A Tauberian Theorem for signed measures

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Abstract

Motivated by asymptotic analysis of value functions in stochastic optimal control problems, we develop Karamata’s Tauberian theorem for generalised signed measures. The result relies on a continuity theorem, which relates the convergence of Laplace transforms of generalised signed measures to the convergence of their distribution functions, and vice versa. The proof hinges on the subtleties of vague convergence, described in Herdegen et al. [16].

Keywords: Tauberian theorem, signed measures, continuity theorem, vague convergence.

1 Introduction

We investigate in what sense the following two payoff functionals of a one-dimensional diffusion process $X$,

$J_T(X) = \mathbb{E} \left[ \int_0^T X_t \, dt \right]$ and $K_\delta(X) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} X_t \, dt \right],$

relate to each other when $T \to \infty$ and $\delta \to 0$. This question is motivated by asymptotic analysis of value functions in stochastic optimal control problems. To be more specific, let $X_t = g(Y_{\alpha t})$ for some payoff function $g(\cdot)$ and controlled diffusion process $Y^\alpha$ associated with an admissible control $\alpha$, it is well understood that the following Cesáro mean and Abel mean coincide when their respective averaging parameters $T \to \infty$ and $\delta \to 0$, i.e.

$$\limsup_{T \to \infty} \left\{ \sup_{\alpha} \left[ \frac{1}{T} J_T(g(Y^\alpha)) \right] \right\} = \limsup_{\delta \to 0} \left\{ \sup_{\alpha} [\delta K_\delta(g(Y^\alpha))] \right\}.$$

The case that the limiting value is independent of the initial state of $X$ was first studied by (Arisawa & Lions, [1, 2]) using ergodic Bellman equations, and has recently been generalised by (Hu et al., [12]) using ergodic backward stochastic differential equations (Furham et al., [9]). On the other hand, the study of the case that the limiting value does depend on the initial state of $X$ was initiated by (Quincampoix & Renault, [17]) in a deterministic control setup, and has been further generalised to a stochastic control setting by (Buckdahn et al., [5, 6]). See also (Goreac, [10]) for generalised means other than Abel and Cesáro. However, some type of dissipative assumption on $Y^\alpha$ is imposed in all the aforementioned works.

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In practice, the dissipative type of assumption does not always hold and, moreover, the corresponding asymptotic values of Cesáro and Abel means may not even exist. Nevertheless, it is still interesting to understand asymptotic values of value functions (with weighted Cesáro and Abel means) and the relationship of their corresponding asymptotic values. For example, such an asymptotic analysis has been found extremely useful in finance. See (Herdegen Et al., [15]) with more references therein.

As a first step, we consider in this paper the case of uncontrolled diffusion process $X_t = g(Y_t)$, or the optimal controls for the above two stochastic optimal control problems coincide. Before proceeding, we introduce some necessary definitions.

### 1.1 Distribution functions and Laplace transforms

Throughout, we let $\mathbb{R}^+ := (0, \infty)$. A positive measure $\mu$ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ is called a Radon measure if it is locally finite and inner regular i.e.,

(i) for any $x \in \mathbb{R}^+$, there exits an open neighbourhood of $x$ such that $\mu(U) < \infty$;

(ii) for each $A \in \mathcal{B}(\mathbb{R}^+)$

$$\mu(A) = \sup \{\mu(K) : K \in \mathcal{B}(\mathbb{R}^+), K \text{ compact}, K \subset A\}. $$

For a signed measure $\mu$ on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, we denote its Hahn-Jordan decomposition by $\mu = \mu^+ - \mu^-$, and its associated variation measure by $|\mu| := \mu^+ + \mu^-$. The total variation of a signed measure $\mu$ is denoted by $||\mu||_{TV} := |\mu|(\mathbb{R}^+)$, and we say that $\mu$ is finite if $||\mu||_{TV} < \infty$. Furthermore, we say $\mu$ is a signed Radon measure if $|\mu|$ is a Radon measure.

**Definition 1.1.** We say that a mapping $\mu : \mathcal{B}(\mathbb{R}^+) \to \mathbb{R} \cup \{\infty\}$ is a generalised signed measure if there exists Radon measures $\mu^\pm$ such that for any $A \in \mathcal{B}(\mathbb{R}^+)$,

$$\mu(A) := \begin{cases} 
\mu^+(A) - \mu^-(A) & \text{if both } \mu^+(A), \mu^-(A) < \infty, \\
+\infty & \text{otherwise.}
\end{cases}$$

We let the space of all generalised signed measures on $\mathbb{R}^+$ be denoted by $\mathcal{M}^*$. The restriction of $\mathcal{M}^*$ to the space of finite signed Radon measures on $\mathbb{R}^+$ is denoted by $\mathcal{M}$.

We will assume without loss of generality that the Radon measures $\mu^\pm$ in Definition 1.1 are mutually singular. Thus, for $\mu \in \mathcal{M}$, $\mu^\pm$ is the Hahn-Jordan decomposition of $\mu$.

**Definition 1.2.** We define the mapping $||\cdot|| : \mathcal{M}^* \to \mathbb{R}^+ \cup \{\infty\}$, such that

$$||\mu|| := \begin{cases} 
||\mu||_{TV} & \text{if } \mu \in \mathcal{M}, \\
+\infty & \text{otherwise.}
\end{cases}$$

For any interval $I \subset \mathbb{R}$, we let $\text{BV}(I)$ denote the space of all functions of bounded variation on $I$. Furthermore, if $F : I \to \mathbb{R}$ such that $F \in \text{BV}([a, b])$ for any $[a, b] \subset I$, then we say $F$ has locally bounded variation, and $F \in \text{BV}_{\text{loc}}(I)$. For any $\mu \in \mathcal{M}^*$ we define its distribution function $F_\mu \in \text{BV}_{\text{loc}}(\mathbb{R}^+)$ such that $F_\mu(x) := \mu((0, x])$. Note that $F_\mu$ is right continuous and for any $a, b \in \mathbb{R}^+$ such that $a \leq b$,

$$F_\mu(b) - F_\mu(a) = \mu((a, b]).$$

Similarly, for any $\mu \in \mathcal{M}^*$, if $\int_{\mathbb{R}^+} e^{-\lambda x} |\mu|(dx) < \infty$ for all $\lambda > 0$, we define its Laplace transform $\Psi_\mu \in \text{BV}_{\text{loc}}(\mathbb{R}^+)$ via $\Psi_\mu(\lambda) := \int_{\mathbb{R}^+} e^{-\lambda x} \mu(dx)$. 


1.2 Karmata’s Tauberian Theorem

Our idea stems from Tauberian characterization of measures via their Laplace transform and vice versa. We may note that by Fubini’s theorem, $J_T(X)$ defines $\mu \in M^*$ via

$$F_\mu(T) := \int_0^T \mathbb{E}[X_t] \, dt.$$  (1.1)

Moreover, $K_\delta(X) = \Psi_\mu(\delta)$. By re-framing the payoff functionals in this way, one might be tempted to deduce an asymptotic relationship between $J_T(X)$ and $K_\delta(X)$ via Karamata’s Tauberian Theorem. It states that, under general conditions, the behaviour of the Laplace transform $\Psi_\mu(\lambda)$ of a positive measure $\mu \in M^*$ at the origin uniquely determines the asymptotic behaviour of $F_\mu$ at infinity, and vice versa (Karamata, [13]).

**Theorem 1.3** (Karamata’s Tauberian Theorem). Let $\rho \geq 0$ and $\mu \in M^*$ be a positive measure such that $\Psi_\mu(\lambda) < \infty$ for all $\lambda > 0$. Then

$$\lim_{\tau \downarrow 0} \frac{\Psi_\mu(\tau \lambda)}{\Psi_\mu(\tau)} = \frac{1}{\lambda^\rho},$$  (1.2)

and

$$\lim_{t \uparrow \infty} \frac{F_\mu(tx)}{F_\mu(t)} = x^\rho,$$  (1.3)

implies the other, as well as

$$\Psi_\mu(t^{-1}) \sim F_\mu(t)\Gamma(\rho + 1),$$

as $t \to \infty$.

According to Feller, Theorem 1.3 has a ‘glorious history’ [8, Section XIII.5], even though it is often omitted from modern books on probability theory. The two implications are usually separated, where Eq. (1.3) $\Rightarrow$ Eq. (1.2) is called an Abelian theorem, whilst Eq. (1.2) $\Rightarrow$ Eq. (1.3) is called a Tauberian theorem. Looking to its origins we see that the Tauberian implication caused the most trouble. It was first proved via lengthy calculations in 1914 by Hardy and Littlewood in their famous paper [11]. Karamata simplified their proof in [13], and subsequently introduced the present day class of regularly varying functions.

**Definition 1.4.** We say that a positive measurable function defined on some neighbourhood $[Z, \infty)$ is regularly varying at infinity (resp. zero) of index $\rho$ if for any $\lambda > 0$,

$$f(\lambda x)/f(x) \to \lambda^\rho$$

as $x \to \infty$ (resp. $x \to 0$), in which case we write $f \in R_\rho$ (resp. $f \in R_\rho^0$). If $\rho = 0$, then we say that $f$ is slowly varying, which is usually denoted by $l$.

**Remark 1.5.** By the representation theorem for slowly varying functions (Bingham Et al., [3, Theorem 1.3.1]), a slowly varying function $l$ must be of the form

$$l(x) = c(x) \exp \left\{ \int_a^x \frac{\epsilon(u)}{u} \, du \right\}$$  (1.4)

for $x \geq a > 0$, where $c$ is measurable and $c(x) \to C \in \mathbb{R}^+$, $\epsilon(x) \to 0$ as $x \uparrow \infty$. 

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Moreover, by [3, Proposition 1.5.1], if \( \rho \neq 0 \), then for any \( f \in R_\rho \),

\[
f(x) \rightarrow \begin{cases} 
\infty & \text{if } \rho > 0, \\
0 & \text{if } \rho < 0,
\end{cases}
\]

and there exists \( l \in R_0 \) such that \( f(x) = x^\rho l(x) \). See [3, Theorem 1.4.1] for further details.

From Theorem 1.3 and Remark 1.5 we have the following corollary which relates asymptotic values of weighted Cesàro and Abel means of a payoff functional.

**Corollary 1.6.** Suppose that \( X \) is a one-dimensional diffusion such that \( \mathbb{E}[X_t] \) is non-negative. Define \( \mu \in \mathcal{M}^* \) via (1.7). If for some \( \rho \geq 0 \) we have either \( (\Psi_\mu)_{-1} \in R_\rho^0 \), or \( F_\mu \in R_\rho \), then there exists some \( l \in R_0 \) such that

\[
\lim_{T \to \infty} \left\{ \frac{\Gamma(\rho + 1)}{T^\rho l(T)} J_T(X) \right\} = \lim_{\delta \to 0} \left\{ \delta^\rho \tilde{l}(\delta) K_\delta(X) \right\},
\]

where \( \tilde{l}(\delta) := 1/l(\delta^{-1}) \).

**Example 1.7.** Let \( W \) be a Brownian motion on a filtered probability space \( (\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P}) \), and define \( X_t := W_t^2 \). Defining \( \mu \in \mathcal{M}^* \) via the density \( g(t) := \mathbb{E}[X_t] = t \), we clearly have \( F_\mu \in R_2 \). Since \( \mu \in \mathcal{M}^* \) is a positive measure, the conditions of Corollary 1.6 are satisfied, whence

\[
\lim_{\tau \to 0} \frac{\Psi_\mu(\tau \lambda)}{\Psi_\mu(\tau)} = \lambda^{-2},
\]

for all \( \lambda > 0 \), and

\[
\lim_{T \to \infty} \left\{ \frac{2}{T^2} J_T(X) \right\} = \lim_{\delta \to 0} \left\{ \delta^2 K_\delta(X) \right\}.
\]

Unfortunately, Corollary 1.6 is of limited use since it is restricted to the class of diffusions where \( \mathbb{E}[X_t] \geq 0 \). Hence, the goal of this paper is to extend Theorem 1.3 such that it holds for general \( \mu \in \mathcal{M}^* \), an extension which is of interest far outside the scope of stochastic control. Below is the main result of this paper.

**Theorem 1.8** (Karamata’s Tauberian Theorem for generalised Signed Measures). Let \( \{\tau_n\} \) be an arbitrary null-sequence and \( \mu \in \mathcal{M}^* \) with decomposition \( \mu^+, \mu^- \) such that \( \Psi_{\mu^\pm}(\lambda) < \infty \) for all \( \lambda > 0 \).

(a) Suppose that

\[
\limsup_{n \to \infty} \frac{\Psi_{|\mu|}(\tau_n \varepsilon)}{|\Psi_{\mu}(\tau_n \varepsilon)|} < \infty
\]

for some \( \varepsilon > 0 \). Moreover, suppose that

\[
\inf_{\delta > 0} \limsup_{\tau \to 0} \frac{|\mu|(|\tau^{-1}(t - \delta), \tau^{-1}(t + \delta))}{|\Psi_{\mu}(\tau)|} = 0,
\]

for each \( t > 0 \). Then

\[
\lim_{\tau \to 0} \frac{\Psi_{\mu}(\tau \lambda)}{\Psi_{\mu}(\tau)} = \frac{1}{\lambda^{\rho}},
\]

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implies

$$\lim_{t \uparrow \infty} F_\mu(tx) = x^\rho,$$

(1.9)
as well as

$$\Psi_\mu(t^{-1}) \sim F_\mu(t) \Gamma(\rho + 1).$$

(1.10)
as $t \to \infty$.

(b) Suppose that

$$\limsup_{n \to \infty} \frac{1}{|F_\mu(\tau_n \lambda)|} < \infty$$

(1.11)

for each $\lambda > 0$. If Eq. (1.9) holds then so do Eq. (1.8) and Eq. (1.10).

Remark 1.9. It will shown in Section 2 that Eqs. (1.6) and (1.11) stem directly from the requirements of Theorem 2.3, Equation (1.7) stems from the conditions of [16, Proposition 3.6]. If the measures in Theorem 1.8 are positive, then Eq. (1.6) is clearly satisfied, Eq. (1.7) is satisfied by [16, Remark 3.7] and Eq. (1.11) is satisfied by Corollary 1.10 below.

The following corollary gives conditions for Eqs. (1.6) and (1.11) in Theorem 1.8 to be satisfied.

Corollary 1.10. Let $\mu \in M^*$ with decomposition $\mu^+, \mu^-$ such that $\Psi_\mu^\pm(\lambda) < \infty$ for all $\lambda > 0$.

(a) Suppose that there exists $\delta \in (0, 1)$ such that for small enough $\tau$ we have $\Psi_{\mu^-}(\tau) \leq \delta \Psi_{\mu^+}(\tau)$ (or vice versa). If

$$\inf_{\tau \downarrow 0} \sup_{\delta > 0} \left| \frac{1}{\mu} \left[ \Psi_{|\mu|}(\tau^{-1}(t-\delta), \tau^{-1}(t+\delta)) \right] \right| = 0.$$ 

(1.12)

and Eq. (1.8) hold, then Eqs. (1.6), (1.9) and (1.10) are satisfied.

(b) Suppose that there exists $\delta \in [0, 1)$ such that for large enough $t$ we have $F_\mu^-(t) \leq \delta F_\mu^+(t)$ (or vice versa). If Eq. (1.9) holds, then Eqs. (1.8), (1.10) and (1.11) are satisfied.

We apply Theorem 1.8 and Corollary 1.10 to the following example.

Example 1.11. Let $W$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, F, \mathbb{P})$ and let $F : \mathbb{R}^+ \to \mathbb{R}$ such that $F$ is equal to $x$ on $[0, 1]$ and $x l(x)$ for $x > 1$, where

$$l(x) := \exp \left\{ \int_1^x \frac{1}{u} \left[ \sin(u^2) \right] du \right\} = \exp \left\{ \int_1^x \frac{\sin(u^2)}{u} \right\}.$$ 

Moreover, let the process $X$ be defined such that

$$\begin{cases} dX_t = \left( \frac{d}{dt} F(t) + \frac{d^2}{dt^2} F(t) \right) dt + dW_t, \\ X_0 = 0. \end{cases}$$

Then, using the integrating factor $e^t$, one sees that

$$d(e^t X_t) = e^t \left[ \frac{d}{dt} F(t) + \frac{d^2}{dt^2} F(t) \right] dt + e^t dW_t$$

$$= \frac{d}{dt} \left[ e^t \frac{d}{dt} F(t) \right] dt + e^t dW_t,$$
whence
\[ \mathbb{E}[X_t] = \frac{d}{dt} F(t). \]
In particular, defining \( \mu \in \mathcal{M}^* \) via (1.1), we see by Eq. (1.4) that
\[ \lim_{t \to \infty} \frac{F_{\mu}(tx)}{F_{\mu}(t)} = x. \]
Moreover,
\[ x e^{x-1} \leq x \exp \left\{ - \int_1^x \frac{\sin(x^2)}{x^2} \, dx \right\} \leq xe^{x-1}, \]
whence
\[ F_{\mu}^-(x) \leq F_{\mu}^+(x) \left( 1 - \frac{xe^{x-1}}{F_{\mu}^+(x)} \right) \leq F_{\mu}^+(x) \left( 1 - e^{-x^{4/5}} \right) \leq F_{\mu}^+(x) \left( 1 - e^{-1} \right) \]
for \( x > 2 \). Thus, by Corollary 1.10(b),
\[ \lim_{\tau \to 0} \frac{\Psi_{\mu}(\tau \lambda)}{\Psi_{\mu}(\tau)} \to \lambda^{-1}, \]
for all \( \lambda > 0 \), and
\[ \lim_{T \to \infty} \left\{ \frac{1}{T \ell(T)} J_T(X) \right\} = \lim_{\delta \to 0} \left\{ \delta \tilde{l}(\delta) K_\delta(X) \right\}. \]
We visualise \( F_{\mu} \) in Fig. 2.

1.3 Organisation of the paper

Section 2 uses results from [16] to prove a continuity theorem (Theorem 2.3), which extends [8, XIII.1, Theorem 2a]. Section 3 proves Theorem 1.8 and Corollary 1.10. Appendix A includes additional referenced theorems and proofs.
Figure 1: A visualisation of $F_\mu$ defined in Example 1.7

Figure 2: A visualisation of $F_\mu$ defined in Example 1.11
2 Continuity Theorem

At its core, Karamata’s Tauberian theorem relies on a continuity theorem for generalised signed measures, which extends [8, XIII.1, Theorem 2a] for positive measures. The proof relies on the concept of vague convergence.

Let \( C(\mathbb{R}^+) \) be the space of all continuous \( \mathbb{R} \)-valued functions on \( \mathbb{R}^+ \), \( C_b(\mathbb{R}^+) \) the subspace of all \( f \in C(\mathbb{R}^+) \) such that \( f \) is bounded, \( C_0(\mathbb{R}^+) \) the subspace of all \( f \in C(\mathbb{R}^+) \) such that for any \( \varepsilon > 0 \), there exists a compact set \( K_\varepsilon \subset \mathcal{B}(\mathbb{R}^+) \) with \( |f| < \varepsilon \) on \( K_\varepsilon \), and \( C_c(\mathbb{R}^+) \) the subspace of all \( f \in C(\mathbb{R}^+) \) such that \( f \) has compact support. Clearly, we have the inclusions \( C_c(\mathbb{R}^+) \subseteq C_0(\mathbb{R}^+) \subseteq C_b(\mathbb{R}^+) \subseteq C(\mathbb{R}^+) \).

**Definition 2.1.** For any \( \mu \in \mathcal{M} \) we define \( I_\mu : C_b(\mathbb{R}^+) \to \mathbb{R} \) via the mapping

\[
f \mapsto \int_{\mathbb{R}^+} f \, d\mu.
\]

Then, we say that a sequence \( \{\mu_n\} \subset \mathcal{M} \) converges:

(a) vaguely to \( \mu \), if \( I_{\mu_n}(f) \to I_\mu(f) \) for all \( f \in C_c(\mathbb{R}^+) \), which we denote by

\[
\text{v-lim}_{n \to \infty} \mu_n = \mu;
\]

(b) weakly* to \( \mu \), if \( I_{\mu_n}(f) \to I_\mu(f) \) for all \( f \in C_b(\mathbb{R}^+) \), which we denote by

\[
\text{w-lim}_{n \to \infty} \mu_n = \mu.
\]

Note that \( C_0(\mathbb{R}^+) \) is also used as test functions for vague convergence in the literature. When \( \sup_{n \in \mathbb{N}} \|\mu_n\| < \infty \), it follows from [10, Proposition 1.3] that the two definitions of vague convergence are equivalent.

The following definition is based on [10, Definition 3.4], which provides a sufficient condition to ensure that distribution functions converge at continuity points.

**Definition 2.2.** We say that \( \{\mu_n\} \subset \mathcal{M}^* \) has no mass at a point \( x \in \mathbb{R}^+ \), if for any \( \varepsilon > 0 \), there exists an open neighbourhood \( (x - \varepsilon, x + \varepsilon) \), such that

\[
\limsup_{n \to \infty} |\mu_n|((x - \varepsilon, x + \varepsilon)) \leq \varepsilon.
\]

**Theorem 2.3 (Continuity Theorem).** Let \( \{\mu_n\} \cup \{\mu\} \subset \mathcal{M}^* \).

(a) Suppose \( \Psi_{\mu_n}(\lambda) \to \Psi_{\mu}(\lambda) \) for all \( \lambda > 0 \). Moreover, assume

\[
\limsup_{n \to \infty} \frac{\Psi_{\mu_n}(\varepsilon)}{|\Psi_{\mu_n}(\varepsilon)|} < \infty \tag{2.1}
\]

for some \( \varepsilon > 0 \). Under these conditions, if \( \{\mu_n\} \) has no mass at the continuity points of \( F_{\mu} \), then \( F_{\mu_n} \to F_{\mu} \) at all continuity points.

1Weak convergence is sometimes referred to as narrow convergence. See [4, Section 8.1].
(b) Suppose that
\[
\limsup_{n \to \infty} \Psi_{|\mu_n|}(\lambda) < \infty \tag{2.2}
\]
for all \(\lambda > 0\). If \(F_{\mu_n} \to F_\mu\) a.e. then \(\Psi_{\mu_n} \to \Psi_\mu\).

Proof. (a) Let \(\{\tilde{\mu}_n\} \subset \mathcal{M}\) such that \(\tilde{\mu}_n(dx) := (e^{-tx}/\Psi_{\mu_n}(\varepsilon))\mu_n(dx)\) for some \(\varepsilon > 0\). Note that Eq. (2.1) implies that

\[
\sup_{n \in \mathbb{N}} \|\tilde{\mu}_n\| = \sup_{n \in \mathbb{N}} \frac{\Psi_{|\mu_n|}(\varepsilon)}{\Psi_{\mu_n}(\varepsilon)} < \infty.
\]

Thus, by [16, Theorem 1.2] we can view \(\{\tilde{\mu}_n\}\) as a bounded family in \((C_0(\mathbb{R}^+))^\ast\). It follows by the Banach-Alaoglu theorem that \(\{\tilde{\mu}_n\}\) is compact in the weak\(^\ast\)-topology \(\sigma((C_0(\mathbb{R}^+))^\ast, C_0(\mathbb{R}^+))\). Furthermore, as a consequence of Theorem A.1, \(C_0(\mathbb{R}^+)\) is separable, whence the sequence is sequentially compact. Thus, there exists a subsequence \(\{n_k\}\) such that \(\nu-lim_{k \to \infty} \tilde{\mu}_{n_k} = \tilde{\mu} \in \mathcal{M}\).

It follows from our hypothesis that

\[
\Psi_{\tilde{\mu}_n}(\lambda) = \frac{\Psi_{\mu_n}(\lambda + \varepsilon)}{\Psi_{\mu_n}(\varepsilon)} \to \frac{\Psi_\mu(\lambda + \varepsilon)}{\Psi_\mu(\varepsilon)}
\]

for any \(\lambda > 0\). Since every element in \(\mathcal{M}^\ast\) is characterised by its Laplace transform by Proposition A.3 by our above argument we have that every subsequence of \(\{\tilde{\mu}_n\}\) has a further vague-convergent subsequence which must converge to \(\tilde{\mu}\). Thus, \(\tilde{\mu} = \nu-lim_{n \to \infty} \mu_n\).

Now take any \(f \in C_c(\mathbb{R}^+)\). By the vague convergence of \(\{\tilde{\mu}_n\}\) and our hypothesis, we have

\[
\int_{\mathbb{R}^+} f \, d\mu_n = \Psi_{\mu_n}(\varepsilon) \int_{\mathbb{R}^+} f(x)e^{tx}\tilde{\mu}_n(dx) \to \Psi_\mu(\varepsilon) \int_{\mathbb{R}^+} f(x)e^{tx}\tilde{\mu}(dx) = \int_{\mathbb{R}^+} f \, d\mu.
\]

Since \(\{\mu_n\}\) has no mass at the continuity points of \(F_\mu\), it follows from a simple generalisation of [16, Proposition 3.6] that \(F_{\mu_n} \to F_\mu\) at all points of continuity of \(F_\mu\).

(b) Define \(\tilde{\mu}_n \in \mathcal{M}^\ast\) such that \(\tilde{\mu}_n(dx) := e^{-tx}\mu_n(dx)\) for some \(\varepsilon > 0\). Using integration by parts we see

\[
F_{\tilde{\mu}_n}(t) = \int_0^t e^{-tx} F_{\mu_n}(dx) = e^{-et} F_{\mu_n}(t) + \varepsilon \int_0^t e^{-tx} F_{\mu_n}(x) \, dx.
\]

Note that for every \(t > 0\), \(\sup_{n \in \mathbb{N}} F_{|\mu_n|}(t) < \infty\). Indeed, if this were not the case for some \(t > 0\), then for any \(\lambda > 0\)

\[
\sup_{n \in \mathbb{N}} \Psi_{|\mu_n|}(\lambda) \geq e^{-\lambda t} \sup_{n \in \mathbb{N}} F_{|\mu_n|}(t) = +\infty,
\]

which contradicts Eq. (2.2). Thus, we may use our hypothesis along with the dominated convergence theorem to deduce that

\[
F_{\tilde{\mu}_n}(t) = e^{-et} F_{\mu_n}(t) + \varepsilon \int_0^t e^{-tx} F_{\mu_n}(x) \, dx \to e^{-et} F_\mu(t) + \varepsilon \int_0^t e^{-tx} F_\mu(x) \, dx = F_\tilde{\mu}(t).
\]

Since \(\sup_{n \in \mathbb{N}} \|\tilde{\mu}_n\| = \sup_{n \in \mathbb{N}} \Psi_{|\mu_n|}(\varepsilon) < \infty\), it follows from [16, Proposition 3.6] that \(\nu-lim_{n \to \infty} \tilde{\mu}_n = \tilde{\mu}\). In particular, for any \(\lambda > 0\),

\[
\Psi_{\tilde{\mu}_n}(\lambda + \varepsilon) = \Psi_{\tilde{\mu}_n}(\lambda) \to \Psi_{\tilde{\mu}}(\lambda) = \Psi_\mu(\lambda + \varepsilon).
\]

Since \(\varepsilon\) is arbitrary, we have \(\Psi_{\tilde{\mu}_n}(\lambda) \to \Psi_\mu(\lambda)\) for any \(\lambda > 0\). \(\Box\)
Remark 2.4. Eq. (2.1) is essential to the proof, since it allows us to construct the vague-convergent subsequence in part (a). The argument is analogous to Helly’s selection theorem, which states that a uniformly bounded sequence of functions of bounded variation admits a convergent subsequence.

If we are dealing with positive measures \( \{ \mu_n \} \subset \mathcal{M}^* \), then Eq. (2.1) is always satisfied, as is the condition that \( \{ \mu_n \} \) has no mass at the continuity points of \( F_\mu \).

For \( \{ \mu_n \} \subset \mathcal{M}^* \), a sufficient condition to satisfy Eq. (2.1) is that for some \( \varepsilon > 0 \) there exists \( \delta \in [0, 1) \) such that \( \Psi_{\mu_n}^-(\varepsilon) < \delta \Psi_{\mu_n}^+(\varepsilon) \). Indeed, if this is the case then

\[
\limsup_{n \to \infty} \frac{\Psi_{\mu_n}(\varepsilon)}{|\Psi_{\mu_n}(\varepsilon)|} \leq \limsup_{n \to \infty} \left\{ \frac{(1 + \delta)\Psi_{\mu_n}^+(\varepsilon)}{|\Psi_{\mu_n}(\varepsilon)|} \right\} \leq \left( \frac{1 + \delta}{1 - \delta} \right) \limsup_{n \to \infty} \frac{\Psi_{\mu_n}(\varepsilon)}{|\Psi_{\mu_n}(\varepsilon)|} < \infty.
\]

3 Proof of Theorem 1.8 and Corollary 1.10

Proof of Theorem 1.8. Throughout let \( \{ \tau_n \} \subset \mathbb{R}^+ \) be an arbitrary null sequence and define the sequence \( \{ t_n \} \) such that \( t_n := \tau_n^{-1} \).

(a) Suppose Eq. (1.8) holds. Define \( \mu_n, \nu_n, \nu \in \mathcal{M}^* \) such that,

\[
F_{\mu_n}(x) := F_\mu(t_n x), \quad \nu_n := \frac{\mu_n}{\Psi_\mu(\tau_n)}, \quad \nu(dx) := \frac{x^{\rho-1}}{\Gamma(\rho)} \, dx.
\]

It follows by Eqs. (1.6) and (1.8) that

\[
\Psi_{\nu_n}(\lambda) = \frac{\Psi_{\mu_n}(\lambda)}{\Psi_\mu(\tau_n)} = \frac{\Psi_\mu(\tau_n \lambda)}{\Psi_\mu(\tau_n)} \to \lambda^{-\rho} = \Psi_\nu(\lambda)
\]

and

\[
\limsup_{n \to \infty} \frac{\Psi_{\nu_n}(\varepsilon)}{|\Psi_{\nu_n}(\varepsilon)|} = \limsup_{n \to \infty} \frac{\Psi_{|\mu|}(\tau_n \varepsilon)}{|\Psi_{|\mu|}(\tau_n \varepsilon)|} < \infty.
\]

Moreover, by Eq. (1.7) we have that \( \{ \nu_n \} \) has no mass at all points of \( \mathbb{R}^+ \). Indeed,

\[
\inf_{\varepsilon > 0} \limsup_{n \to \infty} |\nu_n|(\{ (t - \varepsilon, t + \varepsilon) \}) \leq \inf_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{\Psi_\mu(\tau_n)} \int_{t-\varepsilon}^{t+\varepsilon} |\mu_n|(dx) = \inf_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{\Psi_\mu(\tau_n)} \int_{\tau_n^{-1}(t-\varepsilon)}^{\tau_n^{-1}(t+\varepsilon)} |\mu|(dx) \leq \inf_{\varepsilon > 0} \limsup_{n \to \infty} \frac{|\mu|([\tau_n^{-1}(t-\varepsilon), \tau_n^{-1}(t+\varepsilon)])}{|\Psi_\mu(\tau_n)|} = 0.
\]

We may conclude by Theorem 2.3 a) that \( F_{\nu_n} \to F_\nu \). In particular

\[
F_{\nu_n}(x) = \frac{F_\mu(t_n x)}{\Psi_\mu(\tau_n)} \to F_\nu(x) = \frac{x^\rho}{\Gamma(\rho + 1)}
\]
as \( n \to \infty \). Thus, we have shown Eq. (1.10) by letting \( x = 1 \). Furthermore Eq. (1.9) holds since

\[
\frac{F_\mu(t_n x)}{F_\mu(t_n)} = \frac{F_\mu(t_n x)}{\Psi_\mu(\tau_n)} \Psi_\mu(\tau_n) \to \frac{x^\rho}{\Gamma(\rho + 1)} = x^\rho.
\]
(b) Suppose Eq. (1.9) holds. Let \( \mu_n \) be defined as before, but define \( \nu_n, \nu \in \mathcal{M}^* \) such that
\[
\nu_n := \frac{\mu_n}{F_\mu(t_n)}, \quad \nu(dx) := \rho x^{\rho-1} \, dx.
\]
From Eq. (1.9), it holds that
\[
F_{\nu_n}(x) = \frac{F_\mu(t_n x)}{F_\mu(t_n)} \to x^\rho = F_\nu(x).
\]
By Eq. (1.11), we have
\[
\limsup_{n \to \infty} \Psi_{|\nu_n|}(\lambda) = \limsup_{n \to \infty} \frac{\Psi_{|\mu|}(\tau_n \lambda)}{|F_\mu(\tau_n)|} < \infty
\]
for \( \lambda > 0 \), so by Theorem 2.3(b) we have Eq. (1.10) since
\[
\Psi_{\nu_n}(\lambda) = \frac{\Psi_{\mu}(\tau_n \lambda)}{F_\mu(\tau_n)} \to \frac{\Gamma(\rho + 1)}{\lambda^\rho} = \Psi_\nu(\lambda)
\]
for \( \lambda > 0 \). Moreover, we get Eq. (1.8) since
\[
\frac{\Psi_{\mu}(\tau_n \lambda)}{\Psi_\mu(\tau_n)} = \frac{\Psi_{\mu}(\tau_n \lambda)}{F_\mu(t_n)} \frac{F_\mu(t_n)}{\Psi_\mu(\tau_n)} \to \frac{\Gamma(\rho + 1)}{\lambda^\rho} \frac{1}{\Gamma(\rho + 1)} = \lambda^{-\rho}.
\]

**Proof of Corollary 1.10.** Throughout let \( \{\tau_n\} \subset \mathbb{R}^+ \) be an arbitrary null sequence and define the sequence \( \{t_n\} \) such that \( t_n := \tau_n^{-1} \).

(a) Suppose Eq. (1.8) holds, and there exists \( \delta \in [0,1) \) and \( N_\delta \in \mathbb{N} \) such that \( \Psi_{\mu^*}(\tau_n) \leq \delta \Psi_{\mu^*}(\tau_n) \) for \( n \geq N_\delta \). Define \( \mu_n, \nu_n, \nu \in \mathcal{M}^* \) as in the proof of Theorem 1.8(a). It follows by Eq. (1.8)
\[
\limsup_{n \to \infty} \frac{\Psi_{|\nu_n|}(\varepsilon)}{|\Psi_{|\nu_n|}(\varepsilon)|} = \limsup_{n \to \infty} \frac{\Psi_{|\mu|}(\tau_n \varepsilon)}{|\Psi_{|\mu|}(\tau_n \varepsilon)|} \leq \left| \frac{\Psi_{\mu}(\tau_n \varepsilon)}{\Psi_{\mu}(\tau_n \varepsilon)} \right| \left( \frac{1 + \delta}{1 - \delta} \right) < \infty,
\]
whence Eq. (1.7) is satisfied. Moreover, Eq. (1.7) follows from Eq. (1.12) and \( \Psi_{|\mu|}(\tau_n) \leq \left( \frac{1+\delta}{1-\delta} \right) |\Psi_{\mu}(\tau_n)| \). The remainder of the result follows from the proof of Theorem 1.8.

(b) Suppose Eq. (1.9) holds, and assume there exists \( \delta \in [0,1) \) and \( N_\delta \in \mathbb{N} \) such that \( F_{\mu^*}(t_n) \leq \delta F_{\mu^*}(t_n) \) for \( n \geq N_\delta \). Define \( \mu_n, \nu_n, \nu \in \mathcal{M}^* \) as in the proof of Theorem 1.8(b). The result follows from Theorem 1.8 if we show that
\[
\limsup_{n \to \infty} \Psi_{|\nu_n|}(\lambda) = \limsup_{n \to \infty} \frac{\Psi_{|\mu|}(\tau_n \lambda)}{|F_\mu(t_n)|} < \infty
\]
for \( \lambda > 0 \), i.e. Eq. (1.11) is satisfied. Without loss of generality, take \( \lambda = 1 \). By our hypothesis, there exists some \( N_\delta > 0 \) such that
\[
\Psi_{|\mu|}(\tau_n) = \int_{\mathbb{R}^+} e^{-\tau_n x} |\mu| (dx)
\]
\[
\leq F_{|\mu|}(t_n) + \sum_{k=0}^{\infty} \int_{t_n 2^k}^{t_n 2^{k+1}} e^{-\tau_n x} |\mu| (dx)
\]
\[
\leq \left( \frac{1 + \delta}{1 - \delta} \right) \left\{ |F_\mu(t_n)| + \sum_{k=0}^{\infty} e^{-2k} |F_\mu(t_n 2^{k+1})| \right\}
\]
for $t_n \geq N_\delta$. Moreover, by Eq. (139), there exists $M \in \mathbb{N}$ such that

$$|F_\mu(2t)/F_\mu(t) - 2^\rho| \leq 2^\rho$$

for $t \geq M$. In particular, by induction we have $F_\mu(t(2^{n+1}) \leq 2^{(n+1)(\rho+1)}F_\mu(t)$. Thus, it follows that

$$\frac{\Psi_{\mu|t}(t_n)}{|F_\mu(t_n)|} \leq \left( \frac{1 + \delta}{1 - \delta} \right) \left( 1 + \sum_{k=0}^{\infty} e^{-2^k 2^{(k+1)(\rho+1)}} \right) < \infty$$

for $t_n \geq M \lor N_\delta$. Thus, $\limsup_{n \to \infty} \frac{\Psi_{\mu|t}(t_n)}{|F_\mu(t_n)|} < \infty$. \qed

## A Appendix

### A.1 Stone-Weierstraß

It is useful to note that $C_c(\mathbb{R}^+)$ is a dense subset of $C_0(\mathbb{R}^+)$, which is a direct consequence of the Stone-Weierstraß Theorem as stated by de Branges [7]. Recall that $\mathcal{C} \subset C_0(\mathbb{R}^+)$ vanishes nowhere on $\mathbb{R}^+$ if for all $x \in \mathbb{R}^+$ there exists some $f \in \mathcal{C}$ such that $f(x) \neq 0$. Moreover, $\mathcal{C}$ separates points if for each $x, y \in \mathbb{R}^+$ such that $x \neq y$, there exists $f \in \mathcal{C}$ such that $f(x) \neq f(y)$.

**Theorem A.1** (Stone-Weierstraß Theorem). Let $\mathcal{C}$ be a subalgebra of $C_0(\mathbb{R}^+)$. Then $\mathcal{C}$ is dense in $C_0(\mathbb{R}^+)$ (given the topology of uniform convergence) if and only if it separates points and vanishes nowhere.

### A.2 Characterisation via Laplace Transform

It is crucial in Theorem 13.8 that elements of $\mathcal{M}^*$ are characterised by their Laplace transforms. In order to show the characterisation, we remind ourselves of the notion of a separating class.

**Definition A.2.** Let $\mathcal{F} \subset \mathcal{M}$. We say that a family $\mathcal{C}$ of measurable maps $\mathbb{R}^+ \to \mathbb{R}$ is a separating family for $\mathcal{F}$ if, for any two measures $\mu, \nu \in \mathcal{F}$,

$$\left( \int f \, d\mu = \int f \, d\nu \quad \forall f \in \mathcal{C} \cap L^1(\mu) \cap L^1(\nu) \right) \Rightarrow \mu = \nu.$$

The following is a trivial consequence of [14] Theorem 13.11.

**Theorem A.3.** $C_c((0,\infty)) \cap \text{Lip}_1((0,\infty),[0,1])$ is a separating class for $\mathcal{M}$.

**Corollary A.4.** Let $\mathcal{C} \subset C_0(\mathbb{R}^+)$ be a sub algebra that separates points and vanishes nowhere. Then $\mathcal{C}$ is a separating family for $\mathcal{M}$.

**Proposition A.5** (Characterisation via Laplace Transfrom). Let $\mu \in \mathcal{M}^*$. If $\Psi_\mu$ is well defined, then $\mu$ is uniquely determined by $\Psi_\mu$.

**Proof.** Define $\mathcal{C} := \{ e^{-\lambda x} : \lambda > 0 \}$ and let $\mathcal{C}'$ be the set of finite linear combinations of elements in $\mathcal{C}$. Then $\mathcal{C}'$ is a sub-algebra of $C_0(\mathbb{R}^+)$ which vanishes nowhere. It follows from Corollary A.4 that measures in $\mathcal{M}$ are characterised by their Laplace transform.

Now let $\mu \in \mathcal{M}^*$ such that $\Psi_\mu$ exists. Let $\varepsilon > 0$ and define $\tilde{\mu} \in \mathcal{M}(\mathbb{R}^+)$ via $\tilde{\mu}(dx) := e^{-\varepsilon x} \mu(dx)$. Then for $\lambda > 0$

$$\Psi_\mu(\lambda + \varepsilon) = \Psi_\tilde{\mu}(\lambda). \quad (A.1)$$

Since $\tilde{\mu}$ is characterised by $\Psi_{\tilde{\mu}}$, Eq. (A.1) implies that $\mu$ is characterised by $\Psi_\mu$ since $\varepsilon$ is arbitrary. \qed
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