Abstract

A model for spin-charge separated superconductivity in two dimensions is introduced where the phases of the spinon and holon order parameters couple gauge-invariantly to a statistical gauge-field representing chiral spin-fluctuations. The model is analyzed in the continuum limit and in the low-temperature limit. In both cases we find that physical electronic phase correlations show a superconducting-normal phase transition of the Berezinskii-Kosterlitz-Thouless type, while statistical gauge-field excitations are found to be strictly gapless. It is argued that the former transition is in the same universality class as that of the XY model. We thus predict a universal jump in the superfluid density at this transition. The normal-to-superconductor phase boundary for this model is also obtained as a function of carrier density, where we find that its shape compares favorably with that of the experimentally observed phase diagram for the oxide superconductors.

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I. Introduction

Although the phenomenon of high-temperature superconductivity\cite{1} has sparked a tremendous amount of theoretical activity, a viable theory remains to be uncovered. It is, however, generally agreed upon that strong electron-electron interactions must play an important role in the charge dynamics of the Copper-Oxygen planes that are common to these systems.\cite{2} Among the various attacks on the theoretical problem of strongly interacting electrons in two dimension, those based on the hypothesis of spin-charge separation show promise at the phenomenological level.\cite{2-3} In particular, gauge theories for the unconventional metallic states found in the $t-J$ model, which is considered to be the simplest model containing the essential strong correlation physics of the oxide superconductors,\cite{4} successfully account for many of the unconventional transport and collective mode properties that are common to the corresponding metallic phases of these materials.\cite{5,6} For example, the $T$-linear in-plane resistivity, as well as the paradoxical observations of a hole-type Hall effect in conjunction with a large Luttinger Fermi surface, can be accounted for by such theories.

In a similar spirit, analogous Ginzburg-Landau theories of spin-charge separated superconductivity itself have recently been proposed in the literature.\cite{7,8} It is presumed in such theories that separate superfluid instabilities exists in both the spinon and holon systems in isolation. Here, the spinon superfluidity is assumed to arise from singlet Cooper pairing \textit{à la} the mean-field resonating valence bond (RVB) scenario,\cite{9} while the holon superfluidity is driven by Bose-Einstein condensation. Given the Ioffe-Larkin composition law, $R = R_b + R_f$, which states that the physical resistance is given by the sum of the holon ($b$) and spinon ($f$) contributions, then true superconductivity occurs only when both species are superfluid.\cite{3} However, because such Ginzburg-Landau theories are formulated in the continuum, the analysis of the superconducting transition in two dimensions is complicated by the structure of the vortex cores. In addition, these theories have been only analyzed in the mean-field approximation, to date, where fluctuations of the statistical gauge-field related to the constraint against double occuppancy are neglected. (Note that such excitations physically represent so-called chiral spin fluctuations.\cite{10}) As a result, the precise shape that they obtain for the phase boundary between the normal and the superconducting state is questionable.\cite{8}
In order to address the role played by fluctuations of the statistical gauge-field, we introduce here an alternate spin-charge separated model of strongly correlated superconductivity in which only the phase of each order parameter is allowed to vary. This model is a two-component generalization of a lattice gauge theory model in two euclidean (1+1) dimensions known as the Abelian-Higgs model. Specifically, the theory contains a holon-pair phase order parameter (Higgs field) and a spinon-pair phase order parameter (Higgs field) that couple in a gauge-invariant way to the fluctuating $U(1)$ (Abelian) statistical gauge-field describing chiral spin fluctuations. We henceforth refer to the latter as the two-component Abelian-Higgs (AH$^2$) model. By considering both the continuum and the low temperature limits of this model, we find that it exhibits a normal-to-superconducting transition of the Berezinskii-Kosterlitz-Thouless (BKT) type in strictly two spacial dimensions. Note that the low-temperature analysis is achieved by a Villain duality transformation of this model, which has been successfully employed in the past to study the one-component Abelian Higgs model and the XY model. We obtain the following two major results: that (i) the phase correlations corresponding to the statistical gauge-field are short-range at all temperatures, and that (ii) only phase correlations corresponding to the physical electronic order parameter show algebraic long-range order below a BKT-type transition temperature, $T_c \cong \frac{\pi}{2} (J_b^{-1} + J_f^{-1})^{-1}$, where $J_b$ and $J_f$ are the respective local phase-stiffnesses of the holon-pairs and the spinon-pairs. The former result (i) implies that statistical gauge-field fluctuations do not acquire a gap via the Higgs mechanism. The latter result (ii), on the other hand, implies that this spin-charge separated model shows a true superconducting transition at $T_c$. It is important to remark that the present calculation, which incorporates fluctuations of the statistical gauge-field, results in a transition temperature inferior to the mean-field approximation result, $T_c^{(0)} = \frac{\pi}{2} \min(J_b, J_f)$, as expected. In addition, the presently obtained transition temperature yields a metal-superconductor phase diagram as a function of hole doping that qualitatively resembles that of the oxide superconductors (see Fig. 1). It is also argued that this transition falls into the same universality class as that of the XY model. We thus predict that the present AH$^2$ model for spin-charge separated superconductivity has a universal jump in the superfluid density at the transition. Last, we compute the Wilson loop and find that it shows a perimeter law in the superconducting phase. On
the other hand, by continuity with the corresponding results obtained for the case of the one-component model,\textsuperscript{11} we conclude that the Wilson loop generally shows a “confining” area law in the normal phase. This change in behavior simply reflects the vortex binding-unbinding transition. We then argue, however, that the latter “confinement” effect is a trivial result of “electromagnetism” in one space and one time dimension (1+1), and hence that fluctuations in the statistical gauge-field remain gapless in the normal phase.\textsuperscript{21} This, hence, provides a basis for the calculations of linear-in-$T$ resistance in the “strange” metallic phase of the $t-J$ model,\textsuperscript{5,6} which rely on this property. Note that gapless statistical gauge-field excitations have also been shown to exist in a two-component anyon superconductor saddle-point of the $t-J$ model known as the commensurate flux-phase at low temperature and near half-filling.\textsuperscript{22,23} Yet once dynamical effects are included, an exponentially small gap appears close to half-filling in such case.\textsuperscript{22}

The remainder of the paper is organized as follows. In the next section we introduce the AH$^2$ model in the context of strongly interacting electrons in two dimensions. Section III contains a discussion of various limits of the model, including the continuum limit. In section IV we study the phase correlations in the low-temperature limit via the Villain duality transformation, while the nature of the statistical gauge-field excitations are treated in section V within the same context. A renormalization group analysis of the BKT-transition found in this model is presented in section VI, while we compare the theoretically obtained phase diagram with that observed experimentally in the oxide superconductors as function of hole doping in section VII. The nature of the normal state of the present model is also discussed here. Last, the phase correlators of the model are computed in the “spin-wave” approximations in Appendix A, and those of the one-component model are computed in the low-temperature limit in Appendix B.

\section{II. Two-component Abelian-Higgs Model}

Before introducing the Abelian-Higgs model for spin-charge separated superconductivity in two dimensions, let us first consider the corresponding normal state, which we take to be the unconventional metallic state found in strongly interacting two-dimensional (2D) electron systems first discussed by Ioffe and Larkin in terms of the spin-charge separated language of spinons and holons.\textsuperscript{2,3} The archetypical theoretical description of strong
correlation physics in the context of the oxide superconductors is given by the \( t-J \) model defined over the square lattice,\(^4\) notably where double occupancy of electrons at each site is excluded as a result such correlations. This constraint may be imposed by the introduction of an auxiliary slave-boson field, \( b_i \), such that the electron field is re-expressed as

\[
c_{i\sigma} \rightarrow c_{i\sigma} b_i^\dagger,
\]

along with the constraint \( c_{i\sigma}^\dagger c_{i\sigma} + b_i^\dagger b_i = 1 \) at each site.\(^22\) Certain mean-field treatments of the latter constraint allowing the slave-boson to propagate result in a spin-charge separated description of the \( t-J \) model, with \( c_{i\sigma} \) representing the spinon field and \( b_i \) representing the holon field.\(^3\)–\(^6\) A statistical gauge field, \( A_\mu \), generated by the internal gauge symmetry \((c_{i\sigma}, b_i) \rightarrow (e^{i\theta_i} c_{i\sigma}, e^{i\theta_i} b_i)\) shown in Eq. (1) then mediates interactions between the two species. Within this representation, the “strange” metal is then characterized by a normal spinon Fermi-liquid state with a large Luttinger Fermi surface in conjunction with a normal (uncondensed) holon liquid state. The statistical gauge-field mediates interactions among both species.\(^5\)–\(^6\) In particular, interactions of these fluctuations with the uncondensed holon liquid ultimately give rise to the \( T \)-linear prediction for the resistance characteristic of the “strange” metal phase.\(^5\)

Upon lowering temperature, however, meanfield resonanting-valence-bond (RVB) studies of the \( t-J \) model show that amplitude develops for an order parameter describing a paired spinon state.\(^9\) In addition, amplitude for a separate order parameter representing the Bose-Einstein holon condensate exists at low-temperature. Hence by definition, in the mean-field approximation, the unconventional metallic state can only exist at temperatures above both the highest temperature, \( T_f \), at which any spinon Cooper pairing instability sets in, and above the Bose-Einstein condensation temperature, \( T_b \), for the holons (see Fig. 1). Therefore, as temperature, \( T \), is lowered, the following three situations may arise: (i) \( T_f < T < T_b \); (ii) \( T_b < T < T_f \); and (iii) \( T < T_b, T_f \). Given the Ioffe-Larkin composition law,\(^3\) \( R = R_b + R_f \), which states that the physical resistance is given by the sum of the holon \( (b) \) and spinon \( (f) \) contributions, then only regime (iii) is superconducting. This implies that the critical temperature for the superfluid transition is given by

\[
T_{c}^{(0)} = \min(T_b, T_f)
\]

at the mean-field level. As has been discussed in the literature,\(^8\) regimes (i) and (ii) are
non-superconducting phases that correspond, respectively, to a normal Fermi-liquid state and a to spin-gap state. The various regimes mentioned above are shown in Fig. 1.

The latter phase diagram for strongly interacting electrons in two dimensions is obtained in the mean-field approximation, however, where fluctuations of the statistical gauge-field are absent. In this paper, we shall study what effect such fluctuations have on the previously discussed superfluid transitions occurring in the spin-charge separated metallic state by focusing on the phase degrees of freedom alone of both the spinon and the holon superfluid order parameters. In particular, consider the following partition function for the two-component square lattice Abelian-Higgs model, which we presume models the superfluid sector of the spinon/holon system in the presence of statistical gauge-field fluctuations:

$$Z = \int D\phi_b(r)D\phi_f(r')DA_\mu(r'')\exp(-E/T),$$

where

$$\frac{E}{T} = \beta_b \sum_{r,\mu} \{1 - \cos[\Delta_\mu \phi_b(r) - qA_\mu(r)]\} + \beta_f \sum_{r,\mu} \{1 - \cos[\Delta_\mu \phi_f(r) - qA_\mu(r)]\} + \frac{1}{2g^2} \sum_{r,\mu,\nu} \{1 - \cos[\Delta_\mu A_\nu(r) - \Delta_\nu A_\mu(r)]\}. \quad (4)$$

Here, $\phi_b(r)$ and $\phi_f(r)$ represent the respective phases of the holon and spinon order parameters, $A_\mu(r) = A_{\vec{r},\vec{r}+\vec{\mu}}$ denotes the statistical gauge-field, and $\Delta_\mu$ denotes the lattice difference operator ( $\mu = x, y$). The first two terms above are the lattice versions of the stiffness energies for the phase fluctuations, while the last term is the corresponding stiffness energy for the statistical gauge-field fluctuations. Also, $\beta_b = J_b/T$ and $\beta_f = J_f/T$, where $J_b$ and $J_f$ denote the local phase stiffness of the holon order parameter and the spinon order parameter, respectively, while $g^{-2} = \chi_d/T$, where $\chi_d$ represents the local stiffness for statistical gauge-field fluctuations. (Throughout, we set $k_B$, $\hbar$ and the lattice constant, $a$, to unity, and we sum over repeated indices.) Spin-charge separated treatments of the $t-J$ model in two dimensions give a diamagnetic susceptibility for fluctuations in the statistical gauge-field of $\chi_d = \chi_f + \chi_b$, where $\chi_f$ and $\chi_b$ are the diamagnetic susceptibilities of each species. Within the metallic (fluxless) saddle-point of these treatments, $\chi_f \sim J(1-x)$ and $\chi_b \sim tT_b/T$, where $x$ denotes the hole concentration and $T_b \sim tx$ denotes the ideal Bose-Einstein condensation temperature. Also, $J_b \sim t'x$, where $t' \lesssim t$ denotes
the relevant matrix element for hopping in the holon liquid, and \( J_f \sim J(1 - x/x_0) \), where \( x_0 \lesssim 1 \) denotes the critical hole concentration above which the spinon pairing instability is absent.\(^9\) Finally, \( q \) represents the charge per site of each species. (Throughout, we follow the notations used in refs. 11, 18, and 19.) We remind the reader that Landau-Ginzburg versions of this model, formulated in the 2D continuum, have already been discussed in the literature.\(^7,8\)

Before we begin the analysis of the AH\(^2\) model, a few remarks are in order. First, since physical electrons have no statistical charge by Eq. (1), we have restricted our analysis (4) to statistically charge neutral AH\(^2\) models; i.e., the statistical charge per site, \( q \), of the holons and of the spinons is equal. Second, we implicitly presume that the spinon order parameter, \( \phi_f \), results from Cooper pairing of the spinons \` a la the RVB picture for superconductivity.\(^9\) Hence, from this point on we shall take a charge per site of \( q = 2 \) for the present model, which implies that the holon condensate is made up pairs of holons as well. Note that a holon pairing instability consistent with the previous assignment has been shown to exist in the “strange” metallic phase of the \( t - J \) model,\(^6\) which is the normal state of the present model. Also, in the limit near half-filling relevant to the oxide superconductors, the density of holon pairs is small and the overlap between two pairs of bosons should be negligible. Therefore, a bose condensate of dilute bosonic molecules can form,\(^24\) resulting in the holon order parameter, \( \phi_b \), above. If, on the other hand, the holon condensate where to result from a condensation of unpaired holons as is suggested by certain mean-field treatments,\(^4,5,9\) then such an order parameter must necessarily be defined on a lattice different from that of the spinon order parameter by the requirement of statistical charge neutrality (1). The latter physical situation, however, cannot be described by the present AH\(^2\) model, which has only one underlying lattice.

III. Limits

The nature of the above AH\(^2\) model (4) for spin-charge separated superconductivity on the square lattice can be uncovered, in large part, by an analysis of various limits, as we discuss below.

A. Continuum Limit. Consider the limit where \( q^2 g^2 \beta_b, q^2 g^2 \beta_f \ll 1 \) and where fluctuations in the fields are at longwavelength, which corresponds to the continuum limit [see
Eqs. (4) and (20)]. In this case, the last term in the energy functional (4) is negligible, whereas the first two terms may approximated by
\[
\frac{E}{T} = \frac{1}{2} \int d^2 r [\beta_b (\nabla \phi_b - q \vec{A})^2 + \beta_f (\nabla \phi_f - q \vec{A})^2].
\] (5)

As is customary in the treatment of the XY model, let us now separate the phase configurations into “spin-wave” and vortex components; i.e., let \( \phi_s = \phi_s^{(w)} + \phi_s^{(v)} \) for each species \( s = b, f \), where \( \phi_s^{(w)} \) represents the smooth portion of the configuration such that \( \vec{\nabla} \times (\nabla \phi_s^{(w)}) = 0 \), and where \( \phi_s^{(v)} \) represents the portion of the configuration containing vortex singularities such that \( \vec{\nabla} \cdot (\nabla \phi_s^{(v)}) = 0 \). Taking the Coulomb gauge, \( \vec{\nabla} \cdot \vec{A} = 0 \), we see that statistical gauge-field fluctuations only communicate with the component of the phase configuration containing vorticity, since the statistical gauge-field is purely transverse in this gauge, as are such configurations. Therefore, integrating out first the gauge-field excitations in the corresponding partition function (3) within this gauge leaves us with following effective action for the vortex component:
\[
\frac{E_v}{T} = \frac{\bar{\beta}}{2} \int d^2 r [\nabla \phi_f^{(v)} - \nabla \phi_b^{(v)}]^2,
\] (6)

where
\[
\bar{\beta} = (\beta_b^{-1} + \beta_f^{-1})^{-1}.
\] (7)

The remaining smooth “spin-wave” components result in a trivial Gaussian action,
\[
\frac{E_w}{T} = \frac{1}{2} \int d^2 r [\beta_b (\nabla \phi_b^{(w)})^2 + \beta_f (\nabla \phi_f^{(w)})^2].
\]

Hence, by eqs. (6) and (7) we expect a BKT transition to occur in the physical electronic phase,
\[
\phi_{el} = \phi_f - \phi_b
\] (8)
when \( 2\pi \bar{\beta} = 4 \), which implies a corresponding BKT transition temperature of
\[
T_c \approx \frac{\pi}{2} (J_b^{-1} + J_f^{-1})^{-1}.
\] (9)

In the continuum limit, therefore, statistical gauge-field fluctuations suppress, but do not destroy, the superconductivity in this spin-charge separated system in relation to the Ioffe-Larkin mean-field approximation result; i.e. \( 0 < T_c < \min(T_b, T_f) \), where \( T_b = \frac{\pi}{2} J_b \) and
$T_f = \frac{\pi}{2} J_f$ are the corresponding transition temperatures of each species separately. As we will show in the next section, this result is generally valid outside of the continuum limit, as well.

**B. XY model + Pure Gauge Theory.** Consider now the limit where $\beta_b \ (\text{or} \ \beta_f) \gg 1$. After making the gauge-transformation $A'_\mu = A_\mu - q^{-1} \Delta_\mu \phi_b$, we can rewrite the energy functional (4) as

$$
\frac{E}{T} = \beta_b \sum_{r, \mu} [1 - \cos(q A'_\mu)] + \beta_f \sum_{r, \mu} \{1 - \cos[\Delta_\mu \phi_{el}(r) - q A'_\mu(r)]\} + \\
+ \frac{1}{2g^2} \sum_{r, \mu, \nu} \{1 - \cos[\Delta_\mu A'_\nu(r) - \Delta_\nu A'_\mu(r)]\}.
$$

(10)

Hence, the present limit imposes the constraint $A'_\mu = 2\pi n/q$, where $n$ is any integer. The above energy functional then reduces to

$$
\frac{E}{T} \to \beta_f \sum_{r, \mu} \{1 - \cos[\Delta_\mu \phi_{el}(r)]\} + \frac{1}{2g^2} \sum_{r, \mu, \nu} \{1 - \cos[\Delta_\mu A'_\nu(r) - \Delta_\nu A'_\mu(r)]\}.
$$

(11)

The latter expression, in conjunction with the former constraint, contains an XY model that is decoupled from a pure gauge theory known in the literature as $Z_q$. The XY model part implies a BKT transition at $T_f$ for the physical electronic phase, $\phi_{el}$, which is consistent with the results (9) obtained previously in the continuum limit for the present case, $J_b \to \infty$. The $q$-state ($Z_q$) pure gauge theory part of the energy functional (11), on the other hand, can be easily shown to exhibit no phase transition in two dimensions by choosing to work in the Landau gauge ($A_x = 0$). This, of course, reduces the theory to one dimension.

**C. One-component Abelian-Higgs Model.** Last, let us take the limit $\beta_b = 0$ or $\beta_f = 0$, which corresponds to the well studied one-component Abelian-Higgs model. The latter model sustains no Higgs phenomenon, since the long-range interaction is screened, and in general, it is thought to be free of phase transitions in two dimensions. In fact, the phase correlation length can be shown to be finite in the low-temperature limit (see Appendix B). Again, these results are consistent with those obtained in the continuum limit (9); i.e., $T_c = 0$ in the case that $J_b = 0$ or $J_f = 0$.

In summary, we find that the physical electronic phase (8) undergoes a BKT transition at a critical temperature given by expression (9), while the gauge-field excitations...
experience no phase transitions at all, in the limits considered above. Below, we show that these findings are corroborated by a low-temperature analysis of the model.

IV. Phase Correlations

With the intent of understanding the superconducting properties of our model (4) for spin-charge separated superconductivity in two dimensions, let us consider now the low-temperature limit $\beta_b, \beta_f, g^{-2} \gg 1$. In this limit, the most important configurations of both the phases and the statistical gauge-field lie at extrema of the cosine functions found in expression (4). Thus, we may employ well-known Villain duality transformations that are successful in analysing both the XY model$^{18,19}$ and the one-component Abelian-Higgs model$^{11}$ in the same limit. This approximation amounts to replacing the exponential of a cosine function by the sum of exponentials of parabolas, each situated at the appropriate extrema of the cosine function.$^{17}$ Below, we show that the latter analysis leads to a generalized Coulomb gas ensemble that exhibits a BKT phase transition consistent with the discussion found in the previous section (III.A). Note also that the analogous treatment of the phase auto-correlations function in the special case of the one-component Abelian-Higgs model ($\beta_b = 0$ or $\beta_f = 0$), which is useful in the study of the unconventional normal state corresponding to the present model for spin-charge separated superconductivity,$^{21}$ is found in Appendix B.

A. Coulomb Gas Representation. Application of the above mentioned low-temperature approximation to the presently considered AH$^2$ model reduces to the substitution of the mathematical identity

$$e^{-\beta(1-\cos \theta)} \cong (2\pi \beta)^{-1/2} \sum_{n=-\infty}^{\infty} e^{in\theta} e^{-n^2/2\beta},$$

that is valid in the limit $\beta \to \infty$, into the partition function (3) corresponding to the energy functional (4). By closely following the analogous treat of the one-component version of our model discussed in ref. 11, and performing the integrals that remain over the fields $\phi_b, \phi_f$, and $A_\mu$, we ultimately arrive at the following “roughening model” representation for this partition function:

$$Z = \sum_{\{n_b(r)\}} \sum_{\{n_f(r)\}} \exp \left\{ -\frac{1}{2\beta_b} \sum_{r,\mu} [\Delta_\mu n_b(r)]^2 - \frac{1}{2\beta_f} \sum_{r,\mu} [\Delta_\mu n_f(r)]^2 \right\}$$
\[-\frac{(qg)^2}{2} \sum_r [n_b(r) + n_f(r)]^2 \right\} ,
\]

(13)

where \( n_b(r) \) and \( n_f(r) \) are integer fields that lie on the dual lattice \( r \). Also, after application of the Poisson summation formula, this expression may be transformed further into

\[
Z = \int_{-\infty}^{\infty} \Pi_r d\theta_b(r) \int_{-\infty}^{\infty} \Pi_r d\theta_f(r) \sum_{\{q_b(r)\}} \sum_{\{q_f(r)\}} \exp \left\{ -\frac{1}{2\beta_b} \sum_{r,\mu} (\Delta_{\mu} \theta_b)^2 \right. 
\]

\[
-\frac{1}{2\beta_f} \sum_{r,\mu} (\Delta_{\mu} \theta_f)^2 - \frac{(qg)^2}{2} \sum_r (\theta_b + \theta_f)^2 + 2\pi i \sum_r (q_b \theta_b + q_f \theta_f) \right\},
\]

(14)

where \( q_b(r) \) and \( q_f(r) \) are the respective dual lattice vortex “charges” of each species that range over the integers. After integrating out the fields, \( \theta_b(r) \) and \( \theta_f(r) \), that result from the Poisson summation formula, we find that the present partition function is equal to the product of a trivial gaussian factor with the following generalized 2D Coulomb gas ensemble average:

\[
Z_{\text{Coulomb}} = \sum_{\{q_{\alpha}(r)\}} \exp \left[ -(2\pi)^2 \frac{1}{\beta} \sum_{r,r'} q_{\alpha}(r) G_{\alpha\beta}(r,r') q_{\beta}(r') \right],
\]

(15)

where

\[
G_{\alpha\beta}(r,r') = (2\pi)^{-2} \int_{\text{BZ}} d^2 k e^{ik \cdot (r-r')} G_{\alpha\beta}(\vec{k})
\]

(16)

is the Greens function, such that the matrix inverse of its Fourier transform, \( G(\vec{k}) \), is

\[
G^{-1}_{\alpha\beta}(\vec{k}) = \frac{\bar{\beta}}{\beta_{\alpha}} \vec{k}^2 \delta_{\alpha\beta} + \bar{\beta} q^2 g^2 \vec{u}_{\alpha} \vec{u}_{\beta}.
\]

(17)

The above indices \( \alpha \) and \( \beta \) represent the internal spin/charge label, \( b \) or \( f \), while \( \vec{k}^2 = 4 - 2\cos k_x - 2\cos k_y \) are the eigenvalues of the lattice Laplacian operator. In addition, \( \vec{u} = (1, 1) \) represents the spin/charge component of the statistical degree of freedom [see Eq. (25)]. After inverting the righthand side of Eq. (17), we find that the matrix Greens function is decomposable into long-range and short-range components

\[
G_{\alpha\beta}(\vec{k}) = G^{(\text{lr})}_{\alpha\beta}(\vec{k}) + G^{(\text{sr})}_{\alpha\beta}(\vec{k}),
\]

(18)
such that

\[ G^{(lr)}_{\alpha\beta}(\vec{k}) = v_\alpha v_\beta \frac{1}{k^2}, \quad (19a) \]

\[ G^{(sr)}_{\alpha\beta}(\vec{k}) = \frac{\beta_\alpha \beta_\beta}{\beta_\beta^2} \frac{1}{k^2 + \lambda_{st}^{-2}}, \quad (19b) \]

where \( \vec{v} = (1, -1) \) represents the spin/charge components of the physical electronic degree of freedom (8), and where

\[ \lambda_{st}^{-2} = (qg)^2(\beta_b + \beta_f) \]

yields the characteristic length scale, \( \lambda_{st} \), for fluctuations of the statistical gauge-field.

The conjunction of these last few expressions (Eqs. 15-20) are, in effect, the Coulomb gas representation of the AH\(^2\) model.

The above formulae may be reduced further, however. After substituting in the long-range and short range components of the matrix Greens function discussed above into (15), algebraic manipulation then yields that this generalized Coulomb gas partition function may be expressed as

\[ Z_{\text{Coulomb}} = \sum_{\{q_b(r)\}} \sum_{\{q_f(r)\}} \exp \left\{ 2\pi \vec{\beta} \sum_{(r, r')} [\Gamma_{lr}(\vec{r} - \vec{r}')q_{el}(r)q_{el}(r') + \Gamma_{sr}(\vec{r} - \vec{r}')q_{st}'(r)q_{st}'(r')] \right\}, \]

(21)

where the physical electronic (el) flux-charge and the modified statistical (st) flux-charge are given respectively by

\[ q_{el}(r) = q_f(r) - q_b(r), \quad (22) \]

\[ q_{st}'(r) = \left( \frac{\beta_f}{\beta_b} \right)^{1/2} q_f(r) + \left( \frac{\beta_b}{\beta_f} \right)^{1/2} q_b(r), \quad (23) \]

and where \((r, r')\) denote combinations of points covering the dual lattice.\(^{25}\) Corresponding to these charges are long-range (lr) and short-range (sr) potentials given respectively by

\[ \Gamma_{lr}(\vec{r}) = \int_{\text{BZ}} \frac{d^2 k}{2\pi} \left( 1 - e^{i\vec{k} \cdot \vec{r}} \right) \frac{1}{k^2}, \quad (24a) \]

\[ \Gamma_{sr}(\vec{r}) = \int_{\text{BZ}} \frac{d^2 k}{2\pi} \left( 1 - e^{i\vec{k} \cdot \vec{r}} \right) \frac{1}{k^2 + \lambda_{st}^{-2}}. \quad (24b) \]

Note that for the sake of technical convenience, we have added an overall constant to the matrix Greens function represented by the first term in the above two equations. Due
to the existence of a long range force, charge neutrality is enforced for each species and this constant term therefore sums to zero for all of the relevant charge configurations in the energy functional of the ensemble average [see Eq. (15)]. Since the effective Coulomb gas inverse temperature scale is given by $\bar{\beta}$, we expect there to exist a binding-unbinding transition of the BKT type modified by short-range interactions for the physical electronic flux-charges, $q_{el}(r)$, at a corresponding critical temperature. On the other hand, for the case of purely statistical flux-charge configurations, $q_{el}(r) = 0$ and $q_{st}(r) = q_f(r) + q_b(r) \neq 0$, Eq. (21) indicates that there exists only short-range forces, where $\lambda_{st}$ determines the length scale for such forces in units of the lattice spacing, $a$. Thus, we expect no BKT transition in such a case. Below, we elaborate on the consequences of the latter statements to show that (i) a 2D superconducting transition of the BKT type exists only for the physical electronic phase (8), and that (ii) all other phase auto-correlations, including those among the statistical phase,

$$\phi_{st} = \phi_f + \phi_b,$$

are short-range at all temperatures. Note that exactly the opposite behavior is exhibited by two square-lattice XY models stacked up on top of each other with nearest-neighbor coupling, where the phase difference between layers has short-range order at zero temperature, while the sum of the phases between layers shows quasi long-range order at low temperature.\(^{26}\)

**B. Auto-correlation Functions.** We now wish to probe the phase correlations of the presently considered $AH^2$ model for spin-charge separated superconductivity in two dimensions. The gauge-invariant phase correlation function may be expressed as

$$C_{12} = \left\langle \exp \left\{ i \sum_{\alpha} p_{\alpha} \left[ \phi_{\alpha}(\vec{r}_1) - \phi_{\alpha}(\vec{r}_2) + q \int_{\vec{r}_1}^{\vec{r}_2} A_{\mu} dr_{\mu} \right] \right\} \right\rangle = \frac{Z'}{Z},$$

($\alpha = f, b$) where the partition function $Z'$ differs from $Z$ only by the addition of the exponent $-i \sum_{\alpha} p_{\alpha} [\phi_{\alpha}(\vec{r}_1) - \phi_{\alpha}(\vec{r}_2) + q \int_{\vec{r}_1}^{\vec{r}_2} A_{\mu} dr_{\mu}]$ to the energy functional (4). Again, following refs. 18 and 19, $Z'$ may be computed via the Villain duality transformation. In particular, substitution of the mathematical identity (12) that is valid in the low-temperature limit ultimately yields the following “roughening model” representation for
\[ Z' = \sum_{\{n_b(r)\}} \sum_{\{n_f(r)\}} \exp \left\{ -\frac{1}{2\beta_b} \sum_{r,\mu} [\Delta_\mu n_b(r) + p_b \eta_\mu(r)]^2 - \frac{1}{2\beta_f} \sum_{r,\mu} [\Delta_\mu n_f(r) + p_f \eta_\mu(r)]^2 \right. \\
\left. - \frac{(qg)^2}{2} \sum_r [n_b(r) + n_f(r)]^2 \right\}, \quad (27) \]

where the dual lattice “dipole” vector, \( \eta_\mu(r) \), has a value of \( \pm 1 \) if it intersects a fixed path on the original lattice connecting points \( \vec{r}_1 \) and \( \vec{r}_2 \), and it vanishes otherwise. As before, after the application of the Poisson summation formula, and subsequently integrating over the fields that this formula generates, we find that

\[ Z' = \exp \left[ -\frac{1}{2} \left( \frac{p_b^2}{\beta_b} + \frac{p_f^2}{\beta_f} \right) \sum_{r,\mu} \eta_\mu^2(r) \right] \times \\
\sum_{\{q_\alpha(r)\}} \exp \left\{ -(2\pi)^2 \beta \frac{1}{2} \sum_{r,r'} [q_\alpha(r) + q'_\alpha(r)]G_{\alpha\beta}(r,r')[q_\beta(r') + q'_\beta(r')] \right\}, \quad (28) \]

where the external vortex “charges” corresponding to each species are given by

\[ q'_\alpha(r) = -\frac{p_\alpha}{2\pi i \beta_\alpha} \eta(r), \quad (29) \]

with \( \eta(r) = \Delta_\mu \eta_\mu(r) \).\(^{18,19}\) This expression constitutes the Coulomb gas representation of the modified partition function \( Z' \).

The above expression for the modified partition function can be reduced further by substituting in the explicit form of the matrix Greens function \( G \) [see Eqs. (16-19)]. We then find that the preceding correlation function, \( Z'/Z \), may be factorized into

\[ C_{12} \equiv G_{\text{wave}}(\vec{r}_1 - \vec{r}_2)G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2), \quad (30) \]

where

\[ G_{\text{wave}}(\vec{r}_1 - \vec{r}_2) = \exp \left\{ -\frac{1}{2} \left( \frac{p_b^2}{\beta_b} + \frac{p_f^2}{\beta_f} \right) \sum_{r,\mu} \eta_\mu^2(r) \right. \\
\left. - \frac{1}{\pi \beta_b \beta_f} \sum_{r,r'} \eta(r)[p_\alpha^2 \Gamma_{lr}(\vec{r} - \vec{r}') + p_{st}^2 \Gamma_{sr}(\vec{r} - \vec{r}')] \eta(r') \right\}. \quad (31) \]
and

\[
G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2) = \left\langle \exp \left\{ \frac{i}{2} \frac{2\beta}{(\beta_b\beta_f)^{1/2}} \sum_{r,r'} [p'_{\text{el}} q_{\text{el}}(r) \Gamma_{lr}(\vec{r} - \vec{r}')] + p_{\text{st}} q'_{\text{st}}(r) \Gamma_{sr}(\vec{r} - \vec{r}')] \eta(r') \right\} \right\rangle_{\text{Coulomb}},
\]

(32)

with \( p'_{\text{el}} = \frac{1}{2}[(\beta_b/\beta_f)^{1/2} p_f - (\beta_f/\beta_b)^{1/2} p_b] \) and \( p_{\text{st}} = \frac{1}{2}(p_f + p_b) \). After some manipulation (see Appendix A), it can be shown that in the limit \(|\vec{r}| \gg \lambda_{\text{st}}\),

\[
G_{\text{wave}}(\vec{r}) = \exp \left[ \frac{-p''_{\text{st}}}{2\beta} (r + \pi^{-1}\ln \lambda_{\text{st}}) \right] \exp \left[ \frac{-p''_{\text{el}}}{2\beta} \frac{\Gamma_{lr}(r)}{\pi} \right],
\]

(33)

with

\[
p''_{\text{el}} = \frac{2\beta}{(\beta_b\beta_f)^{1/2}} p'_{\text{el}} = p_{\text{el}} + \beta \left( \frac{1}{\beta_f} - \frac{1}{\beta_b} \right) p_{\text{st}},
\]

(34a)

\[
p''_{\text{st}} = \frac{2\beta}{(\beta_b\beta_f)^{1/2}} p_{\text{st}},
\]

(34b)

and where \( p_{\text{el}} = \frac{1}{2}(p_f - p_b) \) and \( \ln \lambda_{\text{st}} = (2\pi)^{-1} \int_{BZ} d^2 k (\tilde{k}^2 + \lambda_{\text{st}}^{-2})^{-1} \). Hence, all phase correlations are short-range at all temperatures unless \( p_{\text{st}} = 0 \). In particular, purely statistical phase correlations, with \( p_{\text{el}} = 0 \) and \( p_{\text{st}} \neq 0 \), are short-range. This implies that, within the present model, transverse statistical gauge-field fluctuations do not acquire a gap via a Higgs mechanism.

A short-range order to algebraic long-range order transition of the BKT type does, however, exist for purely electronic phase correlations, \( p_{\text{st}} = 0 \) and \( p_{\text{el}} \neq 0 \), as we demonstrate below. But before considering this case in particular, let us first compute \( G_{\text{Coulomb}} \) for the presently considered \( AH^2 \) model on the square-lattice in general. This function may be calculated by following well-known methods that have been successfully employed in the corresponding computation for the case of the XY model. In particular, given the definitions

\[
\sigma_{lr}(r) = \sum_{r'} \Gamma_{lr}(\vec{r} - \vec{r}') \eta(r'),
\]

(35a)

\[
\sigma_{sr}(r) = \sum_{r'} \Gamma_{sr}(\vec{r} - \vec{r}') \eta(r'),
\]

(35b)
we can re-express the latter correlation function (32) as

\[ G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2) = \left\langle \exp \left\{ \frac{-i \beta}{\beta_b \beta_f} \sum_r \langle [p'_{\text{el}} q_{\text{el}}(r) \sigma_{\text{lr}}(r) + p_{\text{st}} q'_{\text{st}}(r) \sigma_{\text{sr}}(r)] \rangle \right\} \right\rangle_{\text{Coulomb}}. \]

(36)

Since charge correlations in the present model are strong as a result of the existence of long-range forces, we can make the following cumulant expansion:

\[
G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2) = \exp \left\{- \frac{1}{2} \sum_{r', r''} \langle [p_{\text{el}} q_{\text{el}}(r') \sigma_{\text{lr}}(r') + p_{\text{st}} q'_{\text{st}}(r') \sigma_{\text{sr}}(r')] \rangle \right\} \times [p'_{\text{el}} q_{\text{el}}(r'') \sigma_{\text{lr}}(r'') + p_{\text{st}} q'_{\text{st}}(r'') \sigma_{\text{sr}}(r'')].
\]

(37)

But since only the lowest energy vortex-antivortex excitations need be accounted for at the low temperatures presently considered, the above expression then reduces to

\[
G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2) = \exp \left[ -\frac{1}{2} \sum_{r', r''} \langle q_i(r') q_i(r'') \rangle \sigma(r') \sigma(r'') \right],
\]

(38)

where

\[
\sigma(r') = \text{sgn}(\beta_b - \beta_f) p''_{\text{el}} \sigma_{\text{lr}}(r') + \min\left[ (\beta_b/\beta_f)^{1/2}, (\beta_f/\beta_b)^{1/2} \right] p''_{\text{st}} \sigma_{\text{sr}}(r'),
\]

(39)

and where the Coulomb charge correlation function for the species labeled by \( i \), such that \( \beta_i = \min(\beta_b, \beta_f) \), is given by

\[
\langle q_i(0) q_i(r) \rangle = -\exp \left\{ -2\pi \beta \left[ \Gamma_{\text{lr}}(r) + \min \left( \frac{\beta_b}{\beta_f}, \frac{\beta_f}{\beta_b} \right) \Gamma_{\text{sr}}(r) \right] \right\}.
\]

(40)

Note, therefore, that vortex-antivortex excitations are the least energetically costly for the latter species.

In the particular case of the physical electronic phase correlator, for which \( p_{\text{st}} = 0 \), the above expression (38) has precisely the same form as that corresponding to the XY model;\(^{18,19}\) i.e., \( \sigma(r') = \pm p_{\text{el}} \sigma_{\text{lr}}(r') \). Due to the fact that vortices are bound to anti-vortices in the present low-temperature limit, we may expand \( \sigma_{\text{lr}}(r') \) and \( \sigma_{\text{lr}}(r'') \) with reference to the center of mass coordinates; i.e.,

\[
\sigma_{\text{lr}}(r') = \sigma_{\text{lr}}(R) + \frac{1}{2} \vec{r} \cdot \vec{\nabla} \sigma_{\text{lr}}(R),
\]

\[
\sigma_{\text{lr}}(r'') = \sigma_{\text{lr}}(R) - \frac{1}{2} \vec{r} \cdot \vec{\nabla} \sigma_{\text{lr}}(R),
\]

\[ p_{\text{st}} = \pm (\beta_f/\beta_b)^{1/2}, \quad \Gamma_{\text{st}} = \pm (\beta_b/\beta_f)^{1/2}. \]

(41)
where $\mathbf{R}$ is the center of mass coordinate and $\mathbf{r}$ is the relative coordinate between points $\mathbf{r}'$ and $\mathbf{r}''$. Due to the charge neutrality condition effectively imposed by the presence of long-range interactions, only the last terms above remain in sum over coordinates found in the exponent of expression (38). This ultimately results in

$$G_{\text{Coulomb}}(\mathbf{r}_1 - \mathbf{r}_2) = \exp \left[ \frac{1}{2} \frac{p_{el}^2}{2} \int \frac{1}{r^2} \langle q_i(0)q_i(r) \rangle \sum_{\mathbf{R}} \frac{1}{4} \sqrt{\nabla \sigma_{lr}(\mathbf{R})^2} \right],$$

(41)

where the factor of $1/2$ found in the sum over the relative coordinate $r$ is due to an angle average. However, it can be shown in a straightforward manner that

$$\sum_{\mathbf{R}} \frac{1}{4} \sqrt{\nabla \sigma_{lr}(\mathbf{R})^2} = \pi \Gamma_{lr}(\mathbf{r}_1 - \mathbf{r}_2).$$

Inserting this identity into (41), we arrive at the final closed form expression

$$G_{\text{Coulomb}}(\mathbf{r}_1 - \mathbf{r}_2) = \exp \left[ \frac{1}{4} \pi p_{el}^2 \Gamma_{lr}(\mathbf{r}_1 - \mathbf{r}_2) \sum_{\mathbf{r}} r^2 \langle q_i(0)q_i(r) \rangle \right].$$

(42)

Last, factoring in above the previous result (33) for the “spin-wave” contribution, we find that in the limit $|\mathbf{r}_1 - \mathbf{r}_2| \gg \lambda_{\text{st}}$, the auto-correlation function (26) for the physical electronic phase is given by

$$\langle e^{i \phi_{el}(1) - \phi_{el}(2)} \rangle \approx \exp \left[ -\frac{p_{el}^2}{2\pi \beta_{\text{eff}}} \Gamma_{lr}(\mathbf{r}_1 - \mathbf{r}_2) \right].$$

(43)

in the low-temperature limit, with an effective temperature scale

$$\frac{1}{\beta_{\text{eff}}} = \frac{1}{\beta} - \frac{\pi^2}{2} \sum_{|\mathbf{r}| > a} r^2 \langle q_i(0)q_i(r) \rangle.$$

(44)

We now recall that $\Gamma_{lr}(\mathbf{r}) \approx \ln(2^{3/2}e^\gamma r)$, where $\gamma$ is Euler’s constant. Hence, in the limit $\lambda_{\text{st}} \to \infty$, where statistical gauge-field fluctuations in the energy functional (4) are “frozen out”, inspection of Eq. (40) reveals that $\beta_{\text{eff}}^{-1}$ diverges at $2\pi \min(\beta_b, \beta_f) = 4$. This implies a BKT transition temperature of $T_c \approx \frac{\pi}{2} \min(J_b, J_f)$ below which algebraic long-range order sets in (see dashed lines, Fig. 1). The latter result is precisely what one expects from employing the Ioffe-Larkin formula for the London penetration length in a spin-charge separated superconductor, i.e., the mean-field approximation result (2). On the other
hand, in the general case of finite $\lambda_{st}$, similar considerations reveal that $\beta_{\text{eff}}^{-1}$ diverges at $2\pi \tilde{\beta} = 4$, which implies a BKT transition temperature given by Eq. (9). Hence, both the previous continuum limit analysis and the present low-temperature analysis arrive at the same answer concerning the nature of the BKT phase-transition in the AH$^2$ model.

Last, we note that the previous analysis for the Coulomb gas factor, $G_{\text{Coulomb}}$, in the particular case of the physical electronic phase correlator ($p_{\text{st}} = 0$) is also valid for the general case in the limit of maximum statistical gauge-field fluctuations, $\lambda_{st} \to 0$ [see Eq. (32)],$^{18,19}$ with the exception that we must replace $p_{\text{el}}$ in Eq. (42) with the orginal expression (34a) for $p_{\text{el}}''$. Factoring in the latter result with the previous result (33) for the “spin-wave” contribution, one finds that the correlation function (26) is given by

$$C_{12} \cong \exp \left( -\frac{p_{\text{st}}^2}{2\tilde{\beta}} |\vec{r}_1 - \vec{r}_2| \right) \exp \left[ -\frac{p_{\text{el}}''^2}{2\pi \beta_{\text{eff}}} \Gamma_{\text{fr}} (\vec{r}_1 - \vec{r}_2) \right].$$

Hence, we see that in the limit of maximum fluctuations in the statistical gauge-field considered here, while the phase auto-correlations are generally short-range, a remnant of the long-range behaviour that exists in the special case of the electronic auto-correlations function (43) persists.

V. Statistical Gauge-field Excitations and the Wilson Loop

We have demonstrated above that the statistical gauge-field does not acquire a gap via the Higgs mechanism, since the corresponding auto-correlations for the statistical phase (25) are short-range. However, a gap in a $U(1)$ gauge-field can also result from confinement effects in 2+1 dimension,$^{27}$ for example. Such effects are conveniently probed by the Wilson loop, $\langle \exp (ie' \oint A_\mu dx_\mu) \rangle$. Below we show that the presently considered AH$^2$ model displays a perimeter law for this quantity at all temperatures below the BKT-transition discussed above, which simply reflects the existence of bound vortices, and which corresponds to a “deconfined” phase. Also, although we do find that the Wilson loop exhibits a “confining” area law at temperatures above the BKT phase-transition, we argue that this behavior is a general property of pure Abelian gauge theories in two euclidean dimensions. We thereby conclude that the statistical gauge-field excitations remain gapless at all temperatures in the present AH$^2$ model.
A. Low-Temperature Phase. Let us first consider the Wilson loop for a large contour $C$ in low temperature limit. As in the previous case of the phase auto-correlation functions, it can be expressed as

$$W(C) = \left\langle \exp \left( ie' \oint_C A_\mu dr_\mu \right) \right\rangle = \frac{Z'}{Z},$$

(46)

where the partition function $Z'$ differs from $Z$ only by the addition of the exponent $-ie' \oint_C A_\mu dr_\mu$ to the energy functional (4). Again, following refs. 18 and 19, $Z'$ may be computed via the Villain duality transformation by substitution of the mathematical identity (12), which is valid in the low-temperature limit. Generalizing the treatment of Jones et al., 11 we find ultimately that $Z'$ has the following “roughening model” representation:

$$Z' = \sum_{\{n_b(r)\}} \sum_{\{n_f(r)\}} \exp \left\{ -\frac{1}{2\beta_b} \sum_{r,\mu} [\Delta_\mu n_b(r)]^2 - \frac{1}{2\beta_f} \sum_{r,\mu} [\Delta_\mu n_f(r)]^2 - \frac{(qg)^2}{2} \sum_r \left[ n_b(r) + n_f(r) + \frac{e'qg}{qg} J(r) \right]^2 \right\},$$

(47)

where $J(r)$ has a value of 1 if the point $r$ is within the contour $C$, and it vanishes otherwise. Again, after the application of the Poisson summation formula, and subsequently integrating over the fields that this formula generates, we find that $Z'$ is given by the product a trivial “spin-wave” factor with the following Coulomb gas ensemble average:

$$Z'_{\text{Coulomb}} = \sum_{\{q_\alpha(r)\}} \exp \left\{ -(2\pi)^2 \beta^2 \frac{1}{2} \sum_{r,r'} q_\alpha(r) G^{\text{sr}}_{\alpha\beta}(r,r') q_\beta(r') + i\pi^{-1} e'qg q_\alpha(r) G^{(sr)}_{\alpha\beta}(r,r') u_\beta J(r') \\
- (2\pi)^{-2} e'^2 (qg)^2 J(r) u_\alpha G^{(sr)}_{\alpha\beta}(r,r') u_\beta J(r') + (2\pi)^{-2} e'^2 \beta^{-1} J(r) \delta_{r,r'} \right\},$$

(48)

again where $\bar{u} = (1,1)$ represents the spin/charge components of the statistical degree of freedom (25), and where $G^{(sr)}_{\alpha\beta}(r,r')$ represents the short-range component of the matrix Greens function [see Eqs. (16-19)]. The above expression constitutes the Coulomb gas representation of the modified partition function $Z'$.

To further reduce expression (48) for $Z'_{\text{Coulomb}}$, we recall that

$$\sum_r G^{(sr)}_{\alpha\beta}(r,0) = \frac{\beta_\alpha \beta_\beta}{\beta_b \beta_f} \lambda^2_{\text{st}}.$$
Hence, in the limit of large contours, $C$, the sum over the relative coordinates in the third term of this expression can be extended to the entire lattice. In this case we find that the last two terms in (48) sum to zero; i.e.,

$$
(2\pi)^{-2} e'^2 \left[ - \sum_r \frac{(qg)^2}{\beta_b \beta_f} (\beta_b + \beta_f)^2 \lambda^2_{st} J(r) + \sum_r \left( \frac{1}{\beta_b} + \frac{1}{\beta_f} \right) J(r) \right] = 0,
$$

by (20). Note that the first term above corresponds to the third term in (48), where the sum over the relative coordinates has been already performed. Similarly, since $G^{(sr)}_{\alpha\beta}(r, r')$ is short-ranged, we have that in the limit of large contours,

$$
\sum_{r, r'} q_i(r) G^{(sr)}_{\alpha\beta}(r, r') u_{ij} J(r') = \frac{\beta_b + \beta_f}{\beta_b \beta_f} \lambda^2_{st} \sum_r [\beta_b q_b(r) + \beta_f q_f(r)] J(r).
$$

The lefthand side of this expression is proportional to the second term in (48), which implies that the Wilson loop, $Z'/Z$, is given by

$$
W(C) = \left\langle \exp \left[ -2\pi i e' qg \lambda^2_{st} \sum_r [\beta_b q_b(r) + \beta_f q_f(r)] J(r) \right] \right\rangle_{\text{Coulomb}}.
$$

(49)

Since vortices are bound to anti-vortices at low temperature due to the existence of long-range interactions, we may employ a cummulant expansion to compute this average, as in the previous calculation of phase correlators [see Eqs. (36) and (37)]. Such an approximation yields

$$
W(C) = \exp \left\{ -\left(2\pi e'/gq\right)^2 \lambda^4_{st} \frac{1}{2} \sum_{r', r''} \langle [\beta_b q_b(r') + \beta_f q_f(r')][\beta_b q_b(r'') + \beta_f q_f(r'')] \rangle J(r') J(r'') \right\}
$$

$$
= \exp \left\{ -\left(\frac{2\pi e'}{gq}\right)^2 \left[ 1 + \max \left( \frac{\beta_b}{\beta_f}, \frac{\beta_f}{\beta_b} \right) \right] \sum_{r', r''} \langle q_i(r') q_i(r'') \rangle J(r') J(r'') \right\},
$$

(50)

again where the species label $i$ is such that $\beta_i = \min(\beta_b, \beta_f)$. Since the charge correlations decrease rapidly, we may expand $J(r')$ and $J(r'')$ with reference to the center of mass coordinates; i.e.,

$$
J(r') = J(R) + \frac{1}{2} \vec{F} \cdot \vec{V} J(R),
$$

$$
J(r'') = J(R) - \frac{1}{2} \vec{F} \cdot \vec{V} J(R),
$$

20
where \( R \) is the center of mass coordinate and \( r \) is the relative coordinate between points \( r' \) and \( r'' \). Due to the charge neutrality condition effectively imposed by the presence of long-range interactions, only the last terms above remain in sum over coordinates found in the exponent of expression (50). This ultimately results in

\[
W(C) = \exp \left\{ \frac{1}{2} \left( \frac{2 \pi e'}{q g} \right)^2 \left[ 1 + \max \left( \frac{\beta_b}{\beta_f}, \frac{\beta_f}{\beta_b} \right) \right]^{-2} \sum_r \frac{1}{2} r^2 (q_i(0)q_i(r)) \sum_R \frac{1}{4} (\nabla J(R))^2 \right\} 
\]

where that prefactor of \( \frac{1}{2} \) in the sum over the relative coordinates arises from an angular average. Above, \( P \) denotes the length of the contour \( C \). The latter expression represents the final result of our manipulations.

We observe, therefore, that at the low temperatures presently considered, the Wilson loop exhibits a perimeter law. And since the vortex charge correlator, \( \langle q_i(0)q_i(r) \rangle \), corresponding to the species with the least costly vortex excitations is given by Eq. (40), we also observe that the coefficient to the above perimeter-law diverges precisely at the BKT phase transition temperature (9). This suggests that a “confining” area law behavior exists above \( T_c \), which we in fact argue for below. Last, it is also of interest to remark that this coefficient vanishes in the limit \( J_b \gg J_f \) or \( J_f \gg J_b \).

**B. High-Temperature Phase.** Clearly, in the high-temperature limit, \( \beta_b = 0 \) and \( \beta_f = 0 \), the present AH\(^2\) reduces to a pure Abelian gauge theory, which is trivially “confining” in 1+1 dimensions, and which therefore shows an area law for the Wilson loop. Also, in either the limits \( \beta_b = 0 \) or \( \beta_f = 0 \) that correspond to the one-component model, it is well know that the Wilson loop follows an area law at all temperatures in the case of fractional probe charges \( e' \).\(^{11}\) It has recently been argued, however, that such “confinement” is trivially due to pure 1+1 dimensional statistical “electro-magnetism” that is renormalized “dielectrically”\(^{21}\). In this case, one finds that gauge-field fluctuations are characterized by a free \( U(1) \) action [see the last term in Eq. (4)], with a renormalized “charge”, \( g/\epsilon_{st}^{1/2} \), that is on the order of \( g \) for temperatures above a cross-over temperature scale,\(^8\) \( T_{c/o} \approx J_s \) (\( s = b \) or \( f \)), and that becomes exponentially small below this cross-over temperature scale.\(^{21}\) This ultimately leads to a cross-over phenomenon between a “strange” metal phase at high temperatures and its absence at low temperature.\(^8\) Given that the only
true phase transition in the present AH$^2$ model is the BKT transition itself, by continuity in the phase diagram (see Fig. 1), (i) the Wilson loop should follow a “confining” area law at all temperatures above $T_c$ for fractional probe charges $e'$ and (ii) we expect that a similar crossover phenomenon occurs in the present AH$^2$ model in this temperature regime, with the exception that the crossover temperature scale is given by $T_{c/a} \approx \max(J_b, J_f)$ in this case.

In summary, the Wilson loop (46) of the statistical gauge-field in the presently considered AH$^2$ model shows a perimeter law at low temperatures $T < T_c$, such that the corresponding coefficient diverges at $T_c$. Also, in the general case of fractional probe charges, $e' \neq nq$ $(n = 0, \pm 1, \pm 2, \ldots)$, the Wilson loop shows an area law for temperatures above the BKT transition, which we argue implies a gapless spectrum for statistical gauge-field excitations. Since we showed in the previous section that there exists no Higgs mechanism for the statistical gauge-field to acquire a gap due to the complete absence of long-range order in the statistical phase (25), and since the perimeter law exhibited by the Wilson loop in the low temperature phase does not indicate the existence of short-range “electromagnetic” interactions, we argue that the statistical gauge-field remains gapless as well in this phase. The occurrence of a perimeter law, therefore, is simply a signal of the vortex binding-unbinding transition present in the system.

VI. Universality Class

By consideration of both the continuum limit (section III) and the low-temperature limit (section IV), we have shown above that a short-range order to algebraic long-range order phase transition occurs at a transition temperature given by Eq. (9) for the physical electronic phase (8) in the present AH$^2$ model for spin-charge separated superconductivity. But what is the nature of this transition in physical terms? Below, we argue that the forementioned transition is in the same universality class as that of the XY model. Hence, we expect a corresponding universal jump in the superfluid density.$^{16}$

Let us now derive the renormalization group equations for the BKT transition in the physical electronic phase of the present model by closely following the treatment of the
corresponding problem in the case of the XY model. From eq. (44), we have that

$$\frac{1}{\beta_{\text{eff}}} = \frac{1}{\beta} + \pi^3 y^2 \int_a^\infty \frac{dr}{ar} \left(\frac{a}{r}\right)^{2\pi \beta(r) - 3}, \quad (52)$$

where by Eq. (40) for the charge correlations, $\beta(r)$ is a smooth function satisfying

$$\beta_i; \quad r \ll \lambda_{\text{st}} \quad (53a)$$

$$\tilde{\beta}(r) = \tilde{\beta}; \quad r \gg \lambda_{\text{st}} \quad (53b)$$

and where the activity at $r = r_0$ is given by

$$y = \exp \left[-\pi \tilde{\beta}(r_0) \ln \frac{a}{r_0}\right]. \quad (54)$$

Let $x = r/a$, and suppose that we make the following approximation to the integral appearing in the right-hand side of Eq. (52):

$$\int_1^\infty dx x^{3-2\pi \tilde{\beta}(x)} \approx \int_1^{\lambda_{\text{st}}} dx x^{3-2\pi \beta_i} + \int_{\lambda_{\text{st}}}^\infty dx x^{3-2\pi \tilde{\beta}}$$

$$= \frac{\lambda_{\text{st}}^4 - 2\pi \beta_i - 1}{4 - 2\pi \beta_i} + \lambda_{\text{st}}^{4-2\pi \tilde{\beta}} \int_1^{\lambda_{\text{st}}} dx x^{3-2\pi \tilde{\beta}}$$

$$\rightarrow \ln \lambda_{\text{st}} + [1 + (4 - 2\pi \tilde{\beta}) \ln \lambda_{\text{st}}] \int_1^{\lambda_{\text{st}}} dx x^{3-2\pi \tilde{\beta}} \quad (55)$$

in the limit $\lambda_{\text{st}} \rightarrow 1$. Hence, in this limit, we have by (52) that

$$\frac{1}{\beta_{\text{eff}}} = \frac{1}{\beta} + \pi^3 y^2 \ln \lambda_{\text{st}} + \pi^3 y^2 [1 + (4 - 2\pi \tilde{\beta}) \ln \lambda_{\text{st}}] \int_1^{\lambda_{\text{st}}} dx x^{3-2\pi \tilde{\beta}}. \quad (56)$$

This formula may be reexpressed as

$$\frac{1}{\beta_{\text{eff}}} = \frac{1}{\beta_\lambda} + \pi^3 y^2 \int_1^{\lambda_{\text{st}}} dx x^{3-2\pi \tilde{\beta}}, \quad (57)$$

where

$$\frac{1}{\beta_\lambda} = \frac{1}{\beta} + \pi^3 y^2 \ln \lambda_{\text{st}}, \quad (58)$$

$$y_\lambda = y + (2 - \pi \beta_\lambda) y \ln \lambda_{\text{st}}. \quad (59)$$
The differential form of the latter two relations are simply the Kosterlitz renormalization group equations\textsuperscript{16,19}

\[
\lambda_{st} \frac{d}{d\lambda_{st}} \beta_{\lambda}^{-1} = \pi^3 y_{\lambda}^2, \tag{60}
\]

\[
\lambda_{st} \frac{d}{d\lambda_{st}} y_{\lambda} = (2 - \pi \beta_{\lambda}) y_{\lambda}, \tag{61}
\]

with the length renormalization scale determined by the model parameter $\lambda_{st}$.

In summary, given the validity of approximation (55), we find the BKT transition obtained previously in the present AH\textsuperscript{2} model for the physical electronic phase is in the same universality class as that of the XY model. We therefore expect a universal jump in the superfluid density, following that predicted for the XY model.\textsuperscript{16}

\textbf{VII. Summary and Discussion}

In this paper, we have introduced a square lattice gauge theory model for spin-charge separated superconductivity in the presence of statistical gauge-field excitations. Such excitations appear naturally in spin-charge separated descriptions of strongly interacting electrons systems, and they represent so-called chiral spin fluctuations.\textsuperscript{10} After analysing this model in various limits (continuum and low-temperature), we found that it explicitly shows a superconducting transition of the BKT type for the physical electronic phase (8) at a critical temperature given by (9). As expected, we see that statistical gauge field excitations suppress the superfluid transition temperature with respect to mean-field; i.e., $T_c < T_c^{(0)}$. It was also argued in the preceding section that this transition shares the same universality class as that of the XY model, and hence that we expect a corresponding universal jump in the superfluid density. All other phases, however, were found to show only short-range auto-correlations at low-temperature. In particular, the fact that statistical phase (25) correlations are short-range at all temperatures implies that the corresponding transverse statistical gauge-field fluctuations do not acquire a gap via the Higgs mechanism. Furthermore, we have argued that because of dimensionality, neither “confinement” effects produce a gap in these excitations. Hence, we claim that the statistical gauge-field excitations remain gapless at all temperature in the present model. It is important to note, however, that gapless statistical gauge-field excitations also have been shown to exist at
the meanfield level in the commensurate flux phase of the $t - J$ model, which is thought to be a spin-charge separated anyonic superconductor, but they were to found acquire a gap once dynamical effects were included.\textsuperscript{22} It is therefore not certain that such gapless excitations will persist in the present model once such effects are accounted for. We hope to address this issue in a future publication.

The superfluid transition found in the AH\textsuperscript{2} model for spin-charge separated superconductivity can be understood \textit{a posteriori} as follows. Let $\Psi_f = \langle c_{i\uparrow}c_{i'\downarrow} \rangle$ and $\Psi_b = \langle b_ib_{i'} \rangle$ be the respective order parameters of each specie. Now suppose that the superconductivity in this spin-charge separated system results from conventional Cooper pairing of electrons, with an order parameter given by $\Psi_{el} = \langle c_{i\uparrow}b_{i'}\dagger c_{i'\downarrow}b_{i'}\dagger \rangle$. Then the mean-field approximation yields that the true superconducting order parameter satisfies

$$\Psi_{el} \simeq \Psi_f \Psi_b^* \propto e^{i(\phi_f - \phi_b)}.$$ 

It is thus not surprising to find that the physical electronic phase (8) develops quasi-long-range order in the present model. What is indeed important to note, however, is that fluctuations in the statistical gauge field do not suppress the transition altogether.

Yet can the present model explain, or at least describe, the phenomenon of high-temperature superconductivity? Mean-field RVB treatments of the $t - J$ model yield a superfluid phase stiffness for spinon pairs of $J_f \sim J(1 - x/x_0)$, where $x$ denotes the hole concentration and $x_0$ denotes the critical concentration beyond which there is no pairing instability.\textsuperscript{9} Also, since we have assumed throughout that the holons form pairs, then their corresponding superfluid phase stiffness is given by $J_b \sim t'x$, where $t'$ denotes the hopping matrix element for such pairs. Now suppose that the latter hopping matrix element satisfies $t' \sim 1000 \text{K}$, which is conceivable since the corresponding matrix elements for a single electron is on the order of $t \sim 5000 \text{K}$. Then since Eq. (9) implies that the superconducting temperature satisfies $T_c \lesssim \min(t'x_0, J)$, and since $J \sim 1000 \text{K}$ while the maximum hole concentrations for oxide superconductivity are typically near $x_0 \cong 0.2$, we have that $T_c \lesssim t'x_0 \sim 200 \text{K}$, in agreement with experiment.\textsuperscript{1} More importantly, however, Eq. (9) for the critical temperature coupled with the above functional dependences for the superfluid stiffnesses of each species with hole concentration imply a phase-diagram for the superconducting versus normal phase that is shown in Fig. 1. The shape of the phase boundary qualitatively resembles that of the high-temperature superconductors.\textsuperscript{20} but it
differs substantially in shape from that predicted by Ginzburg-Landau treatments of spin-charge separated superconductivity. Finally, since \( g^2 = T/\chi_d \) in this case, where \( \chi_d = \chi_f + \chi_b \sim J(1-x) + tT_b/T \) is the sum of the diamagnetic susceptibilities of each species, \(^5, ^6\) with \( T_b \approx tx \), we then have that \( \lambda_{st} = q^{-1} [\chi_d/(J_b + J_f)]^{1/2} \approx 1 \). Hence, the renormalization group analysis discussed in the previous section is valid, and we therefore expect a universal jump in the superfluid density at the critical temperature in the absence of interplanar coupling. Note that the present description of oxide superconductivity differs substantially from that proposed by Anderson and coworkers,\(^28\) where it is assumed that interlayer interactions drive the phenomenon. Although it is proposed here, on the contrary, that oxide superconductivity is primarily a one-layer effect, we nevertheless hope to study the problem of a few coupled layers of square lattice AH\(^2\) models in a future publication. It is quite possible that the nature of the presently discovered superfluid transition changes dramatically once the model is extended into the third dimension, which could be relevant to the layered high-temperature superconductors.\(^8\)

Finally, what is the nature of the normal state above \( T_c \)? It was previously remarked in section II that separate “strange” metal, Fermi-liquid, superconducting, and spin-gap phases exist in certain spin-charge separated treatments of the \( t - J \) model in two dimensions at the mean-field level,\(^8\) as shown by the dashed lines in Fig. 1. We remind the reader that the “strange” metal phase corresponds to the absence of superfluidity in both species, the Fermi-liquid phase to the appearance of superfluidity in the holon species alone, the superconducting phase to the appearance of superfluidity in both species, and the spin-gap phase to the existence of superfluidity in the spinon species alone. In this paper, we have demonstrated explicitly that the only true phase transition that remains beyond the meanfield approximation is the superconducting one depicted by the solid line in Fig. 1. In particular, fluctuations in the statistical gauge field representing chiral spin-fluctuations\(^10\) destroy the meanfield transitions between the “strange” metal phase and the Fermi-liquid phase, and between the “strange” metal phase and the spin-gap phase.\(^8\) This effect can be studied more easily by considering the case when only one of the species has a superfluid instability; i.e., the one-component Abelian-Higgs model corresponding to \( \beta_b = 0 \) or \( \beta_f = 0 \). It has been explicitly shown in this case, by using techniques similar to those employed in this paper, that the BKT superfluid transition present in the absence of statistical gauge-
field fluctuations becomes a cross-over phenomenon. The cross-over temperature scale satisfies \( T_{c/o} \lesssim T_s \), where \( T_s \) denotes the meanfield transition temperature of the species in question, and it vanishes in the limit of maximum gauge-field fluctuations, \( \lambda_{st} \to 0 \). By continuity with the present AH\(^2\) model, we expect that this crossover phenomenon persists in the normal phase above \( T_c \); i.e., \( T_{c/o} \simeq \max(T_b, T_f) \). Furthermore, in the limit \( \lambda_{st} \to 0 \), the normal phase should be entirely “strange”, with a characteristic \( T \)-linear resistivity. As discussed above, however, reasonable estimates based on the \( t - J \) model yield a statistical length scale satisfying \( \lambda_{st} \gtrsim 1 \). Hence, we expect that the cross-over phenomenon from the “strange” metal regime into the Fermi-liquid and spin-gap regimes should persist in strongly interacting 2D electron systems. Such a cross-over scale (see the upper lines in Fig. 1) is consistent with the narrow doping window for linear-in-\( T \) resistivity recently observed in the superconducting oxides.

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Appendix A: “Spin-Wave” Component of Phase Correlator

To calculate the “spin-wave” component of the phase auto-correlation function (31) in the AH$^2$ model at low-temperature, given by

$$G_{\text{wave}}(\vec{r}_{12}) = \exp X_{\text{wave}},$$  \hspace{1cm} (A.1)

with

$$X_{\text{wave}} = -\frac{1}{2} \left( \frac{p_b^2}{\beta_b} + \frac{p_f^2}{\beta_f} \right) \sum_{r,\mu} \eta^2_{\mu}(r)$$

$$- \frac{1}{\pi \beta_b \beta_f} \sum_{r,r'} \eta(r) \left[ p_{\text{el}}^2 \Gamma_{lr}(\vec{r} - \vec{r'}) + p_{\text{st}}^2 \Gamma_{sr}(\vec{r} - \vec{r'}) \right] \eta(r'),$$  \hspace{1cm} (A.2)

we will follow the standard treatment of the corresponding problem for the XY model.$^{18, 19}$

Let us suppose that $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ is directed along the x-axis. Then clearly,

$$X_{\text{wave}} = -\frac{1}{2} \left( \frac{p_b^2}{\beta_b} + \frac{p_f^2}{\beta_f} \right) \vec{r}_{12}$$

$$- \frac{1}{\pi \beta_b \beta_f} \sum_{x,x'=0}^{r_{12}-1} p_{\text{el}}^2 \left[ 2 \Gamma_{lr}(x - x', 0) - \Gamma_{lr}(x - x', 1) - \Gamma_{lr}(x - x', -1) \right]$$

$$- \frac{1}{\pi \beta_b \beta_f} \sum_{x,x'=0}^{r_{12}-1} p_{\text{st}}^2 \left[ 2 \Gamma_{sr}(x - x', 0) - \Gamma_{sr}(x - x', 1) - \Gamma_{sr}(x - x', -1) \right].$$  \hspace{1cm} (A.3)

But since $\Gamma_{lr}(\vec{r})$ and $\Gamma_{sr}(\vec{r}) - \Gamma_{sr}(\infty)$ are proportional to the respective long-range and short-range lattice Greens functions, then by definition [see Eqs. (16), (18) and (19)] they satisfy

$$4 \Gamma_{lr}(x, y) - [\Gamma_{lr}(x + 1, y) + \Gamma_{lr}(x - 1, y)$$

$$+ \Gamma_{lr}(x, y + 1) + \Gamma_{lr}(x, y - 1)] = -2\pi \delta_{x,0} \delta_{y,0},$$  \hspace{1cm} (A.4)

$$4 \Gamma_{sr}(x, y) - [\Gamma_{sr}(x + 1, y) + \Gamma_{sr}(x - 1, y)$$

$$+ \Gamma_{sr}(x, y + 1) + \Gamma_{sr}(x, y - 1)] + \lambda_{\text{st}}^2 [\Gamma_{sr}(x, y) - \Gamma_{sr}(\infty)] = -2\pi \delta_{x,0} \delta_{y,0}. \hspace{1cm} (A.5)$$

Substituting in the latter identities into (A.3) for the Greens function terms at points off of the $y = 0$ axis, and then using the fact that the resulting sum over $x$ and $x'$ is a partially
telescoping series,\textsuperscript{18,19} we find that
\[
X_{\text{wave}} = -\frac{1}{2} \left( \frac{p_b^2}{\beta_b} + \frac{p_f^2}{\beta_f} \right) r_{12} - \frac{1}{\pi} \frac{\bar{\beta}}{\beta_b \beta_f} p_{\text{el}}^2 [2 \Gamma_{\text{lr}}(r_{12}) - 2 \pi r_{12}]
\]
\[
- \frac{1}{\pi} \frac{\bar{\beta}}{\beta_b \beta_f} p_{\text{st}}^2 \left\{ 2 \Gamma_{\text{sr}}(r_{12}) - 2 \pi r_{12} - \lambda_{\text{st}}^{-2} \sum_{x,x'=0}^{r_{12}-1} \left[ \Gamma_{\text{sr}}(x - x') - \Gamma_{\text{sr}}(\infty) \right] \right\},
\]
\[
= - \frac{2 \bar{\beta}}{\beta_b \beta_f} p_{\text{el}}^2 \Gamma_{\text{lr}}(r_{12})
\]
\[
- \frac{2 \bar{\beta}}{\beta_b \beta_f} p_{\text{st}}^2 \left\{ \frac{\Gamma_{\text{sr}}(r_{12})}{\pi} + \lambda_{\text{st}}^{-2} \sum_{x,x'=0}^{r_{12}-1} \frac{1}{2 \pi} \left[ \Gamma_{\text{sr}}(\infty) - \Gamma_{\text{sr}}(x - x') \right] \right\}.
\] (A.6)

But in the limit $r_{12} \gg \lambda_{\text{st}}$, we have the identity
\[
\lambda_{\text{st}}^{-2} \sum_{x,x'=0}^{r_{12}-1} \frac{1}{2 \pi} \left[ \Gamma_{\text{sr}}(\infty) - \Gamma_{\text{sr}}(x - x') \right] \approx \lambda_{\text{st}}^{-2} \sum_{x=0}^{r_{12}-1} \sum_{x'=0}^{\infty} \frac{1}{2 \pi} \left[ \Gamma_{\text{sr}}(\infty) - \Gamma_{\text{sr}}(x - x') \right] = r_{12}.
\]

Substituting this relationship into the last term of (A.6), we find by (A.1) that the “spin-wave” component of the phase correlator in the AH\textsuperscript{2} model on the square-lattice is given by
\[
G_{\text{wave}}(\vec{r}_{12}) = \exp \left[ - \frac{2 \bar{\beta}}{\beta_b \beta_f} p_{\text{el}}^2 (r_{12} + \pi^{-1} \ln \lambda_{\text{st}}) \right] \exp \left[ - \frac{2 \bar{\beta}}{\beta_b \beta_f} p_{\text{st}}^2 \frac{\Gamma_{\text{lr}}(r_{12})}{\pi} \right] \] (A.7)
in the limit $|\vec{r}| \gg \lambda_{\text{st}}$, where

\[
\ln \lambda_{\text{st}} = \Gamma_{\text{sr}}(\infty) = (2\pi)^{-1} \int_{\text{BZ}} d^2 k (\vec{k}^2 + \lambda_{\text{st}}^{-2})^{-1}.
\] (A.8)

Note, of course, that the function defined above coincides with the Napierian logarithm in the limit $\lambda_{\text{st}} \gg 1$.

**Appendix B: Phase Correlations in One-component Model**

Consider the gauge-invariant phase correlation function

\[
C_{12} = \langle \exp \{ i p_b [\phi_b(r_1) - \phi_b(r_2)] + q \int_{r_1}^{r_2} A_\mu dr_\mu \} \rangle = Z'/Z
\] (B.1)
in the case of the one-component Abelian-Higgs model $\beta_f = 0$, where the partition function $Z'$ differs from $Z$ only by the addition of the exponent $-i p_b [\phi_b(r_1) - \phi_b(r_2)] + q \int_{r_1}^{r_2} A_\mu dr_\mu$.
to the energy functional (4). Then by (27), we have that the Villain duality transformation of this partion function is given by

$$Z' = \sum_{\{n_b(r)\}} \exp\left\{ -\frac{1}{2\beta_b} \sum_{r,\mu} [\Delta_\mu n_b(r) + p_b \eta_\mu(r)]^2 - \frac{g^2 q^2}{2} \sum_r n_b^2(r) \right\}. \quad (B.2)$$

As discussed in section IV, $n_b(r)$ ranges over the integers and $r$ covers the dual lattice, while the dual lattice “dipole” vector, $\eta_\mu(r)$, has a value of $\pm 1$ if it intersects a fixed path on the original lattice connecting points $\vec{r}_1$ and $\vec{r}_2$, and it vanishes otherwise.\(^{18,19}\) Therefore, in the limit $\lambda_{st} \ll 1$, where statistical gauge-field fluctuations are maximized, the trivial configuration $n_b(r) = 0$ dominates the ensemble average above, resulting in

$$C_{12} \cong \exp\left( -\frac{p_b^2}{2\beta_b} \mid \vec{r}_1 - \vec{r}_2 \mid \right). \quad (B.3)$$

Notice, as expected [see Eq. (9)], that this correlation function shows no evidence for a phase-transition at non-zero temperature.

It was also shown previously that the application of the Poisson summation formula to the “roughening model” representation expression (B.2) leads ultimately to the factorization $C_{12} \cong G_{\text{wave}}(\vec{r}_1 - \vec{r}_2)G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2)$ of the gauge-invariant phase correlation function. In the present case of $\beta_f = 0$ and $p_f = 0$, Eq. (A.7) yields that

$$G_{\text{wave}}(\vec{r}_1 - \vec{r}_2) \cong \exp \left[ -\frac{p_b^2}{2\beta_b} (\mid \vec{r}_1 - \vec{r}_2 \mid + \pi^{-1} \ln \lambda_{st}) \right] \quad (B.4)$$

for $r \gg \lambda_{st}$, while Eq. (36) yields that

$$G_{\text{Coulomb}}(\vec{r}_1 - \vec{r}_2) = \left\langle \exp \left\{ ip_b \sum_r q_b(r) \sigma_{sr}(r) \right\} \right\rangle_{\text{Coulomb}}, \quad (B.5)$$

where the corresponding partition function (21) for the screened Coulomb gas ensemble is given by\(^{25}\)

$$Z_{\text{Coulomb}} = \sum_{\{q_b(r)\}} \exp \left\{ 2\pi \beta_b \sum_{(r,r')} \Gamma_{sr}(\vec{r} - \vec{r}') q_b(r) q_b(r') \right\}. \quad (B.6)$$

Unlike the previous treatment in the case of the AH\(^2\) model, however, we now take the original definition

$$\Gamma_{sr}(\vec{r}) = -\int_{\text{BZ}} \frac{d^2 k}{2\pi} e^{i\vec{k} \cdot \vec{r}} \frac{1}{k^2 + \lambda_{st}^{-2}}, \quad (B.7)$$
for the interaction energy, since charge conservation is no longer guaranteed due to the absence of long-range forces. Following ref. 11, in the limit \( \lambda_{st} \gg 1 \), in which case the interaction between vortices is negligible at large distances, we may approximate the above ensemble average by

\[
Z_{\text{Coulomb}} \approx \Pi_r \sum_{q_b(r)=-\infty}^{\infty} \exp \left[ -2\pi \beta_b \frac{1}{2} (\ln \lambda_{st}) q_b^2(r) \right].
\] (B.8)

Hence, if we truncate the sum over vortex charge to \( q_b(r) = 0, \pm 1 \), then by (B.5),

\[
G_{\text{Coulomb}} \approx \frac{\Pi_r \{1 + 2 \exp(-\pi \beta_b \ln \lambda_{st}) \cos [p_b \sigma_{sr}(r)]\}}{\Pi_r \{1 + 2 \exp(-\pi \beta_b \ln \lambda_{st})\}}
\approx \exp(2e^{-\pi \beta_b \ln \lambda_{st}} \sum_r \{\cos [p_b \sigma_{sr}(r)] - 1\})
\approx \exp(2\lambda_{st}^{-\pi \beta_b} \sum_r \{\cos [p_b \sigma_{sr}(r)] - 1\})
\rightarrow 1
\]

in the limit \( \lambda_{st} \rightarrow \infty \) and \( \beta_b \rightarrow \infty \). Thus, in these limits, the gauge-invariant phase correlation function, \( C_{12} \), is approximately given by the “spin-wave” contribution (B.4) for \( r \gg \lambda_{st} \).

In the low-temperature limit of the one-component Abelian-Higgs model, therefore, we generally find that the coherence length for the holon phase correlations is given by \( \xi_b \sim \beta_b = J_b/T \) in units of the lattice constant, \( a \), contrary to previous claims in the literature of an exponentially divergent length.8
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Fig. 1. Shown is the phase boundary between the superconducting and the normal phase as a function of hole carrier concentration, $x$. The superconducting transition temperature is determined by Eq. (9), with $\frac{\pi}{2} J_b = (200 \text{K}) \cdot \frac{x}{x_0}$ and $\frac{\pi}{2} J_f = (800 \text{K}) \cdot (1 - \frac{x}{x_0})$, where $x_0 \approx 1$ denotes the critical concentration above which the spinon pairing instability is absent. The upper set of dashed lines represent cross-over regions below which “strange” metallicity begins to dissappear.