Self-adjointness of non-semibounded covariant Schrödinger operators on Riemannian manifolds

Ognjen Milatovic

In the context of a geodesically complete Riemannian manifold $M$, we study the self-adjointness of $\nabla^\dagger \nabla + V$, where $\nabla$ is a metric covariant derivative (with formal adjoint $\nabla^\dagger$) on a Hermitian vector bundle $\mathcal{V}$ over $M$, and $V$ is a locally square integrable section of $\text{End} \mathcal{V}$ such that the (fiberwise) norm of the “negative” part $V^-$ belongs to the local Kato class (or, more generally, local contractive Dynkin class). Instead of the lower semiboundedness hypothesis, we assume that there exists a number $\varepsilon \in [0,1]$ and a positive function $q$ on $M$ satisfying certain growth conditions, such that $\varepsilon \nabla^\dagger \nabla + V \geq -q$, the inequality being understood in the quadratic forms sense over $C^\infty_c (\mathcal{V})$. In the first result, which pertains to the case $\varepsilon \in [0,1)$, we use the elliptic equation method. In the second result, which pertains to the case $\varepsilon = 1$, we use the hyperbolic equation method.

KEYWORDS
covariant Schrödinger operator, non-semibounded, Riemannian manifold, self-adjointness

MSC (2020)
35P05, 47B25, 58J50, 60H30

1 | INTRODUCTION

1.1 | Motivation of the problem

One of the central objects of quantum mechanics is a Schrödinger differential expression

$$S_V := \Delta + V,$$

where $\Delta$ is the (nonnegative) Laplacian on a Riemannian manifold $M$ with a metric $g$, and $V$ is a locally square integrable real-valued function on $M$. Let $C^\infty_c (M)$ denote smooth compactly supported functions on $M$ and let $L^2 (M)$ be the space of square integrable functions on $M$. In the quantum mechanical setting, a self-adjoint extension of $S_V|_{C^\infty_c (M)}$ in $L^2 (M)$ is referred to as a quantum observable associated with $S_V$. In view of Lemma A.1 (see the Appendix), a self-adjoint extension of $S_V|_{C^\infty_c (M)}$ in $L^2 (M)$ always exists, although its uniqueness is not guaranteed (without additional conditions). The essential self-adjointness of $S_V|_{C^\infty_c (M)}$ means that $S_V|_{C^\infty_c (M)}$ has a unique self-adjoint extension in $L^2 (M)$. Furthermore, it is known (see section VI.1.7 in [1]) that a self-adjoint extension of $S_V|_{C^\infty_c (M)}$ is unique if and only if for all $u_0 \in C^\infty_c (M)$, the Cauchy
problem

\[
\frac{1}{i} \frac{\partial u(t)}{\partial t} = S_V u(t), \quad u(0) = u_0
\]

has a unique solution \( u(t) \in L^2(M) \), for all \( t \in \mathbb{R} \). (Here, the \( t \)-derivative is taken in the \( L^2 \)-norm sense, and \( S_V u \) is understood in the sense of distributions.) In this case, the corresponding quantum system is said to have the quantum completeness or quantum confinement property.

Switching to the classical mechanics viewpoint, on the cotangent bundle \( T^*M \) (with the usual symplectic structure), we consider the Hamiltonian

\[
h(p, x) = |p|^2 + V(x), \tag{1.2}
\]

where \( p \in T^*_x M \) and \( |p| \) stands for the length of \( p \) in the metric on \( T^*_x M \) induced by \( g \). Assuming that \( \text{dim} M = n \), in the coordinates \((x^1, \ldots, x^n, p^1, \ldots, p^n)\) the corresponding Hamiltonian system is

\[
\frac{dx_j}{dt} = \frac{\partial h}{\partial p_j}, \quad \frac{dp_j}{dt} = - \frac{\partial h}{\partial x_j}, \quad j = 1, \ldots, n. \tag{1.3}
\]

Assuming that \( V \in C^2(M) \), the local Hamiltonian flow corresponding to (1.2) is well defined. If the solutions to (1.3) with arbitrary initial data are defined for all \( t \), then the corresponding classical system is referred to as classically complete. As seen from the examples in [32], even in the case \( M = \mathbb{R} \) the classical and quantum completeness notions are independent. For a further discussion of the quantum and classical completeness, see [33].

### 1.2 A brief look at the existing literature

In the context of \( M = \mathbb{R}^n \), the essential self-adjointness problem for \(-\Delta + V \) (here, \( \Delta \) is the standard Laplacian) dates back to the work of H. Weyl in the early 1900s, while the corresponding study on Riemannian manifolds was initiated by M. Gaffney in the early 1950s. In particular, the past 25 years have brought quite a few developments on this topic in the context of the operator \( H_V|_{C^\infty} \) in the usual \( L^2 \)-space (square integrable functions or sections of a vector bundle) on a Riemannian manifold \( M \), where \( C^\infty_c \) denotes smooth compactly supported functions (or sections), and \( H_V \) is a Schrödinger-type differential expression

\[
H_V = P + V. \tag{1.4}
\]

Here, \( P \) is the (magnetic) Laplacian on functions or, more generally, Bochner Laplacian (see Section 2.1 for a description) and \( V \) is a locally square integrable (\( L^2_{\text{loc}} \)) real-valued function (or a self-adjoint section of the appropriate endomorphism bundle).

The literature on the essential self-adjointness of \( H_V|_{C^\infty_c} \) showcases two general approaches to the problem—the elliptic equation method and the hyperbolic equation method. For lower semibounded operators, the elliptic equation approach was used by the authors of [5, 38, 42] in the \( \mathbb{R}^n \) setting and in [25, 37] in the Riemannian manifold setting, and—for not necessarily lower semibounded operators—by the authors of [17, 19, 20, 35] in the \( \mathbb{R}^n \) setting and in [3, 4, 21, 27, 36] in the Riemannian manifold setting. (The authors of [3, 21, 27, 35, 36] assumed \( V \in L^\infty_{\text{loc}} \).) For lower semibounded \( H_V|_{C^\infty_c} \), the hyperbolic equation method was used in [30] in the \( \mathbb{R}^n \) setting and by the authors of [6, 8, 12, 16] in the Riemannian manifold setting, and—for not necessarily lower semibounded operators—by the authors of [22, 28] in the \( \mathbb{R}^n \) setting and in [7] in the Riemannian manifold setting. Very recently, in the paper [18], the authors explored the links between the essential self-adjointness of Laplace-type operators on Riemannian manifolds (and discrete graphs) and the so-called \( L^2 \)-Liouville property (the phenomenon that the functions belonging to the kernel of the adjoint operator are constant). For additional pointers to the literature, the reader is referred to the books [9, 13, 33] and papers [4, 24, 39]. (In reference to the papers mentioned above, we should add that the authors of [6, 22] studied the operators with \( V \in L^\infty_{\text{loc}} \) or more regularity, and in [8], the case \( V = 0 \) was considered.)
1.3 Two common types of conditions on $V$

Looking at the works on the essential self-adjointness of $H_V |_{C_c^\infty}$ (some are referenced in Section 1.2), the assumptions on $V \in L^2_{\text{loc}}$ are often tailored in a way that the “negative part” $V^-$ := $\max\{-V, 0\}$ satisfies one of the following properties ($d$ is the usual differential, $\Delta$ is the Laplacian on functions on $M$, $1_G$ is the characteristic function of a set $G$):

(H1) for every compact set $K$, there exist numbers $0 < a_K < 1$ and $b_K \geq 0$ such that $(1_K V^- u, u) \leq a_K \|d u\|^2 + b_K \|u\|^2$, for all $u \in C_c^\infty$;

(H2) for every compact set $K$, there exist numbers $0 < a_K < 1$ and $b_K \geq 0$ such that $\|1_K V^- u\| \leq a_K \|\Delta u\| + b_K \|u\|$, for all $u \in C_c^\infty$.

(Here, $(\cdot, \cdot)$ and $\| \cdot \|$ stand for the inner product and the norm in $L^2$.) When considering (1.4), with $P$ being the Bochner Laplacian, one would replace $d$ by the covariant derivative $V$ in (H1) and $V^-$ by the norm $|V^-|$ in (H2).

Examples of potentials satisfying (H1) and (H2) include those having $V^- \in L^p_{\text{loc}} \cap L^2_{\text{loc}}$ with $p$ as in Remark 2.5, and the examples satisfying (H1) include the potentials with $V^-$ (or the norm $|V^-|$ in the bundle case) belonging to the local Kato class $K_{\text{loc}}$ (or, more generally, local contractive Dynkin class $D_{\text{loc}}$). For a description of $K_{\text{loc}}$ and $D_{\text{loc}}$, see Section 2.2.

If the lower semiboundedness of $H_V |_{C_c^\infty}$ is not assumed (or is not a consequence of the assumptions on $V$), the authors usually impose a condition of the following type: There exists $\varepsilon \in [0, 1]$ such that

$$\varepsilon (Pu, u) + (Vu, u) \geq -(qu, u),$$

for all $u \in C_c^\infty$, where $P$ and $V$ are as in (1.4) and $q(x) = [(\alpha \sigma r)(x)]^2$ with $r(x) := d_g(x, x_0)$. Here, $\alpha(t) > 0$ is a sufficiently regular function tending to $\infty$ not too fast as $t \to \infty$, and the notation $d_g(\cdot, x_0)$ indicates the distance from a reference point $x_0$ in the metric $g$ of $M$.

1.4 A discussion of the first result

This paper is concerned with the operator $H_V = V^\dagger V + V$ on $C_c^\infty$, where $V$ is a metric covariant derivative (with formal adjoint $V^\dagger$) on a Hermitian vector bundle $\mathcal{V}$ over a geodesically complete Riemannian manifold $M$ with metric $g$, and $V \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is a $D_{\text{loc}}$-decomposable self-adjoint operator, that is, $V = V_1 - V_2$, where $V_j \geq 0$ and $|V_2| \in D_{\text{loc}}$ (here, $D_{\text{loc}}$ is the local contractive Dynkin class). As seen in the book [13], the covariant Schrödinger operator $V^\dagger V + V$ has drawn quite a bit of attention in recent years (in particular, as an object of study in stochastic analysis).

Using a variant of the elliptic equation method, in Theorem 2.2 (in particular, in Corollary 2.6) of this paper, we prove that $H_V |_{C_c^\infty(V)}$ is essentially self-adjoint under the following assumptions: $D_{\text{loc}}$ decomposability of $V$, the fulfillment of (1.5) with $0 \leq \varepsilon < 1$ and a function $q \geq 1$ (with an additional hypothesis on $q$), and the geodesic completeness of the metric $g_q := q^{-1}g$. (The latter assumption implies that $(M, g)$ is geodesically complete.) We should indicate that a related result, proven in Theorem 2.7 of [4], assumed the fulfillment of (H2) described earlier in this section. It turns out that the condition (H2) is not necessarily satisfied by the potentials $V$ with $|V_2| \in D_{\text{loc}}$ (or $|V_2| \in K_{\text{loc}}$). Our Theorem 2.2 is capable of handling $D_{\text{loc}}$-decomposable potentials thanks to the “localized self-adjointness” idea from [5] (subsequently implemented in [19] in the non-semibounded case), which enables us to keep the proof of the inclusion $\text{Dom}(H_V |_{C_c^\infty}) \subset W^{1,2}_{\text{loc}}$ within the realm of quadratic-form inequalities. (Here, $T^*$ is the operator adjoint of $T$ and $W^{1,2}_{\text{loc}}$ is a local $L^2$-Sobolev space with differential order 1.) With the latter inclusion at our disposal, the remainder of the proof of Theorem 2.2 follows the lines of the refined integration by parts technique as in [3, 4, 27, 36].

1.5 A discussion of the second result

In Theorem 2.7, we prove that $H_V |_{C_c^\infty(V)}$ has at most one self-adjoint extension in $L^2(V)$ under the following assumptions: geodesic completeness of $(M, g)$, $D_{\text{loc}}$ decomposability of $V$, the fulfillment of (1.5) with $\varepsilon = 1$, and the requirement that $t \cdot \alpha(t)$ (with $\alpha$ coming from $q(x) = [(\alpha \sigma r)(x)]^2$) belongs to the class $\mathcal{V}$ (see Section 2.2 for a description of this class). Examples of $\alpha$ covered by Theorem 2.7 include $\alpha(t) = at + b$ and $\alpha(t) = (at + b)\ln(t + 1)$, with $a > 0$ and $b > 0$. In the case when $\alpha$ is constant (see Corollary 2.8 below), we have a lower semibounded operator $H_V |_{C_c^\infty}$ (which, by an abstract
fact, is guaranteed to have a self-adjoint extension in $L^2(\mathcal{V})$, and Theorem 2.7 then tells us that $H_V|_{C_c^\infty(\mathcal{V})}$ is essentially self-adjoint in $L^2(\mathcal{V})$.

Corollary 2.8 is a generalization of Theorem XII.1 of [13] (see also Theorem 1.1 of [16]), where $|V_2|$ was assumed to belong to the global contractive Dynkin class (which implies lower semiboundedness of $H_V|_{C_c^\infty}$ by Lemma VII.4 in [13]). We should mention that the assumptions on $V$ in Corollary 2.8 are analogous to those of Theorem 1 in [12], where the authors proved (on a geodesically complete manifold), the essential self-adjointness of Schrödinger operators with singular (of the type as in [17, 20]) magnetic potential. Finally, Corollary 2.9 considers $H_V = \Delta + V$ acting on functions, with the same assumptions on $V$ as in Theorem 2.7, and thanks to Lemma A.1 below, in this case, we get the essential self-adjointness of $H_V|_{C_c^\infty(M)}$. Corollary 2.9 is a generalization of Corollary 2.3 in [26], which assumed $V(x) \geq -|\alpha \circ r(x)|^2$ with $\alpha(t) = at + b$, $a > 0$, $b > 0$.

Unlike Corollary 2.3 in [26], which relies on a variant of the elliptic equation method, Theorem 2.7 is obtained using the hyperbolic equation approach, similar to that used in [28] for operators on $\mathbb{R}^n$. The approach uses finite propagation speed to establish the representation formula (6.4) for $\cos(A^{1/2} t)f$, where $f \in C_c^\infty(\mathcal{V})$ and $A$ is a self-adjoint extension of $H_V|_{C_c^\infty}$ in $L^2(\mathcal{V})$ (if such an extension exists). With this formula at our disposal, we show that, thanks to the assumption $t \cdot \alpha(t) \in \mathcal{V}$, the function $t \mapsto (\cos(A^{1/2} t)f, f)$, $f \in C_c^\infty(\mathcal{V})$, is uniquely representable in the class of measures $(E(d\lambda)f, f)$ induced by the spectral resolution $E(\lambda)$ of $A$, which, with the help of Lemma A.3 below, leads to the uniqueness of self-adjoint extension of $H_V|_{C_c^\infty}$ in $L^2(\mathcal{V})$. This variant of the hyperbolic method has a different flavor from that of [7], where an “energy equality” (see eq. (7) in [7]) is established, which is then used (together with the assumption $\alpha(t) = at + b$, $a > 0$ and $b > 0$) to establish $\dim(\ker((H_V|_{C_c^\infty})^* \mp i)) = 0$. (The author of [7] considered second-order elliptic operators in divergence form with real-valued coefficients and zero-order term $V$ satisfying $V = V_1 - V_2$, $0 \leq V_1 \in L^2_{\text{loc}}(M)$, and $0 \leq V_2 \in L^p_{\text{loc}}(M)$, with $p = 2$ for $n < 3$, $p > 2$ for $n = 4$, and $p = n/2$ for $n \geq 5$, where $n = \dim M$.) It is not clear if the procedure of [7] is applicable if we replace $\alpha(t) = at + b$ by more general functions, such as those covered by our Theorem 2.7.

1.6 | Pointers to some recent papers

Recently, the authors of [31] worked in the setting of a not necessarily geodesically complete Riemannian manifold $M$ and, using an ingenious form of the elliptic equation method, obtained, among other things, a far-reaching essential self-adjointness result (see Theorem 1 there) for the operator $\Delta_\omega + V$, where $\Delta_\omega$ is the scalar Laplacian with respect to a smooth measure $\omega$ on $M$ and $V \in L^2_{\text{loc}}(M)$. Theorem 1 in [31] is formulated in terms of the so-called effective potential, which involves the measure $\omega$ and the distance from the metric boundary of $M$. In the geodesically complete case, the requirement imposed on $V$ in Theorem 1 of [31] reduces to $V(x) \geq -(h(x))^2$, where $h(x)$ is a Lipschitz function on $M$. For the case of $H_V = \Delta + V$, where $\Delta$ is the Laplacian on functions on a geodesically complete Riemannian manifold, Corollary 2.9 of our paper handles some potentials $V$ that are not covered by Theorem 1 of [31]. Finally, we mention the recent paper [10], which contains various essential self-adjointness results for sub-Laplacian-type operators on sub-Riemannian manifolds.

1.7 | Organization of our paper

Our paper consists of six sections and an appendix. In Section 2, we give an overview of the notations and state the main results. Section 3 contains some preliminary lemmas. The proofs of Theorem 2.2 and Corollary 2.6 are carried out in Section 4 and Section 5, respectively. The proof of Theorem 2.7 is given in Section 6. Lastly, in the Appendix, we recall some auxiliary results.

2 | RESULTS

2.1 | Notations

We start with an overview of the notations used in the paper. We will work in the context of a smooth connected Riemannian manifold $M$ (without boundary) equipped with the volume measure $d\mu$. The symbol $\mathcal{V} \to M$ denotes a smooth Hermitian vector bundle over $M$ and $\langle \cdot , \cdot \rangle_x$ indicates the corresponding Hermitian structure with the norms $| \cdot |_x$ on
fibers \( V_x \). When there is no confusion, the notation \(| \cdot |\) indicates the norm of a linear operator \( V(x) : V_x \to V_x \). For a fixed \( x_0 \in M \), we denote
\[
r(x) := d_g(x_0, x), \quad x \in M,
\]
(2.1)
and
\[
B_\rho(x_0) := \{ x \in M : r(x) < \rho \},
\]
(2.2)
where \( \rho > 0 \) is a number and \( d_g(\cdot, \cdot) \) is the distance relative to the metric \( g \) on \( M \).

We now move over to function spaces. We begin with the symbols \( C^\infty(\mathbb{R}) \) and \( C^\infty_c(\mathbb{R}) \), which refer to the smooth sections of \( \mathbb{R} \) and smooth compactly supported sections of \( \mathbb{R} \), respectively. The space of square integrable sections of \( \mathbb{R} \) is denoted by \( L^2(\mathbb{R}) \) and the usual inner product and the corresponding norm are indicated, respectively, by \( (\cdot, \cdot) \) and \( \| \cdot \| \).

The symbol \( W^{k,p}(\mathbb{R}) \) stands for the local Sobolev space of sections of \( \mathbb{R} \), where \( k \) and \( p \) indicate, respectively, the highest derivative and the corresponding \( L^p \)-space. Specializing to \( \mathbb{V} = M \times \mathbb{C} \), we have the following notations: \( C^\infty(M), C^\infty_c(M), L^2(M), \) and \( W^{k,p}(M) \). Lastly, the symbol \( \Omega^1(M) \) indicates smooth 1-forms on \( M \).

We now turn to differential operators. We start with a Hermitian covariant derivative on \( \mathbb{R} \)
\[
\nabla : C^\infty(\mathbb{R}) \to C^\infty(T^*M \otimes \mathbb{R})
\]
and its formal adjoint \( \nabla^\dagger \) with respect to \((\cdot, \cdot)\). The composition \( \nabla^\dagger \nabla : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \) is often called Bochner Laplacian. As a special case, we get the magnetic Laplacian on functions \( \Delta_A := d^\dagger_A d_A \).

Here, the symbol \( d_A \) denotes the magnetic differential
\[
d_A u = du + iuA,
\]
where \( d : C^\infty(M) \to \Omega^1(M) \) indicates the standard differential, \( A \in \Omega^1(M) \) stands for a real-valued 1-form, and \( i \) is the imaginary unit. Specializing to the case \( A = 0 \), we get the (nonnegative) Laplace–Beltrami operator \( \Delta = d^\dagger d \).

In this paper, we will be looking at a perturbed Bochner Laplacian
\[
H_V := \nabla^\dagger \nabla + V,
\]
(2.3)
where \( V \) stands for a measurable section \( V \) of the bundle \( \text{End} \mathbb{V} \) and \( V(x) : V_x \to V_x \) is a self-adjoint operator for almost every \( x \in M \). In recent literature (see, for instance, the book \cite{13}), the operator \( H_V \) is called covariant Schrödinger operator.

### 2.2 Statements of results

Before stating the first result, we describe a class of minorizing functions for \( V \). We say that a function \( q : M \to \mathbb{R} \) belongs to the class \( \mathcal{M} \) if
\begin{enumerate}
\item[(i)] \( q \in L^\infty_{\text{loc}}(M) \) and \( q \geq 1 \);
\item[(ii)] there exists a number \( L \) such that \(|q^{-1/2}(x) - q^{-1/2}(y)| \leq L d_g(x, y)\), for all \( x, y \in M \), where \( d_g(\cdot, \cdot) \) is as in (2.1);
\item[(iii)] \( (M, g_q) \) is geodesically complete, where \( g_q \) is a metric on \( M \) defined as \( g_q := q^{-1}g \).
\end{enumerate}

**Remark 2.1.** Item (iii) is equivalent to the following condition: For every curve \( \gamma \) going to infinity, we have \( \int_\gamma q^{-1/2} \, ds_g = \infty \), where \( ds_g \) is the arc length element with respect to the metric \( g \). Since \( q \geq 1 \), the property (iii) implies the geodesic completeness of \( (M, g) \).

We can also look at \( (M, g_q) \) in (iii) as a manifold, which is conformally equivalent to \( (M, g) \) with a conformal factor \( \tau = q^{-1/2} \) in \( g_q = \tau^2 g = q^{-1}g \), which “shrinks” the original manifold (because \( 0 < \tau \leq 1 \)), with the “shrunk” manifold still being complete (with respect to \( g_q \)).
**Theorem 2.2.** Let $(M, g)$ be a Riemannian manifold and let $V$ be a Hermitian vector bundle over $M$ with a Hermitian covariant derivative $\nabla$. Let $V = V_1 - V_2$ with $0 \leq V_j \in L^2_{\text{loc}}(\text{End } V)$, $j = 1, 2$. Assume that for every $x_0 \in M$, there exist numbers $0 < \delta \leq 1$, $C \in \mathbb{R}$, and a function $\phi \in C^\infty_c(M)$ with $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on a neighborhood of $x_0$, such that the following properties are satisfied:

(i) $(\nabla^\dagger \nabla v + \phi V v, v) \geq \delta (\nabla^\dagger \nabla_1 v, v) - C \|v\|^2$, for all $v \in C^\infty_c(V)$;

(ii) $(\nabla^\dagger \nabla + \phi V) |_{C^\infty_c(V)}$ is essentially self-adjoint.

Furthermore, assume that there exists a number $0 \leq \varepsilon < 1$ and a real-valued function $q \in \mathcal{M}$ such that

$$\varepsilon \|\nabla u\|^2 + (V u, u) \geq -(q u, u),$$

for all $u \in C^\infty_c(V)$. Then, the operator $H_V |_{C^\infty_c(V)}$ is essentially self-adjoint.

**Remark 2.3.** In the context of $M = \mathbb{R}^n$, the quadratic form condition (2.4), which is less restrictive than the pointwise condition $V(x) \geq -q(x)$, dates back to the paper [35]. It turns out that some restrictions (as described in Remark 2.1) on the function $q$ in (2.4) are needed. As an illustration, we recall the example III.1.1 from [2]: if $M = \mathbb{R}^n$ and $V = -|x|^\kappa$, where $\kappa > 2$ and $|x|$ is the Euclidean distance from $x$ to $0$ in $\mathbb{R}^n$, then $-\Delta + V$ is not essentially self-adjoint on $C^\infty_c(\mathbb{R}^n)$.

(Here, $\Delta$ is the standard Laplacian on $\mathbb{R}^n$.)

**Remark 2.4.** With regard to the function $q \in \mathcal{M}$ in (2.4), it was pointed out in Remark 1.2 of [36] that the condition (iii) from the definition of $\mathcal{M}$ (see also Remark 2.1) implies the classical completeness of the system (1.3) corresponding to the Hamiltonian $|p|^2 - q$. (Here, we assume, in addition, that $q \in C^2(M)$.) To quickly illustrate this in the case $M = \mathbb{R}$ with the standard metric, we recall the “conservation of energy” property along the classical trajectories of the Hamiltonian:

$$|p|^2 - q(x) = C = \text{const}.$$  

Thus, the condition (iii) from the definition of $\mathcal{M}$ implies the classical completeness of the Hamiltonian system corresponding to $|p|^2 - q$.

Before stating a corollary containing more specific conditions on $V$ that guarantee the essential self-adjointness of $H_V$, we review the concept of (local) Kato and (local) contractive Dynkin classes. We start with the symbol $p(t, x, y), (t, x, y) \in (0, \infty) \times M \times M$, which indicates the minimal positive heat kernel of $M$ as in Theorem 7.13 in [11]. In this paper, $p(t, x, y)$ corresponds to $e^{-t\Delta}$. We can now give a description of the (local) contractive Dynkin class and (local) Kato class. For a Borel function $f : M \to \mathbb{C}$ and $t > 0$, define

$$J(t) := \sup_{x \in M} \int_0^t \int_M p(s, x, y) |f(y)| d\mu(y) ds.$$

Adopting the terminology of Definition VI.1 in [13], we say that a Borel function $f : M \to \mathbb{C}$ is a member of the **contractive Dynkin class** relative to $p(t, x, y)$ and write $f \in D(M)$ if there exists $t > 0$ such that $J(t) < 1$. We say that a Borel function $f : M \to \mathbb{C}$ is a member of the **Kato class** relative to $p(t, x, y)$ and write $f \in \mathcal{K}(M)$ if $\lim_{t \to 0^+} J(t) = 0$. The local contractive Dynkin class $D_{\text{loc}}(M)$ consists of all Borel functions $f : M \to \mathbb{C}$ such that for all compact sets $K \subset M$, we have $1_K f \in D(M)$, where $1_G$ is the indicator function of a set $G$. The local Kato class $\mathcal{K}_{\text{loc}}(M)$ is defined in the same way as $D_{\text{loc}}(M)$, with $\mathcal{K}(M)$ in place of $D(M)$. We observe that $\mathcal{K}(M) \subset D(M)$ and $\mathcal{K}_{\text{loc}}(M) \subset D_{\text{loc}}(M)$.
Remark 2.5. Recently, the author of [14] showed (see Corollary 2.11 there) that the following property holds on an arbitrary (not necessarily geodesically complete) Riemannian manifold $M$ with $\dim M = n$: for every $1 \leq p < \infty$ such that $p \geq 1$ if $n = 1$, and $p > n/2$ if $n \geq 2$, we have $L^p_{\text{loc}}(M) \subset \mathcal{K}_{\text{loc}}(M) \subset D_{\text{loc}}(M)$. To get the analogous global inclusion $L^p(M) \subset \mathcal{K}(M)$, one needs to impose additional requirements on $M$, such as geodesic completeness, positive injectivity radius, and lower semiboundedness of the Ricci curvature; see Theorem 2.9 and Corollary 2.11 in [15].

We are ready to describe a class of potentials $V$ used in the next corollary (and later in the paper). Following the terminology of Definition VII.3 of [13], a section $V$ of $\text{End } \mathcal{V}$ will be called $D_{\text{loc}}$-decomposable if $V = V_1 - V_2$ with $V_j \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ and $V_j \geq 0$ and $|V_2| \in D_{\text{loc}}(M)$. If the latter condition is replaced by $|V_2| \in D(M)$, we say that a section $V$ of $\text{End } \mathcal{V}$ is $D$-decomposable. Analogously, we can define the terms $\mathcal{K}_{\text{loc}}$-decomposable and $\mathcal{K}$-decomposable.

Corollary 2.6. Let $(M, g)$ be a Riemannian manifold and let $\mathcal{V}$ be a Hermitian vector bundle over $M$ with a Hermitian covariant derivative $\nabla$. Assume that $V \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is $D_{\text{loc}}$-decomposable. Furthermore, assume that there exists a number $0 \leq \varepsilon < 1$ and a real-valued function $q \in \mathcal{M}$ such that (2.4) is satisfied. Then, the operator $H_V|_{C_c^\varepsilon(\mathcal{V})}$ is essentially self-adjoint.

Before stating a result concerning the case $\varepsilon = 1$ in (2.4), we describe the class $\mathcal{V}'$ originally introduced by the author of [41]. A function $f : (0, \infty) \to (0, \infty)$ is said to belong to the class $\mathcal{V}'$ if the following conditions are satisfied: $f \in C^1(0, \infty)$; the first derivative $f'$ is increasing on $(0, \infty)$; $\lim_{t \to \infty} \frac{f(t)}{t^\gamma} = \gamma$, where $\gamma$ is finite and $\gamma > 1$; there exists $t_0 > 0$ such that $\int_{t_0}^\infty \frac{g(t)}{t^2} \, dt = \infty$, where $g(t)$ is the inverse function corresponding to $f'(t)$. It is easy to check that if $f \in \mathcal{V}'$, then $f(t + a) + f(at)$, where $a > 0$ is a number, also belong to $\mathcal{V}'$. Furthermore, it can be checked that if $m_1(t) = a + bt$ and $m_2(t) = (a + bt)\ln(t + 1)$, where $a > 0$ and $b > 0$ are numbers and $\ln(\cdot)$ is the natural logarithm, then $f(t) := \int m_j(t), \quad j = 1, 2, \text{ belong to } \mathcal{V}'$. We are now ready to state the second result.

Theorem 2.7. Let $(M, g)$ be a geodesically complete Riemannian manifold. Let $\mathcal{V}$ be a Hermitian vector bundle over $M$ with a Hermitian covariant derivative $\nabla$. Assume that $V \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is $D_{\text{loc}}$-decomposable. Furthermore, assume that

\[ (\nabla^\dagger u + Vu, u) \geq -(qu, u), \tag{2.5} \]

for all $u \in C_c^\infty(\mathcal{V})$, where $q(x) = [(\alpha \circ r)(x)]^2$ with a function $\alpha : (0, \infty) \to (0, \infty)$ such that $t \cdot \alpha(t)$ belongs to the class $\mathcal{V}'$ and with $r(x)$ as in (2.1). Then, one of the following is true: Either $H_V|_{C_c^\varepsilon(\mathcal{V})}$ does not have a self-adjoint extension in $L^2(\mathcal{V})$ or the closure $\bar{H}_V|_{C_c^\varepsilon(\mathcal{V})}$ is the only self-adjoint extension of $H_V|_{C_c^\varepsilon(\mathcal{V})}$ in $L^2(\mathcal{V})$ (i.e., $H_V|_{C_c^\varepsilon(\mathcal{V})}$ is essentially self-adjoint).

If $V \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is a self-adjoint section and

\[ (H_V u, u) \geq -c, \tag{2.6} \]

for all $u \in C_c^\infty(\mathcal{V})$, where $c > 0$ is a constant, then (see section B.2 in [13]) there exists a self-adjoint extension of $H_V|_{C_c^\varepsilon(\mathcal{V})}$ in $L^2(\mathcal{V})$. If $\alpha(t) := \sqrt{c} + bt$, where $c > 0$ is as in (2.6) and $b > 0$ is a constant, then $t \cdot \alpha(t)$ belongs to the class $\mathcal{V}'$. Moreover, the hypothesis (2.6) implies that $H_V$ satisfies the condition (2.5) with $q(x) = [(\alpha \circ r)(x)]^2$, where $\alpha(t) = \sqrt{c} + bt$ and $r(x)$ is as in (2.1). Thus, from Theorem 2.7, we get the following corollary:

Corollary 2.8. Let $(M, g)$, $\nu$, and $\mathcal{V}$ be as in the hypotheses of Theorem 2.7. Assume that $V \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is $D_{\text{loc}}$-decomposable. Furthermore, assume that $H_V$ satisfies the condition (2.6). Then, $H_V|_{C_c^\varepsilon(\mathcal{V})}$ is essentially self-adjoint in $L^2(\mathcal{V})$.

As $(\Delta + V)|_{C_c^\varepsilon(M)}$, with a real-valued $V \in L^2_{\text{loc}}(M)$, has a self-adjoint extension in $L^2(M)$ (see Lemma A.1), the next corollary is a direct consequence of Theorem 2.7:

Corollary 2.9. Let $(M, g)$ be a geodesically complete Riemannian manifold. Assume that $V \in L^2_{\text{loc}}(M)$ is a real-valued and $D_{\text{loc}}$-decomposable function. Assume that (2.5) is satisfied with $\Delta$ in place of $\nabla^\dagger \nabla$. Then, the operator $(\Delta + V)|_{C_c^\varepsilon(M)}$ is essentially self-adjoint in $L^2(M)$.
3 | LEMMAS USED IN THE PROOF THEOREM 2.2

3.1 | Quadratic forms

Assume that $0 \leq Q \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ and consider the sesquilinear forms

$$h_V[u, v] : = \int \langle \nabla u, \nabla v \rangle \, d\mu, \quad h_Q[u, v] : = \int \langle Qu, v \rangle \, d\mu,$$

with the corresponding quadratic forms $h_V[\cdot]$ and $h_Q[\cdot]$ and their domains

$$\text{Dom}(h_V) = \{ u \in L^2(\mathcal{V}) : h_V[u] < \infty \},$$
$$\text{Dom}(h_Q) = \{ u \in L^2(\mathcal{V}) : h_Q[u] < \infty \}.$$

Let $h_1 := h_V + h_Q$ with the domain $\text{Dom}(h_1) = \text{Dom}(h_V) \cap \text{Dom}(h_Q)$. Note that $C_c^\infty(\mathcal{V}) \subset \text{Dom}(h_1)$, which makes $h_1$ densely defined. In the subsequent discussion, the symbol $W^{-1,2}(\mathcal{V})$ refers to the antidual of $\text{Dom}(h_V)$.

3.2 | Lemmas

Having introduced the needed forms, we are ready to state the first lemma, whose proof can be found in Lemma 2.2 of [25].

**Lemma 3.1.** If $(M, g)$ is geodesically complete and $0 \leq Q \in L^2_{\text{loc}}(\text{End } \mathcal{V})$, then $C_c^\infty(\mathcal{V})$ is a form core of $h_1$, that is, $C_c^\infty(\mathcal{V})$ is dense in $\text{Dom}(h_1)$ with respect to the norm $\| \cdot \|_1 := h_V[\cdot] + h_Q[\cdot] + \| \cdot \|^2$, where $\| \cdot \|$ is the norm in $L^2(\mathcal{V})$.

**Lemma 3.2.** Assume that $(M, g)$ is geodesically complete. Let $W = W_1 - W_2$, where $0 \leq W_j \in L^2_{\text{loc}}(\text{End } \mathcal{V})$, $j = 1, 2$. Assume that there exist numbers $0 \leq \delta \leq 1$ and $C \in \mathbb{R}$ such that

$$\langle \nabla^\dagger \nabla v + W_1 v, v \rangle \geq \delta [\langle \nabla^\dagger \nabla v + W_1 v, v \rangle] - C \|v\|^2,$$

for all $v \in C_c^\infty(\mathcal{V})$. Let $u \in \text{Dom}(h_V) \cap \text{Dom}(h_{W_1})$. Then, $\langle W_2 u, u \rangle \in L^1(M)$ and

$$\|\nabla u\|^2 + (W_2 u, u) \geq \delta [\|\nabla u\|^2 + (W_1 u, u)] - C \|u\|^2,$$

with $\delta$ and $C$ as in (3.2).

**Proof.** Denoting $h_1 := h_V + h_{W_1}$, we can rewrite (3.2) as

$$(1 - \delta) h_1[v] + C \|v\|^2 \geq \int_M \langle W_2 v, v \rangle \, d\mu,$$

for all $v \in C_c^\infty(\mathcal{V})$. Since $u \in \text{Dom}(h_V) \cap \text{Dom}(h_{W_1})$, using Lemma 3.1 we can find a sequence $v_k \in C_c^\infty(\mathcal{V})$ approximating $u$ in the norm $\| \cdot \|_1$. In particular, the sequence $\{v_k\}$ has a subsequence, which we also denote by $\{v_k\}$, converging a.e. to $u$. Using (3.4) with $v = v_k$ and applying Fatou’s lemma to the term $\langle W_2 v_k, v_k \rangle$ we see that (3.4) holds with $u$ in place of $v$. This shows that $\langle W_2 u, u \rangle \in L^1(M)$ and that $u$ satisfies the property (3.3). □

In the proof of the next lemma (and later in the paper), we will need the concept of the minimal and maximal operator corresponding to (2.3) with $V \in L^2_{\text{loc}}(\text{End } \mathcal{V})$. The term *minimal operator* corresponding to $H_V$, denoted by $H_{\text{min}}$, refers to the closure of $H_V|_{C_c^\infty(\mathcal{V})}$, while *maximal operator* is defined as $H_{\text{max}} := (H_{\text{min}})^*$, where $T^*$ indicates the adjoint operator corresponding to $T$. It is well known that $\text{Dom}(H_{\text{max}})$ can be described as

$$\text{Dom}(H_{\text{max}}) = \{ u \in L^2(\mathcal{V}) : H_V u \in L^2(\mathcal{V}) \}.$$
Lemma 3.3. Assume that \((M, g)\) is geodesically complete. Let \(W = W_1 - W_2\) with \(0 \leq W_j \in L^2_{\text{loc}}(\text{End} \, \mathcal{V})\), \(j = 1, 2\). Assume that there exist numbers \(0 < \delta \leq 1\) and \(C \in \mathbb{R}\) such that (3.2) is satisfied. Additionally, assume that \((\nabla^\dagger \nabla + W)^\dagger\) is essentially self-adjoint. Let \(w \in L^2(\mathcal{V})\) and \((\nabla^\dagger \nabla + W)w \in W^{-1,2}(\mathcal{V})\). Then, \(w \in \text{Dom}(h \nabla) \cap \text{Dom}(h W_1)\).

Proof. With \(C\) as in (3.2) and \(I\) standing for the identity endomorphism of \(\mathcal{V}\), denote \(S := \nabla^\dagger \nabla + W + (C + 1)I\), and observe that, by hypothesis, \(S|_{C_c^\infty(\mathcal{V})}\) is essentially self-adjoint and (strictly) positive operator (the latter property holds because of (3.2)), and, consequently (by Theorem X.26 in [33]), we have \(\ker(S_{\text{max}}) = \ker((S|_{C_c^\infty(\mathcal{V})})^\dagger) = \{0\}\). Denote \(h_1 = h_V + h_{W_1}\) and note that by Lemma 3.2, on the space \(\text{Dom}(h_1) = \text{Dom}(h \nabla) \cap \text{Dom}(h W_1)\), we can define a sesquilinear form

\[
Y[u_1, u_2] := h_\nabla[u_1, u_2] + h_{W_1}[u_1, u_2] - h_{W_2}[u_1, u_2] + (C + 1)(u_1, u_2),
\]

where \((\cdot, \cdot)\) is the inner product in \(L^2(\mathcal{V})\). Let \((\cdot, \cdot)_{1}\) be the inner product corresponding to the norm \(\| \cdot \|_{1} := h_1[\cdot] + \| \cdot \|^2\). Looking at the property (3.3), as granted by Lemma 3.2, and adding \((C + 1)\| \cdot \|^2\) to both sides, we see that \(Y\) is a bounded and coercive form on the Hilbert space \(\text{Dom}(h_1)\) equipped with the inner product \((\cdot, \cdot)_{1}\). (The form \(Y\) is coercive because in this lemma we assume \(\delta > 0\).)

Let \(w \in L^2(\mathcal{V})\) be as in the hypothesis of this lemma and consider \(F := \nabla^\dagger \nabla w + Ww + (C + 1)w\), with \(C\) as in (3.3). By assumption, we have \(F \in W^{-1,2}(\mathcal{V})\), the antidual of \(\text{Dom}(h_\nabla)\). As \(\text{Dom}(h_1) \subset \text{Dom}(h_\nabla)\), it follows that \(F\) is a bounded linear functional on the Hilbert space \(\text{Dom}(h_1)\) equipped with the inner product \((\cdot, \cdot)_{1}\). Therefore, by Lax–Milgram Theorem (see Theorem 5.8 in [40]), there exists a unique element \(u \in \text{Dom}(h_1)\) such that \(Y(u, s) = F(s)\), for all \(s \in \text{Dom}(h_1)\). In particular, for all \(s \in C_c^\infty(\mathcal{V}) \subset \text{Dom}(h_1)\), the equality \(Y(u, s) = F(s)\) can be written as

\[
(\nabla u, \nabla s) + (W_1 u, s) + (C + 1)(u, s) = (\nabla^\dagger \nabla w + Ww + (C + 1)w, s),
\]

where \((T, s)_d\) stands for the action of a distributional section \(T\) on a section \(s \in C_c^\infty(\mathcal{V})\). Using the integration by parts (see Lemma 8.8 in [4]) in the term \((\nabla u, \nabla s)\) and remembering that \(W_j u \in L^1_{\text{loc}}(\mathcal{V})\), \(j = 1, 2\), the last equation can be rewritten as

\[
(\nabla^\dagger \nabla u + W u + (C + 1)u, s)_d = (\nabla^\dagger \nabla w + Ww + (C + 1)w, s)_d,
\]

that is, \((\nabla^\dagger \nabla + W + (C + 1)I)(u - w) = 0\) in distributional sense, which shows that \((u - w) \in \ker(S_{\text{max}})\). Remembering the property \(\ker(S_{\text{max}}) = \{0\}\), we see that \(u = w\), which means that \(w \in \text{Dom}(h_1) = \text{Dom}(h \nabla) \cap \text{Dom}(h_{W_1})\). □

In the subsequent discussion, we will use the following formula with \(f \in C_c^\infty(M)\) and \(u \in L^2(\mathcal{V})\):

\[
\nabla^\dagger (f u) = f \nabla^\dagger u - 2(\omega^\sharp f)^\dagger u + u \Delta f.
\]

(3.5)

Here, \(\omega^\sharp\) stands for the vector field corresponding to the 1-form \(\omega\) with respect to the metric \(g\).

Lemma 3.4. Assume that \((M, g)\) is geodesically complete. Let \(V = V_1 - V_2\) with \(0 \leq V_j \in L^2_{\text{loc}}(\text{End} \, \mathcal{V})\), \(j = 1, 2\). Assume that \(V\) satisfies the assumptions (i) and (ii) of Theorem 2.2. Let \(u \in \text{Dom}(H_{\text{max}})\), where \(H_V\) is as in (2.3). Then, \(u \in W^{1,2}_{\text{loc}}(\mathcal{V})\) and \((V_1 u, u) \in L^1_{\text{loc}}(M)\).

Proof. Let \(x_0 \in M\) be arbitrary, let \(\phi \in C_c^\infty(M)\) be as in the assumptions (i) and (ii) of Theorem 2.2, and let \(\chi \in C_c^\infty(M)\) be a function satisfying \(0 \leq \chi \leq 1\), \(\chi(x_0) = 1\), and \(\text{supp} \, \chi \subset \{x \in M : \phi(x) = 1\}\), where \(\text{supp} \, \chi\) denotes the support of \(\chi\). If \(u \in \text{Dom}(H_{\text{max}})\), then

\[
\nabla^\dagger (\chi u) + \phi V(\chi u) = \nabla^\dagger V(\chi u) + V(\chi u)
\]

\[
= \chi H_V u - 2(\omega^\sharp \chi)^\dagger u + u \Delta \chi,
\]

(3.6)

where in the first equality, we used the property of \(\text{supp} \, \chi\) and in the second equality we used (3.5) and the definition of \(H_V\). As \(u \in L^2(\mathcal{V})\) and \(H_V u \in L^2(\mathcal{V})\), we infer that

\[
(\nabla^\dagger (\chi u) + \phi V(\chi u)) \in W^{-1,2}(\mathcal{V}).
\]

(3.7)
Since $V$ satisfies the assumptions (i) and (ii) of Theorem 2.2, we see that $W := \phi V$ satisfies the hypotheses of Lemma 3.3. With the properties $\chi u \in L^2(V)$ and (3.7) at our disposal, from Lemma 3.3, we get $\chi u \in \text{Dom}(h_V) \cap \text{Dom}(h_{\phi V^1})$. Since $x_0 \in M$ is arbitrary, the first inclusion means that $u \in W^{1,2}_{\text{loc}}(V)$ and the second one says $\langle \phi V^1 \chi u, \chi u \rangle \in L^1(M)$, that is, $\langle V^1 u, u \rangle \in L^1_{\text{loc}}(M)$. □

We now state the last lemma of this section.

**Lemma 3.5.** Assume that $(M, g)$ is geodesically complete. Let $V = V_1 - V_2$ with $0 \leq V_j \in L^2_{\text{loc}}(\text{End } V)$, $j = 1, 2$. Assume that $V$ satisfies the assumptions (i) and (ii) of Theorem 2.2. Furthermore, assume that there exists a number $0 \leq \varepsilon < 1$ and a function $q \in L^\infty_{\text{loc}}(M)$ with $q \geq 0$ such that

$$\varepsilon \|\nabla v\|^2 + (Vv, v) \geq -(qv, v),$$

(3.8)

for all $v \in C^\infty_c(V)$. Let $H_V$ be as in (2.3). Then, for every Lipschitz compactly supported function $\psi : M \to \mathbb{R}$ and every $u \in \text{Dom}(H_{\text{max}})$, we have

$$\varepsilon (\nabla^\dagger \nabla (\psi u), \psi u) + (V\psi u, \psi u) \geq -(q\psi u, \psi u).$$

(3.9)

**Proof.** Since $u \in \text{Dom}(H_{\text{max}})$, Lemma 3.4 tells us that $u \in W^{1,2}_{\text{loc}}(V)$ and $\langle V_1 u, u \rangle \in L^1_{\text{loc}}(M)$. Therefore, as $\psi$ is a compactly supported Lipschitz function, we have $\psi u \in \text{Dom}(h_V) \cap \text{Dom}(h_{V^1})$. Assuming $0 < \varepsilon < 1$ for the moment, we rewrite (3.8) as follows:

$$\nabla^\dagger (V^1 u, v) + \varepsilon^{-1}((V + q)v, v) \geq 0.$$

Remembering that $\psi u \in \text{Dom}(h_V) \cap \text{Dom}(h_{V^1})$ and taking into account $0 \leq q \in L^\infty_{\text{loc}}(M)$, we obtain $\psi u \in \text{Dom}(h_V) \cap \text{Dom}(h_{W_1})$, where $W_1 = \varepsilon^{-1}(V_1 + q) \geq 0$. Therefore, we can use Lemma 3.2 with $\delta = C = 0$, $W_1 = \varepsilon^{-1}(V_1 + q)$ and $W_2 = \varepsilon^{-1}V_2$ to obtain $\langle W_2 \psi u, \psi u \rangle \in L^1(M)$ and

$$\nabla^\dagger (\psi u, v) + \varepsilon^{-1}((V + q)(\psi u), \psi u) \geq 0,$$

which leads to (3.9). For $\varepsilon = 0$, the inequality (3.8) can rewritten as $((V + q)v, v) \geq 0$ for all $v \in C^\infty_c(V)$. To get (3.9) for $\varepsilon = 0$, we can use an argument based on Friedrichs mollifiers (see the proof of Lemma 2.2 in [25] for details). □

## 4 | PROOF OF THEOREM 2.2

We first recall two “product rule” formulas, which we will use for a Lipschitz compactly supported function $\psi$ and $u \in W^{1,2}_{\text{loc}}(V)$:

$$\nabla (\psi u) = d\psi \otimes u + \psi \nabla u,$$

(4.1)

and

$$\nabla^\dagger (\psi \nabla u) = \psi \nabla^\dagger \nabla u - \nabla ((d\psi)^\sharp u),$$

(4.2)

where $(d\psi)^\sharp$ is understood as in (3.5).

Next, we recall (see, for instance, Lemma 8.9 in [4]) that if $(M, b)$ is a geodesically complete Riemannian manifold with metric $b$, then there exists a sequence of compactly supported Lipschitz functions $\{\phi_k\}$ such that

(L1) $0 \leq \phi_k \leq 1$ and $|d\phi_k|_b \leq \frac{1}{k}$, where $|d\phi_k|_b$ indicates the length of the cotangent vector $d\phi_k$ corresponding to the metric $b$;

(L2) $\lim_{k \to \infty} \phi_k(x) = 1$, for all $x \in M$.

With the preparations carried out in Section 3, the proof of Theorem 2.2 is practically the same as that of Theorem 2.7 of [4]. Nevertheless, as our preparations are different from those in [4] and as we will refer to specific items from Section 3, we
will show the details of the proof. From now on in this section, we assume that the hypotheses of Theorem 2.2 are satisfied. Without stating it explicitly each time, \( q, V, \varepsilon, \text{ and } H_V \) are as in Theorem 2.2.

**Lemma 4.1.** Let \( q \) and \( L \) be as in Theorem 2.2. Then, for all \( u \in \text{Dom}(H_{\max}) \) we have \((q^{-1/2} \nabla u) \in L^2(T^*M \otimes V) \) and

\[
\|q^{-1/2} \nabla u\|^2 \leq \frac{2}{1 - \varepsilon} \left( \left( 1 + \frac{2L^2(1 + \varepsilon)^2}{1 - \varepsilon} \right) \|u\|^2 + \|u\|\|H_V u\| \right).
\] (4.3)

**Proof.** Let \( \psi \) be a Lipschitz compactly supported function satisfying the inequality \( 0 \leq \psi \leq q^{-1/2} \leq 1 \) and let \( K := \text{ess sup}_{x \in M} |d\psi(x)|_g \), where \( |d\psi|_g \) is the length of the cotangent vector \( d\psi \) corresponding to the metric \( g \). We will first show that for all \( u \in \text{Dom}(H_{\max}) \), the inequality (4.3) holds with \( K \) and \( \psi \) in place of \( L \) and \( q^{-1/2} \), respectively. Since \( u \in \text{Dom}(H_{\max}) \), Lemma 3.4 tells us that \( u \in W^{1,2}_{\text{loc}}(V) \). Therefore,

\[
\|\psi \nabla u\|^2 = (\psi \nabla u, \psi \nabla u) = (\nabla(\psi^2 \nabla u), u) = 2(\psi \nabla^\dagger(\psi^2 \nabla u), u) - 2(\psi \nabla (d\psi)^\sharp u, u)
\]

\[
\leq 2\text{Re}(\psi^2 \nabla^\dagger \nabla u, u) + 2K\|\psi \nabla u\|\|u\|,
\] (4.4)

where in the second equality we used integration by parts (see Lemma 8.8 of [4]), the application of which is allowed in our case since \( \psi^2 u \in W^{1,2}_{\text{comp}}(V) \), in the third equality we used (4.2) and the property \( d(\psi^2) = 2\psi d\psi \), and, lastly, we used the definition of \( K \). After rearranging (4.4), we get

\[
\text{Re}(\psi^2 \nabla^\dagger \nabla u, u) \geq (\psi \nabla u, \psi \nabla u) - 2K\|\psi \nabla u\|\|u\|
\]

\[
= (\nabla(\psi u), \nabla(\psi u)) - (d\psi \otimes u, \psi \nabla u) - (\psi \nabla u, d\psi \otimes u) - 2K\|\psi \nabla u\|\|u\|
\]

\[
\geq (\nabla(\psi u), \nabla(\psi u)) - 4K\|\psi \nabla u\|\|u\|,
\] (4.5)

where in the equality we used (4.1) and in the second inequality we used the definition of \( K \).

Multiplying (4.4) by \( 1 - \varepsilon \) and doing further estimates (the steps are explained below), we obtain

\[
(1 - \varepsilon)\|\psi \nabla u\|^2 \leq (1 - \varepsilon)\text{Re}(\psi^2 \nabla^\dagger \nabla u, u) + 2K(1 - \varepsilon)\|\psi \nabla u\|\|u\|
\]

\[
= \text{Re}(\psi^2 H_V u, u) - \varepsilon\text{Re}(\psi^2 \nabla^\dagger \nabla u, u) - (\psi^2 \nabla^\dagger \nabla u, u) + 2K(1 + \varepsilon)\|\psi \nabla u\|\|u\|
\]

\[
\leq \|H_V u\|\|u\| - \varepsilon(V(\psi u), V(\psi u)) - (V(\psi u), \psi u) + 2K(1 + \varepsilon)\|\psi \nabla u\|\|u\|
\]

\[
\leq \|H_V u\|\|u\| + (q \psi u, \psi u) + 2K(1 + \varepsilon)\|\psi \nabla u\|\|u\|
\]

\[
\leq \|H_V u\|\|u\| + \|u\|^2 + 2K(1 + \varepsilon)\|\psi \nabla u\|\|u\|,
\] (4.6)

where in the equality we used the definition of \( H_V \) and in the second estimate we applied Cauchy–Schwarz inequality to \( (\psi^2 H_V u, u) \). Additionally, in the second estimate, the term \( -\varepsilon(V(\psi u), V(\psi u)) \) and the change from \( 2K(1 - \varepsilon) \) to \( 2K(1 + \varepsilon) \) occur as a result of using (4.5) multiplied by \( -\varepsilon \). Furthermore, in the third inequality we used Lemma 3.5, and in the fourth inequality we used the hypothesis \( 0 \leq \psi \leq q^{-1/2} \). With the help of \( 2ab \leq \frac{b^2}{\tau} \), \( a, b \in \mathbb{R} \) with \( \tau = \frac{1 - \varepsilon}{2(1 + \varepsilon)^2} \), we can estimate the last term on the right-hand side of (4.6) to get

\[
2(1 + \varepsilon)K\|\psi \nabla u\|\|u\| \leq \frac{1 - \varepsilon}{2} \|\psi \nabla u\|^2 + 2K^2(1 + \varepsilon)^2 \frac{(1 + \varepsilon)^2}{1 - \varepsilon} \|u\|^2,
\]

which when combined with (4.6) leads to

\[
\frac{1 - \varepsilon}{2} \|\psi \nabla u\|^2 \leq \left( 1 + \frac{2K^2(1 + \varepsilon)^2}{1 - \varepsilon} \right) \|u\|^2 + \|u\|\|H_V u\|.
\] (4.7)
As \((M, g)\) is geodesically complete (see Remark 2.1), there exists a sequence \(\phi_k\) of Lipschitz compactly supported functions satisfying (L1)-(L2) above with \(b = g\). Letting \(\psi_k := \phi_k \cdot q^{-1/2}\), observe that \(0 \leq \psi_k \leq q^{-1/2} \leq 1\) and

\[
|d\phi_k|_g \leq |d\phi_k|_g \cdot q^{-1/2} + \phi_k |dq^{-1/2}|_g.
\]

Therefore, \(|d\psi_k|_g \leq \frac{1}{k} + L\), where \(L\) is as in item (ii) of the definition of \(\mathcal{M}\). Noting that \(\psi_k(x) \to q^{-1/2}(x)\) as \(k \to \infty\) and using the dominated convergence theorem in (4.7) with \(\psi = \psi_k\) we obtain, after dividing both sides by \((1 - \varepsilon)/2\), the inequality (4.3). \(\square\)

Continuation of the proof of Theorem 2.2: To prove that \(H_V|_{C^\infty(\mathcal{V})}\) is essentially self-adjoint, it is enough to show that \(H_{\text{max}}\) is a symmetric operator. Let \(u, v \in \text{Dom}(H_{\text{max}})\) and let \(\psi \geq 0\) be a Lipschitz compactly supported function. As seen in Lemma 3.4, the sections \(u\) and \(v\) belong to \(W^{1,2}_{\text{loc}}(\mathcal{V})\); hence, we can use integration by parts (see Lemma 8.8 in [4]) and (4.1) to get

\[
(u, \nabla^+ \nabla v) = (u, \nabla \nabla v) + (d\psi \otimes u, \nabla v),
\]

and, similarly,

\[
(v, \nabla^+ \nabla u) = (v, \nabla \nabla u) + (\nabla u, d\psi \otimes v),
\]

where \((\cdot, \cdot)\) on the left-hand sides of both equations stands for the antiduality between \(W_{\text{loc}}^{-1,2}(\mathcal{V})\) and \(W^{1,2}_{\text{comp}}(\mathcal{V})\). Keeping in mind (see Lemma 4.1) that \(q^{-1/2} \nabla u\) and \(q^{-1/2} \nabla v\) belong to \(L^2(T^*M \otimes \mathcal{V})\), from the last two equations, we get

\[
|(\psi u, H_V v) - (H_V u, \psi v)| \leq \left| |(d\psi \otimes u, \nabla v)| + |(\nabla u, d\psi \otimes v)|\right|
\]

\[
\leq \text{ess sup}_{x \in M} \left( |d\psi|_g \cdot q^{1/2} \cdot (\|u\|q^{-1/2}\nabla v\| + \|v\|q^{-1/2}\nabla u\|) \right),
\]

where \(|d\psi|_g\) indicates the length of the cotangent vector \(d\psi\) in metric \(g\). Recalling that \((M, g_q)\) is geodesically complete, where \(g_q = q^{-1} g\) is as in (iii) of the definition of \(\mathcal{M}\), we let \(\phi_k\) be a sequence of Lipschitz compactly supported functions on \(M\) satisfying (L1)-(L2) above with \(b = g_q\). From the property \(|d\phi_k|_{g_q} = q^{1/2}|d\phi_k|_g\) and (L2) with \(b = g_q\), we get

\[
\text{ess sup}_{x \in M} \left( |d\phi_k|_{g_q} \cdot q^{1/2}(x) \right) \leq \frac{1}{k},
\]

which, when combined with (4.8) with \(\psi = \phi_k\), leads to

\[
|(\phi_k u, H_V v) - (H_V u, \phi_k v)| \leq \frac{1}{k} \left( \|u\|q^{-1/2}\nabla v\| + \|v\|q^{-1/2}\nabla u\| \right).
\]

Letting \(k \to \infty\) in the last inequality and using the dominated convergence theorem on the left-hand side, we obtain \((H_V u, v) = (u, H_V v)\), for all \(u, v \in \text{Dom}(H_{\text{max}})\), which shows that \(H_{\text{max}}\) is symmetric.

5 | PROOF OF COROLLARY 2.6

It suffices to show that the hypotheses (i) and (ii) of Theorem 2.2 are satisfied. For every \(x_0 \in M\), we can find a function \(\phi \in C^\infty_c(M)\) with \(0 \leq \phi \leq 1\) and \(\phi \equiv 1\) on a neighborhood of \(x_0\). If \(V\) is \(D_{\text{loc}}\)-decomposable with decomposition \(V = V_1 - V_2\), \(0 \leq V_j \in L^2_{\text{loc}}(\text{End } \mathcal{V})\), then from the definition of \(D_{\text{loc}}(M)\) we get \(\phi[V_2] \in D(M)\). Thus, we can use Theorem XII.1 in [13] to conclude that \((\nabla^+ \nabla + \phi V_1)|_{C^\infty_c(\mathcal{V})}\) is essentially self-adjoint, which means that the condition (ii) of Theorem 2.2 is satisfied. Since \(\phi[V_2] \in D(M)\), by Lemma VII.4 in [13] the form \(h_{\phi V_2}\), defined as in (3.1) with \(Q = \phi V_2\), is \(h_{\mathcal{V}}\)-bounded with bound \(< 1\), that is, there exist numbers \(0 \leq a < 1\) and \(C \geq 0\) such that

\[
h_{\phi V_2}[v, v] \leq a h_{\mathcal{V}}[v, v] + C\|v\|^2,
\]

for all \(v \in \text{Dom}(h_{\mathcal{V}})\). Therefore, keeping in mind that \(\phi V_1 \geq 0\), for all \(v \in C^\infty_c(\mathcal{V})\) we have

\[
(\nabla^+ \nabla + \phi V_1, v) = (\nabla^+ \nabla + \phi V_1, v) - h_{\phi V_2}[v, v]
\]

\[
\geq (\nabla^+ \nabla + \phi V_1, v) - ah_{\mathcal{V}}[v, v] - C\|v\|^2
\]
Thus, the condition (i) of Theorem 2.2 is satisfied with $\delta = 1 - a$.

## 6 Proof of Theorem 2.7

We first recall some abstract facts. Let $\mathcal{H}$ be a Hilbert space. Let $A$ be a self-adjoint operator (with domain $\text{Dom}(A)$) in $\mathcal{H}$, let $f \in \text{Dom}(A)$, and let $T > 0$ be a number. By a solution $U(t)$ of the equation

$$U'' + AU = 0, \quad U(0) = f, \quad U'(0) = 0,$$

we mean a $\text{Dom}(A)$-valued function $U(t), t \in [0, T]$, such that $U(t)$ is twice differentiable on $[0, T]$, the equality $U''(t) + AU(t) = 0$ holds for all $t \in [0, T]$, and the initial conditions $U(0) = f$ and $U'(0) = 0$ are satisfied. For the following abstract fact, see Lemma 3 in [29]:

**Lemma 6.1.** Let $A$ be a self-adjoint (not necessarily lower semibounded) operator in a Hilbert space $\mathcal{H}$ and let $T > 0$ be a number. Assume that $U(t), t \in [0, T]$, satisfies (6.1) with some $f \in \text{Dom}(A)$. Then, for all $t \in [0, T]$, we have $f \in \text{Dom}(\cos(A^{1/2}t))$ and $U(t) = \cos(A^{1/2}t)f$.

In the next two lemmas, we consider consider $H_W := \nabla^2 + W$, with $W \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ being $D$-decomposable, where the latter notion was defined before the statement of Theorem 2.2. Under this hypothesis on $W$ and assuming that $(M, g)$ is geodesically complete, the operator $H_W|_{C_c^\infty(\mathcal{V})}$ is lower semibounded and essentially self-adjoint (see Theorem 1.1 in [16] or Theorem XII.1 in [13]). The symbol $\overline{H_W}$ indicates the self-adjoint closure of $H_W|_{C_c^\infty(\mathcal{V})}$, where we dropped $|_{C_c^\infty(\mathcal{V})}$ for simplicity. The following property was proven in Lemma XII.4 (see also eq. XII.13) of [13]:

**Lemma 6.2.** Assume that $(M, g)$ is geodesically complete. Assume that $W \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is $D$-decomposable. Let $f \in \text{Dom}(\overline{H_W})$ be a compactly supported section such that $\text{supp } f \subset B_R(x_0)$, with $B_R(x_0)$ as in (2.2). Then, $u(t, \cdot) := (\cos((\overline{H_W})^{1/2}t)f)(\cdot)$ has the following property: $\text{supp } u(t, \cdot) \subset B_R^+(x_0)$, for all $t \geq 0$.

The following Lemma is analogous to Lemma 4.2 of [7], which was proven for Schrödinger operators acting on scalar-valued functions on a compact Riemannian manifold and assumptions on the potential different from those in this paper.

**Lemma 6.3.** Assume that $(M, g)$ is geodesically complete. Assume that $W \in L^2_{\text{loc}}(\text{End } \mathcal{V})$ is $D$-decomposable. Let $\rho_1 > 0$ and let $f \in \text{Dom}(\overline{H_W})$ be a compactly supported section such that $\text{supp } f \subset B_{\rho_1}(x_0)$. Then, for all $\rho > \rho_1$, there exists a sequence $f_j \in C_c^\infty(\mathcal{V})$ with $\text{supp } f_j \subset B_{\rho}(x_0)$ such that

(i) $\lim_{j \to \infty} f_j = f$,

(ii) $\lim_{j \to \infty} H_W f_j = \overline{H_W} f$, 

where the limits are taken with respect to the norm of $L^2(\mathcal{V})$.

Proof. Let $\rho_1 > 0$ and let $f \in \text{Dom}(\overline{H_W})$ be a compactly supported section such that $\text{supp } f \subset B_{\rho_1}(x_0)$. By the definition of the closure $\overline{H_W}$, there exists a sequence $v_k \in C_c^\infty(M)$ such that $v_k \to f$ and $H_W v_k \to \overline{H_W} f$ in $L^2(\mathcal{V})$. Let $\chi \in C_c^\infty(M)$ be a function satisfying the following properties: $0 \leq \chi \leq 1$; $\chi(x) \equiv 1$ in a neighborhood of $\text{supp } f$; and $\text{supp } \chi \subset B_{\rho}(x_0)$, where $\rho > \rho_1$. Define $u_k := \chi v_k$ and observe that $u_k \in C_c^\infty(\mathcal{V})$ with $\text{supp } u_k \subset B_{\rho}(x_0)$. Furthermore, note that $u_k \to \chi f = f$ in $L^2(\mathcal{V})$. Similarly as in (3.6), we have

$$H_W u_k = H_W(\chi v_k) = \chi H_W v_k - 2\nabla (\chi v_k) v_k + v_k \Delta \chi,$$

(6.2)
from which we see that the first and the third term on the right hand side converge (in the norm of $L^2(\mathcal{V})$) to $\chi \overline{H_W f} = \overline{H_W f}$ and $f \Delta \chi = 0$, respectively, where we have used the definition of $v_k$ and the fact that $\chi(x) \equiv 1$ in a neighborhood of $\text{supp} f$. Next, we show that there is a subsequence of $v_k$, again denoted by $v_k$, such that $\nabla (d\chi)^\# v_k \rightarrow 0$ weakly. For the latter property, it suffices to show that the sequence $\nabla (d\chi)^\# v_k$ is (norm) bounded in $L^2(\mathcal{V})$, after taking into account that $\nabla (d\chi)^\# v_k \rightarrow 0$ in distributional sense:

$$(\nabla (d\chi)^\# v_k, s) = (\nabla v_k, d\chi \otimes s) = (v_k, (\Delta \chi)s) - (v_k, \nabla (d\chi)^\# s)$$

$$\rightarrow (f, (\Delta \chi)s) - (f, \nabla (d\chi)^\# s) = 0,$$

as $k \rightarrow 0$, for all $s \in C_c(\mathcal{V})$, where in the second equality we used integration by parts and in the last equality we used the fact that $\chi(x) \equiv 1$ in a neighborhood of $\text{supp} f$. Finally, thanks to

$$\|\nabla (d\chi)^\# v_k\| \leq c_1 \|\nabla v_k\|,$$

where $c_1$ is a constant, it is enough to show that $\nabla v_k$ is a (norm) bounded sequence in $L^2(T^* M \otimes \mathcal{V})$. Using a decomposition $W = W_1 - W_2$, $0 \leq W_j \in L^2_{\text{loc}}(\text{End} \mathcal{V})$ with $|W_2| \in D(M)$, and proceeding as in (5.1), we see that there exist numbers $0 \leq a < 1$ and $C \geq 0$ such that

$$\|\nabla v_k\|^2 = (\nabla^\top \nabla v_k, v_k) = (H_W v_k, v_k) - (W_1 v_k, v_k) + (W_2 v_k, v_k)$$

$$\leq (H_W v_k, v_k) + a \|\nabla v_k\|^2 + C \|v_k\|^2,$$

which leads to

$$\|\nabla v_k\|^2 \leq (1 - a)^{-1} [ (H_W v_k, v_k) + C \|v_k\|^2].$$

The latter inequality tells us that $\|\nabla v_k\|$ is bounded (because the sequences $H_W v_k$ and $v_k$ converge in $L^2(\mathcal{V})$). Thus, we have shown that there is a subsequence of $u_k \in C_c(\mathcal{V})$, which we again denote by $u_k$, such that $H_W u_k \rightarrow \overline{H_W f}$ weakly in $L^2(\mathcal{V})$. Now we can start with $u_k$ and use Mazur’s Lemma (see Lemma A.2 below) to construct a sequence $f_j \in C_c(\mathcal{V})$ with $\text{supp} f_j \subset B_\rho(x_0)$ satisfying the properties (i) and (ii) of this lemma.

From hereon, we assume that the hypotheses of Theorem 2.7 are satisfied. To make our notations simpler, we drop $x_0$ from the symbol $B_\rho(x_0)$. For $\rho > 0$ define $V_\rho(x) := V(x)$ if $x \in B_\rho$ and $V_\rho(x) := 0$ if $x \notin B_\rho$. Note that $V_\rho$ is $D$-decomposable. Denoting

$$H_{V_\rho} := \nabla^\top \nabla + V_\rho$$

(6.3)

and looking at the comments before Lemma 6.2, we see that $H_{V_\rho} |_{C_c(\mathcal{V})}$ is a lower semibounded and essentially self-adjoint operator. Thus, Lemma 6.2 and Lemma 6.3 are applicable to $H_{V_\rho}$.

Let $H_V$ and $H_{V_\rho}$ be as in (2.3) and (6.3), respectively. As $H_{V_\rho} |_{C_c(\mathcal{V})}$ is lower semibounded and $C_c(\mathcal{V}) \subset C_c(\mathcal{V})$, it follows that $H_V |_{C_c(\mathcal{V})}$ is lower semibounded. Let $F_\rho, \rho > 0$, denote the Friedrichs extension of $H_V |_{C_c(\mathcal{V})}$ in the space $L^2(\mathcal{V})$.

**Lemma 6.4.** Let $H_V$ be as in (2.3). Assume that the hypotheses of Theorem 2.7 are satisfied. Assume that $H_V |_{C_c(\mathcal{V})}$ has a self-adjoint extension $A$ in $L^2(\mathcal{V})$. Let $T_1 > T_0 > 0$ and $R > 0$ be arbitrary real numbers and let $\rho := R + T_1$. Then, for all $t \in [0, T_0]$ and for all $f \in C_c(\mathcal{V})$ such that $\text{supp} f \subset B_R$ we have

$$[\cos(A^{1/2} t)] f(x) = [\cos((F_\rho)^{1/2} t)] f(x),$$

(6.4)

where $F_\rho$ is the Friedrichs extension of $H_V |_{C_c(\mathcal{V})}$ in $L^2(\mathcal{V})$.

**Proof.** To simplify the notations, for the remainder of this proof, we will drop $|_{C_c(\mathcal{V})}$ from $H_{V_\rho} |_{C_c(\mathcal{V})}$ and $H_V |_{C_c(\mathcal{V})}$ and write just $H_{V_\rho}$ and $H_V$. Let $T_1 > T_0 > 0, R > 0, \rho := R + T_1$, and $f \in C_c(\mathcal{V})$ with $\text{supp} f \subset B_R$ be as in the hypothesis of
Define $U(t) := \cos \left( (\sqrt{H_V})^{1/2} t \right) f$ with $t \in [0, T_0]$, and note that $U(t)$ satisfies the equation

$$U'' + \sqrt{H_V} U = 0, \quad U(0) = f, \quad U'(0) = 0. \quad (6.5)$$

Denoting $u(t, \cdot) := U(t)(\cdot)$ and using Lemma 6.2 with $W = V_\rho$ and $t \in [0, T_0]$, we have $u(t, \cdot) \subset B_{R+t} \subset B_{\rho_0}$, where $\rho_0 := R + T_0 < R + T_1 = \rho$.

As $U(t) \in \text{Dom}(\sqrt{H_V})$ with $\text{supp} u(t, \cdot) \subset B_{\rho_0}$, Lemma 6.3 tells us that there exists a sequence $f_j(t, \cdot) \in C^\infty_c(\mathbb{V})$ with $\text{supp} f_j(t, \cdot) \subset B_{\rho}$ such that

$$f_j(t, \cdot) \to U(t), \quad H_V f_j(t, \cdot) \to H_V U(t), \quad (6.6)$$

for all $t \in [0, T_0]$, where both convergence relations are understood in the norm of $L^2(\mathbb{V}|_{B_\rho})$.

As $V_\rho f_j(t, \cdot) = V f_j(t, \cdot)$, we can rewrite (6.6) as

$$f_j(t, \cdot) \to U(t), \quad H_V f_j(t, \cdot) \to H_V U(t), \quad (6.7)$$

for all $t \in [0, T_0]$.

Therefore, $U(t) \in \text{Dom}(\sqrt{H_V})$ and $\overline{H_V U} = \overline{H_V U}$, for all $t \in [0, T_0]$. Recalling (see the hypothesis) that the operator $A$ is a self-adjoint extension of $H_V \mid C^\infty_c(\mathbb{V})$, we infer that $U(t) \in \text{Dom}(A)$ and $\overline{H_V U} = A U$, for all $t \in [0, T_0]$. Thus, from (6.5) we get

$$U'' + A U = 0, \quad U(0) = f, \quad U'(0) = 0, \quad (6.8)$$

and from here, using Lemma 6.1, we obtain

$$U(t) = \cos( A^{1/2} t) f, \quad (6.9)$$

for all $t \in [0, T_0]$.

The argument used in (6.6) and (6.7) shows that $U(t) \in \text{Dom}(F_\rho)$ and

$$\overline{H_V U} = F_\rho U, \quad (6.10)$$

for all $t \in [0, T_0]$, where $F_\rho$ is the Friedrichs extension of $H_V \mid C^\infty_c(\mathbb{V}|_{B_\rho})$ in $L^2(\mathbb{V}|_{B_\rho})$.

Going back to (6.5) and referring to (6.10), we get

$$U'' + F_\rho U = 0, \quad U(0) = f, \quad U'(0) = 0. \quad (6.11)$$

Finally, using Lemma 6.1 we obtain $U(t) = \cos((F_\rho)^{1/2} t)f$ for all $t \in [0, T_0]$. Combining the latter formula with (6.9) leads to (6.4).

$\square$

**Remark 6.5.** Following the notations of Lemma 6.4, let $T_1 > T_0 > 0, R > 0, \rho_0 := R + T_0$, and $\rho := R + T_1$. Furthermore, let $F_\rho, \rho > 0$, be the Friedrichs extension of $H_V \mid C^\infty_c(\mathbb{V}|_{B_\rho})$ in the space $L^2(\mathbb{V}|_{B_\rho})$. Taking into account the proof of Lemma 6.4, we record the following observation: If $f \in C^\infty_c(\mathbb{V})$ is a section such that $\text{supp} f \subset B_R$ and if $U(t) := \cos((F_\rho)^{1/2} t)f$, then $\text{supp} U(t)(\cdot) \subset B_{\rho_0} \subset B_\rho$ for all $t \in [0, T_0]$.

Before proceeding further, we explain some notations used below. If $S$ is a self-adjoint operator in a Hilbert space $\mathcal{H}$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, then (see, for instance, Theorem B.5 in [13]) $S$ has a unique spectral resolution $E(\lambda)$ such that $S = I_\mathbb{R}(E)$, where $I_\mathbb{R} : \mathbb{R} \to \mathbb{R}$ is the identity function and $I_\mathbb{R}(E)$ is understood in the sense of spectral calculus. If $v \in \mathcal{H}$, then the function $\lambda \mapsto (E(\lambda)v, v) = \|E(\lambda)v\|^2$ is right-continuous and increasing, and, hence, gives rise to a Borel measure on $\mathbb{R}$, which we denote by $(E(\lambda)\nu, v)$. For details on the spectral calculus, see section B.1.4 in [13].

**Continuation of the proof of Theorem 2.7:** Let $F_\rho$ be the Friedrichs extension of $H_V \mid C^\infty_c(\mathbb{V}|_{B_\rho})$ in $L^2(\mathbb{V}|_{B_\rho})$. Looking at the hypothesis (2.5) and using $q(x) := [(\alpha r)(x)]^2$, with $r(x)$ as in (2.1), we see that

$$\langle H_V u, u \rangle \geq -((\alpha(\rho))^2)(u, u), \quad (6.11)$$
for all $u \in C_\infty^\infty(\mathcal{V})$ with supp $u \subset B_\rho$, and, therefore, $-(\alpha(\rho))^2$ is a lower bound of $F_\rho$.

In the subsequent discussion, $\cosh(\cdot)$ stands for the hyperbolic cosine. Denoting by $E_F(\lambda)$ the spectral resolution of $F_\rho$, we have,

$$\left| \cos((F_\rho)^{1/2}t)v, v \right| = \left| \int_{-\infty}^{\infty} \cos(\lambda^{1/2}t) (E_F(d\lambda)v, v) \right|$$

$$\leq \int_{-\infty}^{\infty} |\cos(\lambda^{1/2}t)| (E_F(d\lambda)v, v) = \int_{-(\alpha(\rho))^2}^{\infty} |\cos(\lambda^{1/2}t)| (E_F(d\lambda)v, v)$$

$$\leq \cosh(t\alpha(\rho))\|v\|^2_{L^2(\mathcal{V}|_{B_\rho})},$$  \hspace{1cm} (6.12)

for all $t \geq 0$ and all $v \in L^2(\mathcal{V}|_{B_\rho})$, where in the first equality we used the spectral calculus, in the second equality we used the fact that $-(\alpha(\rho))^2$ is a lower bound of $F_\rho$, and in the second inequality we used the property

$$|\cos(\lambda^{1/2}t)| \leq \cosh(\kappa^{1/2}t),$$

which holds for all $t \geq 0$ and $\lambda \geq -\kappa$, with $\kappa > 0$. From (6.12) we obtain

$$\| \cos((F_\rho)^{1/2}t)\|_{2,2,\mathcal{V}|_{B_\rho}} \leq \cosh(t\alpha(\rho)),$$  \hspace{1cm} (6.13)

where $\| \cdot \|_{2,2,\mathcal{V}|_{B_\rho}}$ stands for the norm of a (bounded) operator $L^2(\mathcal{V}|_{B_\rho}) \rightarrow L^2(\mathcal{V}|_{B_\rho})$.

Let $T_1 > T_0 > 0$ and $R > 0$ be arbitrary real numbers and let $\rho := R + T_1$. Assume that there exists a self-adjoint extension $A$ of $H_\mathcal{V}|_{C_\infty^\infty(\mathcal{V})}$ in $L^2(\mathcal{V})$, and let $E(\lambda)$ denote the spectral resolution of $A$. Keeping in mind Remark 6.5, for all sections $f \in C_\infty^\infty(\mathcal{V})$ with $\text{supp} f \subset B_R$ and all $t \in [0, T_0]$, we have

$$\int_{-\infty}^{\infty} \cos^2(\lambda^{1/2}t) (E(d\lambda)f, f) = \| \cos(A^{1/2}t)f \|^2 = \| \cos((F_\rho)^{1/2}t)f \|^2_{L^2(\mathcal{V}|_{B_\rho})}$$

$$\leq \| \cos((F_\rho)^{1/2}t)\|_{2,2,\mathcal{V}|_{B_\rho}}^2 \|f\|_{L^2(\mathcal{V}|_{B_\rho})}^2 \leq \|f\|^2 \cosh^2(t\alpha(\rho))$$

$$\leq \|f\|^2 e^{2t\alpha(\rho)} = \|f\|^2 e^{2t\alpha(T_1+R)},$$  \hspace{1cm} (6.14)

where in the first equality we used the spectral calculus, in the second equality we used the representation (6.4) and Remark 6.5, in the second inequality we used (6.13), and in the third estimate we used the definition of $\cosh(\cdot)$. Before proceeding further, we note that for all $\lambda < 0$ and $t \geq 0$, we have

$$\cos^2(\lambda^{1/2}t) = \cos^2(i|\lambda|^{1/2}t) = \cos^2(|\lambda|^{1/2}t) \geq \frac{e^{2|\lambda|^{1/2}t}}{4},$$  \hspace{1cm} (6.15)

where $i$ is the imaginary unit. Therefore, for all $t \in [0, T_0]$ we have

$$Y(t) := \int_{-\infty}^{0} e^{2|\lambda|^{1/2}t} (E(d\lambda)f, f) \leq 4 \int_{-\infty}^{0} \cos^2(\lambda^{1/2}t) (E(d\lambda)f, f)$$

$$\leq 4 \int_{-\infty}^{\infty} \cos^2(\lambda^{1/2}t) (E(d\lambda)f, f) \leq 4\|f\|^2 e^{2t\alpha(T_1+R)},$$  \hspace{1cm} (6.16)

where in the first inequality we used (6.15) and in the third inequality we used (6.14). Using Cauchy–Schwarz inequality and (6.16) we obtain

$$\int_{-\infty}^{0} e^{\lambda^{1/2}t} (E(d\lambda)f, f) = \int_{-\infty}^{0} e^{\lambda^{1/2}t} [(E(d\lambda)f, f)]^{1/2}[(E(d\lambda)f, f)]^{1/2}$$

$$\leq (Y(t))^{1/2} \left( \int_{-\infty}^{0} (E(d\lambda)f, f) \right)^{1/2} \leq (Y(t))^{1/2}\|f\| \leq 2\|f\|^2 e^{t\alpha(T_1+R)},$$  \hspace{1cm} (6.17)
for all \( t \in [0, T_0] \) and all \( f \in C_c^\infty(V) \) with \( \text{supp} \, f \subset B_R \). In particular, putting \( t = T_0 \) in (6.17), we get

\[
\int_{-\infty}^{0} e^{\frac{1}{2}T_0} (E(d\lambda)f, f) \leq 2\|f\|^2 e^{T_0\alpha(T_1+R)},
\]

which upon letting \( T_1 \to T_0^+ \) leads to

\[
\int_{-\infty}^{0} e^{\frac{1}{2}T_0} (E(d\lambda)f, f) \leq 2\|f\|^2 e^{T_0\alpha(T_0+R)},
\]

for all \( f \in C_c^\infty(V) \) with \( \text{supp} \, f \subset B_R \).

By the assumption on \( \alpha(\cdot) \) and the properties of \( V \) mentioned before Theorem 2.7, it follows that \( p(T_0) := T_0\alpha(T_0+R) \) belongs to \( V \). As \( T_0 > 0 \) and \( R > 0 \) are arbitrary, we see that the measure \( d\sigma_f(\lambda) := (E(\lambda) f, f) \), where \( f \in C_c^\infty(V) \), satisfies the hypothesis (A2) of Lemma A.3.

If \( A_1 \) and \( A_2 \) are two self-adjoint extensions of \( H_V|_{C_c^\infty(V)} \) in \( L^2(V) \) with the corresponding spectral resolutions \( E_1(\lambda) \) and \( E_2(\lambda) \), then by Lemma 6.4 we have \( \cos(A_1^{1/2}t)f = \cos(A_2^{1/2}t)f \), for all \( t > 0 \) and all \( f \in C_c^\infty(V) \). Hence, by spectral calculus, we have

\[
\int_{-\infty}^{\infty} \cos(\lambda^{1/2}t)(E_1(\lambda)f, f) = \int_{-\infty}^{\infty} \cos(\lambda^{1/2}t)(E_2(\lambda)f, f).
\]

Denoting the left-hand side (or the right-hand side) of the last equation by \( Z(t) \) and keeping in mind that \( d\sigma_j(\lambda) := (E_j(\lambda)f, f) \), where \( j = 1, 2 \) and \( f \in C_c^\infty(V) \), satisfy the property (A2), we can use Lemma A.3 (and the fact that \( E_j(\lambda) \) is a spectral resolution) to infer that \( (E_1(\lambda)f, f) = (E_2(\lambda)f, f) \) for all \( f \in C_c^\infty(V) \) and all \( \lambda \in \mathbb{R} \). As \( C_c^\infty(V) \) is dense in \( L^2(V) \), from the last equality, we get \( A_1 = A_2 \). Therefore, if the (densely defined, symmetric) operator \( H_V|_{C_c^\infty(V)} \) has a self-adjoint extension in \( L^2(V) \), then this self-adjoint extension is unique (i.e., \( H_V|_{C_c^\infty(V)} \) is essentially self-adjoint).

**ACKNOWLEDGMENTS**

We are grateful to the referee for valuable comments and suggestions, which have helped us improve the presentation of the paper.

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Before stating the following lemma (see Theorem 2.28 in [40] for the proof), we explain some terminology. Let \( \mathcal{H} \) be a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \). A conjugate linear map \( \tau : \mathcal{H} \to \mathcal{H} \) is called a conjugation if it satisfies \( \tau^2 = I_{\mathcal{H}} \) and \( \langle \tau u, \tau v \rangle = \langle u, v \rangle \), for all \( u, v \in \mathcal{H} \). An operator \( S \) in \( \mathcal{H} \) is called \( \tau \)-real if there exists a conjugation \( \tau : \mathcal{H} \to \mathcal{H} \) such that \( \tau(\text{Dom}(S)) \subset \text{Dom}(S) \) and \( S(\tau u) = \tau(S u) \), for all \( u \in \text{Dom}(S) \). An example of \( \tau \) is provided by the usual complex conjugation in \( \mathcal{H} = L^2(M) \), where \( M \) is a Riemannian manifold and \( L^2(M) \) is the space of square integrable complex-valued functions on \( M \).

**Lemma A.1.** Assume that \( S \) is a symmetric and \( \tau \)-real operator in a Hilbert space \( \mathcal{H} \). Then, \( S \) has a self-adjoint extension in \( \mathcal{H} \).
Next, we recall some known results used in Section 6. We start with Mazur’s Lemma, whose statement can be found in Lemma 10.19 of [34]. In this lemma, the symbol \( \mathbb{Z}_+ \) stands for the set \{1, 2, 3, ... \}.

**Lemma A.2.** Let \( \mathcal{B} \) be a Banach space and suppose that \( u_k \to f \) weakly in \( \mathcal{B} \). Then, there exist a function \( G : \mathbb{Z}_+ \to \mathbb{Z}_+ \) and a sequence of sets of real numbers \( \{(\beta(j))_k\}_{k=j}^{G(j)} \) with \( (\beta(j))_k \geq 0 \) and \( \sum_{k=j}^{G(j)} (\beta(j))_k = 1 \), such that the sequence

\[
    f_j := \sum_{k=j}^{G(j)} (\beta(j))_k u_k
\]

converges strongly to \( f \) in \( \mathcal{B} \).

In the next result, whose proof can be found in [41], the symbol \( \mathcal{V} \) refers to the class of functions defined before the statement of Theorem 2.7.

**Lemma A.3.** Let \( Z : (0, \infty) \to \mathbb{C} \) be a function representable as

\[
    Z(t) = \int_{-\infty}^{\infty} \cos(\lambda^{1/2} t) \, d\sigma(\lambda),
\]

(A1)

for some function \( \sigma : \mathbb{R} \to \mathbb{C} \) of bounded variation satisfying the following property: There exists a number \( C \) and a function \( \xi \in \mathcal{V} \) such that

\[
    \int_{-\infty}^{0} e^{t|\lambda|^{1/2}} |d\sigma(\lambda)| \leq Ce^{\xi(t)},
\]

(A2)

for all \( t > 0 \). Then, the representation (A1) is unique in the following sense: If \( Z \) is representable in the form (A1) with \( \sigma = \sigma_1 \) and \( \sigma = \sigma_2 \), and if \( \sigma_j, j = 1, 2 \), satisfy the property (A2), then \( \sigma_1 \) and \( \sigma_2 \) differ by a constant.

This result of [41] generalized an earlier result of [23], where instead of the assumption \( \xi \in \mathcal{V} \) in (A2), a special case \( \xi(t) = t^\kappa, 0 < \kappa \leq 2 \), was considered.