Non-transversal intersection of the free and fixed boundary is shown to hold and a classification of blow-up solutions is given for obstacle problems generated by fully nonlinear uniformly elliptic operators in two dimensions which appear in the mean-field theory of superconducting vortices.

1. Introduction

Strong $L^2$-solutions are considered for the following PDE

\[
\begin{cases}
  F(D^2 u) = \chi_\Omega & \text{a.e. in } B_1^+
  \\
  u = 0 & \text{on } B_1'
\end{cases}
\]

where $u \in W^{2,2}(B_1^+)$, $F$ is a convex $C^1$ fully nonlinear uniformly elliptic operator, $\Omega \subset \mathbb{R}^2$ is an (a priori unknown) open set, and the free boundary is $\Gamma = \partial \Omega \cap B_1^+$. The assumptions imply $u \in W^{2,p}(B_1^+)$ for all $p < \infty$ and $u$ satisfies (1.1) in the viscosity sense \cite{CCKS96}. Equations of the form

\[ F(D^2 u, x) = g(x, u)\chi_{\{\nabla u \neq 0\}} \]

have been studied in \cite{CS02}; an example is the stationary equation for the mean-field theory of superconducting vortices when the scalar stream is a function of the scalar magnetic potential \cite{Cha95, CRS96, ESS98}.

Problems of interest are the endpoint $W^{2,\infty}$ regularity estimates of the solution and the regularity of $\Gamma$. The regularity of the solution enables convergence of re-scalings to solutions in half-spaces and a classification then yields information on the geometry of the free boundary near contact points in the sense that in some configurations the intersection occurs non-transversally. Thus the free normal points in the $x_n$-direction at the contact point. The endpoint regularity is delicate since it is sensitive to the sign.
of the right-hand-side: there exist solutions to

\[
\begin{cases}
\Delta u = -\chi_{u>0} & \text{a.e. in } B_1 \\
u = g & \text{on } \partial B_1
\end{cases}
\]

for \( g \in C^\infty(\partial B_1) \) which belong to \( C^{1,\alpha} \setminus C^{1,1} \) for all \( \alpha \in (0,1) \) \([\text{AW}06]\). Up to the boundary and interior \( C^{1,1} \) estimates were obtained in \([\text{IM}16\text{a}],\text{IM}16\text{b},\text{FS}14\]\, respectively, for more general versions of \((1.1)\) (valid also in higher dimensions; see also \([\text{LJ}H]\) for applications to double obstacle problems).

The author proved \( C^1 \) regularity of the free boundary for non-negative solutions at contact points for

\[\Omega = (\{\nabla u \neq 0\} \cup \{u \neq 0\}) \cap \{x_n > 0\}\]

without density assumptions and non-transversal intersection was proved without a sign assumption on the solution \([\text{Ind}]\). If \( F(M) = \text{tr}(M) \) this problem was investigated in \([\text{SU}03,\text{And}07]\) and in \([\text{Mat}05]\) when the \( \{u \neq 0\} \) term is removed, cf. \([\text{CSS}04]\).

The class of solutions for which \( \|u\|_{L^\infty(B_1^+)} \leq M \) is denoted by \( P_1^+(0,M,\Omega) \). In what follows, tangential touch is shown to hold for

\[\Omega = \{\nabla u \neq 0\} \cap \{x_2 > 0\} \subset \mathbb{R}^2_+\]

and a classification of blow-up solutions is given.

**Theorem 1.1.** There exists \( r_0 > 0 \) and a modulus of continuity \( \omega \) such that

\[\Gamma(u) \cap B^+_r \subset \{x = (x_1,x_2) : x_2 \leq \omega(|x_1|)|x_1|\}\]

for all \( u \in P_1^+(0,M,\Omega) \) provided \( 0 \in \Gamma(u) \).

**Theorem 1.2.** If \( u \in P_1^+(0,M,\Omega) \), \( 0 \in \{u \neq 0\} \) and \( \nabla u(0) = 0 \), then the blow-up limit of \( u \) at the origin has the form

\[u_0(x) = ax_1x_2 + bx_2^2\]

for \( a,b \in \mathbb{R} \).

2. Preliminaries

\( F \) is assumed to satisfy the following structural conditions.

- \( F(0) = 0 \).
• $F$ is uniformly elliptic with ellipticity constants $\lambda_0, \lambda_1 > 0$ such that

$$\mathcal{P}^-(M - N) \leq F(M) - F(N) \leq \mathcal{P}^+(M - N),$$

where $M$ and $N$ are symmetric matrices and $\mathcal{P}^\pm$ are the Pucci operators

$$\mathcal{P}^-(M) = \inf_{\lambda_0 \leq N \leq \lambda_1} \text{tr}(NM), \quad \mathcal{P}^+(M) = \sup_{\lambda_0 \leq N \leq \lambda_1} \text{tr}(NM).$$

• $F$ is convex and $C^1$.

Let $\Omega$ be an open set. A continuous function $u$ belongs to $P^+_r(0, M, \Omega)$ if $u$ satisfies in the viscosity sense:

1. $F(D^2u) = \chi_\Omega$ a.e. in $B^+_r$;
2. $\|u\|_{L^\infty(B^+_r)} \leq M$;
3. $u = 0$ on $\{x_n = 0\} \cap \overline{B}^+_1 =: B'_r$.

Furthermore, given $u \in P^+_r(0, M, \Omega)$, the free boundary is denoted by $\Gamma = \partial\Omega \cap B^+_r$.

A blow-up limit of $\{u_j\} \subset P^+_1(0, M, \Omega)$ is a limit of the form

$$\lim_{k \to \infty} \frac{u_{jk}(s_k x)}{s_k^2},$$

where $\{j_k\}$ is a subsequence of $\{j\}$ and $s_k \to 0^+$.

### 3. Non-transversal intersection and blow-ups

In the following propositions, the class $P^+_1(0, M, \Omega)$ is in terms of a general $\Omega$ subject to the stated assumptions.

**Proposition 3.1.** [Ind. Proposition 3.6] Let $\{u_j\} \subset P^+_1(0, M, \Omega)$ and suppose $\{\nabla u_j \neq 0\} \cap \{x_n > 0\} \subset \Omega, \ 0 \in \{u_j \neq 0\}$, and $\nabla u_j(0) = 0$. Then one of the following is true:

(i) all blow-up limits of $\{u_j\}$ at the origin are of the form $u_0(x) = bx_n^2$ for some $b > 0$;
(ii) there exists a blow-up limit of $\{u_j\}$ of the form $ax_1x_n + bx_n^2$ for $a \neq 0, b \in \mathbb{R}$.

**Proposition 3.2.** Suppose $\{u_j\} \subset P^+_1(0, M, \Omega)$. If $\{\nabla u_j \neq 0\} \cap \{x_2 > 0\} \subset \Omega, \ 0 \in \{u_j \neq 0\}$ and $\nabla u_j(0) = 0$, then one of the following is true:

(i) all blow-up limits of $\{u_j\}$ at the origin are of the form $u_0(x) = bx_2^2$ for $b > 0$;
(ii) there exists $\{u_{kj}\} \subset \{u_j\}, \ j_1 \in \mathbb{N}, \ \text{and} \ s_j > 0$ such that for all $j \geq j_1$,

$$u_{kj} \in C^{2, \alpha}(B^+_{s_j}).$$
\textbf{Proof.} Either all blow-up limits are of the form \(u_0(x) = bx_2^2\) or there exists a subsequence, say
\[
\tilde{u}_j(x) = \frac{u_{kj}(r_jx)}{r_j^2},
\]
producing a limit of the form \(u_0(x) = ax_1x_2 + bx_2^2\) for \(a > 0\) (up to a rotation). Pick \(R \geq 1\) and note that since \(\tilde{u}_j \to u_0\) in \(C^{1,\alpha}_{loc}\), there exists \(j_R \in \mathbb{N}\) such that for \(j \geq j_R\),
\[
|\nabla \tilde{u}_j| > \tilde{c}
\]
in \(E = \left\{ -R < x_1 < -\frac{R}{2} \right\} \cup \left\{ \frac{R}{2} < x_1 < R \right\} \cap B_{2R}^+\),
and therefore, \(\tilde{u}_j \in C^{2,\alpha}(E)\). In particular, up to a subsequence,
\[
\frac{|\partial_{x_1} \tilde{u}_j - ax_2|}{x_2} = \omega_j \to 0
\]
so that
\[
\partial_{x_1} \tilde{u}_j \geq \frac{a}{2} x_2
\]
in \(E\) for \(j \geq j_R \in \mathbb{N}\). Now pick \(\eta \in (0, R)\) and select \(j_0'\) so that if \(j \geq j_0'\),
\[
\partial_{x_1} \tilde{u}_j(x) \geq \frac{a\eta}{2}
\]
for \(x \in \{x_n \geq \eta\}\). Fix \(s < \min\{\eta, \frac{R}{2}\}\) and suppose that for some \(j \geq \max\{j_0', j_R\}\),
\[
\text{Int}\{\nabla \tilde{u}_j = 0\} \cap B_s^+ \neq \emptyset.
\]
Let \(S = \{0 < x_2 < \eta, -R < x_1 < R\}\) so that \(v := \partial_{x_1} \tilde{u}_j \geq 0\) on \(\partial S\). By differentiating the equation, \(Lv = 0\) on \(S \cap \Omega_j\), where \(L\) is a linear second order uniformly elliptic operator. Since \(v\) vanishes on \(\partial \Omega_j\), it follows that \(v > 0\) in \(S \cap \Omega_j\). Next note that for a disk \(B \subset B_s^+\) such that \(\nabla \tilde{u}_j = 0\) in \(B\), \(u_j = m\) for some \(m \in \mathbb{R}\) and there is a strip \(\tilde{S}\) generated by translating the disk in the \(x_1\)-direction. Select another disk
\[
\tilde{B} \subset \tilde{S} \cap \{-R < x_1 < -\frac{R}{2}\}
\]
and let \(E_t = \tilde{B} + te_1\) for \(t \in \mathbb{R}\). Since \(v > 0\) in \(S \cap \Omega_j\) and \(v = 0\) in \(\Omega_j^c\), it follows that \(u < m\) on \(\tilde{B}\). Denote by \(t^* > 0\) the first value for which \(\partial E_t\) intersects \(\{\tilde{u}_j = m\}\) and let \(y \in \partial E_{t^*} \cap \{\tilde{u}_j = m\}\). Note that \(F(D^2 \tilde{u}_j) \geq 0\) and \(w = \tilde{u}_j - m < 0\) in \(E_{t^*}\) with \(w(y) = 0\), therefore by Hopf’s principle (see e.g. [CLNT13]), \(\partial_n \tilde{u}_j(y) > 0\). Therefore, there is \(\mu > 0\) such that \(B_{\mu}(y) \subset \Omega_j\). In particular, \(v > 0\) on \(B_{\mu}(y)\) and since \(v = 0\) on \(\Omega_j^c\), there is \(p > 0\) such that \(\tilde{u}_j(y + e_1p) > m\) for \(y + e_1p \in \{\tilde{u}_j = m\}\), a contradiction. Therefore, \(\text{Int}\{\nabla \tilde{u}_j = 0\} \cap B_s^+ = \emptyset\) and non-degeneracy implies the claim. \(\Box\)
proof of Theorem 1.1. If not, then there exists \( \epsilon > 0 \) such that for all \( k \in \mathbb{N} \) there exists \( u_k \in P^+_1(0, M, \Omega) \) with
\[
\Gamma(u_k) \cap B^+_{1/k} \cap C_\epsilon \neq \emptyset,
\]
where \( 0 \in \Gamma(u_k) \). If all blow-ups of \( \{u_k\} \) are half-space solutions, let \( x_k \in \Gamma(u_k) \cap B^+_{1/k} \cap C_\epsilon \) and set \( y_k = \frac{x_k}{r_k} \) with \( r_k = |x_k| \). Consider \( \tilde{u}_k(x) = \frac{u_k(r_k x)}{r_k^2} \) so that \( y_k \in \Gamma(\tilde{u}_k) \), \( \tilde{u}_k \to bx^2_2 \), \( y_k \to y \in \partial B_1 \cap C_\epsilon \) (up to a subsequence), and \( y \in \Gamma(u_0) \), a contradiction. Therefore, Proposition 3.2 implies the existence of a subsequence \( \{u_{k_j}\} \) of \( \{u_k\} \) such that for all \( j \geq j_1 \), \( u_{k_j} \in C^{2, \alpha}(B^+_{s_j}) \), where \( j_1 \in \mathbb{N} \). Since \( 0 \in \Gamma(u_{k_j}) \), there exists \( x_j \in \Gamma(u_{k_j}) \cap B^+_{s_j} \) which contradicts the continuity of \( F \).

proof of Theorem 1.2. By Proposition 3.2, either \( u_0(x) = bx^2_2 \) or \( D^2u(0) \) exists and the rescaling of \( u \) is given by
\[
u_j(x) = \frac{u(r_j x)}{r_j^2} = \langle x, D^2u(0)x \rangle + o(1).
\]
Since \( u_0(x_1, 0) = 0 \), it follows that \( u_0 \) has the claimed form (up to a rotation).

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