The study of leading twist light cone wave functions of 2S state charmonium mesons.

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In this paper leading twist light cone wave functions of 2S state charmonium mesons are studied and models of these functions are built.

PACS numbers: 12.38.-t, 12.38.Bx, 13.66.Bc, 13.25.Gv

I. INTRODUCTION.

Charmonium light cone wave functions (LCWF) are universal nonperturbative objects that describe the production of charmonium mesons in hard exclusive processes within light cone formalism [1, 2]. Usually to study hard exclusive processes with quarkonium production one uses NRQCD [2]. So, light cone formalism can be considered as alternative to NRQCD.

There are two very important advantages of light cone formalism in comparison to NRQCD. The first one is connected with the following fact: light cone formalism can be applied to study the production of any meson. For instance, it is possible to study the production of light mesons, such as π mesons, or the production of heavy mesons, such as charmonium mesons, if LCWFs of these mesons are known. From the NRQCD perspective, this implies that light cone formalism resums infinite series of relativistic corrections to the amplitude, what can be very important [3, 4, 5, 6, 7, 8]. The second advantage is that light cone formalism easily resums leading logarithmic radiative corrections to the amplitude \( \sim \alpha_s \log(Q) \) with the help of LCWFs. This is very important advantage since leading logarithmic corrections at high energies can be even more important than relativistic corrections to the amplitude.

From this one can conclude that LCWFs are the key ingredient of light cone formalism. Moreover, the universality of LCWFs and the variety of the processes where these functions can be used make the study of charmonium LCWFs to be a very important task. However, despite the fact that charmonium LCWFs are very important in understanding hard exclusive processes with charmonium production there is a very limited knowledge of the properties of these functions. There are only few papers where this functions were studied [9, 10, 11, 12].

In this paper the procedure developed in papers [3, 4] for the study of charmonium LCWFs will be applied to the study of leading twist LCWFs of Ψ′ and ηc mesons. This paper is organized as follows. In the next section all definitions needed in the calculation will be given. In Section III the moments of LCWFs will be calculated in the framework of Buchmuller-Tye and Cornell potential models. Section IV is devoted to the calculation of the moments within NRQCD. QCD sum rules will be applied to the calculation of the moments in Section V. Using the results obtained in Sections III-V the models of LCWFs will be built in Section VI. In the last section the results of this paper will be summarized.

II. DEFINITIONS.

There is one leading twist light cone wave function (LCWF) of ηc meson \( \phi_\gamma(\xi, \mu) \) and there are two leading twist LCWFs of \( \psi' \) meson \( \phi_L(\xi, \mu) \), \( \phi_T(\xi, \mu) \). The function \( \phi_L(\xi, \mu) \) is twist two LCWF of longitudinally polarized \( \psi' \) meson. The function \( \phi_T(\xi, \mu) \) is twist two LCWF of transversely polarized \( \psi' \) meson. These LCWFs can be defined as follows [1, 2]:

\[
\langle 0 | \bar{Q}(z) \gamma_\alpha \gamma_5 [z, -z] Q(-z) | \eta_c(p) \rangle_\mu = i f_{\eta \gamma} \int_{-1}^{1} d\xi e^{i(pz)\xi} \phi_\eta(\xi, \mu),
\]

\[
\langle 0 | \bar{Q}(z) \gamma_\alpha [z, -z] Q(-z) | \psi'(\epsilon_\lambda=0, p) \rangle_\mu = f_L p_\alpha \int_{-1}^{1} d\xi e^{i(pz)\xi} \phi_L(\xi, \mu),
\]

\[
\langle 0 | \bar{Q}(z) \sigma_{\alpha\beta} [z, -z] Q(-z) | \psi'(\epsilon_\lambda=\pm1, p) \rangle_\mu = f_T(\mu)(\epsilon_\alpha p_\beta - \epsilon_\beta p_\alpha) \int_{-1}^{1} d\xi e^{i(pz)\xi} \phi_T(\xi, \mu),
\]

(1)

where the following designations are used: \( x_1, x_2 \) are the momentum fractions of the whole meson carried by quark and antiquark correspondingly, \( \xi = x_1 - x_2 \), \( p \) is the momentum of corresponding meson, \( \mu \) is an energy scale. The
factor $[z, -z]$, makes the matrix elements to be gauge invariant and the dependence of the LCWFs $\phi_{0, L, T}(x, \mu)$ on scale $\mu$ can be found in [13, 14].

It should be noted here that there is an important distinction between LCWF of $\eta_c'$ and $\psi'$ mesons. Let us, for instance, consider LCWF of longitudinally polarized $\psi'$. Obviously, this function can be written as follows

$$\phi_L(\xi, \mu) = \phi_S^L(\xi, \mu) + \phi_D^L(\xi, \mu),$$  

where $\phi_S^L(\xi, \mu)$ and $\phi_D^L(\xi, \mu)$ are $S$- and $D$-wave contributions to LCWF of $\psi'$ meson. In the case of $\eta_c'$ meson only $S$-wave contributes to the LCWF of this meson. It is not difficult to estimate the contribution of $D$-wave to LCWF $\phi_L(\xi, \mu)$. Evidently, $D$-wave admixture in the LCWF $\phi_L(\xi, \mu)$ is proportional to the factor $\sim \tan(\theta)f_D^L/f_S^L$, where $f_S^L, f_D^L$ are $S$- and $D$-wave contributions to the constant $f_L$, $\theta$ is a mixing angle of $S$ and $D$ waves in $\psi'$ meson. Within potential models this factor can be written as

$$\tan(\theta)f_D^L/f_S^L = \tan(\theta) \frac{5}{\sqrt{8M_c^2}} \frac{R_D^0(0)}{R_S(0)},$$  

(3)

where $M_c$ is a quark mass in the framework of potential model, $R_S(v), R_D(v)$ are radial wave function of $D$ and $S$ waves. Numerical values of parameters $\theta, R_D^0(0), R_S(0), M_c$ needed for the estimation of $D$-wave contribution to LCWF of $\psi'$ meson will be taken from paper [13]: $R_S(0) = 0.734$ GeV$^{3/2}$, $R_D^0(0)/\sqrt{M_c^2} = 0.095$ GeV$^{3/2}, \theta \sim 12^\circ$. Thus one gets rather large suppression of $D$-wave admixture $\tan(\theta)f_D^L/f_S^L \sim 0.03$. On account of the considerable suppression $D$-wave admixture one can disregard its contribution to the LCWFs of $\psi'$ meson. Below this approximation will be used.

Commonly, $\eta_c'$ and $\psi'$ mesons are considered as a nonrelativistic bound states of quark-antiquark pair. At leading order approximation in relative velocity of quark-antiquark pair $\eta_c'$ and $\psi'$ mesons cannot be distinguished. So within this approximation $\eta_c'$ and $\psi'$ mesons have identical LCWFs at scale $\mu \sim M_c$

$$\phi_{\eta}(\xi, \mu) = \phi_L(\xi, \mu) = \phi_T(\xi, \mu) = \phi(\xi, \mu).$$  

(4)

One can expect that in the case of $2S$ mesons corrections to this approximation can be large. However, the accuracy obtained in this paper does not allow one to distinguish LCWFs $\phi_{\eta_{0, 1}, T}(x, \mu)$. For this reason approximation (4) will be used in this paper.

The main goal of this paper is to calculate the LCWFs $\phi_{\eta_{0, 1}, T}(\xi, \mu)$ of $\psi'$ and $\eta_c'$ mesons. These LCWFs will be parameterized by their moments $\langle \xi_n^{\eta_{0, 1}, T}, \mu \rangle$ at some scale. It is worth noting that, the LCWFs (4) are $\xi$-even, so only even moments should be calculated.

III. THE MOMENTS IN THE FRAMEWORK OF POTENTIAL MODELS.

In papers [13, 14] it was shown that the moments of LCWFs of $\eta_c$ and $J/\psi$ mesons can be calculated in the framework of potential models. In comparison with QCD sum rules, such calculation cannot be considered as an accurate one. However, potential models give rather good estimation of the values of the moments.

To calculate the moment of LCWF one can apply Brodsky-Huang-Lepage (BHL) [10] procedure that can be written as

$$\phi(\xi, \mu) \sim \phi_{as}(\xi)\Phi(\xi, \mu) = (1 - \xi^2)\Phi(\xi, \mu),$$

$$\Phi(\xi, \mu) = \int_0^{\mu^2} \frac{dt}{t^2} \psi\left(t + \frac{\xi^2 M_c^2}{1 - \xi^2}\right),$$  

(5)
The statement written above remains true since it is valid for all relations of the type (5) where the function Φ of leading twist and equal time wave function (5). In papers [9, 10] the other relations were proposed. Nevertheless, state has similar extremum at ξ = 0. However, the function Φ(ξ, µ) is much greater than characteristic momentum of relative motion of quark-antiquark pair in meson. This means that the function Φ(ξ, µ) of 2S state has two additional extremums located symmetrically relative to ξ-axis.

It is not difficult to understand why these additional extremums appear. To do this let us differentiate equation (3) over ξ

$$\Phi'(ξ, µ) = \frac{2ξ}{(1 - ξ^2)^2} \left[ (M_c^2 + µ^2) \psi \left( \frac{ξ^2 M_c^2 + µ^2}{1 - ξ^2} \right) - M_c^2 \psi \left( \frac{ξ^2 M_c^2}{1 - ξ^2} \right) \right].$$

Equation (6) can be simplified if one recalls that the scale µ is much greater than characteristic momentum of relative motion of quark-antiquark pair inside the meson. This means that the function ψ((ξ^2 M_c^2 + µ^2)/(1 - ξ^2)) in the first term is much less than ψ((ξ^2 M_c^2)/(1 - ξ^2)) in the second term of equation (3) for not too large ξ. So, the first term gives small correction to the second and can be omitted to the first approximation. Then equation (6) can be written as

$$\Phi'(ξ, µ) = -M_c^2 \frac{2ξ}{(1 - ξ^2)^2} \psi \left( \frac{ξ^2 M_c^2}{1 - ξ^2} \right).$$

Equation (7) is well known that equal time wave function ψ(k^2) of 2S state has one zero at some point k^2. So it is clear that the function Φ'(ξ, µ) changes sign at the points ξ = ±√(k^2/(M_c^2 + M^2)) what corresponds to the two extremums of the function Φ(ξ, µ). Moreover, the function Φ'(ξ, µ) changes sign at ξ = 0 what corresponds to the extremum at ξ = 0. Obviously, if one regards the first term in equation (3) this will just shift the position of extremums.

Applying the same arguments it is not difficult to prove the following statement: **leading twist LCWF of nS state has 2n + 1 extremums.** It should be noted here that our arguments are based on the relation between LCWF of leading twist and equal time wave function (4). In papers [9, 10] the other relations were proposed. Nevertheless, the statement written above remains true since it is valid for all relations of the type (3) where the function ψ(t) can be represented as a product of equal time wave function and some function χ(ξ) ~ 1 + O(ξ^2) for ξ ~ v.
IV. THE MOMENTS IN THE FRAMEWORK OF NRQCD.

To calculate the moments of LCWFs at leading order approximation in relative velocity one can use the following formula [13]:

\[ \langle \xi^n \rangle = \frac{\gamma^n}{n+1}, \]  

where the constant \( \gamma \) can be related to the matrix element of NRQCD operator \( \gamma^2 = \langle v^2 \rangle \). The value of \( \langle v^2 \rangle \) can be calculated using the approach proposed in [19]

\[ \langle v^2 \rangle = 0.65 \pm 0.42. \]  

There are different sources of error to result (9). However, the main source of error is relativistic corrections to formula (8). In (9) the size of these corrections was estimated as \( \sim (\langle v^2 \rangle)^2 \). It is interesting to note that the value \( \langle v^2 \rangle \) obtained at leading order approximation in relative velocity is very close to that obtained at next to leading order approximation \( \langle v^2 \rangle = 0.67 \) [20].

Using (8), (9) one can easily calculate the values of the moments. The results of this calculation are presented in the fourth column of Table I. The central values of the moments were calculated according to formulas (8). The errors of the calculation of the moment \( \langle \xi^{2k} \rangle \) were estimated as \( \sim k(\langle v^2 \rangle \times \langle \xi^{2k} \rangle) \).

It is seen from Table I that within the error NRQCD prediction for the second moment is in agreement with potential model estimation, but the central values are rather far from each other. For higher moments the difference between central values obtained within these approaches becomes more dramatic and the errors of the calculation within NRQCD are very large. From this one can draw a conclusion: although NRQCD can be applied to the calculation of the second moment of \( 2S \) state mesons, the predictions obtained within this approach for higher moments become unreliable due to large relativistic corrections.

It should be noted here that formula (8) is very simple. So it is not difficult to guess that this dependence can be reproduced by the following function

\[ \phi(\xi) = \frac{1}{2\gamma} \theta(\gamma - |\xi|), \]  

where \( \theta(x) \) is the Heaviside step function. Function (10) can be considered as the NRQCD LCWF obtained at leading order in relative velocity. If the velocity of quark-anti-quark pair is infinitely small \( (\gamma \to 0) \) than \( \phi(\xi) \) tends to \( \delta(\xi) \) as it should be.

Function (10) is very simple and it does not reproduce peculiarities of mesons. For instance, the only distinction of LCWFs of \( 1S \) and \( 2S \) states is different constants \( \gamma \) of these mesons. However, from consideration of previous section it is known that the forms of these LCWFs are rather different. Actually, this is not surprising if one recalls that within NRQCD all mesons with the same quantum numbers are described identically by one set of constants and the peculiarities of each meson are contained in the values of these constants. At leading order approximation in relative velocity there is only one constant \( \langle v^2 \rangle \). So it is not possible to reproduce peculiarities of each meson by the only constant. Probably, if one regards relativistic corrections and QCD radiative corrections to the expressions (8) some properties will be restored.
V. THE MOMENTS IN THE FRAMEWORK OF QCD SUM RULES.

A. The moments of $\phi_L(\xi, \mu)$.

In this section QCD sum rules \cite{21, 22} will be applied to the calculation of the moments \cite{1, 23} of LCWF's $\phi_L(\xi, \mu)$.

To do this let us consider two-point correlator:

$$\Pi_L(z, q, n) = i \int d^4 x e^{iqx} \langle 0 | T J_0(x) J_n(0) | 0 \rangle = (zq)^{n+2} \Pi_L(q^2, n), \quad (zq)^2 = 0.$$  \hspace{1cm} \text{(11)}$$

Sum rules for this correlator can be written as follows:

$$\frac{(f_L)^2_{J/\Psi} (\xi_L^n)_{J/\Psi}}{(M_{J/\Psi}^2 + Q^2)^{m+1}} + \frac{(f_L)^2_{\psi'} (\xi_L^n)_{\psi'}}{(M_{\psi'}^2 + Q^2)^{m+1}} = \frac{1}{\pi} \int_{s_0}^{s} ds \frac{\text{Im} \Pi_{\text{pert}}(s, n)}{(s + Q^2)^{m+1}} + \Pi_{\text{pert}}^{(m)}(Q^2, n) = \Pi_L(Q^2, n),$$

where $\text{Im} \Pi_{\text{pert}}(s, n)$ and $\Pi_{\text{pert}}^{(m)}(Q^2, n)$ can be found in paper \cite{14}. $(f_L)_{J/\Psi}$ and $(f_L)_{\psi'}$ are leptonic constants of $J/\Psi$ and $\psi'$ meson, $(\xi_L^n)_{J/\Psi}$ and $(\xi_L^n)_{\psi'}$ are the $n$-th moment of $J/\Psi$ and $\psi'$ mesons’ LCWFs. To remain the designations introduced earlier, below $f_L$ and $\xi_L^n$ will be used instead of $(f_L)_{J/\Psi}$ and $(\xi_L^n)_{J/\Psi}$.

Numerical analysis of QCD sum rules \cite{12} will be done similar to the numerical analysis in paper \cite{13}. To weaken the role of unknown radiative corrections instead of sum rules \cite{12} the ratio of sum rules with different $n$ will be considered:

$$\frac{\langle \xi_L^n \rangle_{J/\Psi} + r \langle \xi_L^n \rangle_{\psi'}}{1 + r a(m)} = \frac{\Pi_L(Q^2, 0)}{\Pi_L(Q^2, n)}, \quad (13)$$

where $r = f_L^2/(f_L)^2_{J/\Psi},$

$$a(m) = \left( \frac{M_{J/\Psi}^2 + Q^2}{M_{J/\Psi}^2 + Q^2} \right)^{m+1}, \quad \text{and} \quad r = \frac{f_L^2}{(f_L)^2_{J/\Psi}} = \frac{M_{\psi'} \Gamma(\psi' \rightarrow e^+e^-)}{M_{J/\Psi} \Gamma(J/\Psi \rightarrow e^+e^-)} \approx 0.53.$$ \hspace{1cm} \text{(14)}$$

To calculate the moments of LCWF $\phi_L(\xi, \mu)$ let us rewrite sum rules \cite{14} as

$$\langle \xi_L^n \rangle_{J/\Psi} = \frac{\Pi_L(Q^2, 0)}{\Pi_L(Q^2, n)} (1 + r a(m)) - \langle \xi_L^n \rangle \left(1 + r a(m)\right).$$

First sum rules \cite{14} for $n = 2$ will be considered. To the first approximation let us disregard the contribution of $\psi'$ meson in the right hand side of equation \cite{14}, as it was done in paper \cite{14} and take the value of the threshold $s_0$ equal to the threshold of $D$-mesons production $\sqrt{s_0} \approx 3.7$ GeV. The left hand side of equation \cite{14} does not depend on $m$. The right hand side of \cite{14} is a function of $m$. This function is plotted in Fig. 2a. It is seen that for too small values of $m$ ($m < 10$) right hand side of equation \cite{14} varies rather rapidly. This happens since there are large contributions from higher resonances disregarded in model of physical spectral density which invalidates sum rules \cite{12}. \cite{14}. Although for $m \gg m_1$ these contributions are strongly suppressed, it is not possible to apply sum rules for too large $m$ ($m > 12$) since the contribution arising from higher dimensional vacuum condensates rapidly grows with $m$(see Fig. 2a) what also invalidates sum rules. It is seen from Fig. 2a that in the region $[10, 12]$ left hand side of equation \cite{14} $m$ varies very slowly. This is the region of applicability of sum rules \cite{12}, \cite{14} where the resonance and the higher dimensional vacuum condensates contributions are not too large. Within the region of applicability the approximation of physical spectral density and the approximation of the contribution of vacuum condensates are valid and one can determine the value of the constant $\langle \xi_L^2 \rangle_{J/\Psi}$. Thus one gets

$$\langle \xi_L^2 \rangle_{J/\Psi} = 0.07.$$ \hspace{1cm} \text{(16)}$$

This value coincides with that found in paper \cite{14}. As it was noted above due to the contribution of higher resonances sum rules \cite{14} is spoiled in low $m$ region. Evidently, the inclusion one resonance succeeding $J/\Psi$-meson will improve sum rules \cite{14} in the region of low $m$. The parameter $\langle \xi_L^2 \rangle$ can be chosen so that to attain best fit of right hand side of equation \cite{14} to the constant $\langle \xi_L^2 \rangle_{J/\Psi}$. The calculation shows that the best fit can be obtained if $\langle \xi_L^2 \rangle = 0.22$. Right hand side of sum rules \cite{14} at $\langle \xi_L^2 \rangle = 0.22$ as a function of $m$ is shown in Fig. 2b.
From Fig. 2b it is seen that if $\psi'$ meson with $\langle \xi_L^2 \rangle = 0.22$ is included into the sum rules, the agreement between right and left hand sides of equation (15) becomes much better. From Fig. 2b one also sees that in the region $m \in [0, 4]$ right hand side of sum rules (15) is rising function of $m$. This seems rather strange since if one includes charmonium meson succeeding $\psi'$ meson to sum rules, right hand side of equation (15) will become decreasing function of $m$. Perhaps, this strange behavior originates from the following fact. In the region of too low $m$ there are large contributions coming from higher resonances not included into physical spectral density. So, if one tries to regard these contributions by the only resonance $- \psi'$ meson, this will lead to an overestimation of the value of $\langle \xi_L^2 \rangle$. This problem can be partially removed if, in addition to the requirement to achieve the best fit of both sides of sum rules, the following requirement will be imposed: right hand side of equation (15) must be decreasing function of $m$. Thus one gets $\langle \xi_L^2 \rangle = 0.18$. The right hand side of sum rules (15) as a function of $m$ with $\langle \xi_L^2 \rangle = 0.18$ is plotted in Fig. 2c.

There are many sources of uncertainty of the calculation fulfilled above. The first one appears due to the uncertainty in sum rules parameters $m_c$ and $\langle \alpha_s G^2/\pi \rangle$. The calculation shows that the uncertainties due to the variation of $m_c$ and $\langle \alpha_s G^2/\pi \rangle$ are not very important (not greater than 10%). For this reason this source of uncertainty will not be considered in the calculation. Probably, the unknown contribution of QCD radiative corrections to the spectral density is much more important, but it is difficult to estimate its contribution. Another very important source of uncertainty results from the unknown value of the threshold parameter $\sqrt{s_0}$. This parameter determines the energy from which continuum contribution to sum rules appears. It is difficult to calculate the value of $s_0$, one can only claim that it is not very far from the threshold of $D$-mesons production $\sqrt{s_0} \approx 3.7$ GeV. In the calculation carried out in this paper it will be assumed that $\sqrt{s_0}$ belongs to the interval $3.7 \pm 0.5$ GeV. The interval chosen in such a way is rather broad and it contains all intervals common for QCD sum rules analysis. It should be noted here that the error due to the variation of $s_0$ within this interval is rather large and below it will be considered as the error of the calculation.

Applying the method discussed above for higher moments one gets the results:

$\langle \xi_L^2 \rangle = 0.18^{+0.05}_{-0.07}$,
$\langle \xi_L^1 \rangle = 0.051^{+0.031}_{-0.031}$,
$\langle \xi_L^0 \rangle = 0.017^{+0.016}_{-0.014}$.

The central values of the moments have been calculated at $\sqrt{s} = 3.7$ GeV. The errors of the calculation appears due to the variation of the threshold parameter $\sqrt{s_0}$ within the interval $3.7 \pm 0.5$ GeV. Physically this variation can be considered as a simulation of the contributions of higher charmonium states and continuum to the moments of LCWF $\phi_L(\xi, \mu)$. From this perspective the error of the calculation is rather large since the contributions from $\psi'$ meson, higher resonances and continuum are not well separated in sum rules (11). All these contributions appear approximately at $\sqrt{s} = 3.7$ GeV. So one can conclude that, although this source of uncertainty can be diminished, it will remain to be the main source of uncertainty of the calculation. From results (17) one sees that the error of the calculation rises as number of the moment increases. Evidently, this happens since the larger the number of the moment the larger the sensitivity of this moment to higher charmonium states and continuum.

Results of the calculation (17) are presented in the fifth column of Table I. It is seen from this table that, although the accuracy of the results obtained within sum rules is better than NRQCD predictions for the moments, the error of the calculation is still rather large. It should be noted also that QCD sum rules predictions for the moments are in better agreement with potential models than with NRQCD results. The central values of NRQCD predictions seems to be overestimated.
It is not difficult to derive sum rules for $\phi_T(\xi, \mu)$ and $\phi_{\eta_c}(\xi, \mu)$. For instance, to calculate the moments of $\phi_{\eta_c}(\xi, \mu)$ one should consider two-point correlator:

$$\Pi_\eta(z, q, n) = i \int d^4x e^{i q x} \langle 0 \mid T J_0(x) J_n(0) \mid 0 \rangle = (zq)^{n+2} \Pi_\eta(q^2, n),$$

where

$$J_0(x) = \bar{Q}(x) \gamma_5 \hat{z} Q(x), \quad J_n(0) = \bar{Q}(0) \gamma_5 \hat{z} (\not{i z^\rho D_\rho})^n Q(0), \quad z^2 = 0.$$  

(B. The moments of $\phi_T(\xi, \mu)$ and $\phi_{\eta_c}(\xi, \mu)$.)

Sum rules for this correlator can be written as

$$\frac{f_{\eta_c}^2 \langle \xi^n \rangle_{\eta_c}}{(M_{\eta_c}^2 + Q^2)^{n+1}} + \frac{f_{\chi_{c1}}^2 \langle \xi^n \rangle_{\chi_{c1}}}{(M_{\chi_{c1}}^2 + Q^2)^{n+1}} + \frac{f_{\eta'_c}^2 \langle \xi^n \rangle_{\eta'_c}}{(M_{\eta'_c}^2 + Q^2)^{n+1}} = \frac{1}{\pi} \int_{4m_c^2}^\infty ds \frac{\text{Im} \Pi_{\text{pert}}(s, n)}{(s + Q^2)^{n+1}} + \Pi_{\text{pert}}^{(m)}(Q^2, n),$$  

where $\langle \xi^n \rangle_{\eta_c}$, $\langle \xi^n \rangle_{\chi_{c1}}$, and $\langle \xi^n \rangle_{\eta'_c}$ are moments of leading twist LCWF of $\eta_c$, $\chi_{c1}$, $\eta'_c$ mesons, the constants $f_{\eta_c}$, $f_{\chi_{c1}}$, $f_{\eta'_c}$ are defined as

$$\langle 0 \mid \bar{Q}(0) \gamma_\alpha \gamma_5 \hat{z} Q(0) \mid M(p) \rangle = i f_M p_\alpha, \quad M = \eta_c, \chi_{c1}, \eta'_c.$$  

One sees that in addition to $\eta'_c$ meson there is contribution of $\chi_{c1}$ meson. Since $\chi_{c1}$ meson is $P$ wave meson, its contribution is a little bit suppressed. Nevertheless, sum rules (19) has one additional unknown parameter $\langle \xi^n \rangle_{\chi_{c2}}$ and this leads to worsening of sum rules predictions in comparison to case considered above. Similar situation takes place for $\phi_T(\xi, \mu)$, where there is contribution of $h_c$ charmonium meson.

From this one can conclude that, unfortunately, QCD sum rules cannot distinguish LCWF $\phi_L(\xi, \mu)$, $\phi_T(\xi, \mu)$ and $\phi_{\eta_c}(\xi, \mu)$ and it is not possible to calculate the moments of $\phi_T(\xi, \mu)$ and $\phi_{\eta_c}(\xi, \mu)$ with the accuracy better than the accuracy of the moments $\langle \xi^n \rangle_{J/\Psi}$. This makes the calculation of the moments of $\phi_T(\xi, \mu)$ $\phi_{\eta_c}(\xi, \mu)$ within QCD sum rules rather pointless. Below hypothesis 41 with moments 17 will be used.

VI. THE MODEL FOR THE FUNCTIONS $\phi_{\eta_c, L, T}(x, \mu)$.

Unfortunately, the methods applied in this paper to the calculation of the moments do not allow one to distinguish LCWFs $\phi_{\eta_c, L, T}(x, \mu)$. For this reason, below these functions are assumed to be equal to some function $\phi(x, \mu)$ at scale $\mu \sim m_c$. This section is devoted to the construction of the model for this function based on the results obtained within QCD sum rules. Results (17) are defined at scale $\mu \sim m_c$ 13. In the calculations it will be assumed that these results are defined at scale $\mu_0 = 1.2$ GeV$\sim m_c$.

In papers 13, 14 it was proposed one parametric model of LCWFs of $\eta_c$ and $J/\Psi$ mesons at scale $\mu_0 = 1.2$ GeV. To reproduce the results obtained in this paper this function can be modified by additional factor $(\alpha + \xi^2)$

$$\phi(\xi, \mu = \mu_0) = c(\alpha, \beta)(1 - \xi^2)(\alpha + \xi^2)\exp\left(-\frac{\beta}{1 - \xi^2}\right) = c(\alpha, \beta)(1 - \xi^2)\Phi(\xi, \mu = \mu_0),$$

Potential model calculation of $\Phi(\xi, \mu = m_c)$ tells us that this function is positive and it has three extrema. Below it will be assumed that these properties remain true for real function $\Phi(\xi, \mu = \mu_0)$. To meet the first requirement
one can suppose that $\alpha \geq 0$. It is not difficult to show that the function $\Phi(\xi, \mu \sim m_c)$ has extremums at $\xi = 0$, $\xi^2 = (2 + \alpha - \sqrt{(2 + \alpha^2) - 4(1 - \alpha^2)})/2$. Two additional extremums of the function $\Phi(\xi, \mu \sim m_c)$ are beyond the physical region $\xi \in [-1, 1]$. So, to meet the second requirement - the function $\Phi(\xi, \mu = \mu_0)$ must have three extremums - one should impose the condition $\alpha\beta < 1$.

Further let us find the region where the constant $\beta$ can vary. This can be done in the framework of Borel version of QCD sum rules [24] where this constant can be expressed through the Borel parameter $M$ as follows $\beta = 4m_c^2/M^2$. The value of Borel parameter cannot be too small ($M > 1$ GeV), otherwise the vacuum condensates contributions become too large. At the same time Borel parameter cannot be too large ($M < 3$ GeV) otherwise the contributions of higher resonances become too large. Thus one gets the assessment of the interval where the constant $\beta$ can vary $\beta \in (0.69, 6.25)$. Now it causes no difficulties to find allowed region of the constants $\alpha, \beta$. This region is painted black in Fig. 3.

The central values of the second and the forth moment can be obtained within model (21) if the values of the constants $(\alpha, \beta)$ are equal to $(0.027, 2.49)$. If one fixes the value of the constant $\alpha = 0.027$ than, to attain the agreement of the model (21) with the results (17) for the second moment, the constant $\beta$ can vary within the interval $\beta \in (1.4, 5.7)$. Similarly if the constant $\beta$ is fixed at $2.49$ than the constant $\alpha$ can vary within the interval $\alpha \in (0, 0.35)$.

Now let us consider model (21) with the central values $\alpha = 0.027, \beta = 2.49$. As it was noted above model (21) with these values of the constants $\alpha, \beta$ is defined at scale $\mu = \mu_0$. It is not difficult to calculate this function at any scale $\mu > \mu_0$ using conformal expansion [1]. This calculation will be done only for the function $\phi_L(x, \mu)$. The function $\phi_L(x, \mu)$ at scales $\mu_0 = 1.2$ GeV, $\mu_1 = 10$ GeV, $\mu_2 = 100$ GeV, $\mu_3 = \infty$ are shown in Fig. 4. The moments of this LCWF at scales $\mu_0 = 1.2$ GeV, $\mu_1 = 10$ GeV, $\mu_2 = 100$ GeV, $\mu_3 = \infty$ are presented in second, third, fourth and fifth columns of Table II.

In papers [13, 14] it was shown that due to evolution LCWFs of $1S$ state have some interesting properties: the violation of nonrelativistic QCD velocity scaling rules, appearance of relativistic tail and improvement of the accuracy of the model. LCWFs of $2S$ states have similar properties and in this paper these properties will not considered.

Now let us consider two different models (21): Model I ($\alpha = 0, \beta = 2.5$) and Model II ($\alpha = 0.2, \beta = 2.5$). LCWF $\phi(\xi, \mu = \mu_0)$ of these models are shown in Fig. 5a. LCWF of Model I has the following moments $\langle \xi^2 \rangle = 0.21, \langle \xi^4 \rangle = 0.061$, Model II has the moments $\langle \xi^2 \rangle = 0.12, \langle \xi^4 \rangle = 0.031$. It is seen that Model I is considerably wider than Model II. In addition, Models I and II are physically different. Really, suppose the meson with momentum $p$ has LCWF of Model I. It is seen from Fig. 5a. that this LCWF has rather sharp extremums at $|\xi| \sim 0.5$. This means that within this model it is not possible to produce $2S$ state charmonium meson from quark-antiquark pair with small relative momentum. Contrary to Model I, within Model II it is possible for quark-antiquar pair to have small relative momentum. Unfortunately, the uncertainties of results [17] are rather large. So, both models are allowed. One can only assert that the model of LCWF with central values of parameters $\alpha = 0.027, \beta = 2.49$ is very similar to Model I. In addition, the forms of LCWF obtained within potential models (see Fig. 1) are similar to Model I. It should be noted here that at leading order approximation of NRQCD quark-antiquark pair has zero relative momentum. So this approximation is in contradiction with Model I.

The effect considered above takes place at scale $\mu = \mu_0$. To understand what happens at larger scales one should evolve Models I and II from scale $\mu_0$ to larger scales. LCWFs of Models I and II at scale $\mu = 10$ GeV are shown in Fig. 5b. It is seen from this plot that the effect is not so dramatic as it is at scale $\mu_0$. This result is in agreement with the property of LCWFs discussed above: the larger the scale the less difference between different models of LCWF.

### Table II

| $n$ | $\phi(\xi, \mu = 1.2 \text{ GeV})$ | $\phi(\xi, \mu = 10 \text{ GeV})$ | $\phi(\xi, \mu = 100 \text{ GeV})$ | $\phi(\xi, \mu = \infty)$ |
|-----|---------------------------------|-------------------------------|-------------------------------|--------------------------|
| 2   | 0.18                            | 0.19                          | 0.19                          | 0.20                     |
| 4   | 0.051                           | 0.068                         | 0.074                         | 0.086                    |
| 6   | 0.018                           | 0.032                         | 0.037                         | 0.048                    |

### VII. CONCLUSION

In this paper the moments of leading twist light cone wave functions (LCWF) of $2S$ state charmonium mesons have been calculated within three approaches. In the first approach Buchmiller-Tye and Cornell potential models were applied to the calculation of the moments of LCWFs. In the second approach the moments of LCWFs were calculated...
in the framework of NRQCD. In the third approach the method QCD sum rules was applied to the calculation of the moments. Although, the results of the calculation are in reasonable agreement with each other, the errors of the calculation are rather large. As the result, it is not possible to distinguish different LCWFs form each other.

Similarly to the study of LCWFs of $1S$ state charmonium mesons [13, 14], the most accurate results were obtained within QCD sum rules. Using these results two parametric model of LCWFs of $2S$ states was proposed. This model can be used in the calculation of different hard exclusive processes with $2S$ charmonium mesons production.

Acknowledgments

The author thanks A.K. Likhoded, V.V. Kiselev and A.V. Luchinsky for useful discussion and help in preparing this paper. The author thanks G.T. Bodwin for useful discussion. This work was partially supported by Russian Foundation of Basic Research under grant 07-02-00417, Russian Education Ministry grant RNP-2.2.2.3.6646, CRDF grant Y3-P-11-05, president grant MK-2996.2007.2 and the Dynasty foundation.

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FIG. 5: LCWFs [21] at scales: fig. a $\mu = 1.2$ GeV; fig. b $\mu = 10$ GeV with different parameters: Model I ($\alpha = 0, \beta = 2.5$) and Model II ($\alpha = 0.2, \beta = 2.5$).