Approximation in law to operator fractional Brownian motion

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Abstract

As an well-known extension of the famous fractional Brownian motion, the operator fractional Brownian motion has been studied extensively. One direction of these studies is to study weak limit theorems for this kind of processes. In this paper, we also go along with this direction. We show that the operator fractional Brownian motion can be approximated in law by some non-linear functions of stationary Gaussian vector-valued variables.

Keywords: Operator fractional Brownian motion, Hermite polynomial, weak convergence.

1. Introduction

Self-similar processes, first studied rigorously by Lamperti \textsuperscript{[11]} under the name “semi-stable”, are stochastic processes that are invariant in distribution under suitable scaling of time and space. There has been an extensive literature on self-similar processes. We refer to Vervaat \textsuperscript{[21]} for general properties, to Samorodnitsky and Taqqu \textsuperscript{[17]} [Chaps.7 and 8] for studies on Gaussian and stable self-similar processes and random fields.

The definition of self-similarity has been extended to allow for scaling by linear operators on $\mathbb{R}^d$, and the corresponding processes are called operator self-similar (o.s.s) processes in the literature. Let $\text{End}(\mathbb{R}^d)$ be the set of linear operators on $\mathbb{R}^d$ (endomorphisms) and let $\text{Aut}(\mathbb{R}^d)$ be the set of invertible linear operators (automorphisms) in $\text{End}(\mathbb{R}^d)$. For convenience, we will not distinguish an operator $D \in \text{End}(\mathbb{R}^d)$ from its associated matrix relative to the standard basis of $\mathbb{R}^d$. Recall that an $\mathbb{R}^d-$ valued stochastic process $\tilde{Y} = \{\tilde{Y}(t)\}$ is said to be operator self-similar if it is stochastically continuous, and there exists a $D \in \text{End}(\mathbb{R}^d)$ such that for every $c > 0$

$$\tilde{Y}(ct) \overset{d}{=} c^D \tilde{Y}(t), \ t \in \mathbb{R},$$

(1.1)

where $\overset{d}{=} \text{ denotes equality of all finite-dimensional distributions, and}$

$$c^D = \exp \((\log c)D) = \sum_{k=0}^{\infty} \frac{1}{k!}(\log c)^k D^k.$$
Any matrix for which (1.1) holds is called an exponent of the o.s.s process $\tilde{Y}$. For more information on this kind of processes, refer to Laha and Rohatgi [12], Hudson and Mason [10], and Sato [16].

One of examples of operator self-similar processes is the operator fractional Brownian motion (OFBM), which corresponds to the fractional Brownian motion in one-dimensional case. OFBMs are mean-zero, o.s.s., Gaussian processes with stationary increments. They are of interests in several areas and for reasons similar to those in the univariate case. For example, see Chung [2], Davidson and de Jong [4], Didier and Pipiras [8, 7] and the references therein.

Weak convergence to FBM processes has been studied extensively since the works of Davydov [3] and Taqqu [20]. As an extension of FBM, some results on approximations of OFBMs also have been established. For example, Marinucci and Robinson [14] presented a weak limit theorem for a special class of OFBMs via a sequence of random variables. Dai [5] shown that a special kind of OFBMs can be approximated in law by a stationary sequence of vector-valued Gaussian random variables. We should point out that, in these work mentioned above, they all studied a special kind of OFBMs, but for the general ones introduced by Didier and Pipiras [7], as far as the author knows, there is no work to study their weak limit theorems. Inspired by this, we present a weak limit theorem for the general OFBMs in this work.

The asymptotical distribution of non-linear functionals of Gaussian fields has been extensively studied. See, for example, Arcones [1] and Sánchez [18, 19]. In particular, Taqqu [20] showed the fractional Brownian motion can be approximated in law by a sequence of non-linear functions of Gaussian random variables. In this paper, we proceed with Taqqu [20], and show that operator fractional Brownian motion can also be approximated by non-linear functions of vector-valued Gaussian random variables.

The rest of this paper is organized as follows. Section 2 is devoted to discussing weak convergence of vector-valued stationary processes. In Section 3, we discuss the weak limit theorems for non-linear functions of random variables. In Section 4, the main result of this paper is presented. A final note at the end of this paper is devoted to discussing the generalization of the method and results appear in this paper.

Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by $K$, which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

2. Conditions for Weak Convergence

In this section, we mainly discuss sufficient conditions for a sequence $Z_N(t), t \in [0, 1]$ of vector-valued functions converging weakly to a process $X(t)$, as $N \to \infty$. Before we state the main result of this section, we need the following preliminaries. Throughout this paper, let $B^*$ be the adjoint operator of $B$. We use $\|x\|_2$ to denote the usual Euclidean norm of $x = (x_1, \ldots, x^d)^* \in \mathbb{R}^d$. For $A \in \text{End}(\mathbb{R}^d)$, let $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2$ denote the operator norm of $A$. It is easy to see that for $A, B \in \text{End}(\mathbb{R}^d)$,

$$\|AB\| \leq \|A\| \cdot \|B\|,$$

and for every $A = (A_{i,j})_{d \times d} \in \text{End}(\mathbb{R}^d)$,

$$\max_{1 \leq i,j \leq d} |A_{i,j}| \leq \|A\| \leq d^2 \max_{1 \leq i,j \leq d} |A_{i,j}|.$$  

(2.2)
Furthermore, let $\sigma(A)$ be the collection of all eigenvalues of $A$. Let

$$\lambda_A = \min\{Re\lambda : \lambda \in \sigma(A)\} \quad \text{and} \quad \Lambda_A = \max\{Re\lambda : \lambda \in \sigma(A)\}. \quad (2.3)$$

We define the following operations between two operators on $\mathbb{R}^d$.

**Definition 2.1** Let $A(n) = \{A_{ij}(n)\} \in \text{End}\{\mathbb{R}^d\}$ and $B(n) = \{B_{ij}(n)\} \in \text{End}\{\mathbb{R}^d\}$, we call $A(n)$ is asymptotically equivalent to $B(n)$, as $n \to \infty$, if $A_{ij}(n)/B_{ij}(n) \to 1$, as $n \to \infty$, for all $i$ and $j$. We denote this by $A(n) \sim B(n)$, as $n \to \infty$.

**Definition 2.2** Let $A = \{A_{i,j}\} \in \text{End}\{\mathbb{R}^d\}$ and $B = \{B_{i,j}\} \in \text{End}\{\mathbb{R}^d\}$. The notation $A \leq B$ means that $\sum_{i,j=1}^{d}|A_{i,j}| \leq \sum_{i,j=1}^{d}|B_{i,j}|$, for all $i$ and $j$.

For the relation $A \leq B$, we have the following two properties.

**Lemma 2.1** If $A \leq B$, then $\|A\| \leq K\|B\|$. 

**Proof:** For any $A = \{A_{i,j}\} \in \text{End}\{\mathbb{R}^d\}$, we define

$$\|A\|_F = \sum_{i,j=1}^{d}|A_{i,j}|.$$ 

One can easily verify that $\|A\|_F$ is a norm. Since all the norms in a finite-dimensional space are equivalent, we easily see that lemma holds. \qed

**Lemma 2.2** If $A \leq B$, then for all $D \in \text{End}\{\mathbb{R}^d\}$,

$$AD \leq KBD, \quad \text{and} \quad DA \leq KDB. \quad (2.4)$$

Lemma 2.2 is clear, so we skip the proof.

Before we introduce the result of this section, we need to introduce some technical lemmas, which play an important role in the proof of the result of this section. The following lemma can be found in Mason and Xiao \[15\].

**Lemma 2.3** Let $D \in \text{End}(\mathbb{R}^d)$. If $\lambda_D > 0$ and $r > 0$, then for any $\delta > 0$, there exist positive constants $K_1$ and $K_2$ such that

$$\|D^r\| \leq \begin{cases} K_1 r^{\lambda_D - \delta}, & \text{for all } r \leq 1, \\ K_2 r^{\lambda_D + \delta}, & \text{for all } r \geq 1. \end{cases} \quad (2.5)$$

In order to prove weak convergence, we need a tightness criterion in the space $\mathcal{D}^d([0, 1]) = \mathcal{D}^d([0, 1] \times \mathbb{R}^d)$. In fact, we have the following lemma which comes from Dai \[5\].

**Lemma 2.4** Let $\{Z_n(t)\}_{n \in \mathbb{N}}$ be a sequence of stochastic processes in $\mathcal{D}^d([0, 1])$ satisfying:

(i) for every $n \in \mathbb{N}$, $Z_n(0) = 0$ a.s.;

(ii) there exist constants $K > 0$, $\beta > 0$, $\alpha > 1$ and an integer $N_0 \in \mathbb{N}$ such that

$$\mathbb{E}\left[\left\|Z_n(t) - Z_n(s)\right\|_2^\beta\right] \leq K(t - s)^\alpha, \quad n \geq N_0. \quad (2.6)$$
Then \( \{Z_n(t)\} \) is tight in \( D^d([0, 1]) \).

Now, we stand at a point where we can introduce the result of this section. However, before we do it, we still need the following.

**Convention:** Empty sums are equal to \((0, \cdots, 0)^*\).

The main result of this section is the following.

**Theorem 2.1** Suppose that the sequence \( Z_N(t), N = 1, 2, \cdots \) of random functions of \( D^d([0, 1]) \) satisfies

\[(i)\quad Z_N(t) = N^{-D}S_N(t), \tag{2.7}\]

with \( 0 < \lambda_D, \Lambda_D < 1 \), and \( S_N(t) = \sum_{i=1}^{[Nt]} Y_i \) for some stationary sequence of vector-valued random variables with zero mean and \( \mathbb{E}[\|Y_i\|^2] < \infty \);

\[(ii) \text{ as } N \to \infty, \quad \mathbb{E}[S_NS_N^*] \sim N^D \Gamma N^{D^*}; \tag{2.8}\]

\[(iii) \text{ as } N \to \infty, \quad \mathbb{E}[\|Z_N\|^{2\alpha}] \sim \|\mathbb{E}[Z_NZ_N^*]\|^\alpha \tag{2.9}\]

with \( \alpha > \frac{1}{2(\lambda_D - \delta)} \) for some \( \delta \in (0, \lambda_D) \);

\[(iv) \text{ the finite-dimensional distributions of } Z_N(t) \text{ converge as } N \to \infty. \]

Then the sequence \( Z_N(t) \) converges weakly, as \( N \to \infty \), to an operator self-similar stationary processes \( X(t) \) with o.s.s exponent \( D \), whose finite-dimensional distributions are the limits of those of \( Z_N(t) \).

**Proof:** In order to prove Theorem 2.1, we first prove that \( \{Z_N(t)\} \) is tight. In fact, since \( \{Y_i\} \) is stationary, \( S_N(t) - S_N(s) = S_N(t - s) \). Hence

\[\mathbb{E}\left[\|Z_N(t) - Z_N(s)\|_2^{2\alpha}\right] = \mathbb{E}\left[\|Z_N(t - s)\|_2^{2\alpha}\right]. \tag{2.10}\]

On the other hand, it follows from (2.9) that there exists a constant \( N_0 \in \mathbb{N} \) such that for all \( N \geq N_0 \)

\[\mathbb{E}\left[\|Z_N(t - s)\|_2^{2\alpha}\right] \leq K\|\mathbb{E}[Z_N(t - s)Z_N^*(t - s)]\|^\alpha. \tag{2.11}\]

We shoulud note that, according to (2.8), we have, as \( N \to \infty \),

\[\mathbb{E}[S_N(t - s)S_N^*(t - s)] \sim N^D(t - s)^D \Gamma(t - s)^{D^*} N^{D^*}. \tag{2.12}\]
Therefore, by (2.10) to (2.12), we can get that there exists \( N_0 \in \mathbb{N} \) such that for all \( N \geq N_0 \),

\[
E \left[ \| Z_N(t) - Z_N(s) \|_2^{2\alpha} \right] \leq K \| (t - s)^D \Gamma (t - s)^{D^*} \|^\alpha. \tag{2.13}
\]

It follows from (2.13) and Lemma 2.3 that there exists a constant \( N_0 \in \mathbb{N} \) such that for all \( N \geq N_0 \)

\[
E \left[ \| Z_N(t) - Z_N(s) \|_2^{2\alpha} \right] \leq K (t - s)^{2\alpha(\lambda_D - \delta)}. \tag{2.14}
\]

Finally, it follows from Lemma 2.4 and (2.14) that \( \{ Z_N(t) \} \) is tight.

The tightness and convergence of the finite-dimensional distributions ((iv) of Theorem 2.1) ensure the weak convergence of \( Z_N(t) \) to some limiting process \( X(t) \). Since \( \{ Y_i \} \) is stationary, \( X(t) \) must be stationary.

Next, we show that \( \{ X(t) \} \) is continuous. From the fact that the finite-dimensional distributions of \( Z_N(t) \) converge to \( X(t) \), we can get there exists \( N_1 \in \mathbb{N} \) such that

\[
E \left[ \| X(t) - X(s) \|_2^{2\alpha} \right] = \lim_{N \to \infty} E \left[ \| Z_N(t) - Z_N(s) \|_2^{2\alpha} \right] \leq K (t - s)^{2\alpha(\lambda_D - \delta)}. \tag{2.15}
\]

It follows from Proposition 10.3 in Ethier and Kurtz [9] that \( X(t) \) is a.s. continuous.

Finally, from the above arguments, we get from Hudson and Mason [10] that \( X(t) \) is operator self-similar. \( \Box \)

**Remark 2.1** From the proof of Theorem 2.1, we have that the condition (ii) in Theorem 2.1 can be replaced by the following condition (ii').

\[(ii') \quad \mathbb{E} \left[ S_N S_N^* \right] \leq K N^D \Gamma N^{D^*}, \text{ as } N \to \infty. \tag{2.17}\]

**Remark 2.2** If \( X(t) \) is a Gaussian process, then we can easily get that \( X(t) \) is a time reversible process.

### 3. Limit Theorems for Non-linear Functions

In this section, we mainly discuss the limit theorems for non-linear functionals of stationary vector-valued Gaussian sequence. In the rest of this paper, in order to simplify our discussion, we only consider this problem in the two-dimensional space. For the higher cases, we can use the same method to get the same result.

#### 3.1. Conditions for Weak Convergence

Let \( \{ X_i = (X_i^1, X_i^2) \} \) be a stationary vector-valued sequence with zero mean. Let \( r(i, j) = r(|i - j|) = \mathbb{E} [X_i X_j^*] = (r_{p,q}(i,j))_{2 \times 2} \) be its correlation function.

Below, we discuss what conditions can be imposed on a function \( G \) and on the sequence of \( r(n) \) such that \( \sum_{i=1}^{[N]} G(X_i) \) converges weakly to a process, as \( N \to \infty \).

Let

\[
H_i(x) = (-1)^i \int e^{\frac{x}{2}} \frac{d^l}{dx^l} e^{-\frac{x^2}{2}}
\]
be the Hermite polynomials. Inspired by Sánchez [18], we let \( e_L(X) \) have one of the following forms
\[
(H_{l_1}(X^1)H_{l_2}(X^2), 0)^*,
\]
and
\[
(0, H_{l_1}(X^1)H_{l_2}(X^2))^*,
\]
where \( X = (X^1, X^2)^* \) and \( L = (l_1, l_2)^* \). Furthermore, let \( G : \mathbb{R}^2 \to \mathbb{R}^2 \) be a measurable vector-valued function. Inspired by Acrones [1], Sánchez [18] and Taqqu [20], we define the Hermite rank of a function \( G \).

**Definition 3.1** If \( \mathbb{E}[\|G\|^2_2] < \infty \), and \( G(X) \) has zero mean, then we define the Hermite rank of \( G \) as follows.

\[
\text{Rank}(G) = \inf \left\{ \tau : \exists L = \{l_1, l_2\} \text{ with } l_1 + l_2 = \tau, \text{ such that } \mathbb{E}\left[ G^*(X)e_L(X) \right] \neq 0 \right\}. \quad (3.1)
\]

Moreover, define:
\[
\mathcal{G}_m = \{ G : \mathbb{E}[G(X)] = (0, 0)^*, \mathbb{E}[\|G(X)\|^2_2] < \infty, \text{ and } \text{Rank}(G) = m \}.
\]

In order to answer the problem mentioned above, we need the following condition.

**Definition 3.2** Let \( D \in \text{End}\{\mathbb{R}^2\} \) with \( \frac{1}{2} < \lambda_D, \Lambda_D < 1 \). We say that a stationary sequence \( \{X_i\} \) of vector-valued random variables with zero mean satisfies **Condition** \( \mathcal{H}(m, D) \), if

(i) as \( N \to \infty \)
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} N^{-D} \Pi N^{-D^*} \to A, \text{ for some operator } A \in \text{End}\{\mathbb{R}^2\},
\]
where \( \Pi = \{\Pi_{p,q}\}_{2 \times 2} \in \text{End}\{\mathbb{R}^2\} \) is an operator with
\[
\Pi_{p,q} = \sum_{k=1}^{2} \sum_{n=1}^{2} [r_{k,n}(|i - j|)]^m;
\]

(ii) as \( N \to \infty \),
\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \left[ (r_{p,q}(i,j))^m \right] \sim N^D \Gamma_N N^{D^*};
\]

(iii) as \( n \to \infty \),
\[
\sum_{p=1}^{2} \sum_{q=1}^{2} r_{p,q}(|n|) \to 0.
\]

The following theorem answers the question we mentioned at the beginning of this section.
Theorem 3.1 Let $G \in \mathbb{G}_m$ and $\{X_i\}$ satisfies Condition $\mathcal{H}(m, D)$. Then for large enough $N \in \mathbb{N}$,

$$
\mathbb{E}\left[ \left( \sum_{i=1}^{N} G(X_i) \right) \left( \sum_{i=1}^{N} G^*(X_i) \right) \right] \leq KN^D \Gamma N^{D^*}.
$$

(3.2)

Proof: Inspired by Major [13] and Sánchez [18], we can expand $G(X_i)$ as follows.

$$
G(X_i) = \sum_{r=0}^{\infty} \sum_{l \in I_r} C_L e_L(X_i),
$$

(3.3)

where $L = (l_1, l_2)$, $I_r = \{l_1 + l_2 = r\}$, and $C_G = \sum_{r=0}^{\infty} \sum_{l \in I_r} C^2_L l_1! l_2! < \infty$.

It follows from Sánchez [18] that for any $i, j \in \mathbb{N}$,

$$
\mathbb{E}\left[ G(X_i) G^*(X_j) \right] = \sum_{r=0}^{\infty} \sum_{K, L \in I_r} C_L C_K \mathbb{E}\left[ e_L(X_i) e^*_K(X_j) \right].
$$

(3.4)

Since the rank of $G$ is $m$, we can rewrite equation (3.4) as follows.

$$
\mathbb{E}\left[ G(X_i) G^*(X_j) \right] = \sum_{K, L \in I_m} \mathbb{E}\left[ e_L(X_i) e^*_K(X_j) \right] + \sum_{r=m+1}^{\infty} \sum_{K, L \in I_r} C_L C_K \mathbb{E}\left[ e_L(X_i) e^*_K(X_j) \right]
$$

$$
= \mathbb{E}[Q_1(i, j)] + \mathbb{E}[Q_2(i, j)],
$$

(3.5)

where

$$
Q_1(i, j) = \sum_{K, L \in I_m} C_L C_K e_L(X_i) e^*_K(X_j),
$$

and

$$
Q_2(i, j) = \sum_{r=m+1}^{\infty} \sum_{K, L \in I_r} C_L C_K e_L(X_i) e^*_K(X_j).
$$

Now we show that there exists a constant $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$
\mathbb{E}\left[ \sum_{i=1}^{N} \sum_{i=1}^{N} Q_1(i, j) \right] \leq KN^D \Gamma N^{D^*}.
$$

(3.6)

In order to simplify the notation, we assume that $L^{(p)}$ belongs to either $I_m$ or $I_0$, so does $K^{(q)}$. Recall that $e_L(X)$ has one of the following forms:

(1) \( \left( H_{l_1}(X^1) H_{l_2}(X^2), 0 \right)^* \) and

(2) \( \left( 0, H_{l_1}(X^1) H_{l_2}(X^2) \right)^* \).

Hence, we have

$$
\mathbb{E}[Q_1(i, j)] = \mathbb{E}\left[ \left( M_{p,q} \right)_{2 \times 2} \right],
$$

(3.7)
where $M_{p,q} = \sum_{K(\varnothing), L(\varnothing)} C_{K(\varnothing)} C_{L(\varnothing)} H_{i(\varnothing)}(X)^{1}_{i} H_{j(\varnothing)}(X)^{2}_{j} H_{k_1(\varnothing)}(X)^{3}_{j} H_{k_2(\varnothing)}(X)^{4}_{j}$. For all $p, q \in \{1, 2\}$, we have
\[
\left| E[M_{p,q}] \right| \leq K \sum_{K(\varnothing), L(\varnothing)} \left| E \left[ \left. C_{L(\varnothing)}^{2} \prod_{n=1}^{2} H_{l_n}^{P}(X)^{n}_{n} H_{k_n}^{B}(X)^{n}_{n} \right]\right. \right| + \\
+ K \sum_{K(\varnothing), L(\varnothing)} \left| E \left[ \left. C_{K(\varnothing)}^{2} \prod_{n=1}^{2} H_{i_n}^{P}(X)^{n}_{n} H_{k_n}^{B}(X)^{n}_{n} \right]\right. \right| \\
\leq K \sum_{K(\varnothing), L(\varnothing)} \left| E \left[ \prod_{n=1}^{2} \frac{H_{i_n}^{P}(X)^{n}_{n} H_{k_n}^{B}(X)^{n}_{n}}{l_n k_n} \right]\right. \right|. \tag{3.8}
\]

It follows from Sánchez [18] that
\[
\sum_{K, L \in \mathcal{L}_m} \left| E \left[ \prod_{n=1}^{2} \frac{H_{i_n}^{P}(X)^{n}_{n} H_{k_n}^{B}(X)^{n}_{n}}{l_n k_n} \right]\right. \right| \leq \frac{1}{m!} \left( \sum_{n,h=1}^{2} \left| E [X^{n}_{i} X^{h}_{j}] \right| \right)^m \\
\leq \frac{1}{m!} \left( \sum_{n,h=1}^{2} \left| r_{n,h}(i,j) \right| \right)^m. \tag{3.9}
\]

Since for $m \in \mathbb{N}, (a + b)^m \leq K(a^{m} + b^{m})$ with $a, b > 0$, we get that
\[
E \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} Q_{1}(i,j) \right] \leq K \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \left[ r_{p,q}(i,j) \right] \right)^m. \tag{3.10}
\]

It follows from the definition of $A \sim B$ and (ii) of Theorem 3.1 that
\[
E \left[ \sum_{i,j=1}^{N} Q_{1}(i,j) \right] \leq K N^{D_{T}} N^{-D_{r}}, \tag{3.11}
\]
as $N \to \infty$.

In order to establish (3.12), it is sufficient to show that as $N \to \infty$,
\[
N^{-D} \mathbb{E} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} Q_{2}(i,j) \right] N^{-D_{r}} \to 0, \tag{3.12}
\]
i.e.,
\[
\left\| N^{-D} \mathbb{E} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} Q_{2}(i,j) \right] N^{-D_{r}} \right\| \to 0. \tag{3.13}
\]

In order to simplify the notation, let us define
\[
B(N, Q) = \{(i,j) : |i - j| \leq Q, 0 \leq i, j \leq N\};
\]
and

\[ \hat{B}(N, Q) = \{(i, j) : |i - j| > Q, 0 \leq i, j \leq N \}. \]

Note that

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E}[Q(i, j)] = \sum_{r=m+1}^{\infty} \sum_{(i, j) \in B(N, Q)} \mathbb{E} \left[ \sum_{L \in \mathcal{I}_r} (C_L e_L(X_i))(\sum_{K \in \mathcal{I}_r} C_K e_K^*(X_i)) \right] \\
+ \sum_{r=m+1}^{\infty} \sum_{(i, j) \in \hat{B}(N, Q)} \mathbb{E} \left[ \sum_{L \in \mathcal{I}_r} (C_L e_L(X_i))(\sum_{K \in \mathcal{I}_r} C_K e_K^*(X_i)) \right] \\
= \sum_{r=m+1}^{\infty} \mathbb{E}[Q_{21}(N)] + \sum_{r=m+1}^{\infty} \mathbb{E}[Q_{22}(N)],
\]

(3.14)

where

\[
\mathbb{E}[Q_{21}(N)] = \sum_{(i, j) \in B(N, Q)} \mathbb{E} \left[ \sum_{L \in \mathcal{I}_r} (C_L e_L(X_i))(\sum_{K \in \mathcal{I}_r} C_K e_K^*(X_i)) \right],
\]

and

\[
\mathbb{E}[Q_{22}(N)] = \sum_{(i, j) \in \hat{B}(N, Q)} \mathbb{E} \left[ \sum_{L \in \mathcal{I}_r} (C_L e_L(X_i))(\sum_{K \in \mathcal{I}_r} C_K e_K^*(X_i)) \right].
\]

We first deal with the first term in the R.H.S of (3.14). Actually,

\[
\mathbb{E}[Q_{21}(N)] = \sum_{(i, j) \in B(N, Q)} \sum_{K, L \in \mathcal{I}_r} C_L C_K \mathbb{E}[e_L(X_i)e_K^*(X_j)].
\]

(3.15)

Using the same method as the proof of (3.17), we can get that

\[
\left\| \mathbb{E}[Q_{21}(N)] \right\| \leq K \max_{p,q} \left\{ \sum_{(i, j) \in B(N, Q)} \left| \mathbb{E}[M_{p,q}] \right| \right\}.
\]

(3.16)

On the other hand, we have that

\[
\left| \mathbb{E}[M_{p,q}] \right| = \left| \sum_{L(p), K(q)} \mathbb{E} \left[ C_L^{(p)} C_K^{(q)} \prod_{n=1}^{2} H_{l_n}(X_i^n)H_{h_n}(X_j^n) \right] \right| \\
\leq K \sum_{L, K \in \mathcal{I}_r} \mathbb{E} \left[ \left( C_L^2 H_{l_1}(X_i^1)H_{j_1}(X_j^2) \right)^2 \right] \\
+ K \sum_{L, K \in \mathcal{I}_r} \mathbb{E} \left[ \left( C_K^2 H_{h_1}(X_j^1)H_{k_2}(X_j^2) \right)^2 \right].
\]

(3.17)

It follows from Lemma 5.1 in Sánchez [18] that

\[
\mathbb{E} \left[ H_{l_1}(X_i^1)H_{l_2}(X_i^2) \right] = l_1!l_2!.
\]

(3.18)
Therefore, by \((3.15)\) to \((3.18)\), we get that
\[
\left\| \mathbb{E}[Q_{21}(N)] \right\| \leq K \sum_{(i,j) \in B(N,Q)} \sum_{L \in I_r} C^2 l_1 l_2!.
\] (3.19)

From the above argument, we get that
\[
\left\| \sum_{r=m+1}^{\infty} \mathbb{E}[Q_{21}(N)] \right\| \leq K \sum_{r=m+1}^{\infty} \sum_{(i,j) \in B(N,Q)} \sum_{L \in I_r} C^2 l_1 l_2!
\leq 2QN \sum_{r=m+1}^{\infty} \sum_{l \in I_r} C^2 l_1 l_2!
\leq 2C_GQN,
\] (3.20)

where \(C_G = \sum_{r=m}^{\infty} \sum_{l \in I_r} C^2 l_1 l_2! < \infty\).

On the other hand,
\[
\left\| \mathbb{E}\left[N^{-D}Q_{21}(N)N^{-D^*}\right] \right\| \leq KN^{-2(\lambda_D-\delta)} \left\| \mathbb{E}[Q_{21}(N)] \right\|,
\] (3.21)

with \(\delta \in (0, 1)\).

Combining \((3.20)\) and \((3.21)\), we get that as \(N \to \infty\)
\[
\left\| \mathbb{E}\left[N^{-D}Q_{21}(N)N^{-D^*}\right] \right\| \to 0,
\] (3.22)

since \(\lambda_D > \frac{1}{2}\).

Now we deal with the second term on the R.H.S of \((3.14)\). Similar to \((3.13)\), we have
\[
\mathbb{E}[Q_{22}(N)] \leq K \sum_{(i,j) \in B(N,Q)} \left( \left| \mathbb{E}[M_{p,q}] \right| \right)_{2 \times 2}.
\] (3.23)

We also note that
\[
\left| \mathbb{E}[M_{p,q}] \right| \leq \sum_{K,L \in I_r} C_L C_K \mathbb{E}\left[ \prod_{n=1}^{2} H_{l_n}(X^n_i)H_{k_n}(X^n_j) \right]
= \sum_{K,L \in I_r} \left\{ C_L C_K \mathbb{E}\left[ r! \prod_{n=1}^{2} \frac{H_{l_n}(X^n_i)H_{k_n}(X^n_j)}{l_n!k_n!} \right] \prod_{n=1}^{2} l_n!k_n! \right\}
\] (3.24)
By the Schwartz inequality, we get that

\[
\sum_{K,L \in I_r} |C_L C_K| \mathbb{E} \left[ \prod_{n=1}^{2} H_{l_{n}}(X_{i}^{n}) H_{k_{n}}(X_{j}^{n}) \right] \\
\leq \left\{ \sum_{K,L \in I_r} (C_L^2 C_K^2) \left( \frac{\prod_{n=1}^{2} l_{n}! k_{n}!}{r!} \right)^2 \right\}^{\frac{1}{2}} \\
\times \left\{ \left( \sum_{K,L \in I_r} r! \mathbb{E} \left[ \prod_{n=1}^{2} H_{l_{n}}(X_{i}^{n}) H_{k_{n}}(X_{j}^{n}) \left( \frac{\prod_{n=1}^{2} l_{n}! k_{n}!}{r!} \right) \right] \right)^2 \right\}^{\frac{1}{2}}.
\]

(3.25)

We observe that

\[
\left\{ \sum_{K,L \in I_r} \left( r! \mathbb{E} \left[ \prod_{n=1}^{2} H_{l_{n}}(X_{i}^{n}) H_{k_{n}}(X_{j}^{n}) \left( \frac{\prod_{n=1}^{2} l_{n}! k_{n}!}{r!} \right) \right] \right)^2 \right\}^{\frac{1}{2}} \\
\leq K \sum_{K,L \in I_r} \left| r! \mathbb{E} \left[ \prod_{n=1}^{2} H_{l_{n}}(X_{i}^{n}) H_{k_{n}}(X_{j}^{n}) \left( \frac{\prod_{n=1}^{2} l_{n}! k_{n}!}{r!} \right) \right] \right|,
\]

(3.26)

and

\[
\left\{ \sum_{K,L \in I_r} C_K^2 C_L^2 \left( \frac{\prod_{n=1}^{2} l_{n}! k_{n}!}{r!} \right)^2 \right\}^{\frac{1}{2}} \leq K \left( \sum_{l \in I_r} C_l^2 l_1! l_2! \right),
\]

(3.27)

since \( k_1! k_2! \leq r! \).

Due to Sánchez [18], we obtain that

\[
\sum_{K,L \in I_r} \left| r! \mathbb{E} \left[ \prod_{n=1}^{2} H_{l_{n}}(X_{i}^{n}) H_{k_{n}}(X_{j}^{n}) \left( \frac{\prod_{n=1}^{2} l_{n}! k_{n}!}{r!} \right) \right] \right| \leq \left( \sum_{p=1}^{2} \sum_{q=1}^{2} |r_{p,q}(i,j)| \right)^r.
\]

(3.28)

It follows from (3.28) to (3.28) that

\[
\mathbb{E} [Q_{22}(N)] \leq K \sum_{r=m+1}^{\infty} \sum_{(i,j) \in B(N,Q)} \left( \sum_{p=1}^{2} \sum_{q=1}^{2} r_{p,q}(i,j) \right)^r \left( \sum_{L \in I_r} C_L^2 l_1! l_2! \right) \Pi
\]

\[
\leq \sum_{r=m+1}^{\infty} \sum_{(i,j) \in B(Q,N)} \left( \sum_{p=1}^{2} \sum_{q=1}^{2} |r_{p,q}(|n|)| \right)^r \left( \sum_{L \in I_r} C_L^2 l_1! l_2! \right) \Pi.
\]

(3.29)

According to (iii) of Condition \( \mathcal{H}(m, D) \), there exists a constant \( Q \in \mathbb{N} \) such that for all \( n \geq Q \)

\[
\sum_{p=1}^{2} \sum_{q=1}^{2} |r_{p,q}(|n|)| \leq \lambda < 1.
\]

(3.30)
By (3.29) and (3.30), we get
\[ E[Q_{22}(N)] \leq K \left( \sum_{p=1}^{2} \sum_{q=1}^{2} r_{p,q}(|n|) \right)^{m+1} \sum_{r=m+1}^{\infty} \sum_{(i,j) \in B(N,Q)} \left( \sum_{L \in I_r} C_{L}^{q} l_{1} ! l_{2} ! \right) \]
\[ \leq KeC_{G} \left( \sum_{p=1}^{2} \sum_{q=1}^{2} r_{p,q}(|n|) \right)^{m} \Pi. \] (3.31)

Since for any \( m \in \mathbb{N} \),
\[ (a + b)^m \leq K(a^m + b^m) \]
with \( a, b > 0 \), we get from (3.31) that
\[ E[Q_{22}(N)] \leq KeC_{G} \left( \sum_{p=1}^{2} \sum_{q=1}^{2} [r_{p,q}(|n|)]^m \right) \Pi, \] (3.32)
as \( N \to \infty \).

Hence, by (i) of Condition \( \mathcal{H}(m,D) \), (3.20), (3.10), and (3.32), we have that
\[ \left\| E \left[ N^{-D}Q_{22}(N) N^{-D^*} \right] \right\| \to 0, \] (3.33)
as \( N \to \infty \).

Therefore, by (3.10), (3.22) and (3.33), we get that the theorem holds. \( \square \)

**Remark 3.1** From the proof of Theorem 3.1 we easily get that as \( N \to \infty \),
\[ \sum_{i=1}^{N} \sum_{j=1}^{N} \left\| E \left[ \sum_{r=m+1}^{\infty} \sum_{K,L \in I_e} C_{L} C_{K} N^{-D} e_{L}(X_i)e_{K}^{*}(X_j) N^{-D^*} \right] \right\| \to 0. \] (3.34)

### 3.2. Reduction Theorem

In this subsection, we assume that \( G \in \mathcal{G}_m \) and \( \{X_i\} \) satisfies Condition \( \mathcal{H}(m,D) \), and study the weak limit theorem for the process
\[ Z_N(t) = N^{-D} \sum_{i=1}^{[Nt]} G(X_i). \] (3.35)

For the sake of convenience, we define the following notations.
\[ Z_{N,m}(t) = N^{-D} \left[ \sum_{i=1}^{[Nt]} \sum_{L \in I_m} C_{L} e_{L}(X_i) \right], \]
\[ = \sum_{i=1}^{[Nt]} \sum_{L \in I_m} C_{L} N^{-D} e_{L}(X_i), \] (3.36)
and

\[ \tilde{Z}_{N,m}(t) = \sum_{i=1}^{[Nt]} \sum_{r=m+1}^{\infty} \sum_{L \in I_r} C_L N^{-D} e_L(X_i), \quad (3.37) \]

where

\[ C_L = \frac{\mathbb{E}[G^*(X)e_L(X)]}{l_1!l_2!}. \]

Before we state our result, we need the following useful lemma.

**Lemma 3.1** If the limits of the finite-dimensional distributions of \((Z_{N,m}(t))\) exist, we denote it by \((Z_m(t_1), \ldots, Z_m(t_p))\), then

\[ \left( Z_{N,m}(t), \cdots, Z_{N,m}(t_p) \right) \overset{d}{\Rightarrow} \left( \sum_{L \in I_m} C_L \tilde{Z}_m(t_1), \cdots, \sum_{L \in I_m} C_L \tilde{Z}_m(t_p) \right), \quad (3.38) \]

where \(\overset{d}{\Rightarrow}\) denotes convergence in distribution.

**Proof:** In order to simplify the discussions, we only prove the case that \(p = 1\). The case that \(p \in \mathbb{N}\) can be done in the same way. According to (3.36) and (3.37), in order to prove (3.38), we only need to prove that

\[ \tilde{Z}_{N,m}(t) \overset{d}{\Rightarrow} 0, \quad (3.39) \]

as \(N \to \infty\). To prove (3.39), it is sufficient to prove that \(\{\tilde{Z}_{N,m}\}\) converges to zero in probability, i.e., as \(N \to \infty\)

\[ \mathbb{P}\left\{ \|\tilde{Z}_{N,m}\|_2 \geq \epsilon \right\} \to 0. \quad (3.40) \]

Note that for an \(\mathbb{R}^d\)-valued random variable \(Q = (Q^1, \cdots, Q^d)^*\), \(\mathbb{E}[\|Q\|_2^2]\) equals the sum of diagonal entries of the correlation matrix. It follows from the above arguments and (2.2) that

\[ \mathbb{E}\left[ \|\tilde{Z}_{N,m}(t)\|_2^2 \right] \leq K \mathbb{E}\left[ \tilde{Z}_{N,m}(t) \tilde{Z}_{N,m}^*(t) \right]. \quad (3.41) \]

Hence,

\[
\mathbb{E}\left[ \|\tilde{Z}_{N,m}(t)\|_2^2 \right] \leq K \mathbb{E}\left[ \tilde{Z}_{N,m}(t) \tilde{Z}_{N,m}^*(t) \right] \\
\leq K \left\| \mathbb{E}\left[ \sum_{i=1}^{[Nt]} \sum_{j=1}^{[Nt]} \left( \sum_{r=m+1}^{\infty} \sum_{L \in I_r} C_L C_K N^{-D} e_L(X_i)^* e_K(X_j) N^{-D^*} \right) \right] \right\| \\
= K \sum_{i,j=1}^{N} \left\| \mathbb{E}\left[ \sum_{r=m+1}^{\infty} \sum_{L \in I_r} C_L C_K N^{-D} e_L(X_i)^* e_K(X_j) N^{-D^*} \right] \right\|, \quad (3.42)
\]

since \(t \in (0, 1)\).

Therefore, we get from Remark 3.1 and the Chebyshev-Markov inequality [6, Chap.1] that (3.42) holds. So the lemma holds. \(\Box\)

By the theorems 2.1, 3.1 and Lemma 3.1 we have
Theorem 3.2 Let \( G \in \mathbb{G}_m \) for some \( m \geq 1 \), and \( \{X_i\} \) satisfies Condition \( H(m, D) \). Define \( Z_N(t) \) as in (3.33) and \( Z_{N,m}(t) \) as in (3.36). If the finite-dimensional distributions of \( Z_{N,m}(t) \) converge to \( Z_m(t) \), then \( Z_N(t) \) converges weakly to some process \( \sum_{L \in I_m} C_L Z_m(t) \).

4. Limit Theorem for Operator Fractional Brownian Motion

In this section, we present the main result of this paper. We show that when the Hermite Rank \( m \) of \( G \) is 1, the limiting process turns out to be proportional to a time reversible operator Brownian motion (OFBM) introduced by Didier and Pipiras [7].

Lemma 4.1 Let \( D \) be a linear operator on \( \mathbb{R}^d \) with \( 0 < \Lambda_D, \lambda_D < 1 \). Let \( X = \{X(t)\} \) be an OFBM with o.s.s. exponent \( D \). Then \( X \) admits the integral representation

\[
X(t) = \int_{\mathbb{R}} e^{itx} - \frac{1}{ix} \left( x^{-(D-rac{1}{2})} A + x^{-(D-rac{1}{2})} \bar{A} \right) W(dx)
\]

for some linear operator \( A \) on \( \mathbb{C}^d \). Here, \( \bar{A} \) denotes the complex conjugate and

\[
W(x) := W_1(x) + iW_2(x)
\]
denotes a complex-valued multivariate Brownian motion such that \( W_1(-x) = W_1(x) \) and \( W_2(-x) = -W_2(x) \), \( W_1(x) \) and \( W_2(x) \) are independent, and the induced random measure \( W(x) \) satisfies

\[
\mathbb{E}\left[ W(dx)W^*(dx) \right] = dx,
\]

where \( W^* \) is the adjoint operator of \( W \).

Inspired by Samorodnitsky and Taqqu [17] [Chap. 7], up to a multiplicative constant from the left, we can rewrite \( \{X(t)\} \) as follows.

\[
X(t) \overset{d}{=} \int_0^\infty G_1(x,t)W_1(dx) + \int_0^\infty G_2(x,t)W_2(dx),
\]

where

\[
G_1(x,t) = \frac{\sin tx}{x} x^{-(D-rac{1}{2})} A_1 + \frac{\cos tx - 1}{x} x^{-(D-rac{1}{2})} A_2,
\]

\[
G_2(x,t) = \frac{\sin tx}{x} x^{-(D-rac{1}{2})} A_2 + \frac{1 - \cos tx}{x} x^{-(D-rac{1}{2})} A_1,
\]

and

\[
A = A_1 + iA_2.
\]

In order to give the main result of this paper, we also need the following lemma, which comes from Dai [5].

Lemma 4.2 Let \( \{Z_i, i = 1, 2, \cdots\} \) be a stationary proper mean-zero Gaussian sequence of \( \mathbb{R}^d \)-valued vectors. We define

\[
r(i,j) = \mathbb{E}[Z_iZ_j'] = \left( r_{k,q}(|i-j|) \right)_{d \times d},
\]

where
Suppose that

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} r(i, j) \sim KBN^D \Gamma N^D B^*, \text{ as } N \to \infty, \tag{4.3}
\]

where \(\Gamma = \mathbb{E}[X(1)X'(1)]\), \(B \in \text{Aut}(\mathbb{R}^d)\) and \(K > 0\) is a positive number. Then

\[
Q_N(t) = d_N \sum_{i=1}^{\lfloor Nt \rfloor} Z_i, \tag{4.4}
\]

with \(d_N \sim CN^{-D}B^{-}\), converges weakly, as \(N \to \infty\) in \(D([0,1])\), up to a multiplicative matrix from the left, to the time reversible OFBM \(X\) given by (4.2) with \(A_2 A_1^* = A_1 A_2^*\), where \(C \in \text{Aut}(\mathbb{R}^d)\).

Now we give the main result of this paper.

**Theorem 4.1** Suppose that \(G \in G_1\) satisfies Condition \(\mathcal{H}(1, D)\). Then

\[
Z_N(t) = N^{-D} \sum_{i=1}^{\lfloor Nt \rfloor} G(X_i), \tag{4.5}
\]

converges weakly to the time reversible OFBM \([C_1 + C_2 + \tilde{C}_1 + \tilde{C}_2]X(t)\) given by (4.2), where

\[
C_i = \mathbb{E}\left[ G^*(X) \left(X^i, 0\right)^* \right],
\]

and

\[
\tilde{C}_i = \mathbb{E}\left[ G^*(X) \left(0, X^i\right)^* \right].
\]

By Theorems 2.1 3.1 3.2 and Lemma 4.2, we can straightly prove Theorem 4.1. Here we omit the proof.

5. A Final Note

In this paper, we present a weak limit theorem for two-dimensional operator fractional Brownian motion. Although we only discuss the problems in the two-dimensional space, our results can be easily extended to the more general \(n\)-dimensional space. In fact, using the same methods and following the same steps, we can trivially extend our results to those in the \(n\)-dimensional space. We should point out that the only difference between the details of studying two-dimensional problems and those of studying \(n\)-dimensional problems is that the calculations involved in \(n\)-dimensional case seems to be more complicated than those in the two-dimensional case. However, the methods involved in both cases are same. So the generalization is trivial and boring. Here, we skip them.

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