Abstract. We show that every block of the category of cuspidal generalized weight modules with finite dimensional generalized weight spaces over the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is equivalent to the category of finite dimensional $\mathbb{C}[\{t_1, t_2, \ldots, t_n\}]$-modules.

1. Introduction and description of the results

We fix the ground field to be the complex numbers. Fix $n \in \{2, 3, \ldots\}$ and consider the symplectic Lie algebra $\mathfrak{sp}_{2n} =: \mathfrak{g}$ with a fixed Cartan subalgebra $\mathfrak{h}$ and root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\Delta$ denotes the corresponding set of roots. For a $\mathfrak{g}$-module $V$ and $\lambda \in \mathfrak{h}^*$ set

$$V_\lambda := \{v \in V : h \cdot v = \lambda(h)v \text{ for any } h \in \mathfrak{h}\},$$

$$V^\lambda := \{v \in V : (h - \lambda(h))^k \cdot v = 0 \text{ for any } h \in \mathfrak{h} \text{ and } k \gg 0\}.$$

A $\mathfrak{g}$-module $V$ is called

- **weight** provided that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$;
- **generalized weight** provided that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$;
- **cuspidal** provided that for any $\alpha \in \Delta$ the action of any nonzero element from $\mathfrak{g}_\alpha$ on $V$ is bijective.

If $V$ is a generalized weight module, then the set $\{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$ is called the support of $V$ and is denoted by supp($V$).

Denote by $\hat{\mathcal{C}}$ the full subcategory in $\mathfrak{g}$-mod which consist of all cuspidal generalized weight modules with finite-dimensional generalized weight spaces, and by $\mathcal{C}$ the full subcategory of $\hat{\mathcal{C}}$ consisting of all weight modules. Understanding the categories $\mathcal{C}$ and $\hat{\mathcal{C}}$ is a classical problem in the representation theory of Lie algebras. The first major step towards the solution of this problem was made in [Mat], where all simple objects in $\hat{\mathcal{C}}$ were classified. In [BKLM] it was shown that the category $\hat{\mathcal{C}}$ is semi-simple, hence completely understood. The aim of the present note is to describe the category $\hat{\mathcal{C}}$.

Apart from $\mathfrak{sp}_{2n}$, cuspidal weight modules with finite dimensional weight spaces exist only for the Lie algebra $\mathfrak{sl}_n$ ([Fe]). In the latter case, simple objects in the corresponding category $\hat{\mathcal{C}}$ are classified in [Mat], the category $\mathcal{C}$ is described in [GS], see also [MS], and the category $\hat{\mathcal{C}}$ is described in [MS]. Taking all these results into account, the present paper completes the study of cuspidal generalized weight modules with finite dimensional generalized weight spaces over semi-simple finite-dimensional Lie algebras.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. The action of $Z(\mathfrak{g})$ on any object from $\hat{\mathcal{C}}$ is locally finite. Using this and the
standard support arguments gives the following block decomposition of \( \hat{\mathcal{C}} \):
\[
\hat{\mathcal{C}} \cong \bigoplus_{\chi : \mathfrak{z}(g) \to \mathbb{C}} \hat{\mathcal{C}}_{\chi, \xi},
\]
where \( \hat{\mathcal{C}}_{\chi, \xi} \) consists of all \( V \) such that \( \text{Supp}(V) \subset \xi \) and \( (z - \chi(z))^k \cdot v = 0 \) for all \( v \in V, \, z \in \mathbb{Z}(g) \) and \( k \gg 0 \). Set \( C_{\chi, \xi} := \mathcal{C} \cap \hat{\mathcal{C}}_{\chi, \xi} \).

From [Mat, Section 9] it follows that each nontrivial \( \hat{\mathcal{C}}_{\chi, \xi} \) contains a unique (up to isomorphism) simple object, in particular, \( \hat{\mathcal{C}}_{\chi, \xi} \) is indecomposable, hence a block. From this and [BKLM] we thus get that every nontrivial block \( C_{\chi, \xi} \) is equivalent to the category of finite-dimensional \( \mathbb{C} \)-modules. Our main result is the following:

**Theorem 1.** Every nontrivial block \( \hat{\mathcal{C}}_{\chi, \xi} \) is equivalent to the category of finite-dimensional \( \mathbb{C}[[t_1, t_2, \ldots, t_n]] \)-modules.

To prove Theorem 1 we use and further develop the technique of extension of the module structure from a Lie subalgebra, originally developed in [MS] for the study of categories of singular and non-integral cuspidal generalized weight \( \mathfrak{sl}_n \)-modules. The proof of Theorem 1 is given in Section 4. In Section 2 we recall the standard reduction to the case of the so-called simple completely pointed modules (i.e. simple weight cuspidal modules for which all nontrivial weight spaces are one-dimensional) and a realization of such modules using differential operators. In Section 3 we define a functor from the category of finite dimensional \( \mathbb{C}[[t_1, t_2, \ldots, t_n]] \)-modules to any block \( \hat{\mathcal{C}}_{\chi, \xi} \) containing a simple completely pointed module. In Section 4 we prove that this functor is an equivalence of categories. In Section 5 we present some consequences of our main result, in particular, we recover the main result from [BKLM] stated above.

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2. Completely pointed simple cuspidal weight modules

A weight \( \mathfrak{g} \)-module \( V \) is called pointed provided that \( \dim V_{\lambda} = 1 \) for some \( \lambda \in \mathfrak{h}^* \). If \( V \) is a pointed simple cuspidal weight \( \mathfrak{g} \)-module, then, obviously, all nontrivial weight spaces of \( V \) are one-dimensional, in which case one says that \( V \) is completely pointed (see [BKLM]). It is enough to consider blocks with completely pointed simple modules because of the following:

**Lemma 2.** All nontrivial blocks of \( \hat{\mathcal{C}} \) are equivalent.

**Proof.** In the case of the category \( \mathcal{C} \) this is proved in [BKLM] Lemma 2]. The same argument works in the case of the category \( \hat{\mathcal{C}} \) as well. \( \square \)

Let us recall the explicit realization of completely pointed simple cuspidal modules from [BL]. Denote by \( W_n \) the \( n \)-th Weyl algebra, that is the algebra of differential operators with polynomial coefficients in variables \( x_1, x_2, \ldots, x_n \). The algebra \( W_n \) is generated by \( x_i \) and \( \frac{\partial}{\partial x_i}, \, i = 1, \ldots, n \), which satisfy the relations \( [\frac{\partial}{\partial x_i}, x_j] = \delta_{i,j} \). Let \( e_1, e_2, \ldots, e_n \) be the vectors of the standard basis in \( \mathbb{C}^n \). Identify \( \mathbb{C}^n \) with \( \mathfrak{h}^* \) such that \( \Delta \) becomes the following standard root system of type \( C_n \):
\[
\{ \pm(e_i \pm e_j) : 1 \leq i < j \leq n \} \cup \{ \pm 2e_i : 1 \leq i \leq n \}.
\]
Then \( H = H_n = \{2\epsilon_1, \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n-1}\} \) is a basis of \( \Delta \). Fix a basis of \( g \) of the form

\[
C := \{X_{\pm \epsilon_i, \pm \epsilon_j} : 1 \leq i < j \leq n\} \cup \{X_{\pm 2\epsilon_i} : i = 1, 2, \ldots, n\} \cup \{H_a : \alpha \in H\}
\]

such that the following map defines an injective Lie algebra homomorphism from \( g \) to the Lie algebra associated with \( W_n \):

\[
\begin{align*}
X_{\epsilon_i, -\epsilon_j} &\mapsto x_i \frac{\partial}{\partial x_j}, & 1 \leq i \neq j \leq n; \\
X_{\epsilon_i, +\epsilon_j} &\mapsto x_i x_j, & i, j = 1, 2, \ldots, n; \\
X_{-\epsilon_i, -\epsilon_j} &\mapsto \frac{\partial}{\partial x_i} x_j, & i, j = 1, 2, \ldots, n; \\
H_{\epsilon_{i+1}, -\epsilon_i} &\mapsto x_{i+1} \frac{\partial}{\partial x_{i+1}} - x_{i} \frac{\partial}{\partial x_i}, & i = 1, 2, \ldots, n-1; \\
H_{2\epsilon_i} &\mapsto \frac{1}{2} \left( x_{i} \frac{\partial}{\partial x_{i}} + \frac{\partial}{\partial x_{i}} x_{i} \right).
\end{align*}
\]

Set \( B := \{(b_1, b_2, \ldots, b_n) \in \mathbb{Z}^n : b_1 + b_2 + \cdots + b_n \in 2\mathbb{Z}\} \).

For \( \mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \) define \( N(\mathbf{a}) \) to be the linear span of \( \mathbf{x}^b = x_{a_1+b_1} x_{a_2+b_2} \cdots x_{a_n+b_n} : b \in B \).

We first define an action of the elements from \( C \) on \( N(\mathbf{a}) \) using the formulas from (1) as follows:

\[
\begin{align*}
X_{\epsilon_i, -\epsilon_j} x^b &\mapsto (a_j + b_j)x^{b+\epsilon_i, -\epsilon_j}, & 1 \leq i \neq j \leq n; \\
X_{\epsilon_i, +\epsilon_j} x^b &\mapsto x^{b+\epsilon_i, +\epsilon_j}, & i, j = 1, 2, \ldots, n; \\
X_{-\epsilon_i, -\epsilon_j} x^b &\mapsto x^{b+\epsilon_i, -\epsilon_j}, & i, j = 1, 2, \ldots, n; \\
X_{2\epsilon_i} x^b &\mapsto (a_i + b_i)(a_i + b_i - 1)x^{b-2\epsilon_i}, & i = 1, 2, \ldots, n; \\
H_{\epsilon_{i+1}, -\epsilon_i} x^b &\mapsto (a_{i+1} + b_{i+1} - a_i - b_i)x^b, & i = 1, 2, \ldots, n-1; \\
H_{2\epsilon_i} x^b &\mapsto \frac{1}{2} \left( x_i x_{i} \frac{\partial}{\partial x_{i}} + \frac{\partial}{\partial x_{i}} x_{i} \right).
\end{align*}
\]

**Theorem 3 (BB).**

(i) For every \( \mathbf{a} \in \mathbb{C}^n \) formulae (2) define on \( N(\mathbf{a}) \) the structure of a completely pointed weight \( g \)-module.

(ii) If \( a_i \notin \mathbb{Z} \) for all \( i = 1, \ldots, n \), then the module \( N(\mathbf{a}) \) is simple and cuspidal.

(iii) Every completely pointed simple cuspidal \( g \)-module is isomorphic to \( N(\mathbf{a}) \) for some \( \mathbf{a} \in \mathbb{C}^n \) such that \( a_i \notin \mathbb{Z} \), \( i = 1, \ldots, n \).

3. The functor \( F \)

This section is similar to [MS Subsection 3.1]. Fix \( \mathbf{a} \in \mathbb{C}^n \) such that \( a_i \notin \mathbb{Z} \), \( i = 1, \ldots, n \). Let \( \mathcal{C}_\mathbf{a} \) denote the block of \( \mathcal{C} \) containing \( N(\mathbf{a}) \). The category \( \mathcal{C}_\mathbf{a} \) is closed under extensions. Denote by \( \mathcal{C}[[t_1, t_2, \ldots, t_n]] \)-mod the category of finite dimensional \( \mathcal{C}[[t_1, t_2, \ldots, t_n]] \)-modules. For \( V \in \mathcal{C}[[t_1, t_2, \ldots, t_n]] \)-mod denote by \( T_i \) the linear operator describing the action of \( t_i \) on \( V \). Set \( 0 = (0, 0, 0, \ldots, 0) \in B \).

For \( \mathbf{b} \in B \) consider a copy \( V^\mathbf{b} \) of \( V \). Define

\[
FV := \bigoplus_{\mathbf{b} \in B} V^\mathbf{b}.
\]

Define the action of elements from \( C \) on the vector space \( FV \) in the following way:

\[
\begin{align*}
X_{\epsilon_i, -\epsilon_j} v &\mapsto (T_j + (a_j + b_j)1)I_d v & \in V^{b+\epsilon_i, -\epsilon_j}; \\
X_{\epsilon_i, +\epsilon_j} v &\mapsto v & \in V^{b+\epsilon_i, +\epsilon_j}; \\
X_{-\epsilon_i, -\epsilon_j} v &\mapsto (T_i + (a_i + b_i)1)(T_j + (a_j + b_j)1)1 v & \in V^{b, -\epsilon_i}; \\
X_{2\epsilon_i} v &\mapsto (T_i + (a_i + b_i)1)(T_i + (a_i + b_i - 1)1)1 v & \in V^{b, 2\epsilon_i}; \\
H_{\epsilon_{i+1}, -\epsilon_i} v &\mapsto (T_{i+1} - T_i + (a_{i+1} + b_{i+1} - a_i - b_i)1)1 v & \in V^b; \\
H_{2\epsilon_i} v &\mapsto \frac{1}{2} (2T_i + (2a_i + 2b_i + 1)1)1 v & \in V^b.
\end{align*}
\]
where \( i \) and \( j \) are as in the respective row of (2). For a homomorphism \( f : V \to W \) of \( \mathbb{C}[[t_1, t_2, \ldots, t_n]] \)-modules denote by \( Ff \) the diagonally extended linear map from \( FV \) to \( FW \), i.e. for every \( b \in B \) and \( v \in V^b \) set
\[
Ff(v) = f(v) \in W^b.
\]

**(Proposition 4).** (i) Formulae (3) define on \( FV \) the structure of a \( g \)-module.
(ii) Every \( V^b \) is a generalized weight space of \( FV \). Moreover, for \( b \neq b' \) the weights of \( V^b \) and \( V^{b'} \) are different.
(iii) The module \( FV \) belongs to \( \hat{C}_a \).
(iv) Formulas (3) and (4) turn \( F \) into a functor
\[
F : \mathbb{C}[[t_1, t_2, \ldots, t_n]] \text{-mod} \to \hat{C}_a.
\]
(v) The functor \( F \) is exact, faithful and full.

**Proof.** Consider the \( g \)-module \( N(a) \) for \( a \) as above. Then, for every \( b \) the defining relations of \( g \) (in terms of elements from \( C \)), applied to \( x^b \), can be written as some polynomial equations in the \( a_i \)'s. Since (2) defines a \( g \)-module for any \( a \) (Theorem 3(ii)), these equations hold for any \( a \), that is they are actual formal identities in the \( a_i \)'s. Write now \( T_j + (a_j + b_j)1 \text{d}_V = A_j + B_j \), a sum of matrices, where \( A_j = T_j + a_j1 \text{d}_V \) and \( B_j = b_j1 \text{d}_V \). Note that all \( A_i \) and \( B_i \) commute with each other and with all \( T_j \)'s. For a fixed \( b \), the defining relations for \( g \) on \( FV \) reduce to our formal identities (in the \( A_i \)'s) and hence are satisfied. This proves claim (i).

Claim (ii) follows from the last two lines in (3) and the fact that all \( T_i \)'s are nilpotent (hence zero is the only eigenvalue).

As \( f \) commutes with all \( T_i \), the map \( Ff \) commutes with the action of all elements from \( C \) and hence defines a homomorphism of \( g \)-modules. By construction we also have \( F(f \circ f') = Ff \circ Ff' \), which implies claim (ii).

By construction, \( F \) is exact and faithful. It sends the simple one-dimensional \( \mathbb{C}[[t_1, t_2, \ldots, t_n]] \)-module to \( N(a) \) (as in this case all \( T_i = 0 \) and hence (3) gives (2)), which is an object of the category \( \hat{C}_a \) closed under extensions. Claim (iii) follows.

To complete the proof of claim (v) we are left to show that \( F \) is full. Let \( \varphi : FV \to FW \) be a \( g \)-homomorphism. Then \( \varphi \) commutes with the action of all elements from \( b \). Using claim (ii), we get that \( \varphi \) induces, by restriction, a linear map \( f : V = V^0 \to W^0 = W \). As \( f \) commutes with all \( H_i \)'s, the map \( f \) commutes with all \( T_i \)'s. As \( f \) commutes with \( H_{2i-1} \), the map \( f \) commutes with \( T_i \). It follows that \( f \) is a homomorphism of \( \mathbb{C}[[t_1, t_2, \ldots, t_n]] \)-modules. This yields \( \varphi = Ff \) and thus the functor \( F \) is full. This completes the proof of claim (v) and of the whole proposition. \( \square \)

4. **Proof of Theorem 1**

Because of Lemma 2 it is enough to fix one particular block and show there that \( F \) is an equivalence. Thus, we may assume that \( a_i + a_j \notin \mathbb{Z} \) for all \( i, j \) (in particular, \( a_i \notin \mathbb{Z} \) for all \( i \)). After Proposition 4 we are only left to show that \( F \) is dense (i.e. essentially surjective). We establish density of \( F \) by induction on \( n \). We first prove the induction step and then the basis of the induction, which is the case \( n = 2 \).

Denote by \( \lambda \) the weight of \( x^0 \in N(a) \) (see Proposition 4(iii)). Let \( M \in \hat{C}_a \). Set \( V := M_\lambda \) and denote by \( M' \) the \( a \)-module \( U(a)V \).

4.1. **Reduction to the case** \( n = 2 \). The main result of this subsection is the following:

**(Proposition 5).** If the functor \( F \) is dense for \( n = 2 \), then it is dense for any \( n \geq 2 \).
Proof. Assume that $n > 2$ and that the functor $F$ is dense in the case of the algebra $\mathfrak{sp}_{2n-2}$. We realize $\mathfrak{sp}_{2n-2}$ as the subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ corresponding to the subset $H_{-1} \subset H$ of simple roots.

Let $Y_1, Y_2, \ldots, Y_n$ be the linear operators representing the action of the elements $H_{2e_1}, H_{e_2-e_1}, H_{e_3-e_2}, \ldots, H_{e_n-e_{n-1}}$ on $V$, respectively. Set

$$
T_1 := Y_1 - \frac{1}{2}(2a_1 + 1)\text{Id}_V;
T_2 := Y_2 + T_1 - (a_2 - a_1)\text{Id}_V;
T_3 := Y_3 + T_2 - (a_3 - a_2)\text{Id}_V;
\vdots
T_n := Y_n + T_{n-1} - (a_n - a_{n-1})\text{Id}_V.
$$

(5)

The $T_i$’s are obviously pairwise commuting nilpotent linear operators.

The module $M'$ is a cuspidal generalized weight $\mathfrak{a}$-module with finite-dimensional weight spaces. Moreover, as all composition subquotients of $M$ are of the form $N(\mathfrak{a})$, all composition subquotients of $M'$ are of the form $N(\mathfrak{a})'$, the latter being a completely pointed simple cuspidal $\mathfrak{a}$-module. By our inductive assumption, the functor $F$ is dense in the case of the algebra $\mathfrak{a}$. Hence $M' \cong N' := \oplus b V^b$, where $b \in B$ is such that $b_n = 0$, and the action of $\mathfrak{a}$ on $N'$ is given by $[3]$.  

Lemma 6. There is a unique (up to isomorphism) $\mathfrak{g}$-module $Q \in \mathcal{C}_n$ such that $Q' = N'$ and which gives the linear operator $T_n$ when computed using [5].

Proof. The existence statement is clear, so we need only to show uniqueness. Assume that $Q \in \mathcal{C}_n$ is such that $Q' = N'$ and the formulae [5], applied to $Q$, produce the linear operator $T_n$. Since $a_n \notin \mathbb{Z}$, the endomorphism $T_n + (a_n + b_n)\text{Id}_V$ is invertible for all $b_n \in \mathbb{Z}$. As the action of $X_{e_n-e_{n-1}}$ on $Q$ is bijective, we can fix a weight basis in $Q$ such that both the $\mathfrak{a}$-action on $Q'$ and the action of $X_{e_n-e_{n-1}}$ on the whole of $Q$ is given by [5]. As $n > 2$, the elements $X_{\pm 2e_1}$ commute with $X_{e_n-e_{n-1}}$ and hence their action extends uniquely to the whole of $Q$ using this commutativity. Similarly for all elements $X_{\pm (e_i-e_{i-1})}$, $i < n - 1$, and for the element $X_{e_{n-2}-e_{n-1}}$. This leaves us with the elements $X_{e_n-e_{n-2}}$ and $X_{e_{n-1}-e_n}$. Note that the simple roots $e_{n-1} - e_{n-2}$ and $e_n - e_{n-1}$ corresponding to the elements $X_{e_{n-1}-e_{n-2}}$ and $X_{e_{n}-e_{n-1}}$ generate a root system of type $A_2$ (this corresponds to the algebra $\mathfrak{sl}_2$). Therefore the fact that the action of $X_{e_{n-1}-e_{n-2}}$ extends uniquely to $Q$ is proved in [MS Lemma 21], and the fact that the action of $X_{e_{n-1}-e_{n}}$ extends uniquely to $Q$ is proved in [MS Lemma 22]. This completes the proof. □

The module $FV$ obviously satisfies $(FV)' = N'$ and defines the linear operator $T_n$ when computed using [5]. Hence Lemma 6 implies $M \cong FV$. Since $M \in \mathcal{C}_n$ was arbitrary, this shows that the functor $F$ is dense, completing the proof. □

4.2. Base of the induction: some $\mathfrak{sl}_2$-theory as preparation. In this subsection we will recall (and slightly improve) some classical $\mathfrak{sl}_2$-theory. We refer the reader to [Maz] for more details. Consider the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ with standard basis

$$
e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Let $V$ be a finite-dimensional vector space and $A$ and $B$ be two commuting linear operators on $V$. For $i \in \mathbb{Z}$ denote by $V^{(i)}$ a copy of $V$ and consider the vector space $V := \oplus_{i \in \mathbb{Z}} V^{(i)}$ (a direct sum of copies of $V$ indexed by $i$). Define the actions of $e, f$, and $h$ on $V$ as follows: for $v \in V^{(i)}$ set

$$
e v := (P - i\text{Id}_V)v \in V^{(i+1)}$$
$$f v := (Q + i\text{Id}_V)v \in V^{(i-1)}$$
$$h v := (Q - P + 2i\text{Id}_V)v \in V^{(i)}.
$$

(6)
This can be depicted as follows (here right arrows represent the action of $e$, left arrows represent the action of $f$ and loops represent the action of $h$):

\[
\cdots \xrightarrow{\mathbb{P} + 2\mathbb{I}_d} V^{(-1)} \xrightarrow{Q - \mathbb{I}_d} V^{(0)} \xrightarrow{Q - P} V^{(1)} \xrightarrow{Q + 2\mathbb{I}_d} \cdots
\]

**Proposition 7.**
(i) Formulae $\mathbb{P}$ define on $V$ the structure of a generalized weight $\mathfrak{sl}_2$-module with finite dimensional generalized weight spaces.
(ii) Every cuspidal generalized weight $\mathfrak{sl}_2$-module with finite dimensional generalized weight spaces is isomorphic to $V$ for some $V$ with $P$ and $Q$ as above.
(iii) The action of the Casimir element $c := (h + 1)^2 + 4fe$ on $V$ is given by the linear operator $(P + Q + \mathbb{I}_V)^2$.
(iv) Let $C^2$ denote the natural $\mathfrak{sl}_2$-module (the unique two-dimensional simple $\mathfrak{sl}_2$-module). Then the linear operator $(c - (P + Q + 2\mathbb{I}_V))(c - (P + Q + 2\mathbb{I}_V))$ annihilates the $\mathfrak{sl}_2$-module $C^2 \otimes V$.
(v) Let $C^3$ denote the unique three-dimensional simple $\mathfrak{sl}_2$-module. Then the linear operator $(c - (P + Q + 3\mathbb{I}_V))(c - (P + Q + 2\mathbb{I}_V))(c - (P + Q + \mathbb{I}_V))^2$ annihilates the $\mathfrak{sl}_2$-module $C^3 \otimes V$.

**Proof.** The fact that $V$ is an $\mathfrak{sl}_2$-module is checked by a direct computation. That $V$ is a generalized weight module follows from the fact that the action of $h$ on $V$ preserves (by $\mathbb{P}$) each $V^i$ and hence is locally finite. Since the category of generalized weight modules is closed under extensions, to prove that $V$ has finite dimensional generalized weight spaces it is enough to consider the case when $h$ has a unique eigenvalue on $V^{(0)}$, say $\lambda$. However, in this case $h$ has a unique eigenvalue on $V^i$, namely $\lambda + 2i$, which implies that $V^{(i)}$ is finite dimensional. Claim (i) follows. To prove Claim (ii) we observe that the action of $c$ on $V^i$ is given by:

\[
(Q - P + (2i + 1)\mathbb{I}_V)^2 + 4(Q + (i + 1)\mathbb{I}_V)(P - i\mathbb{I}_V) = (P + Q + \mathbb{I}_V)^2.
\]

Claim (ii) can be found with all details in [Maz, Chapter 3].

To prove claim (iv) choose a basis $\{e_1, e_2\}$ in $V$, which gives rise to a basis $\{v_1^{(i)}, v_2^{(i)}, i \in \mathbb{Z}\}$ in $V$. Choose the standard basis $\{e_1, e_2\}$ in $C^2$. Since $he_1 = e_1$, $he_2 = -e_2$ and $h$ acts by $Q - P + 2\mathbb{I}_V$ on $V^{(i)}$, we obtain that $h$ acts by $Q - P + (2i + 1)\mathbb{I}_V$ on the vector space $W^{(i)}$ with basis

\[
\{e_1 \otimes v_1^{(i)}, \ldots, e_1 \otimes v_1^{(i)}, e_2 \otimes v_1^{(i)}, \ldots, e_2 \otimes v_1^{(i+1)}\}.
\]

We have $C^2 \otimes V \cong \oplus_{i \in \mathbb{Z}} W^{(i)}$ and one easily computes that in the above basis the actions of $e$ and $f$ on $C^2 \otimes V$ is given by the following picture:

\[
\cdots \xrightarrow{\mathbb{P} + \mathbb{I}_d} W^{(-1)} \xrightarrow{Q - \mathbb{I}_d} W^{(0)} \xrightarrow{Q + 2\mathbb{I}_d} W^{(1)} \xrightarrow{Q + \mathbb{I}_d} \cdots
\]

The action of $c$ on $W^{(0)}$ is now easily computed to be given by the linear operator

\[
G := \begin{pmatrix}
(Q - P + 2\mathbb{I}_d)^2 + 4(Q + \mathbb{I}_d)P & 4(Q + \mathbb{I}_d) \\
4P & (Q - P + 2\mathbb{I}_d)^2 + 4(Q + 2\mathbb{I}_d)(P - 4\mathbb{I}_d) + 4\mathbb{I}_d
\end{pmatrix}.
\]

The characteristic polynomial of $G$ is

\[
\chi_G(\lambda) = (\lambda - (P + Q + 2\mathbb{I}_V)^2)(\lambda - (P + Q)^2).
\]

Claim (iv) now follows from the Cayley-Hamilton theorem.

We have an isomorphism of $\mathfrak{sl}_2$-modules as follows: $C^2 \otimes C^2 \cong C^3 \otimes C$ (here $C$ is the trivial module), and hence claim (v) follows applying claim (iv) twice.
Alternatively, one could do a direct calculation (similar to the proof of (iii)). The proposition follows. □

We note that the statement of Proposition 7(iv) is a special case of a more general result of Gabriel and Drozd describing blocks of the category of (generalized) weight \( \mathfrak{sl}_2 \)-modules, in particular, simple weight \( \mathfrak{sl}_2 \)-modules (see [Dr] 7.8.16 and [Maz]). The statements of Proposition 7(iv) and (v) are \( \mathfrak{sl}_2 \)-refinements of a theorem of Kostant describing possible (generalized) central characters of the tensor product of a finite dimensional module with an infinite dimensional module ([Ko, Theorem 5.1]).

4.3. The case \( n = 2 \). Assume now that \( n = 2 \). We have \( a_1, a_2, a_1 + a_2 \notin \mathbb{Z} \). Let \( \mathfrak{a} \) denote the Lie subalgebra of \( \mathfrak{g} \) generated by \( X_{\pm(e_2-\varepsilon_1)} \). The algebra \( \mathfrak{a} \) is isomorphic to \( \mathfrak{sl}_2 \)

Let \( M \in \tilde{\mathcal{C}}_n \). Denote by \( \lambda \) the weight of \( x^0 \in N(\mathfrak{a}) \) and set \( V := M_\lambda \). Let \( Y_1 \) and \( Y_2 \) be the linear operators representing the actions of the elements \( H_{e_2-\varepsilon_1} \) and \( C := (H_{e_2-\varepsilon_1} + 1)^2 + 4X_{e_1-\varepsilon_2}X_{e_2-\varepsilon_1} \) on \( V \). The element \( C \) is a Casimir element for \( \mathfrak{a} \), in particular, the operators \( Y_1 \) and \( Y_2 \) commute. Our first observation is the following:

Lemma 8. The action of \( C \) on \( V \) is invertible and hence has a square root.

Proof. From (2) we have that \( C \) acts on \( x^0 \) by

\[
(a_2 - a_1 + 1)^2 + 4(a_2 + 1)a_1 = (a_1 + a_2 + 1)^2.
\]

Since \( a_1 + a_2 \notin \mathbb{Z} \) by our assumptions, \( x^0 \) is an eigenvector of \( C \) with a nonzero eigenvalue. As the module \( M \) has a composition series with subquotients isomorphic to \( N(\mathfrak{a}) \), the complex number \((a_1 + a_2 + 1)^2 \neq 0 \) is the only eigenvalue of \( C \) on \( V \). The claim follows. □

Consider the \( \mathfrak{a} \)-module \( M' := U(\mathfrak{a})M_\lambda \). Let \( Y'_2 \) denote any square root of \( Y_2 \), which is a polynomial in \( Y_2 \) (it exists by Lemma 8). Then \( Y'_2 \) commutes with \( Y_1 \). Set

\[
T_1 := \frac{Y'_2 - Y_1 - \text{Id}_V}{2} - a_1\text{Id}_V, \quad T_2 := \frac{Y'_2 + Y_1 - \text{Id}_V}{2} - a_2\text{Id}_V.
\]

Then \( T_1 \) and \( T_2 \) are two commuting nilpotent linear operators (it is easy to check that \( \lambda \) is the unique eigenvalue for both \( T_1 \) and \( T_2 \), hence define on \( V \) the structure of a \( \mathbb{C}[[t_1, t_2]] \)-module. The aim of this subsection is to establish an isomorphism \( FV \cong M \), which would complete the proof of Theorem 1.

Set \( R' := U(\mathfrak{a})(FV)_\lambda \). A direct computation (using (3)) shows that \( H_{e_2-\varepsilon_1} \) and \( C \) act on \( (FV)_\lambda = V^0 \) as the linear operators \( Y_1 \) and \( Y_2 \), respectively. As any cuspidal generalized weight \( \mathfrak{a} \)-module is uniquely determined by the actions of \( H_{e_2-\varepsilon_1} \) and \( C \) (see [Dr] or [Maz] 3.7 for full details), it follows that \( M' \cong R' \). The isomorphism \( FV \cong M \) now follows from the following statement:

Proposition 9. There is at most one (up to isomorphism) \( \mathfrak{g} \)-module \( R \in \tilde{\mathcal{C}}_n \) such that \( U(\mathfrak{a})R_\lambda = R' \).

Proof. Let \( R \in \tilde{\mathcal{C}}_n \) be such that \( U(\mathfrak{a})R_\lambda = R' \). We choose a weight basis in \( R \) such that the action of \( \mathfrak{a} \) on \( R' \) and the action of \( X_{2\varepsilon_1} \) on \( R \) is given by (3) (in other words these actions coincide with the corresponding actions on \( FV \)). Since \( X_{e_1-\varepsilon_2} \) commutes with \( X_{e_2-\varepsilon_1} \), it follows that the action of \( X_{\varepsilon_1-\varepsilon_2} \) on \( R \) is also given by (3).

It is left to show that the action of \( X_{e_2-\varepsilon_1} \) extends uniquely from \( R' \) to \( R \) and then that there is a unique way to define the action of \( X_{-2\varepsilon_1} \). This will be done in the Lemmata 10 and 11 below. □

Lemma 10. There is a unique way to extend the action of \( X_{e_2-\varepsilon_1} \) from \( R' \) to \( R \).
Proof. Let us first show that for every $k \in \{1, 2, \ldots\}$ the action of $X_{2e_1 - e_2}$ extends uniquely from $X_{2e_1 - e_2}^{k-1}R'$ to $X_{2e_1 - e_2}^k R'$ (here $X_{2e_1}^0 R' = R'$).

Consider the following picture:

(7)

Here bullets are weight spaces with some fixed bases. The lower row is a part of $X_{2e_1 - e_2}^{k-1}R'$ where the $\mathfrak{a}$-action is already known by induction. The bases in the weight spaces in the lower row are chosen such that the action of $\mathfrak{a}$ in the lower row is given by $H_{2e_1 - e_2}$. The upper row is a part of $X_{2e_1 - e_2}^k R'$ where the $\mathfrak{a}$-action is to be determined. Arrows pointing up indicate the action of $X_{2e_1}$. The bases of the weight spaces in the upper row are chosen such that the action of $X_{2e_1}^k$ is given by the operator $\text{Id}_V$ (as in (3)). Left arrows indicate the action of $X_{e_1 - e_2}$. The latter commutes with the action of $X_{2e_1}$, and hence is given by the same linear operator in each column. Right arrows indicate the action of $X_{e_1 - e_2}$ (which is known for $X_{2e_1}^{k-1}R'$ and is to be determined for $X_{2e_1}^k R'$). The part to be determined is given by the dashed arrow. Labels $P$ and $Q$ represent coefficients (which are linear operators on $V$) appearing in the corresponding parts of formulae (3). Note that $P$ and $Q$ commute. The action of $X_{e_1 - e_2}$ on $X_{2e_1}^k R'$ which is to be determined is given by some unknown linear operators $X$.

From $H_{e_1 - e_2} = [X_{e_1 - e_2}, X_{e_1 - e_2}]$ we compute that the action of $H_{e_1 - e_2}$ on the middle weight space in the lower row is given by $Q - P$. Using $[H_{e_1 - e_2}, X_{2e_1}] = -2X_{2e_1}$ we get that $H_{e_1 - e_2}$ acts on the right dot of the upper row via $Q - P - 2$. Using $[H_{e_1 - e_2}, X_{-1}] = -2X_{-1}$ we get that $H_{e_1 - e_2}$ acts on the left dot of the upper row via $Q - P - 4$. Hence the action of $C$ on the upper row is given by $(Q - P - 3)^2 + 4XQ$. The action of $C$ on the lower row is given by $(Q - P - 1)^2 + 4(P + 1)Q = (Q + P + 1)^2$.

The elements $X_{2e_1}, X_{2e_2}$ and $X_{e_1 + e_2}$ form a weight basis of a simple three-dimensional $\mathfrak{a}$-module $C^3$ with respect to the adjoint action of $\mathfrak{a}$. Hence the upper row of our picture is a subquotient of the tensor product of the lower row and $C^3$. Therefore, from Proposition (4) we obtain that the linear operator

$$(C - (Q + P - 1)^2)(C - (Q + P + 1)^2)(C - (Q + P + 3)^2)$$

annihilates the upper row. A direct computation using (3) shows that the action of the operators $C - (Q + P - 1)^2$ and $C - (Q + P + 1)^2$ on the part $X_{2e_1}^k N(\mathfrak{a})'$ of the module $N(\mathfrak{a})$ is invertible. As the $g$-module we are working with must have a composition series with subquotients $N(\mathfrak{a})$, it follows that the action of both $C - (Q + P - 1)^2$ and $C - (Q + P + 1)^2$ on $X_{2e_1}^k R'$ is invertible. Hence $C - (Q + P + 3)^2$ annihilates $X_{2e_1}^k R'$, which gives us the equation

$$(Q - P - 3)^2 + 4XQ = (Q + P + 3)^2.$$

This equation has a unique solution, namely $X = Q + 3$, which gives the required extension.

Similarly one shows that for $k \in \{-1, -2, \ldots\}$ the action of $X_{e_1 - e_2}$ extends uniquely from $X_{e_1 - e_2}^{k+1}R'$ to $X_{e_1 - e_2}^k R'$ (here again $X_{e_1 - e_2}^0 R' = R'$). This completes the proof of our lemma. 

\[\square\]
Lemma 11. There is a unique way to define the action of $X_{-2e_1}$ on $N$.

Proof. To determine this action of $X_{-2e_1}$ on $N$ we consider the following extension of the picture (7) with the same notation as in the proof of Lemma (10):

Here all right arrows, representing the action of $X_{-2e_1}$, are now determined by Lemma (10) and we have to figure out the down arrows, representing the action of $X_{-2e_1}$. The two dotted arrows will be used later on in the proof.

Consider the $\mathfrak{sl}_2$-subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ generated by $e := X_{2e_1}$ and $f := X_{-2e_1}$. Set $h := [e, f]$. Denote by $Z$ the action of $h$ in the leftmost weight space of the middle row. Then $Z = x - u$. The element $h$ commutes with both $h$ and $H_{2e_1}$. Therefore, by (3), the operator $Z$ commutes with both $T_1$ and $T_2$ and hence with both $P$ and $Q$.

The algebra algebra $\mathfrak{c}$ has the quadratic Casimir element $C_1$, whose action on the $\mathfrak{c}$-module given by the leftmost column of our picture is given by $x + f(Z)$, where $f$ is some polynomial of degree two. From (3) it follows that the unique eigenvalue of this action is nonzero, in particular, $x + f(Z)$ is invertible. Let $x'$ be a fixed square root $x + f(Z)$, which is a polynomial in $x + f(Z)$.

The elements $X_{2z-e_1}$ and $X_{2+e_1}$ form a basis of a simple two-dimensional $\mathfrak{c}$-module with respect to the adjoint action. Using Proposition 7 and arguments similar to those used in the proof of Lemma (10) we get that $C_1 - (x')^2$ or $C_1 - (x' - 1)^2$ annihilates the middle column (the sign depends on the original choice of $x'$). Note that the middle column equals $X_{2z-e_1}$ applied to the leftmost column.

Similarly, the elements $X_{z_1-e_1}$ and $X_{-e_1}$ form a basis of a simple two-dimensional $\mathfrak{c}$-module with respect to the adjoint action. Applying the same arguments as in the previous paragraph we get that $C_1 - (x')^2$ annihilates any vector of the form $X_{z_1-e_1}X_{2z-e_1}v$, where $v$ is from the leftmost column. This implies that the actions of $C_1$ and $X_{z_1-e_1}X_{2z-e_1}$ and thus the actions of $C_1$ and $C_0$ on the leftmost column commute. As the action of $H$ commutes with the action of $C$, we thus obtain that $x$ commutes with the action of $C$. This implies that $x$ commutes with $T_1 + T_2$. As it obviously commutes with $T_1 - T_2$, we get that $x$ commutes with both $T_1$ and $T_2$ and hence with both $P$ and $Q$.

Similarly one shows that $y$, $u$, $v$ and $w$ commute with both $P$ and $Q$. From the commutativity of $X_{z_1-e_1}$ and $X_{-2e_1}$ we get the following conditions:

$$y(P + 1) = (P - 1)x, \quad v(P + 3) = (P + 1)u, \quad w(P + 2)(P + 3) = P(P + 1)u.$$  

Here everything commutes by the above and $P + 1$, $P + 2$ and $P + 3$ are invertible (as $X_{z_1-e_1}$ acts bijectively). Therefore

$$y = (P - 1)(P + 1)^{-1}x, \quad v = (P + 1)(P + 3)^{-1}u, \quad w = P(P + 1)(P + 3)^{-1}(P + 2)^{-1}u.$$
This implies that \( y, v \) and \( w \) are uniquely determined by \( x \) and \( u \).

Since the actions of both \( X_{-\varepsilon_1} \) and \( X_{2\varepsilon_1} \) are completely determined, we can compute the action of \( X_{2\varepsilon_3} \) and see that it is given (similarly to the action of \( X_{2\varepsilon_3} \)) by \( \text{Id}_V \) (this is depicted by the dotted arrows in the picture). As \( X_{-2\varepsilon_2} \) and \( X_{2\varepsilon_2} \) commute, we obtain that \( w = x \), that is

\[
x = P(P + 1)(P + 3)^{-1}(P + 2)^{-1}u.
\]

Therefore the only parameter left for now is \( u \).

On the one hand, the action of the element \( h \) on the middle dot of the second row is given by

\[
y - v = (P - 1)(P + 1)^{-1}x - (P + 1)(P + 3)^{-1}u.
\]

On the other hand, from \([h, X_{-\varepsilon_1}] = 4X_{\varepsilon_2-\varepsilon_1}\) we have that this action equals \( Z + 4 = x - u + 4 \). This gives us the equation

\[
(P - 1)(P + 1)^{-1}x - (P + 1)(P + 3)^{-1}u = x - u + 4.
\]

Using (9) and (8) we get the equation

\[
\frac{P(P - 1)}{(P + 2)(P + 3)}u + \frac{P + 1}{P + 3}u = \frac{P(P + 1)}{P + 2}(P + 3)u - u + 4.
\]

This is a linear equation with nonzero coefficients and thus it has a unique solution, namely \( u = (P + 3)(P + 2) \). Hence \( u \) is uniquely defined. The claim of the lemma follows.

\[\square\]

5. Consequences

Corollary 12. Let \( a \in \mathbb{C}^n \) be such that \( a_i \not\in \mathbb{Z} \) and \( a_i + a_j \not\in \mathbb{Z} \) for all \( i \) and \( j \). Let \( M \in \tilde{\mathcal{C}} \) and \( \lambda \in \text{supp}(M) \). Denote by \( U_0 \) the centralizer of \( h \) in \( U(\mathfrak{g}) \). Then for any \( A, B \in U_0 \) the actions of \( A \) and \( B \) on \( M_{\lambda} \) commute.

Proof. By Proposition 4 we may assume that \( M \cong FV \). For the module \( FV \) the claim follows from the formulae (3).

\[\square\]

Corollary 13. For any simple weight cuspidal \( \mathfrak{g} \)-module \( L \) with finite dimensional weight spaces we have \( \text{dim} \text{Ext}^1_{\mathfrak{g}}(L, L) = n \).

Proof. This follows from Theorem 4 and the observation that a similar equality is true for the unique simple \( \mathbb{C}[[t_1, t_2, \ldots, t_n]]\)-module.

We also recover the main result of [BKLM]:

Corollary 14 ([BKLM]). The category of all weight cuspidal \( \mathfrak{g} \)-modules is semi-simple.

Proof. By [BKLM] Lemma 2, all blocks of the category of weight cuspidal \( \mathfrak{g} \)-modules are equivalent. Hence it is enough to prove the claim for the block containing \( N(a) \) for some \( a \in \mathbb{C}^n \) such that \( a_i + a_j \not\in \mathbb{Z} \) for all \( i, j \). From (3) it follows that the module \( FV \) is weight if and only if all operators \( T_i \) are semi-simple, hence zero. Therefore from Theorem 4 we get that the block of the category of weight cuspidal modules is equivalent to the category of finite dimensional modules over \( \mathbb{C}[[t_1, t_2, \ldots, t_n]]/(t_1 - 0, t_2 - 0, \ldots, t_n - 0) \cong \mathbb{C} \). The claim follows.

\[\square\]

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