Euler’s factorial series and global relations

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To the memory of Marc Huttner (1947–2015)

Abstract

Using Padé approximations to the series \(E(z) = \sum_{k=0}^{\infty} k! (-z)^k\), we address arithmetic and analytical questions related to its values in both \(p\)-adic and Archimedean valuations.

1 Introduction

In his 1760 paper on divergent series [5], L. Euler introduced and studied the formal (hypergeometric) series

\[ E(z) := \sum_{k=0}^{\infty} k! (-z)^k \]  

(see also [1] and, in particular, [8 Section 2.5]), which specializes at \(z = 1\) to Wallis’ series

\[ W := \sum_{k=0}^{\infty} (-1)^k k! , \]  

which has teased people’s imagination since the times of Euler. Notice that the series (1) is a perfectly convergent \(p\)-adic series in the disc \(|z|_p \leq 1\) for all primes \(p\), so that one can discuss the arithmetic properties of its values, for example, at \(z = 1\) (the Wallis case) and at \(z = -1\). The irrationality of the latter specialization,

\[ K = K_p := \sum_{k=0}^{\infty} k! , \]  

for any prime \( p \) is a folklore conjecture [12, p. 17] (see [11], also for a link of the problem to a combinatorial conjecture of H. Wilf). Already the expectation \( |K_p|_p = 1 \) (so that \( K_p \) is a \( p \)-adic unit) for all primes \( p \), which is an equivalent form of Kurepa’s conjecture [7] from 1971, remains open; see [4] for the latest achievements in this direction.

In what follows, we refer to the series in (1) as Euler’s factorial series and label it \( E_p(\xi) \) when treat it as the function in the \( p \)-adic domain for a given prime \( p \).

**Theorem 1.** Given \( \xi \in \mathbb{Z} \setminus \{0\} \), let \( \mathcal{P} \) be a subset of prime numbers such that

\[
\limsup_{n \to \infty} c^n n! \prod_{p \in \mathcal{P}} |n!|_p^2 = 0, \quad \text{where } c = c(\xi; \mathcal{P}) := 4|\xi| \prod_{p \in \mathcal{P}} |\xi|_p^2.
\]

Then either there exists a prime \( p \in \mathcal{P} \) for which \( E_p(\xi) \) is irrational, or there are two distinct primes \( p, q \in \mathcal{P} \) such that \( E_p(\xi) \neq E_q(\xi) \) (while \( E_p(\xi), E_q(\xi) \in \mathbb{Q} \)).

Because \( \prod_p |n!|_p = 1/n! \) when the product is taken over all primes, condition (4) is clearly satisfied for any subset \( \mathcal{P} \) whose complement in the set of all primes is finite. This also suggests more exotic choices of \( \mathcal{P} \). Furthermore, the conclusion of the theorem is contrasted with Euler’s sum

\[
\sum_{k=0}^{\infty} k \cdot k! = -1,
\]

which is valid in any \( p \)-adic valuation (and follows from \( \sum_{k=0}^{n-1} k \cdot k! = n! - 1 \) — an example of what is called a *global* relation.

The idea behind the proof of Theorem 1 is to construct approximations \( p_n/q_n \) to the number \( \omega_p = E_p(\xi) \) in question, which do not depend on \( p \) and approximate the number considerably well for each \( p \)-adic valuation: \( q_n \omega_p - p_n = r_{n,p} \) for \( n = 0, 1, 2, \ldots \); \( |r_{n,p}|_p \to 0 \) as \( n \to \infty \) and there are infinitely many indices \( n \) for which \( r_{n,p} \neq 0 \) for at least one prime \( p \in \mathcal{P} \). Assume that \( \omega_p = a/b \), the same rational number, for all \( p \in \mathcal{P} \). Then \( q_n a - p_n b \in \mathbb{Z} \setminus \{0\} \), so that \( 0 < |q_n a - p_n b|_p \leq 1 \), for infinitely many indices \( n \) and all primes \( p \), hence

\[
1 = |q_n a - p_n b| \prod_p |q_n a - p_n b|_p \leq |q_n a - p_n b| \prod_{p \in \mathcal{P}} |q_n a - p_n b|_p \\
\leq (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |br_{n,p}|_p \\
\leq (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |r_{n,p}|_p
\]

for those \( n \). This means that the condition

\[
\limsup_{n \to \infty} \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |r_{n,p}|_p = 0
\]

(5)

contradicts the latter estimate, thus making the \( \mathcal{P} \)-global linear relation \( \omega_p = a/b \) impossible.
The result in Theorem 1 can be put in a general context of global relations for Euler-type series; the corresponding settings can be found in the paper [3]. We do not pursue this route here as our principal motivation is a sufficiently elementary arithmetic treatment of an analytical function that have some historical value. The rational approximations to \( E(z) \) we construct in Section 2 are Padé approximations; in spite of being known for centuries, implicitly from the continued fraction for \( E(z) \) given by Euler himself [5] and explicitly from the work of T. J. Stieltjes [13], these Padé approximations remain a useful source for arithmetic and analytical investigations. In Section 3 we revisit Euler’s summation of Wallis’ series (2) using the approximations; we also complement the derivation by providing ‘Archimedean analogue(s)’ for the divergent series \( K = E(-1) \) from [3] — the case when the classical strategy does not work.

Our way of constructing the Padé approximations is inspired by a related Padé construction of M. Hata and M. Huttner in [6]. The construction was a particular favourite of Marc Huttner who, for the span of his mathematical life, remained a passionate advocate of interplay between Picard–Fuchs linear differential equations and Padé approximations. We dedicate this work to his memory.

2 Hypergeometric series and Padé approximations

Euler’s factorial series (1) is the particular \( a = 1 \) instance of the hypergeometric series

\[
_{2}F_{0}(a, 1 \mid z) = \sum_{k=0}^{\infty} (a)_{k} z^{k}.
\]  

Here and in what follows, we use the Pochhammer notation \((a)_{k}\) which is defined inductively by \((a)_{0} = 1\) and \((a)_{k+1} = (a+k)(a)_{k}\) for \(k \in \mathbb{Z}_{\geq 0}\). Our Padé approximations below are given more generally for the function (6).

**Theorem 2.** For \(n, \lambda \in \mathbb{Z}_{\geq 0}\), take

\[
B_{n,\lambda}(z) = \sum_{i=0}^{n} \frac{n!}{i!} \frac{(-1)^i z^{n-i}}{(a)_{i+\lambda}}.
\]

Then \(\deg_{z} B_{n,\lambda} = n\) and for a polynomial \(A_{n,\lambda}(z)\) of degree \(\deg_{z} A_{n,\lambda} \leq n + \lambda - 1\) we have

\[
B_{n,\lambda}(z) \, _{2}F_{0}(a, 1 \mid z) - A_{n,\lambda}(z) = L_{n,\lambda}(z),
\]

where \(\ord_{z=0} L_{n,\lambda}(z) = 2n + \lambda\). Explicitly,

\[
L_{n,\lambda}(z) = (-1)^{n} n! z^{2n+\lambda} \sum_{k=0}^{\infty} k! \binom{n+k}{k} \binom{n+k+a+\lambda-1}{k} z^{k}.
\]

**Proof.** Relation (7) means that there is a ‘gap’ of length \(n\) in the power series expansion

\[
B_{n,\lambda}(z) \, _{2}F_{0}(a, 1 \mid z) = A_{n,\lambda}(z) + L_{n,\lambda}(z).
\]
Write \( B_{n,\lambda}(z) = \sum_{h=0}^{n} b_{h} t^{h} \) and consider the series expansion of the product
\[
B_{n,\lambda}(z) \, _2F_0(a, 1 \mid z) = \sum_{l=0}^{\infty} r_{l} z^{l},
\]
where \( r_{l} = \sum_{h+k=l} b_{h} (a)_{k} \); in particular,
\[
r_{l} = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (a)_{i+\lambda+n} = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} (a+i+\lambda)_{m}
\]
with \( m = l - n - \lambda \) for \( l > n + \lambda - 1 \). To verify the desired ‘gap’ condition,
\[
r_{n+\lambda} = r_{n+\lambda+1} = \cdots = r_{n+\lambda+n-1} = 0,
\]
we introduce the shift operators \( N = N_{a} \) and \( \Delta = \Delta_{a} = N - \text{id} \) defined on functions \( f(a) \) by
\[
N f(a) = f(a+1) \quad \text{and} \quad \Delta f(a) = f(a+1) - f(a).
\]
It follows from
\[
\Delta^{n}(a+\lambda)_{m} = (-1)^{n} (-m)_{n}(a+\lambda+n)_{m-n} \quad \text{for} \quad n \in \mathbb{Z}_{\geq 0}
\]
that
\[
r_{l} = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} N^{i}(a+\lambda)_{m} = (\text{id} - N)^{n}(a+\lambda)_{m}
\]
\[= (-\Delta)^{n}(a+\lambda)_{m} = (-m)_{n}(a+\lambda+n)_{m-n},
\]
which, in turn, implies (9) because of the vanishing of \((-m)_{n} \) for \( m = 0, 1, \ldots, n-1 \).
The explicit expression for \( r_{l} \) just found also gives
\[
r_{2n+\lambda+k} = (-n-k)_{n}(a+\lambda+n)_{k} = (-1)^{n} n! \binom{n+k}{k} \binom{n+k+a+\lambda-1}{k},
\]
hence the closed form (8). We also have
\[
A_{n,\lambda}(z) = \sum_{l=0}^{n+\lambda-1} r_{l} z^{l} = \sum_{l=0}^{n+\lambda-1} z^{l} \sum_{i=0}^{n} (-1)^{i} \frac{\binom{n}{i} (a)_{i+l-n}}{(a)_{i+\lambda}}.
\]
This concludes our proof of the theorem. \( \square \)

From now on we choose \( a = 1, \lambda = 0 \), change \( z \) to \(-z\) and renormalize the corresponding Padé approximations produced by Theorem 2 by multiplying them by
\((-1)^nn!:\)

\[ Q_n(z) := (-1)^nn! B_{n,0}(-z) = \sum_{i=0}^{n} \binom{n}{i} \frac{n!}{i!} z^{n-i} = \sum_{i=0}^{n} i! \binom{n}{i}^2 z^i, \]

\[ P_n(z) := (-1)^n n! A_{n,0}(-z) = (-1)^n \sum_{l=0}^{n-1} (-z)^l \sum_{i=n-l}^{n} (-1)^{n-i} \binom{n}{i} \frac{n! (i + l - n)!}{i!} \]

\[ = \sum_{l=0}^{n-1} (-z)^l \sum_{i=0}^{l} (-1)^i i! (l-i)! \binom{n}{i}^2 \]

and

\[ R_n(z) := (-1)^n n! L_{n,0}(-z) = n!^2 2^n \sum_{k=0}^{\infty} (-1)^k k! \left( \frac{n + k}{k} \right)^2 z^k. \]

In this notation, the Padé approximation formula (7) may be rewritten as

\[ Q_n(z) E(z) - P_n(z) = R_n(z) \] (10)

for \( n \in \mathbb{Z}_{>0} \). Observe that

\[ Q_n(z) P_{n+1}(z) - Q_{n+1}(z) P_n(z) = n!^2 z^{2n}. \] (11)

Indeed, the standard Padé walkabout proves the identity

\[ Q_n(z) P_{n+1}(z) - Q_{n+1}(z) P_n(z) = Q_{n+1}(z) R_n(z) - Q_n(z) R_{n+1}(z), \]

in which the degree of the left-hand side is at most \( 2n \) while the order of the right-hand side is at least \( 2n \).

**Proof of Theorem** We may use the formal series identity (10) to get appropriate numerical approximations for any at \( z = \xi \in \mathbb{Z} \setminus \{0\} \). For \( n = 1, 2, \ldots \), take \( p_n = P_n(\xi) \in \mathbb{Z} \), \( q_n = Q_n(\xi) \in \mathbb{Z} \) and define

\[ r_{n,p} = R_n(\xi) = q_n E_p(\xi) - p_n \]

for each prime \( p \in \mathcal{P} \). Using elementary summation formulas and trivial estimates for binomials we have

\[ |q_n| \leq |\xi|^n n! \sum_{i=0}^{n} \binom{n}{i}^2 < 4^n |\xi|^n n!, \]

\[ |p_n| \leq |\xi|^{n-1} n \sum_{i=0}^{n-1} i! (n - 1 - i)! \binom{n}{i}^2 \leq |\xi|^n n! \sum_{i=0}^{n} \binom{n}{i}^2 < 4^n |\xi|^n n! \]
\begin{align*}
|r_{n,p}|_p &= |\xi_{n!}^2p|^2 \sum_{k=0}^\infty (-1)^k k! \left(\frac{n+k}{k} \right)^2 \xi_k p \\
&\leq |\xi_{n!}^2p|^2.
\end{align*}

Therefore, condition (5) reads
\[
\limsup_{n \to \infty} 4^n |\xi|^n n! \prod_{p \in \mathcal{P}} |\xi_{n!}^2p|^2 = 0
\]
and because for at least one \( p \in \mathcal{P} \) we have either \( r_{n,p} \neq 0 \) or \( r_{n+1,p} \neq 0 \) from (11), the theorem follows.

### 3 Summation of divergent series

Fix a prime \( p \). If we restrict, for simplicity, the sequence of indices \( n \) to the arithmetic progression \( n \equiv 0 \mod p \), then it is not hard to see from the calculation in Section 2 that the sequence of rational approximations \( p_n/q_n = P_n(\xi)/Q_n(\xi) \) converges \( p \)-adically to \( E(\xi) \). Indeed,
\[
q_n = 1 + \sum_{i=1}^{p-1} i! \left(\frac{n}{i} \right)^2 \xi^i + \sum_{i=p}^n i! \left(\frac{n}{i} \right)^2 \xi^i \equiv 1 \mod p
\]
is a \( p \)-adic unit (all the binomials are divisible by \( p \) in the first sum, while \( i! \) is divisible by \( p \) in the second one), and
\[
\left| E(\xi) - \frac{p_n}{q_n} \right|_p = |q_n|_p^{-1} |q_n E(\xi) - p_n|_p = |r_{n,p}|_p \to 0 \quad \text{as } n \to \infty.
\]

When \( \xi > 0 \), the same sequence of rational numbers \( p_n/q_n = P_n(\xi)/Q_n(\xi) \) converges to the value at \( z = \xi \) of the integral
\[
\tilde{E}(z) = \int_0^\infty \frac{e^{-s}}{1 + zs} \, ds \quad (12)
\]
with respect to the Archimedean norm. The integral itself converges for any \( z \in \mathbb{C} \setminus (-\infty, 0] \), though its formal Taylor expansion at \( z = 0 \) (obtained by expanding \( 1/(1 + zs) \) into the power series under the integral sign) is precisely Euler’s factorial series \( E(z) \) as in (11). In particular, relation (10) remains valid with \( E(z) \) replaced by \( \tilde{E}(z) \) and \( R_n(z) \) by
\[
\tilde{R}_n(z) = n! z^{2n} \int_0^\infty \frac{s^n e^{-s}}{(1 + zs)^{n+1}} \, ds \quad \text{for } n = 0, 1, 2, \ldots,
\]
so that
\[
\left| \tilde{E}(\xi) - \frac{p_n}{q_n} \right| = \frac{\tilde{R}_n(\xi)}{Q_n(\xi)} \leq \frac{n^n}{(n+1)^{n+1}} \to 0 \quad \text{as } n \to \infty.
\]
where the estimates \( Q_n(\xi) \geq n! \xi^n \) and
\[
\tilde{R}_n(\xi) \leq n! \xi^{2n} \left( \max_{s>0} \frac{s^n}{(1+\xi s)^{n+1}} \right) \int_0^{\infty} e^{-s} \, ds = n! \xi^n \frac{n^n}{(n+1)^{n+1}}
\]
were used. This computation reveals us that
\[
\lim_{n \to \infty} \frac{p_n}{q_n} = \tilde{E}(\xi) = -xe^x \left( \gamma + \log x + \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!} \right) \bigg|_{x=1/\xi},
\]
where \( \gamma = 0.5772156649 \ldots \) is Euler’s constant. In particular,
\[
W = \tilde{E}(1) = e \left( -\gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \cdot k!} \right) = 0.5963473623 \ldots
\]
for Wallis’ series (2). The resulted quantity is known as the Euler–Gompertz constant [8].

The strategy in the previous paragraph does not apply to \( z = \xi < 0 \), somewhat already observed by Stieltjes in [13]. The analytical continuation of the function \( \tilde{E}(z) \) depends on whether we perform the integration along the upper or lower banks of the ray [0, \( \infty \)) in (12); denote the corresponding values by \( \tilde{E}_+(z) \) and \( \tilde{E}_-(z) \), respectively. By considering the integration of \( e^{-s}/(1+zs) \) along the curvilinear triangle that consists of the segment \([0,R]\) (along a particular bank), the arc \([R, Re^{\sqrt{-1}\theta}]\) followed by the segment \([Re^{\sqrt{-1}\theta}, 0]\), where \( 0 < \theta < \pi/2 \) for the upper bank and \( -\pi/2 < \theta < 0 \) for the lower one, and then taking the limit as \( R \to \infty \) (so that the integral along the arc tends to 0) we conclude that
\[
\tilde{E}_\pm(z) = \int_0^{e^{\sqrt{-1}\theta} \infty} \frac{e^{-s}}{1+zs} \, ds = -xe^x \left( \gamma + \log |x| \pm \sqrt{-1 \pi} + \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!} \right) \bigg|_{x=1/z}, \quad (13)
\]
with the choice of \( \theta \) arbitrary in the interval \( 0 < \theta < \pi/2 \) for \( \tilde{E}_+(z) \) and in the interval \( -\pi/2 < \theta < 0 \) for \( \tilde{E}_-(z) \). In particular,
\[
K = \tilde{E}_\pm(-1) = \frac{1}{e} \left( \gamma + \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} \right) \mp \sqrt{-1 \pi}
\]
\[
= 0.6971748832 \ldots \mp \sqrt{1 \cdot 1.1557273497} \ldots
\]
for the series in (3).

### 4 Final remarks

In [14] we outline a different strategy of proving a result analogous to Theorem 1 on using the Hankel determinants generated by the tails of Euler’s factorial series (1). As the condition on a subset of primes \( \mathcal{P} \) in that result is spiritually similar to (4),
we do not detail the derivation here. However we stress that a potential combination of the two methods, namely, using the Hankel determinants generated by the Padé approximations of Euler’s factorial series, may be a source of further novelties on the topic. A discussion on this type of construction in the Archimedean setting can be found in [15].

One consequence of the formula in (13), which uncovers a pair of complex conjugate values for $E(\xi)$ when $\xi < 0$, is that the rational approximations $p_n/q_n = P_n(\xi)/Q_n(\xi)$ do not converge at all in such cases. Interestingly enough, the Hankel determinants ‘see’ those complex values (13) as experimentally observed in [14].

Finally, we would like to note that nothing is known about the irrationality and transcendence of the Archimedean valuations of Euler’s factorial series (1) at rational $z = \xi$ (see the discussion in [8, Sections 3.15, 3.16]). This is in contrast with its $q$-analogue

$$\sum_{k=0}^{\infty} z^k \prod_{i=1}^{k}(1 - q^i),$$

for which irrationality and linear independence results are known in Archimedean and non-Archimedean places alike—see [9]. Further details on a nice $q$-counterpart of the Padé approximation analysis can be found in [10].

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