Determining quark and lepton mass matrices by a geometrical interpretation

Zhi-Qiang Guo
School of Physics, Peking University, Beijing 100871, China

Bo-Qiang Ma
School of Physics and MOE Key Laboratory of Heavy Ion Physics, Peking University, Beijing 100871, China

Abstract
By designating one eigenvector of the mass matrix, one can reduce the free parameters in the mass matrix effectively. Applying this method to the quark mass matrix and to the lepton mass matrix, we find that this method is consistent with available experimental data. This approach may provide some hints for constructing theoretical models. Especially, in the lepton sector, the Koide’s mass relation is connected to the element of the tribimaximal matrix through Foot’s geometrical interpretation. In the quark sector, we suggest another mass formula and the same procedure also applies.

I. INTRODUCTION
The understanding of quark masses and mixings has posed a major challenge in particle physics for a long time. Recently, the non-zero neutrino masses and their mixings have been confirmed [1], which implies that the mixing also exists in the lepton sector, just like that in the quark sector. A key step to understand the masses and mixings of quarks and leptons is to determine the mass matrices of quarks and leptons. One popular method, suggested by Fritzsch [2], is the texture zero structure. For example, the four texture zero structure can survive current experimental tests [3]. In the lepton sector, other matrices, for example, based on the $\nu_\mu-\nu_\tau$ symmetry and the $A_4$ symmetry, have been suggested [4]. Especially, the nearest-neighbor-interaction form (NNI-form), can be implemented in some grand unified theories [4], and is consistent with the current experimental data from quarks and leptons [4, 5]. Although the progress have been made in these directions, there is still no a commonly granted standard theoretical model for these problems. Therefore, other phenomenological approaches are necessary and worthy to be explored, which may provide some hints for constructing theoretical models. In this paper, we explore a way that can realize Koide’s mass formula [6] through Foot’s geometrical interpretation [7].

The Yukawa sector of the standard model has too many free parameters. In order to make definite predictions, we must make efforts to reduce the redundant parameters effectively. As we have emphasized, many papers have been devoted for this purpose. One common character of these papers is to reduce the redundant parameters by virtue of some symmetry [1, 4]. In this paper, we explore another way, which is different from these approaches. The main ideas are as
follows. First, in the standard model, we can choose the mass matrices to be Hermitian [8]. Then by
designating one eigenvector of a mass matrix, we can reduce the redundant parameters effectively.
In fact, as we will illuminated below, if we make some assumptions and input the values of the
mass parameters, only four free parameters are left. It is well known that the Cabibbo-Kobayashi-
Maskawa (CKM) [9] matrix has four parameters. Therefore, adjusting the values of these left free
parameters, we can fit the experimental data in principle. We find that this way is consistent
with available experimental data. Because in this approach we have the freedom to choose the
eigenvectors, it gives us some advantage to realize extra goals. For example, in the lepton sector,
Koide’s mass formula is connected to the entry 33 of Maki-Nakagawa-Sakata (MNS) matrix [10]
through Foot’s geometrical interpretation, and it is consistent with available experimental data. In
the quark sector, we suggested another mass formula, which can be connected to the entry 33 of
CKM matrix through Foot’s geometrical interpretation.

This paper is composed of five sections. In Sec.II, we introduce the method in detail. In Sec.III
and IV, we apply this method to the quark sector and to the lepton sector respectively. We make
some conclusions in Sec.V.

II. THE METHOD

In the standard model, the mass matrices are complex matrices in general, but we can use the
freedom of right-hand rotation to make them Hermitian [8]. So without loss of generality, we start
our discussion from Hermite mass matrices.

Supposed a general Hermite matrix

\[
\mathbf{M} = \begin{pmatrix}
A & F \exp^i\phi_F & D \exp^i\phi_D \\
F \exp^{-i\phi_F} & B & E \exp^i\phi_E \\
D \exp^{-i\phi_D} & E \exp^{-i\phi_E} & C
\end{pmatrix},
\]

(1)
in which \(A, B, C, \phi_D, \phi_E, \phi_F\) are real and \(D, E, F\) are nonnegative.

The matrix \(\mathbf{M}\) can be written in another way

\[
\mathbf{M} = P^\dagger \mathbf{M} P = P^\dagger \begin{pmatrix}
A & F & D \\
F & B & E \exp^i\alpha \\
D & E \exp^{-i\alpha} & C
\end{pmatrix} P,
\]

(2)
in which \(P = \text{diag}(1, \exp^i\phi_F, \exp^i\phi_D)\), and \(\alpha = \phi_E - \phi_D + \phi_F\). Given that \(\mathbf{M}\) has one eigenvector
\(\mathbf{x}^T = (x_1, x_2 \exp^{i\beta}, x_3 \exp^{i\gamma})^T\) belonging to its eigenvalue \(\lambda\), we have the eigenequation

\[
\mathbf{M} \mathbf{x} = \lambda \mathbf{x},
\]

(3)
in which \(x_1, x_2\) and \(x_3\) are nonnegative real numbers; \(\beta, \gamma\) are real numbers.

After some simplification, it reduces to

\[
\begin{pmatrix}
A - \lambda & F & D \\
F & B - \lambda & E \exp^{i\alpha} \\
D & E \exp^{-i\alpha} & C - \lambda
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \exp^{i(\beta + \phi_F)} \\
x_3 \exp^{i(\gamma + \phi_D)}
\end{pmatrix} = 0.
\]

(4)

We see that Eq. (4) contains complex variables, and it will be difficult to solve them. We notice
that it will be simple in a special case, in which we let \(\beta = -\phi_F\) and \(\gamma = -\phi_D\). This implies that we
designate a real eigenvector to the matrix \(\mathbf{M}\) in Eq. (1). By this choice, all of the equations have real
variables, and they can be solved with little labor. It is obviously that this is only a conventional
choice, which simplifies the equation effectively. Of course, we should consider the possibility that
this choice may be not appropriate, hence the equations have no solutions. However, in Sec.III
and Sec.IV, we will give the numerical results, which imply that our choice is compatible with experimental data. In this paper, we will restrict our discussions on this simple case. Of course, other cases, in which $\beta + \phi_{F} \neq 0$ and $\gamma + \phi_{D} \neq 0$, are not excluded if they are needed. Then we have

$$
\begin{pmatrix}
A - \lambda & F & D \\
F & B - \lambda & E \exp^{i\alpha} \\
D & E \exp^{-i\alpha} & C - \lambda \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} = 0 \iff (M - \lambda I)
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} = 0, \tag{5}
$$

in which $I$ is the identity matrix. In Eq. (5), as $A$, $B$, $C$, $D$, $F$, $\lambda$, $x_1$, $x_2$ and $x_3$ are real numbers, if $E$ and $x_3$ are nonzero, $\alpha$ must equals to $0$ or $\pi$. Therefore, this matrix identity produces three equations. It is well known that two of these equations are independent with each other. We choose the independent equations to be

$$
(A - \lambda)x_1 + Fx_2 + Dx_3 = 0, \tag{6}
$$

$$
Fx_1 + (B - \lambda)x_2 + E \exp^{i\alpha}x_3 = 0, \tag{7}
$$

in which $\alpha = 0$ or $\pi$. In addition to Eq. (6) and Eq. (7), we have three eigenequations

$$
(A - \lambda_1)(B - \lambda_1)(C - \lambda_1) - E^2(A - \lambda_1) - D^2(B - \lambda_1) - F^2(C - \lambda_1) + 2DEFN = 0, \tag{8}
$$

$$
(A - \lambda_2)(B - \lambda_2)(C - \lambda_2) - E^2(A - \lambda_2) - D^2(B - \lambda_2) - F^2(C - \lambda_2) + 2DEFN = 0, \tag{9}
$$

$$
(A - \lambda_3)(B - \lambda_3)(C - \lambda_3) - E^2(A - \lambda_3) - D^2(B - \lambda_3) - F^2(C - \lambda_3) + 2DEFN = 0, \tag{10}
$$

in which $N = \cos \alpha = \pm 1$.

So far, we have five equations, and we have six free parameters, among which only one parameter is still free. We can let $F$ to be the free parameter. Once we fix the value of $F$, all other parameters are fixed. These equations can be solved analytically, but the expressions are too complicated. In order to simplify the expressions of the solutions, we give some analysis in Appendix B. When we apply them to the lepton sector in Sec.III and to the quark sector in Sec.IV, we will give the analytical expressions explicitly in Appendix C and in Appendix D. However, when we adjust the left free parameters to fit the experimental data, the numerical approach is needed. In the text, we display the numerical results. Also, it is possible that these equations have no solutions, if the eigenvalue and the eigenvector are not appropriate. However, in Sec.III and Sec.IV, we will show that for the parameters we choose, the solutions always exist, as we will display explicitly.

The key point of our method is to choose the appropriate eigenvalue and the appropriate eigenvector. In the following application, we will choose the eigenvector and eigenvalue according to physical ground.

With the method we suggested above, if we fix the value of the left free parameter, we can fix the matrix $M$. Now we turn to show how we can use this method to determine the mixing matrix. We take the CKM matrix for example. We let the up-quarks mass matrix to be $\overline{M}_u$, and the down-quarks mass matrix to be $\overline{M}_d$. Like $\overline{M}$, we can write $\overline{M}_u$ and $\overline{M}_d$ as

$$
\overline{M}_u = P^\dagger_u M_u P_u, \quad \overline{M}_d = P^\dagger_d M_d P_d. \tag{11}
$$

We designate $\overline{Z}_T = (y_1, y_2, y_3)^T$ as the eigenvector of $M_u$ belonging to its eigenvalue $\lambda_u$, and $\overline{z}_T = (z_1, z_2, z_3)^T$ as the eigenvector of $M_d$ belonging to its eigenvalue $\lambda_d$. Then we have

$$
(M_u - \lambda_u I) \begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\end{pmatrix} = 0, \quad (M_d - \lambda_d I) \begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{pmatrix} = 0. \tag{12}
$$
According to our analysis above, all the elements of $M_u$ and $M_d$ can be determined except two of them. Suppose that we input the values of these two free parameters, then we can determine $M_u$ and $M_d$. $M_u$ and $M_d$ can be diagonalized by orthogonal transformation

$$V_u^T M_u V_u = \text{diag}(m_u, m_c, m_t), \quad V_d^T M_d V_d = \text{diag}(m_d, m_s, m_b).$$

(13)

By Eq. (11), Eq. (13) can be rewritten as

$$V_u^T P_u M_u P_u^\dagger V_u = \text{diag}(m_u, m_c, m_t), \quad V_d^T P_d M_d P_d^\dagger V_d = \text{diag}(m_d, m_s, m_b).$$

(14)

The CKM matrix can be defined as

$$V_{CKM} = V_u^T P_u P_d^\dagger V_d.$$  

(15)

By our analysis above, $P = P_u P_d^\dagger = \text{diag}(1, \exp(i\xi), \exp(i\eta))$. Therefore, we have four free parameters, i.e., $F_u$ and $F_d$ respectively in $M_u$ and $M_d$, and $\xi$ and $\eta$ in $P$. It is well known that the CKM matrix have four free parameters. Hence in principle it is possible that we can adjust the values of our free parameters to make them consistent with the experimental data. The similar procedure also applies to the lepton sector.

III. THE APPLICATION TO THE LEPTON SECTOR

In recent years, neutrino physics has made great progress. The mixing of neutrinos has been confirmed and the structure of neutrino mixing matrix has been determined to a reasonable precision [1]. It is well known that the neutrino mixing matrix can be expressed with the tribimaximal matrix approximately [11]. It is

$$V_{MNS} = V_{\nu L}^{\dagger} V_{\nu L} = \begin{pmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{2}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$  

(16)

The important step of our method is to choose the eigenvectors appropriately. In order to choose eigenvectors in the lepton sector, we notice some investigations below.

In the lepton sector, Koide [6] ever suggested an accurate formula

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{\sqrt{3\sqrt{m_e} + m_\mu + m_\tau}}.$$  

(17)

Foot [7] gave a geometrical interpretation to it,

$$\cos \theta = \frac{(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})(1, 1, 1)}{(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}) \cdot (1, 1, 1)}.$$  

(18)

where $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ and $(1, 1, 1)$ are interpreted as two vectors, and $\theta$ is the angle between them. If we choose $\theta = \frac{\pi}{4}$, we get the Koide’s mass formula. We have seen that in the tribimaximal matrix, the entry 33 is approximate to be $\frac{1}{\sqrt{2}}$. Therefore there is a natural connection between them. The Koide’s mass formula and Foot’s geometrical interpretation provide hints for choosing the eigenvectors.

We choose $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ and $(1, 1, 1)$ as the eigenvectors we want. It is proper that we choose $(1, 1, 1)$ as the eigenvector of neutrino mass matrix belonging to $m_3$ (the mass of a neutrino)
and $(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ as the eigenvector of the charged lepton matrix belonging to $m_\tau$. This implies that we designate the vectors as follows

$$V_{MNS} = V_{ll}^\dagger V_{eL} = \left( \begin{array}{ccc} * & * & * \\ * & \sqrt{m_\tau} & \sqrt{m_\tau} \\ \sqrt{m_e+m_\mu+m_\tau} & \sqrt{m_e+m_\mu+m_\tau} & \sqrt{m_e+m_\mu+m_\tau} \end{array} \right) P \left( \begin{array}{ccc} * & * & \frac{1}{\sqrt{2}} \\ * & 1 \sqrt{3} & \frac{1}{\sqrt{3}} \\ * & \sqrt{3} & \sqrt{3} \end{array} \right).$$

(19)

where $P = \text{diag}(1, \exp^{i\xi}, \exp^{i\eta})$.

As we described above, we need the values of the mass parameters. For the charged leptons, the masses are known accurately. However, for the neutrinos, only the mass squared differences are measured. The results of global analysis read [1, 12]

$$\Delta m^2_{12} = m_2^2 - m_1^2 = (7.9 \pm 0.4) \times 10^{-5} \text{ eV}^2, \quad (1\sigma)$$

(20)

$$|\Delta m^2_{32}| = |m_3^2 - m_2^2| = (2.4 \pm 0.3) \times 10^{-3} \text{ eV}^2, \quad (1\sigma)$$

(21)

If we assume the normal mass hierarchy, i.e.,

$$m_3 > m_2 > m_1,$$

(22)

then we have

$$\frac{m_2^2}{m_3^2} \approx \frac{m_{31}^2}{m_{33}^2} = 0.033 \pm 0.004,$$

(23)

$$\frac{m_2}{m_3} \geq \sqrt{\frac{m_2^2}{m_{31}^2}} = 0.18 \pm 0.01.$$

(24)

It is obvious that the mixing matrix only depends on the mass ratio [13]. So we only need the ratios of neutrinos masses. It is convenient to normalize the neutrinos masses as follows

$$\lambda_1 = m_1 = 0.1m_3, \quad \lambda_2 = m_2 = 0.2m_3, \quad \lambda_3 = m_3.$$

(25)

Note that these values of mass parameters do not stand for the absolute mass, but just stand for the mass ratio. These ratios are consistent with the experimental data we have displayed above.

Let $x = 1, y = 1$ and $F = 0.279853 m_3$ in the neutrino mass matrix, by Eqs. (8), (9), (10), (13) and (15) in Appendix C and Appendix D, we can get the neutrino mass matrix

$$M_\nu = m_3 \begin{pmatrix} 0.463783 & 0.279853 & 0.256364 \\ 0.279853 & 0.406364 & 0.313783 \\ 0.256364 & 0.313783 & 0.429853 \end{pmatrix}.$$

(26)

For the charged leptons, let

$$\lambda_1 = m_e = 0.511 \text{ MeV}, \quad \lambda_2 = m_\mu = 105.658 \text{ MeV}, \quad \lambda_3 = m_\tau = 1776.97 \text{ MeV},$$

(27)

and

$$x = \frac{\sqrt{m_e}}{\sqrt{m_\tau}}, \quad y = \frac{\sqrt{m_\mu}}{\sqrt{m_\tau}}, \quad F = 55.6898 \text{ MeV},$$

(28)

similarly we can get the charged leptons mass matrix

$$M_l = \begin{pmatrix} 68.2912 & 55.6898 & 15.3959 \\ 55.6898 & 135.509 & 399.315 \\ 15.3959 & 399.315 & 1679.34 \end{pmatrix} \text{ MeV}.$$

(29)
Note that in the course of getting the mass matrices $M_l$ and $M_\nu$ above, we have chosen the eigenvectors of $M_l$ and $M_\nu$ to be real, hence the mass matrices are real matrices according to our analysis in Sec.II. Here we emphasize that this just is a conventional choice, which simplifies the equations effectively.

The matrices that diagonalizes $M_l$ and $M_\nu$ are given as

$$V_l = \begin{pmatrix}
0.599741 & -0.800024 & 0.0164729 \\
-0.779697 & -0.579625 & 0.23687 \\
0.179954 & 0.154905 & 0.971402
\end{pmatrix}, \quad V_\nu = \begin{pmatrix}
0.169807 & -0.798644 & 0.57735 \\
-0.776549 & 0.252625 & 0.57735 \\
0.606743 & 0.546379 & 0.57735
\end{pmatrix}. \quad (30)$$

Obviously the third column of $V_l$ is $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T$, and the third column of $V_\nu$ is $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T$.

The MNS matrix is given as

$$V_{MNS} = V_l^T P_l P_\nu^T V_\nu = V_l^T \begin{pmatrix}
1 & 0 & 0 \\
0 & \exp^{i\xi} & 0 \\
0 & 0 & \exp^{i\eta}
\end{pmatrix} V_\nu \begin{pmatrix}
\exp^{i\alpha_1} & 0 & 0 \\
0 & \exp^{i\alpha_2} & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (31)$$

The phases $\alpha_1$ and $\alpha_2$, known as the Majorana phases, have physical consequences only if neutrinos are Majorana particles. Because there is no clear evidence whether there is CP-violation in the lepton sector, we can not fix the values of $\xi$ and $\eta$. Once we can measure the MNS matrix accurately, we can adjust the free parameters $F$, $\xi$ and $\eta$ to fit the experimental data. If we let $\xi = 0$ and $\eta = 0$, we obtain

$$V_{MNS} = \begin{pmatrix}
0.816499 & -0.577347 & 0 \\
0.408246 & 0.577352 & -0.707107 \\
0.408247 & 0.577352 & 0.707107
\end{pmatrix} \begin{pmatrix}
\exp^{i\alpha_1} & 0 & 0 \\
0 & \exp^{i\alpha_2} & 0 \\
0 & 0 & 1
\end{pmatrix}. \quad (32)$$

The magnitude of the elements are given as

$$|V_{MNS}| = \begin{pmatrix}
0.816499 & 0.577347 & 0 \\
0.408246 & 0.577352 & 0.707107 \\
0.408247 & 0.577352 & 0.707107
\end{pmatrix}, \quad (33)$$

which is very close to the tribimaximal matrix.

**IV. THE APPLICATION TO THE QUARK SECTOR**

The quark mixing matrix has been determined in high precision. The CKM matrix elements can be most precisely determined by a global fit that uses all available measurements and imposes the standard model constraints. The allowed ranges of the magnitudes of all CKM elements are [14]

$$V_{CKM} = V_{UL}^T V_{DL} = \begin{pmatrix}
0.97360 & 0.97407 & 0.2262 & 0.2282 & 0.00387 & 0.00405 \\
0.2261 & 0.2281 & 0.97272 & 0.97320 & 0.04141 & 0.04231 \\
0.0075 & 0.00846 & 0.04083 & 0.04173 & 0.999096 & 0.999134
\end{pmatrix}. \quad (34)$$

Just like that in the lepton sector, we also need two vectors. We find that a numerical relation is well satisfied. It reads

$$\cos \theta = \frac{(\sqrt{m_u}, \sqrt{m_c}, \sqrt{m_t})(m_d, m_s, m_b)}{|(\sqrt{m_u}, \sqrt{m_c}, \sqrt{m_t})|| (m_d, m_s, m_b)|} = 0.999549. \quad (35)$$

The reasons that we choose this formula are displayed in Appendix A.
Hence we might speculate that \( \theta = 0 \), while in the Koide’s mass relation \( \theta = \frac{2}{3} \). Because the elements of \( V_{CKM} \) are given at the scale \( \mu = M_Z \), we use the quark mass given at the scale \( \mu = M_Z \). This well satisfied numerical relation suggests us to designate the vectors as follows

\[
V_{CKM} = V_{uL}^T V_{dL} = \left( \begin{array}{ccc}
* & * & * \\
* & * & * \\
\sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} \\
\sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} \\
\sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} \\
\end{array} \right) \left( \begin{array}{ccc}
* & * & * \\
* & * & * \\
\sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} & \sqrt{m_u + m_c + m_t} \\
\end{array} \right)
\]

where \( P = \text{diag}(1, \exp^{i\xi}, \exp^{i\eta}) \).

It implies that the vector \( (\sqrt{m_u + m_c + m_t}, \sqrt{m_u + m_c + m_t}, \sqrt{m_u + m_c + m_t})^T \) is the eigenvector of \( M_u \) belonging to its eigenvalue \( m_t \), and the vector \( (\sqrt{m_u + m_c + m_t}, \sqrt{m_u + m_c + m_t}, \sqrt{m_u + m_c + m_t})^T \) is the eigenvector of \( M_d \) belonging to its eigenvalue \( m_b \). As we have emphasized before, if we fix the value of the free parameter, other elements of the mass matrix are determined. The equations can be solved analytically. We display the result in Appendix C.

For the up-quark sector, let

\[
\lambda_1 = m_u(M_Z) = 2.33 \text{ MeV}, \quad \lambda_2 = m_c(M_Z) = 677 \text{ MeV}, \quad \lambda_3 = m_t(M_Z) = 18100 \text{ MeV}, \tag{37}
\]

and

\[
x = \frac{\sqrt{m_u}}{\sqrt{m_t}}, \quad y = \frac{\sqrt{m_c}}{\sqrt{m_t}}, \quad F = 133.18 \text{ MeV}, \tag{38}
\]

we can get the up-quark mass matrix by Eqs. \((38), (39), (40)\), \((41)\) and \((42)\) in Appendix C.

\[
M_u = \begin{pmatrix}
17.9519 & 133.18 & 641.198 \\
133.18 & 1335.65 & 10987.5 \\
641.198 & 10987.5 & 180326
\end{pmatrix} \text{ MeV.} \tag{39}
\]

For the down-quark sector, let

\[
\lambda_1 = m_d(M_Z) = 4.69 \text{ MeV}, \quad \lambda_2 = m_s(M_Z) = 93.4 \text{ MeV}, \quad \lambda_3 = m_b(M_Z) = 3000 \text{ MeV}, \tag{40}
\]

and

\[
x = \frac{m}{m_b}, \quad y = \frac{m_s}{m_b}, \quad F = 7.99 \text{ MeV}, \tag{41}
\]

similarly we can get the down-quark mass matrix

\[
M_d = \begin{pmatrix}
5.39717 & 7.99 & 4.43281 \\
7.99 & 95.5146 & 90.4138 \\
4.43281 & 90.4138 & 2997.18
\end{pmatrix} \text{ MeV.} \tag{42}
\]

Again, in the course of getting the mass matrices \( M_u \) and \( M_d \), for simplicity, we choose the eigenvectors to be real, hence the mass matrices are real matrices by the analysis in Sec.II. The matrices that diagonalize \( M_u \) and \( M_d \) are given as

\[
V_u = \begin{pmatrix}
0.990087 & -0.140407 & 0.00358117 \\
-0.140363 & -0.898217 & 0.0610438 \\
0.00503202 & 0.0609414 & 0.998129
\end{pmatrix}, \quad V_d = \begin{pmatrix}
0.996046 & -0.0888212 & 0.00156257 \\
-0.0888268 & -0.995561 & 0.0311182 \\
0.00120832 & 0.0341134 & 0.999514
\end{pmatrix}. \tag{43}
\]
and the invariant measure of CP violation is calculated as
\[ \alpha = \phi_2 = \arg[-V_{ud} V_{ub}^{*} V_{cd} V_{cb}^{*}] = 101.59^\circ, \]
while the experimental data \cite{14} are given as
\[ \alpha = (99^{+13}_{-8})^\circ, \quad \beta = (21.70^{+1.29}_{-1.24})^\circ, \quad \gamma = (63^{+15}_{-13})^\circ, \quad J = (3.08^{+0.16}_{-0.18}) \times 10^{-5}. \]
They are consistent with each other.

V. CONCLUSION

We have illustrated the method in Sec.II, and apply it to the lepton sector in Sec.III and to the quark sector in Sec.IV respectively. The character of this method is to designate the eigenvector and the eigenvalue for the mass matrix appropriately. In the lepton sector, we use the Koide’s formula, and in the quark sector, we use a similar formula that is well satisfied. Now we give some comments about the mass formula we used.

(1) In the lepton sector, the Koide’s mass formula,
\[ \frac{1}{2} = \frac{\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}}{3\sqrt{m_e + m_\mu + m_\tau}}, \]
is satisfied in high precision. It is energy scale insensitive \cite{15}, and its other characters were also discussed \cite{10}. Several explanations that can realize this formula have been given \cite{17}. Among these explanations, Foot’s geometrical interpretation seems fascinating phenomenologically. However, there is still no a theoretical model that can realize it. It seems that our method can implement this interpretation. If there is no CP-violation and the MNS matrix is tribimaximal, the righthand of the

Obviously the third column of \( V_u \) is
\[ \left( \frac{\sqrt{m_e}}{m_e + m_\mu + m_\tau}, \frac{\sqrt{m_\mu}}{m_e + m_\mu + m_\tau}, \frac{\sqrt{m_\tau}}{m_e + m_\mu + m_\tau} \right)^T, \]
and the third column of \( V_d \) is
\[ \left( \frac{m_e}{m_e + m_\mu + m_\tau}, \frac{m_\mu}{m_e + m_\mu + m_\tau}, \frac{m_\tau}{m_e + m_\mu + m_\tau} \right)^T. \]
The CKM matrix is given as
\[ V_{CKM} = V_u^T P_u P_d^T V_d = V_u^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(i\xi) & 0 \\ 0 & 0 & \exp(i\eta) \end{pmatrix} V_d. \] (44)

Given \( \xi = 0.76\pi + \frac{\pi}{4\sqrt{3}} \) and \( \eta = 0.76\pi \), the CKM matrix equals to
\[ V_{CKM} = \begin{pmatrix} 0.973722 + 0.000729014i & -0.227538 + 0.00515366i & 0.00224113 + 0.00318905i \\ -0.227538 + 0.00515366i & 0.971078 + 0.0584958i & -0.0139226 + 0.0399092i \\ 0.00810099 + 0.000510366i & 0.0376987 + 0.0177396i & -0.729142 + 0.683045i \end{pmatrix}. \] (45)
The magnitude of the elements are
\[ | V_{CKM} | = \begin{pmatrix} 0.973722 & 0.227707 & 0.00389778 \\ 0.227596 & 0.972838 & 0.042268 \\ 0.00811705 & 0.041664 & 0.999099 \end{pmatrix}. \] (46)
The quantities of rephasing invariance are calculated as
\[ \alpha = \phi_2 = \arg[-V_{ud} V_{ub}^{*} V_{cd} V_{cb}^{*}] = 101.59^\circ, \]
\[ \beta = \phi_1 = \arg[-V_{ud} V_{ub}^{*} V_{cd} V_{cb}^{*}] = 22.74^\circ, \]
\[ \gamma = \phi_3 = \arg[-V_{ud} V_{ub}^{*} V_{cd} V_{cb}^{*}] = 55.67^\circ, \]
and the invariant measure of CP violation is calculated as
\[ J = -\text{Im}(V_{ud} V_{ub}^{*} V_{cd} V_{cb}^{*}) = 3.01513 \times 10^{-5}, \] (48)
mass formula is connected to the element of the MNS matrix by Foot’s geometrical interpretation. This is an approach that can lead to the Koide’s mass formula.

(2) In the quark sector, the mass formula like Koide’s is unsuccessful [18]. However, because the CKM matrix is very different from the MNS matrix, such a mass formula is useless for us. Alternately, we find another well satisfied mass formula,

$$\cos\theta = \frac{\sqrt{m_u}, \sqrt{m_c}, \sqrt{m_t}}{|(\sqrt{m_u}, \sqrt{m_c}, \sqrt{m_t})|} \cdot (m_d, m_s, m_b) = 0.999549.$$  \hspace{1cm} (51)

This mass formula is connected to the element of the CKM matrix.

(3) There are two reasons that we choose the mass formula as the vectors. First, the vectors are expressed by the mass parameters, so we do not need to introduce extra parameters. Obviously that this is an economic choice. Second, there exists such mass formula. They are excellent and are well satisfied in high precision, like the Koide’s mass formula, but we can not realize them in a concise and convincing way. Our method provides an approach that can realize them, nevertheless in the special situation if there is no CP-violation. Of course other choices of the vectors are also permitted.

Finally we give some comments about the texture zero structure and our method. The differences between our method and the texture zero structure are: in our present approach, at first we choose one eigenvector of the mass matrix, and then we can determine other elements of the mass matrix; in the texture zero structure, some elements of the mass matrix are supposed to zero, which equals to designate the eigenvectors, but we do not know the eigenvector at first. Therefore, in our approach, we have the freedom to choose the eigenvectors to satisfy other request. For example, the Koide’s mass formula can be realized in our approach through Foot’s geometrical interpretation.

As the texture zero structure, our approach is also consistent with current experimental data, and this approach has the merit that it can realize some well satisfied mass formula, for example, the Koide’s mass formula and the mass formula suggested by us. However, just as many texture zero structures, our approach is purely phenomenological, and there is still no a theoretical model to realize it. Therefore it is worthy to investigate whether our approach can be realized in some theoretical models. If this is true, it will provide a new theoretical and phenomenological approach to deal with the mass and mixing problems. Besides, we point out that in our paper we just restrict our discussions in a simple case, in which we choose the eigenvectors to be real, hence we get the real mass matrices. This is just a conventional choice that simplifies the equations. Other cases, in which the eigenvectors are complex, are permitted, and they should be considered if they are needed by some underlying theories.

Acknowledgement This work is partially supported by National Natural Science Foundation of China (Nos. 10421503, 10575003, 10528510), by the Key Grant Project of Chinese Ministry of Education (No. 305001), and by the Research Fund for the Doctoral Program of Higher Education (China).

APPENDIX A

Table: Quark masses at the Z mass scale in the standard model [19]

| Quark | Mass (MeV) |
|-------|------------|
| $m_u$ | $2.33^{+0.42}_{-0.45}$ |
| $m_c$ | 677 $^{+56}_{-61}$ |
| $m_t$ | $181^{+13}_{-13}$ |
| $m_d$ | $4.69^{+0.60}_{-0.66}$ |
| $m_s$ | 93.4 $^{+11.9}_{-13.0}$ |
| $m_b$ | $3.00^{+0.11}_{-0.11}$ |

With the values of the quark masses above,

$$m_u(M_Z) = 2.33 \text{ MeV}, \ m_c(M_Z) = 677 \text{ MeV}, \ m_t(M_Z) = 181 \text{ GeV},$$
As we have shown in Sec.IV, the mass formula is connected to the element of the CKM matrix by, if they are consistent with the experimental data. Our choice is just a convenient one. Of course other choices are permitted we must change the mass parameters that we have used. But the mass formula we have chosen does

\[
\cos \theta = \frac{(m_u, m_c, m_t)(\sqrt{m_d}, \sqrt{m_s}, \sqrt{m_b})}{| (m_u, m_c, m_t) || (m_d, m_s, m_b) |} = 0.999549,
\]

\[
\cos \theta = 0.999549 \text{ is very close to 1.}
\]

Note that we do not choose the mass formula below,

\[
\cos \theta = \frac{(m_u, m_c, m_t)(\sqrt{m_d}, \sqrt{m_s}, \sqrt{m_b})}{| (m_u, m_c, m_t) || (m_d, m_s, m_b) |} = 0.984685,
\]

\[
\cos \theta = \frac{(m_u, m_c, m_t)(\sqrt{m_d}, \sqrt{m_s}, \sqrt{m_b})}{| (m_u, m_c, m_t) || (m_d, m_s, m_b) |} = 0.999624,
\]

\[
\cos \theta = \frac{(m_u, m_c, m_t)(\sqrt{m_d}, \sqrt{m_s}, \sqrt{m_b})}{| (m_u, m_c, m_t) || (m_d, m_s, m_b) |} = 0.992936.
\]

As we have shown in Sec.IV, the mass formula is connected to the element of the CKM matrix by,

\[
V_{33} = \frac{m_u \sqrt{m_d} + \exp i \xi \sqrt{m_s} + \exp i \eta \sqrt{m_b}}{\sqrt{m_u^2 + m_s^2 + m_b^2 + m_d + m_s + m_b}},
\]

\[
V_{33} = \frac{m_u m_d + \exp i \xi m_s m_s + \exp i \eta m_b m_b}{\sqrt{m_u^2 + m_s^2 + m_b^2 + m_d + m_s + m_b}},
\]

\[
V_{33} = \frac{\sqrt{m_u \sqrt{m_d} + \exp i \xi \sqrt{m_s} + \exp i \eta \sqrt{m_b}}}{\sqrt{m_u + m_s + m_b + m_d + m_s + m_b}},
\]

and

\[
0.98385 \leq \frac{m_u \sqrt{m_d} + \exp i \xi \sqrt{m_s} + \exp i \eta \sqrt{m_b}}{\sqrt{m_u^2 + m_s^2 + m_b^2 + m_d + m_s + m_b}} \leq 0.984685,
\]

\[
0.999391 \leq \frac{m_u m_d + \exp i \xi m_s m_s + \exp i \eta m_b m_b}{\sqrt{m_u^2 + m_s^2 + m_b^2 + m_d + m_s + m_b}} \leq 0.999624,
\]

\[
0.971465 \leq \frac{\sqrt{m_u \sqrt{m_d} + \exp i \xi \sqrt{m_s} + \exp i \eta \sqrt{m_b}}}{\sqrt{m_u + m_s + m_b + m_d + m_s + m_b}} \leq 0.992936,
\]

but the experimental value of \( |V_{33}| \) is

\[
0.999096 < |V_{33}| < 0.999134.
\]

They are not consistent. Therefore, in order to make them consistent with the experimental data, we must change the mass parameters that we have used. But the mass formula we have chosen does not have this problem. Our choice is just a convenient one. Of course other choices are permitted if they are consistent with the experimental data.

APPENDIX B

As we have emphasized in the text, the analytical expressions of the solutions will be complicated. In this appendix, we give some analysis that will simplify the expressions effectively.
In order to express the other elements of the mass matrix with the quarks masses and the free parameter $F$, we have to solve the equations displayed below

\[(A - \lambda)x_1 + Fx_2 + Dx_3 = 0, \quad (1)\]
\[Fx_1 + (B - \lambda)x_2 + E \exp^{\alpha x} x_3 = 0, \quad (2)\]
\[(A - \lambda_1)(B - \lambda_1)(C - \lambda_1) - E^2(A - \lambda_1) - D^2(B - \lambda_1) - F^2(C - \lambda_1) + 2DEFN = 0, \quad (3)\]
\[(A - \lambda_2)(B - \lambda_2)(C - \lambda_2) - E^2(A - \lambda_2) - D^2(B - \lambda_2) - F^2(C - \lambda_2) + 2DEFN = 0, \quad (4)\]
\[(A - \lambda_3)(B - \lambda_3)(C - \lambda_3) - E^2(A - \lambda_3) - D^2(B - \lambda_3) - F^2(C - \lambda_3) + 2DEFN = 0. \quad (5)\]

It is well known that

\[\text{Trace}(M) = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow A + B + C = s, \quad (6)\]

in which $s = \lambda_1 + \lambda_2 + \lambda_3$.

If $\lambda_1 \neq \lambda_2$, by Eqs. (3), (4) and (5), we obtain

\[AB + BC + AC = r + (D^2 + E^2 + F^2), \quad (7)\]

in which $r = \lambda_1\lambda_3 + \lambda_1\lambda_2 + \lambda_2\lambda_3$.

By Eqs. (1) and (2), we can express $D$ and $E$ in terms of $A$, $B$ and $F$

\[D = -(A - \lambda)x - Fy, \quad (8)\]
\[E = [-Fx - (B - \lambda)y] \exp^{-\alpha}, \quad (9)\]

in which we have let $x = \frac{2s}{x_3}$ and $y = \frac{2s}{x_3}$, supposing that $x_3 \neq 0$.

We obtain five new equations

\[(A - \lambda_3)(B - \lambda_3)(C - \lambda_3) - E^2(A - \lambda_3) - D^2(B - \lambda_3) - F^2(C - \lambda_3) + 2DEFN = 0, \quad (5)\]
\[A + B + C = s, \quad (6)\]
\[AB + BC + AC = r + (D^2 + E^2 + F^2), \quad (7)\]
\[D = -(A - \lambda)x - Fy, \quad (8)\]
\[E = [-Fx - (B - \lambda)y] \exp^{-\alpha}. \quad (9)\]

Let $N = \cos \alpha = \pm 1$, by these equations, we can express $A$, $B$, $C$, $D$ and $E$ in terms of $F$.

In the following we will deduce another formula, which will simplify the expressions effectively.

By Eq. (10), we obtain

\[C = s - A - B. \quad (10)\]

We have argued that $N = \cos \alpha = \pm 1$ in Sec.II. So by Eq. (1), we have

\[E = \pm [-Fx - (B - \lambda)y]. \quad (11)\]

In Eq. (7), we express $C$, $D$ and $E$ in terms of $A$, $B$ and $F$ by Eqs. (3), (10) and (11). After some simplification, we obtain

\[aB^2 + bB + c = 0, \quad (12)\]

in which
\[a = 1 + y^2, \quad b = A - s - 2\lambda y^2 + 2Fxy, \quad c = (\lambda y - Fx)^2 + [(A - \lambda)x + Fy]^2 + F^2 + r - A(s - A). \]
Then $B$ can be solved as

$$B = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (13)$$

Hence if we know the values of $A$ and $F$, the value of $B$ is determined by the Eq. (13).

**APPENDIX C**

In this appendix, we give the analytical expressions of the solutions for the quark sector.

In the quark sector, let $\lambda = \lambda_3$ and $\alpha = 0$. According to Eqs. (6), (8) and (9), we can express $B$, $D$ and $E$ with $A$, $C$ and $F$ linearly. The Eqs. (5) and (7) are left invariant. We express $B$, $D$ and $E$ in terms of $A$, $C$ and $F$ in Eqs. (5) and (7). Then in Eqs. (5) and (7), only $A$, $C$ and $F$ are present. Therefore by Eqs. (5) and (7), we can solve $A$ and $C$ in terms of $F$. The solution is displayed as follows

$$A = -b' - \sqrt{b'^2 - 4a'c'}, \quad (14)$$

in which $b'$, $a'$, $c'$ are expressed as the same as that in Appendix C. Eqs. (8), (9), (10) and (13) in Appendix C apply similarly.

**APPENDIX D**

In this appendix, we give the analytical expressions of the solutions for the lepton sector.

In the lepton sector, the analysis in Appendix B and C applies similarly. Let $\lambda = \lambda_3$ and $\alpha = 0$. We find that the solutions we need can be analytically expressed as follows

$$A = -b' + \sqrt{b'^2 - 4a'c'}, \quad (15)$$

in which $b'$, $a'$, $c'$ are expressed as the same as that in Appendix C. Eqs. (8), (9), (10) and (13) in Appendix C apply similarly.

**References**

[1] A review with references see, i.e., R. N. Mohapatra, A. Y. Sminov, arXiv: hep-ph/0603118
[2] H. Fritzsch, Phys. Lett. B 73 (1978) 317; Nucl. Phys. B 155 (1979) 182.

[3] H. Fritzsch, Z. Z. Xing, Phys. Lett. B 555 (2003) 63.

[4] K.S. Babu, J. Kubo, Phys. Rev. D 71 (2005) 056006.

[5] Z.-Z. Xing, Phys. Lett. B 550 (2002) 178; M. Fukugita, M. Tanimoto, and T. Yanagida, Phys. Lett. B 562 (2003) 273.

[6] Y. Koide, Lett. Nuovo Cimento. 34 (1982) 201; Phys. Lett. B 120 (1983) 161; Y. Koide, Phys. Rev. D 28 (1983) 252.

[7] R. Foot, hep-ph/9402242.

[8] G. C. Branco, L. Lavoura, Phys. Rev. D 39 (1989) 3443.

[9] N. Cabibbo, Phys. Rev. Lett 10 (1963) 531; M. Kobayashi, T. Maskawa, Prog. Theor. Phys. 49 (1973) 652.

[10] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. 28 (1962) 870.

[11] P. F. Harrison, D. H. Perkins, W. G. Scott, Phys. Lett. B 530 (2002) 167.

[12] A. Strumia, F. Vissani, Nucl. Phys. B 726 (2005) 294; G. L. Fogli et al., hep-ph/0506083.

[13] For example, see, C.Jarlskog, Phys. Rev. Lett. 55 (1985) 1039.

[14] W.-M. Yao, et al. (Particle Data Group), J. Phys. G 33 (2006) 138.

[15] N. Li, B.-Q. Ma, Phys. Rev. D 73 (2006) 013009.

[16] J.-M. Gerard, F. Goffinet, M. Herquet, Phys. Lett. B 633 (2006) 563.

[17] For a review, see: Y. Koide, hep-ph/0508301 Y. Koide, hep-ph/0506247.

[18] S. Esposito and P. Santorelli, Mod. Phys. Lett. A 10 (1995) 3077; N. Li and B.-Q. Ma, Phys. Lett. B 609 (2005) 309; A. Rivero and A. Gsponer, hep-ph/0505220.

[19] H. Fusaoka, Y. Koide, Phys. Rev. D 57 (1998) 3986.