Percolation in Directed Scale-Free Networks

N. Schwartz\(^1\), R. Cohen\(^1\), D. ben-Avraham\(^2\), A.-L. Barabási\(^3\) and S. Havlin\(^1\)

\(^1\)Minerva Center and Department of Physics, Bar-Ilan University, Ramat-Gan, Israel
\(^2\)Department of Physics, Clarkson University, Potsdam NY 13699-5820, USA
\(^3\)Department of Physics, University of Notre Dame, Notre Dame IN 46556, USA

Many complex networks in nature have directed links, a property that affects the network’s navigability and large-scale topology. Here we study the percolation properties of such directed scale-free networks with correlated in and out degree distributions. We derive a phase diagram that indicates the existence of three regimes, determined by the values of the degree exponents. In the first regime we regain the known directed percolation mean field exponents. In contrast, the second and third regimes are characterized by anomalous exponents, which we calculate analytically. In the third regime the network is resilient to random dilution, i.e., the percolation threshold is \(p_c \to 1\).

The structure of a directed graph has been characterized in \([3,4]\), and in the context of the WWW in \([7]\). In general, a directed graph consists of a giant weakly connected component (GWCC) and several finite components. In the GWCC every site is reachable from every other, provided that the links are treated as bidirectional. The GWCC is further divided into a giant strongly connected component (GSCC) and several finite components. The GSCC is the intersection of the IN and OUT components. All sites in the GWCC, but not in the IN and OUT components are referred to as the “tendrils” (see Fig. 1).

\FIG. 1. Structure of a general directed graph.

For a directed random network of arbitrary degree distribution the condition for the existence of a giant component can be deduced in a manner similar to \([6]\). If a site, \(b\), is reached following a link pointing to it from site \(a\), then it must have at least one outgoing link, on average, in order to be part of a giant component. This condition can be written as

\[ \langle k_b | a \to b \rangle = \sum_{j_b, k_b} k_b P(j_b, k_b | a \to b) = 1, \]

where \(j\) and \(k\) are the in- and out- degrees, respectively, \(P(j_b, k_b | a \to b)\) is the conditional probability given that

\[ \langle k_b | a \to b \rangle = \sum_{j_b, k_b} k_b P(j_b, k_b | a \to b) = 1, \]
site a has a link leading to b, and \(\langle k_b \rangle_{a \rightarrow b}\) is the conditional average. Using Bayes rule we get

\[
P(j_b, k_b | a \rightarrow b) = P(j_b, k_b, a \rightarrow b) / P(a \rightarrow b)
\]

\[
= P(a \rightarrow b | j_b, k_b) P(j_b, k_b) / P(a \rightarrow b).
\]

For random networks \(P(a \rightarrow b) = \langle k \rangle / (N - 1)\) and

\[
P(a \rightarrow b | j_b, k_b) = j_b / (N - 1), \text{ where } N \text{ is the total number of nodes in the network.}
\]

The above criterion thus reduces to

\[
\langle jk \rangle \geq \langle k \rangle. \tag{3}
\]

Suppose a fraction \(p\) of the nodes is removed from the network. (Alternatively, a fraction \(q = 1 - p\) of the nodes is retained.) The original degree distribution, \(P(j, k)\), becomes

\[
P'(j, k) = \sum_{j_{0}, k_{0}} P(j_{0}, k_{0}) \left(\begin{array}{c} j_{0} \\ j \end{array}\right) (1 - p)^{j} p^{k_{0} - j} \times \left(\begin{array}{c} k_{0} \\ k \end{array}\right) (1 - p)^{k} p^{k_{0} - k}.
\]

(4)

In view of this new distribution, Eq. (3) yields the percolation threshold

\[
q_c = 1 - p_c = \frac{\langle k \rangle}{\langle jk \rangle}, \tag{5}
\]

where averages are computed with respect to the original distribution before dilution, \(P(j, k)\). Eq. (3) indicates that in directed scale-free networks if \(\langle jk \rangle\) diverges then \(q_c \to 0\) and the network is resilient to random breakdown of nodes and bonds.

The term \(\langle jk \rangle\) may be dramatically influenced by the appearance of correlations between the in- and out-degrees of the nodes. In particular, let us consider scale-free distributions for both the in- and out-degrees:

\[
P_{in}(j) \sim \begin{cases} 
B_c j^{-\lambda_{in}} & j \neq 0, \\
1 - B_c & j = 0,
\end{cases} \tag{6}
\]

and

\[
P_{out}(k) = c_{out} k^{-\lambda_{out}}. \tag{7}
\]

In (3) we choose to add the possible zero value to the in-degree in order to maintain \(\langle j \rangle = \langle k \rangle\). If the in- and out-degrees are uncorrelated, we expect \(\langle jk \rangle = \langle j \rangle \langle k \rangle\).

For several real directed networks this equality does not hold. For example, the network of Notre-Dame University WWW [1], has \(\langle k \rangle = \langle j \rangle \approx 4.6\), and thus \(\langle jk \rangle \approx 21.16\). In contrast, measuring directly we find \(\langle jk \rangle \approx 200\), about an order of magnitude larger than the result expected for the uncorrelated case. This yields an estimate of \(q_c \approx 0.02\), i.e., a very stable directed network.

We obtained similar results also for some metabolic networks [2], indicating that in real directed networks, the in- and out-degrees are correlated.

To address correlations, we model it in the following manner: we first generate the \(j\) values for the entire network. Next, for each site with \(j \neq 0\) with probability \(A\) we generate \(k\) fully correlated with \(j\), i.e., \(k = k(j)\). Assuming that \(k(j)\) is a monotonically increasing function then the requirement \(c_{out} k^{-\lambda_{out}} dk = c_{in} j^{-\lambda_{in}} dj\) — needed to maintain the distributions scale-free — leads to \(k_{\lambda_{out}}^{-1} = j_{\lambda_{in}}^{-1}\). With probability \(1 - A\), the degree \(k\) is chosen independently from:

\[
P(j, k) \sim \begin{cases} 
(1 - A) B_c j^{-\lambda_{in}} c_{out} k^{-\lambda_{out}} & j \neq 0, \\
(1 - B_c) c_{out} k^{-\lambda_{out}} & j = 0,
\end{cases} \tag{8}
\]

where \(j(k) = k_{\lambda_{out}}^{-1}\). With this distribution, any finite fraction \(BA\) of fully correlated sites yields a diverging \(\langle jk \rangle\) whenever

\[
(\lambda_{out} - 2)(\lambda_{in} - 2) \leq 1,
\]

causing the percolation threshold to vanish (see Fig. 2).

![FIG. 2. Phase diagram of the different regimes for the IN component of scale-free correlated directed networks. The boundary between Resilient and Anomalous exponents is derived from Eq. (9) while that between Anomalous exponents and Mean field exponents is given by Eq. (2a) for \(\lambda' = 4\). For the diagram of the OUT component \(\lambda_{in}\) and \(\lambda_{out}\) change roles.](image)

In the case of no correlations between the in- and out-degrees, \(A = 0\), Eq. (8) becomes \(P(j, k) = P_{in}(j) P_{out}(k)\). Then the condition for the existence of a giant component is: \(\langle k \rangle = \langle j \rangle = 1\). Moreover, Eq. (3) reduces to:

\[
q_c = 1 - p_c = \frac{1}{\langle k \rangle}.
\]

(10)

Applying (10) to scale-free networks one concludes that for \(\lambda_{out} > 2\) and \(\lambda_{in} > 2\) a phase transition exists at a finite \(q_c\). Here we concern ourselves with the critical exponents associated with the percolation transition in scale-free network of \(\lambda_{out} > 2\) and \(\lambda_{in} > 2\) which is the most relevant regime (Fig. 2).

Percolation of the GWCC can be seen to be similar to percolation in the non-directed graph created from the
directed graph by ignoring the directionality of the links. The threshold is obtained from the criterion \[ q_c = \frac{\langle k \rangle}{\langle k(k-1) \rangle}. \] (11)

Here the connectivity distribution is the convolution of the in and out distributions

\[ P'(k) = \sum_{l=0}^{k} P(l, k-l). \] (12)

Regardless of correlations, \( P'(k) \) is always dominated by the slower decay-exponent, therefore percolation of the GWCC is the same as in non-directed scale-free networks, with \( \lambda_{eff} = \min(\lambda_{in}, \lambda_{out}) \). Note that the percolation threshold of the GWCC may differ from that of the GSCC and the IN and OUT components.

We now use the formalism of generating functions \([13,18]\) to analyze percolation of the GSCC and IN and OUT components. In \([13,18]\) a generating function is built for the joint probability distribution of outgoing and incoming degrees, before dilution:

\[ \Phi(x,y) = \sum_{j, k} P(j, k)x^j y^k. \] (13)

Using the approach of Callaway et al. \([10]\) let \( q(j, k) \) be the probability that a vertex of degree \( (j, k) \) remains in the network following dilution. The generating function after dilution is then

\[ G(x, y) = \sum_{j, k} P(j, k)q(j, k)x^j y^k. \] (14)

From \([14]\) it is possible to define the generating function for the outgoing degrees \( G_0 \)

\[ G_0(y) = G(1, y) = \sum_{j, k} P(j, k)q(j, k)y^k. \] (15)

The probability of reaching a site by following a specific link is proportional to \( jP(j, k) \), therefore, the probability to reach an occupied site following a specific directed link is generated by

\[ G_1(y) = \frac{\sum_{j, k} jP(j, k)q(j, k)y^k}{\sum_{j, k} jP(j, k)}. \] (16)

Let \( H_1(y) \) be the generating function for the probability of reaching an outgoing component of a given size by following a directed link, after a dilution. \( H_1(y) \) satisfies the self-consistent equation:

\[ H_1(y) = 1 - G_1(1) + yG_1(H_1(y)). \] (17)

Since \( G_0(y) \) is the generating function for the outgoing degree of a site, the generating function for the probability that \( n \) sites are reachable from a given site is

\[ H_0(y) = 1 - G_0(1) + yG_0(H_1(y)) . \] (18)

For the case where correlations exist, and assuming random dilution: \( q(j, k) = q \), Eqs. \((17) \) and \((18) \) reduce to

\[ H_1(y) = 1 - q + \frac{qy}{(j)} \sum_k (BAj(k) + (1 - A)(j))P_{out}(k)H_1(y)^k, \] (19)

and

\[ H_0(y) = 1 - q + qy \sum_k P_{out}(k)H_1(y)^k. \] (20)

If \( A \to 0 \), one expects that \( H_0(y) = H_1(y) \), since there is no correlation between \( j \) and \( k \), thus the probability to have \( k \) outgoing edges is \( P_{out}(k) \) whether we choose the site randomly or weighted by the incoming edges \( j \).

\( H_0(1) \) is the probability to reach an outgoing component of any finite size choosing a site. Thus, below the percolation transition \( H_0(1) = 1 \), while above the transition there is a finite probability to follow a directed link to a site which is a root of an infinite outgoing component: \( P_\infty = 1 - H_0(1) \). It follows that

\[ P_\infty(q) = q(1 - \sum_k P_{out}(k)u^k), \] (21)

where \( u \equiv H_1(1) \) is the smallest positive root of

\[ u = 1 - q + \frac{q}{(j)} \sum_k (BAj(k) + (1 - A)(j))P_{out}(k)u^k. \] (22)

Here \( P_\infty(q) \) is the fraction of sites from which an infinite number of sites is reachable. Eq. \((22) \) can be solved numerically and the solution may be substituted into Eq. \((21) \), yielding the size of the IN component at dilution \( p = 1 - q \).

Near criticality, the probability to start from a site and reach a giant outgoing component follows \( P_\infty \sim (q - q_c)'^{\beta} \). For mean-field systems (such as infinite-dimensional systems, random graphs and Cayley trees) it is known that \( \beta = 1 \) \([19]\). This regular mean-field result is not always valid. Instead, following \([20]\) we study the behavior of Eq. \((22) \) near \( q = q_c \), \( u = 1 \), and find

\[ \beta = \begin{cases} \frac{1}{\frac{\lambda^*}{\lambda^* - 3}} & 2 < \lambda^* < 3, \\ \frac{1}{\lambda^*} & 3 < \lambda^* < 4, \\ \lambda^* > 4, \end{cases} \] (23)

where

\[ \lambda^* = \lambda_{out} + \frac{\lambda_{in} - \lambda_{out}}{\lambda_{in} - 1}. \] (24)

We see that the order parameter exponent \( \beta \) attains its usual mean-field value only for \( \lambda^* > 4 \). As \( \lambda_{out} \to \lambda_{in} \) the correlated fraction \( BA \) of sites resembles non-directed
networks \cite{20,21} (where there is no distinction between incoming and outgoing degrees). In this case we get \( \lambda^* = \lambda_{out} = \lambda_{in} \) for any amount of correlation \( A \). The criterion for the existence of a giant component is then \( \langle k^2 \rangle / \langle k \rangle = 1 \), and not 2 as in the non-directed case. The difference stems from the fact that in the non-directed case one of the links is used to reach the site, while in the directed case there is generally no correlation between the location of the incoming and outgoing links. Therefore, one more outgoing link is available for leaving the site.

Without any correlations, \( A = 0 \), different terms prevail in the analysis and

\[
\beta = \begin{cases} 
\frac{1}{\lambda_{out} - 2} & 2 < \lambda_{out} < 3, \\
\frac{1}{\lambda_{out} - 3} & \lambda_{out} > 3.
\end{cases} \tag{25}
\]

This is the same as Eq. (23) but with \( \lambda^* = \lambda_{out} + 1 \).

The GSCC is the intersection of the IN and OUT components. Therefore, it behaves as the smaller of the two components: \( \beta_{GSCC} = \max(\beta_{in}, \beta_{out}) \). This can be also derived by applying the same methods as for the IN and OUT components to the generating function of the GSCC obtained in \cite{4}. The exponent for the GWCC, on the other hand, is independent of the exponents of the other components, since the transition point is different.

It is known that for a random graph of arbitrary degree distribution the finite clusters follow the scaling form

\[
n(s) \sim s^{-\tau} e^{-s/\lambda^*}, \tag{26}
\]

where \( s \) is the cluster size and \( n(s) \) is the number of clusters of size \( s \). At criticality \( s^* = |q - q_c|^{-\sigma} \) diverges and the tail of the distribution follows a power law.

The probability that \( s \) sites can be reached from a site by following links at criticality follows \( p(s) \sim s^{-\tau} \), and is generated by \( H_0(y) \), where \( H_0(y) = \sum_s p(s)y^s \). As in \cite{20}, \( H_0(y) \) can be expanded from Eq. (18). In the presence of correlations we find

\[
\tau = \begin{cases} 
1 + \frac{1}{\lambda_{in} - 2} & 2 < \lambda^* < 4, \\
1 + \frac{1}{\lambda_{out} - 2} & \lambda^* > 4.
\end{cases} \tag{27}
\]

The regular mean-field exponents are recovered for \( \lambda^* > 4 \). For the uncorrelated case we get

\[
\tau = \begin{cases} 
1 + \frac{1}{\lambda_{out} - 2} & 2 < \lambda_{out} < 3, \\
1 + \frac{1}{\lambda_{out} - 3} & \lambda_{out} > 3.
\end{cases} \tag{28}
\]

Now the regular mean-field results are obtained for \( \lambda > 3 \).

In summary, we calculate the percolation properties of directed scale-free networks. We find that the percolation critical exponents in scale-free networks are strongly dependent upon the existence of correlations and upon the degree distribution exponents in the range of \( 2 < \lambda^* < 4 \). This regime characterizes most naturally occurring networks, such as metabolic networks or the WWW. The regular mean-field behavior of percolation in infinite dimensions is recovered only for \( \lambda^* > 4 \). A connection is found between non-directed and directed scale-free percolation exponents for any finite correlation between the IN- and OUT-degrees. In the uncorrelated case, i.e. \( P(j, k) = P_{in}(j)P_{out}(k) \), the probability to reach an outgoing component does not bear any dependence upon \( P_{in}(j) \). The results are summarized in Table I.

|          | uncorrelated | correlated |
|----------|--------------|------------|
| \( GWCC \) | \( \min(\lambda_{out}, \lambda_{in}) + 1 \) | \( \min(\lambda_{out}, \lambda_{in}) \) |
| \( IN \)   | \( \lambda_{out} + 1 \) | \( \lambda_{out} + \frac{\lambda_{in} - \lambda_{out}}{\lambda_{in} - 1} \) |
| \( OUT \)  | \( \lambda_{in} + 1 \) | \( \lambda_{in} + \frac{\lambda_{out} - \lambda_{in}}{\lambda_{out} - 1} \) |
| \( GSCC \) | \( \min(\lambda_{out}, \lambda_{in}) + 1 \) | \( \min(\lambda^*_out, \lambda^*_in) \) |

TABLE I. Values of \( \lambda^* \) for the different network components for both correlated and uncorrelated cases.
ACKNOWLEDGMENTS

Support from the NSF is gratefully acknowledged (DbA).

[1] R. Albert, and A. L. Barabási, Rev. of Mod. Phys. 74, 47 (2002).
[2] S. N. Dorogovtsev, and J. F. F. Mendes, Adv. in Phys., 51, 1079, (2002).
[3] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E 64, 026118 (2001).
[4] S. N. Dorogovtsev, J. F. F. Mendes and A. N. Samukhin, Phys. Rev. E 64, 025101R, (2001).
[5] A. D. Sánchez, J. M. López and M. A. Rodríguez, Phys. Rev. Lett. 88, 048701, (2002).
[6] A. -L. Barabási, R. Albert, and H. Jeong, Physica A, 281, 2115 (2000).
[7] A. Broder, R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins and J. Wiener, Comput. Netw. 33, 309 (2000).
[8] R. Albert, H. Jeong, and A. L. Barabási, Nature, 406, 6794, 378 (2000).
[9] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. 85, 4626 (2000).
[10] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. 85, 5468 (2000).
[11] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. 86, 3682 (2001).
[12] R. V. Sole and J. M. Montoya, Proc. Roy. Soc. Lond. B Bio. 268, 2039, (2001).
[13] D. Stauffer and A. Aharony, Introduction to Percolation Theory, 2nd edition (Taylor and Francis, London, 1991).
[14] A. Bunde, and S. Havlin (editors), Fractals and Disordered System (Springer, New York, 1996).
[15] H. Hinrichsen, Adv. in Phys. 49, 815 (2000).
[16] H. Jeong, B. Tombor, R. Albert, Z. N. Oltvai and A. L. Barabási, Nature, 407, 651, (2000).
[17] H. S. Wilf, Generatingfunctionology 2nd ed. (Academic Press, London, 1994).
[18] G. H. Weiss, Aspects and Applications of the Random Walk (North-Holland, Amsterdam, 1994).
[19] P. Frojd, M. Howard and K. B. Lauritsen, IJMPB 15, 1761, (2001).
[20] R. Cohen, D. ben-Avraham, and S. Havlin, cond-mat/0202253.
[21] R. Pastor-Satorras, A. Vespignani, Phys. Rev. E, 63, 066117 (2001).