Asymmetric Transmitter Cooperation in Multiuser Wireless Networks: Achievable Rate Regions

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Abstract

In this work, we derive achievable rate regions for the three-user interference channels with asymmetric transmitter cooperation and various decoding capabilities at the receivers. The three-user channel facilitates different ways of message sharing between the transmitters. We introduce two natural ways of extending the concept of unidirectional message sharing from two users to three users - (i) cumulative message sharing and (ii) primary-only message sharing. In addition, we define several cognitive interference channels based on the decoding capability of the receivers. We employ a coding technique, which is a combination of superposition and Gel’fand-Pinsker coding techniques, to derive an achievable rate region for each of the cognitive interference channels. Simulation results, by considering the Gaussian channel case, enables a visual comparison of the two message-sharing schemes considered in this paper. It also provides useful insights into the effect of message-splitting at the transmitters and the decoding capability of the receivers on the achievable rates.

I. Introduction

Most of the current wireless system designs are primarily based on the idea of avoiding interference between users in a given frequency band. This has resulted in an inefficient use of the Radio Frequency (RF) spectrum, which can only be mitigated by introducing some form of cooperation between the users. Cognitive Radio (CR) [1] has recently emerged as a possible solution to this bandwidth scarcity problem, as it tries to improve spectral efficiency by making the users aware of their RF environment and adjusting their transmission and reception parameters accordingly. An overview of the potential benefits offered by the CRs in physical layer research is provided in [2]. In [3], three main CR paradigms have been identified - underlay, overlay and interweave. In the underlay paradigm, the CR users are allowed to operate only if the noncognitive (or primary) users experience an interference (from the CR) which is below a certain threshold. While operating in the overlay paradigm, the CRs transmit their data
simultaneously with the noncognitive users by employing sophisticated techniques that maintain or even improve
the performance of the noncognitive users. In the interweave paradigm, the CRs sense unused frequency bands
called *spectrum holes* to communicate without disrupting the primary transmissions. Of these, the information
theoretic research has focussed primarily on the overlay paradigm with transmitter cooperation introduced through
*unidirectional* message sharing. Here the *primary* user non-causally shares the message it intends to transmit with
the *cognitive* user. Then, the primary and cognitive users simultaneously transmit their messages, but the encoding
is performed in such a way that the the primary user does not suffer in terms of its achievable rates.

We now present a short survey of recent information theoretic work in this area, followed by a summary of our
contributions.

**A. Literature survey**

The concept of cognitive radios has spurred great deal of research in information theory, in addition to other fields
such as signal processing and estimation/detection theory. A recent overview, identifying the three CR paradigms
mentioned above, and exploring some of the fundamental capacity limits and associated transmission strategies for
wireless networks is [3]. In [4], [5], Devroye et al defined the *genie-aided* CR channel and derive an achievable
rate region. The coding scheme comprised a combination of the scheme proposed by Han and Kobayashi for the
interference channel [6], and the one proposed by Gel’fand and Pinsker for channels with side information [7]. Both
senders split their messages such that one of the sub-messages is decodable by the non-pairing receiver. Since the CR
knows the sub-messages and the corresponding codewords of the primary sender, it applies Gel’fand-Pinsker (GP)
coding to encode its own sub-messages by treating the codewords of the primary sender as known interference. In
[8], Wu et al introduced terms like *dumb* and *smart* antennas to refer to primary and cognitive senders, respectively.
They employed a combination of GP and superposition coding [9] techniques to come up with an achievable rate
region for the two-user CR channel where neither sender splits their messages, nor do the receivers decode the
messages from the non-pairing senders. In [20], an achievable rate region for the two-user interference channel with
degraded message sets has been derived by employing a coding scheme which is a combination of superposition
and GP coding techniques.

In [10], Jovičić et al presented the Gaussian CR channel and model the problem such that the primary sender is
oblivious to the presence of the CR. Further, the primary receiver uses a single user decoder, just as it would in the
absence of the CR. They employed *dirty paper (DP) coding* [11] to derive an achievable rate region. In [12], Marić
et al determined the capacity region for interference channels with partially cooperating transmitters, by considering
the strong interference regime in which both receivers decode both messages. In [13], inner and outer bounds on the
capacity region of two-sender, two-receiver interference channels where one transmitter knows both the messages
were established. The decoders were assumed to only decode messages from their intended senders. In [14] - [16],
information theoretic results for interference channels with common information were derived. The sum-capacity of the Gaussian MIMO cognitive radio network was presented in [17], and the results obtained apply to the single-antenna CR channel as well. Capacity scaling laws for CR networks were presented in [18], while achievable rates for channels with different states known non-causally to the encoder were considered in [19]. Interference channels with cognitive and partially-cognitive transmitters were considered in [21] - [23]. Results for strong interference channels with unidirectional cooperation were presented in [24], while [25] and [26] considered strong interference channels with common information. Multiple access channels with conferencing encoders are considered in [27], while [28] presented the capacity region of the Gaussian multiple access channel with conferencing encoders. The capacity region of the three-user multiple access channel with cooperation was presented in [29].

B. Contributions of this paper

In this paper, we consider the case of three-user CR interference channel. The three-user channel facilitates different ways of transmitter cooperation, based on the message-sharing mechanism of the senders. We consider two natural ways of extending the two-user unidirectional message sharing paradigm to the three-user case, which we term (i) cumulative message sharing (CMS) and (ii) primary-only message sharing (PMS) (these notions will be made precise in the next section).

Also, based on the decoding capability of receivers, we define four cognitive channel models with CMS and PMS. We employ a known coding scheme which comprises a combination of GP coding [7] and superposition coding [9] techniques to derive an achievable rate region for each of the four channels. The coding scheme we adopt was first presented in [20] for the two-user case, and is extended here for the three-user CR channel. Initial results of this work have appeared in [30] and [31].

By deriving the rate regions under different message sharing and decoding-capability assumptions, we illustrate the generality of the techniques employed here, and are able to glean useful insights into the rate regions and their characterization. Next, we specialize the achievable rate regions to the Gaussian channel, which enables us to compare the different rate regions both analytically and through simulations. We are able to make several interesting observations on the influence of message sharing and decoding capability assumptions on the achievable rates.

The rest of the paper is organized as follows. In Section II, we introduce the discrete memoryless channel models for CMS and PMS, and lay down the notation used in the paper. We also present the probability distribution functions characterizing these channels. In Section III, we present the achievability theorem for the channel models considered and work out the details of the proof for two of the channel models. In Section IV, we consider the Gaussian channel model and construct the framework for numerical evaluation. Simulation results and related discussions are presented in Section V. We conclude the paper in Section VI. The achievable rate region equations for the four discrete memoryless channels considered in this paper and the proof of achievability theorem for one
channel model is relegated to the Appendix.

II. DISCRETE MEMORYLESS CHANNEL MODEL AND PRELIMINARIES

A schematic diagram of CMS and PMS is shown in Fig. 1. In the case of CMS, the first cognitive radio (CR$_1$) has noncausal knowledge of the message $m_1$ and the corresponding codewords of the primary sender. The second cognitive radio (CR$_2$) has noncausal knowledge of the message $m_1$ of the primary transmitter as well as the message $m_2$ of CR$_1$, and their respective codewords. In the case of PMS, both the cognitive radios CR$_1$ and CR$_2$ have noncausal knowledge of the message $m_1$ and the corresponding codewords of the primary sender. There is no message-sharing mechanism between the cognitive radios themselves. For the two-user case, both CMS and PMS reduce to the models employed in the existing literature.

The three-user discrete memoryless cognitive interference channel is described by $(X_1, X_2, X_3, P, Y_1, Y_2, Y_3)$. We define two channels with CMS and PMS. For $k = 1, 2, 3$,

- the senders and receivers are denoted by $S_k$ and $R_k$, respectively,
- finite sets $X_k$ and $Y_k$ denote the channel input and output alphabets, respectively,
- random variables $X_k \in X_k$ and $Y_k \in Y_k$ are the inputs and outputs of the channel, respectively and
- $P$ denotes the finite set of conditional probabilities $p(y_1, y_2, y_3|x_1, x_2, x_3)$, when $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ are transmitted and $(y_1, y_2, y_3) \in Y_1 \times Y_2 \times Y_3$ are obtained by the receivers.

The channels are assumed to be memoryless. In the usual three-user interference channel, the messages at the senders are given by $m_k \in M_k = \{1, ..., M_k\}$; $M_k$ being finite set with $M_k$ elements. The messages are assumed to be independently generated. For an interference channel having asymmetric transmitter cooperation with cumulative message sharing, $S_1$ has message $m_1$, $S_2$ has messages $(m_1, m_2)$, and $S_3$ has messages $(m_1, m_2, m_3)$. For an interference channel having asymmetric transmitter cooperation with primary-only message sharing, $S_1$ has message $m_1$, $S_2$ has messages $(m_1, m_2)$, and $S_3$ has messages $(m_1, m_3)$. An $(M_1, M_2, M_3, n, P_e^{(n)})$ code exists for these channels, if there exists the following encoding functions:

$$f_1 : M_1 \mapsto X_1^n, \quad f'_1 : M_1 \mapsto X_1^n,$$
$$f_2 : M_1 \times M_2 \mapsto X_2^n, \quad f'_2 : M_1 \times M_2 \mapsto X_2^n,$$
$$f_3 : M_1 \times M_2 \times M_3 \mapsto X_3^n, \quad f'_3 : M_1 \times M_3 \mapsto X_3^n$$

and the following decoding functions:

$$g_1 : Y_1^n \mapsto M_1, \quad g'_1 : Y_1^n \mapsto M_1,$$
$$g_2 : Y_2^n \mapsto M_2, \quad g'_2 : Y_2^n \mapsto M_2,$$
$$g_3 : Y_3^n \mapsto M_3, \quad g'_3 : Y_3^n \mapsto M_3,$$
such that the decoding error probability \( \max\{P_{e,1}^{(n)}, P_{e,2}^{(n)}, P_{e,3}^{(n)}\} \) is \( \leq P_{e,k}^{(n)} \). \( P_{e,k}^{(n)} \) is the average probability of decoding error computed using:

\[
P_{e,k}^{(n)} = \frac{1}{M_1 M_2 M_3} \sum_{m_1, m_2, m_3} p[\hat{m}_k \neq m_k|(m_1, m_2, m_3) \text{ sent}] ; k = 1, 2, 3.
\]

\( f_k \) (or \( g_k \)) correspond to the encoders (or decoders) used by channels with CMS, while \( f'_k \) (or \( g'_k \)) correspond to the encoders (or decoders) used by channels with PMS.

We define two channels denoted \( C_{\text{cms}}^t \) (\( \text{cms} \) for cumulative message sharing) and two channels denoted \( C_{\text{pms}}^t \) (\( \text{pms} \) for primary-only message sharing); \( t = 1, 2 \). A non-negative rate triple \( (R_1, R_2, R_3) \) is achievable for each of the channels \( C_{\text{cms}}^t \) and \( C_{\text{pms}}^t \), if for any \( 0 < P_{e}^{(n)} < 1 \) there exists a \( (2^{[nR_1]}, 2^{[nR_2]}, 2^{[nR_3]}, n, P_{e}^{(n)}) \) code such that \( P_{e}^{(n)} \to 0 \) as \( n \to \infty \). The capacity region for the channels \( C_{\text{cms}}^t \) and \( C_{\text{pms}}^t \) is the closure of the set of all achievable rate triples \( (R_1, R_2, R_3) \). A subset of the capacity region gives an achievable rate region.

As in [6], we will modify the channels \( C_{\text{cms}}^t \) and \( C_{\text{pms}}^t \); \( t = 1, 2 \) to introduce rate splitting. To motivate the discussion, consider the two-user scenario. Han and Kobayashi [6] showed that the achievable rate region of the interference channel can be improved using message splitting, where each user splits its message into two parts. Given that all parts of the message need to be decodable at the intended receiver, this splitting is exhaustive: either the part of the message is decodable at the non-intended receiver, or not. In the three user case, more options exist. For example, consider the message splitting at the primary user. It can split its message into four parts: one decodable at both the cognitive receivers, one decodable at CR\(_1\) but not at CR\(_2\), one decodable at CR\(_2\) but not at CR\(_1\), and finally, one that is not decodable at either CR receiver (i.e., decodable only at the primary receiver). This implies a total of 12 message parts, and the rate region obtained by considering all these possible message splittings would be the largest possible one. Now, each receiver can decode 8 messages, four from its own transmitter and four from the two other transmitters. Of these 8 messages, the receiver is only required to decode four messages correctly, i.e., error events where only the unwanted messages are received in error does not affect the rate region. Thus, out of the possible \( 2^8 - 1 \) events where one or more of the 8 messages are received in error, \( 2^4 - 1 \) events corresponding to one or more of the unintended messages being received in error (and the intended messages being received correctly) do not count towards the rate region description. Therefore, the rate region description would involve \((2^8 - 1) - (2^4 - 1) = 240\) inequalities per receiver, and thus, we would have 720 inequalities in total. Since working out such a rate region would be an arduous task, we make some simplifying assumptions.

In each channel model, we split the message at each transmitter into only two instead of four parts. In \( C_{\text{cms}}^1 \) and \( C_{\text{pms}}^1 \), one part of the message is decodable only at the intended receiver, while the other part is decodable at all receivers. In \( C_{\text{cms}}^2 \) and \( C_{\text{pms}}^2 \), one part of the message is decodable only at the intended receiver, but the other part is decodable at the intended receiver and the primary receiver only. The reason for the choice of \( C_{\text{cms}}^1 \) and \( C_{\text{pms}}^1 \) is mainly to see the effect of allowing each user to decode part of other users’ message to reduce the interference it sees.
It will turn out that unidirectional message sharing mainly benefits the secondary transmitters rather than the primary user, in terms of the achievable rate region. This can be mitigated by allowing the primary receiver to decoding part of the secondary transmitters’ messages, as in $c^2_{cms}$ and $c^2_{pms}$. The notation for describing the achievable rates of these sub-messages and their respective description is tabulated in Table I. The decoding capability of receivers for the channels $c^t_{cms}$; $t = 1, 2$ is summarized in Tables II and III, respectively. We also introduce auxiliary random variables defined on finite sets and tabulate them in Table IV. Depending on the decoding capability of receivers, only a subset of these sub-messages, their corresponding rates, and the corresponding auxiliary random variables will be used to derive an achievable rate region for each channel model. Note that, we do not consider the practical aspects of underlying the physical realization of such models. Also, the computation of the exact capacity region is hard and is beyond the scope of this paper. In this paper, we explicitly show the modification for one channel $(c^2_{cms})$. Referring to the decoding capability of the receivers (see Table III), the messages at the three senders in the modified channel can be written as:

**Sender 1:** $m_{11} \in \mathcal{M}_{11} = \{1, ..., M_{11}\}$,

**Sender 2:** $m_{21} \in \mathcal{M}_{21} = \{1, ..., M_{21}\}$, $m_{22} \in \mathcal{M}_{22} = \{1, ..., M_{22}\}$,

**Sender 3:** $m_{31} \in \mathcal{M}_{31} = \{1, ..., M_{31}\}$, $m_{32} \in \mathcal{M}_{32} = \{1, ..., M_{32}\}$,

with all messages being defined on sets with finite number of elements. Please note that we do not split the message $m_1$, but for consistency in notation we write $m_1$ as $m_{11}$. Define an $(M_{11}, M_{21}, M_{22}, M_{31}, M_{32}, n, P_e(n))$ code for the modified channel as a set of $M_{11}$ codewords for $S_1$, $M_{11}M_{21}M_{22}$ codewords for $S_2$, and $M_{11}M_{21}M_{22}M_{31}M_{32}$ codewords for $S_3$ such that the average probability of decoding error is less than $P_e(n)$. Call a tuple $(R_{11}, R_{21}, R_{22}, R_{31}, R_{33})$ achievable if there exists a sequence of $(2^{[nR_{1t}]}, 2^{[nR_{21}]}, 2^{[nR_{22}]}, 2^{[nR_{31}]}, 2^{[nR_{33}]}, n, P_e(n))$ codes such that $P_e(n) \rightarrow 0$ as $n \rightarrow \infty$. The capacity region for the modified channel is the closure of the set of all achievable rate tuples $(R_{11}, R_{21}, R_{22}, R_{31}, R_{33})$. It can be shown that if the rate tuple $(R_{11}, R_{21}, R_{22}, R_{31}, R_{33})$ is achievable for the modified channel, then the rate triple $(R_{11} + R_{21} + R_{22}, R_{31} + R_{33})$ is achievable for the channel $c^2_{cms}$ (see [6, Corollary 2.1]). In a similar fashion, we can modify all the channel models $c^t_{cms}$ and $c^t_{pms}$; $t = 1, 2$.

### III. An Achievable Rate Region for the Channels

Let $\mathcal{P}^t_{cms}$ denote the set of all joint probability distributions $p^{t}_{cms}(.)$; $t = 1, 2$ respectively, that factor as follows:

$$
p^{t}_{cms}(q, w_0, w_1, x_1, u_0, u_2, x_2, v_0, v_3, x_3, y_1, y_2, y_3) =
$$

$$p(q)p(w_0, w_1, x_1 | q)p(u_0 | w_0, w_1, q)p(u_2 | w_0, w_1, q)
$$

$$p(x_2 | u_0, u_2, w_0, w_1, q)p(v_0 | u_0, u_2, w_0, w_1, q)
$$

$$p(v_3 | u_0, u_2, w_0, w_1, q)p(x_3 | v_0, v_3, u_0, u_2, w_0, w_1, q)$$
\[ x p(y_1, y_2, y_3 | x_1, x_2, x_3), \]  

\[ p^2_{pms}(q, w, x, u_1, x_2, v_1, x_3, y_1, y_2, y_3) = \\
p(q)p(w, x_1 | q)p(u_1 | w, q)p(u_2 | w, q)p(x_2 | u_1, u_2, w, q) \\
p(v_1 | u_1, u_2, w, q)p(v_2 | u_1, u_2, w, q)p(x_3 | v_1, v_3, u_1, u_2, w, q) \\
x p(y_1, y_2, y_3 | x_1, x_2, x_3). \]  

The lower case letters \((q, w, u_2, v_3 \text{ etc.})\) are realizations of their corresponding random variables. An achievable rate region for \(C_{cms}^1 \cup C_{cms}^2\) is described by \(R_{pms}(p_{pms}^1) / R_{pms}(p_{pms}^2)\), defined as the set of all non-negative rate tuples \((R_1, R_2, R_3)\) such that the inequalities given in Appendix A hold simultaneously.

Let \(P^t_{pms}\) denote the set of all joint probability distributions \(p^t_{pms}()\); \(t = 1, 2\) respectively, that factor as follows:

\[ p^1_{pms}(q, w_0, w_1, x_1, u_0, u_2, x_2, v_0, v_3, x_3, y_1, y_2, y_3) = \\
p(q)p(w_0, w_1, x_1 | q)p(u_0 | w_0, w_1, q)p(u_2 | w_0, w_1, q) \\
p(x_2 | u_0, u_2, w_0, w_1, q)p(v_0 | w_0, w_1, q)p(x_3 | w_0, w_1, q) \\
p(x_3 | v_0, v_3, w_0, w_1, q)p(y_1, y_2, y_3 | x_1, x_2, x_3), \]  

\[ p^2_{pms}(q, w, x, u_1, u_2, x_2, v_1, x_3, y_1, y_2, y_3) = \\
p(q)p(w, x_1 | q)p(u_1 | w, q)p(u_2 | w, q)p(x_2 | u_1, u_2, w, q) \\
p(v_1 | w, q)p(v_2 | w, q)p(x_3 | v_1, v_3, w, q) \\
x p(y_1, y_2, y_3 | x_1, x_2, x_3). \]  

An achievable rate region for \(C_{pms}^1 \cup C_{pms}^2\) is described by \(R_{pms}(p_{pms}^1) / R_{pms}(p_{pms}^2)\), defined as the set of all non-negative rate tuples \((R_1, R_2, R_3)\) such that the inequalities given in Appendix B hold simultaneously.

**IV. Achievability Theorem and Proof**

**Theorem 4.1:** Let \(C_{cms}^t\) (or \(C_{pms}^t\)) denote the capacity region of the channel \(C_{cms}^t\) (or \(C_{pms}^t\)); \(t = 1, 2\). Let

\[ R_{cms}^t = \bigcup_{P_{cms}^t} R_{cms}(p_{cms}^t) \text{ and } R_{pms}^t = \bigcup_{P_{pms}^t} R_{pms}(p_{pms}^t). \]

The region \(R_{cms}^t\) (or \(R_{pms}^t\)) is an achievable rate region for the channel \(C_{cms}^t\) (or \(C_{pms}^t\)), i.e., \(R_{cms}^t\) (or \(R_{pms}^t\)) \(\subseteq\) \(C_{cms}^t\) (or \(C_{pms}^t\)).
Proof: We follow the coding scheme presented in [20] to show the achievability of the rate region for the three-user channels. The salient features of this coding scheme are summarized in [20, Remark 1]. We show the proof for $C_{cms}^1$, $C_{cms}^2$, $C_{pms}^1$ and $C_{pms}^2$ in that order.

For the channel $C_{cms}^1$:

A. Codebook generation

Let us fix $p(.) \in \mathcal{P}$. Generate a random time sharing codeword $q_i$ of length $n$, according to the distribution $\prod_{i=1}^n p(q_i)$. For $\gamma = 0, 1$, $\tau = 0, 2$ and $\rho = 0, 3$:

generate $2^{[nR_{\gamma}]}$ independent codewords $W_{\gamma}(j_{\gamma})$, $j_{\gamma} \in \{1, ..., 2^{[nR_{\gamma}]}\}$ according to $\prod_{i=1}^n p(w_{\gamma i}|q_i)$. For every codeword pair $(w_0(j_0), w_1(j_1))$, generate one codeword $X_1(j_0, j_1)$ according to $\prod_{i=1}^n p(x_{1i}|w_i(j, q_i))$.

generate $2^{n(R_{\tau}+I(W_0;W_1;U_{\tau}|Q)+4\epsilon)}$ independent code words $U_{\tau}(l_{\tau})$, according to $\prod_{i=1}^n p(u_{\tau i}|q_i)$. For every codeword tuple $(u_0(l_0), u_2(l_2), w_0(j_0), w_1(j_1))$, generate one codeword $X_2(l_0, l_2, j_0, j_1)$ according to $\prod_{i=1}^n p(x_{2i}|u_{0i}(l_0), u_{2i}(l_2), w_{0i}(j_0), w_{1i}(j_1)|q_i)$. Uniformly distribute the $2^{n(R_{\tau}+I(W_0;W_1;U_{\tau}|Q)+4\epsilon)}$ code words $U_{\tau}(l_{\tau})$ into $2^{nR_{\tau}}$ bins indexed by $l_{\tau} \in \{1, ..., 2^{nR_{\tau}}\}$ such that each bin contains $2^{n(I(W;U_{\tau}|Q)+4\epsilon)}$ codewords.

generate $2^{n(R_{\rho}+I(W_0,W_1,U_0,U_2;V_{\rho}|Q)+4\epsilon)}$ independent code words $V_{\rho}(t_{\rho})$, according to $\prod_{i=1}^n p(v_{\rho i}|q_i)$. For every code word tuple $(v_0(t_0), v_3(t_3), u_0(l_0), u_2(l_2), w_0(j_0), w_1(j_1))$, generate one codeword $X_3(t_0, t_3, l_0, l_2, j_0, j_1)$ according to $\prod_{i=1}^n p(x_{3i}|v_{0i}(t_0), v_{3i}(t_3), u_{0i}(l_0), u_{2i}(l_2), w_{0i}(j_0), w_{1i}(j_1)|q_i)$. Distribute $2^{n(R_{\rho}+I(W_0,W_1,U_0,U_2;V_{\rho}|Q)+4\epsilon)}$ code words $V_{\rho}(t_{\rho})$ uniformly into $2^{nR_{\rho}}$ bins indexed by $t_{\rho} \in \{1, ..., 2^{nR_{\rho}}\}$ such that each bin contains $2^{n(I(W_0,W_1,U_0,U_2;V_{\rho}|Q)+4\epsilon)}$ code words. The indices are given by $j_{\gamma} \in \{1, ..., 2^{[nR_{\gamma}]}\}$, $l_{\tau} \in \{1, ..., 2^{nR_{\tau}}\}$, $t_{\rho} \in \{1, ..., 2^{nR_{\rho}}\}$.

B. Encoding & transmission

Let us suppose that the source message vector generated at the three senders is $(m_{10}, m_{11}, m_{20}, m_{22}, m_{30}, m_{33}) = (j_0, j_1, k_0, k_2, r_0, r_3)$. $S_1$ transmits codeword $x_1(j_0, j_1)$ with $n$ channel uses. $S_2$ first looks for a codeword $u_0(l_0)$ in bin $k_0$ such that $(u_0(l_0), w_0(j_0), w_1(j_1), q) \in A_t^n$, and a codeword $u_2(l_2)$ in bin $k_2$ such that $(u_2(l_2), w_0(j_0), w_1(j_1), q) \in A_t^n$. It then transmits $x_2(l_0, l_2, j_0, j_1)$ through $n$ channel uses. Otherwise, $S_2$ declares an error. $S_3$ first looks for a codeword $v_0(t_0)$ in bin $r_0$ such that $(v_0(t_0), u_0(l_0), u_2(l_2), w_0(j_0), w_1(j_1), q) \in A_t^n$, and a codeword $v_3(t_3)$ in bin $r_3$ such that $(v_3(t_3), u_0(l_0), u_2(l_2), w_0(j_0), w_1(j_1), q) \in A_t^n$. It then transmits $x_3(t_0, t_3, l_0, l_2, j_0, j_1)$ through $n$ channel uses. Otherwise, $S_3$ declares an error. The transmissions are assumed to be perfectly synchronized.
C. Decoding

The three receivers accumulate an  They look for all index tuples \((j_0, j_1, \hat{t}_0, \hat{t}_0)\) such that \((w_0(j_0), w_1(j_1), u_0(l_0), v_0(t_0), y_1, q) \in A^{(n)}_1\). If \(j_0 = j_1\) in all the index tuples found are the same, \(R_1\) determines \((m_{10}, m_{11}) = (j_0, j_1)\) for some \(l_0\) and \(t_0\). Otherwise, it declares an error. Decoder 2 looks for all index tuples \((\hat{t}_0, \hat{t}_2, j_0, \hat{t}_0)\) such that \((w_0(j_0), u_0(l_0), u_2(l_2), v_0(t_0), y_2, q) \in A^{(n)}_2\). If \(\hat{t}_0\) in all the index pairs found are indices of codewords \(u_0(l_0)\) from the same bin with index \(k_0\), and \(\hat{t}_2\) in all the index pairs found are indices of codewords \(u_2(l_2)\) from the same bin with index \(k_2\), then \(R_2\) determines \((m_{20}, m_{22}) = (k_0, k_2)\). Otherwise, it declares an error. Decoder 3 looks for all index pairs \((\hat{t}_0, \hat{t}_3, \hat{t}_0, j_0)\) such that \((w_0(j_0), u_0(l_0), v_3(t_3), y_3, q) \in A^{(n)}_3\). If \(\hat{t}_0\) in all the index pairs found are indices of codewords \(v_0(l_0)\) from the same bin with index \(r_0\), and \(\hat{t}_3\) in all the index pairs found are indices of codewords \(v_3(t_3)\) from the same bin with index \(r_3\), then \(R_3\) determines \((m_{30}, m_{33}) = (r_0, r_3)\). Otherwise, it declares an error.

D. Analysis of probabilities of error

In this section we derive upperbounds on the probabilities of error events, which happens during encoding and decoding processes. We will assume that a source message vector \((m_{10}, m_{11}, m_{20}, m_{22}, m_{30}, m_{33})\) is encoded and transmitted. We will consider the analysis of probability of encoding error at senders \(S_2\) and \(S_3\), and the analysis of probability of decoding error at each of the three receivers \(R_1\), \(R_2\), and \(R_3\) separately.

First, let us define the following events:

(i) \(E_{j_0, j_1, l_0} = \{(W_0(j_0), W_1(j_1), U_0(l_0), q) \in A^{(n)}_1\}\),

(ii) \(E_{j_0, j_1, l_2} = \{(W_0(j_0), W_1(j_1), U_0(l_0), q) \in A^{(n)}_2\}\),

(iii) \(E_{j_0, j_1, l_0, l_0} = \{(W_0(j_0), W_1(j_1), U_0(l_0), U_2(l_2), V_0(t_0), q) \in A^{(n)}_3\}\),

(iv) \(E_{j_0, j_1, l_0, l_3} = \{(W_0(j_0), W_1(j_1), U_0(l_0), U_2(l_2), V_3(t_3), q) \in A^{(n)}_3\}\),

(v) \(E_{j_0, j_1, l_0, l_0, l_0} = \{(W_0(j_0), W_1(j_1), U_0(l_0), V_0(t_0), Y_1, q) \in A^{(n)}_1\}\),

(vi) \(E_{j_0, j_1, l_0, l_2} = \{(W_0(j_0), U_0(l_0), U_2(l_2), V_0(t_0), Y_2, q) \in A^{(n)}_1\}\),

(vii) \(E_{j_0, j_1, l_0, l_3} = \{(W_0(j_0), U_0(l_0), V_3(t_3), Y_3, q) \in A^{(n)}_3\}\).

\(E^c\) \(=\) complement of the event \(E\). Events (i) – (iv) will be used in the analysis of probability of encoding error while events (v) – (vii) will be used in the analysis of probability of decoding error.

1) Probability of error at encoder of \(S_2\): An error is made if (1) the encoder cannot find \(u_0(l_0)\) in bin indexed by \(k_0\) such that \((w_0(j_0), w_1(j_1), u_0(l_0), q) \in A^{(n)}_1\) or (2) it cannot find \(u_2(l_2)\) in bin indexed by \(k_2\) such that \((w_0(j_0), w_1(j_1), u_2(l_2), q) \in A^{(n)}_2\). The probability of encoding error at \(S_2\) can be bounded as

\[
P_{e, S_2} \leq P \left( \bigcap_{u_0(l_0) \in \text{bin}(k_0)} (W_0(j_0), W_1(j_1), U_0(l_0), q) \notin A^{(n)}_1 \right)
\]
Clearly, \( P(\cdot) \) is the probability of an event. Since \( q \) is predetermined,
\[
P(E_{j_0j_1j_0}) = \sum_{(w_0, w_1, u_0, q) \in A_{\varepsilon}^{(n)}} P(W_0(j_0) = w_0, W_1(j_1) = w_1 | q) P(U_0(l_0) = u_0 | q)
\geq 2^n(H(W_0, W_1, U_0 | Q - \varepsilon)) - n(H(W_0, W_1 | Q + \varepsilon)) \cdot 2^{-n(H(U_0 | Q + \varepsilon))} = 2^{-n(I(W_0, W_1; U_0 | Q + 3\varepsilon)).}
\]
Similarly, \( P(E_{j_0j_1j_0}) \geq 2^{-n(I(W_0, W_1; U_2 | Q + 3\varepsilon)).} \) Therefore,
\[
P_{e,S_2} \leq (1 - 2^{-n(I(W_0, W_1; U_0 | Q + 3\varepsilon))})^{2^n(I(W_0, W_1; U_0 | Q + 4\varepsilon))} + (1 - 2^{-n(I(W_0, W_1; U_2 | Q + 3\varepsilon))})^{2^n(I(W_0, W_1; U_2 | Q + 4\varepsilon))}.
\]
Now,
\[
(1 - 2^{-n(I(W_0, W_1; U_0 | Q + 3\varepsilon))})^{2^n(I(W_0, W_1; U_0 | Q + 4\varepsilon))} = e^{2^n(I(W_0, W_1; U_0 | Q + 4\varepsilon)) \ln(1 - 2^{-n(I(W_0, W_1; U_0 | Q + 3\varepsilon))})} \leq e^{2^n(I(W_0, W_1; U_0 | Q + 4\varepsilon))(-2^{-n(I(W_0, W_1; U_0 | Q + 3\varepsilon))})} = e^{-2^n\varepsilon}.
\]
Clearly, \( P_{e,S_2} \to 0 \) as \( n \to \infty \).

2) Probability of error at encoder of \( S_3 \): An error is made if (1) the encoder cannot find \( v_0(t_0) \) in bin indexed by \( r_0 \) such that \((w_0(j_0), w_1(j_1), u_0(l_0), u_2(l_2), v_0(t_0), q) \in A_{\varepsilon}^{(n)} \) or (2) it cannot find \( v_3(t_3) \) in bin indexed by \( r_3 \) such that \((w_0(j_0), w_1(j_1), u_0(l_0), u_2(l_2), v_3(t_3), q) \in A_{\varepsilon}^{(n)} \). The probability of encoding error at \( S_3 \) can be bounded as
\[
P_{e,S_3} \leq P \left( \bigcap_{v_0(t_0) \in \text{bin}(r_0)} (W_0(j_0), W_1(j_1), U_0(l_0), U_2(l_2), V_0(t_0), q) \notin A_{\varepsilon}^{(n)} \right) + P \left( \bigcap_{v_3(t_3) \in \text{bin}(r_3)} (W_0(j_0), W_1(j_1), U_0(l_0), U_2(l_2), V_3(t_3), q) \notin A_{\varepsilon}^{(n)} \right)
\leq (1 - P(E_{j_0j_1l_0j_0})^{2^n(I(W_0, W_1, U_0, U_2, V_0 | Q + 4\varepsilon))} + (1 - P(E_{j_0j_1l_0j_2})^{2^n(I(W_0, W_1, U_0, U_2, V_3 | Q + 4\varepsilon))}.
\]
Since \( q \) is predetermined, we have,
\[
P(E_{j_0j_1l_0j_2}) = \\
\sum_{(w_0, w_1, u_0, u_2, v_0, q) \in A_{\varepsilon}^{(n)}} P(W_0(j_0) = w_0, W_1(j_1) = w_1, U_0(l_0) = u_0, U_2(l_2) = u_2 | q) P(V_0(t_0) = v_0 | q)
\geq 2^n(H(W_0, W_1, U_0, U_2, V_0 | Q - \varepsilon)) - 2^{-n(H(W_0, W_1, U_0, U_2 | Q + \varepsilon))} \cdot 2^{-n(H(V_0 | Q + \varepsilon))} = 2^{-n(I(W_0, W_1, U_0, U_2; V_0 | Q + 3\varepsilon))}.
\]
Similarly, \( P(E_{j_{0}\not=j_{1}}|U_{0}=t_{3}) \geq 2^{-n(I(W_{0},W_{1},U_{0},U_{2};V_{0}|Q)+3\epsilon}) \). Therefore,

\[
P_{e,S_{3}} \leq \left(1 - 2^{-n(I(W_{0},W_{1},U_{0},U_{2};V_{0}|Q)+3\epsilon)}\right)^{2^{n(I(W_{0},W_{1},U_{0},U_{2};V_{0}|Q)+4\epsilon)}} + \left(1 - 2^{-n(I(W_{0},W_{1},U_{0},U_{2};V_{0}|Q)+3\epsilon)}\right)^{2^{n(I(W_{0},W_{1},U_{0},U_{2};V_{0}|Q)+4\epsilon)}} .
\]

Proceeding in a way similar to the encoder error analysis at \( S_{2} \), we get \( P_{e,S_{3}} \to 0 \) as \( n \to \infty \).

3) Probability of error at decoder of \( R_{1} \): There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., \( E_{j_{0}\not=j_{1}} \) happens or (2) there exists some \( j_{0} \not=j_{1} \) and \( j_{1} \not=j_{1} \) such that \( E_{j_{0}\not=j_{1}} \) happens. The probability of decoding error can, therefore, be expressed as

\[
P_{e,R_{1}}^{(n)} = P\left( E_{j_{0}\not=j_{1}}|U_{0}=t_{0} \right) + P\left( \bigcup_{j_{0}\not=j_{1},j_{1}\not=j_{1}} E_{j_{0}\not=j_{1}}|U_{0}=t_{0} \right)
\]

Applying union of events bound, \( [182] \) can be written as,

\[
P_{e,R_{1}}^{(n)} \leq P\left( E_{j_{0}\not=j_{1}}|U_{0}=t_{0} \right) + \sum_{j_{0}\not=j_{1},j_{1}\not=j_{1}} P\left( E_{j_{0}\not=j_{1}}|U_{0}=t_{0} \right)
\]

Let us now evaluate the probability of error events.
\[ P \left( E_{j_0 j_1 l_0 t_0} \right) \text{ can be upper bounded as} \]

\[
P \left( E_{j_0 j_1 l_0 t_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_0(j_0) = w_0 \mid q) P(W_1(j_1) = w_1) P(U_0(l_0) = u_0, V_0(t_0) = v_0) P(Y_1 = y_1 \mid q)
\]

\[
\leq 2^n (H(W_0, W_1, U_0, V_0, Y_1 \mid q) + \epsilon) 2^{-n} (H(W_0 \mid q) - \epsilon) 2^{-n} (H(W_1, U_0, V_0, Y_1 \mid q) - \epsilon)
\]

\[
= 2^{-n} (I(W_0; W_1, U_0, V_0, Y_1 \mid q) - 3\epsilon).
\]

\[ P \left( E_{j_0 j_1 l_0 t_0} \right) \text{ can be upper bounded as} \]

\[
P \left( E_{j_0 j_1 l_0 t_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_1(j_1) = w_1 \mid q) P(W_0(j_0) = w_0, U_0(l_0) = u_0, V_0(t_0) = v_0) P(Y_1 = y_1 \mid q)
\]

\[
\leq 2^n (H(W_0, W_1, U_0, V_0, Y_1 \mid q) + \epsilon) 2^{-n} (H(W_1 \mid q) - \epsilon) 2^{-n} (H(W_0, U_0, V_0, Y_1 \mid q) - \epsilon)
\]

\[
= 2^{-n} (I(W_1; W_0, U_0, V_0, Y_1 \mid q) - 3\epsilon).
\]

\[ P \left( E_{j_0 j_1 l_0 t_0} \right) \text{ can be upper bounded as} \]

\[
P \left( E_{j_0 j_1 l_0 t_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_0(j_0) = w_0 \mid q) P(W_1(j_1) = w_1) P(U_0(l_0) = u_0, V_0(t_0) = v_0) P(Y_1 = y_1 \mid q)
\]

\[
\leq 2^n (H(W_0, W_1, U_0, V_0, Y_1 \mid q) + \epsilon) 2^{-n} (H(W_0 \mid q) - \epsilon) 2^{-n} (H(W_1, U_0, V_0, Y_1 \mid q) - \epsilon)
\]

\[
= 2^{-n} (I(W_0; W_1, U_0, V_0, Y_1 \mid q) + I(W_0; W_1 \mid q) - 4\epsilon).
\]

\[ P \left( E_{j_0 j_1 l_0 t_0} \right) \text{ can be upper bounded as} \]

\[
P \left( E_{j_0 j_1 l_0 t_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_0(j_0) = w_0) P(U_0(l_0) = u_0, V_0(t_0) = v_0) P(W_1(j_1) = w_1) P(Y_1 = y_1 \mid q)
\]

\[
\leq 2^n (H(W_0, W_1, U_0, V_0, Y_1 \mid q) + \epsilon) 2^{-n} (H(W_0 \mid q) - \epsilon) 2^{-n} (H(W_1, U_0, V_0, Y_1 \mid q) - \epsilon)
\]

\[
= 2^{-n} (I(W_0; W_1, U_0, V_0, Y_1 \mid q) + I(W_0; W_1 \mid q) - 4\epsilon).
\]
\[ P\left( E_{j_0j_1l_0} \right) \text{ can be upper bounded as} \]

\[ P\left( E_{j_0j_1l_0} \right) = \sum_{(w_0, w_1, u_0, y_0, q) \in A^n} P(W_0(j_0) = w_0|q)P(V_0(t_0) = v_0|q)P(W_1(j_1) = w_1, U_0(l_0) = u_0, Y_1 = y_1|q) \leq 2^n(H(W_0, W_1, U_0, V_0, Y_1|Q) + \epsilon)2_2^n(H(W_0|Q)\epsilon)2_2^n(H(U_0, Y_1|Q)\epsilon)2_2^n-H(W_1, U_0, Y_1|Q)\epsilon) = 2^n-I(W_0, V_0; W_1, U_0, Y_1|Q) + I(W_0, V_0|Q) - 4\epsilon. \]

\[ P\left( E_{j_0l_0} \right) \text{ can be upper bounded as} \]

\[ P\left( E_{j_0l_0} \right) = \sum_{(w_0, w_1, u_0, y_0, q) \in A^n} P(W_1(j_0) = w_1|q)P(U_0(l_0) = u_0|q)P(W_0(j_0) = w_0, V_0(t_0) = v_0, Y_1 = y_1|q) \leq 2^n(H(W_0, W_1, U_0, V_0, Y_1|Q) + \epsilon)2_2^n(H(W_1|Q)\epsilon)2_2^n(H(U_0, V_0, Y_1|Q)\epsilon) = 2^n-I(W_0, U_0; W_0, V_0, Y_1|Q) + I(W_1, U_0|Q) - 4\epsilon. \]

\[ P\left( E_{j_0l_0} \right) \text{ can be upper bounded as} \]

\[ P\left( E_{j_0l_0} \right) = \sum_{(w_0, w_1, u_0, y_0, q) \in A^n} P(W_1(j_0) = w_1|q)P(V_0(t_0) = v_0|q)P(W_0(j_0) = w_0, U_0(l_0) = u_0, Y_1 = y_1|q) \leq 2^n(H(W_0, W_1, U_0, V_0, Y_1|Q) + \epsilon)2_2^n(H(W_1|Q)\epsilon)2_2^n(H(U_0, V_0, Y_1|Q)\epsilon) = 2^n-I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1, U_0|Q) - 4\epsilon. \]

\[ P\left( E_{j_0l_0} \right) \text{ can be upper bounded as} \]

\[ P\left( E_{j_0l_0} \right) = \sum_{(w_0, w_1, u_0, y_0, q) \in A^n} P(W_0(j_0) = w_0|q)P(W_1(j_1) = w_1|q)P(U_0(l_0) = u_0|q)P(V_0(t_0) = v_0, Y_1 = y_1|q) \leq 2^n(H(W_0, W_1, U_0, V_0, Y_1|Q) + \epsilon)2_2^n(H(W_1|Q)\epsilon)2_2^n(H(U_0, V_0, Y_1|Q)\epsilon) = 2^n-I(W_0, W_1, U_0, V_0, Y_1|Q) + I(W_1, V_0|Q) - 5\epsilon. \]
\[ P\left( E_{j_0j_1l_0l_0} \right) \] can be upper bounded as

\[
P\left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_0(j_0) = w_0 | q)P(W_1(j_1) = w_1 | q)P(V_0(t_0) = v_0 | q)P(U_0(l_0) = u_0, Y_1 = y_1 | q)
\]
\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1(Q)+\epsilon) 2^nH(W_0(Q)-\epsilon) 2^nH(V_0(Q)-\epsilon) 2^nH(U_0, Y_1(Q)-\epsilon)
\]
\[
= 2^n\left( I(W_0, W_1, V_0, U_0, Y_1(Q)+I(W_0, W_1, V_0(Q)+I(W_0, W_1(Q)-5\epsilon) \right).
\]

\[ P\left( E_{j_0j_1l_0l_0} \right) \] can be upper bounded as

\[
P\left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_0(j_0) = w_0 | q)P(U_0(l_0) = u_0 | q)P(V_0(t_0) = v_0 | q)P(W_1(j_1) = w_1, Y_1 = y_1 | q)
\]
\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1(Q)+\epsilon) 2^nH(W_0(Q)-\epsilon) 2^nH(V_0(Q)-\epsilon) 2^nH(U_0, Y_1(Q)-\epsilon)
\]
\[
= 2^n\left( I(W_0, U_0, V_0, W_1, Y_1(Q)+I(W_0, U_0, V_0(Q)+I(W_0, U_0(Q)-5\epsilon) \right).
\]

\[ P\left( E_{j_0j_1l_0l_0} \right) \] can be upper bounded as

\[
P\left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_1(j_1) = w_1 | q)P(U_0(l_0) = u_0 | q)P(V_0(t_0) = v_0 | q)P(W_0(j_0) = w_0, Y_1 = y_1 | q)
\]
\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1(Q)+\epsilon) 2^nH(W_1(Q)-\epsilon) 2^nH(V_0(Q)-\epsilon) 2^nH(U_0, Y_1(Q)-\epsilon)
\]
\[
= 2^n\left( I(W_0, W_1, U_0, Y_0, V_0, Y_1(Q)+I(W_1, U_0, V_0(Q)+I(W_1, U_0(Q)-5\epsilon) \right).
\]

\[ P\left( E_{j_0j_1l_0l_0} \right) \] can be bounded as

\[
P\left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^{(n)}} P(W_0(j_0) = w_0 | q)P(W_1(j_1) = w_1 | q)P(U_0(l_0) = u_0 | q)P(V_0(t_0) = v_0 | q)P(Y_1 = y_1 | q)
\]
\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1(Q)+\epsilon) 2^nH(W_1(Q)-\epsilon) 2^nH(V_0(Q)-\epsilon) 2^nH(U_0, Y_1(Q)-\epsilon)
\]
\[
= 2^n\left( I(W_0, W_1, U_0, V_0, Y_1(Q)+I(W_0, U_1, V_0(Q)+I(W_0, W_1(Q)-6\epsilon) \right).
\]
Substituting these in the probability of decoding error at $R_1$, we note that $P_{e,R_1}^{(n)} \to 0$ as $n \to \infty$ iff the following constraints are satisfied:

\begin{align*}
R_{10} &\leq I(W_0; W_1, U_0, V_0, Y_1|Q), \quad (6) \\
R_{11} &\leq I(W_1; W_0, U_0, V_0, Y_1|Q), \quad (7) \\
R_{10} + R_{11} &\leq I(W_0, W_1; U_0, V_0, Y_1|Q) + I(W_0; W_1), \quad (8) \\
R_{10} + R_{20} &\leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \quad (9) \\
R_{11} + R_{20} &\leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q), \quad (10) \\
R_{11} + R_{30} &\leq I(W_1, V_0; W_0, U_0, Y_1|Q) + I(W_1; V_0|Q) - I(W_0, W_1, U_0, V_0|Q), \quad (11) \\
R_{10} + R_{11} + R_{20} &\leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; U_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1; U_0|Q), \quad (12) \\
R_{10} + R_{11} + R_{30} &\leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; U_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1, U_0, V_0|Q), \quad (13) \\
R_{10} + R_{20} + R_{30} &\leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \quad (14) \\
R_{11} + R_{20} + R_{30} &\leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1, U_0; V_0|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q), \quad (15) \\
R_{11} + R_{20} + R_{30} &\leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1, U_0; V_0|Q) + I(W_1; U_0|Q) - I(W_0, W_1, U_0, V_0|Q), \quad (16) \\
R_{10} + R_{11} + R_{20} + R_{30} &\leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1, U_0; V_0|Q) + I(W_0, W_1; U_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1, U_0, V_0|Q), \quad (17) \\
R_{11} + R_{10} + R_{20} + R_{30} &\leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1, U_0; V_0|Q) + I(W_1; U_0|Q) - I(W_0, W_1, U_0, V_0|Q). \quad (18)
\end{align*}

4) Probability of error at decoder of $R_2$: There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., $E_{j_0,l_0,l_2 \neq l_0}^c$ happens or (2) there exists some $\hat{l}_0 \neq l_0$ and $\hat{l}_2 \neq l_2$ such that $E_{j_0,l_0,l_2 \neq l_0}^c$ happens. The probability of decoding error can, therefore, be expressed as

\begin{equation}
P_{e,R_2}^{(n)} = P \left( E_{j_0,l_0,l_2 \neq l_0}^c \cup \bigcup_{(l_0 \neq l_0, l_2 \neq l_2)} E_{j_0,l_0,l_2 \neq l_0}^c \right)
\end{equation}

Applying union of events bound, \textbf{(19)} can be written as,

\begin{equation*}
P_{e,R_2}^{(n)} \leq P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + P \left( \bigcup_{(l_0 \neq l_0, l_2 \neq l_2)} E_{j_0,l_0,l_2 \neq l_0}^c \right)
\end{equation*}

\begin{equation*}
= P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + \sum_{l_0 \neq l_0} P \left( E_{j_0,l_0,l_2 \neq l_2}^c \right) + \sum_{l_2 \neq l_2} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + \sum_{l_0 \neq l_0, l_2 \neq l_2} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right)
\end{equation*}

\begin{equation*}
+ \sum_{j_0 \neq j_0} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + \sum_{l_0 \neq l_0, l_2 \neq l_2} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + \sum_{l_0 \neq l_0, l_2 \neq l_2} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right)
\end{equation*}

\begin{equation*}
+ \sum_{j_0 \neq j_0, l_0 \neq l_0} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + \sum_{j_0 \neq j_0, l_2 \neq l_2} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right) + \sum_{j_0 \neq j_0, l_0 \neq l_0, l_2 \neq l_2} P \left( E_{j_0,l_0,l_2 \neq l_0}^c \right)
\end{equation*}
$$\leq P \left( E_{j_0l_0t_0}^c \right) + 2^n(R_{20} + I(W_0, W_1; U_0|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$2^n(R_{22} + I(W_0, W_1; U_2|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$+ 2^n(R_{20} + R_{22} + I(W_0, W_1; U_0|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$2^n(R_{10} + R_{20} + I(W_0, W_1; U_1|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$+ 2^n(R_{10} + R_{20} + R_{30} + I(W_0, W_1; U_0|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$+ 2^n(R_{10} + R_{20} + R_{30} + I(W_0, W_1; U_1|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$+ 2^n(R_{10} + R_{20} + R_{30} + I(W_0, W_1; U_2|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

$$+ 2^n(R_{10} + R_{20} + R_{30} + I(W_0, W_1; U_3|Q) + 4\epsilon) P(E_{j_0l_0t_0}^c) +$$

Let us now evaluate the probability of error events.

$P(E_{j_0l_0t_0}^c)$ can be upper bounded as

$$P(E_{j_0l_0t_0}^c) = \sum_{(w_0, u_0, u_2, v_0, y_2, q) \in A_0^{(n)}} P(U_0(l_0) = u_0|q) P(W_0(j_0) = w_0, U_2(l_2) = u_2, V_0(l_0) = v_0, Y_2 = y_2|q)$$

$$\leq 2^n(H(W_0, U_0, U_2, V_0, Y_2|Q) + \epsilon) 2^{-n(H(U_0|Q) - \epsilon)} 2^{-n(H(W_0, U_2, V_0, Y_2|Q) - \epsilon)}$$

$$= 2^{-n(I(U_0; W_0, U_2, V_0, Y_2|Q) - 3\epsilon)}.$$
\[\leq 2^{n(H(U_0,U_2,W_0,Y_2;Q)+\epsilon)}2^{-n(H(U_0;Q)-\epsilon)}2^{-n(H(U_2;Q)-\epsilon)}2^{-n(H(W_0,Y_2;Q)-\epsilon)} = 2^{-n(I(U_0,U_2,W_0,Y_2;Q)+I(U_0,U_2;Q)-4\epsilon)}.
\]

\[P(E_{j_0i_2t_0}) \text{ can be upper bounded as}
\]
\[P(E_{j_0i_2t_0}) = \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0|q)P(U_2(l_2) = u_2)P(U_0(l_0) = u_0, V_0(t_0) = v_0, Y_2 = y_2|q)
\]
\[\leq 2^{n(H(W_0,U_0,U_2,V_0,Y_2;Q)+\epsilon)}2^{-n(H(W_0;Q)-\epsilon)}2^{-n(H(U_2;Q)-\epsilon)}2^{-n(H(U_0,Y_2;Q)-\epsilon)} = 2^{-n(I(W_0,U_0,U_2,V_0,Y_2;Q)+I(W_0,U_2;Q)-4\epsilon)}.
\]

\[P(E_{j_0i_2t_0}) \text{ can be upper bounded as}
\]
\[P(E_{j_0i_2t_0}) = \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_2(l_2) = u_2|q)P(U_0(l_0) = u_0)P(U_0(l_0) = u_0, V_0(t_0) = v_0, Y_2 = y_2|q)
\]
\[\leq 2^{n(H(W_0,U_0,U_2,V_0,Y_2;Q)+\epsilon)}2^{-n(H(W_0;Q)-\epsilon)}2^{-n(H(U_2;Q)-\epsilon)}2^{-n(H(U_0,Y_2;Q)-\epsilon)} = 2^{-n(I(W_0,U_0,U_2,V_0,Y_2;Q)+I(W_0,U_2;Q)-4\epsilon)}.
\]

\[P(E_{j_0i_2t_0}) \text{ can be upper bounded as}
\]
\[P(E_{j_0i_2t_0}) = \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(U_0(l_0) = u_0|q)P(V_0(t_0) = v_0|q)P(W_0(j_0) = w_0)P(U_2(l_2) = u_2, Y_2 = y_2|q)
\]
\[\leq 2^{n(H(W_0,U_0,U_2,V_0,Y_2;Q)+\epsilon)}2^{-n(H(U_0;Q)-\epsilon)}2^{-n(H(U_2;Q)-\epsilon)}2^{-n(H(W_0,Y_2;Q)-\epsilon)} = 2^{-n(I(U_0,V_0,U_2,V_0,Y_2;Q)+I(U_0,V_0;Q)-4\epsilon)}.
\]

\[P(E_{j_0i_2t_0}) \text{ can be upper bounded as}
\]
\[P(E_{j_0i_2t_0}) = \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(U_2(l_2) = u_2|q)P(V_0(t_0) = v_0|q)P(W_0(j_0) = w_0)P(U_0(l_0) = u_0, Y_2 = y_2|q)
\]
\[\leq 2^{n(H(W_0,U_0,U_2,V_0,Y_2;Q)+\epsilon)}2^{-n(H(U_2;Q)-\epsilon)}2^{-n(H(V_0;Q)-\epsilon)}2^{-n(H(W_0,Y_2;Q)-\epsilon)} = 2^{-n(I(U_2,Y_0,V_0,U_2,Y_2;Q)+I(U_2,V_0;Q)-4\epsilon)}.
\]
\( P(E'_{j;\ell;\ell;\ell}) \) can be upper bounded as

\[
P(E'_{j;\ell;\ell;\ell}) = \sum_{(w_0, u_0, u_2, v_0, y_2, q) \in A'_{(\ell)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0 | q) P(U_2(l_2) = u_2 | q) P(V_0(t_0) = v_0, Y_2 = y_2 | q)
\]

\[
\leq 2^{n(H(W_0, U_0, U_2, V_0, Y_2) + \epsilon)} 2^{-n(H(W_0) - \epsilon)} 2^{-n(H(U_0) - \epsilon)} 2^{-n(H(U_2) - \epsilon)} 2^{-n(H(V_0) - \epsilon)} = 2^{-n(I(W_0, U_0, U_2, V_0, Y_2) + I(W_0, U_0, U_2) + I(W_0, U_0) - 5\epsilon)}.
\]

\( P(E'_{j;\ell;\ell;\ell}) \) can be upper bounded as

\[
P(E'_{j;\ell;\ell;\ell}) = \sum_{(w_0, u_0, u_2, v_0, y_2, q) \in A'_{(\ell)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0, Y_2 = y_2 | q)
\]

\[
\leq 2^{n(H(W_0, U_0, U_2, V_0, Y_2) + \epsilon)} 2^{-n(H(W_0) - \epsilon)} 2^{-n(H(U_0) - \epsilon)} 2^{-n(H(U_2) - \epsilon)} 2^{-n(H(V_0) - \epsilon)} = 2^{-n(I(W_0, U_0, U_2, V_0, Y_2) + I(W_0, U_0, U_2) + I(W_0, U_0) - 5\epsilon)}.
\]

\( P(E'_{j;\ell;\ell;\ell}) \) can be upper bounded as

\[
P(E'_{j;\ell;\ell;\ell}) = \sum_{(w_0, u_0, u_2, v_0, y_2, q) \in A'_{(\ell)}} P(U_0(l_0) = u_0 | q) P(U_2(l_2) = u_2, Y_2 = y_2 | q)
\]

\[
\leq 2^{n(H(W_0, U_0, U_2, V_0, Y_2) + \epsilon)} 2^{-n(H(W_0) - \epsilon)} 2^{-n(H(U_2) - \epsilon)} 2^{-n(H(V_0) - \epsilon)} = 2^{-n(I(W_0, U_2, V_0, Y_2) + I(W_0, U_2, V_0) + I(W_0, U_2) - 5\epsilon)}.
\]

\( P(E'_{j;\ell;\ell;\ell}) \) can be upper bounded as

\[
P(E'_{j;\ell;\ell;\ell}) = \sum_{(w_0, u_0, u_2, v_0, y_2, q) \in A'_{(\ell)}} P(U_0(l_0) = u_0 | q) P(U_2(l_2) = u_2, Y_2 = y_2 | q)
\]

\[
\leq 2^{n(H(W_0, U_0, U_2, V_0, Y_2) + \epsilon)} 2^{-n(H(U_0) - \epsilon)} 2^{-n(H(U_2) - \epsilon)} 2^{-n(H(V_0) - \epsilon)} = 2^{-n(I(W_0, U_2, V_0, Y_2) + I(W_0, U_2, V_0) + I(W_0, U_2) - 5\epsilon)}.
\]

\( P(E'_{j;\ell;\ell;\ell}) \) can be upper bounded as

\[
P(E'_{j;\ell;\ell;\ell}) = \sum_{(w_0, u_0, u_2, v_0, y_2, q) \in A'_{(\ell)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0 | q) P(U_2(l_2) = u_2 | q) P(V_0(t_0) = v_0 | q) P(Y_2 = y_2 | q)
\]

\[
\leq 2^{n(H(W_0, U_0, U_2, V_0, Y_2) + \epsilon)} 2^{-n(H(W_0) - \epsilon)} 2^{-n(H(U_0) - \epsilon)} 2^{-n(H(U_2) - \epsilon)} 2^{-n(H(Y_2) - \epsilon)} = 2^{-n(I(W_0, U_0, U_2, V_0, Y_2) + I(W_0, U_2, V_0) + I(W_0, U_0) - 6\epsilon)}.
\]
Substituting these in the probability of decoding error at $R_2$, we note that $P_e^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ iff the following constraints are satisfied:

\begin{align*}
R_{20} & \leq I(U_0;W_0,U_2,V_0,Y_2|Q) - I(W_0,W_1;U_0|Q), \\
R_{22} & \leq I(U_2;W_0,U_0,V_0,Y_2|Q) - I(W_0,W_1;U_2|Q), \\
R_{20} + R_{22} & \leq I(U_0,U_2;W_0,V_0,Y_2|Q) + I(U_0;U_2|Q) - I(W_0,W_1;U_0|Q) - I(W_0,W_1;U_2|Q), \\
R_{10} + R_{20} & \leq I(W_0,U_0;U_2,V_0,Y_2|Q) + I(W_0;U_0|Q) - I(W_0,W_1;U_0|Q), \\
R_{10} + R_{22} & \leq I(W_0,U_2;U_0,V_0,Y_2|Q) + I(W_0;U_2|Q) - I(W_0,W_1;U_2|Q), \\
R_{20} + R_{30} & \leq I(U_0,V_0;W_0,U_2,Y_2|Q) + I(U_0;V_0|Q) - I(W_0,W_1;U_0|Q) - I(W_0,W_1,U_0,U_2;V_0|Q), \\
R_{22} + R_{30} & \leq I(U_2,V_0;W_0,U_0,Y_2|Q) + I(U_2;V_0|Q) - I(W_0,W_1;U_2|Q) - I(W_0,W_1,U_0,U_2;V_0|Q), \\
R_{10} + R_{20} + R_{22} & \leq I(W_0,U_0,U_2;V_0,Y_2|Q) + I(W_0,U_0;U_2|Q) + I(W_0;U_0|Q) + I(W_0,W_1;U_0|Q) - I(W_0,W_1;U_0,U_2|Q), \\
R_{10} + R_{20} + R_{30} & \leq I(W_0,U_0,V_0;U_2,Y_2|Q) + I(W_0,U_0;V_0|Q) + I(W_0;U_0|Q) - I(W_0,W_1;U_0;U_0,U_2|V_0|Q), \\
R_{10} + R_{22} + R_{30} & \leq I(W_0,U_2,V_0;U_0,Y_2|Q) + I(W_0,U_2;V_0|Q) + I(W_0;U_2|Q) - I(W_0,W_1,U_0,U_2;V_0|Q), \\
R_{20} + R_{22} + R_{30} & \leq I(U_0,U_2,V_0;W_0,Y_2|Q) + I(U_0,U_2;V_0|Q) + I(U_0;U_2|Q) - I(W_0,W_1;U_0;U_0,U_2|V_0|Q), \\
R_{10} + R_{20} + R_{22} + R_{30} & \leq I(W_0,U_0,U_2,V_0,Y_2|Q) + I(W_0,U_0,U_2;V_0|Q) + I(W_0,U_0;U_2|Q) - I(W_0,W_1,U_0,U_2;V_0|Q). \\
\end{align*}

5) Probability of error at decoder of $R_3$: There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., $E_{j_0,i_0,t_3}^c$ happens or (2) there exists some $\hat{t}_0 \neq t_0$ and $\hat{t}_3 \neq t_3$ such that $E_{j_0,i_0,l_3}^{\hat{t}_0,\hat{t}_3}$ happens. The probability of decoding error can, therefore, be expressed as

\[ P_{e,R_3}^{(n)} = P \left( E_{j_0,i_0,t_3}^c \cup \bigcup_{(\hat{t}_0 \neq t_0,\hat{t}_3 \neq t_3)} E_{j_0,i_0,l_3}^{\hat{t}_0,\hat{t}_3} \right) \]

(32)

Applying union of events bound, (32) can be written as,

\[
\begin{align*}
P_{e,R_3}^{(n)} & \leq P\left(E_{j_0,i_0,t_3}^c\right) + P\left(\bigcup_{(\hat{t}_0 \neq t_0,\hat{t}_3 \neq t_3)} E_{j_0,i_0,l_3}^{\hat{t}_0,\hat{t}_3}\right) \\
& = P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_0 \neq t_0} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_0 \neq t_0,\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right) \\
& \quad + \sum_{\hat{t}_0 \neq t_0,\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_0 \neq t_0} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_0 \neq t_0,\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_0 \neq t_0} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right) + \sum_{\hat{t}_0 \neq t_0,\hat{t}_3 \neq t_3} P\left(E_{j_0,i_0,t_3}^c\right)
\end{align*}
\]
\[
\sum_{j_0 \neq j_0, l_0 \neq l_0, t_3 \neq t_3} P\left( E_{j_0l_0t_0l_3} \right) + \sum_{j_0 \neq j_0, t_0 \neq t_0, l_3 \neq t_3} P\left( E_{j_0l_0t_0t_3} \right) + \sum_{l_0 \neq l_0, t_0 \neq t_0, t_3 \neq t_3} P\left( E_{j_0l_0l_0l_3} \right) + \sum_{j_0 \neq j_0, l_0 \neq l_0, t_0 \neq t_3} P\left( E_{j_0l_0l_0t_3} \right)
\]

\[
\leq P\left( E_{j_0l_0l_0t_3}^c \right) + 2^n \left( R_{20} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
+ 2^n \left( R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
+ 2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

\[
2^n \left( R_{20} + R_{30} + I(W_0, W_1, U_0, U_2; V_0|Q) + 4\epsilon \right) P(E_{j_0l_0l_0l_3})
\]

Let us now evaluate \( P(E_{j_0l_0l_0t_3}) \), \( P(E_{j_0l_0l_0l_3}) \) and \( P(E_{j_0l_0l_0l_3}) \).

\( P(E_{j_0l_0l_0t_3}) \) can be upper bounded as

\[
P(E_{j_0l_0l_0t_3}) = \sum_{(w_0, u_0, v_0, y_3, q) \in A_1^{(n)}} P(V_0(t_0) = v_0|q) P(W_0(j_0) = w_0, U_0(l_0) = u_0, V_3(t_3) = v_3, Y_3 = y_3|q)
\]

\[
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3|Q) + \epsilon) 2^{-n(H(V_0|Q) - \epsilon)} 2^{-n(H(W_0, U_0, V_0, V_3, Y_3|Q) - \epsilon)}
\]

\[
= 2^{-n(I(V_0; W_0, U_0, V_0, V_3, Y_3|Q) - 3\epsilon)}.
\]

\( P(E_{j_0l_0l_0l_3}) \) can be upper bounded as

\[
P(E_{j_0l_0l_0l_3}) = \sum_{(w_0, u_0, v_0, y_3, q) \in A_1^{(n)}} P(V_3(t_3) = v_3|q) P(W_0(j_0) = w_0, U_0(l_0) = u_0, V_0(t_0) = v_0, V_3 = y_3|q)
\]

\[
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3|Q) + \epsilon) 2^{-n(H(V_3|Q) - \epsilon)} 2^{-n(H(W_0, U_0, V_0, V_3, Y_3|Q) - \epsilon)}
\]

\[
= 2^{-n(I(V_3; W_0, U_0, V_0, Y_3|Q) - 3\epsilon)}.
\]
\( P(E_{julat3}) \) can be upper bounded as

\[
P(E_{julat3}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A^{(n)}} P(V_0(t_0) = v_0 | q) P(V_3(t_3) = v_3 | q) P(W_0(j_0) = w_0, U_0(l_0) = u_0, Y_3 = y_3 | q) \\
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(V_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(W_0, U_0, Y_3 | Q) - \epsilon)} = 2^{-n(I(V_0, V_3 ; W_0, U_0, Y_3 | Q) + I(V_0, V_3 | Q) - 4\epsilon)}.
\]

\( P(E_{julat3}) \) can be upper bounded as

\[
P(E_{julat3}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A^{(n)}} P(W_0(j_0) = w_0 | q) P(V_0(t_0) = v_0 | q) P(U_0(l_0) = u_0, V_3(t_3) = v_3, Y_3 = y_3 | q) \\
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(U_0, V_0, Y_3 | Q) - \epsilon)} = 2^{-n(I(W_0, V_0, U_0, V_3, Y_3 | Q) + I(W_0, V_3 | Q) - 4\epsilon)}.
\]

\( P(E_{julat3}) \) can be upper bounded as

\[
P(E_{julat3}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A^{(n)}} P(W_0(j_0) = w_0 | q) P(V_3(t_3) = v_3 | q) P(U_0(l_0) = u_0, V_0(t_0) = v_0, Y_3 = y_3 | q) \\
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(W_0, V_0, Y_3 | Q) - \epsilon)} = 2^{-n(I(U_0, V_0 ; W_0, V_3, Y_3 | Q) + I(U_0, V_3 | Q) - 4\epsilon)}.
\]

\( P(E_{julat3}) \) can be upper bounded as

\[
P(E_{julat3}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A^{(n)}} P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(W_0(j_0) = w_0, V_3(t_3) = v_3, Y_3 = y_3 | q) \\
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(W_0, V_0, Y_3 | Q) - \epsilon)} = 2^{-n(I(U_0, V_3 ; W_0, V_0, Y_3 | Q) + I(U_0, V_3 | Q) - 4\epsilon)}.
\]

\( P(E_{julat3}) \) can be upper bounded as

\[
P(E_{julat3}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A^{(n)}} P(U_0(l_0) = u_0 | q) P(V_3(t_3) = v_3 | q) P(W_0(j_0) = w_0, V_0(t_0) = v_0, Y_3 = y_3 | q) \\
\leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(V_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(W_0, V_0, Y_3 | Q) - \epsilon)} = 2^{-n(I(U_0, V_3 ; W_0, V_0, Y_3 | Q) + I(U_0, V_3 | Q) - 4\epsilon)}.
\]
\( P(E_{j_0;l_0,t_3}) \) can be upper bounded as

\[
P(E_{j_0;l_0,t_3}) = \sum_{(w_0,u_0,v_0,v_3,y_3) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0|q)P(V_0(t_0) = v_0|q)P(V_3(t_3) = v_3, Y_3 = y_3|q) \\
\leq 2^{n(H(W_0,U_0,V_0,V_3,Y_3|Q)+\epsilon) - n(H(V_0|Q)-\epsilon) - n(H(V_3|Q)-\epsilon) - n(H(V_3,Y_3|Q)-\epsilon)} \\
= 2^{-n(I(W_0,U_0,V_0,V_3|Q)+I(W_0,U_0,V_3|Q)+I(W_0,U_0|Q)-5\epsilon)}.
\]

\( P(E_{j_0;l_0,t_3}) \) can be upper bounded as

\[
P(E_{j_0;l_0,t_3}) = \sum_{(w_0,u_0,v_0,v_3,y_3) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0|q)P(V_3(t_3) = v_3|q)P(V_0(t_0) = v_0, Y_3 = y_3|q) \\
\leq 2^{n(H(W_0,U_0,V_0,V_3,Y_3|Q)+\epsilon) - n(H(V_0|Q)-\epsilon) - n(H(V_3|Q)-\epsilon) - n(H(V_3,Y_3|Q)-\epsilon)} \\
= 2^{-n(I(W_0,U_0,V_0,V_3|Q)+I(W_0,U_0,V_3|Q)+I(W_0,U_0|Q)-5\epsilon)}.
\]

\( P(E_{j_0;l_0,t_3}) \) can be upper bounded as

\[
P(E_{j_0;l_0,t_3}) = \sum_{(w_0,u_0,v_0,v_3,y_3) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(V_0(t_0) = v_0|q)P(V_3(t_3) = v_3|q)P(U_0(l_0) = u_0, Y_3 = y_3|q) \\
\leq 2^{n(H(W_0,U_0,V_0,V_3,Y_3|Q)+\epsilon) - n(H(V_0|Q)-\epsilon) - n(H(V_3|Q)-\epsilon) - n(H(V_3,Y_3|Q)-\epsilon)} \\
= 2^{-n(I(W_0,V_0,V_3|Q)+I(W_0,U_0,V_3|Q)+I(W_0,V_3|Q)-5\epsilon)}.
\]

\( P(E_{j_0;l_0,t_3}) \) can be upper bounded as

\[
P(E_{j_0;l_0,t_3}) = \sum_{(w_0,u_0,v_0,v_3,y_3) \in A_1^{(n)}} P(U_0(l_0) = u_0|q)P(V_0(t_0) = v_0|q)P(V_3(t_3) = v_3|q)P(W_0(j_0) = w_0, Y_3 = y_3|q) \\
\leq 2^{n(H(W_0,U_0,V_0,V_3,Y_3|Q)+\epsilon) - n(H(V_0|Q)-\epsilon) - n(H(V_3|Q)-\epsilon) - n(H(V_3,Y_3|Q)-\epsilon)} \\
= 2^{-n(I(U_0,V_0,V_3|Q)+I(W_0,V_0,V_3|Q)+I(W_0,V_3|Q)-5\epsilon)}.
\]

\( P(E_{j_0;l_0,t_3}) \) can be upper bounded as

\[
P(E_{j_0;l_0,t_3}) = \sum_{(w_0,u_0,v_0,v_3,y_3) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0|q)P(V_0(t_0) = v_0|q)P(V_3(t_3) = v_3|q)P(Y_3 = y_3|q) \\
\leq 2^{n(H(W_0,U_0,V_0,V_3,Y_3|Q)+\epsilon) - n(H(V_0|Q)-\epsilon) - n(H(V_3|Q)-\epsilon) - n(H(V_3,Y_3|Q)-\epsilon)} \\
= 2^{-n(I(W_0,U_0,V_0,V_3|Q)+I(W_0,U_0,V_3|Q)+I(W_0,U_0|Q)-6\epsilon)}.
\]
Substituting these in the probability of decoding error at $\mathcal{R}_3$, we note that $P_e^{(n)} \to 0$ as $n \to \infty$ iff the following constraints are satisfied:

\[ R_{30} \leq I(V_0; W_0, U_0, V_3, Y_3|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \quad (33) \]
\[ R_{33} \leq I(V_3; W_0, U_0, V_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \quad (34) \]
\[ R_{30} + R_{33} \leq I(V_0, V_3; W_0, U_0, Y_3|Q) + I(V_0; V_3|Q) \]
\[ -I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \quad (35) \]
\[ R_{20} + R_{30} \leq I(U_0, V_0; W_0, V_3, Y_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \quad (36) \]
\[ R_{20} + R_{33} \leq I(U_0, V_3; W_0, V_0, Y_3|Q) + I(U_0; V_3|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \quad (37) \]
\[ R_{20} + R_{30} + R_{33} \leq I(U_0, V_0; W_0, V_3, Y_3|Q) + I(U_0, V_0; V_0, Y_3|Q) + I(U_0; V_0|Q) + I(W_0; U_0|Q) \]
\[ -I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \quad (40) \]
\[ R_{20} + R_{30} + R_{33} \leq I(U_0, V_0; V_3; V_0, Y_3|Q) + I(U_0, V_0; V_0; Y_3|Q) + I(W_0; U_0|Q) \]
\[ -I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \quad (41) \]
\[ R_{20} + R_{30} + R_{33} \leq I(U_0, V_0; V_3; U_0, Y_3|Q) + I(U_0, V_0; V_3|Q) + I(W_0; V_0|Q) \]
\[ -I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \quad (42) \]
\[ R_{20} + R_{30} + R_{33} \leq I(U_0, V_0; V_3; W_0, Y_3|Q) + I(U_0, V_0; V_3|Q) + I(W_0; V_0|Q) \]
\[ -I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \quad (43) \]
\[ R_{20} + R_{30} + R_{33} \leq I(U_0, V_0, V_3; Y_3|Q) + I(U_0, V_0, V_0; V_3|Q) + I(U_0, V_0; V_0|Q) + I(W_0; U_0|Q) \]
\[ -I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) - I(W_0, W_1, U_0, U_2; V_3|Q). \quad (44) \]

The achievable rate region for the channel $\mathcal{C}_{cma}$ follows:

\[ R_{10} \leq I(W_0; W_1, U_0, V_0, Y_1|Q), \quad (45) \]
\[ R_{11} \leq I(W_1; W_0, U_0, V_0, Y_1|Q), \quad (46) \]
\[ R_{10} + R_{11} \leq I(W_0, W_1; U_0, V_0, Y_1|Q) + I(W_0; W_1), \quad (47) \]
\[ R_{10} + R_{20} \leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \quad (48) \]
\[ R_{10} + R_{30} \leq I(W_0, V_0; W_1, U_0, Y_1|Q) + I(W_0; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \quad (49) \]
\[ R_{11} + R_{20} \leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q), \quad (50) \]
\[ R_{11} + R_{30} \leq I(W_1, V_0; W_0, U_0, Y_1|Q) + I(W_1; V_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \quad (51) \]
\[ R_{10} + R_{11} + R_{20} \leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; U_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1; U_0|Q), \] (52)

\[ R_{10} + R_{11} + R_{30} \leq I(W_0, W_1, V_0; U_0, Y_1|Q) + I(W_0, W_1; V_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \] (53)

\[ R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0; W_1, Y_1|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) \]
\[ - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \] (54)

\[ R_{11} + R_{20} + R_{30} \leq I(W_1, U_0, V_0; W_0, Y_1|Q) + I(W_1, U_0; V_0|Q) + I(W_1; U_0|Q) \]
\[ - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \] (55)

\[ R_{10} + R_{11} + R_{20} + R_{30} \leq I(W_0, W_1, U_0, V_0; Y_1|Q) + I(W_0, W_1, U_0; V_0|Q) + I(W_0, W_1; U_0|Q) \]
\[ - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \] (56)

\[ R_{20} \leq I(U_0; W_0, U_2, V_0, Y_2|Q) - I(W_0, W_1; U_0|Q), \] (57)

\[ R_{22} \leq I(U_2; W_0, U_0, V_0, Y_2|Q) - I(W_0, W_1; U_0|Q), \] (58)

\[ R_{20} + R_{22} \leq I(U_0, U_2; W_0, V_0, Y_2|Q) + I(U_0; U_2|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q), \] (59)

\[ R_{10} + R_{20} \leq I(W_0, U_0; U_2, V_0, Y_2|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q), \] (60)

\[ R_{10} + R_{22} \leq I(W_0, U_2; U_0, V_0, Y_2|Q) + I(W_0; U_2|Q) - I(W_0, W_1; U_0|Q), \] (61)

\[ R_{20} + R_{30} \leq I(U_0, V_0; W_0, U_2, Y_2|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \] (62)

\[ R_{22} + R_{30} \leq I(U_2, V_0; W_0, U_0, Y_2|Q) + I(U_2; V_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \] (63)

\[ R_{10} + R_{20} + R_{22} \leq I(W_0, U_0, U_2; V_0, Y_2|Q) + I(W_0, U_0; U_2|Q) + I(W_0; U_0|Q) \]
\[ - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q), \] (64)

\[ R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, U_2; V_0, Y_2|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) \]
\[ + I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \] (65)

\[ R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0, U_2, Y_2|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) \]
\[ - I(W_0, W_1; U_0|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \] (66)

\[ R_{10} + R_{22} + R_{30} \leq I(W_0, U_2, V_0; U_0, Y_2|Q) + I(W_0, U_2; V_0|Q) + I(W_0; U_2|Q) \]
\[ - I(W_0, W_1; U_2|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \] (67)

\[ R_{20} + R_{22} + R_{30} \leq I(U_0, U_2; W_0, V_0, Y_2|Q) + I(U_0, U_2; V_0|Q) + I(U_0; U_2|Q) \]
\[ - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1, U_0, U_2; V_0|Q) \] (68)

\[ R_{10} + R_{20} + R_{22} + R_{30} \leq I(W_0, U_0, U_2, V_0; Y_2|Q) + I(W_0, U_0, U_2; V_0|Q) + I(W_0, U_0; U_2|Q) \]
\[ + I(W_0, U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \] (69)

\[ R_{30} \leq I(V_0; W_0, U_0, V_3, Y_3|Q) - I(W_0, W_1, U_0, U_2; V_0|Q), \] (70)

\[ R_{33} \leq I(V_3; W_0, U_0, V_0, Y_3|Q) - I(W_0, W_1, U_0, U_2; V_3|Q), \] (71)

\[ R_{30} + R_{33} \leq I(V_0, V_3; W_0, U_0, Y_3|Q) + I(V_0; V_3|Q) \]
For every codeword quadruple \( \binom{4}{n} \) indexed by \( (i, j, k, l) \) according to the distribution \( \pi_{ij} \), \( R_{10} + R_{30} \leq I(W_0, V_0; U_0, V_3, Y_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1; U_0, V_2; V_3|Q) \) \( (72) \),

\[
R_{10} + R_{33} \leq I(W_0, V_3; U_0, V_0, Y_3|Q) + I(W_0; V_3|Q) - I(W_0, W_1; U_0, U_2; V_3|Q) \tag{73}
\]

\[
R_{20} + R_{30} \leq I(U_0, V_0; W_0, V_3, Y_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0, U_2; V_0|Q) \tag{74}
\]

\[
R_{20} + R_{33} \leq I(U_0, V_3; W_0, V_0, Y_3|Q) + I(U_0; V_3|Q) - I(W_0, W_1; U_0, U_2; V_3|Q) \tag{75}
\]

\[
R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0; V_3, Y_3|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0, U_2; V_3|Q) \tag{76}
\]

\[
R_{10} + R_{20} + R_{33} \leq I(W_0, U_0, V_3; U_0, V_0, Y_3|Q) + I(W_0, U_0; V_3|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0, U_2; V_3|Q) \tag{77}
\]

\[
R_{10} + R_{30} + R_{33} \leq I(W_0, V_0, V_3; U_0, V_0, Y_3|Q) + I(W_0, V_0; V_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1; U_0, U_2; V_3|Q) \tag{78}
\]

\[
R_{20} + R_{30} + R_{33} \leq I(U_0, V_0, V_3; W_0, V_3, Y_3|Q) + I(U_0, V_0; V_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0, U_2; V_3|Q) \tag{79}
\]

\[
R_{10} + R_{20} + R_{30} + R_{33} \leq I(W_0, U_0, V_0, V_3; Y_3|Q) + I(W_0, U_0, V_0; Y_3|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0, U_2; V_3|Q). \tag{80}
\]

For the channel \( C_{cns}^2 \) :

\[
E. \ \text{Codebook generation}
\]

Let us fix \( p(.) \in \mathcal{P} \) and let \( A_{\tau}^{(n)} \) be a typical set. Generate a random time sharing codeword \( q \), of length \( n \), according to the distribution \( \prod_{i=1}^{n} p(w_i|q_i) \). For every \( w(j) \), generate one \( X_1(j) \) codeword according to \( \prod_{i=1}^{n} p(x_{1i}|w_i(j), q_i) \).

For \( \tau = 1, 2 \), generate \( 2^{n(R_{2\tau} + I(W;U;|Q| + 4\epsilon)} \) independent codewords \( U_\tau(l_\tau) \), according to \( \prod_{i=1}^{n} p(u_{\tau i}|q_i) \). For every codeword triple \( [u_1(l_1), u_2(l_2), w(j)] \), generate one codeword \( X_2(l_1, l_2, j) \) according to

\[
\prod_{i=1}^{n} p(x_{2i}|u_{1i}(l_1), u_{2i}(l_2), w_i(j), q_i) \].

Uniformly distribute the \( 2^{n(R_{2\tau} + I(W;U;|Q| + 4\epsilon)} \) codewords \( U_\tau(l_\tau) \) into \( 2^{nR_{2\tau}} \) bins indexed by \( k_\tau \in \{1, ..., 2^{nR_{2\tau}} \} \) such that each bin contains \( 2^{n(I(W;U;|Q| + 4\epsilon)} \) codewords.

For \( \rho = 1, 3 \), generate \( 2^{n(R_{3\rho} + I(W;U;U_2;V_3|Q| + 4\epsilon)} \) independent codewords \( V_\rho(t_\rho) \), according to \( \prod_{i=1}^{n} p(v_{\rho i}|q_i) \). For every codeword quadruple \( [v_1(t_1), v_3(t_3), u_1(l_1), u_2(l_2), w(j)] \), generate one codeword \( X_3(t_1, t_3, l_1, l_2, j) \) according to

\[
\prod_{i=1}^{n} p(x_{3i}|v_{1i}(t_1), v_{3i}(t_3), u_{1i}(l_1), u_{2i}(l_2), w_i(j), q_i) \].

\( V_\rho(t_\rho) \) uniformly into \( 2^{nR_{3\rho}} \) bins indexed by \( r_\rho \in \{1, ..., 2^{nR_{3\rho}} \} \) such that each bin contains \( 2^{n(I(W;U;U_2;V_3|Q| + 4\epsilon)} \) codewords. The indices are given by \( j \in \{1, ..., 2^{nR_1}\} \), \( l_\tau \in \{1, ..., 2^{n(R_{2\tau} + I(W;U;|Q| + 4\epsilon)} \} \) and \( t_\rho \in \{1, ..., 2^{nR_3}\} \)
Let us suppose that the source message vector generated at the three senders is \((m_{11}, m_{21}, m_{22}, m_{31}, m_{33}) = (j, k_1, k_2, r_1, r_3)\). The first component is the message index and the last four components are the bin indices. \(S_2\) looks for a codeword \(u_1(l_1)\) in bin \(k_1\) and a codeword \(u_2(l_2)\) in bin \(k_2\) such that \((u_1(l_1), w(j), q) \in A_\epsilon^{(n)}\) and \((u_2(l_2), w(j), q) \in A_\epsilon^{(n)}\), respectively. \(S_3\) looks for a codeword \(v_1(t_1)\) in bin \(r_1\) and a codeword \(v_3(t_3)\) in bin \(r_3\) such that \((v_1(t_1), u_1(l_1), u_2(l_2), w(j), q) \in A_\epsilon^{(n)}\) and \((v_3(t_3), u_1(l_1), u_2(l_2), w(j), q) \in A_\epsilon^{(n)}\), respectively. \(S_1, S_2\), and \(S_3\) then transmit codewords \(x_1(j), x_2(l_1, l_2, j)\) and \(x_3(t_1, t_3, l_1, l_2, j)\), respectively, through \(n\) channel uses. The transmissions are assumed to be synchronized.

G. Decoding

The three receivers accumulate an \(n\)-length channel output sequence: \(y_1\) at \(R_1\), \(y_2\) at \(R_2\) and \(y_3\) at \(R_3\). Decoders 1, 2 and 3 look for all indices \((\hat{j}, \hat{l}_1, \hat{t}_1), (\hat{l}_1, \hat{l}_2)\) and \((\hat{l}_1, \hat{t}_3)\), respectively, such that \((w(\hat{j}), u_1(\hat{l}_1), v_1(\hat{t}_1), y_1, q) \in A_\epsilon^{(n)}\), \((u_1(\hat{t}_1), u_2(\hat{l}_2), y_2, q) \in A_\epsilon^{(n)}\) and \((v_1(\hat{t}_1), v_3(\hat{t}_3), y_3, q) \in A_\epsilon^{(n)}\). If \(\hat{j}\) in all the index triples found are the same, \(R_1\) declares \(m_{11} = \hat{j}\), for some \(l_1\) and \(t_1\). If \(\hat{l}_1\) in all the index pairs found are indices of codewords \(u_1(\hat{l}_1)\) from the same bin with index \(\hat{k}_1\), and \(\hat{l}_2\) in all the index pairs found are indices of codewords \(u_2(\hat{l}_2)\) from the same bin with index \(\hat{k}_2\), then \(R_2\) determines \(m_{21}, m_{22} = (\hat{k}_1, \hat{k}_2)\). Similarly, if \(\hat{l}_1\) in all the index pairs found are indices of codewords \(v_1(\hat{t}_1)\) from the same bin with index \(\hat{r}_1\), and \(\hat{l}_3\) in all the index pairs found are indices of codewords \(v_3(\hat{t}_3)\) from the same bin with index \(\hat{r}_3\), then \(R_3\) determines \(m_{31}, m_{33} = (\hat{r}_1, \hat{r}_3)\). Otherwise, the receivers \(R_1, R_2\) and \(R_3\) declare an error.

H. Analysis of probabilities of error

In this section we derive upperbounds on the probabilities of error events, which happens during encoding and decoding processes. We will assume that a source message vector \((m_{11}, m_{21}, m_{22}, m_{31}, m_{33})\) is encoded and transmitted. We will consider the analysis of probability of encoding error at senders \(S_2\) and \(S_3\), and the analysis of probability of decoding error at each of the three receivers \(R_1, R_2\), and \(R_3\) separately.

First, let us define the following events:

(i) \(E_{jl_1} \triangleq \{ (W(j), U_1(l_1), q) \in A_\epsilon^{(n)} \} \),

(ii) \(E_{jl_2} \triangleq \{ (W(j), U_2(l_2), q) \in A_\epsilon^{(n)} \} \),

(iii) \(E_{jl_1l_2t_1} \triangleq \{ (W(j), U_1(l_1), U_2(l_2), V_1(t_1), q) \in A_\epsilon^{(n)} \} \),

(iv) \(E_{jl_1l_2t_3} \triangleq \{ (W(j), U_1(l_1), U_2(l_2), V_3(t_3), q) \in A_\epsilon^{(n)} \} \),

\[ 2^n(R_{33}+I(W;U_1,U_2;V,Q)+4\epsilon) \].
(v) \( E_{j_1 t_1} \triangleq \{(W(j), U_1(l_1), V_1(t_1), Y_1, q) \in A_e^{(n)}\} \),

(vi) \( E_{t_1 t_2} \triangleq \{(U_1(l_1), U_2(l_2), Y_2, q) \in A_e^{(n)}\} \),

(vii) \( E_{t_1 t_3} \triangleq \{(V_1(t_1), V_3(t_3), Y_3, q) \in A_e^{(n)}\} \).

\( E_e^{c} \triangleq \) complement of the event \( E_{(.)} \). Events (i) – (iv) will be used in the analysis of probability of encoding error while events (v) – (vii) will be used in the analysis of probability of decoding error.

1) **Probability of error at encoder of \( S_2 \)**: An error is made if (i) the encoder cannot find \( u_1(l_1) \) in bin indexed by \( k_1 \) such that \((w(j), u_1(l_1), q) \in A_e^{(n)}\) or (ii) it cannot find \( u_2(l_2) \) in bin indexed by \( k_2 \) such that \((w(j), u_2(l_2), q) \in A_e^{(n)}\). The probability of encoding error at \( S_2 \) can be bounded as

\[
P_{e, S_2} \leq P \left( \bigcap_{u_1(l_1) \in \text{bin}(k_1)} (W(j), U_1(l_1), q) \notin A_e^{(n)} \right) + P \left( \bigcap_{u_2(l_2) \in \text{bin}(k_2)} (W(j), U_2(l_2), q) \notin A_e^{(n)} \right),
\]

\[
\leq (1 - P(E_{j_1 t_1})) \binom{2n}{l_1} + (1 - P(E_{j_2 t_1})) \binom{2n}{l_2},
\]

where \( P(.) \) is the probability of an event. Since \( q \) is predetermined,

\[
P(E_{j_1 t_1}) = \sum_{(w, u_1, q) \in A_e^{(n)}} P(W(j) = w|q) P(U_1(l_1) = u_1|q)
\]

\[
\geq 2^n(H(W; U_1|Q-e)) - n(H(W|Q+e)) - n(H(U_1|Q+e)) = 2^n(I(W; U_1|Q+3e)).
\]

Similarly, \( P(E_{j_2 t_1}) \geq 2^n(I(W; U_2|Q)+3e) \).

Therefore,

\[
P_{e, S_2} \leq (1 - 2^n(I(W; U_1|Q)+3e))^{2^n(I(W; U_1|Q)+4e)} + (1 - 2^n(I(W; U_2|Q)+3e))^{2^n(I(W; U_2|Q)+4e)}.
\]

Now,

\[
(1 - 2^n(I(W; U_1|Q)+3e))^{2^n(I(W; U_1|Q)+4e)} = e^{2^n(I(W; U_1|Q)+4e) \ln(1 - 2^n(I(W; U_1|Q)+3e))}
\]

\[
\leq e^{2^n(I(W; U_1|Q)+4e)(-2^n(I(W; U_1|Q)+3e))}
\]

\[
= e^{-2^n e}.
\]

Clearly, \( P_{e, S_2} \to 0 \) as \( n \to \infty \).

2) **Probability of error at encoder of \( S_3 \)**: An error is made if (i) the encoder cannot find \( v_1(t_1) \) in bin indexed by \( r_1 \) such that \((w(j), u_1(l_1), u_2(l_2), v_1(t_1), q) \in A_e^{(n)}\) or (ii) it cannot find \( v_3(t_3) \) in bin indexed by \( h_3 \) such that \((w(j), u_1(l_1), u_2(l_2), v_3(t_3), q) \in A_e^{(n)}\). The probability of encoding error at \( S_3 \) can be bounded as

\[
P_{e, S_3} \leq P \left( \bigcap_{v_1(t_1) \in \text{bin}(r_1)} (W(j), U_1(l_1), U_2(l_2), V_1(t_1), q) \notin A_e^{(n)} \right)
\]

\[
+ P \left( \bigcap_{v_3(t_3) \in \text{bin}(r_3)} (W(j), U_1(l_1), U_2(l_2), V_3(t_3), q) \notin A_e^{(n)} \right)
\]

\[
\leq (1 - P(E_{j_1 l_2 t_1}))^{2^n(I(W; U_1, U_2; V_1|Q)+4e)} + (1 - P(E_{j_2 l_2 t_3}))^{2^n(I(W; U_1, U_2; V_3|Q)+4e)}.
\]
Proceeding in a way similar to the encoder error analysis at $S_2$, note that $\hat{c}$ codewords transmitted are not jointly typical i.e., $P_{E_{jl}}(c^{\hat{c}})$ happens. Since $q_{jl} = 2^{-n(I(W;U_1, U_2; V_1 | Q) - \epsilon)}$, Therefore, $P_{e,S_1} \leq \left( 1 - 2^{-n(I(W;U_1, U_2; V_1 | Q) + 3\epsilon)} \right) 2^{n(I(W;U_1, U_2; V_1 | Q) + 4\epsilon)} + \left( 1 - 2^{-n(I(W;U_1, U_2; V_1 | Q) + 3\epsilon)} \right) 2^{n(I(W;U_1, U_2; V_1 | Q) + 4\epsilon)}$.

Proceeding in a way similar to the encoder error analysis at $S_2$, we get $P_{e,S_3} \to 0$ as $n \to \infty$.

3) Probability of error at decoder of $R_1$: There are two possible events which can be classified as errors: (i) The codewords transmitted are not jointly typical i.e., $E_{jl,t_1}$ happens or (ii) there exists some $\hat{j} \neq j$ such that $E_{j\hat{l},t_1}$ happens. Note that $\hat{l}_1$ need not equal $l_1$, and $\hat{l}_1$ need not equal $t_1$, since $R_1$ is not required to decode $\hat{l}_1$ and $\hat{l}_1$ correctly. The probability of decoding error can, therefore, be expressed as

$$P_{e,R_1}^{(n)} = P\left( \bigcup_{j \neq \hat{j}} E_{j\hat{l},t_1} \right)$$

Applying union of events bound, (82) can be written as,

$$P_{e,R_1}^{(n)} \leq P\left( E_{jl,t_1} \right) + P\left( \bigcup_{j \neq \hat{j}} E_{j\hat{l},t_1} \right)$$

$$\leq P\left( E_{jl,t_1} \right) + \sum_{j \neq \hat{j}} P\left( E_{jl,t_1} \right) + \sum_{j \neq \hat{j}, \hat{l}_1 \neq t_1} P\left( E_{j\hat{l},t_1} \right) + \sum_{j \neq \hat{j}, \hat{l}_1 \neq t_1, \hat{l}_1 \neq t_1} P\left( E_{j\hat{l},t_1} \right)$$

$$\leq P\left( E_{jl,t_1} \right) + 2^{R_1} P\left( E_{jl,t_1} \right) + 2^{R_1+R_2} P\left( E_{j\hat{l},t_1} \right) + 2^{R_3+R_4} P\left( E_{j\hat{l},t_1} \right) + 2^{R_1+R_2} P\left( E_{j\hat{l},t_1} \right) + 2^{R_1+R_2} P\left( E_{j\hat{l},t_1} \right) + 2^{R_3+R_4} P\left( E_{j\hat{l},t_1} \right) + 2^{R_3+R_4} P\left( E_{j\hat{l},t_1} \right).$$

Let us now evaluate $P\left( E_{jl,t_1} \right), P\left( E_{j\hat{l},t_1} \right), P\left( E_{j\hat{l},t_1} \right), P\left( E_{j\hat{l},t_1} \right)$. $P\left( E_{jl,t_1} \right)$ can be upper bounded as

$$P\left( E_{jl,t_1} \right) = \sum_{(w,u,v_1,y_1) \in A^{(n)}} P(W(j) = w | q) P(U_1(l_1) = u_1, V_1(t_1) = v_1, Y_1 = y_1 | q) \leq 2^{n(H(W;U_1, V_1; Y_1 | Q) + \epsilon)} 2^{-n(H(W;V_1) - \epsilon)} 2^{-n(H(U_1, V_1; Y_1) - \epsilon)} = 2^{-n(I(W;U_1, V_1; Y_1 | Q) - 3\epsilon)}.$$
\[ P\left( E_{\tilde{i},\tilde{l}} \right) \] can be upper bounded as
\[
P\left( E_{\tilde{i},\tilde{l}} \right) = \sum_{(w,u,v,y,q) \in A_t^{(n)}} P(W(j) = w|q)P(U_1(l_1) = u_1|q)P(V_1(t_1) = v_1, Y_1 = y_1|q)
\leq 2^{n(H(W,U_1,V_1,Y_1|Q)+\epsilon)}2^{-n(H(W|Q)-\epsilon)}2^{-n(H(V_1|Q)-\epsilon)}2^{-n(H(Y_1|Q)-\epsilon)}
= 2^{-n(I(W,U_1;V_1,Y_1|Q)+I(W;V_1|Q)-4\epsilon)).
\]

\[
P\left( E_{\tilde{j},\tilde{i}} \right) \] can be upper bounded as
\[
P\left( E_{\tilde{j},\tilde{i}} \right) = \sum_{(w,u,v,y,q) \in A_t^{(n)}} P(W(j) = w|q)P(V_1(t_1) = v_1|q)P(U_1(l_1) = u_1, Y_1 = y_1|q)
\leq 2^{n(H(W,U_1,V_1,Y_1|Q)+\epsilon)}2^{-n(H(W|Q)-\epsilon)}2^{-n(H(V_1|Q)-\epsilon)}2^{-n(H(Y_1|Q)-\epsilon)}
= 2^{-n(I(W,V_1;U_1,Y_1|Q)+I(W;V_1|Q)-4\epsilon)).
\]

\[
P\left( E_{\tilde{j},\hat{i}} \right) \] can be upper bounded as
\[
P\left( E_{\tilde{j},\hat{i}} \right) = \sum_{(w,u,v,y,q) \in A_t^{(n)}} P(W(j) = w)P(U_1(l_1) = u_1)P(V_1(l_1) = v_1|q)P(Y_1 = y_1|q)
\leq 2^{n(H(W,U_1,V_1,Y_1|Q)+\epsilon)}2^{-n(H(W|Q)-\epsilon)}2^{-n(H(V_1|Q)-\epsilon)}2^{-n(H(Y_1|Q)-\epsilon)}
= 2^{-n(I(W,U_1;V_1,Y_1|Q)+I(W,U_1;V_1|Q)+I(W;U_1|Q)-5\epsilon)).
\]

Substituting these in the probability of decoding error at \( R_1 \), we have,
\[
P^{(n)}_{e,R_1} = \epsilon + 2^{nR_{11}}2^{-n(I(W;U_1,V_1,Y_1|Q)-3\epsilon)} + 2^{n(R_{11}+R_{21}+I(W;U_1|Q)+4\epsilon)}2^{-n(I(W,U_1,V_1,Y_1|Q)+I(W;U_1|Q)-4\epsilon)} +
2^{n(R_{11}+R_{31}+I(W,U_1;V_1|Q)+4\epsilon)}2^{-n(I(W,V_1;U_1,Y_1|Q)+I(W;V_1|Q)-4\epsilon)}+
2^{n(R_{11}+R_{21}+I(W;U_1|Q)+4\epsilon+R_{31}+I(W,U_1,U_2;V_1|Q)+4\epsilon)}2^{-n(I(W,U_1,V_1,Y_1|Q)+I(W,U_1;V_1|Q)+I(W;U_1|Q)-5\epsilon)}.
\]

\( P_{e,R_1}^{(n)} \rightarrow 0 \) as \( n \rightarrow \infty \) if \( R_{11}, R_{21} \) and \( R_{31} \) satisfy the following constraints:
\[
R_{11} \leq I(W;U_1,V_1,Y_1|Q) \tag{83}
\]
\[
R_{11} + R_{21} \leq I(W,U_1;V_1,Y_1|Q) \tag{84}
\]
\[
R_{11} + R_{31} \leq I(W,V_1;U_1,Y_1|Q) + I(W;V_1|Q) - I(W,U_1,U_2;V_1|Q) \tag{85}
\]
\[
R_{11} + R_{21} + R_{31} \leq I(W,U_1,V_1,Y_1|Q) + I(W,U_1;V_1|Q) - I(W,U_1,U_2;V_1|Q). \tag{86}
\]

4) **Probability of error at decoder of \( R_2 \):** The two possible error events are: (i) The codewords transmitted are not jointly typical i.e., \( E_{\tilde{i},\tilde{l}}^{c} \) happens or (ii) there exists some \( \tilde{i}_1 \neq \tilde{l}_1, \tilde{i}_2 \neq \tilde{l}_2 \) such that \( E_{\tilde{i},\tilde{l}} \) happens. The probability of decoding error can be written as
\[
P^{(n)}_{e,R_2} = P\left( E_{\tilde{i},\tilde{l}}^{c} \cup \bigcup_{\tilde{i}_1 \neq \tilde{l}_1,\tilde{i}_2 \neq \tilde{l}_2} E_{\tilde{i},\tilde{l}} \right) \tag{87}
\]
Applying union of events bound, \( \text{[187]} \) can be written as,
\[ P_{e, R_2}^{(n)} \leq P \left( E_{i_1 i_2}^c \right) + P \left( \bigcup_{l_i \neq i_1, l_2 \neq i_2} E_{i_1 l_2} \right) \]
\[ = P \left( E_{i_1 i_2}^c \right) + \sum_{l_i \neq i_1} P \left( E_{i_1 l_2} \right) + \sum_{l_2 \neq i_2} P \left( E_{i_1 i_2} \right) \]
\[ = \sum_{l_i \neq i_1} P \left( U_1(l_1) = u_1 | q \right) P \left( U_2(l_2) = u_2, Y_2 = y_2 | q \right) \]
\[ \leq 2^n(H(U_1, U_2, Y_2) + \epsilon) 2^{-n(H(U_1|Q) - \epsilon)} 2^{-n(H(U_2, Y_2|Q) - \epsilon)} = 2^{-n(I(U_1; U_2, Y_2|Q) - 3\epsilon)}. \]

Let us now evaluate \( P(E_{i_1 i_2}) \), \( P(E_{i_1 i_2}) \) and \( P(E_{i_1 i_2}) \).

\( P(E_{i_1 i_2}) \) can be upper bounded as
\[ P(E_{i_1 i_2}) = \sum_{(u_1, u_2, y_2, q) \in A^n} P(U_2(l_2) = u_2 | q) P(U_1(l_1) = u_1, Y_2 = y_2 | q) \]
\[ \leq 2^n(H(U_1, U_2, Y_2) + \epsilon) 2^{-n(H(U_2|Q) - \epsilon)} 2^{-n(H(U_1, Y_2|Q) - \epsilon)} = 2^{-n(I(U_2; U_1, Y_2|Q) - 3\epsilon)}. \]

\( P(E_{i_1 i_2}) \) can be upper bounded as
\[ P(E_{i_1 i_2}) = \sum_{(u_1, u_2, y_2, q) \in A^n} P(U_1(l_1) = u_1 | q) P(U_2(l_2) = u_2 | q) \]
\[ \leq 2^n(H(U_1, U_2, Y_2) + \epsilon) 2^{-n(H(U_1|Q) - \epsilon)} 2^{-n(H(U_1, Y_2|Q) - \epsilon)} 2^{-n(H(Y_2|Q) - \epsilon)} = 2^{-n(I(U_1, U_2; Y_2|Q) + I(U_1, U_2|Q) - 4\epsilon)}. \]

Substituting these in the probability of decoding error at \( R_2 \), we have,
\[ P_{e, R_2}^{(n)} = \epsilon + 2^n(R_{21} + I(W; U_1|Q) + 4\epsilon) 2^{-n(I(U_1, U_2, Y_2|Q) - 3\epsilon)} + 2^n(R_{22} + I(W; U_2|Q) + 4\epsilon) 2^{-n(I(U_2, U_1|Q) - 3\epsilon)} + 2^n(R_{21} + R_{22} + I(W; U_1|Q) + I(W; U_2|Q) + 4\epsilon) 2^{-n(I(U_1, U_2; Y_2|Q) + I(U_1, U_2|Q) - 4\epsilon)}. \]

\( P_{e, R_2}^{(n)} \rightarrow 0 \) as \( n \rightarrow \infty \) if \( R_{21} \) and \( R_{22} \) satisfy the following constraints:
\[ R_{21} \leq I(U_1; U_2, Y_2|Q) - I(W; U_1|Q) \] (88)
\[ R_{22} \leq I(U_2; U_1, Y_2|Q) - I(W; U_2|Q) \] (89)
\[ R_{21} + R_{22} \leq I(U_1, U_2; Y_2|Q) + I(U_1; U_2|Q) - I(W; U_1|Q) - I(W; U_2|Q). \] (90)
5) **Probability of error at decoder of $R_3$:** The two possible error events are: (i) The codewords transmitted are not jointly typical i.e., $E_{t,t_3}^c$ happens or (ii) there exists some $\left(\hat{t}_1 \neq t_1, \hat{t}_3 \neq t_3\right)$ such that $E_{\hat{t}_1,\hat{t}_3}$ happens. The probability of decoding error can be written as

$$P_{e,R_3}^{(n)} = P\left(E_{\hat{t}_1,\hat{t}_3}^c \cup \bigcup_{(i_1 \neq t_1, i_3 \neq t_3)} E_{i_1,i_3}\right)$$  \hspace{1cm} (91)

Applying union of events bound, (91) can be written as,

$$P_{e,R_3}^{(n)} \leq P\left(E_{\hat{t}_1,\hat{t}_3}^c\right) + P\left(\bigcup_{(i_1 \neq t_1, i_3 \neq t_3)} E_{i_1,i_3}\right)$$

$$\leq P\left(E_{\hat{t}_1,\hat{t}_3}^c\right) + \sum_{i_1 \neq t_1} P\left(E_{\hat{t}_1,\hat{t}_3}\right) + \sum_{i_3 \neq t_3} P\left(E_{\hat{t}_1,\hat{t}_3}\right) + \sum_{i_1 \neq t_1, i_3 \neq t_3} P\left(E_{i_1,i_3}\right)$$

$$\leq P\left(E_{\hat{t}_1,\hat{t}_3}^c\right) + 2^n(R_{31} + I(W;U_1, U_2; V_3|Q) + 4\epsilon) P\left(E_{\hat{t}_1,\hat{t}_3}\right) + 2^n(R_{33} + I(W;U_1, U_2; V_3|Q) + 8\epsilon) P\left(E_{\hat{t}_1,\hat{t}_3}\right)$$

Let us now evaluate $P(E_{\hat{t}_1,\hat{t}_3})$, $P(E_{\hat{t}_1,\hat{t}_3})$ and $P(E_{\hat{t}_1,\hat{t}_3})$.

$P(E_{\hat{t}_1,\hat{t}_3})$ can be upper bounded as

$$P(E_{\hat{t}_1,\hat{t}_3}) = \sum_{(v_1, v_3, y_3, q) \in A_1^{(n)}} P(v_1(t_1) = v_1|q) P(v_3(t_3) = v_3, y_3 = y_3|q)$$

$$\leq 2^n(H(V_1, V_5, Y_3|Q) + \epsilon) 2^{-n(H(V_1|Q) - \epsilon)} 2^{-n(H(V_5|Y_3) - \epsilon)}$$

$$= 2^{-n(I(V_1; V_5, Y_3|Q) - 3\epsilon)}.$$

$P(E_{\hat{t}_1,\hat{t}_3})$ can be upper bounded as

$$P(E_{\hat{t}_1,\hat{t}_3}) = \sum_{(v_1, v_3, y_3, q) \in A_1^{(n)}} P(v_3(t_3) = v_3|q) P(v_1(t_1) = v_1, y_3 = y_3|q)$$

$$\leq 2^n(H(V_1, V_5, Y_3|Q) + \epsilon) 2^{-n(H(V_3|Q) - \epsilon)} 2^{-n(H(V_1, Y_3|Q) - \epsilon)}$$

$$= 2^{-n(I(V_1; V_3, Y_3|Q) - 3\epsilon)}.$$

$P(E_{\hat{t}_1,\hat{t}_3})$ can be upper bounded as

$$P(E_{\hat{t}_1,\hat{t}_3}) = \sum_{(v_1, v_3, y_3, q) \in A_1^{(n)}} P(v_1(t_1) = v_1|q) P(v_3(t_3) = v_3|q) P(y_3|q)$$

$$\leq 2^n(H(V_1, V_3, Y_3|Q) + \epsilon) 2^{-n(H(V_1|Q) - \epsilon)} 2^{-n(H(V_5|Y_3) - \epsilon)} 2^{-n(H(Y_3|Q) - \epsilon)}$$

$$= 2^{-n(I(V_1; V_5, Y_3|Q) + I(V_1; V_3) - 4\epsilon)}.$$
Substituting these in the probability of decoding error at $\mathcal{R}_3$, we have,

$$P^{(n)}_{e,\mathcal{R}_3} = \epsilon + 2^n(R_{31} + I(W;U_1,U_2;V_1|Q) + 4\epsilon)2^{-n(I(V_1;V_3;Y_3|Q) - 3\epsilon)}$$

$$2^n(R_{33} + I(W;U_1,U_2,V_3|Q) + 4\epsilon)2^{-n(I(V_1;V_3;Y_3|Q) - 3\epsilon)}$$

$$2^n(R_{31} + I(W;U_1,U_2;V_1|Q) + R_{33} + I(W;U_1,U_2;V_3|Q) + 8\epsilon)$$

$$\times 2^{-n(I(V_1;V_3;Y_3|Q) + I(V_1;V_3) - 4\epsilon)}$$

$P^{(n)}_{e,\mathcal{R}_3} \rightarrow 0$ as $n \rightarrow \infty$ if $R_{31}$ and $R_{33}$ satisfy the following constraints:

$$R_{31} \leq I(V_1;V_3;Y_3|Q) - I(W;U_1,U_2;V_1|Q), \quad (92)$$

$$R_{33} \leq I(V_3;V_1;Y_3|Q) - I(W;U_1,U_2;V_3|Q), \quad (93)$$

$$R_{31} + R_{33} \leq I(V_1;V_3;Y_3|Q) + I(V_1;V_3|Q) - I(W;U_1,U_2;V_3|Q) - I(W;U_1,U_2;V_1|Q). \quad (94)$$

The achievable rate region for the channel $\mathcal{C}^2_{cma}$ is given by:

$$R_{11} \leq I(W;U_1,V_1,Y_1|Q), \quad (95)$$

$$R_{11} + R_{21} \leq I(W;U_1,V_1,Y_1|Q), \quad (96)$$

$$R_{11} + R_{31} \leq I(W;V_1,U_1,Y_1|Q) + I(W;V_1|Q) - I(W;U_1,U_2;V_1|Q), \quad (97)$$

$$R_{11} + R_{21} + R_{31} \leq I(W;U_1,V_1;Y_1|Q)I(W;U_1,V_1|Q) - I(W;U_1,U_2;V_1|Q), \quad (98)$$

$$R_{21} \leq I(U_1;U_2,Y_2|Q) - I(W;U_1|Q), \quad (99)$$

$$R_{22} \leq I(U_2;U_1,Y_2|Q) - I(W;U_2|Q), \quad (100)$$

$$R_{21} + R_{22} \leq I(U_1,U_2;Y_2|Q) + I(U_1,U_2;Q) - I(W;U_1|Q) - I(W;U_2|Q), \quad (101)$$

$$R_{31} \leq I(V_1;V_3,Y_3|Q) - I(W;U_1,U_2;V_1|Q), \quad (102)$$

$$R_{33} \leq I(V_3;V_1,Y_3|Q) - I(W;U_1,U_2;V_3|Q), \quad (103)$$

$$R_{31} + R_{33} \leq I(V_1,V_3;Y_3|Q) + I(V_1,V_3|Q) - I(W;U_1,U_2;V_3|Q) - I(W;U_1,U_2;V_1|Q). \quad (104)$$

For the channel $\mathcal{C}^1_{pma}$:

I. Codebook generation

Let us fix $p(\cdot) \in \mathcal{P}$. Generate a random time sharing codeword $q$, of length $n$, according to the distribution $\prod_{i=1}^n p(q_i)$. For $\gamma = 0, 1$, $\tau = 0, 2$ and $\rho = 0, 3$:

generate $2^{[nR_{1\gamma}]}$ independent codewords $W_{\gamma}(j_\gamma)$, $j_\gamma \in \{1, \ldots, 2^{[nR_{1\gamma}]}\}$ according to $\prod_{i=1}^n p(w_{\gamma,i}|q_i)$. For every codeword pair $(w_0(j_0), w_1(j_1))$, generate one codeword $X_1(j_0,j_1)$ according to $\prod_{i=1}^n p(x_{1i}|w_1(j), q_i)$. 


generate $2^{n(R_{2r} + I(W_0,W_1;U_r|Q) + 4\epsilon)}$ independent code words $U_r(l_r)$, according to $\prod_{i=1}^{n} p(u_{ri}|q_i)$. For every codeword tuple $(u_0(l_0), u_2(l_2), w_0(j_0), w_1(j_1))$, generate one codeword $X_2(l_0, l_2, j_0, j_1)$ according to $\Pi_{i=1}^{n} p(x_{2i}|u_{0i}(l_0), u_{2i}(l_2), w_{0i}(j_0), w_{1i}(j_1)q_i)$. Uniformly distribute the $2^{n(R_{2r} + I(W_0,W_1;U_r|Q) + 4\epsilon)}$ code words $U_r(l_r)$ into $2^{nR_{2r}}$ bins indexed by $k_r \in \{1, ..., 2^{nR_{2r}}\}$ such that each bin contains $2^{n(I(W_0,W_1;U_r|Q) + 4\epsilon)}$ codewords.

generate $2^{n(R_{3p} + I(W_0,W_1;V_r|Q) + 4\epsilon)}$ independent code words $V_\rho(t_\rho)$, according to $\prod_{i=1}^{n} p(v_{\rho i}|q_i)$. For every codeword tuple $(v_0(t_0), v_3(t_3), w_0(j_0), w_1(j_1))$, generate one codeword $X_3(t_0, t_3, j_0, j_1)$ according to $\prod_{i=1}^{n} p(x_{3i}|v_{0i}(t_0), v_{3i}(t_3), w_{0i}(j_0), w_{1i}(j_1)q_i)$. Distribute $2^{n(R_{3p} + I(W_0,W_1;V_r|Q) + 4\epsilon)}$ code words $V_\rho(t_\rho)$ uniformly into $2^{nR_{3p}}$ bins indexed by $r_\rho \in \{1, ..., 2^{nR_{3p}}\}$ such that each bin contains $2^{n(I(W_0,W_1;V_r|Q) + 4\epsilon)}$ codewords. The indices are given by $j_\gamma \in \{1, ..., 2^{nR_{1r}}\}$, $l_r \in \{1, ..., 2^{n(R_{2r} + I(W_0,W_1;U_r|Q) + 4\epsilon)}\}$, $t_\rho \in \{1, ..., 2^{n(R_{3p} + I(W_0,W_1;V_r|Q) + 4\epsilon)}\}$.

### J. Encoding & transmission

Let us suppose that the source message vector generated at the three senders is $(m_{10}, m_{11}, m_{20}, m_{22}, m_{30}, m_{33}) = (j_0, j_1, k_0, k_2, r_0, r_3)$. $S_1$ transmits codeword $x_1(j_0, j_1)$ with $n$ channel uses. $S_2$ first looks for a codeword $u_0(l_0)$ in bin $k_0$ such that $(u_0(l_0), w_0(j_0), w_1(j_1), q) \in A^e(n)$, and a codeword $u_2(l_2)$ in bin $k_2$ such that $(u_2(l_2), w_0(j_0), w_1(j_1), q) \in A^e(n)$. It then transmits $x_2(l_0, l_2, j_0, j_1)$ through $n$ channel uses. Otherwise, $S_2$ declares an error. $S_3$ first looks for a codeword $v_0(t_0)$ in bin $r_0$ such that $(v_0(t_0), w_0(j_0), w_1(j_1), q) \in A^e(n)$, and a codeword $v_3(t_3)$ in bin $r_3$ such that $(v_3(t_3), w_0(j_0), w_1(j_1), q) \in A^e(n)$. It then transmits $x_3(t_0, t_3, j_0, j_1)$ through $n$ channel uses. Otherwise, $S_3$ declares an error. The transmissions are assumed to be perfectly synchronized.

### K. Decoding

The three receivers accumulate an $n$-length channel output sequence: $y_1$ at $R_1$, $y_2$ at $R_2$ and $y_3$ at $R_3$. Decoder 1 looks for all index tuples $(\hat{j}_0, \hat{j}_1, \hat{l}_0, \hat{l}_0)$ such that $(w_0(\hat{j}_0), w_1(\hat{j}_1), u_0(\hat{l}_0), v_0(\hat{t}_0), y_1, q) \in A^t(n)$. If $\hat{j}_0$ and $\hat{j}_1$ in all the index tuples found are the same, $R_1$ determines $(m_{10}, m_{11}) = (\hat{j}_0, \hat{j}_1)$ for some $l_0$ and $t_0$. Otherwise, it declares an error. Decoder 2 looks for all index tuples $(\hat{l}_0, \hat{l}_2, \hat{j}_0, \hat{l}_0)$ such that $(w_0(\hat{j}_0), u_0(\hat{l}_0), u_2(\hat{l}_2), v_0(\hat{t}_0), y_2, q) \in A^t(n)$. If $\hat{l}_0$ in all the index pairs found are indices of codewords $u_0(\hat{l}_0)$ from the same bin with index $\hat{k}_0$, and $\hat{l}_2$ in all the index pairs found are indices of codewords $u_2(\hat{l}_2)$ from the same bin with index $\hat{k}_2$, then $R_2$ determines $(m_{20}, m_{22}) = (\hat{k}_0, \hat{k}_2)$. Otherwise, it declares an error. Decoder 3 looks for all index pairs $(\hat{l}_0, \hat{l}_3, \hat{l}_0, \hat{j}_0)$ such that $(w_0(\hat{j}_0), u_0(\hat{l}_0), v_3(\hat{l}_3), y_3, q) \in A^t(n)$. If $\hat{l}_0$ in all the index pairs found are indices of codewords $v_0(\hat{t}_0)$ from the same bin with index $\hat{r}_0$, and $\hat{l}_3$ in all the index pairs found are indices of codewords $v_3(\hat{l}_3)$ from the same bin with index $\hat{r}_3$, then $R_3$ determines $(m_{30}, m_{33}) = (\hat{r}_0, \hat{r}_3)$. Otherwise, it declares an error.
L. Analysis of probabilities of error

In this section we derive upper bounds on the probabilities of error events, which happens during encoding and decoding processes. We will assume that a source message vector \((m_{10}, m_{11}, m_{20}, m_{22}, m_{30}, m_{33})\) is encoded and transmitted. We will consider the analysis of probability of encoding error at senders \(S_2\) and \(S_3\), and the analysis of probability of decoding error at each of the three receivers \(R_1\), \(R_2\), and \(R_3\) separately.

First, let us define the following events:

1. \(E_{joj_1l_0} \triangleq \{ (W_0(j_0), W_1(j_1), U_0(l_0), q) \in A_{\epsilon}^{(n)} \} \),
2. \(E_{joj_1l_2} \triangleq \{ (W_0(j_0), W_1(j_1), U_2(l_2), q) \in A_{\epsilon}^{(n)} \} \),
3. \(E_{joj_1t_0} \triangleq \{ (W_0(j_0), W_1(j_1), V_0(t_0), q) \in A_{\epsilon}^{(n)} \} \),
4. \(E_{joj_1t_3} \triangleq \{ (W_0(j_0), W_1(j_1), V_3(t_3), q) \in A_{\epsilon}^{(n)} \} \),
5. \(E_{joj_1t_0|t_0} \triangleq \{ (W_0(j_0), W_1(j_1), U_0(l_0), V_0(t_0), Y_1, q) \in A_{\epsilon}^{(n)} \} \),
6. \(E_{joj_1l_2|t_0} \triangleq \{ (W_0(j_0), U_0(l_0), U_2(l_2), V_0(t_0), Y_2, q) \in A_{\epsilon}^{(n)} \} \),
7. \(E_{joj_1t_3|t_0} \triangleq \{ (W_0(j_0), U_0(l_0), V_0(t_0), V_3(t_3), Y_3, q) \in A_{\epsilon}^{(n)} \} \).

\(E_{\epsilon}^{(n)} \triangleq \) complement of the event \(E_{\epsilon} \). Events (i) – (iv) will be used in the analysis of probability of encoding error while events (v) – (vii) will be used in the analysis of probability of decoding error.

1) Probability of error at encoder of \(S_2\): An error is made if (1) the encoder cannot find \(u_0(l_0)\) in bin indexed by \(k_0\) such that \((w_0(j_0), w_1(j_1), u_0(l_0), q) \in A_{\epsilon}^{(n)}\) or (2) it cannot find \(u_2(l_2)\) in bin indexed by \(k_2\) such that \((w_0(j_0), w_1(j_1), u_2(l_2), q) \in A_{\epsilon}^{(n)}\). The probability of encoding error at \(S_2\) can be bounded as

\[
P_{e,S_2} \leq P \left( \bigcap_{u_0(l_0)\in\text{bin}(k_0)} (W_0(j_0), W_1(j_1), U_0(l_0), q) \notin A_{\epsilon}^{(n)} \right) \\
+ P \left( \bigcap_{u_2(l_2)\in\text{bin}(k_2)} (W_0(j_0), W_1(j_1), U_2(l_2), q) \notin A_{\epsilon}^{(n)} \right),
\]

\[
\leq (1 - P(E_{joj_1l_0}))^{2^n(I(W_0, W_1; U_0|Q) + \epsilon)} + (1 - P(E_{joj_1l_2}))^{2^n(I(W_0, W_1; U_2|Q) + \epsilon)},
\]

where \(P(.)\) is the probability of an event. Since \(q\) is predetermined,

\[
P(E_{joj_1l_0}) = \sum_{(w_0, w_1, u_0, q) \in A_{\epsilon}^{(n)}} P(W_0(j_0) = w_0, W_1(j_1) = w_1|q) P(U_0(l_0) = u_0|q)
\geq 2^{n(H(W_0, W_1; U_0|Q) - \epsilon)} - 2^{-n(H(W_0, W_1|Q) + \epsilon)} 2^{-n(H(U_0|Q) + \epsilon)} = 2^{-n(I(W_0, W_1; U_0|Q) + 3\epsilon)}.
\]

Similarly, \(P(E_{joj_1l_2}) \geq 2^{-n(I(W_0, W_1; U_2|Q) + 3\epsilon)}\). Therefore,

\[
P_{e,S_2} \leq (1 - 2^{-n(I(W_0, W_1; U_0|Q) + 3\epsilon)})^{2^n(I(W_0, W_1; U_0|Q) + \epsilon)} + (1 - 2^{-n(I(W_0, W_1; U_2|Q) + 3\epsilon)})^{2^n(I(W_0, W_1; U_2|Q) + \epsilon)}.
\]
Now,
\[
(1 - 2^{-n(I(W_0, W_1, U_0 | Q) + 3\epsilon)}) e^{2^n(I(W_0, W_1, U_0 | Q) + 4\epsilon)} \leq e^{2^n(I(W_0, W_1, U_0 | Q) + 4\epsilon)} \ln(1 - 2^{-n(I(W_0, W_1, U_0 | Q) + 3\epsilon)}) 
\]
\[
= e^{2^n(I(W_0, W_1, U_0 | Q) + 4\epsilon)} (1 - 2^{-n(I(W_0, W_1, U_0 | Q) + 3\epsilon)}) 
\]
\[
= e^{-2^n}. 
\]
Clearly, \( P_{e, S_2} \to 0 \) as \( n \to \infty \).

2) Probability of error at encoder of \( S_3 \): An error is made if (1) the encoder cannot find \( v_0(t_0) \) in bin indexed by \( r_0 \) such that \((w_0(j_0), w_1(j_1), v_0(t_0), q) \in A^{(n)}_e\) or (2) it cannot find \( v_3(t_3) \) in bin indexed by \( r_3 \) such that \((w_0(j_0), w_1(j_1), v_3(t_3), q) \in A^{(n)}_e\). The probability of encoding error at \( S_3 \) can be bounded as
\[
P_{e, S_3} \leq P \left( \bigcap_{v_0(t_0) \in \text{bin}(r_0)} (w_0(j_0), w_1(j_1), v_0(t_0), q) \notin A^{(n)}_e \right) + P \left( \bigcap_{v_3(t_3) \in \text{bin}(r_3)} (w_0(j_0), w_1(j_1), v_3(t_3), q) \notin A^{(n)}_e \right) 
\]
\[
\leq (1 - P(E_{j_0 j_1 t_0})) e^{2^n(I(W_0, W_1, V_0 | Q) + 4\epsilon)} + (1 - P(E_{j_0 j_1 t_3})) e^{2^n(I(W_0, W_1, V_3 | Q) + 4\epsilon)}. 
\]
Since \( q \) is predetermined, we have,
\[
P(E_{j_0 j_1 t_0}) = \sum_{(w_0, w_1, v_0, q) \in A^{(n)}_e} P(W_0(j_0) = w_0, W_1(j_1) = w_1 | q) P(V_0(t_0) = v_0 | q) 
\]
\[
\geq 2^n(H(W_0, W_1, V_0 | Q - \epsilon)) 2^{-n(H(W_0, W_1 | V_0 | Q) + \epsilon)} 2^{-n(H(V_0 | Q) + \epsilon)} = 2^{-n(I(W_0, W_1, V_0 | Q) + 3\epsilon)}. 
\]
Similarly, \( P(E_{j_0 j_1 t_3}) \geq 2^{-n(I(W_0, W_1, V_3 | Q) + 3\epsilon)} \). Therefore,
\[
P_{e, S_3} \leq (1 - 2^{-n(I(W_0, W_1, V_0 | Q) + 3\epsilon)}) e^{2^n(I(W_0, W_1, V_0 | Q) + 4\epsilon)} + (1 - 2^{-n(I(W_0, W_1, V_3 | Q) + 3\epsilon)}) e^{2^n(I(W_0, W_1, V_3 | Q) + 4\epsilon)}. 
\]

Proceeding in a way similar to the encoder error analysis at \( S_2 \), we get \( P_{e, S_3} \to 0 \) as \( n \to \infty \).

3) Probability of error at decoder of \( R_1 \): There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., \( E^{(n)}_{j_0 j_1 l_0 t_0} \) happens or (2) there exists some \( \tilde{j}_0 \neq j_0 \) and \( \tilde{j}_1 \neq j_1 \) such that \( E^{(n)}_{j_0 \tilde{j}_0 \tilde{j}_1 \\ l_0 t_0} \) happens. The probability of decoding error can, therefore, be expressed as
\[
P^{(n)}_{e, R_1} = P \left( E^{(n)}_{j_0 j_1 l_0 t_0} \bigcup \bigcup_{\tilde{j}_0 \neq j_0, \tilde{j}_1 \neq j_1} E^{(n)}_{j_0 \tilde{j}_0 \tilde{j}_1 l_0 t_0} \right) 
\]
(105)
Applying union of events bound, [182] can be written as,
\[
P^{(n)}_{e, R_1} \leq P \left( E^{(n)}_{j_0 j_1 l_0 t_0} \right) + P \left( \bigcup_{\tilde{j}_0 \neq j_0, \tilde{j}_1 \neq j_1} E^{(n)}_{j_0 \tilde{j}_0 \tilde{j}_1 l_0 t_0} \right) 
\]
\[
\begin{align*}
= P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_0 \neq j_0} P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_1 \neq j_1} P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_0 \neq j_0, j_1 \neq j_1} P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_0 \neq j_0, j_1 \neq j_1} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
+ \sum_{j_0 \neq j_0, l_0 \neq t_0} P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_1 \neq j_1, l_0 \neq t_0} P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_0 \neq j_0, j_1 \neq j_1, l_0 \neq t_0} P \left( E_{j_0j_1l_0t_0}^c \right) + \sum_{j_0 \neq j_0, j_1 \neq j_1, l_0 \neq t_0} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
\leq P \left( E_{j_0j_1l_0t_0}^c \right) + 2^{nR_{10}} P \left( E_{j_0j_1l_0t_0}^c \right) + 2^{nR_{11}} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{10}+R_{11})} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{10}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{11}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{11}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{10}+R_{11}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{10}+R_{11}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{10}+R_{11}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right) + \\
2^{n(R_{10}+R_{11}+R_{20}+I(W_0,W_1;U_0|Q)+4\epsilon)} P \left( E_{j_0j_1l_0t_0}^c \right).
\end{align*}
\]

Let us now evaluate the probability of error events.

\[
P \left( E_{j_0j_1l_0t_0} \right) \text{ can be upper bounded as}
\]

\[
P \left( E_{j_0j_1l_0t_0}^c \right) \leq 2^{n(H(W_0,W_1,U_0,V_1|Q)+\epsilon)} - 2^{n(H(W_0|Q)+\epsilon)} - 2^{n(H(W_1,U_0,V_1|Q)+\epsilon)}.
\]

\[
P \left( E_{j_0j_1l_0t_0} \right) \text{ can be upper bounded as}
\]

\[
P \left( E_{j_0j_1l_0t_0} \right) \leq 2^{n(H(W_0,W_1,U_0,V_1|Q)+\epsilon)} - 2^{n(H(W_1,U_0,V_1|Q)+\epsilon)}.
\]
\[ P \left( E_{3,j_1,l,t_0} \right) \] can be upper bounded as

\[
P \left( E_{3,j_1,l,t_0} \right) = \sum_{(w_0,w_1,u_0,v_0,y_1,q) \in A^{(n)}} P(W_0(j_0) = w_0|q) P(W_1(j_1) = w_1|q) P(U_0(l_0) = u_0, V_0(t_0) = v_0, Y_1 = y_1|q) 
\leq 2^n(H(W_0,W_1,U_0,V_0,Y_1|Q)+\epsilon) 2^{-n(H(W_0|Q)-\epsilon)} 2^{-n(H(U_0,Y_1|Q)-\epsilon)} = 2^{-n(I(W_0,W_1;U_0,V_0,Y_1|Q)+I(W_0;W_1|Q)-4\epsilon)}.
\]

\[ P \left( E_{3,j_1,l,t_0} \right) \] can be upper bounded as

\[
P \left( E_{3,j_1,l,t_0} \right) = \sum_{(w_0,w_1,u_0,v_0,y_1,q) \in A^{(n)}} P(W_0(j_0) = w_0|q) P(U_0(l_0) = u_0) P(W_1(j_1) = w_1, V_0(t_0) = v_0, Y_1 = y_1|q) 
\leq 2^n(H(W_0,W_1,U_0,V_0,Y_1|Q)+\epsilon) 2^{-n(H(U_0|Q)-\epsilon)} 2^{-n(H(W_1,V_0,Y_1|Q)-\epsilon)} = 2^{-n(I(W_0,U_0;W_1,V_0,Y_1|Q)+I(W_0;U_0|Q)-4\epsilon)}.
\]

\[ P \left( E_{3,j_1,l,t_0} \right) \] can be upper bounded as

\[
P \left( E_{3,j_1,l,t_0} \right) = \sum_{(w_0,w_1,u_0,v_0,y_1,q) \in A^{(n)}} P(W_0(j_0) = w_0|q) P(V_0(t_0) = v_0|q) P(W_1(j_1) = w_1, U_0(l_0) = u_0, Y_1 = y_1|q) 
\leq 2^n(H(W_0,W_1,U_0,V_0,Y_1|Q)+\epsilon) 2^{-n(H(W_0|Q)-\epsilon)} 2^{-n(H(U_0,Y_1|Q)-\epsilon)} = 2^{-n(I(W_0,V_0;W_1,U_0,Y_1|Q)+I(W_0;V_0|Q)-4\epsilon)}.
\]

\[ P \left( E_{3,j_1,l,t_0} \right) \] can be upper bounded as

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P \left( E_{3,j_1,l,t_0} \right) = \sum_{(w_0,w_1,u_0,v_0,y_1,q) \in A^{(n)}} P(W_1(j_1) = w_1|q) P(U_0(l_0) = u_0|q) P(W_0(j_0) = w_0, V_0(t_0) = v_0, Y_1 = y_1|q) 
\leq 2^n(H(W_0,W_1,U_0,V_0,Y_1|Q)+\epsilon) 2^{-n(H(W_1|Q)-\epsilon)} 2^{-n(H(U_0,V_0,Y_1|Q)-\epsilon)} = 2^{-n(I(W_1,U_0;W_0,V_0,Y_1|Q)+I(W_1;U_0|Q)-4\epsilon)}.
\]

\[ P \left( E_{3,j_1,l,t_0} \right) \] can be upper bounded as

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P \left( E_{3,j_1,l,t_0} \right) = \sum_{(w_0,w_1,u_0,v_0,y_1,q) \in A^{(n)}} P(W_1(j_1) = w_1|q) P(V_0(t_0) = v_0|q) P(W_0(j_0) = w_0, U_0(l_0) = u_0, Y_1 = y_1|q) 
\leq 2^n(H(W_0,W_1,U_0,V_0,Y_1|Q)+\epsilon) 2^{-n(H(V_0|Q)-\epsilon)} 2^{-n(H(U_0,V_0,Y_1|Q)-\epsilon)} = 2^{-n(I(W_1,V_0;W_0,U_0,Y_1|Q)+I(W_1;V_0|Q)-4\epsilon)}.
\]
\( P \left( E_{j_0j_1l_0l_0} \right) \) can be upper bounded as

\[
P \left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^n} P(W_0(j_0) = w_0 | q) P(W_1(j_1) = w_1 | q) P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0, Y_1 = y_1 | q)
\]

\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(W_1 | Q) - \epsilon)} 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_0, Y_1 | Q) - \epsilon)}
\]

\[
= 2^{-n(I(W_0, W_1, U_0, V_0, Y_1 | Q) + I(W_0, W_1, U_0 | Q) + I(W_0, W_1 | Q) - 5\epsilon)}.
\]

\( P \left( E_{j_0j_1l_0l_0} \right) \) can be upper bounded as

\[
P \left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^n} P(W_0(j_0) = w_0 | q) P(W_1(j_1) = w_1 | q) P(V_0(t_0) = v_0 | q) P(U_0(l_0) = u_0, Y_1 = y_1 | q)
\]

\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(W_1 | Q) - \epsilon)} 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_0, Y_1 | Q) - \epsilon)}
\]

\[
= 2^{-n(I(W_0, W_1, U_0, V_0, Y_1 | Q) + I(W_0, W_1, U_0 | Q) + I(W_0, W_1 | Q) - 5\epsilon)}.
\]

\( P \left( E_{j_0j_1l_0l_0} \right) \) can be upper bounded as

\[
P \left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^n} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(W_1(j_1) = w_1, Y_1 = y_1 | q)
\]

\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(W_1 | Q) - \epsilon)} 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_0, Y_1 | Q) - \epsilon)}
\]

\[
= 2^{-n(I(W_0, U_0, V_0, W_1, Y_1 | Q) + I(W_0, U_0, V_0 | Q) + I(W_0, U_0 | Q) - 5\epsilon)}.
\]

\( P \left( E_{j_0j_1l_0l_0} \right) \) can be upper bounded as

\[
P \left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^n} P(W_1(j_1) = w_1 | q) P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(W_0(j_0) = w_0, Y_1 = y_1 | q)
\]

\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1 | Q) + \epsilon) 2^{-n(H(W_1 | Q) - \epsilon)} 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_0, Y_1 | Q) - \epsilon)}
\]

\[
= 2^{-n(I(W_1, U_0, V_0, W_1, Y_1 | Q) + I(W_1, U_0, V_0 | Q) + I(W_1, U_0 | Q) - 5\epsilon)}.
\]

\( P \left( E_{j_0j_1l_0l_0} \right) \) can be upper bounded as

\[
P \left( E_{j_0j_1l_0l_0} \right) = \sum_{(w_0, w_1, u_0, v_0, y_1, q) \in A^n} P(W_0(j_0) = w_0 | q) P(W_1(j_1) = w_1 | q) P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(Y_1 = y_1 | q)
\]

\[
\leq 2^n(H(W_0, W_1, U_0, V_0, Y_1 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(W_1 | Q) - \epsilon)} 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_0 | Q) - \epsilon)}
\]

\[
= 2^{-n(I(W_0, W_1, U_0, V_0, Y_1 | Q) + I(W_0, W_1, U_0, V_0 | Q) + I(W_0, W_1, U_0 | Q) + I(W_0, W_1 | Q) - 6\epsilon)}.
\]
Substituting these in the probability of decoding error at $\mathcal{R}_1$, we note that \( P_{e,\mathcal{R}_1}^{(n)} \rightarrow 0 \) as \( n \rightarrow \infty \) iff the following constraints are satisfied:

\[
R_{10} \leq I(W_0; W_1, U_0, V_0, Y_1 | Q), \quad (106)
\]

\[
R_{11} \leq I(W_1; W_0, U_0, V_0, Y_1 | Q), \quad (107)
\]

\[
R_{10} + R_{11} \leq I(W_0, W_1; U_0, V_0, Y_1 | Q) + I(W_0; W_1 | Q), \quad (108)
\]

\[
R_{10} + R_{20} \leq I(W_0, U_0; W_1, V_0, Y_1 | Q) + I(W_0; U_0 | Q) - I(W_0, W_1; U_0 | Q), \quad (109)
\]

\[
R_{10} + R_{30} \leq I(W_0, V_0; W_1, U_0, Y_1 | Q) + I(W_0; V_0 | Q) - I(W_0, W_1; V_0 | Q), \quad (110)
\]

\[
R_{11} + R_{20} \leq I(W_1, U_0; W_0, V_0, Y_1 | Q) + I(W_1; U_0 | Q) - I(W_0, W_1; U_0 | Q), \quad (111)
\]

\[
R_{11} + R_{30} \leq I(W_1, V_0; W_0, U_0, Y_1 | Q) + I(W_1; V_0 | Q) - I(W_0, W_1; V_0 | Q), \quad (112)
\]

\[
R_{10} + R_{11} + R_{20} \leq I(W_0, W_1, U_0; V_0, Y_1 | Q) + I(W_0, W_1; U_0 | Q) + I(W_0; W_1 | Q) - I(W_0, W_1; U_0 | Q), \quad (113)
\]

\[
R_{10} + R_{11} + R_{30} \leq I(W_0, W_1, V_0; U_0, Y_1 | Q) + I(W_0, W_1; V_0 | Q) + I(W_0; W_1 | Q) - I(W_0, W_1; V_0 | Q), \quad (114)
\]

\[
R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0; W_1, Y_1 | Q) + I(W_0, U_0; V_0 | Q) + I(W_0; U_0 | Q) - I(W_0, W_1; U_0 | Q), \quad (115)
\]

\[
R_{11} + R_{20} + R_{30} \leq I(W_1, U_0, V_0; W_0, Y_1 | Q) + I(W_1, U_0; V_0 | Q) + I(W_1; U_0 | Q) - I(W_0, W_1; U_0 | Q), \quad (116)
\]

\[
R_{10} + R_{11} + R_{20} + R_{30} \leq I(W_0, W_1, U_0, V_0; Y_1 | Q) + I(W_0, W_1, U_0; V_0 | Q) + I(W_0, W_1; U_0 | Q) + I(W_0; W_1 | Q) - I(W_0, W_1; U_0 | Q), \quad (117)
\]

\[
+ I(W_0, W_1 | Q) - I(W_0, W_1; U_0 | Q) - I(W_0, W_1; V_0 | Q) \quad (118)
\]

4) **Probability of error at decoder of $\mathcal{R}_2$:** There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., $E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0}$ happens or (2) there exists some $\hat{l}_0 \neq l_0$ and $\hat{l}_2 \neq l_2$ such that $E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0}$ happens. The probability of decoding error can, therefore, be expressed as

\[
P_{e,\mathcal{R}_2}^{(n)} = P \left( E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \cup \bigcup_{(l_0 \neq \hat{l}_0, \hat{l}_2 \neq l_2)} E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) \quad (119)
\]

Applying union of events bound, \[[187]\] can be written as,

\[
P_{e,\mathcal{R}_2}^{(n)} \leq P \left( E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + P \left( \bigcup_{(l_0 \neq \hat{l}_0, \hat{l}_2 \neq l_2)} E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) = P \left( E^c_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{j}_0 \neq j_0} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{l}_0 \neq l_0} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{l}_2 \neq l_2} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{l}_0 \neq \hat{l}_2} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{l}_2 \neq l_2} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{j}_0 \neq j_0} \sum_{\hat{l}_0 \neq l_0} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{l}_0 \neq l_0} \sum_{\hat{l}_2 \neq l_2} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{j}_0 \neq j_0} \sum_{\hat{l}_2 \neq l_2} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right) + \sum_{\hat{l}_0 \neq \hat{l}_2} \sum_{\hat{l}_2 \neq l_2} P \left( E_{\hat{j}_0\hat{l}_0\hat{l}_2\hat{i}_0} \right)
\]
Let us now evaluate the probability of error events.

\[ P(E_{j_0,l_2,t_0}) \] can be upper bounded as

\[
P(E_{j_0,l_2,t_0}) = \sum_{(w_0, u_0, u_2, y_0, y_2) \in A_1^{(n)}} P(U_0(l_0) = u_0 | q) P(W_0(j_0) = w_0, U_2(l_2) = u_2, V_0(t_0) = v_0, Y_2 = y_2 | q) \leq 2^n (H(W_0, U_0, U_2, V_0, Y_2 | q) - \epsilon) \]

\[ = 2^n (H(W_0, U_0, U_2, V_0, Y_2 | q) - \epsilon) \]

\[ = 2^n (H(W_0, U_0, U_2, V_0, Y_2 | q) - 3\epsilon) \]

\[ P(E_{j_0,l_2,t_0}) \] can be upper bounded as

\[
\sum_{(w_0, u_0, u_2, y_0, y_2) \in A_1^{(n)}} P(U_0(l_0) = u_0 | q) P(W_0(j_0) = w_0, U_2(l_2) = u_2, V_0(t_0) = v_0, Y_2 = y_2 | q) \leq 2^n (H(W_0, U_0, U_2, V_0, Y_2 | q) - \epsilon) \]
\[ P(E_{j_0\ell_2t_0}) \text{ can be upper bounded as} \]
\[
P(E_{j_0\ell_2t_0}) = \sum_{(w_0, u_0, u_2, y_0, y_2, q) \in A_n^{(n)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0 | q) P(U_2(l_2) = u_2, V_0(t_0) = v_0, Y_2 = y_2 | q)
\leq 2^n(H(W_0, U_0, U_2, V_0, Y_2 | Q) + \epsilon) 2^{n(H(W_0 | Q) - \epsilon)} 2^{n(H(U_2 | Q) - \epsilon)} 2^{n(H(V_0 | Y_2) - \epsilon)}
\leq 2^n(I(W_0, U_0, U_2, V_0, Y_2 | Q) + I(U_0 | U_2 | Q) - 4\epsilon).
\]

\[ P(E_{j_0\ell_2t_0}) \text{ can be upper bounded as} \]
\[
P(E_{j_0\ell_2t_0}) = \sum_{(w_0, u_0, u_2, y_0, y_2, q) \in A_n^{(n)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0, V_0(t_0) = v_0, Y_2 = y_2 | q)
\leq 2^n(H(W_0, U_0, U_2, V_0, Y_2 | Q) + \epsilon) 2^{n(H(W_0 | Q) - \epsilon)} 2^{n(H(U_2 | Q) - \epsilon)} 2^{n(H(V_0 | Y_2) - \epsilon)}
\leq 2^n(I(W_0, U_0, U_2, V_0, Y_2 | Q) + I(U_0 | U_2 | Q) - 4\epsilon).
\]

\[ P(E_{j_0\ell_2t_0}) \text{ can be upper bounded as} \]
\[
P(E_{j_0\ell_2t_0}) = \sum_{(w_0, u_0, u_2, y_0, y_2, q) \in A_n^{(n)}} P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(W_0(j_0) = w_0, U_2(l_2) = u_2, Y_2 = y_2 | q)
\leq 2^n(H(W_0, U_0, U_2, V_0, Y_2 | Q) + \epsilon) 2^{n(H(U_0 | Q) - \epsilon)} 2^{n(H(U_2 | Q) - \epsilon)} 2^{n(H(V_0 | Y_2) - \epsilon)}
\leq 2^n(I(U_0, V_0, U_2, V_2 | Q) + I(U_0 | V_0 | Q) - 4\epsilon).
\]

\[ P(E_{j_0\ell_2t_0}) \text{ can be upper bounded as} \]
\[
P(E_{j_0\ell_2t_0}) = \sum_{(w_0, u_0, u_2, y_0, y_2, q) \in A_n^{(n)}} P(U_2(l_2) = u_2 | q) P(V_0(t_0) = v_0 | q) P(W_0(j_0) = w_0, U_0(l_0) = u_0, Y_2 = y_2 | q)
\leq 2^n(H(W_0, U_0, U_2, V_0, Y_2 | Q) + \epsilon) 2^{n(H(U_2 | Q) - \epsilon)} 2^{n(H(U_0 | Q) - \epsilon)} 2^{n(H(V_0 | Y_2) - \epsilon)}
\leq 2^n(I(U_2, V_0, U_0, V_2 | Q) + I(U_2 | V_0 | Q) - 4\epsilon).
\]

\[ P(E_{j_0\ell_2t_0}) \text{ can be upper bounded as} \]
\[
P(E_{j_0\ell_2t_0}) = \sum_{(w_0, u_0, u_2, y_0, y_2, q) \in A_n^{(n)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0 | q) P(U_2(l_2) = u_2 | q) P(V_0(t_0) = v_0, Y_2 = y_2 | q)
\leq 2^n(H(W_0, U_0, U_2, V_0, Y_2 | Q) + \epsilon) 2^{n(H(W_0 | Q) - \epsilon)} 2^{n(H(U_2 | Q) - \epsilon)} 2^{n(H(V_0 | Y_2) - \epsilon)}
\leq 2^n(I(W_0, U_0, U_2, V_0, Y_2 | Q) + I(W_0, U_0 | Q) - 5\epsilon).
\]
$P(E_{j_1l_0l_2t_0})$ can be upper bounded as

$$P(E_{j_1l_0l_2t_0})$$

$$= \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0|q)P(V_0(t_0) = v_0|q)P(U_2(l_2) = u_2, Y_2 = y_2|q)$$

$$\leq 2^n(H(W_0,U_0,U_2,V_0,Y_2|Q)+\epsilon)2^{-n(H(W_0|Q)-\epsilon)}2^{-n(H(U_0|Q)-\epsilon)}2^{-n(H(U_2|Y_2)-\epsilon)}$$

$$= 2^{-n(I(W_0,U_0,V_0,U_2,V_2|Q)+I(W_0,U_2;V_0)+I(W_0,U_2|Q)-5\epsilon)}.$$

$P(E_{j_1l_0l_2t_0})$ can be upper bounded as

$$P(E_{j_1l_0l_2t_0})$$

$$= \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0, Y_2 = y_2|q)P(U_2(l_2) = u_2, Y_2 = y_2|q)$$

$$\leq 2^n(H(W_0,U_0,U_2,V_0,Y_2|Q)+\epsilon)2^{-n(H(U_0|Q)-\epsilon)}2^{-n(H(U_2|Q)-\epsilon)}2^{-n(H(U_0,Y_2|Q)-\epsilon)}$$

$$= 2^{-n(I(W_0,U_0,V_0,U_2,V_2|Q)+I(W_0,U_2;V_0)+I(W_0,U_2|Q)-5\epsilon)}.$$

$P(E_{j_1l_0l_2t_0})$ can be upper bounded as

$$P(E_{j_1l_0l_2t_0})$$

$$= \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(U_0(l_0) = u_0|q)P(U_2(l_2) = u_2|q)P(V_0(t_0) = v_0|q)P(W_0(j_0) = w_0, Y_2 = y_2|q)$$

$$\leq 2^n(H(W_0,U_0,U_2,V_0,Y_2|Q)+\epsilon)2^{-n(H(U_0|Q)-\epsilon)}2^{-n(H(U_2|Q)-\epsilon)}2^{-n(H(W_0,Y_2|Q)-\epsilon)}$$

$$= 2^{-n(I(U_0,U_2,V_0,Y_2)+I(U_0,U_2;V_0)+I(U_0,U_2|Q)-5\epsilon)}.$$

$P(E_{j_1l_0l_2t_0})$ can be upper bounded as

$$P(E_{j_1l_0l_2t_0})$$

$$= \sum_{(w_0,u_0,u_2,v_0,y_2,q) \in A_1^{(n)}} P(W_0(j_0) = w_0|q)P(U_0(l_0) = u_0|q)P(U_2(l_2) = u_2|q)P(V_0(t_0) = v_0|q)P(Y_2 = y_2|q)$$

$$\leq 2^n(H(W_0,U_0,U_2,V_0,Y_2|Q)+\epsilon)2^{-n(H(W_0|Q)-\epsilon)}2^{-n(H(U_2|Q)-\epsilon)}2^{-n(H(U_0,Y_2|Q)-\epsilon)}$$

$$= 2^{-n(I(W_0,U_0,V_0,U_2,V_2|Q)+I(W_0,U_2;V_0)+I(W_0,U_0,U_2|Q)+I(W_0,U_0|Q)-6\epsilon)}.$$

Substituting these in the probability of decoding error at $R_2$, we note that $P_{e,R_2}^{(n)} \to 0$ as $n \to \infty$ iff the following constraints are satisfied:

$$R_{20} \leq I(U_0;W_0,U_2,V_0,Y_2|Q) - I(W_0,W_1;U_0|Q),$$

$$R_{22} \leq I(U_2;W_0,U_0,V_0,Y_2|Q) - I(W_0,W_1;U_2|Q),$$

$$R_{20} + R_{22} \leq I(U_0,U_2;W_0,V_0,Y_2|Q) + I(U_0;U_2|Q) - I(W_0,W_1;U_0|Q) - I(W_0,W_1;U_2|Q),$$

$$R_{10} + R_{20} \leq I(W_0,U_0,U_2;V_0,Y_2|Q) + I(W_0;U_0|Q) - I(W_0,W_1;U_0|Q).$$
5) Probability of error at decoder of $\mathcal{R}_3$: There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., $E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c$ happens or (2) there exists some $\tilde{i}_0 \neq t_0$ and $\tilde{i}_3 \neq t_3$ such that $E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c$ happens. The probability of decoding error can, therefore, be expressed as

$$P_{e,\mathcal{R}_3}^{(n)} = P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c \cup \bigcup_{\tilde{i}_0 \neq t_0, t_3 \neq t_3} E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right)$$

Applying union of events bound, (187) can be written as,

$$P_{e,\mathcal{R}_3}^{(n)} \leq P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + P\left(\bigcup_{\tilde{i}_0 \neq t_0, t_3 \neq t_3} E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right)$$

$$= P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + \sum_{\tilde{j}_0 \neq j_0} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + \sum_{\tilde{i}_3 \neq t_3} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + \sum_{\tilde{i}_0 \neq t_0, t_3 \neq t_3} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + \sum_{\tilde{j}_0 \neq j_0, \tilde{i}_0 \neq t_0, t_3 \neq t_3} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + \sum_{\tilde{j}_0 \neq j_0, \tilde{i}_0 \neq t_0, \tilde{i}_3 \neq t_3} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + \sum_{\tilde{j}_0 \neq j_0, \tilde{i}_0 \neq t_0, \tilde{i}_0 \neq t_0, t_3 \neq t_3} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right)$$

$$\leq P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right) + 2^{n(R_{30} + I(W_0, W_1; U_0, U_2; V_0|Q) + 4e)} P\left(E_{\tilde{j}_0 \tilde{i}_0 \tilde{i}_3}^c\right)$$
Let us now evaluate $P(E_{j_0l_03})$, $P(E_{j_0l_03})$ and $P(E_{j_0l_03})$. $P(E_{j_0l_03})$ can be upper bounded as

$$P(E_{j_0l_03}) = \sum_{(w_0,u_0,v_0,y_3,q)\in A^{(n)}} P(V_0(t_0) = v_0|q)P(W_0(j_0) = w_0, U_0(l_0) = u_0, V_3(t_3) = v_3, Y_3 = y_3|q) \leq 2^n(H(W_0, U_0, V_0, V_3, Y_3|Q) + \epsilon) 2^{-n(H(V_0|Q) - \epsilon)} 2^{-n(H(W_0, U_0, V_3, Y_3|Q) - \epsilon)} = 2^{-n(I(V_0, W_0, U_0, V_3, Y_3|Q) - 3\epsilon)}.$$

$P(E_{j_0l_03})$ can be upper bounded as

$$P(E_{j_0l_03}) = \sum_{(w_0,u_0,v_0,y_3,q)\in A^{(n)}} P(V_3(t_3) = v_3|q)P(W_0(j_0) = w_0, U_0(l_0) = u_0, V_0(t_0) = v_0, Y_3 = y_3|q) \leq 2^n(H(W_0, U_0, V_0, V_3, Y_3|Q) + \epsilon) 2^{-n(H(V_3|Q) - \epsilon)} 2^{-n(H(W_0, U_0, V_3, Y_3|Q) - \epsilon)} = 2^{-n(I(V_3, W_0, U_0, V_0, Y_3|Q) - 3\epsilon)}.$$

$P(E_{j_0l_03})$ can be upper bounded as

$$P(E_{j_0l_03}) = \sum_{(w_0,u_0,v_0,y_3,q)\in A^{(n)}} P(V_0(t_0) = v_0|q)P(V_3(t_3) = v_3|q)P(W_0(j_0) = w_0, U_0(l_0) = u_0, Y_3 = y_3|q) \leq 2^n(H(W_0, U_0, V_0, V_3, Y_3|Q) + \epsilon) 2^{-n(H(V_0|Q) - \epsilon)} 2^{-n(H(V_3|Q) - \epsilon)} 2^{-n(H(W_0, U_0, Y_3|Q) - \epsilon)} = 2^{-n(I(V_0, Y_3, W_0, U_0, V_3|Q) + I(V_0, V_3|Q) - 4\epsilon)}.$$
\[ P(E_{julot_{3}}) \text{ can be upper bounded as} \]

\[
P(E_{julot_{3}}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A_1^{(n)}} P(W_0(j_0) = w_0 | q) P(V_0(t_0) = v_0 | q) P(U_0(l_0) = u_0, V_3(t_3) = v_3, Y_3 = y_3 | q) \]

\[ \leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(V_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(U_0, V_3, Y_3 | Q) - \epsilon)} = 2^{-n(I(W_0, V_0, U_0, V_3, Y_3 | Q) + I(W_0, V_3 | Q) - 4\epsilon)} . \]

\[ P(E_{julot_{3}}) \text{ can be upper bounded as} \]

\[
P(E_{julot_{3}}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A_1^{(n)}} P(W_0(j_0) = w_0 | q) P(V_3(t_3) = v_3 | q) P(U_0(l_0) = u_0, V_0(t_0) = v_0, Y_3 = y_3 | q) \]

\[ \leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(U_0, V_3, Y_3 | Q) - \epsilon)} = 2^{-n(I(U_0, V_3 ; W_0, V_0, Y_3 | Q) + I(U_0 ; V_3 | Q) - 4\epsilon)} . \]

\[ P(E_{julot_{3}}) \text{ can be upper bounded as} \]

\[
P(E_{julot_{3}}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A_1^{(n)}} P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(W_0(j_0) = w_0, V_3(t_3) = v_3, Y_3 = y_3 | q) \]

\[ \leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(W_0, V_3, Y_3 | Q) - \epsilon)} = 2^{-n(I(U_0, V_3 ; W_0, V_0, Y_3 | Q) + I(U_0 ; V_3 | Q) - 4\epsilon)} . \]

\[ P(E_{julot_{3}}) \text{ can be upper bounded as} \]

\[
P(E_{julot_{3}}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A_1^{(n)}} P(U_0(l_0) = u_0 | q) P(V_3(t_3) = v_3 | q) P(W_0(j_0) = w_0, V_0(t_0) = v_0, Y_3 = y_3 | q) \]

\[ \leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(U_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(W_0, V_3, Y_3 | Q) - \epsilon)} = 2^{-n(I(U_0, V_3 ; W_0, V_0, Y_3 | Q) + I(U_0 ; V_3 | Q) - 4\epsilon)} . \]

\[ P(E_{julot_{3}}) \text{ can be upper bounded as} \]

\[
P(E_{julot_{3}}) = \sum_{(w_0, u_0, v_0, v_3, q) \in A_1^{(n)}} P(W_0(j_0) = w_0 | q) P(U_0(l_0) = u_0 | q) P(V_0(t_0) = v_0 | q) P(V_3(t_3) = v_3, Y_3 = y_3 | q) \]

\[ \leq 2^n (H(W_0, U_0, V_0, V_3, Y_3 | Q) + \epsilon) 2^{-n(H(W_0 | Q) - \epsilon)} 2^{-n(H(V_0 | Q) - \epsilon)} 2^{-n(H(V_3 | Q) - \epsilon)} 2^{-n(H(V_3, Y_3 | Q) - \epsilon)} = 2^{-n(I(W_0, U_0, V_0, V_3, Y_3 | Q) + I(W_0, U_0, V_3 | Q) - 5\epsilon)} . \]
$P(E_{j_0 l_0}|\tilde{f}_3)$ can be upper bounded as

$$P(E_{j_0 l_0}|\tilde{f}_3) = \sum_{(w_0,u_0,\hat{u}_0,\hat{v}_3,\hat{v}_q) \in A_1^{(n)}} P(W_0|q)P(U_0|l_0) = u_0|q)P(V_3|t_3) = v_3|q)P(V_0|t_0) = v_0, Y_3 = y_3|q)$$

$$\leq 2^{n(H(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q+\epsilon)} 2^{-n(H(U_0|Q)-\epsilon)} 2^{-n(H(V_3|Q)-\epsilon)} 2^{-n(H(V_0,\hat{v}_q)|Q)-\epsilon)} = 2^{-n(I(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q)+I(W_0,\hat{u}_0,\hat{v}_q|Q)+I(W_0,\hat{u}_0|Q)-5\epsilon}).$$

$P(E_{j_0 l_0}|\tilde{f}_3)$ can be upper bounded as

$$P(E_{j_0 l_0}|\tilde{f}_3) = \sum_{(w_0,u_0,\hat{u}_0,\hat{v}_3,\hat{v}_q) \in A_1^{(n)}} P(W_0|q)P(V_0|t_0) = v_0|q)P(V_3|t_3) = v_3|q)P(U_0|l_0) = u_0, Y_3 = y_3|q)$$

$$\leq 2^{n(H(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q+\epsilon)} 2^{-n(H(U_0|Q)-\epsilon)} 2^{-n(H(V_3|Q)-\epsilon)} 2^{-n(H(V_0,\hat{v}_q)|Q)-\epsilon)} = 2^{-n(I(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q)+I(W_0,\hat{u}_0,\hat{v}_q|Q)+I(W_0,\hat{u}_0|Q)-5\epsilon}).$$

$P(E_{j_0 l_0}|\tilde{f}_3)$ can be upper bounded as

$$P(E_{j_0 l_0}|\tilde{f}_3) = \sum_{(w_0,u_0,\hat{u}_0,\hat{v}_3,\hat{v}_q) \in A_1^{(n)}} P(U_0|l_0) = u_0|q)P(V_0|t_0) = v_0|q)P(V_3|t_3) = v_3|q)P(W_0|j_0) = w_0, Y_3 = y_3|q)$$

$$\leq 2^{n(H(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q+\epsilon)} 2^{-n(H(U_0|Q)-\epsilon)} 2^{-n(H(V_3|Q)-\epsilon)} 2^{-n(H(V_0,\hat{v}_q)|Q)-\epsilon)} = 2^{-n(I(U_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q)+I(U_0,\hat{u}_0,\hat{v}_q|Q)+I(U_0,\hat{u}_0|Q)-5\epsilon}).$$

$P(E_{j_0 l_0}|\tilde{f}_3)$ can be upper bounded as

$$P(E_{j_0 l_0}|\tilde{f}_3) = \sum_{(w_0,u_0,\hat{u}_0,\hat{v}_3,\hat{v}_q) \in A_1^{(n)}} P(W_0|q)P(U_0|l_0) = u_0|q)P(V_0|t_0) = v_0|q)P(V_3|t_3) = v_3|q)P(Y_3 = y_3|q)$$

$$\leq 2^{n(H(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q+\epsilon)} 2^{-n(H(U_0|Q)-\epsilon)} 2^{-n(H(V_3|Q)-\epsilon)} 2^{-n(H(V_0,\hat{v}_q)|Q)-\epsilon)} = 2^{-n(I(W_0,\hat{u}_0,\hat{v}_3,\hat{v}_q)|Q)+I(W_0,\hat{u}_0,\hat{v}_q|Q)+I(W_0,\hat{u}_0|Q)-6\epsilon}).$$

Substituting these in the probability of decoding error at $R_3$, we note that $P_{e,R_3}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ iff the following constraints are satisfied:

$$R_{30} \leq I(V_0;W_0,U_0,V_3,Y_3|Q) - I(W_0,\hat{W}_0|V_3),$$

$$R_{33} \leq I(V_3;W_0,U_0,V_0,Y_3|Q) - I(W_0,\hat{W}_0|V_3),$$

$$R_{30} + R_{33} \leq I(V_0,V_3;W_0,U_0,V_0,Y_3|Q) + I(V_0,V_3|Q)$$

$$- I(W_0,\hat{W}_0,\hat{V}_0|Q) - I(W_0,\hat{W}_0,V_3|Q).$$
\[ R_{10} + R_{30} \leq I(W_0, V_0; U_0, V_3, Y_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1; V_0|Q), \] (136)

\[ R_{10} + R_{33} \leq I(W_0, V_3; U_0, V_0, Y_3|Q) + I(W_0; V_3|Q) - I(W_0, W_1; V_3|Q). \] (137)

\[ R_{20} + R_{30} \leq I(U_0, V_0; W_0, V_3, Y_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q). \] (138)

\[ R_{20} + R_{33} \leq I(U_0, V_3; W_0, V_0, Y_3|Q) + I(U_0; V_3|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_2|Q). \] (139)

\[ R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0; V_3, Y_3|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q). \] (140)

\[ R_{10} + R_{20} + R_{33} \leq I(W_0, U_0, V_3; V_0, Y_3|Q) + I(W_0, U_0; V_3|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_3|Q). \] (141)

\[ R_{10} + R_{30} + R_{33} \leq I(W_0, V_0, V_3; U_0, Y_3|Q) + I(W_0, V_0; V_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q). \] (142)

\[ R_{20} + R_{30} + R_{33} \leq I(U_0, V_0, V_3; W_0, Y_3|Q) + I(U_0, V_0; V_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_3|Q). \] (143)

\[ R_{10} + R_{20} + R_{30} + R_{33} \leq I(W_0, U_0, V_0, V_3; Y_3|Q) + I(W_0, U_0, V_0; V_3|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q). \] (144)

The achievable rate region follows:

\[ R_{10} \leq I(W_0; W_1, U_0, V_0, Y_1|Q), \] (145)

\[ R_{11} \leq I(W_1; W_0, U_0, V_0, Y_1|Q). \] (146)

\[ R_{10} + R_{11} \leq I(W_0, W_1; U_0, V_0, Y_1|Q) + I(W_0; W_1|Q). \] (147)

\[ R_{10} + R_{20} \leq I(W_0, U_0; W_1, V_0, Y_1|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q). \] (148)

\[ R_{10} + R_{30} \leq I(W_0, V_0; W_1, U_0, Y_1|Q) + I(W_0; V_0|Q) - I(W_0, W_1; V_0|Q). \] (149)

\[ R_{11} + R_{20} \leq I(W_1, U_0; W_0, V_0, Y_1|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q). \] (150)

\[ R_{11} + R_{30} \leq I(W_1, V_0; W_0, U_0, Y_1|Q) + I(W_1; V_0|Q) - I(W_0, W_1; V_0|Q). \] (151)

\[ R_{10} + R_{11} + R_{20} \leq I(W_0, W_1, U_0; V_0, Y_1|Q) + I(W_0, W_1; U_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1; U_0|Q). \] (152)

\[ R_{10} + R_{11} + R_{30} \leq I(W_0, W_1, V_0; U_0, Y_1|Q) + I(W_0, W_1; V_0|Q) + I(W_0; W_1|Q) - I(W_0, W_1; V_0|Q). \] (153)

\[ R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, V_0; W_1, Y_1|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q). \] (154)

\[ R_{11} + R_{20} + R_{30} \leq I(W_1, U_0, V_0; W_0, Y_1|Q) + I(W_1, U_0; V_0|Q) + I(W_1; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q). \] (155)
\[ R_{10} + R_{11} + R_{20} + R_{30} \leq I(W_0, W_1, U_0, V_0; Y_1|Q) + I(W_0, W_1, U_0; V_0|Q) + I(W_0, W_1; U_0|Q) \quad (156) \]
\[ + I(W_0, W_1|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) \quad (157) \]
\[ R_{20} \leq I(U_0; W_0, U_2, V_0, Y_2|Q) - I(W_0, W_1; U_0|Q) \quad (158) \]
\[ R_{22} \leq I(U_2; W_0, U_0, V_0, Y_2|Q) - I(W_0, W_1; U_2|Q) \quad (159) \]
\[ R_{20} + R_{22} \leq I(U_0, U_2; W_0, V_0, Y_2|Q) + I(U_0; U_2|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) \quad (160) \]
\[ R_{10} + R_{20} \leq I(W_0, U_0; U_2, V_0, Y_2|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) \quad (161) \]
\[ R_{10} + R_{22} \leq I(W_0, U_2; U_0, V_0, Y_2|Q) + I(W_0; U_2|Q) - I(W_0, W_1; U_2|Q) \quad (162) \]
\[ R_{20} + R_{30} \leq I(U_0, V_0; W_0, U_2, Y_2|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) \quad (163) \]
\[ R_{22} + R_{30} \leq I(U_2, V_0; W_0, U_0, Y_2|Q) + I(U_2; V_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \quad (164) \]
\[ R_{10} + R_{20} + R_{22} \leq I(W_0, U_0, U_2; V_0, Y_2|Q) + I(W_0, U_0; U_2|Q) + I(W_0, U_2|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) \quad (165) \]
\[ R_{10} + R_{20} + R_{30} \leq I(W_0, U_0, U_2, V_0; Y_2|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) \quad (166) \]
\[ R_{10} + R_{22} + R_{30} \leq I(W_0, U_2, V_0; U_0, Y_2|Q) + I(W_0, U_2; V_0|Q) + I(W_0; U_2|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \quad (167) \]
\[ R_{20} + R_{22} + R_{30} \leq I(U_0, U_2, V_0; W_0, Y_2|Q) + I(U_0, U_2; V_0|Q) + I(U_0; U_2|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \quad (168) \]
\[ R_{10} + R_{20} + R_{22} + R_{30} \leq I(W_0, U_0, U_2, V_0; Y_2|Q) + I(W_0, U_0, U_2; V_0|Q) + I(W_0, U_0; U_2|Q) + I(W_0, U_2|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \quad (169) \]
\[ + I(W_0, U_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; U_2|Q) - I(W_0, W_1; V_0|Q) \quad (170) \]
\[ R_{30} \leq I(V_0; W_0, U_0, V_3, Y_3; Q) - I(W_0, W_1; V_0|Q) \quad (171) \]
\[ R_{33} \leq I(V_3; W_0, U_0, V_0, Y_3|Q) - I(W_0, W_1; V_3|Q) \quad (172) \]
\[ R_{30} + R_{33} \leq I(V_0, V_3; W_0, U_0, Y_3|Q) + I(V_0; V_3|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q) \quad (173) \]
\[ R_{10} + R_{30} \leq I(W_0, V_0; U_0, V_3, Y_3|Q) + I(W_0; V_0|Q) - I(W_0, W_1; V_0|Q) \quad (174) \]
\[ R_{10} + R_{33} \leq I(W_0, V_3; U_0, V_0, Y_3|Q) + I(W_0; V_3|Q) - I(W_0, W_1; V_3|Q) \quad (175) \]
\[ R_{20} + R_{30} \leq I(U_0, V_0; W_0, V_3, Y_3|Q) + I(U_0; V_0|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) \quad (176) \]
\[ R_{20} + R_{33} \leq I(U_0, V_3; W_0, V_0, Y_3|Q) + I(U_0; V_3|Q) - I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_3|Q) \quad (177) \]
\[-I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q),(177)\]

\[R_{10} + R_{20} + R_{33} \leq I(W_0, U_0, V_3; V_0, Y_3|Q) + I(W_0, U_0; V_3|Q) + I(W_0; U_0|Q)\]

\[-I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_3|Q),(178)\]

\[R_{10} + R_{30} + R_{33} \leq I(W_0, V_0, V_3; U_0, Y_3|Q) + I(W_0, V_0; V_3|Q) + I(W_0; V_0|Q)\]

\[-I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q),(179)\]

\[R_{20} + R_{30} + R_{33} \leq I(U_0, V_0, V_3; Y_3|Q) + I(U_0, V_0; V_3|Q) + I(U_0; V_0|Q)\]

\[-I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q),(180)\]

\[R_{10} + R_{20} + R_{30} + R_{33} \leq I(W_0, U_0, V_0, V_3; Y_3|Q) + I(W_0, U_0, V_0; V_3|Q) + I(W_0, U_0; V_0|Q) + I(W_0; U_0|Q)\]

\[-I(W_0, W_1; U_0|Q) - I(W_0, W_1; V_0|Q) - I(W_0, W_1; V_3|Q),(181)\]

For the channel \(C_{\text{samp}}^2\):

**M. Codebook generation**

Let us fix \(p(.) \in \mathcal{P}\). Generate a random time sharing codeword \(q\) of length \(n\), according to the distribution \(\prod_{i=1}^{n} p(q_i)\). Generate \(2^{[nR_{11}]}\) independent codewords \(W(j), j \in \{1, \ldots, 2^{[nR_{11}]}\}\) according to \(\prod_{i=1}^{n} p(w_i|q_i)\). For every \(w(j)\), generate one \(X_1(j)\) codewords according to \(\prod_{i=1}^{n} p(x_{i1}|w_i(j), q_i)\). For \(\tau = 1, 2\), generate \(2^{n(R_{2\tau} + I(W; U_\tau|Q) + 4\epsilon)}\) independent code words \(U_\tau(l_\tau)\), according to \(\prod_{i=1}^{n} p(u_{\tau i}|q_i)\). For every code word triple \((u_1(l_1), u_2(l_2), w(j))\), generate one code word \(X_2(l_1, l_2, j)\) according to \(\prod_{i=1}^{n} p(x_{2i}|u_{1i}(l_1), u_{2i}(l_2), w_i(j), q_i)\). Uniformly distribute the \(2^{n(R_{2\tau} + I(W; U_\tau|Q) + 4\epsilon)}\) code words \(U_\tau(l_\tau)\) into \(2^{nR_{2\tau}}\) bins indexed by \(k_\tau \in \{1, \ldots, 2^{nR_{2\tau}}\}\) such that each bin contains \(2^{n(I(W; U_\tau|Q) + 4\epsilon)}\) codewords. For \(\rho = 1, 3\), generate \(2^{n(R_{2\rho} + I(W; V_\rho|Q) + 4\epsilon)}\) independent code words \(V_\rho(t_\rho)\), according to \(\prod_{i=1}^{n} p(v_{\rho i}|q_i)\). For every code word triple \((v_1(t_1), v_3(t_3), w(j))\), generate one codeword \(X_3(t_1, t_3, j)\) according to \(\prod_{i=1}^{n} p(x_{3i}|v_{1i}(t_1), v_{3i}(t_3), w_i(j), q_i)\). Distribute \(2^{n(R_{2\rho} + I(W; V_\rho|Q) + 4\epsilon)}\) code words \(V_\rho(t_\rho)\) uniformly into \(2^{nR_{2\rho}}\) bins indexed by \(r_\rho \in \{1, \ldots, 2^{nR_{2\rho}}\}\) such that each bin contains \(2^{n(I(W; V_\rho|Q) + 4\epsilon)}\) code words. The indices are given by \(j \in \{1, \ldots, 2^{nR_{11}}\}, l_\tau \in \{1, \ldots, 2^{n(R_{2\tau} + I(W; U_\tau|Q) + 4\epsilon)}\}, t_\rho \in \{1, \ldots, 2^{n(R_{2\rho} + I(W; V_\rho|Q) + 4\epsilon)}\}\). The number of codewords that we need to generate is obtained during the process of driving the encoder-error to zero. The entire codebook is revealed to all senders and receivers.

**N. Encoding & transmission**

Let us suppose that the source message vector generated at the three senders is \((m_{11}, m_{21}, m_{22}, m_{31}, m_{33}) = (j, k_1, k_2, r_1, r_3)\). \(S_1\) transmits codeword \(x_1(j)\) with \(n\) channel uses. \(S_2\) first looks for a codeword \(u_1(l_1)\) in bin \(k_1\) such that \((u_1(l_1), w(j), q) \in A_c^{(n)}\) and a codeword \(u_2(l_2)\) in bin \(k_2\) such that \((u_2(l_2), w(j), q) \in A_c^{(n)}\). It then transmits \(x_2(l_1, l_2, j)\) through \(n\) channel uses. Otherwise, \(S_2\) declares an error. \(S_3\)
first looks for a codeword $v_1(t_1)$ in bin $r_1$ such that $(v_1(t_1), w(j), q) \in A_{e}^{(n)}$, and a codeword $v_3(t_3)$ in bin $r_3$ such that $(v_3(t_3), w(j), q) \in A_{e}^{(n)}$. It then transmits $x_3(t_1, t_3, j)$ through $n$ channel uses. Otherwise, $S_3$ declares an error. The transmissions are assumed to be perfectly synchronized.

O. Decoding

The three receivers accumulate an $n$-length channel output sequence: $y_1$ at $R_1$, $y_2$ at $R_2$ and $y_3$ at $R_3$. Decoder 1 looks for all index triples $(\hat{j}, \hat{l}_1, \hat{t}_1)$ such that $(w(\hat{j}), u_1(\hat{l}_1), v_1(\hat{t}_1), y_1, q) \in A_{e}^{(n)}$. If $\hat{j}$ in all the index triples found are the same, $R_1$ determines $m_{11} = \hat{j}$, for some $l_1$ and $t_1$. Otherwise, it declares an error. Decoder 2 looks for all index pairs $(\hat{l}_1, \hat{l}_2)$ such that $(u_1(\hat{l}_1), u_2(\hat{l}_2), y_2, q) \in A_{e}^{(n)}$. If $\hat{l}_1$ in all the index pairs found are indices of codewords $u_1(\hat{l}_1)$ from the same bin with index $k_1$, and $\hat{l}_2$ in all the index pairs found are indices of codewords $u_2(\hat{l}_2)$ from the same bin with index $k_2$, then $R_2$ determines $(m_{21}, m_{22}) = (\hat{k}_1, \hat{k}_2)$. Otherwise, it declares an error. Decoder 3 looks for all index pairs $(\hat{l}_1, \hat{t}_3)$ such that $(v_1(\hat{l}_1), v_3(\hat{t}_3), y_3, q) \in A_{e}^{(n)}$. If $\hat{l}_1$ in all the index pairs found are indices of codewords $v_1(\hat{l}_1)$ from the same bin with index $\hat{r}_1$, and $\hat{t}_3$ in all the index pairs found are indices of codewords $v_3(\hat{t}_3)$ from the same bin with index $\hat{r}_3$, then $R_3$ determines $(m_{31}, m_{33}) = (\hat{r}_1, \hat{r}_3)$. Otherwise, it declares an error.

P. Analysis of probabilities of error

In this section we derive upperbounds on the probabilities of error events, which happens during encoding and decoding processes. We will assume that a source message vector $(m_{11}, m_{21}, m_{22}, m_{31}, m_{33})$ is encoded and transmitted. We will consider the analysis of probability of encoding error at senders $S_2$ and $S_3$, and the analysis of probability of decoding error at each of the three receivers $R_1$, $R_2$, and $R_3$ separately.

First, let us define the following events:

(i) $E_{j_1t_1} \triangleq \{ (W(j), U_1(l_1), q) \in A_{e}^{(n)} \}$,

(ii) $E_{j_2t_2} \triangleq \{ (W(j), U_2(l_2), q) \in A_{e}^{(n)} \}$,

(iii) $E_{j_3t_3} \triangleq \{ (W(j), V_1(t_1), q) \in A_{e}^{(n)} \}$,

(iv) $E_{j_3t_3} \triangleq \{ (W(j), V_3(t_3), q) \in A_{e}^{(n)} \}$,

(v) $E_{j_3t_3} \triangleq \{ (W(j), U_1(l_1), V_1(t_1), Y_1, q) \in A_{e}^{(n)} \}$,

(vi) $E_{j_3t_3} \triangleq \{ (U_1(l_1), U_2(l_2), Y_2, q) \in A_{e}^{(n)} \}$,

(vii) $E_{j_3t_3} \triangleq \{ (V_1(t_1), V_3(t_3), Y_3, q) \in A_{e}^{(n)} \}$.

$E_c^{(i)} \triangleq$ complement of the event $E_{(i)}$. Events (i) – (iv) will be used in the analysis of probability of encoding error while events (v) – (vii) will be used in the analysis of probability of decoding error.
1) **Probability of error at encoder of \( S_2 \):** An error is made if (1) the encoder cannot find \( u_1(l_1) \) in bin indexed by \( k_1 \) such that \((w(j), u_1(l_1), q) \in A_e^{(n)} \) or (2) it cannot find \( u_2(l_2) \) in bin indexed by \( k_2 \) such that \((w(j), u_2(l_2), q) \in A_e^{(n)} \). The probability of encoding error at \( S_2 \) can be bounded as

\[
P_{e,S_2} \leq P \left( \bigcap_{u_1(l_1) \in \text{bin}(k_1)} (W(j), U_1(l_1), q) \notin A_e^{(n)} \right) + P \left( \bigcap_{u_2(l_2) \in \text{bin}(k_2)} (W(j), U_2(l_2), q) \notin A_e^{(n)} \right),
\]

\[
\leq (1 - P(E_{j_1}))^2 e^{n(I(W;U_1|Q)+4\epsilon)} + (1 - P(E_{j_2}))^2 e^{n(I(W;U_2|Q)+4\epsilon)},
\]

where \( P(.) \) is the probability of an event. Since \( q \) is predetermined,

\[
P(E_{j_1}) = \sum_{(w,u_1,q) \in A_e^{(n)}} P(W(j) = w|q)P(U_1(l_1) = u_1|q)
\]

\[
\geq 2^n(H(W;U_1|\epsilon)) 2^{-n(H(W)+\epsilon)} = 2^{-n(H(W;U_1|Q)+3\epsilon)}.
\]

Similarly, \( P(E_{j_2}) \geq 2^{-n(I(W;U_2|Q)+3\epsilon)} \). Therefore,

\[
P_{e,S_2} \leq (1 - 2^{-n(I(W;U_1|Q)+3\epsilon)})^2 e^{n(I(W;U_1|Q)+4\epsilon)} + (1 - 2^{-n(I(W;U_2|Q)+3\epsilon)})^2 e^{n(I(W;U_2|Q)+4\epsilon)}.
\]

Now,

\[
(1 - 2^{-n(I(W;U_1|Q)+3\epsilon)})^2 e^{n(I(W;U_1|Q)+4\epsilon)} = e^{2n(I(W;U_1|Q)+4\epsilon) \ln(1-2^{-n(I(W;U_1|Q)+3\epsilon)})}
\]

\[
\leq e^{2n(I(W;U_1|Q)+4\epsilon) (-2^{-n(I(W;U_1|Q)+3\epsilon)})} = e^{-2^{n\epsilon}}.
\]

Clearly, \( P_{e,S_2} \to 0 \) as \( n \to \infty \).

2) **Probability of error at encoder of \( S_3 \):** An error is made if (1) the encoder cannot find \( v_1(t_1) \) in bin indexed by \( r_1 \) such that \((w(j), v_1(t_1), q) \in A_e^{(n)} \) or (2) it cannot find \( v_3(t_3) \) in bin indexed by \( r_3 \) such that \((w(j), v_3(t_3), q) \in A_e^{(n)} \). The probability of encoding error at \( S_3 \) can be bounded as

\[
P_{e,S_3} \leq P \left( \bigcap_{v_1(t_1) \in \text{bin}(r_1)} (W(j), V_1(t_1), q) \notin A_e^{(n)} \right) + P \left( \bigcap_{v_3(t_3) \in \text{bin}(r_3)} (W(j), V_3(t_3), q) \notin A_e^{(n)} \right)
\]

\[
\leq (1 - P(E_{j_1}))^2 e^{n(I(W;V_1|Q)+4\epsilon)} + (1 - P(E_{j_3}))^2 e^{n(I(W;V_3|Q)+4\epsilon)}.
\]

Since \( q \) is predetermined, we have,

\[
P(E_{j_1}) = \sum_{(w,v_1,q) \in A_e^{(n)}} P(W(j) = w|q)P(V_1(t_1) = v_1|q)
\]

\[
\geq 2^n(H(W;V_1|\epsilon)) 2^{-n(H(W)+\epsilon)} = 2^{-n(H(W;V_1|Q)+3\epsilon)}.
\]

Similarly, \( P(E_{j_3}) \geq 2^{-n(I(W;V_3|Q)+3\epsilon)} \). Therefore,

\[
P_{e,S_3} \leq (1 - 2^{-n(I(W;V_1|Q)+3\epsilon)})^2 e^{n(I(W;V_1|Q)+4\epsilon)} + (1 - 2^{-n(I(W;V_3|Q)+3\epsilon)})^2 e^{n(I(W;V_3|Q)+4\epsilon)}.
\]

Proceeding in a way similar to the encoder error analysis at \( S_2 \), we get \( P_{e,S_3} \to 0 \) as \( n \to \infty \).
3) **Probability of error at decoder of \( \mathcal{R}_1 \):** There are two possible events which can be classified as errors: (1) The codewords transmitted are not jointly typical i.e., \( E^c_{jl,t_1} \) happens and/or (2) there exists some \( \hat{j} \neq j \) such that \( E^c_{\hat{j}l,\hat{t}_1} \) happens. Note that \( \hat{t}_1 \) need not equal \( t_1 \), and \( \hat{l}_1 \) need not equal \( t_1 \), since \( \mathcal{R}_1 \) is not required to decode \( \hat{t}_1 \) and \( \hat{l}_1 \) correctly. The probability of decoding error can, therefore, be expressed as

\[
P_{e,\mathcal{R}_1}^{(n)} = P \left( E^c_{jl,t_1} \cup \bigcup_{j \neq \hat{j}} E^c_{\hat{j}l,\hat{t}_1} \right)
\]

Applying union of events bound, [182] can be written as,

\[
P_{e,\mathcal{R}_1}^{(n)} \leq P \left( E^c_{jl,t_1} + \bigcup_{j \neq \hat{j}} E^c_{\hat{j}l,\hat{t}_1} \right)
\]

\[
= P \left( E^c_{jl,t_1} + \sum_{j \neq \hat{j}} P \left( E_{jl,t_1} \right) + \sum_{j \neq \hat{j}, l \neq \hat{l}} P \left( E_{\hat{j}l,\hat{t}_1} \right) + \sum_{j \neq \hat{j}, l \neq \hat{l}, \hat{t}_1 \neq t_1} P \left( E_{\hat{j}l,\hat{t}_1} \right) \right)
\]

\[
\leq P \left( E^c_{jl,t_1} + 2^nR_{11} P \left( E_{jl,t_1} \right) + 2^n(R_{11} + R_{21} + I(W;U_1|Q) + 4e) P \left( E_{\hat{j}l,\hat{t}_1} \right) + 2^n(R_{11} + R_{21} + I(W;V_1|Q) + 4e) P \left( E_{\hat{j}l,\hat{t}_1} \right) \right).
\]

Let us now evaluate \( P \left( E^c_{jl,t_1} \right), P \left( E_{jl,t_1} \right), P \left( E_{\hat{j}l,\hat{t}_1} \right), P \left( E_{\hat{j}l,\hat{t}_1} \right). \)

\( P \left( E_{jl,t_1} \right) \) can be upper bounded as

\[
P \left( E_{jl,t_1} \right) = \sum_{(w,u,v,y) \in A^{(n)}} P(W(j) = \text{w} | q) P(U_1(l_1) = \text{u}_1, V_1(t_1) = \text{v}_1, Y_1 = \text{y}_1 | q)
\]

\[
\leq 2^n(H(W;U_1,V_1,Y_1|Q) + \epsilon) 2^{-n(H(W|Q) - \epsilon)} 2^{-n(H(U_1,V_1,Y_1|Q) - \epsilon)}
\]

\[
= 2^{-n(I(W;U_1,V_1,Y_1|Q) - 3\epsilon)}.
\]

\( P \left( E_{\hat{j}l,\hat{t}_1} \right) \) can be upper bounded as

\[
P \left( E_{\hat{j}l,\hat{t}_1} \right) = \sum_{(w,u,v,y) \in A^{(n)}} P(W(j) = \text{w} | q) P(U_1(l_1) = \text{u}_1 | q) P(V_1(t_1) = \text{v}_1, Y_1 = \text{y}_1 | q)
\]

\[
\leq 2^n(H(W;U_1,V_1,Y_1|Q) + \epsilon) 2^{-n(H(W|Q) - \epsilon)} 2^{-n(H(U_1|Q) - \epsilon)} 2^{-n(H(Y_1|Q) - \epsilon)}
\]

\[
= 2^{-n(I(W;U_1,V_1,Y_1|Q) + I(W;V_1|Q) - 4\epsilon)}.
\]

\( P \left( E_{\hat{j}l,\hat{t}_1} \right) \) can be upper bounded as

\[
P \left( E_{\hat{j}l,\hat{t}_1} \right) = \sum_{(w,u,v,y) \in A^{(n)}} P(W(j) = \text{w} | q) P(V_1(t_1) = \text{v}_1 | q) P(U_1(l_1) = \text{u}_1, Y_1 = \text{y}_1 | q)
\]

\[
\leq 2^n(H(W;U_1,V_1,Y_1|Q) + \epsilon) 2^{-n(H(W|Q) - \epsilon)} 2^{-n(H(V_1|Q) - \epsilon)} 2^{-n(H(U_1,Y_1|Q) - \epsilon)}
\]

\[
= 2^{-n(I(W;V_1,U_1,Y_1|Q) + I(W;V_1|Q) - 4\epsilon)}.
\]
$P\left(E_{\tilde{l}_1 \tilde{l}_2}\right)$ can be upper bounded as

$$P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) = \sum_{(w,u_1,v_1,q) \in A^{(n)}} P(W(j) = w)P(U_1(t_1) = u_1)P(V_1(l_1) = v_1|q)P(Y_1 = y_1|q)$$

$$\leq 2^nR_2(I(W;U_1,V_1|Q) + \epsilon) - n(H(U_1|Q) - \epsilon) - n(H(V_1|Q) - \epsilon) - n(H(Y_1|Q) - \epsilon)$$

$$= 2^{-n}I(W;U_1,V_1;Y_1|Q) + I(W;U_1;V_1|Q) + I(W;U_1|Q) - 5\epsilon.$$ 

Substituting these in the probability of decoding error at $R_1$, we have,

$$P_{e,R_1}^{(n)} = \epsilon + 2^nR_1 \leq 2^{-n}(I(W;U_1,V_1;Y_1|Q) - 3\epsilon) + 2^n(R_1 + R_2 + I(W;U_1|Q) + 4\epsilon) - n(I(W;U_1,V_1;Y_1|Q) + I(W;U_1|Q) - 4\epsilon) +$$

$$2^n(R_1 + R_2 + I(W;V_1|Q) + 4\epsilon) - n(I(W;V_1;U_1,Y_1|Q) + I(W;V_1|Q) - 4\epsilon) +$$

$$2^n(R_1 + R_2 + I(W;U_1|Q) + 4\epsilon + R_1 + I(W;V_1|Q) + 4\epsilon) - n(I(W;U_1,V_1;Y_1|Q) + I(W;U_1;V_1|Q) + I(W;U_1|Q) - 5\epsilon).$$

$P_{e,R_1}^{(n)} \to 0$ as $n \to \infty$ if $R_{11}$, $R_{21}$ and $R_{31}$ satisfy the following constraints:

$$R_{11} \leq I(W;U_1,V_1,Y_1|Q) \quad (183)$$

$$R_{11} + R_{21} \leq I(W;U_1,V_1,Y_1|Q) \quad (184)$$

$$R_{11} + R_{31} \leq I(W,V_1;U_1,Y_1|Q) \quad (185)$$

$$R_{11} + R_{21} + R_{31} \leq I(W;U_1,V_1;Y_1|Q) + I(W;U_1;V_1|Q) - I(W;V_1|Q). \quad (186)$$

4) Probability of error at decoder of $R_2$: The two possible error events are: (1) The codewords transmitted are not jointly typical i.e., $E_{\tilde{l}_1 \tilde{l}_2}$ happens and/or (2) there exists some $(\tilde{l}_1 \neq l_1, \tilde{l}_2 \neq l_2)$ such that $E_{\tilde{l}_1 \tilde{l}_2}$ happens. The probability of decoding error can be written as

$$P_{e,R_2}^{(n)} = P\left(E_{\tilde{l}_1 \tilde{l}_2} \cup \bigcup_{(\tilde{l}_1 \neq l_1, \tilde{l}_2 \neq l_2)} E_{\tilde{l}_1 \tilde{l}_2}\right)$$

$$= P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) + \sum_{\tilde{l}_1 \neq l_1} P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) + \sum_{\tilde{l}_2 \neq l_2} P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) + \sum_{\tilde{l}_1 \neq l_1, \tilde{l}_2 \neq l_2} P\left(E_{\tilde{l}_1 \tilde{l}_2}\right)$$

$$\leq P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) + 2^n(R_{21} + I(W;U_1|Q) + 4\epsilon) P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) + 2^n(R_{22} + I(W;U_2|Q) + 4\epsilon) P\left(E_{\tilde{l}_1 \tilde{l}_2}\right) +$$

$$+ 2^n(R_{21} + R_{22} + I(W;U_1|Q) + 4\epsilon + I(W;U_2|Q) + 4\epsilon) P\left(E_{\tilde{l}_1 \tilde{l}_2}\right).$$

Let us now evaluate $P(E_{\tilde{l}_1 \tilde{l}_2})$, $P(E_{\tilde{l}_1 \tilde{l}_2})$ and $P(E_{\tilde{l}_1 \tilde{l}_2})$. 


\( P(E_{l_1l_2}) \) can be upper bounded as

\[
P(E_{l_1l_2}) = \sum_{(u_1, u_2, y_2, q) \in A_n} P(U_1(l_1) = u_1 | q) P(U_2(l_2) = u_2, Y_2 = y_2 | q)
\leq 2^n(H(U_1, U_2, Y_2|Q) + \epsilon) 2^{-n(H(U_1|Q) - \epsilon)} 2^{-n(H(U_2, Y_2|Q) - \epsilon)}
= 2^{-n(I(U_1; U_2, Y_2|Q) - 3\epsilon)}.
\]

\( P(E_{l_1l_2}) \) can be upper bounded as

\[
P(E_{l_1l_2}) = \sum_{(u_1, u_2, y_2, q) \in A_n} P(U_2(l_2) = u_2 | q) P(U_1(l_1) = u_1, Y_2 = y_2 | q)
\leq 2^n(H(U_1, U_2, Y_2|Q) + \epsilon) 2^{-n(H(U_1|Q) - \epsilon)} 2^{-n(H(U_2|Q) - \epsilon)} 2^{-n(H(Y_2|Q) - \epsilon)}
= 2^{-n(I(U_1; U_2, Y_2|Q) - 3\epsilon)}.
\]

\( P(E_{l_1l_2}) \) can be upper bounded as

\[
P(E_{l_1l_2}) = \sum_{(u_1, u_2, y_2, q) \in A_n} P(U_1(l_1) = u_1 | q) P(U_2(l_2) = u_2 | q) P(Y_2 = y_2 | q)
\leq 2^n(H(U_1, U_2, Y_2|Q) + \epsilon) 2^{-n(H(U_1|Q) - \epsilon)} 2^{-n(H(U_2|Q) - \epsilon)} 2^{-n(H(Y_2|Q) - \epsilon)}
= 2^{-n(I(U_1, U_2, Y_2|Q) + I(U_1; U_2) - 4\epsilon)}.
\]

Substituting these in the probability of decoding error at \( \mathcal{R}_2 \), we have,

\[
P_e^{(n)}_{e, R_2} = \epsilon + 2^n(R_{21} + I(W; U_1|Q) + 4\epsilon) 2^{-n(I(U_1; U_2, Y_2|Q) - 3\epsilon)} + 2^n(R_{22} + I(W; U_2|Q) + 4\epsilon) 2^{-n(I(U_2; U_1, Y_2|Q) - 3\epsilon)} +
2^n(R_{21} + R_{22} + I(W; U_1|Q) + 4\epsilon + I(W; U_2|Q) + 4\epsilon) 2^{-n(I(U_1; U_2, Y_2|Q) + I(U_1; U_2) - 4\epsilon)}.
\]

\( P_e^{(n)}_{e, R_2} \) → 0 as \( n \to \infty \) if \( R_{21} \) and \( R_{22} \) satisfy the following constraints:

\[
R_{21} \leq I(U_1; U_2, Y_2|Q) - I(W; U_1|Q) \quad (188)
R_{22} \leq I(U_2; U_1, Y_2|Q) - I(W; U_2|Q) \quad (189)
R_{21} + R_{22} \leq I(U_1, U_2, Y_2|Q) + I(U_1; U_2|Q) - I(W; U_1|Q) - I(W; U_2|Q). \quad (190)
\]

5) Probability of error at decoder of \( \mathcal{R}_3 \): The two possible error events are: (1) The codewords transmitted are not jointly typical i.e., \( E_{t_1t_3}^c \) happens and/or (2) there exists some \( \left(t_1 \neq t_1, \hat{t}_3 \neq t_3\right) \) such that \( E_{t_1\hat{t}_3} \) happens. The probability of decoding error can be written as

\[
P_e^{(n)}_{e, R_3} = P\left(E_{t_1t_3}^c \cup \bigcup_{(t_1 \neq \hat{t}_3, \hat{t}_3 \neq t_3)} E_{t_1\hat{t}_3}\right) \quad (191)
\]

Applying union of events bound, \[191\] can be written as,
\[ P^{(n)}_{e,R_3} \leq P \left( E^c_{t_1 t_3} \right) + P \left( \bigcup_{i_1 \neq t_1, i_3 \neq t_3} E^c_{t_1 i_3} \right) \]
\[ \leq P \left( E^c_{t_1 t_3} \right) + \sum_{i_1 \neq t_1} P \left( E^c_{i_1 t_3} \right) + \sum_{i_3 \neq t_3} P \left( E^c_{t_1 i_3} \right) \]
\[ \leq P \left( E^c_{t_1 t_3} \right) + 2^n (R_{31} + I(W; V_1 | Q) + 4\epsilon) P \left( E^c_{t_1 t_3} \right) \]
\[ + 2^n (R_{33} + I(W; V_3 | Q) + 4\epsilon) P \left( E^c_{t_1 t_3} \right) + 2^n (R_{31} + I(W; V_1 | Q) + R_{33} + I(W; V_3 | Q) + 8\epsilon) P \left( E^c_{t_1 t_3} \right) \]

Let us now evaluate \( P(E^c_{t_1 t_3}) \), \( P(E^c_{t_1 i_3}) \) and \( P(E^c_{t_1 i_3}) \).

\( P(E^c_{t_1 t_3}) \) can be upper bounded as

\[ P(E^c_{t_1 t_3}) = \sum_{(v_1, v_3, y_3, q) \in A^n} P(v_1(t_1) = v_1 | q) P(v_3(t_3) = v_3, y_3 = y_3 | q) \]
\[ \leq 2^n (H(v_1, v_3, y_3 | q) + \epsilon) 2^{-n(H(V_1 | Q) - \epsilon)} 2^{-n(H(V_2, Y_3 | Q) - \epsilon)} \]
\[ = 2^{-n(I(V_1; V_3, Y_3 | Q) - 3\epsilon)}. \]

\( P(E^c_{t_1 i_3}) \) can be upper bounded as

\[ P(E^c_{t_1 i_3}) = \sum_{(v_1, v_3, y_3, q) \in A^n} P(v_3(t_3) = v_3 | q) P(v_1(t_1) = v_1, y_3 = y_3 | q) \]
\[ \leq 2^n (H(v_1, v_3, y_3 | q) + \epsilon) 2^{-n(H(V_5 | Q) - \epsilon)} 2^{-n(H(V_1, Y_3 | Q) - \epsilon)} \]
\[ = 2^{-n(I(V_1; V_3, Y_3 | Q) - 3\epsilon)}. \]

\( P(E^c_{t_1 i_3}) \) can be upper bounded as

\[ P(E^c_{t_1 i_3}) = \sum_{(v_1, v_3, y_3, q) \in A^n} P(v_1(t_1) = v_1 | q) P(v_3(t_3) = v_3 | q) P(y_3 = y_3 | q) \]
\[ \leq 2^n (H(v_1, v_3, y_3 | q) + \epsilon) 2^{-n(H(V_1 | Q) - \epsilon)} 2^{-n(H(V_5 | Q) - \epsilon)} 2^{-n(H(Y_3 | Q) - \epsilon)} \]
\[ = 2^{-n(I(V_1, V_3; Y_3 | Q) + I(V_1; V_3) - 4\epsilon)}. \]

Substituting these in the probability of decoding error at \( R_3 \), we have,

\[ P^{(n)}_{e,R_3} = \epsilon + 2^n (R_{31} + I(W; V_1 | Q) + 4\epsilon) 2^{-n(I(V_1; V_3, Y_3 | Q) - 3\epsilon)} \]
\[ 2^n (R_{33} + I(W; V_3 | Q) + 4\epsilon) 2^{-n(I(V_2; V_3, Y_3 | Q) - 3\epsilon)} \]
\[ 2^n (R_{31} + I(W; V_1 | Q) + R_{33} + I(W; V_3 | Q) + 8\epsilon) \]
\[ \times 2^{-n(I(V_1, V_3; Y_3 | Q) + I(V_1; V_3) - 4\epsilon)} \]

\( P^{(n)}_{e,R_3} \rightarrow 0 \) as \( n \rightarrow \infty \) if \( R_{31} \) and \( R_{33} \) satisfy the following constraints:

\[ R_{31} \leq I(V_1; V_3, Y_3 | Q) - I(W; V_1 | Q), \quad (192) \]
\[ R_{33} \leq I(V_3; V_1, Y_3 | Q) - I(W; V_3 | Q), \quad (193) \]
\[ R_{31} + R_{33} \leq I(V_1, V_3; Y_3 | Q) + I(V_1; V_3 | Q) - I(W; V_3 | Q) - I(W; V_1 | Q). \quad (194) \]
The achievable rate region follows:

\[ R_{11} \leq I(W; U_1, V_1, Y_1 | Q) \]  

\[ R_{11} + R_{21} \leq I(W, U_1; V_1, Y_1 | Q) \]  

\[ R_{11} + R_{31} \leq I(W, V_1; U_1, Y_1 | Q) \]  

\[ R_{11} + R_{21} + R_{31} \leq I(W, U_1, V_1; Y_1 | Q) + I(W, U_1; V_1 | Q) - I(W; V_1 | Q), \]  

\[ R_{21} \leq I(U_1; U_2, Y_2 | Q) - I(W; U_1 | Q) \]  

\[ R_{22} \leq I(U_2; U_1, Y_2 | Q) - I(W; U_2 | Q) \]  

\[ R_{21} + R_{22} \leq I(U_1, U_2; Y_2 | Q) + I(U_1; U_2 | Q) - I(W; U_1 | Q) - I(W; U_2 | Q), \]  

\[ R_{31} \leq I(V_1; V_3, Y_3 | Q) - I(W; V_1 | Q), \]  

\[ R_{33} \leq I(V_3; V_1, Y_3 | Q) - I(W; V_3 | Q), \]  

\[ R_{31} + R_{33} \leq I(V_1, V_3; Y_3 | Q) + I(V_1; V_3 | Q) - I(W; V_3 | Q) - I(W; V_1 | Q). \]  

This complete the proof of Theorem 4.1.

V. THE COGNITIVE GAUSSIAN CHANNEL MODEL

We introduce now the three-user continuous alphabet cognitive Gaussian channel and derive an achievable rate region for this channel. The achievable rate regions described for the discrete memoryless channels can be extended to the Gaussian channels by quantizing the channel inputs and outputs [32]. Let \( C_{G,cms}^1 \) denote the cognitive Gaussian channel with cumulative message sharing and \( C_{G,pms}^1 \) the cognitive Gaussian channel with primary-only message sharing (\( G \) for Gaussian, cms and pms are same as before); \( t = 1, 2 \). We show the extension for only one of the channel models - from \( C_{cms}^2 \) to \( C_{G,cms}^2 \).

The cognitive Gaussian channel is described by an input \( \tilde{X}_k \), a corresponding output \( \tilde{Y}_k \), and a random variable \( \tilde{Z}_k \) denoting noise at the receiver; \( k = 1, 2, 3 \). The channel is time-discrete unless otherwise specified. Following the maximum-entropy theorem [33], the input random variable \( \tilde{X}_k \); \( k = 1, 2, 3 \) is assumed to have a Gaussian distribution. The transmitted codeword \( \tilde{x}_k = (\tilde{x}_{k1}, ..., \tilde{x}_{kn}) \) satisfies the average power constraint given by

\[ \mathbb{E}\{||\tilde{x}_k||^2\} \leq \tilde{P}_k; \; k = 1, 2, 3, \]

where \( n \) is the length of the codeword and \( \mathbb{E}\{\cdot\} \) is the expectation operator. The zero-mean random variable \( \tilde{Z}_k \) is drawn i.i.d from a Gaussian distribution with variance \( \tilde{N}_k; \; k = 1, 2, 3 \), and is assumed to be independent of the signal \( \tilde{X}_k \). The Gaussian CR channel can be converted to standard from using invertible transformations [10],[34].

For the channel \( C_{G,cms}^2 \), we have \( W, U_1, U_2, V_1 \) and \( V_3 \) as the random variables (RV) which describe the sources at the transmitters. We also some consider some additional RVs - \( \tilde{W}, \tilde{U}_1, \tilde{U}_2, \tilde{V}_1 \) and \( \tilde{V}_3 \) - with the following statistics:
\[ \tilde{W} \sim \mathcal{N}(0, \lambda P_1), \]
\[ \tilde{U}_1 \sim \mathcal{N}(0, \tau P_2), \tilde{U}_2 \sim \mathcal{N}(0, \tilde{\tau} P_2), \text{ with } \tau + \tilde{\tau} = 1, \]
\[ \tilde{V}_1 \sim \mathcal{N}(0, \kappa P_3), \tilde{V}_3 \sim \mathcal{N}(0, \tilde{\kappa} P_3), \text{ with } \kappa + \tilde{\kappa} = 1. \]

Further,
\[ W = \tilde{W}, \]
\[ U_1 = \tilde{U}_1 + \alpha_1 X_1, U_2 = \tilde{U}_2 + \alpha_2 X_1, \]
\[ V_1 = \tilde{V}_1 + \alpha_3 X_1 + \beta_1 X_2, V_3 = \tilde{V}_3 + \alpha_4 X_1 + \beta_2 X_2, \]

where the input RV's \( X_1, X_2 \) and \( X_3 \) are given by \( X_1 = \tilde{W}, X_2 = \tilde{U}_1 + \tilde{U}_2 \) and \( X_3 = \tilde{V}_1 + \tilde{V}_3 \). Notice that \( \tilde{W}, \tilde{U}_1, \tilde{U}_2, \tilde{V}_1 \) and \( \tilde{V}_3 \) are mutually independent. Therefore, \( X_1 \sim \mathcal{N}(0, P_1), X_2 \sim \mathcal{N}(0, P_2) \) and \( X_3 \sim \mathcal{N}(0, P_3) \).

The values of \( \tau \) and \( \kappa \) are randomly selected from the interval \([0,1]\). The values of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1 \) and \( \beta_2 \) are repeatedly generated according to \( \mathcal{N}(0, 1) \). The channel outputs are
\[
Y_1 = X_1 + a_{12} X_2 + a_{13} X_3 + Z_1, \\
Y_2 = a_{21} X_1 + X_2 + a_{23} X_3 + Z_2, \\
Y_3 = a_{31} X_1 + a_{32} X_2 + X_3 + Z_3,
\]

where \( Z_1 \sim \mathcal{N}(0, Q_1), Z_2 \sim \mathcal{N}(0, Q_2) \) and \( Z_3 \sim \mathcal{N}(0, Q_3) \) are independent additive noise. Substituting for \( X_1, X_2 \) and \( X_3 \), we get,
\[
Y_1 = \tilde{W} + a_{12}(\tilde{U}_1 + \tilde{U}_2) + a_{13}(\tilde{V}_1 + \tilde{V}_3) + Z_1, \\
Y_2 = a_{21} \tilde{W} + (\tilde{U}_1 + \tilde{U}_2) + a_{23}(\tilde{V}_1 + \tilde{V}_3) + Z_2, \\
Y_3 = a_{31} \tilde{W} + a_{32}(\tilde{U}_1 + \tilde{U}_2) + \tilde{V}_1 + \tilde{V}_3 + Z_3,
\]

where the interference coefficients \( a_{12}, a_{13}, a_{21}, a_{23}, a_{31} \) and \( a_{32} \) are assumed to be real and globally known. The rate region \( \mathcal{R}_{cma}^2 \) for the channel \( C_{cma}^2 \) can be extended to its respective Gaussian channel model by evaluating the mutual information terms. To this end, we construct a covariance matrix and compute its entries. These entries will be used to compute the differential entropy terms which will further be used to evaluate the mutual information.

Let us first define a vector \( \Theta \) as follows:
\[ \Theta = (Y_1 \ Y_2 \ Y_3 \ W \ U_1 \ U_2 \ V_1 \ V_3). \]
The covariance matrix $\Sigma$ is given by $\Sigma = \mathbb{E}[\Theta^T \Theta]$, where $\mathbb{E}(\cdot)$ is the expectation operator.

\[
\Sigma = \begin{bmatrix}
\theta^1_1 & \theta^1_2 & \ldots & \theta^1_8 \\
\theta^2_1 & \theta^2_2 & \ldots & \theta^2_8 \\
\vdots & \ddots & \ddots & \vdots \\
\theta^8_1 & \theta^8_2 & \ldots & \theta^8_8 
\end{bmatrix},
\]

where $\theta^i_j$ represents the matrix-entry of the $i^{th}$ row and $j^{th}$ column; $i = 1 \ldots 8$ and $j = 1 \ldots 8$. The individual entries of the covariance matrix follow:

\[
\begin{align*}
\theta^1_1 &= \mathbb{E}(Y_1Y_1) = \lambda P_1 + a^2_{12} P_2 + a^2_{13} P_3 + Q_1 \\
\theta^1_2 &= \mathbb{E}(Y_1Y_2) = x \\
\theta^1_3 &= \mathbb{E}(Y_1Y_3) = x \\
\theta^1_4 &= \mathbb{E}(Y_1W) = \lambda P_1 \\
\theta^1_5 &= \mathbb{E}(Y_1U_1) = \alpha_1 \lambda P_1 + a_{12} \tau P_2 \\
\theta^1_6 &= \mathbb{E}(Y_1U_2) = \alpha_2 \lambda P_1 + a_{12} \tau P_2 \\
\theta^1_7 &= \mathbb{E}(Y_1V_1) = \alpha_3 \lambda P_1 + a_{12} \beta_1 P_2 + a_{13} \kappa P_3 \\
\theta^1_8 &= \mathbb{E}(Y_1V_3) = \alpha_4 \lambda P_1 + a_{12} \beta_2 P_2 + a_{13} \kappa P_3
\end{align*}
\]

\[
\begin{align*}
\theta^2_1 &= \mathbb{E}(Y_2Y_1) = x \\
\theta^2_2 &= \mathbb{E}(Y_2Y_2) = a^2_{21} \lambda P_1 + P_2 + a^2_{23} P_3 + Q_2 \\
\theta^2_3 &= \mathbb{E}(Y_2Y_3) = x \\
\theta^2_4 &= \mathbb{E}(Y_2W) = x \\
\theta^2_5 &= \mathbb{E}(Y_2U_1) = a_{21} \alpha_1 \lambda P_1 + \tau P_2 \\
\theta^2_6 &= \mathbb{E}(Y_2U_2) = a_{21} \alpha_2 \lambda P_1 + \tau P_2 \\
\theta^2_7 &= \mathbb{E}(Y_2V_1) = a_{21} \alpha_3 \lambda P_1 + \beta_1 P_2 + a_{23} \kappa P_3 \\
\theta^2_8 &= \mathbb{E}(Y_2V_3) = x
\end{align*}
\]

\[
\begin{align*}
\theta^3_1 &= \mathbb{E}(Y_3Y_1) = x \\
\theta^3_2 &= \mathbb{E}(Y_3Y_2) = x \\
\theta^3_3 &= \mathbb{E}(Y_3Y_3) = a^2_{31} \lambda P_1 + a^2_{32} P_2 + P_3 + Q_3 \\
\theta^3_4 &= \mathbb{E}(Y_3W) = a_{31} \lambda P_1 \\
\theta^3_5 &= \mathbb{E}(Y_3U_1) = a_{31} \alpha_1 \lambda P_1 + a_{32} \tau P_2
\end{align*}
\]
\[ \theta^3_6 = \mathbb{E}(Y_3 U_2) = a_{31} \alpha_2 \lambda P_1 + a_{32} \tau P_2 \]
\[ \theta^3_7 = \mathbb{E}(Y_3 V_1) = a_{31} \alpha_3 \lambda P_1 + a_{32} \beta_1 P_2 + \kappa P_3 \]
\[ \theta^3_8 = \mathbb{E}(Y_3 V_3) = a_{31} \alpha_4 \lambda P_1 + a_{32} \beta_2 P_2 + \bar{\kappa} P_3 \]

\[ \theta^4_1 = \mathbb{E}(W Y_1) = \lambda P_1 \]
\[ \theta^4_2 = \mathbb{E}(W Y_2) = x \]
\[ \theta^4_3 = \mathbb{E}(W Y_3) = a_{31} \lambda P_1 \]
\[ \theta^4_4 = \mathbb{E}(W W) = \lambda P_1 \]
\[ \theta^4_5 = \mathbb{E}(W U_1) = \alpha_1 \lambda P_1 \]
\[ \theta^4_6 = \mathbb{E}(W U_2) = \alpha_2 \lambda P_1 \]
\[ \theta^4_7 = \mathbb{E}(W V_1) = \alpha_3 \lambda P_1 \]
\[ \theta^4_8 = \mathbb{E}(W V_3) = \alpha_4 \lambda P_1 \]

\[ \theta^5_1 = \mathbb{E}(U_1 Y_1) = \alpha_1 \lambda P_1 + a_{12} \tau P_2 \]
\[ \theta^5_2 = \mathbb{E}(U_1 Y_2) = a_{21} \alpha_1 \lambda P_1 + \tau P_2 \]
\[ \theta^5_3 = \mathbb{E}(U_1 Y_3) = a_{31} \alpha_1 \lambda P_1 + a_{32} \tau P_2 \]
\[ \theta^5_4 = \mathbb{E}(U_1 W) = \alpha_1 \lambda P_1 \]
\[ \theta^5_5 = \mathbb{E}(U_1 U_1) = \alpha_1^2 \lambda P_1 + \tau P_2 \]
\[ \theta^5_6 = \mathbb{E}(U_1 U_2) = \alpha_1 \alpha_2 \lambda P_1 \]
\[ \theta^5_7 = \mathbb{E}(U_1 V_1) = \alpha_1 \alpha_3 \lambda P_1 + \beta_1 \tau P_2 \]
\[ \theta^5_8 = \mathbb{E}(U_1 V_3) = \alpha_1 \alpha_4 \lambda P_1 + \beta_2 \tau P_2 \]

\[ \theta^6_1 = \mathbb{E}(U_2 Y_1) = \alpha_2 \lambda P_1 + a_{12} \tau P_2 \]
\[ \theta^6_2 = \mathbb{E}(U_2 Y_2) = a_{21} \alpha_2 \lambda P_1 + \tau P_2 \]
\[ \theta^6_3 = \mathbb{E}(U_2 Y_3) = a_{31} \alpha_2 \lambda P_1 + a_{32} \tau P_2 \]
\[ \theta^6_4 = \mathbb{E}(U_2 W) = \alpha_2 \lambda P_1 \]
\[ \theta^6_5 = \mathbb{E}(U_2 U_1) = \alpha_1 \alpha_2 \lambda P_1 \]
\[ \theta^6_6 = \mathbb{E}(U_2 U_2) = \alpha_2^2 \lambda P_1 + \tau P_2 \]
\[ \theta^6_7 = \mathbb{E}(U_2 V_1) = \alpha_2 \alpha_3 \lambda P_1 + \beta_1 \tau P_2 \]
\[ \theta^6_8 = \mathbb{E}(U_2 V_3) = \alpha_2 \alpha_4 \lambda P_1 + \beta_2 \tau P_2 \]

\[ \theta^7_1 = \mathbb{E}(V_1 Y_1) = \alpha_3 \lambda P_1 + a_{12} \beta_1 P_2 + a_{13} \kappa P_3 \]
\[ \theta_2^7 = \mathbb{E}(V_1 Y_2) = a_{21} \alpha_3 \lambda P_1 + \beta_1 P_2 + a_{23} \kappa P_3 \]
\[ \theta_3^7 = \mathbb{E}(V_1 Y_3) = a_{31} \alpha_3 \lambda P_1 + a_{32} \beta_1 P_2 + \kappa P_3 \]
\[ \theta_4^7 = \mathbb{E}(V_1 W) = \alpha_3 \lambda P_1 \]
\[ \theta_5^7 = \mathbb{E}(V_1 U_1) = \alpha_1 \alpha_3 \lambda P_1 + \beta_1 \tau P_2 \]
\[ \theta_6^7 = \mathbb{E}(V_1 U_2) = \alpha_2 \alpha_3 \lambda P_1 + \beta_1 \tau P_2 \]
\[ \theta_7^7 = \mathbb{E}(V_1 V_1) = \alpha_3^2 \lambda P_1 + \beta_1^2 P_2 + \kappa P_3 \]
\[ \theta_8^7 = \mathbb{E}(V_1 V_3) = \alpha_3 \alpha_4 \lambda P_1 + \beta_1 \beta_2 P_2 \]

\[ \theta_8^8 = \mathbb{E}(V_3 Y_1) = \alpha_4 \lambda P_1 + a_{12} \beta_2 P_2 + a_{13} \kappa P_3 \]
\[ \theta_9^8 = \mathbb{E}(V_3 Y_2) = x \]
\[ \theta_9^9 = \mathbb{E}(V_3 Y_3) = a_{31} \alpha_4 \lambda P_1 + a_{32} \beta_2 P_2 + \kappa P_3 \]
\[ \theta_9^8 = \mathbb{E}(V_3 W) = \alpha_4 \lambda P_1 \]
\[ \theta_9^5 = \mathbb{E}(V_3 U_1) = \alpha_1 \alpha_4 \lambda P_1 + \beta_2 \tau P_2 \]
\[ \theta_9^6 = \mathbb{E}(V_3 U_2) = \alpha_2 \alpha_4 \lambda P_1 + \beta_2 \tau P_2 \]
\[ \theta_9^7 = \mathbb{E}(V_3 V_1) = \alpha_3 \alpha_4 \lambda P_1 + \beta_1 \beta_2 P_2 \]
\[ \theta_9^8 = \mathbb{E}(V_3 V_3) = \alpha_4^2 P_1 + \beta_1^2 P_2 + \kappa P_3 \]

With \( x \) representing 'don't care' condition.

Let \( \Gamma(x) = \frac{\log_2(x)}{2} \) and \( \varepsilon = \frac{\log_2(2\pi e)}{2} \). We will express the differential entropy in terms of \( \Gamma \) and \( \varepsilon \).

To compute \( R_{11}, R_{12}, R_{21}, R_{31}, R_{11} + R_{21} + R_{31} \):

\[ h(W|Q) = \varepsilon + \Gamma(\theta_1^4), h(U_1|Q) = \varepsilon + \Gamma(\theta_2^5), h(V_1|Q) = \varepsilon + \Gamma(\theta_3^7), h(Y_1|Q) = \varepsilon + \Gamma(\theta_4^1), \]
\[ h(W, U_1|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_1^4 & \theta_1^5 \\ \theta_2^5 & \theta_5^5 \end{pmatrix}, h(W, V_1|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_1^4 & \theta_1^5 \\ \theta_2^7 & \theta_7^7 \end{pmatrix}, \]
\[ h(U_1, Y_1|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_5^5 & \theta_5^5 \end{pmatrix}, h(V_1, Y_1|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_7^7 & \theta_7^7 \end{pmatrix}, \]
\[ h(U_1, V_1, Y_1|Q) = 3\varepsilon + \Gamma \begin{pmatrix} \theta_1^6 & \theta_1^6 \\ \theta_5^5 & \theta_5^5 \end{pmatrix}, h(W, U_1, U_2|Q) = 3\varepsilon + \Gamma \begin{pmatrix} \theta_1^6 & \theta_1^6 \\ \theta_5^5 & \theta_5^5 \end{pmatrix}, \]
\[ h(W, U_1, V_1|Q) = 3\varepsilon + \Gamma \begin{pmatrix} \theta_1^7 & \theta_1^7 \\ \theta_7^7 & \theta_7^7 \end{pmatrix}, h(W, U_1, V_1|Q) = 3\varepsilon + \Gamma \begin{pmatrix} \theta_1^7 & \theta_1^7 \\ \theta_7^7 & \theta_7^7 \end{pmatrix}. \]
The mutual information terms are computed as follows:

\[ h(W, U_1, V_1, Y_1|Q) = 4\varepsilon + \Gamma \begin{pmatrix} \theta_1^1 & \theta_4^1 & \theta_5^1 & \theta_7^1 \\ \theta_4^1 & \theta_4^4 & \theta_5^4 & \theta_5^2 \\ \theta_5^5 & \theta_4^5 & \theta_5^5 & \theta_2^2 \\ \theta_7^1 & \theta_4^7 & \theta_7^7 & \theta_7^7 \end{pmatrix}, h(W, U_1, U_2, V_1|Q) = 4\varepsilon + \Gamma \begin{pmatrix} \theta_4^1 & \theta_4^4 & \theta_6^4 & \theta_7^4 \\ \theta_5^5 & \theta_5^5 & \theta_6^6 & \theta_7^5 \\ \theta_4^6 & \theta_6^6 & \theta_6^6 & \theta_6^6 \\ \theta_7^1 & \theta_7^7 & \theta_7^7 & \theta_7^7 \end{pmatrix} \]

To compute \( R_{21}, R_{22} \) and \( R_{21} + R_{22} \):

\[ h(U_2|Q) = \varepsilon + \Gamma(\theta_6^5), h(Y_2|Q) = \varepsilon + \Gamma(\theta_2^5), \]

\[ h(U_2, Y_2|Q) = 2\varepsilon + \Gamma\left( \begin{array}{cc} \theta_2^2 & \theta_6^2 \\ \theta_5^2 & \theta_6^6 \end{array} \right), h(U_1, Y_2|Q) = 2\varepsilon + \Gamma\left( \begin{array}{cc} \theta_2^2 & \theta_2^5 \\ \theta_5^2 & \theta_5^5 \end{array} \right), \]

\[ h(W, U_2|Q) = 2\varepsilon + \Gamma\left( \begin{array}{cc} \theta_4^1 & \theta_6^1 \\ \theta_4^5 & \theta_6^5 \end{array} \right), h(U_1, U_2|Q) = 2\varepsilon + \Gamma\left( \begin{array}{cc} \theta_5^5 & \theta_6^5 \\ \theta_6^5 & \theta_6^6 \end{array} \right), \]

\[ h(U_1, U_2, Y_2|Q) = 3\varepsilon + \Gamma\left( \begin{array}{ccc} \theta_2^2 & \theta_2^5 & \theta_2^5 \\ \theta_5^5 & \theta_5^5 & \theta_5^6 \\ \theta_6^5 & \theta_6^6 & \theta_6^6 \end{array} \right) \]

The mutual information terms are computed as follows:

\[ I(U_1; U_2, Y_2|Q) = h(U_1|Q) + h(U_2, Y_2|Q) - h(U_1, U_2, Y_2|Q), \]

\[ I(W; U_1|Q) = h(W|Q) + h(U_1|Q) - h(W, U_1|Q), \]

\[ I(U_2; U_1, Y_2|Q) = h(U_2|Q) + h(U_1, Y_2|Q) - h(U_1, U_2, Y_2|Q), \]

\[ I(W; U_2|Q) = h(W|Q) + h(U_2|Q) - h(W, U_2|Q), \]

\[ I(U_1, U_2; Y_2|Q) = h(U_1, U_2|Q) + h(Y_2|Q) - h(U_1, U_2, Y_2|Q), \]

\[ I(U_1; U_2|Q) = h(U_1|Q) + h(U_2|Q) - h(U_1, U_2|Q). \]

To compute \( R_{30}, R_{33} \) and \( R_{30} + R_{33} \):
The mutual information terms are computed as follows:

\[
\begin{align*}
    h(V_3|Q) &= \varepsilon + \Gamma(\theta_3^3), h(Y_3|Q) = \varepsilon + \Gamma(\theta_3^3), \\
    h(V_3, Y_3|Q) &= 2\varepsilon + \Gamma \begin{pmatrix} \theta_3^3 & \theta_3^3 \\ \theta_3^3 & \theta_3^3 \end{pmatrix}, h(V_1, Y_3|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_3^3 & \theta_3^3 \\ \theta_3^3 & \theta_3^3 \end{pmatrix}, \\
    h(V_1, V_3|Q) &= 2\varepsilon + \Gamma \begin{pmatrix} \theta_3^7 & \theta_3^7 \\ \theta_3^7 & \theta_3^7 \end{pmatrix}, \\
    h(V_1, V_3, Y_3|Q) &= 3\varepsilon + \Gamma \begin{pmatrix} \theta_3^3 & \theta_3^3 & \theta_3^3 \\ \theta_3^3 & \theta_3^3 & \theta_3^3 \\ \theta_3^3 & \theta_3^3 & \theta_3^3 \end{pmatrix}.
\end{align*}
\]

The mutual information terms are computed as follows:

\[
\begin{align*}
    I(V_1; V_3, Y_3|Q) &= h(V_1|Q) + h(V_3, Y_3|Q) - h(V_1, V_3, Y_3|Q), \\
    I(V_3; V_1, Y_3|Q) &= h(V_3|Q) + h(V_1, Y_3|Q) - h(V_1, V_3, Y_3|Q), \\
    I(V_1; V_3|Q) &= h(V_1|Q) + h(V_3|Q) - h(V_1, V_3|Q), \\
    I(V_1; V_3, V_1|Q) &= h(W, U_1, U_2, V_1|Q) + h(V_1|Q) - h(W, U_1, U_2, V_1|Q), \\
    I(W, U_1, U_2, V_1|Q) &= h(W, U_1, U_2, V_1|Q) + h(V_3|Q) - h(W, U_1, U_2, V_1|Q).
\end{align*}
\]

**Theorem 5.1:** Let \( \Upsilon = (\lambda, \tau, \kappa, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2) \). For a fixed \( \Upsilon \), let \( G_{c_m}^2(\Upsilon) \) be achievable. The rate region \( \mathfrak{G}_{c_m}^2 \) is achievable for the Gaussian channel \( C_{G,c_m}^2 \) with

\[
\mathfrak{G}_{c_m}^2 = \bigcup_{\Upsilon} G_{c_m}^2(\Upsilon).
\]

The same procedure follows for the channels \( C_{p_m}^1, C_{p_m}^1 \) and \( C_{p_m}^2 \). Without stating the theorem, we outline the Gaussian channel equivalent for \( C_{p_m}^2 \).

For the channel \( C_{G,p_m}^2 \), we have \( W, U_1, U_2, V_1 \) and \( V_3 \) as the random variables (RV) which describe the sources at the transmitters. We also some consider some additional RVs - \( \bar{W}, \bar{U}_1, \bar{U}_2, \bar{V}_1 \) and \( \bar{V}_3 \) - with the following statistics:

- \( \bar{W} \sim \mathcal{N}(0, \lambda P_1) \),
- \( \bar{U}_1 \sim \mathcal{N}(0, \tau P_2) \), \( \bar{U}_2 \sim \mathcal{N}(0, \bar{\tau} P_2) \), with \( \tau + \bar{\tau} = 1 \),
- \( \bar{V}_1 \sim \mathcal{N}(0, \kappa P_3) \), \( \bar{V}_3 \sim \mathcal{N}(0, \bar{\kappa} P_3) \), with \( \kappa + \bar{\kappa} = 1 \).

Further,

- \( W = \bar{W} \),
\( U_1 = \tilde{U}_1 + \alpha_1 X_1, \) \( U_2 = \tilde{U}_2 + \alpha_2 X_1, \)
\( V_1 = \tilde{V}_1 + \alpha_3 X_1, \) \( V_3 = \tilde{V}_3 + \alpha_4 X_1, \)

where the input RV's \( X_1, X_2 \) and \( X_3 \) are given by \( X_1 = \tilde{W}, X_2 = \tilde{U}_1 + \tilde{U}_2 \) and \( X_3 = \tilde{V}_1 + \tilde{V}_3. \) Notice that \( \tilde{W}, \) \( \tilde{U}_1, \tilde{U}_2, \tilde{V}_1 \) and \( \tilde{V}_3 \) are mutually independent. Therefore, \( X_1 \sim \mathcal{N}(0, P_1), X_2 \sim \mathcal{N}(0, P_2) \) and \( X_3 \sim \mathcal{N}(0, P_3). \)

The values of \( \tau \) and \( \kappa \) are randomly selected from the interval \([0,1].\) The values of \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are repeatedly generated according to \( \mathcal{N}(0,1). \) The channel outputs are:

\[
\begin{align*}
Y_1 &= X_1 + a_{12} X_2 + a_{13} X_3 + Z_1, \\
Y_2 &= a_{21} X_1 + X_2 + a_{23} X_3 + Z_2, \\
Y_3 &= a_{31} X_1 + a_{32} X_2 + X_3 + Z_3,
\end{align*}
\]

where \( Z_1 \sim \mathcal{N}(0, Q_1), Z_2 \sim \mathcal{N}(0, Q_2) \) and \( Z_3 \sim \mathcal{N}(0, Q_3) \) are independent additive noise. Substituting for \( X_1, X_2 \) and \( X_3, \) we get,

\[
\begin{align*}
Y_1 &= \tilde{W} + a_{12}(\tilde{U}_1 + \tilde{U}_2) + a_{13}(\tilde{V}_1 + \tilde{V}_3) + Z_1, \\
Y_2 &= a_{21} \tilde{W} + (\tilde{U}_1 + \tilde{U}_2) + a_{23}(\tilde{V}_1 + \tilde{V}_3) + Z_2, \\
Y_3 &= a_{31} \tilde{W} + a_{32}(\tilde{U}_1 + \tilde{U}_2) + \tilde{V}_1 + \tilde{V}_3 + Z_3,
\end{align*}
\]

where the interference coefficients \( a_{12}, a_{13}, a_{21}, a_{23}, a_{31} \) and \( a_{32} \) are assumed to be globally known.

We now construct a covariance matrix and compute its entries. This will be used to compute the differential entropy terms which will further be used to compute the mutual information terms. Let us first define a vector \( \Theta \) as follows:

\[
\Theta = (Y_1 \ Y_2 \ Y_3 \ W \ U_1 \ U_2 \ V_1 \ V_3)
\]

The covariance matrix \( \Sigma \) is given by \( \Sigma = E[\Theta^T \Theta], \) where \( E(.) \) is the expectation operator.

\[
\Sigma = \begin{bmatrix}
    \theta_1 & \theta_2 & \cdots & \theta_8 \\
    \theta_2 & \theta_2 & \cdots & \theta_8 \\
    \vdots & \ddots & \ddots & \vdots \\
    \vdots & \ddots & \ddots & \vdots \\
    \theta_8 & \theta_8 & \cdots & \theta_8
\end{bmatrix},
\]

where \( \theta^i_j \) represents the matrix-entry of the \( i^{th} \) row and \( j^{th} \) column; \( i = 1 \ldots 8 \) and \( j = 1 \ldots 8. \) The individual entries of the covariance matrix follow:

\[
\begin{align*}
\theta_1 &= E(Y_1 Y_1) = P_1 + a_{12}^2 P_2 + a_{13}^2 P_3 + Q_1 \\
\theta_2 &= E(Y_1 Y_2) = x
\end{align*}
\]
\[
\begin{align*}
\theta_3^1 &= \mathbb{E}(Y_1 Y_3) = x \\
\theta_4^1 &= \mathbb{E}(Y_1 W) = \lambda P_1 \\
\theta_5^1 &= \mathbb{E}(Y_1 U_1) = \alpha_1 \lambda P_1 + a_{12} \tau P_2 \\
\theta_6^1 &= \mathbb{E}(Y_1 U_2) = \alpha_2 \lambda P_1 + a_{12} \bar{\tau} P_2 \\
\theta_7^1 &= \mathbb{E}(Y_1 V_1) = \alpha_3 \lambda P_1 + a_{13} \kappa P_3 \\
\theta_8^1 &= \mathbb{E}(Y_1 V_3) = \alpha_4 \lambda P_1 + a_{13} \bar{\kappa} P_3 \\
\theta_2^2 &= \mathbb{E}(Y_2 Y_1) = x \\
\theta_3^2 &= \mathbb{E}(Y_2 Y_2) = a_{21}^2 \lambda P_1 + P_2 + a_{23}^2 P_3 + Q_2 \\
\theta_4^2 &= \mathbb{E}(Y_2 Y_3) = x \\
\theta_5^2 &= \mathbb{E}(Y_2 W) = a_{21} \lambda P_1 \\
\theta_6^2 &= \mathbb{E}(Y_2 U_1) = a_{21} \alpha_1 \lambda P_1 + \tau P_2 \\
\theta_7^2 &= \mathbb{E}(Y_2 U_2) = a_{21} \alpha_2 \lambda P_1 + \bar{\tau} P_2 \\
\theta_8^2 &= \mathbb{E}(Y_2 V_1) = a_{21} \alpha_3 \lambda P_1 + a_{23} \kappa P_3 \\
\theta_8^3 &= \mathbb{E}(Y_2 V_3) = x \\
\theta_3^3 &= \mathbb{E}(Y_3 Y_1) = x \\
\theta_4^3 &= \mathbb{E}(Y_3 Y_2) = x \\
\theta_5^3 &= \mathbb{E}(Y_3 Y_3) = a_{31}^2 \lambda P_1 + a_{32}^2 P_2 + P_3 + Q_3 \\
\theta_6^3 &= \mathbb{E}(Y_3 W) = a_{31} \lambda P_1 \\
\theta_7^3 &= \mathbb{E}(Y_3 U_1) = a_{31} \alpha_1 \lambda P_1 + a_{32} \tau P_2 \\
\theta_8^3 &= \mathbb{E}(Y_3 U_2) = a_{31} \alpha_2 \lambda P_1 + a_{32} \bar{\tau} P_2 \\
\theta_9^3 &= \mathbb{E}(Y_3 V_1) = a_{31} \alpha_3 \lambda P_1 + \kappa P_3 \\
\theta_8^3 &= \mathbb{E}(Y_3 V_3) = a_{31} \alpha_4 \lambda P_1 + \bar{\kappa} P_3 \\
\theta_4^4 &= \mathbb{E}(W Y_1) = \lambda P_1 \\
\theta_5^4 &= \mathbb{E}(W Y_2) = a_{21} \lambda P_1 \\
\theta_6^4 &= \mathbb{E}(W Y_3) = a_{31} \lambda P_1 \\
\theta_7^4 &= \mathbb{E}(W W) = \lambda P_1 \\
\theta_8^4 &= \mathbb{E}(W U_1) = \alpha_1 \lambda P_1 \\
\theta_9^4 &= \mathbb{E}(W U_2) = \alpha_2 \lambda P_1 \\
\theta_9^4 &= \mathbb{E}(W V_1) = \alpha_3 \lambda P_1
\end{align*}
\]
\( \theta_8 = \mathbb{E}(WV_3) = \alpha_4 \lambda P_1 \)

\( \theta_5 = \mathbb{E}(U_1 Y_1) = \alpha_1 \lambda P_1 + a_{12} \tau P_2 \)
\( \theta_2 = \mathbb{E}(U_1 Y_2) = a_{21} \alpha_1 \lambda P_1 + \tau P_2 \)
\( \theta_3 = \mathbb{E}(U_1 Y_3) = a_{31} \alpha_1 \lambda P_1 + a_{32} \tau P_2 \)
\( \theta_4 = \mathbb{E}(U_1 W) = \alpha_1 \lambda P_1 \)
\( \theta_5 = \mathbb{E}(U_1 U_1) = \alpha_1^2 \lambda P_1 + \tau P_2 \)
\( \theta_6 = \mathbb{E}(U_1 U_2) = \alpha_1 \alpha_2 \lambda P_1 \)
\( \theta_7 = \mathbb{E}(U_1 V_1) = \alpha_1 \alpha_3 \lambda P_1 \)
\( \theta_8 = \mathbb{E}(U_1 V_3) = \alpha_1 \alpha_4 \lambda P_1 \)

\( \theta_5 = \mathbb{E}(U_2 Y_1) = \alpha_2 \lambda P_1 + a_{12} \tau P_2 \)
\( \theta_2 = \mathbb{E}(U_2 Y_2) = a_{21} \alpha_2 \lambda P_1 + \tau P_2 \)
\( \theta_3 = \mathbb{E}(U_2 Y_3) = a_{31} \alpha_2 \lambda P_1 + a_{32} \tau P_2 \)
\( \theta_4 = \mathbb{E}(U_2 W) = \alpha_2 \lambda P_1 \)
\( \theta_5 = \mathbb{E}(U_2 U_1) = \alpha_1 \alpha_2 \lambda P_1 \)
\( \theta_6 = \mathbb{E}(U_2 U_2) = \alpha_2^2 \lambda P_1 + \tau P_2 \)
\( \theta_7 = \mathbb{E}(U_2 V_1) = \alpha_1 \alpha_3 \lambda P_1 \)
\( \theta_8 = \mathbb{E}(U_2 V_3) = \alpha_2 \alpha_4 \lambda P_1 \)

\( \theta_7 = \mathbb{E}(V_1 Y_1) = \alpha_3 \lambda P_1 + a_{13} \kappa P_3 \)
\( \theta_2 = \mathbb{E}(V_1 Y_2) = a_{21} \alpha_3 \lambda P_1 + a_{23} \kappa P_3 \)
\( \theta_3 = \mathbb{E}(V_1 Y_3) = a_{31} \alpha_3 \lambda P_1 + \kappa P_3 \)
\( \theta_4 = \mathbb{E}(V_1 W) = \alpha_3 \lambda P_1 \)
\( \theta_5 = \mathbb{E}(V_1 U_1) = \alpha_1 \alpha_3 \lambda P_1 \)
\( \theta_6 = \mathbb{E}(V_1 U_2) = \alpha_1 \alpha_3 \lambda P_1 \)
\( \theta_7 = \mathbb{E}(V_1 V_1) = \alpha_3^2 \lambda P_1 + \kappa P_3 \)
\( \theta_8 = \mathbb{E}(V_1 V_3) = \alpha_3 \alpha_4 \lambda P_1 \)

\( \theta_8 = \mathbb{E}(V_3 Y_1) = \alpha_4 \lambda P_1 + a_{13} \kappa P_3 \)
\( \theta_8 = \mathbb{E}(V_3 Y_2) = \times \)
\( \theta_8 = \mathbb{E}(V_3 Y_3) = a_{31} \alpha_4 \lambda P_1 + \kappa P_3 \)
\(\theta^8_4 = \mathbb{E}(V_3W) = \alpha_4 \lambda P_1\)
\(\theta^8_5 = \mathbb{E}(V_3U_1) = \alpha_1 \alpha_4 \lambda P_1\)
\(\theta^8_6 = \mathbb{E}(V_3U_2) = \alpha_2 \alpha_4 \lambda P_1\)
\(\theta^8_7 = \mathbb{E}(V_3V_1) = \alpha_3 \alpha_4 \lambda P_1\)
\(\theta^8_8 = \mathbb{E}(V_3V_3) = \alpha_2^2 \lambda P_1 + \kappa P_3\).

Let \(\Gamma(x) = \frac{\log_2(x)}{2}\) and \(\varepsilon = \frac{\log_2(2\pi e)}{2}\). We will express the differential entropy in terms of \(\Gamma\) and \(\varepsilon\).

To compute \(R_{11}, R_{11} + R_{21}, R_{11} + R_{31}, R_{11} + R_{21} + R_{31}\):

\[
h(W|Q) = \varepsilon + \Gamma(\theta^4_1), h(U_1|Q) = \varepsilon + \Gamma(\theta^5_0), h(V_1|Q) = \varepsilon + \Gamma(\theta^7_1), h(Y_1|Q) = \varepsilon + \Gamma(\theta^1_1),
\]
\[
h(W, U_1|Q) = 2\varepsilon + \Gamma \left( \begin{array}{cc}
    \theta^4_1 & \theta^4_3 \\
    \theta^5_0 & \theta^5_3 \\
    \theta^7_0 & \theta^7_3 \\
\end{array} \right), h(W, V_1|Q) = 2\varepsilon + \Gamma \left( \begin{array}{cc}
    \theta^4_1 & \theta^4_4 \\
    \theta^5_0 & \theta^5_7 \\
    \theta^7_0 & \theta^7_7 \\
\end{array} \right),
\]
\[
h(U_1, Y_1|Q) = 2\varepsilon + \Gamma \left( \begin{array}{cc}
    \theta^4_1 & \theta^4_3 \\
    \theta^5_0 & \theta^5_3 \\
    \theta^7_0 & \theta^7_3 \\
\end{array} \right), h(V_1, Y_1|Q) = 2\varepsilon + \Gamma \left( \begin{array}{cc}
    \theta^4_1 & \theta^4_4 \\
    \theta^5_0 & \theta^5_7 \\
    \theta^7_0 & \theta^7_7 \\
\end{array} \right),
\]
\[
h(U_1, V_1, Y_1|Q) = 3\varepsilon + \Gamma \left( \begin{array}{ccc}
    \theta^4_1 & \theta^4_3 & \theta^4_7 \\
    \theta^5_0 & \theta^5_3 & \theta^5_7 \\
    \theta^7_0 & \theta^7_3 & \theta^7_7 \\
\end{array} \right), h(W, U_1, U_2|Q) = 3\varepsilon + \Gamma \left( \begin{array}{ccc}
    \theta^4_1 & \theta^4_3 & \theta^4_6 \\
    \theta^5_0 & \theta^5_3 & \theta^5_6 \\
    \theta^7_0 & \theta^7_3 & \theta^7_6 \\
\end{array} \right),
\]
\[
h(U_1, V_1, V_1|Q) = 3\varepsilon + \Gamma \left( \begin{array}{ccc}
    \theta^4_1 & \theta^4_3 & \theta^4_7 \\
    \theta^5_0 & \theta^5_3 & \theta^5_7 \\
    \theta^7_0 & \theta^7_3 & \theta^7_7 \\
\end{array} \right), h(W, U_1, V_1|Q) = 3\varepsilon + \Gamma \left( \begin{array}{ccc}
    \theta^4_1 & \theta^4_3 & \theta^4_4 \\
    \theta^5_0 & \theta^5_3 & \theta^5_7 \\
    \theta^7_0 & \theta^7_3 & \theta^7_7 \\
\end{array} \right),
\]
\[
h(W, U_1, V_1, Y_1|Q) = 4\varepsilon + \Gamma \left( \begin{array}{cccc}
    \theta^4_1 & \theta^4_3 & \theta^4_7 & \theta^4_4 \\
    \theta^5_0 & \theta^5_3 & \theta^5_7 & \theta^5_7 \\
    \theta^7_0 & \theta^7_3 & \theta^7_7 & \theta^7_7 \\
\end{array} \right), h(W, U_1, V_2, V_1|Q) = 4\varepsilon + \Gamma \left( \begin{array}{cccc}
    \theta^4_1 & \theta^4_3 & \theta^4_6 & \theta^4_4 \\
    \theta^5_0 & \theta^5_3 & \theta^5_6 & \theta^5_7 \\
    \theta^7_0 & \theta^7_3 & \theta^7_6 & \theta^7_7 \\
\end{array} \right),
\]
\[
h(W, U_1, V_1, Y_1|Q) = 4\varepsilon + \Gamma \left( \begin{array}{cccc}
    \theta^4_1 & \theta^4_3 & \theta^4_7 & \theta^4_4 \\
    \theta^5_0 & \theta^5_3 & \theta^5_7 & \theta^5_7 \\
    \theta^7_0 & \theta^7_3 & \theta^7_7 & \theta^7_7 \\
\end{array} \right)
\]

The mutual information terms are computed as follows:

\(I(W; U_1, V_1, Y_1|Q) = h(W|Q) + h(U_1, V_1, Y_1|Q) - h(W, U_1, V_1, Y_1|Q)\).

\(I(W, U_1; V_1, Y_1|Q) = h(W, U_1|Q) + h(V_1, Y_1|Q) - h(W, U_1, V_1, Y_1|Q)\).

\(I(W, V_1; U_1, Y_1|Q) = h(W, V_1|Q) + h(U_1, Y_1|Q) - h(W, U_1, V_1, Y_1|Q)\).

\(I(W; V_1|Q) = h(W|Q) + h(V_1|Q) - h(W, V_1|Q)\).

\(I(W, U_1, U_2; V_1|Q) = h(W, U_1, U_2|Q) + h(V_1|Q) - h(W, U_1, U_2, V_1|Q)\).

\(I(W, U_1, V_1; Y_1|Q) = h(W, U_1, V_1|Q) + h(Y_1|Q) - h(W, U_1, V_1, Y_1|Q)\).

\(I(W, U_1; V_1|Q) = h(W, U_1|Q) + h(V_1|Q) - h(W, U_1, V_1|Q)\).
To compute $R_{21}$, $R_{22}$ and $R_{21} + R_{22}$:

$$h(U_2|Q) = \varepsilon + \Gamma(\theta_6^2), \quad h(Y_2|Q) = \varepsilon + \Gamma(\theta_7^2),$$

$$h(U_2, Y_2|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_2^2 & \theta_6^2 \\ \theta_5^2 & \theta_6^2 \end{pmatrix}, \quad h(U_1, Y_2|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_2^3 & \theta_5^3 \\ \theta_4^3 & \theta_6^3 \end{pmatrix},$$

$$h(W, U_2|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_2^4 & \theta_6^4 \\ \theta_4^4 & \theta_6^4 \end{pmatrix}, \quad h(U_1, U_2|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_2^6 & \theta_5^6 \\ \theta_5^6 & \theta_6^6 \end{pmatrix},$$

$$h(W, U_1, V_1|Q) = 3\varepsilon + \Gamma \begin{pmatrix} \theta_2^7 & \theta_5^7 & \theta_6^7 \\ \theta_2^8 & \theta_5^8 & \theta_6^8 \\ \theta_2^9 & \theta_5^9 & \theta_6^9 \end{pmatrix}.$$

The mutual information terms are computed as follows:

$$I(U_1; U_2, Y_2|Q) = h(U_1|Q) + h(U_2, Y_2|Q) - h(U_1, U_2, Y_2|Q),$$

$$I(W; U_1|Q) = h(W|Q) + h(U_1|Q) - h(W, U_1|Q),$$

$$I(U_2; U_1, Y_2|Q) = h(U_2|Q) + h(U_1, Y_2|Q) - h(U_1, U_2, Y_2|Q),$$

$$I(W; U_2|Q) = h(W|Q) + h(U_2|Q) - h(W, U_2|Q),$$

$$I(U_1, U_2; Y_2|Q) = h(U_1, U_2|Q) + h(Y_2|Q) - h(U_1, U_2, Y_2|Q),$$

$$I(U_1; U_2|Q) = h(U_1|Q) + h(U_2|Q) - h(U_1, U_2|Q).$$

To compute $R_{30}$, $R_{33}$ and $R_{30} + R_{33}$:

$$h(V_3|Q) = \varepsilon + \Gamma(\theta_8^3), \quad h(Y_3|Q) = \varepsilon + \Gamma(\theta_3^8),$$

$$h(V_3, Y_3|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_3^3 & \theta_8^3 \\ \theta_5^3 & \theta_8^3 \end{pmatrix}, \quad h(V_1, Y_3|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_3^4 & \theta_7^4 \\ \theta_3^7 & \theta_7^4 \end{pmatrix},$$

$$h(V_1, V_3|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_7^2 & \theta_8^7 \\ \theta_7^8 & \theta_8^7 \end{pmatrix},$$

$$h(W, V_1|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_4^3 & \theta_7^4 \\ \theta_4^7 & \theta_7^4 \end{pmatrix},$$

$$h(W, V_3|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_4^1 & \theta_7^4 \\ \theta_4^7 & \theta_7^4 \end{pmatrix},$$

$$h(W, V_3|Q) = 2\varepsilon + \Gamma \begin{pmatrix} \theta_4^8 & \theta_7^8 \\ \theta_4^8 & \theta_7^8 \end{pmatrix}.$$
\[
h(V_1, V_3, Y_3|Q) = 3\varepsilon + \Gamma \begin{pmatrix} \theta_3^3 & \theta_3^3 & \theta_3^3 \\ \theta_7^3 & \theta_7^7 & \theta_7^3 \\ \theta_8^3 & \theta_8^8 & \theta_8^3 \end{pmatrix}
\]

The mutual information terms are computed as follows:

\[
I(V_1; V_3, Y_3|Q) = h(V_1|Q) + h(V_3, Y_3|Q) - h(V_1, V_3, Y_3|Q),
\]

\[
I(V_1; V_3, Y_3|Q) = h(V_3|Q) + h(V_1, Y_3|Q) - h(V_1, V_3, Y_3|Q),
\]

\[
I(V_1; V_3, Y_3|Q) = h(V_1, V_3|Q) + h(Y_3|Q) - h(V_1, V_3, Y_3|Q),
\]

\[
I(V_1; V_3|Q) = h(V_1|Q) + h(V_3|Q) - h(V_1, V_3|Q),
\]

\[
I(W; V_1|Q) = h(W|Q) + h(V_1|Q) - h(W, V_1|Q),
\]

\[
I(W; V_3|Q) = h(W|Q) + h(V_3|Q) - h(W, V_3|Q).
\]

VI. SIMULATION RESULTS AND DISCUSSION

A. Setup

We consider the 2-user Gaussian cognitive channel with message sharing and the 3-user Gaussian cognitive channels with CMS and PMS for the simulations. For simplicity, we assume that the input distributions are Gaussian and generate the source and channel symbols as described in the previous page. Also, for ease of generating the plots and presenting the results, we focus on two cases: the case where the primary can decode the public part of the messages from CR_1 and CR_2 but not vice versa, and the case where none of the receivers can decode any part of the other transmitters’ messages.

- The interference coefficients \(a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.55\)
- The values of \(\tau\) and \(\kappa\) are assumed to be randomly selected from the interval \([0,1]\).
- The values of \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1\) and \(\beta_2\) are repeatedly generated according to \(\mathcal{N}(0,1)\).
- The noise variances \(Q_1 = Q_2 = Q_3 = 1\).
- The transmit powers \(P_1 = P_2 = P_3 = 10\)dB unless otherwise specified.

B. Details of simulations

We consider a 3-user Gaussian cognitive channel with CMS and PMS for the simulations. For simplicity we assume that the input distributions are Gaussian and generate the source and channel symbols as described in the previous page.

- The interference coefficients \(a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0.55\)
- The values of \(\tau\) and \(\kappa\) are assumed to be randomly selected from the interval \([0,1]\).
- The values of \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1\) and \(\beta_2\) are repeatedly generated according to \(\mathcal{N}(0,1)\).
- The noise variances \(Q_1 = Q_2 = Q_3 = 1\).
• The transmit powers $P_1 = P_2 = P_3 = 6\text{dB}$ and $10\text{dB}$ as specified.

C. Simulation results and discussion

We now present the simulation results and draw several interesting observations.

1) Figure 2 shows the plot of rate regions for 2-user interference channels with various decoding capability combinations. In (a), we consider the case where both receivers cannot decode any message from the non-pairing transmitter. In (b), receiver of the primary is allowed to decode the public part of the secondary transmitter’s message because of which it performs interference cancelation to improve its rate. In (c), both receivers (primary and secondary) are allowed to decode the public part of the non-pairing transmitter’s message so that both can perform successive interference cancelation. Therefore, it achieves the biggest rate region. Note that (c) is the Han-Kobayashi achievable rate region for the interference channel [6].

2) In Fig.3, we plot the achievable rate regions for the 2-user CR and interference channels. We consider the case where both receivers are unable to decode any message from the non-pairing transmitter. It is clear that the CR channel has a bigger rate region than the interference channel. The CR’s transmitter uses the message of the primary to do superposition coding. Therefore, in this particular manner of decoding capability of the receivers, we notice that message sharing has a beneficial effect on the achievable rates for both the users (primary and the CR). Note that, in all our discussions, we assume that the receivers of the interference channel has the same decoding capability as that of the corresponding CR channels.

3) Figure 4 shows the achievable rate regions for the 2-user CR and interference channels with the assumption that both the primary and the CR’s receiver can decode the public part of the non-pairing transmitter’s message. It is interesting to notice that while both the primary and the CR benefit from the message-sharing mechanism, the maximum achievable rate of the CR (which happens when the rate of the primary is zero, $R_1 = 0$) remains the same as it did when the CR could decode part of the message of the primary, but in the absence of message-sharing. That is, the same rate is achievable by the CR when $R_1 = 0$ in Figs. 2, 3 and 4, although the coding schemes are different. This model was introduced and analyzed in [4]. When the message-sharing mechanism is disabled, the model reduces to the Han-Kobayashi achievable rate region for the interference channel [6], shown in Fig. 2.

4) In Fig. 5, we plot the achievable rate regions for the 2-user CR and interference channels with the assumption that the primary receiver can decode the public part of the CR’s message, while the CR’s receiver can only decode its own messages. The reason for considering this setup is that it corresponds to the decoding capabilities assumed in $c_{cm_s}^2$ and $c_{pm_s}^2$ in the two-user case. As before, both the primary and the CR benefit from the message-sharing mechanism. An improvement in the primary’s rate can be attributed to the fact that (i) the CR does superposition coding (using the message from the primary) and (ii) the primary receiver can
decode the public part of the CR’s message. The figure is also intuitively satisfying, as the achievable rate region is very close to that of Fig. 4, which shows that the additional ability of the CR receiver of being able to decode part of the primary user’s message does not add much to the achievable rate region when the CR transmitter has non-causal knowledge of the message of the primary. This model was presented and analyzed in [20] for the two-user case, but the comparison with the interference channel and the CR channel with dual decoding capabilities presented here provides useful insights. It motivates our choice of decoding capabilities in $C_{\text{cms}}^2$ and $C_{\text{pms}}^2$, i.e., when the CRs have non-causal knowledge of the message of the primary, allowing the CR receivers to be able to decode a part of the primary’s message offers only a marginal improvement in the achievable rate region. Hence, in the three-user case, we focus on the system models in $C_{\text{cms}}^2$ and $C_{\text{pms}}^2$.

5) In Fig. 6, we plot the achievable rate regions for 3-user CR channels with cumulative message sharing (CMS). Similar to our experiments for the 2-user case, we consider two decoding capabilities at the receivers. We compare the achievable rate region when the receivers cannot decode the public parts of the other transmitters’ messages (denoted $C_{\text{cms}}^0$ and $C_{\text{pms}}^0$) with the achievable rate region when only the primary transmitter can decode the ability to decode the public part of the cognitive transmitters’ messages but not vice versa (denoted $C_{\text{cms}}^2$ and $C_{\text{pms}}^2$). We notice that the achievable rate region is significantly improved by allowing the primary receiver to decode part of the messages from the non-pairing senders. Figure 7 shows the achievable rate region for the 3-user CR channel with primary-only message sharing (PMS). Since it is difficult to infer from visual inspection, we resort to numerical tabulation of the maximum achievable rates for each of the three users, for a better understanding of the performance limits.

Table V shows the maximum achievable rates of the primary and the two cognitive users, and the maximum sum rate achieved under cumulative and primary-only message sharing, and with the two decoding capability models assumed in this sub-section. Note that the maximum achievable rate for CR$_1$ and CR$_2$ are the same for $C_{\text{cms}}^0$ and $C_{\text{cms}}^2$ (and similarly for $C_{\text{pms}}^0$ and $C_{\text{pms}}^2$), which is as expected, as the decoding capability of the two cognitive receivers has not been changed. Also, the maximum achievable rate of the primary is significantly improved in going from $C_{\text{cms}}^0$ to $C_{\text{cms}}^2$ (and from $C_{\text{pms}}^0$ to $C_{\text{pms}}^2$), reflecting the benefit of allowing the primary receiver to decode part of the other transmitters’ messages. Comparing $C_{\text{cms}}^0$ with $C_{\text{pms}}^0$, we see that the maximum rate of CR$_2$ in $C_{\text{cms}}^0$ is higher than the corresponding rate in $C_{\text{pms}}^0$, illustrating the benefit of allowing CR$_2$ to have non-causal knowledge of CR$_1$’s message. The same conclusion can be drawn from comparing $C_{\text{cms}}^2$ with $C_{\text{pms}}^2$. Finally, notice that the sum rate in $C_{\text{cms}}^2$ is higher than the individual maximum rate of any of the users (unlike in the case of $C_{\text{cms}}^0$), illustrating that although only the primary receiver has the additional decoding ability, in fact, all users have benefited and the rate region has expanded. A similar remark can be made by observing $C_{\text{pms}}^2$ versus $C_{\text{pms}}^0$. 
VII. Conclusions

In this paper, we introduced multiuser channels with asymmetric transmitter cooperation and presented two different ways of message sharing which we termed cumulative message sharing (CMS) and primary-only message sharing (PMS). We modified the channel model to introduce rate-splitting and considered different ways in which receivers can decode messages. We then derived an achievable rate region for each of the channels by employing a coding scheme which comprised a combination of superposition and Gel’fand-Pinsker coding techniques. Numerical evaluation of the Gaussian case ascertains that the rate regions with CMS is indeed larger than those with PMS, and enables finer comparison between the two message-sharing schemes. Future work could include deriving outer bounds for the three-user CR channel, which would be useful in determining how close the rate regions derived in this paper are to the capacity.

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Achievable rates and their description. For ex. $R_{11}$ is the rate achieved between $S_1$ and $R_1$, while $R_{21}$ is the rate achieved between $S_2$, and $R_2, R_1$, etc.

| Sub-message | Rate | Description |
|-------------|------|-------------|
| $m_{10} \in \{1, \ldots, 2^{nR_{10}}\}$ | $R_{10}$ | Rate achieved: $S_1 \rightarrow (R_1, R_2, R_3)$ |
| $m_{11} \in \{1, \ldots, 2^{nR_{11}}\}$ | $R_{11}$ | Rate achieved: $S_1 \rightarrow R_1$ |
| $m_{20} \in \{1, \ldots, 2^{nR_{20}}\}$ | $R_{20}$ | Rate achieved: $S_2 \rightarrow (R_1, R_2, R_3)$ |
| $m_{21} \in \{1, \ldots, 2^{nR_{21}}\}$ | $R_{21}$ | Rate achieved: $S_2 \rightarrow (R_3, R_2)$ |
| $m_{22} \in \{1, \ldots, 2^{nR_{22}}\}$ | $R_{22}$ | Rate achieved: $S_2 \rightarrow R_2$ |
| $m_{30} \in \{1, \ldots, 2^{nR_{30}}\}$ | $R_{30}$ | Rate achieved: $S_3 \rightarrow (R_1, R_2, R_3)$ |
| $m_{31} \in \{1, \ldots, 2^{nR_{31}}\}$ | $R_{31}$ | Rate achieved: $S_3 \rightarrow (R_1, R_3)$ |
| $m_{33} \in \{1, \ldots, 2^{nR_{33}}\}$ | $R_{33}$ | Rate achieved: $S_3 \rightarrow R_3$ |

TABLE I

Decoding capability of receivers for the channels $C_{cma}^1, C_{pma}^1$. For ex. Receiver $R_2$ can decode messages $m_{10}, m_{20}, m_{22}, m_{30}$.

| Receiver | Decoding capability |
|----------|---------------------|
| $R_1$    | $m_{10}, m_{11}, m_{20}, m_{30}$ |
| $R_2$    | $m_{10}, m_{20}, m_{22}, m_{30}$ |
| $R_3$    | $m_{10}, m_{20}, m_{30}, m_{33}$ |

TABLE II

Decoding capability of receivers for the channels $C_{cma}^2, C_{pma}^2$. For ex. Receiver $R_3$ can decode messages $m_{31}, m_{33}$.

| Receiver | Decoding capability |
|----------|---------------------|
| $R_1$    | $m_{11}, m_{21}, m_{31}$ |
| $R_2$    | $m_{21}, m_{22}$ |
| $R_3$    | $m_{31}, m_{33}$ |
TABLE IV

AUXILIARY RANDOM VARIABLES AND THEIR DESCRIPTION. FOR EX. $U_1$ DENOTES PUBLIC INFORMATION FROM $S_2$ DECODABLE AT $R_1$ AND $R_2$.

| Variable | Description |
|----------|-------------|
| $W_0 \in \mathcal{W}_0$ | Public Information: $S_1 \rightarrow (R_1, R_2, R_3)$ |
| $W_1 \in \mathcal{W}_1$ | Private Information: $S_1 \rightarrow R_1$ |
| $U_0 \in \mathcal{U}_0$ | Public Information: $S_2 \rightarrow (R_1, R_2, R_3)$ |
| $U_1 \in \mathcal{U}_1$ | Public information: $S_2 \rightarrow (R_1, R_2)$ |
| $U_2 \in \mathcal{U}_2$ | Private information: $S_2 \rightarrow R_2$ |
| $V_0 \in \mathcal{V}_0$ | Public information: $S_3 \rightarrow (R_1, R_2, R_3)$ |
| $V_1 \in \mathcal{V}_1$ | Public information: $S_3 \rightarrow (R_1, R_3)$ |
| $V_3 \in \mathcal{V}_3$ | Private information: $S_3 \rightarrow R_3$ |

TABLE V

MAXIMUM RATE THAT CAN BE ACHIEVED BY THE PRIMARY, CR$_1$ AND CR$_2$ WITH CMS AND PMS AND DIFFERENT DECODING CAPABILITIES. $C^0_{cms}$ AND $C^0_{pms}$ CORRESPOND TO CHANNELS WITH CMS AND PMS RESPECTIVELY, WITH NONE OF THE RECEIVERS BEING ABLE TO DECODE ANY PART OF THE NON-PAIRING TRANSMITTERS’ MESSAGES.

![Fig. 1. Three-user cognitive channel with CMS (left) and PMS (right)](image-url)
Fig. 2. Two-user interference channels with different decoding capabilities at the receivers. In (a), both receivers cannot decode any message from the unintended transmitter. In (b), the receiver denoted primary can decode public part of the CR’s message. In (c), both receivers can decode public part of the message from the unintended transmitter. The power at the transmitters are 10dB.

Fig. 3. Two-user CR and interference channels with both receivers unable to decode any message from the unintended transmitter.
Fig. 4. Two-user CR and interference channels with both receivers being able to decode the public part of the unintended transmitter's message.

Fig. 5. Two-user CR and interference channels with the primary's receiver being able to decode the public part of the CR's message. Note that the rate region is nearly the same as in Fig.4.
Fig. 6. Three-user CR channels with CMS where the receivers cannot decode the message of other users (left), and the primary receiver can decode the public part of CR$_1$ and CR$_2$ (right). The power at the transmitters is 10dB.

Fig. 7. Three-user CR channels with PMS where the primary receiver can decode the public part of CR$_1$ and CR$_2$ (right). The power at the transmitters are 8dB (left) and 10dB (right).