Optimal Grid Drawings of Complete Multipartite Graphs and an Integer Variant of the Algebraic Connectivity

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Problem 1
How to draw the vertices of a complete multipartite graph $G$ on different points of a bounded $d$-dimensional integer grid, such that the sum of squared distances between vertices of $G$ is minimized? On each grid point one vertex of the graph is drawn.

Equivalent:
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

A drawing of the vertices of $K_{3,4}$ on a line ($d=1$).

A drawing of the vertices of $K_{12,9,4}$ on a 2-dimensional grid.
A relation between eigenvalues and drawings of a graph

Let $G = (V, E)$ be a graph and let $\lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_N(G)$ be the Laplacian eigenvalues of $G$.

**Embedding Lemma (Spielman-Teng):**

$$\lambda_2(G) = \min \frac{\sum_{i,j \in E} \|\vec{v}_i - \vec{v}_j\|^2}{\sum_{i \in V} \|\vec{v}_i\|^2}$$

where the minimum is taken over all tuples $(\vec{v}_1, \ldots, \vec{v}_N)$ of vectors $\vec{v}_i \in \mathbb{R}^d$ with $\sum_{i=1}^{N} \vec{v}_i = 0$, and not all $\vec{v}_i$ are zero-vectors $0$.

Spielman, D. A., Teng, S.-H. Spectral partitioning works: Planar graphs and finite element meshes. 2007.
Example (1):
An optimal drawing of the complete bipartite graph $K_{3,3}$

The complete bipartite graph $K_{3,3}$

\[
\begin{pmatrix}
a & b & c & x & y & z \\
a & 3 & 0 & 0 & -1 & -1 & -1 \\
b & 0 & 3 & 0 & -1 & -1 & -1 \\
c & 0 & 0 & 3 & -1 & -1 & -1 \\
x & -1 & -1 & -1 & 3 & 0 & 0 \\
y & -1 & -1 & -1 & 0 & 3 & 0 \\
z & -1 & -1 & -1 & 0 & 0 & 3 \\
\end{pmatrix}
\]

The Laplacian matrix of $K_{3,3}$

Laplacian eigenvalues $\{0^{(1)}, 3^{(4)}, 6^{(1)}\}$
Example (2):
An optimal drawing of the complete bipartite graph $K_{3,3}$

$$\lambda_2(G') = \min \frac{\sum_{i,j \in E} \| \vec{v}_i - \vec{v}_j \|^2}{\sum_{i \in V} \| \vec{v}_i \|^2}$$

$$\lambda_2(G') \leq \frac{16 + 2 \cdot 8 + 2 \cdot 4 + 4 \cdot 2}{4 \cdot 2 + 2 \cdot 4} = 3$$

Laplacian eigenvalues $\{0^{(1)}, 3^{(4)}, 6^{(1)}\}$
Problem 1
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

If each color class $C_i$ has the same number of points, then any drawing with $\sum_{\vec{v} \in C_i} \vec{v} = 0$, for all $i$, is optimal.

There is an exponential number of optimal drawings.
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There is an exponential number of optimal drawings.

The number $\mathcal{N}$ of optimal drawings of $K_{1,2m,2m}$ for $d = 1$ is

$$c \cdot \frac{16^m}{m} < \mathcal{N} < 16^m.$$
Problem 1
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

**Lemma:**
For a straight line drawing of $G = (V, E) = K_{n_1, ..., n_r}$ on "a small box of" the grid $\mathbb{Z}^d$, 
\[
\frac{\sum_{i,j \in E} \| \vec{v}_i - \vec{v}_j \|^2}{\sum_{i \in V} \| \vec{v}_i \|^2} = N + \frac{1}{S} \sum_{i=1}^{r} \left( -n_i \sum_{v \in C_i} \| v \|^2 \right) + \frac{1}{S} \sum_{i=1}^{r} \left\| \sum_{v \in C_i} v \right\|^2
\]

where 
\[
N = n_1 + \ldots + n_r \quad \quad S = \sum_{i \in V} \| v_i \|^2 \quad \quad \text{color classes } C_i
\]
Problem 1
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is minimized.

If all color classes $C_i$ have a **different** number of points, then in an optimal drawing, for each $i$, the union of the smallest $i$ color classes, $\bigcup_{j=1}^{i} C_j$, forms a ball centered at 0.

The optimal drawing is essentially unique.
Problem 2
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is maximized.

Drawings obtained with computer simulations, using simulated annealing.

The best way to maximize the sum of squared distances between points of different colors on a $101 \times 101$ integer grid. Left: for $r = 6$ colors, with 1701 purple points and 1700 points of every other color. Right: for $r = 7$ colors, with 1459 yellow points and 1457 points of every other color.
Problem 2
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is maximized.

Embedding lemma for largest Laplacian eigenvalue:

\[ \lambda_N(G) = \max \frac{\sum_{ij \in E} \| \vec{v}_i - \vec{v}_j \|^2}{\sum_{i \in V} \| \vec{v}_i \|^2} \]
Problem 2
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is maximized.

Given a set of points \( \{c_i\}_{i=1}^r \), the Voronoi region of a point \( c_i \) is defined by \( V_i = \{x \in \mathbb{R}^d \mid \|x - c_i\| < \|x - c_j\| \text{ for } j = 1, \ldots, r, j \neq i \} \).

A Voronoi diagram is *centroidal* if each \( c_i \) is the center of mass of its Voronoi region.
Problem 2
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is maximized.

For a set of \( n \) points \( v_1, \ldots, v_n \) in \( \mathbb{R}^d \), with centroid \( c = \frac{1}{n} \sum_{i=1}^{n} v_i \)

\[
\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \|v_i - v_j\|^2 = n \sum_{i=1}^{n} \|v_i - c\|^2
\]
Problem 2
Find the assignment of colors to the grid points such that the sum of squared distances between points of different colors is \textbf{maximized}.

weighted centroidal Voronoi diagrams
if color classes of $K_{n_1,n_2,...,n_r}$ have different sizes
Part 2

An Integer Variant of the Algebraic Connectivity

Consider drawings in dimension $d = 1$ of graphs $G = (V, E)$ with $|V| = N$.

Define

$$\lambda^I_2(G) = \min \frac{\sum_{ij \in E} (v_i - v_j)^2}{\sum_{i \in V} v_i^2}$$

where the minimum is taken over drawings of $G$ on $\{-\lfloor N/2 \rfloor, \ldots, \lceil N/2 \rceil\}$ with $\sum_{i=1}^{N} v_i = 0$

and no two vertices are mapped to the same point.

If $N$ is even, then no vertex is mapped to the origin.

A drawing of vertices of $K_{3,3}$ on $\{-3, \ldots, 3\}$.

$$\lambda^I_2(K_{3,3}) = \lambda_2(K_{3,3}) = 3$$

A drawing of vertices of $K_{3,4}$ on $\{-3, \ldots, 3\}$.

$$\lambda^I_2(K_{3,4}) > \lambda_2(K_{3,4}) = 3$$
An Integer Variant of the Algebraic Connectivity

\[ \lambda_2^I(G) = \min \frac{\sum_{ij \in E} (v_i - v_j)^2}{\sum_{i \in V} v_i^2} \]

where the minimum is taken over drawings of \( G \) on \( \{-[N/2], \ldots, [N/2]\} \) with \( \sum_{i=1}^{N} v_i = 0 \) and no two vertices are mapped to the same point.

For \( N \) odd, \( \lambda_2^I \) is equivalent to the **minimum-2-sum**.

The **minimum-\( p \)-sum-problem**: for \( p > 0 \), for a graph \( G \) and a bijective mapping \( \Psi \) from \( V \) to \( \{1, \ldots, N\} \), define \( \sigma_2(G, \Psi) = \left( \sum_{uv \in E(G)} |\Psi(u) - \Psi(v)|^p \right)^{1/p} \).

The quantity \( \sigma_p(G) = \min_{\Psi} \sigma_p(G, \Psi) \) (where the minimum is taken over all bijective mappings) is then called the minimum-\( p \)-sum of \( G \). The problem of finding \( \sigma_p(G) \) is called the minimum-\( p \)-sum problem.

Juwan, M., Mohar, B. Optimal linear labelings and eigenvalues of graphs. 1992.
An Integer Variant of the Algebraic Connectivity

Properties of $\lambda_2^I$

If $G$ and $H$ are edge-disjoint graphs with the same set of vertices, then

$$\lambda_2^I(G) + \lambda_2^I(H) \leq \lambda_2^I(G \cup H)$$

Denote by $G + e$ the graph obtained from the graph $G$ with $N$ vertices by adding an edge $e$. Then

$$\frac{\lambda_2^I(G) + 1}{2 \sum_{i=1}^{[N/2]} i^2} \leq \lambda_2^I(G + e) \leq \lambda_2^I(G) + \frac{N^2}{2 \sum_{i=1}^{[N/2]} i^2}$$

Analogous properties of $\lambda_2$

$$\lambda_2(G) + \lambda_2(H) = \lambda_2(G \cup H)$$

$$\lambda_2(G) \leq \lambda_2(G + e) \leq \lambda_2(G) + 2$$
The Cartesian product of graphs

\[ \lambda_2^I(G \times H) \text{ can be strictly larger than } \min\{\lambda_2^I(G), \lambda_2^I(H)\} \]

\[ \lambda_2(G \times H) = \min\{\lambda_2(G), \lambda_2(H)\} \]

\[ \lambda_2^I(C_3 \times P_2) = \frac{60}{28} \]

\[ \lambda_2(C_3 \times P_2) = 2 \]
The Cartesian product of graphs

There are graphs $G$ with $\lambda_2^I(G \times G) < \lambda_2^I(G)$.

$\lambda_2(G \times G) = \lambda_2(G)$

$\lambda_2^I(G) = \frac{8}{10}$

$\lambda_2^I(G \times G) < \frac{6}{10}$
The Cartesian product of graphs

Let $G$ and $H$ be graphs of odd order $|G|$ and $|H|$.

\[
\lambda_2^I(G \times H) \leq \lambda_2^I(G) \left( \frac{|G|^2 - 1}{|G|^2|H|^2 - 1} \right) + \lambda_2^I(H) \left( \frac{|G|^2(|H|^2 - 1)}{|G|^2|H|^2 - 1} \right)
\]

A drawing of $C_3 \times P_3$ that attains this bound.
The Cartesian product of graphs

Let $G$ and $H$ be graphs of odd order $|G|$ and $|H|$.

$$\lambda_2^I(G \times H) \leq \lambda_2^I(G) \left( \frac{|G|^2 - 1}{|G|^2|H|^2 - 1} \right) + \lambda_2^I(H) \left( \frac{|G|^2(|H|^2 - 1)}{|G|^2|H|^2 - 1} \right)$$

$$\lambda_2^I(G \times H) \leq \frac{\lambda_2^I(G) + \lambda_2^I(H)}{2}$$

If $\lambda_2^I(G) = \lambda_2(G)$ and $G$ has odd order, then $\lambda_2^I(G \times G) = \lambda_2^I(G)$.
Comparing $\lambda_2^I$ with the minimum 2-sum

There exist graphs $G$ of even order, for which the optimal drawings of $\lambda_2^I(G)$ and $\sigma_2^2(G)$ are different.

The second drawing is better for $\lambda_2^I(G)$

optimal drawing for $\sigma_2^2(G)$
An Integer Variant of the Algebraic Connectivity

Open problem:
Characterize the class of graphs $G$ for which $\lambda_2(G) = \lambda_2^I(G)$.

What we know:

\[ \lambda_2^I = \lambda_2 \]

- for the hypercube
- for complete multi-partite graphs $K_{n,n,...,n}$ for $n > 1$

- If $\lambda_2^I(G) = \lambda_2(G)$ and $|G|$ is odd, then $\lambda_2^I(G \times G) = \lambda_2^I(G)$