Clifford-Klein forms and a-hyperbolic dimension

Maciej Bocheński and Aleksy Tralle

November 26, 2013

Abstract

The purpose of this article is to introduce and investigate properties of a tool (the a-hyperbolic dimension) which enables us to obtain new examples of homogeneous spaces $G/H$ which admit and do not admit almost compact Clifford-Klein forms. We achieve this goal by exploring in greater detail the technique of adjoint orbits developed by Okuda combined with the well-known conditions of Benoist. We find easy-to-check conditions on $G$ and $H$ expressed directly in terms of the Satake diagrams of the corresponding Lie algebras, in cases when $G$ is a real form of a complex Lie group of type $A_n$, $D_{2k+1}$ or $E_6$. One of the advantages of this approach is the fact, that we don’t need to know the embedding of $H$ into $G$. Using the a-hyperbolic dimension we also show, that the homogeneous space $E_6^{IV}/H$ of reductive type admits compact Clifford-Klein forms if and only if $H$ is compact. Also, inspired by the work of Okuda on symmetric spaces $G/H$ we classify all 3-symmetric spaces admitting almost compact Clifford-Klein forms.

1 Introduction

Let $G/H$ be a homogeneous space of a connected semisimple real Lie group $G$ with finite center. We say that $G/H$ admits an almost compact Clifford-Klein form, if $H$ has compact center and there exists a discrete and not virtually abelian subgroup $\Gamma \subset G$ acting discontinously on $G/H$. We say that $G/H$ admits a compact Clifford-Klein form, if there exists a discrete subgroup $\Gamma \subset G$ acting discontinuously on $G/H$ and with compact quotient $\Gamma \backslash G/H$. It is known that if $G/H$ admits compact Clifford-Klein forms, it necessarily admits almost compact ones (but not vice versa). The problem of the existence of Clifford-Klein forms is very important in several
research areas. It is not our intention here to describe the whole topic, therefore, we refer to the excellent survey [8] and papers [10, 11].

The purpose of this note is to demonstrate a simple way of checking when certain types of homogenous spaces $G/H$ admit or do not admit almost compact Clifford-Klein forms. Our method is based on a more detailed exploration of the technique of adjoint orbits from [16], together with the well-known sufficient condition of Benoist [1]. It yields simple conditions for the existence and the non-existence of such forms expressed in terms of the a-hyperbolic dimension, which depends on the Satake diagrams of real Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. As a consequence we get new examples of homogeneous spaces $G/H$ which admit almost compact Clifford-Klein forms as well as of $G/H$ which do not. Our main result is stated in Theorem 8. Although this theorem gives only sufficient conditions, it resolves the problem of existence of an almost compact Clifford-Klein forms in vast classes of examples. As an application we give a classification of all simple and connected real 3-symmetric spaces (i.e. homogeneous spaces $G/H$ generated by automorphisms of order 3 of a simple and connected real Lie group $G$) admitting almost compact Clifford-Klein forms (Table 2). Recall that Okuda classified symmetric spaces with this property in [16]. Generalized symmetric spaces constitute a large class of homogeneous spaces whose properties are relatively close to the properties of symmetric spaces (see [3], [12]). Therefore, it is natural to extend results from [16] onto this class.

Our methods are based on the Lie group theory. We refer to [7], [6] and [17] and use facts from these sources without further explanations. Our notation and terminology is close to [17].

Acknowledgement. The authors thank Andrei Rapinchuk and Dave Witte-Morris for answering their questions. The second author acknowledges partial support of the ESF Research Network "Contact and Symplectic Topology" (CAST).

2 a-hyperbolic dimension

By the a-hyperbolic dimension we will understand a dimension of a specific convex cone defined by the action of the Weyl group of a semisimple Lie group $G$.

Let $V$ be a real vector space of dimension $n$. Choose a set of linearly independent vectors $B \subset V$. A convex cone $A^+$ is a subset of $V$ generated by all linear combinations with non-negative coefficients $A^+ := \text{Span}^+(B)$. The cardinality of $B$ is the dimension of the convex cone $A^+$. In the sequel we will use the simple observation
that for any linear automorphism $f : V \to V$, such that $f(A^+) = A^+$, the set of fixed points of $f$ in $A^+$ is a convex cone.

**Lemma 1.** Let $V_1, \ldots, V_n$ be a collection of vector subspaces of $V$ and let $A^+$ be a convex cone. Assume that

$$A^+ \subset \bigcup_{k=1}^{n} V_k.$$

Then there exists a number $k$, such that $A^+ \subset V_k$.

### 2.1 Antipodal hyperbolic orbits

In this section we are interested in antipodal hyperbolic orbits in absolutely simple Lie algebras. Let $G$ be a real, connected and absolutely simple Lie group with a Lie algebra $\mathfrak{g}$. We say that an element $X \in \mathfrak{g}$ is hyperbolic, if $X$ is semisimple (that is, $ad_X$ is diagonalizable) and all eigenvalues of $ad_X$ are real.

**Definition 1.** An adjoint orbit $O_X := Ad(G)X$ is said to be hyperbolic if $X$ (and therefore every element of $O_X$) is hyperbolic. An orbit $O_Y$ is antipodal if $-Y \in O_Y$ (and therefore for every $Z \in O_Y, -Z \in O_Y$).

We begin with a brief description of an effective way of classifying antipodal hyperbolic orbits in $\mathfrak{g}^\mathbb{C}$ and in $\mathfrak{g}$. For a more detailed treatment of this subject please refer to [16]. Fix a Cartan subalgebra $\mathfrak{j}^\mathbb{C}$ of $\mathfrak{g}^\mathbb{C}$. Let $\Delta = \Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{j}^\mathbb{C})$, be the root system of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{j}^\mathbb{C}$. Consider the subalgebra

$$\mathfrak{j}^+ := \{ X \in \mathfrak{j}^\mathbb{C} | \forall \alpha \in \Delta^+ \alpha(X) \in \mathbb{R} \},$$

which is a real form of $\mathfrak{j}^\mathbb{C}$. Choose a subsystem $\Delta^+$ of positive roots in $\Delta$. Then

$$\mathfrak{j}^+ := \{ X \in \mathfrak{j} | \forall \alpha \in \Delta^+ \alpha(X) \geq 0 \}$$

is the closed Weyl chamber for the Weyl group $W_{\mathfrak{g}^\mathbb{C}}$ of $\Delta$.

**Lemma 2** (Fact 6.1 in [16]). Every complex hyperbolic orbit in $\mathfrak{g}^\mathbb{C}$ meets $\mathfrak{j}$ in a single $W_{\mathfrak{g}^\mathbb{C}}$ orbit. In particular there is a bijective correspondence between complex hyperbolic orbits $O_X$ and elements $X$ of $\mathfrak{j}^+$.

Let $\Pi$ be a simple root system for $\Delta^+$. For every $X \in \mathfrak{j}$ we define

$$\Psi_X : \Pi \to \mathbb{R}, \alpha \to \alpha(X).$$
The above map is called the *weighted Dynkin diagram* of $X \in j$, and the value $\alpha(X)$ is the weight of the node $\alpha$. Since $\Pi$ is a base of the dual space $j^*$, the map

$$\Psi : j \to \text{Map}(\Pi, \mathbb{R}), \ X \to \Psi_X$$

is an linear isomorphism. From Lemma 2 we see, that:

$$\Psi|_j^+ : j^+ \to \text{Map}(\Pi, \mathbb{R}_{\geq 0}), \ X \to \Psi_X$$

is bijective. We also need a tool, that will enable us to distinguish antipodal hyperbolic orbits. Let $w_0$ be the longest element of $W_{gC}$. The action of $w_0$ sends $j^+$ to $-j^+$, $X \to -X$. Define:

$$-w_0 : j \to j, \ X \to -(wX).$$

This is an involutive automorphism of $j$, which preserves $j^+$. Then $\Psi$ and $-w$ induce the linear automorphism $\iota = \Psi \circ (-w) \circ \Psi^{-1}$ of $\text{Map}(\Pi, \mathbb{R})$.

**Theorem 1** (Theorem 6.3 in [16]). *Complex hyperbolic orbit $O$ in $g^C$ is antipodal if and only if the weighted Dynkin diagram of $O$ is invariant with respect to $\iota$.*

The involutive automorphism $\iota$ is non-trivial only for $g^C$ of type $A_n$, $D_{2k+1}$, $E_6$, for $n, k \geq 2$. In such cases the form of $\iota$ is described in Theorem 6.3 (ii) in [16].

Now we should investigate which complex hyperbolic orbits in $g^C$ meet $\mathfrak{g}$. Choose a Cartan involution $\theta$ of $\mathfrak{g}$, and the corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$ 

There exists a Cartan subalgebra $\mathfrak{j}_\mathfrak{g} = \mathfrak{t} + \mathfrak{a}$ of $\mathfrak{g}$ for which $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$, and $\mathfrak{t}$ is in the centralizer $Z_{\mathfrak{g}}(\mathfrak{a})$ (such subalgebra is called split). Notice that

$$j^C = j_\mathfrak{g} + i\mathfrak{j}_\mathfrak{g}$$

and

$$j = it + \mathfrak{a}.$$ 

Consider

$$\Sigma := \{\alpha|_\mathfrak{a} \ | \ \alpha \in \Delta \} - \{0\} \subset \mathfrak{a}^*,$$

the restricted root system of $g$ with respect to $\mathfrak{a}$. The set of positive roots has the form

$$\Sigma^+ := \{\alpha|_\mathfrak{a} \ | \ \alpha \in \Delta^+ \} - \{0\}.$$ 

Define

$$\mathfrak{a}^+ := \{X \in \mathfrak{a} \ | \ \forall_{\xi \in \Sigma^+} \xi(X) \geq 0\}.$$ 

We have $\mathfrak{a}^+ = j^+ \cap \mathfrak{a}$. 

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**Theorem 2** (Lemma 7.2 and Proposition 4.5 in [16]). Complex hyperbolic orbit $O_X = \text{Ad}(G^\mathbb{C})X$ in $\mathfrak{g}^\mathbb{C}$ meets $\mathfrak{g}$ if and only if the corresponding, unique element $X \in j^+$ is also in $\mathfrak{a}^+$. Also
\[ \text{Ad}(G)X = O_X \cap \mathfrak{g}, \]
and every hyperbolic orbit in $\mathfrak{g}$ is obtained this way.

We see that $\iota(\mathfrak{a}^+) = \mathfrak{a}^+$ therefore we can define the convex cone
\[ \mathfrak{b}^+ \subset \mathfrak{a}^+ \]
as the set of all fixed points of $\iota$ in $\mathfrak{a}^+$.

For a semisimple $\mathfrak{g}$ we will take $\mathfrak{b}^+$ to be the convex cone spanned by vectors generating convex cones $\mathfrak{b}_s^+$, defined for every simple part $\mathfrak{s}$ of $\mathfrak{g}$ (we can perform such construction, since the Cartan involution $\theta$ never maps a nonzero vector $X$ to a vector orthogonal to $X$ with respect to the Killing form of the semisimple algebra $\mathfrak{g}$).

**Definition 2.** The dimension of $\mathfrak{a}^+$ is called the real rank ($\text{rank}_\mathbb{R}(\mathfrak{g})$) of $\mathfrak{g}$. The dimension of $\mathfrak{b}^+$ is called the $\mathfrak{a}$-hyperbolic dimension of $\mathfrak{g}$ and is denoted by $\tilde{d}(\mathfrak{g})$.

By Lemma 2, Theorem 1 and Theorem 2 we have:

**Lemma 3.** There is a bijective correspondence between antipodal hyperbolic orbits $O_X$ in $\mathfrak{g}$ and elements $X \in \mathfrak{b}^+$. One also has
\[ O_X = \text{Ad}(G)X \]

In what follows we will use the notion of the Satake diagram. Let us recall this construction. Choose the Cartan decomposition as above. Let $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the Cartan involution defined, as before. Recall that all our constructions are performed with respect to $j$ and $j^\mathbb{C}$. Consider the complex conjugation $\sigma : \mathfrak{g}^\mathbb{C} \rightarrow \mathfrak{g}^\mathbb{C}$. As usual we get the root space decomposition
\[ \mathfrak{g}^\mathbb{C} = j^\mathbb{C} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha. \]

Define the involution $\sigma^*$ on $(j^\mathbb{C})^*$ by the formula
\[ (\sigma^* \varphi)(X) = \overline{\varphi(\sigma(X))}, \forall X \in j^\mathbb{C}. \]
If $\alpha \in \Delta$, then $\sigma^* \alpha \in \Delta$, and $\sigma g_\alpha = g_{\sigma^* \alpha}$. Put $\Delta_0 = \{ \alpha \in \Delta \mid \sigma^* \alpha = -\alpha \}$. One easily checks that

$$\Delta_0 = \{ \alpha \in \Delta \mid \alpha|_a = 0 \}.$$ 

Put $\Delta_1 = \Delta \setminus \Delta_0$. A direct check-up shows that $\sigma^*(\Delta_0) \subset \Delta_0$, and $\sigma^*(\Delta_1) = \Delta_1$. Choose an order in $\Delta$ in a way that $\sigma^*(\Delta^+_1) \subset \Delta^+_1$. Put $\Pi_0 = \Pi \cap \Delta_0$ and $\Pi_1 = \Pi \cap \Delta_1$. Recall that the Satake diagram for $g$ is defined as follows. One takes the Dynkin diagram for $g_C$ and paints vertexes from $\Pi_0$ in black and vertexes from $\Pi_1$ in white. Next, one shows that $\sigma^*$ determines an involution $\tilde{\sigma}$ on $\Pi_1$ defined by the equation

$$\sigma^* \alpha - \beta = \sum_{\gamma \in \Pi_0} k_\gamma \gamma, \quad k_\gamma \geq 0.$$ 

By definition, if the above equality holds for $\alpha$ and $\beta$, then $\tilde{\sigma} \alpha = \beta$. Now the construction of the Satake diagram is completed by joining by arrows the white vertexes transformed into each other by $\tilde{\sigma}$. Recall that semisimple real Lie algebras are uniquely determined by their Satake diagrams up to isomorphism. The table of the Satake diagrams for all real forms of simple complex Lie algebras can be found in [17].

In order to calculate the $a$-hyperbolic dimension we need the following definition and theorems (compare Definition 7.3 in [16]).

**Definition 3.** Let $\Psi_X \in \text{Map}(\Pi, \mathbb{R})$ be the weighted Dynkin diagram of $g_C$ and $S_g$ be the Satake diagram of $g$. We say that $\Psi_X$ matches $S_g$ if all black nodes in $S_g$ have weights equal to 0 in $\Psi_X$ and every two nodes joined by an arrow have the same weights.

**Theorem 3** (Theorem 7.4 in [16]). The weighted Dynkin diagram $\Psi_X \in \text{Map}(\Pi, \mathbb{R}_{\geq 0})$ of a complex hyperbolic orbit $O$ in $g_C$ matches $S_g$ if and only if $O$ meets $g$. There is also a bijective correspondence between elements of $\text{Map}(\Pi, \mathbb{R}_{\geq 0})$ and the set of complex hyperbolic orbits meeting $g$.

**Theorem 4** (Theorem 7.5 in [16]). The map $\Psi : \mathfrak{a} \to \{ \Psi_X \text{ matches } S_g \}, \quad X \mapsto \Psi_X$

The procedure of calculating the $a$-hyperbolic dimension is a straightforward consequence of the cited results and looks as follows.

**Step 1.** We calculate the $a$-hyperbolic dimension separately for every simple part of $g$ and add results.

**Step 2.** We calculate the $a$-hyperbolic dimension for simple $g$ ($\dim(g) = n$) by taking the weighted Dynkin diagrams of hyperbolic orbits in $g_C$ matching $S_g$ and preserved
We interpret weights of a given weighted Dynkin diagram as coordinates of a vector in $\mathbb{R}^n$. All vectors constructed this way give us the convex cone which has dimension equal to $\tilde{d}(\mathfrak{g})$.

**Example 1.** We will show how to calculate the $a$-hyperbolic dimension of $G = E_{6}^{IV}$. We take the weighted Dynkin diagram of $E_{6}^{\mathbb{C}}$ and check how $\iota$ acts on it

$$a \circ b \circ c \circ d \circ e \rightarrow e \circ d \circ c \circ b \circ a \circ f$$

where $a, b, c, d, e, f \geq 0$. Next take the Satake diagram of $E_{6}^{IV}$

$$\circ \cdot \circ \cdot \circ \cdot \circ$$

According to Definition 3 the weighted Dynkin diagram of $E_{6}^{\mathbb{C}}$ matches the Satake diagram of $E_{6}^{IV}$ if and only if it is of the form:

$$a \circ 0 \circ 0 \circ 0 \circ e \circ 0 \circ 0 \circ 0 \circ e$$

Moreover, the above diagram is preserved by $\iota$ if and only if $a = e$. Hence we obtain the following weighted Dynkin diagram:

$$a \circ 0 \circ 0 \circ 0 \circ e \circ 0 \circ 0 \circ 0 \circ e$$

Therefore $\mathfrak{b}^+$ has a dimension equal to the dimension of $\text{Span}^+((1, 0, 0, 0, 1, 0))$. Thus $\tilde{d}(E_{6}^{IV}) = 1$.

Using the described procedure we get the following table of $a$-hyperbolic dimensions which are not equal to the real ranks of the corresponding Lie algebras.
\begin{table}
\begin{center}
\begin{tabular}{|c|c|}
\hline
$\mathfrak{g}$ & a-hyperbolic dim \\
\hline
$\mathfrak{sl}(2k, \mathbb{R})_{k \geq 1}$ & $d(\mathfrak{g}) = k$ \\
\hline
$\mathfrak{sl}(2k+1, \mathbb{R})_{k \geq 1}$ & $d(\mathfrak{g}) = k$ \\
\hline
$\mathfrak{su}^*(4k)_{k \geq 1}$ & $d(\mathfrak{g}) = k$ \\
\hline
$\mathfrak{su}^*(4k+2)_{k \geq 1}$ & $d(\mathfrak{g}) = k$ \\
\hline
$\mathfrak{so}(2k+1, 2k+1)_{k \geq 2}$ & $d(\mathfrak{g}) = 2k$ \\
\hline
$E_6^I$ & $d(\mathfrak{g}) = 4$ \\
\hline
$E_6^{IV}$ & $d(\mathfrak{g}) = 1$ \\
\hline
\end{tabular}
\end{center}
\end{table}

Table 1: This table contains all real forms of simple lie algebras $\mathfrak{g}^C$, for which $\text{rank}_{\mathbb{R}}(\mathfrak{g}) \neq \tilde{d}(\mathfrak{g})$.

We also need the following fact.

**Theorem 5** (Facts 5.1 i 5.3 in [16]). Let $O$ be an antipodal hyperbolic orbit in $\mathfrak{g}$ and $W_\mathfrak{g}$ be the Weyl group of $\mathfrak{g}$. Then $O \cap \mathfrak{a}$ is a single $W_\mathfrak{g}$-orbit in $\mathfrak{a}$.

### 2.2 Almost compact Clifford-Klein forms

Let $G$ be a connected, semisimple, real Lie group with Lie algebra $\mathfrak{g}$ and $H \subset G$ be a closed and connected subgroup. Assume that $G$ has finite center. Also, let $\mathfrak{h}$ be the Lie algebra of $H$ and $W_\mathfrak{g}$ be the Weyl group of $\mathfrak{g}$.

**Definition 4.** The subgroup $H$ is reductive in $G$ if $\mathfrak{h}$ is reductive in $\mathfrak{g}$, that is, there exists a Cartan involution $\theta$ for which $\theta(\mathfrak{h}) = \mathfrak{h}$. The space $G/H$ is called the homogeneous space of reductive type. Moreover, in this setting the Lie algebra $\mathfrak{h}$ is reductive (which means that it is a sum of its center and the semisimple part $[\mathfrak{h}, \mathfrak{h}]$).

Introduce the following definition (motivated by results [1], [2] cited below).

**Definition 5.** The homogeneous space $G/H$ is called an almost compact Clifford-Klein form if $H$ has compact center and $G/H$ admits a discontinuous action of a discrete subgroup $\Gamma \subset G$ which is not virtually abelian (that is, it does not contain an abelian subgroup of finite index).

By [1], if a non-compact space $G/H$ admits a compact Clifford-Klein form then it
also admits a discontinuous action of a discrete and non virtually abelian subgroup of $G$. Also, the following is known.

**Theorem 6** (Corollary 2 in [2]). Assume that $G$ has finite center. If $G/H$ admits compact Clifford-Klein form then $H$ has compact center.

The Lie subalgebra $h$ is reductive in $g$, therefore we can choose a Cartan involution $\theta$ of $g$ preserving $h$. We obtain the Iwasawa decomposition

$$h = \mathfrak{t}_h + \mathfrak{a}_h + \mathfrak{n}_h$$

which is compatible with the decomposition $g = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ (that is $\mathfrak{t}_h \subset \mathfrak{k}$, $\mathfrak{a}_h \subset \mathfrak{a}$ and $\mathfrak{n}_h \subset \mathfrak{n}$). Y. Benoist in [1] gave the following characterization of homogeneous spaces $G/H$ which admit a discontinuous action of a non virtually abelian subgroup $\Gamma \subset G$.

**Theorem 7** (Theorem 1 in [1]). Group $G$ contains a discrete and non virtually abelian subgroup $\Gamma$ which acts discontinuously on $G/H$ if and only if for every $w$ in $W_g$, $w \cdot \mathfrak{a}_h$ does not contain $\mathfrak{b}^+$. Also one can choose $\Gamma$ to be Zariski dense $G$.

In this section we will show an effective way of determining if such action exists. Let $g^\mathbb{C}$ be a complexification of $g$ and assume that a subalgebra $h \subset g$ is reductive in $g$ and has compact center. Since the subalgebra $h$ is reductive we have

$$h = z(h) + [h, h],$$

with the decomposition of the semisimple part

$$h = z(h) + \mathfrak{k}_h^0 + \mathfrak{a}_h^0 + \mathfrak{n}_h^0.$$}

Take the split Cartan subalgebra $j_h$ of $h$. Then

$$j_h = t_h + \mathfrak{a}_h,$$

where $t_h \subset \mathfrak{t}_h \subset \mathfrak{k}$ and $\mathfrak{a}_h \subset \mathfrak{a}$. Therefore

$$j_h = t_h + z_{ah} + \mathfrak{a}_h^0,$$

with $z_{ah} := z(h) \cap \mathfrak{a}_h$, and since the center of $h$ is compact we obtain $z_{ah} = \{0\}$. This gives us

$$\mathfrak{a}_h^0 = \mathfrak{a}_h.$$

Let $\mathfrak{b}_h^+$ be the convex cone constructed according to the procedure described in the previous subsection (for $[h, h]$). Note that $h$ has compact center and therefore every hyperbolic element in $h$ is contained in $[h, h]$. We will also need following lemma.
Lemma 4. Let \( X \in \mathfrak{b}_h^+ \). The orbit \( O_X^g := Ad(G)X \) is an antipodal hyperbolic orbit in \( \mathfrak{g} \).

Proof. The vector \( X \) defines an antipodal hyperbolic orbit in \( \mathfrak{h} \). Therefore we can find \( h \in H \subset G \) such that \( Ad_h(X) = -X \). Since for the decomposition (defined by the Cartan involution \( \theta \))

\[
\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}
\]

the space \( \mathfrak{a} \) consists of vectors for which \( \text{ad} \) is diagonalizable with real values and \( X \in \mathfrak{b}_h^+ \subset \mathfrak{a}_h \subset \mathfrak{a} \), vector \( X \) is hyperbolic in \( \mathfrak{g} \). Thus \( Ad(G)X \) is a hyperbolic orbit in \( \mathfrak{g} \) and \( -X \in AdG(X) \). \( \square \)

2.3 Main result

The following theorem is the main result of this work.

Theorem 8. Let \( G \) be a connected and semisimple Lie group and let \( H \) be a reductive subgroup with a compact center and a finite number of connected components. Let \( \mathfrak{g} \) and \( \mathfrak{h} \) denote the appropriate Lie algebras. Then

1. If \( \tilde{d}(\mathfrak{g}) = \tilde{d}(\mathfrak{h}) \) then \( G/H \) does not admit almost compact (and, therefore, compact) Clifford-Klein forms.

2. If \( \text{rank}_\mathbb{R}(\mathfrak{g}) = \text{rank}_\mathbb{R}(\mathfrak{h}) \), then \( G/H \) does not admit almost compact (and, therefore, compact) Clifford-Klein forms.

3. If \( \tilde{d}(\mathfrak{g}) > \text{rank}_\mathbb{R}(\mathfrak{h}) \) then \( G/H \) admits almost compact Clifford-Klein forms.

Proof. Let us begin with the proof of the first claim of the theorem. By Theorem 7, the non-existence of almost compact Clifford-Klein forms for \( G/H \) can be reformulated as the following sequence of the equivalent conditions:

1. \( \mathfrak{b}_h^+ \subset w \cdot \mathfrak{a}_h \)

2. \( \text{Span}(\mathfrak{b}_h^+) \subset w \cdot \mathfrak{a}_h \)

3. \( w^{-1} \cdot \text{Span}(\mathfrak{b}_h^+) \subset \mathfrak{a}_h \)

Here is the proof of their equivalence. The first two are equivalent, because if a vector space contains \( B \) then it also contains \( \text{Span}(B) \). The equivalence of the second and the third condition is straightforward.
Taking into consideration the above equivalences, one reformulates the condition in Theorem 7 as follows: for every \( w \) in \( W_g \) the space \( w \cdot \text{Span}(b^+) \) is not in \( a_h \). According to Lemma 3 and Lemma 4 the orbit \( AdG(X) \) for \( X \in b^+_h \subset a \) is an antipodal hyperbolic orbit. Therefore we can find \( Y \in b^+ \subset a \) such that \( Ad(G)X = Ad(G)Y \). Theorem 5 implies that \( X, Y \) are in some \( W_g \)-orbit in \( a \), that is \( X = w \cdot Y \) for some \( w \in W_g \). We have

\[
b^+_h \subset W_g \cdot b^+ = \bigcup_{w \in W_g} w \cdot b^+.
\]

Since \( b^+ \subset \text{Span}(b^+) \) we obtain

\[
b^+_h \subset W_g \cdot \text{Span}(b^+) \subset a.
\]

Taking into consideration Lemma 1 we get the inclusion

\[
b^+_h \subset w \cdot \text{Span}(b^+), \tag{1}
\]

for some \( w \in W_g \). Since \( a \)-hyperbolic dimensions are equal one has the equality

\[
\bar{d}(b^+_h) = \dim \text{Span}(b^+).
\]

Since \( w \) is an automorphism one obtains

\[
\text{Span}(b^+_h) = w \cdot \text{Span}(b^+),
\]

which implies

\[
w \cdot \text{Span}(b^+) = \text{Span}(b^+_h) \subset a_h.
\]

The second claim is well known as the Calabi-Markus phenomenon (see [8]).

The third claim is also straightforward. Every element \( w \in W_g \) acts on \( a \) by linear transformations, and, therefore, preserves the dimension \( n \) of the subspace \( a_h \subset a \). Our assumption implies that \( b^+ \) contains subset of more than \( n \) linearly independent vectors, therefore \( b^+ \) can not be a subset of linear subspace \( w \cdot a_h \subset a \) (for any \( w \)).

\[\square\]

3 New examples

Let \( G \) be a semisimple Lie group with Lie algebra \( g \) and \( H \subset G \) a closed subgroup.
Theorem 9 ([13]). If $H$ is semisimple then $H$ is reductive in $G$.

Corollary 1. If $\tilde{d}(H) = \tilde{d}(G)$ for semisimple $H$ and $G/H$ is non-compact then $G/H$ does not admit compact Clifford-Klein forms. If $\text{rank}_\mathbb{R}(H) < \tilde{d}(G)$ then $G/H$ admits almost compact Clifford-Klein forms.

Theorem 10 ([14]). If $G_n \subset G_{n-1} \subset \ldots \subset G_0$ and $G_i/G_{i-1}$ is of reductive type (for $1 \leq i \leq n$) then $G_0/G_n$ is a homogeneous space of reductive type.

Corollary 2. Let $\tilde{d}(G_n) = \tilde{d}(G_0)$. Assume that $G_j$ has compact center and $G_i/G_j$ is non-compact. Then $G_i/G_j$ does not admit compact Clifford-Klein forms for $i < j$. If $\text{rank}_\mathbb{R}(G_j) < \tilde{d}(G_i)$ then $G_i/G_j$ admits almost compact Clifford-Klein forms.

The following examples are obtained by calculating the a-hyperbolic dimensions of the corresponding $G$ and $H$ (according to Table 1).

Example 2. The following homogeneous spaces do not admit compact Clifford-Klein forms:

\[
\begin{align*}
SL(4k + 2l, \mathbb{R})/SO(2k, 2k) \times Sp(l, \mathbb{R}); \\
SL(2k + 2l, \mathbb{R})/Sp(k, \mathbb{R}) \times Sp(l, \mathbb{R}); \\
SL(4k + 4l, \mathbb{R})/SO(2k, 2k) \times SO(2l, 2l); \\
SL(4k + 2l + 1, \mathbb{R})/SO(2k, 2k) \times SO(l, l + 1); \\
SU^*(4k + 2)/U(s, r - s) \times Sp(t, 2k + 1 - r - t), & \text{ for } s + t = k + 1, 1 \leq r \leq 2k + 1; \\
SU^*(4k)/U(s, r - s) \times Sp(t, 2k + 1 - r - t), & \text{ for } s + t = k, 1 \leq r \leq 2k.
\end{align*}
\]

Example 3. The following homogeneous spaces admit almost compact Clifford-Klein forms:

\[
\begin{align*}
SL(2k + 2l + 2, \mathbb{R})/SO(k, k + 1) \times SO(l, l + 1); \\
SL(2k + 2l + 2, \mathbb{R})/SO(k, k) \times SO(l, l); \\
E_6^3/\{SL(3, \mathbb{C}) \times SU(2, 1)\}/\mathbb{Z}_3
\end{align*}
\]

The a-hyperbolic dimension gives us also an easy way of determining if a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ can determine a closed, reductive subgroup $H$ in $G$.

Lemma 5. If $H$ is closed and reductive subgroup of $G$ with compact center then:

$$\tilde{d}(H) \leq \tilde{d}(G).$$

In particular, if $H$ is closed and semisimple subgroup of $G$ then the above condition has to be satisfied.

Proof. The Lemma follows from equation (1) (in the proof of Theorem 8) and Theorem 9. \qed

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Example 4. Lie group $E_6^{IV}$ does not admit $G_2$, $SO(2,3)$, $SO(2,5)$, $SO(2,7)$ and $Sp(2,\mathbb{R})$ as closed subgroups.

We also have the following theorem:

Theorem 11. Assume that $G = E_6^{IV}, SO^*(6), SL(3,\mathbb{R})$ and $H$ is a non-compact subgroup of reductive type. Then $G/H$ does not admit compact Clifford-Klein forms.

Proof. Notice that $\tilde{d}(G) = 1$. If $\tilde{d}(H) = 1$ then $G/H$ does not admit Clifford-Klein forms. On the other hand, if $\tilde{d}(H) = 0$ then $\text{rank}_{\mathbb{R}}(H) = 0$ and thus $H$ is compact. \qed

One should point out, that this property was already known for $G = SL(3,\mathbb{R})$ (compare Proposition 1.10 in [15]).

Now we mention an observation with possible applications to symplectic topology. It is based on the following result.

Theorem 12 ([9]). If $X \in \mathfrak{g}$ is a semisimple element, then the semisimple orbit $G/Z_G(X) \cong \text{Ad}(G)X$ is a homogeneous space of reductive type, where:

$$Z_G(X) := \{g \in G \mid \text{Ad}(g)X = X\}.$$ 

The above property shows us an interesting method of checking if a given elliptic orbit can be compactified.

Example 5. Every elliptic orbit of $SL(4,\mathbb{R})$ with a non-compact isotropy subgroup (with compact center) is an almost compact Clifford-Klein form (please refer to [4] for a classification of the isotropy subgroups of elliptic orbits).

4 3-Symmetric spaces

Definition 6. A regular homogeneous $k$-manifold is a triple $(G, H, \sigma)$, where $G$ is a connected Lie group, $H \subseteq G$ is a closed subgroup, and $\sigma : G \to G$ is an automorphism of $G$ such that:

1. $\sigma^k = id$ and $k \geq 2$ is the least integer with this property,

2. $(G^\sigma)^o \subset H \subset G^\sigma$, where $G^\sigma = \{g \in G \mid \sigma(g) = g\}$, and $(G^\sigma)^o$ is the identity component of $G^\sigma$. 

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In this article we consider the case $k = 3$ introduced for the first time by J. Wolf and A. Gray in [5]. Clearly, if $k = 2$ we get a class of symmetric spaces, that is, homogeneous spaces generated by involutive automorphisms. For the general theory we refer to [12].

As an application of Theorem 8 we give a list of all connected and simple real 3-symmetric spaces in the table below.
Table 2. Non-compact, simple 3-symmetric spaces admitting almost compact Clifford-Klein forms.

| G | K |
|---|---|
| $SL(2n, \mathbb{R})/\mathbb{Z}_2$ | \{ $SL(n, \mathbb{C}) \times T^1$ \} / $\mathbb{Z}_2$ |
| $SO(2n + 1 - 2s - 2t, 2s + 2t)$ | $U(a - s, s) \times SO(2n - 2a + 1 - 2t, 2t)$  
$1 \leq a \leq n, 2 \leq 2s \leq a$ |
| $Sp(n, \mathbb{R})/\mathbb{Z}_2$ | \{ $U(a - s, s) \times Sp(n - a, \mathbb{R})$ \} / $\mathbb{Z}_2$  
$1 \leq a \leq n, 2 \leq 2s \leq a$ |
| $SO(2n - 2s - t, 2s + t)/\mathbb{Z}_2$ | \{ $U(a - s, s) \times SO(2n - 2a - t, t)$ \} / $\mathbb{Z}_2$  
$1 \leq a \leq n, 0 \leq 2s \leq a, 0 \leq t \leq n - a, (s, t) \neq (0, 0)$ |
| $SO^*(2n)/\mathbb{Z}_2$ | \{ $U(a - s, s) \times SO^*(2n - 2a)$ \} / $\mathbb{Z}_2$  
$1 \leq a \leq n, 0 \leq 2s \leq a$ |
| $G_2$ | $U(1, 1), SU(2, 1)$ |
| $F_4^1$ | \{ $Spin(7 - r, r) \times T^1$ \} / $\mathbb{Z}_2$,  
$r = 2, 3$  
\{ $Sp(3, \mathbb{R}) \times T^1$ \} / $\mathbb{Z}_2$,  
\{ $Sp(2, 1) \times T^1$ \} / $\mathbb{Z}_2$  
\{ $SU(3) \times SU(2, 1)$ \} / $\mathbb{Z}_3$,  
\{ $SU(2, 1) \times SU(2, 1)$ \} / $\mathbb{Z}_3$ |
| $E_6^1$ | \{ $SL(3, \mathbb{C}) \times SU(2, 1)$ \} / $\mathbb{Z}_4$ |
| $E_6^{II}$ | \{ $SO^*(10) \times SO(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(5, 1) \times SU(2, 1)$ \} / $\mathbb{Z}_2$  
\{ $SU(2, 1) \times SU(2, 1)$ \} / $\mathbb{Z}_3$  
\{ $SU(2, 1) \times SU(2, 1)$ \} / $\mathbb{Z}_3$ |
| $E_6^{III}$ | \{ $SU(5, 1)$ \} / $\mathbb{Z}_2$,  
\{ $SU(2, 1)$ \} / $\mathbb{Z}_3$  
\{ $SU(3, 3)$ \} / $\mathbb{Z}_2$,  
\{ $SU(2, 1)$ \} / $\mathbb{Z}_3$  
\{ $SU(2, 1)$ \} / $\mathbb{Z}_3$ |
| $E_7^1$ | \{ $E_6^{II} \times T^1$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(3)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$ |
| $E_7^{II}$ | \{ $E_6^{II} \times T^1$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(3)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$ |
| $E_7^{III}$ | \{ $SU(1, 1)$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SO(10, 2)$ \} / $\mathbb{Z}_2$  
\{ $SU(2)$ \} / $\mathbb{Z}_2$  
\{ $SU(5, 1)$ \} / $\mathbb{Z}_2$  
\{ $SU(5, 1)$ \} / $\mathbb{Z}_2$ |
| $E_8^1$ | \{ $SO(8, 6) \times SO(2)$ \} / $\mathbb{Z}_2$,  
\{ $SO(14) \times SO(2)$ \} / $\mathbb{Z}_2$  
\{ $E_6^{II} \times T^1$ \} / $\mathbb{Z}_2$,  
\{ $E_7^1 \times T^1$ \} / $\mathbb{Z}_2$  
\{ $SU(8, 1)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$ |
| $E_8^{II}$ | \{ $SO(8, 6)$ \} / $\mathbb{Z}_2$,  
\{ $SO(14)$ \} / $\mathbb{Z}_2$  
\{ $E_6^{II} \times T^1$ \} / $\mathbb{Z}_2$,  
\{ $E_7^1 \times T^1$ \} / $\mathbb{Z}_2$  
\{ $SU(8, 1)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$,  
\{ $SU(4)$ \} / $\mathbb{Z}_3$ |
| $SO(4, 4)$ | $G_2$ |
| Spin(5,3) | $G_2$ |
| Spin(4,4) | $G_2$ |
References

[1] Y. Benoist, *Actions propres sur les espaces homogenes reductifs*, Ann. of Math. 144 (1996), 315-347.

[2] Y. Benoist, F. Labourie, *Sur les espaces homogenes modeles de varietes compactes*, Publications Mathematiques de I.H.E.S 76 (1992) 99-109.

[3] M. Bocheński, A. Tralle, *Generalized symplectic symmetric spaces*, Geom. Dedicata, 2013, DOI 10.1007/s10711-013-9902-x

[4] N. Boumuki, *Isotropy subalgebras of elliptic orbits in semisimple Lie algebras, and the canonical representatives of pseudo-Hermitian symmetric elliptic orbits*, J. Math. Soc. Japan 59 (2007), 1135-1177.

[5] A. Gray and J. Wolf, *Homogeneous spaces defined by Lie group automorphisms I, II*, J. Differential Geometry 2 (1968), 77-159.

[6] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, AMS, Providence, RI, 2001

[7] A. Knapp *Representation theory of semisimple Lie groups*, Princeton 1986.

[8] T. Kobayashi, T. Yoshino *Compact Clifford-Klein forms of symmetric spaces revisited*, Pure Appl. Math. Quart. 1(2005), 603-684

[9] T. Kobayashi, *Discontinuous groups and Clifford-Klein forms of pseudo-Riemannian homogeneous manifolds*, Perspectives in Mathematics, 17 (1996), 99-165.

[10] T. Kobayashi, *Proper actions on a homogeneous space of reductive type*, Math. Ann. 285(1989), 249-263

[11] T. Kobayashi, *Discrete decomposability of the restriction of \( A_q(\lambda) \) with respect to reductive subgroups and its applications*, Invent. Math. 117(1994), 181-205

[12] O. Kowalski, *Generalized Symmetric Spaces*, Springer, Berlin, 1980.

[13] A. N. Minchenko, *The semisimple subalgebras of exceptional Lie algebras*, Trans. Moscow Math. Soc. 67 (2006), 225-259.

[14] G. D. Mostow, *Self-adjoint groups*, Ann. of Math. 62 (1955), 44-55.

[15] H. Oh, D. Witte *Compact Clifford-Klein forms of homogeneous spaces of \( SO(2,n) \)*, Geom. Dedic. 89 (2002), 25-57.
[16] T. Okuda, *Classification of Semisimple Symmetric Spaces with Proper-Actions*, J. Different. Geom. 2 (2013), 301-342.

[17] A. L. Onishchik, E. B. Vinberg, *Lie Groups and Lie Algebras III*, Springer, 2004.

[18] K. Yosida, *A theorem concerning the semisimple Lie groups*, Tohoku Math. J. 44 (1938), 81-84.

Department of Mathematics and Computer Science
University of Warmia and Mazury
Słoneczna 54, 10-710, Olsztyn, Poland
e-mail: hoh@poczta.onet.pl (MB),
tralle@matman.uwm.edu.pl (AT)