Studying the character of temperature distribution in a field using heat conductivity equation solved by finite-difference method

M A Bryukhanov, N V Tsvetkov, A A Vinogradova

Saint Petersburg Mining University, Saint Petersburg,

E-mail: Vinogradova_AA@pers.spmi.ru

Abstract. Present work solves a heat conductivity equation with initial and boundary conditions. Maximum and minimum temperatures in the area under consideration are calculated through partial differential equations. The sought values are calculated using the derived expressions; in addition the plots of investigated values are built using Microsoft Excel spreadsheet.

1. Introduction
Over the recent decade, the study and application of heat exchange have greatly intensified. A special attention is drawn by investigation of thermophysical properties at micro- and nano-levels, which is very promising in many fields of science and currently actively implemented in industry [1–5]. Theoretical study of heat exchange processes is mostly based on computer numerical simulation. Starting from 1990s, the computer technologies have been used in Russia to solve various R&D problems [6]. Modern level of computer technologies—along with analytical and numerical analytical methods—facilitates the wide application of numerical methods, including the finite difference method (FDM) [7–9]. The numerical simulation of heat exchange processes currently plays increasingly important role, because modern science and technology need adequate estimation of such processes, while their experimental study in laboratory or full-scale conditions is very complex, expensive and in most cases impossible.

2. Heat conductivity equation
The direct study of heat transfer processes in solid bodies began in 1822 with The Analytical Theory of Heat, a prominent book of Joseph Fourier where the Fourier’s Law was formulated. Nonstationary heat transfer via heat conductivity is described by the following equation written in Cartesian axes:

\[ pc \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right) + Q_w(x,y,z,t,T) \]  

This equation (Fourier-Kirchhoff Equation) connects the temporal and spacial change to temperature of any spot in a body. Here \( p \) is density, \( c \) is specific heat capacity, \( \lambda \) is heat conductivity coefficient, \( Q_w(x,y,z,t,T) \) is power of internal heat emission sources, \( T \) is temperature, \( x,y,z \) are space coordinates, \( t \) is time.

The physical conditions determine thermophysical body characteristics \( \lambda, p, c, \). Time (initial) conditions contain temperature distribution in the body at initial moment of time:

\[ t = 0; \ T = f(x,y,z) \]  

The boundary conditions determine the peculiarities of the process on the body surface and can be
set in several ways.
For the process under consideration, we may confine ourselves by one-dimensional equation of
conductive heat transfer:
\[ pc \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) + Q_w(x, t, T) \]  \hspace{1cm} (3)

3. Heat conductivity equation solution
Finite difference method (FDM)
The finite difference method can be treated as the one of the most effective numerical methods; it is
widely used to solve important applied problems of modern science; unlike many computer programs
that have no justified body of mathematics, it has simple interface that requires no additional training of
engineer [7, 10].

Let us consider the numerical solution of heat conductivity equation:
\[ \frac{\partial u(x, t)}{\partial t} = \lambda \frac{\partial^2 u(x, t)}{\partial x^2}, \]  \hspace{1cm} (4)

Where \( u(x, t) \) is temperature, \( x \) is a spacial coordinate, \( t \) is time, \( \lambda \) is heat conductivity coefficient.
The solution of this equation should be obtained for the arguments
\( x \in [a; b], t \in [T_1; T_2] \). At initial moment of time, the temperature distribution is known.
\[ u(x, t) = f(x) \]  \hspace{1cm} (5)

In addition, the function values are set on the ends of the spacial coordinate integration range.
\[ u(a, t) = \varphi(t) \, u(b, t) = \omega(t) \]  \hspace{1cm} (6)

This means that two steps are required to write down the difference scheme, which approximately
describes the differential equation:
1. replace the region of continuous change of the argument with the region of its discreet change
2. replace the differential operator with some difference correlation and formulate the difference
analogue for boundary and initial conditions.

The result of the first step are values \( x_i \) and \( t_j \), which determine the solution \( u(x_i, t_j) \). The second
step results in a system of algebraic equations relative to the values of differential equation solution in
certain spots of the argument domain \( u(x_i, t_j) \).

Let us substitute partial derivatives \( \frac{\partial u(x, t)}{\partial t}, \frac{\partial^2 u(x, t)}{\partial t^2} \) with finite differences:
\[ \frac{\partial u(x, t)}{\partial t} \approx \frac{\Delta_t u(x, t)}{\Delta t} \, \frac{\partial^2 u(x, t)}{\partial x^2} \approx \frac{\Delta^2_x u(x, t)}{(\Delta x)^2}, \]  \hspace{1cm} (7)

where the increment of arguments along time and space coordinates are assumed constant and equal
to \( \Delta t = t_{j+1} - t_j = \tau \) and \( \Delta x = x_{i+1} - x_i = h \). Substitution of (7) into the partial differential equation
(4) results in differential equations for the sought solution \( u(x_i, t_j) \) on the argument value grid in terms
of space \( (x_i) \) and time \( (t_j) \) variables:
\[ \frac{\Delta_t u(x, t)}{\Delta t} = \lambda \frac{\Delta^2_x u(x, t)}{(\Delta x)^2}. \]  \hspace{1cm} (8)

The first-order difference in time is substituted by the following differences:
\[ \Delta_t u(x_i, t_j) = u(x_i, t_j) - u(x_i, t_{j-1}) \]  \hspace{1cm} (9)

The second-order difference in space coordinate is substituted by the following differences:
\[ \Delta^2_x u(x_i, t_j) = u(x_i - h, t_j) - 2 \, u(x_i, t_j) + u(x_i + h, t_j) \]  \hspace{1cm} (10)

Similar difference relations were applied to solve boundary problem of conventional second-order
differential equations. Substitution of (9) and (10) into (8) gives a system of algebraic equations in terms
of temperature values in nodes \( u(x_i, t_j) \):
\[ \frac{u(x_i, t_j) - u(x_i, t_{j-1})}{\tau} = \lambda \frac{u(x_{i-1}, t_j) - 2 \, u(x_i, t_j) + u(x_{i+1}, t_j)}{h^2} \]  \hspace{1cm} (11)

Where \( i = 1, 2, \ldots m - 1 \) \( u j = 1, 2, \ldots n \). Here \( m \) and \( n \) are the numbers of sections of space and time
variables.

Equations (11) allow calculating the solution in the internal spots of the solution domain grid. The number of equation in system (11) is less than the number of unknowns. The lacking equations are found from initial (5) and boundary (6) conditions.

The initial condition (5) at $t=t_1$ in spots $x_i$ are as follows:

$$u(x_i, t_1) = f(x_i)$$ (12)

For the values at the ends of space variable change (7):

$$u(a, t_j) = \varphi(t_j) u(b, t_j) = \omega(t_j)$$ (13)

The difference scheme (12) is referred to as implicit difference scheme. The values of the sought functions at one value of the temporal variable are called a layer. The initial condition (13) sets the solution values at initial (zero) layer. The solution values at zero layer (at $t=t_1$) to do this, system of equations (12) should be solved by substituting the known values at $t=t_1$. The obtained values become the initial values to determine the solution at the second time layer as the solutions of system (12) at known solution value at $t = \tau$ (time variable increment step). This process is repeated until complete calculation of the solution at all layers.

Let us consider the numerical solution for heat conductivity equation (5) at the change of $x$ argument from 3 to 5.2 with step 0.2, and $t$ argument from 2 to 3.2 with step 0.11 at initial condition

$$y(x) = \frac{5x^3}{x}$$ (14)

and boundary conditions

$$u(3, t) = 1.55 + (4 - t)^{-2}, \quad u(5.2, t) = \sin(4t + 1) + 5$$ (15)

The heat conductivity coefficient equals 4. The solution is found on rectangular argument range grid containing twelve values of the space coordinate and thirteen values of time coordinate.

Difference equation (11) is transformed, so the sought solutions of a single layer are grouped on one side of the equation sign.

$$\sigma u(x_{i-1}, t_j) - \alpha u(x_i, t_j) + \sigma u(x_{i+1}, t_j) = -u(x_i, t_{j-1})$$ (16)

Here we introduce the following designations:

$$\sigma = \frac{\lambda \cdot \tau}{h^2}, \quad \alpha = 1 + 2\sigma$$ (17)

At each temporal layer, one should solve system (16) containing 10 equations. Let us rewrite system (16) in explicit form

$$\begin{cases}
\sigma u(x_0, t_j) - \alpha u(x_1, t_j) + \sigma u(x_2, t_j) = -u(x_1, t_{j-1}), \\
\sigma u(x_1, t_j) - \alpha u(x_2, t_j) + \sigma u(x_{i+1}, t_j) = -u(x_2, t_{j-1}), \\
\sigma u(x_0, t_j) - \alpha u(x_1, t_j) + \sigma u(x_{i+1}, t_j) = -u(x_{i+1}, t_{j-1}), \\
\sigma u(x_{i+1}, t_j) - \alpha u(x_{i+2}, t_j) + \sigma u(x_{i+2}, t_j) = -u(x_{i+2}, t_{j-1}).
\end{cases}$$ (18)

Let us transform system (18) with a due consideration that the boundary values are set and known (14), (15). To calculate the solution in the nodes of a single time layer at $i=1...m-1$, we obtained a system of linear algebraic equations with the tridiagonal matrix of coefficients. This system can be solved by any known method.

$$\begin{cases}
-\alpha u(x_1, t_j) + \sigma u(x_2, t_j) = -u(x_1, t_{j-1}) - \sigma u(x_0, t_j), \\
\sigma u(x_1, t_j) - \alpha u(x_2, t_j) + \sigma u(x_{i+1}, t_j) = -u(x_2, t_{j-1}), \\
\sigma u(x_0, t_j) - \alpha u(x_1, t_j) + \sigma u(x_{i+1}, t_j) = -u(x_{i+1}, t_{j-1}), \\
\sigma u(x_{i+1}, t_j) - \alpha u(x_{i+2}, t_j) + \sigma u(x_{i+2}, t_j) = -u(x_{i+2}, t_{j-1}).
\end{cases}$$ (19)

The obtained solution for one layer is used to get the solution for the next time layer. Derivation of the equation solution with initial and boundary conditions. This scheme is referred to as implicit scheme.

We will solve it in Microsoft Excel spreadsheet.
Numerical solution of heat conductivity equation in MS Excel

When solving a system of linear algebraic equations of reciprocal matrix in Microsoft Excel spreadsheet, the initial system is transformed into the form of (20) when the coefficient in the first equation before the unknown equals one. We obtain a tridiagonal coefficient matrix and a matrix of intercept terms where the number of rows and columns is less by two than the number of nodes (because we have boundary conditions). Let us multiply the reciprocal matrix of coefficients and matrix of intercept terms, thus obtaining the temperatures in the section points for the subsequent time layer.

To ensure the availability of all calculation data, let us make calculations in first columns of the table. (Figs. 1, 2 and 3).

**Figure 1.** Initial data for calculation by reciprocal matrix method, calculation of temperature on the rod ends (p, q) and temperature in the rod at initial moment of time (u0), Show Formulas off.

Then, a coefficient matrix is built, which main diagonal elements equal \(-\alpha\); the elements of adjacent diagonals equal \(\sigma\).

**Figure 2.** Coefficient matrix

**Figure 3.** Calculation of u1 by reciprocal matrix method, Show Formulas on
Figure 4. Solution of heat conductivity equation and analysis for maximum and minimum values

Let us plot the graphical representation:

| Time, s | Rod length |
|--------|------------|
| 0      | 1.882695   |
| 2      | 3.89591271 |
| 4      | 5.9091464  |
| 6      | 7.9231794  |
| 8      | 9.9472121  |
| 10     | 11.971244 |
| 12     | 13.995276 |

Figure 5. Graphical representation of the solution

4. Conclusions

The heat conductivity equation solution allows answering a whole number of important questions on the character of temperature distribution in a field. The temperature changes monotonously without discontinuities. One can determine the maximum and minimum temperature and their location in the field. One can track the temperature change in a specific moment of time, and the changes to the temperature in a certain field spot over the necessary time period.

References

[1] Khvesyuk V I, Skryabin A S 2017 TVT 55(3) 447-471
[2] Dmitriev A S 2012 MEI Printing House, 302
[3] Fisher T S 2013 World Scientific 93(2) 793
[4] Cahill D G, Braun P V, Chen G etc 2014 Appl. Phys. Rev. 1(1) 011305
[5] Georgieva E Yu, Sharikov Yu V 2011 Journal of Mining Institute 189 280-283
[6] Kotlov S N 2011 Journal of Mining Institute 189 34-37
[7] Gospodarikov A P, Zatsepin M A, Meleshko A V 2009 Journal of Mining Institute 182 123-134
[8] Ustyugov D I 2013 Journal of Mining Institute 200 332-335
[9] Chernyaev A V, Pavlov A A 2013 Journal of Mining Institute 203 237-241
[10] Gospodarikov A P, Zatsepin M A 2010 Journal of Mining Institute 182 47-54