Improved estimation via model selection method for semimartingale regressions based on discrete data

Evgeny A. Pchelintsev, † Sergey M. Pergamenshchikov, ‡ Maria A. Povzun §

Abstract

We consider the robust adaptive nonparametric estimation problem for a periodic function observed in the framework of a continuous time regression model with semimartingale noises, i.e. by incomplete observations. As an example, we consider the regression model with noise defined by non-Gaussian Ornstein–Uhlenbeck processes. A model selection procedure, based on the shrinkage (improved) weighted least squares estimates, is proposed. Constructive sufficient conditions for the observations frequency are found under which sharp oracle inequalities for the robust risks are obtained. Moreover, on the basis of these inequalities the robust efficiency property has been established in adaptive setting. Finally, the Monte Carlo simulations are given which confirm numerically the obtained theoretical results.

Key words: Improved non-asymptotic estimation, Least squares estimates, Robust quadratic risk, Non-parametric regression, Semimartingale noise, Ornstein–Uhlenbeck–Lévy process, Model selection, Sharp oracle inequality, Asymptotic efficiency.

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†Department of Mathematical Analysis and Theory of Functions, Tomsk State University, e-mail: evgen-pch@yandex.ru
‡Laboratoire de Mathématiques Raphael Salem, Université de Rouen and International Laboratory of Statistics of Stochastic Processes and Quantitative Finance of Tomsk State University, e-mail: Serge.Pergamenchchikov@univ-rouen.fr
§International Laboratory of Statistics of Stochastic Processes and Quantitative Finance of Tomsk State University, e-mail: povzunyasha@gmail.com
1 Introduction

In this paper we consider the following continuous time regression model
\[ dy_t = S(t)dt + d\xi_t, \quad 0 \leq t \leq n, \quad (1.1) \]
where \( S \) is an unknown 1-periodic \( \mathbb{R} \to \mathbb{R} \) function from \( L^2[0,1] \), \( (\xi_t)_{t \geq 0} \) is an unobservable noise which is a square integrated semimartingale with the values in the Skorokhod space \( D[0,n] \) such that, for any function \( f \) from \( L^2[0,n] \), the stochastic integral
\[ I_n(f) = \int_0^n f(s)d\xi_s \quad (1.2) \]
has the following properties
\[ E_Q I_n(f) = 0 \quad \text{and} \quad E_Q I_n^2(f) \leq \kappa_Q \int_0^n f^2(s)ds. \quad (1.3) \]
Here \( E_Q \) denotes the expectation with respect to the distribution \( Q \) of the noise process \( (\xi_t)_{t \geq 0} \) on the space \( D[0,n] \), \( \kappa_Q > 0 \) is some positive constant depending on the distribution \( Q \). As to the noise distribution \( Q \) we assume that it is unknown and belongs to some family \( Q_n \) of probability distributions in \( D[0,n] \).

The problem is to estimate the unknown function \( S \) in the model (1.1) on the basis of observations \( (y_{t_j})_{0 \leq j \leq np} \), \( t_j = j/p \), (1.4)
where the observations frequency \( p \) is some fixed integer number. For this problem we use the quadratic risk, which for any estimate \( \hat{S} \), is defined as
\[ \mathcal{R}_Q(\hat{S},S) := E_{Q,S} \| \hat{S} - S \|^2 \quad \text{and} \quad \| f \|^2 := \int_0^1 f^2(t)dt, \quad (1.5) \]
where \( E_{Q,S} \) stands for the expectation with respect to the distribution \( P_{Q,S} \) of the process (1.1) with a fixed distribution \( Q \) of the noise \( (\xi_t)_{0 \leq t \leq n} \) and a given function \( S \). Moreover, in the case when the distribution \( Q \) is unknown we use also the robust risk
\[ \mathcal{R}^*(\hat{S},S) = \sup_{Q \in Q_n} \mathcal{R}_Q(\hat{S},S). \quad (1.6) \]
Note that if \( (\xi_t)_{t \geq 0} \) is a Brownian motion, then we obtain the well known white noise model (see, for example, [10, 30, 20, 21, 22] and etc.) which is popular in statistical radio-physics. Later, to take into account the dependence structure in the papers [13, 9, 16] it was proposed to use the Ornstein
- Uhlenbeck noise processes, so called color gaussian noises. Then, to study the estimation problem for non-Gaussian observations (1.1) in the papers [12, 1, 25, 19, 17, 18] it was introduced impulse noises defined through the semi-Markov or compound Poisson processes with unknown impulse distributions. However, the semi-Markov or compound Poisson processes can describe the impulse influence of only one fixed frequency. It should be noted that in the telecommunication systems, the noise impulses are without limitations on frequencies and, therefore, such models are too restricted for practical applications. To include all possible impulse noises, in [14, 15] it was proposed to use general non-Gaussian semimartingale processes. Later, for semimartingale models in the papers [26, 27, 28, 29, 31] the authors developed the improved (shrinkage) nonparametric estimation methods. It should be emphasized, that in all these papers the improved estimation problems are studied only for the complete observations cases, i.e. when the all trajectory \((y_t)_{0 \leq t \leq n}\) is accessed to be observed.

Our main goal in this paper is to develop improved estimation methods for the incomplete observations, i.e. when the process (1.1) can be observed only in the fixed time moments (1.4). As an example, we consider the regression model (1.1) with the noise defined by non-Gaussian Ornstein–Uhlenbeck process with unknown distribution. To this end we propose model selection methods based on the improved weighted least squares estimates. For the first time such approach was proposed in [6] for regression models in discrete time and in [16] for Gaussian regression models in continuous time. It should be noted that for the non-Gaussian regression models we can not use directly the well-known improved estimators proposed in [11] for spherically symmetric observations. To apply the improved estimation methods to the non-Gaussian regression models in continuous time one needs to use the modifications of the James - Stein shrinkage procedure proposed in [19, 25] for parametric estimation problems and developed in [26, 27, 28, 29, 31] for nonparametric estimation. Moreover, we develop new analytical tools to provide the improvement effect for the non-asymptotic estimation accuracy. It turns out that in this case the accuracy improvement is much more significant than for parametric models, since according to the well-known James - Stein formula the accuracy improvement increases when dimension of the parameters increases. Recall, that for the parametric models this dimension is always fixed, while for the nonparametric models it tends to infinity, that is, it becomes arbitrarily large with an increase in the number of observations. Therefore, the gain from the application of improved methods is essentially increasing with respect to the parametric case. Then, we find constructive conditions for the observation frequency and the noise distributions under which we show sharp non-asymptotic oracle inequalities for the robust risks (1.6). Then, through the established oracle inequalities we get the efficiency property for the developed model selection methods in adaptive setting. Furthermore, we show that the obtained conditions hold
for the non-Gaussian Ornstein–Uhlenbeck processes defined in Section 2.

The rest of the paper is organized as follows. In Section 3 we construct the shrinkage weighted least squares estimates and study the improvement effect. In Section 4 we construct the model selection procedure on the basis of improved weighted least squares estimates. In Section 5.1 we state the main results in the form of oracle inequalities for the quadratic risk (1.5) and the robust risk (1.6). In Section 5.2 it is shown that the proposed model selection procedure for estimating $S$ in (1.1) is asymptotically efficient with respect to the robust risk (1.6). In Section 6 we illustrate the performance of the proposed model selection procedure through numerical simulations. Section 7 gives the main properties of stochastic integrals for the non-Gaussian Ornstein-Uhlenbeck processes. Section 8 gives the proofs of the main results. Appendix A contains all auxiliary results.

2 Non-Gaussian Ornstein-Uhlenbeck-Lévy process

Now we consider the noise process $(\xi_t)_{t \geq 0}$ in (1.1) defined by a non-Gaussian Ornstein–Uhlenbeck process with the Lévy subordinator. Such processes are used in the financial Black–Scholes type markets with jumps (see, for example, [2], and the references therein). Let the noise process in (1.1) obeys the equation

$$d\xi_t = a\xi_t dt + du_t, \quad \xi_0 = 0,$$

where

$$u_t = \varrho_1 w_t + \varrho_2 z_t \quad \text{and} \quad z_t = x \ast (\mu - \widehat{\mu})_t.$$

Here $(w_t)_{t \geq 0}$ is a standard Brownian motion, "\ast" denotes the stochastic integral with respect to the compensated jump measure $\mu(ds, dx)$ with deterministic compensator $\widehat{\mu}(ds dx) = ds\Pi(dx)$, i.e.

$$z_t = \int_0^t \int_{\mathbb{R}_+} v(\mu - \widehat{\mu})(ds dv) \quad \text{and} \quad \mathbb{R}_+ = \mathbb{R} \setminus \{0\},$$

$\Pi(\cdot)$ is the Lévy measure on $\mathbb{R}_+ = \mathbb{R} \setminus \{0\}$, (see, for example in [5]), such that

$$\Pi(x^2) = 1 \quad \text{and} \quad \Pi(x^8) < \infty.$$

We use the notation $\Pi(|x|^m) = \int_{\mathbb{R}_+} |z|^m \Pi(dz)$. Moreover, we assume that the nuisance parameters $a \leq 0$, $\varrho_1$ and $\varrho_2$ satisfy the conditions

$$-a_{\text{max}} \leq a \leq 0, \quad 0 < \varrho \leq \varrho_1^2 \quad \text{and} \quad \sigma_Q = \varrho_1^2 + \varrho_2^2 \leq \varsigma^*,$$

where the bounds $a_{\text{max}}, \varrho$ and $\varsigma^*$ are functions of $n$, i.e. $a_{\text{max}} = a_{\text{max}}(n)$, $\varrho = \varrho_n$ and $\varsigma^* = \varsigma^*_n$, such that for any $\epsilon > 0$

$$\lim_{n \to \infty} \frac{a_{\text{max}}(n) + \varsigma^*_n}{n^\epsilon} = 0 \quad \text{and} \quad \liminf_{n \to \infty} n^\epsilon \frac{\varrho_n}{\varsigma_n^*} > 0.$$

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We denote by $Q_n$ the family of all distributions of process (1.1) – (2.1) on the Skorokhod space $D[0,n]$ satisfying the conditions (2.4) – (2.5). It should be noted that in view of Corollary 7.2 and the last inequality in (2.4) the condition (1.3) for the process (2.1) holds with $\kappa_Q = 2\varsigma^*$. Note also that the process (2.1) is conditionally-Gaussian square integrated semimartingale with respect to $\sigma$-algebra $G = \sigma\{z_t, t \geq 0\}$ which is generated by jump process $(z_t)_{t \geq 0}$.

### 3 Improved estimation

For estimating the unknown function $S$ in (1.1) we will use its Fourier expansion with respect to an orthonormal basis $(\phi_j)_{j \geq 1}$ in $L^2[0,1]$. We extend these functions by the periodic way on $\mathbb{R}$, i.e. $\phi_j(t) = \phi_j(t + 1)$ for any $t \in \mathbb{R}$. Assume that the basis functions are uniformly bounded, i.e. for some constant $\phi_* \geq 1$, which may be depend on $n$,

$$\sup_{0 \leq j \leq n} \sup_{0 \leq t \leq 1} |\phi_j(t)| \leq \phi_* < \infty. \quad (3.1)$$

Moreover we will use such basis that the restrictions of the functions $(\phi_j)_{1 \leq j \leq p}$, on the sampling lattice

$$T_p = \{t_1, \ldots, t_p\}, \quad t_j = j/p,$$

form an orthonormal basis in the Hilbert space $\mathbb{R}^{T_p}$ with the inner product

$$(x, y)_p = \frac{1}{p} \sum_{j=1}^{p} x(t_j)y(t_j) \quad \text{for} \quad x, y \in \mathbb{R}^{T_p}, \quad (3.2)$$

i.e. $(\phi_i, \phi_j)_p = 1_{i=j}$. We put the norm $\|x\|_p = \sqrt{(x, x)_p}$.

For example, we can take the trigonometric basis defined as $\text{Tr}_1 \equiv 1$ and for $j \geq 2$

$$\text{Tr}_j(x) = \sqrt{2} \begin{cases} 
\cos(2\pi [j/2] x) & \text{for even} \quad j; \\
\sin(2\pi [j/2] x) & \text{for odd} \quad j.
\end{cases} \quad (3.3)$$

Here $[a]$ denotes the integer part of $a$.

We write the discrete Fourier expansion of the unknown function $S$ on the lattice $T_p$ in the form

$$S(t) = \sum_{j=1}^{p} \theta_{j,p} \phi_j(t),$$

where the corresponding Fourier coefficients

$$\theta_{j,p} = (S, \phi_j)_p \quad (3.4)$$
can be estimated from the discrete data \((1.4)\) by the formulae
\[
\hat{\theta}_{j,p} = \frac{1}{n} \int_0^n \psi_{j,p}(t) \, dy_t, \quad \psi_{j,p}(t) = \sum_{k=1}^{np} \phi_j(t_k) \mathbf{1}_{(t_{k-1}, t_k]}(t).
\] (3.5)

We note that the system of the functions \(\{\psi_{j,p}\}_{1 \leq j \leq p}\) is orthonormal in \(L_2[0,1]\) because
\[
(\psi_i, \psi_j) = \int_0^1 \psi_i(t) \psi_j(t) \, dt = (\phi_i, \phi_j)_p = \mathbf{1}_{\{i=j\}}.
\]

Then the Fourier coefficients for the function \(S\) with respect to these functions can be written as
\[
\tilde{\theta}_{j,p} = (S, \psi_j) = \theta_{j,p} + h_{j,p},
\] (3.6)
where
\[
h_{j,p} = h_{j,p}(S) = \sum_{k=1}^{t_k} \int_{t_{k-1}}^{t_k} \phi_j(t_k) (S(t) - S(t_k)) \, dt.
\]

In view of \((1.1)\), one obtains
\[
\hat{\theta}_{j,n} = \tilde{\theta}_{j,p} + \frac{1}{\sqrt{n}} \xi_{j,p}, \quad \xi_{j,p} = \frac{1}{\sqrt{n}} I_n(\psi_{j,p}),
\] (3.7)
where \(I_n(\cdot)\) is given in \((1.2)\). As in \([18]\) we define a class of weighted least squares estimates for \(S(t)\) as
\[
\hat{S}_\gamma(t) = \sum_{j=1}^{p} \gamma(j) \hat{\theta}_{j,n} \psi_{j,p}(t),
\] (3.8)
where the weights \(\gamma = (\gamma(j))_{1 \leq j \leq p} \in \mathbb{R}^p\) belong to some finite set \(\Gamma\) from \([0,1]^p\) for which we set
\[
\nu = \text{card}(\Gamma) \quad \text{and} \quad |\Gamma|_* = \max_{\gamma \in \Gamma} \sum_{j=1}^{p} \gamma(j),
\] (3.9)
where \(\text{card}(\Gamma)\) is the number of the vectors \(\gamma\) in \(\Gamma\). In the sequel we assume that all vectors from \(\Gamma\) satisfies the following condition.

**D1)** Assume that for any vector \(\gamma \in \Gamma\) there exists some fixed integer \(d = d(\gamma) \leq p\) such that their first \(d\) components are equal to one, i.e. \(\gamma(j) = 1\) for \(1 \leq j \leq d\) for any \(\gamma \in \Gamma\).

**D2)** There exists \(p_0 \geq 1\) such that for any \(p \geq p_0\) there exists a \(\sigma\) - field \(\mathcal{G}_p\) for which the random vector \(\check{\xi}_{d,p} = (\xi_{j,p})_{1 \leq j \leq d}\) is the \(\mathcal{G}_p\) - conditionally Gaussian in \(\mathbb{R}^d\) with the covariance matrix
\[
\mathbf{G}_p = \left( \mathbf{E} \xi_{i,p} \xi_{j,p} | \mathcal{G}_p \right)_{1 \leq i,j \leq d}
\] (3.10)
and for some nonrandom constant \( l_p > 0 \)
\[
\inf_{Q \in \mathcal{Q}_n} \left( \text{tr} G_p - \lambda_{\text{max}}(G_p) \right) \geq l_p \quad \text{a.s.,}
\]  
(3.11)
where \( \lambda_{\text{max}}(A) \) is the maximal eigenvalue of the matrix \( A \).

As it is shown in Proposition 7.11 in [27] the condition \( D_2 \) holds for the model (1.1) – (2.1) with \( l_p = \varrho_n \frac{(d - 6)}{2} \) and \( d \geq 7 \).

For the first \( d \) Fourier coefficients in (3.7) we will use the improved estimation method proposed for parametric models in [25]. To this end we set \( \tilde{\theta}_p = (\hat{\theta}_{j,p})_{1 \leq j \leq d} \). In the sequel we will use the norm \( |x|^2_d = \sum_{j=1}^d x_j^2 \) for any vector \( x = (x_j)_{1 \leq j \leq d} \) from \( \mathbb{R}^d \). Now we define the shrinkage estimators as
\[
\theta_{j,p}^* = (1 - g(j)) \tilde{\theta}_{j,p},
\]  
(3.12)
where \( g(j) = (c_n / |\tilde{\theta}_p|_d) 1_{\{1 \leq j \leq d\}} \),
\[
c_n = \frac{l_p}{(r + \sqrt{d \kappa / n}) n} \quad \text{and} \quad \kappa = \sup_{Q \in \mathcal{Q}_n} \kappa_Q.
\]
The positive parameter \( r \) may be dependent of \( n \), i.e. \( r = r_n \), and such that
\[
\lim_{n \to \infty} n^{-\epsilon} r_n = 0 \quad \text{for any} \quad \epsilon > 0.
\]  
(3.13)
Now we set shrinkage estimates for \( S \)
\[
S_{S}^*(t) = \sum_{j=1}^p \gamma(j) \theta_{j,p}^* \psi_{j,p}(t).
\]  
(3.14)
We compare the estimators (3.8) and (3.14) through the difference
\[
\Delta_Q(S) := R_Q(S^*, S) - R_Q(\hat{S}_p, S).
\]
Now we obtain the non asymptotic bound for this comparative risk. Let now we set
\[
p_0 = \frac{\sqrt{d \phi} L}{c_n} + 1,
\]  
(3.15)
where \( L \) is the Lipschitz constant, i.e.
\[
L = \sup_{0 \leq s, t \leq 1} \frac{|S(t) - S(s)|}{|t - s|}.
\]

**Theorem 3.1.** Assume that the conditions \( D_1 \) – \( D_2 \) hold. Moreover, assume that the function \( S \) is Lipschitzian. Then for any \( p \geq p_0 \)
\[
\sup_{Q \in \mathcal{Q}_n} \|S\| \leq r \sup_{Q \in \mathcal{Q}_n} \Delta_Q(S) < 0.
\]  
(3.16)

**Remark 3.1.** This inequality (3.16) means that non-asymptotically, i.e. for any \( p \geq p_0 \) the estimate (3.14) outperforms in mean square accuracy the estimate (3.8).
4 Model selection

This Section gives the construction of a model selection procedure for estimating a function $S$ in (1.1) on the basis of improved weighted least square estimates.

The model selection procedure for the unknown function $S$ in (1.1) will be constructed on the basis of a family of estimates $(S^*_\gamma)_{\gamma \in \Gamma}$.

The performance of any estimate $S^*_\gamma$ will be measured by the empirical squared error

$$\text{Err}_p(\gamma) = \| S^*_\gamma - S \|^2. \quad (4.1)$$

In order to obtain a good estimate, we have to write a rule to choose a weight vector $\gamma \in \Gamma$ in (3.14). It is obvious, that the best way is to minimise the empirical squared error with respect to $\gamma$. Making use the estimate definition (3.14) and the Fourier transformation of $S$ implies

$$\text{Err}_p(\gamma) = \sum_{j=1}^{p} \gamma^2(j)(\theta^*_j)^2 - 2 \sum_{j=1}^{p} \gamma(j)\theta^*_j \tilde{\theta}_j + \| S \|^2. \quad (4.1)$$

Since the Fourier coefficients $(\tilde{\theta}_j,p)_{j \geq 1}$ are unknown, the weight coefficients $(\gamma_j)_{j \geq 1}$ can not be found by minimizing this quantity. To circumvent this difficulty one needs to replace the terms $\theta^*_j \tilde{\theta}_j$ by their estimators $\tilde{\theta}_j,p$. We set

$$\tilde{\theta}_j,p = \theta^*_j \tilde{\theta}_j - \frac{\hat{\sigma}_n}{n}, \quad (4.2)$$

where $\hat{\sigma}_n$ is the estimate for the limiting variance of $E_Q \xi^2_{j,n}$ which we choose in the following form

$$\hat{\sigma}_n = \sum_{j=[\sqrt{n}]+1}^{n} \hat{\tau}^2_{j,n}, \quad \hat{\tau}_{j,n} = \int_0^1 \text{Tr}_j(t)dy_t. \quad (4.3)$$

For this change in the empirical squared error, one has to pay some penalty. Thus, one comes to the cost function of the form

$$J_p(\gamma) = \sum_{j=1}^{p} \gamma^2(j)(\theta^*_j)^2 - 2 \sum_{j=1}^{p} \gamma(j)\tilde{\theta}_j,p + \rho \hat{P}_n(\gamma) \quad (4.4)$$

where $\rho$ is some positive constant, $\hat{P}_n(\gamma)$ is the penalty term defined as

$$\hat{P}_n(\gamma) = \frac{\hat{\sigma}_n |\gamma|^2}{n}. \quad (4.5)$$

Substituting the weight coefficients, minimizing the cost function

$$\gamma^* = \arg\min_{\gamma \in \Gamma} J_p(\gamma), \quad (4.6)$$
in (3.8) leads to the improved model selection procedure
\[ S^* = S^*_\gamma^*. \] (4.7)

It will be noted that $\gamma^*$ exists because $\Gamma$ is a finite set. If the minimizing sequence in (4.6) $\gamma^*$ is not unique, one can take any minimizer.

Now we specify the weight coefficients $(\gamma(j))_{j \geq 1}$ as it is proposed in [7, 8] for a heteroscedastic regression model in discrete time. Firstly, we define the normalizing coefficient $v_n = n/\varsigma^*$. Consider a numerical grid of the form
\[ A_n = \{1, \ldots, k^*\} \times \{r_1, \ldots, r_m\}, \]
where $r_i = \varepsilon^i$, $i = 1, m$ and $m = [1/\varepsilon^2]$. We assume that the parameters $k^* \geq 1$ and $0 < \varepsilon \leq 1$ are functions of $n$, i.e. $k^* = k^*(n)$ and $\varepsilon = \varepsilon(n)$, such that
\[ \lim_{n \to \infty} \left( \frac{1}{k^*(n)} + \frac{k^*(n)}{\ln n} \right) = 0 \quad \text{and} \quad \lim_{n \to \infty} \left( \varepsilon(n) + \frac{1}{n^b \varepsilon(n)} \right) = 0 \] (4.8)
for any $b > 0$. One can take, for example, for $0 < \varepsilon < 1$
\[ \varepsilon(n) = 1/\ln(n + 1) \quad \text{and} \quad k^*(n) = k_0^* + \sqrt{\ln(n + 1)}, \] (4.9)
where $k_0^* \geq 0$ is some fixed constant. For each $\alpha = (\beta, r) \in A_n$ we introduce the weight sequence $\gamma_\alpha = (\gamma_\alpha(j))_{j \geq 1}$ as
\[ \gamma_\alpha(j) = 1_{\{1 \leq j \leq d(\alpha)\}} \left( 1 - \left( j/\omega_\alpha \right)^\beta \right) 1_{\{d(\alpha) < j \leq \omega_\alpha\}} \] (4.10)
where $d(\alpha) = [\omega_\alpha/\ln(n + 1)]$, $\omega_\alpha = (\tau_\beta r v_n)^{1/(2\beta + 1)}$ and
\[ \tau_\beta = \frac{(\beta + 1)(2\beta + 1)}{\pi^{2\beta+1}}. \]

Finally, we set
\[ \Gamma = \{ \gamma_\alpha, \alpha \in A_n \} \] (4.11)
It will be noted that such weight coefficients satisfy the condition $D_1$.

5 Main results

In this Section we obtain the sharp oracle inequalities for the quadratic risk $(1.5)$ and robust risk $(1.6)$ of proposed procedure. Then on the basis of these inequalities the robust efficiency property has been established in adaptive setting.
5.1 Oracle inequalities

To prove the sharp oracle inequality, the following conditions will be needed for the family $Q_n$ of distributions of the noise $(\xi_i)_{i \geq 0}$ in (1.1).

We need to impose some stability conditions for the noise Fourier transform sequence $(\xi_{j,p})_{1 \leq j \leq p}$ introduced in [26].

$C_1)$ There exists a proxy variance $\sigma_Q > 0$ such that for any $\epsilon > 0$

$$\lim_{n \to \infty} \frac{1}{n^\epsilon} \sup_{Q \in Q_n} \frac{\sum_{j=1}^p |E_Q \xi_{j,p}^2 - \sigma_Q|}{\sigma_Q} = 0.$$ 

In the sequel we will use the following notations

$$L_{1,n}(Q) = \sum_{j=1}^p \left| E_Q \xi_{j,p}^2 - \sigma_Q \right| \quad \text{and} \quad L_{1,n}^* = \sup_{Q \in Q_n} L_{1,n}(Q). \quad (5.1)$$

Moreover, we set

$$L_{2,n}(Q) = \sup_{|x| \leq 1} E_Q \left( \sum_{j=1}^p x_j (\xi_{j,p}^2 - E_Q \xi_{j,p}^2) \right)^2,$$ 

where $| \cdot |$ is euclidean norm in $\mathbb{R}^n$, i.e. $|x|^2 = \sum_{j=1}^n x_j^2$ for any $x = (x_j)_{1 \leq j \leq n}$ from $\mathbb{R}^n$.

$C_2)$ Assume that the sequence $L_{2,n}^* = \sup_{Q \in Q_n} L_{2,n}(Q)$ is such that

$$\lim_{n \to \infty} \frac{L_{2,n}^*}{n^\epsilon} = 0 \quad \text{for any} \quad \epsilon > 0.$$ 

First, we obtain the oracle inequalities for the risks (1.5).

**Theorem 5.1.** Assume that the conditions $C_1)$ and $C_2)$ hold. Then, for any $n \geq 1$ and $0 < \rho < 1/2$

$$\mathcal{R}_Q(S^*, S) \leq \frac{1 + 5\rho}{1 - \rho} \min_{\gamma \in \Gamma} \mathcal{R}_Q(S^*_\gamma, S) + \frac{U_n}{\rho n} \left( 1 + |\Gamma| n E_Q |\hat{\sigma}_n - \sigma_Q| \right),$$

where the coefficient $U_n$ is such that for any $\epsilon > 0$

$$\lim_{n \to \infty} \frac{U_n}{n^\epsilon} = 0. \quad (5.3)$$

In the case, when the value of $\sigma_Q$ in $C_1)$ is known, one can take $\hat{\sigma}_n = \sigma_Q$ and

$$P_n(\gamma) = \frac{\sigma_Q |\gamma|^2}{n}. \quad (5.4)$$
Now we study the estimate (4.3). To obtain the oracle inequality for the robust risk (1.6) we need some additional condition on the distribution family $Q_n$. We set
\[
\varsigma^* = \varsigma^*_n = \sup_{Q \in Q_n} \sigma_Q. \tag{5.5}
\]

$C_3$) Assume that the limit equation (5.1) - (5.2) hold uniformly in $Q \in Q_n$ and $\varsigma^*_n/n^\epsilon \to 0$ as $n \to \infty$ for any $\epsilon > 0$.

Now we impose the conditions on the set of the weight coefficients $\Gamma$.

$C_4$) Assume that the set $\Gamma$ is such $\nu/n^\epsilon \to 0$ and $|\Gamma|/n^{1/2+\epsilon} \to 0$ for any $\epsilon > 0$.

As is shown in [27], both the conditions $C_3$) and $C_4$) hold for the model (1.1) with Ornstein-Uhlenbeck noise process (2.1). Using Proposition 4.2 from [27] we can obtain the oracles inequalities for the robust risks (1.6).

**Theorem 5.2.** Assume that the conditions $C_1$)–$C_4$) hold and $S$ is continuously differentiable. Then for any $n \geq 2$ and $0 < \rho < 1/2$
\[
\mathcal{R}^*(S^*, S) \leq \frac{1 + 5\rho}{1 - \rho} \min_{\gamma \in \Gamma} \mathcal{R}^*_n(S^*_\gamma, S) + \frac{1}{\rho n} U_n(1 + \|\hat{S}\|^2),
\]
where the term $U_n$ satisfies the property (5.3).

### 5.2 Asymptotic efficiency

In order to study the asymptotic efficiency we define the following functional Sobolev ball
\[
W_{k,r} = \{ f \in C^k_p[0, 1] : \sum_{j=0}^k \|f^{(j)}\|^2 \leq r \},
\]
where $r > 0$ and $k \geq 1$ are some unknown parameters, $C^k_p[0, 1]$ is the space of $k$ times differentiable 1-periodic $\mathbb{R} \to \mathbb{R}$ functions such that $f^{(i)}(0) = f^{(i)}(1)$ for any $0 \leq i \leq k - 1$. To study the asymptotic efficiency we denote by $\Sigma_n$ all estimators $\hat{S}_n$ i.e. any $\sigma \{y_t, 0 \leq t \leq n\}$ measurable functions. In the sequel we denote by $Q^*$ the distribution of the process $(y_t)_{0 \leq t \leq n}$ with $\xi_t = \varsigma^* w_t$, i.e. white noise model with the intensity $\varsigma^*$.

**Theorem 5.3.** Assume that $Q^* \in Q_n$. The robust risk (1.6) admits the following lower bound
\[
\liminf_{n \to \infty} \inf_{\hat{S}_n \in \Sigma_n} \nu^{2k/(2k+1)} \sup_{S \in W_{k,r}} \mathcal{R}^*(\hat{S}_n, S) \geq l_k(r),
\]
where $l_k(r) = ((2k + 1)r^{1/(2k+1)} (k/\pi(k + 1)))^{2k/(2k+1)}$.

We show that this lower bound is sharp in the following sense.
Theorem 5.4. Assume that $Q^* \in Q_n$ and there exists $\epsilon > 0$ such that
$$\lim_{n \to \infty} n^{5/6+\epsilon}/p = 0.$$ Then the robust risk of the model selection procedure (4.7) with the weight coefficients (4.11) satisfies the following upper bound
$$\limsup_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_{k,r}} R^*(S^*, S) \leq l_k(r).$$

It is clear that these theorems imply the following efficient property.

Corollary 5.5. Assume that $Q^* \in Q_n$. Then the model selection procedure (4.7) with the weight coefficients (4.11) is asymptotically efficient, i.e.
$$\lim_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_{k,r}} R^*(S^*, S) = l_k(r).$$

Theorem 5.3 is shown by the same way as Theorem 1 in [15]. Theorem 5.4 follows from Theorems 5.2, 3.1 and Theorem 5.2 in [18].

6 Monte Carlo simulations

In this section we give the results of numerical simulations to assess the performance and improvement of the proposed model selection procedure (4.6). We simulate the model (1.1) with 1-periodic functions $S$ of the forms

\begin{align*}
S_1(t) &= t \sin(2\pi t) + t^2(1 - t) \cos(4\pi t), \quad (6.1) \\
S_2(t) &= 0.5 - 0.5 - |0.5 - t| \quad (6.2)
\end{align*}
on [0, 1] and the Lévy noise process $\xi_t$ is defined as
$$d\xi_t = -\xi_t dt + 0.5 \, dw_t + 0.5 \, dz_t, \quad z_t = \sum_{j=1}^{N_t} Y_j,$$
where $N_t$ is a homogeneous Poisson process with intensity $\lambda = 1$ and $(Y_j)_{j \geq 1}$ is i.i.d. $\mathcal{N}(0, 1)$ sequence (see, for example, [18]).

We use the model selection procedure (4.6) with the weights (4.10) in which $k^* = 100 + \sqrt{\ln(n + 1)}$, $r_i = i/\ln(n + 1)$, $m = [\ln^2(n + 1)]$, $\zeta^* = 0.5$ and $\rho = (3 + \ln(n))^{-2}$. We define the empirical risk as
$$R(S^*, S) = \frac{1}{p} \sum_{j=1}^{p} \hat{E}\Delta_n^2(t_j) \quad \text{and} \quad \hat{E}\Delta_n^2(t) = \frac{1}{N} \sum_{l=1}^{N} \Delta_n^2(t),$$
where $\Delta_n(t) = S^*_n(t) - S(t)$ and $\Delta_{n,l}(t) = S^*_{n,l}(t) - S(t)$ is the deviation for the $l$-th replication. In this example we take $p = 10001$ and $N = 1000$.

Table 1 gives the values for the sample risks of the improved estimate (4.6) and the model selection procedure based on the weighted LSE (3.15).
Table 1: The sample quadratic risks for different optimal $\gamma$

| $n$ | 100  | 200  | 500  | 1000 |
|-----|------|------|------|------|
| $\mathcal{R}(S^*_\gamma, S_1)$ | 0.0819 | 0.0319 | 0.0098 | 0.0051 |
| $\mathcal{R}(\hat{S}_\gamma, S_1)$ | 0.0787 | 0.0479 | 0.0287 | 0.0178 |
| $\mathcal{R}(\hat{S}_\gamma, S_1)/\mathcal{R}(S^*_\gamma, S_1)$ | 0.9 | 1.5 | 2.9 | 3.5 |
| $\mathcal{R}(S^*_\gamma, S_2)$ | 1.0819 | 0.9119 | 0.1198 | 0.0161 |
| $\mathcal{R}(\hat{S}_\gamma, S_2)$ | 3.0187 | 1.0197 | 0.4287 | 0.0998 |
| $\mathcal{R}(\hat{S}_\gamma, S_2)/\mathcal{R}(S^*_\gamma, S_2)$ | 2.8 | 1.1 | 3.6 | 6.2 |

Table 2: The sample quadratic risks for the same optimal $\hat{\gamma}$

| $n$ | 100  | 200  | 500  | 1000 |
|-----|------|------|------|------|
| $\mathcal{R}(S^*_\hat{\gamma}, S_1)$ | 0.0871 | 0.0397 | 0.0195 | 0.0097 |
| $\mathcal{R}(\hat{S}_\gamma, S_1)$ | 0.0787 | 0.0479 | 0.0287 | 0.0178 |
| $\mathcal{R}(\hat{S}_\gamma, S_1)/\mathcal{R}(S^*_\hat{\gamma}, S_1)$ | 0.9 | 1.2 | 1.5 | 1.8 |
| $\mathcal{R}(S^*_\hat{\gamma}, S_2)$ | 2.0671 | 0.9979 | 0.2195 | 0.0198 |
| $\mathcal{R}(\hat{S}_\gamma, S_2)$ | 3.0187 | 1.0197 | 0.4287 | 0.0998 |
| $\mathcal{R}(\hat{S}_\gamma, S_2)/\mathcal{R}(S^*_\hat{\gamma}, S_2)$ | 1.5 | 1.0 | 2.0 | 5.0 |

Figure 1: Behavior of the regression functions and their estimates for $n = 1000$.

from [17] for different numbers of observation period $n$. Table 2 gives the values for the sample risks of the the model selection procedure based on the weighted LSE (3.15) from [17] and its improved version for different numbers of observation period $n$.

**Remark 6.1.** Figure shows the behavior of the procedures (3.8) and (4.6)
depending on the values of observation periods \( n \). The bold line is the function (6.2), the continuous line is the model selection procedure based on the least squares estimators \( \hat{S} \) and the dashed line is the improved model selection procedure \( S^* \). From the Table 2 for the same \( \gamma \) with various observations numbers \( n \) we can conclude that theoretical result on the improvement effect (3.16) is confirmed by the numerical simulations. Moreover, for the proposed shrinkage procedure, Table 1 and Figures 1–3, we can conclude that the benefit is considerable for non large \( n \). However we note that for the function \( S_1 \) we have an improvement in accuracy for \( n > 100 \).

7 Stochastic calculus for the non-Gaussian Ornstein-Uhlenbeck processes

In this section we study the process (2.1).

Proposition 7.1. Let \( f \) and \( g \) be two nonrandom left continuous \( \mathbb{R}_+ \to \mathbb{R} \) functions with the finite right limits. Then for any \( t > 0 \)

\[
\mathbb{E} \, I_t(f)I_t(g) = \sigma_Q \tau_t(f, g), \tag{7.1}
\]

where

\[
\tau_t(f, g) = \int_0^t (f(s)g(s) + \tilde{\varepsilon}_s(f)g(s) + f(s)\tilde{\varepsilon}_s(g)) \, ds \quad \text{and}
\]

\[
\tilde{\varepsilon}_t(f) = a \int_0^t e^{a(t-s)} f(s) \left( \frac{1 + e^{2as}}{2} \right) \, ds.
\]

Proof. Taking into account the definitions (3.7) and (2.1) we obtain through the Ito formula that

\[
I_t(f)I_t(g) = \sigma_Q \int_0^t f(s)g(s) \, ds + a \int_0^t \Upsilon_s(f, g) \, \xi_t ds + M_t(f, g), \tag{7.2}
\]

where

\[
\Upsilon_s(f, g) = f(s)I_s(g) + g(s)I_s(f),
\]

\[
M_t(f, g) = \int_0^t \Upsilon_{t-s}(f, g) \, du_s + \sigma_2^2 \int_0^t f(s)g(s) \, dm_s
\]

and \( m_t = x^2 * (\mu - \tilde{\mu})_t \). Moreover, using the Ito formula we obtain

\[
\mathbb{E} \, I_t^2(1) = \mathbb{E} \, \xi_t^2 = \sigma_Q \frac{e^{2at} - 1}{2a}. \tag{7.3}
\]

Note now, that

\[
\mathbb{E} \, I_t^2(f) = \mathbb{E} \left( a \int_0^t f(s)\xi_s ds + \int_0^t f(s) \, du_s \right)^2
\]

\[
\leq 2a^2 \int_0^t f^2(s)ds \int_0^t \mathbb{E} \xi_s^2 ds + 2\sigma_Q \int_0^t f^2(s)ds.
\]

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So, from here

\[
\sup_{0 \leq t \leq n} \mathbb{E} I_t^2(f) \leq 2\sigma_Q (|a| + 1) \int_0^n f^2(s) ds < \infty. \tag{7.4}
\]

This implies immediately that \( \mathbb{E} M_t(f, g) = 0 \). Using this in (7.2) yields

\[
\mathbb{E} I_t(f) I_t(g) = \sigma_Q \int_0^t f(s) g(s) ds + a \int_0^t (f(s) \mathbb{E} \zeta_s(g) + g(s) \mathbb{E} \zeta_s(f)) ds, \tag{7.5}
\]

where \( \zeta_t(f) = \xi_t(t) I_t(f) = I_t(1) I_t(f) \). Therefore, putting \( g = 1 \) in (7.5), we obtain that

\[
\mathbb{E} \zeta_t(f) = \sigma_Q \int_0^t f(s) ds + a \int_0^t (f(s) \mathbb{E} \zeta_s(1) + \mathbb{E} \zeta_s(f)) ds.
\]

Taking into account here, that \( \zeta_t(1) = \xi_t^2 \), we obtain that

\[
\mathbb{E} \zeta_t(f) = \sigma_Q \int_0^t e^{a(t-s)} f(s) \frac{1 + e^{2as}}{2} ds = \sigma_Q \dot{E}_t^f.
\]

Therefore, using this in (7.5) we obtain (7.1). \( \square \)

**Corollary 7.2.** For any cadlag function \( f \) from \( L_2[0,n] \)

\[
\mathbb{E} I_n^2(f) \leq 2\sigma_Q \int_0^n f^2(s) ds. \tag{7.6}
\]

**Proof.** Indeed, putting \( f = g \) in (7.1) we get

\[
\mathbb{E} I_n^2(f) = \sigma_Q \int_0^n (f^2(t) + 2\dot{E}_t(f) f(t)) dt.
\]

Moreover, note that

\[
\int_0^n \dot{E}_t(f) f(t) dt = a \int_0^n e^{ax} \int_x^n f(t)(t - x) \frac{1 + e^{2as(t-x)}}{2} dx dt.
\]

By the Cauchy-Bunyakovsky-Schwarz inequality

\[
\int_0^n \dot{E}_t(f) |f(t)| dt \leq |a| \int_0^n e^{ax} dx \int_0^n f^2(t) dt dt \leq \int_0^n f^2(t) dt.
\]

This implies immediately upper bound (7.6). Hence Corollary 7.2. \( \square \)

Now we set

\[
\tilde{I}_t(f) = I_n^2(f) - \mathbb{E} I_n^2(f) \quad \text{and} \quad V_t(f) = \zeta_t(f) - \mathbb{E} \zeta_t(f). \tag{7.7}
\]
Using (7.2) with \( f = g \) we can obtain that
\[
d\tilde{I}_t(f) = 2af(t)V_t(f)dt + d\tilde{M}_t(f),
\]
where \( \tilde{M}_t(f) = M_t(f, f) \). To study this process we need to introduce the following functions
\[
\tilde{\tau}_i(f, g) = f(t)g(t)\tau_i(1, 1) + f(t)\tau_i(1, g) + g(t)\tau_i(1, f) + \tau_i(f, g)
\]
and
\[
A_i(f) = \int_0^te^{3a(t-s)}f(s)v(s)ds + 2\sigma_Q^2\int_0^te^{3a(t-s)}\tilde{\xi}_s(f)ds,
\]
where \( v(s) = \alpha^2E\tilde{\xi}_s^2 + \sigma_Q^2(e^{2as} - 1) + \alpha\tilde{\theta}_2, \tilde{\xi}_s = \xi_s^2 - E\xi_s^2 \) and \( \tilde{\theta}_2 = \theta_2\Pi(x^4) \).

**Proposition 7.3.** For any left continuous functions with finite right limits \( f \) and \( g \)
\[
E V_t(f)V_t(g) = \int_0^t e^{2a(t-s)}H_s(f, g)ds
\]
where \( H_t(f, g) = g(t)A_t(f) + f(t)A_t(g) + \sigma_Q^2\tilde{\tau}_t(f, g) + \tilde{\theta}_2 f(t)g(t) \).

**Proof.** Applying again (7.2) with \( g = 1 \) yields
\[
dV_t(f) = aV_t(f)dt + a f(t)\tilde{I}_t(1)dt + dL_t(f),
\]
where \( L_t(f) = \int_0^t \tilde{I}_{s-}(f)du_s + \tilde{\theta}_2\int_0^t f(s)dm_s \) and \( \tilde{I}_s(f) = f(s)\xi_s + I_s(f) \).
By the Ito formula we get
\[
dV_t(f)V_t(g) = 2aV_t(f)V_t(g)dt + a (g(t)V_t(f) + f(t)V_t(g)) \tilde{I}_t(1)dt + d[L(f), L(g)]_t + V_{t-}(f)L_t(g) + V_{t-}(g)L_t(f).
\]
Now from Lemma A.2 we obtain that
\[
dE V_t(f)V_t(g) = 2aE V_t(f)V_t(g)dt + (g(t)A_t(f) + f(t)A_t(g)) dt + dE[L(f), L(g)]_t,
\]
where \( A_t(f) = aE V_t(f)\tilde{I}_t(1) = aE V_t(f)V_t(1) \). Note that \( E \tilde{I}_s(f)\tilde{I}_s(g) = \sigma_Q^2\tilde{\tau}_s(f, g) \) and
\[
E[L(f), L(g)]_t = \tilde{\theta}_2^2\int_0^t E \tilde{I}_s(f)\tilde{I}_s(g)ds + E \sum_{0 \leq s \leq t} \Delta L_s(f)\Delta L_s(g)
\]
\[
= \sigma_Q^2\int_0^t \tilde{\tau}_s(f, g)ds + \tilde{\theta}_2\int_0^t f(s)g(s)ds.
\]
To find the function $A_t(f)$ we put $g = 1$ in (7.13). Taking into account that $A_t(1) = \tilde{I}^2_t = \xi^2_t$ we get

$$E V_t(f)V_t(1) = \int_0^t e^{3a(t-s)} \left( a f(s) E \tilde{\xi}^2_s + \sigma^2_Q \tilde{\xi}_s(f, 1) + \tilde{\theta}_2 f(s) \right) ds .$$

Using here that

$$a \tau_t(1, 1) = (e^{2at} - 1)/2 \quad \text{and} \quad a \tau_t(1, f) = \tilde{\varepsilon}_t(f) , \quad (7.14)$$

we obtain the representation (7.10). Hence Proposition 7.3.

**Proposition 7.4.** For any left continuous function $f$ with finite right limits

$$E \tilde{I}_t(f) \tilde{I}_t(1) = \int_0^t e^{2a(t-s)} \tilde{\kappa}_s(f) ds , \quad (7.15)$$

where $\tilde{\kappa}_s(f) = 2 f(s) A_s(f) + 4 \sigma^2 Q \tilde{\xi}_s(f, 1) + \tilde{\theta}_2 f^2(s)$.

**Proof.** Using the Ito formula and Lemma A.2 we obtain that for any bounded nonrandom functions $f$ and $g$

$$dE \tilde{I}_t(f)V_t(g) = aE \tilde{I}_t(f)V_t(g) dt + 2af(t)E V_t(f)V_t(g) dt$$

$$+ a g(t) E \tilde{I}_t(f) \tilde{I}_t(1) dt + dE [\tilde{M}(f), L(g)]_t . \quad (7.16)$$

Putting here $g = 1$ and taking into account that $V_t(1) = \tilde{I}_t(1)$, we obtain that

$$dE \tilde{I}_t(f)V_t(1) = 2aE \tilde{I}_t(f)V_t(1) dt + 2af(t)E V_t(f)V_t(1) dt$$

$$+ dE [\tilde{M}(f), L(1)]_t .$$

By the direct calculation we find

$$E [\tilde{M}(f), L(1)]_t = \int_0^t \tilde{\kappa}_s(f) ds .$$

So, we get (7.15) and this proposition.

Further we need the following correlation measures for two integrated $[0, +\infty) \rightarrow \mathbb{R}$ functions $f$ and $g$

$$\varpi_n(f, g) = \max_{0 \leq v + t \leq n} \left( \left| \int_0^t f(u + v)g(u)du \right| + \left| \int_0^t g(u + v)f(u)du \right| \right) \quad (7.17)$$

For any bounded $[0, \infty) \rightarrow \mathbb{R}$ function $f$ we introduce the following uniform norm

$$\|f\|_{*,n} = \sup_{0 \leq t \leq n} |f(t)| .$$
Proposition 7.5. Let \( f \) and \( g \) be two left continuous bounded by \( \phi_s \) functions with finite right limits, i.e. \( \| f \|_{s,n} \leq \phi_s \) and \( \| g \|_{s,n} \leq \phi_s \). Then for any \( 0 \leq t \leq n \)

\[
\left| a \mathbb{E} \tilde{I}_t(f) V_t(g) \right| \leq u_1^* \varpi_t(1, g) + u_2^* \varpi_t(f, g) + u_3^*,
\]

where \( u_1^* = 4\phi_s^2(a_{\text{max}})\tilde{b}_2 + 3\sigma_Q^2, \ u_2^* = 44\phi_s\sigma_Q^2 \) and \( u_3^* = 3\phi_s^2\tilde{b}_2 \).

Proof. First, note that from Ito formula we find

\[
a \mathbb{E} \tilde{I}_t(f) V_t(g) = a^2 \int_0^t e^{a(t-s)} g(s) \left( \mathbb{E} \tilde{I}_s(f) \tilde{I}_s(1) \right) ds
+ 2a^2 \int_0^t e^{a(t-s)} f(s) \left( \mathbb{E} V_s(g) V_s(f) \right) ds
+ a \int_0^t e^{a(t-s)} d\mathbb{E} \left[ \tilde{M}(f), L(g) \right]_s.
\]

(7.19)

Using here Lemma A.4. and Lemma A.6 we can obtain that

\[
|a \mathbb{E} V_t(g) V_t(f)| \leq 15\sigma_Q^2 \varpi_t(f, g) + \tilde{b}_2 \| f \|_{s,t} \| g \|_{s,t}.
\]

(7.20)

One can check directly that

\[
\mathbb{E} \left[ \tilde{M}(f), L(g) \right]_t = 2\sigma_Q \int_0^t g(s)f(s) \left( \mathbb{E} I_s(f) I_s(1) \right) ds
+ 2\sigma_Q \int_0^t f(s) \left( \mathbb{E} I_s(f) I_s(g) \right) ds + \tilde{b}_2 \int_0^t f^2(s) g(s) ds.
\]

From (7.1) we find that

\[
\mathbb{E} \left[ \tilde{M}(f), L(g) \right]_s = 2\sigma_Q^2 \int_0^s g(s)f(s) \tau_s(f, 1) ds
+ 2\sigma_Q^2 \int_0^s f(s) \tau_s(f, g) ds + \tilde{b}_2 \int_0^s f^2(s) g(s) ds.
\]

Using the last equality in (7.14) we obtain that

\[
a \int_0^t e^{a(t-s)} d\mathbb{E} \left[ \tilde{M}(f), L(g) \right]_s = 2\sigma_Q^2 \int_0^t e^{a(t-s)} g(s)f(s) \tilde{\xi}_s(f) ds
+ 2\sigma_Q^2 a \int_0^t e^{a(t-s)} f(s) \tau_s(f, g) ds + a\tilde{b}_2 \int_0^t e^{a(t-s)} f^2(s) g(s) ds.
\]

Note now that

\[
\tilde{\xi}'_t(f) = a\tilde{\xi}_t(f) + af(t)(1 + e^{2at})/2,
\]

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i.e. $\|\tilde{f}\|_{*,t} \leq 2a\|f\|_{*,t}$. Therefore, in view of Lemma A.3 we get

$$\left| \int_0^t e^{a(t-s)} g(s) f(s) \tilde{\epsilon}_s(f) ds \right| \leq 4\varpi_t(f,g)\|f\|_{*,t}.$$  

Moreover, by integrating by parts we can obtain directly that

$$\left| \int_0^t g(s) \tilde{\epsilon}_s(f) ds \right| \leq \varpi_t(f,g),$$

and, therefore,

$$|\tau_t(f,g)| \leq 3\varpi_t(f,g). \tag{7.21}$$

So, the last term in (7.19) can be estimated as

$$\left| a \int_0^t e^{a(t-s)} dE \left[ \tilde{M}(f), \tilde{M}(g) \right] \right| \leq 14\sigma^2_Q \varpi_t(f,g) \|f\|_{*,t} + \vartheta_2 \|f\|_{*,t}^2 \|g\|_{*,t}.$$  

Using Lemma A.5 in (7.19) we come to the bound (7.18). Hence Proposition 7.5. \qed

**Proposition 7.6.** Let $f$ and $g$ be two left continuous bounded by $\phi_*$ functions with finite right limits, i.e. $\|f\|_{*,n} \leq \phi_*$ and $\|g\|_{*,n} \leq \phi_*$. Then for any $t > 0$

$$\left| E \left[ \tilde{M}(f), \tilde{M}(g) \right] \right| \leq \left( 12\sigma^2_Q \phi_*^2 \varpi_t(f,g) + \phi_*^4 \vartheta_2 \right) t. \tag{7.22}$$

**Proof.** First of all note that from (7.1) we obtain that

$$E \left[ \tilde{M}(f), \tilde{M}(g) \right] = 4\sigma^2_Q \int_0^t f(s)g(s)\tau_s(f,g) ds$$

$$+ \vartheta_2 \int_0^t f^2(s)g^2(s) ds. \tag{7.23}$$

Using here the bound (7.21) we obtain (7.22). Hence Proposition 7.6. \qed

**Corollary 7.7.** Let $f$ and $g$ be two left continuous bounded by $\phi_*$ functions with finite right limits, i.e. $\|f\|_{*,n} \leq \phi_*$ and $\|g\|_{*,n} \leq \phi_*$. Then for any $t > 0$

$$\left| E \tilde{I}_t(f)\tilde{I}_t(g) \right| \leq \left( v_1^* (\varpi_t(1,f) + \varpi_t(1,g)) + v_2^* \varpi_t(f,g) + v_3^* \right) t, \tag{7.24}$$

where $v_1^* = 8\phi_*^2 a_{\max} \vartheta_2 + 6\sigma^2_Q$, $v_2^* = 100\phi_*^2 \sigma^2_Q$ and $v_3^* = 13\phi_*^4 \vartheta_2$.  



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Proof. From (7.8) by the Ito formula one finds for $t \geq 0$

$$E \tilde{I}_t(f)\tilde{I}_t(g) = E [\tilde{M}(f), \tilde{M}(g)]_t$$

$$+ 2a \int_0^t \left( f(s)E \tilde{I}_s(g) V_s(f) + g(s)E \tilde{I}_s(f) V_s(g) \right) ds. \quad (7.25)$$

Using here Proposition 7.5 and Proposition 7.6 we come to desire result.

Now we set

$$I_n(x) = \sum_{j=1}^n x_j \tilde{I}_n(\phi_j). \quad (7.26)$$

For this we show the following proposition.

**Proposition 7.8.** Assume that $\phi_1 \equiv 1$. Then for any $n \geq 1$

$$E I_n^2(x) \leq 4n^2 \left( (2\varpi^*_n + 4\phi^*_2)v^*_1 + (\varpi^*_n + 8\phi^*_2)v^*_2 + (1 + 2\phi^*_2)v^*_3 \right), \quad (7.27)$$

where $\varpi^*_n = \sup_{|i-j| \geq 2} \varpi_n(\phi_i, \phi_j)$.

**Proof.** We represent the sum as

$$\sum_{j=1}^n x_j \tilde{I}_n(\phi_j) = J_{1,n} + J_{2,n},$$

where $J_{1,n} = x_1 \tilde{I}_n(\phi_1) + x_2 \tilde{I}_n(\phi_2)$ and $J_{2,n} = \sum_{j=3}^n x_j \tilde{I}_n(\phi_j)$. From here we have

$$E I_n^2(x) \leq 2 \left( E J_{1,n}^2 + E J_{2,n}^2 \right).\quad (7.28)$$

By applying the Cauchy-Bunyakovsky-Schwarz inequality and noting that $x_1^2 + x_2^2 \leq 1$, one gets

$$E J_{1,n}^2 \leq E \tilde{I}_n^2(\phi_1) + E \tilde{I}_n^2(\phi_2).$$

Corollary 7.7 implies

$$E_{Q,S} J_{1,n}^2 \leq 4\phi^2_n (2v^*_1 + v^*_2 + v^*_3).$$

Here we use that each $\varpi_n(\phi_i, \phi_j) \leq 2\phi^*_2 n$.

Applying Corollary 7.7, one gets

$$E J_{2,n}^2 = 2 \sum_{i,j=3}^n x_ix_j E \tilde{I}_n(\phi_i) \tilde{I}_n(\phi_j) \leq 2n \sum_{i,j=3}^n |x_i||x_j|\tilde{c}_{i,j}, \quad (7.29)$$

where $\tilde{c}_{i,j} = v^*_1 (\varpi_n(1, \phi_i) + \varpi_n(1, \phi_j)) + v^*_2 \varpi_n(\phi_i, \phi_j) + v^*_3$. We can estimate the coefficient $\varpi_{i,j} = \varpi_n(\phi_i, \phi_j)$ for any $i \geq 3$ as $\varpi_{i,j} \leq 2\phi^*_2 n 1_{\{|i-j| \leq 1\}} + \ldots$
By making use of this estimate in (7.29) and taking into account that
\[ \sum_{i,j \geq 1} |x_i||x_j| \leq 1 \quad \text{and} \quad \sum_{i,j \geq 3} 1_{\{|i-j| \leq 1\}} |x_i||x_j| \leq 3, \]
one gets
\[ \sum_{i,j \geq 3} |x_i||x_j| \lesssim n \left(2\varpi^* v_1^* + (6\phi_2^* + \varpi^*_n)v_2^* + v_3^*\right). \]

From here and the inequalities (7.28)–(7.29) we come to the desired assertion. Hence Proposition 7.8.

Now we check the conditions \(C_1\) and \(C_2\) for the model with noise process (2.1). Here we will use the trigonometric basis (3.3).

**Proposition 7.9.** For any \(Q \in \mathcal{Q}_n\) and any \(n \geq 1\)
\[ \mathbf{L}_{t,n}(Q) \leq 2\sigma_Q^2 (4a^2 + 15|a| + 2). \]

**Proof.** First we note that
\[ \mathbf{E}_{Q,S}\xi_{i,p}^2 = \sigma_Q (1 + b_{j,n}), \]where \(b_{j,n} = n^{-1}a \int_0^n e^{av} Y_j(v)dv\) and
\[ Y_j(v) = \int_0^n \text{Tr}_j(t+v) \text{Tr}_j(t) \left(1 + e^{2at}\right) dt. \]
If \(j = 1\), one has
\[ |\mathbf{E}_{Q,S}\xi_{1,p}^2 - \sigma_Q| \leq 2\sigma_Q. \]Since for the trigonometric basis (4.1) for \(j \geq 2\)
\[ \text{Tr}_j(t+v) \text{Tr}_j(t) = \cos(a_j v) + (-1)^j \cos(a_j (2t + v)) \]
where \(a_j = 2\pi [j/2]\), therefore,
\[ Y_j(v) = \cos(a_j v) F(v) + (-1)^j Y_{0,j}(v), \quad F(v) = \int_0^n (1 + e^{2at}) dt \]
and
\[ Y_{0,j}(v) = \int_0^n \cos(a_j (2t + v)) (1 + e^{2at}) dt. \]
Integrating by parts one finds
\[ Y_{0,j}(v) = -\frac{2 + e^{2a(n-v)}}{2a_j} \sin(va_j) + \frac{a}{2a_j} Y_{1,j}(v) \]
where 
\[ \Upsilon_{1,j}(v) = \cos(va_j)(e^{2a(n-v)} - 1) - 2a \int_0^{n-v} e^{2at} \cos((2t+v)a_j) \, dt. \]

It is obvious that \(|\Upsilon_{1,j}(v)| \leq 2\). Further we calculate
\[ b_{j,n} = \frac{a}{n} \int_0^n e^{av} F(v) \cos(va_j) \, dv + \frac{a}{n} (-1)^j \int_0^n e^{av} \Upsilon_{0,j}(v) \, dv \]
\[ := aD_{1,j} + a(-1)^j D_{2,j}. \]

Integrating by parts two times yields
\[ D_{1,j} = \frac{1}{na^2_j} \left( e^{an} \dot{F}(n) - \dot{F}(0) - aF(0) - \int_0^n e^{av} F_1(v) \, dv \right), \]
where \( F_1(v) = a^2 F(v) + 2a \dot{F}(v) + \ddot{F}(v) \). Since \( a_j \geq j \) for \( j \geq 2 \), we obtain
\[ |D_{1,j}| \leq \frac{1}{j^2} (4|a| + 10). \]

Similarly, one gets \(|D_{2,j}| \leq 5/j^2\). Substituting these estimates in (7.30) and using the upper bound (7.31), we obtain for all \( j \geq 1 \)
\[ |E_{Q,S}^2 j,p - \sigma_Q| \leq \sigma_Q \frac{(4a^2 + 15|a| + 2)}{j^2}. \]  

(7.32)

Thus we arrive at the inequality
\[ L_{1,n}(Q) \leq 2\sigma_Q (4a^2 + 15|a| + 2). \]

Hence Proposition 7.9.

Proposition 7.9 and the conditions (2.4) – (2.5) imply that the condition \( C_1 \) holds with \( L_{1,n}^* = 2\sigma_n^*(4a_{\text{max}}^2 + 15a_{\text{max}} + 2) \).

**Proposition 7.10.** For any \( n \geq 1 \) and \( Q \in Q_n \)
\[ L_{2,n}(Q) \leq 8M_Q, \]
where \( M_Q = 48 \sqrt{2}|a|\tilde{\varphi}_2 + 918\sigma_Q + 65\tilde{\varphi}_2 \).

**Proof.** We note that for the trigonometric basis (3.3) \( ||\text{Tr}_j||_{s,n} \leq \sqrt{2} \) and \( \varpi_n^* = 2 \). Indeed, for any \( i \geq 3 \),
\[ \text{Tr}_i(v + u) = \kappa_{1,i}(v)\text{Tr}_{i-1}(u) + \kappa_{2,i}(v)\text{Tr}_i(u) + \kappa_{3,i}(v)\text{Tr}_{i+1}(u), \]
where \( \kappa_{i,j}(\cdot) \) are bounded functions. From here in view of the orthonormality and the periodicity of the functions \((\text{Tr}_j)_{j \geq 1}\), it follows that for \(0 \leq t \leq n\) and \(|i - j| \geq 2\)

\[
\left| \int_0^t \text{Tr}_i(u + v)\text{Tr}_j(u)du \right| = \left| \int_0^{\{t\}} \text{Tr}_i(u + v)\text{Tr}_j(u)du \right| \\
\leq \sqrt{\int_0^1 \text{Tr}_i^2(u + v)du} = 1,
\]

where \(\{t\}\) is the fractional part of \(t\). Therefore \(\varpi^*_n \leq 2\) if \(|i - j| \geq 2\). Thus, taking into account Proposition 7.8 we have that \(L_{2,n}^*(Q) \leq 8M_Q\). Hence Proposition 7.10.

Proposition 7.10 and the conditions (2.4) - (2.5) imply that the upper bound for \(L_{2,n}^*(Q)\) in the condition (2.5) is equal to

\[ L_{2,n}^* = 8(48\sqrt{2}a_{\text{max}} + 65)(\varsigma^*)^2\Pi(x^4) + 7344\varsigma^*. \]

8 Proofs

8.1 Proof of Theorem 3.1

Consider the quadratic error of the estimate (3.14)

\[
R_Q(S^*_\lambda, S) = \mathbf{E} \sum_{j=1}^d (\lambda(j)\theta^*_j - \theta_j)^2 = \mathbf{E}|\theta^*_p - \theta|_d^2 + \mathbf{E} \sum_{j=d+1}^p (\lambda(j)\bar{\theta}_j - \theta_j)^2 \\
= \mathbf{E}|\theta^*_p - \bar{\theta}|_d^2 + 2\mathbf{E}(\theta^*_p - \bar{\theta}_p)\bar{h}_p(S) + |h_p(S)|_d^2,
\]

where \(\bar{\theta} = (\bar{\theta}_{1,p}, \ldots, \bar{\theta}_{d,p})'\) (the prime denotes the transposition) with components \(\bar{\theta}_{j,p} = \theta_j + h_{j,p}(S)\) and the first summand on the right-hand side can be represented as

\[
\mathbf{E}|\theta^*_p - \bar{\theta}|_d^2 = \mathbf{E}|\hat{\theta}_p - \bar{\theta}|_d^2 + c_n^2 - 2c_n\mathbf{E}_{Q,S} \sum_{j=1}^d J_{j,n},
\]

where \(J_{j,n} = \mathbf{E}(\lambda_j(\bar{\theta}_n)(\bar{\theta}_{j,n} - \bar{\theta}_{j,p})|\mathcal{G}_n)\) with \(\lambda_j(x) = x_j/|x|_d\) for \(x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d\). Now, taking into account that the vector \(\bar{\theta}_n = (\bar{\theta}_{j,n})_{1 \leq j \leq d}\) is the \(\mathcal{G}_p\) conditionally gaussian vector in \(\mathbb{R}^d\) with mean \(\bar{\theta}_p\) and covariance matrix \(n^{-1}\mathbf{G}_p\), we obtain

\[
J_{j,n} = \int_{\mathbb{R}^d} \lambda_j(x)(x - \bar{\theta}_{j,p})p(x|\mathcal{G}_p)dx.
\]

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In view of the inequality $z$, where $J$, furthermore, integrating by parts, the integral $\int \tilde{t}_{j,n}(u)u_i\exp\left(-\frac{n|u|^2}{2}\right)du$, one finds that

$$J_{j,n} = \frac{n^{d/2}}{(2\pi)^{d/2}} \sum_{l=1}^{\infty} \mathbf{v}_{j,l} \int_{\mathbb{R}^d} \tilde{t}_{j,n}(u)u_i\exp\left(-\frac{n|u|^2}{2}\right)du,$$

where $\tilde{t}_{j,n}(u) = t_j(G_p^{1/2}u + \tilde{\theta}_p)$ and $\mathbf{v}_{ij}$ denotes the $(i,j)$-th element of $G_p^{1/2}$. Furthermore, using the Jensen inequality we can estimate the last expectation.

In view of the inequality $z'Az \leq \lambda_{\max}(A)||z||^2$ and Condition D$_2$) we obtain that

$$E|\theta_p(n) - \tilde{\theta}_p|^2 - E|\tilde{\theta}_p(n) - \tilde{\theta}_p|^2 = c_n^2 - 2c_n n^{-1} E_{Q,S} \left( \frac{\text{tr}G_p \hat{\theta}_n}{|\hat{\theta}_n|^d} - \frac{\hat{\theta}_n G_p \hat{\theta}_n}{|\hat{\theta}_n|^3} \right)$$

$$\leq c_n^2 - 2c_n t_p n^{-1} E_{Q,S} \frac{1}{|\hat{\theta}_n|^d}.$$

Moreover, using the Jensen inequality we can estimate the last expectation from below as

$$E_{Q,S}(|\hat{\theta}_n|_d)^{-1} = E_{Q,S}(|\hat{\theta} + n^{-1/2} \xi_n|_d)^{-1} \geq (|\theta|_d + n^{-1/2} E_{Q,S} |\xi_n|_d)^{-1}.$$

From (1.3) and the condition (2.4) we obtain

$$E_{Q,S} |\xi_n|_d^2 \leq c_n^2 d.$$

So, for $||S|| \leq r$

$$E_{Q,S} |\hat{\theta}_n|_d^{-1} \geq \left(r + \sqrt{d c_n^2 / n}\right)^{-1}$$

and, therefore,

$$E|\theta_p(n) - \tilde{\theta}_p|_d^2 \leq E|\tilde{\theta}_p(n) - \tilde{\theta}_p|_d^2 - c_n^2$$

$$= E|\tilde{\theta}_p(n) - \theta|_d^2 - c_n^2 - 2E(\tilde{\theta}_p(n) - \theta)'h_p(S) + |h_p(S)|_d^2.$$

Substituting this estimate in (8.1), one has

$$\Delta_Q(S) \leq -c_n^2 + 2E(\theta_p(n) - \tilde{\theta}_p(n))'h_p(S). \quad (8.2)$$
Using the elementary inequality
\[ 2|ab| \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}, \quad \varepsilon > 0, \]
we obtain
\[ 2E(\theta_p^*(n) - \hat{\theta}_p(n))'h_p(S) \leq \varepsilon E|\theta_p^*(n) - \hat{\theta}_p(n)|_d^2 + \frac{1}{\varepsilon}|h_p(S)|_d^2 \]
\[ = \varepsilon c_n^2 + \frac{1}{\varepsilon}|h_p(S)|_d^2 \]
Now, from definition of \( h_p(S) \) and by applying the Cauchy-Bunyakovsky-Schwarz inequality, we have the estimate
\[ |h_p(S)|_d^2 = \sum_{j=1}^{d} \left( \sum_{k=1}^{p} \int_{t_{k-1}}^{t_k} \phi_j(t_k) (S(t) - S(t_k)) dt \right)^2 \]
\[ \leq \sum_{j=1}^{d} \sum_{k=1}^{p} \int_{t_{k-1}}^{t_k} \phi_j^2(t_k) dt \sum_{k=1}^{p} \int_{t_{k-1}}^{t_k} (S(t) - S(t_k))^2 dt \leq \frac{d\phi^2 L^2}{p^2}. \]
From here it follows that
\[ 2E(\theta_p^*(n) - \hat{\theta}_p(n))'h_p(S) \leq \varepsilon c_n^2 + \frac{d\phi^2 L^2}{p^2}. \]
Choosing the value of \( \varepsilon \), which minimizes the right-hand side of this inequality, and substituting it in (8.2), and then we obtain the inequality (3.16) for \( p \geq p_0 \) defined in (3.15). Hence Theorem 3.1.

8.2 Proof of Theorem 5.1
Substituting (4.4) in (4.1) yields for any \( \gamma \in \Gamma \)
\[ \text{Err}_p(\gamma) = J_p(\gamma) + 2 \sum_{j=1}^{p} \gamma(j) \left( \theta_{j,p}^* \hat{\theta}_{j,p} - \frac{\hat{\sigma}_n}{n} - \theta_{j,p}^* \bar{\theta}_{j,p} \right) \]
\[ + \|S\|^2 - \rho \hat{P}_n(\gamma). \quad (8.3) \]
Now we set \( L(\gamma) = \sum_{j=1}^{p} \gamma(j), \)
\[ B_{1,n}(\gamma) = \sum_{j=1}^{p} \gamma(j)(\mathbf{E}_Q^2\xi_{j,p}^2 - \sigma_Q), \quad B_{2,n}(\gamma) = \sum_{j=1}^{p} \gamma(j)(\xi_{j,p}^2 - \mathbf{E}_Q^2\xi_{j,p}^2), \]
\[ M(\gamma) = \frac{1}{\sqrt{n}} \sum_{j=1}^{p} \gamma(j)\bar{\theta}_{j,p}\xi_{j,p} \text{ and } B_{3,n}(\gamma) = \frac{1}{\sqrt{n}} \sum_{j=1}^{p} \gamma(j)g(j)\bar{\theta}_{j,p}\xi_{j,p}. \]
Taking into account the definition (4.5), we can rewrite (8.3) as

\[
\text{Err}_n(\gamma) = J_n(\gamma) + 2 \frac{\sigma Q - \bar{\sigma} \sigma L(\gamma)}{n} + 2 \frac{\nu B_{1,n}(\gamma)}{n} \\
+ 2 \sqrt{P_n(\gamma)} \frac{B_{2,n}(\gamma)}{\sqrt{\sigma Q n}} - 2 B_{3,n}(\gamma) + \|S\|^2 - \rho \hat{P}_n(\gamma) \tag{8.4}
\]

with \( \gamma = \gamma / |\gamma|_n \). Let \( \gamma_0 = (\gamma_0(j))_{1 \leq n} \) be a fixed sequence in \( \Gamma \) and \( \gamma^* \) be as in (4.6). Substituting \( \gamma_0 \) and \( \gamma^* \) in (8.4), we consider the difference

\[
\text{Err}_n(\gamma^*) - \text{Err}_n(\gamma_0) \leq 2 \frac{\sigma Q - \bar{\sigma} \sigma L(x) + 2 M(x) + \frac{\nu}{n} B_{1,n}(x)}{n} \\
+ 2 \sqrt{P_n(\gamma^*)} \frac{B_{2,n}(\gamma^*)}{\sqrt{\sigma Q n}} - 2 \sqrt{P_n(\gamma_0)} \frac{B_{2,n}(\gamma_0)}{\sqrt{\sigma Q n}} \\
- 2 B_{3,n}(\gamma^*) + 2 B_{3,n}(\gamma_0) - \rho \hat{P}_n(\gamma^*) + \rho \hat{P}_n(\gamma_0),
\]

where \( x = \gamma^* - \gamma_0 \). Note that \( |L(x)| \leq 2 |\Gamma|^*_n \) and \( |B_{1,n}(x)| \leq L_{1,n}(Q) \). Applying the elementary inequality

\[
2|ab| \leq \varepsilon a^2 + \varepsilon^{-1} b^2 \tag{8.5}
\]

with any \( \varepsilon > 0 \), we get

\[
2 \sqrt{P_n(\gamma)} \frac{B_{2,n}(\gamma)}{\sqrt{\sigma Q n}} \leq \varepsilon P_n(\gamma) + \frac{B_{2,n}(\gamma)}{\varepsilon \sigma Q n} \leq \varepsilon P_n(\gamma) + \frac{B^*_2}{\varepsilon \sigma n},
\]

where

\[
B^*_2 = \max_{\gamma \in \Gamma} \left( B_{2,n}(\gamma) + B_{2,n}(\gamma^2) \right)
\]

with \( \gamma^2 = (\gamma_j^2)_{1 \leq j \leq n} \). Note that from definition the function \( L_{2,n}(Q) \) in the condition \( C_2 \) we obtain that

\[
E_Q B^*_2 \leq \sum_{\gamma \in \Gamma} \left( E_Q B_{2,n}^2(\gamma) + E_Q B_{2,n}^2(\gamma_2) \right) \leq 2 \nu L_{2,n}(Q). \tag{8.6}
\]

Moreover, by the same method we estimate the term \( B_{3,n} \). Note that

\[
\sum_{j=1}^{n} \bar{g}_j^2(j) \hat{\theta}_j^2 = c^2_n \leq \frac{c^*_n}{n}, \tag{8.7}
\]

where \( c^*_n = n \max_{\gamma \in \Gamma} c^*_n \). Therefore, through the Cauchy-Bunyakovsky-Schwarz inequality we can estimate the term \( B_{3,n}(\gamma) \) as

\[
|B_{3,n}(\gamma)| \leq |\gamma|_n \left( \sum_{j=1}^{n} \bar{g}_j^2(j) \xi_j^2 \right)^{1/2} = |\gamma|_n c^*_n \left( \sigma Q + B_{2,n}(\gamma^2) \right)^{1/2}.
\]

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So, applying the elementary inequality (8.5) with some arbitrary \( \varepsilon > 0 \), we have

\[
2|B_{3,n}(\gamma)| \leq \varepsilon P_n(\gamma) + \frac{c^*_n}{\varepsilon \sigma_Q n}(\sigma_Q + B^*_2).
\]

Using the bounds above, one has

\[
\text{Err}_n(\gamma^*) \leq \text{Err}_n(\gamma_0) + \frac{4|\Gamma|_n |\hat{\sigma}_n - \sigma_Q|}{n} + 2M(x) + \frac{2}{n} L_{1,n}(Q)
\]

\[
+ \frac{2}{\varepsilon n \sigma_Q}(\sigma_Q + B^*_2) + \frac{2 B^*_2}{\varepsilon n \sigma_Q}
\]

\[
+ 2\varepsilon P_n(\gamma^*) + 2\varepsilon P_n(\gamma_0) - \rho \hat{P}_n(\gamma^*) + \rho \hat{P}_n(\gamma_0).
\]

The setting \( \varepsilon = \rho/4 \) and the estimating where this is possible \( \rho \) by 1 in this inequality imply

\[
\text{Err}_n(\gamma^*) \leq \text{Err}_n(\gamma_0) + \frac{5|\Gamma|_n |\hat{\sigma}_n - \sigma_Q|}{n} + 2M(x) + \frac{2}{n} L_{1,n}(Q)
\]

\[
+ \frac{16(c^*_n + 1)(\sigma_Q + B^*_2)}{\rho n \sigma_Q} - \frac{\rho}{2} \hat{P}_n(\gamma^*) + \frac{\rho}{2} P_n(\gamma_0) + \rho \hat{P}_n(\gamma_0).
\]

Moreover, taking into account here that

\[
|\hat{P}_n(\gamma_0) - P_n(\gamma_0)| \leq |\Gamma|_n |\hat{\sigma}_n - \sigma_Q| \frac{n}{n}
\]

and that \( \rho < 1/2 \), we obtain that

\[
\text{Err}_n(\gamma^*) \leq \text{Err}_n(\gamma_0) + \frac{6|\Gamma|_n |\hat{\sigma}_n - \sigma_Q|}{n} + 2M(x) + \frac{2}{n} L_{1,n}(Q)
\]

\[
+ \frac{16(c^*_n + 1)(\sigma_Q + B^*_2)}{\rho n \sigma_Q} - \frac{\rho}{2} P_n(\gamma^*) + \frac{3\rho}{2} P_n(\gamma_0). \quad (8.8)
\]

Now we examine the third term in the right-hand side of this inequality. Firstly we note that

\[
2|M(x)| \leq \varepsilon\|S_x\|^2 + \frac{Z^*}{n\varepsilon},
\]

where \( S_x = \sum_{j=1}^p x_j \theta_{j,p} \phi_j \) and

\[
Z^* = \sup_{x \in \Gamma_1} \frac{n M^2(x)}{\|S_x\|^2}.
\]

We remind that the set \( \Gamma_1 = \Gamma - \gamma_0 \). Using Proposition 7.1 from [27], we can obtain that for any fixed \( x = (x_j)_{1 \leq j \leq n} \in \mathbb{R}^n \)

\[
\mathbf{E} M^2(x) = \frac{\mathbf{E} I^2_S (S_x)}{n^2} = \frac{\sigma_Q \|S_x\|^2}{n} = \frac{\sigma_Q}{n} \sum_{j=1}^p x_j^2 \theta_{j,p}^2 \quad (8.10)
\]

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and, therefore,  
\[ E_Q Z^* \leq \sum_{x \in \Gamma_1} \frac{n M^2(x)}{\|S_x\|^2} \leq \sigma_Q \nu. \]  
(8.11)

Moreover, the norm \( \|S^*_{\gamma^*} - S^*_{\gamma_0}\| \) can be estimated from below as  
\[ \|S^*_{\gamma} - S^*_{\gamma_0}\|^2 = \sum_{j=1}^p (x(j) + \beta(j))^2 \tilde{\theta}_{j,p} \]
\[ \geq \|\hat{S}_x\|^2 + 2 \sum_{j=1}^p x(j) \beta(j) \tilde{\theta}_{j,p}^2, \]
where \( \beta(j) = \gamma_0(j) g_j(\gamma_0) - \gamma(j) g_j(\gamma) \). Therefore, in view of (3.7)
\[ \|S_x\|^2 - \|S^*_{\gamma} - S^*_{\gamma_0}\|^2 \leq \|\hat{S}_x\|^2 - 2 \sum_{j=1}^p x(j) \beta(j) \tilde{\theta}_{j,p}^2 \]
\[ \leq -2M(x^2) - 2 \sum_{j=1}^p x(j) \beta(j) \tilde{\theta}_{j,p} \bar{\theta}_{j,p} - \frac{2}{\sqrt{n}} \Upsilon(x), \]
where \( \Upsilon(\gamma) = \sum_{j=1}^p \gamma(j) \beta(j) \tilde{\theta}_{j,p} \xi_j \). Note that the first term in this inequality we can estimate as
\[ 2M(x^2) \leq \varepsilon \|S_x\|^2 + \frac{Z_1^*}{n \varepsilon} \quad \text{and} \quad Z_1^* = \sup_{x \in \Gamma_1} \frac{n M^2(x^2)}{\|S_x\|^2}. \]

Note that, similarly to (8.11) we can estimate the last term as  
\[ E_Q Z_1^* \leq \sigma_Q \nu. \]

From this it follows that for any \( 0 < \varepsilon < 1 \)
\[ \|S_x\|^2 \leq \frac{1}{1 - \varepsilon} \left( \|S^*_{\gamma} - S^*_{\gamma_0}\|^2 + \frac{Z_1^*}{n \varepsilon} \right. \]
\[ \left. -2 \sum_{j=1}^p x(j) \beta(j) \tilde{\theta}_{j,p} \bar{\theta}_{j,p} - \frac{2 \Upsilon(x)}{\sqrt{n}} \right). \]  
(8.12)

Moreover, note now that the property (8.7) yields
\[ \sum_{j=1}^p \beta^2(j) \tilde{\theta}_{j,p}^2 \leq 2 \sum_{j=1}^p g^2_\gamma(j) \tilde{\theta}_{j,p}^2 + 2 \sum_{j=1}^p g^2_{\gamma_0}(j) \hat{\theta}_{j,p}^2 \leq \frac{4 \epsilon^*}{\varepsilon n}. \]  
(8.13)
Taking into account that $|x(j)| \leq 1$ and using the inequality (8.5), we get that for any $\varepsilon > 0$

$$2 \left| \sum_{j=1}^{p} x(j) \beta(j) \hat{\vartheta}_{j,p} \right| \leq \varepsilon \|S_x\|^2 + \frac{4\varepsilon^*}{\varepsilon n}.$$ 

To estimate the last term in the right hand of (8.12) we use first the Cauchy-Bunyakovsky-Schwarz inequality and then the bound (8.13), i.e.

$$\frac{2}{\sqrt{n}} |\Upsilon(\gamma)| \leq \frac{2|\Gamma|}{\sqrt{n}} \left( \sum_{j=1}^{p} \beta^2(j) \hat{\vartheta}_{j,p} \right)^{1/2} \left( \sum_{j=1}^{p} \hat{\gamma}^2(j) \xi_{j,p}^2 \right)^{1/2} \leq \varepsilon P_n(\gamma) + \frac{c^* \varepsilon}{n \varepsilon \sigma Q} \sum_{j=1}^{p} \hat{\gamma}^2(j) \xi_{j,p}^2 \leq \varepsilon P_n(\gamma) + \frac{c^*(\sigma + B_2^*)}{n \varepsilon \sigma Q}.$$ 

Therefore,

$$\frac{2}{\sqrt{n}} |\Upsilon(x)| \leq \frac{2}{\sqrt{n}} |\Upsilon(\gamma^*)| + \frac{2}{\sqrt{n}} |\Upsilon(\gamma_0)| \leq \varepsilon P_n(\gamma^*) + \varepsilon P_n(\gamma_0) + \frac{2c^*(\sigma + B_2^*)}{n \varepsilon \sigma Q}.$$ 

So, using all these bounds in (8.12), we obtain that

$$\|S_x\|^2 \leq \frac{1}{(1-\varepsilon)} \left( \frac{Z^*_1}{n \varepsilon} + \|S^*_{\gamma^*} - S^*_{\gamma_0}\|^2 + \frac{6c^*_n (\sigma + B_2^*)}{n \varepsilon} \right. \right.$$ 

$$\left. + \varepsilon P_n(\gamma^*) + \varepsilon P_n(\gamma_0) \right).$$ 

Using in the inequality (8.9) this bound and the estimate

$$\|S^*_{\gamma^*} - S^*_{\gamma_0}\|^2 \leq 2(\text{Err}_p(\gamma^*) + \text{Err}_p(\gamma_0)),$$

we obtain

$$2|M(x)| \leq \frac{Z^* + Z^*_1}{n(1-\varepsilon)\varepsilon} + \frac{2\varepsilon(\text{Err}_p(\gamma^*) + \text{Err}_p(\gamma_0))}{1-\varepsilon}$$

$$\left. + \frac{6c^*_n (\sigma + B_2^*)}{n \varepsilon \sigma Q (1-\varepsilon)} + \frac{\varepsilon^2}{1-\varepsilon} (P_n(\gamma^*) + P_n(\gamma_0)) \right).$$

Choosing here $\varepsilon \leq \rho/2 < 1/2$ we obtain that

$$2|M(x)| \leq \frac{2(Z^* + Z^*_1)}{n \varepsilon} + \frac{2\varepsilon(\text{Err}_p(\gamma^*) + \text{Err}_p(\gamma_0))}{1-\varepsilon}$$

$$\left. + \frac{12c^*_n (\sigma + B_2^*)}{n \varepsilon \sigma Q} + \varepsilon (P_n(\gamma^*) + P_n(\gamma_0)) \right).$$
From here and (8.8), it follows that
\[
\text{Err}_p(\gamma^*) \leq \frac{1 + \varepsilon}{1 - 3\varepsilon} \text{Err}_p(\gamma_0) + \frac{6|\Gamma_0|_s |\hat{\sigma}_n - \sigma_Q|}{n(1 - 3\varepsilon)} + \frac{2}{n(1 - 3\varepsilon)} L_{1,n}(Q) \\
+ \frac{28(1 + c_n^*)(B_2^* + \sigma_Q)}{\rho(1 - 3\varepsilon)n\sigma_Q} + \frac{2(Z^* + Z_1^*)}{n(1 - 3\varepsilon)} + \frac{2\rho P_n(\gamma_0)}{1 - 3\varepsilon}.
\]
Choosing here \(\varepsilon = \rho/3\) and estimating \((1 - \rho)^{-1}\) by 2 where this is possible, we get
\[
\text{Err}_p(\gamma^*) \leq \frac{1 + \rho/3}{1 - \rho} \text{Err}_p(\gamma_0) + \frac{12|\Gamma_0|_s |\hat{\sigma}_n - \sigma_Q|}{n} + \frac{4L_{1,n}(Q)}{n} \\
+ \frac{56(1 + c_n^*)(B_2^* + \sigma_Q)}{\rho n\sigma_Q} + \frac{4(Z^* + Z_1^*)}{n} + \frac{2\rho P_n(\gamma_0)}{1 - \rho}.
\]
Taking the expectation and using the upper bound for \(P_n(\gamma_0)\) in Lemma 7.1 from [27] with \(\varepsilon = \rho\) yields
\[
\mathcal{R}_Q(S^*, S) \leq \frac{1 + 5\rho}{1 - \rho} \mathcal{R}_Q(S_{\gamma_0}^*, S) + \frac{\hat{U}_{Q,n}}{n\rho} + \frac{12|\Gamma_0|_s |\mathcal{E}_Q|_s |\hat{\sigma}_n - \sigma_Q|}{n},
\]
where \(\hat{U}_{Q,n} = 4L_{1,n}(Q) + 56(1 + c_n^*)(2L_{1,n}(Q)\nu + 1) + 2c_n^*\). The inequality holds for each \(\gamma_0 \in \Lambda\), this implies Theorem 5.1. \qed

A Appendix

In the following lemma we need the well-known Bichteler - Jacod - Novikov inequalities [4, 23, 24] for purely discontinuous martingales. Namely, for any \(p \geq 2\) and for any \(t > 0\)
\[
\mathsf{E} \sup_{u \leq t} |h \ast (\mu - \bar{\mu})_u|^p \leq C_p \mathsf{E} \left( |h|^2 \ast \bar{\mu}_t \right)^{p/2} + \mathsf{E} |h|^p \ast \bar{\mu}_t,
\]
where \(\mu\) is a jump measure and \(\bar{\mu}\) is its compensator.

Lemma A.1. For any bounded on the interval \([0, T]\) nonrandom function \(f\)
\[
\sup_{0 \leq t \leq T} \mathsf{E} I_t^f(f) < \infty.
\]

Proof. Taking into account the definition of the process (2.1) we get
\[
\xi_t = \xi_t^{(1)} + \xi_t^{(2)},
\]
where
\[
d\xi_t^{(1)} = a\xi_t^{(1)} dt + dw_t \quad \text{and} \quad d\xi_t^{(2)} = a\xi_t^{(2)} dt + dz_t.
\]
So, denoting $I^{(i)}_t(f) = \int_0^t f(s) d\xi^{(i)}_s$, we obtain that
\[ I_t(f) = \varrho_1 I^{(1)}_t(f) + \varrho_2 I^{(2)}_t(f). \]  
(A.3)
Since the first term $I^{(1)}_s(f)$ is the gaussian random variable we need to show
the inequality (A.2) only for $I^{(2)}_s(f)$. To this end note, that
\[ I^{(2)}_t(f) = a \int_0^t f(s) \xi^{(2)}_s ds + \int_0^t f(s) dz_s. \]

Note that
\[ \xi^{(2)}_t = \int_0^t e^{a(t-s)} ds. \]

So, by the inequality (A.1) and using the properties (2.3) we obtain that for some constant $C > 0$
\[ \mathbb{E} \left( \xi^{(2)}_t \right)^8 \leq C \left( \left( \int_0^t e^{4a(t-s)} ds \right)^4 + \Pi(x^8) \int_0^t e^{8a(t-s)} ds \right) < \infty. \]
Similarly, we can obtain that
\[ \max_{0 \leq t \leq T} \mathbb{E} \left( \int_0^t f(s) dz_s \right)^8 < \infty. \]
From this it follows the inequality (A.2), hence, Lemma A.1.

Lemma A.2. For any bounded on the interval $[0, T]$ nonrandom function $f$ and $g$
\[ \mathbb{E} \int_0^t V_s(f)dL_s(g) = \mathbb{E} \int_0^t \tilde{I}_s(f)dL_s(g) = \mathbb{E} \int_0^t V_s(f)d\tilde{M}_s(g) = 0. \]  
(A.4)
Proof. By the definition we have
\[ \int_0^t V_s(f)dL_s(g) = \int_0^t V_s(f)(I_s(g) + g(s)I_s(1)) du_s + \varrho_2 \int_0^t V_s(f) g(s) d\mu_s. \]
Note that in this case $< \mu >_t = \Pi(x^4)t$. To this end it suffices to check that
for any bounded non random functions $f$ and $g$
\[ \sup_{0 \leq t \leq T} \mathbb{E} V^2_s(f) I^2_s(g) < \infty. \]
Using the definition of $V$, for the last inequality it suffices to check that for any $t > 0$

$$\sup_{0 \leq s \leq t} E^2_s(1) I^2_s(f) I^2_s(g) < \infty.$$  

Applying here two times the Cauchy-Bunyakovsky-Schwarz inequality and Lemma A.1, we obtain the first equality in (A.4). Similarly, we can obtain the others equalities. Hence Lemma A.2.

**Lemma A.3.** Let $\upsilon$ be a continuously differentiable $\mathbb{R} \to \mathbb{R}$ function. Then, for any $t > 0$, $\alpha > 0$ and for any integrated $\mathbb{R} \to \mathbb{R}$ function $h$,

$$\left| \int_0^t e^{-\alpha(t-s)} h(s) \upsilon(s) \, ds \right| \leq \|h\|_{*,t} \left( 2\|\upsilon\|_{*,t} + \frac{\|\dot{\upsilon}\|_{*,t}}{\alpha} \right),$$

where $\overline{T}_s = \int_0^s h(u) \, du$.

**Proof.** This lemma follows immediately from the integrating by parts.

**Lemma A.4.** For any measurable bounded $[0, +\infty) \to \mathbb{R}$ functions $f$ and $g$, for any $-\infty < a \leq 0$ and for any $t > 0$

$$\left| a \int_0^t e^{2a(t-s)} g(s) A_s(f) \, ds \right| \leq 3\sigma^2_Q \bar{\kappa}_t(f, g) + \hat{\sigma}_2 \|f\|_{*,t} \|g\|_{*,t},$$

where $\hat{\sigma}_2 = \sigma^2_2 \Pi(x^4)$.

**Proof.** First note that

$$A_t = J_t(f) + \sigma^2_Q \hat{J}_t(f)$$  \hspace{1cm} (A.5)

where

$$J_t(f) = \int_0^t e^{3a(t-u)} f(u) \upsilon(u) \, du \quad \text{and} \quad \hat{J}_t(f) = 2 \int_0^t e^{3a(t-u)} \xi_u(f) \, du.$$  

To study these integrals we need to calculate $E_{t}^{\xi^2}$. To this end through the equation (7.8) we can represent this expectation in the following integral form

$$E_{t}^{\xi^2} = \int_0^t e^{4a(t-s)} dE[\widetilde{M}(1), \widetilde{M}(1)]_s.$$  

Moreover, using the definition of $M_s(1, 1)$ in (7.2) we obtain that

$$E[\widetilde{M}(1), \widetilde{M}(1)]_t = 4\sigma_Q \int_0^t E\xi^2_s \, ds + \hat{\sigma}_2 t,$$

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where \( \tilde{\theta}_2 = \theta_2^t \Pi(x^t) \). Therefore,

\[
E \tilde{\xi}_t^2 = \int_0^t e^{4at} (4\sigma_Q E \xi_s^2 + \tilde{\theta}_2) \, ds.
\]

Using here (7.3) we obtain that for \( a < 0 \)

\[
E \tilde{\xi}_t^2 = e^{4at} \frac{2\sigma_Q^2 + a \tilde{\theta}_2}{4a^2} - e^{2at} \frac{\sigma_Q^2}{a^2} + \frac{2\sigma_Q^2 - a \tilde{\theta}_2}{4a^2}. \tag{A.6}
\]

Note now that the function \( \upsilon(\cdot) \) defined in (7.10) can be represented as

\[
\upsilon(s) = a \upsilon_1(s) + \upsilon_2(s) \tag{A.7}
\]

with

\[
\upsilon_1(s) = \frac{\tilde{\theta}_2}{4} (e^{4at} + 3) \quad \text{and} \quad \upsilon_2(s) = \frac{\sigma_Q^2}{2} (e^{4at} - 1). \]

It is clear that

\[
\| \upsilon_1 \|_{s,n} \leq \tilde{\theta}_2 \quad \text{and} \quad 2 \| \upsilon_2 \|_{s,n} + \frac{\| \dot{\upsilon}_2 \|_{s,n}}{2|a|} \leq 2\sigma_Q. \tag{A.8}
\]

Now we have

\[
J_t(f) = J_{1, t}(f) + J_{2, t}(f),
\]

where \( J_{1, t}(f) = a \int_0^t e^{3a(s-u)} f(u) \upsilon_1(u) \, du \) and \( J_{2, t}(f) = \int_0^t e^{3a(s-u)} f(u) \upsilon_2(u) \, du \).

It is clear that

\[
|J_{1, t}(f)| \leq \tilde{\theta}_2 \| f \|_{s,n}/3.
\]

So,

\[
\left| a \int_0^t e^{2a(t-s)} g(s) J_{1, s}(f) \, ds \right| \leq \tilde{\theta}_2 \| f \|_{s,t} \| g \|_{s,t}/6.
\]

Now we represent the corresponding integral for \( J_{2, t}(f) \) as

\[
\int_0^t e^{2a(t-s)} g(s) J_{2, s}(f) \, ds = \int_0^t e^{3am} \tilde{J}_{t-u, u}(f, g) \, du
\]

and

\[
\tilde{J}_{t, u}(f, g) = \int_0^t e^{2a(t-s)} g(s + u) f(s) \upsilon_2(s) \, ds.
\]

Using Lemma A.3 and the last inequality in (A.8) we obtain that

\[
\sup_{0 \leq u \leq t} \left| \tilde{J}_{t-u, u}(f) \right| \leq 2\sigma_Q^2 \Omega_t(g, f),
\]

where

\[
\Omega_t(g, f) = \sup_{u \geq 0, v \geq 0, 0 \leq v+u \leq t} \left| \int_0^v g(s + u) f(s) \, ds \right|.
\]
Therefore,
\[
\left| a \int_0^t e^{2a(t-s)} g(s) J_{2,a}(f) \, ds \right| \leq \frac{2\sigma^2}{3} \Omega_t(g, f) \leq \frac{2\sigma^2}{3} \omega_t(f, g).
\]
Similarly we can get
\[
\left| a \int_0^t e^{2a(t-s)} g(s) \tilde{J}_a(f) \, ds \right| \leq \frac{4}{3} \Omega_t(g, \tilde{\varepsilon}(f)).
\]
Note now that for any fixed \( v > 0 \) and \( \theta \geq 0 \) with \( v + \theta \leq t \) we have
\[
\int_0^v g(u + \theta) \tilde{\varepsilon}_u(f) \, du = \frac{a}{2} \int_0^v e^{as} D_{s,\theta} \, ds,
\]
where \( D_{s,\theta} = \int_0^{v-s} g(y + \theta + s) \tilde{f}(y) \, dy \) and \( \tilde{f}(y) = f(y)(1 + e^{2ay}) \). Integrating par parts yields
\[
D_{s,\theta} = \left(1 + e^{2a(t-s)}\right) \int_0^{v-s} g(z + \theta + s) f(z) \, dz
+ 2a \int_0^{v-s} e^{2az} \int_0^u g(z + \theta + s) f(z) \, dz \, du.
\]
This implies, that
\[
|D_{s,\theta}(f, g)| \leq 3\omega_t(f, g),
\]
i.e. for any \( v > 0 \) and \( \theta \geq 0 \)
\[
\left| \int_0^v g(u + \theta) \tilde{\varepsilon}_u(f) \, du \right| \leq 3\omega_t(f, g)/2. \tag{A.9}
\]
Therefore,
\[
\Omega_t(g, \tilde{\varepsilon}(f)) \leq 3\omega_t(f, g)/2.
\]
Hence Lemma A.4. \( \square \)

**Lemma A.5.** For any measurable bounded \([0, +\infty) \to \mathbb{R}\) functions \( f \) and \( g \), for any \(-a_{\max} \leq a \leq 0\) and for any \( t > 0 \)
\[
a^2 \left| \int_0^t e^{a(t-s)} g(s) \mathbf{E} \tilde{I}_a(f) \tilde{I}_a(1) \, ds \right| \leq 4 \| f \|_{s,t}^2 \left( a_{\max} \tilde{\varepsilon}_2 + 3\sigma_Q^2 \right) \omega_t(1, g).
\]

**Proof.** Firstly, note that if \( a = 0 \) then this bound is obvious. Let now \( |a| > 0 \). Then, taking into account the representation (A.5) and the bound \( |\tilde{\varepsilon}_t(f)| \leq \| f \|_{s,t} \) we obtain that
\[
\| aA(f) \|_{s,t} \leq \| f \|_{s,t} \left( a_{\max} \tilde{\varepsilon}_2 / 3 + \sigma_Q^2 \right). \tag{A.10}
\]

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Thus, from the definition of $\tilde{\kappa}_u(f)$ in (7.15) we obtain that
\[
\|a\tilde{\kappa}(f)\|_{s,t} \leq \|f\|_{s,t}^2 \left(2a_{\max} \hat{\vartheta}_2 + 6\sigma^2_Q\right).
\] (A.11)

Moreover, note now, that
\[
\int_0^t e^{a(t-s)} g(s) E\tilde{I}_s(f) \tilde{I}_s(1) ds = \int_0^t e^{a(t-u)} \tilde{\kappa}_u(f) G_{t-u,u} du,
\]
where $G_{T,u} = \int_0^T e^{az} g(z + u) dz$. The integrating by parts yields
\[
G_{T,u} = \int_0^T g(z + u) dz + a \int_0^T e^{ay} \left(\int_0^y g(v + u) dv\right) dy.
\]
So, for any $T + u \leq t$ we obtain that $|G_{T,u}| \leq 2\varpi_t(1,g)$ and, therefore,
\[
\left|\int_0^t e^{a(t-u)} \tilde{\kappa}_u(f) G_{t-u,u} du\right| \leq \frac{4}{a^2} \|f\|_{s,t}^2 \left(a_{\max} \hat{\vartheta}_2 + 3\sigma^2_Q\right) \varpi_t(1,g).
\]
Hence Lemma A.5.

Lemma A.6. For any measurable bounded $[0, +\infty) \rightarrow \mathbb{R}$ functions $f$ and $g$, for any $-\infty < a \leq 0$ and for any $t > 0$
\[
\left|a \int_0^t e^{2a(t-s)} \tau_s(f,g) ds\right| \leq 9 \varpi_t^*(f,g).
\] (A.12)

Proof. Firstly note, that using the bound (A.9) with $\theta = 0$ we obtain
\[
\left|\int_0^t g(s) \tilde{\vartheta}_s(f,g) ds\right| \leq 3\varpi_t(f,g)/2.
\] (A.13)
So, $|\tau_t(f,g)| \leq 4\varpi_t(f,g)$. Moreover, through the bound (A.13) and Lemma A.3 we obtain that
\[
\left|\int_0^t e^{2a(t-s)} f(s) \tilde{\vartheta}_s(g) ds\right| \leq 3\varpi_t(f,g).
\]
Using again Lemma A.3 and taking into account that $a\tau_t(1,1) = (e^{2at} - 1)/2$ we estimate
\[
\left|a \int_0^t e^{2a(t-s)} f(s) g(s) \tau_s(1,1) ds\right| \leq \varpi_t(f,g).
\]
Thus, from taking into account the definition (7.13) we obtain the bound (A.12). Hence Lemma A.6.

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