Chiral Susceptibility in Hard Thermal Loop Approximation

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The static and dynamic chiral susceptibilities in the quark-gluon plasma are calculated within the lowest order perturbative QCD at finite temperature and the Hard Thermal Loop resummation technique using an effective quark propagator. After regularization of ultraviolet divergences, the Hard Thermal Loop results are compared to QCD lattice simulations.

I. INTRODUCTION

The main aim of relativistic heavy-ion collisions is to unravel the basic properties of QCD associated with, e.g., its phase diagram. Two of the main features characterizing the ground state of the theory, confinement and the spontaneous breaking of chiral symmetry, are expected to cease in a possibly common phase transition at finite temperature and/or density \cite{1}. The dynamical breaking of chiral symmetry is associated with the condensation of quark-antiquark pairs in the QCD vacuum. As the temperature and/or baryon density increase, the QCD vacuum undergoes a phase transition to the chirally symmetric phase where the quark-antiquark condensate, the order parameter of the chiral phase transition, vanishes. The fluctuation of this order parameter is related to the associated susceptibilities \cite{2}. The quantity of interest here is the chiral/scalar density susceptibility which measures the response of the chiral condensate/scalar density to the variation of the current quark mass. The chiral susceptibility was recently measured in lattice QCD \cite{1,3} with two light flavors. The chiral susceptibility has also been studied in chiral perturbation theory \cite{4}, in the multflavor Schwinger model for a small nonzero quark mass \cite{4}, in the NJL model \cite{5}, and using the Dyson-Schwinger equation \cite{6}. In this paper, we will investigate the chiral/scalar density susceptibility following the same line as in the case of the quark number susceptibility \cite{7} and the free energy of the quark-gluon plasma \cite{8,9} within the Hard Thermal Loop (HTL) approximation \cite{10}, which selectively resums higher order corrections corresponding to medium effects such as screening, quasiparticles masses and Landau damping beyond the usual perturbation theory.

II. DEFINITION

A. General

Let $O_\alpha$ be a Heisenberg operator. In a static and uniform external field $F_\alpha$, the (induced) expectation value of the operator $O_\alpha(0, \vec{x})$ is written \cite{2} as

$$\phi_\alpha \equiv \langle O_\alpha(0, \vec{x}) \rangle = \frac{\text{Tr}[O_\alpha(0, \vec{x}) e^{-\beta(H+H_{ex})}]}{\text{Tr}[e^{-\beta(H+H_{ex})}]} = \frac{1}{V} \int d^3 x \langle O_\alpha(0, \vec{x}) \rangle ,$$

where translational invariance is assumed and $H_{ex}$ is given by

$$H_{ex} = -\sum_\alpha \int d^3 x O_\alpha(0, \vec{x}) F_\alpha .$$

The (static) susceptibility $\chi_{\alpha\beta}$ is defined as

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\[ \chi_{\alpha \beta}(T) = \frac{\partial \phi_\alpha}{\partial \mathcal{F}_\beta} \bigg|_{T=0} = \beta \int d^3x \left< \mathcal{O}_\alpha(0, \vec{x}) \mathcal{O}_\beta(0, \vec{0}) \right>, \tag{3} \]

assuming no broken symmetry \( \left< \mathcal{O}_\alpha(0, \vec{x}) \right> = \left< \mathcal{O}_\beta(0, \vec{0}) \right> = 0 \). The quantity \( \left< \mathcal{O}_\alpha(0, \vec{x}) \mathcal{O}_\beta(0, \vec{0}) \right> \) is the two point correlation function with operators evaluated at equal times.

**B. Chiral Susceptibility**

The chiral susceptibility measures the response of the chiral condensate to the infinitesimal change of the current quark mass \( m + \delta m \). Here, \( \mathcal{F}_\alpha \) corresponds to the current quark mass and the operator \( \mathcal{O}_\alpha \) to \( \bar{q}q \). Then the static chiral susceptibility can be obtained \[2\] from

\[ \chi_c(T) = - \left. \frac{\partial \langle \bar{q}q \rangle}{\partial m} \right|_{m=0} = \beta \int d^3x \left< \bar{q}(0, \vec{x}) q(0, \vec{x}) \bar{q}(0, \vec{0}) q(0, \vec{0}) \right> = \beta \int d^3x S(0, \vec{x}), \tag{4} \]

where \( S(0, \vec{x}) = \langle \cdots \cdots \rangle \) is the static correlator of the scalar channel. The chiral condensate is given as

\[ \langle \bar{q}q \rangle = \frac{\text{Tr} \left[ \bar{q}q e^{-\beta H} \right]}{\text{Tr} \left[ e^{-\beta H} \right]} = \frac{\partial \Omega}{\partial m}. \tag{5} \]

Here, \( H \) is the dynamical Hamiltonian of the system containing the quark mass, \( \Omega = -(T/V) \log \mathcal{Z} \) is the thermodynamic potential, and \( \mathcal{Z} \) is the partition function of a quark-antiquark gas.

Taking the Fourier transform of the static correlator \( S(0, \vec{x}) = \langle \cdots \cdots \rangle \), it can be shown that

\[ \chi_c(T) = \lim_{p \to 0} \beta \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} S(\omega, \vec{p}), \tag{6} \]

where \( S(\omega, \vec{p}) \) is the Fourier transformed correlator.

Applying the fluctuation-dissipation theorem \[2\], one finds the static limit of the dynamic chiral susceptibility

\[ \tilde{\chi}_c(T) = \lim_{p \to 0} \beta \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{-2}{1 - e^{-\beta \omega}} \text{Im} \Pi(\omega, \vec{p}), \tag{7} \]

where \( \Pi(\omega, \vec{p}) \) is the scalar self-energy. Using \( \text{Im} \Pi(-\omega, \vec{p}) = -\text{Im} \Pi(\omega, \vec{p}) \), (7) can also be written as

\[ \tilde{\chi}_c(T) = - \lim_{p \to 0} \beta = \frac{\omega}{2} \int_{0}^{\infty} d\omega \coth \left( \frac{\beta \omega}{2} \right) \text{Im} \Pi(\omega, \vec{p}). \tag{8} \]

**III. PERTURBATION THEORY**

**A. Static Susceptibility**

The chiral condensate \( \langle \bar{q}q \rangle \) at finite temperature can be obtained from the tadpole diagram in Fig.1 as

\[ \langle \bar{q}q \rangle = -N_c N_f T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ S(K) \right], \tag{9} \]

where, \( N_f \) and \( N_c \) are, respectively, the numbers of quark flavors and colors. \( S(K) \) is the free quark propagator given as

\[ S(K) = \frac{1}{k - m}. \tag{10} \]
Summing over the Matsubara frequencies $k_0$, we obtain

$$
\langle \bar{q}q \rangle = -2N_c \, N_f \int \frac{d^3k}{(2\pi)^3} \frac{m}{E_k} \left[ 1 - 2n_F(E_k) \right],
$$

(11)

where $n_F(E_k)$ is the Fermi distribution function and $E_k = \sqrt{k^2 + m^2}$.

Fig.1: Tadpole diagram determining the free static chiral susceptibility.

Alternatively, the chiral condensate can be derived from the lowest order thermodynamic potential of the quark-antiquark system [11]

$$
\Omega = -2N_f N_c T \int \frac{d^3k}{(2\pi)^3} \left\{ \beta E_k + \log \left[ 1 + e^{-\beta(E_k - \mu_q)} \right] + \log \left[ 1 + e^{-\beta(E_k + \mu_q)} \right] \right\}.
$$

(12)

Using (5) the condensate (11) is reproduced.

Now, the chiral susceptibility in the free case follows from (4) and (11) as

$$
\chi_f^c(T) = \left. \frac{-\partial \langle \bar{q}q \rangle}{\partial m} \right|_{m=0} = 2N_f N_c \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{k^2}{E_k^3} \left[ 1 - 2n_F(E_k) \right] + 2\beta \frac{m^2}{E_k^2} n_F(E_k) \left[ 1 - n_F(E_k) \right] \right\} \bigg|_{m=0} = 2N_f N_c \int \frac{d^3k}{(2\pi)^3} \frac{1}{k} \left[ 1 - 2n_F(k) \right].
$$

(13)

Note that, in contrast to the quark number susceptibility [7] where the zero temperature contribution vanishes due to the transversality properties of the vector channel [12], the static chiral susceptibility contains a quadratic ultraviolet divergence coming from the zero temperature contribution. Using dimensional regularization the temperature independent term in (13) disappears, as there is no scale associated with it, leading to

$$
\chi_f^c(T) = -\frac{N_f N_c}{6} T^2.
$$

(14)

The next order contribution follows from the two-loop thermodynamic potential given in Ref. [11]. However, the second derivative of this expression with respect to $m$ diverges at $m = 0$ (see also Ref. [13]). The reason for this divergence can be seen in the following way: the second derivative with respect to the quark mass $m$ of the two-loop thermodynamic potential, which is a quark loop with an internal gluon line, introduces two additional quark propagators. Hence the originating diagram looks like the polarization tensor containing an internal quark line. This diagram is known to be logarithmically infrared divergent in the case of a vanishing bare quark mass (see e.g. [14]). Hence, the static chiral susceptibility cannot be calculated consistently in usual perturbation theory beyond leading order but requires the HTL resummation.

**B. Dynamic Susceptibility in the Static Limit**

Next, we calculate the static limit of the dynamical chiral susceptibility from the lowest order self-energy diagram shown in Fig.2:
\[ \Pi(P) = N_f N_c T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ S(K) S(Q) \right]. \tag{15} \]

Summing over \( k_0 \), we obtain

\[
\text{Im} \Pi(\omega, \vec{p}) = -N_f N_c \pi \int \frac{d^3k}{(2\pi)^3} \left\{ \left( 1 - \frac{\vec{k} \cdot \vec{q} + m^2}{E_k E_q} \right) \left[ 1 - n_F(E_k) - n_F(E_q) \right] \left[ \delta(\omega - E_k - E_q) - \delta(\omega + E_k + E_q) \right] \right. \\
+ \left. \left( 1 + \frac{\vec{k} \cdot \vec{q} + m^2}{E_k E_q} \right) \left[ n_F(E_k) - n_F(E_q) \right] \left[ \delta(\omega - E_k + E_q) - \delta(\omega + E_k - E_q) \right] \right\} \tag{16} \]

\[ K \]
\[ P \]
\[ Q = P - K \]

Fig. 2: Self-energy diagram determining the free dynamic chiral susceptibility.

Combining (7) and (16) yields

\[
\tilde{\chi}_f(T) = 2N_f N_c \beta \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\beta k^2}{E_k} \left[ 1 - 2n_F(E_k) + 2n_F^2(E_k) \right] + 2\beta \frac{m^2}{E_k} n_F(E_k) \left[ 1 - n_F(E_k) \right] \right\} \bigg|_{m=0} \\
= 2N_f N_c \beta \int \frac{d^3k}{(2\pi)^3} \left[ 1 - 2n_F(k) + 2n_F^2(k) \right]. \tag{17} \]

Obviously (13) and (17) do not agree. For, in general, the static susceptibility, given in (13) as the second derivative of the thermodynamical potential, does not coincide with the static limit of the dynamical susceptibility in (17) which satisfies the fluctuation-dissipation theorem. There is an exception: if the static susceptibility were associated with a conserved quantity such as the (quark) number density, the conservation law would ensure the equality of the two different quantities \([2,15]\). Thus, as shown in Refs. \([2,7]\), the quark number susceptibility can be derived in two different ways as the imaginary part of the self-energy is proportional to \(\delta(\omega)\) implying a conserved number density. However, the static chiral susceptibility is given by the derivative of the scalar/chiral density in (13) which obeys no conservation law, so it does not coincide with the static limit of the dynamic susceptibility in (17) as represented by the fluctuation-dissipation theorem since the imaginary part of the self energy in the static limit is not proportional to \(\delta(\omega)\).

Note that, in contrast to the static chiral susceptibility, there is no zero temperature contribution as there is an overall factor \(\beta\). The term containing no distribution function in (17) shows a cubic ultraviolet divergence.

**IV. HTL APPROACH**

Here, we want to calculate the chiral susceptibility beyond the free quark approximation by taking into account the in-medium properties of quarks in a QGP. In the weak coupling limit \((g \ll 1)\), a consistent method is to use the HTL-resummed quark propagator and HTL quark-meson vertex\(^1\) if the quark momentum is soft \(\sim gT\). However,

\(^1\)In the case of a scalar meson there is no HTL contribution to the quark-meson vertex \([16]\).
in the following we will use the HTL-resummed quark propagator also for hard momenta, in order to consider, at least to some extent, non-perturbative features of the QGP such as effective quark masses and Landau damping. This strategy is in the same spirit as in the case of the quark number susceptibility [7] and the meson correlators [16], although this approach might not be consistent in the weak coupling limit [17]. However, since we do not want to restrict to the weak coupling limit, but consider temperatures close to $T_c$, we do not aim at a complete leading order calculation, using here the HTL Green functions only for including medium effects in a gauge invariant way.

Another, maybe more convincing, reason for studying the chiral susceptibility in the HTL approximation is that this approach has also been used for the free energy of the QGP, where different implementations of the HTL method and their validity have been discussed. The chiral susceptibility may serve as another test for these questions.

In the rest frame of the medium the most general expression for the fermion self energy in the chiral limit of vanishing current mass has the form [18]

$$\Sigma(K) = aK - b\gamma_0,$$

where the scalar quantities $a$ and $b$ are functions of energy $k_0$ and the magnitude $k$ of the three momentum. The HTL-resummed quark propagator, $S^*(k_0, k)$, can be expressed using the helicity representation in the massless case as [19]

$$S^*(k_0, k) = \frac{\gamma^0 - \hat{k}\gamma^\perp}{2D_+(K)} + \frac{\gamma^0 + \hat{k}\gamma^\perp}{2D_-(K)}$$

with

$$D_\pm(k_0, k) = (-k_0 \pm k)(1 + a) - b$$

$$= -k_0 \pm k + \frac{m_q^2}{k} \left[ Q_0 \left( \frac{k_0}{k} \right) + Q_1 \left( \frac{k_0}{k} \right) \right],$$

where the thermal quark mass is given by $m_q = g(T)T/\sqrt{6}$, and $Q_n(y)$ is the Legendre function of second kind. The zeros $\omega_\pm(k)$ of $D_\pm(K)$ describe the two dispersion relations of collective quark modes in a thermal medium [19]. Furthermore, the HTL-resummed quark propagator acquires a cut contribution below the light cone ($k_0^2 < k^2$) as the HTL quark self-energy, contained in $D_\pm(K)$, has a non-vanishing imaginary part, which can be related to Landau damping for spacelike quark momenta resulting from interactions of the quarks with gluons of the thermal medium. For the coupling constant entering $m_q(T)$, $\alpha_s = g^2/(4\pi)$, we choose the two-loop result of the temperature dependent running coupling$^2$ [9]

$$\alpha_s(\mu_4) = \frac{4\pi}{\beta_0 \mu_4} \left[ 1 - \frac{2\beta_1}{\beta_0^2} \log \frac{\tilde{L}}{\mu_4^2} \right],$$

where $\beta_0 = (11 - \frac{2}{3}N_f), \beta_1 = (51 - \frac{19}{3}N_f)$, $\tilde{L} = \log \left( \frac{\mu_4^2}{\Lambda_{QCD}^2} \right)$ and $\mu_4 \sim T$ is the renormalization scale for the running coupling constant. It should also be noted that the HTL-resummed propagator is chirally symmetric in spite of the appearance of an effective quark mass [18].

The HTL-spectral function of $1/D_\pm(K)$, $\rho_\pm = -\text{Im}(1/D_\pm)/\pi$, can be written as [19]

$$\rho_\pm(k_0, k) = \frac{k_0^2 - k^2}{2m_q^2} \left[ \delta(k_0 - \omega_\pm) + \delta(k_0 + \omega_\pm) \right] + \beta_\pm(k_0, k) \Theta(k^2 - k_0^2)$$

with

$$\beta_\pm(k_0, k) = -\frac{m_q^2}{2} \left[ k(-k_0 \pm k) + m_q^2 \left( \pm 1 - \frac{k_0 - k}{k_0 + k} \log \frac{k_0 + k}{k_0 - k} \right) \right]^2 + \left[ \pm m_q^2 + k_0 - k \right]^2,$$

where the first term describing the quasiparticle dispersion in (22) is due to the poles of the propagator and the second term containing Landau damping corresponds to the cut contribution for spacelike quark momenta.

$^2$In this way, we have combined an infrared (HTL) and an ultraviolet (renormalization group) resummation scheme phenomenologically. A systematic combination of these techniques is not available yet.
A. Static Susceptibility

Now, we compute the chiral condensate \( \langle q\bar{q} \rangle \) from the tadpole diagram in Fig.1, where we use the effective HTL quark propagator, yielding

\[
\langle q\bar{q} \rangle = -N_c N_f T \sum_{k_0} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ S^* (K) \right].
\]  

(24)

A similar calculation has been done in the case of the quark contribution to the free energy by Andersen et al. \[9\]. According to (4), but in contrast to the HTL approximation discussed above, we have to take into account a non-vanishing current quark mass. In the weak coupling limit (HTL approximation) there are two possibilities: 1) \( m \geq T \), i.e., \( m \gg \Sigma_{\text{HTL}} \sim gT \) and 2) \( m < gT \). In the first case, \( \Sigma_{\text{HTL}} \) can be neglected in the effective quark propagator compared to the hard scale \( m \), implying \( S^* (K) = S (K) \), which leads to the free susceptibility. In the second case, \( m \) can be neglected in the calculation of the HTL quark self energy, requiring hard loop momenta, i.e., \( \Sigma = \Sigma_{\text{HTL}} (m = 0) \). Then, the effective HTL quark propagator becomes

\[
S^* (K) = \frac{1}{K(1+a) + b\gamma_0 - m} = \frac{K(1+a) + b\gamma_0 + m}{D - m^2}
\]  

(25)

with

\[
D (K) = K^2 (1+a)^2 + 2k_0b(1+a) + b^2 = D_+ (K)D_- (K),
\]  

(26)

where \( D_{\pm} (K) \) is given by the massless HTL-approximation in (20). Note that the current quark mass, \( m \leq gT \), is of the same order as the HTL contributions \( a \) and \( b \) and cannot be neglected therefore.

Summing over \( k_0 \), we obtain

\[
\langle q\bar{q} \rangle = 4mN_f N_c \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} dx \rho (x, k, m) [1 - n_F (x)]
\]  

(27)

where the spectral function corresponding to the massive HTL propagator is given by

\[
\rho (x, k, m) = -\frac{1}{\pi} \text{Im} \frac{1}{D - m^2}.
\]  

(28)

Following (4), the static chiral susceptibility in HTL-approximation is found as

\[
\chi_c^h (T) = -\frac{\partial \langle q\bar{q} \rangle}{\partial m} = -4N_f N_c \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\omega^2 (k) - k^2}{2m^2_q} \text{Re} D_{\pm}^{-1} (\omega_+ , k) [1 - 2n_F (\omega_+)] + \frac{\omega^2 (k) - k^2}{2m^2_q} \text{Re} D_{\pm}^{-1} (\omega_- , k) [1 - 2n_F (\omega_-)]
\]  

\[+ \int_0^k dx \left[ \beta_+ (x, k) \text{Re} D_{\pm}^{-1} (x, k) + \beta_- (x, k) \text{Re} D_{\pm}^{-1} (x, k) \right] [1 - 2n_F (x)] \right\},
\]  

(29)

where

\[
\text{Re} D_{\pm}^{-1} (k_0, k) = \frac{k^2 (k_0 \pm k) + m^2_q \left( \pm 1 - \frac{k_0 - k}{2k} \log \left| \frac{k + k_0}{k - k_0} \right| \right)}{k (k_0 \pm k) + m^2_q \left( \pm 1 - \frac{k_0 - k}{2k} \log \left| \frac{k + k_0}{k - k_0} \right| \right)^2 + \left( \frac{k^2 - k_0^2}{2k} \theta (k^2 - k_0^2) \right)^2}
\]  

(30)

In the infinite temperature limit, the first term in (29) reduces to the free static susceptibility given in (13) whereas the second and third terms vanish. The integrals without the distribution functions in the first and third term in (29) are ultraviolet divergent (quadratically and logarithmically) because of the asymptotic behavior of the quasiparticle dispersion relation \( \omega_+ (k) \) and \( \text{Re} D_{\pm}^{-1} \). However, the second term is convergent since the plasmino dispersion relation, \( \omega_- (k) \), approaches \( k \) exponentially at large \( k \). The quadratic divergence is temperature independent and can be removed by subtracting the zero temperature part, coinciding with dimensional regularization, or by subtracting the free chiral susceptibility (13) as discussed below. The remaining logarithmic divergence turns out to be temperature dependent as in the case of the gluonic part of the free energy [9]. This might be an artifact of the approximation.
used here and might vanish if one goes beyond the one-loop HTL approximation. Here we will use dimensional regularization to remove this divergence following Ref. [9].

First we subtract the free susceptibility given in (13) from the HTL one in (29):

\[
\chi^h_c - \chi^f_c = \frac{2N_fN_c}{\pi^2} \int_0^\infty dk k n_F(k) + \frac{4N_fN_c}{\pi^2} \int_0^\infty dk k^2 \frac{(\omega_+^2 - k^2)}{2m_q^2} \text{Re} D^{-1}_-(\omega_+, k) n_F(\omega_+)
\]

\[
- \frac{2N_fN_c}{\pi^2} \int_0^\infty dk k^2 \frac{(\omega_+^2 - k^2)}{2m_q^2} \text{Re} D^+_+(\omega_-, k) (1 - 2n_F(\omega_-))
\]

\[
- \frac{N_fN_c}{\pi^2} \int_0^\infty dk \left[ k + 2k^2 \frac{\omega_+^2 - k^2}{2m_q^2} \text{Re} D^{-1}_+(\omega_+, k) \right]
\]

\[
+ \frac{4N_fN_c}{\pi^2} \int_0^\infty dk k^2 \int_0^k dx \left[ \beta_+(x, k) \text{Re} D^{-1}_-(x, k) n_F(x) + \beta_-(x, k) \text{Re} D^+_-(x, k) n_F(x) \right]
\]

\[
- \frac{2N_fN_c}{\pi^2} \int_0^\infty dk k^2 \int_0^k dx \left[ \beta_+(x, k) \text{Re} D^{-1}_-(x, k) + \beta_-(x, k) \text{Re} D^+_-(x, k) \right].
\]

The integrals in 4th and 6th terms are, respectively, logarithmically ultraviolet divergent. Using a momentum cutoff \(k < \Lambda\) and energy cutoff \(x < \Lambda\), the 4th and 6th terms can be seen to be proportional to \(m_q^2 \log \Lambda\). This is in contrast to the quark contribution to the free energy but similar to the case of the gluonic part of the free energy where these terms do not cancel [9]. Dimensional regularization will then replace the remaining logarithmic divergences by poles in \(d - 3\), which finally will require a counter term for cancellation [9].

As pointed out in Ref. [20] it is important to use also \(d\)-dimensional expressions for the HTL propagator. Here we will follow Ref. [21,22] and expand the chiral susceptibility up to order \(m_q^2\). The angular average leading to the Legendre functions in (20) can be generalised to \(d = 3 - 2\epsilon\) spatial dimensions in Minkowski space as

\[
T_d(K) = \frac{w(\epsilon)}{2} \int_{-1}^1 dc \ (1 - c^2)^{-\epsilon} \frac{k^2_0}{k^2_0 - k^2 c^2},
\]

where the weight function \(w(\epsilon)\) is

\[
w(\epsilon) = \frac{\Gamma\left(\frac{4}{2} - \epsilon\right)}{\Gamma\left(\frac{4}{2}\right)\Gamma\left(1 - \epsilon\right)}.
\]

Eq. (32) can also be written as [21]

\[
T_d(K) = w(\epsilon) \int_0^1 dc \ (1 - c^2)^{-\epsilon} \frac{k^2_0}{(k^2_0 - k^2 c^2)} = \left\langle \frac{k^2_0}{(k^2_0 - k^2 c^2)} \right\rangle_\epsilon.
\]

Now, it is convenient to define the functions \(A_0(K)\) and \(A_S(K)\) as

\[
A_0 = k_0 - \frac{m_q^2}{k_0} T_d(K),
\]

\[
A_S = k + \frac{m_q^2}{k} (1 - T_d(K)),
\]

which are related to the HTL-resummed quark propagator (20) by

\[
D_\pm (k_0, k) = - (A_0 \mp A_S).
\]

Using (4), (24), and (25) the chiral susceptibility in one-loop HTL order reads

\[
\chi^h_c(T) = -4N_fN_c \sum_{\{K\}} \frac{1}{D(K)},
\]

where [22]

\[
\sum_{\{K\}} = \left\langle \frac{\epsilon^3 \mu^2}{4\pi} \right\rangle (\epsilon^{-\epsilon}) \sum_{k_0} \int \frac{d^{3-2\epsilon} p}{(2\pi)^{3-2\epsilon}}.
\]
with the arbitrary renormalization scale $\mu$ of the ($\overline{\text{MS}}$) renormalization scheme and Euler’s constant $\gamma$. Expanding $\chi^h(T)$ in a power series of $m_q^2/T^2$, and keeping terms up to second order, we obtain

$$\chi^h(T) = -4N_fN_c\sum_{\{K\}} \left[ \frac{1}{K^2} + 2m_q^2\frac{1}{K^4} + m_q^4 \left( \frac{4}{K^6} + \frac{1}{k^2K^4} - \frac{2}{k^2K^4}T_d(K) + \frac{1}{k^2k_0^2K^2T_d'(K)} \right) \right]. \quad (39)$$

Using the sum-integrals evaluated in Appendix A, we end up with

$$\chi^h(T) = -\frac{N_fN_c}{6}T^2 \left[ 1 + \frac{3}{\pi^2} \left( \frac{\mu}{4\pi T} \right)^2 \left( \frac{1}{\epsilon} + 2\gamma + 4\log 2 \right) \frac{m_q^2}{T^2} + \frac{21\zeta(3)}{4\pi^4} \left( \frac{5 + 8\log 2 - \frac{\pi^2}{6}}{7} \right) \frac{m_q^4}{T^4} \right]. \quad (40)$$

The chiral susceptibility is logarithmic divergent at order $m_q^2$, but finite at order $m_q^4$. We choose a counter term within the minimal subtraction renormalization scheme just to cancel the pole in $1/\epsilon$:

$$\Delta \chi^c_{\text{counter}} = \frac{N_fN_c}{2\pi^2\epsilon}m_q^2. \quad (41)$$

The complete expression for chiral susceptibility is then given by

$$\chi^h(T) = \chi^f_c(T) \left[ 1 + \frac{6}{\pi^2} \left( \log \frac{\mu}{4\pi T} + \gamma + 2\log 2 \right) \frac{m_q^2}{T^2} + \frac{21\zeta(3)}{4\pi^4} \left( \frac{5 + 8\log 2 - \frac{\pi^2}{6}}{7} \right) \frac{m_q^4}{T^4} \right]. \quad (42)$$

Fig 3: The HTL static susceptibility (free susceptibility subtracted) as a function of temperature for $\Lambda_{\overline{\text{MS}}} = 300$ MeV, and $N_f = 2$ with the choices of the renormalization scale $\mu = 2\pi T$ and $4\pi T$ for up to order $m_q^2$ (left) and up to order $m_q^4$ (right).

Note that the $m_q^2$-term is the first finite $\alpha_s$-correction to the free chiral susceptibility, since the two-loop contribution in usual perturbative QCD diverges for $m = 0$.

The quantity $|\chi^h_c(T) - \chi^f_c(T)|/T^2$ as a function of $T$ is shown in Fig.3 for two different choices of $\mu$, where we have also chosen $\mu_k = \mu$ in $\alpha_s$ entering $m_q$. Obviously the $m_q^4$ contribution is only a small correction. Whereas in the $m_q^2$ contribution only the effective quark mass (pole contribution) enters, there is a Landau damping contribution coming from $T_d$ in the $m_q^4$ term (see (39)). To understand the behavior of the susceptibility in Fig.3 we ascertain that the quantity $|\chi^h - \chi^f|/T^2$, plotted in Fig.3, depends on the temperature only via the temperature dependent running coupling constant, i.e. $|\chi^h - \chi^f|/T^2 = F[g(T)]$. This can be seen by scaling the variables $k$ and $x$ in the integrals of (29) by $m_q$. It holds also after dimensional regularization, if a renormalization scale proportional to the
temperature is adopted in the latter case (see e.g. (42)). This dependence has also been verified numerically, where a constant, temperature independent chiral susceptibility has been observed, if a constant, temperature independent coupling has been employed. In other words, the decrease of the susceptibility with temperature in Fig.3 is entirely caused by the temperature dependence of the coupling constant. Due to asymptotic freedom, \( g(T \to \infty) \to 0 \), the HTL susceptibility reduces to the free one in the infinite temperature limit, i.e. \( F[g = 0] = 0 \).

In lattice simulations a peak around the critical temperature associated with chiral symmetry restoration has been observed. In contrast to lattice QCD the HTL approximation, although taking into account medium effects in the plasma due to interactions, does not contain any physics related to the chiral restoration, which is a truly non-perturbative effect. In Fig.3 we observe a strong increase of the magnitude of the HTL chiral susceptibility towards low temperatures, similar as in lattice simulations above \( T_c \). However, this increase is caused only by the fact that for our choice of the renormalization constant \( \mu \) the HTL result is above the free one, \( |\chi_{\|}^c| > |\chi_{\parallel}^c| \), and that the temperature dependent coupling increases at smaller temperatures. If we choose a small enough value for \( \mu \), \( |\chi_{\|}^c| < |\chi_{\parallel}^c| \) holds according to (42), and the HTL susceptibility \( |\chi_{\|}^c| \) approaches the free susceptibility \( |\chi_{\parallel}^c| \) from below at large temperatures. This indicates the strong sensitivity to the choice of the regularization procedure and the renormalization constant similar as in the case of the free energy [21,22]. Hence we understand the difference between the lattice and the HTL approach qualitatively but the different ultraviolet behavior of the HTL and the lattice result, which does not contain a temperature dependent logarithmic divergence [23], renders a direct, quantitative comparison of the both approaches difficult.

**B. Dynamic Susceptibility in Static Limit**

Here, we need to calculate the imaginary part of the self energy diagram given in Fig.2, where both bare quark propagators are replaced by effective HTL quark propagators. It is sufficient to restrict to the massless case, i.e., (19).

Apart from constant factors the dynamic chiral susceptibility agrees with the scalar meson correlator at \( \tau = \beta \) [compare (2.3) of Ref. [16] with (8)]. Hence, we will not repeat the derivation here but just present the final results.

As in the case of the meson correlator [16], the dynamic susceptibility can be decomposed into pole-pole, pole-cut and cut-cut contributions,

\[
\chi_c(T) = \chi_{\|}^{pp}(T) + \chi_{\|}^{pc}(T) + \chi_{\|}^{cc}(T).
\]  

The pole-pole contribution reads

\[
\chi_{\|}^{pp}(T) = 2N_cN_f\beta \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{[\omega_c^2(k) - k^2]^2}{4m_q^4} \left[ 1 - 2n_F(\omega_+) + 2n_F^2(\omega_+) \right] 
+ \frac{[\omega_c^2(k) - k^2]^2}{4m_q^4} \left[ 1 - 2n_F(\omega_-) + 2n_F^2(\omega_-) \right] 
+ \frac{[\omega_c^2(k) - k^2][\omega_c^2(k) - k^2]}{2m_q^4} \left[ n_F(\omega_+)[1 - n_F(\omega_-)] + n_F(\omega_-)[1 - n_F(\omega_+)] \right] \right\}.
\]  

In the infinite temperature limit the plasmino mode decouples from the medium, causing the second and third terms to vanish, whereas the first term reduces to the free susceptibility given in (17). The pole-cut contribution is given as

\[
\chi_{\|}^{pc}(T) = \frac{2N_cN_f\beta}{m_q^2} \int_0^\infty \frac{d\omega}{2} \coth \left( \frac{\beta\omega}{2} \right) \int \frac{d^3k}{(2\pi)^3} 
\left\{ \Theta[k^2 - (\omega - \omega_+)^2]n_F(\omega - \omega_+)n_F(\omega_+)\beta_+(\omega - \omega_+, k)(\omega_+^2 - k^2) 
+ \Theta[k^2 - (\omega - \omega_-)^2]n_F(\omega - \omega_-)n_F(\omega_-)\beta_-(\omega - \omega_-, k)(\omega_-^2 - k^2) 
+ \Theta[k^2 - (\omega + \omega_+)^2]n_F(\omega + \omega_+)n_F(\omega_+)\beta_+(\omega + \omega_+, k)(\omega_+^2 - k^2) 
+ \Theta[k^2 - (\omega + \omega_-)^2]n_F(\omega + \omega_-)n_F(\omega_-)\beta_-(\omega + \omega_-, k)(\omega_-^2 - k^2) \right\},
\]  

and the cut-cut contribution as

\[
\chi_{\|}^{cc}(T) = 2N_cN_f\beta \int_0^\infty \frac{d\omega}{2} \coth \left( \frac{\beta\omega}{2} \right) \int \frac{d^3k}{(2\pi)^3} \int_{-k}^k dx n_F(x)n_F(\omega - x)\Theta(k^2 - (x - \omega)^2) 
\times \left[ \beta_+(x, k)\beta_+(\omega - x, k) + \beta_-(x, k)\beta_-(\omega - x, k) \right].
\]
The dynamic chiral susceptibility in the 1-Loop HTL approximation, considered here, has a cubic ultraviolet divergence, which can be removed, for example, by subtracting the free result (17). We will not discuss the dynamic susceptibility further on as it is not directly related to QCD lattice results [3]. We will just mention here that the (ultraviolet divergent) dynamic susceptibility, (44) to (46), normalized to the free case (17) is equal to unity, as can be seen from the scalar meson correlator at $\tau = \beta$ discussed in Ref. [16].

V. CONCLUSIONS

In the present paper we have considered the chiral susceptibility of the quark-gluon plasma within lowest order perturbation theory (free case) and the HTL approximation. We have shown that there are two possible definitions of the chiral susceptibility leading to different results due to a missing conservation law associated with the chiral susceptibility in contrast to the quark number susceptibility [7]. Using the definition based on the derivative of the chiral condensate with respect to the current quark mass, we have computed this static chiral susceptibility within perturbative QCD and using an effective HTL resummed quark propagator. In the case of the usual perturbation theory we found that the chiral susceptibility cannot be computed beyond the leading order (free case) as it diverges due to setting the bare quark mass equal to zero at the end. Similar as in the case of the free energy [9] but in contrast to the quark number susceptibility [7], the chiral susceptibility using the HTL approach is ultraviolet divergent. Even after subtracting the free susceptibility or the zero temperature part, a logarithmic temperature dependent singularity remains as in the case of the gluonic part of the free energy [9]. Applying dimensional regularization the HTL chiral susceptibility can be larger or smaller than the free one depending on the choice of the renormalization constant $\mu$. Here we expanded the chiral susceptibility up to order $m^4_q$. The $m^2_q$ contribution is the lowest order correction to the free susceptibility, which has no $m = 0$ divergence. The temperature dependence of the HTL chiral susceptibility is entirely determined by the temperature dependence of the running coupling constant that enters through the medium effects (quasiparticle quark mass, Landau damping) contained in the HTL approximation. No indication of a peak at the critical temperature is found, as seen in lattice QCD simulations [3]. This indicates that this peak is due to truly non-perturbative effects such as chiral restoration, not included in the HTL approach.

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APPENDIX A

1. One-loop sum-integrals

The one-loop fermionic sum-integrals necessary for our purpose in (39) are:

\[
\sum_{\{K\}} \frac{1}{K^2} = \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \frac{T^2}{24} \left[ 1 + \left( 2\gamma + 2 \log \pi - 2 \frac{\zeta(2)}{\zeta(2)} \right) \epsilon + O(\epsilon^2) \right]. \tag{A.1}
\]

\[
\sum_{\{K\}} \frac{1}{k^4 K^2} = \sum_{\{K\}} \frac{k^2}{k^6 K^2} = \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \frac{7\zeta(3)}{(2\pi)^4 T^2} \left[ 1 + \left( 2 + \frac{8}{7} \log 2 + 2 \frac{\zeta(3)}{\zeta(3)} \right) \epsilon + O(\epsilon^2) \right]. \tag{A.2}
\]

\[
\sum_{\{K\}} \frac{1}{(K^2)^2} = \frac{1}{(4\pi)^2} \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \left[ \frac{1}{\epsilon} + 2(\gamma + 2 \log 2) + \left( \frac{\pi^2}{4} + 4\gamma \log 2 + 4 \log^2 2 + 4\gamma_1 \right) \epsilon + O(\epsilon^2) \right]. \tag{A.3}
\]

\[
\sum_{\{K\}} \frac{1}{k^2 K^4} = \left( \frac{\mu}{4\pi T} \right)^{2\epsilon} \frac{7\zeta(3)}{2(2\pi)^4 T^2} \left[ 1 + \left( 4 + \frac{16}{7} \log 2 + 2 \frac{\zeta(3)}{\zeta(3)} \right) \epsilon + O(\epsilon^2) \right]. \tag{A.4}
\]
The one-loop sum-integrals are given in (A.2) and (A.4). The angular averages are performed using (34), leading to
\[
\left\langle \frac{1}{1-c^2} \right\rangle_c = -\frac{1}{2\epsilon} + 1, \quad (A.7)
\]
\[
\left\langle \frac{c^2 - \epsilon^{3+2\epsilon}}{(1-c^2)^2} \right\rangle_c = -\frac{1}{4\epsilon} + \left(\frac{1}{4} - \log 2\right) + \left(\frac{3}{4} - \frac{\pi^2}{6} + 2\log 2 - \log^2 2 \right) \epsilon + O(\epsilon^2). \quad (A.8)
\]
Once these expressions are combined, the pole in $1/\epsilon$ cancels and we obtain a finite result:
\[
\sum\limits_{\{K\}} T_d(K) = \frac{7\zeta(3)}{4(2\pi)^4T^2} \left[\frac{20}{7} \log 2 - 1\right]. \quad (A.9)
\]
Following the same strategy, the remaining sum-integral in (39) involving $T^2_d(K)$ can be written as
\[
\sum\limits_{\{K\}} T^2_d(K) = \frac{7\zeta(3)}{4(2\pi)^4T^2} \left[\frac{20}{7} \log 2 - \frac{\pi^2}{12}\right]. \quad (A.11)
\]
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