Thermodynamics of the Pauli oscillators and Lee-Wick partners of the Standard model particles

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The present article is about the statistical mechanics of non-trivial field configurations. The non-trivial fields arise from the negative sign of the commutators and the anticommutators of the bosonic and fermionic field excitations. These kinds of fields were previously studied by Pauli and Lee and Wick. The thermal distribution function of the above mentioned fields are calculated in the article and using the thermal distribution functions the energy density, pressure and entropy density of the non-trivial field configurations are found out. The results match exactly with a previous calculation done by Fornal et. al. for higher derivative Lee-Wick theories showing a deeper similarity with the earlier work. It is assumed that such kinds of non-trivial fields may have existed in the early universe and may have some cosmological relevance.

I. INTRODUCTION

For the last couple of decades there has been a series of papers related to the Lee-Wick kind of field theories. The Lee-Wick construction [1, 2] originally was motivated to tackle the problem of infinities in quantum field theories. By including massive partners of the photon and other fermions present in the theory of Quantum electrodynamics Lee and Wick could produce a theory which was devoid of divergences. Later there has been a flurry of activity concerning the connection of higher derivative theories and the Lee-Wick construction [3, 4]. All these theories assumed the existence of some partners of the Standard model particles. The partners of the Standard model fields arose out of suitable redefinitions of fields in a higher derivative version of the Standard model Lagrangian. In these cases the propagators of the partners of the Standard model fields behaved more like the propagators one would get if they had a Lee-Wick like theory. There has been at least one attempt [5] to use the concepts of these Lee-Wick constructions in cosmology where the authors were able to show the bouncing nature of the universe whose energy is dominated by the energies of a scalar field and its Lee-Wick partner. In Ref. [6] the authors tried to formulate a possible thermodynamic theory of particles which includes the Lee-Wick partners using a method of statistical field theory previously formulated by Dashen, Ma and Bernstein in Ref. [7].

In 1943, Pauli wrote a paper on an unusual field theory [8], where the bosons appearing in the theory were accompanied by partners, whose excitation modes were quantized with the wrong sign of the commutators. The idea of proposing such an unusual theory came from an urge to suppress ultraviolet divergences appearing in normal quantum field theories. Most of the work of Lee and Wick [1, 2] utilize the concepts of Pauli, and his field excitations which will be called Pauli oscillators in this paper. Pauli’s paper did not have any ingredients from a higher derivative theory.

There has not been any attempt to produce a statistical field theory based on Pauli’s work up till now. One of the important outcomes of this article is to show that the results obtained on the thermodynamics of a Lee-Wick ghost infested universe, as done in [8], matches exactly with the thermodynamic quantities of an universe full of Pauli oscillators. We have included the fermionic sector also, where the basic quantization prescription was given by Lee and Wick [2]. In this regard it should be noted that unlike the fermions in Lee and Wick’s theory the fermion quantization presented in this article is simpler and the fermions do not have a complex mass. These facts may have some bearing on the unitarity of the theory but presently the theory predicts thermodynamic results which are in agreement with the results obtained in Ref. [8].

The previous attempt to theorize the thermodynamic properties of a universe which is constituted by the Standard model particles and their possible Lee-Wick partners gave interesting results which may have
very important consequences in cosmology. In the present work we use the basic commutation relations and number operator language as used in the original paper of Lee and Wick, and Pauli [8], and derive the expressions of the thermal distribution functions of the Lee-Wick ghosts or the Pauli oscillators which accompany the normal particles present in the Standard model. The distribution functions of such highly non-trivial field configurations are presented for the first time in this article. The thermal distribution function of the usual field configurations are different in nature from those of the standard Bose-Einstein or Fermi-Dirac distributions. Using the new distribution functions we can reproduce all the thermodynamic quantities previously calculated by Fornal, Grinstein and Wise in Ref. [6].

The material in the present article is presented in the following manner. The next section discusses about the technique to find out the thermal distribution function of the Lee-Wick partners. In section III the energy density, pressure and entropy density of a gas comprising of elementary particles and their unusual field partners are calculated using the thermal distribution functions. The last section IV summarizes the important points of the article.

II. DISTRIBUTION FUNCTIONS OF THE PAULI OSCILLATOR EXCITATIONS AND LEE-WICK FIELDS

A free quantum field theory can be thought of to be made up of an infinite number of linear harmonic oscillators in the momentum space oscillating with frequencies $\epsilon(p) = \sqrt{p^2 + m^2}$ where $p$ is the 3-momentum of the oscillator excitations which can take infinite values, and $m$ is the mass of the excitations over the vacuum. For each of the oscillators, corresponding to any $p$, there corresponds an annihilation operator $a(p)$ and a creation operator $\bar{a}(p)$. The operator $\bar{a}(p)$ is defined as $\bar{a}(p) \equiv \eta^{-1} a(p) \eta$, where $\eta$ is the metric on the Hilbert space of the oscillators which is +1 for most of the cases, but need not be so in general. The oscillators can be bosonic, if the only non-vanishing commutators are of the form $[a(p), a(k)] = \delta^3(p - k)$, or fermionic if the only non-vanishing anti-commutators are of the form $\{a(p), \bar{a}(k)\} = \delta^3(p - k)$. In the above cases the metric on the Hilbert spaces of the bosonic or fermionic oscillators are both +1. But the above mentioned commutation relations do not exhaust all the possibilities of quantizing the oscillators. Instead of the usual bosonic commutation, if we assume

$$[a(p), a(k)] = -\delta^3(p - k),$$

then also we get oscillator like spectrum where the eigenvalues of the number operator [2]:

$$N(p) \equiv -\bar{a}(p)a(p)$$

are all positive, and the minimum value of $N(p)$ is zero. But in this case the metric $\eta$ on the Hilbert space of the oscillators become indefinite, which implies it is not always +1 but its general form is $\eta = (-1)^{-N(p)}$. In the present case we can define a vacuum satisfying $N(p)|0\rangle = 0$. For such field excitations with momentum $p$ the Hamiltonian of a single oscillator can be written as

$$H(p) = \frac{1}{2}\epsilon(p) \left[ a(p)\bar{a}(p) + \bar{a}(p)a(p) \right] = -\epsilon(p) \left[ N(p) + \frac{1}{2} \delta^3(0) \right],$$

The additive term $-\frac{1}{2} \epsilon(p) \delta^3(0)$ is the zero point energy which can be ignored in further calculations. In the present case the Hamiltonian of the above field excitations are completely equivalent to the negative energy Hamiltonian of the Pauli oscillator in Ref. [8]. Lee and Wick in Ref. [1, 2] did show that it is indeed possible to produce a sensible unitary quantum field theory with such kind of excitations. In this article we are more interested about the thermal distribution of the Lee-Wick or the Pauli excitations. Some one interested in the formal structure of such kind of theories can refer to the original work in Ref. [1, 2]. Due to the presence of on-shell excitations the thermal vacuum becomes $|\Omega\rangle \equiv |n(p_1), n(p_2), \cdots \rangle$ where $n(p_1)$ is the number of excitations carrying momentum $p_1$. The action of the number operator on such a vacuum is [2]

$$N(p)|\Omega\rangle = n(p)|\Omega\rangle,$$

where $n(p)$ is the number of particles with momentum $p$ present in the thermal vacuum. In this analysis we will consider non-interacting real scalar field for which the chemical potential $\mu = 0$. In the present case with a Hamiltonian of the form as given in Eq. [8] the single particle partition function will be,

$$z_{LW}^B = \text{Tr} e^{-\beta H} = \sum_{n(p)=0}^{\infty} e^{\beta n(p)\epsilon(p)},$$

(5)
where $\beta = \frac{1}{T}$. From the last equation it is seen if we are finding the single particle partition function for the Lee-Wick partner of a Standard model boson then the series representing $\bar{z}_L^{B}$ does not converge for $\beta > 0$. Consequently we regularize the last expression by cutting off the summation for a finite value of $n(p)$ as:

$$z_{LW}^B = \sum_{n(p)=0}^{M-1} e^{\beta n(p)} c(p) = \frac{1 - e^{\beta \epsilon(p) M}}{1 - e^{\beta \epsilon(p)}}, \quad (6)$$

where $M$ is a phenomenological cut-off which can be made indefinitely big at the end of the calculation.

Next we calculate the thermal distribution function of the field excitations, which behave like Pauli oscillator excitations, from the expression of the single cell partition function of the field excitations as given in Eq. (1). In conventional statistical mechanics we can find the single cell distribution function via

$$f(p) = \frac{1}{\beta} \left( \frac{\partial \ln z}{\partial \mu} \right)_{V,\beta}, \quad (7)$$

where $\mu$ is an auxiliary chemical potential whose exact nature is not important for our purpose. In presence of an auxiliary chemical potential the single particle partition function can be written as

$$z_{LW}^B = \sum_{n(p)=0}^{M-1} e^{\beta n(p)} \{\epsilon(p) - \mu\} = \frac{1 - e^{\beta \epsilon(p) - \mu} M}{1 - e^{\beta \epsilon(p) - \mu}}. \quad (8)$$

Applying conventional methods, the distribution function can also be written as

$$f_B(p) = \frac{1}{\beta} \left( \frac{\partial \ln z_{LW}^B}{\partial \mu} \right)_{V,\beta}, \quad (9)$$

which comes out to be,

$$f_B(p) = -\frac{e^{\beta \epsilon(p) - \mu}}{1 - e^{\beta \epsilon(p) - \mu}} + M \frac{e^{\beta \epsilon(p) - \mu} M}{1 - e^{\beta \epsilon(p) - \mu} M}. \quad (10)$$

Now setting the auxiliary chemical potential to be zero, $\mu = 0$, we get the distribution function of the fields whose excitation spectra is given by the Pauli oscillators as

$$f_B(p) = -\frac{1}{e^{-\beta \epsilon(p)} - 1} + \frac{M}{e^{-\beta \epsilon(p)} M - 1}. \quad (11)$$

This is the distribution function of the fields whose excitation are described by creation and annihilation operators which satisfy Eq. (1). These are not the fields which appear in the Standard model of particle physics. The distribution function as plotted in Fig. 1 shows that the average excitation per energy level is negative definite. Obviously these systems describe a physical theory which is non-trivial and the negative sign of the distribution is only meaningful when compared with the positive definite distribution of the normal Standard model bosons. In general these kind of distributions will produce negative energy density and pressure but once these energy density and pressure is added with the positive energy density and pressure of the Standard model bosons we get a net positive energy density and pressure. The important point to notice about the distributions is that there is no pile up of quanta near $\epsilon(p) = 0$ as is in the case of the Bose-Einstein distribution. The reason being that the Pauli excitation spectra distribution has two infinite spikes as the energy tends to zero and they cancel each other near the origin.

If we assume the anticommutators of the creation and the annihilation operators which define the fermionic excitations as

$$a^2_s(p) = \bar{a}^2_s(p) = 0, \quad \{a_s(p), \bar{a}_{s'}(k)\} = -\delta_{s,s'} \delta^3(p - k), \quad (12)$$

where $s$, $s'$ may be some internal quantum numbers, then we can proceed in a similar way as done before and calculate the thermal distribution of these excitations. If we stick to the definition of the number operator as given in Eq. (2), then in this case the possible eigenvalues of the number operator are simply 0 and 1. The Hamiltonian of a single oscillator is

$$H(p) = \frac{1}{2} \epsilon(p) [a_s(p) \bar{a}_s(p) - \bar{a}_s(p) a_s(p)] = -\epsilon(p) \left[ N(p) - \frac{1}{2} \delta^3(0) \right], \quad (13)$$
where $\frac{1}{2}\epsilon(p)\delta^3(0)$ is the zero point energy. In the above equation we have assumed that the Hamiltonian is independent of $s$. If we use this oscillator Hamiltonian to calculate the single particle partition function, neglecting the zero point energies, there is no problem regarding the convergence of the series. The single particle partition function for the anticommuting fields turns out to be

$$z_{\text{LW}}^F = \sum_{n(p)\geq0} e^{\beta n(p)(\epsilon(p)-\mu)} = 1 + e^{\beta(\epsilon(p)-\mu)},$$

(14)

where $\mu$ is an auxiliary chemical potential. Now applying the formula in Eq. (9) and setting $\mu = 0$ at the end we get the distribution function for the Lee-Wick ghost partners or the Pauli excitations of the Standard model fermions as:

$$f_F(p) = -\frac{1}{e^{-\beta\epsilon(p)} + 1}.$$  

(15)

Unlike the previous case, in the present scenario the distribution function has no dependence on the dimensionless regulator $M$.

Here we point out that the fermionic degrees of freedom are not exactly the same as presented in the paper by Lee and Wick. There they assumed that the fermionic ghosts mix with each other and more over they have complex masses. The description of the fermionic sector in this article is more in line with the fermionic extension of Pauli’s work. It is to be noted that although the Standard model particles are proper bosons or fermions whose thermal distributions are given by the Bose-Einstein or Fermi-Dirac distributions, in the present theory the bosons (fields obeying commutation relations) or the fermions (fields obeying anti-commutation relations) do not follow the Standard Bose-Einstein or Fermi-Dirac statistics.

By the nature of the distribution functions, for both the bosonic and fermionic excitations, it is seen that the average number of particles in an energy cell $\epsilon(p)$ is negative. This is a very counter intuitive result. But once we observe both the bosonic and fermionic Hamiltonians we notice that the energy of any single excitation of these fields are negative. The negative energy is not a result of any particular choice of potential or interaction of the fields but rather an artifact of the way these fields are quantized. The vacuum defined is not stable and there exists much less energetic states than the vacuum itself. These kinds of fields are unstable. The maximum energy of the field configurations can be zero. As the distribution functions are interpreted as average number of excitations of the field quanta in any energy level $\epsilon(p)$, which is positive, we immediately notice a contradiction. In the present case the field configurations cannot afford to have
quantum excitations in positive energy states. Consequently the only way there can be some positive energy out of these systems is by removing some of the oscillator modes, or particles, from these systems. As a result the negative distribution functions indicate that the field configurations must be de-excited to get to positive energy states i.e. the Lee-Wick partners decay into the Standard model particles.

III. ENERGY DENSITY, PRESSURE AND ENTROPY DENSITY FROM THE DISTRIBUTION FUNCTION.

A. The bosonic case

To calculate the relevant thermodynamic quantities for the bosonic field from a statistical mechanical point of view we will employ Eq. (11). The energy density can be calculated using the following known equation

\[
\rho = \frac{g}{(2\pi)^3} \int \epsilon(p) f_B(\epsilon) d^3p = \frac{g}{2\pi^2} \int_0^\infty \epsilon(p) f_B(\epsilon) |p|^2 d|p|, \tag{16}
\]

where \(g\) stands for any intrinsic degree of freedom of the particle. For a relativistic excitation \(\epsilon^2 = p^2 + m^2\) where \(m\) is the mass of the bosonic excitations. Changing the integration variable from \(|p|\) to \(\epsilon\) one gets

\[
\rho = -\frac{g}{2\pi^2} \int_0^{\infty} \left( \epsilon^3 - \frac{m^2}{2} \right) \frac{d\epsilon}{e^{\beta \epsilon} - 1} + \frac{M g}{2\pi^2} \int_0^{\infty} \left( \epsilon^3 - \frac{m^2}{2} \right) \frac{d\epsilon}{e^{\beta \epsilon M} - 1}. \tag{17}
\]

In the above integral it is assumed that \(|p| \gg m\) and to have a closed integral the lower limit of the integral is assumed to be zero. In the extreme relativistic limit the system temperature \(T \gg m\). Both of the integrals can only be done when \(\beta < 0\), and in that case the result of the last integral is

\[
\rho = -\frac{g}{2\pi^2} \left( \frac{\pi^4 T^4}{15} - \frac{m^2 \pi^2 T^2}{12} \right) + \frac{M g}{2\pi^2} \left( \frac{\pi^4 T^4}{15M^3} - \frac{m^2 \pi^2 T^2}{12M} \right). \tag{18}
\]

Analytically continuing the above result for \(\beta > 0\) and taking \(M \to \infty\) we see that for normal temperatures the energy density for extreme relativistic excitations of the bosonic fields is of the following form

\[
\rho = -g \left( \frac{\pi^2 T^4}{30} - \frac{m^2 T^2}{24} \right). \tag{19}
\]
As expected, the energy density turns out to be negative for the excitations in this case. The pressure density of the bosonic field excitations can be found out from

\[ p = -g \left( \frac{\beta}{4} - \frac{m^2 T^2}{12} \right). \]  

(20)

Following similar steps as in the case of the energy density, it is seen that the pressure of extremely relativistic excitations of the bosonic fields turns out to be

\[ p = -g \left( \frac{\beta}{6} - \frac{m^2 T^2}{12} \right). \]  

(21)

The entropy density of the bosonic fields is simply given by

\[ s = \frac{\rho + p}{T} = -g \left( \frac{2\pi^2 T^4}{45} - \frac{m^2 T^2}{12} \right). \]  

(22)

These values of the energy density, pressure and entropy density exactly match the corresponding values calculated for the Lee-Wick partners, in [9], in a different way. In [9] the authors were trying to formulate thermodynamics for a higher-derivative theory. The higher derivative theory was converted into standard theory (theory up to a second derivative) with the introduction of Lee-Wick partners. The authors in the previous work did not quantize the system explicitly but were working with the form of the propagators of the Lee-Wick partners.

If we assume that in the early universe for each bosonic degree of freedom in Standard model there exist a corresponding bosonic degree of freedom whose creation and annihilation operators are quantized as in Eq. (13) then the net energy density, pressure and entropy density of the early universe turns out to be

\[ \rho_B = \rho_{SM} + \rho = \frac{g m^2 T^2}{24}, \quad p_B = p_{SM} + p = \frac{g m^2 T^2}{24}, \quad s_B = s_{SM} + s = \frac{g m^2 T^2}{12}, \]  

(23)

which are all positive as expected. Here the energy density, pressure density and entropy density for the Standard model bosonic particles are \( \rho_{SM} = \frac{g m^2 T^2}{24} \), \( p_{SM} = \frac{g m^2 T^2}{24} \) and \( s_{SM} = \frac{g m^2 T^2}{45} \) respectively [9].

### B. The fermionic case

In this subsection we apply Eq. (15) to find the energy density, pressure and entropy density of the fermionic excitations. In this case the distribution function do not have any dependence on the regulator \( M \). For relativistic excitations the integrals which give the energy density and pressure for the fermionic case are exactly similar with the bosonic case except that now we have to use the distribution for the fermions. The integrals can be easily done, granted \( \beta < 0 \), but the results can be analytically continued for positive temperatures. The results in this case are listed below. The energy density, pressure and entropy density of the Lee-Wick partners are as follows:

\[ \rho = -g \left( \frac{7\pi^2 T^4}{240} - \frac{m^2 T^2}{48} \right), \]  

(24)

\[ p = -g \left( \frac{7\pi^2 T^4}{720} - \frac{m^2 T^2}{48} \right), \]  

(25)

\[ s = -g \left( \frac{7\pi^2 T^3}{180} - \frac{m^2 T^2}{24} \right). \]  

(26)

The energy density and pressure quoted above was derived in [9] in a different way for the special case of \( g = 2 \). If we assume that to each unusual fermionic degree of freedom there corresponds one standard fermionic degree from the Standard model, then the total fermionic contribution is

\[ \rho_F = \rho_{SM} + \rho = \frac{g m^2 T^2}{48}, \quad p_F = p_{SM} + p = \frac{g m^2 T^2}{48}, \quad s_F = s_{SM} + s = \frac{g m^2 T^2}{24} \]  

(27)

which are also all positive. Here the energy density, pressure density and entropy density for the Standard Model fermionic particles are \( \rho_{SM} = \frac{g m^2 T^2}{240} \), \( p_{SM} = \frac{g m^2 T^2}{720} \) and \( s_{SM} = \frac{g m^2 T^3}{180} \) respectively [9].

It is worth pointing out here that higher derivative theories of fermions require two auxiliary Lee-Wick partners (one left-handed and the other right-handed) to eliminate the higher derivative terms. In that case the Lee-Wick degrees of freedom exceeds the one of its Standard model partner yielding negative energy, pressure and entropy density [10]. This issue is yet to be resolved.
IV. DISCUSSION AND CONCLUSION

In this article we have studied a system of bosonic and fermionic fields, whose excitations modes are quantized with the negative sign of the commutator or anticommutator. The bosonic part of the theory was well studied before by Pauli [8] and the fermionic part is studied previously by Lee and Wick [2]. These unusual quantization process naturally produces field configurations whose total energy is negative. The negative energy of the field configuration is not due to any particular form of the potential but solely an outcome of the quantization process. The vacuum of the theory is not the state with the lowest energy, it is rather the state with the maximal energy making the field configuration unstable. The bosonic and fermionic degrees of freedom do still follow commutation and anticommutation relations and specifically the fermionic fields still follow the Pauli exclusion principle. In this article the emphasize had been on the calculation of energy density, pressure and entropy density of the unusual filed configurations. To calculate the above mentioned thermodynamic quantities one requires to have a statistical mechanics of the field excitations. One encounters the difficulty of a diverging sum when calculating the single particle partition function of the bosonic fields. Keeping to conventional ways, where the temperature of the system is positive definite, the partition function can only be summed when one uses a ultraviolet cutoff. The distribution function calculated from the partition function turns out to be negative definite, which is a nontrivial result. The negative nature of the distribution function implies that there must be an average loss of particles in any positive energy level. This fact can be understood by noticing that the system can only have positive energy by loosing negative energy quanta and the negative sign of the distribution function implies such a condition.

It is shown in the article that the entropy density of the non-trivial field configurations turn out to be negative. This is a serious issue as by its very nature entropy is always positive. The problem with the negative entropy of the Lee-Wick partners is avoided if we include the entropy of the Standard model particles also. The total bosonic or fermionic entropy turns out to be positive. We presume that the negative entropy of the Pauli excitations and Lee-Wick partners show that they cannot exist alone, the thermodynamic system is only complete if the Standard model particles are also included.

The energy density, pressure calculated from the distribution functions of the unusual fields discussed in this article are exactly the same as calculated by Fornal, Grinstein and Wise in [6]. The derivation of the new distribution functions and using them to show the apparent connection between the Pauli oscillator thermodynamics and the thermodynamics of the Lee-Wick partners are the main motivations for this work. Apparently the work of the previous authors and the work in the present article has a common conceptual similarity. The higher derivative theories and their lower derivative Lee-Wick partner infested theories were motivated to kill the divergences in a quantum field theory. The original work of Pauli [8] also aimed to produce a quantum field theory devoid of divergences. We strongly believe that the work of Pauli and Grinstein, O’Connell and Wise have an underlying equivalence although the two theories arises from different perspectives. In short, the present article acts as a bridge between higher derivative theories and normal theories with deformed quantization conditions. We hope that similar to the Lee-Wick theory, there can be a theory of the early universe where all the normal fields are accompanied by Pauli bosons or fermions which are quantized with a negative commutator or anticommutator bracket. This kind of a theory has the potential of damping the infinite divergences which crop up in normal quantum field theories. The unusual field configurations discussed in this article may have been present in the very early universe and, like the Lee-Wick ghosts, have decayed to other particles as time went on.

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