WHAT COULD BE A SIMPLE PERMUTATION?

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Abstract. Different ways to describe a permutation, as a sequence of integers, or a product of Coxeter generators, or a tree, give different choices to define a simple permutation. We recollect a few of them, define new types of simple permutations, and analyze their interconnections and some asymptotic and geometrical properties of these classes.

1. Introduction

There are different ways to describe a permutation $\alpha : A \rightarrow A$ of a set $A$ with $n$ elements, see for example the section ”Permutation Statistics” in Stanley’s book [11]. First we ‘coordinatise’ the set $A$ fixing a bijection $\beta : A \rightarrow [n] = \{1, 2, \ldots, n\}$ and replace any $A$ permutation $\alpha$ by the $n$-permutation $\pi = \beta \alpha \beta^{-1}$.

The standard representation of such a permutation is the sequence $[\pi] = [\pi(1), \pi(2), \ldots, \pi(n)]$. The first definition of simple permutation is (see [2], [7], and Definition 2.3):

Definition 1.1. $\pi$ is a segment-simple permutation ($s$-simple) if there is no proper connected subset (or segment) $I \subset [n]$ with a connected image $\pi(I)$.

Another standard representation of a permutation $\pi$ is given by a product of disjoint cycles $\pi = (i, \pi(i), \pi^2(i) \ldots) \ldots (k, \pi(k), \pi^2(k), \ldots)$ (the fixed points are omitted). The standard representation (see [11]) gives a unique form of a cycle decomposition, $\widehat{\pi}$: start to write a cycle (the length one cycles are included now) with the biggest element $k_i$ and put the cycles in increasing order of their biggest elements. Now we obtain another sequence of elements in $[n]$ : $\widehat{\pi} = (k_1 \pi(k_1) \ldots k_2 \pi(k_2) \ldots)$.

Definition 1.2. $\pi$ is a cycle-simple permutation ($c$-simple) if its cycle decomposition contains at most one cycle of length $\geq 2$.

Now we consider the permutation $\pi$ as an element in the symmetric group $\Sigma_n$.

Definition 1.3. $\pi$ is a group-simple permutation ($g$-simple) if the subgroup generated by $\pi$ is a simple group.

\vspace{1em}

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In the classical Coxeter presentation \cite{5} of $\Sigma_n$ the generators are the transpositions $\tau_i = (i, i+1), i = 1, 2, \ldots, n-1$. Any element $\pi \in \Sigma_n$ can be represented in a unique way as a product of $\tau_i$ choosing the smallest such product in the length-lexicographic order given by $\tau_1 < \tau_2 < \ldots < \tau_{n-1}$:

$$\pi = (\tau_{k_1} \tau_{k_1-1} \cdots \tau_{j_1})(\tau_{k_2} \tau_{k_2-1} \cdots \tau_{j_2}) \cdots (\tau_{k_s} \tau_{k_s-1} \cdots \tau_{j_s})$$

with $1 \leq k_1 < k_2 < \ldots < k_s \leq n - 1$ (see \cite{5}).

**Definition 1.4.** $\pi$ is a *braid-simple* permutation ($b$-simple) if, in the word $\tau_{i_1} \tau_{i_2} \cdots \tau_{i_k}$ representing $\pi$, where $k$ is the length of $\pi$, a Coxeter generator $\tau_i$ appears at most once.

There are also graphical descriptions of permutation, see \cite{11}. If $(k_1 \ldots k_2 \ldots k_s \ldots)$ is the standard representation of $\pi$, then the associated ordered tree $T(\pi)$ is defined inductively as follows: choose as the root of the tree the smallest element $s$ of the sequence, next put on the left the tree of the subsequence in front of $s$ and on the right the tree corresponding to the subsequence behind $s$. The vertices of $T(\pi)$ are marked from 1 to $n$, 1 is the root of $T(\pi)$, and along the branches the marking is increasing.

**Definition 1.5.** $\pi$ is a *tree-simple* permutation ($t$-simple) if all the vertices of the associated tree $T(\pi)$ have degree 1 or 2.

For instance, the permutation $\pi = \left( \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 6 & 2 & 5 & 3 \end{array} \right)$ has the following descriptions:

$$[\pi] = [4 \ 1 \ 6 \ 2 \ 5 \ 3]$$

$$\hat{\pi} = (4 \ 2 \ 1)(63)$$

$$\hat{\pi} = (4 \ 2 \ 1 \ 5 \ 6 \ 3)$$

$$\pi = (\tau_3 \tau_2 \tau_1)(\tau_4)(\tau_5 \tau_4 \tau_3)$$

and this permutation is $s$- and $t$-simple, but not $b$-, $c$-, or $g$-simple.

A last couple of definitions: we consider a family of subsets $A_n \subset \Sigma_n$, $n \in \mathbb{N}$.

**Definition 1.6.** The family $(A_n)$ has *exponential growth* if there are constants $a$, $b > 1$ such that $\text{card}(A_n) \geq a \cdot b^n$ for any $n \in \mathbb{N}$.

**Definition 1.7.** The family $(A_n)$ is *rare* if $\lim_{n \to \infty} \frac{\text{card}(A_n)}{\text{card}(\Sigma_n)} = 0$. 
In the next sections we give different characterizations of the corresponding simple subsets of $\Sigma_n$: $ss_n$, $cs_n$, $gs_n$, $bs_n$, $ts_n$ (for instance, in terms of cyclic decomposition), we study enumerative combinatorics of these sets and growth type of these subsets. We prove that

**Theorem 1.8.** All five simple families $ss_n$, $cs_n$, $gs_n$, $bs_n$ and $ts_n$ have an exponential growth.

**Theorem 1.9.** The four simple families $cs_n$, $gs_n$, $bs_n$ and $ts_n$ are rare.

We analyze the subgraphs $\Gamma(bs_n), \Gamma(cs_n)$, and $\Gamma(gs_n)$ of the Cayley graph of $\Sigma_n$ (with Coxeter generators), corresponding to these various simple subsets and also the subcomplex $P(bs_n)$ of the permutahedron $P(\Sigma_n)$, and the simplicial complexes $B(bs_n), W(bs_n)$ associated to Bruhat order and weak order respectively.

The main results are:

**Theorem 1.10.** a) $\Gamma(bs_n)$ is a connected graph which is planar if and only if $n \leq 5$. b) $\Gamma(cs_n)$ is connected if and only if $n \leq 4$. c) $\Gamma(gs_n)$ is connected if and only if $n \leq 3$.

Section 5 contains a complete description of connected components of $\Gamma(cs_n)$ and $\Gamma(gs_n)$.

**Theorem 1.11.** $P(bs_n), B(bs_n)$ and $W(bs_n)$ are contractible spaces.

Our starting point was to understand the permutations corresponding to the simple braids, see [3], [4] for this related notion. As we found in the literature different notions of simple permutations, the aim of this paper is to characterize in an algebraic and combinatorial way braid-simple permutations and to have a complete picture of the relations between $bs_n$ and other families of simple permutations. There are a few natural questions about the other families and these could be interesting problems in the combinatorics of $\Sigma_n$.

## 2. Braid-Simple Permutations

The set of braid-simple permutations, $bs_n \subset \Sigma_n$, is the image of the set of simple braids $s_n \subset B_n$ through the canonical map $\pi : B_n \rightarrow \Sigma_n$ (see [3]). Simple braids can be defined as in Definition [4.4], replacing Coxeter generators by classical Artin generators, therefore the simple braids are square free positive braids or divisors of Garside braid $\Delta_n$ (see [6], [8]): $s_n \subset \text{Div}(\Delta_n) \subset B_n$. As the restriction $\pi : \text{Div}(\Delta_n) \rightarrow \Sigma_n$ is a bijection, we obtain the diagram

\[
\begin{array}{ccc}
S_n & \xrightarrow{\approx} & \text{Div}(\Delta_n) \xrightarrow{\approx} & B_n \\
bs_n & \xrightarrow{\pi} & \Sigma_n & \xleftarrow{\pi} & \Sigma_n
\end{array}
\]
and therefore we can use all the results from [3] and [4].

We denote by $\sigma_{n,i}$ the number of b-simple permutations of length $i$ in $\Sigma_n$ and by $(F_n)_{n\geq 0}$ the Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$:

**Proposition 2.1.** (4) a) The numbers $\sigma_{n,i}$ satisfies the recurrences:

1) $\sigma_{1,0} = 1$ and $\sigma_{1,i} = 0$ for $i \neq 0$;
2) $\sigma_{n,i} = \sigma_{n-1,i} + \sigma_{n-1,i-1} + \sigma_{n-2,i-2} + \ldots + \sigma_{n-i,0}$;
3) $\sigma_{n,i} = 2\sigma_{n-1,i-1} + \sigma_{n-1,i} - \sigma_{n-2,i-1}$.

b) The cardinality of the set of braid-simple permutations is given by $|bS_n| = \sigma_{n,0} + \sigma_{n,1} + \ldots + \sigma_{n,n-1} = F_{2n-1}$.

We obtain an asymmetric Pascal triangle for $(\sigma_{n,i})$:

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 2 \\
1 & 3 & 5 & 4 \\
1 & 4 & 9 & 12 & 8 \\
\cdots \ldots \ldots \ldots \ldots \\
\end{array}
\]

with $2^{n-2}$ on the last position (for $n \geq 2$).

**Corollary 2.2.** The family of braid-simple permutations $bS_n$ is rare and has exponential growth.

**Proof.** Elementary computations show that

$$F_{2n-1} = c_1 \left(\frac{1 + \sqrt{5}}{2}\right)^{2n-1} + c_2 \left(\frac{1 - \sqrt{5}}{2}\right)^{2n-1} = k_1 \left(\frac{3 + \sqrt{5}}{2}\right)^n + k_2 \left(\frac{3 - \sqrt{5}}{2}\right)^n.$$

□

Now we start to characterize b-simple permutations in terms of cyclic decomposition. First two definitions:

**Definition 2.3.** A subset $I \subset [n] = \{1, 2, \ldots, n\}$ is connected (or $I$ is a segment) if $I = \{i, i+1, \ldots, j\}$ for some $1 \leq i \leq j \leq n$. $I$ is proper if its cardinality is not 1 or $n$. A permutation $\pi \in \Sigma_n$ is connected if all its orbits are connected. For instance $(51342)(76)$ is connected but $(42)(513)(76)$ is not.

**Definition 2.4.** A cycle $(k_1 k_2 \ldots k_s)$ is unimodal if there is an index $1 \leq m \leq s$ such that $k_1 > k_2 > \ldots > k_m < k_{m+1} < \ldots < k_s$. A permutation $\pi$ is unimodal if all its cycles are unimodal.

In this definition we use the standard representation convention, $k_1$ is the largest element of the cycle. If we start to write the cycle with the smallest element, we find the sequence $(k_m, k_{m+1}, \ldots, k_s, k_1, k_2, \ldots, k_{m-1})$, which is unimodal in the usual sense (first increasing, next decreasing). In fact we need "cyclic unimodal"
sequences, hence the usual definition and our definition coincide. We will use the notations $D(k, j) = \tau_k \tau_{k-1} \dots \tau_{j+1} \tau_j$ for $1 \leq j \leq k \leq n - 1$ and also $D(K_*, J_*) = D(k_1, j_1)D(k_2, j_2) \dots D(k_s, j_s)$, where $1 \leq k_1 < k_2 < \dots < k_s \leq n - 1$, $j_a \leq k_a$, and $1 \leq s \leq n - 1$. Also $D(k)$ is a short notation for $D(k, k) = \tau_k$.

The new result of this section is:

**Proposition 2.5.** A permutation $\pi \in \Sigma_n$ is braid-simple if and only if $\pi$ is connected and unimodal.

**Proof.** A product $D(k, j) = \tau_k \tau_{k-1} \dots \tau_{j+1} \tau_j$ has the cycle representation $(k+1, k, \ldots, j+1, j)$. A b-simple permutation has the canonical representation as a product of Coxeter generators

$$D(K_*, J_*) = D(k_1, j_1)D(k_2, j_2) \dots D(k_s, j_s)$$

where $1 \leq j_1 \leq k_1 < k_2 < \ldots < j_s \leq k_s \leq n - 1$. This permutation is a cycle if and only if $j_2 = k_1 + 1, \ldots, j_s = k_{s-1} + 1$ and in this case its cycle representation is

$$(k_s + 1, k_s, \ldots, j_s + 1, k_{s-1}, k_{s-1} - 1, \ldots, j_s - 1 + 1, \ldots, k_1, k_1 - 1, \ldots, j_1, j_2, \ldots, j_s)$$

(if one of the factors $D(k_i, j_i)$ contains only one generator, $\tau_{k_i} = \tau_{j_i}$, then this generator appears only in the increasing part of this connected unimodal cycle).

Conversely, a unimodal connected cycle can be written as a product of $D$'s factors starting with the leftmost segment containing the minimal element in the cycle, next reading the segments (from right to left), adding at their ends one element from the increasing part of the unimodal sequence and writing these augmented segments from left to right. As an example, the connected unimodal cycle $(13, 12, 9, 8, 7, 5, 3, 2, 1, 4, 6, 10, 11)$ corresponds to the product of Coxeter generators $D(3, 1)D(5, 4)D(9, 6)D(10)D(12, 11)$. If the product $D(K_*, J_*)$ contains $c - 1$ "jumps" of the form $k_i + 1 < j_{i+1}$, then its cycle decomposition has $c$ disjoint cycles, and all of them are connected and unimodal. □

Direct consequences of the proof and of Proposition 3.1 are the next two characterizations of the braid-simple permutations which are cycle-simple and group-simple, respectively.

**Corollary 2.6.** The following properties of a permutation $\pi \in \Sigma_n$ are equivalent:

i) $\pi \in bS_n \cap gS_n$;

ii) $\pi = Id$ or the cycle representation of $\pi$ contains a unique cycle and this is connected and unimodal;

iii) $\pi = Id$ or the Coxeter representation of $\pi$ is $D(K_*, J_*)$ where $k_i + 1 = j_{i+1}$ for any $i$.

**Corollary 2.7.** The following properties of a permutation $\pi \in \Sigma_n$ are equivalent:

i) $\pi \in bS_n \cap gS_n$;
ii) $\pi = \text{Id}$ or there exist a prime $p$ and $q$ ($pq \leq n$) such that the cycle decomposition of $\pi$ contains $q$ cycles of length $p$, and all of them are connected and unimodal;

iii) $\pi = \text{Id}$ or there exist a prime $p$ and $q$ ($pq \leq n$) such that the Coxeter representation of $\pi$ is $D(K_*, J_*) = D(K^1_*, J^1_*) D(K^2_*, J^2_*) \ldots D(K^a_*, J^a_*)$, where $D(K^a_*, J^a_*) = D(k^a_1, j^a_1) \ldots D(k^a_{s_a}, j^a_{s_a})$ with $k^a_1 + 1 = j^a_1 + 1$, $k^a_{s_a} - j^a_1 + 2 = p$, and $k^a_{s_a} + 1 < j^a_{s_a + 1}$.

3. CYCLIC-SIMPLE AND GROUP-SIMPLE PERMUTATIONS

Let start with the remark that cyclic-simple and group-simple notions are invariant under conjugation, i.e. these notions does not depend on the "coordinatization" of a finite set $A$ with $n$ elements. All the other classes, b-, s-, and t-simple permutations, depend on the coordinatization of the set $A$.

The next characterization is obvious:

Proposition 3.1. A permutation $\pi \in \Sigma_n$ is $g$-simple if and only if $\pi = \text{identity}$ or there is a prime number $p$ and a positive integer $k$ ($pk \leq n$) such that $\pi$ has $k$ cycles of length $p$ (and other elements are fixed).

Enumerative combinatorics of $cS_n$ and $gS_n$ is simple:

Proposition 3.2. a) The number of permutations $\pi \in \Sigma_n$, product of $k$ cycles of equal length $l$, is given by: 

$$\frac{n!}{k!(n - kl)!l^k};$$

b) the set of cyclic-simple permutations $cS_n$ has cardinality:

$$1 + \sum_{l=2}^{n} \frac{n!}{l \cdot (n - l)!};$$

c) the set of group-simple permutations $gS_n$ has cardinality:

$$1 + \sum_{p \text{ prime}} \sum_{k=1}^{\lfloor \frac{n}{p} \rfloor} \frac{n!}{k!(n - kp)!p^k}.$$

Proof. a) There are $\binom{n}{ll \ldots ll l n - kl}$ (with $l$ repeated $k$ times) choices for the sequence of $k$ orbits of length $l$, $\frac{1}{kl} \binom{n}{ll \ldots ll l n - kl}$ choices for the set of these $k$-orbits; for each orbit of length $l$, there are $(l - 1)!$ cycles, hence the result.

b) and c) are consequences of a).

Corollary 3.3. The two families $(cS_n)$ and $(gS_n)$ are rare with exponential growth.
Proof. In the sum representing $|cS_n|$ the last term is $(n-1)! \sim \sqrt{2\pi n} (n-1) e^{n-1}$. By Bertrand postulate [1], there is a prime $p$ between $\left\lfloor \frac{n}{2} \right\rfloor$ and $n$, therefore the last term in the double sum representing $|gS_n|$, 

$$\frac{n!}{(n-p)!p},$$

is greater than $\frac{n!}{\left\lfloor \frac{n}{2} \right\rfloor! n} \sim \frac{\sqrt{2}}{n} \left( \frac{2n}{e} \right)^{\frac{\left\lfloor n/2 \right\rfloor}{2}}$.

We split the sum $\frac{1}{n!} + \sum_{l=2}^{n} \frac{1}{(n-1)!l} = |cS_n|\left|\Sigma_n\right|$ in two parts

$$C_1 = \frac{1}{n!} + \sum_{l=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{(n-l)!l} \leq \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2\left\lfloor \frac{n}{2} \right\rfloor!}$$

and

$$C_2 = \sum_{l=\left\lfloor \frac{n}{2} \right\rfloor+1}^{n} \frac{1}{(n-l)!l} \leq \frac{1}{\left\lfloor \frac{n}{2} \right\rfloor!} \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \right) < \frac{e}{\left\lfloor \frac{n}{2} \right\rfloor}$$

hence $\lim \frac{|cS_n|}{n!} = 0$.

In a similar way we can evaluate the quotient $\frac{|gS_n|}{\left|\Sigma_n\right|}$: for primes $p$ in the interval $\left( \left\lfloor \frac{n}{2} \right\rfloor, n \right]$, we have

$$G_2 = \sum_{\left\lfloor \frac{n}{2} \right\rfloor < p \leq n} \frac{1}{k!(n-kp)!p^k} = \sum_{\left\lfloor \frac{n}{2} \right\rfloor < p \leq n} \frac{1}{(n-p)!p} < \frac{1}{\left\lfloor \frac{n}{2} \right\rfloor!} \left( 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \right) < \frac{e}{\left\lfloor \frac{n}{2} \right\rfloor},$$

otherwise $p \leq \left\lfloor \frac{n}{2} \right\rfloor$ and we split in two parts the contribution of $p$:

$$g_p = \sum_{k=0}^{\left\lfloor \frac{n}{2p} \right\rfloor} \frac{1}{k!(n-kp)!p^k} = \sum_{k \leq \left\lfloor \frac{n}{2p} \right\rfloor} \frac{1}{k!(n-kp)!p^k} + \sum_{k > \left\lfloor \frac{n}{2p} \right\rfloor} \frac{1}{k!(n-kp)!p^k} = g'_p + g''_p.$$

For the first sum we find

$$g'_p = \sum_{k \leq \left\lfloor \frac{n}{2p} \right\rfloor} \frac{1}{k!(n-kp)!p^k} \leq \frac{n}{2p} \frac{1}{\left\lfloor \frac{n}{2p} \right\rfloor! \left\lfloor \frac{n}{2p} \right\rfloor^2} \leq \frac{\left\lfloor \frac{n}{2} \right\rfloor}{2\left\lfloor \frac{n}{2} \right\rfloor!}$$

and for the second sum we find

$$g''_p = \sum_{k > \left\lfloor \frac{n}{2p} \right\rfloor} \frac{1}{k!(n-kp)!p^k} \leq \frac{n}{2p} \frac{1}{\left\lfloor \frac{n}{2p} \right\rfloor! \left\lfloor \frac{n}{2p} \right\rfloor^2} < \frac{\text{constant}}{n^2}.$$
(for the last inequality we use \( \frac{n^2}{c} < \left( \frac{\lfloor \frac{n}{2p} \rfloor}{2} \right) (p - 1)^2 < p^{\lfloor \frac{n}{2p} \rfloor} \), with \( c \) a constant in \( n \) and \( p \)). Finally the inequality for \( G_2 \) and

\[
G_1 = \frac{1}{n!} + \sum_{p \leq \lfloor \frac{n}{2} \rfloor} (g'_p + g''_p) < \frac{1}{n!} + \left[ \frac{n}{2!} (\frac{n}{2!} + \text{constant}) \right]
\]

shows that \( \lim \frac{|g_{S_n}|}{n!} = 0. \)

\[\square\]

4. SEGMENT-SIMPLE AND TREE-SIMPLE PERMUTATIONS

In this section we finish the proofs of Theorems 1.8 and 1.9 and we characterize the permutations from the intersections \( bS_n \cap sS_n \) and \( bS_n \cap tS_n \).

The asymptotics of segment-simple permutations is given by the next result:

**Theorem 4.1.** (12) The cardinality of the set of segment-simple permutations is

\[
|sS_n| = \frac{n!}{e^2} \left( 1 - \frac{4}{n} + \frac{2}{n(n-1)} + O(n^{-3}) \right).
\]

Enumerative combinatorics of tree-simple permutations is elementary:

**Proposition 4.2.** The cardinality of the set of tree-simple permutations is, for \( n \geq 2 \),

\[
|tS_n| = 2^{n-2} + 4^{n-2}.
\]

**Proof.** For \( n \geq 2 \), first we count the number of simple oriented rooted marked trees: marks are from 1 to \( n \), the number 1 is the root, at this point we could have a left branch, or a right branch, or both, at any other node we have at most one branch (a left one or a right one), and all the marks on the branches are in increasing order. If there is only one branch at the root 1, we have \( 2^{n-1} \) choices for left-right orientations at the nodes 1, 2, \ldots, \( n - 1 \). If there are \( k \) nodes on the left branch and \( (n-k-1) \) nodes on the right branch \( (1 \leq k \leq n-2) \), we have \( \left( \begin{array}{c} n-1 \\ k \end{array} \right) \) ways to choose the increasing marking on the left branch, \( 2^{k-1} \) possibilities to choose orientations at the first \( k-1 \) nodes of this branch, and \( 2^{n-k-2} \) possibilities to choose left-right orientations at the first \( n-k-2 \) nodes of the right branch. Therefore the total number is \( (n \geq 2) \):

\[
2^{n-1} + \sum_{k=1}^{n-2} \left( \begin{array}{c} n-1 \\ k \end{array} \right) 2^{k-1} \cdot 2^{n-k-2} = 2^{n-1} + (2^{n-1} - 2)2^{n-3} = 2^{n-2} + 4^{n-2}.
\]

For \( n = 1 \), the last formula (with \( \sum = 0 \)) gives the result \( |tS_1| = 1. \) \[\square\]
In the next table one can see few values of the numbers of ∗-simple permutations. For large values of \(n\), there are more segment-simple permutations than in the union of the other four classes.

| \(n\) | \(bS_n\) | \(cS_n\) | \(gS_n\) | \(sS_n\) | \(tS_n\) | \(\Sigma_n\) |
|-------|--------|--------|--------|--------|--------|--------|
| 1     | 1      | 1      | 1      | 1      | 1      | 1      |
| 2     | 2      | 2      | 2      | 2      | 2      | 2      |
| 3     | 5      | 6      | 6      | 0      | 6      | 6      |
| 4     | 13     | 21     | 18     | 2      | 20     | 24     |
| 5     | 34     | 85     | 70     | 6      | 72     | 120    |
| 6     | 89     | 410    | 300    | 46     | 272    | 720    |

A cycle contains three consecutive elements if it contains a subsequence \((\ldots, i, i+1, i+2, \ldots)\) or \((\ldots, i+2, i+1, i, \ldots)\). For example, the cycle \((7,4,3,1,2,5,6)\) contains three consecutive elements \((5,6,7)\), but \((3,1,2)\) are not consecutive.

**Proposition 4.3.** The following properties of a permutation \(\pi \in \Sigma_n\) are equivalent:

i) \(\pi \in bS_n \cap sS_n\);

ii) \(\pi\) is an unimodal cycle of length \(n\) without three consecutive elements.

**Proof.** If \(\pi\) is braid-simple in \(\Sigma_n\) it is connected and unimodal, therefore if \(\pi\) has a cycle of length \(2 \leq q \leq n-1\), a segment of length \(q\) is \(\pi\) invariant. This shows that \(\pi \in bS_n \cap sS_n\) should be an \(n\)-cycle. If there are three consecutive elements, \((i-1, i, i+1), (i+1, i, i-1)\), then \(\pi\)(segment of length 2) is also a segment: \(\pi[i,i \pm 1] = [i, i \mp 1]\).

Conversely, if \(\pi\) is a unimodal \(n\)-cycle without three consecutive elements, then we have to show that \(\pi\) is segment-simple. Suppose that there are two segments \(a \leq b, c \leq d, b-a = d-c = L, 2 \leq L \leq n-1\), such that \(\pi[a, b] = [c, d]\). The segment \([a, b]\) can not be contained entirely in the decreasing part \((n \ldots 1)\) or in the increasing part \((1 \ldots k)\) of the unimodal cycle \(\pi\) (if \(L \geq 3\), then we have three consecutive elements; if \(L = 2\), then \(a \to a+1\), and now \(\pi[a,a+1] = [a+1,a+2]\), therefore again we find three consecutive elements, and similarly, if \(a+1 \to a\) we find \(a \to a-1\), contradiction). Let us denote by \(d_M\) and \(d_m\) the maximal and minimal elements in the decreasing part of the cycle \(\pi\) contained in the interval \([a, b]\), and by \(i_m\) and \(i_M\) the minimal and the maximal elements of \([a, b]\) contained in the increasing part of \(\pi\):

\[
\pi = (n \ldots d_M \ldots d_m \ldots 1 \ldots i_m \ldots i_M \ldots k).
\]

The value \(\pi(d_m)\) is less than \(d_m\) and not in \([a, b]\), therefore \(\pi(d_m) < a\) and \(\pi(i_M)\) is greater than \(i_M\) and not in \([a, b]\), therefore \(\pi(i_M) > b\), but these inequalities give a contradiction

\[
\pi(i_M) - \pi(d_m) > b - a = L = c - d \geq \pi(i_M) - \pi(d_m).
\]
The special cases \( d_m = 1 \), or \( i_m = 1 \), or \( d_m = i_m = 1 \) can be settled in a similar way.

To represent the rooted tree of a permutation, \( T(\pi) \), without self intersections, we make the following ”geometrical” conventions:

a) all the edges make an angle equal to \( \frac{\pi}{4} \) with the vertical direction;

b) the edges starting at the root 1 have the length 1, the next edges (at most four) have length \( \frac{1}{2} \), the next (at most eight) edges have length \( \frac{1}{4} \), and so on.

In the tree \( T(\pi) \) the ending points of the left and right branches are denoted by \( M_L \) and \( M_R \) (they are the maximal marks on their branches).

**Proposition 4.4.** The following properties of a permutation \( \pi \in \Sigma_n \) are equivalent:

i) \( \pi \in bS_n \cap tS_n \);

ii) there is at most one right angle (different from 1) and this could be only on the right branch. In the case of a right angle on the right branch at the point \( M \), we must have \( M_L < M \).

**Proof.** If \( \pi \) is braid-simple, then \( \pi \) is connected unimodal, in particular the cycle containing 1 contains only the numbers from 1 to \( a \): \((a = a_1, a_2, \ldots, 1, \ldots, a_k)\), therefore the first part of the tree \( T(\pi) \) has (at most) a right angle at 1. Because \( \pi \) is tree-simple, the next value, \( a_1 + 1 \), should be either the successor of \( a_k \) (and hence a fixed point of \( \pi \)) or the last one in the standard representation; in the last case, the cycle containing \( a_1 + 1 \) contains also \( n \) (connectedness condition) and must be \((n, n-1, \ldots, a_1 + 1) \) (\( \pi \) is unimodal). Hence we have a unique right angle at \( a_1 + 1 \).

If \( \pi(a_1 + 1) = a_1 + 1 \), the same argument says that the structure of \( \pi \) is:

\[
\pi = (a_1 \ldots 1 \ldots a_k)(a_1 + 1)\ldots(a_1 + i - 1)(n \ldots a_1 + i)
\]

or

\[
\pi = (a_1 \ldots 1 \ldots a_k)(a_1 + 1)(a_1 + 2)\ldots n,
\]

and the corresponding trees have one or none right angles on the right branch and they have no right angles on the left branch. Using the same argument, in the particular case of \( \pi(1) = 1 \), \( T(\pi) \) has only the right branch with at most one right angle. If \( a_1 = n \), then \( \pi = (n, n-1, \ldots, 2, 1) \) has only a straight left branch.

Conversely, the tree \( T_1 \) corresponds to

\[
\pi_1 = (a_1 \ldots 1 \ldots a_k)(a_1 + 1)\ldots (n),
\]
What Could be a Simple Permutation?

The tree $T_2$ corresponds to the simple permutation ($a_j$ is greater than $a_1$)

$$\pi_2 = (a_1 \ldots 1 \ldots a_k)(a_1 + 1) \ldots (a_j - 1)(n, n - 1 \ldots a_j),$$

and both are b- and t-simple; we have similar results for trees without right angles.

**Corollary 4.5.** The intersection $S_n = bS_n \cap cS_n \cap gS_n \cap sS_n \cap tS_n$ is non empty if and only if $n \leq 2$ or $n$ is a prime number greater or equal to 5.

**Proof.** For $n = 1$ and $n = 2$, the intersection $S_n = bS_n \cap cS_n \cap gS_n \cap sS_n \cap tS_n$ is the entire set $\Sigma_n$. For $n = 3$, $sS_3 = \emptyset$ and for $n = 4$, see the first picture in section 5. Because $\pi$ is braid-simple and segment-simple, $\pi$ should be an $n$-cycle; $\pi$ is group-simple implies that $n$ should be a prime number $\geq 5$.

Conversely, if $p$ is prime and $\geq 5$, the permutation

$$\pi = (p, p - 2, p - 4, \ldots, 5, 3, 1, 2, 4, \ldots, p - 3, p - 1)$$

is an element of the intersection $S_p$. 

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### 5. Geometry of Simple Permutations

If $A$ is a subset of $\Sigma_n$, the corresponding *subgraph* $\Gamma(A)$ of the Cayley graph $\Gamma(\Sigma_n)$ has $A$ as a set of vertices and there is an edge $a - b$ between two elements of $A$ if $b = a\tau_i$ for some Coxeter generator $\tau_i = (i, i + 1)$ ($b = a\tau_i$ is equivalent to $a = b\tau_i$).

In a similar way is defined the *subcomplex* $P(A)$ of the permutahedron $P(\Sigma_n)$: the vertices of $P(A)$ are elements of $A$ and a face of permutahedron is a face of $P(A)$ if and only if all its vertices are in $A$. It is obvious that $\Gamma(A)$ is the 1-dimensional skeleton of $P(A)$. In this last section we analyze subgraphs $\Gamma(\ast S_n)$ corresponding to some simple subsets $\ast S_n$ of the symmetric group $\Sigma_n$ and the subcomplex $P(bS_n)$.

Other geometric objects associated to the Coxeter presentation of $\Sigma_n$ are the simplicial complexes corresponding to the natural poset structures on $\Sigma_n$: Bruhat order and (right) weak order (see [5]). The permutations $\alpha$ and $\beta$ are consecutive in these order relations, $\alpha < \beta$, if and only if $\text{length}(\beta) = \text{length}(\alpha) + 1$ and for some transpositions $\beta = \alpha(i, j)$ in the Bruhat order and $\beta = \alpha\tau_i$ in the weak order. We denote by $B(\Sigma_n)$ and $W(\Sigma_n)$ the corresponding simplicial complexes associated to these two posets and by $B(\ast S_n)$ and $W(\ast S_n)$ the subcomplexes corresponding to the $\ast$-simple permutations in $\Sigma_n$. 

---

$T_1:\quad M_L = a_1 \quad a_i + 1 \quad n = M_R$

$T_2:\quad M_L = a_1 \quad a_1 + 1 \quad a_j = M \quad n = M_R$
Proof of Theorem 1.10. a) The b-simple part $\Gamma(bS_n)$ is connected by the very definition of a braid simple permutation: $D(K_*, J_*)$ can be joined with the unit element by the geodesic

$$Id - \tau_{k_1} - \tau_{k_1-1} - \cdots - D(k_1, j_1) - D(k_1, j_1)\tau_{k_2} - \cdots - D(K_*, J_*)$$

The planarity of $\Gamma(bS_6)$ comes from the next figure, and one can find a $K_{3,3}$ subgraph of $\Gamma(bS_7)$ on the next page.

b) This is a consequence of Proposition 5.4.

c) This is a consequence of Proposition 5.6. □

Remark 5.1. a) The figure contains the 89 braid-simple permutations in $\Gamma(\Sigma_6)$. 

$\Gamma(bS_6)$ and $\Gamma(bS_6) \cap \Gamma(*S_6)$
b) Among them, there are 58 elements in $bS_6 \cap cS_6$ marked with $\ast$, 39 elements in $bS_6 \cap gS_6$ marked with $\bullet$, 4 elements in $bS_6 \cap sS_6$ marked with an inscribed $\star$, and 44 elements in $bS_6 \cap tS_6$ which are underlined.

A $K_{3,3}$ subgraph of $\Gamma(bS_7)$

Direct computations give the next results:

**Lemma 5.2.** Let $\pi$ be a cycle in $\Sigma_n$.

a) If $i + 1$ is not an element of this cycle, then

$$\pi \tau_i = (k \ldots a, i, b \ldots h)(i, i + 1) = (k \ldots a, i, i + 1, b \ldots h).$$

b) If $i$ is not an element of this cycle, then

$$\pi \tau_i = (k \ldots a, i + 1, b \ldots h)(i, i + 1) = (k \ldots a, i + 1, i, b \ldots h).$$

c) The next equalities hold

$$(k \ldots a, i, i \pm 1, b \ldots h)(i, i \pm 1) = (k \ldots a, i, b \ldots h).$$

d) If $i, i + 1$ are non consecutive elements of this cycle, then $\pi \tau_i$ is a product of two disjoint cycles:

$$\pi \tau_i = \pi_{i + 1} = (k \ldots a, i, b \ldots c, i + 1, d \ldots h)(i, i + 1)$$

$$= (\ldots c, i + 1, b \ldots )(k \ldots a, i, d \ldots h).$$

**Corollary 5.3.** If $\pi$ is a product of two disjoint cycles, one containing $i$ and the other containing $i + 1$, then

$$\pi \tau_i = \pi_{i + 1} = (m \ldots c, i + 1, b \ldots l)(k \ldots a, i, d \ldots h)(i, i + 1)$$

$$= (k \ldots a, i, b \ldots c, i + 1, d, \ldots h).$$

Previous computations explain the next definitions, necessary to describe the connected components of $\Gamma(cS_n)$. First we introduce, by two examples, an (oriented) **polygonal representation** of a cycle: see the next two diagrams. A **reduction move** of an oriented $k$-gon ($k \geq 4$) consists in replacing a side $i \rightarrow j$ by the vertex $i$, and the side $j \rightarrow k$ by $i \rightarrow k$, if the following condition is fulfilled: the interval $(\min\{i, j\}, \max\{i, j\})$ does not contain another vertex of the polygon. For instance, the first pentagon, (61425) can be reduced in two steps to the triangle (514) or
(614) or (615), and the second pentagon, (63152), is irreducible. Any polygon can be reduced to a unique irreducible type \((a_1, a_2, \ldots, a_s)\) or can be reduced to a triangle (this is not unique). An irreducible type \((a_1, a_2, \ldots, a_s)\) is a sequence of distinct integers (called vertices) in the interval \([1, n]\) such that \(a_1 = \max(a_i)\) and, for any \(i\), in the interval \([\min(a_i, a_{i+1}), \max(a_i, a_{i+1})]\) there is at least one other vertex \(a_j\). We introduce the neighboring intervals \(I^+(a_i)\) and \(I^-(a_i)\) as follows: \(I^+(a_i) = [a_i + 1, a_j - 1]\), where \(a_j\) is the smallest vertex greater than \(a_i\) (in the special case of \(a_1\), \(I^+(a_1) = [a_1 + 1, n]\)), and \(I^-(a_i) = [a_h + 1, a_i - 1]\), where \(a_h\) is greatest vertex smaller than \(a_i\), (in the special case of \(a_m = \min(a_i)\), \(I^-(a_m) = [1, a_m - 1]\)). In the next picture there are only three non empty neighboring intervals: \(I^+(3) = I^-(5) = \{4\}\) and \(I^+(7) = \{8\}\):

Now it is easy to see that a polygon \(P\) can be reduced to the irreducible type \((a_1, \ldots, a_s)\) if and only if it has the next structure:

\[(a_1, b_1^1, \ldots, b_1^{t_1}, a_2, b_2^1, \ldots, b_2^{t_2}, \ldots, a_s, b_s^1, \ldots, b_s^{t_s}),\]

where, for any \(i\), all the elements \(\{b_i^1, \ldots, b_i^{t_i}\}\) are either in \(I^+(a_i)\) or in \(I^-(a_i)\) (in the case of equality \(I^+(a_i) = I^-(a_j)\), the sets \(\{a_i, b_i^1, \ldots, b_i^{t_i}\}\) should be separated: \(\max b_i^j < \min b_i^j\) and also the polygons \((a_1, a_2, \ldots, a_i, b_i^1, b_i^2, \ldots, b_i^{t_i}, a_{i+1}, \ldots, a_s)\) can be reduced to the irreducible type \((a_1, \ldots, a_i, a_{i+1}, \ldots, a_s)\). The uniqueness of irreducible types \((a_1 \ldots a_s)\) \((s \geq 5)\) comes from the invariance of the unremovable vertices (the leftmost vertices \(a\) in the previous formula).

**Proposition 5.4.** a) The cycle-simple graph \(\Gamma(cS_n)\) is connected if and only if \(n \leq 4\).
b) The connected component of identity in \( \Gamma(cS_n) \) contains only identity, all the transpositions, and all the cycles reducible to a triangle.

c) Any other component contains all the oriented polygons reducible to a given irreducible type \((a_1,a_2,\ldots,a_s)\).

**Proof.** a) For \( n \leq 3 \) every permutation is cycle-simple: \( cS_n = \Sigma_n \) and for \( n = 4 \) the graph \( \Gamma(cS_4) \) can be seen in the diagram of \( P(cS_4) \). But if \( n \geq 5 \), using c), we have at least two pentagonal types, \((52413)\) and \((53142)\):

\[\text{Diagram of } P(cS_4)\]

b) We have to show that two cycles belong to the same connected component of \( \Gamma(cS_n) \) if and only if they have polygons related by reduction moves. Lemma 5.2 shows that if two cycles \( \gamma_1, \gamma_2 \) are connected by \( \tau_i \), \( \gamma_1 = \gamma_2 \tau_i \) then one of them, say \( \gamma_2 \), has the form \( \gamma_2 = (k \ldots a,i,i+1,b \ldots h) \) or \( \gamma_2 = (k \ldots a,i+1,i,b \ldots h) \), hence the polygon \( P(\gamma_2) \) can be reduced to \( P(\gamma_1) \): remove the side \( i \to i+1 \) or \( i+1 \to i \), respectively.

Conversely, a polygonal reduction move

\[ P(\gamma_2) = (\ldots i \to j \to k \ldots) \mapsto P(\gamma_1) = (\ldots i \to k \ldots) \]

(we suppose that \( i < j \)) gives the following path in \( \Gamma(cS_n) \):

\[ \gamma_2 = (\ldots i,j,k \ldots) - (\ldots i,i+1,j,k \ldots) - \ldots - (\ldots i,i+1 \ldots j-1,j,k \ldots) - (\ldots i,i+1 \ldots j-1,k \ldots) - \ldots - (\ldots i,i+1,k \ldots) - (\ldots i,k \ldots) = \gamma_1. \]

The case \( (\ldots i \to j \to k \ldots) \) with \( i > j \) can be treated in the same way. \( \square \)

**Example 5.5.** \( \Gamma(cS_5) \) has three connected components: the connected component of identity and two isolated points: \((52413)\) and \((53142)\).
Proposition 5.6. a) The graph $\Gamma(gS_n)$ is connected if and only if $n \leq 3$.
b) The connected component of identity contains only identity, all the transpositions $	au_i$ and $(i + 2, i)$, products of disjoint transpositions of two types: $\tau_{k_1}\tau_{k_2}\ldots\tau_{k_s}$ and $(i + 2, i)\tau_{k_1}\tau_{k_2}\ldots\tau_{k_s}$, and also three three-cycles of the form:

$$(i + 2, i + 1, i), (i + 2, i, i + 1), (i + 3, i, i + 2), (i + 3, i + 1, i);$$

c) the other components are divided in three classes: $C(j, i), C(J_*, I_*)$ and isolated components:
c1) a component of type $C(j, i)$, where $j \geq i + 3$, contains only $(j, i), (j, i)\tau_{k_1}\tau_{k_2} \ldots \tau_{k_s}$ (disjoint transpositions) and also (at most four) three cycles $(j, i, i + 1), (j, i, i - 1), (j + 1, i, j), (j, j - 1, i)$;

c2) a component of type $C(J_*, I_*)$, where $J_* = (j_1, \ldots, j_A), I_* = (i_1, \ldots, i_A), (A \geq 2), j_a \geq i_a + 2$ for any $a$, and the cycles $(j_1, i_1), \ldots, (j_A, i_A)$ are disjoint, contains only the products of disjoint transpositions $(j_1, i_1) \ldots (j_A, i_A)\tau_{k_1} \ldots \tau_{k_s}$;

c3) singletons of type $C^{+}_{k_1} = (k, j, i)$ or $C^{+}_{k_2} = (k, i, j)$, where $k - 2 \geq j \geq i + 2$ and products of disjoint such cycles $C^{\pm}_{k_1, j_1, i_1} \ldots C^{\pm}_{k_s, j_s, i_s}$, and also singletons of type $C_{p, q}$, where $p \geq 5$ is prime and $pq \leq n$, containing a unique product of disjoint $q$ cycles of length $p$.

Proof. All these are consequences of Lemma \[5.2\] and of the computations of the length of the cycles in those formulae: in Corollary \[5.3\], $\alpha \beta \tau_i = (\ldots i \ldots)(\ldots i + 1 \ldots)(i + 1, i) = \gamma$, we have length($\gamma$) = length($\alpha$)+length($\beta$), without solution in primes (all the lengths should be equal to a unique prime).
In the case a) of Lemma 5.2, \( \alpha \tau_i = (\ldots i \ldots)(i + 1, i) = \tilde{\alpha} \), we have length (\( \tilde{\alpha} \)) = length (\( \alpha \)) \( \pm \) 1, with solutions the primes 2 and 3. These give isolated \( g \)-simple permutations for primes \( \geq 5 \) and connected components corresponding to the primes 2 and 3, and also isolated components containing products of cycles of length 3 of types \( C_{k_i j}^{\pm} \) with \( k - 2 \geq i \geq j + 2 \).

**Proof of Theorem 1.11.** The contractibility of \( B(bS_n) \) and of \( W(bS_n) \) is is a consequence of a general theorem of Quillen [10]: the posets \( (bS_n, \text{Bruhat order}) \) and \( (bS_n, \text{weak order}) \) have \( \text{Id} \) as a smallest element.

The contractibility of \( P(bS_n) \), the braid-simple part of the permutahedron, is given by a double induction, on \( n \) and on \( j \), the index of the next decreasing filtration:

\[ P(bS_n) = F_{n+1} \subset F_n \subset F_{n-1} \subset \ldots \subset F_2 \subset F_1 = P(bS_{n+1}), \]

where the \( j \) stage of the filtration is

\[ F_j = \left\{ \text{faces in } P(bS_{n+1}) \text{ with vertices from } F_{j+1} \text{ and new vertices } D(k_1, j_1) \ldots D(k_{s-1}, j_{s-1})D(n, j) \right\}. \]

The induction on \( n \) starts with \( n = 1, 2 \) where \( P(bS_n) = P(\Sigma_n) \) are homeomorphic to a point (\( \text{Id} \bullet \)) and to a segment (\( \text{Id} \bullet \bullet \tau_1 \)). We show that \( F_{j+1} \) is deformation retract of \( F_j \). The space \( F_j \) is obtained from \( F_{j+1} \) by adding vertices of the form \( D(k_1, j_1) \ldots D(n, j) \) along the one dimensional cells \( D(k_1, j_1) \ldots D(n, j + 1) = D(k_1, j_1) \ldots D(n, j) \). The cells in \( F_j \) are the cells of \( F_{j+1} \) and new cells of two types: cylinders \( C_{\alpha} \times I_j \), where \( C_{\alpha} \) is cell in \( F_{j+1} \) and \( I_j \) corresponds to the edge \( \text{Id} - \tau_j \), and the faces of these cylinders. There is no cell of \( F_j \) ”parallel” to \( F_{j+1}, C_{\alpha} \times I_j \), if \( C_{\alpha} \) is not a cell in \( F_j \): to a cell with vertices \( \{D(K^\alpha, J^\alpha_\lambda)D(n, j)\}_{\lambda \in \Lambda_\alpha} \) where \( k_1^\lambda < k_2^\lambda < \ldots < k_{s-1}^\lambda \leq n - 1 \), corresponds a face \( F_\alpha \) of \( P(\Sigma_n) \), hence \( C_{\alpha} = F_\alpha \times (\tau_n \tau_{n-1} \ldots \tau_{j+1}) \) is a face of \( F_{j+1} \) (on the other hand, there are cells in \( F_{j+1} \) with no parallel correspondent in \( F_j \)).

The deformation retract \( D_t : F_j \to F_j, D_1 = \text{Id}_{F_j}, D_0 : F_j \to F_{j+1} \) is defined by projecting the cylinders \( C_{\alpha} \times I_j \) onto \( C_{\alpha} \).

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