Sigma Functions and Lie Algebras of Schrödinger Operators

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Dedicated to the memory of the remarkable mathematician
Viktor Zelikovich Enolskii (1945–2019)

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Abstract. In a 2004 paper by V. M. Buchstaber and D. V. Leikin, published in “Functional Analysis and Its Applications,” for each $g > 0$, a system of $2g$ multidimensional Schrödinger equations in magnetic fields with quadratic potentials was defined. Such systems are equivalent to systems of heat equations in a nonholonomic frame. It was proved that such a system determines the sigma function of the universal hyperelliptic curve of genus $g$. A polynomial Lie algebra with $2g$ Schrödinger operators $Q_0, Q_2, \ldots, Q_{4g-2}$ as generators was introduced.

In this work, for each $g > 0$, we obtain explicit expressions for $Q_0, Q_2, Q_4$ and recurrent formulas for $Q_{2k}$ with $k > 2$ expressing these operators as elements of a polynomial Lie algebra in terms of the Lie brackets of the operators $Q_0, Q_2, Q_4$.

As an application, we obtain explicit expressions for the operators $Q_0, Q_2, \ldots, Q_{4g-2}$ for $g = 1, 2, 3, 4$.

Key words: Schrödinger operator, polynomial Lie algebra, differentiation of Abelian functions with respect to parameters.

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1. Introduction

The heat equation, which is equivalent to the Schrödinger equation, plays a fundamental role in the theory of Abelian functions. For example, Part I of the monograph [2] begins with characterizing the elliptic theta function as a periodic fundamental solution of the one-dimensional heat equation.

It follows directly from the construction of the function $\theta(z, \Omega)$, where $z \in \mathbb{C}^g$ and $\Omega$ is a symmetric $g \times g$ matrix, that this function satisfies the classical system of multidimensional heat equations. In this system, the time components are the $g(g+1)/2$ independent parameters of the manifold of symmetric $g \times g$ matrices $\Omega$. For this system, the vector fields along these components form a holonomic frame.

For $g > 3$, the dimension of the manifold of symmetric $g \times g$ matrices that are period matrices of a nonsingular Riemann surface of genus $g$ is less than $g(g+1)/2$. There arises the well-known Riemann–Schottky problem, which has been solved by Shiota (see [3]) based on a conjecture of S. P. Novikov in terms of the Kadomtsev–Petviashvili equation.

In the framework of F. Klein’s program, for each nonsingular Riemann surface $V$ of genus $g$, an entire function on $\mathbb{C}^g$, namely, the multidimensional sigma function, is constructed. Its logarithmic derivatives of order 2 and higher generate the whole field of meromorphic functions on the Jacobian of the algebraic curve $V$ (see [4]). In its first constructions, this multidimensional sigma function was introduced as a modification of the multidimensional theta function. The effectiveness of its implementation is achieved through the right choice of the curve model. The result is an entire function whose series expansion in the vector argument $z \in \mathbb{C}^g$ gives a series over the ring of polynomials in the parameters of the equation of the curve. Choosing a family of curves (hyperelliptic or more general $(n, s)$-curves), we can represent the theta functions of these curves as modified sigma functions of these curves. An effective description of such modifications gives a solution to
the Riemann–Schottky problem taking into account the specifics of the family of curves under consideration. But, in this case, we cannot use the classical system of heat equations to characterize the obtained theta functions, since the vector fields along the time components corresponding to variations of the periods may intersect the discriminants of the curves. This is also related to the Riemann–Schottky problem.

In [1] a system of heat equations that characterizes multidimensional sigma functions was constructed. In this system the vector fields along the time components corresponding to variations of the parameters of the equations of the curves form a nonholonomic frame. Schrödinger equations in magnetic fields arise.

This naturally suggests the problem of an effective description of this system of equations. This work is devoted to the solution of this problem in the case of a family of hyperelliptic curves.

2. The Problem in the Case of a Family of Hyperelliptic Curves

We consider hyperelliptic curves of genus \(g \in \mathbb{N}\) in the model

\[
\mathcal{V}_\lambda = \{(x, y) \in \mathbb{C}^2 : y^2 = x^{2g+1} + \lambda_1 x^{2g-1} + \lambda_6 x^{2g-2} + \cdots + \lambda_4 x + \lambda_{4g+2}\}.
\]

Each curve depends on the parameters \(\lambda = (\lambda_1, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}\). Let \(B \subset \mathbb{C}^{2g}\) be the subspace of parameters such that the curve \(\mathcal{V}_\lambda\) is nonsingular for \(\lambda \in B\). Then we have \(B = \mathbb{C}^{2g} \setminus \Sigma\), where \(\Sigma\) is the discriminant hypersurface of the universal curve.

For a meromorphic function \(f\) in \(\mathbb{C}^g\), a vector \(\omega \in \mathbb{C}^g\) is a period if \(f(z + \omega) = f(z)\) for all \(z \in \mathbb{C}^g\). If a meromorphic function \(f\) has \(2g\) independent periods in \(\mathbb{C}^g\), then \(f\) is called an Abelian function. Thus, an Abelian function is a meromorphic function on the complex torus \(T^g = \mathbb{C}^g/\Gamma\), where \(\Gamma\) is the lattice formed by the periods.

For each \(\lambda \in B\), the set of periods of holomorphic differentials on the curve \(\mathcal{V}_\lambda\) generates a lattice \(\Gamma_\lambda\) of rank \(2g\) in \(\mathbb{C}^g\). A hyperelliptic function of genus \(g\) (see [5]–[7]) is a meromorphic function on \(\mathbb{C}^g \times B\) such that, for each \(\lambda \in B\), its restriction to \(\mathbb{C}^g \times \lambda\) is an Abelian function (the torus \(T^g\) is the Jacobian \(\mathcal{J}_\lambda = \mathbb{C}^g/\Gamma_\lambda\) of the curve \(\mathcal{V}_\lambda\)). We denote by \(\mathcal{F}\) the field of hyperelliptic functions of genus \(g\). For the properties of this field, see [6] and [7].

We denote the coordinates in \(\mathbb{C}^g\) by \(z = (z_1, z_3, \ldots, z_{2g-1})\). The indices of the coordinates \(z = (z_1, z_3, \ldots, z_{2g-1}) \in \mathbb{C}^g\) and of the parameters \(\lambda = (\lambda_1, \lambda_6, \ldots, \lambda_{4g}, \lambda_{4g+2}) \in \mathbb{C}^{2g}\) determine their weights: \(\text{wt} z_k = -k\) and \(\text{wt} \lambda_k = k\). We denote by \(P\) the ring of polynomials in \(\lambda \in B \subset \mathbb{C}^{2g}\).

We consider polynomial Lie algebras [8] of vector fields tangent to the discriminant \(\Sigma\) in \(\mathbb{C}^{2g}\). Their generators \(\{L_0, L_2, L_4, \ldots, L_{4g-2}\}\) are the vector fields

\[
L_{2k} = \sum_{s=2}^{2g+1} v_{2k+2,2s-2}(\lambda) \frac{\partial}{\partial \lambda_{2k}}, \quad \text{where} \quad v_{2k+2,2s-2}(\lambda) \in P.
\]

At a point \(\lambda \in B\) these vector fields determine a \(2g\)-dimensional nonholonomic frame. The structure of such a Lie algebra as a \(P\)-module with generators 1, \(L_0, L_2, L_4, \ldots, L_{4g-2}\) is determined by the polynomial matrices \(V(\lambda) = (v_{2i,2j}(\lambda))\), where \(i, j = 1, \ldots, 2g\), and \(C(\lambda) = \{c^{2k}_{2i,2j}(\lambda)\}\), where \(i, j, k = 0, \ldots, 2g - 1\), such that

\[
[L_{2i}, L_{2j}] = \sum_{k=0}^{2g-1} c^{2k}_{2i,2j}(\lambda)L_{2k}, \quad [L_{2i}, \lambda_{2q}] = v_{2i+2, 2q-2}(\lambda), \quad [\lambda_{2q}, \lambda_{2r}] = 0.
\]

Here \(\lambda_q\) is the operator of multiplication by the function \(\lambda_q\) in \(P\).

Explicit expressions for the matrix \(V(\lambda)\) can be found in [9; Sec. 4.1] (see also [8] and [10; Lemma 3.1]). For convenience, we assume that \(\lambda_s = 0\) for \(s \notin \{0, 4, 6, \ldots, 4g, 4g + 2\}\) and \(\lambda_0 = 1\).
For \( k, m \in \{1, \ldots, 2g\} \), \( k \leq m \), we set
\[
v_{2k,2m}(\lambda) = \sum_{s=0}^{k-1} 2(k + m - 2s)\lambda_{2s}\lambda_{2(k+m-s)} - \frac{2k(2g - m + 1)}{2g + 1}\lambda_{2k}\lambda_{2m},
\]
and for \( k > m \), we set \( v_{2k,2m}(\lambda) = v_{2m,2k}(\lambda) \).

The vector field \( L_0 \) is the Euler vector field; namely, since \( \text{wt} \lambda_{2k} = 2k \), we have
\[
[L_0, \lambda_{2k}] = 2k\lambda_{2k}, \quad [L_0, L_{2k}] = 2kL_{2k}.
\]
This determines the weights of the vector fields \( L_k \) namely, \( \text{wt} L_{2k} = 2k \). The structure of the Lie algebra described above gives in this case a graded polynomial Lie algebra \([8]\), which we denote by \( L \).

The well-known Lie–Witt algebra \( W_{\geq} \) over the field \( \mathbb{C} \) of complex numbers is generated by the operators \( l_{2i} \), where \( i = 0, 1, 2, \ldots \), with the commutation relations
\[
[l_{2i}, l_{2j}] = 2(j - i)l_{2(i+j)}.
\]
With respect to the bracket \([\cdot, \cdot]\) the Lie–Witt algebra \( W_{\geq} \) is generated by the three operators \( l_0 \), \( l_2 \), and \( l_4 \). The graded polynomial Lie algebra \( L \) over \( P \) is a deformation of the Lie–Witt algebra \( W_{\geq} \). It is also generated by only three operators, \( L_0 \), \( L_2 \), and \( L_4 \). The following relation holds (see Lemma 4.3):
\[
[L_2, L_{2k}] = 2k - 1)2L_{2k+2} + \frac{4(2g - k)}{(2g + 1)}(\lambda_{2k+2}L_0 - \lambda_4L_{2k-2}).
\]

Now we introduce Schrödinger operators. We consider the space \( \mathbb{C}^{3g} \) with coordinates \((z, \lambda)\) and let \( C(z, \lambda) \) denote the ring of differentiable functions in \( z \) and \( \lambda \). We set
\[
Q_{2k} = L_{2k} - H_{2k}, \quad k = 0, 1, 2, \ldots, 2g - 1,
\]
where
\[
H_{2k} = \frac{1}{2} \sum \left( \alpha_{a,b}^{(k)}(\lambda)\partial_a\partial_b + 2\beta_{a,b}^{(k)}(\lambda)z_a\partial_b + \gamma_{a,b}^{(k)}(\lambda)z_a z_b \right) + \delta^{(k)}(\lambda);
\]
the summation is over odd \( a \) and \( b \) from 1 to \( 2g - 1 \). In \([1]\) a solution to the following problem is given.

**Problem 2.1.** Find sufficient conditions on \( \{\alpha^{(i)}(\lambda), \beta^{(i)}(\lambda), \gamma^{(i)}(\lambda), \delta^{(i)}(\lambda)\} \) for the operators (4) to give a representation of the Lie algebra (2) in the ring of operators on \( C(z, \lambda) \).

**Definition 2.2.** The system of equations
\[
Q_{2k}\varphi = 0
\]
for \( \varphi = \varphi(z, \lambda) \) is called the system of heat equations. The operators \( Q_{2k} \) are called the Schrödinger operators.

We use the theory of hyperelliptic Kleinian functions (see [6], [11], [12], [4], and [13] for elliptic functions). Take the coordinates \((z, \lambda)\) in \( \mathbb{C}^g \times \mathbb{B} \subset \mathbb{C}^{3g} \). Let \( \sigma(z, \lambda) \) be the hyperelliptic sigma function (or the elliptic sigma function in the case of genus \( g = 1 \)). We set \( \partial_k = \frac{\partial}{\partial z_k} \). Following [5], [7], and [14], we use the notation
\[
\zeta_k = \partial_k \ln \sigma(z, \lambda), \quad \varphi_{k_1, \ldots, k_n} = -\partial_{k_1} \cdots \partial_{k_n} \ln \sigma(z, \lambda),
\]
where \( n \geq 2 \) and \( k_s \in \{1, 3, \ldots, 2g - 1\} \). The functions \( \varphi_{k_1, \ldots, k_n} \) give examples of hyperelliptic functions. The field \( \mathcal{F} \) is the field of fractions of the polynomial ring \( \mathcal{P} \) generated by the functions \( \varphi_{k_1, \ldots, k_n} \), where \( n \geq 2 \) and \( k_s \in \{1, 3, \ldots, 2g - 1\} \).
As shown in [1], the system of heat equations (6) for the operators $Q_{2k}$ that give a solution to Problem 1.1 determines the hyperelliptic sigma function $\sigma(z, \lambda)$, which allows us to construct the theory of hyperelliptic Kleinian functions starting from such operators.

The construction of the operators $Q_{2i}$ in [1] uses condition (1.3) of [1], namely, that the commutator $[Q_{2i}, Q_{2j}]$ is determined by a formula over $P$ with the same coefficients as the formula for $[L_{2i}, L_{2j}]$; that is, the polynomial algebra generated by the operators $Q_{2i}$ with $i = 0, 1, \ldots$ is yet another realization of a deformation of the Lie–Witt algebra $W_3$. Therefore, for an effective description of the polynomial Lie algebra $\mathcal{L}_q$, one needs to obtain explicit formulas for $Q_0$, $Q_2$, and $Q_4$. These formulas are the main result of this work. As an application, we give an explicit form of differential operators in the case of the universal hyperelliptic curve of genus 4.

3. Generating Functions for the Schrödinger Operators

In this section, in the case of the model (1), we present an explicit form of the solution to Problem 2.1 given in [1; Theorem 2.6]. We will change the original notation so as to make it consistent with the formulas introduced in this work.

We denote by $q(a(x), b(x))$ the quotient and by $r(a(x), b(x))$ the remainder of the Euclidean division of a polynomial $a(x)$ by a polynomial $b(x)$. We set (see (1))

$$f(x) = x^{2g+1} + \sum_{k=0}^{2g-1} \lambda_{2(2g+1-k)} x^k.$$ 

We define generating functions for the operators $L_{2k}$ and $H_{2k}$ by

$$L(x) = x^{2g-1} \sum_{k=0}^{2g-1} x^{-k} L_{2k}, \quad H(x) = x^{2g-1} \sum_{k=0}^{2g-1} x^{-k} H_{2k}. \quad (8)$$

For $R_i(x) = x^{g-i+1} \partial_x q(f(x), x^{2g-2i+2})$, we set

$$h(x) = \sum_{i=1}^{g} x^{g-i} \partial_{2i-1} + R_i(x) z_{2i-1},$$

$$t(x) = \frac{1}{2} \sum_{i=1}^{g} (g - i + 1) z_{2i-1} q(R_i(x), x^{g-i+2})$$

$$+ \sum_{i=1}^{g} \sum_{j=i+1}^{g} (g - j + 1) z_{2j-1} q(x^{g-i} \partial_{2i-1} + R_i(x) z_{2i-1}, x^{g-j+2}).$$

Then we have the relation (see [1; Theorem 2.6])

$$H(x) = r \left( -\frac{1}{4} f''(x) + 2f(x)t(x) + \frac{1}{2} h(x) \circ h(x), f'(x) \right), \quad (9)$$

where $\circ$ denotes the composition of operators.

The function $Q(x) = L(x) - H(x)$ is generating for the Schrödinger operators.

**Lemma 3.1.** For the Schrödinger operators, the coefficients in (5) are

$$\alpha_{a,b}^{(k)}(\lambda) = 1 \quad \text{for} \quad a + b = 2k, \quad a, b \in 2\mathbb{N} + 1,$$

$$\alpha_{a,b}^{(k)}(\lambda) = 0 \quad \text{for} \quad a + b \neq 2k, \quad a, b \in 2\mathbb{N} + 1,$$

$$\delta^{(k)}(\lambda) = \left( -\frac{1}{4}(2g - k + 1)(2g - k) + \frac{1}{2} \left( g + \left[ \frac{k + 1}{2} \right] - k \right) \left( g - \left[ \frac{k + 1}{2} \right] \right) \right) \lambda_{2k}.$$
Proof. The coefficient of $\partial_\alpha \partial_\beta$ in $H(x)$ comes from the summand $\frac{1}{2}h(x) \circ h(x)$, which expands as $\sum_{i=1}^{g} \sum_{j=1}^{g} x^{2g-i-j} \partial_{2i-1} \partial_{2j-1}$. Thus, from (8) we obtain the required expressions for $\alpha_{a,b}^{(k)}(\lambda)$.

The expression for $\delta^{(k)}(\lambda)$ is obtained from the summands $\frac{1}{4}f''(x)$ and $\frac{1}{2}h(x) \circ h(x)$. The first gives $-\frac{1}{4}(2g - k + 1)(2g - k)\lambda_{2k}$ and the second gives $\frac{1}{2}(g + \lfloor \frac{k+1}{2} \rfloor - k)(g - \lfloor \frac{k+1}{2} \rfloor)\lambda_{2k}$, which leads to the required result.

Remark 3.2. We will show later on that, for Schrödinger operators, in (5)

\[ \beta_{a,b}^{(k)}(\lambda) \text{ is a linear function in } \lambda, \]
\[ \gamma_{a,b}^{(k)}(\lambda) \text{ is a quadratic function in } \lambda; \]

see Lemma 4.2.

4. An Explicit Form of the Schrödinger Operators

Recall that in our notation $\lambda_s = 0$ for all $s \notin \{0, 4, 6, \ldots, 4g, 4g + 2\}$ and $\lambda_0 = 1$.

Theorem 4.1. The following explicit expressions hold:

\[ H_0 = \sum_{s=1}^{g} (2s - 1)z_{2s-1} \partial_{2s-1} - \frac{g(g + 1)}{2}, \]
\[ H_2 = \frac{1}{2} \partial_1^2 + \sum_{s=1}^{g} (2s - 1)z_{2s-1} \partial_{2s+1} - \frac{4}{2g + 1} \lambda_4 \sum_{s=1}^{g-1} (g - s)z_{2s+1} \partial_{2s-1} \]
\[ + \sum_{s=1}^{g} \left( \frac{2s - 1}{2} \lambda_4s - \frac{2(g - s + 1)}{2g + 1} \lambda_4 \lambda_{4s-4} \right) z_{2s-1}^2, \]
\[ H_4 = \partial_1 \partial_3 + \sum_{s=1}^{g-2} (2s - 1)z_{2s-1} \partial_{2s+3} + \lambda_4 \sum_{s=1}^{g-1} (2s - 1)z_{2s+1} \partial_{2s+1} \]
\[ - \frac{6}{2g + 1} \lambda_6 \sum_{s=1}^{g-1} (g - s)z_{2s+1} \partial_{2s-1} \]
\[ + \sum_{s=1}^{g} \left( (2s - 1) \lambda_{4s+2} - \frac{3(g - s + 1)}{2g + 1} \lambda_6 \lambda_{4s-4} \right) z_{2s-1}^2 \]
\[ + \sum_{s=1}^{g-1} (2s - 1) \lambda_{4s+4} z_{2s-1} z_{2s+1} - \frac{g(g - 1)}{2} \lambda_4. \]

Proof. The explicit expressions for $\alpha_{a,b}^{(k)}(\lambda)$ and $\delta^{(k)}(\lambda)$ were obtained in Lemma 3.1. We see that, for $k = 0, 1, 2$, they coincide with the ones given in the theorem. Recall that $\lambda_0 = 1$ and $\lambda_2 = 0$.

In (9) the coefficient of $z_{2j-1} \partial_{2i-1}$ is equal to

\[ r(2(g - j + 1)f(x)q(x^{g-i}, x^{g-j+2}) + x^{g-i}R_j(x), f'(x)). \]

To obtain expressions for $H_0$, $H_2$, and $H_4$, we need the coefficients of $x^{2g-1}$, $x^{2g-2}$, and $x^{2g-3}$, respectively, in this polynomial.

For $j < i + 1$, the polynomial is equal to

\[ r(x^{g-i}R_j(x), f'(x)) = x^{g-i}R_j(x), \]

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because the degree of the polynomial $R_1(x)$ in $x$ is $g + j - 1$ and the degree of $f'(x)$ is $2g$. Therefore, we obtain the coefficient $2j - 1$ if $j = i$ for the expression for $H_0$, if $j = i - 1$ for $H_2$, and if $j = i - 2$ for $H_4$, as well as the coefficient $(2j - 3)\lambda_i$ if $j = i \geq 2$ for $H_4$, all the other coefficients being zero.

For $j = i + 1$, the polynomial takes the form

$$r(x^{g-1}R_j(x), f'(x)) = r(x^{2g-2i}q(f(x), x^{2g-2i}), f'(x))$$

$$= r((2i + 1)x^{2g}, f'(x)) + \sum_{k=2g-2i+1}^{2g-1} (k - 2g + 2i)\lambda_{2g+1-k}x^{k-1}$$

$$= -\frac{2i + 1}{2g+1} \sum_{k=0}^{2g-1} k\lambda_{2g+1-k}x^{k-1} + \sum_{k=2g-2i+1}^{2g-1} (k - 2g + 2i)\lambda_{2g+1-k}x^{k-1}.$$  

Therefore, the coefficient for $H_0$ is zero, the coefficient for $H_2$ is $-\frac{1}{2g+1}(g - i)\lambda_1$, and the coefficient for $H_4$ is $-\frac{1}{2g+1}(g - i)\lambda_6$.

For $j \geq i + 2$, the polynomial is equal to

$$r(2(g - j + 1)x^{j-i-2}f(x) + x^{2g-i-j+1}\partial_x q(f(x), x^{2g-2j+2}), f'(x))$$

$$= r(f'(x)x^{j-i+1} + \sum_{k=0}^{2g-2j+2} (2g - 2j + 2 - k)\lambda_{2g+1-k}x^{k+j-i-2}, f'(x))$$

$$= \sum_{k=0}^{2g-2j+2} (2g - 2j + 2 - k)\lambda_{2g+1-k}x^{k+j-i-2}.$$  

As $j \geq i + 2 \geq 3$, the coefficients for $H_0$, $H_2$, and $H_4$ are zero.

In (9) the coefficient of $z_{2i-1}^2$ is equal to

$$r\left((g - i + 1)f(x)q(R_i(x), x^{g-i+2}) + \frac{1}{2} R_i(x)^2, f'(x)\right).$$

(10)

For $i = 1$, we note that $R_1(x) = x^g$ and the expression (10) takes the form

$$\frac{1}{2} r(x^{2g}, f'(x)) = -\frac{1}{2(2g+1)} \sum_{k=1}^{2g-1} k\lambda_{2g+1-k}x^{k-1}.$$  

In this expression the coefficient of $x^{2g-1}$ is zero and the coefficients of $x^{2g-2}$ and $x^{2g-3}$ give the corresponding coefficients of $z_{2i}^2$ for $H_2$ and $H_4$.

For $i > 1$, we have

$$r\left((g - i + 1)f(x)q(R_i(x), x^{g-i+2}) + \frac{1}{2} R_i(x)^2, f'(x)\right)$$

$$= r\left(\frac{1}{2} R_i(x)x^{-g+i-1}\left(2(g - i + 1)x^{-1}f(x) + x^{g-i+1}R_i(x)\right)
- (g - i + 1)f(x)x^{-1}r(x^{-g+i-1}R_i(x), x), f'(x)\right)$$

$$= r\left(\frac{1}{2} R_i(x)x^{-g+i-1}\left(f'(x) + x^{-1}\sum_{k=0}^{2g-2i+1} (2g - 2i + 2 - k)\lambda_{2g+1-k}x^k\right)
- (g - i + 1)f(x)x^{-1}r(x^{-g+i-1}R_i(x), x), f'(x)\right)$$

$$= r\left(\frac{1}{2} R_i(x)x^{-g+i-1}\left(f'(x) + x^{-1}\sum_{k=0}^{2g-2i+1} (2g - 2i + 2 - k)\lambda_{2g+1-k}x^k\right)
- (g - i + 1)f(x)x^{-1}r(x^{-g+i-1}R_i(x), x), f'(x)\right).$$
\[ \frac{1}{2} R_i(x)x^{-g+i-1} \left( x^{-1} \sum_{k=0}^{2g-2i+1} (2g - 2i + 2 - k) \lambda_2(2g+1-k)x^k \right) \]

\[ - (g - i + 1)r(x^{-g+i-1}R_i(x), x)x^{-1}(f(x) - x^{2g+1}) \]

\[ - (g - i + 1)r(x^{-g+i-1}R_i(x), x)r(x^{2g}, f'(x)) \]

\[ = \frac{1}{2} R_i(x)x^{-g+i-1} \left( x^{-1} \sum_{k=0}^{2g-2i+1} (2g - 2i + 2 - k) \lambda_2(2g+1-k)x^k \right) \]

\[ - (g - i + 1)\lambda_{4(i-1)}x^{-1} \left( \sum_{k=0}^{2g-1} \lambda_2(2g+1-k)x^k \right) \]

\[ + \frac{(g - i + 1)}{2g + 1} \lambda_{4(i-1)} \left( \sum_{k=1}^{2g-1} k \lambda_2(2g+1-k)x^{k-1} \right). \]

This is a polynomial of degree 2g - 2. Therefore, the coefficient for \( H_0 \) is zero. The coefficients for \( H_2 \) and \( H_4 \) coincide with the corresponding coefficients in the statement of the theorem.

In (9) the coefficient of \( z_{2i-1}z_{2j-1} \), \( j > i \), is equal to

\[ r \left( R_i(x)x^{-g+j-2} \left( 2(g - j + 1)f(x) + x^{g-j+2}R_j(x) \right), f'(x) \right) \]

\[ = r \left( R_i(x)x^{-g+j-2} \left( x^j f'(x) + \sum_{k=0}^{2g-2j+1} (2g - 2j + 2 - k) \lambda_2(2g+1-k)x^k \right), f'(x) \right) \]

\[ = R_i(x)x^{-g+j-2} \left( \sum_{k=0}^{2g-2j+1} (2g - 2j + 2 - k) \lambda_2(2g+1-k)x^k \right). \]

The degree of this polynomial is equal to \( 2g + i - j - 2 < 2g - 3 \), and the equality holds for \( j = i + 1 \). Therefore, the coefficients for \( H_0 \) and \( H_2 \) are zero, and the coefficient for \( H_4 \) if \( j = i + 1 \) is \( (2i - 1)\lambda_{4i+4} \). This proves the theorem. \( \square \)

**Lemma 4.2.** For the Schrödinger operators, in (5)

\[ \beta_{a,b}^{(k)}(\lambda) \] is a linear function in \( \lambda \),

\[ \gamma_{a,b}^{(k)}(\lambda) \] is a quadratic function in \( \lambda \).

This lemma follows from the explicit expressions in the proof of Theorem 4.1.

**Lemma 4.3.** The following relation holds:

\[ [L_2, L_{2k}] = 2(k - 1)L_{2k+2} + \frac{4(2g - k)}{(2g + 1)}(\lambda_2 k + L_0 - \lambda_4 L_{2k-2}). \]

**Proof.** We have

\[ L_{2k} = 2 \sum_{m=2}^{k+1} \left( \sum_{s=0}^{m-2} (k + m - 2s)\lambda_2s\lambda_2(k+m-s) \right) \]

\[ - \frac{(m - 1)(2g - k)}{2g + 1} \lambda_2(k+1)\lambda_2(m-1) \frac{\partial}{\partial \lambda_2 m} \]

\[ + 2 \sum_{m=k+2}^{2g+1} \left( \sum_{s=0}^{k} (k + m - 2s)\lambda_2s\lambda_2(k+m-s) \right) \]

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whence we obtain (11) by a direct calculation of the coefficients.

\[ L_2 = \sum_{m=2}^{2g+1} \left( 2(m+1)\lambda_{2(m+1)} - \frac{4(2g-m+2)}{2g+1} \lambda_4 \lambda_{2(m-1)} \right) \frac{\partial}{\partial \lambda_{2m}}, \]

whence we obtain (11) by a direct calculation of the coefficients.

**Corollary 4.4.** For \( k = 3, 4, 5, \ldots, 2g - 1, \)

\[ Q_{2k} = \frac{1}{2(k-2)}[Q_2, Q_{2k-2}] - \frac{2(2g-k+1)}{(k-2)(2g+1)}(\lambda_{2k}Q_0 - \lambda_4 Q_{2k-4}). \quad (12) \]

This relation recursively defines the operators \( Q_{2k} \) for \( k = 3, 4, 5, \ldots, 2g - 1 \) and yields explicit expressions for these operators.

**Proof.** By construction (see Section 3) the operators \( Q_{2k} \) give a solution to Problem 2.1. Relation (12) coincides with (11) rewritten in terms of \( Q_{2k} \).

5. Explicit Formulas for the Schrödinger Operators in the Nonholonomic Frame

The following operators can be found in [1], [5], [15], and [16] for the genus \( g = 1 \), in [5], [15], and [16] for the genus \( g = 2 \), and in [10] and [17] for the genus \( g = 3 \). The formulas for the genus \( g = 4 \) are new. They follow from the formulas given in Section 4.

5.1. The Schrödinger operators for the genus \( g = 1 \). In this case, the explicit formulas for \( \{H_{2k}\} \) in (4) are

\[ H_0 = z_1 \partial_1 - 1, \quad H_2 = \frac{1}{2} \partial_1^2 - \frac{1}{6} \lambda_4 z_1^2. \]

5.2. The Schrödinger operators for the genus \( g = 2 \). In this case, the explicit formulas for \( \{H_{2k}\} \) in (4) are

\[ H_0 = z_1 \partial_1 + 3z_3 \partial_3 - 3, \]
\[ H_2 = \frac{1}{2} \partial_1^2 - \frac{1}{6} \lambda_4 z_3 \partial_1 + z_1 \partial_3 - \frac{3}{10} \lambda_4 z_1^2 + (\frac{1}{3} \lambda_8 - \frac{1}{3} \lambda_1^2) z_3^2, \]
\[ H_4 = \partial_1 \partial_3 - \frac{1}{3} \lambda_6 z_3 \partial_1 + \lambda_1 z_3 \partial_3 - \frac{1}{4} \lambda_6 z_1^2 + \lambda_8 z_1 z_3 + (3 \lambda_{10} - \frac{1}{4} \lambda_4 \lambda_6) z_3^2 - \lambda_4, \]
\[ H_6 = \frac{1}{2} \partial_3^2 - \frac{1}{6} \lambda_6 z_3 \partial_1 + \frac{1}{10} \lambda_8 z_1^2 + 2 \lambda_{10} z_1 z_3 - \frac{1}{10} \lambda_4 \lambda_6 z_3^2 - \frac{1}{2} \lambda_6. \]

5.3. The Schrödinger operators for the genus \( g = 3 \). In this case, the explicit formulas for \( \{H_{2k}\} \) in (4) are

\[ H_0 = z_1 \partial_1 + 3z_3 \partial_3 + 5z_5 \partial_5 - 6, \]
\[ H_2 = \frac{1}{2} \partial_1^2 - \frac{8}{7} \lambda_4 z_3 \partial_1 + (z_1 - \frac{1}{7} \lambda_4 z_5) \partial_3 + 3z_3 \partial_3 - \frac{5}{14} \lambda_4 z_1^2 + (\frac{2}{7} \lambda_8 - \frac{1}{7} \lambda_1^2) z_3^2 \]
\[ + \left( \frac{3}{7} \lambda_{12} - \frac{2}{7} \lambda_4 \lambda_8 \frac{z_3}{z_5} \right)^2, \]
\[ H_4 = \partial_1 \partial_3 - \frac{12}{7} \lambda_6 z_3 \partial_1 + (\lambda_4 z_3 - \frac{6}{7} \lambda_6 z_5) \partial_3 + (z_1 + 3 \lambda_4 z_5) \partial_5 - \frac{2}{7} \lambda_6 z_1^2 + \lambda_8 z_1 z_3 \]
\[ + (3 \lambda_{10} - \frac{1}{7} \lambda_4 \lambda_6) z_3^2 + 3 \lambda_{12} z_3 z_5 + (5 \lambda_{14} - \frac{3}{7} \lambda_6 \lambda_8) z_3^2 - 3 \lambda_4, \]
\[ H_6 = \frac{1}{2} \partial_3^2 + \partial_1 \partial_5 - \frac{9}{7} \lambda_8 z_3 \partial_1 - \frac{8}{7} \lambda_8 z_5 \partial_3 + (\lambda_4 z_3 + 2 \lambda_6 z_5) \partial_5 - \frac{4}{7} \lambda_8 z_1^2 + 2 \lambda_{10} z_1 z_3 \]
\[ + (\frac{2}{7} \lambda_{12} - \frac{6}{7} \lambda_4 \lambda_8) z_3^2 + \lambda_{12} z_1 z_5 + 6 \lambda_{14} z_3 z_5 + (\frac{2}{7} \lambda_4 \lambda_12 - \frac{2}{7} \lambda_6^2) z_5^2 - 2 \lambda_6, \]
\[ H_8 = \partial_3 \partial_5 - (\frac{6}{7} \lambda_{10} z_3 - \lambda_{12} z_5) \partial_1 - \frac{10}{7} \lambda_{10} z_5 \partial_3 + \lambda_8 z_5 \partial_5 - \frac{1}{7} \lambda_{10} z_1^2 + 3 \lambda_{12} z_1 z_3 \]
\[ + (6 \lambda_{14} - \frac{1}{7} \lambda_4 \lambda_{10}) z_3^2 + 2 \lambda_{14} z_1 z_5 + \lambda_4 \lambda_{12} z_3 z_5 \]
\[ + (3 \lambda_4 \lambda_{14} + \lambda_6 \lambda_{12} - \frac{2}{7} \lambda_8 \lambda_{10}) z_3^2 - \lambda_8, \]
\[ H_{10} = \frac{1}{2} \partial_3^2 - \left( \frac{2}{7} \lambda_{12} z_3 - 2 \lambda_{14} z_5 \right) \partial_1 - \frac{5}{7} \lambda_{12} z_5 \partial_3 - \frac{1}{7} \lambda_{12} z_1^2 + 4 \lambda_{14} z_1 z_3 \]
\[ - \frac{3}{14} \lambda_4 \lambda_{12} z_3^2 + 2 \lambda_4 \lambda_{14} z_3 z_5 + (2 \lambda_6 \lambda_{14} - \frac{5}{14} \lambda_8 \lambda_{12}) z_5^2 - \frac{1}{2} \lambda_{10}. \]
5.4. The Schrödinger operators for the genus $g = 4$. In this case, the explicit formulas for $\{H_{2k}\}$ in (4) are

$$H_0 = z_1\partial_1 + 3z_3\partial_3 + 5z_5\partial_5 + 7z_7\partial_7 - 10,$$
$$H_2 = \frac{1}{2}\partial^2 + z_1\partial_3 + 3z_3\partial_5 + 5z_5\partial_7 - \frac{4}{3}\lambda_1(3z_3\partial_1 + 2z_5\partial_3 + z_7\partial_5) - \frac{7}{18}\lambda_4 z_1^2$$
$$+ (\frac{4}{3}\lambda_8 - \frac{3}{2}\lambda_7)^2 z_1^2 + (\frac{7}{2}\lambda_12 - \frac{1}{3}\lambda_4\lambda_8) z_5^2 + (\frac{7}{2}\lambda_16 - \frac{2}{9}\lambda_4\lambda_12) z_7^2,$$
$$H_4 = \partial_1\partial_3 + z_1\partial_5 + 3z_3\partial_7 + \lambda_4(z_3\partial_3 + 3z_5\partial_5 + 5z_7\partial_7)$$
$$- \frac{2}{3}\lambda_6(3z_3\partial_1 + 2z_5\partial_3 + z_7\partial_5) - \frac{1}{3}\lambda_6 z_1^2 + 5\lambda_8 z_1 z_3 + (3\lambda_10 - \lambda_4\lambda_6) z_3^2$$
$$+ 3\lambda_12 z_5 z_5 + (5\lambda_14 - \frac{8}{3}\lambda_6\lambda_8) z_5^2 + 5\lambda_16 z_5 z_7 + (7\lambda_18 - \frac{1}{3}\lambda_6\lambda_12) z_7^2 - 6\lambda_4,$$
$$H_6 = \frac{1}{2}\partial^2 + \partial_3\partial_5 - \frac{5}{8}\lambda_8 z_3\partial_3 - \frac{10}{7}\lambda_8 z_5\partial_5 + \frac{5}{8}(\lambda_4 z_3 + 2\lambda_6 z_5 - \frac{8}{9}\lambda_8 z_7)\partial_5$$
$$+ (z_1 + 3\lambda_4 z_5 + 4\lambda_6 z_7)\partial_7 - \frac{4}{9}\lambda_8 z_5^2 + 2\lambda_10 z_1 z_3 + \lambda_12 z_1 z_5$$
$$- \frac{7}{9}(\frac{4}{3}\lambda_4 - \frac{4}{3}\lambda_12) z_5^2 + 6\lambda_14 z_3 z_5 + 3\lambda_16 z_3 z_7 + (\frac{2}{3}\lambda_4\lambda_12 - \frac{4}{7}\lambda_8^2 + \frac{8}{7}\lambda_16) z_5^2$$
$$+ 10\lambda_18 z_5 z_7 - (\frac{4}{7}\lambda_8\lambda_12 - \frac{5}{3}\lambda_4\lambda_16) z_7^2 - \frac{4}{9}\lambda_6,$$
$$H_8 = \partial_3\partial_5 + \partial_1\partial_7 - (\frac{4}{3}\lambda_10 z_1 z_3 - \lambda_12 z_5)\partial_1 - \frac{20}{7}\lambda_10 z_5\partial_3 + (\lambda_8 z_5 - \frac{10}{9}\lambda_10 z_7)\partial_5$$
$$+ (\lambda_4 z_3 + 2\lambda_6 z_5 + 3\lambda_8 z_7)\partial_7 - \frac{2}{3}\lambda_10 z_1^2 + 3\lambda_12 z_1 z_3 + 2\lambda_14 z_1 z_5 + \lambda_16 z_1 z_7$$
$$- \frac{4}{9}(\frac{4}{3}\lambda_4 - \frac{4}{3}\lambda_12) z_5^2 + (\lambda_4\lambda_12 + 9\lambda_16) z_3 z_5 + 6\lambda_18 z_3 z_7$$
$$- \frac{10}{7}(\frac{4}{7}\lambda_8\lambda_10 - \lambda_6\lambda_12 - 3\lambda_4\lambda_14 - 10\lambda_18) z_5^2 + 3\lambda_4\lambda_16 z_5 z_7$$
$$- \frac{4}{9}(\frac{4}{3}\lambda_10\lambda_12 - 2\lambda_6\lambda_16 - 5\lambda_4\lambda_18) z_7^2 - 3\lambda_8,$$
$$H_{10} = \frac{1}{2}\partial^2 + \partial_3\partial_7 - (\lambda_12 z_3 - 2\lambda_14 z_5 - \lambda_16 z_7)\partial_1 - \frac{5}{3}\lambda_12 z_5\partial_3 - \frac{4}{3}\lambda_12 z_7\partial_5$$
$$+ (\lambda_8 z_5 + 2\lambda_10 z_7)\partial_7 - \frac{1}{6}\lambda_12 z_1^2 + 4\lambda_14 z_1 z_3 + 3\lambda_16 z_1 z_5 + 2\lambda_18 z_1 z_7$$
$$- \frac{4}{9}(\frac{4}{3}\lambda_4\lambda_12 - 6\lambda_14) z_5^2 + (\lambda_4\lambda_12 + 12\lambda_18) z_3 z_5 + 4\lambda_16 z_3 z_7$$
$$- \frac{10}{7}(\frac{4}{7}\lambda_8\lambda_10 - \lambda_6\lambda_12 - 3\lambda_4\lambda_14 - 10\lambda_18) z_5^2 + 3\lambda_4\lambda_16 z_5 z_7$$
$$- (\frac{4}{9}\lambda_4^2 - \frac{4}{3}\lambda_6\lambda_16 - 4\lambda_6\lambda_18) z_7^2 - 2\lambda_10,$$
$$H_{12} = \partial_5\partial_7 - (\frac{2}{3}\lambda_14 z_3 - 3\lambda_16 z_5 - 2\lambda_18 z_7)\partial_1 - (\frac{10}{7}\lambda_14 z_5 - \lambda_16 z_7)\partial_3 - \frac{14}{7}\lambda_14 z_7\partial_5$$
$$+ \lambda_12 z_7\partial_7 - \frac{3}{4}\lambda_14^2 z_3^2 + 5\lambda_16 z_1 z_3 + 4\lambda_18 z_1 z_5 - (\frac{3}{4}\lambda_4\lambda_14 - 9\lambda_18) z_5^2$$
$$+ 3\lambda_16 z_3 z_5 + 2\lambda_4\lambda_18 z_3 z_7 - (\frac{4}{9}\lambda_8\lambda_14 - 3\lambda_6\lambda_16 - 6\lambda_4\lambda_18) z_5^2$$
$$+ (\lambda_8\lambda_16 + 4\lambda_6\lambda_18) z_5 z_7 - (\frac{4}{3}\lambda_12\lambda_14 - \lambda_10\lambda_16 - 3\lambda_8\lambda_18) z_7^2 - \lambda_12,$$
$$H_{14} = \frac{1}{2}\partial^2 - (\frac{4}{3}\lambda_16 z_3 - 4\lambda_18 z_5)\partial_1 - (\frac{7}{9}\lambda_16 z_5 - 2\lambda_18 z_7)\partial_3 - \frac{7}{9}\lambda_16 z_7\partial_5 - \frac{14}{9}\lambda_16 z_1^2$$
$$+ 6\lambda_18 z_3 z_5 - \frac{1}{6}\lambda_4\lambda_16 z_5^2 + 4\lambda_4\lambda_18 z_3 z_5 - (\frac{16}{13}\lambda_8\lambda_16 - 4\lambda_6\lambda_18) z_5^2$$
$$+ 2\lambda_8\lambda_18 z_5 z_7 - (\frac{7}{9}\lambda_12\lambda_16 - 2\lambda_10\lambda_18) z_7^2 - \frac{1}{4}\lambda_14.$$

6. Application: Differentiation Operators for the Genus $g = 4$

Recall that $\mathcal{F}$ is the field of hyperelliptic functions of genus $g$. In this section we consider the problem of constructing the Lie algebra of derivations of $\mathcal{F}$, i.e., of finding $3g$ independent differential operators $\mathcal{L}$ such that $\mathcal{L} \mathcal{F} \subset \mathcal{F}$. The setting of the problem, as well as a general approach to the solution, can be found in [15] and [16]. A survey of results is given in [7]. In [18], [5], and [14] an explicit solution to this problem was obtained for the cases $g = 1, 2, 3$. Here we give an explicit solution to this problem in the case of the genus $g = 4$.

We introduce a ring of functions $\mathcal{R}_\varphi$. Generators of this graded ring over $\mathbb{Q}[\lambda]$ are the functions $\psi_{k_1 \ldots k_n} = -\partial_{k_1} \cdots \partial_{k_n} \ln \varphi$, where $n \geq 2$, $k_1 \in \{1, 3, \ldots, 2g - 1\}$, $\psi_{k_1 \ldots k_n} = k_1 + \ldots + k_n$, and wt $\lambda_k = k$. We introduce the operators

$$\mathcal{L}_0 = L_0 - z_1\partial_1 - 3z_3\partial_3 - 5z_5\partial_5 - 7z_7\partial_7,$$
\[ L_2 = L_2 - \psi_1 \partial_1 + \frac{4}{3} \lambda_4 z_3 \partial_1 - (z_1 - \frac{8}{9} \lambda_4 z_3) \partial_3 - (3 z_3 - \frac{1}{9} \lambda_3 z_7) \partial_5 - 5 z_5 \partial_7, \]
\[ L_4 = L_4 - \psi_3 \partial_1 - \psi_2 \partial_3 - 2 \lambda_6 z_3 \partial_1 - (4 \lambda_3 - \frac{4}{3} \lambda_6 z_5) \partial_3 - (z_1 + 3 \lambda_4 z_5 - \frac{2}{3} \lambda_6 z_7) \partial_5 \\
- (3 z_3 + 5 \lambda_4 z_7) \partial_7, \]
\[ L_6 = L_6 - \psi_5 \partial_1 - \psi_3 \partial_3 - \psi_1 \partial_5 + \frac{5}{3} \lambda_8 z_3 \partial_1 + \frac{16}{9} \lambda_8 z_5 \partial_3 \\
- (\lambda_4 z_3 + 2 \lambda_6 z_5 - \frac{8}{9} \lambda_8 z_7) \partial_5 - (z_1 + 3 \lambda_4 z_5 + 4 \lambda_6 z_7) \partial_7, \]
\[ L_8 = L_8 - \psi_7 \partial_3 - \psi_5 \partial_3 - \psi_3 \partial_5 - \psi_1 \partial_7 + (\frac{4}{3} \lambda_{10} z_3 - \lambda_{12} z_5) \partial_1 + \frac{20}{9} \lambda_{10} z_5 \partial_3 \\
- (\lambda_8 z_5 - \frac{10}{9} \lambda_{10} z_7) \partial_5 - (\lambda_4 z_3 + 2 \lambda_6 z_5 + 3 \lambda_8 z_7) \partial_7, \]
\[ L_{10} = L_{10} - \psi_7 \partial_3 - \psi_5 \partial_3 - \psi_3 \partial_5 + (\lambda_{12} z_3 - 2 \lambda_{14} z_5 - \lambda_{16} z_7) \partial_1 \\
+ \frac{5}{3} \lambda_{12} z_5 \partial_3 + \frac{4}{3} \lambda_{12} z_7 \partial_5 - (\lambda_8 z_5 + 2 \lambda_{10} z_7) \partial_7, \]
\[ L_{12} = L_{12} - \psi_7 \partial_3 - \psi_5 \partial_3 + (\frac{7}{9} \lambda_{14} z_3 - 3 \lambda_{16} z_5 - 2 \lambda_{18} z_7) \partial_1 \\
+ (\frac{10}{9} \lambda_{14} z_5 - \lambda_{16} z_7) \partial_3 + \frac{14}{9} \lambda_{14} z_7 \partial_5 - \lambda_{12} z_7 \partial_7, \]
\[ L_{14} = L_{14} - \psi_7 \partial_3 + (\frac{5}{9} \lambda_{16} z_3 - 4 \lambda_{18} z_5) \partial_1 + (\frac{5}{9} \lambda_{16} z_5 - 2 \lambda_{18} z_7) \partial_3 + \frac{7}{9} \lambda_{16} z_7 \partial_5. \]

Let us denote the Lie algebra with these generators by \( L \).

**Theorem 6.1.** If the function \( \varphi \) satisfies the system of heat equations in the nonholonomic frame for the genus 4, then the algebra \( L \) is the algebra of derivations of the ring \( R_\varphi \).

This theorem follows the proof of Theorem 5.9 in [17].

**Corollary 6.2.** For \( g = 4 \) and \( \varphi = \sigma \), the algebra \( L \) is the algebra of derivations of \( F \).

We set
\[ w_{0,1} = \psi_1, \quad w_{0,3} = 3 \psi_3, \quad w_{2k,5} = 5 \psi_5, \quad w_{0,7} = 7 \psi_7, \]
\[ w_{2,1} = \frac{1}{2} \psi_{111} + \psi_3 - \frac{7}{9} \lambda_4 z_1, \]
\[ w_{2,3} = \frac{1}{2} \psi_{113} - \frac{1}{3} \lambda_4 \psi_1 + 3 \psi_5 + (3 \lambda_8 - \frac{1}{3} \lambda_4^2) z_3, \]
\[ w_{2,5} = \frac{1}{2} \psi_{115} - \frac{8}{9} \lambda_4 \psi_3 + 5 \psi_7 + (5 \lambda_{12} - \frac{8}{9} \lambda_4 \lambda_8) z_5, \]
\[ w_{2,7} = \frac{1}{2} \psi_{117} - \frac{8}{9} \lambda_4 \psi_5 + (7 \lambda_{16} - \frac{2}{9} \lambda_4 \lambda_12) z_7, \]
\[ w_{4,1} = \psi_{113} + \psi_5 - \frac{2}{9} \lambda_6 z_1 + \lambda_8 z_3, \]
\[ w_{4,3} = \psi_{133} - 2 \lambda_6 \psi_1 + \lambda_4 \psi_3 + 3 \psi_7 + \lambda_8 z_1 + (6 \lambda_{10} - 2 \lambda_4 \lambda_6) z_3 + 3 \lambda_{12} z_5, \]
\[ w_{4,5} = \psi_{135} - \frac{4}{9} \lambda_6 \psi_3 + 3 \lambda_4 \psi_5 + 3 \lambda_{12} z_3 + (10 \lambda_{14} - \frac{4}{9} \lambda_6 \lambda_8) z_5 + 5 \lambda_{16} z_7, \]
\[ w_{4,7} = \psi_{137} - \frac{8}{9} \lambda_6 \psi_5 + 5 \lambda_4 \psi_7 + 5 \lambda_{16} z_5 + (14 \lambda_{18} - \frac{8}{9} \lambda_6 \lambda_12) z_7, \]
\[ w_{6,1} = \frac{1}{2} \psi_{133} + \psi_{115} + \psi_7 - \frac{5}{9} \lambda_8 z_1 + 2 \lambda_{10} z_3 + \lambda_{12} z_5, \]
\[ w_{6,3} = \frac{1}{2} \psi_{333} + \psi_{135} - \frac{5}{9} \lambda_8 \psi_1 + \lambda_4 \psi_5 + 2 \lambda_{10} z_1 - (\frac{5}{9} \lambda_4 \lambda_8 - 9 \lambda_{12}) z_3 \\
+ 6 \lambda_{14} z_5 + 3 \lambda_{16} z_7, \]
\[ w_{6,5} = \frac{1}{2} \psi_{335} + \psi_{155} - \frac{16}{9} \lambda_8 \psi_3 + 2 \lambda_6 \psi_5 + 3 \lambda_4 \psi_7 + \lambda_{12} z_1 + 6 \lambda_{14} z_3 \\
+ (3 \lambda_4 \lambda_{12} - \frac{16}{9} \lambda_8^2 + 15 \lambda_{16}) z_5 + 10 \lambda_{18} z_7, \]
\[ w_{6,7} = \frac{1}{2} \psi_{337} + \psi_{157} - \frac{8}{9} \lambda_8 \psi_5 + 4 \lambda_6 \psi_7 + 3 \lambda_{16} z_3 + 10 \lambda_{18} z_5 - (\frac{8}{9} \lambda_8 \lambda_{12} - 5 \lambda_4 \lambda_{16}) z_7, \]
\[ w_{8,1} = \psi_{135} + \psi_{117} - \frac{1}{3} \lambda_10 z_1 + 3 \lambda_{12} z_3 + 2 \lambda_{14} z_5 + \lambda_{16} z_7, \]
\[ w_{8,3} = \psi_{335} + \psi_{137} - \frac{1}{3} \lambda_10 \psi_1 + \lambda_4 \psi_7 \\
+ 3 \lambda_{12} z_1 - (\frac{8}{9} \lambda_10 - 12 \lambda_{14}) z_3 + (\lambda_4 \lambda_{12} + 9 \lambda_{16}) z_5 + 6 \lambda_{18} z_7, \]
\[ w_{8,5} = \psi_{355} + \psi_{157} + \lambda_{12} \psi_1 - \frac{20}{9} \lambda_10 \psi_3 + \lambda_8 \psi_5 + 2 \lambda_6 \psi_7 + 2 \lambda_{14} z_1 \\
+ (\lambda_4 \lambda_{12} + 9 \lambda_{16}) z_3 - (\frac{20}{9} \lambda_8 \lambda_{10} - 2 \lambda_6 \lambda_{12} - 6 \lambda_4 \lambda_{14} - 20 \lambda_{18}) z_5 + 6 \lambda_4 \lambda_{16} z_7, \]
\[ w_{8,7} = \psi_{375} + \psi_{177} - \frac{10}{9} \lambda_10 \psi_5 + 3 \lambda_8 \psi_7 \\
+ \lambda_{16} z_1 + 6 \lambda_{18} z_3 + 3 \lambda_4 \lambda_{16} z_5 - (\frac{10}{9} \lambda_10 \lambda_{12} - 4 \lambda_6 \lambda_{16} - 10 \lambda_4 \lambda_{18}) z_7. \]
For the genus $g = 4$, we obtain a theorem similar to Theorems 4.1, 4.2, and 4.3 in [17].

**Theorem 6.3.** For $g = 4$, a solution $\varphi$ of the system of heat equations (6) gives a solution $(\varphi_1, \varphi_3, \varphi_5, \varphi_7) = (\partial_1 \ln \varphi, \partial_3 \ln \varphi, \partial_5 \ln \varphi, \partial_7 \ln \varphi)$ of the system of nonlinear differential equations, which we call an analogue of the Burgers equation for $g = 4$:

\[
L_k(\varphi_1, \varphi_3, \varphi_5, \varphi_7) = (w_{2k,1,1}, w_{2k,3,1}, w_{2k,5,1}, w_{2k,7,1}), \quad k = 0, 1, 2, 3, 4, 5, 6, 7.
\]

(13)

This theorem is proved by direct computation.

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