Comparative study statically determined trusses with trapezoidal and parabolic shape with large span

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Abstract: In this work the authors propose a parametric study for beams with the opening of 30 m and 60 m. The analysed structures are statically determined and the optimal height and the related weight were established based on strength and stability conditions of structure elements. The structure analysis was performed using a program developed in the Matlab environment based on the displacement method. The analysed beams are manufactured of S355 steel bars with annular section. The beams were analysed for loads applied both the upper and lower nodes of structures. Based on the obtained results a comparison was made concerning the efficient use of material used for the trusses in respect to their weight.

1. Introduction

The statically determined truss type structures have a minimal number of connections required for geometrical invariability and for fixed structure in plan. To determine the sectional efforts only the static equilibrium condition is necessary, but for displacements determination both the static equilibrium condition and the condition of displacements compatibility with the connections are used. The trusses are used to cover large surfaces (sports halls, hangars), for antenna pillars, cranes [1]. The main simplifying hypotheses for truss calculation are:

- the bar axes are concurrent in joints;
- the joints are considered perfect hinged;
- the concentrated forces are applied in joints only.

The consequence of these hypotheses is that only axial effort of tensile or compression appear in the bars composing these structures. The experimental tests and calculations made for beams highlighted that if the largest dimension of cross section is lower than 1/10 of the bar length, the effect of the rigid joint could be considered as a secondary effect in respect to the efforts calculated in the hypothesis of perfect articulated joints [2].

The trusses could have a geometric invariability independently provided by the support base or together with this base. The simplest geometrical form with geometric invariability is the triangle formed by three articulated bars, figure 1. To two triangle joints another joint can be attached using two articulated bars. Therefore, for each joint, except the first three ones, two bars are necessary. Based on this consideration one can determine the relation establishing the condition of the own geometrical invariability of trusses [3].

\[ b + 3 - 2n = 0; \]  

(1)
where \( b \) is the number of bars; \( n \) is the number of joints; 3 represents the three cinematic degrees of freedom which must be blocked to prevent the planar displacement of a rigid body.

![Figure 1. Generation articulated bars.](image)

To attach the structure of trusses to the base, 3 external connections are additionally needed. Therefore, the condition of geometrical invariability together with the connections to the support base can be defined as:

\[
b + r - 2n = 0;
\]

where \( r \) is the number of the simple support;

The condition number 2 is the condition of static determination, which can be defined as:

\[
N = b + r - 2n.
\]

There are three possible cases depending on the \( N \) value:
- \( N = 0 \) statically determined trusses;
- \( N > 0 \) statically undetermined trusses;
- \( N < 0 \) mechanisms.

Because the number of unknowns is equal to the number of the efforts in the \( b \) bars and of the reactions in the \( r \) simple connections to the support base and, on the other hand, the number of equations of static equilibrium is twice the number of joints (two equilibrium equations for each joint), the result is that the equilibrium equation system is a determined compatible system in which the number of equations is equal to the number of unknowns.

Taking into account their composition, the trusses can be classified as follows:
- simple trusses including a succession of triangles; this configuration starts with a basic triangle to which one adds successively segments of one joint and two bars; the optimization of such trusses is discussed in this paper;
- composed trusses consisting of superposition of simple systems in such a way that the assembly has the geometrical invariability ensured; such structures are used when a reduced buckling length is required for the compressed elements;
- complex trusses where at least three bars are connected to each joint and the intersection points of the bars are not considered joints.

To calculate the efforts in the truss bars can use the following methods: the method of joint isolation, the method of simple sections and the method of finite elements. The first two methods are used for simple structures. The main advantage of the statically determined bars is that they are not sensitive to the loads caused by: manufacturing imperfections of the structure bars, structure mounting imperfections and thermal variations applied to the structures. Because all these loads do not introduce additional reactions or efforts in the structure and the sectional efforts in bars do not depend on the relation between the bar stiffnesses, each bar can be dimensioned and tested to the axial tensile and compression independently from the other bars. The main disadvantage of the statically determined structures resides in the fact that there is no structural redundancy because the undetermined degree is zero. Therefore, if the structure has displacements in the post-elastic field when the first plastic articulation is formed, the structure becomes a mechanism.
2. Calculation model

The trusses analyzed in this paper were calculated using a Matlab program which allows the determination of reactions and sectional efforts for the planar truss type structures. Based on the analysis with calculation program one establishes the sectional efforts in the compressed and stretched bars. To optimize the annular bar sections, the calculation is performed for each bar, depending on the sectional efforts developed in each bar.

2.1 Calculation of the stretched bar elements

The structure being a statically determined one, the section area for i element of a stretched bar is calculated with the formula:

$$ f_{yd} = \frac{N_i}{A_i} \rightarrow A_i = \frac{N_i}{f_{yd}} \quad i = 1 \ldots \text{number of bars}, $$

where $f_{yd}$ is the allowable strength and $N_i$ is the stretching sectional effort in the i bar.

For an annular section one supposes that the inner diameter $d$ is a function of outer diameter $D$ according to the formula:

$$ d = kD; \quad k = 0.7. $$

The diameters are calculated according to the obtained area with the formulas:

$$ A = \frac{\pi D^2}{4} (1 - k^2); \quad D = \left( \frac{4A}{\pi (1-k^2)} \right)^{1/2} \quad d = kD. $$

2.2 Calculation of the compressed bar elements

For the compressed bars one should take into account the buckling phenomenon and therefore one calculates the critical force of stability loss using the Euler's formula [4]:

$$ F_{cr} = \frac{\pi^2 E l_{min}}{l_f^2}; $$

where $E$ is the transversal elastic modulus, $l_{min}$ is the minimal axial moment of inertia of the cross section, and $l_f$ is buckling length.

The buckling safety factor for the compressed i bar is:

$$ c_{sf} = \frac{F_{cr}}{N_i}; $$

Based on the condition to not reach the critical force of stability loss and taking into account a buckling safety factor of $c_{sf} = 2$, one can find the minimum axial moment of inertia $l_{min1}$ of the cross section:

$$ \frac{F_{cr}}{N_i} = c_{sf} \rightarrow \frac{\pi^2 E l_{min1}}{l_f^2 A_i} = c_{sf} ; \quad l_{min1} = \frac{l_f^2}{l_f^2 A_i} c_{sf}. $$

Using the condition that the buckling should be in the elastic range, one can find the minimal axial moment of inertia $l_{min2}$:

$$ f_y = \frac{F_{cr}}{A_i} = \frac{\pi^2 E l_{min2}}{l_f^2 A_i} = \frac{\pi^2 E}{\lambda_0^2}; $$

$$ \lambda_0 = \left( \frac{\pi^2 E}{f_y} \right)^{1/2} ; \quad \lambda_0 = \left( \frac{3.14^2 \cdot 210000}{355} \right)^{1/2} \approx 77(355); $$

$$ \lambda = \lambda_0 \sqrt{\frac{1}{l_f}} = l_f \left( \frac{A_i}{l_{min2}} \right)^{1/2} ; \quad \lambda_0^2 = l_f^2 \frac{A_i}{l_{min2}} ; \quad l_{min2} = \frac{A_i l_f^2}{\lambda_0^2}. $$
The moment of inertia is calculated in such a way that the bar remains in the elastic buckling range, using the relation:

\[ I_{min} = \min(I_{min1}, I_{min2}) \]  

(13)

Taking into account an annular section with the outer diameter \( D \) and the inner diameter \( d \) and observing the condition \( d = kD \), the following formulas result for the annular section diameters:

\[ A = \frac{\pi D^2}{4} (1 - k^2); \quad I = \frac{\pi D^4}{64} (1 - k^4); \]

(14)

\[ k^2 = \left(1 - \frac{4A}{\pi D^2}\right); \quad k^4 = \left(1 - \frac{64A}{\pi D^4}\right); \]

(15)

\[ A = \frac{\pi d^2}{4} \left(\frac{1 - k^2}{k^2}\right); \quad I = \frac{\pi d^4}{64} \left(\frac{1 - k^4}{k^4}\right); \]

(16)

\[ D = \frac{B_{l_{min}}}{A} + \frac{2A}{\pi} \left(D^4 - d^4\right); \quad d = \frac{B_{l_{min}}}{A} - \frac{2A}{\pi} \left(D^4 - d^4\right)^{1/2} \]

(18)

The critical force of stability loss and the safety factor for the compressed i bar become:

\[ F_{cr} = \frac{\pi^2 E A_{min}}{l_f^2} = \pi^2 \cdot E \cdot \frac{\pi}{64} \left(D^4 - d^4\right) \cdot \left(l_f^2 \cdot l_f\right)^{-1}; \]

(19)

\[ c_{sf} = \frac{F_{cr}}{N_i} = \pi^2 \cdot E \cdot \frac{\pi}{64} \left(D^4 - d^4\right) \cdot \left(l_f^2 \cdot N_i\right)^{-1}. \]

(20)

3. Studied cases

One assumes that the structure is manufactured of S355 steel according to SR EN 1993-1-1:2006 [5] having the following physical and mechanical characteristics:

- yield strength \( f_y = 355 \text{ MPa} \);
- ultimate strength \( f_u = 510 \text{ MPa} \);
- longitudinal elastic modulus \( E = 210000 \text{ MPa} \);
- transversal elastic modulus \( G = 81000 \text{ MPa} \);
- Poisson coefficient for transversal contraction \( v = 0.3 \).

The cross section of the bars included in the structure has an annular form. The analysed beams are articulated and supported at the ends.

For the parametric study of trusses, one has used trapezoidal and parabolic variations of the elevation along the beam.

For the parabolic variation, one considers the following elevation variation equation of the beam:

\[ y(x) = A \cdot x^2 + B \cdot x + C; \]

(21)

If the beam span is \( d \) and the elevation in the beam middle is \( f \), the following equation system results based on the condition that the parable passes through the \((A,0,0), B\left(\frac{d}{2}, f\right)\) and \(C(d,0)\) points:

\[
\begin{align*}
    y(0) &= 0 = A \cdot 0^2 + B \cdot 0 + C \\
    y\left(\frac{d}{2}\right) &= f = A \cdot \left(\frac{d}{2}\right)^2 + B \cdot \left(\frac{d}{2}\right) + C \\
    y(d) &= A \cdot d^2 + B \cdot d + C 
\end{align*}
\]

(22)

After solving the equation system, the following parabolic equation results:

\[ A = -\frac{4f}{d^2}; B = \frac{4f}{d}; C = 0. \]

(23)
\[ y(x) = -\frac{4f}{d^2} \cdot x^2 + \frac{4f}{d} \cdot x = \frac{4f}{d} \left( 1 - \frac{x}{d} \right) \]  

Figure 2. Beams loaded up.

Figure 3. Beams loaded down.

Figure 4. Beams loaded up.

Figure 5. Beams loaded down.

Figure 6. Beams loaded up.

Figure 7. Beams loaded down.

Figure 8. Beams loaded up.

Figure 9. Beams loaded down.

Figure 10. Beams loaded up.

Figure 11. Beams loaded down.

The parametric study was performed on the following beam types:
- beams with trapezoidal variation on the elevation with span of 30 m - T30-1u and span of 60 m - T60-1u, loaded at the upper side - figure 2;
- beams with trapezoidal variation on the elevation with span of 30 m - T30-1d and span of 60 m - T60-1d, loaded at the lower side - figure 3;
beams with trapezoidal variation on the elevation with span of 30 m - T30-2u and span of 60 m - T60-2u, loaded at the upper side - figure 4;
beams with trapezoidal variation on the elevation with span of 30 m - T30-2d and span of 60 m - T60-2d, loaded at the lower side - figure 5;
beams with parabolic variation on the elevation with span of 30 m - P30-3u and span of 60 m - P60-3u, loaded at the upper side - figure 6;
beams with parabolic variation on the elevation with span of 30 m - P30-3d and span of 60 m - P60-3d, loaded at the lower side - figure 7;
beams with parabolic variation on the elevation with span of 30 m - P30-4u and span of 60 m - P60-4u, loaded at the upper side - figure 6;
beams with parabolic variation on the elevation with span of 30 m - P30-4d and span of 60 m - P60-4d, loaded at the lower side - figure 7;
beams with parabolic variation on the elevation with span of 30 m - P30-5u and span of 60 m - P60-5u, loaded at the upper side - figure 6;
beams with parabolic variation on the elevation with span of 30 m - P30-5d and span of 60 m - P60-5d, loaded at the lower side - figure 7.

The load applied to the trusses with the span of 30 m was determined considering a gravitational afferent of 4 m x 30 m x 3.5 kN/m = 420 KN. The load was applied using concentrated forces in upper or lower side joints according to figure 2 to figure 11.

The load applied to the trusses with the span of 60 m was determined considering a gravitational afferent of 4 m x 60 m x 3.5 kN/m = 840 KN. The load was applied using concentrated forces in upper or lower side joints according to figure 2 to figure 11. The structure analysis was performed with a Matlab program [6] which allows the determination of sectional efforts in the structure bars. To establish the optimal height for the beams having the spans of 30 m and 60 m, the beam height h was varied till the strength $f_{yu}$ was reached and the safety factor of buckling in the compressed bars was $c_{sf} > 2$. The changing process of the height at the beam middle is represented in figure 12 for the beams with span of 30 m and in figure 16 for the span of 60 m. The values for the optimal height are represented in figure 15 and in figure 19 for the beams with the span of 30 m and in figure 17 for the span of 60 m. The buckling calculation for the compressed bars was performed in the elastic range because the slenderness $\lambda > \lambda_p$ was provided for all the compressed bars. The buckling safety factors obtained for the optimal height at the beam are represented in figure 14 for the beams with the span of 30 m and in figure 18 for the beams with the span of 60 m. Comparative analysis of results was performed in Excel program [7].

**Figure 12.** Variation of height-span 30 m.

**Figure 13.** Weight of structure-span 30 m.
Figure 14. Safety coefficient-span 30 m.

Figure 15. Optimal height-span 30 m.

Figure 16. Variation of height-span 60 m.

Figure 17. Weight of structure-span 60 m.

Figure 18. Safety coefficient-span 60 m.

Figure 19. Optimal height-span 60 m.
4. Results and conclusions

From figure 13 and figure 17 one can deduce that the structures having the smallest weight are P30-3d, P30-5d, P60-3d and P60-5d. The reduced weight obtained for these structures is due to the largest number of stretched bars which are dimensioned based on the condition of stretching strength only.

In the process of optimal height determination one can notice that, by increasing the height, at a certain moment one reaches the necessary safety factor of buckling in the compressed bar in the middle of the structure (upper or lower side, depending on the structure and load) and by further increasing of the height one reaches the necessary safety factor of buckling for compressed diagonals from the end of beam.

For trusses with large spans one recommends a greater s step (longer bars) and steels of higher class having a smaller slenderness \( \lambda_0 \) to remain in the range of the elastic buckling.

The advantages resulting from keeping the compressed bars in the range of the elastic buckling are:

- the ratio between the length of a bar element and the largest of the cross-section dimensions is greater than 10, so the rigid joints can be considered articulated joints;
- the truss type structures being statically determined, it results that they have no structural redundancy, therefore the occurrence of the first plastic deformation converts the structure into a mechanism.

References

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