Counting paths, cycles, and blow-ups in planar graphs

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Abstract
For a planar graph \( H \), let \( N_P(n, H) \) denote the maximum number of copies of \( H \) in an \( n \)-vertex planar graph. In this paper, we prove that \( N_P(n, P_7) \sim \frac{4}{27}n^4 \), \( N_P(n, C_6) \sim \frac{(n/3)^3}{n^4} \), \( N_P(n, C_8) \sim \frac{(n/4)^4}{n^4} \), and \( N_P(n, K_4\{1\}) \sim \frac{n}{6} \), where \( K_4\{1\} \) is the 1-subdivision of \( K_4 \). In addition, we obtain significantly improved upper bounds on \( N_P(n, P_{2m+1}) \) and \( N_P(n, C_{2m}) \) for \( m \geq 4 \). For a wide class of graphs \( H \), the key technique developed in this paper allows us to bound \( N_P(n, H) \) in terms of an optimization problem over weighted graphs.

Keywords
extremal graph theory, planar graphs, weighted graphs

1 Introduction

In this paper, we use standard graph theory definitions and notation (cf. [9]): \( P_n \), \( C_n \), and \( K_n \) denote the path, cycle, and clique on \( n \) vertices, respectively. The complete bipartite graph with parts of sizes \( a \) and \( b \) is denoted by \( K_{a,b} \). We use also standard big-oh and little-oh notation.

For graphs \( G \) and \( H \), let \( N(G, H) \) denote the number of (unlabeled) copies of \( H \) in \( G \). For a collection of graphs \( \mathcal{G} \) and a positive integer \( n \), define

\[
N_{\mathcal{G}}(n, H) \overset{\text{def}}{=} \max\{N(G, H) : G \in \mathcal{G}, |V(G)| = n\}.
\]

In this paper, we are concerned with asymptotically determining \( N_P(n, H) \) for various graphs \( H \), where \( \mathcal{P} \) is the set of all planar graphs.
The study of $N_P(n, H)$ was initiated by Hakimi and Schmeichel [6], who determined $N_P(n, C_3)$ and $N_P(n, C_4)$ exactly. Alon and Caro [1] continued this study by determining $N_P(n, K_{2,k})$ exactly for all $k$; in particular, they determined $N_P(n, P_3)$. Győri et al. [4] later gave the exact value for $N_P(n, P_3)$, and the same authors determined $N_P(n, C_5)$ in [5]. Generalizations of some of these results to other surfaces were established by Huynh, Joret, and Wood [8].

The main driving force behind this manuscript is a recent conjecture of Ghosh et al. [2] which posits that

$$N_P(n, P_{2m+1}) = 4m \left( \frac{n}{m} \right)^{m+1} + O(n^m) \quad \text{for all } m \geq 2,$$

the authors construct graphs which meet the lower bound for all $m \geq 2$, and they prove the case of $m = 2$, showing that $N_P(n, P_2) = n^3 + O(n^2)$. We make steps toward this conjecture by proving:

**Theorem 1.1.** The following hold:

$$N_P(n, P_2) = \frac{4}{27} \cdot n^4 + O(n^{4-1/5}) \quad \text{and}$$

$$N_P(n, P_{2m+1}) \leq \frac{n^{m+1}}{2 \cdot (m-1)!} + O(n^{m+4/5}) \quad \text{for all } m \geq 4.$$

This, in particular, establishes the $m = 3$ case of Ghosh et al.’s conjecture, albeit with a worse error term than predicted. Before this result, the best general upper bound that we are aware of is

$$N_P(n, P_{2m+1}) \leq \frac{(6n)^{m+1}}{2} \quad \text{for all } m \geq 3,$$

though this bound does not appear to be in the literature.

The methods used to prove this result extend to even cycles.

**Theorem 1.2.** The following hold:

$$N_P(n, C_6) = \left( \frac{n}{3} \right)^3 + O(n^{3-1/5}),$$

$$N_P(n, C_6) = \left( \frac{n}{4} \right)^4 + O(n^{4-1/5}) \quad \text{and}$$

$$N_P(n, C_{2m}) \leq \frac{n^m}{m!} + O(n^{m-1/5}) \quad \text{for all } m \geq 5.$$

Before this result, the best general upper bound that we are aware of is

$$N_P(n, C_{2m}) \leq \frac{(6n)^m}{4m} \quad \text{for all } m \geq 3.$$

We present also new proofs of some known results.
Theorem 1.3. The following hold:

1. \( N_P(n, P_3) = n^3 + O(n^{14/5}) \) (Ghosh et al. [2]).
2. \( N_P(n, C_4) = \frac{n^2}{2} + O(n^{9/5}) \) (Hakimi–Schmeichel [6]).
3. \( N_P(n, K_{2,k}) = \frac{n^k}{k!} + O(n^{k-1+16/(k+8)}) \) for \( k \geq 9 \) (Alon–Caro [1]).

Although these results are already known and our error terms are worse than those attained in the original papers, these results demonstrate the strength of the method developed in this paper. Indeed, after applying one of a trio of general reduction lemmas (discussed in Section 2), each of these results follows in about one to two lines. Furthermore, our results actually apply to a wider class of graphs than just planar graphs, namely, the class of graphs which have linearly many edges and have no copy of \( K_{3,3} \).

Beyond odd paths and even cycles, our methods allow us to tackle particular blow-ups of graphs.

Definition 1.4. Let \( H = (V, E) \) be a graph and let \( k \) be a positive integer. The \( k \)-edge-blow-up of \( H \) is the graph \( H \{k\} \), which is formed by replacing every edge \( xy \in E \) by an independent set of size \( k \) and connecting each of these \( k \) new vertices to both \( x \) and \( y \).

For example, \( C_m \{1\} = C_{2m} \) for \( m \geq 3 \) and \( K_2 \{k\} = K_{2,k} \) for \( k \geq 1 \). We note that the graph \( C_m \{\ell\} \) where \( \ell = \left\lfloor \frac{n-m}{m} \right\rfloor \) realizes the lower bound in Equation (1).

Alon and Caro [1] determined \( N_P(n, K_2 \{k\}) \) exactly for all \( k \); we extend this to the other two planar cliques by showing:

Theorem 1.5. For all positive integers \( k \),

\[
N_P(n, K_3 \{k\}) = \frac{1}{(k!)^3} \left( \frac{n^k}{3} \right)^m + O(n^{3k-k/(k+4)}) \quad \text{and}
\]

\[
N_P(n, K_4 \{k\}) = \frac{1}{(k!)^6} \left( \frac{n^k}{6} \right)^m + O(n^{6k-k/(k+4)}).
\]

In general, it is not difficult to show that \( N_P(n, H \{k\}) = \Theta(n^{km}) \) if \( H \) is a planar graph on \( m \) edges and \( k \cdot \delta(H) \geq 2 \). Indeed, the graph \( H \{\ell\} \) where \( \ell = \left\lfloor \frac{n-V(H)}{m} \right\rfloor \) shows that

\[
N_P(n, H \{k\}) \geq \left( \frac{\ell}{k} \right)^m = \frac{1}{(k!)^m} \left( \frac{n}{m} \right)^{km} - O(n^{km-1}),
\]

and it is an exercise to bound

\[
N_P(n, H \{k\}) \leq \frac{(6n)^{km}}{\text{Aut } H \cdot (k!)^m},
\]
where Aut $H$ is the automorphism group of $H$. The key step in the proof of this upper bound is the content of Proposition 2.9. In this paper, we significantly improve the leading constant in the upper bound.

**Theorem 1.6.** Let $H$ be a planar graph on $m$ edges and let $k$ be a positive integer. If either

- $k \cdot (\delta(H) - 1) \geq 2$ or
- $\delta(H) = 1$ and $k \geq 9$,

then

$$N_P(n, H[k]) \leq \frac{n^{km}}{(km)!} + o(n^{km}).$$

Compare this result to the naïve bounds in Equations (2) and (3). In fact, provided $k$ is sufficiently large, we are able to asymptotically pin down $N_P(n, H[k])$.

**Theorem 1.7.** Let $H$ be a planar graph on $m$ edges and let $k$ be a positive integer. If either

- $\delta(H) \geq 2$ and $k \geq \frac{\log(m + 1)}{m \log(1 + 1/m)}$ or
- $\delta(H) = 1$ and $k \geq \max\left\{9, \frac{\log(m + 1)}{m \log(1 + 1/m)}\right\}$,

then

$$N_P(n, H[k]) = \frac{1}{(k!)^m} \left(\frac{n}{m}\right)^{km} + o(n^{km}).$$

The requirement that $k \geq \frac{\log(m + 1)}{m \log(1 + 1/m)}$ in the above theorem is necessary for some graphs $H$. As an example, let $I$ denote the skeleton of the icosahedron and let $I^-$ denote the graph formed by deleting any edge from $I$. Since $|E(I^-)| = 29$ and $\delta(I^-) = 4$, Theorem 1.7 implies that

$$N_P(n, I^-[k]) \sim \frac{1}{(k!)^{29}} \left(\frac{n}{29}\right)^{29k}$$

for all $k \geq 4$. However, for $k \in \{1, 2, 3\}$, the graph $I\{\ell\}$ where $\ell = \left\lfloor \frac{n - 12}{30} \right\rfloor$ realizes

$$N_P(n, I^-[k]) \geq 30 \left(\frac{\ell}{k}\right)^{29} \sim \frac{30}{(k!)^{29}} \left(\frac{n}{30}\right)^{29k} > \frac{1.57}{(k!)^{29}} \left(\frac{n}{29}\right)^{29k},$$

since $N(I, I^-) = 30$. The icosahedron is not unique in this regard (see Proposition 4.11).

The paper is organized as follows. In Section 2, we present the key contribution of this paper: a trio of reduction lemmas from which all of our results follow. Section 2.1 contains the proofs of these reduction lemmas. We then, in Section 3, use these reduction lemmas to prove Theorem 1.1 and part 1 of Theorem 1.3. In Section 4, we establish Theorems 1.2, 1.5, 1.6, and 1.7 along with parts 2 and 3 of Theorem 1.3. We conclude with a list of remarks and conjectures in Section 5.
1.1 Notation and preliminaries

We use standard graph theory definitions and notation (cf. [9]). For a graph \( G \), we use \( V(G) \) and \( E(G) \) to denote its vertex-set and edge-set, respectively. When the graph is understood, we omit the parenthetical and simply write \( V \) and \( E \).

For \( v \in V(G) \), we write \( N_G(v) \) to denote the neighborhood of \( v \) in \( G \) and \( \deg_G(v) \) to denote the degree of \( v \) in \( G \). For vertices \( u, v \in V(G) \), we write \( \deg_G(u, v) \) to denote the codegree of \( u \) and \( v \) in \( G \). When the graph \( G \) is understood, we omit the subscript.

For positive integers \( m \leq n \), we write \( \{1, \ldots, n\} \) to denote the set \( \{1, \ldots, n\} \) and write \( \{m, \ldots, n\} \) to denote the set \( \{m, \ldots, n\} \). For a set \( X \), we use \( X^n \) to denote the set of tuples \( x_1, \ldots, x_n \) with \( x_1, \ldots, x_n \) distinct; this notation mirrors that of the falling-factorial. Finally, we will often write \( xy \) to denote the set \( \{x, y\} \) for notational convenience.

We require a special case of the Karush–Kuhn–Tucker (KKT) conditions (cf. [3, Corollaries 9.6 and 9.10]) to prove Lemmas 3.3 and 4.5.

**Theorem 1.8** (Special case of the KKT conditions). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuously differentiable function and consider the optimization problem

\[
\begin{align*}
\max & \quad f(x) \\
\text{s.t.} & \quad \sum_i x_i = 1, \quad x_i, \ldots, x_n \geq 0.
\end{align*}
\]

If \( x^* \) achieves this maximum, then there is some \( \lambda \in \mathbb{R} \) such that, for each \( i \in [n] \), either

\[
x_i^* = 0 \quad \text{or} \quad \frac{\partial f}{\partial x_i}(x^*) = \lambda.
\]

2 THE KEY REDUCTION LEMMAS

Aside from the bounds stated in Section 1, the main contribution of this paper is the technique used in their proofs.

For graphs \( G, H \), let \( C(G, H) \) denote the set of (unlabeled) copies of \( H \) in \( G \); so \( |C(G, H)| = N(G, H) \). For a finite set \( X \), we abbreviate \( C(X, H) \) to \( C(K_X, H) \), where \( K_X \) is the clique on vertex-set \( X \); in other words, \( C(X, H) \) is the set of all copies \( H' \) of \( H \) with \( V(H') \subseteq X \).

The following definition lays out the key quantities used throughout this paper.

**Definition 2.1.** Fix a finite set \( X \) and let \( \mu \) be a probability mass on \( \left( \begin{array}{c} X \\ 2 \end{array} \right) \). We define the following quantities:

1. For \( x \in X \), define
\[ \mu(x) \overset{\text{def}}{=} \sum_{y \in X \setminus \{x\}} \mu(xy), \]

which is the probability that an edge sampled from \( \mu \) is incident to \( x \). It can also be thought of as the weighted degree of \( x \). Note that \( \sum_{x \in X} \mu(x) = 2 \) thanks to the handshaking lemma.

2. For an integer \( m \geq 2 \), define

\[ \rho(\mu; m) \overset{\text{def}}{=} \sum_{x \in (X)_m} \mu(x_1) \left( \prod_{i=1}^{m-1} \mu(x_i, x_{i+1}) \right) \mu(x_m) \quad \text{and} \]

\[ \rho(m) \overset{\text{def}}{=} \sup \left\{ \rho(\mu; m) : \text{supp } \mu \subseteq \binom{X}{2} \text{ for some finite set } X \right\}. \]

The quantity \( \rho(\mu; m) \) is essentially the probability that, upon independently sampling edges \( e_1, \ldots, e_{m+1} \) from \( \mu \), the edges \( e_2, \ldots, e_m \) form a copy of \( P_m \), \( e_1 \) is incident to the first vertex of this path and \( e_{m+1} \) is incident to the last vertex of this path (see the proof of Theorem 3.2 for a more precise interpretation).

3. For a subgraph \( G \subseteq K_X \), define

\[ \mu(G) \overset{\text{def}}{=} \prod_{e \in E(G)} \mu(e), \]

which is essentially the probability that \( |E(G)| \) edges sampled independently from \( \mu \) form the edge-set of \( G \).

4. For a graph \( H \) with no isolated vertices and a positive integer \( k \), define

\[ \beta(\mu; H, k) \overset{\text{def}}{=} \sum_{H' \in \mathcal{E}(X,H)} \mu(H')^k \quad \text{and} \]

\[ \beta(H, k) \overset{\text{def}}{=} \sup \left\{ \beta(\mu; H, k) : \text{supp } \mu \subseteq \binom{X}{2} \text{ for some finite set } X \right\}. \]

The quantity \( \beta(\mu; H, k) \) is essentially the probability that \( k \cdot |E(H)| \) edges sampled independently from \( \mu \) form a copy of \( H \) wherein each edge has multiplicity \( k \) (see the proof of Theorem 4.2 for a more precise interpretation).

While we are primarily concerned with planar graphs, our results apply to a much broader class of graphs.

**Definition 2.2.** For any fixed \( C > 0 \), the collection of graphs \( \mathcal{G}_C \) is defined as follows: \( G \in \mathcal{G}_C \) if and only if

1. \( G \) has no copy of \( K_{3,3} \), and
2. Every subgraph \( H \subseteq G \) satisfies \( |E(H)| \leq C \cdot |V(H)| \).
Observe that $G_{C_1} \subseteq G_{C_2}$ if $C_1 \leq C_2$ and that $P \subseteq G_3$. Furthermore, observe that $H\{k\} \in G_2$ for any graph $H$ and any positive integer $k$. In each of the results discussed in Section 1, $P$ can be replaced by $G_C$ for any $C \geq 2$ (due to monotonicity, all of our upper bounds hold for any $C > 0$, but the lower-bound constructions require $C \geq 2$).

We quickly remark that our results apply to an even wider class of graphs than $G_C$, though we avoid this more general situation for the sake of readability. We discuss these further generalizations in Section 5.

For paths of odd order, we show:

**Lemma 2.3** (Reduction lemma for odd paths). If $m \geq 2$, then

$$N_{G_C}(n, P_{2m+1}) \leq \frac{\beta(m)}{2} \cdot n^{m+1} + O(n^{m+4/5}),$$

where the implicit constant in the big-oh notation depends on $m$ and $C$.

For general edge-blow-ups, we prove:

**Lemma 2.4** (Reduction lemma for edge-blow-ups). Let $H$ be a graph on $m$ edges and let $k$ be a positive integer. If $k \cdot (\delta(H) - 1) \geq 2$, then

$$N_{G_C}(n, H\{k\}) \leq \frac{\beta(H, k)}{(k!)^m} \cdot n^{km} + O(n^{km-k/(k+4)}).$$

If $\delta(H) = 1$ and $k \geq 9$, then

$$N_{G_C}(n, H\{k\}) \leq \frac{\beta(H, k)}{(k!)^m} \cdot n^{km} + O(n^{km-1+16/(k+8)}).$$

In both cases, the implicit constant in the big-oh notation depends on $H, k$, and $C$.

Recall that $C_{2m} = C_m \{1\}$ for $m \geq 3$ and that $C_2 = K_2 \{2\}$. Unfortunately, since $\delta(C_m) = 2$ for $m \geq 3$ and $\delta(K_2) = 1$, we cannot apply Lemma 2.4 to these graphs. However, with a slightly different approach, we can obtain exactly this extension.

**Lemma 2.5** (Reduction lemma for even cycles). The following hold:

$$N_{G_C}(n, C_4) \leq \frac{\beta(K_2, 2)}{2} \cdot n^2 + O(n^{2-1/5}) \quad \text{and}$$

$$N_{G_C}(n, C_{2m}) \leq \beta(C_m, 1) \cdot n^m + O(n^{m-1/5}) \quad \text{for } m \geq 3,$$

where the implicit constant in the big-oh notation depends on $m$ and $C$.

Note that there is still a gap in the reduction lemmas when it comes to $K_2\{k\} = K_{2,k}$, which we can handle only if $k = 2$ or $k \geq 9$; we suspect that this gap can be closed. Granted, at least when dealing with $P$, this result is already superseded by the results of Alon and
Caro [1]. However, we believe that the obvious reduction lemma holds for $H[k]$ provided $k \cdot \delta(H) \geq 2$, though we do not currently have a proof.

While the individual details in each of these reduction lemmas differ, the underlying philosophy is the same. The key idea is to show that the vast majority of the copies of $H$ in $G$ interact predictably with the largest degree vertices of $G$. This being the case, we then argue that $G$ can be suitably approximated by an edge-blow-up of some graph, possibly where each edge is blown up by different amounts. The probability masses $\mu$ discussed in Definition 2.1 are a compact way to represent these edge-blow-ups which approximate $G$.

2.1 Proofs of the reduction lemmas

In this section, we prove Lemmas 2.3–2.5. The approach to the lemmas is very similar, yet each requires separate analysis.

We begin by presenting a simple proposition, pieces of which are used in each proof.

Proposition 2.6. Let $G = (V, E) \in \mathcal{G}_C$ be a graph on $n$ vertices. For $\varepsilon > 0$, define $V' = \{v \in V : \deg(v) \geq \varepsilon n\}$. Then,

$$|V'| \leq \frac{2C}{\varepsilon} \quad \text{and} \quad \sum_{uv \in V'} \deg(u, v) \leq n + 4 \left(\frac{C}{\varepsilon}\right)^4.$$

Proof. We begin by observing that

$$\varepsilon n \cdot |V'| \leq \sum_{v \in V'} \deg(v) \leq \sum_{v \in V} \deg(v) = 2|E| \leq 2Cn \Rightarrow |V'| \leq \frac{2C}{\varepsilon}.$$

For notational convenience set $\tilde{E} \overset{\text{def}}{=} \binom{V'}{2}$ and $S \overset{\text{def}}{=} \sum_{uv \in \tilde{E}} \deg(u, v)$. Since $G$ has no copy of $K_{3,3}$, we know that $|N(u) \cap N(v) \cap N(w)| \leq 2$ for any distinct $u, v, w \in V$. Hence, we can apply the second Bonferroni inequality to bound

$$n \geq \left| \bigcup_{uv \in \tilde{E}} (N(u) \cap N(v)) \right| \geq \sum_{uv \in \tilde{E}} |N(u) \cap N(v)| - \sum_{\{uv, wv\} \in \binom{\tilde{E}}{2}} |N(u) \cap N(v) \cap N(w)|$$

$$\geq S - 2 \left(\binom{\tilde{E}}{2}\right) \geq S - \frac{1}{4} |V'|^4 \geq S - 4 \left(\frac{C}{\varepsilon}\right)^4,$$

which proves the proposition. \qed
2.1.1 Reduction lemma for odd paths

Proof of Lemma 2.3. Fix $G = (V, E) \in \mathcal{G}_C$ on $n$ vertices and fix $v = (v_1, ..., v_m) \in (V)_m$. Label $V(P_{2m+1}) = \{p_1, p_2, ..., p_{2m+1}\}$ in consecutive order and consider the copies of $P_{2m+1}$ in $G$ wherein $v_i$ plays the role of vertex $p_{2i}$. Observe that there are then at most $\deg(v_1)$ choices for the image of $p_1$, at most $\deg(v_i, v_{i+1})$ choices for the image of $p_{2i+1}$ for $i \in [m-1]$ and at most $\deg(v_m)$ choices for the image of $p_{2m+1}$. Thus, there are at most

$$D(v) \overset{\text{def}}{=} \deg(v_1) \prod_{i=1}^{m-1} \deg(v_i, v_{i+1}) \deg(v_m)$$

copies of $P_{2m+1}$ in $G$ wherein $v_1$ plays the role of vertex $p_{2i}$ and so we can bound

$$N(G, P_{2m+1}) \leq \frac{1}{2} \sum_{v \in (V)_m} D(v).$$

Fix $\varepsilon = \varepsilon(n) > 0$ to be chosen later and define the set

$$\widetilde{E} \overset{\text{def}}{=} \left\{ u v \in \binom{V}{2} : \deg(u, v) \geq \varepsilon n \right\}.$$

The set $\widetilde{E}$ induces a graph $\widetilde{G}$ with vertex-set $\widetilde{V} \subseteq V$. Certainly if $v \in \widetilde{V}$, then $\deg(v) \geq \varepsilon n$ and so $|\widetilde{V}| \leq 2C/\varepsilon$, thanks to Proposition 2.6.

Next define

$$\widetilde{P}_m \overset{\text{def}}{=} \{ v \in (V)_m : v_i v_{i+1} \in \widetilde{E} \text{ for all } i \in [m-1] \}$$

and

$$\widetilde{M} \overset{\text{def}}{=} \sum_{v \in \widetilde{P}_m} D(v).$$

We aim to show that $N(G, P_{2m+1}) \approx \widetilde{M}/2$.

For any $u, v \in V$, we have $\deg(u, v) \leq \min\{\deg(u), \deg(v)\}$, so for any $v \in (V)_m$, we can bound

$$D(v) \leq \prod_{i=1}^{j} \deg(v_i) \cdot \deg(v_j, v_{j+1}) \cdot \prod_{i=j+1}^{m} \deg(v_i) \quad \text{for all } j \in [m-1]$$

$$\implies D(v) \leq \left( \min_{i \in [k-1]} \deg(v_i, v_{i+1}) \right) \prod_{i=1}^{m} \deg(v_i).$$

We can therefore bound
2 \cdot N(G, P_{2m+1}) - \widetilde{M} \leq \sum_{v \in (V_m \setminus \widetilde{V}_m)} D(v) \leq \sum_{v \in (V_m \setminus \widetilde{V}_m)} \left( \min_{i \in [\epsilon - 1]} \deg(v_i, v_{i+1}) \right) \prod_{i=1}^{m} \deg(v_i)

\leq \sum_{v \in (V_m \setminus \widetilde{V}_m)} \varepsilon n \cdot \prod_{i=1}^{m} \deg(v_i) \leq \varepsilon n \cdot \sum_{v_1, v_m \in V} \prod_{i=1}^{m} \deg(v_i)

= \varepsilon n \cdot \left( \sum_{v \in V} \deg(v) \right)^{m} \leq \varepsilon n \cdot (2Cn)^m = O(\varepsilon n^{m+1}).

Set \( U = \{ v \in V \setminus \widetilde{V} : |N(v) \cap \widetilde{V}| \geq 3 \} \) and define the subgraph \( G' = (V', E') \) of \( G \) as follows:

- delete all vertices in \( U \),
- delete all vertices \( v \in V \setminus \widetilde{V} \) for which \( N(v) \cap \widetilde{V} = \emptyset \),
- delete all edges induced by \( \widetilde{V} \), and
- delete all edges induced by \( V \setminus \widetilde{V} \).

Since \( G \) has no copy of \( K_{3,3} \),

\[ |U| \leq 2 \left( \frac{|\widetilde{V}|}{3} \right) \leq 2 \left( \frac{2C/\varepsilon}{3} \right) = O(\varepsilon^{-3}). \]

For \( v \in (\widetilde{V})_m \), define

\[ D'(v) \overset{\text{def}}{=} \deg_{G'}(v_1) \prod_{i=1}^{m-1} \deg_{G'}(v_i, v_{i+1}) \deg_{G'}(v_m) \quad \text{and} \]

\[ \overline{M}' \overset{\text{def}}{=} \sum_{v \in \widetilde{V}_m} D'(v). \]

For \( v \in \widetilde{V} \), observe that

\[ \frac{\deg_{G'}(v)}{\deg_{G}(v)} = 1 - \frac{\deg_{G}(v) - \deg_{G'}(v)}{\deg_{G}(v)} \geq 1 - \frac{O(\varepsilon^{-1}) + O(\varepsilon^{-3})}{\varepsilon n} = 1 - O\left( \frac{1}{\varepsilon^4 n} \right). \]

Similarly, for \( uv \in \widetilde{E} \),

\[ \frac{\deg_{G'}(u,v)}{\deg_{G}(u,v)} \geq 1 - \frac{O(\varepsilon^{-1}) + O(\varepsilon^{-3})}{\varepsilon n} = 1 - O\left( \frac{1}{\varepsilon^4 n} \right). \]

Therefore, for any \( v \in \widetilde{V}_m \), we have

\[ \frac{D'(v)}{D(v)} = \frac{\deg_{G'}(v_1) \prod_{i=1}^{m-1} \deg_{G'}(v_i, v_{i+1}) \deg_{G'}(v_m)}{\deg_{G}(v_1) \prod_{i=1}^{m-1} \deg_{G}(v_i, v_{i+1}) \deg_{G}(v_m)} \geq 1 - O\left( \frac{1}{\varepsilon^4 n} \right), \]

and so
\[ \tilde{M}' \geq \left( 1 - O\left( \frac{1}{\epsilon^4 n} \right) \right) \tilde{M}. \]

Next, we can partition \( V' \setminus \tilde{V} = U_1 \cup U_2 \) where \( U_i = \{ v \in V' \setminus \tilde{V} : |N_{G'}(v)| = i \} \). We claim that we may suppose \( U_1 = \emptyset \). Indeed, suppose that \( x \in U_1 \) and that \( xu \in E(G') \), so \( u \in \tilde{V} \). Consider selecting any \( v \) such that \( uv \in \tilde{E} \) and introducing the edge \( xv \). (Note that if \( G \) was planar to begin with, then \( G' \) is still planar after this modification.) Observe that \( \tilde{M}' \) can only increase under this operation and so we may suppose that \( U_1 = \emptyset \).

Thus, set \( S = \sum_{uv \in (\tilde{V})_{\deg_{G'}}(u, v)} \mu(\mu(uv) = \deg_{G'}(u, v)/S) \). Since \( V' = \tilde{V} \cup U_2 \), and \( G' \) has no edges induced by \( \tilde{V} \), we observe that \( S = |U_2| \leq n \). Furthermore, for any \( v \in \tilde{V} \), we have \( \bar{\mu}(v) = \deg_{G'}(v)/S \). Therefore,

\[
\tilde{M}' = \sum_{v \in \tilde{V}} D'(v) \leq \sum_{v \in (\tilde{V})_m} D'(v) = S^{m+1} \cdot \sum_{v \in (\tilde{V})_m} \bar{\pi}(v) \left( \prod_{i=1}^{m-1} \mu(v_i, v_{i+1}) \right) \bar{\pi}(v_m) = \rho(\mu; m) \cdot S^{m+1} \leq \rho(\mu; m) \cdot n^{m+1} \leq \rho(m) \cdot n^{m+1}.
\]

Finally, selecting \( \rho = n^{-1/5} \) yields

\[
\mathcal{N}(G, P_{2m+1}) \leq \frac{1}{2} \tilde{M} + O(\rho n^{m+1}) \leq \frac{1}{2} \left( 1 + O\left( \frac{1}{\epsilon^4 n} \right) \right) \rho(m) \cdot n^{m+1} + O(\rho n^{m+1}) = \frac{\rho(m)}{2} \cdot n^{m+1} + O(n^{m+4/5}).
\]

Before moving on, we make a few remarks.

**Remark 2.7.** It is not difficult to argue that for \( m \geq 2 \) and \( C \geq 2 \),

\[
\mathcal{N}_{\mathcal{W}}(n, P_{2m+1}) \geq \frac{\rho(m)}{2} \cdot n^{m+1} - o(n^{m+1}),
\]

so Lemma 2.3 is asymptotically tight. Indeed, fix a finite set \( X \) and a probability mass \( \mu \) on \( \binom{X}{2} \). For a sufficiently large integer \( n \), let \( G \) be the edge-blow-up of \( K_X \) formed by blowing up each edge \( e \in \binom{X}{2} \) into a set of size \( \lfloor n \cdot \mu(e) \rfloor \). Then, one can show that \( G \in \mathcal{G}_2 \) and

\[
\mathcal{N}(G, P_{2m+1}) \geq \frac{\rho(\mu; m)}{2} \cdot n^{m+1} - O(n^m).
\]

**Remark 2.8.** For a finite set \( X \) and a probability mass \( \mu \) on \( \binom{X}{2} \), let \( G_\mu \) be the graph with vertex-set \( X \) and edge-set \( \text{supp} \mu \). In the proof of Lemma 2.3, if \( G \in \mathcal{P} \), then we can
guarantee also that $G' \in \mathcal{P}$, even after the modification that ensures $U_1 = \emptyset$. Therefore, we can actually establish

$$N_p(n, P_{2m+1}) = \frac{\rho_p(m)}{2} \cdot n^{m+1} + o(n^{m+1}) \quad \text{for } m \geq 2,$$

where

$$\rho_p(m) = \sup\{\rho(\mu; m) : G_\mu \in \mathcal{P}\}.$$

Although this refinement exists, we do not believe it to be helpful here. That is to say, we believe that $\rho_p(m) = \rho(m)$ for all $m \geq 2$.

### 2.1.2 Reduction lemma for edge-blow-ups

We will need another simple proposition to establish Lemma 2.4.

**Proposition 2.9.** Let $H$ be a graph on $m$ edges and let $k$ be a positive integer. If $G = (V, E)$ is any graph and $k \cdot \delta(H) \geq 2$, then

$$\sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} \deg_G(x, y)^k \leq \frac{(2|E|)^{km}}{|\text{Aut } H|}.$$  

**Proof.** Since $k \cdot \delta(H) \geq 2$, we know that for any $x \in \mathbb{R}^+$ and $v \in V(H)$, we have

$$1 + x^{k \cdot \deg_H(v)/2} \leq (1 + x)^{k \cdot \deg_H(v)/2}.$$  

Additionally, for $u \neq v \in V(G)$, observe that $\deg(u, v) \leq \min\{\deg(u), \deg(v)\} \leq \sqrt{\deg(u)\deg(v)}$.

Using these two facts and translating between labeled and unlabeled copies of $H$, we can bound

$$\sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} \deg_G(x, y)^k \leq \sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} (\deg_G(x)\deg_G(y))^{k/2}$$

$$= \sum_{H' \in \mathcal{C}(V, H)} \prod_{x \in V(H')} \deg_G(x)^{k \cdot \deg_H(x)/2}$$

$$= \frac{1}{|\text{Aut } H|} \sum_{g: V(H) \to V, g \text{ injection}} \prod_{v \in V(H)} \deg_G(g(v))^{k \cdot \deg_H(v)/2}$$

$$\leq \frac{1}{|\text{Aut } H|} \sum_{g: V(H) \to V} \prod_{v \in V(H)} \deg_G(g(v))^{k \cdot \deg_H(v)/2}.$$  

From here, we use the fact that
\[
\sum_{x_1, \ldots, x_n \in X} \prod_{i=1}^n f_i(x_i) = \prod_{x \in X} \left( \sum_{i=1}^n f_i(x) \right)
\]

for any finite set \(X\) and any functions \(f_1, \ldots, f_n : X \to \mathbb{R}\) to bound

\[
\sum_{H' \in \mathcal{C}(V,H)} \prod_{xy \in E(H')} \deg_G(x,y)^k \leq \frac{1}{|\text{Aut } H|} \prod_{v \in V(H)} \left( \sum_{x \in V} \deg_G(x)^{k \cdot \deg_G(v)/2} \right)
\leq \frac{1}{|\text{Aut } H|} \prod_{v \in V(H)} \left( \sum_{x \in V} \deg_G(x)^{k \cdot \deg_G(v)/2} \right)
= \frac{(2|E|)^{km}}{|\text{Aut } H|}.
\]

**Proof of Lemma 2.4.** Fix \(G = (V,E) \in \mathcal{G}_C\) on \(n\) vertices. Fix an injection \(g : V(H) \to V\) and consider the copies of \(H[k]\) in \(G\) where, for each \(v \in V(H)\), \(g(v)\) plays the role of vertex \(v\). For each \(uv \in E(H)\), observe that there are at most \(\binom{\deg_G(u,v)}{k}\) choices for the \(k\) common neighbors of \(u,v\) in \(H[k]\); thus there are at most

\[
\prod_{uv \in E(H)} \left( \binom{\deg_G(u,v)}{k} \right)
\]

ways to extend \(g\) to an embedding of \(H[k]\). In particular, we can bound

\[
\mathcal{N}(G, H[k]) \leq \sum_{H' \in \mathcal{C}(V,H)} \prod_{xy \in E(H')} \binom{\deg(x,y)}{k} \leq \frac{1}{(kl)^m} \sum_{H' \in \mathcal{C}(V,H)} \prod_{xy \in E(H')} \deg(x,y)^k.
\]

(4)

Fix \(\varepsilon = \varepsilon(n) > 0\) to be chosen later and define

\[
\widetilde{V} \overset{\text{def}}{=} \{v \in V : \deg(v) \geq \varepsilon n\} \quad \text{and} \quad \widetilde{M} \overset{\text{def}}{=} \sum_{H' \in \mathcal{C}(\widetilde{V},H)} \prod_{xy \in E(H')} \deg(x,y)^k.
\]

We claim that \(\widetilde{M}\) approximates \(\mathcal{N}(G, H[k])\). The proof of this fact depends heavily on the minimum degree of \(H\), so we break the proof into two claims.

**Claim 2.10.** If \(k \cdot (\delta(H) - 1) \geq 2\), then

\[
\mathcal{N}(G, H[k]) \leq \frac{\widetilde{M}}{(kl)^m} + O(\varepsilon^k n^{km}).
\]

**Proof.** Set \(\overline{E} = \left( \begin{array}{c} \widetilde{V} \\ 2 \end{array} \right)\). For \(H' \in \mathcal{C}(V,H)\), observe that \(V(H') \subseteq \widetilde{V}\) if and only if \(E(H') \subseteq \overline{E}\). Furthermore, observe that if \(uv \notin \overline{E}\), then \(\deg(u,v) < \varepsilon n\). Using Equation (4), we begin by bounding
\[(k!)^m \cdot N(G, H \{k\}) - \widehat{M} \leq \sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} \deg(x, y)^k \]

\[= \sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} \deg(x, y)^k \]

\[\leq \sum_{H' \in \mathcal{C}(V, H)} \prod_{e \in E(H') \backslash \overline{E}} (\varepsilon n)^k \cdot \prod_{xy \in E(H') \backslash \{e\}} \deg(x, y)^k \]

\[\leq (\varepsilon n)^k \cdot \sum_{e \in E(H)} \sum_{H' \in \mathcal{C}(V, H - e)} \prod_{xy \in E(H')} \deg(x, y)^k.\]

Now, for any \(e \in E(H)\), we have \(k \cdot \delta(H - e) \geq k \cdot (\delta(H) - 1) \geq 2\), and so we can apply Proposition 2.9 to \(H - e\) to bound

\[(k!)^m \cdot N(G, H \{k\}) - \widehat{M} \leq (\varepsilon n)^k \cdot \sum_{e \in E(H)} \frac{(2|E|)^k(m-1)}{|\text{Aut}(H - e)|} \leq \varepsilon^k \cdot m \cdot (2C)^{k(m-1)} \cdot n^k = O(\varepsilon^k n^k).\]

\[\square\]

**Claim 2.11.** If \(\delta(H) = 1\) and \(k \geq 2\), then

\[N(G, H \{k\}) \leq \frac{\widehat{M}}{(k!)^m} + O(\varepsilon^{k/2} n^{km+1}).\]

**Proof.** The proof of this fact is very similar to the proof of Proposition 2.9. For \(u, v \in V\), certainly \(\deg(u, v) \leq \min\{\deg(u), \deg(v)\} \leq \sqrt{\deg(u)\deg(v)}\). Thus, by applying Equation (4), we can bound

\[(k!)^m N(G, H \{k\}) - \widehat{M} = \sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} \deg(x, y)^k \]

\[\leq \sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} (\deg(x)\deg(y))^{k/2} \]

\[= \sum_{H' \in \mathcal{C}(V, H)} \prod_{x \in V(H')} \deg(x)^{k \cdot \deg_{H'}(x) / 2} \]

\[\leq \sum_{H' \in \mathcal{C}(V, H)} \prod_{y \in V(H') \setminus \{y\}} (\varepsilon n)^{k \cdot \deg_{H'}(y) / 2} \prod_{x \in V(H') \setminus \{y\}} \deg(x)^{k \cdot \deg_{H'}(x) / 2} \]

\[= \sum_{H' \in \mathcal{C}(V, H)} \prod_{xy \in E(H')} (\varepsilon n)^{k \cdot \deg_{H'}(y) / 2} \prod_{x \in V(H') \setminus \{y\}} \deg(x)^{k \cdot \deg_{H'}(x) / 2}.\]

Next, by translating between labeled and unlabeled copies of \(H\), we continue to bound
\[(k!)^m N(G, H \{k\}) - \widetilde{M} \leq \frac{1}{|\text{Aut } H|} \sum_{v \in V(H)} \sum_{g : V(H) \to V \text{ } g(v) \notin \mathcal{V}} (\varepsilon n)^{k \cdot \deg_{\mathcal{V}}(v)/2} \prod_{u \in V(H) \setminus \{v\}} \deg\]

\[\leq \frac{n}{|\text{Aut } H|} \sum_{v \in V(H)} (\varepsilon n)^{k \cdot \deg_{\mathcal{V}}(v)/2} \sum_{g : V(H-v) \to V} \prod_{u \in V(H-v)} \deg\]

\[\leq \frac{n}{|\text{Aut } H|} \left( \sum_{x \in V} \deg(x)^{k \cdot \deg_{\mathcal{V}}(u)/2} \right)\]

From here, we use the fact that \(k \geq 2\) and proceed by the same steps in Proposition 2.9 to bound

\[(k!)^m N(G, H \{k\}) - \widetilde{M} \leq \frac{n}{|\text{Aut } H|} \sum_{v \in V(H)} (\varepsilon n)^{k \cdot \deg_{\mathcal{V}}(v)/2} \prod_{u \in V(H-v)} \left( \sum_{x \in V} \deg(x)^{k \cdot \deg_{\mathcal{V}}(u)/2} \right)\]

\[\leq \frac{n}{|\text{Aut } H|} \sum_{v \in V(H)} \left( \sum_{x \in V} \deg(x)^{k \cdot \deg_{\mathcal{V}}(v)/2} \right) \cdot (2n)^{k \cdot \deg_{\mathcal{V}}(v)/2} \cdot (2C)^{km} \cdot \varepsilon^{km} = O(\varepsilon^{k/2}n^{km+1}).\]

□

We turn our attention now to bounding \(\widetilde{M}\). Set \(S \overset{\text{def}}{=} \sum_{u \in V} \deg(u, v)\) and define the probability mass \(\mu\) on \(\binom{V}{2}\) by \(\mu(uv) = \deg(u, v)/S\). By applying Proposition 2.6, we see that

\[\widetilde{M} = \beta(\mu; H, k) \cdot S^{km} \leq \beta(H, k) \cdot (n + O(\varepsilon^{-4}))^{km} = \beta(H, k) \cdot n^{km} \cdot \left( 1 + O\left( \frac{1}{n^4} \right) \right)^{km}.\]

Therefore, if \(\varepsilon^4 n \to \infty\), we have

\[\widetilde{M} \leq \beta(H, k) \cdot n^{km} + O\left( \frac{n^{km-1}}{\varepsilon^4} \right).\]  

From here, we break into cases to conclude the proof.
Case: \( k \cdot (\delta(H) - 1) \geq 2 \). Select \( \varepsilon = n^{-1/(k+4)} \). Since \( k \geq 1 \), we have \( \varepsilon^4 n \to \infty \); hence we can apply Equation (5) and Claim 2.10 to bound
\[
N(G, H[k]) \leq \frac{\tilde{M}}{(kl)^m} + O(\varepsilon^k n^{km}) \leq \frac{\beta(H, k)}{(kl)^m} \cdot n^{km} + O\left(\frac{n^{k-1}}{\varepsilon^4}\right) + O(\varepsilon^k n^{km})
\]
\[
= \frac{\beta(H, k)}{(kl)^m} \cdot n^{km} + O(n^{km-k/(k+4)}).
\]

Case: \( \delta(H) = 1 \) and \( k \geq 9 \). Select \( \varepsilon = n^{-1/(k+8)} \). Since \( k \geq 9 \), we have \( \varepsilon^4 n \to \infty \); hence we can apply Equation (5) and Claim 2.11 to bound
\[
N(G, H[k]) \leq \frac{\tilde{M}}{(kl)^m} + O(\varepsilon^{k/2} n^{km+1}) \leq \frac{\beta(H, k)}{(kl)^m} \cdot n^{km} + O\left(\frac{n^{k-1}}{\varepsilon^4}\right) + O(\varepsilon^{k/2} n^{km+1})
\]
\[
= \frac{\beta(H, k)}{(kl)^m} \cdot n^{km} + O(n^{km-1+16/(k+8)}).
\]

Before moving on, we make a couple of remarks.

**Remark 2.12.** It is not difficult to argue that if \( H \) is a graph on \( m \) edges with no isolated vertices, \( k \) is a positive integer and \( C \geq 2 \), then
\[
N_{uc}(n, H[k]) \geq \frac{\beta(H, k)}{(kl)^m} \cdot n^{km} - o(n^{km}),
\]
so Lemma 2.4 is asymptotically tight. Indeed, fix a finite set \( X \) and a probability mass \( \mu \) on \( \binom{X}{2} \). For a sufficiently large integer \( n \), let \( G \) be the edge-blow-up of \( K_X \) formed by blowing up each edge \( e \in \binom{X}{2} \) into a set of size \( n \cdot \mu(e) \). Then, one can show that \( G \in G_2 \) and
\[
N(G, H[k]) \geq \frac{\beta(\mu; H, k)}{(kl)^m} \cdot n^{km} - O(n^{km-1}).
\]

**Remark 2.13.** For a finite set \( X \) and a probability mass \( \mu \) on \( \binom{X}{2} \), let \( G_\mu \) be the graph with vertex-set \( X \) and edge-set \( \text{supp} \, \mu \). By following the proof of Lemma 2.3 more diligently, one can show that if \( H \) is planar and \( k \cdot (\delta(H) - 1) \geq 2 \), then
\[
N_P(n, H[k]) = \frac{\beta_P(H, k)}{(kl)^m} \cdot n^{km} + o(n^{km}),
\]
where
\[
\beta_P(H, k) = \sup\{\beta(\mu; H, k) : G_\mu \in P\}.
\]
Despite our beliefs when it comes to this same refinement in Lemma 2.3 (see Remark 2.8), this could actually be an important refinement for certain planar graphs $H$. For instance, we believe that $\beta(K_5^-, 1) > \beta_p(K_5^-, 1)$ where $K_5^-$ is the five-clique minus an edge (we discuss this further in Section 4.3).

In any case, we do not know how to prove a similar refinement in the case that $\delta(H) = 1$ and $k \geq 9$.

2.1.3 | Reduction lemma for even cycles

To prove Lemma 2.5, we will first need a straightforward upper bound on the number of even paths in a graph.

**Proposition 2.14.** If $G = (V, E)$ is any graph and $m$ is a positive integer, then

$$N(G, P_{2m}) \leq \frac{(2|E|)^m}{2}.$$  

**Proof.** Label $V(P_{2m}) = \{p_1, p_2, ..., p_{2m}\}$ in consecutive order. For $(v_1, ..., v_m) \in (V)_m$, consider the copies of $P_{2m}$ in $G$ wherein $v_i$ plays the role of $p_i$. Observe that there are then at most $\deg(v_1)$ choices for the image of $p_1$ and at most $\deg(v_i, v_{i+1})$ choices for the image of $P_{2i+1}$ for all $i \in [m-1]$. Since $\deg(v_i, v_{i+1}) \leq \deg(v_{i+1})$, we can therefore bound

$$N(G, P_{2m-1}) \leq \frac{1}{2} \sum_{v \in (V)_m} \deg(v_1) \left( \prod_{i=1}^{m-1} \deg(v_i, v_{i+1}) \right) \leq \frac{1}{2} \sum_{v \in (V)_m} \prod_{i=1}^{m} \deg(v_i)$$

$$\leq \frac{1}{2} \sum_{v_1, ..., v_m \in V} \prod_{i=1}^{m} \deg(v_i) = \frac{1}{2} \left( \sum_{v \in V} \deg(v) \right)^m = \frac{(2|E|)^m}{2}. \quad \Box$$

We require additionally a simple observation about 2-colorings of $C_m$.

**Proposition 2.15.** Fix $m \geq 2$. For any 2-coloring $\chi : \mathbb{Z} / m\mathbb{Z} \rightarrow \{0, 1\}$, there is some $i \in \mathbb{Z} / m\mathbb{Z}$ for which either $\chi(i) = \chi(i + 2) = 0$ or $\chi(i) = \chi(i + 3) = 1$.

**Proof.** Suppose for the sake of contradiction that the claim does not hold. Since we are done if $\chi \equiv 1$, we may suppose, without loss of generality, that $\chi(0) = 0$. This then implies that $\chi(-2) = \chi(2) = 1$. But then $\chi(1) = \chi(-1) = 0$; a contradiction. \qed

We are now ready to prove the reduction lemma for even cycles.

Recalling Equation (4) from the proof of Lemma 2.4, we know that for a graph $G$,

$$N(G, C_4) \leq \sum_{u,v \in (V)_2} \left( \frac{\deg(u, v)}{2} \right) \quad \text{and}$$

$$N(G, C_{2m}) \leq \sum_{H \in \mathcal{C}(V, C_m)} \prod_{x,y \in E(H)} \deg(x, y) \quad \text{for } m \geq 3.$$
We will not use either of these inequalities directly, but it will be helpful to keep them in mind throughout the following proof.

Proof of Lemma 2.5. Let \( G = (V, E) \in \mathcal{G}_C \) be a graph on \( n \) vertices. Fix \( \varepsilon = \varepsilon(n) > 0 \) to be chosen later and define

\[
\tilde{V} \stackrel{\text{def}}{=} \{ v \in V : \deg(v) \geq \varepsilon n \}.
\]

We denote an element of \( C(G, C_{2m}) \) by a tuple \((u_1, \ldots, u_{2m})\), which is a list of the vertices of the cycle in some cyclic order. We define the following sets:

\[
\begin{align*}
\text{GOOD} & \stackrel{\text{def}}{=} \{(v_1, \ldots, v_{2m}) \in C(G, C_{2m}) : v_1, v_3, \ldots, v_{2m-1} \in \tilde{V} \text{ or } v_2, v_4, \ldots, v_{2m} \in \tilde{V}, v_{2m} \in \tilde{V}\}, \\
\text{BAD} & \stackrel{\text{def}}{=} C(G, C_{2m}) \setminus \text{GOOD}, \\
\text{BIG} & \stackrel{\text{def}}{=} \{(v_1, \ldots, v_{2m}) \in \text{BAD} : v_i, v_{i+2} \in \tilde{V} \text{ for some } i \in [2m]\}, \\
\text{SMALL} & \stackrel{\text{def}}{=} \{(v_1, \ldots, v_{2m}) \in \text{BAD} : v_i, v_{i+3} \notin \tilde{V} \text{ for some } i \in [2m]\}.
\end{align*}
\]

Thanks to Proposition 2.15, we know that \( \text{BAD} = \text{BIG} \cup \text{SMALL} \). We aim to show that \( N(G, C_{2m}) \approx |\text{GOOD}| \). To do so, we must show that both \( \text{BIG} \) and \( \text{SMALL} \) are both of insignificant size.

\( \square \)

Claim 2.16. \( |\text{BIG}| \leq O(\varepsilon n^m) + O(n^{m-1}/\varepsilon^3) \).

Proof. If \( m = 2 \), then \( \text{BIG} = \emptyset \) and so the claim holds. Hence, we may suppose that \( m \geq 3 \). Fix \( H = (u_1, \ldots, u_{2m}) \in \text{BIG} \); without loss of generality, we may suppose that \( u_1, u_3 \in \tilde{V} \). Since \( H \in \text{BAD} \), there must be some \( i \in \{5, 7, \ldots, 2m-1\} \) for which \( \deg(u_i) < \varepsilon n \), and so we bound

\[
\prod_{i=1}^{m} \deg(u_{2i-1}, u_{2i+1}) \leq \varepsilon n \cdot \deg(u_1, u_3) \cdot \prod_{i=3}^{m} \deg(u_{2i-1}).
\]

By appealing additionally to Proposition 2.6, we can therefore crudely bound

\[
|\text{BIG}| \leq \sum_{v \in \langle V \rangle_{m} : v_1, v_3 \in \tilde{V}, v_i \notin \tilde{V} \text{ for some } i \in [3, m]} \prod_{i=1}^{m} \deg(v_i, v_{i+1}) \leq \sum_{v_1 \neq v_2 \in \tilde{V}, v_3, \ldots, v_n \in V} \varepsilon n \cdot \deg(v_1, v_2) \cdot \prod_{i=3}^{m} \deg(v_i)
\]

\[
= \varepsilon n \cdot \left( \sum_{v_1 \neq v_2 \in \tilde{V}} \deg(v_1, v_2) \right) \left( \sum_{v \in V} \deg(v) \right)^{m-2} \leq \varepsilon n \cdot \left( 2n + O\left( \frac{1}{\varepsilon^4} \right) \right) \cdot (2Cn)^{m-2}
\]

\[
= O(\varepsilon n^m) + O\left( \frac{n^{m-1}}{\varepsilon^3} \right).
\]

\( \square \)
Claim 2.17. \(|\text{SMALL}| \leq O(\varepsilon n^m)\).

Proof. Fix \((u_1, \ldots, u_{2m}) \in \text{SMALL}; without loss of generality, we may suppose that \(u_{2m-2}, u_1 \notin \hat{V}\). Observe that \(u_1, \ldots, u_{2m-2}\) forms a copy of \(P_{2m-2}\) and that the edge \(u_{2m-1}u_{2m}\) has both end-points in \(N(u_1) \cup N(u_{2m-2})\). Therefore, by applying Proposition 2.14 and using the fact that \(G \in G_C\), we see that

\[
|\text{SMALL}| \leq 2 \cdot N(G, P_{2m-2}) \cdot \max_{u \notin V \setminus \hat{V}} |E(G[N(u) \cup N(v)])| \\
\leq (2|E|)^{m-1} \cdot \max_{u \notin V \setminus \hat{V}} C \cdot |N(u) \cup N(v)| \\
\leq (2Cn)^{m-1} \cdot C \cdot 2\varepsilon n = O(\varepsilon n^m).
\]

□

We now deal with GOOD. Define \(S \overset{\text{def}}{=} \sum_{uv \in \binom{\hat{V}}{2}} \deg(u, v)\) and let \(\mu\) be the probability mass on \(\binom{\hat{V}}{2}\) defined by \(\mu(uv) = \deg(u, v)/S\). If \(m = 2\), then we can bound

\[
|\text{GOOD}| \leq \sum_{uv \in \binom{\hat{V}}{2}} \left( \frac{\deg(u, v)}{2} \right) \leq \frac{1}{2} \sum_{uv \in \binom{\hat{V}}{2}} \deg(u, v)^2 = \frac{S^2}{2} \sum_{uv \in \binom{\hat{V}}{2}} \mu(uv)^2 \\
= \frac{S^2}{2} \sum_{H \in \mathcal{C}(\hat{V}, K_2)} \mu(H)^2 = \frac{\beta(\mu; K_2, 2)}{2} \cdot S^2 \leq \frac{\beta(K_2, 2)}{2} \cdot S^2.
\]

Similarly, if \(m \geq 3\), then we can bound

\[
|\text{GOOD}| \leq \sum_{H \in \mathcal{C}(\hat{V}, C_m)} \prod_{xy \in E(H)} \deg(x, y) = S^m \sum_{H \in \mathcal{C}(\hat{V}, C_m)} \prod_{xy \in E(H)} \mu(xy) \\
= S^m \sum_{H \in \mathcal{C}(\hat{V}, C_m)} \mu(H) = \beta(\mu; C_m, 1) \cdot S^m \leq \beta(C_m, 1) \cdot S^m.
\]

Thus, by applying Proposition 2.6 and setting \(B_2 = \beta(K_2, 2)/2\) and \(B_m = \beta(C_m, 1)\) for all \(m \geq 3\), we have shown that

\[
|\text{GOOD}| \leq B_m \cdot S^m \leq B_m (n + O(1/\varepsilon^4))^m = B_m \cdot n^m + O\left(\frac{n^{m-1}}{\varepsilon^4}\right),
\]

provided \(\varepsilon^4n \to \infty\). Therefore, by selecting \(\varepsilon = n^{-1/5}\) and applying Claims 2.16 and 2.17, we bound

\[
N(G, C_{2m}) = |\text{GOOD}| + |\text{BAD}| \leq |\text{GOOD}| + |\text{BIG}| + |\text{SMALL}| \\
\leq B_m \cdot n^m + O\left(\frac{n^{m-1}}{\varepsilon^4}\right) + O(\varepsilon n^m) + O\left(\frac{n^{m-1}}{\varepsilon^3}\right) = B_m \cdot n^m + O(n^{m-1/5}).
\]
3 | ODD PATHS

Thanks to Lemma 2.3, to bound $N_P(G, P_{2m+1})$ from above, it suffices to find upper bounds on $\rho(m)$. Recall that for a finite set $X$ and a probability mass $\mu$ on $\binom{X}{2}$,

$$\rho(\mu; m) = \sum_{x \in (X)_m} \bar{\mu}(x_i) \left( \prod_{i=1}^{m-1} \mu(x_i x_{i+1}) \right) \bar{\mu}(x_m),$$

where $\bar{\mu}(x) = \sum_{y \in X \setminus \{x\}} \mu(xy)$.

First, we handle the case of $m = 2$.

**Proposition 3.1.** $\rho(2) = 2$.

**Proof.** The lower bound is realized if $|\text{supp } \mu| = 1$.

For the upper bound, fix a finite set $X$ and a probability mass $\mu$ on $\binom{X}{2}$. Define the matrix $M \in \mathbb{R}^{X \times X}$ by $M_{xy} = \mu(xy)$ under the convention that $\mu(xx) = 0$. Observe that $M$ is a symmetric, nonnegative matrix all of whose row-sums are bounded above by 1. In particular, the largest eigenvalue of $M$ is at most 1 (cf. [7, Lemma 8.1.21]). Thus, by applying standard facts about the Rayleigh quotient (cf. [7, Theorem 4.2.2]) and using the fact that $\sum_{x \in X} \mu(x) = 2$, we bound

$$\rho(\mu; 2) = \sum_{x \neq y \in X} \mu(x) \mu(xy) \mu(y) = \langle \mu, M \mu \rangle \leq \langle \mu, \mu \rangle = \sum_{x \in X} \mu(x)^2 \leq \sum_{x \in X} \mu(x) = 2.$$  

\[\square\]

From here, we have a quick proof of the asymptotic result of Ghosh et al. [2], albeit with a worse error term.

**Proof of part 1 of Theorem 1.3.** By applying Lemma 2.3 and Proposition 3.1, we bound

$$N_P(n, P_3) \leq N_{\mathcal{G}_1}(n, P_3) \leq \frac{\rho(2)}{2} \cdot n^3 + O(n^{14/5}) = n^3 + O(n^{14/5}).$$  

\[\square\]

Next, we establish a general upper bound on $\rho(m)$.

**Theorem 3.2.** For any $m \geq 3$,

$$\rho(m) \leq \frac{1}{(m-1)!}.$$
Proof. Fix a finite set \( X \) and a probability mass \( \mu \) on \( \binom{X}{2} \). The key to this bound is to interpret \( \rho(\mu; m) \) as the probability of some event in a probability space defined by \( \mu \). Intuitively \( \rho(\mu; m) \) is the probability that if we independently sample edges \( e_1, \ldots, e_m \) from \( \mu \), then \( e_2, \ldots, e_m \) form a path with vertices \( x_1, \ldots, x_m \), \( e_1 \) is incident to \( x_1 \) and \( e_{m+1} \) is incident to \( x_m \). We now make this intuition precise.

For a tuple \( \mathbf{x} \in (X)_m \), define the sets

\[
\mathcal{E}(\mathbf{x}) \overset{\text{def}}{=} \left\{ (e_2, \ldots, e_m) \in \left( \binom{X}{2} \right)^{m-1} : \{e_2, \ldots, e_m\} = \{x_i x_2, x_2 x_3, \ldots, x_{m-1} x_m\} \right\}
\]

\[
\mathcal{L}(\mathbf{x}) \overset{\text{def}}{=} \left\{ (e_1, e_{m+1}) \in \left( \binom{X}{2} \right)^2 : e_1 \ni x_i \text{ and } e_{m+1} \ni x_m \right\}.
\]

Observe that \( \Pr_{\mu^1}[\mathcal{L}(\mathbf{x})] = \mu(x_1) \cdot \mu(x_m) \) and that

\[
\mu^{m-1}(\mathbf{e}) = \prod_{i=1}^{m-1} \mu(x_i x_{i+1}) \quad \text{for all } \mathbf{e} \in \mathcal{E}(\mathbf{x}),
\]

where \( \mu^j \) is the product distribution induced on \( \binom{X}{j} \) by \( \mu \).

We can therefore write

\[
\rho(\mu; m) = \sum_{\mathbf{x} \in (X)_m} \mu(x_1) \left( \prod_{i=1}^{m-1} \mu(x_i x_{i+1}) \right) \mu(x_m) = \sum_{\mathbf{x} \in (X)_m} \frac{1}{|\mathcal{E}(\mathbf{x})|} \sum_{\mathbf{e} \in \mathcal{E}(\mathbf{x})} \mu^{m-1}(\mathbf{e}) \cdot \Pr_{\mu^1}[\mathcal{L}(\mathbf{x})]
\]

\[
= \frac{1}{(m-1)!} \sum_{\mathbf{x} \in (X)_m} \Pr_{\mu^{m-1}}[\mathcal{E}(\mathbf{x})] \cdot \Pr_{\mu^1}[\mathcal{L}(\mathbf{x})] = \frac{1}{(m-1)!} \sum_{\mathbf{x} \in (X)_m} \Pr_{\mu^{m+1}}[\mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x})].
\]

For \( \mathbf{x} \in (X)_m \), consider the reverse tuple \( \bar{\mathbf{x}} \in (X)_m \) where \( \bar{x}_i = x_{m+1-i} \). The events \( \mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x}) \) and \( \mathcal{E}(\mathbf{y}) \times \mathcal{L}(\mathbf{y}) \) are almost always disjoint when \( \mathbf{x} \neq \mathbf{y} \). The only circumstance in which they double-count the same event is when \( \mathbf{y} = \bar{\mathbf{x}} \), in which case \( \mathcal{E}(\mathbf{x}) = \mathcal{E}(\bar{\mathbf{x}}) \) and \( \mathcal{L}(\mathbf{x}) \cap \mathcal{L}(\bar{x}) = \{(x_i x_m, x_i x_m)\} \). Indeed, for \( \mathbf{x}, \mathbf{y} \in (X)_m \), we have \( \{x_1, x_2, \ldots, x_{m-1} x_m\} = \{y_1, y_2, \ldots, y_{m-1} y_m\} \) if and only if \( \mathbf{y} \in \{\mathbf{x}, \bar{\mathbf{x}}\} \); thus

\[
\mathcal{E}(\mathbf{x}) \cap \mathcal{E}(\mathbf{y}) = \begin{cases} \mathcal{E}(\mathbf{x}) & \text{if } \mathbf{y} \in \{\mathbf{x}, \bar{\mathbf{x}}\}, \\
\emptyset & \text{otherwise} \end{cases}
\]

and

\[
(\mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x})) \cap (\mathcal{E}(\mathbf{y}) \times \mathcal{L}(\mathbf{y})) = \begin{cases} \mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x}, \\
\mathcal{E}(\mathbf{x}) \times \{x_i x_m, x_i x_m\} & \text{if } \mathbf{y} = \bar{\mathbf{x}}, \\
\emptyset & \text{otherwise.} \end{cases}
\]
Therefore, by grouping together \( \mathbf{x} \) and \( \bar{\mathbf{x}} \), we compute

\[
(m-1)! \cdot \rho(\mu; m) = \sum_{\mathbf{x} \in (X)_{m}} \Pr_{\mu}^{m+1}[\mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x})]
\]

\[
= \Pr_{\mu}^{m+1}\left[ \bigcup_{\mathbf{x} \in (X)_{m}} (\mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x})) \right] + \Pr_{\mu}^{m+1}\left[ \bigcup_{\mathbf{x} \in (X)_{m}} (\mathcal{E}(\mathbf{x}) \times \{(x_1x_m, x_1x_m)\}) \right].
\]

Next, by writing

\[
\mathcal{E} \overset{\text{def}}{=} \bigcup_{\mathbf{x} \in (X)_{m}} \mathcal{E}(\mathbf{x}),
\]

we bound

\[
\Pr_{\mu}^{m+1}\left[ \bigcup_{\mathbf{x} \in (X)_{m}} (\mathcal{E}(\mathbf{x}) \times \mathcal{L}(\mathbf{x})) \right] \leq \Pr_{\mu}^{m+1}\left[ \mathcal{E} \times \bigcup_{\mathbf{x} \in (X)_{m}} \mathcal{L}(\mathbf{x}) \right] \leq \Pr_{\mu}^{m-1}[\mathcal{E}]
\]

and

\[
\Pr_{\mu}^{m+1}\left[ \bigcup_{\mathbf{x} \in (X)_{m}} (\mathcal{E}(\mathbf{x}) \times \{(x_1x_m, x_1x_m)\}) \right] \leq \Pr_{\mu}^{m+1}\left[ \mathcal{E} \times \bigcup_{\mathbf{x} \in (X)_{m}} \{(x_1, x_1x_m)\} \right]
\]

\[
= \Pr_{\mu}^{m-1}[\mathcal{E}] \cdot \Pr_{\mu}^{2}\left[ \bigcup_{e \in (X)} \{(e, e)\} \right]
\]

\[
= \Pr_{\mu}^{m-1}[\mathcal{E}] \cdot \sum_{e \in (X)} \mu(e)^2.
\]

Any member of \( \mathcal{E} \) has the property that its coordinates are distinct members of \((X)_{2}\); hence, since \( m \geq 3 \), we can bound

\[
\Pr_{\mu}^{m-1}[\mathcal{E}] \leq \Pr_{\mu}^{m-1}\left[ \{(e_2, e_3, \ldots, e_m) \in (X)_{2}^{m-1} : e_2, e_3, \ldots, e_m \text{ distinct} \} \right]
\]

\[
= \Pr_{\mu}^{2}\left[ \left\{ (e_2, e_3) \in (X)_{2}^2 : e_2 \neq e_3 \right\} \right] = 1 - \sum_{e \in (X)} \mu(e)^2.
\]

Putting everything together, we have shown that
\[(m - 1)! \cdot \rho(\mu; m) \leq \text{Pr}_{\mu^m \mid [X]} + \text{Pr}_{\mu^m \mid [X]} \cdot \sum_{e \in \left(\begin{array}{c} X \\ 2 \end{array}\right)} \mu(e)^2 \leq \left(1 - \sum_{e \in \left(\begin{array}{c} X \\ 2 \end{array}\right)} \mu(e)^2\right) \left(1 + \sum_{e \in \left(\begin{array}{c} X \\ 2 \end{array}\right)} \mu(e)^2\right)\]

\[= 1 - \left(\sum_{e \in \left(\begin{array}{c} X \\ 2 \end{array}\right)} \mu(e)^2\right)^2 \leq 1,\]

which establishes the claim. \[\Box\]

### 3.1 Paths of order 7

In this section, we prove Theorem 1.1. The main content in this section is the proof that \(\rho(3) = \frac{8}{27}\), which hinges on the following general inequality. We note that the following lemma is a special case of the much more general Theorem 4.10, but we give a direct and self-contained proof here.

**Lemma 3.3.** If \(a_1, ..., a_n \geq 0\), then

\[\left(\sum_i a_i^2\right)^2 - \sum_i a_i^4 \leq \frac{1}{8} \left(\sum_i a_i\right)^4.\]

**Proof.** We notice first that the claim is trivial if \(a_i = 0\) for all \(i\). Furthermore, scaling the \(a_i\)'s by any positive constant leaves the inequality invariant. As such, we may suppose that \(a_1 \geq ... \geq a_n > 0\).

Therefore, noting that \(\left(\sum_i x_i^2\right)^2 - \sum_i x_i^4 = \sum_{i \neq j} x_i^2 x_j^2\), it suffices to show that

\[
\max \sum_{i \neq j} x_i^2 x_j^2
\]

s.t.

\[
x_i = 1,
\]

\[
x_i \geq 0 \quad \text{for all } i \in [n],
\]

is bounded above by \(1/8\). Let \(a_1, ..., a_n\) denote an optimal solution to Equation (6); without loss of generality, we may suppose that \(a_1 \geq ... \geq a_n \geq 0\). Additionally, let \(M\) denote the optimal value, that is, \(M = \sum_{i \neq j} a_i^2 a_j^2\). We may certainly suppose that \(n \geq 2\) since otherwise \(M = 0\).

By applying the KKT conditions (Theorem 1.8) to Equation (6), we find that there is some fixed \(\lambda \in \mathbb{R}\) for which

\[a_i \sum_{j \neq i} a_j^2 = \lambda \quad \text{for all } i \in [n].\]
From here, we use the fact that \( \sum_i a_i = 1 \) to determine
\[
\lambda = \sum_i a_i \lambda_i = \sum_i a_i^2 \sum_{j \neq i} a_j^2 = \sum_{i \neq j} a_i^2 a_j^2 = M.
\] (8)

Now, consider the numbers \( b_1, \ldots, b_{n-1} \) defined by
\[
b_i = a_i / (1 - a_n),
\]
which are well defined since \( n \geq 2 \) and hence \( a_n < 1 \). Note that \( b_i > 0 \) and \( \sum_i b_i = 1 \). Therefore,
\[
M \geq \sum_{i \neq j} b_i^2 b_j^2 = \frac{1}{(1 - a_n)^4} \sum_{i,j \in \{n-1\}^2, i \neq j} a_i^2 a_j^2 = \frac{1}{(1 - a_n)^4} \left( \sum_{i \neq j} a_i^2 a_j^2 - 2a_n \sum_{j=1}^{n-1} a_j^2 \right)
\]
\[
= \frac{1}{(1 - a_n)^4} (M - 2Ma_n) = M \cdot \frac{1 - 2a_n}{(1 - a_n)^4},
\]
where the penultimate equality follows from Equations (7) to (8). We conclude that
\[1 - 2a_n \leq (1 - a_n)^4\]
and thus \( a_n \geq 0.45 \). Since \( a_1 \geq \cdots \geq a_n \), this then implies that \( n = 2 \).
Thus, we apply the arithmetic–geometric (AM–GM) inequality to finally bound
\[
M = 2a_1^2 a_2^2 \leq 2 \left( \frac{a_1 + a_2}{2} \right)^4 = \frac{1}{8}.
\]

\[\square\]

We can now determine \( \rho(3) \).

**Corollary 3.4.** If \( a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0 \), then
\[
\left( \sum_i a_i b_i \right)^2 - \sum_i a_i^2 b_i^2 \leq \frac{1}{8} \left( \sum_i a_i \right)^2 \left( \sum_i b_i \right)^2.
\]

**Proof.** By applying Lemma 3.3 and the Cauchy–Schwarz inequality, we bound
\[
\left( \sum_i a_i b_i \right)^2 - \sum_i a_i^2 b_i^2 = \left( \sum_i (\sqrt{a_i} b_i) \right)^4 - \sum_i (\sqrt{a_i} b_i)^4
\]
\[
\leq \frac{1}{8} \left( \sum_i \sqrt{a_i} b_i \right)^4 \leq \frac{1}{8} \left( \sum_i a_i \right)^2 \left( \sum_i b_i \right)^2.
\]

\[\square\]

We can now determine \( \rho(3) \).
Theorem 3.5. \( \rho(3) = 8/27. \)

Proof. To prove the lower bound, let \( \mu \) be the uniform distribution on \( \binom{[3]}{2} \). Then

\[
\rho(\mu; 3) = \sum_{(x,y,z) \in \binom{[3]}{3}} \pi(x)\mu(xy)\mu(yz)\pi(z) = 3! \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{8}{27}.
\]

To establish the upper bound, fix a finite set \( X \) and let \( \mu \) be any probability distribution on \( \binom{X}{2} \). We begin by writing

\[
\rho(\mu; 3) = \sum_{(x,y,z) \in \binom{X}{3}} \pi(x)\mu(xy)\mu(yz)\pi(z) = \sum_{y \in X} \sum_{x \in X \setminus \{y\}} \pi(x)\mu(xy) \sum_{z \in X \setminus \{x,y\}} \pi(z)\mu(z)
\]

\[
= \sum_{x \in X} \left[ \left( \sum_{y \in X \setminus \{x\}} \pi(x)\mu(xy) \right)^2 - \sum_{y \in X \setminus \{x\}} \pi(x)^2\mu(xy)^2 \right].
\]

Then, by applying Corollary 3.4 to the inner expression and using the fact that \( \sum_{x \in X} \pi(x) = 2 \), we bound

\[
\rho(\mu; 3) \leq \sum_{y \in X} \frac{1}{8} \left( \sum_{x \in X \setminus \{y\}} \pi(x) \right)^2 \left( \sum_{x \in X \setminus \{y\}} \mu(xy) \right)^2 = \frac{1}{8} \sum_{y \in X} (2 - \pi(y))^2 \cdot \pi(y)^2.
\]

We finally observe that the expression \( x(2 - x)^2 \) for \( 0 \leq x \leq 1 \) is maximized when \( x = 2/3 \), yielding a value of 32/27. Therefore,

\[
\rho(\mu; 3) \leq \frac{1}{8} \sum_{y \in X} \pi(y) \cdot \pi(y)(2 - \pi(y))^2 \leq \frac{4}{27} \sum_{y \in X} \pi(y) = \frac{8}{27}.
\]

Now that we know \( \rho(3) \), the proof of Theorem 1.1 follows quickly.

Proof of Theorem 1.1. First, the graph \( K_3[\ell] \) where \( \ell = \left\lfloor \frac{n-3}{3} \right\rfloor \) shows that

\[
\mathbf{N}_P(n, P_7) \geq \frac{4}{27} \cdot n^4 - O(n^3).
\]

Next, we apply Lemma 2.3 to bound

\[
\mathbf{N}_P(n, P_{2m+1}) \leq \mathbf{N}_{v_3}(n, P_{2m+1}) \leq \frac{\rho(m)}{2} \cdot n^{m+1} + O(n^{m+4/5}).
\]
for all \( m \geq 2 \). Finally, Theorem 3.5 tells us that \( \rho(3) = 8/27 \), and Theorem 3.2 tells us that \( \rho(m) \leq 1/(m - 1)! \) for all \( m \geq 4 \); hence the claim follows.

\[ \square \]

4 | EDGE-BLOW-UPS AND EVEN CYCLES

Thanks to Lemmas 2.4 and 2.5, to bound \( n_H(k) \) from above for various \( H, k \), it suffices to prove upper bounds on \( \beta(H, k) \). Recall that for a finite set \( X \) and a probability mass \( \mu \) on \( \binom{X}{2} \),

\[ \beta(\mu; H, k) = \sum_{H' \in \mathcal{C}(X, H)} \mu(H')^k, \]

where \( \mathcal{C}(X, H) \) is the set of copies of \( H \) in \( K_X \), and

\[ \mu(H') = \prod_{e \in E(H')} \mu(e). \]

We deal first with the case of \( H = K_2 \).

**Proposition 4.1.** \( \beta(K_2, k) = 1 \) for all \( k \geq 1 \).

**Proof.** The lower bound is realized if \( |\text{supp} \mu| = 1 \).

Let \( X \) be a finite set and let \( \mu \) be a probability mass on \( \binom{X}{2} \). Then

\[ \beta(\mu; K_2, k) = \sum_{K \in \mathcal{C}(X, K_2)} \mu(K)^k = \sum_{e \in \binom{X}{2}} \mu(e)^k \leq \left( \sum_{e \in \binom{X}{2}} \mu(e) \right)^k = 1. \]

\[ \square \]

Since \( P \subseteq G_3 \), parts 2 and 3 of Theorem 1.3 follow immediately thanks to Lemmas 2.5 and 2.4, respectively.

Next, we prove a general upper bound on \( \beta(H, k) \).

**Theorem 4.2.** If \( H \) is a graph on \( m \) edges with no isolated vertices and \( k \) is a positive integer, then

\[ \beta(H, k) \leq \frac{(k!)^m}{(km)!}. \]

**Proof.** Fix a finite set \( X \) and let \( \mu \) be a probability mass on \( \binom{X}{2} \). The key to this bound is to relate \( \beta(\mu; H, k) \) to an event in a probability space defined by \( \mu \). Intuitively, \( \beta(\mu; H, k) \) is the probability that \( km \) edges sampled independently from \( \mu \) form a copy of \( H \) wherein each edge has multiplicity \( k \). We now make this intuition precise.

For \( H' \in \mathcal{C}(X, H) \), define the set
\[ C(H') = \left\{ e \in \binom{X}{2}^{km} : \text{each } e \in E(H') \text{ occurs exactly } k \text{ times in } e \right\}. \]

Observe that
\[ |C(H')| = \binom{km}{k, \ldots, k} = \frac{(km)!}{(k!)^m}, \]
and that \( \mu(H')^k = \mu^{km}(e) \) for every \( e \in C(H') \), where \( \mu^{km} \) is the product distribution on \( \binom{X}{2}^{km} \) induced by \( \mu \).

Now, the events \( \{C(H') : H' \in C(X, H)\} \) are pairwise disjoint since the entries of any \( e \in C(H') \) uniquely define the edge-set of \( H' \). Consequently, we can bound

\[
\beta(\mu; H, k) = \sum_{H' \in C(X, H)} \mu(H') = \sum_{H' \in C(X, H)} \frac{1}{|C(H')|} \sum_{e \in C(H')} \mu^{km}(e) \\
\leq \frac{(k!)^m}{(km)!} \cdot \sum_{H' \in C(X, H)} \Pr_{\mu^{km}}[C(H')] = \frac{(k!)^m}{(km)!} \cdot \Pr_{\mu^{km}}[\bigcup_{H' \in C(X, H)} C(H')] \\

\]

From here, we can immediately prove Theorem 1.6.

**Proof of Theorem 1.6.** First, Theorem 4.2 tells us that \( \beta(H, k) \leq (k!)^m/(km)! \). Then, thanks to Lemma 2.4, if \( k \cdot (\delta(H) - 1) \geq 2 \) or if \( \delta(H) = 1 \) and \( k \geq 9 \), then

\[
N_p(n, H[k]) \leq N_{G_5}(n, H[k]) \leq \frac{\beta(H, k)}{(k!)^m} \cdot n^{km} + o(n^{km}) \leq \frac{n^{km}}{(km)!} + o(n^{km}).
\]

\[ \square \]

### 4.1 The structure of optimal masses

In this section, we establish structural properties of those masses which achieve \( \beta(H, k) \), which will be used in the next sections to prove Theorems 1.2, 1.5, and 1.7.

Of course, a priori, it is not even clear that \( \beta(H, k) \) is ever achieved. In fact, one can show that \( \beta(K_{1,m}, 1) = 1/m! \) for all \( m \geq 2 \), yet this value is never achieved. Indeed, one can argue that for all \( n \geq m \geq 2 \),

\[
\max \{ \beta(\mu; K_{1,m}, 1) : \text{supp } \mu \leq n \} = \binom{n}{m} \cdot \frac{1}{n^m} < \frac{1}{m!}.
\]

\[ \square \]
The same phenomenon occurs for $\beta(mK_2, 1)$ for $m \geq 2$ where $mK_2$ is the matching on $m$ edges. We conjecture that these are the only situations in which $\beta(H, k)$ is not achieved. See Corollary 4.7 for partial results in this direction.

Despite this, for any fixed, finite set $X$ with at least two elements, the quantity $\max\{\beta(\mu; H, k) : \text{supp } \mu \subseteq \binom{X}{2}\}$ exists, thanks to compactness.

**Definition 4.3.** Let $H$ be a graph with no isolated vertices and let $k$ be a positive integer. For a finite set $X$, we denote by $\text{Opt}(X; H, k)$ the set of all probability masses $\mu$ on $\binom{X}{2}$ satisfying

$$\beta(\mu; H, k) = \max\{\beta(\mu'; H, k) : \text{supp } \mu' \subseteq \binom{X}{2}\}.$$  

In the case that $\beta(H, k)$ is achieved, we denote by $\text{Opt}(H, k)$ the set of all masses $\mu$ satisfying $\beta(\mu; H, k) = \beta(H, k)$.

Fix a finite set $X$ and a probability mass $\mu$ on $\binom{X}{2}$ and let $G_\mu$ be the graph with vertex-set $X$ and edge-set $\text{supp } \mu$. Observe that $\beta(\mu; H, k) > 0$ if and only if $G_\mu$ has a copy of $H$; consequently, if $\beta(\mu; H, k) > 0$, then $|\text{supp } \mu| \geq |E(H)|$ and $\text{supp } \mu \cap |V(H)|$. We see also that if $|X| \geq |V(H)|$ and $\mu \in \text{Opt}(X; H, k)$, then $G_\mu$ must contain a copy of $H$. Additionally, we can determine such an optimal $\mu$ exactly if $|\text{supp } \mu| = |E(H)|$.

**Proposition 4.4.** Let $H = (V, E)$ be a graph on $m$ edges with no isolated vertices and let $k$ be a positive integer. Fix any finite set $X$ with $|X| \geq |V|$ and fix $\mu \in \text{Opt}(X; H, k)$. If $|\text{supp } \mu| = m$, then $\mu$ is the uniform distribution on $E(H')$ for some $H' \in C(X, H)$ and thus $\beta(\mu; H, k) = m^{-km}$.

**Proof:** We know that $G_\mu$ contains a copy of $H$ since $|X| \geq |V|$ and $\mu \in \text{Opt}(X; H, k)$. Since $|\text{supp } \mu| = m$, we conclude that $G_\mu$ must in fact be a copy of $H$, possibly with isolated vertices. We can therefore apply the arithmetic–geometric mean inequality to bound

$$\beta(\mu; H, k) = \prod_{e \in \text{supp } \mu} \mu(e)^k \leq \left(\frac{1}{m} \sum_{e \in \text{supp } \mu} \mu(e)\right)^{km} = \frac{1}{m^{km}},$$

with equality if and only if $\mu(e) = 1/m$ for every $e \in \text{supp } \mu$. $\square$

We next derive regularity conditions for the members of $\text{Opt}(X; H, k)$.

**Lemma 4.5.** Let $H$ be a graph on $m$ edges with no isolated vertices, let $k$ be a positive integer and fix a finite set $X$. If $\mu \in \text{Opt}(X; H, k)$, then...
\[
\mu(e) \cdot m \cdot \beta(\mu; H, k) = \sum_{H' \in \mathcal{C}(X, H) : E(H') \ni e} \mu(H'^k) \quad \text{for every } e \in \binom{X}{2}, \text{ and}
\]
\[
\mathfrak{p}(x) \cdot m \cdot \beta(\mu; H, k) = \sum_{H' \in \mathcal{C}(X, H) : \forall (H') \exists x} \deg_{H'}(x) \cdot \mu(H'^k) \quad \text{for every } x \in X.
\]

**Proof.** By the definition of \( \beta \), we can write
\[
\beta(\mu; H, k) = \max \frac{\sum_{H' \in \mathcal{C}(X, H)} \prod_{e \in E(H')} x_e^k}{\sum_{e \in \binom{X}{2}} x_e = 1, x_e \geq 0 \quad \text{for all } e \in \binom{X}{2}}.
\]
In particular, we can apply the KKT conditions (Theorem 1.8) to \( \mu \). By doing so, we find that there is some fixed \( \lambda \in \mathbb{R} \) such that \( D(e) = \lambda \) for all \( e \in \text{supp} \mu \), where
\[
D(e) \overset{\text{def}}{=} \sum_{H' \in \mathcal{C}(X, H) : E(H') \ni e} \mu(e)^{k-1} \prod_{s \in E(H') \setminus \{e\}} \mu(s)^k.
\]
Of course, whether or not \( e \in \text{supp} \mu \), we always have
\[
\lambda \cdot \mu(e) = D(e) \cdot \mu(e) = \sum_{H' \in \mathcal{C}(X, H) : E(H') \ni e} \mu(H'^k).
\]
Using \( 1[S] \) to denote the indicator function of an event \( S \), we compute
\[
\lambda = \sum_{e \in \binom{X}{2}} \mu(e) \overset{\text{def}}{=}= \sum_{e \in \binom{X}{2}} D(e) \cdot \mu(e) = \sum_{e \in \binom{X}{2}} D(e) \cdot \mu(e) = \sum_{H' \in \mathcal{C}(X, H) : E(H') \ni e} \mu(H'^k)
\]
\[
= \sum_{H' \in \mathcal{C}(X, H)} \mu(H'^k) \cdot \sum_{e \in \binom{X}{2}} 1[e \in E(H')] = m \cdot \beta(\mu; H, k),
\]
and so
\[
\mu(e) \cdot m \cdot \beta(\mu; H, k) = \mu(e) \cdot \lambda = \sum_{H' \in \mathcal{C}(X, H) : E(H') \ni e} \mu(H'^k),
\]
for every \( e \in \binom{X}{2} \).

From here, we see also that for each \( x \in X \),
\[
\overline{\mu}(x) \cdot m \cdot \beta(\mu; H, k) = \sum_{y \in X \setminus \{x\}} \mu(xy) \cdot m \cdot \beta(\mu; H, k) = \sum_{y \in X \setminus \{x\}} H' \in C(X, H): E(H') \ni xy \mu(H')^k \\
= \sum_{H' \in C(X, H): \ V(H') \ni x} \mu(H')^k \cdot \sum_{y \in X \setminus \{x\}} \mathbf{1}[xy \in E(H')] \\
= \sum_{H' \in C(X, H): \ V(H') \ni x} \deg_H(x) \cdot \mu(H')^k.
\]

These regularity conditions allow us to place bounds on the edge- and vertex-masses in an optimal mass.

**Lemma 4.6.** Let \( H \) be a graph on \( m \) edges with no isolated vertices, let \( k \) be a positive integer and fix a finite set \( X \) with \( |X| \geq |V(H)| \). If \( \mu \in \text{Opt}(X; H, k) \), then

\[
1 - m \cdot \mu(e) \leq (1 - \mu(e))^km \quad \text{for all} \ e \in \text{supp} \ \mu \quad \text{and} \\
1 - \frac{m}{\delta(H)}\overline{\mu}(x) \leq (1 - \overline{\mu}(x))^km \quad \text{for all} \ x \in \text{supp} \ \overline{\mu}.
\]

**Proof.** Since \( |X| \geq |V(H)| \), we know that \( \beta(\mu; H, k) > 0 \).

We prove first that \( 1 - m \cdot \mu(e) \leq (1 - \mu(e))^km \) for any \( e \in \text{supp} \ \mu \). Fix any \( e \in \text{supp} \ \mu \). If \( \mu(e) \geq 1/m \), then the claim is trivial; otherwise, \( \mu(e) < 1/m \), and we can define the mass \( \mu' \) on \( \binom{X}{2} \) by

\[
\mu'(s) = \begin{cases} 
1 & \text{if} \ s = e, \\
\frac{1}{1 - \mu(e)} \cdot \begin{cases} 
0 & \text{if} \ s = e, \\
\mu(s) & \text{otherwise}. 
\end{cases}
\end{cases}
\]

Since \( \mu \in \text{Opt}(X; H, k) \), we apply Lemma 4.5 to see that

\[
\beta(\mu; H, k) \geq \beta(\mu'; H, k) = \frac{1}{(1 - \mu(e))^km} \cdot \left( \beta(\mu; H, k) - \sum_{H' \in C(X, H): \ E(H') \ni e} \mu(H')^k \right) \\
= \beta(\mu; H, k) \cdot \frac{1 - m \cdot \mu(e)}{(1 - \mu(e))^km},
\]

which implies that \( 1 - m \cdot \mu(e) \leq (1 - \mu(e))^km \).

We prove next that \( 1 - \frac{m}{\delta}(\overline{\mu}(x) \leq (1 - \overline{\mu}(x))^km \) for any \( x \in \text{supp} \ \overline{\mu} \), where \( \delta = \delta(H) \). Fix any \( x \in \text{supp} \ \overline{\mu} \). If \( \overline{\mu}(x) \geq \delta/m \), then the claim is trivial; otherwise, \( \overline{\mu}(x) < \delta/m \), and we can define the mass \( \mu' \) on \( \binom{X}{2} \) by
\[
\mu'(s) = \frac{1}{1 - \overline{\mu}(x)} \begin{cases} 
0 & \text{if } s \not\exists x, \\
\mu(s) & \text{otherwise.} 
\end{cases}
\]

Since \( \mu \in \text{Opt}(X; H, k) \), we again apply Lemma 4.5 to see that

\[
\beta(\mu; H, k) \geq \beta(\mu'; H, k) = \frac{1}{(1 - \overline{\mu}(x))^{km}} \left( \beta(\mu; H, k) - \sum_{H' \in \mathcal{C}(X, H); \ V(H') \ni x} \mu(H')^{k} \right)
\]

\[
\geq \frac{1}{(1 - \overline{\mu}(x))^{km}} \left( \beta(\mu; H, k) - \frac{\mu(x)}{\delta} \cdot m \cdot \beta(\mu; H, k) \right)
\]

\[
= \beta(\mu; H, k) \cdot \frac{1 - \frac{m}{\delta} \overline{\mu}(x)}{(1 - \overline{\mu}(x))^{km}},
\]

which implies that \( 1 - \frac{m}{\delta} \overline{\mu}(x) \leq (1 - \overline{\mu}(x))^{km} \).\qed

We remark that one can show also that \( \mu(e) \leq 1/m \) for all \( e \in \text{supp } \mu \) and that \( \overline{\mu}(x) \leq \Delta(H)/m \) for all \( x \in \text{supp } \overline{\mu} \); however, we have not found any use for these inequalities.

Lemma 4.6 allows us to place lower bounds on \( \mu(e) \) for \( e \in \text{supp } \mu \) and on \( \overline{\mu}(x) \) for \( x \in \text{supp } \overline{\mu} \) when \( \mu \) is an optimal mass. For instance, consider the inequality \( 1 - m \cdot \mu(e) \leq (1 - \mu(e))^{km} \). This inequality always holds if \( k = 1 \), but if \( k \geq 2 \), then we observe that the curves \( 1 - mx \) and \( (1 - x)^{km} \) intersect at 0 and at a unique \( x^* \in (0, 1] \). Furthermore, \( 1 - mx > (1 - x)^{km} \) for all \( x \in (0, x^*) \) and \( 1 - mx \leq (1 - x)^{km} \) for all \( x \in [x^*, 1] \). Therefore, if we can locate some \( x \in (0, 1] \) for which \( 1 - mx > (1 - x)^{km} \), then we will have shown that \( \mu(e) > x \) for all \( e \in \text{supp } \mu \). Similar reasoning can be applied to the inequality \( 1 - \frac{m}{\delta} \overline{\mu}(x) \leq (1 - \overline{\mu}(x))^{km} \); that is, if we can locate some \( z \in (0, 1] \) for which \( 1 - \frac{m}{\delta} z > (1 - z)^{km} \), then we will have shown that \( \overline{\mu}(x) > z \) for all \( x \in \text{supp } \overline{\mu} \).

Indeed, we will apply precisely this reasoning to establish Theorems 4.8–4.10. However, before we get to this, we first remark on a useful consequence of Lemma 4.6.

**Corollary 4.7.** For a graph \( H \) and a positive integer \( k \), if \( k \cdot \delta(H) \geq 2 \), then \( \beta(H, k) \) is achieved.

**Proof.** Let \( X \) be a finite set with \( |X| \geq |V(H)| \) and fix any \( \mu \in \text{Opt}(X; H, k) \). Bypassing to a subset of \( X \) if necessary, we may suppose that \( \text{supp } \overline{\mu} = X \). Thanks to compactness, to show that \( \beta(H, k) \) is achieved, it suffices to show that \( |X| \) is bounded above by some constant depending only on \( H \) and \( k \).

Set \( \delta = \delta(H), \ m = |E(H)| \) and fix \( x \in X \) with \( \overline{\mu}(x) \) minimum. If \( \overline{\mu}(x) \geq \delta/m \), then

\[
2 = \sum_{y \in X} \overline{\mu}(y) \geq |X| \cdot \overline{\mu}(x) \geq |X| \cdot \frac{\delta}{m} \quad \Rightarrow \quad |X| \leq \frac{2m}{\delta}.
\]
Otherwise, $\bar{\mu}(x) < \delta/m$. We then apply Lemma 4.6 and use the inequalities $e^{-z/(1-z)} < 1 - z < e^{-z}$ for $0 < z < 1$ to bound

$$1 \geq \frac{1 - \frac{m}{\delta}\bar{\mu}(x)}{(1 - \frac{m}{\delta}\bar{\mu}(x))^{km}} > \exp\left\{-\frac{m}{\delta}\bar{\mu}(x) + km \cdot \bar{\mu}(x)\right\}$$

$$= \exp\left\{\frac{m \cdot \bar{\mu}(x)}{1 - \frac{m}{\delta}\bar{\mu}(x)} \left(k - \frac{1}{\delta} - \frac{km}{\delta}\bar{\mu}(x)\right)\right\},$$

and so $\bar{\mu}(x) > \frac{k\delta - 1}{km}$. Therefore, since $k\delta \geq 2$,

$$2 = \sum_{y \in \mathcal{K}} \mu(y) > |\mathcal{K}| \cdot \frac{k\delta - 1}{km} \Rightarrow |\mathcal{K}| < \frac{2km}{k\delta - 1}.$$  

$\square$

### 4.2 Cliques and even cycles

In this section, we prove Theorems 1.2 and 1.5.

We begin by computing $\beta(K_t, k)$.

**Theorem 4.8.** For all $t \geq 2$ and all $k \geq 1$,

$$\beta(K_t, k) = \binom{t}{2}^{-\binom{k}{2}}.$$

**Proof.** The lower bound is realized by the uniform distribution on $E(K_t)$.

For the upper bound, we have already shown that $\beta(K_2, k) = 1$ (Proposition 4.1), so we may suppose that $t \geq 3$. Fix any $\mu \in \text{Opt}(K_t, k)$, which can be done thanks to Corollary 4.7. Note that $\mu^\ast \geq \frac{1}{\mu^\ast_{\supp}}$ and that $|\supp \mu| \geq \binom{t}{2}$.

Set $z = 2/(t + 1)$; we use a version of Bernoulli’s inequality, $(1 - x)^n < 1 - \frac{nx}{1 + (n - 1)x}$ for $0 < x < 1$ and $n > 1$, to bound

$$(1 - z)^{\binom{k}{2}} \leq (1 - z)^{\binom{t}{2}} < 1 - \frac{\binom{t}{2}z}{1 + \left(\binom{t}{2} - 1\right)z} = 1 - \frac{\binom{t}{2}z}{t - 1}.$$

Thus, thanks to Lemma 4.6, we know that $\bar{\mu}(x) > 2/(t + 1)$ for every $x \in \supp \bar{\mu}$. From here, we see that

$$2 = \sum_{x \in \supp \mu} \mu(x) > \frac{2}{|\supp \mu|} \Rightarrow \frac{2}{|\supp \mu|} < t + 1 \Rightarrow |\supp \mu| = t.$$
Therefore, \( \text{supp } \mu l = \left( \frac{1}{2} \right) \), and so the claim follows from Proposition 4.4.

Thus, the proof of Theorem 1.5 follows immediately from Lemma 2.4 (or Lemma 2.5 for \( K_3\{1\} \)) and Theorem 4.8. In fact, we have shown that

\[
N_{\mu l}(n, K_i\{k\}) = \frac{1}{(k!)^{\left( \frac{1}{2} \right)}} \left( \frac{n}{(\frac{i}{2})} \right)^k + O\left( n^{\left( \frac{i}{2} \right) - k/(k+4)} \right),
\]

for all \( t \geq 3, k \geq 1 \) and \( C \geq 2 \).

We next determine \( \beta(C_4, k) \).

**Theorem 4.9.** \( \beta(C_4, k) = 4^{-4k} \) for all \( k \geq 1 \).

**Proof.** The lower bound is achieved by the uniform distribution on the edges of \( C_4 \).

For the upper bound, fix any \( \mu \in \text{Opt}(C_4, k) \), which can be done thanks to Corollary 4.7. Set \( X = \text{supp } \mu \); we claim that \( |X| = 4 \). Indeed, for any \( x \in X \), Lemma 4.6 tells us that

\[
1 - 2\mu(x) = (1 - \mu(x))^4 \leq (1 - \mu(x)) \Rightarrow \mu(x) > 0.45.
\]

Therefore,

\[
2 = \sum_{x \in X} \mu(x) > 0.45 \cdot |X| \Rightarrow |X| < 4.45,
\]

and so \( |X| = 4 \). We can therefore decompose \( \binom{X}{2} = \{e_1, f_1\} \cup \{e_2, f_2\} \cup \{e_3, f_3\} \) where \( e_i, f_i \) are parallel edges, that is, \( e_i \cap f_i = \emptyset \). Since every copy of \( C_4 \) is uniquely determined by a pair of these parallel edges, we can write

\[
\beta(\mu; C_4, k) = \sum_{[i,j] \in \binom{[3]}{2}} \mu(e_i)^k \mu(f_j)^k \mu(e_j)^k \mu(f_j)^k \leq \left( \sum_{[i,j] \in \binom{[3]}{2}} \mu(e_i) \mu(f_j) \mu(e_j) \mu(f_j) \right)^k
\]

\[
= \frac{1}{2^k} \left( \left( \sum_{i=1}^{3} \mu(e_i) \mu(f_i) \right)^2 - \sum_{i=1}^{3} \mu(e_i)^2 \mu(f_i)^2 \right)^k.
\]

We finally apply Corollary 3.4 and the AM–GM inequality to bound

\[
\beta(\mu; C_4, k) \leq \frac{1}{2^k} \left( \frac{1}{8} \left( \sum_{i=1}^{3} \mu(e_i) \cdot \sum_{i=1}^{3} \mu(f_i) \right)^2 \right)^k \leq \frac{1}{4^{2k}} \left( \frac{1}{2} \sum_{i=1}^{3} (\mu(e_i) + \mu(f_i)) \right)^{4k} = \frac{1}{4^{4k}}.
\]
The proof of Theorem 1.2 now follows quickly.

**Proof of Theorem 1.2.** The lower bounds are given in Equation (2).

Now, by applying Lemma 2.5, we know that

\[ N_P(n, C_{2m}) \leq N_G(n, C_{2m}) \leq \beta(C_m, 1) \cdot n^m + O(n^{m-1/2}), \]

for \( m \geq 3 \). Finally, Theorem 4.8 gives \( \beta(C_3, 1) = 3^{-3} \), Theorem 4.9 gives \( \beta(C_4, 1) = 4^{-4} \) and Theorem 4.2 gives \( \beta(C_m, 1) \leq 1/m! \) for all \( m \geq 5 \); hence the claim follows. \( \square \)

### 4.3 Sufficiently large edge-blow-ups

We conclude our study of \( \beta(H, k) \) by proving Theorem 1.7.

**Theorem 4.10.** Let \( H \) be a graph on \( m \) edges with no isolated vertices and let \( k \) be a positive integer. If \( k \geq \frac{\log(m+1)}{m \log(1 + 1/m)} \), then \( \beta(H, k) = m^{-km} \).

**Proof.** We begin by observing that if \( k = \frac{\log(m+1)}{m \log(1 + 1/m)} \), then \((m + 1)^{k/m-1} = m^{km}\). Since \( k, m \) are positive integers and \( m, m + 1 \) are coprime, this can happen only if \( k = m = 1 \).

This situation was covered in Proposition 4.1, so we may suppose that \( k > \frac{\log(m+1)}{m \log(1 + 1/m)} \).

Fix any \( \mu \in \text{Opt}(H, k) \), which can be done thanks to Corollary 4.7 since \( k > \frac{\log(m+1)}{m \log(1 + 1/m)} \). Fix any \( x \in \mu \) with \( \supp \mu > 1/(m + 1) \) and observe that

\[
1 = \sum_{e \in \supp \mu} \mu(e) > \frac{|\supp \mu|}{m + 1} \implies |\supp \mu| < m + 1 \implies |\supp \mu| = m,
\]

and so the claim follows from Proposition 4.4. \( \square \)

The proof of Theorem 1.7 then follows immediately from Lemma 2.4 and Theorem 4.10. The lower bound of \( k \geq \frac{\log(m+1)}{m \log(1 + 1/m)} \) in Theorem 4.10 is tight for infinitely many graphs.

**Proposition 4.11.** Let \( H \) be any edge-transitive graph on \( m + 1 \geq 3 \) edges. If \( H^e \) is an \( m \)-edge subgraph of \( H \) with no isolated vertices, then \( \beta(H^e, k) > m^{-km} \) for all positive integers \( k < \frac{\log(m+1)}{m \log(1 + 1/m)} \).

Proof. First, note that \( \frac{\log(m+1)}{m \log(1 + 1/m)} > 1 \) since \( m \geq 2 \); hence the range for \( k \) is nontrivial.

Let \( \mu \) denote the uniform distribution on \( E(H) \). Since \( H \) is edge-transitive, we know that \( N(H, H^{-}) = m + 1 \) and so

\[
\frac{\beta(H^{-}, k)}{m^{-km}} \geq \frac{\beta(\mu; H^{-}, k)}{m^{-km}} = (m + 1) \cdot \left( \frac{m}{m + 1} \right)^{km} > (m + 1) \cdot \left( \frac{m}{m + 1} \right)^{\frac{\log(m+1)}{\log(m/(m+1))}} = 1.
\]

\( \square \)

We remark that this is the reason that it is likely necessary to use the refined \( \beta_{p}(H, k) \) mentioned in Remark 2.13 to determine \( N_{p}(n, H[k]) \) for \( H \in \{K_{5}^{-}, K_{5,5}^{-}\} \) and \( k \) small. For example, the proof of Proposition 4.11 shows that \( \beta(K_{5}^{-}, 1) \geq 10^{-8} \), yet we think it is likely that \( \beta_{p}(K_{5}^{-}, 1) = 9^{-9} \) since \( K_{5} \) is not planar.

5 | REMARKS AND OPEN PROBLEMS

The techniques introduced in this paper are far reaching. Although we were able to compute \( \rho(m) \) and \( \beta(H, k) \) for certain \( m \) and \( H \), there is much we could not do.

5.1 | Odd paths and even cycles

The main question left open by this paper is that of determining \( \rho(m) \) for \( m \geq 4 \).

**Conjecture 5.1.** For all \( m \geq 2 \), \( \rho(m) \) is achieved by the uniform distribution on \( E(C_{m}) \).

In particular, \( \rho(m) = 8 \cdot m^{-m} \).

If true, then

\[
N_{p}(n, P_{2m+1}) = 4m \left( \frac{n}{m} \right)^{m+1} + O(n^{m+4/5}) \quad \text{for all } m \geq 2,
\]

which would verify a conjecture of Ghosh et al. [2], albeit with a worse error term than predicted. Currently, we have only a proof for the cases of \( m = 2 \) and \( m = 3 \).

Even if Conjecture 5.1 is true, the methods developed in this paper are likely too crude to achieve the posited error term of \( O(n^{m}) \), which would verify the conjecture of Ghosh et al. in full.

Turning to even cycles, we conjecture the following:

**Conjecture 5.2.** For all \( m \geq 3 \), \( \beta(C_{m}, 1) \) is achieved by the uniform distribution on \( E(C_{m}) \). In particular, \( \beta(C_{m}, 1) = m^{-m} \).
If true, then
\[
N_F(n, C_{2m}) = \left( \frac{n}{m} \right)^m + O(n^{m-1/5}) \quad \text{for all } m \geq 3.
\]

Currently, we have only a proof for the cases of \( m = 3 \) and \( m = 4 \).

It is likely that proving \( \beta(C_m, 1) = m^{-m} \) is well within reach for \( m \in \{5, 6\} \). Indeed, for these values of \( m \), one can use Lemma 4.6 to show that \( \beta(C_m, 1) \) is achieved by a mass \( \mu \) spanning exactly \( m \) vertices. Furthermore, one can show that \( \bar{\mu}(x) = 2/m \) for each \( x \in \text{supp } \mu \). We have not explored either of these cases any further. Unfortunately, applying Lemma 4.6 to \( \beta(C_7, 1) \) only allows us to say that this quantity is achieved by a mass spanning at most 8 vertices.

5.2 Edge-blow-ups

The question of determining \( \beta(H, k) \) is wide open for most graphs \( H \). One obvious lower bound on \( \beta(H, k) \) is the value achieved by the uniform distribution on \( E(H) \).

**Question 5.3.** For which graphs \( H = (V, E) \) is \( \beta(H, 1) \) achieved by the uniform distribution on \( E \)? That is, for which graphs \( H \) is \( \beta(H, 1) = |E|^{-1/2} \)?

We have already noted that this is not the case for infinitely many graphs (Proposition 4.11).

Even though \( \beta(H, 1) \) is not always achieved by the uniform distribution on \( E(H) \), it seems reasonable to expect that, given a finite set \( X \), the quantity \( \max \{ \beta(\mu; H, 1) : \text{supp } \mu \subseteq \binom{X}{2} \} \) is achieved by the uniform distribution on the edges of some graph. If true, this leads to the following question, which could be interesting in its own right.

**Question 5.4.** For a graph \( H \) on \( m \) edges with no isolated vertices, what bounds can be placed on the quantity
\[
\sup_G \frac{N(G, H)}{|E(G)|^m}.
\]

Certainly this quantity is at least \( 1/m^m \) and is at most \( 1/m! \) Additionally, we believe that the supremum can be replaced by a maximum unless \( H = K_{1,m} \) or \( H = mK_2 \) for some \( m \geq 2 \) where \( mK_2 \) is the matching on \( m \) edges.

Finally, we still do not even know if \( \beta(H, 1) \) is achieved for many graphs. Recall that \( \beta(K_{1,m}, 1) \) and \( \beta(mK_2, 1) \) are never achieved for \( m \geq 2 \), yet \( \beta(H, k) \) is achieved provided that \( k \cdot \delta(H) \geq 2 \) (Corollary 4.7).

**Conjecture 5.5.** If \( H \) is a graph with no isolated vertices and \( k \) is a positive integer, then \( \beta(H, k) \) is not achieved if and only if \( k = 1 \) and either \( H = K_{1,m} \) or \( H = mK_2 \) for some \( m \geq 2 \).
5.3  The reduction lemmas

Finally, we discuss the reduction lemmas in general. First, as mentioned in Section 2 after the statement of Lemma 2.5, we believe the following to be true:

**Conjecture 5.6.** Let $H$ be a graph on $m$ edges and let $k$ be a positive integer. If $k \cdot \delta(H) \geq 2$, then

$$N_{\leq c}(n, H[k]) = \frac{\beta(H, k)}{(k!)^m} \cdot n^{km} + o(n^{km}).$$

Beyond this, it is natural to wonder if there is an analogous reduction lemma for even paths and odd cycles. For example, the conjectured (asymptotic) extremal example for $N_P(n, P_{2m+2})$ is a modification of $C_m \{n/m\}$ wherein a path is placed among the interior vertices of each blown-up edge (see [2, Conjecture 2]); hence, we expect that the techniques used in this paper can be modified to tackle this question. It is probably necessary to use more about the planar structure of the host-graph to extend the reduction lemmas to this situation.

Interestingly, the reduction lemmas did not explicitly require the host-graph to have only linearly many edges. By playing with the error terms, one can extend each of the reduction lemmas to the collection of graphs $G$ which have no $K_{3,3}$ and $|E(H)| \leq C \cdot |V(H)|^{1+c}$ for each subgraph $H \subseteq G$, where $C > 0$ is any fixed constant and $c > 0$ depends on the particular situation at hand. We opted to avoid this more general situation for the sake of readability.

Furthermore, it was not crucial that the host-graph avoided copies of $K_{3,3}$. Indeed each of the reduction lemmas can be reworked to handle the case when the host-graph avoids copies of $K_{3,t}$ for some fixed $t \geq 3$. In particular, the reduction lemmas apply to the class graphs which can be embedded onto any surface of a fixed genus. However, the fact that one side of this forbidden biclique has size 3 appears to be necessary for each of our arguments. It seems unlikely that similar reduction lemmas could be pushed through if the host-graph only avoids copies of, say, $K_{4,4}$.

Finally, it is pertinent to point out that the techniques developed in this paper can likely be extended to prove stability results for $N_P(n, H)$ for various graphs $H$. This would, however, likely require a few new ideas.

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