Laser Cooling of two trapped ions: Sideband cooling beyond the Lamb–Dicke limit.

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(November 4, 2021)

We study laser cooling of two ions that are trapped in a harmonic potential and interact by Coulomb repulsion. Sideband cooling in the Lamb–Dicke regime is shown to work analogously to sideband cooling of a single ion. Outside the Lamb–Dicke regime, the incommensurable frequencies of the two vibrational modes result in a quasi-continuous energy spectrum that significantly alters the cooling dynamics. The cooling time decreases nonlinearly with the linewidth of the cooling transition, and the effect of trapping states which may slow down the cooling is considerably reduced. We show that cooling to the ground state is possible also outside the Lamb-Dicke regime. We develop the model and use Quantum Monte Carlo calculations for specific examples. We show that a rate equation treatment is a good approximation in all cases.

I. INTRODUCTION

The emergence of schemes that utilize trapped ions or atoms for quantum information, and the interest in quantum statistics of ultra cold atoms, have provided renewed interest and applications for laser cooling techniques [1]. The present goal is to laser cool several atoms to a pure quantum state (to the motional ground state), and experimental and theoretical efforts are made in this direction. The cooling of a large number of particles using lasers is a prerequisite for coherent control of atomic systems [2,3]. In quantum information, for example, laser cooling to the motional ground state is a fundamental step in the preparation of trapped ions for quantum logic [4]. Coherent control and manipulation of information requires that each ion be individually addressable with a laser, and thus restricts the choice of the trap frequency, and consequently the regime in which cooling must work, to relatively shallow traps.

Laser cooling of single ions in traps has been extensively studied [5] and in particular sideband cooling has been demonstrated to be a successful technique for cooling single ions to the ground state of a harmonic trap [6]. Sufficient conditions for sideband cooling a two level system are: (i) the radiative linewidth $\gamma$ is smaller than the trap frequency $\nu$, such that motional sidebands, i.e. optical transitions that involve the creation or annihilation of a specific number of motional quanta, can be selectively excited; (ii) the Lamb–Dicke limit is fulfilled, i.e. the ion’s motional excursion is much smaller than the laser wavelength. The first condition can be achieved through an adequate choice of atomic transition or a manipulation of the internal atomic structure [7], while the Lamb–Dicke regime requires the trap frequency to be much larger than the recoil frequency of the optical transition.

For more than one ion, as required in quantum logic schemes, individual addressing imposes small trap frequencies, whereas sideband cooling imposes high trap frequencies. Furthermore, the Coulomb interaction between the particles makes the problem much more complex, and it is not obvious whether the techniques developed for single ions can be transferred directly to this situation. Experimentally, sideband cooling of two ions to the ground state has been achieved in a Paul trap that operates in the Lamb–Dicke limit [8]. However in this experiment the Lamb–Dicke regime required such a high trap frequency that the distance between the ions does not allow their individual addressing with a laser. For this purpose, and also for an extension beyond two ions, linear ion traps are the most suitable systems, and for their physical parameters laser cooling to the ground state is a goal yet to be achieved. Laser cooling of two ions into the ground state is the problem that we address in this paper.

Theoretical studies on cooling of single ions outside the Lamb–Dicke regime exist [9,10], while laser cooling of more than one ion has been analyzed only in the Lamb–Dicke regime [11,12]. In this paper, we investigate laser cooling, as developed for single ions, when it is applied to two or more ions. We focus our attention to sideband cooling, showing and discussing new physical effects which arise because of the presence of two interacting particles. Doppler cooling will be discussed in a future work. We show that cooling of two ions outside the Lamb–Dicke regime presents novel features with respect to single ion cooling, and we show how the preparation of the two ions in a pure quantum state is possible. The results will allow us to get some insight into the more general...
problem of cooling a string of $N$ ions. This is not only relevant for quantum logic with $N$–ion strings but also for laser cooling of ion clusters in Paul and Penning traps.

The paper is organized as follows. In section II we introduce and discuss the model which we will use throughout the paper, and discuss some concepts developed in sideband cooling of one ion in relation to the presence of more than one ion. In section III we study and discuss sideband cooling of two ions inside and outside the Lamb–Dicke regime, and compare the two different behaviours. Finally in the conclusions we summarize the main results, and discuss the problem of cooling $N > 2$ ions.

II. MODEL

We consider two ions of mass $m$ and charge $e$ placed in a one–dimensional harmonic potential of frequency $\nu$. We assume the ions to be strongly trapped in the other spatial dimensions so that their motion in those directions is frozen out. Their internal structure is described by a two level system with ground state $|g\rangle$, excited state $|e\rangle$ and resonance frequency $\omega_0$. The ions interact with laser light at frequency $\omega_L$ and wavevector $k$. For classical laser light and in the Rotating Wave Approximation the Hamiltonian of the system is:

$$H = H_i + H_{\text{mec}} + V. \quad (1)$$

Here $H_i$ is the internal energy in the rotating frame

$$H_i = -\delta \sum_{j=1,2} |e\rangle_j \langle e| \quad (2)$$

where $\delta = \omega_L - \omega_0$ is the detuning, $j$ labels the ion ($j = 1, 2$) and we have taken $\hbar = 1$. $H_{\text{mec}}$ is the mechanical Hamiltonian,

$$H_{\text{mec}} = \frac{p_i^2}{2m} + \frac{p_j^2}{2m} + \frac{1}{2} m \nu^2 x_1^2 + \frac{1}{2} m \nu^2 x_2^2 + \frac{e^2}{4\pi\epsilon_0 |x_1 - x_2|} \quad (3)$$

with $x_j$, $p_j$ position and momentum of the $j$th ion ($j = 1, 2$), and $V$ describes the interaction between laser and atoms,

$$V = \sum_{j=1,2} \frac{\Omega(x_j)}{2} |\sigma_j^\dagger \sigma_j - \nu \cos \theta x_j + \text{h.c.}|. \quad (4)$$

Here $\Omega(x_j)$ is the Rabi frequency at the position $x_j$, $\sigma_j^\dagger$, $\sigma_j$ are the raising and lowering dipole operators respectively defined on the $j$th ion ($j = 1, 2$), and $\theta$ is the angle between the laser wavevector and the trap axis.

Using the center of mass (COM) and relative coordinates, the mechanical Hamiltonian in (3) is composed of two separate terms: one for the COM motion which describes a particle of mass $M = 2m$ interacting with a harmonic potential of frequency $\nu$; the other for the relative motion which describes a particle of mass $\mu = m/2$ interacting with a potential, which is the sum of a harmonic potential of frequency $\nu$ and a Coulomb-type central potential. This potential may be approximated by a harmonic oscillator potential of frequency $\nu_r = \sqrt{3} \nu$, obtained through the truncation at the second order of its Taylor expansion around the equilibrium distance $x_0 = (2e^2/4\pi\epsilon_0 M \nu)^{1/2}$ between the ions. In appendix A we discuss this approximation and we show that it is valid in the regime that we are going to study. With this approximation the term (3) becomes (apart for a constant)

$$H_{\text{mec}} = \frac{p_x^2}{2M} + \frac{1}{2} M \nu^2 X^2 + \frac{p_y^2}{2\mu} + \frac{1}{2} \mu \nu_r^2 x^2, \quad (5)$$

where $X = (x_1 + x_2)/2$, $P = p_1 + p_2$ are position and momentum of the COM respectively, and $x = x_1 - x_2 - x_0$, $p = (p_1 - p_2)/2$ are position and momentum of the relative motion. Thus the term (3), apart for a constant, can be rewritten as

$$H_{\text{mec}} = \nu \sigma_0^\dagger \sigma_0 + \nu_r \sigma_r^\dagger \sigma_r, \quad (6)$$

where we have defined $X = \sqrt{1/2M}(a_0^\dagger + a_0)$, $P = i \sqrt{M \nu_0/2}(a_0^\dagger - a_0)$, $x = x_0 + \sqrt{1/2\mu \nu_r}(a_r^\dagger + a_r)$, $p = i \sqrt{\mu \nu_r/2}(a_r^\dagger - a_r)$, with $a_0$, $a_0^\dagger$ the annihilation and creation operators for the COM mode.
respectively, and \( a_\nu, a_\nu^\dagger \) the corresponding ones for the relative motion (stretch) mode. We stress that in this new representation the mechanical problem of two ions interacting through Coulomb forces is reduced to the one of two harmonic oscillators, while the interaction of each ion with the radiation is now transformed into a nonlinear coupling between the harmonic oscillators. In general, \( N \) ions in a trap can be described by a set of \( N \) harmonic oscillators, coupled by laser light \( [7] \). The master equation for the density matrix \( \rho \) of the two ion system is \( [3] \)

\[
\frac{d}{dt}\rho = -\frac{i}{\hbar}[H, \rho] + L\rho.
\]

Here \( L \) is the Liouvillian describing the incoherent evolution of the system:

\[
L\rho = \frac{\gamma}{2} \sum_{j=1,2} [2\sigma_j \rho \sigma_j^\dagger - \sigma_j^\dagger \sigma_j \rho - \rho \sigma_j^\dagger \sigma_j],
\]

where \( \gamma \) is the decay rate out of the internal excited state \( |e\rangle \), and \( \rho_j \) describes the density matrix after a spontaneous emission for the \( j \)th ion:

\[
\rho_j = \int_{-1}^{1} duN(u)e^{ikux_j} \rho e^{-ikux_j}
\]

with \( N(u) \) being the dipole pattern for the decay. In this treatment we have neglected both dipole–dipole interaction between the ions and quantum statistical properties. This approximation is justified in the regime that we investigate, which is the Lamb–Dicke regime, corresponding to the condition \( \eta \ll 1 \), where \( \eta \) is:

\[
\eta = k\sqrt{\frac{1}{2m\nu}} = \sqrt{\frac{\omega_{\text{rec}}}{\nu}}.
\]

where \( \omega_{\text{rec}} = k^2/2m \) is the recoil frequency. The parameter \( \eta \) appears in the kick operator \( \exp(ikx) \) in the term describing the exchange of momentum between radiation and atoms, which using the relation \( x = \sqrt{1/2m\nu}(a^\dagger + a) \) and the definition \( [11] \) is rewritten as \( \exp(ikx) = \exp(i\eta(a^\dagger + a)) \). The Lamb–Dicke regime corresponds to the condition \( \sqrt{\nu\eta} \ll 1 \), with \( n \) vibrational number; in other words, to the situation in which, during a spontaneous emission, a change in the vibrational
number of the atomic state is unlikely due to energy conservation. In this regime the kick operator may be expanded in powers of $\eta$, and with good approximation the expansion may be truncated at the first order. Another important parameter, as known from cooling of single ions, is the ratio between the radiative linewidth $\gamma$ and the trap frequency $\nu$. In the so-called strong confinement regime $\gamma/\nu \ll 1$ the laser can selectively excite sidebands of the optical transition which involve a well-defined change of the vibrational number $n$. In this regime, together with the Lamb-Dicke regime, sideband cooling works efficiently: when the laser is red detuned with $\delta = -\nu$ the system is cooled by approximately one phonon of energy $\nu$ in each fluorescence cycle, finally reaching the vibrational ground state $n = 0$. In contrast, in the weak confinement regime $\gamma/\nu \geq 1$, transitions which involve different changes of the vibrational number $n$ are excited simultaneously. This is the Doppler cooling regime, where the achievable minimum energy for a single ion is approximately $\gamma/2$ for a detuning $\delta = -\gamma/2$.

Having introduced these basic concepts and methods of laser cooling of single ions in traps, we now turn back to the problem of two ions, to discuss how those techniques may be applied, and whether the same concepts are still valid. For $N$ harmonic oscillator modes we can define a Lamb-Dicke parameter for each mode in an analogous way to (11). For our case, $N = 2$, the Lamb-Dicke parameters $\eta_0$ for the COM mode and $\eta_v$ for the stretch mode are defined as:

$$
\eta_0 = k \frac{\hbar}{2N\nu} \approx \frac{\eta}{\sqrt{2}},
$$

$$
\eta_v = k \frac{\hbar}{2\nu \eta_v} = \frac{\eta}{\sqrt{2} \sqrt{3}}.
$$

so that the kick operator for the $j$th ion ($j = 1, 2$) is written as:

$$
e^{i k x_j} = e^{i \eta_0 (a_0^+ + a_0)} e^{(-1)^{j-1} \eta_v (a_v^+ + a_v)}. \tag{13}
$$

In general for $N$ ions $\eta_0 = \eta/\sqrt{N}$, $\eta_v$. In the following, when we refer to the Lamb-Dicke regime we will consider the situation where the conditions $\sqrt{N} \eta_0 \ll 1$, $\sqrt{N} \eta_v \ll 1$ are fulfilled. From (13) we see that in Eq. (1) the Lamb-Dicke parameters appear multiplied by the factor $\cos \theta$. Therefore the Lamb-Dicke parameters for the coherent excitation, $\eta \cos \theta$, are always less than or equal to the Lamb-Dicke parameters defined in (12), which characterize the spontaneous emission. To discuss the importance of the ratio $\gamma/\nu$ in the case of two ions, we first consider the bare spectrum of energies of our system with frequencies $\nu$ and $\sqrt{3} \nu$. In Fig. 1 we plot the density of states $D(E)$ vs. the energy $E$, defined as the number of states in the interval of energy $[E, E + \delta E]$. From this figure we see that due to the incommensurate character of the frequencies the spectrum does not exhibit a well-distinguished series of energy levels, rather it tends to a quasi-continuum. Therefore, the strong confinement requirement for sideband cooling needs to be reconsidered. The main question which we will address in the following is whether it is still possible to cool one mode to the ground state by means of sideband cooling. As we will show, the Lamb-Dicke parameter distinguishes two regimes which exhibit dramatic differences.

### III. SIDEBAND COOLING OF TWO IONS

In the following we study sideband cooling of two ions, first in the Lamb-Dicke regime and then outside of this regime. We will show that in this latter case two-ionic effects appear due to the dense spectrum of energy levels. In our calculations we first consider sideband cooling when laser light excites only one of the two ions directly. Thus in (1) we take $\Omega(x_1) = \Omega$, and $\Omega(x_2) = 0$. Afterwards we compare this case to the one in which both are driven by light, i.e. $\Omega(x_1) = \Omega(x_2) = \Omega$, showing that the only difference between the two cases is the cooling time, which in the latter case scales by a factor $1/2$. In the following we assume the laser wavevector parallel to the trap axis, i.e. $\cos \theta = 1$. This assumption facilitates the analysis and it is justified by the simple scaling just described. Furthermore it corresponds to the case in which the two ions are two different ionic isotopes, of which one is driven by light (15). At the end of this section we will briefly discuss cooling of two identical ions when the wavevector is not parallel to the trap axis.
A. Lamb–Dicke regime.

In the Lamb–Dicke regime the Franck–Condon coefficients $\langle n | \exp(ikx) | l \rangle$ in the numerators of the right hand side terms of Eq. (14) may be expanded in terms of the Lamb–Dicke parameters $\eta_0$, $\eta_r$. The response of the system to laser light is governed by its spectrum of resonances $I(\delta)$, which is evaluated by summing all contributions to laser–excited transitions at frequency $\omega_L$,

$$I(\delta) = \sum_{(n-l):\nu=\delta} |\langle n | \exp(ikx) | l \rangle|^2 P(n)$$

where $P(n)$ is a normalized distribution of the states $|n\rangle$. In the Lamb-Dicke regime, we find that $I(\delta)$ exhibits two main pairs of sidebands around the optical frequency $\omega_0$: one at frequencies $\omega_0 \pm \nu_0$ corresponding to the transition $n_0 \rightarrow n_0 \pm 1$; the other at frequencies $\omega_0 \pm \nu_r$ corresponding to $n_r \rightarrow n_r \pm 1$ (see Fig. 2(a)). The strength of these sidebands relative to the carrier $n \rightarrow n$ is proportional to $\eta_0^2$ and to $\eta_r^2$, respectively. All the other sidebands have strengths of higher orders in $\eta_0^2$, $\eta_r^2$. This implies that by selecting one of these four sidebands by laser excitation we will induce the corresponding phononic transition; for example by choosing the sideband corresponding to $n_0, n_r \rightarrow n_0 - 1, n_r$ we can cool the COM mode to its vibrational ground state, as for a single ion. This has been experimentally demonstrated by the NIST group at Boulder [10]. In Fig. 2 we plot the results of a Quantum Monte Carlo wavefunction simulation (QMC) of Eq. (7) for two ions in a trap with Lamb–Dicke parameter $\eta_0 = 0.1$, radiative linewidth $\gamma = 0.2 \nu$, detuning $\delta = -\nu$ and an initially flat distribution for the states with energy $E \leq 15 \nu$. In Fig. 2(b) the average vibrational numbers of the COM mode (solid line) and of the stretch mode (dashed line) are plotted as a function of time in unit of fluorescence cycles $t_F = 2 \gamma/\Omega^2$. The system behaves as if the two modes were decoupled, since only one mode is cooled while the other remains almost frozen. Nevertheless the stretch mode is cooled on a much longer time scale, as an effect of off–resonant excitation. In Fig. 2(c) the populations of the vibrational states of the two modes are plotted at time $t = 600t_F$, showing the COM mode in the ground state and the nearly uncooled stretch mode. In this limit we can neglect the coupling of the population to the coherences in Eq. (10) as defined in Eq. (14) for two ions in a harmonic trap with $\eta_0 = 0.6$. As a further proof that the two modes can be considered decoupled during the time in which the COM motion is cooled, in Fig. 3(b) we compare the time dependence of the average vibrational number of the COM mode with the one of a single trapped ion which is cooled under the same Lamb–Dicke parameter, radiative linewidth, trap frequency, Rabi frequency and initial distribution as the COM mode. We see that the two curves overlap appreciably, justifying the picture of sideband cooling of two ions in the Lamb–Dicke regime as if the modes were decoupled from one another.

B. Outside the Lamb–Dicke regime

To illustrate the physical features of the system outside the Lamb–Dicke regime, we plot in Fig. 4(a) the spectrum of resonances $I(\delta)$ as defined in Eq. (14) for two ions in a harmonic trap with $\eta_0 = 0.6$. We see that the spectrum exhibits many sidebands whose density increases as the detuning increases. The main consequence is that we cannot select a given sideband by choosing the laser frequency, but rather excite a group of resonances that correspond to transitions to a set of quasi–degenerate states. The range of transitions that are excited increases with $\gamma$. In figure 4(b),(c) we consider sideband cooling for Lamb–Dicke parameter $\eta_0 = 0.6$ and detuning $\delta = -2\nu$. Where
the other parameters are the same as in Fig. 2(b),(c). As one can see, the two modes are coupled and cooled together. Thus, as a first big difference with respect to single ion cooling, we see that here the energy is not taken away from one mode only, rather it is subtracted from the system as a whole. Another striking difference appears in the cooling time, which is significantly longer in comparison with the time necessary to cool one single ion outside the Lamb–Dicke regime [13]. This slowing down is partly due to the increase of the dimension of the phase space where the cooling takes place: the presence of two modes makes the problem analogous to cooling in a two dimensional trap, whose axis are coupled by the laser. The ions thus make a random walk in a larger phase space, and the cooling gets slower. However the cooling time is even considerably longer than one would expect taking the dimensionality into account. This can be explained by looking again at the spectrum of resonances in Fig. 4(a): despite the high density of resonances, the coupling between the states is still governed by the Franck–Condon coupling, i.e. by the terms in the numerator of Eq. (14) which outside the Lamb–Dicke limit oscillate with the vibrational numbers of the states. In the limit of linewidth \( \gamma \ll \nu \), where a single sideband can be selected, we may encounter trapping states like in cooling of single ions [19], i.e. states whose coupling to the resonantly excited state is very small since their motional wavefunction after the absorption of a laser photon happens to have a very small overlap with the motional wavefunction of the excited states. This effect limits the cooling efficiency, since the atoms may remain trapped in these states and not be cooled further, or much more slowly, towards the ground state. For two ions the probability of finding zeroes of the Franck Condon coupling is larger than for one ion, as the coupling to the excited state is constituted by two integrals, one for the COM and the other for the relative motion wavefunctions. Thus the probability of having trapping states is higher. To illustrate this phenomenon we plot in Fig. 5(a) the occupation of the states \( P_n \) as a function of the COM and relative vibrational numbers \( n_0 \) and \( n_r \), respectively, at a time \( t = 1000t_F \) after sideband cooling of the COM with \( \delta = -2\nu \). Here \( \gamma = 0.02\nu \), and we are in the limit in which the single resonances are resolved. As a consequence the most likely coherent transitions are \( n_0 \rightarrow n_0 - 2, n_r \rightarrow n_r \). The effect of the trapping states is visible in the tail of occupied states of \( P_n \), with \( n_r = 6, 7 \). In Fig. 5(b),(c) we plot the modulus square of the Franck Condon coefficients for the relative motion corresponding to the coupling of the states \( n_r = 6, 7 \) to the other motional states respectively: here it is clearly shown that for the transition \( n_r = 6 \rightarrow 6, n_r = 7 \rightarrow 7 \), the coupling is reduced nearly to zero. As the linewidth \( \gamma \) increases, the number of states to which a single state is coupled increases. Thus the number of channels through which the atom may be cooled is larger. As an effect the trapping states disappear. This is shown in Fig. 6, where the population \( P_n \) is plotted for \( t = 600t_F \) and \( \gamma = \nu \), and otherwise the same parameters as before. Here we see that the system is cooled homogeneously. The effect of varying \( \gamma \) is summarized in Fig. 7, where we compare the average COM vibrational number vs. time in unit \( t_F \) for various values of \( \gamma \). The results of Fig. 7 show clearly that as the linewidth increases the number of fluorescence cycles needed for cooling the system decreases dramatically. It is important to note that in this diagram the time is measured in units of fluorescence cycles for each \( \gamma \), so that the absolute cooling time clearly reduces more strongly. We stress that this strong dependence on the linewidth is a two-ion effect. In contrast, in sideband cooling of single ions, the fluorescence time \( t_F \) determines the cooling time scale for \( \gamma / \nu \leq 1 \), and the curves for different values of \( \gamma \) vs. the time in unit of the respective \( t_F \) do not show striking differences.

The presence of trapping states and the coupling of each state to more than one state at almost the same transition frequency might lead to the formation of dark coherences between quasi-degenerate states, i.e. to superpositions of states which decouple from laser excitation because of quantum interference. However for the considered systems those dark states do not play any significant role. We prove this numerically in Fig. 8, where we plot the comparison between a QMC and a rate equation simulation. We see that there are no striking differences between the two curves. We point out that outside the Lamb–Dicke regime a rate equations treatment is not justified in principle, since secular approximation arguments and Lamb–Dicke limit arguments cannot be applied. Here the rate equations are used to highlight the effect of neglecting the coherences in the dynamical evolution of the cooling, while these coherences are fully accounted for in the QMC treatment. In order to see why coherences do not play any significant role in the cooling dynamics, we look at the definition of a dark state. Let us consider a state \( |\alpha\rangle \) at \( t = 0 \) defined for simplicity as linear superposition of two quasi-degenerate states \( |\alpha\rangle = a_1 |n\rangle + a_2 e^{i\phi} |m\rangle \), with \( a_1, a_2, \phi \) real coefficients and with \( n, m \) states almost degenerate in energy, so that \( (|n - m\rangle \cdot v) = \Delta E \) with \( \Delta E \leq \gamma \). In principle a dark state can be a linear superposition of any number of states. However as we will see from the arguments below our restriction to two states does not affect the generality of the result. The evolution \( |\alpha(t)\rangle \) in the Schroedinger picture, apart from a global phase factor, is written as

\[
|\alpha(t)\rangle = a_1 |n\rangle + a_2 e^{i\phi(t)} |m\rangle,
\]
with $\phi(t) = \phi_0 + \Delta E t$. The state is dark when the following condition is fulfilled:

$$\langle l | e^{ik(X+x/2)} | \alpha(t) \rangle \sim 0$$

for any state $|l\rangle$ belonging to the set of states $\{|l\rangle\}$ to which it is resonantly or almost resonantly coupled. If the condition (17) holds at $t = 0$, it will hold up to a time $t$ such that $\Delta E t \sim \pi/2$. For the system we are dealing with we do not have exact degeneracy, thus we check whether the state $|\alpha\rangle$ can remain dark for a time sufficiently long to affect the cooling dynamics appreciably. The smallest possible value of $\Delta E$ in the range of energies of our calculations is $\Delta E = 0.07\nu$, and we find that $\phi$ rotates by an angle $\pi/2$ in less than one fluorescence cycle for the values of $\gamma$ that we have considered. The dark coherences are then washed away during the evolution as an effect of the incommensurate frequencies between the two modes. This result together with the numerical results suggests that rate equations can be used in the study of cooling [2].

In order to highlight that the absence of dark coherences is a signature of the peculiar spectrum of the system we plot in Fig. 9 the cooling of one mode outside the Lamb–Dicke regime for the case of a discrete spectrum where we have exact degeneracy. More precisely we consider two harmonic oscillators with commensurate frequencies $\nu$ and $2\nu$, where all the other physical parameters are the same as before. In this case the different outcome between the QMC and the rate equations treatment is dramatic, giving evidence to the role of the coherences in the evolution.

C. Light on both ions

The calculations that we have shown refer to the case in which only one ion is illuminated. As we have seen, although light interacts with one ion it couples with both modes simultaneously, as shown in Eqs. (1) and (13). When both ions are excited by laser light, the system is described by a 4–level scheme, corresponding to the 4 internal states $|a_1, b_2\rangle$ with $a, b = e, g$, where we assume that when a photon is emitted, we detect from which ion the event has occurred, as a consequence of the spatial resolution of the ions. We expect that the effect on the cooling will be a doubling in the number of quantum jumps and hence of the cooling rate. This is shown in Fig. 10, where we compare the time dependence of the COM vibrational number for the cases in which only one ion is illuminated (dashed line) or both ions are illuminated with the same laser intensity (solid line with label 1). In the latter case cooling is visibly faster, and the time dependence scales with a factor of two with respect to the case with one ion illuminated, as we can see when we replot the solid line 1 vs. $t/2t_F$ (solid line with index 2). The two–ions effects found above are clearly independent of the number of scattering points, with the only difference that the dark state condition is now written as $\langle l \{ e^{ik(X+x/2)} + e^{ik(X-x/2)} \} | \alpha(t) \rangle \sim 0$. We point out again that effects due to interference between the internal excitation paths have been neglected, as we consider the ions to be two distinguishable particles [2].

We would like to stress that in the above calculations we have considered the case of only one ion driven by radiation while the laser wavevector is parallel to the trap axis. However if one wants to cool two identical ions by shining light on one of them the laser beam must necessarily be at a certain angle $\theta$ with respect to the ion string. Thus the Lamb–Dicke parameter characterizing the coherent excitation will be smaller than the Lamb–Dicke parameter for the spontaneous decay, and depending on the minimum angle $\theta$ required, the coherent laser will excite with some selectivity one of the two modes. However for cooling purposes it is preferable to have the two Lamb–Dicke parameters values, corresponding to the spontaneous emission and the coherent excitation, as close as possible.

IV. CONCLUSIONS

In this paper we have studied the question of cooling two ions in a linear trap to the ground state by means of sideband cooling. We have studied sideband cooling in the Lamb–Dicke regime, and we have shown that in this limit the two harmonic oscillators can be considered decoupled when one of the two is cooled by means of sideband cooling. We have found that the cooling dynamics in the low intensity limit may be described by rate equations, and essentially that all the considerations developed for sideband cooling of single ions apply. This regime has been considered by Vogt et al. [3] for studying the effects of dipole–dipole interaction in laser cooling of two ions and by Javanainen
in [14] in his study on laser cooling of ion clusters.

We have then investigated laser cooling outside the Lamb–Dicke regime, finding striking differences with the Lamb–Dicke limit. Here the energy spectrum may be considered a quasi–continuum, though the coupling between the states is still governed by the Franck–Condon coupling. A consequence is that the cooling efficiency depends strongly on the radiative linewidth. For very small ratio \( \gamma/\nu \) the effect of trapping states is appreciable, and manifests itself in a drastic increase of the cooling time, i.e. the number of fluorescence cycles needed to cool the system. For \( \gamma/\nu < 1 \), but large enough, the effect of trapping states is washed away, the modes are cooled simultaneously, and the cooling time is considerably shorter and comparable to the time needed for cooling single ions outside of the Lamb–Dicke regime. These effects are all consequences of the density of states in the energy spectrum. A further property of the system is the absence of dark coherences, as a consequence of the incommensurate frequencies of the harmonic oscillators, i.e. of the absence of perfect degeneracy. This implies that in the low intensity limit rate equations provide still a good description of the cooling dynamics. Finally we have compared cooling when light is shone on one ion only or on both ions, finding a simple difference of a factor two in the rate of cooling.

On the basis of the obtained results we would like to comment on cooling of a string of \( N > 2 \) ions. \( N \) ions in a harmonic trap may be described by \( N \) harmonic oscillators. The mode frequencies \( \nu_0, \nu_1, \ldots, \nu_{N-1} \) are all incommensurate, and the number of states in the interval of energy \( \left[E, E + \delta E\right] \) is \( D(E) = E^{N-1}/\nu_0\nu_1\ldots\nu_{N-1}\delta E \). Outside the Lamb–Dicke regime the spectrum of resonances is then even more dense than in the two–ions case, and the probability of trapping states will be larger. However a suitable increase of the linewidth will cancel their effects, since each state will see an even more dense states, on the other hand for values of \( \gamma \) large enough the coherent effects will wash out because of the coupling to a “continuum” of states. From these considerations we expect laser cooling to the ground state to be still possible for \( N > 2 \) ions. Furthermore, one can cool a given set of modes to the ground state through the choice of the laser detuning outside the Lamb–Dicke regime. For example taking a detuning \( \delta = -\nu_k \) the modes with frequency \( \nu_j \geq \nu_k \) may be cooled to their vibrational ground states. It should be noticed that as the number of ions \( N \) increases, the Lamb–Dicke parameter of each mode decreases approximately as \( 1/\sqrt{N} \), allowing to reach the Lamb–Dicke regime also when this condition is not fulfilled for single ions. In this limit the modes may be considered decoupled and sideband cooling is particularly efficient.

As a last consideration, we note that the behaviour of a number of ions \( N \geq 3 \) cooled by light depends on which ions of the string are driven. In fact each position of the string couples with the different modes with amplitudes that depend on the position itself [17]. Only if all the ions are illuminated we may consider all the modes as coupled and cooled simultaneously. But this “coupling” changes as we select and drive only certain ions of the string. In this case a certain amount of modes may be cooled, while the others will remain hot or get cooled on a longer time scale. In this respect the system can be considered as having a reduced dimensionality, and the time of the cooling will be accordingly shorter with respect to the case in which the laser couples to all modes.

V. ACKNOWLEDGEMENTS

We would like to thank D. Leibfried, F. Schmidt–Kaler, H. Baldauf and W. Lange for many stimulating discussion. One of us (G.M.) whishes to thank S. Stenholm for stimulating discussions. This work was supported by the Austrian Fond zur Förderung der wissenschaftlichen Forschung and the TMR network ERBFMRX-CT96-0002.

APPENDIX A: HARMONIC APPROXIMATION

We consider the term [3] and rewrite it in COM and relative motion canonical variables:

\[
H_{\text{inc}} = \frac{P^2}{2M} + \frac{1}{2} M \nu^2 X^2 + \frac{P^2}{2 \mu} + \frac{1}{2} \nu^2 x^2 + \frac{e^2}{4 \pi \epsilon_0 |x|} = H(X, P) + H(x, p).
\]  
(A1)
with $\mu = m/2$ reduced mass, $M = 2m$ total mass. The mechanical problem is separable into center of mass motion and relative motion, where $H(X, P)$ describes the harmonic motion of a particle of mass $M$ interacting with a harmonic oscillator of frequency $\nu$, and $H(x, p)$ the motion of a particle of mass $\mu$ interacting with a potential $V(x) = \mu^2 x^2/2 + e^2/4\pi\epsilon_0|x|$, i.e. a harmonic potential of frequency $\nu$ and a central repulsive Coulomb–type potential.

We focus our attention on the potential $V(x)$, restricting its domain on the semiaxis $x > 0$. The equilibrium point is found to be $x_0 = (e^2/4\pi\epsilon_0\mu\nu^2)^{1/3}$. Expanding $V(x)$ around $x_0$ we find:

$$V(x) = \frac{3}{2} \left( \frac{e^2}{4\pi\epsilon_0\mu\nu^2} \right)^{\frac{1}{3}} + \frac{3}{2} \mu^2 (x - x_0)^2 + \sum_{n=3}^{\infty} \frac{e^2}{4\pi\epsilon_0 x_0^{n+1}} (x - x_0)^n$$

(A2)

$$= \frac{3}{2} \left( \frac{e^2}{4\pi\epsilon_0\mu\nu^2} \right)^{\frac{1}{3}} + \frac{1}{2} \mu^2 (x - x_0)^2 + A(x),$$

(A3)

where $\nu_r = \sqrt{3}\nu$ and $A(x)$ sum over the higher order terms, which we call the anharmonic terms.

We quantize the oscillation around the equilibrium position $x_0$:

$$x = x_0 + \sqrt{\frac{1}{2m\nu_r}} \left( a_r^+ + a_r \right),$$

(A4)

where $a_r$, $a_r^+$ are annihilation and creation operators respectively. From perturbation theory we may consider $V(x)$ harmonic when the following conditions are fulfilled:

$$\langle j|A(x)|j \rangle \ll \hbar \nu_r$$

$$\langle j|A(x)|j \pm 1 \rangle \ll \hbar \nu_r$$

(A5)

with $|j\rangle$ eigenstate of the harmonic oscillator of frequency $\nu_r$ with eigenvalue $j\hbar\nu_r$. The first condition means that the energy shift due to the anharmonic term is much smaller than the spectrum separation, whereas the second condition means that the coupling between the states is a small perturbation, and we will show that it may be neglected. The coupling between the state $j$ and the state $j + k$ is not taken here into account for simplicity, but it may be shown that it is much smaller than $k\hbar\nu_r$ in a similar way to the one we discuss below. Let us rewrite the relations in (A3) as:

$$\langle j|A(x)|j \rangle = \frac{e^2}{4\pi\epsilon_0 x_0} \sum_{m=2}^{\infty} \zeta_{j,2m}$$

$$\langle j|A(x)|j + 1 \rangle = -\frac{e^2}{4\pi\epsilon_0 x_0} \sum_{m=1}^{\infty} \zeta_{j,2m+1}$$

(A6)

with

$$\zeta_{j,2m} = \left( \frac{1}{x_0^{2}\mu\nu_r} \right)^{2m} \langle j| (a_r^+ + a_r)^{2m} |j \rangle.$$  

(A7)

From the following relation

$$\frac{(j+m)!}{j!} < \langle j| (a_r^+ + a_r)^{2m} |j \rangle < 2^{2m} \frac{(j+m)!}{j!},$$

$$\sqrt{j + m + 1} \sqrt{j + m + 1} < \langle j| (a_r^+ + a_r)^{2m+1} |j \pm 1 \rangle < 2^{2m+1} \frac{(j+m)!}{j!} \sqrt{j + m + 1}$$

(A8)

we find that

$$\frac{\zeta_{j,2m+1}}{\zeta_{j,2m}} \sim \left( 2 \frac{1}{x_0^{2}\mu\nu_r} \right)^{2m} \sqrt{j + m + 1}$$

(A9)

From (A9) we see that the series does not converge, as the term of the expansion depends on the term $m$. However for a certain interval corresponding to $j, m \leq M_0$ such that $\psi = \sqrt{1/x_0^{2}2\mu\nu_r}\sqrt{M_0} < 1$, each term is bounded by the corresponding term of a geometrical series with factor $\psi < 1$. For typical values of a linear ion trap $\psi \sim \sqrt{M_0}/100$, so that $M_0$ can assume very large values, $M_0 \ll 10^4$. 

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The divergence at orders \( m > N_0 \) is a signature of the divergence of the Coulomb potential at \( x = 0 \): such divergence requires the wavefunctions to be zero in \( x \leq 0 \), whereas the harmonic oscillator wavefunctions are different from zero on all the space, although their occupation on the semiaxis \( x \leq 0 \) is very small for \( \psi \ll 1 \). The divergence at \( x = 0 \) is a mathematical consequence, and does not correspond to the physical situation, in which the ions move in a three dimensional space, although the confinement in the other two dimensions is relatively tight. In principle one could build up a potential that does not diverge in a specific point of the \( x \) axis, whose behaviour for \( x > 0 \) tends to the Coulomb one. In this limit the series should converge for low number states \( j \). However such a detailed study is beyond the scope of this paper \[23\]. We restrict to the case in which \( \psi \ll 1 \), showing that in the chosen range the harmonic approximation is quite good.

From this consideration it is then sufficient to compare the third order term with the harmonic potential in order to show that the approximation is sensible. The relation to be satisfied is then

\[
\hbar \nu_r \gg \frac{e^2}{4\pi \epsilon_0 x_0} \left( \sqrt{\frac{1}{2JN}} \right)^3 j^{3/2},
\]

(A10)

with \( j \leq N_0 \), which poses a further condition on \( j, N_0 \). Manipulating the expression we find

\[
j \ll j_{\text{max}} \approx \left( \frac{\hbar \nu_r}{e^2/4\pi \epsilon_0 x_0} \right)^{2/3} \left( \frac{x_0}{\sqrt{\hbar/2JN}} \right)^2
\]

(A11)

which substantially agrees with the qualitative estimate in \[22\]. For \( j \ll j_{\text{max}} \) the potential can be considered harmonic. For linear traps \[6\] \( j_{\text{max}} \sim 120 \), and the harmonic approximation is valid for the region of energy we are considering. As an example the case \( j = 100 \) corresponds to a correction of the order of \( 10^{-3} \hbar \nu_r \).

**APPENDIX B: ADIABATIC ELIMINATION OF THE EXCITED STATE**

We rewrite (7) in the following way

\[
\frac{d}{dt} \rho = -i\hbar[H_{\text{eff}} \rho - \rho H_{\text{eff}}^\dagger] + J\rho,
\]

(B1)

where \( H_{\text{eff}} \) is the effective Hamiltonian

\[
H_{\text{eff}} = H_i + H_{\text{mech}} - i\frac{\gamma}{2} \sum_{i=1,2} \sigma_i^\dagger \sigma_i
\]

(B2)

and \( J\rho \) the jump operator

\[
J\rho = \gamma \sum_{i=1,2} \sigma_i^\dagger \rho \sigma_i
\]

(B3)

We introduce the Liouvillians:

\[
L_0 \rho = \frac{i}{\hbar} [H_{\text{eff}} \rho - \rho H_{\text{eff}}^\dagger]
\]

(B4)

\[
L_1 \rho = \frac{i}{\hbar} [V, \rho]
\]

\[
L_2 \rho = J\rho
\]

(B5)

so that (B1) can be rewritten as:

\[
\frac{d}{dt} \rho = [L_0 + L_1 + L_2] \rho
\]

(B6)

In the limit \( \Omega \ll \gamma \) we can eliminate the excited state in second order perturbation theory. Calling \( P \) the projector onto the internal ground state \( |g\rangle \), and using a standard derivation based on projectors \[24\], we obtain the following equation for the ground state of the system:

\[
\frac{d}{dt} P(t) = PL_0 P(t) + \int_0^t d\tau_1 PL_1 (1 - P) \exp (L_0 \tau_1) (1 - P) L_1 P(t - \tau_1)
\]

\[
+ \int_0^t d\tau_2 \int_0^\tau_2 d\tau_1 PL_2 (1 - P) \exp (L_0 \tau_2) (1 - P) L_1 \exp (L_0 (\tau_2 - \tau_1)) P(t - \tau_2)
\]

(B7)
where \( P \) is a projector so defined on a density operator \( \rho \). Markov approximation can be applied in the limit in which we may consider the coupling to the excited state to evolve at a higher rate with respect to the time scale which characterizes the ground state evolution. This is true once we have moved to the interaction picture with respect to the trap frequency. We define

\[
v_1(t) = e^{iH_{\text{mecl}}t} \rho(t) e^{-iH_{\text{mecl}}t}
\]

and in the interaction picture \([B7]\) the form:

\[
\frac{d}{dt} v_1(t) = e^{iH_{\text{mecl}}t} \left[ \int_0^t d\tau P L_1 \exp \left( L_0 \tau \right) L_1 e^{-iH_{\text{mecl}}(t-\tau)} v_1(t-\tau) e^{iH_{\text{mecl}}(t-\tau)} \right] e^{-iH_{\text{mecl}}t} + e^{iH_{\text{mecl}}t} \left[ \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 P L_2 \exp \left( L_0 \tau_1 \right) L_1 \exp \left( L_0 (\tau_2 - \tau_1) \right) C L_1 e^{-iH_{\text{mecl}}(t-\tau_2)} v_1(t-\tau_2) e^{iH_{\text{mecl}}(t-\tau_2)} \right] e^{-iH_{\text{mecl}}t}
\] (B8)

Next we can neglect the change of \( v_1 \) during the time \( \tau \) on which the excited state evolves. Going back to the original reference frame we have now the equation in Markov approximation:

\[
\frac{d}{dt} \rho(t) = \int_0^\infty d\tau P L_1 \exp \left( L_0 \tau \right) L_1 e^{iH_0 \tau} \rho(t) e^{-iH_0 \tau} + \int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 P L_2 \exp \left( L_0 \tau_1 \right) L_1 \exp \left( L_0 (\tau_2 - \tau_1) \right) \left[ e^{-iH_{\text{mecl}} \tau_2} \rho_1(t) e^{iH_{\text{mecl}} \tau_2} - \rho_2(t) e^{-iH_{\text{mecl}} \tau_2} \right]
\] (B9)

We substitute now in \([B10]\) the explicit form of the operators. After some algebra and application of the commutation rules we obtain the following equation (where we have neglected the interference terms between the two ions):

\[
\frac{d}{dt} \rho(t) = P L_0 \rho(t) - \sum_{i=1,2} \frac{\Omega^2}{4} \int_0^\infty d\tau \left[ e^{-i(\Delta+\gamma/2)\tau} \rho_1(t) e^{-iH_{\text{mecl}}} \right] e^{i(\Delta+\gamma/2)\tau} + \sum_{i=1,2} \frac{\Omega^2}{4} \int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 \int_{-1}^1 d\Omega \left( e^{-i\Delta}\rho_2(t) \right) e^{i\Omega}\rho_1(t) e^{-i\Omega}
\] (B10)

Projecting \([B11]\) on the basis of states \( \{|n\} \), we obtain Eq. \([B11]\).
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FIG. 1. Density of states \( D(E) \) plotted as a function of energy \( E \) in units of \( \nu \). The grid is \( \delta E = \nu / 3 \).

FIG. 2. (a) Spectrum of resonances for \( \eta_\nu = 0.1 \) for a thermal distribution with average energy per mode \( \bar{\nu} \nu = 7.5 \nu \), plotted on a grid of width \( \nu / 10 \). (b) Plot of \( \langle n_\nu \rangle \) (solid line) and \( \langle n_\nu \rangle \) (dashed line) vs. time in units \( t_F = 2 \gamma / \Omega^2 \), for \( \eta_\nu = 0.1 \), \( \gamma = 0.2 \nu \), \( \Omega = 0.034 \nu \), \( \delta = - \nu \), and atoms initially in a flat distribution on the states with energy \( E \leq 15 \nu \). (c) Population of COM mode \( P_{n_0} \) (onset) and of stretch mode \( P_{n_\nu} \) (inset) vs. the respective vibrational state number at \( t = 600 t_F \).

FIG. 3. (a) Comparison between rate equation (solid line) and QMC calculation (dashed line). Same parameters as in Fig. 2(b). (b) Comparison between the time dependence of the COM average vibrational number as in Fig. 2(b) (solid line) and the average vibrational number for the case in which a single ion is cooled (dashed line). For the single ion the mass has been rescaled so that \( \eta^{(1)} = \eta_\nu \), \( \nu^{(1)} = \nu \), \( \gamma = 0.2 \nu \), \( \Omega = 0.034 \nu \), with an initially flat distribution for the first 15 states.

FIG. 4. (a) Spectrum of resonances for \( \eta_\nu = 0.6 \) of ions in a thermal distribution with average energy per mode \( \bar{\nu} \nu = 7.5 \nu \). (b) Plot of \( \langle n_\nu \rangle \) (solid line) and \( \langle n_\nu \rangle \) (dashed line) vs. time in units \( t_F = 2 \gamma / \Omega^2 \), for \( \eta_\nu = 0.6 \), \( \gamma = 0.2 \nu \), \( \Omega = 0.034 \nu \), \( \delta = - 2 \nu \), and atoms initially in a flat distribution for states with energy \( E \leq 15 \nu \). (c) Population of COM (onset) and of relative motion (inset) vs. the respective vibrational number state at \( t = 600 t_F \).

FIG. 5. (a) Population \( P_\nu \) as a function of \( n_\nu \) and \( n_\nu \) at a time \( t = 1000 t_F \) and for \( \gamma = 0.02 \nu \), \( \Omega = 0.17 \gamma = 0.0034 \nu \), \( t_F = 2 \gamma / \Omega^2 = 3460 / \nu \). All the other physical parameters are the same as in Fig. 4(b),(c). (b),(c) Modulus square of the Franck Condon coefficients for the relative motion \( F_{l,n_\nu} = |\langle l| e^{i \nu_0 (a_\nu + a_\nu^\dagger)}|n_\nu \rangle|^2 \) with \( l = 6, 7 \).
FIG. 6. Population $P_n$ as a function of $n_0$ and $n_r$ at a time $t = 600t_F$ and for $\gamma = \nu$, $\Omega = 0.17\gamma = 0.17\nu$, $t_F = 2\gamma/\Omega^2 = 69/\nu$. All the other physical parameters are the same as in Fig. 4(b),(c).

FIG. 7. Time dependence of the average vibrational number of the COM mode for $\gamma = 0.02\nu$ (dashed line), $\gamma = 0.2\nu$ (solid line), $\gamma = 0.4\nu$ (dashed-dotted line) and $\gamma = \nu$ (dotted line), keeping constant the ratio $\Omega/\gamma = 0.17$. The time is in unit $t_F(\gamma) = 2\gamma/\Omega^2 \approx 70/\gamma$. All the other parameters are the same as in Fig. 4(b),(c).

FIG. 8. Comparison between rate equation (solid line) and QMC calculation (dashed line). Same physical parameters as in Fig. 4(b). The onset refers to the COM and the inset to the relative motion vibrational number.

FIG. 9. Plot of the average vibrational number vs. time for the harmonic oscillator of frequency $\nu$ coupled to a second one with frequency $2\nu$. Comparison between rate equation (solid line) and QMC calculation (dashed line). Lamb–Dicke parameter for the mode $\nu$ is $\eta_\nu = 0.6$, $\gamma = 0.2\nu$, $\Omega = 0.034\nu$, $\delta = -2\nu$, and atoms initially flat distributed on the states with energy $E \leq 15\nu$.

FIG. 10. Plot of the average vibrational number of the COM mode vs. the time in unit $t_F$ for the case in which both ions are illuminated (solid line, index 1) and only one ion is illuminated (dashed line). The solid line with index 2 corresponds to line 1 rescaled, where the time has been divided by a factor 2. The other parameters are the same as in Fig. 4(b),(c).
