PARAMETER ESTIMATION FOR AN ORNSTEIN-UHLENBECK PROCESS DRIVEN BY A GENERAL GAUSSIAN NOISE WITH HURST PARAMETER $H \in (0, \frac{1}{2})$

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Abstract. In Chen and Zhou 2021, they consider an inference problem for an Ornstein-Uhlenbeck process driven by a general one-dimensional centered Gaussian process $(G_t)_{t \geq 0}$. The second order mixed partial derivative of the covariance function $R(t, s) = \mathbb{E}[G_t G_s]$ can be decomposed into two parts, one of which coincides with that of fractional Brownian motion and the other is bounded by $\lvert ts \rvert^{H-1}$ with $H \in (\frac{1}{2}, 1)$, up to a constant factor. In this paper, we investigate the same problem but with the assumption of $H \in (0, \frac{1}{2})$. It is well known that there is a significant difference between the Hilbert space associated with the fractional Gaussian processes in the case of $H \in (\frac{1}{2}, 1)$ and that of $H \in (0, \frac{1}{2})$. The starting point of this paper is a new relationship between the inner product of $\mathcal{H}_1$ associated with the Gaussian process $(G_t)_{t \geq 0}$ and that of the Hilbert space $\mathcal{H}_1$ associated with the fractional Brownian motion $(B^H_t)_{t \geq 0}$. Then we prove the strong consistency with $H \in (0, \frac{1}{2})$, and the asymptotic normality and the Berry-Esséen bounds with $H \in (0, \frac{3}{8})$ for both the least squares estimator and the moment estimator of the drift parameter constructed from the continuous observations. A good many inequality estimates are involved in and we also make use of the estimation of the inner product based on the results of $\mathcal{H}_1$ in Hu, Nualart and Zhou 2019.

Keywords: Fourth Moment theorems; Ornstein-Uhlenbeck process; fractional Gaussian process; Berry-Esséen bounds; Malliavin calculus.

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1. Introduction

In [1], the statistical inference problem of the unknown parameter $\theta$ is considered for the Ornstein-Uhlenbeck process defined by the following stochastic differential equation (SDE)

$$dX_t = -\theta X_t dt + \sigma dG_t, \quad t \in [0, T], \quad T > 0$$ (1.1)

where $X_0 = 0$ and $(G_t)_{t \geq 0}$ is a general one-dimensional centered Gaussian process satisfying the following Hypothesis 1.1.

HYPOTHESIS 1.1. For $H \in (\frac{1}{2}, 1)$, the covariance function $R(t, s) = \mathbb{E}[G_t G_s]$ for any $t \neq s \in [0, \infty)$ satisfies

$$\frac{\partial^2}{\partial t \partial s} R(t, s) = H(2H - 1) |t - s|^{2H-2} + \Psi(t, s),$$ (1.2)

with

$$|\Psi(t, s)| \leq C_H |ts|^{H-1},$$ (1.3)

where $C_H$ is a constant.
where the constant $C'_H \geq 0$ do not depend on $T$. Moreover, for any $t \geq 0$, $R(0, t) = 0$.

Without loss of generality, $\sigma = 1$ is also assumed. Suppose that only one trajectory $(X_t, t \in [0, T])$ can be obtained. When $\theta > 0$, i.e., in the ergodic case, the least squares estimator (LSE) and the moment estimator (ME) are respectively constructed from the continuous observations as follows:

$$\hat{\theta}_T = -\frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \theta - \frac{\int_0^T X_t dG_t}{\int_0^T X_t^2 dt},$$  \hspace{1cm} (1.4)

$$\tilde{\theta}_T = \left( \frac{1}{\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}},$$  \hspace{1cm} (1.5)

where the integral with respect to $G$ is interpreted in the Skorohod sense (or say a divergence-type integral). Then the strong consistency, the asymptotic normality, and the Berry-Esséen bounds are obtained in [1].

In fact, the statistical inference problem about the parameter $\theta$ has been intensively studied over the past decades (see [2], [3] and the references therein) when the Gaussian process is Brownian motion. In the fractional Brownian motion case, the consistency property for the maximum likelihood estimation (MLE) method was obtained in [4], [5], and the central limit theorem was proved in [6], [7], and the LSE and the ME and their asymptotic behavior were studied in [8], [9]. The ME in the case of general stationary-increment Gaussian processes was considered in [10]. The MLE in the case of sub-fractional Brownian motion case was investigated in [11] and, recently, the LSE in the case of mixed sub-fractional Brownian motion was studied in [12]. We would like to mention some work for the non-ergodic case as well, i.e., $\theta < 0$. MLE was studied in [13], [14] and the limiting distribution is Cauchy for the Brownian motion case, and the LSE in the case of fractional Brownian motion and other Gaussian processes was considered in [15], [16], [17], [18] and the references therein. We also mention some work for the Ornstein-Uhlenbeck process driven by the non-Gaussian Hermite processes with periodic mean in [19], [20] and the references therein.

Clearly, the following four types of fractional Gaussian processes satisfy Hypothesis 1.1. The covariance function of the fractional Brownian motion $\{B_t^H, t \geq 0\}$ is

$$R^B(s, t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right).$$  \hspace{1cm} (1.6)

The sub-fractional Brownian motion $\{S_t^H, t \geq 0\}$ with parameter $H \in (0, 1)$ has the covariance function

$$R(t, s) = s^{2H} + t^{2H} - \frac{1}{2} \left( (s + t)^{2H} + |t - s|^{2H} \right).$$

The bi-fractional Brownian motion $\{B_t^{H,K}, t \geq 0\}$ with parameters $H, K \in (0, 1)$ has the covariance function

$$R(t, s) = \frac{1}{2K} \left( (s^{2H} + t^{2H})^K - |t - s|^{2HK} \right).$$
The generalized sub-fractional Brownian motion \( \{ S_{t,K}^{H,K}, t \geq 0 \} \) with parameters \( H \in (0,1) \), \( K \in [1,2] \) and \( HK \in (0,1) \) has the covariance function
\[
R(t, s) = (s^{2H} + t^{2H})^K - \frac{1}{2}[(t + s)^{2HK} + |t - s|^{2HK}]
\]
The Hurst parameters should be understood as \( HK \) for both of the last two Gaussian processes.

When \( \theta > 0 \), i.e., in the ergodic case, for the above LSE (1.4) and ME (1.5), most of the results in literatures are restricted to the Hurst parameter \( H \in (\frac{1}{2}, 1) \) except [9] and [21] as far as we know. This is mainly due to the significant difference between the Hilbert space associated with the fractional Gaussian processes and the representation of their inner products in the case of \( H \in (\frac{1}{2}, 1) \) and that in the case of \( H \in (0, \frac{1}{2}) \). We would like to point out a remarkable fact that the monotonicity of the norm may not hold in the case of \( H \in (0, \frac{1}{2}) \), please see Remark 2.3 in Section 2 or refer to [22] for details.

However, it is clearly that for all the above four types fractional Gaussian processes, the identity (1.1) and the inequality (1.3) are valid for both \( H \in (0, \frac{1}{2}) \) and \( H \in (\frac{1}{2}, 1) \). Then the question naturally arises if the asymptotic properties of LSE (1.4) and ME (1.5) are also valid for \( H \in (0, \frac{1}{2}) \). We have pointed out in the last paragraph that this problem have been solved in [9] and [21] in the case of fractional Brownian motion. In this paper, we will partly give an affirmative answer to this question in the case of general fractional Gaussian processes. For simplicity, we would like to discuss only the ergodic case, i.e., \( \theta > 0 \).

The starting point of the present paper is to establish the key inequalities (2.4) and (2.5) which relate the inner product of the Hilbert space associated with the general fractional Gaussian process to that associated with the fractional Brownian motion. To this aim, we need improve Hypothesis 1.1 into the following form [23]:

**HYPOTHESIS 1.2.** For \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \), the covariance function \( R(t, s) = E[G_tG_s] \) satisfies that
1. for any \( s \geq 0 \), \( R(0, s) = 0 \).
2. for any fixed \( s \in (0, T) \), \( R(t, s) \) is a continuous function on \([0, T]\) which is differentiable with respect to \( t \) in \((0, s) \cup (s, T)\), such that \( \frac{\partial}{\partial t} R(t, s) \) is absolutely integrable.
3. for any fixed \( t \in (0, T) \), the difference
\[
\frac{\partial R(t, s)}{\partial t} - \frac{\partial R^B(t, s)}{\partial t}
\]
is a continuous function on \([0, T]\) which is differentiable with respect to \( s \) in \((0, T)\) such that \( \Psi(t, s) \), the partial derivative with respect to \( s \) of the difference, satisfies
\[
|\Psi(t, s)| \leq C_H |ts|^{H-1},
\]
where the constant \( C_H \geq 0 \) do not dependent on \( T \), and \( R^B(t, s) \) is the covariance function of the fractional Brownian motion given in (1.6).

It is easy to see that fractional Brownian motion, sub-fractional Brownian motion, bi-fractional Brownian motion, generalized sub-fractional Brownian motion and some other Gaussian processes are special examples to satisfy the Hypothesis 1.2. For example, the mixed Gaussian process [12]
which is a linear combination of independent centered Gaussian processes satisfies Hypothesis 1.2 as long as each Gaussian process satisfies it. In this case, the mixed Gaussian process fails to be self-similar.

In this paper, we will prove the strong consistency and the central limit theorems for the two estimators. The Berry-Esséen bounds will be also obtained. These results are stated in the following theorems.

**Theorem 1.3.** Let the least squares estimator $\hat{\theta}$ and the moment estimator $\tilde{\theta}_T$ be as (1.4) and (1.5) respectively. When Hypothesis 1.2 is satisfied, both $\hat{\theta}$ and $\tilde{\theta}_T$ are strongly consistent, i.e.,

$$\lim_{T \to \infty} \hat{\theta}_T = \theta, \quad \lim_{T \to \infty} \tilde{\theta}_T = \theta, \quad a.s..$$

A famous theorem of Pickands, see e.g. [24], states that if the covariance $\text{Cov}(\xi_0, \xi_t)$ of a stationary Gaussian process $\xi_t$ with unit variance satisfies

$$\text{Cov}(\xi_0, \xi_t) = 1 - c|t|^\alpha + o(|t|^\alpha)$$

as $t \to 0$ for some $c$, $0 < c < \infty$ and some $\alpha$, $0 < \alpha \leq 2$, then, for any $\gamma > 0$, $\xi_t$ converges almost surely to zero as $t \to \infty$. For the case of the fractional Brownian motion noise, the strong consistency is obtained from applying that theorem to the stationary Gaussian process $\xi_t$ with unit variance satisfies

$$\xi_t := \int_{-\infty}^{t} e^{-\theta(t-s)} dW_s$$

in [8, 9]. For our case, $\xi_t$ is not a stationary Gaussian process any more and thus we can not apply the above theorem of Pickands. We get around this difficulty by means of combining three techniques together: the hypercontractivity of multiple Wiener-Itô integrals, Kolmogorov’s continuity theorem and the Garsia-Rodemich-Rumsey inequality. Then we obtain the similar stochastic integrals (see Propositions 3.3 and 3.5 below) converge almost surely to zero as $t$ tends to infinity. This method has been used to replace the theorem of Pickands in the literatures such as [1, 12, 25, 26, 27]. We would also like to mention [28] where they use the moment generation function method to obtain a similar result under some assumptions different to ours. Those two methods are similar since for multiple Wiener-Itô integrals, the hypercontractivity can be used to obtain the exponential integrability, see e.g. [29]. Please refer to [29, 30] for the Garsia-Rodemich-Rumsey inequality.

**Theorem 1.4.** Assume $H \in (0, \frac{1}{2})$ and Hypothesis 1.2 is satisfied. Then, both $\sqrt{T}(\hat{\theta}_T - \theta)$ and $\sqrt{T}(\tilde{\theta}_T - \theta)$ are asymptotically normal as $T \to \infty$. Namely,

$$\sqrt{T}(\hat{\theta}_T - \theta) \xrightarrow{f.a.} \mathcal{N}(0, \theta \sigma_H^2),$$

$$\sqrt{T}(\tilde{\theta}_T - \theta) \xrightarrow{f.a.} \mathcal{N}(0, \theta \sigma_H^2/4H^2),$$

where

$$\sigma_H^2 = (4H - 1) + \frac{2\Gamma(2 - 4H)\Gamma(4H)}{\Gamma(2H)\Gamma(1 - 2H)}.$$
Remark 1.5. Because of the calculation complication of the inequality (4.1) in the case of $H \in (0, \frac{1}{2})$ (see Proposition 4.1), we only obtain the CLT for $H \in (0, \frac{3}{8})$ in the present paper. A more sharp estimate of the inequality (4.1) is still awaiting. We will investigate this topic in other works.

Theorem 1.6. Let $\Phi(z)$ be the standard normal distribution function. Assume $H \in (0, \frac{3}{8})$ and Hypothesis 1.2 is satisfied. Then, when $H \neq \frac{1}{4}$, there exists a constant $C_{\theta,H} > 0$ such that when $T$ is large enough,

$$\sup_{z \in \mathbb{R}} \left| P\left( \frac{T}{\theta \sigma^2_H} (\hat{\theta}_T - \theta) \leq z \right) - \Phi(z) \right| \leq \frac{C_{\theta,H}}{T^3}, \quad (1.10)$$

and

$$\sup_{z \in \mathbb{R}} \left| P\left( \frac{4H^2 T}{\theta \sigma^2_H} (\hat{\theta}_T - \theta) \leq z \right) - \Phi(z) \right| \leq \frac{C_{\theta,H}}{T^3}, \quad (1.11)$$

where

$$\delta = \left\{ \begin{array}{ll} \frac{1}{4}, & \text{when } H \in (0, \frac{1}{2}), \\ \frac{3}{2} - 4H, & \text{when } H \in (\frac{1}{2}, \frac{3}{8}). \end{array} \right. \quad (1.12)$$

and when $H = \frac{1}{4}$, the upper bound can be replaced by $\frac{\log T}{\sqrt{T}}$.

Remark 1.7. For the moment estimator $\hat{\theta}_T$, when the driven noise $G_t$ degenerates to the fractional Brownian $B_t^H$ with $H \in (0, \frac{1}{4})$, the Berry-Essèen upper bound in (1.11) is shown to be also $\frac{1}{\sqrt{T}}$, see Proposition 4.1 of [10]. Thus, it is reasonable to conjecture that when $H \in (\frac{1}{4}, \frac{1}{2})$, a better bound should also be $\frac{1}{\sqrt{T}}$ for the two estimators. This requires us to find two more sharp estimates than those given by Lemma 3.11 of [21] and Proposition 4.1 respectively.

In the remaining part of this paper, $C, c$ will be a generic positive constant independent of $T$ whose value may differ from line to line.

2. Preliminary

Denote $G = \{G_t, t \in [0, T]\}$ as a continuous centered Gaussian process with covariance function

$$\mathbb{E}(G_s G_t) = R(s, t), \quad s, t \in [0, T],$$

defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The filtration $\mathcal{F}$ is generated by the Gaussian family $G$. Suppose in addition that the covariance function $R$ is continuous. Let $\mathcal{E}$ denote the space of all real valued step functions on $[0, T]$. The Hilbert space $\mathcal{H}$ is defined as the closure of $\mathcal{E}$ endowed with the inner product

$$\langle 1_{[a,b]}, 1_{[c,d]} \rangle_\mathcal{H} = \mathbb{E}\left( (G_b - G_a)(G_d - G_c) \right).$$

Abusing the notation slightly, we also write $G = \{G(h), h \in \mathcal{H}\}$ as the isonormal Gaussian process on the probability space $(\Omega, \mathcal{F}, P)$, indexed by the elements in the Hilbert space $\mathcal{H}$. In other words, $G$ is a Gaussian family of random variables such that

$$\mathbb{E}(G) = \mathbb{E}(G(h)) = 0, \quad \mathbb{E}(G(g)G(h)) = \langle g, h \rangle_\mathcal{H},$$

for any $g, h \in \mathcal{H}$. 

In particular, when $G$ is exact the fractional Brownian motion $B^H$, we denote by $\mathcal{H}_1$ the associated Hilbert space.

**Notation 1.** Denote by $\mathcal{V}_{[0,T]}$ the set of bounded variation functions on $[0,T]$. For functions $f, g \in \mathcal{V}_{[0,T]}$, we define two products as

$$\langle f, g \rangle_{\mathcal{H}_1} = - \int_{[0,T]^2} f(t) \frac{\partial R^B(t,s)}{\partial t} d\nu_g(ds),$$

$$\langle f, g \rangle_{\mathcal{H}_2} = C'_{H} \int_{[0,T]^2} |f(t)g(s)| (ts)^{H-1} dtds = C'_{H} \mu(|f|)\mu(|g|),$$

where $\nu_g$ is given below and $R^B(t,s)$ is the covariance function of the fractional Brownian motion as (1.6) and $\mu(f) = \int f d\mu$ with $\mu(dx) = x^{2-1}dx$. We also denote for any $f \in L^2(R^2, \mu \times \mu)$

$$(\mu \times \mu)(f) = \int f d(\mu \times \mu).$$

The following proposition is an extension of [22, Theorem 2.3] and [21, Proposition 2.2], which gives the inner products representation of the Hilbert space $\mathcal{H}$:

**Proposition 2.1.** $\mathcal{V}_{[0,T]}$ is dense in $\mathcal{H}$ and we have

$$\langle f, g \rangle_{\mathcal{H}} = \int_{[0,T]^2} R(t,s)\nu_f(dt)\nu_g(ds), \quad \forall f, g \in \mathcal{V}_{[0,T]},$$

where $\nu_g$ is the restriction to $([0,T],\mathcal{B}([0,T]))$ of the Lebesgue-Stieljes signed measure associated with $g^0$ defined as

$$g^0(x) = \begin{cases} g(x), & \text{if } x \in [0,T), \\ 0, & \text{otherwise}. \end{cases}$$

Furthermore, if the covariance function $R(t,s)$ satisfies Hypothesis 1.2, then

$$\langle f, g \rangle_{\mathcal{H}} = - \int_{[0,T]^2} f(t) \frac{\partial R(t,s)}{\partial t} d\nu_g(ds), \quad \forall f, g \in \mathcal{V}_{[0,T]},$$

and

$$|\langle f, g \rangle_{\mathcal{H}} - \langle f, g \rangle_{\mathcal{H}_2}| \leq \langle f, g \rangle_{\mathcal{H}_2}, \quad \forall f, g \in \mathcal{V}_{[0,T]}.$$  

**Corollary 2.2.** Let $g = h \cdot 1_{(a,b)}(\cdot)$ with $h$ a continuously differentiable function. Then we have [21]

$$\langle g, g \rangle_{\mathcal{H}_1} = - \int_{(a,b)^2} h(t)h'(s) \frac{\partial R^B(t,s)}{\partial t} dtds + \int_{(a,b)} h(t) \left( h(b) \frac{\partial R^B(t,b)}{\partial t} - h(a) \frac{\partial R^B(t,a)}{\partial t} \right) dt.$$

**Remark 2.3.** When $H \in (\frac{1}{2}, 1)$, the identity (2.1) is also equal to the following formula:

$$\langle f, g \rangle_{\mathcal{H}_1} = H(2H-1) \int_{[0,T]^2} f(t)g(s) |t-s|^{2H-2} dtds, \quad \forall f, g \in \mathcal{V}_{[0,T]},$$

and hence if $0 \leq f \leq g$ then the monotonicity of the norm holds:

$$\langle f, f \rangle_{\mathcal{H}_1} \leq \langle g, g \rangle_{\mathcal{H}_1}.$$ 

A remarkable fact is that this monotonicity of the norm may not hold in the case of $H \in (0, \frac{1}{2})$. This is one of the reasons why it is more difficult to deal with the problems of $H \in (0, \frac{1}{2})$. 

The following proposition is an immediately consequence of the identity (2.3).

**Proposition 2.4.** Suppose that Hypothesis 1.2 holds. Then for any \( \varphi, \psi \in (\mathcal{V}_{0,T})^{\otimes r} \),

\[
|\langle \varphi, \psi \rangle_{H^2} - \langle \varphi, \psi \rangle_{H^r}^2| \leq (C_H')(\mu \times \mu)(|\varphi|)(\mu \times \mu)(|\psi|) + 2C_H'(\mu \times \mu)(|\varphi \otimes \psi|),
\]

where \( \varphi \otimes \psi \) is the 1-th contraction between \( \varphi \) and \( \psi \) in \( H^2 \), see (2.6) below.

Denote \( H^p, \) and \( H^p \) as the pth tensor product and the pth symmetric tensor product of the Hilbert space \( H \). Let \( H_p \) be the pth Wiener and \( H^p \) with respect to \( G \). It is defined as the closed linear subspace of \( L^2(\Omega) \) generated by the random variables \( \{H_p(G(h)) : h \in H, \|h\|_H = 1\} \), where \( H_p \) is the pth Hermite polynomial defined by

\[
H_p(x) = \frac{(-1)^p}{p!} e^{\frac{x^2}{2}} \frac{d^p}{dx^p} e^{-\frac{x^2}{2}}, \quad p \geq 1,
\]

and \( H_0(x) = 1 \). We have the identity \( I_p(h^{\otimes p}) = H_p(G(h)) \) for any \( h \in H \) where \( I_p(\cdot) \) is the generalized Wiener-Itô stochastic integral. Then the map \( I_p \) provides a linear isometry between \( H^p \) (equipped with the norm \( \|\cdot\|_{H^p} \)) and \( H_p \). Here \( H_0 = \mathbb{R} \) and \( I_0(x) = x \) by convention.

We choose \( \{e_k, k \geq 1\} \) to be a complete orthonormal system in the Hilbert space \( H \). Given \( f \in H^{\otimes m}, g \in H^{\otimes n} \), the q-th contraction between \( f \) and \( g \) is an element in \( H^{\otimes (m+n-2q)} \) that is defined by

\[
f \otimes_q g = \sum_{i_1, \ldots, i_q = 1}^{\infty} \langle f, e_{i_1} \otimes \cdots \otimes e_{i_q} \rangle_{H^{\otimes q}} \otimes \langle g, e_{i_1} \otimes \cdots \otimes e_{i_q} \rangle_{H^{\otimes q}},
\]

for \( q = 1, \ldots, m \wedge n \). Especially, when \( G \) is exact the fractional Brownian motion \( B^H \), we denote by \( f \otimes_q g \) the q-th contraction between \( f \) and \( g \) is an element in \( H^{\otimes (m+n-2q)} \).

For \( g \in H^p \) and \( h \in H^q \), we have the following product formula for the multiple integrals,

\[
I_p(g)I_q(h) = \sum_{r=0}^{p+q-1} \binom{p}{r} \binom{q}{r} I_{p+q-2r}(g \otimes_r h),
\]

where \( g \otimes_r h \) is the symmetrization of \( g \otimes_r h \) (see [31]).

The following Theorem 2.5, known as the fourth moment theorem, provides necessary and sufficient conditions for the convergence of a sequence of random variables to a normal distribution (see [31, 32]).

**Theorem 2.5.** Let \( p \geq 2 \) be a fixed integer. Consider a collection of elements \( \{f_T, T > 0\} \) such that \( f_T \in H^p \) for every \( T > 0 \). Assume further that

\[
\lim_{T \to \infty} \mathbb{E}[I_p(f_T)^2] = \lim_{T \to \infty} p!\|f_T\|_{H^p}^2 = \sigma^2.
\]

Then the following conditions are equivalent:

1. \( \lim_{T \to \infty} \mathbb{E}[I_p(f_T)^4] = 3\sigma^4 \).
2. For every \( q = 1, \ldots, p-1 \), \( \lim_{T \to \infty} \|f_T \otimes_q f_T\|_{H^{2(p-q)}} = 0 \).
3. As \( T \) tends to infinity, the p-th multiple integrals \( \{I_p(f_T), T \geq 0\} \) converge in distribution to a Gaussian random variable \( N(0, \sigma^2) \).
3. Strong Consistency: Proof of Theorem 1.3

We will discuss exclusively the case $H \in (0, \frac{1}{2})$ since the case of $H > \frac{1}{2}$ has been treated in [1]. We first define some important functions that will be used in the proof. Denote

$$f_T(t, s) = e^{-\theta |t-s|} \mathbb{1}_{[0 \leq s, t \leq T]}, \quad h_T(t, s) = e^{-\theta (T-t)-\theta (T-s)} \mathbb{1}_{[0 \leq s, t \leq T]}, \quad g_T(t, s) = \frac{1}{2\theta T} (f_T - h_T).$$

The solution to the SDE (1.1) with $X_0 = 0, \sigma = 1$ is

$$X_t = \int_0^t e^{-\theta (t-s)} dG_s = I_1(f_T(t, \cdot) \mathbb{1}_{[0, t]}(\cdot)).$$

We apply the product formula of multiple integrals (2.7) and stochastic Fubini theorem to obtain

$$\frac{1}{T} \int_0^T X_t^2 dt = I_2(g_T) + b_T,$$

where

$$b_T = \frac{1}{T} \int_0^T \left\| e^{-\theta (t-\cdot)} \mathbb{1}_{[0,t]}(\cdot) \right\|^2_{\mathcal{H}} dt.$$

From the equation (1.4), we can write

$$\sqrt{T} (\bar{\theta}_T - \theta) = -\frac{1}{2 \sqrt{T}} I_2(f_T) \frac{I_2(g_T) + b_T}{I_2(f_T) + b_T}.$$

For the kernel $f_T(r, s) \mathbb{1}_{[0 \leq r, s \leq t]}$, it’s double Wiener-Itô integral

$$F_t := I_2(f_T(r, s) \mathbb{1}_{[0 \leq r, s \leq t]}), \quad t \in [0, T]$$

is a stochastic process which is named as a chaos (stochastic) process [33]. The next two propositions are about the asymptotic behaviors of the second moment of $F_t$ and the increment $F_t - F_s$ respectively, which are used to estimate the modulus of continuity of the chaos process $\{F_t\}$ on $[n, n+1]$ for any integer $n \geq 1$ and then to get the asymptotic growth of the process $F_t$ as $t \to \infty$ (see Proposition 3.5). Those types of sample path properties of chaos processes are the key tools to show the strong consistency in the present paper.

**Proposition 3.1.** When $H \in (0, \frac{1}{2})$,

$$\lim_{t \to \infty} \frac{1}{4\theta \sigma_H^2 t} \mathbb{E}[|F_t|^2] = (H \Gamma(2H) \theta^{-2H})^2.$$

**Proof.** The inequality (2.5) implies that

$$\left| \|f_t\|_{\mathcal{H}^2} - \|f_t\|_{\mathcal{H}^2} \right| \leq (C_H(\mu \times \mu)(f_t))^2 + 2C_H(\mu \times \mu)(f_t \otimes \mathcal{V} f_t).$$

Lemma 17 in [9] implies that when $H \in (0, \frac{1}{2})$,

$$\lim_{t \to \infty} \frac{1}{2\theta \sigma_H^2 t} \|f_t\|_{\mathcal{H}^2}^2 = (H \Gamma(2H) \theta^{-2H})^2.$$


Lemma 5.1 implies that there exists a positive constant $C$ independent on $T$ such that

$$
(\mu \times \mu)(f_t) = \int_{[0,t]^2} e^{-\theta |r-s|} (rs)^{H-1} \, dr \, ds = 2 \int_0^t e^{-\theta r} H-1 \, dr \int_0^r e^{\theta s} H-1 \, ds \leq C. \quad (3.11)
$$

Then it follows from Lemma 5.7 that when $H \in (0, \frac{1}{2})$

$$
\lim_{t \to \infty} \frac{1}{t} \|f_t\|_{\mathfrak{B}^2}^2 - \|f_t\|_{\mathfrak{B}^2}^2 = 0,
$$

which together with (3.10) and the Itô’s isometry implies that there exists a positive constant

$$
\mathbb{E}[|F_t|^2] = 2 \|f_t\|_{\mathfrak{B}^2}^2
$$

implies the desired (3.8). \qed

**Remark 3.2.** Together with Lemma 3.11 of [21], we have that when $H \in (0, \frac{1}{2})$, the speed of convergence

$$
\frac{1}{2\theta \sigma^4 \Gamma^3} \|f_t\|_{\mathfrak{B}^2}^2 \rightarrow (H \Gamma(2H)^{\theta^{-2H}})^2
$$

is at least $\frac{1}{\theta^2}$ as $t \to \infty$.

**Notation 2.** Let $0 \leq s < t \leq T$. Denote

$$
\phi_1(u,v) = e^{-\theta |u-v|} 1_{\{s \leq u,v \leq t\}}, \quad (3.12)
$$

$$
\phi_2(u,v) = e^{-\theta |u-v|} (1_{\{0 \leq u \leq s, s \leq v \leq t\}} + 1_{\{0 \leq v \leq s, s \leq u \leq t\}}). \quad (3.13)
$$

**Proposition 3.3.** If Hypothesis 1.2 is satisfied, there exists a constant $C > 0$ independent of $T$ such that for all $s,t \geq 0$,

$$
\mathbb{E}[|F_t - F_s|^2] \leq C \left(|t-s|^{4H+2} + |t-s|^{4H+1} + |t-s|^{2H+1} + |t-s|^{2H} + |t-s|^{4H} \right). \quad (3.14)
$$

Moreover, for any real number $p > \frac{4}{H}$, $q > 1$ and integer $n \geq 1$, there exists a random constant $R_{p,q}$ independent of $n$ such that

$$
|F_t - F_s| \leq R_{p,q} n^{q/p}, \quad \forall \ t,s \in [n, n+1]. \quad (3.15)
$$

**Proof.** Itô’s isometry implies that

$$
\mathbb{E}[|F_t - F_s|^2] = 2 \|f_t - f_s\|_{\mathfrak{B}^2}^2 \leq 4(\|\phi_1\|_{\mathfrak{B}^2}^2 + \|\phi_2\|_{\mathfrak{B}^2}^2). \quad (3.16)
$$

For simplicity, we can assume that $\theta = 1$. Lemma 5.1 implies that there exists a positive constant $C$ independent on $T$ such that

$$
(\mu \times \mu)(|\phi_1|) = 2 \int_s^t e^{-u} u^{H-1} \, du \int_s^u e^v v^{H-1} \, dv \leq 2 \int_s^t u^{H-1} \, du \leq C |t-s|^H,
$$

$$
(\mu \times \mu)(|\phi_2|) \leq 2 \int_0^s e^{-(s-u)} u^{H-1} \, du \int_s^t e^{-(t-v)} v^{H-1} \, dv \leq 2 \int_s^t v^{H-1} \, dv \leq C |t-s|^H.
$$

Hence, it follows from the inequality (2.5) and Lemma 5.11 that

$$
\|\phi_1\|_{\mathfrak{B}^2}^2 \leq \|\phi_1\|_{\mathfrak{B}^2}^2 + (C_H'(\mu \times \mu)(|\phi_1|))^2 + 2C_H''(\mu \times \mu)(\phi_1 \otimes \phi_1)
$$

$$
\leq C \left(|t-s|^{2H} + |t-s|^{4H} + |t-s|^{4H+1} + |t-s|^{2H+2} \right). \quad (3.17)
$$
and similarly, the inequality (2.5) and Lemma 5.12 implies that
\[
\|\phi_2\|_{\ddot{H}^2}^2 \leq \|\phi_2\|_{\ddot{H}^2}^2 + (C_H'(\mu \times \mu)(|\phi_2|))^2 + 2C_H'(\mu \times \mu)(\phi_2 \otimes \phi_2) 
\leq C(t-s)^{2H+1} + |t-s|^{2H} + |t-s|^H.
\]
(3.18)

By plugging the inequalities (3.17) and (3.18) into (3.16), we obtain the desired estimate (3.14).

The inequality (3.15) is then obtained from the Garsia-Rodemich-Rumsey inequality (see Proposition 3.4 of [25]). For the reader’s convenience, we rewrite it as follows: By the inequality (3.14), there exists a positive constant \(C\) independent of \(T\) such that, for any \(|t-s| \leq 1\),
\[
\mathbb{E}[|F_t - F_s|^2] \leq C |t-s|^H.
\]

The hypercontractivity of multiple Wiener-Itô integrals implies that for any \(p \geq 2\) and any \(n \leq t < s \leq n + 1\),
\[
\mathbb{E}[|F_t - F_s|^p] \leq C \times (p-1)^{\frac{p}{2H}} |t-s|^{\frac{pH}{2}}.
\]

Then \(F_t\) has a continuous realization on \([n, n+1]\) for all integer \(n \geq 1\) by Kolmogorov’s continuity theorem. Next, take \(\Psi(x) = x^p\) and \(\rho(x) = x^{\frac{H}{2}}\). Denote
\[
B_n = \int_{[n, n+1]^2} \Psi\left(\frac{|F_t - F_s|}{\rho(|t-s|)}\right) \, dt \, ds.
\]

The above inequality implies that for any \(q > 1\),
\[
\mathbb{E}\left(\sum_{n=1}^{\infty} \frac{B_n}{n^q}\right) = \sum_{n=1}^{\infty} \frac{\mathbb{E}(B_n)}{n^q} \leq C \times (p-1)^{\frac{p}{2H}} \sum_{n=1}^{\infty} \frac{1}{n^q} < \infty.
\]

Hence, there exists a random constant \(R_{p,q}\) such that
\[
\sum_{n=1}^{\infty} \frac{B_n}{n^q} \leq R_{p,q},
\]
which implies that for all positive \(q > 1\) and integer \(n \geq 1\),
\[
B_n \leq R_{p,q}n^q.
\]
(3.19)

An application of the Garsia-Rodemich-Rumsey inequality, see e.g. [29, 30], implies that when \(p > \frac{4}{H}\), we have for all \(s, t \in [n, n+1]\),
\[
|F_t - F_s| \leq 8 \int_{0}^{[t-s]} \Psi^{-1}\left(\frac{4B_n}{u^2}\right) \rho'(u) \, du = 2H \left(\frac{4B_n}{H - \frac{p}{2}}\right) |t-s|^{\frac{H}{2}} < c_p B_n^{\frac{1}{p}}.
\]

This combined with (3.19) shows the proposition. \(\square\)

We denote a chaos process
\[
J_t = I_2(h_T(r, u)1_{(0 \leq r, u \leq t)}), \quad t \in [0, T],
\]
(3.20)
and apply the similar computations as above to obtain its sample path properties as follows:
Proposition 3.4. If Hypothesis 1.2 is satisfied, there exists a constant $C > 0$ independent of $T$ such that

$$\sup_{\epsilon \geq 0} E[|J_\epsilon|^2] < C, \quad (3.21)$$

and there exist two positive constants $C$ and $\alpha \in (0, 1)$ independent of $T$ such that, for any $|t - s| \leq 1$,

$$E[|J_t - J_s|^2] \leq C |t - s|^\alpha. \quad (3.22)$$

Moreover, for any real number $p > \frac{2}{\alpha}, q > 1$ and integer $n \geq 1$, there exists a random constant $R_{p,q}$ independent of $n$ such that

$$|J_t - J_s| \leq R_{p,q} n^{q/p}, \quad \forall t, s \in [n, n + 1]. \quad (3.23)$$

Proposition 3.5. Let $F_T$ and $J_T$ be given in (3.7) and (3.20) respectively. If Hypothesis 1.2 is satisfied, we have

$$\lim_{T \to \infty} F_T = 0 \quad \text{and} \quad \lim_{T \to \infty} J_T = 0, \quad a.s. \quad (3.24)$$

for any $\alpha > 0$.

Proof. The proof is similar to that in [1] or [25]. We will only show $\lim_{T \to \infty} \frac{F_T}{T} = 0$, and the other is similar. When $H \in (0, \frac{1}{2})$, Chebyshev’s inequality, the hypercontractivity of multiple Wiener-Itô integrals and Proposition 3.1 imply that for any $\epsilon > 0$,

$$P \left( \frac{|F_n|}{n} > \epsilon \right) \leq \frac{E|F_n|^4}{n^4 \epsilon^4} \leq \frac{C (E(F_n^2))^2}{n^\epsilon} \leq C n^{-2},$$

which implies that $\frac{F_n}{n}$ converges to 0 almost surely as $n \to \infty$ by the Borel-Cantelli lemma. Since

$$\left| \frac{F_T}{T} \right| \leq \frac{1}{T} |F_T - F_n| + \frac{n |F_n|}{n},$$

where $n = \lfloor T \rfloor$ is the biggest integer less than or equal to a real number $T$, we have $\frac{F_T}{T}$ converges to 0 almost surely as $T \to \infty$ by Proposition 3.3. \hfill \Box

Proposition 3.5 implies the following chaos process

$$I_2(g_T) = \frac{F_T - J_T}{2\theta T} \to 0$$

as $T \to \infty$ almost surely. Next we study the term $b_T$.

Proposition 3.6. Let $b_T$ be given by (3.5). Suppose that Hypothesis 1.2 holds. We have

$$\lim_{T \to \infty} b_T = H\Gamma(2H)\theta^{-2H} > 0, \quad (3.25)$$

Proof. The L’Hôpital’s rule implies that

$$\lim_{T \to \infty} b_T = \lim_{T \to \infty} \left\| e^{-\theta(T-\cdot)} \mathbb{1}_{[0,T]}(\cdot) \right\|_{L^2_B}^2. \quad (3.26)$$

It follows from from [9] or [21] that
\[
\lim_{T \to \infty} \left\| e^{-\theta(T-\cdot)} \mathbb{1}_{[0,T]}(\cdot) \right\|_{H_1}^2 = \Gamma(2H)\theta^{-2H}.
\]
The identity (2.4) implies that when \( T \geq 1 \),
\[
\left\| e^{-\theta(T-\cdot)} \mathbb{1}_{[0,T]}(\cdot) \right\|_{H_1}^2 - \left\| e^{-\theta(T-\cdot)} \mathbb{1}_{[0,T]}(\cdot) \right\|_{H_1}^2 \leq C \int_0^T e^{-\theta(T-u)} u^{H-1} du^2 \leq CT^{2(H-1)}. \quad (3.26)
\]
where in the last line we use Lemma 5.1. By substituting the above limit and estimate into the identity (3.25), we obtain the desired limit (3.24).

**Remark 3.7.** Lemma 3.2 of [21] implies that the speed of convergence
\[
\frac{1}{T} \int_0^T e^{-\theta(T-\cdot)} \mathbb{1}_{[0,T]}(\cdot) \, dt \to \Gamma(2H)\theta^{-2H}
\]
is at least \( \frac{1}{T} \). Clearly, \( 2(H-1) < -1 \) when \( H \in (0, \frac{1}{2}) \). Then by the inequality (3.26), there exists a positive constant \( C \) such that when \( T \) is large enough,
\[
|\hat{b}_T - \Gamma(2H)\theta^{-2H}| \leq \frac{C}{T}. \quad (3.27)
\]

**Proof of Theorem 1.3.** From(3.4), (3.3), (3.7), and (3.20),
\[
\frac{1}{T} \int_0^T X_t^2 \, dt = I_2(\vartheta_T) + b_T = \frac{1}{2\theta} \left[ F_T - J_T \right] + b_T.
\]
Proposition 3.5 and 3.6 imply that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T X_t^2 \, dt = \Gamma(2H)\theta^{-2H}, \quad \text{a.s.,}
\]
which implies that the moment estimator \( \hat{\theta}_T \) is strongly consistent by the continuous mapping theorem. Since
\[
\hat{\theta}_T - \theta = \frac{-1}{2\theta} \frac{F_T}{I_2(\vartheta_T) + b_T},
\]
Proposition 3.5 and 3.6 imply that the least squares estimator \( \hat{\theta}_T \) is also strongly consistent. \( \square \)

4. The Asymptotic normality and the Berry-Esséen bound

4.1. The Asymptotic normality.

**Proposition 4.1.** Let \( \delta \) be given as in (1.12). When \( H \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{3}{8}) \), there exists a constant \( C_{\theta,H} > 0 \) such that
\[
\frac{1}{T} \left\| f_T \otimes_1 f_T \right\|_{\mathbb{R}^{\otimes 2}} \leq \frac{C_{\theta,H}}{T^{\delta}}, \quad (4.1)
\]
and when \( H = \frac{1}{4} \), the upper bound can be replaced by \( \frac{\log T}{\sqrt{T}} \).
Proof. Without loss of generality, we assume \( \theta = 1 \). Recall that

\[
(f_T \otimes_1 f_T)(u_1, u_2) := -\int_{[0,T]^2} f_T(u_1, v_1) \frac{\partial}{\partial v_2} f_T(u_2, v_2) \frac{\partial}{\partial v_1} R(v_1, v_2) dv_1 dv_2
\]

\[
= \int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} \text{sgn}(v_2-u_2) \frac{\partial}{\partial v_1} R(v_1, v_2) dv_1 dv_2
\]

\[
+ \int_{[0,T]} e^{-|u_1-v_1|-(T-u_2)} \frac{\partial}{\partial v_1} R(v_1, T) dv_1.
\]

The triangle inequality implies

\[
\|f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2 \leq \|f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2 + \|f_T \otimes_1 f_T - f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2.
\]

It follows from the inequality (2.5) that when \( H \neq \frac{1}{4} \),

\[
\|f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2 \leq \|f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2 + \left[ C \left[ \left( \mu + \mu \right) \left| f_T \otimes_1 f_T \right| \right] \right]^2 + \left[ \left( \mu + \mu \right) \left| f_T \otimes_1 f_T \otimes_1 f_T \otimes_1 f_T \right| \right]^2
\]

\[
\leq C[T + T^{2\gamma_1} + T^\gamma],
\]

where in the last line, we use the inequality (3.17) of [9] and Lemma 5.7 and Lemma 5.10 respectively. The formula of integration by parts implies that

\[
f_T \otimes_1 f_T - f_T \otimes_1 f_T = \int_{[0,T]^2} f_T(u_1, v_1) f_T(u_2, v_2) \frac{\partial^2}{\partial v_1 \partial v_2} (R(v_1, v_2) - R^B(v_1, v_2)) dv_1 dv_2,
\]

which, together with Lemma 5.8, implies that

\[
\|f_T \otimes_1 f_T - f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2 \leq C_{\theta, H} T^{2\gamma_1}.
\]

Comparing three values \( 1, 2\gamma_1, \) and \( \gamma, \) we see that the largest one is

\[
\delta_0 = \max \{1, 2\gamma_1, \gamma\} = \begin{cases} 
1, & \text{when } H \in (0, \frac{1}{4}], \\
8H - 1, & \text{when } H \in (\frac{1}{4}, \frac{1}{2}).
\end{cases}
\]

Then we have

\[
\|f_T \otimes_1 f_T\|_{\mathcal{H}_{\otimes 2}}^2 \leq C T^{\delta_0}.
\]

Clearly, \( \delta = 1 - \frac{2}{9} \). Hence, we have the desired (4.1). The case of \( H = \frac{1}{4} \) is similar. \( \square \)

Proof of Theorem 1.4. Denote a constant that depends on \( \theta \) and \( H \) as

\[
a := H \Gamma(2H)\theta^{-2H}.
\]

First, Proposition 3.1, Proposition 4.1 and Theorem 2.5 imply that when \( H \in (0, \frac{2}{3}) \), as \( T \to \infty \),

\[
\frac{1}{2\sqrt{T}} F_T \overset{\text{law}}{\to} N(0, \theta a^2 \sigma H^2).
\]

Second, recall the identity (3.6):

\[
\sqrt{T}(\hat{\theta}_T - \theta) = \frac{1}{2\sqrt{T}} F_T + b_T.
\]

The Slutsky’s theorem, Proposition 3.6 and the convergence result (4.4) imply that the asymptotic normality (1.8) holds when \( H \in (0, \frac{2}{3}) \).
Third, we can show the asymptotic normality of
\[
\sqrt{T} \left( \frac{1}{T} \int_0^T X_s^2 ds - a \right) \xrightarrow{law} \mathcal{N}(0, a^2 \sigma_H^2 / \theta)
\] (4.5)
when \( H \in (0, \frac{3}{8}) \). In fact, we have
\[
\sqrt{T} \left( \frac{1}{T} \int_0^T X_s^2 ds - a \right) = \frac{1}{2\theta} \left[ \frac{F_T}{\sqrt{T}} - \frac{J_T}{\sqrt{T}} \right] + \sqrt{T} (b_T - a).
\] (4.6)

The inequality (3.27) implies that
\[
\lim_{T \to \infty} \sqrt{T} (b_T - a) = 0.
\]
Proposition 3.4 and Proposition 3.5 imply that \( J_T \sqrt{T} \rightarrow 0 \) a.s. as \( T \to \infty \). Thus, the Slutsky’s theorem implies that (4.5) holds. Finally, since
\[
\tilde{\theta}_T = \left( \frac{1}{H!} \frac{1}{2H} \right) T \int_0^T X_s^2 ds - \frac{1}{2} H \frac{1}{H} \frac{1}{(2H)^{\frac{1}{2}}} \frac{1}{H} \theta,
\]
the delta method implies that the asymptotic normality (1.9) holds when \( H \in (0, \frac{3}{8}) \). \( \square \)

4.2. the Berry-Esséen bound.

**Proposition 4.2.** Let the constant \( \delta \) be given as in (1.12) and the double Wiener-Itô integrals \( F_T, J_T \) be as in (3.7) and (3.20) respectively. Denote the double Wiener-Itô integral
\[
Q_T := \frac{F_T - J_T}{\sqrt{T}}.
\]
Then there exists a constant \( C_{\theta, H} \) such that when \( H \neq \frac{1}{4} \) and \( T \) is large enough
\[
\sup_{z \in \mathbb{R}} \left| P \left( \frac{1}{\sqrt{4\theta a^2 \sigma_H^2}} Q_T \leq z \right) - \Phi(z) \right| \leq \frac{C_{\theta, H}}{T^8},
\] (4.7)
where \( \Phi(u) \) stands for the standard normal distribution function. When \( H = \frac{1}{4} \), the upper bound can be replaced by \( \frac{\log T}{\sqrt{T}} \).

**Proof.** We follow the line of the proof of Proposition 5.3 of [1]. It follows from Remark 3.2, the inequality (3.21) and Lemma 5.13 that
\[
\frac{||E[Q_T^2] - 4\theta a^2 \sigma_H^2||}{E[Q_T^2] \vee (4\theta a^2 \sigma_H^2)} \leq C \times (T^{2H-1} + T^{-1} + T^{2H-1} \log T) \leq CT^{2H-1}.
\]
Clearly, \( 1 - 2H \geq \delta \). Then the the Fourth moment Berry-Esséen bound (see, for example, Corollary 5.2.10 of [31]) implies that we only need to show that when \( T \) is large enough,
\[
E[Q_T^4] - 3E[Q_T]^2 \leq \frac{C}{T^{28}},
\] (4.8)
which is the consequence of the following three estimates from the identity (5.8) of [1].

Proposition 4.1 implies that
\[
E \left[ \left( \frac{F_T}{\sqrt{T}} \right)^4 \right] - 3 \left[ E \left( \frac{F_T}{\sqrt{T}} \right)^2 \right]^2 \leq C \left( \frac{1}{T} \| f_T \otimes_1 f_T \|_{\mathcal{S}^2} \right)^2 \leq \frac{C}{T^{23}}.
\]
The Cauchy-Schwarz inequality and Lemma 5.13 imply that when $H \neq \frac{1}{4}$

$$\left| \mathbb{E}[Q_T^2] \right|^2 - \left( \mathbb{E}\left[ \frac{F_T}{\sqrt{T}} \right] \right)^2 \leq \mathbb{E}\left[ Q_T^2 + \frac{F_T}{\sqrt{T}} \right] \mathbb{E}\left[ \frac{H_T}{\sqrt{T}} \left( \frac{H_T - 2F_T}{\sqrt{T}} \right) \right] \leq C \times \frac{1}{T^{1-(4H-1)+}} \leq \frac{C}{T^{2\delta}}.$$ 

The identity (5.9) of [1], Lemma 5.13 and Proposition 4.1 imply that when $H \neq \frac{1}{4}$

$$\frac{1}{T^2} \left| \mathbb{E}[F_T^3 J_T] \right| \leq \frac{2}{T^2} \left[ \| f_T \|_{\mathcal{B}^{2\delta}} \| f_T \|_{\mathcal{B}^{2\delta}} \| h_T \|_{\mathcal{B}^{2\delta}} + \| f_T \|_{\mathcal{B}^{2\delta}}^2 \| (f_T, h_T)_{\mathcal{B}^{2\delta}} \right] \leq \frac{C}{T^{2\delta}}.$$ 

**Proof of Theorem 1.6.** Denote $a = HT(2H)\theta^{-2H}$ and $b_T$ be given by (3.5). Then Remark 3.7 and Remark 3.2 imply that there exists a constant $C > 0$ such that for $T$ large enough,

$$\left| b_T^2 - \frac{\| f_T \|_{\mathcal{B}^{2\delta}}^2}{2\theta \sigma_H^2 T} \right| \leq \left| b_T^2 - a^2 \right| + \left| \frac{\| f_T \|_{\mathcal{B}^{2\delta}}^2}{2\theta \sigma_H^2 T} - a^2 \right| \leq C \times \left| \frac{1}{T} + \frac{1}{T^{1-2H}} \right| \leq \frac{C}{T^{1-2H}}.$$ 

Recalling the definition of $g_T$ in (3.3), from Propositions 3.1 and 3.4, we have that there exists a constant $C > 0$ such that for $T$ large enough,

$$\| g_T \|_{\mathcal{B}^{2\delta}} \leq \frac{C}{\sqrt{T}}.$$ 

Hence, Corollary 1 of [34] implies that when $H \neq \frac{1}{4}$, there exists a positive constant $C$ independent on $T$ such that, for $T$ large enough,

$$\sup_{z \in \mathbb{R}} \left| P \left( \left. \frac{T}{\theta \sigma_H^2} \left( \hat{\theta}_T - \theta \right) \leq z \right| - \Phi(z) \right| \leq C \times \max \left( \left| b_T^2 - \frac{\| f_T \|_{\mathcal{B}^{2\delta}}^2}{2\theta \sigma_H^2 T} \right|, \frac{1}{T} \| f_T \|_{\mathcal{B}^{2\delta}} \| f_T \|_{\mathcal{B}^{2\delta}}, \frac{\| f_T \|_{\mathcal{B}^{2\delta}}^2}{\sqrt{T}} \right) \leq C \times \left| \frac{1}{T^{1-2H}} + \frac{1}{T^3} + \frac{1}{\sqrt{T}} \right|,$n

where in the last line we use Propositions 3.1 and 4.1. Comparing three values $1/2$, $1 - 2H$, and $\delta$, we see that the smallest one is $\delta$. Hence, we obtain the Berry-Essèen type bound (1.10) of the LSE $\hat{\theta}_T$.

The Berry-Essèen type bound (1.11) of the ME $\hat{\theta}_T$ can be obtained along the similar arguments of Theorem 3.2 in [10]. Please refer to the proof of Theorem 1.3 in [1] for details. We only point out that (1.11) is a consequence of Proposition 4.2 together with the following basic estimate:

$$\sup_{z > -c \sqrt{T}} \left| \Phi \left( \sqrt{\frac{\theta \sigma_H^2}{a^2 \sigma_H^2}} (\hat{\theta}_T - a) \right) - \Phi(\nu(z)) \right| \leq \frac{C}{\sqrt{T}}. \tag{4.9}$$
where \( c = \frac{2H\sqrt{T}}{n} \) and \( \Phi(z) = 1 - \Phi(z) \) and

\[
\nu(z) = \frac{c}{2H} \sqrt{T} \left[ (1 + \frac{z}{c\sqrt{T}})^{-2H} - 1 \right].
\]

The above estimate (4.9) is a direct consequence of Lemma 5.4 in [1], Remark 3.7 and a well known inequality for the Gaussian distribution: \( |\Phi(z_1) - \Phi(z_2)| \leq |z_1 - z_2| \). \( \square \)

5. Appendix

In this section, we will make use of the following inequalities repeatedly. We ignore the proof since it is very elementary.

**Lemma 5.1.** Assume \( \beta > 0 \) and \( \theta > 0 \). Denote

\[
A(s) = \int_0^s e^{-\theta r s^{-1}} \, dr, \quad \bar{A}(s) = e^{-\theta s} \int_0^s e^{\theta r s^{-1}} \, dr.
\]

Then there exists a constant \( C > 0 \) such that for any \( s \in [0, \infty) \),

\[
A(s) \leq C \times \left( s^3 \mathbb{1}_{[0,1]}(s) + \mathbb{1}_{(1,\infty)}(s) \right) \leq C \times (1 \wedge s^3),
\]

\[
\bar{A}(s) \leq C \times \left( s^3 \mathbb{1}_{[0,1]}(s) + s^{3-1} \mathbb{1}_{(1,\infty)}(s) \right) \leq C \times (s^{3-1} \wedge s^3).
\]

Especially, when \( \beta \in (0,1) \), there exists a constant \( C > 0 \) such that for any \( s \in [0, \infty) \),

\[
\bar{A}(s) \leq C \times (1 \wedge s^{3-1}),
\]

\[
\int_0^\infty e^{-\theta [r-s]} s^{-1} \, dr \leq C \times (1 \wedge s^{3-1}),
\]

\[
\int_0^s e^{-\theta [r-t]} (s-r)^{3-1} \, dr \leq C \times (1 \wedge (s-t)^{3-1}), \quad t \in [0,s].
\]

The following two corollaries are consequences of Lemma 5.1. We ignore the first one's proof since it is very direct and simple.

**Corollary 5.2.** When \( H \in (0, \frac{1}{2}) \), there exists a positive constant \( C \) independent on \( T \) such that for any \( t \in [0,T] \),

\[
\int_0^T e^{-|r-t|} \frac{\partial}{\partial r} R^B(r, T) \, dr \leq C \times [1 \wedge t^{2H-1} + 1 \wedge (T-t)^{2H-1}],
\]

and for any \( s \in [0,T] \),

\[
\int_0^s e^{-|r-s|} \frac{\partial}{\partial r} R^B(r, T) \, dr \leq C \times [1 \wedge s^{2H-1} + 1 \wedge (T-s)^{2H-1}] \leq C \times \frac{\partial}{\partial s} R^B(s, T),
\]

\[
\int_s^T e^{-|r-s|} \frac{\partial}{\partial r} R^B(r, s) \, dr \leq C \times [1 \wedge s^{2H-1} + 1 \wedge (T-s)^{2H-1}] \leq C \times \frac{\partial}{\partial s} R^B(s, T),
\]

\[
\int_s^T e^{-|r-s|} \frac{\partial}{\partial s} R^B(r, s) \, dr \leq C \times [1 \wedge (T-s)^{2H-1}],
\]

\[
\int_0^s e^{-|r-s|} \frac{\partial}{\partial s} R^B(r, s) \, dr \leq C \times [1 \wedge (T-s)^{2H-1}],
\]
\[
\int_s^T e^{v-T} \left| \frac{\partial}{\partial s} R^B(r, s) \right| \, dr \leq C \times [s^{2H-1} + 1 \wedge (T-s)^{2H-1}] \leq C \times \frac{\partial}{\partial s} R^B(s, T).
\]

**Notation 3.** For any \( v, w \in [0, T] \), define
\[
\psi(w, T) = \mu(e^{-|w|} \mathbb{1}_{[0,T]}(\cdot)) = \int_0^T e^{-|w-u|} u^{H-1} \, du,
\]
\[
\phi(v, T) = \int_0^T \psi(w, T) \left| \frac{\partial}{\partial v} R^B(v, w) \right| \, dw = \int_{[0,T]^2} e^{-|w-u|} u^{H-1} \left| \frac{\partial}{\partial v} R^B(v, w) \right| \, dudw.
\]

It is clear that
\[
\frac{\partial}{\partial T} \psi(w, T) = e^{w-T} T^{H-1},
\]
and Lemma 5.1 implies that there exists a positive constant \( C \) independent on \( T \) such that
\[
\psi(T, T) \leq CT^{H-1},
\]
and
\[
\int_0^T \psi(w, T) ((T-w)^{2H-1} + w^{2H-1}) \, dw \leq C \times \begin{cases} T^{(3H-1)_+}, & \text{if } H \in (0, \frac{1}{2}) \cup (\frac{1}{3}, \frac{1}{2}), \\ \log T, & \text{if } H = \frac{1}{3} \end{cases}
\]
where \( a_+ = \max \{a, 0\} \).

**Corollary 5.3.** When \( H \in (0, \frac{1}{2}) \), there exists a positive constant \( C \) independent on \( T \) such that for any \( v \in [0, T] \),
\[
0 < \frac{\partial \phi(v, T)}{\partial T} \leq C \times T^{H-1} \times \frac{\partial}{\partial v} R^B(v, T).
\]

**Proof.** It is clear that
\[
\frac{\partial \phi(v, T)}{\partial T} = \left| \frac{\partial}{\partial v} R^B(v, T) \right| \int_0^T e^{-(T-u)} u^{H-1} \, du + T^{H-1} \int_0^T e^{-(T-w)} \left| \frac{\partial}{\partial v} R^B(v, w) \right| \, dw.
\]
Lemma 5.1 and Corollary 5.2 imply that when \( H \in (0, \frac{1}{2}) \),
\[
\left| \frac{\partial}{\partial v} R^B(v, T) \right| \int_0^T e^{-(T-u)} u^{H-1} \, du \leq C \times T^{H-1} \times \frac{\partial}{\partial v} R^B(v, T);
\]
and
\[
\int_0^T e^{-(T-u)} \left| \frac{\partial}{\partial v} R^B(v, w) \right| \, dw = \left( \int_0^v + \int_v^T \right) e^{-(T-w)} \left| \frac{\partial}{\partial v} R^B(v, w) \right| \, dw \leq C \times \frac{\partial}{\partial v} R^B(v, T).
\]
By plugging these two inequalities into (5.7), we obtain the desired (5.6). \( \square \)

**Lemma 5.4.** Let \( B(\cdot, \cdot) \) be the Beta function and \( \phi(\cdot, T) \) be given as in (5.2). We have that when \( H \in (0, \frac{1}{2}) \),
\[
\lim_{T \to \infty} \frac{\phi(T, T)}{T^{3H-1}} = \lim_{T \to \infty} \frac{1}{T^{3H-1}} \int_0^T \phi(v, T) e^{v-T} \, dv = 2 [B(2H, H) H - 1].
\]

(5.8)
In addition, when \( \beta > -1 \), there exist a positive constant \( C \) independent on \( T \geq 1 \) such that

\[
\int_1^T \phi(v, T)v^\beta \, dv \leq C \times \begin{cases} 
T^H, & \text{when } \beta \in (-1, -2H), \\
T^H \log T, & \text{when } \beta = -2H, \\
T^{3H + \beta}, & \text{when } \beta > -2H.
\end{cases}
\] (5.9)

Moreover, when \( \beta > -2H \), there exist positive constants \( c \) and \( C \) independent on \( T \) such that

\[
c \leq \liminf_{T \to \infty} \frac{1}{T^{3H + \beta}} \int_0^T \phi(v, T)v^\beta \, dv \leq \limsup_{T \to \infty} \frac{1}{T^{3H + \beta}} \int_0^T \phi(v, T)v^\beta \, dv \leq C.
\] (5.10)

**Proof.** It is clear that

\[
\frac{\phi(T, T)}{T^{3H - 1}} = \frac{H}{T^{3H - 1}} \left( \int_{0 \leq u \leq w \leq T} + \int_{0 \leq w \leq u \leq T} \right) e^{-\beta} |u-w| u^{H-1} (T-w)^{2H-1} - T^{2H-1} du dw,
\]

and as \( T \to \infty \),

\[
\frac{1}{T^{3H-1}} \times T^{2H-1} \left( \int_{0 \leq u \leq w \leq T} + \int_{0 \leq w \leq u \leq T} \right) e^{-\beta} |u-w| u^{H-1} du dw = \frac{2}{H}.
\] (5.11)

Denote the other terms as

\[
\frac{1}{T^{3H-1}} \left( \int_{0 \leq u \leq w \leq T} + \int_{0 \leq w \leq u \leq T} \right) e^{-\beta} |u-w| u^{H-1} (T-w)^{2H-1} du dw := J_1 + J_2.
\]

The change of variables \( a = T-w, z = a+u \) and the L'Hôpital’s rule imply that

\[
\lim_{T \to \infty} J_1 = \lim_{T \to \infty} \frac{1}{e^{T} T^{3H-1}} \int_0^T e^{z} \int_0^z a^{2H-1}(z-a)^{-1} da = B(2H, H).
\]

Similarly, the change of variables \( a = T-w \) and the L'Hôpital’s rule imply that

\[
\lim_{T \to \infty} J_2 = \lim_{T \to \infty} \frac{1}{e^{-T} T^{3H-1}} \int_0^T e^{-u} u^{H-1} du \int_{T-u}^T e^{-a} a^{2H-1} da = B(2H, H).
\]

Substituting the above three limits into (5.11), we obtain the first limit of (5.8).

The L'Hôpital’s rule and Corollary 5.3 imply that

\[
\lim_{T \to \infty} \int_0^T \phi(v, T)e^{v-T} \, dv = \lim_{T \to \infty} \frac{1}{e^{T} T^{3H-1}} \left[ \phi(T, T)e^T + \int_0^T \frac{\partial \phi(v, T)}{\partial v} e^v \, dv \right] = \lim_{T \to \infty} \frac{\phi(T, T)}{T^{3H-1}} = 2 [B(2H, H)H - 1],
\]

and when \( \beta > -2H \),

\[
\liminf_{T \to \infty} \frac{1}{T^{3H+\beta}} \int_0^T \phi(v, T)v^\beta \, dv \geq c \lim_{T \to \infty} \frac{\phi(T, T)}{T^{3H-1}} \geq c,
\]

\[
\limsup_{T \to \infty} \frac{1}{T^{3H+\beta}} \int_0^T \phi(v, T)v^\beta \, dv \leq C \lim_{T \to \infty} \frac{1}{T^{3H+\beta}} \left[ \phi(T, T)T^\beta + T^{H-1} \int_0^T \left| \frac{\partial R_B(v, T)}{\partial v} \right| v^\beta \, dv \right] < \infty.
\]

With a minor modification of the last limit, we can also obtain the inequality (5.9). \( \square \)
Corollary 5.5. Let the function \( \phi(v, T) \) be given as in (5.2). When \( H \in (0, \frac{1}{2}) \), there exists a positive constant \( C \) independent on \( T \geq 1 \) such that
\[
\int_0^T \phi(v, T)(v^{2H-1} \wedge 1)dv \leq C \times \begin{cases} 
T^{\gamma_2}, & \text{when } H \neq \frac{1}{4}, \\
T^{\gamma_2} \log T, & \text{when } H = \frac{1}{4},
\end{cases}
\]
where
\[
\gamma_2 = \begin{cases} 
H, & \text{when } H \in (0, \frac{1}{4}], \\
5H - 1, & \text{when } H \in (\frac{1}{4}, \frac{1}{2}).
\end{cases}
\]

Proof. Clearly,
\[
\int_0^T \phi(v, T)(v^{2H-1} \wedge 1)dv = \int_0^1 \phi(v, T)dv + \int_1^T \phi(v, T)v^{2H-1}dv.
\]
The inequality (5.9) implies that when \( H \in (0, \frac{1}{2}) \),
\[
\int_1^T \phi(v, T)v^{2H-1}dv \leq C \times \begin{cases} 
T^{\gamma_2}, & \text{when } H \neq \frac{1}{4}, \\
T^{\gamma_2} \log T, & \text{when } H = \frac{1}{4},
\end{cases}
\]
By the L'Hôpital's rule and Corollary 5.3, we have for any \( T \geq 1 \),
\[
\limsup_{T \to \infty} \frac{1}{T^H} \int_0^1 \phi(v, T)dv \leq \limsup_{T \to \infty} \frac{C}{T^{H-1}} \int_0^1 \frac{\partial}{\partial v} R^B(v, T)dv 
\leq C \limsup_{T \to \infty} \int_0^1 v^{2H-1} + (1 - v)^{2H-1}dv < \infty,
\]
which implies that there exists a positive constant \( C \) independent on \( T \geq 1 \) such that
\[
\int_0^1 \phi(v, T)dv \leq CT^H.
\]
Plugging the inequalities (5.14)-(5.15) into the identity (5.13), we obtain the desired estimate (5.16).

\( \square \)

Lemma 5.6. Let the function \( \phi(v, T) \) be given as in (5.2). Denote
\[
\chi(T) = \frac{1}{T^{5H-1}} \int_0^T \phi(v, T)(T - v)^{2H-1}dv.
\]
When \( H \in (0, \frac{1}{2}) \), there exist two positive constant \( c, C \) independent on \( T \) such that
\[
c \leq \liminf_{T \to \infty} \chi(T) \leq \limsup_{T \to \infty} \chi(T) \leq C.
\]

Proof. First, we rewrite \( \chi(T) \) and then divide the domain \([0, T]^3\) of the integral into several parts by the order of the variables \( u, v, w \) as follows:
\[
\chi(T) = \frac{1}{T^{5H-1}} \int_{[0, T]^3} e^{-|u-w|}u^{H-1} \left| \frac{\partial}{\partial v} R^B(v, w) \right| (T - v)^{2H-1}dudwdv
= \frac{1}{T^{5H-1}} \left( \int_{T>u>v,w>0} + \int_{T>u>v,w>0} + \int_{T>v,w>u} + \int_{T>v,w>u} \right) 
\times e^{-|u-w|}u^{H-1} \left| \frac{\partial}{\partial v} R^B(v, w) \right| (T - v)^{2H-1}dudwdv
:= H \times (I_1 + I_2 + I_3 + I_4 + I_5),
\]
where
\[ I_1 = \frac{1}{T^{5H-1}} \int_{T > u > 0} e^{w-u} u^{H-1} (v^{2H-1} + (w-v)^{2H-1}) (T-v)^{2H-1} \, dv \, du := I_{11} + I_{12}. \]

By the L'Hôpital's rule, we have

\[
\lim_{T \to \infty} I_{11} \leq \lim_{T \to \infty} \frac{1}{T^{5H-1}} \int_0^T e^{-w} \int_0^w e^{u} u^{H-1} du \int_0^T v^{2H-1} (T-v)^{2H-1} \, dv = \frac{B(2H,2H)}{H}.
\]  
(5.17)

By the change of variables \( a = w-v, b = T-v \) and the L'Hôpital's rule again, we have

\[
\lim_{T \to \infty} I_{12} = \lim_{T \to \infty} \frac{1}{T^{5H-1}} \int_0^T b^{2H-1} db \int_0^b a^{2H-1} da \int_0^{T+b+a} u^{H-1} e^{u+b-a} du
\]
\[
= \lim_{T \to \infty} \frac{1}{T^{5H-1}} \int_0^T b^{2H-1} db \int_0^b a^{2H-1} (T-b+a-1)^{-1} da
\]
\[
= \int_0^1 y^{2H-1} dy \int_0^y x^{2H-1} (1-y+x)^{-1} \, dx.
\]  
(5.18)

Second, in the same vein, we have

\[
\lim_{T \to \infty} I_2 = \lim_{T \to \infty} \frac{1}{T^{5H-1}} \int_{T > a > 0} e^{w-a} a^{H-1} (v^{2H-1} + (w-v)^{2H-1}) (T-v)^{2H-1} \, dv \, da
\]
\[
\leq \frac{B(2H,2H)}{H} + \lim_{T \to \infty} \frac{1}{T^{5H-1}} \int_0^T b^{2H-1} db \int_0^b a^{2H-1} da \int_{T+b-a}^{T} e^{a-b-a} e^{H-1} du
\]
\[
= \frac{B(2H,2H)}{H} + \int_0^1 y^{2H-1} dy \int_0^y x^{2H-1} (1-y+x)^{-1} \, dx.
\]  
(5.19)

Third, it is clear that as \( T \to \infty, \)

\[
I_3 \leq \frac{1}{T^{5H-1}} \int_0^T u^{H-1} du \int_0^u e^{v-u} (T-v)^{2H-1} dv \int_0^v e^{u-v} (v-w)^{2H-1} \, dw
\]
\[
\leq \frac{1}{T^{5H-1}} \int_0^T u^{H-1} (T-u)^{2H-1} du = B(2H,H) \frac{1}{T^{2H}} \to 0,
\]  
(5.20)

and Lemma 5.1 implies that

\[
I_4 \leq \frac{1}{T^{5H-1}} \int_0^T (T-v)^{2H-1} dv \int_0^v (v-w)^{2H-1} dw \int_0^w e^{u-w} u^{H-1} du
\]
\[
\leq \frac{C}{T^{5H-1}} \int_0^T (T-v)^{2H-1} dv \int_0^v (v-w)^{2H-1} u^{H-1} dw = C \times \frac{\Gamma(2H) \Gamma(2H) \Gamma(H)}{\Gamma(5H)},
\]  
(5.21)

and

\[
I_5 \leq \frac{1}{T^{5H-1}} \int_0^T (T-v)^{2H-1} dv \int_0^v u^{H-1} du \int_0^u e^{w-u} (v-w)^{2H-1} \, dw
\]
\[
\leq \frac{1}{T^{5H-1}} \int_0^T (T-v)^{2H-1} dv \int_0^v (v-w)^{2H-1} u^{H-1} du = \frac{\Gamma(2H) \Gamma(2H) \Gamma(H)}{\Gamma(5H)},
\]  
(5.22)

Finally, combining the limits (5.17)-(5.22) together, we obtain the desired estimate (5.16).  

\[ \square \]

**Notation 4.** Denote by \( \delta_a(\cdot) \) the Dirac delta function centered at a point \( a \).
Notation 5. Let the function $f_T(u,v)$ be given as in (3.1). Denote the 1-th contraction between $f_T$ and $f_T$ in $F_{12}$ as a function 
$$\kappa(u_1,u_2) = f_T \otimes_1 f_T.$$ 

Lemma 5.7. When $H \in (0,\frac{1}{2})$, there exist a positive constant $C$ independent on $T \geq 1$ such that 
$$\left(\mu \times \mu\right)(|\kappa|) \leq CT^{\gamma_1},$$ 
where for any $\epsilon > 0$, 
$$\gamma_1 = \begin{cases} 
H, & \text{when } H \in (0,\frac{1}{4}), \\
H + \epsilon, & \text{when } H = \frac{1}{4}, \\
4H - 1, & \text{when } H \in (\frac{1}{4},\frac{1}{2}). 
\end{cases}$$ 

Proof. For simplicity, we assume that $\theta = 1$. First, since for any $-\infty < a < b < \infty$, 
$$\frac{d}{dx}1_{[a,b]}(x) = \delta_a(x) - \delta_b(x),$$ 
we have when $u_1,u_2 \in [0,T]$, 
$$\kappa(u_1,u_2) = -\int_{[0,T]^2} f_T(u_1,v_1) \frac{\partial}{\partial v_2} f_T(u_2,v_2) \frac{\partial R_B(v_1,v_2)}{\partial v_1} dv_1 dv_2$$ 
$$= -\int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} \frac{\partial R_B(v_1,v_2)}{\partial v_1} dv_1 dv_2$$ 
$$-\int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} \frac{\partial R_B(v_1,v_2)}{\partial v_1} (\delta_0(v_2) - \delta_T(v_2)) dv_1 dv_2$$ 
$$:= H \times (I_1 + I_2).$$ 
It is clear that 
$$|I_1| \leq \int_{0 \leq v_1 \leq v_2 \leq T} e^{-|u_1-v_1|-|u_2-v_2|}((v_2-v_1)^{2H-1} + v_1^{2H-1}) dv_1 dv_2$$ 
$$+ \int_{0 \leq v_2 \leq v_1 \leq T} e^{-|u_1-v_1|-|u_2-v_2|}(v_1-v_2)^{2H-1} dv_1 dv_2$$ 
$$:= I_{11} + I_{12},$$ 
and 
$$I_2 = e^{-(T-v_2)} \int_{[0,T]} e^{-|u_1-v_1|((T-v_1)^{2H-1} + v_1^{2H-1})} dv_1.$$ 
Hence, we have 
$$\left(\mu \times \mu\right)(|\kappa|) \leq H \times (\mu \times \mu)(I_{11} + I_{12} + I_2).$$ 
(5.26)

Second, let $\psi(\cdot,T)$ be given as in (5.1). The L'Hôpital's rule, the identity (5.3), the inequalities (5.4)-(5.5) and Corollary 5.2 imply that when $H \in (0,\frac{1}{2})$,
$$\limsup_{T \to \infty} \frac{1}{T^{\gamma_1}}(\mu \times \mu)(I_{11})$$ 
$$= \limsup_{T \to \infty} \frac{1}{T^{\gamma_1}} \int_{0 \leq v_1 \leq v_2 \leq T} \psi(v_1,T)\psi(v_2,T)((v_2-v_1)^{2H-1} + v_1^{2H-1}) dv_1 dv_2$$ 
(5.27)
and

\[ \limsup_{T \to \infty} \gamma T^{-1} \psi(T, T) \int_0^T \psi(v_1, T)(v_1 - v_1)^{2H-1} + v_1^{2H-1}) \, dv_1 \]

\[ + \limsup_{T \to \infty} \gamma T^{-1-H} \int_{0 \leq v_1 \leq v_2 \leq T} \left[ \left( e^{v_2-T} \psi(v_1, T) + e^{v_1-T} \psi(v_2, T) \right) \left( (v_2 - v_1)^{2H-1} + v_1^{2H-1} \right) \right] \, dv_1 \, dv_2 \]

\[ \leq \limsup_{T \to \infty} C T^{-1-H} \int_0^T \psi(v, T)(v - v)^{2H-1} + v^{2H-1}) \, dv < \infty, \]

and

\[ \limsup_{T \to \infty} \gamma T^{-1} (\mu + \mu)(I_2) = \lim_{T \to \infty} \gamma T^{-1} \psi(T, T) \int_0^T \psi(v, T) \left( (T - v)^{2H-1} + v^{2H-1} \right) \, dv = 0. \]

Plugging the above three limits into (5.26), we obtain the desired (5.23).

\[ \square \]

**Lemma 5.8.** Let the function \( f_T(u, v) \) be given as in (3.1). Suppose that \( u_1, u_2 \in [0, T] \). Denote

\[ \varphi(u_1, u_2) = \int_{[0, T]^2} f_T(u_1, v_1)f_T(u_2, v_2) \frac{\partial^2}{\partial v_1 \partial v_2}(R(v_1, v_2) - R_B(v_1, v_2)) \, dv_1 \, dv_2. \]

When \( H \in (0, \frac{1}{2}) \), there exists a constant \( C_{\theta, H} > 0 \) such that

\[ \| \varphi \|^2_{\delta_1 \otimes \delta_2} \leq C_{\theta, H} T^{2\gamma_1}, \]

where \( \gamma_1 \) is given in (5.24).

**Proof.** First, without loss of generality, we assume \( \theta = 1 \). The inequality (2.5) implies that

\[ \| \varphi \|^2_{\delta_1 \otimes \delta_2} \leq \| \varphi \|^2_{\delta_1 \otimes \delta_2} + \left( C_H' \left( \mu \times \mu \right) \left( \| \varphi \| \right) \right)^2 + 2C_H' \left( \mu \times \mu \right) \left( \| \varphi \| \otimes \| \varphi \| \right). \]

(5.30)

Second, it is clear that

\[ \| \varphi \|^2_{\delta_1 \otimes \delta_2} = \int_{[0, T]^4} \frac{\partial^2}{\partial u_1 \partial w_2} \varphi(u_1, u_2) \varphi(w_1, w_2) \frac{\partial}{\partial w_1} R_B(u_1, w_1) \frac{\partial}{\partial u_2} R_B(u_2, u_2) \, du_1 \, dw_1 \, dw_2 \, dw_1. \]

and

\[ \frac{\partial}{\partial u_1} \varphi(u_1, u_2) = \int_{[0, T]^2} e^{-|u_1 - v_1| - |u_2 - v_2|} \frac{\partial^2}{\partial v_1 \partial v_2} (R(v_1, v_2) - R_B(v_1, v_2)) \, dv_1 \, dv_2 \times [\chi_{[0, T]^2}(u_1, u_2) \text{sgn}(v_1 - u_1) + \chi_{[0, T]}(u_2)(\delta_1(u_1) - \delta_T(u_1))]. \]
Under Hypothesis 1.2, from the assumed inequality (1.7), we have

\[
\|\varphi\|_{\mathcal{B}^{1,2}}^2 \leq C \times (I_1 + I_2 + I_3),
\]  

(5.31)

where

\[
I_1 = \int_{[0,T]^2} \int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} (v_1 v_2)^{H-1} dv_1 dv_2 \left( \int_{[0,T]^2} e^{-|w_1-w_1'|-|w_2-w_2'|} (v_1' v_2')^{H-1} dw_1 dw_2 \right) \\
\times \left| \frac{\partial}{\partial u_1} R^B(u_1, w_1) \right| \left| \frac{\partial}{\partial w_2} R^B(u_2, w_2) \right| du_1 dw_2 dw_1 \\
= \left[ \int_{[0,T]^2} \psi(u_1, T) \psi(w_1, T) \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| du_1 dw_1 \right]^2 \leq C T^{2\gamma_1},
\]

where \(\psi(u, T)\) is given as in (5.1) and the last inequality is from (5.27)-(5.28). In the same vein, the inequalities (5.4)-(5.5) and the identities (5.27)-(5.28) imply that

\[
I_2 = \int_{[0,T]^2} \int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} (v_1 v_2)^{H-1} dv_1 dv_2 \left( \int_{[0,T]^2} e^{-|w_1-w_1'|-(T-w_2')} (v_1' v_2')^{H-1} dw_1 dw_2 \right) \\
\times \left| \frac{\partial}{\partial u_1} R^B(u_1, w_1) \right| \left| \frac{\partial}{\partial w_2} R^B(u_2, T) \right| du_1 dw_2 dw_1 \\
= \int_{[0,T]^2} \psi(u_1, T) \psi(w_1, T) \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| du_1 dw_1 \int_{[0,T]} \psi(u_2, T) \psi(T, T) \left| \frac{\partial}{\partial w_2} R^B(u_2, T) \right| du_2 \\
\leq C T^{2\gamma_1 - 1},
\]

and

\[
I_3 = \int_{[0,T]^2} \int_{[0,T]^2} e^{-(T-w_1)-|u_2-v_2|} (v_1 v_2)^{H-1} dv_1 dv_2 \left( \int_{[0,T]^2} e^{-|w_1-w_1'|-(T-w_2')} (v_1' v_2')^{H-1} dw_1 dw_2 \right) \\
\times \left| \frac{\partial}{\partial u_1} R^B(T, w_1) \right| \left| \frac{\partial}{\partial w_2} R^B(u_2, T) \right| du_1 dw_1 \\
= \int_{[0,T]} \psi(u_2, T) \psi(T, T) \left| \frac{\partial}{\partial w_2} R^B(u_2, T) \right| du_2 \right]^2 \leq C T^{2(\gamma_1 - 1)}.
\]

Plugging the above three estimates into (5.31), we have

\[
\|\varphi\|_{\mathcal{B}^{1,2}}^2 \leq C T^{2\gamma_1}.
\]  

(5.32)

Third, it is clear that when \(H \in (0, \frac{1}{2})\)

\[
(\mu \times \mu)(\|\varphi\|) \leq C \times \int_{[0,T]^2} \left( \int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} (v_1 v_2)^{H-1} dv_1 dv_2 \right) (u_1 u_2)^{H-1} du_1 du_2 \\
= C \times \left[ \int_{[0,T]^2} e^{-|u_1-v_1|} (u_1 v_1)^{H-1} du_1 dv_1 \right]^2 \\
\leq C \times \left[ \int_{0}^{T} (1 \wedge v_1^{H-1}) v_1^{H-1} dv_1 \right]^2 \leq C,
\]

(5.33)

where we use Lemma 5.1 in the last line. It is clear that

\[
(\varphi \otimes_1 \varphi)(u_2, w_2) = \int_{[0,T]^2} \frac{\partial}{\partial u_1} \phi(u_1, u_2) \phi(w_1, w_2) \frac{\partial}{\partial w_1} R^B(u_1, w_1) du_1 dw_1,
\]
and

\[(\mu \times \mu)(|\varphi \otimes \varphi|) \leq C \times (J_1 + J_2),\]

where

\[J_1 = \int_{[0,T]^4} \int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} e^{-|w_1-v'_1|-|w_2-v'_2|} \text{d}v_1 \text{d}v_2 \text{d}u_1 \text{d}u_2 \]

\[\times \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| (u_2 w_2)^{H-1} \text{d}v_1 \text{d}v_2 \text{d}u_1 \text{d}w_1 \text{d}w_2 \]

\[\leq C \times \int_{[0,T]^2} \psi(u_1, T) \psi(w_1, T) \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| \text{d}u_1 \text{d}w_1 \leq CT^{\gamma_1},\]

where we use the inequality (5.33) in the last line. In the same vein, we have

\[J_2 = \int_{[0,T]^3} \int_{[0,T]^2} e^{-(T-v_1)-|u_2-v_2|} e^{-|w_1-v'_1|-|w_2-v'_2|} \text{d}v_1 \text{d}v_2 \text{d}u_1 \text{d}w_2 \text{d}w_1 \]

\[\times \left| \frac{\partial}{\partial w_1} R^B(T, w_1) \right| (u_2 w_2)^{H-1} \text{d}v_1 \text{d}w_1 \text{d}w_2 \]

\[\leq C \times \psi(T, T) \times \int_{0}^{T} \psi(w_1, T) \left| \frac{\partial}{\partial w_1} R^B(T, w_1) \right| \text{d}w_1 \leq CT^{\gamma_1-1}.\]

Hence,

\[(\mu \times \mu)(|\varphi \otimes \varphi|) \leq C \times T^{\gamma_1}. \quad (5.34)\]

Finally, substituting (5.32)-(5.34) into (5.30), we obtain the desired (5.29). \hfill \square

**Lemma 5.9.** Let \( \phi(\cdot, T) \) be given as in (5.2). Denote

\[\gamma = \begin{cases} 4H, & \text{when } H \in (0, \frac{1}{2}), \\ 8H - 1, & \text{when } H \in (\frac{1}{2}, \frac{1}{2}). \end{cases} \quad (5.35)\]

When \( H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1}{2}) \), there exists a positive constant \( C \) independent on \( T \geq 1 \) such that

\[\int_{T > u > v_1, v_1' > 0} e^{-|u-v_1|-|w-v_1'|} \phi(v_1, T) \left| \frac{\partial R^B(v_1', T)}{\partial v_1} \right| \text{d}v_1 \text{d}v_1' \text{d}u \text{d}w \leq CT^{\gamma-H}, \quad (5.36)\]

\[\int_{T > u > v_1, v_1' > 0} e^{-|u-v_1|-|w-v_1'|} \phi(v_1, T) \left| \frac{\partial R^B(u, w)}{\partial v_1} \right| \text{d}v_1 \text{d}v_1' \text{d}u \text{d}w \leq CT^{\gamma-H}, \quad (5.37)\]

\[\int_{T > u > v_1', v_1' > 0} e^{-|u-v_1'|-|w-v_1|} \phi(v_1', T) \left| \frac{\partial R^B(u, w)}{\partial v_1} \right| \text{d}v_1 \text{d}v_1' \text{d}u \text{d}w \leq CT^{\gamma-H}, \quad (5.38)\]

\[\int_{T > u > v_1', v_1' > 0} e^{-|u-v_1'|-|w-v_1|} \phi(v_1', T) \left| \frac{\partial R^B(u, w)}{\partial w} \right| \text{d}v_1 \text{d}v_1' \text{d}u \text{d}w \leq CT^{\gamma-H}. \quad (5.39)\]

When \( H = \frac{1}{2} \), all of the upper bounds can be replaced by \( T^{\gamma-H} \log T \).
Proof. We will the inequality (5.36) firstly. To this end, we claim that there exists a positive constant $C$ independent on $T \geq 1$ such that for any fixed $v_1 \in (0,T)$,

$$
\int_{T > v_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} v_1^{2H-1} |\frac{\partial R^B(u, w)}{\partial w}| dv'_1 dw du \leq CT^{2H} (v_1^{2H-1} \land 1) \quad (5.40)
$$

and

$$
\int_{T > v'_1 > v_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} (T - v'_1)^{2H-1} |\frac{\partial R^B(u, w)}{\partial w}| dv'_1 dw du \leq CT^{2H} (T - v_1)^{2H-1} \quad (5.41)
$$

In fact, we can obtain them by dividing $\{T > v'_1 > v_1, u, w > 0\}$, the domain of the triple integral, into six sub-domain according to the distinct order of $v_1, u, w$, doing suitable changes of variables and then applying Lemma 5.1 to these triple integrals. Since this calculation is very elementary, we ignore the details.

Next, it follows from the inequalities (5.40)-(5.41) that when $H \in (0, \frac{1}{2})$,

$$
\int_{T > v'_1 > v_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} \phi(v_1, T) |\frac{\partial R^B(v'_1, T)}{\partial v'_1}| |\frac{\partial R^B(u, w)}{\partial w}| dv_1 dv'_1 dw du
$$
\begin{align*}
&= H \int_{T > v'_1 > v_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} \phi(v_1, T) v_1^{2H-1} |\frac{\partial R^B(u, w)}{\partial w}| dv_1 dv'_1 dw du \\
&+ H \int_{T > v'_1 > v_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} \phi(v_1, T) (T - v'_1)^{2H-1} |\frac{\partial R^B(u, w)}{\partial w}| dv_1 dv'_1 dw du \\
&\leq C \times T^{2H} \int_0^T \phi(v_1, T) [(v_1^{2H-1} \land 1) + (T - v_1)^{2H-1}] dv_1 \leq C \times T^{\gamma - H},
\end{align*}

where the last line is from Corollary 5.5 and Lemma 5.6.

In the same vein, we have that there exists a positive constant $C$ independent on $T \geq 1$ such that for any fixed $v'_1 \in (0, T)$,

$$
e^{-v'_1} \int_{(0,v'_1)^3} e^{-|u-v_1| + w(T - v_1)^{2H-1} |\frac{\partial R^B(u, w)}{\partial w}|} dv_1 dw du \leq C \times (v'_1)^{(4H-1)}_+,$$

where $a_+ = \max\{a, 0\}$, and

$$
e^{-v'_1} \int_{(0,v'_1)^3} e^{-|u-v_1| + w(T - v_1)^{2H-1} |\frac{\partial R^B(u, w)}{\partial w}|} dv_1 dw du \leq C \times T^{2H} (T - v'_1)^{2H-1},
$$

which, together with Lemma 5.4 and Lemma 5.6, implies that the inequality (5.37) holds.

Similarly, the inequalities (5.38)-(5.39) are from Corollary 5.5 and the following estimates: there exists a positive constant $C$ independent on $T \geq 1$ such that for any fixed $v_1 \in (0, T)$,

$$
\int_{T > v_1, v'_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} \frac{\partial R^B(v'_1, T)}{\partial v'_1} |\frac{\partial R^B(u, w)}{\partial w}| dv_1 dv'_1 dw du \leq C,
$$

and respectively, for any fixed $v'_1 \in (0, T)$,

$$
\int_{T > v_1, v'_1, u, w > 0} e^{-|u-v_1| - |w-v'_1|} \frac{\partial R^B(v_1, T)}{\partial v_1} |\frac{\partial R^B(u, w)}{\partial w}| dv_1 dv'_1 dw du
\leq C \times T^{2H} \times [(T - v'_1)^{2H-1} + ((v'_1)^{2H-1} \land 1)].$$
Lemma 5.10. Let $\kappa(u_1, u_2)$ be given in Notation 5 and $\gamma$ be given as in (5.35). There exists a positive constant $C$ independent on $T \geq 1$ such that

$$
(\mu \times \mu)(|\kappa \otimes_1^\gamma \kappa|) \leq C \times \begin{cases} T^\gamma, & \text{when } H \neq \frac{1}{4}, \\ T^\gamma \log T, & \text{when } H = \frac{1}{4}. \end{cases}
$$

(5.42)

Proof. For simplicity, we only show the case of $H \neq \frac{1}{4}$. First, recall that

$$
\kappa(u_1, u_2) = -\int_{[0,T]^2} f_T(u_1, v_1) \frac{\partial}{\partial v_2} f_T(u_2, v_2) \frac{\partial}{\partial v_1} R^B(v_1, v_2) dv_1 dv_2
$$

$$
= \left[ \int_{[0,T]^2} e^{-|u_1-v_1|-|u_2-v_2|} \text{sgn}(v_2-u_2) \frac{\partial}{\partial v_1} R^B(v_1, v_2) dv_1 dv_2 + \int_{[0,T]} e^{-|v_1-v_2|-|v_2-u_1|} \frac{\partial}{\partial v_1} R^B(v_1, T) dv_1 \right] 1_{[0,T]^2}(u_1, u_2).
$$

Hence,

$$
(\kappa \otimes_1^\gamma \kappa)(u_2, w_2) = -\int_{[0,T]^2} \frac{\partial}{\partial u_1} \kappa(u_1, u_2) \frac{\partial}{\partial w_1} R^B(u_1, w_1) du_1 dw_1,
$$

and similar to the inequality (5.26), we have

$$
(\mu \times \mu)(|\kappa \otimes_1^\gamma \kappa|) \leq C \times (K_1 + K_2 + K_3 + K_4 + K_5 + K_6),
$$

(5.43)

where

$$
K_1 = \int_{[0,T]^4} e^{-|u_1-v_1|-|u_2-v_2|} \left| \frac{\partial}{\partial v_1} R^B(v_1, v_2) \right| e^{-|w_1-v'_1|-|w_2-v'_2|} \left| \frac{\partial}{\partial v'_1} R^B(v'_1, v'_2) \right| \left| \frac{\partial}{\partial u_1} R^B(u_1, w_1) \right| (u_2 w_2)^H \left| \frac{\partial}{\partial u_2} R^B(u_2, w_2) \right| dv_1 dv_2 dw_1 dw_2 dv_1 dv_2 dv'_1 dv'_2.
$$

We divide the integral $K_1$ by $T^\gamma$ and separate the domain of the integration into four parts as follows:

$$
\frac{1}{T^\gamma} \int_{[0,T]^4} \phi(v_1, T) \phi(v'_1, T) e^{-|u_1-v_1|-|w-v'_1|} \left| \frac{\partial}{\partial u} R^B(u, w) \right| dv_1 dv'_1 dw
$$

$$
= \frac{1}{T^\gamma} \left( \int_{T^{v'_1} > v_2, u, u > 0} + \int_{T^{v'_1} > v_2, u, u > 0} + \int_{T^{v_2} > v_1, v'_1, w > 0} + \int_{T^{v_2} > v_1, v'_1, w > 0} \right) \phi(v_1, T) \phi(v'_1, T) e^{-|u_1-v_1|-|w-v'_1|} \left| \frac{\partial}{\partial u} R^B(u, w) \right| dv_1 dv'_1 dw
$$

$$
:= L_1 + L_2 + L_3 + L_4.
$$

(5.44)

The L'Hôpital's rule, Lemma 5.1, Corollary 5.2, Lemma 5.4 and Corollary 5.3 imply that

$$
\lim_{T \to \infty} L_1
$$

(5.45)
\[
\begin{aligned}
&\leq c \times \limsup_{T \to \infty} \left[ \frac{1}{T^{\gamma-3H+1}} \int_{[0,T]^3} \phi(v_1, T)e^{-|u-v_1|+w} \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v u \text{d}w u \\
&+ \frac{1}{T^{\gamma-H}} \int_{T_{v_1'}, v_1', u, w > 0} e^{-|u-v_1|+|w-v_1'|} \phi(v_1, T) \left| \frac{\partial R^B(v_1', T)}{\partial v_1'} \right| \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v' \text{d}v u \text{d}w u \\
&+ \frac{1}{T^{\gamma-H}} \int_{T_{v_1'}, v_1', u, w > 0} e^{-|u-v_1|+|w-v_1'|} \phi(v_1', T) \left| \frac{\partial R^B(v_1, T)}{\partial v_1} \right| \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v' \text{d}v u \text{d}w u \\
\right] \\
&:= L_{11} + L_{12} + L_{13} + L_{14} + L_{15},
\end{aligned}
\]
where
\[
L_{11} = \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} e^{-|u-v_1|+w} \phi(v_1, T) \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v u \text{d}w u 
\]
\[
\leq H \times \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{[0,T]^2} \phi(v_1, T)e^{v_1+w}(u^{2H-1} + (T - u)^{2H-1}) \text{d}v_1 \text{d}w u 
\]
+ \[C \times \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} e^{-|u-v_1|+w} \frac{\partial R^B(v_1, T)}{\partial v_1} \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v' \text{d}v u \text{d}w u 
\]
= 0,
\]
and
\[
L_{12} = \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} e^{v_1+w} \phi(v_1, T) \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v u \text{d}w u 
\]
\[
\leq C \times \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{[0,T]^2} e^{v_1+w} \frac{\partial R^B(u, w)}{\partial w} \text{d}w u 
\]
+ \[C \times \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} e^{-|u-v_1|+w} \frac{\partial R^B(v_1, T)}{\partial v_1} \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v' \text{d}v u \text{d}w u 
\]
= 0,
and
\[
L_{13} = \limsup_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} \phi(v_1, T)e^{-|u-v_1|+w} \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v u \text{d}w u 
\]
\[
\leq \limsup_{T \to \infty} \frac{1}{T^{\gamma-H}} \int_{[0,T]^2} \phi(v_1, T)e^{-|u-v_1|} (T - u)^{2H-1} \text{d}v_1 \text{d}w u 
\]
+ \[C \times \lim_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} e^{-|u-v_1|+w} \frac{\partial R^B(v_1, T)}{\partial v_1} \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v' \text{d}v u \text{d}w u 
\]
\[
\leq C \times \limsup_{T \to \infty} \frac{\int_{[0,T]} \phi(v, T)(T - v)^{2H-1} d v}{T^{\gamma-H}} < \infty,
\]
where the last inequality is from Lemma 5.6. Lemma 5.9 implies that
\[
L_{14} + L_{15} = \limsup_{T \to \infty} \frac{1}{e^{T_{\gamma-3H+1}}} \int_{T_{v_1', v_1', u, w > 0}} e^{-|u-v_1|+|w-v_1'|} \left| \frac{\partial}{\partial w} R^B(u, w) \right| \text{d}v_1 \text{d}v' \text{d}v u \text{d}w u
\]
\[
\times \left[ \phi(v_1, T) \frac{\partial R^B(v_1', T)}{\partial v_1'} + \phi(v_1', T) \frac{\partial R^B(v_1, T)}{\partial v_1} \right] < \infty.
\]
Hence,

$$\limsup_{T \to \infty} L_1 = L_{11} + L_{12} + L_{13} + L_{14} + L_{15} < \infty.$$  \hfill (5.49)

In the same vein, we have

$$\limsup_{T \to \infty} L_2 < \infty.$$  \hfill (5.50)

The L'Hôpital's rule, Corollary 5.3 and Lemma 5.4 imply that

$$\limsup_{T \to \infty} L_3 \leq \limsup_{T \to \infty} \frac{1}{\gamma T^{\gamma-1}} \int_0^T \phi(v_1, T)e^{-(T-v_1)}dv_1 \int_{[0,T]^2} \phi(v_1', T)e^{-|v_1'|}\left|\frac{\partial}{\partial v_1}R^B(T, w)\right|dv_1'dw$$

$$+ \varepsilon \limsup_{T \to \infty} \frac{1}{\gamma T^{-H}} \left[ \int_{T^2} (\phi(v_1, T)\frac{\partial}{\partial v_1'}R^B(v_1', T) + \phi(v_1', T)\frac{\partial}{\partial v_1}R^B(v_1, T))e^{-|v_1-v'|}\left|\frac{\partial}{\partial v_1'}R^B(u, w)\right|dv_1'dv_1'dudw \right]$$

$$\leq \limsup_{T \to \infty} \frac{1}{\gamma T^{\gamma-3H}} \int_{[0,T]^2} \phi(v_1', T)e^{-|v_1'|}(T-w)^{2H-1}dv_1'dw + C < \infty,$$  \hfill (5.51)

where we use (5.48) and the inequalities (5.38)-(5.39) in the last line. In the same vein, we have

$$\limsup_{T \to \infty} L_4 < \infty.$$  \hfill (5.52)

Substituting the limits (5.49)-(5.52) into (5.44), we have that when $H \in (0, \frac{1}{2})$, there exist a positive constant $C$ such that

$$\limsup_{T \to \infty} \frac{K_1}{T^{\gamma}} = \limsup_{T \to \infty}[L_1 + L_2 + L_3 + L_4] \leq C,$$

which implies that there exist a positive constant $C$ such that

$$K_1 \leq CT^\gamma.$$

Second, it is clear that when $H \in (0, \frac{1}{2})$,

$$K_2 = \int_{[0,T]^7} e^{-|T-v_1|-|w_2-v_2|} \left|\frac{\partial}{\partial v_1}R^B(v_1, v_2)\right| e^{-|w_1-v_1|-|w_2-v_2|} \left|\frac{\partial}{\partial v_1'}R^B(v_1', v_2)\right|$$

$$\times \left|\frac{\partial}{\partial w_1}R^B(T, w_1)\right| (w_2v_2)^{H-1}dw_1dw_2dv_1dv_2dv_1'dv_2$$

$$= \int_0^T \phi(v_1, T)e^{-(T-v_1)}dv_1 \int_{[0,T]^3} \phi(v_1', T)e^{-|v_1'|}\left|\frac{\partial}{\partial w}R^B(T, w)\right|dv_1'dw,$$

$$\leq CT^{\gamma-1},$$

where the last line is from the expression of $L_3$ and its limit (5.51).

Third, it is clear that there exists a positive constant $C$ independent of $T$ such that

$$\int_0^T \left|\frac{\partial}{\partial w_1}R^B(u_1, w_1)\right| du_1 \leq C \times T \times \left|\frac{\partial}{\partial w_1}R^B(u_1, T)\right|,$$
which, combined with Lemma 5.1 and Corollary 5.2, implies that when $H \in (0, \frac{1}{2})$,

$$K_3 = \int_{[0,T]^2} e^{-|u_1-v_1|-|T-u_2|} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| e^{-|w_1-v'_1|-|w_2-v'_2|} \left| \frac{\partial}{\partial v_1'} R^B(v'_1, v'_2) \right|$$

$$\times \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| (u_2w_2)^{H-1} dw_1 dw_2 dv_1 dv'_1 dv'_2.$$ 

$$\leq CT^{H-1} \int_{[0,T]} \phi(v'_1, T) dv'_1 \int_{[0,T]^3} e^{-|w_1-v'_1|-|w_1-v'_1|} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| dv_1 dw_1$$

$$\leq CT^{H} \int_{[0,T]} \phi(v'_1, T) dv'_1 \int_{[0,T]} e^{-|w_1-v'_1|} \left| \frac{\partial}{\partial w_1} R^B(w_1, T) \right| dv_1$$

$$\leq CT^{H} \int_{[0,T]} \phi(v'_1, T) dv'_1 \leq CT^\gamma,$$

where the last inequality is from Lemma 5.4.

Fourth, Lemma 5.1 and Corollary 5.2 imply that when $H \in (0, \frac{1}{2})$,

$$K_4 = \int_{[0,T]^3} e^{-(T-v_1)-|T-u_2|} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| e^{-|w_1-v'_1|-|w_2-v'_2|} \left| \frac{\partial}{\partial v_1'} R^B(v'_1, v'_2) \right|$$

$$\times \left| \frac{\partial}{\partial w_1} R^B(T, w_1) \right| (u_2w_2)^{H-1} dw_1 dw_2 dv_1 dv'_1 dv'_2.$$ 

$$\leq CT^{H-1} \int_{[0,T]^3} \phi(v'_1, T) e^{-|v_1-v'_1|-|v_1-v'_1|} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| \left| \frac{\partial}{\partial w_1} R^B(T, w_1) \right| dv_1 dv_1 dv'_1 dv'_1$$

$$\leq CT^{H-1} \int_{[0,T]} \phi(v'_1, T) dv'_1 \leq CT^\gamma,$$

where the last inequality is from Lemma 5.4.

Fifth, Lemma 5.1 and Corollary 5.2 imply that when $H \in (0, \frac{1}{2})$,

$$K_5 = \int_{[0,T]^6} e^{-|u_1-v_1|-|T-u_2|} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| e^{-|w_1-v'_1|-|T-w_2|} \left| \frac{\partial}{\partial v_1'} R^B(v'_1, T) \right|$$

$$\times \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| (u_2w_2)^{H-1} dw_1 dw_2 dv_1 dv'_1 dv'_1$$

$$\leq CT^{2(H-1)} \int_{[0,T]^2} \left| \frac{\partial}{\partial w_1} R^B(u_1, w_1) \right| dw_1$$

$$\leq CT^\gamma,$$

and

$$K_6 = \int_{[0,T]^3} e^{-(T-v_1)-|T-u_2|} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| e^{-|w_1-v'_1|-|T-w_2|} \left| \frac{\partial}{\partial v_1'} R^B(v'_1, T) \right|$$

$$\times \left| \frac{\partial}{\partial w_1} R^B(T, w_1) \right| (u_2w_2)^{H-1} dw_1 dw_2 dv_1 dv'_1$$

$$\leq CT^{2(H-1)} \int_{[0,T]} \left| \frac{\partial}{\partial w_1} R^B(T, w_1) \right| dw_1.$$
Substituting these upper bounds of $K_i$, $i = 1, \ldots, 6$ into (5.43), we obtain the desired (5.42).

\[ \leq CT^\gamma. \]

Lemma 5.11. Let $0 \leq s < t \leq T$ and $\phi_1(u,v)$ be given as in (3.12). Then there exists a constant $C > 0$ independent of $T$ such that for all $s, t \geq 0$,

\[ \|\phi_1\|_2^2 \leq C \left( |t-s|^{4H} + |t-s|^{4H+1} + |t-s|^{4H+2} \right), \]

\[ (\mu \times \mu)(|\phi_1 \otimes_1 \phi_1|) \leq C \left( |t-s|^{4H} + |t-s|^{4H+1} \right). \]

Proof. For simplicity, we can assume that $\theta = 1$. Denote $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. First, by the identity (5.25), we have

\[ \|\phi_1\|_2^2 = \int_{[0,T]^4} \frac{\partial^2}{\partial u_1 \partial v_2} \left[ e^{-|u_1-v_1|} e^{-|u_2-v_2|} \mathbb{1}_{[s,t]}(u_1, u_2, v_1, v_2) \right] \frac{\partial}{\partial u_2} R^B(u_1, u_2) \frac{\partial}{\partial v_1} R^B(v_1, v_2) \, \, \, d\vec{u} d\vec{v} \]

\[ = \int_{[0,T]^4} e^{-|u_1-v_1|} e^{-|u_2-v_2|} \mathbb{1}_{[s,t]}(u_1, v_1) \frac{\partial}{\partial u_2} R^B(u_1, u_2) \frac{\partial}{\partial v_1} R^B(v_1, v_2) \]

\[ \times \mathbb{1}_{[s,t]}(u_2, v_2) \text{sgn}(u_1 - v_1) \text{sgn}(v_2 - u_2) - \mathbb{1}_{[s,t]}(u_1) \mathbb{1}_{[s,t]}(v_1) \delta_s(u_1 - v_1)(\delta_t(v_2) - \delta_t(v_2)) \]

\[ - \mathbb{1}_{[s,t]}(v_2) \text{sgn}(v_2 - u_2)(\delta_s(u_1) - \delta_s(u_1)) + (\delta_s(v_2) - \delta_t(v_2))(\delta_s(u_1) - \delta_t(u_1)) \]

\[ =: I_1 + I_2 + I_3 + I_4. \]

It is cleat that

\[ |I_1| \leq \int_{[s,t]^4} \left| \frac{\partial}{\partial u_2} R^B(u_1, u_2) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, v_2) \right| \, \, \, d\vec{u} d\vec{v} = \left( \int_{[s,t]^2} \left| \frac{\partial R^B(u_1, u_2)}{\partial u_2} \right| d\vec{u} \right)^2, \]

and

\[ \int_{s \leq u_1 \leq u_2 \leq t} \left| \frac{\partial R^B(u_1, u_2)}{\partial u_2} \right| d\vec{u} \leq H \int_{s \leq u_1 \leq u_2 \leq t} (u_2 - u_1)^{2H-1} d\vec{u} = \frac{1}{2(2H+1)} (t-s)^{2H+1}. \]

By the fact $1-x^\beta \leq (1-x)^\beta$ for any $x \in [0,1]$ and $\beta \in (0,1)$, it is clear that when $H \in (0, \frac{1}{2})$,

\[ \int_{s \leq u_2 \leq u_1 \leq t} \left| \frac{\partial R^B(u_1, u_2)}{\partial u_2} \right| d\vec{u} \leq H \int_{s \leq u_2 \leq u_1 \leq t} ((u_1 - u_2)^{2H-1} + u_2^{2H-1}) d\vec{u} \]

\[ \leq \frac{1}{2H+1} (t-s)^{2H+1}. \]

Plugging the above two formula into (5.56), we have

\[ |I_1| \leq C |t-s|^{4H+2}. \]

In the same vein, we have

\[ |I_2| + |I_3| + |I_4| \leq C \left( |t-s|^{4H} + |t-s|^{4H+1} \right). \]

Hence, we obtain the inequality (5.53).

Next, similar to the inequality (5.26), we have

\[ (\mu \times \mu)(|\phi_1 \otimes_1 \phi_1|) \leq H \times (\mu \times \mu)(J_{11} + J_{12} + J_{2} + J_{3}). \]
where
\[
J_{11} = \int_{s \leq u_1 \leq v_2 \leq t} e^{-|u_1 - v_1| - |u_2 - v_2|} ((v_2 - v_1)^{2H-1} + v_1^{2H-1}) d
v_1 dv_2 \|_{s \leq t}^{\theta}(u_1, u_2)
\]
\[
J_{12} = \int_{s \leq u_2 \leq v_1 \leq t} e^{-|u_1 - v_1| - |u_2 - v_2|} ((v_1 - v_2)^{2H-1} - v_1^{2H-1}) d
v_1 dv_2 \|_{s \leq t}^{\theta}(u_1, u_2)
\]
\[
J_2 = e^{-(t-u_2)} \int_s^t e^{-|u_1 - v_1|} ((t - v_1)^{2H-1} + v_1^{2H-1}) d
v_1 \|_{s \leq t}^{\theta}(u_1, u_2)
\]
\[
J_3 = e^{-(u_2 - s)} \int_s^t e^{-|u_1 - v_1|} ((v_1 - s)^{2H-1} - v_1^{2H-1}) d
v_1 \|_{s \leq t}^{\theta}(u_1, u_2)
\]

It is clear that
\[
(\mu \times \mu)(J_{11}) = \int_{s \leq u_1 \leq v_2 \leq t} e^{-|u_1 - v_1| - |u_2 - v_2|} (u_1 u_2)^{H-1} d
u_1 d
u_2 (v_2 - v_1)^{2H-1} + v_1^{2H-1}) d v_1 d v_2 \leq \frac{1}{H^2(t-s)^{2H+1}},
\]
where the last line is from the inequality (5.57). In the same vein, we have
\[
(\mu \times \mu)(J_{12}) \leq \frac{1}{2H^2(2H+1)} (t-s)^{4H+1},
\]
\[
(\mu \times \mu)(J_2 + J_3) \leq \frac{2}{H^3(t-s)^{4H}}
\]
Plugging the above three inequalities into (5.58), we obtain the desired (5.54).

**Lemma 5.12.** Let \(0 \leq s < t \leq T\) and \(\phi_2(u, v)\) be given as in (3.13). Then there exists a constant \(C > 0\) independent of \(T\) such that for all \(s, t \geq 0\),
\[
\|\phi_2\|^2_{\mathcal{H}^{\theta}_2} \leq C \left( |t-s|^{2H} + |t-s|^{2H+1} \right),
\]
\[
(\mu \times \mu)(|\phi_2 \otimes \nu_1, \phi_2|) \leq C \left( |t-s|^H + |t-s|^{2H} + |t-s|^{2H+1} \right).
\]

**Proof.** We only give the sketch of the proof since it is similar to that of Lemma 5.11. For simplicity, we can assume that \(\theta = 1\). Denote \(\vec{u} = (u_1, u_2)\) and \(\vec{v} = (v_1, v_2)\). First, by the identity (5.25), we have
\[
\|\phi_2\|^2_{\mathcal{H}^{\theta}_2} \leq \left\| e^{-|u-v|} I_{\{0 \leq u \leq s, s \leq v \leq t\}} \right\|^2_{\mathcal{H}^{\theta}_2}
\]
\[
= \int_{[0,T]^4} \frac{\partial^2}{\partial u_1 \partial v_2} \left[ e^{-|u_1 - v_1|} e^{-|u_2 - v_2|} I_{\{0 \leq u \leq s, s \leq v \leq t\}} \right] \frac{\partial R^B(u_1, u_2)}{\partial u_2} \frac{\partial R^B(v_1, v_2)}{\partial v_1} d\vec{u} d\vec{v}
\]
\[
= \int_{[0,T]^4} e^{-|u_1 - v_1|} e^{-|u_2 - v_2|} I_{\{0 \leq u \leq s, s \leq v \leq t\}} \frac{\partial R^B(u_1, u_2)}{\partial u_2} \frac{\partial R^B(v_1, v_2)}{\partial v_1}
\]
\[
\times \left[ I_{\{0 \leq u \leq s\}}(u_1, v_1) \frac{\partial}{\partial u_1} (\delta_0(u_1) - \delta_s(u_1)) + (\delta_0(v_1) - \delta_s(v_1)) (\delta_0(u_2) - \delta_s(u_2)) \right] d\vec{u} d\vec{v}
\]
\[
:= I_1 + I_2 + I_3 + I_4.
\]
Corollary 5.2 implies that there exists a constant $C > 0$ independent of $T$ such that

$$|I_1| \leq \int_{[0,s]^2} e^{-(s-u_1)-(s-u_2)} \left| \frac{\partial R^B(u_1, u_2)}{\partial u_2} \right| \, du \times \int_{[s,t]^2} e^{-(v_1-s)-(v_2-s)} \left| \frac{\partial R^B(v_1, v_2)}{\partial v_1} \right| \, dv \leq C \times \int_{[s,t]^2} \left| \frac{\partial R^B(v_1, v_2)}{\partial v_1} \right| \, dv \leq C |t-s|^{2H+1}.$$ 

In the same vein, we have that there exists a constant $C > 0$ independent of $T$ such that

$$|I_2| + |I_3| + |I_4| \leq C \left( |t-s|^{2H} + |t-s|^{2H+1} \right).$$

Combining the above two estimates together, we obtain (5.59).

Next, we have

$$(\mu \times \mu)(|\phi_2 \otimes \phi_2|) \leq H \times (\mu \times \mu)(J_{11} + J_{12} + J_{21} + J_{22}),$$

where

$$J_{11} = \int_{s \leq u_1 \leq u_2 \leq t} e^{-(v_1-u_1)-(v_2-u_2)} \left( v_1^{2H-1} + (v_2-v_1)^{2H-1} \right) dv_1 dv_2 \mathbb{1}_{[0,s]^2}(u_1, u_2)$$

$$+ \int_{s \leq v_2 \leq u_1 \leq t} e^{-(v_1-u_1)-(v_2-u_2)} (v_1 - v_2)^{2H-1} dv_1 dv_2 \mathbb{1}_{[0,s]^2}(u_1, u_2)$$

$$+ \int_s^t \left[ e^{-(v_1-u_1)-(t-u_2)} \left( v_1^{2H-1} + (t-v_1)^{2H-1} \right) + e^{-(v_1-u_1)-(s-u_2)} (v_1-s)^{2H-1} \right] dv_1 \mathbb{1}_{[0,s]^2}(u_1, u_2)$$

$$J_{12} = \int_s^t dv_1 \int_0^s dv_2 e^{-(v_1-u_1)-(u_2-v_2)} (v_1-v_2)^{2H-1} \mathbb{1}_{[s,t]}(u_1) \mathbb{1}_{[s,t]}(u_2)$$

$$+ \int_s^t e^{-(v_1-u_1)-(u_2-s)} (v_1-s)^{2H-1} dv_1 \mathbb{1}_{[0,s]}(u_1) \mathbb{1}_{[s,t]}(u_2)$$

$$J_{21} \leq \int_0^s dv_1 \int_s^t dv_2 e^{v_1-u_1+u_2-v_2} \left( v_1^{2H-1} + (v_2-v_1)^{2H-1} \right) \mathbb{1}_{[s,t]}(u_1) \mathbb{1}_{[0,s]}(u_2)$$

$$+ 2 \int_0^s e^{v_1-u_1+u_2-s} \left( v_1^{2H-1} + (s-v_1)^{2H-1} \right) dv_1 \mathbb{1}_{[s,t]}(u_1) \mathbb{1}_{[0,s]}(u_2)$$

$$J_{22} = \int_{0 \leq u_1 \leq u_2 \leq s} e^{-(u_1-v_1)-(u_2-v_2)} \left( v_1^{2H-1} + (v_2-v_1)^{2H-1} \right) dv_1 dv_2 \mathbb{1}_{[s,t]}(u_1, u_2)$$

$$+ \int_{0 \leq v_2 \leq u_2 \leq s} e^{-(u_1-v_1)-(u_2-v_2)} (v_1-v_2)^{2H-1} dv_1dv_2 \mathbb{1}_{[s,t]}(u_1, u_2)$$

$$+ \int_0^s e^{-(u_1-v_1)-(u_2-s)} \left( v_1^{2H-1} + (s-v_1)^{2H-1} \right) dv_1 \mathbb{1}_{[s,t]}(u_1, u_2).$$

Lemma 5.1 and Corollary 5.2 imply that there exists a constant $C > 0$ independent of $T$ such that

$$(\mu \times \mu)(J_{11}) \leq C ((t-s)^{2H+1} + (t-s)^{2H}), \quad (\mu \times \mu)(J_{12}) \leq C (t-s)^{2H},$$

$$(\mu \times \mu)(J_{21}) \leq C (t-s)^H, \quad (\mu \times \mu)(J_{22}) \leq C (t-s)^{2H}.$$ 

This combined with the inequality (5.61) proves the proposition. \qed
Lemma 5.13. Denote $a_+ = \max \{a, 0\}$. Let $f_T$, $h_T$ be given in (3.1)-(3.2) respectively. There exists a constant $C > 0$ independent on $T \geq 1$ such that

$$
\langle f_T, h_T \rangle_{\mathcal{S}^2} \leq C \times \begin{cases} 
T^{(4H-1)+}, & \text{if } H \in (0, \frac{1}{4}) \cup (\frac{1}{4}, \frac{1}{2}), \\
\log T, & \text{if } H = \frac{1}{4}.
\end{cases}
$$

(5.62)

Proof. For simplicity, we assume that $\theta = 1$. Similar to the proof of Lemma 5.11 and Lemma 5.12, we have

$$
\langle f_T, h_T \rangle_{\mathcal{S}^2} \leq I_1 + I_2 + I_3 + I_4,
$$

where

$$
I_1 = \int_{[0,T]^4} e^{-|u_1-v_1|-|T-u_2|-(T-v_2)} \left| \frac{\partial}{\partial u_2} R^B(u_1, u_2) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, v_2) \right| du_1 du_2 dv_1 dv_2,
$$

$$
I_2 = \int_{[0,T]^3} e^{-|T-u_1|-(T-u_2)-(T-v_2)} \left| \frac{\partial}{\partial u_2} R^B(T, u_2) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, v_2) \right| du_2 dv_1 dv_2,
$$

$$
I_3 = \int_{[0,T]^3} e^{-|u_1-v_1|-(T-u_2)} \left| \frac{\partial}{\partial u_2} R^B(u_1, u_2) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| dv_1 du_2 dv_1,
$$

$$
I_4 = \int_{[0,T]^2} e^{-|v_1|+(T-v_2)} \left| \frac{\partial}{\partial u_2} R^B(T, u_2) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| dv_2.
$$

It follows from Corollary 5.2, Lemma 5.1 and Lemma 5.4 that when $H \neq \frac{1}{4}$, there exists a positive constant $C$ independent on $T$ such that

$$
I_1 \leq \int_{[0,T]^2} e^{-|u_1-v_1|} \left| \frac{\partial}{\partial u_1} R^B(u_1, T) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| du_1 dv_1 \leq CT^{(4H-1)+},
$$

$$
I_2 \leq \int_{[0,T]^2} \int_T e^{-|u_1-v_1|+(T-u_2)-(T-v_2)} \left| \frac{\partial}{\partial v_1} R^B(v_1, v_2) \right| dv_1 dv_2 \leq C \int_T \int_0^T e^{-|v_1|+(T-v_2)} \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| dv_1 dv_2 \leq C,
$$

$$
I_3 \leq \int_{[0,T]^2} \left| \frac{\partial}{\partial u_1} R^B(u_1, T) \right| \left| \frac{\partial}{\partial v_1} R^B(v_1, T) \right| du_1 dv_1 \leq CT^{(4H-1)+},
$$

$$
I_4 \leq C \left( \int_0^T e^{-|v_1|+(T-v_2)} \left| \frac{\partial}{\partial v_1} R^B(T, v_1) \right| dv_1 \right)^2 \leq C,
$$

and

$$
(\mu \times \mu)(|f_T|) = \int_{[0,T]^2} e^{-|u_1-v_1|} |uv|^{H-1} du dv \leq C \int_0^T e^{H-1}(1 \wedge v^{-H}) dv \leq C,
$$

$$
(\mu \times \mu)(|h_T|) = \left( \int_0^T e^{-|v_1|} v^{-H-1} dv \right)^2 \leq C
$$

$$
(\mu \times \mu)(|f_T \otimes v_T h_T|) \leq \left( \int_0^T e^{-|v_1|+(T-v_2)} \left| \frac{\partial}{\partial u_2} R^B(u_2, T) \right| dv_2 \right)^2 \leq CT^{4H-2},
$$

where $\phi(\cdot, T)$ is given as in (5.2).
Comparing three values $0$, $(4H - 1)_+$, and $4H - 2$, we see that the largest one is $(4H - 1)_+$. Hence,

$$
\left|\langle f_T, h_T \rangle_{H} \right|^2 + \left(C_H'(\mu \times \mu)\langle |f_T| \rangle(\mu \times \mu)\langle |h_T| \rangle + 2C_H'(\mu \times \mu)\langle |f_T \otimes_1 h_T| \rangle \right) \leq CT^{(4H-1)_+}.
$$

This combined with the inequality (2.5) proves the proposition. The case of $H = \frac{1}{4}$ is in the same vein. 

\[\square\]

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References

[1] Chen Y, Zhou H. Parameter estimation for an Ornstein-Uhlenbeck process driven by a general gaussian noise. Acta Mathematica Scientia, 2021, 41 B(2): 573-595.

[2] Kutoyants Y A. Statistical Inference for Ergodic Diffusion Processes. Springer, 2004

[3] Liptser R S, Shiryaev A N, Statistics of Random Processes: II Applications, second ed. In: Applications of Mathematics. Springer, 2001

[4] Kleptsyna M L, Le Breton A. Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Stat Inference Stoch Process, 2002, 5: 229-248

[5] Tudor C, Viens F. Statistical aspects of the fractional stochastic calculus. Ann. Statist, 2007, 35(3): 1183-1212

[6] Bercu B, Coutin L, Savy N. Sharp large deviations for the fractional Ornstein-Uhlenbeck process. Theory Probab Appl, 2011, 55(4): 575-610

[7] Brouste A, Kleptsyna M. Asymptotic properties of MLE for partially observed fractional diffusion system. Stat Inference Stoch Process, 2010, 13(1): 1-13

[8] Hu Y, Nualart D. Parameter estimation for fractional Ornstein-Uhlenbeck processes. Stat Probab Lett, 2010, 80 (11-12): 1030-1038

[9] Hu Y, Nualart D, Zhou H. Parameter estimation for fractional Ornstein-Uhlenbeck processes of general Hurst parameter. Stat Inference Stoch Process, 2019, 22: 111-142

[10] Sottinen T, Viitasaari L. Parameter estimation for the Langevin equation with stationary-increment Gaussian noise. Stat Inference Stoch Process, 2018, 21(3): 569-601

[11] Dietz H M, Kutoyants Y A. Parameter estimation for some non-recurrent solutions of SDE. Statistics and Decisions, 2003, 21(1): 29-46

[12] Belfadli R, Es-Sebaiy K, Ouknine Y. Parameter estimation for fractional Ornstein-Uhlenbeck processes: Non-ergodic case. Front Sci Eng Int J Hassan II Acad Sci Technol, 2011, 1(1): 1-16

[13] El Machkouri M, Es-Sebaiy K, Ouknine Y. Least squares estimator for non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian processes. J Korean Stat Soc, 2016, 45: 329-341

[14] Mendy I. Parametric estimation for sub-fractional Ornstein-Uhlenbeck process. Journal of Statistical Planning and Inference, 2013, 143 (4): 663-674

[15] Alazemi F, Alsenafi A, Es-Sebaiy K. Parameter estimation for Gaussian mean-reverting Ornstein-Uhlenbeck processes of the second kind: non-ergodic case. Stoch. Dyn. 2020, 20(2): 2050011, 25 pp.

[16] Shevchenko R, Tudor C. A. Parameter estimation for the Rosenblatt Ornstein-Uhlenbeck process with periodic mean. Stat. Inference Stoch. Process, 2020, 23 (1): 227-247
[20] Shen G, Yu Q, Tang Z. The Least Squares Estimator for an Ornstein-Uhlenbeck Process Driven by a Hermite Process with a Periodic Mean. Acta Math. Sci. Ser. 2021, 41 B(2): 517-534.

[21] Chen, Y, Li, Y. Berry-Esseen bound for the Parameter Estimation of Fractional Ornstein-Uhlenbeck Processes with \( H \in (0, \frac{1}{2}) \). Communications in Statistics-Theory and Methods, 2021, 50(13): 2996-3013.

[22] Jolis M. On the Wiener integral with respect to the fractional Brownian motion on an interval. J Math Anal Appl, 2007, 330: 1115-1127.

[23] Chen Y, Ding Z, Li Y. Berry-Esseen bounds and almost sure CLT for the quadratic variation of a general Gaussian noise. arXiv: 2106.01851.

[24] Pickands J Asymptotic properties of the maximum in a stationary Gaussian process. Trans Am Math Soc, 1969, 145:75-86.

[25] Chen Y, Hu Y, Wang Z. Parameter Estimation of Complex Fractional Ornstein-Uhlenbeck Processes with Fractional Noise. ALEA Lat Am J Probab Math Stat, 2017, 14: 613-629.

[26] Shen G, Tang Z, Yin X. Least-squares estimation for the Vasicek model driven by the complex fractional Brownian motion, Stochastics, 2021. to appear. DOI: 10.1080/17442508.2021.1959587.

[27] Chen Y, Li Y, Pei X. Parameter estimation for Vasicek model driven by a general Gaussian noise. Communications in Statistics-Theory and Methods, 2022. to appear. DOI: 10.1080/03610926.2021.1967399.

[28] Kozachenko Y, Melnikov A, Mishura Y. On drift parameter estimation in models with fractional Brownian motion, Statistics: A Journal of Theoretical and Applied Statistics, 2015, 49(1): 35-62.

[29] Hu Y. Analysis on Gaussian spaces. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. 2017.

[30] Stroock D W, Varadhan S R S. Multidimensional diffusion processes. Classics in Mathematics. Springer-Verlag, Berlin, 1979.

[31] Nourdin I, Peccati G. Normal approximations with Malliavin calculus: from Stein’s method to universality (Vol. 192). Cambridge University Press. 2012.

[32] Nualart D, Peccati G. Central limit theorems for sequences of multiple stochastic integrals. Ann Probab, 2005, 33(1): 177-193.

[33] Houdré C, Pérez-Abreu V, Üstünel A. S. Multiple Wiener-Itô integrals: an introductory survey. Chaos expansions, multiple Wiener-Itô integrals and their applications (Guanajuato, 1992), 1-33, Probab. Stochastics Ser., CRC, Boca Raton, FL, 1994.

[34] Kim Y T, Park H. S. Optimal Berry-Esseen bound for statistical estimations and its application to SPDE. Journal of Multivariate Analysis, 2017, 155: 284-304.