A New Approach to $\eta'$ on the Lattice
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We perform an $\eta'$ mass analysis based on a total of 1130 dynamical gauge field configurations, with 5 different quark mass values on lattices of size $16^3 \times 32$ (SESAM) and $24^3 \times 40$ (TχL) at $\beta = 5.6$. We employ the stochastic estimator technique and spectral methods to deal with the disconnected piece of the flavour singlet correlation function. We demonstrate that very early plateau formation in the local $\eta'$ mass can be achieved by first ground state projecting the connected piece of its correlator.

1. Introduction

The masses of light non-singlet mesons have been determined quite accurately in full QCD simulations. The status of the $\eta'$ meson, however, is much less satisfactory, since it is affected for various reasons by severe noise problems: (a) the $\eta'$-propagator requires the evaluation of fermion loop operators, which so far has been achieved only with stochastic estimator methods; (b) the Zweig-rule forbidden piece, which is a loop-loop correlator, suffers from gauge field fluctuations; (c) from Wick contraction the $\eta'$ propagator $C_{\eta'}(\Delta t)$ is computed as the numerical difference between its connected and disconnected pieces, leading to loss of signal at large time separations $\Delta t$.

The challenge is to fight noise by exploiting the information from the $\eta'$ correlator at small time separations. In this contribution, we intend to revisit the issue of the $\eta'$ signal by testing the potential of a spectral representation. The question is whether in the current regime of full QCD calculations (we use SESAM and TχL data), low eigenmodes are actually saturating the fermion loop expressions. Another concern is to improve the ground state projection in the $\eta'$-channel, aiming at very early mass plateauing.

2. The spectral approach

Our truncated eigenmode approach (TEA) applies the Lanczos method on the Hermitian form of Wilson-Dirac matrix, $M: Q \equiv \gamma_5 M$ and computes the lowest (in terms of their moduli) eigenmodes, $|\psi_i\rangle: Q|\psi_i\rangle = \lambda_i |\psi_i\rangle$.

In general we need the spectral representation of the quark propagator, $P$, which implies the eigenfunctions

$$P(x,y) = \sum_{i=1}^{V} \frac{1}{\lambda_i} \frac{\gamma_5 |\psi_i(x)\rangle \langle \psi_i(y)|}{\langle \psi_i|\psi_i\rangle}.$$ (1)

We use the 300 lowest-lying eigenmodes, which at our lightest quark mass amounts to about an equal computational effort as the use of 400 stochastic sources. For details of our analysis, see the parameters given in Table 1.

The $\eta'$ propagator has the form

$$C_{\eta'}(\Delta t) = C_\pi(\Delta t) - 2T(\Delta t),$$ (2)

where the connected piece, $C_\pi$, is readily computed by use of linear solvers. The two-loop spec-

1For details see [4]
Local Masses

Figure 1. Upper frame: Comparison of mass plateaus from TEA and SET. Lower frame: Mass plateaus from SET, top down for $\kappa = 0.156$ S, 0.1565 S, 0.157 S, 0.1575 S and T, 0.158 T, where S stands for SESAM and T for T$\chi$L. The horizontal lines show the fitted mass values.

We propose to project out excited states from the connected piece by the replacement $C_\pi \rightarrow C_\pi^g$, where $C_\pi^g$ stands for the fitted ground state correlation function, and to extract local $\eta'$ masses from the combination of ‘pseudodata’

$$\tilde{C}_{\eta'}(\Delta t) = C_{\eta'}^g(\Delta t) - 2T(\Delta t).$$

Using Eq. (4) we obtain local mass plateaus that start at very small values of $\Delta t$, as illustrated in Fig. 1, where both TEA and stochastic estimator results are shown. Note that the statistical accuracy is sufficient to discriminate the flavour non-singlet pseudoscalar mass very well from the singlet one. We find the data from the TEA analysis to be less noisy than the SET one. Given the very early onset of the $\eta'$ mass plateaus, we can perform single cosh fits to the flavour singlet correlator with safe control on the underlying $t$-range. The fit ranges in $t$ as well as the $\eta'$-masses in lattice units for SET and TEA are listed in Table 1. We find perfect agreement between the results from the SET and TEA analyses.

Table 1. Parameter settings. Smeared sources and sinks are used. $N_{s/l}$ denotes the number of stochastic sources/low eigenmodes and S (T) refers to SESAM (T$\chi$L) lattices. The $\eta'$ masses result from a single cosh-fit over the $t$-range quoted with a jackknife analysis.

| $\kappa$ | $N_{s/l}$ | #confs | t-range | $m_{\eta'}$ |
|---------|-----------|--------|---------|-------------|
| SET:    |           |        |         |             |
| .1560/S | 400       | 195    | 2-4     | .4648(29)   |
| .1565/S | 400       | 200    | 2-6     | .4326(39)   |
| .1570/S | 400       | 200    | 2-10    | .3775(72)   |
| .1575/S | 400       | 200    | 2-8     | .3071(87)   |
| .1575/T | 100       | 180    | 2-8     | .3068(69)   |
| .1580/T | 100       | 155    | 2-8     | .241(12)    |
| TEA:    |           |        |         |             |
| .1560/S | 300       | 195    | 1-4     | .4645(28)   |
| .1575/S | 300       | 200    | 1-8     | .3080(70)   |

Figure 2. Linear chiral extrapolation of $m_{\eta'}^2$ in the quark mass, $m_q$. The two flavor estimate in is denoted by $m_{pe}$. 

*Note: The table and figures contain specific data and analyses that are not transcribed here due to the nature of the image. The text provides a summary of the results and methods used in the analysis.*
3. Chiral extrapolation

We translate the lattice mass values to physical units through the lattice spacing $a^{-1} (\kappa_{\text{light}}) = 2.802(64) \text{GeV}$ derived from our previous light spectrum analysis \cite{6}. The critical and physical light quark hopping parameters are $\kappa_c = 0.158507(44)$ and $\kappa_{\text{light}} = 0.158462(42)$, respectively.

We perform the chiral extrapolation along $\kappa_{\text{sea}} = \kappa_{\text{val}}$, assuming either $m_{\eta'}$ or $m_{\eta'}^2$ to be linear in the quark mass. Our fits favour the dependence $m_{\eta'}^2 = c + c'm_q$ from the $\chi^2/d.o.f.$ in Table \(2\).

To compare with the physical $\eta'$ mass, we have to transform the mass from the real world of three flavours into our $N_f = 2$ setting. This can be done by making use of the experimental mass splitting between flavour singlet and non-singlet states

$$ M_{\eta';N_f=3}^2 = M_{\eta';N_f=3}^2 - (2M_K^2 - M_{\pi}^2) \quad (5) $$

and the Witten-Veneziano formula\cite{7}.

$$ M_{\pi}^2 = 2N_f\chi/F_{\pi}^2. \quad (6) $$

For $N_f = 2$ all non-singlet masses are degenerate. Hence the above equations translate to

$$ M_{\eta';N_f=2}^2 = 2/3M_{\eta';N_f=3}^2 + M_{\pi}^2. \quad (7) $$

Inserting the physical mass values on the right hand side of Eq. (7) leads to the two flavour $\eta'$ mass

$$ M_{\eta';N_f=2} = 715\text{MeV}. \quad (8) $$

By inspection of Table \(2\), we find that (at our lattice spacing!) the linear (in quark mass) chiral extrapolations of both $m_{\eta'}$ and $m_{\eta'}^2$ are definitely above the pion mass, yet significantly below the two-flavour pseudoexperimental value estimated in Eq. (8).

4. Summary and Conclusion

We have presented a new approach to the computation of flavour singlet masses, where the two-loop correlators are estimated by a spectral representation. We have shown that $O(300)$ low eigenmodes suffice to saturate the fermion loop contribution to the $\eta'$ propagator in the regime of quark masses of the SESAM and $T_\chi L$ settings.

We found that, in both the TEA and SET analyses, most of the excited state contamination to the $\eta'$ propagator is related to its connected contribution: by a conventional ground state projection of the latter, one can achieve a strikingly early onset of plateau behaviour in the effective $\eta'$ mass. This then leads to 14 % statistical accuracy of $M_{\eta'}$ after chiral extrapolation.

In our setting, TEA and SET are equally cost effective. It is obvious that TEA is bound to win as we shall go to lighter quark masses in the future; there iterative solvers – unlike the Lanczos procedure – will suffer severe convergence problems due to large condition numbers, while eigenmode expansions will converge even better.

It appears worthwhile to investigate the potential of TEA for disconnected matrix elements. Research in this direction is under way.

### Table 2

| Fit | $m_{\eta'}$ | $M_{\eta'}$ [MeV] | $\chi^2$/d.o.f. |
|-----|-------------|-------------------|-----------------|
| $m$-fit | 0.214(7) | 493(30) | 5.6 |
| $m^2$-fit | 0.138(15) | 318(43) | 2.2 |

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