A FAMILY OF FRACTAL FOURIER RESTRICTION ESTIMATES WITH IMPLICATIONS ON THE KAKEYA PROBLEM

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Abstract. In a recent paper [4], Du and Zhang proved a fractal Fourier restriction estimate and used it to establish the sharp $L^2$ estimate on the Schrödinger maximal function in $\mathbb{R}^n$, $n \geq 2$. In this paper, we show that the Du-Zhang estimate is the endpoint of a family of fractal restriction estimates such that each member of the family (other than the original) implies a sharp Kakeya result in $\mathbb{R}^n$ that is closely related to the polynomial Wolff axioms. We also prove that all the estimates of our family are true in $\mathbb{R}^2$.

1. Introduction

Let $Ef = E Pf$ be the extension operator associated with the unit paraboloid $P = \{\xi \in \mathbb{R}^n : \xi_n = \xi_1^2 + \ldots + \xi_{n-1}^2 \leq 1\}$ in $\mathbb{R}^n$:

$$Ef(x) = \int_{B_{n-1}} e^{-2\pi i x \cdot (\omega, |\omega|^2)} f(\omega) d\omega,$$

where $B_{n-1}$ is the unit ball in $\mathbb{R}^{n-1}$.

Our starting point is the following fractal restriction theorem of Du and Zhang [4]. (Throughout this paper, we denote a cube in $\mathbb{R}^n$ of center $x$ and side-length $r$ by $\tilde{B}(x, r)$.)

Theorem 1-A (Du and Zhang [4, Corollary 1.6]). Suppose $n \geq 2$, $1 \leq \alpha \leq n$, $R \geq 1$, $X = \cup_k \tilde{B}_k$ is a union of lattice unit cubes in $\tilde{B}(0, R) \subset \mathbb{R}^n$, and

$$\gamma = \sup \{\tilde{B}_k : \tilde{B}_k \subset \tilde{B}(x', r)\},$$

where the sup is taken over all pairs $(x', r) \in \mathbb{R}^n \times [1, \infty)$ satisfying $\tilde{B}(x', r) \subset \tilde{B}(0, R)$. Then to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that

$$(1) \quad \int_X |Ef(x)|^2 dx \leq C_\epsilon R^\epsilon \gamma^{2/n} R^{\alpha/n} \|f\|_{L^2(B_{n-1})}^2$$

for all $f \in L^2(\mathbb{R}^{n-1})$.

In [4], Theorem 1-A was used to derive the sharp $L^2$ estimate on the Schrödinger maximal function (see [4, Theorem 1.3] and the paragraph following the statement of [4, Corollary 1.6]). The authors of [4], also used Theorem 1-A to obtain new results on the Hausdorff dimension of the sets where Schrödinger solutions diverge (see [11]), achieve progress on Falconer’s distance set conjecture in geometric measure theory (see [6]), and improve on the decay estimates of spherical means of Fourier transforms of measures (see [11]).

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The purpose of this paper is threefold:

- Show that Theorem 1-A is a borderline sharp Kakeya result in the sense that (1) is the endpoint of a family of estimates (see (2) in the statement of Conjecture 1.1) such that each member of the family (other than (1)) implies a certain sharp Kakeya result that we will formulate in §3 below.
- Show that the sharp Kakeya result is true in certain cases in $\mathbb{R}^3$; see Theorem 4.1.
- Prove Conjecture 1.1 in $\mathbb{R}^2$; see Theorem 5.1.

**Conjecture 1.1** (when $\beta = 2/n$ or $n = 2$, this is a theorem). Suppose $n, \alpha, R, X,$ and $\gamma$ are as in the statement of Theorem 1-A.

Let $\beta$ be a parameter satisfying $1/n \leq \beta \leq 2/n$, and define the exponent $p$ by

$$p = 2 + \frac{n - \alpha}{n - 1} \left( \frac{2}{n} - \beta \right).$$

Then to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that

$$\int_X |Ef(x)|^p dx \leq C_\epsilon R^\gamma R^{\alpha/n} \|f\|_{L^p(B^n-1)}^p$$

for all $f \in L^p(B^n-1)$.

We note that when $\beta = 2/n$, (2) becomes (1), so, to prove Conjecture 1.1 we need to perform the following trade: lower the power of $\gamma$ in (1) from $2/n$ to $\beta$ in return for raising the Lebesgue space exponent from 2 to $p$.

We will show below that if (2) holds for any $\beta < 2/n$, then we obtain the sharp Kakeya result of §3.

As noted above, in dimension $n = 2$, (2) is true for all $1/2 \leq \beta \leq 1$ (and hence Conjecture 1.1 is a theorem in the plane). We will prove this in the last three sections of the paper by using weighted bilinear restriction estimates and the broad-narrow strategy of [1].

Before we discuss the implications of Conjecture 1.1 to the Kakeya problem, it will be convenient to write (2) in an equivalent form, which is, perhaps, more user-friendly. This is the purpose of the next section.

### 2. Writing (2) in an Equivalent Form

Suppose $n \geq 1$ and $0 < \alpha \leq n$. Following [12] (see also [3] and [13]), for Lebesgue measurable functions $H : \mathbb{R}^n \to [0,1]$, we define

$$A_\alpha(H) = \inf \left\{ C : \int_{B(x_0,R)} H(x)dx \leq CR^\alpha \text{ for all } x_0 \in \mathbb{R}^n \text{ and } R \geq 1 \right\},$$

where $B(x_0,R)$ denotes the ball in $\mathbb{R}^n$ of center $x_0$ and radius $R$. We say $H$ is a **weight of fractal dimension $\alpha$** if $A_\alpha(H) < \infty$. We note that $A_\beta(H) \leq A_\alpha(H)$ if $\beta \geq \alpha$, so we are not really assigning a dimension to the function $H$; the phrase “$H$ is a weight of dimension $\alpha$” is merely another way for us to say that $A_\alpha(H) < \infty$.

**Proposition 2.1.** Suppose $n, \alpha, R, X, \gamma, \beta,$ and $p$ are as in the statement of Conjecture 1.1. Then the estimate (2) holds if and only if to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that

$$\int_{B(0,R)} |Ef(x)|^p H(x)dx \leq C_\epsilon R^\gamma A_\alpha(H)^\beta R^{\alpha/n} \|f\|_{L^p(B^n-1)}^p$$

for all $f \in L^p(B^n-1)$. 


for all functions $f \in L^p(\mathbb{R}^{n-1})$ and weights $H$ of fractal dimension $\alpha$.

Proof. Let $H$ be the characteristic function of $X$. By the definition of $\gamma$, we have

$$\int_{\tilde{B}(x_0,r)} H(x) dx \leq \gamma (r + 2)^\alpha \leq \gamma (3r)^\alpha$$

for all $x_0 \in \mathbb{R}^n$ and $r \geq 1$. Thus $H$ is a weight on $\mathbb{R}^n$ of fractal dimension $\alpha$, and $A_\alpha(H) \leq 3^\alpha \gamma$. This immediately shows that (3) implies (2).

To prove the reverse implication, we follow [4, Proof of Theorem 2.2].

We consider a covering $\{\tilde{B}\}$ of $B(0,R)$ by unit lattice cubes. Since every unit cube is contained in a ball of radius $\sqrt{n}$, we have $\int_{\tilde{B}} H(x) dx \leq A_\alpha(H)n^{\alpha/2}$, so, if we define $v(\tilde{B}) = A_\alpha(H)^{-1} \int_{\tilde{B}} H(x) dx$ and $V_k = \{\tilde{B} : 2^{k-1} < n^{-\alpha/2}v(\tilde{B}) \leq 2^k\}$, then

$$B(0,R) \subset \cup \tilde{B} \subset \cup_{k=-\infty}^0 V_k.$$

We note that

$$\int_{\tilde{B}} H(x) dx \leq \left( \int_{\tilde{B}} H(x)^{1/\beta} dx \right)^\beta \leq \left( \int_{\tilde{B}} H(x) dx \right)^\beta,$$

or (4)

$$= \left( A_\alpha(H)v(\tilde{B}) \right)^\beta \leq n^{\alpha/2}A_\alpha(H)^\beta 2^{\beta k}$$

for all $\tilde{B} \in V_k$, where we have used the assumptions $\beta \leq 2/\alpha \leq 1$ and $\|H\|_{L^\infty} \leq 1$.

The vast majority of the sets $V_k$ are negligible for us. In fact, letting $k_1$ be the sup of the set $\{k \in \mathbb{Z} : 2^k \leq R^{-1000n/\beta}\}$, we see that

$$\int_{\cup_{k=-\infty}^{k_1} \tilde{B} \in V_k} |Ef(x)|^p H(x) dx \leq \|f\|_{L^1(\mathbb{R}^{n-1})}^p \sum_{k=-\infty}^{k_1} \sum_{\tilde{B} \in V_k} \int_{\tilde{B}} H(x) dx$$

$$\leq CA_\alpha(H)^\beta \|f\|_{L^1(\mathbb{R}^{n-1})}^p \sum_{k=-\infty}^{k_1} R^n 2^{\beta k}$$

$$\leq CR^{-999n}A_\alpha(H)^\beta \|f\|_{L^1(\mathbb{R}^{n-1})}^p,$$

where we used (4) on the line before the last, and the fact that $2^{k_1} \leq R^{-1000n/\beta}$ on the last line. Therefore, we only need to estimate

$$\int_{\cup_{k=k_1+1}^{0} \tilde{B} \in V_k} |Ef(x)|^p H(x) dx = \sum_{k=k_1+1}^{0} \sum_{\tilde{B} \in V_k} \int_{\tilde{B}} |Ef(x)|^p H(x) dx.$$

Letting $k_0 \in \{k_1 + 1, k_1 + 2, \ldots, 0\}$ be the integer satisfying

$$\sum_{\tilde{B} \in V_{k_0}} \int_{\tilde{B}} |Ef(x)|^p H(x) dx = \max_{k_1+1 \leq k \leq 0} \left[ \sum_{\tilde{B} \in V_k} \int_{\tilde{B}} |Ef(x)|^p H(x) dx \right],$$

we see that

$$\int_{B(0,R)} |Ef(x)|^p H(x) dx \leq (-k_1) \sum_{\tilde{B} \in V_{k_0}} \int_{\tilde{B}} |Ef(x)|^p H(x) dx + CR^{-999n}A_\alpha(H)^\beta \|f\|_{L^1(\mathbb{R}^{n-1})}^p.$$
Since $-k_1 \lesssim \log(2R)$, it follows that we only need to estimate
\[
\sum_{\tilde{B} \in V_{k_0}} \int_{\tilde{B}} |Ef(x)|^p H(x) dx.
\]

We start by using the uncertainty principle in the following form. Let $d\sigma$ be the pushforward of the $(n-1)$-dimensional Lebesgue measure under the map $T : \mathbb{B}^{n-1} \to \mathcal{P}$ given by $T(\omega) = (\omega, |\omega|^2)$. Since the measure $d\sigma$ is compactly supported and $Ef = g d\sigma$, where $g$ is the function on $\mathcal{P}$ defined by the equation $f = g \circ T$, it follows that there is a non-negative rapidly decaying function $\psi$ on $\mathbb{R}^n$ such that
\[
\sup_{\tilde{B}} |Ef|^p \lesssim |Ef|^p * \psi(c(\tilde{B})),
\]
where $c(\tilde{B})$ is the center of $\tilde{B}$. Thus
\[
\int_{\tilde{B}} |Ef(x)|^p H(x) dx \lesssim \left( \int_{\tilde{B}} H(x) dx \right) |Ef|^p * \psi(c(\tilde{B})).
\]
From (4) we know that $\int_{\tilde{B}} H(x) dx \lesssim A_\alpha(H)^{\beta 2^{\lambda_0} \beta}$ for all $\tilde{B} \in V_{k_0}$. Also,
\[
|Ef|^p * \psi(c(\tilde{B})) = \int_{B(c(\tilde{B}), R^*)} |Ef(x)|^p \psi(c(\tilde{B}) - x) dx
\]
\[
+ \int_{B(c(\tilde{B}), R^*)^c} |Ef(x)|^p \psi(c(\tilde{B}) - x) dx \lesssim \int_{B(c(\tilde{B}), R^*)} |Ef(x)|^p dx + R^{-1000n} \| f \|_{L^1(\mathbb{B}^{n-1})}^p
\]
and
\[
(6) \sum_{\tilde{B} \in V_{k_0}} \chi_{B(c(\tilde{B}), R^*)} \lesssim R^{n_\epsilon},
\]
so
\[
\sum_{\tilde{B} \in V_{k_0}} \int_{\tilde{B}} |Ef(x)|^p H(x) dx \lesssim \sum_{\tilde{B} \in V_{k_0}} |Ef|^p dx + A_\alpha(H)^{\beta} 2^{\lambda_0} \beta \int_{V} |Ef(x)|^p dx + A_\alpha(H)^{\beta} R^{-999n} \| f \|_{L^1(\mathbb{B}^{n-1})}^p,
\]
where $V = \bigcup_{\tilde{B} \in V_{k_0}} B(c(\tilde{B}), R^*)$.

We now let $\{B^*\}$ be the set of all the unit lattice cubes that intersect $V$, and $X = \bigcup \tilde{B}^*$. We plan to apply (2) on this set $X$, but we first need to estimate $\gamma$.

Let $B_r$ be a ball in $\mathbb{R}^n$ of radius $r \geq R^*$ (if $1 \leq r \leq R^*$, then, clearly, $\# \{\tilde{B}^* : \tilde{B}^* \subset B_r\} \lesssim R^{n_\epsilon}$), and $V_r$ the subset of $V_{k_0}$ that consists of all unit cubes $\tilde{B}$ such that $B(c(\tilde{B}), 2R^*) \cap B_r \neq \emptyset$. If $B_r$ intersects any of the cubes $\tilde{B}^*$ that make up $X$, then $B_r$ intersect $B(c(\tilde{B}), 2R^*)$ for some $\tilde{B} \in V_r$. Therefore, $\gamma \lesssim R^{n_\epsilon} \#(V_r)$.

Our assumption $r \geq R^*$, tells us that
\[
\bigcup_{\tilde{B} \in V_r} B(c(\tilde{B}), 2R^*) \subset B_{5r},
\]
so (using (6))
\[
\int_{B(\epsilon(\hat{B}), 2R)} H(x) dx \geq \sum_{\hat{B} \in V_r} \int_{\hat{B}} H(x) dx
\]
\[
\geq \sum_{\hat{B} \in V_r} \int_{\hat{B}} H(x) dx = \sum_{\hat{B} \in V_r} v(\hat{B}) A_{\alpha}(H) \geq (V_r) n^{\alpha/2} 2^{-k_0} A_{\alpha}(H).
\]
On the other hand,
\[
\int_{B(5R)} H(x) dx \leq A_{\alpha}(H)(5^\alpha, 2R)
\]
so \((V_r) \lesssim R^{2^{\alpha}2^{-k_0}}\), and so \(\gamma \lesssim R^{2^{\alpha}2^{-k_0}}\).

Applying (2), we now obtain
\[
\int_{\Omega} |E(f)(x)|^p dx \leq \int_{\mathbb{R}^n} |E(f)(x)|^p dx \leq R^5 2^{-k_0} \beta R^{\alpha/n} \|f\|^p_{L^p(B_{(0, 5L)})}
\]
which, combined with (6) and (7), implies that
\[
\int_{B(0, R)} |E(f)(x)|^p H(x) dx \leq R^{(n+6)\epsilon (2^{k_0})^{\beta-\beta} A_{\alpha}(H)^\beta R^{\alpha/n} \|f\|^p_{L^p(B_{(0, 5L)})}
\]
\[
= R^{(n+6)\epsilon} A_{\alpha}(H)^\beta R^{\alpha/n} \|f\|^p_{L^p(B_{(0, 5L)})}
\]
which is our desired estimate (3).

3. Conjecture [11] implies a sharp Kakeya result

Let \(\Omega\) be a subset of \(\mathbb{R}^n\) that obeys the following property: there is a number \(\alpha\) between 1 and \(n\) such that
\[
|\Omega \cap B_R| \leq CR^n
\]
for all balls \(B_R\) in \(\mathbb{R}^n\) of radius \(R \geq 1\). (Here, and throughout the paper, \(|\cdot|\) = Lebesgue measure of the set.)

For large \(L\), we divide the unit paraboloid \(\mathbb{P}\) into finitely overlapping caps \(\theta_j\) each of radius \(L^{-1}\), and we associate with each \(\theta_j\) a family \(T_j\) of parallel \(1 \times L\) tubes that tile \(\mathbb{R}^n\) and point in the direction normal to \(\theta_j\) at its center. We let \(N\) be the cardinality of the set
\[
J = \{j : \text{there is a tube of } T_j \text{ that lies in } \Omega \cap B(0, 5L)\}.
\]

It is easy to see that the Kakeya conjecture (in its maximal operator form) implies the following bound on \(N\): to every \(\epsilon > 0\) there is a constant \(C_\epsilon\) such that
\[
N \leq C_\epsilon L^\epsilon L^{n-1}
\]
for all \(L \geq 1\). In fact, [2, Proposition 2.2] presents a proof of the fact that the Kakeya conjecture implies (10) in the case when \(\Omega\) is a neighborhood of an algebraic variety. This proof easily extends to general sets \(\Omega\) satisfying (6). (For the connection between neighborhoods of algebraic varieties and the condition (8), we refer the reader to [14].)

We note that (10) implies that if \(\Omega \cap B(0, 5L)\) contains at least one tube from each direction (i.e. at least one tube from each of the \(~ L^{n-1}\) families \(T_j\)), then \(\alpha = n\).

In the special case when \(\Omega\) is a neighborhood of an algebraic variety, this bound on \(N\) was proved by Guth [7] in \(\mathbb{R}^3\), conjectured by Guth [8] to be true in \(\mathbb{R}^n\) for
all $n \geq 3$, and proved by Zahl [17] in $\mathbb{R}^4$; see also [9]. The conjecture of [8] was then settled in all dimensions by Katz and Rogers in [10].

In this section we prove that if (3) (or equivalently (2)) holds for any $1/n \leq \beta < 2/n$, then (10) will follow.

We first write the set $J$ as $\{j_1, j_2, \ldots, j_N\}$, and for each $1 \leq l \leq N$, we let $T_l$ be a tube from $T_j$ that lies in $\Omega \cap B(0, 5L) = \Omega \cap B_{5L}$. Then

$$NL = \sum_{l=1}^{N} |T_l| = \sum_{l=1}^{N} \int_{B_{5L} \cap \Omega} \chi_{T_l}(x) \, dx = \int_{B_{5L} \cap \Omega} \sum_{l=1}^{N} \chi_{T_l}(x) \, dx$$

$$= L^{2(n-1)} \sum_{l=1}^{N} \left( \frac{1}{L_{n-1}} \chi_{T_l}(x) \right)^2 \, dx \lesssim L^{2(n-1)} \int_{B_{5L} \cap \Omega} \sum_{l=1}^{N} |Ef_l(Lx)|^2 \, dx,$$

where $f_l$ is supported in $\theta_l$ (in fact, $f_l$ is supported in the projection of $\theta_l$ into $\mathbb{R}^{n-1}$) and $|Ef_l| \geq |\theta_l| \chi_{T^*}$, where $T^*$ is an appropriate $L \times L^2$ tube of the same direction as $T_l$. More precisely,

$$|Ef_l(Lx)| \sim |\theta_l| \chi_{T^*}(Lx) = |\theta_l| \chi_{T_l}(x) \gtrsim \frac{1}{L_{n-1}} \chi_{T_l}(x)$$

for all $x \in \mathbb{R}^n$. Letting $H = \chi_{\Omega}$, we arrive at

$$NL \lesssim L^{2(n-1)} \int_{B_{5L}} \sum_{l=1}^{N} |Ef_l(Lx)|^2 H(x) \, dx.$$

Next, we let $\epsilon_l = \pm 1$ be random signs, define the function $f : \mathbb{R}^{n-1} \to \mathbb{C}$ by $f = \sum_{l=1}^{N} \epsilon_l f_l$, and use Khintchin’s inequality to get

$$NL \lesssim L^{2(n-1)} \mathcal{E} \left( \int_{B_{5L}} |Ef(Lx)|^2 H(x) \, dx \right),$$

where $\mathcal{E}$ is the expectation sign. Since $p \geq 2$, we can apply Hölder’s inequality in the inner integral to get

$$NL \lesssim L^{2(n-1)} \left( \int_{B_{5L}} H(x) \, dx \right)^{1-(2/p)} \mathcal{E} \left( \int_{B_{5L}} |Ef(Lx)|^p H(x) \, dx \right)^{2/p}$$

$$\lesssim L^{2(n-1)} L^a(1-(2/p)) \mathcal{E} \left( \int_{B_{5L}} |Ef(Lx)|^p H(x) \, dx \right)^{2/p}.$$

Applying the change of variables $u = Lx$ and defining the weight $H^*$ by $H^*(u) = H(x) = H(u/L)$, this becomes

$$NL \lesssim L^{2(n-1)} L^a(1-(2/p)) L^{-2n/p} \mathcal{E} \left( \int_{B_{5L}^2} |Ef(u)|^p H^*(u) \, du \right)^{2/p},$$

so that

$$NL^{3-n} \lesssim L^{(n+a)(1-(2/p))} \mathcal{E} \left( \int_{B_{5L}^2} |Ef(u)|^p H^*(u) \, du \right)^{2/p}. \tag{11}$$

We note that

$$\int_{B(u_0/R)} H^*(u) \, du = L^n \int_{B(u_0/L,R/L)} H(x) \, dx$$

$$\leq L^n A_\alpha(H) \left( \frac{R}{L} \right)^\alpha = L^{n-\alpha} A_\alpha(H) R^\alpha.$$
if $R \geq L$. On the other hand, if $R \leq L$, then
\[
\int_{B(u_0, R)} H^*(u) du \lesssim R^n = R^{n-\alpha} R^\alpha \leq L^{n-\alpha} R^\alpha.
\]
Therefore,
\[
A_\alpha(H^*) \lesssim L^{n-\alpha}.
\]
We are now in a good shape to apply \(3\), which tells us that
\[
\int_{B_{3L}^2} |E f(u)|^p H^*(u) du \lesssim (L^2)^\epsilon A_\alpha(H^*)^\beta (L^2)^{\alpha/n} \|f\|_{L^p(B_{3n-1})}^p \lesssim L^{2\epsilon} L^{(n-\alpha)\beta} L^{2\alpha/n} \frac{N}{L^{n-1}}.
\]
Inserting this back in \((11)\), we get
\[
N L^{3-n} \lesssim L^{2\epsilon} L^{(n+\alpha)(1-2/p)} \left( L^{(n-\alpha)\beta} L^{2\alpha/n} L^{1-n} N \right)^{2/p},
\]
so that
\[
N^{1-(2/p)} L^{3-n} \lesssim L^{2\epsilon} L^{(n+\alpha)(1-2/p)} \left( L^{(n-\alpha)\beta} L^{2\alpha/n} L^{-2} \right)^{2/p},
\]
so that
\[
N^{1-(2/p)} \lesssim L^{2\epsilon} L^{(n-3)(1-2/p)} L^{(n+\alpha)(1-2/p)} L^{(n-\alpha)\beta} L^{2\alpha/n} L^{-2}.
\]
Therefore,
\[
N \lesssim L^{O(\epsilon)} L^{n-3} L^n L^{\frac{(n-\alpha)(\beta-\frac{2}{n})(\frac{\alpha}{n-1})}{2\epsilon} \gamma} = L^{O(\epsilon)} L^{2n-3+n} L^{\frac{(n-\alpha)(\beta-\frac{2}{n})(\frac{\alpha}{n-1})}{2\epsilon} \gamma}.
\]
But
\[
\frac{(n-\alpha)(\beta-\frac{2}{n})}{2\epsilon-1} = (n-\alpha)(\beta-\frac{2}{n}) \frac{2(n-1)}{(n-\alpha)(\frac{\alpha}{n-1})} = -2(n-1) = 2-2n,
\]
so
\[
N \lesssim L^{O(\epsilon)} L^{2n-3+n-2} = L^{O(\epsilon)} L^{\alpha-1}.
\]

4. Proof of \((10)\) in the Regime $1 \leq \alpha \leq 2$ in $\mathbb{R}^3$

The fact that the Kakeya conjecture is true in $\mathbb{R}^2$ tells us that \((10)\) is also true there. In this section, we use Wolff’s hairbrush argument from \(15\), as adapted by Guth in \(7\), to prove the following bound on $N$.

**Theorem 4.1.** In $\mathbb{R}^3$, we have
\[
N \lesssim \begin{cases} (\log L)^{L^{\alpha-1}} & \text{if } 1 \leq \alpha \leq 2, \\ (\log L)^{L^{2n-3}} & \text{if } 2 \leq \alpha \leq 3. \end{cases}
\]

**Proof.** Let $\Omega$ be a subset of $\mathbb{R}^3$ that obeys \(8\). As we did in the previous section, for large $L$, we consider a decomposition $\{\theta_j\}$ of $\mathcal{P}$ into finitely overlapping caps each of radius $L^{-1}$, and we associate with each $\theta_j$ a family $T_j$ of parallel $1 \times L$ tubes that tile $\mathbb{R}^3$ and point in the direction of the normal vector $v_j$ of $\mathcal{P}$ at the center of $\theta_j$. The quantity $N$ that we need to estimate is the cardinality of the set $J$ as defined in \(3\).

For each $j \in J$, we let $T_j$ be the member of $T_j$ that lies in $\Omega \cap B(0, 5L)$, and $S = \{T_j\}$. Of course, $N = \#(S)$.
We tile $\Omega \cap B(0, 5L)$ by unit lattice cubes $\tilde{B}$. Then (8) tells us that
\begin{equation}
\#(\{\tilde{B}\}) \lesssim L^\alpha.
\end{equation}

Also, each tube $T_j$ intersects $\sim L$ of the cubes $\tilde{B}$.

We now define the function $f : \{\tilde{B}\} \to \mathbb{Z}$ by
\[ f(\tilde{B}) = \#\{T_j \in S : T_j \cap \tilde{B} \neq \emptyset\}. \]

Then
\[ \sum_{\tilde{B}} f(\tilde{B}) \sim NL. \]

So, by Cauchy-Schwarz and (12),
\[ NL \lesssim \left( \sum_{\tilde{B}} f(\tilde{B})^2 \right)^{1/2} \left( \#(\{\tilde{B}\}) \right)^{1/2} \lesssim \left( \sum_{\tilde{B}} f(\tilde{B})^2 \right)^{1/2} L^{\alpha/2}, \]

and so
\[ \sum_{\tilde{B}} f(\tilde{B})^2 \gtrsim N^2 L^{2-\alpha}, \]

which means that the set
\[ \{(\tilde{B}, T_1, T_j) : T_1, T_j \in S, T_i \cap \tilde{B} \neq \emptyset, \text{ and } T_j \cap \tilde{B} \neq \emptyset\} \]

has cardinality $\gtrsim N^2 L^{2-\alpha}$. Therefore, the set
\[ X = \{(\tilde{B}, T_1, T_j) : T_1, T_j \in S, T_i \cap \tilde{B} \neq \emptyset, \text{ and } T_j \cap \tilde{B} \neq \emptyset \text{ and } i \neq j\} \]

has cardinality
\[ \geq C_1 N^2 L^{2-\alpha} - \sum_{\tilde{B}} f(\tilde{B}) \geq C_1 N^2 L^{2-\alpha} - C_2 NL. \]

If $C_1 N^2 L^{2-\alpha} \leq 5C_2 NL$, then $N \lesssim (5C_2/C_1)L^{\alpha-1}$ and the theorem will be proved. So, we may assume that $N \gtrsim C_3 L^{\alpha-1}$ for some large constant $C_3$. Therefore,
\[ \#(X) \gtrsim N^2 L^{2-\alpha}. \]

For $l \in \mathbb{N}$, we define $X_l$ to be the subset of $X$ for which
\[ \frac{2^{l-1}}{L} \leq \text{Angle}(v_i, v_j) \leq \frac{2^l}{L}. \]

Since the angle between any two tubes in our set $S$ ranges between $L^{-1}$ and $1$, it follows by the pigeonhole principle that $\#(X) \lesssim (\log L)\#(X_{l_0})$ for some $l_0 \in \mathbb{N}$. Denoting $2^{l_0}L^{-1}$ by $\theta$, and $X_{l_0}$ by $X'$, we have $L^{-1} \leq \theta \leq 1$ and $\#(X') \gtrsim N^2 L^{2-\alpha}(\log L)^{-1}$.

There are $N$ tubes in $S$. By the pigeonhole principle, one of the tubes must appear in $\gtrsim N^2 L^{2-\alpha}(\log L)^{-1}/N = NL^{2-\alpha}(\log L)^{-1}$ of the elements of $X'$. We call this tube $T$, and we define
\[ \mathcal{H} = \{T_j \in S : (\tilde{B}, T, T_j) \in X'\}. \]

Let $v$ be the direction of the tube $T$. Since the angle between $v$ and $v_j$ is $\sim \theta$, it follows that $|T \cap T_j| \lesssim \theta^{-1}$. So, the set $\{\tilde{B} : (\tilde{B}, T, T_j) \in X'\}$ has cardinality $\lesssim \theta^{-1}$, and so
\[ \#(\mathcal{H}) \gtrsim \frac{NL^{2-\alpha}(\log L)^{-1}}{\theta^{-1}} = \theta NL^{2-\alpha}(\log L)^{-1}. \]
To finish the proof, we need to also have an upper bound on $|(H)\, |$. We first observe that
$$
\bigcup_{T_j \in H} T_j \subset \Omega \cap B,
$$
where $B$ is a box in $\mathbb{R}^3$ of dimensions $L \times \theta L \times \theta L$. Since $B$ can be covered by $\sim L/(\theta L)$ balls of radius $\theta L$, and since $\theta L \geq 1$, the dimensionality property (8) tells us that
$$
\bigg| \bigcup_{T_j \in H} T_j \bigg| \lesssim \theta^{-1}(\theta L)^\alpha.
$$
Next, we use the (by now) standard fact that the tubes $T_j$ in $H$ are morally disjoint (see [7, Lemma 4.9] for a very nice explanation of this idea) to see that
$$
\bigg| \bigcup_{T_j \in H} T_j \bigg| \gtrsim \#(H) |T_j| = \#(H) L.
$$
Therefore,
$$
\#(H) \lesssim \theta^{-1} L^{-1}(\theta L)^\alpha = (\theta L)^{\alpha-1}.
$$
Comparing the lower and upper bounds we now have on the cardinality of $H$, we conclude that
$$
\theta N L^{2-\alpha}(\log L)^{-1} \lesssim (\theta L)^{\alpha-1}.
$$
Therefore,
$$
N \lesssim (\log L)^{\alpha-3} L^{2\alpha-3}.
$$
If $\alpha \geq 2$, then the fact that $\theta \leq 1$ tells us that
$$
N \leq (\log L) L^{2\alpha-3}.
$$
If $1 \leq \alpha < 2$, then the fact that $\theta \geq 1/L$ tells us that
$$
N \lesssim (\log L)(L)^{2-\alpha} L^{2\alpha-3} = (\log L) L^{\alpha-1}.
$$
It might be interesting for the reader to observe that the sharp result that we get in the case $1 \leq \alpha < 2$ is due to the fact that we are using ‘substantial’ information about $\theta$ (namely, $\theta \geq 1/L$), whereas in the $2 \leq \alpha \leq 3$ we only can use the relatively ‘unsubstantial’ information that $\theta \leq 1$.

5. Proof of Conjecture 1.1 in the plane

The rest of the paper is concerned in proving that Conjecture 2.1 is true in $\mathbb{R}^2$. In view of Proposition 2.1 this task will be accomplished as soon as we prove Theorem 5.1 below.

We alert the reader that the extension operator in Theorem 5.1 is the one associated with the unit circle $S^1 \subset \mathbb{R}^2$ and is given by
$$
Ef(x) = \int e^{-2\pi i x \cdot \xi} f(\xi) d\sigma(\xi)
$$
for $f \in L^1(\sigma)$, where $\sigma$ is induced Lebesgue measure on $S^1$. The proof for the extension operator associated with the unit parabola is similar (and a little easier).
Theorem 5.1. Suppose \( 1 \leq \alpha \leq 2 \) and \( R \geq 1 \). Let \( \beta \) be a parameter satisfying \( 1/2 \leq \beta \leq 1 \), and define the exponent \( p \) by

\[
p = 2 + (2 - \alpha)(1 - \beta).
\]

Then to every \( \epsilon > 0 \) there is a constant \( C_\epsilon \) such that

\[
\int_{B(0,R)} |Ef(x)|^p H(x) dx \leq C_\epsilon R^\alpha A_\alpha(H)^{\beta} R^{\alpha/2} \|f\|_{L^p(\sigma)}^p
\]

for all functions \( f \in L^p(\sigma) \) and weights \( H \) of fractal dimension \( \alpha \).

The proof of Theorem 5.1 will use ideas from [16], [5], [12], and [4]. The overarching idea, however, is the broad-narrow strategy of [1]. Implementing this strategy involves

- proving a bilinear estimate (see (24) in Subsection 7.1 below) that will be used to control \( Ef \) on the broad set
- proving a linear estimate (see (25) in Subsection 7.2 below) that will be used to establish (13) when the function \( f \) is supported on an arc of small size (i.e. \( \sigma \)-measure), which will provide the base of an induction argument
- carrying out an induction on the size of the function’s support argument that will establish (13) for general \( f \).

The main new idea in the proof of Theorem 5.1 is a localization of the weight argument that will help us in deriving the bilinear estimate (24). We use this argument to take advantage of the locally constant property of the Fourier transform, and we will end this section by describing the intuition that lies behind it.

Suppose \( R > K^2 \geq 1 \), \( Q \) is a box in \( \mathbb{R}^2 \) of dimensions \( R/K \times R \) (boxes of such dimensions are a common feature in this context; see [1] Subsection 3.2 and Subsection 6.2 below), \( f \) is a non-negative function supported in \( Q \) that is essentially constant at scale \( K \), and we are seeking an estimate of the form

\[
\int_Q f(x) H(x) dx \lesssim K^{-m} A_\alpha(H)^{\beta} R^{\alpha/2} \|f\|_{L^2(Q)}
\]

for some \( m \geq 0 \), where \( \beta \) is as in the statement of Theorem 5.1.

We tile \( \mathbb{R}^2 \) by cubes \( \tilde{B}_l \) of center \( c_l \) and side-length \( K \), and write

\[
\int_Q f(x) H(x) dx = \sum_l \int_{\tilde{B}_l} f(x) H(x) dx \sim \sum_l f(c_l) \int_{\tilde{B}_l} H(x) dx
\]

\[
= \sum_l K^{-2} \int_{\tilde{B}_l} f(c_l) H'(y) dy \sim K^{-2} \int_Q f(y) H'(y) dy
\]

(recall that \( f \) is zero outside \( Q \)), where \( H' : \mathbb{R}^2 \to [0,\infty) \) is given by

\[
H'(y) = \int_{\tilde{B}_l} H(x) dx \quad \text{for} \quad y \in \tilde{B}_l.
\]

For \( y \in \tilde{B}_l \), we have

\[
H'(y) = \left( \int_{\tilde{B}_l} H(x) dx \right)^{1-\theta} \left( \int_{\tilde{B}_l} H(x) dx \right)^{\theta}
\]

\[
\leq K^{2(1-\theta)} A_\alpha(H)^{\theta}(\sqrt{2}K)^{\theta},
\]

where \( 0 \leq \theta \leq 1 \) is a parameter that will be determined later in the argument.
Next, we define the function $\mathcal{H} : \mathbb{R}^2 \to [0, 1]$ by
\[
\mathcal{H}(y) = 2^{-\alpha/2} A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} H'(y)
\]
and observe that
\[
\int_{B(x_0,r)} \mathcal{H}(y) dy \leq K^2 A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} \int_{B(x_0,2r)} H'(y) dy
\]
\[
\leq K^2 A_\alpha(H)^{-\theta} K^{-2(1-\theta)-\alpha\theta} A_\alpha(H) r^\alpha = A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)} r^\alpha
\]
for all $x_0 \in \mathbb{R}^2$ and $R \geq 1$, which means that $\mathcal{H}$ is a weight on $\mathbb{R}^2$ of fractal dimension $\alpha$ with
\[
A_\alpha(\mathcal{H}) \leq A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)}.
\]

Going back to our integral, we now have
\[
\int_Q f(x) H(x) dx \sim A_\alpha(H)^{\theta} K^{\theta(\alpha-2)} \int_Q f(y) \mathcal{H}(y) dy.
\]
Bounding the integral on the right-hand side by Cauchy-Schwarz, this becomes
\[
\int_Q f(x) H(x) dx \lesssim A_\alpha(H)^{\theta} K^{\theta(\alpha-2)} \left( \int_Q \mathcal{H}(y) dy \right)^{1/2} \|f\|_{L^2(Q)}.
\]
But $Q$ can be covered by $\sim K$ balls of radius $R/K$, so
\[
(14) \quad \int_Q \mathcal{H}(y) dy \lesssim K A_\alpha(\mathcal{H})(K^{-1} R)^\alpha
\]
\[
\lesssim A_\alpha(H)^{1-\theta} K^{\theta(2-\alpha)} (K^{-1} R)^\alpha,
\]
and so
\[
\int_{B(0,R)} f(x) H(x) dx \lesssim A_\alpha(H)^{(1+\theta)/2} K^{\theta(\alpha-2)/2} (K^{-1} R)^{\alpha/2} \|f\|_{L^2(B(0,R))}.
\]
We now determine $\theta$ by solving the equation $(1+\theta)/2 = \beta$, which gives $\theta = 2\beta - 1$, and we arrive at
\[
\int_{B(0,R)} f(x) H(x) dx \lesssim K^{-m} A_\alpha(H)^{\beta} R^{\alpha/2} \|f\|_{L^2(B(0,R))}
\]
with $m = \beta + (1 - \beta)(\alpha - 1)$.

6. Preliminaries for the proof of Theorem [5.1]

This section contains basic facts that we need to prove Theorem [5.1] that we include to make the paper as self-contained as possible.

6.1. The $L^1$ norm of a rapidly decaying function over a box. In the rigorous version of the localization argument that we described in the previous section, instead of integrating over a proper $R/K \times R$ box, we will be integrating against a Schwartz function that is essentially supported on such a box. It is easy to see that (14) continues to be true in this case. Here are the details.

Suppose $R_1, \ldots, R_n > 0$ and $\Psi$ is a non-negative Schwartz function. For $l = 0, 1, 2, \ldots$, we let $\chi_l$ be the characteristic function of the box in $\mathbb{R}^n$ of center 0 and
dimensions $2^{l+1}R_1 \times \ldots \times 2^{l+1}R_n$, and $B_l = B(0, 2^l)$. Then

$$
\Psi \left( \frac{x_1 - \nu_1}{R_1}, \ldots, \frac{x_n - \nu_n}{R_n} \right)
\leq \left( \sup_{B_0} \Psi \right) \chi_{B_0} \left( \frac{x_1 - \nu_1}{R_1}, \ldots, \frac{x_n - \nu_n}{R_n} \right)
+ \sum_{l=1}^{\infty} \left( \sup_{B_l \setminus B_{l-1}} \Psi \right) \chi_{B_l \setminus B_{l-1}} \left( \frac{x_1 - \nu_1}{R_1}, \ldots, \frac{x_n - \nu_n}{R_n} \right)
\lesssim \sum_{l=0}^{\infty} 2^{-Nl} \chi_l(x - \nu)
$$

for all $x, \nu \in \mathbb{R}^n$ and $N \in \mathbb{N}$, so that

$$
\int \Psi \left( \frac{x_1 - \nu_1}{R_1}, \ldots, \frac{x_n - \nu_n}{R_n} \right) H(x) dx \lesssim \sum_{l=0}^{\infty} 2^{-Nl} \int_{P_l} H(x) dx,
$$

where $P_l$ is the box in $\mathbb{R}^n$ of center $\nu$ and dimensions $2^{l+1}R_1 \times \ldots \times 2^{l+1}R_n$.

In the special case $R_1 = \ldots = R_{n-1} = R/K$ and $R_n = R$ with $R \geq K^2 \geq 1$ (as in [14]), this gives

$$
\int \Psi \left( \frac{x_1 - \nu_1}{R K^{-1}}, \ldots, \frac{x_{n-1} - \nu_{n-1}}{R K^{-1}}, \frac{x_n - \nu_n}{R} \right) H(x) dx \lesssim K A_\alpha(H)(K^{-1} R)^\alpha
$$

for all weights $H$ on $\mathbb{R}^n$ of fractal dimension $\alpha$.

### 6.2. A property of $R/K \times \cdots \times R/K \times R$ boxes

Suppose $R \geq K^2 \geq 1$, $Q$ is an $R/K \times \cdots \times R/K \times R$ box in $\mathbb{R}^n$, and $Q^*$ is one of the dual boxes of $Q$ that is tangent to the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Let $\delta = K^{-1}$. Then $Q^*$ has dimensions $(R \delta)^{-1} \times \ldots \times (R \delta)^{-1} \times R^{-1}$ and its $(R \delta)^{-1} \times \ldots \times (R \delta)^{-1}$ is tangent to $S^{n-1}$ at some point $e$. It is to be proved that $Q^*$ lies in the $R^{-1}$-neighborhood of $S^{n-1}$.

Without any loss of generality, we may assume that $e = (0, \ldots, 0, 1)$.

Suppose $y \in Q^*$. Then

$$
|y|^2 = y_1^2 + \ldots + y_{n-1}^2 + (y_n - 1 + 1)^2 = y_1^2 + \ldots + y_{n-1}^2 + (y_n - 1)^2 + 2(y_n - 1) + 1
$$

so that

$$
||y|^2 - 1| \leq y_1^2 + \ldots + y_{n-1}^2 + |y_n - 1|^2 + 2|y_n - 1|
$$

so that

$$
||y| - 1| |y| + 1| \leq y_1^2 + \ldots + y_{n-1}^2 + 3|y_n - 1|
$$

so that

$$
||y| - 1| \leq y_1^2 + \ldots + y_{n-1}^2 + 3|y_n - 1| \leq \frac{n - 1}{(R \delta)^2} + \frac{3}{R} \lesssim \frac{1}{R},
$$

where we have used the fact that

$$
\frac{1}{(R \delta)^2} = \frac{1}{R^2} \leq \frac{1}{R}.
$$
6.3. The Kakeya information underlying the bilinear estimate. Suppose \( \delta > 0, R \geq \delta^{-1}, \) and \( J_1 \) and \( J_2 \) are subsets of the circular arc \( \{ e^{i\theta} : \pi/4 \leq \theta \leq 3\pi/4 \} \) such that \( \text{Dist}(J_1, J_2) \geq 3\delta \).

Let \( N_1 \) and \( N_2 \) be the \( R^{-1} \)-neighborhoods of \( J_1 \) and \( J_2 \), respectively. In this subsection, we derive a bound on the Lebesgue measure of the set \( (x + N_1) \cap N_2 \) for \( x \in \mathbb{R}^2 \).

Since we are interested in the \( L^\infty \)-norm of the function

\[
x \mapsto \int \chi_{x + N_1}(y) \chi_{N_2}(y) dy,
\]

we let \( h \in L^1(\mathbb{R}^2) \) be a non-negative function and consider the integral

\[
I = \int \int \chi_{x + N_1}(y) \chi_{N_2}(y) dh(x) dy.
\]

Writing

\[
I = \int \int \chi_{N_1}(y - x) \chi_{N_2}(y) h(x) dy dx = \int \chi_{N_1}(y) \int \chi_{N_1}(y - x) h(x) dx dy,
\]

and applying the change of variables \( u = y - x \) in the inner integral, we see that

\[
I = \int \chi_{N_2}(y) \int \chi_{N_1}(u) h(y - u) du dy = \int_{N_2} \int_{N_1} h(y - u) du dy.
\]

Changing into polar coordinates, this becomes

\[
I = \int_{J_1}^{1+R^{-1}} \int_{J_2}^{1+R^{-1}} \int_{J_1}^{J_2} \int_{J_1}^{J_2} h(re^{i\theta} - se^{i\varphi}) rs dr ds d\theta d\varphi,
\]

where \( J_1 = N_1 \cap S^1 \) and \( J_2 = N_2 \cap S^1 \).

We define

\[
T(\theta, \varphi) = re^{i\theta} - se^{i\varphi} = (r \cos \theta - s \cos \varphi, r \sin \theta - s \sin \varphi).
\]

The Jacobian of this transformation is

\[
J_T(\theta, \varphi) = \begin{vmatrix} -r \sin \theta & s \sin \varphi \\ r \cos \theta & -s \cos \varphi \end{vmatrix} = rs \sin(\theta - \varphi).
\]

So

\[
\int_{J_1}^{J_2} \int_{J_1}^{J_2} rsh(re^{i\theta} - se^{i\varphi}) dr d\varphi = \int_{J_1 \times J_2} \frac{h(T(\theta, \varphi)) |J_T|}{\sin(\theta - \varphi)} d(\theta, \varphi).
\]

But \( |\theta - \varphi| \leq \pi/2 \), so

\[
|\sin(\theta - \varphi)| \geq \frac{2}{\pi} |\theta - \varphi| \geq \frac{2}{\pi} \text{Dist}(\tilde{J}_1, \tilde{J}_2) \geq \frac{2\delta}{\pi},
\]

and so

\[
\int_{J_1}^{J_2} \int_{J_1}^{J_2} rsh(re^{i\theta} - se^{i\varphi}) dr d\varphi \leq \frac{\pi}{2\delta} \int_{J_1 \times J_2} h \circ T(\theta, \varphi) |J_T(\theta, \varphi)| d(\theta, \varphi) = \frac{\pi}{2\delta} \int_X h(x, y) d(x, y) \leq \frac{\pi}{2\delta} \|h\|_{L^1}.
\]

Thus

\[
I \leq \int_{1-R^{-1}}^{1+R^{-1}} \int_{1-R^{-1}}^{1+R^{-1}} \frac{\pi}{2\delta} \|h\|_{L^1} dr ds = \frac{\pi}{2\delta R^2} \|h\|_{L^1}.
\]
Therefore, by duality,
\[
|x + N_1 \cap N_2| \leq \frac{\pi}{2R^2 \delta}
\]
for a.e. \(x \in \mathbb{R}^2\).

6.4. Calculation giving the right exponent for the restriction estimate.
(The reader is advised to skip this section until we refer back to it in Subsections 7.1 and 7.2.)
Suppose \(0 < \delta \leq 1\), \(1 \leq \alpha \leq n\), \(1/n \leq \beta \leq 2/n\), \(\sigma\) is induced Lebesgue measure on the unit sphere \(S^{n-1} \subset \mathbb{R}^n\), and \(f, g \in L^1(\sigma)\) are functions satisfying \(\sigma(\text{supp} f), \sigma(\text{supp} g) \leq \delta^{n-1}\). We are looking for an exponent \(p \geq 2\) so that
\[
\|f\|_{L^1(\sigma)}^{p-2} \|f\|_{L^2(\sigma)}^2 \leq \delta^{(n-\alpha)((2/n)-\beta)} \|f\|_{L^p(\sigma)}^p
\]
and
\[
\|f\|_{L^1(\sigma)}^{(p/2)-1} \|f\|_{L^2(\sigma)} \|g\|_{L^1(\sigma)}^{(p/2)-1} \|g\|_{L^2(\sigma)} \leq \delta^{(n-\alpha)((2/n)-\beta)} \|f\|_{L^p(\sigma)}^{p/2} \|g\|_{L^2(\sigma)}^{p/2}.
\]
We have
\[
\|f\|_{L^1(\sigma)} \leq \sigma(\text{supp} f)^{1-(1/p)} \|f\|_{L^p(\sigma)} \leq \delta^{(n-1)/(p-1)/p} \|f\|_{L^p(\sigma)}
\]
and
\[
\|f\|_{L^2(\sigma)}^2 \leq \sigma(\text{supp} f)^{1-(2/p)} \left(\int |f|^{2(p/2)} d\sigma\right)^{2/p} \leq \delta^{(n-1)/(p-2)/p} \|f\|_{L^p(\sigma)}^2,
\]
so
\[
\|f\|_{L^1(\sigma)}^{p-2} \|f\|_{L^2(\sigma)}^2 \leq \delta^{(n-1)/(p-2)/(p-1)/p} \|f\|_{L^1(\sigma)}^{p-2} \|f\|_{L^p(\sigma)} \delta^{(n-1)/(p-2)/p} \|f\|_{L^p(\sigma)}^2 = \delta^{(n-1)/(p-2)/p} \|f\|_{L^p(\sigma)}^2,
\]
so \((n - 1)(p - 2) = (n - \alpha)((2/n) - \beta)\), and so
\[
p = 2 + \frac{n - \alpha}{n - 1} \left(\frac{2}{n} - \beta\right).
\]
Therefore, (17) holds with the above value of \(p\). Using (17), we now have
\[
\|f\|_{L^1(\sigma)}^{(p-2)/2} \|f\|_{L^2(\sigma)}^{(p-2)/2} \|g\|_{L^1(\sigma)} \|g\|_{L^2(\sigma)} \leq \left(\delta^{(n-\alpha)((2/n)-\beta)} \|f\|_{L^p(\sigma)}^{p/2}\right)^{1/2} \left(\delta^{(n-\alpha)((2/n)-\beta)} \|g\|_{L^p(\sigma)}^{p/2}\right)^{1/2},
\]
which is the inequality in (18).

7. Proof of Theorem 5.1
As the paragraph following the statement of Theorem 5.1 says, our proof of this theorem relies on ideas from [16], [5], [1], [12], and [4].
7.1. The bilinear estimate. Following [11] pages 1281–1283, we write the ball $B(0, R)$ as a disjoint union of two sets, one broad, the other narrow (see Subsection 7.3 below for the definition of these two sets). To estimate the $L^p(Hdx)$-norm of $Ef$ on the broad set, we consider a bilinear estimate.

For the rest of the paper, we will use the following notation. If $\phi$ is a function on $\mathbb{R}^2$ and $\rho > 0$, then $\phi_\rho$ is the function given by $\phi_\rho(\cdot) = \rho^{-2}\phi(\rho^{-1}\cdot)$.

Suppose $f$ is supported in an arc $I$ and $g$ is supported in an arc $J$ with $\sigma(I) \sim \sigma(J) \sim \delta$ and $\delta \leq \text{Dist}(I, J) \leq R\delta$. In this subsection, we shall always assume that

$$\frac{1}{10} R^\varepsilon \leq \frac{1}{\delta} \leq \frac{R\delta}{10}. \tag{19}$$

Let $\eta$ be a $C^\infty_0$ function on $\mathbb{R}^2$ satisfying $|\hat{\eta}| \geq 1$ on $B(0, 1)$. Then

$$\int_{B(0,R)} |Ef(x)Eg(x)||H(x)|dx = \int_{B(0,R)} |\hat{f}\sigma(x)\hat{g}\sigma(x)||H(x)|dx \leq \int_{B(0,R)} |\hat{f}\sigma(x)\hat{g}\sigma(x)||\hat{\eta}(x/R)|^2|H(x)|dx = \int_{B(0,R)} |(\eta_{R^{-1}} * f\sigma)(x)(\eta_{R^{-1}} * g\sigma)(x)||H(x)|dx = \int_{B(0,R)} |\hat{F}(x)\hat{G}(x)||H(x)|dx,$$

where $F = \eta_{R^{-1}} * f\sigma$ and $G = \beta_{R^{-1}} * g\sigma$.

Applying the Cauchy-Schwarz inequality in the convolution integral with respect to the measure $|\eta_{R^{-1}}(\xi - \cdot)|d\sigma$, we see that

$$\|F\|_{L^2}^2 \leq \int \left( \int |f(\theta)|^2 |\eta_{R^{-1}}(\xi - \theta)|d\sigma(\theta) \right) \left( \int |\eta_{R^{-1}}(\xi - \theta)|d\sigma(\theta) \right) d\xi \lesssim R \int |f(\theta)|^2 |\eta_{R^{-1}}(\xi - \theta)|d\sigma(\theta)d\xi = R \|\eta\|_{L^1} \|f\|_{L^2(\sigma)}^2,$$

where in the second inequality we used the fact that

$$\int |\eta_{R^{-1}}(\xi - \theta)|d\sigma(\theta) \lesssim R^2 \sigma(B(\xi, R^{-1})) \lesssim R.$$

Therefore,

$$\|F\|_{L^2} \lesssim R^{1/2}\|f\|_{L^2(\sigma)} \quad \text{and} \quad \|G\|_{L^2} \lesssim R^{1/2}\|g\|_{L^2(\sigma)}.$$

Since $F$ is supported in the $R^{-1}$-neighborhood of $I$ and $G$ is supported in the $R^{-1}$-neighborhood of $J$, we see (via (19)) that $F$ is supported in a ball of radius $(\delta/2) + (\delta/10) = (3\delta/5)$ and similarly for $G$. So $F * G$ is supported in a ball of radius $(6\delta/5)$, say $B(\xi_0, (6\delta/5))$. Via the locally constant property of the Fourier transform, this fact tells us that the Fourier transform of $F * G$ is essentially constant at scale $K = \delta^{-1}$, and hence allows us to implement the localization of the weight argument that we described in Section 5 at the intuitive level, and which we now carry out rigorously.
Let $\phi$ be a Schwartz function which is equal to 1 on $B(0,6/5)$. Then $\phi_\delta(x - \xi_0) = \delta^{-2}$ on $B(\xi_0, \frac{6\delta}{5})$, so that

$$F * G = \delta^2 \phi_\delta(\cdot - \xi_0)(F * G)$$

and

$$\hat{F}(x)\hat{G}(x) = \delta^2 \left(\phi_\delta(\cdot - \xi_0)(F * G)\right)(x) = \delta^2 \left(\phi_\delta(\cdot - \xi_0)\right) * \hat{F}(x),$$

Since $(\phi_\delta(\cdot - \xi_0))(x) = e^{-2\pi i x \cdot \xi_0} \hat{\phi}(\delta x)$, it follows that

$$\hat{F}(x)\hat{G}(x) = \delta^2 \int (\phi_\delta(\cdot - \xi_0))(x-y)\hat{F}(y)\hat{G}(y)dy$$

$$= \delta^2 \int e^{-2\pi i (x-y) \cdot \xi_0} \hat{\phi}(\delta(x-y))\hat{F}(y)\hat{G}(y)dy,$$

so that

$$|\hat{F}(x)\hat{G}(x)| \leq \delta^2 \int |\hat{\phi}(\delta(x-y))| |\hat{F}(y)\hat{G}(y)|dy.$$

Therefore,

$$\int_{B(0,\frac{6}{5})} |Ef(x)Eg(x)|H(x)dx \leq \delta^2 \int |\hat{F}(y)\hat{G}(y)|\int |\hat{\phi}(\delta(x-y))|H(x)dydx.$$

For $l = 0, 1, 2, \ldots$, we let $B_l = B(y, 2^l\delta^{-1})$ and write

$$\int |\hat{\phi}(\delta(x-y))|H(x)dx$$

$$= \int_{B_0} |\hat{\phi}(\delta(x-y))|H(x)dx + \sum_{l=1}^{\infty} \int_{B_l \setminus B_{l-1}} |\hat{\phi}(\delta(x-y))|H(x)dx$$

$$\leq \int_{B_0} \frac{CNH(x)}{(1 + \delta|x-y|)^N}dx + \sum_{l=1}^{\infty} \int_{B_l \setminus B_{l-1}} \frac{CNH(x)}{(1 + \delta|x-y|)^N}dx$$

$$\leq CN\int_{B_0} H(x)dx + \sum_{l=1}^{\infty} \frac{CN}{(1 + \delta^{2l-1})^N} \int_{B_l} H(x)dx.$$
Also,\
\[
\int_{B(x_0, r)} |\hat{\phi}(\delta(x - y))|H(x)dx dy = \int \int \chi_{B(x_0, r)}(y)|\hat{\phi}(\delta(x - y))|dy H(x)dx.
\]
Applying the change of variables \( z = \delta(x - y) \) in the inner integral, we get\
\[
\int_{B(x_0, r)} |\hat{\phi}(\delta(x - y))|H(x)dx dy = \frac{1}{\delta^2} \int \int \chi_{B(x_0, r)}(x - \frac{z}{\delta})|\hat{\phi}(z)|dz H(x)dx
\]
\[
= \frac{1}{\delta^2} \int |\hat{\phi}(z)| \int \chi_{B(x_0, r)}(x - \frac{z}{\delta})H(x)dx dz.
\]
But\
\[
\int \chi_{B(x_0, r)}(x - \frac{z}{\delta})H(x)dx = \int_{B(x_0 + \frac{1}{\delta}r, r)} H(x)dx \leq A_\alpha(H) r^\alpha
\]
for all \( x_0 \in \mathbb{R}^n \) and \( r \geq 1 \), so\
\[
\int_{B(x_0, r)} |\hat{\phi}(\delta(x - y))|H(x)dx dy \leq \frac{1}{\delta^2} \left\| \hat{\phi} \right\|_{L^1} A_\alpha(H) r^\alpha
\]
for all \( x_0 \in \mathbb{R}^2 \) and \( r \geq 1 \).

For \( y \in \mathbb{R}^2 \), define\
\[
\mathcal{H}(y) = \frac{\delta^{2(1-\theta)+\alpha \theta}}{C_{N, \theta} A_\alpha(H)^\theta} \int |\hat{\phi}(\delta(x - y))|H(x)dx.
\]
In view of the above discussion, we have\
\[
\left\| \mathcal{H} \right\|_{L^\infty} \leq 1 \quad \text{and} \quad \int_{B(x_0, r)} \mathcal{H}(y)dy \leq C A_\alpha(H)^{1-\theta} \delta^{(\alpha-2)\theta} r^\alpha
\]
for all \( x_0 \in \mathbb{R}^2 \) and \( r \geq 1 \). Thus \( \mathcal{H} \) is a weight on \( \mathbb{R}^2 \) of fractal dimension \( \alpha \) with\
\[
A_\alpha(\mathcal{H}) \leq C A_\alpha(H)^{1-\theta} \delta^{(\alpha-2)\theta}.
\]

Going back to (21), we now have\
\[
\int_{B(0, R)} |EF(x)EG(x)|H(x)dx \leq \delta^2 \frac{C_{N, \theta} A_\alpha(H)^\theta}{\delta^{2(1-\theta)+\alpha \theta}} \int |\hat{F} \ast \hat{G}(y)| \mathcal{H}(y)dy
\]
\[
= C_{N, \theta} \delta^{(2-\alpha)\theta} A_\alpha(H)^\theta \int |\hat{F} \ast \hat{G}(y)| \mathcal{H}(y)dy.
\]
(22)

Next, we let \( Q^* \) be the box in frequency space (where the circle is located) of dimensions \((R\delta)^{-1} \times R^{-1}\), centered at the origin, and with the \((R\delta)^{-1}\)-side (i.e. the long side) parallel to the line segment that connects the midpoint of \( I \) to that of \( J \). We also let \( \{Q_1\} \) be a tiling of \( \mathbb{R}^2 \) by boxes dual to \( Q^* \) (i.e. each \( Q_1 \) is an \( R\delta \times R \) box whose \( R\delta \)-side is parallel to the \((R\delta)^{-1}\)-side of \( Q^* \)) with centers \( \{\nu_1\} \), \( \psi \) be a \( C_0^\infty \) function on \( \mathbb{R}^2 \), and we define\
\[
\psi_1(\xi) = (R\delta)R \psi(R\delta \xi_1, R\xi_2) e^{2\pi i \nu_1 \xi}.
\]
In the definition of \( \psi_1 \), we are assuming that the line joining the midpoint of \( I \) to that of \( J \) is horizontal (i.e. parallel to the \( \xi_1 \)-axis). This assumption makes the presentation a little smoother and, of course, does not cost us any loss of generality.

We assume further that the Fourier transform of \( \psi \) is non-negative and satisfies \( \psi \geq 1/2 \) on \([-1/2, 1/2] \times [-1/2, 1/2] \). Then\
\[
\hat{\psi}_1(x) = \hat{\psi}(\frac{x_1 - \nu_{1,1}}{R\delta}, \frac{x_2 - \nu_{1,2}}{R}) \geq \frac{1}{2} \quad \text{if} \quad x \in Q_1.
\]
By the Schwartz decay of \( \hat{\psi} \), we have \( \sum_{m \in \mathbb{Z}^2} \hat{\psi}(-m)^k \lesssim 1 \) for any \( k \in \mathbb{N} \). Also, \( \{ \nu_l \} \) is basically \( \mathcal{R} \delta \mathbb{S} \times \mathcal{R} \mathbb{Z} \), so
\[
\sum_{l=1}^{\infty} \hat{\psi}_l(R\delta x_1, R\delta x_2)^k = \sum_{l=1}^{\infty} \hat{\psi} \left( \frac{R\delta x_1 - \nu_{l,1}}{R\delta}, \frac{R\delta x_2 - \nu_{l,2}}{R} \right)^k = \sum_{m \in \mathbb{Z}^2} \hat{\psi}(x - m)^k \lesssim 1,
\]
and so
\[
(23) \quad \sum_{l=1}^{\infty} \hat{\psi}_l(x)^k \lesssim 1
\]
for all \( x \in \mathbb{R}^2 \).

Going back to (22), we can now write
\[
\int_{B(0,R)} |E f(x) E g(x)| H(x) dx \lesssim \delta^{(2-\alpha)\theta} A_\alpha(H)^\theta \sum_{l=1}^{\infty} \int |\hat{F}(x) \hat{G}(x)| \hat{\psi}_l(x)^3 H(x) dx.
\]
Letting \( \hat{F_l} = \hat{\psi}_l * F \) and \( \hat{G_l} = \hat{\psi}_l * G \), this becomes
\[
\int_{B(0,R)} |E f(x) E g(x)| H(x) dx \lesssim \delta^{(2-\alpha)\theta} A_\alpha(H)^\theta \sum_{l=1}^{\infty} \int |\hat{F_l}(x) \hat{G_l}(x)| \hat{\psi}_l(x) H(x) dx.
\]
By Cauchy-Schwarz,
\[
\int |\hat{F_l}(x) \hat{G_l}(x)| \hat{\psi}_l(x) H(x) dx \leq \| \hat{F_l} \|_{L^2} \| \hat{G_l} \|_{L^2} \| \hat{\psi}_l(x) H(x) \|_{L^2}.
\]
Applying (15) from Subsection 6.1 with \( n = 2 \) and \( K = \delta^{-1} \), we have
\[
\int \hat{\psi}_l(x)^2 H(x)^2 dx \lesssim \int \hat{\psi}_l(x) H(x) dx \lesssim A_\alpha(H) \frac{R}{R\delta} (R\delta)^\alpha \lesssim A_\alpha(H) R^{1-\delta(\alpha-2)\theta+\alpha-1} R^{-\alpha},
\]
so that
\[
\| \hat{\psi}_l(x) H(x) \|_{L^2} \lesssim A_\alpha(H)^{(1-\theta)/2} R^{(\alpha-2)\theta+\alpha-1/2} R^{\alpha/2}.
\]
Therefore,
\[
\int_{B(0,R)} |E f(x) E g(x)| H(x) dx \lesssim A_\alpha(H)^{(1+\theta)/2} R^{(\alpha-2)\theta+\alpha-1/2} R^{\alpha/2} \sum_{l=1}^{\infty} \| \hat{F_l} \|_{L^2}.
\]
Letting \( \beta = (1 + \theta)/2 \) (since \( 0 \leq \theta \leq 1 \), we have \( 1/2 \leq \beta \leq 1 \)), this becomes
\[
\int_{B(0,R)} |E f(x) E g(x)| H(x) dx \lesssim A_\alpha(H)^{\beta} R^{(\alpha-2)\beta+\alpha-(3/2)} R^{\alpha/2} \sum_{l=1}^{\infty} \| \hat{F_l} \|_{L^2}.
\]
We now let \( A_l \) be the support of \( F_l \), \( B_l \) be the support of \( G_l \), and define the function \( \lambda_l : \mathbb{R}^2 \to [0, \infty) \) by \( \lambda_l(\xi) = |(\xi - A_l) \cap B_l| \). Applying Plancherel’s theorem followed by Cauchy-Schwarz, we see that
\[
\| \hat{F_l} \|_{L^2}^2 = \int |F_l * G_l(\xi)|^2 d\xi \leq \| \lambda_l \|_{L^\infty} \int |F_l|^2 * |G_l|^2(\xi) d\xi.
\]
By Young’s inequality,
\[
\int |F_l|^2 * |G_l|^2(\xi) d\xi \leq \| |F_l| \|_{L^2} \| |G_l|^2 \|_{L^1} = \| |F_l| \|_{L^2}^2 \| |G_l| \|_{L^1}^2 = \| |F_l| \|_{L^2}^2 \| |G_l| \|_{L^1}^2,
\]
so the only problem is to estimate \( \| \lambda_l \|_{L^\infty} \). We will do this by using the Kakeya bound (10) of Subsection 6.3.
Our assumptions on the arcs \( I \) and \( J \) imply that the angle between any two points in \( I \cup J \) is \( \lesssim R^\delta \). Also, for each \( l \), the function \( \psi_l \) is supported in the \((R\delta)^{-1} \times R^{-1}\) box \( Q^* \) of center \((0,0)\) and with the long side parallel to the line joining the midpoints of \( I \) and \( J \). So, if \( e \in I \cup J \), then the translate \( Q^* + e \) of \( Q^* \) is contained in an \((R\delta)^{-1} \times R^{-1}\) box with the \((R\delta)^{-1}\)-side tangent to \( S^1 \) at \( e \).

Therefore, the property of boxes of this form that was presented in Subsection 6.2 tells us that \( Q^* + e \) is contained in the \( R^{-1}\)-neighborhood of \( S^1 \). Therefore, the sets \( A_l \) and \( B_l \) satisfy the requirements needed for us to apply (10) and conclude

\[
\|\lambda_l\|_{L^\infty} \lesssim \frac{R^\delta}{R^2\delta}.
\]

Putting together what we have proved in the previous two paragraphs, we obtain

\[
\|\hat{F}_l\hat{G}_l\|_{L^2}^2 \lesssim \frac{R^\delta}{R^2\delta} \|F_l\|_{L^2}^2 \|G_l\|_{L^2}^2,
\]

and hence

\[
\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim R^\delta A_\alpha(H)^\beta \delta(2-\alpha)(\beta+\alpha-(3/2)) \frac{R^{\alpha/2}}{(R^2\delta)^{1/2}} \sum_{l=1}^{\infty} \|F_l\|_{L^2} \|G_l\|_{L^2}
\]

\[
= R^\delta A_\alpha(H)^\beta \delta(2-\alpha)(\beta-1) \frac{R^{\alpha/2}}{R} \sum_{l=1}^{\infty} \|F_l\|_{L^2} \|G_l\|_{L^2}.
\]

By Cauchy-Schwarz and Plancherel,

\[
\sum_{l=1}^{\infty} \|F_l\|_{L^2} \|G_l\|_{L^2} \leq \left( \sum_{l=1}^{\infty} \|\hat{F}_l\|_{L^2}^2 \right)^{1/2} \left( \sum_{l=1}^{\infty} \|\hat{G}_l\|_{L^2}^2 \right)^{1/2}.
\]

Also, by (23),

\[
\sum_{l=1}^{\infty} \|\hat{F}_l\|_{L^2}^2 = \int |\hat{F}(x)|^2 \sum_{l=1}^{\infty} |\hat{\psi}_l(x)|^2 dx \lesssim \|\hat{F}\|_{L^2}^2 = \|F\|_{L^2}^2
\]

and similarly for \( \sum_{l=1}^{\infty} \|\hat{G}_l\|_{L^2}^2 \), so

\[
\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim R^\delta A_\alpha(H)^\beta \delta(2-\alpha)(\beta-1) \frac{R^{\alpha/2}}{R} \|F\|_{L^2} \|G\|_{L^2}.
\]

Recalling (20), our bilinear estimate becomes

\[
\int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx \lesssim R^\delta A_\alpha(H)^\beta \delta(2-\alpha)(\beta-1) \frac{R^{\alpha/2}}{R} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}.
\]

Writing

\[
\int_{B(0,R)} |Ef(x)Eg(x)|^{p/2}H(x)dx = \int_{B(0,R)} |Ef(x)Eg(x)|^{(p/2)-1} |Ef(x)Eg(x)|H(x)dx
\]

\[
\leq \|f\|_{L^{(p/2)-1}(\sigma)} \|g\|_{L^{(p/2)-1}(\sigma)} \int_{B(0,R)} |Ef(x)Eg(x)|H(x)dx
\]

\[
\leq C_B R^\delta A_\alpha(H)^\beta R^{\alpha/2} \delta(2-\alpha)(\beta-1) \|f\|_{L^2(\sigma)}^{(p/2)-1} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}^{(p/2)-1} \|g\|_{L^2(\sigma)}
\]
and applying (13), we arrive at

\begin{equation}
\int_{B(0,R)} |Ef(x)Eg(x)|^{p/2}H(x)dx \leq R^cC_BA_\alpha(H)^{\beta}R^{\alpha/2}\|f\|_{L^p(\sigma)}^{p/2}\|g\|_{L^p(\sigma)}^{p/2}.
\end{equation}

7.2. The linear estimate. In this subsection, we work in \(\mathbb{R}^n\) with \(n \geq 2\).

Suppose \(f\) is supported in a cap of radius \(\delta/2\). The purpose of this section is to establish the estimate

\begin{equation}
\int_{B(0,R)} |Ef(x)|^{p}H(x)dx \leq C_LA_\alpha(H)^{\beta}\delta^{-2n/\beta}\delta^{-2}(\delta^2R)\|f\|_{L^p(\sigma)}^{p}
\end{equation}

under the condition

\begin{equation}
(10)R^c \leq \frac{1}{\delta} \leq \frac{R}{10}.
\end{equation}

Let \(\eta\) be a \(C_0^\infty\) function on \(\mathbb{R}^n\) satisfying \(|\hat{\eta}| \geq 1\) on \(B(0,1)\), and \(F = \eta_{R-1} \ast f \sigma\). Then

\[\int_{B(0,R)} |Ef(x)|^{p}H(x)dx \leq \int_{B(0,R)} |\hat{F}(x)|^{p}H(x)dx.\]

Also, let \(\psi\) be a \(C_0^\infty\) function on \(\mathbb{R}^n\), and \(\{B_l\}\) be a tiling of \(\mathbb{R}^n\) by balls dual to \(B(0,\delta)\) (i.e. \(\delta^{-1}\)-balls) with centers \(\{\nu_l\}\), and set

\[\psi_l(\xi) = \delta^{-n}\psi(\delta^{-1}\xi)e^{2\pi i \nu_l \xi}.
\]

We assume further that \(\hat{\psi}\) is non-negative and \(\geq 1/2\) on the unit ball. Then

\[\hat{\psi}_l(x) = \hat{\psi}(\delta(x - \tau_l)) \geq \frac{1}{2}
\]

if \(|\delta(x - \tau_l)| \leq 1\), i.e. if \(x \in B_l\). Thus

\[\int_{B(0,R)} |Ef(x)|^{p}H(x)dx \lesssim \int_{\mathbb{R}^n} |\hat{F}(x)| \hat{\psi}(x) |\hat{\psi}_l(x)|H(x)dx.
\]

Since \(1/n \leq \beta \leq 2/n\), we can apply Hölder’s inequality with the dual exponents \(1/(1 - \beta)\) and \(1/\beta\) to get

\[\int_{B(0,R)} |Ef(x)|^{p}H(x)dx \lesssim \sum_{l=1}^{\infty} \|F \ast \hat{\psi}_l\|_{L^2(1-\beta)}\|\hat{\psi}_lH\|_{L^{1/\beta}}.
\]

Since \(\|H\|_{L^\infty} \leq 1\), we have

\[\|\hat{\psi}_lH\|_{L^{1/\beta}}^{1/\beta} \leq \int \hat{\psi}_l(x)^{1/\beta}H(x)dx,
\]

and hence (by the proof of (13))

\[\|\hat{\psi}_lH\|_{L^{1/\beta}}^{1/\beta} \leq A_\alpha(H)\left(\frac{1}{\delta}\right)^{\alpha}.
\]

Also, by Hausdorff-Young,

\[\|F \ast \hat{\psi}_l\|_{L^2(1-\beta)}^{1/\beta} \leq \|F \ast \psi_l\|_{L^2(1+\beta)}.
\]

Therefore,

\[\int_{B(0,R)} |Ef(x)|^{p}H(x)dx \leq A_\alpha(H)^{\beta}\delta^{-n}\sum_{l=1}^{\infty} \|F \ast \psi_l\|_{L^2(1+\beta)}^{2}.
\]
Since (20) tells us $1/R \leq \delta/10$, it follows that $F$ is supported in a ball of radius $(\delta/2) + (\delta/10) = (3/5)\delta$, say $B(x_0, 3\delta/5)$. Moreover, since $\psi_l$ is supported in $B(0, \delta)$, it follows by Hölder’s inequality and Plancherel’s theorem that

$$\|F * \psi_l\|_{L^2/(1+\beta)} \lesssim \delta^{\alpha\beta} \|F * \psi_l\|_{L^2} = \delta^{\alpha\beta} \|\hat{F} \psi_l\|_{L^2}.$$ 

Thus

$$\int_{B(0,R)} |Ef(x)|^2 H(x) dx \lesssim A_\alpha(H)^\beta \delta^{-\alpha\beta} \delta^{\alpha\beta} \sum_{i=1}^\infty \int |\hat{F}(\xi)\hat{\psi}_i(\xi)|^2 d\xi$$

$$= A_\alpha(H)^\beta \delta^{(n-\alpha)\beta} \sum_{i=1}^\infty |\hat{\psi}_i(\xi)|^2 d\xi$$

$$\lesssim A_\alpha(H)^\beta \delta^{(n-\alpha)(\beta-(2/n))} \delta^{2-2(n/\alpha)} \|F\|_{L^2}^2.$$ 

But we know from (20) (whose proof shows that it is true in $\mathbb{R}^n$ for all $n \geq 2$) that $\|F\|_{L^2} \lesssim \sqrt{R} \|f\|_{L^{2}(\sigma)}$, so

$$\int_{B(0,R)} |Ef(x)|^2 H(x) dx \lesssim A_\alpha(H)^\beta \delta^{-2\alpha/n} \delta^2 R \delta^{(n-\alpha)(\beta-(2/n))} \|f\|_{L^2(\sigma)}^2.$$ 

Writing $|Ef(x)|^p = |Ef(x)|^{p-2}|Ef(x)|^2 \leq \|f\|_{L^1(\sigma)}^{p-2} \|Ef(x)\|^2$ and using (17), we now see that

$$\int_{B(0,R)} |Ef(x)|^p H(x) dx$$

$$\lesssim A_\alpha(H)^\beta \delta^{-2\alpha/n} \delta^2 R \delta^{(n-\alpha)(\beta-(2/n))} \|f\|_{L^1(\sigma)}^{p-2} \|f\|_{L^2(\sigma)}^2$$

$$\lesssim A_\alpha(H)^\beta \delta^{-2\alpha/n} \delta^2 R \|f\|_{L^p(\sigma)}^p,$$

which proves (25).

7.3. The induction argument. We let $0 < \epsilon < 10^{-2}$ and $R \geq 1$ be two numbers satisfying $R \geq (1000)^{1/(1-4\epsilon)}$. We also let $\delta$ be as in (19). We’re going to prove our estimate by inducting over $\delta$.

Base of the induction: Here $\delta = R^{-1/2}$. Plugging this value of $\delta$ into (25) in dimension $n = 2$, we get

$$\int_{B(0,R)} |Ef(x)|^p H(x) dx \leq C_L A_\alpha(H)^\beta R^{\alpha/2} \|f\|_{L^p(\sigma)}^p.$$ 

The inductive step: Suppose $\delta$ satisfies the condition (19):

$$(10)R^\epsilon \leq \frac{1}{\delta} \leq \frac{R\delta}{10},$$

and the estimate is true for $\delta$, i.e.

$$\int_{B(0,R)} |Ef(x)|^p H(x) dx \leq CR^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|f\|_{L^p(\sigma)}^p,$$

whenever $f \in L^1(\sigma)$, $f$ is supported on an arc $I_\delta \subset S^1$, and $\sigma(I_\delta) \leq \delta$. We are going to show that

$$\int_{B(0,R)} |Eg(x)|^p H(x) dx \leq C'R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \|g\|_{L^p(\sigma)}^p$$ 

for all $g \in L^1(\sigma)$.
whenever \( g \in L^1(\sigma) \), \( f \) is supported on an arc \( I_{R,\delta} \subset S^1 \), and \( \sigma(I_{R,\delta}) \leq R^\epsilon \delta \), where
\[
C' = 3^p C + (10)^p R^{(p+2)\epsilon} C_B.
\]

We let \( K = R^\epsilon \) and cover the support of \( g \) by \( K \) arcs \( \tau \) each of measure \( \delta \). We then write \( g = \sum \tau f_\tau \) with each function \( f_\tau \) supported in the arc \( \tau \).

Following [1] and [7], for \( x \in \mathbb{R}^2 \), we define the significant set of \( x \) by
\[
S(x) = \{ \tau : |Ef_\tau(x)| \geq \frac{1}{10K}|Ef_g(x)| \}.
\]
Then
\[
|Ef_g(x)| \leq \sum_{\tau \in S(x)} |Ef_\tau(x)| + \frac{1}{10}|Ef_g(x)|,
\]
so that
\[
|Ef_g(x)| \leq \frac{10}{9} \sum_{\tau \in S(x)} |Ef_\tau(x)|.
\]

The narrow set \( \mathcal{N} \) and the broad set \( \mathcal{B} \) are now defined as
\[
\mathcal{N} = B(0, R) \cap \{ x \in \mathbb{R}^2 : \#S(x) \leq 2 \} \quad \text{and} \quad \mathcal{B} = B(0, R) \setminus \mathcal{N}.
\]

We will estimate \( \int_{\mathcal{N}} |Ef_g(x)|^p H(x)dx \) by induction and \( \int_{\mathcal{B}} |Ef_g(x)|^p H(x)dx \) by using the bilinear estimate.

By (27) and (29),
\[
\int_{\mathcal{N}} |Ef_g(x)|^p H(x)dx \leq 2^{p-1} \left( \frac{10}{9} \right)^p \int_{\mathcal{N}} \sum_{\tau \in S(x)} |Ef_\tau(x)|^p H(x)dx
\]
\[
\leq \left( \frac{20}{9} \right)^p \int_{\mathcal{N}} \sum_{\tau} |Ef_\tau(x)|^p H(x)dx
\]
\[
\leq 3^p \sum_{\tau} CR^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \| f_\tau \|_{L^p(\sigma)}^p
\]
\[
= 3^p CR^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \| g \|_{L^p(\sigma)}^p.
\]

To every \( x \in \mathcal{B} \) there are two caps \( \tau_x, \tau'_x \in S(x) \) so that \( \text{Dist}(\tau_x, \tau'_x) \geq \delta \). Writing
\[
|Ef_g(x)|^p = |Ef_g(x)|^{p/2} |Ef_g(x)|^{p/2} \leq (10K|Ef_{\tau_x}(x)|)^{p/2} (10K|Ef_{\tau'_x}(x)|)^{p/2},
\]
we see that
\[
|Ef_g(x)|^p \leq (10K)^p \sum_{\tau, \tau' : \text{Dist}(\tau, \tau') \geq \delta} |Ef_\tau(x)|^{p/2} |Ef_{\tau'}(x)|^{p/2}.
\]

Using the bilinear estimate (23), it follows that
\[
\int_{\mathcal{B}} |Ef_g(x)|^p H(x)dx
\]
\[
\leq (10K)^p \sum_{\tau, \tau' : \text{Dist}(\tau, \tau') \geq \delta} \int_{\mathcal{B}} |Ef_\tau(x)|^{p/2} |Ef_{\tau'}(x)|^{p/2} H(x)dx
\]
\[
\leq (10K)^p C_B R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \sum_{\tau, \tau' : \text{Dist}(\tau, \tau') \geq \delta} \| f_\tau \|_{L^p(\sigma)}^{p/2} \| f_{\tau'} \|_{L^p(\sigma)}^{p/2}
\]
\[
\leq (10)^p K^p C_B R^\epsilon A_\alpha(H)^\beta R^{\alpha/2} \sum_{\tau, \tau' : \text{Dist}(\tau, \tau') \geq \delta} \| g \|_{L^p(\sigma)}^{p/2} \| g \|_{L^p(\sigma)}^{p/2}.
\]
Therefore,

\[ \int_B |Eg(x)|^p H(x) dx \leq (10)^p K^{p+2} C_B A_\alpha(H)^\beta R^{\alpha/2} \|g\|_{L^p(\sigma)}^p. \]

Combining the narrow and broad estimates, we arrive at (28).

The iteration: In the inductive step we proved that if we were given that the estimate

\[ \int_{B(0,R)} |Ef(x)|^p H(x) dx \leq CR' A_\alpha(H)^\beta R^{\alpha/2} \|f\|_{L^p(\sigma)}^p \]

holds for every function \( f \in L^1(\sigma) \) that is supported in an arc of \( \sigma \)-measure \( \leq \delta \), and \( \delta \) obeys (25), then we can produce the estimate

\[ \int_{B(0,R)} |Eg(x)|^p H(x) dx \leq C' R' A_\alpha(H)^\beta R^{\alpha/2} \|g\|_{L^p(\sigma)}^p \]

for every function \( g \in L^1(\sigma) \) that is supported in an arc of \( \sigma \)-measure \( \leq R' \delta \), where

\[ C' = 3^p C + (10)^p R^{(p+2)\epsilon} C_B. \]

Starting with the base of the induction, where \( \delta = R^{-1/2} \) and \( C = C_L \), and applying the inductive step \( k \) times, we arrive at an estimate that holds for every function \( f \in L^1(\sigma) \) that is supported on an arc of \( \sigma \)-measure \( \leq \delta_k = R^{k\epsilon} \delta = R^{k\epsilon}/\sqrt{R} \), with constant

\[ C_k = 3^p C_L + (10)^p R^{(p+2)\epsilon} C_B \frac{3^p - 1}{1 - 3^p}. \]

At the step before the last, \( k = (1/(2\epsilon)) - 2 \) and \( \delta_k = R^{(1/(2\epsilon)) - 2\epsilon}/\sqrt{R} = R^{-2\epsilon} \), which is a valid value of \( \delta \) (i.e. \( \delta_k = R^{-2\epsilon} \) obeys (25), because \( 10R^\epsilon \leq 1/R^{-2\epsilon} \leq R^{-2\epsilon}/10 \)). Applying the inductive step one last time, we get the estimate

\[ \int_{B(0,R)} |Ef(x)|^p H(x) dx \leq CR' A_\alpha(H)^\beta R^{\alpha/2} \|f\|_{L^p(\sigma)}^p \]

for every function \( f \in L^1(\sigma) \) that is supported on an arc of \( \sigma \)-measure \( \leq R^{-\epsilon} \), where the constant \( C \) satisfies

\[ C \leq 3^{p/(2\epsilon)} \left( C_L + \frac{(10)^p R^{(p+2)\epsilon}}{3^p - 1} C_B \right). \]

Since the circle \( S^1 \) can be covered by \( \sim R^\epsilon \) such arcs, (13) follows and Theorem 5.1 is proved.

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