FOUR-DIMENSIONAL EINSTEIN MANIFOLDS WITH
SECTIONAL CURVATURE BOUNDED FROM ABOVE

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ABSTRACT. Given an Einstein structure with positive scalar curvature on a four-dimensional Riemannian manifolds, that is $\text{Ric} = \lambda g$ for some positive constant $\lambda$. For convenience, the Ricci curvature is always normalized to $\text{Ric} = 1$. A basic problem is to classify four-dimensional Einstein manifolds with positive or nonnegative curvature and $\text{Ric} = 1$. In this paper, we firstly show that if the sectional curvature satisfies $K \leq \frac{\sqrt{3}}{2} \approx 0.866025$, then the sectional curvature will be nonnegative. Next, we prove a family of rigidity theorems of Einstein four-manifolds with nonnegative sectional curvature, and satisfies $K_{ik} + sK_{ij} \geq K_s = \frac{1 + \sqrt{2}}{3} + \frac{\sqrt{4+2\sqrt{2}}}{4} + \frac{2-s}{6} s$ for every orthonormal basis $\{e_i\}$ with $K_{ik} \geq K_{ij}$, where $s$ is any nonnegative constant. Indeed, we will show that these Einstein manifolds must be isometric either $S^4$, $RP^4$ or $CP^2$ with standard metrics. As a corollary, we give a rigidity result of Einstein four-manifolds with $\text{Ric} = 1$, and the sectional curvature satisfies $K \leq M_2 = \frac{2-\sqrt{2}}{6} + \frac{\sqrt{4+2\sqrt{2}}}{4} \approx 0.750912$.

1. INTRODUCTION

A Riemannian manifold $(M, g)$ is called Einstein if the Ricci curvature satisfies the Einstein equation

$$\text{Ric} = \lambda g$$

for some constant $\lambda$. In differential geometric, a basic problem is to classify Einstein manifolds with positive or nonnegative sectional curvature in the category of either topology, diffeomorphism, or isometry. By the work of Berger [2], four-dimensional Einstein manifold with nonnegative sectional curvature and positive scalar curvature must have its Euler characteristic $\chi(M)$ bounded by $1 \leq \chi(M) \leq 9$. Furthermore, Hitchin [15] had shown that $|\tau(M)| \leq \left(\frac{3}{4}\right)^{3/2} \chi(M)$, where $\tau(M)$ is the signature. Later, Gursky and Lebrun [11] improved this result, showed that $\chi(M) > \frac{15}{4} |\tau(M)|$ if the manifold is not half conformally flat.

These results suggest that there are few four-dimensional manifolds can carry Einstein structure with nonnegative sectional curvature and positive scalar curvature. Indeed, Up to now, the only known examples of oriented four-dimensional Einstein manifolds with nonnegative sectional curvature and positive scalar curvature are the sphere $S^4$, the product of 2-spheres $S^2 \times S^2$, and the complex projective space $CP^2$.

In the isometric category, Hitchin’s classification theorem (see [10] Theorem 13.30) states that half conformally flat Einstein four-manifolds with positive scalar curvature are isometric to either $S^4$, $CP^2$. If Einstein manifolds have positive

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curvature, Tachibana [18] proved that Einstein manifolds with positive curvature operator are space forms. Recently, Brandle [4] improved this result and showed that Einstein manifolds with positive isotropy curvature are space forms, and with nonnegative isotropy curvature are locally symmetric.

On the other hand, if Einstein manifolds have positive or nonnegative sectional curvature, Gursky and LeBrun [11] showed that compact Einstein four-manifolds of nonnegative sectional curvature and positive intersection form are $\mathbb{C}P^2$. While Yang [20] considered Einstein four-manifold with $Ric = 1$, and with nonnegative sectional curvature. If the sectional curvature further satisfies condition (a):

$$K \geq (\sqrt{1249} - 23)/120 \approx 0.102843$$

or condition (b):

$$2K_{ik} + K_{ij} \geq \frac{9}{14}$$

for every orthonormal basis $\{e_i\}$ with $K_{ik} \geq K_{ij}$, then it must be isometric either $S^4$, $RP^4$ or $CP^2$ with standard metrics. Later, Costa [9] improve Yang’s result, showed that if $K \geq \epsilon_0 = (2 - \sqrt{2})/6 \approx 0.097631$, then Yang’s result remains true.

In this paper, we will consider Einstein four-manifolds with sectional curvature bounded from above by a constant less than 1. Firstly, we obtain an important observation of Einstein four-manifolds as follow.

**Theorem 1.1.** Let $(M, g)$ be a complete four-dimensional Einstein manifold with $Ric = 1$, and the sectional curvature satisfies

$$K \leq M_1 = \frac{\sqrt{3}}{2} \approx 0.866025.$$

Then, $(M, g)$ must have nonnegative sectional curvature.

In the next part, we consider the rigidity of Einstein four-manifolds with nonnegative sectional curvature.

**Theorem 1.2.** For any constant $s \geq 0$, let $K_s = \frac{1 + \sqrt{3}}{3} - \frac{\sqrt{4 + 2\sqrt{2}} + 2 - \sqrt{2}}{6}s$, and we have the following property. Suppose $(M, g)$ be a complete four-dimensional Einstein manifold with nonnegative sectional curvature, and $Ric = 1$. If we further assume that

$$K_{ik} + sK_{ij} \geq K_s$$

for every orthonormal basis $\{e_i\} \subset T_o M$, which satisfies $K_{ik} \geq K_{ij}$. Then, $(M, g)$ must be isometry to either the Euclidean sphere $S^4$, the real projective space $RP^4$ with constant sectional curvature $K = \frac{1}{4}$, or the complex projective space $CP^2$ with the normalized Fubini-Study metric.

Take $s = \frac{1}{2}$, then we have the following rigidity theorem, which generalize Yang’s result [20].

**Corollary 1.3.** Let $(M, g)$ be a complete four-dimensional Einstein manifold with nonnegative sectional curvature, and $Ric = 1$. If we further assume that

$$2K_{ik} + K_{ij} \geq \frac{2 + \sqrt{2}}{2} - \frac{\sqrt{4 + 2\sqrt{2}}}{4} \approx 0.400543$$

for every orthonormal basis $\{e_i\} \subset T_o M$, which satisfies $K_{ik} \geq K_{ij}$. Then, $(M, g)$ must be isometry to either the Euclidean sphere $S^4$, the real projective space $RP^4$ with constant sectional curvature $K = \frac{1}{4}$, or the complex projective space $CP^2$ with the normalized Fubini-Study metric.

**Remark 1.4.** In [20], Yang showed the rigidity theorem under the condition that $2K_{ik} + K_{ij} \geq \frac{9}{14} \approx 0.642857$. 
Combining the above two theorems, we obtain the following rigidity theorem of Einstein four-dimensional with sectional curvature bounded from above.

**Theorem 1.5.** Let $(M, g)$ be a complete four-dimensional Einstein manifold with $\text{Ric} = 1$, and the sectional curvature $K$ satisfies

$$K \leq M_2 = \frac{2 - \sqrt{2}}{6} + \frac{\sqrt{4 + 2\sqrt{2}}}{4} \approx 0.750912.$$

Then, $(M, g)$ must be isometry to either the Euclidean sphere $S^4$, the real projective space $\mathbb{R}P^4$ with constant sectional curvature $K = \frac{1}{3}$, or the complex projective space $\mathbb{C}P^2$ with the normalized Fubini-Study metric.

**Remark 1.6.** In [9], Costa showed a similar rigidity theorem under the condition that the sectional curvature satisfies $K \leq \frac{2}{3}$. Our theorem can be considered as a generalization of Costa’s result.

A crucial idea on the classification theorem of Einstein four-manifolds is the Weitzenböck formula. Usually one needs to prove a Kato type inequality to get some curvature estimates for the self-dual and anti-dual Weyl tensor $W^\pm$ in terms of the length function $|W^\pm|$.

However, our proof will base on the Ricci flow theory. In 1982, Hamilton [12] introduced Ricci flow to study compact three-manifolds with positive Ricci curvature. Later, Ricci flow theory have become an important tool in differential geometry. For example, by using the Ricci flow, we obtained the classification theorem of manifolds with positive curvature operator [13, 8], Poincaré and geometrization conjecture [16, 17, 6], 1/4-pinched differential sphere theorem [5], the classification theorem of manifolds with positive isotropy curvature [14, 8, 7], etc.

In this paper, we will use the advanced maximum principle to construct some pinching sets that invariant under the Ricci flow, which suggest that the curvature will become better and better along the Ricci flow. But since manifolds we consider are Einstein, the metric only change by scaling along the Ricci flow, so the initial metric must be good enough, and we get our rigidity theorem.

The rest of the paper is organized into five sections. In Section 2, we introduce some basic facts about Einstein four-manifolds that will be used throughout the paper, and obtain a key estimate Lemma 2.2. We will introduce some basic facts and ODE system about Ricci flow on four-manifolds in Section 3, and then we prove Theorem 1.1 by using Lemma 2.2. In Section 4 and Section 5, we will prove rigidity results Theorem 1.2 and Theorem 1.5, respectively. The arguments are base on the pinched estimate Claim 4.2, which will be proved by using a key estimate Lemma 4.1.

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2. Preliminaries

Let $(M, g)$ be a closed oriented four-dimensional Einstein manifold. The space $\Lambda^2_\pm$ of self-dual and anti-self dual 2-forms on $M$ are the eigenspaces of the eigenvalues $+1$ and $-1$ of the Hodge star operator $\ast$ on 2-forms, respectively. This gives an orthogonal decomposition

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-.$$
Furthermore, this decomposition induces a block decomposition of the curvature operator matrix as

$$M_{\alpha\beta} = \begin{pmatrix} A & B \\ \dagger B & C \end{pmatrix},$$

and $B \equiv 0$ on $M$ since $M$ is Einstein.

For any point $o \in M$, we will denote by $K(\pi)$ the sectional curvature of the plane $\pi \subset T_o M$. It is well known that

$$K(\pi) = K(\pi^\perp),$$

where $\pi^\perp \subset T_o M$ is the plane perpendicular to $\pi$. More precisely, by choosing a positive oriented orthonormal basis $\{e_i\}$ of $T_o M$, we have $K_{12} = K_{34}$, $K_{13} = K_{24}$ and $K_{14} = K_{23}$, where $K_{ij}$ is the sectional curvature of the plane spanned by $e_i, e_j$.

Let $\{\theta_i\}$ be the dual orthonormal coframe, then a basis of $\Lambda^2_o$ is

$$\varphi_1 = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4, \quad \varphi_2 = \theta_1 \wedge \theta_3 + \theta_4 \wedge \theta_2, \quad \varphi_3 = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3,$$

while a basis for $\Lambda^2_o$ is

$$\psi_1 = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4, \quad \psi_2 = -\theta_1 \wedge \theta_3 + \theta_4 \wedge \theta_2, \quad \psi_3 = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$

In order to give a representation more specific for the curvature operator matrix $A$ and $C$, we shall need the following lemma due to Berger [2].

**Proposition 2.1.** Let $(M, g)$ be an oriented four-dimensional Einstein manifold. For a point $o \in M$, there exists a positive oriented orthonormal basis $\{e_i\}$ of $T_o M$, such that the curvature tensor $\{R_{ijkl}\}$ satisfies the following properties.

1. $K_{12} = R_{1212} = \min\{K(\pi)|\pi \subset T_o M\}$.
2. $K_{14} = R_{1414} = \max\{K(\pi)|\pi \subset T_o M\}$.
3. $R_{ijkj} = 0$ for all $i \neq j$.
4. $|R_{1342} - R_{1234}| \leq K_{13} - K_{12}, |R_{1423} - R_{1342}| \leq K_{14} - K_{13}$, and $|R_{1423} - R_{1234}| \leq K_{14} - K_{12}$.

Throughout this paper, we always choose the above basis $\{e_i\}$ of $T_o M$ to get some estimates. A simple fact is the following observation of curvature operator matrix. For the above basis $\{e_i\}$ of $T_o M$, we obtain the dual coframe $\{\theta_i\}$, and then a basis $\{\varphi_i\}$ of $\Lambda^2_o$, a basis $\{\psi_i\}$ of $\Lambda^2_o$.

A direct computation shows that $A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$, and $C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$, where $a_1 = 2(K_{12} + R_{1234}) \leq a_2 = 2(K_{13} + R_{1342}) \leq a_3 = 2(K_{14} + R_{1423})$ and $c_1 = 2(K_{12} - R_{1234}) \leq c_2 = 2(K_{13} - R_{1342}) \leq c_3 = 2(K_{14} - R_{1423})$.

Define a quadratic function of curvature operator as follow

$$I = (c_2 - c_1)c_3 + (c_3 - c_1)c_2 + (a_2 - a_1)a_3 + (a_3 - a_1)a_2.$$

Now we can pose our key lemma in this section.

**Lemma 2.2.** Let $(M, g)$ be a complete oriented four-dimensional Einstein manifold with $\text{Ric} = 1$, and with the sectional curvature bounded from above by $K \leq \frac{\sqrt{2}}{2}$. Fixed a point $o \in M$, if at this point the least sectional curvature $K_{12} \leq -\epsilon < 0$ for some positive constant $\epsilon$. Then the following hold.
(1). If $a_2 \geq 0$ and $c_2 \geq 0$, then
\[ I \geq \frac{16}{3} \varepsilon. \]

(2). If $a_2 < 0$ or $c_2 < 0$, then
\[ I > \frac{1}{4} \varepsilon. \]

Proof. Choose the basis $\{e_i\}$ of Proposition 2.1. Note that $K_{12} + K_{13} = 1 - K_{14} \geq 1 - \sqrt{\frac{\varepsilon}{3}} > 0$, the manifold has two-positive sectional curvature.

(1). Since $a_2 \geq 0$ and $c_2 \geq 0$, we have $I \geq (c_2 - c_1)c_3 + (a_2 - a_1)a_3$. Now since $K_{12} = 1, a_1 + a_2 + a_3 = c_1 + c_2 + c_3 = \frac{4}{3} = 2$. And then $a_3 \geq \frac{1}{3}(a_1 + a_2 + a_3) = \frac{2}{3}$. Similarly, $c_3 \geq \frac{2}{3}$. So we can obtain
\[
I \geq \frac{2}{3}[(a_2 + c_2) - (a_1 + c_1)]
\[
= \frac{8}{3}(K_{13} - K_{12}) = \frac{8}{3}(K_{12} + K_{13} - 2K_{12})
\[
> \frac{16}{3} \varepsilon. \]

(2). Without lose of generality, we can assume $a_2 < 0$. By direct computation, we have
\[
\frac{1}{4} I = [(K_{13} - R_{1342}) - (K_{12} - R_{1234})](K_{14} - R_{1423})
\[
+ [(K_{13} + R_{1342}) - (K_{12} + R_{1234})](K_{14} + R_{1423})
\[
+ [(K_{14} - R_{1423}) - (K_{12} - R_{1234})](K_{13} - R_{1342})
\[
+ [(K_{14} + R_{1423}) - (K_{12} + R_{1234})](K_{13} + R_{1342})
\[
= 2(K_{13} - K_{12})K_{14} + 2(R_{1342} - R_{1234})R_{1423}
\[
+ 2(K_{14} - K_{12})K_{13} + 2(R_{1423} - R_{1234})R_{1342}.
\]

Hence by the first Bianchi identity,
\[
\frac{1}{8} I = (K_{13} - K_{12})K_{14} + (K_{14} - K_{12})K_{13} + R_{1234}^2 + 2R_{1342}R_{1423}.
\]

The condition of two-positive sectional curvature implies that $K_{13} \geq -K_{12} \geq \epsilon > 0$. It is easy to see that $K_{14} \geq \frac{K_{14} + K_{13}}{2} \geq \frac{K_{14} + K_{13} + K_{12}}{2} = \frac{1}{2}$. Denote by $x = -R_{1234}, y = -R_{1342}$. Then $R_{1423} = x + y$, and hence
\[
R_{1234}^2 + 2R_{1342}R_{1423} = x^2 - 2y(x + y).
\]

In the following, we divide the argument into three cases.

Case 1: $K_{14} - K_{13} \leq \frac{1}{2}(K_{13} - K_{12})$.

In this case, $K_{14} - K_{13} < 7K_{13},$ and then $8K_{13} > K_{14}$.

Note that
\[
R_{1234}^2 + 2R_{1342}R_{1423} = \frac{1}{3}[2(x - y)^2 + 2(x - y)(x + 2y) - (x + 2y)^2]
\[
> -\frac{1}{2}(x + 2y)^2.
\]

Now since $a_2 < 0$, we obtain $a_1 < 0$, and hence
\[
x > K_{12}, \quad y > K_{13} > 0.
\]
This implies that $x + y > K_{12} + K_{13} > 0$, and $x + 2y > 0$. Furthermore, by Proposition 2.1, we have

$$x + 2y = R_{1423} - R_{1342} \leq K_{14} - K_{13},$$

so we have

$$\frac{1}{8} I \geq (K_{13} - K_{12})K_{14} + (K_{14} - K_{12})K_{13} - \frac{1}{2}(K_{14} - K_{13})^2$$

$$= (K_{13} - K_{12})(K_{14} + K_{13}) + (K_{14} - K_{13})K_{13} - \frac{1}{2}(K_{14} - K_{13})^2$$

$$\geq (K_{13} - K_{12})(K_{14} + K_{13}) - \frac{1}{2} - \frac{1}{7}(K_{14} - K_{13})^2$$

$$\geq (K_{13} - K_{12})(K_{14} + K_{13}) - \frac{5}{4}(K_{13} - K_{12})(K_{14} - K_{13})$$

$$= \frac{1}{4}(K_{13} - K_{12})(9K_{13} - K_{14})$$

$$\geq \frac{1}{4} 2(-K_{12}) \cdot \frac{1}{8} K_{14} > \frac{1}{32} \epsilon.$$

**Case 2:** $K_{14} - K_{13} > \frac{7}{2}(K_{13} - K_{12})$, and $x \leq 0$.

By Proposition 2.1, we have

$$|y - x| = |R_{1342} - R_{1234}| \leq K_{13} - K_{12}.$$

And then

$$R_{1234}^2 + 2R_{1342}R_{1423} = 3x^2 - 6xy - 2(x - y)^2$$

$$\geq -2(K_{13} - K_{12})^2.$$

so we have

$$\frac{1}{8} I \geq (K_{13} - K_{12})K_{14} + (K_{14} - K_{12})K_{13} - 2(K_{13} - K_{12})^2$$

$$\geq (K_{13} - K_{12})(K_{14} + K_{13}) + (K_{14} - K_{13})K_{13} - 2(K_{13} - K_{12})^2$$

$$\geq (K_{13} - K_{12})[K_{14} + \frac{1}{2}K_{13} + 2(K_{13} + K_{12})]$$

$$\geq 2(-K_{12})K_{14} > \epsilon.$$

**Case 3:** $K_{14} - K_{13} > \frac{7}{2}(K_{13} - K_{12})$, and $x > 0$.

Similarly, we have $x + 2y \leq K_{14} - K_{13}$, $y > 0$, and

$$R_{1234}^2 + 2R_{1342}R_{1423} = \frac{1}{3}[2(x - y)^2 + 2(x - y)(x + 2y) - (x + 2y)^2]$$

$$\geq \frac{1}{3}[2(x - y)^2 + 2(x - y)(K_{14} - K_{13}) - (K_{14} - K_{13})^2].$$

Furthermore, by $|y - x| \leq K_{13} - K_{12}$, and

$$x - y > -\frac{1}{2}(x + 2y) \geq -\frac{1}{2}(K_{14} - K_{13}),$$

so we have

$$R_{1234}^2 + 2R_{1342}R_{1423} \geq \frac{1}{3}[2(K_{13} - K_{12})^2 - 2(K_{13} - K_{12})(K_{14} - K_{13}) - (K_{14} - K_{13})^2].$$
Denote by $m = K_{12}$, $M = K_{14}$, then $z = K_{13} = 1 - (m + M)$, and

\[
\frac{3}{8} I = 3M(z - m) + 3z(M - m) + [2(z - m)^2 - 2(z - m)(M - z) - (M - z)^2]
\]

\[= 3M(1 - M - 2m) + 3(1 - M - m)(M - m) + 2(1 - M - 2m)^2
- 2(1 - M - 2m)(2M + m - 1) - (2M + m - 1)^2
\]

\[= -4M^2 + 3 + 8mM - 15m + 14m^2
\]

\[> 7\varepsilon,
\]

the last inequality holds because $-4M^2 + 3 \geq 0$.

Combining the above argument, we complete the proof of Lemma 2.2.

□

3. **Einstein four-manifolds with two-positive sectional curvature**

Let $(M, g)$ be a closed Riemannian manifold, and $g_{ij}(t)$ is the unique short time solution of Ricci flow

\[\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t).
\]

In terms of moving frames \[13\], the curvature operator $M_{\alpha\beta}$ evolves by

\[\frac{\partial}{\partial t} M_{\alpha\beta} = \Delta M_{\alpha\beta} + M_{\alpha\beta}^2 + M_{\alpha\beta}^\#,
\]

where $M_{\alpha\beta}^\#$ is the Lie algebra adjoint of $M_{\alpha\beta}$. And the ODE corresponding to the above equations is

\[\frac{d}{dt} M_{\alpha\beta} = M_{\alpha\beta}^2 + M_{\alpha\beta}^\#.
\]

(ODE)

Suppose $(M, g)$ is a closed oriented four-dimensional normalized Einstein manifold with positive scalar curvature, it is easy to see that the unique solution of Ricci flow is a self-similar solution given by

\[g(t) = (1 - 2t)g, \quad t \in (-\infty, \frac{1}{2}).
\]

Follow by the discussion in the last section, we have a good block decomposition of the curvature operator matrix

\[M_{\alpha\beta} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix},
\]

where $A$ and $C$ are diagonalized by the eigenvalues $\{a_i\}$ and $\{c_i\}$, respectively. Furthermore, the ODE corresponding to the Ricci flow of $A$ and $C$ becomes

\[\frac{d}{dt} A = A^2 + 2A^\varepsilon, \quad \frac{d}{dt} C = C^2 + 2C^\varepsilon.
\]

So the ODE system of eigenvalues are given by

\[
\begin{align*}
\frac{d}{dt} a_1 &= a_1^2 + 2a_2a_3, \\
\frac{d}{dt} a_2 &= a_2^2 + 2a_1a_3, \\
\frac{d}{dt} a_3 &= a_3^2 + 2a_1a_2,
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} c_1 &= c_1^2 + 2c_2c_3, \\
\frac{d}{dt} c_2 &= c_2^2 + 2c_1c_3, \\
\frac{d}{dt} c_3 &= c_3^2 + 2c_1c_2,
\end{align*}
\]

Now we can prove Theorem 1.1.
Proof. of Theorem 1.1 If $M$ is not orientable, we can lift the Einstein metric onto an oriented 2-cover of $M$. So we always assume $(M, g)$ is an oriented four-dimensional Einstein manifolds with $Ric = 1$, and $K \leq \frac{3\sqrt{3}}{2}$.

Let $\kappa = \min \frac{K}{R}$, where $R$ is the scalar curvature. We only need to prove $\kappa \geq 0$. If not, then there exist some constant $\epsilon > 0$, such that $K_{12} = \min \{K(\pi) < -\epsilon$. Now for the self-similar solution of Ricci flow $g(t)$, we have the following claim.

Claim 3.1. There exist a constant $\delta > 0$, such that

$$a_1 + c_1 \geq (\kappa + \delta t)R$$

is preserved under the Ricci flow for all $t \geq 0$.

proof of Claim 3.1. Consider the set $\Omega$ of matrices defined by the above inequality. It is easy to see that $\Omega$ is closed, convex and $O(n)$-invariant. By using the advanced maximum principle, we only need to show the set $\Omega$ is preserved by the (ODE) system. Indeed, we only need to look at points on the boundary of the set. Suppose at some point $(o, t)$, $a_1 + c_1 = (\kappa + \delta t)R$. Then

$$\frac{d}{dt}(a_1 + c_1) = a_1^2 + 2a_2a_3 + c_1^2 + 2c_2c_3$$

$$= a_1(a_1 + a_2 + a_3) + c_1(c_1 + c_2 + c_3) + I$$

$$= (a_1 + c_1) \cdot \frac{R}{2} + I$$

$$= (\kappa + \delta t)\frac{R^2}{2} + I$$

On the other hand, since $g(t)$ is just a scaling of Einstein matric $g$, $R_{ij}(t) = \lambda(t)g_{ij}(t) = \frac{R(t)}{4}g_{ij}(t)$,

$$\frac{d}{dt}\left[(\kappa + \delta t)R\right] = (\kappa + \delta t)\frac{d}{dt}R + \delta R$$

$$= (\kappa + \delta t) \cdot 2|Ric|^2 + \delta R$$

$$= (\kappa + \delta t) \cdot \frac{R^2}{2} + \delta R$$

Furthermore, follow by Lemma 2.2,

$$I > C(\epsilon)R$$

for some positive constant $C = C(\epsilon)$. Take $\delta = C(\epsilon)$, and we complete the proof of Claim 3.1.

By Claim 3.1, we obtain that

$$a_1 + c_1 > \kappa R$$

for all small $t > 0$, but this is impossible, since $g(t)$ is a scaling of $g$, and then $a_1 + c_1 = \kappa R$ at some point for all $t$. \qed
4. Rigidity of Einstein four-manifolds with curvature bounded from below

In this section, we will consider the rigidity of Einstein four-manifolds with curvature bounded from below. Given any constant \( s \geq 0 \), let
\[
K_s = \frac{1 + \sqrt{2}}{3} - \frac{\sqrt{4 + 2\sqrt{2}}}{4} + \frac{2 - \sqrt{2}}{6} s.
\]
Then we can get the following key lemma.

Lemma 4.1. Suppose \((M, g)\) is a complete oriented four-dimensional Einstein manifold with nonnegative sectional curvature and \( \text{Ric} = 1 \). If we further assume that \( K_{ik} + sK_{ij} \geq K_s \) for every orthonormal basis \( \{e_i\} \) with \( K_{ik} \geq K_{ij} \), and the least sectional curvature \( K_{12} \leq \epsilon_0 - \epsilon \), where \( \epsilon_0 = \frac{\sqrt{2}}{6} \). Then the following hold.

(1). If \( a_2 \geq 0 \) and \( c_2 \geq 0 \), then

\[
I > \frac{8}{3} \epsilon.
\]

(2). If \( a_2 < 0 \) or \( c_2 < 0 \), then

\[
I > \epsilon.
\]

Proof. Choose the basis \( \{e_i\} \) of Proposition 2.1. It is easy to see that
\[
K_{13} + sK_{12} < 1 + \frac{2 - \sqrt{2}}{6} s < \frac{1 + \sqrt{2}}{3} + \frac{\sqrt{4 + 2\sqrt{2}}}{4} + \frac{2 - \sqrt{2}}{6} s.
\]

(1). Follow the same argument as Lemma 2.2,
\[
I \geq (c_2 - c_1)c_3 + (a_2 - a_1)a_3 \\
\geq \frac{2}{3}[(a_2 + c_2) - (a_1 + c_1)] \\
= \frac{8}{3}(K_{13} - K_{12}) = \frac{8}{3}[K_{13} + sK_{12} - (1 + s)K_{12}] \\
\geq \frac{8}{3}[K_s - (1 + s)\epsilon_0 + (1 + s)\epsilon] \\
\geq \frac{8}{3}\frac{\sqrt{2}}{2} - \frac{\sqrt{4 + 2\sqrt{2}}}{4} + (1 + s)\epsilon > \frac{8}{3} \epsilon.
\]

(2). Similarly, we can assume \( a_2 < 0 \). Then \( a_1 < 0 \). Let \( x = -R_{1234} \), \( y = -R_{1342} \). We have
\[
x > K_{12} \geq 0, \ y > K_{13} > 0.
\]
Furthermore,
\[
0 < x + 2y = R_{1423} - R_{1342} \leq K_{14} - K_{13},
\]
and
\[
|y - x| = |R_{1342} - R_{1234}| \leq K_{13} - K_{12}.
\]
Since
\[
\frac{1}{8} I = (K_{13} - K_{12})K_{14} + (K_{14} - K_{12})K_{13} + R_{1234}^2 + 2R_{1342}R_{1423},
\]
and
\[ R_{1234}^2 + 2R_{1342}R_{1423} = x^2 - 2y(x + y) \]
\[ = \frac{1}{3}[2(x - y)^2 + 2(x - y)(x + 2y) - (x + 2y)^2] \]
\[ \geq \frac{1}{3}[2(x - y)^2 + 2(x - y)(K_{14} - K_{13}) - (K_{14} - K_{13})^2] \]

Note that
\[ x - y \geq -\frac{1}{2}(x + 2y) \geq -\frac{1}{2}(K_{14} - K_{13}), \]
so we have
\[ R_{1234}^2 + 2R_{1342}R_{1423} \geq \frac{1}{3}[2(K_{13} - K_{12})^2 - 2(K_{13} - K_{12})(K_{14} - K_{13}) - (K_{14} - K_{13})^2]. \]

Denote by \( m = K_{12} \), \( z = K_{13} + sK_{12} \), then \( K_{14} = 1 - K_{12} - K_{13} \), and
\[ \frac{3}{8}I = 3(K_{13} - K_{12})(1 - K_{12} - K_{13}) + 3(1 - 2K_{12} - K_{13})K_{13} \]
\[ + 2(K_{13} - K_{12})^2 - 2(K_{13} - K_{12})(1 - K_{12} - 2K_{13}) - (1 - K_{12} - 2K_{13})^2 \]
\[ = -1 + K_{12} + 8K_{13} + 2K_{12}^2 - 4K_{13}^2 - 16K_{12}K_{13} \]
\[ = -4z^2 + 8(1 + sm - 2m)z + [-1 + (1 - 8s)m + 2(1 + 8s - 2s^2)m^2] \]

It is easy to see that
\[ \frac{\partial}{\partial m} \left( \frac{3}{8}I \right) = 8z(s - 2) + 1 - 8s + 4m(1 + 8s - 2s^2) \]
\[ = - (16z - 1 - 4m) - 8s(1 + ms - z - 4m). \]

Since \( s \) is nonnegative, \( z \geq K_s \geq \frac{1 + \sqrt{\frac{1}{3}}}{3} - \frac{\sqrt{1 + 2\sqrt{3}}}{4} \approx 0.151456 \), and \( m \leq \epsilon_0 \approx 0.097631 \), we have
\[ 16z - 1 - 4m > 1. \]

On the other hand,
\[ 1 + ms - z - 4m = K_{14} - 3m \geq \frac{1}{3} - 3\epsilon_0 > 0. \]

Hence
\[ \frac{\partial}{\partial m} \left( \frac{3}{8}I \right) < -1. \]

So
\[ \frac{3}{8}I \geq \frac{3}{8}I \bigg|_{m=\epsilon_0} > \frac{3}{8}I \bigg|_{m=\epsilon_0} + \epsilon. \]

While
\[ \frac{3}{8}I \bigg|_{m=\epsilon_0} = -4z^2 + 8(1 + s\epsilon_0 - 2\epsilon_0)z \]
\[ + [-1 + (1 - 8s)\epsilon_0 + 2(1 + 8s - 2s^2)\epsilon_0^2] \geq 0, \]
because the facts that $z = K_{13} + sK_{12} < \frac{1+\sqrt{2}}{3} + \frac{2-\sqrt{2}}{6}s + \frac{\sqrt{4+2\sqrt{2}}}{4}$, and

$$z \geq 1 - 2\epsilon_0 + s\epsilon_0 - \frac{1}{2}\sqrt{3(1 - 2\epsilon_0)(1 - 3\epsilon_0)}$$

$$= \frac{1 + \sqrt{2}}{3} + \frac{2 - \sqrt{2}}{6}s - \frac{\sqrt{4 + 2\sqrt{2}}}{4}.$$ 

Now we can following a similar argument to prove our main theorem 1.2.

Proof. of Theorem 1.2 Follow the same argument as Theorem 1.1, we can assume $(M, g)$ is an oriented four-dimensional Einstein manifolds with nonnegative sectional curvature and $Ric = 1$. In the following, we will divide the argument into two cases.

Case 1. $\min K \geq \epsilon_0$. In this case, by the theorem of Costa [9], the theorem holds.

Case 2. $\min K < \epsilon_0$. But this is impossible. We will argue by contradiction. Since $M$ is compact, there exist a constant $\epsilon > 0$, such that $\min K \leq \epsilon_0 - \epsilon$.

Let $\kappa = \min K$, where $R$ is the scalar curvature. We have the following assertion.

Claim 4.2. There exist a constant $\delta > 0$, such that

$$a_1 + c_1 \geq (\kappa + \delta t)R$$

is preserved under the Ricci flow for all $t \geq 0$.

proof of Claim 4.2. Similarly, we only need to show that along the (ODE) system,

$$\frac{d}{dt}(a_1 + c_1) \geq \frac{d}{dt}[(\kappa + \delta t)R]$$

at the boundary $a_1 + c_1 = (\kappa + \delta t)R$. Note that

$$\frac{d}{dt}(a_1 + c_1) = (\kappa + \delta t)\frac{R^2}{2} + I,$$

and

$$\frac{d}{dt}[(\kappa + \delta t)R] = (\kappa + \delta t)\cdot \frac{R^2}{2} + \delta R.$$ 

Then by Lemma 4.1,

$$I > C(\epsilon)R$$

for some positive constant $C = C(\epsilon)$. Take $\delta = C(\epsilon)$, and we get our assertion.

But Claim 4.2 develops a contradiction that

$$a_1 + c_1 > \kappa R$$

for all small $t > 0$. And we complete the proof of Theorem 1.2. 

□
5. **Rigidity of Einstein four-manifolds with two-positive sectional curvature**

In this section, we will consider the rigidity of Einstein four-manifolds with two-positive sectional curvature.

Take $s = 1$ in Lemma 4.1. Then the assumption of curvature becomes $K_{ik} + K_{ij} \geq K_1$, that means the sectional curvature satisfies

$$K \leq 1 - K_1 = \frac{2 - \sqrt{2}}{6} + \frac{\sqrt{4 + 2\sqrt{2}}}{4} = M_2.$$  

Now we can prove our main theorem 1.5.

**Proof.** Similarly, we can assume $(M,g)$ is an oriented four-dimensional Einstein manifolds with $Ric = 1$. And the sectional curvature has a upper bound $M_2$ means

$$K_{ik} + K_{ij} \geq K_1.$$  

On the other hand, $M_2 < M_1$, so Theorem 1.1 implies that the sectional curvature must be nonnegative.

Hence $\min K \geq \epsilon_0$ (Otherwise, we can follow the same argument as Theorem 1.2, by using Lemma 4.1 to construct a pinched set invariant under the Ricci flow, and develop a contradiction.) So we can apply Costa’s result [9] to get our rigidity theorem.

□

**References**

1. M. Berger, *Sur quelques varietes riemanniennes suffisamment pincees*, Bull. Soc. Math. France **88** (1960), 57-71.
2. M. Berger, *Sur quelques varietes d’Einstein compacts*, Ann. Mat. Pur. Appl. **53** (1961), 89-96.
3. C. Böhm, and B. Wilking, *Manifolds with positive curvature operators are space forms*, Ann. of Math. (2) **167** (2008), no. 3, 1079-1097.
4. S. Brendle, *Einstein manifolds with nonnegative isotropy curvature are locally symmetric*, Duke. Math. J., **151**(1) (2010), 1-21.
5. S. Brendle, and R. Schoen, *Manifolds with 1/4-pinched curvature are space forms*, J. Amer. Math. Soc. **22** (2009), no. 1, 287-307.
6. H. D. Cao, and X. P. Zhu, *A complete proof of the Poincaré and geometrization conjecture – application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. **10** (2006), no. 2, 165-492.
7. B. L. Chen, S.H. Tang and X. P. Zhu, *Complete classification of compact four-manifolds with positive isotropic curvature*, J. Differential Geom., **91** (2012), 1-169.
8. B. L. Chen, and X. P. Zhu, *Ricci Flow with Surgery on Four-manifolds with Positive Isotropic Curvature*, J. Differential Geometry, **74** (2006), 177-264.
9. E. Costa, *On Einstein four-manifolds*, J. Geom. Phys **51**(2) (2004), 244-255.
10. A. L. Besse, *Einstein manifolds*, volume 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1987.
11. M. J. Gursky, and C. LeBrun, *On Einstein manifolds of positive sectional curvature*, Ann. Global Anal. Geom. **17**(1999), 315-328.
12. R. S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Diff. Geom. **17** (1982), 255-306.
13. R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153-179.
14. R. S. Hamilton, *Four manifolds with positive isotropy curvature*, Comm. Anal. Geom. **5** (1997), 1-92. (or see, Collected Papers on Ricci Flow, Edited by H. D. Cao, B. Chow, S. C. Chu and S. T. Yau, International Press 2002).
15. N. J. Hitchin, On compact four-dimensional Einstein manifolds, J. Diff. Geom. 9 (1974), 435-442.
16. G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159 v1 November 11, 2002. preprint.
17. G. Perelman, Ricci flow with surgery on three manifolds, arXiv:math.DG/0303109 v1 March 10, 2003. preprint.
18. S. Tachibana, A theorem of Riemannian manifolds of positive curvature operator, Proc. Japan Acad., 50 (1974), 301-302.
19. G. Tsagas, A relation between Killing tensor fields and negative pinched Riemannian manifolds, Proc. Amer. Math. Soc. 22 (1969), 476-478.
20. D. G. Yang, Rigidity of Einstein 4-manifolds with positive curvature, Invent. Math 142 (2000), 435-450.

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