The \( \ell \)-good-neighbor edge connectivity of graphs

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Abstract. -The \( \ell \)-good-neighbor edge connectivity is an useful parameter to measure the reliability and tolerance of interconnection networks. For a graph \( H \) with order \( p \) and an integer \( \ell \ (\ell \geq 0) \), an edge subset \( X \subseteq E(H) \) is called a \( \ell \)-good-neighbor edge-cut if \( H - X \) is disconnected and the minimum degree of every component of \( H - X \) is at least \( \ell \). The order of the minimum \( \ell \)-good-neighbor edge-cut of \( H \) is called the \( \ell \)-good-neighbor edge connectivity of \( H \), denoted by \( \lambda_\ell(H) \). In this paper, we show \( \lambda_\ell(H) \leq \lambda_{\ell+1}(H) \), obtain the bounds of \( \lambda_\ell(H) \) when \( 0 \leq \ell \leq \left\lfloor \frac{p-2}{2} \right\rfloor \), character some graphs with the small \( \lambda_\ell(H) \) and get some results about the Erdös-Gallai-type problem about \( \lambda_\ell(H) \).

1. Introduction
Super-large scale systems of multi-processors and multi-computers connected with various complex networks are made up of tens of thousands. In network theory, the network reliability and fault tolerability are measured by the connectivity \( \kappa(H) \) and the edge connectivity \( \lambda(H) \) of a graph \( H \). The connectivity (the edge connectivity) of a graph \( H \) are integer numbers, such that at least \( \kappa \) vertices (\( \lambda \) edges) are removed to result disconnect \( H \). Nowadays, massive-scale multiprocessor and multi-computer systems are bound to gain more popularity. These two parameters can not the best reflection the network reliability and fault tolerability. Due to this fact, Fàbrega and Fiol generalized the classical connectivity parameters \( \kappa(H) \) and \( \lambda(H) \) of a graph \( H \), and \( \kappa^1(H) \) and \( \lambda^1(H) \) are proposed [2].

2. Terminology and Concepts
Let \( H = (V, E) \) to be a graph, \( V(H) \) and \( E(H) \) denote the vertex set and the edge set of \( H \) respectively. A subgraph \( H' \) of \( H \), denoted by \( H' \subseteq H \), is a graph in which \( V(H') \subseteq V(H) \) and \( E(H') \subseteq E(H) \). When \( X \) is any subset of \( V(H) \), the notation \( H - X \) denotes a graph obtained by removing all vertices in \( X \) from \( H \) and deleting those edges with at least one end - vertex in \( X \), simultaneously. Similarly, for any \( X \) of \( E(H) \), the notation \( H - X \) denotes a graph obtained by removing all edges in \( X \) from \( H \). For any vertex \( v \) of \( H \), the neighborhood \( N_H(v) \) is defined as the set of all vertices which are adjacent to \( v \). We also denote the degree of the vertex \( v \) by \( \deg(v) \) (simply \( d(v) \)), and \( \delta(H) \) and \( \Delta(H) \) denote the minimum degree and the maximum degree of \( H \), respectively. For the graph theoretical notation and terminology which are not mentioned, we refer to [1].
2.1 $\ell$-good-neighbor edge connectivity
In 1996, Fábrega and Fiol [2] generalized the classical connectivity parameters $\kappa(H)$ and $\lambda(H)$ of a graph $H$, and $\kappa^\ell(H)$ and $\lambda^\ell(H)$ are proposed. For a graph $H = (V, E)$ and an integer $\ell$, $\ell$-good-neighbor-cut of $H$ is a subset $X$ resulting that $H - X$ is disconnected and the minimum degree of every component is at least $\ell$. The $\ell$-good-neighbor connectivity of $H$ is the minimum order of $\ell$-good-neighbor-cut, denoted by $\kappa^\ell(H)$. Note $\kappa^0(H) = \kappa(H)$. Some results about $\kappa^\ell(H)$ have been obtained in recent years [3-7].

The same as $\kappa^\ell(H)$, an edge subset $X \subseteq E(H)$ is called a $\ell$-good-neighbor edge-cut if $H - X$ is disconnected and the minimum degree of each component is at least $\ell$. We use $\lambda^\ell(H)$ to represent the $\ell$-good-neighbor edge connectivity of $H$, which is the minimum order of a $\ell$-good-neighbor edge-cut. Note that $\lambda^0(H) = \lambda(H)$.

**Observation 2.1** Let $X$ be any $\ell$-good-neighbor edge cut of $H$, then $H - X$ has only two components.

**Proposition 2.1** For any $\ell$-good-neighbor edge connected graph $H$,
\[
0 \leq \ell \leq \min\{\delta(H), \left\lfloor \frac{p-2}{2} \right\rfloor \}.
\]

**Proof:** Since $H$ is $\ell$-good-neighbor edge connected, there is the subset $X \subseteq E$ such that $|X| = \lambda^\ell(G)$ and $H - X$ is disconnected and each component with the minimum degree at least $\ell$. In other words, each component contains at least $\ell + 1$ vertices. Thus $p \geq 2(\ell + 1)$, namely, $\ell \leq \left\lfloor \frac{p-2}{2} \right\rfloor$. Hence $0 \leq \ell \leq \min\{\delta(H), \left\lfloor \frac{p-2}{2} \right\rfloor \}$.

**Proposition 2.2** For any $\ell$-good-neighbor edge connected graph $H$,
\[
\lambda^\ell(H) \leq \lambda^{\ell+1}(H).
\]

**Proof:** By deleting $\lambda^{\ell+1}(H)$ edges, the graph $H$ is disconnected and each component with the minimum degree at least $\ell + 1 > \ell$, hence $\lambda^\ell(H) \leq \lambda^{\ell+1}(H)$.

**Proposition 2.3** Let $Y$ be a subgraph of $H$ with $Y(V) = V(H)$, then $\lambda^0(Y) \leq \lambda^0(H)$.

**Remark 2.1** Let $\ell$ be obtained from two cliques $K_{\ell+1}$ and $K_{\ell+2}$ and one vertex $v$ by adding $E_H[v, K_{\ell+1}] \cup E_H[v, K_{\ell+2}]$, and $Y$ be obtained by deleting the edge $e$ from $K_{\ell+1}$.

By deleting the edges $E_H[v, K_{\ell+1}]$ from $H$, there are two components with the minimum degrees at least $\ell$, hence $\lambda^\ell(H) \leq \ell + 1$. Deleting $\ell$ edges from $H$, the graph is still connected, so $\lambda^{\ell+1}(H) \geq \ell + 1$. Thus, $\lambda^{\ell}(H) = \ell + 1$.

Suppose $\lambda^\ell(Y) = \ell + 1$. Then there exist $\ell + 1$ edges such that the resulting graph by deleting the edges is disconnected and each component with the minimal degree at least $\ell$. If deleting the edges $E_Y[v, K_{\ell+1}]$, then we have a component with the minimum degree $\ell - 1$ in the resulting graph, a contradiction. If deleting any $\ell + 1$ edges from $H$, then the resulting graph is still connected. So $\lambda^{\ell}(H) \geq \ell + 2$. Moreover, by deleting the edges $E_Y[v, K_{\ell+2}]$ from $H$, there are two components with the minimum degrees at least $\ell$, we have $\lambda^\ell(H) \leq \ell + 2$. So $\lambda^{\ell}(Y) = \ell + 2 > \lambda^\ell(H)$. 

3. Results for special graphs

**Proposition 3.1** Let $H$ be a graph and $0 \leq \ell \leq \min\{\delta, \lfloor \frac{p-2}{2} \rfloor \}$, then

$$\lambda(H) \leq \lambda^\ell(H) \leq \frac{p(p-1) - p\ell}{2}.$$  

Furthermore, the bounds are tight.

**Proof:** By definition of $\lambda^\ell(H)$, $\lambda^\ell(H) \geq \lambda(H)$ and

$$\lambda^\ell(H) \leq E(H) - \frac{1}{2}p\ell \leq \left(\frac{p}{2}\right)^2 - \frac{1}{2}p\ell = \frac{p(p-1) - p\ell}{2}.$$  

**Proposition 3.2** Let $\ell$ be an integer and $\ell > 0$.

1. Let $P_p$ be a path with order $p$. Then $\lambda^\ell(P_p) = 1$ for $\ell = 0, 1$.
2. Let $C_p$ be a path with order $p$. Then $\lambda^\ell(C_p) = 2$ for $\ell = 0, 1$.

**Proof:** (1) From $\lambda^\ell(P_p)$, there exists $X \subseteq E(P_p)$, and $P_p - X$ is disconnected and each component of $P_p - X$ with the minimum degree at least $\ell$. The two components are a path or a vertex, so $\ell < 1$. When $\ell = 0$, $\lambda^0(P_p) = \lambda(P_p) = 1$. When $\ell = 1$, $\lambda^1(P_p) \geq \lambda(P_p) = 1$ by Proposition 2.1. Let $P_p = v_1v_2 \cdots v_p$. Choosing an edge $e$ which is not suspending edge, $H - e$ is disconnected and each component of $H - e$ with the minimum degree at least $\ell$. So $\lambda^\ell(P_p) \leq 1$. Hence $\lambda^\ell(P_p) = 1$.

(2) The same as the proof of (1), $\ell \leq 1$. When $\ell = 0$, $\lambda^\ell(C_p) = \lambda(C_p) = 2$. When $\ell = 1$, $\lambda^\ell(C_p) \geq \lambda(C_p) = 2$ from Proposition 2.1. Let $C_p = v_1v_2 \cdots v_p$. Choosing two edges $e_1, e_2$ which are nonadjacent, we know that $H - \{e_1, e_2\}$ is not connect and the two components of $H - \{e_1, e_2\}$ of each component with the minimum degree at least $\ell$. So $\lambda^\ell(C_p) \leq 2$. Hence $\lambda^\ell(C_p) = 2$.

**Proposition 3.3** For any tree $T_p$ with order $p (p \geq 3)$, $\lambda^\ell(T_p) = 1$ for $\ell = 0, 1$.

**Proof:** From $\lambda^\ell(T_p)$, there exists $X \subseteq E(T_p)$, and $T_p - X$ is disconnected and each component of $T_p - X$ with the minimum degree at least $\ell$. The two components are $T_p - X$ are a tree or a vertex. Then $\ell < 1$. When $\ell = 0, 1$, $\lambda^\ell(T_p) \geq \lambda(T_p) = 1$. Choosing an edge $e$ which is not suspending edge of $T_p$, we know that $T_p - e$ not connected and the two components of $T_p - e$ with the minimum degree at least $\ell$. So $\lambda^\ell(T_p) \leq 1$. Hence $\lambda^\ell(T_p) = 1$.

**Proposition 3.4** For any a complete graph $K_p$, $0 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$ and $\lambda^\ell(K_p) = (\ell + 1)(p - \ell - 1)$.

**Proof:** There exists $X \subseteq E(K_p)$ with $|X| = \lambda^\ell(K_p)$ and $K_p - X$ has the two components with the minimum degrees at least $\ell$, then $p \geq 2(\ell + 1)$. Thus $0 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$. The graph $K_{\ell+1} \vee K_{p-\ell-1}$ is isomorphic with $K_p$. By deleting the edges $E[V(K_{\ell+1}), V(K_{p-\ell-1})]$ in $K_{\ell+1} \vee K_{p-\ell-1}$, the two
components with the minimum degree at least \( \ell \) are obtained. So 
\[ \lambda^\ell(K_p) \leq |E(V(K_{\ell+1}), V(K_{p-\ell-1}))| = (\ell + 1)(p - \ell - 1). \]
Let \( X \subseteq E(K_n) \) with \( |X| = \lambda^\ell(K_n) \) and assume that \( K_p - X \) has components \( C_1 \) and \( C_2 \), \( |V(C_1)| = x \), then \( |V(C_2)| = p - x \) and \( \ell + 1 \leq x \leq p - \ell - 1 \). Hence
\[
\lambda^\ell(K_p) = \left( \frac{p}{2} - \frac{x}{2} \right) - \left( \frac{p - x}{2} \right) = \frac{p(p - 1)}{2} - \frac{x(x - 1)}{2} - \frac{(p - x)(p - x - 1)}{2} = px - x^2 = -(x - \frac{p}{2})^2 + \frac{p^2}{4} \]
\[ \geq -(p - \ell - 1 - \frac{p}{2})^2 + \frac{p^2}{4} = -\frac{(p - \ell - 1)^2}{2} + \frac{p^2}{4} = \frac{p\ell + p - \ell^2 - 2\ell - 1}{2} = (\ell + 1)(p - \ell - 1). \]
It means that \( \lambda^\ell(K_n) \geq (\ell + 1)(p - \ell - 1) \). Thus \( \lambda^\ell(K_n) = (\ell + 1)(p - \ell - 1) \).

4. Graphs with small \( \ell \)-good-neighbor edge connectivity

**Observation 4.1** For any positive integer \( \ell \) with \( 0 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor \), if there exists a cut-edge \( e \) in \( H \) such that each component of \( H - e \) with the minimum degree at least \( \ell \), then \( \lambda^\ell(H) = 1 \) is hold.

**Theorem 4.1** For any \( \ell \) with \( 0 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor \), the necessary and sufficient conditions for \( \lambda^\ell(H) = 2 \) is that one of the two is available to \( H \):

1. \( \lambda(H) = 2 \) and \( H \) has an edge-cut \( \{e_1, e_2\} \) such that the two components of \( H - \{e_1, e_2\} \) with the minimum degree at least \( \ell \);
2. \( \lambda(H) = 1, \ell \geq 1 \), and
   1. for any cut edge \( e, H - e \) has a component with the minimal degree at most \( \ell - 1 \);
   2. for two non-cut edges \( e_1, e_2 \) of \( H, H - \{e_1, e_2\} \) is disconnected and has the two components with the minimum degree at least \( \ell \).

**Proof:** If (1) is available to \( H \), then \( \lambda^\ell(H) \leq 2 \). According to the definition of \( \lambda^\ell(G) \), \( \lambda^\ell(H) \geq \lambda(H) = 2 \) is true from Proposition 2.1. So \( \lambda^\ell(H) = 2 \).

If (2) is available to \( H \). From (2.1), \( \lambda^\ell(H) \geq 2 \). And from (2.2), \( \lambda^\ell(H) \leq 2 \). Thus \( \lambda^\ell(H) = 2 \).

Now suppose \( \lambda^\ell(H) = 2 \). From Proposition 2.1, \( \lambda(H) \leq 2 \), and \( \lambda^\ell(H) \geq 3 \), this is impossible.

5. Extremal problems

In the paper, we study some results about the Erdös-Gallai-type problem about \( \lambda^\ell(H) \) of a graph.

Assume that \( p, t, \ell \) are positive integers and \( 2 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor \). The graphs \( S_1, S_2 \) are \( \ell \)-regular graphs with order \( x \) and \( p - x \), respectively. Let \( S_p^\ell \) obtained by adding the \( t \) edges \( E[v, S_p^\ell] \) between \( S_1 \) and \( S_2 \) where \( v \) is any vertex of \( V(S_1) \).

**Lemma 5.1** For the positive integers \( p, t, \ell \) and \( 2 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor \), \( \lambda^\ell(S_p^\ell) = t \).

**Proof:** \( X = E[v, S_p^\ell] \). Then \( S_p^\ell - X \) is not connected and has the two components with the minimum degree at least \( \ell \). Hence \( \lambda^\ell(S_p^\ell) \leq t \). By the definition of \( \lambda^\ell(S_p^\ell) \), choosing any edge set \( X_1 \subseteq E(G) \) and \( X_1 \neq X \), we know that \( H - X_1 \) is connected or has the two components with the minimum degree less than \( \ell \). So \( \lambda^\ell(S_p^\ell) = t \).
For two positive integers $p$ and $t$, $\mathcal{H}(p,t)$ is a family of all graphs with order $p$ and $\lambda^0(G) = t$, $s(p,t) = \min\{|E(H)| : H \in \mathcal{H}(p,t)\}$. $f(p,t)$ is the minimum integer if $|E(H)| \geq f(p,t)$ then $\lambda^H(H) \geq t$ for any connected graph $H$ with order $p$. $g(p,t)$ is the maximum integer if $|E(G)| \leq g(p,t)$ then $\lambda^H(H) \leq t$ for any connected graph $H$ with order $p$.

**Theorem 5.1** For the integers $p, t, \ell$ and $2 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$.

1. When $\ell = 1$, $s(p, t) = p - 1$.
2. When $\ell \geq 2$, $s(p, t) = \frac{1}{2}p\ell + t$.

**Proof:** $s(p, t) = 1$ when $H = T_p$.

Note that $g(p, t) = s(p, t + 1) - 1$, we can obtain the following result.

**Theorem 5.2** For the positive integers $p, t, \ell$ and $2 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$.

1. When $\ell = 1$, $g(p, t) = p - 1$.
2. When $\ell \geq 2$, $g(p, t) = \frac{1}{2}p\ell + t$.

The graph $\mathcal{Q}$ is obtained by adding $t - 1$ edges $E[v, K_{p-\ell+1}]$ between $K_{\ell+1}$ and $K_{p-\ell-1}$, where $v$ is any vertex of $V(K_{\ell+1})$.

**Lemma 5.2** For the positive integers $p, t, \ell$ and $2 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$, $\lambda^H(Q) = t - 1$.

**Proof:** we can obtain $\lambda^H(Q)$ easily.

**Theorem 5.3** For the positive integers $p, t, \ell$ and $2 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$,

$$f(p, t) = \binom{p}{2} - (\ell + 1)(p - \ell - 1) + t.$$

**Proof:** Since $e(H) = \binom{p}{2} - (\ell + 1)(p - \ell - 1) + t - 1$ and by Lemma 4.2, $f(p, t) \geq \binom{p}{2} - (\ell + 1)(p - \ell - 1) + t - 1$. In order to show $f(p, t) \leq \binom{p}{2} - (\ell + 1)(p - \ell - 1) + t$, we assume that $\lambda^H(H) = t - 1$ for any $H$ with $e(H) \geq \binom{p}{2} - (\ell + 1)(p - \ell - 1) + t$. Then there exists $X \subseteq E(H)$ and $|X| = t - 1$ such that the two components of $H - X$ with the minimum degree at least $\ell$. We may as well suppose that one component of $H - X$ is $C_1$ and $|V(C_1)| = x$. Then

$$e(H) \leq \binom{x}{2} + \binom{p-x}{2} + t - 1$$

$$= \frac{x(x-1)}{2} + \frac{(p-x)(p-x-1)}{2} + t - 1$$

$$= \frac{p^2 - p}{2} + x(x-p) + t - 1$$

$$\leq \frac{p^2 - p}{2} + (p - \ell - 1)(-\ell - 1) + t - 1$$

$$= \frac{p}{2} - (p - \ell - 1)(\ell + 1) + t - 1.$$

Hence $f(p, t) = \binom{p}{2} - (\ell + 1)(p - \ell - 1) + t$.

**6. Conclusion**

In this paper, we determine the $\ell$-good-neighbor edge connectivity of graphs. And we show $\lambda^H(H) \leq \lambda^{H+1}(H)$, obtain the bounds of $\lambda^H(H)$ when $0 \leq \ell \leq \lfloor \frac{p-2}{2} \rfloor$, character some graphs with the small $\lambda^H(H)$ and get some results about the Erdős-Gallai-type problem about $\lambda^H(H)$. We can consider other conditional edge connectivity of graphs in our future research.
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