Canonical quantization of systems with 
time-dependent constraints

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Abstract

The Hamilton - Jacobi method of constrained systems is dis- 
cussed. The equations of motion for a singular system with time 
dependent constraints are obtained as total differential equations in 
many variables. The integrability conditions for the relativistic par-
ticle in a plane wave lead us to obtain the canonical phase space 
coordinates with out using any gauge fixing condition. As a result 
of the quantization, we get the Klein-Gordon theory for a particle 
in a plane wave. The path integral quantization for this system is 
obtained using the canonical path integral formulation method.

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1 Introduction

There are many singular theories with constraints which depend on time manifestly. The relativistic particle theories, string theories and theories of gravity are some examples. The presence of first class constraints in these theories forces one to impose time dependent gauge fixing condition for each first class constraint such that the whole set of constraints become converted in to second-class constraints. The rules of canonical quantization of systems with second-class constraints [1, 2], which depend on time should be modified [3]. However, there exist cases where one can make canonical transformation to make these constraints time-independent and then use the ordinary rules of quantization.

Recently, the canonical method [4-7] has been developed to investigate constrained systems. The equations of motion are obtained as total differential equations in many variables which require the investigation of integrability conditions. If the system is integrable, one can solve the equations of motion without using any gauge fixing conditions.

In this paper we consider the quantization of the relativistic spinless particle in the external field of a plane wave using the Hamilton-Jacobi method [8-10]. in fact, this work is a continuation of a previous work [11], where we have obtained the path integral for a relativistic charged particle in an external electromagnetic field and it is shown that the problems which arise from identifying the measure of the path integral when applying methods [1,2] and [12,13], are solved naturally by the canonical path integral formulation [8-10].

Now we would like to give a brief discussion of the canonical method. This method gives the set of Hamilton - Jacobi partial differential equations [HJPDE] as

\[ H'_\alpha(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_a}) = 0, \]

\[ \alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r, \]

(1)

where

\[ H'_\alpha = H(\alpha, q_a, p_a) + p_a, \]

(2)
and $H_0$ is defined as

$$H_0 = p_a w_a + p_\mu \dot{q}_\mu \big|_{p_\nu = -H_\nu} - L(t, q_i, \dot{q}_i, \dot{q}_a = w_a),$$

$$\mu, \nu = n - r + 1, ..., n. \quad (3)$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\beta = -\frac{\partial H'_\alpha}{\partial t_\beta} dt_\alpha. \quad (4)$$

$$dz = (-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}) dt_\alpha; \quad (5)$$

$$\alpha, \beta = 0, n - r + 1, ..., n, a = 1, ..., n - r$$

where $z = S(t_\alpha; q_a)$. The set of equations (4,5) is integrable [6,7] if

$$dH'_0 = 0, \quad dH'_\mu = 0, \mu = n - p + 1, ..., n. \quad (6)$$

If condition (6) are not satisfied identically, one considers them as new constraints and again testes the consistency conditions. Hence, the canonical formulation leads to obtain the set of canonical phase space coordinates $q_a$ and $p_a$ as functions of $t_\alpha$, besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians $H'_\alpha$ are considered as the infinitesimal generators of canonical transformations given by parameters $t_\alpha$ respectively.

## 2 Quantization of constrained systems

For the quantization of constrained systems we can use the Dirac’s method of quantization [1,2], or the path integral quantization method [8-10].

Now will shall give a brief information about these two methods.

### 2.1 Operator quantization

For the Dirac’s quantization method we have

$$H'_\alpha \Psi = 0, \quad \alpha = 0, n - r + 1, ..., n, \quad (7)$$
where $\Psi$ is the wave function. The consistency conditions are

$$[H'_\mu, H'_\nu] \Psi = 0, \quad \mu, \nu = 1, \ldots, r,$$  

(8)

where $[,]$ is the commutator. The constraints $H'_\alpha$ are called first-class constraints if they satisfy

$$[H'_\mu, H'_\nu] = C'_{\mu\nu} H'_\gamma,$$  

(9)

In the case when the Hamiltonians $H'_\mu$ satisfy

$$[H'_\mu, H'_\nu] = C_{\mu\nu},$$  

(10)

with $C_{\mu\nu}$ do not depend on $q_i$ and $p_i$, then from (8) there arise naturally Dirac’ brackets and the canonical quantization will be performed taking Dirac’s brackets into commutators.

### 2.2 Path integral quantization method

The path integral quantization is an alternative method to perform the quantization of constrained systems.

Now we shall give a brief review of the canonical path integral formulation of constrained systems [8-10].

If the set of equations (4) is integrable then one can solve them to obtain the canonical phase-space coordinates as

$$q_a \equiv q_a(t, t), \quad p_a \equiv p_a(t, t), \quad \mu = 1, \ldots, r,$$  

(11)

In this case, the path integral representation may be written as [8-10]

$$\langle \text{Out} | S | \text{In} \rangle = \int \prod_{a=1}^{n-r} dq^a dp^a \exp\left[i \int_{t_\alpha}^{t_\alpha'} \left(-H_{\alpha} + p_a \frac{\partial H'_a}{\partial p_a} \right) dt_a \right],$$

$$a = 1, \ldots, n - r, \quad \alpha = 0, n - r + 1, \ldots, n.$$  

(12)

One should notice that the integral (12) is an integration over the canonical phase-space coordinates $(q_a, p_a)$. 

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3 Quantization of the relativistic particle in a plane wave

Motion of spinless particle in the external field of a plane wave is defined by a singular Lagrangians. The usual approach that authors investigate is the Dirac’s approach [1,2].

Let us consider the reparametrization invariant action of a spinless particle in the external electromagnetic field of a plane wave directed along the axis $x^3$ as

$$S = -\int (m\sqrt{\dot{x}^2} + e\dot{x}A)d\tau, \quad (13)$$

$$= -\int [m\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2} + e\dot{x}_a A_a(x_-)]d\tau,$$

$$A_\mu = (0, A^a, 0), \quad x_\pm = (x^0x^3)/\sqrt{2}, \quad x_\perp = (x^a), \quad a = 1, 2.$$

The momenta conjugated to $x_\pm$ and $x^a$ are

$$\pi_+ = \frac{\partial L}{\partial \dot{x}_+} = -\frac{m\dot{x}_-}{\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2}}, \quad (14)$$

$$\pi_- = \frac{\partial L}{\partial \dot{x}_-} = -\frac{m\dot{x}_+}{\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2}}, \quad (15)$$

$$\pi_a = \frac{\partial L}{\partial \dot{x}_a} = \frac{m\dot{x}_a}{\sqrt{2\dot{x}_+\dot{x}_- - (\dot{x}_\perp)^2}} - eA_a(x_-), \quad (16)$$

the primary constraint is in the form

$$\phi^{(1)} = \pi_- - \left[\frac{\pi_a + eA_a(x_-)^2}{2\pi_+} + \frac{m^2}{2\pi_+}\right] = 0, \quad \pi_\pm \neq 0. \quad (17)$$

The canonical Hamiltonian (3) vanishes identically. So the primary Hamiltonian is the constraint itself multiplied by a function $\lambda(\tau)$, i.e

$$H^{(1)} = \lambda\phi^{(1)}. \quad (18)$$
The equations of motion read as

\[ \dot{x}_+ = \{x_+, H^{(1)}\} = \lambda \frac{[\pi_a + eA_a(x_-)]^2 + m^2}{\pi_+}, \]  
(19)

\[ \dot{x}_- = \{x_-, H^{(1)}\} = \lambda, \]  
(20)

\[ \dot{x}_a = \{x_a, H^{(1)}\} = \lambda \frac{[\pi_a + eA_a(x_-)]}{\pi_+}, \]  
(21)

\[ \dot{\pi}_+ = \{\pi_+, H^{(1)}\} = 0, \]  
(22)

\[ \dot{\pi}_a = \{\pi_a, H^{(1)}\} = 0, \]  
(23)

\[ \dot{\pi}_- = \{\pi_-, H^{(1)}\} = -\lambda e \frac{[\pi_a + eA_a(x_-)] \partial A_a}{\pi_+} \partial x_-, \]  
(24)

Since we have only one first class constraint, the theory is degenerate and it is impossible to find \( \lambda \) in the Dirac’s procedure. To determine \( \lambda \), one should impose time-dependent gauge fixing of the form

\[ \Phi^G = x_- - \tau = 0. \]  
(25)

From the condition of conservation of this gauge in the time \( \tau \)

\[ \frac{d\Phi^G}{d\tau} = \frac{\partial \Phi^G}{\partial \tau} + \{\Phi^G, H^{(1)}\} = (\lambda - 1) = 0, \]  
(26)

we obtain \( \lambda = 1 \). No other constraints arise.

Now we would like to study the quantization of the same problem using the canonical method [4-7]. Equations (14) and (16) lead us to obtain the expressible velocities \( \dot{x}_+ \) and \( \dot{x}_a \) in terms of the primary unexpressible velocity \( \dot{x}_- \) as

\[ \dot{x}_+ = \frac{\pi_-}{\pi_+} \dot{x}_- = w_+, \quad \dot{x}_a = \frac{(\pi_a + eA_a(x_-))}{\pi_+} \dot{x}_- = w_a. \]  
(27)

Substituting (27) in (15), one gets

\[ \pi_- = \frac{[\pi_a + eA_a(x_-)]^2 + m^2}{2\pi_+} = -H_{x_-}, \quad \pi_+ \neq 0. \]  
(28)

The canonical Hamiltonian \( H_{\tau} \) can be written as

\[ H_{\tau} = -L + \pi_a w_a + \pi_+ w_+ - \dot{x}_- H_{x_-}. \]  
(29)
Explicit calculations show that \( H_\tau \) vanishes identically.

Making use of equations (2) and (28,29), one obtains the set of Hamilton-Jacobi partial differential equations as

\[
H'_\tau = \pi_\tau = 0, \quad \pi_\tau = \frac{\partial S}{\partial \tau}, \quad (30)
\]
\[
H'_{x_-} = \pi_- + H_{x_-} = 0, \quad \pi_- = \frac{\partial S}{\partial x_-}. \quad (31)
\]

The equations of motion are obtained as total differential equations in many variables as follows:

\[
dx_+ = \frac{\partial H'_\tau}{\partial \pi_+} d\tau + \frac{\partial H'_{x_-}}{\partial \pi_+} dx_- = \left[ \pi_a + e A_a(x_-) \right]^2 + m^2 \pi_+ dx_- , \quad (32)
\]
\[
dx_a = \frac{\partial H'_\tau}{\partial \pi_a} d\tau + \frac{\partial H'_{x_-}}{\partial \pi_a} dx_- = \left( \pi_a + e A_a(x_-) \right) \pi_+ dx_- , \quad (33)
\]
\[
d\pi_+ = - \frac{\partial H'_\tau}{\partial x_+} d\tau - \frac{\partial H'_{x_-}}{\partial x_+} dx_- = 0, \quad (34)
\]
\[
d\pi_a = - \frac{\partial H'_\tau}{\partial x_a} d\tau - \frac{\partial H'_{x_-}}{\partial x_a} dx_- = 0, \quad (35)
\]
\[
d\pi_- = - \frac{\partial H'_\tau}{\partial x_-} d\tau - \frac{\partial H'_{x_-}}{\partial x_-} dx_- = 0, \quad (36)
\]
\[
d\pi_\tau = - \frac{\partial H'_\tau}{\partial \tau} d\tau - \frac{\partial H'_{x_-}}{\partial \tau} dx_- = 0. \quad (37)
\]

In order to have a consistent theory, one should consider the variations of constraints in general. Since the total variations of the constraints \( H'_\tau \) and \( H'_{x_-} \) are identically zero, no further constraints arise and the equations of motion are integrable. Hence, the canonical phase space coordinates \((x_+, \pi_+)\) and \((x_a, \pi_a)\) are obtained in terms of parameters \( \tau \) and \( x_- \).
To obtain the operator quantization of this system one can follow the procedure discussed in section (2.1). In this case one takes the constraint equation as an operator whose action on the allowed Hilbert space vectors is constrained to zero, i.e., \( H'_{x-} \Psi = 0 \), we obtain

\[
\frac{i}{2\pi_+} \frac{\partial \Psi}{\partial (x_-)} = \frac{[-i \frac{\partial}{\partial x_-} - eA^a(x_\mu)]^2 + m^2}{2\pi_+} \Psi
\] (39)

One should notice that (39) is the Klein-Gordon equation. In fact, the Klein-Gordon equation

\[
(\mathcal{P}^2 - m^2)\Psi(x) = 0, \quad \mathcal{P}_\mu = i\partial_\mu - eA_\mu(x)
\] (40)

in the external field of a plane wave (13), being written in the light cone variables \( (x_\pm, x_\perp) = (x^\alpha) \), reads as

\[
(2 \frac{\partial^2}{\partial x_- \partial x_+} + [-i \frac{\partial}{\partial x_-} - eA^a(x_-)]^2 + m^2)\Psi(x) = 0.
\] (41)

Now to obtain the path integral quantization of this system, we can use equation (5) to obtain the canonical action as

\[
S = \int \left[ \frac{\pi_a(x_-) + eA^a(x_-)}{2\pi_+} \frac{\partial}{\partial (x_-)} + \frac{\partial}{\partial x_+} \Psi(x_+ - x_-) \right] - \frac{\partial}{\partial x_+} \pi_+ + \pi_a\dot{x}_a dx_-
\] (42)

Making use of (42) and (12) the path integral for the system (13) is obtained as

\[
\langle x_+, x_a, \tau, x_-; x'_+, x'_a, \tau', x'_- \rangle = \int_{x_+, x_-} dx_a dx_+ d\pi_+ d\pi_a \exp\left\{ i \int_{x_-}^{x'_-} \left[ \frac{\pi_a(x_-) + eA^a(x_-)}{2\pi_+} \frac{\partial}{\partial (x_-)} + \frac{\partial}{\partial x_+} \Psi(x_+ - x_-) \right] - \frac{\partial}{\partial x_+} \pi_+ + \pi_a\dot{x}_a dx_-ight\}.
\] (43)

In fact, the path integral representation (43) is an integration over the canonical phase-space coordinates This path integral representation is an integration over the canonical phase space coordinates \( (x_+, \pi_+, x_a, \pi_a) \).

Now for a system with \( n \) degrees of freedom and \( r \) first class constraints \( \phi^\alpha \), the matrix element of the \( S \)-matrix is given by Faddeev and Popov \([12,13] \) as

\[
\langle \text{Out} | S | \text{In} \rangle = \int d\mu(q_j, p_j) \exp\left\{ i \int_{-\infty}^{\infty} dt (p_j \dot{q}_j - H_0) \right\}, \quad j = 1, \ldots, n,
\] (44)
where the measure of integration is given as

\[
d\mu = \det |\{\phi^\alpha, \chi^\beta\}| \prod_{\alpha=1}^{r} \delta(\chi^\alpha) \delta(\phi^\alpha) \prod_{j=1}^{n} dq_j dp_j,
\]

(45)

and \(\chi^\alpha\) are \(r\)-gauge constraints.

If we perform now the usual path integral quantization [12,13] using (44) for the system (13), one must choose one time dependent gauge fixing (equation (25)) condition to obtain the path integral quantization over the canonical phase-space coordinates.

4 Conclusion

Path integral quantization for relativistic particle in plane wave is obtained by using the canonical path integral formulation [8-10]. In this approach, since the integrability conditions \(dH'_\tau = 0\) and \(dH'_{x_-} = 0\) are satisfied identically, this system is integrable. Hence, the canonical phase-space coordinates \((x_+, \pi_+, x_a, \pi_a)\) are obtained in terms of parameter \((x_-)\). In this case the path integral, then follows directly as given in (43) without using any gauge fixing conditions.

When applying the analog of Faddeev’s method [12,13] to this model, one may choose gauge fixing of the form \(x_- - \tau = 0\), so that one can integrate over the extended phase space coordinates \((x_-, \pi_-, x_+, \pi_+, x_a, \pi_a)\) and after integration over the redundant variables \((x_-, \pi_-)\), one can arrive at the result (43). Besides different choice of gauge fixing will lead to different quantum theories. In this case there are many measures one can use to define an integral. In other words considerable care must be taken with the choice of gauge fixing conditions in order to arrive at the correct result.

As a conclusion, the path integral for relativistic particle in a plane wave is obtained without using any gauge fixing conditions and without the problems of the measure. Besides the canonical action integral is obtained with the consequence constraints (30,31) are enforced explicitly without Lagrange multipliers.

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