A sharp Leibniz rule for BV functions in metric spaces

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Abstract
We prove a Leibniz rule for BV functions in a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality. Unlike in previous versions of the rule, we do not assume the functions to be locally essentially bounded and the end result does not involve a constant $C \geq 1$, and so our result seems to be essentially the best possible. In order to obtain the rule in such generality, we first study the weak* convergence of the variation measure of BV functions, with quasi semicontinuous test functions.

Keywords Function of bounded variation · Leibniz rule · Metric measure space · Weak* convergence · Quasi semicontinuity

Mathematics Subject Classification 30L99 · 31E05 · 26B30

1 Introduction

The Leibniz rule for functions of bounded variation (BV functions) says that if $u, v \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the variation measures satisfy

$$dD(uv) = \bar{u} dDv + \bar{v} dDu,$$

where $\bar{u}, \bar{v}$ are the so-called precise representatives of $u$ and $v$; see [31] or [32, Section 4.6.4]. More precisely, this result is proved in the above references with somewhat weaker assumptions; in particular, the boundedness assumption can be weakened to only one of the functions being locally (essentially) bounded.

In the past two decades, a theory of BV functions as well as other topics in analysis has been developed in the abstract setting of metric measure spaces. The standard assumptions in this setting are that $(X, d, \mu)$ is a complete metric space equipped with
a Borel regular, doubling outer measure \(\mu\), and that \(X\) supports a Poincaré inequality. See Sect. 2 for definitions. In this setting, the following Leibniz rule for BV functions was proved in [17].

**Proposition 1.1** ([17, Proposition 4.2]) Let \(u, v \in \text{BV}(X) \cap L^\infty(X)\) be nonnegative functions. Then \(uv \in \text{BV}(X) \cap L^\infty(X)\) such that

\[
d\|D(uv)\| \leq C v^\vee d\|Du\| + Cu^\vee d\|Dv\|
\]

for some constant \(C \geq 1\) that depends only on the doubling constant of the measure and the constants in the Poincaré inequality.

Note that in metric spaces, one cannot talk about the vector measure \(Du\), only the total variation \(\|Du\|\). In the above Leibniz rule, we see that again the functions are assumed to be in \(L^\infty(X)\). Additionally, there is a multiplicative constant \(C \geq 1\) that arises from the use of a discrete convolution technique in the proof of the Leibniz rule. This is a common technique in metric space analysis, and sometimes the constant \(C\) appearing in an end result cannot be removed, see e.g. [13, Remark 4.7, Example 4.8]. On the other hand, for the upper gradients of Newton–Sobolev functions (a generalization of Sobolev functions to metric spaces), one has the Leibniz rule

\[
g_{uv} \leq ug_v + vg_u,
\]

which does not involve a constant \(C\). Thus it is natural to ask whether the constant \(C\), as well as the \(L^\infty\)-assumption, can be dropped from the BV Leibniz rule, and in this paper we show that this is indeed the case. Our main result is the following.

**Theorem 1.2** Let \(\Omega \subset X\) be an open set and let \(u, v \in L^1_{\text{loc}}(\Omega)\). Then

\[
\|D(uv)(\Omega)\| \leq \int_\Omega |u|^\vee d\|Dv\| + \int_\Omega |v|^\vee d\|Du\|.
\]

Since neither of the functions \(u, v\) is assumed to be in \(L^\infty_{\text{loc}}(\Omega)\), we do not automatically even have \(uv \in L^1_{\text{loc}}(\Omega)\), but we are able to prove this assuming that the right-hand side is finite, and then we also obtain \(\|D(uv)\|(\Omega) < \infty\). Hence the result may be of use already in establishing membership of a given product of two functions in the BV class—which by contrast could be obtained almost immediately from the stronger assumptions of Proposition 1.1. Moreover, we do not assume the functions \(u, v\) to be BV functions even locally, so the measures \(\|Du\|, \|Dv\|\) could be large; it is only necessary that the two integrals are finite. If this is the case, then we can obtain

\[
d\|D(uv)\| \leq |u|^\vee d\|Dv\| + |v|^\vee d\|Du\|, \quad (1.2)
\]

as measures on \(\Omega\), improving on Proposition 1.1 — see Remark 4.7. In fact, due to our minimal assumptions, our result seems to give a slight improvement on what is known even in Euclidean spaces. On the other hand, in Example 4.8 we show that unlike in Euclidean spaces, in metric spaces it is not possible to replace the representatives \(u^\vee, v^\vee\) by \(\overline{u}, \overline{v}\), and that equality may hold in (1.2). Thus our Leibniz rule appears to be essentially the best possible in every respect.
It can be said that the Leibniz rules for BV functions that are found in the literature are already quite sufficient for most applications, typically in the calculus of variations. Thus perhaps the main interest of this paper is in the methods that we employ. Indeed, to prove the Leibniz rule in the above generality, we use several rather strong tools and also develop a few new ones. To avoid having to assume that \( u, v \) are BV functions, we use an extension property that relies on Federer’s characterization of sets of finite perimeter proved in [18]. The most significant effort is required in ensuring that the constant \( C \) does not appear in the end result. For this, we use two tools: one is a result on the pointwise convergence of BV functions given in [22]. The other is a result on the weak* convergence of variation measures in the case of quasi semicontinuous test functions, which we derive in Sect. 3. This is based on results in [21], and we expect it to have applications also in future work.

2 Preliminaries

In this section we present the necessary notation, definitions, assumptions, and a few background results.

Throughout this paper, \((X, d, \mu)\) is a complete metric space that is equipped with a metric \(d\) and a Borel regular outer measure \(\mu\) satisfying a doubling property, meaning that there exists a constant \(C_d \geq 1\) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for every ball \(B(x, r) := \{y \in X : d(y, x) < r\}\). When a property holds outside a set of \(\mu\)-measure zero, we say that it holds almost everywhere, abbreviated a.e.

As a complete metric space equipped with a doubling measure, \(X\) is proper, that is, closed and bounded sets are compact. All functions defined on \(X\) or its subsets will take values in \([-\infty, \infty]\). Given a \(\mu\)-measurable set \(A \subset X\), we define \(L^1_{\text{loc}}(A)\) as the class of functions \(u\) on \(A\) such that for every \(x \in A\) there exists \(r > 0\) such that \(u \in L^1(A \cap B(x, r))\). Other local spaces of functions are defined analogously. For an open set \(\Omega \subset X\), a function is in the class \(L^1_{\text{loc}}(\Omega)\) if and only if it is in \(L^1(\Omega')\) for every open \(\Omega' \Subset \Omega\). Here \(\Omega' \Subset \Omega\) means that \(\Omega'\) is a compact subset of \(\Omega\).

For any set \(A \subset X\) and \(0 < R < \infty\), the restricted Hausdorff content of codimension one is defined as

\[
\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.
\]

The codimension one Hausdorff measure of \(A \subset X\) is then defined as

\[
\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).
\]

By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into \(X\). A nonnegative Borel function \(g\) on \(X\) is an upper gradient of a function \(u\) on \(X\) if for all curves \(\gamma\), we have
\[ |u(x) - u(y)| \leq \int_{\gamma} g \, ds, \]  

(2.1)

where \(x\) and \(y\) are the end points of \(\gamma\) and the curve integral is defined by using an arc-length parametrization, see [15, Section 2] where upper gradients were originally introduced. We interpret \(|u(x) - u(y)| = \infty\) whenever at least one of \(|u(x)|, |u(y)|\) is infinite.

We say that a family of curves \(\Gamma\) is of zero 1-modulus if there is a nonnegative Borel function \(\rho \in L^1(X)\) such that for all curves \(\gamma \in \Gamma\), the curve integral \(\int_{\gamma} \rho \, ds\) is infinite. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If \(g\) is a nonnegative \(\mu\)-measurable function on \(X\) and (2.1) holds for 1-almost every curve, we say that \(g\) is a 1-weak upper gradient of \(u\).

Given a \(\mu\)-measurable set \(H \subset X\), we let

\[ \|u\|_{N^{1,1}(H)} := \|u\|_{L^1(H)} + \inf \|g\|_{L^1(H)}, \]

where the infimum is taken over all 1-weak upper gradients \(g\) of \(u\) in \(H\). The substitute for the Sobolev space \(W^{1,1}\) in the metric setting is the Newton–Sobolev space \(N^{1,1}(H) := \{ u : \|u\|_{N^{1,1}(H)} < \infty\}\), which was first introduced in [30]. We understand a Newton–Sobolev function to be defined at every \(x \in H\) (even though \(\|\cdot\|_{N^{1,1}(H)}\) is then only a seminorm). It is known that for any \(u \in N^{1,1}_{\text{loc}}(H)\) there exists a minimal 1-weak upper gradient of \(u\) in \(H\), always denoted by \(g_u\), satisfying \(g_u \leq g\) a.e. in \(H\) whenever \(g \in L^1_{\text{loc}}(H)\) is a 1-weak upper gradient of \(u\) in \(H\), see [4, Theorem 2.25].

We will assume throughout the paper that \(X\) supports a \((1, 1)\)-Poincaré inequality, meaning that there exist constants \(C_p > 0\) and \(\lambda \geq 1\) such that for every ball \(B(x, r)\), every \(u \in L^1_{\text{loc}}(X)\), and every upper gradient \(g\) of \(u\), we have

\[ \int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_p r \int_{B(x, \lambda r)} g \, d\mu, \]

where

\[ u_{B(x, r)} := \int_{B(x, r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu. \]

The 1-capacity of a set \(A \subset X\) is defined as

\[ \text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)}, \]

where the infimum is taken over all functions \(u \in N^{1,1}(X)\) such that \(u \geq 1\) in \(A\).

**Definition 2.1** Let \(H \subset X\). We say that a set \(A \subset H\) is 1-quasiopen in \(H\) if for every \(\varepsilon > 0\) there is a relatively open set \(G \subset H\) such that \(\text{Cap}_1(G) < \varepsilon\) and \(A \cup G\) is relatively open in \(H\). When \(H = X\), we omit mention of it.
We say that a function \( u \) is 1-quasi (lower/upper semi-)continuous on \( H \) if for every \( \varepsilon > 0 \) there exists a relatively open set \( G \subset H \) such that \( \text{Cap}_1(G) < \varepsilon \) and \( u|_{H \setminus G} \) is real-valued (lower/upper semi-)continuous.

It is a well-known fact that a Newton–Sobolev function \( u \in N^{1,1}(\Omega) \) is 1-quasicontinuous on an open set \( \Omega \), see [7, Theorem 1.1] or [4, Theorem 5.29].

The variational 1-capacity of a set \( A \subset D \) with respect to a set \( D \subset X \) is defined as

\[
\text{cap}_1(A, D) := \inf \int_X g_u \, d\mu,
\]

where the infimum is taken over functions \( u \in N^{1,1}(X) \) such that \( u \geq 0 \) on \( A \) and \( u = 0 \) on \( X \setminus D \).

Next we present the basic theory of functions of bounded variation on metric spaces. This was first developed in [1,28]; see also the monographs [2,9,10,12,33] for the classical theory in Euclidean spaces. We will always denote by \( \Omega \) an open subset of \( X \). Given a function \( u \in L^1_{\text{loc}}(\Omega) \), we define the total variation of \( u \) in \( \Omega \) as

\[
\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(\Omega), \ u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},
\]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( \Omega \). If \( u \not\in L^1_{\text{loc}}(\Omega) \), we interpret \( \|Du\|(\Omega) = \infty \). (In [28], local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function \( u \in L^1(\Omega) \) is of bounded variation, and denote \( u \in BV(\Omega) \), if \( \|Du\|(\Omega) < \infty \). For an arbitrary set \( A \subset X \), we define

\[
\|Du\|(A) := \inf \{ \|Du\|(W) : A \subset W, \ W \subset X \text{ is open} \}.
\]

In general, we understand the expression \( \|Du\|(A) < \infty \) to mean that there exists some open set \( \Omega \supset A \) such that \( u \) is defined on \( \Omega \) with \( u \in L^1_{\text{loc}}(\Omega) \) and \( \|Du\|(\Omega) < \infty \).

**Theorem 2.2** ([28, Theorem 3.4]) If \( u \in L^1_{\text{loc}}(\Omega) \), then \( \|Du\|(\cdot) \) is a Borel measure on \( \Omega \).

Note that this result does not require that \( \|Du\|(\Omega) < \infty \), as can be seen from the proof in [28]. We call \( \|Du\| \) the variation measure of \( u \). A \( \mu \)-measurable set \( E \subset X \) is said to be of finite perimeter if \( \|D\chi_E\|(X) < \infty \), where \( \chi_E \) is the characteristic function of \( E \). The perimeter of \( E \) in a set \( A \subset X \) is also denoted by

\[
P(E, A) := \|D\chi_E\|(A).
\]

The measure-theoretic boundary \( \partial^*E \) of a set \( E \subset X \) is defined as the set of points \( x \in X \) at which both \( E \) and its complement have strictly positive upper density, i.e.

\[
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.
\]
For a $\mu$-measurable set $E \subset X$ with $P(E, \Omega) < \infty$, we know that for any Borel set $A \subset \Omega$,

$$P(E, A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H}, \quad (2.2)$$

where $\theta_E : X \to [\alpha, C_d]$ with $\alpha = \alpha(C_d, C_P, \lambda) > 0$, see [1, Theorem 5.3] and [3, Theorem 4.6]. The following coarea formula is given in [28, Proposition 4.2]: if $u \in L^1_{\text{loc}}(\Omega)$, then

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega) \, dt. \quad (2.3)$$

If $\|Du\|(\Omega) < \infty$, then the formula holds with $\Omega$ replaced by any Borel set $A \subset \Omega$. From this combined with (2.2), we obtain the absolute continuity

$$\|Du\| \ll \mathcal{H} \text{ on } \Omega. \quad (2.4)$$

Since Lip$_{\text{loc}}(\Omega)$ is dense in $N^{1,1}(\Omega)$, see [4, Theorem 5.47], it follows that

$$N^{1,1}(\Omega) \subset \text{BV}(\Omega) \quad \text{with} \quad \|Du\|(\Omega) \leq \int_{\Omega} g_u \, d\mu \text{ for every } u \in N^{1,1}(\Omega). \quad (2.5)$$

If we apply the $(1,1)$-Poincaré inequality to sequences of approximating locally Lipschitz functions in the definition of the total variation, we get the following BV version: for every ball $B(x, r)$ and every $u \in L^1_{\text{loc}}(X)$, we have

$$\int_{B(x, r)} |u - u_{B(x,r)}| \, d\mu \leq C_P r \frac{\|Du\|(B(x, \lambda r))}{\mu(B(x, \lambda r))}. \quad (2.6)$$

The lower and upper approximate limits of a function $u$ on $X$ are defined respectively by

$$u^\wedge(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\}$$

and

$$u^\vee(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\}.$$
Proposition 2.3 Let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \|Du\|(\Omega) < \infty \). Then \( u^\wedge \) is 1-quasi lower semicontinuous on \( \Omega \) and \( u^\vee \) is 1-quasi upper semicontinuous on \( \Omega \).

Proof This follows from [23, Corollary 4.2] (which is based on [26, Theorem 1.1]). \( \square \)

For \( D \subset \Omega \subset X \) (recall that we always assume \( \Omega \) to be open) we define the class of BV functions with zero boundary values as

\[
\text{BV}_0(D, \Omega) := \{ u|_D : u \in \text{BV}(\Omega), \ u^\wedge(x) = u^\vee(x) = 0 \text{ for } \mathcal{H}\text{-a.e. } x \in \Omega \setminus D \}. \tag{2.7}
\]

This class was previously considered in [25]. It follows rather easily from the coarea formula (2.3) that for \( u \in \text{BV}_0(D, \Omega) \), defining \( u = 0 \) (a.e.) on \( \Omega \setminus D \), we have

\[
\|Du\|(\Omega \setminus D) = 0, \tag{2.8}
\]

see [25, Proposition 3.14].

Next we define the fine topology in the case \( p = 1 \).

Definition 2.4 We say that \( A \subset X \) is 1-thin at the point \( x \in X \) if

\[
\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.
\]

We also say that a set \( U \subset X \) is 1-finely open if \( X \setminus U \) is 1-thin at every \( x \in U \). Then we define the 1-fine topology as the collection of 1-finely open sets on \( X \).

We denote the 1-fine interior of a set \( H \subset X \), i.e. the largest 1-finely open set contained in \( H \), by \( \text{fine-int } H \). We denote the 1-fine closure of \( H \), i.e. the smallest 1-finely closed set containing \( H \), by \( \overline{H}^1 \).

See [19, Section 4] for a proof of the fact that the 1-fine topology is indeed a topology. The following fact is given in [18, Proposition 3.3]:

\[
\text{Cap}_1(A) = \text{Cap}_1(A^{-1}) \text{ for any } A \subset X. \tag{2.9}
\]

Theorem 2.5 ([24, Corollary 6.12]) A set \( U \subset X \) is 1-quasiopen if and only if it is the union of a 1-finely open set and a \( \mathcal{H} \)-negligible set.

Throughout this paper we assume that \( (X, d, \mu) \) is a complete metric space that is equipped with a doubling measure \( \mu \) and supports a \((1, 1)\)-Poincaré inequality.

3 Weak* convergence of the variation measure

In this section we prove some new results concerning the weak* convergence of the variation measure. First we collect some necessary existing results.
It follows almost directly from the definition of the total variation that this quantity is lower semicontinuous with respect to $L^1$-convergence in open sets. We also have the following stronger fact.

**Theorem 3.1** ([21, Theorem 4.5]) Let $U \subset X$ be a 1-quasiopen set. If $\|Du\|(U) < \infty$ and $u_i \to u$ in $L^1_{\text{loc}}(U)$, then

$$\|Du\|(U) \leq \liminf_{i \to \infty} \|Du_i\|(U).$$

Recall that we understand the expression $\|Du\|(U) < \infty$ to mean that there is some open set $\Omega \supset U$ such that $u \in L^1_{\text{loc}}(\Omega)$ and $\|Du\|(\Omega) < \infty$.

We also have the following.

**Theorem 3.2** ([21, Theorem 4.3]) Let $U \subset X$ be a 1-quasiopen set. If $\|Du\|(U) < \infty$, then

$$\|Du\|(U) = \inf \left\{ \liminf_{i \to \infty} \int_U g_{u_i} \, d\mu, \ u_i \in N_{\text{loc}}^1(U), \ u_i \to u \text{ in } L^1_{\text{loc}}(U) \right\},$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $U$.

From this it follows that if $U$ is 1-quasiopen, $\|Du\|(U) < \infty$, and $u_i \to u$ in $L^1_{\text{loc}}(U)$, then

$$\|Du\|(U) \leq \liminf_{i \to \infty} \int_U g_{u_i} \, d\mu,$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $U$; naturally any other 1-weak upper gradient can also be used. Note that integrals over a 1-quasiopen set $U$ make sense, since every such set is $\mu$-measurable, see [5, Lemma 9.3].

The variation measure is always absolutely continuous with respect to the 1-capacity, in the following sense.

**Lemma 3.3** ([23, Lemma 3.8]) Let $\Omega \subset X$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \subset \Omega$ with $\text{Cap}_1(A) < \delta$, then $\|Du\|(A) < \varepsilon$.

**Lemma 3.4** Let $\Omega \subset X$ be an open set and let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$. Then every 1-quasiopen subset of $\Omega$ is $\|Du\|$-measurable.

**Proof** Let $U \subset \Omega$ be 1-quasiopen. For each $j \in \mathbb{N}$, choose an open set $G_j \subset \Omega$ such that $U \cup G_j$ is open and $\text{Cap}_1(G_j) \to 0$ as $j \to \infty$. Let $H := \bigcap_{j=1}^\infty (U \cup G_j)$. Then $U \subset H$ and

$$\|Du\|(H \setminus U) \leq \|Du\|(G_j) \to 0 \text{ as } j \to \infty$$

by Lemma 3.3. The set $H$ is $\|Du\|$-measurable since it is a Borel set, and then also $U$ is $\|Du\|$-measurable. □

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Lemma 3.5 Let $U \subset X$ be 1-quasiopen and let $V \subset U$ be 1-quasiopen in $U$. Then $V$ is 1-quasiopen.

This is proved as part of [6, Proposition 3.3], but we also give a short proof here.

**Proof** Let $\varepsilon > 0$. We find a relatively open set $G_1 \subset U$ and an open set $G_2 \subset X$ such that $\text{Cap}_1(G_1) < \varepsilon / 2$, $\text{Cap}_1(G_2) < \varepsilon / 2$, $V \cup G_1$ is relatively open in $U$, and $U \cup G_2$ is open. Then clearly $V \cup G_1 \cup G_2$ is open. \(\square\)

It is known that if $\Omega \subset X$ is an open set, $u_i \to u$ in $L^1_{\text{loc}}(\Omega)$, and

$$\lim_{i \to \infty} \| Du_i \|(\Omega) = \| Du \|(\Omega) < \infty,$$

then $\| Du_i \| \to \| Du \|$ weakly* as measures in $\Omega$. Using Theorem 3.1 we will show that in fact the measures converge in a stronger topology, namely in the dual of quasicontinuous functions instead of continuous ones. This result may naturally be of independent interest and so we prove it in somewhat greater generality than is necessary for our purposes.

**Proposition 3.6** Let $U \subset X$ be 1-quasiopen and let $\| Du \|(U) < \infty$. If $u_i \to u$ in $L^1_{\text{loc}}(U)$ such that

$$\| Du \|(U) = \lim_{i \to \infty} \| Du_i \|(U),$$

then

$$\int_U \eta \, d\| Du \| = \lim_{i \to \infty} \int_U \eta \, d\| Du_i \|$$

for every bounded 1-quasicontinuous function $\eta$ on $U$.

Similarly, if $u_i \to u$ in $L^1_{\text{loc}}(U)$ such that

$$\| Du \|(U) = \lim_{i \to \infty} \int_U g_{u_i} \, d\mu,$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $U$, then

$$\int_U \eta \, d\| Du \| = \lim_{i \to \infty} \int_U \eta g_{u_i} \, d\mu$$

for every bounded 1-quasicontinuous function $\eta$ on $U$.

**Proof** We follow an argument that can be found e.g. in [2, Proposition 1.80]. Let $\eta$ be a 1-quasicontinuous bounded function on $U$. By replacing $\eta$ by $a\eta + b$ for suitable $a, b \in \mathbb{R}$, we can assume that $0 \leq \eta \leq 1$. Note that every super-level set $\{ \eta > t \}$, $t \in \mathbb{R}$, is 1-quasiopen in $U$, and thus 1-quasiopen by Lemma 3.5. By Lemma 3.4, 1-quasiopen sets are $\| Du \|$-measurable, and so the integrals in the formulation of
the proposition make sense. Let $\rho$ be a nonnegative real-valued 1-quasicontinuous function on $U$. By Cavalieri’s principle, Fatou’s lemma, and Theorem 3.1 we have

$$
\liminf_{i \to \infty} \int_U \rho \, d\|Du_i\| = \liminf_{i \to \infty} \int_0^\infty \|Du_i\|((\rho > t)) \, dt \\
\geq \int_0^\infty \liminf_{i \to \infty} \|Du_i\|((\rho > t)) \, dt \\
\geq \int_0^\infty \|Du\|((\rho > t)) \, dt \\
= \int_U \rho \, d\|Du\|.
$$

(3.2)

It is easy to check that if $(a_i)$ and $(b_i)$ are sequences of numbers such that

$$
\liminf_{i \to \infty} a_i \geq a, \quad \liminf_{i \to \infty} b_i \geq b, \quad \text{and} \quad \lim_{i \to \infty} (a_i + b_i) = a + b,
$$

then $\lim_{i \to \infty} a_i = a$ and $\lim_{i \to \infty} b_i = b$. Choosing

$$
a_i = \int_U \eta \, d\|Du_i\|, \quad a = \int_U \eta \, d\|Du\|,
$$

$$
b_i = \int_U (1 - \eta) \, d\|Du_i\|, \quad b = \int_U (1 - \eta) \, d\|Du\|,
$$

and using the fact that $\|Du_i\|(U) \to \|Du\|(U)$ as well as (3.2) with the choices $\rho = \eta$ and $\rho = 1 - \eta$, we obtain the first claim. The second claim if proved analogously, using (3.1) instead of Theorem 3.1. \qed

Now we get the following result which we will use in the sequel.

**Proposition 3.7** Let $U \subset X$ be 1-quasiopen. If $\|Du\|(U) < \infty$ and $u_i \to u$ in $L^1_{loc}(U)$ such that

$$
\|Du\|(U) = \lim_{i \to \infty} \int_U g_{u_i} \, d\mu,
$$

(3.3)

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $U$, then

$$
\int_U \eta \, d\|Du\| \geq \limsup_{i \to \infty} \int_U \eta g_{u_i} \, d\mu
$$

for every nonnegative bounded 1-quasi upper semicontinuous function $\eta$ on $U$.

**Proof** Take $M > 0$ such that $0 \leq \eta \leq M$ on $U$. For each $j \in \mathbb{N}$ we find a relatively open set $G_j \subset U$ such that $\text{Cap}_1(G_j) < 1/j$ and $\eta|_{U \setminus G_j}$ is upper semicontinuous. Then each $\eta_j := \chi_{U \setminus G_j} \eta$ is upper semicontinuous on $U$. Let

$$
\eta_{j,k}(x) := \sup \{\eta_j(y) - kd(y, x) : y \in U\}, \quad x \in U, \ k \in \mathbb{N}.
$$
It is easy to check that $\eta_j \leq \eta_{j,k} \leq M$, $\eta_{j,k} \in \text{Lip}(U)$, and $\eta_{j,k} \searrow \eta_j$ pointwise as $k \to \infty$. Fix $\varepsilon > 0$. By Lemma 3.3 we find $0 < \delta < \varepsilon$ such that whenever $A \subset U$ with $\text{Cap}_1(A) < \delta$, then $\|Du\|(A) < \varepsilon$. Choose $j \in \mathbb{N}$ such that $\text{Cap}(G_j) < \delta$. Then, using Lebesgue’s dominated convergence theorem, choose $k \in \mathbb{N}$ such that

$$\int_U \eta_j \, d\|Du\| \geq \int_U \eta_{j,k} \, d\|Du\| - \varepsilon.$$  

By Theorem 2.5 we know that $X \setminus \overline{G_j}$ is a 1-quasiopen set, and then so is $U \setminus \overline{G_j}$, since it is easy to check that the intersection of two 1-quasiopen sets is 1-quasiopen (this fact is also proved in [11, Lemma 2.3]). Thus by (3.3) and (3.1) we have that

$$\|Du\|(U \cap \overline{G_j}) = \|Du\|(U) - \|Du\|(U \setminus \overline{G_j})$$

$$\geq \lim_{i \to \infty} \int_U \eta_{j,k} \, d\mu - \liminf_{i \to \infty} \int_{U \setminus \overline{G_j}} \eta_{j,k} \, d\mu$$

$$\geq \limsup_{i \to \infty} \int_{U \cap G_j} \eta_{j,k} \, d\mu.$$  

Moreover, by (2.9), $\text{Cap}_1(U \cap \overline{G_j}) \leq \text{Cap}_1(G_j) < \delta$ and then $\|Du\|(U \cap \overline{G_j}) < \varepsilon$. Now

$$\int_U \eta \, d\|Du\| \geq \int_U \eta_j \, d\|Du\|$$

$$\geq \int_U \eta_{j,k} \, d\|Du\| - \varepsilon$$

$$\geq \lim_{i \to \infty} \int_U \eta_{j,k} g_{u_i} \, d\mu - \varepsilon \quad \text{by Proposition 3.6}$$

$$\geq \limsup_{i \to \infty} \int_U \eta_{j} g_{u_i} \, d\mu - \varepsilon$$

$$\geq \limsup_{i \to \infty} \int_U \eta g_{u_i} \, d\mu - M \limsup_{i \to \infty} \int_{U \cap G_j} g_{u_i} \, d\mu - \varepsilon$$

$$\geq \limsup_{i \to \infty} \int_U \eta g_{u_i} \, d\mu - M \|Du\|(U \cap \overline{G_j}) - \varepsilon \quad \text{by (3.4)}$$

$$\geq \limsup_{i \to \infty} \int_U \eta g_{u_i} \, d\mu - M \varepsilon - \varepsilon.$$  

Since $\varepsilon > 0$ was arbitrary, we have the result. 

\[ \square \]

4 Proof of the Leibniz rule

In this section we prove the Leibniz rule for BV functions, Theorem 1.2. Again, $\Omega$ always denotes an open set.

First we note that the total variation is lower semicontinuous not only with respect to $L^1_{\text{loc}}$-convergence, but also pointwise convergence.
Proposition 4.1 Let $u$ be finite a.e. on $\Omega$ and let $(u_i) \subset L^1_{loc}(\Omega)$ such that $u_i \rightharpoonup u$ a.e. on $\Omega$. Then

$$\|Du\|(\Omega) \leq \liminf_{i \to \infty} \|Du_i\|(\Omega).$$

In particular, if the right-hand side is finite, then $u \in L^1_{loc}(\Omega)$.

Proof We can assume that the right-hand side is finite. Let $B(x, r)$ be a ball such that $B(x, \lambda r) \subset \Omega$. By the Poincaré inequality (2.6),

$$\infty > \liminf_{i \to \infty} \|Du_i\|(B(x, \lambda r)) \geq \liminf_{i \to \infty} \|D(u_i)_+\|(B(x, \lambda r)) \geq \frac{1}{C_{Pr}} \liminf_{i \to \infty} \int_{B(x, r)} |(u_i)_+ - ((u_i)_+)_{B(x, r)}| d\mu$$

by Fatou’s lemma. Since $\lim_{i \to \infty} (u_i)_+$ exists and is finite a.e., we conclude that $\liminf_{i \to \infty} ((u_i)_+)_{B(x, r)}$ is finite. By another application of Fatou’s lemma, it follows that $(u_+)_+ \in L^1_{loc}(\Omega)$. Similarly we show that $u_- \in L^1_{loc}(\Omega)$. In conclusion, $u \in L^1_{loc}(\Omega)$. If $M > 0$ and $u_M := \min\{M, \max\{-M, u\}\}$, we have $(u_i)_M \rightharpoonup u_M$ in $L^1_{loc}(\Omega)$ by Lebesgue’s dominated convergence theorem, and so

$$\|Du_M\|(\Omega) \leq \liminf_{i \to \infty} \|D(u_i)_M\|(\Omega) \leq \liminf_{i \to \infty} \|Du_i\|(\Omega).$$

On the other hand, since we now know that $u \in L^1_{loc}(\Omega)$, we also know that $u_M \rightharpoonup u$ in $L^1_{loc}(\Omega)$ as $M \to \infty$, and so

$$\|Du\|(\Omega) \leq \liminf_{M \to \infty} \|Du_M\|(\Omega).$$

The result follows. \[ \square \]

Now we turn to the proof of the Leibniz rule. As we recall from the introduction, the Leibniz rule for Newton–Sobolev functions is a standard result also in metric spaces. We begin by noting that the rule is easy to extend to the case where one of the functions is a BV function.

Lemma 4.2 Let $u \in L^\infty(\Omega)$ with $\|Du\|(\Omega) < \infty$ and let $\eta \in N^{1,1}(\Omega) \cap L^\infty(\Omega)$. Then

$$\|D(\eta u)\|(\Omega) \leq \int_\Omega |\eta| d\|Du\| + \int_\Omega |u|g_\eta d\mu.$$

\[ \square \text{ Springer} \]
Proof By the definition of the total variation, we find a sequence \((u_i) \subset \text{Lip}_{\text{loc}}(\Omega)\) such that \(u_i \to u\) in \(L^1_{\text{loc}}(\Omega)\) and
\[
\lim_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu = \|Du\|(\Omega).
\]
By truncating if necessary, we can assume that \(\|u_i\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}\), and by passing to a subsequence (not relabeled) we can assume that \(u_i \to u\) a.e. in \(\Omega\). Also, \(\eta u_i \to \eta u\) in \(L^1_{\text{loc}}(\Omega)\), and so by lower semicontinuity and the Leibniz rule for Newton–Sobolev functions (see [4, Theorem 2.15]),
\[
\|D(\eta u)\|(\Omega) \leq \liminf_{i \to \infty} \int_{\Omega} g_{\eta u_i} \, d\mu \\
\leq \liminf_{i \to \infty} \left( \int_{\Omega} |\eta| g_{u_i} \, d\mu + \int_{\Omega} |u_i| g_{\eta} \, d\mu \right) \\
= \int_{\Omega} |\eta| \, d\|Du\| + \int_{\Omega} |u| g_{\eta} \, d\mu
\]
by the second part of Proposition 3.6 (recall that the Newton–Sobolev function \(\eta\) is quasicontinuous by e.g. [4, Theorem 5.29]) and by Lebesgue’s dominated convergence theorem.

This first step of proving the BV Leibniz rule was essentially the same as in [17]. However, to handle the case where both functions are BV functions (or even more generally locally integrable functions), we will rely on the theory of Sect. 3 as well as the following results.

Theorem 4.3 ([22, Theorem 3.2]) Let \(u_i, u \in \text{BV}(\Omega)\) such that \(u_i \to u\) in \(L^1(\Omega)\) and \(\|Du_i\|(\Omega) \to \|Du\|(\Omega)\). Then there exists a subsequence (not relabeled) such that for \(\mathcal{H}\)-a.e. \(x \in \Omega\),
\[
u^\wedge(x) \leq \liminf_{i \to \infty} u_i^\wedge(x) \leq \limsup_{i \to \infty} u_i^\vee(x) \leq u^\vee(x).
\]

Next we note that Federer’s characterization of sets of finite perimeter holds also in metric spaces.

Theorem 4.4 ([20, Theorem 1.1]) Let \(\Omega \subset X\) be open, let \(E \subset X\) be \(\mu\)-measurable, and suppose that \(\mathcal{H}(\partial^* E \cap \Omega) < \infty\). Then \(P(E, \Omega) < \infty\).

Recall the definition of the class \(\text{BV}_0(D, \Omega)\) from (2.7). The following result and its proof are similar to [27, Theorem 6.1], which was originally based on [16, Theorem 1.1].

Theorem 4.5 Let \(W \subset \Omega \subset X\) be open sets and let \(u \in \text{BV}(W)\) such that
\[
\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap W} |u| \, d\mu = 0 \quad (4.1)
\]
for \(\mathcal{H}\)-a.e. \(x \in \Omega \cap \partial W\). Then \(u \in \text{BV}_0(W, \Omega)\).
Theorem 4.6

Let $u \in L^1_{loc}(\Omega)$ and $v \in L^1_{loc}(\Omega)$. Then

$$\|D uv\|_1(\Omega) \leq \int_{\Omega} |u|^{\gamma} d\|Dv\| + \int_{\Omega} |v|^{\gamma} d\|Du\|.$$  

(4.2)

Remark 4.7

Note that since $|u|^{\gamma}, |v|^{\gamma}$ are Borel functions and $\|D u\|, \|D v\|$ are Borel measures by Theorem 2.2, the integrals are always well defined.

Given functions $u, v \in L^1_{loc}(\Omega)$, it is of course not always true that $uv \in L^1_{loc}(\Omega)$, but implicit in the theorem is the fact that if the right-hand side is finite, then necessarily $uv \in L^1_{loc}(\Omega)$. Suppose this is the case. Theorem 2.2 implies that $\|D uv\|$, $\|D v\|$, and $\|Du\|$ are all Borel measures on $\Omega$, and since $|u|^{\gamma}$ and $|v|^{\gamma}$ are Borel functions,
it is a standard result (see e.g. [29, Theorem 1.29]) that $|u|^\gamma d\|Dv\|$ and $|v|^\gamma d\|Du\|$ are Borel measures on $\Omega$. They are also of finite mass (note that $\|Du\|$ and $\|Dv\|$ themselves might not be even locally finite), and thus by e.g. [2, Proposition 1.43] we know that the measure of Borel sets can be approximated from the outside by open sets. Inequality (4.2) holds of course also with $\Omega$ replaced by any open set $W \subset \Omega$, and then

$$d\|D(uv)\| \leq |u|^\gamma d\|Dv\| + |v|^\gamma d\|Du\|$$

as Borel measures on $\Omega$.

**Proof of Theorem 4.6** First assume that $\Omega$ is bounded, that $u, v \in L^\infty(\Omega)$, and that $\|Du\| (\Omega) < \infty$ and $\|Dv\| (\Omega) < \infty$. Take a sequence $(u_i) \subset \text{Lip}_{loc}(\Omega)$ such that $u_i \to u$ in $L^1_{loc}(\Omega)$ and

$$\lim_{i \to \infty} \int_{\Omega} g_{u_i} d\mu = \|Du\| (\Omega).$$

We can assume that $\|u_i\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$ for all $i \in \mathbb{N}$. Under our assumptions, we can in fact also assume that $(u_i) \subset N^{1,1}(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$. By (2.5) and by the lower semicontinuity of the total variation, it follows that also $\lim_{i \to \infty} \|Du_i\| (\Omega) = \|Du\| (\Omega)$. By Lemma 4.2 we have for all $i \in \mathbb{N}$

$$\|D(u_i v)\| (\Omega) \leq \int_{\Omega} |u_i| d\|Dv\| + \int_{\Omega} |v| g_{u_i} d\mu$$

$$= \int_{\Omega} |u_i| d\|Dv\| + \int_{\Omega} |v|^\gamma g_{u_i} d\mu. \quad (4.3)$$

We pass to a subsequence of $(u_i)$ (not relabeled) for which the conclusion of Theorem 4.3 holds. Note that now also for $\mathcal{H}$-a.e. $x \in \Omega$,

$$\limsup_{i \to \infty} (-u_i)^\gamma (x) = \limsup_{i \to \infty} -u_i^\wedge (x) = - \liminf_{i \to \infty} u_i^\wedge (x) \leq -u^\wedge (x) = (-u)^\gamma (x). \quad (4.4)$$

Since the $u_i$ are continuous functions, of course $u_i = u_i^\wedge = u_i^\vee$. Now by Theorem 4.3 we have for $\mathcal{H}$-a.e. $x \in \Omega$, and thus also for $\|Dv\|$-a.e. $x \in \Omega$ (recall (2.4))

$$\limsup_{i \to \infty} u_i (x) = \limsup_{i \to \infty} u_i^\vee (x) \leq u^\vee (x) \leq |u|^\vee (x)$$

and

$$\limsup_{i \to \infty} (-u_i) (x) = \limsup_{i \to \infty} (-u_i)^\gamma (x) \leq (-u)^\gamma (x) \leq |u|^\gamma (x),$$

so that $\limsup_{i \to \infty} |u_i| (x) \leq |u|^\vee (x)$. Recall that $|u|^\vee$ is a Borel function and thus $\|Dv\|$-measurable. Now we have by Fatou’s lemma
Since \( \|Dv\| (\Omega) < \infty \), clearly also \( \|D|v|\| (\Omega) < \infty \), and so by Proposition 2.3, \( |v|^\gamma \) is a bounded 1-quasi upper semicontinuous function on \( \Omega \). Thus we have by Proposition 3.7

\[
\limsup_{i \to \infty} \int_\Omega |u_i| d\|Dv\| \leq \int_\Omega \limsup_{i \to \infty} |u_i| d\|Dv\| \leq \int_\Omega |u|^\gamma d\|Dv\|.
\]

Since \( u_i v \to uv \) in \( L^1 (\Omega) \), by lower semicontinuity and by (4.3) we now have

\[
\|D(uv)\| (\Omega) \leq \liminf_{i \to \infty} \|D(u_i v)\| (\Omega) \leq \int_\Omega |u|^\gamma d\|Dv\| + \int_\Omega |v|^\gamma d\|Du\|,
\]

which is the desired result.

Now we drop the assumption that \( \|Du\| (\Omega) < \infty \) and \( \|Dv\| (\Omega) < \infty \). We can assume that the right-hand side of (4.2) is finite. Let

\[
A_k := \{ x \in \Omega : |u|^\gamma (x) > 1/k \text{ and } |v|^\gamma (x) > 1/k \}, \quad k \in \mathbb{N}.
\]

Then necessarily \( \|Du\| (A_k) < \infty \) and \( \|Dv\| (A_k) < \infty \), and so for some open set \( W_k \) with \( A_k \subset W_k \subset \Omega \) we have \( \|Du\| (W_k) < \infty \) and \( \|Dv\| (W_k) < \infty \). Then the theorem holds in \( W_k \), that is,

\[
\|D(uv)\| (W_k) \leq \int_{W_k} |u|^\gamma d\|Dv\| + \int_{W_k} |v|^\gamma d\|Du\| \leq \int_\Omega |u|^\gamma d\|Dv\| + \int_\Omega |v|^\gamma d\|Du\|.
\]

Letting \( k \to \infty \), by Theorem 2.2 we get for \( W := \bigcup_{k=1}^\infty W_k \)

\[
\|D(uv)\| (W) \leq \int_\Omega |u|^\gamma d\|Dv\| + \int_\Omega |v|^\gamma d\|Du\|. \tag{4.5}
\]

Also, \( (uv)|_W \in L^1 (W) \) since \( W \) is bounded and since \( u, v \in L^\infty (\Omega) \), and so \( (uv)|_W \in BV(W) \). Since \( W \supset \{|u|^\gamma > 0\} \cap \{|v|^\gamma > 0\} \) and since \( u, v \in L^\infty (\Omega) \), we have

\[
\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x, r) \cap W} |uv| d\mu = 0
\]

for all \( x \in \partial W \cap \Omega \). By Theorem 4.5 we find that \( (uv)|_W \in BV_0 (W, \Omega) \). We have \( uv = 0 \) a.e. on \( \Omega \setminus W \) by Lebesgue’s differentiation theorem (see e.g. [14, Chapter 1]).

Thus by (2.8) and (4.5),

\[
\|D(uv)\| (\Omega) = \|D(uv)\| (W) \leq \int_\Omega |u|^\gamma d\|Dv\| + \int_\Omega |v|^\gamma d\|Du\|.
\]
Next we drop the assumption that $\Omega$ is bounded. For a fixed point $x \in X$, we have by Theorem 2.2

$$\|D(uv)(\Omega)\| = \lim_{R \to \infty} \|D(uv)(\Omega \cap B(x, R))\|$$

$$\leq \limsup_{R \to \infty} \left( \int_{\Omega \cap B(x, R)} |u|^\gamma d\|Dv\| + \int_{\Omega \cap B(x, R)} |v|^\gamma d\|Du\| \right)$$

$$\leq \int_{\Omega} |u|^\gamma d\|Dv\| + \int_{\Omega} |v|^\gamma d\|Du\|.$$ 

Finally we drop the assumption $u, v \in L^\infty(\Omega)$. Let

$$u_M := \min\{M, \max\{-M, u\}\}, \quad M > 0.$$ 

By the above, we have

$$\|D(u_M v_M)(\Omega)\| \leq \int_{\Omega} |u_M|^\gamma d\|Dv_M\| + \int_{\Omega} |v_M|^\gamma d\|Du_M\|$$

$$\leq \int_{\Omega} |u|^\gamma d\|Dv\| + \int_{\Omega} |v|^\gamma d\|Du\|.$$ 

Now we can use the lower semicontinuity of Proposition 4.1 to get

$$\|D(uv)(\Omega)\| \leq \liminf_{M \to \infty} \|D(u_M v_M)(\Omega)\| \leq \int_{\Omega} |u|^\gamma d\|Dv\| + \int_{\Omega} |v|^\gamma d\|Du\|. \quad \Box$$

**Example 4.8** Recall that in Euclidean spaces we have the Leibniz rule (1.1), which for nonnegative $u, v$ yields the scalar version

$$d\|D(uv)\| \leq \overline{u} d\|Dv\| + \overline{v} d\|Du\|.$$ 

Here $\overline{u}(x) := \limsup_{r \to 0} \frac{1}{\mu_B(x, r)} u d\mathcal{L}^n$, where $\mathcal{L}^n$ is the $n$-dimensional Lebesgue measure. In metric spaces this version of the Leibniz rule does not hold; in [17, Example 4.3] (only in the arxiv version) the following counterexample was given: equip $\mathbb{R}^2$ with the weighted Lebesgue measure $d\mu := w d\mathcal{L}^2$, where $w := 2 - \chi_{B(0,1)}$ and the origin $(0, 0)$ is denoted by 0. Let $u := v := \chi_{B(0,1)}$. Then $\overline{u} = \overline{v} = 1/3$ on $\partial B(0, 1)$, and it follows that

$$d\|D(uv)\| = d\|Du\| > \frac{2}{3} d\|Du\| = \overline{v} d\|Du\| + \overline{v} d\|Dv\|.$$ 

On the other hand, sometimes one also defines $\overline{u} := (u^\wedge + u^\vee)/2$. With this definition we would have $\overline{u} = \overline{v} = 1/2$ on $\partial B(0, 1)$ and then

$$d\|D(uv)\| = \overline{v} d\|Du\| + \overline{u} d\|Dv\|.$$
Thus with this definition of the representative \( \pi \), we seem to need a different type of counterexample, which we construct as follows. Consider the space

\[
X := \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0 \}
\]

consisting of the two coordinate axes. Equip this space with the Euclidean metric inherited from \( \mathbb{R}^2 \), and let \( \mu \) be the 1-dimensional Hausdorff measure. It is straightforward to check that this measure is doubling and supports a \((1, 1)\)-Poincaré inequality.

Let

\[
u := \chi_X - \chi_{\{x_2 \geq 0\}} \in \text{BV}_{\text{loc}}(X)
\]

and

\[
v := \chi_X - \chi_{\{x_1 > 0\}} \in \text{BV}_{\text{loc}}(X).
\]

For brevity, denote the origin \((0, 0)\) by \(0\). It is straightforward to check that

\[
\|Du\| (X) = \|Du\| (0) = \|Dv\| (X) = \|Dv\| (0) = 1,
\]

and that \(\|D(uv)\| (X) = \|D(uv)\| (0) \leq 2\). To see that in fact equality holds, take a sequence \((u_i) \subset \text{Lip}_{\text{loc}}(X)\) such that \(u_i \to uv\) in \(L^1_{\text{loc}}(X)\). Passing to a subsequence (not relabeled) we have also \(u_i \to uv\) a.e. in \(X\). Thus we find \(0 < t < 1\) such that for the points \(x_1 := (t, 0)\), \(x_2 := (0, t)\), \(x_3 := (-t, 0)\), and \(x_4 := (0, -t)\) we have \(u_i(x_1) \to 0\), \(u_i(x_2) \to 0\), \(u_i(x_3) \to 1\), and \(u_i(x_4) \to 1\) as \(i \to \infty\). Note that in this one-dimensional setting, each pair \((u_i, g_{u_i})\) satisfies the upper gradient inequality for every curve in the space. Let \(\varepsilon > 0\). Now

\[
\int_{B(0,1)} g_{u_i} \, d\mu \geq |u_i(x_1) - u_i(0)| + |u_i(x_2) - u_i(0)|
\]

\[
+ |u_i(x_3) - u_i(0)| + |u_i(x_4) - u_i(0)|
\]

\[
\geq 2|u_i(0)| + 2|1 - u_i(0)| - \varepsilon \quad \text{for large } i \in \mathbb{N}
\]

\[
\geq 2 - \varepsilon.
\]

Since \(\varepsilon > 0\) was arbitrary, it follows that \(\|D(uv)\| (X) \geq 2\) and so in fact equality holds.

In conclusion,

\[
\|D(uv)\| (0) = 2 > 1 = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = \pi(0) \|Dv\| (0) + \overline{\pi}(0) \|Du\| (0).
\]

Here we have in fact exactly

\[
\frac{d}{d} \|D(uv)\| = u^\vee \frac{d}{d} \|Dv\| + v^\vee \frac{d}{d} \|Du\|,
\]

demonstrating in this respect the sharpness of our Leibniz rule.
Remark 4.9 Consider inequality (4.3) in the proof of Theorem 4.6,
\[ \| D(u_i v) \| (\Omega) \leq \int_{\Omega} |u_i| d\|Dv\| + \int_{\Omega} |v|^\vee g_{u_i} d\mu. \]
In the proof of the Leibniz rule given in [17] (recall Proposition 1.1), the functions \( u_i \) were taken to be discrete convolution approximations of \( u \), because such approximations and their upper gradients can be described by explicit formulas which enables analysis of limiting behavior. However, this produces the constant \( C \geq 1 \) in the end result. We are able to avoid this constant by exploiting the convenient way in which two BV functions “pair up” on the right-hand side of (4.3): the pointwise convergence of \( (u_i) \) is \( \|Dv\| \)-almost everywhere, and the weak* convergence of \( g_{u_i} \) is in a strong enough topology that \( |v|^\vee \) can act as a test function.

References
1. Ambrosio, L.: Fine properties of sets of finite perimeter in doubling metric measure spaces, calculus of variations, nonsmooth analysis and related topics. Set Valued Anal. 10(2–3), 111–128 (2002)
2. Ambrosio, L., Fusco, N., Pallara, D.: Functions of Bounded Variation and Free Discontinuity Problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2000)
3. Ambrosio, L., Miranda, M., Jr., Pallara, D.: Special Functions of Bounded Variation in Doubling Metric Measure Spaces, Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Università di Napoli, Caserta (2004)
4. Björn, A., Björn, J.: Nonlinear Potential Theory on Metric Spaces, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich. xii+403 pp (2011)
5. Björn, A., Björn, J.: Obstacle and Dirichlet problems on arbitrary nonopen sets in metric spaces, and fine topology. Rev. Mat. Iberoam. 31(1), 161–214 (2015)
6. Björn, A., Björn, J., Malý, J.: Quasiconvex and p-path open sets, and characterizations of quasicontinuity. Potential Anal. 46(1), 181–199 (2017)
7. Björn, A., Björn, J., Shanmugalingam, N.: Quasicontinuity of Newton–Sobolev functions and density of Lipschitz functions on metric spaces. Houston J. Math. 34(4), 1197–1211 (2008)
8. Carrero, M., Dal Maso, G., Leaci, A., Pascali, E.: Relaxation of the nonparametric plateau problem with an obstacle. J. Math. Pures Appl. (9) 67(4), 359–396 (1988)
9. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics Series. CRC Press, Boca Raton (1992)
10. Federer, H.: Geometric Measure Theory, Die Grundlehren der mathematischen Wissenschaften, Band vol. 153. Springer, New York Inc., New York, xiv+676 pp (1969)
11. Fuglede, B.: The quasi topology associated with a countably subadditive set function. Ann. Inst. Fourier 21(1), 123–169 (1971)
12. Giusti, E.: Minimal Surfaces and Functions of Bounded Variation, Monographs in Mathematics, vol. 80. Birkhäuser Verlag, Basel. xii+240 pp (1984)
13. Hakkarainen, H., Kinnunen, J., Lahtii, P., Lehtelä, P.: Relaxation and integral representation for functions of linear growth on metric measure spaces. Anal. Geom. Metr. Spaces 4, 288–313 (2016)
14. Heinonen, J.: Lectures on Analysis on Metric Spaces, Universitext. Springer, New York, x+140 pp (2001)
15. Heinonen, J., Koskela, P.: Quasiconformal maps in metric spaces with controlled geometry. Acta Math. 181(1), 1–61 (1998)
16. Kinnunen, J., Korte, R., Shanmugalingam, N., Tuominen, H.: A characterization of Newtonian functions with zero boundary values. Calc. Var. Partial Differ. Equ. 43(3–4), 507–528 (2012)
17. Kinnunen, J., Korte, R., Shanmugalingam, N., Tuominen, H.: Pointwise properties of functions of bounded variation in metric spaces. Rev. Mat. Complut. 27(1), 41–67 (2014)
18. Lahtii, P.: A Federer-style characterization of sets of finite perimeter on metric spaces. In: Calculus of Variations and Partial Differential Equations, vol. 56, no. 5, Art. 150, 22 pp (2017)
19. Lahti, P.: A notion of fine continuity for BV functions on metric spaces. Potential Anal. 46(2), 279–294 (2017)
20. Lahti, P.: Federer’s characterization of sets of finite perimeter in metric spaces, to appear in Analysis & PDE
21. Lahti, P.: Quasiopen sets, bounded variation and lower semicontinuity in metric spaces. In: Potential Analysis (to appear)
22. Lahti, P.: Strict and pointwise convergence of BV functions in metric spaces. J. Math. Anal. Appl. 455(2), 1005–1021 (2017)
23. Lahti, P.: Strong approximation of sets of finite perimeter in metric spaces. Manuscripta Math. 155(3–4), 503–522 (2018)
24. Lahti, P.: The Choquet and Kellogg properties for the fine topology when $p = 1$ in metric spaces. J. Math. Pures Appl. 126, 195–213 (2019)
25. Lahti, P.: The variational 1-capacity and BV functions with zero boundary values on metric spaces. In: Advances in Calculus of Variations (to appear)
26. Lahti, P., Shanmugalingam, N.: Fine properties and a notion of quasicontinuity for BV functions on metric spaces. J. Math. Pures Appl. 107(2), 150–182 (2017)
27. Lahti, P., Shanmugalingam, N.: Trace theorems for functions of bounded variation in metric spaces. J. Funct. Anal. 274(10), 2754–2791 (2018)
28. Miranda Jr., M.: Functions of bounded variation on “good” metric spaces. J. Math. Pures Appl. (9) 82(8), 975–1004 (2003)
29. Rudin, W.: Real and Complex Analysis, 3rd edition. McGraw-Hill Book Co., New York, (1987). xiv+416 pp
30. Shanmugalingam, N.: Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. Rev. Mat. Iberoam. 16(2), 243–279 (2000)
31. Volpert, A.I.: Spaces BV and quasilinear equations. (Russian). Mat. Sb. (N.S.) 73(115), 255–302 (1967)
32. Volpert, A.I., Hudjaev, S.I.: Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics, Mechanics: Analysis, vol. 8. Martinus Nijhoff Publishers, Dordrecht, xviii+678 pp (1985)
33. Ziemer, W.P.: Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation, Graduate Texts in Mathematics, vol. 120. Springer, New York (1989)

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