ON EISENSTEIN SERIES IN $M_{2k}(\Gamma_0(N))$ AND THEIR APPLICATIONS

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Abstract. Let $k, N \in \mathbb{N}$ with $N$ square-free and $k > 1$. Let $f(z) \in M_{2k}(\Gamma_0(N))$ be a modular form. We prove an orthogonal relation, and use this to compute the coefficients of Eisenstein part of $f(z)$ in terms of sum of divisors function. In particular, if $f(z) \in E_{2k}(\Gamma_0(N))$, then the computation will yield an expression for Fourier coefficients of $f(z)$. We give two applications of the results. First, we give formulas for convolution sums of the divisor function to extend the result by Ramanujan. Second, we give formulas for number of representations of integers by certain families of quadratic forms. We finish the paper with some observations on determination of the Fourier coefficients of eta quotients in $E_{2k}(\Gamma_0(N))$.

Keywords: sum of divisors function, convolution sums, theta functions, representations by quadratic forms, Eisenstein series, Dedekind eta function, eta quotients, modular forms, cusp forms, Fourier series.

Mathematics Subject Classification: 11A25, 11E20, 11E25, 11F11, 11F20, 11F27, 11F30, 11Y35.

1. Introduction

Let $N, N_0, Z, Q, \mathbb{C}$ and $\mathbb{H}$ denote the sets of positive integers, non-negative integers, integers, rational numbers, complex numbers and the upper half plane, respectively. Throughout the paper we let $z \in \mathbb{H}$ and $q = e^{2\pi iz}$. Let $\Gamma_0(N) (N \in \mathbb{N})$ be the modular subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1, \ c \equiv 0 \pmod{N} \right\}.$$

An element $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(1)$ acts on $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by

$$M(z) = \begin{cases} \frac{az+b}{c} & \text{if } z \neq \infty, \\ \frac{a}{c} & \text{if } z = \infty. \end{cases}$$

Let $k, N \in \mathbb{N}$. We write $M_k(\Gamma_0(N))$ to denote the space of modular forms of weight $k$ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ to denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N))$, respectively. It is known (see for example [16] and [17, p. 83]) that

$$M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N)).$$
In the remainder of the paper, unless otherwise stated, we assume \( N \in \mathbb{N} \) is square-free, \( p \) always stands for prime numbers, and all the divisors considered are positive divisors.

A set of representatives of cusps of \( \Gamma_0(N) \) is given by

\[
R(N) = \left\{ \frac{1}{c} : c \mid N \right\}
\]

when \( N \) is square-free, see [8, Proposition 2.6], [11].

Let

\[
A_c = \begin{bmatrix} -1 & 0 \\ c & -1 \end{bmatrix},
\]

(1.2)

then the Fourier series expansion of \( f(z) \in M_k(\Gamma_0(N)) \) at the cusp \( \frac{1}{c} \in \mathbb{Q} \cup \{\infty\} \) is given by the Fourier series expansion of \( f(A_c^{-1}z) \) at the cusp \( \infty \), see [10, pg. 35]. Let the Fourier series expansion of \( f(z) \) at the cusp \( \frac{1}{c} \) be given by the infinite sum

\[
f(A_c^{-1}z) = (cz + 1)^k \sum_{n \geq 0} a_c(n)e^{2\pi iz/h},
\]

where \( h \), the width of \( \Gamma_0(N) \) at the cusp \( \frac{1}{c} \). Then we use the notation \([n]_c f(z)\) to denote \( a_c(n)\). We write \([n]\), instead of \([n]_0\), at the cusp \( \infty \) (\( = 1/0 \)). If we say Fourier series expansion (or Fourier coefficients) without specifying the cusp, we mean the expansion (or coefficients) at the cusp \( \infty \). And, for modular forms, ‘the first term of the Fourier expansion of \( f(z) \) at cusp \( \frac{1}{c} \)’ refers to the term \([0]_c f(z)\).

We define \( v_c(f) \), the order of \( f(z) \) at \( \frac{1}{c} \), to be the smallest \( n \) such that \([n]_c f(z) \neq 0\).

The Dedekind eta function \( \eta(z) \) is the holomorphic function defined on the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) by the product formula

\[
\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).
\]

Let \( N \in \mathbb{N} \) and \( r_\delta \in \mathbb{Z} \) for all \( 1 \leq \delta \mid N \), where the \( r_\delta \) are not all zero. Let \( z \in \mathbb{H} \). We define an eta quotient by the product formula

(1.3)

\[
f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z).
\]

For \( k \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \) we define the sum of divisors function by

\[
\sigma_k(n) = \sum_{1 \leq d \mid n} d^k.
\]

If \( n \notin \mathbb{N} \) we set \( \sigma_k(n) = 0 \). We define the Eisenstein series \( E_{2k}(z) \) by

(1.4)

\[
E_{2k}(z) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,
\]
where $B_{2k}$ are Bernoulli numbers, defined by the generating function

$$
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.
$$

For all $d \mid N$ and $1 < k \in \mathbb{N}$ we have

$$E_{2k}(dz) \in E_{2k}(\Gamma_0(N)),$$

see [17, Theorem 5.8]. Noting that $\omega(d)$ stands for the number of distinct prime divisors of $d$, we state our main theorem.

**Theorem 1.1.** Let $N \in \mathbb{N}$ be square-free and $k > 1$ be an integer. Let $f(z) \in M_{2k}(\Gamma_0(N))$. Then there exists a cusp form $C_N(f) \in S_{2k}(\Gamma_0(N))$ such that

$$f(z) = \sum_{d \mid N} \left( \sum_{c \mid N} \left( \prod_{p \mid |N|} (p^{2k} - 1) \left( \frac{N \gcd(c, d)}{c} \right)^2 [0]_{c} f(z) \right) \right) E_{2k}(dz) + C_N(f).$$

(1.5)

In particular, if $f(z) \in E_{2k}(\Gamma_0(N))$ then we have

$$f(z) = \sum_{d \mid N} \left( \sum_{c \mid N} \left( \prod_{p \mid |N|} (p^{2k} - 1) \left( \frac{N \gcd(c, d)}{c} \right)^2 [0]_{c} f(z) \right) \right) E_{2k}(dz).$$

(1.6)

Comparing coefficients of $q^n$ on both sides of equations (1.5) and (1.6), for $n > 0$ we have

$$[n] f(z) = -\frac{4k}{B_{4k}} \left( \sum_{c \mid N} \left( \prod_{p \mid |N|} (p^{2k} - 1) \left( \frac{N \gcd(c, d)}{c} \right)^2 [0]_{c} f(z) \right) \right) \sigma_{2k-1}(n/d)$$

(1.7)

$$+ [n] C_N(f),$$

in particular, if $f(z) \in E_{2k}(\Gamma_0(N))$ then for $n > 0$ we have

$$[n] f(z) = -\frac{4k}{B_{4k}} \left( \sum_{c \mid N} \left( \prod_{p \mid |N|} (p^{2k} - 1) \left( \frac{N \gcd(c, d)}{c} \right)^2 [0]_{c} f(z) \right) \right) \sigma_{2k-1}(n/d).$$

(1.8)

If $f(z) \in M_{2k}(\Gamma_0(N))$, then it is analytic at all cusps. Thus, it is possible to calculate $[0]_{c} f(z)$ for all $c \mid N$. That is, Theorem 1.1 can be applied to any $f(z) \in M_{2k}(\Gamma_0(N))$ to obtain its Eisenstein part. In most applications of modular forms in number theory, we realized, only the expansion at infinity is considered. This approach is pretty useful, however fails to provide general results. Consideration of expansions at other cusps, as we do in this paper, allows us to derive more general
results using modular forms. Theorem 1.1 and its applications given in Sections 4 and 5 are an indicator of the power of the techniques used in this paper.

In the next section, we compute Fourier series expansions of certain modular forms (specifically, the modular forms \( f(dz) \) where \( f(z) \in M_{2k}(\Gamma_0(1)) \) at the cusps \( \frac{1}{c} \in \mathbb{Q} \). In Section 3 we use these expansions, together with the orthogonal relation given by Lemma 3.1 to prove Theorem 1.1. In Sections 4 and 5 we apply Theorem 1.1 to convolution sums of the divisor function and representations by certain quadratic forms, respectively. All the theorems and corollaries stated in this paper are new, and if we fix \( N \) and/or \( k \), our formulas agree with the previously known formulas. We discuss the details of Sections 4 and 5 below.

Let \( a, b, l, m \in \mathbb{N} \) and let us define the convolution sum

\[
W(a^l, b^m; n) = \sum_{ar + bs = n} \sigma_l(r)\sigma_m(s)
\]

for \( n > 0 \). This convolution sum is a generalized version of the convolution sum defined by Ramanujan in [14], where he gave a formula for \( W(1^{l-1}, 1^{m-1}; n) \). For some history on the formulas for \( W(a^l, b^m; n) \), see [9, 18]. In Section 4 we use Theorem 1.1 to give the following formula:

\[
W(a^{2l-1}, b^{2m-1}; n) =
\]

\[
-\frac{k}{4}B_2B_2m \sum_{d|N} \left( \sum_{c|N} \frac{(-1)^{\omega(d) + \omega(c)}}{\prod_{p|N} (p^{2k} - 1)} \left( \frac{N\gcd(c, d)}{c} \right)^{2k} \left( \frac{\gcd(a, c)}{a} \right)^{2l} \left( \frac{\gcd(b, c)}{b} \right)^{2m} \right)^{\sigma_{2k-1}(n/d)}
\]

\[
+ \frac{B_2}{4l} \sigma_{2m-1}(n/b) + \frac{B_{2m}}{4m} \sigma_{2l-1}(n/a) + \frac{B_2B_2m}{16lm} [\zeta \mid C_{\text{lcm}(a, b)}(E_{2l}(az)E_{2m}(bz))]
\]

where \( a, b \in \mathbb{N} \) are square-free, \( l, m > 1, k = l + m \). This formula extends the result given by Ramanujan in [14], except the cases \( l = 1 \) or \( m = 1 \). This is due to the complicated behaviour of weight 2 Eisenstein series at cusps. (See [3], for a treatment of the case when \( l = 1 \) and \( m = 1 \).) At the end of Section 4 we let the level to be 6 to illustrate this formula, we then describe \( C_{\text{lcm}(a, b)}(E_{2l}(az)E_{2m}(bz)) \) (for all \( \text{lcm}(a, b) \mid 6 \)) in terms of eta quotients and finally we let \( a, b = 1 \) to deduce that (1.9) agrees with the formula given by Ramanujan.

Let \( m \in \mathbb{N} \), \( r_i \in \mathbb{N}_0 \), and \( a_i \in \mathbb{N} \) for all \( 1 \leq i \leq m \). Let

\[
N(a_1^{r_1}, a_2^{r_2}, \ldots, a_m^{r_m}; n)
\]

denote the number of representations of \( n \) by the quadratic form

\[
\sum_{i=1}^{m} r_i a_i x_i^2.
\]

In Section 5 we apply Theorem 1.1 to obtain the following formula for representations of an integer by a family of quadratic forms with odd square-free coefficients:

\[
N(1^{s_1}, \ldots, g^{s_g}, \ldots, N^{s_N}; n) =
\]
\[
\frac{4k}{B_{2k}} \sum_{d \mid N} \left( \sum_{e \mid N} (-1)^{w(d) + w(e) + 1} \left( \frac{2N \gcd(c, d)}{e} \right)^{2k} \prod_{\delta \mid N} \left( \frac{\gcd(c, \delta)}{\delta} \right)^{4\sigma_{2k-1}(n/2d)} \right) \\
+ (-1)^n \frac{4k}{B_{2k}} \sum_{d \mid N} \left( \sum_{e \mid N} (-1)^{w(d) + w(e)} \left( \frac{N \gcd(c, d)}{e} \right)^{2k} \prod_{\delta \mid N} \left( \frac{\gcd(c, \delta)}{\delta} \right)^{4\sigma_{2k-1}(n/d)} \right) \\
+ (-1)^n [n] C_{2N} \left( \prod_{\delta \mid N} g^{8r_{s}(\delta z)} \right).
\]

Then we compare our formula with recent formulas from Cooper et al. \[7\] and classical results by Ramanujan \[6, 12, 14\]. We also illustrate our formula with \(2N = 6\) and give a description of \(C_6 \left( g(z)^{8r_{1}} g(3z)^{8r_{3}} \right) \) in terms of eta quotients.

One might expect to use \(1.8\) to determine Fourier coefficients of eta quotients in \(E_{2k}(\Gamma_0(N))\). In \[1, 2\] we take advantage of a similar idea, for fixed weight and level, to determine Fourier coefficients of certain families of weight 2 eta quotients. It seems, if \(N \in \mathbb{N}\) square-free and the weight is greater than 2, then there are only two eta quotients in \(E_{2k}(\Gamma_0(N))\). We try to explain the reason for this in Section 6. In Section 6 we also discuss two interesting relations between zeros of a modular form and the Fourier coefficients of \(f(z) \in E_{2k}(\Gamma_0(N))\).

### 2. Preliminary results

Let \(f(z) \in M_{2k}(\Gamma_0(1))\). In this section we state and prove Theorem 2.1, which gives the Fourier series expansions of \(f(dz) \in M_{2k}(\Gamma_0(d))\) at cusps \(1/c \in \mathbb{Q}\). We use Theorem 2.1 to compute first terms of Fourier series expansions of \(E_{2k}(dz)\) at the cusps \(1/c\). Together with the fact that the first terms of Fourier series expansions of cusp forms are always 0, Theorem 2.1 will be used to prove Theorem 1.1.

**Theorem 2.1.** Let \(k \in \mathbb{N}\). Let \(f(z) \in M_{2k}(\Gamma_0(1))\), with the Fourier series expansion given by

\[
f(z) = \sum_{n \geq 0} a_n q^n.
\]

Then for \(d \in \mathbb{N}\), the Fourier series expansion of \(f_d(z) = f(dz)\) at cusp \(1/c \in \mathbb{Q}\) is given by

\[
f_d(A_c^{-1}z) = \left( \frac{g}{d} \right)^{2k} (cz + 1)^{2k} f \left( \frac{g^2}{d} z + \frac{yg}{d} \right) = \left( \frac{g}{d} \right)^{2k} (cz + 1)^{2k} \sum_{n \geq 0} a_n q_c^n,
\]

where \(g = \gcd(d, c)\), \(y\) is some integer, \(A_c\) is the matrix given by \[1.2\] and \(q_c = e^{2\pi i \left( \frac{g^2}{d} z + \frac{yg}{d} \right)}\).
Proof. The Fourier series expansion of \( f_d(z) \) at the cusp \( 1/c \) is given by the Fourier series expansion of \( f_d(A_c^{-1}z) \) at the cusp \( \infty \). We have

\[
f_d(A_c^{-1}z) = f_d \left( \frac{-z}{-cz - 1} \right) = f \left( \frac{-dz}{-cz - 1} \right) = f(\gamma z),
\]
where \( \gamma = \begin{bmatrix} -d & 0 \\ -c & -1 \end{bmatrix} \). As \( \gcd(d/g, c/g) = 1 \), there exist \( y, v \in \mathbb{Z} \) such that

\[
\frac{-dv}{g} + \frac{cy}{g} = 1.
\]
Thus \( L := \begin{bmatrix} -d/g & y \\ -c/g & v \end{bmatrix} \in SL_2(\mathbb{Z}) \). Then for \( k \geq 1 \), we have

\[
f_d(A_c^{-1}z) = f(LL^{-1} \gamma z)
\]

\[
= \left( -c\left(\frac{-vd + cy}{d}z + y\right) + v\right) f\left(\frac{(-vd + cy)z + y}{d/g}\right)
\]

\[
= (g/d)^{2k}\left(\frac{vd - cy}{g}z + \frac{vd - cy}{g}\right) f\left(\frac{g^2z + yg}{d}\right)
\]

\[
= (g/d)^{2k}(cz + 1)^{2k} f\left(\frac{g^2z + yg}{d}\right),
\]
which completes the proof. Note that, the value of \( e^{2\pi i y} \) is independent of choice of \( y \). □

It follows from Theorem 2.1 and (1.4) that the terms of the Fourier coefficients of \( E_{2k}(dz) \) at the cusp \( 1/c \) are

\[
(2.1) \quad [0_c] \hspace{0.5em} E_{2k}(dz) = \left( \frac{\gcd(c, d)}{d} \right)^{2k},
\]

\[
(2.2) \quad [n_c] \hspace{0.5em} E_{2k}(dz) = \frac{-4k}{B_{2k}} e^{\frac{2\pi i y n}{\gcd(c, d)^2 N}} \left( \frac{\gcd(c, d)}{d} \right)^{2k} \sigma_{2k-1} \left( \frac{cd}{\gcd(c, d)^2 N} \right)^n, \quad (n \geq 1),
\]
for all \( k > 1 \), \( d \in \mathbb{N} \).

It is worth noting that it is possible to give statements equivalent to Theorem 2.1 for other spaces, i.e., let \( f(z) \in M_k(\Gamma_0(N), \chi) \), then we can determine Fourier series expansion of \( f(dz) \) at cusps of \( \Gamma_0(dN) \), by looking at the Fourier series expansion of \( f(z) \) at cusps of \( \Gamma_0(N) \).

3. Proof of Theorem 1.1

Let \( N \in \mathbb{N} \) be square-free and \( k > 1 \) be an integer. Assume that \( f(z) \in M_{2k}(\Gamma_0(N)) \). By [17, Theorem 5.9] and (1.1), we have

\[
f(z) = \sum_{d|N} a_d E_{2k}(dz) + C_N(f),
\]
for some \( a_d \in \mathbb{C} \) and \( C_N(f) \in S_{2k}(\Gamma_0(N)) \). First terms of Fourier series expansions of cusp forms are always 0. Then by (2.1), for each \( c \mid N \), we have

\[
(3.1) \quad [0] \lambda f(z) = \sum_{d \mid N} a_d \left( \frac{\gcd(c, d)}{d} \right)^{2k} + 0.
\]

In Lemma 3.1 below we give an orthogonal relation, which is useful for computing the inverse of the coefficient matrix of system of linear equations given by (3.1). Then using this inverse matrix we compute \( a_d \) in terms of \([0] \lambda f(z)\). This completes the proof of Theorem 1.1.

**Lemma 3.1.** Let \( N \in \mathbb{N} \) be square-free and \( k > 1 \) be an integer. Then for all \( c, d \mid N \) we have

\[
\sum_{t \mid N} (-1)^{\omega(t)+\omega(c)} \left( \frac{N \gcd(c, t)}{t} \cdot \frac{\gcd(t, d)}{d} \right)^{2k} = \begin{cases} 
\prod_{p \mid N} (p^{2k} - 1) & \text{if } c = d, \\
0 & \text{if } c \neq d.
\end{cases}
\]

**Proof.** Let \( N \in \mathbb{N} \) be square-free and \( k > 1 \) be an integer and \( c, d \mid N \). Let

\[
T(c, d; N) = \sum_{t \mid N} (-1)^{\omega(t)+\omega(c)} \left( \frac{N \gcd(c, t)}{t} \cdot \frac{\gcd(t, d)}{d} \right)^{2k}.
\]

Let us fix a prime \( p \mid N \). Then for all \( c, d \mid N \) we have

\[
T(c, d; N) = \sum_{t \mid N/p} (-1)^{\omega(t)+\omega(c)} \left( \frac{N \gcd(c, t)}{t} \cdot \frac{\gcd(t, d)}{d} \right)^{2k} + \sum_{t \mid N/p} (-1)^{\omega(tp)+\omega(c)} \left( \frac{N \gcd(c, tp)}{tp} \cdot \frac{\gcd(tp, d)}{d} \right)^{2k}
\]

(3.2)

\[
= (p^{2k} - (\gcd(c, p) \gcd(d, p))^{2k}) T(c, d; N/p),
\]

where in the last step we use the equation (for \( p \nmid t \))

\[
\gcd(c, tp) \frac{\gcd(d, tp)}{tp} = \gcd(c, p) \frac{\gcd(d, p)}{p} \cdot \gcd(c, t) \frac{\gcd(d, t)}{t}.
\]

If \( c \neq d \), then there exists a prime \( p \mid N \), such that \( \gcd(c, p) \gcd(d, p) = p \). Then by (3.2) we have \( T(c, d; N) = 0 \).

For \( c = d \), by (3.2), we have

\[
T(c, c; N) = T(c, c; 1) \prod_{p \mid N} (p^{2k} - \gcd(c, p)^{4k})
\]

\[
= (-1)^{\omega(c)} \left( \frac{1}{c} \right)^{2k} \prod_{p \mid N} (p^{2k} - \gcd(c, p)^{4k})
\]
Let \( S \in \mathbb{N} \) be square-free, \( l, m > 1 \) be integers and \( a, b \in \mathbb{N} \) be such that \( \text{lcm}(a, b) \mid N \). Then we have

\[
E_{2l}(az)E_{2m}(bz) \in M_{2(l+m)}(\Gamma_0(N)).
\]

Thus, combining Theorem 1.1 and (2.1), we obtain the following theorem.

**Theorem 4.1.** Let \( N \in \mathbb{N} \) be square-free. Let \( l, m > 1 \) be integers and \( a, b \in \mathbb{N} \) be such that \( \text{lcm}(a, b) \mid N \). Then there exists a cusp form \( C_N(E_{2l}(az)E_{2m}(bz)) \in S_{2k}(\Gamma_0(N)) \) such that

\[
E_{2l}(az)E_{2m}(bz) = \sum_{d \mid N} \left( \frac{-1}{d} \right)^{\omega(d)+\omega(c)} \left( N \gcd(c, d) \right)^{2k} \left( \frac{\gcd(a, c)}{a} \right)^{2l} \left( \frac{\gcd(b, c)}{b} \right)^{2m} E_{2k}(dz) + C_N(E_{2l}(az)E_{2m}(bz)),
\]

where \( k = l + m \). Comparing coefficients of \( q^n \) \((n > 0)\) in (4.1), we obtain the formula for \( W(a^{2l-1}, b^{2m-1}, n) \) given by (1.9).

In the remainder of this section we let \( \text{lcm}(a, b) \mid 6 \) and give the description of cusp forms \( C_6(E_{2l}(az)E_{2m}(bz)) \in S_{2k}(\Gamma_0(6)) \) for all \( k > 1 \) in terms of eta quotients. Let \( k, i \in \mathbb{N} \), and let us define the following eta quotient

\[
S(2k, 6, i; z) = \left( \frac{\eta^6(z)\eta(6z)}{\eta^3(2z)\eta^7(3z)} \right)^{2k} \left( \frac{\eta(2z)\eta^5(6z)}{\eta^3(z)\eta(z)} \right)^{i} \left( \frac{\eta^{13}(2z)\eta^{11}(3z)}{\eta^{17}(z)\eta^{17}(6z)} \right).
\]

We note that the order of \( S(2k, 6, i; z) \) at \( \infty \) is \( i \), i.e. we have \([n]S(2k, 6, i; z) = 0 \) when \( n < i \). By [10, Corollary 2.3, p. 37] and [13, Theorem 1.64] we have \( S(2k, 6, i; z) \in S_{2k}(\Gamma_0(6)) \) for all \( k > 1 \) and \( 1 \leq i \leq 2k - 3 \). Since the dimension of \( S_{2k}(\Gamma_0(6)) \) is \( 2k - 3 \) for all \( k > 1 \), and \( S(2k, 6, i; z) \) \((1 \leq i \leq 2k - 3)\) are linearly independent, we obtain a basis for \( S_{2k}(\Gamma_0(6)) \).

**Theorem 4.2.** Let \( k > 1 \) be an integer. Then the set of eta quotients

\[
\{S(2k, 6, i; z) : 1 \leq i \leq 2k - 3\}
\]

form a basis for \( S_{2k}(\Gamma_0(6)) \).

Theorem 4.2 gives an example of a family of modular form spaces which is generated by eta quotients, a question raised by Ono in [13] and answered by Rouse and Webb in [15]. We can now express \( C_6(E_{2l}(az)E_{2m}(bz)) \) for \( \text{lcm}(a, b) \mid 6 \) as linear combinations of \( S(2k, 6, i; z) \).
**Theorem 4.3.** Let $l, m > 1$ be integers and $a, b \in \mathbb{N}$ be such that $\text{lcm}(a, b) \mid 6$. Then we have

$$C_6(E_{2l}(az)E_{2m}(bz)) = \sum_{i=1}^{2k-3} b_i S(2k, 6, i; z),$$

where $k = l + m \geq 4$, and

$$b_i = \frac{4k}{B_{2k}} \sum_{d|6} \left( \sum_{c|6} \frac{(-1)^{\omega(d)+\omega(c)}}{(2^{2k} - 1)(3^k - 1)} \left( \frac{6 \gcd(c, d)}{c} \right)^{2k} \left( \frac{\gcd(a, c)}{a} \right)^{2l} \left( \frac{\gcd(b, c)}{b} \right)^{2m} \right) \sigma_{2k-1}(i/d)$$

$$- \frac{4m}{B_{2m}} \sigma_{2m-1}(i/b) - \frac{4l}{B_{2l}} \sigma_{2l-1}(i/a) + \frac{16lm}{B_{2l}B_{2m}} W(\omega(2^{2l-1}, b^{2m-1}; i)) - \sum_{j=1}^{i-1} |d| \sigma_{2l-1}(i/d).$$

Below we give two examples to show that coefficients of $\sigma_{2k-1}(n/d)$ in (1.9) lead to simple and beautiful expressions. We first let $a, b = 1$ in (1.9). Then we have

$$W(1^{2l-1}, 1^{2m-1}; n) = -2B_{2l}B_{2m} \sigma_{2l-1}(n) + \frac{B_{2l}}{4l} \sigma_{2m-1}(n) + \frac{B_{2m}}{4m} \sigma_{2l-1}(n)$$

$$+ \frac{B_{2l}B_{2m}}{16lm} \sum_{i=1}^{2k-3} [n]b_i S(2k, 6, i; z)$$

which, for $l, m > 1$, is a similar expression to the one given by Ramanujan in [14]. In (1.3) the cusp part vanishes when $l = 2, m = 2; l = 2, m = 3; l = 2, m = 5; l = 3, m = 4$, in agreement with Ramanujan’s results in [14].

Second, we let $a = 2, b = 3$ in (1.9), and $1 < l = m$. Then we have

$$W(2^{2l-1}, 3^{2l-1}; n) = -2B_{2l} \sigma_{2l-1}(n) - 2^{2l} \sigma_{2l-1}(n/2) + 3^{2l} \sigma_{2l-1}(n/3) - 6^{2l} \sigma_{2l-1}(n/6)$$

$$+ \frac{B_{2l}}{4l} \sigma_{2l-1}(n/3) + \frac{B_{2l}}{4l} \sigma_{2l-1}(n/2) + \frac{B_{2l}B_{2l}}{16l^2} \sum_{i=1}^{4l-3} [n]b_i S(4l, 6, i; z).$$

Note that, in (1.4), the cusp part seems to be never vanishing.

5. **APPLICATION: REPRESENTATIONS BY QUADRATIC FORMS**

Let $N \in \mathbb{N}$ be an odd square-free number. Let $g(z) = \frac{\eta^2(z)}{\eta(2z)}$. Then for $r_\delta \in \mathbb{N}_0$ ($\delta \mid N$), not all zeros, we have

$$\prod_{\delta \mid N} g^{r_\delta}(\delta z) \in M_{2k}(\Gamma_0(2N)),$$

where $k = 2 \sum_{\delta \mid N} r_\delta > 0$. $2N$ is square-free, thus we can apply Theorem 1.4 to (5.1).
Theorem 5.1. Let $N \in \mathbb{N}$ be an odd square-free number. Let $r_\delta \in \mathbb{N}_0$ ($\delta \mid N$, not all zeros), and $k = 2 \sum_{\delta \mid N} r_\delta > 0$. Then there exists a cusp form $C_{2N} \left( \prod_{\delta \mid N} g^{8r_\delta}(\delta z) \right) \in S_{2k}(\Gamma_0(2N))$, such that

$$
\prod_{\delta \mid N} g^{8r_\delta}(\delta z) = \sum d_{\mid N} \left( \sum_{\epsilon \mid N} \left( \frac{N \gcd(2c, d)}{c} \right)^{2k} \prod_{\delta \mid N} \left( \frac{\gcd(2c, \delta)}{\delta} \right)^{4r_\delta} \right) E_{2k}(dz) + C_{2N} \left( \prod_{\delta \mid N} g^{8r_\delta}(\delta z) \right).
$$

Proof. Let $N \in \mathbb{N}$ be an odd square-free number, $r_\delta \in \mathbb{N}_0$ ($\delta \mid N$, not all zeros), and $k = 2 \sum_{\delta \mid N} r_\delta > 0$. We use [10, Proposition 2.1] to compute

$$[0]_c g^8(\delta z) = \begin{cases} 
(\gcd(c, \delta))^4 & \text{if } 2 \mid c, \\
0 & \text{if } 2 \nmid c.
\end{cases}
$$

Thus we have

$$[0]_c \prod_{\delta \mid N} g^{8r_\delta}(\delta z) = \begin{cases} 
\prod_{\delta \mid N} \left( \frac{\gcd(c, \delta)}{\delta} \right)^{4r_\delta} & \text{if } 2 \mid c, \\
0 & \text{if } 2 \nmid c.
\end{cases}
$$

Then the result follows from Theorem 1.1 by putting the values of $[0]_c \prod_{\delta \mid N} g^{8r_\delta}(\delta z)$ in (1.5). □

Below we apply Theorem 5.1 to give formulas concerning representations by quadratic forms. Following Ramanujan’s notation, let us define

$$\varphi(z) = \sum_{n=-\infty}^{\infty} q^{n^2}.
$$

Then we have

$$
\sum_{n=0}^{\infty} N(a_1^{r_1}, a_2^{r_2}, \ldots, a_m^{r_m}; n)q^n = \prod_{i=1}^{m} \varphi^{r_i}(a_{i}z).
$$

On the other hand, by [3, (1.3.13)], we have $\varphi(z) = g(z + 1/2)$, (or equivalently $\varphi(q) = g(-q)$). Replacing $z$ by $z + 1/2$ in Theorem 5.1 we have the following statement.

Theorem 5.2. Let $N \in \mathbb{N}$ be an odd square-free number, $r_\delta \in \mathbb{N}_0$, $\delta \mid N$, not all zeros, and $k = 2 \sum_{\delta \mid N} r_\delta > 0$. Then we have

$$
\prod_{\delta \mid N} \varphi^{8r_\delta}(\delta z) = \sum d_{\mid N} \left( \sum_{\epsilon \mid N} \left( \frac{-1)^{\omega(\epsilon)+\omega(c)}}{c} \right) \prod_{\delta \mid N} \left( \frac{\gcd(c, \delta)}{\delta} \right)^{4r_\delta} \right) E_{2k}(2dz)
$$

10
Theorem 5.3. Let \( N \) be a positive integer.

Let \( N \) be a positive integer.

Recently in [7], Cooper, Kane and Ye obtained formulas for \( N(1^{s_{r_1}}, \ldots, N^{s_{r_N}}; n) \) given by (5.2).

We compare coefficients of \( q^n \) on both sides of (5.2) to obtain the formula for \( N(1^{s_{r_1}}, \ldots, N^{s_{r_N}}; n) \) given by (1.10).

Below we illustrate Theorem 5.2 for the case \( N = p, p \) a prime. Let \( r_1, r_p \in \mathbb{N}_0 \) and \( k = 2(r_1 + r_p) \). Then we have

\[
N(1^{8r_1}, p^{8r}; n) = \frac{4k}{B_{2k}} \left( \frac{(-1)^n(p^{4r_1} - 1)}{(2k - 1)(p^{2k} - 1)} \sigma_{2k-1}(n) - \frac{2^{2k}(p^{4r_1} - 1)}{(2k - 1)(p^{2k} - 1)} \sigma_{2k-1}(n/2) \right.
\]

\[
+ \frac{(-1)^n(p^{2k} - p^{4r_1})}{(2k - 1)(p^{2k} - 1)} \sigma_{2k-1}(n/p) - \frac{2^{2k}(p^{2k} - p^{4r_1})}{(2k - 1)(p^{2k} - 1)} \sigma_{2k-1}(n/2p)
\]

\[
+ [n]C_{2p} \left( \prod_{p \mid n} g^{8r}(\delta z + 1/2) \right).
\]

Letting \( r_1 = r_p = l \) in (5.3), we obtain

\[
N(1^{8l}, p^{8l}; n) = \frac{16l}{B_{8l}} \left( \frac{(-1)^n \left( \sigma_{8l-1}(n) + p^{4l} \sigma_{8l-1}(n/p) \right) - 2^{8l} \left( \sigma_{8l-1}(n/2) + p^{4l} \sigma_{8l-1}(n/2p) \right)}{(2^{8l} - 1)(p^{4l} + 1)} \right)
\]

\[
+ [n]C_{2p} \left( \prod_{p \mid n} g^{8l}(\delta z + 1/2) \right).
\]

Recently in [7], Cooper, Kane and Ye obtained formulas for \( N(1^k, p^k; n) \), valid for all \( k \in \mathbb{N} \) and \( p = 3, 7, 11, 23 \). For \( t \in \{1, p\} \), we have

\[
(5.5)
\]

\[
(-1)^n \sigma_{8l-1}(n/t) - 2^{8l} \sigma_{8l-1}(n/2t) = -\sigma_{8l-1}(n/t) + 2\sigma_{8l-1}(n/2t) - 2^{8l} \sigma_{8l-1}(n/4t).
\]

We use (5.5) to show that the Eisenstein parts of the formula in [7] and of (5.4) agrees when \( k = 8l \). Our formula is valid for all primes, but fails when \( 8 \nmid k \).

Recalling the eta quotients \( S(2k, 6, i; z) \) defined by (1.12), we can express the cusp part of (5.3) in terms of eta quotients, when \( p = 3 \).

**Theorem 5.3.** Let \( r_1, r_3 \in \mathbb{N}_0 \) and \( k = 2(r_1 + r_3) \). Then we have

\[
(5.6)
C_6 \left( g^{8r_1}(z + 1/2)g^{8r_3}(3z + 1/2) \right) = \sum_{i=1}^{2k-3} b_i S(2k, 6, i; z + 1/2),
\]

where

\[
b_i = (-1)^i N(1^{8r_1}, 3^{8r_3}; i)
\]
\[ + \frac{4k}{B_{2k}} \left( \frac{(1 - 3^{4r_1})}{(2^{2k} - 1)(3^{2k} - 1)} \sigma_{2k-1}(i) + \frac{2^{2k}(3^{4r_1} - 1)}{(2^{2k} - 1)(3^{2k} - 1)} \sigma_{2k-1}(i/2) \right. \\
\left. + \frac{(3^{4r_1} - 3^{2k})}{(2^{2k} - 1)(3^{2k} - 1)} \sigma_{2k-1}(i/3) + \frac{2^{2k}(3^{2k} - 3^{4r_1})}{(2^{2k} - 1)(3^{2k} - 1)} \sigma_{2k-1}(i/6) \right) \\
- \sum_{j=1}^{i-1} b_j[i] S(2k, 6, j; z). \]

If we let \( p = 3, r_1 = k \) and \( r_3 = 0 \) in (5.3), we have

\[ N(1^{8k}, 3^0; n) = -\frac{4k}{B_{2k}} \left( \frac{(-1)^{n+1}}{2^{4k} - 1} \sigma_{4k-1}(n) + \frac{2^{4k}}{2^{4k} - 1} \sigma_{4k-1}(n/2) \right) \\
+ \sum_{i=1}^{4k-3} (-1)^n b_i[n] S(4k, 6, i; z), \]

which, after an algebraic manipulation with an equation similar to (5.5), agrees with the Ramanujan-Mordell formula, see [6, 12, 14].

6. Further discussions

In this section we discuss three consequences of previous results, first we give a ‘Sturm bound’ for Eisenstein series; second we discuss an application of Theorem 1.1 to eta quotients; and third we give an interesting relationship between the Fourier coefficients of \( f(z) \in E_{2k}(\Gamma_0(N)) \) and \( v_c(f) \).

Let \( k > 1 \) be an integer, \( N \in \mathbb{N} \) be square-free and \( p_1 \) be the smallest prime that divides \( N \). Then, using (2.2), it is not hard to see

\[ 0 = [n]_c \sum_{d \mid N} a_d E_{2k}(dz), \quad n \leq \frac{N}{p_1}, \tag{6.1} \]

if and only if \( a_d = 0 \) for all \( d \mid N \). This gives us the following theorem, which could be viewed as a Sturm Theorem for Eisenstein forms. Note that this bound is much smaller than the Sturm bound, which is to be expected.

**Theorem 6.1.** Let \( k > 1 \) be an integer, \( N \in \mathbb{N} \) be square-free and \( p_1 \) be the smallest prime that divides \( N \). Let \( f(z) \in E_{2k}(\Gamma_0(N)) \) be a non-zero function. Then for all \( c \mid N \) we have \( v_c(f) \leq \frac{N}{p_1} \).

The second part of Theorem 1.1 can be applied to finding Fourier coefficients of eta quotients. The statement is as follows.
Theorem 6.2. Let $N \in \mathbb{N}$ be square-free and $k > 1$ be an integer. Let $r_\delta \in \mathbb{Z}$ $(\delta \mid N)$, not all zero. If $f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z) \in E_{2k}(\Gamma_0(N))$ then for $n > 0$ we have

$$[n]f = -\frac{4k}{B_{2k}} \sum_{d \mid N} \left( \sum_{c \mid N, v_c(f) = 0} \frac{(-1)^{\omega(d)+\omega(c)}}{\prod_{p \mid N}(p^{2k} - 1)} \left( \frac{N \gcd(c,d)}{c} \right)^{2k} \prod_{\delta \mid N} \frac{\nu(r_\delta,c)\gcd(c,\delta)^{r_\delta/2}}{\delta^{r_\delta/2}} \right) \sigma_{2k-1}(n/d).$$

However this theorem is not as useful as we expected. One can use Theorem 6.1 (6.6) below and pigeonhole principle to show that there are only finitely many eta quotients in $E_{2k}(\Gamma_0(N))$ for each $N \in \mathbb{N}$ square-free. We believe, disregarding the repetitions, the following well known equations are the only examples:

\begin{align}
(6.2) & \quad \frac{\eta^{16}(z)}{\eta^8(2z)} = 1 + \sum_{n>0} (-16\sigma_3(n) + 256\sigma_3(n/2))q^n, \\
(6.3) & \quad \frac{\eta^{16}(2z)}{\eta^8(z)} = \sum_{n>0} (\sigma_3(n) - \sigma_3(n/2))q^n.
\end{align}

Below we try to explain the reason of this. Before we start, note that there are two types of repetitions, first if $f(z) \in E_{2k}(\Gamma_0(N))$, then we also have $f(z) \in E_{2k}(\Gamma_0(N'))$ for any $N \mid N'$. Second, if we have $f(z) \in E_{2k}(\Gamma_0(N))$, then $f(dz) \in E_{2k}(\Gamma_0(dN))$. Note that this is slightly different than the concept of oldforms (which is defined for cusp forms). In the following arguments we assume $f(z)$ is not a repetition. Assuming $f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z) \in E_{2k}(\Gamma_0(N))$, we have

$$f(z) = \sum_{d \mid N} a_d E_{2k}(dz).$$

We compare both sides of (6.4) at different cusps using (2.1) and (2.2). The zeros of the eta quotient yield to the equations

$$0 = [n]c \sum_{d \mid N} a_d E_{2k}(dz), \text{ for } n < v_c(f).$$

On the other hand for sum of orders of zeros of $f(z)$ we have

$$S(k,N) = \sum_{r \in R(N)} v_{1/r}(f(z)) = \sum_{c \mid N} \frac{N}{24\gcd(c^2,N)} \sum_{\delta \mid N} \frac{\gcd(c,\delta)^2r_\delta}{\delta} = \frac{k}{6} \prod_{p \mid N}(p+1),$$

see [4] (4.2.9) for the details, we also have the number of cusps of $\Gamma_0(N)$ is $|R(N)| = \sigma_0(N)$. It appears, if $S(k,N) > |R(N)|$, then the number of linearly independent equations coming from (6.5) which equal to zero are equal to the number of variables $a_d$. This forces $f(z) = 0$. We have $S(k,N) \leq |R(N)|$ for the couples $(k,N) = (2,1), (2,2), (2,3), (2,5), (2,6), (3,1), (3,2), (3,3), (4,1), (4,2), (5,1), (5,2)$. Finally, we use similar arguments used in [1] to find out (6.2) and (6.3) are the only eta quotients in spaces corresponding to the couples above.
To stress the intimate relationship between a modular form and its orders of zeros at cusps we note the following relation. Let $N \in \mathbb{N}$ be square-free and $k > 1$ be an integer, and $f(z) = \sum_{d|N} a_d E_{2k}(dz) \in E_{2k}(\Gamma_0(N))$. Then from (2.2) we deduce that if $v_c(f) > 1$ then $a_{N/c} = 0$.

**Acknowledgements**

I would like to thank Dr. Song Heng Chan for his helpful comments throughout the course of this research. I am grateful to Dr. Heng Huat Chan, whose feedback helped to give a more elegant proof of Lemma 3.1. The author was supported by the Singapore Ministry of Education Academic Research Fund, Tier 2, project number MOE2014-T2-1-051, ARC40/14.

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