Optimized Aaronson-Gottesman stabilizer circuit simulation through quantum circuit transformations

Dmitri Maslov

National Science Foundation, Arlington, VA 22230, USA

In this paper we improve the layered implementation of arbitrary stabilizer circuits introduced by Aaronson and Gottesman in Phys. Rev. A 70(052328), 2004. In particular, we reduce their 11-stage computation -H-C-P-C-P-C-H-P-C-P-C- into an 8-stage computation of the form -H-CZ-P-H-P-CZ-C-. We show arguments in support of using -CZ- stages over the -C- stages: not only the use of -CZ- stages allows a shorter layered expression, but -CZ- stages are simpler and appear to be easier to implement compared to the -C- stages. Relying on the 8-stage decomposition we develop a two-qubit depth-(14n − 4) implementation of stabilizer circuits over the gate library {P, H, CNOT}, executable in the LNN architecture, improving best previously known depth-25n circuit, also executable in the LNN architecture. Our constructions rely on folding arbitrarily long sequences (-P-C-)^m into a 3-stage computation -P-CZ-C-, as well as efficient implementation of the -CZ- stage circuits.

I. INTRODUCTION

Stabilizer circuits are of particular interest in quantum information processing (QIP) due to their prominent role in fault tolerance [4, 7, 12, 14], the study of entanglement [3, 12], and in evaluating quantum information processing proposals via randomized benchmarking [9], to name a few. For the purpose of this paper, we define stabilizer circuits to be those composed with the single-qubit quantum Hadamard, H, and the well-known formula to calculate the number of elements in the respective symplectic group, |Sp(2n, F2)| = 2^{2n^2 + O(n)}.

In this paper, we rely on the phase polynomial representation of {P, CNOT} circuits. Specifically, arbitrary quantum circuits over P and CNOT gates can be described in an alternate form, which we refer to as phase polynomial description, and vice versa, each phase polynomial description can be written as a P and CNOT gate circuit. We use this result to induce circuit transformations via rewriting the respective phase polynomials. We adopt the phase polynomial expression result from [2] to this paper as follows:

**Theorem 1.** Any circuit C on n qubits over {P, CNOT} library with k Phase gates can be described by the combination of a phase polynomial \( p = f_1(x_1, x_2, ..., x_n) + f_2(x_1, x_2, ..., x_n) + \cdots + f_k(x_1, x_2, ..., x_n) \) and a linear reversible function \( g(x_1, x_2, ..., x_n) \), such that the action of C can be constructed as \( C(x_1, x_2, ..., x_n) = i^p|g(x_1, x_2, ..., x_n)\), where i is the complex number i. Functions \( f_j \) corresponding to the \( j^{th} \) Phase gate are

*Electronic address: dmitri.maslov@gmail.com
obtained from the circuit $C$ via devising Boolean linear functions computed by the CNOT gates in the circuit $C$ leading to the position of the respective Phase gate.

In the following we focus on finding a short layered sequence of gates capable of representing an arbitrary stabilizer circuit over $n$ primary inputs. The layers are defined as follows:

- **-H-** layer contains all unitaries representable by arbitrary circuits composed of the Hadamard gates. Since $H^2 = I$, and Hadamard gate is a single-qubit gate, -H- layer has zero or one gates acting on each of the respective qubits. The number of distinct layers -H- on $n$ qubits is thus $2^n$. We say -H- has $n$ Boolean degrees of freedom.

- **-P-** layer is composed of an arbitrary set of Phase gates. Since $P^4 = I$, and Phase gate is also a single-qubit gate, -P- layer has anywhere between zero to three gates on each of the respective qubits. Note that $P^2 = Z$, and therefore the gate sequence $PP$ may be better implemented as the Pauli-Z gate; $P^3 = P^1$, and frequently $P^1$ is constructible with the same cost as $P$. This means that -P- layer is essentially analogous to the -H- layer in the sense that it consists of at most $n$ individual single-qubit gates. The number of different unitaries represented by -P- layers on $n$ qubits is $2^{2n}$. We say -P- has $2n$ Boolean degrees of freedom.

- **-C-** layer contains all unitaries computable by the CNOT gates. The number of different -C- layers corresponds to the number of affine linear reversible functions, and it is equal to $\prod_{j=0}^{n-1} (2^n - 2^j) = 2^{n^2 + O(n)}$.

  We say -C- has $n^2 + O(n)$ Boolean degrees of freedom.

- **-CZ-** layer contains all unitaries computable by the CZ gates. Since all CZ gates commute, and due to CZ being self-inverse, i.e., $CZ^2 = I$, the number of different unitaries computable by -CZ- layers is $\prod_{j=1}^{n} 2^{n-j} = 2^{\frac{n^2}{2} + O(n)}$. We say -CZ- has $\frac{n^2}{2} + O(n)$ Boolean degrees of freedom.

Observe that the above count of the degrees of freedom suggests that -P- and -H- layers are “simple”. Indeed, each requires no more than the linear number of single-qubit gates to be constructed via a circuit. The number of the degrees of freedom in -C- and -CZ- stages is quadratic in $n$. Other than two-qubit gates often being more expensive than the single-qubit gates $[5, 8]$, the comparison of the degrees of freedom suggests that we will need more of the respective gates to construct each such stage. The -CZ- layer has roughly half the number of the degrees of freedom compared to the -C- layer. We may thus reasonably expect that the -CZ- layer can be easier to obtain.

Unlike the -C- circuits, the -CZ- circuits have not been studied in the literature. Part of the reason could be due to CZ gate complexity of the -CZ- circuits being a very inconspicuous to study problem: indeed, worst case optimal circuit has $(n-1)n$ CZ gates, and optimal circuits are easy to construct, as they are determined by the presence of lack of CZ gates acting on the individual pairs of qubits. However, we claim that using only CZ gates to construct -CZ- layer is not the best solution, and a better approach would be to also employ CNOT and P gates as well. Indeed, both CNOT and CZ gates must have a comparable cost of the implementation, since they are related by the formula $\text{CNOT}(a, b) = H(b)CZ(a, b)H(b)$, and single-qubit gates are “easy” $[5, 8]$. CZ is furthermore the elementary gate in superconducting QIP $[6]$, and as such, technically, costs less than the CNOT, and in the trapped ions QIP (Quantum Information Processing) the costs of the two are comparable $[11]$. Further discussion of the relation of implementation costs between -C- and -CZ- layers is postponed to Section III.

The different layers can be interleaved to obtain stabilizer circuits not computable by a single layer. A remarkable result of $[1]$ shows that 11 stages over a computation of the form $-H-C-P-C-P-C-P-C-P-C-P-C-\ldots-\text{H-C-P-C-P-C-}$ suffices to compute an arbitrary stabilizer circuit. The number of Boolean degrees of freedom in the group of stabilizer unitaries, defined as the logarithm base-2 of their total count, is given by the formula $\log_{2} |\text{Sp}(2n, F_2)| = 2n^2 + O(n)$. This suggests that the 11-stage circuit by Aaronson and Gottesman $[1]$ could be suboptimal, as it relies on $5n^2 + O(n)$ degrees of freedom, whereas only $2n^2 + O(n)$ are necessary. Indeed, we find a shorter 8-stage decomposition of the form $-H-CZ-P-H-P-CZ-C-\ldots-\text{H-CZ-C-}$ relying on $3n^2 + O(n)$ degrees of freedom.

II. **(-P-C-)**$^n$ **CIRCUITS**

In this section we show that an arbitrary length $n$-qubit computation described by the stages -P-C-P-C-..-P-C- folds into an equivalent three-stage computation -P-CZ-C-.
Theorem 2. \((-P - C -)^m = -P - CZ - C -\).

Proof: \((-P - C -)^m\) circuit has \(k \leq nm\) Phase gates. Name those gates \(P_{j=1..k}\), denote Boolean linear functions they apply phases to as \(f_j = f_{j=1..k}(x_1, x_2, ..., x_n)\), and name the reversible linear function computed by \((-P - C -)^m\) (Theorem 1) as \(g(x_1, x_2, ..., x_n)\). Phase polynomial computed by the original circuit is \(f_1(x_1, x_2, ..., x_n) + f_2(x_1, x_2, ..., x_n) + ... + f_k(x_1, x_2, ..., x_n)\). We will next transform phase polynomial to an equivalent one, that will be easier to write as a compact circuit. To accomplish this, observe that \(f^3 + b + c(a+b) + (a+c) + (b+c) + a + b + c = i^4 = 1\), where \(a, b, c\) are arbitrary Boolean linear functions of the primary variables. This equality can be verified by inspection through trying all 8 possible combinations for Boolean values \(a, b, c\).

The transformed phase polynomial description of the original circuit now has the following form: phase polynomial computed by the original circuit has \(k \leq nm\) EXOR gates; it implements phase polynomial \(\sum_{j=1}^{k} \sum_{k=1}^{n} u_{j,k}(x_j \oplus x_k)\), where \(u, u, z \in \mathbb{Z}_4\), and the linear reversible function \(g(x_1, x_2, ..., x_n)\). We next show how to implement such a unitary as \(-P - CZ - C\) circuit, focusing separately on the phase polynomial and the linear reversible part. We synthesize individual terms in the phase polynomial as follows.

- For \(j = 1..n\), \(u_j x_j\) is obtained as the single-qubit gate circuit \(P^u_x(x_j)\);
- For \(j = 1..n, k = j + 1..n\), \(u_{j,k}(x_j \oplus x_k)\) is obtained as follows:
  - if \(u_{j,k} \equiv 2 \mod 4\), by the circuit \(P^2(x_j)P^2(x_k) = Z(x_j)Z(x_k)\);
  - if \(u_{j,k} \equiv 1 \mod 4\), by the circuit \(P^{u_{j,k}}(x_j)P^{u_{j,k}}(x_k)CZ(x_j, x_k)\).

The resulting circuit contains \(P\) and \(CZ\) gates; it implements phase polynomial \(\sum_{j=1}^{n} u_j x_j + \sum_{j=1}^{n} \sum_{k=1}^{n} u_{j,k}(x_j \oplus x_k)\) and the identity linear reversible function. Since all \(P\) and \(CZ\) gates commute, Phase gates can be collected on the left side of the circuit. This results in the ability to express phase polynomial construction as a \(-P - CZ\) circuit. We conclude the entire construction via obtaining the linear reversible function \(g(x_1, x_2, ..., x_n)\) as a \(-C\) stage, with the overall computation described as a \(-P - CZ - C\) circuit. □

Note that \(-P - CZ - C\) can also be written as \(-C - P - CZ\), if one first synthesizes the linear reversible function \(g(x_1, x_2, ..., x_n) = (g_1(x_1, x_2, ..., x_n), g_2(x_1, x_2, ..., x_n), ..., g_n(x_1, x_2, ..., x_n))\), and expresses the phase polynomial in terms of degree-2 terms over the set \(\{g_1, g_2, ..., g_n\}\). Other ways to write such a computation include \(-C - P - CZ\) and \(-CZ - P\), that are obtained from the first two by commuting \(-P\) - and \(-CZ\) - stages.

Corollary 1. \(-H - C - P - C - H - P - C - C -\) circuit has \(\|I\| = -H - C - CZ - P - H - P - CZ - C -\).
III. -C- VS -CZ-

We have previously noted that CNOT and CZ gates have a comparable cost as far as their implementation within some QIP proposals is concerned. In this section, we study \{P,CZ,CNOT\} implementations of the stages -C- and -CZ-. The goal is to provide further evidence in support of the statement that -CZ- can be thought of as a simpler stage compared to the -C- stage, and going beyond counting the degrees of freedom argument.

Lemma 1. Optimal quantum circuit over \{CZ\} library for a -CZ- stage has at most \(\frac{n(n-1)}{2}\) CZ gates.

Indeed, all CZ gates commute, which limits the expressive power of the circuits over CZ gates. However, once we add the non-commuting CNOT gate, and after that the Phase gate, the situation changes. We can now implement -CZ- circuits more efficiently, such as illustrated by the following circuit identities.

\[
\begin{align*}
\text{CNOT} & \quad \text{Phase} & \quad \text{CNOT} \\
\text{CZ} & \quad \text{Phase} & \quad \text{CNOT} \\
\text{CZ} & \quad \text{Phase} & \quad \text{CNOT}
\end{align*}
\]

The unitary implemented by the circuitry shown above requires 7 CZ gates as a \{CZ\} circuit, 6 gates as a \{CZ,CNOT\} circuit, and only 5 two-qubit gates as a \{P,CZ,CNOT\} circuit. This illustrates that the CNOT and P gates are important in constructing efficient -CZ- circuits.

We may consider adding the P and CZ gates to the \{CNOT\} library in hopes of constructing more efficient circuits implementing the -C- stage. However, this does not help.

Lemma 2. Any \{P,CZ,CNOT\} circuit implementing an element of the layer -C- using a non-zero number of P and CZ gates is suboptimal.

Proof: Each P gate applied to a qubit \(x\) can be expressed as a phase polynomial \(1 \cdot x\) over the identity reversible linear function. Each CZ gate applied to a set of qubits \(y, z\) can be expressed as a phase polynomial \(y + z + 3(y \oplus z)\) and the identity reversible function. Removing all P and CZ gates from the given circuit thus modifies only the phase polynomial part of its phase polynomial description. Removing all P and CZ gates from the \{P,CZ,CNOT\} circuit guarantees that the phase polynomial of the resulting circuit equals to the identity, such as required in the -C- stage. This results in the construction of a shorter circuit in cases when the original P and CZ gate count was non-zero. \(\Box\)

We next show a table of optimal counts and upper bounds on the number of gates it takes to synthesize most difficult function from the sets -CZ- and -C- for some small \(n\), Table I. Observe how the two-qubit gate counts for -CZ- stage, when constructed as a circuit over \{P,CZ,CNOT\} library, remain lower than those for the -C- stage.

| \(n\) | -CZ- \{P,CZ,CNOT\} | -C- \{P,CZ,CNOT\} |
|------|---------------------|---------------------|
| 2    | 1                   | 3                   |
| 3    | 3                   | 6                   |
| 4    | 6                   | 5                   |
| 5    | 10                  | 7                   |
| 6    | 12                  | 12                  |

TABLE I: Gate counts required to implement arbitrary -CZ- and -C- stages for some small \(n\): optimal -CZ-stage gate counts as circuits over \{CZ\}, upper bounds on the two-qubit gate count for -CZ- over \{P,CZ,CNOT\} (achieved based on the application of identities (2) applied to circuits with CZ gates), optimal \{CNOT\} and \{P,CZ,CNOT\} two-qubit gate counts for stage -C-.

Literature encounters an asymptotically optimal algorithm (13) for \{CNOT\} synthesis of arbitrary -C- stage functions. The gate complexity is \(O\left(\frac{n^2}{\log n}\right)\). It is possible that an asymptotically optimal algorithm for \{P,CZ,CNOT\} circuits implementing arbitrary -CZ- stage functions can be developed, at which point its complexity has to be \(O\left(\frac{n^2}{\log n}\right)\). To determine which of the two results in shorter circuits, one has to develop constants in front of the leading complexity terms.
and the linear reversible function $g(x_1, x_2, ..., x_n) : |x_1x_2...x_n⟩ \mapsto |x_nx_{n-1}...x_1⟩$. Observe that $y_j \oplus y_k = x_j \oplus x_{j+1} \oplus ... \oplus x_k$, and thereby this linear function can be encoded by the integer segment $[j, k]$. The primary variable $x_j$ admits the encoding $[j, j]$. We use this notation next. In the following we implement the pair of the phase polynomial and the reversal of qubits (a linear reversible function) via a quantum circuit.
Observe that the swapping operation \( g(x_1, x_2, \ldots, x_n) : \{ x_1x_2 \ldots x_n \} \mapsto \{ x_nx_{n-1} \ldots x_1 \} \) can be implemented as a circuit similar to the one from Theorem 5.1 [11] in depth \( 2n + 2 \). The rest of the proof concerns the ability to insert Phase gates in the circuit accomplishing the reversal of qubits such as to allow the implementation of each term in the phase polynomial \( \sum_{j=1}^{n} u_jy_j + \sum_{j=1}^{n} \sum_{k=j+1}^{n} u_{j,k}(y_j \oplus y_k) \).

Since our qubit reversal circuit is slightly different from the one used in [11], and we explore its structure more extensively, we describe it next. It consists of \( n \) alternating stages, \( S_1 \) and \( S_2 \), where

\[
S_1 = \text{CNOT}(x_1; x_2)\text{CNOT}(x_3; x_4)\ldots\text{CNOT}(x_{n-2}; x_{n-1})
\]

- \( \text{CNOT}(x_3; x_2)\text{CNOT}(x_5; x_4)\ldots\text{CNOT}(x_{n}; x_{n-1}) \) for odd \( n \), and

\[
S_1 = \text{CNOT}(x_1; x_2)\text{CNOT}(x_3; x_4)\ldots\text{CNOT}(x_{n-1}; x_n)
\]

- \( \text{CNOT}(x_3; x_2)\text{CNOT}(x_5; x_4)\ldots\text{CNOT}(x_{n-1}; x_{n-2}) \) for even \( n \) is a depth-2 circuit composed with the CNOT gates, and

\[
S_2 = \text{CNOT}(x_2; x_1)\text{CNOT}(x_4; x_3)\ldots\text{CNOT}(x_{n-1}; x_n)
\]

- \( \text{CNOT}(x_4; x_3)\text{CNOT}(x_6; x_5)\ldots\text{CNOT}(x_{n}; x_{n-1}) \) for odd \( n \), and

\[
S_2 = \text{CNOT}(x_2; x_1)\text{CNOT}(x_4; x_3)\ldots\text{CNOT}(x_{n-2}; x_{n-1})
\]

- \( \text{CNOT}(x_4; x_3)\text{CNOT}(x_6; x_5)\ldots\text{CNOT}(x_{n-2}; x_{n-1}) \) for even \( n \) is also a depth-2 circuit composed with the CNOT gates. We refer to the concatenation of \( S_1 \) and \( S_2 \) as \( S \).

The goal is to show that after \( \left\lceil \frac{n}{2} \right\rceil \) applications of the circuit \( S \) we are able to cycle through all \( \frac{n(n+1)}{2} \) linear functions \([j,k], j \leq k\). The remainder of the proof works slightly differently depending on the parity of \( n \). First, choose odd \( n = 2m + 1 \). Consider two patterns of length \( 2n - 3 \),

\[
P_j = (n - 1, n - 3, n - 3, \ldots, 4, 4, 2, 2, 1, n, 3, \ldots, n - 2, n - 2) \text{ and}
\]

\[
P_k = (3, 3, 5, 5, \ldots, n, n - 1, n - 1, n - 3, n - 3, \ldots, 6, 4, 4, 2).
\]

Observe by inspection that the \( i \)th linear function computed by the single application of the stage \( S \) is given by the formula \([P_j(n - 3 + i), P_k(i)]\), where \( P_j(i) \) and \( P_k(i) \) return \( i \)th component of the respective pattern. It may further be observed, via direct multiplication by the linear reversible matrix corresponding to the transformation \( S \), that the \( i \)th component upon \( t \) \((t \leq m)\) applications of the circuit \( S \) is computable by the following formula,

\[
[P_j(n - 1 - 2t + i), P_k(2t - 2 + i)] = [P_j(n - 3 - 2(t - 1) + i), P_k(2(t - 1) + i)].
\]

A simple visual explanation can be given: at each application of \( S \) pattern \( P_j \) is shifted by two positions to the left (down, Figure 1), whereas pattern \( P_k \) gets shifted by two positions to the right (up, Figure 1).

Observe that every \([j,k], j = 1, k = 1, n, j \leq k\) is being generated. Indeed, a given \([j,k]\) may only be generated at most once by the \( 0 \) to \( m \) applications of the circuit \( S \). This is because once a given \( j \) meets a given \( k \) for the first time, at each following step, the respective value \( k \) gets shifted away from \( j \) to never meet again. We next employ the counting argument to show that all functions \([j,k]\) are generated. Indeed, the total number of functions generated by \( 0 \) to \( m \) applications of the stage \( S \) is \( (m + 1)n = \left(\frac{n(n+1)}{2}\right) \), each linear function generated is of the type \([j,k]\) \((j = 1, k = 1, n, j \leq k)\), none of which can be generated more than once, and their total number is \( \frac{n(n+1)}{2} \). This means that every \([j,k]\) is generated.

We illustrate the construction of the circuit implementing \(-\text{CZ}\)- for \( n = 7 \) in Figure 1.

For even \( n = 2m \) the construction works similarly. The patterns \( P_j \) and \( P_k \) are \( (n, n - 2, n - 2, n - 4, n - 4, \ldots, 2, 2, 1, 1, 3, 3, \ldots, n - 3, n - 3, n - 1) \) and \( (3, 3, 5, 5, \ldots, n - 1, n, n - 1, n, n - 2, n - 2, \ldots, 4, 4, 2, 2) \), respectively. The formulas for computing linear function \([j,k]\) for \( i \)th coordinate after \( t \) applications of \( S \) is \([P_j(n - 2t + i), P_k(2t - 2 + i)]\). After \( m \) applications of the circuit \( S \) we generate linear functions \( x_n, x_{n-1}, \ldots, x_4, x_2 \) in addition to the \( m \) new linear functions of the form \([j,k]\) \((j < k)\).

Circuit depth makes most sense when applied to measure depth across most computationally intensive operations. In both of the two leading approaches to quantum information processing, and limiting the attention to fully programmable universal quantum machines, superconducting \( \S \) and trapped ions \( \S \), the two-qubit gates take longer to execute and are associated with lower fidelity. As such, they constitute the most expensive resource and motivate our choice to measure depth in terms of the two-qubit operations. The selection of the LNN architecture to measure the depth over is motivated by the desire to restrict arbitrary interaction patterns to a reasonable set. Both superconducting and trapped ions qubit-to-qubit connectivity patterns \( \S, \S \) are furthermore such that they allow embedding the linear chain in them.

A further concern is that the two-qubit CNOT gate may not be native to a physical implementation, and therefore the CNOT implementation may likely use correcting single-qubit gates before and after using a specific two-qubit interaction. This means that interleaving the two-qubit gates with the single-qubit gates
such as done in the proof of Theorem 3 may not increase the depth, and restricting depth calculation to just the two-qubit stages is appropriate. We did, however, report enough detail to develop depth figure over both single- and two-qubit gates for the implementations of stabilizer circuits relying on our construction.

Corollary 2. Arbitrary $n$-qubit stabilizer unitary can be executed in two-qubit depth $14n - 4$ as a \{P, H, CNOT\} circuit over LNN architecture.

Proof: Firstly, observe that $-H-C-Z-P-H-P-C-Z-C = -H-C-ZP-H-P-CZC$. This is because both $-CZ$-stages reverse the order of qubits, and therefore the effect of the qubit reversal cancels out. The two-qubit depth of the -$C$- stage is $5n$ \[18\], and the two-qubit depth of the -$CZ$- stage is $2n + 2$, per Theorem 3. This means that the overall two-qubit depth is $14n + 4$. This number can be reduced somewhat by the following two observations. Name individual stages in the target decomposition as follows, $-H-C_1-Z_1-P-H-P-CZ_2-Z_2$. Using the construction in Theorem 3, we can implement -$CZ_1$- without the first $S$ circuit through applying Phase gates at the end of it (see Figure 1 for illustration). The first $S$ circuit can then be combined with the -$C_1$- stage preceding it. This results in the saving of 4 layers of two-qubit computations. Similarly, -$CZ_2$- can be implemented up to $S$ if it is implemented in reverse, and phases are applied in the beginning (the end, but invert the circuit). This allows to merge depth-4 computation $S$ with the stage -$C_2$- that follows. These two modifications result in the improved depth figure of $14n - 4$. \[\square\]

Observe how the aggregate contribution to the depth from both -$CZ$- stages used in this paper, ~$4n$, is less than that from a single -$C$- stage, $5n$. The result of \[18\] can be applied to the 11-stage decomposition -$H-C-P-C-P-H-P-C-P-C-\[29\]$ of \[1\] to obtain a two-qubit depth-$25n$ LNN-executable implementation of an arbitrary stabilizer unitary. In comparison, our reduced 8-stage decomposition -$H-C-Z-P-H-P-CZ-C-\[30\]$ allows execution in the LNN architecture in only $14n - 4$ two-qubit stages.

IV. DISCUSSION

In this paper, we reduced the 11-stage computation -$H-C-P-C-P-H-P-CZ-C-\[30\]$ to an 8-stage computation of the form -$H-C-Z-P-H-P-CZ-C-$. The optimized implementation relies on the stage -$CZ$- not previously considered by \[1\]. We showed evidence that the -$CZ$- stage is likely superior to the comparable -$C$- stage. Indeed, the number of the Boolean degrees of freedom in the -$CZ$- stage is only about a half of that in the -$C$- stage, two-qubit gate counts for optimal implementations of -$CZ$- circuits remain smaller than those for -$C$-circuits (see Table IV), and -$CZ$- computations were possible to implement in a factor of 2.5 less depth than that for -$C$- computations over LNN architecture.

We reported a two-qubit depth-(14n – 4) implementation of stabilizer unitaries over the gate library \{P, H, CNOT\}, executable in the LNN architecture. This improves previous result, a depth-25n circuit \[1\] \[10\] executable over LNN architecture.

Our 8-stage construction can be written in 16 different ways, by observing that -$C-Z-P$- can be written in 4 different ways ( -$C-Z-P$, -$C-P-Z$, -$P-C-Z$, and -$C-Z-P$-).

For the purpose of practical implementation we believe a holistic approach to the implementation of the 3-layer stage -$P-C-Z-C$- may be due, where the linear reversible function $g(x_1, x_2, ..., x_n)$ is implemented by the CNOT gates such that the intermediate linear Boolean functions generated go through the set that allows implementation of the phase polynomial part.

V. ACKNOWLEDGEMENTS

I thank Dr. Yunseong Nam from the University of Maryland–College Park and Dr. Martin Roetteler from Microsoft Research–Redmond for their helpful discussions.

Circuit diagrams were drawn using qcircuit.tex package, http://physics.unm.edu/CQuIC/Qcircuit/.

This material was based on work supported by the National Science Foundation, while working at the Foundation. Any opinion, finding, and conclusions or recommendations expressed in this material are those of
the author and do not necessarily reflect the views of the National Science Foundation.

[1] S. Aaronson and D. Gottesman. Improved simulation of stabilizer circuits. Phys. Rev. A, 70, 052328, 2004, quant-ph/0406196.

[2] M. Amy, D. Maslov, M. Mosca, and M. Roetteler. A meet-in-the-middle algorithm for fast synthesis of depth-optimal quantum circuits. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems 32(6):818–830, 2013. arXiv:1206.0758.

[3] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters. Mixed state entanglement and quantum error correction. Phys. Rev. A 54:3824–3851, 1996. quant-ph/9604024.

[4] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane. Quantum error correction and orthogonal geometry. Phys. Rev. A 54:405–408, 1997. quant-ph/9605005.

[5] S. Debnath, N. M. Linke, C. Figgatt, K. A. Landsman, K. Wright, and C. Monroe. Demonstration of a programmable quantum computer module. Nature 536:63–66, 2016. arXiv:1603.04512.

[6] J. Ghosh, A. Galiautdinov, Z. Zhou, A. N. Korotkov, J. M. Martinis, and M. R. Geller. High-fidelity controlled-σZ gate for resonator-based superconducting quantum computers. Phys. Rev. A 87, 022309, 2013. arXiv:1301.1719.

[7] M. Grassl. Code Tables: Bounds on the parameters of various types of codes. http://www.codetables.de/; last accessed February 27, 2017.

[8] IBM Research Quantum Experience. [http://www.research.ibm.com/quantum/] last accessed September 30, 2016.

[9] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. B. Blakestad, J. D. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland. Randomized benchmarking of quantum gates. Phys. Rev. A 77, 012307, 2008. arXiv:0707.0963.

[10] S. A. Kutin, D. P. Moulton, and L. M. Smithline. Computation at a distance. 2007. quant-ph/0701194.

[11] D. Maslov. Basic circuit compilation techniques for an ion-trap quantum machine. New J. Phys. 19, 023035, 2017. arXiv:1603.07678.

[12] M. A. Nielsen and I. L. Chuang. Quantum Computation and Quantum Information, Cambridge University Press, New York, 2000.

[13] K. N. Patel, I. L. Markov, and J. P. Hayes. Optimal synthesis of linear reversible circuits. Quantum Information and Computation 8(3&4):282–294, 2008.

[14] A. M. Steane. Quantum computing and error correction. 2003. quant-ph/0304016.