Skyrme-type Non-linear sigma Models via The Higher Dimensional Landau Models

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Abstract

A curious correspondence has been known between Landau models and non-linear sigma models in low dimensions: Reinterpreting the base-manifold of Landau models as a field manifold, the Landau models are transformed to non-linear sigma models with same global and local symmetries. With the idea of the dimensional hierarchy of the higher dimensional Landau models, we exploit this correspondence to present a systematic procedure for construction of non-linear sigma models in higher dimensions. We explicitly derive $O(2k+1)$ non-linear sigma models in $2k$ dimension based on the parent tensor gauge theories that originate from non-Abelian monopoles. The obtained non-linear sigma models turn out to be Skyrme-type non-linear sigma models with hidden $O(2k+1)$ local symmetries. By a dimensional reduction based on the Chern-Simons tensor field theory, we also derive Skyrme-type $O(2k)$ non-linear sigma models in $2k−1$ dimension. As a unified description, we explore Skyrme-type $O(d+1)$ non-linear sigma models and clarify their basic properties, such as stability of soliton configurations, scale invariant solutions, and topological field configurations of higher winding number.
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1 Introduction

Non-linear sigma (NLS) models were originally introduced for a description of mesons in hadron physics around 1960 [11, 23, 45, 55, 60]. Skyrme proposed his celebrated NLS model with a higher derivative term [27] to describe baryons as solitonic excitations of meson fluid. We refer to such non-linear sigma models with a higher derivative term as the Skyrme-type non-linear sigma model (S-NLS) in this paper. The Skyrmions, or more generally the NLS model topological solitons, accommodate interesting mathematical structure related to gauge theories. In particular, relationship between the quaternionic projective non-linear sigma model and $SU(2)$ gauge theory was investigated intensively around 1970 [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The self-dual equations of higher dimensional gauge theories were also revealed in 1980s [18, 19, 20, 21, 22, 23, 24, 25, 26]. An explicit recipe for the derivation of the Skyrmion field configuration from the Yang-Mills gauge theories was proposed by Atiyah and Manton [27, 28], which stimulated recent studies about mathematical connections of topological solitons in different dimensions [29, 30, 31, 32, 33, 34] and [35, 36, 37, 38, 39, 40, 41]. Apart from such deep mathematical structures, Skyrmions now appear ubiquitously in many branches of theoretical physics [42] and are also observed in daily nanoscale magnetic experiments (see [43] and references therein).

One of the most prominent early experiments about Skyrmions, more precisely $O(3)$ NLS model solitons, is the NMR Knight shift measurement of the spin texture in quantum Hall ferromagnets [44]. Besides of the quantum Hall ferromagnets, we often come across the $O(3)$ NLS model solitons in various contexts of the quantum Hall effect. One example is about anyonic excitations of the fractional quantum Hall effect. The effective field theory of the fractional quantum Hall effect is the Chern-Simons topological field theory [45, 46, 17]. The Chern-Simons statistical field coupled to the $O(3)$ NLS model solitons provides a field theoretical description of anyons [48, 49] and such anyons are realized as fractionally charge excitations of the fractional quantum Hall effect [50, 51]. Another important example is about their analogous mathematical structures. The Haldane’s formulation of the quantum Hall effect [52] is based on the $SO(3)$ Landau model [53, 54] in the Dirac monopole background [55], in which the base-manifold or physical space is given by $S^2$ and the gauge symmetry is $U(1)$. Meanwhile in the $O(3)$ NLS model [56, 57] or equivalently the $CP^1$ model [58, 59, 60], the target-manifold manifold or the field-space is $S^2 \simeq CP^1$ and the hidden local symmetry is $U(1)$. One may find a curious correspondence between the Landau model and the NLS model: The base-manifold $S^2$ of the Landau model is identical to the target-manifold of the $O(3)$ NLS model, and their local symmetries are also given by $U(1)$. We will refer to this correspondence as the Landau/NLS model correspondence.

The Landau/NLS model correspondence is not a special property in 2D, but holds in 4D. In the 4D quantum Hall effect [61], the Landau model is given by the $SO(5)$ Landau model [62, 63] whose base-manifold is $S^4$ and magnetic field background is given by the Yang’s $SU(2)$ monopole [64]. Meanwhile in the $O(5)$ NLS model or the $\mathbb{H}P^1$ model [8, 9, 13, 14, 15, 16, 17], the field-manifold is $S^4$ and the hidden local symmetry is $SU(2)$. Besides, anyonic excitations in

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1 We used “$SO(3)$” for the Landau models, since the Landau model Hamiltonian is constructed by the angular momentum operators of the $SO(3)$ group, while “$O(3)$” for the NLS model since the NLS model Hamiltonian is invariant under the $O(3)$ transformation, i.e., $SO(3)$ rotations and $Z_2$ reflection of the NLS field.
the 4D quantum Hall effect are known to be membrane-like objects whose internal space is $S^4$ which is described by the field-manifold of the $O(5)$ NLS model \[65, 66\]. Thus, the Landau/NLS model correspondence is naturally generalized from 2D to 4D. Arbitrary $2kD$ generalization of the quantum Hall effect has been constructed in our previous works \[67, 68, 69\]. The mathematical set-up of the $2kD$ quantum Hall effect is the $SO(2k + 1)$ Landau model on $S^{2k}$ in the $SO(2k)$ monopole background. The excitations are $(2k-2)$-dimensionally extended anyonic objects whose fractional statistics are well investigated in \[70, 71, 72, 73, 74\]. Besides, the effective field theory is given by a tensor-type Chern-Simons field theory coupled to the $(2k-2)$-brane with $S^{2k}$ internal space, which is identified with the field manifold of $O(2k+1)$ NLS models \[68\].

While NLS model solitons play crucial roles in the higher dimensional quantum Hall effect, a systematic analysis of the $O(2k+1)$ NLS model to host membrane excitations is still lacking. To be more precise, there are numerous possible NLS models with field-manifold being $S^{2k}$, but there is no criterion to choose better models or hopefully the best model among them. A main purpose of this paper is to provide a systematic procedure to construct appropriate NLS models based on the Landau/NLS model correspondence [Fig.1]. The idea of the dimensional hierarchy of the higher dimensional Landau models \[75, 68, 67\] is essential in the construction, and the obtained NLS models necessarily inherit structures of the differential geometry of the Landau models. For a concrete construction of NLS model Hamiltonians, we adopt the idea originally suggested by Tchrakian \[18\] and recently made manifest by Adam et al. \[76\] where a BPS equation is firstly given and the Hamiltonian is later derived so that the Hamiltonian may satisfy the BPS equation.

The paper is organized as follows. Sec.2 reviews the differential geometry associated with non-Abelian monopoles in the higher dimensional Landau models. In Sec.3 we reconsider geometric meanings of the Skyrme’s NLS field and the $O(5)$ S-NLS model in the light of the Landau/NLS model correspondence. We present a systematic method for the derivation of $O(2k+1)$ S-NLS

**Figure 1**: The Landau/NLS model correspondence. The differential topological structure of the $SO(2k+1)$ Landau model is same as of the $O(2k+1)$ NLS model. The Landau model is transformed to the NLS model under identification of the base-manifold with the field-manifold.
# Differential Geometry of the Higher Dimensional Landau Model

In this section, we review the differential geometry of the \(SO(2k+1)\) Landau models and discuss extended objects that are realized as the \(O(2k+1)\) NLS model solitons.

## 2.1 Non-Abelian monopole configuration of the \(SO(2k+1)\) Landau model

The \(SO(5)\) Landau model is formulated on \(S^4\) embedded in \(\mathbb{R}^5\) \([61, 62, 63]\), and the background magnetic field is given by the Yang’s \(SU(2)\) monopole \([64]\)

\[
A = -\frac{1}{2r(r + r_5)}\eta^i_{mn} r_n \sigma_i dr_m, \quad (m, n = 1, 2, 3, 4)
\]

where \(\eta^i_{mn} = \epsilon_{mn4} + \delta_{mi} \delta_{n4} - \delta_{m4} \delta_{ni}\) denotes the 't Hooft symbol \([77]\). The 1D reduction of the \(SO(5)\) Landau model reproduces the \(SO(4)\) Landau model \([78, 79]\) on the \(S^3\)-equator of \(S^4\) \([63, 84]\). In Sec.3 we will consider the reverse 1D promotion process to derive the \(O(5)\) S-NLS model from the Skyrme’s field-manifold \(S^3\).

Generalizing the \(SU(2)\) \((\otimes SU(2) \simeq SO(4))\) to the \(SO(2k)\) group \([80]\), the \(SO(2k+1)\) Landau model is introduced on a base-manifold \(S^{2k}\) in the \(SO(2k)\) monopole background \([67, 68]\) \([Table 1]\). Notice that the gauge group is uniquely determined by the dimension of the base-manifold.

The \(SO(2k)\) monopole gauge field is represented as

\[
A = \sum_{a=1}^{2k+1} A_a dr_a - \frac{1}{r(r + r_{2k+1})} \sum_{m,n=1}^{2k} \sigma_{mn} r_n dr_m, \quad (2)
\]

\(\text{To be precise, Spin}(2k)\) group.

| Landau model | \(SO(3)\) | \(SO(5)\) | \(SO(2k+1)\) |
|--------------|-----------|-----------|--------------|
| Base-manifold | \(S^2\) | \(S^4\) | \(S^{2k}\) |
| Global symmetry | \(SO(3) \simeq SU(2)\) | \(SO(5)\) | \(SO(2k+1)\) |
| Monopole gauge group | \(SO(2) \simeq U(1)\) | \(SO(4) \simeq SU(2) \otimes SU(2)\) | \(SO(2k)\) |
| Chern number | 1st | 2nd | \(k\)th |
| Topological map | \(\pi_1(U(1)) \simeq \mathbb{Z}\) | \(\pi_3(SU(2)) \simeq \mathbb{Z}\) | \(\pi_{2k-1}(SO(2k)) \simeq \mathbb{Z}\) |

Table 1: Geometric and topological features of the Landau models. The monopole gauge group \(SO(2k)\) is chosen so that it is identical to the holonomy group of the base-manifold \(S^{2k} \simeq SO(2k+1)/SO(2k)\) \([67]\). In the \(SO(5)\) Landau model, the holonomy of \(S^4\) is \(SO(4) \simeq SU(2) \otimes SU(2)\) and one \(SU(2)\) is adopted as the gauge group.

models and explicitly construct the \(O(7)\) NLS model and \(O(2k+1)\) NLS model Hamiltonians in Sec.4. In Sec.5 we derive \(O(2k)\) S-NLS models using the Chern-Simons term of pure gauge fields. We explore a general construction of \(O(d+1)\) S-NLS models and analyze their basic properties in Sec.6. Sec.7 is devoted to summary and discussions.
or
\[ A_m = -\frac{1}{r(r + r_{2k+1})} \sigma_{mn} r_n, \quad A_{2k+1} = 0. \quad (m, n = 1, 2, \cdots, 2k) \tag{3} \]
which is regular except for the south pole. Here, \( \sigma_{mn} \) are the \( \text{SO}(2k) \) matrices in the spinor representation, i.e. \( \text{Spin}(2k) \) matrix generators:
\[ \sigma_{ij} = -i \frac{1}{4} [\gamma_i, \gamma_j], \quad \sigma_{i, 2k} = -\sigma_{2k, i} = \frac{1}{2} \gamma_i \tag{6} \]
that satisfy
\[ [\sigma_{mn}, \sigma_{pq}] = i(\delta_{mp}\sigma_{nq} - \delta_{mq}\sigma_{np} + \delta_{nq}\sigma_{mp} - \delta_{np}\sigma_{mq}). \tag{7} \]
\( \gamma_i \) \((i = 1, 2, \cdots, 2k - 1)\) stand for the \( \text{SO}(2k - 1) \) gamma matrices. The \( \text{SO}(2k) \) monopole field strength is derived as
\[ F = dA + iA^2 = \frac{1}{2} F_{ab} \, dr_a \wedge dr_b, \tag{8} \]
where \( F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b] \) are
\[ F_{mn} = \frac{1}{r^2} \sigma_{mn} - \frac{1}{r^2} (r_m A_n - r_n A_m), \quad F_{m, 2k+1} = -F_{2k+1, m} = \frac{1}{r^2} (r + r_{2k+1}) A_m. \tag{9} \]
(3) and (9) satisfy the field equations of motion of the pure Yang-Mills theory in \((2k + 1)D\):
\[ D_a F_{ab} = \partial_a F_{ab} + i[A_a, F_{ab}] = 0. \tag{10} \]
One may need only the algebraic property of the \( \text{SO}(2k) \) generators (7) to verify (10), and so the monopole gauge field (3) of any \( \text{Spin}(2k) \) representation realizes a solution of the pure Yang-Mills field equation. The monopole configuration carries unit Chern number. Indeed, substituting (9) into the \( k \)th Chern number
\[ c_k = \frac{1}{k!(2\pi)^k} \int \text{tr}(F^k), \tag{11} \]
we have
\[ N_{2k} = \frac{1}{A(S^2_{\text{phys}})} \int_{S^2_{\text{phys}}} \frac{1}{(2k)!} \epsilon_{a_1 a_2 \cdots a_{2k+1}} r^{a_1 a_2} dr_{a_1} dr_{a_2} \cdots dr_{a_{2k}} = 1, \tag{12} \]
where \( A(S^{2k}) \) denotes the area of \( S^{2k} \):
\[ A(S^{2k}) = \frac{2^{k+1}}{(2k - 1)!!} \pi^k. \tag{13} \]

\footnote{At \( r_{2k+1} = 0 \), the \( \text{SO}(2k) \) monopole configuration, (3) or (9), is reduced to the meron configuration on \( \mathbb{R}^{2k} \) [31]:
\[ A_\mu = -\frac{1}{x_2} \sigma_{\mu\nu} x_\nu, \quad F_{\mu\nu} = \frac{1}{x^2} \sigma_{\mu\nu} - \frac{1}{x^2} (x_\mu A_\nu - x_\nu A_\mu), \tag{4} \]
which satisfies the pure Yang-Mills field equation on \( \mathbb{R}^{2k} \) [32, 33]:
\[ \frac{\partial}{\partial x_\mu} F_{\mu\nu} + i[A_\mu, F_{\mu\nu}] = 0. \tag{5} \]}

\footnote{We will give an alternative verification in Appendix A.4.}
Another expression of the SO gauge with (15), are related by a gauge transformation on the which is regular except for the north-pole. The two expressions of the monopole gauge fields, (2) and (15), are related by a gauge transformation on the $S^{2k-1}$-equator of $S^{2k}$:

$$A' = g^\dagger Ag - ig^\dagger dg,$$

where $g$ denotes a transition function of the form

$$g = \frac{1}{\sqrt{r^2 - r_{2k+1}^2}} 1_{2k-1} + i \frac{1}{\sqrt{r^2 - r_{2k+1}^2}} \sum_{i=1}^{2k-1} \hat{r}_i \gamma_i = \cos \theta 1_{2k-1} + i \sin \theta \sum_{i=1}^{2k-1} \hat{r}_i \gamma_i = e^{i\theta \sum_{i=1}^{2k-1} \hat{r}_i \gamma_i}.$$  

(17)

Here, $\hat{r}_i = \frac{1}{\sqrt{r^2 - r_{2k+1}^2 - r^2}} r_i$ ($i = 1, 2, \cdots, 2k - 1$) represent the normalized-latitude $S^{2k-2}$ at an azimuth angle $\theta$ on the $S^{2k-1}$-latitude. Since $\gamma_i = 2\sigma_i, 2k \in Spin(2k), g = e^{2i\theta \sum_{i=1}^{2k-1} \hat{r}_i \sigma_i, 2k}$ describes a map from $S^{2k-1}$ to $SO(2k)$ group and satisfies

$$g(x)\dagger g(x) = \frac{1}{r^2 - r_{2k+1}^2} \sum_{\mu=1}^{2k} r_\mu r_\mu = 1_{2k-1}, \quad \det(g(x)) = (\frac{1}{r^2 - r_{2k+1}^2} r_\mu r_\mu)^{2k-2} = 1.$$  

(19)

With $g$, $A$ and $A'$ are simply represented as

$$A = \frac{1}{2r}(r - r_{2k+1})dgg^\dagger, \quad A' = -\frac{1}{2r}(r + r_{2k+1})g^\dagger dg,$$

where

$$-idgg^\dagger = \frac{2}{r^2 - r_{2k+1}^2} \sigma_{mn} r_n dr_m, \quad -ig^\dagger dg = -\frac{2}{r^2 - r_{2k+1}^2} \bar{\sigma}_{mn} r_n dr_m.$$  

(21)

The $k$th Chern number (11) can be expressed by the transition function as $[68]

$$c_k = \frac{(-i)^{k-1}}{(2k-1)!} A(S^{2k-1}) \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1} = (-i)^{k-1} \frac{1}{(2\pi)^k (2k-1)!} \int_{S^{2k-1}} \text{tr}(-ig^\dagger dg)^{2k-1}$$

(22)

where $A(S^{2k-1})$ signifies the area of $(2k - 1)$-sphere:

$$A(S^{2k-1}) = \frac{2\pi^k}{(k-1)!}.$$  

(23)

$^5 \theta$ is given by

$$\tan \theta = \frac{1}{r_{2k}} \sqrt{r^2 - r_{2k+1}^2 - r_{2k}^2}.$$  

(18)
The associated topology is indicated by
\[ \pi_{2k-1}(SO(2k)) \simeq \mathbb{Z}. \] (24)

Substituting (17) into (22), we have
\[ N_{2k-1} = \frac{1}{A(S^{2k-1})} \int_{S^{2k-1}} \frac{1}{(2k-1)!} r_{a_1 a_2 \cdots a_{2k}} dr_{a_1} dr_{a_2} \cdots dr_{a_{2k-1}} = 1, \] (25)

which reproduces the previous result (12), as it should be. The equivalence between (12) and (25) holds for other higher dimensional representations of gauge group matrix generators [75]. We thus find that there are the two equivalent but superficially different representations of the \( k \)th Chern number for the monopole field configuration:

1. Winding number associated with \( \pi_{2k}(S^{2k}) \simeq \mathbb{Z} \)
2. Winding number associated with \( \pi_{2k-1}(S^{2k-1}) \simeq \mathbb{Z} \).

We utilize the first observation in the construction of the \( O(2k+1) \) S-NLS models, while the second one in the construction of the \( O(2k) \) S-NLS models. This equivalence will also be important in the discussions of topological field configurations (Sec.6.2).

### 2.2 Tensor gauge fields and extended objects

The Chern number (11) can be expressed as
\[ c_k = \frac{1}{k!(2\pi)^k} \int G_{2k}, \] (26)

where \( G_{2k} \) denotes a \( 2k \) rank tensor field strength
\[ G_{2k} = \text{tr}(F^k) = \frac{1}{(2k)!} G_{a_1 a_2 \cdots a_{2k}} dr_{a_1} dr_{a_2} \cdots dr_{a_{2k}} \] (27)

or
\[ G_{a_1 a_2 \cdots a_{2k}} = \frac{1}{2k} \text{tr}(F_{[a_1 a_2} F_{a_3 a_4} \cdots F_{a_{2k-1} a_{2k}]}) = \frac{1}{2k} \text{tr}(F_{[a_1 a_2 \cdots a_{2l-1} a_{2l}} F_{a_{2l+1} a_{2l+2} \cdots a_{2k-1} a_{2k}]}) \] (28)

Here, we introduced the antisymmetric tensor field strength [18]
\[ F_{a_1 a_2 \cdots a_{2l}} \equiv \frac{1}{(2l)!} F_{[\mu_1 \mu_2 a_3 a_4 \cdots F_{a_{2l-1} a_{2l}]}. \] (29)

There are \( [k/2] \) ways in the decomposition (28) in correspondence with \( l = 1, 2, \cdots, [k/2] \). The antisymmetric tensor field strength (29) will play a crucial role in constructing higher dimensional NLS models in Sec.4.

\( [k/2] \) signifies the maximum integer that does not exceed \( k/2 \). Apparently, there exists a local degree of freedom in the decomposition [57]:
\[ F_{a_1 a_2 \cdots a_{2l}} \cdot F_{a_{2l+1} a_{2l+2} \cdots a_{2k}} = \lambda(x) F_{a_1 a_2 \cdots a_{2l}} \cdot \frac{1}{\lambda(x)} F_{a_{2l+1} a_{2l+2} \cdots a_{2k}}. \] (30)
For the non-Abelian monopole gauge field \[68\], we can evaluate \((28)\) as
\[
G_{2k} = \frac{1}{2^k + 1} \tau_{2k+1} \epsilon_{a_1 a_2 \cdots a_{2k+1}} dr_{a_1} dr_{a_2} \cdots dr_{a_{2k+1}},
\]
(31)
or
\[
G_{a_1 a_2 \cdots a_{2k}} = \frac{(2k)!}{2^k + 1} \tau_{2k+1} \epsilon_{a_1 a_2 \cdots a_{2k+1}} \tau_{a_{2k+1}},
\]
(32)
which signifies the \(2k\)-rank tensor monopole field strength in its own right \[90, 91\], and the \((2k-1)\)-rank tensor gauge field \((dC_{2k-1} = G_{2k})\) \[92\] couples to \((2k-2)\)-dimensionally extended objects, \(i.e., (2k-2)\)-branes. In the higher dimensional quantum Hall effect, the size of the gauge space is comparable with the size of the base-manifold \(S^{2k}\) \[68\], and the whole system is regarded as a \((4k-1)D\) space-time. The \((2k-2)\)-brane current in \((4k-1)D\) space-time is simply given by
\[
J_{\mu_1 \mu_2 \cdots \mu_{2k-1}} = \frac{1}{(2k)!} \tau_{\mu_1 \mu_2 \cdots \mu_{4k-1}} \epsilon_{a_1 a_2 \cdots a_{2k+1}} n_{a_1} \partial_{\mu_{2k}} n_{a_2} \partial_{\mu_{2k+1}} n_{a_3} \cdots \partial_{\mu_{4k-1}} n_{a_{2k+1}},
\]
(33)
where \(n_a\) denote the internal field coordinates of the \((2k-2)\)-brane, which is depicted as the blue sphere of the left-figure of Fig.1. A simple subtraction, \((4k-1) - (2k-2) = 2k + 1\), implies that the dimension of the internal space of the \((2k-2)\)-brane is \(2kD\) and is naturally described by the \(S^{2k}\) field-manifold of \(O(2k+1)\) NLS models. Indeed, \((33)\) is identical to the topological current of the \(O(2k+1)\) NLS model soliton in \((4k-1)D\) space-time with coordinates \(n_a\) subject to
\[
\sum_{a=1}^{2k+1} n_a n_a = 1.
\]
The \((2k-2)\)-brane current is coupled to the \((2k-1)\)-rank tensor Chern-Simons field and used to describe anyonic excitations in higher dimensions. In this way, the \(O(2k+1)\) NLS model solitons necessarily appear in the context of the higher dimensional quantum Hall effect.

3 1D promotion and the \(O(5)\) S-NLS model

In Sec.2 we first introduced the two monopole gauge field configurations on \(S^{2k}\) and later gave the transition function connecting them on the \(S^{2k-1}\)-equator of \(S^{2k}\). In this section, we apply the reverse process to construct the \(O(5)\) S-NLS model from the Skyrme’s \(S^3\) field-manifold.

3.1 Translation to the field manifold and 1D Promotion

Recall that the base-manifold of the \(SO(5)\) Landau model is \(S^4\) with its equator being \(S^3\). We reinterpret \(S^4\) and \(S^3\) as field manifolds in the NLS model side.

3.1.1 Skyrme’s Field-manifold \(S^3\)

The Skyrme’s field \(n_m\) \((m = 1, 2, 3, 4)\) takes its values on \(S^3\) field:
\[
\sum_{m=1}^{4} n_m n_m = 1.
\]
(34)
Instead of using \( n_m \) directly, we will represent the field as the SU(2) group element

\[
g = \sum_{m=1}^{4} n_m \bar{q}_m, \tag{35}
\]

where \( \bar{q}_m \equiv \{-q_{i=1,2,3,4}\} \) are quaternions that satisfy

\[
q_i^2 = -1, \quad q_i q_j = -q_j q_i = q_k \quad (i \neq j). \tag{36}
\]

In a matrix representation, \( q_i \) can be represented as

\[
q_i = -i\sigma_i. \tag{37}
\]

The associated gauge field is simply a pure gauge on \( S^3_{\text{field}} \):

\[
A = -ig^\dagger dg = -\tilde{\eta}^i_{mn} \sigma_i n_m dn_m, \quad F = dA + iA^2 = \tilde{\eta}^i_{mn} \sigma_i dn_m \wedge dn_n \quad (1 - 1) = 0, \tag{38}
\]

where \( \tilde{\eta}^i_{mn} \equiv \epsilon_{mn4} - \delta_m \delta_n^4 + \delta_m^4 \delta_n \) and we used \( \sum_{m=1}^{2k} n_m n_m = 1 \). Suppose \( n_m \) is a field on \( x_\alpha \in \mathbb{R}^3 \), the Skyrme’s higher derivative term can be expressed as

\[
(\partial_\alpha n_m)^2 (\partial_\beta n_n)^2 - (\partial_\alpha n_m \cdot \partial_\beta n_m)^2 = -\frac{1}{8} \text{tr}([A^\alpha, A_\beta]^2) = \frac{1}{8} \text{tr}((\partial_\alpha A_\beta - \partial_\beta A_\alpha)^2). \tag{39}
\]

### 3.1.2 1D promotion

Stacking \( S^3_{\text{field}} \) along a virtual 5th direction, we form a virtual \( S^4_{\text{field}} \) (the middle figure of Fig. 2), in which the radii of \( S^3_{\text{field}} \)s are continuously tuned as

\[
n_m \rightarrow \frac{1}{\sqrt{1 - n_5^2}} n_m. \tag{40}
\]

\( n_{a=1,2,3,4,5} \) realize the coordinates of \( S^4_{\text{field}} \):

\[
\sum_{a=1}^{5} n_a n_a = 1. \tag{41}
\]

This process demonstrates 1D promotion from 3D to 4D to manifest the idea of the dimensional hierarchy \([75][84]\). The SU(2) group element \((35)\) now turns to

\[
g = \frac{1}{\sqrt{1 - n_5^2}} \sum_{m=1}^{4} n_m \bar{q}_m. \tag{42}
\]

We regard \( g \) as a transition function connecting two gauge fields on the \( S^3_{\text{field}} \)-equator of the virtual field manifold \( S^4_{\text{field}} \):

\[
A' = g^\dagger Ag - ig^\dagger dg. \tag{43}
\]

\( \text{\footnotesize \textsuperscript{35}} \) is known as the principal chiral field of mesons in hadron physics.
Figure 2: We first promote the $S^3_{\text{field}}$ to $S^4_{\text{field}}$. Secondly, we construct a gauge field theory on the field manifold $S^4_{\text{field}}$. Lastly, we derive $O(5)$ S-NLS model Hamiltonian.

Such gauge fields are given by (20):

\[
A = i \frac{1}{2} (1 - n_5) d g g^\dagger = - \frac{1}{2(1 + n_5)} \eta^i_{mn} n_n \sigma_i d n_m, \quad A' = - \frac{1}{2} (1 + n_5) g^\dagger d g = - \frac{1}{2(1 - n_5)} \tilde{\eta}^i_{mn} n_n \sigma_i d n_m. \tag{44}
\]

Let us assume that $n_a$ denote the field representing a map from $x^\mu \in \mathbb{R}^4_{\text{phys.}}$ to $n_a \in S^4_{\text{field}}$, and then (44) becomes

\[
A = - \frac{1}{2(1 + n_5)} \eta^i_{mn} \partial_\mu n_m \sigma_i d x_\mu, \quad A' = - \frac{1}{2(1 - n_5)} \tilde{\eta}^i_{mn} n_n \sigma_i \partial_\mu n_m d x_\mu. \tag{45}
\]

Notice that (45) can be regarded as field configurations on $\mathbb{R}^4_{\text{phys.}}$:

\[
A_\mu(n_a(x)) = - \frac{1}{2(1 + n_5)} \eta^i_{mn} n_n \partial_\mu n_m \sigma_i, \quad A'_\mu(n_a(x)) = - \frac{1}{2(1 - n_5)} \tilde{\eta}^i_{mn} n_n \partial_\mu n_m \sigma_i. \tag{46}
\]

The corresponding field strengths on $\mathbb{R}^4_{\text{phys.}}$ are

\[
F_{\mu\nu}(n_a(x)) = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu],
\]

\[
= \frac{1}{2} \eta^i_{mn} \partial_\mu n_m \partial_\nu n_n \sigma_i \quad - \frac{1}{2(1 + n_5)} \eta^i_{mn} n_n (\partial_\mu n_m \partial_\nu n_5 - \partial_\nu n_m \partial_\mu n_5) \sigma_i,
\]

\[
F'_{\mu\nu}(n_a(x)) = \partial_\mu A'_\nu - \partial_\nu A'_\mu + i [A'_\mu, A'_\nu],
\]

\[
= \frac{1}{2} \eta^i_{mn} \partial_\mu n_m \partial_\nu n_n \sigma_i \quad - \frac{1}{2(1 - n_5)} \tilde{\eta}^i_{mn} n_n (\partial_\mu n_m \partial_\nu n_5 - \partial_\nu n_m \partial_\mu n_5) \sigma_i. \tag{47}
\]

When $n_a$ are given by the inverse stereographic coordinates on $S^4_{\text{phys.}}$ from $\mathbb{R}^4_{\text{phys.}}$:

\[
r_a = \{ r_\mu, r_5 \} \equiv \left\{ \frac{2}{1 + x^2} x_\mu, \frac{1 - x^2}{1 + x^2} \right\}, \tag{48}
\]
(46) and (47) realize the BPST instanton configuration [93]:

\[
A_\mu|_{n_a=r_a} = -\frac{1}{x^2 + 1} \eta^i_{\mu \nu} x^\nu \sigma_i, \quad F_{\mu \nu}|_{n_a=r_a} = 2 \frac{1}{(x^2 + 1)^2} \eta^i_{\mu \nu} \sigma_i, \quad (49)
\]

which carries unit 2nd Chern number. (49) simply corresponds to the stereographic projection of the Yang’s SU(2) monopole gauge field (11) on \( S^4 \) [94] (see Appendix A for details).

3.2 From the non-Abelian gauge theory to \( O(5) \) S-NLS model

The next step is to adopt an appropriate gauge theory action to construct NLS model Hamiltonian. As the field strength is represented by the NLS field, we readily construct NLS model Hamiltonian, provided a gauge theory action was given. A natural choice is to adopt the pure Yang-Mills action

\[
S = \frac{1}{6} \int_{\mathbb{R}^4} d^4 x \, \text{tr}(F_{\mu \nu}^2). \quad (50)
\]

The previous studies [8, 13, 14, 15, 16, 17] already showed that substitution of \( F_{\mu \nu} \) (47) into (50) yields the \( O(5) \) S-NLS model Hamiltonian

\[
H = \frac{1}{12} \int_{\mathbb{R}^4} d^4 x \left( (\partial_\mu n_a)^2 (\partial_\nu n_b)^2 - (\partial_\mu n_a \partial_\nu n_a)^2 \right). \quad (51)
\]

One may notice that (51) is a straightforward 4D generalization of the Skyrme term (39). We reconsider this result below.

3.2.1 BPS inequality and Yang-Mills action

[12] and [76, 85, 86, 87, 88, 89] indicate a procedure to construct an action from a given BPS inequality. Usually for a given system we have an action at first, and the BPS inequality is later derived, but here the process is reversed: BPS inequality is firstly given, and then an appropriate action is constructed so that the action can satisfy the given BPS inequality. We discuss how this idea works in the 4D Yang-Mills gauge theory. The BPS inequality is given by

\[
\text{tr}((F_{\mu \nu} - \bar{F}_{\mu \nu})^2) \geq 0 \quad (52)
\]

or

\[
\text{tr}(F_{\mu \nu}^2) + \text{tr}(\bar{F}_{\mu \nu}^2) \geq 2 \text{tr}(F_{\mu \nu} \bar{F}_{\mu \nu}), \quad (53)
\]

where \( \bar{F}_{\mu \nu} \) are defined as

\[
\bar{F}_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F_{\rho \sigma}. \quad (54)
\]

The integral of the right-hand side signifies the second Chern number:

\[
c_2 = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} d^4 x \, \text{tr}(F_{\mu \nu} \bar{F}_{\mu \nu}), \quad (55)
\]

\( ^8 \)The author is indebted to Dr. Amari for the information.
and (53) implies
\[ S_{4,2} = \frac{1}{12} \int_{\mathbb{R}^4} \left( \text{tr}(F_{\mu\nu}^2) + \text{tr}(\tilde{F}_{\mu\nu}^2) \right) \geq A(S^4) \cdot c_2, \] (56)
where \( A(S^4) = \frac{8}{3} \pi^2 \). From the special property in 4D,
\[ \tilde{F}_{\mu\nu} = F_{\mu\nu}^2, \] (57)
\( S_{4,2} \) “accidentally” coincides with the pure Yang-Mills action (50):
\[ S_{4,2} = \frac{1}{6} \int_{\mathbb{R}^4} d^4x \text{tr}(F_{\mu\nu}^2). \] (58)
In even higher dimensions, actions are no longer Yang-Mills type but higher tensor-field type as we shall see in Sec.4.

### 3.2.2 Construction of the \( O(5) \) S-NLS model

We next substitute (47) into the parent gauge theory action (58) to obtain
\[ F_{\mu\nu} \rightarrow F_{\mu\nu}(n_a) \]
\[ S_{4,2} \quad H_{4,2} = \frac{1}{12} \int_{\mathbb{R}^4} d^4x \partial_\mu n_a \partial_\nu n_b \cdot \partial_\mu n_c [\partial_\rho n_d] = \frac{1}{24} \int_{\mathbb{R}^4} d^4x \left( \partial_\mu n_a \partial_\nu n_b \right)^2, \] (59)
which is nothing but (51). Hereafter, \([ \cdots ]\) denotes the totally antisymmetric combination only about the Latin indices. For instance,
\[ \partial_\mu n_a \partial_\nu n_b \partial_\rho n_c \equiv \partial_\mu n_a \partial_\nu n_b - \partial_\mu n_b \partial_\nu n_a, \]
\[ \partial_\mu n_a \partial_\nu n_b \partial_\rho n_c \equiv \partial_\mu n_a \partial_\nu n_b \partial_\rho n_c - \partial_\mu n_a \partial_\nu n_c \partial_\rho n_b + \partial_\mu n_b \partial_\nu n_c \partial_\rho n_a - \partial_\mu n_b \partial_\nu n_a \partial_\rho n_c + \partial_\mu n_c \partial_\nu n_a \partial_\rho n_b - \partial_\mu n_c \partial_\nu n_b \partial_\rho n_a. \] (60)
Note that the antisymmetry of the Latin indices inherits the antisymmetry of the Greek indices of the parent tensor field strengths. Similarly, the 2nd Chern number (55) turns to the winding number:
\[ c_2 = \frac{1}{16\pi^2} \int_{\mathbb{R}^4} d^4x \text{tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}) \] \[ \quad N_4 = \frac{1}{A(S^4)} \int_{\mathbb{R}^4} d^4x \epsilon_{\mu\nu\rho\sigma} \frac{1}{4!} \epsilon_{abcde} n_e \partial_\mu n_a \partial_\nu n_b \partial_\rho n_c \partial_\sigma n_d, \] (61)
which indicates the homotopy
\[ \pi_4(S^4) \simeq \mathbb{Z}. \] (62)
Since we started from the BPS inequality of the gauge field (53), the obtained \( O(5) \) S-NLS model Hamiltonian satisfies the BPS inequality:
\[ H_{4,2} \geq A(S^4) \cdot N_4. \] (63)

Some technical comments are added here. It is a rather laborious task to derive (59) by directly substituting (47) into (58), but fortunately there exists a much easier way. First, we temporally

\[ \text{If one adopted } F_{\mu\nu}(n_a) \text{ instead of } F_{\mu\nu}(n_a), \text{ the obtained Hamiltonian would be the same since the parent action is gauge invariant.} \]
Table 2: Geometric features of the $O(5)$ NLS model are naturally generalized in even higher dimensions.

| NLS model | $O(5)$ | $O(7)$ | $O(2k + 1)$ |
|-----------|--------|--------|-------------|
| Base-manifold | $\mathbb{R}^4$ | $\mathbb{R}^6$ | $\mathbb{R}^{2k}$ |
| Target manifold | $S^4$ | $S^6$ | $S^{2k}$ |
| Global symmetry | $SO(5)$ | $SO(7)$ | $SO(2k + 1)$ |
| Hidden local symmetry | $SO(4) \simeq SU(2)(\otimes SU(2))$ | $SO(6) \simeq SU(4)$ | $SO(2k)$ |
| Winding number | $\pi_4(S^4) \simeq \mathbb{Z}$ | $\pi_6(S^6) \simeq \mathbb{Z}$ | $\pi_{2k}(S^{2k}) \simeq \mathbb{Z}$ |

Table 2: Geometric features of the $O(5)$ NLS model are naturally generalized in even higher dimensions.

neglect the clumsy parts associated with $n_5$ in (17); $F_{\mu\nu} \sim \frac{1}{2} n_{\mu}^m \sigma_i \partial_\mu n_m \partial_\nu n_n$. With such simplified $F_{\mu\nu}$, we next evaluate the Yang-Mills action $\text{tr}(F_{\mu\nu}^2)$ to have $\frac{1}{2} (\partial_\mu n_m \partial_\nu n_n \cdot \partial_\mu n_{[m} \partial_\nu n_{n]}).$ Lastly, we just recover $n_5$-component in such a way that $\frac{1}{2} (\partial_\mu n_a \partial_\nu n_b \cdot \partial_\mu n_{[a} \partial_\nu n_{b]}).$ This short-cut method will be useful in deriving S-NLS model Hamiltonians in even higher dimensions.

From (59), the equations of motion for the $O(5)$ NLS field are derived as

$$\partial_\mu (\partial_\nu n_b \partial_\mu n_{[a} \partial_\nu n_{b]}) - \frac{\lambda}{2} n_a = 0. \quad (64)$$

Here, $\lambda$ denotes the Lagrange multiplier and is given by

$$\lambda = 2 n_a \partial_\mu (\partial_\nu n_b \partial_\mu n_{[a} \partial_\nu n_{b]}). \quad (65)$$

(64) is highly non-linear, but a solution is simply given by $n_a = r_a$ with $r_a$ being the coordinates on $S^4_{\text{phys}}$ (48). The solution also carries the winding number $N_4 = 1$, which is expected from the discussions around (19).

4 $O(2k + 1)$ S-NLS Models

In this section, we present a general procedure to construct S-NLS models in arbitrary even dimension and demonstrate the procedure to derive $O(7)$ S-NLS and $O(2k + 1)$ S-NLS model Hamiltonians, respectively (Table 2).

4.1 General Procedure

The basic steps for the construction of higher dimensional S-NLS models are as follows.

1. Promote $S_{\text{field}}^{2k-1}$-coordinates $n_m$ to $S_{\text{field}}^{2k}$-coordinates $n_a$.

First prepare a normalized field, $n_{m = 1, 2, \ldots, 2k}$, representing a manifold $S_{\text{field}}^{2k-1}$. We assume that $S_{\text{field}}^{2k-1}$ is realized as a latitude of a virtual $S_{\text{field}}^{2k}$:

$$n_m \rightarrow \frac{1}{\sqrt{1 - n_{2k+1}^2}} n_m, \quad (66)$$
where \( n_m \) and \( n_{2k+1} \) on the right-hand side denote the coordinates on \( S_{\text{field}}^{2k} \):

\[
\sum_{a=1}^{2k+1} n_a n_a = 1.
\]

(67)

We also suppose that NLS field \( n_a(x) \) represents a map from \( x_\mu \in \mathbb{R}^{2k_{\text{phys}}} \) to \( n_a \in S_{\text{field}}^{2k} \). Note that the dimension of the physical space is same as the dimension of the field space.

2. Derive \( SO(2k) \) gauge fields on the field-manifold \( S_{\text{field}}^{2k} \) from the transition function.

The \( \text{Spin}(2k) \) group element is expressed as

\[
g = \sum_{m=1}^{2k} n_m \bar{g}_m,
\]

(68)

where \( \bar{g}_m \) denote some higher dimensional counterpart of the quaternions:

\[
g_m = \{-i\gamma_i, 1\}, \quad \bar{g}_m = \{i\gamma_i, 1\},
\]

(69)

which we call the \( g \) matrices in this paper. In the \( O(5) \) NLS model, \( \gamma_i \) were given by the Pauli matrices, i.e. the \( SO(3) \) gamma matrices \([37]\). Therefore, to take \( \gamma_i \) \((i = 1, 2, \ldots, 2k-1)\) as the \( SO(2k-1) \) gamma matrices will be a natural choice and is also implied by the expression of the \( SO(2k) \) transition function \([17]\). The basic properties of the \( g \) matrices are given by [see Appendix B.1 also]

\[
g_m \bar{g}_n + g_n \bar{g}_m = \bar{g}_m g_n + \bar{g}_n g_m = 2\delta_{mn},
g_m \bar{g}_n - g_n \bar{g}_m = 4i\bar{\sigma}_{mn}, \quad \bar{g}_m g_n - \bar{g}_n g_m = 4i\sigma_{mn},
\]

(70)

where either of \( \sigma_{mn} \) and \( \bar{\sigma}_{mn} \) denote \( \text{Spin}(2k) \) matrix generators. By the 1D promotion \([66], [68]\) becomes

\[
g = \frac{1}{\sqrt{1 - n_{2k+1}}} \sum_{m=1}^{2k} n_m \bar{g}_m,
\]

(71)

which acts as a transition function that connects the \( SO(2k) \) monopole gauge fields defined on the field manifold \( S_{\text{field}}^{2k} \): \( A' = g^\dagger A g - i g^\dagger dg \).

(72)

The gauge field is expressed as

\[
A_\mu(n_a(x)) = i \frac{1}{2} (1 - n_{2k+1}) \partial_\mu g \; g^\dagger = - \frac{1}{1 + n_{2k+1}} \sigma_{mn} n_m \partial_\mu n_m,
\]

(73)

and the field strength \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \) is

\[
F_{\mu\nu}(n_a(x)) = \sigma_{mn} \partial_\mu n_m \partial_\nu n_n - \frac{1}{1 + n_{2k+1}} \sigma_{mn} n_n (\partial_\mu n_m \partial_\nu n_{2k+1} - \partial_\nu n_m \partial_\mu n_{2k+1}).
\]

(74)
3. Make use of the BPS inequality to construct tensor field theory actions.

With the totally antisymmetric tensor field strength

\[ F_{\mu_1 \mu_2 \cdots \mu_{2l}} \equiv \frac{1}{(2l)!} F_{[\mu_1 \mu_2} F_{\mu_3 \mu_4} \cdots F_{\mu_{2l-1} \mu_{2l}]}, \]  

(75)

and its dual tensor field strength\(^{10}\)

\[ \tilde{F}_{\mu_1 \mu_2 \cdots \mu_{2l}} \equiv \frac{1}{(2k - 2l)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}} F_{\mu_{2l+1} \mu_{2l+2} \cdots \mu_{2k}}, \]  

(77)

the \(k\)th Chern number can be expressed as

\[ c_k = \frac{1}{k!(4\pi)^k} \int d^{2k} x \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}} \text{tr}(F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} \cdots F_{\mu_{2k-1} \mu_{2k}}) \]

\[ = \frac{(2k - 2l)!}{k!(4\pi)^k} \int d^{2k} x \text{tr}(F_{\mu_1 \mu_2 \cdots \mu_{2l}} \tilde{F}_{\mu_1 \mu_2 \cdots \mu_{2l}}), \]  

(78)

where

\[ l = 1, 2, \cdots, [k/2]. \]  

(79)

Following to the idea of \(^{13}\) and \(^{76}\), we construct tensor gauge theory action so that the action can satisfy the BPS inequality:

\[ S_{2k,2l} \geq A(S_{\text{phys.}}^{2k}) \cdot c_k, \]  

(80)

which is\(^{11}\)

\[ S_{2k,2l} = \frac{(2k - 2l)!}{(2k)!} \int_{\mathbb{R}^{2k}} d^{2k} x \text{tr}\left( \frac{1}{2^{k-2l}} F_{\mu_1 \mu_2 \cdots \mu_{2l}}^2 + 2^{k-2l} \tilde{F}_{\mu_1 \mu_2 \cdots \mu_{2l}}^2 \right). \]  

(81)

According to the distinct decompositions of the \(k\)th Chern number \(^{79}\), there exist \([k/2]\) different tensor gauge theory actions\(^{12}\). From

\[ \frac{1}{(2l)!} F_{\mu_1 \mu_2 \cdots \mu_{2l}}^2 = \frac{1}{(2k - 2l)!} \tilde{F}_{\mu_1 \mu_2 \cdots \mu_{2k-2l}}^2, \]  

(82)

we can find that \(^{81}\) has the symmetry

\[ S_{2k,2l} = S_{2k,2k-2l}, \]  

(83)

and hence there are \([k/2]\) independent actions \(S_{2k,2l}\) in accordance with \(^{79}\).

---

\(^{10}\) \(^{76}\) satisfies

\[ \tilde{F}_{\mu_1 \mu_2 \cdots \mu_{2l}} = F_{[\mu_1 \mu_2 \cdots \mu_{2l}].} \]  

(76)

\(^{11}\) Here, we added the coefficients in front of \(F^2\) and \(\tilde{F}^2\) for the later convenience. Recall that there exists the local degree of freedom indicated by \(\lambda(x)\) in \(^{79}\).

\(^{12}\) See Appendix \(^{C}\) for details about the tensor gauge field theory.
4. Express the tensor gauge theory action by the NLS field.

Substitute (74) into (81) to express \( S_{2k,2l} \) with the NLS field:

\[
S_{2k,2l} \rightarrow H_{2k,2l} = \left( \frac{(2k - 2l)!}{(2k)!} \right) \int_{\mathbb{R}^{2k}} d^{2k}x \left( \frac{1}{2^{k - 2l}} \frac{1}{\mu_1 \mu_2 \cdots \mu_{2l}} \right) F_{\mu_1 \mu_2 \cdots \mu_{2l}}^2 \right) \bigg|_{F_{\mu \nu} = F_{\mu \nu}^{(na_2)}}
\]

which signifies our \( O(2k + 1) \) S-NLS model Hamiltonian. Similarly, \( k \)th Chern number turns to

\[
c_k \rightarrow c_k F_{\mu \nu}^{(na_2)} = \frac{1}{A(S^{2k})} \int_{\mathbb{R}^{2k}_{\text{phys.}}} d^{2k}x \frac{1}{(2k)!} \epsilon_{a_1 a_2 \cdots a_{2k+1}} \partial_1 n_{a_1} \partial_2 n_{a_2} \cdots \partial_2 n_{a_{2k}},
\]

which stands for the \( O(2k + 1) \) NLS model winding number associated with \( \pi_{2k}(S^{2k}) \simeq \mathbb{Z} \).

The BPS inequality (80) is rephrased as

\[
H_{2k,2l} \geq A(S^{2k}_{\text{phys.}}) \cdot N_{2k}.
\]

Two important features of the tensor field gauge theory are inherited to the S-NLS models. One is the local symmetry and the other is the BPS inequality. As the tensor field strength action (81) enjoys the \( \text{SO}(2k) \) gauge symmetry, the S-NLS model Hamiltonian necessarily possesses the hidden local \( \text{SO}(2k) \) symmetry. Similarly, as the tensor gauge field action is constructed so as to satisfy the BPS inequality, the S-NLS model Hamiltonian automatically satisfies the BPS inequality.

4.2 \( O(7) \) S-NLS model

From the general procedure, we explicitly construct the \( O(7) \) S-NLS model Hamiltonian. The steps 1 and 2 are obvious. From (74), the \( \text{SO}(6) \) gauge field strength is given by

\[
F_{\mu \nu} = \sigma_{mn} \partial_\mu n_m \partial_\nu n_n - \frac{1}{1 + n_7} \sigma_{mn} n_n (\partial_\mu n_m \partial_\nu n_7 - \partial_\nu n_m \partial_\mu n_7),
\]

where \( \sigma_{mn} \) denote the \( \text{Spin}(6) \) generators, and (75) yields the totally antisymmetric four-rank tensor

\[
F_{\mu \nu \rho \sigma} = \frac{1}{4!} F_{\mu [\nu \rho \sigma]} = \frac{1}{6} (\{ F_{\mu \nu}, F_{\rho \sigma} \} + \{ F_{\mu \rho}, F_{\nu \sigma} \} + \{ F_{\mu \sigma}, F_{\nu \rho} \}),
\]

and its dual

\[
\tilde{F}_{\mu \nu} = \frac{1}{4!} \epsilon_{\mu \nu \rho \sigma \kappa \tau} F_{\rho \sigma} F_{\kappa \tau} = \frac{1}{4!} \epsilon_{\mu \nu \rho \sigma \kappa \tau} F_{\rho \sigma \kappa \tau}.
\]

The BPS inequality,

\[
S_{6,2} \geq A(S^6) \cdot c_3,
\]

introduces the tensor gauge field action:

\[
S_{6,2} = \frac{1}{60} \int_{\mathbb{R}^6} d^6x \left( F_{\mu \nu}^2 + 4 \tilde{F}_{\mu \nu}^2 \right) = \frac{1}{60} \int_{\mathbb{R}^6} d^6x \left( F_{\mu \nu}^2 + \frac{1}{3} F_{\mu \nu \rho \sigma}^2 \right)
\]

\[
= \frac{1}{60} \int_{\mathbb{R}^6} d^6x \left( F_{\mu \nu}^2 + \frac{1}{18} (F_{\mu \nu}^2)^2 - \frac{2}{9} F_{\mu \nu} F_{\rho \sigma} F_{\mu \rho} F_{\nu \sigma} + \frac{1}{18} (F_{\mu \nu} F_{\rho \sigma})^2 \right).
\]
Here, we used $A(S^6) = \frac{16}{15} \pi^3$ and
\[ c_3 = \frac{1}{3! (4\pi)^3} \int d^6 x \, \epsilon_{\mu\nu\rho\sigma\kappa\tau} \text{tr}(F_{\mu\nu} F_{\rho\sigma} F_{\kappa\tau}) = \frac{1}{2 (2\pi)^3} \int d^6 x \, \text{tr}(F_{\mu\nu} \tilde{F}_{\mu\nu}). \] (92)

(91) is essentially the 6D action constructed by Tchrakian [18].

With (87) and the properties of the $Spin(6)$ generators
\[ \text{tr}(\tilde{\sigma} \sigma) = \delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np}, \quad \sigma_{[mn][pq]} = 3 \epsilon_{mpnqst} \sigma_{st}, \] (93)
we can express the two terms of $S_{6,2}$ as
\[ \text{tr}(F_{\mu\nu}^2) |_{F_{\mu\nu} = F_{\mu\nu}(n_a)} = (\partial_{\mu} n_a)^2 (\partial_{\nu} n_b)^2 - (\partial_{\mu} n_a (\partial_{\nu} n_a))^2 = \partial_{\mu} n_a \partial_{\nu} n_b \cdot \partial_{\mu} n_a (\partial_{\nu} n_b) = \frac{1}{2} (\partial_{\mu} n_a (\partial_{\nu} n_b))^2, \]
\[ \text{tr}(F_{\mu\nu}^2) |_{F_{\mu\nu} = F_{\mu\nu}(n_a)} = \frac{1}{2 \cdot 4!} \partial_{\mu} n_a \partial_{\nu} n_b \partial_{\rho} n_c \partial_{\sigma} n_d \cdot \partial_{\mu} n_a (\partial_{\nu} n_b) \partial_{\rho} n_c \partial_{\sigma} n_d = \frac{1}{2 \cdot 4!} (\partial_{\mu} n_a (\partial_{\nu} n_b) (\partial_{\rho} n_c \partial_{\sigma} n_d))^2. \] (94)

and then
\[ H_{6,2} = \frac{1}{60} \int d^6 x \left( \partial_{\mu} n_a \partial_{\nu} n_b \cdot \partial_{\mu} n_a (\partial_{\nu} n_b) + \frac{1}{12} \cdot \partial_{\mu} n_a \partial_{\nu} n_b (\partial_{\rho} n_c \partial_{\sigma} n_d) \cdot \partial_{\mu} n_a (\partial_{\nu} n_b) (\partial_{\rho} n_c \partial_{\sigma} n_d) \right). \] (95)

The third Chern number $c_3$ also turns to the $O(6)$ NLS model winding number associated with $\pi_6(S^6) \simeq \mathbb{Z}$:
\[ N_6 = \frac{1}{A(S^6_{\text{phys}})} \int_{S^6_{\text{phys}}} d^6 x \frac{1}{6!} \epsilon_{\mu\nu\rho\sigma\kappa\tau} \epsilon_{abcdefg} n_g \partial_{\mu} n_a \partial_{\nu} n_b \partial_{\rho} n_c \partial_{\sigma} n_d \partial_{\tau} n_e \partial_{\kappa} n_f. \] (96)

Notice that the second term of $H_{6,2}$ is the octic derivative term and is expanded as
\[ \partial_{\mu} n_a \partial_{\nu} n_b \partial_{\rho} n_c \partial_{\sigma} n_d \cdot \partial_{\mu} n_a (\partial_{\nu} n_b) \partial_{\rho} n_c \partial_{\sigma} n_d \]
\[ = ((\partial_{\mu} n_a)^2)^4 + 3((\partial_{\mu} n_a (\partial_{\nu} n_a))^2)^2 - 6((\partial_{\mu} n_a)^2)^2 (\partial_{\rho} n_b (\partial_{\sigma} n_b))^2 \]
\[ - 6(\partial_{\mu} n_a (\partial_{\nu} n_a)) (\partial_{\rho} n_b (\partial_{\sigma} n_b)) (\partial_{\rho} n_c \partial_{\sigma} n_c) (\partial_{\sigma} n_d (\partial_{\mu} n_d)) + 8(\partial_{\mu} n_a)^2 (\partial_{\rho} n_b (\partial_{\sigma} n_b)) (\partial_{\rho} n_c \partial_{\sigma} n_c) (\partial_{\sigma} n_d (\partial_{\mu} n_d)). \] (97)

The first quartic derivative term of $H_{6,2}$ acts to shrink a soliton configuration, while the second term acts to expand the configuration just like the original Skyrme term model.

### 4.3 $O(2k + 1)$ S-NLS models

In low dimensions, the numbers of the S-NLS model Hamiltonians are counted as
\[ O(5) : 1, \; O(7) : 1, \; O(9) : 2, \; O(11) : 2. \] (98)

For the previous $O(5)$ and $O(7)$ cases, we have single S-NLS model Hamiltonian, but for $O(2k + 1)$, we have $\lfloor k/2 \rfloor$ Hamiltonians. In the following, we construct $O(2k + 1)$ NLS model Hamiltonians for two typical cases, $2 + (2k - 2)$ and $k + k$.

---

13Saclioglu constructed another 6D action [22] of a triple form of the field strengths, $\frac{1}{4} f^{abc} F_{\mu\nu}^{ab} F_{\nu\gamma}^{bc} F_{\gamma\mu}^{ca}$, which is not positive definite in general. Meanwhile, $S_{6,2}$ only with even powers of the field strengths does not have such a problem.
4.3.1 $2 + (2k - 2)$ decomposition

In $2 + (2k - 2)$ decomposition, the tensor gauge theory action is given by

$$S_{2k,2} = \frac{1}{2k(2k - 1)} \int d^{2k}x \, tr \left( \frac{1}{2k-2} F_{\mu\nu}^2 + 2^{k-2} \tilde{F}_{\mu\nu}^2 \right)$$

$$= \frac{1}{(2k)!} \int d^{2k}x \, tr \left( \frac{1}{2k-2}(2k-2)! \, F_{\mu\nu}^2 + 2^{k-2} \, 2! \, F_{\mu_1\mu_2\mu_3\cdots\mu_{2k-2}}^2 \right).$$

(99)

From the properties of the $Spin(2k)$ generators

$$tr(\sigma_{mn}\sigma_{pq}) = 2^{k-3}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}),$$

$$\sigma_{[m_1m_2}\sigma_{m_3m_4}\cdots\sigma_{m_{2k-3}m_{2k-2}]} = \frac{(2k-2)!}{2^{k-1}} \epsilon_{m_1m_2\cdots m_{2k}} \sigma_{m_{2k-1}m_{2k}}$$

the two terms of $S_{2k,2}$ (99) can be represented as

$$tr(F_{\mu\nu}^2)|_{F_{\mu\nu}=F_{\mu\nu}(n_a)} = 2^{k-3} (\partial_{\mu}n_a)^2 (\partial_{\nu}n_b)^2 - (\partial_{\mu}n_a \partial_{\nu}n_b)^2) = 2^{k-3} \partial_{\mu}n_a \partial_{\nu}n_b - \partial_{\mu}n_{[a} \partial_{\nu}n_{b]}$$

$$tr(\tilde{F}_{\mu\nu}^2)|_{F_{\mu\nu}=F_{\mu\nu}(n_a)} = \frac{1}{2^{k-2} (2k-2)!} \partial_{\mu_1}n_{a_1} \partial_{\mu_2}n_{a_2} \cdots \partial_{\mu_{2k-2}}n_{a_{2k-2}} \cdot \partial_{\mu_1}n_{[a_1} \partial_{\mu_2}n_{a_2} \cdots \partial_{\mu_{2k-2}}n_{a_{2k-2}]},$$

(100)

and so we have

$$H_{2k,2} = \frac{1}{4k(2k - 1)} \int_{R^{2k}} d^{2k}x \times$$

$$\left( \partial_{\mu_1}n_{a_1} \partial_{\mu_2}n_{a_2} \cdots \partial_{\mu_{2k}}n_{a_{2k}} + \frac{2}{(2k-2)!} \partial_{\mu_1}n_{a_1} \partial_{\mu_2}n_{a_2} \cdots \partial_{\mu_{2k-2}}n_{a_{2k-2}} \cdot \partial_{\mu_1}n_{[a_1} \partial_{\mu_2}n_{a_2} \cdots \partial_{\mu_{2k-2}}n_{a_{2k-2}]}. \right)$$

(101)

Notice that the first term is a quartic derivative term while the second term is the $4(k-1)$th derivative term. Their competing scaling effect determines the size of soliton configurations (except for the scale invariant case $k = 2$). For $k = 2$ and 3, (102) indeed reproduces the previous $O(5)$ and $O(7)$ NLS model Hamiltonians, respectively.

4.3.2 $k + k$ decomposition for even $k$

In the special case $(d,2l) = (2k,k)$:

$$(d,k) = (4,2), (8,4), (12,6), (16,4), \cdots,$$

(103)

$$F_{\mu_1\mu_2\cdots\mu_k}^2 = F_{\mu_1\mu_2\cdots\mu_k}^2$$

holds, and so (84) is reduced to

$$S_{2k,k} = \frac{k!}{(2k)!} \int d^{2k}x \, tr(F_{\mu_1\mu_2\cdots\mu_k}^2).$$

(104)

\footnote{For $O(5)$ ($k = 2$), the first and second terms on the right-hand side of (102) coincide, and (102) is reduced to (59).}
The equations of motion are derived as

\[ D_{\mu_1} F_{\mu_1 \mu_2 \cdots \mu_k} = \partial_{\mu_1} F_{\mu_1 \mu_2 \cdots \mu_k} + i[A_{\mu_1}, F_{\mu_1 \mu_2 \cdots \mu_k}] = 0. \]  

The tensor gauge field strength \( F_{\mu_1 \mu_2 \cdots \mu_k} = \frac{1}{k!} F_{[\mu_1 \mu_2 \cdots \mu_{k-1} \mu_k]} \) made of the \( SO(2k) \) “instanton” configuration\(^{15}\)

\[ F_{\mu \nu} |_{\text{anti-static}} = \frac{4}{(x^2 + 1)^2} \sigma_{\mu \nu}, \]

is given by

\[ F_{\mu_1 \mu_2 \cdots \mu_k} = \frac{1}{k!} \left( \frac{2}{x^2 + 1} \right)^k \sigma_{[\mu_1 \mu_2 \sigma_{\mu_3 \mu_4} \cdots \sigma_{\mu_{k-1} \mu_k]}, \]  

which carries unit \( k \)th Chern number. \((107)\) satisfies the self-dual equation \(20\) \(21\) \(23\) \(40\)

\[ \tilde{F}_{\mu_1 \mu_2 \cdots \mu_k} = F_{\mu_1 \mu_2 \cdots \mu_k}, \]  

due to the property of the \( Spin(2k) \) matrix generators\(^{16}\)

\[ \sigma_{[\mu_1 \mu_2 \sigma_{\mu_3 \mu_4} \cdots \sigma_{\mu_{k-1} \mu_k}] = \frac{1}{k!} \epsilon_{\mu_1 \mu_2 \mu_3 \cdots \mu_{2k}} \sigma_{[\mu_{k+1} \mu_{k+2} \cdots \mu_{2k-1} \mu_{2k}]. \]  

Because of the Bianchi identity for tensor fields, the self-dual tensor field \((107)\) is a solution of the equations of motion \((105)\) (see Appendix \(C\) for details). Note that while \((107)\) realizes a solution of \((105)\), \((106)\) is not a solution of the pure Yang-Mills field equation except for \(k = 2\) (Appendix \(A.4\)). In low dimensions, one may directly confirm that \((107)\) satisfies \((105)\) with

\[ A_\mu = \frac{2}{x^2 + 1} \sigma_{\mu \nu} x_\nu. \]  

To rewrite the tensor gauge theory action by \(O(2k + 1)\) NLS field, we utilize the short-cut method mentioned in Sec\(3.2.2\). We truncate the field strength \( F_{\mu \nu} \rightarrow \sigma_{mn} \partial_\mu n_m \partial_\nu n_n \) to have

\[ \text{tr} \left( F_{\mu_1 \mu_2 \cdots \mu_k}^2 \right) \rightarrow \left( \frac{1}{k!} \right)^2 \text{tr} \left( \sigma_{m_1 m_2 \cdots m_{k-1} m_k} \sigma_{m'_1 m'_2 \cdots m'_{k-1} m'_k} \right) \partial_\mu n_{m_1} \partial_\nu n_{m_2} \cdots \partial_\mu n_{m_k} \partial_\nu n_{m'_1} \partial_\nu n_{m'_2} \cdots \partial_\mu n_{m'_k}, \]  

\(113\)

\(^{15}\)The \(SO(2k)\) instanton configuration \((106)\) is a stereographic projection of the \(SO(2k)\) monopole field configuration on \(S^{2k}\) \((9)\) (Appendix \(A\)).

\(^{16}\)Generally, the \(Spin(2k)\) generators satisfy

\[ \frac{1}{(2l)!} \sigma_{[\mu_1 \mu_2 \sigma_{\mu_3 \mu_4} \cdots \sigma_{\mu_{2l-1} \mu_{2l}}] = 2^{k-2l} \frac{1}{((2k - 2l)!)^2} \epsilon_{\mu_1 \mu_2 \mu_3 \cdots \mu_{2k}} \sigma_{[\mu_{2l+1} \mu_{2l+2} \cdots \mu_{2k-1} \mu_{2k}}, \]  

which is reduced to \((111)\) in the special case \(k = 2l\). The tensor instanton configuration \((107)\) also satisfies

\[ F_{\mu_1 \mu_2 \cdots \mu_{2l}} = \left( \frac{x^2 + 1}{2} \right)^{k-2l} \tilde{F}_{\mu_1 \mu_2 \cdots \mu_{2l}}, \]  

\(110\) and \((110)\) is reduced to \((108)\) when \(k = 2l\).
\[ \partial_{\mu_1} n_{[m_1} \partial_{\mu_2} n_{m_2} \cdots \partial_{\mu_k} n_{m_k]} \] denotes \( k! \) terms of totally antisymmetric combination about the Latin indices, \( m_1, m_2, \ldots, m_k \). The Spin(2) matrix part of \((113)\) can be expressed as

\[
\text{tr}(\sigma_{m_1 m_2} \sigma_{m_3 m_4} \cdots \sigma_{m_{2k-1} m_{2k}}) = \frac{1}{2} \left( -\frac{i}{4} \right)^k \text{tr}(\gamma_{m_1} \gamma_{m_2} \gamma_{m_3} \cdots \gamma_{m_{2k}}) \cdot (1 - P_{m_1 m_2}) (1 - P_{m_3 m_4}) \cdots (1 - P_{m_{2k-1} m_{2k}}) + \frac{1}{2} \epsilon_{m_1 m_2 m_3 \cdots m_{2k}}.
\]

(114)

Here, \( P_{mn} \) signifies an operation that interchanges \( m \) and \( n \), i.e. \( P_{mn} (\gamma_m \gamma_n) = \gamma_n \gamma_m \), and in the present case, due to the antisymmetricity of \( m \), \( n \), we can just replace \( (1 - P_{mn}) \) with 2. Besides the epsilon tensor part of \((113)\) obviously has no effect in \((113)\), and thereby

\[
\text{tr}(\sigma_{m_1 m_2} \sigma_{m_3 m_4} \cdots \sigma_{m_{2k-1} m_{2k}}) \rightarrow \frac{1}{2} \left( -\frac{i}{2} \right)^k \text{tr}(\gamma_{m_1} \gamma_{m_2} \gamma_{m_3} \cdots \gamma_{m_{2k}}) \rightarrow \frac{1}{2} k! \delta_{m_1 m_2} \delta_{m_3 m_4} \cdots \delta_{m_{2k-1} m_{2k}}.
\]

(115)

In the last arrow we assumed that \( k \) is even. Eventually, we obtain

\[
\text{tr}(F_{\mu_1 \mu_2 \cdots \mu_k}^2) = \text{tr}(\tilde{F}^2_{\mu_1 \mu_2 \cdots \mu_k}) = \frac{1}{2} (\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]}) \cdot (\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]}),
\]

(116)

which implies

\[
H_{2k, k} = \frac{k!}{(2k)!} \int_{\mathbb{R}^{2k}} d^{2k} x \frac{1}{2} (\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]}) \cdot (\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]})
\]

\[
= \frac{1}{(2k)!} \int_{\mathbb{R}^{2k}} d^{2k} x \frac{1}{2} (\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]})^2.
\]

(117)

\( H_{2k, k} \) accommodates scale invariant soliton solutions as we shall discuss in Sec 6.3. For \( k = 2 \), \((117)\) is reduced to the \( O(5) \) S-NLS model Hamiltonian \((52)\).

### 5 \( O(2k) \) S-NLS Models

In this section, based on the Chern-Simons term expression of the \( k \)th Chern number, we construct \( O(2k) \) S-NLS model Hamiltonians in \((2k - 1)D\). The dimensional hierarchy of the Landau models \((75, 63)\) suggests that the dimensional reduction of the \( O(2k) \) NLS model may yield the \( O(2k + 1) \) NLS model \((13)\). More specifically, the 1D reduction of \( H_{2k, 2l} \) gives rise to two \( O(2k) \) Hamiltonians, \( H_{2k-1, 2l-1} \) and \( H_{2k-1, 2l} \). By removing duplications from the symmetry \( H_{2k-1, 2l} = H_{2k-1, 2k-1-2l} \), we obtain \((k - 1)\) distinct \( O(2k) \) Hamiltonians in \((2k - 1)D\). In low dimensions \(\text{[17]}\)

\[
\begin{align*}
 k = 2 : & \quad O(5) \text{ S-NLS model : } H_{4, 2} \quad \rightarrow \quad O(4) \text{ S-NLS model : } H_{3, 1}, \\
 k = 3 : & \quad O(7) \text{ S-NLS model : } H_{6, 2} \quad \rightarrow \quad O(6) \text{ S-NLS model : } H_{5, 1}, \quad H_{5, 2}, \\
 k = 4 : & \quad O(9) \text{ S-NLS model : } H_{8, 2}, \quad H_{8, 4} \quad \rightarrow \quad O(8) \text{ S-NLS model : } H_{7, 1}, \quad H_{7, 2}, \quad H_{7, 3}.
\end{align*}
\]

(118)

\[^{17}\text{The soliton configuration of \( O(2) \) NLS model is given by the Nielsen-Olsen vortex \((96)\).}\]
5.1 The Chern-Simons term and the action of pure gauge fields

As is well known, the Chern number (density) can be expressed by

$$\text{tr}(F^k) = dL_{CS}^{(2k-1)}[A]$$

(119)

where $L_{CS}^{(2k-1)}[A]$ signifies the $(2k-1)$D Chern-Simons term

$$L_{CS}^{(2k-1)}[A] = k \int_0^1 dt \text{tr}(A(tdA + it^2 A^2)^{k-1}).$$

(120)

In low dimensions, $L_{CS}^{(2k-1)}$ reads as

$$L_{CS}^{(1)}[A] = \text{tr}A, \quad L_{CS}^{(3)}[A] = \text{tr}(AF - \frac{1}{3} iA^3), \quad L_{CS}^{(5)}[A] = \text{tr}(AF^2 - \frac{1}{2} iA^3F - \frac{1}{10} A^5).$$

(121)

We make use of the Chern-Simons field description of the Chern number to construct $O(2k)$ S-NLS model Hamiltonians. Recall that the transition function $[68]$ represents $S_{\text{field}}^{2k-1}$, and the associated gauge field is given by a pure gauge

$$A = -ig^i dg, \quad F = dA + iA^2 = 0.$$  

(122)

For the pure gauge $[122]$, the Chern-Simons term $[120]$ is reduced to

$$L_{CS}^{(2k-1)}[A] = (-i)^{k-1} \frac{k!(k-1)!}{(2k-1)!} \text{tr}(A^{2k-1})$$

$$= (-i)^{k-1} \frac{k!(k-1)!}{(2k-1)!} d^{2k-1}x \epsilon_{\alpha_1\alpha_2\cdots\alpha_{2k-1}} \text{tr}(A_{\alpha_1}A_{\alpha_2} \cdots A_{\alpha_{2k-1}}),$$

(123)
where we used \( \int_0^1 dt (t - t^2)^{k-1} = \frac{(k-1)!}{(2k-1)!} \) and assumed that \( \mathcal{A} \) is one-form on \( x_\alpha \in \mathbb{R}^{2k-1}_{\text{phys.}} \):

\[
\mathcal{A} = \sum_{\alpha=1}^{2k-1} A_\alpha dx_\alpha. \tag{124}
\]

We introduce tensor field for the pure gauge as

\[
\mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p} \equiv (-i)^{\frac{1}{2}p(p-1)} \frac{1}{p!} A_{[\alpha_1} A_{\alpha_2} \cdots A_{\alpha_p]},
\]

and its dual

\[
\bar{\mathcal{A}}_{\alpha_1 \alpha_2 \cdots \alpha_p} \equiv \frac{1}{(d-p)!} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} A_{\alpha_{p+1} \alpha_{p+2} \cdots \alpha_d}
= (-i)^{\frac{1}{2}(d-p)(d-p-1)} \frac{1}{(d-p)!} \epsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} A_{\alpha_{p+1}} A_{\alpha_{p+2}} \cdots A_{\alpha_d}. \tag{126}
\]

\([126]\) satisfies

\[
\frac{1}{p!} \bar{\mathcal{A}}_{\alpha_1 \alpha_2 \cdots \alpha_p}^2 = \frac{1}{(2k-1-p)!} A_{\alpha_1 \alpha_2 \cdots \alpha_{2k-1-p}}^2. \tag{127}
\]

In \([125]\), \((-i)^{\frac{1}{2}p(p-1)}\) is added so that \( \mathcal{A}_{\alpha_1 \alpha_2 \cdots \alpha_p} \) can be Hermitian. For instance,

\[
A_{\alpha \beta} = -\frac{1}{2} \{ A_\alpha, A_\beta \} = -\frac{1}{2} \partial_{[\alpha} A_{\beta]};
\]

\[
A_{\alpha \beta \gamma} = \frac{i}{3!} A_{[\alpha A_\beta A_\gamma]} = -\frac{1}{3} (A_{\alpha A_\beta \gamma} + A_{\beta A_\gamma \alpha} + A_{\gamma A_\alpha \beta}),
\]

\[
A_{\alpha \beta \gamma \delta} = -\frac{1}{4!} A_{[\alpha A_\beta A_\gamma A_\delta]} = \frac{1}{6} (\{ A_{\alpha \beta}, A_{\gamma \delta} \} - \{ A_{\alpha \gamma}, A_{\beta \delta} \} + \{ A_{\alpha \delta}, A_{\beta \gamma} \}). \tag{128}
\]

In a similar manner to Sec.4.1, we represent the Chern-Simons action as

\[
S_{\text{CS}}^{(2k-1)}[\mathcal{A}] \equiv \frac{1}{k!(2\pi)^k} \int L_{\text{CS}}^{(2k-1)}[\mathcal{A}]
= \frac{1}{(2\pi)^k} \frac{(k-1)!(2k-1-p)!}{(2k-1)!} \int_{\mathbb{R}^{2k-1}_{\text{phys}}} d^{2k-1}x \, \text{tr}(A_{\alpha_1 \alpha_2 \cdots \alpha_p} \bar{A}_{\alpha_1 \alpha_2 \cdots \alpha_p}), \tag{129}
\]

where

\[
p = 1, 2, \ldots, k - 1. \tag{130}
\]

In low dimensions, \([129]\) yields

\[
S_{\text{CS}}^{(3)}[\mathcal{A}] = \frac{1}{12\pi^2} \int d^3x \, \text{tr}(A_\alpha \bar{A}_\alpha),
\]

\[
S_{\text{CS}}^{(5)}[\mathcal{A}] = \frac{1}{20\pi^3} \int d^5x \, \text{tr}(A_\alpha \bar{A}_\alpha) = \frac{1}{80\pi^3} \int d^5x \, \text{tr}(A_{\alpha \beta} \bar{A}_{\alpha \beta}). \tag{131}
\]

From the BPS inequality

\[
S_{2k-1,p}[\mathcal{A}] \geq A(S^{2k-1}) \cdot S_{\text{CS}}^{(2k-1)}[\mathcal{A}], \tag{132}
\]
we introduce an action of the pure gauge tensor field as
\[
S_{2k-1,p}[A] = \frac{1}{2^k} \frac{(2k-1-p)!}{(2k-1)!} \int_{\mathbb{R}^{2k-1}} d^{2k-1}x \left( \text{tr}(A_{\alpha_1\alpha_2\cdots\alpha_p})^2 + \text{tr}(A_{\alpha_1\alpha_2\cdots\alpha_p}^2) \right)
\]
\[
= \frac{1}{2^k(2k-1)!} \int_{\mathbb{R}^{2k-1}} d^{2k-1}x \left( (2k-1-p)! \text{tr}(A_{\alpha_1\alpha_2\cdots\alpha_p})^2 + p! \text{tr}(A_{\alpha_1\alpha_2\cdots\alpha_{2k-1-p}}^2) \right).
\]
(133)

Unlike the 2kD action $S_{2k,2l}$ (51), $S_{2k-1,p}$ (133) is made of the tensor gauge field itself (not of the field strength), and so $S_{2k,p}$ does not have the $SO(2k)$ gauge symmetry.

5.2 Explicit constructions

For (68), the pure gauge field (122) can be represented as
\[
A_\alpha(n_m) = -ig\sigma_\alpha g = 2\sigma_{mn}n_m\partial_\alpha n_n,
\]
(134)
where $\sigma_{mn}$ denote the $Spin(2k)$ matrix generators. Substituting (134) into (125), we can derive the NLS field expression of $A_{\alpha_1\alpha_2\cdots\alpha_p}$. For instance
\[
A_{\alpha\beta} \bigg|_{A_\alpha = A_\alpha(n_m)} = -2i\sigma_{mp}\sigma_{nq}n_p\partial_\alpha n[\beta n] = -\sigma_{mn}\partial_\alpha n(n_\beta n).
\]
(135)

Just as in the tensor gauge field strength in Sec.4, the antisymmetry of the Greek indices of the pure gauge field is inherited to that of the Latin indices of the NLS field. With such substitutions, the $O(2k)$ S-NLS model Hamiltonian is obtained from $S_{2k-1,p}$:
\[
\begin{align*}
S_{2k-1,p} \rightarrow & \\
H_{2k-1,p} &= \frac{1}{2^k(2k-1)!} \int_{\mathbb{R}^{2k-1}} d^{2k-1}x \left( (2k-1-p)! \text{tr}(A_{\alpha_1\alpha_2\cdots\alpha_p})^2 + p! \text{tr}(A_{\alpha_1\alpha_2\cdots\alpha_{2k-1-p}}^2) \right) \bigg|_{A_\alpha = A_\alpha(n_m)}.
\end{align*}
\]
(136)

Similarly, the Chern-Simons term (129) turns to the winding number of $\pi_{2k-1}(S^{2k-1}) \simeq \mathbb{Z}$:
\[
S_{CS}^{2k-1} \rightarrow N_{2k-1} = \frac{1}{A(S^{2k-1})} \int_{\mathbb{R}^{2k-1}} d^{2k-1}x \epsilon_{m_1m_2\cdots m_{2k}} n_{m_1} \partial_1 n_{m_2} \partial_2 n_{m_3} \cdots \partial_{2k-1} n_{m_{2k}}.
\]
(137)

As in the previous $O(2k+1)$ S-NLS models, the parent BPS inequality (132) guarantees the BPS inequality of the $O(2k)$ S-NLS models:
\[
H_{2k-1,p} \geq A(S^{2k-1}) \cdot N_{2k-1}.
\]
(138)

Since the pure gauge field actions (133) do not have gauge symmetries, the corresponding $O(2k)$ S-NLS models do not either. This is a higher dimensional analogue of the non-existence of the gauge symmetry of the Skyrme model. In the following, we explicitly derive the $O(2k)$ S-NLS model Hamiltonian for $d = 3$ and $d = 5$.  

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5.2.1 The Skyrme model: $O(4)$ S-NLS model

For $d = 3$, the pure gauge field action is given by

$$ S_{3,1} = \frac{1}{12} \int_{\mathbb{R}^3_{\text{phys.}}} d^3x \, \text{tr}(A^2_\alpha + \tilde{A}^2_\alpha) = \frac{1}{12} \int_{\mathbb{R}^3_{\text{phys.}}} d^3x \, (\text{tr}(A^2_\alpha) + \frac{1}{2} \text{tr}(A_{\alpha\beta}^2)) = S_{3,2}, \quad (139) $$

where $A_\alpha$ and its dual field $\tilde{A}_\alpha$ are represented as

$$ A_\alpha = 2\tilde{\sigma}_{mn}n_m\partial_\alpha n_n, \quad \tilde{A}_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma}A_{\beta\gamma} = \epsilon_{\alpha\beta\gamma}\tilde{\sigma}_{mn}\partial_\beta n_m\partial_\gamma n_n, \quad (140) $$

with $\text{Spin}(4)$ matrix generators:

$$ \tilde{\sigma}_{mn} = \frac{1}{2}\eta_{mn}\sigma_1. \quad (141) $$

From the following formula\textsuperscript{18}

$$ \tilde{\sigma}_{mn}\tilde{\sigma}_{pq} = \frac{1}{4}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np} - \epsilon_{mpnq})12 + \frac{1}{2}(\delta_{mp}\tilde{\sigma}_{nq} - \delta_{mq}\tilde{\sigma}_{np} + \delta_{nq}\tilde{\sigma}_{mp} - \delta_{np}\tilde{\sigma}_{mq}), \quad (142) $$

we can readily show

$$ \text{tr}(A^2_\alpha)|_{A=A(n_m)} = 2(\partial_\alpha n_m)^2, \quad \text{tr}(\tilde{A}^2_\alpha)|_{A=A(n_m)} = \frac{1}{2}(\partial_\alpha n_m[\partial_\beta n_n])^2, \quad (143) $$

and so

$$ H_{3,1} = \frac{1}{6} \int_{\mathbb{R}^3_{\text{phys.}}} d^3x \, \left((\partial_\alpha n_m)^2 + \frac{1}{4}(\partial_\alpha n_m[\partial_\beta n_n])^2\right). \quad (144) $$

This $O(4)$ S-NLS model Hamiltonian is nothing but the Skyrme Hamiltonian. As mentioned before, the anti-symmetricity of the indices of $A_{\alpha\beta}$ is inherited to the anti-symmetricity of the Latin indices of $O(4)$ NLS field of the Skyrme term.

5.2.2 $O(6)$ S-NLS models

Next we consider the case $d = 5$. There exist two distinct actions in this case:

$$ S_{5,1} = \frac{1}{40} \int_{\mathbb{R}^5_{\text{phys.}}} d^5x \, \text{tr}(A^2_\alpha + \tilde{A}^2_\alpha) = \frac{1}{40} \int_{\mathbb{R}^5_{\text{phys.}}} d^5x \, \text{tr}(A^2_\alpha + \frac{1}{4!}A_{\alpha\beta\gamma\delta}^2), \quad (145a) $$

$$ S_{5,2} = \frac{1}{160} \int_{\mathbb{R}^5_{\text{phys.}}} d^5x \, \text{tr}(A_{\alpha\beta}^2 + \tilde{A}^2_{\alpha\beta}) = \frac{1}{80} \int_{\mathbb{R}^5_{\text{phys.}}} d^5x \, \left(\frac{1}{2}\text{tr}(A_{\alpha\beta}^2) + \frac{1}{3!}\text{tr}(A_{\alpha\beta\gamma})^2\right), \quad (145b) $$

$A_\alpha$ is given by\textsuperscript{134} with $Spin(6)$ matrix generators $\tilde{\sigma}_{mn}$. From the isomorphism $Spin(6) \simeq SU(4)$, we can express the $Spin(6)$ matrices $\tilde{\sigma}_{mn}$ as a linear combination of the $SU(4)$ Gell-Mann matrices matrices $\lambda_{A=1,2,\ldots,15}^{102}$:

$$ \tilde{\sigma}_{mn} = \frac{1}{2} \sum_{A=1}^{15} \tilde{\eta}_{mn}^{A} \lambda_{A}. \quad (146) $$

\textsuperscript{18}The $U(2)$ generators (the Pauli matrices and the unit matrix) span the $2 \times 2$ matrix space, and so the product of two $SU(2)$ Pauli matrices or $Spin(4)$ matrix generators can be represented as a linear combination of the $U(2)$ generators.
As the expansion coefficients, we introduced an $SU(4)$-generalized 't Hooft symbol, $\tilde{\eta}_{mn}^{A} = \text{tr}(\lambda_{A} \bar{\sigma}_{mn})$ (see Appendix [12] for its detail properties). With unit matrix, the $SU(4)$ matrix generators span the $4 \times 4$ matrix space, and then the product of two $SU(4)$ Gell-Mann matrices can be expressed by a linear combination of the $U(4)$ matrix generators$^{19}$ Indeed, the product of two $Spin(6)$ generators is explicitly given by

$$
\tilde{\sigma}_{mn}\tilde{\sigma}_{pq} = \frac{1}{4}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np})1_4 + \frac{1}{2}(\delta_{mp}\tilde{\sigma}_{nq} - \delta_{mq}\tilde{\sigma}_{np} + \delta_{nq}\tilde{\sigma}_{mp} - \delta_{np}\tilde{\sigma}_{mq}) - \frac{1}{4}\epsilon_{mnpqrs}\tilde{\sigma}_{rs}. 
$$

(147)

From this formula, the pure tensor gauge fields are expressed as

$$
A_{\alpha\beta} = -\tilde{\sigma}_{mn}\partial_{\alpha}[n]_{\beta\gamma} \partial_{\gamma}[m]_{\beta\gamma},
$$

$$
A_{\alpha\beta\gamma} = -\frac{1}{3}(\mathcal{A}_{\alpha\beta\gamma\alpha} + \mathcal{A}_{\beta\gamma\alpha\alpha} + \mathcal{A}_{\alpha\beta\alpha\beta}) = \epsilon_{mnpqrs}\partial_{\alpha}[n]_{\gamma\delta} \partial_{\gamma}[m]_{\beta\gamma} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \tilde{\sigma}_{rs},
$$

$$
\mathcal{A}_{\alpha\beta\gamma\delta} = \frac{1}{3}(\{\mathcal{A}_{\alpha\beta}, A_{\gamma\delta}\} - \{A_{\alpha\gamma}, A_{\beta\delta}\} + \{A_{\alpha\delta}, A_{\beta\gamma}\}) = -\epsilon_{mnpqrs}\tilde{\sigma}_{rs}\partial_{\alpha}[n]_{\gamma\delta} \partial_{\gamma}[m]_{\beta\gamma} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n},
$$

(148)

where we used

$$
A_{\alpha\beta\gamma} = 2i\partial_{\alpha}[n]_{\gamma\delta} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} - \epsilon_{mnpqrs}\partial_{\alpha}[n]_{\beta\gamma} \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n},
$$

$$
\{A_{\alpha\beta}, A_{\gamma\delta}\} = 2(\partial_{\alpha}[n]_{\gamma\delta} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} - \partial_{\alpha}[n]_{\beta\gamma} \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n})1_4 - 2\epsilon_{mnpqrs}\partial_{\alpha}[n]_{\beta\gamma} \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n}.
$$

(149)

Substituting (148) into (145), we can derive $O(6)$ S-NLS model Hamiltonians as

$$
H_{5,1} = \frac{1}{10} \int_{\mathbb{R}^5_{\text{phys}}} d^5 x \left( (\partial_{\alpha}[n]_{\gamma\delta})^2 + \frac{1}{4!} (\partial_{\alpha}[n]_{\gamma\delta} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n})^2 \right),
$$

$$
H_{5,2} = \frac{1}{80} \int_{\mathbb{R}^5_{\text{phys}}} d^5 x \left( (\partial_{\alpha}[n]_{\beta\gamma})^2 + \frac{1}{9} (\partial_{\alpha}[n]_{\beta\gamma} \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n})^2 \right).
$$

(150a)

(150b)

The octic derivative term of $H_{5,1}$ is similarly given by (147) and the sextic derivative term of $H_{5,2}$ is

$$
(\partial_{\alpha}[n]_{\beta\gamma} \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n})^2 = 6((\partial_{\alpha}[n]_{\gamma\delta})^2 - 18(\partial_{\alpha}[n]_{\beta\gamma})^2 \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n})^2 + 12(\partial_{\alpha}[n]_{\beta\gamma} \partial_{\beta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n} \partial_{\gamma}[\gamma\delta]_{n} \partial_{\delta}[\gamma\delta]_{n}).
$$

(151)

The mathematical structure of the $O(6)$ S-NLS model Hamiltonians is quite similar to that of the Skyrme’s $O(4)$ Hamiltonian (144). Each partial derivative acts to every component of the NLS field and all of the Latin indices of the components are totally antisymmetrized to build the constituent terms of the S-NLS model Hamiltonian. The $O(2k + 1)$ S-NLS model Hamiltonians also exhibited the similar structures. Common structures are actually expected by the dimensional hierarchy of the S-NLS models (118), which implies that the $O(2k)$ S-NLS models are obtained by a dimensional reduction of the $O(2k + 1)$ S-NLS model. Besides, the common structures also suggest the existence of a unified formulation that covers all S-NLS models, which we shall discuss in Sec [6].

---

$^{19}$The $SU(4)$ Gell-Mann matrices are ortho-normalized as $\text{tr}(\lambda_{A} \lambda_{B}) = 2\delta_{AB}$, and with the $4 \times 4$ unit matrix they constitute the $U(4)$ matrix generators that span the whole $4 \times 4$ matrix space.

$^{20}$$H_{5,1}$ and $H_{5,2}$ respectively correspond to Type I and Type II Hamiltonians in [80].
6 O(d + 1) S-NLS Models

We demonstrate a general construction of the S-NLS models from the expression of the higher winding number. This method actually reproduces all of the S-NLS model Hamiltonians previously derived and also supplements other S-NLS models that eluded the tensor gauge theory based constructions.

6.1 O(d + 1) S-NLS models and their basic properties

6.1.1 General O(d + 1) S-NLS model Hamiltonians

The winding number of the O(d + 1) NLS model associated with \( \pi_d(S^d) \) is given by [95]

\[
N_d = \frac{1}{A(S^d_{\text{phys.}})} \frac{d!}{d!} \int_{\mathbb{R}^d} d^d x \epsilon_{a_1 a_2 \cdots a_d+1} \epsilon_{\mu_1 \mu_2 \cdots \mu_d} n_{a_{d+1}} \partial_{\mu_1} n_{a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_d} n_d
\]

\[
\equiv \frac{1}{A(S^d_{\text{phys.}})} \int_{\mathbb{R}^d} d^d x \epsilon_{a_1 a_2 \cdots a_{d+1}} n_{a_{d+1}} \partial_1 n_{a_1} \partial_2 n_{a_2} \cdots \partial_d n_d,
\]

(153)

where \( n_a(x) \) are the O(d + 1) NLS model field on \( x_\mu \in \mathbb{R}^d \) subject to

\[
\sum_{a=1}^{d+1} n_a n_a = 1 : S^d.
\]

As in the previous cases, we first decompose the winding number [155]:

\[
N_d = \frac{1}{A(S^d_{\text{phys.}})} \frac{p!(d-p)!}{d!} \int_{\mathbb{R}^d} d^d x N_{\mu_1 \mu_2 \cdots \mu_p}^a n_{a_{d+1}} \tilde{N}_{\mu_1 \mu_2 \cdots \mu_p}^a n_{a_{d+1}}
\]

(155)

where

\[
N_{\mu_1 \mu_2 \cdots \mu_p}^a \equiv \frac{1}{p!} \partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_p} n_{a_p]};
\]

\[
\tilde{N}_{\mu_1 \mu_2 \cdots \mu_p}^a \equiv \frac{1}{p!(d-p)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_d} \epsilon_{a_1 a_2 \cdots a_{d+1}} n_{a_{d+1}} N_{\mu_p+1 \mu_{p+2} \cdots \mu_d}^{a_{d+1} a_{d+2} \cdots a_d} \]

\[
= \frac{1}{p!(d-p)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_d} \epsilon_{a_1 a_2 \cdots a_{d+1}} n_{a_{d+1}} \partial_{\mu_p+1} n_{a_{p+1}} \partial_{\mu_{p+2}} n_{a_{p+2}} \cdots \partial_{\mu_d} n_d.
\]

(156)

The BPS inequality, \((N_{\mu_1 \mu_2 \cdots \mu_p}^a - \tilde{N}_{\mu_1 \mu_2 \cdots \mu_p}^a)^2 \geq 0\), or

\[
H_{d,p} \geq A(S^d) \cdot N_d,
\]

(157)

yields the O(d + 1) S-NLS model Hamiltonian:

\[
H_{d,p} = H_{d,p}^{(1)} + H_{d,p}^{(2)}
\]

(158)
with

\[ H_{d,p}^{(1)} = \frac{(d - p)!}{2 \cdot d!} \int_{\mathbb{R}_{\text{phys.}}^{d}} d^{d}x \ (\partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \cdots \partial_{\mu_{p}} n_{a_{p}}})^{2}, \quad (159a) \]

\[ H_{d,p}^{(2)} = \frac{p!}{2 \cdot d!(d - p)!} \int_{\mathbb{R}_{\text{phys.}}^{d}} d^{d}x \ (\partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \cdots \partial_{\mu_{d-p}} n_{a_{d-p}}})^{2}, \quad (159b) \]

and the BPS equation, \( N_{\mu_{1}\mu_{2}\cdots\mu_{p}}^{a_{1}a_{2}\cdots a_{p}} = \tilde{N}_{\mu_{1}\mu_{2}\cdots\mu_{p}}^{a_{1}a_{2}\cdots a_{p}} \), or

\[ \partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \cdots \partial_{\mu_{p}} n_{a_{p}}} = \frac{1}{(d - p)!} \epsilon_{\mu_{1}\mu_{2}\cdots\mu_{d}} \epsilon_{a_{1}a_{2}\cdots a_{d+1}} n_{a_{d+1}} \partial_{\mu_{p+1}} n_{a_{p+1}} \partial_{\mu_{p+2}} n_{a_{p+2}} \cdots \partial_{\mu_{d}} n_{a_{d}}. \quad (160) \]

Notice that the \( O(d + 1) \) Hamiltonian is invariant under the interchange \( p \leftrightarrow d - p \):

\[ H_{d,p} = H_{d,d-p}, \quad (161) \]

Therefore, there are \([d/2]\) distinct Hamiltonians in correspondence with \( p = 1, 2, \ldots [d/2] \). One may readily check that \([158]\) reproduces the \( O(2k + 1) \) S-NLS model Hamiltonians, \([102]\) and \([117]\), and also the \( O(2k) \) S-NLS model Hamiltonians, \([144]\) and \([150]\), meaning that \( H_{d,p} \) covers all of the previously derived S-NLS model Hamiltonians. Besides, \( H_{d,p} \) provides other S-NLS model Hamiltonians that eluded the previous derivations. In low dimensions, from \([158]\) such S-NLS model Hamiltonians are readily derived as

\[ H_{2,1} = \frac{1}{2} \int_{\mathbb{R}_{\text{phys.}}^{2}} d^{2}x \ (\partial_{\mu} n_{m})^{2}, \]

\[ H_{4,1} = \frac{1}{8} \int_{\mathbb{R}_{\text{phys.}}^{4}} d^{4}x \ (\partial_{\mu} n_{m})^{2} + \frac{1}{36} (\partial_{\mu} n_{m} \partial_{\nu} n_{n} \partial_{\rho} n_{p})^{2}, \]

\[ H_{6,1} = \frac{1}{12} \int_{\mathbb{R}_{\text{phys.}}^{6}} d^{6}x \ (\partial_{\mu} n_{m})^{2} + \frac{1}{14400} (\partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \cdots \partial_{\mu_{5}} n_{a_{5}}})^{2}, \]

\[ H_{6,3} = \frac{1}{720} \int_{\mathbb{R}_{\text{phys.}}^{6}} d^{6}x \ (\partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \cdots \partial_{\mu_{6}} n_{a_{6}}})^{2}. \quad (162) \]

Note that \( H_{2,1} \) represents the well known \( O(3) \) NLS model Hamiltonian.

6.1.2 Equations of motion and the scaling arguments

From \([158]\), it is not difficult to derive the equations of motion:

\[ \partial_{\mu_{1}} \left( (d - p - 1)! \partial_{\mu_{2}} n_{a_{2}} \partial_{\mu_{3}} n_{a_{3}} \cdots \partial_{\mu_{p}} n_{a_{p}} \partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \partial_{\mu_{3}} n_{a_{3}} \cdots \partial_{\mu_{p}} n_{a_{p}}} \right. \]

\[ + \left. (p - 1)! \partial_{\mu_{2}} n_{a_{2}} \partial_{\mu_{3}} n_{a_{3}} \cdots \partial_{\mu_{d-p}} n_{a_{d-p}} \partial_{\mu_{1}} n_{[a_{1}] \partial_{\mu_{2}} n_{a_{2}} \partial_{\mu_{3}} n_{a_{3}} \cdots \partial_{\mu_{d-p}} n_{a_{d-p}}} \right) - \lambda n_{a_{1}} = 0, \quad (163) \]
where λ denotes the Lagrange multiplier

\[ \lambda = n_{a_1} \partial_{\mu_1} \left( (d - p - 1)! \partial_{\mu_2} n_{a_2} \partial_{\mu_3} n_{a_3} \cdots \partial_{\mu_p} n_{a_p} \cdot \partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \partial_{\mu_3} n_{a_3} \cdots \partial_{\mu_p} n_{a_p]} \right) + (p - 1)! \partial_{\mu_2} n_{a_2} \partial_{\mu_3} n_{a_3} \cdots \partial_{\mu_{d-p}} n_{a_{d-p}} \cdot \partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \partial_{\mu_3} n_{a_3} \cdots \partial_{\mu_{d-p}} n_{a_{d-p}]}. \]

(164)

For \((d, p) = (2k, 2l)\), (163) signifies the equations of motion of the \(O(2k + 1)\) S-NLS model Hamiltonian \(\mathcal{H}_\lambda\). In particular For \((d, p) = (2k, 2)\), (163) becomes

\[ \partial_{\mu_1} \left( \partial_{\mu_2} n_b \cdot \partial_{\mu_1} n_{[a} \partial_{\mu_2} n_{b]} + \frac{1}{(2k - 3)!} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_{2k-2}} n_{a_{2k-2}} \partial_{\mu_1} n_{[a} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_{2k-2}} n_{a_{2k-2}]} \right) = \frac{1}{(2k - 3)!} \lambda n_a = 0, \]

which represents the equations of motion of \(\mathcal{H}_\lambda\). In low dimensions, (163) reads as

\[
\begin{align*}
(d, p) = (2, 1) : & \quad 2\partial_\mu n_a - \lambda n_a = 0, \\
(d, p) = (3, 1) : & \quad 1! \partial_\mu n_a + \partial_\mu (\partial_\nu n_b \partial_\mu n_{[a} \partial_\mu n_{b]} - \lambda n_a) = 0, \\
(d, p) = (4, 1) : & \quad 2! \partial_\mu n_a + \partial_\mu (\partial_\nu n_\rho \partial_\mu n_{[a} \partial_\mu n_{\rho b]} n_{b]} - \lambda n_a) = 0, \\
(d, p) = (4, 2) : & \quad 2\partial_\mu (\partial_\nu n_b \partial_\mu n_{[a} \partial_\mu n_{b]} - \lambda n_a) = 0, \\
(d, p) = (5, 1) : & \quad 3! \partial_\mu n_a + \partial_\mu (\partial_\nu n_b \partial_\mu n_{c} \partial_\sigma n_{d} \partial_\rho n_{[a} \partial_\mu n_{b]} n_{c]} - \lambda n_a) = 0, \\
(d, p) = (5, 2) : & \quad 2\partial_\mu (\partial_\nu n_b \partial_\mu n_{[a} \partial_\mu n_{b]} + \partial_\mu (\partial_\nu n_b \partial_\mu n_{c} \partial_\sigma n_{d} \partial_\rho n_{[a} \partial_\mu n_{b]} n_{c]} - \lambda n_a) = 0.
\end{align*}
\]

(166)

The equations of motion of the \(O(3)\) NLS model and the \(O(5)\) S-NLS model are realized as \((d, p) = (2, 1)\) and \((4, 2)\) in (166), respectively. For the \(O(3)\) NLS model soliton solutions with arbitrary winding number are derived in \([56, 57]\), but for other S-NLS models, it is a formidable task to solve the equations of motion \((163)\) generally.

Instead of solving the equations of motion \((163)\) exactly, we prepare one-scale-parameter family of a field configuration and evaluate the size of the configuration. The mass dimensions of the quantities inside the integrals of \(H_{\lambda, p}^{(1)}\) and \(H_{\lambda, p}^{(2)}\) \((159)\) are respectively given by \(2p - d\) and \(d - 2p\). Suppose that the energy of field configuration \(n_a(x)\) is given by \(E_{\lambda, p} = E_{\lambda, p}^{(1)} + E_{\lambda, p}^{(2)}\). Under the scale transformation

\[ n_a(x) \to n_a^{(R)}(x) \equiv n_a(x/R), \]

(167)

\(E_{\lambda, p}^{(1)}\) and \(E_{\lambda, p}^{(2)}\) are then transformed as

\[ E_{\lambda, p} = E_{\lambda, p}^{(1)} + E_{\lambda, p}^{(2)} \to E_{\lambda, p}(R) = R^{d-2p} E_{\lambda, p}^{(1)} + \frac{1}{R^{d-2p}} E_{\lambda, p}^{(2)}. \]

(168)

The scale parameter \(R\) can be considered as a variational parameter of the size of the field configuration. Since \(p \geq \lfloor d/2 \rfloor\), as \(R\) increases, the first term monotonically increases while the second term monotonically decreases (except for \(p = \lfloor d/2 \rfloor\)). This means that the first term

\[^{21}\text{Both } H_{\lambda, p}^{(1)} \text{ and } H_{\lambda, p}^{(2)} \text{ should have mass dimension one, and so, to be precise, some dimensionful parameters are needed in front of them to adjust the dimension counting.}\]
energetically favors a smaller size field configuration while while the second term favors a larger size configuration. These two competing effects determine an optimal size of the field configuration. More specifically, we take the derivative of \( E_{d,p}(R) \) with respect to \( R \) to obtain the local energy minimum, and the size is determined as

\[
R_{d,p} = \left( \frac{E^{(2)}_{d,p}}{E^{(1)}_{d,p}} \right)^{\frac{1}{2(d-2p)}}.
\]

The present S-NLS models thus realize field configurations with the finite size given by (169) (except for the scale invariant case).

### 6.1.3 Scale invariant solutions

Next let us consider the case \((d, p) = (2k, k)\), in which the two Hamiltonians coincide, \( H^{(1)}_{2k,k} = H^{(2)}_{2k,k} \), and their competing effects balance to give scale invariant field solutions. The S-NLS model Hamiltonian (158) becomes

\[
H_{2k,k} = \frac{1}{(2k)!} \int_{\mathbb{R}^d} dx \ (\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]})^2.
\]

When \( k \) is even, (170) is exactly equal to the former scale invariant Hamiltonian (117). The equations of motion (169) and the BPS equation (160) are reduced to

\[
\begin{align*}
\partial_{\mu_1} \left( \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k} \right) \cdot \left( \partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]} \right) &- \frac{1}{2(k-1)!} \lambda n_{a_1} = 0, \\
\partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_k} n_{a_k]} &- \frac{1}{k!} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}} \epsilon_{a_1 a_2 \cdots a_{2k+1}} n_{a_{2k+1}} \partial_{\mu_{k+1}} n_{a_{k+1}} \partial_{\mu_{k+2}} n_{a_{k+2}} \cdots \partial_{\mu_{2k}} n_{a_{2k}} = 0.
\end{align*}
\]

Especially for \( d = 4 \), (171a) reproduces the \((d, p) = (4, 2)\) equation of (166). The equations of motion (171a) are highly non-linear equations, but inverse stereographic coordinate configuration

\[
n_{a}(x) = r_a \equiv \{ \frac{2}{1 + x^2} x_{\mu_1}, \frac{1 - x^2}{1 + x^2} \},
\]

realizes a simple solution of (171a) and also satisfies the BPS equation (171b). When \( k \) is odd, (171b) \( \rightarrow \) (172) carries a topological configuration of unit winding number \( N_d = 1 \). Since from the one-to-one correspondence between the points on \( \mathbb{R}^{2k} \) and those on \( S^{2k} \), it may be obvious that (172) represents a field configuration of the winding number \( 1 \). One can explicitly confirm this by

\[
N_d|_{n_a=r_a} = \frac{1}{A(S^d)} \int_{\mathbb{R}^d_{\text{phys}}} d^dx \epsilon_{a_1 a_2 \cdots a_{d+1}} \partial r_{a_1} \partial r_{a_2} \cdots \partial r_{a_d} = \frac{A(S^{d-1})}{A(S^d)} \int_0^\infty dx \ x^{d-1} \frac{2^d}{(1 + x^2)^d} = 1.
\]

The energy density of (172) is also evaluated as

\[
\frac{1}{d!} \left| \partial_{\mu_1} n_{[a_1} \partial_{\mu_2} n_{a_2} \cdots \partial_{\mu_p} n_{a_{d/2]} \right|^2 \bigg|_{n_a=r_a} = \frac{2^d}{(1 + x^2)^d},
\]

which implies that (172) signifies a solitonic configuration localized around the origin.

---

22 Also recall that in Sec. 4.3.3, we saw that the tensor instanton configuration satisfies the BPS equation and the equations of motion.
6.2 Topological field configurations

Recall that the \( k \)th Chern number has two equivalent expressions, \( N_{2k-1} \) and \( N_{2k} \) (Sec. 2.1), which implies intimate relations between topological field configurations of \( O(2k) \) and \( O(2k + 1) \) S-NLS models of same winding number. In this Section, we demonstrate this idea to construct topological field configurations of higher winding numbers. The size of the topological field configurations is variationally determined by the scaling arguments.

6.2.1 Topological field configurations in odd \( d \)

The transition function \( g \) \(^{23} \) represents \( N_{2k-1} = 1 \) associated with the homotopy \( \pi_{2k-1}(S^{2k-1}) \simeq \mathbb{Z} \). Using \( \text{(175)} \), a map from \( r_\mu \in S^{2k-1}_{\text{phys.}} \) to \( n_\mu \in S^{2k-1}_{\text{field}} \) with arbitrary winding number \( N \) is obtained as

\[
g^N = e^{i(N\theta)\sum_{i=1}^{2k-1} \gamma_i \hat{r}_i} = \sum_{\mu=1}^{2k} n_\mu r_\mu.
\] (176)

Here, \( n_\mu \) is given by

\[
n_\mu = \{ n_i, n_{2k} \} \equiv \{ \sin(N\theta) \ r_i \ , \ \cos(N\theta) \}.
\] (177)

The argument of the trigonometric function in \( \text{(177)} \) is \( N \cdot \theta \), meaning that when the azimuthal angle \( \theta \) sweeps \( S^{2k-1}_{\text{phys.}} \) once, it wraps \( S^{2k-1}_{\text{field}} \) \( N \) times. For small \( N \), \( \text{(177)} \) is given by

\[
\begin{align*}
N = 1 & : \quad n_\mu = \{ n_i, n_{2k} \} = \{ \sin(\theta) \ \hat{r}_i, \ \cos(\theta) \} = r_\mu, \\
N = 2 & : \quad n_\mu = \{ n_i, n_{2k} \} = \{ \sin(2\theta) \ \hat{r}_i, \ \cos(2\theta) \} = \{ 2r_{2k}r_i, -r_i^2 + r_{2k}^2 \}, \\
N = 3 & : \quad n_\mu = \{ n_i, n_{2k} \} = \{ \sin(3\theta) \ \hat{r}_i, \ \cos(3\theta) \} = \{ -(r_j^2 - 3r_{2k}^2)r_i, -(r_j^2 - 2r_{2k}^2)r_{2k} \}.
\end{align*}
\] (178)

Notice that the map with the winding number \( N \) is expressed by the \( N \)th polynomials of \( r_\mu \). \( N_{2k-1} \) \(^{23} \) is actually evaluated for \( \text{(177)} \) as

\[
N_{2k-1} = \frac{1}{A(S^{2k-1}_{\text{phys.}})} \int_{S^{2k-1}_{\text{phys.}}} N \sin^{2k-2}(N\theta) \ d\theta \ d\Omega_{2k-2} = \frac{1}{A(S^{2k-1}_{\text{phys.}})} \int_{S^{2k-1}_{\text{phys.}}} d\Omega_{2k-1} = N,
\] (179)

where we used

\[
\int_0^\pi d\theta \sin^{2k}(N\theta) = \frac{\pi(2k-1)!!}{(2k)!!} = \int_0^\pi d\theta \sin^{2k}(\theta).
\] (180)

Regarding \( n_\mu \) as the \( O(2k) \) NLS field, we treat \( \text{(177)} \) as topological field configuration on \( S^{2k-1}_{\text{phys.}} \) with the the winding number \( N \). To construct topological field configurations on \( S^{2k-1}_{\text{phys.}} \), we apply the stereographic projection in the physical space:

\[
r_\mu \in S^{2k-1}_{\text{phys.}} \quad \rightarrow \quad x_i = \frac{R}{R + r_{2k}} \hat{r}_i \in \mathbb{R}^{2k-1}_{\text{phys.}} \quad (i = 1, 2, \cdots, 2k - 1)
\] (181)

\(^{23} \) \( \text{(175)} \) can be regarded as a non-linear representation of the coset \( S^{2k-1} \simeq SO(2k)/SO(2k - 1) \) of the broken generators, \( \sigma_{i, 2k} = 2\gamma_i \).
or
\[ r_i = \frac{2R^2}{R^2 + x^2} x_i, \quad r_{2k} = \frac{R^2 - x^2}{R^2 + x^2} R. \] (182)

Here, we took the radius of \( S_{\text{phys.}}^{2k-1} \) as \( R \). Substituting (182) into the expressions of \( n_\mu \) such as (178), we obtain one-parameter family \( O(2k) \) NLS field configuration on \( \mathbb{R}^{2k-1}_\text{phys.} \):
\[ n^{(R)}_\mu(x_i) = n_\mu(x_i/R). \] (183)

For instance,
\[
N = 1 : \quad n^{(R)}_1(x) = \frac{2R}{x^2 + R^2} x_i, \quad n^{(R)}_{2k}(x) = -\frac{x^2 - R^2}{x^2 + R^2},
\]
\[
N = 2 : \quad n^{(R)}_1(x) = -\frac{4R}{(x^2 + R^2)^2} (x^2 - R^2)x_i, \quad n^{(R)}_{2k}(x) = \frac{1}{(x^2 + R^2)^2} (-4R^2 x^2 + (x^2 - R^2)^2),
\]
\[
N = 3 : \quad n^{(R)}_1(x) = -\frac{2R}{(x^2 + R^2)^3} (4R^2 x^2 - 3(x^2 - R^2)^2)x_i,
\]
\[ n^{(R)}_{2k}(x) = \frac{1}{(x^2 + R^2)^3} (12R^2 x^2 - (x^2 - R^2)^2)(x^2 - R^2). \] (184)

Substituting (184) into (183), one can explicitly confirm that (184) represents the topological field configurations of \( N_{2k-1} = 1, 2, 3 \). Obviously, the radius of the sphere \( R \) corresponds to the size of the soliton configuration. Intuitively, when the size of the sphere becomes larger, the “concentration” of the soliton field around the origin will be thinner, and subsequently the size of the soliton becomes larger. Treating \( R \) as a variational parameter of \( n^{(R)}_\mu(x) \), we consider minimal energy configuration in each topological sector. The previous scaling argument (169) indicates
\[ R_{2k-1, p}(N) = \left( \frac{E^{(2)}_{2k-1, p}(N)}{E^{(1)}_{2k-1, p}(N)} \right)^{\frac{1}{2k-1}}, \] (185)
which is the optimal size of the \( O(2k) \) NLS field configuration of the topological number \( N \).

### 6.2.2 Topological field configurations in even \( d \)

With the idea of the dimensional hierarchy, we construct \( O(2k + 1) \) topological field configuration in \( 2kD \) from the set-up of \( (2k - 1)D \):
\[ \pi_{2k}(S^{2k}) \simeq \mathbb{Z}. \] (186)

We add a radial direction to \( S_{\text{phys.}}^{2k-1} \) and consider 1D higher space, \( \mathbb{R}^{2k}_\text{phys.} \) (the left-figure of Fig 4). The original map from \( r_\mu \in S_{\text{phys.}}^{2k-1} \) to \( n_\mu \in S_{\text{field}}^{2k-1} \) is now transformed to (Fig 4)
\[ x_\mu \in \mathbb{R}^{2k}_\text{phys.} \rightarrow h_\mu = n_\mu(x) \in \mathbb{R}^{2k}_\text{field}. \] (187)

The radial direction has no effect about the winding in (186), and the winding number associated with the map (187) is accounted for by the winding from \( S_{\text{phys.}}^{2k-1} \) on \( \mathbb{R}^{2k}_\text{phys.} \) to the \( S_{\text{field}}^{2k-1} \) on \( \mathbb{R}^{2k}_\text{field} \).
(Fig. 4), which is nothing but the previous \((2k - 1)D\) winding, \(\pi_{2k-1}(S^{2k-1}) \simeq \mathbb{Z}\) (Fig. 4). In correspondence with (178), we have

\[
N = 1 : h_\mu = \frac{1}{R} x_\mu, \quad \mu = 1, 2, \ldots, 2k,
\]

\[
N = 2 : h_\mu = \{h_i, h_{2k}\} = \frac{1}{R^2} \{2x_{2k} x_i - x_i^2 + x_{2k}^2\},
\]

\[
N = 3 : h_\mu = \{h_i, h_{2k}\} = \frac{1}{R^3} \{-(x_j^2 - 3x_{2k}^2)x_i - (3x_j^2 - 2x_{2k}^2)x_{2k}\}. \quad (188)
\]

Figure 4: The \(O(2k+1)\) NLS field of the winding number \(\pi_{2k}(S^{2k}) \simeq \mathbb{Z}\) is constructed by the \(O(2k)\) NLS field of the winding number \(\pi_{2k-1}(S^{2k-1}) \simeq \mathbb{Z}\).

To realize topological field configurations with field-manifold \(S_{\text{field}}^{2k}\), we apply the inverse stereographic projection in the field space (the right figure of Fig. 4):

\[
h_\mu \in \mathbb{R}^{2k}_{\text{field}} \quad \rightarrow \quad n_\mu = \frac{2}{1 + h_\nu^2} h_\mu, \quad n_{2k+1} = \frac{1 - h_\nu^2}{1 + h_\nu^2} \in S_{\text{field}}^{2k}. \quad (189)
\]

Substituting (188) into (189), we obtain the \(O(2k+1)\) topological field configurations on \(\mathbb{R}^{2k}_{\text{phys}}:\)

\[
N = 1 : n^{(R)}_i(x) = \frac{2R}{(x_\nu^2 + R^2)x_\nu^2}, \quad n^{(R)}_{2k+1}(x) = -\frac{x_\nu^2 - R^2}{x_\nu^2 + R^2},
\]

\[
N = 2 : n^{(R)}_i(x) = \frac{4R^2}{(x_\nu^2)^2 + R^4} x_{2k} x_i, \quad n^{(R)}_{2k}(x) = \frac{2R^2}{(x_\nu^2)^2 + R^4} (-x_i^2 + x_{2k}^2), \quad n^{(R)}_{2k+1}(x) = -\frac{(x_\nu^2)^2 - R^4}{(x_\nu^2)^2 + R^4},
\]

\[
N = 3 : n^{(R)}_i(x) = -\frac{2R^3}{(x_\nu^2)^3 + R^6} (x_j^2 - 3x_{2k}^2)x_i, \quad n^{(R)}_{2k}(x) = -\frac{2R^3}{(x_\nu^2)^3 + R^6} (3x_j^2 - x_{2k}^2)x_{2k}, \quad n^{(R)}_{2k+1}(x) = -\frac{(x_\nu^2)^3 - R^6}{(x_\nu^2)^3 + R^6}. \quad (190)
\]

One can explicitly check that (190) describes topological field configurations of \(N_{2k} = 1, 2, 3\) by (153). The scaling argument (169) determines the parameter \(R\) as

\[
R_{2k,p}(N) = \left(\frac{E^{(2)}_{2k,p}(N)}{E^{(1)}_{2k,p}(N)}\right)^{\frac{1}{p(k-p)}}. \quad (191)
\]

For the \(O(3)\) NLS model, soliton solutions of arbitrary topological numbers are given by the holomorphic functions on \(\mathbb{C} \simeq \mathbb{R}^2\) [56, 57], and the power of the complex coordinates corresponds
to the winding number \([97, 56]\). Meanwhile for the \(O(5)\) S-NLS model (of \(H_{4,2}\)), though the topological field configuration is simply obtained by the multiple of quaternionic analytic function \([12, 16]\), it is not easy to derive the soliton solutions except for \(N_4 = 1\). Similarly as obtained above, the \(O(2k + 1)\) topological field configurations are a solution of the equations of motion \([171a]\) of \(N_{2k} = 1\) but other configurations of higher winding number do not satisfy the equations of motion.

7 Summary

We proposed a systematic procedure to construct higher dimensional S-NLS models based on the Landau/NLS model correspondence. Exploiting the differential geometry of the Landau models, we introduced the \([k/2]\) distinct parent tensor gauge theories on the field manifold \(S^{2k}\) and subsequently derived the \([k/2]\) \(O(2k + 1)\) S-NLS models on \(\mathbb{R}^{2k}_{\text{phys}}\). The gauge symmetry and the BPS inequality of the parent tensor gauge theories are necessarily inherited to the obtained \(O(2k + 1)\) S-NLS models. As a dimensional reduction from \(2kD\) to \((2k - 1)D\), we adopted the Chern-Simons term description of the Chern number. Representing the transition function by \(O(2k)\) NLS field, we constructed the \(O(2k)\) S-NLS model Hamiltonians from pure tensor gauge fields, which indeed include the Skyrme model as its \(O(4)\) model. The obtained \(O(2k)\) S-NLS models do not possess gauge symmetries unlike the \(O(2k + 1)\) S-NLS models. From the NLS field expression of the higher winding number, we discussed a unified formulation of the S-NLS models. We derived the equations of motion and constructed an exact scale invariant solution with unit winding number. The topological field configurations with arbitrary winding number are also constructed by exploiting the idea of the dimensional hierarchy. The topological field configurations depend on the variational scaling parameter which is determined by the scaling arguments. A particular feature of the present construction is that the decomposition of the topological number necessarily yields two competing terms in the S-NLS model Hamiltonian to realize finite size soliton configurations.

Though we obtained the equations of motion, their explicit solutions have not been generally derived. The derivation of explicit solutions is not easy even for the original “simple” Skyrme model\(^{24}\). One apparent direction is to evaluate the field configurations by using numerical methods. Another direction will be a generalization of the S-NLS models based on different symmetries. While in this work we were focused on the \(O(N)\) S-NLS models that are closely related to the \(SO(N)\) Landau models, many Landau models with different symmetries, including supersymmetric generalizations \([99, 100]\), have been constructed with the developments of the higher dimensional quantum Hall effect in the past two decades. In view of the topological insulator \([101]\), the present Landau models are categorized as A-class or AIII class. The topological table accommodates various cousins of the Landau models with different symmetries. It is tempting to construct other NLS models from such various Landau models. The NLS models not only exhibits deep mathematical structures but appear in important physical applications. As is well known, the Skyrme model plays a crucial role in the non-perturbative analysis of QCD. As the S-NLS

\(^{24}\)For \(O(6)\) S-NLS models, explicit solutions were recently derived in toroidal coordinates \([89]\). 

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model solitons emerge as anyonic collective excitations in the higher dimensional quantum Hall effect, their roles will be indispensable in understanding topological phases in higher dimensions.

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A Stereographic projection and $SO(2k)$ instanton configurations

Here, we review the stereographic projection from $S^{2k}$ to $\mathbb{R}^{2k}$ and explore the relationship between the monopole gauge field on $S^{2k}$ and the instanton field on $\mathbb{R}^{2k}$ [103, 23, 94, 104].

A.1 Map from $\mathbb{R}^{2k}$ to $S^{2k}$

First we introduce a general map from $\mathbb{R}^{2k}$ to $S^{2k}$:

$$x_\mu \in \mathbb{R}^{2k} \to n_a(x) \in S^{2k}$$

(192)

where $n_a$ are subject to

$$\sum_{a=1}^{2k+1} n_a n_a = 1.$$  

(193)

We give gauge fields $A_\mu$ on $\mathbb{R}^{2k}$ and $A_a$ on $S^{2k}$ as

$$A = A_\mu dx_\mu = A_a dn_a, \quad F = dA + iA^2 = \frac{1}{2} F_{\mu\nu} dx_\mu dx_\nu = \frac{1}{2} F_{ab} dn_a dn_b.$$  

(194)

Since $dn_a = \frac{\partial n_a}{\partial x_\mu} dx_\mu$, they are related as

$$A_\mu = \frac{\partial n_a}{\partial x_\mu} A_a, \quad F_{\mu\nu} = \frac{\partial n_a}{\partial x_\mu} \frac{\partial n_b}{\partial x_\nu} F_{ab}.$$  

(195)

The $SO(2k)$ monopole gauge field on $S^{2k}$ is expressed as

$$A_m = -\frac{1}{1 + n_{2k+1}} \sigma_{mn} n_n, \quad A_{2k+1} = 0,$$  

(196)

and the monopole field strength $F_{ab} = \partial_a A_b - \partial_b A_a + i[A_a, A_b]$ is

$$F_{mn} = \sigma_{mn} - n_m A_n + n_n A_m, \quad F_{m,2k+1} = -F_{2k+1,m} = (1 + n_{2k+1}) A_m.$$  

(197)

(196) and (197) are related to (73) and (74) though (195).
A.2 Stereographic projection and gauge theory on a sphere

We choose \( n_a \) as the inverse stereographic coordinates on \( S^d \), \( r_a = \{ r_\mu, r_{d+1} \} \):

\[
  r_\mu = \frac{2}{1 + x^2} x_\mu, \quad r_{d+1} = \frac{1 - x^2}{1 + x^2} \in S^d.
\]

(198)

Through (195), the monopole configuration on \( S^{2k} \)

\[
  \hat{A}_\mu = -\frac{1}{1 + r_{d+1}} \sigma_{\mu\nu} r_\nu, \quad \hat{A}_{d+1} = 0
\]

\[
  \hat{F}_{\mu\nu} = -r_\mu \hat{A}_\nu + r_\nu \hat{A}_\mu + \hat{\sigma}_{\mu\nu}, \quad \hat{F}_{\mu,d+1} = -\hat{F}_{d+1,\mu} = (1 + r_{2k+1}) \hat{A}_\mu,
\]

(199)

is transformed to the “instanton” configuration on \( \mathbb{R}^{2k} \), (112) and (106):

\[
  A_\mu = -\frac{1}{x^2 + 1} \sigma_{\mu\nu} x_\nu, \quad F_{\mu\nu} = \frac{1}{(x^2 + 1)^2} \sigma_{\mu\nu}.
\]

(200)

(200) represents the BPST instanton configuration for \( k = 2 \). Even for arbitrary \( k \), in this paper we call (200) the “instanton” configuration, although (200) is not a solution of the pure Yang-Mills field equations except for \( k = 2 \) (Appendix A.4). Notice that the moduli size-parameter of the instanton (200) is identified with the radius of \( S^{2k} \) on which the monopole gauge field lives.

Indeed, under the scale transformation to change the radius of sphere from 1 to \( R \):

\[
  r_a \rightarrow R r_a
\]

or \( x \rightarrow \frac{1}{R} x \), (200) is transformed as

\[
  A \rightarrow -\frac{2}{x^2 + R^2} \sigma_{\mu\nu} x_\nu dx_\mu.
\]

(202)

Since the instanton configuration can be obtained by the stereographic projection of the monopole configuration on the sphere, it may be obvious that the size of the instanton is determined by the size of the sphere.

From (199), we can obtain the tensor monopole field strength on \( S^{2k} \) [68]:

\[
  G_{a_1 a_2 \cdots a_{2k}} = \frac{1}{2^k} \text{tr} \left( \hat{F}_{[a_1 a_2} \hat{F}_{a_3 a_4} \cdots \hat{F}_{a_{2k-1} a_{2k}]} \right) = \frac{(2k)!}{2^{k+1}} \epsilon_{a_1 a_2 \cdots a_{2k+1}} r_{a_{2k+1}},
\]

(203)

and similarly the tensor instanton field strength on \( \mathbb{R}^{2k} \):

\[
  G_{\mu_1 \mu_2 \cdots \mu_{2k}} = \frac{1}{2^k} \text{tr} \left( F_{[\mu_1 \mu_2} F_{\mu_3 \mu_4} \cdots F_{\mu_{2k-1} \mu_{2k}]} \right)_{r_a = R} = (2k)! 2^{k-1} \left( \frac{1}{1 + x^2} \right)^{2k} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}},
\]

(204)

where we used

\[
  \text{tr} (\sigma_{[\mu_1 \mu_2} \sigma_{\mu_3 \mu_4} \cdots \sigma_{\mu_{2k-1} \mu_{2k}}] ) = \frac{1}{2} (2k)! \epsilon_{\mu_1 \mu_2 \mu_3 \cdots \mu_{2k}}.
\]

(205)
\( \hat{C}_{a_1 a_2 \cdots a_{2k-1}} \) and \( C_{\mu_1 \mu_2 \cdots \mu_{2k-1}} \) that satisfy
\[
\hat{C}_{a_1 a_2 \cdots a_{2k}} = \frac{1}{(2k-1)!} \hat{\partial}_{[a_1} \hat{C}_{a_2 a_3 \cdots a_{2k}],} \tag{206a}
\]
\[
C_{\mu_1 \mu_2 \cdots \mu_{2k}} = \frac{1}{(2k-1)!} \partial_{\mu_1} C_{\mu_2 \mu_3 \cdots \mu_{2k}],} \tag{206b}
\]
are obtained from the Chern-Simons term:
\[
\frac{1}{(2k-1)!} \hat{C}_{a_1 a_2 \cdots a_{2k-1}} dx_{a_1} \wedge dx_{a_2} \cdots dx_{a_{2k-1}} = L_{\text{CS}}^{(2k-1)} [\hat{A}], \tag{207a}
\]
\[
\frac{1}{(2k-1)!} C_{\mu_1 \mu_2 \cdots \mu_{2k-1}} dx_{\mu_1} \wedge dx_{\mu_2} \cdots dx_{\mu_{2k-1}} = L_{\text{CS}}^{(2k-1)} [A]. \tag{207b}
\]

In low dimensions, (207b) is expressed as
\[
k = 1 : C_{\mu} = \text{tr} A_{\mu},
\]
\[
k = 2 : C_{\mu \rho} = \text{tr}(A_{[\mu} \partial_{\rho]} A_{\rho]} + \frac{2}{3} i A_{[\mu} A_{\nu} A_{\rho]}), \frac{1}{2} \text{tr}(A_{[\mu} F_{\nu \rho]} - \frac{2}{3} i A_{[\mu} A_{\nu} A_{\rho]}),
\]
\[
k = 3 : C_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} = \frac{1}{4} \text{tr}(A_{[\mu_1 \mu_2 \mu_3} F_{\mu_4 \mu_5]} - i A_{[\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} F_{\mu_5]} - \frac{2}{5} A_{[\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} A_{\mu_5]}).
\]

For the instanton configuration (200), (208) becomes
\[
k = 1 : C_{\mu} = \frac{1}{1 + x^2} \epsilon_{\mu \nu} x_{\nu},
\]
\[
k = 2 : C_{\mu \rho} = -\left(\frac{2}{1 + x^2}\right)^3 \frac{3}{2} \epsilon_{\mu \rho \sigma} x_{\sigma},
\]
\[
k = 3 : C_{\mu_1 \mu_2 \cdots \mu_5} = -9 \left(\frac{2}{1 + x^2}\right)^5 \left(1 + \frac{1}{2} x^2 + \frac{2}{3} \left(\frac{1}{2} x^2\right)^2\right) \epsilon_{\mu_1 \mu_2 \cdots \mu_6} x_{\mu_6}. \tag{209}
\]

(195) implies the transformation between the monopole and instanton tensor fields as
\[
G_{\mu_1 \mu_2 \cdots \mu_{2k}} = \hat{G}_{a_1 a_2 \cdots a_{2k}} \frac{\partial r_{a_1}}{\partial x_{\mu_1}} \frac{\partial r_{a_2}}{\partial x_{\mu_2}} \cdots \frac{\partial r_{a_{2k}}}{\partial x_{\mu_{2k}}} = \left(\frac{2}{1 + x^2}\right)^{4k} K_{a_1}^{\mu_1} K_{a_2}^{\mu_2} \cdots K_{a_{2k}}^{\mu_{2k}} \hat{G}_{a_1 a_2 \cdots a_{2k}},
\]
\[
C_{\mu_1 \mu_2 \cdots \mu_{2k-1}} = \hat{C}_{a_1 a_2 \cdots a_{2k-1}} \frac{\partial r_{a_1}}{\partial x_{\mu_1}} \frac{\partial r_{a_2}}{\partial x_{\mu_2}} \cdots \frac{\partial r_{a_{2k-1}}}{\partial x_{\mu_{2k-1}}} = \left(\frac{2}{1 + x^2}\right)^{(2k-1)} K_{a_1}^{\mu_1} K_{a_2}^{\mu_2} \cdots K_{a_{2k-1}}^{\mu_{2k-1}} \hat{C}_{a_1 a_2 \cdots a_{2k-1}}, \tag{210}
\]

which can be explicitly confirmed with the expressions of the fields. In (210), we introduced an important quantity
\[
K_{a}^{\mu} \equiv \left(\frac{1 + x^2}{2}\right)^2 \frac{\partial r_{a}}{\partial x_{\mu}}, \tag{211}
\]
or
\[
K_{\mu}^{\mu} = \frac{1 + x^2}{2} \delta_{\nu}^{\mu} - x_{\mu} x_{\nu}, \quad K_{2k+1}^{\mu} = -x_{\mu}. \tag{212}
\]

\footnote{The explicit forms of \( \hat{C}_{a_1 a_2 \cdots a_{2k-1}} \) are derived in [68].}
$K_a^\mu$ are known as the conformal Killing vectors \[103\] that satisfy the conformal Killing equations

$$\partial^\mu K^\nu + \partial^\nu K^\mu = \frac{2}{d} \partial^\lambda K^\lambda \delta_{\mu\nu}, \quad (\mu, \nu = x_1, x_2, \cdots, x_d).$$  \hspace{1cm} (213)$$

and the transversality condition

$$r_a K_a^\mu = 0.$$  \hspace{1cm} (214)$$

The conformal Killing vectors have the following properties:

$$K_a^\mu K_a^\nu = \left( \frac{1 + x^2}{2} \right)^2 \delta^{\mu\nu}, \quad K_a^\mu K_b^\nu = \left( \frac{1 + x^2}{2} \right)^2 (\delta_{ab} - r_a r_b),$$  \hspace{1cm} (215)$$

For more detail properties about $K_a^\mu$, see \[103\].

We here discuss somewhat in detail about the formulation of the field theory on sphere by adding some more informations to \[103, 104\]. Apparently, gauge fields on $\mathbb{R}^{2k}$ and on $S^{2k}$ are generally related as

$$A_\mu = \left( \frac{2}{1 + x^2} \right)^2 K_a^\mu \hat{A}_a, \quad F_{\mu\nu} = \left( \frac{2}{1 + x^2} \right)^4 K_a^\mu K_b^\nu \hat{F}_{ab},$$  \hspace{1cm} (216)$$

or

$$\hat{A}_a = K_a^\mu A_\mu, \quad \hat{F}_{ab} = K_a^\mu K_b^\nu F_{\mu\nu}. \hspace{1cm} (217)$$

The derivative on $S^{2k}$ is constructed as

$$\hat{\partial}_a \equiv K_a^\mu \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial r_a} - r_a r_b \frac{\partial}{\partial r_b} = i r_a L_{ba},$$  \hspace{1cm} (218)$$

where

$$L_{ab} = - i r_a \frac{\partial}{\partial r_b} + i r_b \frac{\partial}{\partial r_a} = - i r_a \hat{\partial}_b + i r_b \hat{\partial}_a = - i K_a^\mu \frac{\partial K_b^\nu}{\partial x_\mu} \frac{\partial}{\partial x_\nu} + i K_b^\mu \frac{\partial K_a^\nu}{\partial x_\mu} \frac{\partial}{\partial x_\nu}. \hspace{1cm} (219)$$

Although $r_a$ are the coordinates on $S^d$ subject to $\sum_{a=1}^{d+1} r_a r_a = 1$, we can treat $r_a$ as if they are independent parameters in using \[218\]. $\hat{\partial}_a$ are non-commutative operators and satisfy the $SO(d+1,1)$ algebra with $L_{ab}$\[26\]

$$[- i \hat{\partial}_a, - i \hat{\partial}_b] = - i L_{ab}, \quad [L_{ab}, - i \hat{\partial}_c] = i \delta_{ac} (- i \hat{\partial}_b) - i \delta_{bc} (- i \hat{\partial}_a),$$

$$[L_{ab}, L_{cd}] = i \delta_{ac} L_{bd} - i \delta_{ad} L_{bc} + i \delta_{bd} L_{ac} - i \delta_{bc} L_{ad}. \hspace{1cm} (221)$$

\[26\] Under the identification $L_{a,d+2} = - i \hat{\partial}_a$ ($a = 1, 2, \cdots, d+1$), $L_{AB} (A, B = 1, 2, \cdots, d+2)$ realize the $SO(d+1,1)$ operators that satisfy

$$[L_{AB}, L_{CD}] = i \eta_{ABC} L_{BD} - i \eta_{ABD} L_{BC} + i \eta_{BDC} L_{AC} - i \eta_{BCD} L_{AD}. \hspace{1cm} (220)$$

with $\eta_{AB} = \text{diag}(+, +, \cdots, +, -)$. 

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The field strength on $S^{2k}$ is given by

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a + i[\hat{A}_a, \hat{A}_b] + ir_c L_{ab} \hat{A}_c. \quad (224)$$

Notice the last term on the right-hand side of (224). For tensor fields, (224) may be generalized as

$$\hat{G}_{a_1 a_2 \cdots a_{2k}} = \frac{1}{(2k-1)!} \hat{\partial}_{a_1} \hat{C}_{a_2 \cdots a_{2k}} + i \frac{1}{2(2k-2)!} r_{a_{2k+1}} L_{[a_1 a_2} \hat{C}_{a_3 \cdots a_{2k}]a_{2k+1}}. \quad (225)$$

(224) can be easily confirmed for the monopole and instanton configurations. Substituting (217) and (218) into (224), we have

$$\hat{F}_{ab} = K^\mu_a K^\nu_b F_{\mu\nu} + K^\mu_a (\partial_\mu K^\nu_b) A_\nu - K^\mu_b (\partial_\mu K^\nu_a) A_\nu + ir_c L_{ab} \hat{A}_c. \quad (226)$$

For the monopole field (199) and the instanton field (200), we can show

$$K^\mu_a (\partial_\mu K^\nu_b) A_\nu - K^\mu_b (\partial_\mu K^\nu_a) A_\nu = r_a \hat{A}_b - r_b \hat{A}_a = -ir_c L_{ab} \hat{A}_c. \quad (227)$$

Therefore, only the first term on the right-hand side of (226) survives to yield

$$\hat{F}_{ab} = K^\mu_a K^\nu_b F_{\mu\nu}, \quad (228)$$

which is (216).

### A.3 Yang-Mills action and Chern number

With the area element of $S^d$

$$d\Omega_d = \left(\frac{2}{1+x^2}\right)^d d^d x \quad (228)$$

and

$$\hat{F}_{ab}^2 = \left(\frac{1+x^2}{2}\right)^4 F_{\mu\nu}^2 \quad (229)$$

the Yang-Mills action is expressed as

$$\int_{S^{2k}} d\Omega_{2k} \, \text{tr}(\hat{F}_{ab}^2) = \int_{\mathbb{R}^{2k}} d^{2k} x \, \left(\frac{1+x^2}{2}\right)^{4-2k} \text{tr}(F_{\mu\nu}^2). \quad (230)$$

For the special case $2k = 4$, the conformal factor on the right-hand side of (230) vanishes and (230) becomes

$$\int_{S^4} d\Omega_4 \, \text{tr}(\hat{F}_{ab}^2) = \int_{\mathbb{R}^4} d^4 x \, \text{tr}(F_{\mu\nu}^2), \quad (231)$$

which yields the equations of motion:

$$\hat{D}_a \hat{F}_{ab}|_{2k=4} = D_\mu F_{\mu\nu}|_{2k=4} = 0. \quad (232)$$

---

$^{27}$ (224) is simply related to the three-rank antisymmetric field strength

$$\hat{F}_{abc} = i(L_{ab} \hat{A}_c + L_{bc} \hat{A}_a + L_{ca} \hat{A}_b) + i(r_a [\hat{A}_b, \hat{A}_c] + r_b [\hat{A}_c, \hat{A}_a] + r_c [\hat{A}_a, \hat{A}_b]) \quad (222)$$

as

$$\hat{F}_{ab} = r_c \hat{F}_{abc}. \quad (223)$$

---
Meanwhile, the $k$th Chern number is expressed as
\[
c_k = \frac{1}{(2\pi)^k k!} \int_{S^{2k}} \text{tr}(\hat{F}^k) = \frac{1}{(4\pi)^k k!} \int_{S^{2k}} \text{tr}(\hat{F}_{a_1 a_2} \cdots \hat{F}_{a_{2k-1} a_{2k}}) \epsilon_{a_1 a_2 \cdots a_{2k+1}} r_{a_{2k+1}} d\Omega_{2k} = \frac{1}{(4\pi)^k k!} \int_{\mathbb{R}^{2k}} \text{tr}(F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} \cdots F_{\mu_{2k-1} \mu_{2k}} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}} d^{2k}x) = \frac{1}{(2\pi)^k k!} \int_{\mathbb{R}^{2k}} \text{tr}(F^k). \tag{233}
\]

In the third equation, we used (215) and (217). The Chern number of the instanton configuration on $\mathbb{R}^{2k}$ is exactly equal to that of the monopole configuration on $S^{2k}$. Indeed for instance, (199) and (200) yield $c_k = 1$ in (233).

A.4 Equations of motion for the monopole fields and the instanton fields

For the monopole gauge field $\hat{A}_a$ (199), the field strength is obtained from (224):
\[
\hat{F}_{\mu \nu} = -r_\mu \hat{A}_\nu + r_\nu \hat{A}_\mu + \sigma_{\mu \nu}, \quad \hat{F}_{\mu, d+1} = -\hat{F}_{d+1, \mu} = -\sigma_{\mu \nu} r_\nu = (1 + r_{d+1}) \hat{A}_\mu, \tag{234}
\]
where we used
\[
\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + i [\hat{A}_\mu, \hat{A}_\nu] = \sigma_{\mu \nu}, \quad i r_\mu L_{\mu \nu} \hat{A}_\rho = -r_\mu \hat{A}_\nu + r_\nu \hat{A}_\mu, \quad \partial_\mu \hat{A}_{d+1} - \partial_{d+1} \hat{A}_\mu + i [\hat{A}_\mu, \hat{A}_{d+1}] = \hat{A}_\mu, \quad i r_\mu L_{\mu, d+1} \hat{A}_\rho = r_{d+1} \hat{A}_\mu. \tag{235}
\]
(234) is consistent with (199). We can check that the monopole gauge field satisfies the pure Yang-Mills equation on $S^{2k}$:
\[
\hat{D}_a \hat{F}_{ab} \equiv \hat{\partial}_a \hat{F}_{ab} + i [\hat{A}_a, \hat{F}_{ab}] = 0, \tag{236}
\]
where we used
\[
\hat{\partial}_a \hat{F}_{ab} = (2 - d) \hat{A}_b = -i [\hat{A}_a, \hat{F}_{ab}]. \tag{237}
\]
(236) is expected from the previous result (10).

Meanwhile, the instanton configuration (200) satisfies
\[
D_\mu F_{\mu \nu} + \left( \frac{2}{1 + x^2} \right)^2 (4 - 2k) A_\nu = 0, \tag{238}
\]
where
\[
D_\mu F_{\mu \nu} \equiv \frac{\partial}{\partial x_\mu} F_{\mu \nu} + i [A_\mu, F_{\mu \nu}]. \tag{239}
\]
Notice that for the special case $2k = 4$, the second term on the left-hand side of (238) vanishes, and the instanton configuration is a solution of the pure Yang-Mills field equation:
\[
D_\mu F_{\mu \nu}|_{2k=4} = 0, \tag{240}
\]

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but for general $k$, the instanton configuration (200) does not satisfy the pure Yang-Mills equation. Using the explicit expressions, (199) and (200), we can directly show

$$\hat{D}_a \hat{F}_{ab} = \left(\frac{1 + x^2}{2}\right)^2 \frac{K'_{\nu}}{K_{\nu}} \left(\frac{2}{1 + x^2}\right)^2 (4 - 2k) A_{\nu}$$ (241)

or

$$D_\mu F_{\mu\nu} + \left(\frac{2}{1 + x^2}\right)^2 (4 - 2k) A_{\nu} = \left(\frac{2}{1 + x^2}\right)^4 K'_{\nu} \hat{D}_a \hat{F}_{ab}.$$ (242)

Here, we used

$$\hat{D}_a \hat{F}_{ab} = \left(\frac{1 + x^2}{2}\right)^2 \left(\frac{K'_{\nu}}{K_{\nu}} D_\mu F_{\mu\nu} + R_b\right)$$ (243)

with

$$R_a \equiv \frac{\partial}{\partial x_\mu} K'_{\nu} + \frac{2(2 - d)}{1 + x^2} x_\mu K'_{\nu} F_{\mu\nu} = \left(\frac{2}{1 + x^2}\right)^2 (4 - d) K'_{\nu} A_{\nu}.$$ (244)

(241) or (242) implies that

$$\hat{D}_a \hat{F}_{ab} = 0 \leftrightarrow D_\mu F_{\mu\nu} + \left(\frac{2}{1 + x^2}\right)^2 (4 - 2k) A_{\nu} = 0,$$ (245)

which is consistent with the above results, (236) and (238).

## B  $g$ matrices and the $SU(4)$ generalized ’t Hooft symbol

### B.1 Properties of $g$ matrices

$g$ matrices are a higher dimensional counterpart of the quaternions:

$$g_m \equiv (-i\gamma_i, 1_{2k-1}), \quad (m = 1, 2, \cdots, 2k)$$ (246)

and

$$\bar{g}_m \equiv (i\gamma_i, 1_{2k-1}) = g_m^\dagger,$$ (247)

where $\gamma_i$ ($i = 1, 2, \cdots, 2k - 1$) are the $SO(2k - 1)$ gamma matrices:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}.$$ (248)

The $SO(2k + 1)$ gamma matrices, $\Gamma_a$, and $SO(2k + 1)$ matrix generators, $\Sigma_{ab} = -i\frac{1}{4}[\Gamma_a, \Gamma_b]$, are constructed as

$$\Gamma_m = \begin{pmatrix} 0 & \bar{g}_m \\ g_m & 0 \end{pmatrix}, \quad \Gamma_{2k+1} = \begin{pmatrix} 1_{2k-1} & 0 \\ 0 & -1_{2k-1} \end{pmatrix},$$

$$\Sigma_{mn} = \begin{pmatrix} \sigma_{mn} & 0 \\ 0 & \bar{\sigma}_{mn} \end{pmatrix}, \quad \Sigma_m,2k+1 = -\Sigma_{2k+1,m} = i\frac{1}{2} \begin{pmatrix} 0 & \bar{g}_m \\ -g_m & 0 \end{pmatrix}.$$ (249)
where Spin(2k) generators are given by

\[
\sigma_{mn} = -\frac{i}{4}(\bar{g}_m g_n - g_n \bar{g}_m), \quad \bar{\sigma}_{mn} = -\frac{i}{4}(g_m \bar{g}_n - g_n \bar{g}_m). \tag{250}
\]

Several properties of \(g\) matrices are

\[
g_m g_n + g_n g_m = 2\delta_{mn},
\]

\[
g_m \sigma_{np} - \sigma_{np} g_m = -i(\delta_{mm} g_p - \delta_{np} g_m), \quad \bar{g}_m \bar{\sigma}_{np} - \bar{\sigma}_{np} \bar{g}_m = -i(\delta_{mm} \bar{g}_p - \delta_{np} \bar{g}_m). \tag{251b}
\]

B.2 Generalized 't Hooft symbol

B.2.1 The original 't Hooft symbol

The SO(4) gamma matrices and matrix generators are expressed as

\[
\gamma_m = \begin{pmatrix} 0 & \bar{q}_m \\ q_m & 0 \end{pmatrix}, \quad (q_i = -i\sigma_i, \ q_4 = 12)
\]

\[
\Sigma_{mn} = \begin{pmatrix} \sigma_{mn} & 0 \\ 0 & \bar{\sigma}_{mn} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \eta_{mn} \sigma_i & 0 \\ 0 & \bar{\eta}_{mn} \bar{\sigma}_i \end{pmatrix}, \tag{253}
\]

where \(\eta_{mn}\) and \(\bar{\eta}_{mn}\) are the 't Hooft symbols \[77\]:

\[
\eta_{mn} = \epsilon_{mni4} + \delta_m \delta_n 4 - \delta_{m4} \delta_{ni}, \quad \bar{\eta}_{mn} = \epsilon_{mni4} - \delta_m \delta_n 4 + \delta_{m4} \delta_{ni}. \tag{254}
\]

The Pauli matrices are inversely represented as

\[
\sigma_i = \frac{1}{4} \eta_{mn} \sigma_{mn} = \frac{1}{4} \bar{\eta}_{mn} \bar{\sigma}_{mn}. \tag{255}
\]

The Spin(4) matrix generators satisfy the self-dual and the anti-self-dual equations,

\[
\sigma_{mn} = \frac{1}{2} \epsilon_{mnpq} \sigma_{pq}, \quad \bar{\sigma}_{mn} = \frac{1}{2} \epsilon_{mnpq} \bar{\sigma}_{pq}, \tag{256}
\]

and

\[
\sigma_{mn} \sigma_{pq} = \frac{1}{2} (\delta_{mp} \sigma_{nq} - \delta_{mq} \sigma_{np} + \delta_{nq} \sigma_{mp} - \delta_{np} \sigma_{mq}) + \frac{1}{4} (\delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np})12 + \frac{1}{4} \epsilon_{mnpq}12,
\]

\[
\bar{\sigma}_{mn} \bar{\sigma}_{pq} = \frac{1}{2} (\delta_{mp} \bar{\sigma}_{nq} - \delta_{mq} \bar{\sigma}_{np} + \delta_{nq} \bar{\sigma}_{mp} - \delta_{np} \bar{\sigma}_{mq}) + \frac{1}{4} (\delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np})12 - \frac{1}{4} \epsilon_{mnpq}12,
\]

\[
\sigma_{mn} \bar{\sigma}_{mn} = \bar{\sigma}_{mn} \sigma_{mn} = \frac{1}{4} (3 - 3)12 = 0_2. \tag{257}
\]

\[28\] The components of \(\sigma_{mn}\) and \(\bar{\sigma}_{mn}\) are

\[
\sigma_{ij} = \sigma_{ij} = \frac{1}{2} \epsilon_{ijk} \sigma_k, \quad \sigma_{i4} = -\sigma_{i4} = \frac{1}{2} \sigma_i. \quad (i, j = 1, 2, 3) \tag{252}
\]
The above relations are rephrased as the properties of the 't Hooft symbol:

\[ \eta^i_{mn} = \frac{1}{2} \epsilon_{mnpq} \eta^i_{pq}, \]  
\[ \eta^i_{mn} \eta^i_{pq} = \delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np} + \epsilon_{mnpq}, \]  
\[ \epsilon_{ijk} \eta^i_{mn} \eta^j_{op} = \delta_{mp} \eta^k_{nq} - \delta_{mq} \eta^k_{np} - \delta_{np} \eta^k_{mq}, \]  
and

\[ \eta^i_{mn} \eta^j_{mn} = 4 \delta^{ij}, \quad \eta^i_{mn} \eta^i_{np} \eta^k_{pm} = 4 \epsilon^{ijk}. \]  

(258a)

(258b)

(258c)

(259)

Note that \( \epsilon_{ijk} = -\frac{1}{2} \text{tr}(\sigma_i \sigma_j \sigma_k) \) are the structure constants of the SU(2). Except for (258c) and (258b), all relations also hold for \( \bar{\eta}^i_{mn} \):

\[ \bar{\eta}^i_{mn} = -\frac{1}{2} \epsilon_{mnpq} \bar{\eta}^i_{pq}, \]  
\[ \bar{\eta}^i_{mn} \bar{\eta}^i_{pq} = \delta_{mp} \delta_{nq} - \delta_{mq} \delta_{np} - \epsilon_{mnpq}. \]  

(260a)

(260b)

\( \eta^A_{mn} \) and \( \bar{\eta}^B_{mn} \) satisfy

\[ \eta^i_{mn} \bar{\eta}^j_{mn} = 0. \]  

(261)

B.2.2 The SU(4) generalized 't Hooft symbol

The SO(6) gamma matrices are represented as

\[ \Gamma_{m=1,2,\ldots,6} = \begin{pmatrix} 0 & \tilde{g}_m \\ g_m & 0 \end{pmatrix}, \]  

(262)

with

\[ g_m = \{g_{i=1,2,\ldots,5}, g_6\} = \{-i \gamma_i, 1_4\}, \quad \tilde{g}_m = \{\tilde{g}_{i=1,2,\ldots,5}, \tilde{g}_6\} = \{+i \gamma_i, 1_4\}. \]  

(263)

Here, \( \gamma_{i=1,2,3,4,5} \) are the SO(5) gamma matrices; \( \gamma_{i=1,2,3,4} \) and \( \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \). The SO(6) matrix generators, \( \Sigma_{mn} = -i \frac{1}{4} [\Gamma_m, \Gamma_n] \), take the form of

\[ \Sigma_{mn} = \begin{pmatrix} \sigma_{mn} & 0 \\ 0 & \bar{\sigma}_{mn} \end{pmatrix}, \]  

(264)

where \( \sigma_{mn} \) and \( \bar{\sigma}_{mn} \) are the Spin(6) matrix generators:

\[ \sigma_{ij} = \bar{\sigma}_{ij} = -i \frac{1}{4} [\gamma_i, \gamma_j], \quad \sigma_{i6} = -\bar{\sigma}_{i6} = \frac{1}{2} \gamma_i. \]  

(265)

\( \sigma_{mn} \) and \( \bar{\sigma}_{mn} \) satisfy the generalized self-dual and anti-self-dual equations,

\[ \sigma_{mn} = \frac{1}{12} \epsilon_{mnpqr} \sigma_{pq} \sigma_{rs}, \quad \bar{\sigma}_{mn} = -\frac{1}{12} \epsilon_{mnpqr} \bar{\sigma}_{pq} \bar{\sigma}_{rs}, \]  

(266)
and
\[
\sigma_{mn}\sigma_{pq} = \frac{1}{4}\left(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}\right)\lambda_4 + i\frac{1}{2}\left(\delta_{mp}\sigma_{nq} - \delta_{mq}\sigma_{np} + \delta_{nq}\sigma_{mp} - \delta_{np}\sigma_{mq}\right) + \frac{1}{4}\epsilon_{mnpqr}\sigma_{rs},
\]
\[
\bar{\sigma}_{mn}\bar{\sigma}_{pq} = \frac{1}{4}\left(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}\right)\lambda_4 + i\frac{1}{2}\left(\delta_{mp}\bar{\sigma}_{nq} - \delta_{mq}\bar{\sigma}_{np} + \delta_{nq}\bar{\sigma}_{mp} - \delta_{np}\bar{\sigma}_{mq}\right) - \frac{1}{4}\epsilon_{mnpqr}\bar{\sigma}_{rs},
\]
\[
\sigma_{mn}\bar{\sigma}_{mn} = \bar{\sigma}_{mn}\sigma_{mn} = 2\frac{1}{4}(10 - 5)\lambda_4 = \frac{5}{2}\lambda_4.
\]

Since Spin(6) \(\cong SU(4)\), \(\sigma_{mn}\) and \(\bar{\sigma}_{mn}\) can be expressed as linear combinations of the \(SU(4)\) Gell-Mann matrices \[102\] \(\lambda^A\) \((A = 1, 2, \cdots, 15)\):
\[
\sigma_{mn} = \frac{1}{2}\eta^A_{mn}\lambda_A, \quad \bar{\sigma}_{mn} = \frac{1}{2}\bar{\eta}^A_{mn}\lambda_A.
\]

Here, we introduced \(\eta^A_{mn}\) and \(\bar{\eta}^A_{mn}\) as the expansion coefficients, which we refer to as the \(SU(4)\) generalized 't Hooft symbols. The \(SU(4)\) Gell-Mann matrices are inversely represented as
\[
\lambda_A = \frac{1}{4}\eta^A_{mn}\sigma_{mn} = \frac{1}{4}\bar{\eta}^A_{mn}\bar{\sigma}_{mn}.
\]

The \(SU(4)\) Gell-Mann matrices have the following properties
\[
[\lambda_A, \lambda_B] = 2if^{ABC}\lambda_C, \quad \{\lambda_A, \lambda_B\} = \delta^{AB}\lambda_4 + 2d^{ABC}\lambda_C,
\]

or
\[
\lambda_A\lambda_B = \frac{1}{2}\delta^{AB}\lambda_4 + i(f^{ABC} - id^{ABC})\lambda_C,
\]

where \(f^{ABC}\) are the structure constants (totally antisymmetric tensors) and \(d^{ABC}\) are the totally symmetric tensors \[102\]:
\[
f^{ABC} = -i\frac{1}{12}\text{tr}(\lambda_A\lambda_B\lambda_C) = -i\frac{1}{4}\text{tr}[\{\lambda_A, \lambda_B\}\lambda_C],
\]
\[
d^{ABC} = \frac{1}{12}\text{Str}(\lambda_A\lambda_B\lambda_C) = \frac{1}{4}\text{tr}(\{\lambda_A, \lambda_B\}\lambda_C).
\]

From (265), we obtain
\[
\eta^A_{ij} = \bar{\eta}^A_{ij}, \quad \eta^A_{i6} = -\bar{\eta}^A_{i6} = -\bar{\eta}^A_{6i} = \bar{\eta}^A_{6i}.
\]

Substituting (268) into the equations of the Spin(6) matrix generators, one may find properties of the \(SU(4)\) generalized 't Hooft symbol:
\[
\eta^A_{mn} = \frac{1}{24}\epsilon_{mnpqr}d^{ABC}\eta^B_{pq}\eta^C_{rs},
\]
\[
\eta^A_{mn}\eta^A_{pq} = 2(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np}),
\]
\[
(f^{ABC} - id^{ABC})\eta^A_{mn}\eta^B_{pq} = (\delta_{mp}\eta^A_{nq} - \delta_{mq}\eta^A_{np} - \delta_{nq}\eta^A_{mp} - \delta_{np}\eta^A_{mq}) - i\frac{1}{2}\epsilon_{mnpqr}\bar{\eta}^A_{rs},
\]
and
\[
\eta^A_{mn}\eta^B_{mn} = 4\delta^{AB}, \quad \eta^A_{mn}\eta^B_{mp}\eta^C_{nm} = 4f^{ABC}, \quad \epsilon_{mnpqr}\eta^A_{mn}\eta^B_{pq}\eta^C_{rs} = 32d^{ABC}.
\]
Similar relations also hold for $\bar{\eta}^{i}_{mn}$ except for (274a) and (274c):

$$\bar{\eta}^{A}_{mn} = -\frac{1}{24} \epsilon_{mnpqr} d_{ABC} \bar{\eta}^{B}_{pq} \bar{\eta}^{C}_{rs};$$  \hspace{1cm} (276a)

$$(f_{ABC} - id_{ABC}) \bar{\eta}^{B}_{mn} \bar{\eta}^{C}_{pq} = (\delta_{mp} \bar{\eta}^{A}_{nq} - \delta_{mq} \bar{\eta}^{A}_{np} - \delta_{np} \bar{\eta}^{A}_{mq} + \frac{i}{2} \epsilon_{mnpqr} \bar{\eta}^{A}_{rs}).$$ \hspace{1cm} (276b)

The last equation of (267) yields

$$\eta^{A}_{mn} \bar{\eta}^{A}_{mn} = 20, \quad d_{ABC} \eta^{B}_{mn} \bar{\eta}^{C}_{mn} = 0.$$ \hspace{1cm} (277)

C  Tensor gauge field theory

Here, we review tensor gauge field theories in even dimensions mainly based on [18, 21, 23].

C.1  Basic properties of the tensor field

From the following property of the anti-commutator

$$M_1 M_2 M_3 M_4 \cdots M_{2l} = \frac{1}{2^l (2l - 2)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2l}} \{M_{[\mu_1} M_{\mu_2]}, M_{[\mu_3} M_{\mu_4} \cdots M_{\mu_{2l]}},$$ \hspace{1cm} (278)

we have

$$F_{123\ldots2l} = \frac{1}{(2l)!} F_{[12} F_{34} \cdots F_{2l-1,2]}$$

$$= \frac{1}{2 (2l)!} \epsilon_{\mu_1 \mu_2 \mu_3 \cdots \mu_{2l}} \{F_{\mu_1 \mu_2}, F_{\mu_3 \mu_4 \cdots \mu_{2l}}\}$$

$$= \frac{1}{2 (2l - 1)} \left(\{F_{12}, F_{34\ldots2l}\} = \{F_{13}, F_{24\ldots2l}\} + \cdots + \{F_{1,2l}, F_{23\ldots2l-1}\}\right) + \cdots + \{F_{1,2l}, F_{23\ldots2l-1}\}. \hspace{1cm} (279)$$

A covariant fashion of (279) yields

$$F_{\mu_1 \mu_2 \cdots \mu_{2l}} = \frac{1}{(2l)!} F_{[\mu_1 \mu_2} F_{\mu_3 \mu_4} \cdots \mu_{2l-1,2]}$$

$$= \frac{1}{2 (2l - 1)} \sum_{i=2}^{2l} (-1)^i \{F_{\mu_1 \mu_i}, F_{\mu_{2i+1} \cdots \mu_{2l-1}}\} \{F_{\mu_{2i+1}, F_{\mu_2 \mu_4 \cdots \mu_{2l}}\} + \cdots + \{F_{\mu_1, \mu_{2l}}, F_{\mu_2 \mu_3 \cdots \mu_{2l-1}}\}. \hspace{1cm} (280)$$

For instance,

$$F_{\mu\nu} = \frac{1}{2!} F_{[\mu\nu]},$$

$$F_{\mu \nu \rho \sigma} = \frac{1}{4!} F_{[\mu \nu} F_{\rho \sigma]} = \frac{1}{6} \{\{F_{\mu \nu}, F_{\rho \sigma}\} - \{F_{\mu \rho}, F_{\nu \sigma}\} + \{F_{\mu \sigma}, F_{\nu \rho}\}\},$$

$$F_{\mu \nu \rho \sigma \kappa \tau} = \frac{1}{6!} F_{[\mu \nu \rho \sigma} F_{\kappa \tau]} = \frac{1}{10} \{\{F_{\mu \nu}, F_{\rho \sigma \kappa \tau}\} - \{F_{\mu \rho}, F_{\nu \sigma \kappa \tau}\} + \{F_{\mu \sigma}, F_{\nu \rho \kappa \tau}\} - \{F_{\mu \kappa}, F_{\nu \rho \sigma \tau}\} + \{F_{\mu \tau}, F_{\nu \rho \sigma \kappa}\}\}. \hspace{1cm} (281)$$
One may observe that lower rank tensor fields hierarchically constitute higher rank tensor fields. The squares of the four-rank and six-rank tensor field strengths are respectively given by

\[
\begin{align*}
\text{tr}(F_{\mu\nu\rho\sigma}^2) &= \frac{1}{6} \text{tr}((F_{\mu\nu}^2)^2) - \frac{2}{3} \text{tr}(F_{\mu\nu} F_{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}) + \frac{1}{6} \text{tr}((F_{\mu\nu} F_{\rho\sigma})^2), \\
\text{tr}(F_{\mu\rho\sigma\kappa\tau}^2) &= \frac{1}{15} \text{tr}((F_{\mu\rho} F_{\sigma\kappa\tau})^2) - \frac{116}{225} \text{tr}(F_{\mu\rho} F_{\sigma\kappa\tau} F_{\mu\rho} F_{\nu\sigma\kappa\tau}) + \frac{94}{225} \text{tr}(F_{\mu\rho} F_{\sigma\kappa\tau} F_{\rho\sigma} F_{\mu\nu\kappa\tau}).
\end{align*}
\]

C.2 Gauge Symmetry and covariant derivatives

Under the gauge transformation

\[
\begin{align*}
A_\mu \to g(x)^\dagger A_\mu g(x) - i g(x)^\dagger \partial_\mu g(x), \quad (g(x)^\dagger g(x) = 1) \\
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] \to g(x)^\dagger F_{\mu\nu} g(x),
\end{align*}
\]

the tensor field strength \( \text{(280)} \) is transformed as

\[
F_{\mu_1\mu_2\cdots\mu_{2l}} \to g(x)^\dagger F_{\mu_1\mu_2\cdots\mu_{2l}} g(x).
\]

The covariant derivative of the tensor field strength is introduced so as to satisfy

\[
D_\mu F_{\mu_1\mu_2\cdots\mu_{2l}} \to g(x)^\dagger D_\mu F_{\mu_1\mu_2\cdots\mu_{2l}} g(x),
\]

and such a covariant derivative is simply constructed as

\[
D_\mu F_{\mu_1\mu_2\cdots\mu_{2l}} \equiv \partial_\mu F_{\mu_1\mu_2\cdots\mu_{2l}} + i [A_\mu, F_{\mu_1\mu_2\cdots\mu_{2l}}].
\]

One may easily check that \( \text{(286)} \) transforms as \( \text{(285)} \) under \( \text{(283a)} \) and \( \text{(284)} \). Note that the covariant derivative linearly acts to the original constituent 2-rank field strength of the tensor field strength. For instance,

\[
D_\mu F_{\nu\rho\sigma\tau} = \frac{1}{4!} (D_\mu F_{[\nu\rho}, F_{\sigma\tau]} + F_{[\nu\rho} : D_\mu F_{\sigma\tau]}),
\]

where index \( \mu \) in the second term is not included in the antisymmetrization.

C.3 Bianchi Identity and equations of motion

The original Bianchi identity

\[
D_{[\mu} F_{\rho]\sigma] = 0,
\]

is readily verified from the definition of the field strength, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu] \). For tensor field strength, \( \text{(288)} \) is generalized as

\[
D_{[\mu} F_{\mu_1\mu_2\cdots\mu_{2l}]} = 0.
\]

\footnote{\( \text{(282a)} \) was utilized in 8D tensor gauge theory of \[39\] to realize a 7(+1)D Skyrmion from the Atiyah-Manton construction.}
One may easily verify (289) with the linearity of the covariant derivative (287) and the original Bianchi identity (288).

We introduce the (Euclidean) tensor field theory action as

$$ S = \frac{1}{4} \int d^d x \, \text{tr} \left( F_{\mu_1 \mu_2 \cdots \mu_2l}^2 \right). $$

Since tensor field strength is originally made of the field strength, we should take a variation of $S$ with respect to $A_\mu$ to derive equations of motion:

$$ \frac{\delta}{\delta A_\nu} S = -D_\mu G_{\mu \nu} = 0, \quad (291) $$

where

$$ G_{\mu_1 \mu_2} \equiv \sum_{p=1}^{k} F_{\mu_3 \mu_4 \cdots \mu_2p} F_{\mu_1 \mu_2 \cdots \mu_2l} F_{\mu_2p+1 \mu_2p+2 \cdots \mu_2l} $$

$$ = F_{\mu_1 \cdots \mu_2l} F_{\mu_3 \cdots \mu_2l} + F_{\mu_3 \mu_4} F_{\mu_1 \cdots \mu_2l} F_{\mu_5 \cdots \mu_2l} + F_{\mu_3 \mu_4 \mu_5 \mu_6} F_{\mu_1 \cdots \mu_2l} F_{\mu_7 \cdots \mu_2l} + \cdots + F_{\mu_3 \cdots \mu_2l} F_{\mu_1 \cdots \mu_2l}. $$

For instance,

$$ l = 1 : G_{\mu \nu} = F_{\mu \nu}, $$

$$ l = 2 : G_{\mu \nu} = F_{\mu \rho \sigma} F_{\rho \sigma} + F_{\rho \sigma} F_{\mu \rho \sigma} = \{ F_{\mu \rho \sigma}, F_{\rho \sigma} \}, $$

$$ l = 3 : G_{\mu \nu} = F_{\mu \rho \sigma \kappa \tau} F_{\rho \sigma \kappa \tau} + F_{\rho \sigma} F_{\mu \rho \sigma \kappa \tau} F_{\rho \sigma \kappa \tau} + F_{\rho \sigma} F_{\mu \rho \sigma \kappa \tau} F_{\rho \sigma \kappa \tau}. $$

From the Bianchi identity (289) and the linearity of the covariant derivative (287), we have

$$ D_{\mu_1} G_{\mu_1 \mu_2} = \sum_{p=1}^{k} F_{\mu_3 \mu_4 \cdots \mu_2p} (D_{\mu_1} F_{\mu_1 \mu_2 \cdots \mu_2l}) F_{\mu_2p+1 \mu_2p+2 \cdots \mu_2l} $$

$$ = (D_{\mu_1} F_{\mu_1 \cdots \mu_2l}) F_{\mu_3 \cdots \mu_2l} + F_{\mu_3 \mu_4} (D_{\mu_1} F_{\mu_1 \cdots \mu_2l}) F_{\mu_5 \cdots \mu_2l} $$

$$ + F_{\mu_3 \mu_4 \mu_5 \mu_6} (D_{\mu_1} F_{\mu_1 \cdots \mu_2l}) F_{\mu_7 \cdots \mu_2l} + \cdots + F_{\mu_3 \cdots \mu_2l} (D_{\mu_1} F_{\mu_1 \cdots \mu_2l}), $$

which implies

$$ D_{\mu_1} F_{\mu_1 \mu_2 \mu_3 \cdots \mu_2l} = 0 \rightarrow D_{\mu_1} G_{\mu_1 \mu_2} = 0. \quad (295) $$

### C.4 Self-dual equations

The tensor field Bianchi identity (289) can be expressed as

$$ D_{\mu_1} \tilde{F}_{\mu_1 \mu_2 \cdots \mu_2l} = 0, \quad (296) $$

where

$$ \tilde{F}_{\mu_1 \mu_2 \cdots \mu_2l} \equiv \frac{1}{(2k-2l)!} \epsilon_{\mu_1 \mu_2 \cdots \mu_{2k}} F_{\mu_1 \mu_2 \cdots \mu_{2k-2l}}, \quad (297) $$
For \( l = k/2 \) (\( k \): even), the self-dual equation is given by
\[
\tilde{F}_{\mu_1\mu_2\cdots\mu_{2l}} = F_{\mu_1\mu_2\cdots\mu_{2l}}. \tag{298}
\]
When (298) holds, its dual equation automatically follows:
\[
\tilde{F}_{\mu_1\mu_2\cdots\mu_{2k-2l}} = F_{\mu_1\mu_2\cdots\mu_{2k-2l}}., \tag{299}
\]
and then there are \([k/2]\) independent self-dual equations in 2\( k \)D. In low dimensions, the independent self-dual equations are
\[
\begin{align*}
k &= 2 : & \tilde{F}_{\mu\nu} &= F_{\mu\nu}, \\
k &= 3 : & \tilde{F}_{\mu\nu} &= F_{\mu\nu}, \\
k &= 4 : & \tilde{F}_{\mu\nu} &= F_{\mu\nu}, & \tilde{F}_{\mu\nu\rho\sigma} &= F_{\mu\nu\rho\sigma}. \tag{300}
\end{align*}
\]
The self-dual tensor field satisfies
\[
D_{\mu_1} F_{\mu_1\mu_2\cdots\mu_{2l}} = D_{\mu_1} \tilde{F}_{\mu_1\mu_2\cdots\mu_{2l}} = 0. \tag{301}
\]
From (295), the self-dual tensor field realizes a solution of the equations of motion (291).

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