Interplay between complex symmetry and Koenigs eigenfunctions

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Abstract
We investigate the relationship between the complex symmetry of composition operators $C_\phi f = f \circ \phi$ induced on the classical Hardy space $H^2(\mathbb{D})$ by an analytic self-map $\phi$ of the open unit disk $\mathbb{D}$ and its Koenigs eigenfunction. A generalization of orthogonality known as conjugate-orthogonality will play a key role in this work. We show that if $\phi$ is a Schröder map (fixes a point $a \in \mathbb{D}$ with $0 < |\phi'(a)| < 1$) and $\sigma$ is its Koenigs eigenfunction, then $C_\phi$ is complex symmetric if and only if $(\sigma^n)_{n \in \mathbb{N}}$ is complete and conjugate-orthogonal in $H^2(\mathbb{D})$. We study the conjugate-orthogonality of Koenigs sequences with some concrete examples. We use these results to show that commutants of complex symmetric composition operators with Schröder symbols consist entirely of complex symmetric operators.

Keywords: Complex symmetric operator, composition operator, Koenigs eigenfunction.

1. Introduction
Let $X$ be a vector space of analytic functions on the open unit disk $\mathbb{D}$ and $\phi$ an analytic self-map of $\mathbb{D}$. The composition operator $C_\phi$ with symbol $\phi$ is defined as

$$C_\phi f = f \circ \phi \text{ for } f \in X.$$  

Operators of this type have been studied on a variety of spaces and in great detail. The objective is to study the interaction between the operator theoretic properties of $C_\phi$ and function theoretic properties of $\phi$. It is well-known that $C_\phi$ is bounded on the classical Hardy Hilbert space $H^2(\mathbb{D})$ whenever $\phi$ is an analytic self-map of $\mathbb{D}$. The monographs of Shapiro [21] and Cowen and McCluer [5] contain detailed accounts of the subject. An analytic self map $\phi$ of $\mathbb{D}$ is called a Schröder map if it fixes a point $a \in \mathbb{D}$ with $0 < |\phi'(a)| < 1$. Let $H(\mathbb{D})$ denote Frechet space of all analytic functions on $\mathbb{D}$. In 1884, Koenigs [13] showed that if $\phi$ is a Schröder map with fixed point $a \in \mathbb{D}$, then the eigenvalues of the operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ are

$$1, \phi'(a), \phi'(a)^2, \phi'(a)^3, \ldots.$$  

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These eigenvalues are *simple* (that have multiplicity one) and the eigenfunction \( \sigma \) corresponding to the eigenvalue \( \phi' (a) \) is called the *Koenigs eigenfunction* for \( \phi \). It follows that \( \sigma^n \) is an eigenfunction corresponding to \( \phi' (a) ^n \) for each \( n \in \mathbb{N} \), and all such eigenfunctions are scalar multiples of \( \sigma^n \). The sequence \( (\sigma^n)_{n \in \mathbb{N}} \) will be referred to as the *Koenigs sequence* for \( \phi \). The eigenfunction \( \sigma \) does not necessarily belong to any of the Hardy spaces \( H^p (\mathbb{D}) \) for \( p > 0 \). Characterizations of \( H^p (\mathbb{D}) \)-membership of \( \sigma \) have been obtained by Bourdon and Shapiro \([2, 3]\) and Poggi-Corradini \([19]\). In particular, they show that \( \sigma \in H^p (\mathbb{D}) \) for all \( p > 0 \), or equivalently \( \sigma^n \in H^2 (\mathbb{D}) \) for all \( n \in \mathbb{N} \) if and only if the composition operator \( C_\phi \) is a *Riesz operator*, that is when the *essential spectrum* of \( C_\phi \) is the singleton \( \{0\} \).

On the other hand a bounded operator \( T \) on a separable Hilbert space \( \mathcal{H} \) is *complex symmetric* if \( T \) has a self-transpose matrix representation with respect to some orthonormal basis for \( \mathcal{H} \). An equivalent definition also exists. A *conjugation* is a conjugate-linear operator \( J : \mathcal{H} \rightarrow \mathcal{H} \) that satisfies the conditions

(a) \( J \) is *isometric*: \( \langle Jf, Jg \rangle = \langle f, g \rangle \) \( \forall f, g \in \mathcal{H} \),

(b) \( J \) is *involutive*: \( J^2 = I \).

Then \( T \) is *\( J \)-symmetric* if \( JT = T^*J \), and is called *complex symmetric* if \( T \) is \( J \)-symmetric for some conjugation \( J \) on \( \mathcal{H} \). A sequence \( (f_n)_{n \in \mathbb{N}} \) is *conjugate-orthogonal* in \( \mathcal{H} \) if there exists a conjugation \( J \) on \( \mathcal{H} \) such that \( \langle Jf_n, f_m \rangle = 0 \) for all \( n \neq m \). An orthonormal basis \( (e_n)_{n \in \mathbb{N}} \) for \( \mathcal{H} \) is always conjugate-orthogonal. Indeed, just define the conjugation \( Je_n = e_n \) for all \( n \in \mathbb{N} \) and extend to all of \( \mathcal{H} \) by conjugate-linearity. Hence \( \langle e_n, e_m \rangle = \delta_{m,n} \) where \( \delta_{m,n} \) is the Kronecker delta. Complex symmetric operators are natural generalizations of complex symmetric matrices and normal operators, and their general study was initiated by Garcia, Putinar, and Wogen \([7, 8, 11, 12]\). Conjugate-orthogonal systems were studied by Garcia and Putinar \([6, 10]\) as eigenvectors of complex symmetric operators.

The study of complex symmetric composition operators on \( H^2 (\mathbb{D}) \) was initiated by Garcia and Hammond \([6]\). They showed that involutive disk automorphisms induce *non-normal* complex symmetric composition operators. This was the first such example. Bourdon and Noor \([1]\) proved that if \( C_\phi \) is complex symmetric then \( \phi \) must necessarily fix a point in \( \mathbb{D} \). They further showed that if \( \phi \) is a disk automorphism that is not elliptic of order two (involutive) or three, then \( C_\phi \) is never complex symmetric. Then Narayan, Sievärinto, and Thompson \([13]\) discovered the first non-automorphic composition operators that are complex symmetric. Recently Narayan, Sievärinto, and Tjani \([16]\) characterized all non-automorphic linear fractional \( \phi \) for which \( C_\phi \) is complex symmetric. The analogous problem for the Hardy space of the half-plane \( H^2 (\mathbb{C}_+) \) was recently solved by Noor and Severiano \([18]\). For more general symbols the problem still remains open. It follows that the non-automorphic symbols \( \phi \) of interest from the point of view of complex symmetry are those that fix a point in \( \mathbb{D} \), and that are also univalent (see \([6\text{, Prop. 2.5}]\)). These \( \phi \) are precisely the *univalent Schröder maps* and the Koenigs eigenfunction \( \sigma \) for \( \phi \) is also univalent. In this case if \( C_\phi \) is complex symmetric then it is a Riesz operator and the Koenigs sequence \( (\sigma^n)_{n \in \mathbb{N}} \) is contained in \( H^2 (\mathbb{D}) \) \( ([6\text{, Prop. 2.7}] \).
The objective of this article is to study properties of the Koenigs sequence \((\sigma_n)_{n \in \mathbb{N}}\) and how they interact with the complex symmetry of \(C_\phi\). The main result of this article can now be stated as follows (see Theorem 3.1).

**Theorem 1.1.** Let \(\phi\) be a Schröder map with Koenigs eigenfunction \(\sigma\). Then \(C_\phi\) is complex symmetric if and only if the Koenigs sequence \((\sigma_n)_{n \in \mathbb{N}}\) is complete and conjugate-orthogonal in \(H^2(\mathbb{D})\).

A Riesz basis is similar to an orthonormal basis (ONB), that is, it is the image of an ONB under an invertible operator. But Riesz bases are not necessarily conjugate-orthogonal. Let \(\phi_a(z) = \frac{a - z}{1 - \overline{a} z}\) be the involutive disk automorphism for some \(a \in \mathbb{D}\). Then \((\phi_a^n)_{n \in \mathbb{N}}\) is a Riesz basis and is the Koenigs sequence for the Schröder map

\[
\Phi_{a,\lambda}(z) = \phi_a(\lambda \phi_a(z))
\]

with fixed point \(a\) and \(\lambda \in \mathbb{D}\). We then apply Theorem 1.1 to show that this Riesz basis \((\phi_a^n)_{n \in \mathbb{N}}\) is not conjugate-orthogonal (see Theorem 4.1).

**Theorem 1.2.** For \(a, \lambda \in \mathbb{D} \setminus \{0\}\), the composition operator \(C_{\Phi_{a,\lambda}}\) is not complex symmetric on \(H^2(\mathbb{D})\), and therefore \((\phi_a^n)_{n \in \mathbb{N}}\) is not conjugate-orthogonal.

We next provide an example of a complete and conjugate-orthogonal sequence that is not an ONB in \(H^2(\mathbb{D})\) (see Theorem 5.2).

**Theorem 1.3.** The sequence \(((z - a)^n)_{n \in \mathbb{N}}\) is complete and conjugate-orthogonal for all \(a \in \mathbb{D}\), and is not an ONB when \(a \neq 0\).

If \(\phi(z) = cz + d\) is an affine self-map of \(\mathbb{D}\) with fixed point \(a \in \mathbb{D}\), then \(((z - a)^n)_{n \in \mathbb{N}}\) is the Koenigs sequence for \(\phi\). Therefore by Theorems 1.1 and 1.3 we get a simple proof for one of the main results of Narayan, Sievewright and Thompson [15] (see Corollary 5.3).

**Corollary 1.4.** If \(\phi\) is an affine self-map with a fixed point in \(\mathbb{D}\), then \(C_\phi\) is complex symmetric on \(H^2(\mathbb{D})\).

The set of bounded operators that commute with an operator \(T\) is called the commutant of \(T\) and is denoted by \(T'\). Our main result about commutants of complex symmetric composition operators is the following (see Theorem 6.1).

**Theorem 1.5.** Let \(\phi\) be a Schröder map such that \(C_\phi\) is complex symmetric. Then each \(A \in C_\phi'\) is also complex symmetric.

This is not true in general. If \(\phi\) is an elliptic automorphism of order 2, then \(C_\phi\) is complex symmetric (see [6] and [17]), but \(C_\phi'\) contains composition operators that are not complex symmetric. On the other hand if \(T\) is complex symmetric, then \(T^n\) for \(n \in \mathbb{N}\) are also complex symmetric. For composition operators with Schröder symbols we obtain a strong converse of this (see Corollary 6.2).
Corollary 1.6. Let \( \phi \) be a Schröder map. Then \( C_\phi \) is complex symmetric if and only if \( C_\phi^n \) is complex symmetric for some integer \( n \geq 2 \).

This is not true in general even for composition operators, since if \( \phi \) is an elliptic automorphism of order 4 then \( C_\phi \) is not complex symmetric (see [1, Prop. 3.3]), but \( C_\phi^2 \) is elliptic of order two and is hence complex symmetric.

2. Preliminaries

2.1. The Hardy space \( H^2(D) \)

The Hardy space \( H^2 := H^2(D) \) is the space of all analytic functions in \( D \) given by

\[
  f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n,
\]

for which the norm

\[
  \|f\|_2^2 := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.
\]

The Hardy space \( H^2 \) is a Hilbert space with the following inner product:

\[
  \langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)}
\]

where \( f, g \in H^2 \). For each non-negative integer \( n \) and \( a \in D \), the evaluation of the \( n \)th derivative of \( f \in H^2 \) at \( a \) is

\[
  f^{(n)}(a) = \langle f, K_a^{(n)} \rangle
\]

where

\[
  K_a^{(n)}(z) = \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} \frac{a^{m-n} z^m}{1-az}.
\]

We write \( f = f^{(0)} \) and hence \( K_a(z) = K_a^{(0)}(z) = \frac{1}{1-az} \) is the usual reproducing kernel for \( H^2 \) at \( a \in D \). Let \( \phi \) be a Schröder map with fixed point \( a \in D \) and \( \sigma \) its Koenigs eigenfunction normalized with \( ||\sigma|| = 1 \). If \( C_\phi \) is complex symmetric, then Garcia and Hammond [6, p.176] proved the following relation we will use

\[
  |\sigma(0)| = \frac{|K_a^{(1)}(a)|}{\|K_a\| \|K_a^{(1)}\|}.
\]

2.2. Conjugate-orthogonal vectors

Complex symmetric operators satisfy the following spectral symmetry property. Let \( T \) be \( J \)-symmetric on a Hilbert space \( H \) and \( \lambda \in \mathbb{C} \), then

\[
  f \in \text{Ker}(T - \lambda I) \iff Jf \in \text{Ker}(T^* - \overline{\lambda} I).
\]

A sequence \( (f_n)_{n \in \mathbb{N}} \) is conjugate-orthogonal in \( H \) if \( \langle Jf_n, f_m \rangle = 0 \) for \( n \neq m \) and some conjugation \( J \) on \( H \). Alternatively we shall just say \( (f_n)_{n \in \mathbb{N}} \) is \( J \)-orthogonal. Note that if \( (f_n)_{n \in \mathbb{N}} \) is also complete in \( H \), then \( \langle Jf_n, f_n \rangle \neq 0 \) for all \( n \in \mathbb{N} \). This implies that \( (Jf_n)_{n \in \mathbb{N}} \) is the unique sequence (upto scalar multiples) that is biorthogonal to \( (f_n)_{n \in \mathbb{N}} \) (see [4, Lemma 3.3.1]. In [10], Garcia proved the following.
Lemma 2.1. The eigenvectors of a $J$-symmetric operator $T$ corresponding to distinct eigenvalues are $J$-orthogonal.

Proof. Let $Tf_i = \lambda_i f_i$ for $i = 1, 2$ where $\lambda_1 \neq \lambda_2$. Then

$$\bar{\lambda}_1 \langle Jf_1, f_2 \rangle = \langle JTf_1, f_2 \rangle = \langle T^*Jf_1, f_2 \rangle = \langle Jf_1, Tf_2 \rangle = \bar{\lambda}_2 \langle Jf_1, f_2 \rangle$$

implies that $\langle Jf_1, f_2 \rangle = 0$.

If $\phi$ is a Schröder map with fixed point $a \in \mathbb{D}$ and $\lambda = \phi'(a)$, then the eigenvalues $(\lambda^n)_{n \in \mathbb{N}}$ of $C_{\phi}$ are all distinct since $|\lambda| < 1$. Therefore Lemma 2.1 gives us

Proposition 2.2. Let $\phi$ be a Schröder map with Koenigs eigenfunction $\sigma$. If $C_{\phi}$ is complex symmetric then $(\sigma^n)_{n \in \mathbb{N}}$ is conjugate-orthogonal in $H^2$.

3. Complex symmetry and Koenigs eigenfuctions

In this section, we characterize the complex symmetry of $C_{\phi}$ with Schröder symbol $\phi$ in terms of the conjugate-orthogonality of its Koenigs sequence $(\sigma^n)_{n \in \mathbb{N}}$.

Theorem 3.1. Let $\phi$ be a Schröder map with Koenigs eigenfunction $\sigma$. Then $C_{\phi}$ is complex symmetric if and only if the Koenigs sequence $(\sigma^n)_{n \in \mathbb{N}}$ is complete and conjugate-orthogonal in $H^2$.

Proof. Let $a \in \mathbb{D}$ be the fixed point of $\phi$ and denote $\lambda = \phi'(a)$. First suppose $(\sigma^n)_{n \in \mathbb{N}}$ is complete and $J$-orthogonal. Then for $n, m \in \mathbb{N}$, we have

$$\langle (C_{\phi}^*J - JC_{\phi})\sigma^n, \sigma^m \rangle = \langle J\sigma^n, C_{\phi}\sigma^m \rangle - \bar{\lambda}^n \langle J\sigma^n, \sigma^m \rangle = (\bar{\lambda}^m - \bar{\lambda}^n) \langle J\sigma^n, \sigma^m \rangle.$$

Since $\langle J\sigma^n, \sigma^m \rangle = 0$ for all $n \neq m$, the completeness of $(\sigma^n)_{n \in \mathbb{N}}$ implies that $C_{\phi}^*J = JC_{\phi}$, and therefore that $C_{\phi}$ is $J$-symmetric. Conversely suppose $C_{\phi}$ is $J$-symmetric. By Proposition 2.2 we know that $(\sigma^n)_{n \in \mathbb{N}}$ is conjugate-orthogonal. Hence we only need to show that $(\sigma^n)_{n \in \mathbb{N}}$ is complete in $H^2$. For each $n \in \mathbb{N}$, the function $K_a^{(n)}$ can be written in the form

$$K_a^{(n)} = a_n v_n + a_{n-1} v_{n-1} + \ldots + a_0 v_0, \quad (4)$$

where $v_j$ is an eigenvector for $C_{\phi}^*$ corresponding to the eigenvalue $\bar{\lambda}^j$, and the $a_j$ are scalars (see [6, Prop. 2.6]). Now apply $J$ on both sides of (4) and note that each $Jv_j$ is a scalar multiple of $\sigma^j$ by spectral symmetry (see [3]) and the fact that the $\lambda^j$ are simple eigenvalues of $C_{\phi}$. We then get

$$JK_a^{(n)} = c_n \sigma^n + c_{n-1} \sigma^{n-1} + \ldots + c_0 \sigma.$$

To conclude the completeness of $(\sigma^n)_{n \in \mathbb{N}}$, we note that if some $f \in H^2$ is orthogonal to $(\sigma^n)_{n \in \mathbb{N}}$ then $f \perp JK_a^{(n)}$ and hence $Jf \perp K_a^{(n)}$ for all $n \in \mathbb{N}$. But $(Jf)^{(n)}(a) = 0$ for all $n$ implies $Jf \equiv 0$ and hence $f \equiv 0$. Therefore $(\sigma^n)_{n \in \mathbb{N}}$ is complete in $H^2$. 

4. A Riesz basis that is not conjugate-orthogonal

Let \( \phi_a(z) = \frac{a - z}{1 - a\bar{z}} \) be the involutive disk automorphism for some \( a \in \mathbb{D} \setminus \{0\} \). The goal of this section is to prove that although \( (\phi^n_a)_{n \in \mathbb{N}} \) is a Koenigs sequence and a Riesz basis, it is not conjugate-orthogonal. The sequence \( (\phi^n_a)_{n \in \mathbb{N}} \) is a Riesz basis since \( \phi^n_a = C_{\phi_a} z^n \) for \( n \in \mathbb{N} \) and it is also the Koenigs sequence for the Schröder map

\[
\Phi_{a,\lambda}(z) = \phi_a(\lambda\phi_a(z))
\]

since \( C_{\Phi_{a,\lambda}} \phi_a = \lambda \phi_a \) with \( \lambda \in \mathbb{D} \setminus \{0\} \) and fixed point \( a \).

**Theorem 4.1.** For \( a, \lambda \in \mathbb{D} \setminus \{0\} \), the composition operator \( C_{\Phi_{a,\lambda}} \) is not complex symmetric on \( H^2 \), and therefore \( (\phi^n_a)_{n \in \mathbb{N}} \) is not conjugate-orthogonal.

**Proof.** First let \( 0 < a < 1 \). Observe that by (1) we have

\[
K_a^{(1)}(z) = \sum_{n=1}^{\infty} na^{n-1} z^n.
\]

Since \( \phi_a \) is inner we have \( \|\phi_a\| = 1 \). Moreover, simple computations show that

\[
|K_a^{(1)}(a)| = \frac{a}{(1 - a^2)^2} \quad \text{and} \quad \|K_a^{(1)}\| = \frac{(a^2 + 1)^{1/2}}{(1 - a^2)^{3/2}}.
\]

(5)

Now suppose on the contrary that \( C_{\Phi_{a,\lambda}} \) is complex symmetric. Then (2) with \( \sigma = \phi_a \) and \( \phi = \Phi_{a,\lambda} \) gives

\[
|K_a^{(1)}(a)| = |\phi_a(0)| \|K_a\| \|K_a^{(1)}\|.
\]

(6)

By combining (3) and (6), we obtain

\[
\frac{a}{(a^2 - 1)^2} = \frac{a(1 + a^2)^{1/2}}{(a^2 - 1)^{3/2}}
\]

which is possible only if \( a = 0 \). However, this contradicts the hypothesis that \( a \in (0, 1) \) and therefore \( C_{\Phi_{a,\lambda}} \) is not complex symmetric. For the general case \( a \in \mathbb{D} \setminus \{0\} \), let \( \theta \) be a real number such that \( a = |a| e^{i\theta} \). Then

\[
e^{-i\theta} \big( \phi_a \big( e^{i\theta} z \big) \big) = e^{-i\theta} \left( \frac{a - e^{i\theta} z}{1 - a e^{i\theta} z} \right) = \frac{|a| - z}{1 - |a| z} = \phi_{|a|}(z).
\]

(7)

By using (7), we can express \( \Phi_{a,\lambda} \) in term of \( \Phi_{|a|,\lambda} \)

\[
e^{-i\theta} \Phi_{a,\lambda}(e^{i\theta} z) = e^{-i\theta} \phi_a(\lambda \phi_a(e^{i\theta} z)) = e^{-i\theta} \phi_a(\lambda e^{i\theta} e^{-i\theta} \phi_a(e^{i\theta} z))
\]

\[
= e^{-i\theta} \phi_a(e^{i\theta} \lambda \phi_{|a|}(z)) = \phi_{|a|}(\lambda \phi_{|a|}(z))
\]

\[
= \Phi_{|a|,\lambda}(z).
\]

This implies that \( C_{e^{i\theta} z} C_{\Phi_{a,\lambda}} C_{e^{i\theta} z}^* = C_{\Phi_{|a|,\lambda}} \). Since \( C_{e^{i\theta} z} \) is a unitary operator on \( H^2 \), it follows that \( C_{\Phi_{a,\lambda}} \) is unitarily equivalent to \( C_{\Phi_{|a|,\lambda}} \) and therefore \( C_{\Phi_{a,\lambda}} \) is not complex symmetric. Finally since \( (\phi^n_a)_{n \in \mathbb{N}} \) is the Koenigs sequence for \( \Phi_{a,\lambda} \) and is complete in \( H^2 \), by Theorem 3.1 it follows that \( (\phi^n_a)_{n \in \mathbb{N}} \) is not conjugate-orthogonal.
5. A complete and conjugate-orthogonal Koenigs sequence

Let \( a \in \mathbb{D} \) and \( \sigma(z) = z - a \). In this section we show that \((\sigma^n)_{n \in \mathbb{N}}\) is a complete and conjugate-orthogonal sequence that is not an ONB when \( a \neq 0 \). First recall that for \( r \in \mathbb{R} \) and \( j \in \mathbb{N} \), the generalized binomial coefficient is defined as

\[
\binom{r}{j} = \frac{r(r-1) \ldots (r-(j-1))}{j!}
\]

if \( j > 0 \) and \( \binom{r}{j} = 1 \) if \( j = 0 \). If \( r \in \mathbb{N} \) with \( 0 \leq j \leq r \), then \( \binom{r}{j} = \frac{r!}{j!(r-j)!} \) is the usual binomial coefficient. In general, the binomial series for \((1 - z)^r\) is

\[
(1 - z)^r = \sum_{j=0}^{\infty} \binom{r}{j} (-z)^j
\]

for each \( z \in \mathbb{D} \). We begin by showing that \((\sigma^n)_{n \in \mathbb{N}}\) has a biorthogonal sequence \((z^nK_{a+1}^n)_{n \in \mathbb{N}}\) when \( a \in (-1, 1) \).

**Lemma 5.1.** For each \( a \in (-1, 1) \) and non-negative integers \( n \) and \( m \), we have

\[
\langle z^nK_{a+1}^n, (z-a)^m \rangle = \delta_{nm},
\]

where \( \delta \) is the Kronecker delta function.

**Proof.** First observe that \( K_{a+1}^n(z) = (1 - az)^{-(n+1)} \). By using (8), we get

\[
z^nK_{a+1}^n(z) = \sum_{j=0}^{\infty} \binom{r}{j} (-a)^j z^{j+n}.
\]

Then (9) holds for \( m \leq n \) since the smallest term in (10) is \( z^n \), whereas the largest term in \((z-a)^m\) is \( z^m \). Therefore let \( m > n \) and defining \( k = m - n \) we get

\[
\langle z^nK_{a+1}^n, (z-a)^m \rangle = \binom{m}{1} \sum_{j=0}^{\infty} \binom{(m-1)(j)}{(n+j)} (-a)^j z^{j+n}
\]

\[
= \binom{(m-1)(k)}{(n+k)} (-a)^k z^k
\]

\[
= \frac{(-1)^k (n+1) (n+2) \ldots (n+j) m!}{j! (n+j)!(k-j)!}
\]

\[
= \frac{(-1)^k m!}{n! k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0.
\]
The last sum vanishes by the identity \( \sum_{j=0}^{k} (-1)^j \binom{k}{j} = 0 \) for \( k > 0 \) \cite[Cor. 2]{20}.

Let \( J \) be the conjugation on \( H^2 \) defined by \((Jf)(z) = \overline{f(z)}\), \( T_a \) the operator of multiplication by the normalized reproducing kernel \( K_a/\|K_a\| \) and \( \phi_a(z) = \frac{a-z}{1-az} \). If \( a \in (-1, 1) \), then the operator defined by

\[
J_a = JT_a C_{\phi_a}
\]

is a conjugation on \( H^2 \) by \cite[Prop. 2.3]{15}. We arrive at the main result.

**Theorem 5.2.** If \( a \in \mathbb{D} \), then \( ((z-a)^n)_{n \in \mathbb{N}} \) is complete and conjugate-orthogonal in \( H^2 \). It is not an ONB when \( a \neq 0 \).

**Proof.** First suppose \( a \in (-1, 1) \). It is easy to see that

\[
\phi_a(z) - a = -(1-a^2)zK_a(z) = -zK_a(z)/\|K_a\|^2.
\]

Then we get

\[
J_a(z-a)^n = K_a(z)(\phi_a(z)-a)^n/\|K_a\| = (-1)^n||K_a||^{-2n-1}z^nK_a^{n+1}(z). \tag{11}
\]

Therefore \( J_a(z-a)^n \) is a scalar multiple of \( z^nK_a^{n+1} \) for \( n \in \mathbb{N} \) and hence \( (z-a)^n \) is \( J_a \)-orthogonal by Lemma 5.1 for \( a \in (-1, 1) \). For the general case \( a \in \mathbb{D} \), let \( \theta \) be a real number such that \( a = e^{i\theta}|a| \). Since \( J_{|a|} \) is a conjugation for each \( a \in \mathbb{D} \) and \( U_{\theta} = C_{e^{i\theta}z} \) is a unitary operator, it follows that \( U_{\theta}^* J_{|a|} U_{\theta} \) is also a conjugation. Therefore we get

\[
\langle U_{\theta}^* J_{|a|} U_{\theta}(z-a)^m, (z-a)^n \rangle = \langle J_{|a|} U_{\theta}(z-a)^n, U_{\theta}(z-a)^m \rangle = \langle J_{|a|}(e^{i\theta}z-a)^n, (e^{i\theta}z-a)^m \rangle = e^{-i\theta(n+m)} \langle J_{|a|}(z-|a|)^n, (z-|a|)^m \rangle
\]

for \( m, n \in \mathbb{N} \). Now since \( (z-|a|)^n \) is \( J_{|a|} \)-orthogonal by the first part, it follows that \( ((z-a)^n)_{n \in \mathbb{N}} \) is \( U_{\theta}^* J_{|a|} U_{\theta} \)-orthogonal. We now prove the completeness of \( ((z-a)^n)_{n \in \mathbb{N}} \). Note that the operator \( U : f(z) \rightarrow f(z-a) \) is unbounded on \( H^2 \) and has the polynomials \( \mathbb{C}[z] \) in its domain. By \( U z^n = (z-a)^n \) for \( n \in \mathbb{N} \) it is enough to show that \( U \mathbb{C}[z] = \mathbb{C}[z] \). This is clear since for any polynomial \( p \) we have \( U(p(z+a)) = p(z) \) where \( p(z+a) \) is also a polynomial. The non-orthonormality is clear since \( 1 \not\in ((z-a)^n) \) for \( n > 0 \) when \( a \neq 0 \).

As an immediate application of Theorem 5.2, we obtain one of the main results of Narayan, Sievwright and Thompson \cite{15}.

**Corollary 5.3.** If \( \phi \) is an affine self-map with a fixed point in \( \mathbb{D} \), then \( C_{\phi} \) is complex symmetric on \( H^2 \).

**Proof.** Let \( \phi(z) = cz + d \) with fixed point \( a = \frac{d}{1-c} \in \mathbb{D} \). Then we see that

\[
C_{\phi}(z-a)^n = c^n(z-a)^n
\]

where \( |c| < 1 \) necessarily and hence \( ((z-a)^n)_{n \in \mathbb{N}} \) is the Koenigs sequence for \( \phi \). Therefore \( C_{\phi} \) is complex symmetric by Theorems 3.1 and 5.2.
6. Commutants of complex symmetric composition operators

In this section, we study how the complex symmetry of \( C_\phi \) affects the structure of its commutant \( C'_\phi \) when \( \phi \) is a Schröder symbol. The main result is the following.

**Theorem 6.1.** Let \( \phi \) be a Schröder map such that \( C_\phi \) is complex symmetric on \( H^2 \). Then each \( A \in C'_\phi \) is complex symmetric.

**Proof.** Let \( a \in \mathbb{D} \) be the fixed point of \( \phi \), \( \sigma \) its Koenigs eigenfunction and \( \lambda = \phi'(a) \). If \( A \) commutes with \( C_\phi \) then \( C_\phi A \sigma^n = \lambda^n A \sigma^n \) for each \( n \in \mathbb{N} \). By Koenigs’ Theorem there must exist \( a_n \in \mathbb{C} \) such that \( A \sigma^n = a_n \sigma^n \) for \( n \in \mathbb{N} \). If \( C_\phi \) is \( J \)-symmetric, then for \( n, m \in \mathbb{N} \) we have

\[
\langle \sigma^m, (A^* - \bar{a}_n I) J \sigma^n \rangle = \langle (A - a_n I) \sigma^m, J \sigma^n \rangle = (a_m - a_n) \langle \sigma^m, J \sigma^n \rangle.
\]

Since \( (\sigma^n)_{n \in \mathbb{N}} \) is complete \( J \)-orthogonal by Theorem 3.1, we obtain \( A^* J \sigma^n = \overline{a}_n J \sigma^n \). Hence

\[
A^* J \sigma^n = \overline{a}_n J \sigma^n = J (a_n \sigma^n) = JA \sigma^n
\]

and the completeness of \( (\sigma^n)_{n \in \mathbb{N}} \) implies \( A^* J = JA \).

The conclusion of Theorem 6.1 does not hold in general. Let \( \phi = \phi_a \circ (-\phi_a) \) with \( a \in \mathbb{D} \setminus \{0\} \). Then \( \phi \) is elliptic of order 2 and \( C_\phi \) is complex symmetric. If \( \Phi_{a,\lambda} = \phi_a \circ (\lambda \phi_a) \) where \( \lambda \in \mathbb{D} \setminus \{0\} \), then \( \Phi_{a,\lambda} \) commutes with \( \phi \) and hence \( C_{\Phi_{a,\lambda}} \in C'_\phi \) even though \( C_{\Phi_{a,\lambda}} \) is not complex symmetric by Theorem 4.1. An immediate implication of Theorem 6.1 is the following interesting result.

**Corollary 6.2.** Let \( \phi \) be a Schröder map. Then \( C_\phi \) is complex symmetric if and only if \( C^n_\phi \) is complex symmetric for some integer \( n \geq 2 \).

**Proof.** If \( C_\phi \) is complex symmetric then clearly so is \( C^n_\phi \) for all \( n \in \mathbb{N} \). Conversely suppose \( C^n_\phi \) is complex symmetric for some \( n \geq 2 \). Clearly the \( n \)-th composite \( \phi^{[n]} \) of \( \phi \) is also a Schröder map. Since \( C_{\phi^{[n]}} = C^n_\phi \) is complex symmetric and \( C_\phi \) commutes with \( C_{\phi^{[n]}} \), it follows from Theorem 6.1 that \( C_\phi \) is complex symmetric.

Again this is not true in general, since if \( \phi \) is an elliptic automorphism of order 4 then \( C_\phi \) is not complex symmetric (see [1, Prop. 3.3]), but \( C^2_\phi \) is elliptic of order two and is hence complex symmetric.

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