Tight Wavelet Frame Sets in Finite Vector Spaces

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Abstract

Let $q \geq 2$ be an integer, and $\mathbb{F}_q^d$, $d \geq 1$, be the vector space over the cyclic space $\mathbb{F}_q$. The purpose of this paper is two-fold. First, we obtain sufficient conditions on $E \subset \mathbb{F}_q^d$ such that the inverse Fourier transform of $1_E$ generates a tight wavelet frame in $L^2(\mathbb{F}_q^d)$. We call these sets (tight) wavelet frame sets. The conditions are given in terms of multiplicative and translational tilings, which is analogous with Theorem 1.1 ([20]) by Wang in the setting of finite fields. In the second part of the paper, we exhibit a constructive method for obtaining tight wavelet frame sets in $\mathbb{F}_q^d$, $d \geq 2$, $q$ an odd prime and $q \equiv 3 \pmod{4}$.

Key words and phrases. Prime fields; finite vector space; wavelet frames; wavelet frame sets; tight frames; translational tiling; multiplicative tiling; rotational tiling; spectral set; spectrum; spectral pair

1 Introduction

A countable subset $\{x_k\}_{k \in I}$ of a Hilbert space $\mathcal{H}$ is said to be a frame if there exists two positive constants $A \leq B$ such that for any $x \in \mathcal{H}$

$$A\|x\|_\mathcal{H}^2 \leq \sum_{k \in I} |\langle x, x_k \rangle_\mathcal{H}|^2 \leq B\|x\|_\mathcal{H}^2.$$ 

The positive constants $A$ and $B$ are called lower and upper frame bounds, respectively. The frame is called a tight frame when we can take $A = B$ and it is called a Parseval frame if $A = B = 1$. One of most significant features of the frames that makes them practical and useful is their redundancy which has an important role, for example, in robustness. The frames also allow a localized representation of elements in the Hilbert space and they have been used for a number of years by engineers and applied mathematicians for purposes of signal processing and data compression. The notion of frames was first introduced by Duffin and Schaeffer [6]. Amongst the frames, tight frames play a fundamental role in the applications of frames due to their numerical stability. In this paper we aim to construct tight frames on the finite vector spaces over the finite fields which arise from dilation and translation of a function whose Fourier transform is characteristic function of a non-empty set.

In the classical setting, a function $\psi \in L^2(\mathbb{R}^d)$ is said to generate a orthonormal wavelet basis (resp. wavelet frame) if there is a set of $d \times d$ matrices $D \subset GL(d, \mathbb{R})$ and a subset $T \subset \mathbb{R}^d$ such
that the family
\[
\{|\det(D)|^{1/2}\psi(Dx - t) : D \in \mathcal{D}, t \in T\},
\]  
forms an orthonormal basis (resp. frame) for \(L^2(\mathbb{R}^d)\). Then we say \(\psi\) is an orthonormal wavelet (resp. frame wavelet) and every element \(|\det(D)|^{1/2}\psi(Dx - t)\) is a dilation and translation copy of \(\psi\) with respect to the invertible matrix \(D\) and translation \(t\), respectively. The structures of \(\mathcal{D}\) and \(T\) and associated to which there exists orthonormal and frame wavelets for \(L^2(\mathbb{R}^d)\) have been studied by many authors, for example [9, 5, 19, 20]. See also [12] for an alternative perspective on wavelets in vector spaces over finite fields and connections with combinatorial problems.

Let \(q\) be an odd prime and \(\mathbb{F}_q\) be the prime field with \(q\) elements. Then \(\mathbb{F}_q^d\) is the vector space of dimension \(d\) over the finite field \(\mathbb{F}_q\). In analogue to \(\mathbb{R}^d\), the purpose of this paper is to study tight wavelet frames on \(\mathbb{F}_q^d\) and its subspace for \(d \geq 1\). Let \(\text{Aut}(\mathbb{F}_q^d)\) be the set of all automorphisms on \(\mathbb{F}_q^d\). Following the same spirit of the frame wavelets on \(\mathbb{R}^d\), we say that a function \(\psi : \mathbb{F}_q^d \to \mathbb{C}\) is a wavelet or a frame wavelet for \(L^2(\mathbb{F}_q^d)\) if there exists a set of automorphisms \(\mathcal{A} \subset \text{Aut}(\mathbb{F}_q^d)\) and a subset \(\Lambda \subset \mathbb{F}_q^d\) such that the family
\[
\{\psi(ax - \lambda) : a \in \mathcal{A}, \lambda \in \Lambda\}.
\]  
is an orthonormal basis (resp. frame) for \(L^2(\mathbb{F}_q^d)\). Note that in the continuous case, for the matrix \(D\), the factor \(\Delta_D := |\det(D)|^{1/2}\) makes the dilation map
\[
f \to |\det(D)|^{1/2}f(Dx)
\]  
an isometry. In the discrete case \(\mathbb{F}_q^d\) the Haar measure on \(\mathbb{F}_q^d\) is discrete and invariant under the dilation, therefore the dilation factor \(\Delta_a\) is 1.

A common way to construct a wavelet frame on \(\mathbb{R}^d\) is to choose a function whose Fourier transform is the indicator of a measurable set, and then consider the system (1.1) of dilations and translations of the function. This leads to the traditional definition of frame wavelet sets as follows: A set \(\Omega \subset \mathbb{R}^d\) is called a frame wavelet set with respect to \(\mathcal{D}\) and \(T\) if for the function \(\psi\) with \(\hat{\psi} = 1\Omega\), the system (1.1) is a frame in \(L^2(\mathbb{R}^d)\). If the system is an orthonormal basis for \(L^2(\mathbb{R}^d)\), then the function \(\psi\) is called minimally supported frequency wavelet (MSF wavelet) and \(\Omega\) is called a wavelet set. The wavelet sets and minimally supported frequency wavelets were introduced in [19] and studied exclusively, for example, in [10, 11] and by many other authors. The existence of wavelet sets in \(\mathbb{R}^d\) for any expansive matrix was given in [5]. The wavelet theory and wavelet sets are studied in a constructive way in locally compact abelian groups with compact open subgroups, such as \(p\)-adic groups \(\mathbb{Q}_p\), in [2, 3]

A well-known example of a wavelet set is the Shannon set given by
\[
\Omega = [-2\pi, -\pi] \cup [\pi, 2\pi].
\]  
The orthonormal wavelet for \(\Omega\) is then given by
\[
\psi(x) = 2 \text{sinc}(2x - 1) - \text{sinc}(x)
\]  
with \(\widehat{\psi} = 1\Omega\), where \(1\Omega\) is the indicator of set \(\Omega\). For more examples and constructions of wavelet sets in \(\mathbb{R}^d\) we invite the reader to see [1, 3, 4, 15, 16].
In [20], Wang tied the existence of wavelet sets with the notion of spectral sets. We say that a measurable set of positive measure $\Omega$ is a spectral set if there exists a countable set $\Gamma$ such that the collection of exponentials $\{e^{2\pi i \gamma \cdot x} : \gamma \in \Gamma\}$ forms an orthonormal basis for $L^2(\Omega)$. In this case we say that $(\Omega, \Gamma)$ forms a spectral pair. Spectral sets were first introduced by Fuglede [8] and he proposed an infamous conjecture asserting that spectral sets are exactly translational tiles on $\mathbb{R}^d$. However, this conjecture was proven to be false in its full generality by Tao [18]. Nowadays, the exact relationship between spectral sets and translational tiles are mysterious. We refer to [7, 13] for some recent progress.

Wang proved the following result in the classical setting which characterizes the wavelet sets by multiplicative and translational tiling. Let $D^T$ denote the transpose of matrix $D$, and $\hat{g}$ denote the inverse Fourier transform of $g$.

**Theorem 1.1** (Theorem 1.1, [20]). Let $D \subset GL(d, \mathbb{R})$ and $\Gamma \subset \mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ with positive and finite Lebesgue measure. If $\{D_t(\Omega) : D \in D\}$ is a tiling of $\mathbb{R}^d$ and $(\Omega, \Gamma)$ is a spectral pair, then $\psi = \hat{1}_\Omega$ is a wavelet with respect to the dilation set $D$ and the translation set $\Gamma$. Conversely, if $\psi = \hat{1}_\Omega$ is a wavelet with respect to $D$ and $\Gamma$ and $0 \in \Gamma$, then $\{D^T(\Omega) : D \in D\}$ is a tiling of $\mathbb{R}^d$ and $(\Omega, \Gamma)$ is a spectral pair.

Inspired by the result of the theorem, it is natural for us to ask for what degree one can extend the notion and concept of wavelet sets and multiplicative tiling in $\mathbb{F}^d_q$. In what follows, we shall study this. For the multiplicative tiling purpose, we have to remove the origin and let $Y := \mathbb{F}^d_q \setminus \{0\}$.

**Definition 1.2** (Multiplicative and translational tiling). Let $E$ be a subset of $\mathbb{F}^d_q$. We say $E$ is a multiplicative tiling set for $\mathbb{F}^d_q$ if there is a set of automorphisms $A$ in Aut($\mathbb{F}^d_q$) such that $E$ tiles $Y$ multiplicatively by $A$, i.e.,

$$Y = \bigcup_{\alpha \in A} \alpha(E) \text{ (disjoint union).}$$

As a result, a multiplicative tiling set does not include the origin $\vec{0}$. This is a natural requirement since $\alpha(\vec{0}) = \vec{0}$ for any automorphism $\alpha$.

We say that a subset $F \subset \mathbb{F}^d_q$ is a translational tiling set for $\mathbb{F}^d_q$ if there exists $\Lambda \subset \mathbb{F}^d_q$ such that

$$\mathbb{F}^d_q = \bigcup_{\lambda \in \Lambda} (F + \lambda) \text{ (disjoint union).}$$

We say a set $E$ has a spectrum $L$ if the characters $\{\chi_l\}_{l \in L}$ is an orthonormal basis for $L^2(E)$. In this case, we say $E$ is a spectral and $(E, L)$ is a spectral pair.

Our first result shows that there exists no wavelet sets in the traditional sense for $L^2(\mathbb{F}^d_q)$. More precisely, there is no Parseval frame (thus no orthonormal basis) of type (1.2) for $L^2(\mathbb{F}^d_q)$ generated by any function of type $\psi := \hat{1}_E$ (Theorem 2.3). However, later in this paper we prove the existence of subsets $E$ in $\mathbb{F}^d_q$ for which $\psi := \hat{1}_E$ generates a tight wavelet frame for a subspace of $L^2(\mathbb{F}^d_q)$. We shall call these sets tight wavelet frame sets. We will then present an explicit construction of a class of wavelet frame sets when $q \equiv 3 \pmod{4}$.
We organize the paper as follows: In Section 2 we prove an analogy of Theorem 1.1 (Theorem 1.1, [20]) in $\mathbb{F}_d^q$ and we study necessary and sufficient conditions for a set $E$ such that $E$ is a tight wavelet frame set for a subspace of $L^2(\mathbb{R}_d^q)$. These results are collected in Theorems 2.9 and 2.10. In this section we also provide a counter example where the disjointness of the sets does not necessarily hold for tight wavelet frame sets. In Section 3, we will show the existence of a multiplicative tiling set in $\mathbb{F}_d^q$ for $q$ prime and $q \equiv 3 \pmod{4}$. In Section 4 we present a constructive approach in Theorem 4.4 to prove the existence of tight wavelet frame sets in $\mathbb{F}_d^q$, when $d = 2$, $q$ prime and $q \equiv 3 \pmod{4}$.

In the sequel we shall assume that the subset $E$ is non trivial, i.e., $1 < |E| < q^d$.

## 2 Sufficient Conditions for Tight Wavelet Frame Sets

Here we first review the Fourier transform on $\mathbb{F}_d^q$. If $f$ is a function on $\mathbb{F}_d^q$, then the Fourier coefficients, $\hat{f}(\xi)$, of $f$ are given by

$$
\hat{f}(\xi) = q^{-d} \sum_{m \in \mathbb{F}_d^q} a_m \chi_m(\xi), \quad \xi \in \mathbb{F}_d^q
$$

where $\chi_m(\xi) = e^{2\pi i m \cdot \xi}$ is the character and $m \cdot \xi$ is the usual inner product and $a_m = f(m)$. The function $\hat{f}$ is called the Fourier transform of $f$. With the above notations,

$$f(x) = \sum_{m \in \mathbb{F}_d^q} c_m \chi_m(x),$$

where $c_m = \hat{f}(m)$. By the above definition, the Fourier transform $\mathcal{F} : L^2(\mathbb{F}_d^q) \to L^2(\mathbb{F}_d^q)$ given by $f \to \hat{f}$ is a unitary map. For any $g \in L^2(\mathbb{F}_d^q)$, we shall denote by $\hat{g}$ the inverse Fourier transform of $g$.

Let $\text{Aut}(\mathbb{F}_d^q)$ be the set of all automorphisms on $\mathbb{F}_d^q$. Let $\psi$ be a function defined on $\mathbb{F}_d^q$. For $a \in \text{Aut}(\mathbb{F}_d^q)$ and $t \in \mathbb{F}_d^q$ define the associated dilation and translation operators $\delta_a$ and $\tau_t$ by

$$
\delta_a \psi(x) = \psi(ax),
$$

and

$$
\tau_t \psi(x) = \psi(x - t),
$$

respectively. These operators are unitary.

**Lemma 2.1.** Let $\psi : \mathbb{F}_d^q \to \mathbb{C}$. Then for given automorphism $a \in \text{Aut}(\mathbb{F}_d^q)$ and $t \in \mathbb{F}_d^q$ we have

$$
\hat{\delta_a \tau_t \psi}(m) = \chi_{a^{-1} t}(m) \hat{\psi}(a^* m), \quad (2.1)
$$

where $a^* = (a^t)^{-1} = (a^{-1})^t$ is the inverse transpose of $a$. If $\hat{\psi} = 1_E$, then

$$
\hat{\delta_a \tau_t \psi}(m) = \chi_{a^{-1} t}(m) 1_{a^*(E)}(m). \quad (2.2)
$$
Proof. Let $t \in \mathbb{F}_q^d$ and $a \in \text{Aut} \big( \mathbb{F}_q^d \big)$. By applying the Fourier transform and using a change of variable in the definition of the Fourier transform, for all $m \in \mathbb{F}_p^d$ we have

$$\hat{\tau}_t \psi(m) = q^{-d} \sum_{n \in \mathbb{F}_q^d} \psi(n - t) \overline{\chi_m(n)}$$

$$= q^{-d} \sum_{n \in \mathbb{F}_q^d} \psi(n) \overline{\chi_m(n + t)}$$

$$= \overline{\chi_m(t)} \left( q^{-d} \sum_{n \in \mathbb{F}_q^d} \psi(n) \overline{\chi_m(n)} \right)$$

$$= \overline{\chi_m(t)} \hat{\psi}(m)$$

$$= \chi_t(m) \hat{\psi}(m),$$

and

$$\hat{\delta}_a \psi(m) = q^{-d} \sum_{n \in \mathbb{Z}_p^d} \psi(an) \overline{\chi_m(n)}$$

$$= q^{-d} \sum_{n \in \mathbb{Z}_p^d} \psi(n) \overline{\chi_m(a^{-1}n)}$$

$$= q^{-d} \sum_{n \in \mathbb{Z}_p^d} \psi(n) \chi_{a^*m}(n)$$

$$= \hat{\psi}(a^*m).$$

Now, a combination of (2.4) and (2.3) yields the assertion of the lemma:

$$\hat{\delta}_a \hat{\tau}_t \psi(m) = \hat{\tau}_t \hat{\psi}(a^*m) = \overline{\chi_{a^*m}(t)} \hat{\psi}(a^*m) = \overline{\chi_{a^{-1}t}(m)} \hat{\psi}(a^*m).$$

By the equality $1_E(a^*m) = 1_{a^*(E)}(m)$, the second part of the lemma also holds true.

The proof of the following result is straightforward using the Fourier transform.

**Lemma 2.2.** Let $\mathcal{A} \subseteq \text{Aut} \big( \mathbb{F}_q^d \big)$ and $T \subseteq \mathbb{F}_q^d$, and the family

$$\{ \delta_a \tau_t \psi : a \in \mathcal{A}, \ t \in T \}$$

is an orthonormal basis for $L^2(\mathbb{F}_q^d)$ if and only if the family

$$\{ \overline{\chi_{a^{-1}t}(m)} \hat{\psi}(a^*m) : \ t \in T, \ a \in \mathcal{A} \}$$

is an orthonormal basis for $L^2(\mathbb{F}_q^d)$. Here, $m$ is the generic variable.

The next result proves the existence of no Parseval wavelet frame for $L^2(\mathbb{F}_q^d)$ generated by $\psi := 1_E$. 


Theorem 2.3. There exists no non-empty subset $E \subseteq \mathbb{F}_q^d$ such that $\psi := \hat{1}_E$, the inverse Fourier transform of the indicator function $1_E$, is the generator of a Parseval wavelet frame for $L^2(\mathbb{F}_q^d)$.

Proof. We shall prove this theorem by a contradiction argument. Assume that for a subset $E$ there is an automorphism set $\mathcal{A} \subset \text{Aut}(\mathbb{F}_q^d)$ and a subset $\Lambda$ of $\mathbb{F}_q^d$ such that the set

$$\{ \delta_a \tau_\lambda \hat{1}_E : a \in \mathcal{A}, \lambda \in \Lambda \}$$

is a Parseval frame for $L^2(\mathbb{F}_q^d)$. By Lemma 2.2, this is equivalent to say that the family

$$\{ \chi_{a^t(E)}(m) : \lambda \in \Lambda, a \in \mathcal{A} \} = \{ \chi_{a^{-1}(m)} \hat{1}_{a^t(E)}(m) : \lambda \in \Lambda, a \in \mathcal{A} \} \quad (2.8)$$

is a Parseval frame for $L^2(\mathbb{F}_q^d) = L^2(\mathbb{F}_q^d)$. (Here, $a^t$ is the transpose of $a$.) Let $\hat{g} = 1_{\overline{0}} \in L^2(\mathbb{F}_q^d)$ be the indicator function for the set $\{ 0 \}$. Then

$$1 = \| \hat{g} \|^2 = \sum_{a \in \mathcal{A}, \lambda \in \Lambda} | \langle g, \delta_a \tau_\lambda \psi \rangle |^2 \quad (2.9)$$

$$= \sum_{a \in \mathcal{A}, \lambda \in \Lambda} | \langle \hat{g}, \delta_a \tau_\lambda \psi \rangle |^2$$

$$= \sum_{a \in \mathcal{A}, \lambda \in \Lambda} | \sum_{m \in \mathbb{F}_q^d} \hat{g}(m) \chi_{a^t(E)}(m) \overline{\chi_{a^{-1}(m)}}(\lambda) |^2$$

$$= \sum_{a \in \mathcal{A}, \lambda \in \Lambda} | \hat{1}_{a^t(E)}(0) |^2$$

$$= \hat{\sharp}(\Lambda) \sum_{a \in \mathcal{A}} | \hat{1}_{a^t(E)}(0) |^2.$$

Here, we shall consider two cases: If $0 \in E$, then $0 \in a^t(E)$ for all $a \in \mathcal{A}$. Thus the above calculation implies that $1 = \hat{\sharp}(\mathcal{A}) \hat{\sharp}(\Lambda)$. This means that the wavelet system must have only one element. Let $\mathcal{A} = \{ a \}$ and $\Lambda = \{ \lambda \}$. Then all the vectors in $L^2(\mathbb{F}_q^d)$ must be a constant multiple of $\delta_a \tau_\lambda \psi$. We show that this is not the case since $E \neq \mathbb{F}_q^d$. Let $f \neq 0$ in $L^2(\mathbb{F}_q^d)$ such that $\text{supp}(\hat{f}) \cap a^t(E) = \emptyset$. Such function exists since $E$ is not the whole $\mathbb{F}_q^d$. Then there is no constant $c$ such that $f = c \delta_a \tau_\lambda \psi$. This shows that the Parseval wavelet frame can not have only one element when $E \neq \mathbb{F}_q^d$, thus $\hat{\sharp}(\mathcal{A}) \hat{\sharp}(\Lambda) > 1$.

Now let us now assume that $0 \not\in E$. By the equations in (2.9) we obtain $1 = 0$ which is impossible. This completes the proof our assertion.

As a result of Theorem 2.3, we have the following corollary.

Corollary 2.4. There is no orthonormal basis of form $\{ \delta_a \tau_\lambda \psi \}_{a \in \mathcal{A}, \lambda \in \Lambda}$ for $L^2(\mathbb{F}_q^d)$ where $\psi = 1_E$ and $E \subsetneq \mathbb{F}_q^d$.

Remark. Notice when $E = \mathbb{F}_q^d$, by a similar calculation (2.9) for $\hat{g} = 1_{\mathbb{F}_q^d}$, we can conclude that the translation set $\Lambda$ must contains $0$ and $\hat{\sharp}(\mathcal{A}) = 1$. Then by the divisibility we must have $\Lambda = \mathbb{F}_q^d$. This implies that the set $\{ \tau_\lambda \psi : \lambda \in \mathbb{F}_q^d \}$, with $\psi = 1_{\mathbb{F}_q^d}$, forms an orthonormal basis for
\(L^2(\mathbb{F}_q^d)\). But this is already well-known by the Fourier transform. Therefore, it is reasonable to assume in Theorem 2.3 that \(E \neq \mathbb{F}_q^d\).

As we observed above, Corollary 2.4 implies the existence of no orthonormal basis of form (2.8) for \(L^2(\mathbb{F}_q^d)\). However, in the following theorem, we prove that if we choose \(E\) appropriately in \(\mathbb{F}_q^d\), then the family (2.8) for \(E^* = E \setminus \{0\}\) is a tight wavelet frame for \(L^2((\mathbb{F}_q^d)^*)\).

For the rest, we use the notation \(Y := (\mathbb{F}_q^d)^*\).

**Definition 2.5.** Given \(E\) and \(L\) subsets of \(\mathbb{F}_q^d\), we say \((E, L)\) is a (tight) frame spectral pair if the set of “exponentials” \(\{\chi_l : l \in L\}\) is a (tight) frame for \(L^2(E)\).

**Theorem 2.6.** Let \(E\) and \(L\) be subsets of \(\mathbb{F}_q^d\) and \((E, L)\) is a spectral pair. Assume that \(E^*\) is a multiplicative tiling set with respect to a set of automorphisms \(A \subset \text{Aut}(\mathbb{F}_q^d)\). Then the following hold true:

1. \((E^*, L)\) is tight frame spectral pair with the frame bound \(\|h\|(E)\).
2. \(\forall a \in A, \ (a(E^*), (a^{-1})^t(L))\) is tight frame spectral pair with the frame bound \(\|h\|(E)\).
3. The family \(\{1_{a(E^*)}\chi_l(a^{-1})^t(l) : l \in L, a \in A\}\) is a tight frame for \(L^2(Y)\) with the frame bound \(\|h\|(E)\), where \(Y = (\mathbb{F}_q^d)^*\).
4. If \(0 \notin E\), then \(\{(\sqrt{2}a(E)^{-1/2}1_{a(E)}\chi_l(a^{-1})^t(l)) : l \in L, a \in A\}\) is an orthonormal basis for \(L^2(Y)\).

**Proof.** Assume that \(E\) has a spectrum \(L\). Then \(\{\chi_l : l \in L\}\) is an orthonormal basis for \(L^2(E)\). To prove (1), note that the map \(f \mapsto f1_{E^*}\) is a projection of \(L^2(E)\) onto \(L^2(E^*)\), thus the image of the orthonormal basis \(\{\chi_l : l \in L\}\) by this map is a Parseval frame for \(L^2(E^*)\). This proves the statement (1). To prove (2), we instead prove the following: Let \(F \subset \mathbb{F}_q^d\) and \(\{\chi_l : l \in L\}\) be a frame for \(L^2(F)\) with the frame bounds \(0 < A \leq B < \infty\). Then \(\{\chi_l : l \in L\}\) is a frame for \(L^2(a(F))\) with the unified frame bounds, \(A\) and \(B\).

To prove this, define the map \(T_a : L^2(F) \rightarrow L^2(a(F))\) by \(f \mapsto f \circ a^{-1}\). The image of \(\{\chi_l : l \in L\}\) under \(T_a\) is \(\{1_{a(F)}\chi_l(a^{-1})^t(l) : l \in L\}\) and the map is unitary. Therefore, \(\{1_{a(F)}\chi_l(a^{-1})^t(l) : l \in L\}\) forms a frame for \(L^2(a(F))\) with the same frame bounds.

To prove (3), note that by the assumption on the multiplicative tiling property of \(E^*\) we have

\[L^2(Y) = \oplus_{a \in A} L^2(a(E^*))\].

To complete (3), we shall prove the following, instead.

Let \(X\) be measurable set and \(A\) be an index set such that \(X = \bigcup_{a \in A} X_a\) (disjoint). Assume that for all \(a \in A\), \(L^2(X_a)\) has a tight frame \(\{f_{n,a} : n \in I_a\}\) with frame bound \(C\). We claim that the family \(\{f_{n,a} : a \in A, n \in I_a\}\) is a tight frame for \(L^2(X)\) with the frame bound \(C\). To prove that, let \(g \in L^2(X)\). Sine \(\{X_a : a \in A\}\) is a partition for \(X\), then \(g = \oplus_{a \in A} g_a\), \(g_a := g1_{X_a}\), and
we have

\[ \|g\|^2 = \sum_{a \in A} \|g_a\|^2_{L^2(X_a)} \tag{2.10} \]

\[ = \sum_{a \in A} \left( C^{-1} \sum_{n \in I} |\langle g_a, f_{n,a} \rangle|_{L^2(X_a)}|^2 \right) \]

\[ = C^{-1} \sum_{a \in A, n \in I_a} |\langle g, 1_{X_a} f_{n,a} \rangle|_{L^2(X)}|^2. \]

This completes the proof of the assertion, thus (3). For (4), note that \( E = E^* \) when \( 0 \notin E \). Then the proof can be obtained directly from the fact that any Parseval frame with normalized frame elements is an orthonormal basis.

We state the following result from [14] which characterizes tiling set and spectral sets in \( \mathbb{F}_q^2 \) and proves the Fuglede conjecture for \( \mathbb{F}_q^2 \).

**Theorem 2.7** (Fuglede conjecture for \( \mathbb{F}_q^2 \)). A subset \( \emptyset \neq E \) of \( \mathbb{F}_q^2 \) tiles \( \mathbb{F}_q^2 \) with its translations if and only if there is a set \( L \subseteq \mathbb{F}_q^2 \) such that \( (E, L) \) is a spectral pair.

As the corollary of Theorems 2.7 and Theorem 2.6 (3) we have the following result.

**Corollary 2.8.** Assume that \( E \) is a translation tiling for \( \mathbb{F}_q^2 \) and \( E^* \) is a multiplicative tiling with respect to the automorphisms \( A \subset \text{Aut}(\mathbb{F}_q^2) \). Then there is a set \( L \) in \( \mathbb{F}_q^2 \) such that the family

\[ \{ (\sharp E)^{-1/2} \chi_{a^{-1}(l)} 1_{a(E^*)} : l \in L, a \in A \} \]

is a Parseval frame for \( L^2(Y) \). The system is an orthonormal basis if \( 0 \notin E \).

Given a subset \( F \) of \( \mathbb{F}_q^d \) we say a function \( f \in L^2(\mathbb{F}_q^d) \) belongs to \( PW_F \) (The Paley-Wiener space) if its Fourier transform \( \hat{f} \) has support in \( F \). Then

\[ PW_F := \{ f \in L^2(\mathbb{F}_q^d) : \hat{f}(m) = 0 \ \forall m \notin F \}. \tag{2.11} \]

Note that if \( F = (\mathbb{F}_q^d)^* \), then \( PW_Y \) contains the all functions \( f \in L^2(\mathbb{F}_q^d) \) for which \( \sum_{m \in \mathbb{F}_q^d} f(m) = 0 \).

Our next result is analogous with Theorem 1.1. in [20] in \( \mathbb{F}_q^d \). For \( A \subset \text{Aut}(\mathbb{F}_q^d) \), we denote by \( A^t \) the set of transpose of all matrices in \( A \).

**Theorem 2.9.** Assume that \( E \subseteq \mathbb{F}_q^d \) has a spectrum \( L \) and \( \emptyset \neq E^* \) is multiplicative tiling with respect to \( A^t \) for some \( A \subseteq \text{Aut}(\mathbb{F}_q^d) \). Take \( \psi := (\sharp E)^{-1/2} 1_{E^*} \) and \( Y := (\mathbb{F}_q^d)^* \). Then the family \( W := \{ \delta_a \tau_l \psi : a \in A, l \in L \} \) is a Parseval frame for \( PW_Y \). The system \( W \) is an orthonormal basis if \( E = E^* \). Conversely, if \( \psi = (\sharp E)^{-1/2} 1_E \) is a Parseval wavelet for \( PW_Y \) with respect to dilation set \( A \) and translation set \( L \), then \( Y = \cup_{a \in A} a \delta_l(E^*) \). The sets \( a^t(E^*) \) are disjoint, then \( (E^*, L) \) is a tight frame spectral pair.

**Proof.** Assume that \( (E, L) \) is a spectral pair and \( E^* \) is a multiplicative tiling with respect to \( A^t \) for some \( A \subseteq \text{Aut}(\mathbb{F}_q^d) \). By an application of the Fourier transform, to prove that \( W \) is a
Parseval frame for $PW_Y$ is equivalent to say that $\hat{W} := \{(\hat{E})^{-1/2}\chi_{a^{-1}(l)}1_{a^{-1}(E^*)} : l \in L, a \in A\}$ is a Parseval frame for $L^2(Y)$. By Theorem 2.6 (3) we know that $\hat{W}$ is a Parseval frame for $L^2(Y)$. Therefore by inverse of the Fourier transform, which is unitary, we conclude that the wavelet system $W$ is a Parseval frame for $PW_Y$ and this completes the proof of "$\Leftarrow$".

Now assume that $W$ is a Parseval frame for $L^2(Y)$. To prove the union of the sets $a^t(E^*)$, $a \in A$, covers $Y$, we use a contradiction argument. Assume that there is a non empty set $M \subseteq Y$ such that $M \cap a^t(E^*) = \emptyset$ for all $a \in A$. Take $f := 1_M$. Then

$$
\|f\|^2 = (\hat{E})^{-1/2} \sum_{a \in A, l \in L} |\langle f, 1_{a^{-1}(E^*)}l\rangle|^2
$$

(2.12)

Since $M \cap a^t(E^*) = \emptyset$ for all $a \in A$, then the right side in the preceding equation must be zero, while the left side is $\hat{I}(M)$. This is a contradiction to our assumption that $M$ is non-empty, therefore $Y = \bigcup_{a \in A} a^t(E^*)$. To show that the pair $(E^*, L)$ is a tight frame spectral pair, note that by the hypothesis on the disjointness of the sets $a^t(E^*)$, for any $a \in A$ the system $\{\chi_{a^{-1}(l)} : l \in L\}$ is a tight frame for $L^2(a^t(E^*))$ with frame bound $(\hat{E})^{-1/2}$. The dilation operator $T_a : L^2(a^t(E^*)) \to L^2(E^*)$ given by $f \to f \circ a^t$, $f \circ a^t(x) = f(a^t(x))$, $x \in E^*$, is unitary, therefore $\{\chi_l : l \in L\}$, image of $\{\chi_{a^{-1}(l)} : l \in L\}$ under $T_a$, is a tight frame for $L^2(E^*)$ with the unified frame bound and we are done. \hfill \Box

The disjointness of the sets $a^t(E^*)$, $a \in A$, in Theorem 2.9 can be obtained under some additional assumptions on $W$.

**Theorem 2.10.** If the system $W$ is an orthogonal basis for $PW_Y$ and $0 \in L$, then the sets $a^t(E^*)$, $a \in A$, are mutual disjoint and $(E^*, L)$ is a spectral pair:

**Proof.** Now assume that $W$ is an orthogonal basis and $0 \in L$. Take $l = 0$. Then the functions in the family $\{1_{a^t(E^*)} : a \in A\}$ are orthogonal and $\hat{I}(a^t(E^*) \cap a^2(E^*)) = 0$ for any distinct $a_1$ and $a_2$. From the other side, the exponentials $\{\chi_{a^{-1}(l)} : l \in L\}$ are orthogonal basis for $L^2(a^t(E^*))$.

By a similar argument as above, we conclude that the family $\{\chi_l : l \in L\}$ is also an orthogonal basis for $L^2(E^*)$. This completes the proof of the theorem. \hfill \Box

**Question:** Must $0 \in E^c$ when the system is an orthogonal system?

We conclude this section with an example of a tight wavelet frame associated to a set $E$ and automorphisms $A$, where the sets $a^t(E)$, $a \in A$, are not necessarily disjoint. Let $W$ be a tight wavelet frame for $PW_Y$ with frame bound $A$, where $\hat{\psi} = 1_E$. Take $W_1 = W \cup \hat{W}$. Then $W_1$ is a tight frame for $PW_Y$ with the frame bound $A/2$ and the sets $a^t(E^*)$ are not disjoint.

**3 Existence of Multiplicative tiling sets in $\mathbb{F}^d_q$**

Let $q$ be an odd prime and $d \geq 1$. In this section we shall prove the existence of non-trivial (non-trivial here simply means that the set $E$ is neither the whole space nor one point) multiplicative tiling sets in the finite vector space $\mathbb{F}^d_q$ when $q \equiv 3 \pmod{4}$. 
When $d = 1$, non-trivial multiplicative tiling set on $\mathbb{F}_q$, $q$ odd, exists. Notice that $\mathbb{F}_q$ can be identified as $\{-\frac{(q-1)}{2}, \ldots, -1, 0, 1, \ldots, \frac{q-1}{2}\}$ and $q - 1$ is an even number. Define the automorphisms $\alpha_1$ and $\alpha_2$ to be $\alpha_1(x) = x$ and $\alpha_2(x) = -x$. Take $E = \{1, \ldots, (q - 1)/2\}$. Then we immediately see that

$$\mathbb{F}_q \setminus \{0\} = \alpha_1(E) \cup \alpha_2(E).$$

The union is disjoint and this yields naturally a non-trivial multiplicative tiling. If $d > 1$, it is not immediately clear that why non-trivial multiplicative tiling sets exist. We first prove this for $d = 2$.

Let us first consider the problem of multiplicative tiles on $\mathbb{R}^2$ and gain some motivations. Indeed, if we define $R_\theta$ be the rotation matrix of angle $\theta$ and $E_p$ be the sector without the origin with aperture $2\pi/p$, then $\mathbb{R}^2 \setminus \{0\}$ is naturally partitioned into

$$\mathbb{R}^2 \setminus \{0\} = \bigcup_{k=0}^{p-1} R_{2\pi k/p}(E_p).$$

We can make the set compact by considering annulus of sectors with inner and outer radii equal to 1 and 2, respectively, and taking also dilation matrices into account. However, sectors can never be a translational tile and hence we cannot produce wavelet sets on $\mathbb{R}^2$ using sectors. Nonetheless, we will see our construction of wavelet sets are produced by rotation on the finite field and it can also be a translational tile on $\mathbb{F}_q^2$.

For $r \in \mathbb{F}_q$, we consider the circle $S_r$ on $\mathbb{F}_q^2$ with radius $r$ as follows:

$$S_r := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{F}_q^2 : x^2 + y^2 = r \right\}$$

We recall that $a$ is a quadratic residue (mod $q$) if $x^2 \equiv a \pmod{q}$ has a solution in $\mathbb{F}_q$, otherwise, it is called a quadratic non-residue. On $\mathbb{F}_q$, there exist exactly $(q - 1)/2$ of non-zero quadratic residue and $(q - 1)/2$ are quadratic non-residue. By studying the tiling properties of the quadratic residues, we have the following lemma:

**Lemma 3.1.** Let $q \equiv 3 \pmod{4}$. Then

$$\#S_r = \begin{cases} 1, & \text{if } r = 0; \\ q + 1, & \text{if } r \neq 0 \end{cases}$$

**Proof.** By Theorem 1.2 in [17], every quadratic residue [non-residue] can be written as a sum of two quadratic residues [non-residues] in exactly $d_q - 1$ ways, and every quadratic residue [non-residue] can be written as a sum of two quadratic non-residues [residues] in exactly $d_q$ ways, where

$$d_q = \frac{q + 1}{4}, \quad \text{when} \quad q \equiv 3 \pmod{4}$$

Suppose that $r$ is a quadratic residue mod $q$. Then each sum of $r$ as quadratic residues $a_1 + a_2$ induces four points in $S_r$. Indeed, there are $a_1 = x^2$ has two solutions $x_1, x_2$ and $a_2 = x^2$ has two solutions $y_1, y_2$. Thus there are four distinct points $(x_1, y_1), (x_1, y_2), (x_2, y_1)$ and $(x_2, y_2)$. \\
Furthermore, as \( r = x^2 \) also has two solutions \( z_1, z_2 \). It induces 4 more solutions \((z_1, 0), (z_2, 0), (0, z_1)\) and \((0, z_2)\) on the axes. Hence, by the theorem, when \( q \equiv 3 \pmod{4} \) we have
\[
\#S_r = 4(d_q - 1) + 4 = 4d_q = q + 1. \tag{3.1}
\]
Suppose that \( r \) is a quadratic non-residue of \( q \). Then \( r \) can be written as exactly \( d_q \) ways as sum of quadratic residues. As each sum induces 4 pairs, \( \#S_r = 4d_q \) which is the same answer as in (3.1).

Finally, it follows directly that
\[
\#S_0 = q^2 - \sum_{r=1}^{q-1} \#S_r = q^2 - (q - 1)(4d_q) = 1,
\]
as required.

In the following lemma we prove that there exists an orthogonal matrix such that the multiplication of its exponents with a vector in a circle \( S_r \), \( r \neq 0 \), generates the circle.

**Lemma 3.2.** Suppose that \( q \equiv 3 \pmod{4} \). Then there exists an orthogonal matrix
\[
R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
\]
such that \( a^2 + b^2 = 1 \pmod{q} \), \( R^{q+1} = I \), and for any \( e \in S_r \), \( Re, R^2e, ..., R^{#S_r}e \) generates \( S_r \) for all \( r \neq 0 \).

**Proof.** Since \( q \equiv 3 \pmod{4} \), then \(-1\) is not a square in \( \mathbb{F}_q \). In particular, we define \( i \) to be the imaginary solution of \( i^2 = -1 \pmod{q} \). Thus, we can identify \( \begin{pmatrix} x \\ y \end{pmatrix} \) in \( \mathbb{F}_q \) as \( x + yi \). In this sequel, \( \mathbb{F}_q^2 \) is isomorphic to the finite field of \( q^2 \) elements, denoted as \( \mathbb{F}_{q^2} \). Note that the multiplicative group \( \mathbb{F}_{q^2}^\times \) is a cyclic group and the circle
\[
S_1 = \{ x + yi : x^2 + y^2 = 1 \}
\]
is a subgroup of \( \mathbb{F}_{q^2}^\times \). As \( \#S_1 = q + 1 \), there exists \( a + bi \in S_1 \) such that \( S_1 = \{ 1, a + bi, (a + bi)^2, ..., (a + bi)^q \} \). In other words, \( a + bi \) generates the group \( S_1 \).

Then it follows that for any \( e \in S_r \) we can write \( e = c + di \), and we have
\[
S_r = \{ c + di, (a + bi)(c + di), ..., (a + bi)^q(c + di) \}.
\]

Define the matrix \( R \) by \( \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \). Observe that \( Re = (ac - bd, bc + ad)^T \) in \( \mathbb{F}_q^2 \) is equal to \( (ac - bd) + (ad + bc)i = (a + bi)(c + di) \) in \( \mathbb{F}_{q^2} \). Then we have
\[
S_r = \{ e, Re, ..., R^q e \}.
\]
This completes the proof of the lemma.
Our next result proves the existence of non-trivial multiplicative tiling sets $\mathbb{F}_q^d$ for $d > 1$.

**Theorem 3.3.** There exists multiplicative tiling set $E$ in $\mathbb{F}_q^d$ for $q \equiv 3 \pmod{4}$.

**Proof.** We first consider $d = 2$. Take the automorphisms to be $\{I, R, R^2, ... , R^q\}$ where $R$ is defined in Lemma 3.2. Define the set $E$ by taking one point from each $S_r$, for $r \neq 0$. Then Lemma 3.2 shows that $\mathbb{F}_q^2 \setminus \{0\} = E \cup R(E) \cup ... \cup R^q(E)$ and the unions are disjoint since each $R^j(E)$ intersects $S_r$ exactly once.

If $d > 2$, we may take $\tilde{E} := E \times \mathbb{F}_q^{d-2}$ and $\tilde{R} := \begin{pmatrix} R & O \\ O & I \end{pmatrix}$, where $I$ is $(d-1) \times (d-2)$ identity matrix. Then $\tilde{E}$ is a multiplicative tiling set associated to the automorphisms $\{\tilde{R}^j : 1 \leq j \leq q+1\}$. $\square$

We will call the multiplicative tiling in Proposition 3.3 rotational tiling. Therefore, any rotational tiling is a set of $q-1$ elements which has only one vector in the intersection with any circle of non-zero radius. In the sequel, we will consider the wavelet sets originated by the rotational tilings.

### 4 Construction of tight wavelet frame sets

As we will see in this section, the construction of tight wavelet frame sets on $\mathbb{F}_q^2$ requires us to find a set $E$ such that $0 \in E$, $E^*$ is a multiplicative tiling and $E$ tiles $\mathbb{F}_q^2$ by translations. Here we consider rotational tilings. We recall the following characterization of translational tiles on $\mathbb{F}_q^2$.

**Theorem 4.1.** [14] Let $E$ be a set that tiles $\mathbb{F}_q^2$ by translation. Then $\sharp E = 1, q$ or $q^2$ and $E$ is a graph if $\sharp E = q$, i.e.

$$E = \{xe_1 + f(x)e_2 : x \in \mathbb{F}_q\}$$

for some basis $e_1, e_2$ in $\mathbb{F}_q^2$ and function $f : \mathbb{F}_q \to \mathbb{F}_q$.

Note that, by our construction in Proposition 3.3, a rotational tiling set has $q-1$ elements. Therefore, by the classification of translational tiles in Theorem 4.1, to construct tight wavelet frame sets, we must consider graph of functions defined on $\mathbb{F}_q$ that are simultaneously rotational tiling sets. We will provide a systematic way to construct such sets as we prove Theorem 4.4.

First we need some key lemmas.

**Lemma 4.2.** There exists $k \in \mathbb{F}_q$, $0 < k \leq \frac{q-1}{2}$ such that $1 + k^2$ is a quadratic non-residue.

**Proof.** Assume that such $k$ does not exist. This means that $1 + k^2$ are quadratic residues for all $0 < k \leq \frac{q-1}{2}$. We note that all $1 + k^2$ are in distinct residue classes (mod $q$), otherwise $1 + k^2 \equiv 1 + k'^2$ would imply $k = k'$ or $k = -k' \equiv q - k'$ (mod $q$). The latter is not possible since $0 < k, k' \leq \frac{q-1}{2}$. Since $\sharp QR = \frac{q-1}{2}$ and $1 + k^2$ are distinct for different $k$, then we must have $QR = \{1 + k^2 : 0 < k \leq \frac{q-1}{2}\}$. However, we also know that $1 \in QR$. Then for some $k$ we have $1 + k^2 \equiv 1$ (mod $q$). This implies that $k = 0$ or $k = q$, which contradicts the assumption. $\square$
As an example for $k$, if $q = 7, 19, \text{or } 23$, then $k$ equals to 1, 1, or 2, respectively. The existence of $k$ in Lemma 4.2 allows us to represent the quadratic non-residue numbers as follows.

**Lemma 4.3.** For the $k$ defined in Lemma 4.2,

$$QNR = \{(1 + k^2)x^2 : (q + 1)/2 \leq x \leq q - 1\},$$

and $QR = \{x^2 : 0 \leq x < (q - 1)/2\}$ and

**Proof.** There is nothing to prove about the statement about $QR$. For the statement about $QNR$, due to the multiplicative property of Legendre symbol we have

$$\left(\frac{(1 + k^2)x^2}{q}\right) = \left(\frac{1 + k^2}{q}\right)\left(\frac{x^2}{q}\right) = (-1)(1) = -1$$

(Recall that Legendre symbol is equivalent to the Euler's criterion and $(a/q) = a^{(q-1)/2} = 1$ if $a$ is a quadratic residue and $(a/q) = -1$ if $a$ is a quadratic non-residue). Hence, all $(1 + k^2)x^2$ are quadratic non-residues. Moreover, they are all distinct since if $(1 + k^2)x^2 = (1 + k^2)y^2$ and $x \neq y$, then $x = q - y$, which means $x, y$ can't be in $\{(q + 1)/2, ..., q - 1\}$ at the same time. Hence, the set $\{(1 + k^2)x^2 : (q + 1)/2 \leq x < q\}$ contains all $(q - 1)/2$ quadratic non-residues. This proves the second statement.

The main result of this section follows.

**Theorem 4.4.** Assume that $q$ is an odd prime congruent to 3 (mod 4). Then there exists a subset $E$ of $\mathbb{F}_q^d$ such that $E$ is a translational tiling, has a spectrum and $E^\ast$ is a multiplicative tiling in $\mathbb{F}_q^d$.

**Proof.** We first prove the case $d = 2$. By Theorem 4.1 and Theorem 3.3, it is sufficient to construct a set $E$ in $\mathbb{F}_q^2$ for $q \equiv 3$ (mod 4) which is a graph of the form

$$E = \{(x, f(x)) : x \in \mathbb{F}_q\}$$

for some function $f : \mathbb{F}_q \to \mathbb{F}_q$ and $\nu(E \cap S_r) = 1$ for all $r \in \mathbb{F}_q$. For this, we construct $E$ with the following properties:

- $\vec{0} = (0, 0) \in E$
- $(x, 0) \in E$ where $0 < x \leq \frac{q - 1}{2}$
- For $k$ in Lemma 4.2, let $(x, kx) \in E$ where \(\frac{q + 1}{2} \leq x < q\).

Set $E$ is clearly a graph of function $f : \mathbb{F}_q \to \mathbb{F}_q$ with $f(0) = 0$, $f(x) = 0$ when $0 < x \leq \frac{q - 1}{2}$, and $f(x) = kx$ when $\frac{q + 1}{2} \leq x < q$. So, it tiles $\mathbb{F}_q^2$ by translations with respect to some coordinate system and the tiling partner $A = \{(0, t) : t \in \mathbb{F}_q\}$, and by Theorem 4.1 it has a spectrum. We show that $E$ is a rotational tiling by showing that $\nu(E \cap S_r) = 1$, $r \neq 0$. Indeed, if $r$ is a quadratic residue, then there exists unique $x$ satisfying $0 < x \leq \frac{q - 1}{2}$ such that $x^2 + 0^2 = r$. If $r$ is a quadratic non-residue, then Lemma 4.3 implies the existence of the unique $x$ satisfying
\( \frac{q+1}{2} < x \leq q \) such that \( x^2 + (kx)^2 = (1 + k^2)x^2 = r \). Hence, \( \sharp(E \cap S_\tau) = 1 \). This completes the proof for \( d = 2 \).

When \( d > 2 \), we take \( E \) in \( \mathbb{F}_q^2 \) be the set we just constructed above and define \( \tilde{E} = E \times \mathbb{F}_q^{d-2} \). Then \( \tilde{E} \) is a multiplicative tiling set on \( \mathbb{F}_q^d \) by Theorem 3.3. Moreover, \( \tilde{E} \) is also a translational tile with respect to a coordinate system. For example if we choose \( e_2 = (0, 1, 0, \ldots, 0) \), the tiling set we obtain is \( \{re_2 : 0 \leq r \leq q-1\} \). To prove that \( \tilde{E} \) has a spectrum, let \( L \) be a spectrum for \( E \). A simple calculation shows that \( L \times \mathbb{F}_q^{d-2} \) is a spectrum for \( \tilde{E} \), and this completes the proof of the theorem.

**Remark.** Notice in Theorem 4.4 the tiling and spectral sets for \( d > 2 \) are given by \( \tilde{E} = E \times \mathbb{F}_q^{d-2} \). However, other natural candidate is \( \tilde{E} := E \times E \times \cdots \times E \times \mathbb{F}_q^k \) where \( 0 \leq k \leq d - 2 \). Clearly, the assertions of the theorem holds for \( \tilde{E} \).

As a corollary of Theorem 4.4 and Theorem 2.9 we have the following result.

**Corollary 4.5.** Let \( q \) be an odd prime congruent to 3 (mod 4) and \( d \geq 2 \). Let \( Y := (\mathbb{F}_q^d)^* \). Then there exists tight wavelet frame sets for \( PW_Y \).

Our result settles the existence of wavelet sets when \( q \) is an odd prime and \( q \equiv 3 \) (mod 4). The following proposition shows however that our construction method cannot work for \( q \equiv 1 \) (mod 4). The reason is that the circle of radius 0, \( S_0 \), contains more than one point.

**Proposition 4.6.** If \( q \equiv 1 \) (mod 4), there cannot be wavelet sets obtained by rotational tilings and translations in \( \mathbb{F}_q^2 \).

**Proof.** Take \( Y := \mathbb{F}_q^2 \setminus \{0\} \). If \( E \) is a non-trivial set in \( \mathbb{F}_q^2 \) which tiles \( Y \) by a set of rotations and translations, then \( \#E = q \) and \( E \) is a graph. Also, \( \sharp(E \setminus \{0\}) = q - 1 \) and it contains exactly one point from each circle. However, there are \( q - 1 \) circles of positive radius. Taking one point from each circle of positive radius would occupy all points in \( E \setminus \{0\} \). Taking union of all possible rotations, the rotational tiling covers only points of non-zero radius. Therefore the points in the circle with zero radius \( S_0 \) are not covered. Note that for \( q \equiv 1 \) (mod 4), \( \sharp S_0 = 2q - 1 \). Therefore there cannot be rotational tiling sets thus wavelet sets obtained by rotation and translation when \( q \equiv 1 \) (mod 4). \( \square \)

# 5 Open problems

We end this paper with two open questions.

**Question #1:** Does there exist tight frame wavelet sets when \( q \equiv 1 \) (mod 4)?

To handle the case \( q \equiv 3(mod4) \) we exploited the fact that circle of 0 radius has only one element. We would have to come to grips with such circles to extend our results to the case \( q \equiv 1(mod4) \).

**Question #2:** To what extent is it possible to generalize the results of this paper to the case \( \mathbb{F}_q^d \), for \( q = p^\alpha, \alpha > 1 \)?
By restricting ourselves to the case where \( q \) is prime, congruent to 1 mod 4, we limited the impact of arithmetic intricacies on the problem. The situation becomes quite fascinating when \( \alpha > 1 \) due to the existence of subfields. We shall address this issue in a sequel.

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