2 t), \quad \lambda > 0,
\]
significantly considering (1.1) in the so-called critical spaces, i.e., the function space invariant under the scaling \( u(\cdot) \rightarrow \lambda u(\lambda \cdot) \) is very important. In Theorem 1.1, the function spaces containing solutions are critical spaces.

For the proof of Theorem 1.1, it is necessary to study the initial-boundary value problem of the Stokes equations in \( \mathbb{R}_+^n \times (0, \infty) \) as follows:

\[
u_t - \Delta \nu + \nabla p = f, \quad \text{div } u = 0 \text{ in } \mathbb{R}_+^n \times (0, \infty), \quad u|_{t=0} = h, \quad u|_{x_n=0} = g.
\]

(1.4)
Various studies have been conducted on the solvability of the Stokes equations (1.4) with homogeneous or nonhomogeneous boundary data. Refs. 10, 23, 24, 30, 31, 37, 39 and the references can be referred to for Stokes problems with homogeneous boundary data, whereas Refs. 25, 26, 27, 28, 30, 37, 38, 34 and the references therein can be referred to for Stokes problems with nonhomogeneous boundary data.

Koch and Solonnikov [29] showed the unique local in time existence of solution $u \in W^{1,\frac{1}{2}}_p(\Omega \times (0,T))$ of Stokes equations in a bounded convex domain $\Omega$ with $C^2$ boundaries when $f = \text{div} F$, $F \in L^p(\Omega \times (0,T))$, $h = 0$, and $g \in W^{1-\frac{1}{2p},\frac{1}{2p}}(\partial \Omega \times (0,T))$ with $g_n = \text{div} A$ and $A \in W^{1,\frac{1}{2}}_{\rho_0}(0,T; W^{1-\frac{1}{2p},\frac{1}{2p}}(\partial \Omega))$. See also [30, 37, 39].

In [27], an interior or exterior domain problem was considered for $h = 0$ and for $\alpha q > 1$ if $h = 0$ and for $\alpha q > 2$ if $h \neq 0$. See also [25, 26, 28, 38]. In [18, 34], Stokes equations were solved for rough data including distributions. The following theorem states our result on the unique solvability of the Stokes equations (1.4).

**Theorem 1.2.** Let $1 < q < \infty$, $0 < \alpha < 2$. In addition, let $h$ and $g$ be the same as those given in Theorem 1.1, and $f = \text{div} F$. Assume that $F \in L^p(0,\infty; B^0_q(\mathbb{R}^n))$ for some $(\beta, p)$ satisfying conditions $p \leq q$, $0 < \beta < \alpha \leq \beta + 1 < 2$, $0 = 1 - \alpha + \beta - (n+2)(\frac{1}{p} - \frac{1}{q})$, and $\frac{n+1}{p} > \frac{n+2}{q} - \alpha$. Then, there is a weak solution $u \in B^0_q(\mathbb{R}^n \times (0,\infty))$ satisfying

$$
\|u\|_{B^0_q(\mathbb{R}^n \times (0,\infty))} \leq c \left( \|h\|_{B^0_q(\mathbb{R}^n)} + \|g\|_{B^0_q(\mathbb{R}^n) \cap L^q(\mathbb{R}^{n-1} \times (0,\infty))} + \|g_n\|_{B^0_q(0,\infty; B^0_q(\mathbb{R}^{n-1}))} + \|F\|_{L^p(0,\infty; B^0_q(\mathbb{R}^n))} \right).
$$

Moreover, if $\alpha - \frac{n+2}{q} = \frac{n+2}{q}$ for some $r$ with $1 < r < \infty$, then the solution is unique in the class $B^0_q(\mathbb{R}^n \times (0,\infty))$.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations and function spaces. Section 3 presents the preliminary estimates in homogeneous anisotropic Besov spaces for the Riesz and Poisson heat operators. In Section 4, we consider Stokes equations (1.4) with zero force and zero initial velocity, and provide proof of Theorem 1.1. Sections 5 show the proof of Theorem 1.2 with the help of Theorem 4.1 and the preliminary estimates in Section 3. In Section 6, we give the proof of Theorem 1.1 by constructing approximate solutions.

Our arguments in this paper are based on the elementary estimates of heat and Laplace operators.

## 2. Notations and Definitions

The points of spaces $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ are denoted by $x'$ and $x = (x', x_n)$, respectively. In addition, multiple derivatives are denoted by $D^k_i D^m_t = \frac{\partial^{k+1}f^m}{\partial x_n^k \partial t^m}$ for multi-index $k$ and nonnegative integer $m$. For vector field $f = (f_1, \cdots, f_n)$ on $\mathbb{R}^n$, we write $f' = (f_1, \cdots, f_{n-1})$ and $f = (f', f_n)$. Throughout this paper, we
denote various generic constants by using $c$. Let $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$, $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n \geq 0\}$, and $\mathbb{R}_+ = (0, \infty)$.

For the Banach space $X$ and interval $I$, we denote by $X'$ the dual space of $X$, and by $L^p(I; X), 1 \leq p \leq \infty$ the usual Bochner space. For $0 < \theta < 1$ and $1 < p < \infty$, denote by $(X, Y)_{\theta,p}$ the real interpolation space of the Banach space $X$ and $Y$. For $1 \leq p \leq \infty$, we write $p' = \frac{p}{p-1}$. For $s \in \mathbb{R}$, we write $[s]$ is the largest integer less than $s$.

Let $\Omega$ be a $m$-dimensional Lipschitz domain, $m \geq 1$. Let $1 \leq p \leq \infty$ and $k$ be a nonnegative integer. The norms of usual Lebesque space $L^p(\Omega)$, the usual homogeneous Sobolev space $W^k_p(\Omega)$ are written by $\|\cdot\|_{L^p(\Omega)}$, $\|\cdot\|_{W^k_p(\Omega)}$, respectively.

Note that $W^0_p(\Omega) = L^p(\Omega)$.

For $s \in \mathbb{R}$, we denote by $\hat{B}^s_{p,q}(\mathbb{R}^m), 1 \leq p, q \leq \infty$ the usual homogeneous Besov space in $\mathbb{R}^m$ and denote by $\hat{B}^s_{p,q}(\Omega)$ the restriction of $\hat{B}^s_{p,q}(\mathbb{R}^m)$ to $\Omega$. For the simplicity, set $\hat{B}^s_{p}(\Omega) = \hat{B}^s_{p,p}(\Omega)$.

It is known that $\hat{B}^s_{p}(\Omega) = (L^p(\Omega), W^k_p(\Omega))_{\theta,p}$ for $0 < s < k$ and $\hat{B}^s_{p}(\Omega) = (\hat{B}^{s+1}_{p}(\Omega), \hat{B}^s_{p}(\Omega))_{\theta,p}$ for $s = (1-\theta)s_1 + \theta s_2, 0 < \theta < 1$ and $1 < p < \infty$. In particular, $\hat{B}^s_{p}(\Omega) = \left(\hat{B}^{s-\theta s_1}_p(\Omega)\right)'$ if $-1 + \frac{1}{p} < s < \frac{1}{p}$ and $1 < p < \infty$. See [33] for the reference.

Denote by $\hat{B}^s_{q,\sigma}(\mathbb{R}^n) = \{f \in \hat{B}^s_{q}(\mathbb{R}^n) | \text{div } f = 0\}$ and $\hat{B}^s_{q,\sigma}(\Omega)$ is the restriction of $\hat{B}^s_{q,\sigma}(\mathbb{R}^n)$ to $\Omega$.

Now, we introduce homogeneous anisotropic Besov space and its properties (See Chapter 5 of [33], and Chapter 3 of [3] for the definition of homogeneous anisotropic spaces and their properties, although different notations were used in each books).

Define homogeneous anisotropic Besov space $\hat{B}^{s,\sharp}_{p}(\mathbb{R}^n \times \mathbb{R})$ by

$$
\hat{B}^{s,\sharp}_{p}(\mathbb{R}^n \times \mathbb{R}) = \begin{cases} 
L^p(\mathbb{R}; \hat{B}^s_{p}(\mathbb{R}^n)) \cap L^p(\mathbb{R}^n; \hat{B}^s_{p}(\mathbb{R})) & \text{if } s > 0, \\
L^p(\mathbb{R}; \hat{B}^s_{p}(\mathbb{R}^n)) + L^p(\mathbb{R}^n; \hat{B}^s_{p}(\mathbb{R})) & \text{if } s < 0,
\end{cases}
$$

The above definition is equivalent to the definitions in [33]. Denote by $\hat{B}^{s,\sharp}_{q,\sigma}(\Omega \times I)$ the restriction of $\hat{B}^{s,\sharp}_{q,\sigma}(\mathbb{R}^n \times \mathbb{R})$ to $\Omega \times I$, with norm

$$
\|f\|_{\hat{B}^{s,\sharp}_{q,\sigma}(\Omega \times I)} = \inf \{\|F\|_{\hat{B}^{s,\sharp}_{q,\sigma}(\mathbb{R}^n \times \mathbb{R})} : F \in \hat{B}^{s,\sharp}_{q,\sigma}(\mathbb{R}^n \times \mathbb{R}) \text{ with } F|_{\Omega \times I} = f\}.
$$

For $k \in \mathbb{N} \cup \{0\}$, denote by $\hat{W}^{2k,s}_{2k}(\Omega \times I)$ the usual homogeneous anisotropic Sobolev space.

The properties of the homogeneous anisotropic Besov spaces are comparable with the properties of Besov spaces: In particular, the following properties can be used in this paper.

**Proposition 2.1.** (1) The real interpolation method gives

$$
\hat{B}^{s,\sharp}_{p}(\mathbb{R}^n \times \mathbb{R}) = (L^p(\mathbb{R}^n \times \mathbb{R}), \hat{W}^{2k,s}_{2k}(\mathbb{R}^n \times \mathbb{R}))_{\theta,p}, 0 < s < 2k;
$$

$$
\hat{B}^{s,\sharp}_{p}(\mathbb{R}^n \times \mathbb{R}) = (\hat{B}^{s,\sharp}_{p}(\mathbb{R}^n \times \mathbb{R}), \hat{B}^{s,\sharp}_{p}(\mathbb{R}^n \times \mathbb{R}))_{\theta,p}, 0 < \theta < 1, s = (1-\theta)s_1 + \theta s_2, s_1 < s_2,
$$

for any real number $1 < p < \infty$. 

(2) For $s > 0$
$$\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}^n \times I) = L^p(I; \dot{B}^{s, \frac{n}{p}}_p(\Omega)) \cap L^p(\Omega; \dot{B}^{s, \frac{n}{p}}_p(I)).$$

(3) Let $1 < p_0 \leq p_1 < \infty$, $1 < q_0 \leq q_1 < \infty$ and $s_0 \geq s_1$ with $s_0 - \frac{n+2}{p_0} = s_1 - \frac{n+2}{p_1}$. Then, the following inclusions hold
$$\dot{B}^{s_0, \frac{n}{p_0}}_{p_0q_0}((\mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{R}^n \times \mathbb{R})) \subset \dot{W}^{s_1, \frac{n}{p_1}}_{p_1q_1}((\mathbb{R}^n \times \mathbb{R}), \quad \dot{B}^{s_0, \frac{n}{p_0}}_{p_0q_0}((\mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{R}^n \times \mathbb{R})).$$

(4) For $f \in \dot{W}^{\alpha, \frac{s}{p}}_p(\mathbb{R}^n \times \mathbb{R})$ and $f \in \dot{B}^{s_0, \frac{n}{p_0}}_{p_0q_0}((\mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{R}^n \times \mathbb{R}))$ with $n \notin [\frac{1}{p}, 1)$, $f|_{t=0} \in \dot{B}^{\alpha-\frac{1}{p}, \frac{s}{p}}_p(\mathbb{R}^n)$ with
$$\|f\|_{\dot{B}^{\alpha-\frac{1}{p}, \frac{s}{p}}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{W}^{\alpha, \frac{s}{p}}_p(\mathbb{R}^n \times \mathbb{R})}, \quad \|f\|_{\dot{B}^{\alpha-\frac{1}{p}, \frac{s}{p}}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{B}^{s_0, \frac{n}{p_0}}_{p_0q_0}((\mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{R}^n \times \mathbb{R})).$$

(5) For $f \in \dot{W}^{\alpha, \frac{s}{p}}_p(\mathbb{R}^n \times \mathbb{R})$ and $f \in \dot{B}^{s_0, \frac{n}{p_0}}_{p_0q_0}((\mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{R}^n \times \mathbb{R}))$ with $n \notin [\frac{1}{p}, 1)$, $f|_{t=0} \in \dot{B}^{\alpha-\frac{1}{p}, \frac{s}{p}}_p(\mathbb{R}^n) with
$$\|f\|_{t=0}\dot{B}^{\alpha-\frac{1}{p}, \frac{s}{p}}_p(\mathbb{R}^n) \leq c\|f\|_{\dot{W}^{\alpha, \frac{s}{p}}_p(\mathbb{R}^n \times \mathbb{R})}, \quad \|f\|_{t=0}\dot{B}^{\alpha-\frac{1}{p}, \frac{s}{p}}_p(\mathbb{R}^n) \leq c\|f\|_{\dot{B}^{s_0, \frac{n}{p_0}}_{p_0q_0}((\mathbb{R}^n \times \mathbb{R}) \cap (\mathbb{R}^n \times \mathbb{R})).$$

For the proof of (1), refer to page 169 in [11] or (a) of Theorem 2.4.2.1 in [11], and Theorem 6.4.5 in [11]. For the proof of (2), refer to the proof of Theorem 3 in [15]. For the proof of (3), refer to the proof of Theorem 6.5.1 in [11], and for the proofs of (4) and (5), refer to Theorem 6.6.1 in [11].

**Remark 2.2.** The properties in Proposition 2.2 of the homogeneous anisotropic Besov spaces in $\mathbb{R}^n \times \mathbb{R}$ hold for the homogeneous Besov spaces in $\mathbb{R}_+^n \times \mathbb{R}$ and $\mathbb{R}^n_+ \times \mathbb{R}_+$ (see [31]).

Denote by $\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}^n \times \mathbb{R}_+)$ the completion of $C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ in $\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}^n \times \mathbb{R}_+)$. It is known that
$$\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}^n \times \mathbb{R}_+) = \begin{cases} \{g \in \dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+) : g|_{t=0} = 0\} & \text{if } 2 > s > \frac{2}{p}, \\ \dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+) & \text{if } 0 \leq s < \frac{2}{p}, \\ \left(\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+)\right)' & \text{if } -2 < s < 0, \end{cases}$$

where $p'$ is the Hölder conjugate of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

**Remark 2.3.**

(1) Let $f \in \dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+)$. If $0 \leq s$, then its zero extension $\tilde{f}$ to $\mathbb{R}_+^{n-1} \times \mathbb{R}$ is contained in $\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R})$ such that $\|\tilde{f}\|_{\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R})} \approx \|f\|_{\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+)'}$. If $s < 0$, then the zero extension $\tilde{f}$ of the distribution $f$ is defined by
$$\langle \langle \tilde{f}, \phi \rangle \rangle := \langle f, \phi|_{\mathbb{R}_+^{n-1} \times \mathbb{R}_+} \rangle,$$
for any $\Phi \in \dot{B}^{-s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R})$, where $p'$ is the Hölder conjugate of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. Then, $\|\tilde{f}\|_{\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+ \times \mathbb{R})} \approx \|f\|_{\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+ \times \mathbb{R}_+)'}$. Here $\langle \cdot, \cdot \rangle$ is the duality pairing between $\dot{B}^{s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+) and \dot{B}^{-s, \frac{n}{p}}_p(\mathbb{R}_+^{n-1} \times \mathbb{R}_+)$.\]
3.1. Basic Theories. Let $P_{x_n}$ be the Poisson operator defined by

$$P_{x_n}f(x) = c_n \int_{\mathbb{R}^{n-1}} \frac{x_n}{(x' - y')^2 + x_n^2} f(y')dy'.$$

Note that $P_{x_n}f$ is a harmonic function in $\mathbb{R}^n$ and $P_{x_n}f|_{x_n=0} = f$ on $\mathbb{R}^{n-1}$.
Proposition 3.1. Let 1 < p < ∞ and α > 0. If \( f \in \dot{B}^{\alpha-\frac{1}{p}}_p(\mathbb{R}^{n-1}) \), then \( P_{x_n} f \in \dot{B}^{\alpha}_p(\mathbb{R}^{n}) \) and

\[
\|P_{x_n} f\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{\dot{B}^{\alpha-\frac{1}{p}}_p(\mathbb{R}^{n-1})}, \quad \|P_{x_n} f\|_{\dot{B}^\alpha_p(\mathbb{R}^n)} \leq c\|f\|_{\dot{B}^{\alpha-\frac{1}{p}}_p(\mathbb{R}^{n-1})}. \tag{3.1}
\]

It is well known that \( P_{x_n} \) is bounded from \( \dot{B}^{\alpha-\frac{1}{p}}_p(\mathbb{R}^{n-1}) \) to \( L^p(\mathbb{R}^n) \); this was proved in [30, Lemma 2.1]. Through an interpolation argument, we obtained the second inequality of (3.1) for \( \alpha > 0 \).

In addition, for a solenoidal vector field \( u \in L^p(\mathbb{R}^n) \), the following trace theorem holds (see [20] for the proof).

Proposition 3.2. Let 0 < p < ∞. Let \( u \in L^p(\mathbb{R}^n) \) such that \( \text{div } u = 0 \).

Then, \( u_n \in \dot{B}^{\frac{1}{p}}_{p'}(\mathbb{R}^{n-1}) \),

\[
\|u_n\|_{\dot{B}^{\frac{1}{p}}_{p'}(\mathbb{R}^{n-1})} \leq c\|u\|_{L^p(\mathbb{R}^n)}, \quad \|u_n\|_{\dot{B}^{-\frac{1}{p'}}_{p'}(\mathbb{R}^{n-1})} \leq c\|u\|_{\dot{B}^{\frac{1}{p}}_p(\mathbb{R}^n)}. \]

Proof. The first inequality is a well-known result (see [20] for the proof).

For \( \alpha > \frac{1}{p} \), the second inequality is obtained from a usual trace theorem.

Let 0 < \( \alpha \leq \frac{1}{p} \) and \( f \in \dot{B}^{\alpha+\frac{1}{p}}_p(\mathbb{R}^{n-1}) \), where \( p' \) is the Hölder conjugate of \( p \), that is, \( \frac{1}{p} + \frac{1}{p'} = 1 \). Let \( \dot{B}^{\alpha}_{p_0}(\mathbb{R}^n) \) be a dual space of \( \dot{B}^{\alpha}_p(\mathbb{R}^n) \) and \( \langle \cdot, \cdot \rangle \) be a duality pairing between \( \dot{B}^{\alpha}_p(\mathbb{R}^n) \) and \( \dot{B}^{-\alpha}_{p_0}(\mathbb{R}^n) \). Then, from Proposition 3.1 we have

\[
\langle u_n|_{x_n=0}, f \rangle = \int_{\mathbb{R}^n} u(x) \cdot \nabla P_{x_n} f(x) \, dx
\]

\[
\leq \|u\|_{\dot{B}^{\alpha}_p(\mathbb{R}^n)} \|\nabla P_{x_n} f\|_{\dot{B}^{-\alpha}_{p_0}(\mathbb{R}^n)}
\]

\[
\leq \|u\|_{\dot{B}^{\alpha}_p(\mathbb{R}^n)} \|P_{x_n} f\|_{\dot{B}^{-\alpha+1}_{p_0}(\mathbb{R}^n)}
\]

Hence, the proof of the second inequality of Proposition 3.2 is completed. \( \square \)

It is well known that the Riesz transforms in \( \mathbb{R}^n \), \( R_i \), where 1 ≤ i ≤ n, are bounded from \( \dot{B}^{\alpha}_p(\mathbb{R}^n) \) to \( \dot{B}^\alpha_p(\mathbb{R}^n) \) for \( s \in \mathbb{R} \) [10]. According to the definition of the homogeneous anisotropic Besov space \( \dot{B}^{\alpha}_q(\mathbb{R}^n \times \mathbb{R}) \) and the multiplier theorem, the following boundedness property holds true for homogeneous anisotropic Besov spaces.

Proposition 3.3. Let 1 < q < ∞. Then,

\[
\|R_i f\|_{\dot{B}^\alpha_q(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{B}^\alpha_q(\mathbb{R}^n \times \mathbb{R})}, \quad s \in \mathbb{R}, \ 1 \leq i \leq n.
\]

We say that a distribution \( f \) in \( \mathbb{R}^{n-1} \times \mathbb{R}_+ \) is contained in function space \( \dot{B}^{\alpha}_q(\mathbb{R}_+; \dot{B}^{-\frac{1}{p'}}_q(\mathbb{R}^{n-1})) \), 0 < \( \alpha < 2 \) if \( f \) satisfies

\[
\|f\|_{\dot{B}^{\alpha}_{q}(\mathbb{R}_+; \dot{B}^{-\frac{1}{p'}}_q(\mathbb{R}^{n-1}))} := \left( \int_{0}^{\infty} \int_{0}^{\infty} \|f(\cdot, t) - f(\cdot, s)\|_{\dot{B}^{-\frac{1}{p'}}_q(\mathbb{R}^{n-1})}^q \, ds dt \right)^{\frac{1}{q}} < \infty.
\]
3.2. Estimates of the heat operator.

Proposition 3.4. Let $0 < \alpha < 2$ and $1 < q < \infty$. If $f \in L^q(\mathbb{R}^n; \dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n)) \cap \dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n)$, then, $P_{\alpha,n} f \in \dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n)$ with

$$\|P_{x,n} f\|_{\dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R}^n)} \leq c(\|f\|_{L^q(\mathbb{R}^n; \dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n))} + \|f\|_{\dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n; \dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n))}).$$

Proof. From Proposition 3.1, we have

Lemma 3.6. Let $\dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n)$, then, $f \in \dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n)$. Hence we have

Proposition 3.1 also gives the estimate

$$\|P_{x,n} f(x', t) - P_{x,n} f(x', s)\|_{L^q(\mathbb{R}^n)} \leq c\|f(\cdot, t) - f(\cdot, s)\|_{\dot{B}_{q}^{-\frac{1}{2}}(\mathbb{R}^n)}.$$
The following estimates for $\Gamma_t * h := \int_{\mathbb{R}^n} \Gamma(x - y, t)h(y)dy$ and $\Gamma_t * h|_{x_n=0} := \int_{\mathbb{R}^n} \Gamma(x' - y', y_n, t)h(y)dy$ are derived in Appendix C.

**Lemma 3.7.** Let $1 < q < \infty$ and $0 < \alpha < 2$. Then,

$$\|\Gamma_t * h\|_{B^\alpha_q(\mathbb{R}^n)} \leq c\|h\|_{B^{-\frac{2}{q}}_q(\mathbb{R}^n)}.$$

Moreover, $\Gamma_t * h|_{x_n=0} \in B^\alpha_q(\mathbb{R}^n)$ with

$$\|\Gamma_t * h|_{x_n=0}\|_{B^\alpha_q(\mathbb{R}^n)} \leq c\|h\|_{B^{-\frac{2}{q}}_q(\mathbb{R}^n)}.$$

The following estimates for $D_x \Gamma * f := \int_0^t \int_{\mathbb{R}^n} D_x \Gamma(x - y, t - s) f(y, s)dyds$ and $D_x \Gamma * f|_{x_n=0} := \int_0^t \int_{\mathbb{R}^n} D_x \Gamma(x' - y', y_n, t - s) f(y, s)dyds$ are derived in Appendix D.

**Lemma 3.8.** Let $1 < q < \infty$ and $0 < \alpha < 2$. Further, let $f \in L^p(\mathbb{R}^n; \dot{B}^\beta_p(\mathbb{R}^n))$ for some $(\beta, p)$ with $p \leq q$, $0 < \beta < \alpha \leq \alpha + 1 < 2$, and $1 - \alpha + \beta - (n + 2)(\frac{1}{p} - \frac{1}{q}) = 0$. Then, $D_x \Gamma * f \in B^\alpha_{q(0)}(\mathbb{R}^n)$ with

$$\|D_x \Gamma * f\|_{B^\alpha_{q(0)}(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n; \dot{B}^\beta_p(\mathbb{R}^n))}.$$

Moreover, if $\alpha + \frac{n+1}{p} - \frac{n+2}{q} > 0$, then $D_x \Gamma * f|_{x_n=0} \in B^\alpha_{q(0)}(\mathbb{R}^n)$ with

$$\|D_x \Gamma * f|_{x_n=0}\|_{B^\alpha_{q(0)}(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n; \dot{B}^\beta_p(\mathbb{R}^n))}.$$

4. **Stokes equations** (1.4) with $f = 0$ and $h = 0$ and $g_n = 0$

Let

$$K_{ij}(x, t) = -2\delta_{ij} D_{x_n} \Gamma(x, t) + 4D_{x_j} \int_0^t \int_{\mathbb{R}^n} D_{x_n} \Gamma(z, t) D_{x_i} N(x - z) dz.$$

In [1, 7], an explicit formula was formulated for the solution $w$ of the Stokes equations (1.4) with $f = 0$, $h = 0$, and boundary data $g = (g', 0)$ by

$$w_i(x, t) = \sum_{j=1}^{n-1} \int_0^t \int_{\mathbb{R}^n} K_{ij}(x' - y', x_n, t - s) g_j(y', s) dy'ds. \quad (4.1)$$

**Theorem 4.1.** Let $0 < \alpha < 2$ and $1 < q < \infty$. In addition, let $w$ be the vector field defined by (4.1) for $g \in B^{\alpha}_{q(0)}(\mathbb{R}^n)$ with $g_n = 0$. Then, $w \in \dot{B}^\alpha_q(\mathbb{R}^n)$ with

$$\|w\|_{\dot{B}^\alpha_q(\mathbb{R}^n)} \leq c\|g\|_{B^{\alpha}_{q(0)}}.$$

**Proof.** From (1) of Remark 2.3, the zero extension $\tilde{g}$ of $g$ satisfies

$$\|\tilde{g}\|_{\dot{B}^\alpha_q(\mathbb{R}^n)} \leq c\|g\|_{B^{\alpha}_{q(0)}}.$$

$w$ can be rewritten through the following form

$$w_i = -D_{x_n} T_2 \tilde{G}_i - 4\delta_{in} \left( \sum_{j=1}^{n-1} R_{ij} D_{x_n} T_2 \tilde{g}_j \right) + 4 \frac{\partial}{\partial x_i} S \left( \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} D_{x_n} T_2 \tilde{g}_j \right), \quad (4.2)$$

where $S$ is the surface integral operator.
i = 1, \ldots, n$, where $R' = (R'_1, \ldots, R'_{n-1})$ is the $n-1$ dimensional Riesz operator and $S$ is defined by

$$Sf(x) := \int_0^x \int_{\mathbb{R}^{n-1}} N(x-y)f(y)dy. \quad (4.3)$$

Based on the property of the Riesz operator, we have

$$\| \sum_{j=1}^{n-1} R'_j D_{x_n} T_2 \hat{g}_j \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \sum_{j=1}^{n-1} \| D_{x_n} T_2 \hat{g}_j \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \quad (4.4)$$

Let $f = \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} D_{x_n} T_2 \hat{g}_j$. Direct computation also shows that $Sf$ solves

$$\Delta Sf(t) = \text{div} \frac{F}{t} \text{ in } \mathbb{R}_+^\times \text{ for each } t > 0, \quad Sf|_{t=0} = 0, \quad (4.5)$$

where

$$F_j := -\frac{1}{2} D_{x_n} T_2 \hat{g}_j, \quad j = 1, \ldots, n-1, \quad F_n := \sum_{j=1}^{n-1} R'_j D_{x_n} T_2 \hat{g}_j(x, t). \quad (4.6)$$

According to the well-known result for the elliptic partial differential equation $[2, 3]$, solution $Sf$ of the Laplace equation $\Delta f = 0$ satisfies the estimate as

$$\| D_x Sf \|_{L^q(\mathbb{R}_+^n \times \mathbb{R})} \leq \| Sf \|_{L^q(0, \infty; W^{2, q}_2(\mathbb{R}_+^n))} \leq c \| F \|_{L^q(\mathbb{R}_+^n \times \mathbb{R})} \quad (4.7)$$

and

$$\| D_x Sf \|_{L^q(0, \infty; W^{2, q}_2(\mathbb{R}_+^n))} \leq \| Sf \|_{L^q(0, \infty; W^{2, q}_2(\mathbb{R}_+^n))} \leq c \| F \|_{L^q(0, \infty; W^{2, q}_2(\mathbb{R}_+^n))}. \quad (4.8)$$

However, as $D_x Sf$ also satisfies elliptic equation $\Delta f = \text{div} F$ with the right hand side $\text{div} D_x F$, we have

$$\| D_x D_x Sf \|_{L^q(\mathbb{R}_+^n \times \mathbb{R})} \leq \| D_x Sf \|_{L^q(0, \infty; W^{2, q}_2(\mathbb{R}_+^n))} \leq c \| D_x F \|_{L^q(\mathbb{R}_+^n \times \mathbb{R})}. \quad (4.9)$$

By combining (4.8) and (4.9), we obtain

$$\| D_x Sf \|_{W^{2, q}_2(\mathbb{R}_+^n \times \mathbb{R})} \leq c \| F \|_{W^{2, q}_2(\mathbb{R}_+^n \times \mathbb{R})}. \quad (4.10)$$

The interpolation of (4.7) and (4.10) (see (1) of Proposition (2.1)) gives

$$\| D_x Sf \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \| F \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \| D_x T_2 \hat{g} \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \quad (4.11)$$

for $0 < \alpha < 2$. Based on (4.2), (4.3), and (4.11), we conclude that

$$\| w \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \| D_x T_2 \hat{g} \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \quad 0 < \alpha < 2. \quad (4.12)$$

From Lemma 3.6, we have

$$\| D_x T_2 \hat{g} \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \| \hat{g} \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \| g \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \leq c \| g \|_{B^\alpha_q \mathbb{R}_+^{n-1 \times \mathbb{R}}} \quad (4.13)$$

Hence, the proof of Theorem 4.1 is completed. \hfill \Box

5. Proof of Theorem 1.2

Let us consider the Stokes equations $[1, 4]$ with nonhomogeneous data $h, f = \text{div} \mathcal{F}, g$. We represent the solution of Stokes equations $[1, 4]$ according to four vector fields, $v, V, \nabla \phi$, and $w$ as follows.
5.1. Solution Representation. Let \( \tilde{\mathcal{F}} \) be the extension of \( \mathcal{F} \) over \( \mathbb{R}^n \times \mathbb{R}_+ \) such that \( \tilde{\mathcal{F}} \in L^p(\mathbb{R}^n; \dot{B}_p^\beta(\mathbb{R}^n)) \) with \( \| \tilde{\mathcal{F}} \|_{L^p(\mathbb{R}^n; \dot{B}_p^\beta(\mathbb{R}^n))} \leq c\| \mathcal{F} \|_{L^p(\mathbb{R}^n; \dot{B}_p^\beta(\mathbb{R}^n))} \). Set \( \hat{f} = \text{div} \tilde{\mathcal{F}} \). Define \( V \) by

\[
V(x, t) = \int_0^t \int_{\mathbb{R}^n} \Gamma(x - y, t - s)\mathbb{P}\hat{f}(y, s)dyds. \tag{5.1}
\]

Here, \( \mathbb{P} \) is the Helmholtz projection operator defined on \( \mathbb{R}^n \) defined as

\[
[\mathbb{P}\hat{f}]_j(x, t) = \delta_{ij} \hat{f}_i + D_{x_i}D_{x_j} \int_{\mathbb{R}^n} N(x - y) \hat{f}_i(y, t)dy = \delta_{ij} \hat{f}_i + R_i R_j \hat{f}_i.
\]

Observe that \( \text{div} \mathbb{P}\hat{f} = 0 \) in \( \mathbb{R}^n \times \mathbb{R}_+ \) and \( \hat{f} = \mathbb{P}\hat{f} + \nabla \hat{Q}f \), where

\[
\hat{Q}f = -D_{x_i} \int_{\mathbb{R}^n} N(x - y) \hat{f}_i(y, t)dy.
\]

In addition, \( V \) satisfies the following equations:

\[
V_t - \Delta V = \mathbb{P}\hat{f}, \quad \text{div} V = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+, \quad V|_{t=0} = 0 \quad \text{on} \quad \mathbb{R}^n. \tag{5.2}
\]

Furthermore, \( V \) can be rewritten as

\[
V_j(x, t) = -\int_0^t \int_{\mathbb{R}^n} D_y \Gamma(x - y, t - s) \left( \delta_{ij} \tilde{F}_{ki} + R_i R_j \tilde{F}_{ki} \right)(y, s)dyds. \tag{5.3}
\]

Let \( \tilde{h} \in \dot{B}_q^{-\frac{\beta}{p}}(\mathbb{R}^n) \) be the solenoidal extension of \( h \) with \( \| \tilde{h} \|_{\dot{B}_q^{-\frac{\beta}{p}}(\mathbb{R}^n)} \leq c\| h \|_{\dot{B}_q^{-\frac{\beta}{p}}(\mathbb{R}^n)} \).

We define \( v \) as

\[
v(x, t) = \int_{\mathbb{R}^n} \Gamma(x - y, t)\tilde{h}(y)dy. \tag{5.4}
\]

Note that \( v \) satisfies the following equations:

\[
v_t - \Delta v = 0, \quad \text{div} v = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+, \quad v|_{t=0} = \tilde{h} \quad \text{on} \quad \mathbb{R}^n. \tag{5.5}
\]

Next, we define \( \phi \) as

\[
\phi(x, t) = 2 \int_{\mathbb{R}^n \times \mathbb{R}^n} N(x' - x, t) \left( g_n(x', t) - v_n(x', 0, t) - V_n(x', 0, t) \right)dy'. \tag{5.6}
\]

In addition,

\[
\Delta \phi = 0, \quad \nabla \phi|_{z_n=0} = (\mathcal{R}'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}), g_n - v_n|_{x_n=0} - V_n|_{x_n=0}).
\]

Note that \( \nabla \phi|_{t=0} = 0 \) if \( g_n|_{t=0} = h_n|_{x_n=0} \).

Let \( G = (G', 0) \), where

\[
G' = (G_1, \ldots, G_{n-1}) = g' - v'|_{x_n=0} - V'|_{x_n=0} - \mathcal{R}'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}). \tag{5.7}
\]

Note that \( G'|_{t=0} = 0 \) if \( g|_{t=0} = h|_{x_n=0} \). Let \( w \) be the vector field defined using (5.1) with boundary data \( G = (G', 0) \) for \( G' \), as defined in (5.7). Then,

\[
u = w + \nabla \phi + v + V \quad \text{and} \quad p = r - \phi_t + \hat{Q}f \tag{5.8}
\]

formally satisfies the nonstationary Stokes equations (1.4).
5.2. Estimates of \( u = v + V + \nabla \phi + w \). \( \bullet \) By applying Proposition 3.2 to \( V(t) - V(s) \) and \( v(t) - v(s) \), we also have
\[
\|V_n(t)|_{x_n=0} - V_n(s)|_{x_n=0}\|_{B_q^1(\mathbb{R}^{n-1})} \leq c\|V(t) - V(s)\|_{L^p(\mathbb{R}^n)}, \quad (5.9)
\]
\[
\|v_n(t)|_{x_n=0} - v_n(s)|_{x_n=0}\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1})} \leq c\|v(t) - v(s)\|_{L^p(\mathbb{R}^n)}. \quad (5.10)
\]
From (5.9) and (5.10), we have
\[
\|V_n(0)|_{x_n=0}\|_{B_q^1(\mathbb{R}^{n-1})} \leq \|V(0)\|_{L^p(\mathbb{R}^n)} + \|v(0)\|_{L^p(\mathbb{R}^n)}, \quad (5.11)
\]
\[
\|V_n(0)|_{x_n=0}\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1})} \leq \|V(0)\|_{L^p(\mathbb{R}^n)} + \|v(0)\|_{L^p(\mathbb{R}^n)} \quad (5.12)
\]
\bullet Note that
\[
D_x\phi(x, t) = P_{x_n}(g_n - v_n|_{y_n=0} - V_n|_{y_n=0}),
\]
\[
D_{x'}\phi(x, t) = P_{x_n}R'(g_n - v_n|_{y_n=0} - V_n|_{y_n=0}).
\]
From Proposition 3.3 and according to the properties of Riesz operator, (5.11) and (5.12), we have
\[
\|\nabla \phi\|_{B_q^{\frac{3}{2}}(\mathbb{R}^n \times \mathbb{R}_+)} \leq \|g_n - v_n|_{x_n=0} - V_n|_{x_n=0}\|_{L^p(\mathbb{R}^n; B_q^{\frac{3}{2}}(\mathbb{R}^{n-1})})
\]
\[
+ c\|g_n - v_n|_{x_n=0} - V_n|_{x_n=0}\|_{B_q^{\frac{3}{2}}(\mathbb{R}^n; B_q^{\frac{3}{2}}(\mathbb{R}^{n-1})})
\]
\[
\leq c\left(\|g_n\|_{L^p(\mathbb{R}^n; B_q^{\frac{3}{2}}(\mathbb{R}^{n-1})}) + \|g_n\|_{B_q^{\frac{3}{2}}(\mathbb{R}^n; B_q^{\frac{3}{2}}(\mathbb{R}^{n-1})})
\]
\[
+ \|v\|_{B_q^{\frac{3}{2}}(\mathbb{R}^n; \mathbb{R}_+)} + \|V\|_{B_q^{\frac{3}{2}}(\mathbb{R}^n; \mathbb{R}_+)}\right). \quad (5.13)
\]
From Lemma 3.7 and Lemma 3.8, we have \( v|_{x_n=0} \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \) and \( V|_{x_n=0} \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \). Together with (1.3) and (2.1), we conclude that \( g - v|_{x_n=0} - V|_{x_n=0} \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \). This again implies that \( R'(g - v|_{x_n=0} - V|_{x_n=0}) \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \). In the end, we conclude that \( G' = g' - v'|_{x_n=0} - V|_{x_n=0} - R'(g_n - v_n|_{x_n=0} - V_n|_{x_n=0}) \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \) with
\[
\|G\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+)} \leq c\left(\|g\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+)} + \|v\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+)} \right. \quad (5.14)
\]
\bullet By applying Theorem 2.4 to the fact that \( G = G', 0 \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \), we conclude that \( w \in B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+) \) with
\[
\|w\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+)} \leq c\|G\|_{B_q^{\frac{3}{2}}(\mathbb{R}^{n-1} \times \mathbb{R}_+)} \quad (5.14)
\]
\bullet From (5.8), (5.13), and (5.14), as well as Lemma 3.7 and from Lemma 3.8 the proof of the estimate in Theorem 1.2 for smooth \( (h, g) \) with \( h|_{x_n=0} = g|_{t=0} \) is completed.
5.3. **Uniqueness.** Let \( \tilde{u} \in \dot{B}^{\alpha,q}_{q}((\mathbb{R}^{n}_{+} \times \mathbb{R}_{+}) \) be another solution of the Stokes equations with the same data. Then
\[
-\int_{0}^{\infty} \int_{\mathbb{R}^{n}_{+}} (u - \tilde{u}) \cdot (\Delta \Phi + D_{t} \Phi) dx dt = 0
\]
for any \( \Phi \in C_{c}^{\infty}((\mathbb{R}^{n}_{+} \times [\mathbb{R}_{+}]) \) with \( \text{div}_{x} \Phi = 0, \ |\Phi|_{x_{n}=0} = 0 \).

Suppose that \( \alpha - \frac{n+2}{q} = -\frac{n+2}{r} \) for some \( r \) with \( 1 < r < \infty \), then \( \dot{B}^{\alpha,q}_{q}((\mathbb{R}^{n}_{+} \times \mathbb{R}_{+}) \) is well known (See Lemma 2.2 in [11]).

6. **Nonlinear problem**

In this section, we give the proof of Theorem 1.1. Accordingly, we construct approximate solutions and then derive uniform convergence in homogeneous anisotropic Besov spaces \( \dot{B}^{\alpha,q}_{q}((\mathbb{R}^{n}_{+} \times \mathbb{R}_{+}) \). For the uniform estimates, bilinear estimates should be preceded.

6.1. **Hölder type inequality.** The following Hölder type inequality in Besov space is well known (See Lemma 2.2 in [11]).

**Proposition 6.1.** Let \( \beta > 0, \ \frac{1}{q_{i}} + \frac{1}{s_{i}} = \frac{1}{p}, \) and \( i = 1, 2 \). Then,
\[
\|fg\|_{\dot{B}^{\beta,q}_{q}((\mathbb{R}^{n} \times \mathbb{R}) \leq c(\|f\|_{\dot{B}^{\beta,q}_{q}((\mathbb{R}^{n} \times \mathbb{R})} \|g\|_{L^{1}(\mathbb{R}^{n} \times \mathbb{R})} + c\|f\|_{L^{\frac{q}{\beta}}(\mathbb{R}^{n} \times \mathbb{R})} \|g\|_{\dot{B}^{\beta,q}_{q}((\mathbb{R}^{n} \times \mathbb{R})} ) \).}
\]

Let \( f \) and \( g \) be functions defined in \( \mathbb{R}^{n}_{+} \times \mathbb{R}_{+} \). Further, let \( \tilde{f} \) and \( \tilde{g} \) be the reflective extensions over \( \mathbb{R}^{n} \times \mathbb{R} \) with respect to space and time of \( f \) and \( g \), respectively. Then, by applying Proposition 6.1 to \( \tilde{f} \) and \( \tilde{g} \) for \( 0 < \beta < 1 \), we obtain
\[
\|fg\|_{\dot{B}^{\beta,q}_{q}((\mathbb{R}^{n}_{+} \times \mathbb{R}_{+}) \leq c(\|f\|_{\dot{B}^{\beta,q}_{q}((\mathbb{R}^{n}_{+} \times \mathbb{R}_{+})} \|g\|_{L^{1}(\mathbb{R}^{n}_{+} \times \mathbb{R}_{+})} + c\|f\|_{L^{\frac{q}{\beta}}(\mathbb{R}^{n}_{+} \times \mathbb{R}_{+})} \|g\|_{\dot{B}^{\beta,q}_{q}((\mathbb{R}^{n}_{+} \times \mathbb{R}_{+})} ) \}.}
\]

6.2. **Proof of Theorem 1.1.** In this section, we show the construction of a solution of Navier–Stokes equations (1.1).

6.2.1. **Approximate solutions.** Let \( (u^{1}, p^{1}) \) be the solution of the equations
\[
u^{1} - \Delta u^{1} + \nabla p^{1} = 0, \quad \text{div} u^{1} = 0, \quad \text{in} \ \mathbb{R}^{n}_{+} \times \mathbb{R}_{+},
u^{1}|_{t=0} = h, \quad u^{1}|_{x_{n}=0} = g. \quad (6.2)
\]

Let \( m \geq 1 \). After obtaining \( (u^{1}, p^{1}), \cdots, (u^{m}, p^{m}) \), construct \( (u^{m+1}, p^{m+1}) \), which satisfies the equations
\[
u^{m+1} - \Delta u^{m+1} + \nabla p^{m+1} = f^{m}, \quad \text{div} u^{m+1} = 0, \quad \text{in} \ \mathbb{R}^{n}_{+} \times \mathbb{R}_{+},
u^{m+1}|_{t=0} = h, \quad u^{m+1}|_{x_{n}=0} = g, \quad (6.3)
\]
where \( f^{m} = \nabla m = -\text{div}(u^{m} \otimes u^{m}). \)
6.2.2. Uniform boundedness. Let $0 < \alpha < 2$ and $q = \frac{n+2}{n+1}$. Moreover, let $h$ and $g$ satisfy the hypothesis in Theorem 1.1. Hence, $h, g$, and $F^m$ satisfy the hypothesis in Theorem 1.2. Set

$$M_0 = \|h\|_{B_q^{\alpha}((\mathbb{R}_+^n \times \mathbb{R}_+))} + \|g\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}^{n-1}_+ \times \mathbb{R}_+)} + \|g_n\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}^{n-1}_+ \times \mathbb{R}_+)} + \|g_{n+1}\|_{L^q(\mathbb{R}_+; B_q^{\alpha - \frac{2}{q}}(\mathbb{R}^{n-1}_+))}.$$ 

Observe that $0 < -\alpha + \frac{n+2}{q} < n + 2$, so the solution of (6.2) exists uniquely in $B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)$. By applying Theorem 1.2 to the solution of (6.2), we have

$$\|u^1\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \leq c_1 M_0. \quad (6.4)$$

Take $1 < p < \min(q, \frac{n+2}{q})$ and let $\beta = -2 + \frac{n+2}{q}$. Then

$$1 < p \leq q, \quad 1 - \alpha + \beta - \frac{n+2}{p} + \frac{n+2}{q} = 0,$$

$$0 < \beta < \alpha < \beta + 1 < 2, \quad -\alpha + \frac{n+1}{p} - \frac{n+2}{q} > 0.$$ 

Hence, $(\beta, p)$ satisfies the assumption of Theorem 1.2.

From Besov embedding theorem (see (3) of Proposition 2.1), it holds that

$$B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+) \hookrightarrow L^{n+2}(\mathbb{R}_+^n \times \mathbb{R}_+),$$

$$B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+) \hookrightarrow \tilde{B}_{p,2}^{\beta,2}(\mathbb{R}_+^n \times \mathbb{R}_+).$$

In Proposition 6.1 by considering $s_2 = r_2 = n + 2$ and $r_1 = s_2 = \frac{n(n+2)}{n+2-p}$ and based on (3) of Proposition 2.1, we have

$$\|u^m \otimes u^m\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \leq c \left(\|u^m\|_{B_{p,2}^{\beta,2}(\mathbb{R}_+^n \times \mathbb{R}_+)} \|u^m\|_{L^1(\mathbb{R}_+^n \times \mathbb{R}_+)} + \|u^m\|_{L^2(\mathbb{R}_+^n \times \mathbb{R}_+)} \|u^m\|_{B_{p,2}^{\beta,2}(\mathbb{R}_+^n \times \mathbb{R}_+)} \right) \leq c \|u^m\|^2_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)}. \quad (6.5)$$

As $\|u^m \otimes u^m\|_{L^p(\mathbb{R}_+; B_{p,2}^{\beta,2}(\mathbb{R}_+^n \times \mathbb{R}_+))} \leq c \|u^m \otimes u^m\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)}$, according to Theorem 1.2 there is $u^{m+1} \in B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)$ satisfying that

$$\|u^{m+1}\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \leq c \left(M_0 + \|u^m \otimes u^m\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \right) \leq c_1 \left(M_0 + \|u^m\|^2_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \right). \quad (6.6)$$

Observe that $0 < -\alpha + \frac{n+2}{q} < n + 2$, so the solution of (6.3) exists uniquely in $B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)$. Under the condition that $\|u^m\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \leq M$ and from (6.6), we have

$$\|u^{m+1}\|_{B_q^{\alpha - \frac{2}{q}}(\mathbb{R}_+^n \times \mathbb{R}_+)} \leq c_1 \left(M_0 + M^2 \right).$$
Choose $c_1 M \leq \frac{1}{2}$ and $M_0$ with $2c_1 M_0 \leq M$. Then, based on the mathematical induction argument, we can conclude that
\[
\|u^m\|_{\dot{B}^{\alpha}_{q} \dot{\mathcal{F}}(\mathbb{R}^n \times \mathbb{R}_+)} \leq M \quad \text{for all } m = 1, 2, \ldots.
\]

6.2.3. Uniform convergence. Let $U^m = u^{m+1} - u^m$ and $P^m = p^{m+1} - p^m$. Then, $(U^m, P^m)$ satisfies the equations
\[
U^m_t - \Delta U^m + \nabla P^m = -\nabla (u^m \otimes U^m - U^{m-1} \otimes u^{m-1}) \quad \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+,
\]
\[
\text{div } U^m = 0 \quad \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+,
\]
\[
U^m|_{t=0} = 0, \quad U^m|_{x_n = 0} = 0.
\]

(6.7)

Since $0 < -\alpha + \frac{n+2}{q} < n+2$, so the solution of (6.7) exists uniquely in $\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)$. From Theorem 1.2 we have
\[
\|U^m\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)} \leq c_2 (\|u^m\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)} + \|u^{m-1}\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)}). \|U^{m-1}\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)}.
\]
\[
\leq 2c_2 M \|U^{m-1}\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)}. \]

Choose $M$ so that $c_2 M < \frac{1}{2}$, then, the above-mentioned estimate results in
\[
\|U^m\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)} \leq \frac{1}{2} \|U^{m-1}\|_{\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)}. \quad (6.8)
\]

(6.8) implies that the infinite series $\sum_{k=1}^\infty U^k$ converges in $\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)$. Hence, $u^n = u^1 + \sum_{k=1}^n U^k, m = 2, 3, \cdots$ converges to $u^1 + \sum_{k=1}^\infty U^k$ in $\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)$. Now, set $u := u^1 + \sum_{k=1}^\infty U^k$.

6.3. Existence. Let $u$ be the vector field constructed in the previous section. In this section, we show that $u$ satisfies a weak formulation of the Navier–Stokes equations. Let $\Phi \in C^\infty_0(\mathbb{R}^n_+ \times [0, T])$ with div $\Phi = 0$ and $\Phi|_{x_n = 0} = 0$. Note that
\[
- \int_0^\infty \int_{\mathbb{R}^n_+} u^{m+1} \cdot (\Delta \Phi + D_t \Phi) dx dt = \int_0^\infty \int_{\mathbb{R}^n_+} (u^m \otimes u^m) : \nabla \Phi dx dt - \int_{\mathbb{R}^n_+} h(x) \cdot \Phi(x, 0) dx
\]
\[
- \int_0^\infty \int_{\mathbb{R}^{n-1}} g(x', t) \cdot \frac{\partial \Phi}{\partial x_n} (x', t) dx' dt.
\]

As $\alpha = -1 + \frac{n+2}{q}$, by using (3) of Proposition 2.1 we have $\dot{B}^\alpha_q (\mathbb{R}^n_+ \times \mathbb{R}_+) \subset L^{n+2} (\mathbb{R}^n_+ \times \mathbb{R}_+)$. Now, send $m$ to infinity, then, as $u^m \to u$ in $\dot{B}^\alpha_q \dot{\mathcal{F}}(\mathbb{R}^n_+ \times \mathbb{R}_+)$, we have
\[
- \int_0^\infty \int_{\mathbb{R}^n_+} u \cdot (\Delta \Phi + D_t \Phi) dx dt = \int_0^\infty \int_{\mathbb{R}^n_+} (u \otimes u) : \nabla \Phi dx dt - \int_{\mathbb{R}^n_+} h(x) \cdot \Phi(x, 0) dx
\]
\[
- \int_0^\infty \int_{\mathbb{R}^{n-1}} g(x', t) \cdot \frac{\partial \Phi}{\partial x_n} (x', t) dx' dt.
\]

Therefore, we conclude that $u$ is a weak solution of (1.1).
6.4. Uniqueness. Let \( v \in \tilde{L}^p_q(\mathbb{R}^n_+ \times \mathbb{R}_+) \) be another solution of Naiver–Stokes equations \([11]\) with pressure \( q \). Then, \( u - v \) satisfies the equations

\[
(u - v)_t - \Delta(u - v) + \nabla(p - q) = -\text{div}(u \otimes (u - v) + (u - v) \otimes v) \quad \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+, \\
\text{div}(u - v) = 0, \quad \text{in } \mathbb{R}^n_+ \times \mathbb{R}_+, \\
(u - v)|_{t=0} = 0, \quad (u - v)|_{x_n=0} = 0.
\]

Again, applying the estimate of Theorem 1.2 in \([14]\) to the above Stokes equations, we have

\[
\|u - u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (0, \tau))} \leq c\|u \otimes (u - u_1) + (u - u_1) \otimes u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (0, \tau))},
\]

where \( c \) is a constant. Therefore, we conclude the proof of the global in time uniqueness.

Note that \( u, u_1 \in L^{n+2}(\mathbb{R}^n_+ \times \mathbb{R}_+) \). Applying the estimate of Theorem 1.2 in \([14]\) to the above Stokes equations, we have

\[
\|u - u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (0, \tau))} \leq c\|u \otimes (u - u_1) + (u - u_1) \otimes u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (0, \tau))}.
\]

This implies that \( \|u - u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (0, \delta))} = 0 \), that is, \( u \equiv u_1 \) in \( \mathbb{R}^n_+ \times (0, \delta) \). Observe that \( u - u_1 \) satisfies the Stokes equations

\[
(u - u_1)_t - \Delta(u - u_1) + \nabla(p - p_1) = -\text{div}(u \otimes (u - u_1) + (u - u_1) \otimes u_1) \quad \text{in } \mathbb{R}^n_+ \times (\delta, \infty),
\]

\[
\text{div}(u - u_1) = 0, \quad \text{in } \mathbb{R}^n_+ \times (\delta, \infty),
\]

\[
(u - u_1)|_{t=\delta} = 0, \quad (u - u_1)|_{x_n=0} = 0.
\]

Again, applying the estimate of Theorem 1.2 in \([14]\) to the above Stokes equations, we have

\[
\|u - u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (\delta, 2\delta))} \leq c\|u \otimes (u - u_1) + (u - u_1) \otimes u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (\delta, 2\delta))}.
\]

This implies that \( \|u - u_1\|_{L^{n+2}(\mathbb{R}^n_+ \times (\delta, 2\delta))} = 0 \), that is, \( u \equiv u_1 \) in \( \mathbb{R}^n_+ \times (\delta, 2\delta) \). After iterating this procedure finitely many times, we obtain the conclusion that \( u = u_1 \) in \( \mathbb{R}^n_+ \times \mathbb{R}_+ \). Therefore, we conclude the proof of the global in time uniqueness.

Appendix A. Proof of Lemma \([6,7]\)

In \([32]\), it was determined that

\[
\|T_1 f\|_{\dot{W}^{-1,1}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R})}.
\]

Note that \( T_1^* \) is the adjoint operator of \( T_1 \). Hence, \([A.1]\) implies that

\[
\|T_1^* f\|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{\dot{W}^{-1,1}_p(\mathbb{R}^n \times \mathbb{R})}.
\]

Further, note that \( D^2_{\eta} T_1^* f \) and \( D_\xi T_1^* f \) comprise \( L^p \) Fourier multipliers as the Fourier transform of \( T_1^* f \) is \( \hat{T}_1^* f(\xi, \eta) = \frac{1}{\sqrt{1 - i\eta}} \hat{f}(\xi, \eta) \). Hence, we have

\[
\|T_1^* f\|_{\dot{W}^{-1,1}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R})}, \quad 1 < p < \infty.
\]
As $T_1^*$ is the adjoint of $T_1$, (A.3) implies that
\[
\|T_1 f\|_{L^p(R^n \times \mathbb{R})} \leq c \|f\|_{W^{2-\frac{2}{p}, 1}_p(R^n \times \mathbb{R})}. \tag{A.4}
\]
By applying the real interpolation theory to (A.1) and (A.4), and (A.2) and (A.3), we obtain estimates of $T_1 f$ and $T_1^* f$ in $\dot{B}^{\frac{1}{2}, 2}_q(R^n \times \mathbb{R})$ for $0 < \alpha < 2$.

**Appendix B. Proof of Lemma 3.7**

First, let us derive the estimate of $T_2 g$. From [32], we have the following estimate
\[
\|T_2 g\|_{W^{2-\frac{2}{q}, 1}_q(R^n \times \mathbb{R})} \leq c \|g\|_{\dot{B}^{-\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}}_q(R^n \times \mathbb{R})}. \tag{B.1}
\]
Note that the identity
\[
\int_{\mathbb{R}^n} T_2 g(x, t) \phi(x, t) dx dt = \langle g, T_1^* \phi \rangle_{Y_n, 0} \tag{B.2}
\]
holds for $\phi \in C_0^\infty(R^n \times \mathbb{R})$, where $T_1^* \phi$ is defined in Section 3 with zero extension $\phi$ of $\phi$ and $< \cdot, \cdot > \,$ is the duality pairing between $\dot{B}^{-\frac{1}{q'}, \frac{1}{q'} - \frac{2}{q}}_q(R^n \times \mathbb{R})$ and $\dot{B}^{1+\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}_q(R^n \times \mathbb{R})}$. Based on (4) of Proposition 2.1 and (A.3), we have
\[
\|T_1^* \phi\|_{Y_n, 0} \leq c \|T_1^* \phi\|_{W^{2-\frac{2}{q}, 1}_q(R^n \times \mathbb{R})} \leq c \|\phi\|_{\hat{L}^{q'}(R^n \times \mathbb{R})}. \tag{B.3}
\]
By applying the estimates in (B.2) to (B.3), we have
\[
\|T_2 g\|_{\hat{L}^{q'}(R^n \times \mathbb{R})} \leq c \|g\|_{\dot{B}^{-\frac{1}{q'}, \frac{1}{q'} - \frac{2}{q}_q(R^n \times \mathbb{R})}}. \tag{B.4}
\]
Further, by applying the real interpolation theory to (B.1) and (B.4), we obtain the estimate of $T_2 g$ in $\dot{B}^{\frac{1}{2}, 2}_q(R^n \times \mathbb{R})$ for $0 < \alpha < 2$.

Analogously, we can derive the estimate of $T_2^* g$ by observing that the identity
\[
\int_{\mathbb{R}^n} T_2^* g(y, s) \phi(y, s) dy ds = \langle g, T_1 \phi \rangle_{X_n, 0} \tag{B.5}
\]
holds for $\phi \in C_0^\infty(R^n \times \mathbb{R})$, where $T_1 \phi$ is defined in Section 3 with zero extension $\tilde{\phi}$ of $\phi$, and $< \cdot, \cdot > \,$ is the duality pairing between $\hat{\dot{B}}^{-\frac{1}{q'}, \frac{1}{q'} - \frac{2}{q}_q(R^n \times \mathbb{R})}$ and $\hat{\dot{B}}^{1+\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}_q(R^n \times \mathbb{R})}$. By using the same procedure as that used for the estimate of $T_2 g$, we can obtain the estimate of $T_2^* g$ as
\[
\|T_2^* g\|_{\hat{L}^{q'}(R^n \times \mathbb{R})} \leq c \|g\|_{\dot{B}^{1-\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}_q(R^n \times \mathbb{R})}}. \tag{B.6}
\]
(As the procedure is the same as that for $T_2 g$, we omitted the details). As $D_s T_2 g = T_2^* (D_s g)$, we have
\[
\|D_s T_2^* g\|_{\hat{L}^{q'}(R^n \times \mathbb{R})} \leq c \|D_s g\|_{\dot{B}^{1-\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}_q(R^n \times \mathbb{R})}} \leq c \|g\|_{\dot{B}^{1-\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}_q(R^n \times \mathbb{R})}}. \tag{B.7}
\]
In addition, as $\Delta_s T_2 g = -D_s T_2^* g$ and $\frac{\partial}{\partial y_n} T_2 g|_{y_n = 0} = g$, based on the well-known elliptic theory [3, 4], we have
\[
\|T_2 g(s)\|_{W^{2/q}_q(R^n)} \leq c \|D_s T_2^* g(s)\|_{\hat{L}^{q'}(R^n \times \mathbb{R})} + c \|g(s)\|_{\dot{B}^{1-\frac{1}{q'}, \frac{1}{q'} + \frac{2}{q}_q(R^n \times \mathbb{R})}}.
\]
This implies that
\[ ||Tg||_{L^q(R^2(W^{2,2}(R^n)))} \leq c ||g||_{L^q(B_1^{\frac{1}{2}+\frac{n}{2}}(R^n \times R))}. \]  
(B.8)

By combining (B.7) and (B.8), we have
\[ ||Tg||_{W^{2,1}(R^n \times R)} \leq c ||g||_{L^q(B_1^{\frac{1}{2}+\frac{n}{2}}(R^n \times R)).} \]  
(B.9)

By applying the real interpolation theory to (3.6) and (3.9), we obtain the estimate of $Tg$ in $B_q^{\frac{n}{2}}(R^n \times R)$ for $0 < \alpha < 2$. Thus, we complete the proof of Lemma 3.6.

APPENDIX C. PROOF OF LEMMA 3.7

From (3.2), the following estimate is known:
\[ ||\Gamma_t h||_{W^{2,1}(R^n \times R)} \leq c ||h||_{B_q^{\frac{n}{2}}(R^n)}. \]  
(C.1)

Let us consider the case where $h \in B_q^{-\frac{n}{2}}(R^n)$. Note that the identity $\int_0^\infty \int_{R^n} \Gamma_t h(x, t) \phi(x, t) dxdt = \langle h, T^*_1 \phi \rangle_{s=0}$ holds for $\phi \in C^\infty_0(R^n \times R)$, where $T^*_1 \phi(y, s) = \int_s^\infty \int_{R^n} \Gamma(x - y, t - s) \phi(x, t) dxdt$. and $\langle \cdot, \cdot \rangle$ is the duality pairing between $B_q^{-\frac{n}{2}}(R^n)$ and $B_q^1(R^n)$. From (A.3), we have
\[ ||T^*_1 \phi||_{W^{2,1}(R^n \times R)} \leq c ||\phi||_{L^q(R^n \times R)}. \]

By using (5) of Proposition 2.11 this implies that
\[ ||T^*_1 \phi||_{L^q(R^n \times R)} \leq c ||T^*_1 \phi||_{W^{2,1}(R^n \times R)} \leq c ||\phi||_{L^q(R^n \times R)}. \]

(See (11) and (3.4).) Hence, we have
\[ < h, T^*_1 \phi|_{s=0} > \leq c ||h||_{B_q^{-\frac{n}{2}}(R^n)} ||T^*_1 \phi||_{B_q^{\frac{n}{2}}(R^n)} \leq c ||h||_{B_q^{-\frac{n}{2}}(R^n)} ||\phi||_{L^q(R^n \times R)}. \]

Again, this leads to the following conclusion
\[ ||\Gamma_t h||_{L^q(R^n \times R)} \leq c ||h||_{B_q^{-\frac{n}{2}}(R^n)}. \]  
(C.2)

By interpolating (C.1) and (C.2), we have
\[ ||\Gamma_t h||_{B_q^{\alpha} (R^n \times R)} \leq c ||h||_{B_q^{-\frac{n}{2}}(R^n)}, \quad 0 < \alpha < 2. \]  
(C.3)

Now, we will derive the estimate of $\Gamma_t h|_{x_n=0}$.

1) Let $\alpha > \frac{1}{q}$. Then by (5) of Proposition 2.11 $\Gamma_t h \in B_q^{\frac{n}{2}}(R^n \times R)$ implies that $\Gamma_t h|_{x_n=0} \in B_q^{\alpha-\frac{n}{2}-\frac{1}{q}}(R^{n-1} \times R)$ with
\[ ||\Gamma_t h|_{x_n=0}||_{B_q^{\alpha-\frac{n}{2}-\frac{1}{q}}(R^{n-1} \times R)} \leq c ||\Gamma_t h||_{B_q^{\frac{n}{2}}(R^n \times R)} \leq c ||h||_{B_q^{-\frac{n}{2}}(R^n)}. \]

2) Let $0 < \alpha < \frac{1}{q}$. In this case, usual trace theorem does not hold any more. For $h \in B_q^{\frac{n}{2}}(R^n)$ the following identity holds:
\[ < \Gamma_t h|_{x_n=0}, \phi > = < h, T^*_2 \phi|_{s=0} >. \]  
(C.4)
holds for any \( \phi \in C^\infty_0(\mathbb{R}^{n-1} \times \mathbb{R}) \), where \( T_2^s \phi(y,s) = \int_s^\infty \int_{\mathbb{R}^{n-1}} \Gamma(x' - y', y_n, t - s) \phi(x', t) dx' dt \) and \( \langle \cdot, \cdot \rangle \) is the duality pairing between \( \dot{B}^\alpha_{q,2}(\mathbb{R}^n) \) and \( \dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n) \).

From the result of Lemma 3.6, \( T_2^s \phi \in \dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n) \times \mathbb{R} \) with

\[
\|T_2^s \phi\|_{\dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n) \times \mathbb{R}} \leq c\|\phi\|_{\dot{B}^\alpha_{q,2}(\mathbb{R}^n) \times \mathbb{R}}.
\]

By Proposition 2.1, this implies that \( T_2^s \phi \big|_{s=0} \in \dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n) \) with

\[
\|T_2^s \phi\big|_{s=0}\|_{\dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n)} \leq c\|\phi\|_{\dot{B}^\alpha_{q,2}(\mathbb{R}^n)}.
\]

Hence

\[
|< h, T_2^s \phi|_{s=0}> | \leq c\|h\|_{\dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n)} \|\phi\|_{\dot{B}^\alpha_{q,2}(\mathbb{R}^n)}
\]

Applying the above estimate to (C.3), \( \Gamma_t * h|_{x_n=0} \in \dot{B}^{-\alpha + \frac{2}{q}}_{q,2}((\mathbb{R}^{n-1} \times \mathbb{R}) \) with

\[
\|\Gamma_t * h|_{x_n=0}\|_{\dot{B}^{-\alpha + \frac{2}{q}}_{q,2}((\mathbb{R}^{n-1} \times \mathbb{R})} \leq c\|h\|_{\dot{B}^{-\alpha + \frac{2}{q}}_{q,2}(\mathbb{R}^n)}.
\]

3) Finally let us consider the case \( \alpha = \frac{1}{q} \). Using the real interpolation, we get the case of \( \alpha = \frac{1}{q} \).

**APPENDIX D. PROOF OF LEMMA 3.8**

- Let \( \tilde{f} \in L^p(\mathbb{R}^n) \) be the zero extension of \( f \) to \( \mathbb{R}^n \times \mathbb{R} \). Note that \( D_x \Gamma \ast \tilde{f} = \Gamma \ast D_x \tilde{f} \). From (A.1), we have

\[
\|D_x \Gamma \ast \tilde{f}\|_{W^{\beta,1}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|D_x \tilde{f}\|_{L^p(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)}
\]

and

\[
\|D_x \Gamma \ast \tilde{f}\|_{W^{\beta,1}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|\Gamma \ast \tilde{f}\|_{W^{\beta,1}_p(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^p(\mathbb{R}^n)}.
\]

By interpolating these two estimates, we can obtain

\[
\|D_x \Gamma \ast f\|_{\dot{B}^\beta_{q,p}(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^p(\mathbb{R}^n)} \quad 0 < \beta < 1.
\]

Further, by applying Besov imbedding (see (3) of Proposition 2.1), for \( 1 - \alpha + \beta - (n + 2)(\frac{1}{p} - \frac{1}{q}) = 0 \), we have

\[
\|D_x \Gamma \ast f\|_{\dot{B}^\beta_{q,p}(\mathbb{R}^n \times \mathbb{R})} \leq c\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R})}.
\]

Note that \( D_x \Gamma \ast f(x,t) = 0 \) for \( t \leq 0 \). Hence, \( D_x \Gamma \ast f \in \dot{B}^\alpha_{q,2}(\mathbb{R}^n \times \mathbb{R}^+ \)

- Now, we derive the estimate of \( D_x \Gamma \ast f \big|_{x_n=0} \).

1) Let \( \alpha > \frac{1}{q} \). Then, according to the usual trace theorem, \( D_x \Gamma \ast f \in \dot{B}^\alpha_{q,2}(\mathbb{R}^n \times \mathbb{R}^+) \) implies that \( D_x \Gamma \ast f \big|_{x_n=0} \in \dot{B}^\alpha_{q,2}(\mathbb{R}^n \times \mathbb{R}^+) \) with

\[
\|D_x \Gamma \ast f \big|_{x_n=0}\|_{\dot{B}^\alpha_{q,2}(\mathbb{R}^n \times \mathbb{R}^+)} \leq c\|D_x \Gamma \ast f\|_{\dot{B}^\alpha_{q,2}(\mathbb{R}^n \times \mathbb{R}^+)} \leq c\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}^+)}.
\]

2) Let \( 0 < \alpha \leq \frac{1}{q} \). In this case, the usual trace theorem does not hold true.
If $\alpha + \frac{n+1}{p} - \frac{n+2}{q} > 0$, we can choose $r$ with $p < r < q$, $\alpha + \frac{n+1}{p} - \frac{n+2}{q} > 0$. Set $\gamma = \alpha + \frac{n+1}{r} - \frac{n+2}{q}$, then $\alpha - \frac{1}{q} - \frac{n+1}{r} = \gamma - \frac{1}{r} - \frac{n+1}{r}$ and $\alpha - \frac{1}{q} < \gamma - \frac{1}{r}$. Hence, by using the Besov embedding theorem,

$$
\|D_x \Gamma \ast f|_{x_n=0}\|_{\dot{B}^\gamma_{q,0}} \leq c\|D_x \Gamma \ast f|_{x_n=0}\|_{\dot{B}^\gamma_{q,0}}.
$$

As $\gamma > \frac{1}{r}$, the use of the usual trace theorem gives

$$
\|D_x \Gamma \ast f|_{x_n=0}\|_{\dot{B}^\gamma_{q,0}} \leq c\|D_x \Gamma \ast f\|_{\dot{B}^\gamma_{\infty,0}} \leq c\|f\|_{L^p(R^n ; \dot{B}^0_\infty(R^n))}.
$$

Hence, the proof of Lemma 3.8 is completed.

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