Research Article

Xiaoxiao Wang, Chaoqian Li*, and Yaotang Li

A Geršgorin-type eigenvalue localization set with $n$ parameters for stochastic matrices

https://doi.org/10.1515/math-2018-0030
Received September 29, 2017; accepted February 13, 2018.

Abstract: A set in the complex plane which involves $n$ parameters in $[0, 1]$ is given to localize all eigenvalues different from 1 for stochastic matrices. As an application of this set, an upper bound for the moduli of the subdominant eigenvalues of a stochastic matrix is obtained. Lastly, we fix $n$ parameters in $[0, 1]$ to give a new set including all eigenvalues different from 1, which is tighter than those provided by Shen et al. (Linear Algebra Appl. 447 (2014) 74-87) and Li et al. (Linear and Multilinear Algebra 63(11) (2015) 2159-2170) for estimating the moduli of subdominant eigenvalues.

Keywords: Stochastic Matrix, Geršgorin set, Subdominant eigenvalue

MSC: 65F15, 15A18, 15A51

1 Introduction

Stochastic matrices and eigenvalue localization of stochastic matrices play key roles in many application fields, such as Computer Aided Geometric Design [1], Birth-Death Processes [2–5], and Markov chains [6]. An entrywise nonnegative matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is called row stochastic (or simply stochastic) if all its row sums are 1, that is,

$$\sum_{j=1}^{n} a_{ij} = 1, \text{ for each } i \in N = \{1, 2, \ldots, n\}.$$

Let us denote the $i$th deleted column sum of the moduli of off-diagonal entries of $A$ by

$$C_i(A) = \sum_{j \neq i} |a_{ji}|.$$

Obviously, 1 is an eigenvalue of a stochastic matrix with a corresponding eigenvector $e = [1, 1, \ldots, 1]^T$. From the Perron-Frobenius Theorem [7], for any eigenvalue $\lambda$ of $A$, that is, $\lambda \in \sigma(A)$, we have $|\lambda| \leq 1$ [8]. Here we call $|\lambda|$ a moduli of subdominant eigenvalue of a stochastic matrix $A$ if $1 > |\lambda| > |\eta|$ for every eigenvalue $\eta$ different from 1 and $\lambda$ [8–10].

Since the subdominant eigenvalue of a stochastic matrix is crucial for bounding the convergence rate of stochastic processes [8, 11–14], it is interesting to give a set to localize all eigenvalues different from 1, or an upper bound for the moduli of its subdominant eigenvalue [8, 15].

One can use the well-known Geršgorin circle set [16] to localize all eigenvalues for a stochastic matrix. However, this set always includes the trivial eigenvalue 1, and thus it is not always precise for capturing...
all eigenvalues different from 1 of a stochastic matrix. Therefore, several authors have tried to modify the Geršgorin circle set to localize more precisely all eigenvalues different from 1. In [8], Cvetković et al. gave the following set.

**Theorem 1.1** ([8, Theorem 3.4]). Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A) \setminus \{1\}$, then

$$\lambda \in \Gamma(A) = \{z \in \mathbb{C} : |z - \gamma(A)| < 1 - trace(A) + (n - 1) \gamma(A)\},$$

where $\gamma(A) = \max_{i \in \mathbb{N}} (a_{ii} - l_i(A))$, $l_i(A) = \min_{j \neq i} a_{ji}$ and $trace(A)$ is the trace of $A$.

However, the set provided by Theorem 1.1 is not effective in some cases, such as, for the class of stochastic matrices

$$SM_0 = \{A \in \mathbb{R}^{n \times n} : A is stochastic, and a_{ii} = l_i = 0, for each i \in \mathbb{N}\},$$

for more details, see [15]. To overcome this drawback, Li and Li [15] provided another set as follows.

**Theorem 1.2** ([15, Theorem 6]). Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A) \setminus \{1\}$, then

$$\lambda \in \Gamma(A) = \{z \in \mathbb{C} : |z - \tilde{\gamma}(A)| < trace(A) + (n - 1) \tilde{\gamma}(A) - 1\},$$

where $\tilde{\gamma}(A) = \max_{i \in \mathbb{N}} (L_i(A) - a_{ii})$ and $L_i(A) = \max_{j \neq i} a_{ji}$.

Recently, by taking respectively

$$l_i(A) = \min_{j \neq i} a_{ji}, \nu_i(A) = \max \left\{0, \frac{1}{2} \min_{k \neq i, m \neq i} \{a_{ki} + a_{mi}\}\right\} = \frac{1}{2} \min_{k \neq i, m \neq i} \{a_{ki} + a_{mi}\},$$

and

$$q_i(A) = \frac{1}{n - 1} \sum_{j \neq i} a_{ji}$$

to modify the Geršgorin circle set, Shen et al. [12], and Li et al. [11] gave three sets to localize all eigenvalues different from 1.

**Theorem 1.3** ([11, 12]). Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A) \setminus \{1\}$, then

$$\lambda \in \Gamma_{stol}(A) = \bigcup_{i \in \mathbb{N}} \left( I_{stol}^i(A) = \{z \in \mathbb{C} : |a_{ii} - z - l_i(A)| < Cl_i(A)\}\right),$$

$$\lambda \in \Gamma_{stov}(A) = \bigcup_{i \in \mathbb{N}} \left( I_{stov}^i(A) = \{z \in \mathbb{C} : |a_{ii} - z - v_i(A)| < Cv_i(A)\}\right)$$

and

$$\lambda \in \Gamma_{stolq}(A) = \bigcup_{i \in \mathbb{N}} \left( I_{stolq}^i(A) = \{z \in \mathbb{C} : |a_{ii} - z - q_i(A)| < Cq_i(A)\}\right),$$

where

$$Cl_i(A) = \sum_{j \neq i} |a_{ji} - l_i(A)| = \sum_{j \neq i} a_{ji} - \sum_{j \neq i} l_i(A) = Cl_i(A) - (n - 1)l_i(A),$$

$$Cv_i(A) = \sum_{j \neq i} |a_{ji} - v_i(A)| = Cv_i(A) - (n - 3)v_i(A) - 2l_i(A)$$

and

$$Cq_i(A) = \sum_{j \neq i} |a_{ji} - q_i(A)|.$$
Theorem 1.4 ([11, Theorems 3.3 and 3.8]). Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A) \setminus\{1\}$, then
\[
\lambda \in I^{\text{stol}}(A) = \bigcup_{i \in \mathbb{N}} \left( I^{\text{stol}}_i(A) = \{ z \in \mathbb{C} : |L_i(A) - a_{ii} - z| < CL_i(A) \} \right)
\]
and
\[
\lambda \in I^{\text{stov}}(A) = \bigcup_{i \in \mathbb{N}} \left( I^{\text{stov}}_i(A) = \{ z \in \mathbb{C} : |V_i(A) - a_{ii} + z| < CV_i(A) \} \right),
\]
where
\[
CL_i(A) = \sum_{j \neq i} |L_i(A) - a_{ij}| = (n - 1)L_i(A) - C_i(A)
\]
and
\[
CV_i(A) = \sum_{j \neq i} |V_i(A) - a_{ij}| = (n - 3)V_i(A) + 2L_i(A) - C_i(A).
\]

Note that $l_i(A), v_i(A), q_i(A), V_i(A)$ and $L_i(A)$ are all in the interval $[\min_{j \neq i} a_{ij}, \max_{j \neq i} a_{ij}]$. So it is natural to ask whether or not there is an optimal value in $[\min_{j \neq i} a_{ij}, \max_{j \neq i}]$ such that the set, which is obtained by using this value to modify the Geršgorin circle set, captures all eigenvalues different from 1 of a stochastic matrix most precisely. To answer this question, we give a set in Section 2 with $n$ parameters to localize all eigenvalues different from 1 for a stochastic matrix, and show that this set would reduce to $I^{\text{stol}}(A)$, $I^{\text{stov}}(A)$, $I^{\text{stol}}(A)$ and $I^{\text{stov}}(A)$ by taking some fixed parameters. And we use this set in Section 3 to give an upper bound for the moduli of its subdominant eigenvalue for a stochastic matrix. In section 4, by choosing special values of these $n$ parameters in $[0, 1]$ for the upper bound obtained in Section 3, we give a new set including all eigenvalues different from 1, which is better than $I^{\text{stol}}(A)$ and $I^{\text{stov}}(A)$ in the sense of estimating the moduli of subdominant eigenvalues.

2 A Geršgorin-type eigenvalue localization set with $n$ parameters

We first begin with an important lemma, which is used to give some modifications of the Geršgorin circle set.

Lemma 2.1 ([8, 11, 12]). Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. For any $d = [d_1, d_2, \ldots, d_n]^T \in \mathbb{R}^n$, if $\mu \in \sigma(A) \setminus\{1\}$, then $-\mu$ is an eigenvalue of the matrix
\[
B = ed^T - A.
\]

Lemma 2.1 shows that once an eigenvalue localization set for $B = ed^T - A$ is given, we can get a set to localize all eigenvalues different from 1 for the stochastic matrix $A$ [11]. Now we present the following choice of $d$:
\[
d = L^{\alpha_i}(A),
\]
where $L^{\alpha_i}(A) = [L^{\alpha_i}_1(A), L^{\alpha_i}_2(A), \ldots, L^{\alpha_i}_n(A)]^T$, $\alpha_i \in [0, 1]$ for $i \in \mathbb{N}$ and
\[
L^{\alpha_i}_i(A) = \alpha_i L_i(A) + (1 - \alpha_i) I_i(A) = \alpha_i \max_{j \neq i} a_{ij} + (1 - \alpha_i) \min_{j \neq i} a_{ij}, i \in \mathbb{N}.
\]

By Lemma 2.1 and (1), we can obtain the following set to localize all eigenvalues different from 1 of a stochastic matrix.

Theorem 2.2. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A) \setminus\{1\}$, then for any $\alpha_i \in [0, 1], i \in \mathbb{N}$,
\[
\lambda \in I^{\text{stol}}^{\alpha_i}(A) = \bigcup_{i \in \mathbb{N}} I^{\text{stol}}_i^{\alpha_i}(A),
\]
where
\[
I^{\text{stol}}_i^{\alpha_i}(A) = \{ z \in \mathbb{C} : |\alpha_i L_i(A) + (1 - \alpha_i) I_i(A) - a_{ii} + z| \leq CL_i^{\alpha_i}(A) \}.
\]
Furthermore, note that for any $\lambda \in \sigma(B^{\alpha})$, 
\[
\tilde{\lambda} \in \bigcup_{i \in \mathbb{N}} \{ z \in \mathbb{C} : |b_{ii}^{\alpha} - z| \leq C_i(B^{\alpha}) \}.
\]
By Lemma 2.1, we have for $\lambda \in \sigma(A) \setminus \{1\}$, then $-\lambda \in \sigma(B^{\alpha})$, that is,
\[
-\lambda \in \bigcup_{i \in \mathbb{N}} \{ z \in \mathbb{C} : |b_{ii}^{\alpha} - z| \leq C_i(B^{\alpha}) \}.
\]
Furthermore, note that for any $i \in \mathbb{N}$,
\[
b_{ii}^{\alpha} = L_i^{\alpha}(A) - a_{ii} = \alpha_i L_i(A) + (1 - \alpha_i) l_i(A) - a_{ii}
\]
and
\[
C_i(B^{\alpha}) = \sum_{j=1}^n |L_j^{\alpha}(A) - a_{ji}| = CL_i^{\alpha}(A).
\]
Hence,
\[
\lambda \in \Gamma_{\text{stol}}^{\alpha}(A) = \bigcup_{i \in \mathbb{N}} \Gamma_i^{\text{stol}}(A),
\]
where $\Gamma_i^{\text{stol}}(A) = \{ z \in \mathbb{C} : |\alpha_i L_i(A) + (1 - \alpha_i) l_i(A) - a_{ii} + z| \leq CL_i^{\alpha}(A) \}$. \hfill \qed

**Example 2.3.** Consider the first 50 stochastic matrices generated by the MATLAB code

\[
k = 10; A = \text{rand}(k, k); A = \text{inv}((\text{diag}((\text{sum}(A^2))) * A),
\]
and take $\alpha_i \in [0, 1]$ for $i = 1, 2, \ldots, 10$ by the MATLAB code
\[
alpha = \text{rand}(1, k).
\]
By drawing the sets $\Gamma_{\text{stol}}^{\alpha}(A)$ in Theorem 2.2 and
\[
\Gamma = (\Gamma(A) \cap \tilde{\Gamma}(A))
\]
in Theorems 1.1 and 1.2, it is not difficult to see that the number of $\Gamma_{\text{stol}}^{\alpha}(A) \cap \Gamma$ is 46, that if $1 \notin \Gamma$, then $1 \notin \Gamma_{\text{stol}}^{\alpha}(A)$, and that if $1 \in \Gamma$, then $\Gamma_{\text{stol}}^{\alpha}(A)$ may not contain the trivial eigenvalue 1 (also see Table 1). So, by these examples, we conclude that the set in Theorem 2.2 captures all eigenvalues different from 1 of a stochastic matrix more precisely than the sets in Theorem 1.1 and Theorem 1.2 in some cases.

**Table 1. Comparisons of $\Gamma_{\text{stol}}^{\alpha}(A)$ and $\Gamma = (\Gamma(A) \cap \tilde{\Gamma}(A))$**

| $1 \in \Gamma_{\text{stol}}^{\alpha}(A)$ | $1 \notin \Gamma$ | $\Gamma_{\text{stol}}^{\alpha}(A) \cap \Gamma$ | $\Gamma_{\text{stol}}^{\alpha}(A) \cap \Gamma$ |
|----------------|-----------------|----------------|----------------|
| Number | 8 | 2 | 4 | 46 |
| The i-th happens | 5, 8, 7, 14, 20, 39, 59, 66, 62 | 25, 31 | 3, 7, 11, 35 | otherwise |

**Remark 2.4.** (i) When $\alpha_i = 1$ for each $i \in \mathbb{N}$, then $\Gamma_i^{\alpha}(A) = L_i(A)$ and $CL_i^{\alpha}(A) = CL_i(A)$ for any $i \in \mathbb{N}$, which implies $\Gamma_{\text{stol}}^{\alpha}(A)$ reduces to $\Gamma_{\text{stol}}^{1}(A)$ in Theorem 1.4;
(II) When \( \alpha_i = \frac{V_i(A) - l_i(A)}{l_i(A) - l_i(A)} \in [0, 1] \) and \( L_i(A) > l_i(A) \) for each \( i \in N \), then \( I_{CL}^{\alpha_i}(A) = CV_i(A) \) and \( CL_i^{\alpha_i}(A) = C V_i(A) \) for any \( i \in N \). On the other hand, if for some \( i \in N \), \( L_i(A) = l_i(A) \), then for any \( \alpha_i \in [0, 1] \) we also have \( L_i^{\alpha_i}(A) = V_i(A) \) and \( CL_i^{\alpha_i}(A) = CV_i(A) \). These imply \( I^{stol^{\alpha_i}}(A) \) reduces to \( I^{stol^{\alpha_i}}(A) \) in Theorem 1.4;

(III) When \( \alpha_i = \frac{V_i(A) - l_i(A)}{l_i(A) - l_i(A)} \in [0, 1] \) and \( L_i(A) > l_i(A) \) for each \( i \in N \), then \( L_i^{\alpha_i}(A) = q_i(A) \) and \( CL_i^{\alpha_i}(A) = C q_i(A) \) for any \( i \in N \). On the other hand, if for some \( i \in N \), \( L_i(A) = l_i(A) \), then for any \( \alpha_i \in [0, 1] \) we also have \( L_i^{\alpha_i}(A) = q_i(A) \) and \( CL_i^{\alpha_i}(A) = C q_i(A) \). These imply \( I^{stol^{\alpha_i}}(A) \) reduces to \( I^{stol^{\alpha_i}}(A) \) in Theorem 1.3;

(IV) When \( \alpha_i = \frac{V_i(A) - l_i(A)}{l_i(A) - l_i(A)} \in [0, 1] \) and \( L_i(A) > l_i(A) \) for each \( i \in N \), then \( L_i^{\alpha_i}(A) = V_i(A) \) and \( CL_i^{\alpha_i}(A) = C V_i(A) \) for any \( i \in N \). On the other hand, if for some \( i \in N \), \( L_i(A) = l_i(A) \), then for any \( \alpha_i \in [0, 1] \) we also have \( L_i^{\alpha_i}(A) = V_i(A) \) and \( CL_i^{\alpha_i}(A) = C V_i(A) \). These imply \( I^{stol^{\alpha_i}}(A) \) reduces to \( I^{stol^{\alpha_i}}(A) \) in Theorem 1.3;

(V) When \( \alpha_i = 0 \) for each \( i \in N \), then \( L_i^{\alpha_i}(A) = l_i(A) \) and \( CL_i^{\alpha_i}(A) = C l_i(A) \) for any \( i \in N \), which implies \( I^{stol^{\alpha_i}}(A) \) reduces to \( I^{stol^{\alpha_i}}(A) \) in Theorem 1.3.

Hence, we say that the set \( I^{stol^{\alpha_i}}(A) \) is a generalization of \( I^{stol^{\alpha_i}}(A) \), \( I^{stol^{\alpha_i}}(A) \) and \( I^{stol^{\alpha_i}}(A) \) in Theorem 1.3, and \( I^{stol^{\alpha_i}}(A) \) and \( I^{stol^{\alpha_i}}(A) \) in Theorem 1.4. Moreover, according to \( \alpha_i \in [0, 1] \) in Theorem 2.2, we can get the following result easily.

**Remark 2.5.** Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be a stochastic matrix. If \( \lambda \in \sigma(A) \backslash \{1\} \), then
\[
\lambda \in I^{[0,1]}(A) = \bigcap_{\alpha \in [0,1]} I^{stol^{\alpha}}(A).
\]
Furthermore, \( I^{[0,1]}(A) \subseteq \left( I^{stol^{\alpha}}(A) \cap I^{stol^{\alpha}}(A) \cap I^{stol^{\alpha}}(A) \cap I^{stol^{\alpha}}(A) \right) \).

The set \( I^{[0,1]}(A) \) in Remark 2.5 is not of much practical use because it involves some parameters \( \alpha_i \). In fact, we can take some special \( \alpha_i \) in practice, which is illustrated by the following example.

**Example 2.6.** Consider the third stochastic matrix \( A \) in Example 2.3. By Table 1, we have that
\[
1 \in I^{stol^{\alpha}}(A), \quad I^{stol^{\alpha}}(A) \subseteq I^{\alpha} \quad \text{and} \quad I^{\alpha} \subseteq I^{stol^{\alpha}}(A),
\]
which is shown in Figure 1, where \( I^{stol^{\alpha}}(A) \) is drawn slightly thicker than \( I^{\alpha} \). Furthermore, we take the first 3 vectors
\[
\alpha^{(j)} = [\alpha^{(j)}_1, \alpha^{(j)}_2, \ldots, \alpha^{(j)}_{10}], \quad j = 1, 2, 3
\]
generated by the MATLAB code \( \text{alpha} = \text{rand}(1, 10) \), that is,
\[
\alpha^{(1)} = [0.8147, 0.9058, 0.1270, 0.9134, 0.6324, 0.0975, 0.2785, 0.5469, 0.9575, 0.9649],
\]
\[
\alpha^{(2)} = [0.1576, 0.9706, 0.9572, 0.4854, 0.8003, 0.1419, 0.4218, 0.9157, 0.7922, 0.9595],
\]
and
\[
\alpha^{(3)} = [0.6557, 0.0357, 0.8491, 0.9340, 0.6787, 0.7577, 0.7431, 0.3922, 0.6555, 0.1712].
\]
By Remark 2.5, we have that for any \( \lambda \in \sigma(A) \backslash \{1\} \),
\[
\lambda \in \left( I^{stol^{\alpha^{(1)}}}(A) \cap I^{stol^{\alpha^{(2)}}}(A) \cap I^{stol^{\alpha^{(3)}}}(A) \right).
\]
We draw this set in the complex plane, see Figure 2. It is easy to see
\[
1 \notin \left( I^{stol^{\alpha^{(1)}}}(A) \cap I^{stol^{\alpha^{(2)}}}(A) \cap I^{stol^{\alpha^{(3)}}}(A) \right)
\]
and
\[
\left( I^{stol^{\alpha^{(1)}}}(A) \cap I^{stol^{\alpha^{(2)}}}(A) \cap I^{stol^{\alpha^{(3)}}}(A) \right) \subset I^{\alpha}.
\]
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**Fig. 1.** \(I^{\text{stol}^n}(A) \notin \bar{\Gamma}, \text{ and } \bar{\Gamma} \notin I^{\text{stol}^n}(A)\)

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**Fig. 2.** \(I^{\text{stol}^n}(A) \cap I^{\text{stol}^n}(A) \cap I^{\text{stol}^n}(A) \subset \bar{\Gamma}\)

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This example shows that we can take some special \(\alpha_i\) to get a set which is tighter than the sets in Theorems 1.1 and 1.2.

It is well-known that an eigenvalue inclusion set leads to a sufficient condition for nonsingular matrices, and vice versa [12, 16]. Hence, from Theorem 2.2 or Remark 2.5, we can get a nonsingular condition for stochastic matrices.

**Proposition 2.7.** Let \(A = [a_{ij}] \in \mathbb{R}^{n \times n}\) be a stochastic matrix. If for some \(\tilde{\alpha}_i \in [0, 1], i \in N,\)

\[
|\tilde{\alpha}_i l_i(A) + (1 - \tilde{\alpha}_i) l_i(A) - a_{ii}| > CL^{\tilde{\alpha}_i}_i(A), \ i \in N,
\]

where \(CL^{\tilde{\alpha}_i}_i\) is defined as (2), then \(A\) is nonsingular.
Proof. Suppose that $A$ is singular, that is, $0 \in \sigma(A)$. From Theorem 2.2, we have that for any $\alpha_i \in [0, 1]$, $i \in N$,

$$0 \in I_{stol}^{\alpha_i} (A) = \bigcup_{i \in N} I_{stol}^{\alpha_i} (A).$$

In particular,

$$0 \in I_{stol}^{\alpha_i} (A) = \bigcup_{i \in N} I_{stol}^{\alpha_i} (A).$$

Hence, there is an index $i_0 \in N$ such that

$$|\tilde{a}_{i_0}L_{i_0}(A) + (1 - \tilde{a}_{i_0})I_{i_0}(A) - a_{i_0}| \leq CL_{i_0}^{\tilde{a}_{i_0}}(A).$$

This contradicts (3). The conclusion follows.  \hfill \square

3 An upper bound for the moduli of subdominant eigenvalues

By using the set $I_{stol}(A)$ in Theorem 2.2, we can give a bound to estimate the moduli of subdominant eigenvalues of a stochastic matrix.

Theorem 3.1. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A) \setminus \{1\}$, then

$$|\lambda| \leq \rho^{[0,1]},$$

where $\rho^{[0,1]} = \max_{i \in N} \min_{\alpha_i \in [0,1]} \left\{ \sum_{j=1}^{n} |\alpha_jL_j(A) + (1 - \alpha_j)I_j(A) - a_{ij}| \right\}.$

Proof. Let

$$f_i(\alpha_i) = \sum_{j=1}^{n} |\alpha_jL_j(A) + (1 - \alpha_i)I_i(A) - a_{ij}|$$

$$= CL_i^{\alpha_i}(A) + |\alpha_jL_j(A) + (1 - \alpha_j)I_j(A) - a_{ij}|, \ \alpha_i \in [0, 1], \ \alpha_i \in N,$$

where

$$CL_i^{\alpha_i}(A) = \sum_{j=1}^{n} |\alpha_jL_j(A) + (1 - \alpha_i)I_i(A) - a_{ij}|.$$

Therefore, each $f_i(\alpha_i)$, $i \in N$ is a continuous function of $\alpha_i \in [0, 1]$, and there are $\tilde{\alpha}_i \in [0, 1], \ i \in N$ such that

$$f_i(\tilde{\alpha}_i) = \min_{\alpha_i \in [0,1]} \left\{ CL_i^{\alpha_i}(A) + |\alpha_jL_j(A) + (1 - \alpha_j)I_j(A) - a_{ij}| \right\}, \ i \in N.$$

For these $\tilde{\alpha}_i \in [0, 1], \ i \in N$, by Theorem 2.2 we have

$$\lambda \in I_{stol}^{\alpha_i}(A) = \bigcup_{i \in N} I_{stol}^{\alpha_i}(A).$$

Hence, there is an index $i_0 \in N$ such that

$$|\tilde{a}_{i_0}L_{i_0}(A) + (1 - \tilde{a}_{i_0})I_{i_0}(A) - a_{i_0}| \leq CL_{i_0}^{\tilde{a}_{i_0}}(A),$$

which gives

$$|\lambda| \leq CL_{i_0}^{\tilde{a}_{i_0}}(A) + |\tilde{a}_{i_0}L_{i_0}(A) + (1 - \tilde{a}_{i_0})I_{i_0}(A) - a_{i_0}|.$$  \hfill (6)

By (5) we have

$$|\lambda| \leq \min_{\alpha_i \in [0,1]} \left\{ CL_{i_0}^{\alpha_i}(A) + |\alpha_iL_i(A) + (1 - \alpha_i)I_i(A) - a_{i_0}| \right\},$$
Theorem 3.2. Let \( A \) be a stochastic matrix. If \( \lambda \in \sigma(A) \setminus \{1\} \), then
\[
|\lambda| \leq \min \{ \rho_L, \rho_V, \rho_q, \rho_r, \rho_l \},
\]
where
\[
\rho_L = \max_{i \in N} \{ a_{ii} + nL_i(A) - C_i(A) \},
\]
\[
\rho_V = \max_{i \in N} \{ a_{ii} + (n - 2)V_i(A) + 2L_i(A) - C_i(A) \},
\]
\[
\rho_q = \max_{i \in N} \left\{ \sum_{j=1}^{n} |a_{ji} - q_i(A)| \right\},
\]
\[
\rho_r = \max_{i \in N} \{ a_{ii} - (n - 4)V_i(A) + 2L_i(A) + C_i(A) \}
\]
and
\[
\rho_l = \max_{i \in N} \{ a_{ii} - (n - 2)l_i(A) + C_i(A) \}.
\]

Proof. We first prove \( |\lambda| \leq \rho_L \). From Theorem 1.4,
\[
\lambda \in I^{stol}(A) = \bigcup_{i \in N} I^{stol}_i(A).
\]
As in the proof of Theorem 3.1, we have that there is an index \( i_0 \in N \) such that
\[
|L_{i_0}(A) - a_{i_0i_0} + \lambda| \leq CL_{i_0}(A),
\]
and
\[
|\lambda| \leq |a_{i_0i_0} - L_{i_0}(A)| + CL_{i_0}(A) \leq a_{i_0i_0} + L_{i_0}(A) + (n - 1)L_{i_0}(A) - C_{i_0}(A) = a_{i_0i_0} + nL_{i_0}(A) - C_{i_0}(A) \leq \max_{i \in N} \{ a_{ii} + nL_i(A) - C_i(A) \},
\]
i.e., \( |\lambda| \leq \rho_L \). Similarly, by
\[
\lambda \in I^{stov}(A), \lambda \in I^{stol}(A), \lambda \in I^{stov}(A), \text{ and } \lambda \in I^{stol}(A),
\]
we can get respectively
\[
|\lambda| \leq \rho_V, |\lambda| \leq \rho_q, |\lambda| \leq \rho_r, \text{ and } |\lambda| \leq \rho_l.
\]
The conclusion follows.
Proposition 3.4. Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be a stochastic matrix. Then
\[
\rho_{[0,1]} \leq \min\{\rho_L, \rho_V, \rho_q, \rho_r, \rho_t\},
\]
where \( \rho_{[0,1]}, \rho_L, \rho_V, \rho_q, \rho_r, \) and \( \rho_t \) are defined in Theorem 3.1 and Theorem 3.2, respectively.

As in the proof of Theorem 3.2, by Theorems 1.1 and 1.2 two upper bounds for the subdominant eigenvalue of a stochastic matrix are obtained easily.

Theorem 3.3. Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be a stochastic matrix and \( \lambda \in \sigma(A) \setminus \{1\} \) be its subdominant eigenvalue. Then
\[
|\lambda| \leq 1 - \text{trace}(A) + n\gamma(A), \text{ and } |\lambda| \leq \text{trace}(A) + n\tilde{\gamma}(A) - 1,
\]
consequently,
\[
|\lambda| \leq \min\{1 - \text{trace}(A) + n\gamma(A), \text{trace}(A) + n\tilde{\gamma}(A) - 1\}. \tag{8}
\]

For the comparison of \( \rho_{[0,1]} \) and the upper bound
\[
\Lambda := \min\{1 - \text{trace}(A) + n\gamma(A), \text{trace}(A) + n\tilde{\gamma}(A) - 1\}
\]
in (8), we conclude here that by taking some special \( \alpha_i \) and the fact that \( \Lambda \) is given by Theorems 1.1 and 1.2, an upper bound can be obtained, which is better than
\[
\min\{1 - \text{trace}(A) + n\gamma(A), \text{trace}(A) + n\tilde{\gamma}(A) - 1\}.
\]

4 Special choices of \( \alpha_i \) for the set \( I^{stol^\alpha}(A) \)

In this section, we choose \( \alpha_i \) for the set \( I^{stol^\alpha}(A) \) to give a set, which is tighter than the sets \( I^{stol}(A) \) and \( I^{stol^\alpha}(A) \) by determining the optimal value of \( \alpha_i \) for estimating the moduli of subdominant eigenvalues of a stochastic matrix.

For a given stochastic matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \), let
\[
N^+(A) = \{i \in N : \Delta_i(A) \geq 0\}
\]
and
\[
N^-(A) = \{i \in N : \Delta_i(A) < 0\},
\]
where \( \Delta_i(A) = n L_i(A) + (n - 2) i_i(A) - 2C_i(A) \). Obviously, \( N = N^+(A) \cup N^-(A) \).

Proposition 4.1. Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be a stochastic matrix. If \( \lambda \in \sigma(A) \setminus \{1\} \), then
\[
|\lambda| \leq \rho_{[0,1]}, \tag{9}
\]
where
\[
\rho_{[0,1]} = \max \left\{ \max_{i \in N^+(A)} \left\{ a_{ii} - (n - 2) i_i(A) + C_i(A) \right\}, \max_{i \in N^-(A)} \left\{ a_{ii} + n L_i(A) - C_i(A) \right\} \right\}.
\]

Proof. Note that
\[
CL_{i_i}^\alpha(A) + |\alpha_i L_i(A) + (1 - \alpha_i) i_i(A) - a_{ii}|
= \sum_{j \neq i} |\alpha_i L_i(A) + (1 - \alpha_i) i_i(A) - (\alpha_i a_{ji} + (1 - \alpha_i) a_{jj})|
+ |\alpha_i L_i(A) + (1 - \alpha_i) i_i(A) - a_{ii}|
\leq \alpha_i \sum_{j \neq i} |L_i(A) - a_{ji}| + (1 - \alpha_i) \sum_{j \neq i} |i_i(A) - a_{ji}|
\]
Proof. By the proof of Proposition 4.1, we have that
\[ +\alpha_i L_i(A) + (1 - \alpha_i) l_i(A) + a_{ii} \]
\[ = (n - 1) (\alpha_i L_i(A) - (1 - \alpha_i) l_i(A)) + (1 - 2\alpha_i) C_i(A) + \alpha_i l_i(A) + a_{ii} \]
\[ = a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i (nL_i(A) + (n - 2) l_i(A) - 2\alpha_i C_i(A)) \]
\[ = a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A). \]

Hence, from Theorem 3.1, we have
\[
|\lambda| \leq \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ \sum_{j=1}^{n} |\alpha_i L_i(A) + (1 - \alpha_i) l_i(A) - a_{ii}| \right\} 
\leq \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ CL_i^{\alpha_i}(A) + |\alpha_i L_i(A) + (1 - \alpha_i) l_i(A) - a_{ii}| \right\}
\leq \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\}
\leq \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\},
\]
\[ = \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\}. \tag{10} \]

Furthermore, let
\[ f(\alpha) = a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A), \quad \alpha \in [0, 1]. \]

Then when \( \Delta_i(A) \geq 0 \), \( f(\alpha) \) reaches its minimum \( a_{ii} - (n - 2) l_i(A) + C_i(A) \) at \( \alpha = 0 \), and when \( \Delta_i(A) < 0 \), \( f(\alpha) \) reaches its minimum
\[ a_{ii} - (n - 2) l_i(A) + C_i(A) + \Delta_i(A) = a_{ii} + nL_i(A) - C_i(A) \]
at \( \alpha = 1 \). Therefore, Inequality (10) is equivalent to
\[
|\lambda| \leq \max_{i \in N} \left\{ \max_{\alpha \in [0, 1]} \left\{ a_{ii} - (n - 2) l_i(A) + C_i(A) \right\} , \right.
\left. \max_{i \in N} \left\{ a_{ii} + nL_i(A) - C_i(A) \right\} \right\}.
\]
The conclusion follows. \( \square \)

By the proof of Proposition 4.1, it is not difficult to see that the upper bound \( \rho_{0, 1}^{0, 1} \) is larger than \( \rho_{0, 1}^{[0, 1]} \) in Theorem 3.1, but \( \rho_{0, 1}^{[0, 1]} \) depends only on the entries of a stochastic matrix. Moreover, \( \rho_{0, 1}^{0, 1} \leq \rho_L \) and \( \rho_{0, 1}^{0, 1} \leq \rho_L \), which are given as follows.

**Proposition 4.2.** Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) be a stochastic matrix. Then
\[ \rho_{0, 1}^{0, 1} \leq \min\{\rho_L, \rho_T\}, \]
where \( \rho_T, \rho_L, \) and \( \rho_{0, 1}^{0, 1} \) are defined in Theorem 3.2 and Proposition 4.1, respectively.

**Proof.** By the proof of Proposition 4.1, we have that \( \rho_{0, 1}^{0, 1} \) is equivalent to the last of Inequality (10), that is,
\[
\rho_{0, 1}^{0, 1} = \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\},
\]
\[ \leq \max_{i \in N} \min_{\alpha \in [0, 1]} \left\{ a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \right\}. \]

Also let
\[ f(\alpha) = a_{ii} - (n - 2) l_i(A) + C_i(A) + \alpha_i \Delta_i(A), \quad \alpha \in [0, 1]. \]
Then when \( \Delta_i(A) \geq 0 \), \( f(\alpha) \) is a monotonically increasing function of \( \alpha \), and when \( \Delta_i(A) < 0 \), \( f(\alpha) \) is a monotonically decreasing function of \( \alpha \).

For the case that \( L_i(A) = \max_{i \in \mathbb{N}} a_{ii} > l_i(A) = \min_{i \in \mathbb{N}} a_{ii}, i \in \mathbb{N} \), we will prove \( \rho^{0.1} < \rho_L \). Note that

\[
 f(0) = a_{ii} - (n - 2)l_i(A) + C_i(A), \quad \text{and} \quad f(1) = a_{ii} + nL_i(A) - C_i(A).
\]

Since \( f(\alpha) \) is increasing when \( \Delta_i(A) \geq 0 \), we have

\[
 \max_{i \in \mathbb{N}^+(A)} \{ a_{ii} - (n - 2)l_i(A) + C_i(A) \} = \max_{i \in \mathbb{N}^+(A)} \min_{\alpha, \epsilon [0, 1]} \{ a_{ii} - (n - 2)l_i(A) + C_i(A) + \alpha \Delta_i(A) \} \\
\leq \max_{i \in \mathbb{N}^+(A)} \{ a_{ii} - (n - 2)l_i(A) + C_i(A) + \Delta_i(A) \} \\
= \max_{i \in \mathbb{N}^+(A)} \{ a_{ii} + nL_i(A) - C_i(A) \},
\]

which implies

\[
 \rho^{0.1} \leq \max_{i \in \mathbb{N}} \{ a_{ii} + nL_i(A) - C_i(A) \} = \rho_L.
\]

Then similarly as in the proof of \( \rho^{0.1} \leq \rho_L \), we can obtain easily \( \rho^{0.1} \leq \rho_L \).

For the case that \( L_i(A) = l_i(A) \) for some \( i \in \mathbb{N} \), we have \( \Delta_i(A) = 0 \) and

\[
a_{ii} + nL_i(A) - C_i(A) = a_{ii} + (n - 2)v_i(A) + 2L_i(A) - C_i(A) \\
= \sum_{j=1}^{n} |a_{ji} - q_i(A)| \\
= a_{ii} - (n - 2)v_i(A) - 2l_i(A) + C_i(A) \\
= a_{ii} - (n - 2)l_i(A) + C_i(A).
\]

Similarly as in the case \( L_i(A) > l_i(A) \), \( i \in \mathbb{N} \), we can also obtain easily

\[
 \rho^{0.1} \leq \rho_L \quad \text{and} \quad \rho^{0.1} \leq \rho_L.
\]

The conclusion follows. \( \Box \)

By propositions 4.1 and 4.2, we know that the optimal values of \( \alpha_i, i \in \mathbb{N} \) for the bound

\[
 \max_{i \in \mathbb{N}} \min_{\alpha_i \epsilon [0, 1]} \{ a_{ii} - (n - 2)l_i(A) + C_i(A) + \alpha_i \Delta_i(A) \},
\]

which could be obtained by using the set \( I^{\text{stoL}^+}(A) \) in Theorem 2.2, are \( \alpha_i = 0 \) for \( i \in \mathbb{N}^+(A) \) and \( \alpha_i = 1 \) for \( i \in \mathbb{N}^-(A) \) such that

\[
 \rho^{0.1} = \max \left\{ \max_{i \in \mathbb{N}^+(A)} \{ a_{ii} - (n - 2)l_i(A) + C_i(A) \}, \max_{i \in \mathbb{N}^-(A)} \{ a_{ii} + nL_i(A) - C_i(A) \} \right\}.
\]

is less than or equal to the bounds obtained by using the sets in Theorem 1.3 and 1.4 respectively. This provides a choice of \( \alpha_i, i \in \mathbb{N} \) for the set \( I^{\text{stoL}^+}(A) \) to localize all eigenvalues different from 1 of a stochastic matrix.

For a stochastic matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) and

\[
d = L^{\alpha_i}(A) = [L_{i}^{\alpha_i}(A), L_2^{\alpha_i}(A), \ldots, L_{n}^{\alpha_i}(A)]^T
\]

defined as (1), we take \( \alpha_i = 0 \) for \( i \in \mathbb{N}^+(A) \) and \( \alpha_i = 1 \) for \( i \in \mathbb{N}^-(A) \), that is,

\[
 L_{i}^{\alpha_i}(A) = \begin{cases} 
 L_0^i(A) = l_i(A), & i \in \mathbb{N}^+(A), \\
 L_1^i(A) = l_i(A), & i \in \mathbb{N}^-(A).
\end{cases}
\]

For this choice, the set \( I^{\text{stoL}^+}(A) \) reduces to

\[
 I^{\text{stoL}^+}(A) := \left( \bigcup_{i \in \mathbb{N}^+(A)} I_i^{\text{stoL}^+}(A) \right) \cup \left( \bigcup_{i \in \mathbb{N}^-(A)} I_i^{\text{stoL}^+}(A) \right),
\]

where \( I_i^{\text{stoL}^+}(A) = I_i^{\text{stoL}}(A) \) and \( I_i^{\text{stoL}^+}(A) = I_i^{\text{stoL}}(A) \). Hence, we have the following result.
Theorem 4.3. Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a stochastic matrix. If $\lambda \in \sigma(A)\backslash \{1\}$, then

$$
\lambda \in \Gamma_{\text{stol}^{0.1}}(A) = \left( \bigcup_{i \in N^+(A)} \Gamma_i^{\text{stol}}(A) \right) \cup \left( \bigcup_{i \in N^-(A)} \Gamma_i^{\text{stol}}(A) \right).
$$

Example 4.4. Consider the stochastic matrix

$$
A = \begin{bmatrix}
0.2656 & 0.0471 & 0.1452 & 0.0758 & 0.2199 & 0.2463 \\
0.2634 & 0.3368 & 0.0475 & 0.1143 & 0.1354 & 0.1026 \\
0.0591 & 0.2002 & 0.1831 & 0.1916 & 0.1814 & 0.1846 \\
0.2699 & 0.2753 & 0.1655 & 0.1941 & 0.0788 & 0.0165 \\
0.1443 & 0.0598 & 0.1205 & 0.2582 & 0.2839 & 0.1332 \\
0.2355 & 0.1027 & 0.1399 & 0.2358 & 0.2111 & 0.0750
\end{bmatrix},
$$

By computations, we have that $N^+(A) = \{2, 4, 6\}$, $N^-(A) = \{1, 3, 5\}$, and

$$
\Gamma_{\text{stol}^{0.1}}(A) = \Gamma_2^{\text{stol}}(A) \cup \Gamma_4^{\text{stol}}(A) \cup \Gamma_6^{\text{stol}}(A) \cup \Gamma_4^{\text{stol}}(A) \cup \Gamma_3^{\text{stol}}(A) \cup \Gamma_5^{\text{stol}}(A).
$$

By drawing the sets $\Gamma_{\text{stol}}(A)$, $\Gamma_{\text{stol}^{0.1}}(A)$ and $\Gamma_{\text{stol}^{0.1}}(A)$ in the complex plane (see Figure 3), it is not difficult to see that for any $\lambda \in \sigma(A)\backslash \{1\}$,

$$
\lambda \in \Gamma_{\text{stol}^{0.1}}(A),
$$

and that although $\Gamma_{\text{stol}^{0.1}}(A) \subseteq \Gamma_{\text{stol}}(A)$ and $\Gamma_{\text{stol}}(A) \subseteq \Gamma_{\text{stol}^{0.1}}(A)$, the set $\Gamma_{\text{stol}^{0.1}}(A)$ is better than $\Gamma_{\text{stol}}(A)$ and $\Gamma_{\text{stol}}(A)$ for estimating the moduli of subdominant eigenvalues.

Fig. 3. $\Gamma_{\text{stol}}(A)$, $\Gamma_{\text{stol}^{0.1}}(A)$ and $\Gamma_{\text{stol}^{0.1}}(A)$

5 Conclusions

In this paper, a set with $n$ parameters in $[0, 1]$ is given to localize all eigenvalues different from 1 for a stochastic matrix $A$, that is,

$$
\sigma(A)\backslash \{1\} \subseteq \Gamma_{\text{stol}^{\alpha}}(A), \text{ for any } \alpha_i \in [0, 1], i \in N.
$$
In particular, when $\alpha_i = 0$ for each $i \in N$, $I_{\text{stol}}^{\alpha}(A)$ reduces to the set $I_{\text{stol}}(A)$, which consists of $n$ sets $I_i^{\text{stol}}(A)$, and when $\alpha_i = 1$ for each $i \in N$, $I_{\text{stol}}^{\alpha}(A)$ reduces to the set $I_{\text{stol}}(A)$, which consists of $n$ sets $I_i^{\text{stol}}(A)$. The sets $I_{\text{stol}}(A)$ and $I_{\text{stol}}(A)$ are used to estimate the moduli of subdominant eigenvalues, that is, for any $\lambda \in \sigma(A) \setminus \{1\}$,

$$|\lambda| \leq \rho_l = \max_{i \in N} \{a_{ii} - (n - 2)I_i(A) + C_i(A)\}$$

and

$$|\lambda| \leq \rho_L = \max_{i \in N} \{a_{ii} + nL_i(A) - C_i(A)\}.$$ 

Moreover, by taking $\alpha_i = 0$ for each $i \in N^+(A)$ and $\alpha_i = 1$ for each $i \in N^-(A)$, we give a set $I_{\text{stol}}^{0.1}(A)$, which consists of $|N^+(A)|$ sets $I_i^{\text{stol}}(A)$ and $|N^-(A)|$ sets $I_i^{\text{stol}}(A)$ where $|N^+(A)| + |N^-(A)| = n$. By using $I_{\text{stol}}^{0.1}(A)$, we can get an upper bound for the moduli of subdominant eigenvalues which is better than $\rho_l$ and $\rho_L$, i.e., for any $\lambda \in \sigma(A) \setminus \{1\}$,

$$|\lambda| \leq \rho^{0.1} \leq \min\{\rho_l, \rho_L\},$$

where

$$\rho^{0.1} = \max\left\{ \max_{i \in N^+(A)} \{a_{ii} - (n - 2)I_i(A) + C_i(A)\}, \max_{i \in N^-(A)} \{a_{ii} + nL_i(A) - C_i(A)\} \right\}.$$ 

Acknowledgement: The authors are grateful to the referees for their useful and constructive suggestions. This work is supported by National Natural Science Foundations of China (11601473) and CAS "Light of West China" Program.

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