PARISIAN RUIN WITH POWER-ASYMMETRIC VARIANCE NEAR THE OPTIMAL POINT WITH APPLICATION TO MANY-INPUTS PROPORTIONAL REINSURANCE

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Abstract: This paper investigates the Parisian ruin probability for processes with power-asymmetric behavior of the variance near the unique optimal point. We derive the exact asymptotics as the ruin boundary tends to infinity and extend the previous result [1] to the case when the length of Parisian interval is of Pickands scale. As a primary application, we extend the recent result [2] on the many inputs proportional reinsurance fractional Brownian motion risk model to the Parisian ruin.

Key Words: Parisian ruin, ruin probability, fractional Brownian motion

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Consider the following reinsurance scheme: \( d \) companies share premiums and one claim process proportionally. Suppose that the risk process \( R(t) \) is composed of a large number of i.i.d. sub-risk processes \( R^{(i)}(t) \) representing independent businesses, and let each \( R^{(i)} \) be driven by a fractional Brownian motion \( B_H(t) \). That is, let

\[
R_N(t) = \sum_{i=1}^{N} R^{(i)}(t), \quad \text{where} \quad R^{(i)}(t) = \alpha + \mu t - \sigma B_H^{(i)}(t),
\]

where \( B_H^{(i)}, i \geq 1 \) are independent fractional Brownian motions and \( \alpha, \mu, \sigma \in \mathbb{R}^d \). In a recent contribution [2], the authors derived the exact asymptotics of the simultaneous ruin probability

\[
P \{ \exists t \in [0,T]: R_N(t) < 0 \}, \quad N \to \infty
\]

in the case of \( d = 2 \). In the present work we shall concentrate on the simultaneous Parisian ruin probability

\[
(1) \quad \Pi(N) = P \{ \mathcal{P}_{[0,S],[0,T_N]}(R_N) < 0 \}, \quad \text{where} \quad \mathcal{P}_{E,F}(Z) = \max_{i=1,\ldots,d} \sup_{t \in E} \inf_{s \in F} Z_i(t+s).
\]

Parisian stopping times have been first introduced in relation to barrier options in mathematical finance, see [3], and since then attracted substantial interest. For the applications to actuarial risk theory, we refer to [4], where risk process is treated as a surplus process of an insurance company with initial capital \( u \).

In opposition to the well-studied classical ruin, when the failure is recognized at the moment of surplus hitting zero, the Parisian ruin is recognized only if the surplus process has spent a sufficient, pre-specified amount of time below zero. We refer to [5, 6, 7, 8, 9] and the references therein for analysis of Parisian ruin in the one-dimensional Lévy surplus model.

In the univariate Gaussian setup, Parisian ruin has been investigated in [10] for self-similar Gaussian processes and in [1] for general Gaussian processes, satisfying some standard assumptions (see [11]). Another interesting univariate case is of Parisian ruin over discrete sets. In [12], the authors have proved that for the Brownian motion and equidistant grid the asymptotics differs from the continuous one by some constant factor.

There are many possible extensions of the notion of the Parisian ruin to multivariate risk processes, such as simultaneous Parisian ruin, when all the components of a multivariate process should plunge below zero at the same time and remain there long enough for the ruin to be attested. This problem has recently been studied in [13] for the case when the risk process consists of two correlated Brownian motions. Another possible extension is...
the joint or non-simultaneous Parisian ruin, when ruin is attested if all the components experience Parisian ruin during some interval of time, but not necessarily at the same time. This problem has been studied in [14], also for the bivariate Brownian motion with \( \rho \in (-1, 1) \). A third possible extension would be the notion of “at least one” ruin, suggested in the classical ruin context in [2]: the ruin is declared if either of the processes has a Parisian ruin over time.

In this paper, we derive the exact asymptotics as \( u \to \infty \) of the one-dimensional Parisian ruin probability

\[
\Pi(u) = \mathbb{P}\{\mathcal{P}_{[-S,S],[0,T_u]}(Z) > u\}
\]

for \( T_u \to 0 \) at some specified rate and a class of Gaussian processes with correlation structure

\[
\text{corr}(Z(t), Z(s)) = 1 - D|t-s|^\alpha + o(|t-s|^\alpha)
\]

and a unique optimal point \( t_* = 0 \) of the variance with asymmetric behaviour near this point:

\[
\sigma(t) = 1 - A_\pm |t|^\gamma \left(1 + o(1)\right) \quad \text{as } t \to \pm 0,
\]

from which we further derive the exact asymptotics of the many-inputs Parisian ruin probability (1).

The asymptotic behaviour of \( \Pi(u) \) for such class of processes is of interest by itself. Similar problems have recently been studied in [10] and [1]. Our findings account for the previously discarded type of the Talagrand case with \( T > 0 \) (see Section 2) and discover the new type of asymptotics therein:

\[
\Pi(u) = e^{-\min\{A_-T^{\gamma_-}, A_+T^{\gamma_+}\}} \Psi(u),
\]

which rather surprisingly happens only if \( \gamma_+ < \gamma_- , \alpha \) and does not happen if \( \gamma_- < \gamma_+, \alpha \).

The paper is organized as follows. In Section 2 we present our main findings. Theorem 1 provides the exact asymptotics of \( \Pi(u) \) for the general Gaussian process with power-asymmetric behaviour of the variance near the optimal point and under some assumption on the speed \( T_u \to 0 \) convergence. It covers the previously unaccounted for case when the size of Parisian interval is equivalent to the Pickands scale of the process. Corollary 1 contains the exact asymptotics of the many-inputs Parisian ruin probability. The proof of Theorem 1 is presented in a separate Section 3. All known results and technical details are relegated to the Appendix.

2. Main results

In this section, we first explain how to rewrite many-inputs ruin probability in a form suitable for applying Theorem 1, then specify the assumptions under which the general theorem works and conclude with deriving the exact asymptotics of the many-inputs proportional reinsurance ruin probability 1.

Observe that by properties of Gaussian distribution

\[
R_N \overset{d}{=} \alpha N + \mu N t - \sigma \sqrt{N} B_H(t),
\]

we can rewrite the ruin probability (1) as

\[
\Pi_{[0,S]}(N) = \mathbb{P}\left\{\mathcal{P}_{[0,S],[0,T_N]}(Z) > \sqrt{N}\right\}, \quad \text{where} \quad Z(t) = \frac{B_H(t)}{D(t)}, \quad D(t) = \max_{i=1,...,d} (\alpha_i + \mu_i t).
\]

Next, we state a result concerning Parisian ruin probabilities

\[
\Pi(u) = \mathbb{P}\{\mathcal{P}_{E,[0,T_u]}(Z) > u\}
\]

for some large class of Gaussian processes and then apply it to \( \Pi_{[0,S]}(N) \), rewritten in the latter form.
2.1. Assumptions. Let $E$ be a compact subset of $\mathbb{R}$, containing point 0 in its interior, and let $Z(t), t \in E$ be a centered Gaussian process with a.s. continuous sample paths satisfying the following two assumptions:

**Assumption A1:** The standard variance function $\sigma_Z$ of the Gaussian process $Z$ attains its maximum on $E$ at the unique point $\hat{r} = 0$. Further, there exist positive constants $\gamma_\pm$ and $A_\pm$ such that

$$
\sigma_Z(t) = 1 - A_\pm |t|^\gamma_\pm + o(|t|^\gamma_\pm) \quad \text{as} \quad t \to \pm 0.
$$

**Assumption A2:** There exists some positive constant $\alpha \in (0, 2]$ such that

$$
corr(Z(t), Z(s)) = 1 - D |t-s|^\alpha + o(|t-s|^\alpha) \quad \text{as} \quad t, s \to 0.
$$

**Remark 1.** Note that it follows from A2 that there exists such $\delta > 0$ that

$$
\mathbb{E} \left( \left( Z(t) - Z(s) \right)^2 \right) < C|t-s|^\alpha
$$

for all $t, s < \delta$.

As it turns out, there are two numbers

$$
\nu = \min\{\alpha, \gamma_-, \gamma_+\} \quad \text{and} \quad \gamma = \max\{\gamma_-, \gamma_+\}
$$

which determine the type of the asymptotics, but before proceeding to that, we also need the following assumption on the convergence rate of $T_u \to 0$:

**Assumption B:** $T_u = T u^{-2/\nu}$ for some $T \in [0, \infty)$.

Next, we introduce two well-known and important constants in the theory of Gaussian extremes, see [15, 10, 1]. Define for $T \geq 0$ and $\alpha \in (0, 2]$ the generalized Pickands and Piterbarg constants

$$
\mathcal{H}_\alpha^P(T) = \lim_{\lambda \to \infty} \mathcal{H}_\alpha^P(\lambda, T) \quad \text{and} \quad \mathcal{H}_\alpha^P(T) = \lim_{\lambda \to \infty} \mathcal{H}_\alpha^P(T),
$$

where

$$
\mathcal{H}_\alpha^P(T) = \mathbb{E} \exp \left( \sup_{t \in [-\lambda, \lambda]} \inf_{s \in [0, T]} \left( \sqrt{2} B_{\alpha/2}(t + s) - |t+s|^{\alpha} - h(t+s) \right) \right)
$$

for such continuous $h$ that the limit exists. We are in a position to formulate our main theorem.

**Theorem 1.** Let $(Z(t))_{t \geq 0}$ be a centered Gaussian process satisfying assumptions A1 and A2, and let $T_u$ be a positive measurable function of $u$ satisfying assumption (B). Then

In the Pickands case $\nu = \alpha \neq \gamma$: we have

$$
\Pi(u) = C_S \mathcal{H}_\alpha^P(D^{1/\alpha} T) u^{2/\nu - 2/\gamma} \Psi(u) \left( 1 + o(1) \right),
$$

with

$$
C_S = A_+^{-1/\gamma_+} D^{1/\alpha} \Gamma \left( \frac{1}{\gamma_+} + 1 \right) 1_{\gamma_+ = \gamma_+} + A_-^{-1/\gamma_-} D^{1/\alpha} \Gamma \left( \frac{1}{\gamma_-} + 1 \right) 1_{\gamma_+ = \gamma_-}.
$$

In the Piterbarg case $\nu = \gamma = \alpha$: we have

$$
\Pi(u) = \mathcal{H}_\alpha^P(D^{1/\alpha} T) \Psi(u) \left( 1 + o(1) \right),
$$

where $h(t) = A_- D^{-1/\alpha} |t|^{\gamma_-} 1_{t \leq 0} + A_+ D^{-1/\alpha} |t|^{\gamma_+} 1_{t \geq 0}$.

In the Talagrand-1 case $\gamma = \nu \neq \alpha$:

$$
\Pi(u) = C \Psi(u) \left( 1 + o(1) \right), \quad C = \begin{cases} 1, & \gamma_+ \geq \gamma_-, \\ \exp(- \min\{A_- T^{\gamma_-}, A_+ T^{\gamma_+}\}), & \gamma_+ < \gamma_- \end{cases}
$$
Now we proceed with our initial problem, to which end we first have to study the behaviour of $\var B_H(t)/D(t)$. Note that the derivative $\sigma'$ of the variance function

$$\sigma(t) = \frac{t^H}{D(t)}, \quad D(t) = \max_{i=1,\ldots,d} (\alpha_i + \mu_i t)$$

changes its sign exactly once, since

$$\sigma'(t) = \frac{t^{H-1}}{D^2(t)} G(t), \quad G(t) = HD(t) - tD'(t),$$

where $G(t)$ is monotone and decreasing, $G(0) > 0$ and $G(t) \to -\infty$ as $t \to \infty$. Since $\sigma'$ must change sign (possibly in a discontinuous manner) at the optimal point $t_*$ of $\sigma$, we have thus proved that such point is unique. Let us assume that $t_* \in (0, S)$ or $S = \infty$, since to account for the boundary maxima case an approach slightly different to ours is needed.

The maximum can either be caused by intersection of some two lines $l_\pm(t) = \alpha_\pm + \mu_\pm t$ from $D(t)$, that is,

$$t_* = \frac{\alpha_+ - \alpha_-}{\mu_- - \mu_+}$$

in which case $\sigma'$ is discontinuous at $t_*$, or by a point away from the lines’ intersections, satisfying $\sigma'(t_*) = 0$, that is,

$$t_* = \frac{H\alpha}{\mu(1-H)}.$$

Finally, these two types of maxima can coincide, giving rise to a power-asymmetric behavior near $t_*$

$$\frac{\sigma(t)}{\sigma(t_*)} = 1 - A_\pm |t - t_*|^\gamma_\pm \left(1 + o(1)\right)$$

with $\gamma_\pm \in \{1, 2\}$. Precisely, if

$$-C_\pm^{(1)} = \frac{\sigma''(t_*)}{\sigma(t_*)} = \frac{H}{t_*} - \frac{\mu_+}{\alpha_+ + \mu_+ t_*} < 0,$$

then $\gamma_\pm = 1$ and $A_\pm = C_\pm^{(1)}$. If on the other hand $C_\pm^{(1)} = 0$, then under the following non-degeneracy assumption

$$-C_\pm^{(2)} = \frac{\sigma''(t_*)}{\sigma(t_*)} = -\frac{H}{t_*^2} + \frac{\mu_+^2}{(\alpha_+ + \mu_+ t_*)^2} < 0$$

we have $\gamma_\pm = 2$ and $A_\pm = C_\pm^{(2)}$.

Now we may introduce the natural asymptotic parameter

$$\hat{N} = \frac{\sqrt{N}}{\sigma_Z(t_*)}$$

and formulate the corollary on the MIPR asymptotics.

**Corollary 1.** Let $T_N$ satisfy the condition

$$\lim_{N \to \infty} T_N \hat{N}^{1/H} = T \in [0, \infty).$$

- If either $\gamma_+$ or $\gamma_-$ equals 2, then

  $$\Pi(N) = \sqrt{\pi} \frac{1}{\sqrt{A}} \left(1_{\gamma_+ = 2} + 1_{\gamma_- = 2}\right) \frac{\mathcal{H}_{2H}^T \left(T/2^{1/2H} t_*\right)}{2^{1/2H} t_*} \hat{N} \cdot \Psi(\hat{N}) \left(1 + o(1)\right).$$

- If both $\gamma_\pm = 1 > 2H$, then

  $$\Pi(N) = \left(\frac{1}{A_-} + \frac{1}{A_+}\right) \frac{\mathcal{H}_{2H}^T \left(T/2^{1/2H} t_*\right)}{2^{1/2H} t_*} \hat{N} \cdot \Psi(\hat{N}) \left(1 + o(1)\right).$$

- If both $\gamma_\pm = 1 = 2H$, then

  $$\Pi(N) = \mathcal{H}_{2H,h}^T(T/2^{1/2H} t_*) \Psi(\hat{N}) \left(1 + o(1)\right).$$

- If $\gamma_\pm = 1 < 2H$, then

  $$\Pi(N) = \Psi(\hat{N}) \left(1 + o(1)\right).$$
3. Proof of Theorem 1

This section is dedicated to the proof of Theorem 1.

3.1. Large vicinities. Looking ahead, we shall prove that only a small vicinity of the optimal point contributes to the first order asymptotics, and to evaluate its contribution we shall divide this small vicinity into even smaller parts of some size $q(u)$ (referred to as the Pickands scale of the process $Z$, determined only by the covariance structure of $Z$), on which the uniform local Pickands lemma may be applied. It follows directly from the Piterbarg inequality and the following obvious but important property of the Parisian functional:

$$P_{E,F}(f) = \sup_{t \in E} \inf_{s \in F} f(t+s) \leq \sup_{t \in E} f(t)$$

that

$$\limsup_{u \to \infty} \frac{\Pi_{E\setminus[-\delta_-(u),\delta_+(u)]}(u)}{u^\gamma \Psi(u)} = 0$$

for $\delta_\pm(u) = u^{-2/\gamma_{\pm}} \ln^{2/\gamma_{\pm}} u$ and all $\kappa > 0$. Since we intend to prove that $\Pi(u) \sim Hu^\gamma \Psi(u)$ for some $H, \kappa > 0$, from this inequality will follow that $\Pi(u) \sim \Pi_{[-\delta_-(u),\delta_+(u)]}(u)$ as $u \to \infty$. We can narrow the vicinity even further by once again using (3) and applying Lemma 4.3 (Lemma 5.4 from [16])

$$\limsup_{u \to \infty} \frac{\Pi_{[-\delta_-(u),\delta_+(u)]\setminus[-\delta u^{-2/\gamma_{\pm}},\delta u^{-2/\gamma_{\pm}}]}(u)}{u^\gamma \Psi(u)} \leq Ce^{-c\gamma}$$

where $\gamma = \max\{\gamma_-, \gamma_+\}$. Due to this inequality, we may concentrate on the exact asymptotics of

$$\Pi_{\Delta(u, \Lambda)}(u) \quad \text{where} \quad \Delta(u, \Lambda) = \Delta^+(u, \Lambda) \cup \Delta^-(u, \Lambda), \quad \Delta^\pm(u, \Lambda) = \pm[0, \Lambda u^{-2/\nu}]$$

and then let $\Lambda \to \infty$.

3.2. Pickands intervals. Next, we introduce the left and right Pickands intervals $\Delta^\pm_k(u, \lambda)$ with some additional parameter $\lambda > 0$

$$\Delta^\pm_k(u, \lambda) = \pm \lambda \nu q(u)[k, k+1], \quad \text{where} \quad \nu = \min\{\alpha, \gamma_+, \gamma_-\}, \quad q(u) = u^{-2/\nu},$$

and the number of those fitting into the large vicinity $\Delta^\pm(u, \Lambda)$:

$$N^\pm(u, \lambda, \Lambda) = \left\lfloor \frac{\Delta^\pm(u, \lambda)}{\lambda u^{-2/\nu}} \right\rfloor = \left\lfloor \frac{u^{\zeta_\pm}}{\lambda} \right\rfloor, \quad \zeta_\pm = \frac{2}{\nu} - \frac{2}{\gamma_\pm} = \max\left\{\frac{2}{\alpha} - \frac{2}{\gamma_+}, \frac{2}{\gamma_-} - \frac{2}{\gamma_+}, 0\right\} \geq 0.$$

We shall split the proof in four cases:

**Pickands case:** $\gamma \neq \nu = \alpha$

**Piterbarg case:** $\gamma = \nu = \alpha$

**Talagrand-1 case:** $\gamma = \nu \neq \alpha$

**Talagrand-2 case:** $\gamma \neq \nu \neq \alpha$

In the Pickands case at least one of $\zeta_\pm$ is nonzero, therefore $N^\pm$ grows as $u^{\zeta_\pm}$. In both Piterbarg and Talagrand-1 cases $\zeta_+ = \zeta_- = 0$, hence $u \to N^\pm$ is constant and we can set $\lambda = \Lambda$, in which case $N^\pm = 1$ – the zeroth Pickands interval coincides with the informative vicinity. The Talagrand-2 case is to be treated separately.

3.3. Pickands case. To deal with the Pickands case, we employ the so-called double sum method, which is based on the Bonferroni inequality

$$\Sigma_1(u, \lambda, \Lambda) - \Sigma_2(u, \lambda, \Lambda) \leq \Pi_{\Delta(u, \Lambda)}(u) \leq \Sigma_1(u, \lambda, \Lambda),$$

where

$$\Sigma_1(u, \lambda, \Lambda) = \sum_{k=1}^{N^+(u,\lambda,\Lambda)} \Pi_{\Delta^+(k,\lambda)}(u) + \sum_{k=1}^{N^-(u,\lambda,\Lambda)} \Pi_{\Delta^-(k,\lambda)}(u) + \Pi_{\Delta_0(u,\lambda)}(u),$$

and

$$\Sigma_2(u, \lambda, \Lambda) = \sum_{k=1}^{N^+(u,\lambda,\Lambda)} \Pi_{\Delta^+(k,\lambda)}(u) + \sum_{k=1}^{N^-(u,\lambda,\Lambda)} \Pi_{\Delta^-(k,\lambda)}(u) + \Pi_{\Delta_0(u,\lambda)}(u).$$
and $\Sigma'_1 = \Sigma^{\uparrow}_1 + \Sigma^{\downarrow}_1 - \Sigma_0$ denote the same $\Sigma_1$ but with $N^\pm + 1$ instead of $N^\pm$ (so that the collection of Pickands intervals indeed cover $\Delta(u, \Lambda)$), and finally

$$\Sigma_2(u, \lambda, \Lambda) = \sum_{\kappa, \kappa' \in \{\uparrow, \downarrow\}, 1 \leq k \leq N^\kappa(u, \Lambda, \Lambda), 1 \leq k' \leq N^{\kappa'}(u, \Lambda, \Lambda), (\kappa, k) \neq (\kappa', k')} P \left\{ \mathcal{P}_{\Delta^{\kappa}_v(u, \Lambda, \Lambda), 0, T_u} | Z > u, \mathcal{P}_{\Delta^{\kappa'}_w(u, \Lambda, \Lambda), 0, T_u} | Z > u \right\} .$$

Since the Parisian functional is bounded by sup functional, we can reduce the double sum estimate to the classical (sup) case

$$\Sigma_2(u, \lambda, \Lambda) \leq \sum_{\kappa, \kappa' \in \{\uparrow, \downarrow\}, 1 \leq k \leq N^\kappa(u, \Lambda, \Lambda), 1 \leq k' \leq N^{\kappa'}(u, \Lambda, \Lambda), (\kappa, k) \neq (\kappa', k')} \mathbb{P} \left\{ \sup_{\Delta_v^\kappa(u, \Lambda, \Lambda)} Z(t) > u, \sup_{\Delta_w^{\kappa'}(u, \Lambda, \Lambda)} Z(t) > u \right\} ,$$

and using similar arguments to those in [17] obtain

$$\lim_{\lambda \to \infty} \lim_{u \to \infty} \limsup_{\Lambda \to \infty} \frac{\Sigma_2(u, \lambda, \Lambda)}{u^k \Psi(u)} = 0 \quad \text{for all } k > 0. \tag{4}$$

We shall prove that if there exist two constants $C_\pm > 0$ such that

$$\lim_{\lambda \to \infty} \lim_{u \to \infty} \lim_{\Lambda \to \infty} \frac{\Sigma_\pm(u, \lambda, \Lambda)}{u^k \Psi(u)} = C_\pm ,$$

which together with the double sum estimate above and

$$\lim_{\lambda \to \infty} \lim_{u \to \infty} \frac{\Sigma_0(u, \lambda)}{\Psi(u)} = H_0 \in (0, \infty), \tag{5}$$

yields

$$\lim_{\lambda \to \infty} \lim_{u \to \infty} \frac{\Pi_{\Delta(u, \Lambda)}(u)}{u^k \Psi(u)} = C,$$

where $\zeta = \max\{\zeta_+, \zeta_-\} > 0$ and $C = C_+ 1_{\zeta = \zeta_+} + C_- 1_{\zeta = \zeta_-}$. Finally, we obtain

$$\Pi(u) \sim C u^\zeta \Psi(u).$$

3.4. Piterbarg and Talagrand-1 cases. As noted before, in both Piterbarg and Talagrand-1 cases $u \mapsto N^\pm(u, \lambda, \Lambda)$ is constant. We may set $\lambda = \Lambda$, which makes this constant equal one, and therefore

$$\Pi_{\Delta(u, \Lambda)} = \Sigma_0(u, \Lambda).$$

By (5), we have

$$\lim_{\lambda \to \infty} \lim_{u \to \infty} \frac{\Sigma_0(u, \lambda)}{\Psi(u)} = H_0$$

for some $H_0 > 0$, which together with Lemma 4.3 yields $\Pi(u) \sim H_0 \Psi(u)$. In the Talagrand case $\nu \neq \alpha$ we shall see that $H_0 = 1$, and therefore $\Pi(u) \sim \Psi(u)$.

3.5. Talagrand-2 case. In the Talagrand-2 case we shall directly (that is, without appealing to Pickands intervals) prove that there exists positive and finite limit

$$\lim_{u \to \infty} \frac{\Pi_{\Delta(u, \Lambda)}(u)}{\Psi(u)} = \begin{cases} 1, & \gamma_- < \gamma_+, \\ \exp(- \min\{A_- T^{\gamma_-}, A_+ T^{\gamma_+}\}), & \gamma_+ < \gamma_- , \end{cases}$$

which ceases to depend on $\lambda$ as long as $\lambda > T$. 


3.6. Asymptotics of \(\Sigma_0(u, \lambda)\) and the Piterbarg and Talagrand-1 cases. Denote

\[ q(u) = u^{-2/\nu}, \quad \nu = \min\{\alpha, \gamma_+, \gamma_-\}. \]

Note that

\[ u^{2/\alpha}q(u) \xrightarrow[\nu=\alpha]{} 1 \quad \text{and} \quad u^{2/\gamma\pm}q(u) \xrightarrow[\nu=\gamma\pm]{} 1. \]

To apply the uniform local Pickands Lemma 2, let us rewrite the probability \(\Sigma_0(u, \lambda)\) in terms of a standardized process as follows:

\[ \Sigma_0(u, \lambda) = \mathbb{P}\left\{ \mathcal{P}_{\Delta_0^+(u, \lambda) \cup \Delta_0^-(u, \lambda)}(Z) > u \right\} = \mathbb{P}\left\{ \mathcal{P}'(\xi_{u, 0}) > u \right\}, \]

where we have defined \(\mathcal{P}' = \mathcal{P}_{[\lambda, \lambda]}[0, 1]\) and the family \(\{\xi_{u, 0} : u > 0\}\) of centered Gaussian processes by

\[ \xi_{u, 0}(t, s) = \frac{Z(q(u)t + T_us)}{1 + g(q(u)t + T_us)}, \quad 1 + g(t) = \frac{1}{\sigma_Z(t)}, \]

Note that in contrast to \(\xi_{u,k}, k > 0\) (see below), the process \(\xi_{u,0}\) is defined for \(t \in [-\lambda, \lambda]\), not \([0, \lambda]\). This is neither a coincidence, nor a technical decision: the two adjacent intervals near the optimal point cannot be treated separately as it will be evident from the result.

By (2) and the definition of \(g\) we have

\[ u^2g(q(u)t + T_us) \rightarrow h(t + Ts) = h^+(t + Ts)1_{\nu=\gamma_+} + h^-(t + Ts)1_{\nu=\gamma_-}. \]

where

\[ h^\pm(\mu) = A_{\pm}|\mu|^\gamma\pm 1_{\pm\mu>0}. \]

By assumption \(A2\) we have

\[ u^2\mathbb{E}\left\{ |Z(q(u)t + T_us) - Z(q(u)t' + T_us')|^\alpha \right\} \rightarrow 2D1_{\nu=\alpha}\left|(t - t') + (s - s')\right|^\alpha, \]

which means that the condition \(C2\)

\[ \eta(t, s) = B_{\alpha/2}(t + s)1_{\nu=\alpha}, \quad (t, s) \in [-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda] \times [0, D^{1/\alpha}T] \]

for \(T \geq 0\). By the uniform local Pickands Lemma 2 (condition (C3) is obviously satisfied) we have

\[ \lim_{u \rightarrow \infty} \frac{\Sigma_0(u, \lambda)}{\Psi(u)} = \mathcal{H}^{\nu}_{\eta, \lambda}\left([-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda] \times [0, D^{1/\alpha}T]\right). \]

where \(\mathcal{H}^{\nu}_{\eta, \lambda}(E) = \mathbb{E}\left\{ e^{\mathcal{P}(a^h)} \right\}, \mathcal{P}_0 = \mathcal{P}_{[-\lambda, \lambda],[0,T]}\) and

(6) \(\eta^h(t, s) = \left(\sqrt{2}B_{\alpha/2}(t + s) - |t + s|^\alpha\right)1_{\nu=\alpha} - A_\pm D_{\gamma+/\alpha} |t + s|^{\gamma_\pm 1_{t+s\leq 0, \nu=\gamma_-}} - A_\pm D_{\gamma+/\alpha} |t + s|^{\gamma_\pm 1_{t+s\geq 0, \nu=\gamma_+}}. \)

3.6.1. Piterbarg case. To prove the main theorem in the Piterbarg case \(\gamma = \alpha = \nu\), that is when \(\eta^h\) has all the terms

\[ \eta^h(t, s) = \left(\sqrt{2}B_{\alpha/2}(t + s) - |t + s|^\alpha\right) - A_- D_{\gamma-/\alpha} |t + s|^{\gamma_- 1_{t+s\leq 0, \nu=\gamma_-}} - A_+ D_{\gamma+/\alpha} |t + s|^{\gamma_+ 1_{t+s\geq 0}}, \]

it remains to apply the standard result on the existence of Piterbarg constants to see that

\[ \lim_{\lambda \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{\Sigma_0(u, \lambda)}{\Psi(u)} = \lim_{\lambda \rightarrow \infty} \mathcal{H}^{\nu}_{\eta, \lambda}\left([-D^{1/\alpha}\lambda, D^{1/\alpha}\lambda] \times [0, D^{1/\alpha}T]\right) = \mathcal{H}^{\nu}_{\eta, \lambda}(D^{1/\alpha}T) \]

exists and is finite. This ends the proof of the main theorem in the Piterbarg case.
3.6.2. **Talagrand-1 case.** In the Talagrand-1 case \( \gamma \neq \nu \) and, therefore, the random part disappears from (6), whereas all non-random terms are present:

\[
\eta^h(t, s) = -A_+ D^{-\gamma/\alpha} |t + s|^{\gamma + 1} 1_{t + s \geq 0} - A_- D^{-\gamma - \gamma/\alpha} |t + s|^{\gamma - 1} 1_{t + s \leq 0}
\]

It remains to calculate \( \mathcal{P}_0(\eta^h) \) explicitly: if \( \lambda > T \), we have

\[
\mathcal{P}_0(\eta^h) = \inf_{t \in [-\lambda, \lambda]} \left( -A_+ |t + s|^{\gamma + 1} 1_{t + s \geq 0} - A_- |t + s|^{\gamma - 1} 1_{t + s \leq 0} \right) = \max \left\{ \sup_{t \in [-\lambda, \lambda]} \inf_{s \in [0, T]} (-A_- |t|^{\gamma - 1}) \right\} = \max \left\{ \sup_{t \in [-\lambda, \lambda]} (-A_- |t + T|^{\gamma - 1}) \right\} = \max \{-A_- T^{\gamma - 1}, 0\} = 0.
\]

Therefore, by lemma above in the Talagrand case

\[
\mathcal{H}^{\mathcal{P}_0}_h([-D^{1/\alpha} \lambda, D^{1/\alpha} \lambda] \times [0, D^{1/\alpha} T]) = 1
\]

for all \( \lambda > T \). Thus, we have proved that

\[
\lim_{\lambda \to \infty} \lim_{u \to \infty} \frac{\Sigma_0(u, \lambda)}{\Psi(u)} = 1.
\]

This ends the proof of the main theorem in the Talagrand case.

3.7. **Talagrand-2 case.** To apply the uniform local Pickands Lemma 2, let us rewrite the probability \( \Pi_{\Delta(u, \lambda)}(u) \) in terms of a standardized process as follows. First, observe that the trivial equality

\[
t + s = (t + s)1_{t + s \geq 0} + (t + s)1_{t + s \leq 0}, \quad (t, s) \in [-\lambda u^{-2/\gamma -}, \lambda u^{-2/\gamma +}] \times [0, T_u]
\]

may be rewritten as

\[
t + s = q(u, t', s') = q_+(u, t', s') + q_-(u, t', s'), \quad (t', s') \in [-\lambda, \lambda] \times [0, 1].
\]

where

\[
q_{\pm}(u, t', s') = \left(u^{-2/\gamma \pm t'} + T_u s'\right)1_{\pm(u^{-2/\gamma \pm t'} + T_u s') \geq 0}.
\]

Using this reparametrization, we rewrite

\[
\Pi_{\Delta(u, \lambda)}(u) = \mathbb{P} \left\{ \mathcal{P}_{[-\lambda u^{-2/\gamma -}, \lambda u^{-2/\gamma +}]}(Z) > u \right\} = \mathbb{P} \left\{ \mathcal{P}'(\xi_{u, 0}) > u \right\},
\]

where we have defined \( \mathcal{P}' = \mathcal{P}_{[-\lambda, \lambda], [0, 1]} \) and the family \( \{\xi_{u, 0} : u > 0\} \) of centered Gaussian processes by

\[
\xi_{u, 0}(t', s') = \frac{Z(q(u, t', s'))}{1 + g(q(u, t', s'))}, \quad 1 + g(t) = \frac{1}{\sigma_Z(t)}.
\]

Note that

\[
u^{2/\gamma \pm} q_{\pm}(u, t', s') \to t' + 1_{\nu = \gamma \pm} T_s 1_{\pm(t' + 1_{\nu = \gamma \pm} T_s) \geq 0}
\]

uniformly in \((t', s')\). By (2) and the definition of \( g \) we have

\[
u^{2} g(q(u, t', s')) \sim A_+ |u^{2/\gamma +} q_+(u, t', s')|^{\gamma +} + A_- |u^{2/\gamma -} q_-(u, t', s')|^{\gamma -} \sim
\]

\[
A_+ |t' + 1_{\nu = \gamma \pm} T_s|^{\gamma +} 1_{t' + 1_{\nu = \gamma \pm} T_s \geq 0} + A_- |t' + 1_{\nu = \gamma \pm} T_s|^{\gamma -} 1_{t' + 1_{\nu = \gamma \pm} T_s \geq 0}.
\]

By assumption A2 we have

\[
u^{2} \mathbb{E} \left\{ |Z(q(u, t'_1, s'_1)) - Z(q(u, t'_2, s'_2))|^2 \right\} \to 0,
\]
which means that the condition $C2 \eta(t, s) = 0$.

By the uniform local Pickands Lemma 2 (condition $C3$ is obviously satisfied) we have

$$
\lim_{u \to \infty} \frac{\Pi_{\Delta(u, \lambda)}(u)}{\Psi(u)} = H_{0,h}^P([-\lambda, \lambda] \times [0, T]),
$$

where $H_{0,h}^P(E) = e^{P_h(-h)}$, $P_0 = P_{[-\lambda, \lambda], [0, T]}$ and

$$
h(t, s) = A_+ |t + 1, t + 1 + \gamma s|^{-1} t + s \leq 0 + A_+ |t + 1, t + 1, t + 1 + \gamma s|^{-1} t + s \geq 0.
$$

Since we are looking at a case where $\gamma \neq \nu$, either $1, \gamma$ or $1, \gamma$ is zero. Suppose, $\nu = \gamma_+ \neq \gamma_-$. Then

$$
h(t, s) = -A_- |t + s|^{-1} t + s \leq 0 - A_+ |t|^{-1} t \geq 0.
$$

therefore, we have

$$
\mathcal{P}(h) = \max \left\{ \sup_{t \in [-\lambda, 0]} \inf_{s \in [0, t]} (-a_- |t + s|^{-1} t + s \leq 0), \sup_{t \in [0, \lambda]} \inf_{s \in [0, t]} (-a_+ |t|^{-1} t \geq 0) \right\} =
$$

$$
= \max \left\{ \sup_{t \in [-\lambda, 0]} \inf_{\mu \in [t, t + T]} (-\mu^{-1} \mu_{\gamma \leq 0}), 0 \right\} = 0
$$

since the first term is non-positive.

If, on the other hand, $\nu = \gamma_+ \neq \gamma_-$, we have

$$
h(t, s) = -a_- |t|^{-1} t \leq 0 - a_+ |t + s|^{-1} t \geq 0
$$

and therefore for $\lambda > T$ we have

$$
\mathcal{P}(h) = \max \left\{ \sup_{t \in [-\lambda, 0]} \inf_{s \in [0, t]} (-A_- |t|^{-1} - A_+ |t + s|^{-1} t \geq 0), \sup_{t \in [0, \lambda]} \inf_{s \in [0, t]} (-A_+ |t + s|^{-1} t \geq 0) \right\} =
$$

$$
= \max \left\{ \sup_{t \in [-\lambda, 0]} \inf_{\mu \in [t, t + T]} (-A_- |t|^{-1} - A_+ |t + s|^{-1} \mu \geq 0), -A_+ |t|^{-1} \right\} =
$$

$$
= \max \left\{ \sup_{t \in [-\lambda, 0]} \inf_{\mu \in [0, \max(T, 0)]} (-A_- |t|^{-1} - A_+ |t + s|^{-1} \mu), -A_+ |t|^{-1} \right\} =
$$

$$
= \max \left\{ \sup_{t \in [-\lambda, 0]} (A_- |t - t + s|^{-1} - A_+ \max(T, 0)|^{-1} t + s), -A_+ |t|^{-1} \right\} =
$$

$$
= \max \left\{ \sup_{t \in [-\lambda, -T]} (A_- |t|^{-1} - A_+ |t + T|^{-1} t + s), -A_+ |t|^{-1} \right\} =
$$

$$
= \max \left\{ \sup_{t \in [-T, 0]} (A_- |t|^{-1} - A_+ |t + T|^{-1} t + s), -A_+ |t|^{-1} \right\} =
$$

$$
= \max \left\{ \sup_{\mu \in [0, \max(T, 0)]} (A_- |t - t + s|^{-1} - A_+ |T - \mu|^{-1} t + s), -A_+ |t|^{-1} \right\} =
$$

$$
= - \min \{A_- |t - t + s|^{-1} - A_+ |T - \mu|^{-1} t + s, -A_+ |t|^{-1} \}
$$

We have thus proved that

$$
H_{0,h}^P([-\lambda, \lambda] \times [0, T]) = e^{P_h(-h)} = \begin{cases} 1, & \gamma_- < \gamma_+ \\ \exp(-\min \{A_- |T - \mu|^{-1} t + s, -A_+ |T - \mu|^{-1} t + s\}), & \gamma_+ < \gamma_- \end{cases}
$$

for all $\lambda > T$. 
3.8. Pickands case. Now we proceed to the Pickands case. To find the aforementioned asymptotics of $\Sigma^\pm_i(u, \lambda, \Lambda)$ we shall first find the uniform in $k \in \{1, \ldots, N^\pm(u, \lambda, \Lambda)\}$ asymptotics of each summand $\Pi_{\Delta^\pm(u, \lambda, \Lambda)}(u)$ and then sum them up. To this end, let us rewrite the probability in the form required for the uniform local Pickands Lemma 2

$$\Pi_{\Delta^\pm(u, \lambda, \Lambda)}(u) = \mathbb{P}\left\{ \mathcal{P}_{\Delta^\pm(u, \lambda, \Lambda),[0, T]}(Z) > u \right\} = \mathbb{P}\left\{ \mathcal{P}''(\xi_{u,k}^\pm) > u \right\},$$

where we have defined $\mathcal{P}'' = \mathcal{P}_{[0, \lambda], [0, 1]}$ and the family

$$\{\xi_{u,k}^\pm: \kappa \in \{+, -\}, 1 \leq k \leq N^\pm(u, \lambda, \Lambda), u > 0\}$$

of centered Gaussian processes by

$$\xi_{u,k}^\pm(t, s) = \frac{\tilde{Z}_{u,k}^\pm(t, s)}{1 + h_{u,k}^\pm(t, s)} = 1 + h_{u,k}^\pm(t, s) = \frac{1}{\sigma Z(\pm q(u)(\lambda k + t) + T_u)}$$

and

$$\tilde{Z}_{u,k}^\pm(t, s) = \frac{Z(\pm q(u)(\lambda k + t) + T_u)}{\sigma Z(\pm q(u)(\lambda k + t) + T_u)},$$

$t \in [0, \lambda], s \in [0, 1]$. Note that $\tilde{Z}_{u,k}^\pm$ is a centered Gaussian random field with unit variance and continuous paths. Besides, $h_{u,k}^\pm \in C_0([0, \lambda] \times [0, T])$, that is, it is a continuous function on $[0, \lambda] \times [0, T]$, such that $h_{u,k}^\pm(0, 0) = 0$. There is, however, a pitfall in trying to apply uLP directly to $\xi_{u,k}^\pm$. Even though $u^2 h_{u,k}^\pm$ may have a limit $h$ for each $k$, it is never uniform. In other words, the condition (C1)

$$\lim_{u \to \infty} \sup_{k \in K_u, (t, s) \in [0, \lambda] \times [0, T]} |u^2 h_{u,k}(t, s) - h(t, s)| = 0$$

is not satisfied. To get around this inconvenience, we shall coarsen the inequality describing the event by taking $1 + h_{u,k}^\pm$ out of $\mathcal{P}''$

$$\mathbb{P}\left\{ \mathcal{P}''(\tilde{Z}_{u,k})/U(1 + h_{u,k}^\pm) > u \right\} \leq \Pi_{\Delta^\pm(u, \lambda)}(u) = \mathbb{P}\left\{ \mathcal{P}''(\xi_{u,k}^\pm) > u \right\} \leq \mathbb{P}\left\{ \mathcal{P}''(\tilde{Z}_{u,k})/L(1 + h_{u,k}^\pm) > u \right\},$$

where

$$U(f) = \sup_{t \in [0, \lambda]} \sup_{s \in [0, T]} f(t + s) \quad \text{and} \quad L(f) = \inf_{t \in [0, \lambda]} \inf_{s \in [0, T]} f(t + s).$$

Let us rewrite it in a handier fashion as

$$\mathbb{P}\left\{ \mathcal{P}''(\tilde{Z}_{u,k}) > u \left(1 + U(h_{u,k}^\pm)\right) \right\} \leq \Pi_{\Delta^\pm(u, \lambda)} = \mathbb{P}\left\{ \mathcal{P}''(\xi_{u,k}^\pm) > u \right\} \leq \mathbb{P}\left\{ \mathcal{P}''(\tilde{Z}_{u,k}) > u \left(1 + L(h_{u,k}^\pm)\right) \right\}.$$

Using the assumption that $A_{\pm} > 0$, we get

$$L(h_{u,k}^+) = A_+ |\lambda q(u)k|^{\gamma_+}, \quad U(h_{u,k}^+) = A_+ |\lambda q(u)(k + 1) + T_u|^{\gamma_+}$$

and

$$L(h_{u,k}^-) = A_- |\lambda q(u)k + T_u|^{\gamma_-}, \quad U(h_{u,k}^-) = A_- |\lambda q(u)(k + 1)|^{\gamma_-}.$$

All four bounds can be rewritten as follows:

$$L(h_{u,k}^\pm) = A_\pm |\lambda q(u)k - q(u)m_{\pm}(u)|^{\gamma_\pm}, \quad U(h_{u,k}^\pm) = A_\pm |\lambda q(u)k + q(u)(\lambda + m_{\pm}(u))|^{\gamma_\pm},$$

where $m_+(u) = 0$ and $m_-(u) = T_u/q(u) \to T$ as $u \to \infty$. It is important for us that $m_{\pm}$ do not depend on $k$ and have finite limits as $u \to \infty$.

In order to apply the uniform local Pickands Lemma 2 to the upper bound, we set

$$g_{u,k}(\lambda) = u \left(1 + A_\pm |\lambda q(u)k - q(u)m_{\pm}(u)|^{\gamma_\pm}\right)$$

and note that the condition (C2) remains valid with $g_{u,k}$ instead of $u$. It now follows directly from the uniform local Pickands Lemma 2 that with $\mathcal{P} = \mathcal{P}_{[0, \lambda], [0, T]}$ we have

$$\mathbb{P}\left\{ \mathcal{P}''(\tilde{Z}_{u,k}) > g_{u,k}(\lambda) \right\} = \mathcal{H}_{2H}^{\mathcal{P}} \left( D^{1/\alpha} \lambda, D^{1/\alpha}T \right) \Psi(u) \exp \left( -A_\pm |\lambda u^{-\gamma_\pm k} - u^{-\gamma_\pm m_{\pm}(u)}|^{\gamma_\pm} \right) (1 + o(1)),$$
where \( o(1) \) is uniform in \( k \in \{1, \ldots, N^\pm(u, \lambda, \Lambda)\} \). Thus,

\[
\Sigma_1^\pm(u, \lambda, \Lambda) \leq \sum_{k=1}^{N^\pm(u, \lambda, \Lambda)} \mathbb{P}\left\{ \mathcal{P}(Z_{u,k}) > g_{u,k}(\lambda) \right\} \sim
\]

\[
\sim \mathcal{H}_2^P\left(D^{1/\alpha}\lambda, D^{1/\alpha}T\right) \Psi(u) \sum_{k=1}^{N^\pm(u, \lambda, \Lambda)} \exp\left(-A_{\pm}\left|\lambda u^{-\xi_z} k - u^{-\xi_z} m_{\pm}(u)\right|^{\gamma_z}\right) \sim
\]

\[
\sim \mathcal{H}_2^P\left(D^{1/\alpha}\lambda, D^{1/\alpha}T\right) u^{\xi_z} \Psi(u) \int_0^\Lambda e^{-A_{\pm} x^{\gamma_z}} \, dx
\]

and, letting \( u \to \infty \), then \( \Lambda \to \infty \), and ultimately \( \lambda \to \infty \), we see that for large enough \( u \), holds

\[
\lim_{\lambda \to \infty} \lim_{\Lambda \to \infty} \lim_{u \to \infty} \frac{\Sigma_1^\pm(u, \lambda, \Lambda)}{u^{\xi_z} \Psi(u)} \leq \Gamma \left(\frac{1}{\gamma_z} + 1\right) A_{\pm}^{-1/\gamma_z} D^{1/\alpha} \mathcal{H}_2^P(D^{1/\alpha}T)
\]

where

\[
\mathcal{H}_2^P(T) = \lim_{\lambda \to \infty} \frac{\mathcal{H}_2^P(\lambda, T)}{\lambda} \in (0, \infty).
\]

Using lower bound in much the same fashion, we obtain

\[
\lim_{\lambda \to \infty} \lim_{\Lambda \to \infty} \lim_{u \to \infty} \frac{\Sigma_1^\pm(u, \lambda, \Lambda)}{u^{\xi_z} \Psi(u)} = \Gamma \left(\frac{1}{\gamma_z} + 1\right) A_{\pm}^{-1/\gamma_z} D^{1/\alpha} \mathcal{H}_2^P(D^{1/\alpha}T).
\]

The same formula obviously holds for \( \Sigma_1^\pm \). To conclude the proof in the Pickands case, it remains to notice that

\[
\lim_{\lambda \to \infty} \lim_{\Lambda \to \infty} \lim_{u \to \infty} \frac{\Sigma_1(u, \lambda, \Lambda)}{u^{\xi_0} \Psi(u)} =
\]

\[
= \lim_{\lambda \to \infty} \lim_{\Lambda \to \infty} \lim_{u \to \infty} \left( u^{\xi_+} \frac{\Sigma_1^+(u, \lambda, \Lambda)}{u^{\xi_+} \Psi(u)} + u^{\xi_-} \frac{\Sigma_1^-(u, \lambda, \Lambda)}{u^{\xi_-} \Psi(u)} + u^{-\xi_0} \frac{\Sigma_0(u, \lambda)}{\Psi(u)} \right) =
\]

\[
= \left( \Gamma \left(\frac{1}{\gamma_+} + 1\right) A_+^{-1/\gamma_+} 1_{\xi_z = \xi_+} + \Gamma \left(\frac{1}{\gamma_-} + 1\right) A_-^{-1/\gamma_-} 1_{\xi_z = \xi_-} \right) D^{1/\alpha} \mathcal{H}_2^P(D^{1/\alpha}T),
\]

that the same is obviously true for \( \Sigma_1^\pm \), and that \( 1_{\xi_z = \xi_\pm} = 1_{\gamma = \gamma_\pm} \).

4. Appendix

In this appendix, we recall some known results necessary for the proofs of previous section.

4.1. Parisian functional continuity. Let us show that the Parisian functional \( \mathcal{P} : C(E \times F) \to \mathbb{R} \) is continuous in uniform topology. To this end, we take an arbitrary function \( f \in C(E \times F) \) and a family \( \{f_\varepsilon, \varepsilon > 0\} \subset C(E \times F) \), which converges to a function \( f \in C(E \times F) \) uniformly

\[
\sup_{(t,s) \in E \times F} \left| f(t,s) - f_\varepsilon(t,s) \right| < \varepsilon
\]

as \( \varepsilon \to 0 \), from which we obtain

\[
\begin{cases}
  f(t,s) - f_\varepsilon(t,s) < \varepsilon, \\
  f_\varepsilon(t,s) - f(t,s) < \varepsilon
\end{cases}
\]

for all \( (t,s) \in E \times F \).

Hence,

\[
\sup_{t \in E} \inf_{s \in F} f(t,s) < \varepsilon + \sup_{t \in E} \inf_{s \in F} f_\varepsilon(t,s) \quad \text{and} \quad \sup_{t \in E} \inf_{s \in F} f_\varepsilon(t,s) < \varepsilon + \sup_{t \in E} \inf_{s \in F} f(t,s),
\]

or, equivalently,

\[
\left| \sup_{t \in E} \inf_{s \in F} f(t,s) - \inf_{t \in E} \sup_{s \in F} f_\varepsilon(t,s) \right| = \left| \mathcal{P}(f) - \mathcal{P}(f_\varepsilon) \right| < \varepsilon.
\]
4.2. Uniform local Pickands lemma. The following lemma is from [15], it is reproduced here for the reader’s convenience. Let
\[\xi_{u,\tau_u}(t) = \frac{Z_{u,\tau_u}(t)}{1 + h_{u,\tau_u}(t)}, \quad t \in E, \tau_u \in K_u,\]
be a family of centered Gaussian random fields with \(Z_{u,\tau_u}\), a centered Gaussian random field with unit variance and continuous paths, and \(h_{u,\tau_u}\) belonging to \(C_0(E)\), that is, \(h_{u,\tau_u}\) is a continuous function on \(E\), such that \(h_{u,\tau_u}(0) = 0\). We assume that \(E\) is a compact subset of \(\mathbb{R}^d\) and \(0 \in E\).

The Parisian functional \(\Gamma_{E,F}\)
\[\Gamma_{E,F}(X) = \sup_{t \in E} \inf_{s \in F} X(t + s)\]
satisfies the conditions

(F1): there exists \(c > 0\) such that \(\Gamma(f) \leq c \sup_{t \in E} f(t)\) for any \(f \in C(E)\)

(F2): \(\Gamma(af + b) = a\Gamma(f) + b\) for any \(f \in C(E)\) and \(a > 0, b \in \mathbb{R}\)

of the paper [15]. Therefore, under conditions (C0) – (C3)

(C0): \(\lim_{u \to \infty} \inf_{\tau_u \in K_u} g_{u,\tau_u} = \infty\)

(C1): there exists \(h \in C_0(E)\) such that
\[\lim_{u \to \infty} \sup_{\tau_u \in K_u, t \in E} \left| g_{u,\tau_u}^2 h_{u,\tau_u}(t) - h(t) \right| = 0\]

(C2): there exists \(\theta_{u,\tau_u}(t, s)\) such that
\[\lim_{u \to \infty} \sup_{\tau_u \in K_u, t \in E} \sup_{s \neq t \in E} \left| g_{u,\tau_u}^2 \frac{\mathbb{E}\left\{ |Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)|^2 \right\}}{2\theta_{u,\tau_u}(t, s)} - 1 \right| = 0,\]

where for some centered Gaussian random field \(\eta(t), t \in \mathbb{R}^d\) with continuous paths and \(\eta(0) = 0\),
\[\lim_{u \to \infty} \sup_{\tau_u \in K_u} \left| \theta_{u,\tau_u}(t, s) - \mathbb{E}\left\{ |\eta(t) - \eta(s)|^2 \right\} \right| = 0.\]

(C3): there exists \(a > 0\) such that
\[\lim_{u \to \infty} \sup_{\tau_u \in K_u, t \in E} \sup_{s \neq t \in E} \frac{\theta_{u,\tau_u}(t, s)}{\sum_{i=1}^d |s_i - t_i|^a} < \infty,\]
\[\lim_{\epsilon \to 0} \lim_{u \to \infty} \sup_{\tau_u \in K_u} \sup_{||t-s|| \in \epsilon, t, s \in E} \sup_{\sum_{i=1}^d |s_i - t_i|^a} g_{u,\tau_u}^2 \mathbb{E}\left\{ |Z_{u,\tau_u}(t) - Z_{u,\tau_u}(s)| Z_{u,\tau_u}(0) \right\} = 0.\]

we have

**Theorem 2.** Under assumptions (C0)-(C3), if, further, \(\mathbb{P}(\Gamma_{E,F}(\xi_{u,\tau_u}) > g_{u,\tau_u}) > 0\) for all \(\tau_u \in K_u\) and all large \(u\), then
\[\lim_{u \to \infty} \sup_{\tau_u \in K_u} \left| \frac{\mathbb{P}(\Gamma(\xi_{u,\tau_u}) > g_{u,\tau_u})}{\mathbb{P}(g_{u,\tau_u})} - \mathcal{H}^G_{n,h}(E) \right| = 0,\]
where
\[\mathcal{H}^G_{n,h}(E) = \mathbb{E}\left\{ e^{\gamma^h(t)} \right\}, \quad \gamma^h(t) = \sqrt{2} \eta(t) - \sigma^2_{\eta}(t) - h(t).\]

4.3. Large vicinity cut-off lemma. Next lemma is from [16] (Lemma 5.4), but instead of the version therefrom, we give a version suitable for our needs. This lemma allows one to get rid of the complement of the Piterbarg vicinity in all three (Piterbarg, Pickands and Talagrand) cases (see proof of Theorem 1 below).

**Lemma 1.** There exist positive constants \(C, c, u_0\) and \(\Lambda_0\) such that for \(\Lambda \geq \Lambda_0\) and \(u \geq u_0\)
\[\mathbb{P}\left\{ \exists t \in \left[-\delta_-(u), \Lambda u^{-2/\gamma_-}\right] \cup \left[\Lambda u^{-2/\gamma_+}, \delta_+(u)\right] : Z(t) > u \right\} \leq Ce^{-c\Lambda} \mathbb{P}\{Z(0) > u\} .\]

where \(\delta_\pm(u) = u^{-2/\gamma_\pm} \ln^{2/\gamma_\pm} u.\)
Financial support by SNSF Grant 200021-196888 is kindly acknowledged.

References

[1] K. Dębicki, E. Hashorva, and L. Ji, “On Parisian ruin over a finite-time horizon,” Sci. China Math., vol. 59, pp. 557–572, Mar. 2016.
[2] K. Kępczyński, “Proportional reinsurance for fractional Brownian risk model,” ArXiv201000222 Math, Oct. 2020.
[3] M. Chesney, M. Jeanblanc-Picquè, and M. Yor, “Brownian excursions and Parisian barrier options,” Adv. in Appl. Probab., vol. 29, no. 1, pp. 165–184, 1997.
[4] A. Dassios and S. Wu, “Semi-Markov model for excursions and occupation time of Markov processes,” 2011.
[5] A. Dassios and S. Wu, “Parisian ruin with exponential claims,” LSE Research Online Documents on Economics, 2008.
[6] I. Czarna and Z. Palmowski, “Ruin probability with Parisian delay for a spectrally negative Lévy risk process,” Journal of Applied Probability, vol. 48, no. 4, pp. 984–1002, 2011.
[7] R. Loeffen, I. Czarna, and Z. Palmowski, “Parisian ruin probability for spectrally negative Lévy processes,” Bernoulli, vol. 19, no. 2, 2011.
[8] I. Czarna and Z. Palmowski, “Dividend problem with Parisian delay for a spectrally negative Lévy risk process,” Theory Appl., vol. 161, pp. 239–256, 2014.
[9] I. Czarna, Z. Palmowski, and P. Świątek, “Binomial discrete time ruin probability with Parisian delay,” Scandinavian Actuarial Journal, 2014.
[10] K. Dębicki, E. Hashorva, and L. Ji, “Parisian ruin of self-similar Gaussian risk processes,” ArXiv14052958 Math, May 2014.
[11] V. I. Piterbarg, Asymptotic methods in the theory of Gaussian processes and fields. American Mathematical Society, 1996.
[12] G. Jasnowidov, “Approximation of ruin probability and ruin time in discrete Brownian risk models,” Scand. Actuar. J., no. 8, pp. 718–735, 2020.
[13] N. Kriukov, “Parisian and cumulative Parisian ruin probability for two-dimensional Brownian risk model,” An International Journal of Probability and Stochastic Processes, 2021.
[14] K. Krystecki, “Parisian ruin probability for two-dimensional Brownian risk model,” 2021.
[15] K. Dębicki, E. Hashorva, and P. Liu, “Uniform tail approximation of homogenous functionals of Gaussian fields,” Adv. Appl. Probab., vol. 49, pp. 1037–1066, Dec. 2017.
[16] K. Dębicki, E. Hashorva, and L. Wang, “Extremes of Vector-Valued Gaussian processes,” ArXiv191106350 Math, Nov. 2019.
[17] L. Ji and S. Robert, “Ruin problem of a two-dimensional fractional Brownian motion risk process,” Stochastic Models, vol. 34, pp. 73–97, Jan. 2018.

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