SIMILARITIES OF GAUGE AND GRAVITY AMPLITUDES

N. E. J. Bjerrum-Bohr, David C. Dunbar and Harald Ita

Department of Physics
University of Wales Swansea
Swansea, SA2 8PP, UK

Abstract

We review recent progress in computations of amplitudes in gauge theory and gravity. We compare the perturbative expansion of amplitudes in $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity and discuss surprising similarities.
1 Introduction

Perturbative gauge theory and gravity in four dimensions are quite dissimilar from a dynamical viewpoint. Gauge theory (e.g. pure Yang-Mills theory) is a renormalisable theory that is strongly coupled in the infrared and asymptotically free in the ultraviolet. Gravity on the other hand is a weakly coupled theory in the infrared but strongly coupled in the ultraviolet. By power counting, gravity in four dimensions is potentially a non-renormalisable theory. Pure gravity scattering amplitudes are finite at one-loop with the first divergence occurring at two-loops [1].

Supersymmetry generally softens the UV behaviour in a quantum field theory. For example, maximally supersymmetric Yang-Mills is a finite theory [2] and supergravity theories have a finite S-matrix until at least three loops [3]. Although four-dimensional power counting and counter-term arguments suggest that supergravity theories are non-renormalisable [4] this has, so far, not been tested by direct computations. Arguments based on power counting within unitary cuts suggest that the first counter term in maximal supergravity [5] is expected at five loops [6, 7].

Recently, initiated by the duality between gauge theories and a twister string theory [8], there has been much progress in the computation of amplitudes in gauge theory. In this talk we discuss how these ideas may be applied to gravity calculations and the results thereof. We will first review the recent progress in computing physical on-shell tree amplitudes for gravity theories particularly focusing on the on-shell recursion relations [9, 10] and the MHV-vertex construction [11, 12, 13, 14, 15]. Later we will discuss one-loop amplitudes. A surprising result is that the one-loop amplitudes of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity [16, 17] occur to be expressible in terms of scalar box integral functions - despite the expectation from power counting. Supergravity multi-loop amplitudes are not directly addressed, however, the structure of amplitudes at tree-level and one-loop have, through factorisation and unitarity, important consequences on the structure of higher loop amplitudes.

2 Old and New Techniques for Gravity Tree Amplitudes

Graviton scattering amplitudes are extremely difficult to evaluate using conventional Feynman diagram techniques. In this section we review alternative methods: 1) the Kawai-Lewellen-Tye relations, 2) on shell recursion relations and 3) MHV vertex constructions.

1) Gravity amplitudes can be constructed through the Kawai, Lewellen and Tye (KLT)-relations [18] as squares of gauge theory amplitudes. The KLT relations are inspired by the naive string theory relation

$$\text{closed string} \sim (\text{left-mover}) \times (\text{right-mover}) ,$$  \hspace{1cm} (2.1)
and have the explicit form, up to five points,

\[
\begin{align*}
M_3^{\text{tree}}(1, 2, 3) &= -iA_3^{\text{tree}}(1, 2, 3)A_3^{\text{tree}}(1, 2, 3), \\
M_4^{\text{tree}}(1, 2, 3, 4) &= -is_{12}A_4^{\text{tree}}(1, 2, 3, 4)A_4^{\text{tree}}(1, 2, 4, 3), \\
M_5^{\text{tree}}(1, 2, 3, 4, 5) &= is_{12}s_{34}A_5^{\text{tree}}(1, 2, 3, 4, 5)A_5^{\text{tree}}(2, 1, 4, 3, 5) \\
&\quad + is_{13}s_{24}A_5^{\text{tree}}(1, 3, 2, 4, 5)A_5^{\text{tree}}(3, 1, 4, 2, 5),
\end{align*}
\]

where \( A_n^{\text{tree}} \) are the tree-level colour-ordered gauge theory partial amplitudes. We suppress factors of \( g^{n-2} \) in the \( A_n^{\text{tree}} \) and \( (\kappa/2)^{n-2} \) in the \( M_3^{\text{tree}} \).

The KLT relations are helpful in the calculation of gravity tree amplitudes, however they have some undesirable features. The factorisation structure is not manifest and the expressions do not tend to be compact, as the permutation sums grow rather quickly with \( n \). In fact, the Berends, Giele and Kuijf (BGK) form of the MHV gravity amplitude \cite{19},

\[
M_n^{\text{tree}}(1^-, 2^-, 3^+, \ldots, n^+) = -i \langle 12 \rangle^8 \times \left[ \frac{[2]}{1} \frac{[n - 2 - n - 1]}{N(n)} \right] \times
\]

\[
\prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} \langle i j \rangle \prod_{l=3}^{n-3} (-[n|K_{l+1,n-1}|l]) + \mathcal{P}(2, 3, \ldots, n - 2),
\]

is rather more compact than that of the KLT sum (as is the expression in \cite{10}.). In the above we use the definitions, \([k|K_{i,j}|l] \equiv \langle k^+|K_{i,j}|l^+ \rangle \equiv \langle l^-|K_{i,j}|k^- \rangle \equiv \langle l|K_{i,j}|k \rangle \equiv \sum_{a=1}^J [k|a \rangle \langle a l| \), and \( N(n) = \prod_{1 \leq i < j \leq n} \langle i j \rangle \). In terms of the above Weyl spinors we often use twistor variables \( \lambda_i \equiv |k_i^+ \rangle \) and \( \bar{\lambda}_i \equiv |k_i^- \rangle \). The MHV amplitudes for graviton scattering display a feature not shared by the Yang-Mills expressions, they depend not only on the holomorphic variables \( \lambda \), but also on the anti-holomorphic \( \bar{\lambda} \) variables (within the \( s_{ij} \) for the KLT expression).

2) In a recent computational approach for amplitudes, Britto, Cachazo, Feng and Witten \cite{9} obtained on-shell recursion relations for trees. The recursion relations are based on factorisation properties of amplitudes and are thus applicable to a wide range of theories and in particular to gravity \cite{9, 10}. The technique is based on analytically shifting a pair of external legs, \( \lambda_i \rightarrow \lambda_i + z \lambda_j, \bar{\lambda}_j \rightarrow \bar{\lambda}_j - z \bar{\lambda}_i \), and on determining the physical amplitude, \( M_n(0) \), from the poles in the shifted amplitude, \( M_n(z) \). This leads to a recursion relation of the form,

\[
M_n(0) = \sum_\alpha \tilde{M}_{n-k_\alpha+2}(z_\alpha) \times \frac{i}{P_\alpha^2} \times \tilde{M}_{k_\alpha}(z_\alpha),
\]

where the factorisation is only on these poles, \( z_\alpha \), where legs \( i \) and \( j \) are connected to different sub-amplitudes. An essential condition for the recursion relations is that the shifted amplitude \( M_n(z) \) vanishes for large \( z \). Whereas proven for gauge theory amplitudes a general proof (for arbitrary helicities \cite{10}) in gravity is an open problem.

Recursion relations based on the analyticity in the complex plane can also be used at loop level both to calculate rational terms \cite{20} and the coefficients of integral functions \cite{21}. 

2
Finally, we would like to mention the CSW construction [11, 22] of amplitudes and its generalisation to gravity [15]. In this approach MHV-amplitudes are treated as fundamental vertices and generic scattering amplitudes are expanded in terms of these MHV-vertices.

Considering the $N^s$MHV amplitude with $n$ external legs. One would begin by drawing all diagrams which may be constructed using MHV vertices.

$$
\begin{array}{c}
k_{i_3}^+ \quad \ldots \quad k_{i_2}^+ \quad k_{i_1}^-
\end{array}
\times
\begin{array}{c}
\frac{1}{p_{j_1}^2}
\end{array}
\times
\begin{array}{c}
\ldots
\end{array}
\times
\begin{array}{c}
p_{j_i}^2
\end{array}
\times
\begin{array}{c}
\text{MHV vertices and propagators as indicated above.}
\end{array}

The contribution from each diagram is a product of $(s + 1)$ MHV vertices and $s$ propagators.

The contribution of a given diagram to the total amplitude can be calculated by evaluating the product of MHV amplitudes and propagators,

$$
M_n^s \big|_{\text{CSW-diagram}} = \left( \prod_{i=1,s+1} M_{N_i}^{\text{MHV}}(\hat{K}_i) \right) \prod_{j=1,s+1} \frac{i}{p_j^2},
$$

where the propagators are computed on the set of momenta $k_i$ and $p_j$, and the MHV vertices are evaluated at shifted momenta $\hat{k}_i$ and $\hat{p}_j$. The momenta $k_i$ are external and the momenta $p_j$ internal, and given by momentum conservation at each MHV-vertex. A key feature is the interpretation of the MHV amplitudes for internal legs. For Yang-Mills where the MHV vertices only depend on $\lambda$ the correct interpretation is [11]

$$
\lambda(p)_a = p_{a\dot{a}} \eta^\dot{a};
$$

for an arbitrary reference spinor $\eta$. For gravity amplitudes we must also solve for $\bar{\lambda}(p)$ which is less obvious [23] and will be a function of the momentum of the negative helicity legs. It turns out that all spinors are uniquely defined in terms of the shifted momenta $\hat{k}_i$ and $\hat{p}_j$ if we demand that they are null vectors obeying momentum conservation at each vertex.

Explicitly they are given by shifting the negative helicity legs $\bar{\lambda}_i$ by

$$
\bar{\lambda}_i \rightarrow \bar{\lambda}_i + a_i \bar{\eta} ,
$$

and leaving $\bar{\lambda}_j$ of the positive helicity legs $k_{j+}$ untouched. The $s + 2$ parameters, $a_i^-$ are uniquely fixed [15] by demanding a) overall momentum conservation, b) momentum conservation at each vertex and finally c) that the internal momenta, $\hat{p}_j$, are massless $\hat{p}_j^2 = 0$.

As an example of how the MHV-vertex constructions works for gravity, we can consider the $n$-point NMHV amplitude with three negative helicity legs $m_i$. The MHV-vertex approach gives the amplitude in the form,

$$
\sum_{\text{perms, } r} M^{\text{MHV}}(\hat{k}_1^-, k_4^+, \ldots k_r^+, p^-) \times \frac{1}{p^2} \times M^{\text{MHV}}(\hat{k}_2^-, \hat{k}_3^- k_{r+1}^+, \ldots k_{n-3}^+, \hat{p}^+) ,
$$
where the sum runs over diagrams involving all choices of \( r > 0 \) and all permutations of the negative and positive helicity legs. To illustrate the correct continuation, the three negative helicity \( \lambda \) must be shifted \( \lambda_i \rightarrow \lambda_i + a_i \bar{\eta} \). Imposing the momentum constraints leaves us with a shift,
\[
\bar{\lambda}_1 \rightarrow \bar{\lambda}_1 + z(k_2, k_3) \bar{\eta},
\]
together with the cyclic shifts of the other two legs. Momentum is conserved for any value of the parameter \( z \). Requiring \( \hat{p}^2 = 0 \) then fixes \( z \) uniquely as
\[
 z = p^2 / (|\eta||p|k_1) \]
and the MHV vertex expansion is completely determined.

3 One-Loop Amplitudes in \( N = 8 \) Supergravity

In a Yang-Mills theory, the loop momentum polynomial in a one-loop \( n \)-point diagram will generically be of degree \( \leq n \). \( N = 4 \) one-loop amplitudes exhibits considerable simplification and the loop momentum integral will be of degree \( n - 4 \)\([24, 25]\). Consequently, from a Passarino-Veltman reduction\([26]\), the amplitudes can be expressed as a sum of scalar box integrals with rational coefficients,
\[
A^{1-\text{loop}} = \sum_a c_a I_a^4.
\]
Determining the amplitude then reduces to determining the rational coefficients \( c_a \). Inspired by the duality in\([8]\), considerable progress has recently been made in determining such coefficients, \( c_a \), using a variety of methods based on unitarity\([25, 27, 28, 29]\).

For \( N = 8 \) supergravity the equivalent power counting arguments\([30]\) give a loop momentum polynomial of degree
\[
2(n - 4),
\]
which is consistent with Eq. (2.1). Reduction for \( n > 4 \) leads to a sum of tensor box integrals with integrands of degree \( n - 4 \) which would then reduce to scalar boxes and triangle, bubble and rational functions,
\[
M^{1-\text{loop}} = \sum_a c_a I_a^4 + \sum_a d_a I_a^3 + \sum_a e_a I_a^2 + R,
\]
where the \( I_3 \) are present for \( n \geq 5 \), \( I_2 \) for \( n \geq 6 \) and \( R \) for \( n \geq 7 \).

There is evidence that all one-loop amplitudes of \( N = 8 \), like the \( N = 4 \) amplitudes Eq. (3.1), can be expressed as a sum over scalar box integrals, the so called “no-triangle hypothesis”\([16, 17]\). Firstly, in the few definite computations at one-loop level, triangle or bubble functions do not appear. The first computation was of the four-point amplitude\([31]\) where only box functions appear (although this is consistent with power counting). Beyond this computations of the five and six point MHV-amplitudes yielded only scalar box-functions\([32]\).

Secondly, the factorisation properties of physical amplitudes do not demand the presence of these functions. Since the four and five point amplitudes are triangle-free then in any factorisation limit of a higher point function the triangles must vanish. In
this spirit an ansatz for the $n$-point MHV amplitude was constructed [32] entirely of box functions consistent in all soft limits.

Thirdly, one can check whether the amplitudes composed purely from box-functions precisely give the expected soft divergence in a $n$ graviton amplitude [33],

$$M_{\text{one-loop}}^{1,2,...,n} = i\epsilon \kappa^2 \left[ \sum_{i<j} s_{ij} \ln[-s_{ij}] \right] \times M_{\text{tree}}^{1,2,...,n}.$$ 

In [17] the box coefficients were explicitly computed for the six-point NMHV amplitudes confirming the above claims.

The “no-triangle” hypothesis applies to one-loop amplitudes. However, by factorisation it should have implications beyond one-loop suggesting the UV behaviour of maximal supergravity may be significantly milder than expected from power counting.

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