Research Article

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Extensions of Three Matrix Inequalities to Semisimple Lie Groups

Abstract: We give extensions of inequalities of Araki-Lieb-Thirring, Audenaert, and Simon, in the context of semisimple Lie groups.

Keywords: Araki-Lieb-Thirring inequality, positive definite matrices, semisimple Lie groups, log majorization, Kostant’s pre-order

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Dedicated to In memory of our colleague Professor William Ullery who passed away on January 1, 2012.

1 Introduction

Let $\mathbb{C}^{n \times n}$ denote the vector space of $n \times n$ complex matrices. A norm $\| \cdot \|$ on $\mathbb{C}^{n \times n}$ is said to be unitarily invariant if $\| UAV \| = \| A \|$ for all unitary $U, V \in \mathbb{C}^{n \times n}$. For example, the spectral norm is unitarily invariant. The characterization of unitarily invariant norms in terms of symmetric gauge functions has been given by von Neumann [16] (see [3, p.91]).

A famous inequality with many applications is the Araki-Lieb-Thirring inequality [1, 9]:

**Theorem 1.1.** (Araki-Lieb-Thirring) Suppose $A, B \in \mathbb{C}^{n \times n}$ are positive semidefinite. If $r \geq 1$, then

$$\text{tr} (ABA)^r q \leq \text{tr} (A^r B^r A^r)^q,$$

(1.1)

for all $q \geq 0$. If $0 \leq r \leq 1$, the inequality is reversed.

For $A \in \mathbb{C}^{n \times n}$, let $|A| = (A^* A)^{1/2}$ denote the positive semidefinite part of $A$ in the polar decomposition $A = U|A|$, where $U \in \mathbb{C}^{n \times n}$ is unitary. Audenaert [2] obtained the following inequality as a generalization of Araki-Lieb-Thirring inequality.

**Theorem 1.2.** (Audenaert [2, Proposition 3]) Suppose $A, B \in \mathbb{C}^{n \times n}$ with $B$ Hermitian. If $r \geq 1$, then for any unitarily invariant norm $\| \cdot \|$:

$$\| ABA^r \| \leq \| A^r B^r A^r \| .$$

(1.2)

If $0 \leq r \leq 1$, the inequality is reversed.

The case $0 \leq r \leq 1$ in Theorem 1.2 is not stated in [2], but it can be derived by a similar method as for the case $r \geq 1$.

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Let $|A| = (AA^*)^{1/2} = |A^*|$. So $A = |A^*|V$ is also a polar decomposition for some unitary $V \in \mathbb{C}_{n,n}$. Since $ABA^*$ is Hermitian, (1.2) is equivalent to

$$
\| |ABA^*| \| = \| |ABA^*|' \| \leq \| |A^*|B|A| \| = \| |A^*|B|A| \|'.
$$

(1.3)

The following result of Simon [11, p.95] is also interesting (see [3, p.253, p.285] for historical remarks).

**Theorem 1.3.** (Simon [11]) Let $A, B \in \mathbb{C}_{n,n}$ such that the product $AB$ is normal. For any unitarily invariant norm $\| \cdot \|$ on $\mathbb{C}_{n,n}$,

$$
\| AB \| \leq \| BA \|.
$$

(1.4)

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Let $x^i = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ denote a rearrangement of the components of $x$ such that $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. We say that $x$ is *majorized* by $y$, denoted by $x \prec y$, if

$$
\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n - 1, \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.
$$

We say that $x$ is *weakly majorized* by $y$, denoted by $x \prec_w y$, if the equality becomes inequality. An equivalent condition for $x \prec y$ is

$$
\text{conv } S_n \cdot x \subset \text{conv } S_n \cdot y,
$$

where $\text{conv } S_n \cdot x$ denotes the convex hull of the orbit of $x$ under the action of the symmetric group $S_n ([6, 10])$. When $x$ and $y$ are nonnegative, we say that $x$ is *log majorized* by $y$, denoted by $x \prec_{\log} y$, if

$$
\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n - 1, \quad \text{and} \quad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.
$$

In other words, when $x$ and $y$ are positive, we have $x \prec_{\log} y$ if and only if $\log x \prec y$.

For $X \in \mathbb{C}_{n,n}$, let $s(X)$ and $\lambda(X)$ be the vector of singular values of $X$ in decreasing order and the vector of eigenvalues of $X$ whose absolute values are in decreasing order, respectively. If $X, Y \in \mathbb{C}_{n,n}$, then $\| X \| \leq \| Y \|$ for all unitarily invariant norms $\| \cdot \|$ if and only if $s(X) \prec_w s(Y)$, according to Ky Fan’s Dominance Theorem (see [3, p.93]). Since both $|ABA^*|$ and $|A^*|B|A|$ are positive semidefinite, Theorem 1.2 amounts to say

$$
\lambda(|ABA^*|) \prec_w \lambda(|A^*|B|A|), \quad \text{if } r \geq 1.
$$

(1.5)

We are going to extend Theorem 1.1, Theorem 1.2, and Theorem 1.3 in the context of noncompact connected semisimple Lie groups. We need to be cautious since norm is a concept for vector spaces but not for groups. Indeed, there is another way to express the relationship between the vectors of eigenvalues of $|ABA^*|$ and $|A^*|B|A|$:

$$
\lambda(|ABA^*|) \prec_{\log} \lambda(|A^*|B|A|), \quad \text{if } r \geq 1.
$$

(1.6)

In fact, (1.5) and (1.6) are equivalent.

**Theorem 1.4.** Suppose $A, B \in \mathbb{C}_{n,n}$ with $B$ Hermitian. If $r \geq 1$, the following statements are valid and are equivalent:

1. $\| |ABA^*|' \| \leq \| |A^*|B|A| \|'$ for all unitarily invariant norms $\| \cdot \|$ on $\mathbb{C}_{n,n}$.
2. $\lambda(|ABA^*|') \prec_w \lambda(|A^*|B|A|')$.
3. $\lambda(|ABA^*|') \prec_{\log} \lambda(|A^*|B|A|')$.

If $0 \leq r \leq 1$, the above inequalities are reversed.

**Proof.** It suffices to show that (2) and (3) are equivalent for $r \geq 1$. Notice that $a \prec_{\log} b$ implies $a \prec_w b$ for all nonnegative vectors $a, b \in \mathbb{R}^n$ [5, Proposition 1.3] or [17, Theorem 2.7]. So (3) implies (2). To establish the converse, we first notice that $\lambda_k(|ABA^*|') \leq \lambda_k(|A^*|B|A|')$. Then we use the $k$th compound matrix $C_k(\cdot)$ [10, p.502–504] argument. Note that if $M = UP$ (where $P = (A^*A)^{1/2}$ and $U$ unitary) is the polar decomposition of
$M \in \mathbb{C}_{n,n}$, so is $C_k(M) = C_k(U)C_k(P)$. Since the $k$th compound matrix is multiplicative and respects complex conjugate transpose, we have

$$C_k(|ABA^*|') = |C_k(ABA^*)|^' = |C_k(A)C_k(B)C_k(A)|',$$

and

$$C_k(|A'|B'|A|') = [C_k(|A|)]'[C_k(|B|)]'[C_k(|A|)]' = |C_k(A)|' |C_k(B)|' |C_k(A)|'.$$

Moreover, for any positive semidefinite $Y \in \mathbb{C}_{n,n}$, the eigenvalues of $C_k(Y)$ are the $\binom{n}{2}$ products $\prod_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_k}(Y)$ of the eigenvalues $\lambda_i(Y) \geq \cdots \geq \lambda_n(Y)$ of $Y$. Thus

$$\prod_{1 \leq i \leq k} \lambda_i(|ABA^*|') = \lambda_i(C_k(|ABA^*|')) \leq \lambda_i(C_k(|A'|B'|A|')) = \prod_{1 \leq i \leq k} \lambda_i(|A'|B'|A|')$$

for $k = 1, \ldots, n$. They are equal when $k = n$ because $\det|ABA^*'| = \det(|A'|B'|A|')$. 

**Remark 1.5.** It is claimed in [2] that Theorem 1.2 is a generalization of Theorem 1.1, but no proof is given there. We now offer a proof. If $A, B \in \mathbb{C}_{n,n}$ are positive semidefinite, so are $ABA^* = ABA$ and $|A'|B'|A|' = A'B'A'$. By Theorem 1.1 and Weyl-Horn's theorem [15], there exists a matrix in $\mathbb{C}_{n,n}$ with eigenvalues $\lambda_i(|ABA|') \geq \cdots \geq \lambda_n(|ABA|')$ and singular values $\lambda_1(|ABA') \geq \cdots \geq \lambda_n(|ABA')$. Theorem 1.1 then follows by [3, Theorem II.3.6].

**Remark 1.6.** Notice that (1.4) can be stated as any of the following forms:

$$||AB|| \leq ||BA||, \quad ||AB|| \leq ||BA||, \quad ||B^*A' AB|| \leq ||A'B^*BA||.$$

The group version of Theorem 1.3 takes the following stronger form: If $A, B \in \mathbb{C}_{n,n}$ are nonsingular with $A$ normal, then

$$||A|| \leq ||BAB^{-1}||,$$

for all unitarily invariant norm. Here is a short proof: $s(A) = |\lambda(A)| = |\lambda(BAB^{-1})| \leq_w s(BAB^{-1}).$

The equivalent condition (1.6) is the one that suits our extensions. In other words, we will express our group results in terms Kostant’s pre-order instead of unitarily invariant norm or weak majorization. Kostant’s preorder and log majorization are equivalent when all matrices are nonsingular, as we will see this in Example 2.2.

## 2 Main results

Our goal is to establish a generalized form of (1.6) in the context of noncompact connected semisimple Lie groups. To do so, we need to introduce some basic concepts. The reader is referred to [4, 8] for the standard notation.

Let $G$ be a noncompact connected semisimple Lie group with Lie algebra $\mathfrak{g}$, let $\Theta: G \to G$ be a nontrivial Lie group isomorphism with $\Theta^2$ being the identity map on $G$, called a involution, and let $K$ be the fixed point set of $\Theta$, which is an analytic subgroup of $G$. The differential map $d\Theta: \mathfrak{g} \to \mathfrak{g}$ of $\Theta$ has eigenvalues $\pm 1$. The eigenspace of $d\Theta$ corresponding to $1$ is the Lie algebra $\mathfrak{t}$ of $K$, and the eigenspace of $d\Theta$ corresponding to $-1$ is an $\text{Ad}K$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ complementary to $\mathfrak{t}$. Since $G$ is semisimple, the Killing form $B$ on $\mathfrak{g}$ is nondegenerate. Let $\Theta$ be chosen such that $B$ is negative definite on $\mathfrak{t}$ and positive definite on $\mathfrak{p}$. This is equivalent to say that the bilinear form $B_\Theta$ defined by

$$B_\Theta(X, Y) = -B(X, d\Theta Y), \quad X, Y \in \mathfrak{g}$$

is an inner product on $\mathfrak{g}$. In this case, the decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$ is called Cartan decomposition of $\mathfrak{g}$, and $d\Theta$ is called Cartan involution of $\mathfrak{g}$, and $\Theta$ is called Cartan involution of $G$. Then the map $\mathfrak{p} \times K \to G$, $(X, k) \to$
g = \exp Xk, is a diffeomorphism, where \( \exp : X \rightarrow G \) is the Lie group exponential map ([4, VI. Theorem 1.1]). Let \( P := \{ e^X : X \in p \} \). Then every \( g \in G \) can be uniquely written as \( g = p(g)k(g) = pk \) with \( p = p(g) \in P \) and \( k = k(g) \in K \). The decomposition \( G = PK \) is called Cartan decomposition of \( G \). Let \( * : G \rightarrow G \) be the diffeomorphism defined by \( g^* = \Theta(g^{-1}) \). Then \( k^* = k^{-1} \) for \( k \in K \) and \( p^* = p \) for \( p \in P \). We remark that \( P \) is a subset of the fixed point set of \( \Theta \). An element \( g \in G \) is said to be normal if \( g \) and \( g^* \) commute.

An element \( X \in g \) is called real semisimple (resp., nilpotent) if \( \text{ad} X \) is diagonalizable over \( \mathbb{R} \) (resp., \( \text{ad} X \) is nilpotent). An element \( g \in G \) is called hyperbolic (resp., unipotent) if \( g = \exp X \) for some real semisimple (resp., nilpotent) \( X \in g \); in either case \( X \) is unique and we write \( X = \log g \). An element \( g \in G \) is called elliptic if \( \text{Ad} g \) is diagonalizable over \( \mathbb{C} \) with eigenvalues of modulus 1.

The following important result, due to Kostant [8], is called the complete multiplicative Jordan decomposition, abbreviated as CMJD.

**Theorem 2.1.** (Kostant [8, Proposition 2.1]) Each \( g \in G \) can be uniquely written as \( g = ehu \), where \( e \) is elliptic, \( h \) is hyperbolic, \( u \) is unipotent, and the three elements \( e, h, u \) commute.

Let \( a \) be a maximal abelian subspace of \( p \) and let \( A \) be the analytic subgroup generated by \( a \). We have \( p = \text{Ad}(K)a \) ([7, p. 378]). The Weyl group \( W \) of \( (g, a) \) acts simply transitively on \( a \), and also on \( A \) through the exponential map \( \exp : a \rightarrow A \).

For any real semisimple element \( X \in g \), let \( W(X) \) denote the set of elements in \( a \) that are conjugate to \( X \):

\[
W(X) = \text{Ad} G(X) \cap a.
\]

It is known that \( W(X) \) is a single \( W \)-orbit in \( a \) ([8, Proposition 2.4]). Let \( \text{conv} W(X) \) be the convex hull in \( a \) generated by \( W(X) \). For any \( g \in G \), define

\[
A(g) := \exp \text{conv} W(\log h(g)),
\]

where \( h(g) \) is the hyperbolic component of \( g \) in its CMJD.

Kostant’s pre-order \( \prec \) on \( G \) ([8, p. 426]) is defined by setting \( f \prec g \) if

\[
A(f) \subset A(g).
\]

This pre-order induces a partial order on the conjugacy classes of \( G \).

**Example 2.2.** See [14, Proposition 2.2] for the example \( G = \text{SL}_n(\mathbb{C}) \) in which \( \prec \) becomes log majorization: For \( A, B \in \text{SL}_n(\mathbb{C}) \), \( A \prec B \) if and only if \( |\lambda(A)| \prec_\log |\lambda(B)| \), where \( |\lambda(A)| = (|\lambda_1(A)|, \ldots, |\lambda_n(A)|) \) with the components in decreasing order. When \( A \) and \( B \) are positive definite, \( \lambda(A) = \sigma(A) > 0 \) and \( \lambda(B) = \sigma(B) > 0 \), so \( A \prec B \) if and only if \( \lambda(A) \prec_\log \lambda(B) \).

Kostant’s pre-order does not depend on the choice of \( a \) due to the following result.

**Theorem 2.3.** (Kostant [8, Theorem 3.1]) Let \( f, g \in G \). Then \( f \prec g \) if and only if

\[
\rho(\pi(f)) \leq \rho(\pi(g))
\]

for any irreducible finite dimensional representation \( \pi \) of \( G \), where \( \rho(\pi(g)) \) denotes the spectral radius of the operator \( \pi(g) \).

The following is our first main result, as an extension of Theorem 1.2.

**Theorem 2.4.** Suppose \( g, h \in G \) and \( h^* = h \). If \( r \geq 1 \), then

\[
p^r(ggh^*) \prec p^r(g^*)p^r(h)p^r(g^*),
\]

where \( p^r(g) := [p(g)]^r \). If \( 0 \leq r \leq 1 \), the inequality is reversed.
Proof. Let $G = PK$ be a Cartan decomposition of $G$. Let $\hat{G}$ denote the set of all irreducible finite dimensional representations of $G$. For each $\pi \in \hat{G}$, let $V_\pi$ be the representation space. We can fix once and for all an inner product on $g$ such that $\pi(p) \in \text{Aut} V_\pi$ is positive definite for all $p \in P$ and $\pi(k) \in \text{Aut} V_\pi$ is unitary for all $k \in K$ [8, p.435]. For $g \in G$, let $g = pk$ be such that $p \in P$ and $k \in K$. Then $\pi(g) = \pi(p)\pi(k)$ is the polar decomposition of $\pi(g)$. Also, $\pi^* = k^{-1}p$ and thus $\pi(g^*) = [\pi(k)]^{-1}\pi(p)$. Thus we have

$$|\pi(g)| = (\pi(g)\pi^*(g))^{1/2} = \pi(p). \tag{2.2}$$

On the other hand, $\pi^*(g) = [\pi(g)]^* = (\pi(p)\pi(k))^* = [\pi(k)]^{-1}\pi(p)$.

In other words,

$$\pi^*(g) = \pi(g^*) \tag{2.3}$$

where $\pi^*(g)$ is the adjoint of $\pi(g) \in \text{Aut} V_\pi$. By Theorem 2.3 it suffices to show that for all $\pi \in \hat{G}$

$$\rho(\pi(p^*(ghg^*))) \leq \rho(\pi(p^*(g^*)p'(h)p^*(g^*))) \tag{2.4}$$

where $\rho(\cdot)$ denotes the spectral radius. From (2.2) and the fact that $\rho(g)$ is positive definite, we have

$$\|\pi(g)|| = \|\pi(g)|| = \rho(\pi(p)) \quad \text{for all } g \in G, \tag{2.5}$$

where $\|\cdot\|$ denotes the spectral norm on $\text{End} V_\pi$. Thus

$$\rho(\pi(p^*(ghg^*))) = \|\pi(p^*(ghg^*))\| \quad \text{by (2.5)}$$

$$= \|\pi^*(p^*(ghg^*))\| \quad \text{since } \pi \in \hat{G}$$

$$= \|\pi^*(ghg^*)\| \quad \text{by (2.2)}$$

$$= \|\pi(g)\pi(h)\pi^*(g)\| \quad \text{by (2.3)}$$

$$\leq \|\pi^*(g)|\pi(h)|\pi^*(g)| \| \quad \text{by (1.3)}$$

$$= \|\pi(g^*)|\pi(h)|\pi(g^*)| \| \quad \text{by (2.3)}$$

$$= \rho(\pi(g^*)|\pi(h)|\pi(g^*)|) \quad \text{since } |\pi(g^*)|^r|\pi(h)|^r|\pi(g^*)|^r \text{ is p.d.}$$

$$= \rho(\pi^*(p^*(g^*))\pi^*(p^*(h))\pi^*(p^*(g^*))) \quad \text{by (2.2)}$$

$$= \rho(\pi(p^*(g^*)p'(h)p^*(g^*))) \quad \text{since } \pi \in \hat{G}.$$

Thus (2.4) is established. The case $0 \leq r \leq 1$ is similar. \hfill \Box

We remark that similar technique was first used in [13].

The following is an extension of Theorem 1.1 in the context of semisimple Lie groups, i.e., we can obtain the result for $GL_n(\mathbb{C})$ by appropriate scaling on the semisimple Lie group $SL_n(\mathbb{C})$.

**Theorem 2.5.** Suppose $g, h \in G$ and $h^* = h$. If $r \geq 1$, then for any finite dimensional character $\chi$ of $G$,

$$\chi(p^*(ghg^*)) \leq \chi(p^*(g^*)p'(h)p^*(g^*)).$$

If $0 \leq r \leq 1$, then the inequality is reversed.

**Proof.** Since $p^*(ghg^*), p'(h)p^*(g) \in P$, they are both hyperbolic by [8, Proposition 6.2]. Then apply Theorem 2.4 and [8, Theorem 6.1] to have

$$\chi(p^*(ghg^*)) \leq \chi(p'(h)p^*(g)), \quad \text{if } r \geq 1.$$

The case $0 \leq r \leq 1$ is similar. \hfill \Box

Finally we want to extend Theorem 1.3 (in the form of (1.7)) in the context of semisimple Lie groups. The following theorem asserts that a normal element is the “smallest” in its conjugacy class.

**Theorem 2.6.** If $g, h \in G$ such that $g$ is normal, then $p(g) \prec p(gh^{-1})$. In particular $\chi(p(g)) \leq \chi(p(gh^{-1}))$ for any finite dimensional character $\chi$ of $G$. 

Proof. It is easy to see that $g$ being normal implies that $\pi(g) \in \text{End } V_n$ is a normal operator for any finite dimensional representation $\pi$ of $G$. Thus the spectral radius and the spectral norm of $\pi(g)$ are the same. For any $\pi \in \hat{G}$, by (1.7)

$$\rho(\pi(p(g))) = \|\pi(g)\| = \rho(\pi(g)) = \rho(\pi(h)\pi(g)\pi(h)^{-1}) \leq \|\pi(hgh^{-1})\| = \rho(p(hgh^{-1})),$$

where $\| \cdot \|$ denotes the spectral norm on $\text{End } V_n$. By Theorem 2.3 we have the desired result.

We remark the above relevant results are also true for the Cartan decomposition $G = KP$ with appropriate adjustment.

The following example shows that there are cases in which the pre-order is not necessarily log majorization.

**Example 2.7.** Let $G = \text{SO}_{2n}(\mathbb{C}) := \{ g \in \text{SL}_{2n}(\mathbb{C}) : g^Tg = 1 \}$. It is a connected noncompact simple Lie group \cite[p.449]{4} when $n \geq 2$ and its Lie algebra is

$$\mathfrak{g} := \mathfrak{so}_{2n}(\mathbb{C}) = \{ X \in \mathbb{C}_{2n \times n} : X^T = -X \}.$$

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with

$$\mathfrak{k} = \{ X \in \mathbb{R}_{2n \times n} : X^T = -X \}, \quad \mathfrak{p} = i\mathfrak{k},$$

that is, the corresponding Cartan involution on $\mathfrak{g}$ is $d\Theta(Y) = -Y^*$ for all $Y \in \mathfrak{g}$ and on $G$ is $\Theta(g) = (g^{-1})^*$ for all $g \in G$. So $K = \text{SO}(2n)$, the special orthogonal group. Pick

$$a = \left\{ \left( \begin{array}{cc} 0 & it_1 \\ -it_1 & 0 \end{array} \right) \oplus \left( \begin{array}{cc} 0 & it_2 \\ -it_2 & 0 \end{array} \right) \oplus \cdots \oplus \left( \begin{array}{cc} 0 & it_n \\ -it_n & 0 \end{array} \right) : t_1, \ldots, t_n \in \mathbb{R} \right\},$$

and

$$a^+ = \left\{ \left( \begin{array}{cc} 0 & it_1 \\ -it_1 & 0 \end{array} \right) \oplus \left( \begin{array}{cc} 0 & it_2 \\ -it_2 & 0 \end{array} \right) \oplus \cdots \oplus \left( \begin{array}{cc} 0 & it_n \\ -it_n & 0 \end{array} \right) : t_1 \geq \cdots \geq t_{n-1} \geq |t_n| \right\}.$$

Now

$$A^+ = \exp a^+,$$

$$A = \exp a, \quad p = \text{Ad}(K)a, \quad P = \exp p.$$

Notice that $\text{SO}_{2n}(\mathbb{C}) \subset \text{SL}_{2n}(\mathbb{C})$ and $P$ is a subset of the set of $n \times n$ positive definite matrices in $\text{SL}_n(\mathbb{C})$. Each $f \in P$ is $K$-conjugate to a unique

$$f_+ = \left( \begin{array}{cc} \cosh f_1 & i \sinh f_1 \\ -i \sinh f_1 & \cosh f_1 \end{array} \right) \oplus \left( \begin{array}{cc} \cosh f_2 & i \sinh f_2 \\ -i \sinh f_2 & \cosh f_2 \end{array} \right) \oplus \cdots \oplus \left( \begin{array}{cc} \cosh f_n & i \sinh f_n \\ -i \sinh f_n & \cosh f_n \end{array} \right) \in A^+,$$

where $f_1 \geq \cdots \geq f_{n-1} \geq |f_n|$. Its eigenvalues are $e^{\xi_i}, e^{-\xi_i}, i = 1, \ldots, n$, which are the singular values of $f$. We now identify $a$ with $\mathbb{R}^n$. With this identification, the Weyl group $W$ acts on $a$ in the following way:

$$(t_1, \ldots, t_n) \mapsto (xt_{\sigma(1)}, \ldots, xt_{\sigma(n)}),$$

in which the total number of negative sign is even. By definition, if $f, g \in P$, then $f \prec g$ means $f_+ \in \text{conv } Wg^+$, where $f_+, g_+ \in a^+$ are described as above. It means that \cite{12}

$$\sum_{i=1}^{n-1} f_i - |f_n| \prec w (g_1, \ldots, g_{n-1}, |g_n|),$$

$$\sum_{i=1}^{n-1} g_i - |g_n| \leq \sum_{i=1}^{n-1} f_i - |f_n|,$$
and in addition

\[ \sum_{i=1}^{n-1} f_i + |f_n| \leq \sum_{i=1}^{n-1} g_i - |g_n|. \]

if one and only one of \( f_n \) and \( g_n \) is negative. This relation has more structure than the majorization that is the pre-order for \( \text{SL}_n(\mathbb{C}) \) on the algebra level while the group level description is log majorization.

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