A REMARK ON THE HOMOLOGY OF FINITE COVERINGS OF A SURFACE

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ABSTRACT. Let $p: S \to S_g$ be a finite covering of an orientable closed surface of genus $g$. We prove that, for $g \geq 3$, the rational homology group $H_1(S; \mathbb{Q})$ is generated by cycles supported on simple closed curves $\gamma \subset S$ such that $p(\gamma)$ is contained in a 3-punctured, genus 0 subsurface of $S_g$. In particular, this answers positively, for $g \geq 3$ and rational coefficients, a question by Autumn Kent (cf. [4]).

1. INTRODUCTION

Let $p: S \to S_g$ be a finite, possibly branched, covering of an orientable closed surface of genus $g$. A question which has recently received a lot of attention (cf. [7], [5], [3], [6]) is whether $H_1(S; A) = H_{1cc}(S; A)$, for $A = \mathbb{Z}$ or $A = \mathbb{Q}$, where the simple closed curve homology $H_{1cc}(S; A)$ is the submodule of the first homology group generated by the cycles supported on the inverse images, via $p$, of simple closed curves on $S_g \smallsetminus B$, where $B$ is the branch locus of $p$.

For $A = \mathbb{Z}$, a negative answer has been given by Koberda and Santharoubane (cf. [3]). For $A = \mathbb{Q}$ and under the additional hypothesis that the branch locus $B$ of $p$ is nonempty, a negative answer has been given by Malestein and Putman (cf. [6]). However, as observed by Kent, there is a weaker question which is open and of interest (cf. [4] and Remark 1.5 in [6]), namely whether $H_1(S; A)$ is generated by cycles supported on simple closed curves $\gamma$ on $S$ such that $p(\gamma)$ does not fill $S_g$. In this paper, for $A = \mathbb{Q}$ and $g \geq 3$, we will prove that a stronger property actually holds. Let us give the following definition:

Definition 1.1. Let $p: S \to S_g$ be a finite covering of an orientable closed surface of genus $g$ with branch locus $B$. The tripod homology $H_1^{\text{trip}}(S; A)$ is the submodule of $H_1(S; A)$ generated by cycles supported on simple closed curves $\gamma$ on $S$ such that $p(\gamma)$ is contained in a 3-punctured genus 0 subsurface (a tripod) of $S_g \smallsetminus B$.

We will then prove:

Theorem A. For $g \geq 3$, we have $H_1^{\text{trip}}(S; \mathbb{Q}) = H_1(S; \mathbb{Q})$.

Note that, by replacing the 3-punctured genus 0 subsurface in Definition 1.1 with a 2-punctured genus 0 subsurface (an annulus), we just get back the definition of simple closed curve homology. Hence, from Theorem B in [6], it follows that Theorem A, from this perspective, is optimal. Note, however, that the results which we are going to prove in the next section are somewhat stronger than Theorem A.

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2. Generating the homology of finite coverings of a surface

In this section, we are going to prove a stronger result which, in particular, implies Theorem A. In order to formulate this result, we need to introduce the curve complex of the punctured surface $S_g \setminus B$, where $B$ is a finite set of points of $S_g$ such that $S_g \setminus B$ has negative Euler characteristic. The curve complex $C(S_g \setminus B)$ is the abstract simplicial complex whose simplices consist of sets $\sigma$ of distinct isotopy classes of essential simple closed curves on $S_g \setminus B$ which admit disjoint representatives.

The mapping class group $\text{Mod}(S_g \setminus B)$ of the surface $S_g \setminus B$ acts naturally on $C(S_g \setminus B)$. Let us denote by $o(\sigma)$ the $\text{Mod}(S_g \setminus B)$-orbit of a simplex $\sigma \in C(S_g \setminus B)$.

**Definition 2.1.** Let $p: S \to S_g$ be a finite covering of an orientable closed surface of genus $g$ with branch locus $B$ and let $\sigma \in C(S_g \setminus B)$.

(i) The $\sigma$-homology $H_1^\sigma(S; A)$ is the submodule of $H_1(S; A)$ generated by cycles supported on simple closed curves $\gamma$ on $S$ such that $p(\gamma)$ is contained in $S_g \setminus (B \cup \sigma)$.

(ii) The $o(\sigma)$-homology $H_1^{o(\sigma)}(S; A)$ is the submodule $\sum_{\eta \in o(\sigma)} H_1^\eta(S; A)$ of $H_1(S; A)$.

(iii) The $\sigma$-simple closed curve homology $H_1^{ssc}(S; A)$ is the submodule of $H_1(S; A)$ generated by cycles supported on the inverse images of simple closed curves $\gamma$ on $S_g \setminus B$ such that their isotopy class belongs to some $\eta \in o(\sigma)$.

**Theorem 2.2.** With the notations of Definition 2.1, for $g \geq 3$ and any $\sigma \in C(S_g \setminus B)$, we have $H_1(S; \mathbb{Q}) = H_1^{ssc}(S; \mathbb{Q}) + H_1^\sigma(S; \mathbb{Q})$.

Since, obviously, both $H_1^{ssc}(S; \mathbb{Q})$ and $H_1^\sigma(S; \mathbb{Q})$ are contained in $H_1^{o(\sigma)}(S; \mathbb{Q})$, an immediate consequence of Theorem 2.2 is:

**Corollary 2.3.** For $g \geq 3$, we have $H_1^{o(\sigma)}(S; \mathbb{Q}) = H_1(S; \mathbb{Q})$.

Let us observe that, if $\sigma$ is a facet of $C(S_g \setminus B)$, its complement $S_g \setminus (B \cup \eta)$, for $\eta \in o(\sigma)$, is a disjoint union of 3-punctured spheres and hence $H_1^{o(\sigma)}(S; \mathbb{Q}) \subseteq H_1^{\text{trip}}(S; \mathbb{Q})$.

Theorem A then immediately follows from Corollary 2.3.

**Proof of Theorem 2.2.** We can easily reduce to the case when the covering $p: S \to S_g$ is Galois with covering transformation group $G$. Let then $p_{\text{alg}}: C \to C_g$ be a Galois covering of smooth projective complex curves such that the given $p$ is the covering induced on the underlying differential manifolds. There is also a $G$-covering of semistable curves $p_{\text{alg}}^\sigma: C^\sigma \to C_g^\sigma$ whose topological model is the covering $p^\sigma: S^\sigma \to S_g^\sigma$ obtained from $p$ collapsing to points, in a $G$-equivariant way, simple closed curves on $S_g \setminus B$ and $S \setminus p^{-1}(B)$ whose isotopy classes are contained in $\sigma$ and $p^{-1}(\sigma)$, respectively. In this way, we also get a specialization map $q_\sigma: S_g \to C_g^\sigma$ and a $G$-equivariant specialization map $\tilde{q}_\sigma: S \to C^\sigma$ which lifts $q_\sigma$.

Let $\tilde{C}_g^\sigma$ and $\tilde{C}^\sigma$ be the normalizations, respectively, of $C_g^\sigma$ and $C^\sigma$. These are disjoint unions of smooth projective complex curves whose topological models are obtained from the connected components of $S_g \setminus \sigma$ and $S \setminus p^{-1}(\sigma)$, respectively, by glueing a disc to the each of their punctures. Note that, since $\tilde{C}^\sigma = \tilde{C}_g^\sigma \times_{C_g^\sigma} C^\sigma$, the curve $\tilde{C}^\sigma$ comes with a natural $G$-action such that the normalization morphism $\tilde{C}^\sigma \to C^\sigma$ is $G$-equivariant.
The dual graph $\Gamma(X)$ of a semistable curve $X$ is defined taking for vertices the set of irreducible components of $X$ and for edges its singular points. Two vertices are joined by an edge if and only if the corresponding irreducible components share the singularity parameterized by the edge. In particular, a singular point contained in only one irreducible component is parameterized by a loop based on the vertex corresponding to that irreducible component. Note that also the connected components of the normalization $\tilde{X}$ of $X$ are parameterized by the vertex set $\nu(\Gamma(X))$ of $\Gamma(X)$.

There is a continuous map $r: X \to \Gamma(X)$ from the analytic space underlying $X$ to the dual graph. This map collapses a cone around each singularity of $X$ to the corresponding edge of $\Gamma(X)$ and retracts their complement in each irreducible component of $X$ to the corresponding vertex of $\Gamma(X)$. In our case, we get a continuous map $r_\sigma: C_\sigma \to \Gamma(C_\sigma)$ and a $G$-equivariant continuous map $\tilde{r}_\sigma: C_\sigma \to \Gamma(C_\sigma)$.

Note that there is a natural bijective correspondence between the set of simple closed curves on $\Gamma(X)$ and for edges its singular points. Two vertices are joined by an edge if and only if the corresponding irreducible components share the singularity parameterized by the corresponding edge. Moreover, since irreducible components are self-dual, the image by $\nu(\Gamma(C_\sigma))$ of the corresponding irreducible component is parameterized by a loop based on the vertex corresponding to that irreducible component. Note that the image by $\nu(\Gamma(C_\sigma))$ of the corresponding irreducible component is parameterized by a loop based on the vertex corresponding to that irreducible component. Note that the image by $\nu(\Gamma(C_\sigma))$ of the corresponding irreducible component is parameterized by a loop based on the vertex corresponding to that irreducible component.

Proposition 2.4 (Brylinski). With the above notations:

(i) There is a natural (split) short exact sequence of $\mathbb{Q}G$-modules:

\begin{equation}
0 \to H_1(\Gamma(C_\sigma); \mathbb{Q}) \overset{j}{\to} H_1(S; \mathbb{Q}) \overset{\tilde{r}_\sigma}{\to} H_1(C_\sigma; \mathbb{Q}) \to 0,
\end{equation}

where the map $j$ is defined by the assignment $e^v \mapsto \gamma_e$, for all $e \in e(\Gamma(C_\sigma))$.

(ii) There is a natural (split) short exact sequence of $\mathbb{Q}G$-modules:

\begin{equation}
0 \to \bigoplus_{v \in \nu(\Gamma(C_\sigma))} H_1(C_\sigma; \mathbb{Q}) \to H_1(C_\sigma; \mathbb{Q}) \overset{\tilde{r}_\sigma}{\to} H_1(\Gamma(C_\sigma); \mathbb{Q}) \to 0.
\end{equation}

(iii) The $G$-invariant three steps filtration on $H_1(S; \mathbb{Q})$ determined by the two above short exact sequences is self-dual for the cup-product.

Fixed splittings (in the category of $\mathbb{Q}G$-modules) of the short exact sequences (1) and (2) then determine an isomorphism of $\mathbb{Q}G$-modules:

$$\Phi: H_1(\Gamma(C_\sigma); \mathbb{Q}) \oplus H_1(\Gamma(S); \mathbb{Q}) \oplus \bigoplus_{v \in \nu(\Gamma(C_\sigma))} H_1(C_\sigma; \mathbb{Q}) \sim \to H_1(S; \mathbb{Q}).$$

Note that the image by $\Phi$ of $H_1(\Gamma(C_\sigma); \mathbb{Q})$ in $H_1(S; \mathbb{Q})$ does not depend on the choice of the splittings. Moreover, since irreducible $\mathbb{Q}G$-modules are self-dual, the image by $\Phi$ of $H_1(\Gamma(C_\sigma); \mathbb{Q})$ in $H_1(S; \mathbb{Q})$ lies in the sum of the isotypic components of the irreducible $\mathbb{Q}G$-modules afforded by $H^1(\Gamma(C_\sigma); \mathbb{Q})$.

The images by $\Phi$ of both $H_1(\Gamma(C_\sigma); \mathbb{Q})$ and $\bigoplus_{v \in \nu(\Gamma(C_\sigma))} H_1(C_\sigma; \mathbb{Q})$ are contained in the $\sigma$-homology $H^\sigma_1(S; \mathbb{Q})$, independently from the chosen splittings. Of course, the image by $\Phi$ of $H_1(\Gamma(C_\sigma); \mathbb{Q})$ is also contained in the $\sigma$-simple closed curve homology $H^\text{occ}_1(S; \mathbb{Q})$. 

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Since, by definition, the subspace $H^1_{\text{sssc}}(S; \mathbb{Q})$ of $H_1(S; \mathbb{Q})$ is preserved by the group $G$ and by the centralizer $\text{Mod}(S)^G$ of $G$ in the mapping class group $\text{Mod}(S)$, Corollary 3.5 in [1] then implies that $H^1_{\text{sssc}}(S; \mathbb{Q})$ contains the isotypic components of all irreducible $\mathbb{Q}G$-modules afforded by $H^1(\Gamma(C^\sigma); \mathbb{Q})$. In particular, it contains the image by $\Phi$ of its dual $H^1(\Gamma(C^\sigma); \mathbb{Q})$. Theorem 2.2 then follows. □

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