Percolating sets in bootstrap percolation on the Hamming graphs

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Abstract

For any integer \( r \geq 0 \), the \( r \)-neighbor bootstrap percolation on a graph is an activation process of the vertices. The process starts with some initially activated vertices and then, in each round, any inactive vertex with at least \( r \) active neighbors becomes activated. A set of initially activated vertices leading to the activation of all vertices is said to be a percolating set. Denote the minimum size of a percolating set in the \( r \)-neighbor bootstrap percolation process on a graph \( G \) by \( m(G, r) \). In this paper, we present upper and lower bounds on \( m(K^d_n, r) \), where \( K^d_n \) is the Cartesian product of \( d \) copies of the complete graph \( K_n \) which is referred as the Hamming graph. Among other results, we show that 

\[
m(K^d_n, r) = \frac{1+o(1)}{(d+1)r^d} \text{ when both } r \text{ and } d \text{ go to infinity with } r < n \text{ and } d = o(\sqrt{r}).
\]

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1 Introduction

Bootstrap percolation process on graphs can be interpreted as a cellular automaton, a concept was introduced by von Neumann \cite{13}. It has been
extensively investigated in several diverse fields such as combinatorics, probability theory, statistical physics and social sciences. The r-neighbor model is the most studied version of this process in the literature. It was introduced in 1979 by Chalupa, Leith and Reich [7]. In the r-neighbor bootstrap percolation process on a graph, first some vertices are initially activated and then, in each phase, any inactive vertex with at least r active neighbors becomes activated. Once a vertex becomes activated, it remains active forever. This process has also been treated in the literature under other names like irreversible threshold, influence propagation and dynamic monopoly.

Throughout this paper, all graphs are assumed to be finite, undirected, without loops and multiple edges. For a graph $G$, we denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. For a vertex $v$ of $G$, we set $N(v) = \{ x \in V(G) \mid x \text{ is adjacent to } v \}$. The degree of $v$ is defined to be $|N(v)|$. Given a nonnegative integer $r$ and a graph $G$, the $r$-neighbor bootstrap percolation process on $G$ begins with a subset $A_0$ of $V(G)$ whose elements are initially activated and then, at step $i$ of the process, the set $A_i$ of active vertices is

$$A_i = A_{i-1} \cup \left\{ v \in V(G) \mid |N(v) \cap A_{i-1}| \geq r \right\}$$

for any $i \geq 1$. We say $A_0$ is a percolating set of $G$ if $\bigcup_{i \geq 0} A_i = V(G)$. The main extremal problem here is to determine the minimum size of a percolating set which is denoted by $m(G,r)$. The size of percolating sets has been studied for various families of graphs such as hypercubes [12], grids [4,10], tori [10], trees [14] and random graphs [8,11].

Let us fix some notation and terminology. The Cartesian product of two graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent if and only if either $g_1 = g_2$ and $h_1$ is adjacent to $h_2$ or $h_1 = h_2$ and $g_1$ is adjacent to $g_2$. We denote the complete graph on $n$ vertices by $K_n$ and we consider $[n] = \{0, 1, \ldots, n-1\}$ as the vertex set of $K_n$. Denote by $K_n^d$ the Cartesian product of $d$ vertex disjoint copies of $K_n$, that is, the Hamming graph of dimension $d$.

In this paper, we present upper and lower bounds on $m(K_n^d,r)$. In particular, we establish that $m(K_n^d,r) = \frac{1+o(1)}{(d+1)!} r^d$ when both $r$ and $d$ go to infinity with $r < n$ and $d = o(\sqrt{r})$. It is worth to mention that a random version of the $r$-neighbor bootstrap percolation process on the Hamming graphs has been investigated in [9].
2 Two-dimensional Hamming graphs

For every integers \( n \geq 1 \) and \( r \geq 0 \), it is clear that \( m(K_n, r) = \min\{n, r\} \).

In this section, we deal with the first nontrivial case, that is, the Hamming graph of dimension 2. We derive an exact formula for \( m(K_{2n}, r) \). If \( n \leq \lceil r/2 \rceil \), then the degree of any vertex of \( K_{2n}^2 \) is at most \( r - 1 \), implying that \( m(K_{2n}^2, r) = n^2 \). The following theorem resolves the remaining cases.

**Theorem 2.1.** For every nonnegative integers \( n \) and \( r \) with \( n \geq \lceil r/2 \rceil + 1 \),

\[
m(K_{2n}^2, r) = \left\lfloor \frac{(r + 1)^2}{4} \right\rfloor.
\]

**Proof.** Let

\[
V_{n, r} = \left\{ (x, y) \in [n]^2 \mid x + (n - 1 - y) < \left\lceil \frac{r}{2} \right\rceil \text{ or } (n - 1 - x) + y < \left\lfloor \frac{r}{2} \right\rfloor \right\}.
\]

As an example, \( V_{6, 5} \) is shown in Figure 1. It is well known that the number of solutions of \( x_1 + \cdots + x_k < m \) for the nonnegative integers \( x_1, \ldots, x_k \) is \( (m + k - 1) \). As \( n \geq \lceil r/2 \rceil + 1 \), we have

\[
|V_{n, r}| = \left( \left\lceil \frac{r}{2} \right\rceil + 1 \right) + \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) = \left\lfloor \frac{(r + 1)^2}{4} \right\rfloor.
\]

Note that \( V_{n, r} \cap [n - 1]^2 = V_{n-1, r-2} \). We prove by induction on \( r \) that \( V_{n, r} \) is a percolating set in the \( r \)-neighbor bootstrap percolation process on \( K_n^2 \). The statement is trivial for \( r = 0, 1 \). Let \( r \geq 2 \) and assume that the vertices in \( V_{n, r} \) are initially activated. The points on the lines \( x = n - 1 \) and \( y = n - 1 \) become activated from top to bottom and from right to left, respectively. Remove from \( K_n^2 \) all the vertices in the set

\[
L = \left\{ (x, y) \in [n]^2 \mid x = n - 1 \text{ or } y = n - 1 \right\}
\]

to get \( K_{n-1}^2 \). By the induction hypothesis, \( V_{n-1, r-2} = V_{n, r} \cap [n - 1]^2 \) is a percolating set of \( K_{n-1}^2 \) in the \( (r-2) \)-neighbor bootstrap percolation process. Since each vertex in \( [n - 1]^2 \) has two additional activated neighbors in \( L \), we conclude that \( V_{n-1, r-2} \cup L \) is a percolating set of \( K_n^2 \) in the \( r \)-neighbor bootstrap percolation process. This proves the assertion.

We next use induction on \( r \) to establish that any percolating set of \( K_n^2 \) in the \( r \)-neighbor bootstrap percolation process has at least \( \lceil (r + 1)^2/4 \rceil \) elements. The statement is trivially true for \( r = 0, 1 \). Let \( r \geq 2 \) and
consider a percolating set $A$ in the \( r \)-neighbor bootstrap percolation process on \( K_n^2 \). Without loss of generality, one may assume that \((n-1, n-1)\) is the first vertex in \([n]^2 \setminus A\) that becomes activated. So, \((n-1, n-1)\) must have at least \( r \) initially activated neighbors in \( L \), meaning that \(|A \cap L| \geq r\). Remove from \( K_n^2 \) all vertices in \( L \) to get \( K_n^2 \setminus L \). Since \( A \cup L \) is a percolating set in the \( r \)-neighbor bootstrap percolation process on \( K_n^2 \) and each vertex in \([n-1]^2\) has exactly two neighbors in \( L \), we deduce that \( A \cap [n-1]^2 \) is a percolating set of \( K_{n-1}^2 \) in the \((r-2)\)-neighbor bootstrap percolation process. It follows from the induction hypothesis that

\[
|A| \geq |A \cap L| + |A \cap [n-1]^2| \geq r + \left[ \frac{(r-1)^2}{4} \right] = \left[ \frac{(r+1)^2}{4} \right].
\]

Figure 1. The set \( V_{6,5} \) is outlined with circles drawn around its elements.

3 Polynomial method

Closely related to the \( r \)-neighbor bootstrap percolation is the notion of graph bootstrap percolation which was introduced by Bollobás in 1968 under the name of ‘weak saturation’ [6] and was later studied in 2012 by Balogh, Bollobás and Morris [3]. We recall the formal definition. Given two graphs \( G \) and \( H \), the \( H \)-bootstrap percolation process on \( G \) begins with a subset \( E_0 \) of \( E(G) \) whose elements are initially activated and then, at step \( i \) of the process, the set of activated edges is

\[
E_i = E_{i-1} \cup \left\{ e \in E(G) \mid \begin{array}{l}
\text{There exists a subgraph } H_e \text{ of } G \text{ such that } H_e \text{ is isomorphic to } H, e \in E(H_e) \\
\text{and } E(H_e) \setminus \{e\} \subseteq E_{i-1}.
\end{array}\right\}
\]

for any \( i \geq 1 \). The set \( E_0 \) is called a percolating set of \( G \) provided \( \bigcup_{i \geq 0} E_i = E(G) \). The minimum size of a percolating set in the \( H \)-bootstrap percolation
Proof. We introduce an edge coloring $W_a$ proper edge coloring of $K$ in the next section. We first recall the following definition from [10].

**Definition 3.1.** Let $r$ be a positive integer and let $G$ be a graph equipped with a proper edge coloring $c : E(G) \rightarrow \mathbb{R}$. Let $W_c(G, r)$ be the vector space over $\mathbb{R}$ consisting of all functions $\phi : E(G) \rightarrow \mathbb{R}$ for which there exist polynomials $\{P_v(x)\}_{v \in V(G)}$ satisfying

(i) $\deg P_v(x) \leq r - 1$ for any vertex $v \in V(G)$;

(ii) $P_u(c(uv)) = P_v(c(uv)) = \phi(uv)$ for each edge $uv \in E(G)$.

It is said that the polynomials $\{P_v(x)\}_{v \in V(G)}$ recognize $\phi$.

The following theorem provides an interesting linear algebraic lower bound on $m_e(G, r)$.

**Theorem 3.2** (Hambardzumyan, Hatami, Qian [10]). Let $r$ be a positive integer and let $c : E(G) \rightarrow \mathbb{R}$ be a proper edge coloring of a graph $G$. Then $m_e(G, r) \geq \dim W_c(G, r)$.

**Lemma 3.3.** For every positive integers $n$ and $r$ with $n \geq r + 1$, there exists a proper edge coloring $c : E(K_n) \rightarrow \mathbb{R}$ such that $\dim W_c(K_n, r) \geq \left(\frac{r + 1}{2}\right)$.

**Proof.** We introduce an edge coloring $c$ and $\left(\frac{r + 1}{2}\right)$ independent vectors in $W_c(K_n, r)$. Fix arbitrary distinct nonzero real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ and let $c(ij) = \gamma_i \gamma_j$ for any edge $ij \in E(K_n)$. Obviously, $c : E(K_n) \rightarrow \mathbb{R}$ is a proper edge coloring of $K_n$. For each edge $uv \in E(K_n)$ with $u, v \in \{r + 1\}$, we define polynomials $P^0_{u} (x), P^1_{u} (x), \ldots, P^{n-1}_{u} (x)$ as follows. For any $i \in [n]$, let

$$P^i_{uv} (x) = \left\{ \begin{array}{ll}
0, & \text{if } i \in [r + 1] \setminus \{u, v\}; \\
\prod_{k \in \{r + 1\} \setminus \{u, v\}} \frac{x - \gamma_i \gamma_k}{\gamma_u \gamma_v - \gamma_i \gamma_k}, & \text{if } i \in \{u, v\}; \\
\prod_{k \in \{r + 1\} \setminus \{u, v\}} \frac{(x - \gamma_i \gamma_k)(\gamma_i - \gamma_k)}{\gamma_i(\gamma_u - \gamma_k)(\gamma_v - \gamma_k)}, & \text{if } i \in \{r + 1, \ldots, n - 1\}. 
\end{array} \right.$$
Proof. Consider arbitrary distinct nonzero real numbers \( c \) according to Definition 3.1, there exist polynomials \( \{t, \phi \} \) for every \( i \) and \( j \) of \( G \). From this, it follows that \( \{\phi_{uv}\}_{u,v \in [r+1]} \) is a linearly independent subset of \( W_c(K_n, r) \). This completes the proof.

**Lemma 3.4.** Let \( n, r \) be two positive integers and let \( c : E(G) \rightarrow \mathbb{R} \) be a proper edge coloring of a graph \( G \). Then, there is a proper edge coloring \( \hat{c} : E(G \square K_n) \rightarrow \mathbb{R} \) such that

\[
\dim W_{\hat{c}}(G \square K_n, r) \geq \sum_{t=0}^{n-1} \dim W_c(G, r - t),
\]

where \( W_c(G, i) \) is defined to be \( \{0\} \) if \( i \leq 0 \).

Proof. Consider arbitrary distinct nonzero real numbers \( \gamma_0, \gamma_1, \ldots, \gamma_{n-1} \) such that none of the numbers \( \gamma_i \gamma_j \) is in the image of \( c \). For every two adjacent vertices \( u = (g, i) \) and \( v = (h, j) \) of \( G \square K_n \), define

\[
\hat{c}(uv) = \begin{cases} 
  c(gh), & \text{if } i = j; \\
  \gamma_i \gamma_j, & \text{if } g = h.
\end{cases}
\]

Fix \( t \in [n] \), a basis \( B_t \) for \( W_c(G, r - t) \) and a function \( \phi \in B_t \). According to Definition 3.1 there exist polynomials \( \{P^t_i(x)\}_{g \in V(G)} \) recognizing \( \phi \). Define polynomial \( Q^t_u(x) \) for any vertex \( u = (g, i) \in V(G \square K_n) \) as

\[
Q^t_u(x) = P^t_g(x) \Gamma^t_i(x),
\]

where

\[
\Gamma^t_i(x) = \prod_{\ell=0}^{t-1} \left( \frac{x}{\gamma_i - \gamma_\ell} \right).
\]

Note that \( \Gamma^t_i(\gamma_i \gamma_j) = \Gamma^t_j(\gamma_i \gamma_j) \) for all \( i \) and \( j \). Also, we know from Definition 3.1 that \( P^t_g(c(gh)) = P^t_h(c(gh)) \) for each edge \( gh \in E(G) \). Hence, \( Q^t_u(x) \) and \( Q^t_v(x) \) have the same value on \( \hat{c}(uv) \) for any edge \( uv \in E(G \square K_n) \). This implies that \( \{Q^t_u\}_{u \in V(G \square K_n)} \) recognize a function \( \hat{\Psi}_{t,\phi} \in W_c(G \square K_n, r) \).

Since we may choose the pair \( (t, \phi) \) in \( \sum_{t=0}^{n-1} \dim W_c(G, r - t) \) different ways, it remains to show that all functions \( \hat{\Psi}_{t,\phi} \) are linearly independent. Suppose that \( \sum_{t,\phi} \lambda_{t,\phi} \hat{\Psi}_{t,\phi} = 0 \) for some scalars \( \lambda_{t,\phi} \in \mathbb{R} \). Towards a contradiction, assume that \( \tau \) is the smallest value such that \( \lambda_{t,\phi} \neq 0 \) for some \( \phi \). Obviously, \( \Gamma^t_i = 0 \) for any \( i < t \). This yields that \( Q^t_{(g, \tau)} = 0 \) for every
integer $t > \tau$ and vertex $g \in V(G)$. Thus, for every two adjacent vertices $u = (g, \tau)$ and $v = (h, \tau)$ in $G \square K_n$, we have
\[
\sum_{t, \phi} \lambda_{t, \phi} \Psi_{t, \phi}(uv) = \sum_{t, \phi} \lambda_{t, \phi} Q_{t, \phi}^c(\bar{c}(uv)) = \sum_{\phi \in \mathcal{B}_\tau} \lambda_{\tau, \phi} P_{\phi}^g(c(gh)) \Gamma_{\tau}^r(c(gh)) = 0.
\]
Our assumption on $\gamma_0, \gamma_1, \ldots, \gamma_{n-1}$ implies that $\Gamma_{\tau}^r(c(gh)) \neq 0$. Therefore,
\[
\left( \sum_{\phi \in \mathcal{B}_\tau} \lambda_{\tau, \phi} \phi \right) (gh) = \sum_{\phi \in \mathcal{B}_\tau} \lambda_{\tau, \phi} P_{\phi}^g(c(gh)) = 0
\]
for each edge $gh \in E(G)$. This is a contradiction, since $\mathcal{B}_\tau$ is a basis for $W_c(G, r - \tau)$.

\section*{Lemma 3.5}
Let $n, r$ be two positive integers and let $G$ be a graph all whose vertices are of degree at least $r$. Then
\[
m_e(G \square K_n, r) \leq \sum_{t=0}^{n-1} m_e(G, r - t),
\]
where $m_e(G, i)$ is defined to be 0 if $i < 0$.

\section*{Proof}
For any $t$ with $0 \leq t \leq \min\{r, n - 1\}$, consider the subgraph $G_t$ of $G \square K_n$ induced by $\{(v, t) \in V(G \square K_n) \mid v \in V(G)\}$ which is clearly isomorphic to $G$. Also, consider a percolating set $U_t$ of the minimum possible size in the $S_{r-t+1}$-bootstrap percolation process on $G_t$ and activate its elements. We show that the edges of $G_0, \ldots, G_{n-1}$ become activated in the $S_{r+1}$-bootstrap percolation process consecutively. At first, the edges of $G_0$ become activated in $S_{r+1}$-bootstrap percolation process, according to the definition of $U_0$. Let $t \geq 1$ and assume that the edges of $G_0, \ldots, G_{t-1}$ are activated. Since any vertex $(v, t) \in V(G_t)$ is incident to $t$ activated edges with endpoints in $\{(v, i) \mid 0 \leq i \leq t - 1\}$, we conclude that the edges of $G_t$ become activated in the $S_{r+1}$-bootstrap percolation process on $G_t$ by considering $U_t$ as the set of initially activated vertices. Hence, $\bigcup_{t=0}^{n-1} U_t$ is a percolating set of size $\sum_{t=0}^{n-1} m_e(G, r - t)$ in the $S_{r+1}$-bootstrap percolation process on $G \square H$.

\section*{Theorem 3.6}
Let $n, r, d$ be positive integers with $n \geq r+1$. Then $m_e(K_n^d, r) = \frac{(d+r)}{(d+1)}$. 

7
Proof. First, we prove by induction on $d$ that there exists a proper edge coloring $c_d : E(G) \rightarrow \mathbb{R}$ such that $\dim W_{c_d}(K_n^d, r) \geq \binom{d+r}{d+1}$. In view of Lemma 3.3 there is nothing to prove for $d = 1$. By Lemma 3.4 and the induction hypothesis, there is a proper edge coloring $c_d : E(K_n^d) \rightarrow \mathbb{R}$ such that

$$\dim W_{c_d}(K_n^d, r) \geq \sum_{t=0}^{n-1} \dim W_{c_{d-1}}(K_{n}^{d-1}, r-t) \geq \sum_{t=0}^{r-1} \binom{d-1+r-t}{d} = \binom{d+r}{d+1}.$$ 

It follows from Theorem 3.2 that $m_e(K_n^d, r) \geq \binom{d+r}{d+1}$. Now, we establish by induction on $d$ that $m_v(K_n^d, r) \leq \binom{d+r}{d+1}$. The edges of $K_n$ with two endpoints in $[r+1]$ clearly form a percolating set in the $S_{r+1}$-bootstrap percolation process on $K_n$ and so there is nothing to prove for $d = 1$. By applying Lemma 3.5 and the induction hypothesis, we obtain that

$$m_v(K_n^d, r) \leq \sum_{t=0}^{r-1} m_v(K_{n}^{d-1}, r-t) \leq \sum_{t=0}^{r-1} \binom{d-1+r-t}{d} = \binom{d+r}{d+1}.$$ 

4 Multi-dimensional Hamming graphs

Balister, Bollobás, Lee and Narayanan [11] gave the lower bound $(r/d)^d$ and the approximate upper bound $r^d/(2d!)$ on $m(K_n^d, r)$. In this section, we improve both bounds which result in an asymptotic formula for $m(K_n^d, r)$. To begin with, let us fix the notation we shall use throughout this section. We set $d \geq 2$ and $\delta = (d-2)/(d-1)$. For a point $t = (t_1, \ldots, t_d) \in \{0,1\}^d$ and a subset $P \subseteq [n]^d$, we define

$$P(t) = \left\{ (x_1, \ldots, x_d) \in [n]^d \left| \begin{array}{l} \text{There exists } (p_1, \ldots, p_d) \in P \text{ such that } \\
 x_i = t_i(n-1-p_i) + (1-t_i)p_i \text{ for all } i. \end{array} \right. \right\}.$$ 

8
Roughly speaking, $P(t)$ is a region in $[n]^d$ congruent to $P$ around the point $(n-1)t$ instead of the origin. For the sets

$$A_r^d = \left\{(x_1, \ldots, x_d) \in [n]^d \mid \sum_{i=1}^{d} x_i \leq \left\lfloor \frac{r}{2} \right\rfloor - 1 \right\},$$

$$B_r^d = \left\{(x_1, \ldots, x_d) \in [n]^d \mid x_1 + x_2 + \delta \sum_{i=3}^{d} x_i < \delta \left( \left\lfloor \frac{r}{2} \right\rfloor - 1 \right) \right\}$$

and $C_r^d = A_r^d \setminus B_r^d$, we define

$$A_r^d = \bigcup_{t \in T} A_r^d(t), \quad B_r^d = \bigcup_{t \in T} B_r^d(t) \quad \text{and} \quad C_r^d = \bigcup_{t \in T} C_r^d(t),$$

where $T = \{(t_1, \ldots, t_d) \in \{0,1\}^d \mid t_1 = t_2\}$.

**Lemma 4.1.** Let $n, r, d$ be positive integers with $n \geq r + 1$ and $d \geq 2$. Then $A_r^d$ is a percolating set of $K_n^d$ in the $r$-neighbor bootstrap percolation process.

**Proof.** Let $s = \lfloor r/2 \rfloor$. We use an induction argument on $d$. Theorem 2.1 concludes the assertion for $d = 2$. Let $d \geq 3$ and assume that the assertion holds for $d-1$. Set $P_i = \{(x_1, \ldots, x_d) \in [n]^d \mid x_i = i\}$ and $Q_i = P_i \cap A_r^d$. It is not hard to check that, after ignoring the last coordinate, both $Q_i$ and $Q_{n-1-i}$ are exactly $A_{r-2i}^{d-1}$ for any $i \in [s]$.

We consider the following iterative procedure for any $i \in [s]$. At step $i$, we show that the vertices in $P_i \cup P_{n-1-i}$ become activated. The induction hypothesis implies that all vertices in $P_0$ and $P_{n-1}$ are activated by $Q_0$ and $Q_{n-1}$, respectively. Hence, there is nothing to prove for $i = 0$. Assume that $i \geq 1$. Each vertex in $P_i \cup P_{n-1-i}$ has already $2i$ activated neighbors from the previous steps. So, in order to activate the vertices in $P_i \cup P_{n-1-i}$, it is enough to consider the $(r-2i)$-neighbor bootstrap percolation process on $P_i \cup P_{n-1-i}$. This is done by the induction hypothesis and by considering $Q_i \cup Q_{n-1-i}$ as the initially activated set, since both $Q_i$ and $Q_{n-1-i}$ are copies of $A_{r-2i}^{d-1}$.

Finally, we observe that any vertex in $\bigcup_{i=s}^{n-s} P_i$ has at least $r$ neighbors in $\bigcup_{i=0}^{s-1} (P_i \cup P_{n-1-i})$ and so it becomes activated. This completes the proof, since $\bigcup_{i=0}^{n-1} P_i = [n]^d$ and $\bigcup_{i=0}^{n-1} Q_i = A_r^d$. \qed

**Lemma 4.2.** Let $n, r, d$ be positive integers with $n \geq r + 1$ and $d \geq 2$. Then $C_r^d$ is a percolating set of $K_n^d$ in the $r$-neighbor bootstrap percolation process.
Proof. By Lemma 4.1 it suffices to prove that all vertices in \( B_r^d \) become activated in the \( r \)-neighbor bootstrap percolation process on \( K_n^d \). Note that once a vertex in \( B_r^d \) becomes activated, the corresponding vertices in all other \( B_r^d(t) \) become simultaneously activated, due to symmetry. So, it is sufficient to show that any vertex in \( B_r^d \) becomes activated in the \( r \)-neighbor bootstrap percolation process on \( K_n^d \). Since \( B_r^2 = \emptyset \), we may assume that \( d \geq 3 \). Fix an arbitrary vertex \( x = (x_1, x_2, ..., x_d) \in B_r^d \) and denote by \( \eta^i_x \), the number of neighbors of \( x \) in \( C^d_r \) differing from \( x \) in the coordinate \( i \). Let \( \eta_x = \eta^1_x + \cdots + \eta^d_x \) and \( \sigma_x = x_3 + \cdots + x_d \). It straightforwardly follows from the definitions of \( A^d_r \), \( B^d_r \) and \( C^d_r \) that \( \eta^1_x = \eta^2_x = x - \sigma_x - \lfloor \delta(s - 1 - \sigma_x) \rfloor \) and

\[
\eta^3_x = \cdots = \eta^d_x = 2 \left( \left\lfloor \frac{x_1 + x_2}{d - 2} \right\rfloor + 1 \right),
\]

where \( s = \lceil r/2 \rceil \). Therefore,

\[
\eta_x = 2(s - \sigma_x - \lfloor \delta(s - 1 - \sigma_x) \rfloor) + 2(d - 2) \left( \left\lfloor \frac{x_1 + x_2}{d - 2} \right\rfloor + 1 \right).
\]

Since \( s \geq r/2 \) and

\[
\left\lfloor \frac{x_1 + x_2}{d - 2} \right\rfloor \geq \frac{x_1 + x_2 - (d - 3)}{d - 2},
\]

we obtain that \( \eta_x \geq r - 2(\rho_x + \sigma_x) \), where \( \rho_x = \lfloor \delta(s - 1 - \sigma_x) \rfloor - (x_1 + x_2 + 1) \). Note that \( \rho_x \geq 0 \) in view of the definition of \( B^d_r \).

We now prove by induction on \( \tau_x = \rho_x + 2\sigma_x \) that any vertex \( x \in B_r^d \) becomes activated in the \( r \)-neighbor bootstrap percolation process on \( K_n^d \). If \( \tau_x = 0 \), then \( \rho_x = \sigma_x = 0 \) and it follows from \( \eta_x \geq r - 2(\rho_x + \sigma_x) \) that \( x \) has at least \( r \) activated neighbors, we are done. So, we may assume that \( \tau_x \geq 1 \). In view of the inequality \( \eta_x \geq r - 2(\rho_x + \sigma_x) \), it is sufficient to show that at least \( 2(\rho_x + \sigma_x) \) neighbors of \( x \) in \( B_r^d \) have been activated during the previous induction steps. For this, consider the sets

\[
P_x = \bigcup_{i=1}^2 \left\{ w \in [n]^d \mid x \text{ and } w \text{ coincide in all components except the } \text{ith component and } w_i \in \{x_i + 1, \ldots, x_i + \rho_x\} \right\},
\]

\[
Q_x = \bigcup_{i=3}^d \left\{ w \in [n]^d \mid x \text{ and } w \text{ coincide in all components except the } \text{ith component and } w_i \in [x_i] \right\},
\]

and

\[
Q'_x = \bigcup_{i=3}^d \left\{ w \in [n]^d \mid x \text{ and } w \text{ coincide in all components except the } \text{ith component and } n - 1 - w_i \in [x_i] \right\},
\]

10
where \( w = (w_1, \ldots, w_d) \). Clearly, \( P_x \cup Q_x \cup Q'_x \subseteq N(x) \cap B_r^d \). Further, \( \tau_w < \tau_x \) for any vertex \( w \in P_x \cup Q_x \). Therefore, by the induction hypothesis and the symmetry of \( B_r^d \), we deduce that \( P_x \cup Q_x \cup Q'_x \) is a set of activated vertices of size \( 2(\rho_x + \sigma_x) \). Thus, \( x \) becomes activated, as required.

We need the following theorem in order to prove our result about the upper bound on \( m(K_n^d, r) \).

**Theorem 4.3** (Beged-Dov [5]). Let \( a_1, \ldots, a_k, b \) be positive numbers with \( b \geq \min\{a_1, \ldots, a_k\} \) and let \( N \) be the number of solutions of \( a_1 x_1 + \cdots + a_k x_k \leq b \) for the nonnegative integers \( x_1, \ldots, x_k \). Then
\[
\frac{b^k}{k! a_1 \cdots a_k} \leq N \leq \frac{(a_1 + \cdots + a_k + b)^k}{k! a_1 \cdots a_k}.
\]

**Theorem 4.4.** Let \( n, r, d \) be positive integers with \( n \geq r + 1 \) and \( d \geq 2 \). Then
\[
\frac{1}{r} \left( \frac{d + r}{d + 1} \right) \leq m(K_n^d, r) \leq \frac{(r + 2d - 1)^d - \delta^2(r - 2)^d}{2d!}.
\]

**Proof.** The lower bound is obtained from Theorem 3.6 and the fact that \( m_e(G, r) \leq rm(G, r) \). For the upper bound, note that \( C_r^d \) is a percolating set in the \( r \)-neighbor bootstrap percolation process on \( K_n^d \) by Lemma 4.2.

It follows from \( B_r^d \subseteq A_r^d \) and Theorem 4.3 that
\[
\left| C_r^d \right| = \left| A_r^d \right| - \left| B_r^d \right| \leq \frac{(d + \left\lceil \frac{r}{2} \right\rceil - 1)^d}{d!} - \frac{(\delta(\left\lceil \frac{r}{2} \right\rceil - 1))^d}{d! \delta^{d-2}} \leq \frac{(r + 2d - 1)^d - \delta^2(r - 2)^d}{2d!}.
\]

As \( |T| = 2^d - 1 \), we have
\[
\left| C_r^d \right| \leq \sum_{t \in T} \left| C_r^d(t) \right| \leq \frac{(r + 2d - 1)^d - \delta^2(r - 2)^d}{2d!}.
\]

This proves the upper bound.

**Corollary 4.5.** Let \( r \to \infty, n \geq r + 1 \) and \( d = o(\sqrt{r}) \). Then
\[
\frac{r^d}{(d + 1)!} (1 + o(1)) \leq m(K_n^d, r) \leq \frac{r^d(2d - 3)}{2d!(d - 1)^2} (1 + o(1)).
\]

In particular, if in addition \( d \to \infty \), then \( m(K_n^d, r) = \frac{1 + o(1)}{(d+1)!} r^d \).
5 Line Graphs

The line graph of a graph \(G\), written \(L(G)\), is the graph whose vertex set is \(E(G)\) and two vertices are adjacent if they share an endpoint. We determined \(m(K^2_{2n}, r)\) in Section 2. One may think of \(K^2_{2n}\) as the line graph of \(K_{n,n}\), the complete bipartite graph with parts of size \(n\). Inspired by this observation, we study \(m(L(K_n), r)\), where \(L(K_n)\) is the line graph of the complete graph on \(n\) vertices. Note that the \(r\)-neighbor bootstrap percolation on \(L(K_n)\) can be viewed as an edge percolation process on \(K_n\) and so it is somehow similar to the \(S_{r+1}\)-bootstrap percolation on \(K_n\). In the former, an edge of \(K_n\) becomes activated if the number of activated edges incident with either of its end points is at least \(r\) while in the latter, an edge of \(K_n\) becomes activated when there are at least \(r\) activated edges all incident with one of its end points.

By Theorem 3.6, \(m_e(K_n, r) = (r+1)^2\) for \(n \geq r + 1\) which resolves the minimum size of a percolating set in the \(S_{r+1}\)-bootstrap percolation on \(K_n\).

In this section, we compute \(m(L(K_n), r)\) using our interpretation of the \(r\)-neighbor bootstrap percolation on \(L(K_n)\) as the edge percolation process on \(K_n\).

Definition 5.1. Let \(r, n\) be nonnegative integers with \(n \geq \lceil \frac{r}{2} \rceil + 2\). Define the graph \(G^n_r\) as follows. Let \([n]\) be the vertex set and for \(i = 0, \ldots, \lceil r/2 \rceil - 1\), connect \(i\) to the last \(\lceil r/2 \rceil - i\) vertices. If \(r\) is even, then also connect \(n - 3 + 2j - r/2\) to \(n - 2 + 2j - r/2\) for \(1 \leq j \leq \lceil r/4 \rceil\).

The condition \(n \geq \lceil \frac{r}{2} \rceil + 2\) ensures that \(G^n_r\) is a simple graph with

\[
|E(G^n_r)| = \sum_{i=0}^{\lceil \frac{r}{2} \rceil - 1} \left( \left\lfloor \frac{r}{2} \right\rfloor - i \right) + \epsilon \sum_{j=1}^{\lceil r/4 \rceil} 1 = \left( \left\lfloor \frac{r}{2} \right\rfloor + 1 \right) + \epsilon \left\lfloor \frac{r}{4} \right\rfloor = \left\lfloor \frac{(r + 2)^2}{8} \right\rfloor,
\]

where \(\epsilon = 1\) if \(r\) is even and 0 otherwise.

Lemma 5.2. If \(n \geq \lceil \frac{r}{2} \rceil + 2\), then \(m(L(K_n), r) \leq \left\lfloor \frac{(r + 2)^2}{8} \right\rfloor\).

Proof. We show that the activation of \(E(G^n_r)\) leads to the activation of \(E(K_n)\) in the edge percolation process on \(K_n\). From the definition, we see
that the subgraph of $G^n_r$ induced on $[n-1]$ is $G^{n-1}_{r-2}$. This proposes to use an induction argument on $r$. The assertion trivially holds for $r = 0, 1$. Assume that $r \geq 2$. By the definition and some calculations, one can find that $\deg(n - i) = \max\{0, \lceil r \rceil - i + 2\}$ for $i = 2, \ldots, n - \lceil r \rceil$. Also, $\deg(n - 1) = \lceil r \rceil + \epsilon$, where $\epsilon$ is 1 if $r \equiv 2 \pmod{4}$ and 0, otherwise. Note that the set of vertices of $G^n_r$ which are not adjacent to $n - 1$ is $\{\lceil r \rceil, \ldots, n - 2 - \epsilon\}$. As $\deg(n - 1) + \deg(n - 2 - \epsilon) = \lceil r \rceil + \epsilon + \lceil r \rceil - \epsilon = r$, the edge between $n - 1$ and $n - 2 - \epsilon$ becomes activated and the degree of $n - 1$ increases by 1. Using the same argument, the edge between $n - 1$ and $n - 3 - \epsilon$ becomes activated and so on. Once every edge incident to $n - 1$ percolates, we may omit $n - 1$ and consider the subgraph of $G^n_r$ induced on $[n - 1]$ which is $G^{n-1}_{r-2}$. As each end point of every edge in this graph is adjacent to $n - 1$ through an activated edge, we may consider the edge bootstrap percolation process with parameter $r - 2$ on $K_{n-1}$. The hypothesis of the induction implies that the activation of $E(G^{n-1}_{r-2})$ leads to the activation of $E(K_{n-1})$, completing the proof. □

We next find a lower bound on $m(L(K_n), r)$.

**Lemma 5.3.** If $n \geq \lceil r \rceil + 2$, then $m(L(K_n), r) \geq \lceil (r + 2)^2/8 \rceil$.

**Proof.** Fix positive integers $r, n$ with $n \geq \lceil r \rceil + 2$. Let $A \subset E(K_n)$ be a minimum size set whose activation leads to the activation of $E(K_n)$ in the edge percolation process on $K_n$. Lemma 5.2 implies that $|A| \leq \lceil (r + 2)^2/8 \rceil$. Let $e = (e_0, e_1, \ldots, e_{t-1})$ be an order in which the edges of $E(G) \setminus A$ become activated, where $t = \lceil (r + 2)^2/8 \rceil$. We find a maximal subsequence $f = (e_{i_0}, e_{i_1}, \ldots, e_{i_{k-1}})$ of $e$ as follows. Let $e_{i_0} = e_0$. If $e_{i_0}, \ldots, e_{i_{k-1}}$ are chosen, then let $e_{i_j}$ be the first edge in $e$ after $e_{i_{j-1}}$ which is independent from $e_{i_0}, \ldots, e_{i_{j-1}}$.

We show that $k > \lceil r/4 \rceil$. To prove it, we find an upper bound on $t$. First note that by the definition of $f$, every edge in $e$ is incident with some edge in $f$. Assume that $e_{i_j} = x_jy_j$ for $0 \leq j \leq k - 1$. Since $e_{i_j}$ becomes activated after the activation of $e_{i_{j-1}}$, the vertices $x_j$ and $y_j$ are incident with at least $r$ edges in $A \cup \{e_0, e_1, \ldots, e_{i_{j-1}}\}$. Hence the number of edges in $\{e_{i_j}, \ldots, e_{i_{j-1}}\}$ with one end point in $\{x_j, y_j\}$ is at most $2n - 3 - r$. It follows that $t \leq k(2n - 3 - r)$. On the other hand, $t \geq \binom{n}{2} - [(r + 2)^2/8]$. An easy calculation shows that $k > \lceil r/4 \rceil$.

By the definition of $f$, $x_j$ (similarly $y_j$) is incident with at most $2j$ edges of $\{e_0, e_1, \ldots, e_{i_{j-1}}\}$. Hence, the set of edges in $A$ incident with either $x_j$ or $y_j$, say $E_{i_j}$, is of the size at least $r - 4j$. Since the end points of all edges in
f are distinct, the sets $E_j$ are pairwise disjoint and therefore

$$|A| \geq \sum_{j=0}^{k-1} |E_j| \geq \sum_{j=0}^{\lfloor r/4 \rfloor} r - 4j = \left\lfloor \frac{(r + 2)^2}{8} \right\rfloor,$$

as desired.

Since the upper and lower bounds on $m(L(K_n), r)$ coincide, we have the following result.

**Theorem 5.4.** Let $n, r$ be two positive integers. Then

$$m(L(K_n), r) = \begin{cases} \left\lfloor \frac{(r + 2)^2}{8} \right\rfloor, & n \geq \left\lceil \frac{r}{2} \right\rceil + 2; \\ \frac{n}{2}, & \text{o.w.} \end{cases}$$

6 Concluding remarks

For $n \geq r + 1$, as we have seen, $m(K^n_d, r)$ is independent of $n$. For $n \leq r$, it seems that $m(K^n_d, r)$ depends on $n$ and so in this case it would probably be much harder to derive a formula for $m(K^n_d, r)$. The special case $n = 2$ has been asymptotically determined in [10, 12]. It is easily checked that $m(K^n_2, 1) = 1$ and $m(K^n_2, 2) = \lceil r/2 \rceil + 1$. Using the result $m(K^n_2, 3) = \lfloor d(d+3)/6 \rfloor + 1$ of [12], one may show that $m(K^n_2, 3) \leq \lceil (d+1)(d+5)/6 \rceil + 1$. On the other hand, by Theorem 3.6, $m(K^n_2, 3) \geq \lfloor d(d+5)/6 \rfloor + 1$. It would be challenging to find $m(K^n_2, 3)$ for $n \geq 3$. Another interesting problem is the determination of $m_c(L(K_n), r)$ using the polynomial method.

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