EXISTENCE OF FINITE TIME BLOW-UP SOLUTIONS IN A NORMAL FORM OF THE SUBCRITICAL HOPF BIFURCATION WITH TIME-DELAYED FEEDBACK FOR SMALL INITIAL FUNCTIONS

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Abstract. We study a normal form of the subcritical Hopf bifurcation subjected to time-delayed feedback. An unstable periodic orbit is born at the bifurcation in the normal form without the delay and it can be stabilized by the time-delayed feedback. We show that there exist finite time blow-up solutions for small initial functions, near the bifurcation point, when the feedback gains are small. This can happen even if the origin is stable or the unstable periodic orbit of the normal form is stabilized by the delay feedback. We give numerical examples to illustrate the theoretical result.

1. Introduction. We consider the two-dimensional time-delayed system

\[
\begin{align*}
\dot{x}_1(t) &= -\alpha x_1(t) - x_2(t) + (x_1(t)^2 + x_2(t)^2)(x_1(t) - \gamma x_2(t)) \\
&\quad - k_{11}(x_1(t) - x_1(t - \tau)) - k_{12}(x_2(t) - x_2(t - \tau)), \\
\dot{x}_2(t) &= x_1(t) - \alpha x_2(t) + (x_1(t)^2 + x_2(t)^2)(\gamma x_1(t) + x_2(t)) \\
&\quad - k_{21}(x_1(t) - x_1(t - \tau)) - k_{22}(x_2(t) - x_2(t - \tau)),
\end{align*}
\]

where \(\alpha, \tau > 0\), \(\gamma\) and \(k_{jl}, j, l = 1, 2\), are constants. When \(k_{jl} = 0\) for \(j, l = 1, 2\), Eq. (1.1) becomes a normal form of the subcritical Hopf bifurcation [7]. So it represents the normal form subjected to time-delayed feedback when \(k_{jl} \neq 0\) for some \(j, l = 1, 2\), where \(k_{jl}, j, l = 1, 2\), represent the feedback gains. When \(k_{jl} = 0\), \(j, l = 1, 2\), the dynamics of (1.1) is very simple: there exist a stable equilibrium at the origin and an unstable periodic orbit given by

\[
(x_1, x_2) = (\sqrt{\alpha} \cos \Omega t, \sqrt{\alpha} \sin \Omega t), \quad \Omega = \alpha \gamma + 1,
\]

and the other trajectories converge to the equilibrium or diverge to infinity, depending on whether their initial points are inside or outside of the periodic orbit. See Fig. 1.

Time-delayed feedback is one of useful tools to control nonlinear dynamical systems. Pyragas [14] demonstrated that unstable periodic orbits can be stabilized by time-delayed feedback control when the dynamical systems exhibit chaotic motions. In particular, the unstable periodic orbits do not have to be detected in advance,
and they are embedded in chaotic invariant sets. Thus, time-delayed feedback control enables us to successfully avoid undesired chaotic motions in many dynamical systems. Moreover, the technique can be applied even when chaotic dynamics does not occur. Since his pioneer work there have been extensive studies on time-delayed feedback control. See [15] and references therein. Among them, Fiedler et al. [4] and Just et al. [9] considered a special case of (1.1) in which

\[ k_{11} = k_{22}, \quad k_{12} = -k_{21}, \]  

(1.3)

and showed that the unstable periodic orbit, which has one positive Floquet multiplier, can also be stabilized by time-delayed feedback control although the claim known as the odd number limitation [10, 11] that unstable periodic orbits having an odd number of Floquet multipliers with moduli greater than one cannot be stabilized by time-delayed feedback control had been believed true for one decade till then. After their work, several researches on this subject have been done in this direction [1, 5, 6, 12, 13]. In particular, the existence of infinitely many periodic orbits in (1.1) with (1.3) was shown in [13]. We remark that from a viewpoint of applications the assumption (1.3) is a little too restrictive. On the other hand, a generalized technique called extended time-delay feedback control was proposed to improve its performance in [17]. It was shown in [16] that one can find control gains to stabilize periodic orbits satisfying a genericity condition of linear controllability by the extended time-delayed feedback control.

More recently, Eremin et al. [3] studied the time-delayed system

\[
\begin{align*}
\dot{x}_1 &= x_1(t) - x_2(t) - x_1(t - \tau)(x_1(t)^2 + x_2(t)^2), \\
\dot{x}_2 &= x_1(t) + x_2(t) - x_2(t - \tau)(x_1(t)^2 + x_2(t)^2),
\end{align*}
\]  

(1.4)

which has a globally asymptotically stable periodic solution for \( \tau = 0 \), and showed the following for \( \tau > 0 \):

(i) There exist finite-time blow-up solutions, which go to infinity in finite time;
(ii) There are infinitely many periodic solutions.

Their arguments rely on the form of the system (1.4), especially in which the time-delay appears in the nonlinear terms. So we raise a question on the existence of finite time blow-up solutions in (1.1), in which the delay appears in the linear terms.
In this paper, we study the normal form (1.1) of the subcritical Hopf bifurcation subjected to time-delayed feedback and show that there exist finite time blow-up solutions for small initial functions, near the bifurcation point (i.e., \(\alpha > 0\) is small), when the feedback gains \(k_{jl}, j, l = 1, 2,\) are small. We give some details on our result below.

Let
\[
\varphi(t) = (\rho_1 \cos(\omega t + \phi_1), \rho_2 \cos(\omega t + \phi_2)) \quad \text{for} \quad t \in [-\tau, 0],
\]
and let \(x_{10} = \rho_1 \cos \phi_1\) and \(x_{20} = \rho_2 \cos \phi_2\). We choose (1.5) as the initial functions and
\[
(x_1(0), x_2(0)) = (x_{10}, x_{20})
\]
as the initial condition, so that the delay terms are continuous at \(t = 0\). Thus, \(\rho_j, \phi_j\) and \(\omega\), respectively, represent the amplitude, phase (at \(t = 0\)) and frequency of the initial function for \(j = 1, 2\). So Eq. (1.1) becomes
\[
x_1 = -\alpha x_1 - x_2 + (x_1^2 + x_2^2)(x_1 - \gamma x_2) - k_{11}(x_1 - \rho_1 \cos(\omega t + \phi_1)) - k_{12}(x_2 - \rho_2 \cos(\omega t + \phi_2)),
\]
\[
x_2 = x_1 - \alpha x_2 + (x_1^2 + x_2^2)(\gamma x_1 + x_2) - k_{21}(x_1 - \rho_1 \cos(\omega t + \phi_1)) - k_{22}(x_2 - \rho_2 \cos(\omega t + \phi_2)),
\]
for \(t \in [0, \tau]\), where \(\dot{\phi}_j = \phi_j - \omega \tau, j = 1, 2\). Here we prove the following theorem.

**Theorem 1.1.** Let
\[
c(\phi) = \frac{1}{2}(k_{12} \rho_2 \cos \phi - k_{22} \rho_2 \sin \phi + k_{11} \rho_1),
\]
\[
s(\phi) = \frac{1}{2}(k_{22} \rho_2 \cos \phi + k_{12} \rho_2 \sin \phi + k_{21} \rho_1),
\]
where \(\phi = \phi_2 - \phi_1\), and let
\[
\chi = \sqrt{c(\phi)^2 + s(\phi)^2}.
\]
Suppose that \(\alpha, \rho_j > 0\) as well as \(|k_{jl}|, j, l = 1, 2\), are sufficiently small such that \(|k_{jl}| = O(\alpha)\), \(\rho_j = O(\sqrt{\alpha})\) and
\[
\chi^2 > \frac{2\delta_1^3}{27}.
\]
where \(\delta_1 = \alpha + \frac{1}{2}(k_{11} + k_{22})\). Let \(x(t)\) be the solution to (1.1) for the initial functions (1.5) with \(\omega = 1 + \frac{1}{2}(k_{12} - k_{21})\). For \(|\gamma|\) sufficiently small there exists \(\tau_0 > 0\) such that if \(\tau > \tau_0\) and \(|x_0|\) is sufficiently small, then the solution \(x(t)\) blows up in finite time, i.e., there exists a positive constant \(T\) such that \(\lim_{t \to T} |x(t)| = \infty\). Moreover, \(T > \tau\) holds if the delay \(\tau\) is appropriately chosen.

Theorem 1.1 means that for \(\delta_1 > 0\), if \(\alpha\) and \(|k_{jl}|, j, l = 1, 2,\) are sufficiently small and of the same order, say \(O(\delta_1)\), then we can choose small initial functions (1.5) of \(O(\sqrt{\delta_1})\) (satisfying (1.9)) such that for sufficiently large delay \(\tau\) and small value \(|\gamma|\) the solution to (1.1) blows up in finite time. Recall that the amplitude of the unstable periodic orbit (1.2) is \(\sqrt{\alpha} = O(\sqrt{\delta_1})\). When \(\delta_1 \leq 0\), finite-time blow-up necessarily occurs in (1.1) for sufficiently large delay \(\tau\) and small value \(|\gamma|\) if the initial functions are sufficiently small and \(\omega = 1 + \frac{1}{2}(k_{12} - k_{21})\). We also remark that the blow-up time for the solutions to (1.4) found in [3] is less than the delay, i.e., the delay has no influence on their behavior.

The rest of the paper is as follows: In Section 2 we apply the averaging method [7] to (1.6) and present some preliminary results. In Section 3 we give the proof of Theorem 1.1 based on the preliminary results. Finally, we give numerical examples.
to illustrate the theoretical result in Section 4. For the reader’s convenience, some discussions on the stability of the origin and the periodic orbit (1.2) in (1.1) are provided in Appendices A and B, respectively. They are important to understand the numerical results of Section 4.

2. Averaging. In this section we give some preliminary results for the proof of Theorem 1.1.

Let \( \varepsilon \) be a small parameter such that \( 0 < \varepsilon \ll 1 \). Scaling the state variables and coefficients as \( (x_j, \rho_j) = (\sqrt{\varepsilon}x_j, \sqrt{\varepsilon}\rho_j) \), \( j = 1, 2 \), and shifting the time as \( t \to t - \tau + \phi_1/\omega \) in (1.6), we have

\[
\dot{x}_1 = -\alpha \bar{x}_1 - \bar{x}_2 + \varepsilon (\bar{x}_1^2 + \bar{x}_2^2)(\bar{x}_1 - \gamma \bar{x}_2) \\
- k_{11}(\bar{x}_1 - \bar{\rho}_1 \cos \omega t) - k_{12}(\bar{x}_2 - \bar{\rho}_2 \cos (\omega t + \phi)), \\
\dot{x}_2 = \bar{x}_1 - \alpha \bar{x}_2 + \varepsilon (\bar{x}_1^2 + \bar{x}_2^2)(\gamma \bar{x}_1 + \bar{x}_2) \\
- k_{21}(\bar{x}_1 - \bar{\rho}_1 \cos \omega t) - k_{22}(\bar{x}_2 - \bar{\rho}_2 \cos (\omega t + \phi)),
\]

where \( \phi = \phi_2 - \phi_1 \). Assume that \( \omega \approx 1 \) and set \( \nu = \omega - 1 \). Let \( \bar{x}_1 = r \cos (\omega t + \theta) \) and \( \bar{x}_2 = r \sin (\omega t + \theta) \). Changing the coordinates from \((\bar{x}_1, \bar{x}_2)\) to \((r, \theta)\) in (2.1), we have

\[
\dot{r} = \varepsilon r^3 - \delta_1 r - \frac{1}{2} k_{11}(r \cos (2\omega t + \theta) - \bar{\rho}_1 (\cos \theta + \cos (2\omega t + \theta))) \\
- \frac{1}{2} k_{12}(r \sin (2\omega t + \theta) - \bar{\rho}_2 (\sin \theta \cos (2\omega t + \theta))) \\
- \frac{1}{2} k_{21}(r \sin (2\omega t + \theta) - \bar{\rho}_1 (\sin \theta + \sin (2\omega t + \theta))) \\
+ \frac{1}{2} k_{22}(r \cos (2\omega t + \theta) + \bar{\rho}_2 (\sin \theta - \sin (2\omega t + \theta))),
\]

\[
\dot{\theta} = \varepsilon r^3 - \delta_2 r + \frac{1}{2} k_{11}(r \sin (2\omega t + \theta) - \bar{\rho}_1 (\sin \theta + \sin (2\omega t + \theta))) \\
- \frac{1}{2} k_{12}(r \cos (2\omega t + \theta) + \bar{\rho}_2 (\sin \theta - \sin (2\omega t + \theta))) \\
- \frac{1}{2} k_{21}(r \cos (2\omega t + \theta) - \bar{\rho}_1 (\cos \theta + \cos (2\omega t + \theta))) \\
- \frac{1}{2} k_{22}(r \sin (2\omega t + \theta) - \bar{\rho}_2 (\cos \theta - \cos (2\omega t + \theta))),
\]

where \( \delta_2 = \nu + \frac{1}{2} (k_{21} - k_{12}) \). Carrying out the averaging procedure [7], we obtain

\[
\dot{r} = \varepsilon r^3 - \delta_1 r + c(\phi) \cos \theta + \bar{s}(\phi) \sin \theta, \\
\dot{\theta} = \varepsilon r^3 - \delta_2 r + \bar{s}(\phi) \cos \theta - c(\phi) \sin \theta,
\]

where \( c(\phi) = c(\phi)/\sqrt{\varepsilon} \) and \( \bar{s}(\phi) = s(\phi)/\sqrt{\varepsilon} \) (see (1.7) for the definitions of \( c(\phi) \) and \( s(\phi) \)). Let \( \tilde{\phi}(\phi) \in [0, 2\pi] \) be a constant depending on \( \phi \) such that \( \tilde{c}(\phi) = \tilde{c}(\phi) \) and \( \tilde{s}(\phi) = \tilde{s}(\phi) \), where \( \tilde{\chi} = \chi/\sqrt{\varepsilon} \) (see (1.8) for the definition of \( \chi \)). Changing the variable as \( \theta \to \theta - \tilde{\phi}(\phi) \), we rewrite (2.3) as

\[
\dot{r} = (\varepsilon r^2 - \delta_1)r + \tilde{\chi} \cos \theta, \\
\dot{\theta} = (\varepsilon r^2 - \delta_2)r - \tilde{\chi} \sin \theta.
\]

Let \( \gamma = 0 \) and \( \delta_2 = 0 \). Then the averaged system (2.4) becomes

\[
\dot{r} = (\varepsilon r^2 - \delta_1)r + \bar{\chi} \cos \theta, \\
\dot{\theta} = -\bar{\chi} \sin \theta.
\]

We see that if \( r = r_* \) satisfies

\[
(\varepsilon r^2 - \delta_1)r - \bar{\chi} = 0 \quad \text{(resp. } \varepsilon r^2 - \delta_1)r + \bar{\chi} = 0),
\]

then \((r, \theta) = (r_*, \pi) \) (resp. \((r_*, 0)\)) is an equilibrium in (2.5).
Proposition 2.1. Assume that
\[ \bar{\chi}^2 > \frac{4\delta_1^3}{27\varepsilon}. \] (2.7)

Then Eq. (2.5) has only one equilibrium at \((r_*, \pi)\), which is a source, where \(r = r_*\) is a root of the first equation in (2.6). Moreover, no closed (periodic) orbit exists.

Proof. We easily see that if condition (2.7) holds, then in (2.6) the first equation has only one root \(r = r_*\) and the second equation has no root on \((0, \infty)\). So there exists only one equilibrium at \((r_*, \pi)\) in (2.5). We also compute the Jacobian matrix as
\[ J = \begin{pmatrix} 3\varepsilon r_*^2 - \delta_1 & 0 \\ 0 & \bar{\chi} \end{pmatrix}, \]
the trace and determinant of which become
\[ \text{tr} \ J = 3\varepsilon r_*^2 - \delta_1 + \bar{\chi} > 0, \quad \det J = \bar{\chi}(3\varepsilon r_*^2 - \delta_1) > 0 \]
since
\[ 3\varepsilon r_*^2 - \delta_1 = \frac{\bar{\chi}}{r_*} + 2\varepsilon r_*^2 > 0. \]
This means that the equilibrium is a source.

It remains to prove the second part. Assume that a closed orbit exists in (2.5), equivalently in the system of the Cartesian coordinates,
\[ \dot{u}_1 = -\delta_1 u_1 + \varepsilon(u_1^2 + u_2^2)u_1 + \bar{\chi}, \quad \dot{u}_2 = -\delta_1 u_2 + \varepsilon(u_1^2 + u_2^2)u_2. \]
Then we show that inside the closed orbit there must be an equilibrium (see, e.g., Corollary 1.8.5 of [7]). However, this is impossible since in (2.5) \((r_*, \pi)\) is the only equilibrium and \(\dot{r} > 0\) when \(r > r_*\). Thus, no closed orbit exists in (2.5). \(\square\)

Remark 2.2. In Proposition 2.1 the equilibrium \((r_*, \pi)\) is hyperbolic. Hence, the conclusion also holds when \(\gamma, \delta_2 \neq 0\) are sufficiently small.

By the averaging theorem (e.g., Theorem 4.1.1 of [7]), we see that the distance of the solutions to (2.5) and (2.2) with \(\gamma, \delta_2 = 0\) satisfying the same initial conditions remains \(O(\varepsilon)\) on any finite time interval, no matter when the initial time is. In particular, there exists a hyperbolic unstable periodic orbit near the equilibrium \((r_*, \pi)\) in (2.2). It follows from Proposition 2.1 that trajectories starting anywhere other than on the periodic orbit, for example, even near the origin, go to infinity in finite time. We immediately obtain the following estimate on the growth of solutions in (2.2).

Lemma 2.3. If \(r(t_0)\) satisfies
\[ \varepsilon r^3 - \left( \frac{1}{2}(|k_{11}| + |k_{12}| + |k_{21}| + |k_{22}|) + |\delta_1| \right) r \]
\[ > (|k_{11}| + |k_{21}|)\bar{\rho}_1 + (|k_{12}| + |k_{22}|)\bar{\rho}_2 \] (2.8)
at some \(t = t_0\), then \(r(t)\) is monotonically increasing for \(t > t_0\) in (2.2).

Proof. We see that the right-hand-side of the first equation in (2.2) is positive if condition (2.8) holds. Moreover, the left-hand-side of (2.8) is monotonically increasing when it is positive and \(r > 0\). Thus, we obtain the desired result. \(\square\)

For (1.6) we replace the region (2.8) with
\[ r^3 - \left( \frac{1}{2}(|k_{11}| + |k_{12}| + |k_{21}| + |k_{22}|) + |\delta_1| \right) r \]
\[ > (|k_{11}| + |k_{21}|)\bar{\rho}_1 + (|k_{12}| + |k_{22}|)\rho_2, \] (2.9)
where \( r = \sqrt{x_1^2 + x_2^2} \) by abuse of notation, to have the same conclusion as in Lemma 2.3.

3. Proof of Theorem 1.1. In this section we give a proof of Theorem 1.1. We first remark that on \([0, \tau], \gamma, \delta_3.\]

**Lemma 2.3.**

\[
(x_1(0), x_2(0)) = \sqrt{e}(r(0) \cos(\theta(0) - \tilde{\phi}(\phi)), r(0) \sin(\theta(0) - \tilde{\phi}(\phi))).
\]

This fact is not affected by the time shift \( t \to t - \tau + \phi_1/\omega \) done for (1.1) in Section 2.

Letting \( x_1 = r \cos \theta \) and \( x_2 = r \sin \theta \) by abuse of notation, we transform (1.1) as

\[
\begin{align*}
\dot{r}(t) &= r(t)^3 - \delta_1 r(t) \\
&= \frac{1}{2} k_{11}(r(t) \cos 2\theta(t) - 2r(t - \tau) \cos \theta(t) \cos (t - \tau)) \\
&\quad - \frac{1}{2} k_{12}(r(t) \sin 2\theta(t) - 2r(t - \tau) \cos \theta(t) \sin (t - \tau)) \\
&\quad - \frac{1}{2} k_{21}(r(t) \sin 2\theta(t) - 2r(t - \tau) \sin \theta(t) \cos (t - \tau)) \\
&\quad + \frac{1}{2} k_{22}(r(t) \cos 2\theta(t) + 2r(t - \tau) \sin \theta(t) \sin (t - \tau)),
\end{align*}
\]

\[
(3.1)
\]

like (2.2). As in Lemma 2.3 we obtain the following.

**Lemma 3.1.** Assume that for some \( t_0 \geq \tau, \dot{r}(t) > 0 \) or \( r = r(t) \) does not satisfy

\[
r^3 - \left( \frac{3}{2}(|k_{11}| + |k_{12}| + |k_{21}| + |k_{22}|) + |\delta_1| \right) r \geq 0 \tag{3.2}
\]

for any \( t \in [t_0 - \tau, t_0] \). If \( r(t_0) \) satisfies (3.2), then \( r(t) \) is monotonically increasing for \( t > t_0 \) and blows up in finite time in (3.1).

**Proof.** Assume that the hypothesis of Lemma 3.1 holds and \( r(t_0) \) satisfies (3.2) additionally. Then we see that if \( r(t) \) satisfies (3.2) at some \( t = t_1 \in [t_0 - \tau, t_0] \) and does not for \( t \in [t_0 - \tau, t_1] \), then \( r(t) \) is monotonically increasing on \([t_1, t_0] \). Moreover, the left-hand-side of (3.2) is monotonically increasing when it is positive and \( r > 0 \). Hence, we have \( r(t_0) > r(t_0 - \tau) \), so that the right-hand-side of the first equation in (3.1) is positive at \( t = t_0 \). This argument can apply continuously for \( t > t_0 \), so that \( \dot{r}(t) > 0 \).

Let \( \beta = \frac{3}{2}(|k_{11}| + |k_{12}| + |k_{21}| + |k_{22}|) + |\delta_1| > 0 \). For \( t > t_0 \), it follows from the first equation of (3.1) that

\[
\dot{r} > r^3 - \beta r,
\]

which yields

\[
\log \frac{r(t)^2 - \beta}{r(t)^2} > \log \frac{r_0^2 - \beta}{r_0^2} > 2\beta t,
\]

where \( r_0 = r(t_0) \). Hence,

\[
r(t)^2 > \frac{\beta r_0^2}{r_0^2 - (r_0^2 - \beta)e^{2\beta t}},
\]

which means that \( r(t) \) blows up in finite time. \(\square\)
Note that the region (3.2) is included by (2.9) when ρ₁, ρ₂ are sufficiently small. We can now easily prove Theorem 1.1.

Proof of Theorem 1.1. Suppose that the hypotheses of Theorem 1.1 holds with γ = 0. As stated in Section 2, by Proposition 2.1 and the averaging theorem [7], any trajectory starting near the origin such that
\[ |x₀| < \sqrt{ɛrₚ} \quad (3.3) \]
enters the region given by (2.9) by \( t = \tau_0 \) if \( \tau_0 \) is taken sufficiently large. Let \( \tau > \tau_0 \). Obviously, if \( r(t) \to ∞ \) as \( t \to t₁ \) for some \( t₁ ≤ τ \), then the statement of Theorem 1.1 holds. Assume that \( r(τ) < ∞ \). From Lemma 2.3 we see that \( r(t) > 0 \) or \( r(t) \) does not satisfy (2.9) and consequently (3.2) on \( [0, τ) \). Hence, it follows from Lemma 3.1 that \( r(t) \) blows up in finite time in (3.1). In particular, the blow-up time is greater than \( τ \). By Remark 2.2, these statements also hold for \( |γ| > 0 \) sufficiently small. □

Remark 3.2.

(i) As in Proposition 2.1, the statement of Theorem 1.1 also holds when \( ω \) is not equal but close to 1 + \( \frac{1}{2} (k₁₂ - k₂₁) \) (see Remark 2.2). Moreover, even if condition (3.3) does not hold, the solution blows up in finite time when it does not start on the periodic orbit of source-type in (1.6) at \( t = 0 \).

(ii) From the proof of Theorem 1.1 we see that \( τ₀ = O(1/α) \) since the right-hand-side of the averaged system (2.3) is \( O(α) \) in the scaling of the parameters in Theorem 1.1.

4. Numerical Examples. We carried out numerical simulations for (1.1) with \( α = 0.02, k₁₁ = k₂₂ = 0.02, k₁₂ = -0.02, k₂₁ = 0.06 \) and \( φ₁ = φ₂ = \frac{1}{2} π \), so that \( x₁₀ = x₂₀ = 0, φ = 0 \) and \( δ₁ = 0.04 \) and \( ω = 0.96 \). The delay \( τ = 2nπ/Ω \) was chosen for some positive integer \( n \). Condition (3.2) holds if \( r > \sqrt{0.22} = 0.46904157 \ldots \)

We used the fourth-order Runge-Kutta method and estimated the state variables at the midpoint of the time interval \([t, t + h]\) via the dense output [8] as
\[ x(t + \frac{1}{2} h) - x(t) = \frac{h}{2π} k₁ + \frac{1}{2} (k₂ + k₃) - \frac{1}{2π} k₄, \]
where \( h \) is the step size and
\[ k₁ = f(t, x(t)), \quad k₂ = f(t + \frac{1}{2} h, x(t) + \frac{1}{2} hk₁), \]
\[ k₃ = f(t + \frac{1}{2} h, x(t) + \frac{1}{2} hk₂), \quad k₄ = f(t + h, x(t) + hk₃). \]

Here \( f(t, x) \) represents the right-hand-side of (1.1). Note that the state variables are estimated at the end point of the time interval as
\[ x(t + h) - x(t) = \frac{1}{6} k₁ + \frac{1}{3} (k₂ + k₃) + \frac{1}{6} k₄ \]
in the Runge-Kutta method.

Figure 2 shows numerical simulation results for \( γ = 0, -0.5 \) and \(-5 \). Here \( ρ₁ = ρ₂ = 0.1 \) are taken in Figs. 2(a) and (b), and \( ρ₁ = ρ₂ = 0.15 \) are taken in Fig. 2(c). Recall that \( r = \sqrt{x₁₀² + x₂₀²} \). We see that condition (1.9) holds as shown in Fig. 3, as well as the origin is stable (see Fig. A.1 in Appendix A). In Fig. 2(a), where \( γ = 0 \), we observe that the solution blows up in finite time after it enters the region (3.2) by \( t = τ \) when the delay is large (\( τ = 46π \)), as predicted from the theory, while it converges to zero when the delay is small (\( τ = 20π \)). We make similar observations in Fig. 2(b), where \( γ = -0.5 \): The solution blows up in finite time after it enters the region (3.2) by \( t = τ \) when the delay is large (\( τ = 46π/0.99 \)) while it converges to zero when the delay is small (\( τ = 20π/0.99 \)).
This observation is also consistent with the theory. On the other hand, in Fig. 2(c), where $\gamma = -5$, some different observations are made: The solution converges to zero when $\tau = 2\pi/0.9$ or $\tau = 22\pi/0.9$; it converges to the periodic orbit (1.2), which is stable (see Fig. B.1 in Appendix B), when $\tau = 4\pi/0.9$; it converges to another stable periodic orbit when $\tau = 8\pi/0.9$; and it blows up in finite time but continues to stay outside of the region (3.2) for a long time $(> \tau)$ when $\tau = 46\pi/0.9$.

Figure 4 shows the loci of periodic orbits in (1.6) for $\alpha = 0.02$, $\omega = 0.96$, $k_{11} = k_{22} = 0.02$, $k_{12} = -0.02$, $k_{21} = 0.06$ and $\phi_1 = \phi_2 = \frac{1}{2}\pi$. The computer software
Figure 3. Condition (1.9) for $\alpha = 0.02$, $k_{11} = k_{22} = 0.02$, $k_{12} = -0.02$ and $\phi = 0$ when $\rho_1 = \rho_2 (= \rho)$. It holds above the curves. Figure (b) is an enlargement of Fig. (a).

Figure 4. Periodic orbits in (1.6) for $\alpha = 0.02$, $\omega = 0.96$, $k_{11} = k_{22} = 0.02$, $k_{12} = -0.02$, $k_{21} = 0.06$ and $\bar{\phi}_1 = \bar{\phi}_2 = \frac{1}{2}\pi$. Stable and unstable periodic orbits are, respectively, plotted as solid and broken lines and torus bifurcations points are denoted by the symbol ‘•’.

**AUTO** [2] was used to draw this figure. We observe that the periodic orbits are unstable when $\gamma \approx 0$, as shown by the averaging analysis in Section 2, and they change their stability through torus bifurcations at which unstable tori are born. The invariant torus on the Poincaré section \{\omega t = 0 \mod 2\pi\} is plotted along with the stable periodic orbit for $\gamma = -5$ and $\rho_1 = \rho_2 = 0.15$ in Fig. 5. The other parameter values are the same as those of Fig. 4. Thus, the dynamics of (1.6) are different from those predicted by the averaging analysis in Section 2 after the torus bifurcation occurs. This is one of the reasons why our observations of Fig. 2(c) were different from those of Figs. 2(a) and (b).

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Appendix A. Stability of the origin in (1.1). We take the delay $\tau$ as a control parameter. Linearizing (1.1) around the origin, we have

$$
\begin{align*}
\dot{\xi}_1(t) &= -\alpha \xi_1(t) - \xi_2(t) - k_{11}(\xi_1(t) - \xi_1(t - \tau)) - k_{12}(\xi_2(t) - \xi_2(t - \tau)), \\
\dot{\xi}_2(t) &= \xi_1(t) - \alpha \xi_2(t) - k_{21}(\xi_1(t) - \xi_1(t - \tau)) - k_{22}(\xi_2(t) - \xi_2(t - \tau)).
\end{align*}
$$

(A.1)

The characteristic equation for (A.1) is given by

$$
\det \begin{pmatrix} -\alpha - k_{11}(1 - e^{-\lambda \tau}) & -1 - k_{12}(1 - e^{-\lambda \tau}) \\ 1 - k_{21}(1 - e^{-\lambda \tau}) & -\alpha - k_{22}(1 - e^{-\lambda \tau}) \end{pmatrix} - \lambda \text{id} = 0,
$$

(A.2)

where id is the 2 × 2 identity matrix. So we see that a Hopf bifurcation occurs if for some $s > 0 \lambda = is$ is a root of (A.2), i.e.,

$$
\begin{align*}
-s^2 - (k_{11} + k_{22}) s \sin st + (k_{11} k_{22} - k_{12} k_{21})(\cos^2 st - \sin^2 st) \\
-((k_{11} + k_{22}) \alpha + 2(k_{11} k_{22} - k_{12} k_{21}) + k_{12} - k_{21}) \cos st \\
+ \alpha^2 + (k_{11} + k_{22}) \alpha + (k_{11} k_{22} - k_{12} k_{21}) + k_{12} - k_{21} + 1 = 0
\end{align*}
$$

(A.3)

and

$$
\begin{align*}
(2\alpha + (k_{11} + k_{22})(1 - \cos st)) s - 2(k_{11} k_{22} - k_{12} k_{21}) \sin st \cos st \\
+ ((k_{11} + k_{22}) \alpha + 2(k_{11} k_{22} - k_{12} k_{21}) + k_{12} - k_{21}) \sin st = 0.
\end{align*}
$$

(A.4)

When $k_{11} = k_{22}$ and $k_{12} = -k_{21}$ and they are sufficiently small, Eqs. (A.3) and (A.4) reduce to

$$
\begin{align*}
s &= -k_{11} \sin st + (1 + k_{12} - k_{12} \cos st), \\
\alpha &= -k_{11} \cos st - k_{12} \sin st.
\end{align*}
$$

Figure A.1 shows the stability region of the origin in (1.1) for $\alpha = 0.02$, $k_{11} = k_{22} = 0.02$ and $k_{12} = -0.02$. Equations (A.3) and (A.4) were numerically solved to obtain the boundary. It was confirmed by numerical simulations that the origin is stable below the curve in Fig. A.1.
Appendix B. Stability of the periodic orbit (1.2) in (1.1). Let $\tau = 2n\pi/\Omega$. Linearizing (1.1) around the periodic orbit (1.2), we have

$$
\begin{align*}
\dot{\xi}_1(t) &= \alpha(\cos 2\Omega t - \gamma \sin 2\Omega t + 1)\xi_1(t) \\
&\quad + (\alpha(\sin 2\Omega t + \gamma \cos 2\Omega t - 2\gamma) - 1)\xi_2(t) \\
&\quad - k_{11}(\xi_1(t) - \xi_1(t - \tau)) - k_{12}(\xi_2(t) - \xi_2(t - \tau)), \\
\dot{\xi}_2(t) &= (\alpha(\gamma \cos 2\Omega t + \sin 2\Omega t + 2\gamma) + 1)\xi_1(t) \\
&\quad + \alpha(\gamma \sin 2\Omega t - \cos 2\Omega t + 1)\xi_2(t) \\
&\quad - k_{21}(\xi_1(t) - \xi_1(t - \tau)) - k_{22}(\xi_2(t) - \xi_2(t - \tau)).
\end{align*}
$$

(B.1)

Letting

$$
\begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix} =
\begin{pmatrix}
\cos \Omega t & -\sin \Omega t \\
\sin \Omega t & \cos \Omega t
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix},
$$

we rewrite (B.1) as

$$
\begin{pmatrix}
\dot{\eta}_1(t) \\
\dot{\eta}_2(t)
\end{pmatrix} =
\begin{pmatrix}
2\alpha & 0 \\
2\alpha \gamma & 0
\end{pmatrix}
\begin{pmatrix}
\eta_1(t) \\
\eta_2(t)
\end{pmatrix}
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix} + 
\begin{pmatrix}
\cos 2\Omega t & -\sin 2\Omega t \\
-\sin 2\Omega t & -\cos 2\Omega t
\end{pmatrix}
\begin{pmatrix}
\eta_1(t) - \eta_1(t - \tau) \\
\eta_2(t) - \eta_2(t - \tau)
\end{pmatrix} + 
\begin{pmatrix}
\sin 2\Omega t & -\cos 2\Omega t \\
\cos 2\Omega t & -\sin 2\Omega t
\end{pmatrix}
\begin{pmatrix}
\eta_1(t) - \eta_1(t - \tau) \\
\eta_2(t) - \eta_2(t - \tau)
\end{pmatrix},
$$

(B.2)

where

$$
k = \frac{1}{2}(k_{11} + k_{22})^2 + (k_{21} - k_{12})^2, \quad \tan \beta = \frac{k_{21} - k_{12}}{k_{11} + k_{22}},
$$

$$
\Delta_1 = \frac{1}{2}(k_{11} - k_{22}), \quad \Delta_2 = \frac{1}{2}(k_{21} + k_{12}).
$$
For $\lambda \neq 0$, let $\Phi_\lambda(t)$ be a fundamental solution to
\[
\begin{pmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{pmatrix} = \begin{pmatrix} 2\alpha & 0 \\
2\alpha\gamma & 0 \\
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}
- (1 - \lambda^{-1}) \left(
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
+ \Delta_1 \left(
\begin{pmatrix}
\cos 2\Omega t & -\sin 2\Omega t \\
-\sin 2\Omega t & \cos 2\Omega t
\end{pmatrix}
\right)
\right.

+ \Delta_2 \left(
\begin{pmatrix}
\sin 2\Omega t & -\cos 2\Omega t \\
\cos 2\Omega t & -\sin 2\Omega t
\end{pmatrix}
\right)
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix},
\] (B.3)

with
\[
\Phi_\lambda(0) = \text{id}. \quad \text{(B.4)}
\]

The characteristic equation for (B.2) is given by
\[
\det(\lambda \text{id} - \Phi_\lambda(\tau)) = 0. \quad \text{(B.5)}
\]

We easily see that $\lambda = 1$ is an eigenvalue of (B.5) since
\[
\Phi_1(t) = \begin{pmatrix}
e^{2\alpha t} & 0 \\
\gamma(e^{2\alpha t} - 1) & 1
\end{pmatrix}.
\]

The periodic orbit is stable if the other eigenvalues of (B.5) have moduli less than one, and it is unstable if an eigenvalue has a modulus greater than one.

Let $k_{11} = k_{22}$ and $k_{12} = -k_{21}$, i.e., $\Delta_1, \Delta_2 = 0$. Then Eq. (B.3) reduces to
\[
\begin{pmatrix}
\dot{\eta}_1 \\
\dot{\eta}_2
\end{pmatrix} = \left(
\begin{pmatrix} 2\alpha & 0 \\
2\alpha\gamma & 0 \\
\end{pmatrix} - k(1 - \lambda^{-1}) \begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\right)
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix},
\] (B.6)

In particular, when $\beta = 0$ or $\pi$,
\[
\Phi_\lambda(t) = \begin{pmatrix}
\text{exp}((2\alpha \mp k(1 - \lambda^{-1}))t) & 0 \\
\text{exp}(\mp k(1 - \lambda^{-1})t)(e^{2\alpha t} - 1) & \text{exp}(\mp k(1 - \lambda^{-1})t)
\end{pmatrix},
\] (B.7)

is a fundamental solution to (B.6) satisfying (B.4), where the upper and lower signs are taken for $\beta = 0$ and $\pi$, respectively. Letting $\lambda = e^{\sigma \tau}$ with $\sigma \in \mathbb{C}$, we see that the trivial solution to (B.6) is unstable or stable, depending on whether
\[
g(\sigma) = \det \left(\sigma \text{id} - \begin{pmatrix} 2\alpha & 0 \\
2\alpha\gamma & 0 \\
\end{pmatrix} + k(1 - e^{-\sigma \tau}) \begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix} \right)
\]
\[
= \sigma(\sigma - 2\alpha) + k(1 - e^{-\sigma \tau})(k(1 - e^{-\sigma \tau}) + 2((\sigma - \alpha) \cos \beta - \alpha \gamma \sin \beta))
\]

has a root with a positive real part or not. A candidate for the boundary of stability for the trivial solution to (B.6) is given by $g'(0) = 0$ with $s \in \mathbb{R}$, i.e.,
\[
k = -\frac{1}{\tau(\cos \beta + \gamma \sin \beta)}.
\]

Another one is given by $g(is) = 0$ with $s > 0$, i.e.,
\[
- s^2 + 2k(\alpha(\cos \beta + \gamma \sin \beta) - \cos s\tau
- 2ks \cos \beta \sin s\tau + k^2 \cos 2s\tau + k^2 - 2k\alpha(\cos \beta + \gamma \sin \beta) = 0,
- 2\alpha s + 2ks \cos \beta - 2ks \cos \beta \cos s\tau
- 2k(\alpha(\cos \beta + \gamma \sin \beta) - \cos s\tau - k^2 \sin 2s\tau = 0,
\]

which yields
\[
k = \mp \frac{s}{\sin s\tau}, \quad s \tan \frac{1}{2}s\tau + 2\alpha = 0
\]
when $\beta = 0$ and $\pi$, where the upper and lower signs are taken for $\beta = 0$ and $\pi$, respectively.

Figure B.1 shows parameter regions in which the periodic orbit (1.2) may be stable in (1.1) for $\alpha = 0.02$ and $k_{11} = k_{22} = 0.02$ when $\gamma = -0.5$, $-1$ or $-5$. The boundary was computed by numerically solving the boundary value problem of (B.3) with (B.4) and (B.5). To carry out necessary computations, the computer software AUTO [2] was used and the solution (B.7) was chosen as the starting solution. In each figure, the characteristic equation (B.5) has an eigenvalue $\lambda = 1$ of multiplicity two on the red line and a pair of complex eigenvalues with moduli one on the blue line. Here $\tau = 2\pi n/\Omega$ with $n = 1$ and 2 are taken for the solid and dashed lines, respectively.

Figure B.1. Stability regions of the periodic orbit (1.2) in (1.1) for $\alpha = 0.02$ and $k_{11} = k_{22} = 0.02$: (a) $\gamma = -0.5$; (b) $-1$; (c) $-5$.

In each figure, the characteristic equation (B.5) has an eigenvalue $\lambda = 1$ of multiplicity two on the red line and a pair of complex eigenvalues with moduli one on the blue line. Here $\tau = 2\pi n/\Omega$ with $n = 1$ and 2 are taken for the solid and dashed lines, respectively.
nonlinear system (1.1), according to the previous results [4, 9, 13], transcritical and torus bifurcations occur on the red and blue lines, respectively. We also remark that in a similar numerical computation the periodic orbit was observed to be unstable when $\gamma = 0$ even if $k_{12} \neq -k_{21}$, as in $k_{12} = -k_{21}$ [13].

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