A Note on Quadratic Funding under Constrained Matching Funds

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Abstract

In this note I show that quadratic funding achieves decentralized social efficiency in the extent there are enough (donor) matching funds to cover the quadratic funding objective. If individual backers internalize that matching funds will not be sufficient to reach the quadratic level, allocation will be biased towards the capitalist allocation, the more so, the less matching funds are available. This result emerges even when individual contributors are not required to finance the deficit (i.e., the difference between total contributions and available matching funds). I also show properties of the level of required matching fund, in order to better understand under which conditions social efficiency will most likely be compromised.

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Introduction

Buterin, Hitzig, and Weyl (2018) (BHW) propose a matching funding mechanism that has the property of achieving (nearly) optimal provision of public goods. The mechanism, they named Liberal Radicalism (LR), replicates the (centralized) social optimum investment level, by means of decentralized coordination in the presence of an external philanthropic donor.

This promising mechanism has attracted a lot of public attention, and is currently under experimentation in Gitcoin Grants, an open source software and Ethereum community-related projects financing platform.

Buterin, Hitzig, and Weyl (2018) have shown that if individual backers are required to finance the deficit between the LR rule financing level and the actual funds committed by them, the mechanism will fail to achieve social efficiency. This implies that the mechanism requires the availability of external or philanthropic funds to reach efficiency. In Gitcoin Grants, for example, the pool of matching funds is collected from donors, and backers are not required to finance the deficit.

My purpose in this note is to examine some properties of this mechanism, particularly in relation to its behavior under a limited pool of matching funds. First, I note that even when project backers are not required to fund a deficit, their investment will be lower than the optimal investment if they perceive that available matching funds will not be enough to reach the optimal LR financing levels.

In practice, as is the case with Gitcoin Grants, when the sum of required payments to each project exceeds available matching funds committed by donors, the subsidies to each project are scaled down by a constant in order that totals add up to the subsidy pool’s budget[^1]. This procedure actually lowers the individual optimal contribution, biasing this response from the socially optimal level.

I then examine the question of what determines the increase in the required size of matching funds. I show how required funds increase non-linearly with the number of contributors, a property that will probably put the efficiency of the mechanism under stress in most applications. I also show that required funds are maximized when backers have perfectly correlated investment shares across projects.

[^1]: This is explained by Buterin in a blog post here: https://vitalik.ca/general/2019/12/07/quadratic.html
1 Sub-optimal investment under constrained matching funds

The Liberal Radical Mechanism is a rule to allocate funding for public goods. Assuming there are \( p \in P \) public goods candidates (and competing) to receive financing, the LR mechanism is intended to allocate, for each project \( p \), the amount of funds \( F_{p,LR} \), where this amount is a function of (decentralized) individual (\( i \in I \)) contributions to project \( p \), \( C_i^p \):

\[
F_{p,LR} = \left( \sum_i \sqrt{C_i^p} \right)^2
\]

This implies that the target LR matched subsidy to be provided to a project \( p \) is:

\[
M_{p,LR} = F_{p,LR} - C^p
\]

Where \( C^p = \sum_i C_i^p \). In the context of Gitcoin Grants, matching funds are collected from donors, conforming a subsidy pool we will denote \( D \).

In the practice, in order to meet the budget constrain, the actual matched amounts received by each project are scaled down by a constant we will denote \( k \). So \( k \) results to satisfy the matching funds budget constrain, defined as:

\[
\frac{1}{k} \sum_p M_{p,LR} = D
\]  

(1)

This implies that the actual funds to be received by the project \( F_p \) are:

\[
F_p = \frac{1}{k} (F_{p,LR} - C^p) + C^p = \frac{1}{k} F_{p,LR} + (1 - \frac{1}{k}) C^p
\]

1.1 Individual contributor problem

As in Buterin, Hitzig, and Weyl (2018), let \( V_i(F^p) \) to be the currency-equivalent utility citizen \( i \) receives if the funding level of public good \( p \) is \( F^p \). I also maintain here the assumptions regarding the independence among values generated by public goods across citizens, simultaneous timing, and a setting of complete information.

The problem that defines the optimal individual contribution from the perspective of backer \( i \) is

\[
\max_{\{c_i\}} V_i \left( \frac{1}{k} \left( \sum_i \sqrt{C_i} \right)^2 + (1 - \frac{1}{k}) C^p \right) - c_i
\]

2In the current Gitcoin rounds, the available pool is subsequently pre-divided into categories, such as infrastructure, applications (dapps), and community, etc. For simplicity we will abstract away from this.
This problem has a first order condition given by:
\[ V'_i(F_p)(\frac{1}{k} \sum_i \sqrt{c_i} + (1 - \frac{1}{k})) = 1 \quad (2) \]

Notice that if \( k \to 1 \) then the condition converges to:
\[ V'_i(F_p) = \frac{\sqrt{c_i}}{\sum_i \sqrt{c_i}} \]

Which summing across individuals gives the socially optimal condition:
\[ \sum_i V'_i(F_p) = 1 \]

In other words, the marginal cost of investing 1 unit of contribution equals the aggregate marginal benefit for the community.

Also notice that if \( k \to \infty \) in Equation 2 then
\[ V'_i(C^p) = 1 \]

Which is the individual capitalist solution in the terminology of BHW.

**2 What determines the size of the required matching pool?**

Since individual contribution suboptimality depends on the extent there are insufficient funds available, it is useful to note under what conditions this will most likely happen. Rearranging Equation 1, \( k \) is a function of
\[ k = \sum_p \frac{M_p^{LR}}{D} \]

so for a given amount of contributed funds, \( k \) will be higher the higher the target LR matched funding amount.

It is useful to notice that the level of LR subsidy a project \( p \) receives can also be expressed as:
\[ M_p^{LR} = (\sum_i \sqrt{c_i^p})^2 - \sum_i c_i^p = \sum_i c_i^p + 2 \sum_{i \neq j} \sqrt{c_i^p c_j^p} - \sum_i c_i^p = 2 \sum_{i \neq j} \sqrt{c_i^p c_j^p} \quad (3) \]

Where the second equality results from a well known property of the sum of squares. This expression is useful because the summation has a number of terms equal to the number of pairs of contributors. So while the subsidy amount scales linearly in the contributions of individuals, it scales non-linearly in the number of contributors, which follow the combinatorial number \( \binom{n}{2} = \frac{n(n-1)}{2} \).
Therefore, it follows that, in order to keep $k$ small, the level of contributions by donors should scale as fast as the number of contributors, and this is certainly demanding (if not impossible) for any pool of philanthropic funds.

A second observation is related to what determines the total target subsidy to increase, considering, for example, if individual contributors have a given budget to invest across projects. The answer in this case is that the subsidy requirements will be maximized if the investment preferences, as measure by their share of invested wealth, are perfectly correlated across individuals.

To see this, denote $m_i$ the total amount of available funds for individual $i$ and $\alpha^p_i$ the share contributed to project $p$, so $\sum_p \alpha^p_i = 1$

The maximum subsidy results from solving the problem:

$$\max_{\alpha^p_i, i \in I, p \in P} 2 \sum_p \sum_{i \neq j} \sqrt{\alpha^p_i m_i \alpha^p_j m_j} + \sum_i \lambda_i (1 - \sum_p \alpha^p_i) \quad (4)$$

And the first order condition with respect to $\alpha^p_i$ is

$$2 \frac{1}{2} (\alpha^p_i)^{-\frac{1}{2}} \sqrt{m_i \alpha^p_j m_j} - \lambda_i = 0, \forall i, p$$

In particular, if we take two conditions for projects $p$ and $p'$ we have

$$(\alpha^p_i)^{-\frac{1}{2}} \sqrt{m_i \alpha^p_j m_j} = \lambda_i (\alpha^{p'}_i)^{-\frac{1}{2}} \sqrt{m_i \alpha^{p'}_j m_j} = \lambda_i$$

And dividing each side of the each equation gives:

$$\left( \frac{\alpha^p_i}{\alpha^{p'}_i} \right)^{-\frac{1}{2}} \left( \frac{\alpha^p_j}{\alpha^{p'}_j} \right)^{\frac{1}{2}} = 1$$

Or alternatively:

$$\frac{\alpha^p_i}{\alpha^{p'}_i} = \frac{\alpha^p_j}{\alpha^{p'}_j}$$

Therefore, the total required subsidy is maximized when the invested shares across individuals are perfectly correlated.

This result has the corollary that subsidy will be maximized under complete coordination, for instance, if investments by all backers are allocated to a single project, if all backers are coordinated to invest half of the investments in two projects, and so on. In any of these cases the total (maximum) amount of required funds will be given by

$$M^{LR, \text{MAX}} = 2 \sum_{i \neq j} \sqrt{m_i m^p_j}$$

Which is a similar expression to Equation (3), and conserves the properties of scaling linearly in terms of individuals wealth, and scaling non-linearly (quadratically) in the number of contributors.
In contrast, the total subsidy is minimized in the case the invested shares are perfectly non-correlated, as in a case where each individual invest in a separate project. It is easy to see in Equation 4 that case would imply 0 matching funds.

3 Summarizing and preliminary conclusions

Buterin, Hitzig, and Weyl (2018) show a financing mechanism for public goods with some promising features. My interest in this note is to further understand some of its properties and practical limitations.

While BHW have shown that inefficiency is compromised under a requirement to finance the matching pool, my focus here is in the case where funds come from a separate philanthropic fund (no taxes to contributors), similarly to what is currently taking place at Gitcoin Grants.

Up to this point I’ve noted that, keeping the assumptions of preferences in relation to the public good, the ability of the mechanism to achieve social efficiency is tied to achieving a sufficiently large pool of matching funds. The lower the restriction of funds, the greater its social efficiency. Given the quadratic characteristics of the mechanism, however, this is especially difficult to occur since the target LR financing increases non-linearly in the number of contributors—following the combinatorial (n²/2)_. Asking for a similar increase in the philanthropic pool of funds seems certainly difficult.

If conditions under which quadratic funding achieves social efficiency are so demanding, particularly under growth in the number of contributors, then perhaps these results speaks about the impossibility of implementing practical mechanisms that manage to replicate social efficiency in first place. But perhaps some of the social efficiency assumptions made here might not be the most appropriate in the first place. In particular in relation to the monotonically increasing nature of the utility function or the absence of interdependences among projects.

References

Buterin, Vitalik, Zoë Hitzig, and Eric Glen Weyl. 2018. “Liberal Radicalism: Formal Rules for a Society Neutral Among Communities.” SSRN Electronic Journal. [https://doi.org/10.2139/ssrn.3243656]