Twisted superYangians and their representations

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Abstract

Starting with the superYangian $Y(M|N)$ based on $gl(M|N)$, we define twisted superYangians $Y(M|N)^\pm$. Only $Y^+(M|2n)$ and $Y^-(2m|N)$ can be defined, and appear to be isomorphic one with each other. We study their finite-dimensional irreducible highest weight representations.

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1 Introduction

Quite recently, a revival of interest has been put on coideal algebras of Hopf algebras, both from mathematical and physical point of view. Among these algebras, let us note the twisted Yangians $Y^\pm(N)$, introduced by Olshanski [1] and widely studied (see for instance [2] and references therein), or the reflection algebras, introduced by Sklyanin [3] and studied in [4, 5].

From the mathematical point of view, it seems that such coideal condition is quite restrictive, leading to a very small class of subalgebras, for a given Hopf algebra [6, 7]. Indeed, for quantum algebras $U_q(gl_N)$, it has been proven that they are natural deformations of symmetric spaces [6].

From a physical point of view, such ideals appear to play an important role in integrable systems with boundaries [4, 8, 9]. They appear to be the integrals of motion of such systems [4, 9], and also naturally deduced from the boundary condition [8].

It appears thus natural to look for such coideals when the underlying algebra is not $gl_N$ anymore. Such types of algebras have been introduced in [10] for the case of Yangians based on orthogonal and symplectic algebras, and orthosymplectic superalgebras. They are defined as the “twist” of the (super)Yangian based on the corresponding Lie (super)algebra.

The aim of the present work is to complete the picture with the case of superYangians based on $gl(M|N)$. After recalling the basic definitions and properties of the superYangians $Y(gl(M|N)) \equiv Y(M|N)$ in section 2, we will define the twisted superYangians $Y(M|N)^+$ in section 3. Their finite-dimensional irreducible representations are studied in section 4. We conclude in section 5.

2 Super Yangians $Y(M|N)$

The superYangian $Y(M|N)$ based on the $gl(M|N)$ superalgebra has been introduced in [11], and its irreducible finite-dimensional representations studied in [12]. Since it is a $\mathbb{Z}_2$-graded (Hopf) algebra, different conventions can be chosen: the ones we choose are given below. We will rephrase the properties given in [12] in this context.

2.1 Graded spaces

We start with $K \times K$ matrices acting on the vector space $\mathbb{C}^K$, and introduce a $\mathbb{Z}_2$-grading [1] on these spaces. We will denote by $E_{ij}$ the usual $K \times K$ matrices which have 1 in position $(i, j)$, and $e_i$ the basic vectors of $\mathbb{C}^K$ which have 1 in position $i$.

$$E_{ij}e_k = \delta_{jk}e_i \quad (2.1)$$
The $\mathbb{Z}_2$-grade is defined by

$$[E_{ij}] = [i] + [j] ; \ [e_i] = [i] \quad \text{and} \quad [i] \in \{0, 1\} \ \forall \ i, j = 1, \ldots, K \quad (2.2)$$

We will call even the matrices and vectors such that

$$A = A_{ij} E_{ij} \quad \text{with} \quad [A_{ij}] = [i] + [j] ; \quad u = u_i e_i \quad \text{with} \quad [u_i] = [i] \quad (2.3)$$

The tensor product of graded matrices is chosen to be graded:

$$(E_{ij} \otimes E_{kl})(E_{ab} \otimes E_{cd}) = (-1)^{([k]+[l])([a]+[b])}(E_{ij} E_{ab}) \otimes (E_{kl} E_{cd}) \quad (2.4)$$

On tensor product of $\mathbb{C}^K$ vector spaces, one has

$$(E_{ij} \otimes E_{kl})(e_a \otimes e_b) = (-1)^{([k]+[l])([a]+[b])} (E_{ij} e_a) \otimes (E_{kl} e_b) \quad (2.5)$$

We introduce the graded permutation operator:

$$P_{12} = \sum_{i,j} (-1)^{[j]} E_{ij} \otimes E_{ji} \quad (2.6)$$

which obeys $P^2 = I$ and

$$P(e_i \otimes e_j) = (-1)^{[i][j]} e_j \otimes e_i \ , \ P(E_{ij} \otimes E_{kl})P = (-1)^{([i]+[j])([k]+[l])} E_{kl} \otimes E_{ij} \quad (2.7)$$

### 2.2 Definition and first properties of $Y(M|N)$

We set $K = M + N$ and define the $\mathbb{Z}_2$-grade by:

$$[i] = 0 \quad \text{for} \quad 1 \leq i \leq M \quad \text{and} \quad [i] = 1 \quad \text{for} \quad M + 1 \leq i \leq M + N \quad (2.8)$$

The super Yangian $Y(M|N)$ has generators $T_{(n)}^{ab}$ (of $\mathbb{Z}_2$-grade $[a] + [b]$), gathered in:

$$T(u) = \sum_{a,b=1}^K \sum_{n \geq 0} u^{-n} T_{(n)}^{ab} E_{ab} = \sum_{a,b=1}^K T_{(n)}^{ab}(u) E_{ab} = \sum_{n \geq 0} u^{-n} T_{(n)} \quad \text{with} \quad T_{(0)} = I_K \quad (2.9)$$

The even matrix $T(u) \in M_K[Y(M|N)]$ obeys:

$$R_{12}(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R_{12}(u - v) \quad (2.10)$$

with $R_{12}(x) = I \otimes I - \frac{1}{u-v} P_{12} \ , \ P_{12} = \sum_{i,j} (-1)^{[j]} E_{ij} \otimes E_{ji} \quad (2.11)$

or equivalently

$$[T^{ab}(u) , T^{cd}(v)] = \frac{(-1)^{[a][b]+([a]+[b])[c]}}{u-v} \left( T^{cb}(u) T^{ad}(v) - T^{cb}(v) T^{ad}(u) \right) \quad (2.12)$$
where the graded commutator is defined by

\[ [A, B] = AB - (-1)^{[A][B]} BA \]  

(2.13)

It is a Hopf algebra:

\[ \Delta(T^{ab}(u)) = \sum_{c=1}^{M+N} T^{ac}(u) \otimes T^{cb}(u); \, \epsilon(T^{ab}(u)) = \delta^{ab}; \, S(T^{ab}(u)) = (T^{-1}(u))^{ab} \]  

(2.14)

Note that \( Y(M|N) \) contains several subalgebras:

Property 2.1 (Subalgebras of \( Y(M|N) \))

(i) The generators \( T^{ab}(u) \) with \( a, b = 1, \ldots, M \) (resp. \( T^{ab}(u) \), \( a, b = M+1, \ldots, M+N \)) define the algebra inclusion \( Y(M) \subset Y(M|N) \) (resp. \( Y(N) \subset Y(M|N) \)).

(ii) The generators \( T^{ab}_{(1)} \) with \( a, b = 1, \ldots, M+N \) form a \( gl(M|N) \) Lie sub-super-algebra of \( Y(M|N) \).

(iii) The generators \( T^{ab}(u) \) with \( a, b \in \{M, M+1\} \) define the algebra inclusion \( Y(1|1) \subset Y(M|N) \).

Remark: The above subalgebras are not Hopf subalgebras of \( Y(M|N) \). Indeed, the coproduct (2.14) is based on all the generators of \( Y(M|N) \), so that it does not induce the coproduct of \( Y(M) \) and \( Y(N) \).

We have also the property

Property 2.2 (Isomorphism between \( Y(M|N) \) and \( Y(N|M) \))

Let

\[ \tilde{T}^{ab}(u) = (-1)^{[a][b]+1} T^{ba}(u) \]  

with \( a' = K + 1 - a \)

\[ [a]' = [\bar{a}] + 1 \]  

(2.15)

\( \tilde{T}(u) \) obey the Hopf algebra relations of \( Y(N|M) \), with \( \Delta'(x) = P \Delta(x) P \). We have thus a Hopf algebra isomorphism between \( Y(M|N) \) and \( Y(N|M) \).

Proof: One first proves that \( \tilde{T}(u) \) satisfies the commutation relations of \( Y(N|M) \).

To prove that, we need:

\[ T^{cb}(u) T^{ad}(v) - T^{cb}(v) T^{ad}(u) = -(-1)^{[a]+[d][b]+[c]} (T^{ad}(u) T^{cb}(v) - T^{ad}(v) T^{cb}(u)) \]

which can be proven either by a direct calculation, or by using the graded antisymmetry of the commutator:

\[ [T^{ab}(u) , T^{cd}(v)] = -(-1)^{[a]+[d][b]+[c]} [T^{cd}(v) , T^{ab}(u)] \]  

(2.16)

and computing \( [T^{cd}(v) , T^{ab}(u)] \) using (2.12) with the replacements \( (a, b, u) \leftrightarrow (c, d, v) \).
With the help of the above calculation one gets:
\[
[T^{ab}(u), T^{cd}(v)] = (-1)^{[a][b]+[c][d]} [T^{ba}(u), T^{dc}(v)] = (-1)^{[a][b]+[c][d]+[b][a]+[c][b]+[d][a]} \left( T^{ba}(u)T^{bc}(v) - T^{cb}(v)T^{ba}(u) \right)
\]
which is the correct expression for the commutator in \(Y(N|M)\), since \([\ ]'\) is the correct gradation of \(Y(N|M)\).

For the Hopf structure, one has:
\[
\Delta'_{M|N} T^{ab}(u) = \sum_{c=1}^{K} (-1)^{[a][b]+[c][d]} T^{bc}(u) \otimes T^{ca}(u) = \sum_{c=1}^{K} (-1)^{[a][b]+[c][d]+[a][c][d]+[b][c][d]} T^{cb}(u) \otimes T^{ac}(u) = P \tilde{T}^{ac}(u) \otimes \tilde{T}^{cb}(u) \frac{P}{\epsilon_{M|N}(\tilde{T}^{ab}(u))} = \Delta'_{N|M} \tilde{T}^{ab}(u)
\]
where we have denoted by \(\Delta'_{M|N}\) (resp. \(\Delta'_{N|M}\)) the coproduct on \(Y(M|N)\) (resp. on \(Y(N|M)\)).

A simple calculation shows
\[
\epsilon_{M|N}(\tilde{T}^{ab}(u)) = \epsilon'_{N|M}(\tilde{T}^{ab}(u)) \quad (2.17)
\]
where \(\epsilon' = \epsilon\) is the counit associated to \(\Delta'\), while for the antipode
\[
S'_{M|N}(\tilde{T}^{ab}(u)) = (-1)^{[a][b]+[c]} \theta_{a} \theta_{b} (T^{-1}(u))^{ab} = S'_{N|M}(\tilde{T}^{ab}(u)) = \left( \tilde{T}^{-1}(u) \right)^{ab} \quad (2.18)
\]
where in the last equality, the inverse \(\tilde{T}^{-1}(u)\) is computed using \(m'\) instead of \(m\):
\[
m'(\tilde{T}^{ab}(u) \otimes \tilde{T}^{-1}(u)^{bc}) = (-1)^{[a'][b'][c']} \right)^{bc} \tilde{T}^{-1}(u)^{abc}. \tilde{T}^{ab}(u) = \delta^{ac} \quad (2.19)
\]
\(S'_{M|N}\) obeys the relations
\[
m'(S' \otimes id)\Delta' = \epsilon' = m'(id \otimes S')\Delta' \quad (2.20)
\]
2.3 Finite dimensional irreducible representations of $Y(M|2n)$

The finite dimensional irreducible representations of the superYangian $Y(M|N)$ have been studied in [12]. We recall here its main results, referring to [12] for the proofs.

We will specify to the case $N = 2n$, for it is the only case that is needed for twisted superYangians. Moreover, in order to be able to deal with the twisted superYangians, we need to choose a positive roots system different from the one chosen in [12]. Indeed, the situation in analogous to the one encountered in the case of simple Lie superalgebras, which admit different inequivalent systems of simple roots (see for instance [13]). For our purpose, we define:

**Definition 2.3 (Positive roots)**

Let $\Phi^{+,0} \in \mathbb{N}^2_K$, where $\mathbb{N}_K = \mathbb{N}_{M+2n} = [1, M + 2n] \cap \mathbb{N}$, be defined by

\[
\Phi^{+} = \left\{ (a, b) \in \mathbb{N}^2_K, \text{ with either } \begin{array}{l}
1 \leq a < b \leq M \\
M + 1 \leq a < b \leq M + 2n \\
1 \leq a \leq M, M + n + 1 \leq b \leq M + 2n \\
M + n + 1 \leq a \leq M + 2n, 1 \leq b \leq M
\end{array} \right\}
\]

(2.21)

\[
\Phi^{-} = \{ (a, b) \in \mathbb{N}^2_{M+2n}, \text{ such that } (b, a) \in \Phi^{+} \} 
\]

(2.22)

\[
\Phi^{0} = \{ (a, a), \text{ with } a \in \mathbb{N}_{M+2n} \}
\]

(2.23)

We have $\mathbb{N}_{M+2n} = \Phi^{-} + \Phi^{0} + \Phi^{+}$, and the positive roots will be associated to $T^{ab}(u)$ with $(a,b) \in \Phi^{+}$.

Once the positive roots system is chosen, one can introduce the notion of highest weight vectors:

**Definition 2.4 (Highest weight vectors)**

Let $\mathcal{M}$ be a module of $Y(M|2n)$. A highest weight vector $v \in \mathcal{M}$ is defined by

\[
T^{aa}(u)v = \lambda^a(u)v, \quad \forall a = 1, \ldots, M + 2n \\
T^{aa}(u)v = 0, \quad \forall (a,b) \in \Phi^{+}
\]

(2.24)

$\lambda(u) = (\lambda^1(u), \ldots, \lambda^{M+2n}) \in \mathbb{C}[[u^{-1}]]$ is the highest weight associated to $v$.

The notion of highest weight vectors grounds in the to following properties:

**Property 2.5** Any irreducible finite dimensional representation of $Y(M|N)$ admits a unique (up to multiplication by scalars) highest weight vector.

**Property 2.6** The irreducible representation of $Y(M|N)$ with highest weight $\lambda(u)$ is finite dimensional if and only if we have:

\[
\frac{\lambda^a(u)}{\lambda^{a+1}(u)} = \frac{P_a(u+1)}{P_a(u)}, \quad 1 \leq a \leq M + N - 1, \ a \neq M 
\]

(2.25)

\[
\frac{\lambda^M(u)}{\lambda^{M+1}(u)} = \frac{P_M(u)}{P_{M+N}(u)}
\]

(2.26)

where $P_a(u)$ are monic polynomials.
Let us remark that, some signs differ between our presentation and the presentation given in [12] because of the definition for \( T(u) \): the relation between these two notations is given by \( T_{(n)}^{ab} = (-1)^{|b|} t_0^n [a] \).

**Definition 2.7 (Evaluation representations)**

Let \( J^{ab} \) be the generators of the \( gl(M|N) \) superalgebra and \( \pi^{ab} = \pi(J^{ab}) \) a finite dimensional representation of \( gl(M|N) \). Then, the morphism

\[
ev(T(u)) = 1 + \frac{E}{u} \quad \text{with} \quad E = \pi^{ab} \bar{E}_{ab}
\]  

provides a finite dimensional representation of \( Y(M|N) \), called an evaluation representation.

The usefulness of evaluation representations reveals in the following theorem:

**Theorem 2.8** Any irreducible finite dimensional representation of \( Y(M|N) \) is isomorphic to the irreducible part of tensor products of evaluation representations.

## 3 Twisted superYangians

### 3.1 Introduction to \( Y(M|N)^\tau \)

We now introduce the notion of twisted super Yangian, in the same way twisted Yangians have been defined from the Yangians \( Y(N) \).

We first introduce the transposition \( t \) on matrices:

\[
E_{ar{a}b}^t = (-1)^{|a|(|b|+1)} \theta_a \theta_b E_{ab} \quad \text{with} \quad \theta_a = \pm 1 \\
\bar{a} = M + 1 - a \quad \text{for} \quad 1 \leq a \leq M \\
\bar{a} = 2M + N + 1 - a \quad \text{for} \quad M + 1 \leq a \leq M + N
\]  

which satisfies

\[
(AB)^t = B^t A^t
\]  

Demanding the transposition to be of order 2 leads to the constraint

\[
(-1)^{|a|} \theta_a \theta_{\bar{a}} = \theta_0 = \pm 1 \quad \forall \ a
\]  

Let us stress that \( \bar{a} \) has not the same meaning as in section 2: from now on, we will use this notation to denote \( [a] \). These new \( \bar{a} \) satisfy \( [a] = [\bar{a}] \).

Let us also note that there is a freedom on the definition of the transposition, \( \theta_a \to (-1)^{|a|} \theta_a \): this freedom is fixed when imposing \( I^t = I \). Remark also the useful identity

\[
(-1)^{|a|(|b|+1)} \theta_a \theta_b = (-1)^{|[\bar{a}]|([\bar{b}]+1)} \theta_{\bar{a}} \theta_{\bar{b}}
\]
Then, we define on $Y(M|N)$:

$$
\tau[T(u)] = \sum_{a,b} \tau[T^{ab}(u)] E_{ab} = \sum_{a,b} T^{ab}(-u) E^t_{ab}
$$

which reads for the superYangian generators:

$$
\tau(T^{ab}(u)) = (-1)^{[a][b]+1} \theta_a \theta_b T^{\bar{a}\bar{b}}(-u)
$$

**Property 3.1** $\tau$ is an algebra automorphism for $Y(M|2n)$ and $Y(2m|N)$ only.

In that case, one must choose $\theta_0 = +1$ for $Y(M|2n)$ and $\theta_0 = -1$ for $Y(2m|N)$.

**Proof:** Considering subalgebras mentioned in property 2.1, one can see that $\tau$ acts as an automorphism of $Y(M)$ of the type defined in [14], with $\theta_a \theta_{\bar{a}} = \theta_0$, and an automorphism of $Y(N)$ with $\theta_a \theta_{\bar{a}} = -\theta_0$. Using the results of [14], where it is proved that when $M$ is odd, one must have $\theta_0 = +1$ in $Y(M)$, one immediately concludes that we cannot have $MN$ odd, and that the values for $\theta_0$ are the ones given in the property.

Then, it is a simple matter of calculation to show that $\tau$ is an automorphism of the superalgebra $Y(M|N)$:

$$
\tau \left( \{ T^{ab}(u), T^{cd}(v) \} \right) = \{ \tau(T^{ab}(u)), \tau(T^{cd}(v)) \}
$$

One defines in $Y(M|N)$ (we take $MN$ even):

$$
S(u) = T(u) \tau[T(u)] = \sum_{a,b=1}^{M+N} S^{ab}(u) E_{ab} = \mathbb{I} + \sum_{a,b=1}^{M+N} \sum_{n>0} u^{-n} S^{ab}_{(n)}
$$

$$
S^{ab}_{(n)} = \sum_{c=1}^{M+N} \sum_{p=0}^{n} (-1)^{c} (-1)^{[b][c]+1} \theta_c \theta_{\bar{c}} T^{ac}_{(n-p)} T^{\bar{b}\bar{c}}_{(p)}
$$

$$
S^{ab}(u) = \sum_{c=1}^{M+N} (-1)^{[c][b]+1} \theta_c \theta_{\bar{b}} T^{ac}(u) T^{\bar{c}}(-u)
$$

$S(u)$ defines a subalgebra of the superYangian:

**Theorem 3.2** $S(u)$ obey the following relations:

$$
R_{12}(u-v) S_1(u) R'_{12}(u+v) S_2(v) = S_2(v) R'_{12}(u+v) S_1(u) R_{12}(u-v)
$$

$$
\tau(S(u)) = S(u) + \frac{\theta_0}{2u} (S(u) - S(-u))
$$
where $R(x)$ is the super Yangian $R$-matrix,

$$R'(x) = \mathbb{I} + \frac{1}{x}Q = R^{t_1}(-x) \quad \text{with} \quad Q = P^{t_1}$$

(3.13)

and $t_1$ is the transposition (3.7) in the first space. These two relations uniquely define a subalgebra $Y(M|N)^r$ in the super Yangian.

**Proof:** One starts with the relation (2.10), applies the transposition $t_1$ and the sign operation $(u, v) \rightarrow (-u, -v)$ to get

$$\tau[T_1(u)]R'_{12}(u + v)T_2(v) = T_2(v)R'_{12}(u + v)\tau[T_1(u)]$$

(3.14)

A direct calculation shows that

$$PQ = QP = \theta_0 Q \ ; \ P^2 = \mathbb{I} \quad \text{and} \quad Q^2 = (M - N)Q$$

(3.15)

Thus, applying $P(.)P$ on (3.14) leads to (after the exchange $u \leftrightarrow v$):

$$T_1(u)R'_{12}(u + v)\tau[T_2(v)] = \tau[T_2(v)]R'_{12}(u + v)T_1(u)$$

(3.16)

Finally, applying once again the transposition $t_1$ and $(u, v) \leftrightarrow (-u, -v)$, we obtain:

$$R_{12}(u - v)\tau[T_1(u)]\tau[T_2(v)] = \tau[T_2(v)]\tau[T_1(u)]R_{12}(u - v)$$

(3.17)

which is another way to prove that $\tau$ is an automorphism. A simple calculation using (2.10), (3.14), (3.16) and (3.17) shows then that (3.11) is satisfied.

The second relation is also proved directly:

$$(\tau[S(u)])^{ab} = \sum_{c=1}^{M+N} (-1)^{|a|(|b|+1)+|c|(|a|+1)}\theta_a\theta_b\theta_cT^{bc}(-u)T^{ac}(u)$$

(3.18)

$$= S(u) + \sum_{c=1}^{M+N} (-1)^{|a|(|b|+|c|)}\theta_b\theta_c[T^{bc}(-u), T^{ac}(u)]$$

(3.19)

$$= \left(S(u) + \frac{\theta_0}{2u} (S(u) - S(-u))\right)^{ab}$$

(3.20)

where, in the last step, we have used the graded commutator (2.12).

Conversely, let us start with a subalgebra $A$ of $Y(M|2n)$ whose generators $\sigma^{ab}_{(n)}$ obey (3.11) and (3.12). There is an obvious surjective morphism $j$ from $A$ to $Y(M|2n)^+$. Thus, it remains to show that this morphism is injective. We follow the argumentation done in [13] for the case of (non super) twisted Yangians.

We first introduce a filtration on $Y(M|2n)$ induced by

$$\deg(T^{ab}_{(p)}) = p \quad \text{and} \quad \deg(XY) = \deg(X)\deg(Y), \ \forall X, Y \in Y(M|2n)$$

(3.21)
The graded algebra $grY(M|2n)$ is defined as usual by:

\[
\begin{align*}
Y_p(M|2n) &= \{ X \in Y(M|2n), \text{ with } \deg(X) \leq p \}; \ p > 0 \quad (3.22) \\
Y_0(M|2n) &= \mathbb{C}; \ gr_0Y(M|2n) = \mathbb{C} \quad (3.23) \\
gr_pY(M|2n) &= Y_p(M|2n)/Y_{p-1}(M|2n); \ p > 0 \quad (3.24) \\
grY(M|2n) &= \bigoplus_{p \geq 0} gr_pY(M|2n) \quad (3.25)
\end{align*}
\]

Since for $X \in gr_pY(M|2n)$ and $Y \in gr_qY(M|2n)$, we have $[X, Y] \in gr_{p+q-1}Y(M|2n)$, we deduce that $grY(M|2n)$ is commutative. The same is true for $grY(M|2n)^+$, here the filtration is induced by the $Y(M|2n)$ one.

Similarly, on $A$, we introduce a filtration given by

\[
\deg(\sigma_{(n)}^{ab}) = n \quad (3.26)
\]

This makes $grA$ a commutative algebra, for the same reasons as above. Moreover, since the morphism $\bar{j}$ preserves the filtration, it is enough to show that the induced morphism $\bar{j}$ between graded algebras is injective.

Let $T_{(p)}^{ab}$ and $S_{(p)}^{ab}$ be the image of $T_{(p)}^{ab}$ and $S_{(p)}^{ab}$ in $grY(M|2n)^+$. The expression (3.9) is still valid for the elements of $grY(M|2n)^+$, so that we deduce

\[
\bar{S}_{(p)}^{ab} = (-1)^p(-1)^{|a||b|} \theta_a \theta_b S_{(p)}^{ba} \quad (3.27)
\]

Thus, we conclude that $grY(M|2n)^+$ is isomorphic to the algebra of polynomials in the $(\mathbb{Z}_2$-graded) letters $x_{(p)}^{ab}$ submitted to the constraints (3.27).

Finally, the symmetry relation (3.12) in the algebra $grA$ just takes the form (3.27), so that $grA$ is also isomorphic to the algebra of polynomials in the $(\mathbb{Z}_2$-graded) letters $x_{(p)}^{ab}$ submitted to the constraints (3.27). Hence, $\bar{j}$ is injective.

**Corollary 3.3 (PBW basis for $Y(M|2n)^+$)**

*During this proof, and to avoid confusion with the gradation $\deg$, we will write commutative and commutator where one should has written $\mathbb{Z}_2$-graded commutative and $\mathbb{Z}_2$-graded commutator.*

Given an arbitrary linear order on the following set of generators (for $n = 1, 2, \ldots$):

\[
\begin{align*}
&\bar{S}_{(2n)}^{ij} \quad \text{for } 1 \leq i, j \leq M & \text{and } i + j \leq M + 1 \\
&\bar{S}_{(2n+1)}^{ij} \quad \text{for } 1 \leq i, j \leq M & \text{and } i + j < M + 1 \\
&\bar{S}_{(2n)}^{ij} \quad \text{for } M + 1 \leq i, j \leq M + 2n & \text{and } i + j < 2M + 2 + 2n \quad (3.28) \\
&\bar{S}_{(2n+1)}^{ij} \quad \text{for } M + 1 \leq i, j \leq M + 2n & \text{and } i + j \leq 2M + 2 + 2n \\
&\bar{S}_{(2n)}^{ij} \quad \text{for } M + 1 \leq i \leq M + 2n & \text{and } 1 \leq j \leq M
\end{align*}
\]

any element of $Y(M|2n)^+$ is uniquely written as a linear combination of the ordered monomials in these generators.
Proof: It is a direct consequence of the proof of theorem 3.2. Indeed, considering $gr Y(M|2n)^*$, it is sufficient to find a basis for it, i.e. for the algebra of polynomials in the ($\mathbb{Z}_2$-graded) letters $x^{ab}_{\langle p \rangle}$ submitted to the constraints (3.27). From the property $a + b \leq M + 1 \Leftrightarrow \bar{a} + \bar{b} \geq M + 1, \bar{a} = b \Leftrightarrow a + b = M + 1$ when $a, b \leq M$, and $a + b \leq M + n + 2 \Leftrightarrow \bar{a} + \bar{b} \geq M + n + 2, \bar{a} = b \Leftrightarrow a + b = M + n + 2$ when $a, b \geq M + 1$, an analysis of these constraints lead to the above basis.

Although several automorphisms $\tau$ can be defined (depending upon the choices for the $\theta$’s), they all lead to the same subalgebra $Y(M|N)^*$:

Property 3.4 All the $\theta_a$ dependence can be removed in the commutation relations of $Y(M|2n)^*$.

Proof: We prove the property by exhibiting a basis in which the $\theta$ dependence has disappeared. When restricted to the bosonic part, it is the same basis as the one given in [14] for (bosonic) twisted Yangians.

For $Y^-(2n)$:

$$J^{ij}(u) = \theta^i \theta^j S^{ij}(u) ; \ K^{ij}(u) = \theta^i S^{i,j}(u) ; \ K^{ij}(u) = \theta^j S^{i,j}(u) ; \ i, j = 1, \ldots, n \quad (3.29)$$

For $Y^+(M)$, the redefinition is the same as before, plus for the remaining generators which appear when $M = 2m + 1$:

$$J_0(u) = \theta^\bar{m} S^{\bar{m},\bar{m}}(u) ; \ L^i(u) = \theta^i S^{\bar{m},i}(u) ; \ L^i(u) = \theta^\bar{m} \theta^i S^{\bar{m},i}(u) ; \ i, j = 1, \ldots, m \quad (3.30)$$

This proves that the commutation relations among generators of $Y^-(2n)$, and those among $Y^+(M)$ are free from $\theta$’s in this basis. Commuting an element of $Y^-(2n)$ with one of $Y^+(M)$ provide the change of basis for the fermionic generators:

$$F^{ai}(u) = \theta_a \theta_i S^{ai}(u) \quad F^{ia}(u) = \theta_a \theta_i S^{ai}(u) \quad i = M + 1, \ldots, M + n$$
$$G^{ai}(u) = \theta_a S^{ai}(u) \quad G^{ia}(u) = \theta_i S^{ai}(u) \quad a = 1, \ldots, M$$
$$H^i(u) = \theta_i S^{m+1,i}(u) \quad H^i(u) = \theta_{m+1} \theta_i S^{i,m+1}(u) \quad \text{if } M = 2m + 1 \quad (3.31)$$

All the $Y(M|2n)^+$-generators are expressible in terms of the generators (3.29), (3.30) and (3.31) using the symmetry relation (3.12). Then, one can check that all the graded commutators in this basis are free from $\theta$.

Definition 3.5 The twisted superYangian $Y(M|2n)^+ \equiv Y(2n|M)^-$ is the subalgebra generated by $S(u) = T(u) \tau[T(u)]$, with $\tau$ given in (3.4) and

$$\theta_a = 1 \quad \text{for } 1 \leq a \leq M$$
$$\theta_a = sg(\frac{2M+2a+1}{2} - a) \quad \text{for } M + 1 \leq a \leq M + 2n \quad (3.32)$$
3.2 Few properties of $Y(M|2n)^+$

**Property 3.6** The relation (3.11) is equivalent to the following commutator:

$$[S_1(u), S_2(v)] = \frac{1}{u-v} \left( P_{12}S_1(u)S_2(v) - S_2(v)S_1(u)P_{12} \right) +$$

$$- \frac{1}{u+v} \left( S_1(u)Q_{12}S_2(v) - S_2(v)Q_{12}S_1(u) \right) +$$

$$+ \frac{1}{u^2-v^2} \left( P_{12}S_1(u)Q_{12}S_2(v) - S_2(v)Q_{12}S_1(u)P_{12} \right)$$

(3.33)

and also to

$$[S^{ab}(u), S^{cd}(v)] = \frac{(-1)^{[a]+[b]+[c]}}{u-v} \left( S^{cb}(u)S^{ad}(v) - S^{cb}(v)S^{ad}(u) \right) +$$

$$- \frac{(-1)^{[a]+[b]+[c]}}{u+v} \left( (1)^{[a][b]} \theta_b \theta_c S^{ac}(u)S^{bd}(v) - (1)^{[b][d]} \theta_a \theta_d S^{ca}(v)S^{bd}(u) \right) +$$

$$+ \frac{(-1)^{[a]+[b]+[c]}}{u^2-v^2} \left( S^{ca}(u)S^{bd}(v) - S^{ca}(v)S^{bd}(u) \right)$$

(3.34)

**Proof:** (3.33) follows from a direct calculation using (3.11), (2.11) and (3.13).

As an obvious consequence, we get

**Corollary 3.7** The twisted super-Yangian $Y(M|2n)^+$ contains $osp(M|2n)$ as Lie sub-superalgebra. It is generated by

$$S^{ab}_{(1)} = T^{ab}_{(1)} - (1)^{[a]+[b]+1} \theta_a \theta_b \overline{T}^{b\overline{a}}_{(1)} \quad a, b = 1, \ldots, M + 2n$$

(3.35)

which obey

$$S^{ab}_{(1)} = - (1)^{[a]+[b]+1} \theta_a \theta_b \overline{S}^{b\overline{a}}_{(1)}$$

(3.36)

and defines a morphism of algebra $\mathcal{U}(osp(M|2n)) \to Y(M|2n)^+$. The action of the $osp(M|2n)$ generators on the twisted Yangian is given by:

$$[S^{ab}_{(1)}, S^{cd}(v)] = \left( -1 \right)^{[a]+[b]+[c]} \left\{ (1)^{[a][b]} \left( \delta_{cb} S^{ad}(v) - \delta_{ad} S^{cb}(v) \right) + \right.$$}

$$\left. - \theta_a \theta_b \left( \delta_{ac} S^{bd}(v) - \delta_{bd} S^{ca}(v) \right) \right\}$$

(3.37)

**Proof:** Expanding $(u \pm v)^{-1} = u^{-1}(1 \mp vu^{-1} + \ldots)$ and taking the coefficient of $u^{-1}v^{-1}$ in (3.33) leads to:

$$[S_{1(1)}, S_{2(1)}] = P_{12}S_{2(1)} - S_{2(1)}P_{12} - Q_{12}S_{2(1)} + S_{2(1)}Q_{12}$$

(3.38)

where the subscript $(1)$ refers to the coefficient of $u^{-1}$ and $v^{-1}$, while the indices 1, 2 label the auxiliary spaces. The symmetry relation projected on the $u^{-1}$ term reads

$$S^{t}_{(1)} = -S_{(1)}$$

(3.39)
(3.38) and (3.39) are just the defining relations of $\text{osp}(M|2n)$.

Starting now from (3.34) and taking the coefficient of $u^{-1}$ gives the relation (3.37). Note that taking the coefficient of $v^{-1}$ in this last relation gives again the commutation relations of $\text{osp}(M|2n)$.

Let us denote the $\text{osp}(M|2n)$ generators by $J^{ab}$ and gather them in the matrix

$$F = \sum_{a,b=1}^{M+N} J^{ab} F_{ab} \quad \text{with} \quad F_{ab} = E_{ab} - (-1)^{[a][b]+1} \theta_{a} \theta_{b} E_{ba}$$

(3.40)

It satisfies:

$$F^t = -F$$

$$\{F_1, F_2\} = P_{12} F_2 - F_2 P_{12} + F_2 Q_{12} - Q_{12} F_2$$

(3.41)

where $t$ is the transposition (3.1).

Property 3.8 The following map defines an algebra inclusion:

$$Y(M|2n)^+ \to U[\text{osp}(M|2n)]$$

$$S(u) \to \mathbb{F}(u) = I + \frac{1}{u + \frac{1}{2}} F$$

(3.42)

Proof: We have to prove that $\mathbb{F}$ obeys to the relations (3.11) and (3.12). A direct calculation shows:

$$\mathbb{F}^t(-u) = I + \frac{1}{u - \frac{1}{2}} F = \mathbb{F}(u) + \frac{1}{2u} (\mathbb{F}(u) - \mathbb{F}(-u))$$

Moreover, using (3.15), the commutator (3.41) and the relations

$$P_{12} F_2 = F_1 P_{12} \Rightarrow Q_{12} F_2 = -Q_{12} F_1$$

$$P_{12} F_1 = F_2 P_{12} \Rightarrow F_1 Q_{12} = -F_2 Q_{12}$$

(3.43)

one proves that we have

$$\{\mathbb{F}_1(u), \mathbb{F}_2(v)\} = \frac{1}{u - v} \left( P_{12} \mathbb{F}_1(u) \mathbb{F}_2(v) - \mathbb{F}_2(v) \mathbb{F}_1(u) P_{12} \right) +$$

$$- \frac{1}{u + v} \left( \mathbb{F}_1(u) Q_{12} \mathbb{F}_2(v) - \mathbb{F}_2(v) Q_{12} \mathbb{F}_1(u) \right) +$$

$$+ \frac{1}{u^2 - v^2} \left( P_{12} \mathbb{F}_1(u) Q_{12} \mathbb{F}_2(v) - \mathbb{F}_2(v) Q_{12} \mathbb{F}_1(u) P_{12} \right)$$

(3.44)

The relation between $\text{osp}(M|2n)$ and $Y(M|2n)^+$ also reveals in
Property 3.9 \( Y(M|2n)^+ \) is a deformation of \( \mathcal{U}(osp(M|2n)[x]) \), the (positive modes) loop algebra based on \( osp(M|2n) \).

Proof: We start with \( S(u) = I + s(u) \), and make a change of basis \( \tilde{s}(u) = h^{-1}s(u/h) \). In this basis, the commutation relations read:

\[
[\tilde{s}_1(u), \tilde{s}_2(v)] = \frac{1}{u - v} \left( P_{12}\tilde{s}_1(u) + P_{12}\tilde{s}_2(v) - \tilde{s}_1(u)P_{12} - \tilde{s}_2(v)P_{12} \right) + \\
- \frac{1}{u + v} \left( \tilde{s}_1(u)Q_{12} + Q_{12}\tilde{s}_2(v) - Q_{12}\tilde{s}_1(u) - \tilde{s}_2(v)Q_{12} \right) + \\
+ \frac{\hbar}{u - v} \left( P_{12}\tilde{s}_1(u)\tilde{s}_2(v) - \tilde{s}_2(v)\tilde{s}_1(u)P_{12} \right) + \\
- \frac{\hbar}{u + v} \left( \tilde{s}_1(u)Q_{12}\tilde{s}_2(v) - \tilde{s}_2(v)Q_{12}\tilde{s}_1(u) \right) + \\
+ \frac{\hbar}{u^2 - v^2} \left( P_{12}\tilde{s}_1(u)Q_{12} + P_{12}\tilde{s}_2(v) - Q_{12}\tilde{s}_1(u)P_{12} - \tilde{s}_2(v)P_{12} \right) + \\
+ \frac{\hbar^2}{u^2 - v^2} \left( P_{12}\tilde{s}_1(u)Q_{12}\tilde{s}_2(v) - \tilde{s}_2(v)Q_{12}\tilde{s}_1(u)P_{12} \right)
\]

For \( \hbar \neq 0 \) all the algebras \( Y_h(M|2n)^+ \) are isomorphic, while in the limit \( \hbar \to 0 \), \( Y_{\hbar=0}(M|2n)^+ \) reduces to \( \mathcal{U}(osp(M|2n)[x]) \).

Note also the isomorphism

Property 3.10 (Automorphism of \( Y((M|2n)^+) \))

The transformations

\[
S(u) \to g(u)S(u) \quad \text{with } g(u) \text{ even } \mathbb{C}\text{-function} \quad (3.44)
\]

are automorphisms of \( Y(M|2n)^+ \).

Proof: Multiplying (3.11) by \( g(u)g(v) \) shows that it is invariant under the transformation (3.44), for any function \( g \). The symmetry relation (3.12) is preserved for \( g(u) \) even only.

There is another of type of automorphism that we will need when looking at the representations of twisted superYangians:

Definition 3.11 (# involution)

For any index \( i = 1, \ldots, M + N \), we define \( i' \) by

\[
i' = \begin{cases} 
    m' = \bar{m} + 1, & (\bar{m} + 1)' = m \\
    i & \text{otherwise}
\end{cases} \quad (3.45)
\]

\# defined by

\[
S^i_{\#}(u) = S^{i'}(u) \quad (3.46)
\]

is an order 2 automorphism of \( Y(M|2n)^+ \).
Proof: Obvious direct calculation from the relations (3.34) and (3.12).

For the Hopf structure, and mimicking again the case of twisted Yangians, one can show:

**Property 3.12** \( Y(M|2n)^+ \) is a left coideal of \( Y(M|2n) \):

\[
\Delta(Y(M|2n)^+) \subset Y(M|2n) \otimes Y(M|2n)^+
\]

More precisely:

\[
\Delta(S_{(p)}^{ab}) = \sum_{y=0}^{p} \sum_{q=0}^{M+2n} \sum_{d,e=1} (-1)^q (-1)^{[d][e]+[b]} \theta_b \theta_e T_{(y-q)}^{ad} T_{(q)}^{be} \otimes S_{(y)}^{de}
\]

\[
\Delta[S_{(u)}^{ab}] = \sum_{d,e=1}^{M+2n} (-1)^{[d][e]+[b]} \theta_b \theta_e T^{ad}(u) T^{be}(-u) \otimes S^{de}(u)
\]

Proof: Direct calculation using (2.14) and (3.9).

---

### 3.3 Subalgebras of \( Y(M|2n)^+ \)

**Property 3.13** The twisted superYangian \( Y(M|N)^+ \) contains as subalgebras \( Y(M)^+ \), \( Y(N)^- \) and \( osp(M|N) \).

Proof: A direct examination on the commutator (2.12) and the symmetry relation (3.12) shows that \( S_{(u)}^{ab} \) with \( a,b = 1,\ldots,M \) (resp. \( a,b = M+1,\ldots,M+N \) generates the twisted Yangian \( Y(M)^+ \) (resp. \( Y(N)^- \)). The last inclusion has been proved in the corollary 3.7.

**Property 3.14** As algebra embeddings, we have:

\[
Y(1|2)^+ \subset Y(2m+1|2n)^+ \quad \text{and} \quad Y(2|2)^+ \subset Y(2m|2n)^+
\]

Proof: We consider the generators \( S^{ij}(u) \), with \( i,j = m+1,2m+n+1,2m+n+2 \) in \( Y(2m+1|2n)^+ \) and \( S^{ij}(u) \), with \( i,j = m,m+1,2m+n,2m+n+1 \) in \( Y(2m|2n)^- \): they obey the commutation and symmetry relations of \( Y(1|2)^+ \) and \( Y(2|2)^+ \) respectively.

Note that there is no regular embedding of \( Y(1|2)^+ \) into \( Y(2m|2n)^+ \). The circumstances are here different from both simple superalgebras and non-super twisted Yangians cases: in the first case, it always exists a regular \( osp(1|2) \) embedding, and in the second case, one can always construct a regular \( Y(2)^\pm \) embedding in the twisted Yangian \( Y(M)^\pm \). It is the symmetry relation which causes this unusual situation.
Property 3.15 As algebra embedding, we have:

\[ Y(2m|2n)^+ \subset Y(2m + 1|2n)^+ \quad (3.49) \]

Let us stress however that \( Y(2m|2n)^+ \) is not a Hopf coideal of \( Y(2m + 1|2n)^+ \).

The same results apply for \( Y(2m)^+ \) and \( Y(2m + 1)^+ \).

Proof: Let \( s^{ij}(u) \) be the generators of \( Y(2m + 1|2n)^+ \). We set \( M = 2m + 1 \) and introduce

\[
\begin{align*}
\sigma^{ij}(u) &= s^{ij}(u) \quad \text{for } 1 \leq i, j \leq m \text{ and } M + 1 \leq i, j \leq M + n + 1 \\
\sigma_{i,j}(u) &= s_i^{-1}s_j^{-1}(u) \quad \text{for } m + 2 \leq i, j \leq M \text{ and } M + n + 2 \leq i, j \leq M + 2n \\
\sigma^{ii}(u) &= s_i^{i-1} \quad \text{for } \begin{cases} 1 \leq j \leq m \text{ or } M + 1 \leq j \leq M + n + 1 \\ m + 2 \leq i \leq M \text{ or } M + n + 2 \leq i \leq M + 2n \end{cases} \\
\sigma^{ii}(u) &= s_i^{i-1} \quad \text{for } \begin{cases} 1 \leq i \leq m \text{ or } M + 1 \leq i \leq M + n + 1 \\ m + 2 \leq j \leq M \text{ or } M + n + 2 \leq j \leq M + 2n \end{cases}
\end{align*}
\]

We prove that the generators \( \sigma^{ij}(u) \) generates \( Y(2m|2n)^+ \). We denote by \( x \to \bar{x} \) the "bar" operator introduced in (3.43) for \( Y(2m + 1|2n)^+ \), and by \( x \to \tilde{x} \) this "bar" operator for \( Y(2m|2n)^+ \). In the same way, we call \( \tau \) and \( \theta \) (resp. \( \bar{\tau} \) and \( \bar{\theta} \)) the corresponding operations in \( Y(2m + 1|2n)^+ \) (resp. \( Y(2m|2n)^+ \)). It is easy to see that

\[
\bar{i} = \tilde{i} \quad \theta_i = \tilde{\theta}_i \quad \text{for } i \leq m \text{ and } M + 1 \leq i \leq M + n + 1 \\
\bar{i} - 1 = \tilde{i} \quad \theta_{i-1} = \tilde{\theta}_i \quad \text{for } m + 2 \leq i \leq M \text{ and } M + n + 2 \leq i, j \leq M + 2n
\]

so that the action of \( \tau \) on \( s(u) \) is equivalent to the action of \( \bar{\tau} \) on \( \sigma(u) \). It also proves that the symmetry relation of \( s(u) \) (coming from \( Y(2m + 1|2n)^+ \)) implies the symmetry relation for \( \sigma(u) \) (as \( Y(2m|2n)^+ \) generator).

In the same way, one shows, starting with the commutation relations of \( s(u) \), that the commutation relations of \( \sigma(u) \) are those of \( Y(2m|2n)^+ \).

Finally, computing \( \Delta \sigma^{ij}(u) \) as it is induced from the \( Y(2m + 1|2n)^+ \) coproduct does not lead to the coproduct formula for \( Y(2m|2n)^+ \).

\[ \blacksquare \]

4 Finite dimensional irreducible representations of \( Y(M|2n)^+ \)

We study here the finite dimensional irreducible representations of \( Y(M|2n)^+ \) starting from \( Y(M|2n) \) in the same way those of \( Y(M)^\pm \) have been studied starting from \( Y(M) \).

As a short hand writing, we note irreps for irreducible representations.
4.1 Generalities

Definition 4.1 (Highest weight vector)
Let $\mathcal{M}$ be a module of $Y(M|2n)^+$. A nonzero vector $v \in \mathcal{M}$ is called highest weight if it satisfies

\begin{align*}
S_{ij}^u(v) &= 0 \quad \text{for} \quad (i, j) \in \Phi^+ \quad (4.1) \\
S_{ii}^u(v) &= \mu_i(u)v \quad \text{for} \quad i = 1, \ldots, M + 2n \quad (4.2)
\end{align*}

for some formal series $\mu_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The set $\mu(u) \equiv (\mu_1(u), \ldots, \mu_{M+2n}(u))$ is the highest weight of $\mathcal{M}$.

Remark 1: Due to the symmetry relation (3.12), some of the relations (4.1) are redundant, and one could reduce $\Phi^+$: we keep it as it is to make the comparison with the $Y(M|N)$ case.

Note also that, in the basis of [12], the symmetry relation would have led to $S_{ij}^u(v) = 0$, $\forall i \neq j$, hence the present choice for the positive roots system.

Remark 2: The symmetry relation also implies for the highest weight:

$$
\mu_a'(u) = \frac{1}{2u}\mu_a(u) + \frac{2u-1}{2u}\mu_a(-u) \quad (4.3)
$$

so that, in the $Y(2m+1|2n)^+$ case, $\mu_{m+1}(u)$ is an even function of $u$.

Definition 4.2 (Highest weight representations)
A representation $V$ of the twisted super Yangian $Y(M|2n)^+$ is called highest weight if it is generated by a highest weight vector. If $\mu(u)$ is the highest weight of $v$, we will use the notation $V[\mu(u)]$ for $V$.

Theorem 4.3 Every finite-dimensional irrep $V$ of $Y(M|2n)^+$ is highest weight. Moreover, $V$ contains a unique (up to scalar multiples) highest weight vector.

Proof: We define

$$
V_+ = \{v \in V | S_{ab}^u(v) = 0, \forall (a, b) \in \Phi^+ \text{ and } p > 0\} \quad (4.4)
$$

We first prove that $V_+ \neq \emptyset$.

Let $m \equiv \lfloor \frac{M}{2} \rfloor$. The generators $S_{(1)}^{11}, \ldots, S_{(1)}^{mm}, S_{(1)}^{M+1,M+1}, \ldots, S_{(1)}^{M+n,M+n}$ form a Cartan subalgebra of $Osp(M|2n)$, so there exists a least one eigenvector $v$ common to all $S_{(1)}^{aa}$ and with eigenvalue $\mu = (\mu_1^{(1)}, \ldots, \mu_{M+2n}^{(1)})$.

If $v \in V_+$, then we have $V_+ \neq \emptyset$. If $v \not\in V_+$, by applying $S_{(p)}^{ab}$, $(a, b) \in \Phi^+$, to $v$ we obtain an other common eigenvector of the $S_{(1)}^{aa}$ with eigenvalue $\mu + \omega$, where $\omega$ is a $\mathbb{Z}_{>0}$-linear combination of the positive roots. As $V$ is finite-dimensional, repeated
applications of generators $S_{(a)}^{ab}$, $(a, b) \in \Phi^+$, $n > 0$, will lead to a non-vanishing vector $v_+ \in V$ such that
\begin{align}
S_{(a)}^{ab}v_+ &= 0 \forall (a, b) \in \Phi^+ \quad (p > 0) \\
S_{(1)}^{aa}v_+ &= \lambda^{(1)}v_+ \forall a
\end{align}

So $v_+ \in V_+$ and $V_+$ contains at least one nonzero element.

One defines
\[ T_\pm = \{ S^{ab}(u), \forall (a, b) \in \Phi^\pm \} \]
and $L$ (resp. $R$) the left (resp. right) ideal generated by $T_+$ (resp. $T_-$). We also introduce the subalgebra
\[ V_0 = \{ y \in Y(M|2n)^+, \text{ such that } [S_{(1)}^{aa}, y] = 0 \forall a = 1, \ldots, M + 2n \} \]
and correspondingly
\[ L_0 = V_0 \cap L \quad \text{and} \quad R_0 = V_0 \cap R \]

Using the PBW theorem [3.3], one shows that $L_0 = R_0 \equiv I_0$ is a two-sided ideal so that $G = V_0/I_0$ is an algebra. From the commutation relations (3.34), one gets that $[S^{aa}(u), S^{bb}(v)] \in I_0$, i.e. $G$ is a commutative algebra.

By construction, $\forall v \in V_+$ and $i \in I_0$, one has $iv = 0$, so that $V_+$ is a $G$-module. Since $G$ is commutative, there exists a nonzero common eigenvector $\xi \in V_+$. Now, let $V' = U(T_-)\xi$: it is a non-zero submodule of $V$, and since $V$ is supposed irreducible, it must equal $V$. Thus, $\xi$ is the highest weight vector of $V$.

Finally, if there is another highest weight vector $\xi'$, the above construction ensures that $V = U(T_-)\xi = U(T_-)\xi'$ which is possible only for $\xi$ and $\xi'$ proportional.

---

**Theorem 4.4 (Necessary conditions for finite-dimensional irreps)**

If the irreducible highest weight representation $V[\mu(u)]$ of $Y(M|2n)^+$ is finite-dimensional then the following relations hold:

\[ \frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_{i+1}(u + 1)}{P_{i+1}(u)} \quad \text{for} \quad \begin{cases} 
    m + 2 \leq i \leq M - 1 \\
    M + n + 1 \leq i \leq M + 2n - 1 
\end{cases} \]

\[ \frac{\mu_{M+n+1}(-u)}{\mu_{M+n+1}(u)} = \frac{P_{M+n+1}(u + 1)P_{M+n+1}(-u)}{P_{M+n+1}(u)P_{M+n+1}(1 - u)} \]

If $M = 2m + 1$, one among these two relations also holds:

\[ \frac{\mu_{m+1}(u)}{\mu_{m+2}(u)} = \frac{P_{m+1}(u + 1)}{P_{m+1}(u)}, \quad \text{with} \quad \gamma(u) = 1 \quad \text{or} \quad \frac{2u}{2u + 1} \]

If $M = 2$, we have the supplementary condition

\[ \frac{\mu_2(-u)}{\mu_2(u)} = \frac{P(u + 1)P(-u)(u + \gamma)(2u - 1)}{P(u)P(1 - u)(u - \gamma)(2u + 1)}, \quad \text{with} \quad P(-\gamma)P(\gamma + 1) \neq 0 \]

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Finally, for \( M = 2m > 2 \), we have the relations

\[
\gamma(u) \left\{ \begin{array}{c}
\left. \frac{\mu^0_{m+1}(u)}{\mu^0_{m+1}(-u)} \right| \mu^0_{m+2}(u) \\
\left. \frac{\mu^0_{m+1}(u)}{\mu^0_{m+1}(-u)} \right| \mu^0_{m+2}(u)
\end{array} \right\} = \left\{ \begin{array}{c}
P_{m+2}(u+1) \\
P_{m+2}(u)
\end{array} \right\} P_{m+1}(u+1) P_{m+1}(-u) \frac{P_{m+1}(u)}{P_{m+1}(1-u)}
\]

(4.14)

with \( \gamma(u) = 1 \) or \( \frac{2u-1}{2u+1} \), and \( \mu^0_{m+1}(u) = \mu_{m+1}(u) \) or \( \mu^#_{m+1}(u) \). We have introduce \( \mu^#_{m+1}(u) \), which is deduced from \( \mu_{m+1}(u) \) by the action of the \# automorphism (see definition [3.1] and [13] for more details).

Proof: It is a direct consequence of the classification of finite-dimensional irreps for the algebras \( Y(M)^\pm \) done in [13]. Since \( Y(M)^+ \) and \( Y(2n)^- \) are subalgebras of \( Y(M|2n)^+ \), starting with an \( Y(M|2n)^+ \)-irrep with highest weight \( \xi \), and considering the cyclic span of \( \xi \) with each of these subalgebras leads to the result.

4.2 Finite-dimensional irreps of \( Y(1|2)^+ \)

Let \( V[\mu(u)] \) be an irrep of \( Y(1|2)^+ \) with highest weight \( \mu(u) \equiv (\mu_1(u), \mu_2(u), \mu_3(u)) \). From the symmetry relation (1.3) we obtain that \( \mu_1(u) \) is an even series in \( u^{-1} \) and \( \mu_2(u) \) can be deduced from \( \mu_3(u) \).

Property 4.5 If \( V[\mu_1(u), \mu_3(u)] \) is finite dimensional then there exists a formal even series \( \psi(u) \) in \( u^{-1} \) such that

\[
\psi(u) \mu_1(u) = (1 - \alpha_1^2 u^{-2}) \cdots (1 - \alpha_k^2 u^{-2}) \quad (4.15)
\]

\[
\psi(u) \mu_3(u) = (1 - \alpha_1 u^{-1}) \cdots (1 - \alpha_k u^{-1})(1 + \beta_1 u^{-1}) \cdots (1 + \beta_k u^{-1}) \quad (4.16)
\]

Proof: Let \( \xi \) be the highest weight vector of \( V[\mu(u)] \).

Under \( (S_{(1)}^{22}, S_{(1)}^{33}), S_{(1)}^{32} \cdots S_{(p_1)}^{32} \cdots S_{(p_r)}^{31} \cdots S_{(j)}^{31} \xi \) has weight \( (\mu_2^{(1)} + 2s + r, \mu_3^{(1)} - 2s - r) \) whereas \( S_{(j)}^{31} \xi \) has weight \( (\mu_2^{(1)} + 1, \mu_3^{(1)} - 1) \). So \( S_{(j)}^{31} \xi \) can only be written as a linear combination of vectors \( S_{(j)}^{31} \xi \). Let \( k \) be the minimum non-negative integer such that \( S_{(k+1)}^{31} \xi \) is a linear combination of the vectors \( \xi_1 \equiv S_{(1)}^{31} \xi, \ldots, \xi_k \equiv S_{(k)}^{31} \xi \) (such \( k \) exists because \( V[\mu(u)] \) is finite-dimensional).

We will prove that for any vector \( S_{(r)}^{31} \xi \) with \( r \geq k + 1 \) we have:

\[
S_{(r)}^{31} \xi = a_1^{(r)} \xi_1 + \cdots + a_k^{(r)} \xi_k \quad (4.17)
\]

where the \( a_i^{(r)} \) are complex numbers.

Equation (1.7) is true for \( r = k + 1 \) by definition of \( k \). Taking \( i = k = l = 3, j = 1 \) in the commutation relation and exchanging \( u \) and \( v \), we get:
\[ [S^{33}(u), S^{31}(v)] = -\frac{1}{u-v}(S^{31}(v)S^{33}(u) - S^{31}(u)S^{33}(v)) \] 
\[ -\frac{1}{u+v}S^{32}(u)S^{21}(v) + \frac{1}{u^2-v^2}S^{32}(u)S^{13}(v) \]
\[ + \frac{1-u-v}{u^2-v^2}S^{32}(v)S^{13}(u) \]  

We multiply by \((u^2-v^2)\) and take the coefficient at \(u^0v^p\) \((p \geq 1)\). Using the fact that \(S^{21}(u)\xi = S^{13}(u)\xi = 0\) we obtain:
\[ S^{31}_{(2)} S^{31}(\xi) = -S^{31}_{(p+1)}\xi + S^{31}_{(1,2)} S^{31}_p \xi + S^{31}_v (S^{32}_v - S^{33}_v) \xi \]  
For \(i = 1, \ldots, k-1\) \((4.19)\) gives:
\[ S^{31}_{(2)} \xi_i = -\xi_{i+1} + \mu^1_3 \xi_1 + \mu^2_3 \xi_1 + \mu^3_3 \xi_1 \]  
For \(i = k\), using \(S^{31}_{(k+1)}\xi = a^{(k+1)}_1 \xi_1 + \ldots + a^{(k+1)}_k \xi_k\) in \((4.19)\) gives:
\[ S^{31}_{(2)} \xi_k = -(a^{(k+1)}_1 \xi_1 + \ldots + a^{(k+1)}_k \xi_k) + \mu^1_3 \xi_1 + \mu^2_3 \xi_1 + \mu^3_3 \xi_1 \]  
So \(\forall \xi \in \{1, \ldots, k\}, S^{31}_{(2)} \xi_i\) is a linear combination of the \(\{\xi_j\}_{j=1,\ldots,k}\).

Now suppose that \(\forall r \in \{k+1, \ldots, p\}\) \((p \geq k + 1)\), equation \((4.17)\) holds. We then have:
\[ S^{31}_{(p+1)}\xi = -S^{31}_{(2)} S^{31}_p + \mu^1_3 \xi_1 + \mu^2_3 \xi_1 + \mu^3_3 \xi_1 \]  
so \(S^{31}_{(p+1)}\xi\) is a linear combination of the \(\{\xi_j\}_{j=1,\ldots,k}\) and equation \((4.17)\) is proved by induction on \(p\). We can therefore write:
\[ S^{31}(u)\xi = a_1(u)\xi_1 + \ldots + a_k(u)\xi_k \]  
where \(a_i(u) = u^{-i} + \sum_{s=k+1}^{\infty} a^{(s)}_i u^{-s}\).

We can rewrite \((4.19)\) as:
\[ S^{31}_{(2)} S^{31}(v) = -v S^{31}(v)\xi + \mu^3_3 \xi_1 + \mu^2_3 \xi_1 + \mu^3_3 \xi_1 \]  
On the other hand, applying \(S^{31}_{(2)}\) on \((4.23)\) and using \((4.20)\) and \((4.21)\) we have:
\[ S^{31}_{(2)} S^{31}(v) \xi = \sum_{i=1}^{k} a_i(v) S^{31}_{(2)} \xi_i \]  
\[ = \left( \sum_{i=1}^{k} (a_i(v)\mu^1_3) + (\mu^2_3 - \mu^3_3) - a_k(v)a^{(k+1)}_1 \right) \xi_1 \]
\[ + \sum_{i=2}^{k} (-a_{i-1}(v) + (\mu^2_3 - \mu^3_3)a_i(v) - a_k(v)a^{(k+1)}_1) \xi_i \]
Taking the coefficient at $\xi_i$ for $i = 2, \ldots, k$ in (4.24) and (4.25) leads to:

$$-a_{i-1}(v) + va_i(v) - a_i^{k+1}a_k(v) = 0$$  \hspace{1cm} (4.26)

so that:

$$a_i(v) = a_k(v) \left( v^{k-i} - v^{k-1-i}a_{k+1}^{(k+1)} - v^{k-i-2}a_{k-1}^{(k+1)} - \ldots - va_{k+2}^{(k+1)} - a_i^{(k+1)} \right)_{\phi_i(v)}$$  \hspace{1cm} (4.27)

for $i = 1, \ldots, k - 1$. The coefficient at $\xi_1$ in (4.24) and (4.25) leads to:

$$\mu_3(v) = \sum_{i=1}^{k} \mu_3^{(i)} a_k(v) A_i(v) - a_i^{(k+1)}a_k(v) + va_k(v) A_1(v)$$  \hspace{1cm} (4.28)

So $\mu_3(v) = a_k(v) B(v)$ where $B(v)$ is a monic polynomial in $v$ of degree $k$.

For $\mu_1(v)$ we use:

$$[S^{11}(u), S^{31}(v)] = \frac{u + v + 1}{u^2 - v^2} S^{31}(u) S^{11}(v) - \frac{1}{u + v} S^{12}(u) S^{11}(v)$$

$$- \frac{2v + 1}{u^2 - v^2} S^{31}(v) S^{11}(u)$$

Notice that $S^{11}(u)$ is an even series in $u^{-1}$. Using the same procedure we find that

$\mu_1(v) = a_k(v) C(v)$ where $C(v)$ is a monic polynomial in $v$ of degree $k$.

Defining $\varphi(u) = (a_k(u) u^k)^{-1}$, we have:

$$\varphi(u) \mu_1(u) = (1 + \alpha_1 u^{-1}) \ldots (1 + \alpha_k u^{-1})$$  \hspace{1cm} (4.29)

$$\varphi(u) \mu_3(u) = (1 + \beta_1 u^{-1}) \ldots (1 + \beta_k u^{-1})$$  \hspace{1cm} (4.30)

where the $\alpha_i$'s and the $\beta_i$'s are complex numbers.

The formal series

$$\psi(u) \equiv \frac{\varphi(u)(1 - \alpha_1 u^{-1}) \ldots (1 - \alpha_k u^{-1})}{\mu_1(u)}$$

$$= \frac{(1 - \alpha_1^2 u^{-2}) \ldots (1 - \alpha_k^2 u^{-2})}{\mu_1(u)}$$

is an even series in $u^{-1}$. The composition of the automorphism $S(u) \rightarrow \psi(u) S(u)$ with $V[\mu(u)]$ is an irrep with the following highest weight which we shall again denote by $\mu(u)$

$$\mu_1(u) = (1 - \alpha_1^2 u^{-2}) \ldots (1 - \alpha_k^2 u^{-2})$$  \hspace{1cm} (4.32)

$$\mu_3(u) = (1 - \alpha_1 u^{-1}) \ldots (1 - \alpha_k u^{-1})(1 + \beta_1 u^{-1}) \ldots (1 + \beta_k u^{-1})$$  \hspace{1cm} (4.33)

Thus, up to an automorphism of $Y(1|2)^+$, we can assume that $\mu_1(u)$ and $\mu_3(u)$ are polynomials in $u^{-1}$. 

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Theorem 4.6 Let $V[\mu_1(u), \mu_3(u)]$ be an irrep of $Y(1|2)^+$. Suppose $\mu_1(u)$ and $\mu_3(u)$ satisfy

\[
\frac{\mu_1(u)}{\mu_3(u)} = \frac{P(u + 1) R(u)}{P(u) Q(u)} \quad (4.34)
\]

\[
\frac{\mu_3(-u)}{\mu_3(u)} = \frac{P(u + 1) P(-u)}{P(u) P(1 - u)} \quad (4.35)
\]

where $P(u)$, $Q(u)$ and $R(u)$ are a monic polynomial, $Q(u)$ and $R(u)$ are even in $u$ and of same degree.

Then $V$ is finite-dimensional.

Proof: We call $p$ (resp. $2r$) the degree of $P(u)$ (resp. $Q(u)$ and $R(u)$). Since $R(u)$ and $Q(u)$ are even, they write

\[
R(u) = R_0(u) R_0(-u) \quad ; \quad Q(u) = Q_0(u) Q_0(-u) \quad \text{with} \quad dg(R_0) = dg(Q_0) = r
\]

We introduce:

\[
\lambda_1(u) = u^{-s-r} P(u + 1) R_0(u) \quad (4.37)
\]

\[
\lambda_2(u) = u^{-s-r} P(u + 1) Q_0(u) \quad (4.38)
\]

\[
\lambda_3(u) = u^{-s-r} P(u) Q_0(u) \quad (4.39)
\]

Let $L[\lambda(u)]$ be the corresponding irreducible highest weight module of $Y(1|2)$. Since $\lambda_1(u)/\lambda_2(u) = R_0(u)/Q_0(u)$ and $\lambda_2(u)/\lambda_3(u) = P(u + 1)/P(u)$, according to [12], $L[\lambda(u)]$ is finite-dimensional. The cyclic $Y(1|2)^+$-span of its highest weight vector is a finite-dimensional representation $V[\mu'(u)]$ of $Y(1|2)^+$ with $\mu_1'(u) = \lambda_1(u) \lambda_1(-u)$ and $\mu_3'(u) = \lambda_3(u) \lambda_2(-u)$. By construction, the polynomials $\mu_i'(u)$ satisfy (4.34–4.35). This implies that:

\[
\psi(u) \equiv \frac{\mu_3(u)}{\mu_3'(u)} = \frac{\mu_3(-u)}{\mu_3'(-u)} \quad (4.40)
\]

is an even series in $u^{-1}$ and

\[
\mu_1(u) = \frac{\mu_3(u)}{\mu_3'(u)} \mu_1'(u) = \psi(u) \mu_1'(u) \quad (4.41)
\]

Thus, there exists an automorphism $S(u) \rightarrow \psi(u) S(u)$ of $Y(1|2)^+$ such that its composition with the representation $V[\mu'(u)]$ is isomorphic to $V[\mu(u)]$. $V[\mu(u)]$ is therefore finite-dimensional.

\[\blacksquare\]

Conjecture 1 The sufficient condition (4.34) of theorem 4.6 for the existence of finite-dimensional irreps of $Y(1|2)^+$ is also a necessary condition.

We remind that the condition (4.33) has been proved to be necessary (see theorem 4.6), so that the conjecture just says that the theorem 4.6 states necessary and sufficient conditions for finite-dimensional irreps of $Y(1|2)^+$. 

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4.3 The general case \(Y(2m+1|2n)^{+}\)

**Theorem 4.7** Let \(V = V[\mu_{m+1}(u), \ldots \mu_{2m+1}(u), \mu_{M+n+1}(u), \ldots, \mu_{M+2n}(u)]\) be an irrep of \(Y(2m+1|2n)^{+}\). We take \(m \geq 1\) and note \(M = 2m + 1\).

Suppose the weights \(\mu_i(u)\) obey

\[
\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_{i+1}(u + 1)}{P_{i+1}(u)} \quad \text{for} \quad \begin{cases} \ m + 2 \leq i \leq 2m \\ M + n + 1 \leq i \leq M + 2n - 1 \end{cases} \tag{4.42}
\]

\[
\frac{\mu_{M+n+1}(u)}{\mu_{M+n+1}(u)} = \frac{P_{M+n+1}(u + 1)P_{M+n+1}(-u)}{P_{M+n+1}(u)P_{M+n+1}(1 - u)} \tag{4.43}
\]

\[
\gamma(u) \frac{\mu_{m+1}(u)}{\mu_{m+2}(u)} = \frac{P_{m+2}(u + 1)}{P_{m+2}(u)} \tag{4.44}
\]

\[
\gamma(u) \frac{\mu_{m+1}(u)}{\mu_{m+2}(u)} = \frac{P_{M+n+1}(u + 1) R(u)}{P_{M+n+1}(u) Q(u)} \tag{4.45}
\]

with \(R(u)\) and \(Q(u)\) even and of same degree. In the above formulas, \(\gamma(u)\) is either 1, and the corresponding relations will be called case (a), or \(\frac{2u}{2u+1}\), case (b).

Then \(V\) is finite-dimensional.

Under the assumption of conjecture \(7\), the above sufficient conditions are also necessary ones.

Proof: First, let case (a) hold. We note \(s_i\) the degree of the polynomials \(P_{i}(u)\), decompose \(R(u)\) and \(Q(u)\) as in (4.30), and note:

\[
P_{+}(u) = \prod_{a=m+2}^{M} P_{a}(u) \quad ; \quad P_{-}(u) = \prod_{a=M+n+1}^{M+2n} P_{a}(u) \quad ; \quad s_0 = r + \sum_{i=m+1}^{M} s_i + \sum_{i=M+n+1}^{M+2n} s_i \tag{4.46}
\]

We also define:

\[
\lambda_i(u) = u^{-s_0} P_+(u + 1) P_-(u + 1) R_0(u), \quad i = 1, \ldots, m + 1 \tag{4.47}
\]

\[
\lambda_i(u) = u^{-s_0} P_-(u + 1) R_0(u) \prod_{a=m+2}^{i} P_{a}(u) \prod_{a=i+1}^{2m+1} P_{a}(u + 1), \quad i = m + 2, \ldots, M \tag{4.48}
\]

\[
\lambda_i(u) = u^{-s_0} P_+(u + 1) P_-(u + 1) Q_0(u), \quad i = M + 1, \ldots, M + n \tag{4.49}
\]

\[
\lambda_i(u) = u^{-s_0} P_+(u + 1) Q_0(u) \prod_{a=M+n+1}^{i} P_{a}(u) \prod_{a=i+1}^{M+2n} P_{a}(u + 1), \quad i = M + n + 1, \ldots, M + 2n \tag{4.50}
\]
We therefore have:

\[
\begin{align*}
\frac{\lambda_i(u)}{\lambda_{i+1}(u)} &= \frac{P_{i+1}(u+1)}{P_{i+1}(u)} \quad \text{for} \quad \begin{cases} i = m + 1, \ldots, M - 1 \\
       i = M + n, \ldots, M + 2n - 1 \end{cases} \quad (4.51) \\
\frac{\lambda_i(u)}{\lambda_{i+1}(u)} &= 1 \quad \text{for} \quad \begin{cases} i = 1, \ldots, m \\
       i = M + 1, \ldots, M + n - 1 \end{cases} \quad (4.52) \\
\frac{\lambda_M(u)}{\lambda_{M+1}(u)} &= \frac{R_0(u)}{Q_0(u)} \frac{P_n(u)}{P_+(u+1)} \quad (4.53)
\end{align*}
\]

We consider the highest weight irrep \(L[\lambda(u)]\) of \(Y(2m+1|2n)\). According to property 2.6, the relations (4.51), (4.52) and (4.53) ensure that \(L[\lambda(u)]\) is finite-dimensional. The cyclic \(Y(2m+1|2n)^+\)-span of its highest weight vector is a finite-dimensional representation with highest weights \(\mu'_i(u) = \lambda_i(u)\lambda_i(-u)\) for \(i = m + 1, \ldots, 2m + 1\) and \(i = M + n + 1, \ldots, M + 2n\). Its irreducible quotient is a finite-dimensional irrep \(V[\bar{\mu}(u)]\) of \(Y(2m|2n)^+\).

Moreover, the \(\mu'_i(u)\) verify:

\[
\begin{align*}
\frac{\mu'_i(u)}{\mu'_{i+1}(u)} &= \frac{P_{i+1}(u+1)}{P_{i+1}(u)} = \frac{\mu_i(u)}{\mu_{i+1}(u)} \quad \text{for} \quad \begin{cases} i = m + 1, \ldots, M - 1 \\
       i = M + n + 1, \ldots, M + 2n - 1 \end{cases} \quad (4.54) \\
\frac{\mu'_{M+n+1}(-u)}{\mu'_{M+n+1}(u)} &= \frac{P_{M+n+1}(u+1)P_{M+n+1}(-u)}{P_{M+n+1}(u)P_{M+n+1}(1-u)} = \frac{\mu_{M+n+1}(-u)}{\mu_{M+n+1}(u)} \quad (4.55) \\
\frac{\mu'_{m+1}(u)}{\mu'_{M+n+1}(u)} &= \frac{P_{M+n+1}(u+1)R(u)}{P_{M+n+1}(u)Q(u)} = \frac{\mu_{m+1}(u)}{\mu_{M+n+1}(u)} \quad (4.56)
\end{align*}
\]

The formal series

\[
\psi(u) = \frac{\mu_{M+n+1}(-u)}{\mu'_{M+n+1}(u)} \quad (4.57)
\]

is an even series in \(u^{-1}\) and we have

\[
\frac{\mu_i(u)}{\mu'_i(u)} = \psi(u), \quad \forall i \quad (4.58)
\]

Hence there exists an automorphism \(S(u) \to \psi(u)S(u)\) of \(Y(2m + 1|2n)^+\) such that its composition with \(V[\mu'(u)]\) is isomorphic to \(V[\mu(u)]\). This later is thus finite-dimensional.

Now, let case (b) hold. We introduce the \(osp(2m + 1|2n)\) representation \(V_0\), of highest weight \(l_i = -\frac{1}{2}\) for \(i = m+2, \ldots, 2m+1\) and \(l_i = 0\) for \(i = m+1, M+n+1, \ldots, M+2n\) and promote it to a \(Y(2m + 1|2n)^+\) representation using the evaluation map. The corresponding highest weight has components \(l_i(u) = \frac{2u}{2u+1}\) for \(i = m + 2, \ldots, 2m + 1\), and \(l_i(u) = 1\) for \(i = m + 1, M + n + 1, \ldots, M + 2n\).

Moreover, making the same construction as for case (a), and considering the tensor product \(L[\lambda(u)] \otimes V_0\), we get a finite dimensional representation \(V[\mu''(u)]\) obeying the
relations of case (b). Its irreducible subquotient is isomorphic to \( V[\mu(u)] \), which is therefore finite-dimensional.

Conversely, let us suppose that the irrep \( V[\mu(u)] \) is finite dimensional. From theorem 4.4, one already knows that the conditions (4.42), (4.43) and (4.44) must be satisfied.

Suppose also that the conjecture \( 1 \) holds. The subalgebra generated by the coefficients of \( S_{ij}(u) \), \( i, j = m + 1, M + n, M + n + 1 \) is isomorphic to \( Y(1|2)^+ \). The cyclic span of the highest weight vector of \( V[\mu(u)] \) with respect to this subalgebra is a representation with highest weight \( (\mu_{m+1}(u), \mu_{M+n+1}(u)) \). Its irreducible quotient is finite-dimensional and so, we have relation (4.45).

4.4 Finite-dimensional irreps of \( Y(2|2)^+ \)

Let \( V = V[\mu(u)] \) be an irrep of \( Y(2|2)^+ \) with highest weight \( \mu(u) \).

Property 4.8 If \( V[\mu(u)] \) is finite dimensional then there exists a formal even series \( \psi(u) \) in \( u^{-1} \) such that

\[
\psi(u)\mu_2(u) = (1 - \alpha_1 u^{-1}) \ldots (1 - \alpha_k u^{-1}) \quad (4.59)
\]

\[
\psi(u)\mu_4(u) = (1 - \beta_1 u^{-1}) \ldots (1 - \beta_k u^{-1}) \quad (4.60)
\]

Proof: The proof is very similar to the case of \( Y(1|2)^+ \), and we leave it to the reader. Note that the calculation being achieved using the fermionic generator \( S_{13}(u) \) (instead of the even bosonic one \( S_{12}(u) \)), there is no difference in the proof for \( Y(1|2)^+ \) and \( Y(2|2)^+ \), in opposition with the \( Y(2)^+ \) and \( Y(2)^- \) cases [13].

Theorem 4.9 Let \( V[\mu_2(u), \mu_4(u)] \) be an irrep of \( Y(2|2)^+ \). If \( \mu_2(u) \) and \( \mu_4(u) \) satisfy

\[
\frac{\mu_2(u)}{\mu_4(u)} = \frac{u - \gamma P_4(u + 1) P_2(u) R(u)}{u + \frac{1}{2} P_4(u) P_2(u + 1) Q(u)} \quad (4.61)
\]

\[
\frac{\mu_4(-u)}{\mu_4(u)} = \left( \frac{u + \frac{1}{2}}{u - \frac{1}{2}} \right)^2 \frac{P_4(u + 1) P_4(-u)}{P_4(u) P_4(1 - u)} \quad (4.62)
\]

then \( V \) is finite dimensional.

In the above formulas, \( P_2(u), P_4(u), Q(u) \) and \( R(u) \) are monic polynomials, \( Q(u) \) and \( R(u) \) are even in \( u \) and of same degree, and \( \gamma \in \mathbb{C} \).
Proof: We call $p_2$ (resp. $p_4$, resp. $2r$) the degree of $P_2(u)$ (resp. $P_4(u)$, resp. $Q(u)$ and $R(u)$) and decompose $Q(u)$ and $R(u)$ as in (4.38). Let $L[\lambda(u)]$ be the irrep of $Y(2|2)$ with weights

$$
\lambda_1(u) = u^{-p_2-p_4}P_4(u)P_2(u)R_0(u) \quad (4.63)
$$

$$
\lambda_2(u) = u^{-p_2-p_4}P_4(u)P_2(u)R_0(u) \quad (4.64)
$$

$$
\lambda_3(u) = u^{-p_2-p_4}P_4(u)P_2(u+1)Q_0(u) \quad (4.65)
$$

$$
\lambda_4(u) = u^{-p_2-p_4}P_4(u)P_2(u+1)Q_0(u) \quad (4.66)
$$

Since we have

$$
\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P_2(u+1)}{P_2(u)} \quad ; \quad \frac{\lambda_2(u)}{\lambda_3(u)} = \frac{P_2(u+1)R_0(u)}{P_2(u)Q_0(u)} \quad ; \quad \frac{\lambda_3(u)}{\lambda_4(u)} = \frac{P_4(u+1)}{P_4(u)} \quad (4.67)
$$

$L[\lambda(u)]$ is finite dimensional. The $Y(2|2)^+$-cyclic span of its highest weight vector is a finite dimensional $Y(2|2)^+$-representation $V[\mu'(u)]$ of weights $\mu_2'(u) = \lambda_2(u)\lambda_1(-u)$ and $\mu_4'(u) = \lambda_4(u)\lambda_3(-u)$. These weights obey the relations

$$
\frac{\mu_2'(u)}{\mu_4'(u)} = \frac{P_4(u+1)P_2(u)R(u)}{P_4(u)P_2(u+1)Q(u)} \quad (4.68)
$$

$$
\frac{\mu_4'(-u)}{\mu_4'(u)} = \frac{P_4(u)P_4(-u)}{P_4(u)P_4(1-u)} \quad (4.69)
$$

We now consider the $osp(2|2)$ finite-dimensional irrep $V_0$ with weights $l_2 = -\gamma - \frac{1}{2}$ and $l_4 = -1$. Through the evaluation map, its provides a finite-dimensional representation of $Y(2|2)^+$ with weights

$$
l_2(u) = \frac{u - \gamma}{u + \frac{1}{2}} \quad \text{and} \quad l_4(u) = \frac{u - \frac{1}{2}}{u + \frac{1}{2}} \quad (4.70)
$$

The tensor product $L[\lambda(u)] \otimes V_0$ is thus a finite-dimensional representation of $Y(2|2)^+$, with weights $\mu_i''(u) = \mu_i'(u)l_i(u)$, $i = 2, 4$. They obey to relations (4.61) and (4.62), so that the irreducible quotient provide a finite-dimensional irrep isomorphic to $V$. Thus, $V$ finite-dimensional.

Note that the polynomial $P(u) = (u - \frac{1}{2})^2$ obeys the relation $P(1 - u) = P(u)$, so that the condition on $\mu_4(u)$ does not differ from the one obtained for $Y(2)^-$. 

**Conjecture 2** The sufficient condition (4.64) of theorem (4.3) for the existence of finite-dimensional irreps of $Y(2|2)^+$ is also a necessary condition.
4.5 The general case \(Y(2m|2n)^+\)

**Theorem 4.10 (Case of \(Y(2|2m)^+\))**

Let \(V = V[\mu_2(u), \mu_{n+3}(u), \ldots, \mu_{2n}(u)]\) be an irrep of \(Y(2|2n)^+\).

Suppose the weights \(\mu_i(u)\) obey

\[
\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_{i+1}(u + 1)}{P_{i+1}(u)} \quad \text{for} \quad n + 3 \leq i \leq 2n + 1 \tag{4.71}
\]

\[
\frac{\mu_{n+3}(-u)}{\mu_{n+3}(u)} = \left(\frac{u - \frac{1}{2}}{u + \frac{1}{2}}\right)^2 \frac{P_{n+3}(u + 1)P_{n+3}(-u)}{P_{n+3}(u)P_{n+3}(1-u)} \tag{4.72}
\]

\[
\frac{\mu_2(u)}{\mu_{n+3}(u)} = \frac{u - \gamma}{u + \frac{1}{2}} \frac{P_{2}(u + 1)}{P_{n+3}(u)P_{n+3}(u + 1)} \tag{4.73}
\]

with \(R(u)\) and \(Q(u)\) even and of same degree, and \(\gamma \in \mathbb{C}\).

Then \(V\) is finite-dimensional.

Under the assumption of conjecture 3, the conditions (4.71)-(4.73) are necessary and sufficient conditions for \(V\) to be a finite-dimensional irrep.

**Proof:** The proof is similar to the previous ones. One constructs a finite-dimensional irrep for \(Y(2|2n)^+\) which fulfills the conditions (4.71)-(4.73). It takes the form \(V' = L[\lambda(u)] \otimes V_{\gamma}. L[\lambda(u)]\) is constructed as in theorem 4.1. The finite-dimensional \(osp(2|2n)\)-irrep \(V_{\gamma}\) has weight

\[
l_2 = -\gamma - \frac{1}{2} \quad \text{and} \quad l_i = -1, \quad \text{for} \quad n + 3 \leq i \leq 2n \tag{4.74}
\]

Looking at the \(osp(2|2n)\)-span of the highest weight vector and taking the irreducible subquotient, we get a finite-dimensional irrep \(V'\) with highest weight obeying (4.71)-(4.73). \(V\) being isomorphic to \(V'\), it is therefore finite dimensional.

Conversely, assuming the conjecture 3, and looking at the subalgebras \(Y(2|2)^+\) and \(Y(2n)^-\), one easily proves that the conditions (4.71)-(4.73) are necessary conditions. ■

**Theorem 4.11 (Case of \(m > 1\))**

Let \(V = V[\mu_{m+1}(u), \ldots, \mu_{M}(u), \mu_{M+n+1}(u), \ldots, \mu_{M+2n}(u)]\) be an irrep of \(Y(2m|2n)^+\). We note \(M = 2m\) and take \(m > 1\).

Suppose the weights \(\mu_i(u)\) obey

\[
\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_{i+1}(u + 1)}{P_{i+1}(u)} \quad \text{for} \quad \begin{cases} m + 1 \leq i \leq 2m - 1 \\ M + n + 1 \leq i \leq M + 2n - 1 \end{cases} \tag{4.75}
\]

\[
\frac{\mu_{M+n+1}(-u)}{\mu_{M+n+1}(u)} = \frac{P_{M+n+1}(u + 1)P_{M+n+1}(-u)}{P_{M+n+1}(u)P_{M+n+1}(1-u)} \tag{4.76}
\]

\[
\frac{\gamma(u) \mu_{m+1}(u)}{\mu_{M+n+1}(u)} = \frac{P_{M+n+1}(u + 1)P_{m+1}(u)R(u)}{P_{M+n+1}(u)P_{m+1}(u + 1)} \tag{4.77}
\]

\[
\text{for} \quad n + 3 \leq i \leq 2n + 1
\]
with \(R(u)\) and \(Q(u)\) even and of same degree, and \(\gamma(u) = 1\) or \(\gamma(u) = \frac{2u-1}{2u+1}\).

Then \(V\) is finite-dimensional.

Proof: We start with the case \(\gamma(u) = 1\), and do the same construction as in theorem 4.7, to get weights \(\lambda_i(u)\) defined as in equations (4.47)-(4.50), with now \(M = 2m\). We introduce:

\[
\lambda_i'(u) = P_{m+1}(u+1)\lambda_i(u) \quad \text{for} \quad \begin{cases} i = 1, \ldots, m \\ i = M+1, \ldots, M+2n \end{cases} \tag{4.78}
\]

\[
\lambda_{M}'(u) = P_{m+1}(u)\lambda_i(u) \quad \text{for} \quad i = m+1, \ldots, M \tag{4.79}
\]

For these new weights, the relations (4.51)-(4.53) are still valid when \(i \neq m, M\). In these later cases, we get

\[
\frac{\lambda_m'(u)}{\lambda_{m+1}'(u)} = \frac{P_{m+1}(u+1)}{P_{m+1}(u)} \tag{4.80}
\]

\[
\frac{\lambda_M'(u)}{\lambda_{M+1}'(u)} = \frac{P_{m+1}(u)}{P_{m+1}(u+1)} \frac{P_{+}(u)}{P_{+}(u+1)} \frac{R_0(u)}{Q_0(u)} \tag{4.81}
\]

Thus, the \(Y(2m|2n)\)-irrep \(L[\lambda'(u)]\) is still finite-dimensional. The cyclic \(Y(2m|2n)^+\)-span of its highest weight vector is a representation with highest weight \(\bar{\mu}_i(u) = \lambda_i'(u)\lambda_i'(-u)\) for \(i = m+1, \ldots, 2m\) and \(i = M+n+1, \ldots, M+2n\). Its irreducible quotient is a finite-dimensional irrep \(V[\bar{\mu}(u)]\) of \(Y(2m|2n)^+\).

Moreover, the weights \(\bar{\mu}_i(u)\) for \(i = m+1, \ldots, M-1\) on the one hand, and \(i = M+n+1, \ldots, M+2n-1\) on the other hand, verify the same relations as the \(\mu_i'(u)\), i.e. the conditions (4.75)-(4.77). For the remaining relation, one computes;

\[
\frac{\bar{\mu}_{m+1}(u)}{\bar{\mu}_{M+n+1}(u)} = \frac{P_{m+1}(u)}{P_{m+1}(u+1)} \frac{\mu_{m+1}(u)}{\mu_{M+n+1}(u)} \tag{4.82}
\]

which gives the relation (4.77).

The weights \(\mu_i(u)\) and \(\bar{\mu}_i(u)\) obeying both the relations (4.75)-(4.77), there exists an automorphism \(S(u) \rightarrow \psi(u)S(u)\) of \(Y(2m|2n)^+\) such that its composition with \(V[\bar{\mu}(u)]\) is isomorphic to \(V[\mu(u)]\). This later is thus finite-dimensional.

If now \(\gamma(u) = \frac{2u}{2u+1}\), we construct the tensor product of the above representation by the \(osp(2m|2n)\) finite-dimensional irrep \(V_0\) with weights \(l_i = -\frac{1}{2}\) for \(m+1 \leq i \leq 2m\) and \(l_i = -1\) for \(2m+n+1 \leq i \leq 2m+2n\). \(V_0\) provides a finite-dimensional representation for \(Y(2m|2n)^+\) with weights \(l_i(u) = \frac{2u}{2u+1}\) for \(m+1 \leq i \leq 2m\) and \(l_i(u) = \frac{2u}{2u-1}\) for \(2m+n+1 \leq i \leq 2m+2n\). The weights of the tensor product obey the relations (4.73)-(4.77), and we conclude as in theorem 4.7. ■
Remark:
Conversely, let us suppose that the irrep $V[\mu(u)]$ is finite dimensional and that the conjecture \(2\) holds. From theorem \([1.4]\), one already knows that the conditions \((1.73)\) and \((1.76)\) must be satisfied (for $i \neq m + 1$). Moreover, we get also

$$\gamma(u) \frac{\mu_{m+1}(u)}{\mu_{m+1}(u)} = \frac{P_m(u + 1)P_m(-u)}{P_m(u)P_m(1-u)} \quad (4.83)$$

with $\gamma(u)$ and $\mu_{m+1}(u)$ defined as in the theorem \([1.4]\).

We suppose also that $\mu_{m+1}(u) = \mu_{m+1}(u)$, which turns out to suppose that \((1.73)\) is valid for $i = m + 1$.

The subalgebra generated by the coefficients of $S_{ij}(u)$, $i, j = m, m + 1, M + n, M + n + 1$ is isomorphic to $Y(2|2)$. The cyclic span of the highest weight vector of $V[\mu(u)]$ with respect to this subalgebra is a representation with highest weight $(\mu_{m+1}(u), \mu_{M+n+1}(u))$. Its irreducible quotient is finite-dimensional and so, we have

$$\frac{\mu_{m+1}(u)}{\mu_{M+n+1}(u)} = \frac{u - \gamma}{u + \frac{1}{2}} P_{M+n+1}(u) P_{m+1}(u + 1) Q(u) \quad (4.84)$$

We look at the $osp(2m|2n)$ irrep induced by the generators $S_{ab}^{ij}(u)$ acting on the highest weight vector. It is finite dimensional, so that we must have

$$l_{i+1} - l_i \in \mathbb{Z}_+ \quad \text{for} \quad \begin{cases} m + 2 \leq i \leq 2m \\ M + n + 1 \leq i \leq M + 2n - 1 \end{cases} \quad (4.85)$$

$$-(l_{m+2} + l_{m+1}) \in \frac{1}{2}\mathbb{Z}_+ \quad \text{and} \quad -l_{M+n+1} \in \mathbb{Z}_+ \quad (4.86)$$

$$l_{m+1} - l_{M+n+1} \in \frac{1}{2}\mathbb{Z}_+ \quad (4.87)$$

where $\mu_i(u) = 1 + u^{-1}l_i + \ldots$. The above relations (on the weights $\mu_i(u)$) imply the following constraints:

$$l_{i+1} - l_i \in \mathbb{Z}_+ \quad \text{for} \quad \begin{cases} m + 2 \leq i \leq 2m \\ M + n + 1 \leq i \leq M + 2n - 1 \end{cases} \quad (4.88)$$

$$-(l_{m+2} + l_{m+1}) \in \mathbb{Z}_+ \quad \text{and} \quad -l_{M+n+1} \in \mathbb{Z}_+ \quad (4.89)$$

$$l_{m+1} - l_{M+n+1} \in \mathbb{Z}_+ + (-\gamma - \frac{1}{2}) \quad (4.90)$$

This implies in particular that $\gamma \in -\frac{1}{2}\mathbb{Z}_+$, so that we are back to the conditions \((1.77)\). Thus, we are led to the following:

**Conjecture 3** The sufficient conditions of theorem \([4.11]\) for the existence of finite-dimensional irreps of $Y(2m|2n)^+$ are also necessary conditions.
5 Conclusion

We have defined the notion of twisted Yangians in the context of superalgebra $gl(M|N)$. It appears that most of the properties of the twisted Yangians $Y^\pm(N)$ can be exhibited in the superalgebra case. However, only $Y^+(M|2n)$ and $Y^-(2m|N)$ can be defined, and appear to be isomorphic. We thus concentrate on $Y^+(M|2n)$. Its finite dimensional irreducible representations have being studied. $Y^+(M|2n)$ is also a coideal subalgebra of $Y(M|2n)$, and is a deformation of the polynomial superalgebra $U(osp(M|2n)[x])$.

From a mathematical point of view, the centre of this algebra remains to be studied, and in particular the notion of Sklyanin determinant (which appear in the context of twisted Yangians) has to be generalised to this case. Note that the notion of quantum Berezinian, which generates central element of $Y(M|N)$ has been introduced in [11].

From the physical of view, the integrable systems with boundary that could be relevant for such an algebra has to be determined. Nonlinear sigma models based on a supergroup seem to be a good candidate.

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