Multiscale cosmology and structure–emerging Dark Energy:
A plausibility analysis

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Cosmological backreaction suggests a link between structure formation and the expansion history of the Universe. In order to quantitatively examine this connection, we dynamically investigate a volume partition of the Universe into over- and underdense regions. This allows us to trace structure formation using the volume fraction of the overdense regions $\lambda_M$ as its characterizing parameter. Employing results from cosmological perturbation theory and extrapolating the leading mode into the nonlinear regime, we construct a three-parameter model for the effective cosmic expansion history, involving $\lambda_{M0}$, the matter density $\Omega_m^{\text{D}}$, and the Hubble rate $H_{D0}$ of today’s Universe. Taking standard values for $\Omega_m^{\text{D}}$ and $H_{D0}$ as well as a reasonable value for $\lambda_{M0}$, that we derive from $N$–body simulations, we determine the corresponding amounts of backreaction and spatial curvature. We find that the obtained values that are sufficient to generate today’s structure also lead to a $\Lambda$CDM-like behavior of the scale factor, parametrized by the same parameters $\Omega_m^{\text{D}}$ and $H_{D0}$, but without a cosmological constant. However, the temporal behavior of $\lambda_M$ does not faithfully reproduce the structure formation history. Surprisingly, however, the model matches with structure formation with the assumption of a low matter content, $\Omega_m^{\text{D0}} \approx 3\%$, a result that hints to a different interpretation of part of the backreaction effect as kinematical Dark Matter.

A complementary investigation assumes the $\Lambda$CDM fit–model for the evolution of the global scale factor by imposing a global replacement of the cosmological constant through backreaction, and also supposes that a Newtonian simulation of structure formation provides the correct volume partition into over– and underdense regions. From these assumptions we derive the corresponding evolution laws for backreaction and spatial curvature on the partitioned domains. We find the correct scaling limit predicted by perturbation theory, which allows us to rederive higher–order results from perturbation theory on the evolution laws for backreaction and curvature analytically. This strong backreaction scenario can explain structure formation and Dark Energy simultaneously.

We conclude that these results represent a conceptually appealing approach towards a solution of the Dark Energy and coincidence problems. Open problems are the still too large amplitude of initial perturbations that are required for the scenarios proposed, and the role of Dark Matter that may be partially taken by backreaction effects. Both drawbacks point to the need of a reinterpretation of observational data in the new framework.

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I. Introduction

One decade has passed since the first long distance SN measurements (see [1,2] for the latest data) led to the belief in an accelerating expansion of our Universe and the postulation of Dark Energy (even if the effect is model dependent [3,4] and there are some possibilities that do not need acceleration [5,6]). There is still no convincing theoretical understanding of the experimental result, and the creativity of theoretical physicists has led to a plethora of proposals on the nature of this new energy component, see e.g. [8].

In the same decade, however, initially independent of the development in the Dark Energy sector, our knowledge about another, long-standing problem in relativistic cosmology has grown to an extent that allowed to connect it to the Dark Energy problem: the question of how effectively inhomogeneities in matter and geometry would influence the global behavior of the Universe [9]. As general relativity is far too complex to simply solve the Einstein equations for an inhomogeneous matter distribution and calculate the effect through tensorial averaging techniques, there have been several approaches to get a handle on this problem [10]. The one which lies at the basis of this paper was introduced in [11] and uses a truncated hierarchy of the averaged scalar parts of Einstein’s equations. As will be recalled in Section IIA this strategy reduces the problem to a set of three coupled equations for four independent variables characterizing the state of some spatial domain in the Universe, its volume, mean density, mean scalar curvature and backreaction, linked to the degree of inhomogeneity of the domain’s deformation. For details see Sec. IIA and [11,12].

Based on this framework the connection of the two
above-mentioned problems was made in \[16\] and \[13\], where the authors proposed that the backreaction of the inhomogeneities developing in the era of structure formation could lead to the effect interpreted as accelerated expansion. This would also solve the coincidence problem, i.e. answer the question why the acceleration takes place at a moment of the evolution when the Universe leaves its homogeneous initial state. For this and other more theoretically founded reasons, there have already been some applications of the formalism of the averaged equations to structure formation. One has been carried out by Räsänen \[17\] (generalizing his earlier work on a two–scale model \[18\]). He used a model of structure formation in which the extreme points of the initial overdensity field are evolved separately using a spherical LTB model. His treatment did not give evidence for an accelerated expansion, but he could identify a signal of onset of an eventually strong backreaction regime at the right time. Another study that also follows a multiscale approach as in the present paper has been performed by Wiltshire \[19, 20\]. He separated the universe model into “voids” and “walls” and studied the backreaction effects resulting from fluctuations due to this partitioning. To solve the equations he assumed the backreaction term to vanish on each of the subregions and set the curvature on the overdense regions to zero (as in the two–scale model of Räsänen \[18\]). For the underdense regions he considered a Friedmann–like constant curvature term. Under these assumptions he also did not find volume acceleration, but described an effect that should result from the different lapse of time in underdense void regions and overdense matter dominated regions. This allowed him to fit his model to a fair number of data \[19, 21, 22\], despite his assumption that the backreaction effect on the individual domains was absent. We shall come back to his approach later. Other interesting considerations about a multiscale approach to self–gravitating systems may be found in \[24\].

We wish to take one step back and return to the more general problem (i.e. without using Räsänen’s or Wiltshire’s assumptions), by presenting the general framework for separations of the averaged equations into subdomains of the Universe. This has been begun in \[24\] by investigating a general volume partition of cosmological hypersurfaces. We will show that this separation is consistently possible also for the evolution equations. The advantage of not neglecting any effect is impaired by the caveat not to be able to close the system of averaged equations without further assumptions on evolution laws for backreaction and curvature. We are going to investigate a volume partition of the Universe into initially overdense \(\mathcal{M}\)–regions and underdense \(\mathcal{E}\)–regions in analogy with Wiltshire’s approach. However, instead of setting the backreaction on \(\mathcal{M}\) and \(\mathcal{E}\) to zero, we will first use perturbative results on the early evolution of backreaction \[25, 28, 29\]: in perturbation theory one finds that the backreaction term on a subdomain of a matter dominated universe model decays only as \(a^{-1}\) with the scale factor (a result found in various papers in relativistic cosmology \[15, 14, 26, 28\] that confirms the formally analogous situation for the backreaction term in Newtonian cosmology \[25\]). The latter work also shows that it is the leading term in a nonperturbative approximation based on the exact averaged equations and the Zel’dovich approximation \[29, 32\] for the fluctuations. Furthermore, this work selects the leading term as the dominant contribution on large, “quasilinear” scales. Note here that in some work this leading mode is dismissed due to the property that its coefficient function is a full divergence and must hence vanish by imposing periodic boundary conditions. While this latter remark is true \[33\], the mode is nevertheless there in the interior of a periodic Newtonian or quasi–Newtonian simulation box. Moreover, in a proper relativistic theory these terms are not full divergences and there are no boundary conditions to be satisfied apart from the constraints on the initial hypersurface.

As a result of this behavior of the leading mode the importance of the backreaction term, although being a second–order quantity, quickly rises with respect to the matter density which goes as \(a^{-3}\). This perturbative mode falls on an exact scaling solution \[34\] of the general problem and we, hence, consider evolution laws for backreaction and curvature by extrapolation of the perturbative mode along this particular scaling solution. We emphasize that this extrapolation of the leading perturbative mode already furnishes a nonperturbative model due to the fact that the mode is referred to an evolving background that includes backreaction \[35, 36, 37\]. As an aside we here note that a perturbation theory on the evolving background given by the averaged equations has not yet been concisely investigated; forthcoming work will especially focus on the formulation of such a theory, which will be key to an adequate generalization of perturbation theories in the averaging framework.

Our second approach uses the general formalism and is specialized to a concrete model by making use of \(N\)–body simulation data and the assumption that backreaction globally acts like a cosmological constant. This latter approach illustrates a strong backreaction scenario that simultaneously describes accelerated expansion and the structure formation history. \(N\)–body simulation data have recently also been used to estimate the backreaction in a quasi–Newtonian setting through an estimation of the post–Newtonian potential \[35\]. We here allow for a more general reinterpretation of Newtonian simulations that – combined with the relativistic equations – takes scalar curvature into account.

This paper is organized as follows: Section \[11\] shortly reviews the averaged equations and the occurrence of the backreaction terms. Then the relevant separation formulae are presented and the consistency of the resulting evolution equations is checked. The question of how accelerated expansion may be understood in the context of these equations is addressed in Subsection \[A\]. Section \[11\] uses the assumption of the \(a^{-1}\)–scaling of the backreac-
tion terms to clarify the connection between structures today, the initial density fluctuations and the accelerated expansion. The comparison of [11] shows that the specific model is not sufficient to explain structure formation and points out why this could also not have been expected. The discussion will concentrate on how one has to devise the evolution of inhomogeneities in the context of the averaged equations in order to fully explain the phenomenon of Dark Energy.

Section [14] investigates how the perturbative $a^{-1}$-behavior has to be extended to combine structure formation and accelerated expansion. In [14], we motivate the choice of the $a^{-1}$-scaling for the backreaction term on $\mathcal{M}$ and $\mathcal{E}$ and rederive the results of [23, 24, 25] in our context. We also perform a quantitative estimation of the amount of initial backreaction necessary for structure formation. Finally, we give quantitative conclusions in [14B, and discuss in Sec. [14] what extensions could be made to the models considered and comment on the need for a substantial reinterpretation of observational data.

II. VOLUME PARTITIONS OF THE AVERAGE UNIVERSE

A. The averaged Einstein equations for compact spatial domains

The basis for our investigation of the link of structure formation to the expansion history of the Universe is to consider integral properties of scalar variables in a pressureless – i.e. dust – universe model, given in [11], later refined to be applicable to perfect fluid matter models in [13], where also compact formulations of the averaged equations of [11] can be found. The space time is foliated into flow-orthogonal hypersurfaces featuring the line-element

$$ds^2 = -dt^2 + g_{ij}dX^i dX^j,$$

where the proper time $t$ labels the hypersurfaces and $X^i$ are Gaussian normal coordinates (locating free-falling fluid elements or generalized fundamental observers) in the hypersurfaces. $g_{ij}$ is the full inhomogeneous three-metric of the hypersurfaces of constant proper time. Such a split corresponds in the ADM formalism to the choices of a constant lapse and a vanishing shift. Extensions of this formalism to nonvanishing shift and tilted foliations have been given in [40] and [39], in [11] with applications to perturbation theory. For considerations of covariance and gauge issues see [12].

On these hypersurfaces we want to study the evolution of compact spatial domains $\mathcal{D}$, comoving with the fluid. This latter property ensures that the domain is frozen into the general three-metric, i.e. its shape encodes the geometrical structure of the local inhomogeneities. Note that in Newtonian spacetimes it can be shown that the morphology of the boundary of a comoving domain already contains comprehensive information of the matter distribution, including all higher-order correlations. The morphology-characterizing set of integral geometrical measures is known as the Minkowski functionals, from which the backreaction term can be built (see [14] for details). One fundamental quantity characterizing such a domain is its volume, which is the only such measure used here, since we wish to address questions related to the size of the domains and its time-derivatives only,

$$|\mathcal{D}|_g := \int_{\mathcal{D}} d\mu_g,$$  \hspace{1cm} (2)

where $d\mu_g := \sqrt{\det g(t,x^i,x^j)} dx^1 dx^2 dx^3$. From the domain’s volume one may define a scale factor

$$a_D(t) := \left(\frac{|\mathcal{D}|}{|\mathcal{D}|_g}\right)^{\frac{1}{3}},$$  \hspace{1cm} (3)

encoding the average stretch of all directions of the domain. For wild changes of the shape of the initial domain $D_i$, one might want to know more about the evolution of other morphological characteristics to deduce directional expansion information, and would therefore have to extend the analysis.

Concentrating on the volume and the effective scale factor alone, one can derive, from the Einstein equations with a pressureless fluid source, the following set of equations governing its evolution:

$$3\frac{\dot{a}_D}{a_D} = -4\pi G \langle\rho\rangle_D + Q_D + \Lambda$$  \hspace{1cm} (4)

$$3H_D^2 = 8\pi G \langle\rho\rangle_D - \frac{1}{2} \langle\mathcal{R}\rangle_D - \frac{1}{2} Q_D + \Lambda$$  \hspace{1cm} (5)

$$0 = \partial_t \langle\rho\rangle_D + 3H_D \langle\rho\rangle_D,$$  \hspace{1cm} (6)

where the average over scalar quantities is defined as

$$\langle f \rangle_D(t) := \frac{\int_D f(t,x^1,x^2,x^3) d\mu_g}{\int_D d\mu_g},$$  \hspace{1cm} (7)

and where $\rho$, $\mathcal{R}$ and $H_D$ denote the local matter density, the Ricci scalar of the three-metric $g_{ij}$, and the domain dependent Hubble rate $H_D := \dot{a}_D/a_D$, respectively. The kinematical backreaction $Q_D$ is defined as

$$Q_D := \frac{2}{3} \left(\langle\theta^2\rangle_D - \langle \theta^2 \rangle_D \right) - 2 \langle \sigma^2 \rangle_D,$$  \hspace{1cm} (8)

where $\theta$ is the local expansion rate and $\sigma^2 := 1/2\sigma_{ij}\sigma^{ij}$ is the squared rate of shear. Note that $H_D$ is defined as $H_D = 1/3 \langle \theta \rangle_D$. $Q_D$ is composed of the variance of the local expansion rates, $\langle \theta^2 \rangle_D - \langle \theta \rangle_D^2$, and the averaged shear scalar $\langle \sigma^2 \rangle_D$ on the domain under consideration. For a homogeneous domain it is zero. It therefore encodes the departure from homogeneity and is supposed to be particularly important in the late, inhomogeneous phase of the Universe at the epoch of structure formation.

The integrability condition connecting Eqs. (4) and (5) reads

$$\dot{a}_D^{-2} \partial_t \left(a_D^{-2} \langle \mathcal{R} \rangle_D \right) = -a_D^{-6} \partial_t \left( a_D^2 Q_D \right),$$  \hspace{1cm} (9)

This equation expresses the conservation of energy-momentum of the fluid. The pressure anisotropy $Q_D$ appears in this equation, which shows the complexity of the interplay between expansion and backreaction. The next section will discuss how this evolution modifies the perturbations in the linear regime.
which already shows an important feature of the averaged equations as they in general couple the evolution of the backreaction term, and hence extrinsic curvature inhomogeneities (or in this picture matter inhomogeneities), to the average intrinsic curvature. Unlike in the case of a standard Friedmannian evolution the curvature term here is not restricted to an $a_D^{-2}$ scaling behavior but is dynamical in the sense that it may be an arbitrary function of $a_D$. We emphasize that the essential effect of backreaction models is not a large magnitude of $Q_D$, but a dynamical coupling of a nonvanishing $Q_D$ to the averaged scalar curvature, changing the temporal behavior of this latter.

One may also define cosmological parameters on the domain investigated. In analogy to the Friedmannian case they are derived from the Hamilton constraint by a division by $3H_D^2$. For different spatial domains, which will be denoted by $D$, $M$ and $E$, we formulate the cosmological parameters in a generic way by taking $F$ to denote one of the domains $D$, $M$, $E$ (this will hold for the whole text, but depending on the context $F$ may also only denote $M$ and $E$). The definitions are

$$\Omega_m^F := \frac{8\pi G}{3H_D^2} \langle q \rangle_F ; \quad \Omega_k^F := \frac{\Lambda}{3H_D^2} ;$$

$$\Omega_Q^F := \frac{\langle R \rangle_F}{6H_D^2} ; \quad \Omega_R^F := -\frac{Q_F}{6H_D^2},$$

where the decision to divide by $H_D^2$ for every $F$ will become clear from the definition of $M$, $E$ and $D$ in the next section. Using those definitions, the Hamilton constraint directly provides the expressions for the scalar valued function $f$ on the domain $D$,

$$\langle f \rangle_D := |D|^{-1} \int_D f d\mu_g ,$$

which means that the dimensionless parameters $\Omega$ only add up to 1 for the domain $D$. On other domains they may add up to a more or less arbitrary positive value, depending on whether the corresponding region $F$ expands faster or slower than the $D$ region.

To point out the analogy with the $k = 0$ Friedmann equations one may recast combining the backreaction and the curvature terms to an effective density and pressure:

$$3 \frac{\ddot{a}_D}{a_D} = \Lambda - 4\pi G (\varepsilon_{\text{eff}}^D + 3p_{\text{eff}}^D) ; \quad 3H_D^2 = \Lambda + 8\pi G \varepsilon_{\text{eff}}^D$$

$$0 = \varepsilon_{\text{eff}}^D + 3H_D^2 (\varepsilon_{\text{eff}}^D + p_{\text{eff}}^D) ,$$

where the effective densities are defined as

$$\varepsilon_{\text{eff}}^D := \langle q \rangle_D - \frac{1}{16\pi G} Q_D - \frac{1}{16\pi G} \langle R \rangle_D$$

$$p_{\text{eff}}^D := -\frac{1}{16\pi G} Q_D + \frac{1}{48\pi G} \langle R \rangle_D .$$

In this sense, $Q_D$ and $\langle R \rangle_D$ may be combined to some kind of dark fluid component that is commonly referred to as $X$-matter. One quantity characterizing this $X$-matter is its equation of state

$$w_{X,\text{eff}}^D := \frac{p_{\text{eff}}^D}{\varepsilon_{\text{eff}}^D - \langle q \rangle_D} = \frac{\Omega_Q^D - \frac{1}{6}\Omega_R^D}{\Omega_Q^D + \frac{1}{3}\Omega_R^D}, \quad (14)$$

which is an effective one due to the fact that backreaction and curvature give rise to an effective energy density and pressure.

### B. Separation formulas for arbitrary partitions

After the short review of the averaging formalism we present in the following how the resulting equations can be separated, if one wants to consider the behavior of subdomains of some “global” region $D$, which we may assume to be associated with a (postulated) scale of homogeneity. We consider, a priori, arbitrary partitions of the spatial hypersurfaces into subregions $F_\ell$, which themselves consist of elementary space entities $F_\ell^{(\alpha)}$ that may be associated with some averaging length scale. The idea will be to choose entities $F_\ell^{(\alpha)}$ in a way that they share some common properties. In our case this property will be the value of the overdensity field, but the separation is general and also valid for other choices. To be more precise we want to have $D = \cup \ell F_\ell$ where $F_\ell := \cup_\alpha F_\ell^{(\alpha)}$ and $F_\ell^{(\alpha)} \cap F_\ell^{(m)} = \emptyset$ for all $\alpha \neq \beta$ and $\ell \neq m$. In the sequel we follow closely the investigation in [23] of a general volume partitioning of cosmological hypersurfaces.

The average of the scalar valued function $f$ on the domain $D$,

$$\langle f \rangle_D := |D|^{-1} \int_D f d\mu_g ,$$

defined in [2], where $|D|_g$ denotes its volume $|D|_g := \int_D d\mu_g$, may then be split into the averages of $f$ on the subregions $F_\ell$, in the form

$$\langle f \rangle_D = \sum_\ell \frac{|F_\ell|_g}{|D|_g} \langle f \rangle_{F_\ell} = \sum_\ell \frac{|F_\ell|_g}{|D|_g} \sum_\alpha \langle f \rangle_{F_\ell^{(\alpha)}} ,$$

where we have introduced

$$\lambda_\ell := \frac{|F_\ell|_g}{|D|_g} , \quad (17)$$

i.e. the volume fraction $\lambda_\ell$ of the subregion $F_\ell$. Regarding the dynamical equations we wish to separate, i.e. [3], [4] and [4], Equation (18) directly provides the expressions for the scalar functions $q$, $R$ and $H_D := 1/3 \langle \dot{\theta} \rangle_D$. Only $Q_D$ as defined in [5] does not split in this simple way due to the $\langle \dot{\theta} \rangle_D^2$-term. A short calculation rather provides

$$Q_D = \sum_\ell \lambda_\ell Q_\ell + 3 \sum_{\ell \neq m} \lambda_\ell \lambda_m (H_\ell - H_m)^2 \quad (18)$$
as the correct formula. \( Q_\ell \) is defined as in (3) with \( \mathcal{F}_\ell \) instead of \( \mathcal{D} \). The shear part \( \langle a^2 \rangle_\ell \) is completely absorbed in \( Q_\ell \), whereas the variance of the local expansion rates, \( \langle \theta^2 \rangle_\ell - \langle \theta \rangle_\ell^2 \), is partly contained in \( Q_\ell \) but also generates the extra term \( 3 \sum_{\ell \neq m} \lambda_\ell \lambda_m (H_\ell - H_m)^2 \). This is because the part of the variance that is present in \( Q_\ell \), namely \( \langle \theta^2 \rangle_\ell - \langle \theta \rangle_\ell^2 \), only takes into account points inside \( \mathcal{F}_\ell \). To restore the variance that comes from combining points of \( \mathcal{F}_\ell \) with others in \( \mathcal{F}_m \), the extra term containing the averaged Hubble rates emerges.

One may also define a scale factor \( a_\ell \) analogous to \( a \) for each of the subregions \( \mathcal{F}_\ell \). As their definition implies that they are disjoint, it follows that \( \mathcal{D}_{|y} = \sum_\ell |\mathcal{F}_{\ell}|_y \) and we may therefore define \( a^3_\ell = \sum_{\ell \neq m} a^3_\ell \). An important detail when using the equation in this form is that, at the initial time when \( a_\ell \) is equal to one, the scale factors \( a_\ell \) of the subregions will not be equal to one as well. This different normalization has to be taken into account when scaling the local \( \mathcal{M} \) and \( \mathcal{E} \) parameters.

Differentiating \( a^3_\ell = \sum_{\ell \neq m} a^3_\ell \) twice with respect to the foliation time and using the result for \( \dot{a}_\ell \), we finally provide the relation that links the acceleration of the scale factors of the subdomains to the global one:

\[
\frac{\dot{a}_\ell}{a_\ell} = \sum_\ell \lambda_\ell \frac{\dot{a}_\ell(t)}{a_\ell(t)} + \sum_{\ell \neq m} \lambda_\ell \lambda_m (H_\ell - H_m)^2 . \tag{19}
\]

As an immediate consequence one can see that even when the subregions decelerate, the second term of (19) may counterbalance the first one to lead to global accelerated expansion. We will examine this property, which appears generically in averaged models due to correlations of local expansion rates, in Appendix A.

In the following we will be mainly considering the case where we have only two types of subregions \( \mathcal{F}_\ell \): over- and underdense regions. We will define them at some initial time and call \( \mathcal{M} := \mathcal{F}_1 \) the part of the hypersurfaces that consist of all elementary regions \( \mathcal{M}^{(o)} := \mathcal{F}_1^{(o)} \) with an initial overdensity and \( \mathcal{E} := \mathcal{F}_2 \) the complementary part \( \mathcal{E} := \mathcal{D} \cap \mathcal{M} \), i.e. the one with the initially underdense regions. The formulae (16) and (17) then simplify to give

\[
H_D = \lambda_M H_M + (1 - \lambda_M) H_E , \tag{20}
\]

with the same expression valid also for \( \langle \varrho \rangle_D \) and \( \langle R \rangle_D \), and

\[
Q_D = \lambda_M Q_M + (1 - \lambda_M) Q_E + 6\lambda_M (1 - \lambda_M) (H_M - H_E)^2 . \tag{21}
\]

In both cases we used \( \sum_\ell \lambda_\ell = 1 \), which results from the fact that the \( \mathcal{F}_\ell \) are disjoint, and defined \( \lambda_M := |\mathcal{M}|/|\mathcal{D}| \) and \( \lambda_E := |\mathcal{E}|/|\mathcal{D}| \). This general separation formula allows us to illustrate the simplifications applied by Wiltshire in his two-scale model (19). As he assumes the underdense \( \mathcal{M} \)-regions to have no curvature, by Eq. (19) the only possible nonzero \( Q_M \)-term would have a strongly decaying \( a^{-6}_M \)-behavior and may therefore equally well be set to zero. For the underdense \( \mathcal{E} \)-regions the situation is similar as he assumes them to have the Friedmann–like \( a^{-2}_E \) constant curvature term. This again makes the left–hand side of Eq. (19) vanish and leads to the choice of \( Q_E = 0 \). This reduces the general formula (21) to its third term on the right–hand side, which corresponds to his Eq. (31) in [19], that he derived for the case of vanishing shear. This is consistent, as we remarked below (15), because the shear term of \( Q_D \) is completely contained in \( Q_M \) and \( Q_E \). Considering the overall formalism his choice is conservative in the sense that one would indeed expect the biggest backreaction effect to stem from the variance of the expansion rates between the \( \mathcal{M} \)- and \( \mathcal{E} \)-regions and not from those on \( \mathcal{M} \) and \( \mathcal{E} \) themselves, as they are chosen to have similar properties and therefore similar expansion rates. On the other hand, setting the local backreaction to zero restricts the new interesting feature of the averaging formalism, i.e. the coupling of backreaction to curvature, to the \( \mathcal{D} \)-regions only. Furthermore, as we argue in Subsection III C, it may be eligible to allow for shear on the \( \mathcal{M} \)-regions, which may (and will) imply a negative \( Q_M \) (i.e., a Dark Matter behavior over \( \mathcal{M} \)). We will therefore not restrict ourselves to the assumptions of constant curvature, but use other well–motivated conditions in Section III.

C. Consistent split of the dynamical equations

With the knowledge of the relation between the quantities on the subregions with those on the global domain \( \mathcal{D} \), one may now ask, how the separation affects the evolution equations for those quantities. Therefore, we insert the expressions (15) for \( H_D \), \( \langle \varrho \rangle_D \) and \( \langle R \rangle_D \) for \( Q_D \) into (4), (5) and (9) for \( \mathcal{M} \) and \( \mathcal{E} \). A straightforward calculation shows that the equations take the following form:

\[
0 = \sum_\ell \lambda_\ell \left[ 8\pi G \langle \varrho \rangle_\ell - \frac{\varrho_\ell + \langle R \rangle_\ell}{2} + \Lambda - 3H_\ell^2 \right] \tag{22}
\]

\[
0 = \sum_\ell \lambda_\ell \left[ -4\pi G \varrho_\ell + H_\ell - \frac{\ddot{a}_\ell(t)}{a_\ell(t)} \right] \tag{23}
\]

\[
0 = \sum_\ell \lambda_\ell \left[ a_\ell^{-2} \partial_\ell \left( a_\ell^2 \langle R \rangle_\ell \right) + a_\ell^{-6} \partial_\ell (a_\ell^6 Q_\ell) \right] , \tag{24}
\]

i.e. they may be split into a sum in which the equations on the subregions take the same form as those of the global region (4)–(6) and their contribution is weighted with the volume fraction of the respective subregion. As Equations (4)–(6) hold for an arbitrary domain \( \mathcal{D} \) and therefore also for the subregions \( \mathcal{F}_\ell \), Equations (22)–(24) show that the separation advocated in the previous section is also consistent at the level of the evolution equations. This is not surprising as the separation procedure is straightforward and the equations are supposed to hold on any domain, but was, especially in view of the nonlinear form of (15), not entirely clear when just looking at the formulas. The consistent split assures that, if we have
found a solution for the quantities on the subregions $F_i$ and use the relations of the previous section to calculate those on the global domain, then we will automatically obtain a solution of the averaged Equations (4)–(6) on this global domain.

III. BACKREACTION SCENARIO BASED ON EXTRAPOLATING THE LEADING PERTURBATIVE MODE

After setting up the equations that link different subregions of a certain partitioning of space to the whole in the preceding section, we will now have a closer look at the simplest partitioning. As already mentioned at the end of II B, we are interested in a subdivision into overdense and underdense regions. These will be labeled by $M$ and $E$, respectively, and the classification is made at some initial time and then kept fixed. As the averaged equations are derived for a certain region $F_i$, that is assumed to keep its identity, an exchange of elementary space entities $F_i^{(a)}$ between the two classes of regions $M$ and $E$ would spoil the applicability of (4)–(6) on $M$ and $E$, and therefore would lead to some complicated modified form which would introduce even more unknown parameters. This is the reason for keeping the identification of $F_i$–regions, even if this will introduce some problems with the clearcut interpretation of $M$ and $E$, for initially overdense regions may become underdense in the course of evolution. A refined description that helps to evade this problem will be mentioned in the discussion in Sec. VI.

The motivation of distinguishing between the $M$– and $E$–classes of regions is to get an explicit handle on structure formation. Implicitly it is already present in the averaging formalism mainly via the $\left((\theta^2)_D - (\theta^2)_{pE}\right)$–term in the kinematical backreaction defined by (8). As structure formation implies a different evolution of the local expansion rate at different places in the Universe, this variance is expected to grow. The distinction now offers the possibility to examine the development of parts that are in the focus of experimental interest, namely "voids" (see e.g. [43] and references therein) and clusters of galaxies. Moreover, it allows us to trace the process of structure formation by the simplest parameter characterizing its history, i.e. the volume fraction of the overdense regions. The current section presents the outcome of this investigation for the special case of exact scaling solutions.

A. Exact scaling solutions of the averaged equations

As we have seen in Section II A, the averaged equations do not form a closed system. One closure strategy is to impose a specific equation of state. This is similar to closing the Friedmannian equations, but here the equation of state is dynamical. As done in quintessence models for the $X$–matter content inserted as source into the standard Friedmann equations, the simplest assumption is to choose a constant equation of state parameter $w$. This means that we have a single scaling law for the corresponding matter component. If we proceed in a similar way for the backreaction and curvature terms in the averaged equations, we may choose (following [44]):

$$Q_F = Q_{F,p} a_F^w; \langle R \rangle_F = R_{F,p} a_F^w,$$

(25)

where $F$ stands for $M$ or $E$ and where $F$ indicates the initial state of the domain $F$. By the integrability condition (9), $Q_F$ and $\langle R \rangle_F$ are related. Inserting the above ansatz leads to two types of solutions: first, there is a solution with $n = -6$ and $p = -2$. This one is not very interesting as this is the only case where the generic coupling of backreaction and curvature is absent. This case would correspond to fluctuation histories that are decoupled from a constant–curvature (Friedmann–like) behavior. Second, (9) will be satisfied if $n = p$. This is the case we are going to study in the following. It implies that $Q_F$ and $\langle R \rangle_F$ are related by a constant $r_F$

$$Q_F = r_F \langle R \rangle_F = r_F R_{F,p} a_F^w,$$

(26)

which is determined by (9) to be

$$r_F = -\frac{n+2}{n+6}.$$  

(27)

The effective equation of state for the $X$–matter (backreaction and curvature) component, defined in (14), is for scaling solutions simply

$$w_{X,\text{eff}}^F = -\frac{1}{3} (n+3).$$  

(28)

For the explicit solution of the equations on $M$ and $E$ in this paragraph we will further specify $n$ to $-1$. The motivation for this choice is, that Li and Schwarz [26] found, in a calculation using second–order perturbation theory, that the backreaction term may be expressed as a Laurent series with a leading term going like $a_F^{-1}$. We will rederive this behavior, under similar conditions as in their case, in Section VII. There we also argue that the leading term of the scalar curvature, going like $a_F^{-2}$ is expected to vanish for plausible initial conditions. We will show that extrapolating the $a_F^{-1}$–behavior up to today runs into trouble and that higher terms in the series are expected to make an important contribution at the epoch of structure formation. Nevertheless, the $a_F^{-1}$ scaling solution provides a first and simple example of why one would expect that backreaction through structure formation might cause accelerated expansion [14].

We may argue that we did not have to rely on assumptions on the effective equation of state, if we exploited the knowledge of structure formation further. Such an attempt runs into problems as we do not have an inhomogeneous relativistic metric model at our disposal that
would cover the details of nonlinear structure formation. On large scales we probably understand what is happening, but we are already using this information by the known $a_F^{-1}$ scaling behavior on these scales. In the late Universe with density contrasts of order 1, the closest to a realistic model at present is the one by Räsänen [17]. But also this has to rely on some unrealistic assumptions. We therefore try to derive the equation of state from the $N$–body simulation of structure formation in our second approach in Section IV where we use the knowledge on nonlinear structure formation encoded in the volume fraction of the simulation.

B. Free parameters and constraints

In order to derive the evolution of the global domain $D$ under the assumption of the $a_F^{-1}$--scaling behavior for $M$ and $E$, we have to solve Equation (5) (the other independent Equation (9) is already satisfied by the scaling ansatz) which simplifies to

$$H_{D_0}^2 \left[ \Omega_m \frac{a_F^3}{a_F} + \Omega_{RQ} \left( \frac{a_F}{a_F} \right)^{-n} \right] = \left( \frac{a_F}{a_F} \right)^2$$ (29)

for $F$ out of $M$ and $E$, where we have defined $\Omega_{RQ} := \left( \Omega_R + \Omega_Q \right)$. Instead of using $a_M$ and $a_E$ as free parameters we may equally well work with $\lambda_M$ and $\lambda_E$. To specify these and the other free parameters, namely $H_{D_0}^2$, $\Omega_m$, $\lambda_E$, and $\Omega_{RQ}$ – with the index 0 denoting the value of today – we have to make some further assumptions. One is that the density fluctuations in the Early Universe around $z = 1000$, where we choose our initial time $t_i$, were Gaussian and very small as indicated by the results of the CMB experiments. For our parameters this means that $\lambda_M \approx 0.5$ as Gaussian fluctuations imply that there were as many over– as underdense regions and we set $a_M = a_E = \sqrt{1/2}$. The assumption of being close to homogeneity implies $\langle \phi \rangle_D \approx \langle \phi \rangle_{M,E}$. It then follows that

$$\Omega_m = \frac{8 \pi G}{3 H_{D_0}^2} \frac{a_F^3}{a_F^3} \langle \phi \rangle_F \approx \lambda_M \frac{a_P^3}{a_P^3} \Omega_{m0}$$, (30)

which allows us to simplify Equation (29) by replacing $\Omega_m$ and $\Omega_{m0}$ with $\Omega_{m0}$. To reduce the set of unknown parameters even further, we may use (29) today which gives

$$\frac{\lambda_M}{\lambda_m} \Omega_m + \Omega_{RQ} = \frac{H_{D_0}^2}{H_{D_0}^2}; \quad \lambda_M \Omega_m + \Omega_{RQ} = \frac{H_{D_0}^2}{H_{D_0}^2},$$ (31)

and may be used together with (29) to eliminate $\Omega_{RQ}$. Another consistency condition finally allows us to get rid of the parameter $\Omega_{RQ}$ as well, enabling us to fix our model without having to know anything about the values of the backreaction or curvature terms. The argument is as follows: consider the integral of Equation (29)

$$H_{D_0} \int_0^t \frac{dt'}{\sqrt{1/2}} \left[ \Omega_m \frac{a_F^3}{a_F} + \Omega_{RQ} a_F a_F' \right]^{-1}$$ (32)

which gives two functions $t_M (a_M)$ and $t_E (a_E)$ depending on the parameters in the integral. This may be stated as $t_F (a_F) = t_F \left( a_F; \Omega_{RQ}, H_{D_0}, \Omega_{m0}, a_{D0}, \lambda_M, \lambda_E \right)$. As there is only one time even in the separated equations, which is the time of the foliation, $t_M$ and $t_E$ have to be equal. (Note that here we do not introduce a phenomenological lapse between times of different regions as done by Wiltshire, since this would require another (very involved) implementation of a multiscale foliation; we so must imply this spirit we will use for $M$, $E$–body simulation of large–scale structure formation. These are in our context not the best sources of data either, as they are Newtonian and therefore do not include scalar curvature being a major player in the averaged equations. The reason why we think that the domain dependent $\Omega_{m0}$ of the averaged equations should have the same value as the $\Omega_m$ parameter whose value is determined from the data under the assumption of a homogeneous and isotropic universe model, this is the only thing we can do before the analysis of observations in the context of the averaged model, begun in [45], is completed. In this spirit we will use for $\lambda_{M0}$ some reasonable value derived from a $N$–body simulation of large–scale structure formation. These are in our context not the best sources of data either, as they are Newtonian and therefore do not include scalar curvature being a major player in the averaged equations. The reason why we think that the value of the parameter $\lambda_{M0}$ is nevertheless derivable from them, is first, that they are designed in a way to match best today’s Universe and secondly, that, for the calculation of the volume, even a substantial curvature deviation from a Euclidean space does not change the value very much. This is because the calculation of volumes is performed with the metric and its spatial derivatives that are linked to curvature do not enter in the comoving foliation used here (see [46] for clarifying remarks on that point).
Figure 1: Plots of some quantities derived from the single scaling model with $\lambda_{M_0} = 0.09$ and $\Omega^D_m = 0.27$. Top left: Comparison of the evolution of the different scale factors of our model and the one in the $\Lambda$CDM-case. On the right the corresponding Hubble rates. The one on $M$ goes to zero in the end whereas $H_D$ approaches $H_E$ asymptotically. $H_\Lambda$ and $H_D$ are very close to each other. The middle row shows the $\Omega$-parameters on $D$ and $E$. Those for $M$ are left at the bottom. Finally, on the right the behavior of the effective equation of state as defined in (14). The $a^{-1}$-scaling implies that $w_{\Lambda,\text{eff}}^M$ and $w_{\Lambda,\text{eff}}^E$ both have a value of $-\frac{1}{3}$.

We will explain further details of our analysis of the $N$-body data in Appendix B.1. The value we will be using in this section has been obtained by dividing the analysis volume into blocks with sidelength $5h^{-1}\text{Mpc}$ and evaluating whether they are over- or underdense. This leads to $\lambda_{M_0} \approx 0.09$. Together with $\Omega^D_m \approx 0.27$ and $a_{D_0} \approx 1000$ we may numerically evaluate Equation (29) which results in the curves of Fig. 1. The value of $a_{D_0} \approx 1000$ is used to identify “today” in our plots. It can be interpreted in line with the standard model, if the global volume is
considering the evolution of \(a_M\) and \(a_\mathcal{E}\) we find that they behave as one would expect from their nature. The overdense regions \(\mathcal{M}\) begin their evolution as the underdense \(\mathcal{E}\)-regions due to their assumed similar matter content. Then their density increases by gravitational instability and the expansion slows down. The underdense regions which contain most of the voids continue their expansion. This may also be seen from the Hubble rates in the plot on the right. The expansion rate of the overdense regions drops down to zero rather rapidly whereas the one of the underdense regions slows down much less. The shrinking percentage of the overdense regions causes \(H_D\) to bend up again implying acceleration. The difference to the FLRW Hubble rate is not yet visible, but as the evolution proceeds the cosmological constant of the FLRW model will force \(H_\Lambda\) to rise in the future whereas \(H_D\) will join the evolution of \(H_\mathcal{E}\).

2. Energy content

The plot of the \(\Omega\)-parameters in Fig. 1 is consistent with the conclusion drawn from Eq. (1) that \(\Omega_\mathcal{D}^Q\) has to be less than \(-\frac{1}{4}\Omega_m^D\) to give accelerated expansion. In addition, the magnitude of \(\Omega_\mathcal{D}^Q\) underlines the statement that curvature is an important player in our model. The combination of these two facts has an interesting consequence: if, even for a small but nonzero backreaction parameter, the average scalar curvature parameter is of order 1, and therefore the actual matter parameter \(\Omega_m^D\) is small, the model may explain Dark Energy without a large amount of backreaction. A small \(\Omega_\mathcal{D}^Q\) larger than
\(-\frac{1}{3}\Omega_{\text{m}}^{D}\) does, however, not explain an accelerated expansion, but this latter property may actually not be needed to explain observations. This remark is particularly interesting in view of the observation in Subsection III D 3.

That curvature is in fact more important than suggested by the restricted constant curvature FLRW models has been shown recently in [16]. The plots in the following row clarify its origin. It stems from the underdense regions which are supposed to contain most of the large voids that have developed negative curvature in the course of evolution. The value for the matter parameter of about 0.15 makes clear that one may not think of \(\mathcal{E}\) being composed of voids only. From its origin as the underdense half of the initial universe volume, it should be clear that it also has to contain regions that have developed overdensities.

For the overdense regions the situation is different in that in the course of their evolution, positive curvature emerges. This is what one would expect due to the fact that they contain many gravitationally bound systems like clusters. Another observation that coincides with the intuitive picture we have, is the fact, that the backreaction contribution is negative [note again the sign in the definition (11)]. According to definition (3), negative backreaction means that the region over which we average is shear dominated. In view of the filamentary structure of the matter distribution of the Universe this is also what we would expect.

When interpreting the \(\Omega\)-parameters of \(M\) and \(\mathcal{E}\) one should have in mind that they do not have to add up to unity but, due to the definition of the \(\Omega\)'s using \(H_D\) and not \(H_M\) and \(H_E\), will add up to \(H_M^2/H_D^2\) and \(H_E^2/H_D^2\). This means for \(M\) that the sum at the end will be close to 0, whereas the value on \(\mathcal{E}\) approaches 1.

Finally, the plot in the last row on the right shows the evolution of the effective equation of state for the X-Matter component. For \(\Omega_{\text{M}}\) and \(\Omega_{\text{E}}\) the value is, according to (28), simply \(-2/3\). For the global domain \(D\) we have an evolution from this limiting value to \(-0.95\) at the maximum of structure formation which then relaxes back to \(\approx 0.9\) today. It is interesting to note that the value stays relatively constant throughout the period of structure formation.

This observation, i.e. that the \(a^{-1}\)-scaling model on the partitioned domains naturally leads to an approximate cosmological constant behavior on the homogeneity scale, may also be seen from Fig. 3. There it is shown, how the backreaction and curvature terms deviate from their initial \(a^{-1}\)-scaling when structure formation sets in. They stay approximately constant for quite a long period of the evolution before the \(\mathcal{E}\)-regions finally dominate and the backreaction and curvature terms begin to decay faster again.

If it was easily possible to interpret the quantities of the averaged model in terms of standard model ones, one might well imagine the value of the equation of state today of Fig. 1 to lie within the \(1-\sigma\) boundary derived by the WMAP team for a time-dependent dark energy equation of state [47]. We do not show the corresponding plot however, because the comparison is not expected to be very meaningful as the analysis uses the FLRW model as a prior. We again have to ask the reader to await a more reliable interpretation of the data in the context of the averaged model.

3. Dark Matter

The comparison of the \(\Omega\)-evolution of the \(M\)- and \(\mathcal{E}\)-regions in Fig. 1 illustrates nicely the property of averaged models that, depending on the domain under consideration, the effective energy content may vary. Seen in terms of X-matter this means that this component behaves in a more Dark Energy-like way when we consider only the \(\mathcal{E}\)-regions, since the backreaction is positive and drives acceleration. For the \(M\)-regions, however, the backreaction term is negative and slows down the evolution in the same manner as a Dark Matter component would act [18]. If backreaction is an important contribution in today’s Universe, this might imply that parts of the effect known as Dark Matter could be due to a combination of the backreaction and curvature term on \(M\)-regions. This would perhaps open the possibility to reinterpret Dark Matter models in which the particles decay and the Dark Matter responsible for the acoustic oscillations might be replaced through its cosmological effect by a rising X-matter component.

It is to be expected that only parts of the effect attributed to Dark Matter is modeled by the effect of a
negative backreaction, since there are many independent lines of evidence for the existence of Dark Matter. While part of the Dark Matter problem related to the expansion history might well be attributed to backreaction, this would only alter the Late Universe constraints. The relative abundances of Dark Matter to baryonic matter in the Early Universe, determined by the CMB, are not expected to change if we assume a near–homogeneous initial state of the Universe. All CMB constraints that stem from a combination of primordial information and constraints derived from observations of the Late Universe, however, will be subject to changes in course of a reinterpretation of observational data. To show what conditions the CMB data really impose, and that will have to be satisfied also by a backreaction model, we refer to a recent paper\cite{49} that analyses the CMB in a model independent way.

If we choose for the $X$–matter component the description of\cite{38} in terms of an effective scalar field, the “morphon”, we would have two effective potentials depending on whether we are in an over– or underdense region. They are of the form

$$u_M(s_M) = -\frac{4}{\sqrt{5}} H_M^2 \Omega_m^M \left( -\gamma_{Rm}^M \right) \frac{2}{3} \sin^{-1}(s_M) \tag{33}$$

$$u_E(s_E) = \frac{4}{\sqrt{5}} H_E^2 \Omega_{mE} \left( \gamma_{Rm}^E \right) \frac{2}{3} \sinh^{-1}(s_E) \tag{34}$$

where $\gamma_{Rm}^X := \Omega_{Rm}^X / \Omega_{m}^X$, $s_X(t) := \sqrt{8\pi G \Phi_X(t)}$ and $u_X(s) := 8\pi G U_X(\Phi)$. These potentials are simply the specialization of the general result (37) of\cite{34} to the $a^{-1}_X$–scaling and the different negative sign for $Q_M$. The $\sin^{-1}$–behavior reflects the fact that formally the $M$–regions recollapse. In reality the description as a comoving perfect fluid will, however, break down before so that only the rising branch is expected to be physical. The values of the parameters on $M$ (more precisely the fact that $\langle R \rangle_D$ is positive and $n = -1$, see\cite{34} for details) imply, that we have an effective phantom field for the overdense regions. In spite of the usual interpretation in terms of Dark Energy, the different signs lead to a positive effective pressure and therefore to a decelerating component, which hence rather acts in a matterlike way. The form of the potential for the parameters of the model of Fig.\textbf{1} has been plotted in\cite{50}. The approach to describe the effect of inhomogeneities by an effective scalar field allows us to connect the results obtained by the quintessence and scalar field community to the backreaction formalism and to reinterpret their fields and potentials in terms of physical quantities, i.e. the parameters of the averaged model. This is potentially interesting as $\sinh$–potentials have been shown to be able to behave like a Dark Energy component, e.g. in\cite{51} or\cite{52} where their $\left( \cosh - 1 \right)^p$ potential is just a $\sinh$ one by $\cosh (2x) - 1 = 2 \sinh^2 (x)$. For more detailed information to possible quintessence potentials, see\cite{53} and references therein.

Note that scalar fields have also been used to model Dark Matter halos\cite{54} and that other potentials for a scalar field have been advocated to implement the cosmological evolution of such scalar field Dark Matter\cite{55,56}. Consequently, it has been proposed to unify these Dark Matter and Dark Energy scalar fields into a single description e.g. in\cite{57} and\cite{58}. Questions related to unification have been also addressed in a recent work on employing the Chaplygin gas as effective equation of state in the present context\cite{59}. Note also that the common strangeness of scalar field models, i.e. requiring particles with masses of order of $10^{-26}$eV is not present in the morphon picture as there the notion of a “particle” is only an effective one.

D. Comparison with other calculations

In order to judge the quality of the model presented in the previous section, we will now compare its predictions with other calculations in the literature.

1. Magnitude of the initial backreaction

We first consider the perturbative results of\cite{38,41}. There it has been shown that, at the recombinant epoch, the magnitude of the effective energy contribution of backreaction and curvature is expected to be of the order of $2 \times 10^{-8}$ on the scale of the horizon for a Hubble rate of $h = 0.7$\cite{41} that has to be compared to a cosmological constant contribution in the standard model of $3 \times 10^{-7}$ (as is demanded in the strong backreaction scenario below). The corresponding term in our model (on the scale of the simulation box), $\Omega_Q + \Omega_R$, becomes, for the special parameters of the previous section (the $1/a_F$ scaling scenario and at an initial time where $a_D = 1$), $8 \times 10^{-7}$ compared to the matter density parameter. This comparison illustrates, that even if the Early Universe is in a near to homogeneous state, where one can apply perturbation theory, the Late Universe can look strongly different by the effect of the leading order perturbative $a^{-1}_F$–mode alone. When higher order –growing– terms of their Laurent series play an important role, or when nonperturbative terms enter due to a substantial injection of backreaction at the stage of nonlinear structure formation, the situation will even become more different. As this comparison shows, the initial amplitude needed is still too large in the $1/a_F$ scaling model. However, our value was measured on the scale of structure, and may still drop by going to the horizon scale, which would reduce the discrepancy.

2. Structure formation in the concordance model

The second comparison we want to present is the one for the evolution of $\lambda_M$.

In the corresponding figure, Fig.\textbf{1} the line shows $\lambda_M(z)$ as calculated from our model, where we have
Figure 4: Top left: Comparison of the evolution of the parameter $\lambda_M$ calculated from the scale factors shown in Fig. 1 (straight line), with the values derived from the $N$-body simulation (dots). The graph shows that for the $N$-body simulation the evolution of $\lambda_M$ is faster at the beginning and slower at the end compared to the one for our model.

Top right: Comparison of $\lambda_M$ as calculated from Eq. (35) for $\Omega^\text{D}_0 m \approx 0.03$ (straight lines) with the data (dots) derived from the $N$-body simulation for grid sizes of $R = 5, 10, 15, 20, 30, 50$ h$^{-1}$Mpc (from bottom to top). The only input used to generate these curves was the last point of the $N$-body data and the value of $\Omega^\text{D}_0$.

Bottom left: Same plot as top right but with $R = 5, 10, 15$ h$^{-1}$Mpc only and $\Omega^\text{D}_0 m = 0.018$.

Bottom right: Same plot as top right but with $R = 20, 30, 50$ h$^{-1}$Mpc only and $\Omega^\text{D}_0 m = 0.035$.

chosen to present the results in terms of $z$ to provide a more intuitive picture of the actual evolution of structure formation. The points result from our analysis of the $N$-body simulation [60] using a simple separation into blocks described in Appendix B.1. The figure shows a clear discrepancy between the Newtonian $N$-body simulation and our averaged model. This is not surprising as we are approximating the whole Laurent series of $Q_F$ and $\langle R \rangle_F$ by their leading terms only. We will see in Section IV how the higher terms are connected to the shape of $\lambda_M (z)$. Of course it may also be that a genuine relativistic simulation might add corrections to the values of $\lambda_M$, but for the moment we have to work with the models we have. Therefore, we will present, in Section IV the characteristic features of a model that describes structure formation and accelerated expansion in the present framework of averaged models in a consistent manner.

3. Structure formation in general $a_F^{-1}$-scaling models

We finally want to report on an observation that might become of interest in the upcoming reinterpretation of the observational data in the backreaction context. As described in Appendix B.1, we conducted the analysis of the $N$-body data using blocks of various side lengths. In addition to the grid size of 5h$^{-1}$Mpc, being at the basis of the result of the previous section, we also used spacings of 10 – 50h$^{-1}$Mpc. The resulting values for $\lambda_M (z)$ are shown as dots in the graph on the right-hand side of Fig. 4. As we have seen on the left-hand side of this same figure, the $a_F^{-1}$-scaling model using the concordance value of $\Omega^\text{D}_m \approx 0.27$ is not able to reproduce the structure formation history as observed in the $N$-body simulation. We may however try to derive the evolution of $\lambda_M (z)$ for different choices of $\Omega^\text{D}_m$. The differential equation
determining \( \lambda_M(z) \) may be derived from the definition of the latter and reads:

\[
\frac{1}{3} a_D \partial_{a_D} \lambda_M = \frac{H_M(a_D, \lambda_M)}{H_D(a_D, \lambda_M)} - 1 ,
\]

where \( H_D = \lambda_M H_M + (1 - \lambda_M) H_\xi \) and \( H_M(a_D, \lambda_M) \), respectively. \( H_\xi(a_D, \lambda_M) \) may be found by replacing \( a_M \to a_D \lambda_M \) and \( a_D \to a_D (1 - \lambda_M) \) in Equation (29). Using the relations between the different parameters discussed in Subsection III B, we may numerically solve (35) after having specified the two remaining parameters \( \Omega_M^0 \) and \( \lambda_M^0 \). The result for \( \Omega_M^0 \approx 0.03 \) is shown by the lines in the graph on the right-hand side of Fig. 8. There we have fixed, for each curve \( \lambda_M(z; \Omega_m^0, \lambda_M^0) \), \( \lambda_M^0 \) to the end value of the result from the \( N \)-body simulation. It is interesting to notice, that the common value of \( \Omega_m^0 \approx 0.03 \) allows us to describe the evolution on different scales quite well and that the shape of the function as defined by Equation (35) meets the actual form without having to use the theoretically unmotivated fitting ansatz that will be presented in Subsection V B. The fit may even be improved if one allows for a weak scale dependence of \( \Omega_m^0 \) as for \( \Omega_m^0 \approx 0.018 \) the small scales are fitted in an exquisite manner and for \( \Omega_m^0 \approx 0.035 \) this is true for the large scale evolution.

If this is just a coincidence or if this points to a deeper physical interpretation of the backreaction effect in terms of its capability to unify Dark Energy and Dark Matter in one effective fluid cannot be finally decided from this investigation. It is clear that the backreaction terms imply the emergence of both dark components, however, in an entangled way that does not allow us to assign Dark Matter to over–dense and Dark Energy to underdense regions uniquely due to the changing variance between the two types of regions. We may speculate that, in the course of a reinterpretation of observations in the context of the averaged equations, the role of Dark Matter in structure formation on large and intermediate scales might be attributed to average curvature and backreaction components that effectively yield the same distribution of visible matter as CDM–models (but see the constraining remarks of Sec. III C.3). But it may also be that a low value of \( \Omega_m^0 \approx 0.03 \) turns out to be incompatible with other observations. These questions will be addressed in upcoming work when testing the \( a^{-1}_D \)-scaling model against observations as done for a simple \( a^{-3}_D \)-model in [15] (with an improved template metric). Apart from observational issues the model has to be compared to \( N \)-body simulations with baryonic and neutrino content only in order to quantitatively support this necessarily speculative aspect of our model. A naive interpretation of the low- \( \Omega_m^0 \) model also runs into the same difficulties of high initial amplitudes of perturbations as in the old studies of baryonic universe models. Furthermore, since independent evidence for Dark Matter exists [61], especially on small scales and unlike the situation for Dark Energy, we cannot conclude from this work that Dark Matter may be fully identified with backreaction effects.

But, our study clearly indicates the need to exploit this aspect of the backreaction effect.

### E. Discussion

It may be helpful to hold in for a moment and put into perspective what we have seen in this section. It has been shown that in a model that possesses the \( a^{-1}_D \)-limiting behavior of the backreaction components and that starts from almost homogeneous initial conditions as in the standard model, the simple fact that there is a lot of structure today in terms of volume dominance of devoid regions (manifesting itself in our model through a value of \( \lambda_M \) around 0.1) implies some \( \Lambda \)-like accelerated expansion of the volume scale factor \( a_D \). Even though this model is too basic as is indicated by the discrepancy seen in Fig. 8, it clearly illustrates the way in which one would expect backreaction to act. (Here we do not want to over–emphasize the result that a pure baryonic matter content would even do the job.) To avoid the impression that we just replaced the mysterious Dark Energy component by some other mysterious component, i.e. the backreaction term, let us emphasize the different physical situation. In the general relativistic framework that lies at the basis of the averaged equations the emergence of structure is associated with a geometrical deformation of the underlying spacetime. In the comoving time–synchronous slicing chosen to describe the dust universe there is no movement of matter particles with respect to the space. All inhomogeneities emerge from the distortions of space itself. This results in the emergence of intrinsic curvature of the spatial hypersurface, reflected in the three Ricci scalar \( R \), as well as extrinsic curvature of this same hypersurface. This extrinsic curvature is what makes up the backreaction as it has been shown that it may be defined instead of kinematical quantities as in Eq. (3), in terms of invariants of the extrinsic scalar curvature (see [11]). To gain some intuition it may be helpful to see Equation (4) not in an active sense stating that there is some fundamental component that forces the scale factor to accelerate, but rather in a passive sense that it traces the complex evolution resulting from the full Einstein equations with the only active component being the perfect fluid dust matter. Then one may picture the evolution to be governed by gravitational instability that causes inhomogeneities to grow, which manifest itself in extrinsic curvature showing up in the backreaction term, but also – and this is the key issue – in intrinsic curvature of space in which structures emerge. How acceleration in this context may be understood will be discussed in Appendix A.

In this general–relativistic picture one may also see that, even without a cosmological constant, we have some kind of "Dark Energy" in our model universe that now has a clear physical interpretation: It is the curvature energy of the spatial hypersurface, communicated by an effective potential energy in the morphon field correspon-
dence of backreaction \[34\]. This new kind of "Dark Energy" therefore necessarily emerges when structure forms and it is this "Dark Energy" the title of this paper is referring to.

Having emphasized the role of intrinsic curvature one cannot argue that this picture should yield the same results as the Newton–inspired picture of structure forming in space and matter moving around. If the averaged intrinsic curvature evolves differently as compared with a constant curvature model – and this is a generic outcome of the fact that the kinematical backreaction term and the averaged scalar curvature are coupled – then curvature plays a substantial role for the structure formation history, but also for the interpretation of observational data. While metrical deviations from a flat space may be small, derivatives of the metric can be substantial, as has recently been shown by estimating its magnitude in \[16\].

In addition the existence of a strong average scalar curvature term today is not excluded by the data. A study on light propagation in statistically homogeneous universes \[62\] has shown, that the position of the first acoustic peak in the CMB spectrum is consistent with a non–negligible amount of curvature if the expansion history of the scale factor \(a_D\) follows a \(\Lambda\)CDM–evolution (see also \[63\]). As this condition is fulfilled in our model, the occurrence of \(\Omega_R^2 \approx 1\) today is not problematic, if observational data are interpreted in the backreaction context.

\section*{IV. Modeling Structure Formation and Accelerated Expansion: Strong Backreaction Scenario}

After having explored the partitioning approach with the help of a simple \(a_F^{-1}\)–scaling solution in the previous section, which provided the result that this leading perturbative mode alone is able to account for a \(\Lambda\)–like accelerated expansion if we insert what we know about the matter content and the structures of today’s Universe, it may be interesting to learn how the higher terms in the Laurent series of \(Q_F\) and \(\langle R \rangle_F\) have to contribute to give rise to the evolution of structures as derived from the \(N\)–body simulation. In other words we are searching for the nonperturbative behavior of \(Q_F\) and \(\langle R \rangle_F\) that will match the simulated structures. One would like to get a handle on the expansion behavior from simply tracing this evolution with the help of \(\lambda_M(\varepsilon)\) but as the system will still not close one has to find another condition to come to results not being based on assumptions on the scaling of the unknown backreaction and curvature terms themselves. Even if, in principle, all the quantities that figure in the averaged equations are measurable, this is very difficult in practice and we do not yet have a precise idea how they evolve. For the backreaction and curvature terms, Ref. \[24\] provides strategies on how they could be determined, but there are no quantitative results yet.

We will therefore only show here that the averaged equations are able to describe accelerated expansion and structure formation in a model using the partitioning introduced in Section \[II\]. The philosophy is that we want to show for which evolution of the regional backreaction and curvature terms the phenomenological global \(\Lambda\)CDM model, that provides a good fit to a large number of data, may be understood to be the result of the evolution of the physical variables extrinsic and intrinsic curvature.

\subsection{A. Evolution of parameters}

In order to construct the nonperturbative model discussed above, we first have to translate the assumptions of \(\Lambda\)CDM evolution and \(N\)–body structure formation into quantities of our model. The \(\Lambda\)CDM behavior means \(\Omega^0_Q + \Omega^0_R = \Omega^0_{\text{Friedmann}}\), which results in \(Q_D(t) = \Lambda\) and \(\langle R \rangle_D = -3\Lambda\). \(\Lambda\) is determined by requiring \(\Omega^0_Q + \Omega^0_R \approx 0.7\). This is possible, since the cosmological constant is a particular exact scaling solution of the averaged equations. To find \(\lambda_M(a_D)\) we analyzed the \(N\)–body simulation \[60\] using this time a separation into subvolumes based on a Voronoi tessellation of the simulation volume. The resulting data points for five different redshifts are shown in Fig. \ref{fig:5}. They are fitted with a functional ansatz of the form

\begin{equation}
\lambda_M(a_D) := \frac{1}{2} \left( \frac{1}{1 + \left( \frac{\sigma_M}{\lambda_M} \right)^2 + \left( \frac{\sigma_M}{\beta_M} \right)^2} \right) \tag{36}
\end{equation}

and the best–fit parameters providing the curve of Fig. \ref{fig:5} were \(\sigma_M \approx 191\) and \(\beta_M \approx 419\).

\subsubsection{1. Results for the scaling of \(Q_F\) and \(\langle R \rangle_F\)}

The resulting scale factors of the domains are shown in the upper right panel of Fig. \ref{fig:5}. Comparing them to the evolution of the scale factors for the \(a_F^{-1}\)–scaling it becomes clear, that a faster structure formation in the \(N\)–body case, which was found in Fig. \ref{fig:4} requires \(Q_M\) to slow down much earlier. The size of the \(M\)–regions is therefore nearly constant throughout a long period. This is again qualitatively as expected, because the overdense regions virialize and decouple from the overall expansion. \(a_F\) evolves similar to the \(a_F^{-1}\)–case, but finally has to take over the accelerated expansion and therefore has a limiting behavior of a cosmological constant evolution. This limiting behavior might not be realistic as it implies a constant variance of expansion rates which is assumed to shrink as the fastest expanding regions will dominate the volume in the late–time limit. If there is an intermediate state providing this evolution is speculative.

The lower panels of Fig. \ref{fig:5} show the evolution of \(Q_M(a_M)\) resp. \(\langle R \rangle_M(a_M)\) (left) and \(Q_F(a_F)\) resp. \(\langle R \rangle_F(a_F)\) (right). They were multiplied by \(a_M\) resp. \(a_F\) to point out the \(a_F^{-1}\)–behavior at the beginning of the evolution. It is interesting that this limiting behavior, which is – as already mentioned – in accord with the
perturbative result of Li and Schwarz [26, 28], arises naturally in the present setup. The reason why this is the case will be explained in Section V. It should be noted that at the beginning, the curvature on $M$ is positive, the backreaction negative. For $E$ it is just the opposite. Regarding the further evolution it can be seen that the backreaction changes sign (on $E$ even twice) and that $Q_{E}$ and $\langle R \rangle_{E}$ shrink faster than $a_{E}^{-1}$. The reason for this behavior can be seen in the shape of $\lambda_{M}$ in Fig. 3. First we need a strongly different behavior of $M$– and $E$–regions to assure that $\lambda_{M}$ shrinks rapidly. Therefore $Q_{M}$ is negative and acts like matter to slow down the expansion of $M$, whereas $Q_{E}$ is positive to lead to a faster growth. Then the growth on $E$ is slowed down by a negative $Q_{E}$, whereas $Q_{M}$ becomes positive to counterbalance the deceleration of the matter component to lead to the nearly constant part of the evolution of the $a_{M}$–scale factor of Fig. 5 upper right. The linear rise for $E$ at the end encodes the cosmological constant behavior, meaning that

\[ Q_{E} (a_{E}) = -\frac{1}{3} \langle R \rangle_{E} (a_{E}) = \text{const.} \]

which is reflected by an increasing line in this figure as we are multiplying it by $a_{E}$.

2. Derived quantities

Figure 6 finally shows the functional form of the dimensionless parameters. The upper left graph simply shows the evolution imposed by the required cosmological constant behavior of $D$ going from the matter dominated era $\Omega_{m}^{D} = 1$ to today’s value of $\Omega_{m}^{D} = 0.27$. The rest, which is dubbed Dark Energy in the standard concordance model is now represented by our $X$–matter and split into backreaction and curvature. The $E$–regions in the upper right graph develop rapidly “strong” negative curvature, which is however not due to a rise in the $\langle R \rangle_{E}$–term (as may be seen from Fig. 4), but to the faster decrease of the $\langle \theta \rangle_{E}$–contribution. The evolution on $M$
shows that one has to be careful in the interpretation of the $\Omega$–parameters. Due to the approximately constant scale factor $a_M$, which results in a nearly constant $\langle \rho \rangle_M$, the division by $H_D$ which is decreasing makes $\Omega_M$ increase rather strongly. If one interprets $3H_D^2/(8\pi G)$ as the critical density, this increase in $\Omega_M$ reflects the emergence of a strong density contrast between the $M$–regions and the averaged universe model.

Another interesting plot is shown in Fig. 6 in the lower right panel. It presents the effective equation of state of the $X$–matter component of our model composed of $Q_F$ and $\langle R \rangle_F$ as $X_F := Q_F + \langle R \rangle_F$. $w_{\Lambda,\text{eff}} = -1$ reflects the imposed cosmological constant behavior on $D$, whereas the initial values on $E$ and $M$, $w_{\Lambda,\text{eff}} = w_{\Lambda,\text{eff}} = -2/3$, are related to the $a_F^{-1}$–limit shown in Fig. 5. $w_{\Lambda,\text{eff}}$ approaches $-1$ and $w_{\Lambda,\text{eff}}$ rapidly goes to zero. This means that it is scaling like a matter contribution with $a_M^{-2}$ in the end, but as the sign of $Q_M$ is positive, it acts as “repulsive matter”, counterbalancing the deceleration of the matter component to lead to a nearly constant $a_M$. Again, this is expected physically. While our model does not describe the relevant small–scale effects like velocity dispersion and rotation, effects that stabilize the virialized regions, the $N$–body simulation does describe these effects. The evolution of the $Q_M$–term thus rather reflects the behavior of a more general backreaction term as expressed in \[13\].

Another interesting plot is shown in Fig. 6 in the lower right panel. It presents the effective equation of state of the $X$–component as defined in \[14\]. The implications are discussed in Section IV A 2.

B. Cosmic phase space

Another possibility to characterize the evolution of $Q_F(a_F)$ and $\langle R \rangle_F(a_F)$ is to trace their solutions in a “phase space” introduced in \[34\]. Its dimension is two, i.e. there is room for many homogeneous, almost–isotropic effective states, while the homogeneous–isotropic solutions just form a line in this space. Every curve in this space represents a solution of the averaged equations, while straight lines represent the class of scaling solutions. It is shown schematically on the right–hand side of Fig. 7. The different sections are exhaustively discussed in \[34\]. We will repeat what is necessary for our purpose in the following. The coordinates of this space are chosen to be
the matter parameter $\Omega^F_m$ and the deceleration parameter $q^F$, defined as

$$q^F := -\frac{\ddot{a}_F}{a_F H^2_F} = \frac{H^2_F}{H^2_D} \left[ \frac{1}{2} \Omega^F_m + 2 \Omega^F_D \right].$$  \hspace{1cm} (37)

This latter was chosen instead of $\Omega^F_Q$ alone in order to have an additional intuitive meaning. In this space, every straight line passing through the center, which represents the EdS model, corresponds to an elementary scaling solution $a^r_F$. It has been shown in [34] that the EdS model $(\Omega^F_m, q^F) = (1, 1/2)$ is in addition a saddle point for the dynamics of a universe model described by the averaged equations. Even a small initial backreaction – which is always present due to the observed emergence of structure – will drive the expansion away from it and will result in accelerated expansion if the deviation goes in the corresponding sector, which is the one on the lower left–hand side. Besides the EdS model, further special solutions are the line $r = 1/3$ ($r$ has been introduced in Eq. (27)), corresponding to models with Friedmannian kinematics but rescaled cosmological parameters, and $r = 0$ representing models without backreaction. In this case curvature reduces to a constant curvature $a^{-2}_F$ behavior and they are therefore scale–dependent Friedmannian models. The line $r = -1/3$ comprises models for which backreaction acts like a scale–dependent cosmological constant. The introduction of a genuine cosmological constant would simply shift the diagram down by its value if it is positive.

Figure 7 shows the form of our solutions in this space. The dashed curve is the one for the $E$–regions, the dotted one is for $M$. Both begin at the EdS model in the center as they are initially matter dominated. Because of their $a^{-1}_F$–limit they lie on the line corresponding to $r = -1/5$. Their opposite signs that are responsible for the vanishing initial backreaction on $D$ make them evolve from the center in different directions. Both lines get shallower and $E$ approaches the line with Friedmannian kinematics ($r = 1/3$) until structure formation sets in and the growing variance of expansion rates drives it to the line of a scale–dependent cosmological constant ($r = -1/3$). The deceleration parameter is for $M$ not the best quantity as it is not guaranteed that it does not diverge for $\dot{a}_F = 0$.

V. POWER SERIES OF THE BACKREACTION TERMS

A. Calculation

In Section IV we have seen that the backreaction and curvature terms show an initial $a^{-1}_F$–behavior. In the following we will explore where this comes from and rederive this behavior in our present context of the partitioning into two subregions. We first present the result in the form of a proposition:

Proposition. For every evolving spacetime that possesses a flat Einstein–de Sitter limit at the beginning of its evolution, the backreaction and curvature dependence on the volume scale factor $a_F$ on arbitrary subregions $F$ stemming from a metric–compatible partition of compact domains in the hypersurfaces of constant time $D$ and that are evolving differently from one another, may
be expressed as a Laurent series beginning with $a_F^{-1}$ for backreaction and with $a_F^{-2}$ for curvature.

To prove this proposition we will have to find the expressions for $Q_F(a_F)$ and $\langle R \rangle_F(a_F)$. They can be derived from (41) and (42) which gives

$$Q_M = \frac{2}{3} \lambda_M k^3 \left( \frac{a_M}{a_M(t)} \right)^3 + 3 \frac{\ddot{a}_M(t)}{a_M(t)}$$

(38)

$$\langle R \rangle_M = \frac{2}{3} \lambda_M k^3 \left( \frac{a_M}{a_M(t)} \right)^3 - \frac{6}{3} \frac{\ddot{a}_M(t)}{a_M(t)} - 3 \frac{\ddot{a}_M(t)}{a_M(t)},$$

(39)

where $k = \left( \frac{2}{3} \alpha D a_M^2 H_D^2 \right)^{\frac{3}{2}}$. To calculate $Q_F(a_F)$ and $\langle R \rangle_F(a_F)$ we therefore have to derive $\dot{a}_M(a_M)$ and $\ddot{a}_M(a_M)$. We achieve this by the assumption of a flat matter dominated initial universe model that leads to

$$a_D(t) = k t^\frac{3}{2}; \quad \ddot{a}_D(t) = \frac{2}{3} \left( \frac{k^3}{a_D(t)} \right); \quad \ddot{a}_D(t) = -2 \frac{k^3}{9 a_D^2(t)}.$$

(40)

To connect $a_M$ to $a_D$ we use the relation $a_M = a_D^2 a_M^\frac{1}{2}$ and expand $\lambda_M$ with a small parameter $\alpha$ around $\alpha = 0$ which gives

$$a_M = a_D^2 a_M^\frac{1}{2} (ad \alpha) = \lambda_M^\frac{1}{2} a_D (1 + \lambda_M a_D \alpha + \lambda_M a_D^2 \alpha^2 + O(\alpha^3)),$$

where we have defined $\lambda_M$ and $\lambda_M^\prime$ as $\lambda_M := \lambda_M(0)/(3 \lambda_M)$ and $\lambda_M^\prime := (2 \lambda_M^2(0) + 3 \lambda_M^4(0) \lambda_M) / (18 \lambda_M^3)$. A prime stands for a derivative with respect to the argument and $\lambda_M(0)$ is denoted as $\lambda_M := \lambda_M(0)$. Differentiating (41) with respect to time gives $\dot{a}_M(\dot{a}_D)$ and $\ddot{a}_M(\dot{a}_D, \ddot{a}_D)$. Using (40) we find $\dot{a}_M(a_D)$ and $\ddot{a}_M(a_D)$, which leads to the finally necessary $\dot{a}_M(a_M)$ and $\ddot{a}_M(a_M)$ by an inversion of (41). This provides

$$Q_M = \frac{2}{3} \lambda_M^\frac{1}{2} k^3 (3 \lambda_M^2 + 7 \lambda_M^2) a_M^2,$$

(42)

$$\langle R \rangle_M = \frac{40}{3} \lambda_M^\frac{1}{2} k^3 \lambda_M a_M^2,$$

(43)

and therefore proves the proposition if we take into account that the expansion (41) may be extended to arbitrary order. The third- and fourth-order terms of $Q_M$ are

$$-4 \frac{k^3}{3} (2 \lambda_M^3 - 3 \lambda_M^2 \lambda_M - 9 \lambda_M) \alpha^3$$

(44)

$$2k^3 (3 \lambda_M^4 - 6 \lambda_M^2 \lambda_M^2 - 4 \lambda_M^3 \lambda_M) \lambda_M^\frac{1}{2} a_M^2,$$

(45)

and all higher-order terms may be calculated in a straightforward way, but have increasingly complicated coefficients. By Equation (42), $Q_F(a_F)$ and $\langle R \rangle_F(a_F)$ are linked and the terms $a_F^{-3}$ of $\langle R \rangle_F(a_F)$ are the same as those for $Q_F(a_F)$ simply multiplied by $r^{-1} = -(n + 2)^{-1}$. For the second–order term this may be seen in (42) and (43) where $r^{-1} = -5$. For the third– and fourth–order terms above one arrives at $\langle R \rangle_F(a_F)$ when multiplying the third–order term with $r^{-1} = -3$ and the fourth–order one with $r^{-1} = -7/3$.

These results show why we had to include the condition of a different evolution of the subregions. If they expand in the same manner, the ratio $\lambda_M$ stays constant which means that $\lambda_M = 0$ and therefore $Q_F = \langle R \rangle_F = 0$. Another way to phrase it is, that there is no backreaction in a completely homogeneous universe. But if there are structures developing at any scale, we will have a non-vanishing backreaction term and it will go as $a_F^{-1}$ in a matter dominated state.

### B. Quantitative conclusions

Two remarks are in order here. First we want to emphasize that the result of Equations (42) and (43) coincides with what has been found by Li and Schwarz [26, 28] in second–order cosmic perturbation theory. They also found that on a region $D$ on a flat matter dominated background the backreaction term is of second order and scales as $a_D^{-1}$. The third–order term is a (cosmological) constant.

Second, we want to explore what the result means quantitatively for the initial backreaction. For simplicity let us first take the fit to the $N$–body simulation data of Section IV. For the functional form (46) we obtain

$$a_M = \lambda_M^\frac{1}{2} a_D^2 \left( 1 - \frac{1}{3} a_D^2 \right)^2 + O(\alpha^4),$$

(46)

and we may derive the expansion coefficients to be $\lambda_M = 0$ and $\lambda_M = -1/3$. The first observation is that $\lambda_M = 0$ causes the $a_F^{-2}$ term of the averaged scalar curvature $\langle R \rangle_F$ to be zero. In view of the definition of $\lambda_M$ below Eq. (41) it becomes clear that this is the case whenever $\lambda_M(0) = 0$. This means that, if the difference between the subregions at the beginning is not very important so that they evolve initially in a similar way, the term that evolves as a constant curvature term in a FLRW model with $a_F^{-2}$ will be vanishing, provided the background does not have a constant curvature term by itself.

If we demand that the initial matter densities are similar on all the subregions $\Omega_m^i = \Omega_m^D$, we find to leading order

$$\frac{\Omega_m^i}{\Omega_m^D} = \frac{7}{12} a^2,$$

(47)

which gives the initial value of the backreaction compared to the initial matter density. This relation is interesting since it encodes the relation between the evolution of structure in the Universe and the initial value
of the backreaction term. In (36) $\alpha := \alpha^{-1}_{M}$ characterizes the rate at which the $M$–regions decouple from the common evolution of the subregions. For a larger $\alpha$, $\lambda_{M}$ decreases more rapidly. This faster evolution requires a bigger initial amount of backreaction as it encodes the in-homogeneities that force the system to deviate from the initial near to homogeneous state. The larger the in-homogeneities are, the faster this deviation takes place.

To give a quantitative estimate we will however use a slightly different fitting function than (36) that is adopted to the shape of $\lambda_{M}(a_{D})$ on a larger range of scales. We analyze to this end the $N$–body data provided by [60] with a mesh of varying grid size. The result are the data points of the left graph of Fig. 8.

They are shown together with a fit using the two parameter model

$$
\lambda_{M}(a_{D}) = \left( \beta + \frac{1/2 - \beta}{1 + (aa_{D})^2} \right) e^{-\frac{aa_{D}}{\gamma}},
$$

(48)

where $\gamma(\alpha, \beta)$ is determined by the requirement that $\lambda_{M}(a_{D})$ is flat for $a_{D} = 1$. The form of this function is motivated in Appendix B.1.

An interesting outcome from this figure is that it confirms the idea that $\lambda_{M}$ may be a good parameter characterizing the formation of structure. This is because the different scales evolve differently as expected from the hierarchical formation of structure. On small scales, e.g. for $R = 1h^{-1}\text{Mpc}$, $\lambda_{M}$ has already dropped to half of its initial value at a redshift of approximately $z = 6$. For scales of $R = 10h^{-1}\text{Mpc}$ this happens only at $z \approx 0.7$. Therefore, we recover at least qualitatively the fact that structures start forming at small scales and only then assemble to bigger ones.

Using an expansion of the form (46) and the general form of (47)

$$
\frac{\Omega^{\lambda_{M}}_{i}}{\Omega^{m}_{i}} = -\frac{1}{4} (3\lambda^{2}_{M_i} + 7\lambda_{M_i}) \alpha^2,
$$

(49)

we finally derive the initial abundances of the $a^{-1}_{M}$ backreaction term that are shown on the right–hand side of Fig. 8. We have evaluated grid lengths from $1 - 50h^{-1}\text{Mpc}$. The line of Fig. 8 indicating an $R^{-1}$–behavior of the scale dependence shows that in our evaluation of the $N$–body structure formation there seems to be a discrepancy with results from [27], where the authors found that the values for the backreaction term as derived from the power spectrum should scale as $R^{-4}$ with the length scale $R$. In their case, however, they used a pure Harrison–Zel’dovich spectrum scaling as $k^{1}$ to derive this result. For a more realistic CDM power spectrum, the small–scale behavior is rather $k^{-2}$ which in turn means that the spatial dependence changes from $R^{-4}$ to $R^{-1}$. The region that we could evaluate with scales up to $25h^{-1}\text{Mpc}$ (top hat), is in this small–scale regime. It would be interesting to evaluate a bigger simulation to go beyond this scale and to see the change in the scaling behavior of the backreaction term.

Quantitatively, the percentage of initial backreaction shown in Fig. 8 ranges from $10^{-5} - 10^{-7}$ depending on the scale (see Fig. 8). These specific values show that backreaction is indeed perturbatively small in the quasi-homogeneous epoch of the Universe. On large scales it seems insufficient to change the overall behavior as indicated by the value of $2 \times 10^{-8}$ on the scale of the horizon, found for example in the perturbative calculation of [41]. On the scale of structures, however, this perturbative contribution grows with respect to the matter content due to its $a^{-1}$–scaling and is sufficient to lead to the diverse structures we see in today’s Universe.
VI. CONCLUDING REMARKS AND OUTLOOK

Let us discuss what we have seen in this paper. Section II showed that a consistent split of the dynamical equations governing the averaged universe model is possible. This split also allows, in Appendix A, to shed light on the property of the volume scale factor to show accelerated expansion, even in a case where there was no acceleration in the evolution of its components. The split of the equations enabled us to construct an averaged universe model without having to know the initial magnitude of backreaction. Instead this latter follows from the strength of structure formation that we put in in the form of today’s volume fraction $\lambda_\mathcal{M}$ of the initially overdense regions $\mathcal{M}$. This model implies that the volume scale factor $a_D$ evolves quite similarly to the scale factor in a flat Friedmann model with about $70\%$ cosmological constant. An obvious shortcoming of this model is a mismatch with the actual time–evolution of the best–fit volume fraction with that predicted by the $N$–body data. Surprisingly, this mismatch disappears neatly, if we reinterpret the fundamental Dark Matter fraction in the matter parameter by the backreaction effect, leaving only the baryon content as fundamental. We do not want, however, to emphasize this result, since the Dark Matter issue has to be investigated much beyond our simple model.

Having found that there may be more to the evolution of $Q_\mathcal{F}$ than just extrapolating the perturbative $a_F^{-1}$–behavior, we investigated in Sec. IV how to connect structure formation and accelerated expansion within a strong backreaction scenario. By globally imposing the particular scaling solution that mimics a cosmological constant and by assuming the structure formation history of a Newtonian $N$–body simulation, the initial $a_F^{-1}$–limit on the subdomains is obtained, but this also provided a non-perturbative extension of the $Q_\mathcal{F}$–scaling to later times. The emergence of the $a_F^{-1}$–scaling was then closer investigated in Sec. V where it became clear that it is generic for any matter dominated universe model that starts out close to homogeneity. This also confirmed that our choice to invoke the $a_F^{-1}$–behavior on the evolution on $\mathcal{M}$ and $\mathcal{E}$ instead of $D$ was the right one. We could instead have identified our $D$–region with the one that Li and Schwarz looked at, but Section V showed that our $D$–region corresponds rather to their background. Finally, we showed that if the Universe may be described by the average equations and we require it to have the structure we see today, this means that the backreaction component has to be of the order of $10^{-5} - 10^{-7}$ (as compared to the matter density and depending on the scale one is considering) in the initial near to homogeneous phase that may be treated by perturbation theory. In the later non-perturbative stages of the evolution of the Universe, this tiny initial fraction is sufficient to give rise to structures and inhomogeneities that we see in the recent epoch.

A striking result of this work is the fact that both complementary scenarios lead to qualitatively similar evolution laws for the backreaction terms on the largest scales, while the models only differ in the details, e.g. in the concrete form of the structure formation histories and in the behavior of the variables on $\mathcal{M}$–regions. Since the assumptions underlying these two scenarios are quite orthogonal, we are confident that the $a_F^{-1}$–scaling behavior at early stages of backreaction evolution on the subdomains (imposed in the first scenario and derived from the latter) should be close to the actual physical evolution that has to be confirmed by future investigations of perturbation theory on an evolving background and by nonperturbative models that contain exact solutions as limiting cases.

One out of a number of problems that remain is to give a clearcut interpretation of the quantities calculated in terms of observables. As we argued it is difficult to link the initially underdense $\mathcal{E}$–regions directly to today’s voids as one might wish to. To bypass this ambiguity one could imagine to extend the analysis to three regions $\mathcal{M}$, $\mathcal{A}$, and $\mathcal{E}$ where one could choose which overdensity they possess. It would be natural to consider as $\mathcal{M}$–regions the ones with overdensities of typical clusters, the $\mathcal{E}$–regions with those of voids and put all the astrophysically un–spectacular rest into the $\mathcal{A}$–regions. The advantage of probably being able to give a more refined meaning to for example the $\Omega$–parameters is, however, counterbalanced by the inconvenience to have to introduce another parameter in the scaling model or even a free function $Q_A(a_A)$ in the general case. First results along the lines of the scaling of Section III and the fit of Section IV have been obtained in [50, but further analysis would be needed to find out whether the identification is possible and if it is therefore worthwhile to go this way.

A further refinement of the presented model would be to give up the preservation of the identity of the initially characterized subregions. Whether this is necessary could be checked through a detailed analysis of $N$–body simulations. As our model is effective one could think of implementing reaction rates between over– and underdense regions in the spirit of nonequilibrium transitions in chemical reactions. A mass–action law could be devised that governs an equilibrium state between elementary subregions in which the reaction rates are equal but nonvanishing, and the transition laws could be determined as done in nonequilibrium chemical reactions (details of such an approach in the cosmological context may be found in [65]).

We shall, however, first investigate other models that are based on relativistic Lagrangian perturbations, generalizing the Newtonian nonperturbative model investigated in [22, 64]. This systematic attempt – that is currently in preparation – will provide the first generic relativistic evolution model for structure formation, including the spacetime metric for implementing observables as measured along the light cone.
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Appendix A: Accelerated expansion in averaged
models

We want to use this appendix to elaborate on the pos-
sibility of accelerated expansion with local deceleration.
This property of averaged models often leads to confu-
sion so it may be worthwhile to give a simple example, in
which one can understand intuitively its origin, and then
recap some arguments of the literature for the general
case. One popular paper that gave rise to the confusion
is, for example, [60]. Apart from the acceleration issue
discussed in this section, it also postulates the negligibil-
ity of backreaction effects due to the applicability of the
post–Newtonian metric. The reasons of why this argu-
ment is false, are by now given in various papers in the
literature, e.g. [46, 67, 68].
the thickness of the shell. In between, the expansion of the total radius $a_D$ experiences an acceleration as is argued in the text.

1. A toy model

Let us consider the setup of Fig. 9 where we have identified two distinct regions $M$ and $E$.

The spherical $E$–region in the middle is assumed to be devoid of matter and to expand only due to its negative curvature. The shell however is assumed to be matter dominated and virialized such that the net expansion rate $H_M$ is zero. This means that the volume of the $M$ region is constant, i.e. $\dot{a}_M = 0$, and that $\dot{a}_E = 0$. Let us consider the case when $\dot{a}_E$ is also vanishing which leads with (19) to

$$\frac{\ddot{a}_D}{a_D} = 2 \lambda_M (1 - \lambda_M) H^2_E. \quad (A1)$$

As long as the inner region keeps expanding we will also have acceleration of the global expansion, even if none of the two regions themselves accelerate. In our case this can be understood as a volume effect. As $\dot{a}_E = 0$, $\dot{a}_E$ is constant and therefore the growth in radius within a time interval $\Delta t$ is $\Delta a_E$ and also constant. That this does not result in a constant expansion rate for the total radius $a_D$ is due to the condition that the volume of $M$ is constant. With an increasing inner radius, the shell must become thinner to satisfy this condition. In a more intuitive way one may think of the growth of the inner radius as if it grew into the shell so that we had to subtract a shell with thickness $\Delta a_E$ from the inner border of the $M$–shell and to add the missing volume at the outer border of the whole configuration. As the radius of the shell to add there is bigger than at the inner border, we need not to add $\Delta a_E$ but a shell of a smaller thickness to keep $|M|$ constant. This means that the radius $a_D$ of the total configuration does not grow by $\Delta a_E$ but by $\Delta a_D < \Delta a_E$. With increasing size of the expanding inner sphere and the corresponding reduction of the thickness of the $M$–shell, as shown in Fig. 9 on the right, the difference of the thickness of the shell taken away at the inner border of $M$ and added at the outer border decreases. $\Delta a_D$ becomes more and more $\Delta a_E$, i.e. it increases. Therefore, $a_D$ experiences accelerated expansion as the growth $\Delta a_D$ in a fixed interval $\Delta t$ is growing. From this explanation it becomes also clear that accelerated expansion can only be a temporary effect. In Eq. (A1) this is reflected by the fact that with the growth of the $E$–region the parameter $\lambda_M$ is going to zero and with it $\dot{a}_D$. This explanation also shows that the effect depends on the number of dimensions in which one is evaluating it. In a one–dimensional setup, it will not appear at all as the two scale factors just add linearly. It then grows with every dimension one adds what may be seen in the growth of the second term in (19). Adding generic volumes of the same shape but evolving differently, one has geometric effects on the evolution of the global scale factor (e.g. combine 4 squares to a bigger square but, as soon as you evolve the individual squares differently, the overall shape will not be a square).

If we generalize the setup a little to include a decelerated expansion of the $E$–region, Eq. (A1) acquires an extra $\dot{a}_E$ term. If it is smaller than the $H^2_E$ term, one has acceleration of the whole configuration despite deceleration of the inner constituent.

Note that the acceleration described above also shows up in a configuration where the inner part is contracting, i.e. $H_E < 0$. Then, the difference between the shell that has to be taken away at the exterior of the outer $M$–region and the one that has to be added on the inner side of the $M$–shell will be growing. First, they are similar, but finally, a reduction of the diameter of the $E$–region by $\Delta a_E$ will result in a smaller reduction of the overall diameter $a_D$, i.e. $\Delta a_D < \Delta a_E$. This is to be interpreted as acceleration of the $D$–region in the sense of shrinking deceleration.

To go one step further one could imagine to have a toy model of the Universe build up out of balls of the first type. Each of them expands in an accelerated way. If the accelerated phase sets in during a narrow interval of time for all of the balls, one would find that the effect adds up and can cause global acceleration. The setup in this extended case is a bit similar to the Swiss cheese models, but without the restriction of the embedding in an overall FLRW evolution.

2. More general cases

The emergence of accelerated volume expansion despite local deceleration has been already widely discussed in the literature. Essentially it is due to the fact that the local contributions are correlated in an average [14]. To give a more intuitive explanation, Räsänen, for example, argued in [14] that the physical reason for this volume acceleration is that the volume fraction of the faster expanding regions rises. In our partitioning approach this
is reflected in Eq. (19), which may be recast in the form
\[
\frac{\dot{a}_D}{a_D} = \lambda_M \frac{\dot{a}_M}{a_M} + (1 - \lambda_M) \frac{\dot{a}_E}{a_E} + \frac{2}{9} \left[ \lambda_M (1 - \lambda_M) \right]^{-1} \lambda_M^2.
\] (A2)

It underlines the statement that there can be accelerated expansion if the change of the volume fraction of the faster expanding regions \(\lambda_M\) is sufficient. It also shows that the possibility of acceleration holds beyond the toy model case. If structure formation is rapid enough so that \(\lambda_M\) is able to counterbalance \(\dot{a}_M/a_M\) and \(\dot{a}_E/a_E\), it will drive acceleration.

From the averaged equations themselves one can derive the condition for an acceleration of the volume scale factor. For strong backreaction (See [67] for the distinction between weak and strong backreaction) Equation [4] tells us that \(Q_D\) must be positive and bigger than \(4\pi G (\bar{\rho})_D\) to lead to acceleration. In view of the definition [8] this means that the term \((\bar{\rho}^2)_D - (\bar{\theta}^2)_D\) has to be sufficiently bigger than the shear term \((\bar{\theta}^2)_D\), i.e. the Universe should be dominated by expansion fluctuations rather than by fluctuations of the averaged rate of shear.

**Appendix B: Evaluation of the N–body simulation**

Here we want to describe how the results on the evolution of the \(\lambda_M\) parameter in Figs. 4 and 5 were obtained. The data that we analyzed were obtained from a simulation of the Virgo Supercomputing Consortium [22]. They trace the structure formation in a cube of a sidelength of 479h^{-1}Mpc. The underlying cosmological model was a \(\Lambda\)CDM model with \(\Omega_M = 0.7\) and \(H_0 = 70\) km s^{-1}Mpc^{-1}. The simulation contained 512^3 \approx 134Mio particles.

1. A simple mesh method

To obtain a rough approximation of the overdensity field, we separated the data cube with a mesh with a fixed grid size and determined the number of data points in the cells. Then we added up the volume of the most dense cells until the sum of the points contained in the cells added reached half the number of total points of the simulation volume. This is because we decided to fix the characterization of the volumes in the initial, near to homogeneous state. Under the assumption of a Gaussian distribution of the density field with only small density fluctuations, approximately one half of the mass in the Universe will be in overdense \(M\)– resp. underdense \(E\)– regions, if their typical size was nearly the same at that epoch. The volume obtained by the addition described was suspected to be \(V_M\), and used to determine \(\lambda_M\) by dividing by the total box volume. Of course this identification is a source of error because it may happen that in a region that was underdense in the beginning structure formation leads to a density peak that belongs to the densest half of the matter distribution. But as the considerations in this paper are mostly concerned with the influence of structure formation on the global expansion history, we expect the error to be tolerable.

Apart from this problem the determination is well–defined even if the \(M\)–regions scale differently than the global \(a_{\Lambda\text{CDM}}\)–scaling of the simulation. This is because we are only interested in volume fractions and assume our background region \(D\) to scale as \(a_{\Lambda\text{CDM}}\) even if we assume different physical reasons for this behavior. Then
\[
\lambda_M = \frac{V_M}{V_D} = \frac{N_M V_{\text{box},\text{initial}} a_{\text{\Lambda CDM}}^3}{N_D V_{\text{box},\text{initial}} a_{\text{\Lambda CDM}}^3} = \frac{N_M}{N_D},
\] (B1)
where \(N_M\) and \(N_D\) are the number of boxes of the overdense half and the total number of boxes respectively, and the simulation volume just drops out.

To derive a more reliable estimate for \(\lambda_M\) than the one obtained by the procedure described above, one would have to consider Lagrangian domains as required by the averaging formalism. This would imply to have a clear definition of the volume associated to the particles in the initial state and to follow exactly this volume throughout the evolution. As in the real world, at least at small scales vorticity is not absent, it would be complicated to put this into practice, especially in a relativistic framework. Other complications include particle crossing and domain merging. We suppose, however, that this would not be completely different, since averaging strategies to find extensive quantities over smoothed–out singularities are implementable and would likely not change the partitioning. It is clear that there is a huge potential to improve on our rough evaluation.

The above procedure was carried out for different grid sizes to get an impression of the dependence of \(\lambda_M\) on the grid length. The result is shown on the left–hand side of Fig. 5. We used values of 1–50h^{-1}Mpc. The plot makes clear that there is in fact a strong dependence of \(\lambda_M\) on the grid size. This is not surprising as structure formation in the Universe proceeds in a hierarchical way. The small–scale structure forms first and then starts to combine to larger structures. This is reflected in the steeper decrease of \(\lambda_M\) for smaller grid lengths.

To motivate the choices of grid resolution in Sec. 11 we consider that a scale of 1h^{-1}Mpc, as used for the lowest curve of Fig. 10 may already be too small as the simulation is used for large–scale structure. The highest curves above 10h^{-1}Mpc grid length is on the other hand already merging typical overdense structures like clusters with underdense voids. If one takes into account that a typical cluster of galaxies is in the range of 2 – 8Mpc one would expect that grid sizes of 2 – 8h^{-1}Mpc are perhaps best adopted to the considered problem. Therefore, we chose a grid length of 5h^{-1}Mpc for the comparison of Fig. 11.

For the results of Section 10B we had to find an adequate functional form for the evolution of \(\lambda_M(a_D)\). From the theoretical point of view it should meet several re-
requirements. First of all, we expect $\lambda_M$ to start at an initial value of about 0.5 on all scales, as the density fluctuations of the Early Universe are assumed to be Gaussian. The second condition should be that it does not deviate rapidly from this value at the beginning. So $\lambda_M$ should have a flat tangent at $a_D = 1$. Third, we expect that $\lambda_M$ tends to 0 on all scales as matter clusters strongly and the voids keep expanding. For a standard ΛCDM model this is even more true as then the cosmological constant will accelerate the growth of the empty regions between the matter filaments. These three conditions lead to the ansatz of (48).

2. The Voronoi method and structure formation

As the fixed mesh is not adapted to the real matter distribution in the simulation volume, we used in a second step a partitioning of the volume into Voronoi regions. This approach has been advocated by van de Weygaert [69, 70] and assigns each simulated point the region that is closer to it than to any other point. In condensed matter physics a Voronoi cell is better known as the Wigner Seitz cell. To calculate the Voronoi cells we used the program “qhull” [71], and derived $\lambda_M$ in the same way as described above by adding the densest Voronoi cells. The resulting points are shown in Fig. 10 on the left together with the curves for 0.5, 1 and 5$h^{-1}$Mpc, as derived by the mesh method. The plot shows that the Voronoi results are closest to the 1$h^{-1}$Mpc-mesh ones. The shape, however, shows a steeper behavior in the beginning, which one would interpret as a faster structure formation. The Voronoi result was used for the fit of $\lambda_M$ of Fig. 5.

We finally use the separation into Voronoi cells to show the formation of structure in the distribution of the overdensity field.

To this end we plot in Fig. 10 on the right the evolution of the distribution of the density contrasts of the Voronoi cells. For our first data at a redshift of $z = 5$ the distribution still is strongly peaked around zero. The maximum is then moving to underdense regions. In the meantime a second maximum emerges for high density regions. The mean overdensity, however, stays zero all the time. It is interesting to see that the distribution spreads and reaches an extension of 8 orders of magnitude in the density contrast. These extreme differences in the local densities and their expected different evolution is one of the main motivation for the idea of a structure formation effect on the global evolution.