Bounding the Bethe and the Degree-\(M\) Bethe Permanents

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Abstract

In [1], it was conjectured that the permanent of a \(P\)-lifting \(\theta^\text{IP}\) of a matrix \(\theta\) of degree \(M\) is less than or equal to the \(M\th\) power of the permanent \(\perm(\theta)\), i.e., \(\perm(\theta^\text{IP}) \leq \perm(\theta)^M\) and, consequently, that the degree-\(M\) Bethe permanent \(\perm_{M,B}(\theta)\) of a matrix \(\theta\) is less than or equal to the permanent \(\perm(\theta)\) of \(\theta\), i.e., \(\perm_{M,B}(\theta) \leq \perm(\theta)\). In this paper, we prove these related conjectures and show in addition a few properties of the permanent of block matrices that are lifts of a matrix. As a corollary, we obtain an alternative proof of the inequality \(\perm_{M,B}(\theta) \leq \perm(\theta)\) on the Bethe permanent of the base matrix \(\theta\) that uses only the combinatorial definition of the Bethe permanent (the first proof was given by Gurvits in [2]).

Index Terms

Bethe free energy, Bethe permanents, permanents, matrix lifts, protographs.

I. INTRODUCTION

A. Permanents and Bethe permanents

The concept of the Bethe permanent was introduced in [3], [4] to denote the approximation of a permanent of a non-negative matrix by solving a certain minimization problem of the Bethe free energy with the sum-product algorithm. In his paper [1], Vontobel uses the term Bethe permanent to denote this approximation and provides reasons why the approximation works well by showing that the Bethe free energy is a convex function and that the sum-product algorithm finds its minimum efficiently. Although its definition looks simpler than that of the determinant, the permanent does not have the properties of the determinant that enable efficient computation [5], [6]. Whereas the arithmetic complexity (number of real additions and multiplications) needed to compute the determinant is in \(O(n^3)\), Ryser's algorithm, one of the most efficient known algorithms for computing the permanent, requires \(\Theta(n \cdot 2^n)\) arithmetic operations [7]. This clearly improves upon the brute-force complexity \(O(n \cdot n!) = O(n^{3/2} \cdot (n/e)^n)\) for computing the permanent, but it is still exponential in the matrix size. In terms of complexity classes, the computation of the permanent is in the complexity class \#P [8], where \#P is the set of the counting problems associated with the decision problems in the class NP. Even the computation of the permanent of 0-1 matrices restricted to have only three ones per row is \#P-complete [9]. However, for circulant matrices, one can exactly calculate the permanent in polynomial time [10]–[13]. Later these results were strengthened in various ways in [14]–[17]. In contrast to the permanent, the Bethe permanent can be computed efficiently (i.e., in polynomial time).

In the recent paper [2], Gurvits shows that the permanent of a matrix is lower bounded by the Bethe permanent of that matrix, \(\perm_B(\theta) \leq \perm(\theta)\), and discusses conjectures on the constant \(C\) in the inequality \(\perm(\theta) \leq C \cdot \perm_B(\theta)\). Also, in [18], Ruozzi showed an analogous result for log-supermodular graphical models. Related to these results, Vontobel [1] formulates a conjecture that the permanent of an \(M\)-lift \(\theta^\text{IP}\) of a non-negative matrix \(\theta\) is less than or equal to the \(M\th\) power of the permanent \(\perm(\theta)\), i.e., \(\perm(\theta^\text{IP}) \leq \perm(\theta)^M\), and that the degree \(M\)-Bethe permanent \(\perm_{M,B}(\theta)\) of a matrix \(\theta\) is less than or equal to the permanent \(\perm(\theta)\) of \(\theta\), i.e., \(\perm_{M,B}(\theta) \leq \perm(\theta)\) and proves it for the particular case of \(\theta\) equal to the all-one matrix. A proof of his general conjecture would imply an alternative proof of the inequality \(\perm_B(\theta) \leq \perm(\theta)\) that uses only the combinatorial definition of the Bethe permanent.

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1A non-negative matrix contains only non-negative real entries.
2The formal definition of the Bethe and \(M\)-Bethe permanents is given in Definition.
In this paper, we prove this conjecture in its generality. In addition, we prove certain structural properties of the permanent of block matrices that are lifts of a matrix; these matrices are the matrices of interest when studying the degree-$M$ Bethe permanent.

B. Related work

The literature on permanents and on adjacent areas (of counting perfect matchings, counting 0-1 matrices with specified row and column sums, etc.) is vast. Apart from the previously mentioned papers, the most relevant papers to our work are the one by Chertkov & Yedidia [4] that studies the so-called fractional free energy functionals and resulting lower and upper bounds on the permanent of a non-negative matrix, the papers [19] (on counting perfect matchings in random graph covers), [20] (on counting matchings in graphs with the help of the sum-product algorithm), and [3], [21], [22] (on max-product/min-sum algorithms based approaches to the maximum weight perfect matching problem). Relevant is also the line of work on approximating the permanent of a non-negative matrix using Markov-chain-Monte-Carlo-based methods [23], polynomial-time randomized approximation schemes [24], and Bethe-approximation based methods or sum-product-algorithm (SPA) based method [3], [25].

C. Paper outline

The remainder of the paper is structured as follows. In Section I-D, we list basic notations and definitions, provide the necessary background, and formally define the Bethe permanent and degree-$M$ Bethe permanent. The following Section II contains the results of this paper and consists of three subsections. In Section II-A we show that the bound \( \text{perm}(\theta^{1P}) \leq \text{perm}(\theta)^M \) is tight, i.e., for every matrix \( \theta \), there exists a \( P \)-lifting of \( \theta \) for which the bound is satisfied with equality. Sections II-B–II-G present some useful results on the structure of the permanent of a \( P \)-lifting of degree \( M \), \( \theta^{1P} \). Section II-H contains the proof of the conjecture on the permanent of a matrix lifting which follows immediately from these first results, and Section III contains the bounding results on the Bethe permanent and the degree-$M$ Bethe permanent. We conclude the paper in Section IV and present a few extra examples of our techniques in the appendix.

D. Notations and definitions

Rows and columns of matrices and entries of vectors are indexed starting at 1. For an integer \( M \), we use the common notation \( [M] = \{1, \ldots, M\} \). We use the common notation \( h_{ij} \) or \( H_{ij} \) to denote the \( (i,j) \)th entry of a matrix \( H \). For a set \( \alpha \), |\( \alpha \)\| is the cardinality of \( \alpha \) (the number of elements in the set \( \alpha \)). The set of all \( M \times M \) permutation matrices is denoted by \( \mathcal{P}_M \), and the set of all permutations on the set \( [m] \) is denoted by \( S_m \).

**Definition 1.** Let \( \theta = (\theta_{ij}) \) be an \( m \times m \)-matrix over the integers. Its determinant and permanent, respectively, are defined to be

\[
\det(\theta) \triangleq \sum_{\sigma \in S_m} \text{sgn}(\sigma) \prod_{i \in [m]} \theta_{i\sigma(i)}; \quad \text{perm}(\theta) \triangleq \sum_{\sigma \in S_m} \prod_{i \in [m]} \theta_{i\sigma(i)},
\]

where \( \text{sgn}(\sigma) \) is the signature operator.

We call the products \( \prod_{i \in [m]} \theta_{i\sigma(i)} \), \( \sigma \in S_m \), permanent-products of \( \theta \).

The following combinatorial description of the Bethe permanent can be found in [1]. We use it here as a definition.

**Definition 2.** Let \( \theta \) be a non-negative (with non-negative real entries) \( m \times m \) matrix and \( M \) be a positive integer. Let \( \mathcal{P}_M^m \) be the set of all \( mM \times mM \) matrices whose blocks are permutation matrices, i.e.,

\[
\mathcal{P}_M^m \triangleq \{ P = (P_{ij}) \mid P_{ij} \in \mathcal{P}_M, \forall i \in [m] \}.
\]

For a matrix \( P \in \mathcal{P}_M^m \), the \( P \)-lifting of \( \theta \) is defined as the \( mM \times mM \) matrix of weighted permutation matrices:

\[
\theta^{1P} \triangleq \begin{bmatrix}
\theta_{11}P_{11} & \ldots & \theta_{1m}P_{1m} \\
\vdots & & \vdots \\
\theta_{m1}P_{m1} & \ldots & \theta_{mm}P_{mm}
\end{bmatrix},
\]

\(^{3}\text{Computing the permanent is related to counting perfect matchings.}\)

\(^{4}\text{See [1] for a more detailed account of these and other related papers.}\)
and the degree-$M$ Bethe permanent of $\theta$ is defined as

$$\text{perm}_{B,M}(\theta) \triangleq \left\langle \text{perm}(\theta^{\uparrow P}) \right\rangle^{1/M},$$

where the angular brackets $\left\langle \text{perm}(\theta^{\uparrow P}) \right\rangle$ represent the arithmetic average of $\text{perm}(\theta^{\uparrow P})$ over all $P \in \mathcal{P}_M^m$.

The Bethe permanent of $\theta$ is defined as

$$\text{perm}_B(\theta) \triangleq \lim_{M \to \infty} \text{perm}_{B,M}(\theta).$$

Since the permanent operator is invariant to the elementary operations of interchanging rows or columns, we can assume, when taking the permanent, without loss of generality, that matrices $P \in \mathcal{P}_M^m$ have $P_{1j} = P_{i1} = I_M$, for all $i, j \in [m]$, where $I_M$ is the identity matrix of size $M \times M$. We call such matrices reduced.

Definition 3. A matrix $P = (P_{ij}) \in \mathcal{P}_M^m$ is reduced if $P_{1j} = P_{i1} = I_M$, for all $i, j \in [m]$. The set of all reduced matrices $P$ is denoted by $\mathcal{P}_M^m$.

Remark 4. Note that a $P$-lifting of a matrix $\theta$ corresponds to an $M$-graph cover of the protograph (base graph) described by $\theta$. Therefore we can consider $\theta^{\uparrow P}$ to represent a protograph-based LDPC code and $\theta$ to be its protomatrix (also called its base matrix or its mother matrix) [26].

II. THE PERMANENT OF A MATRIX-LIFT

In [1], it was conjectured that for any non-negative square matrix $\theta$ and for any $P \in \mathcal{P}_M^m$,

$$\text{perm}(\theta^{\uparrow P}) \leq \text{perm}(\theta)^M.$$  

In this section we prove this conjecture and several related lemmas on the structure of the $\text{perm}(\theta^{\uparrow P})$ of the lift $\theta^{\uparrow P}$ of the matrix $\theta$, for any non-negative matrix $\theta$.

A. Tightness of the bound

We start by showing that there exists at least one lifting for which the bound is tight. The following example shows a lift of the matrix $\theta$ of degree 2 that has maximum permanent $\text{perm}(\theta)^2$.

Example 5. Let $\theta, P \in \mathcal{P}_2^3$ and $\theta^{\uparrow P}$ as follows:

$$\theta \triangleq \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad P \triangleq \begin{bmatrix} I_2 & I_2 & I_2 \\ I_2 & I_2 & I_2 \\ I_2 & I_2 & I_2 \end{bmatrix}, \quad \theta^{\uparrow P} \triangleq \begin{bmatrix} a & 0 & b & 0 & c & 0 \\ 0 & a & 0 & b & 0 & c \\ d & 0 & e & 0 & f & 0 \\ 0 & d & 0 & e & 0 & f \\ g & 0 & h & 0 & i & 0 \\ 0 & g & 0 & h & 0 & i \end{bmatrix}.$$

Equivalently, $\theta^{\uparrow P} = \theta \otimes I$, where $\otimes$ denotes the Kronecker product of matrices. After row and column permutations, which leave the permanent invariant, the matrix $\theta^{\uparrow P}$ can be rewritten as

$$I \otimes \theta = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

where the empty blocks contain only zero entries. This last matrix is a block-diagonal matrix with permanent equal to the product of the permanents of the matrices on the diagonal, therefore

$$\text{perm}(\theta^{\uparrow P}) = (\text{perm}(\theta))^2.$$
In fact, a stronger result was shown by Brualdi in [27].

**Theorem 6** (Theorem 3.1, [27]). Let $\theta$ a non-negative matrix of size $m \times m$ and $P$ a matrix of size $M \times M$. Then
\[
\text{perm}(\theta \otimes P) = \text{perm}(P \otimes \theta) \geq \text{perm}(\theta)^M \text{ perm}(P)^m
\]
with equality if and only if $P$ or $\theta$ has at most one non-zero permanent-product.

Since a permutation matrix has exactly one non-zero permanent-product, the bound holds with equality when $P$ is a permutation matrix. The following corollary follows from this theorem.

**Corollary 7.** Let $\theta = (\theta_{ij})$ be a non-negative matrix of size $m \times m$ and let $\tilde{P} = (P_{ij}) \in \mathcal{P}_M^m$ such that $P_{ij} = P$ for all $i, j \in [m]$, where $P$ is a permutation matrix of size $M$. Then
\[
\text{perm}(\theta^T \tilde{P}) = \text{perm}(\theta)^M.
\]

This corollary applies, in particular, for $P = I_M$, the identity matrix of size $M$.

**B. The exponent matrix of a permanent-product**

Let $\theta = (\theta_{ij})$ be a non-negative matrix of size $m \times m$ and $P = (P_{ij}) \in \mathcal{P}_M^m$. Let $\tau \in S_{mM}$ be a permutation on the set $[mM]$ and let
\[
A_\tau \triangleq \prod_{i \in [mM]} (\theta^T P)^{\tau(i)}
\]
be the permanent-product of $\theta^T P$ pertaining to permutation $\tau$.

**Definition 8.** We say that $A_\tau$ is trivially zero if there exists $i \in [mM]$ such that $(\theta^T P)^{\tau(i)} = 0$ and $(\theta^T P)^{\tau(i)} \neq \theta_{jl}$, for all $j, l \in [m]$, and we say that $A_\tau$ is non-trivially zero if $(\theta^T P)^{\tau(i)} = \theta_{jl}$ for some $j, l \in [m]$ and $\theta_{jl} = 0$. □

In the rest of the paper, all permanent-products considered will be assumed not to be trivially zero. Since for each $i \in [mM]$, there exist $j, l \in [m]$ such that $i \in \{(j-1)M + 1, \ldots, jM\}$ and $\tau(i) \in \{(l-1)M + 1, \ldots, lM\}$, $(\theta^T P)^{\tau(i)}$ is an entry in the weighted permutation matrix $\theta_{jl} P_{jl}$ of $\theta^T P$. By the assumption that $A_\tau$ is not trivially zero, we have $(\theta^T P)^{\tau(i)} = \theta_{jl}$. Let
\[
\alpha^\tau_{jl} \triangleq \{i \in \{(j-1)M + 1, \ldots, jM\} \mid \tau(i) \in \{(l-1)M + 1, \ldots, lM\}\},
\]
\[
r^\tau_{jl} \triangleq |\alpha^\tau_{jl}|.
\]

Then, $(\theta^T P)^{\tau(i)} = \theta_{jl}$, for all $i \in \alpha^\tau_{jl}$ and for all $j, l \in [m]$, therefore
\[
\prod_{i=(j-1)M+1}^{jM} (\theta^T P)^{\tau(i)} = \theta_{jl}^{\tau_{j1}} \theta_{jl}^{\tau_{j2}} \cdots \theta_{jl}^{\tau_{jm}} = \prod_{l=1}^{m} \theta_{jl}^{\tau_{jl}}, \quad \forall j \in [m].
\]

Since each row and each column of $\theta^T P$ must contribute to the product exactly once, the matrix $\alpha^\tau \triangleq (\alpha^\tau_{jl})_{j,l}$ with its $(j, l)$ entry the set $\alpha^\tau_{jl}$ satisfies
\[
\alpha^\tau_{jl} \bigcap \alpha^\tau_{jl'} = \emptyset, \forall j, l, l' \in [m], l \neq l', \quad \bigcup_{l=1}^{m} \alpha^\tau_{jl} = \{(j-1)M + 1, \ldots, jM\},
\]
from which we obtain that $\sum_{l=1}^{m} r^\tau_{jl} = M$, for all $j \in [m]$, and $\sum_{j=1}^{m} r^\tau_{jl} = M$, for all $l \in [m]$.

Therefore, the matrix $R^\tau \triangleq (r^\tau_{jl})_{j,l \in [m]}$ corresponding to $A_\tau$ has the property that all its entries are positive, $r^\tau_{jl} \geq 0$, and the sums of all entries on each row and each column equal $M$.

We state this fact in the following lemma.
Lemma 9. Let \( \theta = (\theta_{ij}) \) be a non-negative matrix of size \( m \times m \) and let \( P = (P_{ij}) \in \mathcal{P}_M^m \). Let \( \tau \in \mathcal{S}_{mM} \) be a permutation on the set \( [mM] \) and let \( A_\tau \triangleq \prod_{i \in [mM]} (\theta^{i\tau})_{\tau(i)} \) be a permanent-product of \( \theta^{i\tau} \). Then, there exists a unique non-negative integer matrix \( R_\tau = (r_{ij}^\tau) \) of size \( m \times m \) with the properties

\[
\sum_{l=1}^m r_{jl}^\tau = M, \quad \forall j \in [m], \tag{4}
\]
\[
\sum_{j=1}^m r_{jl}^\tau = M, \quad \forall l \in [m], \tag{5}
\]

such that

\[
A_\tau = \prod_{j=1}^m \prod_{l=1}^m (\theta_{jl})^{r_{jl}^\tau}. \tag{6}
\]

We call the matrix \( R_\tau \) the exponent matrix of \( A_\tau \).

C. Decomposing the permanent-products of lifts of matrices

In this subsection, we present a lemma and an algorithm that allows us to rewrite the permanent-products of a \( P \)-lifting of \( \theta \) into a form useful for proving the conjecture, namely, as a product of \( M \) permanent-products in \( \theta \) that are not necessarily distinct.

Lemma 10. Let \( \theta = (\theta_{ij}) \) be a non-negative matrix of size \( m \times m \) and let \( P = (P_{ij}) \in \mathcal{P}_M^m \). Let \( \tau \in \mathcal{S}_{mM} \) be a permutation on the set \( [mM] \) and let \( A_\tau \triangleq \prod_{i \in [mM]} (\theta^{i\tau})_{\tau(i)} \) be a permanent-product of \( \theta^{i\tau} \). Then, there exists, not necessarily uniquely, a set of integers \( 0 \leq t_{\tau\sigma} \leq M \) such that \( \sum_{\sigma \in \mathcal{S}_m} t_{\tau\sigma} = M \) and

\[
A_\tau = \prod_{\sigma \in \mathcal{S}_m} (\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)})^{t_{\tau\sigma}}. \tag{7}
\]

Proof: Let \( R_\tau \) be the exponent matrix of \( A_\tau \). For each \( \sigma \in \mathcal{S}_m \), let \( P_\sigma \in \mathcal{P}_m \) be the \( m \times m \) permutation matrix corresponding to \( \sigma \) and let \( t_{\tau\sigma} \triangleq \min \{ r_{1\sigma(1)}^\tau, r_{2\sigma(2)}^\tau, \ldots, r_{m\sigma(m)}^\tau \} \geq 0 \). Then \( R_\tau - t_{\tau\sigma}P_\sigma \) is a positive matrix with the sums of all entries on each row and each column equal to \( M - t_{\tau\sigma} \) and with all its entries equal to the ones on the same positions of \( R_\tau \) except for the entries corresponding to the permutation \( \sigma \), which decreased by the same amount \( t_{\tau\sigma} \). We can index the set \( \{ \sigma \in \mathcal{S}_m \} \triangleq \{ \sigma_k \in \mathcal{S}_m, k \in [m!] \} \) and compute sequentially

\[
R_{\tau,1} \triangleq R_\tau,
\]
\[
R_{\tau,k+1} \triangleq R_{\tau,k} - t_{\tau\sigma_k}P_{\sigma_k} = R_\tau - \sum_{s=1}^k t_{\tau\sigma_s}P_{\sigma_s}, \quad k \geq 2,
\]

where the sums of all entries on each row and each column of \( R_{\tau,k+1} \) are all equal to \( M - \sum_{s=1}^k t_{\tau\sigma_s} \). Note that after one round corresponding to a permutation \( \sigma \), the entries are either the same if \( t_{\tau\sigma} = 0 \) or, if \( t_{\tau\sigma} \neq 0 \), at least one non-zero entry in the matrix \( R_{\tau,k} \) (corresponding to \( t_{\tau\sigma} \)) gets changed to a zero entry in the matrix \( R_{\tau,k+1} \) and all the other entry values on the positions corresponding to the permutation \( \sigma_k \) decrease by the same amount \( t_{\tau\sigma_k} \). The algorithm runs until all non-zero entries get changed into zero entries, see Example 12 for an illustration of this process. Consequently, the matrix \( R_\tau - \sum_{\sigma \in \mathcal{S}_m} t_{\tau\sigma}P_\sigma = 0 \).

This yields \( R = \sum_{\sigma \in \mathcal{S}_m} t_{\tau\sigma}P_\sigma \), leading to

\[
A_\tau = \prod_{i \in [mM]} (\theta^{i\tau})_{\tau(i)} = \prod_{j=1}^m \prod_{l=1}^m (\theta_{jl})^{r_{jl}^\tau} = \prod_{\sigma \in \mathcal{S}_m} (\theta_{1\sigma(1)}\theta_{2\sigma(2)}\cdots\theta_{m\sigma(m)})^{t_{\tau\sigma}}
\]

and \( \sum_{\sigma \in \mathcal{S}_m} t_{\tau\sigma} = M \).

Note that this described decomposition always works, i.e., the steps presented above can be always performed until all the entries are changed into zero entries. This is due to the Birkhoff-von Neumann theorem on the decomposition of doubly stochastic matrices into a convex combination of permutation matrices that insures that the doubly stochastic matrix \( \frac{1}{M} R_\tau \) can be...
be decomposed indeed as a convex sum of permutation matrices\footnote{A matrix is doubly stochastic if it has positive entries and both its rows and columns sum to 1.} The decomposition algorithm is basically the one presented above.

\[ \begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 4 \\ 4 & 2 & 1 \end{bmatrix}, \quad \theta^{R_\tau} = \begin{bmatrix} a^3 & b^2 & c^2 \\ e^3 & f^4 & g^4 \\ h^2 & i^1 \end{bmatrix}. \] (8)

Following the algorithm we obtain

\[ R_\tau = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 4 \\ 4 & 2 & 1 \end{bmatrix} \rightarrow (aei) \rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 4 \\ 4 & 2 & 0 \end{bmatrix} \rightarrow (bfg)^2 \rightarrow \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} \rightarrow (ceg)^2 \rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow (afh)^2. \]

So \( a^3b^2c^2f^4g^4h^2i = (aei)(bfg)^2(ceg)^2(afh)^2. \)

\[ \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \quad \theta^{R_\kappa} = \begin{bmatrix} a^3 & b^2 & c^2 \\ d^2 & e^3 & f^2 \\ g^2 & h^2 & i^3 \end{bmatrix}. \] (9)

Following the algorithm we obtain

\[ R_\kappa = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \rightarrow (aei) \rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \rightarrow (bfg)^2 \rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix} \rightarrow (cdh)^2. \]

So \( a^3b^2c^3f^4g^4h^2i = (aei)^3(bfg)^2(ceg)^2(afh)^2. \)

However, we can also group the entries in the following way:

\[ R_\kappa = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \rightarrow (afh)^2 \rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 2 & 0 & 3 \end{bmatrix} \rightarrow (aei) \rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow (cdh)^2. \]

So \( a^3b^2c^3f^4g^4h^2i = (afh)^2(aei)(ceg)^2(bdi)^2. \) Similarly, we can also group them in the following way:

\[ R_\kappa = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \rightarrow (afh) \rightarrow \begin{bmatrix} 2 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{bmatrix} \rightarrow (aei)^2 \rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \rightarrow (ceg) \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow (bdh) \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow (bfg) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow (cdh). \]

So \( a^3b^2c^3f^4g^4h^2i = (afh)(aei)^2(ceg)(bdh)(bfg)(cdh). \) It can be easily seen that these three decompositions are the only possible ones. \[ \square \]

\[ \text{Remark 11. In the rest of the paper, we will refer to the algorithm in the proof of Lemma\textsuperscript{10} as the decomposition algorithm.} \]

\[ \square \]

\textbf{Example 12.} Let \( M = 7 \) and \( \theta \) as in Example\textsuperscript{5} and suppose that \( A_\tau \triangleq \bar{a}^3b^2c^2e^3f^4g^4h^2i^4 \) is a product in \( \text{perm}(\theta^{1P}) \). Then this product corresponds to the following exponent matrix \( R_\tau \) and the corresponding matrix \( \theta^{R_\tau} \triangleq (\theta^{R_\tau})_{ij} \)

\[ R_\tau = \begin{bmatrix} 3 & 2 & 2 \\ 0 & 3 & 4 \\ 4 & 2 & 1 \end{bmatrix}, \quad \theta^{R_\tau} = \begin{bmatrix} a^3 & b^2 & c^2 \\ e^3 & f^4 & g^4 \\ h^2 & i^1 \end{bmatrix}. \]

It can be easily checked that the decomposition in Example\textsuperscript{12} is unique. However, this is not always the case. Next we show an example where there are 3 possible decompositions.

\textbf{Example 13.} Let \( M = 7 \) and \( \theta \) as in Example\textsuperscript{5}

Suppose that \( A_\kappa \triangleq \bar{a}^3b^2c^2e^3f^4g^4h^2i^3 \) is a permanent-product in \( \text{perm}(\theta^{1P}) \) that corresponds to the following exponent matrix \( R_\kappa \) and the corresponding matrix \( \theta^{R_\kappa} \triangleq (\theta^{R_\kappa})_{ij} \)

\[ R_\kappa = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \quad \theta^{R_\kappa} = \begin{bmatrix} a^3 & b^2 & c^2 \\ d^3 & e^2 & f^2 \\ g^2 & h^2 & i^3 \end{bmatrix}. \]

\[ \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \rightarrow (aei) \rightarrow \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \rightarrow (bfg)^2 \rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \rightarrow (cdh)^2. \]

So \( a^3b^2c^3f^4g^4h^2i = (aei)^3(bfg)^2(ceg)^2(afh)^2. \)
Therefore, the decomposition is not always unique, i.e., there are exponent matrices for which the decomposition is unique and there are matrices for which the decomposition is not unique. We will refer to a decomposition of an exponent matrix obtained by the decomposition algorithm as a standard decomposition of the exponent matrix. Similarly, we will refer to a decomposition of a product \( \prod_{j=1}^{m} \prod_{l=1}^{n} (\theta_{jl})^{r_{jl}} \) into some product \( \prod_{\sigma \in S_m} (\theta_{1\sigma(1)} \theta_{2\sigma(2)} \cdots \theta_{m\sigma(m)})^{r_{\sigma}} \) as a standard decomposition of the permanent-product, as it corresponds to a standard decomposition of the exponent matrix.

D. Same-index decomposition of a permanent-product

The algorithm presented in the proof of Lemma 10 provides a way to decompose the product \( \prod_{j=1}^{m} \prod_{l=1}^{n} (\theta_{jl})^{r_{jl}} \) into a new product \( \prod_{\sigma \in S_m} (\theta_{1\sigma(1)} \theta_{2\sigma(2)} \cdots \theta_{m\sigma(m)})^{r_{\sigma}} \) but does not tell us exactly how to combine the entries \( (\theta_{1\sigma(1)} \theta_{2\sigma(2)} \cdots \theta_{m\sigma(m)})^{r_{\sigma}} \) for all \( \sigma \in S_m \). The answer is yes, as we explain in the next example of a concrete \( \mathbf{P} \)-lifting of \( \theta \) from Example 12 with \( \mathbf{P} \) reduced.

Before presenting it, let us introduce a new matrix \( \overline{\alpha}_\tau \triangleq (\overline{\alpha}_{jl})_{ij} \) obtained from \( \alpha_\tau \) by substituting each index \((j-1)M+k\) in an entry set by \( k, k \in [M] \). Then the properties (3) of the matrix \( \overline{\alpha}_\tau \) translate into the following properties of the matrix \( \overline{\alpha}_\tau \):

\[
\overline{\alpha}_{jl} \cap \overline{\alpha}_{jl'} = \emptyset, \quad \forall j, l, l' \in [m], l \neq l', \quad \bigcup_{l=1}^{m} \overline{\alpha}_{jl} = [M].
\]

(10)

The following example uses the matrix \( \overline{\alpha}_\tau \) and provides a unique method of combining the indices \( \overline{\alpha}_{jl} \) to obtain the desired decomposition of the product \( A_\tau \). This method follows the steps of the algorithm that we described in Example 12 for modifying the matrix \( R_\tau \).

Example 14. Let \( \theta \) be the \( 3 \times 3 \) matrix in Example 5 \( \mathbf{P} = (P_{ij}) \in \mathbb{P}_{3}, \theta^{\mathbf{P}} \) and \( A_\tau = a^2bdf^2h^2i \) as follows:

\[
\theta = \begin{pmatrix}
\alpha & 0 & 0 \\
0 & a & 0 \\
0 & 0 & a
\end{pmatrix},
\]

\[
Q = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]

\[
\theta^{\mathbf{P}} = \begin{pmatrix}
\alpha_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{pmatrix},
\]

\[
\mathbf{P} = (P_{ij}) \triangleq \begin{pmatrix}
I_3 & I_3 & I_3 \\
I_3 & Q & Q^2 \\
I_3 & I_3 & Q^2
\end{pmatrix},
\]

\[
\mathbf{A}_\tau = \begin{pmatrix}
a_1 & 0 & 0 \\
0 & b_1 & 0 \\
0 & 0 & c_1
\end{pmatrix},
\]

\[
\mathbf{Q}^2 \triangleq \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]

\[
\mathbf{A}_\tau \triangleq \begin{pmatrix}
a_1 & 0 & 0 \\
0 & b_1 & 0 \\
0 & 0 & c_1
\end{pmatrix},
\]

\[
\mathbf{P}_3 \triangleq \begin{pmatrix}
I_3 & I_3 & I_3 \\
I_3 & Q & Q^2 \\
I_3 & I_3 & Q^2
\end{pmatrix},
\]

\[
\mathbf{A}_\tau \triangleq \begin{pmatrix}
\begin{bmatrix}
1 & 3 \\
2 & \emptyset \\
0 & 8
\end{bmatrix} \\
\begin{bmatrix}
1 & 3 \\
2 & \emptyset \\
0 & 8
\end{bmatrix}
\end{pmatrix},
\]

\[
R_\tau = \begin{pmatrix}
2 & 1 & 0 \\
1 & 0 & 2 \\
0 & 2 & 1
\end{pmatrix},
\]

where \( I_3 \) denotes the identity matrix of size 3 and the entries boxed in (11) (left matrix) correspond to the permutation \( \tau \) that gives the product \( A_\tau = a^2bdf^2h^2i \). In (11) (right matrix), we wrote the matrix \( \theta^{\mathbf{P}} \) with its entries indexed by their row, e.g., \( a_1 = a_2 = a_3 = a \) and \( a_i \) is on the \( i \)th row of the first block \( P_{11} \).

The matrices \( \alpha_\tau, \overline{\alpha}_\tau \) and \( R_\tau \) are

\[
\alpha_\tau = \begin{pmatrix}
\{ 1, 3 \} \\
\{ 2 \} \\
\emptyset
\end{pmatrix},
\]

\[
\overline{\alpha}_\tau = \begin{pmatrix}
\{ 1 \} & \{ 2 \} \\
\{ 0 \} & \{ 4, 6 \} \\
\emptyset & \{ 7, 9 \} & \{ 8 \}
\end{pmatrix},
\]

\[
R_\tau = \begin{pmatrix}
2 & 1 & 0 \\
1 & 0 & 2 \\
0 & 2 & 1
\end{pmatrix},
\]

where, for simplicity in writing, we omit the set parentheses in \( \overline{\alpha}_\tau \). Note that \( \overline{\alpha}_\tau \) corresponds to the row indices of the boxed entries in (11) (left matrix) that are illustrated through indexed entries in (11) (right matrix). In the matrix \( \overline{\alpha}_\tau \), we use circles, boxes and shaded boxes to show how to group the entries of (11) (left matrix) that appear in \( A_\tau \), as follows. We group together
entries in $\theta^{1\mathbf{P}}$ in rows indexed by the circled entries in $\overline{\mathbf{r}}$, and we group together entries in $\theta^{1\mathbf{P}}$ in rows indexed by the boxed entries in $\overline{\mathbf{r}}$, thus obtaining a unique rewriting of the product $A_{\mathbf{r}}$ as $A_{\mathbf{r}} = (afh)^2(bdi)$, in correspondence to the rewriting steps of matrix $R_{\mathbf{r}}$. In terms of the indexed entries of $\theta^{1\mathbf{P}}$, the above grouping corresponds to $A_{\mathbf{r}} = (a1f1h1)(a3f3h3)(b2d2i2)$ which is exemplified through circles, boxes and shaded boxes in the version of matrix $\overline{\mathbf{r}}$. Therefore, for each of the first column, or, equivalently, $d2$, none of the entries $a2$ or $g2$ on that column can be part of the permanent-product anymore and, therefore, the second row of matrix $P_{11}$ (where $a2$ is positioned) and the second row of the matrix $P_{31}$ (where $g2$ is positioned) must contribute each with exactly one entry other than the entries $a2$ and $g2$ that are not allowed. These are the boxed entries $b2$ and $i2$. We group these entries with $d2$ uniquely and continue the same way to group each of the $a$ entries with the entries $f$ and $h$ that are on the two rows associated with the other two entries on the columns of the entries $a$ to obtain $(a1f1h1)$ and $(a3f3h3)$.

In terms of the entries of the matrix $\overline{\mathbf{r}}$, this corresponds to the grouping we showed in Example 14 because the matrix $\mathbf{P}$ is reduced, so the first matrices $P_{11}$ in each row and $P_{1l}$ in each column are equal to the identity matrix, for all $l \in [m]$. Therefore, for each of the first $M$ columns, the nonzero entries on the $j$th column are all positioned on the $j$th row of the matrices $P_{1j}$, for all $l \in [m]$. Of course, this is not valid for a column that is not among the first $M$. Indeed, the boxed $i$ of $\theta^{1\mathbf{P}}$ in (11) is on row 2 of matrix $P_{33}$ and has the nonzero entries on rows 3 of matrix $P_{13}$ and 2 of matrix $P_{23}$. However, it still holds that the rows corresponding to these non-zero entries must contribute to the product with one entry exactly that cannot be on the column of $i$. In this case, $d2$ on position $(2,2)$ in $P_{21}$ and $a3$ on position $(3,3)$ of $P_{11}$ are these entries. We can group these together as well. In fact any such grouping of three where two of them are on the rows corresponding to the non-chosen entries of the column of the third of the group is a good association; the permanent-product $A_{\mathbf{r}}$ is then a product of some of these three-products with the property that the entries in the products are taken only once and they cover all the entries in the permanent-product $A_{\mathbf{r}}$ (i.e., they form a partition). Such a partition is surely given by the three-sets of the boxed entries in the first $M$ columns, because each of these sets must be disjoint and they are exactly $M$, the number of boxed entries from the first $M$ columns, so the union of all entries in these products is equal to all entries in the product $A_{\mathbf{r}}$. In fact, any three-sets associated to the boxed entries in a set $(j-1)M+1,\ldots,jM$ of columns corresponds to a partition of the entries in $A_{\mathbf{r}}$. For simplicity, however, we choose the partition corresponding to the first $M$ columns, or, equivalently, to the matrix $\overline{\mathbf{r}}$. We call this decomposition same-index decomposition.

Therefore, the same-index decomposition of a permanent-product in $\theta^{1\mathbf{P}}$ is the writing of the permanent-product as a product of $M$ sub-products of $m$ entries in $\theta$ each indexed by the same row index, e.g., $(a1f1h1)(b2d2i2)(a3f3h3)$.

E. The relation between the exponent matrix decomposition and the permanent-product same-index decomposition

In this section we will revisit the setting of Example 13 in order to understand how the grouping described in Section II-D determines the type of the decomposition into $M$ products of permanent-products of $\theta$ in the decomposition algorithm of the exponent matrix described in Section II-C.

Example 15. Let $M = 7$ and let $\theta$ and $A_{\mathbf{r}} \triangleq a^3b^2c^2e^2f^3g^2h^2i^3$ as in Example 13. We saw that there were three possible decompositions of $A_{\mathbf{r}}$ in permanent-products as follows

$$A_{\mathbf{r}} = (afh)^2(ace)(ced)^2(bdi)^2 = (afh)(ace)^2(ced)(bdi)(bfg)(cdh) = (aei)^3(bfg)^2(cd)h.$$

How are these three possible decompositions of the exponent matrix visible in the same-index decomposition of a given permanent-product described in Section II-D? We can assume for simplicity (and without loss of generality) that the exponent $r_{11} = 3$ corresponds to the row indices $\{1,2,3\}$ of the entries in $P_{11}$ that appear in $A_{\mathbf{r}}$. We have three possible scenarios for how these row indices can be combined with the indices of the entries in $\begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix}$ with associated exponent matrix $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.
(modulo some permutations of indices) such that the overall exponent matrix is \( R_\kappa \):

\[
\overline{\sigma}_\kappa = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix} ;
\begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
3 & 1 & 2 \\
\end{bmatrix} ;
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix} .
\]

These correspond to the following exponent matrices:

\[
R_{\kappa,a^3e^3i^3} = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix} ;
R_{\kappa,a^3e^2fhi^2} = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix} ;
R_{\kappa,a^3e^2f^2hi^2} = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix} .
\]

The remaining indices are uniquely determined in the way shown in Example 13 so we omit them from the matrices above.

Equivalently, we have the following possible same-index decompositions:

\[
a_1a_2a_3 \ e_1e_2e_3 \ i_1i_2i_3 = (a_1e_1i_1)(a_2e_2i_2)(a_3e_3i_3)
\]

\[
a_1a_2a_3 \ e_1e_2f_i \ i_1i_2h_3 = (a_1e_1i_1)(a_2e_2i_2)(a_3f_ih_3)
\]

\[
a_1a_2a_3 \ e_1f_2f_3 \ i_1h_2h_3 = (a_1e_1i_1)(a_2f_2h_2)(a_3f_3h_3).
\]

Therefore, when fixing the indices of \( P_{11} \) to \( \{1, 2, 3\} \), there are 3 non-equivalent ways in which the exponent matrix \( R_\kappa \) in (9) can occur, where by non-equivalent we mean that the matrices \( \overline{\sigma}_\kappa \) do not map into each other after applying some permutation on the set of row indices \( \{M\} \).

However, in the case of \( A_\tau = a^3b^2c^2d^0 e^3 f^4 g^1 h^2 i^1 \) in Example 12 with the exponent \( R_\tau \) given in (8) we can only have

\[
\overline{\sigma}_\kappa = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{bmatrix}
\]

(or equivalent matrices) due to the entry of 1 in the position \( (3, 3) \) of \( R_\tau \) and 0 in the position \( (2, 1) \). Indeed, as explained in Section 1.1.12 if the entry from \( P_{3,3} \) is, for example, on row 1, the entries on the row 8 (the first of the second row of blocks) must contribute to the permanent-product with an entry from the matrices \( P_{2,1} \) or \( P_{2,2} \). Since \( r_{21} = 0 \), it implies that the entry on the first row of \( P_{2,2} \) is also in the permanent-product. Similarly, the entry on the first row of \( P_{1,1} \) will also appear in the product, and thus we have the unique scenario (modulo permutations) presented above and the product

\[
a_1a_2a_3 \ e_1f_2f_3 \ h_2h_3i_1 = (a_1e_1i_1)(a_2f_2h_2)(a_3f_3h_3).
\]

\[\Box\]

So far, in all our examples the same-index decomposition of a permanent-product is equal to its standard decomposition. In the following section, we see that this is not always the case.

\(F. \) Decompositions that contain illegal sub-products

Note that in Example 15 one of the following decompositions in \( \overline{\sigma}_\kappa \) associated with the entry 3 in the position \( (1, 1) \) of \( R_\kappa \) could also occur (and their equivalent version):

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
1 & 2 & 3 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
3 & 2 & 1 \\
\end{bmatrix} ,
\begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
3 & 2 & 1 \\
\end{bmatrix},
\]

yielding the following permanent-products of \( g^1P \):

\[
a_1a_2a_3 \ e_1e_2f_3 \ h_1i_2i_3 = (a_1e_1i_1)(a_2e_2i_2)(a_3f_3i_3),
\]

\[
a_1a_2a_3 \ e_1e_2f_3 \ i_1i_2h_3 = (a_1e_1i_1)(a_2e_2i_2)(a_3e_3i_3),
\]

\[
a_1a_2a_3 \ e_1f_2f_3 \ h_1i_2i_3 = (a_1e_1i_1)(a_2f_2h_2)(a_3f_3i_3),
\]

\[
a_1a_2a_3 \ e_1f_2f_3 \ h_2h_3i_1 = (a_1e_1i_1)(a_2f_2h_2)(a_3e_3i_3).}
\]
In this case, not all of the products of 3 entries of the same index correspond to permanent-products in the matrix $\theta$; we marked with $\dagger$ the ones that do not, for example, $(a_1e_1h_1)\dagger$ corresponds to $aei$ in $\theta$ which is not a permanent-product. We call such a product *illegal*. This illegal three-product needs to be grouped with another illegal three-product in the same grouping, in this case $(a_3f_3i_3)\dagger$, and rearranged as $(a_1e_1i_3)(a_3f_3h_1)$ to obtain a standard decomposition, i.e., a product of permanent-products of $\theta$. We call these sub-products that correspond to a permanent-product in $\theta$ *legal*.

**Example 16.** Let $\theta$ the $3 \times 3$ matrix from Example 5 let $M = 6$, and let

$$
\begin{bmatrix}
\theta_{11} & 0 & 0 & 0 & 0 & b_1 \\
0 & a_2 & 0 & 0 & 0 & 0 \\
0 & 0 & a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & a_4 & 0 & 0 \\
0 & 0 & 0 & 0 & a_5 & 0 \\
0 & 0 & 0 & 0 & 0 & a_6 \\
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_4 \\
\tau_5 \\
\tau_6 \\
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 \\
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5 \\
c_6 \\
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6 \\
\end{bmatrix}
$$

with the entries of the product $A_{\tau} \triangleq a_1 a_0 b_0 c_0 d_0 e_0 f_0 g_0 h_0 i_0 j_0 k_0 l_0 m_0 n_0 o_0 p_0 q_0 r_0 s_0 t_0 u_0 v_0 w_0 x_0 y_0 z_0$ highlighted in the matrix $\theta^{1P}$. All entries in a block $P_{ij}$ are equal to the entry $\theta_{ij}$, for example, $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a$, etc. We use the index $l$ for an entry in $P_{ij}$ to denote the row position of that entry in $P_{ij}$, for example, $a_4$ is on the 4th row of $P_{11}$. The indices for the entries are helpful when describing $\tau$. We use boxes, shades, circles and bold faced with circles, boxes, and shades, respectively, to draw the entries of $A_{\tau}$ so that the decomposition according to the matrix $\pi_{\tau}$, i.e., the same-index decomposition, is visible. This means that all entries in the permanent-product $A_{\tau}$ of the same index will have the same shape/color. The following matrices can be computed.

$$
R_{\tau} = \begin{bmatrix}
3 & 1 & 2 \\
1 & 2 & 3 \\
2 & 3 & 1 \\
\end{bmatrix} ; \quad \pi_{\tau} = \begin{bmatrix}
1 & 4 & 5 & 6 \\
2 & 3 \\
3 & 6 & 1 & 4 & 5 \\
\end{bmatrix}
$$

The same-index decomposition corresponding to the grouping of the matrix $\pi_{\tau}$ is

$$
A_{\tau} = (a_1 f_1 h_1)(d_2 c_2 h_2)(g_3 c_3 e_3)(a_4 f_4 i_4)^{\dagger}(a_5 f_5 h_5)(g_6 b_6 e_6)^{\dagger}.
$$

Here we have an example in which not all products of 3 entries of the same index in a permanent-product of $\theta^{1P}$ are legal, i.e., they correspond to permanent-products in the matrix $\theta$. As before, we marked with $\dagger$ the illegal ones, for example, $(a_4 f_4 i_4)^{\dagger}$ corresponds to $afi$, which is not a permanent-product in $\theta$. This three-product needs to be grouped with another illegal three-product in the same decomposition and rearranged as follows.

$$(a_4 f_4 i_4)^{\dagger}(g_6 b_6 e_6)^{\dagger} = (a_4 e_6 i_4)(g_6 b_6 f_4) = (aei)(gbf).$$

This results in the following standard decomposition of the permanent-products $A_{\tau}$ that is not equal to its same-index decomposition, i.e., into a product that contains only legal terms, although some of them contain sub-products with indices that are not all the same:

$$
A_{\tau} = (a_1 f_1 h_1)(d_2 c_2 h_2)(g_3 c_3 e_3)(a_4 e_6 i_4)(a_5 f_5 h_5)(g_6 b_6 f_4).
$$
G. Mapping illegal products into legal products

In this section we will show that we can always assume that all permanent-products in $\theta^{\text{IP}}$ are products of $\theta$-permanent-products by showing that any permanent-product of $\theta^{\text{IP}}$ containing some illegal sub-products can be mapped uniquely into some product of $M$ same-index permanent-products of $\theta$. In addition, this product has the same exponent matrix as the original permanent-product but is not a permanent-product of $\theta^{\text{IP}}$. This way, we establish a one-to-one correspondence between permanent-products of $\theta^{\text{IP}}$ and products of $M$ permanent-products in $\theta$.

We revisit Example 16 to exemplify this correspondence.

Example 17. Let $\theta$, $\theta^{\text{IP}}$ and $A_t$ be like in Example 16. Recall that the same-index decomposition of $A_t$ in Example 16 was $A_t = (a_1f_1h_1)(d_2e_2h_2)(g_3c_3e_3)\{(a_4f_4i_4)\}^1(g_6b_6c_6)^1$, which contained two illegal sub-products $\{(a_4f_4i_4)\}$ and $(g_6b_6c_6)^1$ that were combined to obtain $(a_4f_4i_4)(g_6b_6c_6)$. Note that this combination is unique; no other combination resulting in legal sub-products, i.e., in permanent-products in $\theta$, is possible between the two products. Each $\theta$-permanent-products $(a_4f_4i_4)$ and $(g_6b_6c_6)$ contains the combined indices 4 and 6.

Let $A_t' \triangleq (a_1f_1h_1)(d_2e_2h_2)(g_3c_3e_3)(a_5f_5h_5)(a_3e_4i_4)(g_6b_6c_6)$ be the (unique) product of $M$ same-index $\theta$-permanent-products starting with $a_1, d_2, g_3, a_4, a_5$ and $g_6$ with the same exponent matrix $R_t$. Map $A_t \mapsto A_t'$. We observe that $A_t'$ cannot be a permanent-product if $A_t$ is. Indeed, the two products are equal in all but 2 positions, therefore, if the two were both permanent-products, then $\theta^{\text{IP}}$ would need to have a $2 \times 2$ submatrix $\begin{bmatrix} c_4 & f_4 \\ c_6 & f_6 \end{bmatrix}$, which is not allowed as no two $e$ entries (and no two $f$ entries) are on the same row or column (they are entries in $eF_{22}$, respectively, $fP_{23}$, where $P_{22}$ and $P_{23}$ are permutation matrices). Hence the correspondence $A_t \mapsto A_t'$ is an instance of the desired correspondence between the permanent-products in $\theta^{\text{IP}}$ that have illegal sub-products in their same-index decompositions, and products of $M$ permanent-products in $\theta$ that are not permanent-products in $\theta^{\text{IP}}$. In addition, since a permanent-product in $\theta^{\text{IP}}$ that does not contain any illegal sub-products has its same-index decomposition equal to its standard decomposition, it can be mapped trivially into itself. This way, we obtain a map from the set of all permanent-products in $\theta^{\text{IP}}$ into the set of all $M$ products of permanent-products in $\theta$. □

This correspondence illustrated in the previous example can be generalized to all permanent-products of $\theta^{\text{IP}}$ with same-index decompositions that contain some illegal sub-products in the following way.

- Let $\theta$ be an $m \times m$ non-negative matrix and $\theta^{\text{IP}}$ be a reduced matrix of degree $M$.
- Let $\tau$ be a permutation on $[mM]$ and $A_t$ be a permanent-product in $\theta^{\text{IP}}$ that is not trivially zero. Let $R_t$ be its exponent matrix.
- Write $A_t$ as the same-index decomposition; $A_t$ can or not contain illegal same-index sub-products, i.e., products of $m$ entries in $\theta$ of the same index that are not permanent-products in $\theta$.
- List all distinct products of $M$ same-index permanent-products in $\theta$ corresponding to all standard decompositions of $R_t$ that start with the entries in $A_t$, that are in the first $M$ columns of $\theta^{\text{IP}}$. Call them $A_{t,1}, \ldots, A_{t,t}$ and reorder, if needed, the entries in the sub-products of $A_t$ and $A_{t,1}, \ldots, A_{t,t}$ such that the entries from the first $M$ columns are always first in the subproduct, followed by the entries ordered by the row index in $\theta$ increasingly from 1 to $m$ and such that the indices of the $\theta$-permanent-products are ordered increasingly from 1 to $M$.

This procedure, henceforth called standard mapping, is formalized in the following lemma. Several examples can be found in Appendix A.

**Lemma 18 (Standard mapping).** Initially, set $\mathcal{L} := \{ A_{t,1}', \ldots, A_{t,t}' \}$.

**Start** Let $0 \leq s \leq M$ and $1 \leq t < m$ be such that

- $A_t$ and each $A_{t,j}' \in \mathcal{L}$ have their first $s$ $\theta$-permanent-products equal and
- $A_t$ and each $A_{t,j}' \in \mathcal{L}$ have their $(s+1)$th $\theta$-permanent-products either equal in the first $t$ entries or have all of the first $t$ entries distinct except for the first entry and
- $A_t$ and $A_{t,i}' \in \mathcal{L}$ have their $(s+1)$th $\theta$-permanent-product equal in the $(t+1)$th entry, while there exists $A_{t,j}' \neq A_{t,v}'$, such that $A_t$ and $A_{t,j}'$ have the $(s+1)$th $\theta$-permanent-product distinct in the $(t+1)$th entry.

Let $\{A_{t,j,1}', \ldots, A_{t,j,k}' \} \subset \{ A_{t,1}', \ldots, A_{t,t}' \}, 1 \leq k < l$, such that $A_t$ and each $A_{t,j,n}'$, $n \in [k]$, have their $(s+1)$th $\theta$-permanent-product equal in the $(t+1)$th entry.

Map $A_t \mapsto A_{t,j}$ if $k = 1$, otherwise update $\mathcal{L} := \mathcal{L}_k$ and repeat the steps from **Start**.
Then, this map is a well-defined one-to-one (injective) map from the set of all permanent-products of $\theta^1 \mathbf{P}$ of a certain exponent matrix to the set of all products of $M \theta$-permanent-products of the same exponent matrix. This gives a one-to-one map from the set of all permanent-products in $\theta^1 \mathbf{P}$ to the set of all products of $M \theta$-permanent-products.

Proof: The fact that the map is well-defined is easy to see since there can only be one matrix $A'_{\tau,i}$ satisfying the conditions, while the existence of this matrix is ensured by the decomposition algorithm presented in Section II-C. Indeed, the exponent matrix decomposing algorithm guarantees the existence of the list of products of $\theta$-permanent-product, which has its cardinality at least one, and at the same time, guarantees the existence of a standard decomposition of the permanent-product into legal sub-products not necessarily of the same index obtained from its same-index decomposition; this can be mapped into a product of same-index $\theta$-permanent-products, thus guaranteeing the existence. The fact that no two permanent-products can be mapped into the same $A'_{\tau,i}$ is also ensured by the conditions of the mapping; if two different permanent products $A_\tau$ and $A_{\nu}$ map into the same $A'_{\tau,i}$, then they must have a first entry in which they differ; this entry must be necessarily after the first $s$ entries. This means, however, that there must exist an $A'_{\tau,j}$ that shares with $A_{\nu}$ that entry but not with $A'_{\tau,i}$. Therefore, $A_{\nu}$ cannot get mapped into the same $A'_{\tau,i}$ as $A_{\tau}$, proving that the function is one-to-one. In addition, if $A_{\tau}$ contains illegal same-index sub-products, then $A'_{\tau,i}$ such that $A_{\tau} \mapsto A'_{\tau,i}$ cannot be a permanent-product in $\theta^1 \mathbf{P}$. To see this, erase from $\theta^1 \mathbf{P}$ all rows and columns corresponding to the entries that the two share. Suppose that there are $k$ entries in which the two products are different, say, $x_1, x_2, \ldots, x_k$ in $A_{\tau}$ and $x'_1, x'_2, \ldots, x'_k$ in $A'_{\tau,j}$. Because the two products $A_{\tau}, A'_{\tau,i}$ have the same exponent matrix, so do the two products $x_1 x_2 \ldots x_k$ and $x'_1 x'_2 \ldots x'_k$. Therefore, in each block in which there exists some $x_i, i \in [k]$, there must exist also a $j \in [k]$ such that $x'_j$ is also in that block. We can reorder $x'_1, x'_2, \ldots, x'_k$ so that each $x'_j$ is in the same block as $x'_i$. Note that there can be more entries in one block, but to each entry $x_i$ corresponds a unique entry $x'_i$ in the same block. Since there is only one column in the $k \times k$ submatrix crossing the term $x_i$ and since $x'_j \notin \{x_1, \ldots, x_k\}$, we obtain that $x_i$ and $x'_j$ must be on the same column which contradicts the fact that the block is a weighted permutation matrix.

Therefore, if $A_{\tau}$ contains illegal same-index sub-products, then it is mapped through the above mapping into a product $A'_{\tau,i}$ that is not a permanent-product in $\theta^1 \mathbf{P}$. This also implies that an all-legal permanent-product $A_{\tau}$ and a permanent-product in containing some illegal same-index sub-products $A_{\kappa}$ do not map into the same product of $M \theta$-permanent-products, which in this case would be $A_{\tau}$. Indeed, if $A_{\tau}$ does not contain any illegal sub-products, i.e., it is a product of $M \theta$-permanent-products, then $A_{\tau} = A'_{\tau,i}$, for some $i$, and the mapping corresponds to $A_{\tau} \mapsto A_{\tau}$ as expected.

Such a mapping can be defined for each exponent matrix, which proves the existence of the overall one-to-one map from the set of all permanent-products in $\theta^1 \mathbf{P}$ to the set of all products of $M \theta$-permanent-products.

H. Upper bounding the permanent of a lifting of a matrix

The mapping in Section II-C allows us to compute, for a fixed exponent matrix $R = (r_{ij})$, the coefficient of $\prod_{j=1}^{m} \prod_{i=1}^{m} (\theta_{ji})^{r_{ij}}$ in $\text{perm}(\theta^1 \mathbf{P})$, or, equivalently, the maximum possible number of permutations $\tau \in \mathcal{S}_{m,M}$ such that $A_{\tau} = \prod_{j=1}^{m} \prod_{i=1}^{m} (\theta_{ji})^{r_{ij}}$ is a permanent-product with exponent matrix $R$ that is not trivially-zero, and, using this, to prove the upper bound $\text{perm}(\theta^1 \mathbf{P}) \leq \text{perm}(\theta^M)$.

The following corollary is an immediate consequence of the one-to-one mapping.

Corollary 19. Let $R = (r_{ij})$ be an exponent matrix of some permanent-product in $\text{perm}(\theta^1 \mathbf{P})$. For each $\tau \in \mathcal{S}_{m,M}$ with $A_{\tau} = \prod_{j=1}^{m} \prod_{i=1}^{m} (\theta_{ji})^{r_{ij}}$, let $A'_{\tau,1}, \ldots, A'_{\tau,l}$ be the possible products of $M \theta$-permanent-products associated with $R$. For each $j \in [l]$, denote by $N_{\tau,j}$ the number of products of $M \theta$-permanent-products that are equivalent to $A'_{\tau,j}$, i.e., they can be obtained from $A'_{\tau,j}$ by applying an $M$-permutation on the indices. Then, the coefficient of $\prod_{j=1}^{m} \prod_{i=1}^{m} (\theta_{ji})^{r_{ij}}$ in $\text{perm}(\theta^1 \mathbf{P})$ is upper bounded by $\sum_{j=1}^{l} N_{\tau,j}$.

The following lemma determines the number $N_{\tau,j}$ for all $j \in [l]$.

Lemma 20. For each $j \in [l]$ and $\sigma \in \mathcal{S}_{m}$, let $0 \leq t_{j,\sigma} \leq M$ such that $\sum_{\sigma \in \mathcal{S}_{m}} t_{j,\sigma} = M$ and $A'_{\tau,j} = \prod_{\sigma \in \mathcal{S}_{m}} (\theta_{1\sigma(1)} \theta_{2\sigma(2)} \cdots \theta_{m\sigma(m)})^{t_{j,\sigma}}$. Then $N_{\tau,j} = \binom{M}{t_j}$ where $\binom{M}{t_j}$ is the multinomial coefficient associated with the vector $t_j \triangleq (t_{j,\omega})_{\omega \in \mathcal{S}_{m}}$. 


Proof: The entries that lie in the first \( M \) columns of \( \theta^{1 P} \) uniquely determine the way the products of \( \theta \)-permanent-products
\((\theta_{\sigma(1)} \theta_{\sigma(2)} \cdots \theta_{\sigma(m)})^{t_j, \sigma}\) are formed. We can choose these in \( \binom{M}{t_j} \) ways.

The main result of the paper now follows immediately.

**Theorem 21.** Let \( \theta = (\theta_{ij}) \) be a non-negative matrix of size \( m \times m \) and let \( P = (P_{ij}) \in \mathcal{P}_M^m \). Then

\[
\text{perm}(\theta^{1 P}) \leq \text{perm}(\theta)^M.
\]

**Proof:** The upper bound follows immediately from Lemma 20 and the expansion of \( \text{perm}(\theta)^M \) as

\[
\text{perm}(\theta)^M = \left( \sum_{\sigma \in S_m} \theta_{\sigma(1)} \theta_{\sigma(2)} \cdots \theta_{\sigma(m)} \right)^M = \sum_{|t_j| = M} \binom{M}{t_j} \prod_{\sigma \in S_m} (\theta_{\sigma(1)} \theta_{\sigma(2)} \cdots \theta_{\sigma(m)})^{t_j, \sigma}.
\]

In the next example, we illustrate the upper bound for the exponent matrix \( R = R_\tau \) in Example 14.

**Example 22.** Let \( M = 3 \), and let \( \theta, \theta^{1 P}, R = R_\tau \) as in Example 14 and \( a^2 b d f^2 h^2 i \) be the product corresponding to \( R \).

How many permanent-products could exist in \( \theta^{1 P} \) that lead to the product \( a^2 b d f^2 h^2 i \)? Note that \( R \) has the unique standard decomposition \( a^2 b d f^2 h^2 i = (a f h)^2 (d b i) \). Therefore, by Cor. 19 we expect no more than \( \binom{3 + 3}{3} = \frac{3!}{1!1!1!} = 3 \) permutations to result in this product. Indeed, there are exactly three combinations of same-index permanent-products in \( \theta \) mapping into \( (a f h)^2 (d b i) \), namely \( (a_1 f_1 h_1) (d_2 b_2 i_2) (a_3 h_3 f_3) \), \( (a_1 f_1 h_1) (a_2 h_2 f_2) (d_3 b_3 i_3) \) and \( (d_1 b_1 i_1) (a_2 h_2 f_2) (a_3 h_3 f_3) \), giving 3 possible products of \( M = 3 \) permanent-products in \( \theta \) with the standard decomposition \( (a f h)^2 (d b i) \), i.e., a maximum of \( N_1 \triangleq \binom{3 + 3}{3} = 3 \) possible products.

Note that, in fact, all three above products of permanent products in \( \theta \) are valid permanent-products in \( \theta^{1 P} \), resulting in the coefficient of \( a^2 b d f^2 h^2 i \) being equal to the upper bound 3. One of these products was \( A_\tau \triangleq a_1 d_2 a_3 h_1 b_2 h_3 f_3 i_2 = (a_1 f_1 h_1) (d_2 b_2 i_2) (a_3 h_3 f_3) = a^2 b d f^2 h^2 i \) of Example 14.

Let us now compute the maximum coefficient of \( abcd ef gh i \) in \( \theta^{1 P} \). Observe that its exponent matrix has two possible standard decompositions: \( (a f h) (c e g) (b d i) \) and \( (a e i) (b f g) (c s h) \). Therefore, the maximum possible coefficient of \( abcd ef gh i \) in \( \text{perm}(\theta^{1 P}) \) is equal to the sum of two equal multinomial coefficients associated with the vector \( (1, 1, 1) \), i.e., \( \binom{3}{1,1,1} + \binom{3}{1,1,1} = 2 \frac{3!}{1!1!1!} = 12 \). The actual coefficient of \( abcd ef gh i \) in \( \text{perm}(\theta)^M \) is 0, which satisfies the upper bound trivially.

In Example 30 in Appendix B we used Maple to compute the actual permanent of \( \theta^{1 P} \). We also expanded \( (\text{perm}(\theta))^3 \) to illustrate the upper bound.

\[\square\]

1. **Bounding the degree \( M \)-Bethe and Bethe permanents**

The bound \( \text{perm}(\theta^{1 P}) \leq \text{perm}(\theta)^M \) gives the following inequality conjectured in [1] by applying the bound to the permanent of each of the \( M \)-lifts of \( \theta \), and hence also to their average, and then taking the \( M \)th root.

**Theorem 23.** Let \( \theta = (\theta_{ij}) \) be a non-negative matrix of size \( m \times m \) and \( M \geq 1 \) an integer. Then

\[
\text{perm}_{B,M}(\theta) \leq \text{perm}(\theta).
\]

Taking the limit we obtain the following theorem.

**Theorem 24.** Let \( \theta = (\theta_{ij}) \) be a non-negative matrix of size \( m \times m \) and \( M \geq 1 \) an integer. Then

\[
\text{perm}_B(\theta) \leq \text{perm}_{B,M}(\theta) \leq \text{perm}(\theta).
\]

Note that the inequality \( \text{perm}_B(\theta) \leq \text{perm}(\theta) \) was proved by Gurvits in [2] using a very different method. Our proof is a simple alternative that uses only the combinatorial definition of the Bethe permanent.

**III. Conclusions**

In this paper we proved two related conjectures posed by Vontobel in [1] on the permanent of an \( M \)-lift \( \theta^{1 P} \) of a matrix \( \theta \) and on the degree \( M \) Bethe permanent \( \text{perm}_{B,M}(\theta) \) of \( \theta \), namely, we show that \( \text{perm}(\theta^{1 P}) \leq \text{perm}(\theta)^M \) and, consequently, that \( \text{perm}(\theta) \) of \( \theta \), i.e., \( \text{perm}_{M,B}(\theta) \leq \text{perm}(\theta) \). As a corollary, our proof of these conjectures provides an alternative proof
of the inequality \( \text{perm}_B(\theta) \leq \text{perm}(\theta) \) on the Bethe permanent of the base matrix \( \theta \), one that uses only the combinatorial Definition 2 of the Bethe permanent from [1]. The first proof was given by Gurvits in [2].

The consequences of the results in this paper are more than just purely theoretical. Apart from showing that it is possible to give a purely combinatorial proof that \( \text{perm}_B(\theta) \leq \text{perm}(\theta) \) on the Bethe permanent of the base matrix \( \theta \) (the earlier proof [2] used different techniques), they provide new insight into the structure of the permanent of a \( \mathbf{P} \)-lifting of a matrix, which can be exploited algorithmically to decrease the computational complexity of the permanent of the \( \mathbf{P} \)-liftings. Such an algorithm can search for products of groups of entries according to the decompositions presented in this paper to check if they form valid permanent-products.

In addition, the structure of the permanent-products of \( \mathbf{P} \)-liftings of a matrix may have some implications on the constant \( C \) in the inequality \( \text{perm}(\theta) \leq C \cdot \text{perm}_B(\theta) \) in the conjectures stated by Gurvits in [2]. Lastly, since a \( \mathbf{P} \)-lifting of a matrix \( \theta \) corresponds to an \( M \)-graph cover of the protograph (base graph) described by \( \theta \), which, in turn, correspond to LDPC codes, these results may help explain the performance of these codes through the techniques presented in [28] and extended and refined in [29]–[34] for upper bounding the minimum Hamming distance and the minimum pseudo-weight [35] of a binary linear code that is described by an \( m \times n \) parity-check matrix \( \mathbf{H} \). This is done based on explicitly constructing codewords and pseudo-codewords with components equal to determinants or permanents of some \( m \times m \) submatrices of \( \mathbf{H} \) over the binary field or the ring of integers.

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APPENDIX A

EXAMPLES OF STANDARD MAPPING

In this appendix, we illustrate the standard mapping from Lemma 18 by a few diverse examples.

Example 25. Let \( m = 3 \), \( M = 2 \), and \( \theta = (\theta_{ij}) \) as in Example 5 and let \( \theta^{\mathbf{P}} \) be defined as follows:

\[
\theta^{\mathbf{P}} \triangleq \begin{bmatrix}
  a_1 & 0 & b_1 & 0 & c_1 & 0 \\
  0 & a_2 & 0 & b_2 & 0 & c_2 \\
  d_1 & 0 & e_1 & 0 & f_1 & 0 \\
  0 & d_2 & 0 & e_2 & 0 & f_2 \\
  g_1 & 0 & h_1 & 0 & i_1 & 0 \\
  0 & g_2 & h_2 & 0 & i_2 & 0
\end{bmatrix}
\]

\[=
\begin{bmatrix}
  (a_1) & 0 & b_1 & 0 & c_1 & 0 \\
  0 & a_2 & 0 & b_2 & 0 & c_2 \\
  d_1 & 0 & e_1 & 0 & f_1 & 0 \\
  0 & d_2 & 0 & e_2 & 0 & f_2 \\
  g_1 & 0 & h_1 & 0 & i_1 & 0 \\
  0 & g_2 & h_2 & 0 & i_2 & 0
\end{bmatrix},
\]

where in \([12]\) (left matrix) we highlighted the permanent product \( A_\tau \triangleq (a_1 e_1 h_1)^\dagger (a_2 f_2 i_2)^\dagger = (aei)(afh) \) and in \([12]\) (right matrix) we highlighted the permanent product \( A_\kappa \triangleq (a_1 f_1 i_1)^\dagger (a_2 e_2 h_2)^\dagger = (aei)(afh) \). These are both products of two illegal sub-products and have the same exponent matrix. In order to map these products, we need to list the possible same-index decompositions for the two products:

\[A_{\tau,1} \triangleq (a_1 e_1 i_1) (a_2 f_2 h_2),\]

\[A_{\tau,2} \triangleq (a_1 f_1 h_1) (a_2 e_2 i_2)\]

and

\[A_{\kappa,1} \triangleq (a_1 e_1 i_1) (a_2 f_2 h_2),\]

\[A_{\kappa,2} \triangleq (a_1 f_1 h_1) (a_2 e_2 i_2).\]

Note that \( A_{\tau,i} = A_{\kappa,i} \), for all \( i = 1, 2 \). Note also that the products are indexed from 1 to \( M = 2 \) and that the entries in the sub-products are listed from top row to bottom row.

Then \( A_\tau \mapsto (a_1 e_1 i_1) (a_2 f_2 h_2) \) and \( A_\kappa \mapsto (a_1 f_1 h_1) (a_2 e_2 i_2) \) because the first product \( (a_1 e_1 h_1) \) of \( A_\tau \) has its first two entries equal to those of \( (a_1 e_1 i_1) \) and the first product \( (a_1 f_1 i_1) \) of \( A_\kappa \) has its first two entries equal to those of \( (a_1 f_1 h_1) \).
Example 26. Let \( m = 5, \ M = 3, \) and let

\[
\begin{bmatrix}
  a_1 & 0 & 0 & d_1 & 0 & 0 \\
  0 & a_2 & 0 & b_2 & 0 & 0 \\
  0 & 0 & a_3 & 0 & b_3 & 0 \\
  f_1 & 0 & 0 & 0 & y_1 & 0 \\
  0 & f_2 & 0 & g_2 & 0 & 0 \\
  0 & 0 & f_3 & 0 & g_3 & 0
\end{bmatrix}
\quad \begin{bmatrix}
  c_1 & 0 & 0 \\
  0 & c_2 & 0 \\
  0 & 0 & c_3 \\
  i_1 & 0 & 0 \\
  0 & i_2 & 0 \\
  0 & 0 & i_3
\end{bmatrix}
\quad \begin{bmatrix}
  d_1 & 0 & 0 \\
  0 & d_2 & 0 \\
  0 & 0 & d_3 \\
  j_1 & 0 & 0 \\
  0 & j_2 & 0 \\
  0 & 0 & j_3
\end{bmatrix}
\]

with the entries of the product \( A_r \triangleq a_2 a_3 d_1 h_1 i_3 j_2 l_1 l_2 l_3 p_1 r_2 t_3 w_2 x_3 y_1 = a^2 d h i j l^3 p r t w x y \) highlighted in the matrix \( \theta^{1P} \). All entries in a \((i,j)\) block are equal to the entry \( \theta_{ij} \), for all \( l \in [3] \), e.g., \( a_1 = a_2 = a_3 = a \), where the index denotes the row position of that entry in \( P_{ij} \), for example, \( a_2 \) is on the 2nd row of \( a P_{11} \). The following matrices can be computed:

\[
R_r = \begin{bmatrix}
  2 & 0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 1 & 1 \\
  0 & 3 & 0 & 0 & 0 \\
  1 & 0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 1 & 1
\end{bmatrix}
\quad \bar{r}_r = \begin{bmatrix}
  2 & 3 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}
\]

which gives the same-index decomposition \( A_r = (p_1 d_1 h_1 l_1 y_1) (a_{2j2l2r2w2})^t (a_{3i3l3t3x3})^t \).

The following is the list of possible same-index products of permanent-products of \( \theta \) that have the exponent matrix \( R_r \):

\[
A_{r,1} \triangleq (p_1 d_1 h_1 l_1 y_1) (a_{2j2l2r2w2}) (a_{3i3l3t3w3}),
\]

\[
A_{r,2} \triangleq (p_1 d_1 h_1 l_1 y_1) (a_{2j2l2t2w2}) (a_{3j3l3t3x3}),
\]

\[
A_{r,3} \triangleq (p_1 d_1 j_1 l_1 w_1) (a_{2j2l2r2w2}) (a_{3j3l3t3x3}),
\]

\[
A_{r,4} \triangleq (p_1 d_1 j_1 l_1 w_1) (a_{2j2l2t2x2}) (a_{3i3l3t3y3}).
\]

The first term \((d_1 h_1 l_1 p_1 y_1)\) of \( A_r \) is equal to the first term of \( A_{r,1} \) and \( A_{r,2} \) and not equal to the first term of \( A_{r,3} \) and \( A_{r,4} \). The second term \((a_{2j2l2r2w2})^t \) has the first three terms equal to the first three terms of both \((a_{2j2l2r2w2}) \) of \( A_{r,1} \) and \( (a_{2j2l2r2w2}) \) of \( A_{r,2} \) and the forth term is equal to that of \((a_{2j2l2w2}) \) of \( A_{r,1} \) and not equal to that of \((a_{2j2l2t2w2}) \) of \( A_{r,2} \). Therefore, we map \( A_r \mapsto A_{r,1} \). Note that \( A_{r,1} \) cannot be a permanent-product in \( \theta^{1P} \) (otherwise we would have a 2 \times 2 sub-matrix of \( \theta^{1P} \) with \( x_2 \) and \( x_3 \) on the same column).
Example 27. Let us now take $A_\gamma \triangleq a_1a_3d_2h_1i_3j_2l_1l_2l_3p_2r_1t_3w_3x_2y_1 = a^2dhijt^3prtwxy$ illustrated through the highlighted entries in the matrix

$$
\theta^{IP} \triangleq \begin{bmatrix}
  a_1 & 0 & 0 & d_1 & 0 & 0 & e_1 & 0 & 0 \\
  0 & a_2 & 0 & b_2 & 0 & c_2 & 0 & d_2 & 0 \\
  0 & 0 & a_3 & b_3 & c_3 & 0 & e_2 & 0 & 0 \\
  f_1 & 0 & 0 & 0 & g_1 & 0 & h_1 & 0 & j_1 \\
  0 & f_2 & 0 & g_2 & 0 & h_2 & 0 & i_2 & 0 \\
  0 & 0 & f_3 & g_3 & 0 & h_3 & i_3 & 0 & j_3 \\
  k_1 & 0 & 0 & 0 & l_1 & 0 & m_1 & 0 & n_1 \\
  0 & k_2 & 0 & l_2 & 0 & m_2 & 0 & n_2 & 0 \\
  0 & 0 & k_3 & l_3 & m_3 & 0 & n_3 & 0 & o_3 \\
  p_1 & 0 & 0 & 0 & q_1 & r_1 & 0 & s_1 & 0 \\
  0 & p_2 & 0 & q_2 & 0 & r_2 & 0 & s_2 & t_2 \\
  0 & 0 & p_3 & q_3 & 0 & r_3 & 0 & s_3 & t_3 \\
  u_1 & 0 & 0 & 0 & v_1 & 0 & w_1 & 0 & x_1 \\
  0 & u_2 & 0 & v_2 & 0 & w_2 & 0 & x_2 & y_1 \\
  0 & 0 & u_3 & v_3 & 0 & w_3 & 0 & x_3 & y_3 \\
\end{bmatrix},
$$

$A_\gamma$ has the same exponent matrix $R_\tau$ in (12) and the index-matrix

$$
\pi_\gamma = \begin{bmatrix}
  1 & 3 & 0 & 0 & \Theta & 0 & 3 & 2 \\
  0 & \Theta & 0 & 1 & 2 & 3 & 0 & \Theta & 0 \\
  2 & 0 & \Theta & 0 & 1 & 3 & 2 & \Theta \\
  0 & \Theta & 0 & 3 & 2 & 1 & 0 & \Theta \\
\end{bmatrix},
$$

which gives the same-index decomposition $A_\gamma = (a_1h_1l_1r_1y_1)^\dagger(p_2d_2j_2l_2x_2)(a_3i_3l_3t_3w_3)$. The set associated with $A_\gamma$ is

$A_{\gamma,1} \triangleq (a_1h_1l_1l_1x_1)(p_2d_2j_2l_2w_2)(a_3i_3l_3l_3y_3),$

$A_{\gamma,2} \triangleq (a_1i_1l_1r_1y_1)(p_2d_2j_2l_2w_2)(a_3h_1l_3l_3t_3),$

$A_{\gamma,3} \triangleq (a_1i_1l_1t_1w_1)(p_2d_2h_2l_2y_2)(a_3j_3l_3l_3x_3),$

$A_{\gamma,4} \triangleq (a_1j_1l_1t_1x_1)(p_2d_2h_2l_2y_2)(a_3i_3l_3l_3w_3).$

We see that $A_\gamma \mapsto A_{\gamma,1}$ since the products $(a_1h_1l_1r_1y_1)$ and $(a_1h_1l_1l_1x_1)$ have the first 3 entries in common. Note that $A_{\gamma,1}$ is not a permanent-product.

\[\square\]

Example 28. Let $m = 5$, $M = 3$, $\theta$ and $\theta^{IP}$ be as in Example 26. Let $A_\kappa \triangleq p_1a_2u_3g_2l_3d_2e_1g_1h_3n_2o_1r_2t_3x_2y_1 = (p_1e_1g_1o_1y_1)\dagger(a_2g_2n_2r_2x_2)\dagger(u_3d_3h_3l_3r_3)\dagger = ade^2bhnopr^2axy$ as highlighted below, together with its exponent matrix and its index matrix.
Example 29. Let

\[ \nu = (p_1 d_1 h_1 l_1 y_1)(a_2 g_2 o_2 r_2 x_2)(u_3 e_3 g_3 n_3 r_3), \]

\[ A_{\nu,1} \triangleq (p_1 e_1 g_1 o_1 y_1)(a_2 g_2 o_2 r_2 x_2)(u_3 e_3 g_3 n_3 r_3). \]

We see that \( A_{\nu,1} \) is the product highlighted in \((15)\).

The set associated with \( A_{\nu} \) is

\[ A_{\nu,1} \triangleq (p_1 d_1 h_1 l_1 y_1)(a_2 g_2 o_2 r_2 x_2)(u_3 e_3 g_3 n_3 r_3). \]

\[ A_{\nu,2} \triangleq (p_1 e_1 h_1 l_1 x_1)(a_2 g_2 n_2 r_2 y_2)(u_3 d_3 g_3 o_3 r_3). \]

We see that \( A_{\nu,1} \) is the product highlighted in \((15)\). Lastly, let

\[ A_{\nu} = p_1 a_2 g_2 o_2 r_2 x_2 y_2 = (p_1 d_1 g_1 o_1 x_1)(a_2 g_2 n_2 r_2 y_2)(u_3 e_3 g_3 h_3 l_3 r_3), \]

which is the product highlighted in \((15)\).
The exponent matrix of $A_\nu$ is $R_\nu = R_\kappa$, its index matrix is

$$\mathbf{r}_\nu = \begin{bmatrix}
2 & 0 & 0 & 1 & 3 \\
\emptyset & 1 & 2 & 3 & 0 & 0 \\
0 & 1 & 3 & 0 & 0 & 1 & 2 \\
3 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\
\end{bmatrix},$$

and the associated set of products $A_\nu$ is

$$A_{\nu,1} \triangleq (p_1 d_1 h_1 l_1 y_1)(a_2 g_2 o_2 r_2 x_2)(u_3 e_3 g_3 n_3 r_3) = A_{\kappa,1},$$

$$A_{\nu,1} \triangleq (p_1 e_1 h_1 l_1 x_1)(a_2 g_2 n_2 r_2 y_2)(u_3 d_3 g_3 o_3 r_3) = A_{\kappa,2}.$$

We see that $A_\nu \rightarrow A_{\kappa,1}$ since the products $(p_1 d_1 g_1 o_1 x_1)$ and $(p_1 d_1 h_1 l_1 y_1)$ have the first 2 entries in common while $(p_1 d_1 g_1 o_1 x_1)$ and $(p_1 e_1 h_1 l_1 x_1)$ do not. Note that $A_{\kappa,1}$ is not a permanent-product.

\textbf{APPENDIX B}

\textbf{EXAMPLE FOR PERMANENT BOUNDS}

Here we give an example comparing $\text{perm}(\theta^{1P})$ with $(\text{perm}(\theta))^3$.

\textbf{Example 30.} Computing $\text{perm}(\theta^{1P})$ we get

$$\text{perm}(\theta^{1P}) = a^3 e^3 i^3 + a^3 f^3 h^3 + 3 a^2 b d f^2 h^2 i + 3 a^2 b f^3 g h^2 + 3 a^2 c d e^2 h^2 i + 3 a^2 c e^2 f g h i + 3 a b^2 d^2 f h i^2 + 6 a b^3 d f^2 g h i + 3 a b^2 f^3 g^2 h + 3 a b c d e^2 g^2 i + 3 a b c e^2 f g h i + 3 a c^2 d^2 e h^2 i + 3 a c^2 d e f g h + 3 b^3 d^3 i^3 + 3 b^3 d^2 g f i^2 + 3 b^3 f^2 g^2 i + 3 b^3 f^3 g^3 + 3 b^2 c^2 d e g h i + 3 b c^2 d e f g h + c^3 d^3 h^3 + c^3 e^3 g^3. \quad (17)$$

Computing $(\text{perm}(\theta))^3$, we can easily verify that all the products in $\text{perm}(\theta^{1P})$ appear in $(\text{perm}(\theta))^3$ with a larger or equal coefficient, as predicted by Theorem 21.

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