THE DRAZIN SPECTRUM IN BANACH ALGEBRAS

ENRICO BOASSO

Abstract. Several basic properties of the Drazin spectrum in Banach algebras will be studied. As an application, some results on meromorphic Banach space operators will be obtained.

1. Introduction

The Drazin spectrum of Banach space operators is closely related to other several spectra such as the ascent, the descent and the left and the right Drazin spectra, see for example [9, 3]. In [3] the relationships among all the aforementioned spectra and the multiplication operators were studied both for Banach space bounded and linear maps and for Banach algebra elements; what is more, a characterization of the spectrum in terms of the Drazin spectrum and the poles of the resolvent was also presented. On the other hand, in two recent articles the descent and the left Drazin spectra of Banach space operators were considered, see [6, 1]. The main objective of this work is to study several basic properties of the Drazin spectrum in Banach algebras similar to the ones presented in [6, 1].

In fact, in section 2 conditions equivalent to the emptyness of the Drazin spectrum will be given. In addition, it will be characterized when the Drazin spectrum is countable. Moreover, if $a$ and $b$ belong to a Banach algebra $A$, then the Drazin spectra of $ab$ and $ba$ will be proved to coincide. On the other hand, in [6 page 265] was pointed out that there is no relation between the descent spectra of $R_T$ and $T$, where $T \in L(H)$ is an operator defined on the Hilbert space $H$, and $R_T \in L(L(H))$ is the right multiplication operator. In the next section a description of the ascent and the descent spectra of the multiplication operators in Hilbert space will be given. To this end, the relationships among the involution of a $C^*$-algebra and the ascent, the descent, and the Drazin spectra of the left and the right multiplication operators will be characterized, see also [3] for more results on the aforementioned spectra of the multiplication operators. Finally, as an application, in section 3 the Drazin spectrum will be used to prove several results on meromorphic Banach space operators.

The results of this work are related to the ones presented at the 23th International Conference on Operator Theory held in Timisoara, Romania, from June 29 to July 4, 2010. It is a pleasure for the author to acknowledge the stimulating atmosphere and the warm hospitality of the Conference. Moreover, the author wish to express his fully indebtedness to the organizers of the Conference, specially to Professor Dan Timotin, for the help to attend the Conference.

Before going on, several definitions and some notation will be recalled.
From now on, $A$ will denote a unital Banach algebra and $e \in A$ will stand for
the unit element of $A$. If $a \in A$, then $L_a : A \to A$ and $R_a : A \to A$ will denote
the maps defined by left and right multiplication respectively, namely $L_a(x) = ax$ and
$R_a(x) = xa$, where $x \in A$.

Next follows the key notions of the present work. Given a Banach algebra $A$,
an element $a \in A$ is said to be Drazin invertible, if there exist a necessarily unique
$b \in A$ and some $m \in \mathbb{N}$ such that
\[a^m b a = a^m, \quad bab = b, \quad ab = ba.\]

If the Drazin inverse of $a$ exists, then it will be denoted by $a^D$, see for example
[7, 2, 3].

On the other hand, the notion of regularity was introduced in [11]. Recall that,
given a unital Banach algebra $A$ and a regularity $R \subseteq A$, the spectrum derived from
the regularity $R$ is defined by $\sigma_R(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R\}$, where $a \in A$
and $a - \lambda$ stands for $a - \lambda e$ (see [11]). Next consider the set $\mathcal{DR}(A) = \{a \in A : a \text{ is Drazin invertible}\}$. According to [2, Theorem 2.3], $\mathcal{DR}(A)$ is a regularity.

This fact led to the following definition [2].

**Definition 1.1.** Let $A$ be a unital Banach algebra. The Drazin spectrum of an
element $a \in A$ is the set
\[\sigma_{\mathcal{DR}}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin \mathcal{DR}(A)\}.\]

Note that $\sigma_{\mathcal{DR}}(a) \subseteq \sigma(a)$, the spectrum of $a$, and according to [11, Theorem 1.4],
the Drazin spectrum of a Banach algebra element satisfies the spectral mapping
theorem for analytic functions defined on a neighbourhood of the usual spectrum
which are non-constant on each component of its domain of definitoin. In addition,
according to [2, Proposition 2.5], $\sigma_{\mathcal{DR}}(a)$ is a closed subset of $\mathbb{C}$.

Recall that according to [3, Theorem 12],
\[\sigma(a) = \sigma_{\mathcal{DR}}(a) \cup \Pi(a), \quad \sigma_{\mathcal{DR}}(a) \cap \Pi(a) = \emptyset, \quad \sigma_{\mathcal{DR}}(a) = \text{acc } \sigma(a) \cup \text{IES}(a)\]
where $\Pi(a)$ is the set of poles of the resolvent of $a$ (see [3, Remark 10]), $\text{IES}(a) = \text{iso } \sigma(a) \setminus \Pi(a)$, and if $K \subseteq \mathbb{C}$, then $\text{iso } K$ denote the set of isolated points of $K$
and $\text{acc } K = K \setminus \text{ iso } K$.

When $X$ is a Banach space, $A = L(X)$ will denote the Banach algebra of all
operators defined on and with values in $X$. In addition, if $T \in L(X)$, then $N(T)$
and $R(T)$ will stand for the null space and the range of $T$ respectively. Recall that
the descent and the ascent of $T \in L(X)$ are $d(T) = \inf\{n \geq 0 : R(T^n) = R(T^{n+1})\}$
and $a(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$ respectively, where if some of the above
sets is empty, its infimum is then defined as $\infty$, see for example [9, 13, 6, 11, 3].

Given a Banach space $X$, the left and the right Drazin spectra of an operator
have been introduced. The mentioned spectra are the ones derived from the
regularities $\mathcal{LD}(X) = \{T \in L(X) : a(T) \text{ is finite and } R(T^{a(T)+1}) \text{ is closed}\}$
and $\mathcal{RD}(X) = \{T \in L(X) : d(T) \text{ is finite and } R(T^{d(T)}) \text{ is closed}\}$ respectively, and
they will be denoted by $\sigma_{\mathcal{LD}}(T)$ and $\sigma_{\mathcal{RD}}(T)$ respectively. In addition, concerning
the basic properties of the left and the right Drazin spectra, see [13, 11].

Furthermore, if $X$ is a Banach space, then according again to [13], the sets
$\mathcal{R}^L_1(X) = \{T \in L(X) : d(T) \text{ is finite}\}$ and $\mathcal{R}^s_2(X) = \{T \in L(X) : a(T) \text{ is finite}\}$
are regularities with well-defined spectra derived from them, namely the descent
spectrum and the ascent spectrum respectively. The descent spectrum was also
studied in [6] and it will be denoted by $\sigma_{\text{dsc}}(T)$, $T \in L(X)$. On the other hand, $\sigma_{\text{asc}}(T)$ will stand for the ascent spectrum.

2. Properties of the Drazin spectrum

Given a unital Banach algebra $A$ and $a \in A$, according to [12 Proposition 1.5], necessary and sufficient for the Drazin spectrum of $a$ to be empty is the fact that the usual spectrum of $a$ consists in a finite number of poles of the resolvent of $a$.

In the following theorem several other equivalent conditions of this property will be presented, compare with [6 Theorem 1.5] and [1 Theorem 2.7]. To this end, recall first that an element $a \in A$ is said to be algebraic, if there exists a polynomial $P$ with coefficients in $\mathbb{C}$ such that $P(a) = 0$. In addition, $\partial \sigma(a)$ will stand for the topological boundary of $\sigma(a)$ and $\rho_{\text{DRAZIN}}(a)$ for the Drazin resolvent set of $a$, i.e., $\rho_{\text{DRAZIN}}(a) = C \setminus \sigma_{\text{DRAZIN}}(a)$.

**Theorem 2.1.** Let $A$ be a unital Banach algebra and consider $a \in A$. Then, the following statements are equivalent.

(i) $\sigma_{\text{DRAZIN}}(a) = \emptyset$,
(ii) $\partial \sigma(a) \subseteq \rho_{\text{DRAZIN}}(a)$,
(iii) $a$ is algebraic.

**Proof.** Note that $\sigma_{\text{DRAZIN}}(a) = \emptyset$ if and only if $\rho_{\text{DRAZIN}}(a) = C$. Then, it is clear that the first statement implies the second.

Suppose that $\partial \sigma(a) \subseteq \rho_{\text{DRAZIN}}(a)$. Then according to [3 Theorem 12], $\Pi(a) \subseteq \text{iso } \sigma(a) \subseteq \partial \sigma(a) \subseteq \rho_{\text{DRAZIN}}(a) \cap \sigma(a) = \Pi(a)$. Consequently, $\Pi(a) = \text{iso } \sigma(a) = \partial \sigma(a)$. What is more, since acc $\sigma(a) = \sigma(a) \setminus \text{iso } \sigma(a)$, acc $\sigma(a)$ coincides with the interior set of $\sigma(a)$. Thus, since acc $\sigma(a)$ is closed and open, acc $\sigma(a) = \emptyset$. In particular, according to [3 Theorem 12(iv)], $\sigma_{\text{DRAZIN}}(a)$ is empty.

To prove the equivalence between statements (i) and (iii), in first place the case $A = L(X)$, $X$ a Banach space, will be considered.

Let $T \in L(X)$ be an algebraic operator and consider $P \in \mathbb{C}[X]$ the minimal polynomial such that $P(T) = 0$. Then, from this condition and the spectral mapping theorem follows that $\sigma(T) = \{ \lambda \in \mathbb{C}; P(\lambda) = 0 \}$. In particular, if $\sigma(T) = \{ \lambda_i; 1 \leq i \leq n \}$, then according to [15 Theorem 5.7-A] and [15 Theorem 5.7-B], there exist $(M_i)_{1 \leq i \leq n}$ such that $M_i \subseteq X$ is an invariant closed subspace for $T$, $1 \leq i \leq n$, $X = \oplus_{1 \leq i \leq n} M_i$, and if $T_i = T \upharpoonright_{M_i}$, then $\sigma(T_i) = \{ \lambda_i \}$.

Now well, given a fixed $i$, $1 \leq i \leq n$, since $\lambda_j \notin \sigma(T_i)$, $1 \leq j \leq n$, $j \neq i$, and $P(X) = \Pi_{k=1}^n (X - \lambda_k)_{\leq n}$ for some $n_k \in \mathbb{N}$, $1 \leq k \leq n$, $(T_i - \lambda_i)^{\leq n_i} = 0$. Therefore, $T_i - \lambda$ is Drazin invertible in $M_i$ for all $\lambda \in \mathbb{C}$, $1 \leq i \leq n$. Since $X = \oplus_{1 \leq i \leq n} M_i$, and $T - \lambda = \oplus_{i=1}^n (T_i - \lambda)$, $T - \lambda$ is Drazin invertible for all $\lambda \in \mathbb{C}$.

On the other hand, if $\sigma_{\text{DRAZIN}}(T) = \emptyset$, then according to [3 Theorem 12(i)], $\sigma(T) = \Pi(T)$, actually $\sigma(T)$ is a finite set of poles. If $\sigma(T) = \{ \lambda_i; 1 \leq i \leq n \}$, then as in the previous paragraph consider $(M_i)_{1 \leq i \leq n}$ such that $M_i \subseteq X$ is an invariant closed subspace for $T$, $1 \leq i \leq n$, $X = \oplus_{1 \leq i \leq n} M_i$, and if $T_i = T \upharpoonright_{M_i}$, then $\sigma(T_i) = \{ \lambda_i \}$. Moreover, according to [15 Theorem 5.8-A], a straightforward calculation proves that $\lambda_i$ is a pole of $T_i$. Then, $M_i = N(T_i - \lambda_i)^{\leq n_i} \oplus R(T - \lambda)^{\leq n_i}$ for some $m_i \in \mathbb{N}$, what is more, $T_i - \lambda_i$ restricted to $R(T - \lambda)^{\leq n_i}$ is invertible, see [13 Theorem 5.8-A]. Now well, given $i$ such that $1 \leq i \leq n$, since $N_i = N(T_i - \lambda)^{m_i}$ and $R_i = R(T - \lambda)^{m_i}$ are closed invariant subspaces for $T_i - \lambda_i$, if $T_i^1$ and $T_i^2$ are the restrictions of $T_i$ to $N_i$ and $R_i$ respectively, then $\sigma(T_i) = \sigma(T_i^1) \cup \sigma(T_i^2)$, see [15]
Theorem 5.4-C]. However, since \( \sigma(T_i) = \{ \lambda_i \} \), \( N_i \neq 0 \) and \( \lambda_i \in \sigma(T_i^1) \setminus \sigma(T_i^2) \), \( \sigma(T_i^2) = \emptyset \). Therefore \( (T_i - \lambda_i)^m = 0 \). Define \( P(X) = \Pi_{i=1}^n (X - \lambda_i)^{m_i} \). Since \( X = \oplus_{1 \leq n} M_i \), it is not difficult to prove that \( P(T) = 0 \).

To conclude the proof, use [3, Theorem 4(iv)] and the fact that \( a \in A \) is algebraic if and only if \( L_a \in L(A) \) is algebraic.

Next at most countable Drazin spectrum will be characterized, see also [6, Corollary 1.8] and [1, Corollary 2.10].

**Theorem 2.2.** Let \( A \) be a unital Banach algebra and consider \( a \in A \). Then, the following statements are equivalent.

(i) \( \sigma(a) \) is at most countable,  
(ii) \( \sigma_{DR}(a) \) is at most countable.

Furthermore, in this case

\[
\sigma_{DR}(a) = \sigma_{LD}(L_a) = \sigma_{RD}(L_a) = \sigma_{dsc}(L_a)
\]

\[
= \sigma_{LD}(R_a) = \sigma_{RD}(R_a) = \sigma_{dsc}(R_a).
\]

**Proof.** Clearly, the first statement implies the second. On the other hand, since according to [3, Theorem 12(i)], \( \sigma(a) = \sigma_{DR}(a) \cup \Pi(a) \), being \( \Pi(a) \) a countable set, the second statement implies the first.

Concerning the remaining identities, since \( \sigma(a) = \sigma(L_a) \), [4, Chapter I, section 5, Proposition 4], according to [6, Corollary 1.8] and [1, Corollary 2.10],

\[
\sigma(L_a) = \sigma_{dsc}(L_a) \cup \Pi(L_a) = \sigma_{LD}(L_a) \cup \Pi(L_a).
\]

Moreover, according to [6, Theorem 1.5] and [1, Theorem 2.7], \( \sigma_{dsc}(L_a) \cap \Pi(L_a) = \emptyset = \sigma_{LD}(L_a) \cap \Pi(L_a) \). Therefore, according to [3, Theorem 12] and to the fact that \( \sigma_{LD}(L_a) \subseteq \sigma_{DR}(L_a) \) and \( \sigma_{dsc}(L_a) \subseteq \sigma_{RD}(L_a) \subseteq \sigma_{DR}(L_a) \) ([3, Theorem 3]),

\[
\sigma_{DR}(L_a) = \sigma_{LD}(L_a) = \sigma_{RD}(L_a) = \sigma_{dsc}(L_a).
\]

Since, according to [3, Theorem 4(iv)], \( \sigma_{DR}(a) = \sigma_{DR}(L_a) \), the first identity holds. To prove the identity involving \( R_a \), interchange \( L_a \) with \( R_a \) and use the same argument. \( \square \)

Given a Banach algebra \( A \) and \( a \) and \( b \in A \), the identity

\[
\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\},
\]

is well known ([4, Chapter I, section 5, Proposition 3]). However, in the case of the Drazin spectrum, both spectra coincide, see also [1, Theorem 2.11].

**Theorem 2.3.** Let \( A \) be a unital Banach algebra and consider \( a \) and \( b \in A \). Then,

\[
\sigma_{DR}(ab) = \sigma_{DR}(ba).
\]

**Proof.** First of all the Banach space operator case will be studied.

Let \( X \) be a Banach space and consider \( S \) and \( T \in L(X) \). Then, according to [3, Theorem 3] and [5, Theorem 1],

\[
\sigma_{DR}(ST) \setminus \{0\} = \sigma_{DR}(TS) \setminus \{0\}.
\]

Therefore, in order to conclude that the Drazin spectra of \( ST \) and \( TS \) coincide, it is enough to prove that \( ST \) is Drazin invertible if and only if \( TS \) is Drazin invertible. However, this statement is a consequence of the equivalence between [14, Proposition 2.1(ii)] and [14, Proposition 2.1(iii)].
Finally, the general case can be proved applying [3, Theorem 4(iv)]. □

**Remark 2.4.** Let \( A \) be a unital Banach algebra. Note that according to Theorem [23, 3, Theorem 12(i)] and the identity \( \sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\} \),
\[
\Pi(ab) \setminus \{0\} = \Pi(ba) \setminus \{0\},
\]
where \( a \) and \( b \) belong to \( A \). Furthermore, this identity cannot be improved. Consider for example the Banach space \( X = l^2(\mathbb{N}) \) and the operator \( S, T \in L(X) \) defined by
\[
S((x_n)_{n \geq 1}) = (0, x_1, x_2, \ldots, x_n, \ldots), \quad T((x_n)_{n \geq 1}) = (x_2, x_3, \ldots, x_n, \ldots),
\]
where \( (x_n)_{n \geq 1} \in X \). Then, a straightforward calculation proves that
\[
\Pi(ST) = \{0, 1\}, \quad \Pi(TS) = \{1\}.
\]

Next the relationships among the Drazin spectra of the multiplication operators and the adjoint of a \( C^* \)-algebra will be studied. First of all the operator case will be considered.

**Remark 2.5.** Given \( X \) a Banach space and \( T \in L(X) \), note that \( \sigma_{\mathcal{DR}}(T^*) = \sigma_{\mathcal{DR}}(T) \), where \( T^* \in L(X^*) \) is the adjoint of \( T \) and \( X^* \) is the dual space of \( X \). Consequently, according to [3, Theorem 4(iv)], all the sets \( \sigma_{\mathcal{DR}}(T), \sigma_{\mathcal{DR}}(T^*), \sigma_{\mathcal{DR}}(L_T), \sigma_{\mathcal{DR}}(L_{T^*}), \sigma_{\mathcal{DR}}(R_T), \text{ and } \sigma_{\mathcal{DR}}(R_{T^*}) \) coincide.

In addition, according to [13, page 139], \( \sigma_{\mathcal{LD}}(T^*) = \sigma_{\mathcal{RD}}(T) \) and \( \sigma_{\mathcal{RD}}(T^*) = \sigma_{\mathcal{LD}}(T) \). However, when \( X \) is a Hilbert space, according to [13, page 139] and [3, Theorem 9]
\[
(i) \quad \sigma_{\mathcal{RD}}(T) = \sigma_{\mathcal{RD}}(L_T) = \sigma_{\mathcal{LD}}(R_T) = \overline{\sigma_{\mathcal{LD}}(T^*)} = \sigma_{\mathcal{LD}}(T^*) = \overline{\sigma_{\mathcal{RD}}(T)}.
\]
\[
(ii) \quad \sigma_{\mathcal{LD}}(T) = \sigma_{\mathcal{LD}}(L_T) = \sigma_{\mathcal{RD}}(R_T) = \overline{\sigma_{\mathcal{RD}}(T^*)} = \sigma_{\mathcal{RD}}(T^*) = \overline{\sigma_{\mathcal{LD}}(T)}.
\]
where if \( \lambda \in \mathbb{C} \), then \( \overline{\lambda} \) denotes the complex conjugate of \( \lambda \), and if \( A \subseteq \mathbb{C} \), then \( \overline{A} = \{ \overline{\alpha} : \alpha \in A \} \).

What is more, if \( T = T^* \), then a straightforward calculation proves that the sets considered in (i) and (ii) are contained in the real line and they all coincide. In the following theorem, similar identities will be proved for \( C^* \)-algebra elements. However, before going on three facts should be mentioned.

First, given a Banach algebra \( A \), recall that \( a \in A \) is said to be regular, if there exists \( b \in A \) such that \( a = aba \), see [8]. Second, given a \( C^* \)-algebra \( A \) and \( a \in A \), note that \( \sigma_{\mathcal{DR}}(a^*) = \overline{\sigma_{\mathcal{DR}}(a)} \). Third, some notation. Given a \( C^* \)-algebra \( A \), if \( B \subseteq A \), then \( B^* = \{ \beta^* : \beta \in B \} \).

**Theorem 2.6.** Let \( A \) be a \( \mathbb{C}^* \)-algebra. Then, the following statements hold.

\[
(i) \quad \sigma_{\mathcal{dsc}}(L_{a^*}) = \overline{\sigma_{\mathcal{dsc}}(R_a)}, \quad \sigma_{\mathcal{dsc}}(R_{a^*}) = \overline{\sigma_{\mathcal{dsc}}(L_a)}.
\]
\[
(ii) \quad \sigma_{\mathcal{RD}}(L_{a^*}) = \sigma_{\mathcal{RD}}(R_a), \quad \sigma_{\mathcal{RD}}(R_{a^*}) = \sigma_{\mathcal{RD}}(L_a).
\]
\[
(iii) \quad \sigma_{\mathcal{asc}}(L_{a^*}) = \overline{\sigma_{\mathcal{asc}}(R_a)}, \quad \sigma_{\mathcal{asc}}(R_{a^*}) = \overline{\sigma_{\mathcal{asc}}(L_a)}.
\]
(iv) \[ \sigma_{LD}(L_a^\ast) = \sigma_{LD}(R_a), \quad \sigma_{LD}(R_a) = \sigma_{LD}(L_a). \]

Furthermore, when \( a \) is a hermitian element of \( A \), all the spectra considered in statements (i)-(iv) are contained in the real line, and

(v) \[ \sigma_{asc}(L_a) = \sigma_{asc}(R_a), \quad \sigma_{LD}(L_a) = \sigma_{LD}(R_a), \]

(vi) \[ \sigma_{DR}(a) = \sigma_{dsc}(L_a) = \sigma_{dsc}(R_a) = \sigma_{RD}(L_a) = \sigma_{RD}(R_a). \]

Proof. Let \( a \in A \). According to the fact that \((R(L_a))^* = R(R_a^\ast)\), a straightforward calculation proves that necessary and sufficient for \( d(L_a) \) to be finite is the fact that \( d(R_a^\ast) \) is finite. Moreover, in this case \( d(L_a) = d(R_a^\ast) \). However, from this identity the first statement can be easily deduced.

On the other hand, due to the fact that \( b \in A \) is a regular element of \( A \) if and only if \( b^* \) is regular, if \( 0 \notin \sigma_{dsc}(L_a) \), \( d = d(L_a) \), and \( R(L_a^\ast) \) is closed, then according to [X] Theorem 2, [X] Theorem 8, and what has been proved, \( 0 \notin \sigma_{dsc}(R_a^\ast) \), \( d = d(R_a^\ast) \), and \( R(R_a^\ast) \) is closed. Therefore, \( \sigma_{RD}(R_a^\ast) \subseteq \overline{\sigma_{\mathcal{P}}(L_a)} \). A similar argument proves the other inclusion. Interchanging \( a \) with \( a^\ast \), the remaining identity can be proved.

According to the identity \((N(L_a))^* = N(R_a^\ast)\), it is not difficult to prove that necessary and sufficient for \( a(L_a) \) to be finite is the fact that \( a(R_a^\ast) \) is finite. Furthermore, in this case \( a(L_a) = a(R_a^\ast) \). As in the case of the descent spectrum, from this identity the third statement can be proved.

In order to prove the fourth statement, apply an argument similar to the one used to prove the second identity of statement (ii), considering the ascent spectrum instead of the descent spectrum.

As regard the last statements, if \( a = a^\ast \), then an easy calculation proves that \( \sigma_{DR}(a) \) is contained in the real line. However, according to [X] Theorem 3 and [X] Theorem 4(iv), all the spectra in statements (i)-(iv) are contained in \( \mathbb{R} \). Moreover, \( \sigma_{dsc}(L_a) = \sigma_{dsc}(R_a), \quad \sigma_{RD}(L_a) = \sigma_{RD}(R_a), \quad \sigma_{asc}(L_a) = \sigma_{asc}(R_a), \quad \sigma_{LD}(L_a) = \sigma_{LD}(R_a) \). Finally, statement (vi) can be easily derived from [X] Theorem 5(ii) and what has been proved. \( \square \)

Recall that, according to the example [6] page 265, there is no relation that lies the descent of \( R_T \) to the descent of \( T, T \in L(H), H \) a Hilbert space. In the following theorem the ascent and the descent spectra of \( R_T \) will be characterized. See also [3] Theorem 8 and [3] Theorem 9.

Theorem 2.7. Let \( H \) be a Hilbert space. Then, the following statement hold.

(i) \[ \sigma_{dsc}(R_T) = \overline{\sigma_{dsc}(L_T^\ast)} = \sigma_{dsc}(T^\ast), \quad \sigma_{dsc}(R_T^\ast) = \overline{\sigma_{dsc}(L_T)} = \sigma_{dsc}(T). \]

(ii) \[ \sigma_{asc}(R_T) = \overline{\sigma_{asc}(L_T^\ast)} = \sigma_{asc}(T^\ast), \quad \sigma_{asc}(R_T^\ast) = \overline{\sigma_{asc}(L_T)} = \sigma_{asc}(T). \]
Furthermore, when $T = T^*$, $\sigma_{\text{asc}}(R_T)$ and $\sigma_{\text{asc}}(L_T)$ are subsets of the real line, and

\begin{align*}
(iii) & \quad \sigma_{\text{asc}}(R_T) = \sigma_{\text{asc}}(L_T) = \sigma_{\text{asc}}(T), \\
(iv) & \quad \sigma_{\text{DR}}(T) = \sigma_{\text{dsc}}(R_T) = \sigma_{\text{dsc}}(L_T) = \sigma_{\text{dsc}}(T).
\end{align*}

**Proof.** Apply Theorem 2.6 [3, Theorem 8] and [3, Theorem 9]. \qed

### 3. Meromorphic Banach space operators

Recall that given $X$ a Banach space and $T \in L(X)$, the operator $T$ is said to be **meromorphic**, if $\sigma(T) \setminus \{0\} \subseteq \Pi(T)$. As an application of the properties of the Drazin spectrum, in the following theorem several results on meromorphic Banach space operators will be proved, see [6, Corollary 1.9] and the paragraph that follows [1, Corollary 2.10].

**Theorem 3.1.** Let $X$ be a Banach space and consider $T \in L(X)$.

(a) The following statements are equivalent.

\begin{enumerate}
  \item $T$ is meromorphic,
  \item $\sigma_{\text{DR}}(T) \subseteq \{0\}$,
  \item $L_T \in L(L(X))$ is meromorphic,
  \item $R_T \in L(L(X))$ is meromorphic.
\end{enumerate}

(b) Let $S$ and $T \in L(X)$ . Then, necessary and sufficient for $ST$ to be meromorphic is the fact that $TS$ is meromorphic.

(c) Let $F \in L(X)$ and suppose that there exists a positive integer $n$ such that $F^n$ has finite dimensional range and $F$ commutes with $T$. Then, if $T$ is meromorphic, $T + F$ is meromorphic.

**Proof.** To prove the equivalence between statements a(i) and a(ii), apply [3, Theorem 12]. Next, to prove the equivalence between statements a(ii)-a(iv), apply [3, Theorem 4(iv)].

To prove the second statement, apply Theorem 2.3 and statement (a)(ii).

Concerning the third statement, given $U \in L(X)$ and $F$ as in statement (c), according to [3, Theorem 4] and [10, Theorem 2.2], $U$ is Drazin invertible if and only if $U + F$ is Drazin invertible. In particular, $\sigma_{\text{DR}}(U) = \sigma_{\text{DR}}(U + F)$. Clearly, statement (c) follows from this identity and a(ii). \qed

**References**

[1] O. Bel Hadj Fredj, M. Burgos and M. Oudghiri, *Ascent spectrum and essential ascent spectrum*, Studia Math. 187 (2008), 59-73.

[2] M. Berkani and M. Sarih, *An Atkinson-type theorem for B-fredholm operators*, Studia Math. 148 (2001), 251-257.

[3] E. Boasso, *Drazin spectra of Banach space operators and Banach algebra elements*, J. Math. Anal. Appl. 359 (2009), 48-55.

[4] F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin-New York, 1973.

[5] J. J. Buoni and J. D. Faires, *Ascent, descent, nullity and defect of products of operators*, Indiana Univ. Math. J. 25 (1976), 703-707.

[6] M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri, *The descent spectrum and perturbations*, J. Operator Theory 56 (2006), 259-271.

[7] M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly 65 (1958), 506-514.
[8] R. Harte and M. Mbekhta, *On generalized inverses in $C^*$-algebras*, Studia Math. 103 (1992), 71-77.
[9] C. King, *A note on Drazin inverses*, Pacific J. Math. 70 (1977), 383-390.
[10] M. A. Kaashoek and D. C. Lay *Ascent, descent and commuting perturbations*, Trans. Amer. Math. Soc. 169 (1972), 35-47.
[11] V. Kordula and V. Müller, *On the axiomatic theory of spectrum*, Studia Math. 119 (1996), 109-128.
[12] R. A. Lubansky, *Koliha-Drazin invertibles form a regularity*, Math. Proc. Roy. Ir. Acad. 107A (2007), 137-141.
[13] M. Mbekhta and V. Müller, *On the axiomatic theory of spectrum II*, Studia Math. 119 (1996), 129-147.
[14] V. Rakocević and Y. Wei, *A weighted Drazin inverse and applications*, Linear Algebra Appl. 350 (2002), 25-39.
[15] A. E. Taylor, *Introduction to Functional Analysis*, Wiley and Sons, New York, 1958.

Enrico Boasso
E-mail address: enrico.odisseo@yahoo.it