NUMBER OF VERTICES IN GELFAND–ZETLIN POLYTOPES

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Abstract. We discuss the problem of counting vertices in Gelfand–Zetlin polytopes. Namely, we deduce a partial differential equation with constant coefficients on the exponential generating function for these numbers. For some particular classes of Gelfand-Zetlin polytopes, the number of vertices can be given by explicit formulas.

1. Introduction and statement of results

Gelfand–Zetlin polytopes play an important role in representation theory [GZ, O] and in algebraic geometry (see [KST]). Let \( \lambda_1 \leq \ldots \leq \lambda_n \) be a non-decreasing finite sequence of integers, i.e. an integer partition. The corresponding Gelfand–Zetlin polytope is a convex polytope in \( \mathbb{R}^{\frac{n(n-1)}{2}} \) defined by an explicit set of linear inequalities depending on \( \lambda_i \). It will be convenient to label the coordinates \( u_{i,j} \) in \( \mathbb{R}^{\frac{n(n-1)}{2}} \) by pairs of integers \((i,j)\), where \( i \) runs from 1 to \( n-1 \), and \( j \) runs from 1 to \( n-i \). The inequalities defining the Gelfand–Zetlin polytope can be visualized by the following triangular table.

\[
\begin{array}{cccccc}
\lambda_1 & & & & & \lambda_n \\
& \lambda_2 & & & \ldots & \lambda_n \\
& & \ldots & \ldots & \ldots & \lambda_n \\
& & & \ldots & \lambda_n & \lambda_n \\
& & & & u_{1,1} & \ldots & u_{1,n-1} \\
& & & & u_{2,1} & \ldots & u_{2,n-2} \\
& & & & \vdots & \ldots & \vdots \\
& & & & u_{n-2,1} & \ldots & u_{n-2,2} \\
& & & & u_{n-1,1} & \\
\end{array}
\]

where every triple of numbers \( a, b, c \) that appear in the table as vertices of the triangle

\[
\begin{array}{ccc}
a & b \\
& c \\
\end{array}
\]

are subject to the inequalities \( a \leq c \leq b \).

In this paper, we discuss generating functions for the number of vertices in Gelfand–Zetlin polytopes. We will use the multiplicative notation for partitions, e.g. \( 1^{i_1}2^{i_2}3^{i_3} \) will denote the partition consisting of \( i_1 \) copies of 1, \( i_2 \) copies of 2, and \( i_3 \) copies of 3. Given a partition \( p \), we write \( GZ(p) \) for the corresponding Gelfand–Zetlin polytope, and \( V(p) \) for the number of vertices in \( GZ(p) \).

Fix a positive integer \( k \), and consider all partitions of the form \( 1^{i_1}\ldots k^{i_k} \), where a priori some of the powers \( i_j \) may be zero. We let \( E_k \) denote the exponential
generating function for the numbers \( V(1^i \ldots k^i) \), i.e. the formal power series

\[
E_k = \sum_{i_1, \ldots, i_k \geq 0} V(1^i \ldots k^i) \frac{z_1^{i_1} \ldots z_k^{i_k}}{i_1! \ldots i_k!}.
\]

Our first result is a partial differential equation on the function \( E_k \):\

**Theorem 1.1.** The formal power series \( E_k \) satisfies the following partial differential equation with constant coefficients:

\[
\left( \frac{\partial^k}{\partial z_1 \ldots \partial z_k} - \left( \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) \ldots \left( \frac{\partial}{\partial z_{k-1}} + \frac{\partial}{\partial z_k} \right) \right) E_k = 0.
\]

E.g. we have

\[
E_1(z_1) = e^{z_1}, \quad E_2(z_1, z_2) = e^{z_1+z_2} I_0(2\sqrt{z_1z_2}),
\]

where \( I_0 \) is the modified Bessel function of the first kind with parameter 0. This function can be defined e.g. by its power expansion

\[
I_0(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!^2}.
\]

It is also useful to consider ordinary generating functions for the numbers \( V(1^i \ldots k^i) \):

\[
G_k(y_1, \ldots, y_k) = \sum_{i_1, \ldots, i_k \geq 0} V(1^i \ldots k^i) y_1^{i_1} \ldots y_k^{i_k}.
\]

We will also deduce equations on \( G_k \). These will be difference equations rather than differential equations. For any power series \( f \) in the variables \( y_1, \ldots, y_k \), define the action of the divided difference operator \( \Delta_i \) on \( f \) as

\[
\Delta_i(f) = \frac{f - f|_{y_i=0}}{y_i}.
\]

**Theorem 1.2.** The ordinary generating function \( G_k \) satisfies the following equation

\[
(\Delta_1 \ldots \Delta_k - (\Delta_1 + \Delta_2) \ldots (\Delta_{k-1} + \Delta_k)) G_k = 0.
\]

It is known that the ordinary generating functions \( G_k \) can be obtained from exponential generating functions \( E_k \) by the Laplace transform. Thus Theorem 1.2 can in principle be deduced from Theorem 1.1 and the properties of the Laplace transform. However, we will give a direct proof.

For \( k = 1, 2 \) and 3, the generating functions \( G_k \) can be computed explicitly. It is easy to see that

\[
G_1(y_1) = \frac{1}{1 - y_1}, \quad G_2(y_1, y_2) = \frac{1}{1 - y_1 - y_2}.
\]

We will prove the following theorem:
Theorem 1.3. The function $G_3(x, y, z)$ is equal to
\[
\frac{2xz - y(1 - x - z) - y\sqrt{1 - 2(x + z) + (x - z)^2}}{2(1 - x - z)((x + y)(y + z) - y)}.
\]

The numbers $V_{k,\ell,m} = V(1^k 2^\ell 3^m)$ can be alternatively expressed as coefficients of certain polynomials:

Theorem 1.4. The number $V_{k,\ell,m}$ for $k > 0$, $\ell > 0$, $m > 0$ is equal to the coefficient with $x^k z^m$ in the polynomial
\[
\frac{1 - xz}{1 + xz} \left( (1 + x)^{k + \ell + m} (1 + z)^{k + \ell + m} - (x + z)^{k + \ell + m} \right).
\]

Set $s = k + \ell + m$. Note that, since the term $(x + z)^s$ is homogeneous of degree $s$, the number $V_{k,\ell,m}$, where $k, \ell, m > 0$, is also equal to the coefficient with $x^k z^m$ in the power series
\[
\frac{(1 - xz)(1 + x)^s(1 + z)^s}{1 + xz}.
\]
This implies the following explicit formula for the numbers $V_{k,\ell,m}$ ($k, \ell, m > 0$):
\[
V_{k,\ell,m} = \binom{s}{k} \binom{s}{m} + 2 \sum_{i=1}^{k} (-1)^i \binom{s}{k-i} \binom{s}{m-i}.
\]

Note that the sum $\sum_{i=1}^{k} (-1)^i \binom{s}{k-i} \binom{s}{m-i}$ can be expressed as the value of the generalized hypergeometric function $\binom{s}{k-1} \binom{s}{m-1} 3F_2(1, 1 - k, 1 - m; 2 + \ell + m, 2 + k + \ell; -1)$.

2. Recurrence relations

Let $R$ be the polynomial ring in countably many variables $x_1, x_2, x_3, \ldots$. Define a linear operator $A : R \to R$ by the following formula:
\[
A(x_{i_1} x_{i_2} \ldots x_{i_k} f) = (x_{i_1} + x_{i_2})(x_{i_2} + x_{i_3}) \ldots (x_{i_{k-1}} + x_{i_k}) f.
\]
In this formula, $x_{i_1}, \ldots, x_{i_k}$ is any finite subset of variables, and $f(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ is any polynomial in these variables. The formula displayed above defines a linear action of $A$ on the entire $R$. Indeed, any polynomial $P \in R$ can be represented as the sum $P = \sum_S P_S$, where $S$ runs through all finite subsets of variables, and $P_S$ denotes the sum of all terms (monomials together with coefficients) that involve exactly all variables from $S$ and no other variables. The polynomial $A(P_S)$ is defined above, and we extend $A$ by linearity. We also set $A(1) = 1$ by definition.

The operator $A$ thus defined reduces the degrees of all nonconstant polynomials. Therefore, for any polynomial $P$, there exists a positive integer $N$ such that $A^N(P)$ is a constant, which is independent of the choice of $N$ provided that $N$ is sufficiently large. We let $A^\infty(P)$ denote this constant.
Proposition 2.1. We have

\[ V(1^{i_1} \ldots k^{i_k}) = A^\infty(x_1^{i_1} \ldots x_k^{i_k}). \]

Proof. We will argue by induction on the degree \( i_1 + \cdots + i_k \), equivalently, on the dimension of the Gelfand–Zetlin polytope \( GZ(1^{i_1} \ldots k^{i_k}) \). Let \( \pi \) be the linear projection of \( GZ(1^{i_1} \ldots k^{i_k}) \) to the cube \( C \) given in coordinates \((u_1, \ldots, u_{k-1})\) by the inequalities

\[ 1 \leq u_1 \leq 2 \leq u_2 \leq \ldots \leq k - 1 \leq u_{k-1} \leq k. \quad (C) \]

Namely, we set \( u_1 = u_{1,i_1}, u_2 = u_{1,i_1+i_2}, \ldots, u_{k-1} = u_{1,i_1+\cdots+i_{k-1}} \). Observe that all vertices of \( GZ(p) \) project to vertices of the cube \( C \). Thus it suffices to describe the fibers of the projection \( \pi \) over the vertices of the cube \( C \).

It will be convenient to label the vertices of the cube \( C \) by the monomials in the expansion of the polynomial \( A(x_1 \ldots x_k) \). Namely, to fix a vertex of \( C \), one needs to specify, for every \( j \) between 1 and \( k - 1 \), which of the two inequalities \( j \leq u_j \) or \( u_j \leq j + 1 \) turns to an equality. Similarly, to fix a monomial in the polynomial \( A(x_1 \ldots x_k) \), one needs to specify, for every \( j \) between 1 and \( k - 1 \), which term is taken from the factor \((x_j + x_{j+1})\), the term \( x_j \) or the term \( x_{j+1} \). This description makes the correspondence clear.

Let \( v \) be the vertex of the cube \( C \) corresponding to a monomial \( x_1^{\alpha_1} \ldots x_k^{\alpha_k} \). It is not hard to see that the polytope \( \pi^{-1}(v) \) is combinatorially equivalent to

\[ GZ(1^{i_1-1+\alpha_1} \ldots k^{i_k-1+\alpha_k}). \]

Suppose that

\[ A(x_1 \ldots x_k) = \sum_{\alpha_1, \ldots, \alpha_k} c_{\alpha_1 \ldots \alpha_k} x_1^{\alpha_1} \ldots x_k^{\alpha_k}. \]

Then we have

\[ V(1^{i_1} \ldots k^{i_k}) = \sum_{\alpha_1, \ldots, \alpha_k} c_{\alpha_1 \ldots \alpha_k} V(1^{i_1-1+\alpha_1} \ldots k^{i_k-1+\alpha_k}). \]

Since for any \( k \)-tuple of indices \( \alpha_1, \ldots, \alpha_k \), for which the corresponding coefficient \( c_{\alpha_1 \ldots \alpha_k} \) is nonzero, the Gelfand–Zetlin polytope \( GZ(1^{i_1-1+\alpha_1} \ldots k^{i_k-1+\alpha_k}) \) has smaller dimension than \( GZ(1^{i_1} \ldots k^{i_k}) \), we can assume by induction that

\[ V(1^{i_1-1+\alpha_1} \ldots k^{i_k-1+\alpha_k}) = A^\infty(x_1^{i_1-1+\alpha_1} \ldots x_k^{i_k-1+\alpha_k}). \]

Hence we have

\[ V(1^{i_1} \ldots k^{i_k}) = \sum_{\alpha_1, \ldots, \alpha_k} c_{\alpha_1 \ldots \alpha_k} A^\infty(x_1^{i_1-1+\alpha_1} \ldots x_k^{i_k-1+\alpha_k}) = A^\infty(A(x_1^{i_1} \ldots x_k^{i_k})). \]

The desired statement follows. \( \square \)
3. Equations on Generating Functions $E_k$ and $G_k$

In this section, we deduce equations on the generating functions $E_k$ and $G_k$. In particular, we prove Theorems 1.1 and 1.2.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k)$, we let $z^\alpha$ denote the monomial $z_1^{\alpha_1} \cdots z_k^{\alpha_k}$, and $\alpha!$ denote the product $\alpha_1! \cdots \alpha_k!$. The partial derivation with respect to $z_\ell$ will be written as $\partial_\ell$. The power $\partial^\alpha$ will mean $\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k}$. We will write $I_\ell$ for the operator of integration with respect to the variable $z_\ell$. This operator acts on the power series $\sum_{n=0}^{\infty} a_n z_\ell^n$, where $a_n$ are power series in the other variables, as follows:

$$I_\ell \left( \sum_{n=0}^{\infty} a_n z_\ell^n \right) = \sum_{n=0}^{\infty} a_n \frac{z_\ell^{n+1}}{\ell + 1}.$$  

We will use the expansion

$$(x_1 + x_2) \cdots (x_{k-1} + x_k) = \sum_{\alpha} c_\alpha x_\alpha,$$

in which the coefficients $c_\alpha$ can be computed explicitly. Let $E^*_k$ be the sum of all terms in $E_k$ divisible by $z_1 \cdots z_k$. Then we have ($i$, $j$, $\alpha$ being multi-indices of dimension $k$)

$$E^*_k = \sum_{i>0} A^\infty(x^i) \frac{z^i}{i!} = \sum_{i>0} \sum_{\alpha} c_\alpha A^\infty(x^{i-1+\alpha}) \frac{z^i}{i!} =$$

$$= \sum_{\alpha} c_\alpha \partial^\alpha I_1 \cdots I_k \sum_{i>0} A^\infty(x^{i-1+\alpha}) \frac{z^{i+1+\alpha}}{(i-1+\alpha)!} = \sum_{\alpha} c_\alpha \partial^\alpha I_1 \cdots I_k \sum_{j>\alpha} A^\infty(x^j) \frac{z^j}{j!}.$$  

Apply the differential operator $\partial_1 \cdots \partial_k$ to both sides of this equation. Note that $\partial_1 \cdots \partial_k(E^*_k) = \partial_1 \cdots \partial_k(E_k)$. Thus we have

$$\partial_1 \cdots \partial_k(E_k) = \sum_{\alpha} c_\alpha \partial^\alpha \sum_{j>\alpha} A^\infty(x^j) \frac{z^j}{j!}.$$  

Observe also that, since $\alpha \geq 0$ whenever $c_\alpha \neq 0$, we have

$$\partial^\alpha \sum_{j>\alpha} A^\infty(x^j) \frac{z^j}{j!} = \partial^\alpha E_k.$$  

This implies Theorem 1.1.

Example: $k = 1$ and $k = 2$. In the case $k = 1$, we have $E_1 = e^{z_1}$. Consider now the case $k = 2$. Set $E = E_2$, $x = z_1$ and $y = z_2$. By Theorem 1.1, the function $E$ satisfies the following partial differential equation:

$$E_{xy} = E_x + E_y.$$  

This equation can be simplified by setting $E = e^{x+y}u$, then the function $u$ satisfies the equation

$$u_{xy} = u.$$
and the boundary value conditions \( u(x, 0) = u(0, y) = 1 \). We can now look for solutions \( u \) that have the form \( v(xy) \), where \( v \) is some smooth function. This function must satisfy the initial condition \( v(0) = 1 \) and the ordinary differential equation
\[
tv''(t) + v'(t) - v(t) = 0.
\]
It is known that the only analytic solution of this initial value problem is \( I_0(2\sqrt{t}) \), where \( I_0 \) is the modified Bessel function of the first kind. Thus \( I_0(2\sqrt{xy}) \) is a partial solution of the boundary value problem \( u_{xy} = u, \ u(x, 0) = u(0, y) = 1 \).

The solution of this boundary value problem is unique (note that the boundary values are defined on characteristic curves!). Therefore, we must conclude that \( E(x, y) = e^{x+y}I_0(2\sqrt{xy}) \).

The proof of Theorem 1.2 is very similar to the proof of Theorem 1.1. Let \( G^k \) be the sum of all terms in \( G_k \) that are divisible by \( y_1 \ldots y_k \), i.e.
\[
G^*_k = \sum_{i>0} V(1^{i_1} \ldots k^{i_k})y^i.
\]
Then, similarly to a formula obtained for \( E^*_k \), we have
\[
G^*_k = \sum_{\alpha} c_\alpha y_1^{1-\alpha_1} \ldots y_k^{1-\alpha_k} \sum_{j>\alpha} A^\infty(x^j)y^j.
\]
Applying the operator \( \Delta_{x_1} \ldots \Delta_{x_k} \) to both sides of this equation, we obtain Theorem 1.2 Similarly to the proof of Theorem 1.1 we need to use that
\[
\Delta_{x_1} \ldots \Delta_{x_k}(G^*_k) = \Delta_{x_1} \ldots \Delta_{x_k}(G_k)
\]
and that
\[
\Delta_{x_1}^{\alpha_1} \ldots \Delta_{x_k}^{\alpha_k}(G_k) = y_1^{-\alpha_1} \ldots y_k^{-\alpha_k} \sum_{j>\alpha} A^\infty(x^j)y^j.
\]

We will now discuss several examples.

**Example**: \( k = 1 \) and \( k = 2 \). For \( k = 1 \), we have the following equation: \( \Delta_1 G_1 = G_1 \), i.e. \( G_1(y_1) - G_1(0) = y_1 G_1(y_1) \). Knowing that \( G_1(0) = 1 \), this gives
\[
G_1(y_1) = 1 + y_1 + y_1^2 + \cdots = \frac{1}{1 - y_1}.
\]
Suppose that \( k = 2 \). Set \( G = G_2, \ x = y_1, \ y = y_2 \). The function \( G \) satisfies the following equation
\[
\Delta_x \Delta_y G = \Delta_x G + \Delta_y G.
\]
Note that \( G(x, 0) = G_1(x) \) and \( G(0, y) = G_1(y) \). Therefore, the right-hand side can be rewritten as
\[
\frac{G - \frac{1}{1-x}}{x} + \frac{G - \frac{1}{1-y}}{y}.
\]
The left-hand side is
\[
\Delta_x \left( \frac{G - \frac{1}{1-x}}{y} \right) = \frac{1}{x} \left( \frac{G - \frac{1}{1-x}}{y} - \frac{1}{1-y} - 1 \right).
\]
Solving the linear equation on $G$ thus obtained, we conclude that

$$G = \frac{1}{1-x-y}.$$  

**Example:** $k = 3$. We set $G = G_3$, $x = y_1$, $y = y_2$ and $z = y_3$. The function $G$ satisfies the following equation: $\Delta_x \Delta_y \Delta_z G = (\Delta_x + \Delta_y)(\Delta_y + \Delta_z)G$. This equation can be rewritten as follows:

$$\Delta^2_y G = \frac{G(1-x-y-z) - 1}{xyz}.$$  

Suppose that $G = G(x, 0, z) + A(x, z)y + \ldots$, where dots denote the terms divisible by $y^2$. Then we have

$$\Delta^2_y G = G - G(x, 0, z) - A(x, z)y = G - \frac{1}{1-x-z} - A(x, z)y.$$  

Substituting this into the equation, we can solve the equation for $G$ in terms of $A$:

$$G = \frac{-xz + y(1-x-z)(1-A(x, z)xz)}{(1-x-z)(y-(x+y)(y+z))}.$$  

Since the power series $1-x-z$ is invertible, it follows that $G$ has the form

$$a + by \over y-(x+y)(y+z),$$

where $a$ and $b$ are some power series in $x$ and $z$. Let $\lambda$ and $\mu$ be the two solutions of the equation $y = (x+y)(y+z)$, namely,

$$\lambda, \mu = 1-x-z \pm \sqrt{1-2(x+z) + (x-z)^2} \over 2.$$  

The signs are chosen so that, at the point $x = z = 0$, we have $\lambda = 1$ and $\mu = 0$. Then

$$1 \over y-(x+y)(y+z)} = c \over y-\lambda + d \over y-\mu,$$  

where $c$ and $d$ are some power series in $x$ and $z$. Note that, since $(y-\lambda)^{-1}$ makes sense as a power series, $c(a+by)/(y-\lambda)$ can be represented as a power series in $x$, $y$ and $z$. Thus the function $d(a+by)/(y-\mu)$ must also be representable as a power series in $x$, $y$ and $z$. However, this is only possible if the numerator is a multiple of the denominator, i.e. $(a+by) = e(y-\mu)$, where the coefficient $e$ is a power series of $x$ and $z$. It follows that $G$ is equal to $e(y-\lambda)^{-1}$. The coefficient $e$ can be found from the condition $G(x, 0, z) = 1 \over 1-x-z$:

$$G = \frac{1}{1-x-z} \frac{\lambda}{\lambda-y} = \frac{2xz - y(1-x-z) - y\sqrt{1-2(x+z) + (x-z)^2}}{2(1-x-z)((x+y)(y+z) - y).}$$
4. PROOF OF THEOREM 1.4

In this section, we will prove Theorem 1.4, which expresses the numbers \( V_{k,\ell,m} \) as coefficients of certain Laurent polynomials. The numbers \( V_{k,\ell,m} \) satisfy the following recurrence relation:

\[
V_{k,\ell,m} = V_{k-1,\ell,m} + V_{k,\ell-1,m} + V_{k,\ell,m-1} + V_{k-1,\ell+1,m-1}
\]

provided that \( k, \ell, m > 0 \), and the following initial conditions:

\[
V_{0,\ell,m} = V_{\ell,m}, \quad V_{k,0,m} = V_{k,m}, \quad V_{k,\ell,0} = V_{k,\ell}.
\]

Set \( V_{k,m}^s = V_{k,s-k-m,m} \). Then we can write the following recurrence relations on the numbers \( V_{k,m}^s \):

\[
V_{k,m}^s = V_{k-1,m}^{s-1} + V_{k,m-1}^{s-1} + V_{k-1,m-1}^{s-1} + V_{k,m}^{s-1}
\]

provided that \( k \geq 1, m \geq 1, k + m \leq s - 1 \), and

\[
V_{k,m}^s = V_{k-1,m}^{s-1} + V_{k,m-1}^{s-1} + V_{k-1,m-1}^{s-1}
\]

provided that \( k + m = s \).

For a fixed \( s \), we can arrange the numbers \( V_{k,m}^s \) into a triangular table \( T_s \) of size \( s \) as shown on Figure 1. Namely, the number \( V_{k,m}^s \) is placed into the cell, whose southwest (lower left) corner is at position \((k, m)\). The next table \( T_{s+1} \) can be obtained from the table \( T_s \) as follows. First, we add to every element of \( T_s \) its south, west and southwest neighbors. Next, we add a line of cells, whose positions \((k, m)\) satisfy the equality \( k + m = s \). In every cell of this line, we put the sum of the south and west neighbors. Note that, by construction, the boundary of every table \( T_s \) consists of binomial coefficients.

Consider the generating function \( G = G_3 \) for the numbers \( V_{k,\ell,m} \). The splitting of \( G \) into homogeneous components can be obtained by expanding the function \( G(xy, y, zy) \) into powers of \( y \). We set

\[
G(xy, y, zy) = \sum_{s=0}^{\infty} g_s(x, z)y^s
\]
Then we have

\[
g_s(x, z) = \sum_{k=0}^{s} \sum_{m=0}^{s-k} V_{s,k} x^k z^m.
\]

Thus the coefficients of the polynomial \(g_s\) are precisely elements of the table \(T^s\).

The recurrence relations on the numbers \(V_{s,k}\) displayed above imply the following property of the generating functions \(g_s\):

**Proposition 4.1.** The polynomials \(g_s\) satisfy the following recurrence relations:

\[
g_{s+1} = (1 + x + z)g_s + \tau_{s\leq s}(x z g_s),
\]

where the truncation operator \(\tau_{s\leq s}\) acts on a polynomial by removing all terms, whose degrees exceed \(s\).

Consider the polynomials

\[
h_s(x, z) = g_s(x, z) - (x z)^s g_s(z^{-1}, x^{-1}).
\]

Geometrically, these polynomials can be described as follows. Let \(\tilde{T}^s\) denote the table, into which we put all coefficients of the polynomial \(h_s\), see Figure 2. The lower left triangle of size \(s - 1\) is the same in the tables \(T^s\) and \(\tilde{T}^s\). The table \(\tilde{T}^s\) is skew-symmetric with respect to the main diagonal. These two properties give a unique characterization of the tables \(\tilde{T}^s\).
The rules, by which the tables $\tilde{T}^s$ are formed, are the following (see Figure 3). The first table $\tilde{T}^1$ is by definition the left-most table shown on Figure 2. The next table $\tilde{T}^{s+1}$ is obtained inductively from the preceding table $\tilde{T}^s$ in two steps. In the first step, we add to every element of $\tilde{T}^s$ its immediate west, south and southwest neighbors. In the second step, we modify elements in two diagonals of the table, namely, the elements, whose positions (measured by southwest corners) $(k, m)$ satisfy the equality $k + m = s$ or $k + m = s + 2$. To the cell at position $(k, m)$, where $k + m = s$, we add the binomial coefficient $\binom{k+m}{m}$. From the cell at position $(k+1, m+1)$, we subtract this binomial coefficient.

We have the following recurrence relation on the polynomials $h^s$:

$$h_{s+1} = h_s(1 + x)(1 + z) + (1 - xz)(x + z)^s,$$

which does not contain truncation operators. Therefore, the generating function $H = \sum_{s=0}^{\infty} h_s y^s$ satisfies the following linear equation:

$$H = 1 + y((1 + x)(1 + z)H + (1 - xz)(1 - y(x + z))^{-1}).$$

Solving this equation, we find that

$$H = \frac{y(1 - xz)}{(1 - y(x + z))(1 - y(1 + x)(1 + z))}.$$

Knowing the generating function $H$, we can now obtain an explicit formula for the polynomials $h_s$, namely,

$$h_s(x, z) = \frac{1 - xz}{1 + xz} ((1 + x)^s(1 + z)^s - (x + z)^s).$$

Theorem 1.4 is thus proved.

Open problems.

1. Prove or disprove: the generating function $G_4$ is algebraic. Note that $G_1$ and $G_2$ are rational, and $G_3$ is algebraic.

2. Deduce differential or difference equations on the generating functions for the $f$-vectors and for the modified $h$-vectors of Gelfand–Zetlin polytopes.

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