Comment on Inconsistency of the basic nonadditivity of q-nonextensive statistical mechanics, \[\text{arXiv:0910.3826v1}\]

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Abstract

In a recent paper, Wang, et al. (2009) claim that Tsallis’ nonadditivity of \(q\)-nonextensive statistical mechanics (Gell-Mann and Tsallis 2004, Tsallis 2009) is mathematically inconsistent and hence one should carefully review Tsallis’ ideas and theory in toto. This present comment is to point out that the conclusions of Wang et al. (2009) were arrived at by misinterpreting two basic items and hence the advice of Wang et al. should be carefully reviewed.

1. Introduction

Wang et al. (2009) state that Tsallis claims that “\(\alpha\) is a real coupling constant characterizing nonadditivity”. This is incorrect if “characterization” is used in the mathematical sense. No such statement is implied in Tsallis’ theory, as the present authors understand the subject matter (Gell-Mann and Tsallis 2004, Tsallis 2009). The parameter \(\alpha\) characterizes nonadditivity as claimed by Tsallis, and no claim is made that \(\alpha\) is unique. The parameter can be estimated, giving rise to different values in different contexts, as shown by the many papers on the topic. It is never claimed to be a universal constant. The parameter is not given a universal physical interpretation. The present authors do not believe that nonadditivity parameter \(\alpha\) can be given a universal physical interpretation but it is possible to give physical interpretations in terms of moments which will have different physical meanings in different contexts.

The mathematical inconsistencies pointed out in Wang et al. (2009) stem from the misreading of statistical independence and misusing the concept of statistical independence to describe subsystems of a given system. “Independence” is a misnomer and the events depend on each other through a product probability property (PPP). Mathai and Haubold (2007) have suggested to replace the phrase “statistical independence” with the statement of events satisfying “product probability property (PPP)” in order to avoid misuse when the concept is applied to practical situations. For two events \(A\) and \(B\), when \(P(A \cap B) = P(A)P(B)\), where \(P(\cdot)\) represents the probability of the event \((\cdot)\), the dependence of \(A\) and \(B\) is through this product property. When PPP holds and if logarithm is taken then, naturally,

\[
\ln P(A \cap B) = \ln P(A) + \ln P(B). \tag{a}
\]

Thus, a sum is obtained. Shannon’s entropy being a logarithmic function, for any two events satisfying PPP, the entropy in \(A \cap B\) will be the sum of the entropies in \(A\) and \(B\). In terms of random variables or probability/density functions, the correct interpretation is that when the joint probability/density function is the product of the marginal probability/density functions then all events on the product space will have the property in (a). In this case, Shannon’s entropy on the joint distribution will be the sum of the entropies on the individual probability/density functions.

Part of the misinterpretations come from the use of misleading notations + for addition by Tsallis and \(\cup\) for union by Wang et al. It is the entropy on the joint distribution, and if \(X(i)\) is used for the entropy
on the probability/density function of the $i$-th variable, then the entropy on the joint probability/density of $k$ random variables may be denoted as $X(1, 2, ..., k)$. If the variables have a general joint distribution, the entropy of the joint distribution cannot be expected, and not claimed by anyone, to be the sum of the entropies on the individual distributions. The claims, and thereby controversies, are only connected to the situation when the $k$ variables are mutually independently distributed or when PPP holds. Then only the question of additivity or nonadditivity becomes meaningful.

But Tsallis’ entropy and many other $\alpha$-generalized entropies do not have this additivity property because, basically, they are not logarithmic functions. For a discussion of such nonadditive entropies and their mathematical characterizations, see Mathai and Rathie (1975, 1976).

Inconsistencies also come from misinterpreting subsystem as subsets of a given set, ignoring the fact that such subsystems need not hold the property (a). Result corresponding to equation (1) of Wang et al. (2009) will hold for any number of distributions when the random variables involved are mutually independently distributed in the sense of joint probability/density satisfying PPP. Corresponding results are given in Mathai and Haubold (2007) for the case of more than two variables, along with mathematical characterizations of Tsallis’ type entropies. Mathematical inconsistencies pointed out in Wang et al. (2009) are not inconsistencies but misinterpretations of two items that (1) $\alpha$ is taken as a unique universal constant, (2) statistical independence of random variables is misinterpreted by drawing parallel or by comparing to a system and its subsystems, ignoring PPP. One can have a system and its subsystems satisfying PPP.

Only when $\alpha \to 1$ Tsallis’ entropy and many other $\alpha$-generalized entropies will have additivity property. In this sense, physics coming from an $\alpha$-generalized entropy goes beyond the physics coming from Shannon’s entropy or Boltzmann-Gibbs entropy, giving rise to a wider coverage. If someone wants to call this wider coverage as “nonadditivity” there is no harm in accepting this terminology. Tsallis’ statistics is proved to be a good model to describe many practical situations, as admitted in Wang et al. (2009), and hence one should go forward with Tsallis’ ideas or any other new idea of “nonadditivity” but at the same time giving proper interpretations which are mathematically and statistically sound.

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