SOME REMARKS ON THE GROUP OF DERIVED AUTOEQUIVALENCES

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Abstract. We prove that the neutral component of the group of derived autoequivalences of a smooth projective variety is the semi-direct product of the neutral component of its Picard group and its group of automorphisms. We use this result to prove several results concerning pairs of derived equivalent varieties.

1. Introduction

After the seminal work of Mukai, Orlov, Kawamata and many other people after them, it is now clear that for two algebraic varieties having equivalent derived categories is a very interesting relation. We know that when it is the case, the two algebraic varieties share many geometric properties, like their dimension, the order of their canonical sheaf, their canonical ring, to name a few. It is clear that the study of this relation is closely related to the study of the group of (triangulated) autoequivalences of the derived category of a variety. Thanks to the work of Toën and Vaquié [17], this group is in a natural way an algebraic group scheme. In this paper we precise the structure of this group. More accurately we determine its neutral component. We can sum up everything in the following statement:

Theorem 1.1 (see theorem 2.12). Let \( k \) be an algebraically closed field and \( X \) a smooth and projective \( k \)-scheme. Let \( D(X) \) be the bounded derived category of coherent sheaves on \( X \). Then the neutral component of the group of triangulated autoequivalence of \( D(X) \) is \( \text{Aut}^\circ \rtimes \text{Pic}^\circ(X) \). (Where the notation \( G^\circ \) stands for the neutral component of the group \( G \))

The paper is organised as follows, in section 2 we recall some basic definition about autoequivalences of derived categories and use this to explain a new construction of the group \( \text{Aut}^D \) of triangulated autoequivalences. Then we prove the main theorem. In section 3 we give several applications of the theorem 2.12 with a particular emphaze to the case of abelian varieties. We close this section with a reformulation of a conjecture of Kawamata on the cardinality of classes of derived equivalent algebraic variety.

2. The algebraic group of derived autoequivalences

Throughout this section we fix an algebraically closed field \( k \) and a smooth projective variety \( X \) over \( k \). We denote by \( D(X) \) the bounded derived category of coherent sheaves on \( X \). Our aim is to explain how to associate to \( X \) a locally algebraic group scheme \( \text{Aut}^D_X \) whose points are naturally identified with exact autoequivalences of \( D(X) \). This was first done by Toën and Vaquié in [17] in the
context of DG-categories. Our construction is different but leads to the same algebraic group. Our new result is theorem 2.12 which describes the neutral component of this group. Applications are discussed in section 3.

2.1. A quick review of integral transform. Let $S$ be a $k$-scheme and $X \rightarrow S$ and $Y \rightarrow S$ be two smooth and projective $S$-schemes.

**Definition 2.1.** To any object $F \in D(X \times S Y)$ we associate a functor, called an integral transform, by the formula:

$$
\phi^F_{X \rightarrow Y} : D(X) \rightarrow D(Y) \quad F \mapsto Rp^*_Y(F' \otimes^L p^*_X(F))
$$

When this functor is an equivalence we call it a Fourier-Mukai transform. The composite of two integral functors is again an integral functor. More precisely let $Z$ be a smooth and projective $S$-scheme and $G' \in D(Y \times S Z)$. Let $p_{X,Y}$ (resp $p_{Y,Z}$, $p_{X,Z}$) the projection of $X \times S Y \times S Z$ on $X \times S Y$ (resp $Y \times S Z$, $X \times S Z$). We define

$$
G' \ast F = Rp_{X,Z}^*(p_{X,Y}^*(F') \otimes^L p_{Y,Z}^*(G'))
$$

**Lemma 2.2.** The composite $\phi^{G'} \circ \phi^F$ is an integral transform. Its kernel is given by the following formula:

$$
\phi^{G'}_{Y \rightarrow Z} \circ \phi^F_{X \rightarrow Y} \simeq \phi^{G' \ast F}_{X \rightarrow Z}
$$

This is well known (e.g. see [5]).

An integral functor always have a right adjoint.

**Lemma 2.3.** With notation as above the integral functor with kernel $F'$ has a right adjoint, the integral transform with kernel

$$
F'^{-1} = R\text{Hom}(F', \mathcal{O}_{X \times S Y}) \otimes^L p_{X}^*(\omega_{X/S}[n])
$$

where $\omega_{X/S}$ stands for the relative canonical sheaf of $X$ over $S$.

Finally we recall that an integral transform commutes with a base change.

Let $h : T \rightarrow S$ a morphisme of schemes. With evident notations we have the commutative diagramme:

$$
\begin{array}{ccc}
X_T & \times_T Y_T & \\
\downarrow \scriptstyle p_{X_T} & \downarrow \scriptstyle p_{Y_T} & \\
X_T & \times_T Y_T & \\
\downarrow \scriptstyle h_X & \downarrow \scriptstyle h_Y & \\
X & \times_S Y & \\
\downarrow \scriptstyle p_X & \downarrow \scriptstyle p_Y & \\
X & \times_S Y & \\
\end{array}
$$

Let $F_T = Lh_X^*(X_T \times T Y_T)$.

**Lemma 2.4.** The functors $Lh^*_Y \circ \phi_F$ and $\phi_{F_T} \circ Lh^*_X$ are isomorphic.

The proof is a straightforward computation using projection and base change formulæ.
2.2. The algebraic group of derived autoequivalences. We now construct a presheaf on the site Aff/k of affine k-schemes:

\[ \text{Aff}/k \to \text{Ens} \quad A \mapsto \{ \text{perfect complexes } F \in D_{qc}(X \times X \times A) | F \ast F^{-1} = F^{-1} \ast F = \mathcal{O}_\Delta \} \]

Let’s recall that a perfect complex on a scheme X is one locally isomorphic (in the derived category) to a bounded complex of locally free sheaves of finite rank and that as usual \( \mathcal{O}_\Delta \) stands for the structure sheaf of the diagonal embedding \( X \hookrightarrow X \times X \).

**Theorem 2.5.** The sheaf \( \text{Aut}^D \) associated to the presheaf above (for the etale topology) is an algebraic space locally of finite presentation. Moreover we can endow it with the composition law \( \ast \) of \( \Box \), making it into a group object in the category of algebraic spaces over k, hence a locally algebraic group scheme.

We don’t give a proof of this theorem here, instead we just explain the (simple) idea of the proof. The reader interested in a proof can consult either my thesis \[14\], or the paper \[17\] where this result is proven (in a quite different context). Before we go on we need a last result due to Inaba, Lieblich and Toën and Vaquié(see \[9\], \[11\], \[17\]). Let \( F \) be a perfect complex in \( D_{qc}(X \times S) \). For any point closed point \( s \) of \( S \) we denote by \( i_s \) the closed immersion \( s \hookrightarrow S \) and \( X_s \) the fiber.

**Definition 2.6.** We say that \( F \) is simple iff for any closed point \( s \) of \( S \) the complex \( Li_s^*(F) \) is simple, that is:

\[ \text{Hom}(Li_s^*(F), Li_s^*(F)) = k \]

We say that \( F \) is universally gluable (rigid in the terminology of \[17\]) iff:

\[ \text{Rhom}(Li_s^*(F), Li_s^*(F)) \text{ is concentrated in degrees greater or equal to 0} \]

**Theorem 2.7.** (\[9\], \[11\], \[17\]) The etale sheaf associated to the presheaf of perfect, simple, universally gluable complexes is an algebraic space locally of finite presentation. We denote it \( \text{Perf}_X \)

**Proof.** (Idea of the proof of theorem \[2.5\]) Let \( F \in D_{qc}(X \times X) \) a perfect complex such that \( F \ast F^{-1} = F^{-1} \ast F = \mathcal{O}_\Delta \). Then keeping in mind that the integral transform associated to \( \mathcal{O}_\Delta \) is the identity it follows that \( \phi^F \) is an equivalence. The same argument works if we base change to \( X \) as in the lemma \[2.4\]. Then a direct computation gives:

\[ \phi^F_x(\mathcal{O}_\Delta) = F \]

So that

\[ \text{Hom}^i(F, F) = \text{Hom}^i(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \]

The right member of this last equation is 0 for \( i < 0 \) and \( k \) for \( i = 0 \). So that we have morphism \( j : \text{Aut}^D_X \to \text{Perf}_{X \times X} \), induced by an obvious monomorphism at the level of the presheaves defining these two sheaves. As sheafification is left exact it follows that the morphism \( j \) is a monomorphism. The idea of the proof is then to show that it is formally smooth. It implies that \( j \) is an open immersion, thus proving the representability of \( \text{Aut}^D(X) \). To end the proof we appeal to a result of Artin to the effect that a group object in the category of algebraic spaces over a field is a group scheme (\[1\], lemma 4.2).
Remark 2.8.  

By a result of Orlov [13], it is straightforward to check that the $k$-points of $\mathrm{Aut}^D_X$ are in one to one correspondence with the exact equivalence of $D(X)$.  

By a deep result in [17], this algebraic group has countably many connected components, we will come back on this latter.

We now come to our main result.

2.3. The neutral component of $\mathrm{Aut}^D_X$. Let $\mathrm{Aut}$ be the group scheme of $k$-automorphisms of $X$ and Pic the Picard scheme of $X$. Let $S \to k$ be an affine scheme. Let’s consider an invertible sheaf $L$ on $X_S$ and an $S$-automorphism $f$ of $X_S$. To this data we associate a sheaf $\mathcal{F} = \Psi_S(f, L)$ on $X \times X \times S$ in the following manner. Identifying $X \times X \times S$ with the scheme $X_S \times_S X_S$, we denote by $\Gamma_f$ the section of $X_S \times_S X_S \xrightarrow{p_1} X_S$ induced by $f^{-1}$, we then define

$$\mathcal{F} = \Psi_S(f, L) = \Gamma_f(L)$$

(2.1)  

It’s clear that $\mathcal{F}$ is perfect ($X$ is smooth). A straightforward computation then shows that the integral transform associated to $\mathcal{F}$ is nothing but $Rf^{-1}_*(L \otimes^L (\cdot))$ which is an equivalence.

If $(f_1, L_1)$ and $(f_2, L_2)$ are as above, the integral transform associated with $\Psi_S(f_1, L_1) \ast \Psi_S(f_2, L_2)$ is $Rf^{-1}_{2*}(Rf^{-1}_{1*}(L_1) \otimes L_2)$; by the projection formula we thus have

$$\Psi_S(f_1, L_1) \ast \Psi_S(f_2, L_2) = \Psi_S(f_1 \circ f_2, L_1 \otimes f^{-1}_{1*}(L_2))$$

(2.2)  

So that $\Psi_S$ induce a group morphism between the semi-direct product of the picard group by the group of $S$-automorphisms (the former acting on the first by $f \mapsto f^{-1*}$) on the one side, and the group of invertible integral transform on the other side. As $\Psi_S$ is injective we can sum up everything in the

Proposition 2.9. The group morphisms $\Psi_S$ induce a sheaf monomorphism

$$\Psi : \mathrm{Aut} \ltimes \mathrm{Pic} \to \mathrm{Aut}^D$$

which is a group morphism.

Proposition 2.10. The morphism $\Psi$ is open.

Proof. As $\Psi$ is a monomorphism between schemes of finite type over a field, it’s enough to check that $\Psi$ is formally smooth.

So that we have to prove that if

- $A$ is a local artinian ring with maximal ideal $\mathfrak{m}$
- $A' \to A$ is an infinitesimal extension, i.e the kernel $I$ is a square zero principal ideal
- $(f, L) \in \mathrm{Aut} \ltimes \mathrm{Pic}(A)$ and $\mathcal{F} \in \mathrm{Aut}^D(A')$ are such that $\mathcal{F} \otimes^L I = A = \Psi(f, L)$ then there exist $(f', L') \in \mathrm{Aut} \ltimes \mathrm{Pic}(A')$ such that $\mathcal{F}' = \Psi(f', L')$.

It’s clear that $\Psi(f, L)$ is a sheaf flat over $A$. Let $Z_A$ be its support. Lemma 2.11 implies that $\mathcal{F}$ is in fact a sheaf flat over $A'$. Let $Z_{A'}$ be the support of $\mathcal{F}'$. The projection $p_1 : X_{A'} \times_{A'} X_{A'} \to X_{A'}$ induce a morphism $Z_{A'} \to X_{A'}$ whose reduction to $A$ is $p_1 : Z_{A} \to X_{A}$. Hence $p_1 : Z_{A'} \to X_{A'}$ is also an isomorphism. On its support $\mathcal{F}$ is a deformation of $L$ which is locally free of rank 1. As $\mathcal{F}$ is flat over $A'$, it follows that $\mathcal{F}$ restricted to its support is also free of rank 1. To sum up there exist a morphism $f'$ and an invertible sheaf $\mathcal{L}'$ such that $\mathcal{F}' = \Psi(f', \mathcal{L}')$ with the slight abuse that we don’t know yet that $f'$ is an automorphism. But it’s clear as $f'$ lifts $f$. 

□
Lemma 2.11. (Theorem 4.3) Let $\pi : S \to T$ be a morphism of schemes, and for each point $t \in T$, let $i_t : S_t \to S$ denote the inclusion of the fibre $\pi^{-1}(t)$. Let $\mathcal{F}$ be an object of $D(S)$, such that for all $t \in T$, $Li_t^*(\mathcal{F})$ is a sheaf on $S_t$. Then $\mathcal{F}$ is a sheaf on $S$, flat over $T$.

As a corollary we get the following theorem summing up all the previous discussion

Theorem 2.12. There is an exact sequence of locally algebraic group

$$0 \to \text{Aut}^c \times \text{Pic}^c \to \text{Aut}^D \to G \to 0.$$ 

The group $G$ has at most countably many elements. The neutral component of $\text{Aut}^D$ is $\text{Aut}^c \times \text{Pic}^c$.

Remark 2.13. The existence of $\text{Aut}^D$ and the fact that there are countably connected components in it are already proven in [17].

Proof. As $\text{Aut} \times \text{Pic}$ is an open subgroup of $\text{Aut}^D$, they have the same neutral component, and it’s clear that the neutral component of $\text{Aut} \times \text{Pic}$ is $\text{Aut}^c \times \text{Pic}^c$. At this point the only non trivial point remaining is the countability assertion which results from [17, corollaries 3.31 and 3.32]).

In the next section we give several applications of the theorem 2.12.

3. Applications

3.1. The group $\text{Aut}^c \times \text{Pic}^c$ is a derived invariant.

Theorem 3.1. Let $X$ and $Y$ be two smooth projective varieties over an algebraically closed field $k$. Suppose we have an equivalence of triangulated categories:

$$D^b(\text{Coh}(X)) \simeq D^b(\text{Coh}(Y)).$$

Then

$$\text{Aut}^c_X \times \text{Pic}^c_X \simeq \text{Aut}^c_Y \times \text{Pic}^c_Y.$$ 

Proof. From [18, theorem 2.2] it follows that the equivalence is given by a Fourier-Mukai transform. Let $\mathcal{F}$ be its kernel and $\mathcal{G}$ the kernel of the inverse transform. One then defines a morphism $\text{Aut}^D_X \to \text{Aut}^D_Y$ by sending $C \in \text{Aut}^D_X(S)$ to $G \star C \star \mathcal{F}$. It’s clear that this morphism is an isomorphism. So that $\text{Aut}^D_X \simeq \text{Aut}^D_Y$, in particular their neutral component are isomorphic. Now use the fact from theorem 2.12 that the neutral components respectively are $\text{Aut}^c_X \times \text{Pic}^c_X$ and $\text{Aut}^c_Y \times \text{Pic}^c_Y$.

This fact was already announced by Rouquier in [19], where the author does not give a proof. In the same paper the following theorem is conjectured.

Theorem 3.2. Let $A$ be an abelian variety over $k$ an algebraically closed field of characteristic 0. Suppose we have an equivalence of triangulated categories

$$D^b(\text{Coh}(X)) \simeq D^b(\text{Coh}(A)),$$

Then $X$ is also an abelian variety.

This theorem has been already proved in [8], we give a new proof using theorem 3.1. Actually our proof shows directly that $X$ is an abelian subvariety of $A \times \hat{A}$. It is interesting to compare with [12].
Proof. From theorem [2] $\text{Aut}_{X}^\circ \times \text{Pic}_{X}^\circ \simeq \text{Aut}_{A}^\circ \times \text{Pic}_{A}^\circ$. It follows that $\text{Aut}_{X}^\circ$ and $\text{Pic}_{X}^\circ$ are abelian varieties. The order of the canonical sheaf and the dimension are derived invariants (see e.g. [4, lemma 2.1]). So $\omega_X$ is trivial and if $n = \dim X = \dim A$, then $\dim(\text{Aut}_{X}^\circ \times \text{Pic}_{X}^\circ) = 2n$. It is known that the tangent space of $\text{Aut}_{X}^\circ$ is $H^0(X,T_X)$ and the tangent space of $\text{Pic}_{X}^\circ$ is $H^1(X,O_X)$. As $\omega_X$ is trivial an easy computation using the symmetry of Hodge numbers (Here we use the hypothesis that the characteristic of $k$ is 0 ) and Serre duality shows that $\dim H^1(X,O_X) = \dim H^0(X,T_X)$. As $\text{Aut}_{X}^\circ$ and $\text{Pic}_{X}^\circ$ are smooth, it follows that $\dim \text{Aut}_{X}^\circ = \dim \text{Pic}_{X}^\circ = n$. Let $x$ be a point of $X$. From lemma 3.3 the stabilizer of $x$ under the action $\text{Aut}_{X}^\circ$ is finite. Hence the orbit of $x$ is the whole $X$. To conclude we use the well known fact that the quotient of an abelian variety $A$ by a finite subgroup of $A$ is again an abelian variety. \hfill \Box

Lemma 3.3. Let $X$ be a scheme and $G$ an algebraic subgroup of $\text{Aut}(X)$ which acts on $X$ fixing a point $x$. Then $G$ is affine. In particular if $G$ is complete then $G$ is finite.

Proof. We have a dual action of $G$ on $O_x$ and so on the $k$-vectorial space $O_x/m_x^{n+1} := V_n$ ($O_x$ is the local ring at $x$ and $m_x$, is the maximal ideal of this ring). Let $G_n := \ker(G \rightarrow GL(V_n))$. We have $\cdots \subset G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$. More over if $g \in \bigcap_n G_n$ then $g$ is the identity on $O_x$ because $O_x$ is separated, $\bigcap_n m_{n+1} = 0$. Hence $\bigcap_n G_n = 1$. Now $G$ being algebraic this sequence is stationnary, hence there exist $n_0$ such that $G_{n_0} = 1$ and $G$ is a closed subgroup of $GL(O_x/m_x^{n_0+1})$. \hfill \Box

3.2. Derived equivalence classes of algebraic varieties. Let’s say that two smooth projective varieties $X$ and $Y$ over a field $k$ are derived equivalent if there exist an equivalence of triangulated categories $D^b(X) \simeq D^b(Y)$. In [10] Kawamata conjecture the following:

Conjecture 3.4 (conjecture 1.5, [10]). There are up to isomorphism finitely many smooth projective varieties derived equivalent to a given one.

It has been proven in [2] that there are at most countably many derived equivalence classes when $k = \mathbb{C}$. We give here a new proof of a slightly better result, namely the only assumption on the field is that it is algebraically closed.

Theorem 3.5. Let $X$ be a smooth projective variety over a field $k$ algebraically closed. Then there are at most countably many (up to isomorphism) smooth projective varieties derived equivalent to $X$.

Before we proceed to the proof we need a lemma due to Favero in [7]. Suppose we have $X$ and $Y$ derived equivalent. Fix an equivalence $F : D^b(X) \rightarrow D^b(Y)$. Then has we have seen in the proof of 3.1 $F$ induces an isomorphism $F^* : \text{Aut}^D_{Y} \rightarrow \text{Aut}^D_{X}, G \mapsto F^{-1} \circ G \circ F$.

Lemma 3.6 (lemma 4.1 [7]). Let $X$ and $Y$ be a smooth projective varieties. Let $A$ be an ample line bundle on $Y$, $\tau \in \text{Aut}(X)$, and $L \in \text{Pic} X$ and suppose we have an equivalence $F : D^b(X) \simeq D^b(Y)$ and $F^*((\_ \_ \_ \_ \_ \_ \_ A)) = (\tau, L)[r]$ for some $r \in \mathbb{Z}$. Then $F \cong (\gamma, N)[s]$ for some line bundle $N \in \text{Pic}(Y)$, an isomorphism $\gamma : X \rightarrow Y$, and $s \in \mathbb{Z}$.

Combining the above lemma with theorem 2.12 we get:
Lemma 3.7. Let $X$, $Y$ and $Z$ be smooth projective varieties. Let $A_Y$ and $A_Z$ be ample line bundle on $Y$ and $Z$ respectively. Let $C_{A_Y}$ (resp $C_{A_Z}$) be the connected component of $\text{Aut}^D Y$ (resp $(\text{Aut}^D Z)$ that contains $((\cdot) \otimes A_Y)$ (resp $((\cdot) \otimes A_Y)$). Let $G : D^b(X) \simeq D^b(Y)$ and $H : D^b(X) \simeq D^b(Z)$ be derived equivalence. Finally suppose

$$G^*(C_{A_Y}) = H^*(C_{A_Z}).$$

Then $Y$ and $Z$ are isomorphic.

Proof. The hypothesis imply that $F = H \circ G^{-1} : D^b(Y) \to D^b(Z)$ is an equivalence and that $F^*(C_{A_Z}) = C_{A_Y}$. Now theorem 2.12 implies that any element in $C_{A_Y}$ is a $(\sigma, L)$, for some automorphism $\sigma$ of $Y$ and some line bundle $L$ on $Y$. Applying lemma 3.6 the conclusion follows.

Corollary 3.8. The number of smooth projective varieties derived equivalent to $X$ is bounded by the number of connected components of $\text{Aut}^D X$.

Corollary 3.9. Theorem 3.7 holds true.

Proof. By theorem 2.12 there are countably many connected components in $\text{Aut}^D X$.

### 3.3. A new conjecture.

Using the previous discussion and a new ingredient we introduce a new conjecture, we prove equivalent to conjecture 3.4. We begin with some notations. Let $(X_i)_{i \in \mathbb{N}}$ be the set of varieties derived equivalent to $X$. Let $A_i$ be an ample line bundle on $X_i$ and $F_i : D^b(X) \to D^b(X_i)$ be a triangulated equivalence. Let $K_i \subset D^b(X \times X)$ be the kernel of the Fourier-Mukai transform $F_i^*((\cdot) \otimes A_i)$. Let $L$ be a very ample line bundle on $X \times X$. Finally let $d \in \mathbb{N}$ such that $L \oplus L^{-1} \oplus \ldots \oplus L^{-d}$ is a strong generator of $D^b_{qc}(X \times X)$, recall from [6] that such a $d$ exist. Then conjecture 3.3 is equivalent to the following

Conjecture 3.10. We can choose the $A_i$ and the $F_i$ such that there exist a function $\nu : \mathbb{Z} \to \mathbb{N}$ with finite support such that

\[
\dim \mathbb{H}^j(X, K_i \otimes L^l) \leq \nu(j) \forall j, \forall l \in [0, d] \forall i
\]

Proof. It’s clear that assuming 3.3 conjecture 3.10 holds. Only the converse needs a proof. So now assume 3.10 holds. Then from lemma 3.7 it’s enough to prove that there are only finitely many elements in $\{C_{A_i}\}$, the set of connected components of $\text{Aut}^D X$ containing the $F_i^*((\cdot) \otimes A_i)$. Now the condition (3.1) in conjunction with proposition 3.11 implies

the family $K_i$ is bounded hence quasi-compact.

Proposition 3.11 (corollary 3.32 in [17]). Let $k$ be a field and $X$ be a smooth and projective variety over $k$. Let $O(1)$ be a very ample line bundle on $X$. Then, there exists an integer $d$, such that the following condition is satisfied:

A family of perfect complexes $\{E_i\}_{i \in I}$ on $X$ is bounded if and only if there exists a function $\nu : \mathbb{Z} \to \mathbb{N}$ with finite support, such that

$$\text{Dim}_k \mathbb{H}^j(X, E_i(j)) \leq \nu(k) \quad \forall k, \forall j \in [0, d] \forall i \in I.$$

Remark 3.12. 

- Any integer $d$ such that $\bigoplus_{i=0}^d O(-i)$ is a compact generator is good for proposition 3.11.
- It follows from [16] that one can choose $d \leq 2\dim X$. 

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