COMPLEX DYNAMICS IN A QUASI-PERIODIC PLASMA PERTURBATIONS MODEL

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(Communicated by Miguel Sanjuan)

ABSTRACT. In this paper, the complex dynamics of a quasi-periodic plasma perturbations (QPP) model, which governs the interplay between a driver associated with pressure gradient and relaxation of instability due to magnetic field perturbations in Tokamaks, are studied. The model consists of three coupled ordinary differential equations (ODEs) and contains three parameters. This paper consists of three parts: (1) We study the stability and bifurcations of the QPP model, which gives the theoretical interpretation of various types of oscillations observed in [Phys. Plasmas, 18(2011):1-7]. In particular, assuming that there exists a finite time lag $\tau$ between the plasma pressure gradient and the speed of the magnetic field, we also study the delay effect in the QPP model from the point of view of Hopf bifurcation. (2) We provide some numerical indices for identifying chaotic properties of the QPP system, which shows that the QPP model has chaotic behaviors for a wide range of parameters. Then we prove that the QPP model is not rationally integrable in an extended Liouville sense for almost all parameter values, which may help us distinguish values of parameters for which the QPP model is integrable. (3) To understand the asymptotic behavior of the orbits for the QPP model, we also provide a complete description of its dynamical behavior at infinity by the Poincaré compactification method. Our results show that the input power $h$ and the relaxation of the instability $\delta$ do not affect the global dynamics at infinity of the QPP model and the heat diffusion coefficient $\eta$ just yield quantitative, but not qualitative changes for the global dynamics at infinity of the QPP model.

1. Introduction. The Tokamak is a toroidal fusion plasma physics device which uses magnetic confinement to realize controlled nuclear fusion. Its name Tokamak comes from toroidal, kamera, magnet and kotushka, which was first invented by Artsimovich et. al in 1954 [1, 2]. Plasma is a form of matter, which mainly consisting of free electrons and charged ions. The basic principle of nuclear fusion in the Tokamak device is as follows. Under the high constraint mode of Tokamak, the region at the plasma edge usually has a relatively steep pressure distribution and a

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high current peak value. When a certain threshold value is reached, the instability of edge local modes (ELMs) will be triggered, and the plasma at the plasma edge will collapse. Some particles and energy will be released to make the pressure gradient or current peak value lower than the threshold value. With the injection of external heating power, the pressure gradient and current peak value will gradually recover until trigger the next burst of the instability for ELMs, and so on in a circle [3, 4].

In the past few decades, Tokamak plasma physics has been widely studied in various research fields such as physical experiment research and engineering technology applications. Especially, the dynamical behaviors of interaction between drive and relaxation processes in a deterministic or perturbed toroidal Tokamaks magnetic field have been investigated by many scholars [5, 6, 7].

In 2011, Constantinescu et. al [8] provided a minimal possible nonlinear model for quasi-periodic plasma perturbations to describe drive and relaxation processes under the simplest and general assumptions. The model can lead to behavior that is qualitatively consistent with the experiments for sawteeth or ELMs. To motivate this plasma perturbation model to be studied, we give a brief account of the modeling. The model consists of two equations. The first one is due to magnetohydrodynamics (MHD) force balance and the second one is due to the energy balance. By the standard linear MHD, we have the linearized force balance

\[ \rho_m \frac{d^2 \xi}{dt^2} = -K \cdot \xi, \]

where the linear operator \( K = K(\nabla p, \nabla j) \) describes the driving or damping terms, and \( \xi \) is the amplitude of the displacement of the magnetic field, \( \nabla p \) is the pressure gradient and \( \nabla j \) is the current gradient. For simplicity, one can assume that either the pressure gradient or the current gradient is dominant (for instance, \( K = K(\nabla p) \) for ELMs and \( K = K(\nabla p) \) for sawteeth). In the one-dimensional case, we take a simple function for \( K(p') \) and rewrite (1) into

\[ \frac{d^2 \xi}{dt^2} = -\gamma_0^2 \left( 1 - \frac{p'}{p'_{\text{crit}}} \right) \xi, \]

where \( \gamma_0 \) is the growth rate of the instability and \( p'_{\text{crit}} \) is the critical point for marginal stability. In addition, there exists two typical dissipation mechanisms: viscosity as a dissipation mechanism or resistivity as a dissipation mechanism. It is shown that the general force in two both cases can has the following form:

\[ F = \delta^* \frac{d \xi}{dt}, \]

with \( \delta^* \) accounting for the two dissipation mechanisms [8]. In a word, we get the first equation for the pressure gradient and the magnetic field

\[ \frac{d^2 \xi}{dt^2} + \gamma_0^2 \left( 1 - \frac{p'}{p'_{\text{crit}}} \right) \xi + \delta^* \frac{d \xi}{dt} = 0, \]

or in dimensionless form

\[ \frac{d^2 \xi_n}{dt_n^2} + (1 - p'_n) \xi_n + \delta \frac{d \xi_n}{dt_n} \xi_n = 0, \]

with \( p'_n = p'/p'_{\text{crit}}, \xi_n = \xi/\xi_{\text{unit}} \) (\( \xi_{\text{unit}} = 1m \)), \( t_n = \gamma_0 t \) and \( \delta = \delta^*/\gamma_0 \). Next, we consider the toroidal skin region with major radius \( R \) and minor radius \( a \). It follows
from the power balance that
\[ 2\pi Ra^2 l_{rad} \frac{d}{dt} p' = H_{tot} - 4\pi^2 Ra p' \chi_0 - 4\pi^2 Ra p' \chi_{anom} \xi_n^2. \] (4)

Here, the first term on the right-hand side is the total input power, the second term is the regular loss term without magnetic perturbation, and the third term is the additional loss term due to the magnetic perturbation. In dimensionless form, (4) becomes
\[ \frac{d}{dt} n_p' \eta = \left( h - p'_n - p'_n \chi_{anom} \chi_0 \xi_n^2 \right) \] (5)
with \( \eta = 2\chi_0/(\gamma_0 a l_{rad}) \) and \( h = H_{tot}/(4\pi^2 Ra p'_{crit} \chi_0) \). In short, the plasma perturbation model consists of the equation (3) and (5) with respect to the amplitude of the magnetic field displacement \( \xi_n \) and the plasma pressure gradient \( p'_n \), which contains only three parameters: the power input \( h \), the relaxation of the instability \( \delta \), the influence of the instability on the heat diffusion coefficient \( \eta \).

By introducing the variables \( y = \sqrt{\chi_{anom}/\chi_0} \xi_n \) and \( z = p'_n \), equations (3) and (5) could be transformed into a system of three coupled first order differential equations,
\[
\begin{align*}
\frac{dx}{dt} &= (z - 1)y - \delta x, \\
\frac{dy}{dt} &= x, \\
\frac{dz}{dt} &= \eta(h - z - y^2 z),
\end{align*}
\] (6)
called the QPP model.

In spite of its simplicity, the model has rich variety of dynamics depending on the model parameters. By numerical methods, Constantinescu et. al [8] showed that the plasma instability cannot occur in case the input power \( h \) is small, saying less than one, and the plasma perturbation model may admit various types of oscillations in case that the input power \( h \) is large, saying more than one. These oscillations include damped oscillations, double periodic oscillations, chaotic oscillations and ELM/sawtooth oscillations. The aim of this work is to give some new insights into the understanding of the complex dynamics of QPP model, including providing the theoretical interpretation of numerical results in [8].

In Section 2, we focus on the stability and bifurcations of QPP model, especially the effect of time delay on the QPP model. The stability and Hopf bifurcation of system (6) have been studied by Elsadany et. al [9]. In fact, time delay plays an essential role for the occurrence of Hopf bifurcation. To show that, we propose a delayed QPP model due to negative feedback, consider the existence of Hopf bifurcation (Theorem 2.3), and derive the explicit formulas for determining the stability and direction of periodic solutions (Theorem 2.4). In addition, some numerical simulations and conclusions are given to illustrate the theoretical predictions.

In Section 3, we turn to understand the complexity and topological structure of the dynamics of the QPP model from the point of integrability. For a three dimensional system of ODEs with parameters, it may have a first integral for some values of parameters, which implies it admits the regular and predictable dynamics, that is, non-chaotic dynamics. It may also be non-integrable for other value of parameters. In this case, this system may have chaotic dynamics such as strange attractors, positive Lyapunov exponents or metric entropy. The fact that the QPP model admits chaotic behaviors for a large range of its parameters [9] leads us to expect it is non-integrable. By analyzing the property of differential Galois group
along a straight line solution, we prove its rational non-integrability in an extended Liouville sense for almost all parameter values (Theorem 3.1).

In Section 4, we contribute to investigating the global dynamics of the QPP model by studying its behavior at infinity. Since the QPP model is a polynomial vector field in \( \mathbb{R}^3 \), by the Poincaré compactification one can extend the QPP model into an analytic system defined on a closed ball of radius one, called the Poincaré ball, whose interior is diffeomorphic to \( \mathbb{R}^3 \) and its invariant boundary, the two-dimensional spherical shell \( S^2 = \{(x, y, z)|x^2 + y^2 + z^2 = 1\} \) plays the role of the infinity, called the Poincaré sphere [10], [11], [12], [13]. Using this compactification technique, we give a complete description of the dynamical behavior of the QPP model on the sphere at infinity (Theorem 4.1). Lastly, we give the conclusions and Appendix (the proof of Proposition 3 and proof of Theorem 2.3) in Section 5 and 6, respectively.

2. Stability and bifurcations in the QPP model. When the input heating power is small than one, i.e., \( h < 1 \), the QPP model has only one trivial equilibrium state \( E_0 = (0, 0, h) \). By Routh-Hurwitz stability criterion [14], one can easily obtain the following result.

**Proposition 1.** If \( h < 1 \), then the trivial equilibrium state \( E_0 = (0, 0, h) \) of system (6) is asymptotically stable.

Proposition 1 shows that if the input power is small the amplitude of the magnetic field displacement will ultimately arrives at zero and the plasma instability cannot occur.

There are two possibilities for the equilibrium state to loss its stability: one is that a zero eigenvalue is created and the other is a pair of pure imaginary eigenvalues. When the input power is one, \( E_0 \) has a zero eigenvalue and the pitchfork bifurcation occurs. Hence, when the input power is large than one, system (6) admits two new equilibrium state \( E_+ = (0, \sqrt{h-1}, 1) \) and \( E_- = (0, -\sqrt{h-1}, 1) \). By Routh-Hurwitz stability criterion [14] again, one can get the stability of \( E_+ \) and \( E_- \).

**Proposition 2.** Assume \( h > 1 \) and \( \delta h(\delta + h\eta) > 2(h - 1) \). Then the equilibrium \( E_+ = (0, \sqrt{h-1}, 1) \) of system (6) is asymptotically stable.

Proposition 2 shows that system (6) will ultimately arrives at the equilibrium state \( E_+ \) when the initial value \((x_0, y_0, z_0)\) lies in the domain of attraction of \( E_+ \) under some conditions. For instance, setting \((h, \delta, \eta) = (1.5, 0.5, 0.668)\), Constantinescu et. al [8] observed the damped oscillations with the initial value \((x_0, y_0, z_0) = (0, 1.70, 1.01)\).

When \( h > 1 \) and \( \delta \eta h^2 + (\delta^2 - 2)h + 2 = 0 \), the equilibriums state \( E_+ \) has a pair of pure imaginary eigenvalues and loss their stability. Furthermore, taking the parameter \( \eta \) as the bifurcation parameter, one can prove the existence of codimension one Hopf bifurcations and get the direction, stability and period of bifurcating periodic solutions for the QPP system (6) as follows.

**Proposition 3.** Let \( h > 1 \) and \( \eta_0 = (2h - 2 - \delta^2 h)/(\delta h^2) > 0 \). Then the following statements hold for the QPP model.

(a) As the parameter \( \eta \) passes through the critical \( \eta_0 \), the QPP model undergoes a Hopf bifurcation at the equilibrium \( E_+ = (0, \sqrt{h-1}, 1) \).

(b) Bifurcating periodic solutions exist for sufficient small \( \eta - \eta_0 < 0 \). Moreover, the period solutions of the QPP model from Hopf bifurcation at \( E_+ \) are non-degenerate,
subcritical and orbitally unstable.
(c) The period and characteristic exponent of the bifurcating periodic solution are:
\[ T = \frac{2\pi}{\omega_0} \left( 1 + \tau_2 \varepsilon^2 + o(\varepsilon^4) \right), \quad \beta = \beta_2 \varepsilon^2 + o(\varepsilon^4), \]
where
\[ \omega_0 = \frac{\sqrt{h(2h - 2 - \delta^2h)}}{h}, \quad \tau_2 = -\frac{\text{Im} C_1(0) + \mu_2 \omega'(0)}{\omega_0}, \]
\[ \beta_2 = 2 \text{Re} C_1(0), \quad \varepsilon^2 = \frac{\eta - \eta_0}{\mu_2} + o \left( (\eta - \eta_0)^2 \right), \quad \mu_2 = -\frac{\text{Re} C_1(0)}{\alpha'(0)}. \]

For the proof of Proposition 3 and the notations of \( \mu_2, \beta_2 \) and \( T_2 \) in details, please see Appendix.

Note that system (6) is invariant under the transformation
\[(x, y, z) \rightarrow (-x, -y, z),\]
that is, if \((x(t), y(t), z(t))\) is a solution of system (6), then \((-x(t), -y(t), z(t))\) is also a solution of system (6). This implies dynamics in the neighbourhood of a point \((x, y, z)\) looks the same as that of a point \((-x, -y, z)\). Therefore, due to the symmetry with respect to the \(z\)-axis of the system (6), it is sufficient to study the existence of local Hopf bifurcations occurring at \(E_+\) and Proposition 3 is also true for \(E_- = (0, -\sqrt{h - 1}, 1)\). Hence, we obtain two period solutions of the QPP model from Hopf bifurcation at \(E_+\) and \(E_-\), respectively, which corresponds to the double periodic oscillations observed in [8].

2.1. Hopf Bifurcation of the QPP model with delayed feedback. In this section, we mainly discuss the effect of negative delayed feedback on the QPP model. As a first step, we consider the plasma pressure gradient at the plasma edge has negative feedback for the velocity of the magnetic field, that is, \(z(t)\) has delay feedback to the \(x(t)\) term, the system is expressed as follows
\[
\frac{dx}{dt} = (z(t - \tau) - 1)y - \delta x, \\
\frac{dy}{dt} = x, \\
\frac{dz}{dt} = \eta(h - z - y^2z),
\]
where \(\tau\) is a positive constant time delay term. We will investigate the stability of equilibrium for system (7), by taking the time delay \(\tau\) as the bifurcation parameter. Our results show that when \(\tau\) passes through the critical values \(\tau'_k \) (\(k = 1, 2\) and \(j = 0, 1, 2, \cdots\)), the equilibrium loses its stability and the Hopf bifurcation occurs.

2.2. Stability analysis. System (7) is rotationally symmetric under the transformation \((x, y, z) \rightarrow (-x, -y, z)\), which is symmetry with respect to the \(z\)-axis. When \(h > 1\), system (7) has three equilibrium \(E = (0, 0, h), E_- = (0, -\sqrt{h - 1}, 1)\) and \(E_+ = (0, \sqrt{h - 1}, 1)\). Due to the symmetry of the system, it is sufficient to study the existence of local Hopf bifurcations occurring at \(E_+\).

We translate the equilibrium \(E_+ = (0, \sqrt{h - 1}, 1)\) to the origin via the coordinate shift \(x_1(t) = x(t), x_2(t) = y(t) - \sqrt{h - 1}, x_3(t) = z(t) - 1, x_3(t - \tau) = z(t - \tau) - 1\), so that system (7) becomes
\[
\begin{align*}
\dot{x}_1(t) &= x_3(t - \tau)(x_2 + \sqrt{h - 1}) - \delta x_1, \\
\dot{x}_2(t) &= x_1, \\
\dot{x}_3(t) &= \eta \left[ h - x_3 - 1 - (x_2 + \sqrt{h - 1})^2(x_3 + 1) \right],
\end{align*}
\]
the linearization of system (8) is

\[
\begin{align*}
\dot{x}_1(t) &= -\delta x_1 + \sqrt{h - 1} x_2(t - \tau), \\
\dot{x}_2(t) &= x_1, \\
\dot{x}_3(t) &= \eta(-2\sqrt{h - 1} x_2 - h x_3),
\end{align*}
\]  

(9)

and the associated characteristic equation of (9) is

\[
\begin{align*}
\begin{vmatrix}
\lambda + \delta & 0 & -\sqrt{h - 1}e^{-\lambda \tau} \\
-1 & \lambda & 0 \\
0 & 2\eta \sqrt{h - 1} & \lambda + \eta h
\end{vmatrix} &= 0,
\end{align*}
\]

i.e.

\[
\lambda^3 + (\delta + \eta h)\lambda^2 + \delta \eta h\lambda + 2(h - 1)\eta e^{-\lambda \tau} = 0. 
\]  

(10)

When \( \tau = 0 \), Eq.(10) becomes

\[
\lambda^3 + (\delta + \eta h)\lambda^2 + \delta \eta h\lambda + 2(h - 1)\eta = 0.
\]

If a pair of complex roots with negative real parts and non-zero imaginary parts cross the imaginary axis as the time delay \( \tau \) increases, this potentially results in Hopf bifurcation and the equilibrium \( E_+ \) loses stability. The following we discuss the existence of a local Hopf bifurcation occurring at \( E_+ \). If \( i\omega(\omega > 0) \) is a root of Eq.(10), then

\[
-i\omega^3 - (\delta + \eta h)\omega^2 + i\omega \delta \eta h + 2(h - 1)\eta(\cos \omega \tau - i \sin \omega \tau) = 0.
\]

Separating the real and imaginary parts, we obtain

\[
\begin{align*}
\sin(\omega \tau) &= \frac{\omega \delta \eta h - \omega^2}{2\eta(h - 1)^2}, \\
\cos(\omega \tau) &= \frac{\omega^2(h \eta + \delta)}{2\eta(h - 1)^2}.
\end{align*}
\]  

(11)

By \( \sin^2(\omega \tau) + \cos^2(\omega \tau) = 1 \), (11) is equivalent to

\[
\omega^6 + (h^2 \eta^2 + \delta^2) \omega^4 + \delta^2 h^2 \eta^2 \omega^2 - 4\eta^2(h - 1)^2 = 0. 
\]  

(12)

Let \( z = \omega^2 \), Eq.(12) becomes

\[
f(z) = z^3 + (h^2 \eta^2 + \delta^2) z^2 + \delta^2 h^2 \eta^2 z - 4\eta^2(h - 1)^2 = 0. 
\]  

(13)

Then, \( f'(z) = 3z^2 + 2(h^2 \eta^2 + \delta^2)z + \delta^2 h^2 \eta^2 > 0 \), the equation \( f'(z) = 0 \) has two real negative roots

\[
z_1^* = \frac{-2(h^2 \eta^2 + \delta^2) + \sqrt{\Delta}}{6}, \quad z_2^* = \frac{-2(h^2 \eta^2 + \delta^2) - \sqrt{\Delta}}{6}. 
\]

In addition, \( h(0) < 0, \lim_{z \to +\infty} h(z) = +\infty \), hence Eq.(13) has one positive real root \( z^* \), and Eq.(12) has two real roots \( \omega_1 = -\sqrt{z^*}, \omega_2 = \sqrt{z^*} \). Substituting \( \omega_k(k = 1, 2) \) into (11), we have

\[
\tau_k^j = \frac{1}{\omega_k} \left[ \arccos \left( \frac{\omega_k^2(h \eta + \delta)}{2\eta(h - 1)} \right) + 2j\pi \right],
\]

where \( k = 1, 2 \) and \( j = 0, 1, 2, \ldots \), then \( \omega_k(k = 1, 2) \) is a pair of pure imaginary roots of Eq.(10) with \( \tau_k^j \). Set

\[
\tau_0 = \tau_0^{(0)} = \min_{k \in \{1, 2\}} \{ \tau_k^{(0)} \}.
\]

**Lemma 2.1.** When \( \tau = \tau_k^j(k = 1, 2 \) and \( j = 0, 1, 2, \ldots \)), then (10) has a pair of pure imaginary roots \( \omega_k \), and all other roots of (10) have nonzero real parts.
Let $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ be the roots of (10) near $\tau = \tau_k^j$ satisfying $\alpha(\tau_k^j) = 0, \omega(\tau_k^j) = \omega_k (j = 0, 1, 2, \cdots)$. By the theory of functional differential equation [15], for $\forall \tau \in \tau_k^j$, $\exists \varepsilon > 0$ such that $\lambda(\tau) \in |\tau - \tau_k^j| < \varepsilon$ about $\tau$ is continuous and differentiable.

**Lemma 2.2.** The transversality condition

$$\left[ \frac{d(\Re(\lambda(\tau)))}{d\tau} \right]_{\tau = \tau_k^j} > 0.$$

**Proof.** Substituting $\lambda(\tau)$ into Eq. (10) and taking the derivative with respect to $\tau$, we have

$$\left( \frac{d\lambda(\tau)}{d\tau} \right)^{-1} = \frac{[3\lambda^2 + 2\lambda(\delta + h\eta) + \delta h\eta]e^{\lambda\tau}}{2\eta(h - 1)\lambda} - \frac{\tau}{\lambda},$$

it follows from (11) that

$$\left[ \frac{d(\Re(\lambda(\tau)))}{d\tau} \right]_{\tau = \tau_k^j} = \Re\left[ \frac{[3\lambda^2 + 2\lambda(\delta + h\eta) + \delta h\eta]e^{\lambda\tau}}{2\eta(h - 1)\lambda} - \frac{\tau}{\lambda} \right]_{\tau = \tau_k^j}$$

$$= \Re\left[ \frac{-3\omega^2 + 2i\omega(\delta + h\eta) + \delta h\eta(\cos\omega\tau + i\sin\omega\tau)}{2i\omega\eta(h - 1)} \right]_{\tau = \tau_k^j}$$

$$= \frac{\eta^2h^2\delta^2 + 3\omega^4 + 2(\delta^2 + h^2\eta^2)\omega^2}{4\eta^2(h - 1)^2} > 0.$$

This completes the proof. \hfill \Box

By Lemma 2.1 and 2.2, we have the following theorem which are due to Beretta and Kuang [16].

**Theorem 2.3 (Existence of Hopf Bifurcation).** Assume that $(H_1) \delta h(\delta + h\eta) > 2(h - 1)$ holds.

$$\tau_k^j = \frac{1}{\omega_k} \left\{ \arccos \left( \frac{\omega_k^2(h\eta + \delta)}{2\eta(h - 1)} \right) + 2j\pi \right\}$$

are Hopf bifurcation critical values at $E_+$, where $i\omega_k (k = 1, 2)$ are the roots of (10). Furthermore, $E_+$ is locally asymptotically stable for $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$.

The explicit formulae determining the stability and the direction of periodic solutions bifurcating from Hopf bifurcations are necessary studied, we will give them as follows.

**Theorem 2.4.** If $\mu_2 > 0 \ (\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical); If $\beta_2 < 0 \ (\beta_2 > 0)$, then bifurcating periodic solution is stable (unstable); If $T_2 > 0 \ (T_2 < 0)$, then periods of periodic solutions increase (decrease).

For the proof of Theorem 2.4 and the notations of $\mu_2$, $\beta_2$ and $T_2$, please see Appendix.

Our research results reveal a classical Hopf bifurcation in the studied system, and play an important role for the better understanding of the complex dynamics of QPP model subject to time delay. Finally, we mention that when the delay is continuous and modeled by a convolution, the problem on the periodic phenomenon can be restricted on the critical manifold, the limit cycle can be detected by the zeros of Melnikov function, see [17], [18], [19].
2.3. Numerical simulations. In this part, we shall give some numerical simulations to support the theoretical analysis discussed in the previous part. We study the following specific model

\[
\begin{align*}
\frac{dx}{dt} &= (z(t - \tau) - 1)y - x, \\
\frac{dy}{dt} &= x, \\
\frac{dz}{dt} &= 3(2 - z - y^2z),
\end{align*}
\]

with initial values \((x(t), y(t), z(t)) = (0.1, 0.1, 0.1)\). The parameter values we chose satisfies \((H_1)\). By computing, \(E_+ = (0.1, 1, 1)\) and \(h(z) = z^3 + 17z^2 + 144z - 16\) has only one positive root \(z \approx 0.6106\), we get \(\omega_1 \approx -0.7814, \omega_2 \approx 0.7814, \tau_0 \approx 0.9955\). By Theorem 2.3, we get that \(E_+\) is asymptotically stable when \(0 \leq \tau < \tau_0\) as Figure 1 illustrates, and \(E_+\) is unstable when \(\tau > \tau_0\) as shown by Figure 2.

![Figure 1](image1.png)

**Figure 1.** The equilibrium \(E_+ = (0, 1, 1)\) is asymptotically for system (7) with \(\tau = 0.9 < \tau_0\).

In Figure 3, the blue solid line represents the stable equilibrium point, the red dotted line is an unstable equilibrium point, the filled green circle represents a stable periodic orbit, and open blue circles are unstable periodic orbits.
The approximate frequency is 0.7814, that is the approximate period is 8.0409. The critical value of the supercritical Hopf bifurcation is \( \tau_0 \approx 0.9955 \) and bifurcates the stable periodic orbit, the first subcritical Hopf bifurcation critical value is \( \tau \approx 9.0369 \) and the second subcritical Hopf bifurcation critical value is \( \tau \approx 17.0780 \), which bifurcate the unstable periodic orbit as shown by Figure 3.

By Theorem 2.3 and 2.4, from the physical aspect, the most interesting results are the following: if the plasma pressure gradient at the plasma edge has shorter negative feedback for the velocity of the magnetic field, the amplitude of the magnetic field displacement and the plasma pressure gradient at the plasma edge are at the fixed level, the system is stable and will not trigger the edge localized mode instability explosion. As the delay increased, they asymptotically vary in a periodic or in a quasi-periodic way which implied the edge localized mode instability explosion periodicity or quasi-periodicity. Moreover, the short term data observed in experiments may be misleading to make predictions dues to the complex dynamical behaviors. Our results might be useful for the physicists who work with Tokamaks plasmas.
2.4. Canard solution of the QPP model as a singularly perturbed system. Constantinescu et al. [8] observed that when \( \eta \) is small, the oscillation of the QPP model admits a long rise time and a short crash time, called the ELMs/sawtooth oscillation. This numerical result implies the QPP model has a separation of time scales. Indeed, we set \( \eta = \varepsilon \ll 1 \) and rewrite the QPP model as a fast-slow system

\[
\begin{align*}
\dot{x} &= (z-1)y - \delta x = g_1(x, y, z), \\
\dot{y} &= x = g_2(x, y, z), \\
\dot{z} &= \varepsilon(h - z - y^2z = \varepsilon f_1(x, y, z),
\end{align*}
\]  

(15)

with \((x, y)\) being fast variables and \(z\) being slow fast variable and the dot denotes the derivation with respect to \(t\). We can change (15) from the fast time scale \(t\) to the slow time scale \(\tau = \varepsilon t\), yielding

\[
\begin{align*}
\varepsilon \frac{dx}{d\tau} &= g_1(x, y, z), \\
\varepsilon \frac{dy}{d\tau} &= g_2(x, y, z), \\
\frac{dz}{d\tau} &= f_1(x, y, z).
\end{align*}
\]  

(16)

Let us mention that the fast system (15) and the slow system (16) are equivalent if \(\varepsilon \neq 0\). However, insight can be gained by looking at the \(\varepsilon = 0\) limit. As \(\varepsilon \to 0\),

\textbf{Figure 3.} Bifurcation diagram at \(E_+\) in \((\tau, x)\), \((\tau, y)\) and \((\tau, z)\) space, respectively.
system (15) approaches the so-called layer system
\[
\begin{align*}
\frac{dx}{dt} &= g_1(x, y, z), \\
\frac{dy}{dt} &= g_2(x, y, z), \\
\frac{dz}{dt} &= 0,
\end{align*}
\]
which is a complete integrable system (see Proposition 5 in Section 3), and system (16) approaches the so-called reduced system
\[
\begin{align*}
0 &= g_1(x, y, z), \\
0 &= g_2(x, y, z), \\
\frac{dz}{d\tau} &= f_1(x, y, z),
\end{align*}
\]
which is a differential-algebraic system. The critical manifold
\[
M_0 = \{(x, y, z) : g_1 = g_2 = 0\} = \{(x, y, z) : x = 0, y(z - 1) = 0\}
\]
is special since it is the set of equilibrium points of (17) and the algebraic set on which (18) is defined. By Fenichel theory [20], if \(\varepsilon\) is sufficiently small, then there exists an invariant manifold \(M_\varepsilon\) closed to \(M_0\) under the flow of system (16), and there also exits the local stable invariant manifold \(M_s\) and unstable invariant manifold \(M_u\) for the slow invariant manifold \(M_\varepsilon\).

In this subsection, we will show the existence of canard solutions for the QPP model by analyzing the stability of pseudo singular points of the normalized slow dynamics. Recall that a canard is a solution of a singularly perturbed dynamical system (16) following the stable invariant manifold \(M_s\), passing near a bifurcation point located on the fold of the slow invariant manifold \(M_0\), and then following the unstable invariant manifold \(M_u\).

Due to
\[
\begin{align*}
\frac{d^2g_1}{d\tau^2} &= \frac{\partial g_1}{\partial x} \frac{\partial x}{d\tau} + \frac{\partial g_1}{\partial y} \frac{\partial y}{d\tau} + \frac{\partial g_1}{\partial z} \frac{\partial z}{d\tau} = 0, \\
\frac{d^2g_2}{d\tau^2} &= \frac{\partial g_2}{\partial x} \frac{\partial x}{d\tau} + \frac{\partial g_2}{\partial y} \frac{\partial y}{d\tau} + \frac{\partial g_2}{\partial z} \frac{\partial z}{d\tau} = 0,
\end{align*}
\]
we have
\[
\begin{align*}
\dot{x} &= -\left(\frac{\partial g_1}{\partial y} \frac{\partial z}{\partial y} - \frac{\partial g_1}{\partial y} \frac{\partial z}{\partial z}\right) \frac{\dot{z}}{\det[J(x, y)]} = 0, \\
\dot{y} &= -\left(\frac{\partial g_1}{\partial y} \frac{\partial z}{\partial z} - \frac{\partial g_2}{\partial z} \frac{\partial z}{\partial z}\right) \frac{\dot{z}}{\det[J(x, y)]} = \frac{y\dot{z}}{1 - z},
\end{align*}
\]
where \(\det[J(x, y)]\) is the Jacobi matrix of the vector function \((g_1, g_2)\) with respect to \((x, y)\) and the dot is the derivative with respect to \(\tau\). Then we get the following \textit{constrained system}
\[
\begin{align*}
\dot{x} &= -\left(\frac{\partial g_1}{\partial z} \frac{\partial z}{\partial y} - \frac{\partial g_1}{\partial y} \frac{\partial z}{\partial z}\right) \frac{\dot{z}}{\det[J(x, y)]} = 0, \\
\dot{y} &= -\left(\frac{\partial g_1}{\partial z} \frac{\partial z}{\partial z} - \frac{\partial g_2}{\partial z} \frac{\partial z}{\partial z}\right) \frac{\dot{z}}{\det[J(x, y)]} = \frac{y(h - z - y^2z)}{1 - z}, \\
\dot{z} &= f_1(x, y, z) = h - z - y^2z, \\
0 &= g_1(x, y, z), \\
0 &= g_2(x, y, z).
\end{align*}
\]
We introduce the time rescaling $d\tau = -\det [J_{(x,y)}] \, dT = (z-1)\,dT$ and transform (19) into the normalized slow dynamics
\[
\frac{dx}{dT} = \left( \frac{\partial g_1}{\partial z} \frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y} \frac{\partial g_2}{\partial z} \right) f_1(x, y, z) = 0,
\]
\[
\frac{dy}{dT} = \left( \frac{\partial g_1}{\partial x} \frac{\partial g_2}{\partial z} - \frac{\partial g_1}{\partial z} \frac{\partial g_2}{\partial x} \right) = -y(h - z - y^2 z),
\]
\[
\frac{dz}{dT} = f_1(x, y, z) = (z-1)(h - z - y^2 z).
\]

0 = g_1(x_1, y_1, y_2)
0 = g_2(x_1, y_1, y_2).

Finally, we try to analyze the stability of the pseudos singular points, which will give rise to a condition for the existence of canard solutions in the original system (16). Recall that the pseudos singular points are defined by
\[
\det [J_{(y_1, y_2)}] = 0,
\]
\[
\frac{\partial g_2}{\partial z} \frac{\partial g_2}{\partial y} = 0,
\]
\[
\frac{\partial g_1}{\partial x} \frac{\partial g_2}{\partial z} - \frac{\partial g_1}{\partial z} \frac{\partial g_2}{\partial x} = 0,
\]
\[
g_1(x, y, z) = 0,
\]
\[
g_2(x, y, z) = 0.
\]

Solving the above equations gives a solution $P_0 = (0, 0, 1)$, and the eigenpolynomial of the Jacobi matrix at $P_0$ has the form
\[
\lambda^2 + \sigma_2 = 0,
\]
with $\sigma_2 = -(1-h)^2 < 0$. Hence the normalized slow dynamics has a pseudosingular point of saddle-type. It follows from Proposition 3.4 in [21] that the singularly perturbed system admit a canard solution.

**Proposition 4.** Assume $h > 1$ and $\eta$ is sufficiently small. Then the QPP model (6) admits a canard solution.

### 3. Chaotic dynamics and non-integrability of the QPP model

In the QPP model, the parameters represent tools for influencing the drive and relaxation processes. The aim of the analysis is to find such quantities which may lead to the optimal control for these processes. Whereas if the QPP model admits chaotic dynamics, it makes things complicated since the high sensitivity of chaotic systems to a change the initial conditions makes it impossible to predict correctly. Hence, it is important to ask whether there exists any set of values of parameters for which dynamics is regular and a considered system is integrable.

Above all, we provide some numerical indices for identifying chaotic properties of the QPP system. The system received much attention due to its stability to describe bifurcation of the associated Lorenz like strange attractors [22], e.g., taking $\delta = 0.6$, $\eta = 0.1$ and $h = 3$, the two and three dimensional phase portraits of system (6), illustrating its chaotic behavior, are shown in Figure 4.

Figure 5 shows the largest Lyapunov exponents of the system (6) versus parameters $h$, $\eta$, and $\delta$ varying, respectively. Obviously, system (6) can exhibit chaotic behavior, periodicity, quasiperiodicity and stable equilibria. In conclusion, from the dynamical behaviors aspect, the three parameters $(h, \delta, \eta)$ play an important role in the QPP model.
Figure 4. The two and three dimensional phase portraits of system (6), illustrating its chaotic behavior, are shown for values of parameters $\delta = 0.6$, $\eta = 0.1$ and $h = 3$.

Obviously, it is impossible to make a numerical analysis for all values of parameters. In what follows, we can prove the non-integrability of the QPP model in a wide class of functions. This result may help us distinguish values of parameters for which the QPP model is suspected to be integrable. Roughly speaking, we say that a system of differential equations is integrable if and only if it has sufficient first integrals, symmetric vector, $n$-forms or other invariant tensors such that it is solvable by quadratures.

We shall use the extended Morales-Ramis theory [23, 24] to investigate the necessary conditions for system (6) to be integrable in Bogoyavlenskij sense. Here, we say that an $n$-dimensional system $\dot{x} = v(x)$ is integrable in a Bogoyavlenskij sense if there exist $q \in \{1, 2, \cdots, n\}$ vector fields $v_1(x) := v(x), v_2(x), \ldots, v_q(x)$ and $(n - q)$ scalar-valued functions $F_1(x), \ldots, F_{n-q}(x)$ such that the following conditions hold:

(i) $v_1, \ldots, v_q$ are linearly independent almost everywhere and commute with each other, that is,

$$[v_j, v_k] := \frac{\partial v_k}{\partial x} v_j - \frac{\partial v_j}{\partial x} v_k = 0.$$

(ii) $\partial F_1/\partial x, \ldots, \partial F_{n-q}/\partial x$ are linearly independent almost everywhere and $F_1, \ldots, F_{n-q}$ are first integrals of $v_1, \ldots, v_q$. 
Figure 5. The corresponding graphs of Lyapunov exponents for
(a) $\delta = 0.5, \eta = 0.1$ and $0 < h \leq 50$; (b) $\delta = 0.5, h = 2$ and
$0 < \eta \leq 50$ and (c) $h = 2, \eta = 0.1$ and $0 < \delta \leq 50$.

This definition is introduced by Bogoyavlenskij [25] and is considered as a generalization of the famous Liouville integrability for Hamiltonian systems. Recently, using the differential Galois theory, many scholars have studied non-integrability in Bogoyavlenskij sense of models that appear in physics, economics, biology, chemistry and others [26], [27], [28], [29], [30].

As mentioned above, when $\eta = 0$, system (6) can be reduced into a linear system. Moreover, system (6) with $\eta = 0$ can also be viewed as a completely integrable system:

**Proposition 5.** Assume $\eta = 0$, then the QPP model has two functionally independent first integrals

$$ H_1 = z \quad \text{and} \quad H_2 = \begin{cases} (x-\lambda_2 y)^{\lambda_2}, & \text{if } z = 1 - \frac{\delta^2}{4}, \\ (x-\lambda_1 y)^{\lambda_1}, & \text{if } z \neq 1 - \frac{\delta^2}{4} \end{cases} $$

where

$$ \lambda_1 = -\frac{\delta}{2} - \frac{\sqrt{\delta^2 + 4z - 4}}{2}, \quad \lambda_2 = -\frac{\delta}{2} + \frac{\sqrt{\delta^2 + 4z - 4}}{2}.$$ 

In what follows, we consider the case $\eta \neq 0$ and apply the Galoisan approach to investigate the rational non-integrability of system (6): to find a non-equilibrium
solution for the considered system firstly, and then get the normal variational equations along the obtained particular solution, analyze the differential Galois group of the normal variational equations.

Clearly, system (6) has a non-equilibrium solution \((x(t), y(t), z(t)) = (0, 0, e^{-\eta t} + h)\). Denote the phase curve associated with this particular solution by \(\Gamma_1\). Setting \((x, y, z) = (X, Y, Z + e^{-\eta t} + h)\) in (6) and neglecting quadratic terms of \(X, Y\) and \(Z\), we obtain the variational equations (VE) along \(\Gamma_1\),

\[
\begin{pmatrix}
\dot{X} \\
\dot{Y} \\
\dot{Z}
\end{pmatrix} =
\begin{pmatrix}
-\delta & e^{-\eta t} + h - 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -\eta
\end{pmatrix}
\begin{pmatrix}
X \\
Y \\
Z
\end{pmatrix},
\]

the corresponding normal variational equations (NVE) is given by

\[
\begin{pmatrix}
\dot{X} \\
\dot{Y}
\end{pmatrix} =
\begin{pmatrix}
-\delta & e^{-\eta t} + h - 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
X \\
Y
\end{pmatrix}.
\] (20)

Eliminating \(X\) in (20) leads to

\[
\ddot{Y} + \delta \dot{Y} - (e^{-\eta t} + h - 1)Y = 0,
\] (21)

where the dot is the derivation with respect to \(t \in \mathbb{C}\). We make the time scale transformation \(\tilde{t} = e^{-\eta t}\) to transform (21) into an equation with rational coefficients. By

\[
\frac{d}{dt} = -\eta \frac{d}{d\tilde{t}},
\]

\[
\frac{d^2}{dt^2} = \eta^2 \frac{d}{d\tilde{t}} + \eta^2 \tilde{t}^2 \frac{d^2}{d\tilde{t}^2},
\]

(21) is converted into

\[
\frac{dY^2}{d\tilde{t}^2} = \frac{\delta - \eta}{\eta} \frac{dY}{d\tilde{t}} + \frac{\tilde{t} + h - 1}{\eta^2 \tilde{t}^2} Y.
\] (22)

In order to eliminate the first order term of (22), we introduce

\[
Y(\tilde{t}) = \varepsilon(\tilde{t}) e^{\frac{\tilde{t} + h - 1}{\eta}}
\]

and transform (22) into

\[
\frac{d\varepsilon^2(\tilde{t})}{d\tilde{t}^2} = \frac{\delta^2 - \eta^2 + 4(\tilde{t} + h - 1)}{4\eta^2 \tilde{t}^2} \varepsilon(\tilde{t}).
\] (23)

Note that the linear differential equation (23) has two singular points: \(\tilde{t} = 0\) with order two and \(\tilde{t} = \infty\) with order one. Hence, we may transform it into the Bessel equation whose differential Galois group is known. Indeed, set \(s = 2i\sqrt{\tilde{t}/\eta}\), where \(i = \sqrt{-1}\). Then (23) becomes

\[
s^2 \varepsilon''(s) - s \varepsilon'(s) - \left( \frac{\delta^2 - \eta^2 + 4h - 4}{\eta^2} - s^2 \right) \varepsilon(s) = 0,
\] (24)

where the prime is the derivation with respect to \(s \in \mathbb{C}\). Moreover, let \(\varepsilon(s) = \tilde{\varepsilon}(s) \cdot s\). Then (24) becomes the Bessel equation

\[
s^2 \tilde{\varepsilon}'' + s \tilde{\varepsilon}' + (s^2 - n^2) \tilde{\varepsilon} = 0,
\]

where \(n = \sqrt{\delta^2 + 4h - 4}/\eta\). (25)

Our result is as follows, which shows that system (6) is, in fact, non-integrable in nearly all parameters.
Theorem 3.1. Assume $2\sqrt{\delta^2 + 4h - \frac{4}{\eta}}$ is not an odd number. Then system (6) is not rationally integrable in Bogoyavlenskij sense.

Proof. Suppose system (6) is rationally integrable in the Bogoyavlenskij sense. By Theorem 1 in [24], the identity component of the differential Galois group of NVEs (20) is Abelian. Recall that an Abelian group is also a solvable group. Hence, the identity component of the differential Galois group of NVEs (20) is solvable i.e., it has a Liouville solution. Let us point out that the above transformations do not change the solvability of the identity component of the differential Galois group of (20). Therefore the identity component of (25) is solvable, too. However, it is well known that the identity component of the Bessel equation (25) is solvable if and only if $n + 1/2$ belongs to $\mathbb{Z}$, see [23] for instance. This contradicts the assumption in Theorem 3.1.

Remark 1. When $2\sqrt{\delta^2 + 4h - \frac{4}{\eta}}$ is an odd number, Theorem 3.1 does not answer whether system (6) is rationally integrable in the Bogoyavlenskij sense. There are two possible solutions to deal with this question. One can use the higher-order theory of [31] and analyze the differential Galois group of higher variational equations along $\Gamma$. In addition, the Darboux theory of integrability plays a central role in the integrability of the polynomial differential models. Hence, to find all possible polynomial and rational first integrals of system (6), one may use the characteristic curve method to get the information about invariant algebraic surfaces of this system, see [32], [33] for instance.

4. Dynamical behavior of system (6) at infinity. For system (6), the final evolution of the orbit $\{x(t), y(t), z(t)\}$ plays an important part in understanding its dynamics. As a first step for characterizing the final evolution, we need to clarify all finite equilibrium points, limit cycles and infinite equilibrium points. In this section, we shall make an analysis of the global dynamical behaviors of system (6), that is the flow of system (6) at infinity, which yields all information about infinite equilibrium points. To do this, in the next three subsections we shall analyze the Poincaré compactification of system (6) in the local charts $U_i$ and $V_i (i = 1, 2, 3)$. The detailed theory of Poincaré compactification can be found in [10], [11], [12], [13].

Let $S^3 = \{r = (r_1, r_2, r_3, r_4) \in \mathbb{R}^4 ||r|| = 1\}$ be a Poincaré unit sphere [34]. We divide the above sphere into three parts, that is the northern hemisphere $S_+ = \{r \in S^3, r_4 > 0\}$, the southern hemisphere $S_- = \{r \in S^3, r_4 < 0\}$ and the equator $S^1 = \{r \in S^3, r_4 = 0\}$. Denote the tangent hyperplanes at the point $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$ and $(0, 0, 0, \pm 1)$ by the local chart $U_i$ and $V_i (i = 1, 2, 3, 4)$ where $U_i = \{r \in S^3, r_i > 0\}$ and $V_i = \{r \in S^3, r_i < 0\}$. Define the central projections $f^+ : \mathbb{R}^3 \to S^3$ and $f^- : \mathbb{R}^3 \to S^3$ by

$$f^+(x, y, z) = \pm\left(\frac{x}{\Delta}, \frac{y}{\Delta}, \frac{z}{\Delta}, \frac{1}{\Delta}\right), \text{ where } \Delta = \sqrt{1 + x^2 + y^2 + z^2},$$

and define $\varphi_k : U_k \to \mathbb{R}^3$, $\phi_k : V_k \to \mathbb{R}^3$ by $\varphi_k = -\phi_k = (\frac{r_i}{r_k}, \frac{r_m}{r_k}, \frac{r_n}{r_k})$ for $k = 1, 2, 3, 4$ with $1 \leq l, m, n \leq 4$ and $l, m, n \neq k$. We only consider the local charts $U_i$ and $V_i$ for $i = 1, 2, 3$ to get the dynamics at $x$, $y$ and $z$ infinity shown as Figure 6.
4.1. Dynamics in the local charts $U_1$ and $V_1$. Doing the change $x = \frac{1}{z_3}, y = \frac{z_1}{z_3}, z = \frac{z_2}{z_3}$, and rescaling the time $t = z_3^2 \hat{\tau}$, the expression of the Poincaré compactification $p(X)$ of the system (6) in the local chart $U_1$ is given by

$$
\begin{align*}
\frac{dz_1}{d\hat{\tau}} &= [z_4 - (z_2 - z_3)z_1^2 + \delta z_1 z_3]z_3, \\
\frac{dz_2}{d\hat{\tau}} &= \eta(hz_3^3 - z_2 z_1^2 - z_1^2 z_2) - z_2 z_3[(z_2 - z_3)z_1 - \delta z_3], \\
\frac{dz_3}{d\hat{\tau}} &= z_3^2[\delta z_3 - z_1(z_2 - z_3)].
\end{align*}
$$

(26)

For $z_3 = 0$ (which corresponds to the points on the sphere $S^2$ at infinity), system (26) has a line of equilibria given by the $z_1$–axis and the linear part of the system at these equilibria has three null eigenvalues. We know that the $z_1z_2$–plane is invariant under the flow of system (28), so we can completely describe the dynamics on the sphere at infinity.

For $z_3 = 0$, system (26) reduces to

$$
\begin{align*}
\frac{dz_1}{d\hat{\tau}} &= 0, \\
\frac{dz_2}{d\hat{\tau}} &= -\eta z_1^2 z_2. 
\end{align*}
$$

(27)

Given initial value $(z_1, z_2) = (z_1(0), z_2(0))$, the solution of system (27) is, the solution of system (27) is

$$
z_1 = z_1(0), \quad z_2 = z_2(0)e^{-\eta z_1^2(0)\hat{\tau}},
$$

hence the nonhyperbolic equilibria on the $z_1$–axis are normally asymptotically stable if $\eta > 0$ and normally asymptotically unstable if $\eta < 0$ as shown in Figure 7(a).

The flow in the local chart $V_1$ is the same as the flow in the local chart $U_1$ reversing the time, because the compactified vector field $p(X)$ in $V_1$ coincides with the vector field $p(X)$ in $U_1$ multiplied by $(-1)^{n-1}$, where $n = 3$ is the degree of system (6). Hence the phase portrait of system (26) on the chart $V_1$ is the same as shown in Figure 7(a).
4.2. Dynamics in the local charts \( U_2 \) and \( V_2 \). Doing the change \( x = \frac{z_1}{z_3}, y = \frac{z_2}{z_3}, z = \frac{z_3}{z_3} \), and rescaling the time \( t = z_3^2 \tau \), the expression of the Poincaré compactification \( p(X) \) of system (6) in the local chart \( U_2 \) is given by

\[
\begin{aligned}
\frac{dz_1}{dt} &= (z_2 - z_3 - \delta z_1 z_3 - z_1^2 z_3) z_3, \\
\frac{dz_2}{dt} &= \eta (h z_3^3 - z_2 z_3^2 - z_2) - z_1 z_2 z_3^2, \\
\frac{dz_3}{dt} &= -z_1 z_3^3.
\end{aligned}
\] (28)

For \( z_3 = 0 \), system (28) has a line of equilibria given by the \( z_1 \)-axis and the linear part of the system at these equilibria has three null eigenvalues. We know that the \( z_1 z_2 \)-plane is invariant under the flow of system (28), so we can completely describe the dynamics on the sphere at infinity. In fact, for \( z_3 = 0 \), system (28) restricted to the \( z_1 z_2 \)-plane is given by

\[
\begin{aligned}
\frac{dz_1}{dt} &= 0, \\
\frac{dz_2}{dt} &= -\eta z_2.
\end{aligned}
\] (29)

Given initial value \((z_1, z_2) = (z_1(0), z_2(0))\), the solution of system (29) is

\[
\begin{aligned}
z_1 &= z_1(0), \\
z_2 &= z_2(0)e^{-\eta \tau}.
\end{aligned}
\]

Hence the nonhyperbolic equilibria on the \( z_1 \)-axis are normally asymptotically stable if \( \eta > 0 \) and normally asymptotically unstable if \( \eta < 0 \) as shown in Figure 7(b).

The flow in the local chart \( V_2 \) is the same as the flow in the local chart \( U_2 \) reversing the time, because the compactified vector field \( p(X) \) in \( U_2 \) coincides with the vector field \( p(X) \) in \( U_2 \) multiplied by \((-1)^n\), where \( n = 3 \) is the degree of the system (6). Hence the phase portrait of the system (28) on the chart \( V_2 \) is the same as shown in Figure 7(b).

4.3. Dynamics in the local charts \( U_3 \) and \( V_3 \). Doing the change \( x = \frac{z_1}{z_3}, y = \frac{z_2}{z_3}, z = \frac{1}{z_3} \), and rescaling the time \( t = z_3^2 \tau \), the expression of the Poincaré compactification \( p(X) \) of system (6) in the local chart \( U_3 \) is given by

\[
\begin{aligned}
\frac{dz_1}{dt} &= z_2 z_3 (1 - z_3) - \delta z_1 z_3^2 - \eta z_1 (h z_3^3 - z_3^2 - z_3^2), \\
\frac{dz_2}{dt} &= z_1 z_3^2 - \eta z_2 (h z_3^3 - z_3^2 - z_3^2), \\
\frac{dz_3}{dt} &= -\eta (h z_3^3 - z_3^2 - z_3^2 z_3).
\end{aligned}
\] (30)

For \( z_3 = 0 \), system (30) restricted to the \( z_1 z_2 \)-plane is given by

\[
\begin{aligned}
\frac{dz_1}{dt} &= \eta z_1 z_2^2, \\
\frac{dz_2}{dt} &= \eta z_3^2.
\end{aligned}
\] (31)

Given initial value \((z_1, z_2) = (z_1(0), z_2(0))\), the solution of system (31) is

\[
\begin{aligned}
z_1 &= \frac{z_1(0) \sqrt{\frac{1}{z_2(0)^2}}}{\sqrt{-2\eta \tau + \frac{1}{z_2(0)^2}}}, \\
z_2 &= \frac{1}{\sqrt{-2\eta \tau + z_2(0)}}.
\end{aligned}
\]

The phase portrait of the system (28) on the chart \( V_2 \) is the same as shown in Figure 7(c).
Figure 7. Dynamics of system (6) on the Poincaré sphere at infinity in the local charts $U_i$ and $V_i (i = 1, 2, 3)$: the solutions tend toward the equilibria in the $z_1$-axis for $\eta > 0$ and outward of this line for $\eta < 0$ as $t \to +\infty$.

4.4. **Dynamics of system (6) on the sphere at infinity.** Based on the analysis made in the previous sections, we get a global structure of the dynamical behavior of (6) on the Poincaré sphere at infinity as follows:

**Theorem 4.1.** For any values $\delta, h \in \mathbb{R}$ and $\eta \in \mathbb{R}/\{0\}$, the phase portrait of the QPP model on the Poincaré sphere is shown in Figure 8: when the parameter $\eta > 0 (< 0)$, the model (6) has two stable (unstable) nodes localized at the endpoints of the $z$-axis, and one line of equilibria contained in the endpoints of the $xy$-plane which are normally asymptotically stable (unstable).

When $\eta = 0$, the system (6) can be reduced into a linear system and no complex dynamics occur, so we omit it.

Theorem 4.1 shows that the input power $h$ and the relaxation of the instability $\delta$ do not effect the global dynamics at infinity of the QPP model (6). In addition, the global dynamics at the infinity for different values of $\eta$ are topologically equivalent although it depends on the parameter $\eta$. This means that the heat diffusion coefficient just yield quantitative, but not qualitative changes for the global dynamics at infinity of the QPP model (6).
5. Conclusions. The dynamics of a three dimensional chaotic QPP model is studied in this paper. We considered the delayed QPP model, the significant effect of time delay is carried out, including: the stability of equilibrium is investigated and existence of Hopf bifurcations is demonstrated. By applying center manifold theory and the normal form method, the explicit formulae determining the stability and the direction of periodic solutions are obtained. The dynamical behavior of QPP system on the sphere at infinity is studied by the Poincaré compactification method. Necessary numerical simulations are also examined for illustrating the theoretical results. For the QPP model, we apply the differential Galoisan approach to investigate the meromorphic non-integrability in the Bogoyavlenskij sense, the results show that the system is non-integrable in nearly all parameters. The theoretical analysis and interesting numerical observations in this manuscript may be useful both in mathematical and physical research areas. More profound discussions and research results will be provided in the forthcoming study.

6. Appendix.

6.1. Proof of Proposition 3. When \( h > 1 \) and \( \eta = \eta_0 = \frac{2h - 2 - \delta^2 h}{\delta h} > 0 \), system (6) possesses a negative real root \( \lambda_1 = \frac{2(1-h)}{\delta h} \) and a pair of conjugate purely imaginary roots \( \lambda_{2,3} = \pm \frac{i\sqrt{h(2h - 2 - \delta^2 h)}}{h} \). Under this condition, the transversality condition

\[
\text{sign} \left( \text{Re}(\lambda'(\eta_0)) \right)_{\lambda = \frac{i\sqrt{h(2h - 2 - \delta^2 h)}}{h}} = \text{sign} \left( 2\delta(2h - 2 - \delta^2 h)(\delta h + \delta + \frac{2h - 2 - \delta^2 h}{\delta h}) \right) = 1,
\]

is also satisfied. Accordingly, Hopf bifurcation at \( E_+ \) occurs.

The above analysis is summarized as follows: Let \( \omega_0 = \frac{\sqrt{h(2h - 2 - \delta^2 h)}}{h} \) and \( \eta = \eta_0 = \frac{2h - 2 - \delta^2 h}{\delta h} \). Then, we have

\[
v_1 = \begin{pmatrix} i\omega_0 \sqrt{h - 1} - \omega_0^2 + i\omega_0 \delta \\ \sqrt{h - 1} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{2(1-h)}{h^2} \sqrt{h - 1} \\ \frac{2(1-h)}{h\delta} \left( \frac{2(1-h)}{h\delta + \delta} \right) \end{pmatrix},
\]
which satisfy

\[ Av_1 = i\omega_0 v_1, \quad Av_2 = \frac{2(1 - h)}{\delta h} v_2, \]

where \( A \) is the Jacobian matrix of system (6) at \( E_+ \), and

\[
A = \begin{bmatrix}
-\delta & 0 & \sqrt{h - 1} \\
1 & 0 & 0 \\
0 & 2\sqrt{h-1}(2+\delta^2 h - 2h) & \frac{2(1-h)}{\delta h^2} h^2 - \frac{2(1-h)}{\delta h^2} h \\
\end{bmatrix}.
\]

From system (6), define

\[
P = (Re v_1, - Im v_1, v_2) = \begin{bmatrix}
0 & -\omega_0 \sqrt{h - 1} & \frac{2(1-h)}{\delta h^2} h^2 \\
\omega_0^2 & -\omega_0 \delta & \frac{2(1-h)}{\delta h^2} h^2 + \delta \\
\end{bmatrix},
\]

and

\[
X = E_+ + PY,
\]

where \( X = (x, y, z)^T \) and \( Y = (x_1, y_1, z_1)^T \). By computation, we obtain

\[
x = \sqrt{h - 1} \left[ \frac{2(1 - h)}{\delta h} z_1 - \omega_0 y_1 \right],
\]

\[
y = \sqrt{h - 1} \left[ (1 + x_1 + z_1) \right],
\]

\[
z = 1 - \omega_0^2 x_1 - \omega_0 \delta y_1 + \frac{2(1 - h)}{\delta h} \left( \frac{2(1 - h)}{\delta h^2} + \delta \right) z_1.
\]

\[
\begin{pmatrix}
x_1 \\
y_1 \\
z_1
\end{pmatrix} = P^{-1} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \frac{1}{\omega_0 \sqrt{h - 1} \left[ \frac{4(1-h)^2}{\delta^2 h^2} + \omega_0^2 \right]} \begin{pmatrix}
f_1 \\
f_2 \\
f_3
\end{pmatrix}
\]

where

\[
f_1 = \omega_0 \delta [(z - 1)y - \delta x] + \frac{4(1 - h)^2 \omega_0}{\delta^2 h^2} x - \omega_0 \eta \sqrt{h - 1} (h - z - y^2 z),
\]

\[
f_2 = -\omega_0^2 + \frac{4(1 - h)^2}{\delta^2 h^2} + \frac{2(1 - h)}{h} \left[ (z - 1)y - \delta x \right] + \frac{2(1 - h) \omega_0^2}{\delta h} x + \frac{2(1 - h) \omega_0}{\delta h} \eta (h - z - y^2 z),
\]

\[
f_3 = -\omega_0 \delta [(z - 1)y - \delta x] + \omega_0^3 x + \omega_0 \eta \sqrt{h - 1} (h - z - y^2 z).
\]

Thus

\[
\begin{cases}
\dot{x}_1 = -\omega_0 y_1 + F_1 (x_1, y_1, z_1), \\
\dot{y}_1 = \omega_0 x_1 + F_2 (x_1, y_1, z_1), \\
\dot{z}_1 = \frac{2(1-h)}{\delta \kappa} z_1 + F_3 (x_1, y_1, z_1),
\end{cases} \quad (32)
\]
where

\[
F_1(x_1, y_1, z_1) = \omega_0 \delta \sqrt{h - 1} \left\{ -\omega_0^2 x_1^2 - \omega_0 \delta x_1 y_1 + \left[ \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) - \omega_0^2 \right] x_1 z_1 - \omega_0 \delta y_1 z_1 \\
+ \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) z_1^2 \right\} + \omega_0 \eta_0 \sqrt{h - 1} \left\{ (1 - 2\omega_0^2) x_1^2 + \left[ \frac{4(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right] + \delta \right\} z_1^2 + \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right\}.
\]

\[
F_2(x_1, y_1, z_1) = -\sqrt{h - 1} \left[ \omega_0^2 + \frac{4(1-h)^2}{h^2 \delta^2} + \frac{2(1-h)}{h \delta} \left\{ -\omega_0^2 x_1^2 - \omega_0 \delta x_1 y_1 + \left[ \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) - \omega_0^2 \right] x_1 z_1 - \omega_0 \delta y_1 z_1 \\
- \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) z_1^2 \right\} - \omega_0 \eta_0 \sqrt{h - 1} \left\{ (1 - 2\omega_0^2) x_1^2 + \left[ \frac{4(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right] + \delta \right\} z_1^2 + \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right\}.
\]

\[
F_3(x_1, y_1, z_1) = -\omega_0 \delta \sqrt{h - 1} \left\{ -\omega_0^2 x_1^2 - \omega_0 \delta x_1 y_1 + \left[ \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) - \omega_0^2 \right] x_1 z_1 - \omega_0 \delta y_1 z_1 \\
+ \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) z_1^2 \right\} - \omega_0 \eta_0 \sqrt{h - 1} \left\{ (1 - 2\omega_0^2) x_1^2 + \left[ \frac{4(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right] + \delta \right\} z_1^2 + \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right\}.
\]

Furthermore,

\[
g_{11} = \frac{1}{4} \left[ \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial y_1^2} + \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial y_1^2} \right] \]

\[
= \frac{1}{2} \omega_0 \sqrt{h - 1} \left\{ -\omega_0^2 \delta + \eta_0 \left[ 1 - 2\omega_0^2 - 3\omega_0^2 x_1 - \omega_0 \delta y_1 + \left( \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right) \right] \right\} + \frac{1}{2} \omega_0 \sqrt{h - 1} \left[ \omega_0^2 + \frac{4(1-h)^2}{h^2 \delta^2} + \frac{2(1-h)}{h \delta} \right] - \omega_0 \eta_0 \sqrt{h - 1} \left[ \left( \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) \right) \right] - \frac{1}{2} \omega_0^2 \left[ 1 - 2\omega_0 \right] - 3\omega_0^2 x_1 - \omega_0 \delta y_1 + \left( \frac{2(1-h)}{h \delta} \left( \frac{2(1-h)}{h \delta} + \delta \right) - 2\omega_0 \right) z_1 \right\}.\]
By solving the following equations

\[\lambda_1 w_{11} = -h_{11},\]
\[(\lambda_1 - 2\omega_0) w_{20} = -h_{20},\]

we obtain:

\[g_{20} = \frac{1}{4} \left[ \left( \frac{\partial^2 F_1}{\partial x_1^2} - \frac{\partial^2 F_2}{\partial y_1^2} \right) + \frac{1}{2} \left( \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial y_1^2} \right) \right] + \frac{1}{2} \left\{ \left( \omega_0^2 - \omega_0 \eta_0 \right) \frac{\partial^2}{\partial h^2} - \left( \omega_0^2 - \omega_0 \eta_0 \right) \frac{\partial}{\partial h} + \left( \omega_0^2 - \omega_0 \eta_0 \right) \delta_0 \frac{\partial}{\partial h} \right\}.

\[g_{12} = \frac{1}{4} \left[ \left( \frac{\partial^2 F_1}{\partial x_1^2} - \frac{\partial^2 F_2}{\partial y_1^2} \right) - \frac{1}{2} \left( \frac{\partial^2 F_2}{\partial x_1^2} + \frac{\partial^2 F_1}{\partial y_1^2} \right) \right] + \frac{1}{2} \left\{ \left( \omega_0^2 - \omega_0 \eta_0 \right) \frac{\partial^2}{\partial h^2} + \left( \omega_0^2 - \omega_0 \eta_0 \right) \frac{\partial}{\partial h} - \left( \omega_0^2 - \omega_0 \eta_0 \right) \delta_0 \frac{\partial}{\partial h} \right\}.

\[G_{21} = \frac{1}{8} \left[ \frac{\partial^3 F_1}{\partial x_1^2} + \frac{\partial^3 F_2}{\partial x_1^2} + \frac{\partial^3 F_3}{\partial y_1^2} + \frac{\partial^3 F_3}{\partial y_1^2} \right] + \frac{1}{8} \left[ \frac{\partial^3 F_2}{\partial x_1^2} + \frac{\partial^3 F_1}{\partial y_1^2} - \frac{\partial^3 F_1}{\partial x_1^2} - \frac{\partial^3 F_2}{\partial y_1^2} \right].

Meanwhile, one has:

\[h_{11} = \frac{1}{4} \left( \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial y_1^2} \right)\]
\[= \frac{1}{2} \omega_0 \sqrt{h - 1} \left\{ \delta \omega_0^2 - \eta_0 (h - 1) \left[ 1 - 2 \omega_0^2 - 3 \omega_0^2 x_1 - \delta \omega_0 y_1 + \frac{2(1 - h)}{h \delta} \left( \frac{2(1 - h)}{h \delta} + \delta \right) - 2 \omega_0^2 \right] \right\}.

\[h_{20} = \frac{1}{4} \left( \frac{\partial^2 F_3}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial y_1^2} - 2 \frac{\partial^2 F_2}{\partial x_1^2} \right)\]
\[= \frac{1}{2} \omega_0 \sqrt{h - 1} \left\{ \delta \omega_0^2 + \eta_0 (h - 1) \left[ 1 - 2 \omega_0^2 - 3 \omega_0^2 x_1 - \delta \omega_0 y_1 + \frac{2(1 - h)}{h \delta} \left( \frac{2(1 - h)}{h \delta} + \delta \right) - 2 \omega_0^2 \right] \right\} - \frac{1}{2} \delta \omega_0^2 \sqrt{h - 1} \left[ \delta + 2 \eta_0 (h - 1)(1 + x_1 + z_1) \right].
Based the above calculation and analysis, one can compute the following quantities:

\[
C_1(0) = \frac{i}{2\omega_0} \left( g_{20}g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{1}{2} g_{21},
\]

\[
\mu_2 = -\frac{\text{Re} C_1(0)}{\alpha'(0)},
\]

\[
\tau_2 = -\frac{\text{Im} C_1(0) + \mu_2\omega'(0)}{\omega_0},
\]

\[
\beta_2 = 2 \text{Re} C_1(0).
\]
where
\[ \alpha'(0) = \text{Re} \left( \lambda'(\eta_0) \right) = \frac{2h^3\delta^3(2h - 2 - \delta^2h)(2h - 2 + \delta^2h^2)}{4h\delta^2(2 - 2h + \delta^2h^2)^2 + 16(h - 1)^2(2h - 2 - \delta^2h)}, \]
\[ \omega_0 = \frac{\sqrt{h(2h - 2 - \delta^2h)}}{h}, \]
\[ \omega'(\eta_0) = \frac{2\delta^3h}{\omega_0(2h + \delta^2h - 2)}, \]
\[ T = \frac{2\pi}{\omega_0} \left( 1 + \tau_2\varepsilon^2 + o \left( \varepsilon^4 \right) \right), \]
\[ \beta = \beta_2\varepsilon^2 + o \left( \varepsilon^4 \right), \]
\[ \varepsilon = \frac{\eta - \eta_0}{\mu_2} + o \left( (\eta - \eta_0)^2 \right). \]

And the expression of the bifurcating periodic solution is (except for an arbitrary phase angle):
\[ X = E_+ + PY, \]
where the matrix \( P \) is defined as in (32),
\[ x_1 = \text{Re} u, \quad y_1 = \text{Im} u, \quad z_1 = w_{11}|u|^2 + \text{Re} \left( w_{20}u^2 \right) + o \left( |u|^3 \right), \]
and
\[ u = \varepsilon e^{\frac{2\pi i}{\omega_0} + \frac{1}{6}\varepsilon^3 \left( g_{02}e^{-\frac{4\pi i}{\omega_0}} - 3g_{20}e^{\frac{4\pi i}{\omega_0}} + 6g_{11} \right) + o \left( \varepsilon^3 \right). \]

6.2. **Proof of Theorem 2.3.** Theorem 2.3 shows that system (7) undergoes Hopf bifurcation under certain conditions. In the following, we will proof the Theorem 2.4, and derive explicit formulae determining the direction of Hopf bifurcation and the stability of the periodic solutions bifurcating from \( E_+ \) at \( \tau^j \) if \( j = 0, 1, 2, \ldots \), by employing center manifold theory and normal form method [35], [36], [37], [38]. For convenience, denote \( \bar{\tau} \) by \( \bar{\tau} \), and \( \tau = \bar{\tau} + \mu, \mu \in \mathbb{R}, \) then \( \mu = 0 \) is the Hopf bifurcation value for system (7).

The proof of Theorem 2.4 will be divided into five steps.

**Step 1. Transform system (7) into the abstract ODE.**

The system (7) can locally be represented as the following DDE in \( C = C([-1, 0], \mathbb{R}^3) \)
\[ \dot{u}(t) = L_\mu(u) + F(\mu, u), \quad (33) \]
where \( u(t) = (u_1(t), u_2(t), u_3(t))^T, u_t(\theta) = u(t + \theta), L_\mu : C \to \mathbb{R} \) is a bounded linear operator and \( F : \mathbb{R} \times C \to \mathbb{R} \) is continuous and differentiable with
\[ L_\mu \phi = (\bar{\tau} + \mu) \begin{pmatrix} -\delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2\eta\sqrt{\mu - 1} - \eta h \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \sqrt{h - 1}, \]
and
\[ F(\mu, \phi) = (\bar{\tau} + \mu) \begin{pmatrix} -\phi_2(0)\phi_3(-1) \\ 0 \\ -\phi_2(0)[\phi_2(0) + \phi_2(0)\phi_3(0) + 2\sqrt{h - 1}\phi_3(0)] \end{pmatrix}, \]
where \( \phi = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C. \)
By the Riesz representation theorem, there exists a $3 \times 3$ matrix whose elements are bounded variation function $\eta(\theta, \mu)$ in $\theta \in [-1, 0]$ such that

$$L_\mu \phi = \int_{-1}^{0} d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C,$$

where $\eta(\theta, \mu)$ can be chosen as

$$\eta(\theta, \mu) = (\bar{\tau} + \mu) \begin{pmatrix} -\delta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2\eta\sqrt{h-1} & -\eta h \end{pmatrix} \delta(\theta) + \begin{pmatrix} 0 & 0 & \sqrt{h-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + 1),$$

where $\delta(\theta)$ is a Dirac delta function [39] and $\theta \in [-1, 0]$.

For $\phi \in C$, let

$$A(\mu)\phi(\theta) = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\mu, \theta) \phi(\theta), & \theta = 0, \end{cases}$$

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0, \end{cases}$$

then system (33) is equivalent to the following abstract operator equation

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t. \tag{34}$$

**Step 2. Calculate the eigenfunctions of $A = A(0)$ and the adjoint operator $A^*$ corresponding to $\omega_{0\bar{\tau}}$ and $-\omega_{0\bar{\tau}}$.**

For $\psi \in C([0, 1], (C^3)^*)$, where $(C^3)^*$ is the 3-dimensional complex space of row vectors, we define the adjoint operator $A^*$ of $A$

$$A^*\psi(s) = \begin{cases} \frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta(\mu, t) \psi(-t), & s = 0, \end{cases}$$

and the bilinear form is given by

$$\langle \psi(s), \phi(\theta) \rangle = \overline{\psi(0)}\phi(0) - \int_{-\tau}^{0} \int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,$$

where $\eta(\theta) = \eta(\theta_0, 0)$. Then $A = A(0)$ and $A^*(0)$ are adjoint operators.

By [40], $\pm \omega_{0\bar{\tau}}$ are eigenvalues of $A(0)$, so they are also eigenvalues of $A^*(0)$. Suppose that $q(\theta) = (1, \alpha, \beta)^T e^{i\omega_{0\bar{\tau}}}$ is the eigenfunction of $A(0)$ corresponding to the eigenvalue $\omega_{0\bar{\tau}}$ and $q^*(s) = G(1, \alpha^*, \beta^*) e^{i\omega_{0\bar{\tau}}}$ is the eigenfunction of $A^*$ corresponding to the eigenvalue $-\omega_{0\bar{\tau}}$, where

$$\alpha = 1, \beta = \frac{(i\omega_0 + \delta)e^{i\omega_{0\bar{\tau}}}}{\sqrt{h-1}}, \quad \alpha^* = \delta - i\omega_0, \beta^* = \frac{\sqrt{h-1}e^{-i\omega_{0\bar{\tau}}}}{\eta h - i\omega_0},$$

$$G = (1 + \alpha\beta^* + \beta\beta^* + \tau\beta\sqrt{h-1}e^{-i\omega_{0\bar{\tau}}})^{-1},$$

which assures that $\langle q^*(s), q(\theta) \rangle = 1$, $\langle q^*(s), \bar{q}(\theta) \rangle = 0$.

**Step 3. Obtain the reduced system on the center manifold.**

In this part, we will use the same notations as in [41] and compute the coordinates to describe the center manifold $C_0$ at $\mu = 0$ (A local center manifold is in general not unique). Let $u_t \in C$ be the solution of system (34) when $\mu = 0$, and define

$$z(t) = \langle q^*, u_t \rangle, W(t, \theta) = u_t(\theta) - z(t)q(\theta) - \tau(t)\bar{q}(\theta), \tag{35}$$
where $z$ and $\overline{z}$ are local coordinates for center manifold $C_0$ in the direction of $q^*$ and $\overline{q}^*$. On the center manifold $C_0$, we have $W(t, \theta) = W(z(t), \overline{z}(t), \theta)$, where

$$W(z, \overline{z}, \theta) = \frac{W_{20}(\theta)z^2}{2} + W_{11}(\theta)z\overline{z} + \frac{W_{02}(\theta)\overline{z}^2}{2} + \cdots. \quad (36)$$

The existence of the center manifold enables us to reduce (34) to an ODE on $C_0$. Note that $W$ is real if $u_t$ is real, we consider only real solutions. For solution $u_t \in C_0$ of system (34) at $\mu = 0$,

$$\dot{z}(t) = \langle q^*, u_t \rangle \triangleq g(z, \overline{z}).$$

Rewrite (37), we obtain the reduced system on $C_0$ is described by

$$\dot{z}(t) = i\omega_0 \overline{z}(t) + g(z, \overline{z}), \quad (38)$$

where

$$g(z, \overline{z}) = \frac{g_{20}(\theta)z^2}{2} + g_{11}(\theta)z\overline{z} + \frac{g_{02}(\theta)\overline{z}^2}{2} + \frac{g_{21}(\theta)z^2\overline{z}}{2} + \cdots. \quad (39)$$

We will mainly discuss Eq. (38) in the following step.

**Step 4. Get the value of $g_{20}, g_{11}, g_{02}, g_{21}$ in (39).**

In this part, we will calculate the coefficient $W_{20}(\theta), W_{11}(\theta), W_{02}(\theta), \cdots$, then substitute them in (37), and get the reduced system (38) on $C_0$.

It follows from (35) that

$$x_t(\theta) = x(t + \theta) = W(t, \theta) + 2\text{Re}\{z(t), q(\theta)\}$$

$$= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + (1, \alpha, \beta)^T e^{i\omega_0 \tau z}$$

$$+ (1, \alpha, \beta)^T e^{-i\omega_0 \tau \overline{z}} + \cdots. \quad (40)$$

We obtain

$$x_2(t) = \alpha z + \overline{\alpha} \overline{z} + W^{(2)}(0) + \cdots,$$

$$x_3(t) = \beta z + \overline{\beta} \overline{z} + W^{(3)}(0) + \cdots,$$

$$x_3(t - \tau) = \beta z e^{-i\omega \tau} + \overline{\beta} \overline{z} e^{i\omega \tau} + W^{(3)}(-\tau) + \cdots. \quad (40)$$

It follows together with $F(\mu, \phi)$, we get

$$f(0, x_t) = \tau \begin{pmatrix} x_2(t) x_3(t - \tau) \\
0 \\
x_2(t) [x_2(t) + x_2(t) x_3(t) + 2\sqrt{h - 1} x_3(t)] \end{pmatrix}. \quad (41)$$
Substituting (40) into (41) (37) and comparing the coefficients with (39), we obtain

\[ g_{20} = 2\tilde{G}(\alpha e^{-i\omega \tau} - \beta(\alpha^2 + 2\sqrt{h - 1}\alpha \beta)), \]
\[ g_{11} = \overline{\tilde{G}}(\alpha \overline{\tilde{G}}(\alpha e^{-i\omega \tau} - \beta(\alpha^2 + 2\sqrt{h - 1}\alpha \beta)), \]
\[ g_{21} = \overline{\tilde{G}}(\alpha \overline{\tilde{G}}(\alpha e^{-i\omega \tau} - \beta(\alpha^2 + 2\sqrt{h - 1}\alpha \beta)) \}

Since there are \( W_{20}(\theta) \) and \( W_{11}(\theta) \) in \( g_{21} \), we still need to compute them.

From (34) and (35), we have

\[ \dot{W} = iu - \dot{z}q - \dot{\bar{z}}q \begin{cases} AW - 2\text{Re}\{gg(\theta)\}, & \theta \in [-1, 0), \\ AW - 2\text{Re}\{gg(0)\} + f_0, & \theta = 0, \end{cases} \]  
(2.7.15)

where

\[ f_0 = f_{z2} \frac{\dot{z}^2}{2} + f_{z\bar{z}} z\bar{z} + f_{z2} \frac{\dot{z}^2}{2} + f_{z2} z\bar{z} \frac{\dot{z}^2}{2} + \cdots. \]

On the other hand, near the origin, on the center manifold \( C_0 \), according to (36)

\[ \dot{W} = W_{20}(\dot{z}) + W_{11}(\dot{z}) \bar{z} + W_{02}(\theta) \bar{z} \]

Substituting (36) into right side of (42), equating terms of \( \dot{z}^2 \) and \( z\bar{z} \) of (42) with (43), we obtain

\[ (2i\omega_0 I - A)W_{20}(\theta) = \begin{cases} -g_{20} q(\theta) - \overline{\gamma}_{02} \overline{\gamma}(\theta), & \theta \in [-\tau, 0), \\ -g_{20}(0) - \overline{\gamma}_{02}\overline{\gamma}(0) + f_{z2}, & \theta = 0, \end{cases} \]  
(4.4.15)

\[ -AW_{11}(\theta) = \begin{cases} -g_{11} q(\theta) - \overline{\gamma}_{11} \overline{\gamma}(\theta), & \theta \in [-\tau, 0), \\ -g_{11}(0) - \overline{\gamma}_{11}\overline{\gamma}(0) + f_{z\bar{z}}, & \theta = 0. \end{cases} \]  
(4.4.15)

According to the definition of \( A \) and from (44)(45) for \( \theta \in [-\tau, 0), \) we get

\[ W_{20}(\theta) = 2i\omega_0 W_{20}(\theta) + g_{20}(\theta) + \overline{\gamma}_{02}\overline{\gamma}(\theta), \]
\[ W_{11}(\theta) = g_{11}(\theta) + \overline{\gamma}_{11}\overline{\gamma}(\theta), \]

solving for \( W_{20}(\theta) \) and \( W_{11}(\theta) \), we obtain

\[ W_{20}(\theta) = ig_{20}/\omega_0 \cdot q(0)e^{i\omega_0 \theta} + ig_{02}/3\omega_0 \cdot \overline{\gamma}(0)e^{-i\omega_0 \theta} + E_1e^{2i\omega_0 \theta}, \]  
(5.15.15)

\[ W_{20}(\theta) = -ig_{11}/\omega_0 \cdot q(0)e^{i\omega_0 \theta} + ig_{11}/\omega_0 \cdot \overline{\gamma}(0)e^{-i\omega_0 \theta} + E_2, \]  
(5.15.15)

where \( E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)})^T \in \mathbb{R}^3 \) and \( E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)})^T \in \mathbb{R}^3 \) are constant vectors.

In what follows we shall seek appropriate \( E_1 \) and \( E_2 \) in (46) and (47), respectively. According to the definition of \( A \) and (44)(45) for \( \theta = 0 \), we have

\[ \int_{-\tau}^{0} dq(\theta) W_{20}(\theta) = 2i\omega_0 W_{20}(0) + g_{20}(0) + \overline{\gamma}_{02}\overline{\gamma}(0) - f_{z2}, \]
\[ \int_{-\tau}^{0} dq(\theta) W_{11}(\theta) = g_{11}(0) + \overline{\gamma}_{11}\overline{\gamma}(0) - f_{z\bar{z}}, \]

(5.15.15)
where $\eta(\theta) = \eta(0, \theta)$ and

$$f_{z2} = \begin{pmatrix}
\alpha \beta e^{-i\omega \tau} \\
0 \\
-\alpha(\alpha + 2\sqrt{h} - 1) \\
\end{pmatrix},
\quad f_{z\tau} = \begin{pmatrix}
\alpha \beta e^{i\omega \tau} + \bar{\alpha} \beta e^{-i\omega \tau} \\
0 \\
-2\alpha \bar{\alpha} - 2\sqrt{h} - 1(\alpha \beta + \bar{\alpha} \beta) \\
\end{pmatrix}.$$

Substituting (46) into (48), we obtain

$$\left(2i\omega_0 I - \int_{-\bar{\tau}}^{0} e^{2i\omega_0 \theta} d\eta(\theta)\right) E_1 = f_{z^2},$$

that is

$$\left(\begin{array}{ccc}
2i\omega_0 + \delta & 0 & -\sqrt{h} - 1 e^{-2i\omega_0 \tau} \\
-1 & 2i\omega_0 & 0 \\
0 & 2\eta\sqrt{h} - 1 & 2i\omega_0 + \eta h
\end{array}\right) E_1 = f_{z^2}.$$

(50)

Similarly, substituting (47) into (49), we get

$$\int_{-\bar{\tau}}^{0} d\eta(\theta) E_2 = f_{z\tau},$$

that is

$$\left(\begin{array}{ccc}
\delta & 0 & -\sqrt{h} - 1 \\
-1 & 0 & 0 \\
0 & 2\eta\sqrt{h} - 1 & \eta h
\end{array}\right) E_2 = f_{z\tau}.$$

(51)

We have got the value of $E_1$ and $E_2$ as (50) and (51) shown and the reduced system (38) finally.

**Step 5. Obtain the key values $\mu_2, \beta_2, T_2$ to judge the property of the Hopf bifurcation.**

Similar to calculate the Hopf bifurcation parameter of the ODE and as in [41], according to the analysis above and the expressions of $g_{20}, g_{11}, g_{02}$ and $g_{21},$ we can compute the following values

$$c_1(0) = i/2\omega_0 \bar{\tau} \left(g_{11} g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\text{Re}\{c_1(0)\}}{\alpha'(\bar{\tau})},$$

$$\beta_2 = 2\text{Re}\{c_1(0)\},$$

$$T_2 = -\frac{\text{Im}\{c_1(0)\} + \mu_2 \omega'(\bar{\tau})}{\omega_0},$$

where $\lambda(\tau) = \alpha(\tau) \pm i\omega(\tau)$ is the characteristic root of (10), which is a continuous differentiable family. $\alpha'(\bar{\tau})$ and $\omega'(\bar{\tau})$ can be obtained by taking the derivation of two sides of (10) and taking values at $\bar{\tau}$.

These formulae give a description of the Hopf bifurcation periodic solution of system (7) at $\tau = \tau_j (j = 0, 1, 2 \cdots)$ on the center manifold. Thus, we can complete the proof of Theorem 2.4 according to the discussion about properties of Hopf bifurcating periodic solutions of dynamical system in [41].
Acknowledgments. This work was supported by the Education Department of Jilin Province (JJKH20170454KJ) and NNSFC (National Natural Science Foundation of China) under Grant Number 11461074. Thanks Matthew J. Wade of Newcastle University for assistance with the scientific English. Thanks Yifeng Wang of Institute of Plasma Physics, Chinese Academy of Sciences for refining the physical background of the model.

Conflict of Interest The authors declare that they have no conflict of interest.

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Received January 2020; revised July 2020.

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