A Conjugate Gradient Method for Inverse Problems of Non-linear Coupled Diffusion Equations

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Abstract. In many fields there are many problems called inverse problems, which infer the reasons from the observations. The inverse problem of nonlinear diffusion equations plays a crucial role in the numerical simulation of reservoirs. This article constructs a conjugate gradient method to solve the inverse problem of a nonlinear diffusion equation within oil reservoir simulation. The numerical simulation is performed, and the results show the effectiveness of the method.

1. Introduction
The nonlinear diffusion equation can approximately describe the flow process of multiphase porous media, and the inverse problems based on them are widely used in reservoir simulation. Therefore, the study of the inverse problem of nonlinear diffusion equation has important theoretical significance and practical value. With the development of science and technology, scholars at home and abroad have established various effective numerical inversion methods. Wang Yan[1] gives a new method to solve the inverse problem of parameter operator identification of diffusion equation by simulated annealing method, that is, the inverse problem of parameter operator identification is transformed into an optimization problem and solved by simulated annealing method. Gu Wenjuan[2] put forward the numerical inversion problem of fractional differential order and diffusion coefficient when studying the one-dimensional time fractional diffusion equation, and applied the optimal perturbation regularization algorithm to carry out the numerical simulation of the parameter inversion problem. Liu Tao and Song Jie[3] use the nonlinear multigrid method to solve the inverse problem of nonlinear diffusion equation. By adjusting the cost functional on different grids to be consistent, we can ensure that the method has good convergence. Nilssen[4] transformed the inverse problem of nonlinear diffusion equation into a constrained optimization problem, and introduced the augmented Lagrangian method to solve it.

In this paper, the conjugate gradient method is constructed for the inverse problem of nonlinear diffusion equation in reservoir numerical simulation, and it is regularized due to the ill posed of the inverse problem. Finally, a large number of numerical examples are used to verify that the conjugate gradient method is very effective and stable.

2. Inverse problem of nonlinear diffusion equation
Under zero gravity effect, the model of two-phase incompressible fluid immiscible flow in porous media can be described by the following partial differential equation system[4]:
where: $u$ is the wet phase fluid saturation, $p$ is the wet phase fluid pressure, $\phi$ is the porosity, $N$ is the absolute permeability, $P$ is the capillary pressure, $K_u$ is the non wet phase relative permeability, $\mu_w$ is the wet phase fluid viscosity, $\mu_n$ is the non wet phase fluid viscosity, $f_w$ is the wet phase fluid source, $f_n$ is the non wet phase fluid source.

Equation (2.1) has a very important application in many fields, such as reservoir numerical simulation, geological exploration and so on. Therefore, it is of great significance for reservoir workers to study its inverse problems[5,6]. It is difficult to study the inverse problem of equation (2.1) directly, so we simplify equation (2.1) appropriately, and get the following nonlinear diffusion equation:

$$
\phi(x,y) \frac{\partial u}{\partial t} - \nabla \cdot (N(x,y)\mu^{-1}_w(p)K_u(u)\nabla p) = f_w(x,y,t)
$$

(2.1)

$$
-\phi(x,y) \frac{\partial u}{\partial t} - \nabla \cdot (N(x,y)\mu^{-1}_w(p)K_u(u)(\nabla p + P'(u)\nabla u)) = f_w(x,y,t)
$$

Equation (2.2) has a very important application in many fields, such as reservoir numerical simulation, geological exploration and so on. Therefore, it is of great significance for reservoir workers to study its inverse problems[5,6]. It is difficult to study the inverse problem of equation (2.1) directly, so we simplify equation (2.1) appropriately, and get the following nonlinear diffusion equation:

$$
\frac{\partial u}{\partial t} - \nabla \cdot (p(x,y)K(u,\nabla u)\nabla u) = f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T)
$$

(2.2)

Nonlinear function $K(u,\nabla u)$ is used to express the nonlinearity related to the coefficient function in equation (2.1). In equation (2.1), $u$ is to influence the nonlinearity in equation (2.1) by the value of $u$, $p$ is to influence the nonlinearity in equation (2.1) by $\nabla p$. Therefore, the nonlinear function $K(u,\nabla u)$ of the diffusion term in equation (2.2) contains two independent variables $u$ and $\nabla u$.

We add initial condition and boundary condition to equation (2.2), and describe the two equations separately, then we can get the specific mathematical model of the definite solution of nonlinear diffusion equation:

$$
\phi(x,y) \frac{\partial u}{\partial t} - \nabla \cdot (N(x,y)\mu^{-1}_w(p)K_u(u)\nabla p) = f_w(x,y,t), \quad (x,y,t) \in \Omega \times (0,T)
$$

(2.3)

or

$$
-\phi(x,y) \frac{\partial u}{\partial t} - \nabla \cdot (N(x,y)\mu^{-1}_w(p)K_u(u)(\nabla p + P'(u)\nabla u)) = f_w(x,y,t)
$$

(2.4)

boundary condition:

$$
u(x,y,t) = 0, \quad (x,y,t) \in \partial \Omega \times (0,T)
$$

(2.5)

initial condition:

$$
u(x,y,0) = c(x,y), \quad (x,y) \in \Omega
$$

(2.6)

Equations (2.3) - (2.6) constitute the initial and boundary value problem of the nonlinear diffusion equation, which is usually called forward problem. It is to find the density function $u$ from the known coefficient function $p(x,y)$, where the coefficient function $p(x,y)$ represents the permeability. If the permeability coefficient function $p(x,y)$ is unknown, but we know the value of Part $u$, such as

$$
u(x^m,y^m,t) = g(x^m,y^m,t), \quad m = 1,2,L,M, \quad t \in (0,T)
$$

(2.7)

So it is an inverse problem to find the unknown permeability coefficient function $p(x,y)$ by these known density functions $u$. Therefore, equation (2.3) or (2.4), boundary condition (2.5), initial condition (2.6), and additional condition (2.7) constitute the inverse problem of nonlinear diffusion equation.

3. Conjugate gradient inversion method

By using the implicit finite difference scheme[7], equation (2.3) - (2.7) can be discretized into
\[
\begin{align*}
\frac{u^n_{i,j} - u^{n-1}_{i,j}}{\Delta t} - \nabla \cdot (p_{i,j} K_{i,j} \nabla u^n_{i,j}) &= f^n_{i,j}, \\
n = 1, 2, L, T / \Delta t; & \quad i = 1, 2, L, n_i; \\
\quad j = 1, 2, L, n_j, \\
u^n_{i,j} = u^n_{i,j} = u^n_{j,j} = 0, & \quad i = 0, 1, L, n_i; \\
\quad j = 0, 1, L, n_j, \\
u^n_{i,j} = c(i \Delta x, j \Delta y), & \quad i = 0, 1, L, n_i; \\
\quad j = 0, 1, L, n_j, \\
u^n_{i,j} = g^n_{i,j}, & \quad m = 1, 2, L, M, \\
\quad n = 1, 2, L, T / \Delta t.
\end{align*}
\]

$\Delta x$ and $\Delta y$ each represent the discrete step on the $x$, $y$ axis, and $\Delta t$ represents the time discrete step, $u^n_{i,j} = u(i \Delta x, j \Delta y, n \Delta t)$, $f^n_{i,j} = f(i \Delta x, j \Delta y, n \Delta t)$, $p_{i,j} = p(i \Delta x, j \Delta y)$, $K_{i,j} = K(u^n_{i,j})$ or $K_{i,j} = K(\nabla u^n_{i,j})$.

In fact, the definite solution of nonlinear diffusion equation can be written as $p$ to $u$ operator equation.

If:

\[
P^T = \left( p_{11}, p_{12}, L, p_{1n_1}, p_{21}, p_{22}, L, p_{2n_1}, p_{n_11}, p_{n_12} L, p_{n_1n_1} \right)
\]

\[
U^T = \left( u^1_{i,j}, u^2_{i,j}, L, u^1_{i,j}, u^1_{i,j}, L, u^2_{i,j}, L, u^2_{i,j}, L, u^2_{i,j}, L, u^2_{i,j}, L, u^2_{i,j}, L, u^2_{i,j}, L \right)
\]

$P^T$ and $U^T$ represent permeability coefficient function $p(x)$ and partial density function $u$ after finite difference dispersion, respectively. Then $p$ to $u$ operator equation can be written as $P$ to $U$ vector value function. Let's assume that this vector valued function is $F$.

\[
G^T = \left( g^1_{i,j}, g^2_{i,j}, L, g^1_{i,j}, g^2_{i,j}, L, g^2_{i,j}, L, g^2_{i,j}, L, g^2_{i,j}, L, g^2_{i,j}, L, g^2_{i,j}, L, g^2_{i,j}, L \right)
\]

$G^T$ represents observation data. Thus, the inverse problem of nonlinear diffusion equation can be transformed into the following optimization problem:

\[
\min \left\| F(P) - G \right\|^2
\]  (3.1)

Because observation data $G$ usually contains noise, and the inverse problem is ill posed, that is, a small disturbance in the problem can have a great impact on the solution. Therefore, in order to solve the optimization problem (3.1) effectively, we must first regularize it. Tikhonov regularization method is a classical regularization method, which can effectively solve the instability in the process of solving inverse problems. Therefore, by introducing Tikhonov regularization into equation (3.1), we can get the following optimization problems:

\[
\min \left\| F(P) - G \right\|^2 + b_1 \left\| M_1 P \right\|^2 + b_2 \left\| M_2 P \right\|^2
\]  (3.2)

Among them, $b_1$ and $b_2$ are used to represent the regularization parameters, $M_1$ and $M_2$ are used to represent the second-order smooth matrix in $x$ direction and $y$ direction.
The objective function of optimization problem (3.2) is

\[
J(P) = \|F(P) - G\|^2 + b_1 \|M_1 P\|^2 + b_2 \|M_2 P\|^2
\]  

(3.3)

Then the gradient of the objective function can be expressed as

\[
\nabla J(P^k) = \frac{\partial J(P^k)}{\partial P^k} = 2[F(P^k) - G]^T \frac{\partial F(P^k)}{\partial P^k} + 2b_1 M_1^T M_1 P^k + 2b_2 M_2^T M_2 P^k
\]

(3.4)

Let \( \lambda^k \) and \( d^k \) represent step size and search direction of step \( k \). in principle, \( \lambda^k \) should satisfy

\[
\frac{\partial J(P^k - \lambda^k d^k)}{\partial \lambda^k} = 0
\]

(3.5)

Taylor expansion of \( F(P^k - \lambda^k d^k) \) at \( P^k \) and ignore the second and higher order terms, we can get

\[
F(P^k - \lambda^k d^k) = F(P^k) - \frac{\partial F(P^k)}{\partial P^k} \lambda^k d^k
\]

(3.6)

From (3.3) and (3.4), we can get the following facts

\[
(F(P^k) - G)^T \frac{\partial F(P^k)}{\partial P^k} d^k + b_1 (M_1 P^k)^T d^k + b_2 (M_2 P^k)^T d^k
\]

\[
= \lambda^k \left[ \frac{\partial F(P^k)}{\partial P^k} \right]^T d^k + b_1 [M_1 d^k]^T d^k + b_2 [M_2 d^k]^T d^k
\]

(3.7)

At this point, the conjugate gradient algorithm for solving the inverse problem of nonlinear diffusion equation can be constructed:

I. Initial value \( P^0, d_0 = \nabla J(P^0), \) set \( k = 1, P^1 = P^0 - \lambda^0 d^0 .\)

Where step \( \lambda^0 \) is obtained from equation (3.7).

II. Calculation of conjugate coefficient: \( \xi^k = \frac{\nabla J^T(P^k) \nabla J(P^k)}{\nabla J^T(P^{k-1}) \nabla J(P^{k-1})} ,\)
Search direction: \( d^k = \nabla J^k(P^k) + \epsilon_k d^{k-1} \).

III Calculation \( P^k = P^{k-1} - \lambda_k d^k \), where \( \lambda_k \) is obtained from equation (3.7).

IV The objective function \( J(P^k) \) is calculated from equation (3.3). If non-converge; Set \( k = k + 1 \) and back to step 2.

4. Numerical simulation

In this section, the conjugate gradient method is used to simulate the inverse problem of the nonlinear diffusion equation. The objective equation is the two-dimensional nonlinear diffusion equation. In the numerical calculation, the parameter is selected as \( c(x,y) = \sin(\pi x) \sin(\pi y) \), \( f(x,y,t) = 0 \), \( T = 0.06 \), \( b_1 = b_2 = 10^{-12} \). In the first two numerical examples, we use the parameter model with lower resolution; in the last two numerical examples, we use the parameter model with higher resolution. In the observation data, 2% Gaussian noise is added uniformly. For the initial value of conjugate gradient method, \( P^0 = 5 \).

In the first example, the nonlinear diffusion equation (1.3) is numerically simulated, and the nonlinear diffusion function selected is \( K(u) = u^2 - u + 1 \). Permeability model is shown in Fig. 1 (a) and inversion result is shown in Fig. 1 (b).

![Figure1. Permeability model 1: (a) real model; (b) inverse result](image1)

In the second example, the nonlinear diffusion equation (1.4) is simulated. The nonlinear diffusion function is \( K(\nabla u) = 1 + 0.1|\nabla u|^2 \). Permeability model is shown in Fig. 2 (a) and inversion result is shown in Fig. 2 (b).

![Figure2. Permeability model 2: (a) real model; (b) inverse result](image2)

In the third example, the nonlinear diffusion equation (1.3) is simulated, and the nonlinear diffusion function is \( K(u) = u^2 + u + 1 \). Permeability model is shown in Fig. 3 (a) and inversion result is shown in Fig. 3 (b).
In the fourth example, the nonlinear diffusion equation (1.4) is simulated. The nonlinear diffusion
function is \( K(Vu) = 1/(1 - 0.1|Vu|)\). Permeability model is shown in Fig 4 (a) and inversion result is
shown in Fig4 (b).

5. Conclusion
In this paper, the conjugate gradient inversion method is successfully constructed for the inverse
problem of nonlinear diffusion equation. Through numerical simulation, it is clear that the proposed
method is effective for a variety of nonlinear diffusion functions \( K(u, Vu) \). In the first two numerical
examples of two-dimensional nonlinear diffusion equation, although noise is added to the observation
data, the inversion result is still practical, which shows that the conjugate gradient method has good
stability and anti noise ability.

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Reference
[1] Wang Yan. A simulated annealing algorithm for parameter operator identification of inverse
problems in diffusion equation J Journal of Sichuan Institute of Technology, 2007, 20(6): 23-26.
[2] Gu Wenjuan, Li Gongsheng, Yin Fenglan, Chi Guangsheng. Parameter inversion of a time fractional diffusion equation. Journal of Shandong University of science and technology, 2010, 24(6): 22-25.

[3] Liu T, Song J. Estimation of a permeability field within the two-phase porous media flow using nonlinear multigrid method. Math Probl Eng, 2017, Article ID 2945712.

[4] Nilssen T K, Mannseth T, Tai X C. Permeability estimation with the augmented Lagrangian method for a nonlinear diffusion equation. J Computat Geosci, 2003, 7(1): 27-47.

[5] Gottlieb J, Dietrich P. Identification of the permeability distribution in soil by hydraulic tomography. J Inverse Probl, 1995, 11: 353-60.

[6] Paillet F L. Flow modeling and permeability estimation using borehole flow logs in heterogeneous fractured formations. J Water Resour Res, 1998, 34: 997-1010.

[7] Zhao J J, Liu T, Liu S S. Identification of space-dependent permeability in nonlinear diffusion equation from interior measurements using wavelet multiscale method. J Inverse Probl Sci Eng, 2014, 22(4): 507-29.

[8] Engl H W, Zou J. A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction. J Inverse Probl, 2000, 16: 1907-23.