Source localization in reverberant rooms using sparse modeling and narrowband measurements

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Abstract—We study two cases of acoustic source localization in a reverberant room, from a number of point-wise narrowband measurements. In the first case, the room is perfectly known. We show that using a sparse recovery algorithm with a dictionary of sources computed a priori requires measurements at multiple frequencies. Furthermore, we study the choice of frequencies for these measurements, and show that one should avoid the modal frequencies of the room. In the second case, when the shape and the boundary conditions of the room are unknown, we propose a model of the acoustical field based on the Vekua theory, still allowing the localization of sources, at the cost of an increased number of measurements. Numerical results are given, using simple adaptations of standard sparse recovery methods.

Index Terms—Source localization, sparsity, microphone array, reverberation.

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I. INTRODUCTION

Acoustic or electromagnetic source localization is an inverse problem for which numerous methods have been developed, based on various models and algorithms. A common assumption is that a low number of sources are present. This assumption can be modeled in various ways, such as the low-dimensionality of a subspace build from the measurements in subspace methods (e.g. MUSIC [1]), or as the sparsity of the measurements in a pre-defined dictionary. We will use here this latter model, introduced for source localization by Malioutov et al. [2].

Another frequent assumption is that the wave propagation occurs in free-field, i.e. that the acoustic field verifies the Sommerfeld radiation condition. In this case, the Green function of the medium is known, but this limits the range of application to particular cases such as open environment or anechoic chambers, or at least to rooms with small reverberation.

The particular focus of this paper is the localization of a small number of sources in a reverberant environment, from frequency-domain measurements. There has been an increased activity in the last few years, where localization in this framework was improved by explicitly taking into account the specifics of the environment, e.g. the shape of the room and the reflective properties of its walls, assuming these are known [3], [4]. The goal of this paper is to compare the performance of this approach with known environment, to a more generic model of reverberation based on a sparse model of the wavefield itself, that does not require the knowledge of the propagation environment. We first recall some existing methods for different cases: known or unknown environment, and time or frequency domain.

In the case of known environments and in the time domain, by using the fact that the wave equation is invariant by reversing the time parameter, so-called time reversal techniques [5], [6] allow a robust localization of one source. After recording the sound radiated by a source on an array of transducers, these recording are played backwards by the microphones. The resulting soundfield (that can be either produced experimentally or simulated) focuses back to the location of the original source. However, its resolution is limited by the standard wave diffraction limit, and this method does not easily take into account prior knowledge on the source.

Another method in the known environment/time domain case is the use of cosparsity [7]. The soundfield created by a low number \( J \) of sources is solution to the wave equation with Neumann boundary conditions (in the ideal case of rigid walls)

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u &= \sum_{j=1}^{J} s_j \delta_{x_j} \\
\frac{\partial u}{\partial n} &= 0 \text{ on } \partial \Omega_0
\end{align*}
\]

where \( c \) is the wave velocity, and \( s_j \) the sound emitted by the \( j \)-th source, located at \( x_j \). After discretization (using e.g. finite differences), this can be interpreted as a cosparse model [8] for the pressure, allowing the recovery of the positions of the sources and the signals they emit.

In the frequency domain case, Dokmanic and Vetterli proposed a method allowing the localization of sources in a known reverberant environment using measurements in multiple frequency bands [3]. They replaced the free-field impulse response of the sources by the impulse response computed using the finite element method, and used this as a dictionary. They used a multichannel Orthogonal Matching Pursuit (OMP) to consider the different frequency bands. This method has interesting properties, such as localization of sources without direct line of sight, but as will be shown in this paper, needs more and more frequency bands as the number of sources increases. A variant of this method in the time domain was proposed by Le Roux et al. [4].

For the localization in unknown environment and using narrowband measurements, we introduce a sparse model for soundfields based on the Vekua theory [9], [10], and the associated sparse recovery algorithms. While this method needs more measurements than the method introduced by Dokmanic and Vetterli, it does not need any prior information on the shape of the room or its boundary conditions, and can be used with only one arbitrarily chosen frequency. For the sake of simplicity and readability, most of the results will be...
given for propagation in 2D domains. The adaptation to 3D domains is however straightforward.

The paper is structured as follows. We recall in section 2 the application of sparsity to source localization in free-field using narrowband measurements. We recall some results about Green functions in section 3. In section 4 we recall the method developed by Dokmanic and Vetterli in a known reverberant, and study the necessity of multiple bands measurements. In section 5, we introduce our sparse model for reverberated sources and the associated algorithms and we show corresponding numerical results.

II. SPARSITY AND SOURCE LOCALIZATION

Sparsity is a signal processing paradigm used in various domains, such as compression, machine learning, inverse problems, etc. The application to the particular problem of source localization has been introduced by Malioutov et al. [2]. We give here the basic concepts of sparsity applied to source localization.

The soundfield is measured on an array of $M$ microphones located at points $x_m$, and is assumed to be emitted by $N$ sources located at points $y_n$. The pressure emitted by the $n$-th source measured at the $m$-th microphone is given by $a_n G(y_n, x_m)$ where $a_n$ is the complex amplitude of the $n$-th source, and $G$ the Green function of the medium. In a 2D free-field, the Green function writes

$$G_0(y, x) = \frac{i}{4} H_0(k\|x - y\|). \quad (1)$$

Using $g(y) = (G_0(y, x_m))_m$, we can write the vector $p$ of measurements on the array as

$$p = \sum_n a_n g(y_n). \quad (2)$$

If we further assume that the points $y_n$ are located on a predefined grid of $L$ points $z_l$, we can build a dictionary $G$ of possible sources, with $g(z_l)$ as the $l$-th column. Using this dictionary, we can write

$$p = Ga$$

where $a$ is a $L$-dimensional vector, with only $N$ non-zero coefficients whose indexes correspond to the positions of the sources.

The sparsity of $a$ allows the use of sparse recovery methods, although the number of pressure measurements is much lower than $L$ the dimension of the space (equation 2 is underdetermined). More precisely, the compressed sensing paradigm indicates that, under some conditions on the measurements, the required number of measurements scales as $N$, the number of non-zero coefficients in $a$ (i.e., the number of sources to recover), with $N << L$. Many algorithms exist for such sparse recovery problem, including Basis Pursuit [11], or iterative algorithms (Matching Pursuit (MP) [12], Orthogonal Matching Pursuit (OMP) [13], etc.).

A critical point of this framework is the computation of the dictionary and its numerical properties. While the construction of the dictionary is straightforward in the free-field case, its computation in the case of a room is more involved, and the case of the unknown room requires a more complex model. Tools useful to the treatment of these two cases are given in the next section.

III. PROPERTIES OF GREEN FUNCTIONS

In this section, we review some results on the Green functions. These results will be used for the design of dictionaries for sparse source localization in reverberant environments.

A. Green function in a closed room

The Green function used in the previous section, in the case of free-field propagation, is solution to the Helmholtz equation with a point source as a right-hand side and with the Sommerfeld radiation condition. This radiation condition models the fact that the energy is radiating to infinity, and that no energy is coming from the infinity to the sources.

In a closed room, this condition is replaced by boundary conditions, usually expressed using the values of the pressure and its derivatives at the boundaries. In the ideal case of rigid walls, where there is no normal displacement of the air relative to the walls, the boundary conditions used are the Dirichlet boundary conditions. The normal displacement is proportional to the normal derivative of the pressure, and the Green function is solution to the boundary value problem

$$\left\{ \begin{array}{l} \Delta G + k^2 G = \delta \\ \frac{\partial G}{\partial n} = 0 \text{ on } \partial \Omega_0 \end{array} \right. \quad (3)$$

In the general case, the solution (and thus, the dictionary of sources) cannot be obtained analytically, but we can still obtain decompositions that will be useful for computing and modeling the Green functions.

The Green function (actually any solutions of 3 with some constraints on the right hand side) can be expanded, in the $L_2$ sense, on the modal basis of the room. The vectors of this basis are the eigenmodes of the Laplace operator with Neumann boundary conditions, i.e. non-zero solutions to

$$\left\{ \begin{array}{l} \Delta u + k^2 u = 0 \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega_0 \end{array} \right. \quad (4)$$

The eigenmodes $u_n$ are associated to their spatial eigenfrequencies $k_n$.

Using properties of these eigenmodes, it can be shown that the Green function has the following modal expansion:

$$G(x, y) = \sum_n \frac{u_n(x)u_n(y)}{k^2 - k_n^2}.$$

The Green function can also be decomposed as the sum of a particular solution of the Helmholtz equation with right hand side, but no boundary conditions, and of a solution to the homogeneous equation, such that the sum satisfies the boundary conditions. A typical choice for the particular solutions is the free-field Green function $G_0$. The homogeneous term $G_h$ is then the solution to

$$\left\{ \begin{array}{l} \Delta G_h + k^2 G_h = 0 \\ \frac{\partial G_h}{\partial n} = -\frac{\partial G_0}{\partial n} \text{ on } \partial \Omega_0 \end{array} \right. \quad (5)$$

and the Green function writes

$$G = G_0 + g_h.$$
Numerous approximations schemes exist for the treatment of solutions of the homogeneous Helmholtz equation (Method of Fundamental Solutions [14], Boundary Element Methods [15], etc). We will here use the Vekua theory [9], [10]. A result of this theory is that, in a 2D domain, a solution \( u \) to the homogeneous Helmholtz equation be approximated by sums of Fourier-Bessel functions:

\[
  u \approx \sum_{l=-L}^{L} \alpha_l J_l(kt) e^{in\theta}
\]

or sum of plane waves:

\[
  u \approx \sum_{l=-L}^{L} \beta_l e^{ikx_l y}
\]

where the wavevectors share their wavenumbers. Similar results are available in 3D using spherical harmonics.

These decompositions are valid in any star-convex domain, and do not depend on the boundary conditions. As the order needed to achieve a good approximation is significantly lower than e.g. the number of samples needed to apply the Shannon-Nyquist sampling theorem to these quantities (linear with respect to \( kL \) vs. quadratic), these approximations provide a low-dimensional model for acoustical waves, in particular for the homogeneous component \( G_h \).

**IV. LOCALIZATION IN A KNOWN ROOM**

In the case of source localization in a reverberant room, the Green function no longer has the simple form of the free-field case, but is dependent on the shape of the room and on the boundary conditions. For sake of simplicity, we will consider ideal rigid walls, i.e. Neumann boundary conditions. To treat this case, Dokmanic and Vetterli suggest to replace the free-field dictionary by an ad-hoc dictionary computed a priori. The sound field emitted by a source at location \( y \) is solution to the Helmholtz equation with a second right hand side and Neumann boundary conditions:

\[
  \begin{cases}
    \Delta p + k^2 p = \delta_y \\
    p = 0 \text{ on } \partial \Omega
  \end{cases}
\]

By sampling these solutions at the positions of the measurements and, as in the free-field case, sampling the domain where the sources are assumed to be, we can build a dictionary for the localization of the sources in this particular room. Dokmanic and Vetterli combined this dictionary with joint sparsity of the sources in different frequencies, and a simple adaptation of the OMP algorithm.

We will show that, in this framework:

- it is actually necessary to use measurements at multiple frequencies to locate more than one source,
- the choice of the frequencies is critical.

We first analyze how the reverberant case relates to the free-field case. As pointed out above, the Green function can be decomposed as a free-field term \( G_0 \) added to a homogeneous term \( G_h \). Near the modal frequencies, that is when a nonzero solution with zero normal derivative on the border exists and the relative power of \( G_h \) increases, this corrective term can be much larger than the free-field contribution, which changes the properties of the dictionary.

A more precise characterization of this perturbation can be obtained using the modal basis. As pointed out in section [III] the Green function can be decomposed as

\[
  G(x, y) = \sum u_n(x) u_n(y) \frac{\delta_{x,y}}{k^2 - k_n^2}
\]

Near some eigenfrequency \( k_n \), the mode \( u_n \) (if the eigenfrequency is non-degenerate) will dominate the Green function, which be almost constant with respect to the position of the source. A consequence on the dictionary is that, if at least a measurement is not on a nodal line, its coherence (i.e. the maximal scalar product between columns) tends to 1 as the frequency approaches an eigenfrequency. This is detrimental to the localization, as the coherence of the dictionary has to be low to ensure reconstruction by a sparse recovery algorithm such as OMP [16]. In our case, it means that it is in practice impossible to locate more that one source in this case.

Measuring the impulse response at frequencies near an eigenfrequency will provide no information on the location of the sources, and between two eigenfrequencies, the Green function will be dominated by two modes associated to these frequencies. This limits the amount of information available at a given frequency, making it necessary to use multiple frequencies, which are not close to an eigenfrequency.

To illustrate this, we simulate the propagation in the room pictured on figure [1] described by the parametric equations

\[
  \begin{align*}
    x &= \cos t \\
    y &= \sin t + \frac{1}{3} \sin 2t \\
  \end{align*}
\]

The dimensions of this room are approximately \( 2 \times 2.3 \). We plot on figure [2] the correlations between the dictionary and the signals emitted by two different sources, at two different frequencies: an eigenfrequency of the room, and a frequency at mid-point between eigenfrequencies. For the eigenfrequency, the correlations for the two different sources are very similar, while for the other case, the correlations are different, but do
not exhibit a clear maximum allowing the identification of the sources. Using multiple frequencies allows a clear localization. This is visible on figure 3 that shows the correlations between the dictionary and the measurements for the first step of OMP, using 10 measurements and 6 frequencies, in order to locate 3 sources. A clear maximum is visible at the location of the most powerful source, and all 3 sources are then identified in an iterative way.

In order to evaluate the localization problem in a known room, comprehensive simulations are run with varying numbers of sources, microphones, frequencies, and three choices of frequencies:
- random draw of frequencies within a given interval.
- modal frequencies of the room
- means of two successive modal frequencies

We use the FreeFem++ software to simulate the data, and a least-square method based on (1) and the Vekua approximations to compute the dictionary [17]. The modal frequencies are computed using the Method of Particular Solutions [18]. At each trial, the positions of the measurements and of the sources are drawn randomly inside the 2-dimensional domain $\Omega_0$ pictured on figure 1.

Localization results for 2 sources are shown on figure 4. The experiment is repeated 20 times, for a number of measurements from 1 to 42, and a number of frequencies from 1 to 31. A source is considered localized if an estimated source is at a distance less than $\epsilon = 0.2$. The respective performances of the three choices of frequencies are clearly different, and coherent with our analysis. These results highlight clear differences in the three possible strategies for choosing the frequencies: while the random choice has mediocre performance and the use of the modal frequencies does not yield exploitable results, using frequencies between these modal frequencies makes the localization possible, although a large number of frequencies is needed to achieve robust localization. Even in this case, using too few frequencies prevents the localization of the sources.
At least 10 frequencies are needed to recover the two sources with a success rate of at least 80%. We see that both a minimal number of measurements, as well as a minimal number of well chosen frequencies are needed to localize sources in a known reverberant room.

V. UNKNOWN ROOM

We know turn to the case of an unknown room, for which we do not know the shape and/or the boundary conditions. This makes the direct problem, and thus the computation of the dictionary, impossible. We propose an alternative sparse modelization of the soundfield radiated by sources, which allows localization from narrowband measurements in unknown rooms using simple adaptations of sparse recovery algorithms.

A. Sparse modeling of the soundfield

We assume that the sources and the measurements are located in a domain of interest $\Omega$, contained in the domain of the room $\Omega_0$ (see figure 1). The pressure $p$ radiated by the sources is solution to the Helmholtz equation with a right hand side, and boundary conditions which are unknown to us. We can however decompose $p$ as the sum of a particular solution to the Helmholtz equation $p_s$, arbitrarily chosen, and a solution to the homogeneous equation $p_0$.

Using the Vekua theory, and although the boundary conditions are unknown to us, the homogeneous component $p_0$ can be approximated by sums of Fourier-Bessel functions or plane waves:

$$p_0 \approx \sum_{l=-L}^{L} \alpha_l J_l(kr) e^{i\omega t}$$

where the wavevectors $\vec{k}$ are uniformly sampled on the circle or the sphere of radius $k$. Note that sources in the room, but outside the domain of interest $\Omega$ will be included in this component. This can be viewed both as a feature of the method, which is able to neglect unwanted sources outside of its domain of interest, but also as a disadvantage, as it makes localization of sources possible only inside the convex hull of the sampling points.

Although a natural choice for $p_s$ is to use the free-field Green function, we use its real part only, i.e. the Bessel function of second kind $Y_0$: The pressure produced by $N$ sources at positions $y_n$ with complex amplitudes $a_n$ is given by:

$$p_s = \sum_{n=1}^{N} \frac{i a_n}{4} Y_0(k\|x - y_n\|).$$

Indeed, the imaginary part, a Bessel function of the first kind, can be included in the homogeneous component. This also emphasizes that we cannot really locate sources, but more precisely right hand sides of the Helmholtz equation. The component $p_s$ has a sparse decomposition in a dictionary similar to the one introduced in section 2.

After sampling at the location of the measurements, the model writes

$$p \approx S\alpha + W\beta$$

where $S$ is a dictionary of sources, with $S_{kl} = Y_0(k\|x_k - y_l\|)$ and $W$ a dictionary of Fourier-Bessel functions $W_{kl} = J_l(k\|x_k\|)$ or plane waves $W_{kl} = \exp(i\vec{k}\cdot\vec{x}_k)$. $\alpha$ is the vector containing the coefficients of the sources (assumed to be sparse), and $\beta$ the coefficients of the Fourier-Bessel functions (not sparse, but low-dimensional).

B. Algorithms

We propose two numerical methods to localize sources using this sparse model.

The first is based on Orthogonal Matching Pursuit. In standard OMP, the sources would be localized by correlating the measurements with the source dictionary $S$. Here however, these correlations would be corrupted by the homogeneous term $p_0$. To avoid this, the measurements are projected on the orthogonal of the space spanned by the columns of $W$. At each iterations, the residual is obtained by projecting the measurement vector $p$ on the space spanned by $W$ and the estimated sources. This scheme is equivalent to using a dictionary consisting of the union of $W$ and $S$, and forcing the first steps of OMP to choose the atoms of $W$, or to using standard OMP after projecting the dictionary of sources on the orthogonal complement of $W$.

This algorithm takes as inputs a dictionary of sources, a dictionary used to approximate the homogeneous field (e.g. plane waves or Fourier-Bessel functions), the measurements, and a stopping criterion (e.g. number of sources to be localized).

**Algorithm 1 Greedy source localization algorithm**

**Input:** measurements $p$, number of sources $n$, plane waves or Fourier-Bessel dictionary $W$, source atoms $s_y$

**Output:** estimated positions of the sources $y_j$

1. $p_s \leftarrow p - W^\dagger p$
2. for $j = 1$ to $n$ do
   1. $y_j \leftarrow \text{max}_y \{ |s_y, p_s| \}$
   2. $W \leftarrow (W|s_{y_j})$
   3. $p_s \leftarrow p - W^\dagger p$
3. end for

The second method is a variant of Basis Pursuit. In equation 5, the vector $\alpha$, containing the activations of the sources, is assumed to be sparse, while no assumption is made on the vector $\beta$ of coefficients of the decomposition of the homogeneous field. To recover the location of the sources, we minimize the $\ell_1$-norm of $\alpha$, promoting its sparsity, added to the $\ell_2$-norm of $\beta$:

$$\text{argmin}_{\alpha, \beta} \|\alpha\|_1 + \|\beta\|_2 \text{ s.t. } \|p - S\alpha - W\beta\| \leq \epsilon$$

where $\epsilon$ is the expected noise level. This minimization problem is a particular case of the Group Basis Pursuit problem.

While we here use measurements at only one frequency, which is sufficient in this case, both these methods can be extended to the case of multiple frequencies, through the use of an adaptation of OMP to treat multiple frequencies, or a mixed norm for the minimization based method.
C. Numerical results

We test the OMP-based method for various frequencies and numbers of sources. For each case, we estimate the probability of localizing the sources as a function of the number of measurements and the number of Fourier-Bessel functions used to approximate the homogeneous field. We assume that a source is successfully localized if one estimated source lies within a tolerance region of radius $\epsilon = 0.2$.

We use the same domain as above, but restrict the domain of interest to a disk $\Omega$ of diameter $R = 1.4$, in which the sources are randomly drawn.

1) Distribution of the samples: We first test three different sampling strategies, for which the sampling points are drawn using three different probability densities:
   - uniform density in the domain,
   - uniform density on its border,
   - 50% in the domain, 50% on its border.

Results for these 3 densities are given on figure 5 for the case of two sources with $k = 10$.

Sampling only on the border fails, as it is actually impossible to distinguish the field created by a source from a homogeneous field using only measurements on the border. Indeed, if one or more sources radiate a pressure $p_s$ on the border, the solution to
\[
\begin{align*}
\Delta p_0 + k^2 p_0 &= 0 \\
p_0 &= p_s \text{ on } \partial \Omega
\end{align*}
\]
is an homogeneous field with the same value on the border of $\Omega$.

Mixed sampling has slightly better performances than interior sampling, in particular for high numbers of Fourier-Bessel functions. This is likely due to the fact the higher order Fourier-Bessel functions are better identified with mixed sampling.

2) Number of measurements and Fourier-Bessel functions: We here explain the particular shape of the domain of parameters for which the method works.

Fig. 5. Unknown room - Probability of successful localization for the three different sampling densities, with 2 sources and $k = 10$, for 20 trials as a function of the number of measurements and the number of Fourier-Bessel functions. a) uniform density in the domain, b) uniform density on its border, c) mixture between these densities. Black = no source localized, white = all sources localized, $\epsilon = 0.2$.
On figure 6 we compare the performance for varying number of sources and measurements with \( k = 10 \) and \( N_{fb} = 21 \). When \( N_m < N_{fb} \), the localization fails (the growing percentages of localization as the number of sources increases when \( N_m < N_{fb} \) is an artefact of the way we measure the performance of the method, as more and more sources get localized due to pure chance).

On figure 7 we show the performance of the method with the same parameters as on figure 5c, but as a function of the number of measurements minus the number of Fourier-Bessel functions instead of the number of measurements. This figure shows that as long as there are enough Fourier-Bessel functions to capture the reverberant part of the acoustic field, the performance of the method is more or less independent of the number of Fourier-Bessel functions.

d) Wavenumber: In the known room case, the particular structure of the dictionary makes the localization possible only when using multiple frequencies that were not modal frequencies of the room. The proposed method for the unknown room case is less sensitive to the frequency of the measurements. Results of the proposed methods are given on figure 8 for two different frequencies, an eigenfrequency and a frequency between two modes. These frequencies are chosen close enough so that the main difference in behavior is due to their modal or not character, and not to their respective magnitude. The main difference between the two cases is that a larger number of Fourier-Bessel functions has to be used to capture the homogeneous part in the case of a modal frequency. This is expected as the homogeneous part has more energy in this case, and high-order components that were not large enough to perturb the localization have to be eliminated. This obviously makes the minimal number of measurements higher, but no unreasonably so. While some differences can be seen figure 8 the overall performance is similar, if slightly better for the non-modal case. While the choice of the frequency is not as critical as in the case of the known room, it is still preferable to use frequencies between modes to locate sources in this case.

We test, for different wavenumbers \( (k = 5, 10, 15, 20) \) and fixed number of measurements (60) and sources (2), the performance of the method as a function of the number of Fourier-Bessel functions. As seen on figure 9 the minimal number of Fourier-Bessel functions required to localize sources depends on the wavenumber. This minimal number is approximately \( kD \) where \( D \) is the diameter of \( \Omega_0 \). Using too many Fourier-Bessel functions is detrimental to the localization, as projecting on the orthogonal of the space spanned by these functions diminishes the effective number of measurements available for the localization.

e) Basis Pursuit based method: To close this section, we give an example of localization of 4 sources using the \( l_1 \)-minimization scheme introduced above (implemented using SPGL1 [19], [20]). In this example, we localize 4 sources, with 50 measurements, at \( k = 10 \) and 21 Fourier-Bessel functions. The output of the \( l_1 \)-minimization is drawn on figure 11 along with the considered domain and the domain of propagation.

VI. CONCLUSION

Our experiments confirm that narrowband localization of sources in known or unknown reverberant rooms is possible using adequate models. However, the two cases have quite different requirements in terms of measurements. While the known room case can deal with a small number of measurements, the numerical properties of the dictionaries require
the use of multiple and carefully chosen frequencies. More precisely, we show that the measurements should be done between the modal frequencies of the room.

Localizing sources in a unknown reverberant environment is possible using only one frequency, but needs a larger number of measurements, used to separate the direct response from the reverberation. Another difference is that this scheme can only localize sources inside the convex hull of the antenna.

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