On convex hulls and pseudoconvex domains generated by $q$-plurisubharmonic functions, part II

Thomas Pawlaschyk¹ · Eduardo S. Zeron²

Abstract We study the properties of hulls of compact sets in $\mathbb{C}^n$ that are generated by certain subfamilies of $q$-plurisubharmonic functions. We consider in particular those functions that are plurisubharmonic on the level sets of holomorphic mappings defined from $\mathbb{C}^n$ into $\mathbb{C}^q$. We also compare the above-mentioned hulls against the generalised polynomially and rationally convex hulls already defined in the literature. Our main result yields that the hulls defined by $q$-plurisubharmonic functions or $q$-pseudoconvex sets are all $q$-maximum, so that their complement is relatively $(n-q-2)$-pseudoconvex.

To Professor Sergei Grudsky on the occasion of his 60th anniversary.

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1 Introduction

The present paper is a sequel of our previous work [12] and it has multiple purposes: we analyse the properties of those functions that are $r$-plurisubharmonic on the level sets of holomorphic mappings defined from $\mathbb{C}^n$ into $\mathbb{C}^q$. Moreover, we also make a review of the properties of (and the relations between) the $q$-maximum sets, the hulls that are created via the subfamilies of $q$-plurisubharmonic functions described above, and the generalised polynomially and rationally convex hulls already defined in the literature.

We enlist in Sect. 2 the definition and basic properties of the $q$-plurisubharmonic functions. We include in particular some of the results given in [1,5,9,13]. We also characterise those continuous mappings $h : U \to \mathbb{C}^m$, for which every upper semi-continuous function $\psi : U \to (\mathbb{R}^+)\cup \{\text{min} \}$ is $q$-plurisubharmonic on $U$, whenever $\psi$ is already 0-plurisubharmonic on the inverse fibres $h^{-1}(c)$ for every $c \in h(U)$. The family of $q$-plurisubharmonic functions $\psi$ constructed in this form has very interesting properties, as it is shown in Sect. 4 of this work. This family includes, for example the upper semi-continuous compositions $\varphi(h)$; and all holomorphic mappings $h$ trivially have the characteristics announced above.

Section 3 is devoted to enlist the definitions and some properties of the relative $q$-pseudoconvex and $k$-maximum sets defined by Słodkowski in [14]. The $k$-maximum sets have the important property that every $k$-plurisubharmonic function admits the local maximum principle on them. Moreover, we believe that the original proof of Corollary 3.11 in [12] has some minor gaps, so that we present now a complete and new proof.

Some of the principal results of this work are presented in Sect. 4. We analyse here some of the properties and relations existing between the following sets: the hulls $\hat{K}^q_U$ defined via the family of functions $q$-plurisubharmonic on $U$ (and which are used in the characterisation of the $q$-pseudoconvex sets); the hulls defined via the families of functions constructed in Sect. 2; the hulls $\mathcal{H}_q(K)$ defined as the intersection of all $q$-pseudoconvex neighbourhoods of $K$ in $\mathbb{C}^n$; and the generalised polynomially and rationally convex hulls introduced by Basener in [1] and studied by Lupacciolu and Stout in [10]. We show, for example that the generalised Basener polynomially convex hull $h_q(K)$ indeed coincides with the hull generated by those upper semi-continuous functions that are plurisubharmonic on the inverse fibres $p^{-1}(c)$ of some holomorphic polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^q$.

We close this paper by showing that $\hat{K}^q\setminus K$ and $\mathcal{H}_q(K)\setminus K$ are both $q$-maximum for any compact set $K \subset \mathbb{C}^n$; i.e. every $q$-plurisubharmonic function admits the local maximum principle on $\hat{K}^q\setminus K$ and $\mathcal{H}_q(K)\setminus K$. This result is proved in Sect. 5 and is inspired by Słodkowski’s results published in [14]. As a consequence, given any $q$-pseudoconvex domain $D$ in the complement of $K$, the difference of $D$ minus $\hat{K}^{n-q-2}$ or $\mathcal{H}_{n-q-2}(K)$ is again $q$-pseudoconvex. The importance of this result comes from
the fact that the 0-pseudoconvex domains are Stein ones; and moreover it generalises a classical result for the case when \( n = 2 \) and \( q = 0 \); see for example Theorem 5.2.18 in [15].

### 2 \( q \)-plurisubharmonic functions

In this section, we begin recalling the basic properties of \( q \)-plurisubharmonic functions. In particular, we introduce functions that are plurisubharmonic on the inverse fibres \( h^{-1}(c) \) of a holomorphic mapping with image in \( \mathbb{C}^m \). This kind of functions is interesting because they are naturally \( q \)-plurisubharmonic on their domain of definition. Here, the integer \( q \geq 0 \) is the supremum of the codimension of all inverse fibres of \( h \). Moreover, the composition \( \varphi(h) \) is \( q \)-plurisubharmonic for every upper semi-continuous function \( \varphi \) defined on the image of \( h \). Later in this paper we shall use this type of functions to characterise the so-called generalised polynomially convex hulls.

We proceed with the main definitions: from now on \( K \subset \mathbb{C}^n \) denotes a compact set, while \( D \) stands for a domain in \( \mathbb{C}^n \) and \( U \) for an arbitrary open set in \( \mathbb{C}^n \). We also use the standard notation for the balls and polydisks centred at a point \( p \in \mathbb{C}^n \) and with radius \( r > 0 \),

\[
B^n_r(p) = \{ z \in \mathbb{C}^n : \| z - p \| < r \} \quad \text{and} \quad \Delta^n_r(p) = \{ z \in \mathbb{C}^n : | z - p |_\infty < r \},
\]

where \( | z |_\infty = \max_{1 \leq k \leq n} | z_k | \) is the maximum norm and \( \| z \|^2 = \sum_{k=1}^{n} | z_k |^2 \) is the square of the Euclidean norm. Moreover, given a complex function \( f \) defined on an arbitrary set \( X \subset \mathbb{C}^n \), we write \( \| f \|_X \) as an abbreviation to \( \sup_X | f | \).

We begin recalling the definition of a \( q \)-plurisubharmonic function introduced by Hunt and Murray; see for example [9].

**Definition 2.1** Let \( u : U \rightarrow [-\infty, \infty) \) be an upper semi-continuous function defined on some open set \( U \subset \mathbb{C}^n \), and \( q \) be an integer such that \( 0 \leq q < n \).

1. The function \( u \) is said to be \((n-1)\)-plurisubharmonic on \( U \) if and only if for every plurisuperharmonic function \( h \) defined in a neighbourhood of a compact ball \( \overline{B} \subset U \) the inequality \( u \leq h \) holds on \( \overline{B} \), whenever \( u \leq h \) already holds on the boundary \( \partial B \).
2. We also say that \( u \) is \( q \)-plurisubharmonic on \( U \) if and only if it is \( q \)-plurisubharmonic on the intersection \( D \cap \pi \) with every possible \((q+1)\)-dimensional complex plane \( \pi \) in \( \mathbb{C}^n \).
3. Finally, we say by convention that any upper semi-continuous function is \( N \)-plurisubharmonic whenever \( N \geq n \).
4. The set of all \( q \)-plurisubharmonic functions on \( U \) is denoted by \( \text{PSH}_q(U) \). We simply write \( \text{PSH}_q \) instead of \( \text{PSH}_q(\mathbb{C}^n) \) when \( U = \mathbb{C}^n \).

Słodkowski showed in Lemma 4.4 of [14] that conditions in the point (1) above can be relaxed; namely, it is sufficient to consider pluriharmonic polynomials (globally defined on \( \mathbb{C}^n \)) instead of plurisuperharmonic functions. Recall that a pluriharmonic polynomial (or function) is the real part \( \pm \text{Re}(h) \) of a holomorphic one \( h \). Moreover,
once the point (1) is rewritten, the whole Definition 2.1 can be restated as well in terms of pluriharmonic polynomials. We present below a collection of properties of $q$-plurisubharmonic functions which can be found in various papers and are needed along this work; see for example the references [5,7,9,12,13].

**Proposition 2.2** Let $q$ and $r$ be a pair of non-negative integers, and $U$ be an arbitrary open set in $\mathbb{C}^n$.

1. The 0-plurisubharmonic functions are the classical plurisubharmonic ones. Moreover, $\text{PSH}_q(U)$ is contained in $\text{PSH}_r(U)$ whenever $q \leq r$.
2. The $q$-plurisubharmonicity is a local property for every $q \geq 0$.
3. Given $c > 0$ in $\mathbb{R}$ and two functions $u \in \text{PSH}_q(U)$ and $v \in \text{PSH}_r(U)$,
   
   \[ cu \in \text{PSH}_q(U), \quad \max\{u, v\} \in \text{PSH}_{\max\{q, r\}}(U), \quad u + v \in \text{PSH}_{q+r}(U), \quad \min\{u, v\} \in \text{PSH}_{q+r+1}(U). \]
4. A $C^2$-smooth function $u$ is $q$-plurisubharmonic on $U$ if and only if its complex Hessian $\frac{\partial^2 u}{\partial z_k \partial \bar{z}_\ell}$ has at most $q$ negative eigenvalues at each point in $U$.
5. If $u_1 \geq u_2 \geq \ldots$ is a non-increasing sequence of functions in $\text{PSH}_q(U)$, then the point-wise limit $\lim_{k \to \infty} u_k$ lies in $\text{PSH}_q(U)$.
6. Given a collection $\{u_j\}_{j \in J}$ in $\text{PSH}_q(U)$ which is locally bounded from above at each point in $U$, the upper semi-continuous regularisation of $v = \sup_{j \in J} u_j$,

   \[ z \mapsto v^*(z) := \limsup_{\zeta \to z} v(\zeta) \quad \text{lies in} \quad \text{PSH}_q(U). \]
7. Assume that the open set $U \subset \mathbb{C}^n$ is bounded and that the integer $q \geq 0$ satisfies $q < n$. Given any function $u \in \text{PSH}_q(U)$ that is upper semi-continuous up to the boundary $bU$, it satisfies the maximum principle; i.e.

   \[ \max_{\overline{U}} u = \max_{bU} u. \]

8. Consider $u \in \text{PSH}_q(U)$ and $w \in \text{PSH}_q(W)$, such that $U$ is properly contained in a second open set $W \subset \mathbb{C}^n$. If the following inequality holds

   \[ w(z) \geq \limsup_{\zeta \to z} u(\zeta) \quad \text{for every point} \quad z \in W \cap bU; \]

   then, the function $\psi$ below lies in $\text{PSH}_q(W)$,

   \[ \psi = \begin{cases} w & \text{on } W \setminus U, \\ \max\{u, w\} & \text{on } U. \end{cases} \]

9. If $\psi$ lies in $\text{PSH}_q(U)$ and $g : V \to \mathbb{R}$ is a non-decreasing and convex function defined on a neighbourhood $V$ of $\psi(U)$ in $\mathbb{R}$, then the composition $g \circ \psi$ lies again in $\text{PSH}_q(U)$. 

Given any $\psi \in \text{PSH}_q(U)$, the composition $\psi \circ h$ lies in $\text{PSH}_q(W)$ for every holomorphic mapping $h$ defined on an open set $W \subset \mathbb{C}^k$ and with image in $U$. Moreover, an upper semi-continuous function $\varphi : U \to [-\infty, \infty)$ is $q$-plurisubharmonic if and only if $\varphi \circ h$ is $q$-plurisubharmonic for every holomorphic mapping $h$ defined on a domain $D \subset \mathbb{C}^{q+1}$ and with image in $U$.

Notice that the composition $\psi \circ h$ in the statement (10) above is trivially $q$-plurisubharmonic on the open set $W \subset \mathbb{C}^k$ whenever the dimension $k \leq q$, because of the point (3) in Definition 2.1, so that the result present in the statement (10) above is only relevant when $k > q$. We now introduce the main definition of this section: we define special subfamilies of upper semi-continuous functions which are $q$-plurisubharmonic on the inverse fibres $h^{-1}(c)$ of particular mappings $h$.

**Definition 2.3** Let $h : U \to \mathbb{C}^m$ be a continuous mapping defined on an open subset $U \subset \mathbb{C}^n$, and $q \geq 0$ be any non-negative integer.

1. The mapping $h$ is said to have $q$-special inverse fibres $h^{-1}(c)$ if and only if the characteristic function below is $q$-plurisubharmonic on the variable $z \in U$ for each fixed point $c \in h(U)$,

$$z \mapsto \chi_c(z) := \begin{cases} 0 & \text{if } z \in h^{-1}(c), \\ -\infty & \text{otherwise.} \end{cases}$$

2. Let $\psi : X \to [-\infty, \infty)$ be an upper semi-continuous function defined on a closed subset $X$ of $U$ (so that $X$ is equal to $\overline{X} \cap U$). We say that $\psi$ is $q$-plurisubharmonic on $X$ if and only if for every point $x \in X$ there are an open neighbourhood $V$ of $x$ in $U$ and a $q$-plurisubharmonic function $\Psi$ in $\text{PSH}_q(V)$ such that $\psi \equiv \Psi$ holds on $V \cap X$.

3. The notation $\text{PSH}(q, h, U)$ stands for the family of all upper semi-continuous functions well defined on $U$ and $q$-plurisubharmonic on the inverse fibre $h^{-1}(c)$ for every $c \in h(U)$; so that the restriction $\psi|_{h^{-1}(c)}$ is $q$-plurisubharmonic on the closed subset $h^{-1}(c)$ of $U$ according to the previous point (2). We write $\text{PSH}(h, U)$ instead of $\text{PSH}(0, h, U)$ when $q = 0$.

The following proposition is the main result of this section; it gives a natural and useful characterisation of the mappings with $q$-special inverse fibres.

**Proposition 2.4** Let $h : U \to \mathbb{C}^m$ be a continuous mapping defined on an open subset $U \subset \mathbb{C}^n$, and $q \geq 0$ be a non-negative integer. The following statements are all equivalent.

1. The mapping $h$ has $q$-special inverse fibres.
2. Every function $\psi \in \text{PSH}(h, U)$ is $q$-plurisubharmonic on $U$.
3. The composition $\varphi(h)$ is $q$-plurisubharmonic on $U$ for every upper semi-continuous function $\varphi$ defined on the image $h(U) \subset \mathbb{C}^m$.
4. The functions $z \mapsto -\|h(z) - c\|^2$ are $q$-plurisubharmonic on $U$ for all vectors $c$ in $h(U)$. 
Proof (1) $\Rightarrow$ (2) Let $\psi : U \rightarrow [-\infty, \infty)$ be a function in $\text{PSH}(h, U)$, so that the restriction of $\psi$ to $h^{-1}(c)$ is $0$-plurisubharmonic for every $c \in h(U)$. Recall points (2) and (3) in Definition 2.3. Consider as well the characteristic function restriction of $\psi$ of $h$. Given any upper semi-continuous function in $\text{PSH}$, the $q$-plurisubharmonic on $U$ for each fixed point $c \in h(U)$; and we assert that $\psi + \chi_c$ is also $q$-plurisubharmonic on $U$. Given any point $x$ in $h^{-1}(c)$, there exist an open neighbourhood $V$ of $x$ in $U$ and a $0$-plurisubharmonic function $\Psi$ on $V$ such that $\Psi \equiv \Psi$ holds on the intersection $V \cap h^{-1}(c)$. Since $\chi_c$ is identically equal to $-\infty$ on the complement of $h^{-1}(c)$, we have that $\Psi + \chi_c$ is both equal to $\psi + \chi_c$ and $q$-plurisubharmonic on $V$ according to point (3) in Proposition 2.2.

The previous statements actually mean that $\psi + \chi_c$ is locally $q$-plurisubharmonic, and so it is $q$-plurisubharmonic on $U$ according to the same Proposition 2.2. Finally, since $\psi \in \text{PSH}(h, U)$ is upper semi-continuous, we have that $\psi$ is also $q$-plurisubharmonic on $U$ because it is equal to the supremum:

$$\psi(z) = \sup_{c \in h(U)} (\psi(z) + \chi_c(z)) \text{ for every } z \in U.$$  

(2) $\Rightarrow$ (3) The composition $\varphi(h)$ belongs to $\text{PSH}(h, U)$, because it is constant (and equal to $\varphi(c)$) on the inverse fibre $h^{-1}(c)$ for every $c \in h(U)$, so that $\varphi(h)$ is $q$-plurisubharmonic on $U$.

(3) $\Rightarrow$ (4) This implication is obvious.

(4) $\Rightarrow$ (1) Given any $c \in h(U)$ and $k \geq 1$, the function $\varphi_k = -k \| h - c \|^2$ is $q$-plurisubharmonic on $U$ and the sequence $\{ \varphi_k \}_{k \geq 1}$ is decreasing and converges pointwise to the function $\chi_c$ in (1) as $k$ tends to infinity. The point (5) in Proposition 2.2 yields the $q$-plurisubharmonicity for $\chi_c$ on $U$, and so the mapping $h$ has $q$-special inverse fibres according to Definition 2.3. \(\blacksquare\)

The following two corollaries are really simple and useful. It is quite interesting to compare the result below against Theorem 2.5 and Propositions 5.2 of [14].

**Corollary 2.5** Every holomorphic mapping $h : U \rightarrow \mathbb{C}^q$ defined on an open subset $U \subset \mathbb{C}^n$ (and with image in $\mathbb{C}^q$) has $q$-special inverse fibres, so that the function below is $q$-plurisubharmonic on $U$ for each fixed point $c \in h(U)$,

$$z \mapsto \chi_c(z) := \begin{cases} 0 & \text{if } z \in h^{-1}(c), \\ -\infty & \text{otherwise}, \end{cases} \text{ for } z \in U.$$  

In particular, any function $\psi \in \text{PSH}(h, U)$ is $q$-plurisubharmonic on $U$.

**Proof** Given any upper semi-continuous function $\varphi$ defined on a neighbourhood $\Omega$ of $h(U)$ in $\mathbb{C}^q$, it is $q$-plurisubharmonic on $\Omega$ because of point (3) in Definition 2.1. The composition $\varphi(h)$ is then $q$-plurisubharmonic on $U$ because of point (10) in Proposition 2.2. Proposition 2.4 then implies that $h$ has $q$-special inverse fibres, as we wanted to prove. \(\blacksquare\)

**Corollary 2.6** Let $h : U \rightarrow \mathbb{C}^m$ be a continuous mapping defined on an open subset $U$ of $\mathbb{C}^n$ and with $q$-special inverse fibres for some integer $q \geq 0$. Every function $\psi$ in $\text{PSH}(r, h, U)$ is $(q+r)$-plurisubharmonic on $U$.  

\(\blacksquare\)
Proof The result can be shown mutatis mutandis following the steps involved in the proof of (1) \(\Rightarrow\) (2) in Proposition 2.4. \(\square\)

The following examples of \(q\)-plurisubharmonic functions are a direct and easy consequence of the corollaries above.

**Example 2.7** Let \(U\) be an open subset in the product \(\mathbb{C}^q \times \mathbb{C}^r\) for some non-negative integers \(q\) and \(r\). Given any fixed point \(x \in \mathbb{C}^q\), define \(U_x := \{y \in \mathbb{C}^r : (x, y) \in U\} \subset \mathbb{C}^r\).

An upper semi-continuous function \(\psi : U \rightarrow (-\infty, \infty)\) is \(q\)-plurisubharmonic on \(U\) whenever the compositions \(y \mapsto \psi(x, y)\) are all plurisubharmonic with respect to the variable \(y \in U_x\) for each fix point \(x \in \mathbb{C}^q\) with \(U_x \neq \emptyset\).

Another example comes from complex norms on \(\mathbb{C}^n\).

**Example 2.8** Let \(U \subset \mathbb{C}^n\) be an open set, and \(\mathcal{N}\) be a complex norm on \(\mathbb{C}^n\), so that \(\mathcal{N}(\lambda z)\) is equal to \(|\lambda|\mathcal{N}(z)\) for all \(\lambda \in \mathbb{C}\) and \(z \in \mathbb{C}^n\). Define the boundary distance function on \(U\) with respect to \(\mathcal{N}\) by the formula

\[
 z \mapsto d_{\mathcal{N}}(z, bU) := \inf\{\mathcal{N}(z - w) : w \in bU\} \quad \forall \ z \in U. \tag{2}
\]

The function \(z \mapsto -\ln d_{\mathcal{N}}(z, bU)\) is then \((n-1)\)-plurisubharmonic on \(U\). We write \(d(\cdot)\) instead of \(d_{\mathcal{N}}(\cdot)\) when \(\mathcal{N}(z) = \|z\|\) is the Euclidean norm. Indeed, the function (2) is continuous on \(U\) with respect to the topology induced by the Euclidean norm, because all norms on \(\mathbb{C}^n\) are equivalent. Moreover, every complex line passing through the origin in \(\mathbb{C}^n\) is the closure of the inverse fibre \(h_j^{-1}(c)\) for some point \(c \in \mathbb{C}^{n-1}\) and holomorphic mapping

\[
 z \mapsto h_j(z) := (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)/z_j \in \mathbb{C}^{n-1}
\]
defined on the open set \(U_j = \{z \in \mathbb{C}^n : z_j \neq 0\}\) with \(j = 1, \ldots, n\). Recall that identities below hold for all \(z \in \mathbb{C}^n\) and \(\lambda \neq 0\) in \(\mathbb{C}\),

\[
 h_j(\lambda z) = h_j(z) \quad \text{and} \quad -\ln \mathcal{N}(\lambda z) = -\ln |\lambda| - \ln \mathcal{N}(z).
\]

Whence \(-\ln \mathcal{N}\) is harmonic on the complex line \(h_j^{-1}(c)\) minus the origin (parametrised by \(\lambda\)). Proposition 2.4 and Corollary 2.5 imply that \(-\ln \mathcal{N}\) is \((n-1)\)-plurisubharmonic on all the open sets \(U_j\) and \(\mathbb{C}^n\\setminus\{0\}\). Finally, we can use points (10) and (6) of Proposition 2.2 in order to show that the function

\[
 z \mapsto -\ln d_{\mathcal{N}}(z, bU) = \sup\{-\ln \mathcal{N}(z - w) : w \in bU\}
\]
is \((n-1)\)-plurisubharmonic with respect to the variable \(z \in U\).
3 Relative $q$-pseudoconvex and $k$-maximum sets

There are several families of sets naturally related to the $q$-plurisubharmonic functions. We shall extensively use two of these families in this paper: the (relative) $q$-pseudoconvex sets and the $k$-maximum ones. The (relative) $q$-pseudoconvex sets generalise the concept of a Stein or pseudoconvex set. This generalisation seems to be a natural one, as it is implicated by the results obtained from the Grauert–Andreotti theory [8] or the foliation of continuous graphs in Chapter 4.7 of [11]. We begin including the formal definitions.

**Definition 3.1** Let $U$ and $V$ be open sets in $\mathbb{C}^n$ with $U \subset V$.

1. The set $U$ is called $q$-pseudoconvex (in the general sense) if and only if it admits a $q$-plurisubharmonic exhaustion function $\psi \in \text{PSH}_q(U)$; i.e. if and only if the closure of $\{z \in U : \psi(z) < c\}$ is compactly contained in $U$ for every $c \in \mathbb{R}$.

2. The set $U$ is called $q$-pseudoconvex in $V$ (or relative to $V$) if and only if for each point $p$ in $V \cap bU$ there is a ball $B \subset V$ centred in $p$ such that $B \cap U$ is $q$-pseudoconvex (in the general sense).

We must mention some examples and properties of $q$-pseudoconvex sets.

**Remark 3.2** Let $U$ and $V$ be open sets in $\mathbb{C}^n$ with $U \subset V$.

(a) Słodkowski showed in Definition 4.1, Theorem 4.3, and Corollary 4.7 of [14] that $U$ is $q$-pseudoconvex in $V$ if and only if there exist an open neighbourhood $\Omega$ of $V \cap bU$ in $\mathbb{C}^n$ and a function $u$ in $\text{PSH}_q(U \cap \Omega)$ such that

$$\limsup_{\zeta \to z, \zeta \in U \cap \Omega} u(\zeta) = +\infty$$

for all $z \in V \cap bU$. (3)

In fact, the different results presented in Section 4 of [14] can be used as an alternative to Definition 3.1, point (2) above.

(b) Clearly, the open set $U$ is $q$-pseudoconvex relative to $\mathbb{C}^n$ if and only if its is $q$-pseudoconvex (in the absolute sense). Moreover, the open set $V \subset \mathbb{C}^n$ is always $(n-1)$-pseudoconvex; see Proposition 4.6 in [14] or consider the exhaustion function $\|z\|^2 - \ln d_N(z, bV)$ defined according to (2) in Example 2.8.

We refer the interested reader to Proposition 3.10 in [12], where it is compiled a long list of different characterisations for the $q$-pseudoconvexity. We include below a short list of results we will need later.

**Proposition 3.3** Let $U \subset \mathbb{C}^n$ be an open set, and $d_N(z, bU)$ be the boundary distance function given in (2). The following statements are all equivalent.

1. The set $U$ is $q$-pseudoconvex (in $\mathbb{C}^n$) for some integer $q \geq 0$.

2. $z \mapsto -\ln d_N(z, bU)$ lies in $\text{PSH}_q(U)$ for some complex norm $N$ on $\mathbb{C}^n$.

3. $z \mapsto -\ln d_N(z, bU)$ is in $\text{PSH}_q(U)$ for every complex norm $N$ on $\mathbb{C}^n$.

4. The $q$-plurisubharmonic hull $\hat{K}^q_U$ of every compact set $K \subset U$ is again compactly contained in $U$, where

$$\hat{K}^q_U := \left\{ z \in U : \psi(z) \leq \max_K \psi, \ \forall \ \psi \in \text{PSH}_q(U) \right\}.$$ (4)
(5) There is an increasing sequence \( U_1 \subset U_2 \subset \cdots \) of \( q \)-pseudoconvex open subsets of \( U \), which union \( \bigcup_{k=1}^\infty U_k \) is equal to \( U \).

(6) There is an increasing sequence \( U_1 \Subset U_2 \Subset \cdots \) of \( q \)-pseudoconvex and relatively compact open sets in \( U \), which union \( \bigcup_{k=1}^\infty U_k \) is equal to \( U \).

**Proof**  The results trivially hold when \( q \geq n - 1 \) because of point (b) in Remark 3.2. We assume from now on that \( q \leq n - 2 \). The statements (1), (4), (5) and (6) are all equivalent according to Definition 3.3 and Propositions 3.10 and 3.13 of [12]; but the proofs shall not be repeated here.

(3) \( \Rightarrow \) (2) This implication is obvious.

(2) \( \Rightarrow \) (1) It is easy to verify that \( \psi(z) = \|z\|^2 - \ln d_N(z, bU) \) is a \( q \)-plurisubharmonic exhaustion function for \( U \).

(1) \( \Rightarrow \) (3) Given \( z \in U \) and \( \omega \in \mathbb{C}^n \) with \( \|\omega\| = 1 \), define

\[
R_U(z, \omega) := \inf\{\|y-z\| : y = z + s \omega \in bU, \text{ for some } s \in \mathbb{C}\}
\]  

(5) as the Euclidean distance between the point \( z \) and the intersection of the boundary \( bU \) with the complex line \( z + \omega \mathbb{C} \) in \( \mathbb{C}^n \). We can easily verify that the following identity holds for every complex norm \( N \) on \( \mathbb{C}^n \),

\[
d_N(z, bU) = \inf\{N(\omega)R_U(z, \omega) : \omega \in \mathbb{C}^n, \|\omega\| = 1\}.
\]  

(6)

Recall that \( \|y-z\|N(\omega) = |s|N(\omega) = N(s\omega) = N(y-z) \) for \( y = z + s \omega \) as in equation (5). The distance \( d_N(z, bU) \) is continuous on \( U \), because the complex norms on \( \mathbb{C}^n \) are all equivalent to the Euclidean one. Finally, it is shown in Théorème 1 of [6, p. 643] that, if \( U \) is \( q \)-pseudoconvex, each composition \( -\ln R_U(z, \omega) \) is \( q \)-plurisubharmonic with respect to the variable \( z \in U \) and for every fixed \( \omega \in \mathbb{C}^n \) with \( \|\omega\| = 1 \) (see also Proposition 3.10 of [12]). Equation (6) and the point (6) in Proposition 2.2 then imply that \( -\ln d_N(z, bU) \) is \( q \)-plurisubharmonic, as we wanted to prove, because it is the supremum of the family of \( q \)-plurisubharmonic functions

\[ -\ln R_U(z, \omega) - \ln N(\omega) \]

indexed by the vector \( \omega \in \mathbb{C}^n \) with \( \|\omega\| = 1 \).

**Corollary 3.4** Let \( U \) and \( V \) be open sets in \( \mathbb{C}^n \), such that \( U \) is contained and \( q \)-pseudoconvex in \( V \). If \( V \) is \( q \)-pseudoconvex, then \( U \) is also \( q \)-pseudoconvex.

**Proof**  Point (6) in Proposition 3.3 implies the existence of an increasing sequence \( V_1 \Subset V_2 \Subset \cdots \) of \( q \)-pseudoconvex and relatively compact open sets in \( V \), which union \( \bigcup_{j=1}^\infty V_j \) is equal to \( V \). Moreover, since \( U \) is contained and \( q \)-pseudoconvex in \( V \), point (a) in Remark 3.2 yields that the existence of an open neighbourhood \( \Omega \) of \( V \cap bU \) in \( \mathbb{C}^n \) and a function \( u(\zeta) \) in \( \text{PSH}_q(U \cap \Omega) \) such that equation (3) holds. We may easily take a smaller neighbourhood \( \Omega \), if it is necessary, so that for every index \( j \geq 1 \) there exists a positive constant \( \beta_j > 0 \) with

\[
\beta_j > \limsup_{\zeta \to z, \xi \in \overline{U} \cap \Omega} u(\zeta) \quad \text{for all } z \in \overline{V_j \cap U} \cap b\Omega \quad \text{and} \quad j \geq 1.
\]
Recall that each $V_j \subset V$ is compact. Since $V_j$ is also $q$-pseudoconvex, it admits an exhaustion function $w_j \in \text{PSH}_q(V_j)$. Points (3) and (8) of Proposition 2.2 imply that the following function lies in $\text{PSH}_q(V_j \cap U)$,

$$
\psi_j = \begin{cases} 
\max\{\beta_j, w_j\} & \text{on } (V_j \cap U) \setminus \Omega, \\
\max\{u, \beta_j, w_j\} & \text{on } V_j \cap U \cap \Omega, 
\end{cases} \quad \text{for } j \geq 1.
$$

It is easy to verify that every $\psi_j$ is an exhaustion function for $V_j \cap U$, because the function $u$ (resp. $w_j$) goes to plus infinity when $\zeta$ converges to any point in the intersection $V_j \cap bU$ (resp. $U \cap bV_j$). Recall equation (3) and the fact that each $V_j \subset V$ is $q$-pseudoconvex in $\mathbb{C}^n$. Hence, the set $V_j \cap U$ is $q$-pseudoconvex for all $j \geq 1$. Point (5) in Proposition 3.3 then implies that $U$ is $q$-pseudoconvex, as we wanted to prove, because $U$ is equal to $\bigcup_{j=1}^{\infty}(V_j \cap U)$.

\[\blacksquare\]

### 3.1 $k$-maximum sets

The second family of sets (related to the $q$-plurisubharmonic functions) that we use in this manuscript are the $k$-maximum sets introduced by Słodkowski’s in [14, p. 108].

**Definition 3.5** Let $X$ be a non-empty locally closed set in $\mathbb{C}^n$, so that $X$ is equal to the intersection $V \cap X$ for some open set $V$ in $\mathbb{C}^n$.

(1) The set $X$ is said to be 0-maximum if and only if, given any holomorphic polynomial $p : \mathbb{C}^n \to \mathbb{C}$, the absolute value $|p|$ has the local maximum principle on $X$; i.e. if and only if the following inequality holds for all compact subsets $K$ of $X$,

$$
\|p\|_K \leq \|p\|_{b_X K}, \quad \text{where } \|p\|_{b_X K} := \sup_{b_X K} |p|.
$$

(7)

The boundary $b_X K$ is calculated in the relative topology of $X$. We simply write $bK$ instead of $b_{\mathbb{C}^n} K$ when $X = \mathbb{C}^n$.

(2) Given any integer $k \geq 0$ such that $k \leq n-1$. The set $X$ is said to be $k$-maximum if and only if the intersection $X \cap \pi$ is either empty or a 0-maximum set for all possible complex affine subspaces $\pi \subset \mathbb{C}^n$ of codimension $k$.

Definition 3.5.(1) above can be reformulated by using the closure of bounded open sets instead of general compact ones. This reformulation is very useful, because one only needs to verify that the local maximum modulus principle holds for a smaller family of sets.

**Theorem 3.6** (Słodkowski) Let $X$ be a non-empty locally closed set in $\mathbb{C}^n$, so that $X$ is equal to $V \cap \overline{X}$ for some open set $V$ in $\mathbb{C}^n$. The set $X$ is 0-maximum if and only if the following inequality holds for all compactly contained open sets $W \subseteq V$ and all holomorphic polynomials $p : \mathbb{C}^n \to \mathbb{C}$,

$$
\|p\|_{X \cap \overline{W}} \leq \|p\|_{X \cap b W}.
$$

(8)
Proof. Recall that $bW$ is the boundary of $W$ calculated in the standard topology of $\mathbb{C}^n$. See Proposition 2.3 and its proof in pages 109 and 110 in [14].

The $k$-maximum sets have some simple properties.

Remark 3.7 (a) Only the open sets in $\mathbb{C}^n$ can be $(n-1)$-maximum.
(b) Given any $(k+1)$-maximum set $X$, it is easy to verify from Definition 3.5 and Theorem 3.6 that $X$ is also $k$-maximum.
(c) An interesting consequence of Definition 3.5 is that no compact set $X$ in $\mathbb{C}^n$ can be 0-maximum, because the relative boundary $b_X X$ is empty after taking $K = X$ in Eq. (7). It can be proved in a similar way that no 0-maximum set can have compact connected components.

Słodkowski has deduced several interesting characterisations of the $q$-maximum sets in terms of $q$-plurisubharmonic functions and $q$-pseudoconvex sets.

Theorem 3.8 (Słodkowski) Let $X$ be a non-empty locally closed set in $\mathbb{C}^n$, so that $X$ is equal to $V \cap \overline{X}$ for some open set $V \subset \mathbb{C}^n$. The following statements are all equivalent for any integer $k \in \{0, \ldots, n-1\}$.

1. The set $X$ is $k$-maximum.
2. The intersection $X \cap p^{-1}(0)$ is 0-maximum for every holomorphic polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^k$.
3. The characteristic function $\chi_X$ below is $(n-k-1)$-plurisubharmonic on $V$,

$$
\chi_X(z) := \begin{cases} 
0 & \text{if } z \in X; \\
-\infty & \text{otherwise.}
\end{cases}
$$

4. Under the extra condition $k \leq n-2$, the set $V \backslash X$ is $(n-k-2)$-pseudoconvex relative to $V$; see Definition 3.1.
5. For each compact set $K \subset X$ and $k$-plurisubharmonic function $\psi \in \text{PSH}_k(U)$ defined on some neighbourhood $U$ of $K$ in $\mathbb{C}^n$, we have that

$$
\sup \psi_K = \sup \psi_{b_X K}.
$$

Notice that the boundaries $b_{X \backslash U} K = b_X K$ of $K$, with respect to $X \cap U$ and $X$, coincide because the compact set $K$ is completely contained in $U$.

Proof. See Theorem 2.5, Corollary 2.6, Theorem 4.2, and Theorem 5.1 in [14].

Definition 2.3 and Corollary 2.5 yield a useful result. It is quite interesting to compare this corollary against Theorem 2.5 and Propositions 5.2 of [14].

Corollary 3.9 Let $h : U \to \mathbb{C}^m$ be a continuous mapping defined on an open subset $U$ of $\mathbb{C}^n$, and $q \in \{0, \ldots, n-1\}$. The mapping $h$ has $q$-special fibres if and only if its inverse fibres $h^{-1}(c)$ are all $(n-q-1)$-maximum for every $c \in h(U)$. Moreover, if $h : U \to \mathbb{C}^m$ is a holomorphic mapping, then the fibres $h^{-1}(c)$ are all $k$-maximum for every $c \in h(U)$ and $k < n - m$. 
We close this section presenting the following result, which is an extension of Corollary 3.11 in [12] and will be extensively used in the following chapters. We believe that the original proof of Corollary 3.11 in [12] has some minor gaps, so that we present now an alternative proof.

**Corollary 3.10** Let $\psi \in \text{PSH}_q(V)$ be any $q$-plurisubharmonic function defined on an open subset $V$ of $\mathbb{C}^n$. The set $U = \{z \in V : \psi(z) < c\}$ is $q$-pseudoconvex in $V$ for every real number $c \in \mathbb{R}$. Furthermore, if $V$ is itself $q$-pseudoconvex (in $\mathbb{C}^n$), then $U$ is $q$-pseudoconvex (in $\mathbb{C}^n$) as well.

**Proof** The results trivially hold when $U = V$ or $q \geq n - 1$, because every open set in $\mathbb{C}^n$ is $q$-plurisubharmonic in this case according to Remark 3.2 (b). Thus, we assume from now on that $V \setminus U$ is not empty and $q \leq n - 2$. We assert that the characteristic function of $V \setminus U$ is $(q+1)$-plurisubharmonic; and this result follows from points (3) and (5) of Proposition 2.2 after observing that for $j \geq 1$ the function $\phi_j = j \min\{0, \psi - c\}$ is $(q+1)$-plurisubharmonic and the sequence $\{\phi_j\}_{j \geq 1}$ is decreasing and converges pointwise to $\chi_{V \setminus U}$ as $j$ tends to infinity.

Proposition 3.8 then implies that both $V \setminus U$ is a $(n-q-2)$-maximum set and $U$ is $q$-pseudoconvex in $V$. We can conclude that $U$ is $q$-pseudoconvex, when $V$ is also $q$-pseudoconvex, after applying Corollary 3.4. 

\[\square\]

4 Generalised convex hulls

The main objective of this section is to define and present some of the properties of the $q$-plurisubharmonic hulls $\hat{K}^q$, which are used for characterising the $q$-pseudoconvex sets in $\mathbb{C}^n$; see (4) in Proposition 3.3. We also analyse their relations with other convex hulls already defined in the literature; e.g. as in [10].

**Definition 4.1** Let $U \subset \mathbb{C}^n$ be an open set, and $K \subset U$ be a compact one.

(1) Given a family $\mathcal{A}$ of upper semi-continuous functions well defined on $U$, we introduce the hull of $K$ in $U$ with respect to $\mathcal{A}$ to be

$$\hat{K}_\mathcal{A} := \left\{ z \in U : \psi(z) \leq \max_K \psi \quad \forall \psi \in \mathcal{A} \right\}.$$  \hspace{1cm} (9)

In particular, if $B$ is a subset of $\mathcal{A}$, the hull $\hat{K}_\mathcal{A}$ is contained in $\hat{K}_B$.

(2) When $\mathcal{A} = \text{PSH}_q(U)$ is the family of $q$-plurisubharmonic functions on $U$, the resulting hull $\hat{K}^q_U := \hat{K}_\mathcal{A}$ is called the $q$-plurisubharmonic hull of $K$ in $U$. We simply write $\hat{K}^q$ instead of $\hat{K}^q_U$ if the open set $U = \mathbb{C}^n$.

(3) Let $\mathcal{P}$ be the family of all holomorphic polynomials on $\mathbb{C}^n$ (and image in $\mathbb{C}$). Define the family $|\mathcal{P}| = \{|p| : p \in \mathcal{P}\}$; the resulting hull $\hat{K}_{|\mathcal{P}|}$ is called the polynomially convex hull of $K$. We write $\hat{K}$ or $K^\wedge$ instead of $\hat{K}_{|\mathcal{P}|}$.

(4) Given an integer $q \geq 0$, the hull $\mathcal{H}_q(K)$ denotes the intersection $\bigcap_{j \in J} U_j$ of all open neighbourhoods $U_j$ of $K$ which are $q$-pseudoconvex in $\mathbb{C}^n$. We call it the $q$-pseudoconvex hull of $K$ and it is related to the Nebenhülle [4].
The relation between the hulls $\hat{K}$ and $\hat{K}^q$ above is established in Theorem 1.3.11 of [15, p. 27], where it is proved that the 0-plurisubharmonic hull $\hat{K}^0$ coincides with the polynomial convex hull $\hat{K}$. The $q$-plurisubharmonic hull $\hat{K}^q$ is compact when the open set $U$ is pseudoconvex; see for example point (4) of Proposition 3.3. In the similar form, the $q$-pseudoconvex hull $\mathcal{H}_q(K)$ is always a compact set.

**Lemma 4.2** Given a compact set $K$ in $\mathbb{C}^n$, its hull $\mathcal{H}_q(K)$ is also compact. Moreover, for every neighbourhood $V$ of $\mathcal{H}_q(K)$ in $\mathbb{C}^n$ there exists another open neighbourhood $U$ of $\mathcal{H}_q(K)$ such that $U \subset V$ and $U$ is $q$-pseudoconvex in $\mathbb{C}^n$. Finally, the hull $\mathcal{H}_r(K)$ is equal to $K$ when $r \geq n-1$.

**Proof** The hull $\mathcal{H}_r(K)$ is equal to $K$ when $r \geq n-1$, because every open set in $\mathbb{C}^n$ is $r$-pseudoconvex in this case according to Remark 3.2.(b). On the other hand, $\mathcal{H}_q(K)$ is obviously bounded in $\mathbb{C}^n$, because we can always enclose $K$ by a (pseudoconvex) open ball $B^p_r(0)$ centred at the origin of $\mathbb{C}^n$ and of radius $r > 0$ large enough. By definition, the hull $\mathcal{H}_q(K)$ is also contained in $B^p_r(0)$.

We show that the hull $\mathcal{H}_q(K)$ is closed as well. Assume that it is not the case, so that there exists a point $p \notin \mathcal{H}_q(K)$ which lies in the closure of $\mathcal{H}_q(K)$. By the definition of the hull we can find an open neighbourhood $W$ of $K$ that is $q$-pseudoconvex in $\mathbb{C}^n$, but $p \notin W$. In particular, $W$ admits a $q$-plurisubharmonic exhaustion function $\psi$. Notice that $\psi$ attains its maximum in $K$, because it is upper semi-continuous. Define the set

$$D := \left\{ z \in W : \psi(z) < 1 + \max_K \psi \right\}. \tag{10}$$

It is easy to see that $K \subset D \subset W$ because $\psi$ is an exhaustion function for $W$. Corollary 3.10 yields that $D$ is $q$-pseudoconvex in $\mathbb{C}^n$, and so the hull $\mathcal{H}_q(K)$ is contained in $D$. Since $p \notin W$ and the closure $\overline{D}$ is contained in $W$, we can conclude that $p$ is not in the closure of $\mathcal{H}_q(K)$. This is a contradiction to the hypotheses, so that the hull $\mathcal{H}_q(K)$ is closed and compact.

Finally pick an arbitrary neighbourhood $V$ of $\mathcal{H}_q(K)$ in $\mathbb{C}^n$. We know from the previous paragraphs that the set $\mathcal{H}_q(K)$ is compact, so that we may assume that $V$ is bounded; i.e. its boundary $bV$ is also compact. Let $y$ be any point in $bV$. Since $y$ does not lie in $\mathcal{H}_q(K)$, there is an open neighbourhood $W_y$ of $\mathcal{H}_q(K)$ that is $q$-pseudoconvex in $\mathbb{C}^n$, but $y \notin W_y$. We now proceed as in the paragraphs above producing a $q$-plurisubharmonic exhaustion function $\psi$ for $W_y$ and an open set $D_y$ like in (10), such that $D_y$ is $q$-pseudoconvex in $\mathbb{C}^n$, $\mathcal{H}_q(K)$ is contained in $D_y$, but $D_y$ does not meet an open ball $B(y)$ centred at $y \in bV$ and of radius $r(y) > 0$ small enough.

The collection $\{B(y)\}_{y \in bV}$ is a cover for the compact set $bV$, so that it has a finite subcover $\{B(y_k)\}_{k=1,...,m}$ as well. Proposition 3.13 in [12, p. 404] yields that the finite intersection $U = \bigcap_{k=1}^m D_{y_k}$ is open and $q$-pseudoconvex in $\mathbb{C}^n$. Moreover, $U$ is a relatively compact neighbourhood of $\mathcal{H}_q(K)$ in $V$, as we wanted. $\square$

Definition 4.1 enables the construction of different hulls related to the $q$-plurisubharmonic ones $\hat{K}^q$. One may, for example construct hulls using the family of functions introduced in Definition 2.3 and Proposition 2.4, where the concept of mappings $h$ with
q-special fibres and the family of functions PSH(h, U) are defined and characterised. The relations between these families and the hulls given in the Definition 4.1 are presented in the proposition below.

**Proposition 4.3** Let $S = \{h_v : U \to \mathbb{C}^m\}_{v \in J}$ be a family of continuous mappings defined on a $(q-1)$-pseudoconvex open set $U \subset \mathbb{C}^m$ and with q-special fibres for some integer $q \geq 1$. Assume that the family $S$ is invariant under addition by constants. Given any compact set $K \subset U$, we have that

$$
\hat{K}_U^q \subset \mathcal{H}_{q-1}(K)
$$

for

$$
\hat{K}_A \subset \hat{K}_{-\|S\|}
$$

Moreover,

$$
\hat{K}_{-\|S\|} = \{z \in U : h(z) \in h(K) \quad \forall \, h \in S\}.
$$

**Proof** We obviously have that $-\|S\|$ is contained in $\mathcal{A}$, because the composition $-\|h\|$ lies in $\text{PSH}(h, U)$ for every $h \in S$. Indeed, $-\|h\|$ is constant (and so 0-plurisubharmonic) on the inverse fibres $h^{-1}(c)$ for every $c \in h(U)$. Moreover, Proposition 2.4 implies that $\mathcal{A}$ and $-\|S\|$ are both subfamilies of $q$-plurisubharmonic functions, so that

$$
\hat{K}_U^q \subset \hat{K}_A \subset \hat{K}_{-\|S\|}.
$$

We now prove the identity in (12). Let $z \in U$ be any point such that $h(z)$ lies in $h(K)$ for every $h \in S$. The value $-\|h(z)\|$ is obviously less than or equal to the maximum of $-\|h\|$ on $K$, so that $z$ lies in the hull $\hat{K}_{-\|S\|}$, because $h \in S$ is arbitrary. On the other hand, take any point $z \in U$ such that $h(z)$ does not lie in $h(K)$ for some element $h \in S$. Since $S$ is invariant under addition by constants, we can assume without loss of generality that $h(z) = 0$ is equal to the origin, but 0 is not contained in $h(K)$. Hence, the value $-\|h(z)\|$ is both equal to zero and strictly larger than the maximum of $-\|h\| < 0$ on the compact set $K$, so that $z$ does not lie on $\hat{K}_{-\|S\|}$ and the identity in (12) holds.

The contention $\mathcal{H}_{q-1}(K) \subset \hat{K}_{-\|S\|}$ is trivial when $q \geq n$, because $\mathcal{H}_{q-1}(K)$ is equal to $K$ in this case according to Lemma 4.2. This contention follows from (12) when $q \leq n-1$. Indeed, given any point $z \in U$ in the complement of $\hat{K}_{-\|S\|}$, there is an element $h \in S$ such that $h(z)$ does not lie in $h(K)$. Since $h$ has $q$-special fibres and $U$ is $(q-1)$-pseudoconvex, point 1 of Definition 2.3, Theorem 3.8, and Corollary 3.4 imply that the inverse fibre $h^{-1}(c)$ is $(n-q-1)$-maximum and the difference $U \setminus h^{-1}(c)$ is $(q-1)$-pseudoconvex for $c = h(z)$. The set $K$ is obviously not contained in the complement $U \setminus h^{-1}(c)$, so that $z$ is not contained in $\mathcal{H}_{q-1}(K)$ because of point 4 of Definition 4.1. We can conclude that $\mathcal{H}_{q-1}(K)$ is contained in $\hat{K}_{-\|S\|}$, as we wanted.

It remains to show that $\hat{K}^q$ is contained in $\mathcal{H}_{q-1}(K)$. This contention is trivial when $q \geq n$, because every upper semi-continuous function is $q$-plurisubharmonic in this case, and so $\hat{K}^q$ is equal to $K$. To verify that this contention holds when
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$q \leq n-1$, we pick any point $p \notin \mathcal{H}_{q-1}(K)$. By Lemma 4.2 there is a relatively compact neighbourhood $W$ of $K$ in $\mathbb{C}^n$, such that $p \notin W$ and $W$ is $(q-1)$-pseudoconvex in $\mathbb{C}^n$. In view of the definition of $q$-pseudoconvexity, the set $W$ admits a $(q-1)$-plurisubharmonic exhaustion function $\psi$, and so we can assume without loss of generality that $K$ is contained in the open set $\{\psi < 0\}$. Define the function $\varphi := \min\{\psi, 1\}$. We easily have that $\varphi = \psi < 0$ on $K$ and that $\varphi$ lies in $\text{PSH}_q(W)$ according to point (3) in Proposition 2.2.

Since $\psi(z)$ tends to $+\infty$ whenever $z$ converges to any point in the boundary $bW$, the function $\varphi$ is identically equal to 1 in a neighbourhood of $bW$, and so $\varphi$ can be trivially extended as a $q$-plurisubharmonic function $\widehat{\varphi}$ defined on the whole space $\mathbb{C}^n$; we only need to define $\widehat{\varphi}$ identically equal to 1 in the complement of $W$. Finally, notice that $\widehat{\varphi} = \varphi = \psi$ is strictly negative on $K$, but $\widehat{\varphi}(p) = 1$, because $p \notin W$. Hence, $p$ does not lie in the hull $\mathcal{H}_q$, and so $\mathcal{H}_q$ is contained in $\mathcal{H}_{q-1}(K)$, as we wanted to prove. Recall that $p \notin \mathcal{H}_{q-1}(K)$ is arbitrary.

As it is pointed out in Corollaries 2.5 and 3.9, every holomorphic polynomial mapping defined from $\mathbb{C}^n$ into $\mathbb{C}^k$ has $q$-special fibres, so that the family of polynomial mappings $\mathcal{P}$ is the perfect example for applying Proposition 4.3 above. The resulting hulls $\mathcal{K}_\Delta$ and $\mathcal{K}_{\|\cdot\|}$ are exactly the hulls introduced by Basener in [2] and analysed by Lupacciolu and Stout in [10].

**Definition and Remark 4.4** Let $q \geq 0$ be a fixed integer.

1. The generalised polynomially convex hull of a compact set $K \subset \mathbb{C}^n$ in the sense of Basener is defined by

$$h_q(K) = \left\{ z \in \mathbb{C}^n : z \in [K \cap p^{-1}(0)]^\wedge \text{ for all polynomial mappings } p : \mathbb{C}^n \to \mathbb{C}^q \text{ with } p(z) = 0 \right\}. \tag{13}$$

2. The topologically convex hull $\text{Top-hull}(L)$ of a compact set $L \subset \mathbb{C}^{q+1}$ is the union of $L$ and all the bounded connected components of its complement $\mathbb{C}^{q+1} \setminus L$. Corollary 3.4 in page 327 of [10] yields an alternative definition for the generalised polynomially convex hull of the compact $K \subset \mathbb{C}^n$,

$$h_q(K) = \left\{ z \in \mathbb{C}^n : p(z) \in \text{Top-hull}(p(K)) \text{ for all polynomials } p : \mathbb{C}^n \to \mathbb{C}^{q+1} \right\}. \tag{14}$$

3. The generalised rationally convex hull of a compact set $K \subset \mathbb{C}^n$ in the sense of Basener is given by,

$$r_q(K) = \left\{ z \in \mathbb{C}^n : p(z) \in p(K) \text{ for every polynomial } p : \mathbb{C}^n \to \mathbb{C}^{q+1} \right\}. \tag{15}$$

Basener’s hulls are identical to those generated via Proposition 4.3 with the family of holomorphic polynomial mappings.
Lemma 4.5 Let $\mathcal{P}$ be the family of all holomorphic polynomial mappings defined from $\mathbb{C}^n$ into $\mathbb{C}^q$ with $q \geq 1$. Given any compact set $K \subset \mathbb{C}^n$, we have that

\[
\hat{K}_A = h_q(K) \quad \text{for} \quad A := \bigcup_{p \in \mathcal{P}} \text{PSH}(p, \mathbb{C}^n),
\]

Moreover, $\hat{K}^0 = h_0(K) = \hat{K}$.

Proof The identities $\hat{K}^0 = h_0(K) = \hat{K}$ holds after setting $\mathbb{C}^0 = \{0\}$ in (13) and applying Theorem 1.3.11 of [15, p. 27]. The hull $\hat{K}_{-\|P\|}$ and $r_{q-1}(K)$ coincide after comparing Eq. (12) in Proposition 4.3 against (15) in Definition 4.4.

We finally prove that $\hat{K}_A = h_q(K)$. Given any point $x \notin h_q(K)$, there is a polynomial mapping $p : \mathbb{C}^n \to \mathbb{C}^q$ such that $p(x) = 0$ and $x$ does not lie in the polynomially convex hull of $p^{-1}(0) \cap K$. Hence, there exists a second holomorphic polynomial $\eta : \mathbb{C}^n \to \mathbb{C}$ such that $\eta(x) = 1$ and $|\eta(y)| < 1$ for every $y \in K$ that also satisfies $p(y) = 0$. Let $\beta > 0$ be large enough such that $\text{Re}(\eta(y)) < 1 + \beta\|p(y)\|^2$ for every $y$ in the compact set $K$. Define the following function on $\mathbb{C}^n$,

\[
\psi(z) := \frac{\text{Re}(\eta(z))}{1 + \beta\|p(z)\|^2}.
\]

The function $\psi$ lies in $\text{PSH}(p, \mathbb{C}^n) \subset A$, because $\text{Re}(\eta)$ is plurisubharmonic on $\mathbb{C}^n$ and $p$ is indeed constant on the inverse fibres $p^{-1}(c)$ for every $c \in \mathbb{C}^q$. Moreover, we easily have that $\max_K \psi < 1$ and $\psi(x) = 1$, so that $x \notin \hat{K}_A$. Hence, the hull $\hat{K}_A$ is contained in $h_q(K)$.

Finally, take $z_0 \in h_q(K)$ and $\psi \in A$ arbitrary. Let $p_1 : \mathbb{C}^n \to \mathbb{C}^q$ be a polynomial mapping such that $\psi \in \text{PSH}(p_1, \mathbb{C}^n)$; i.e. $\psi$ is plurisubharmonic on the analytic set $p_1^{-1}(c)$ for every $c \in \mathbb{C}^q$. One may assume without loss of generality that $p_1(z_0) = 0$, so that $z_0$ lies in the polynomially convex hull of $K \cap p_1^{-1}(0)$ according to (13). Colțoiu’s Proposition 2 in [3, p. 547] implies that there is an entire plurisubharmonic function $\Psi$ on $\mathbb{C}^n$ whose restriction to $p_1^{-1}(0)$ coincides with $\psi$. Since the convex hulls with respect to polynomials and entire plurisubharmonic functions coincide (see the first paragraph of this proof or Theorem 1.3.11 of [15, p. 27]), the inequalities below hold,

\[
\psi(z_0) = \Psi(z_0) \leq \max_{K \cap p_1^{-1}(0)} \Psi = \max_{K \cap p_1^{-1}(0)} \psi \leq \max_K \psi.
\]

Whence: $z_0 \in \hat{K}_A$. Recall that $\psi$ in $A$ is arbitrary. The analysis done in the previous paragraphs yields that $h_q(K)$ is equal to $\hat{K}_A$, as we wanted to prove.

Proposition 4.3 can be extended as follows.
Proposition 4.6 For all compact sets $K \subset \mathbb{C}^n$ and integers $q \geq 0$ we have the following families of contentions:

\[
\hat{K}^{q+1} \subset \mathcal{H}_q(K) \subset \hat{K}^q \quad \cap \quad \cap \quad \cap
\]

\[h_{q+1}(K) \subset r_q(K) \subset h_q(K)\] (17)

Moreover, $\hat{K}^r = h_r(K) = \mathcal{H}_{r-1}(K) = r_{r-1}(K) = K$ for every $r \geq n$.

Proof Proposition 4.3 and Lemma 4.5 yield almost all the contentions in (17), we only need to prove that $\mathcal{H}_q(K) \subset \hat{K}^q$ and $r_q(K) \subset h_q(K)$. The latter contention can be easily verified after comparing Eqs. (14) and (15) in Definition 4.4. Moreover, we can verify that $\mathcal{H}_q(K)$ is contained in $\hat{K}^q$ by considering any point $y \notin \hat{K}^q$. By definition of the hull $\hat{K}^q$ there exists a $q$-plurisubharmonic function $\psi$ defined on $\mathbb{C}^n$ such that $\psi(y)$ is strictly greater than $\psi(x)$ for every $x \in K$. The following set is then an open neighbourhood of $K$ that does not contain $y$,

\[U := \{z \in \mathbb{C}^n : \psi(z) < \psi(y)\}.\]

Corollary 3.10 yields that $U$ is $q$-pseudoconvex in $\mathbb{C}^n$, so that $\mathcal{H}_q(K)$ is contained in $U$ and $y \notin \mathcal{H}_q(K)$. Since $y \notin \hat{K}^q$ is arbitrary, we have that $\mathcal{H}_q(K) \subset \hat{K}^q$.

Finally, the identities $\hat{K}^r = h_r(K) = \mathcal{H}_{r-1}(K) = r_{r-1}(K) = K$ are easily deduced from (17) when $r \geq n$; we only need to recall that $\mathcal{H}_{r-1}(K) = K$ according to Lemma 4.2 and to observe that $r_{r-1}(K) = K$ after using a mapping of the form $p(z) = (z, 0)$ in equation (15). \[\square\]

Remark 4.7 The sets $\mathcal{H}_0(K)$ and $r_0(K)$ are different in general. Consider the graph

\[M := \{(s, f(s)) \in \mathbb{C}^2 : |s| \leq 1\} \quad \text{for} \quad f(s) = (1+i)s - is\bar{s}^2 - s^2\bar{s}^3.\]

It is easy to see that $M$ is totally real, so that it has a system of Stein open neighbourhoods. By definition, $\mathcal{H}_0(M)$ is equal to $M$. Consider the disk

\[D_0 := \{(s, 0) \in \mathbb{C}^2 : |s| \leq 1\}.\]

The boundary of $D_0$ is contained in $M$, because $f(s)$ vanishes in $|s| = 1$. We assert that $D_0$ is contained in $r_0(M)$. Let $y$ be any point in $D_0 \setminus M$. If $y$ does not lies in $r_0(M)$, there is a polynomial $p : \mathbb{C}^2 \to \mathbb{C}$ such that $p(y)$ vanishes, but the origin $0 \notin p(M)$. The function $p$ has a well-defined logarithm in $M$ because it is diffeomorphic to a disk. Therefore, $p$ vanishes at $y \in D_0$ and has a well-defined logarithm at the boundary of $D_0$; recall that the latter is contained in $M$. This is a contradiction to the argument principle, so that $D_0$ is contained in the rational convex hull $r_0(M)$.

The following lemma is a direct application of Proposition 2.2.
Lemma 4.8 Let $U \subset \mathbb{C}^n$ be open, and $K$ and $L$ be a pair of compact sets contained in $U$. Given two non-negative integers $q$ and $r$, we have

$$
\widehat{K} \cup L^{1+q+r} \subset \widehat{K}^q_U \cup \widehat{L}^r_U \subset \widehat{K}^{\min(q,r)}_U \cup L^r_U.
$$

(18)

Proof The first contention above is verified by taking any point $z \in \mathbb{C}^n$ that does not lie in the union of $\widehat{K}^q_U$ and $\widehat{L}^r_U$, so that there is a pair of functions $\psi_1$ $q$-plurisubharmonic on $U$ and $\psi_2$ $r$-plurisubharmonic on $U$ such that:

$$
\max_K \psi_1 < 0, \quad \max_L \psi_2 < 0, \quad \text{and} \quad \psi_1(z) = \psi_2(z) = 0.
$$

Point (3) in Proposition 2.2 implies that $\varphi := \min\{\psi_1, \psi_2\}$ is $(1+q+r)$-plurisubharmonic on $U$. Moreover: $\varphi(z) = 0$ and $\varphi(y) < 0$ for every $y$ in $K \cup L$, so that $z$ does not lie either in the $(1+q+r)$-plurisubharmonic hull of $K \cup L$. One can conclude that the first contention in (18) holds. Assume from now on and without loss of generality that $q \leq r$, so that $\text{PSH}^q_y(U)$ is contained in $\text{PSH}^r_y(U)$.

The following result follows directly from Definition 4.1 and automatically imply the second contention in (18),

$$
\widehat{K}^q_U \subset \widehat{K} \cup \widehat{L}^q_U \quad \text{and} \quad \widehat{L}^r_U \subset \widehat{K} \cup L^q_U.
$$

We can easily deduce a simple but very interesting consequence.

Corollary 4.9 Let $\Omega$ and $U$ be two open sets in $\mathbb{C}^n$ such that $n \geq 2$ and $\Omega$ is compactly embedded in $U$. Assume that $q$ and $r$ are two non-negative numbers that satisfy the condition $1+q+r < n$. Given any point $y \in \Omega$, there cannot exist a pair of compact sets $K$ and $L$ in $\mathbb{C}^n$ whose union is equal the compact boundary $b\Omega$ and such that the point $y \notin \widehat{K}^q_U$ and $y \notin \widehat{L}^r_U$.

5 Relation between $q$-maximum sets and generalised hulls

We may now present the principal result of this paper: the complements of the hulls $\widehat{K}^q$ and $\text{H}^q(K)$ in $\mathbb{C}^n$ are all $(n-q-2)$-pseudoconvex in $\mathbb{C}^n \setminus K$ for every compact set $K$ and the hulls given in points (2) and (4) of Definitions 4.1. Nevertheless, we do not give a direct proof of this result; instead, we show in Proposition 5.5 that $q$-plurisubharmonic functions have the local maximum principle on $\widehat{K}^q$ and $\text{H}^q(K)$; and the main result is then presented in Corollary 5.6 as a direct application of the Slodkowski’s characterisations enlisted in Theorem 3.8. Before we proceed, we shall mention the motivation of this result.

Remark 5.1 Let $D$ be a pseudoconvex domain in a two-dimensional Stein manifold $\mathcal{M}$. It is known that if $K \subset \mathcal{M} \setminus D$ is a compact set and $\widehat{K}_{\mathcal{M}}(\mathcal{M})$ is the hull of $K$ with respect to holomorphic functions on $\mathcal{M}$, then $D \setminus \widehat{K}_{\mathcal{M}}(\mathcal{M})$ is again pseudoconvex; see e.g. Theorem 5.2.8 in [15]. This result is no longer true for higher dimensions. Let $A$ be a one-dimensional analytic subset of $\mathbb{C}^3$ and $D$ be any ball $B^3_r(p)$ in $\mathbb{C}^3$ centred at
some point \( p \in A \). If \( K \) is the intersection \( A \cap bD \), the hull \( \hat{K}_O(C^n) \) is then equal to \( A \cap \overline{D} \), but \( D \setminus A \) is not pseudoconvex. Recall that the hulls with respect to polynomials, plurisubharmonic functions on \( C^n \), and holomorphic functions on \( C^n \) are all identical.

We present the relations between \( k \)-maximum sets and Basener’s hulls.

**Theorem 5.2** Let \( X \neq \emptyset \) be a \( k \)-maximum set in \( C^n \) for some index \( k \geq 0 \), so that \( X \) is equal to \( V \cap \overline{X} \) for a given open set \( V \) in \( C^n \). If \( X \) is bounded, it is contained in Basener’s hull \( h_k(X \setminus V) \). In particular \( X \setminus V \) is compact and non-empty.

**Proof** The difference \( X \setminus V \) is compact, because \( X \) is bounded and \( V \) is open. We also have that \( X \setminus V \) is not empty. Otherwise, we would have that \( X \) is completely contained in \( V \), and so \( X = \overline{X} \) would be compact and could not be \( k \)-maximum set according to Remark 3.7.

Assume that \( X \) is \( 0 \)-maximum and let \( \{W_k\}_{k \in \mathbb{N}} \) be a collection of open bounded subsets of \( V \), such that their union \( \bigcup_{k=1}^{\infty} W_k \) is equal to \( V \) each \( W_k \) is compactly contained in \( W_{k+1} \). Take any holomorphic polynomial \( p : C^n \to C \) and point \( y \in X \). We can suppose without lost of generality that \( y \in W_0 \), so that \( y \in W_k \) for every \( k \in \mathbb{N} \). Since each closed set \( \overline{W}_k \subset V \) is compact, Theorem 3.6 implies that

\[
|p(y)| \leq \|p\|_{X \cap W_k} \leq \|p\|_{X \cap bW_k} \leq \|p\|_{\overline{X} \setminus W_k} \quad \text{for all} \quad k \in \mathbb{N}.
\]

We have that \( |p(y)| \leq \|p\|_{\overline{X} \setminus V} \), because \( \overline{X} \setminus V = \bigcap_{k=1}^{\infty} (\overline{X} \setminus W_k) \). Hence, \( X \) is contained in the polynomially convex hull of \( \overline{X} \setminus V \), because \( y \in X \) and the polynomial \( p \) are arbitrary. The proof is complete when \( X \) is \( 0 \)-maximum.

For case when \( X \) is \( k \)-maximum with \( k \geq 1 \), we take any point \( y \in X \) and polynomial mapping \( p : C^n \to C^k \) such that \( p(y) = 0 \). Point (2) in Theorem 3.8 yields that \( X \cap p^{-1}(0) \) is \( 0 \)-maximum. The analysis done above yields that \( y \in X \) lies in the polynomial convex hull of the following compact set

\[
(\overline{X} \setminus V) \cap p^{-1}(0) = (X \cap p^{-1}(0)) \setminus V,
\]

We may so conclude that \( X \) lies in the hull \( h_q(\overline{X} \setminus V) \) defined in (13), because the point \( y \in X \) and the mapping \( p \) are both arbitrary with \( p(y) = 0 \).

**Remark 5.3** The previous result does not hold in general when \( X \) is unbounded. For example, it is easy to see that the real axis \( X = \mathbb{R} \) is closed and \( 0 \)-maximum in the complex plane \( C \) according to Definition 3.5, but the difference \( \overline{X} \setminus V \) is empty for every open set \( V \) that contains \( X \).

The following result was originally proved by Dieu and it is central for showing Proposition 5.5.

**Theorem 5.4** (Dieu) Let \( D \) be a \( q \)-pseudoconvex domain in \( C^n \), and \( K \) be a compact subset of \( D \) (so that the \( q \)-plurisubharmonic hull \( \hat{K}_D^q \) in (4) is compact in \( D \) according to Proposition 3.3). Given any neighbourhood \( V \) of \( \hat{K}_D^q \) in \( D \), there exists a piecewise-smooth (strictly) \( q \)-plurisubharmonic function \( \psi \) on \( D \), such that \( \psi < 0 \) on \( K \) and \( \psi > 0 \) on \( D \setminus V \). Furthermore, \( \{z \in D : \psi(z) < c\} \) is relatively compact in \( D \) for every \( c \in \mathbb{R} \).
Proof. See Theorem 4.1 and Corollary 4.2 in pages 363 to 365 of [5]. \(\square\)

We show that \(q\)-plurisubharmonic functions have the local maximum principle on the hulls \(\hat{K}^q\) and \(\mathcal{H}_q(K)\) given in points (2) and (4) of Definitions 4.1.

**Proposition 5.5** Let \(K\) be a compact set in \(\mathbb{C}^n\), and \(\Pi\) be one of the compact hulls \(\hat{K}^q\) or \(\mathcal{H}_q(K)\) in \(\mathbb{C}^n\). Given a compact set \(E \subset \Pi\) and a function \(\psi \in \text{PSH}_q(U)\) defined on a neighbourhood \(U\) of \(E\) in \(\mathbb{C}^n\), the following equality holds

\[
\sup\{\psi(z) : z \in E\} = \sup\{\psi(z) : z \in (E \cap K) \cup b_{\Pi} E\},
\]

where \(b_{\Pi} E\) is the boundary of \(E\) with respect to the topology of \(\Pi\).

**Proof** The identity (19) obviously holds when \(\Pi = K\) or \(E \setminus K\) has empty interior with respect to the topology of \(\Pi\), because in both cases \(E\) is equal to the union of the intersection \(E \cap K\) and the boundary \(b_{\Pi} E\). We assume from now on that \(E \setminus K\) has non-empty interior and follow the proof of Theorem 2.18 in [15, p. 78], which is due to Rosay for the case when \(q = 0\). Suppose the result is false, then there exist a constant \(C_0\) and a point \(\hat{p}\) in the interior of \(E \setminus K\) (with respect to the topology of \(\Pi\)) such that the following inequalities hold,

\[
\psi(\hat{p}) > C_0 > \sup\{\psi(z) : z \in (E \cap K) \cup b_{\Pi} E\}.
\]

We can assume that \(\psi(\hat{p}) > C_0 > 0\). Choose a small enough open neighbourhood \(U_0\) of the compact set \(E\) in \(\mathbb{C}^n\), such that \(U_0 \subset U\) and the inequality \(\psi < C_0\) still holds on both sets \(K \cap U_0\) and \(\Pi \cap bU_0\). Define the function

\[
\psi_0 := \begin{cases} 
C_0 & \text{on } \mathbb{C}^n \setminus U_0, \\
\max\{\psi, C_0\} & \text{on } U_0.
\end{cases}
\]

Notice that \(\psi_0(\hat{p}) = \psi(\hat{p}) > C_0\), but \(\psi_0 \leq C_0\) on \(K\). Since \(\psi < C_0\) on the compact set \(\Pi \cap bU_0\), one can choose another small enough open neighbourhood \(V\) of \(\Pi\), such that \(V \subset \mathbb{C}^n\) and \(\psi < C_0\) still holds on \(V \cap bU_0\). Point (8) in Proposition 2.2 implies that \(\psi_0\) lies in \(\text{PSH}_q(V)\). Unfortunately, we may not assert that \(\psi_0\) is \(q\)-plurisubharmonic on \(\mathbb{C}^n\), because we only know that \(\psi < C_0\) on \(V \cap bU_0\); i.e. we do not know yet the behaviour of \(\psi\) on \(V \setminus U_0\).

We now proceed by cases. If \(\Pi = \mathcal{H}_q(K)\), Lemma 4.2 yields that we can find a neighbourhood \(D\) of \(\mathcal{H}_q(K)\) such that \(D \subset V\) and \(D\) is \(q\)-pseudoconvex in \(\mathbb{C}^n\). The following set is obviously a neighbourhood of \(K\) that does not contains \(\hat{p}\),

\[
D^* := \{z \in D : \psi_0(z) < \psi_0(\hat{p})\}.
\]

Recall that \(\psi_0(\hat{p}) > C_0\), but \(\psi_0 \leq C_0\) on \(K\). Corollary 3.10 implies that \(D^*\) is also \(q\)-pseudoconvex in \(\mathbb{C}^n\), and so \(\mathcal{H}_q(K) \subset D^*\) according to Definition (4). Hence, the point \(\hat{p} \notin \mathcal{H}_q(K)\), but this is a contradiction to the fact that \(\hat{p}\) lies in the interior of \(E \subset \mathcal{H}_q(K)\). One can conclude that (19) holds.
On the other hand, if \( \Pi = \tilde{K}^q(K) \), we may proceed as follows. Recall that \( V \) is a neighbourhood of \( \tilde{K}^q(K) \), the function \( \psi_0 \leq C_0 \) on \( K \), and the constant \( C_0 > 0 \). By Theorem 5.4 there is a continuous function \( \varphi \) in \( \text{PSH}_q(\mathbb{C}^n) \) such that \( \varphi < 0 \) on \( K \) and \( \varphi > 0 \) on the complement \( \mathbb{C}^n \setminus V \). Define

\[
\psi_1 := \begin{cases} 
C_2\varphi & \text{on } \mathbb{C}^n \setminus V, \\
\max\{\psi_0, C_2\varphi\} & \text{on } V,
\end{cases}
\]

for \( C_2 > 0 \).

Notice that \( \psi_1(\hat{p}) \geq \psi(0) > C_0 \), but \( \psi_1 \leq C_0 \) on \( K \) according to equations (20) and (21). Point (8) in Proposition 2.2 implies that \( \psi_1 \in \text{PSH}_q(\mathbb{C}^n) \) when \( C_2 > 0 \) is large enough. Therefore, the point \( \hat{p} \notin \tilde{K}^q \). This is again a contradiction to the fact that \( \hat{p} \) lies in \( E \subset \tilde{K}^q \). Hence, we can conclude that (19) holds. \( \square \)

We conclude this work presenting an extension to Theorem 5.2.8 in [15]. Recall the hulls \( \tilde{K}^q \) and \( \mathcal{H}_q(K) \) given in points (2) and (4) of Definitions 4.1.

**Corollary 5.6** Let \( \Pi \) be one of the hulls \( \mathcal{H}_{n-q-2}(K) \) or \( \tilde{K}^{n-q-2} \), where \( K \) is any compact set in \( \mathbb{C}^n \) and \( q \geq 0 \) is a non-negative integer with \( q \leq n-2 \). The complement \( \mathbb{C}^n \setminus \Pi \) is then \( q \)-pseudoconvex in \( \mathbb{C}^n \setminus K \). Moreover, if \( D \subset \mathbb{C}^n \) is a \( q \)-pseudoconvex domain that does not meet \( K \), then \( D \setminus \Pi \) is \( q \)-pseudoconvex in \( \mathbb{C}^n \) as well.

**Proof** The set \( X := \Pi \setminus K \) is \((n-q-2)\)-maximum in \( V := \mathbb{C}^n \setminus K \) because of point (5) in Theorem 3.8 and the previous Proposition 5.5. Point (4) in the same Theorem 3.8 then implies that \( V \setminus X = \mathbb{C}^n \setminus \Pi \) is \( q \)-pseudoconvex in \( V \), as we wanted. Finally, if \( D \subset \mathbb{C}^n \) is a \( q \)-pseudoconvex domain that does not meet \( K \), we only need to repeat the same arguments presented above, using \( D \) instead of \( \mathbb{C}^n \), so as to conclude that \( D \setminus \Pi \) is \( q \)-pseudoconvex in \( D \setminus K = D \). Corollary 3.4 finally implies that the set \( D \setminus \Pi \) is \( q \)-pseudoconvex in \( \mathbb{C}^n \). \( \square \)

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