1. Introduction

One of the basic constructions in convex geometry is the surface area measure of a convex body. To recall the definition, let $K \subseteq \mathbb{R}^n$ be a convex body, i.e a compact convex set with non-empty interior. Then the Gauss map $n_K : \partial K \to \mathbb{S}^{n-1}$ exists $\mathcal{H}^{n-1}$-almost everywhere, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure and $\mathbb{S}^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$ denotes the unit sphere. We then define the surface area measure $S_K$ as the push-forward $S_K = (n_K)_\sharp (\mathcal{H}^{n-1}|_{\partial K})$. More explicitly, for every measurable function $\varphi : \mathbb{S}^{n-1} \to \mathbb{R}$ we have

$$\int_{\mathbb{S}^{n-1}} \varphi dS_K = \int_{\partial K} (\varphi \circ n_K) d\mathcal{H}^{n-1}.$$ 

The surface area measure can be equivalently defined using the first variation of volume. For convex bodies $K$ and $L$, let $K + L$ denotes the usual Minkowski sum, and let $|K|$ denote the (Lebesgue) volume of $K$. Then we have

$$\lim_{t \to 0^+} \frac{|K + tL| - |K|}{t} = \int_{\mathbb{S}^{n-1}} h_L dS_K,$$

where $h_L : \mathbb{S}^{n-1} \to \mathbb{R}$ is the support function of $L$ which is defined by $h_L(\theta) = \max_{x \in L} \langle x, \theta \rangle$. For a proof of this fact the reader may consult a standard reference book in convex geometry such as \cite{17} or \cite{9}.

In this paper we will be interested in functional extensions of the formula (1.1). Recall that a function $f : \mathbb{R}^n \to [0, \infty)$ is called log-concave if

$$f((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. We denote by $\text{LC}_n$ the class of all upper semi-continuous log-concave functions. Note that the class of convex bodies in $\mathbb{R}^n$ embeds naturally into $\text{LC}_n$ using the map

$$K \leftrightarrow 1_K(x) = \begin{cases} 1 & x \in K \\ 0 & \text{otherwise}. \end{cases}$$

We would like to consider log-concave functions as “generalized convex bodies”, and extend the notion of the surface area measure to this setting. To achieve this goal we first need to recall the

\begin{thebibliography}{9}
\bibitem{17} The author is partially supported by ISF grant 1468/19 and BSF grant 2016050.
\end{thebibliography}
standard operations on log-concave functions: The sum of two log-concave functions is given by the sup-convolution, i.e.
\[(f \ast g)(x) = \sup_{y \in \mathbb{R}^n} (f(y)g(x-y)).\]
The associated dilation operation is given by \((\lambda \cdot f)(x) = f\left(\frac{x}{\lambda}\right)^\lambda\) – note that we have for example \(f \ast f = 2 \cdot f\). Finally, the “volume” of \(f\) will be given by the Lebesgue integral \(\int f\). Using these constructions we may define:

**Definition 1.1.** Fix \(f, g \in \text{LC}_n\). The first variation of the integral of \(f\) in the direction of \(g\) is given by
\[
\delta(f, g) = \lim_{t \to 0^+} \frac{\int f \ast (t \cdot g) - \int f}{t}.
\]
The study of log-concave functions as geometric objects has become a major idea in convex geometry, with useful applications even if eventually one is only interested in convex bodies. Due to the very large number of papers in this direction we will not survey all of them, but only mention the ones that directly deal with the first variation \(\delta(f, g)\). In the case when \(f = e^{-\frac{|x|^2}{2}}\) is a Gaussian, \(\delta(f, g)\) was studied under the name “the mean width of \(g\)” by Klartag and Milman ([11]) in one of the papers that began the geometric study of log-concave functions. This mean width was further studied in [14]. In particular it was proved there that in this case we have
\[
\delta(f, g) = \int_{\mathbb{R}^n} h_g(x)e^{-|x|^2/2}dx.
\]
Here \(h_g : \mathbb{R}^n \to \mathbb{R}\) is the support function of \(g\), defined by \(h_g = (-\log g)^*\) where
\[
\varphi^*(x) = \sup_{y \in \mathbb{R}^n} ((x, y) - \varphi(y))
\]
is the Legendre transform.

The case of a general function \(f\) was studied by Colesanti and Fragała in [4]. In particular they showed that the limit in (1.2) always exists when \(0 < \int f < \infty\), though it may be equal to \(+\infty\). To further explain their results we will need some important definitions:

**Definition 1.2.** Fix a log-concave function \(f : \mathbb{R}^n \to \mathbb{R}\) with \(0 < \int f < \infty\), and write \(f = e^{-\varphi}\) for a convex function \(\varphi : \mathbb{R}^n \to (-\infty, \infty]\). Then:

1. The measure \(\mu_f\) is a measure on \(\mathbb{R}^n\) defined as the push-forward \(\mu_f = (\nabla \varphi)^*_x(fdx)\).
2. The measure \(\nu_f\) is a measure on the sphere \(S^{n-1}\), defined as the push-forward \(\nu_f = (n_{K_f})_\sharp \left(fd\mathcal{H}^{n-1}\right)\). Here \(K_f\) is a shorthand notation for the support of \(f\), i.e. \(K_f = \{x \in \mathbb{R}^n : f(x) > 0\}\), and \(n_{K_f}\) denotes the Gauss map \(n_{K_f} : \partial K_f \to S^{n-1}\).

For example, for \(f(x) = e^{-|x|^2/2}\) we have \(\mu_f = e^{-|x|^2/2}dx\) and \(\nu_f \equiv 0\), as \(\partial K_f = \partial \mathbb{R}^n = \emptyset\). For a convex body \(K\) we have \(\mu_{1_K} = |K|\delta_0\) and \(\nu_{1_K} = S_K\), the usual surface area measure.

It will be important for us to observe that no regularity is required for the definitions of \(\mu_f\) and \(\nu_f\). Indeed, as \(\varphi = -\log f\) is a convex function it is differentiable Lebesgue-almost-everywhere on the set \(\{x : \varphi(x) < \infty\} = K_f\). Therefore the push-forward \((\nabla \varphi)^*_x(fdx)\) is well-defined. Similarly since \(K_f\) is a closed convex set its boundary \(\partial K_f\) is a Lipschitz manifold, so in particular the Gauss map \(n_{K_f}\) is defined \(\mathcal{H}^{n-1}\)-almost-everywhere and the push-forward is again well-defined.

While regularity is not needed for the definitions of \(\mu_f\) and \(\nu_f\), it was definitely needed for the representation theorem of [4]:
Theorem 1.3 (Colesanti–Fragalà). Fix $f, g \in \text{LC}_n$, and assume that:

1. The supports $K_f, K_g$ are $C^2$ smooth convex bodies with everywhere positive Gauss curvature.
2. The functions $\psi = -\log f$ and $\varphi = -\log g$ are continuous in $K_f$ and $K_g$ respectively, $C^2$ smooth in the interior of these sets, and have strictly positive-definite Hessians.
3. We have $\lim_{x \to \partial K_f} |\nabla \psi(x)| = \lim_{x \to \partial K_g} |\nabla \varphi(x)| = \infty$.
4. The difference $h_f - c \cdot h_g$ is convex for small enough $c > 0$.

Then

$$\delta(f, g) = \int_{\mathbb{R}^n} h_g \, d\mu_f + \int_{\mathbb{S}^n-1} h_{K_g} \, d\nu_f.$$ 

Based on Theorem 1.3 we can make the following definition:

Definition 1.4. Given $f \in \text{LC}_n$ with $0 < \int f < \infty$, we call the pair $(\mu_f, \nu_f)$ the surface area measures of the function $f$.

We emphasize that unlike a convex body, a log-concave function has two surface area measures: one defined on $\mathbb{R}^n$, and one defined on $\mathbb{S}^{n-1}$.

While the regularity assumptions of Theorem 1.3 are sufficient, it has always been clear that they are not necessary. For example, we already saw that for the function $f(x) = e^{-|x|^2/2}$ no regularity assumptions on $g$ are needed. In fact, it was proved in [15] that if $0 < \int f < \infty$ and $\nu_f = 0$ then we have

$$\delta(f, g) = \int_{\mathbb{R}^n} h_g \, d\mu_f$$

with no regularity assumptions. Note that since $f$ is log-concave and upper semi-continuous it is only discontinuous at points $x \in \partial K_f$ such that $f(x) \neq 0$. Therefore the condition $\nu_f = 0$ is equivalent to the statement that $f$ is continuous $\mathcal{H}^{n-1}$-almost everywhere. This property was dubbed essential continuity by Cordero-Erausquin and Klartag ([5]). In their paper, the authors studied the moment measure of a convex function $\varphi$, which in our terminology is simply the surface area measure $\mu_{e-\varphi}$. One of the main results of their paper is a functional analogue of Minkowski’s existence theorem: Given a measure $\mu$ on $\mathbb{R}^n$, they provide a necessary and sufficient condition for the existence of a function $f \in \text{LC}_n$ with $\mu_f = \mu$ and $\nu_f = 0$. They also prove the uniqueness of such an $f$. We remark that when the functions involved are not necessarily essentially continuous, but are sufficiently regular in the sense of Theorem 1.3, a similar uniqueness result was previously proved by Colesanti–Fragalà in [1]. We will further discuss this issue of uniqueness in Section 3 after proving our main theorem, and explain the results of both papers. We also remark that another proof of the same existence theorem was given by Santambrogio in [16].

The main goal of this paper is to prove the most general form of Theorem 1.3, which requires no regularity assumptions:

Theorem 1.5. Fix $f, g \in \text{LC}_n$ such that $0 < \int f < \infty$. Then

$$\delta(f, g) = \int_{\mathbb{R}^n} h_g \, d\mu_f + \int_{\mathbb{S}^n-1} h_{K_g} \, d\nu_f.$$ 

While the improvement over previous results is simply the elimination of the various technical assumptions, we do believe Theorem 1.5 is of real value. For example, we will see as a corollary that the pair of measures $(\mu_f, \nu_f)$ determines $f$ uniquely, and for this result it is very useful not to have any technical conditions for the validity of formula (1.3). Moreover, we believe our proof sheds some light on the reason this formula holds. In particular, we will see an interesting connection between Theorem 1.5 and the notions of anisotropic total variation and anisotropic perimeter.
main point will be that when \( g = 1_L \) and \( f \in \text{LC}_n \) is arbitrary, Theorem 1.5 can be viewed as an anisotropic version of the coarea formula.

The rest of this note is dedicated to the proof of the theorem. In Section 2 we will introduce the anisotropic coarea formula, and explain why it is in fact equivalent to Theorem 1.5 in the case when \( g = 1_L \) is the indicator of a convex body. Then in Section 4 we will discuss the case of a general function \( g \), and conclude the proof. Some of the ingredients that were used in previous results (mostly in [15]) can be used in the proof of Theorem 1.5 with few changes, and in these cases we will either give an exact reference or briefly sketch the argument.

Before we start with the main proof let us prove a finiteness property of the measures \( \mu_f \) and \( \nu_f \):

**Proposition 1.6.** Assume \( f \in \text{LC}_n \) and \( 0 < \int f < \infty \). Then the measure \( \mu_f \) is finite with a finite first moment. The measure \( \nu_f \) is also finite.

**Proof.** \( \mu_f \) is finite by definition, as \( \int_{\mathbb{R}^n} d\mu_f = \int f \, dx \). Moreover, we have

\[
\int |x| \, d\mu_f = \int |\nabla \varphi| \, f \, dx = \int |\nabla f| \, dx = \int |\nabla f| \, dx < \infty,
\]

where the last inequality is part of Lemma 4 of [5].

Next we show that \( \nu_f \) is finite. Note that this is not entirely trivial since \( \int d\nu_f = \int_{\partial K_f} f \, d\mathcal{H}^{n-1} \), and while \( f \) is clearly bounded we can have \( \mathcal{H}^{n-1} (\partial K_f) = \infty \). We therefore adapt a simple argument of Ball ([2]). Since \( f \) is log-concave and integrable, there exists constants \( A, c > 0 \) such that \( f(x) \leq Ae^{-c|x|} \) (see e.g. Lemma 2.1 of [10]). Note that for all \( x \in \mathbb{R}^n \) we have

\[
e^{-c|x|} = c \int_0^\infty e^{-ct} 1_{tB}(x) \, dt,
\]

where \( B = \{ x : |x| \leq 1 \} \) is the unit ball. We may therefore compute

\[
\int d\nu_f = \int_{\partial K_f} f \, d\mathcal{H}^{n-1} \leq A \int_{\partial K_f} e^{-c|x|} \, d\mathcal{H}^{n-1} = Ac \int_0^\infty \int_{\partial K_f} e^{-ct} 1_{tB}(x) \, d\mathcal{H}^{n-1}(x) \, dt
\]

\[
= Ac \cdot \int_0^\infty e^{-ct} \mathcal{H}^{n-1} (\partial K_f \cap tB) \, dt \leq Ac \cdot \int_0^\infty e^{-ct} \mathcal{H}^{n-1} (\partial (K_f \cap tB)) \, dt
\]

\[
\leq Ac \cdot \int_0^\infty e^{-ct} \mathcal{H}^{n-1} (tB) \, dt = Ac \cdot \mathcal{H}^{n-1} (S^{n-1}) \cdot \int_0^\infty t^{n-1} e^{-ct} \, dt < \infty,
\]

which is what we wanted to prove. Note that the last inequality holds since \( K_f \cap tB \subseteq tB \) and surface area is monotone for convex bodies. \( \square \)

In particular, it follows that the equality in (1.3) is an equality of finite quantities whenever \( g \) is compactly supported, as in this case \( h_g \leq A |x| + B \) and \( h_{K_g} \) is bounded on \( S^{n-1} \). When \( g \) is not compactly supported it is possible to have \( \delta(f,g) = \infty \), as already mentioned.

## 2. Anisotropic total variations

In order to start our proof, we need the notion of the anisotropic total variation. First recall the classical (isotropic) total variation: An integrable function \( f : \mathbb{R}^n \to \mathbb{R} \) is said to have bounded variation if

\[
\sup \left\{ \int_{\mathbb{R}^n} f \, \text{div} \Phi \, dx : \Phi : \mathbb{R}^n \to \mathbb{R}^n \text{ is } C^1\text{-smooth, compactly supported and } |\Phi(x)| \leq 1 \text{ for all } x \in \mathbb{R}^n \right\} < \infty.
\]
This supremum is then known as the total variation of \( f \), which we will denote by \( \text{TV}(f) \). Moreover, if \( f \) is of bounded variation then there exists a vector-valued measure \( Df \) on \( \mathbb{R}^n \) such that
\[
\int_{\mathbb{R}^n} f \, \text{div} \, \Phi \, dx = -\int_{\mathbb{R}^n} \langle \Phi, d(Df) \rangle,
\]
and \( \text{TV}(f) = |Df| (\mathbb{R}^n) \). Here \( |Df| \) denotes the total variation (in the sense of measures) of \( Df \).

A set \( A \subseteq \mathbb{R}^n \) is said to have finite perimeter if \( 1_A \) has finite variation, and we define \( \text{Per}(A) = \text{TV}(1_A) \). Finally, the coarea formula states that if \( f \) has bounded variation then \( F_s = \{ x : f(x) \geq s \} \) has finite perimeter for almost every \( s \), and \( \text{TV}(f) = \int_{-\infty}^{\infty} \text{Per}(F_s) \, ds \). All of these facts are standard – see e.g. Chapter 5 of \([7]\) for proofs.

It is less well known that the role of Euclidean norm in the definition of \( \text{TV}(f) \) is not essential. Fix a convex body \( L \subseteq \mathbb{R}^n \) and assume that 0 belongs to the interior of \( L \). Then the (non-symmetric) norm
\[
\|x\|_L = \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in L \right\}
\]
is equivalent to the Euclidean norm. We then define:

**Definition 2.1.**

1. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be an integrable function. Then the \( L \)-total variation of \( f \) is given by
   \[
   \text{TV}_L(f) = \sup \left\{ \int_{\mathbb{R}^n} f \, \text{div} \, \Phi \, dx : \Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is } C^1 \text{-smooth, compactly supported and } \|\Phi(x)\|_L \leq 1 \text{ for all } x \in \mathbb{R}^n \right\}.
   \]

2. Let \( A \subseteq \mathbb{R}^n \) be a measurable set. The \( L \)-perimeter of \( A \) is defined by \( \text{Per}_L(A) = \text{TV}_L(1_A) \).

Since \( \|\cdot\|_L \) and \( |\cdot| \) are equivalent the notion of “bounded variation” does not depend on \( L \), and \( \text{TV}_L(f) < \infty \) if and only if \( \text{TV}(f) < \infty \). Of course, the variation itself does depend on \( L \).

The theory of anisotropic total variations is analogous to the standard theory. We now cite two results that we will require. We were only able to find as reference the technical report \([8]\), where these results are proven by Grasmair, but the results can be proved in the same way as the classical proofs that appear e.g. in \([7]\):

**Proposition 2.2.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be of bounded variation. Write the vector valued measure \( Df \) as \( Df = \sigma \mu \) where \( \mu \) is a positive measure and \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies \( h_L(\sigma(x)) = 1 \) for all \( x \in \mathbb{R}^n \). Then \( \text{TV}_L(f) = \mu(\mathbb{R}^n) \).

Note that when \( L \) is the Euclidean ball the measure \( \mu \) is exactly \( |Df| \), the usual total variation of \( Df \). In the general case \( \mu \) can be considered as “total variation of \( Df \) with respect to \( L \)”. Also note that \( \text{TV}_L(f) \) was defined using the norm \( \|\cdot\|_L \), but in Proposition 2.2 the norm that appears in the dual norm \( h_L \).

We will also need the anisotropic coarea formula:

**Theorem 2.3.** Fix an integrable \( f : \mathbb{R}^n \rightarrow \mathbb{R} \), and denote its level sets by \( F_s = \{ x : f(x) \geq s \} \). Then
\[
\text{TV}_L(f) = \int_{-\infty}^{\infty} \text{Per}_L(F_s) \, ds.
\]

Our goal for this section is to prove Theorem 1.5 when \( g = 1_L \), the indicator of a convex body. We will do so by proving that in this case Theorem 1.5 is essentially equivalent to Theorem 2.3. We begin with finding an alternative formula for \( \delta(f,g) \) in this case. Recall that if \( K, L \subseteq \mathbb{R}^n \) are convex bodies then the volume \( |K + tL| \) for \( t \geq 0 \) can be written as a polynomial,

\[
|K + tL| = \sum_{k=0}^{\infty} \binom{n}{k} W_k(K,L)t^k.
\]

(2.1)
The non-negative coefficients $W_k(K, L)$ are known in our normalization as the relative quermassintegrals of $K$ with respect to $L$. Formula (2.1) is a special case of the celebrated Minkowski theorem, and for the proof and basic properties of the relative quermassintegrals we refer the reader again to [17] or [9]. For now we just note that $W_0(K, L) = |K|$.

We now prove:

**Proposition 2.4.** Fix $f \in \text{LC}_n$ with $0 < \int f < \infty$ and fix a compact convex body $L \subseteq \mathbb{R}^n$. For every $s > 0$ we write $F_s = \{x \in \mathbb{R}^n : f(x) \geq s\}$. Then

$$\delta(f, 1_L) = n \int_0^\infty W_1(F_s, L) \, ds.$$  

This result essentially appears in [3], at least in the case when $L$ is the unit ball. Nonetheless we present its short proof:

**Proof.** For brevity we define $f_t = f \ast (t \cdot 1_L) = f \ast 1_{tL}$ and $F_s^{(t)} = \{ x \in \mathbb{R}^n : f_t(x) \geq s \}$. It is immediate from the definition of $f_t$ that $F_s^{(t)} = F_s + tL$. By layer cake decomposition we have

$$\delta(f, g) = \lim_{t \to 0^+} \frac{\int f_t - \int f}{t} = \lim_{t \to 0^+} \frac{\int_0^\infty \left| F_s^{(t)} \right| \, ds - \int_0^\infty \left| F_s \right| \, ds}{t} = \lim_{t \to 0^+} \frac{\int_0^\infty \left| F_s + tL \right| - \left| F_s \right| \, ds}{t}.$$ 

Using (2.1) we see that for $0 < t < 1$ we have

$$0 \leq \frac{|F_s + tL| - |F_s|}{t} = \sum_{k=1}^n \binom{n}{k} W_k(F_s, L) t^{k-1} \leq \sum_{k=1}^n \binom{n}{k} W_k(F_s, L) = |F_s + L| - |F_s|.$$ 

Since $\int_0^\infty (|F_s + L| - |F_s|) \, ds = \int f_1 - \int f < \infty$, we can use the dominated convergence theorem to conclude that

$$\delta(f, g) = \int_0^\infty \left( \lim_{t \to 0^+} \frac{|F_s + tL| - |F_s|}{t} \right) \, ds = n \int_0^\infty W_1(F_s, L) \, ds.$$ 

$\square$

We will also need one more identity, which was proved in [15] as part of Theorem 3.2:

**Proposition 2.5.** Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ denote a $C^1$-smooth compactly supported vector field. Fix $f \in \text{LC}_n$ with $0 < \int f < \infty$. Then

$$\int_{\mathbb{R}^n} f \, \text{div} \, \Phi \, dx = -\int_{\mathbb{R}^n} \langle \nabla f, \Phi \rangle \, dx + \int_{\partial K_f} f \langle \Phi, n_{K_f} \rangle \, d\mathcal{H}^{n-1}.$$ 

Armed with these tools, we can finally prove:

**Proposition 2.6.** Fix $f \in \text{LC}_n$ with $0 < \int f < \infty$, and set $g = c \cdot 1_L$ for a compact convex set $L \subseteq \mathbb{R}^n$ and $c > 0$. Then

$$\delta(f, g) = \int_{\mathbb{R}^n} h_g \, d\mu_f + \int_{\mathbb{S}^{n-1}} h_{K_g} \, d\nu_f.$$ 

**Proof.** First note that if the result holds for a function $g$, it also holds for $\tilde{g}(x) = e^c \cdot g(x)$ for all $c \in \mathbb{R}$. Indeed, it is easy to check by the chain rule that $\delta(f, \tilde{g}) = \delta(f, g) + c \int f$ (see Proposition
3.4 in [15]). Since \( h_\bar{g} = h_g + c \) and \( h_{K_g} = h_{\bar{K}_g} \), we have

\[
\delta(f, \bar{g}) = \delta(f, g) + c \int f = \int_{\mathbb{R}^n} h_g d\mu_f + \int_{\mathbb{S}^{n-1}} h_{K_g} d\nu_f + c \int f \\
= \int_{\mathbb{R}^n} (h_g + c) d\mu_f + \int_{\mathbb{S}^{n-1}} h_{K_g} d\nu_f = \delta(f, g) = \int_{\mathbb{R}^n} h_g d\mu_f + \int_{\mathbb{S}^{n-1}} h_{K_g} d\nu_f
\]
as claimed. Therefore from now on we can (and will) assume that \( g = 1_L \).

Next, assume that 0 belongs to the interior of \( L \) so the theory of anisotropic total variations applies. Proposition 2.5 immediately implies that

\[
d(Df) = -\nabla f dx + fn_{\partial K_f} dH^{n-1}|_{\partial K_f}.
\]

Therefore the measure \( \mu \) from Proposition 2.2 is \( d\mu = h_L (-\nabla f) dx + fh_L (n_{\partial K_f}) dH^{n-1}|_{\partial K_f} \). By the same proposition we then have

\[
TV_L(f) = \int d\mu = \int_{\mathbb{R}^n} h_L (-\nabla f) dx + \int_{\partial K_f} fh_L (n_{\partial K_f}) dH^{n-1} \\
= \int_{\mathbb{R}^n} h_L (\nabla \varphi) f dx + \int_{\partial K_f} h_L (n_{\partial K_f}) f dH^{n-1} \\
= \int h_L d\mu_f + \int h_L d\nu_f = \int h_g d\mu_f + \int h_{K_g} d\nu_f,
\]

where of course \( \varphi = -\log f \).

As (2.2) holds for all \( f \in LC_n \) with \( 0 < \int f < \infty \) we can in particular apply this formula to the indicator \( 1_F \) of a convex body \( F \). We then obtain

\[
\text{Per}_L(F) = TV_L (1_F) = \int h_L d(\langle F \rangle \delta_0) + \int h_L dS_F = \int h_L dS_F = nW_1(F, L),
\]

where the last equality is a standard (and follows immediately from (1.1) and (2.1)). Therefore, using in order Proposition 2.3, equation (2.3), Theorem 2.3 and equation (2.2) we have

\[
\delta(f, g) = n \int_0^\infty W_1(F_s, L) ds = \int_0^\infty \text{Per}_L (F_s) ds = TV_L(f) = \int h_g d\mu_f + \int h_{K_g} d\nu_f.
\]

This concludes the proof in the case \( 0 \in \text{int}(L) \).

For the general case, fix a large Euclidean ball \( B \) centered at the origin such that \( B + L \) contains the origin in its interior. From Proposition 2.4 and standard properties of quermassintegrals it follows that \( \delta(f, 1_L) \) is linear in \( L \) with respect to the Minkowski addition. Therefore

\[
\delta(f, 1_L) = \delta(f, 1_{L+B}) - \delta(f, 1_B) \\
= \left( \int h_{L+B} d\mu_f + \int h_{L+B} d\nu_f \right) - \left( \int h_B d\mu_f + \int h_B d\nu_f \right) \\
= \int h_L d\mu_f + \int h_L d\nu_f
\]

and the proof is complete. \( \square \)

Note that as a corollary we obtain the following result:

**Corollary 2.7.** For \( f \in LC_n \) with \( 0 < \int f < \infty \) the sum \( \mu_f + \nu_f \) is centered, i.e. for all \( v \in \mathbb{R}^n \) we have

\[
\int_{\mathbb{R}^n} \langle x, v \rangle d\mu_f + \int_{\mathbb{S}^{n-1}} \langle x, v \rangle d\nu_f = 0.
\]
Proof. Simply take $g = 1_{v}$ in Proposition 2.6 and note that $\delta(f, g) = 0$. \qed

The fact that $\mu_f$ is centered when $\nu_f = 0$ was observed already in [5].

## 3. Completing the proof

In this section we finish the proof of Theorem 1.5. We start with the case of compactly supported $g$. The following lemma from [15] will be crucial:

**Lemma 3.1.** Fix $f, g \in \text{LC}_n$ such that $0 < \int f < \infty$ and $g$ is compactly supported. Then for (Lebesgue) almost every $x \in \mathbb{R}^n$ we have

$$
\lim_{t \to 0^+} \frac{(f \star (t \cdot g))(x) - f(x)}{t} = h_g(\nabla \varphi(x)) f(x).
$$

Here $\varphi = -\log f$, and the right hand side is interpreted at 0 whenever $f(x) = 0$.

This lemma is proved in [15] as part of the proof of Lemma 3.7 (The condition $h_g(y) \leq m|y| + c$ in the statement of that lemma is exactly equivalent to $g$ being compactly supported). If all functions involved are sufficiently regular Lemma 3.1 follows from the standard formula for the first variation of the Legendre transform (see more information in [15]). To prove this result without regularity assumptions does take a bit of work which we will not reproduce here.

We will now prove:

**Proposition 3.2.** Fix $f, g \in \text{LC}_n$ such that $0 < \int f < \infty$ and $g$ is compactly supported. Then

$$
\delta(f, g) = \int_{\mathbb{R}^n} h_g d\mu_f + \int_{S^{n-1}} h_{K_g} d\nu_f.
$$

**Proof.** To simplify our notation let us define $f_t = f \star (t \cdot g)$. We also choose $A > 0$ such that $0 \leq g(x) \leq A$ for all $x \in \mathbb{R}^n$, and we define $\tilde{g} = A \cdot 1_{K_g}$ and $\tilde{f}_t = f \star (t \cdot \tilde{g})$. Note that $g \leq \tilde{g}$, so $f_t \leq \tilde{f}_t$ for all $t > 0$. Also note that

$$
\lim_{t \to 0^+} \frac{\tilde{f}_t - f_t}{t} = \lim_{t \to 0^+} \left( \frac{\tilde{f}_t - f}{t} - \frac{f_t - f}{t} \right) = h_{\tilde{g}}(\nabla \varphi) f - h_g(\nabla \varphi) f
$$

almost everywhere, where we used Lemma 3.1 twice. We may therefore apply Fatou’s lemma and deduce that

$$
\liminf_{t \to 0^+} \left( \int_{\mathbb{R}^n} \frac{\tilde{f}_t - f}{t} - \frac{f_t - f}{t} \, dx \right) = \liminf_{t \to 0^+} \int_{\mathbb{R}^n} \frac{\tilde{f}_t - f}{t} \, dx \geq \int (h_{\tilde{g}}(\nabla \varphi) f - h_g(\nabla \varphi) f) \, dx
$$

$$
= \int (h_{\tilde{g}} - h_g) \, d\mu_f.
$$

However, by Proposition 2.6 we know that

$$
\lim_{t \to 0^+} \int_{\mathbb{R}^n} \frac{\tilde{f}_t - f}{t} \, dx = \delta(f, \tilde{g}) = \int h_{\tilde{g}} d\mu_f + \int h_{K_g} d\nu_f,
$$

where we used the fact that $K_{\tilde{g}} = K_g$. Combining the last two formulas we see that

$$
\int h_{\tilde{g}} d\mu_f + \int h_{K_g} d\nu_f - \limsup_{t \to 0^+} \left( \frac{\int f_t - f}{t} \right) \geq \int (h_{\tilde{g}} - h_g) \, d\mu_f,
$$

$$
\int h_{\tilde{g}} d\mu_f + \int h_{K_g} d\nu_f - \limsup_{t \to 0^+} \left( \frac{\int f_t - f}{t} \right) \geq \int (h_{\tilde{g}} - h_g) \, d\mu_f,
$$
\[ (3.1) \limsup_{t \to 0^+} \frac{\int f_t - \int f}{t} \leq \int h_g d\mu_f + \int h_{K_g} d\nu_f. \]

Note that we were allowed to cancel \( \int h_\tilde{g} d\mu_f \) from both sides since this expression is finite by Proposition 1.6.

The proof of the opposite inequality is similar. Fix \( m \in \mathbb{N} \) and consider \( K_m = \{ x \in K_g : g(x) \geq \frac{1}{m} \} \). This time we define \( \tilde{g} = \frac{1}{m} 1_{K_m} \) and \( \tilde{f}_t = f \circ (t \cdot \tilde{g}) \), and we have the opposite inequality \( f_t \geq \tilde{f}_t \). Applying Fatou’s lemma in the same way we have

\[ \liminf_{t \to 0^+} \left( \int f_t - \int f - \int \tilde{f}_t \right) \geq \int (h_g (\nabla \varphi) f - h_{\tilde{g}} (\nabla \varphi) f) \]

and this time we have

\[ \lim_{t \to 0^+} \frac{\int \tilde{f}_t - \int f}{t} = \delta(f, \tilde{g}) = \int h_\tilde{g} d\mu_f + \int h_{K_m} d\nu_f, \]

so we obtain

\[ \liminf_{t \to 0^+} \int f_t - \int f \geq \int h_g d\mu_f + \int h_{K_g} d\nu_f. \]

Since \( K_m \subseteq K_{m+1} \) for all \( m \) and \( \bigcup_{m=1}^{\infty} K_m = K_g \), we have \( h_{K_g} = \sup_m h_{K_m} = \lim_{m \to \infty} h_{K_m} \). We may therefore let \( m \to \infty \) in the last formula and deduce that

\[ \liminf_{t \to 0^+} \int f_t - \int f \geq \int h_g d\mu_f + \int h_{K_g} d\nu_f, \]

which together with (3.1) completes the proof.

Now we can finally prove Theorem 1.5. The final step is an approximation argument, which is essentially the same as the one in [15]. Therefore we repeat the argument briefly without repeating some of the computations:

**Proof of Theorem 1.5** Define a sequence \( \{ g_m \}_{m=1}^{\infty} \subseteq \text{LC}_n \) by

\[ g_m(x) = \begin{cases} g(x) & |x| \leq m \\ 0 & \text{otherwise.} \end{cases} \]

A computation shows that \( h_{g_m} \not\to h_g \) as \( m \to \infty \). Moreover, since \( K_{g_m} \subseteq K_{g_{m+1}} \) for all \( m \) and \( \bigcup_{m=1}^{\infty} K_{g_m} = K_g \) we also have \( h_{K_{g_m}} \not\to h_{K_g} \). If we also define \( f_t = f \circ (t \cdot g) \) and \( f_{t,m} = f \circ (t \cdot g_m) \) then another computation shows that \( f_{t,m}(x) \not\to f_t(x) \) for all \( t > 0 \) and \( x \in \mathbb{R}^n \). This implies that \( \int f_{t,m} \not\to \int f_t \) (see e.g. Lemma 3.2 of [1]).

Using the chain rule for derivatives we may write

\[ \delta(f,g) = \int f \cdot \lim_{t \to 0^+} \frac{\log \int f_t - \log \int f}{t}. \]
This formula has the advantage that by the Prékopa-Leindler inequality (\cite{13, 12}) the function $t \mapsto \log \int f_t$ is concave, so we may replace the limit by a supremum. It follows that

\[
\lim_{m \to \infty} \delta(f, g_m) = \sup_m \delta(f, g_m) = \int f \cdot \sup_{t>0} \frac{\log \int f_{t,m} - \log \int f}{t} = \int f \cdot \sup_{t>0} \frac{\log \int f - \log \int f}{t} = \delta(f, g).
\]

Therefore, applying Proposition 3.2 and the monotone convergence theorem we conclude that

\[
\delta(f, g) = \lim_{m \to \infty} \delta(f, g_m) = \lim_{m \to \infty} \left( \int h_{g_m} d\mu_f + \int h_{K_{g_m}} d\nu_f \right) = \int h_g d\mu_f + \int h_{K_g} d\nu_f,
\]

and the proof is complete. \qed

As a corollary of the theorem we now prove that the measures $\mu_f$ and $\nu_f$ characterize the function $f$ uniquely up to translations:

**Corollary 3.3.** Fix $f, g \in \text{LC}_n$ with $0 < \int f, \int g < \infty$ and assume that $\mu_f = \mu_g$ and $\nu_f = \nu_g$. Then there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = g(x - x_0)$.

**Proof.** Corollary 5.3 of \cite{4} states that if $f, g \in \text{LC}_n$ satisfy $0 < \int f = \int g < \infty$, $\delta(f, g) = \delta(g, f)$ and $\delta(g, f) = \delta(f, f)$, then there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = g(x - x_0)$. This is proved by showing that we have equality in the Prékopa-Leindler inequality, and using a characterization of the equality case by Dubuc (\cite{6}). A similar strategy was used in \cite{5}, and indeed in the classical proof that the surface area measure $S_K$ determines the body $K$ uniquely.

In our case we have $\int f = \mu_f (\mathbb{R}^n) = \mu_g (\mathbb{R}^n) = \int g$, and since

\[
\delta(f, g) = \int_{\mathbb{R}^n} h_g d\mu_f + \int_{\mathbb{R}^{n-1}} h_{K_g} d\nu_f,
\]

we clearly have $\delta(f, g) = \delta(g, g)$ and similarly $\delta(g, f) = \delta(f, f)$. The result follows immediately. \qed

In \cite{4} the same argument was used but with Theorem 1.3 replacing Theorem 1.5 so uniqueness was only proved under the regularity assumptions of that theorem. In \cite{5} there was no explicit representation formula for $\delta(f, g)$, but a weaker statement that in the essentially continuous case was also sufficient in order to reduce the uniqueness result to the equality case of Prékopa-Leindler inequality (see also \cite{15} for an explanation of why the result of \cite{5} is a weak representation theorem for $\delta(f, g)$). We see that in order to get a clean uniqueness result in the general case one indeed needs the full strength of Theorem 1.5.

Of course, Corollary 3.3 raises the question of existence: Given measures $\mu$ and $\nu$, when is there a function $f \in \text{LC}_n$ with $\mu_f = \mu$ and $\nu_f = \nu$? We believe this question can be answered by essentially the same argument as the argument of \cite{5}, which handled the case $\nu_f \equiv 0$, but the full details are beyond the scope of this paper.

**References**

\begin{enumerate}
\item Shiri Artstein-Avidan, Bo'az Klartag, and Vitali Milman. The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika*, 51(1-2):33–48, feb 2010.
\item Keith Ball. The reverse isoperimetric problem for Gaussian measure. *Discrete & Computational Geometry*, 10(4):411–420, dec 1993.
\item Sergey Bobkov, Andrea Colesanti, and Ilaria Fragala. Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities. *Manuscripta Mathematica*, 143(1-2):131–169, jan 2014.
\end{enumerate}
[4] Andrea Colesanti and Ilaria Fragnà. The first variation of the total mass of log-concave functions and related inequalities. *Advances in Mathematics*, 244:708–749, sep 2013.

[5] Dario Cordero-Erausquin and Bo’az Klartag. Moment measures. *Journal of Functional Analysis*, 268(12):3834–3866, 2015.

[6] Serge Dubuc. Critères de convexité et inégalités intégrales. *Annales de l’institut Fourier*, 27(1):135–165, 1977.

[7] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. CRC Press, New York, NY, 1992.

[8] Markus Grasmair. A Coarea Formula for Anisotropic Total Variation Regularisation. Technical report, FWF National Research Network S92, No. 103, 2010.

[9] Daniel Hug and Wolfgang Weil. *Lectures on Convex Geometry*, volume 286 of *Graduate Texts in Mathematics*. Springer International Publishing, Cham, 2020.

[10] Bo’az Klartag. Uniform almost sub-gaussian estimates for linear functionals on convex sets. *Algebra i Analiz*, 19(1):109–148, dec 2007.

[11] Bo’az Klartag and Vitali Milman. Geometry of log-concave functions and measures. *Geometriae Dedicata*, 112(1):169–182, apr 2005.

[12] László Leindler. On a Certain Converse of Hölder’s Inequality II. *Acta Scientiarum Mathematicarum*, 33(3-4), 1972.

[13] András Prékopa. Logarithmic concave measures with application to stochastic programming. *Acta Scientiarum Mathematicarum*, 32(3-4):301–316, 1971.

[14] Liran Rotem. On the mean width of log-concave functions. In Bo’az Klartag, Shahar Mendelson, and Vitali Milman, editors, *Geometric Aspects of Functional Analysis, Israel Seminar 2006-2010*, volume 2050 of *Lecture Notes in Mathematics*, pages 355–372. Springer, Berlin, Heidelberg, 2012.

[15] Liran Rotem. Surface area measures of log-concave functions. *arXiv:2006.16933*, jun 2020.

[16] Filippo Santambrogio. Dealing with moment measures via entropy and optimal transport. *Journal of Functional Analysis*, 271(2):418–436, jul 2016.

[17] Rolf Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, second edition, 2014.

Department of Mathematics, Technion - Israel Institute of Technology, Israel

*Email address: lrotem@technion.edu*