Persistent currents in a circular array of Bose-Einstein condensates

Gh.-S. Paraoanu

1Department of Physics, University of Jyväskylä, P.O.Box 35 (YFL), 40014 Jyväskylä, Finland and Department of Physics, Loomis Laboratory, 1110 W. Green Street, University of Illinois at Urbana-Champaign, Urbana IL61801, USA

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A ring-shaped array of Bose-Einstein condensed atomic gases can display circular currents if the relative phase of neighboring condensates becomes locked to certain values. It is shown that, irrespective of the mechanism responsible for generating these states, only a restricted set of currents are stable, depending on the number of condensates, on the interaction and tunneling energies, and on the total number of particles. Different instabilities due to quasiparticle excitations are characterized and possible experimental setups for testing the stability prediction are also discussed.

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I. INTRODUCTION

The existence of flow without dissipation is a signature of superfluidity and superconductivity. This phenomenon has been studied in various condensed-matter systems, most notably in helium and metallic superconductors. The experimental achievement of Bose-Einstein (BEC) condensation in alkali-metal atomic gases opened the possibility of investigating the superfluid properties of dilute BEC gases trapped in various external potentials generated by magnetic or laser fields. The geometry of the trapping external potentials can be engineered into various shapes: from almost spherically symmetric to highly anisotropic, from double wells to lattices.

We study, in the present paper, the stability of currents for a system of \( M \) condensates in a topologically restricted configuration, with tunneling allowed only between neighboring condensates. This type of trap can be created, for instance, for not too large \( M \)'s, by using an experimental setup similar to that of the MIT interference experiment but with a spatial shape of the blue-detuned laser beam, which would ensure the separation of the gas into three or more pieces. Another possibility involves the use of two-dimensional optical lattices in magnetic traps of the same type as those recently employed to study the superfluid-Mott insulator transition. Overlapping a highly repulsive optical potential (for example, generated by a blue-detuned cylindrical laser beam) in the middle of the magnetic trap results in the creation of a Mexican-hat potential for the atoms already confined in the lattice.

A simple application of the concept of spontaneously broken gauge symmetry to this system has as a consequence the appearance of well-defined phase relations between consecutive condensates and the flow of a circular current around the ring. Nowadays there is a variety of experimental methods for inducing currents into an already formed condensate: tilting the lattice in a gravitational field, or accelerating it, displacing the enveloping external potential created by magnetic trapping, phase imprinting, coherent conversion between two hyperfine states (one serving as the pinning potential), or stirring the condensate with a laser beam.

In this paper we show that, no matter what mechanism is chosen to attempt to generate these circular flow states, some of them are in fact unstable, either dynamically or thermodynamically. The states that are stable and produce persistent currents are identified as those with circulation below a certain critical value, which in general depends on the parameters related to on-site interaction such as density and scattering length, on the tunneling rate, and on the number \( M \) of condensates. The maximum critical circulation (vorticity) of these states is \( M/4 \).

In the following we will briefly discuss the relevance of our study for the two processes that involve at different points the concept of broken phase symmetry: the Kibble-Zurek mechanism and the non-adiabatic quantum phase transitions.

The Kibble-Zurek mechanism predicts the appearance of topological defects in systems undergoing a rapid quench through a continuous phase transition. For the case of dilute alkali atomic gases it has been already argued that during a fast quench vortex lines will be formed. It is likely however that a Kibble-Zurek experiment with atomic gases will not be realized in the way originally imagined, but rather by crafting traps simulating a closed array of condensates which can tunnel from one site to another, in a way similar to the experiments done using Josephson junctions. This is precisely the system analyzed in this paper. A fast evaporative cooling of a gas of bosons in this type of trap would result in the formation of domains with broken phase symmetry in the wells of the potential (due to higher particle densities). The overlap of the order parameters leads to transfer of atoms between neighboring wells. But in the end, the probabilities assigned to different outcome states by any microscopic analysis (time-dependent Ginzburg-Landau
GL, kinetic theory, etc.) of the non-equilibrium problem have to be supplemented with the constrain that only the stable currents will survive - indeed, as shown in this paper, modes that otherwise would pass unsuppressed into the order parameter can decay by the emission of quasiparticles.

A similar conclusion can be drawn for the second process mentioned above, namely quantum phase transitions. The superfluid - Mott insulator transition has been achieved and studied experimentally intensively in recent times in two-dimensional optical lattices: the same ideas can be applied for an array of the type described in this paper. In this case crossing the quantum critical point results in a coherent superposition of states with different circulations. The mechanism that breaks the phase symmetry is then any decoherence process: a measurement, particle losses, etc. But, roughly, as shown below, only half of the macroscopically occupied states that result in this way will be persistent currents.

II. A SIMPLE MODEL

Let us consider a simple model that captures the essential stability features of a macroscopically occupied mode in the topologically constrained configuration described above.

We consider $M$ identical small condensates in contact with each other and a cylindrical system of coordinates $(r, \theta, z)$. The total number of atoms is $N$. The centers of the condensates are positioned at $\vec{R}_\lambda = (R, \theta, z = 0)$, where $\theta_\lambda = 2\pi \lambda / M$ and $\lambda$ runs from 0 to $M - 1$.

The Hamiltonian of the system is

$$H = \int d\vec{r} \hat{\psi}^\dagger(\vec{r}) \left[ -\frac{\hbar^2}{2m} \Delta + V(\vec{r}) + \frac{g}{2} \hat{\psi}(\vec{r})^\dagger \hat{\psi}(\vec{r}) \right] \hat{\psi}(\vec{r}),$$

where $g = 4\pi \hbar^2 / m$ and $a$ is the scattering length.

The external potential $V(r)$ is crafted to be high enough around the origin, so that the atoms cannot penetrate there; it also has a number of $M$ minima at $(R, \theta_\lambda)$, around which the condensed atoms tend to localize. The delocalization effect comes from the possibility of tunneling between nearby wells. We also assume that each condensate has a small enough number of particles so that they are not in the Thomas-Fermi regime (for wells of dimension $\approx 1\mu m$ and the scattering length of Na or Rb, the number of particles in each condensate can be at most in the hundreds). In this situation, the wavefunctions of each condensate depend weakly on the number of atoms in the well, and one can apply a $M$-mode approximation for the field operator

$$\hat{\psi}(\vec{r}) = \sum_{\lambda=0}^{M-1} \phi(\vec{r} - \vec{R}_\lambda) \hat{a}(\lambda),$$

where $\phi$ is a solution of the Schrödinger equation for each

well; the Hamiltonian takes the Bose-Hubbard form

$$H = -t \sum_{\lambda=0}^{M-1} \left[ \hat{a}^\dagger(\lambda) \hat{a}(\lambda + 1) + \hat{a}^\dagger(\lambda + 1) \hat{a}(\lambda) \right]$$

$$+ \frac{w}{2} \sum_{\lambda=0}^{M-1} \hat{a}^\dagger(\lambda) \hat{a}^\dagger(\lambda) \hat{a}(\lambda) \hat{a}(\lambda).$$

Here the constant terms are omitted; the tunneling matrix element is given by

$$t = \int d\vec{r} \phi^*(\vec{r} - \vec{R}_\lambda) \left[ \frac{\hbar^2}{m} \Delta - V(\vec{r}) \right] \phi(\vec{r} - \vec{R}_{\lambda+1}).$$

and the on-site energy is

$$w = g \int d\vec{r} |\phi(\vec{r})|^4.$$
\[
\hat{b}_k^\dagger = \frac{1}{\sqrt{M}} \sum_{\lambda=0}^{M-1} e^{i(2\pi/M)k\lambda} \hat{a}_\lambda^\dagger.
\] (11)

The result is, with the indices \(k, k',\) and \(l\) taking all integer values in the interval \((-M/2, M/2),\)

\[
H = -2t \sum_k \cos \left(\frac{2\pi}{M} k\right) \hat{b}_k^\dagger \hat{b}_{k'} + \frac{w}{2M} \sum_{k,k',l} \hat{b}_{k+l}^\dagger \hat{b}_{k'}^\dagger \hat{b}_{k'} \hat{b}_k,
\]

which is formally the Hamiltonian for a uniform system of free bosons with kinetic energy \(-2t \cos (2\pi k/M)\) instead of the usual \(\hbar^2 k^2 / 2m\).

Equation (8) has solutions

\[
\chi_q(\lambda) = \frac{1}{\sqrt{M}} e^{i(2\pi/M)q\lambda},
\] (12)

\[
\mu_q = -2t \cos \frac{2\pi}{M} q + \frac{N}{2M} w,
\] (13)

which is precisely a circular current state with circulation quantized by \(q\).

In obtaining the Gross-Pitaevskii equation we have assumed that there is already a phase relation established between the \(M\) condensates, in other words the state is superfluid and not fragmented (Mott insulator) [3].

\[
E_k u_k(\lambda) = -t [u_k(\lambda + 1) + u_k(\lambda - 1)] + [-\mu_q + 2Nw|\chi_q(\lambda)|^2] u_k(\lambda) + Nw\chi_q^2(\lambda) u_k(\lambda),
\] (14)

\[
-E_k v_k(\lambda) = -t [v_k(\lambda + 1) + v_k(\lambda - 1)] + [-\mu_q + 2Nw|\chi_q(\lambda)|^2] v_k(\lambda) + Nw\chi_q^2(\lambda) u_k(\lambda).
\] (15)

The coherence between adjacent sites is achieved when \(w \ll tN/M,\) which corresponds to fluctuations in the relative phase between neighboring sites much smaller than 1 (see the Appendix). If this condition is satisfied, we can distinguish two limits, depending on how the kinetic energy per particle and the interaction energy per particle compare to each other: \(t \gg wN/M\) defines the Rabi regime, and \(t \ll wN/M\) defines the Josephson regime. These limits were initially introduced in connection with the two-well problem [15], where they have been shown to correspond to number-phase coherent and squeezed ground states but their generalization to lattices is straightforward (see [4, 16] and the Appendix). Experimentally, achieving the Rabi regime and at the same time preserving the validity of the tight-binding approximation can be done by tuning the scattering length in a magnetic field (Feshbach resonance) rather than decreasing the depth of the potential wells.

Let us now turn to the problem of the excitation spectrum. To derive the Bogoliubov - de Gennes equations we start with the time-dependent Gross-Pitaevskii equation and linearize it for small fluctuations of the order parameter around the macroscopically occupied mode. We obtain

\[
E_k u_k(\lambda) = -t [u_k(\lambda + 1) + u_k(\lambda - 1)] + [-\mu_q + 2Nw|\chi_q(\lambda)|^2] u_k(\lambda) + Nw\chi_q^2(\lambda) u_k(\lambda),
\] (14)

\[
-E_k v_k(\lambda) = -t [v_k(\lambda + 1) + v_k(\lambda - 1)] + [-\mu_q + 2Nw|\chi_q(\lambda)|^2] v_k(\lambda) + Nw\chi_q^2(\lambda) v_k(\lambda).
\] (15)

where we have used the notation

\[
\epsilon_k = 2t \cos \frac{2\pi}{M} q \left(1 - \cos \frac{2\pi}{M} k\right).
\] (21)

Replacing the solution for \(E_k\) in Eqs. (18, 19) we obtain

\[
\left[\epsilon_k \pm \sqrt{\epsilon_k \left(\epsilon_k + 2Nw\right) + Nw}\right] u_k(\pm) = -Nw v_k(\pm),
\] (22)

\[
\left[\epsilon_k \pm \sqrt{\epsilon_k \left(\epsilon_k + 2Nw\right) + Nw}\right] v_k(\pm) = -Nw u_k(\pm),
\] (23)

with the restriction imposed by the normalization condition \(|u_k|^2 - |v_k|^2 = 1\) which eventually will force us to make a choice between the two possible solutions indexed by (±). An interesting observation concerns the negative-energy eigenstates; assuming that we found a positive eigenenergy \(E_k^{(\pm)}\) with eigenvalues \(\begin{pmatrix} u_k^{(\pm)}(\lambda) \\ v_k^{(\pm)}(\lambda) \end{pmatrix}\), the corresponding negative-energy eigenstate is \(-E_k^{(\pm)}\).
with eigenvalues \( u_k^{(\pm)}(\lambda) \). But \( -E_k^{(\pm)} = E_{-k}^{(\mp)} \) and
\[
\left( \begin{array}{c} u_k^{(\pm)}(\lambda) \\ v_k^{(\pm)}(\lambda) \end{array} \right) = \left( \frac{1}{\sqrt{M}} e^{-\frac{i}{2w}(kq)q} u_k^{(\pm)} \right) = \left( \begin{array}{c} u_{-k}^{(\mp)}(\lambda) \\ v_{-k}^{(\mp)}(\lambda) \end{array} \right),
\]
where we have used \( u_k^{(\pm)} = u_{-k}^{(\mp)} \) and \( v_k^{(\pm)} = v_{-k}^{(\mp)} \); these last two relations can be proved readily from Eqs. (18) and (19). They show that if, say, \((+)\) for a certain \( k \) is a positive-energy eigenstate, then its corresponding negative-energy eigenstate is a positive-energy eigenstate, which is satisfied for any \( k \).

III. STABILITY ANALYSIS

The stability of the states can be checked out by considering small perturbations of the condensate state - by small alterations of the phase and the number of particles on each site. It is useful to distinguish between two types of stabilities.

A. Dynamical stability

If the eigenvalues \( E_k \) are complex, then the system is in a dynamically unstable state (or in a point of Lyapunov instability). The reason is that any perturbation will be exponentially magnified - the system tends to go as fast as it can as far as possible from that point. Dynamically stable states are those for which the condition
\[
\epsilon_k \left( \epsilon_k + \frac{2N}{M} \right) \geq 0
\]
(24)
is satisfied for any \( k \). Let us first note that \( \cos 2\pi q/M = 0 \) does not yield properly normalized solutions, according to Eqs. (22) and (23), so we will exclude from the beginning the states \( q = \pm M/4 \). We distinguish two cases:

1\(^{\circ}\) \( \epsilon_k > 0 \) which is equivalent to \( \cos (2\pi q/M) > 0 \), or \( q \) is in the interval \((-M/4, M/4)\).

2\(^{\circ}\) \( \epsilon_k < 0 \) and \( -\epsilon_k \geq 2wN/M \). Since the minimum value of \(-\epsilon_k\) is reached when \( k = \pm 1 \), it follows that
\[
\cos \frac{2\pi}{M} q \leq -\frac{2wN}{M} \left( 1 - \cos \frac{2\pi}{M} \right).
\]
(25)
This inequality implies also \( wN/M \leq t(1 - \cos 2\pi/M) \).

In conclusion, the system is in a dynamically stable state if either condition 1\(^{\circ}\) or 2\(^{\circ}\) is satisfied. In the Josephson regime, only condition 1\(^{\circ}\) can be satisfied, so only the modes from \(-M/4\) to \( M/4\) are dynamically stable. In the Rabi regime, \( Nw/Mt \ll 1 \) so we can distinguish two cases: when \( M \) is of the order of unity, clearly all the modes are stable; however, when \( M \) is large, of the order of \( t/Nw \), there can be dynamically unstable modes located at \( |q| > M/4 \); the relative number of these modes, as a fraction of the total number \( M \) of modes, is extremely small, since it is limited by \( \sqrt{Nw/Mt} \ll 1 \). We conclude that, in the Rabi regime, most of the states are dynamically stable. The dynamically unstable states can be treated in a linearized theory only for periods of time that are logarithmic with respect to the initial state \([17]\), since the quantum fluctuations will trigger an exponentially divergent evolution away from the initial state.

B. Thermodynamical stability

A point of thermodynamical instability (or energetic instability) is a circular flow state which is still not a local minimum of the energy functional; however, small perturbations do not dynamically bring the system far from the initial state in the absence of dissipation. If dissipation is introduced (or, as in our case, if the system suffers thermalization by collisions), the system will not stay arbitrarily close to the initial state but instead decay to a thermodynamically stable state. The name "thermodynamical instability" is thus justified by the fact that such a state cannot be in thermodynamic equilibrium. The thermodynamically unstable states have a real excitation spectrum which is characterized by the existence of eigenenergies with negative \( E_k \) (with the normalization corresponding to positive eigenstates, \( |u_k|^2 - |v_k|^2 = 1 \), and are known to produce interesting effects in optical lattices \([18]\). For vortices in harmonic traps they are responsible for the precession of the vortex core around the condensate axis \([19]\). This occurs because the system in a thermodynamically unstable vortex state can reduce its energy by transferring particles from the condensate to the negative-energy modes; for an un-pinned vortex for example this happens by few-particle excitations to the core mode \([20]\). Let us now study the case of thermodynamical stability in situations 1\(^{\circ}\) and 2\(^{\circ}\) described above.

1\(^{\circ}\) \( \cos 2\pi q/M > 0 \). In this case only \( E_k^{(\mp)} \) is a valid solution. Indeed, from Eq. (22), the condition \( |u_k| > |v_k| \) can be satisfied if
\[
|\epsilon_k + \sqrt{\epsilon_k \left( \epsilon_k + \frac{2N}{M} \right) + \frac{N}{M} w} | < \frac{N}{M} w,
\]
(26)
and clearly the solution with the lower sign cannot satisfy it.

Let us show that the upper sign solution always satisfies this inequality. If the expression under the modulus in Eq. (26) is positive, the inequality is fulfilled trivially. If it is negative, it becomes \( \epsilon_k - \sqrt{\epsilon_k \left( \epsilon_k + 2wN/M \right) - 2wN/M} \); or in another form \( \epsilon_k \left( \epsilon_k + 2wN/M \right) < 1 \), which is satisfied for any values of the parameters.

The values of the Bogoliubov amplitudes \( u_k \) and \( v_k \) are

\[
\begin{align*}
\left| u_k \right|^2 = |\epsilon_k + 2wN/M | & < 1, \\
\left| v_k \right|^2 = |\epsilon_k - 2wN/M | & > 1.
\end{align*}
\]
obtained from Eq. (22) as
\begin{align}
  v_k &= \frac{\sqrt{\epsilon_k + 2wN/M} - \sqrt{\epsilon_k}}{2(\epsilon_k (\epsilon_k + 2wN/M))^{1/4}} \\
  u_k &= \frac{\sqrt{\epsilon_k + 2wN/M} + \sqrt{\epsilon_k}}{2(\epsilon_k (\epsilon_k + 2wN/M))^{1/4}}
\end{align}

We now have to impose the condition of thermodynamical stability: $E_k^{(+)} \geq 0$ for any $k$. This condition is certainly satisfied when $2\pi k/M$ is in the first and second quadrants, but not necessarily when $\sin 2\pi k < 0$. Enforcing $E_k^{(+)} \geq 0$ for $k < 0$ yields
\[ t \left[ 1 + \cos \frac{2\pi}{M} k - 2\cos \frac{2\pi}{M} q \right] \leq \frac{N}{M} w \cos \frac{2\pi}{M} q. \]  
(29)

The maximum of the left hand side (LHS) is achieved when $k = -1$. With some algebraic manipulations the constraint can be put in the form
\[ 2t \sin \frac{\pi}{M} (2q - 1) \sin \frac{\pi}{M} (2q + 1) \leq \frac{N}{M} w \cos \frac{2\pi}{M} q. \]  
(30)

If this condition is satisfied we have a so-called persistent current flowing in our geometry - this current does not decay in time (say by thermal fluctuations) to a lower energy state. As it should be, the ground state (obtained for $q = 0$) trivially satisfies this inequality since in this case the LHS turns negative. The thermodynamically stable states with $q \neq 0$ are local minima of the energy functional; in their case, it costs energy to create excitations. We expect persistent currents to be those that will form via symmetry-breaking mechanisms, and survive enough to be finally detected.

\[ 2q \]  
(29)

There are no persistent currents in this case, as one can readily check. Indeed, this $\epsilon_k \leq -2wN/M$, and in order to have $|u_k|^2 > |v_k|^2$, we need to impose, from (22), the condition
\[ 0 < |\epsilon_k| \geq \sqrt{|\epsilon_k| (|\epsilon_k| - \frac{2N}{M} w)} < \frac{2N}{M} w. \]  
(31)

Only the lower sign ensures the validity of these inequalities. The corresponding energy will be then $E_k^{(-)}$,
\[ E_k^{(-)} = 2t \sin \frac{2\pi}{M} k \sin \frac{2\pi}{M} q - \sqrt{\epsilon_k (\epsilon_k + \frac{2N}{M} w)}. \]  
(32)

But now it is obvious that we cannot have $E_k^{(-)} \geq 0$ for all $k$'s, because if $q < 0$ then the modes with $k > 0$ will be thermodynamically unstable, and if $q > 0$ then the modes with $k < 0$ will not be stable. Thus, the only case in which we can have thermodynamical stability is case $1^o$.

In conclusion, the number of stable states of circular currents depends, in general, on the number of particles $N$, on the number of condensates $M$, and on the ratio $t/w$ (which can be controlled experimentally by varying the barrier potentials between the condensates). The precise form of this dependence is given by the condition $|q| < M/4$, combined with Eq. (30). For example, according to Eq. (30) there cannot be stable currents with $M = 3$ and $M = 4, M = 5, 6, 7, 8$ give at most two stable circular currents, $M = 9, 10, 11, 12$ give at most four stable ones, etc.

It is instructive to see what happens in the Rabi regime, defined as $t \gg wN/M$, and in the Josephson regime, where $tM/N \ll w \ll tN/M$. In the Josephson regime it is clear that Eq. (30) is satisfied, so we have both thermodynamical and dynamical stability for $2\pi q/M > 0$. More interesting is what happens in the Rabi regime. If $M$ is of the order of unity clearly Eq. (30) cannot be fulfilled; when $M$ becomes of the order of $2t/Nw$, there can be stable modes with momenta given by $2\pi q/M < \sqrt{Nw/2M + (\pi/M)^2}$. They are then a very small fraction of the total number of modes $M$ and they lead to small momenta of the circulating fluid. For almost all practical purposes these modes can be assimilated with $q = 0$ - in a real experiment they would be difficult to be distinguished from the ground state. The fact that the rest of the modes (with larger $q$) in the Rabi regime are thermodynamically unstable can be understood physically in a simple way. For these modes, we can simply put $w = 0$ and search for solutions with $u_k = 1$ and $v_k = 0$. If we look at Eq. (22), it is clear that we need to take the upper sign only into account. The energy in this case is
\[ E_k = 2t \sin \frac{2\pi}{M} k \sin \frac{2\pi}{M} q + 2t \cos \frac{2\pi}{M} q \left(1 - \cos \frac{2\pi}{M} k \right), \]
which can also be written in the more relevant form
\[ E_k = 2t \cos \frac{2\pi}{M} q - 2t \cos \frac{2\pi}{M} (k + q). \]  
(33)

This shows that $E_k$ is the difference between the phase-twisting energy per particle corresponding to a circulation $q$ and the energy of an excitation around the macroscopically occupied mode. In other words, to create an elementary excitation $k$ relative to the circulation current, we need to transfer an atom from the condensate to the single-particle state $k + q$. This expression is positive for all $k$'s only if $q = 0$ - so all the flow states are thermodynamically unstable. Instead, we have no dynamical instability. It is interesting to note that it is indeed the interaction that stabilizes in the end the system: a large number of persistent currents cannot exist in its absence. Thus, phase-symmetry breaking mechanisms, for instance, can effectively work only for systems in which there is some extra energy to compensate for the loss of kinetic energy due to phase fluctuations. These results are summarized in Table 1. In the Fock regime $(w \gg tN/M)$ we cannot write Bogoliubov equations, so our treatment would not be valid.

For thermodynamically stable states (persistent cur-
TABLE I: Stability of circular currents: the Rabi and Josephson regimes. We find that a relatively large number (compared with $M$) of persistent currents can exist only above a certain critical value of the interaction (which turns out to be in between the Rabi and Josephson regimes) and only for certain circulations (or angular momenta).

| REGIME | RABI | JOSEPHSON |
|--------|------|------------|
| defined by $t \gg wN/M$ | $tM/N \ll w \ll tN/M$ | |
| dynamically stable | most | $|q| < M/4$ |
| thermodynamically stable | very few | $|q| < M/4$ |

rents) the quasiparticle spectrum obtained,

$$E_k^{(+)} = 2t \sin \frac{2\pi}{M} k \sin \frac{2\pi}{M} q + \sqrt{\epsilon_k \left( \epsilon_k + \frac{2N}{M} w \right)},$$

is different from that of the ground state, which can serve as a method to detect a circular flow state [21]. Also, similar to a phenomenon called the Sagnac effect in relativistic physics, we note that the quasiparticles $k$ and $-k$ propagate with different speeds around the loop, because they are excitations on top of a state with a flow in a certain direction. We have then a condensed-matter equivalent of the Sagnac effect which is related to the lifting of the degeneracy of the excitations $k$, $-k$ due to the rotation of the condensate.

IV. CONCLUSIONS

We have obtained the excitation spectrum of a circular flow state in a gas of weakly interacting bosons and analyzed its dynamical and thermodynamical stability. Testing these stability predictions, by changing the tunneling, the interaction, or the number of particles is within the reach of present-day experimental technology.

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APPENDIX: NUMBER FLUCTUATIONS

We give here a justification of the limits that define the Rabi and Josephson regimes. As in the case of the double well [15], these are set respectively by the condition of Poissonian or sub-Poissonian fluctuations of the number of atoms in each well.

Let us look now at the ground state of the system. We will use the same notations as throughout the main part of the paper with a superscript 0 to indicate that they are taken at $q = 0$. Since all sites are equivalent, it is enough to look at one of them, say $\lambda = 0$. If $N_0$ is the number of atoms condensed on the mode $q = 0$, then $b_0 \simeq \sqrt{N_0}$, and from Eqs. (10,11) one gets

$$\hat{n}(0) \equiv \hat{a}^\dagger(0)\hat{a}(0) \simeq \frac{N_0}{M} + \sqrt{\frac{N_0}{M}} \sum_{k \neq 0} \left( \hat{b}_k + \hat{b}_k^\dagger \right) + \frac{1}{M} \sum_{k \neq 0, k' \neq 0} \hat{b}_k \hat{b}_{k'}.$$  

which implies $n \equiv \langle \hat{n}(0) \rangle = N/M$. In other words, the splitting of the particle number operator at any point in the lattice (the density operator in the limit $M \gg 1$) into a mean-field value and a fluctuating field of zero average results naturally from the Bogoliubov theory. Now we can calculate the fluctuations on the ground state

$$\sigma_n^2 = \langle \hat{n}^2(0) \rangle - \langle \hat{n}(0) \rangle^2 \simeq \frac{N}{M^2} \sum_{k \neq 0} \frac{\epsilon_k^0}{E_k^{(+)}},$$

Here $\epsilon_k^0 = 4t^2(\sin \pi k/M)^2$ and $E_k^{(+)} = \sqrt{\epsilon_k^0 (\epsilon_k^0 + 2wN/M)}$. If the “size” $M$ of the system is large the sum can be transformed into an integral and calculated analytically. The result is

$$\sigma_n^2 \simeq \frac{2n}{\pi} \arctan \sqrt{\frac{2t}{nw}},$$

the same as obtained in Ref. [15] for a similar Hamiltonian by using a phonon approach. The parameter that controls the fluctuations of the density (and phase) is then $t/nw$; the Rabi regime, with Poissonian fluctuations $\sigma_n \simeq \sqrt{n}$, is obtained for $t \gg nw$, while the Josephson regime, with sub-Poissonian fluctuations $\sigma_n \simeq (8nt/\pi w)^{1/4} \ll \sqrt{n}$ is obtained in the limit $t \ll nw$. The condition $\sigma_n \gg 1$ (or, equivalently, phase fluctuations much smaller than 1) sets the limit of validity of the Bogoliubov approach: $tn \gg w$. In the Josephson regime, the excitation spectrum is $E_k^0 \simeq 8ntw \sin \pi k/M$. In the Rabi regime, $E_k^0 \simeq \epsilon_k^0$ for momenta $2\pi k/M$ much larger than a critical value $\sqrt{nw/t} \ll 1$ (that is, for most of the modes available in the system). Below this critical value, if there are modes available (in other words if the minimum momenta allowed by quantization $2\pi/M$ is of
the order of or less than $\sqrt{nw/t} \ll 1$), the spectrum is phonon-like, $E_k^0 \approx \sqrt{8nw/\pi k/M}$. One can regard the Rabi regime as being essentially a noninteracting limit since the fraction of excitation modes that do not have a spectrum as given by simply setting $w = 0$ is negligible both in the case of $M \gg 1$ and $M$ of the order of unity (when, since $\sqrt{nw/t} \ll 2\pi/M$, all the modes correspond in fact to a noninteracting spectrum).

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