Conformally Dual to Inflation

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Abstract

It is showed by a conformal rescaling that the inflationary background can be dual to a slowly expanding background, which is almost Minkowski and described by a conformal field theory conformally coupled to gravity. It is proved that the primordial curvature perturbation and tensor perturbation generated during these two conformally equivalent backgrounds are completely equal, and the scale invariance of perturbations is determined by the conformal invariance of field theory in slowly expanding background. In dual slowly expanding scenario, the primordial universe is asymptotical to a static state in infinite past. We discuss the implication of the results obtained.

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The inflation scenario is the current paradigm of the early universe, which is highly favored by observations. However, the inflatons are diversified in diversified inflation models [1]. What is inflaton is still a significant issue. Here, it will be disposed in an insightful perspective.

The perturbation on large scale consists of a constant mode and a mode dependent of time [2]. When the scale factor is rapidly changed while $\epsilon \ll 1$ is nearly constant, the constant mode is responsible for the scale invariance of the primordial perturbations in inflation scenario [3].

However, the scale invariance of curvature perturbation can also be obtained for $|\epsilon| \gg 1$. The evolution with $\epsilon \gg 1$ is the slowly contracting, which is that of ekpyrotic universe [4]. While $\epsilon \ll -1$ gives the slow expansion [5], which has been applied for island universe [6]. In slowly evolving scenario, the tensor perturbation generated is generally strongly blue, which implies that it is negligible on large scale. While in inflationary scenario, the tensor perturbation is scale invariant, which can be detected.

Recently, it has been found in [7] that, in a background of slowly expanding with the rapidly changed gravitational coupling, both the curvature perturbation and the tensor perturbation can be scale invariant, and the ratio of the tensor to scalar is in a regime consistent with that of inflation scenario. This result is actually a reflection that the perturbation is conformal invariant fully nonperturbatively [8]. The conformal invariance of perturbations implies that the background evolution having coincident predictions with that of inflation might be designed by a conformal rescaling of inflationary background. Thus it might be significant to contemplate inflation in its conformally dual background, which might bring a fruitful insight to the inflation scenario itself and also the early universe.

Here, we will show that the inflationary background can be conformally dual to a slowly expanding background, which is described by a conformal field theory conformally coupled to gravity. We prove that the primordial curvature perturbation and tensor perturbation generated in both pictures are be completely equal, and the scale invariance of perturbations is determined by the conformal invariance of field theory in slowly expanding background. We discuss the implication of the results obtained.

We firstly will show the slowly expanding background, which is described by a conformal field theory conformally coupled to gravity, is actually conformally dual to the inflationary
background. We begin with

$$\mathcal{L} \sim \frac{M_{\text{P}}^2}{2} R + \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{4} \lambda \varphi^4 - \Delta,$$

(1)

where $M_\text{P}^2 = 1$, $M_{\text{P}}^2 = \xi \varphi^2$, and $\xi \simeq 1/6$ will be determined. In [9], the perturbation of the field conformally coupled to the gravity has been studied, however, there the field is a light field which hardly affect the cosmological evolution, and the potential of field is negative. Here, the potential is positive, and the field is ghostlike, which is required for implementing the slow expansion. However, as will be showed, there is not the ghost instability.

The calculation is highly similar to that in [5]. The background evolution of the slow expansion is [5],[10],

$$a \sim \frac{1}{(t_* - t)^{1/|\epsilon|}}, \quad H = \frac{1}{|\epsilon|(t_* - t)},$$

(2)

for $\epsilon \ll -1$. When $3H^2 \varphi^2 \ll 6H \dot{\varphi} \varphi$ is neglected, the Fridmann equation is simplified as

$$H \simeq \frac{-\frac{1}{2} \dot{\varphi}^2 + \frac{1}{4} \lambda \varphi^{4-\Delta}}{3 \frac{4}{\pi} M_{\text{P}}^2}, \quad \sim \frac{1}{|\epsilon|(t_* - t)}.$$

(3)

This requires $\Delta \equiv \frac{2}{|\epsilon|}$, and

$$\varphi \sim \left(1 + \mathcal{O}\left(\frac{1}{|\epsilon|}\right)\right) \frac{\sqrt{2/\lambda}}{(t_* - t)^{1/|\epsilon|+1}},$$

(4)

where the deviation $\sim 1/|\epsilon|$ is required to accurately give Eq.(3). Thus $\varphi \sim \frac{1}{(t_* - t)}$ for $\epsilon \ll -1$. The result is consistent with $3H^2 \varphi^2 \ll 6H \dot{\varphi} \varphi$, since

$$H \varphi \sim \frac{1}{|\epsilon|(t_* - t)^2} \ll \dot{\varphi} \sim \frac{1}{(t_* - t)^2}$$

(5)

for $\epsilon \ll -1$. The equation of $\dot{H}$ is given by,

$$2M_{\text{P}}^2 \dot{H} - \left(\frac{d}{dt} M_{\text{P}}^2\right) H = \dot{\varphi}^2 - \frac{d^2}{dt^2} M_{\text{P}}^2,$$

(6)

where $M_{\text{P}}^2 = \xi \varphi^2$. This requires

$$\xi \simeq \frac{1}{6}(1 + \frac{1}{3|\epsilon|}).$$

(7)

Thus for $|\epsilon| \gg 1$, $\xi = 1/6$ is just that of conformal coupling. This implies that the field theory [11] is a conformal field theory, conformally coupled to the gravity, however, there is a slight deviation of $\sim 1/|\epsilon|$. We will see it is this deviation that determines the tilt of the spectrum of primordial perturbation.
We conformally rescale $g_{\mu\nu}$ as

$$g_{E\mu\nu} = M_{P,eff}^2 g_{\mu\nu}. \quad (8)$$

We follow Ref. [11]. The line element is $ds_E^2 = M_{P,eff}^2 ds^2$, which gives $dt_E = M_{P,eff} dt$. Thus we have

$$t_E = \int M_{P,eff} dt = \frac{|\epsilon|}{\sqrt{3}\lambda (t_\ast - t)^{1/|\epsilon|}} \quad (9)$$

where $t_E$ is positive and Eq. (7) is applied. $a_E = M_{P,eff} a$ brings $H_E$,

$$H_E = \frac{1}{M_{P,eff}} \left( H + \frac{\dot{M}_{P,eff}}{M_{P,eff}} \right) \simeq \frac{|\epsilon|}{t_E} \quad (10)$$

where Eq. (9) is applied. Thus $a_E$ is given by $a_E \sim exp \int H_E dt_E$, which is

$$a_E \sim t_E^{|\epsilon|}. \quad (11)$$

This is just inflation since $|\epsilon| \gg 1$. The field $\varphi_E$ after this conformal rescaling is canonical

$$d\varphi_E^2 = \frac{1}{M_{P,eff}^2} \left( 6M_{P,eff}^2 \varphi - 1 \right) d\varphi^2 \simeq \frac{2}{|\epsilon|\varphi^2} d\varphi^2. \quad (12)$$

Thus $\varphi_E = \sqrt{\frac{2}{|\epsilon|}} \ln \varphi$. Thus (1) after the conformal rescaling is given by

$$\mathcal{L}_E \sim \frac{1}{2} R_E - \frac{1}{2} (\partial_E \varphi_E)^2 - 9 \lambda e^{-\sqrt{\frac{2}{|\epsilon|}}\varphi_E}, \quad (13)$$

which is actually consistent with Eq. (11). Thus it can be found

$$|\epsilon| = 1/\epsilon_E. \quad (14)$$

Thus the slowly expanding background $|\epsilon| \gg 1$, which here is described by a conformal field theory conformally coupled to gravity, is conformally dual to the inflationary background $\epsilon_E \ll 1$.

The curvature perturbation $\mathcal{R}$ and the tensor perturbation $h_{ij}$ are actually conformal invariant. Here, we will prove it in detail, which helps to clarify what determines the scale invariance of the perturbations.

In inflationary background $a_E \sim t_E^{1/\epsilon_E}$, the results of the primordial perturbations generated are familiar, which have been listed in Table I. In its conformal dual background given by Eq. (2), the case is slightly complicated. We will calculate $\mathcal{R}$ and $h_{ij}$ in this picture. The
quadratic action of curvature perturbation $\mathcal{R}$ for the most general single field, including the nonminimal coupling case \[12, 13\], is \[14\]

$$
S_2 \sim \int d\eta d^3 x \frac{a^2 Q_{\mathcal{R}}}{c_{\mathcal{R}}^2} \left( \mathcal{R}^2 - c_{\mathcal{R}}^2 (\partial \mathcal{R})^2 \right), \tag{15}
$$

where $Q_{\mathcal{R}} > 0$ and $c_{\mathcal{R}}^2 > 0$ are required to avoid the ghost and gradient instabilities. Thus the equation of $\mathcal{R}$ is given by \[15, 16\],

$$
u_k'' + \left( c_{\mathcal{R}}^2 k^2 - \frac{z_{\mathcal{R}}''}{z_{\mathcal{R}}} \right) u_k = 0, \tag{16}
$$

after $u_k \equiv z_{\mathcal{R}} R_k$ is defined, where $z_{\mathcal{R}} = \sqrt{2} a Q_{\mathcal{R}}^{1/2}/c_{\mathcal{R}}$ and the prime is the derivative for $\eta$.

The scale invariance of $\mathcal{R}$ requires

$$z_{\mathcal{R}} \sim \frac{a Q_{\mathcal{R}}^{1/2}}{c_{\mathcal{R}}} \sim \frac{1}{\eta_* - \eta} \text{ for constant mode} \quad \text{or} \quad (\eta_* - \eta)^2 \text{ for increasing mode} \tag{17, 18}
$$

has to be satisfied. In certain sense, both evolutions are dual \[17\]. Here, $c_{\mathcal{R}}^2$ is not changed. However, the changed $c_{\mathcal{R}}^2$ will alter the result \[18, 19, 20, 21, 22, 23, 24, 25\]. During slow evolution, the scale invariant curvature perturbation can be induced by either its constant mode \[26, 27, 28, 29\], or its increasing mode \[30, 31\]. In inflation scenario, the scale invariant perturbation is induced by the constant mode of perturbation, thus in its dual slow expansion, the perturbation certainly must be that induced by the constant mode.

Here, $c_{\mathcal{R}}^2 = 1$ and $Q_{\mathcal{R}} = \frac{1}{6|\epsilon|} \phi^2 > 0$ \[7\]. Thus $z_{\mathcal{R}}$ is

$$
z_{\mathcal{R}} = \sqrt{2} \frac{a Q_{\mathcal{R}}^{1/2}}{c_{\mathcal{R}}} \sim \frac{\sqrt{2}}{\sqrt{3\lambda|\epsilon| (t_* - t)^2 / |\epsilon| + 1}}
\sim \frac{\sqrt{2}}{\sqrt{3\lambda|\epsilon| (\eta_* - \eta)^1/|\epsilon| + 1}}, \tag{19}
$$

where $t \sim \eta^{1-1/|\epsilon|}$ for $\epsilon \ll -1$. When $k^2 \gg z_{\mathcal{R}}'' / z_{\mathcal{R}}$, the perturbation is deep inside the $\mathcal{R}$ horizon $1/\mathcal{H}_{\mathcal{R}}^\text{freeze}$, $u_{\mathcal{R}} \sim \frac{1}{\sqrt{2k}} e^{ik\eta}$. Here, the $\mathcal{R}$ horizon

$$
\mathcal{H}_{\mathcal{R}}^\text{freeze} = \sqrt{\frac{z_{\mathcal{R}}''}{z_{\mathcal{R}}}} \sim \frac{1}{\eta_* - \eta} = \mathcal{H}_{E}^\text{freeze} \tag{20}
$$

which is actually a reflection of the conformal invariance of $z_{\mathcal{R}}(\eta)$. When $k^2 \ll z_{\mathcal{R}}'' / z$,

$$
u_{\mathcal{R}} \sim \frac{1}{\sqrt{2k}} \frac{1}{(-k\eta)^1/|\epsilon| + 1}. \tag{21}
$$
TABLE I: The $n_R$, $r$ and $n_T$ for inflation with $\epsilon_E \ll 1$ and slow expansion $|\epsilon| \gg 1$, respectively. Here, the conformal rescaling by which the slow expansion is dual to inflation implies $|\epsilon| = 1/\epsilon_E$.

Thus $P_R$ is given by

$$P_R = \frac{k^3}{2\pi^2} \left| \frac{u_R}{z_R} \right|^2 \simeq \frac{3}{8\pi^2} \lambda |\epsilon|.$$  

(22)

The tilt is $n_R \simeq 1 - \frac{2}{|\epsilon|}$. Thus the only adjusted parameter $\lambda$ of the model is normalized by $P_R \sim 1/10^{10}$, and there is not additional finetuning.

The quadratic action of the tensor perturbation $h_{ij}$ is

$$S_2 \sim \int d\eta d^3x \frac{a^2 Q_T}{c_T^2} \left( h_{ij}^{'2} - c_T^2 (\partial h_{ij})^2 \right),$$  

(23)

where $Q_T > 0$ and $c_T^2 > 0$ are required to avoid the ghost and gradient instabilities. The shape of this action is similar to that of the curvature perturbation.

The spectrum of $h_{ij}$ is determined by $Q_T$, which is given by $Q_T = M_{Peff}^2 \simeq \varphi^2/6$, and $c_T^2 = 1/7$. Thus similarly

$$z_T = 0.5 \frac{aQ_T^{1/2}}{c_T} \sim \frac{1}{2\sqrt{3} \lambda (\eta_\ast - \eta)^{1/|\epsilon|+1}}.$$  

(24)

When $k^2 \gg z'' / z_T$, the perturbation is deep inside the $h_{ij}$ horizon $1/\mathcal{H}_{freeze}^T$, $u_T \sim \frac{1}{\sqrt{2k}} e^{i\eta}$. Here, the $h_{ij}$ horizon

$$\mathcal{H}_{freeze}^T \sim \frac{1}{\eta_\ast - \eta} = \mathcal{H}_{Efreeze}^T$$  

(25)

which is actually a reflection of the conformal invariance of $z_T(\eta)$. When $k^2 \ll z'' / z_T$,

$$u_T \simeq \frac{1}{\sqrt{2k}} \frac{1}{(-k\eta)^{1/|\epsilon|+1}}.$$  

(26)

Thus $P_T$ is given by

$$P_T = \frac{k^3}{\pi^2} \left| \frac{u_T}{z_T} \right|^2 \simeq \frac{6}{\pi^2} \lambda.$$  

(27)
The tilt is \( n_T \simeq -\frac{2}{|\epsilon|} \). Thus the ratio of tensor to scalar \( r \equiv \frac{P_T}{P_R} \) is given by

\[
    r \simeq \frac{16}{|\epsilon|},
\]

which is only dependent of \(|\epsilon|\). Here, if (14) is applied for Eq. (28), \( r \) obtained is just that of inflation with constant \( \epsilon_E \). Thus the inflation and the slow expansion have equal predictions, which are listed in Table I.

Thus it is obviously found that the scale invariance of primordial perturbations is determined by the conformal invariance of field theory in slowly expanding background. However, this conformal field is actually a slightly deformed conformal field, there is a slight deviation of \( \sim 1/|\epsilon| \) from the conformal invariance, which naturally brings a slight red tilt of the primordial perturbation.

In [9],[32], the similar conformal field has been applied, though for a picture of slow contraction. However, in [9], the field conformally coupled to the gravity is a light field, which hardly affect the cosmological evolution. While in [32],[33], though the conformal field is dominated, it is not conformally coupled with the gravity. Thus both fail to conformally dual to inflation. In [9],[32],[33], the conformal invariance is only related to the scale invariance of the perturbation of a light field. While here what the conformal invariance determines is that of metric perturbations, which is adiabatically generated.

Though the predictions of inflation picture and its dual slow expansion picture for the perturbations are indistinguishable, the evolutions of their backgrounds are completely different. The inflation spacetime is geodesically incomplete, which thus is initially singular [34]. However, in its conformally dual background, the case is different, the primordial universe is slowly expanding, which is asymptotical to a static state in infinite past, and thus is singular free. This is illustrated in Fig.1.

Thus this conformal duality might imply an alternative to the origin of observational universe. The primordial universe might be in a slowly expanding conformal phase, in which the full theory depicting the universe is conformal invariant. As a result of the conformal invariance of underlying theory, both the curvature perturbation and tensor perturbation emerged in this conformal phase are naturally scale invariant. When the conformal phase ends, the energy of conformal field will be released to heat the universe. Here, the heating mechanism might be similar to that after inflation. However, after the heating, \( M_{P,\text{eff}}^2 = M_P^2 \) should be fastened, or there will be conflicts with observations, which will be studied.
FIG. 1: The evolutions of $a$ and $H$ in inflation picture (dashed line) and its conformally dual, i.e. slow expansion picture (solid line), respectively. Before $t_*$, which is $t_{E*}$ for inflation, due to the rapidly change of $M_{\text{Peff}}$, both evolutions are completely different, in which $t$ is from $-\infty$ to $t_*$ while $t_E$ is from 0 to $t_{E*}$, in term of Eq.(9), noting that the conformal time $\eta$ is same for both evolutions, which is from $-\infty$ to $\eta_*$. However, after $t_*$, since $M_{\text{Peff}} = M_{\text{P}}$ is fastened, both evolutions coincide. $t_*$ is the heating time, after which the evolution of hot big bang model begins.

elsewhere. Hereafter, the evolution of universe coincides with that of hot big bang model, see Fig.1.

Here, the picture is as if homological with pre big bang picture \[35],\[36]. However, there the superinflationary expanding phase in string frame, which but is not the slow expansion, is conformally dual to a contracting phase. While here the conformal rescale of the slow expanding phase is inflation, thus the scale invariance of the curvature perturbation and the tensor perturbation is simply obtained.

Here, with Fig.1, it might be designed that this infinite past could be jointed to the infinite latetime of previous universe. In principle, after doing this joint an infinite number of times, we will have a cyclic universe. Here, the infinite past is actually a conformally dual to inflation initial singularity. In this sense, this conformal duality might imply a feasible implementing of the conformal cyclic cosmology \[37], which is left for the aftertime.

In conclusion, we show that the inflationary background can be conformally dual to a slowly expanding background. We prove that the primordial curvature perturbation and tensor perturbation generated during both backgrounds are completely equal, and the scale invariance of perturbations is determined by the conformal invariance of field theory in slowly expanding background. Though we only consider a simple case, we conjecture that
all inflation models can be conformally dual to a slightly deformed conformal field theory in slowly expanding background, which might imply that though the inflatons are diversified in diversified inflation models, all actually have a common origin, when being performed in its conformally dual background. In dual slow expansion, the primordial universe is asymptotical to a static state in infinite past, and thus is singular free. Here, this slowly expanding phase is depicted by a conformal field theory, thus it might has a dual description of AdS$_5$ gravity [38], which might be interesting for studying.

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