We provide a theoretical framework for encoding arbitrary logical states of a quantum bit (qubit) into a continuous-variable quantum mode through quantum walks. Starting with a squeezed-vacuum state of the quantum mode, we show that quantum walks of the state in phase space can generate output states that are variants of codeword states originally put forward by Gottesman, Kitaev, and Preskill (GKP) [Phys. Rev. A 64, 012310 (2001)]. In particular, with a coin-toss transformation that projects the quantum coin onto the diagonal coin-state, we show that the resulting dissipative quantum walks can generate qubit encoding akin to the prototypical GKP encoding. We analyze the performance of these codewords for error corrections and find that even without optimization our codewords outperform the GKP ones by a narrow margin. Using the circuit representation, we provide a general architecture for the implementation of this encoding scheme and discuss briefly its possible realization.
I. INTRODUCTION

Computing based on quantum mechanical principles (i.e., quantum computing) requires exquisite control of quantum systems [1]. Thanks to advancements in experimental techniques, tremendous progress has been made for achieving this goal during the past few years [2]. For large scale quantum computing, it is indispensable to have an architecture that enables efficient detection and correction of errors during the computing [3]. Recently, there has been significant progress towards this direction in the field of continuous-variable (CV) measurement-based quantum computing, which seeks to achieve quantum computing by sequence of adaptive local measurements over highly entangled resource states in a state space with continuous spectrum [4, 5]. In particular, Menicucci has shown that fault-tolerant quantum computing can be achieved in this scheme provided resource states with squeezing above 20.5 dB are available [6]. Recently, with the aid of topological codes, this squeezing threshold has been reduced to less than 10 dB [7, 8]. Essential to these breakthroughs is a quantum error-correcting scheme due to Gottesman, Kitaev, and Preskill (GKP) [9]. In this approach, quantum information is encoded through a “hybrid” quantum bit (qubit) embedded in the (quantum mechanical) phase space of a quantum harmonic oscillator. Despite the importance of the GKP scheme, existing proposals for the experimental generation of GKP qubits remain to pose major challenges [10–18] (see, however, the recent report in Ref. 19). In this paper we propose a new scheme for preparing GKP qubits through quantum walks (QWs) of an oscillator mode in phase space [20, 21]. As we will show, our encoding scheme indeed also provides a framework for experimentally accessing other general “grid states”, namely, codeword states with grid-like structures in phase space.

In the CV approach to quantum computing, quantum information are carried by quantum modes (aka “qumodes”) with the logical states encoded via eigenstates of the canonical coordinates of the field mode, which are usually likened to the position and momentum of a harmonic oscillator [22, 23]. Decoherence of the qumode then manifests as shift errors in these basis states. In order to correct such errors, GKP propose to invoke “hybrid” qubits
that consist of superposition of uniformly spaced position eigenstates separated by $2\sqrt{\pi}$ \[|0\rangle_L = \sum_{s=0}^{\infty} |2s\sqrt{\pi}\rangle_x = \frac{1}{\sqrt{2}} \sum_{s=-\infty}^{\infty} |s\sqrt{\pi}\rangle_p ,
|1\rangle_L = \sum_{s=0}^{\infty} |(2s+1)\sqrt{\pi}\rangle_x = \frac{1}{\sqrt{2}} \sum_{s=-\infty}^{\infty} (-1)^s |s\sqrt{\pi}\rangle_p, \tag{1}\]

where $|x\rangle_x$ and $|p\rangle_p$ are, respectively, position and momentum eigenstates. Thus the position-space wavefunctions for the codewords $|0\rangle_L$ and $|1\rangle_L$ comprise combs of delta functions located at, respectively, even and odd multiples of $\sqrt{\pi}$. In the presence of shift errors, it is then possible to correct sufficiently small errors in the encoded qubits through position and momentum measurements \[9\]. However, the GKP codeword states (1) require infinite squeezing, and hence infinite energy. In practice, therefore, one must approximate (1) with finitely squeezed states, such as uniformly spaced Gaussian spikes modulated by Gaussian envelopes \[9\]

$$x\langle x| \tilde{l}\rangle_L \propto \sum_n' e^{-\frac{n^2 \Delta_x^2}{2}} \exp \left[ -\frac{(x - n\sqrt{\pi})^2}{2\Delta_x^2} \right] ,$$

$$p\langle p| \tilde{l}\rangle_L \propto \sum_n (-1)^n e^{-\frac{\Delta_p^2 p^2}{2}} \exp \left[ -\frac{(p - n\sqrt{\pi})^2}{2\Delta_p^2} \right] , \tag{2}\]

where $l = \{0, 1\}$ are the logical bit values, $\sum_n'$ in the position-space wavefunctions indicate summations over even/odd integers $n$ for $l = 0/1$, and $\Delta_x, \Delta_p$ specify the widths of the Gaussians. It is our goal in the present paper to provide experimentally feasible schemes for engineering approximate GKP codewords such as (2) by implementing QWs in phase space for a qumode. In contrast to its classical counterpart, QW takes place in accordance with a quantum coin that admits superposition of orthogonal coin-states \[20, 21\]. Utilizing QWs of a qumode initially in a squeezed vacuum state, we will demonstrate that GKP-type codewords can be generated under appropriate “coin-toss rules”. As we will show, by changing the nature of the coin toss, one can attain GKP-type encodings with different characteristics. Our approach thus offers not only a promising pathway to the preparation of GKP qubits, but also opens up new dimensions to the GKP encoding.

In the following, we will start in Sec. II by first explaining how the features of QWs in phase space for a qumode can be exploited to encode a squeezed vacuum state into a GKP qubit. We will then illustrate with two instances: One involving generic unitary QWs, and the other \textit{dissipative}, non-unitary QWs. Performance of our codewords for error
correction will then be analyzed for the dissipative case. We will then discuss in Sec. III the implementation for our encoding scheme by first establishing a protocol using the circuit model and next touching on its possible experimental realizations. Finally, we conclude in Sec. IV with a summary and brief discussions for our findings. For presentational clarity, we relegate detailed calculations and formulas to the Appendices.

II. FROM QUANTUM WALK TO THE GKP ENCODING

Let us consider one-dimensional (1D) QW in phase space for a qumode with mode operator \( \hat{a} = (\hat{x} + i\hat{p})/\sqrt{2} \), where \( \hat{x} \) and \( \hat{p} \) are, respectively, the “position” and “momentum” quadrature operators of the qumode. From the commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\) for the mode operator, it follows that \([\hat{x}, \hat{p}] = i\), which corresponds to setting \( \hbar = 1 \) for us. For the QW, we will be concerned with position-squeezed states for the qumode, which will be denoted as \(|q\rangle_r\). Here the subscript \( r \) indicates the squeezing parameter, \( q \) is the expectation value \( \langle \hat{x} \rangle \) of the state in units of \( \sqrt{2} \) times the QW step length \( \Delta \xi \) [24]. Each step of the QW is conditioned on the configuration \(|\epsilon\rangle\) of a two-state quantum coin with \( \epsilon = \{R, L\} \) corresponding to rightward (\( R \)) and leftward (\( L \)) displacements by one single step length. More precisely, in terms of the phase-space displacement operator \( \hat{D}(\xi) \equiv \exp\{\xi(\hat{a}^\dagger - \hat{a})\} \) for real \( \xi \) and the squeezing operator \( \hat{S}(r) \equiv \exp\{r^2(\hat{a}^2 - \hat{a}^\dagger^2)\} \) for real \( r \), we will be considering squeezed coherent states

\[
|q\rangle_r \equiv \hat{D}(q\Delta \xi) \hat{S}(r) |\text{vac.}\rangle
\]

with \(|\text{vac.}\rangle\) the vacuum state of the qumode. Therefore, in the language of QW, the product state \(|q\rangle_r|\epsilon\rangle\) indicates a walker at position \( q \times \sqrt{2} \Delta \xi \equiv q \Delta x \) with a coin configuration \( \epsilon \) [24].

Let us suppose the QW has the coin-toss operator \( \hat{C} \). The corresponding “walk operator” would then read in the state space of (qumode)\( \otimes \) (coin)

\[
\hat{W} = \hat{T}(\Delta \xi) \left( \hat{I} \otimes \hat{C} \right).
\]

Here \( \hat{I} \) is the qumode identity operator and \( \hat{T} \) is a translation operator whose action is conditioned on the coin configuration

\[
\hat{T}(\Delta \xi) = \hat{D}(+\Delta \xi) \otimes |R\rangle\langle R| + \hat{D}(-\Delta \xi) \otimes |L\rangle\langle L|.
\]
In order to prepare GKP qubits for the qumode, we shall consider 1D QW of a position-squeezed vacuum state along with a coin qubit in an arbitrary configuration

$$|\psi_{\text{in}}\rangle = |0\rangle_r (\alpha |R\rangle + \beta |L\rangle ) ,$$

(6)

where $|\alpha|^2 + |\beta|^2 = 1$. As we will demonstrate below, by means of 1D QW in phase space, one can transcribe the “logical state” imprinted in the coin configuration in (6) onto the qumode “coordinate” degrees of freedom $\{|q\rangle_r\}$. Since different choices for the coin-toss transformation $\hat{C}$ can lead to rather distinct walk patterns in the QW [20, 21], it can be anticipated that different encodings can be achieved through different coin-toss transformations. In the following, we will consider first the case of a “Hadamard coin-toss”, which induces unitary evolution of the input state (6). As we shall find out, the consequent codewords will be quite different from the approximate GKP codewords in (2). We will then turn to another coin-toss transformation, which will engender codeword states that are similar to the “standard” ones in (2).

A. Generic (unitary) quantum-walk encoding

For a walker that localizes initially at the origin, after even (odd) steps of QW the wavefunction of the walker would become coherent superposition of localized states over even (odd) multiples of the step length. In view of the structure of the ideal GKP codewords (1), it is clear that one can exploit this feature of QWs to prepare GKP qubits. The conceptual plan for our encoding scheme is outlined in Fig. 1. In order to simplify our theoretical formulation, let us suppose for now that when implementing the QW, we are able to separate and combine the components, e.g. $|0\rangle_r |R\rangle$ and $|0\rangle_r |L\rangle$ (which, for brevity, will be referred to as the $R$ and $L$-components, respectively) of the input state (6) in the way shown in Fig. 1. As we shall find out, this serves only as a convenience to help present our theoretical ideas in a clear way, but won’t be a necessity in physical implementations for the scheme, which we shall discuss in Sec. III.

In our encoding scheme, as shown in Fig. 1, prior to the QW we separate the $R$ and the $L$-components of the input state (6) to produce a time gap between the them. In the scenario of Fig. 1, we delay the $R$-component so that when it begins its QW, the $L$-component would have completed exactly one step of QW. The two components are then combined
FIG. 1. Conceptual layout for our QW-encoding scheme, where each thin line represents the respective component of the input state $|\psi_{\text{in}}\rangle$ [see Eq. (6)] as indicated, while heavy lines stand for the full state. Each passing (dashed line) of the state through the QW-setup implements for the qumode one single step of QW in phase space. Before entering the QW-setup, the input state $|\psi_{\text{in}}\rangle$ is split into its $R$ and $L$-components, with the $R$-component subsequently delayed behind the $L$-component. After the $L$-component has completed one single step of QW while none for the $R$-component, the two are combined for the ensuing QWs. The encoded state is generated after the combined state has completed the designated steps of QW.

for all subsequent QWs. Since the $R$-component always lags behind the $L$-component by one single step, in the course of the QW coherent superposition of localized spikes at sites of opposite parities are generated progressively for the two components of the input state. Therefore, a GKP-type encoding can be furnished after the state has completed the desired number of steps of QW.

As an illustration, let us consider an encoding with the following coin-toss transformation in the coin basis $\{|R\rangle, |L\rangle\}$

$$\hat{C}_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

(7)

which corresponds to a Hadamard coin-toss for the QW [21]. Suppose the initial state (6) undergoes the encoding process of Fig. 1, so that its $R$ and $L$-components would complete, respectively, $N$ and $(N + 1)$ steps of QW upon output. It then leads to the state

$$|\psi_{\text{out}}\rangle = \alpha |\psi_N^{(R)}\rangle + \beta |\psi_N^{(L)}\rangle,$$

(8)

where $|\psi_N^{(\epsilon)}\rangle$ with $\epsilon = \{R, L\}$ are the resulting states for $|0\rangle, |\epsilon\rangle$ after $N$ steps of QW. One
FIG. 2. Probability densities in position space and momentum space for generic QW-codewords with \(N = 8\) at a squeezing with \(e^{-r} = 0.2\) (~13.98 dB). Panels (a) and (b) illustrate the results for \(|0\rangle_{\text{QW}}\) (solid curves) and \(|1\rangle_{\text{QW}}\) (dashed curves), while panels (c) and (d) show those for \(|+\rangle_{\text{QW}} \equiv (|0\rangle_{\text{QW}} + |1\rangle_{\text{QW}})/\sqrt{2}\) (solid curves) and \(|-\rangle_{\text{QW}} \equiv (|0\rangle_{\text{QW}} - |1\rangle_{\text{QW}})/\sqrt{2}\) (dashed curves).

can find analytically that [25]

\[
|\psi^{(e)}_{N}\rangle = \sum_{n=-N}^{N'} |n\rangle_{r} \left( u^{(e)}_{N}(n)|R\rangle + v^{(e)}_{N}(n)|L\rangle \right),
\]

where \(\sum_{n}^{t}\) indicates summation over every other integers, that is, \(n = \{-N, -(N-2), \ldots, (N-2), N\}\). To avoid distractions, we supply explicit expressions for the amplitudes \(u^{(e)}_{N}(n)\) and \(v^{(e)}_{N}(n)\) in Appendix A 1. From (9), it is then clear that when \(N\) is, for instance, even \(|\psi^{(R)}_{N}\rangle\) in (8) would cover only even sites, while \(|\psi^{(L)}_{N+1}\rangle\) only odd sites. By defining the encoded logical basis states here

\[
|0\rangle_{\text{QW}} \equiv |\psi^{(R)}_{N}\rangle \quad \text{and} \quad |1\rangle_{\text{QW}} \equiv |\psi^{(L)}_{N+1}\rangle,
\]

we then have from (8) the encoded state for the input state (6)

\[
|\psi_{\text{encd}}\rangle = \alpha|0\rangle_{\text{QW}} + \beta|1\rangle_{\text{QW}}.
\]

Our scheme thus furnishes a GKP-type encoding for an arbitrary input state.

To find the position-space and the momentum-space wavefunctions for the codewords \(|0\rangle_{\text{QW}}\) and \(|1\rangle_{\text{QW}}\), we note that from (3) one can obtain the wavefunctions for the squeezed
coherent states $|n\rangle_r$

$$x\langle x|n\rangle_r = \frac{e^{+\frac{1}{2}r}}{\pi^{\frac{1}{4}}} \exp \left[ \frac{-(x - n\Delta x)^2}{2 e^{-2r}} \right],$$

$$\nu\langle p|n\rangle_r = \frac{e^{-\frac{1}{2}r}}{\pi^{\frac{1}{4}}} \exp \left[ +inp\Delta x - \frac{p^2}{2 e^{+2r}} \right],$$

(12)

where $\Delta x \equiv \sqrt{2}\Delta \xi$, as before [24]. Making use of (12), one can obtain the wavefunctions for the codeword states $|0\rangle_{QW} = |\psi^{(R)}_N\rangle$ and $|1\rangle_{QW} = |\psi^{(L)}_{N+1}\rangle$ using (9) and the expressions for the amplitudes $u_N^{(e)}$, $v_N^{(e)}$, and etc. in Appendix A1. Setting $\Delta x = \sqrt{\pi}$ for the QW, we show in Fig. 2 the probability densities in position and momentum spaces for the QW-codewords $|0\rangle_{QW}$ and $|1\rangle_{QW}$, together with the conjugate states $|+\rangle_{QW} \equiv (|0\rangle_{QW} + |1\rangle_{QW})/\sqrt{2}$ and $|−\rangle_{QW} \equiv (|0\rangle_{QW} − |1\rangle_{QW})/\sqrt{2}$ for the case of $N = 8$ at a squeezing with $e^{-r} = 0.2$ (corresponding to $\sim 13.98$ dB squeezing).

From Fig. 2, we see that, despite the unusual envelopes, the probability densities of the QW-codewords exhibit features characteristic of approximate GKP codewords that are essential for correcting shift errors [11]. In particular, the position distributions of the codewords $|0\rangle_{QW}$ and $|1\rangle_{QW}$ consist of Gaussian spikes at, respectively, even and odd multiples of $\sqrt{\pi}$, while the momentum distributions of the codewords $|+\rangle_{QW}$ and $|−\rangle_{QW}$ manifest peaks at, respectively, even and odd multiples of $\sqrt{\pi}$. Therefore, in principle, the QW-codewords here can be adopted to correct shift errors in accordance with the GKP scheme, despite the issues with their performance and probably also efficiencies [6]. The key drawbacks of the QW-codewords (10) reside in their momentum distributions, such as the featureless Gaussians in Fig. 2(b) for $|0\rangle_{QW}$ and $|1\rangle_{QW}$, and the broad peaks in Fig. 2(d) for $|+\rangle_{QW}$ and $|−\rangle_{QW}$, which would render the correction for $p$-errors ineffective. We find numerically that these features are independent of the QW steps $N$, likely due to unitarity of the QW here.

Since different coin-toss transformations for QW can lead to very different walk patterns [20, 21], one possible remedy for the present dilemma is to replace the coin-toss operation (7) with a new one. Alternatively, since the difficulty with the codewords $|0\rangle_{QW}$ and $|1\rangle_{QW}$ lies in the lack of (coin-state independent) structures in their momentum distributions, one can attempt to implement periodic structures in both position and momentum directions through two-dimensional QWs [21]. As our point here is to demonstrate the feasibility of the proposed QW-scheme for generating GKP-type encodings in a generic setting, we shall not pursue these issues further.
In addition to the “unconventional” profiles in the probability densities of the QW-codewords in Fig. 2, it should be noted that here the codewords $|0\rangle_{\text{QW}}$ and $|1\rangle_{\text{QW}}$ are entangled states between the qumode and the ancillary coin-qubit [see (9)]. This is in stark contrast to codewords from the original GKP-scheme, where the encoding resides entirely in the qumode. To disentangle the qumode from the coin qubit in the QW-codewords, one can choose to project them onto any given coin state, such as the symmetric “diagonal” coin-state $|D\rangle \equiv (|R\rangle + |L\rangle)/\sqrt{2}$. However, we find this would lead to states that are plagued by fast oscillations of intervals below $\sqrt{\pi}$ in their momentum-space distributions, which are unfavorable for correcting $p$-errors. Therefore, here we choose to retain the form (9) for the QW-codewords with its full generality. At this point, it is then natural to ask whether our scheme is capable of producing QW-codewords akin to the “standard” GKP ones as in (2). As we will now show, this is indeed possible if appropriate coin-toss transformation is used.

**B. Dissipative (non-unitary) quantum-walk encoding**

In order to generate codewords similar to (2), it necessitates implementing QWs with Gaussian probability distributions in our scheme. Intuitively, one might expect decoherence of the state must be incorporated, so that the QW would become classical and yield the desired probability distributions [26]. However, this would inevitably lead to mixed states, which are unfavorable for our purposes here, as the codeword states must be pure states. To find the way out, we note that the nonclassical nature of the QW arises from the interference between the coin-toss outcomes for the $R$ and the $L$-components. Therefore, if we project the state vector at each step of the QW, so that the two coin-state components won’t interfere, it would then be possible to generate coherent superposition of “classically” distributed Gaussian spikes. In other words, by “resetting” the coin state of the walker to a symmetric combination of the $|R\rangle$ and the $|L\rangle$ states in each step of the QW, one can then generate the desired walk pattern here. It thus follows that one should replace the coin-toss transformation (7) with the projection operator for the diagonal coin-state $|D\rangle = (|R\rangle + |L\rangle)/\sqrt{2}$, i.e.,

$$\hat{C}_D = |D\rangle\langle D| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

(13)
With this change, as we shall show below, we are then able to achieve the targeted codeword states. Since the projection operator can reduce the total probability of the state it acts on, the QW here becomes nonunitary, and we shall refer to it as the “dissipative” QW.

For the QW-encoding modified with the new coin-toss operation (13), the calculation for the corresponding encoded states proceeds in exactly the same manner as before. After the $R$-component and the $L$-component of the input state (6) have completed, respectively, $N$ and $(N + 1)$ steps of QW, one can find an output state with the same structure as Eqs. (8) and (9), but now with different explicit forms for the amplitudes $u^\epsilon_N(n)$ and $v^\epsilon_N(n)$ (see Appendix A 2). At this point, it is tempting to conclude immediately that the encoding can then be done in exactly the same way as in (10). This is, however, incorrect because we now have nonunitary, dissipative QWs and thus must take extra care for the normalization of the state vectors. Moreover, in order that the codeword states would resemble better the approximate GKP codewords (2), we find it advantageous to project the final state of the QW onto the diagonal coin-state $|D\rangle$, which also serves to disentangle the qumode from the coin qubit here. For the initial state $|0\rangle_r|\epsilon\rangle$ the resulting unnormalized state vector after $N$ steps of dissipative QW (including the action of the $|D\rangle$-projector upon output) takes the form

$$|\psi_N\rangle = \sum_{n=-N}^{N} w_N(n) |n\rangle_r |D\rangle,$$

(14)

where

$$w_N(n) = \frac{1}{2^{N+1}} \left( \frac{N}{N+n} \right).$$

(15)

Note that here the qumode and the coin qubit are fully disentangled. Also, since the $R$ and the $L$-components are now symmetrical, we have dropped the superscript $\epsilon$ for the initial coin configurations in (14) and (15) [cf. Eq. (9) for the unitary case]. In terms of the normalized state vectors for (14)

$$|\phi_N\rangle \equiv Z_N^{-1/2} |\psi_N\rangle,$$

(16)

with $Z_N \equiv \langle \psi_N |\psi_N\rangle$, we find the output state for the dissipative QW

$$|\psi_{out}\rangle = \alpha \sqrt{Z_N} |\phi_N\rangle + \beta \sqrt{Z_{N+1}} |\phi_{N+1}\rangle$$

$$\propto \alpha' |\phi_N\rangle + \beta' |\phi_{N+1}\rangle,$$

(17)
where we have denoted in the second line $\alpha' \equiv \alpha / \sqrt{|\alpha|^2 + \gamma^2 |\beta|^2}$ and $\beta' \equiv \gamma \beta / \sqrt{|\alpha|^2 + \gamma^2 |\beta|^2}$ with $\gamma \equiv \sqrt{Z_{N+1}/Z_N}$. Therefore, identifying

$$|0\rangle_{dQW} \equiv |\phi_N\rangle \quad \text{and} \quad |1\rangle_{dQW} \equiv |\phi_{N+1}\rangle,$$

we arrive at the following encoding for the input state (6)

$$|\psi_{\text{encd}}\rangle = N ( \alpha' |0\rangle_{dQW} + \beta' |1\rangle_{dQW} )$$

with $N \equiv \sqrt{|\alpha|^2 Z_N + |\beta|^2 Z_{N+1}}$. Note that the coefficients $\alpha$, $\beta$ in the original state (6) have been modified in the final encoding (19) due to the dissipative nature of the QW. Therefore, when applying this encoding scheme, one must prepare the input state properly, so that the desired encoded states can be obtained at the output.

To find the wavefunctions of the codewords here, one can again use (12) in (18) [together with Eqs. (14)–(16)]. For the momentum-space wavefunction, the summation over the site index $n$ can be done analytically. We find

$$p \langle p|\phi_N\rangle = \left( \frac{e^{-r}}{2\sqrt{\pi Z_N}} \right)^{1/2} \exp \left[ -\frac{p^2}{2e^{2r}} \right] \cos^N(p\Delta x) |D\rangle.$$

For the position-space wavefunction, it is of particular interest to examine its large $N$ limit, for which the binomial distribution would tend to a Gaussian. Applying Stirling’s formula, we find for large $N$

$$x \langle x|\phi_N\rangle \approx \left( \frac{2e^{+r}}{\pi \sqrt{N}} \right)^{1/2} \sum_{n=-N}^{N} e^{-\frac{x^2}{2\pi}} \exp \left[ -\frac{(x-n\Delta x)^2}{2e^{-2r}} \right] |D\rangle.$$

For the normalization in (21), we have taken the squeezed coherent states $\{|n\rangle_r\}$ here to be approximately orthogonal, so that $Z_N$ in (16) can be approximated as

$$Z_N \approx \frac{1}{22^{N+1}} \left( \begin{array}{c} 2N \\ N \end{array} \right) \approx \frac{1}{2\sqrt{\pi N}}.$$

Taking $\Delta x = \sqrt{\pi}$ in (21) and comparing the expression with the approximate GKP codewords (2), we see that in the large $N$ limit the dissipative QW (or dQW, for short) codewords (18) correspond to approximate GKP codewords with width $\Delta_x \approx e^{-r}$ and $\Delta_p \approx 1/\sqrt{\pi N}$. Therefore, for shift errors symmetric in the position and the momentum quadratures, following GKP [9], the choice for encodings with $\Delta_x = \Delta_p$ becomes here $e^{+r} = \sqrt{N\pi}$. As an
FIG. 3. (a) Position and (b) momentum wavefunctions for the dissipative QW-codewords $|0\rangle_{dQW}$ (solid curves) and $|1\rangle_{dQW}$ (dashed curves) with $N = 8$ at a squeezing with $e^{-r} = 1/\sqrt{8\pi} \sim 0.199$ ($\sim 14.00$ dB squeezing).

Illustration, we plot in Fig. 3 the wavefunctions for the dQW-codewords for the case with $N = 8$ and $e^{-r} = 1/\sqrt{8\pi} \sim 0.199$ (corresponding to $\sim 14.00$ dB squeezing), which carry the hallmarks of approximate GKP codewords (2).

In order to evaluate the performance of the dQW-codewords (18), we consider the error-correcting scheme of Ref. 27 and find the probability $P_{\text{no error}}$ for repeated error corrections using the dQW-codewords without incurring Pauli errors (see Appendix B for details). For this calculation, we consider $N$-step dissipative QW-encoding with width $e^{-r} = 1/\sqrt{N\pi} \equiv \Delta$. The results are shown in Fig. 4, where we also plot the results for the approximate GKP codewords (2) with $\Delta_x = \Delta_p = \Delta$ for comparison. It is encouraging to find that the dQW-codewords in fact outperform their GKP counterparts for all $\Delta$ by a small margin. In particular, for the $N = 8$ dQW-codewords we get $P_{\text{no error}} \approx 0.936$ [vs. $\approx 0.929$ for the GKP case] at the squeezing $\sim 14.00$ dB, which is within current experimental capabilities [28]. Although for dQW-codewords with $N = 10$ one can even attain $P_{\text{no error}} \approx 0.966$ at a squeezing $\sim 14.97$ dB, which lies barely below the 15 dB squeezing achieved in Ref. 28, implementing 10 and 11 steps of QW are even more challenging experimentally.

III. IMPLEMENTATIONS

Let us now look into the physical implementations for our QW-encoding scheme. To begin with, we note that if the coin states $|R\rangle$ and $|L\rangle$ are taken the logical states $|0\rangle_c$ and $|1\rangle_c$ (with subscript “c” for “coin” or “control”), respectively, of a control qubit for a controlled-displacement gate, we can then write the walk operator $\hat{W}$ of (4) as

$$\hat{W} = \left[ I \otimes |0\rangle_c\langle 0| + \hat{D}(-2\Delta \xi) \otimes |1\rangle_c\langle 1| \right] \left( \hat{D}(+\Delta \xi) \otimes \hat{C} \right).$$

(23)
FIG. 4. The probability for repeated error corrections without Pauli errors using dissipative QW-codeword $|0\rangle_{dQW}$ (dots) with $\Delta = e^{-r} = 1/\sqrt{N\pi}$, where the solid line serves as guide to the eyes. For comparison, results for the approximate GKP codeword $|\tilde{0}\rangle_L$ (dashed line) with $\Delta_x = \Delta_p = \Delta$ are also shown. The arrow indicates the point for $N = 8$ and $\Delta = 1/\sqrt{8\pi} \approx 0.199$ (corresponding to $\sim 14.0$ dB squeezing) in the dissipative QW-codeword.

It then follows that the walk operator $\hat{W}$ can be implemented through the quantum circuit depicted in Fig. 5(a). To prepare logical basis states in the QW-scheme [i.e., (10) or (18)], one can thus supply the states $|0\rangle_r|R\rangle$ and $|0\rangle_r|L\rangle$ to the circuit separately and cycle for the corresponding numbers of rounds for the QW. Note that for the dissipative QW-encoding of Sec. II B, an additional projection operator $|D\rangle\langle D|$ over the coin degree of freedom has to be applied at the output of the final round of QW [see above Eq. (14)], which is not included in the circuit of Fig. 5(a). To encode qubits with arbitrary logic, however, it requires additional efforts, as one has to delay either the $R$ or the $L$-component of the general input state (6) for the encoding.

As previously, let us delay the $R$-component of the input state one step behind its $L$-component for the QW. It then suffices if we are able to prepare from the input state $|\psi_{\text{in}}\rangle$ the corresponding “delayed state” $|\tilde{\psi}_{\text{in}}\rangle$, which has the $L$-component already taken one step of QW, while none for the $R$-component, that is,

$$|\tilde{\psi}_{\text{in}}\rangle = \alpha |0\rangle_r |R\rangle + \beta \hat{W} |0\rangle_r |L\rangle.$$  

For instance, in the case of the unitary QW-encoding with a Hadamard coin (7) discussed in Sec. II A, the desired delayed state would then be

$$|\tilde{\psi}_{\text{in}}\rangle_{\text{QW}} = \alpha |0\rangle_r |R\rangle + \frac{\beta}{\sqrt{2}} (|+\rangle_r |R\rangle - |-\rangle_r |L\rangle).$$  

As shown in Appendix C, this state can be produced from the input state $|\psi_{\text{in}}\rangle$ making
FIG. 5. Quantum circuits for (a) implementing one single step of QW and (b) preparing the properly delayed state $|\tilde{\psi}_{in}\rangle$ from the input state $|\psi_{in}\rangle$. Here $\hat{D}$ is the phase-space displacement operator, $\hat{C}$ the coin-toss operator for the QW, and $\hat{B}$ the “biased” coin-toss operator for state preparation (see text).

use of the circuit illustrated in Fig. 5(b), which has two controlled-displacement operators separated by a “biased” coin-toss operator $\hat{B} = \hat{B}_H$ given by

$$\hat{B}_H = \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$  

Comparing (26) with (7), we see that $\hat{B}_H$ does not flip the R-component of the quantum coin, while tosses its L-component in the way of the Hadamard coin $\hat{C}_H$ (thus, a “biased” coin-toss). It is therefore not surprising that the delayed state can be duly prepared this way.

Similarly, for the dissipative QW-encoding of Sec. II B the necessary delayed state can be obtained using (24) and the corresponding walk operator. The result reads

$$|\tilde{\psi}_{in}\rangle_{dQW} = a|0\rangle_r |R\rangle + \frac{\beta}{2} (|+\rangle_r |R\rangle + |-\rangle_r |L\rangle).$$  

Again, this state can be prepared through the circuit of Fig. 5(b) with the following biased coin-toss

$$\hat{B}_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$
Once the delayed state is available, for both encoded states (11) and (19) discussed in Sec. II, the remaining $N$ steps of QW can then be implemented by sending the respective $|\tilde{\psi}_{in}\rangle$ into the circuit of Fig. 5(a) and cycling for $N$ rounds. As before, in the dissipative case an additional projector $|D\rangle\langle D|$ over the coin qubit must be incorporated into the circuit of Fig. 5(a) at the output of the final round.

With the architecture for implementing QW-encodings in place, it is then of interest to look into its possible experimental realizations. One possible route that may be accessible to current technologies is offered by circuit quantum-electrodynamics (cQED) systems similar to that used in realizing the cat code [29–32]. The system shall consist mainly of a (superconducting) transmon qubit coupled to one single mode of a microwave cavity in the strong dispersive regime, where the transmon acts as the quantum coin and the cavity mode the qumode to be encoded. The controlled-displacement operations in Fig. 5 can then be implemented either directly or via compositions of conditional phase rotations, which have been analyzed in great detail in earlier works [16, 31–33]. For the biased coin-toss operations such as (26) and (28), one can implement through, for instance, realizations for the corresponding positive operator-valued measures (POVM’s) [34]. Here, however, one must account for effects from the Kerr nonlinearity intrinsic to cQED systems, which can degrade the fidelities of the gate operations [16, 33] and thus also of the resulting codeword states. Detailed analysis along these lines of investigation is underway.

IV. CONCLUSION AND DISCUSSIONS

In summary, we have shown that by implementing QWs in phase space for a qumode, it is possible to furnish GKP-type encodings for quantum error-corrections in CV quantum computing. In addition to demonstrating an encoding through generic unitary QWs that produces codewords with “unconventional” profiles, we show further that an encoding via dissipative, nonunitary QWs can generate codeword states similar to the standard GKP ones. We examine the performance of the dissipative QW-codewords for error corrections and find that they do better than the standard GKP codewords by a small amount. In view of this result, it is promising that with optimized coin-toss transformations, one may find QW-codewords that perform even better. We provide an architecture for implementing the QW-encoding in terms of quantum circuits, which is likely realizable through circuit
quantum-electrodynamics systems.

Although we have focused primarily on engineering the GKP codeword states, our work indeed uncovers a new avenue to accessing general GKP-type “grid states”. For instance, as pointed out earlier, extending our encoding scheme to the case of two-dimensional QWs can be useful for rectifying the undesirable momentum distributions of the codewords in Fig. 2. At the same time, this may also help enhance the versatility of the QW-encoding scheme. In view of the multitude of walk patterns available for QWs [20, 21], the QW-encoding scheme proposed in this work thus opens up a new dimension for the GKP encoding that is yet to be explored. In particular, it may offer a unified framework for experimentally generating grid states for quantum error correcting codes in the CV regime.

We are very grateful for Prof. Stephen Barnett’s kind help and insightful suggestions. We also thank Profs. Tzu-Chieh Wei and Dian-Jiun Han for valuable discussions. This research is supported by the Ministry of Science and Technology of Taiwan through grants MOST 107-2112-M-194-002 and MOST 107-2627-E-008-001.

Appendix A: Formulas for the amplitudes \( u_N^{(R)}(n) \) and \( v_N^{(R)}(n) \) in the codeword states

We provide here explicit expressions for the amplitudes \( u_N^{(R)}(n) \) and \( v_N^{(R)}(n) \) in (9) for constructing the generic and the dissipative QW codeword states.

1. Generic QW codeword states

In the generic (unitary) case if the initial state is a squeezed-vacuum state \( |0\rangle_r \) along with the coin configuration \( |R\rangle \), after \( N \) steps of QW one has for \( n \neq \pm N \) [25]

\[
\begin{align*}
\quad u_N^{(R)}(n) &= \frac{1}{\sqrt{2^N}} \sum_{k=0}^{k_u} \left( \frac{N-n-2}{2} \right) \left( \frac{N+n}{2} \right) \frac{(-1)^{N-n-2}}{k+1} , \\
\quad v_N^{(R)}(n) &= \frac{1}{\sqrt{2^N}} \sum_{k=0}^{k_v} \left( \frac{N-n-2}{2} \right) \left( \frac{N+n}{2} \right) \frac{(-1)^{N-n-2}}{k} ,
\end{align*}
\]

(A1)

where the upper bounds for the summations are

\[
\begin{align*}
k_u &\equiv \min \left\{ \frac{N-n-2}{2}, \frac{N+n-2}{2} \right\} \quad \text{and} \quad k_v \equiv \min \left\{ \frac{N-n-2}{2}, \frac{N+n}{2} \right\} . \quad (A2)
\end{align*}
\]
For the boundary points \( n = \pm N \), one finds

\[
\begin{align*}
  u_N^{(R)}(+N) &= \frac{1}{\sqrt{2N}}, & u_N^{(R)}(-N) &= 0, \\
  v_N^{(R)}(+N) &= 0, & v_N^{(R)}(-N) &= \frac{(-1)^{N-1}}{\sqrt{2N}}. 
\end{align*}
\] (A3)

In the case of a squeezed-vacuum state with coin configuration \(|L\rangle\) initially, one finds after \( N \) steps of QW for \( n \neq \pm N \)

\[
\begin{align*}
  u_N^{(L)}(n) &= \frac{1}{\sqrt{2N}} \sum_{k=0}^{k_u'} \binom{N+n-2}{k} \binom{N-n}{k} (-1)^{N-n-k}, \\
  v_N^{(L)}(n) &= \frac{1}{\sqrt{2N}} \sum_{k=0}^{k_v'} \binom{N+n-2}{k} \binom{N-n}{k+1} (-1)^{N-n-k-1}, 
\end{align*}
\] (A4)

where the upper bounds for the summations are

\[
  k_u' \equiv \min \left\{ \frac{N+n-2}{2}, \frac{N-n}{2} \right\} \quad \text{and} \quad k_v' \equiv \min \left\{ \frac{N+n-2}{2}, \frac{N-n-2}{2} \right\}. 
\] (A5)

For the boundary points \( n = \pm N \), one gets

\[
\begin{align*}
  u_N^{(L)}(+N) &= \frac{1}{\sqrt{2N}}, & u_N^{(L)}(-N) &= 0, \\
  v_N^{(L)}(+N) &= 0, & v_N^{(L)}(-N) &= \frac{(-1)^N}{\sqrt{2N}}. 
\end{align*}
\] (A6)

2. Dissipative QW codeword states

In the dissipative case, as pointed out in the text, the \( R \) and the \( L \)-components are now symmetric. Therefore, for both types of initial states \(|0\rangle_r|R\rangle\) and \(|0\rangle_r|L\rangle\), one has the same result for the amplitudes after \( N \) steps of QW. We can thus drop the superscripts \( \epsilon \) for coin configurations in the amplitudes \( u_N^{(\epsilon)} \) and \( v_N^{(\epsilon)} \) here. For \( n \neq \pm N \) we find

\[
\begin{align*}
  u_N(n) &= \frac{1}{2^N} \binom{N-1}{\frac{N+n-2}{2}}, & v_N(n) &= \frac{1}{2^N} \binom{N-1}{\frac{N+n}{2}}. 
\end{align*}
\] (A7)

In the case of the boundary points \( n = \pm N \), we get

\[
\begin{align*}
  u_N(+N) &= \frac{1}{2^N}, & u_N(-N) &= 0, \\
  v_N(+N) &= 0, & v_N(-N) &= \frac{1}{2^N}. 
\end{align*}
\] (A8)

For the amplitudes \( w_N(n) \) in (15) that incorporates the projection onto the diagonal coin-state \(|D\rangle\), one can then obtain through (A7) and (A8) accordingly.
Appendix B: Calculation for $P_{\text{no error}}$

Here we explain briefly how $P_{\text{no error}}$ are calculated for the data plotted in Fig. 4. It has been shown by Glancy and Knill in Ref. 27 that when the shift errors in the codewords are sufficiently bounded, it is possible to repeatedly recover the corrupted qubits without Pauli errors. To find the corresponding probabilities, the codewords are projected onto the basis set $\{|s,t\rangle\}$ that consist of the ideal GKP codeword $|0\rangle_L$ in (1) shifted in both $x$ and $p$ [27]

$$|s,t\rangle \equiv \pi^{-\frac{1}{4}} \sum_{m=-\infty}^{\infty} e^{-2im\sqrt{\pi}} |2m\sqrt{\pi} + s\rangle_x,$$

(B1)

where $-\sqrt{\pi} \leq s \leq \sqrt{\pi}$ and $-\sqrt{\pi}/2 \leq t \leq \sqrt{\pi}/2$ due to the periodicity of the $|0\rangle_L$ state. The probability density for a state $|\psi\rangle$ to have shifts $s$ and $t$ relative to the ideal GKP codeword $|0\rangle_L$ is then given by $|\langle s,t|\psi\rangle|^2$. As demonstrated in Ref. [27], when the shift errors of a codeword are bounded within the range $[-\sqrt{\pi}/6, +\sqrt{\pi}/6]$ for both $x$ and $p$ quadratures, it is then ensured that repeated error corrections will be successful. Namely, for the dQW-codeword $|0\rangle_{dQW}$, the probability for repeated error corrections without incurring errors is

$$P_{\text{no error}} \equiv \int_{-\sqrt{\pi}/6}^{\sqrt{\pi}/6} ds \int_{-\sqrt{\pi}/6}^{\sqrt{\pi}/6} dt \ |\langle s,t|0\rangle_{dQW}|^2 .$$

(B2)

Following Eqs. (14)–(16) and (B1), one finds upon invoking (12) for $\Delta x = \sqrt{\pi}$

$$\langle s,t|0\rangle_{dQW} = \left( \frac{e^{+r}}{\pi Z_N} \right)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \sum_{n=-N}^{N} w_N(n) e^{2im\sqrt{\pi}} \exp \left[ -\frac{s + (2m - n)\sqrt{\pi}}{2e^{-2r}} \right] .$$

(B3)

Substituting (B3) into (B2), one can obtain the results displayed in Fig. 4 accordingly.

Appendix C: Circuit for preparing the delayed state $|\tilde{\psi}_{\text{in}}\rangle$

Here we explain how the circuit shown in Fig. 5(b) for generating the delayed state (24) can be derived. To prepare the delayed state, we first displace the input state $|\psi_{\text{in}}\rangle$ of (6) conditionally in the following way

$$\left[ \hat{D}(\Delta \xi) \otimes |R\rangle\langle R| + \hat{I} \otimes |L\rangle\langle L| \right] |\psi_{\text{in}}\rangle = \alpha |1\rangle_r |R\rangle + \beta |0\rangle_r |L\rangle.$$

(C1)

Let us next consider a “biased” coin-toss operation $\hat{B}$, which induces the map

$$|R\rangle \xrightarrow{\hat{B}} |R\rangle \quad \text{and} \quad |L\rangle \xrightarrow{\hat{B}} \hat{C} |L\rangle$$

(C2)

with $\hat{C}$ the original coin-toss operator for the QW. Suppose one single step of QW is enacted for the state (C1) with the biased coin-operator (C2) in place of the original $\hat{C}$. Namely,
here we have the “biased” walk operator
\[
\hat{W}_B \equiv \left[ \hat{D}(+\Delta \xi) \otimes |R\rangle \langle R| + \hat{D}(-\Delta \xi) \otimes |L\rangle \langle L| \right] \left( \hat{I} \otimes \hat{B} \right). \tag{C3}
\]
It is easy to check that the resulting state would then be the delayed state $|\tilde{\psi}_{\text{in}}\rangle$ of (24).
Effectively, here the combined action of the controlled-displacement (C1) and the biased walk-operator (C3) can be reduced as follows
\[
\left[ \hat{D}(+\Delta \xi) \otimes |R\rangle \langle R| + \hat{D}(-\Delta \xi) \otimes |L\rangle \langle L| \right] \left( \hat{I} \otimes \hat{B} \right) \left[ \hat{D}(-\Delta \xi) \otimes |R\rangle \langle R| + \hat{I} \otimes |L\rangle \langle L| \right] \\
= \left[ \hat{I} \otimes |R\rangle \langle R| + \hat{D}(-2\Delta \xi) \otimes |L\rangle \langle L| \right] \left( \hat{I} \otimes \hat{B} \right) \left[ \hat{I} \otimes |R\rangle \langle R| + \hat{D}(+\Delta \xi) \otimes |L\rangle \langle L| \right], \tag{C4}
\]
which has the circuit representation of Fig. 5(b).

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Here the extra factor $\sqrt{2}$ comes from defining the Hermitian part of the mode operator $\hat{a}$ to be $\hat{x}/\sqrt{2}$ (and hence $h = 1$). If one uses instead $\hat{x}$ for the Hermitian part of $\hat{a}$ (thus $h = 1/2$), such factor of $\sqrt{2}$ would then disappear. Here we are following conventions used commonly in the GKP literatures.