Quantum mechanical instabilities of Cauchy horizons in two dimensions — a modified form of the blueshift instability mechanism.

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There are several examples known of two dimensional spacetimes which are linearly stable when perturbed by test scalar classical fields, but which are unstable when perturbed by test scalar quantum fields. We elucidate the mechanism behind such instabilities by considering minimally coupled, massless, scalar, test quantum fields on general two dimensional spacetimes with Cauchy horizons which are classically stable. We identify a geometric feature of such spacetimes which is a necessary condition for obtaining a quantum mechanical divergence of the renormalized expected stress tensor on the Cauchy horizon for regular initial states. This feature is the divergence of the affine parameter length of a certain one parameter family of null geodesics which lie parallel to the Cauchy horizon, where the affine parameter normalization is determined by parallel transport along a fixed, transverse null geodesic which intersects the Cauchy horizon. (By contrast, the geometric feature of such spacetimes which underlies classical blueshift instabilities is the divergence of a holonomy operator). We show that the instability can be understood as a “delayed blueshift” instability, which arises from the infinite blueshifting of an energy flux which is created locally and quantum mechanically. The instability mechanism applies both to chronology horizons in spacetimes with closed timelike curves, and to the inner horizon in black hole spacetimes like two dimensional Reissner-Nordstrom-de Sitter.

I. INTRODUCTION AND SUMMARY

A. Cauchy horizon instabilities : background and motivation

The physical question addressed by the Haifa workshop can be summarized as “What is the generic physical nature of black hole interiors?” Current attempts to answer this question involve using all of the laws of physics which are well understood today, including classical gravity and semiclassical gravity. As remarked by Valerie Frolov at the workshop, we theorists who study this issue are fortunate in two separate ways: First, we are protected from possibly embarrassing confrontations with experimental data by the fact that the gravitational singularities inside black holes are, very likely, always hidden behind event horizons. Second, some of the most interesting and deep issues involving the singularities inside black holes such as their possible traversability [1] and such as the information loss paradox [2] are ultimately obscured by the “Planck fog” of Planck scale physics where both classical and semiclassical gravity break down. Rather than attempt to grapple with these deep issues, we can justifiably throw up our hands once Planck scale curvatures are reached and declare the subsequent evolution to be beyond our purvue, given the absence of a well understood theory of quantum gravity.

Nevertheless, it seems to me that it is still of interest to investigate the structure of black hole interiors within the domains of validity of classical and semiclassical gravity. Black hole interiors can in principle be probed experimentally, by observers who venture inside the event horizon, and possibly even by external observers if weak cosmic censorship turns out not to be valid. Moreover, the structure of singularities even in the sub-Planckian regime is a deep and complicated question, and is interesting both in its own right and because understanding this structure is presumably a necessary prerequisite to the eventual understanding of black holes in full quantum gravity.

The study of black hole interiors is impeded by the fact that under certain circumstances general relativity breaks down; indeed, it predicts its own demise. As is well known, given initial data specified on some spacelike surface Σ, this breakdown can take two forms. First, in the maximal Cauchy evolution $D^+(Σ)$ of the initial data, the predicted gravitational field strength (curvature scalars or components of curvature tensors on parallel propagated bases along curves) can grow without limit. This prediction is presumably valid only up to the regime of Planckian curvatures, at which point our understanding breaks down. The prediction of infinities or singularities is not particularly worrisome; similar situations, where a continuum theory predicts its own breakdown, occur in other contexts in physics. A good example, as Amos Ori pointed out during the workshop, is the formation of shocks in fluids. There, the hydrodynamic equations predict that the fluid variables diverge; in actuality, the structure of shocks is determined by microscopic physics for which the fluid approximation is invalid. Presumably, something similar occurs at gravitational singularities: their structure is ruled by the as-yet-unknown laws of quantum gravity.

The second, well-known type of breakdown of general relativity is where the maximal Cauchy evolution $D^+(Σ)$ of the initial data is geodesically incomplete without any curvature singularities. For example, if the initial data on Σ were to consist of a spherically symmetric, collapsing charged star, the maximal Cauchy evolution would be a spacetime consisting of an interior solution describ-
ing the star, together with an exterior solution consisting of a portion of the Reissner-Nördstrom spacetime (the “initial globally hyperbolic region”). In such spacetimes, certain observers, after a finite amount of their proper time, come to the “edge” of the maximal Cauchy evolution; the theory fails to predict what such observers subsequently measure. Clearly this is a serious breakdown of the theory.

Mathematically, the breakdown is characterized by the fact that one can extend the maximal Cauchy evolution, which we will denote by \((M, g_{ab})\), to a larger spacetime \((M', g'_{ab})\). In any such larger spacetime, the future Cauchy horizon \(H^+(\Sigma)\) is the boundary of \(M\) in \(M'\); the breakdown is thus signified by the existence of a Cauchy horizon. Note, however, that the extension spacetime is not uniquely determined by the initial data and thus is not physical (even in situations where a natural extension is determined by analytic continuation).

The disturbing aspect of this second type of breakdown is that it can occur within the (apparent) domain of validity of general relativity, entirely at low curvature scales, as in the Reissner-Nördstrom example. Of course, here we are assuming that classical general relativity is a good approximation at any point \(\mathcal{P}\) in spacetime whenever the curvature is sub-Planckian everywhere in the past lightcone of \(\mathcal{P}\), which seems like a reasonable assumption. In order to highlight how disturbing such breakdowns are, consider the hypothetical analogous situation in fluid hydrodynamics. Suppose one were given a solution of the hydrodynamic equations in which all of the lengthscales and timescales determined by the solution are much larger than the relevant microscopic lengthscales and timescales (so that the continuum approximation should be good) but where the solution is nevertheless incomplete, cannot be uniquely extended, and fails to predict the complete future evolution of the fluid. Such a situation would be paradoxical, and of course does not occur.

In the context of general relativity, if such breakdowns were ubiquitous one would apparently be forced to abandon general relativity as a viable description of Nature even in the regime of low curvatures. The traditional refuge of theorists has been to assert that such breakdowns should be non-generic; that is, that only “isolated” initial data give rise to geodesically incomplete spacetimes without curvature singularities. In other words, Cauchy horizons in spacetimes without singularities should always be unstable. This is the essential content of the strong cosmic censorship hypothesis.

The subject of this contribution to the proceedings is the instability properties of Cauchy horizons in classical and semi-classical gravity. To summarize the above discussion, one of the main motivations for studying the stability of Cauchy horizons is to show that the breakdowns of general relativity are not so serious as to render it not viable as a description of Nature. However, there are additional motivations. In attempting to determine the interior structure of black holes, one finds that the well-known analytic solutions posses Cauchy horizons in the interior, and in order to determine the generic interior structure one must investigate solutions in a neighborhood of the solution with the Cauchy horizon. (Such investigations were one of the focuses of the Haifa workshop.) Moreover, the stability of Cauchy horizons is also relevant to the question “Does Nature permit the occurrence of closed timelike curves?”

**B. Meaning of stability/instability**

One would like to show that spacetimes with Cauchy horizons and without curvature singularities are not generic, in the sense that “generic” perturbations to the initial data \(I\) for the gravitational and matter fields on some initial Cauchy surface \(\Sigma\) will give rise to spacetimes without Cauchy horizons (i.e., inextendible spacetimes). To make the notion of genericity precise would involve defining a topology and/or measure on the set of such initial data, generic then meaning either “all initial data sets \(I'\) in some open set containing \(I\)” or “all initial data sets \(I'\) in some set whose complement is of measure zero”. It has become customary to say that a Cauchy horizon is unstable when this non-genericity property is satisfied.

A somewhat different notion of instability is linear instability: a Cauchy horizon is linearly unstable if generic perturbations to the initial data (both matter and gravitational), when evolved forward using the linearized Einstein-matter equations, yield a singularity of the perturbed metric on the Cauchy horizon. By continuity, one would expect stability to imply linear stability, and thus linear instability should be a sufficient condition for a true nonlinear Cauchy horizon instability. It need not be a necessary condition as instabilities need not show up in linearized analyses, but this has not occurred in most investigations to date in the context of black holes.

In this contribution, we will use a still weaker notion of instability. We will call a Cauchy horizon test field unstable if, when one evolves linearized test matter fields on the spacetime, either classical or quantum mechanical, the stress-energy tensor of the test field diverges on the Cauchy horizon (in the sense that observers who cross the Cauchy horizon measure diverging stress tensor components). One would expect that test field instability should imply linear instability, since the behavior of test matter fields should presumably be similar to the behavior of linearized metric perturbations, and also the test field’s stress tensor should act as a source for the leading order metric perturbation in a coupled perturbation analysis. Again, in cases that have been examined, test field instability has been a good indicator of linear instability. (For a detailed analysis of the relation between various types of test field instability and nonlinear instability, see Ref. and references therein).

In the remainder of this contribution, we shall for the
most part adopt a conventional abuse of terminology and abbreviate “test field unstable” as simply “unstable”. Thus, by stability or instability we shall not mean the notion of full, nonlinear stability or instability discussed above. Our focus shall not be on understanding the effects of Cauchy horizon instabilities, for which purpose one typically needs to perform nonlinear analyses; rather we shall focus on trying to understand when and why Cauchy horizon instabilities occur. Also, we shall restrict attention to the simplified context of two dimensional spacetimes.

C. Purpose and overview of this contribution

Many investigations have found Cauchy horizons to be classically linearly unstable, for example, Cauchy horizons in black holes in asymptotically flat spacetimes [11,12]. However, there is some evidence for the existence of stable Cauchy horizons. The first evidence for stable Cauchy horizons of which I am aware is the work of Morris, Thorne and Yurtsver, who showed that the Cauchy horizons in wormhole spacetimes with closed timelike curves were classically test field stable [13]; their analysis was later generalized by Hawking [14]. However, the spacetimes in these analyses were not solutions of Einstein’s equation for a given matter model; rather they were simply posited background spacetimes. Also, the Cauchy horizon in the two dimensional version of the Reissner-N"{o}rdstrom-deSitter spacetime is classically linearly test field stable [11].

However, it has been shown in Refs. [15,16] that the two dimensional Reissner-N"{o}rdstrom-deSitter spacetime is always semiclassically unstable (see Sec. V A below for more details). Similarly, several researchers have shown that Cauchy horizons in spacetimes with closed timelike curves are semiclassically test field unstable [20–23], even in those cases which are classically test field stable [22].

In the case of classically unstable Cauchy horizons, the physical mechanism causing the instability is well understood — it is just the blueshifting of radiation to higher and higher energies [11,12,13]. Moreover, one can identify a simple geometric feature of the background spacetime — whether or not the blueshift factor diverges — which allows one to predict whether or not the Cauchy horizon is classically stable.

By contrast, our understanding of the nature of semiclassical instabilities has been far less detailed. We have had no simple and general physical explanation of how or why such instabilities operate in classically stable spacetimes. The main purpose of this contribution is to show that, in the simplified context of two dimensional spacetimes, semiclassical instabilities can be understood in a simple and intuitive way, and that one can identify a geometric property of the background spacetime which allows one to predict instabilities.

We examine conformally coupled, massless, scalar test quantum fields on general two dimensional spacetimes with Cauchy horizons and without singularities. In such contexts, it is well known that the expected stress tensor can be split in a unique way into the sum of two terms [Eq. (2.5) below]: an “initial data” piece which depends just on the initial values of the stress tensor on some initial surface, and a “locally generated” piece which describes local particle creation and/or vacuum polarization effects, and which depends only on the spacetime geometry and not on the initial data. Thus, there are two different types of semiclassical instabilities: (i) Instabilities for which the “initial data” piece of the stress tensor diverges on the Cauchy horizon. This type of instability arises classically as well as quantum mechanically; it is just the blueshift instability. (ii) Instabilities for which the “locally generated” piece diverges on the Cauchy horizon. We will call such instabilities locally created energy flux instabilities.

We now define a particular geometric property of spacetimes which is relevant to locally created energy flux instabilities. In a neighborhood of any point $\mathcal{P}$ on the Cauchy horizon, we construct a family of null geodesics which lie parallel to the Cauchy horizon. We normalize the affine parameter $\lambda$ on these geodesics by demanding that the vector field $d/d\lambda$ be parallel transported along a fixed, transverse null geodesic $\Lambda$ which intersects the Cauchy horizon at $\mathcal{P}$. Let $\Delta \lambda$ denote the total affine parameter length along any of these null geodesics parallel to the Cauchy horizon, from $\Lambda$ back to the initial data surface (see Fig. 2 below). Then, if $\Delta \lambda$ diverges as one moves closer and closer to the point $\mathcal{P}$ on the Cauchy horizon, we will say that the spacetime has the property of “divergence of affine parameter length”.

The main result of this contribution is that, under suitable mild assumptions, the divergence of affine parameter length is a necessary condition for a locally created energy flux instability. Roughly speaking, if the total affine parameter length is bounded above and there are no curvature singularities, then the locally small semiclassical corrections do not have enough time to accumulate and cause a divergence of the stress tensor.

The divergence of affine parameter length is not a sufficient condition for semiclassical test field instabilities, as there are locally flat spacetimes such as Misner space, which satisfy the divergence of affine parameter length condition, but which do not suffer from the locally created energy flux instability. However, locally flat spacetimes are not generic; spacetimes close to Misner space but with small amounts of curvature on the chronology horizon will suffer from the locally created energy flux instability. Hence, in suitable dynamical two-dimensional versions of semiclassical gravity [25], one would expect the locally created energy flux instability to be apparent in second order semiclassical perturbation theory about locally flat backgrounds which satisfy the divergence of affine parameter length property. We therefore conjecture that, quite generally, the divergence of affine parameter length is a sufficient condition for the full nonlinear
instability of Cauchy horizons in suitable dynamical two-dimensional versions of semiclassical gravity.

The properties of the blueshift and locally created energy flux instability mechanisms are summarized and contrasted in Table I below.

The concept of divergence of affine parameter length meshes nicely with our understanding of Cauchy horizon instabilities in the special case of spacetimes with closed timelike curves [20,7,14,23,26–31]. In such spacetimes, it is easy to see intuitively that the property of divergence of affine parameter length will always be satisfied: geodesics starting from points near the Cauchy horizon (which will be a closed null geodesic in two dimensions) will circle around very close to this closed null geodesic many times before eventually making their way back to the initial data surface. We make this argument more precise in Sec. IV B below [32].

Why should the property of divergence of affine parameter length be relevant to divergences of the expected stress tensor? There are two different types of intuitive explanation which, although apparently quite different, are not necessarily incompatible. First, in spacetimes in which the curvature is low everywhere, it is well known that classical general relativity coupled to classical fields is locally a good approximation. However, in a global context, small semiclassical corrections can “accumulate” and eventually become important. A good example of this is the Hawking evaporation of macroscopic black holes, which takes place over long timescales. Now when the property of divergence of affine parameter length is satisfied, in some sense the quantum field perceives points very near the Cauchy horizon to lie at “asymptotically late times”. Thus, there is enough time for semiclassical corrections which are locally small to accumulate and to become large at the Cauchy horizon.

The second intuitive explanation is that the locally created energy flux instability can be understood as a modified type of blueshift instability, a “delayed blueshift instability”. In the normal blueshift instability mechanism, an energy flux that is present in the initial data surface. We make this argument more precise in Sec. IV B below [32].

In Sec. IV D below we argue that locally generated energy flux instabilities should also occur in four-dimensional spacetimes. We also conjecture that, in the four-dimensional context, locally generated energy flux instabilities need not always correspond to delayed blueshift instabilities.

D. Organization of this contribution

The organization of this contribution to the proceedings is as follows. Section II is devoted to general analyses that underly both the blueshift and locally created energy flux instabilities. In Sec. III A we define the class of spacetimes which we analyze. In Sec. III C we define a basis of null vectors which is adapted to the local geometry near a given point on the Cauchy horizon. This null basis serves as a convenient tool throughout our calculations, and it allows us to avoid having to introduce a coordinate system to describe the spacetime. Using this basis, we derive in Sec. II B a (well-known) general formula for the expected stress tensor in terms of its initial values on some initial surface $\Sigma$ [Eq. (2.1)]. That formula exhibits the split of the stress tensor, referred to above, into an “initial data” piece plus a “locally generated” piece. We show in Sec. II D that all observers of bounded acceleration measure finite energy densities while crossing the Cauchy horizon if and only if the components of the expected stress tensor on the null basis are finite. Thus, in investigating the stability of the Cauchy horizon, we can focus attention on the components of the stress tensor on the null basis.

In Sec. II E we review and discuss the well-known blueshift instability mechanism, in order to contrast it with the locally created energy flux instability mechanism. In Sec. III A we give a very general definition of the blueshift factor, and recall that it can be interpreted in terms of a holonomy operator around a certain closed loop in spacetime, as illustrated in Fig. 2. A discussion of necessary and sufficient conditions for the instability is given in Sec. III B. Finally, in Sec. III C, we review the well-known fact that the instability acts only on radiation propagating parallel to the Cauchy horizon and not to radiation which crosses the Cauchy horizon; and we show that the radiation which crosses the Cauchy horizon is always finite, semiclassically as well as classically.

Section IV is devoted to the locally created energy flux instability mechanism. In Sec. IV A we show that the property of divergence of affine parameter length is a necessary condition for a locally created energy flux instability, when one assumes that the blueshift factor is globally bounded and one makes some other mild assumptions about the spacetime and the initial slice $\Sigma$. In Sec. IV B we explain that the divergence of affine pa-
rameter length property is not a sufficient condition for a linear instability (giving the counterexample of Misner space), and conjecture that it should be a sufficient condition for instabilities in a full, nonlinear analyses. In Sec. V D, we argue that some of the key ideas which underly, in two dimensions, the classification of instabilities into blueshift and locally created energy flux instabilities, should also generalize to four dimensions. We speculate that the divergence of affine parameter length might be relevant to Cauchy horizon instabilities in four dimensions.

In Sec. V C we show that the divergence of the stress tensor is always caused by an infinite blueshifting of a portion of the locally created piece of the stress tensor, thus showing the instability mechanism can be understood as a “delayed blueshift” instability.

In Section V we apply the general analyses of Secs. I, II and V to several different spacetimes and classes of spacetimes, in order to clarify and illustrate the results. In Sec. V A we show that the property of divergence of affine parameter length is satisfied in two-dimensional Reissner-Nördstrom-deSitter spacetimes, and reproduce the result of Marković and Poisson that such spacetime are semiclassically unstable. In Sec. V B we show, by adapting an argument due to Hawking, that two dimensional spacetimes with closed timelike curves always satisfy the property of divergence of affine parameter length. A particular example of such a spacetime, Misner space, is analyzed in Sec. V C. We review the well-known fact that Misner space is blueshift unstable. We also show that Misner space does not suffer from the locally created energy flux instability mechanism; we argue, however, that generic spacetimes “close to” Misner space should suffer from the instability.

Finally, section V D summarizes our main conclusions. Appendix A shows that the property of divergence of affine parameter length will be satisfied at a point on the Cauchy horizon whenever the generator of the Cauchy horizon through that point has infinite affine parameter length in the past direction.

We use units in which \( G = c = \hbar = 1 \), and we use the \((+, +, +)\) sign convention in the notation of Ref. [34].

II. CAUCHY HORIZON STABILITY: FOUNDATIONAL ANALYSES

A. Class of spacetimes and matter models

We start by specifying the class of spacetimes which we shall discuss. We shall be interested in globally hyperbolic, two-dimensional spacetimes \((M, g_{ab})\), i.e., spacetimes in which there exists a partial Cauchy surface \(\Sigma\) whose domain of dependence \(D(\Sigma)\) is the entire spacetime \(M\). The reason we restrict attention to such spacetimes is the following. Physically realistic spacetimes should be obtainable by specifying initial data for the gravitational and matter fields on some initial slice \(\Sigma\) and by solving the classical (or semiclassical) Einstein equation together with the matter equations of motion in order to recover the entire spacetime. In this way one recovers the maximal Cauchy evolution of the initial data, which will be just \(D(\Sigma)\). Thus, spacetimes obtainable by this procedure will all be globally hyperbolic.

We will also assume that the spacetime \((M, g_{ab})\) is extendible to a larger spacetime \((M', g'_{ab})\), and thus is geodesically incomplete. The spacetime \((M', g'_{ab})\) is useful from a mathematical point of view — using it one can define the future Cauchy Horizon \(H^+(\Sigma)\), which is just the boundary in \(M'\) of the future domain of dependence \(D^+(\Sigma)\). Note, however, that the extension is not unique and thus not physically meaningful. Our arguments and conclusions will be independent of the choice of extension \((M', g'_{ab})\).

The initial surface \(\Sigma\) may either be spacelike, or it may consist of two null segments as indicated in Fig. I below. Also, we do not restrict the topology of \(\Sigma\): it may be compact (topology of a circle) or non-compact (topology of the real line). In Sec. V below we discuss several specific examples of spacetimes to which our analyses apply; most of these spacetimes will be spatially non-compact, but one of them, Misner space, will be spatially compact.

We will also assume that the spacetime \((M, g_{ab})\) does not have any curvature singularities, either scalar or parallel propagated. Thus, we assume that (i) all local curvature invariants are globally bounded on \((M, g_{ab})\), and (ii) there are no causal geodesic curves along which the components of local curvature tensors on parallel propagated bases diverge. These assumptions are valid, for example, for the initial globally hyperbolic regions \((M, g_{ab})\) of the Reissner-Nördstrom and Reissner-Nördstrom-de Sitter spacetimes.

Our matter model will consist of a massless, minimally coupled scalar quantum field \(\Phi\) on \((M, g_{ab})\). We shall be concerned with the expected stress-energy tensor \(T_{ab}\) of the field \(\Phi\) in some quantum state in the vicinity of some point \(P\) on the Cauchy horizon. For ease of notation, we will denote this expected stress tensor simply as \(T_{ab}\).

In this model, the test field stability of Cauchy horizons which we are investigating should provide a good indication to the full, nonlinear stability properties of two dimensional spacetimes, in the context of a suitable two dimensional version of semiclassical gravity. Alternatively, one can regard two dimensional, semiclassical, test field instability results as implying that test field instabilities are also likely in similar four dimensional, spherically symmetric spacetimes; such four dimensional test field instabilities would be directly relevant to nonlinear instabilities in standard four dimensional semiclassical gravity.
B. General formula for stress tensor

The stress tensor $T_{ab}$ obeys the conservation equation

$$\nabla^a T_{ab} = 0 \quad (2.1)$$

together with the trace anomaly equation

$$T^a_a = \frac{1}{24 \pi} R, \quad (2.2)$$

where $R$ is the two dimensional Ricci scalar. As is well known, these conservation and trace anomaly equations are sufficient to determine the evolution of $T_{ab}$ from its initial value on $\Sigma$; one does not need to specify the details of the quantum state, unlike the situation in four dimensions. We can therefore calculate the behavior of $T_{ab}$ near the Cauchy horizon in terms of its initial data on $\Sigma$.

We split up the stress tensor into a traceless part and a trace part:

$$T_{ab} = \hat{T}_{ab} + \frac{1}{48 \pi} R g_{ab}, \quad (2.3)$$

where $g^{ab} \hat{T}_{ab} = 0$. It follows from Eq. (2.1) that the traceless part $\hat{T}_{ab}$ obeys the equation

$$\nabla^a \hat{T}_{ab} = -\frac{1}{48 \pi} \nabla_b R. \quad (2.4)$$

This equation has a well posed initial value formulation; initial values of $\hat{T}_{ab}$ determine uniquely its evolution. The general solution of Eq. (2.4) can be written as

$$\hat{T}_{ab} = \hat{T}_{ab}^{\text{(initial data)}} + \hat{T}_{ab}^{\text{(locally generated)}}, \quad (2.5)$$

where $\hat{T}_{ab}^{\text{(initial data)}}$ is the solution of the homogeneous version of Eq. (2.4) with the same initial data on $\Sigma$, and $\hat{T}_{ab}^{\text{(locally generated)}}$ is the solution of Eq. (2.4) with vanishing initial conditions.

We now derive explicit formulae for these two pieces of the general solution. These formulae are not new, but are usually derived and expressed in a “double-null” coordinate systems $(u, v)$ in which the metric is conformally flat:

$$ds^2 = -2e^{\sigma(u,v)} du dv; \quad (2.6)$$

see, for example, Ref. [33]. Here, however, we derive and express the formulae using a basis of null vectors $\{\vec{k}, \vec{l}\}$, which has the minor advantage that one does not need to assume the global existence of a coordinate system of the form (2.6).

Let $\Gamma$ be any null geodesic in the spacetime, going from some point $Q$ to some point $R$. Let $k^a$ denote the future directed, null tangent to the geodesic, with associated affine parameter $\lambda$, so that $\vec{k} = \left(\frac{d}{d\lambda}\right)$. Let $l^a$ be the vector field on $\Gamma$ which is null, future directed and which satisfies $l^a k_a = -1$, from which it follows that $\vec{l}$ is parallel transported along $\Gamma$. Using the relation

$$g_{ab} = -2l(a k_b) \quad (2.7)$$

in Eq. (2.4) yields

$$-2k(ab) l^c \nabla_c T_{cb} = -\frac{1}{48 \pi} l^b \nabla_b R. \quad (2.8)$$

Contracting this equation with $l^b$ yields

$$-k^a l^c b^d \nabla_a \hat{T}_{cb} - k^c l^b \nabla_a \hat{T}_{cb} = -\frac{1}{48 \pi} l^b \nabla_b R. \quad (2.9)$$

The second term on the left hand side in Eq. (2.9) vanishes by Eq. (2.7). Also the first term can be written as $-d/d\lambda(\hat{T}_{ab}^l l^b)$, since the vector $\vec{l}$ is parallel transported along the geodesic. Integrating with respect to $\lambda$ and using $\hat{T}_{ab}^l l^b = \hat{T}_{ab} k^b l^b$ therefore yields

$$T_{ab} l^a l^b (R) = T_{ab} l^a l^b (Q) + \frac{1}{48 \pi} \int_Q^R d\lambda \nabla_a \nabla_b R. \quad (2.10)$$

Note that the first term in Eq. (2.10) clearly corresponds to the first term in Eq. (2.3), and similarly for the second term.

The formula (2.10) allows one to determine the entire stress tensor from its initial value on $\Sigma$. Namely, given any point $R$ in $M$, one can choose a pair of future directed null vectors $\vec{k}$ and $\vec{l}$ at $R$ with $k^a l_a = -1$. Then, the stress tensor at $R$ can be written as

$$T_{ab}(R) = \rho l_a l_b + \sigma k_a k_b + \frac{1}{24 \pi} R g_{ab}, \quad (2.11)$$

where $\rho = T_{ab} k^a k^b$ and $\sigma = T_{ab} l^a l^b$. One can determine $\sigma(R)$ by shooting off a past-directed null geodesic $\Gamma$ from $R$ with initial tangent $-\vec{k}$, extending it until it intersects the initial surface $\Sigma$, parallel transporting $\vec{l}$ along $\Gamma$, and then applying the formula (2.10). Similarly one can determine $\rho(R)$ by shooting off a null geodesic from $R$ with initial tangent $-\vec{l}$, and applying the formula (2.10) with $\vec{k}$ and $\vec{l}$ interchanged.

Note that, in the formula (2.10), one has the freedom to perform the rescaling

$$\vec{k} \to e^\mu \vec{k} \quad \vec{l} \to e^{-\mu} \vec{l} \quad (2.12)$$

where $\mu$ is a constant. Both sides of Eq. (2.10) then get multiplied by $e^{-2\mu}$. If one has a family of geodesics (which corresponds to a specification of the basis $\vec{k}$, $\vec{l}$ in an open region), then $\mu$ can be any function on the manifold $M$ satisfying $k^a \nabla_a \mu = 0$. In terms of double null coordinates $(u, v)$ [cf. Eq. (2.6) above], $\mu$ can be an arbitrary function of the null coordinate $v$, when we adopt the convention $\vec{k} \propto \partial/\partial u$ and $\vec{l} \propto \partial/\partial v$. 


C. A null basis adapted to the local geometry near the Cauchy horizon

Fix a point $P$ in the Cauchy horizon $H^+(\Sigma)$. We can construct a natural basis $\{\vec{k}, \vec{l}\}$ in the intersection of the past light cone of $P$ with a neighborhood $U$ of $P$, in which we resolve the “gauge freedom” \( \frac{2.12}{2} \) where $\mu = \mu(v)$, as follows (see Fig. 1). Let $\vec{t}(P)$ be the future-directed tangent to the null generator of the Cauchy horizon at $P$, with some arbitrary choice of normalization. Then there is a unique, null, future directed vector $\vec{l}(P)$ with $\vec{l} \cdot \vec{k} = -1$. Let $\Lambda$ be the null geodesic which starts from $P$ with initial tangent $-\vec{l}$ and extends into the past, and extend $\vec{l}$ and $\vec{k}$ along $\Lambda$ by parallel transport. Finally, at an arbitrary point $R$ on $\Lambda$ in $U$, shoot out a geodesic $\Gamma$ into the past with initial tangent $-\vec{k}$ and continue it until it reaches that initial surface $\Sigma$, and extend $\vec{k}$ and $\vec{l}$ along $\Gamma$ by parallel transport \([36]\). The resulting dreibein (set of basis vectors) $\{\vec{k}, \vec{l}\}$ is unique up to transformations of the form \( \frac{2.12}{2} \) where now $\mu$ is a constant. Also, it follows from the construction that Eq. \( \frac{2.7}{2} \) holds in the domain of definition of the null basis.

D. Measurements made by observers crossing the Cauchy horizon

Our aim is to characterize when the stress tensor $T_{ab}$ diverges at the Cauchy horizon. Now, as is well known, to detect divergences it is insufficient in general to examine coordinate invariant scalar quantities such as $T_{ab}T^{ab}$ near the Cauchy horizon. Instead, one must examine the behavior of components of $T_{ab}$ with respect to parallel propagated bases along curves of bounded proper acceleration which cross the Cauchy horizon. In other words, one must examine what physical observers who cross the Cauchy horizon would measure. Now, we have constructed above a basis of null vectors which is naturally adapted to the spacetime geometry in the vicinity of a point $P$ on the Cauchy horizon. Therefore, one would expect that the stress tensor is regular in the above parallel-propagated sense at $P$ if and only if the components of $T_{ab}$ on this basis, i.e., the quantities $\sigma$ and $\rho$ defined by Eq. \( \frac{2.11}{2} \), are regular at $P$. In this section we give a brief proof that this is indeed the case. More precisely, we shall show that the measured energy densities will be finite for all observers of finite proper acceleration who cross the Cauchy horizon at $P$ if and only if both $\sigma$ and $\rho$ are bounded above in a neighborhood of $P$.

Let $C$ be the timelike worldline of an observer who passes through $P$. Her velocity can be expressed as

$$\vec{u} = \frac{1}{\sqrt{2}} \left[ e^{\chi} \vec{l} + e^{-\chi} \vec{k} \right] \tag{2.13}$$

for some $\chi = \chi(\tau)$, where $\tau$ is her proper time with $\tau = 0$ at $P$. Now it follows from our construction of the basis $\{\vec{k}, \vec{l}\}$ that

$$\nabla_\vec{k} \vec{k} = \nabla_\vec{k} \vec{l} = 0 \quad \nabla_\vec{l} \vec{l} = \kappa \vec{l} \quad \nabla_\vec{l} \vec{k} = -\kappa \vec{k} \tag{2.14}$$

where $\kappa$ is a spin coefficient analogous to Newman-Penrose quantity $\gamma$ in four dimensions. Combining Eqs. \(2.13\) and \(2.14\) we obtain for the acceleration

$$a^a = \left( \frac{1}{2} e^{2\chi} \vec{l}^a - \kappa \vec{l}^a \right) \left( \kappa + \sqrt{2} e^{-\chi} u^b \nabla_b \chi \right). \tag{2.15}$$

Squaring this equation yields
\[
\frac{d\chi}{d\tau} = \pm \frac{1}{\sqrt{2}} a(\tau) - \frac{1}{\sqrt{2}} e^{\chi} \kappa, \tag{2.16}
\]

where \(a(\tau) \equiv \sqrt{a_a a^a}\). It follows from Eq. (2.16) that the boost parameter \(\chi\) will be regular in a neighborhood of \(\tau = 0\), since by assumption the geometry is regular in a neighborhood of \(\mathcal{P}\), so \(\kappa\) will be regular, and also the proper acceleration \(a(\tau)\) is bounded by assumption.

Next, the energy density \(\rho_c = T_{ab} u^a u^b\) that the observer measures is given by, from Eqs. (2.11) and (2.13),
\[
\rho_c = \frac{1}{2} \rho e^{-2\chi} + \frac{1}{2} \sigma e^{2\chi} - \frac{1}{24\pi} R. \tag{2.17}
\]

The third term here is always bounded since the background geometry is regular in a neighborhood of \(\mathcal{P}\). Now if both \(\sigma\) and \(\rho\) are finite at \(\mathcal{P}\), then it follows that \(\rho_c\) will be finite for all observers of bounded acceleration, since for such observers \(\chi\) is bounded. Conversely, if either \(\rho\) or \(\sigma\) diverged at \(\mathcal{P}\), then there will be some choice of worldline \(\mathcal{C}\) for which \(\rho_c\) will be divergent [37].

To summarize, the Cauchy horizon will be stable if \(\sigma\) and \(\rho\) are finite on the Cauchy horizon, and unstable otherwise.

E. Comparison between classical and semiclassical theories

The classical version of the theory of a massless, conformally coupled scalar field differs from the above semiclassical version in only two respects:

- In the equation of motion (2.4), the source term on the right-hand side is not present in the classical theory; it describes local particle creation and/or vacuum polarization effects. Correspondingly, in the split of the stress tensor into a “locally generated” piece and an “initial data” piece, the locally generated term is absent in the classical theory.

- The set of allowed initial data for \(\mathcal{T}_{ab}\) on \(\Sigma\) is larger in the semiclassical theory than in the classical theory [33].

Therefore, the piece \(\hat{T}_{ab}^{\text{(initial data)}}\) of the general solution is purely classical — it has exactly the same behavior in semiclassical solutions as in classical solutions (except for the greater freedom in initial data in semiclassical solutions). In particular, if the Cauchy horizon is stable classically but unstable quantum mechanically, the instability must be due to the locally generated term \(\hat{T}_{ab}^{\text{(locally generated)}}\). In the following two sections we turn to a discussion of the two different instability mechanisms discussed in the Introduction: the well-known blueshift instability which manifests itself as a divergence of the term \(\hat{T}_{ab}^{\text{(initial data)}}\) and which can cause both classical and semiclassical instabilities, and the locally created energy flux instability mechanism which manifests itself as a divergence of the term \(\hat{T}_{ab}^{\text{(locally generated)}}\) in Eq. (2.13).

III. THE BLUESHIFT INSTABILITY

The blueshift instability mechanism is responsible for the classical instability of Cauchy horizons in charged and/or rotating black hole spacetimes [33,39], which is nicely reviewed in the contributions by Poisson and Chambers to this volume [33,39]. In this section, we review this mechanism in the language we have developed above, in order to contrast the blueshift mechanism with the locally created energy flux instability mechanism we discuss in Sec. IV below. Our discussion will apply both to classical and quantum mechanical analyses.

A. Preliminary definitions and constructions

As discussed in the Introduction, when we say that the Cauchy horizon is unstable we mean that the stress tensor diverges on the Cauchy horizon for generic, regular initial data on the initial Cauchy surface \(\Sigma\). We now discuss what is a suitable meaning of “regular” in our context of very general two-dimensional spacetimes. To this end, we define a basis of null vectors \(\{\vec{k}_\Sigma, \vec{l}_\Sigma\}\) on the initial surface \(\Sigma\), according to the following prescription: pick, at some point \(Q_0\) on \(\Sigma\), two future directed, null vectors \(\vec{k}_\Sigma(Q_0)\) and \(\vec{l}_\Sigma(Q_0)\) with \(\vec{k}_\Sigma(Q_0) \cdot \vec{l}_\Sigma(Q_0) = -1\), and extend this basis to all of \(\Sigma\) by parallel transport. If \(\Sigma\) has the topology of a circle, as is the case for Misner space discussed in Sec. V below, this definition of the basis \(\{\vec{k}_\Sigma, \vec{l}_\Sigma\}\) is ambiguous. We resolve the ambiguity by demanding that the basis be continuous at all points on \(\Sigma\) except possibly at \(Q_0\); a non-unit holonomy around \(\Sigma\) will yield a discontinuity at \(Q_0\). The resulting basis \(\{\vec{k}_\Sigma, \vec{l}_\Sigma\}\) is unique up to an overall constant boost of the form (2.13).

We will say that initial data for \(\mathcal{T}_{ab}\) on \(\Sigma\) is regular if the components \(T_{ab}^\kappa k^\kappa k^b\) and \(T_{ab}^\kappa k^b l^\kappa\) of the stress tensor are continuous (at all points except possibly \(Q_0\)) and bounded. It is clear that all physically reasonable initial data should be regular in this sense. In some spacetimes, the set of “physically reasonable” initial data will be a proper subset of the set of regular initial data (due, eg, to asymptotic fall-off conditions); see the discussion of various examples of spacetimes in Sec. V.

We now extend the basis \(\{\vec{k}_\Sigma, \vec{l}_\Sigma\}\) to the entire spacetime \((M, g_{ab})\) by parallel transporting along null geodesics parallel to \(\vec{k}_\Sigma\). This basis will be related to the basis \(\{\vec{k}, \vec{l}\}\) discussed in Sec. II above by a spacetime dependent boost:
\[
\vec{k} = e^{-\Psi} \vec{k}_\Sigma \\
\vec{l} = e^{\Psi} \vec{l}_\Sigma. \tag{3.1}
\]
Since the vector fields $\vec{k}$ and $\vec{k}_\Sigma$ are geodesic, we have $\kappa^a \nabla_a \Psi = 0$, i.e., $\Psi$ depends only on one of the two null coordinates $(u, v)$. For convenience, we will always choose the boost-normalization of the basis $\{\vec{k}, \vec{l}\}$ so that $\Psi(\Sigma_0) = 0$.

We now describe the blueshift instability mechanism. Spacetimes in which this instability operates have the property that, for some point $\mathcal{P}$ on the Cauchy horizon, the “blueshift factor” $e^{\Psi(\mathcal{R})}$ diverges as the point $\mathcal{R}$ approaches $\mathcal{P}$. From Eqs. (2.11), (2.10) and (3.1) we find that the quantity $\sigma$ (describing radiation propagating parallel to the Cauchy horizon) evaluated at the point $\mathcal{R}$ near the Cauchy horizon is given by

$$\sigma(\mathcal{R}) = e^{2\Psi(\mathcal{R})} \sigma_\Sigma(\mathcal{Q}(\mathcal{R})) + \text{(locally generated term)}. \quad (3.3)$$

Here, $\mathcal{Q}(\mathcal{R})$ denotes the unique point $\mathcal{Q}$ on $\Sigma$ determined by $\mathcal{R}$ according to the construction described in Sec. 1C, and $\sigma_\Sigma = T_{ab} P_a^{\Sigma} P_b^{\Sigma}$ is the initial data on $\Sigma$ which we have assumed is bounded. In this section we will ignore the locally generated term in Eq. (3.3); it will be absent in a classical treatment, and even in a semiclassical treatment, a divergence of the first term in Eq. (3.3) should be sufficient to produce an instability [40].

From Eq. (3.3) the quantity $\sigma(\mathcal{R})$ will diverge at $\mathcal{P}$ unless the initial data $\sigma_\Sigma(\mathcal{Q}(\mathcal{R}))$ goes to zero as $\mathcal{R} \to \mathcal{P}$ faster than the divergence of the blueshift factor $e^{\Psi(\mathcal{R})}$. Thus, whether or not the spacetime is unstable depends on the precise specification of the class of “physically reasonable” initial data on $\Sigma$ (cf. Sec. 1A above).

There are several possibilities for the behavior of $\mathcal{Q}(\mathcal{R})$ on $\Sigma$ as $\mathcal{R} \to \mathcal{P}$:

- When $\Sigma$ is non-compact, it can happen that $\mathcal{Q}(\mathcal{R}) \to \infty$ as $\mathcal{R} \to \mathcal{P}$. Then the behavior of

$\Psi(\mathcal{R}) \to \infty$ as $\mathcal{R} \to \mathcal{P}$.

The blue-shift instability can be understood as the ratio of differential proper times of freely falling observers near $\mathcal{P}$ and $\mathcal{Q}$, as explained in detail in Refs. [15, 39].

Note that the quantity $\Psi$ can be understood as characterizing a holonomy operator around a closed loop in spacetime, in the special case where the topology of $\Sigma$ is that of the real line (which excludes examples like Misner space). More specifically, consider the construction of the basis $\{\vec{k}, \vec{l}\}$ described in Sec. 1C above. That construction describes a family of null geodesics $\Gamma$ joining points $\mathcal{Q}$ on the initial slice $\Sigma$ to points $\mathcal{R}$ on the geodesic $\Lambda$ which emanates from the Cauchy horizon at the point $\mathcal{P}$ (see Fig. 2). Pick a particular null geodesic $\Gamma_0$ joining some points $\mathcal{R}_0$ and $\mathcal{Q}_0$. Then it follows that at any point $\mathcal{R}$ on $\Lambda$, the holonomy around the loop $\mathcal{R} \mathcal{R}_0 \mathcal{Q}_0 \mathcal{Q} \mathcal{R}$ indicated in Fig. 3 is simply the following boost with rapidity parameter $\Psi(\mathcal{R})$:

$$\alpha \vec{k}(\mathcal{R}) + \beta \vec{l}(\mathcal{R}) \to \alpha e^{\Psi(\mathcal{R})} \vec{k}(\mathcal{R}) + \beta e^{-\Psi(\mathcal{R})} \vec{l}(\mathcal{R}). \quad (3.2)$$

Here $\alpha$ and $\beta$ are arbitrary real numbers. To derive Eq. (3.2), consider starting at $\mathcal{R}$ with the vector $\vec{k}(\mathcal{R})$. When one parallel transports this vector to $\mathcal{Q}_0$ along $\Lambda$ and then to $\mathcal{Q}_0$ along $\Gamma_0$, the result is just $\vec{k}(\mathcal{Q}_0)$ by the definition of the basis $\{\vec{k}, \vec{l}\}$ from Eq. (3.1) and the fact that $\Psi(\mathcal{Q}_0) = 0$. When one then parallel transports this vector along $\Sigma$ to $\mathcal{Q}$ and then along $\Gamma$ back to $\mathcal{R}$, the result is $\vec{k}_\Sigma(\mathcal{R})$ by the definition of the basis $\{\vec{k}_\Sigma, \vec{l}_\Sigma\}$, which by Eq. (3.1) is the same as $e^{\Psi(\mathcal{R})} \vec{k}(\mathcal{R})$. A similar argument applies when one starts at $\mathcal{R}$ with $\vec{l}(\mathcal{R})$, and one obtains in this way Eq. (3.2).

Note also that the blueshift factor $e^{\Psi}$ can be understood as the ratio of differential proper times of freely falling observers near $\mathcal{P}$ and on $\Sigma$, as explained in detail in Refs. [15, 39].

B. Instability mechanism

![Diagram](image)

FIG. 2. A specialization of the construction shown in Fig. 1 to illustrate the difference between the blueshift and locally created energy flux instability mechanisms. Here $\Sigma$ is the initial Cauchy surface, $\mathcal{P}$ is a point on the future Cauchy horizon $H^+(\Sigma)$, and $\mathcal{R}$ and $\mathcal{R}_0$ are points on the null geodesic $\Lambda$ emanating from the Cauchy horizon at $\mathcal{P}$, which are connected to points $\mathcal{Q}$ and $\mathcal{Q}_0$ on $\Sigma$ via null geodesics. The blueshift instability mechanism is characterized by a divergence of the total holonomy around the loop $\mathcal{R} \mathcal{R}_0 \mathcal{Q}_0 \mathcal{Q} \mathcal{R}$ as $\mathcal{R} \to \mathcal{P}$. By contrast, what we call the locally created energy flux instability mechanism is characterized by the divergence of the affine parameter length of the geodesic $\mathcal{Q} \mathcal{R}$ as $\mathcal{R} \to \mathcal{P}$, where the normalization of the affine parameter $\lambda$ is such that $d/d\lambda$ is parallel transported along the geodesic $\Lambda$. This property of divergence of affine parameter length is a necessary but not sufficient condition for a locally created energy flux instability.
σ_Σ |Q(R)| depends on the asymptotic fall-off behavior of the initial data. In the case of classically unstable black hole spacetimes, the fall off rate of σ_Σ |Q(R)| for physically reasonable initial data is sufficiently slow that, although σ_Σ |Q(R)| → 0, the quantity |3.3| still diverges [13].

- It can happen that Q(R) → Q_1 as R → P, where Q_1 is some fixed point on Σ [1]. In this case, generic initial data will have σ_Σ(Q_1) ≠ 0, so the spacetime will be unstable.

- In the case where Σ is compact, it can happen that Q(R) “circles around and around” Σ as R → P, as occurs for example in Misner space. In this case also, it follows from Eq. (3.3) that σ(R) will be unbounded near P for all non-zero initial data, and thus the spacetime will be unstable.

To summarize, when the background spacetime (M, g_ab) has the property that the holonomy-like quantity e^{Ψ(R)} diverges, and when in the non-compact case the fall-off rate of the class of “physically reasonable” initial data is sufficiently slow, then the spacetime will be (classically and quantum mechanically) unstable. The condition on the fall-off rate of the initial data depends on the spacetime and on the choice of initial surface Σ; it is satisfied in black hole spacetimes [12].

C. Radiation crossing the Cauchy horizon

So far in this section we have considered only the term σ_kab in Eq. (2.11), which describes radiation propagating parallel to the Cauchy horizon. What of the other term ρ_{λab} in that equation? This term describes radiation which crosses the Cauchy horizon. We now show that such radiation can never give rise to an instability by showing that the quantity ρ is always bounded near the point P.

Extend the geodesic Λ into the past until it intersects the initial surface Σ at some point S. Then, since the vectors ̂k and ̂l are both parallel transported along Λ, it follows from Eq. (2.10) with ̂k and ̂l interchanged that

\[ \rho(R) = (T_{ab} k^a k^b)(S) + \frac{1}{48\pi} \int_S^R dξ \ k^a \nabla_a R, \]  

where ξ is an affine parameter along Λ such that ̂l = d/dξ. The first term in Eq. (3.4) is just

\[ e^{-2ψ(S)} \rho_{Σ}(S), \]  

where ρ_{Σ} ≡ T_{ab} k_a k^b, from Eq. (3.1). Now the point S does not vary as R → P, and thus the quantity (3.5) is fixed and finite as R → P. The second term in Eq. (3.4) will converge to the finite quantity

\[ \frac{1}{48\pi} \int_S^P dξ \ k^a \nabla_a R \]  

as R → P, and so ρ(R) is bounded as R → P.

Thus, when the blueshift factor is finite, it follows from Eq. (3.3) and the analysis of this subsection that the spacetime is stable except possibly for a divergence of the semiclassical, locally generated piece of σ(R). We now turn to a discussion of this possibility.

IV. THE LOCALLY CREATED ENERGY FLUX INSTABILITY

In this section we discuss a second instability mechanism which can cause a divergence of the second term \( t_{ab}^{(\text{locally generated})} \) in Eq. (2.5). This mechanism is a purely quantum mechanical effect as the term \( t_{ab}^{(\text{locally generated})} \) is absent in classical analyses.

Let us focus attention on spacetimes in which the blueshift factor e^Ψ is globally bounded, and in which therefore the blueshift instability mechanism does not operate. For example, the two dimensional Reissner-Nördstrom-de Sitter spacetime has this property in a certain region of parameter space [14], when the initial surface Σ is chosen to lie outside the event horizon (see Sec. V A below). Such spacetimes will be classically stable but may be semiclassically unstable.

From Eq. (2.10), the locally generated piece of σ is given by

\[ σ^{(\text{locally generated})(R)} = \frac{1}{48\pi} \int_{Q(R)}^R dλ l^a \nabla_a R. \]  

Now by assumption the background geometry is nonsingular, so the Ricci scalar R is bounded; however the quantity l^a \nabla_a R may still be unbounded and may give rise to a divergence of the integral (4.1) as R → P. Alternatively, one might imagine that the integrand l^a \nabla_a R could be bounded, but that the total affine parameter length in Λ between Q(R) and R could diverge as R → P giving rise to a divergence of the integral. In Sec. V below, we examine several spacetimes for which the integral (4.1) diverges, and we find that in these example spacetimes, both the integrand and the total affine parameter length diverge.

We will say that a spacetime has the property of “divergence of affine parameter length” if, for some point P on the Cauchy horizon,

\[ Δλ(R) → \infty \quad \text{as} \quad R → P. \]  

Here Δλ(R) is the affine parameter length of the null geodesic from R to Q(R), where the affine parameter λ is normalized according to ̂k = d/dλ (see Fig. 2 above). In Appendix A we show that the property (4.2) will hold if the generator of the Cauchy horizon through the point P has infinite affine parameter length towards the past.
[Note that, although any such generator must be inex-
tendible in the past direction \( \Gamma \), it may have finite affine
parameter length in the past direction if the spacetime
\((M', g_{ab}')\) is geodesically incomplete; see, for example,
Fig. 8.2 of Ref. 41.]

A. A necessary condition for instability

In this subsection section we prove that when one
makes certain mild assumptions about the spacetime
\((M, g_{ab})\) and the initial slice \(\Sigma\), the property of diver-
gence of affine parameter length is a necessary condition
for an instability of the Cauchy horizon mediated by a
divergence of the quantity (4.1).

A precise statement of our result is the following. Assume
that (i) The spacetime \((M, g_{ab})\) is non-singular in
the sense discussed in Sec. IA above; (ii) The holonomy-
like quantity \(\Psi(R)\) is globally bounded; (iii) The total
affine parameter length \(\Delta \lambda (R)\) is globally bounded by
some maximum \(\Delta \lambda_{\text{max}}\); (iv) The initial slice \(\Sigma\) is regular
in the sense that the quantity \(\int\xi \nabla_a R\) is bounded on \(\Sigma\).
Then the quantity (4.1) is bounded and thus the Cauchy
horizon is stable. In other words, assuming the condi-
tions (i), (ii) and (iv), the divergence of affine parameter
length is a necessary condition for an instability.

Assumption (iv) is only relevant when \(\Sigma\) is non-
compact; it is satisfied automatically in compact cases
like Misner space. In the non-compact case, the condition
(iv) is a reasonable assumption as the vector \(\xi\) is parallel
transported along \(\Sigma\), and moreover if \(\Sigma\) is asymptotically
spacelike and the spacetime is asymptotically flat, then \(R\)
will go to zero at large distances. The condition should
thus be satisfied by slices \(\Sigma\) which are asymptotically null or asymptotically spacelike in spacetimes that are
asymptotically flat.

We now turn to a proof of the above result. Consider the quantity
\(k^a \nabla_a l^b \nabla_b R\), which we can write as
\[
\begin{align*}
k^a \nabla_a l^b \nabla_b R &= k^a \nabla_a l^b \nabla_b R + (k^a \nabla_a l^b) \nabla_b R \\
&= -\frac{1}{2} \Box R. \quad (4.3)
\end{align*}
\]

Here the second term on the first line vanishes by
Eq. (2.14), and the second equality follows from Eq. (2.7).
Now integrating with respect to \(\lambda\) along the geodesic \(\Gamma\)
in Fig. 1 using \(k^a \nabla_a = d/d\lambda\) and Eq. (3.3), we obtain
\[
(l^a \nabla_a R)(\mathcal{R}') = (l^a \nabla_a R)(\mathcal{Q}) - \frac{1}{2} \int_{\mathcal{Q}}^{\mathcal{R}'} d\lambda \Box R \\
= e^{\Psi(\mathcal{Q})} \left( l^a \nabla_a R \right)(\mathcal{Q}) - \frac{1}{2} \int_{\mathcal{Q}}^{\mathcal{R}'} d\lambda \Box R, \quad (4.4)
\]

where \(\mathcal{R}'\) is some point on \(\Gamma\) between \(\mathcal{Q}\) and \(\mathcal{R}\). Integrating once more with respect to \(\lambda\) and using Eq. (4.1) yields
\[
|\sigma^{\text{locally generated}}(\mathcal{R})| \leq \frac{1}{48\pi} \Delta \lambda_{\text{max}} e^{\Psi_{\text{max}}} ||l^a \nabla_a R||_{\infty} + \frac{1}{96\pi} (\Delta \lambda_{\text{max}})^2 ||\Box R||_{\infty}, \quad (4.5)
\]

where \(\Psi_{\text{max}}\) is the maximum value of \(\Psi\), \(||l^a \nabla_a R||_{\infty} < \infty\) is the maximum value of \(l^a \nabla_a R\) on \(\Sigma\), and \(||\Box R||_{\infty} < \infty\) is the maximum value of \(\Box R\) on \(M\). It follows from Eq. (4.5) that \(\sigma^{\text{locally generated}}(\mathcal{R})\) is bounded.

B. Discussion of instability

As discussed in the Introduction, one can intuitively
understand the reason for the instability in the following
way. The locally generated term in Eq. (2.10) describes
local particle creation and/or vacuum polarization effects
due to the background gravitational field. [In general dy-
namic spacetimes it is not meaningful to distinguish be-
tween particle creation and vacuum polarization effects,] Radiation propagating parallel to the Cauchy horizon is
generated all along the geodesic \(\Gamma\), and since the length
of this geodesic is becoming infinite, an infinite amount of
radiation can be accumulated at the Cauchy horizon.

Consider now the issue of under what conditions the
divergence of affine parameter length is a sufficient con-
dition for a locally created energy flux instability of the
Cauchy horizon. First, as we discuss in Sec. V C below,
there are spacetimes such as Misner space which are
locally flat so that the locally generated term (4.1) vanishes identically, but for which nevertheless the condi-
tion of divergence of affine parameter length is satisfied.
Therefore, the divergence of affine parameter length is not a sufficient condition for test field locally created en-
ergy flux instabilities. However, the example of Misner
is very special — it is locally flat. In general spacetimes,
one might imagine that if the curvature were very small
or vanishing near the Cauchy horizon in the background
dspace, then the locally created energy flux instability
might be present in second order semiclassical pertur-
bation if not in first order: the first order perturbation
would give rise to some curvature near the Cauchy hori-
zon, and this curvature would then act as a source and
give rise to an divergence of the expected stress tensor
at second order. Therefore, we conjecture that the di-
vergence of affine parameter length is a sufficient condi-
tion for a full, nonlinear instability of Cauchy horizons in
dynamical, two dimensional semiclassical theories with
backreaction 23.

C. Explanation of instability as a “delayed blueshift”
instability

Our classification of Cauchy horizon instabilities is
based on the split (2.3) of the stress tensor into “ini-
tial data” and “locally generated” pieces. However, this
split depends on the location of the initial data surface \( \Sigma \). An instability which is a locally generated energy flux instability from the point of the initial data surface \( \Sigma \) can instead appear to be an initial-data-related blueshift instability from the point of view of some later surface \( \Sigma' \). An example of this behavior is given in Sec. IV A below. We now show that this situation always occurs when the spacetime is blueshift stable but semiclassically unstable: there is always some initial data surface \( \Sigma' \) with respect to which the instability is a blueshift instability.

The blueshift factor \( \Psi_P(\mathcal{R}) \) defined in Sec. III A above depends not only on the point \( \mathcal{R} \) but also implicitly on the point \( P \) on the Cauchy horizon. In this section we will write the blueshift factor as

\[
\Psi_P(\mathcal{R})
\]

to make the dependence on \( P \) explicit. Consider now the construction illustrated in Fig. 2. Let \( \mathcal{R}' \) be an arbitrary point on the geodesic \( \Gamma \) between \( \mathcal{R} \) and \( \mathcal{Q} \), and let \( \mathcal{R}'_0 \) and \( P' \) be corresponding points on the geodesics \( \Gamma_0 \) and on the Cauchy horizon respectively, so that \( \mathcal{R}'_0 \) and \( \mathcal{R}' \) all lie on a null geodesic parallel to \( \Gamma \). Let \( \lambda_0 \) be the affine parameter along the geodesic \( \Gamma_0 \), so that \( \kappa = d/d\lambda_0 \) along \( \Gamma_0 \). Then it is possible to show that the affine parameter length \( \Delta \lambda(\mathcal{R}) \) of the geodesic \( \Gamma \) is given by the following integral along the geodesic \( \Gamma_0 \):

\[
\Delta \lambda(\mathcal{R}) = \int_{\Gamma_0} d\lambda_0 \frac{e^{\Psi_P(\mathcal{R})}}{e^{\Psi_{P'}(\mathcal{R}')}}.
\]  

(4.6)

Now, if the total blueshift factor \( \Psi_P(\mathcal{R}) \) is globally bounded, and if \( \Delta \lambda(\mathcal{R}) \to \infty \) as \( \mathcal{R} \to \mathcal{P} \), it follows from Eq. (4.6) that the quantity \( e^{\Psi_{P'}(\mathcal{R}')/\mathcal{R}' - \mathcal{P}} \) cannot be globally bounded below by a positive constant. Therefore, for some point \( \mathcal{P}' \) on the Cauchy horizon (passing to some conformal completion of the spacetime if necessary), there is a diverging redshift, i.e. \( e^{\Psi_{P'}(\mathcal{R}')/\mathcal{R}' - \mathcal{P}} \to 0 \) as \( \mathcal{R}' \to \mathcal{P}' \).

Consider now an “initial data surface” \( \Sigma' \) for which the rightmost portion is a null geodesic parallel to \( \Gamma \) which intersects the Cauchy horizon at \( \mathcal{P}' \). Denote by \( \Psi_{P'}(\mathcal{R}) \) the blueshift factor given by the construction of Sec. III A, with respect to the surface \( \Sigma' \). Then one has

\[
\Psi_{P'}(\mathcal{R}) = \Psi_P(\mathcal{R}) - \Psi_{P'}(\mathcal{R}'),
\]

(4.7)

which diverges as \( \mathcal{R} \to \mathcal{P} \). Therefore, the Cauchy horizon is blueshift unstable with respect to the surface \( \Sigma' \). As explained in the Introduction, any energy flux present on the initial surface \( \Sigma \) will not give rise to a divergence of the stress tensor on the Cauchy horizon, since the net blueshift is finite. By contrast, an energy flux which is created locally near \( \Sigma' \) will suffer an infinite blueshift and give rise to an instability.

D. Relevance to four dimensional spacetimes

We now discuss the issue of to what extent this two dimensional locally created energy flux instability mechanism is relevant to four dimensional spacetimes. Suppose one has a four dimensional spacetime on which propagates a quantized scalar field \( \Phi \). In this context, the expected stress tensor is not determined by its value on an initial surface \( \Sigma \), but the split (2.3) of the expected stress tensor in any state into a “locally generated” piece plus an “initial data” piece would not seem to have an analog in four dimensions. However, we now show that this splitting, and in addition some of the features of the two dimensional semiclassical theory which underly the instability mechanism, do generalize to at least a certain class of four dimensional spacetimes.

Consider a four dimensional spacetime \( (M, g_{ab}) \) which, to the past of some Cauchy surface \( \Sigma \), is isometric to a portion of Minkowski spacetime. For any quantum state, let \( G(x, y) = \langle \Phi(x) \Phi(y) \rangle \) be that state’s two point function, and let \( G_0(x, y) \) be the two point function of the state which to the past of \( \Sigma \) is just the usual Minkowski vacuum. Let

\[
F(x, y) = G(x, y) - G_0(x, y),
\]

(4.8)

which is a smooth bisolution of the wave equation. Then, as explained in detail in Ref. [44], the total expected stress tensor can be written in a form analogous to Eq. (2.3):

\[
T_{ab} = T_{ab}^{\text{(initial data)}} + T_{ab}^{\text{(locally generated)}},
\]

(4.9)

where now

- The “initial data” piece is given by \( T_{ab}^{\text{(initial data)}} = \lim_{y \to x} D_{ab} F(x, y) \), where \( D_{ab} \) is a second order differential operator. This is the same stress tensor as would be obtained in a purely classical theory of a statistical ensemble of scalar fields, for which the stress tensor is determined by the expected value \( F(x, y) = \langle \Phi(x) \Phi(y) \rangle \) of \( \Phi(x) \Phi(y) \) with respect to the classical ensemble. Moreover, the quantity \( F(x, y) \) is uniquely determined (both classically and semiclassically) by its initial data and derivatives on \( \Sigma \times \Sigma \) from the classical equations of motion [44]. Just as in two dimensions, the main difference between the classical and semiclassical theories is that the class of allowed initial data (here, initial data for \( F \) and its derivatives on \( \Sigma \times \Sigma \)) is larger in the semiclassical case than in the classical case.

- The remaining “locally generated” piece \( T_{ab}^{\text{(locally generated)}} \) of the stress tensor is the same for all quantum states, and is just a unique functional of the spacetime geometry, just as in two dimensions. An explicit formula for this piece of the stress tensor is known for the special case of metrics that are linearized perturbations off Minkowski spacetime [44].
These similarities between the two dimensional and four dimensional semiclassical theories indicate that

- As in two dimensions, so also in four dimensions, one can probably classify instabilities at Cauchy horizons into two types (i) “blueshift type” instabilities that cause a divergence of the term $T_{ab}^{(\text{initial data})}$ and that operate both classically and semiclassically; and (ii) intrinsically quantum mechanical, “locally created energy flux type” instabilities that cause a divergence of the term $T_{ab}^{(\text{locally generated})}$.

- The property of divergence of affine parameter length might be relevant to divergences of the term $T_{ab}^{(\text{locally generated})}$ in four dimensions. On the other hand, other issues such as the focusing/defocusing along any geodesic, which enter in four dimensions but not in two, probably complicate the situation.

In summary, it is an interesting open question whether or not any of the ideas and results discussed here can be extended to four dimensions; it seems conceivable that some of them could be.

V. EXAMPLE SPACETIMES

In this section, in order to illustrate and clarify the above discussions and results, we examine several specific spacetimes — the two dimensional Reissner-Nördstrom-deSitter spacetime in Sec. V A, general two dimensional spacetimes with closed timelike curves in Sec. V B, and Misner space in Sec. V C.

A. The two dimensional Reissner-Nördstrom-deSitter spacetime

The metric for a spherically symmetric black hole of mass $m$ and charge $e$ in de-Sitter space can be written as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2,$$  \hspace{1cm} (5.1)

where

$$f(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2} - \frac{1}{3}\Lambda r^2,$$  \hspace{1cm} (5.2)

and $\Lambda$ is the cosmological constant. There are three positive roots of the equation $f(r) = 0$, which we will denote as $r_i$, $r_e$ and $r_c$, following Ref. [19]. Here $r_i < r_e < r_c$, and these three roots correspond to the inner or Cauchy horizon, the event horizon, and the cosmological horizon, respectively (see Fig. 3). We also define $\kappa_j = |f'(r_j)|/2$, which is the surface gravity at the $j$th horizon, for $r_j = r_i$, $r_e$ or $r_c$. We will discuss for the most part only the two dimensional version of the spacetime in which the last term in Eq. (5.1) is omitted.

In Ref. [13], Brady and Poisson showed that the Cauchy horizon is classically test field stable in the region $\kappa_i < \kappa_e$ of parameter space, when the matter model is taken to consist of an infalling null fluid (or equivalently a minimally coupled scalar field in the two dimensional context), and they argued that the stability result should also hold for more realistic matter models and for gravitational perturbations. In Ref. [19], Marković and Poisson showed that the two dimensional spacetime is semiclassically unstable, by showing that the stress tensor diverges on the Cauchy horizon in the Marković-Unruh state [47], and in Ref. [63] Poisson extended this result to show that the stress tensor must diverge on the Cauchy horizon in any state which is regular on the cosmological and event horizons. (See also earlier calculations by Davies and Moss [18] which indicated that the Cauchy horizon is unstable, and see Ref. [39] for a detailed overview of this literature).

In this section, we will focus attention on some of the above results, and for the most part simply verify that they are reproduced by the general formalism of Secs. I – IV. We will also show that in the integral (4.1), both the integrand $l^a\nabla_a R$ and the total affine parameter length diverge, as claimed in Sec. IV.
We take the initial surface to consist of two null lines, as illustrated in Fig. 3, where the rightmost null line is outside the event horizon. The null basis \{\vec{k}, \vec{l}\} must have the form

\[
\vec{k} = \alpha \left[ \frac{1}{f} \frac{\partial}{\partial r} + \frac{\partial}{\partial \tau} \right],
\]

\[
\vec{l} = \frac{f}{2\alpha} \left[ \frac{1}{f} \frac{\partial}{\partial r} - \frac{\partial}{\partial \tau} \right],
\]

(5.3)

where \(\alpha\) is some positive function on spacetime, since the vectors \(\vec{k}, \vec{l}\) are future directed and null. Now the two dimensional version of the metric (2.3) can be written in the usual way as \(-f(d\tau^2 - dr^2)\), and since in any spacetime of the form (2.3) the vector fields \(e^{-\sigma}\partial_u\) and \(e^{-\sigma}\partial_v\) are geodesic, it follows that the vector fields \(\pm f^{-1}\partial_t \pm \partial_r\) are locally geodesic. Therefore since \(\vec{l}\) is geodesic along the geodesic \(\Lambda\) in Fig. 3, it follows from Eq. (5.3) that \(f/\alpha\) must be constant along \(\Lambda\), and without loss of generality we can take this constant to be \(-1:\)

\[
\alpha(\mathcal{R}) = -f(\mathcal{R}), \quad \mathcal{R} \text{ on } \Lambda.
\]

(5.4)

Similarly, \(\vec{k}\) is geodesic along \(\Gamma\), so \(\alpha\) is constant along \(\Gamma\),

\[
\alpha(\mathcal{R}') = \alpha(\mathcal{R})
\]

(5.5)

where \(\mathcal{R}'\) is any point along \(\Gamma\). Combining Eqs. (5.3) – (5.7) we find that the integrand in Eq. (4.1) is given by

\[
(\mathcal{L}^a\nabla_a \mathcal{R})(\mathcal{R}') = -\frac{f(\mathcal{R}')}{2f(\mathcal{R})} R'(r),
\]

(5.6)

where \(R = R(r)\) is the Ricci scalar which depends only on \(r\).

Next, from Eq. (5.3) we find that the relationship between the affine parameter \(\lambda\) along \(\Gamma\) and the coordinate \(r\) is given by \(d\lambda = -dr/\alpha\), which from Eq. (5.3) gives

\[
\lambda = -\frac{1}{f(\mathcal{R})} \left[ r + \text{const} \right].
\]

(5.7)

Combining Eqs. (4.1), (5.6) and (5.7) now gives

\[
\sigma(\text{locally generated})(\mathcal{R}) = \frac{1}{96\pi f(\mathcal{R})^2} \int_{r(\mathcal{R})}^{r[Q(\mathcal{R})]} dr \int_{r(\mathcal{R})}^{r[Q(\mathcal{R})]} f(r') R'(r) .
\]

(5.8)

Here as before \(Q = Q(\mathcal{R})\) denotes the unique point \(Q\) on the initial surface \(\Sigma\) determined by \(\mathcal{R}\) according to the construction of Sec. III C. Using the formula \(R(r) = -f''(r)\) and integrating by parts gives

\[
\int_{r(\mathcal{R})}^{r[Q(\mathcal{R})]} dr f(r') R'(r) = \left[ \frac{1}{2} f'' - f f'' \right]_{r(\mathcal{R})}^{r[Q(\mathcal{R})]}.
\]

(5.9)

Finally, using that fact that as \(\mathcal{R} \to \mathcal{P}\), the limits of integration behave as \(r(\mathcal{R}) \to r_i\), \(r[Q(\mathcal{R})] \to r_e\) and that \(f(r_i) = f(r_e) = 0\) gives

\[
\sigma(\text{locally generated})(\mathcal{R}) \approx \frac{\text{(const)}}{f(\mathcal{R})^2} \left[ \kappa_c^2 - \kappa_i^2 \right].
\]

(5.10)

Equation (5.10) reproduces the result of Refs. \[19\] that \(\sigma\) is divergent on the Cauchy horizon in the classically stable region \(\kappa_c > \kappa_i\) of parameter space, since \(f(\mathcal{R}) \to 0\) as \(\mathcal{R} \to \mathcal{P}\). The agreement with Ref. \[19\] can be made more explicit by noting that in this limit \(f(\mathcal{R}) \approx \text{(const)} \exp[-\kappa_i v(\mathcal{R})]\), where \(v\) is the advanced time coordinate \(t + r\) which goes to infinity as \(\mathcal{R} \to \mathcal{P}\).

Several points should be noted about the above derivation. First, from Eq. (5.3) it can be seen that the affine parameter length is divergent:

\[
\Delta \lambda(\mathcal{R}) \approx \frac{1}{f(\mathcal{R})} (r_e - r_i) \approx e^{\kappa_i v} (r_e - r_i);
\]

(5.11)

and that the integrand (5.4) itself also has an overall factor of \(1/f(\mathcal{R})\) which is divergent as \(\mathcal{R} \to \mathcal{P}\). Second, the instability result applies to all quantum states for which the initial data on \(\Sigma\) is regular. The quantity \(\sigma(\mathcal{R})\) will be a sum of the divergent, locally generated piece (5.10), together with the piece (5.3) which depends on the initial data and which will be finite as the blueshift factor \(e^\Psi\) is finite. It is easy to verify that up to an overall constant factor the blueshift factor is

\[
e^{\Psi(\mathcal{R})} = \frac{f(Q(\mathcal{R}))}{f(\mathcal{R})} \approx \frac{e^{\kappa_i v(\mathcal{R})}}{e^{-\kappa_c v(\mathcal{R})}} \text{ as } \mathcal{R} \to \mathcal{P}.
\]

(5.12)

This is finite for \(\kappa_c > \kappa_i\), as first noted in Ref. \[15\]. Third, the derivation allows us to understand why the divergent piece of the quantity \(\sigma(\mathcal{R})\) in semiclassical analyses does not contain a cosmological redshift factor \(e^{-2\kappa_c v}\) as it does in classical analyses \[19\]. The reason is that the radiation giving rise to the divergence (5.10) is produced in the vicinity of the event horizon and propagates from there inward to the Cauchy horizon, and thus is not affected by the redshift factor describing propagation from the cosmological horizon to the event horizon.

Note that it is straightforward to show that in the classically stable region \(\kappa_c > \kappa_i\) of parameter space, the blueshift factor \(e^\Psi(\mathcal{R})\) is globally bounded, not just inside the black hole but also along the event horizon. Thus, the class of spacetimes to which the general result of Sec. IV A applies is not empty.

We have explained the above instability as a locally created energy flux instability and not as a blueshift instability. However, it can also be thought of as a “delayed blueshift” instability, as discussed in Sec. IV C above. If one chooses the right-most portion of the initial data surface \(\Sigma\) to lie on the event horizon, then the corresponding blueshift factor given by the construction of Sec. III A is proportional to \(e^{2\kappa_c v}\) and so is unbounded. In classical analyses, the initial data on the event horizon must fall off like \(e^{-2\kappa_c v}\) (due to having propagated inward from some
initial surface outside the event horizon), thus giving rise to a finite value of $\sigma$ on the Cauchy horizon. However, in a semiclassical analysis, things are different. Consider an ingoing mode of the quantum field whose mode function is concentrated near some geodesic that is very close to the Cauchy horizon. Any ingoing quanta in this mode from outside the black hole experience first a large redshift (from the outside to the event horizon), then a large blueshift (from the event horizon to the interior) which is smaller than the redshift. The crucial feature in the semiclassical analysis is that ingoing quanta are created in the vicinity of the event horizon; these propagate inwards and suffer only the large blueshift. Thus, the “initial data” for the stress tensor on the event horizon need not fall off like $e^{-2\kappa_c v}$ in the semiclassical theory. Correspondingly, near the Cauchy horizon $\sigma$ diverges like $e^{2\kappa_c v}$ by the usual blueshift effect.

Thus, the distinction between blueshift and locally created energy flux instabilities is dependent on the choice of location of the initial data surface, as discussed in Sec. IV C above

B. General spacetimes with closed timelike curves

A special case of the general class of spacetimes described in Sec. IIA above is when the spacetime $(M', g_{ab}')$ contains closed timelike curves, so that the Cauchy horizon $H^+(\Sigma)$ is also a chronology horizon. A simple example of such a spacetime is given by Yurtsever [28], in which the lightcones on a cylinder “tip over” to produce a closed null geodesic around the cylinder [49]. This closed null geodesic coincides with the chronology horizon. In the four dimensional context, closed null geodesics or “fountains” will form a small subset of the full chronology horizon [6].

In Ref. [44], Hawking gives a general argument, in the context of chronology horizons in four dimensional spacetimes which are compactly generated, that the total affine parameter length of all fountains should be infinite in the past direction (although generically finite in the future direction): If $\vec{k}$ is a future directed, null vector which is tangent to the geodesic at some point, then the total holonomy around the fountain in the future direction will map $\vec{k}$ to $e^{h} \vec{k}$ for some constant $h$. Hawking shows that if $h < 0$, then there must exist a closed timelike curve to the past of the chronology horizon, which is a contradiction. Therefore $h \geq 0$ and so the the total affine parameter length in the past direction of the closed null geodesic is proportional to

$$\sum_{n=0}^{\infty} e^{nh} = \infty,$$  \hspace{1cm} (5.13)

(although the total affine parameter length in the future direction is proportional to $\sum e^{-nh} < \infty$ as long as $h \neq 0$). It can be checked that this argument applies equally well to two dimensional spacetimes. In the two dimensional context, the generator of the Cauchy horizon through any point $\mathcal{P}$ will therefore have infinite affine parameter length in the past direction, and it follows from the argument of appendix A that all such spacetimes satisfy the property (4.2) of divergence of affine parameter length.

However, it does not follow that all such spacetimes are subject to the locally created energy flux instability. We now turn to a specific spacetime containing closed timelike curves which illustrates this point.

C. Misner space

Misner space is a well-known locally flat spacetime with topology $S^1 \times R [50,51]$. As is well known, Misner space is both classically and semiclassically unstable [2]. These instability properties of the spacetime are reviewed in Ref. [52]. In this section we show how the construction of Secs. IIA and IVC applies to Misner space. The construction reproduces the well-known fact that Misner space suffers from the blueshift instability, which explains both the classical and semiclassical instabilities. We also show that Misner space does not suffer from the locally created energy flux instability, despite the fact that the spacetime does satisfy the property of divergence of affine parameter length.

Misner space can be constructed as follows [52]. In two dimensional Minkowski spacetime with coordinates $(t, x)$, identify the worldline $x = 0$ with the worldline $x = \xi = 0$, for some $L > 0$, $\beta > 0$ (see Fig. 4). Let $\tau$ be the proper time measured by a clock on one of these worldlines; $\tau = t$ for the leftmost worldline, whereas $\tau = t - \beta t$ for the rightmost worldline, where $\gamma$ is the usual special relativistic time dilation factor. An observer crossing the rightmost worldline at coordinate time $t = t_0$ therefore emerges from the leftmost line at coordinate time $t = t_0 + \beta t$. Hence, at large $t$, there will be closed timelike curves in the spacetime, after the closed null geodesic marked in Fig. 4. This closed null geodesic is the Cauchy horizon or chronology horizon of Misner space.

Consider now how the constructions of Secs. IVC and IVA apply to Misner space. First, note that if any vector is parallel transported through the leftmost worldline to emerge from the rightmost worldline, it undergoes the boost given by $\partial_u \rightarrow e^{\xi} \partial_u$, $\partial_v \rightarrow e^{-\xi} \partial_v$, where $\cosh \xi = \gamma$ and $\sinh \xi = \beta$. From the diagram of Misner space shown in Fig. 4, we can fairly easily see that (i) if we take the initial surface $\Sigma$ to be the surface $t = 0$, then $\Sigma$ has the topology of a circle. The basis of the null geodesic $\Gamma$ through a point $\mathcal{P}$ near the Cauchy horizon passes through the selected worldlines some number $n$ of times before it reaches the initial surface, where $n$ grows
without limit as $R \to P$. Thus, the initial point $Q(R)$ circles around and around $\Sigma$, as claimed in Sec. III B above. (iii) When one parallel transports the vector $k$ along $\Gamma$ to $\Sigma$, it is boosted $n$ times, and thus the redshift factor is
\[ e^{\Psi(R)} = e^{n\xi}, \] (5.14)
which diverges as $R \to P$. This redshift factor changes discontinuously by a factor of $e^\xi$ every time $Q(R)$ crosses $Q_0$ on $\Sigma$, as the basis $\{ \hat{k}_\Sigma, \hat{l}_\Sigma \}$ is discontinuous there. (iv) The total affine parameter length $\Delta \lambda(R)$ of $\Gamma$ also diverges as $R \to P$. [This also follows from the general argument of Sec. V B above].

Since the blueshift factor diverges, the spacetime is classically and quantum mechanically unstable, from the arguments of Sec. III above. In the classical case, this means that the stress tensor diverges on the Cauchy horizon for generic initial data of the scalar field [53]. Of course, for the exceptional case of vanishing initial data on $\Sigma$, the stress tensor vanishes identically. Nevertheless, one still regards the Cauchy horizon as being unstable as vanishing initial data is not generic or stable in any physical sense.

The situation is very similar in the semiclassical theory. Originally it was conjectured in Ref. [14] that the stress tensor must diverge on the Cauchy horizon for all quantum states. The argument of Sec. III shows that the stress tensor must diverge unless the initial data $\sigma_\Sigma$ on $\Sigma$ is vanishing; thus, the conjecture was effectively that there is no state on Misner space for which the expected stress tensor vanishes identically on the initial surface. However, it is now known that there are quantum states on both two dimensional and four dimensional Misner space for which the stress tensor does vanish identically [29–31]. Nevertheless, states for which the initial data for the stress tensor is vanishing are clearly in some sense non-generic, and therefore one is still justified in regarding Misner space as unstable in the semiclassical theory, just as in the classical theory.

Finally, we note that the locally created energy flux instability mechanism does not operate in Misner space, since it is locally flat and so the integrand in Eq. (4.1) is identically vanishing. This does not contradict the fact that the condition (4.2) is satisfied by Misner space, since we have only shown that the condition (4.2) is a necessary condition for the instability, and not a sufficient condition. Note, however, that all spacetimes which are “close” to Misner space but which differ from it by having an arbitrarily small amount of spacetime curvature on the Cauchy horizon should suffer from the locally created energy flux instability. See also the related discussion in Sec. IV B above.

VI. CONCLUSION

We have shown that in the context of semiclassical gravity in two dimensions, there are two different types of instabilities of Cauchy horizons. The first is a divergence of the piece of the stress tensor which is determined by the initial data; it operates classically as well as semiclassically, and is just the well-understood blueshift instability. The second, which we call the locally created energy flux instability, is a divergence of the locally generated piece of the expected stress tensor. This instability is characterized by the divergence of the affine parameter lengths of null geodesics parallel to and close to the Cauchy horizon.

It is natural to conjecture that for all Cauchy horizons in two dimensional spacetimes without singularities, one...
or other of these two instability mechanism always applies, if not in linear perturbation theory, then at least in nonlinear analyses.

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APPENDIX A: DIVERGING AFFINE PARAMETER LENGTH CONDITION FOLLOWS FROM CAUCHYHorizon GENERATOR BEING OF INFINITE AFFINE PARAMETER LENGTH IN THE PAST

In this appendix we show that if the generator of the Cauchy horizon through the point \( P \) has infinite affine parameter length in the past, then the condition (4.2) must be satisfied by the spacetime.

Let \( TM' \) be the tangent bundle over the spacetime \( (M', g'_{ab}) \), and let

\[
exp : D \subset TM' \rightarrow TM'
\]

be the exponential map, where the domain of definition \( D \) of the exponential map is an open subset of \( TM' \). If \( (M', g'_{ab}) \) were geodesically complete we would have \( D = TM' \). In other words, for any point \( B \) in \( M' \) and any vector \( \vec{v} \) at \( B \), \( \exp(B, \vec{v}) \) will be the pair \( (B', \vec{v}') \) in \( TM' \) such that the geodesic starting at \( B \) with initial tangent \( \vec{v} \) reaches \( B' \) after one unit of parameter length, and such that \( \vec{v}' \) is the tangent to the geodesic at \( B' \).

Let \( U_+ = M' - \overline{D^{-}(\Sigma)} \), the complement of the closure of the past domain of dependence of \( \Sigma \), and let \( W_+ \subset TM' \) be the tangent bundle over \( U_+ \). It is clear that the set \( W_+ \) is open in \( TM' \). Consider now the construction outlined in Sec. 11C. It can be seen that, for any \( l > 0 \) and for any \( R \) on \( \Lambda \),

\[
\exp[R, -l \vec{k}(R)] \in W_+ \quad \text{implies} \quad \Delta \lambda(R) \geq l, \quad (A1)
\]

by the definition of \( \Delta \lambda(R) \) given in Sec. 11V above and using the fact that \( \Sigma \) is a Cauchy surface for \( (M, g_{ab}) \).

Suppose now that the affine parameter length towards the past of the generator of the Cauchy horizon through \( P \), normalized with respect to \( \vec{k}(P) \), is greater than some number \( \beta \). It follows that \( [P, -\beta \vec{k}(P)] \) lies in the domain of definition \( D \) of the exponential map. Also it follows that \( \exp[P, -\beta \vec{k}(P)] \) lies in \( W_+ \), and thus by continuity of the exponential map there is some open neighborhood \( U \) of \( [P, -\beta \vec{k}(P)] \) in \( D \) whose image under the exponential map lies in \( W_+ \). Hence there is some neighborhood \( V \) of \( P \) in \( M' \) such that for all points \( R \) on \( \Lambda \) and in \( V, \exp[R, -\beta \vec{k}(R)] \) is defined and lies in \( W_+ \) and so \( \Delta \lambda(R) \geq \beta \). Since this is true for all \( \beta \) the result follows.

Note that the converse of this result is not true: if the diverging affine parameter condition holds for a given point \( P \) on the Cauchy horizon, it does not follow that the generator of the Cauchy horizon through \( P \) has infinite affine parameter length in the past. This is because one always has the freedom to redefine the spacetime \( (M', g'_{ab}) \) by excising points on the Cauchy horizon.

We have chosen to express the condition in terms of the behavior of the spacetime \( (M, g_{ab}) \) [divergence of affine parameter length of null geodesics parallel to and close to the Cauchy horizon] rather than the behavior of the larger spacetime \( (M', g'_{ab}) \) [generator of Cauchy horizon having infinite affine parameter length in the past] in order that the condition be manifestly independent of which extension we pick.

[1] See the contribution of A. Ori to this proceedings.
[2] For a review of the information loss paradox, see, for example, J. Preskill, hep-th/9209058.
[3] We assume that the initial surface is a complete Riemannian manifold to exclude examples, such as the region \( |u| < 1, |v| < 1 \) of two dimensional Minkowski spacetime, where the predicted spacetime is incomplete simply because the initial data is incomplete.
[4] Of course, it could also happen that one can have maximal Cauchy evolutions with both curvature singularities and Cauchy horizons. However, this possibility is not so disturbing. Suppose we define the “classical region” of spacetime to consist of all points \( P \) for which the curvature is sub-Planckian everywhere to the past of \( P \); then if the Cauchy horizon is outside the “classical region” there is less cause for concern, as one would not believe the predictions of the classical theory outside the classical region anyway.
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(25) As is well known, it is necessary to couple in additional scalar fields as part of the gravitational sector of the theory to get a non-trivial, dynamical theory of gravity in two dimensions. One can then add quantum scalar fields to get a coupled semiclassical theory. See, for example, T.M. Fiola, J. P. Preskill, A. D. Strominger and S. P. Trivedi, Phys. Rev. D 50, 3987 (1994) (also hep-th/9403137).
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(36) The resulting basis can be expressed as

\[ \tilde{u}(u, v) = e^{-\sigma(u, v)} \frac{\partial}{\partial u} \]

and

\[ \tilde{v}(u, v) = e^{\sigma(u, v)} \frac{\partial}{\partial u}, \]

for any double null coordinates \((u, v)\) for which the metric takes the form (2.1) and which are regular at \(P\), where \(u_P\) is the value of the \(u\) coordinate at \(P\).
(37) One might worry that if the behavior of, say, \(r\) near \(P\) were very direction dependent, then it might happen that \(\rho(\pi^\pm(\lambda))\) would diverge for some curves \(\pi^\pm(\lambda)\) while nevertheless remaining always bounded for all timelike curves of bounded proper acceleration. This possibility is excluded by the equation of motion (2.4) for the stress tensor: since the geometry is regular near \(P\), in a sufficiently small neighborhood of \(P\) the effect of the source term will be unimportant and to a good approximation \(\rho\) will depend only on one of the two null coordinates \(u\) and \(v\).
(38) Specifically, consider the special case where \(\Sigma\) consists of two intersecting null lines as illustrated in Fig. 8. Then, the free data on one of these two half-lines consists of the function \(\sigma(\lambda) \equiv T_{\mu\nu}(\partial/\partial \lambda)^\mu(\partial/\partial \lambda)^\nu\), where \(\lambda\) is an affine parameter along the line. In the classical theory, \(\sigma(\lambda)\) can be any non-negative smooth function. In the semiclassical theory in the special case of two dimensional Minkowski spacetime, it follows from the analyses of Ref. [5] that the class of allowed functions \(\sigma(\lambda)\) consists of those smooth functions which satisfy

\[ \int d\lambda f(\lambda) \sigma(\lambda) \geq \frac{1}{48\pi} \int d\lambda \frac{f'(\lambda)^2}{f(\lambda)}, \]

for all smooth non-negative functions \(f(\lambda)\) [where \(f'(\lambda)^2/f(\lambda)\) is interpreted to mean zero when \(f(\lambda) = 0\)].
This characterization also applies to spacetimes which are globally conformal to two dimensional Minkowski spacetime, as can be seen from Eq. (6.133) of Ref. [53].

[54] C. Chambers, The Cauchy Horizon in Black Hole-de Sitter Spacetimes, contribution to this volume (also q-c/9709023).

[55] This is not quite true as one could envisage a situation where the “locally generated” and “initial data” terms in Eq. (2.3) both diverge but where they cancel each other out so that the sum is finite. We ignore this possibility here. See however Ref. [18] for a scenario where such a cancelation could occur.

[56] An example of a spacetime where this occurs is the following: Let $M_0$ be the region $|u| < 1$, $|v| < 1$ of Minkowski spacetime, where $u$ and $v$ are the usual null coordinates, and let $M = M_0 - I_\pm(P)$, where $P$ is the point $(u, v) = (1/2, 1/2)$.

[57] This statement assumes that the initial slice is outside the event horizon. When considering classically stable Reissner-Nordstrom-de Sitter spacetimes ($\kappa_c > \kappa$) in Sec. V A, we shall take the rightmost half of our initial data slice $\Sigma$ to be a null line outside the event horizon as depicted in Fig. 1. In this case the blueshift factor is finite. If one took the rightmost half of the initial slice to be the event horizon itself, then the blueshift factor would be infinite, but the spacetime would still be (classically) stable because of the falloff rate of the initial data on the event horizon.

[58] Suppose that one uses null coordinates $(u, v)$ such that $\tilde{t} \propto \partial/\partial\nu$ and $K \propto \partial/\partial\mu$. Then one might think that the quantity $\Psi(R)$ would depend on the four coordinate values $u(P)$, $v(P)$, $u(R)$ and $v(R)$. However, $v(P)$ is fixed by the requirement that $P$ lie on the Cauchy horizon, and $u(R) = u(P)$ by construction. Thus, $\Psi(R)$ depends only on two independent real parameters.

[59] E. E. Flanagan and R. M. Wald, Phys. Rev. D, 54, 6233 (1996) (gr-qc/9602052).

[60] G. T. Horowitz, Phys. Rev. D 21, 1445 (1980).

[61] B. Carter, in Black Holes, edited by C. DeWitt and B.S. DeWitt (Gordon and Breach, New York, 1973).

[62] D. Marković and W.G. Unruh, Phys. Rev. D 43, 332 (1991).

[63] This does not contradict the fact that the integrand is finite on the Cauchy horizon itself near $P$. As a function of the affine parameter $\lambda$ and the retarded time coordinate $v$, the integrand in Eq. (11) at large $v$ is $t^a \nabla_a R \propto f[\tau(\lambda)] f''[\tau(\lambda)] \exp[\kappa_i v]$, with $\tau(\lambda) \approx r_i + \exp[-\kappa_i v] \lambda$. Therefore in the limit $v \to \infty$, $t^a \nabla_a R \to f'(r_i) f''(r_i) \lambda$, which is finite at each fixed $\lambda$ but which grows linearly without limit as one moves down the Cauchy horizon. A similar result is found on the cosmological horizon.

[64] Note that in four dimensional contexts, one needs to distinguish between smoothly closed null geodesics, and self-intersecting null geodesics in which the direction of the tangent to the geodesic is discontinuous at some point. In the two dimensional context, however, no such distinction is necessary, if we assume that the spacetime $(M', g_{ab})$ is orientable and time orientable. This is because given a future directed, null tangent vector $k^a$ at a point on a self-intersecting null geodesic, when one parallel trans-
TABLE I. A table contrasting the two different instability mechanisms discussed in this contribution, in the context of two dimensional spacetimes: the *blueshift instability* characterized by the divergence on the Cauchy horizon of the “initial data” piece of the expected stress tensor, and what we call the *locally created energy flux instability* mechanism, which is characterized by the divergence on the Cauchy horizon of the “locally generated” piece of the expected stress tensor.

|                        | Blueshift instability          | Locally created energy flux instability |
|------------------------|--------------------------------|----------------------------------------|
| Applies classically?   | Yes                            | No                                     |
| Applies semiclassically?| Yes                            | Yes                                    |
| Necessary condition for instability | Divergence of holonomy         | Divergence of affine parameter length   |
| Sufficient condition for instability | Divergence of holonomy (with some conditions on initial data) | Divergence of affine parameter length is probably a sufficient condition for nonlinear instability, but is not for linear instability |
| Spacetimes in which applies | Reissner-Nördstrom             | some Reissner-Nördstrom-deSitter spacetimes |
|                         | Misner space                   | spacetimes                             |
| Spacetimes in which does not apply | some Reissner-Nördstrom-deSitter spacetimes | Misner space: linear instability does not apply |