Hadronic decays of the (pseudo-)scalar charmonium states $\eta_c$ and $\chi_{c0}$ in the extended Linear Sigma Model

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We study the phenomenology of the ground-state (pseudo-)scalar charmonia $\eta_c$ and $\chi_{c0}$ in the framework of a $U(4)_s \times U(4)_f$ symmetric linear sigma model with (pseudo-)scalar and (axial-)vector mesons. Based on previous results for the spectrum of charmonia and the spectrum and (OZI-dominant) strong decays of open charmed mesons, we extend the study of this model to OZI-suppressed charmonia decays. This includes decays into ‘ordinary’ mesons but also particularly interesting channels with scalar-isoscalar resonances $f_0(1370), f_0(1500), f_0(1710)$ that may include sizeable contributions from a scalar glueball. We study the variation of the corresponding decay widths assuming different mixings between glueball and quark-antiquark states. We also compute the decay width of the pseudoscalar $\eta_c$ into a pseudoscalar glueball. In general, our results for decay widths are in reasonable agreement with experimental data where available. Order of magnitude predictions for as yet unmeasured states and channels are potentially interesting for BESIII, Belle II, LHCb as well as the future PANDA experiment at the FAIR facility.

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I. INTRODUCTION

In recent years, the charm quark energy region has been the focus of many theoretical and experimental investigations [1]. In particular the plethora of recently discovered exotic states, i.e. states with quark content beyond $q\bar{q}$ and $qqq$ or states with quantum numbers not accounted for in non-relativistic quark models, have raised enormous interest. With the copious production of ‘ordinary’ charmonia in experiments such as BESIII, Belle (II) and LHCb, however, also their potentially rare decays into light hadrons gains interest. These transitions occur via gluon-rich processes and are therefore interesting with respect to the interplay and transition of perturbative and non-perturbative QCD. Consequently, theoretical work in the past has concentrated on the application and limits of perturbative QCD and the construction of non-perturbative models for the (OZI-suppressed) production of $q\bar{q}$-pairs in the decays, see e.g. [2, 3] and Refs. therein.

Exotic states, however not only occur in the charm quark region, but may be present already in the light quark sector. In particular the light scalar meson sector is discussed frequently, with a putative scalar isoscalar glueball in the energy region around 1.5 GeV. The decay of charmonia into pairs of these states may offer the interesting possibility to study the nature of these states using information from the gluon-rich decay processes.

In order to study these decays we employ the extended Linear Sigma Model (eLSM) [8, 9], an effective model describing the vacuum phenomenology of (pseudo-)scalar and (axial-)vector mesons in the cases of $N_f = 2$ [10–12] and $N_f = 3$ [13]. Recently, the framework has been generalised to $N_f = 4$ [14, 19] and applied to the calculation of masses of open and hidden charmed mesons as well as the decays of open charmed mesons in the low energy limit. Although the model is rooted in chiral symmetry and its breaking pattern in the light quark sector, these applications have been (perhaps unexpectedly) quite successful. In this work we extend these studies to also include the decays of hidden charmonia.

The construction of the eLSM is based on a global chiral symmetry $U(N_f)_s \times U(N_f)_f$ as well as the classical dilation symmetry. In the vacuum, global chiral symmetry is broken spontaneously by a non-vanishing expectation value of the quark condensate, anomalously by quantum effects, and explicitly by non-vanishing quark masses. Furthermore, the dilation symmetry is broken explicitly. As a consequence, besides the usual meson multiplets the eLSM also includes two glueballs composed of two gluons each: a scalar one (denoted as $G$) [24–26] and a pseudoscalar one (denoted as $\tilde{G}$). The identification of the scalar glueballs with experimentally observed states is notoriously complicated, see e.g. [24–32] for discussions. In general, there is a mixing between states: the non-strange $\sigma_N \equiv (u\bar{u} + d\bar{d})/\sqrt{2}$, hidden-strange $\sigma_S \equiv s\bar{s}$, and scalar glueball $G \equiv q\bar{q}$ [4, 6, 7]. Within the eLSM this three-body mixing in the scalar-isoscalar channel was resolved in Ref. [12, 33] and generated the physical resonances $f_0(1370), f_0(1500)$ and $f_0(1710)$. Of these, the $f_0(1710)$ has the largest overlap with the scalar glueball. Its counterpart in the pseudoscalar sector has been studied within the eLSM in Refs. [34, 35]. In the charm sector, the eLSM contains so far four charmonium states, which are (pseudo-)scalar and (axial-)vector ground states $\eta_c, \chi_{c0}, J/\psi(1S)$ and $\chi_{c1}$ [18]. In the present work we study the
OZI-suppressed decays of the two (pseudo-)scalar charmonia.  
We use eLSM parameters in the light, strange and charm quark sectors fixed previously\textsuperscript{[13,18]}. In addition we introduce two new parameters, $\lambda_c^2$, $h_c^2$, which control the decay widths of $\eta_c$ and $\chi_{c0}$. This, together with a summary of the construction of the eLSM is discussed in section \textsuperscript{[11,13]} In section \textsuperscript{[13,18]} we present the results of the two- and three-body decay widths of $\eta_c$ and $\chi_{c0}$, and discuss their significance. In section \textsuperscript{[LV]} we conclude. Details of the calculations are relegated to the Appendices.

II. THE $U(4)_L \times U(4)_R$ LSM INTERACTION WITH GLUEBALLS

The $U(4)_L \times U(4)_R$ linear sigma model with (pseudo)scalar and (axial-)vector mesons, a scalar $G$ and a pseudoscalar glueball $\tilde{G}$ is given by \textsuperscript{[18]}

\[
\mathcal{L} = \mathcal{L}_{det} + \text{Tr}[(D^\mu \Phi)^{\dagger}(D^\mu \Phi)] - m_0^2 \left( \frac{G}{G_0} \right)^2 \text{Tr}(\Phi^\dagger \Phi) - \lambda^S_1 \left[ \text{Tr}((1 - \Pi_C)(1 - \Pi_C)) \right]^2 - \lambda^S_2 \left[ \text{Tr}(\Phi^\dagger \Phi) \right]^2
\]

\[
+ \text{Tr}[H(\Phi^\dagger \Phi)] + c(\text{det}\Phi - \text{det}\Phi^{\dagger})^2 - \delta c(\text{det}\Phi - \text{det}\Phi^{\dagger})^2 \text{Tr}(\Pi_C \Phi^\dagger \Pi_C \Phi) + i\theta \tilde{G} \left( \text{det}\Phi - \text{det}\Phi^{\dagger} \right)
\]

\[
+ \frac{h^S_1}{2} \text{Tr}((1 - \Pi_C)(1 - \Pi_C)) \text{Tr}[(L^\mu)^2 + (R^\mu)^2] + \frac{h^C_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}[(L^\mu)^2 + (R^\mu)^2]
\]

\[
+ h_2 \text{Tr}(\Phi^\dagger R^\mu) + 2h_3 \text{Tr}(\Phi R^\mu \Phi^{\dagger} L^\mu)
\]

\[
i \frac{G_2}{2} \{ \text{Tr}(L_{\mu \nu}[L^\mu, L^\nu]) + \text{Tr}(R_{\mu \nu}[R^\mu, R^\nu]) \} + \ldots,
\]

(1)

where $D^\mu \Phi \equiv \partial^\mu \Phi - ig_1 (L^\mu \Phi - \Phi R^\mu)$ is the covariant derivative; $L^\mu \nu \equiv \partial^\mu L^\nu - \partial^\nu L^\mu$, and $R^\mu \nu \equiv \partial^\mu R^\nu - \partial^\nu R^\mu$ are the left- and right-handed field strength tensors. In addition to chiral symmetry it features dilatation invariance and invariance under the discrete symmetries $C$ and $P$. In the following we summarise the most important features of the so defined extended linear sigma model (eLSM).

In our framework with $N_f = 4$ quark flavours the field $\Phi$ represents the $4 \times 4$ (pseudo)scalar multiplets

\[
\Phi = (S^a + i P^a)^\dagger \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{(\sigma_N + a_0^0 + i(\eta_N + \pi^0))}{\sqrt{2}} & a_0^+ + i\pi^+ & K_0^+ + iK^+ & D_0^+ + iD^+
\frac{(\sigma_N - a_0^0 + i(\eta_N - \pi^0))}{\sqrt{2}} & K_0^- + iK^- & D_0^- + iD^-
\end{pmatrix},
\]

(2)

where $t^a$ are the generators of the group $U(N_f)$. The multiplet $\Phi$ transforms as $\Phi \rightarrow U_L \Phi U_R^\dagger$ under $U_L(4) \times U_R(4)$ chiral transformations with $U_{L,R} = e^{-i\omega\xi_{\phi}(t,\vec{x})}$, with $\Phi(t, \vec{x}) \rightarrow \Phi^\dagger(t, -\vec{x})$ under parity transformations, and $\Phi \rightarrow \Phi^\dagger$ under charge conjugation. The determinant of $\Phi$ is invariant under $SU(4)_L \times SU(4)_R$, but not under $U(1)_A$ because $\det \Phi \rightarrow \det U_A \Phi U_A = e^{-i\omega_{\pi}^0 \sqrt{2N_f}} \det \Phi \neq \det \Phi$.

Next we present the left- and right-handed (axial)vector multiplets $L^\mu$ and $R^\mu$ containing the vector and axial-vector degrees of freedom $V^a$ and $A^a$

\[
L^\mu = (V^a + i A^a)^\mu \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{(\omega_N + a_0^0 + i(\eta_N + \pi^0))}{\sqrt{2}} & \rho^+ + a_1^+ & K^+ + K^+ & D_0^+ + D_1^+
\frac{(\omega_N - a_0^0 + i(\eta_N - \pi^0))}{\sqrt{2}} & K_0^- + K^- & D_0^- + D_1^-
\end{pmatrix}^\mu,
\]

(3)

\textsuperscript{1} Note that the current set-up of the eLSM does not contain decay channels of the two (axial-)vector charmonia. These, however, have been considered in other approaches, see e.g. \textsuperscript{[39,41]} and Refs. therein.
In the vector sector \(V^a \sim i A^a \mu \epsilon^\mu\) to the scalars is small. The charm sectors \(D_A\) contributions of the physical states \(\eta, \phi\), and \(\chi_{C0}\) by their vacuum expectation values \(G_0, \phi_N, \phi_S, \phi_C\) ~

\[G \rightarrow G + G_0, \quad \sigma_N \rightarrow \sigma_N + \phi_N, \quad \sigma_S \rightarrow \sigma_S + \phi_S, \quad \chi_{C0} \rightarrow \chi_{C0} + \phi_C,\]

where \(\phi_N, \phi_S\) and \(\phi_C\) are the corresponding chiral shifts, which read

\[\phi_N = 164.6 \text{ MeV}, \quad \phi_S = 126.2 \text{ MeV}, \quad \phi_C = 176 \text{ MeV}.\]

In order to be consistent with the full effective chiral Lagrangian of the extended Linear Sigma Model \[18\] we also shift the axial-vector fields and thus redefine the wave functions of the pseudoscalar fields

\[\pi^{\pm, 0} \rightarrow Z_{\pi^{\pm, 0}}, \quad K^{\pm, 0, \bar{0}} \rightarrow Z_K K^{\pm, 0, \bar{0}},\]

\[\eta_{N/S/C} \rightarrow Z_{\eta_{N/S/C}}, \quad K_0^{\pm, 0, \bar{0}} \rightarrow Z_K K_0^{\pm, 0, \bar{0}},\]

\[D^{\pm, 0, \bar{0}} \rightarrow Z_D D^{\pm, 0, \bar{0}}, \quad D_0^{\pm} \rightarrow Z_{D_0^{\pm}} D_0^{\pm},\]

\[D_0^{0, \bar{0}} \rightarrow Z_{D_0^{0, \bar{0}}} D_0^{0, \bar{0}}, \quad D_0^{* \pm} \rightarrow Z_{D_0^{* \pm}} D_0^{* \pm},\]

where \(Z_i\) are the renormalization constants of the corresponding wave functions \[18\].

The terms involving the matrices \(H, \epsilon\) and \(\Delta\) correspond to explicit breaking of the dilaton and chiral symmetry due to non-zero current quark masses. They are all diagonal with constant entries (see \[18\] for details):

\[H = \frac{1}{2} \text{diag}(h_N, h_N, \sqrt{2}h_S, \sqrt{2}h_C)\]

\[\Delta = \text{diag}(\delta_N, \delta_N, \delta_S, \delta_C)\]

\[\epsilon = \text{diag}(\varepsilon_N, \varepsilon_N, \varepsilon_S, \varepsilon_C)\]

where \(h_i \sim m_i, \delta_i \sim m_i^2\) and \(\varepsilon_i \sim m_i^2\).

In order to make contact with experiment one needs to assign the various fields of the model to physical states: (i) In the pseudoscalar sector \(P^a\) the fields \(\pi\) and \(K\) correspond to the physical pion iso-triplet and the kaon isodoublet, respectively \[32\]. The bare fields \(\eta_{N/S/C} \equiv |\bar{u}u + \bar{d}d| / \sqrt{2}\) and \(\eta_{S} \equiv |\bar{s}s|\) are the non-strange and strange mixing contributions of the physical states \(\eta\) and \(\eta'\) with mixing angle \(\varphi \approx -44.6^\circ\) \[12, 33, 34\]:

\[\eta = \eta_{N} \cos \varphi + \eta_{S} \sin \varphi, \quad \eta' = -\eta_{N} \sin \varphi + \eta_{S} \cos \varphi.\]

In the pseudoscalar charm sector, we have the well-established \(D\) resonances, the open strange-charmed \(D_s\) states, and the charm-anticharm state \(\eta_{c}(1S)\).

(ii) In the vector sector \(V^a\) the iso-triplet fields \(\rho\), the kaon states \(K^*\), and the isoscalar states \(\omega_N\) and \(\omega_S\) correspond to the \(\rho(770), K^*(892), \omega\), and \(\phi\) mesons, respectively. Notice that the mixing between strange and non-strange isoscalars is small. The charm sectors \(D^{\pm, 0, \bar{0}}, D_0^{0, \bar{0}}\), and charmonium state \(J/\psi\) correspond to the open-charm sectors \(D^{*}(2007)^0, D^{*}(2010)^0, D_s^{*+}\) (with mass \(= 2112.3 \pm 0.5\) MeV), and \(J/\psi(1S)\), respectively.

(iii) In the axial-vector sector \(A^a\), the iso-triplet field \(\tilde{a}_1(1260)\), the kaon states \(K_1\), the isoscalar fields \(f_{1,N}\) and \(f_{1,S}\), the open-charm sector \(D_1\) and the strange-charmed doublet \(D_{S_1}^\pm\) are assigned to \(a_1(1260), K_1(1270)\), or \(K_1(1400)\) mesons, \(f_1(1285), f_1(1420), D_1(2420)^0,^\pm,\) and \(D_{S_1}(2536)^-\), respectively. The charm-anticharm state \(\chi_{c1}\) represent the ground-state charmonium resonance \(\chi_{c1}(1P)\). For more detail of strange-non-strange fields assignment see Refs.
mimics the trace anomaly of the pure Yang-Mills sector of QCD [13, 20–23]. The rationale behind this approach is that some previous works the corresponding parameters \( \lambda^C \) and \( h^C_1 \) in the light quark sector have either been set to zero [10, 18, 19] or have been determined from decays of states with light quarks [13, 33]. In contrast to these, the decays in the heavy quark sector always come with charm quark-antiquark annihilations. In order to reflect the different physics of these processes we use the projection operator \( P_C = \text{diag}\{0, 0, 0, 1\} \) onto the charmed states to introduce separate parameters \( \lambda^C \) and \( h^C_1 \) in this sector. In addition, it is possible to improve the description of charmonia decays by taking into account interaction terms that break chiral symmetry explicitly [18].

The energy scale of low-energy QCD is described by the dimensionful parameter \( \Lambda \) which is identical to the minimum \( G_0 \) of the dilaton potential \( (G_0 = \Lambda) \). The scalar glueball mass \( m_G \) has been evaluated by lattice QCD which gives a mass of about (1.5-1.7) GeV [43–48]. The dilatation symmetry or scale invariance, \( x^\mu \rightarrow x^\mu \), is realized at the classical level of the Yang-Mills sector of QCD and explicitly broken due to the logarithmic term of the dilaton potential. This breaking leads to the divergence of the corresponding current: \( \partial_\mu J^\mu_{dil} \approx T^\mu_{dil, \mu} = -\frac{1}{4} m_G^2 \Lambda^2 \) [12].

The identification of the scalar (isoscalar) glueball in the experimental spectrum is highly controversial. Here, the quark-antiquark states \( \sigma_N \equiv |\bar{u}u + \bar{d}d|/\sqrt{2} = |n\bar{n}|, \sigma_S \equiv |s\bar{s}| \), and the scalar glueball \( G \) mix and generate the three resonances \( f_0(1370), f_0(1500), \) and \( f_0(1710) \). In the flavour singlet basis this mixing is expressed via

\[
\begin{pmatrix}
|f_0(1710)\rangle \\
|f_0(1500)\rangle \\
|f_0(1370)\rangle \\
\end{pmatrix} = U_n \begin{pmatrix}
|G\rangle \\
|\sigma_N\rangle \\
|n\bar{n}\rangle \\
\end{pmatrix} ; \quad U_1 = \begin{pmatrix}
0.93 & 0.26 & -0.27 \\
-0.17 & 0.94 & 0.30 \\
-0.33 & 0.24 & -0.91 \\
\end{pmatrix},
\]

where \( U_1 \) represents the best result in the extended linear sigma model [33] from a systematic fit to the spectrum and decay properties of light mesons. Thus the resonance \( f_0(1710) \) is identified as being predominantly a scalar glueball. This matrix will be used below in our consistent evaluation of the decays of the \( \chi_{CO} \) into scalar isoscalar states. However, in order to contrast the resulting decay widths with the ones of potentially different assignments we will also play with other mixing matrices. Since the charm quark sector of the linear sigma model does not feed back into the light quark sector, results using different mixing matrices may be used as a self-consistency check of the model in comparison with corresponding experimental data. This will be discussed in more detail in section [III]. In particular we use the mixing matrix \( U_2 \) from Ref. [4] and \( U_{3, 5} \) from Ref. [5].

\[
U_2 = \begin{pmatrix}
0.36 & 0.93 & 0.09 \\
-0.84 & 0.35 & -0.41 \\
0.40 & -0.07 & -0.91 \\
\end{pmatrix},
\]
III. PARAMETERS AND RESULTS

All the parameters and wave-function renormalization constants of the Lagrangian \[11\] have been fixed in Refs. \[13, 18\]. Their values are summarized in Table I.

| parameter | value | parameter | value | renormalization factor | value |
|-----------|-------|-----------|-------|------------------------|-------|
| \(m_1^2\) | \(0.413 \times 10^6\) MeV\(^2\) | \(\omega_{\pi_1}\) | 0.00068384 | \(Z_\pi = Z_{\eta_N}\) | 1.70927 |
| \(m_2^2\) | \(-0.918 \times 10^6\) MeV\(^2\) | \(\omega_{\eta_N}\) | 0.00068384 | \(Z_K\) | 1.60406 |
| \(\delta_S\) | \(0.151 \times 10^6\) MeV\(^2\) | \(\omega_{\eta_S}\) | 0.0005538 | \(Z_{\eta_S}\) | 1.11892 |
| \(\delta_C\) | \(3.91 \times 10^6\) MeV\(^2\) | \(\omega_{D_S}\) | \(-0.0000523\) | \(Z_{D_S}\) | 1.00649 |
| \(c\) | \(1.987 \times 10^{-8}\) MeV\(^{-2}\) | \(\omega_{D_{S1}}\) | 0.000203 | \(Z_{D_{S0}}\) | 1.00649 |
| \(g_1\) | 5.84 | \(\omega_{D^{*}} = \omega_{D^{0}}\) | \(-0.0000523\) | \(Z_{\eta_{S0}}\) | 1.53854 |
| \(h_2\) | 9.88 | \(\omega_{D_{S2}}\) | \(-0.0000423\) | \(Z_{K_{S0}}\) | 1.00105 |
| \(\lambda_2\) | 68.3 | \(\omega_{D_1}\) | 0.000209 | \(Z_{D}\) | 1.15256 |
| \(h_3\) | 3.87 | \(\omega_{\chi_{c1}}\) | 0.000138 | \(Z_{D_{S0}}\) | 1.00437 |

TABLE I: Parameters and wave-function renormalization constants.

The wave-function renormalization constants for \(\pi\) and \(\eta_N\) are equal because of isospin symmetry, similar for \(D_S^0\) and \(D^{*0}\). The gluon condensate \(G_0\) is equal to \(\Lambda \approx 3.3\) GeV \[33\] in pure YM theory, which is also used in the present discussion.

The parameters \(\lambda_1^C, \lambda_1^C\) and \(h_1^C, h_1^C\) have either been set to zero (or not present at all) in previous studies \[10, 18, 19\] in agreement with large-\(N_c\) expectations. In the latter works, masses of charmed mesons and the (OZI-dominant) strong decays of open charmed mesons have been considered, but not OZI-suppressed decays. However, as we will explain in the following, in these cases small but non-zero values of \(\lambda_1^C\) and \(h_1^C\) are mandatory.

The decay of the charmonium states \(\eta_c\) and \(\chi_{c0}\) into hadrons is mediated by gluon annihilation. For the two-body decays, this annihilation can occur via two-gluon exchange diagrams for all contributions and additional so-called double OZI-suppressed three-gluon exchange diagrams for decays into pairs of iso-singlet states, cf. Fig. \[1\]. Similar diagrams contribute to the three-particle decays. In these decays, gluons carry all the energy. Therefore, the interaction is relatively weak due to asymptotic freedom, which leads to OZI-suppression. As a consequence, the decay of the charmonium states \(\eta_c\) and \(\chi_{c0}\) into (axial-)vector and (pseudo)scalar mesons or scalar glueballs is dynamically suppressed. In the eLSM, this is reflected by small but non-zero values of the large-\(N_c\) suppressed parameters \(\lambda_1^C\) and \(h_1^C\).

We determine their size using experimental decay widths of \(\chi_{c0}\) listed by the PDG \[1\] via a \(\chi^2\) fit. To this end we employ the five known decay widths into non-isosinglet final states listed in table III. Using

\[
\chi^2(\lambda_1^C, h_1^C) = \sum_i^6 \left(\frac{\Gamma_{i}^{th} - \Gamma_{i}^{exp}}{\Gamma_{i, error}^{th}}\right)^2,
\]

we obtain

\[
\lambda_1^C = -0.161 \pm 0.005 \quad \text{and} \quad h_1^C = 0.046 \pm 0.003.
\]
with a reasonable $\chi^2/d.o.f = 3.8$. The values of the parameters $\lambda_C^C$ and $h_C^C$ are indeed small. A posteriori we therefore justify the assumptions of Refs. 18, 19 concerning the heavy quark sector. Reviewing the fit results, it is apparent that the decay into the iso-singlet $\phi$-mesons is not very well reproduced by the model (albeit with a large error). It dominates by far the $\chi^2$-value. Since this decay is distinguished from the others by the additional contributions from double OZI-suppressed diagrams (cf. Fig. 1), we conclude that these may not be well represented by the current model setup. One could proceed by excluding the $\phi$-meson decay from the fitting procedure and strictly restrict the whole approach (in the current setup) to the non-iso-singlet sector. However, since we are particularly interested in decays to scalar isoscalar states, we decided to keep the information from the $\phi$-meson decay in the fit and accept a large error margin in our predictions for these channels. As a result we may expect (semi-)quantitative predictions in the non-iso-singlet sector (where the fits works excellent), whereas our results for the iso-singlet decays should be taken as order of magnitude estimates. Certainly, this situation should be improved in the future.

| Decay Channel          | theoretical result [MeV] | Experimental result [MeV] |
|------------------------|--------------------------|---------------------------|
| $\Gamma_{\chi_c^0 \to K^0 K^0}$ | 0.010 ± 0.003            | 0.010 ± 0.004             |
| $\Gamma_{\chi_c^0 \to K^- K^+}$ | 0.059 ± 0.008            | 0.062 ± 0.005             |
| $\Gamma_{\chi_c^0 \to \pi \pi}$ | 0.090 ± 0.011            | 0.087 ± 0.006             |
| $\Gamma_{\chi_c^0 \to K^{*0} K^{*0}}$ | 0.014 ± 0.007            | 0.018 ± 0.006             |
| $\Gamma_{\chi_c^0 \to \omega \omega}$ | 0.012 ± 0.006            | 0.010 ± 0.001             |
| $\Gamma_{\chi_c^0 \to \phi \phi}$ | 0.0035 ± 0.0036          | 0.0081 ± 0.0009           |
| $\Gamma_{\chi_c^0 \to \eta \eta}$ | 0.022 ± 0.002            | 0.031 ± 0.003             |
| $\Gamma_{\chi_c^0 \to \eta' \eta'}$ | 0.021 ± 0.001            | 0.021 ± 0.002             |
| $\Gamma_{\eta \to \eta \pi \pi}$ | 0.12 ± 0.02              | 0.54 ± 0.16               |
| $\Gamma_{\eta \to \eta' \pi \pi}$ | 0.081 ± 0.019            | 1.30 ± 0.54               |

TABLE II: The partial decay widths of $\chi_c^0$ used to fix three model parameters. The upper six decays fix $\lambda_C^C$ and $h_C^C$, whereas the lower four decays are used to fix $\delta \bar{c}$.

Furthermore, we adjust the coefficient $\delta \bar{c}$ of the modification of the axial anomaly term to fit the results of the decay widths of $\chi_c^0$ into flavour singlet mesons. Here we use the decay widths of $\chi_c^0$ into the experimentally known $\eta$ and $\eta'$-channels given in table III. We perform a fit by minimizing the $\chi^2$-function,

$$\chi^2(CD) \equiv \left( \frac{\Gamma_{\chi_c^0 \to \eta \eta}}{\Gamma_{\chi_c^0 \to \eta \eta}^{\text{error}}} - \frac{\Gamma_{\chi_c^0 \to \eta \eta}}{\Gamma_{\chi_c^0 \to \eta \eta}^{\text{exp}}} \right)^2 + \left( \frac{\Gamma_{\chi_c^0 \to \eta' \eta'}}{\Gamma_{\chi_c^0 \to \eta' \eta'}^{\text{error}}} - \frac{\Gamma_{\chi_c^0 \to \eta' \eta'}}{\Gamma_{\chi_c^0 \to \eta' \eta'}^{\text{exp}}} \right)^2$$

FIG. 1: Decay of charmonium state into two mesons. $q$ refers to the up (u), down (d), and strange (s) quark flavours.
where \(c_D = (c - \frac{2}{3}\phi_C \delta C)\). We obtain
\[
c_D = (7.255 \pm 0.0001) \times 10^{-10}\text{MeV}^{-4}
\] (20)
with a \(\chi^2/d.o.f = 5.4\). This value is equally dominated from the decays into two \(\eta\) mesons and the two three-body decays. Also this result indicates that we may only expect order of magnitude estimates in the channels involving iso-singlets.

| Decay Channel | theoretical result [MeV] | Experimental result [MeV] |
|---------------|---------------------------|---------------------------|
| \(\Gamma_{\chi_c^0 \rightarrow a_0 a_0}\) | 0.0036±0.0019 | - |
| \(\Gamma_{\chi_c^0 \rightarrow K^+ K^-}\) | 0.000069±0.000049 | - |
| \(\Gamma_{\chi_c^0 \rightarrow pp}\) | 0.010±0.006 | - |
| \(\Gamma_{\chi_c^0 \rightarrow \eta \eta'}\) | 0.0012±0.0005 | <0.0024 |
| \(\Gamma_{\chi_c^0 \rightarrow K^0_S K^-}\) | 0.00042±0.00015 | - |
| \(\Gamma_{\chi_c^0 \rightarrow K^0_S K^-}\) | 0.00021±0.00013 | - |

TABLE III: The partial decay widths of \(\chi_c^0\) predicted by the model.

| Decay Channel | \(U_1\) (Ref. [33]) | \(U_2\) (Ref. [4]) | \(U_3\) (Ref. [5]) | \(U_4\) (Ref. [5]) | \(U_5\) (Ref. [5]) | Experimental result [1] |
|---------------|-----------------------|---------------------|---------------------|---------------------|---------------------|--------------------------|
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) f_0(1370)}\) | \(5 \times 10^{-3}\) | \(5 \times 10^{-3}\) | \(4 \times 10^{-3}\) | \(5 \times 10^{-3}\) | \(1 \times 10^{-2}\) | \(< 3 \times 10^{-3}\) |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1500) f_0(1500)}\) | \(4 \times 10^{-3}\) | \(2 \times 10^{-3}\) | \(4 \times 10^{-3}\) | \(2 \times 10^{-3}\) | \(3 \times 10^{-3}\) | \(< 5 \times 10^{-4}\) |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) f_0(1710)}\) | \(1 \times 10^{-5}\) | \(2 \times 10^{-4}\) | \(9 \times 10^{-6}\) | \(4 \times 10^{-4}\) | \(7 \times 10^{-4}\) | \(< 2 \times 10^{-3}\) |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) f_0(1710)}\) | \(2 \times 10^{-4}\) | \(3 \times 10^{-5}\) | \(3 \times 10^{-4}\) | \(1 \times 10^{-6}\) | \(5 \times 10^{-3}\) | \((6.9^{+3.7}_{-2.4}) \times 10^{-3}\) |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1500) f_0(1710)}\) | \(3 \times 10^{-5}\) | \(1 \times 10^{-4}\) | \(2 \times 10^{-5}\) | \(5 \times 10^{-6}\) | \(1 \times 10^{-4}\) | \(< 7 \times 10^{-4}\) |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) \eta \eta'}\) | \(3 \times 10^{-6}\) | \(3 \times 10^{-8}\) | \(9 \times 10^{-7}\) | \(5 \times 10^{-7}\) | \(5 \times 10^{-6}\) | - |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1500) \eta \eta'}\) | \(1 \times 10^{-5}\) | \(3 \times 10^{-6}\) | \(2 \times 10^{-5}\) | \(9 \times 10^{-11}\) | \(5 \times 10^{-6}\) | - |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) \eta' \eta'}\) | \(1 \times 10^{-5}\) | \(2 \times 10^{-5}\) | \(2 \times 10^{-5}\) | \(2 \times 10^{-5}\) | \(3 \times 10^{-5}\) | - |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) \eta' \eta'}\) | \(4 \times 10^{-10}\) | \(9 \times 10^{-6}\) | \(2 \times 10^{-5}\) | \(2 \times 10^{-6}\) | \(7 \times 10^{-5}\) | - |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1500) \eta' \eta'}\) | \(6 \times 10^{-6}\) | \(4 \times 10^{-6}\) | \(4 \times 10^{-5}\) | \(2 \times 10^{-6}\) | \(6 \times 10^{-6}\) | - |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1710) \eta \eta'}\) | \(8 \times 10^{-7}\) | \(8 \times 10^{-6}\) | \(6 \times 10^{-7}\) | \(1 \times 10^{-5}\) | \(6 \times 10^{-6}\) | - |
| \(\Gamma_{\chi_c^0 \rightarrow f_0(1710) \eta' \eta'}\) | \(4 \times 10^{-7}\) | \(1 \times 10^{-5}\) | \(2 \times 10^{-6}\) | \(1 \times 10^{-5}\) | \(1 \times 10^{-5}\) | - |

TABLE IV: The partial decay widths of \(\chi_c^0\) into scalar isoscalar mesons in units of MeV. The corresponding mixing matrices in the scalar isoscalar sector are given in Eqs. (14)-(19).

Having fixed all model parameters we now discuss the predictions of the model starting with the two- and three-body decays in table [III] (all relevant expressions for the calculations are presented in the Appendix). As discussed above we do expect very reasonable predictions for the channels not involving iso-singlet states, i.e. the first three entries in the table, whereas we regard the second three entries as order of magnitude estimates. In this respect it is satisfactory to see that the decay into the \(\eta \eta'\) pair is in agreement with the experimental bounds.

The two- and three-body decays of the \(\chi_c^0\) into the scalar-isoscalar resonances \(f_0(1370), f_0(1500), f_0(1710)\) are shown in table [IV]. Considering the intrinsic uncertainties of the model, we regard these results as order of magnitude estimates\(^2\). Nevertheless it is interesting to compare the results obtained with the mixing matrix of the eLSM, \(U_1\) [33], with the ones obtained by other mixing patters. Compared with the experimental bounds we find that none of the mixing scenarios are in agreement with all experimental bounds. However, considering the current experimental and theoretical uncertainties this statement is not very rigorous. Comparing the five different scenarios we find that in some channels the deviations are within an order of magnitude, whereas in other channels like \(\Gamma_{\chi_c^0 \rightarrow f_0(1370) \eta \eta'}\) or

\(^2\) Therefore we only give one significant digit and refrain from giving the error due to the fitting procedure, since this error is at least an order of magnitude smaller than the systematic error due to model uncertainties.
Theoretical result [MeV]

\[
\Gamma_{\eta_c \to f_0(1370)\eta} = 0.010 \pm 0.007 \\
\Gamma_{\eta_c \to f_0(1500)\eta} = 0.054 \pm 0.015 \\
\Gamma_{\eta_c \to \eta'\eta} = 0.0024 \pm 0.0032 \\
\Gamma_{\eta_c \to \eta''\eta} = 0.045 \pm 0.014 \\
\Gamma_{\eta_c \to \eta''\eta} = 0.0036 \pm 0.0039 \\
\Gamma_{\eta_c \to \eta K K} = 0.16 \pm 0.03 \\
\Gamma_{\eta_c \to \eta' K K} = 0.43 \pm 0.04 \\
\Gamma_{\eta_c \to K K \pi} = 0.006 \pm 0.020
\]

Experimental result [1]

\[
\Gamma_{\eta_c \to f_0(1370)\eta} = 0.007 \pm 0.032 \\
\Gamma_{\eta_c \to f_0(1500)\eta} = 0.014 \pm 0.036 \\
\Gamma_{\eta_c \to \eta'\eta} = 0.054 \pm 0.054 \\
\Gamma_{\eta_c \to \eta''\eta} = 0.007 \pm 0.007 \\
\Gamma_{\eta_c \to \eta''\eta} = 0.036 \pm 0.036 \\
\Gamma_{\eta_c \to \eta K K} = 0.16 \pm 0.03 \\
\Gamma_{\eta_c \to \eta' K K} = 0.43 \pm 0.04 \\
\Gamma_{\eta_c \to K K \pi} = 0.006 \pm 0.020
\]

Differences of two or more magnitudes arise which may be resolved by future experiments. Thus, in general, decays of charmonia offer the interesting possibility to involve the heavy quark sector to further constrain and explore glueball physics in the light quark sector.

Finally, let us discuss the decay of \( \eta_c \) into a pseudoscalar glueball \( \tilde{G} \). It proceeds via the channel \( \eta_c \to \pi \pi \tilde{G} \). The width \( \Gamma_{\eta_c \to \pi \pi \tilde{G}} \) depends on the coupling constant \( c_{\tilde{G} \Phi} \) which can be determined by the following relation

\[
c_{\tilde{G} \Phi} = \frac{\sqrt{2} c_{\tilde{G} \Phi}(N_f=3)}{\phi_C} \tag{21}
\]

where \( c_{\tilde{G} \Phi}(N_f=3) = 4.48 \pm 0.46 \) has been determined in Ref. [34] via the decay of the pseudoscalar glueball into scalar and pseudoscalar mesons. We then obtain \( c_{\tilde{G} \Phi} = 0.036 \) in the present four-flavour case. In order to determine the decay width, the mass of the resulting pseudoscalar glueball is important. In the literature, sometimes the state \( \eta(1405) \) has been considered as a candidate for a light pseudoscalar glueball, see e.g. [49]. However, this identification may be questionable, in particular since the (quenched) lattice results point to much heavier masses [44, 47]. Here we consider two cases, \( m_{\tilde{G}} = 2.6 \) GeV as found on the lattice and a putative lower mass of \( m_{\tilde{G}} = 2.37 \) GeV as suggested e.g. by the identification of the pseudoscalar glueball with the resonance \( X(2370) \). We then find

\[
\Gamma_{\eta_c \to \pi \pi \tilde{G}(2600)} = 0.124 \text{ MeV}, \quad \Gamma_{\eta_c \to \pi \pi \tilde{G}(2370)} = 0.160 \text{ MeV}, \tag{22}
\]

showing a not too large variation of the decay width with glueball mass.
IV. CONCLUSION AND OUTLOOK

In this work we have further extended the extended linear sigma model to be able to deal with the strong decays of the $\eta_c$ and the $\chi_{c0}$. Encouraged by the unexpected success of the model to deal with the spectrum of charmonium and open charm decays [18], we determined various two- and three-body decays into (pseudo-)scalar and vector mesons and a putative pseudoscalar glueball. Unfortunately, the study is not as conclusive as one could wish for. We obtain excellent results for the non-iso-singlet two-body decays of the $\chi_{c0}$, where we also presented predictions for some as yet unmeasured channels. The singlet decays, however, give mixed results, with some very reasonable but others only correct on the order of magnitude level. Consequently, the predictive power of the model in this sector remains on the order of magnitude level. Since, on the other hand, experimental results are entirely missing for a large range of decays, we nevertheless consider these order of magnitude estimates to be a helpful guidance. Particularly interesting for future experimental and theoretical studies could be the decay channels into isoscalar scalar mesons, which are supposed to contain admixtures from a scalar glueball. Different mixing matrices lead to a range of different decay patterns for the $\eta_c$ and the $\chi_{c0}$, which could be used to discriminate between different mixing scenarios. To this end one needs to further refine the model and focus on these channels in ongoing and future experiments such as BESIII, Belle II, LHCb and the PANDA experiment at the FAIR facility.

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Appendix A: Decay widths

The general formula for the two-body decay width is given by [1]:

$$\Gamma_{A \to BC} = S_{A \to BC} \frac{k(m_A, m_B, m_C)}{8\pi m_A^2} |M_{A \to BC}|^2.$$  \hspace{2cm} (A1)

The center-of-mass momentum $k(m_A, m_B, m_C)$ of the decay products B,C is given by

$$k(m_A, m_B, m_C) = \frac{1}{2m_A} \sqrt{m_A^4 + (m_B^2 - m_C^2)^2 - 2m_A^2 (m_B^2 + m_C^2) \theta(m_A - m_B - m_C)}.$$ \hspace{2cm} (A2)

$M_{A \to BC}$ is the corresponding tree-level decay amplitude, and $S_{A \to BC}$ denotes a symmetrization factor (it equals 1 if B and C are different and it equals 1/2 for two identical particles in the final state).

Using the notation of Fig. 2 and the definition $m_{ij} = m_i + m_j$ the corresponding expression for the three-body decay

\begin{center}
\begin{tikzpicture}
  \node[circle,draw] (0) at (0,0) {};
  \node at (1,1) {$P_1, m_1$};
  \node at (-1,1) {$P_2, m_2$};
  \node at (1,-1) {$P_3, m_3$};
  \node at (0,1) {$P, M$};
  \draw[->] (0) to (1,0);
  \draw[->] (0) to (0,1);
  \draw[->] (0) to (0,-1);
\end{tikzpicture}
\end{center}
width of the process $A \to B_1 B_2 B_3$ reads \[^{[1]}\]:

$$
\Gamma_{A \to B_1 B_2 B_3} = \frac{S_{A \to B_1 B_2 B_3}}{32(2\pi)^3 M^3} \int_{(m_1+m_2)^2}^{(M-m_3)^2} \left| -i M_{A \to B_1 B_2 B_3} \right|^2 \\
\times \sqrt{\frac{(-m_1+m_12-m_2)(m_1+m_12-m_2)(-m_1+m_12+m_2)(m_1+m_12+m_2)}{m_{12}^2}} \\
\times \sqrt{\frac{(-M+m_12-m_3)(M+m_12-m_3)(-M+m_12+m_3)(M+m_12+m_3)}{m_{12}^2}} \, dm_{12}^2, \tag{A3}
$$

where $M_{A \to B_1 B_2 B_3}$ is the corresponding tree-level decay amplitude, and $S_{A \to B_1 B_2 B_3}$ is a symmetrization factor (equal to 1 if the $B_i$ are all different, equal to 2 for two identical particles in the final state, and equal to 6 for three identical particles).

**Appendix B: Decay rates for $\chi_{c0}$**

We present the explicit expressions for the two- and three-body decay rates for the scalar hidden-charmed meson $\chi_{c0}$.

1. **Two-body decay rates for $\chi_{c0}$**

The explicit expressions for the two-body decay rates of $\chi_{c0}$ are extracted from the Lagrangian \[^{[1]}\] and are presented in the following.

**Decay channel** $\chi_{c0} \to \overline{K}_0^{*0} K_0^{*0}$

The corresponding interaction part of the Lagrangian \[^{[1]}\] reads

$$
L_{\chi_{c0} \overline{K}_0^{*0} K_0^{*0}} = -2\lambda^C_{\overline{K}_0^{*0} K_0^{*0}} Z^2_{K_0^{*0}} \phi_C \chi_{c0}(\overline{K}_0^{*0} K_0^{*0} + K_0^{*0} - K_0^{*0} + K_0^{*0}) \\
- h_1^C \phi_C Z^2_{K_0^{*0}} \omega^2_{K_0^{*0}} \chi_{c0}(\partial_\mu K_0^{*0} \partial^\mu K_0^{*0} + \partial_\mu K_0^{*0} \partial^\mu K_0^{*0}). \tag{B1}
$$

Consider only the $\chi_{c0} \to \overline{K}_0^{*0} K_0^{*0}$ decay channel, the $\chi_{c0} \to K_0^{*0} - K_0^{*0}$ will give the same contribution due to isospin symmetry,

$$
L_{\chi_{c0} \overline{K}_0^{*0} K_0^{*0}} = -2\lambda^C_{\overline{K}_0^{*0} K_0^{*0}} Z^2_{K_0^{*0}} \phi_C \chi_{c0}(\overline{K}_0^{*0} K_0^{*0} - K_0^{*0} + K_0^{*0}) \\
- h_1^C \phi_C Z^2_{K_0^{*0}} \omega^2_{K_0^{*0}} \chi_{c0}(\partial_\mu K_0^{*0} \partial^\mu K_0^{*0} + \partial_\mu K_0^{*0} \partial^\mu K_0^{*0}). \tag{B2}
$$

Let us denote the momenta of $K_0^{*0}$ and $\overline{K}_0^{*0}$ as $P_1$ and $P_2$, respectively. The energy-momentum conservation on the vertex implies $P = P_1 + P_2$, where $P$ denotes the momenta of the decaying particle $\chi_{c0}$. Given that our particles are on-shell, we obtain

$$
P_1 \cdot P_2 = \frac{P^2 - P_1^2 - P_2^2}{2} = \frac{m_{\chi_{c0}}^2 - 2m_{K_0^{*0}}^2}{2}. \tag{B3}
$$

Upon substituting $\partial_\mu \to -iP^\mu$ for the decay particle and $\partial_\mu \to +iP^\mu_{1,2}$ for the outgoing particles, one obtains

$$
L_{\chi_{c0} \overline{K}_0^{*0} K_0^{*0}} = \phi_C Z^2_{K_0^{*0}} \left[ -2\lambda^C_{\overline{K}_0^{*0} K_0^{*0}} + h_1^C \omega^2_{K_0^{*0}} \frac{m_{\chi_{c0}}^2 - 2m_{K_0^{*0}}^2}{2} \right] \chi_{c0} \overline{K}_0^{*0} K_0^{*0}. \tag{B4}
$$

Consequently, the decay amplitude is given by

$$
- i M_{\chi_{c0} \to \overline{K}_0^{*0} K_0^{*0}} = i \phi_C Z^2_{K_0^{*0}} \right[ 2\lambda^C_{\overline{K}_0^{*0} K_0^{*0}} + h_1^C \omega^2_{K_0^{*0}} \frac{m_{\chi_{c0}}^2 - 2m_{K_0^{*0}}^2}{2} \right]. \tag{B5}
$$

The decay width is obtained as

$$
\Gamma_{\chi_{c0} \to \overline{K}_0^{*0} K_0^{*0}} = \frac{1}{8\pi m_{\chi_{c0}}} \left| - i M_{\chi_{c0} \to \overline{K}_0^{*0} K_0^{*0}} \right|^2. \tag{B6}
$$
Decay channel $\chi_{C0} \to K^- K^+$

The corresponding interaction part of the Lagrangian reads

$$\mathcal{L}_{\chi_{C0}KK} = -2\lambda_1^C Z_K^2 \phi_C \chi_{C0} (\overline{K}^0 K^0 + K^- K_0^+) + h_1^C \phi_C Z_{K,1}^2 \omega_{K,1} \chi_{C0} (\partial_{\mu} K^0 \partial^\mu \overline{K}^0 + \partial_{\mu} K^- \partial^\mu K^+).$$ (B7)

In a similar way as the previous case, one can obtain the decay width for the channel $\chi_{C0} \to K^- K^+$ as

$$\Gamma_{\chi_{C0}K^- K^+} = \frac{\vec{k}_1}{8\pi m_{\chi_{C0}}^2} \phi_C^2 Z_K^4 \left[ 2\lambda_1^C + h_1^C \omega_{K,1}^2 \left( \frac{m_{\chi_{C0}}^2 - 2m_K^2}{2} \right) \right]^2.$$ (B8)

Decay channel $\chi_{C0} \to \pi \pi$

The corresponding interaction part of the Lagrangian reads

$$\mathcal{L}_{\chi_{C0}\pi\pi} = -\lambda_1^C \phi_C Z_{\pi,1}^2 \chi_{C0} (\overline{\pi}^0 \pi^0 + 2\pi^- \pi^+) + \frac{1}{2} h_1^C \phi_C Z_{\pi,1}^2 \omega_{\pi,1} \chi_{C0} ([\partial_{\mu} \pi^0]^2 + 2\partial_{\mu} \pi^- \partial^\mu \pi^+),$$ (B9)

and leads to the decay width

$$\Gamma_{\chi_{C0}\pi\pi} = \frac{3}{2} \frac{\vec{k}_1}{8\pi m_{\chi_{C0}}^2} \phi_C^2 Z_{\pi,1}^4 \left[ 2\lambda_1^C + h_1^C \omega_{\pi,1}^2 \left( \frac{m_{\chi_{C0}}^2 - 2m_{\pi}^2}{2} \right) \right]^2.$$ (B10)

Decay channel $\chi_{C0} \to \overline{K}^{*0} K^{*0}$

The corresponding interaction Lagrangian is extracted as

$$\mathcal{L}_{\chi_{C0}\overline{K}^{*0} K^{*0}} = h_C^C \phi_C \chi_{C0} (K^{*-} K^{*+} + K^{*+} K^{*-}) + h_C^C \phi_C \chi_{C0} (K^{*0} \overline{K}^{*0}).$$ (B11)

Consider only the $K^{*\mu} \overline{K}^{*0\mu}$ decay channel, then

$$\mathcal{L}_{\chi_{C0}\overline{K}^{*0} K^{*0}} = h_C^C \phi_C \chi_{C0} K^{*0\mu} \overline{K}^{*0\mu}.$$ (B12)

Put

$$A_{\chi_{C0}\overline{K}^{*0} K^{*0}} = h_C^C \phi_C.$$ (B13)

Let us denote the momenta of $\chi_{C0}$, $\overline{K}^{*0}$, and $K^{*0}$ as $P$, $P_1$, and $P_2$, respectively, while the polarisation vectors are denoted as $\varepsilon_{\mu}^{(a)}(P_1)$ and $\varepsilon_{\nu}^{(b)}(P_2)$. Then, upon substituting $\partial_{\mu} \rightarrow iP_{\mu,2}$ for the outgoing particles, we obtain the following Lorentz-invariant $\chi_{C0} \overline{K}^{*0} K^{*0}$ scattering amplitude $\chi_{C0} \overline{K}^{*0} K^{*0}$:

$$-iM_{\chi_{C0} \overline{K}^{*0} K^{*0}}^{(a,b)} = \varepsilon_{\mu}^{(a)}(P_1) \varepsilon_{\nu}^{(b)}(P_2) h_{\chi_{C0} \overline{K}^{*0} K^{*0}}^{\mu\nu},$$ (B14)

with

$$h_{\chi_{C0} \overline{K}^{*0} K^{*0}}^{\mu\nu} = iA_{\chi_{C0} \overline{K}^{*0} K^{*0}}^\nu \eta^{\mu\nu},$$ (B15)
where $h^\mu_\chi^{c0\bar{K}^{*0}}$ denotes the $\chi_{c0}\bar{K}^{*0}$ vertex.

The averaged squared amplitude $|\bar{\mathcal{M}}|^2$ is determined as follows:

$$
|\bar{\mathcal{M}}_{\chi_{c0}\rightarrow K^{*0}}|^2 = \frac{1}{3} \sum_{\alpha, \beta = 1}^{3} |\bar{\mathcal{M}}^{(\alpha, \beta)}_{\chi_{c0}\rightarrow K^{*0}}|^2
$$

$$
= \frac{1}{3} \sum_{\alpha, \beta = 1}^{3} \varepsilon^{(\alpha)}(P_1)\varepsilon^{(\beta)}(P_2)h^\mu_\chi^{c0\bar{K}^{*0}}(P_1)\varepsilon^{(\alpha)}(P_1)
\times \varepsilon^{(\beta)}(P_2)h^{*\kappa\lambda}_{\chi_{c0}\bar{K}^{*0}}(P_1)
.$$  \hspace{1cm} (B16)

Then,

$$
|\bar{\mathcal{M}}_{\chi_{c0}\rightarrow K^{*0}}|^2 = \frac{1}{3} \left[ \left| h^\mu_\chi^{c0\bar{K}^{*0}}P_{1\mu} \right|^2 - \frac{h^\mu_\chi^{c0\bar{K}^{*0}}P_{1\mu}}{m_{V_1}} \frac{h^{*\kappa\lambda}_{\chi_{c0}\bar{K}^{*0}}P_{2\nu}}{m_{V_2}} \right].
$$  \hspace{1cm} (B17)

From Eq. (B15) we obtain

$$
h^\mu_\chi^{c0\bar{K}^{*0}}P_{1\mu} = iA_{\chi_{c0}\bar{K}^{*0}}P_{1\mu},
$$

$$
h^\mu_\chi^{c0\bar{K}^{*0}}P_{2\nu} = iA_{\chi_{c0}\bar{K}^{*0}}P_{2\nu},
$$

and

$$
h^{*\kappa\lambda}_{\chi_{c0}\bar{K}^{*0}}P_{1\mu}P_{2\nu} = iA_{\chi_{c0}\bar{K}^{*0}}P_{1\mu}P_{2\nu},
$$

and consequently

$$
|\bar{\mathcal{M}}_{\chi_{c0}\rightarrow K^{*0}}|^2 = \frac{1}{3} \left[ \frac{2 + (P_1 \cdot P_2)^2}{m_{K^{*0}}^2} \right] A^2_{\chi_{c0}\bar{K}^{*0}}.
$$  \hspace{1cm} (B18)

For on-shell states, $P^2_{1,2} = m_{K^{*0}}^2$ and Eq. (B18) reduces to

$$
|\bar{\mathcal{M}}_{\chi_{c0}\rightarrow K^{*0}}|^2 = \frac{1}{3} \left[ \frac{2 + (P_1 \cdot P_2)^2}{m_{K^{*0}}^2} \right] A^2_{\chi_{c0}\bar{K}^{*0}}.
$$  \hspace{1cm} (B19)

Consequently, the decay width is

$$
\Gamma_{\chi_{c0}\rightarrow K^{*0}} = \frac{|\bar{\mathcal{M}}_{\chi_{c0}\rightarrow K^{*0}}|^2}{8\pi m_{\chi_{c0}}}.
$$  \hspace{1cm} (B20)

Decay channel $\chi_{c0} \rightarrow \omega \omega$
The corresponding interaction Lagrangian is extracted as

$$\mathcal{L}_{\chi_C 0 \omega} = \frac{1}{2} h^C_1 \phi C \chi_C 0 \bar{\omega} \omega_N \mu, \quad (B21)$$

which also has the same form as the interaction Lagrangian $\mathcal{L}_{\chi_C 0 \rightarrow K^+ K^-}$, thus one can obtain the decay width for $\chi_C 0 \rightarrow \omega \omega$ as

$$\Gamma_{\chi_C 0 \rightarrow \omega \omega} = 2 \frac{[m_{\chi_C 0}^4 - 4 m_{\phi}^2 m_{\chi_C 0}^2]^{1/2}}{16 \pi m_{\chi_C 0}^2} \frac{1}{4} (h^C_1)^2 \phi C \left[ 2 + \frac{(m_{\chi_C 0}^2 - 2 m_{\phi}^2)^2}{4 m_{\phi}^4} \right]. \quad (B22)$$

**Decay channel $\chi_C 0 \rightarrow \phi \phi$**

The corresponding interaction Lagrangian is extracted as

$$\mathcal{L}_{\chi_C 0 \phi \phi} = \frac{1}{2} h^C_1 \phi C \chi_C 0 \bar{\phi} \phi S \mu. \quad (B23)$$

Similar to the decay width for $\chi_C 0 \rightarrow \omega \omega$, the decay width for $\chi_C 0 \rightarrow \phi \phi$ is

$$\Gamma_{\chi_C 0 \rightarrow \phi \phi} = 2 \frac{[m_{\chi_C 0}^4 - 4 m_{\phi}^2 m_{\chi_C 0}^2]^{1/2}}{16 \pi m_{\chi_C 0}^2} \frac{1}{4} (h^C_1)^2 \phi C \left[ 2 + \frac{(m_{\chi_C 0}^2 - 2 m_{\phi}^2)^2}{4 m_{\phi}^4} \right]. \quad (B24)$$

**Decay channel $\chi_C 0 \rightarrow \rho \rho$**

The corresponding interaction Lagrangian is extracted as

$$\mathcal{L}_{\chi_C 0 \rho \rho} = \frac{1}{2} h^C_1 \phi C \chi_C 0 (\rho^0 \rho^0 + 2 \rho^{-} \rho^+), \quad (B25)$$

which also has the same form as $\mathcal{L}_{\chi_C 0 \omega \omega}$. We thus obtain the decay width as

$$\Gamma_{\chi_C 0 \rightarrow \rho \rho} = 3 \Gamma_{\chi_C 0 \rightarrow \phi \phi} = 3 \frac{[m_{\chi_C 0}^4 - 4 m_{\rho}^2 m_{\chi_C 0}^2]^{1/2}}{16 \pi m_{\chi_C 0}^2} \frac{1}{12} (h^C_1)^2 \phi C \left[ 2 + \frac{(m_{\chi_C 0}^2 - 2 m_{\rho}^2)^2}{4 m_{\rho}^4} \right]. \quad (B26)$$

**Decay channel $\chi_C 0 \rightarrow a_0 a_0$**

The corresponding interaction Lagrangian has the form

$$\mathcal{L}_{\chi_C 0 a_0 a_0} = - \chi^C_1 \phi C \chi_C 0 (a_0^2 + 2 a_0^+ a_0^-). \quad (B27)$$

The decay width of $\chi_C 0$ into $a_0 a_0$ can be obtained as

$$\Gamma_{\chi_C 0 \rightarrow a_0 a_0} = 3 \Gamma_{\chi_C 0 \rightarrow a_0 a_0} = 3 \frac{[m_{\chi_C 0}^4 - 4 m_{a_0}^2 m_{\chi_C 0}^2]^{1/2}}{16 m_{\chi_C 0}^2} \frac{1}{2m_{\chi_C 0}} \frac{(h^C_1)^2 \phi C}{8 \pi m_{\chi_C 0}^2} \left[ \frac{m_{\chi_C 0}^4 - 4 m_{a_0}^2 m_{\chi_C 0}^2}{2m_{\chi_C 0}} \right]. \quad (B28)$$

**Decay channel $\chi_C 0 \rightarrow K^+ K^- 0$**

The corresponding interaction Lagrangian from the Lagrangian $\mathcal{L}$ reads

$$\mathcal{L}_{\chi_C 0 K^+ K^- 0} = Z_{K^0} w_{K^0} h^C_1 \phi C \chi_C 0 \bar{K}^0 \partial_\mu K^0, \quad (B29)$$
which obtain from the corresponding interaction Lagrangian

$$\mathcal{L}_{\chi_{C0}K^*\overline{K}_0^0} = Z_{K^0} w_{K^*} h_C^0 \phi_C \chi_{C0}(\overline{K}_0^0 0 0 - \partial_\mu K_0^0 - 0 - 0 K_0^0 + 0 - K_{3\mu - \mu} + 0 - 0 K_{0^0}^0 K_{0^0}^0).$$  (B30)

We compute the decay width as

$$\Gamma_{\chi_{C0}K^*\overline{K}_0^0} = \frac{\sqrt{\kappa_1}}{8\pi m^2_{\chi_{C0}}} w^2_{K^*} Z_{K^*} (h_C^0)^2 \left| -m^2_{K^*0} + \frac{(m^2_{\chi_{C0}} - m^2_{K^*0} - m^2_{K_{0^0}})^2}{4m^2_{K_{0^0}}^2} \right|. \quad (B31)$$

Note that we considered only the decay channel $\chi_{C0}K^*\overline{K}_0^0$ because the other decay channels contribute the same of isospin symmetry reasons. Thus,

$$\Gamma_{\chi_{C0}K^*\overline{K}_0^0} = \Gamma_{\chi_{C0}K^*\overline{K}_0^0} + \Gamma_{\chi_{C0}K^*\overline{K}_0^0} + \Gamma_{\chi_{C0}K^*\overline{K}_0^0} + \Gamma_{\chi_{C0}K^*\overline{K}_0^0}. \quad (B32)$$

**Decay channels $\chi_{C0} \to \eta, \eta'$**

The corresponding interaction Lagrangian of $\chi_{C0}$ with the $\eta'$ and the $\eta$ resonances reads

$$\mathcal{L}_{\chi_{C0}\eta\eta', \eta_3, \eta_3, \eta_3, \eta_3} = \left[ -\lambda_*^C (Z_{\eta_3}^2 \cos^2 \varphi_{\eta} + Z_{\eta_3}^2 \sin^2 \varphi_{\eta}) - \frac{1}{2} c \phi_3^0 \phi_3^2 Z_{\eta_3} \cos^2 \varphi_{\eta} \right]
+ \frac{1}{8} \phi_3^0 \phi_3^2 Z_{\eta_3}^2 \sin^2 \varphi_{\eta} + \phi_3 Z_{\eta_3} Z_{\eta_3} \sin \varphi_{\eta} \cos \varphi_{\eta} ;
+ \frac{1}{2} h_C^0 w_{f_{j_3}}^2 Z_{\eta_3} \cos^2 \varphi_{\eta} + w_{f_{j_3}}^2 Z_{\eta_3}^2 \sin^2 \varphi_{\eta} \right) \chi_{C0} \partial_\mu \eta \partial^\mu \eta $$

$$\left[ -\lambda_*^C (Z_{\eta_3}^2 \cos^2 \varphi_{\eta} + Z_{\eta_3}^2 \sin^2 \varphi_{\eta}) \right]
+ \frac{1}{2} c \phi_3^0 \phi_3^2 Z_{\eta_3}^2 \cos^2 \varphi_{\eta} \n+ \phi_3 Z_{\eta_3} Z_{\eta_3} \sin \varphi_{\eta} \cos \varphi_{\eta} ;
+ \frac{1}{2} h_C^0 w_{f_{j_3}}^2 Z_{\eta_3} \cos^2 \varphi_{\eta} + w_{f_{j_3}}^2 Z_{\eta_3}^2 \cos^2 \varphi_{\eta} \right) \chi_{C0} \partial_\mu \eta \partial^\mu \eta $$

By using Eq. (11), the interaction Lagrangian (B33) will transform to a Lagrangian which describes the interaction of $\chi_{C0}$ with $\eta$ and $\eta'$,

$$\mathcal{L}_{\chi_{C0}\eta^2, \eta'^2, \eta'^2} = \left[ -\lambda_*^C (Z_{\eta_3}^2 \cos^2 \varphi_{\eta} + Z_{\eta_3}^2 \sin^2 \varphi_{\eta}) \right]
+ \frac{1}{2} c \phi_3^0 \phi_3^2 Z_{\eta_3} \cos^2 \varphi_{\eta} \n+ \phi_3 Z_{\eta_3} Z_{\eta_3} \sin \varphi_{\eta} \cos \varphi_{\eta} ;
+ \frac{1}{2} h_C^0 w_{f_{j_3}}^2 Z_{\eta_3} \cos^2 \varphi_{\eta} + w_{f_{j_3}}^2 Z_{\eta_3} \cos^2 \varphi_{\eta} \right) \chi_{C0} \partial_\mu \eta \partial^\mu \eta $$

which contains three different decay channels, $\chi_{C0} \to \eta \eta$, $\chi_{C0} \to \eta' \eta'$, and $\chi_{C0} \to \eta \eta'$, with the following vertices

$$A_{\chi_{C0}\eta} = -\lambda_*^C \phi_C (Z_{\eta_3}^2 \cos^2 \varphi_{\eta} + Z_{\eta_3}^2 \sin^2 \varphi_{\eta})$$

$$B_{\chi_{C0}\eta} = \frac{1}{2} h_C^0 \phi_C (w_{f_{j_3}}^2 Z_{\eta_3}^2 \cos^2 \varphi_{\eta} + w_{f_{j_3}}^2 Z_{\eta_3} \sin^2 \varphi_{\eta}), \quad (B35)$$

$$C_{\chi_{C0}\eta} = \frac{1}{2} h_C^0 \phi_C (w_{f_{j_3}}^2 Z_{\eta_3}^2 \cos^2 \varphi_{\eta} + w_{f_{j_3}}^2 Z_{\eta_3} \sin^2 \varphi_{\eta}), \quad (B36)$$
Then, we compute the decay widths for all these channels by using the formula of the two-body decay (A1).

The corresponding interaction Lagrangian is extracted from the Lagrangian (1)

\[
A_{\chi_{\text{co}}} = -\lambda_C^\prime \phi_C Z_{\eta N}^2 \sin^2 \varphi_\eta + Z_{\eta S}^2 \cos^2 \varphi_\eta \\
- \frac{1}{2} \phi_N^2 \phi_C (\phi_S^2 Z_{\eta N}^2 \sin^2 \varphi_\eta + \frac{1}{4} \phi_N^2 Z_{\eta S}^2 \cos^2 \varphi_\eta - \phi_N \phi_S Z_{\eta N} Z_{\eta S} \sin \varphi_\eta \cos \varphi_\eta),
\]

(B37)

\[
B_{\chi_{\text{co}}} = \frac{1}{2} h_C^\prime \phi_C (w_{11N}^2 Z_{\eta N}^2 \sin^2 \varphi_\eta + w_{11S}^2 Z_{\eta S}^2 \cos^2 \varphi_\eta),
\]

(B38)

\[
A_{\chi_{\text{co}}} = -2 \lambda_C^\prime \phi_C (-Z_{\eta N}^2 + Z_{\eta S}^2) \sin \varphi_\eta \cos \varphi_\eta + \frac{1}{2} \phi_N^2 \phi_C (2 \phi_S^2 Z_{\eta N}^2 - \frac{1}{2} \phi_N^2 Z_{\eta S}^2) \cos \varphi_\eta \sin \varphi_\eta \\
- \frac{1}{2} \phi_N^3 \phi_S \phi_C Z_{\eta N} Z_{\eta S} (\cos^2 \varphi_\eta - \sin^2 \varphi_\eta \cos \varphi_\eta),
\]

(B39)

\[
B_{\chi_{\text{co}}} = h_C^\prime \phi_C \cos \varphi_\eta \sin \varphi_\eta (w_{11S}^2 Z_{\eta S}^2 - w_{11N}^2 Z_{\eta N}^2).
\]

(B40)

Let us firstly consider the channel $\chi_{\text{co}} \rightarrow \eta \eta$. We denote the momenta of the two outgoing $\eta$ particles as $P_1$ and $P_2$, and $P$ denotes the momentum of the decaying $\chi_{\text{co}}$ particle. Given that our particles are on shell, we obtain

\[
P_1 \cdot P_2 = \frac{P^2 - P_1^2 - P_2^2}{2} = \frac{m_{\chi_{\text{co}}}^2 - 2m_\eta^2}{2}.
\]

(B41)

After replacing $\partial_\mu \rightarrow +iP^\mu$ for the outgoing particles, one obtains the decay amplitude as

\[-iM_{\chi_{\text{co}} \rightarrow \eta \eta} = i \left[ A_{\chi_{\text{co}}} - B_{\chi_{\text{co}}} \frac{m_{\chi_{\text{co}}}^2 - 2m_\eta^2}{2} \right].
\]

(B42)

Then the decay width is

\[
\Gamma_{\chi_{\text{co}} \rightarrow \eta \eta} = \frac{1}{8\pi} \frac{\tilde{k}_1}{m_{\chi_{\text{co}}}^2} \left| -iM_{\chi_{\text{co}} \rightarrow \eta \eta} \right|^2.
\]

(B43)

Similarly, the decay width of $\chi_{\text{co}}$ into $\eta' \eta'$ is obtained as

\[
\Gamma_{\chi_{\text{co}} \rightarrow \eta' \eta'} = \frac{1}{8\pi} \frac{\tilde{k}_1}{m_{\chi_{\text{co}}}^2} \left| A_{\chi_{\text{co}}} - B_{\chi_{\text{co}}} \frac{m_{\chi_{\text{co}}}^2 - 2m_{\eta'}^2}{2} \right|^2.
\]

(B44)

In a similar way, the decay width of $\chi_{\text{co}}$ into $\eta \eta'$ can be obtained as

\[
\Gamma_{\chi_{\text{co}} \rightarrow \eta \eta'} = \frac{1}{8\pi} \frac{\tilde{k}_1}{m_{\chi_{\text{co}}}^2} \left| A_{\chi_{\text{co}}} - B_{\chi_{\text{co}}} \frac{m_{\chi_{\text{co}}}^2 - m_\eta^2 - m_{\eta'}^2}{2} \right|^2.
\]

(B45)

**Decay channels $\chi_{\text{co}} \rightarrow f_0 f_0$**

The corresponding interaction Lagrangian is extracted from the Lagrangian (1)

\[
L_{\chi_{\text{co}} f_0 f_0} = -\lambda_C^\prime \phi_C \chi_{\text{co}} (\sigma_N^2 + \sigma_S^2) - \frac{m_\eta^2}{G_0^2} \phi_C \chi_{\text{co}} G^2.
\]

(B46)

By using the mixing matrices (14), (13) and (16), we obtain the interaction Lagrangian (B46) as a function of all the following channels:

\[
\chi_{\text{co}} \rightarrow f_0(1370)f_0(1370), \chi_{\text{co}} \rightarrow f_0(1500)f_0(1500),
\]

\[
\chi_{\text{co}} \rightarrow f_0(1370)f_0(1500), \chi_{\text{co}} \rightarrow f_0(1370)f_0(1710),
\]

\[
\chi_{\text{co}} \rightarrow f_0(1500)f_0(1710).
\]

Then, we compute the decay widths for all these channels by using the formula of the two-body decay (A1).
2. Three-body decay rates for $\chi_{C0}$

Decay channel $\chi_{C0} \rightarrow K^*_0 K \eta, \eta'$

The corresponding interaction Lagrangian can be obtained from the Lagrangian (1) as

$$\mathcal{L}_{\chi_{C0}K^*_0K\eta,\eta'} = \frac{1}{\sqrt{2}} (c - \frac{1}{2} \phi^2 \delta \bar{c}) Z_K Z_{K^*_0} \phi^2 N \phi_S \phi_C \chi_{C0} \eta_N (K^*_0 K) + \bar{K}^*_0 K^{0} + K^* - K^- + K^{*+} - K^-$$

Consequently, the amplitude decay for the decay channels $\chi_{C0} \rightarrow K^*_0 K \eta, \eta'$ can be obtain as

$$- i \mathcal{M}_{\chi_{C0} \rightarrow K^*_0 K \eta} = - \frac{1}{\sqrt{2}} (c - \frac{1}{2} \phi^2 \delta \bar{c}) \phi_C \phi^2 N Z_K Z_{K^*_0} (\phi_S Z_{\eta_N} \cos \varphi_\eta + \frac{1}{2} \phi_N Z_{\eta_S} \sin \varphi_\eta),$$  \hspace{1cm} (B49)

and

$$- i \mathcal{M}_{\chi_{C0} \rightarrow K^*_0 K \eta'} = \frac{1}{\sqrt{2}} (c - \frac{1}{2} \phi^2 \delta \bar{c}) \phi_C \phi^2 N Z_K Z_{K^*_0} (- \phi_S Z_{\eta_N} \sin \varphi_\eta + \frac{1}{2} \phi_N Z_{\eta_S} \cos \varphi_\eta),$$  \hspace{1cm} (B50)

which are used to compute $\Gamma_{\chi_{C0} \rightarrow K^*_0 K \eta}$ and $\Gamma_{\chi_{C0} \rightarrow K^*_0 K \eta'}$ by Eq. (A3).

Decay channel $\chi_{C0} \rightarrow f_0 \eta, \eta'$

The corresponding interaction Lagrangian is extracted from the Lagrangian (1) and given by

$$\mathcal{L}_{\chi_{C0}f_0 \eta, \eta'} = - \frac{3}{2} (c - \frac{1}{2} \phi^2 \delta \bar{c}) Z_Q Z_{\eta_S} \phi_S \phi_S \phi_C \chi_{C0} \sigma_N \eta_N \eta_S - c Z^2 \eta_S \phi^2 N \phi_S \phi_C \chi_{C0} \sigma \eta^2 N$$

$$- (c - \frac{1}{2} \phi^2 \delta \bar{c}) Z^2 \eta_S \phi^2 N \phi_S \phi_C \chi_{C0} \sigma \eta^2 N - \frac{1}{2} (c - \frac{1}{2} \phi^2 \delta \bar{c}) Z^2 \eta_S \phi^3 N \phi_S \phi_C \chi_{C0} \sigma \eta^2 S$$

$$- \frac{1}{2} (c - \frac{1}{2} \phi^2 \delta \bar{c}) Z_{\eta_N}, Z_{\eta_S} \phi^3 N \phi_C \chi_{C0} \sigma \eta_S \eta_N.$$  \hspace{1cm} (B51)

We get the decay amplitudes for the following channels:

$$\chi_{C0} \rightarrow f_0(1370) \eta, \chi_{C0} \rightarrow f_0(1370) \eta' \eta'$$

$$\chi_{C0} \rightarrow f_0(1370) \eta', \chi_{C0} \rightarrow f_0(1500) \eta$$

$$\chi_{C0} \rightarrow f_0(1500) \eta', \chi_{C0} \rightarrow f_0(1710) \eta$$

$$\chi_{C0} \rightarrow f_0(1710) \eta'',$$

which used in Eq. (A3) to compute the decay widths for all these channels.
Appendix C: Decay rates for \( \eta_C \)

We present the explicit expressions for the two- and three-body decay rates for the pseudoscalar hidden-charmed meson \( \eta_C \).

The chiral Lagrangian contains the tree-level vertices for the decay processes of the pseudoscalar \( \eta \) into (pseudo)scalar mesons, through the chiral anomaly term

\[
L_{U(1)\phi} = c (\text{det}\Phi - \text{det}\Phi^\dagger)^2 - \delta c (\text{det}\Phi - \text{det}\Phi^\dagger)^2 \text{Tr}(\mathcal{P}_C\Phi\Phi^\dagger\mathcal{P}_C\Phi). \tag{C1}
\]

The terms in the Lagrangian \( (C1) \) which correspond to decay processes of \( \eta \) meson through the chiral anomaly term

\[
\Gamma = \sum_{\text{two-body decay}} \text{Re} \text{Im} \mathcal{M}^2 \] 

The corresponding interaction Lagrangian can be obtained from the Lagrangian \( (C2) \) as

\[
L_{\eta_C} = \frac{1}{8}(c - \frac{1}{2} \phi_C^2 \bar{\phi}_0) \phi_N^2 \phi_C Z_{\eta_C} \eta_C \{ \sqrt{2} \phi_N Z_K Z_{K_0^*} (K_0^0 K_0^0 + K_0^*-K^+ + K_0^{**} K^-) + 2Z_\pi^2 \phi_N^2 (a_0^0) + a_0^+ \phi^+ + a_0^- \phi^- \} - 4 \phi_N \phi_S (Z_{\eta_S} \eta_S + Z_{\eta_N} \eta_N) 
\]

The decay widths obtained as

\[
\Gamma = \frac{1}{8}(c - \frac{1}{2} \phi_C^2 \bar{\phi}_0) \phi_N^2 \phi_C Z_{\eta_C} \eta_C \{ \sqrt{2} \phi_N Z_K Z_{K_0^*} (K_0^0 K_0^0 + K_0^*-K^+ + K_0^{**} K^-) \} \tag{C2}
\]

1. Two-body decay expressions for \( \eta_C \)

The explicit expressions for the two-body decay widths of \( \eta_C \) are given by

**Decay channel \( \eta_C \to K_0^+K \)**

The corresponding interaction Lagrangian can be obtained from the Lagrangian \( (C2) \) as

\[
L_{\eta_C} = \frac{1}{8}(c - \frac{1}{2} \phi_C^2 \bar{\phi}_0) \phi_N^2 \phi_C Z_K Z_{K_0^*} \eta_C \{ \sqrt{2} \phi_N Z_K Z_{K_0^*} (K_0^0 K_0^0 + K_0^*-K^+ + K_0^{**} K^-) \}. \tag{C3}
\]

The decay width obtained as

\[
\Gamma_{\eta_C \to K_0^+K} = \frac{\sqrt{2}}{8} \frac{1}{68\pi m_{\eta_C}^2} (c - \frac{1}{2} \phi_C^2 \bar{\phi}_0)^2 \phi_N^2 \phi_C^2 Z_K Z_{K_0^*} \eta_C. \tag{C4}
\]

**Decay channel \( \eta_C \to a_0 \pi \)**

The corresponding interaction Lagrangian is extracted as

\[
L_{\eta_C} = \frac{1}{4}(c - \frac{1}{2} \phi_C^2 \bar{\phi}_0) \phi_N^2 \phi_C Z_{\eta_C} \eta_C \{ a_0^0 \phi^0 + a_0^+ \phi^+ + a_0^- \phi^- \}. \tag{C5}
\]

The decay width obtained as

\[
\Gamma_{\eta_C \to a_0 \pi} = \frac{3\sqrt{2}}{128\pi m_{\eta_C}^2} (c - \frac{1}{2} \phi_C^2 \bar{\phi}_0)^2 \phi_N^2 \phi_C^2 Z_K Z_{K_0^*} \eta_C. \tag{C6}
\]

**Decay channel \( \eta_C \to f_0 \eta, \eta' \)**

The corresponding interaction Lagrangian can be obtained from the Lagrangian \( (C2) \) as

\[
L_{\eta_C \eta} = \frac{1}{8}(c - \frac{1}{2} \phi_C^2 \bar{\phi}_0) \phi_N^2 \phi_C Z_{\eta_C} \eta_C \{ -4 \phi_N \phi_S (Z_{\eta_S} \eta_S \sigma_N + Z_{\eta_N} \eta_N \sigma_S) 
- 6 \phi_S^2 Z_{\eta_S} \eta_N + \phi_N^2 \eta_N \sigma_S \} \tag{C7}
\]
By substituting from Eqs. (11) and mixing matrices (14), (15) and (16), then using the two-body decay \(A1\). We get the results of several scenarios of the decay widths:

\[
\Gamma_{\eta C \to f_0(1370)\eta}, \quad \Gamma_{\eta C \to f_0(1500)\eta}, \quad \Gamma_{\eta C \to f_0(1710)\eta},
\]

\[
\Gamma_{\eta C \to \eta'\eta}, \quad \Gamma_{\eta C \to \eta'(1500)\eta}, \quad \Gamma_{\eta C \to \eta(1710)\eta'}.
\]

2. Three-body decay expressions for \(\eta_C\)

The corresponding interaction Lagrangian, contains the three-body decay rates for the \(\eta_C\) meson, is extracted as

\[
\mathcal{L}_{\eta C} = \frac{1}{8} (c - \frac{1}{2} \phi_C^2 \phi_C^3) \phi_N^2 \phi_C \left( 2 \phi_N Z_{qs}^2 Z_{\eta N}^2 \eta S + 6 \phi_S Z_{\eta N}^2 \eta S \right)
\]

\[
- \sqrt{2} \phi_N Z_{qs}^2 Z_{K}^2 \left( K^0 K^0 + K^- K^+ \right) \eta S - 3 \sqrt{2} \phi_S Z_{\eta N}^2 Z_{K}^2 \left( K^0 K^0 + K^- K^+ \right) \eta N
\]

\[
+ \sqrt{2} \phi_S Z_{\pi} Z_{K}^2 \left[ \sqrt{2} (K^0 K^+ + K^0 K^-) \eta S - (K^0 K^0 - K^- K^+) \eta N \right]
\]

\[
- 2 \phi_S \eta S Z_{qs}^2 Z_{K}^2 \left( \eta S + 2 \eta S \right)
\].

(C8)

Using the general formula for the three-body decay width for \(\eta_C\) \(A3\), which the corresponding tree-level decay amplitudes for \(\eta_C\) are obtained as follows:

Decay channel \(\eta_C \to \eta^3\): \(m_1 = m_2 = m_3 = m_{\eta}\) and \(S = 6\)

\[
| -iM_{\eta C \to \eta^3} |^2 = \left[ \frac{1}{4} (c - \frac{1}{2} \phi_C^2 \phi_C^3) \phi_N^2 \phi_C \sin \phi \cos \phi (Z_{qs} \phi_N \sin \phi + 3 \phi_N \phi_S \cos \phi) Z_{\eta C} Z_{\eta N} Z_{\eta S} \right]^2.
\]

(C9)

Decay channel \(\eta_C \to \eta^3\): \(m_1 = m_2 = m_3 = m_{\eta'}\) and \(S = 6\)

\[
| -iM_{\eta C \to \eta^3} |^2 = \left[ \frac{1}{4} (c - \frac{1}{2} \phi_C^2 \phi_C^3) \phi_N^2 \phi_C \sin \phi \cos \phi (Z_{qs} \phi_N \cos \phi - 3 \phi_N \phi_S \sin \phi) Z_{\eta C} Z_{\eta N} Z_{\eta S} \right]^2.
\]

(C10)

Decay channel \(\eta_C \to \eta' \eta^2\): \(m_1 = m_{\eta'}, \ m_2 = m_3 = m_{\eta}\) and \(S = 2\)

\[
| -iM_{\eta C \to \eta' \eta^2} |^2 = \frac{1}{16} (c - \frac{1}{2} \phi_C^2 \phi_C^3) \phi_N^2 \phi_C \left[ \phi_N Z_{qs} \sin \phi \eta (2 \cos^2 \phi - \sin^2 \phi) \right.
\]

\[
+ \left. 3 \phi_N \phi_S \sin \phi \eta (2 \cos^2 \phi - \sin^2 \phi) \right]^2.
\]

(C11)

Decay channel \(\eta_C \to \eta^2 \eta^2\): \(m_1 = m_2 = m_{\eta'}, \ m_3 = m_{\eta}\) and \(S = 2\)

\[
| -iM_{\eta C \to \eta^2 \eta^2} |^2 = \frac{1}{16} (c - \frac{1}{2} \phi_C^2 \phi_C^3) \phi_N^2 \phi_C \left[ Z_{qs} \phi_N \cos \phi \eta (2 \cos^2 \phi - 2 \cos^2 \phi) \right.
\]

\[
+ \left. 3 \phi_N \phi_S \sin \phi \eta (2 \cos^2 \phi - 2 \cos^2 \phi) \right]^2.
\]

(C12)
Decay channel $\eta K\bar{K}$: $m_1 = K^0$, $m_2 = m_{\eta K}$, $m_3 = m_{\eta}$

$$\Gamma_{\eta C \to \eta K \bar{K}} = \Gamma_{\eta C \to \eta K^+ K^-} + \Gamma_{\eta C \to \eta K^0 K^0} = 2\Gamma_{\eta C \to \eta K^0 K^0}. \quad (C13)$$

with the average modulus squared decay amplitude

$$|\overline{M_{\eta C \to \eta K \bar{K}}}|^2 = \frac{1}{32}(c - \frac{1}{2}\phi_C^2 \phi_C \phi_S^2 \phi_S Z_{\eta C}^2 Z_{\eta S}^2 |Z_K| (\phi_N Z_{\eta S} \sin \varphi_\eta + 3\phi S Z_{\eta S} \cos \varphi_\eta)^2. \quad (C14)$$

Decay channel $\eta_C \to \eta' K\bar{K}$: $m_1 = K^0$, $m_2 = m_{\eta K}$, $m_3 = m_{\eta'}$

$$\Gamma_{\eta C \to \eta' K \bar{K}} = 2\Gamma_{\eta C \to \eta' K^0 K^0}. \quad (C15)$$

The average modulus squared decay amplitude for this process reads

$$|\overline{M_{\eta C \to \eta' K \bar{K}}}|^2 = \frac{1}{32}(c - \frac{1}{2}\phi_C^2 \phi_C \phi_S^2 \phi_S Z_{\eta C}^2 Z_{\eta S}^2 |Z_K| (\phi_N Z_{\eta S} \cos \varphi_\eta - 3\phi S Z_{\eta S} \sin \varphi_\eta)^2. \quad (C16)$$

Decay channel $\eta_C \to \eta \pi \pi$: $m_1 = \eta$, $m_2 = m_3 = m_{\pi 0}$ and $S = 2$

$$\Gamma_{\eta C \to \eta \pi \pi} = 3\Gamma_{\eta C \to \eta \pi 0 \pi 0}. \quad (C17)$$

where the average modulus squared decay amplitude for this process is obtained from the Lagrangian (C8) as

$$|\overline{M_{\eta C \to \eta \pi \pi}}|^2 = \frac{1}{16}(c - \frac{1}{2}\phi_C^2 \phi_C \phi_S^2 \phi_S Z_{\eta C}^2 Z_{\eta S}^2 |Z_K| \sin^2 \varphi_\eta. \quad (C18)$$

Decay channel $\eta_C \to \eta' \pi \pi$: $m_1 = \eta'$, $m_2 = m_3 = m_{\pi 0}$ and $S = 2$

$$\Gamma_{\eta C \to \eta' \pi \pi} = 3\Gamma_{\eta C \to \eta' \pi 0 \pi 0}. \quad (C19)$$

where the average modulus squared decay amplitude for this process is

$$|\overline{M_{\eta C \to \eta' \pi \pi}}|^2 = \frac{1}{16}(c - \frac{1}{2}\phi_C^2 \phi_C \phi_S^2 \phi_S Z_{\eta C}^2 Z_{\eta S}^2 |Z_K| \cos^2 \varphi_\eta. \quad (C20)$$

Decay channel $\eta_C \to K K \pi$: $m_1 = K^+$, $m_2 = K^-$, $m_3 = m_{\pi 0}$ and $S = 2$

$$\Gamma_{\eta C \to K K \pi} = \Gamma_{\eta C \to K^+ K^- \pi^0} + \Gamma_{\eta C \to K^0 K^0 \pi^0} + \Gamma_{\eta C \to K^0 K^+ \pi^-} + \Gamma_{\eta C \to K^0 K^- \pi^+} = 4\Gamma_{\eta C \to K^+ K^- \pi^0}. \quad (C21)$$

with the average modulus squared decay amplitude

$$|\overline{M_{\eta C \to K K \pi}}|^2 = \frac{1}{32}(c - \frac{1}{2}\phi_C^2 \phi_C \phi_S^2 \phi_S Z_{\eta C}^2 Z_{\eta S}^2 |Z_K| \sin^2 \varphi_\eta. \quad (C22)$$

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