A Corrected Simplified Proof of Chen’s Theorem

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Abstract

In 1973, J.-R. Chen [2] showed that every large even integer is a sum of a prime and a product of at most two primes. In this paper, the author indicates and fixes the issues in a simplified proof of this result given by Pan et al. [7]

Keywords: Chen’s theorem, Goldbach’s problem, Sieve theory, Switching principle

I. Introduction

For brevity, let \( \{1, c\} \) denote the following proposition:

There exists a constant \( N_0 \) such that for all even \( N > N_0 \), there exists a prime \( p < N - 1 \) such that \( N - p \) is a product of at most \( c \) primes.

Initially, the study of \( \{1, c\} \) assumes the truthfulness of the Generalized Riemann Hypothesis for Dirichlet L-function:

Proposition 1 (GRH). Let \( \pi(x; q, l) \) denote the number of primes \( p \) such that \( p \leq x, p \equiv l \pmod{q} \)

If \( (l, q) = 1 \), then

\[
\pi(x; q, l) = \frac{\text{li} x}{\varphi(q)} + O(\sqrt{x \log x}) \tag{1}
\]

where \( \text{li} x \) denotes the logarithmic integral function.

Assuming GRH, Estermann [4] proves \( \{1, 6\} \) in 1932. Subsequently, this result is improved to \( \{1, 4\} \) by Wang [11] using the sieves of Buchstab [13] and Selberg [9]. Then in 1962, Wang [12] applies Kuhn’s methods [6] and shows \( \{1, 3\} \) holds under GRH.

In 1965, Bombieri [1] and Vinogradov [10] apply the large sieve method [3] and prove the following unconditional mean value theorem:

Theorem 1 (Bombieri-Vinogradov). For all fixed \( A > 0 \), there exists \( B = B(A) \) such that for all large \( x \):

\[
\sum_{q \leq x} \max_{\nu \leq x} \max_{(l, q) = 1} \left| \pi(x; q, l) - \frac{\text{li} x}{\varphi(q)} \right| < \frac{x}{\log^A x}
\]
By replacing (1) with Theorem 1 while estimating error terms, Estermann’s \{1, 6\} and Wang’s \{1, 4\} and \{1, 3\} become unconditional results.

To prove \{1, 2\}, Chen applies the Jurkat–Richert theorem\(^*\) and proves\(^†\) that if \(W(N)\) denotes the number of primes \(p\) such that \(N - p\) has no prime factors \(\leq N^{\frac{1}{3}}\) and has most one prime factor in \((N^{\frac{1}{3}}, N^{\frac{1}{2}}]\), then for sufficiently large even \(N\)

\[
W(N) \geq 2.6408 \mathfrak{S}(N) \frac{N}{\log^2 N} \tag{2}
\]

where \(\mathfrak{S}(N)\), the singular series for the Goldbach’s problem, is defined as

\[
\mathfrak{S}(N) = \prod_{p|N} \frac{p - 1}{p - 2} \prod_{p > 2} \left(1 - \frac{1}{(p - 1)^2}\right)
\]

It follows from pigeonhole principle that \(W(N)\) gives a lower bound for the number of ways to express \(N\) as a sum of a prime and a product of at most three primes, and \(N - p\) counted by \(W(N)\) has exactly three prime factors if and only if

\[
N - p = p_1 p_2 p_3, \quad N^{\frac{1}{10}} < p_1 \leq N^{\frac{1}{3}} < p_2 < p_3 \tag{3}
\]

To estimate primes satisfying (3), Chen devised the following switched sum

\[
\Omega = \sum_a \sum_{N - a p_3 \text{ prime}} f(a)
\]

where \(f(a)\) is the characteristic function for the condition that

\[
a = p_1 p_2, \quad N^{\frac{1}{10}} < p_1 \leq N^{\frac{1}{3}} < (N/p_1)^{\frac{1}{2}} \tag{4}
\]

Then, he applies the Selberg’s sieve [9] and the large sieve to obtain

\[
\Omega \leq 3.9404 \mathfrak{S}(N) \frac{N}{\log^2 N} \tag{5}
\]

By symmetry, it is evident that the number of primes \(p\) satisfying (3) is bounded by half of \(\Omega\), so \{1, 2\} is deduced as the number of ways to express \(N\) as a sum of a prime and a product of at most two primes is no less than

\[
W(N) - \frac{\Omega}{2} \geq \left(2.6408 - \frac{3.9404}{2}\right) \mathfrak{S}(N) \frac{N}{\log^2 N} > 0.67 \mathfrak{S}(N) \frac{N}{\log^2 N}
\]

Soon after Chen, Pan et al. [7] simplified the proof of (5) by introducing the following new mean value theorem:

**Theorem 2** (Pan et al.). Let \(\pi(x; a, q, l)\) denote the number of primes satisfying

\[
ap \leq x, ap \equiv l \pmod{q}
\]

and \(\Delta(x; a, q, l)\) be defined by

\[
\Delta(x; a, q, l) = \pi(x; a, q, l) - \frac{\li}{\varphi(q)}
\]

\[^*\]Chen himself regards this as Richert’s sieve [8], but the sieve method in Richert’s paper is essentially the Jurkat–Richert theorem [5].

\[^†\]See Lemma 9 of [2]
Then for every fixed $A > 0$, there exists $B = B(A)$ such that

$$
\sum_{q \leq x} \mu^2(q) \frac{3^{\omega(q)}}{q} \max_{y \leq x} \max_{l(q) = 1} \left| \sum_{(a,q) = 1} f(a) \Delta(a; a, q, l) \right| \ll \frac{x}{\log A}.
$$

Theorem 2 is devoted to give upper estimate for the error terms emerged from the estimation of $\Omega$. However, this result does not serve to estimate all the error terms appeared in the upper bound sieve for $\Omega$.

In this paper, we present a corrected version of Pan et al.’s proof of (5). The issues in their original proof are discussed and resolved in section IV.

II. AUXILIARY LEMMAS

**Lemma 1.** Let $\beta > \alpha > 0$ be fixed. Then for all $x \geq 2$, the sum

$$
\sum_{x^\alpha < p \leq x^\beta} \frac{1}{p}
$$

is bounded.

**Proof.** Mertens’ second theorem asserts that there exists a fixed constant $B_1$ such that

$$
\sum_{p \leq x} \frac{1}{p} = \log \log x + B_1 + O \left( \frac{1}{\log x} \right)
$$

so we have

$$
\sum_{x^\alpha < p \leq x^\beta} \frac{1}{p} = \log \frac{\beta}{\alpha} + O \left( \frac{1}{\log x} \right)
$$

This suggests the sum is bounded. Q.E.D.

**Lemma 2.** Let $A > 0$ be a fixed constant. Then for any integer $n \geq 1$

$$
\sum_{d|n} \frac{\mu^2(d) A^{\omega(d)}}{\varphi(d)} \ll (\log \log 3n)^A
$$

where $\omega(n)$ denotes the number of distinct prime factors of $n$. In particular, if $n$ is the product of distinct primes $\leq y$, then

$$
\sum_{d|n} \frac{\mu^2(d) A^{\omega(d)}}{\varphi(d)} = \sum_{d|n} \frac{A^{\omega(d)}}{\varphi(d)} \ll (\log y)^A
$$

**Proof.** It is evident that the left hand side is multiplicative, so by definition we have

$$
\sum_{d|n} \frac{\mu^2(d) A^{\omega(d)}}{\varphi(d)} = \prod_{p|n} \left( 1 + \frac{A}{p - 2} \right) \leq \exp \left\{ \sum_{p|n} \frac{A}{p - 1} \right\}
$$

To estimate the remaining sum, we replace the denominator:

$$
\sum_{p|n} \frac{1}{p - 1} = \sum_{p|n} \frac{1}{p} + \sum_{p|n} \frac{1}{p(p - 1)} = \sum_{p|n} \frac{1}{p} + O(1)
$$

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Now, we introduce a parameter $1 \leq u \leq n$ so that

$$\sum_{\nu | n} \frac{1}{p} \leq \sum_{p \leq u} \frac{1}{p} + \sum_{\nu | n} \frac{1}{p} \leq \sum_{p \leq u} \frac{1}{p} + \frac{\omega(n)}{u}$$

$$= \log \log u + O(1) + O\left(\frac{\log n}{u}\right)$$

where the last line follows from (7) and the fact that $\omega(n) = O(\log n)$. Plugging this result back with $u = \log 3n$, we obtain

$$\sum_{d | n} \mu_2(d) A^{\omega(d)} \leq \exp\left\{A \log \log \log 3 n + O(1)\right\} \ll (\log \log 3 n)^{A}$$

If $n$ is the product of primes $\leq y$, then by Chebyshev’s estimates we have

$$\log n = \sum_{p \leq y} \log p \ll y$$

Thus, (8) is also proven.

**Q.E.D.**

**Lemma 3** (Selberg’s sieve). Let $Q$ denote the product of primes $\leq z = N^{\frac{1}{4}}$ that do not divide $N$. There exists real sequence $\lambda_d$ satisfying

1. $\lambda_1 = 1$.
2. $\lambda_d = 0$ for $d > z$ or $d \nmid Q$.
3. $|\lambda_d| \leq 1$.

such that

$$\sum_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2}}{\phi([d_1, d_2])} = [8 + O(\epsilon)] N / \log N$$

**Proof.** See Lemma 2 and §4 of [12]. Q.E.D.

**Lemma 4.** For large $N$, we have

$$\sum_{a} f(a) \frac{1}{a \log(N/a)} \leq 0.49254 \log N$$

**Proof.** According Equation 4 and Equation 7, we can use partial summation to transform the leftmost sum into

$$= \sum_{N^{\frac{1}{4}} < p_1 \leq N^{\frac{1}{4}}} \sum_{N^{\frac{1}{4}} < p_2 \leq (N/p_1)^{\frac{1}{4}}} \frac{1}{p_1 p_2 \log(x/p_1 p_2)}$$

$$\sim \int_N^{N^{\frac{3}{4}}} \frac{dx}{x \log x} \int_N^{N^{\frac{1}{4}}} \frac{1}{\log N - \log x - \log y} \frac{dy}{y \log y}$$

$$= \frac{1}{\log N} \int_0^{\frac{1}{4}} \frac{d\alpha}{\alpha} \int_{\frac{1}{2}}^{1 - \alpha} \frac{d\beta}{\beta(1 - \alpha - \beta)} \ll 0.49254 \frac{1}{\log N}$$

where the last inequality follows the numerical calculations in (28) of [2]. Q.E.D.
III. Preliminary treatments for $\Omega$

Since every prime number is either co-prime to $Q$ or a divisor of $Q$, we have

$$\Omega \leq \sum_a f(a) \sum_{ap \leq N} \frac{\lambda_d}{d(N-ap)}$$

where $\lambda_d$ is defined as in Lemma 3. By an interchanging of summation, there is

$$M = \sum_{a} f(a) \sum_{ap \leq N} \sum_{d_1, d_2(N-ap,Q)} \lambda_{d_1} \lambda_{d_2}$$

$$= \sum_{d_1, d_2} \lambda_{d_1} \lambda_{d_2} \sum_{a} f(a)$$

$$= \sum_{d_1, d_2} \lambda_{d_1} \lambda_{d_2} \sum_a f(a) \pi(N; a, [d_1, d_2], N) = M_1 + R$$

where $M_1$ and $R$ satisfy:

$$M_1 = \sum_{d_1, d_2} \frac{\lambda_{d_1} \lambda_{d_2}}{\varphi([d_1, d_2])} \sum_a f(a) \ln \frac{N}{a}$$

$$|R| \leq \sum_{d \leq N^{1/2}} 3^{\omega(d)} \left| \sum_a f(a) \pi(N; a, d, N) \right|$$

where the factor $3^{\omega(d)}$ follows from the fact that there are exactly $3^{\omega(d)}$ pairs of $d_1, d_2$ satisfying $[d_1, d_2] = d$. To estimate the main term, notice that

$$\ln x = \frac{x}{\ln x} + O \left( \frac{x}{\ln^2 x} \right)$$

so plugging Lemma 3 and Lemma 4 into (9) gives

$$M_1 \leq [8 + O(\varepsilon)] \frac{\Theta(N)}{\log N} \frac{0.49254 N}{\log N} \leq 3.94033 \Theta(N) \frac{N}{\log^4 N}$$

Thus the remaining task is to estimate the error term in (10).

IV. Salvaging the estimation of $R$

The authors of [7] concluded

$$R \ll \frac{N}{\log^4 N}$$
merely from Theorem 2 because they assume \( f(a) \neq 0 \) automatically ensures \( a \) is co-prime to \( d \). However, due to (4), it is possible for \( a \) to possess a prime divisor \( \leq N^{\frac{1}{10} - \frac{1}{2}} \), so their arguments for (12) are incorrect.

To salvage (12) and their proof of Chen’s theorem, we consider each case separately. After applying Theorem 2 to estimate the sum over \( (a, d) = 1 \), all we need is to estimate the remaining situation where \( (a, d) > 1 \):

\[
R \ll \frac{N}{\log^4 N} + \sum_{d \mid Q, d \leq N^{1/2}} 3^{\omega(d)} \sum_{(a,d) > 1} f(a) |\Delta(N; a, d, N)|
\]  

(13)

Since \( (a, d) > 1 \) implies \( \pi(N; a, d, N) \leq 1 \), it follows from (6) and Lemma 2 that

\[
R_1 \ll \sum_{d \mid Q} 3^{\omega(d)} \max_{d \leq N^{1/2}} \sum_{(a,d) > 1} f(a) \frac{N}{a \log(N/a)}
\]

\[
\ll (\log N)^3 \max_{d \leq N^{1/2}} \sum_{(a,d) > 1} f(a) \frac{N}{a \log(N/a)}
\]

\[
\ll N(\log N)^2 \max_{d \leq N^{1/2}} \sum_{(a,d) > 1} \frac{f(a)}{a}
\]

where the last \( \ll \) follows from the fact that \( f(a) \neq 0 \) implies \( a \leq N^{\frac{2}{5}} \). Due to (4), \( (a, d) > 1 \) if and only if \( p_1 \mid d \), so

\[
\sum_{(a,d) > 1} \frac{f(a)}{a} \leq \sum_{p \mid p_1} \sum_{N^{\frac{2}{10} - \frac{1}{5}} < p \leq N^{\frac{2}{10} - \frac{1}{5}}} \frac{1}{p} \sum_{p_2 \mid p_1} \frac{1}{p_2}
\]

\[
\leq \sum_{p \mid p_1} \frac{1}{p_1} \sum_{N^{\frac{2}{10} - \frac{1}{5}} < p_2 \leq N^{\frac{2}{10} - \frac{1}{5}}} \frac{1}{p_2}
\]

Now, we apply Lemma 1 with \( x = N, \alpha = \frac{1}{4}, \beta = \frac{1}{4} \), so that

\[
\sum_{(a,d) > 1} \frac{f(a)}{a} \ll \sum_{p \mid p_1} \frac{1}{p} < 10N^{-\frac{1}{10}} \frac{\log N}{\log 2} \sum_{p \mid d} \log p \ll N^{-\frac{1}{10}}
\]

Plugging this result back, we obtain \( R_1 \ll N^{\frac{1}{10} \log^2 N} \). Finally, combining (11) and (13), we conclude that for large \( N \)

\[
\Omega \leq M_1 + R + N^{\frac{1}{10} \log^2 N} \leq 3.94033 \mathbb{S}(N) \frac{N}{\log^2 N}
\]

\[
+ O \left( \frac{N}{\log^4 N} + N^{\frac{1}{10} \log^2 N} + N^{\frac{1}{11}} \right)
\]

This gives (5), so the proof of \( \{1,2\} \) is now complete.
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