ALMOST UNIMODULAR SYSTEMS
ON COMPACT GROUPS WITH DISJOINT SPECTRA

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ABSTRACT. We emulate the Rademacher functions on any non-commutative
compact group requiring the resulting system to have pairwise disjoint spectra.

INTRODUCTION

Let $G$ be a compact topological group with normalized Haar measure $\mu$ and dual
object $\hat{G}$. Let $\pi : G \to U(\mathcal{H}_\pi)$ be an irreducible unitary representation of $G$. Then,
given an integrable function $f : G \to \mathbb{C}$, the Fourier coefficient of $f$ at $\pi$ is defined
as follows

$$\hat{f}(\pi) = \int_G f(g)\pi(g)^* \, d\mu(g) \in \mathcal{B}(\mathcal{H}_\pi).$$

It is well-known how to construct $n$ functions $f_1, f_2, \ldots, f_n : G \to \mathbb{C}$ satisfying

(P1) $f_1, f_2, \ldots, f_n$ have pairwise disjoint supports on $G$.
(P2) The norm of $\hat{f}_k(\pi)$ on $S^*_{q'}$ does not depend on $k$ for any $\pi \in \hat{G}$.

Just take disjoint translations of a common function on $G$ with sufficiently small
support. Also, one can consider the dual properties with respect to the Fourier
transform on $G$. Namely,

(\hat{P}1) $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ have pairwise disjoint supports on $\hat{G}$.
(\hat{P}2) The absolute value $|f_k(g)|$ does not depend on $k$ for almost all $g \in G$.

If $G$ is abelian we can easily construct such a system by taking $n$ irreducible
characters of $G$. Moreover, many other constructions are available. Namely, we can
take $n$ disjoint translations of a common function on $\hat{G}$. Then we get the desired
property by Pontrjagin duality. This kind of systems have been applied to study
the Fourier type constants of finite-dimensional Lebesgue spaces: if $1 \leq p < q \leq 2$,
the Fourier $q$-type constant of $\ell_p(n)$ with respect to $\hat{G}$ is $n^{1/p - 1/q}$, which is optimal
among the family of $n$ dimensional Banach spaces. In particular, this provides
sharp results about the Fourier type of infinite-dimensional Lebesgue spaces.

However, irreducible characters on non-commutative compact groups are no
longer unimodular. Furthermore, the dual object is not a group anymore and consequently
Pontrjagin duality does not hold. Besides, Tannaka’s theorem (the
non-commutative counterpart of Pontrjagin theorem) does not fit properly in this
context. Hence, it is natural to wonder if it is possible to construct a system

$$\Phi = \left\{ f_m : G \to \mathbb{C} \mid m \geq 1 \right\}$$

made up of functions $f_1, f_2, \ldots$ satisfying similar properties to (P1) and (P2). In
this paper, we construct an almost unimodular system of trigonometric polynomials
f_m : G → C with non-overlapping ranges of frequencies. The work [2] is the main motivation for this note.

1. Almost unimodular systems

**Theorem 1.1.** Let G be an infinite compact group and let f_0 be a continuous function in \( \ell^2(G) \). Let \( \varepsilon_1, \varepsilon_2, \ldots \) be any sequence of positive numbers decreasing to 0. Then there exists a collection \( \Omega_1, \Omega_2, \ldots \) of measurable subsets of G such that \( \mu(\Omega_m) \to 1 \) as \( m \to \infty \), and a system \( \Phi = \{ f_m \mid m \geq 1 \} \) of trigonometric polynomials in G satisfying:

i) \( \hat{f}_1, \hat{f}_2, \ldots \) have pairwise disjoint supports on \( \hat{G} \).

ii) \( |f_m(g)| < |f_0(g)| + \varepsilon_m \) for all \( g \in G \) and all \( m \geq 1 \).

iii) \( |f_m(g)| > |f_0(g)| - \varepsilon_m \) for all \( g \in \Omega_m \) and all \( m \geq 1 \).

If \( f_0 \equiv 1 \) we have an almost unimodular system on G with pairwise disjoint spectra.

**Proof.** First we recall that it is essentially no restriction to assume that a compact topological group is Hausdorff, see e.g. Corollary 2.3 in [1]. Then we point out that the normalized Haar measure \( \mu \) of an infinite Hausdorff compact group G has no atoms. This is an easy consequence of the translation invariance and finiteness of \( \mu \). Indeed, let us assume that \( \mu \) has an atom \( \Omega \) with \( \mu(\Omega) = \alpha \) so that \( 0 < \alpha \leq 1 \).

Let us consider an open neighborhood \( U \) of G. Since G is non-finite and Hausdorff, we can choose \( U \) small enough so that there exists \( m_0 \) pairwise disjoint translations \( g_1 U, g_2 U, \ldots, g_{m_0} U \) with \( m_0 > 1/\alpha \). On the other hand, the compactness of G allows us to write

\[
G = \bigcup_{k=1}^{m_0} h_k U
\]

as a finite union of translations of \( U \). In particular, there must exists \( 1 \leq k_0 \leq m_0 \) such that \( \mu(h_{k_0} U \cap \Omega) > 0 \). Moreover, we must have \( \mu(h_{k_0} U \cap \Omega) = \alpha \) since \( \Omega \) is an atom. Then, by the translation invariance of the Haar measure, we have

\[
\mu(g_j U \cap g_j h_{k_0}^{-1} \Omega) = \mu(h_{k_0} U \cap \Omega) = \alpha.
\]

Then, since \( g_1 U, g_2 U, \ldots, g_{m_0} U \) are pairwise disjoint, we obtain

\[
\mu(G) \geq \sum_{j=1}^{m_0} \mu(g_j U) \geq m_0 \alpha > 1,
\]

which contradicts the assumption that \( \mu \) has mass 1 and proves our claim.

**Step 1.** Since the Haar measure \( \mu \) has no atoms, it is clear that the same holds for the measure \( d\nu = |f_0|^2 d\mu \). Then, by the absence of \( \nu \)-atoms we can define a family of \( \nu \)-measurable dyadic sets

\[
\left\{ D_j^k \mid k \geq 1, 1 \leq j \leq 2^k \right\}
\]

in G satisfying the standard conditions:

(a) \( D_j^k = D_{j-1}^{k+1} \cup D_{2j}^{k+1} \) for all \( j, k \).

(b) \( D_1^k, D_2^k, \ldots, D_{2^k}^k \) are pairwise disjoint and

\[
G = \bigcup_{j=1}^{2^k} D_j^k \quad \text{for all} \quad k \geq 1.
\]
(c) The sets $D_j^k$ have the same $\nu$-measure for $k$ fixed
\[
\int_{D_j^k} |f_0(g)|^2 d\mu(g) = 2^{-k} \int_G |f_0(g)|^2 d\mu(g).
\]
Then, if $1_\Omega$ denotes the characteristic function of $\Omega$, we define the system
\[
\Delta = \{ \delta_k : G \to \mathbb{C} \mid k \geq 1 \}
\]
as follows
\[
\delta_k(g) = f_0(g) \sum_{j=1}^{2^k} (-1)^{j+1} 1_{D_j^k}(g).
\]
Since every finite Radon measure is regular, we can consider compact sets $K_j^k \subset D_j^k$ such that $\mu(D_j^k \setminus K_j^k) < 2^{-2k}$ for all $k \geq 1$ and all $1 \leq j \leq 2^k$. Then, by Uryshon’s lemma we can construct continuous functions $\gamma_j^k$ on $G$ such that $1_{K_j^k} \leq \gamma_j^k \leq 1_{D_j^k}$.

Then, we define the system
\[
\Psi = \{ \psi_k : G \to \mathbb{C} \mid k \geq 1 \}
\]
as follows
\[
\psi_k(g) = f_0(g) \sum_{j=1}^{2^k} (-1)^{j+1} \gamma_j^k(g).
\]

**Step 2.** Let $\pi : G \to B(H_\pi)$ be an irreducible unitary representation of degree $d_\pi$ and let us fix $1 \leq i, j \leq d_\pi$. Since $\pi(g)$ can be identified with a unitary $d_\pi \times d_\pi$ matrix, we have
\[
\left( \sum_{k=1}^{\infty} \left| \hat{\psi}_k(\pi)_{ij} \right|^2 \right)^{1/2}
\]
\[
= \left( \sum_{k=1}^{\infty} \left| \int_G \psi_k(g) \overline{\pi_{ji}(g)} \mu(g) \right|^2 \right)^{1/2}
\]
\[
\leq \left( \sum_{k=1}^{\infty} \left| \int_G (\psi_k - \delta_k) \overline{\pi_{ji}(g)} \mu(g) \right|^2 \right)^{1/2} + \left( \sum_{k=1}^{\infty} \left| \int_G \delta_k(g) \overline{\pi_{ji}(g)} \mu(g) \right|^2 \right)^{1/2}
\]
Applying Hölder inequality for the first part and Bessel inequality for the second (note that the system $\|f_0\|^{-2}_{L_2(G)} \Delta$ is orthonormal in $L_2(G)$) we get
\[
\left( \sum_{k=1}^{\infty} \left| \hat{\psi}_k(\pi)_{ij} \right|^2 \right)^{1/2} \leq \|\pi_{ji}\|_{L_2(G)} \left( \sum_{k=1}^{\infty} \|\psi_k - \delta_k\|_{L_2(G)}^2 \right)^{1/2} + \|f_0\|_{L_2(G)} \|\pi_{ji}\|_{L_2(G)}
\]
\[
\leq \frac{1}{\sqrt{d_\pi}} \left( \|f_0\|_{L_\infty(G)} + \|f_0\|_{L_2(G)} \right) < \infty.
\]
The last inequality uses the estimate
\[
\|\psi_k - \delta_k\|_{L_2(G)}^2 \leq \sum_{j=1}^{2^k} \int_{D_j^k \setminus K_j^k} |f_0(g)|^2 d\mu(g) \leq 2^{-k} \|f_0\|_{L_\infty(G)}^2.
\]
which follows since \( |\psi_k - \delta_k|^2 \) is supported in

\[
\bigcup_{j=1}^{2^k} D_j^k \setminus K_j^k
\]

and bounded above by \( |f_0|^2 \). In particular, \( |\hat{\psi}_k(\pi)_{ij}| \to 0 \) as \( k \to \infty \). Therefore, given any \( \delta > 0 \) and any finite subset \( \Lambda \subset \hat{G} \), there exists a positive integer \( M(\Lambda, \delta) \) such that for all \( k \geq M(\Lambda, \delta) \) we have

\[
\sum_{\pi \in \Lambda} d_{\pi} \sum_{i,j=1}^{d_{\pi}} |\hat{\psi}_k(\pi)_{ij}| < \delta.
\]

**Step 3.** For any \( \pi \in \hat{G} \) let us denote by \( E_\pi \) the linear span of the entries of \( \pi \). Also, we shall write \( E \) for the linear span of the union of the spaces \( E_\pi \) when \( \pi \) runs over \( \hat{G} \). That is, \( E \) is the space of trigonometric polynomials on \( G \). By the Peter-Weyl theorem we know that \( E \) is dense in the space \( C(G) \) of continuous functions on \( G \) with respect to the uniform norm. Then we construct the system

\[
\Phi = \left\{ f_m : G \to \mathbb{C} \mid m \geq 1 \right\}
\]

as follows:

1. Let \( f_1 \in E \) be such that \( \sup_{g \in G} |f_1(g) - \psi_1(g)| < \varepsilon_1 \).
2. For \( m > 1 \) we proceed by induction. Let

\[
\Lambda_m = \bigcup_{k=1}^{m-1} \text{supp} \hat{f}_k \subset \hat{G} \quad \text{and} \quad \delta_m = \min \left( \frac{\varepsilon_m}{3}, \left[ \sum_{\pi \in \Lambda_m} d_{\pi}^{5/2} \right]^{-1} \right).
\]

Let \( k_m = M(\Lambda_m, \delta_m) \) and let \( \xi_m \in E \) be such that

\[
\sup_{g \in G} |\xi_m(g) - \psi_{k_m}(g)| < \delta_m^2.
\]

Then we define

\[
f_m(g) = \xi_m(g) - \sum_{\pi \in \Lambda_m} d_{\pi} \text{tr}(\pi(g)\hat{\xi}_m(\pi)).
\]

**Step 4.** Kunze’s Hausdorff-Young inequality and \( (1) \) give

\[
\sup_{g \in G} |f_m(g) - \psi_{k_m}(g)| \leq \sup_{g \in G} |\xi_m(g) - \psi_{k_m}(g)| + \sup_{g \in G} \left\| \sum_{\pi \in \Lambda_m} d_{\pi} \text{tr}(\pi(g)\hat{\xi}_m(\pi)) \right\|_{L^1(G)}
\]

\[
< \delta_m^2 + \sum_{\pi \in \Lambda_m} d_{\pi} \| \hat{\xi}_m(\pi) \|_{G^*}
\]

\[
\leq \delta_m^2 + \sum_{\pi \in \Lambda_m} d_{\pi} \sum_{i,j=1}^{d_{\pi}} |\hat{\xi}_m(\pi)_{ij}|
\]

\[
< \delta_m^2 + \delta_m + \sum_{\pi \in \Lambda_m} d_{\pi} \sum_{i,j=1}^{d_{\pi}} \left| \int_G (\xi_m - \psi_{k_m})(g) \overline{\pi_{ij}(g)} \, d\mu(g) \right|
\]

\[
\leq \delta_m^2 + \delta_m + \|\xi_m - \psi_{k_m}\|_{L^1(G)} \sum_{\pi \in \Lambda_m} d_{\pi} \sum_{i,j=1}^{d_{\pi}} \|\pi_{ij}\|_{L^2(G)}
\]

\[
\sum_{\pi \in \Lambda_m} d_{\pi} \sum_{i,j=1}^{d_{\pi}} \|\pi_{ij}\|_{L^2(G)}
\]
≤ \delta^2_m + \delta_m + \sup_{g \in G} |\xi_m(g) - \psi_k_m(g)| \sum_{\pi \in \Lambda_m} d_{\pi}^{5/2} \\
< \delta^2_m + 2\delta_m \leq \varepsilon_m.

We finally show that \Phi satisfies all the properties stated above. Taking
\[ \Omega_m = \bigcup_{1 \leq j \leq 2^k m} K_{j}^{k_m}, \]
we have
\[ \mu(\Omega_m) = 1 - \sum_{j=1}^{2^k m} \mu(D_j^{k_m} \setminus K_j^{k_m}) \geq 1 - 2^{-k_m} \rightarrow 1 \quad \text{as} \quad m \rightarrow \infty. \]

Then, the announced properties follow. Indeed, we have
i) \( \widehat{f_1}, \widehat{f_2}, \ldots \) have pairwise disjoint supports on \( \widehat{G} \), see the definition of \( f_m \).
ii) If \( g \in G \), we have
\[ |f_m(g)| \leq |\psi_k_m(g)| + |f_m(g) - \psi_k_m(g)| < |f_0(g)| + \varepsilon_m. \]
iii) If \( g \in \Omega_m \), we have
\[ |f_m(g)| \geq |\psi_k_m(g)| - |f_m(g) - \psi_k_m(g)| > |f_0(g)| - \varepsilon_m. \]

This concludes the proof. \( \square \)

Note from the author. The original purpose of this paper was to study the sharp Fourier type of \( L_p \) spaces with respect to non-commutative compact groups. This is what I did in the preliminary version of this paper. Unfortunately, I found an stupid mistake but with serious implications in the proof. What I present here is the material which remains alive from the previous version.

References
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