Entanglement description in nilpotent quantum mechanics

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Abstract. Recently proposed formalism based on nilpotent commuting variables can effectively be used to the description of multiqubit entanglement. We show that separability/entanglement questions can be formulated within the spaces of functions of such variables with use of the appropriate generalization of differential calculus.

1. Introduction

The study of quantum entanglement phenomenon [1], being extremely interesting experimentally, having potentially crucial applications, also triggered interest in many mathematical tools, old and new. Some of them, like classical invariant theory are rooted in the Nineteen’s century mathematics, many are invented recently. In the present work we want to show, how formalism based on the commuting nilpotents [2] can be used to effectively study the entanglement. Such approach has been proposed recently [3, 4]. Here the aim is to bring the links of the new formalism to the classical invariant theory and to describe how the entanglement versus factorization of many-qubit states can be addressed in terms of the factorization of relevant generalized wave functions. The generalization of wave function we shall understand in the following way [4].

The states of a two level system, which we shall call qubit, are generated by nilpotent variable \( \eta \), which has values:

\[
\eta^0 = 1, \quad \eta \quad \text{and} \quad \eta^2 = 0 \quad \text{i.e. it is a first order nilpotent).}
\]

Explicitly for one qubit we shall have

\[
\psi(x, \eta) = \psi_0(x) + \eta \psi(x)
\]

and similarly for two qubits

\[
\psi^{(2)}(x, \eta_1, \eta_2) = \psi_0(x) + \eta_1 \psi_1(x) + \eta_2 \psi_2(x) + \eta_1 \eta_2 \psi_{12}(x)
\]

Using ordered multi-indices, analogous expansion for three qubits will read as

\[
\psi^{(3)}(x, \eta_1, \eta_2, \eta_3) = \psi_0(x) + \eta_i \psi_i(x) + \eta_i \eta_j \psi_{ij}(x) + \eta_1 \eta_2 \eta_3 \psi_{123}(x),
\]

where for general ordered multi-index \( J = j_1 j_2 \ldots j_n \) we assume \( j_1 < j_2 \ldots < j_n \). To make contact with common binary basis notation let as rewrite above functions in equivalent form

\[
\psi^{(1)}(x) = \psi_0(x)|0\rangle + \psi(x)|1\rangle
\]

\[
\psi^{(2)}(x) = \psi_0(x)|00\rangle + \psi_1(x)|10\rangle + \psi_2(x)|01\rangle + \psi_{12}(x)|11\rangle
\]

\[
\psi^{(3)}(x) = \psi_0(x)|000\rangle + \psi_1(x)|100\rangle + \psi_2(x)|010\rangle + \psi_{12}(x)|001\rangle + 
\psi_{13}(x)|110\rangle + \psi_{13}(x)|101\rangle + \psi_{23}(x)|011\rangle + \psi_{123}(x)|111\rangle
\]
In general above \( \eta \)-functions can depend on real or complex variables, in the following we shall drop such a dependence to focus only on the \( \eta \) dependence. The new way of writing a multi-qubit state by \( \psi(\eta) \) seems to be merely a change of notation, but viewing it as a function is very powerful, we can formulate the questions of factorization of state as a problem of the factorization of an \( \eta \)-function, moreover emerging within this approach \( \eta \)-function Wronskians will also serve as natural entities measuring the degree of entanglement of relevant multi-qubit states. It is very interesting that complementarity relations known for two-qubit and three-qubit states, in this formalism come as natural geometric relations in the relevant \( \eta \)-function spaces.

2. Space of \( \eta \)-wavefunctions

In studying the properties of (differential) functions crucial role plays differential calculus. For the \( \eta \) variable we define derivation as

\[
\partial_i \eta^j = \delta^j_i, \quad \partial_i 1 = 0, \quad \partial_i \partial_j = \partial_j \partial_i,
\]  

where \( \partial_j = \frac{\partial}{\partial \eta^j} \). Analogously the \( \eta \)-integral is introduced as the following contraction

\[
\int \eta^i d\eta_j = \delta^j_i, \quad \int d\eta_i = 0
\]

Using the \( \eta \)-integral let us define the scalar product in the space of the \( \eta \)-functions (wave \( \eta \)-functions) and the structure mimicking that of Hilbert space. We shall call it the \( \eta \)-Hilbert space. Its natural realization is given by the space of \( \eta \)-wavefunctions. The scalar product is given in the following form

\[
<\psi, \phi> = \int \psi^* (\vec{\eta}) \phi(\vec{\eta}) e^{-\eta^* . \vec{\eta}} d\eta^* d\vec{\eta}, = \int \psi^* (\vec{\eta}) \phi(\vec{\eta}) d\mu(\vec{\eta}^*, \vec{\eta}),
\]

where \( \vec{\eta} = (\eta_1, \eta_2, \ldots, \eta_n) \) for the given \( n \), and

\[
\psi^* (\vec{\eta}) = \sum_{k=0}^{n} \sum_{I_k} \psi^*_{I_k} \eta^{I_k*}
\]

with * denoting the complex conjugation. Explicitly we have

\[
<\psi, \phi> = \sum_{k=0}^{n} \sum_{I_k} \psi^*_{I_k} \phi_{I_k}
\]

In a simplest case of the 1-qubit algebra \( F[\eta] \) (set of \( \eta \)-functions of one variable), the \( \eta \)-scalar product of \( \psi(\eta) \) and \( \phi(\eta) \) functions takes simple form

\[
<\psi, \phi> = \psi_0^* \phi_0 + \psi_1^* \phi_1
\]

and has good properties. The qubit algebra let be given by mutually conjugated operators

\[
d^+ = \eta \cdot, \quad d = \partial / \partial \eta
\]

i.e. operators \( d^+ \) and \( d \) are conjugated with respect to the above scalar. For example Heisenberg’s \( \sigma_3 \) matrix is realized as

\[
\sigma_3 = 1 - 2\eta \partial_\eta.
\]
In the space of $\eta$-functions there exists an analog of the operator known for exterior differential forms, namely, the Hodge $\star$. Here the duality operator $\star$ we shall define as follows

$$\star(\eta_k) = (-1)^k \eta_{n-k}, \quad I_k \cup I_{n-k} = I_n,$$

(15)

naturally $\star 1 = \eta_1 \eta_2 \ldots \eta_n$ and $\star^2 = (-1)^n \text{id}$.

To address the question of factoriality of the multiqubit wave functions let us recall that a factorable function of two real variables $f(x, y) = h(x)g(y)$ has to satisfy the d’Alember condition

$$\frac{\partial^2 \ln f}{\partial x \partial y} = 0.$$

(16)

which can be written also in the following form

$$\left| \begin{array}{cc} f & \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial y} & \frac{\partial^2 f}{\partial x \partial y} \end{array} \right| = 0.$$

(17)

Above criterion can be generalized to the functions of $\eta$-variables. Namely, by the $w_{12}$ we denote the $\eta$-Wronskian with respect to $\eta_1$ and $\eta_2$ variables of the form

$$w_{12}(\psi(\eta_1, \eta_2)) = \det W_{12} = \left| \begin{array}{cc} \psi & \frac{\partial \psi}{\partial \eta_2} \\ \frac{\partial \psi}{\partial \eta_1} & \frac{\partial^2 \psi}{\partial \eta_1 \partial \eta_2} \end{array} \right| = \psi_0 \psi_{12} - \psi_1 \psi_2.$$

(18)

In our new case we get strong criterion that any $\psi(\eta_1, \eta_2)$ function of two $\eta$ variables factorizes iff $w_{12}(\psi) = 0$. This means that for the two-qubit pure states described in $\eta$-function space we easily determine presence of entanglement which is here detected by nonzero value of the $w_{12} = \psi_0 \psi_{12} - \psi_1 \psi_2$. Moreover, the explicit form of this Wronskian yields the definition of the concurrence

$$C(\psi) = 2|w_{12}(\psi(\eta_1, \eta_2))|, \quad < \psi, \psi >= 1,$$

(19)

where

$$< \psi, \psi >= |\psi_0|^2 + |\psi_1|^2 + |\psi_2|^2 + |\psi_{12}|^2.$$

(20)

equivalent to the conventional one. In the case of three qubits, $\eta$-functions depend on three variables and we have to consider several Wronski matrices, for all distinct pairs of variables. Such bipartite information can be used to find a type of nonfactorability of the $\eta$-wavefunction $\psi(\eta_1, \eta_2, \eta_3)$. For three qubits Wronsksians depend explicitly on complementary $\eta$-variable

$$w_{12}(\psi)(\eta_3) = w_{12}(\psi|_{\eta_3=0}) + (\mathcal{H} + 2\psi \partial \psi_{12}) \eta_3 \equiv w_{12}(\psi|_{\eta_3=0}) + \tilde{\mathcal{H}}_3 \eta_3$$

(21)

$$w_{13}(\psi)(\eta_2) = w_{13}(\psi|_{\eta_2=0}) + (\mathcal{H} + 2\psi \partial \psi_{13}) \eta_2 \equiv w_{13}(\psi|_{\eta_2=0}) + \tilde{\mathcal{H}}_2 \eta_2$$

(22)

$$w_{23}(\psi)(\eta_1) = w_{23}(\psi|_{\eta_1=0}) + (\mathcal{H} + 2\psi \partial \psi_{23}) \eta_1 \equiv w_{23}(\psi|_{\eta_1=0}) + \tilde{\mathcal{H}}_1 \eta_1,$$

(23)

where $\mathcal{H}$ is defined as $\mathcal{H} = \psi_0 \psi_{123} - \psi_1 \psi_{23} - \psi_2 \psi_{13} - \psi_3 \psi_{12}$. What is interesting, one can express all terms in the above expansions of $\eta$-function determinants in terms of the determinants and traces of Wronsks matrices, since

$$\tilde{\mathcal{H}}_k(\psi) = tr W_{ij}(\psi|_{\eta_k=0}) \cdot tr W_{ij}(\partial_k \psi) - tr W_{ij}(\psi|_{\eta_k=0}) W_{ij}(\partial_k \psi).$$

(24)

Effective check of bipartite factorization of two subsystems $(i - k)(j)$, with full separation of dependence on one variable is based on mutual relations between chosen variable $\eta_j$ and the
two remaining \( \eta_i \) and \( \eta_k \). Let be arbitrary three-qubit pure state \( \psi = \psi(\eta_1, \eta_2, \eta_3) \), then it can be factorized into product of \( \phi(\eta_i, \eta_k) \) and \( \phi(\eta_j) \) iff:

\[
\begin{align*}
    w_{ij}(\psi)(\eta_k) &= 0, & w_{ik}(\psi)(\eta_i) &= 0, \\
    w_{ij}(\partial_k \psi) &= 0, & w_{ik}(\partial_i \psi) &= 0,
\end{align*}
\]

where \( (i, j, k = 1, 2, 3, \text{are all different and fixed}) \).

Factorization properties of the \( \eta \)-function \( \psi(\eta_1, \eta_2, \eta_3) \) and its Hodge dual \( \psi^\star(\eta_1, \eta_2, \eta_3) \) are closely related, because

\[
    w_{ij}(\psi|_{\eta_k = 0}) = w_{ij}(\partial_k(\psi^\star)), \quad \mathcal{H}_i(\psi) = \mathcal{H}_i(\psi^\star)
\]

Examples:

- two-qubit Werner state is represented by the function \( \psi_W(\eta_1, \eta_2) = \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \), and we have that \( w_{12}(\psi_W) = -\frac{1}{2} \).
- for two-qubit GHZ state \( \psi_{\text{GHZ}}(\eta_1, \eta_2) = \frac{1}{\sqrt{2}}(1 + \eta_1 \eta_2) \) and \( w_{12}(\psi_{\text{GHZ}}) = \frac{1}{2} \),
- three-qubit Werner state \( \psi_W = \frac{1}{\sqrt{3}}(\eta_1 + \eta_2 + \eta_3) \), it is obviously nonseparable, and we have

\[
    w_{ij}(\psi_W|_{\eta_k = 0}) = -\frac{1}{3}, \quad \mathcal{H}_i = 0, \\
    w_{ij}(\partial_k \psi_W) = 0.
\]

Let us observe that mentioned above invariance with respect to the Hodge \( \star \) is not trivial, because nonzero contributions come from different components

\[
    w_{ij}(\psi_W|_{\eta_k = 0}) = 0, \quad \mathcal{H}_i = 0, \\
    w_{ij}(\partial_k \psi_W) = -\frac{1}{3}.
\]

3. Entanglement monotones for \( \eta \)-functions

The \( \eta \)-functions turn out to be suitable tool to describe entanglement of multiqubit states. In this work we focus only on pure states. Entanglement monotones can be defined here by means of the natural geometric structure present in the \( \eta \)-function spaces.

There are accepted monotones for \( n = 2, 3, 4 \) qubits [5], but there is still search for other functions which suit special needs. To show how our new formalism works let us discuss known entanglement monotones within the \( \eta \)-function formalism. It is remarkable, that the entanglement monotones for pure states can be expressed in terms of Wronskians of the \( \eta \)-functions representing relevant state.

3.1. 2-qubits

The principal entanglement monotone for two-qubit system is the concurrence. As we already mentioned above, it can be expressed using the Wronskian \( C(\psi(\eta_1, \eta_2)) = |w_{12}(\psi(\eta_1, \eta_2))| \), \( < F, F > = 1 \). But in the conventional approach it is known that using the comb [7] one can define an antilinear mapping \( \psi \mapsto \psi^c \), where \( \psi^c = (\sigma^y \otimes \sigma^y)\psi^* \) and write concurrence as

\[
    C(\psi) = |< \psi^c, \psi>| = 2|\psi_0 \psi_{12} - \psi_1 \psi_2|.
\]

The comb operator \( \sigma^y \otimes \sigma^y \) in \( \eta \)-formalism is realized as \( \eta \)-differential operator

\[
    \sigma^y \otimes \sigma^y = -(\partial_1 \partial_2 + \eta_1 \eta_2 - \eta_2 \partial_1 - \eta_1 \partial_2).
\]
However it turns out it is just representation of the action of the $\eta$-Hodge star. What means that $\psi^c = \star \psi^*$, for $\psi(\eta_1, \eta_2)$

$$C(\psi) = | \langle \star \psi^*, \psi \rangle |.$$  \hspace{1cm} (30)

Taking despite of the $\psi^c$ also $\eta$-derivatives $\partial_i \psi$ and images of $\psi$ under gradation mappings $J_i$ i.e. $J_1(\psi(\eta_1, \eta_2)) = \psi(-\eta_1, \eta_2)$ and $J_2(\psi(\eta_1, \eta_2)) = \psi(\eta_1, -\eta_2)$, we are able to define two aditional entities

- visibility
  $$\mathcal{V}_i = 2 | \langle \partial_i \psi, \psi \rangle |.$$  \hspace{1cm} (31)

- predictability
  $$\mathcal{P}_i = | \langle \psi, J_i(\psi) \rangle |.$$  \hspace{1cm} (32)

These three naturally defined in the function space objects combine into the so called complementarity relations

$$C^2(F) + \mathcal{V}_i^2(F) + \mathcal{P}_i^2(F) = \langle F, F \rangle^2 = 1, \quad i = 1, 2$$  \hspace{1cm} (33)

Such relations are fundamental in optics, and here emerge from natural operations in the $\eta$-function space.

3.2. 3-qubits

The ”degree” of entanglement/non-separability of 3-qubit systems is characterized by the 3-tangle \[6\], in this case the principal entanglement monotone measure. It is defined with the use of the hyperdeterminant known in invariants theory for a long time

$$\tau_{123} = 4 | \text{Det}(\psi) |,$$  \hspace{1cm} (34)

where

$$\text{Det}(\psi) = (\psi_0^2\psi_{123}^2 + \psi_1^2\psi_{12}^2 + \psi_2^2\psi_{13}^2 + \psi_3^2\psi_{23}^2) + 4(\psi_0\psi_{12}^3\psi_{13}\psi_{12} + \psi_1\psi_2\psi_{13}\psi_{23} + \psi_2\psi_3\psi_{23}\psi_{12} + \psi_3\psi_1\psi_{12}\psi_{23})$$

$$-2(\psi_0\psi_{12}\psi_{12}\psi_{12} + \psi_1\psi_2\psi_{13}\psi_{12} + \psi_0\psi_1\psi_{12}\psi_{23} + \psi_2\psi_3\psi_{23}\psi_{12} + \psi_1\psi_3\psi_{12}\psi_{23} + \psi_3\psi_1\psi_{12}\psi_{23})$$

As it is known this entanglement monotone detects the GHZ state entanglement, which is maximal, but for the Werner state gives zero. In their paper, Coffman Kundu and Wooters relate the 3-tangle $\tau_{123}$ and mutual concurrences of bipartite systems of three qubits as follows

$$C^2_{1(23)} = C^2_{12} + C^2_{13} + \tau_{123}$$  \hspace{1cm} (36)

We shall use here symmetric form of above formula, averaged over possible configurations of qubits i.e.

$$Q(\psi) = \frac{1}{3}(C^2_{1(23)} + C^2_{2(13)} + C^2_{3(12)}) = \frac{2}{3}(C^2_{12} + C^2_{13} + C^2_{23}) + \tau_{123}$$  \hspace{1cm} (37)

In fact, the $Q$ is the $n = 3$ realization of global entanglement measure introduced by Meyer and Wallach \[8\] for arbitrary $n$. Now in terms of $\eta$-functions we have

$$\text{Det}(\psi) = \frac{1}{3} \sum_k (\bar{\mathcal{H}}_k^2 - 4w_{ij}(\psi|_{\eta_k = 0})w_{ij}(\partial_k \psi)), \quad i \neq j \neq k$$  \hspace{1cm} (38)

From the $\eta$-realization of $\sigma$ matrices it follows that

$$\langle \sigma^2_{ij} \psi, \psi \rangle = -2(\psi_{ij}(\psi|_{\eta_i = 0}) + w_{ij}(\partial_k \psi)),$$  \hspace{1cm} (39)
where bar denotes complex conjugation and $\sigma_{ij}^2$ is a tensor product of $I$ and $\sigma_2$ matrices on $i^{th}$ and $j^{th}$ positions e.g. $\sigma_{23}^2 = I \otimes \sigma_2 \otimes \sigma_2$

\begin{equation}
C_{ij} = 2|w_{ij}(\psi|_{\eta_1=0}) + w_{ij}(\partial_k \psi)|
\end{equation}

\begin{equation}
Q(\psi) = \frac{2}{3} \left( \sum_{i<j} |<\sigma_{ij}^2 \tilde{\psi}, \psi>|^2 + \tau_{123} \right)
\end{equation}

\begin{equation}
Q(\psi) = \frac{4}{3} \left( 2 \sum_k |w_{ij}(\psi|_{\eta_k=0}) + w_{ij}(\partial_k \psi)|^2 + |\sum_k (\tilde{H}_k^2 - 4w_{ij}(\psi|_{\eta_k=0})w_{ij}(\partial_k \psi)) \right)
\end{equation}

The $Q(\psi)$ can be expressed solely in terms of determinants and traces of the Wronskian matrices $W_{ij}(\psi)$, $W_{ij}(\partial_k \psi)$ namely,

\begin{equation}
Q(\psi) = \frac{4}{3} \left( 2 \sum_k |w_{ij}(\psi|_{\eta_k=0}) + w_{ij}(\partial_k \psi)|^2 + |\sum_k (\text{tr}W_{ij}(\psi|_{\eta_k=0}) \cdot \text{tr}W_{ij}(\partial_k \psi) - \right.

\left. -\text{tr}(W_{ij}(\psi|_{\eta_k=0})W_{ij}(\partial_k \psi))^2 - 4w_{ij}(\psi|_{\eta_k=0})w_{ij}(\partial_k \psi)|\right).
\end{equation}

Above relations shows explicitly how entanglement and factorability properties are intertwined.

From our previous discussion on Wronskians and $\tilde{H}_i$ it follows that duality transformation preserves the value of $\mu$ i.e. $\mu(F) = \mu(\star F)$. In the Table 1, we collect values of the $Q$ for selected entangled states, illustrating properties of this entanglement monotone.

**Table 1.** Values of the entanglement monotone $Q$ for pure entangled states.

| State       | Q               |
|-------------|-----------------|
| $\psi_W$    | $\frac{1}{\sqrt{3}}(\eta_1 + \eta_2 + \eta_3)$ | $\frac{8}{9}$ |
| $\psi_{GHZ}$| $\frac{1}{\sqrt{3}}(1 + \eta_1 \eta_2 \eta_3)$ | 1             |
| $\psi_{CW}$ | $\frac{1}{\sqrt{3}}(\eta_1 \eta_2 + \eta_1 \eta_3 + \eta_2 \eta_3)$ | $\frac{8}{9}$ |

**Final Comments**

In this work we have presented some issues of the new formalism capable of effective description of entanglement. Here focus was only on pure state entanglement, to show the moments where possibility of using $\eta$-differential calculus gives new input, relevant functional determinants select appropriate sets of invariants, what is crucial to study of $n$-qubit entanglement with $n \geq 4$.

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