Computable axiomatizability of elementary classes

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The goal of this paper is to generalise Alex Rennet’s proof of the non-axiomatizability of the class of pseudo-o-minimal structures. Rennet showed that if \( L \) is an expansion of the language of ordered fields and \( K \) is the class of pseudo-o-minimal \( L \)-structures (\( L \)-structures elementarily equivalent to an ultraproduct of o-minimal structures) then \( K \) is not computably axiomatizable. We give a general version of this theorem, and apply it to several classes of structures.

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1 Introduction

Given a class \( K \) of \( L \)-structures, we write \( \text{Th}(K) \) for the first order theory of \( K \); that is, the set of all \( L \)-sentences that are true in every structure of \( K \). Recall that a class \( K \) is called elementary when \( M \models \text{Th}(K) \) if and only if \( M \) is an element of \( K \), and that this holds if and only if \( K \) is closed under ultraproducts and ultraroots [7, Corollary 8.5.13]. We say that an elementary class \( K \) is computably axiomatizable if there is a computable axiomatization of \( \text{Th}(K) \). With this terminology, Rennet proved that the class of pseudo-o-minimal fields (fields which are elementarily equivalent to an ultraproduct of o-minimal structures) is not computably axiomatizable [13].

Rennet’s paper was motivated by a number of results, among them Ax’s proof [1] that the theory of finite fields is decidable, and hence that the class of pseudo-finite fields is computably axiomatizable. As with the class of finite fields in the language of rings, the class of o-minimal structures in a language with an ordering and an extra unary predicate is not elementary. For each \( n \in \mathbb{N} \), let \( M_n \) be a copy of the real numbers in this language, where the ordering is interpreted by the usual ordering and the unary predicate is interpreted as \( \{0, 1, \ldots, n\} \). It is easy to see that each \( M_n \) is o-minimal, but that the ultraproduct has a copy of the natural numbers as a definable set; this is clearly not a finite union of points and intervals, and hence the ultraproduct is not o-minimal. Thus, the class of o-minimal structures is not closed under ultraproducts, and so is not elementary.

Multiple proposals were made for possible axiomatizations of the class of pseudo-o-minimal structures (cf., e.g., [4, 14]). However, Rennet showed that in the case where the language expands that of ordered rings, there is no computable axiomatization for the theory of o-minimality, and hence the class of pseudo-o-minimal structures is not computably axiomatizable.

In [5], Haskell and Macpherson developed the notion of C-minimality, a generalization of o-minimality obtained by replacing the binary ordering by a ternary relation. Haskell and Macpherson looked at another generalization of o-minimality in [6], P-minimality, which is defined so that P-minimal fields are \( p \)-adically closed, just as o-minimal fields are real closed. Given the similarities between these settings and o-minimality, they are both contexts in which it is natural to ask whether Rennet’s theorem applies.

In this paper, we adapt Rennet’s proof to give a more general theorem, which can then be applied to other classes, including those of C-minimal and P-minimal structures. § 2 contains the preliminaries and proof of the generalized theorem, while § 3 contains several examples.

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2 Preliminaries and the generalized theorem

Recall the notion of a provability relation which plays a fundamental role in the proof of Gödel’s Second Incompleteness Theorem (cf., e.g., [2]): if $\Gamma$ is a computable list of sentences in the language of arithmetic then there exists a binary relation $\text{prov}(s, d)$ such that in the standard model of Peano Arithmetic, $\text{prov}(s, d)$ if and only if $d$ is the code number of a sentence and $s$ is the code number for a proof of that sentence from $\Gamma$.

**Theorem 2.1** Fix any computable language $L$ containing a unary predicate $N$ and a distinguished $L$-sentence $\sigma$. Suppose $\mathbb{K}$ is a class of $L$-structures such that for all $A \in \mathbb{K}$, $A \models \sigma$ if and only if $N_A$ is finite. Let $\Lambda$ be any computable subset of $\text{Th}(\mathbb{K})$.

Fix distinguished $L$-formulas $\alpha, \mu$, and $\leq$ which define subsets of $N^3, N^3$, and $N^2$, respectively, without parameters. Also fix $\varnothing$-definable constants $0, 1 \in N$. Let $T$ be the $L$-theory described below:

- $(I) \ (N, \alpha, \mu, \leq, 0, 1)$ is a model of the relational theory of Peano Arithmetic, $PA$.
- $(II)$ For each $\psi \in A \cup \{\sigma\}$, $T$ contains $\forall x \in N \exists y \psi^x$, where $\psi^x$ is the sentence $\psi$ with any occurrence of $i$ replaced by $N(i)$ replaced by $N(i) \land i \leq x$.

If $T$ is consistent then there is an $L$-structure $R^x$ which satisfies $\Lambda$, but is not elementarily equivalent to an ultraproduct of structures in $\mathbb{K}$.

It follows that the class $\{ M : M \models \text{Th}(\mathbb{K}) \}$ is not computably axiomatizable, since given any potential axiomatization $\Lambda$, the structure $R^x$ obtained in the theorem satisfies $\Lambda$ but not $\text{Th}(\mathbb{K})$.

**Proof.** Assume that $T$ is consistent. In every model of $T$, the interpretation of $N$ is a model of Peano Arithmetic, and so by Gödel’s Second Incompleteness Theorem, $T + \neg \text{Con}(T)$ is also consistent. Thus, there exists a model $A$ of $T + \neg \text{Con}(T)$. In particular, if $\text{prov}(s, d)$ is the provability relation for $T$ and $c$ is the Gödel number for the statement $0 = 1$ then $A \models \exists x \text{prov}(s, c); that is, there exists $a \in N$ with $A \models \text{prov}(a, c)$.

Note that the provability relation is $\Delta_0$, meaning that all quantifiers in the formula are bounded. Fix $x \in N$ sufficiently large; at the very least, larger than each $t(z)$, where $t$ is a term occurring in $\text{prov}(s, c)$ and $z$ is some combination of $a, c$, and bounds on quantifiers in $\text{prov}(s, c)$. Such an $x \in N$ exists since $\text{prov}(s, d)$ is a finite formula with bounded quantifiers. Define $A_i$ to be the structure which is identical to $A_i$ except that $N_i$ is replaced by the initial segment $\{n \in N_A : n \leq x\}$. Then $A_i \models \exists s \text{prov}(s, c)$. Moreover, if $\psi \in \Lambda$ then $A_i \models \psi^x$ by $(II)$, so $A_i \models \psi$, and hence $A_i$ is a model of $\Lambda$.

We claim that $A_i$ is the desired structure $R^x$. Suppose for contradiction that $A_i$ is elementarily equivalent to an ultraproduct of structures in $\mathbb{K}$:

$$A_i \equiv A' = \prod_{i \in I} A_i / U$$

where $U$ is a non-principal ultrafilter on $I$, and every $A_i$ is a structure in $\mathbb{K}$. Note that $N_A = N^x$, so $A_i \models \sigma$ since $A \models \sigma^x$. Then by elementary equivalence $A' \models \sigma$, and so by Łos’s theorem $U$-most of the $A_i \models \sigma$. But each $A_i \in \mathbb{K}$, and hence each $A_i$ which models $\sigma$ has finite $N_A$ by choice of $\sigma$.

Let $\beta(y)$ be the formula $\exists x \alpha(y, 1, z)$; then $A_i \models \beta(b)$ if and only if the successor of $b$ is an element of $N_A$. Let $\gamma$ be the $L$-sentence which asserts that $N$ satisfies $PA^-$, the formulas in $PA$ other than the induction schema, and that there is a unique $y \in N$ such that $\beta(y)$ does not hold. Since the greatest element $y$ of $N$ will always satisfy $\neg \beta(y)$, $\gamma$ says that there are no “gaps” in $N$. Moreover, since the induction schema is only relevant in infinite structures, if $A_i$ is finite then $A_i \models \gamma$ if and only if it is an initial segment of a model of $PA$.

Clearly, $A_i \models \gamma$, and so again by elementary equivalence and Łos’s theorem, $U$-most of the $A_i$ are finite initial segments of a model of $PA$. These $A_i$ are therefore isomorphic to a substructure of $\mathbb{N}$ with universe $I_n = \{0, 1, \ldots, n\}$ for some $n \in \mathbb{N}$. That is, $U$-most $N_A$ are isomorphic, for some $n_i$, to the structure

$$N_{n_i} = (I_{n_i}, \{(x, y, z) \in I_{n_i} : x = y = z\}, \{(x, y, z) \in I_{n_i} : xy = z\}, \{(x, y) \in I_{n_i} : x \leq y\})$$

Let $c' \in N_A$ be a code for $0 = 1$ and $\text{prov}(d, s)$ the provability relation for $T$. Since $N_A \equiv N_A^*$, we have $N_A \models \exists s \text{prov}(s, c')$. Choose an index $i$ such that $N_A \models \exists s \text{prov}(s, c')$ and $N_A$ is isomorphic to some $N_{n_i}$ as above.

Then, since $N_A \equiv N_{n_i} \subset \mathbb{N}$, there exists $b \in \mathbb{N}$ such that $\mathbb{N} \models \text{prov}(b, c)$, where $c \in \mathbb{N}$ is the image of
c′ ∈ NA. Because of the interpretation of prov(b, c) in the standard model N, this b corresponds to an actual proof of 0 = 1 in T. Hence T is inconsistent, contradicting our assumption, and so A, cannot be elementarily equivalent to an ultraproduct of structures in K.

Note that the requirement of the predicate N being included in the language is merely a convenience. Any occurrence of N could be replaced by a distinguished formula in one variable and the proof would be unaffected.

By adding additional restrictions to the language L and the class K, we can simplify the task of finding a model of T. Suppose L expands the language Lring of rings by adding at least a unary predicate N and let A be an L-field of characteristic zero. Write ϕ : Q → A for the usual embedding of the rationals into A. Then we can take N_A = ϕ(N), α and μ to be restrictions of the field addition and multiplication to N_A, and x ≤ y to mean ∃z ∈ N (y = x + z). Then N_A is isomorphic (as a structure in the relational language of arithmetic) to N, and hence is a model of PA. This leads to a useful corollary to the theorem:

**Corollary 2.2** Suppose L expands Lring by adding at least a unary predicate N, and let K and σ be as in the theorem. Moreover, suppose there exists an L\{N\}-structure A such that A is a field of characteristic zero and for all n ∈ N, the L-structure A_n = (A, ϕ([0, . . . , n])) is an element of K. Then (A, ϕ(N)) ⊨ T and so the class K′ = {B : B ⊨ Th(K)} is not computably axiomatizable.

**Proof.** Clearly, N_A = ϕ(N) ⊨ PA. Since N_A = N^{≤ x}_A where x = ϕ(n) ∈ N_A, we have A_n ⊨ ψ if and only if A ⊨ ψ^{≤ x}; thus, since A_n ⊨ ψ for all ψ ∈ L ∪ {σ} and all n ∈ N, we have A ⊨ ∀x ψ^{≤ x}. Therefore (A, ϕ(N)) ⊨ T.

### 3 Consequences

The examples below are all straightforward consequences of the theorem, which amount to choosing an appropriate class for K, an appropriate sentence σ, and showing that the theory T from the theorem is consistent. In each example below, we assume for convenience that L contains a unary relation symbol N, although as discussed, we could replace N by a distinguished formula without changing any of the results.

The first pair of examples, P-minimality and C-minimality, are variations of o-minimality designed for valued fields. While more detailed descriptions can be found in [6] and [5], for our purposes we need only a single example of each to use in our construction of a model of T.

Fix a prime p. Then any rational number can be written in the form p^n a / p^m b where n, a, b ∈ Z and p ∤ ab. We define a valuation va : Q → Z by v_p(p^n a / p^m b) = n. With appropriate choices of language, the completion of Q with respect to the norm |x| = p^{-n(x)} is an example of a P-minimal structure, denoted Q_p. In Chapter III of [9], Koblitz shows that the metric completion of the algebraic closure of Q_p, denoted Ω_p, is an algebraically closed valued field; it then follows from [5, Theorem C] that Ω_p is an example of a C-minimal structure.

**Example 3.1** Let L_D = {+, −, 0, 1, Div, {P_n}_{n∈N}} be the language used in [6], let L be any proper expansion of L_D, and let K be the class of P-minimal L-structures. Then the class K′ = {A : A ⊨ Th(K)} is not computably axiomatizable.

**Proof.** Let σ be the following sentence:

∀x ∈ N ∃a ∃b ∀y ∈ N (¬Div(b, y − a) ↔ y = x),

which says that N is discrete in the valuation topology. As with o-minimality, a definable subset of a P-minimal structure is discrete if and only if it is finite, and so σ holds in a P-minimal structure precisely when N is finite in that structure.

Take A to be the L\{N\} structure with underlying set Q_p, the usual interpretation for L_D, and trivial interpretations of all other symbols in L\{N\}. Then each A_n = (A, [0, . . . , n]) is P-minimal, since adding finite relations to the language does not affect which sets are definable. Thus, by Corollary 2.2, A = (A, N) is a model of T and so by the theorem, K′ is not computably axiomatizable.

**Example 3.2** Let L_C = {+, −, ·, 0, 1, C} be the language of C-minimal fields described in [5], let L be any proper expansion of L_C, and let K be the class of C-minimal L-structures. Then the class K′ = {A : A ⊨ Th(K)} is not computably axiomatizable.
Proof. Let \( \sigma \) be the following sentence:
\[
\forall x \in N \exists a \exists y \in N (C(a, y, b) \leftrightarrow y = x),
\]
which says that \( N \) is discrete in the valuation topology. As with o-minimality and P-minimality, a definable subset of a C-minimal structure is discrete if and only if it is finite, and so this choice of \( \sigma \) satisfies the condition of the theorem.

Take \( A \) to be the \( \mathcal{L}\backslash \{N\} \) structure with underlying set \( \Omega_1 \), the usual interpretation for \( \mathcal{L}_{\mathcal{C}} \), and trivial interpretations of all other symbols in \( \mathcal{L}\backslash \{N\} \). Then each \( A_n = (A, \{0, \ldots, n\}) \) is C-minimal, since adding finite relations to the language does not affect which sets are definable. Thus, by the corollary, \( \mathcal{A'} = (A, \mathbb{N}) \) is a model of \( T \) and so by the theorem, \( \mathcal{K'} \) is not computably axiomatizable.

In the next two examples, we use a result about rings with no zero divisors, whose theory satisfies the exchange property for acl. If \( A \) is such a ring, and \( x_1, x_2, x_3, x_4 \in A \) with \( x_1 \neq x_3 \), then there is at most one \( t \in M \) such that \( t x_1 + x_3 = t x_3 + x_3 \); define \( F : A^4 \to A \) by \( F(x_1, \ldots, x_4) = t \) if \( x_1 \neq x_3 \) and such a \( t \) exists, and \( F(x_1, \ldots, x_4) = 0 \) otherwise. In [3], it is shown that for any definable \( N \subset A \), \( F(N^4) = A \) if and only if \( D \) is infinite. Thus, if \( \sigma \) is the sentence
\[
\exists y \forall x_1, x_2, x_3, x_4 \in N (y \neq F(x_1, \ldots, x_4))
\]
then \( A \models \sigma \) if and only if \( N \) is finite.

**Example 3.3** Let \( \mathcal{L} \) be any proper expansion of \( L_{\text{o-ring}} = L_{\text{ring}} \cup \{<\} \) and let \( \mathcal{K} \) be the class of o-minimal \( \mathcal{L} \)-structures. Then the class \( \mathcal{K}' = \{A : A \models \text{Th}(\mathcal{K})\} \) is not computably axiomatizable.

**Proof.** We again apply Corollary 2.2. Let \( A \) be the \( \mathcal{L}\backslash \{N\} \)-structure with underlying set \( \mathbb{R} \), the usual interpretation for \( L_{\text{o-ring}} \), and trivial interpretations of all other symbols in \( \mathcal{L}\backslash \{N\} \). Then each \( A_n = (A, \{0, \ldots, n\}) \) is o-minimal, since adding finite relations to the language does not affect which sets are definable. Thus, taking \( \sigma \) as above, \( \mathcal{A'} = (A, \mathbb{N}) \) is a model of \( T \) and \( \mathcal{K}' \) is not computably axiomatizable.

In Rennet’s original proof, he used the topology of o-minimal structures; Example 3.3 gives an alternate method which does not use any topological properties.

**Example 3.4** Let \( \mathcal{L} \) be any proper expansion of \( L_{\text{ring}} \) and let \( \mathcal{K} \) be the class of strongly minimal \( \mathcal{L} \)-structures. Then the class \( \mathcal{K}' = \{A : A \models \text{Th} (\mathcal{K})\} \) is not computably axiomatizable.

**Proof.** Proceed as in the o-minimal case, but take the underlying set of \( A \) to be \( \mathbb{C} \) instead of \( \mathbb{R} \).

We look to Pillay’s paper [12] for the next pair of examples. In section 3 of that paper, Pillay defines a dimension rank \( D_A \) for first order topological structures, which we shall not repeat here. He notes that every stable first order topological structure has the discrete topology, and so Theorem 2.1 cannot be applied to stable structures. However, he introduces a different notion of stability for topological structures, which provides another pair of classes to investigate:

**Definition 3.5** A first order topological structure \( A \) is said to be *topologically totally transcendental*, or t.t.t., if it satisfies the following properties:

(A) Every definable set \( X \subseteq A \) is a boolean combination of definable open sets.

(B) Every definable set \( X \subseteq A \) has \( d(X) < \infty \), where \( d(X) \) is the maximum choice of \( d \) such that \( X \) can be written as a disjoint union of nonempty definable sets \( X_1, \ldots, X_d \) with each \( X_i \) both closed and open in \( X \).

(C) \( A \) has dimension, meaning \( D_A (A) < \infty \).

(D) The topology on \( A \) is Hausdorff.

Moreover, \( A \) is said to be t-minimal if \( A \) is t.t.t. and \( D_A(A) = d(A) = 1 \).

In the case of an ordered structure, t-minimality is equivalent to o-minimality [12, Proposition 6.2], and since the ordering on the reals is definable in the field language, \( (\mathbb{R},+,\cdot) \) with the usual topology is t-minimal. The structure \( (\mathbb{C},+,-,\cdot,P) \) with the usual topology and \( P \) interpreted as a predicate for the positive reals is an example of a t.t.t. structure which is not t-minimal.
Example 3.6 Let $\mathcal{L}$ be any proper expansion of $\mathcal{L}_{ring} \cup \{B\}$, where $B$ is an $n$-ary relation symbol for some $n \geq 2$, and let $\mathbb{K}$ be the class of t.t.t. $\mathcal{L}$-structures in which $B(x, \bar{y})$ gives a basis for a topology. Then the class $\mathbb{K}' = \{A : A \models \text{Th}(K)\}$ is not computably axiomatizable.

Proof. As in the P-minimal and C-minimal cases, we take $\sigma$ to be the sentence which asserts $N$ is discrete, namely:

$$\forall x \in N \exists y \forall z \in N (B(z, \bar{y}) \leftrightarrow x = z).$$

If $\mathcal{A} \in \mathbb{K}$ with $\mathcal{A} \models \sigma$ then since $\mathcal{A}$ is Hausdorff, each point $x \in N$ is closed, and since $N$ is discrete, each $a \in N$ is open in $N$. Thus, $|N| = d(N)$ is finite by condition (B). Conversely, if $N_{\mathcal{A}}$ is finite then since $\mathcal{A}$ is Hausdorff, $N_{\mathcal{A}}$ is discrete and $\sigma \models \mathcal{A}$.

It remains to find a model of $T$. Take $\mathcal{A}$ to be the $\mathcal{L}'\{N\}$-structure with underlying set $\mathbb{R}$, the usual interpretation of $\mathcal{L}_{ring}$, and a trivial interpretation for every other symbol in $\mathcal{L}'\{N, B\}$. For $B$, note that there exists a bijection $\varphi : \mathbb{R} \to I$, where $I$ is the set of open intervals with endpoints in $\mathbb{R}$, so we can take $B(x, \bar{y})$ to mean $x \in \varphi(y_1)$.

Each $\mathcal{A}_n = (\mathcal{A}, \{0, \ldots, n\})$ is t.t.t., since adding a predicate for a finite relation does not affect which sets are definable, and so by Corollary 2.2, $\mathcal{A}' = (\mathcal{A}, \mathbb{N})$ is a model of $T$. Hence, by the theorem, $\mathbb{K}'$ is not computably axiomatizable.

As with $N$, the inclusion of $B$ in the language is merely a convenience. Given a distinguished formula for $B$ that satisfies the assumptions for the structure to be t.t.t., we could (with more difficulty) interpret the function and relation symbols in such a way that we obtain essentially the same model of $T$ given above.

Example 3.7 Let $\mathcal{L}$ be any proper expansion of $\mathcal{L}_{ring} \cup \{B\}$, where $B$ is an $n$-ary relation symbol for some $n \geq 2$, and let $\mathbb{K}$ be the class of $t$-minimal $\mathcal{L}$-structures in which $B(x, \bar{y})$ gives a basis for a topology. Then the class $\mathbb{K}' = \{A : A \models \text{Th}(K)\}$ is not computably axiomatizable.

Proof. In the previous example, we have already shown everything necessary except that each structure $\mathcal{A}_n$ has $d(A) = 1$. But this is equivalent to saying that $\mathbb{R}$ (with its usual topology) is connected, which is clearly true.

In [8], Hrushovski and Zilber developed a generalization of the Zariski topology on an algebraically closed field, in order to define a class of structures in which a field can be interpreted. We can use this class of structures, called Zariski structures, to find examples where the field structure is not explicit in the language, but Theorem 2.1 can still be applied.

Hrushovski and Zilber were not particular with their choice of language, merely requiring that all constructible sets be definable. If $F$ is a field, then the set of (Zariski) closed subsets of $F^{n}$ (for any fixed $n$) must have the same cardinality as $F$, since each closed subset is defined by a finite collection of polynomials. Thus, for each $n$ there is a bijection $\varphi$ between $F$ and the set of closed subsets of $F^{n}$. Let $\mathcal{L}$ be an expansion of $\{C_n\}_{n \in \mathbb{N}}$, with each $C_n$ an $n + 1$-ary relation. We call an $\mathcal{L}$-structure $\mathcal{A}$ a Zariski $\mathcal{L}$-structure if there is a field structure on $A$ such that $C_n(A^n, b) = \varphi(b)$ for all $b \in A$.

Example 3.8 Let $\mathcal{L}$ be any expansion of $\{+, \cdot, (C_n)_{n \in \mathbb{N}}\}$, where $+$ is a binary function symbol, $0$ is a constant symbol, and each $C_n$ is an $n + 1$-ary relation symbol. Let $\mathbb{K}$ be the class of Zariski $\mathcal{L}$-structures where $+$ is a group operation with identity $0$. Then the class $\mathbb{K}' = \{A : A \models \text{Th}(K)\}$ is not computably axiomatizable.

Proof. Let $\sigma$ be the sentence

$$\exists x \exists y (\neg C_1(x, y) \land \forall z (N(z) \rightarrow C_1(z, y))).$$

If $\mathcal{A} \models \sigma$ then $N$ is contained in a proper closed subset of $A^1$, and by basic properties of Zariski structures, a proper subset of $A^1$ is closed if and only if it is finite. Let $\mathcal{A}$ be the Zariski $\mathcal{L}\{N\}$-structure with underlying set $\mathbb{C}$, each $C_n$ interpreted as a predicate for the Zariski closed subsets of $\mathbb{C}^n$ (as discussed above), the usual interpretation of $+ \models 0$, and trivial interpretations of all other symbols in $\mathcal{L}\{N\}$. By [8], $\mathcal{A}$ interprets an infinite field, and so $\mathbb{C} \rightarrow \mathbb{C}$ is definable in $\mathcal{A}$ by [11, Corollary 1.8].

Let $\mathcal{A}' = (\mathcal{A}, N)$, where $N$ is the usual embedding of $\mathbb{N}$ in $\mathbb{C}$. Since $+ \text{ and } \cdot$ are definable in $\mathcal{A}$, we may take $\alpha$ and $\mu$ to be the restrictions of their graphs to $N$, and $x \leq y$ to mean $\exists z \in N(y = x + z)$. Then $N$ is clearly isomorphic to $\mathbb{N}$. Moreover, since $\{0, \ldots, n\}$ is a proper Zariski-closed set of $\mathbb{C}$ for every $n \in \mathbb{N}$, each $\mathcal{A}_n = (\mathcal{A}, \{0, \ldots, n\})$ is a Zariski group that satisfies $\sigma$, and hence $\mathcal{A}' \models T$. We may then apply the theorem to see that $\mathbb{K}'$ is not computably axiomatizable.
In each example so far, we have defined $N$ to be the standard embedding of $\mathbb{N}$ into some field of characteristic zero. The final example uses the $p$-adic exponential to give a different definition of $N$.

Fix a prime $p$, and define $\Omega$ to be the completion of the algebraic closure of $\mathbb{Q}_p$, the $p$-adic numbers. In [9, Theorem III.13], Koblitz shows that $\Omega$ is algebraically closed. Define $\exp : \Omega \to \Omega$ by the usual power series. In [9, § IV.1], it is shown that $\exp$ is convergent and bijective on the set $m_\sigma = \{ x \in \Omega : v(x) > 0 \}$. Thus, by [5, Theorem C] and [10, Theorem 1.6], $\Omega$ with an added symbol for the exponential function restricted to $m_\sigma$ is $C$-minimal.

Example 3.9 Let $\mathcal{L}_C = \{+, -, \cdot, 0, 1, C\}$ be the language of $C$-minimal fields, and let $\mathcal{L}$ be any proper expansion of $\mathcal{L}_C \cup \{\exp\}$. Let $\mathbb{K}$ be the class of $C$-minimal $\mathcal{L}$-structures in which $\exp$ is interpreted as the restricted exponential function (such structures must have characteristic zero). Then the class $\mathbb{K}' = \{ \mathcal{A} : \mathcal{A} \models \text{Th}(\mathbb{K}) \}$ is not computably axiomatizable.

**Proof.** Let $\sigma$ be the following sentence:

$$\forall x \in N \exists a \exists b \forall y \in N \left( C(a, y, b) \leftrightarrow y = x \right),$$

which says that $N$ is discrete in the valuation topology. As noted in Example 3.2, a definable subset of a $C$-minimal structure is discrete if and only if it is finite, and so this choice of $\sigma$ satisfies the condition of the theorem.

Take $\mathcal{A}$ to be the $\mathcal{L}_\sigma \setminus \{N\}$ structure with underlying set $\Omega$, the usual interpretation for $\mathcal{L}_C \cup \{\exp\}$, and trivial interpretations of all other symbols in $\mathcal{L}_\sigma \setminus \{N\}$. Then each $\mathcal{A}_v = \langle \mathcal{A}, \{1, t, t^2, \ldots, t^n\} \rangle$ is $C$-minimal, since adding finite relations to the language does not affect which sets are definable, so $\mathcal{A}_v = \langle \mathcal{A}, \{t^n : n \in \mathbb{N}\} \rangle$ models $\forall \psi \psi^{S_\mathbb{K}}$ for each $\psi \in \mathcal{L} \cup \{\sigma\}$. All that remains to show is that $\mathcal{N}_v$ has a definable arithmetical structure.

Set $0_N = t^0$, $1_N = t^1$, and for $x, y, z \in N$, set $\alpha(x, y, z)$ to be $x \cdot y = z$; then $\alpha(t^i, t^j, t^k)$ holds if and only if $i + j = k$. As before, set $x \leq y$ to mean $\exists z \in N (y = x + z)$. Now, take $\mu(x, y, z)$ to mean

$$z = \exp \left( \frac{\log(x) \log(y)}{\log(1_N)} \right)$$

where $\log$ denotes the inverse of $\exp$, which is clearly definable in $\mathcal{L}$. As shown in [9, § IV.1], we retain the usual real number identity that $\log(ab) = \log(a) + \log(b)$, and hence $\mu(t^i, t^j, t^k)$ holds if and only if $k = ij$. Then $\langle N, 0_N, 1_N, \cdot, \leq, 0, 1 \rangle$ is isomorphic to $\mathbb{N}$, which means $\mathcal{A}_v$ is a model of $T$, and so applying the theorem gives that $\mathbb{K}'$ is not recursively axiomatizable.

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