Lump, multi-lump, cross kinky-lump and manifold periodic-soliton solutions for the (2+1)-D Calogero–Bogoyavlenskii–Schiff equation

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ABSTRACT
A bilinear form of the (2+1)-dimensional nonlinear Calogero–Bogoyavlenskii–Schiff (CBS) model is derived using a transformation of dependent variable, which contain a controlling parameter. This parameter can control the direction, wave height and angle of the traveling wave. Based on the Hirota bilinear form and ansatz functions, we build many types of novel structures and manifold periodic-soliton solutions to the CBS model. In particular, we obtain entirely exciting periodic-soliton, cross-kinky-lump wave, double kinky-lump wave, periodic cross-kinky-lump wave, periodic two-solitary wave solutions as well as breather style of two-solitary wave solutions. We present their propagation features via changing the existence parametric values in graphically. In addition, we estimate a condition that the waves are propagated obliquely for $\eta \neq 0$, and orthogonally for $\eta = 0$.

1. Introduction

The nonlinear partial differential equations (NPDEs) have remained a subject of international research interest in physics, chemistry, biology and nonlinear sciences, especially, in nonlinear optics, photonics, Bose-Einstein condensate, harbor and coastal designs (Bruzon et al., 2003; Peng, 2006; Kobayashi and Toda, 2006; Li and Chen, 2004; Wang and Yang, 2012; Chen and Ma, 2018; Wazwaz, 2008; Ullah et al., 2020; Roshid and Roshid, 2018; Hossen et al., 2018; Ming et al., 2013; Roshid and Roshid, 2018; Khatun et al., 2020). To realize the physical mechanism of phenomena for the NPDEs in physics and engineering, their exact solutions are highly investigated. One of a significant nonlinear evolution equation is the Calogero–Bogoyavlenskii-Schiff (CBS) equation, which extensively used in various purposes. The CBS model is developed via dissimilar techniques (Peng, 2006; Kobayashi and Toda, 2006; Bruzon et al., 2003) and obtained its exact solutions (Li and Chen, 2004; Wang and Yang, 2012; Chen and Ma, 2018; Wazwaz, 2008) via the dint of symbolic computation.

Recently many authors were worked on the CBS Eq. (1). The multiple-soliton solutions of the CBS model were obtained by Wazwaz, (2008). Zhang et al. (Zhang et al., 2009) did research on the CBS equation and they established substantially abundant symmetries and symmetry reduction of the (2+1)-dimensional generalized CBS equation. Moreover, Wazwaz, (2010) formed multiple soliton solutions and multiple singular soliton solutions for the (2+1)-dimensional as well as the (3+1)-dimensional CBS equations. Quasi-periodic wave solutions for the (2+1)-dimensional generalized CBS equation was incorporated in literature by Wang and Yang (Wang and Yang (2012)). More recently, Chen and Ma (2018) explored lump wave solutions of the generalized CBS equation.

In this article, we aim to determine a new bilinear form and determine innovative periodic-soliton solutions, periodic cross-kink-wave, cross-double kink-periodic wave, periodic two-solitary wave as well as breather style of two-solitary wave of the CBS model.

2. Bilinear forms of the Calogero-Bogoyavlenskii-Schiff equation

In this section, we shall build a bilinear form of the CBS Eq. (1). To do that, at first makes over the (2+1)-dimensional nonlinear CBS (1) into the bilinear forms through the dependent variable transformations (Wang 2012):
\[ u = \eta y + \frac{3}{\sigma} \ln \tau(x, y, t), \]  

(2)

The beyond (2+1)-dimensional nonlinear evolution (1) is drawn into the Hirota D-operator equivalence (Wang 2012) as:

\[ (\sigma D_x D_t + D_x D_y - 3\eta D_x^3 + c) \tau, \tau = 0 \]  

(3)

Now, we consider the relation between Hirota D-operator and its bilinear form via

\[ \prod_{i=1}^{N} D_{xi} \tau = \prod_{i=1}^{N} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} \right) \tau(x(u(x))), \]  

(4)

where \( x = (x_1, \ldots, \ldots, x_N) \), \( \bar{x} = (\bar{x}_1, \ldots, \ldots, \bar{x}_N) \) are nonzero vectors and \( n_1, \ldots, n_N \) are arbitrary nonnegative integers. Under formula (4), (3) can be converted (Wang 2012) to

\[ \tau \left( 2\sigma \alpha + 2\sigma \beta - 3\eta \beta \right) - 2\sigma \alpha \tau_i + 6\sigma \eta \tau_i - 2\sigma \alpha \tau_i + 6\sigma \eta \tau_i + \sigma \tau_i = 0 \]  

(5)

3. Solutions of the Calogero-Bogoyavlenskii-Schiff (CBS) equations

In this section, we present the dynamical behaviors of soliton solutions such as lump wave, multi-lump wave, interaction between kink and lump waves, and interactions of multi-lump and periodic wave for the CBS model in various subsections.

3.1. Lump solutions of the CBS equations

Through the support of the symbolic computational software Maple, we can determine positive quadratic solution to the CBS equation from its bilinear arrangement. Upon the 2-dimensional universe, we are going to determine positive quadratic solution to the CBS equation in various subsections.

3.1.1. Parameters to be determinant.

We now consider the relation between Hirota D-operator and its bilinear form via

\[ \prod_{i=1}^{N} D_{xi} \tau = \prod_{i=1}^{N} \left( \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_i} \right) \tau(x(u(x))), \]  

(4)

where \( x = (x_1, \ldots, \ldots, x_N) \), \( \bar{x} = (\bar{x}_1, \ldots, \ldots, \bar{x}_N) \) are nonzero vectors and \( n_1, \ldots, n_N \) are arbitrary nonnegative integers. Under formula (4), (3) can be converted (Wang 2012) to

\[ \tau \left( 2\sigma \alpha + 2\sigma \beta - 3\eta \beta \right) - 2\sigma \alpha \tau_i + 6\sigma \eta \tau_i - 2\sigma \alpha \tau_i + 6\sigma \eta \tau_i + \sigma \tau_i = 0 \]  

(5)

Thus the solutions are

\[ \ell_1 = 0, \ell_2 = -\ell_1, \ell_3 = \ell_1, \ell_4 = \ell_1, \ell_5 = 0, \ell_6 = -\ell_1, \ell_7 = \ell_1, \ell_8 = \ell_1, \ell_9 = \ell_1, \ell_{10} = -\ell_1, \ell_{11} = \ell_{12} = h_1 = h_i \]  

where \( \ell_1; (i = 1, 2, \ldots, 12) \) produces two set of constraint equations:

\[ \ell_1 = 0, \ell_2 = -\ell_1, \ell_3 = \ell_1, \ell_4 = \ell_1, \ell_5 = 0, \ell_6 = -\ell_1, \ell_7 = \ell_1, \ell_8 = \ell_1, \ell_9 = \ell_1, \ell_{10} = -\ell_1, \ell_{11} = \ell_{12} = h_1 = h_i \]  

(9)

(7)

(1)
Thus the solution

\[ u(x, t) + \eta y + \frac{3}{\delta} \ln \tau, \]

(10)

where \( \tau = \left( -\frac{\ell_5}{t_6^2} y + \ell_8 + \ell_9 \right)^2 + \left( -\ell_8 \ell_9 y + \ell_7 t + \ell_9 \right)^2 + \ell_9 + \h_1 e^{\left( \ell_1 x + \frac{\ell_2}{t_6} y + \ell_3 t \right)/\tau_{10}} \cdot \mathbf{f}_{10} \neq 0 \) and \( \ell_3, \ell_4, \ell_5, \ell_7, \ell_8, \ell_9, \ell_{10}, \ell_{12} \) are arbitrary constants. We see that angle of flow can be controlled via the parameter \( \eta \), which explained in the previous subsection 3.1. The motion of particle describes in a curvy path for \( \eta \neq 0 \), but tend to diminish into a linear path as \( \eta \to 0 \) and exactly through line for \( \eta + 0 \) (see contour plot of Figure 3).

3.3. Multi lump solutions of the CBS equation

Let us pick the trial solution for the superposition of two exponential functions and a cosine function:

Figure 1. Profile of the Eq. (7) for \( \ell_4 = \ell_5 = \ell_6 = \ell_9 = 1, \sigma = 1, \delta = 0.3: \) 3D plot (upper) and corresponding contour plot (below) at \( t = 0 \) where images (a) for \( \eta = 0.5 \), (b) for \( \eta = 0.2 \), and (c) for \( \eta = 0.02 \).

Figure 2. Profile of the Eq. (8) for \( \ell_4 + \ell_5 + \ell_6 + \ell_8 + \ell_9 = 1, \sigma = 1, \delta = 0.3: \) 3D plot (upper) and corresponding contour plot (below) at \( t = 0 \) where images (a) Real part of the Eq. (8) and (b) Imaginary part of the Eq. (8).
\[ \tau + e^{-\delta x} + h_1 e^{\delta x} + h_2 (\cos (d_2 h)) \text{ where } g(x,y,t) + x + \ell_1 y + w_1 t, \]
\[ h(x,y,t), \]
\[ + x + \ell_2 y + w_2 t, \]
\[ \theta \]  
where \( \ell_1, \ell_2, w_1, w_2 \) are real parameters to be calculated.

Inserting Eq. (11) into Eq. (5), and solving for the unknown parameters \( \ell_1, \ell_2, w_1, w_2, d_1, d_2 \) yield a set of constraints:

\[ \{ \ell_1 = \frac{-1}{2} \sigma w_2 - 3 \eta, \quad \ell_2 = 0, \quad d_1 = d_1, d_2 \}, \]
\[ h_1 = h_1 + h_2, \quad w_1 = w_1 + \frac{1}{2} \frac{3 \eta d_1^2 + \sigma w_2}{\delta \sigma}, \quad w_2 = w_2 \}

Thus the solution
\[ u(x,t) = \eta y + \frac{3}{h} \ln \tau, \]

3.4. Interaction of the multi-lump and periodic solutions of the CBS equation

Let us take the trial solution for the superposition of the sum and product of sine, cosine and their hyperbolic functions:
\( \tau = 1 + \cosh(\zeta_1) \sigma_1 \cos(\zeta_2) + \cosh(\zeta_1) \sigma_2 \sin(\zeta_2) + \sigma_3 \sinh(\zeta_1) + \sigma_4 \cosh(\zeta_1) \)  

(13)

where \( \zeta_1(x, y, t) = \ell_1 x + \wp_1 y + \omega_1 t, \zeta_2(x, y, t) = \ell_2 x + \wp_2 y + \omega_2 t \) and \( \ell_1, \ell_2, \wp_1, \wp_2, \omega_1, \omega_2, \) are parameters to be calculated.

Putting Eq. (13) into Eq. (5), and resolving for unknown parameters \( \ell_1, \ell_2, \wp_1, \wp_2, \omega_1, \omega_2 \) yield eight set of constraints:

**Set-1:**

\[ \begin{align*}
\ell_1 &= \ell_1, \\
\ell_2 &= \ell_2, \\
\wp_1 &= \frac{-1/4 \eta \ell_1}{\ell_1}, \\
\wp_2 &= \frac{1/8 - 9 \omega_1^2 \omega_1 \ell_2^2 + 3 \omega_2 \ell_1^2 - 9 \omega_1 \ell_1^2 + 3 \omega_2 \ell_2^2}{\ell_1 \ell_2}, \\
\omega_1 &= \omega_1, \\
\omega_2 &= \frac{1/8 - 6 \omega_1^2 \omega_1 \ell_2^2 + 3 \omega_2 \ell_1^2 - 6 \omega_1 \ell_1^2 - 9 \omega_1 \ell_1^2 - 3 \ell_1^2 \sigma}{\ell_1 \ell_2}.
\end{align*} \]

**Set-2:**

Figure 5. Profile of the Eq. (14) for \( \sigma = -1.9, \eta = 0.9, \omega_1 = -2.0, \omega_2 = 2.0, \sigma_1 = 0, \sigma_4 = 3 \) : 3D plots (upper) and corresponding contour plots (below) at \( t = 0 \) where images (a) Real part of the Eq. (14) and (b) Imaginary part of the Eq. (14).

Figure 6. Profile of the Eq. (15) for \( \ell_1 = \wp_2 = \sigma_1 = \sigma_2 = 0, \ell_2 = \omega_1 = \sigma = \delta = \sigma_4 = 1, \eta = 0.4 \) : 3D plot (upper) and contour plot (below) at \( t = 0 \) where images (a) for \( \eta = 0.4 \) and (b) for \( \eta = 0 \).
Figure 7. Profile of interaction of lump and periodic wave solution of the Eq. (16) for $\ell_1 = \sigma_4 = 0, \ell_2 = W_1 = 1, W_2 = \sigma_1 = \sigma_2 = 2, \eta = 0.04, \delta = 3$ : 3D plot (upper) and corresponding contour plot (below) at $t = 0$ where images (a) Real part of the Eq. (16) and (b) Imaginary part of the Eq. (16).

Figure 8. Profile of two bell solitons and two kink solitons of the Eq. (17) for $\ell_1' = -1, \ell_2' = 1, W_1 = \sigma_1 = \sigma_2 = 0, \sigma_4 = 0.05, \eta = 0.05, \delta = 3$ : 3D plot (upper) and corresponding contour plot (below) at $t = 0$ where images (a) Real part of the Eq. (17) and (b) Imaginary part of the Eq. (17).

Figure 9. Profile of the Eq. (18) for $\ell_1' = -1, \ell_2' = 2, \sigma_1 = \sigma_2 = 0, \sigma_4 = 0.05, W_1 = \sigma = 1, \eta = 0.05, \delta = 3$ : (a) 3D plot (left), (b) contour plot (right) at $t = 0$. 

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\[ \ell_1 = 0, \ell_2 = \ell_2, \varphi_1 = -\frac{\sigma w_1}{\ell_2}, \varphi_2 = 0, \sigma_1 = 0, \sigma_2 = 0, \sigma_3 = \sigma_1, \sigma_4 = \sigma_1, \]
\[ w_1 = w_1, w_2 = \frac{3\nu \ell_2}{\sigma}. \]

Set-3:
\[ \ell_1 = 0, \ell_2 = \ell_2, \varphi_1 = -\frac{3\nu \ell_2 + \sigma w_1}{\ell_2}, \varphi_2 = 0, \sigma_1 = 0, \]
\[ \sigma_2 = \sigma_2, \sigma_3 = \sigma_1, \sigma_4 = 0, w_1 = w_1, w_2 = w_2. \]

Set-4:
\[ \ell_1 = \ell_1, \ell_2 = -\ell_1, \varphi_1 = -\frac{3\nu \ell_1 + \sigma w_1}{\ell_1}, \varphi_2 = 0, \sigma_1 = 0, \]
\[ \sigma_2 = 0, \sigma_3 = \sigma_1, \sigma_4 = \sigma_1, w_1 = w_1, w_2 = \frac{-6\nu \ell_1 + \sigma w_1}{\sigma}. \]

Set-5:
\[ \ell_1 = \ell_1, \ell_2 = \ell_1, \varphi_1 = -\frac{3\nu \ell_1 + \sigma w_1}{\ell_1}, \varphi_2 = 0, \sigma_1 = 0, \]
\[ \sigma_2 = 0, \sigma_3 = \sigma_1, \sigma_4 = \sigma_2, w_1 = w_1, w_2 = \frac{-6\nu \ell_1 + \sigma w_1}{\sigma}. \]

Set-6:
\[ \ell_1 = \ell_1, \ell_2 = \ell_2, \varphi_1 = -\frac{3\nu \ell_2 + \sigma w_1}{\ell_2}, \varphi_2 = 0, \sigma_1 = 0, \]
\[ \sigma_2 = 0, \sigma_3 = \sigma_1, \sigma_4 = \sigma_1, w_1 = w_1, w_2 = \frac{-6\nu \ell_2 + \sigma w_1}{\sigma}. \]

For the Set-1, the solution
\[ u(x, t) = \eta y + \frac{1}{\sigma} \left( x + \frac{1}{\nu} \log \tau \right), \]
where
\[ \tau = 1 + \cosh \left( \ell_1 x - \frac{3\nu \ell_1 + \sigma w_1}{\ell_1} y + w_1 t \right) I \sigma_2 \cos (\ell_2 x + \varphi_2 y + w_2 t) \]
\[ + \cosh \left( \ell_1 x - \frac{3\nu \ell_1 + \sigma w_1}{\ell_1} y + w_1 t \right) I \sigma_2 \sin (\ell_2 x + \varphi_2 y + w_2 t). \]
with \( w_2 = -\frac{1}{2} \ell_3 w_1 \). \( \varphi_2 \) = \( \frac{1}{\gamma} - \frac{3\eta \ell_2 + 3\eta \ell_2 \gamma}{\ell_3 \gamma}, \) \( \varphi_2 = \) \( \frac{1}{\gamma} - \frac{3\eta \ell_2 + 3\eta \ell_2 \gamma}{\ell_3 \gamma}, \) \( \alpha \neq 0, \ell_1, \ell_2, w_1, \) and \( \varphi_2 = \) arbitrary constants.

The solution Eq. (14) from the combinations of hyperbolic and sinusoidal functions gives kinky wave whose lumps occurs periodically, known as kinky-periodic-lump wave (Figure 5(a) for the real part, and Figure 5(b) for the imaginary part). The contour path of the solution Eq. (17) comes from the hyperbolic functions only whose lump occurs periodically known as the cross-double kinky-lump wave (See Figure 6(a)), and orthogonally for \( \eta = 0 \) with water surface (See Figure 6(b)).

For the Set-2, the solution
\[
\begin{align*}
 u(x, t) &= ny + \frac{3}{\delta} \ln t, \tag{15}
\end{align*}
\]
where \( \tau = 1 + \sigma_3 \sinh \left( -\frac{\sigma_1}{\ell_1} y + w_1 t \right) + \sigma_4 \cosh \left( \ell_2 x + \frac{3\eta \ell_2}{\ell_2} t \right), \) \( \ell_1, \) \( \sigma \neq 0 \) and \( \ell_2, w_2, \sigma_3, \sigma_4 \) are arbitrary constants. This solution Eq. (19) from the combinations of hyperbolic and sinusoidal functions gives interaction of double kinky wave, whose lump waves occurs periodically known as the cross-double kinky-lump wave (See Figure 10).

For Set-4, the solution
\[
\begin{align*}
 u(x, t) &= ny + \frac{3}{\delta} \ln t, \tag{17}
\end{align*}
\]
where \( \tau = 1 + I \sigma_3 \sinh \left( a_1 x - \frac{-3\eta \ell_1 \gamma + a_1}{\ell_1} y + w_1 t \right) + \sigma_4 \cosh \left( -\ell_1 x - \frac{-3\eta \ell_1 \gamma + a_1}{\ell_1} y + w_1 t \right), \) \( \ell_1, \sigma \neq 0 \) and \( \ell_1, w_1 \) are arbitrary constants. This solution Eq. (21) gives a periodic lump wave, which is going to vanish its wave after a certain times.

4. Conclusions

In this paper, we mainly focused the nature of the traveling wave of the (2+1)-dimensional nonlinear CBS model using a dependent variable transformation including a controlling parameter. We explicitly presented the wave interactions such as periodic-soliton, cross-kinky-lump wave, double kinky-lump wave, periodic cross-double kinky-lump wave, periodic two-solitary wave solutions and the breather style of solutions were investigated. Moreover, we obtained two conditions that made the waves propagated obliquely and orthogonally. Let us point out that the bilinear form of the CBS model and such structural solutions will be useful to investigate many nonlinear dynamics of interaction phenomena in fluids and plasmas fields.
Declarations

Author contribution statement

Harun-Or-Roshid: Conceived and designed the experiments; Analyzed and interpreted the data.
Md. Mahbub Hassan: Performed the experiments; Wrote the paper.
Abdul-Majid Wazwaz: Contributed reagents, materials, analysis tools or data.

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