On the number of hyperelliptic limit cycles of Liénard systems

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Abstract

In this paper, we study the maximum number, denoted by $H(m, n)$, of hyperelliptic limit cycles of the Liénard systems

$$
\dot{x} = y, \quad \dot{y} = -f_m(x)y - g_n(x),
$$

where, respectively, $f_m(x)$ and $g_n(x)$ are real polynomials of degree $m$ and $n$, $g_n(0) = 0$. The main results of the paper are as follows: We obtain the upper bound and lower bound of $H(m, n)$ in all the cases with $n \neq 2m + 1$. When $n = 2m + 1$, we derive the lower bound of $H(m, n)$. Furthermore, these upper bound can be reached in some cases.

Keywords: Hyperelliptic Limit Cycles, Liénard Systems, Configuration

1. Introduction

Consider the following Liénard differential system

$$
\dot{x} = y, \quad \dot{y} = -f_m(x)y - g_n(x),
$$

where $f_m(x)$ and $g_n(x)$ are polynomials of degrees $m$ and $n$, respectively, with the following explicit expressions

$$
f_m(x) = \sum_{i=0}^{m} a_i x^i, \quad g_n(x) = \sum_{i=1}^{n} b_i x^i, \quad a_m b_n \neq 0.
$$

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We shall call this system a *Liénard system of type* \((m,n)\), or simply a Liénard system if no confusion arises.

This paper is primarily devoted to a study of the maximum number \(H(m, n)\) of hyperelliptic limit cycles of the Liénard system in terms of \(m\) and \(n\).

Here we adopt the conventional definition of a limit cycle. Namely, by a *limit cycle* of a polynomial system we mean that it is an isolated closed orbit of the system. It is called an *algebraic limit cycle* if it is a limit cycle and is contained in an invariant algebraic curve \(\{(x,y) \mid F(x,y) = 0\}\). In particular, if \(F(x,y)\) takes the form \(F(x,y) = (y + P(x))^2 - Q(x)\), where \(P\) and \(Q\) are polynomials, then we call the invariant curve hyperelliptic. Correspondingly, a limit cycle is called a hyperelliptic limit cycle if it is contained in a hyperelliptic curve.

The investigation of limit cycles of the Liénard system has been one of the most interesting topics for decades (see [2], [7]). In the most general setting, however, it is a very hard subject and the problem of existence is quite elusive. Therefore certain assumptions are reasonably imposed, and special categories are technically restricted. Among them, the algebraic and hyperelliptic versions of the problem have caught particular attention of the study. A brief survey of the situation is as follows.

Odani [5] in 1995 proved that if \(n \leq m\) and \(f_m g_n \left(\frac{f_m}{g_n}\right)' \neq 0\), then any Liénard system of \((m,n)\)-type has no invariant algebraic curves. Therefore in this case, it is impossible to have any hyperelliptic limit cycles.

Chavarriga et al. [1], Zoladek [10], and Makoto Hayashi [11] proved that any Liénard systems of the types \((0,n)\), \((1,n)\), \((2,4)\) and \((m,m+1)\) have no algebraic limit cycles, hence there are no hyperelliptic limit cycles.

In 2008, Llibre and Zhang [4] proved that no Liénard system of \((3,5)\)-type has hyperelliptic limit cycles. On the other hand, in the same paper [4], they found that in the following cases there are Liénard systems of \((m,n)\)-type which can possess hyperelliptic limit cycles:

(i) \((m,n)\)-type, for \(m \geq 2\) and \(n \geq 2m + 1\);

(ii) \((m,2m)\)-type for \(m \geq 3\);

(iii) \((m,2m-1)\)-type for \(m \geq 4\);

(iv) \((m,2m-2)\)-type for \(m \geq 4\).
Fig. 1. The maximum number of hyperelliptic limit cycles.

An individual type $(5, 7)$ of the Liénard system is discussed in [9], where Yu and Zhang clarified that there exist Liénard systems of $(5, 7)$-type which have hyperelliptic limit cycles.

A recent paper [3] is conclusive, where the authors considered the remaining types of the systems and proved that, in all these cases, there always exist Liénard systems of $(m, n)$-type which have hyperelliptic limit cycles. Thus the problem of the existence of hyperelliptic limit cycles for all the possible types of the Liénard systems is completely answered.

Collecting all the known results mentioned above and arranging them into Fig. 1, we can provide a visual way to exhibit the distribution of the hyperelliptic limit cycles. Namely, in the $(m, n)$-plane, there is a clearly-cut boundary dividing all the types of the Liénard systems into two regions: Systems falling in region 1 can never have any hyperelliptic limit cycle which means $H(m, n) = 0$, and in the other region, for each pair of $(m, n)$, there always exists such a Liénard system which admits at least one hyperelliptic limit cycle, thus $H(m, n) \geq 1$. Systems falling on the boundary are also unambiguously specified.

The present paper grows from a very casual observation. If one looks at Figure and takes region 1 as land and region 2 as sea, and if we walk from the land to the sea, we are in fact traveling from a region where systems have no hyperelliptic limit cycle to a region where such limit cycles start to appear. A very natural question like this can pop up: when we walk from the land to the sea, does the water become deeper and deeper? In other words, does the maximum number of hyperelliptic limit cycles increase as we walk into the sea further and further? Such curiosity leads us to explore this problem and to see if there is any algebraic mechanism behind this. The investigation turns out to be quite interesting: While only those Liénard systems falling in the sea can
have hyperelliptic limit cycles, we prove that those systems in “deeper” water indeed can have larger $H(m,n)$. A detailed classification is summarized in the following theorem. Notice that we also consider the configuration of these limit cycles, another one of very important aspects of the subject.

**Main Theorem:** Consider Liénard systems of the type $(m,n)$ where $m \geq 2$, the maximum number of hyperelliptic limit cycles admits the following estimations:

$$H(m,n) \geq \begin{cases} n - m - 1 & m + 2 \leq n \leq \left[\frac{4m+2}{3}\right] \\ \left\lceil \frac{n-1}{4} \right\rceil & \left\lceil \frac{4m+2}{3} \right\rceil + 1 \leq n \leq 2m, m \geq 4 \\ \left\lceil \frac{m}{2} \right\rceil & n \geq 2m + 1 \end{cases}$$

and

$$H(m,n) \leq \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil & m + 2 \leq n \leq 2m - 2, m \geq 4 \\ \left\lceil \frac{n-1}{4} \right\rceil & n = 2m - 1, or n = 2m, m \geq 4 \\ \left\lceil \frac{n}{2} \right\rceil & n > 2m + 1 \end{cases}$$

In all the cases with $n \neq 2m + 1$ and $H(m,n) > 1$, the hyperelliptic limit cycles of the system can only have non-nested configuration.

**Remark:** It immediately follows from the main theorem that

(i) When $1 + \left\lceil \frac{4m+2}{3} \right\rceil \leq n \leq 2m - 2$, if $n - 1 \equiv 0 \pmod{4}$ or $n - 1 \equiv 1 \pmod{4}$, then $H(m,n) = \left\lceil \frac{n-1}{4} \right\rceil$;

(ii) If $n = 2m - 1$ or $n = 2m$, $m \geq 4$, then $H(m,n) = \left\lceil \frac{n-1}{4} \right\rceil$;

(iii) If $n > 2m + 1$, then $H(m,n) = \left\lceil \frac{n}{2} \right\rceil$.

The paper is organized as follows: In section 2, we shall introduce some preliminaries including definitions, notation and basic methods. In section 3, 4 and 5, we present a detailed proof of the results.

2. Preliminaries

In this section, we shall collect some related properties of Liénard systems and introduce a complete discrimination system for polynomials. For the proof of these results, we refer the reader to (3, 4, 9, 10, 11, 12, 13) for details.
2.1 Hyperelliptic limit cycles of Liénard systems

Recall that the Liénard system takes the form

\[ \dot{x} = y, \quad \dot{y} = -f_m(x)y - g_n(x). \]

Assume that the system has a hyperelliptic invariant curve

\[ F(x, y) = (y + P(x))^2 - Q(x) = 0. \]  \(2\)

The following properties hold, whose proof is standard and hence omitted.

**Lemma 2.1.** There exists \( K(x, y) \in \mathbb{R}[x, y] \) such that

\[ y \frac{\partial F}{\partial x} - (f_m(x)y + g_n(x)) \frac{\partial F}{\partial y} = K(x, y)F. \]

**Lemma 2.2.** If relation \(2\) holds, then the degree of polynomial \( P(x) \) is \( m + 1 \), and the polynomials \( f_m \) and \( g_n \) can be expressed in terms of \( P \) and \( Q \) as follows.

\[ f_m = P' + \frac{PQ'}{2Q}, \quad g_n = \frac{Q'(P^2 - Q)}{2Q}. \]  \(3\)

Since any singular point of system \(1\) must be located on the \( x \)-axis, thus when a hyperelliptic curve \( F(x, y) = 0 \) contains a limit cycle of system \(1\), the limit cycle should intersect the \( x \)-axis at two different points, denoted by \((s_1, 0)\) and \((s_2, 0)\). The following properties hold.

**Lemma 2.3.** (i) \( s_1 \) and \( s_2 \) are real simple roots of \( Q(x) \). (ii) Any root of \( Q(x) \) must be a root of \( P(x) \).

**Lemma 2.4.** If \( s_1 \) and \( s_2 \) are simple roots of \( Q(x) \) and \( Q(x) > 0 \) in \((s_1, s_2)\), then the hyperelliptic curve \(2\) contains a closed curve in the strip \( s_1 \leq x \leq s_2 \).

Now one step further: assume that (i) the hyperelliptic curve \( F(x, y) = 0 \) contains a closed curve \( C \) in the strip \( s_1 \leq x \leq s_2 \), where \( s_1 \) and \( s_2 \) are simple roots of \( Q(x) \), (ii) this closed curve \( C \) surrounds only one singularity \((\alpha, 0)\) of system \(1\), (iii) the singularity \((\alpha, 0)\) is a focus or node. Then this closed curve \( C \) is a limit cycle.

We have the following criteria to recognize the type of the singular point.

**Lemma 2.5.** If \( g_n(\alpha) = 0, g'_n(\alpha) > 0 \) and \( f_m(\alpha) \neq 0 \), then \((\alpha, 0)\) is a focus or a node of system \(1\).
Combining all the known result, we give the following lemma which is very useful in determining if an algebraic curve is a hyperelliptic limit cycle of the Lienard system.

**Lemma 2.6.** An algebraic curve \( \{2\} \) in the strip \( x \in [s_1, s_2] \) is a hyperelliptic limit cycle if the following sufficient conditions are met.

(i) \( f_m \) and \( g_n \) satisfy \([3]\),
(ii) All the roots of \( Q(x) = 0 \) are real and \( s_1, s_2 \) are simple root and \( Q(x) > 0 \) for \( x \in (s_1, s_2) \).
(iii) \( P^2(x) - Q(x) < 0 \) for \( x \in (s_1, s_2) \).
(iv) If \( \alpha \in (s_1, s_2) \) such that \( Q'(\alpha) = 0 \), then \( f_m(\alpha) \neq 0 \).

Proof of Lemma: Condition (i) means that \( F = 0 \) is the invariant curve of the system, and all the roots of \( Q(x) \) are the roots of \( P(x) \). From (iii) we know that the curve \( F(x, y) = 0 \) in the strip bounded by \( x = s_1 \) and \( x = s_2 \) intersects the \( x \) axis only at these two endpoints. Condition (ii) means that \( Q'(s_i) \neq 0 \). It follows that the curve in the strip has no singular points and is closed. From (ii) we also know that \( Q(x) \) has only one real root \( \alpha \) for \( [s_1, s_2] \). Again, from (iii) we see that \( g_n(x) \) has a unique real root \( \alpha \). Therefore the system has only one singular point inside the closed orbit formed by \( F = 0 \) when restricted to the strip. We can even see that this singular point is either a focus type or a node. In fact, \( g_n'(\alpha) = Q''(\alpha) \cdot \frac{P^2(\alpha) - Q(\alpha)}{2Q(\alpha)} + Q'(\alpha)(\frac{P^2 - Q}{2Q})'(\alpha) \). Notice that the second term vanishes, and since \( \alpha \) is the maximal value point of \( Q \), therefore \( Q''(\alpha) < 0 \). It follows that \( g'(\alpha) > 0 \). Condition (iv) says that \( f_m(\alpha) \neq 0 \), therefore \( (\alpha, 0) \) is a focus or a node. Therefore the closed orbit is hyperelliptic limit cycle of the system.

### 2.2 Algorithm for root classification

Given a polynomial
\[
f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,
\]
we write the derivative of \( f(x) \) as
\[
f'(x) = na_0x^{n-1} + (n - 1)a_1x^{n-2} + \cdots + a_{n-1}.
\]
For the \( n \)-degree polynomial \( f(x) \), \( \alpha_1, \alpha_2, \cdots, \alpha_n \) denote all the roots of it.
Let \( s_p = \sum_{j=1}^{n} a^p_j \), \( p = 0, 1, 2, \cdots, n \), \( S_k = |s_{i+j}|, i, j = 0, 1, \cdots, k - 1 \), that is,

\[
S_k = \begin{vmatrix}
  s_0 & s_1 & \cdots & s_{k-1} \\
  s_1 & s_2 & \cdots & s_k \\
  \vdots & \vdots & \ddots & \vdots \\
  s_{k-1} & s_k & \cdots & s_{2k-2} \\
\end{vmatrix}.
\]  \hspace{1cm} (4)

**Definition 2.1.** (discrimination matrix) The Sylvester matrix of \( f(x) \) and \( f'(x) \), denoted by \( \text{Discr}(f) \)

\[
\begin{pmatrix}
  a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\
  0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & 0 & \cdots & 0 \\
  0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & 0 \\
  0 & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_n \\
  0 & 0 & 0 & \cdots & 0 & na_0 & \cdots & a_{n-1} \\
\end{pmatrix}
\]

is called the discrimination matrix of \( f(x) \).

**Definition 2.2.** (discriminant sequence) Denoted by \( D_k \), the determinant of the submatrix of \( \text{Discr}(f) \), formed by the first \( 2k \) rows and the first \( 2k \) columns, for \( k = 1, \cdots, n \). We call the \( n \)-tuple \( (D_1, D_2, \cdots, D_n) \) the discriminant sequence of polynomial \( f(x) \).

**Definition 2.3.** (sign list) we call the list

\[
[\text{sign}(D_1), \text{sign}(D_2), \cdots, \text{sign}(D_n)]
\]

the sign list of the discrimination sequence \( (D_1, D_2, \cdots, D_n) \).

**Definition 2.4.** (revised sign list) Given a sign list \([s_1, s_2, \cdots, s_n]\), we construct a new list \([\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]\) as follows:

- If \([s_1, s_2, \cdots, s_n]\) is a section of given list, where \( s_i \neq 0, s_{i+1} = \cdots = s_{i+j-1} = 0, s_{i+j} \neq 0 \), then we replace the subsection \([s_{i+1}, s_{i+2}, \cdots, s_{i+j-1}]\)
  by \([-s_i, -s_i, s_i, -s_i, -s_i, s_i, -s_i, \cdots]\).
  i.e. let \( \varepsilon_{i+r} = (-1)^{\frac{r-1}{2}} s_i \), for \( r = 1, 2, \cdots, j-1 \).
• Otherwise, let $\varepsilon_k = s_k$, there are no changes for other terms.

From [13], we already know the following lemma.

**Lemma 2.7.** For $k = 1, 2, \cdots, n$, we have $D_k = S_k$.

**Lemma 2.8.** Given a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ with real coefficients, if the number of the sign changes of the revised sign list of

$$\{D_1(f), D_2(f), \cdots, D_n(f)\}$$

is $v$, then the number of the pairs of distinct conjugate imaginary roots of $f(x)$ equals $v$. Furthermore, if the number of non-vanishing members of the revised sign list is $l$, then the number of the distinct real roots of $f(x)$ equals $l - 2v$.

3. The Proof of the Results about Lower Bounds

According to all the possible pairs $(m, n)$ where $m \geq 2$, we divide the proof into the following cases.

(i) $m + 2 \leq n \leq \left\lfloor \frac{4m+2}{3} \right\rfloor$;

(ii) $\left\lfloor \frac{4m+2}{3} \right\rfloor + 1 \leq n \leq 2m$ and $(m, n)$ is not in $\{(3, 5), (2, 4)\}$;

(iii) $n \geq 2m + 1$.

3.1 Case (i)

When $m + 2 \leq n \leq \left\lfloor \frac{4m+2}{3} \right\rfloor$, it suffices to construct a Liénard system of type $(m, n)$ which can have $n - m - 1$ hyperelliptic limit cycles on invariant curve [2].

Suppose $n$ is odd. Now let $t = \frac{4m-3n+3}{2}$. By Corollary 3.1 in [3], there exist a positive constant $c$ and a polynomial

$$Q_1(x) = (x - x_0)(x - 1) \prod_{i=1}^{t} (x - x_i)^2 \prod_{i=1}^{n-m-2} (x - y_i)^2,$$

such that

$$P_1(x) = Q_1(x) + c = \prod_{i=1}^{t} (x - z_i)^2 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i),$$
where \( x_0 < z_1 < x_1 < z_2 < \ldots < z_t < x_t \) and \( x_t < a_1 < b_1 < y_1 < a_2 < b_2 < y_2 < \ldots < a_{n-m-1} < b_{n-m-1} < 1 \). We set

\[
G(x) = (x - x_0)^2(x - 1)^2 \prod_{i=1}^{t} (x - x_i)^2 \prod_{i=1}^{n-m-2} (x - y_i)^2 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i),
\]

\[
P(x) = \sqrt{G(x)}P_1(x)
\]

\[
= (x - x_0)(x - 1)^3 \prod_{i=1}^{t} (x - x_i)(x - z_i) \prod_{i=1}^{n-m-2} (x - y_i)^4 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i),
\]

\[
Q(x) = G(x)Q_1(x)
\]

\[
= (x - x_0)^3(x - 1)^3 \prod_{i=1}^{t} (x - x_i)^4 \prod_{i=1}^{n-m-2} (x - y_i)^4 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i).
\]

then

\[
f_m(x) = P'(x) + \frac{P(x)Q'(x)}{2Q(x)}, \quad g_n(x) = \frac{Q'(x)(P^2(x) - Q(x))}{2Q(x)}
\]

are polynomials of degree \( m \) and \( n \) respectively.

We claim, for each \( i, j = 1, 2, \ldots, n - m - 1 \), when \( x \in [a_i, b_i] \), the closed curve given by (2) is a hyperelliptic limit cycle of the system.

1. In fact, it is easy to see that the condition(i), (ii), (iii) of Lemma 2.6 is satisfied.

2. Let us verify condition(iv) by contradiction. Assume \( Q'(x) \) and \( f_m(x) \) have a common root \( \alpha \) in \( (a_i, b_i) \), then \( P'(\alpha) = 0 \). With \( G(\alpha) \neq 0 \), then \( G'(\alpha) = P_1'(\alpha) = Q_1'(\alpha) = 0 \), and \( \left. \frac{G(x)}{P_1(x)Q_1(x)} \right|_{x=\alpha} = 0 \).

\[
\left. \left( \frac{G(x)}{P_1(x)Q_1(x)} \right)' \right|_{x=\alpha} = \frac{(x - x_0)(x - 1)}{\prod_{i=1}^{t}(x - z_i)^2} \left( \frac{1}{x - x_0} + \frac{1}{x - 1} - 2 \sum_{i=1}^{t} \frac{1}{x - z_i} \right),
\]

we have \( (G/P_1Q_1)'(\alpha) > 0 \), this leads to a contradiction.

By Lemma 2.6 we can prove the system has \( n - m - 1 \) hyperelliptic limit cycles.

Suppose \( n \) is even, let \( t = (4m - 3n + 2)/2 \). By Corollary 3.2 in [3], there exist a positive constant \( c \) and a polynomial

\[
Q_1(x) = (x - 1)^t \prod_{i=1}^{t} (x - x_i)^2 \prod_{i=1}^{n-m-1} (x - y_i)^2,
\]
such that

\[ P_1(x) = Q_1(x) + c = (x - x_0) \prod_{i=1}^{t} (x - z_i)^2 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i), \]

where \( x_0 < x_1 < z_1 < x_2 < \ldots < x_t < z_t \) and \( z_t < y_1 < a_1 < b_1 < y_2 < a_2 < b_2 < \ldots < y_{n-m-1} < a_{n-m-1} < b_{n-m-1} < 1 \). We set

\[ G(x) = (x - x_0)(x - 1)^2 \prod_{i=1}^{t} (x - x_i)^2 \prod_{i=1}^{n-m-1} (x - y_i)^2 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i), \]

\[ P(x) = \sqrt{G(x)}P_1(x) \]

\[ = (x - x_0)(x - 1)^2 \prod_{i=1}^{t} (x - x_i)(x - z_i) \prod_{i=1}^{n-m-1} (x - y_i) \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i), \]

\[ Q(x) = G(x)Q_1(x) \]

\[ = (x - x_0)(x - 1)^3 \prod_{i=1}^{t} (x - x_i)^4 \prod_{i=1}^{n-m-1} (x - y_i)^4 \prod_{i=1}^{n-m-1} (x - a_i)(x - b_i). \]

then

\[ f_m(x) = P'(x) + \frac{P(x)Q'(x)}{2Q(x)}, \quad g_n(x) = \frac{Q'(x)(P^2(x) - Q(x))}{2Q(x)} \]

are polynomials of degree \( m \) and \( n \) respectively.

We claim, for each \( i, i = 1, 2, \ldots, n-m-1 \), when \( x \in [a_i, b_i] \), the closed curve given by (2) is a hyperelliptic limit cycle of the system.

1. In fact, it is easy to see that the condition(i), (ii), (iii) of Lemma 2.6 is satisfied.

2. Let us verify condition(iv) by contradiction. Assume \( Q'(x) \) and \( f_m(x) \) have a common root \( \alpha \) in \( (a_i, b_i) \). Analogous the argument above, we can get \( \alpha \) is a root of \( (G/(P_1Q_1))' \), while

\[ \left( \frac{G(x)}{P_1(x)Q_1(x)} \right)' = \frac{(x - 1)}{\prod_{i=1}^{t} (x - z_i)^2} \left( \frac{1}{x - 1} - 2 \sum_{i=1}^{t} \frac{1}{x - z_i} \right), \]

for each \( i \), we can observe \( x_0 < z_i < \alpha < 1 \), thus \( (G/P_1Q_1)'(\alpha) > 0 \), this leads to a contradiction.

By Lemma 2.6 we can prove the system has \( n - m - 1 \) hyperelliptic limit cycles.
3.2 Case (ii)

Now we come to case (ii), when $\left\lfloor \frac{4m+2}{3} \right\rfloor +1 \leq n \leq 2m−1$, we shall construct a Liénard system \([1]\) that can have $\left\lfloor \frac{n−1}{4} \right\rfloor$ hyperelliptic limit cycles on invariant curve \([2]\), from which we can infer that $H(m, n) \geq \left\lfloor \frac{n−1}{4} \right\rfloor$.

In the proof of case (i), we perturbed the polynomial with a constant to transform repeated roots into single roots, but this perturbation doesn't work in case (ii). To prove case (ii), firstly we divide the case (ii) into the following cases:

(ii-i) $n−1 \equiv 0$ (mod 4);

(ii-ii) $n−1 \equiv 1$ (mod 4);

(ii-iii) $n−1 \equiv 2$ or $n−1 \equiv 3$ (mod 4);

Case (ii-i): $n−1 \equiv 0$ (mod 4)

Lemma 3.1. For $h, l \in N$, define the polynomial

$$Q_1(x) = (x−s)x^{2h+1} \prod_{i=1}^{l}(x−i)^2,$$

where $s > l+1$, then there exists a polynomial $c(x)$ of degree $2h$ which is positive in $[0, s]$ and such that

$$Q_1(x) + c(x) = (x−y_{l+1}) \prod_{i=1}^{l}(x−y_i)(x−z_i) \prod_{i=1}^{2h+1}(x−x_i)$$

where $0 < x_1 < x_2 < \cdots < x_{2h+1} < y_1, y_1 < 1 < z_1 < y_2 < \cdots < z_l < y_{l+1} < s$.

**Proof:** We prove this lemma by mathematical induction. For $h = 0$, let $c(x)$ be a positive constant $\epsilon$. It easily follows that the proposition for $h = 0$ holds, if $\epsilon$ is sufficiently small. Assume the proposition holds for $h = k$, it must been shown that the proposition holds for $h = k + 1$.

Decompose $Q_1(x)$ into two fractions $x^2$ and $Q_1^*(x)$, then 0 is a repeated root of degree of $2k + 1$ of $Q_1^*(x)$. Using the induction hypothesis, there exists a polynomial $c^*(x)$ of degree $2k$ which is positive in $[0, s]$, (we can choose $c^*(x)$ which satisfied the maximum absolute value of its coefficients is sufficient small) and such that

$$Q_1(x) + x^2c^*(x) = x^2(Q_1^*(x) + c^*(x))$$

$$= x^2(x−y_{l+1}) \prod_{i=1}^{l}(x−y'_i)(x−z'_i) \prod_{i=3}^{2k+3}(x−x'_i),$$

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where \( 0 < x'_3 < \cdots < x'_{2k+3} < y'_1, \ y'_1 < 1 < z'_1 < y'_2 < \cdots < z'_l < y_{l+1} < s. \)

Choose a sufficiently small \( d \) which satisfied \( d > 0 \) and \( xc^*(x) - d \) has only one root \( \alpha \ll 1 \) in \([0, s]\). For the maximum absolute value of coefficients of \( c^*(x) \) is sufficient small, the local maximum of \( Q_1(x) + x^2c^*(x) \) in \((0, x'_3)\) is the least maximum among all the maxima of \( Q_1(x) + x^2c^*(x) \) in \([0, s]\). Perturbing \( Q_1(x) + x^2c^*(x) \) with \(-dx\), we get a polynomial

\[
Q_1(x) + x^2c^*(x) - dx = x(x - y''_1)\prod_{i=1}^{l}(x - y''_i)(x - z'_{i}) \prod_{i=2}^{2k+3}(x - x''_i),
\]

where \( 0 < x''_2 < x''_3 < \cdots < x''_{2k+3} < y''_1, \ y''_1 < z''_1 < y''_2 < \cdots < z''_l < y''_{l+1}. \)

Since \( \alpha \ll 1 \) is the only root of \( xc^*(x) - d \) in \([0, s]\), we have \( Q_1(s) + s^2c^*(s) - ds > 0 \) and \( Q_1(i) + i^2c^*(i) - di > 0, \) \( 1 \leq i \leq l \), then \( y''_{l+1} < s \) and \( y''_i < i < z''_i + 1, \) \( 1 \leq i \leq l. \) When \( 0 < x < \alpha \), we have \( x^2c^*(x) - dx < 0, \) while \( x''_i \) is the root of \( Q_1(x) + x^2c^*(x) - dx, \) for \( Q_1(x''_2) < 0, \) then \( \alpha < x''_2. \)

Assume \( \gamma \) is minimum point of \( x^2c^*(x) - dx \) in \([0, s]\), then \( 0 < \gamma < \alpha < x''_2. \) Choose \( b > 0 \) satisfying \( \gamma^2c^*(\gamma) - d\gamma + b > 0, \) \( Q_1(\gamma) + \gamma^2c^*(\gamma) - d\gamma + b < 0. \) (The existence of \( b \) relies on \( Q_1(\gamma) < 0. \)) Now we start to proof all roots of \( Q_1(x) + x^2c^*(x) - dx + b \) are real. Assume \( \beta \) is minimum point of \( Q_1(x) + x^2c^*(x) - dx + b \) in \([0, x''_2]\), we obtain \( Q_1(\beta) + \beta^2c^*(\beta) - d\beta + b \leq Q_1(\gamma) + \gamma^2c^*(\gamma) - d\gamma + b < 0. \)

For \( d \) is sufficiently small, the local minimum \( Q_1(\beta) + \beta^2c^*(\beta) - d\beta + b \) in \((0, x''_2]\) is the largest minimum among all the minima of \( Q_1(x) + x^2c^*(x) - dx \) in \([0, s]\), we know that all roots of \( Q_1(x) + x^2c^*(x) - dx + b \) are real.

Perturbing \( Q_1(x) + x^2c^*(x) - dx \) with \( b \), we get a polynomial

\[
Q_1(x) + x^2c^*(x) - dx + b = (x - y_{l+1})\prod_{i=1}^{l}(x - y_i)(x - z_i) \prod_{i=1}^{2k+3}(x - x_i),
\]

where \( 0 < x_1 < x_2 < \cdots < x_{2k+3} < y_1, \ y_1 < y''_1 < z'_1 < z_1 < y_2 < \cdots < z_l < y_{l+1} < y''_l. \) For \( y''_{l+1} < s \) and \( y''_i < i < z''_i + 1, \) \( 1 \leq i \leq l, \) we have \( 0 < x_1 < x_2 < \cdots < x_{2k+3} < y_1, y_1 < 1 < z_1 < y_2 < \cdots < z_l < y_{l+1} < s. \) On the other hand, we know the degree of \( c(x) = x^2c^*(x) - dx + b \) is \( 2k + 2, \) and \( c(x) \geq \gamma^2c^*(\gamma) - d\gamma + b > 0 \) in \([0, s]\). This completes the proof of lemma.

Denote \( \frac{m-1}{4} = t \), We set

\[
Q_1(x) = (x - 2m + 2t)x^{6t-2m+1} \prod_{i=1}^{m-2t-1}(x - i)^2,
\]

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by Lemma 3.1, we can perturb \(Q_1(x)\) with a polynomial \(c(x)\) of degree 
\(6t - 2m\) which is positive in \([0, s]\), then

\[
P_1(x) = Q_1(x) + c(x) = \prod_{i=1}^{t} (x - a_i)(x - b_i),
\]

where \(0 < a_1 < b_1 < \cdots < a_{3t-m+1},\ a_{3t-m+1} < b_{3t-m+1} < 1 < a_{3t-m+2} < \cdots < m - 2t + 1 < a_t < b_t < 2m - 2t\). We define

\[
G(x) = \prod_{i=1}^{t} (x - a_i)(x - b_i) \prod_{i=0}^{m-2t-1} (x - i)^2(x - 2m + 2t)^2.
\]

\[
P(x) = \sqrt{G(x)P_1(x)}, \quad Q(x) = G(x)Q_1(x),
\]

then

\[
P(x) = (x - 2m + 2t) \prod_{i=1}^{t} (x - a_i)(x - b_i) \prod_{i=0}^{m-2t-1} (x - i),
\]

\[
Q(x) = (x - 2m + 2t)^3x^{6t-2m+3} \prod_{i=1}^{t} (x - a_i)(x - b_i) \prod_{i=1}^{m-2t-1} (x - i)^4,
\]

and

\[
f_m(x) = P'(x) + \frac{P(x)Q'(x)}{2Q(x)}, \quad g_n(x) = \frac{Q'(x)(P^2(x) - Q(x))}{2Q(x)}
\]

are polynomials of degree \(m\) and \(n\) respectively.

We claim, for each \(i, i = 1, 2, ..., t\), when \(x \in [a_i, b_i]\), the closed curve given by (2) is a hyperelliptic limit cycle of the system.

1. In fact, it is easy to see that the condition (i), (ii), (iii) of Lemma 2.6 is satisfied.

2. Let us verify condition (iv) by contradiction. Assume \(Q'(x)\) and \(f_m(x)\) have a common root \(a\) in \((a_i, b_i)\).

Suppose \(6t - 2m = 0\), then \(n = \frac{4m+3}{3}\), which is possible when \(\frac{4m+3}{3} = \lfloor \frac{4m+2}{3} \rfloor + 1\).

For \(G(a) \neq 0\), we get \(\alpha\) is the common root of \(P_1', Q_1', G'\) and \((G/(P_1Q_1))'\), then

\[
\left( \frac{G}{P_1Q_1} \right)' = \frac{x^2(x - 2m + 2t)/x^{6t-2m+1}}{2} = 2x - 2m + 2t,
\]

thus \(\alpha = m - t\), but it is impossible for \(Q_1'(m - t) < 0\) and this leads to a contradiction.
By Lemma 2.6, we can prove the system has $t$ hyperelliptic limit cycles.

On the other hands, $6t - 2m > 0$, for $6t - 2m$ is even, then $6t - 2m \geq 2$.
With $f_m(\alpha) = Q'(\alpha) = 0$, then $P'(\alpha) = 0$. For $G(\alpha) \neq 0$, and $\frac{P}{Q_1} = \frac{Q^2}{Q}$, we get
\[ \left( \frac{Q}{P} \right)' \bigg|_{x=\alpha} = \left( \frac{P}{Q_1} \right)' \bigg|_{x=\alpha} = 0. \]
Since $\frac{Q(x)}{P(x)} = x^{6t-2m+2}(x-2m+2t)^2 \prod_{i=1}^{m-2t-1} (x-i)^3$,
we know $\alpha$ is irrelevant of $c(x)$. Differentiating $P_1/Q_1$, we have
\[ \left( \frac{P_1(x)}{Q_1(x)} \right)' = \frac{c'(x)Q_1(x) - Q_1'(x)c(x)}{Q_1^2(x)} \quad (5) \]
For $Q_1(\alpha) \neq 0$, we have
\[ \frac{c'(\alpha)}{c(\alpha)} = \frac{Q_1'(\alpha)}{Q_1(\alpha)}. \quad (6) \]
With the degree of $c(x)$ is more than 2 and the right side of (6) is irrelevant of $c(x)$, we can change the polynomial coefficients of $c(x)$ to make the left hand side of (6) doesn’t equal the right hand side, such that the root of (5) in $(a_j, b_j)$ is different to the root of equation $(Q/P)'$. Therefore, such $\alpha$ doesn’t exist and this verifies condition(iv).

By Lemma 2.6 we prove the system has $t$ hyperelliptic limit cycles. This completes the proof of the case $n - 1 \equiv 0 \pmod{4}$.

**Case (ii-ii):** $n - 1 \equiv 1 \pmod{4}$

For the proof of Lemma 3.2 is similar to Lemma 3.1 we omit it.

**Lemma 3.2.** For $h \in N^+, l \in N$, define the polynomial
\[ Q_1(x) = (x-s_1)(x-s_2)x^{2h} \prod_{i=1}^{l}(x-i)^2, \]
where $s_1 < -1, s_2 > l + 1$, then there exists a polynomial $c(x)$ of degree $2h - 1$ which is positive in $[s_1, s_2]$ and such that
\[ Q_1(x) + c(x) = (x-z_{-1})(x-y_{l+1}) \prod_{i=1}^{l}(x-y_i)(x-z_i)^{2h} \prod_{i=1}^{l}(x-x_i) \]
where $s_1 < z_{-1} < x_1 < 0 < x_2 < x_3 < \cdots < x_{2h} < y_1, y_1 < 1 < z_1 < y_2 < \cdots < z_l < y_{l+1} < s_2$. 

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Denote $\frac{s-2}{4} = t$. We set
\[
Q_1(x) = (x + 2)(x - s)x^{6t-2m+2} \prod_{i=1}^{m-2t-2} (x - i)^2,
\]
where $s >> m - 2t - 2$. Since $6t - 2m + 2 \geq 2$, by lemma 3.2 we can perturb $Q_1(x)$ with a polynomial $c(x)$ of degree $6t - 2m + 1$ which is positive in $[-2, s]$, then
\[
P_1(x) = Q_1(x) + c(x) = \prod_{i=1}^t (x - a_i)(x - b_i),
\]
where $-2 < a_1 < b_1 < 0 < a_2 < \cdots < a_{3t-m+2} < b_{3t-m+2}, b_{3t-m+2} = 1 < a_{3t-m+3} < b_{3t-m+3} < 2 < a_{3t-m+4} < \cdots < m - 2t - 2 < a_t < b_t < s$. Define
\[
G(x) = \prod_{i=1}^t (x - a_i)(x - b_i) \prod_{i=0}^{m-2t-2} (x - i)^2(x + 2)^2(x - s)^2,
\]
\[
P(x) = \sqrt{G(x)}P_1(x), \quad Q(x) = G(x)Q_1(x),
\]
we have
\[
P(x) = (x + 2)(x - s) \prod_{i=1}^t (x - a_i)(x - b_i) \prod_{i=0}^{m-2t-2} (x - i),
\]
\[
Q(x) = (x + 2)^3(x - s)^3x^{6t-2m+4} \prod_{i=1}^t (x - a_i)(x - b_i) \prod_{i=1}^{m-2t-2} (x - i)^4.
\]
then
\[
f_m(x) = P'(x) + \frac{P(x)}{2Q(x)} Q'(x), \quad g_n(x) = \frac{Q'(x)(P^2(x) - Q(x))}{2Q(x)},
\]
are polynomials of degree $m$ and $n$ respectively.

We claim, for each $i, i = 1, 2, \ldots, t$, when $x \in [a_i, b_i]$, the closed curve given by [2] is a hyperelliptic limit cycle of the system.

1. In fact, it is easy to see that the condition (i), (ii), (iii) of Lemma 2.6 is satisfied.

2. Let us verify condition (iv) by contradiction. Assume $Q'(x)$ and $f_m(x)$ have a common root $\alpha$ in $(a_i, b_i)$, then $P'(\alpha) = 0$. Note that $\frac{P'}{Q'} = \frac{f_m}{Q}$, and $G(\alpha) \neq 0$, we get $(\frac{Q'}{P'})|_{x=\alpha} = (\frac{P'}{Q'})|_{x=\alpha} = 0$. Since
\[
Q(x) = x^{6t-2m+3}(x + 2)^2(x - s)^2 \prod_{i=1}^{m-2t-2} (x - i)^3,
\]
we get $\alpha$ is irrelevant of $c(x)$ immediately. Differentiating $P_1/Q_1$, then
\[
\left(\frac{P_1(x)}{Q_1(x)}\right)' = \frac{c'(x)Q_1(x) - Q_1'(x)c(x)}{Q_1^2(x)},
\]
for $Q_1(\alpha) \neq 0$, we have
\[
\frac{c'(\alpha)}{c(\alpha)} = \frac{Q_1'(\alpha)}{Q_1(\alpha)}.
\]
While $6t - 2m + 1 > 0$, the degree of $c(x)$ is more than 1 and the right side of (8) is irrelevant of $c(x)$, we can change the polynomial coefficients of $c(x)$ to make the left hand side of (8) doesn’t equal the right hand side, such that the root of (7) in $(a_j, b_j)$ is different to the root of equation $(Q/P)'$. Therefore, such $\alpha$ doesn’t exist and this verifies condition(iv).

By Lemma 2.6, we can prove the system has $t$ hyperelliptic limit cycles. Since $t = \frac{n-2}{4} = \lfloor \frac{n-1}{4} \rfloor$, we complete the proof.

**Case (ii-iii):** $n - 1 \equiv 2$ or $n - 1 \equiv 3$ (mod 4)

**Lemma 3.3.** If a Liénard system of $(m,n)$-type has $t$ hyperelliptic limit cycles on invariant curve $(y+P(x))^2 - Q(x) = 0$ and for each limit cycle the conditions of Lemma 2.6 are met, then there exists Liénard system of $(m+1,n+2)$-type with at least $t$ hyperelliptic limit cycles.

**Proof.** It suffices that, based on the system in the assumption, we construct a new Liénard system of $(m+1,n+2)$-type in the form of
\[
\dot{x} = y, \quad \dot{y} = -f_{m+1}(x)y - g_{n+2}(x)
\]
with the same number of hyperelliptic limit cycles. We take $\tilde{P}_s(x)$ and $\tilde{Q}_s(x)$ in the form
\[
\tilde{P}_s(x) = P(x)(x-s), \quad \tilde{Q}_s(x) = Q(x)(x-s)^2.
\]
Changing $P(x)$ and $Q(x)$ in equation (3) to $\tilde{P}_s(x)$ and $\tilde{Q}_s(x)$ respectively, we get
\[
f_{m+1}(x) = \tilde{P}'_s(x) + \frac{\tilde{P}_s(x)\tilde{Q}'_s(x)}{2\tilde{Q}_s(x)}, \quad g_{n+2}(x) = \frac{\tilde{Q}'_s(x)(\tilde{P}^2_s(x) - \tilde{Q}_s(x))}{2\tilde{Q}_s(x)}.
\]
Note they are polynomials of $m + 1$, $n + 2$ degree respectively.

Consider a hyperelliptic limit cycle of the original system on the invariant curve (2) that intersect with $x$-axis on $a_1$ and $b_1$. We claim there exists a
sufficient large \( s_1 \), which satisfied the closed curve with \( x \in [a_1, b_1] \) on invariant curve \((y + \tilde{P}_{s_1}(x))^2 - \tilde{Q}_{s_1}(x) = 0\) is a hyperelliptic limit cycle of the new system.

We observe that the condition (i), (ii) and (iii) of Lemma 2.6 are trivially verified when \( s \) is larger than all the roots of \( Q(x) \). Then we just have to consider condition (iv). Differentiating \( \tilde{Q}_s(x) \), we get

\[
\tilde{Q}_s'(x) = (x - s)^2(Q'(x) + \frac{2}{x - s}Q(x)).
\]

It follows that, \( \tilde{\alpha}_s \to \alpha \) as \( s \to \infty \), where \( \alpha \) and \( \tilde{\alpha}_s \) denote the root of \( Q'(x) \) and \( \tilde{Q}_s'(x) \) in \((a_1, b_1)\) respectively. Differentiating \( \tilde{P}_s(x) \), we get

\[
\tilde{P}_s'(x) = (x - s)(P'(x) + \frac{1}{x - s}P(x)).
\]

Hence, \( \tilde{P}_s'(\tilde{\alpha}_s)/(\tilde{\alpha}_s - s) \to P'((\tilde{\alpha}_s) \to P'(\alpha) \) as \( s \to \infty \). Furthermore, \( P'(\alpha) \neq 0 \) which follows from the assumption that \( f'_m(\alpha) \neq 0 \). Thus, we can find a sufficient large \( s_1 \) satisfied \( \tilde{P}_{s_1}'(\tilde{\alpha}_{s_1}) \neq 0 \) to make \( f_{m+1}(\tilde{\alpha}_{s_1}) \neq 0 \). By Lemma 2.6, we can prove the system

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -f_{m+1}(x)y - g_{n+2}(x)
\end{align*}
\]

has at least \( t \) hyperelliptic limit cycles, where

\[
\begin{align*}
f_{m+1}(x) &= \tilde{P}_{s_1}'(x) + \frac{\tilde{P}_{s_1}(x)\tilde{Q}_{s_1}'(x)}{2\tilde{Q}_{s_1}(x)}, \\
g_{n+2}(x) &= \frac{\tilde{Q}_{s_1}'(x)(\tilde{P}_{s_1}'(x) - \tilde{Q}_{s_1}(x))}{2\tilde{Q}_{s_1}(x)}.
\end{align*}
\]

This completes the proof of the lemma.

**Proof of the case \( n - 1 \equiv 2 \pmod{4} \):** Suppose \((m - 1, n - 2)\) is still in case (ii), then \((m - 1, n - 2)\) is in the case (ii-i), use the above argument, we have a Liénard system of \((m - 1, n - 2)\)-type that has \( \lceil \frac{n - 3}{4} \rceil \) hyperelliptic limit cycles. Since \( n - 1 \equiv 2 \pmod{4} \), we have \( \lceil \frac{n - 3}{4} \rceil = \lceil \frac{n - 1}{4} \rceil \). By the argument of Lemma 3.3, we can construct a new Liénard system of \((m, n)\)-type with \( \lceil \frac{n - 1}{4} \rceil \) hyperelliptic limit cycle based on the system of \((m - 1, n - 2)\)-type.

On the other hand, \((m - 1, n - 2)\) is in case (i), then \( \lceil \frac{4m + 3}{3} \rceil = \lceil \frac{4m + 5}{3} \rceil = n \), but \( n - 1 \equiv 2 \pmod{4} \), which yields a contradiction. Therefore \((m - 1, n - 2)\) can only in case (ii), this completes the proof.

**Proof of the case \( n - 1 \equiv 3 \pmod{4} \):** Suppose \((m - 1, n - 2)\) is still in case (ii), then \((m - 1, n - 2)\) is in the case (ii-i), use the above argument, we have a Liénard system of \((m - 1, n - 2)\)-type that has \( \lceil \frac{n - 3}{4} \rceil \) hyperelliptic limit cycles. Since \( n - 1 \equiv 3 \pmod{4} \), we have \( \lceil \frac{n - 3}{4} \rceil = \lceil \frac{n - 1}{4} \rceil \). By the argument of
Lemma 3.3, we can construct a new Liénard system of \((m, n)\)-type with at least \(\left\lceil \frac{n-1}{4} \right\rceil\) hyperelliptic limit cycles based on the system of \((m-1, n-2)\)-type, thus \(H(m, n) \geq \left\lceil \frac{n-1}{4} \right\rceil\).

On the other hand, \((m-1, n-2)\) is in case (i), we can construct a Liénard system of \((m-1, n-2)\) type that has \(n - m - 2\) hyperelliptic limit cycles. For \((m-1, n-2)\) is in case (i), and \(n-1 \equiv 3 \pmod{4}\), we have \(n-2 = \left\lceil \frac{n-1}{4} \right\rceil\).

This completes the proof of the case \(n-1 \equiv 3 \pmod{4}\).

When \(n = 2m\), we define

\[ P(x) = \prod_{i=1}^{m} (x - i)(x + s) \quad Q(x) = \prod_{i=1}^{m} (x - i)(x + s)^{m+2}, \]

where \(s \gg m\) is sufficiently large. If \(m\) is odd, for each \(i = 1, 2, \ldots, \frac{m-1}{2}\), when \(x \in [2i-1, 2i]\), the closed curve given by (2) is a hyperelliptic limit cycle of the system, therefore \(H(m, 2m) \geq \frac{m-1}{2} = \left\lceil \frac{2m-1}{4} \right\rceil\).

On the other hand, \(m\) is even, for each \(i = 1, 2, \ldots, \frac{m-2}{2}\), when \(x \in [2i, 2i+1]\), the closed curve given by (2) is a hyperelliptic limit cycle of the system, therefore \(H(m, 2m) \geq \frac{m-2}{2} = \left\lceil \frac{2m-1}{4} \right\rceil\).

3.3 Case (iii)

We set

\[ P(x) = \prod_{i=1}^{m} (x - i)(x + s), \]

\[ Q(x) = -s \prod_{i=1}^{m} (x - i)(x + s)^{n-m+1}, \]

where \(s \gg m\) is sufficiently large, we take \(f_m(x)\) and \(g_n(x)\) in system (1) in the form of equation (3). It is easy to see \(f_m(x)\) and \(g_n(x)\) are polynomials of degree \(m\) and \(n\) respectively.

Suppose \(m\) is even. We claim, for each \(i = 1, 2, \ldots, \frac{m}{2}\), when \(x \in [2i-1, 2i]\), the closed curve given by (2) is a hyperelliptic limit cycle of the system.

1. In fact, it is easy to see that the condition(i), (ii), (iii) of Lemma 2.6 is satisfied.

2. Let us verify condition(iv) by contradiction. Assume \(Q'(x)\) and \(f_m(x)\) have a common root \(\alpha\) in \((2i-1, 2i)\), then \(P'(\alpha) = 0\). With

\[ R'(x) = \left( \frac{Q(x)}{P(x)} \right)' = -s(n-m)(x+s)^{n-m-1}, \]

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we would have \( R'(\alpha) = 0 \), but \( R'(x) \) only have one root \(-s\), this leads to a contradiction.

By Lemma 2.6 we can prove the system has \( \frac{m}{2} \) hyperelliptic limit cycles.

Suppose \( m \) is odd. In an analogous way, when \( x \in [2i, 2i+1] \), we can prove the closed curve given by (2) is a hyperelliptic limit cycle of the system for each \( i = 1, 2, \ldots, \frac{m-1}{2} \).

Therefore, we obtain \( H(m, n) \geq \left\lceil \frac{m}{2} \right\rceil \), when \( m \geq 2 \) and \( n \geq 2m+1 \).

4. Configuration Of Hyperelliptic Limit Cycles

Lemma 4.1. If an \((m, n)\)-Lienard system (1) has a hyperelliptic curve
\[
(y + P(x))^2 - Q(x) = 0,
\]
where \( n \neq 2m + 1 \), then the system only has this one hyperelliptic curve.

Proof. From equation (3), we have
\[
2Q(x)f_m(x) = 2Q(x)P'(x) + P(x)Q'(x),
\]
and
\[
2Q(x)g_n(x) = Q'(x)(P^2(x) - Q(x)).
\]
Therefore, we know that the degree of \( P(x) \) is \( m+1 \), while the degree of \( P^2(x) - Q(x) \) is \( n+1 \). Let \( f_m(x) \) and \( g_n(x) \) take the form
\[
f_m(x) = \sum_{i=0}^{m} a_i x^i, \quad g_n(x) = \sum_{i=0}^{n} b_i x^i.
\]
If \( n > 2m + 1 \), the degree of \( P^2(x) - Q(x) \) equals the degree of \( Q(x) \). Let us denote \( P(x) \) and \( Q(x) \) by
\[
P(x) = \sum_{i=0}^{m+1} p_i x^i, \quad Q(x) = \sum_{i=0}^{n+1} q_i x^i.
\]
then the coefficients of the highest degree terms of the each side of equations (9) and (10) are:
\[
2a_m q_{n+1} = (2m + n + 3)p_{m+1}q_{n+1}, \quad 2q_{n+1}b_n = -(n + 1)q_{n+1}^2.
\]
Thus \( p_{m+1} = \frac{2a_m}{2m + n + 3} \) and \( q_{n+1} = \frac{-2b_n}{n + 1} \) are uniquely determined.
Comparing the coefficients of the second highest degree terms of the polynomials on each side of equations (9) and (10) respectively, we have

\[2a_m q_n + 2a_{m-1} q_{n+1} = (2m + 2 + n)p_{m+1} q_n + (2m + n + 1)p_m q_{n+1},\]

\[2b_n q_n + 2b_{n-1} q_{n+1} = -(2n + 1)q_n q_{n+1} + (n + 1)q_{n+1}c,\]

where \(c = 0\) or \(c = p_{m+1}^2\). For the coefficients of \(p_m\) and \(q_n\) in the linear equations mentioned above which derive from comparing the coefficients of the second highest degree terms of the polynomials on each side of equations (9) and (10) are \((2m + n + 1)q_n q_{n+1} + (2m + n + 1)\) respectively, we have the values of \(p_m\) and \(q_n\) are uniquely defined.

More generally, by comparing the coefficients of \(x^{n+i}\) and \(x^{2n+i-m}\) of the equation (9) and (10) respectively, we can get the values of \(p_i\) and \(q_{n+i-m}\) are uniquely defined, where \(i = 0, 1, \ldots, m - 1\). We also can derive the value of \(q_j\) is uniquely defined, where \(j = 1, 2, \ldots, n - m - 1\).

For the value of \(q_0\), We compare the coefficients of \(x^{2m+1}\) and \(x^{n+2m+2}\) of the equation (10) respectively, we have

\[2q_{2m+1} b_0 + \cdots + 2q_0 b_{2m+1} = ((2m + 2)q_{2m+2}(p_{m+1}^2 - q_0) + \cdots + q_1(2p_m p_{m+1} - q_{2m+2})) \quad (11)\]

\[2b_{2m+1} q_{n+1} + \cdots + 2b_n q_{2m+2} = ((n + 1)q_{n+1}(p_{m+1}^2 - q_{2m+2}) + \cdots + (-2m - 2)q_{2m+2} q_{n+1}) \quad (12)\]

the coefficient of \(q_0\) in the linear equation (11) is \(2b_{2m+1} + (2m + 2)q_{2m+2}\), while \(2b_{2m+1} + (2m + 2)q_{2m+2} = (n + 1)p_{m+1}^2 \neq 0\) which derive from the equation (12). Therefore the value of \(q_0\) is uniquely defined. Finally the polynomial \(P(x)\) and \(Q(x)\) are determined, we complete the proof of the lemma in the case \(n > 2m + 1\).

If \(n < 2m + 1\), the degree of \(P^2(x) - Q(x)\) is smaller than the degree of \(P^2(x)\). Thus, the coefficients of some higher terms of \(P^2(x)\) and \(Q(x)\) are same, namely,

\[(P^2)^{(n+i)}(x) = (Q)^{(n+i)}(x), \quad 2 \leq i \leq 2m + 2 - n. \quad (13)\]

Let us separate \(P(x)\) and \(Q(x)\) in the form

\[P(x) = \sum_{i=0}^{m+1} p_i x^i, \quad Q(x) = \sum_{i=0}^{2m+2} q_i x^i.\]
Comparing the coefficients of the highest degree terms of polynomials on each side of equation (9) and (13), we have

\[ 2a_m q_{2m+2} = (4m + 4)p_{m+1} q_{2m+2}, \quad p^2_{m+1} = q_{2m+2}. \]

Thus \( p_{m+1} = \frac{a_m \cdot 2}{2m+1} \) are uniquely defined.

More generally, by comparing the coefficients of \( x^{3m+2-i} \) and \( x^{2m+2-i} \) of the equation (9) and (13) respectively, we have \( p_{m+1-i} = \frac{a_{m-i}}{2(m+1-i)} \) and the value of \( q_{2m+2-i} \) is uniquely defined, where \( 0 \leq i \leq 2m-n \).

Then we compare the coefficients of \( x^{n+m+1} \) and \( x^{n+2m+2} \) of the equation (9) and (10), we have \( p_{n-m} = \frac{a_{n-m-1}}{2(n-m)} + \frac{(n-2m-1)b_n}{2(n-m)a_m} \), and the value of \( q_{n+1} \) is uniquely defined. Repeating the above process, we derive that \( p_{n-m-2} \ldots p_1 \) and \( q_{n-1} \ldots q_{m+2} \) are uniquely defined.

For the value of \( p_0 \), we can compare the coefficient of \( x^n \) of equation (9), then we have a linear equation for \( p_0 \) while the coefficient of \( p_0 \) can only be \(-b_n \) or \( \frac{n-4m-3}{m+1}b_n \), therefore the value of \( p_0 \) is uniquely defined, then the values of \( q_{m+1}, q_{m+2}, \ldots, q_0 \) which are depend on the value of \( p_0 \) are uniquely defined. Finally the polynomial \( P(x) \) and \( Q(x) \) are determined, we complete the proof of the lemma.

We know from the above discuss, if an \((m, n)\)-Lienard system \( \Pi \), where \( n \neq 2m+1 \), has a hyperelliptic curve \( (y + P(x))^2 - Q(x) = 0 \), then the system can only has this hyperelliptic curve. Thus, there are at most two points in the hyperelliptic limit cycles of the system when we fix the value of \( x \) which means no hyperelliptic limit cycle can contained other hyperelliptic limit cycle. Therefore, the hyperelliptic limit cycles only have non-nested configuration. (see Fig.2)

5. The Proof of the Results about Upper Bounds

By the argument of Lemma 4.1 we know a system \( \Pi \) in the case \( n \neq 2m+1 \) has a hyperelliptic curve \( (y + P(x))^2 - Q(x) = 0 \), then the system can only has
this hyperelliptic curve. Take the polynomial \( P, Q \) of the hyperelliptic curve in the form

\[
P(x) = \prod_{i=1}^{a} (x - x_i)^{\alpha_i+1} \prod_{j=1}^{b} (x - y_j)^{\beta_j+1} \prod_{l=1}^{c} (x - z_l)^{\gamma_l+1}
\]

\[
Q(x) = \prod_{i=1}^{a} (x - x_i) \prod_{j=1}^{b} (x - y_j)^{\omega_j+2},
\]

where \( a, b, c, \alpha_i, \beta_j, \gamma_l, \omega_j \geq 0, \alpha = \sum_{i=1}^{a} \alpha_i, \beta = \sum_{j=1}^{b} \beta_j, \gamma = \sum_{l=1}^{c} \gamma_l, \) and \( \omega = \sum_{j=1}^{b} \omega_j. \) We set \( x_i \neq y_j \neq z_l, x_1 \neq x_2 \neq \cdots \neq x_a, y_1 \neq y_2 \neq \cdots \neq y_b \) and \( z_1 \neq z_2 \neq \cdots \neq z_c. \) If \( 2\beta_i > \omega_i, i = 1, 2, \cdots, b, \) then we replace \( \beta_i \) and \( \omega_i \) with \( \beta_i^- \) and \( \omega_i^- \) respectively, and use \( b^- \) denotes the number of \( i \) which satisfied \( 2\beta_i > \omega_i. \) Otherwise, we replace \( \beta_i \) and \( \omega_i \) with \( \beta_i^+ \) and \( \omega_i^+ \), and use \( b^+ \) denotes the number of \( i \) which satisfied \( 2\beta_i \leq \omega_i, \) then \( b = b^+ + b^-, \omega = \omega_i^+ + \omega_i^-, \beta = \beta_i^+ + \beta_i^- \).

For proving the result of upper bounds, firstly, we discuss the case \( m + 2 \leq n \leq 2m - 2. \)

If \( F(x, y) = (y + P(x))^2 - Q(x) = 0 \) is an invariant algebraic curve of system (1), it is necessary that \( P(x) \) has degree \( m + 1, \) and \( P^2(x) - Q(x) \) has degree \( n + 1, \) thus \( a + b + c + \alpha + \beta + \gamma = m + 1, a + 2b + \omega = 2m + 2, \) and \( \ln(P^2/Q) = O(x^{n-2m-1}), \) which implies that

\[
\sum_{i=1}^{a} (2\alpha_i + 1)x_i^j + \sum_{i=1}^{b^-} (2\beta_i^- - \omega_i^-)y_i^j + \sum_{i=1}^{c} (2\gamma_i + 2)z_i^j = \sum_{i=1}^{b^+} (\omega_i^+ - 2\beta_i^+)y_i^j,
\]

where \( j = 1, 2, \cdots, 2m - n. \) Assume

\[
f(x) = \prod_{i=1}^{a} (x - x_i)^{2\alpha_i+1} \prod_{j=1}^{b^-} (x - y_j)^{2\beta_j^- - \omega_j} \prod_{l=1}^{c} (x - z_l)^{2\gamma_l+2},
\]

\[
g(x) = \prod_{j=1}^{b^+} (x - y_j)^{\omega_j^+ - 2\beta_j^+}.
\]

We use \( k \) denotes the number of the distinct roots of \( f(x), t \) denotes the number of the distinct roots of \( g(x), s \) denotes the degree of \( f(x), \tau \) denotes the degree of \( f(x) - g(x), \) and \( t_0 \) denotes the number of the distinct real roots of \( Q(x). \) It
is easy to see $s = 2\alpha + a + 2\beta - \omega + 2\gamma + 2c$, $\tau = n + 1 - a - 2b - 2\beta^+ - \omega^-$, and $t = b^+$, $k = a + b^+ + c$.

From [13], we know the discrimination sequence $(D_1, D_2, \cdots, D_n)$ of $f(x)$ satisfied $D_k \neq 0, D_{k+1} = D_{k+2} = \cdots = D_n = 0$. When $n$ is even, if $t \geq \frac{s-\tau+1}{2}$, then $t_0 \leq k \leq m + 1 - b^+ \leq \frac{n}{2}$. Otherwise, $t \leq \frac{s-\tau-1}{2}$, for the n-degree polynomial $f(x)$, we have

$$S_i = \begin{bmatrix}
\sum_{i=1}^{b^+} (\omega_i^+ - 2\beta^+) y_i & \cdots & \sum_{i=1}^{b^+} (\omega_i^+ - 2\beta^+) y_i^{l-1} \\
\sum_{i=1}^{b^+} (\omega_i^+ - 2\beta^+) y_i & \cdots & \sum_{i=1}^{b^+} (\omega_i^+ - 2\beta^+) y_i^l \\
\cdots & \cdots & \cdots \\
\sum_{i=1}^{b^+} (\omega_i^+ - 2\beta^+) y_i^{l-1} & \cdots & \sum_{i=1}^{b^+} (\omega_i^+ - 2\beta^+) y_i^{2l-2}
\end{bmatrix},$$

where $0 \leq l \leq \frac{s-\tau+1}{2}$.

From Lemma 2.7, we have $D_{t+1} = D_{t+2} = \cdots = D_{\frac{s-\tau+1}{2}} = 0$. If $k \leq \frac{s-\tau+1}{2}$, then $t_0 \leq k \leq m + 1 - b^+ \leq \frac{n}{2}$. Otherwise, from Lemma 2.8, we have

$$t_0 \leq k - \left(\left(\frac{s-\tau+1}{4}\right) \times 2 + \left(\frac{s-\tau+1}{4} - \frac{s-\tau+1}{4} \times 4 + 1\right)\right) \times 2 \leq k - \frac{n}{2} + \frac{n}{2} - \frac{1}{2} + t \leq a + b + c + \frac{n+1}{2} - (a + b + c) - \frac{1}{2} \leq \frac{n}{2} + \frac{n}{2} - \frac{1}{2} + t \leq a + b + c + \frac{n+1}{2} - (a + b + c) - \frac{1}{2} \leq \frac{n+1}{2}.$$

When $n$ is odd, if $t \geq \frac{s-\tau}{2}$, then $t_0 \leq k \leq m + 1 - b^+ \leq \frac{n+1}{2}$. Otherwise, $t \leq \frac{s-\tau-2}{2}$, we have $D_{t+1} = D_{t+2} = \cdots = D_{\frac{s-\tau+2}{2}} = 0$. If $k \leq \frac{s-\tau-2}{2}$, then $t_0 \leq k < \frac{n+1}{2}$. Otherwise, from Lemma 2.8, we have

$$t_0 \leq k - \left(\left(\frac{s-\tau+2}{4}\right) \times 2 + \left(\frac{s-\tau+2}{4} - \frac{s-\tau+2}{4} \times 4 + 1\right)\right) \times 2 \leq k - \frac{n}{2} + \frac{n}{2} + t \leq a + b + c + \frac{n+1}{2} - (a + b + c) \leq \frac{n+1}{2}.$$

Since the system [1] can have at most one hyperelliptic limit curves, and the hyperelliptic limit cycle should intersect the x-axis at two different points $x_1, x_2$, where $x_1, x_2$, are simple root of $Q(x)$, we have $H(m, n) \leq \frac{t_0}{2}$. This completes the proof of case $m + 2 \leq n \leq 2m - 2$.

When $n = 2m - 1$, if $m$ is odd, we know from the preliminaries, any root of $Q(x)$ must be a root of $P(x)$, while the degree of $P(x)$ is $m + 1$, $Q(x)$ can have at most $m$ simple roots, then $H(m, n) \leq \frac{m-1}{2} = \left[\frac{m-1}{2}\right]$. For $m$ is even, if $Q(x)$ have $m$ simple roots, then there are at most $\frac{m-2}{2}$ intervals which satisfied $Q(x) > 0$ in the interval. Otherwise $Q(x)$ can have at most $m - 1$ simple roots, then $H(m, n) \leq \frac{m-2}{2} = \left[\frac{m-1}{2}\right]$. When $n = 2m$, the proof is similar to the case $n = 2m - 1$, so we omit it.
Recall that we want to prove $H(m, n) \leq \left\lceil \frac{m}{2} \right\rceil$ when $n > 2m + 1$. Since the system (1) can have at most one hyperelliptic limit curves, and $Q(x)$ can have not more than $m$ simple roots when $n > 2m + 1$, we obtain the upper bound of $H(m, n)$ is $\left\lceil \frac{m}{2} \right\rceil$.

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