Special Riemannian geometries and the Magic Square of Lie Algebras

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Summary

The present work is a slightly revised version of the author’s M.Sc. thesis presented at the Physics department of Warsaw University, investigating nonintegrable Riemannian geometries modelled after certain symmetric spaces related to the Freudenthal-Tits Magic Square. The collection of four such geometries investigated by Nurowski [1] has been extended by further eight, together with a unified description provided in terms of rank three Jordan algebras and associated constructions. In particular, symmetric tensors reducing the orthogonal group to adequate structure groups have been found and used to describe geometric properties of corresponding $G$-structures on manifolds. The results obtained this way include: conditions for existence of a natural complex or quaternionic Kähler structure; equivalence of the existence of a characteristic connection and the Killing condition on the tensor defining a $G$-structure, which holds for four of the new geometries, thus extending an analogous result of Nurowski. Moreover, the geometries which admit a characteristic connection have been classified. The paper is concluded by an algebraic construction of locally reductive examples.
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Introduction

Special geometries with characteristic torsion

It is common in differential geometry to express a specific geometric structure on a manifold in terms of a reduction of the frame bundle to a subbundle with some structure group \( G \subset \text{GL}(n) \) (\( n \) being the dimension of the manifold), i.e. a \( G \)–structure. Such a reduction naturally distinguishes a class of compatible connections, namely those connections on the frame bundle, which restrict to a connection on the \( G \)–structure. We call the latter integrable iff it admits a torsion-free compatible connection.

In particular, an \( \text{O}(n) \)–structure is always integrable, with unique torsion-free connection, namely the Levi-Civita connection of the corresponding Riemannian metric. If we thus wish to reduce the orthonormal frame bundle to a \( G \)–structure with \( G \subset \text{O}(n) \), we readily notice that the latter is integrable iff the Levi-Civita connection is compatible with respect to it. It then follows that integrability of a \( G \)–structure implies reduction of the Riemannian holonomy group to a subgroup of \( G \).

Such reduced Riemannian holonomy groups are classified by the celebrated Berger’s theorem, stating that the holonomy of an irreducible, simply connected, and not (locally) symmetric Riemannian manifold is either the entire special orthogonal group, or one of the groups corresponding to a Calabi-Yau, Kähler, hyper-Kähler, quaternion-Kähler, \( G_2 \), or \( \text{Spin}(7) \) structure. These are usually referred to as (integrable) special Riemannian geometries and have been extensively studied (in the context of holonomy) throughout last fifty years.

Willing to consider more general geometries, one needs to relax the torsion triviality condition. Observe that the bundle \( \Lambda^2 TM \otimes TM \), the torsion of a metric connection on a Riemannian manifold \( M \) is a section of, decomposes into \( \text{O}(n) \) irreducibles as follows:

\[
\Lambda^2 TM \otimes TM = \Lambda^3 TM \oplus TM \oplus T.
\]

One thus sees that there are in general \( 2^3 = 8 \) classes of metric connections with respect to their torsion. In what follows, we shall focus on \( G \)–structures admitting a compatible connection whose torsion is completely skew, i.e. a section of \( \Lambda^3 TM \).

The first structures studied in this context where those present on Berger’s list, and it is a common feature of these, that a \( G \)–connection with skew torsion, provided it exists, is unique. It has been thus called the characteristic connection, and its torsion tensor (determining the connection itself) – the characteristic torsion.
Let us now return for a moment to a general $G$-structure on a manifold $M$. Being able to describe $G \subset \text{GL}(n)$ as a stabilizer of certain set of $\mathbb{R}^n$ tensors, one can introduce the $G$-structure by means of an analogous set of tensors on $M$: the subbundle is then defined to consist of frames mapping the distinguished $\mathbb{R}^n$ tensors into the distinguished tensors on $M$ (point-wise).

To be specific, let us for the sake of simplicity assume that $G$ is the stabilizer of a single tensor $\mathcal{Y} \in \otimes^p(\mathbb{R}^n)^*$; a $G$-structure on an $n$-dimensional manifold $M$ is then defined by a tensor $\mathcal{Y}_M \in \text{Sec}(\otimes^p T^*M)$ such that (locally) in some adapted coframe $\theta : TM \to \mathbb{R}^n$ one has

$$\mathcal{Y}_M(X_1, \ldots, X_p) = \mathcal{Y}(\theta(X_1), \ldots, \theta(X_p))$$

for all $X_1, \ldots, X_p \in TM$. The fibre above $x \in M$ of the corresponding frame subbundle is the set of frames $e^*_x : T_xM \to \mathbb{R}^n$ such that $e^*_x \mathcal{Y} = \mathcal{Y}_M(x)$. The structure group is clearly the stabilizer of $\mathcal{Y}$, i.e. $G$. Finally, the compatible connections, viewed as connections in the tangent bundle, are those with respect to which $\mathcal{Y}_M$ is parallel.

Proceeding again to the Riemannian case, with $G \subset \text{O}(n)$ being the orthogonal stabilizer of $\mathcal{Y}$, we easily see that the $G$-structure is integrable iff $\mathcal{Y}_M$ is parallel with respect to the Levi-Civita connection $\nabla^{\text{LC}}$. One is then tempted to ask whether a weaker condition on $\nabla^{\text{LC}} \mathcal{Y}_M$ would guarantee existence of a $G$-connection with skew torsion. It is not difficult to check, as it has been noticed by Nurowski [1], that the existence of such a connection implies vanishing of the symmetric part of the derivative

$$(\nabla^{\text{LC}} \mathcal{Y}_M)(X, \ldots, X) = 0 \quad \forall X \in TM,$$

a condition we shall call the nearly-integrability of (the $G$-structure defined by) $\mathcal{Y}_M$. One may hope that the converse would also hold in some cases (whether it does, is a purely algebraic question referring to $\mathcal{Y}$). As only the symmetric part of $\mathcal{Y}_M$ enters the latter equation, it is clear that we should restrict our attention to symmetric tensors $\mathcal{Y}$ and their orthogonal stabilizers.

Examples of geometries defined by a symmetric tensor

The simplest interesting example, thoroughly investigated by Bobienski and Nurowski [2], involves an irreducible $\text{SO}(3)$ structure on a five-dimensional Riemannian manifold $(M, g_M)$. The authors first consider the symmetric space $\text{SU}(3)/\text{SO}(3)$ and the corresponding symmetric pair:

$$\text{su}(3) = \text{so}(3) \oplus V, \quad \dim V = 5.$$ 

The adjoint action of $G = \text{SO}(3)$ on $V$ defines an irreducible 5-dimensional representation of the group, which is moreover self-adjoint, so that $V \simeq V^*$ as $G$-modules via an invariant scalar product $g : V \to V^*$.

It can be further shown, that $\text{SO}(3) \subset \text{O}(5)$ is the stabilizer of a tensor $\Upsilon \in S^3V$ satisfying the relation

$$\Upsilon_{m(ij)} T_{kl} m = g_{(ij} g_{kl)},$$

\[ (2) \]

\[ i.e. \] demanding that $\mathcal{Y}_M$ be a Killing tensor.
where abstract index notation is assumed, together with the identification of $V$ with its dual $V^\ast$. Since the representation is irreducible, the tensor is also obviously required to satisfy $\Upsilon_{\mu\nu\rho\sigma} = 0$.

As it has been already described, one defines the $G$--structure by means of a tensor $\Upsilon_M$ on $M$ such that in a (local) adapted coframe $\theta : TM \to V$ one has

$$\Upsilon_M(X, X, X) = \Upsilon(\theta(X), \theta(X), \theta(X)) \quad \& \quad g_M(X, X) = g(\theta(X), \theta(X)).$$

Equivalently, one may simply demand that $\Upsilon_M$ satisfy the analog of equation (2) with respect to $g_M$.

One then finds that such five-dimensional $SO(3)$ geometries indeed behave in the way we are looking for: nearly integrability of $\Upsilon_M$ (recall equation (1)) implies existence of a compatible $SO(3)$ connection with skew torsion, and the latter is moreover unique (that is, characteristic) [2].

It is now natural to ask, whether a similar setting can be found in other dimensions. Guided by the defining identity on $\Upsilon$, i.e. equation (2), Nurowski checked that the latter can be satisfied for a symmetric third-rank tensor exactly in four distinguished dimensions, namely: 5, 8, 14 and 26 [1]. The corresponding symmetric spaces are:

$$SU(3) \times SU(3), \quad SU(6) \times Sp(3), \quad E_6(-78) \times F_4(-52)$$

(all four appearing on Cartan’s list of irreducible symmetric Riemannian spaces, cf. [3, 4]). The result of Nurowski is then that nearly integrability implies existence of compatible connection with skew torsion in dimensions 5, 8 and 14, while such a connection is unique in dimensions 5, 14 and 26.

The next step would thus be to consider geometries modelled after other symmetric spaces from Cartan’s list. These in particular include symmetric spaces related to a construction known as the Magic Square of Lie algebras, and investigation of corresponding special geometries is the task proposed in [1]. As we shall soon see, there are three families of such spaces, the first one being exactly (3). One may expect the geometries modelled after all of these spaces to exhibit similar properties regarding nearly integrability and characteristic connection.

### Freudenthal-Tits Magic Square and related symmetric spaces

The Magic Square of Lie algebras is an outcome of the constructions developed by Freudenthal and Tits mainly in effort to provide a direct construction of the exceptional groups. When performed over the reals, the Tits construction yields

---

2 Equation (2) is equivalent to demanding that the trivial representation of $G$ appear only once in the decomposition of $S^4V$.

3 Actually, the problem of uniqueness of the characteristic connection has been recently completely solved by Nagy [5].
algebras corresponding to the following ‘magic’ square of compact Lie groups:

\[
\begin{array}{cccc}
SO(3) & \longrightarrow & SU(3) & \longrightarrow & Sp(3) & \longrightarrow & F_4 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SU(3) & \longrightarrow & SU(3) \times SU(3) & \longrightarrow & SU(6) & \longrightarrow & E_6 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Sp(3) & \longrightarrow & SU(6) & \longrightarrow & SO(12) & \longrightarrow & E_7 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
F_4 & \longrightarrow & E_6 & \longrightarrow & E_7 & \longrightarrow & E_8
\end{array}
\]

together with natural inclusions denoted by the arrows (the word ‘magic’ referring to the symmetry of the table, which is not explicit in its original construction, see later).

Let us now consider the ‘quotient’ of the second column by the first one (with respect to the inclusions), namely the homogeneous spaces:

\[
\begin{array}{llll}
SU(3) & SU(3) \times SU(3) & SU(6) & E_6 \\
SO(3) & Sp(3) & F_4
\end{array}
\]

Observe that these are exactly the spaces we have already considered in context of a symmetric third rank tensor. Repeating this procedure for the next pair of columns (2&3), we obtain spaces which are unfortunately not irreducible: their isotropy representations possess a one-dimensional invariant subspace. This can be resolved by augmenting the subgroup by an extra $U(1)$, so that the corresponding four spaces are:

\[
\begin{array}{llll}
Sp(3) & SU(6) & SO(12) & E_7 \\
U(3) & SU(3) \times U(3) & U(1) & E_6 \times U(1)
\end{array}
\]

These are another four irreducible symmetric Riemannian spaces from Cartan’s list, and thus the second of three advertised families (the way this extra $U(1)$ sits in the groups is to be explained in the sequel). Moreover, the generator of the additional $U(1)$ defines a complex structure making these symmetric spaces into Kähler manifolds.

Similar situation occurs for the last pair of columns (3&4). Here however one has to add extra $Sp(1)$, so that the third family consists of the following irreducible symmetric spaces:

\[
\begin{array}{llll}
F_4 & E_6 & E_7 & E_8 \\
Sp(3) \times Sp(1) & SU(6) \times Sp(1) & SO(12) \times Sp(1) & E_7 \times Sp(1)
\end{array}
\]

In this case the generators of the additional $Sp(1)$ define a quaternionic structure making these symmetric spaces into quaternion-Kähler manifolds.

These are the three families of irreducible compact symmetric Riemannian spaces. Nurowski \[\text{[4]}\] worked out the first one, and proposed the task of investigating the next two:

\[\text{[4]}\] The product with $Sp(1)$ is taken with respect to the adjoint (i.e. isotropy) representation. In fact, $G_0 Sp(1) \cong (G_0 \times Sp(1))/\mathbb{Z}_2$
It is interesting if all these geometries admit characteristic connection. Also, we do not know what objects in $\mathbb{R}^{\dim M}$ reduce the orthogonal groups $\text{SO}(\dim M)$ to the above mentioned structure groups. Are these symmetric tensors as it was in the case of the groups $H_k$ [i.e. $\text{SO}(3)$, $\text{SU}(3)$, $\text{Sp}(3)$ and $\text{F}_4$]?"

We answer these questions and provide a systematic approach rooted in the theory of Jordan algebras.

Overview

The work is split into two parts (chapters), which we have tried to make possibly independent of each other.

The first one begins citing known results on Jordan algebras and the Tits construction, providing the minimal theory needed to make sense of the symmetric pairs corresponding to the spaces of our interest, fitting them into a single structure. We then review some further constructions, mainly due to Freudenthal, again introducing just the minimum set of objects needed to describe the isotropy representations of the symmetric spaces. Having the latter done, we finally find the symmetric invariants giving the desired reductions (on a Lie-algebraic level) and prove some of their properties.

The second part, building on the results outlined above, deals directly with the main subject of the work, i.e. geometries related to the symmetric spaces. We first summarize the results on isotropy representations and the invariant tensors: we claim their existence and properties they satisfy, in such a way that no reference to the Jordan-related objects is needed. Then, we review the subject of intrinsic torsion and characteristic torsion of $G$-structures: these results are known, but not particularly accessible in the literature, so that we prove most of them, also in order to make the reader familiar with the general setting. Finally, $g(\mathbb{K}, \mathbb{K}')$-geometries are defined and their properties are investigated. Additionally, we provide a classification of $g(\mathbb{K}, \mathbb{K}')$-geometries with characteristic torsion and prove existence of naturally reductive examples, whose characteristic torsion does not vanish.

The most important original results obtained in the present work are:

- Proposition 3, describing the isotropy representations.
- Proposition 4, describing symmetric invariants giving desired reductions.
- Propositions 7, 9 and 12, expressing intrinsic torsion of $G(\mathbb{K}, \mathbb{K}')$-structures in terms of geometric data.
- Propositions 10 and 13, providing a geometric condition on existence of a natural (quaternion-)Kähler structure on the geometries.

The problem of symmetric defining invariants is addressed in a very original and self-contained way by Cvitanović in his remarkable book on Group Theory [6], where he also discusses the Magic Square. While we do not use his approach and results, we have to acknowledge that the latter work has been an important source of inspiration. In particular, many of the technical calculations contained in the present paper have been initially carried using Cvitanović’s graphical notation, only later to be translated to the conventional one.
• Theorem 1, stating the equivalence of nearly-integrability of the symmetric tensor defining a second-family geometry and existence of a characteristic connection.

• Theorem 2 and Proposition 25, providing a simple criterion for existence of naturally reductive examples with nontrivial characteristic torsion.

Strominger’s superstrings with torsion

Having argued about the geometric significance of studying special geometries with characteristic torsion, one should not overlook an inspiring motivation coming from physics, found by Strominger in his 1986 paper [7].

Strominger considers the geometric setting for compactification of the common sector of type II Superstring Theory, i.e. a 6-dimensional spin manifold \((M,g)\) equipped with a global nonvanishing spinor field \(\epsilon\). In absence of Yang-Mills fields and with constant dilaton, the conditions for \(\epsilon\) to generate supersymmetry transformations read [7]

\[
\nabla^{LC}_X \epsilon + \frac{1}{4} H(X) \cdot \epsilon = 0 \quad \text{for each } X \in TM
\]

\[
H \cdot \epsilon = 0,
\]

where \(H \in \Omega^3(M)\) is the strength of the Kalb-Ramond \(B\)-field, i.e. the background field coupling to massless skew-symmetric excitations of the string, and the dot indicates Clifford action of differential forms on spinor fields (\(H\) moreover obeys certain integrability condition involving the Riemannian curvature).

Strominger now introduces a tangent bundle connection \(\nabla^H\) whose torsion is \(H\), i.e.

\[
\nabla^H_X Y = \nabla^{LC}_X Y + \frac{1}{2} H(X,Y)
\]

for each \(X \in TM\). In terms of this new connection, equation (4) reads simply

\[
\nabla^H \epsilon = 0,
\]

i.e. \(\epsilon\) is parallel with respect to \(\nabla^H\). This in turn is equivalent to a reduction of the holonomy \(\hat{\text{G}}\) of \(\nabla^H\) to (a subgroup of) \(SU(3)\) (in its 6-dimensional representation, not to be confused with the 8-dimensional one mentioned earlier). We thus end up with a \(SU(3)\)-structure with characteristic torsion, namely \(H\) (indeed a compatible connection with skew torsion for a \(SU(3)\)-structure is unique). Moreover, all further equations can be cast in terms of the \(SU(3)\) structure (i.e. an almost hermitian structure and a complex 3-form), so that the latter describes both supersymmetry and the \(B\)-field [7].

Thus, Strominger found that 6-dimensional backgrounds for supersymmetric compactification are naturally described by \(SU(3)\)-structures admitting characteristic torsion. Just as a Yang-Mills field strength is interpreted as the curvature of a connection on a principal bundle, the Kalb-Ramond field strength

\[\text{\footnote{6}{Indeed, existence of a parallel spinor in dimension six implies holonomy reduction to } SU(3). In the conventional approach this condition is applied to the Levi-Civita connection, leading to compactification on Calabi-Yau threefolds.}}\]
is then identified with a torsion of the unique connection associated with the $\text{SU}(3)$-structure\footnote{A much more elaborate approach to a geometric interpretation of the $B$ field has been developed in the last decade in terms of gerbes with connection \footnote{Nevertheless, the homogeneous spaces related to the Magic Square also find their place in supergravity-related physics \cite{10,11}, although without direct relation to the characteristic torsion problem.}}.

This development attracted the attention of mathematicians, who started studying the problem of characteristic torsion, parallel spinors and relation of torsion to curvature, initially for $G$-structures related to Berger’s classical list of irreducible Riemannian holonomies (see \cite{9} and references therein). In the present work we address only the question of characteristic torsion. Moreover, one should note that the dimensions of the geometries we wish to investigate situate them rather remotely from the usual area of interest of fundamental theories.

**Conventions**

All the vector spaces and algebras considered in this work are over the reals, unless stated otherwise. All manifolds and maps are assumed to be smooth. Stating general results, we shall often use $\mathbb{R}^n$ as a generic $n$-dimensional vector space, possibly equipped with a generic positive definite scalar product $\langle \cdot, \cdot \rangle$.

We will often make use of the abstract index notation \cite{12}, using various sets of letters to index both spaces and tensors, so that a homomorphism of vector spaces $f : E \rightarrow F$ can be written as $f_{ai} \in E_i^* \otimes F^a$, while its contraction with a vector $X \in E$ is $f(X)^a = f_{ai}X^i \in F^a$. These indices never refer to any specific frame. They are simply labels, which do not assume any (numerical) values.

Most of the spaces we deal with are equipped with a symmetric scalar product, and we explicitly declare that the latter is used to identify a space with its dual. Accordingly, if such an identification has been performed, the position of indices becomes irrelevant. The map $f$ given as an example above is then written $f_{ai} \in F^a \otimes E_i$ and as such it needn’t be distinguished from the adjoint $f^* : F \rightarrow E$ (of course if we have identified $E \simeq E^*$ and $F \simeq F^*$).

We will often encounter tensors $\mathcal{Y} \in \otimes^p E$, with $E^* \simeq E$. Then by $\mathcal{Y}(X_1, \ldots, X_p)$, where $X_1, \ldots, X_p \in E$, we mean $\mathcal{Y}_{i_1 \ldots i_p}X_1^{i_1} \cdots X_p^{i_p} \in \mathbb{R}$ and the order of the vectors $\mathcal{Y}$ is contracted with does matter (unless $\mathcal{Y}$ is symmetric). We sometimes use also partial application as in

$$\mathcal{Y}(X_1, \ldots, X_q) \in \otimes^{p-q} E,$$

which means

$$\mathcal{Y}_{i_1 \ldots i_p}X_1^{i_1} \cdots X_q^{i_q} \in E_{i_{q+1}} \otimes \cdots \otimes E_{i_p}$$

in this very order (i.e. the components of a tensor product are contracted from the left).

Being given a group $G \subset \text{GL}(E)$, with a Lie algebra $\mathfrak{g} \subset \text{End}E$, we denote both the action of $G$ and $\mathfrak{g}$ on arbitrary tensor products of $E$ and $E^*$ simply by $g(\cdot)$ and $A(\cdot)$, where $g \in G$ and $A \in \mathfrak{g}$. This should be clarified by the example of $\mathcal{Y} \in \otimes^p E^*$, where

$$g(\mathcal{Y})(X_1, \ldots, X_p) = \mathcal{Y}(g^{-1}(X_1), \ldots, g^{-1}(X_p))$$
\[ A(Y)(X_1, \ldots, X_p) = Y(-A(X_1), X_2, \ldots, X_p) + \cdots + Y(X_1, \ldots, X_{p-1}, -A(X_p)). \]

Whether a map is to be considered as acting via a group action, or Lie algebra action, should be clear from context. An exception is e.g. a complex structure, which can be viewed both as an orthogonal map, and as spanning a \(u(1)\) algebra: in this case, we shall by default assume the group action.

Special indexing conventions are introduced for spaces that are to be considered over \(\mathbb{C}\). These are described in Remark 1, which shall be recalled explicitly whenever needed.

Finally, being given a manifold \(M\) we will also use letters to index tensor products of \(TM\) and \(T^*M\) (being identified when a metric tensor is given). This is extended to the \(C^\infty(M)\)-modules of tangent-bundle-valued differential forms, so that for example a local metric connection form can be written as \(\Gamma_{ab} \in \Omega^1(M, (\Lambda^2 TM)_{ab})\).

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Finally, I would like to thank Andrzej Trautman, both for agreeing to write a review of my thesis and, even more importantly, for the beautiful lectures he gave each year in Warsaw. The latter, unique for their clarity and elegance, were a major influence on my interest in differential geometry.
Chapter 1

Algebraic part

1.1 The Tits construction and symmetric pairs

The construction of Magic Square\(^1\) algebras given by Tits \[15\] produces a Lie algebra out of two basic ‘building blocks’: a normed division algebra, i.e. \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) or \(\mathbb{O}\), and a Jordan algebra of 3x3 hermitian matrices with entries in a second normed division algebra. Eventually thus, it gives a Lie algebra for every pair of normed division algebras, fitting into a 4x4 table with rows and columns labelled by \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) and \(\mathbb{O}\).

We shall first introduce the usual algebraic structures on aforementioned ‘building blocks’ (in particular, their automorphism groups and derivation algebras), whose combination will – in a fairly natural way – lead to a Lie algebra structure on the outcome of Tits’ construction.

All algebras and representations used in this section are to be considered over the reals.

1.1.1 Four normed division algebras \(\mathbb{K}\)

As a warm-up we will now recall some well-known facts which hold in general for all the four algebras (see e.g. \[16, 17\]). Although these are rather obvious, we expose them quite carefully to show that analogous structure appears in the Jordan case. As the octonions are the most complex, with their noncommutativity and nonassociativity, it is useful to have this algebra in mind when reading subsequent statements. For their simpler subalgebras, some terms and spaces become trivial, however the formulas, given in their most general form, are still correct.

We thus let \(\mathbb{K}\) be one of the four algebras: \(\mathbb{R}, \mathbb{C}, \mathbb{H}\) and \(\mathbb{O}\). We introduce the commutator

\[\left[p, q\right] = pq - qp,\]

nontrivial for the quaternions and octonions, and the associator

\[\left[p, q, r\right] = p(qr) - (pq)r,\]

nontrivial only for the octonions (where alternativity of the latter implies that the associator is antisymmetric in its three arguments).

\(^1\) Interesting generalisations can be found in \[14, 16\].
These maps are useful when describing the automorphism group $\text{Aut}(K)$, i.e. a subgroup of $\text{GL}(K)$ preserving the product. Its Lie algebra, the derivations $\text{der} K$, i.e. a subalgebra of maps in $\text{End} K$ satisfying the Leibniz rule, is then made accessible by the following Lemma (see [16] for a proof):

**Lemma 1.** Let us introduce a map

$$D : \Lambda^2 K \rightarrow \text{End} K$$

$$D_{p,q}(r) = [[p, q], r] - 3[p, q, r]$$

for $p, q, r \in K$.

Then the algebra $\text{der} K$ is exactly the image of $D$. Moreover, $D$ is equivariant with respect to $\text{Aut}(K)$.

The automorphism groups are found to be [17]:

- $\text{Aut}(\mathbb{R}) \simeq \{e\}$
- $\text{Aut}(\mathbb{C}) \simeq \mathbb{Z}_2$
- $\text{Aut}(\mathbb{H}) \simeq \text{SO}(3)$
- $\text{Aut}(\mathbb{O}) \simeq \text{G}_2$.

More structure on $K$ is provided by a natural trace, namely the real part $\text{Re} : K \rightarrow \mathbb{R}$. Combined with the product, it gives rise to a positive definite scalar product

$$\langle \cdot, \cdot \rangle : K \times K \rightarrow \mathbb{R}$$

$$\langle p, q \rangle = \text{Re}(\overline{pq}).$$

One can decompose $K$ into the real line, spanned by the unit, and its orthogonal complement, namely the imaginary subspace (trivial for the reals):

$$K = \mathbb{R}1 \oplus \text{Im} K.$$  \hspace{1cm} (1.1)

It is then easy to check that:

**Lemma 2.** The automorphisms $\text{Aut}(K)$ preserve the scalar product and the decomposition (1.1), acting irreducibly on $\text{Im} K$.

Observe moreover, that the map $D$ is nontrivial only on $\Lambda^2 \text{Im} K$. Finally, it is convenient to introduce an antisymmetric product on the imaginary subspace, being simply a restriction of the usual one:

$$\times : \text{Im} K \times \text{Im} K \rightarrow \text{Im} K$$

$$p \times q = \frac{1}{2} [p, q].$$

Clearly, this map is also preserved by the automorphisms of $K$.

### 1.1.2 Four Jordan algebras $\mathfrak{h}_3 K$

Jordan algebras emerged in an attempt to axiomatize the properties of hermitian operators representing observables in quantum mechanics. While the original program turned out to be unsuccessful, it was realized that these algebras are interesting in their own right, allowing a rich theory as somewhat a counterpart of Lie algebras [18,19]. Indeed, they are characterized by a commutative product satisfying an identity which may be thought of as playing a role analogous to that of the Jacobi identity for a Lie bracket:
**Definition 1.** A commutative (yet possibly not associative) algebra $J$ is called Jordan iff for arbitrary two elements $a, b \in J$ the following holds:

$$(ab)a^2 = a(ba^2) \quad \text{(Jordan identity)}.$$ 

In particular, the algebras of $n$ by $n$ hermitian matrices with entries in either the reals, complex numbers or quaternions, equipped with anticommutator as the product, are Jordan. The three infinite families are denoted $h_n \mathbb{R}$, $h_n \mathbb{C}$ and $h_n \mathbb{H}$. For the octonions, however, there is only a single one, called the exceptional Jordan algebra, or Albert algebra, namely $h_3 \mathbb{O}$.

In what follows we will only use 3x3 matrices, so that a corresponding Jordan algebra exists for all four normed division algebras. Let thus $K$ be one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We define

$$h_3 K = \left\{ \begin{pmatrix} \alpha & a & b \\ \bar{a} & \beta & c \\ \bar{b} & \bar{c} & \gamma \end{pmatrix} | a, b, c \in K; \alpha, \beta, \gamma \in \mathbb{R} \right\},$$

together with a product

$$X \circ Y = \frac{1}{2}(XY + YX)$$

where the right hand side features usual matrix multiplication. We then have:

**Lemma 3** (Jordan [20]). The algebra $(h_3 K, \circ)$ is Jordan.

We shall now follow the way we used to describe the algebra $K$ itself. We thus begin with the automorphism group $\text{Aut}(h_3 K)$ and its Lie algebra, the derivations $\text{der} h_3 K$. For the latter, we again have a convenient map, denoted with a slight abuse of notation by the same letter:

**Lemma 4** (cf. [19]). Let us introduce a map

$$D : \Lambda^2 h_3 K \to \text{End} h_3 K$$

$$D_{X,Y}(Z) = X \circ (Y \circ Z) - \ Y \circ (X \circ Z) \quad \text{for } X, Y, Z \in h_3 K.$$ 

Then the algebra $\text{der} h_3 K$ is exactly the image of $D$. Moreover, $D$ is equivariant with respect to $\text{Aut}(h_3 K)$.

The automorphism groups are found to be [21][22]:

$$\text{Aut}(h_3 \mathbb{R}) \simeq \text{SO}(3)$$
$$\text{Aut}(h_3 \mathbb{C}) \simeq \text{SU}(3)$$
$$\text{Aut}(h_3 \mathbb{H}) \simeq \text{Sp}(3)$$
$$\text{Aut}(h_3 \mathbb{O}) \simeq F_4(-52).$$

More structure on $h_3 K$ is given by a natural trace, namely the usual matrix trace $\text{tr} : h_3 K \to \mathbb{R}$. Combined with the product, it gives rise to a positive definite scalar product

$$\langle \cdot, \cdot \rangle : h_3 K \times h_3 K \to \mathbb{R}$$

$$\langle X, Y \rangle = \text{tr}(X \circ Y).$$
CHAPTER 1. ALGEBRAIC PART

Multiplication in the algebra is then symmetric with respect to the scalar product:

$$\langle X \circ Y, Z \rangle = \langle Y, X \circ Z \rangle.$$  

One can decompose $\mathfrak{h}_3 K$ into the real line, spanned by the unit, and its orthogonal complement, namely the traceless subspace:

$$\mathfrak{h}_3 K = \mathbb{R} 1 \oplus \mathfrak{sh}_3 K,$$

where $\mathfrak{sh}_3 K = \ker \text{tr}$. One then finds that:

**Lemma 5.** The automorphisms $\text{Aut}(\mathfrak{h}_3 K)$ preserve the scalar product and the decomposition (1.2), acting irreducibly on $\mathfrak{sh}_3 K$.

Observe moreover, that the map $\mathcal{D}$ is nontrivial only on $\Lambda^2 \mathfrak{sh}_3 K$. Finally, it is convenient to introduce a symmetric product on the traceless subspace, being simply a restriction of the usual one:

$$\times: \mathfrak{sh}_3 K \times \mathfrak{sh}_3 K \to \mathfrak{sh}_3 K$$

$$X \times Y = X \circ Y - \frac{1}{3} \langle X, Y \rangle.$$

Clearly, this map is also preserved by the automorphisms of $\mathfrak{h}_3 K$.

1.1.3 The Magic Square of Lie algebras

We are now ready to construct the Magic Square. As the algebras to be constructed are parametrized by a pair of normed division algebras, let us introduce two symbols, $K$ and $K'$, allowing each of them to be one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

Taking $K'$ and $\mathfrak{h}_3 K$ as our building blocks, we can form the product algebra $K' \otimes \mathfrak{h}_3 K$. The automorphism group of the latter is simply the product of $\text{Aut}(K')$ and $\text{Aut}(\mathfrak{h}_3 K)$, so that the corresponding derivation algebra is

$$\text{der}(K' \otimes \mathfrak{h}_3 K) = \text{der} K' \oplus \text{der} \mathfrak{h}_3 K.$$

Recalling the decompositions of $K'$ and $\mathfrak{h}_3 K$, we have in particular the largest irreducible subspace $\text{Im} K' \otimes \mathfrak{sh}_3 K \subset K' \otimes \mathfrak{h}_3 K$. Let us consider the direct sum of the derivation algebra and its irreducible module:

$$\mathfrak{M}(K, K') = \text{der} K' \oplus \text{der} \mathfrak{h}_3 K \oplus \text{Im} K' \otimes \mathfrak{sh}_3 K.$$

Our aim will be to equip the latter with a Lie algebra structure. We keep $\text{der} K' \oplus \text{der} \mathfrak{h}_3 K$ as a subalgebra with its original Lie bracket. Its action on the module provides the mixed bracket:

$$[d + D, p \otimes X] = d(p) \otimes X + p \otimes D(X)$$

for $d \in \text{der} K'$, $D \in \text{der} \mathfrak{h}_3 K$, $p \in \text{Im} K'$ and $X \in \mathfrak{sh}_3 K$.

We still need a bracket of two elements of $\text{Im} K' \otimes \mathfrak{sh}_3 K$. Recalling the structure introduced so far, one sees that there are three natural (equivariant w.r.t. the derivations) maps form $\Lambda^2(\text{Im} K' \otimes \mathfrak{sh}_3 K)$ to $\mathfrak{M}(K, K')$:

$$(p \otimes X) \wedge (q \otimes Y) \mapsto \langle X, Y \rangle \mathcal{D}_{p,q}$$

$$(p \otimes X) \wedge (q \otimes Y) \mapsto \langle p, q \rangle \mathcal{D}_{X,Y}$$

$$(p \otimes X) \wedge (q \otimes Y) \mapsto (p \times q) \otimes (X \times Y),$$
and the most general derivation-equivariant bracket is a linear combination thereof. The factors are determined by demanding that the Jacobi identity be satisfied, so that finally we arrive at the following

**Lemma 6** (Tits [15], cf. [23]). The space \( M(K, K') \) becomes a Lie algebra with the bracket defined by:

1. The natural bracket on \( \text{der} K' \oplus \text{der} h_3 K \)

2. Equation (1.3) for an element of \( \text{der} K' \oplus \text{der} h_3 K \) and an element of \( \text{Im} K' \otimes h_3 K \):

3. The following bracket for two elements \( p \otimes X \) and \( q \otimes Y \) of \( \text{Im} K' \otimes h_3 K \):

\[
[p \otimes X, q \otimes Y] = \frac{1}{12} \langle X, Y \rangle D_{p,q} - \langle p, q \rangle D_{X,Y} + (p \times q) \otimes (X \times Y). \tag{1.4}
\]

The outcome of this procedure is summarized in the following

**Proposition 1** (Tits [15]). The algebras \( M(K, K') \) are isomorphic (over the reals) to the ones in the table, with columns indexed by \( K' \) and rows by \( K \):

|        | R | C | H | O |
|--------|---|---|---|---|
| R      | \( \text{so}(3) \) | \( \text{su}(3) \) | \( \text{sp}(3) \) | \( f_4 \) |
| C      | \( \text{su}(3) \) | \( \text{su}(3) \oplus \text{su}(3) \) | \( \text{su}(6) \) | \( e_6 \) |
| H      | \( \text{sp}(3) \) | \( \text{su}(6) \) | \( \text{so}(12) \) | \( e_7 \) |
| O      | \( f_4 \) | \( e_6 \) | \( e_7 \) | \( e_8 \) |

Two remarks have to be made in this place. First, the algebras in the square appear in their compact form, an important virtue of Tits’ construction. Second, as we have warned before, certain parts of the construction become trivial for \( K' \) being \( R \) or \( C \). In particular, we have \( \text{der} R = 0 \) and \( \text{Im} R = 0 \), so that simply \( M(K, R) = \text{der} h_3 K \).

1.1.4 Cayley-Dickson and induced decompositions

We shall now recall the Cayley-Dickson decomposition of the complex numbers, quaternions and octonions, namely:

\[ C = R \oplus iR, \quad H = C \oplus jC, \quad O = H \oplus lH, \]

and check that, when applied to \( K' \), they naturally lead to decompositions of the magic square algebras \( M(K, K') \) into symmetric pairs. Let us then from now on assume \( K' \neq R \). One easily checks that the Cayley-Dickson construction defines a \( \mathbb{Z}_2 \) grading on \( K' \), that is:

\[
K' = K'_0 \oplus K'_1
\]

\[
K'_i \cdot K'_j = K'_{i+j}
\]

where \( i, j \in \mathbb{Z}_2 \), and moreover \( K'_i \) and \( K'_j \) are orthogonal with respect to \( \langle \cdot, \cdot \rangle \) whenever \( i \neq j \).

Using the latter and Lemma [1] we can decompose the derivation algebra of \( K' \) into a symmetric pair:
Lemma 7. Let us define
\[ \text{der}_0 \mathbb{K}' = \mathcal{D}(\Lambda^2 \mathbb{K}'_0 \oplus \Lambda^2 \mathbb{K}'_1) \]
\[ \text{der}_1 \mathbb{K}' = \mathcal{D}(\mathbb{K}'_0 \wedge \mathbb{K}'_1). \]
Then
\[ \text{der} \mathbb{K}' = \text{der}_0 \mathbb{K}' \oplus \text{der}_1 \mathbb{K}' \]
with
\[ [\text{der}_i \mathbb{K}', \text{der}_j \mathbb{K}'] = \text{der}_{i+j} \mathbb{K}' \] for all \( i, j \in \mathbb{Z}_2 \).

Proof. Using the formula for \( \mathcal{D} \), we have
\[ \mathcal{D}(\mathbb{K}'_{i_1} \wedge \mathbb{K}'_{i_2})(\mathbb{K}'_{j}) \subset \mathbb{K}'_{i_1 + i_2 + j}, \]
which implies (1.6). Then, using equivariance of \( \mathcal{D} \), (1.5) follows immediately. Now, since \( \text{der} \mathbb{K}' \) is the image of \( \Lambda^2 \mathbb{K}' \) under \( \mathcal{D} \), it is clear that \( \text{der}_0 \mathbb{K}' + \text{der}_1 \mathbb{K}' = \text{der} \mathbb{K}' \). It remains to check that the intersection of these spaces is zero. But that already follows from (1.6).

We are now interested in identifying the algebras \( \text{der}_0 \mathbb{K}' \). Note first, that it is a property of the Cayley-Dickson construction, that every element of the even subspace may be represented as a product of two elements of the odd subspace – thus the representation of \( \text{der}_0 \mathbb{K}' \) on \( \mathbb{K}'_1 \) is necessarily faithful and there exists an injective homomorphism \( \text{der}_0 \mathbb{K}' \rightarrow \mathfrak{so}(\mathbb{K}'_1, \langle \cdot, \cdot \rangle) \). In fact, one finds that it is also surjective:

Lemma 8. \( \text{der}_0 \mathbb{K}' \cong \mathfrak{so}(\mathbb{K}'_1, \langle \cdot, \cdot \rangle) \)

Proof. Trivial for \( \mathbb{K}' = \mathbb{C} \), and straightforward for \( \mathbb{K}' = \mathbb{H} \), with \( \text{der}_0 \mathbb{H} \) generated by \( \text{ad}_i = \frac{1}{2} \mathcal{D}_{i,k} \).

The octonionic case, however, is more involved. As \( \mathfrak{so}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \), we shall construct two homomorphisms form \( \mathfrak{sp}(1) \) to \( \text{der}_0 \mathbb{O} \) and check that they combine into an isomorphism from \( \mathfrak{so}(4) \) to \( \text{der}_0 \mathbb{O} \).

To this end, we consider the usual Lie algebra \( \mathfrak{sp}(1) \) of imaginary quaternions with the commutator as a bracket. This algebra acts with left and right multiplications on \( \mathbb{O} = \mathbb{H} \oplus l\mathbb{H} \), preserving the decomposition \( (l \text{ is some imaginary octonion orthogonal } \mathbb{H} \subset \mathbb{O}) \). This gives rise to four irreducible four-dimensional, and necessarily equivalent, representations of the algebra, namely:

\[ q \mapsto L_q|\mathbb{H}, \quad q \mapsto L_q|l\mathbb{H}, \quad q \mapsto R_q|\mathbb{H}, \quad q \mapsto R_q|l\mathbb{H} \]

for \( q \in \text{Im}\mathbb{H} \cong \mathfrak{sp}(1) \). There in particular exists an orthogonal intertwiner of left multiplications
\[ \varphi : \mathbb{H} \rightarrow l\mathbb{H} \]
\[ p\varphi(x) = \varphi(px) \] for all \( p \in \text{Im}\mathbb{H} \) and \( x \in \mathbb{H} \). Orthogonality implies \( \varphi(x)^2 = -|x|^2 \).

\[ \text{Indeed, assume that } d \in \text{der}_0 \mathbb{K}' \text{ acts trivially on } \mathbb{K}'_1. \text{ Then it acts trivially on entire } \mathbb{K}' \text{ and thus } d = 0. \]
1.1. THE TITS CONSTRUCTION AND SYMMETRIC PAIRS

We now define the maps

$$E, E' : \text{Im} \mathbb{H} \to \mathfrak{so}(\mathbb{O})$$

$$E(q)|_{\mathbb{H}} = \text{ad}_q \quad E'(q)|_{\mathbb{H}} = 0$$

$$E(q)|_{\mathbb{O}} = L_q \quad E'(q)|_{\mathbb{O}} = \varphi \circ R_q \circ \varphi^{-1}.$$ 

One readily checks that each of these is a Lie algebra homomorphism and that $E(q)$ commutes with $E'(q')$ for $q, q' \in \text{Im} \mathbb{H}$ (this is because left and right multiplications commute, and $\varphi$ is left-equivariant). It is also clear that the kernel of $E \oplus E' : \text{Im} \mathbb{H} \oplus \text{Im} \mathbb{H} \to \mathfrak{so}(\mathbb{O})$ is trivial. Thus we find that $E \oplus E'$ is an injective Lie algebra homomorphism from $\mathfrak{so}(4) \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ to $\mathfrak{so}(\mathbb{O})$. Moreover, the elements of its image explicitly preserve the decomposition $\mathbb{O} = \mathbb{H} \oplus i\mathbb{H}$.

The last step is to show that $E_q$ and $E'_q$ for $q \in \text{Im} \mathbb{H}$ are derivations (for the moment we write the first argument in a subscript). We have to check the Leibniz formula patiently for the products $xy$, $x\varphi(y)$, and $\varphi(x)\varphi(y)$, where $x, y \in \mathbb{H}$. The first case is obvious (clearly, $E_q$ and $E'_q$ restricted to $\mathbb{H}$ are derivations of the quaternions). Evaluating the next one, we have:

$$E_q(\varphi(xy)) = \varphi(qxy) = [q, x] \varphi(y) + x\varphi(qy) = E_q(x)\varphi(y) + xE_q(\varphi(y))$$

$$E'_q(\varphi(xy)) = \varphi(xyq) = E'_q(x)\varphi(y) + xE'_q(\varphi(y)).$$

To handle expressions like $\varphi(x)\varphi(y)$, we use the orthogonality of $\varphi$ and multiply equation [1.7] with $\varphi(pz)$ on the right and $p^{-1}$ on the left to obtain $\varphi(x)\varphi(pz) = -p^{-1}|p|^2|x|^2$. Then, setting $p = yx^{-1}$, we get

$$\varphi(x)\varphi(y) = -xy\bar{y}.$$ 

We can now check the following:

$$E_q(-xy) = -[q, xy] = -qx\bar{y} + x\bar{y}q = E_q(\varphi(x))\varphi(y) + \varphi(x)E_q(\varphi(y))$$

$$E'_q(-xy) = 0 = -xq\bar{y} - x\bar{y}q = E'_q(\varphi(x))\varphi(y) + \varphi(x)E'_q(\varphi(y)).$$

This way we have checked that the image of $E \oplus E'$ is indeed in $\text{der}_0 \mathbb{O}$. Finally then, there is an injective homomorphism

$$E \oplus E' : \mathfrak{so}(4) \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \text{der}_0 \mathbb{O},$$

and thus an isomorphism. 

The maps $E$ and $E'$ will be used once again in the sequel, so let us mention them in a separate

**Corollary 1.** There is an isomorphism of Lie algebras

$$E \oplus E' : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \to \text{der}_0 \mathbb{O},$$

where $E$ and $E'$ have been introduced in the proof of Lemma 8. When restricted to $\mathbb{H} \subset \mathbb{O}$, the image of $E$ is $\text{der} \mathbb{H} \simeq \mathfrak{sp}(1)$ while $E'$ is trivial.
1.1.5 Symmetric pairs related to the Magic Square

Let us recall the definition of our magic square algebras:

\[ \mathcal{M}(\mathbb{K},\mathbb{K}') = \text{der} \mathbb{K}' \oplus \text{der} \mathfrak{h}_3 \mathbb{K} \oplus \text{Im} \mathbb{K}' \otimes \mathfrak{sh}_3 \mathbb{K}, \]

where, as before, \( \mathbb{K}' \neq \mathbb{R} \). We can now apply the Cayley-Dickson and induced decompositions to \( \mathbb{K}' \) and \( \text{der} \mathbb{K}' \) in the latter formula, and try to group the resulting subspaces into a symmetric pair. A natural candidate for the latter is easily found, and we have the following

Lemma 9. The decomposition \( \mathcal{M}(\mathbb{K},\mathbb{K}') = \mathfrak{g}(\mathbb{K},\mathbb{K}') \oplus \mathfrak{V}(\mathbb{K},\mathbb{K}') \), with

\[
\begin{align*}
\mathfrak{g}(\mathbb{K},\mathbb{K}') &= \text{der} \mathfrak{h}_3 \mathbb{K} \oplus \text{der}_0 \mathbb{K}' \oplus \text{Im} \mathbb{K}'_0 \otimes \mathfrak{sh}_3 \mathbb{K} \\
\mathfrak{V}(\mathbb{K},\mathbb{K}') &= \text{der}_1 \mathbb{K}' \oplus \mathbb{K}'_1 \otimes \mathfrak{sh}_3 \mathbb{K}
\end{align*}
\]

yields a symmetric pair.

Proof. Let us for the sake of brevity omit \((\mathbb{K},\mathbb{K}')\), writing simply \( \mathfrak{g} \) and \( \mathfrak{V} \). What has to be checked is that:

\[ [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}, \quad [\mathfrak{g}, \mathfrak{V}] \subset \mathfrak{V}, \quad [\mathfrak{V}, \mathfrak{V}] \subset \mathfrak{g}. \]

Note first, that \( \text{der} \mathfrak{h}_3 \mathbb{K} \) commutes with \( \mathbb{K}' \) and acts only on the \( \mathfrak{sh}_3 \mathbb{K} \) factor in \( \text{Im} \mathbb{K}' \otimes \mathfrak{sh}_3 \mathbb{K} \), so we already have

\[ [\text{der} \mathfrak{h}_3 \mathbb{K}, \mathfrak{g}] \subset \mathfrak{g} \quad \& \quad [\text{der} \mathfrak{h}_3 \mathbb{K}, \mathfrak{V}] \subset \mathfrak{V}. \]

The other conditions basically follow from Lemma 7. Indeed, the latter implies that \( \text{der}_0 \mathbb{K}' \oplus \text{der}_1 \mathbb{K}' \) is itself a symmetric pair. Moreover, as \( \text{der}_0 \mathbb{K}' \) preserves the Cayley-Dickson decomposition, we have

\[ [\text{der}_0 \mathbb{K}', \mathfrak{g}] \subset \mathfrak{g} \quad \& \quad [\text{der}_0 \mathbb{K}', \mathfrak{V}] \subset \mathfrak{V}. \]

Furthermore, as \( \text{der}_1 \mathbb{K}' \) maps \( \mathbb{K}'_0 \) into \( \mathbb{K}'_1 \), we have \( \text{Im} \mathbb{K}'_0 \otimes \mathfrak{sh}_3 \mathbb{K} \) and \( \text{der}_1 \mathbb{K}' \) is in the odd subalgebra of the derivations, and that it maps \( \mathbb{K}'_1 \) to \( \text{Im} \mathbb{K}'_0 \), we have \( [\text{der}_1 \mathbb{K}', \mathfrak{g}] \subset \mathfrak{g} \) and \( [\text{der}_1 \mathbb{K}', \mathfrak{V}] \subset \mathfrak{V} \). Thus we have checked that \([\mathfrak{V}, \mathfrak{V}] \subset \mathfrak{g}\), which completes the proof. \( \square \)

What we claim now, is that the pairs \( \mathfrak{g}(\mathbb{K},\mathbb{K}') \oplus \mathfrak{V}(\mathbb{K},\mathbb{K}') \) are exactly the ones corresponding to the symmetric spaces described in the introduction. Recall, that we wanted the subgroups in the three families to be, respectively, those of the first column, those of the second column times an additional \( U(1) \), and those of the third column times an additional \( \text{Sp}(1) \). We will first show that the algebras \( \mathfrak{g}(\mathbb{K},\mathbb{K}') \) are indeed of this form.

Proposition 2. There are isomorphisms of Lie algebras:
\[
\begin{align*}
\mathfrak{g}(\mathbb{K}, \mathbb{C}) & \cong \mathfrak{M}(\mathbb{K}, \mathbb{R}) \\
\mathfrak{g}(\mathbb{K}, \mathbb{H}) & \cong \mathfrak{M}(\mathbb{K}, \mathbb{C}) \oplus \mathfrak{u}(1) \\
\mathfrak{g}(\mathbb{K}, \mathbb{O}) & \cong \mathfrak{M}(\mathbb{K}, \mathbb{H}) \oplus \mathfrak{sp}(1).
\end{align*}
\]

Proof. In the first case, we have immediately \(\mathfrak{g}(\mathbb{K}, \mathbb{C}) = \text{der}\ h_3 \mathbb{K} = \mathfrak{M}(\mathbb{K}, \mathbb{R})\). In the second one, we recall that \(\text{der}_0 \mathbb{H} \cong \mathfrak{u}(1)\) is generated by \(\text{ad}_1\), and thus preserves \(\text{Im}\mathbb{H}_0 = i\mathbb{R} \subset \mathbb{H}\). It then follows that
\[
\mathfrak{g}(\mathbb{K}, \mathbb{H}) = \mathfrak{u}(1) \oplus \text{der}\ h_3 \mathbb{K} \oplus i\mathbb{R} \otimes \text{sh} h_3 \mathbb{K} = \mathfrak{u}(1) \oplus \mathfrak{M}(\mathbb{K}, \mathbb{C}),
\]
with the rightmost expression being a direct sum of Lie algebras. Finally, in the last case, we recall from corollary 1, that \(\text{der}_0 \mathbb{O} \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)\) with the first \(\mathfrak{sp}(1)\) acting on \(\mathbb{H} \subset \mathbb{O}\) as the derivations \(\text{der}\mathbb{H}\), and the other one trivially. We then have
\[
\mathfrak{g}(\mathbb{K}, \mathbb{O}) = \mathfrak{sp}(1) \oplus \text{der}\ \mathbb{H} \oplus \text{der}\ h_3 \mathbb{K} \oplus \text{Im}\mathbb{H} \otimes \text{sh} h_3 \mathbb{K} = \mathfrak{sp}(1) \oplus \mathfrak{M}(\mathbb{K}, \mathbb{H}),
\]
with the rightmost expression being again a direct sum of Lie algebras. \(\square\)

Our next goal is to describe the adjoint representations of \(\mathfrak{g}(\mathbb{K}, \mathbb{K}')\) on \(V(\mathbb{K}, \mathbb{K}')\), showing, in particular, that they are faithful and irreducible. It will then turn out, that the connected subgroups of \(\text{GL}(V(\mathbb{K}, \mathbb{K}'))\) resulting from exponentiating the adjoint representation of \(\mathfrak{g}(\mathbb{K}, \mathbb{K}')\) are indeed the magic square groups (admitting a nice description in terms of invariants), possibly augmented by an additional \(U(1)\) or \(\text{Sp}(1)\), corresponding to a complex or quaternionic structure on \(V(\mathbb{K}, \mathbb{K}')\).

1.2 More structure on Jordan algebras

Before we can perform what has just been indicated, we need to introduce some further structure on the Jordan algebras \(h_3 \mathbb{K}\), and their related Freudenthal Triple Systems (FTS, to be defined).

1.2.1 The determinant and Freudenthal product on \(h_3 \mathbb{K}\).

If the dimension of a (commutative) algebra is finite, it is clear that subsequent powers of an element of the algebra cannot be linearly independent. Instead, they satisfy a characteristic equation, polynomial in the element of the algebra. In case of the algebras of our interest, \(h_3 \mathbb{K}\), the characteristic equation is of degree three:

Lemma 10 (cf. [21][13]). There exist natural maps \(T, S, N : h_3 \mathbb{K} \to \mathbb{R}\), respectively linear, quadratic and cubic, such that for each \(X \in h_3 \mathbb{K}\) the following holds:
\[
X^3 - T(X)X^2 + S(X, X)X - N(X, X, X) = 0.
\]
They may be expressed using the product and trace, as follows:
\[
\begin{align*}
T(X) &= \text{tr}X \\
S(X, X) &= \frac{1}{2}(\text{tr}X)^2 - \frac{1}{2}\text{tr}X^2 \\
N(X, X, X) &= \frac{1}{3}\text{tr}X^3 - \frac{1}{2}\text{tr}X^2\text{tr}X + \frac{1}{6}(\text{tr}X)^3.
\end{align*}
\]
The cubic form $N$ is usually referred to as the norm. One can however check, that for all four $K$ it exactly coincides with the matrix determinant, and we will use this more familiar notion in the sequel. To avoid explaining the subtleties of an octonionic determinant and proving the identity, we simply define the symbol $\det$ using the formula given in the Lemma:

$$\det X := N(X, X, X)$$

for $X \in h_3 K$.

We are now naturally interested in the isotropy groups of the determinant. These are related to what is called the structure algebra $\text{str}(h_3 K) \subset \text{End}(h_3 K)$, namely the Lie algebra generated by all multiplications $L_X : Y \mapsto X \circ Y$ for $X \in h_3 K$.

Recalling Lemma 4, one has $[L_X, L_Y] \in \text{der} h_3 K$, and naturally $[d, L_X] = L_{d(x)}$ for a derivation $d \in \text{der} h_3 K$. It thus follows, that

$$\text{str}(h_3 K) = \text{der} h_3 K \oplus L_{h_3 K},$$

and $L_{h_3 K} = \{L_X \mid X \in h_3 K\}$. Multiplications by multiples of identity generate the centre, and one can decompose $\text{str}(h_3 K) = L_{1 \mathbb{R}} \oplus \text{str}_0(h_3 K)$, where

$$\text{str}_0(h_3 K) = \text{der} h_3 K \oplus L_{sh_{h_3 K}},$$

(1.8)

These groups are found to be:

$$\begin{align*}
\text{Str}_0(h_3 \mathbb{R}) &\simeq \text{SL}(3, \mathbb{R}) \\
\text{Str}_0(h_3 \mathbb{C}) &\simeq \text{SL}(3, \mathbb{C}) \\
\text{Str}_0(h_3 \mathbb{H}) &\simeq \text{SL}(3, \mathbb{H}) \\
\text{Str}_0(h_3 \mathbb{O}) &\simeq \mathbb{E}_{6(-26)}
\end{align*}$$

(recall that $\text{SL}(3, \mathbb{H})$ is the group of quaternionic matrices with the quaternionic determinant equal one, so that their Lie algebra $\mathfrak{sl}(3, \mathbb{H})$ consists of matrices with vanishing real part of the trace). Note that these are non-compact forms of the second column of the magic square algebras.

Another incarnation of the natural cubic form, useful in formulas, is the symmetric Freudenthal product,

$$\cdot : h_3 K \times h_3 K \to h_3 K$$

\[\text{Wangberg} \] interestingly argues, that in certain sense $\mathbb{E}_{6(-26)} = "\text{SL}(3, \mathbb{O})"$. 

---

\[\text{Wangberg} \]
\[ \langle X \bullet Y, Z \rangle = 3N(X, Y, Z) \]
for \( X, Y, Z \in \mathfrak{h}_3 \mathbb{K} \), together with a corresponding multiplication map
\[
L_X^\bullet : Y \mapsto X \bullet Y
\]
and a quadratic map
\[
\sharp : \mathfrak{h}_3 \mathbb{K} \to \mathfrak{h}_3 \mathbb{K}
\]
\[
X^\sharp = X \bullet X.
\]
Expressing the latter using powers of \( X \) and their traces, and comparing with the formulae of Lemma 10, one finds that
\[
X^{\sharp \sharp} = (\det X)X.
\]

### 1.2.2 FTS \( \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \) and the triple product.

Freudenthal [25,26,27,28] originally constructed the group \( E_7(-25) \) as the automorphism group of certain algebraic object, namely the space \( \mathfrak{h}_3 \mathbb{O} \oplus \mathfrak{h}_3 \mathbb{O} \oplus \mathbb{R} \oplus \mathbb{R} \) equipped with a symmetric triple product, mapping three elements of the space to a fourth one and satisfying certain identities with respect to a natural symplectic form on this space. This construction has been extended in many ways (see [29,30,31,32]). We shall be interested in four FTS associated with the four Jordan algebras \( \mathfrak{h}_3 \mathbb{K} \) (where the case \( \mathbb{K} = \mathbb{O} \) corresponds to the original construction).

Let us thus define the space of the FTS associated with \( \mathfrak{h}_3 \mathbb{K} \) to be
\[
\mathcal{F}(\mathfrak{h}_3 \mathbb{K}) = (\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}) \otimes \mathbb{R}^2
\]
and equip it with a natural symplectic form
\[
\omega\left( \begin{pmatrix} x & \xi_1 \\ X & \xi_2 \end{pmatrix} \otimes \begin{pmatrix} y & \eta_1 \\ Y & \eta_2 \end{pmatrix} \right) = (xy + \langle X, Y \rangle)(\xi_1 \eta_2 - \xi_2 \eta_1),
\]
where \( X, Y \in \mathfrak{h}_3 \mathbb{K} \) and \( x, \xi_1, \xi_2, y, \eta_1, \eta_2 \in \mathbb{R} \).

Next, we introduce a quartic map \( Q : \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \to \mathbb{R} \), such that for
\[
F = \begin{pmatrix} x & \xi_1 \\ X & \xi_2 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{x} \\ \tilde{X} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
where \( X, \tilde{X} \in \mathfrak{h}_3 \mathbb{K} \) and \( x, \tilde{x} \in \mathbb{R} \), one has
\[
Q(F, F, F, F) = \langle X^2, \tilde{X}^2 \rangle - x \det X - \tilde{x} \det \tilde{X} - \frac{1}{4}(\langle X, \tilde{X} \rangle - x\tilde{x})^2.
\]
The symmetric triple product
\[
\tau : \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \times \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \times \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \to \mathcal{F}(\mathfrak{h}_3 \mathbb{K})
\]
is then defined by
\[
\omega (F'', \tau(F, F, F)) = Q(F'', F, F, F)
\]
for \( F, F' \in \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \).
We are now interested in the automorphism groups $\text{Aut}(\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}))$, defined to be the ones preserving both the symplectic form $\omega$ and triple product $\tau$, so that

$$
\tau(a(F), a(F), a(F)) = a(\tau(F, F, F))
$$

for $a \in \text{Aut}(\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}))$ and $F \in \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$. The corresponding Lie algebra, called the derivations $\text{der} \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$, is given by the following

**Lemma 12** (cf. [23][25]). The derivations of the FTS $\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$ are the following subspace of $\text{End} \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$:

$$
\text{der} \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}) = \mathcal{H}_0(\text{der} \mathfrak{h}_3^3 \mathbb{K}) \oplus \mathcal{H}_1(\mathfrak{h}_3^3 \mathbb{K})
$$

where the maps

$$
\mathcal{H}_0 : \text{der} \mathfrak{h}_3^3 \mathbb{K} \to \text{End} \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})
$$

are given by

$$
\mathcal{H}_0(D, C) = \left( -\text{tr} C \atop L_C \right) \otimes \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ D \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

and

$$
\mathcal{H}_1(A, B) = \left( \begin{pmatrix} 0 & \langle A, \cdot \rangle \\ A & -2L_A \end{pmatrix} \right) \otimes \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} \otimes \begin{pmatrix} 0 & \langle B, \cdot \rangle \atop 2L_B \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix}
$$

as operators on $\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}) = (\mathbb{R} \oplus \mathfrak{h}_3^3 \mathbb{K}) \otimes \mathbb{R}^2$.

**Sketch of a Proof.** This form of the derivation algebra of $\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$ is usually presented for $\mathbb{K} = \mathbb{O}$, as a construction of the exceptional algebra $\mathfrak{e}_7$. However, the same reasoning works equally well for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Indeed, let us write a generic element of $\text{End} \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$ as

$$
E = \sum_{k=1}^4 \left( \begin{array}{c} \lambda_k \\ K_k \end{array} \right) \left( \begin{array}{c} H_k, \cdot \\ E_k \end{array} \right) \otimes \epsilon_k,
$$

for $\lambda_k \in \mathbb{R}$, $H_k, K_k \in \mathfrak{h}_3^3 \mathbb{K}$ and $E_k \in \text{End} \mathfrak{h}_3^3 \mathbb{K}$, where $\epsilon_k$ is a basis in $\text{End} \mathbb{R}^2$ consisting of matrices with one nonzero element. Then, expanding the equation

$$
Q(E(F), F, F, F) = 0
$$

for each $F \in \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K})$, and expressing $F$ as in (1.9), one can solve it using the fact that a general map in $\text{End} \mathfrak{h}_3^3 \mathbb{K}$ stabilizing $\text{det}$ is $D + L_C$ for $C \in \mathfrak{h}_3^3 \mathbb{K}$, $D \in \text{der} \mathfrak{h}_3^3 \mathbb{K}$, and its conjugate w.r.t. $\langle \cdot, \cdot \rangle$ is $D - L_C$. A general solution turns out to be the one given in the Lemma. One then checks, that it also preserves $\omega$, and thus is a derivation. Conversely, every derivation, preserving $\omega$ and $\tau$, must also preserve $Q$. \(\blacksquare\)

The argument we have just sketched leads to the following

**Corollary 2** (cf. [23]). The stabilizer of the map $Q : \mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}) \to \mathbb{R}$ in $\text{GL}(\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}))$ is precisely the automorphism group $\text{Aut}(\mathcal{F}(\mathfrak{h}_3^3 \mathbb{K}))$. 
These groups are found to be $[^{33}]$

\[
\begin{align*}
\text{Aut}(\mathcal{F}(\mathfrak{h}_3 \mathbb{R})) & \simeq \text{Sp}(6, \mathbb{R}) \\
\text{Aut}(\mathcal{F}(\mathfrak{h}_3 \mathbb{C})) & \simeq \text{SU}(3,3) \\
\text{Aut}(\mathcal{F}(\mathfrak{h}_3 \mathbb{H})) & \simeq \text{SO}^*(12) \\
\text{Aut}(\mathcal{F}(\mathfrak{h}_3 \mathbb{O})) & \simeq E_{7(-25)}.
\end{align*}
\]

Note that these are non-compact forms of the third column of magic square algebras.

For later use, we also equip $\mathcal{F}(\mathfrak{h}_3 \mathbb{K})$ with a scalar product

\[
\langle \begin{pmatrix} x \\ X \end{pmatrix} \otimes \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \begin{pmatrix} y \\ Y \end{pmatrix} \otimes \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \rangle = (xy + (X,Y)) (\xi_1 \eta_1 + \xi_2 \eta_2),
\]

where $X, Y \in \mathfrak{h}_3 \mathbb{K}$ and $x, \xi_1, \xi_2, y, \eta_1, \eta_2 \in \mathbb{R}$.

1.3 Identifying the isotropy representations

We have so far encountered three families of isotropy groups for certain structures, namely: Jordan algebra automorphisms $\text{Aut}(\mathfrak{h}_3 \mathbb{K})$, stabilizer of the determinant $\text{Str}_0(\mathfrak{h}_3 \mathbb{K})$ and $\text{FTS}$ automorphisms $\text{Aut}(\mathcal{F}(\mathfrak{h}_3 \mathbb{K}))$. While the first family forms exactly the first column of the magic square, the other two are non-compact forms of the second and third column. In what follows, we shall construct their compact forms in terms of unitary defining representations.

We will need the following simple fact:

**Lemma 13.** Let $V$ be a real vector space equipped with a scalar product $\langle \cdot, \cdot \rangle$ and $\mathcal{A} \subset \text{End}V$ a semisimple Lie algebra acting irreducibly on $V$. The scalar product gives rise to a decomposition

\[
\mathcal{A} = \mathcal{A}_\Lambda \oplus \mathcal{A}_S,
\]

corresponding to $\text{End}V \simeq \Lambda^2V \oplus S^2V$.

Consider now the complexification $V^\mathbb{C} = \mathbb{C} \otimes V$ equipped with a hermitian inner product given by a sesquilinear extension of $\langle \cdot, \cdot \rangle$, as well as the complexification of the algebra, $\mathcal{A}^\mathbb{C} = \mathbb{C} \otimes \mathcal{A} \subset \mathbb{C} \otimes \text{End}V \simeq \text{End}_\mathbb{C}(V^\mathbb{C})$. Let now

\[\mathcal{A}_u = \mathcal{A} \cap \mathfrak{u}(V^\mathbb{C})\]

be the antihermitian subalgebra of $\mathcal{A}^\mathbb{C}$. Then

\[\mathcal{A}_u = \mathcal{A}_\Lambda \oplus i\mathcal{A}_S\]

and is the compact real form of $\mathcal{A}^\mathbb{C}$. It moreover acts complex-irreducibly on $V^\mathbb{C}$.

**Proof.** The form of $\mathcal{A}_u$ follows simply from the consideration of $\mathcal{A}^\mathbb{C} = \mathcal{A}_\Lambda \oplus i\mathcal{A}_S \oplus i\mathcal{A}_\Lambda \oplus \mathcal{A}_S$ where first two summands are antihermitean, while the other two are hermitian. Moreover, being a semisimple subalgebra of the compact algebra $\mathfrak{u}(V^\mathbb{C})$, $\mathcal{A}_u$ is compact.

Finally, assume that there is a complex subspace of $V^\mathbb{C}$ preserved by $\mathcal{A}_u$. The subspace is necessarily of the form $W + iW$ with $W \subset V$. An element $f \in \mathcal{A}_\Lambda$ maps $w + iw'$ to $f(w) + if(w')$ while $i \in \mathcal{A}_S$ maps the same to $-s(w') + is(w)$. Clearly, $f(w), s(w') \in W$ for each $w, w' \in W$. Hence, $W \subset V$ is fixed by the original $\mathcal{A}$. But then $W$ is either empty of $V$. Thus $V^\mathbb{C}$ is irreducible. \qed
Our isotropy representations will turn out to be described by the following spaces:

\[ \mathcal{V}_1(\mathbb{K}) = \mathfrak{so}(3) \]
\[ \mathcal{V}_2(\mathbb{K}) = \mathbb{C} \otimes \mathfrak{h}_3 \mathbb{K}, \quad \mathcal{V}_3(\mathbb{K}) = \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}), \]

where \( \mathcal{V}_2(\mathbb{K}) \) and \( \mathcal{V}_3(\mathbb{K}) \) are complex vector spaces equipped with hermitian inner products being sesquilinear extensions of the scalar products on, respectively, \( \mathfrak{h}_3 \mathbb{K} \) and \( \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \), defined in the previous sections. The space \( \mathcal{V}_3(\mathbb{K}) \) is moreover equipped with a symplectic form being the linear extension of \( \omega \). The interesting algebras acting on these spaces are:

\[ \mathcal{G}'_1(\mathbb{K}) = \text{der} \mathfrak{h}_3 \mathbb{K} \subset \text{End} \mathcal{V}_1(\mathbb{K}) \]
\[ \mathcal{G}'_2(\mathbb{K}) = \text{str} \mathfrak{h}_3 \mathbb{K} \subset \mathfrak{su}(\mathcal{V}_2(\mathbb{K})) \]
\[ \mathcal{G}'_3(\mathbb{K}) = \text{der} \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \subset \mathfrak{sp}(\mathcal{V}_3(\mathbb{K}), \omega), \]

where \( \mathcal{G}'_2(\mathbb{K}) \) and \( \mathcal{G}'_3(\mathbb{K}) \) are the antihermitian subalgebras of, respectively, \( \mathbb{C} \otimes \mathfrak{so}(\mathfrak{h}_3 \mathbb{K}) \) and \( \mathbb{C} \otimes \text{der} \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \), as described in Lemma 13.

Using the Lemma and examining (anti)symmetry of elements of algebras defined in the previous section, we have the explicit forms:

\[ \mathcal{G}'_2(\mathbb{K}) = \text{der} \mathfrak{h}_3 \mathbb{K} \oplus i \mathcal{L}_{\mathfrak{so}(3)} \]
\[ \mathcal{G}'_3(\mathbb{K}) = \mathcal{H}_0(\text{der} \mathfrak{h}_3 \mathbb{K} \oplus i \mathfrak{h}_3 \mathbb{K}) \oplus \mathcal{H}_1(\mathfrak{h}_3 \mathbb{K} \oplus i \mathfrak{h}_3 \mathbb{K}), \]

where the maps \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) of Lemma 12 have been extended by linearity.

We now exponentiate \( \mathcal{G}'_n(\mathbb{K}), n = 1, 2, 3, \) to obtain connected subgroups \( \mathcal{G}_n(\mathbb{K}) \subset \text{GL}(\mathcal{V}_n(\mathbb{K})) \). In particular, \( \mathcal{G}_2(\mathbb{K}) \) and \( \mathcal{G}_3(\mathbb{K}) \) are compact forms of the ones described in the previous section, as given in the following table:

| \( \mathbb{K} \) | \( \mathcal{G}_1(\mathbb{K}) \) | \( \mathcal{G}_2(\mathbb{K}) \) | \( \mathcal{G}_3(\mathbb{K}) \) |
|---------------|-----------------|-----------------|-----------------|
| \( \mathbb{R} \) | \text{SO}(3) | \text{SU}(3) | \text{Sp}(1) |
| \( \mathbb{C} \) | \text{SU}(3) | \text{SU}(3) \times \text{SU}(3) | \text{SU}(6) |
| \( \mathbb{H} \) | \text{Sp}(1) | \text{SU}(6) | \text{SO}(12) |
| \( \mathbb{O} \) | \text{F}_4 | \text{E}_6(−52) | \text{E}_7(−133) |

with

\[ \mathcal{G}_2(\mathbb{K}) \subset \text{SU}(\mathcal{V}_2(\mathbb{K})) \]
\[ \mathcal{G}_3(\mathbb{K}) \subset \text{Sp}(\mathcal{V}_3(\mathbb{K}), \omega). \]

We now arrive at the main point of this section:

**Proposition 3.** The isotropy representations of the magic square symmetric spaces are extensions of those of \( \mathcal{G}_n(\mathbb{K}) \) on \( \mathcal{V}_n(\mathbb{K}) \) with \( n = 1, 2, 3 \) in, respectively, first, second and third family, i.e.:

1. There exist isomorphisms (respectively of Lie algebras and of vector spaces)

\[ F^{\mathfrak{g}}_1 : \mathcal{G}'_1(\mathbb{K}) \rightarrow \mathfrak{g}(\mathbb{K}, \mathbb{C}) \]
\[ F^V_1 : \mathcal{V}_1(\mathbb{K}) \rightarrow V(\mathbb{K}, \mathbb{C}) \]

such that, for each \( A \in \mathcal{G}'_1(\mathbb{K}) \) and \( X \in \mathcal{V}_1(\mathbb{K}) \),

\[ [F^{\mathfrak{g}}_1(A), F^V_1(X)] = F^V_1(A(X)). \]
2. There exist isomorphisms (respectively of Lie algebras and of vector spaces)
\[ F^g_2 : G'_2(\mathbb{K}) \to \mathfrak{M}(\mathbb{K}, \mathbb{C}) \subset \mathfrak{g}(\mathbb{K}, \mathbb{H}) \]
\[ F^V_2 : V_2(\mathbb{K}) \to V(\mathbb{K}, \mathbb{H}) \]
such that, for each \( A \in G'_2(\mathbb{K}) \) and \( X \in V_2(\mathbb{K}) \),
\[ [F^g_2(A), F^V_2(X)] = F^V_2(A(X)). \]
Moreover, as Lie algebras,
\[ \mathfrak{g}(\mathbb{K}, \mathbb{H}) = F^g_2(G'_2(\mathbb{K})) \oplus \mathfrak{u}(1). \]
Once we use \( F^V_2 \) to identify \( \mathbb{C} \otimes F(\mathfrak{h}_3 \mathbb{K}) \) with \( V(\mathbb{K}, \mathbb{H}) \), a generator \( i \) of this extra \( \mathfrak{u}(1) \subset \mathfrak{g}(\mathbb{K}, \mathbb{H}) \) acts on \( \mathbb{C} \otimes \mathfrak{h}_3 \mathbb{K} \) as \( \sqrt{-1} \in \mathbb{C} \). In other words, it defines a complex (hermitian) structure, with respect to which \( \mathfrak{g}(\mathbb{K}, \mathbb{H}) \subset \mathfrak{u}(\dim \mathfrak{h}_3 \mathbb{K}) \).

3. There exist isomorphisms (respectively of Lie algebras and of vector spaces)
\[ F^g_3 : G'_3(\mathbb{K}) \to \mathfrak{M}(\mathbb{K}, \mathbb{H}) \subset \mathfrak{g}(\mathbb{K}, \mathfrak{O}) \]
\[ F^V_3 : V_3(\mathbb{K}) \to V(\mathbb{K}, \mathfrak{O}) \]
such that, for each \( A \in G'_3(\mathbb{K}) \) and \( X \in V_3(\mathbb{K}) \),
\[ [F^g_3(A), F^V_3(X)] = F^V_3(A(X)). \]
Moreover, as Lie algebras,
\[ \mathfrak{g}(\mathbb{K}, \mathfrak{O}) = F^g_3(G'_3(\mathbb{K})) \oplus \mathfrak{sp}(1). \]
Once we use \( F^V_3 \) to identify \( \mathbb{C} \otimes F(\mathfrak{h}_3 \mathbb{K}) \) with \( V(\mathbb{K}, \mathfrak{O}) \), the generators \( i, j, k \) of this extra \( \mathfrak{sp}(1) \subset \mathfrak{g}(\mathbb{K}, \mathbb{H}) \) act on \( \mathbb{C} \otimes F(\mathfrak{h}_3 \mathbb{K}) \) as \( I, J, K \) given explicitly by:
\[ I(X) = iX, \quad \langle \dot{X}, JY \rangle = \omega(X, Y), \quad \langle \dot{X}, KY \rangle = i\omega(X, Y). \quad (1.12) \]
In other words, they define a quaternionic (hermitian) structure, with respect to which \( \mathfrak{g}(\mathbb{K}, \mathfrak{O}) \subset \mathfrak{sp}(\dim \mathfrak{h}_3 \mathbb{K} + 1) \oplus \mathfrak{sp}(1) \).

**Corollary 3.**

1. The adjoint representation
\[ \text{ad} : \mathfrak{g}(\mathbb{K}, \mathfrak{K}') \to \text{End} V(\mathbb{K}, \mathfrak{K}') \]
is faithful and irreducible.

2. Let \( G(\mathbb{K}, \mathfrak{K}') \) be the connected subgroup of \( \text{GL}(V(\mathbb{K}, \mathfrak{K}')) \) obtained by exponentiation of the image of \( \mathfrak{g}(\mathbb{K}, \mathfrak{K}') \) under the adjoint representation on \( V(\mathbb{K}, \mathfrak{K}'). \) Let us moreover identify \( V(\mathbb{K}, \mathfrak{K}') \) with \( \mathcal{V}_0(\mathbb{K}) \) using \( F^V_n \) with \( n = 1, 2, 3 \) for, respectively, \( \mathfrak{K}' = \mathbb{C}, \mathbb{H}, \mathfrak{O} \). Then:
\[ G(\mathbb{K}, \mathbb{C}) = G_1(\mathbb{K}) \]
\[ G(\mathbb{K}, \mathbb{H}) = G_2(\mathbb{K}) \cdot U(1) \]
\[ G(\mathbb{K}, \mathfrak{O}) = G_2(\mathbb{K}) \cdot \mathfrak{sp}(1) \]
where the \( U(1) \) is generated by the natural complex structure on \( V_2(\mathbb{K}) \) and the \( \mathfrak{sp}(1) \) is generated by the natural quaternionic structure (induced by \( \omega \) and the hermitian inner product) on \( V_3(\mathbb{K}) \).
The rest of this section is devoted to proving the Proposition (and may be skipped by the inpatient reader). We first need to complete the discussion of \( \text{der} \, \mathfrak{O} \) started in Lemma 8. Recall that in Corollary 1 we have stated that the subalgebra \( \text{der}_0 \, \mathfrak{O} \) is isomorphic to \( \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \cong \text{Im} \mathbb{H} \oplus \text{Im} \mathbb{H} \) via the maps \( \mathcal{E} \) and \( \mathcal{E}' \). To parametrize \( \text{der}_1 \, \mathfrak{O} \), we introduce for each \( q \in \text{Im} \mathbb{H} \) a map

\[
B_q : \mathbb{H} \to \text{der}_1 \, \mathfrak{O}
\]

\[
B_q(x) = D_{q,qq}(x)
\]

for \( x \in \mathbb{H} \).

Recall that \( \text{der}_1 \, \mathfrak{O} \) maps \( \mathbb{H} \) to \( l \mathbb{H} \). The action on \( \text{Im} \mathbb{H} \) is given in a useful form by the following

**Lemma 14.** Let \( r, s \) be orthogonal unit imaginary quaternions. Then for \( x \in \mathbb{H} \) the following holds:

\[
B_s(x)(r) = 4r \varphi(x)
\]

\[
B_s(x)(r) = -2r \varphi(x)
\]

**Proof.** First, we have

\[
D_{s,s}(s) = [[s, s \varphi(x)], s] = 2[s(s \varphi(x)), s] = -2[\varphi(x), s] = 4s \varphi(x),
\]

where we used the alternativity of \( \mathfrak{O} \) (i.e. antisymmetry of the associator, equivalent to \( a^2b = a(ab) \)) and the fact that elements of \( \text{Im} \mathbb{H} \) anticommute with elements of \( l \mathbb{H} \). Then, we have

\[
D_{s,s}(r) = [[s, s \varphi(x)], r] = -3[s, s \varphi(x), r]
\]

\[
= 4r \varphi(x) - 3s \varphi(x) - 3s(r(s \varphi(x)))
\]

\[
= -2r \varphi(x),
\]

where we used the fact that orthogonal imaginary quaternions anticommute, and checked explicitly that \( s(r(sx')) = rx' \) for \( x' \in \mathfrak{O} \). This proves the Lemma. \( \square \)

It is clear that the maps \( B_i, B_j \) and \( B_k \) cannot be independent. Indeed, one can check that \( B_i + B_j + B_k = 0 \) (note that the map \( q \mapsto B_q \) is quadratic in \( q \), so that the latter sum is simply a polarised version of \( B \) evaluated on the \( \mathfrak{sp}(1) \)-invariant quadratic element). We choose two combinations of \( B_i \) and \( B_j \), adapted to the goal of constructing the intertwiner \( F^V_3 \) :

**Lemma 15.** Let us define two maps \( B', B'' : \mathbb{H} \to \text{der}_1 \, \mathfrak{O} \),

\[
B' = B_i - B_j, \quad B'' = 3B_k = -3(B_i + B_j).
\]

Then the map

\[
B' \times B'' : \mathbb{H} \oplus \mathbb{H} \to \text{der}_1 \, \mathfrak{O}
\]

is a bijection.

The proof is by direct calculation with help of computer algebra. Finally, the adjoint action of \( \text{der}_0 \, \mathfrak{O} \) on \( \text{der}_1 \, \mathfrak{O} \) is given by the following
Lemma 16. Let us use $B' \times B''$ to identify $\text{der}_1 \mathcal{O}$ with $\mathbb{H} \oplus \mathbb{H} = \mathbb{R}^2 \oplus \mathbb{H}$. Then the adjoint action of $\text{der}_0 \mathcal{O}$ on $\text{der}_1 \mathcal{O} \simeq \mathbb{R}^2 \oplus \mathbb{H}$ is:

\[
\begin{align*}
\text{ad}_{\mathcal{E}(i)} &= \begin{pmatrix} 0 & -3 \\ -1 & -2 \end{pmatrix} \otimes L_i \\
\text{ad}_{\mathcal{E}(j)} &= \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix} \otimes L_j \\
\text{ad}_{\mathcal{E}(k)} &= \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \otimes L_k
\end{align*}
\]

and, for $q \in \text{Im} \mathbb{H}$,

\[
\text{ad}_{\mathcal{E}(q)} = \text{id} \otimes R_q.
\]

Proof. Let us compute the commutator of $\mathcal{E}(s)$ and $\mathcal{B}_r(x)$ for $s, r$ being unit imaginary quaternions:

\[
\begin{align*}
[\mathcal{E}(s), \mathcal{D}_{r, r\varphi}(x)] &= \mathcal{D}_{s, r, r\varphi(x)} + \mathcal{D}_{r, [s, r] \varphi(x)} + \mathcal{D}_{r, r\varphi(sx)} \\
&= 2\mathcal{D}_{rs, r\varphi(sx)} - \mathcal{D}_{r, r\varphi(sx)} + 2(r, s)\mathcal{D}_{r, r\varphi(rx)},
\end{align*}
\]

where we used the equivariance of $\mathcal{D}$ and the identity $[r, s] = 2rs + 2(r, s)$. In particular,

\[
\begin{align*}
[\mathcal{E}(s), \mathcal{B}_r(x)] &= \mathcal{B}_r(sx) \\
[\mathcal{E}(s), \mathcal{B}_r(x)] &= 2\mathcal{B}_{sr}(sx) - \mathcal{B}_r(sx) \quad \text{for } s \perp r.
\end{align*}
\]

The commutator of $\mathcal{E}'(s)$ and $\mathcal{B}_r(x)$ is simply

\[
[\mathcal{E}'(s), \mathcal{D}_{r, r\varphi}(x)] = \mathcal{D}_{r, r\varphi(sx)} = \mathcal{B}_r(xs).
\]

This proves the Lemma. \qed

We are now ready to prove the Proposition.

Proof of Proposition. We first notice, that it is enough to check the intertwining property

\[
[F^\varphi_{1,2,3}(A), F^V_{1,2,3}(X)] = F^V_{1,2,3}(A(X))
\]

– then the fact that $F^V_{1,2,3}$ is an isomorphism between the vector spaces of two faithful representations already implies that $F^\varphi_{1,2,3}$ is an isomorphism of Lie algebras.

In the first family, the algebra is $\mathcal{G}'_1(\mathbb{K}) = \text{der} \mathfrak{h}_3 \mathbb{K}$ and the representation space is $\mathcal{V}_1(\mathbb{K}) = \mathfrak{h}_3 \mathbb{K}$. On the other hand, since $\text{der} \mathbb{C}$ is trivial, we have $\mathcal{V}(\mathbb{K}, \mathbb{C}) = i\mathbb{R} \otimes \mathfrak{h}_3 \mathbb{K}$, which is naturally identified with $\mathfrak{h}_3 \mathbb{K}$ and $g(\mathbb{K}, \mathbb{C}) = \text{der} \mathfrak{h}_3 \mathbb{K}$. The maps $F^V$ and $F^\varphi$ can be simply set to identity, and point 1 of the Proposition follows.

In the second family, the algebra is $\mathcal{G}'_2(\mathbb{K}) = \text{der} \mathfrak{h}_3 \mathbb{K} \oplus iL_{\mathfrak{h}_3 \mathbb{K}}$ and the representation space is $\mathcal{V}_2(\mathbb{K}) = \mathbb{C} \otimes \mathfrak{h}_3 \mathbb{K}$. On the other hand, since

\[
\text{der}_0 \mathfrak{h} \oplus \text{der}_1 \mathfrak{h} = \text{Span}\{\text{ad}_i\} \oplus \text{Span}\{\text{ad}_j, \text{ad}_k\},
\]

we have

\[
g(\mathbb{K}, \mathfrak{h}) = \text{der} \mathfrak{h}_3 \mathbb{K} \oplus \text{Span}\{\text{ad}_i\} \oplus i\mathbb{R} \otimes \mathfrak{h}_3 \mathbb{K}
\]
CHAPTER 1. ALGEBRAIC PART

Thus point 2 of the Proposition follows.

for \( z \in \mathfrak{h}_3 \mathbb{K} \) and the representation space is

Now, we check that, for \( D \in \text{der} \mathfrak{h}_3 \mathbb{K}, \ X \in \mathfrak{sh}_3 \mathbb{K}, \ z \in \mathbb{C} \) and \( \mathfrak{h}_3 \mathbb{K} \ni Y = Y_0 1 + Y_1 \) where \( Y_0 \in \mathbb{R}, \ Y_1 \in \mathfrak{sh}_3 \mathbb{K} \), the intertwining property holds:

On the other hand, we have \( \mathfrak{g}(\mathbb{K}, \mathfrak{O}) = \text{der} \mathfrak{h}_3 \mathbb{O} \oplus \text{der}_0 \mathfrak{O} \oplus (\text{Im} \mathfrak{h}_3 \mathbb{K}) \) and the representation space is

Thus point 2 of the Proposition follows.

In the third family, the algebra is

\[ G'_3(\mathbb{K}) = \mathcal{H}_0(\text{der} \mathfrak{h}_3 \mathbb{K} \oplus i\mathfrak{h}_3 \mathbb{K}) \oplus \mathcal{H}_1(\mathfrak{h}_3 \mathbb{K} \oplus i\mathfrak{h}_3 \mathbb{K}) \]

and the representation space is

\[ V_0(\mathbb{K}) = \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \cong (\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}) \otimes \mathbb{C}^2. \]

On the other hand, we have

\[ g(\mathbb{K}, \mathfrak{O}) = \text{der} \mathfrak{h}_3 \mathbb{O} \oplus \text{der}_0 \mathfrak{O} \oplus (\text{Im} \mathfrak{h}_3 \mathbb{K}) \]

\[ V(\mathbb{K}, \mathfrak{O}) = \text{der}_1 \mathfrak{O} \oplus (\text{Im} \mathfrak{h}_3 \mathbb{K}), \]
where the derivations are identified with certain quaternionic spaces:
\[
\begin{align*}
der_0 \mathcal{O} & = (\mathcal{E} \times \mathcal{E}') (\text{Im} \mathbb{H} \oplus \text{Im} \mathbb{H}) \\
der_1 \mathcal{O} & = (\mathcal{B} \times \mathcal{B}'') (\mathbb{H} \oplus \mathbb{H}).
\end{align*}
\]

To relate the latter spaces to the former ones, we note that the space \( C^2 \) becomes an irreducible left \( \mathbb{H} \)-module with the action assigning to \( q \in \mathbb{H} \) an operator \( \tau_q : C^2 \to C^2 \) given by
\[
\tau_i = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \tau_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau_k = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tau_1 = \text{id}.
\]
As its dimension is four, there exists an intertwiner \( \psi : C^2 \to \mathbb{H} \)
\[
\psi(\tau_q \xi) = L_q \psi(\xi)
\]
for \( q \in \mathbb{H}, \xi \in C^2 \), where \( L_q \) is the left multiplication by \( q \) in \( \mathbb{H} \).

As the reader may expect, \( \psi \) will relate the \( C^2 \) factor in \( \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \) to the quaternionic factors in \( V(\mathbb{K}, \mathcal{O}) \). One also needs to note that the \( \mathfrak{h}_3 \mathbb{K} \) spaces on the FTS side must be decomposed into \( \mathbb{R} \oplus \mathfrak{sh}_3 \mathbb{K} \) as the Tits construction features only the traceless subspaces. Clearly, the traces are then to be identified with certain elements of \( \text{der} \mathcal{O} \).

It is convenient to identify the two representation spaces with an intermediate one, namely \((\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}) \otimes \mathbb{H} \), so that we have the isomorphisms
\[
\mu : (\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}) \otimes C^2 \to (\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{sh}_3 \mathbb{K}) \otimes \mathbb{H}
\]
\[
(x, X) \otimes \xi \mapsto (x, X_0, X_1) \otimes \psi(\xi)
\]
for \( X = X_01 + X_1 \) where \( x, X_0 \in \mathbb{R}, \ X_1 \in \mathfrak{sh}_3 \mathbb{K} \), and
\[
\nu : (\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{sh}_3 \mathbb{K}) \otimes \mathbb{H} \to V(\mathbb{K}, \mathcal{O})
\]
\[
(y, y', Y) \otimes q \mapsto B'(y'q) + B''(y''q) - 12\nu(q) \otimes Y
\]
for \( y, y' \in \mathbb{R}, \ Y \in \mathfrak{sh}_3 \mathbb{K} \) and \( q \in \mathbb{H} \). We define the intertwiner to be their composition,
\[
F_3^V = \nu \circ \mu.
\]
Then, for \( A = A_01 + A_1 \) where \( A_0 \in \mathbb{R}, \ A_1 \in \mathfrak{sh}_3 \mathbb{K} \), and \( B = B_01 + B_1, \ C = C_01 + C_1 \) in the same manner, and \( D \in \text{der} \mathfrak{h}_3 \mathbb{K} \), we define
\[
F_3^3 : G_3^3(\mathbb{K}) \to \text{der} \mathfrak{h}_3 \mathbb{K} \oplus \text{der}_0 \mathcal{O} \oplus (\text{Im} \mathbb{H} \otimes \mathfrak{sh}_3 \mathbb{K})
\]
\[
F_3^3(\mathcal{H}_0(D, iC) + \mathcal{H}_1(A, iB)) = D - B_0 \mathcal{E}(i) + A_0 \mathcal{E}(j) + C_0 \mathcal{E}(k)
+ i \otimes 2B_1 - j \otimes 2A_1 + k \otimes C_1.
\]
Clearly, the image of \( F_3^3 \) is
\[
\mathfrak{g}(\mathbb{K}, \mathcal{O}) \supset \mathfrak{M}(\mathbb{K}, \mathbb{H}) = \text{der} \mathfrak{h}_3 \mathbb{K} \oplus \mathcal{E}(\text{Im} \mathbb{H}) \oplus (\text{Im} \mathbb{H} \otimes \mathfrak{sh}_3 \mathbb{K}),
\]
so that \( \mathfrak{g}(\mathbb{K}, \mathcal{O}) = \mathfrak{M}(\mathbb{K}, \mathbb{H}) \oplus \mathcal{E}'(\text{Im} \mathbb{H}) \).
To check the intertwining property, we must find the action of both $G'_3(\mathbb{K})$ and $F_3^I(G'_3(\mathbb{K})) = \mathfrak{m}(\mathbb{K}, \mathbb{H}) \subset \mathfrak{g}(\mathbb{K}, \mathbb{O})$ on $\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K} \otimes \mathbb{H}$ (via the identifications $\mu, \nu$) and show that they agree.

As the action on the FTS side contains the maps $L^*$ and $L$ on $\mathfrak{h}_3 \mathbb{K}$, we have to decompose them with respect to $\mathfrak{h}_3 \mathbb{K} = \mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}$ and express in terms of $L^*$, the multiplication map corresponding to the product $\times : \mathfrak{h}_3 \mathbb{K} \times \mathfrak{h}_3 \mathbb{K} \to \mathfrak{h}_3 \mathbb{K}$.

Recalling $\langle X \cdot Y, Z \rangle = 3N(X, Y, Z)$ and using the defining identity for $N$ (Lemma 10), we find that

$$X \cdot Y = X \circ Y + \frac{1}{2} \text{tr} X \text{tr} Y - \frac{1}{2}[\langle X, Y \rangle + (\text{tr} Y)X + (\text{tr} X)Y]$$

for $X, Y \in \mathfrak{h}_3 \mathbb{K}$. Now, regarding the expressions on the r.h.s as elements of $\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}$, we have:

$$X \circ Y = \left( X_0 Y_0 + \frac{1}{3}(X_1, Y_1), \ X_1 \times Y_1 + X_0 Y_1 + Y_0 X_1 \right)$$

$$X \cdot Y = \left( X_0 Y_0 - \frac{1}{6}(X_1, Y_1), \ X_1 \times Y_1 - \frac{1}{2}X_0 Y_1 - \frac{1}{2}Y_0 X_1 \right)$$

and

$$\langle X, Y \rangle = 3X_0 Y_0 + \langle X_1, Y_1 \rangle$$

for $X = X_0 1 + X_1$ and $Y = Y_0 1 + Y_1$ where $X_0, Y_0 \in \mathbb{R}$ and $X_1, Y_1 \in \mathfrak{h}_3 \mathbb{K}$.

Using these formulas and the definitions of $\mathcal{H}_0$ and $\mathcal{H}_1$, we find that the element

$$\mathcal{H}_0(D, i\mathcal{C}) + \mathcal{H}_1(A, iB) \in G'_3(\mathbb{K})$$

corresponds via $\mu$ to the following operator on $\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K} \otimes \mathbb{H}$:

$$\mu \circ [\mathcal{H}_0(D, i\mathcal{C}) + \mathcal{H}_1(A, iB)] \circ \mu^{-1} =$$

$$\begin{pmatrix}
-B_0 & 0 & -3 \\
 & -1 & -2 \\
 & & 1
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & h(B_1) \\
0 & 0 & -\frac{1}{2}h(B_1) \\
B_1 & -B_1 & 2L_{B_1}^X
\end{pmatrix} \otimes L_i$$

$$+ \begin{pmatrix}
A_0 & 0 & 3 \\
0 & 1 & -2 \\
1 & & 1
\end{pmatrix} +
\begin{pmatrix}
0 & 0 & h(A_1) \\
0 & 0 & \frac{1}{2}h(A_1) \\
A_1 & A_1 & -2L_{A_1}^X
\end{pmatrix} \otimes L_j \ (1.13)$$

$$+ \begin{pmatrix}
C_0 & -3 \\
0 & 1 \\
1 & & 1
\end{pmatrix} +
\begin{pmatrix}
0 & h(C_1) \\
0 & \frac{1}{2}h(C_1) \\
R_1 & L_{C_1}^X & D
\end{pmatrix} \otimes L_k$$

$$+ \begin{pmatrix}
0 & 0 \\
0 & 0 \\
D & & 1
\end{pmatrix} \otimes 1.$$

We now have to find the action of the image of $\mathcal{H}_0(D, i\mathcal{C}) + \mathcal{H}_1(A, iB)$ under $F_3^I$, namely

$$D - B_0 \mathcal{E}(i) + A_0 \mathcal{E}(j) + C_0 \mathcal{E}(k) + i \otimes 2B_1 - j \otimes 2A_1 + k \otimes C_1,$$

an element of $\mathfrak{g}(\mathbb{K}, \mathbb{O})$, while $V(\mathbb{K}, \mathbb{O})$ is identified with $\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K} \otimes \mathbb{H}$ via $\nu$. 
1.3. IDENTIFYING THE ISOTROPY REPRESENTATIONS

Acting with the der $\mathfrak{h}_3\mathbb{K} \oplus \text{der}_0 \mathbb{O}$ part is easy, as we have already found all the necessary formulas while discussing the octonionic derivations in the present section. We still need to compute the following commutator in $\mathfrak{M}(\mathbb{K}, \mathbb{O})$:

$$[r \otimes A_1, B'(q') + B'(q'') + \varphi(q) \otimes X_1] = -\frac{1}{12} h(A_1, X_1) B_r(rq) + \varphi(rq) \otimes (A_1 \times X_1) - [B'(q')(r) + B''(q'')(r)] \otimes A_1$$

for $q, q', q'' \in \mathbb{H}$, $A_1, X_1 \in \mathfrak{sh}_3 \mathbb{K}$ and $r$ a unit imaginary quaternion. Having all that done, we arrive at precisely the same operator as in (1.13):

$$\nu^{-1} \circ \text{ad}_{D-B_0E(i)+A_0E(j)+C_0E(k)+i \otimes 2B_1-j \otimes 2A_1+k \otimes C_1} \circ \nu =$$

$$-B_0 \begin{pmatrix} 0 & -3 \\ -1 & -2 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\frac{1}{2} h(B_1) \\ 0 & 0 & -\frac{1}{2} h(B_1) \\ B_1 & -B_1 & 2L_{B_1} \end{pmatrix} \otimes L_i + A_0 \begin{pmatrix} 0 & 3 \\ 1 & -2 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & h(A_1) \\ 0 & 0 & h(A_1) \\ A_1 & A_1 & -2L_{A_1} \end{pmatrix} \otimes L_j + C_0 \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} h(C_1) \\ 0 & L_{C_1} \\ R_1 & 0 \end{pmatrix} \otimes L_k + \begin{pmatrix} 0 \\ 0 \\ D \end{pmatrix} \otimes 1.$$

so that

$$\text{ad}_{F^2_+}[\mathcal{H}_0(D, iC) + \mathcal{H}_1(A, iB)] \circ (\nu \circ \mu) = (\nu \circ \mu) \circ [\mathcal{H}_0(D, iC) + \mathcal{H}_1(A, iB)]$$

as claimed.

Finally, we need to check the action of the remaining $\mathcal{E}'(\text{Im}\mathbb{H}) \simeq \mathfrak{sp}(1)$ on $\mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K})$. Using once again the formulas for octonionic derivations, we find that $\mathcal{E}'(q)$ acts on $(\mathbb{R} \oplus \mathbb{R} \oplus \mathfrak{sh}_3 \mathbb{K}) \otimes \mathbb{H}$ via

$$\text{id} \otimes R_q$$

where $R$ is the right multiplication map on the quaternions.

Thus the corresponding algebra $\mathfrak{sp}(1)$ acting on $(\mathbb{R} \oplus \mathfrak{h}_3 \mathbb{K}) \otimes \mathbb{C}^2$ is generated by three complex structures, which we choose to be

$$I = \text{id} \otimes (\psi^{-1} \circ R_i \circ \psi)$$

$$J = \text{id} \otimes (\psi^{-1} \circ R_{-j} \circ \psi)$$

$$K = \text{id} \otimes (\psi^{-1} \circ R_k \circ \psi).$$

These can be brought to the form demanded by the Proposition once the intertwiner $\psi : \mathbb{C}^2 \rightarrow \mathbb{H}$ is set to:

$$\psi(1, 1) = 1, \quad \psi(i, i) = i, \quad \psi(-1, 1) = j, \quad \psi(i, -i) = k.$$
Then we indeed have, for \( x \in \mathbb{R} \), \( X \in \mathfrak{h}_3 \mathbb{K} \) and \( z_1, z_2 \in \mathbb{C} \):

\[
\begin{align*}
I((x, X) \otimes (z_1, z_2)) &= (x, X) \otimes (iz_1, iz_2) \\
J((x, X) \otimes (z_1, z_2)) &= (x, X) \otimes (\bar{z}_2, -\bar{z}_1) \\
K((x, X) \otimes (z_1, z_2)) &= (x, X) \otimes (i\bar{z}_2, -i\bar{z}_1).
\end{align*}
\]

Compared with the formulas for the inner product and symplectic form, this completes the proof.

1.4 Constructing symmetric invariants

Now that we have found that the isotropy representations of our symmetric spaces are in fact the natural representations of certain automorphism or isotropy groups (up to complexification), extended by the action of a natural complex or quaternionic structure, we can get the symmetric invariants almost for free.

While the situation is clear in case of the first family, the spaces \( V_2(\mathbb{K}) = \mathbb{C} \otimes \mathfrak{h}_3 \mathbb{K}, \ V_3(\mathbb{K}) = \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \) of other two families can be viewed either over \( \mathbb{R} \) or over \( \mathbb{C} \). Since we will ultimately be dealing with real geometry, the next lemma and proposition are referring to the real vector spaces \( V_{2,3}(\mathbb{K}) \) (although the proofs will in turn utilize the complex point of view). These are naturally equipped with a positive-definite scalar product, denoted in both cases by \( g \) and defined as follows:

\[
g : S^2 V_2(\mathbb{K}) \to \mathbb{R}
\]

\[
g(z \otimes X, z \otimes X) = \langle z, z \rangle \langle X, X \rangle
\]

for \( z \in \mathbb{C}, \ X \in \mathfrak{h}_3 \mathbb{K} \) and

\[
g : S^2 V_3(\mathbb{K}) \to \mathbb{R}
\]

\[
g(z \otimes X, z \otimes X) = \langle z, z \rangle \langle X, X \rangle
\]

for \( z \in \mathbb{C}, \ X \in \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \). This is the scalar product the word ‘orthogonal’ will be referring to. The spaces \( V_3(\mathbb{K}) \) are moreover left \( \mathbb{H} \)-modules, with the action of a quaternion \( q \in \mathbb{H} \) being \( L_q \in \text{End} V_3(\mathbb{K}) \), such that

\[
L_i = I, \ L_j = J, \ L_k = K, \ L_1 = \text{id},
\]

where \( I, J, K \) are the maps defined in point 3 of Proposition 3.

Observe that the problem is to turn the invariants of \( G_{2,3}(\mathbb{K}) \) into maps invariant under the action of the complex or quaternionic structure. This is fairly simple in the second family. The same task for the third family invariants is fulfilled with help of the following lemma, which produces an \( \mathbb{H} \)-equivariant tensor on \( \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \) from a symmetric tensor on \( \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \):

**Lemma 17.** Given a tensor \( t \in S^p \mathcal{F}(\mathfrak{h}_3 \mathbb{K})^* \), define the map

\[
t_L : S^p V_3(\mathbb{K}) \to S^p \mathbb{H}
\]
to be the dual, with respect to the natural scalar products, of the map
\[ t^*_L : S^p \mathbb{H} \otimes S^p V_3(\mathbb{K}) \to \mathbb{R} \]
where \( t^*_L (q^{\otimes p}, Z^{\otimes p}) = \tilde{t}((L_q Z)^{\otimes p}) \),
where \( \tilde{t} \in S^p V_3(\mathbb{K})^* \) is defined by
\[ \tilde{t}(Z^{\otimes p}) = \text{Re}(z^p) \cdot t(X^{\otimes p}) \]
for \( Z = z \otimes X \in V_3(\mathbb{K}) \), where \( z \in \mathbb{C} \) and \( X \in F(\mathfrak{h}_3 \mathbb{K}) \).

Then \( t_L \) is equivariant in the following sense:
\[ t_L \circ L_q = R^*_q \circ t_L \quad \forall q \in H. \]

Proof. Let \( x, w \in \mathbb{H} \) and \( Z = z \otimes X \in V_3(\mathbb{K}) \). We have
\[ \langle t_L((L_x Z)^{\otimes p}), w^{\otimes p} \rangle = t^*_L(w^{\otimes p}, (L_x Z)^{\otimes p}) \]
\[ = \tilde{t}((L_{wx} Z)^{\otimes p}) \]
\[ = t^*_L((wx)^{\otimes p}, Z^{\otimes p}) \]
\[ = \langle t_L(Z^{\otimes p}), (wx)^{\otimes p} \rangle, \]
where \( \langle \cdot, \cdot \rangle \) denotes the natural scalar product on \( S^p \mathbb{H} \) (induced by the one on \( \mathbb{H} \)).
Nondegeneracy of the latter implies
\[ t_L((L_x Z)^{\otimes p}) = R^*_x t_L(Z^{\otimes p}), \]
where \( R^*_x = R_x^\ast \) is dual to \( R_x \) with respect to the scalar product. \( \square \)

Finally, the invariants defining \( G(\mathbb{K}, \mathbb{K}') \) are given by the following

**Proposition 4.** Let us identify \( V(\mathbb{K}, \mathbb{K}') \) with \( V_n(\mathbb{K}) \), where \( n = 1, 2, 3 \) for, respectively, \( \mathbb{K}' = \mathbb{C}, \mathbb{H}, \mathbb{O} \), as in Corollary \( \mathbb{C} \) consider all of them as real vector spaces. Let \( c_1, c_2, c_3 \) be some nonzero constants (introduced to simplify further formulas). Then:

1. \( G(\mathbb{K}, \mathbb{C}) \) is the isotropy group of the tensor
\[ \Upsilon : S^3 V_1(\mathbb{K}) \to \mathbb{R} \]
\[ \Upsilon(X, X, X) = c_1 (X \times X, X) \]
for \( X \in V_1(\mathbb{K}) \).

2. \( G(\mathbb{K}, \mathbb{H}) \) is a connected component of the orthogonal isotropy group of the tensor
\[ \Xi : S^6 V_2(\mathbb{K}) \to \mathbb{R} \]
\[ \Xi(Z, Z, Z, Z, Z) = c_2 |z|^6 (\det X)^2 \]
for \( Z = z \otimes X \in V_2(\mathbb{K}) \), where \( z \in \mathbb{C} \) and \( X \in \mathfrak{h}_3 \mathbb{K} \).

3. \( G(\mathbb{K}, \mathbb{O}) \) is a connected component of the orthogonal isotropy group of the tensor
\[ \Omega : S^8 V_3(\mathbb{K}) \to \mathbb{R} \]
\[ \Omega(Z, Z, Z, Z, Z, Z, Z) = c_3 \| Q_L(Z, Z, Z, Z) \| ^2 \]
for \( Z \in V_3(\mathbb{K}) \), where \( \| \cdot \| \) is the natural norm on \( S^4 \mathbb{H} \) (induced by the scalar product) and \( Q_L \) is defined as in Lemma \( \mathbb{C} \).
CHAPTER 1. ALGEBRAIC PART

The rest of this section is occupied by the proof. We first need to set up some notation.

Remark 1. Consider on \( V_2(K) = \mathbb{C} \otimes h_2 K \) and \( V_3(K) = \mathbb{C} \otimes F(h_3 K) \) a complex structure given by multiplication by \( \sqrt{-1} \) in the \( \mathbb{C} \) factor. (corresponding on \( V_3(K) \) to \( I \) defined in point 3 of Proposition 3). In the following, it will be the distinguished complex structure on these spaces, the terms complex-linear and antilinear refer to.

Let us from now on assume \( n = 2, 3 \) and \( K \) to be fixed. The (complex) space of complex-valued oneforms on \( V_n(K) \) decomposes into the complex-linear and antilinear part, which we denote using, respectively, \( V^*_c \) and \( \bar{V}^*_c \):

\[
\mathbb{C} \otimes V^*_n(K) = V^*_c \oplus \bar{V}^*_c. \tag{1.14}
\]

One extends the decomposition to complex-valued alternating and symmetric forms, so that

\[
\Lambda_p^c V^*_n(K) = \bigoplus_{r+s=p} \Lambda^{r,s}
\]

\[
S_p^c V^*_n(K) = \bigoplus_{r+s=p} S^{r,s}
\]

where the elements of \( \Lambda^{r,s} \) (resp. \( S^{r,s} \)) are alternating (resp. symmetric) forms expressed as sums of expressions linear in \( r \) and antilinear in \( s \) arguments. We shall make use of the natural isomorphisms

\[
\Lambda^{r,s} \simeq \Lambda^{r,0} \otimes \Lambda^{0,s}
\]

\[
S^{r,s} \simeq S^{r,0} \otimes S^{0,s} \tag{1.15}
\]

defined as orthogonal projections in \( \otimes^{r+s}[\mathbb{C} \otimes V^*_n(K)] \).

The duals to \( V^*_c \) and \( \bar{V}^*_c \) are the genuinely complex spaces \( V_c \) and \( \bar{V}_c \), and there is a complex-linear isomorphism \( V_n(K) \simeq V_c \) and an anti-isomorphism \( \bar{\cdot} : V_c \to \bar{V}_c \), i.e. complex conjugation. One can view \( V_c \) (resp. \( \bar{V}_c \)) as the space of linear (resp. antilinear) maps from \( \mathbb{C} \) to \( V_n(K) \).

The hermitian inner product on \( V_n(K) \) is represented by a tensor \( h \in \bar{V}_c^* \otimes V_c^* \), inverse to \( h^{-1} \in V_c \otimes \bar{V}_c \):

\[
h(Z, Z') = \bar{z}z' \cdot \langle X, X' \rangle \quad \text{for} \quad Z = z \otimes X, \ Z' = z' \otimes X'
\]

and defines isomorphisms \( V_c \simeq \bar{V}_c^* \) and \( V_c^* \simeq \bar{V}_c \). We now introduce the abstract index notation with lower latin indices \( a, b, \ldots \) indexing copies of \( \mathbb{C} \otimes V^*_n(K) \) and unbarred and barred greek indices, resp. \( \alpha, \beta, \ldots \) and \( \bar{\alpha}, \bar{\beta}, \ldots \), indexing copies of resp. \( V_c^* \) and \( \bar{V}_c^* \). Upper indices of the same kind naturally index dual spaces. Greek indices corresponding to latin ones denote subspaces. In this manner, the real scalar product \( g \) defined previously, is

\[
g_{ab} \in S_{ab}^{1,1}, \quad g_{ab} = h_{\alpha\beta} + h_{\bar{\alpha}\bar{\beta}}.
\]

Via the scalar product, we have \( \mathfrak{so}(V_n(K)) \simeq \Lambda^2 V_n^*(K) \), so that

\[
\mathfrak{so}_c(V_n(K)) \simeq \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}.
\]
In particular, $\text{Re}\Lambda^{1,1} \simeq \mathfrak{u}(V_3(\mathbb{K}))$. The hermitian product $h_{\alpha\beta}$ and the inverse $h^{\alpha\beta}$ are used to lower and raise indices. The generator of $\mathfrak{u}(1)$ corresponding to the complex structure is

$$\theta_{ab} = i h_{\alpha\beta} - i h_{\beta\alpha} \in \mathfrak{u}(1) \subset \text{Re}\Lambda^{1,1}.$$ 

On $V_3(\mathbb{K})$ there is moreover a distinguished two-form, denoted by slight abuse of notation by

$$\omega_{\alpha\beta} + \bar{\omega}_{\alpha\beta} \in \text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2}),$$

where $\omega_{\alpha\beta} \in \Lambda^{2,0}$ is a linear extension of the symplectic form $\omega$ on $F(\mathfrak{h}_3\mathbb{K})$:

$$\omega(Z, Z') = z z' \omega(X, X')$$

for $Z = z \otimes X$, $Z' = z' \otimes X'$, $z, z' \in \mathbb{C}$, and $X, X' \in F(\mathfrak{h}_3\mathbb{K})$. The subgroup of $U(V_3(\mathbb{K}))$ preserving $\omega$ is $\text{Sp}(V_3(\mathbb{K}), \omega)$, with the Lie algebra $\mathfrak{sp}(V_3(\mathbb{K}), \omega) \subset \text{Re}\Lambda^{1,1}$ consisting of maps $E_{ab} = b_{\alpha\beta} + b_{\beta\alpha}$ such that $b^\mu_{[\alpha\beta]} = 0$ and $b_{\alpha\beta} = -b_{\beta\alpha}$.

In general, $\omega$ defines an isomorphism

$$\Lambda^{1,1} \rightarrow V^*_c \otimes V^*_c = S^{2,0} \oplus \Lambda^{0,2}$$

$$\Lambda^{1,1}_{ab} \ni b_{\alpha\beta} - b_{\beta\alpha} \mapsto b^\gamma_{\gamma\mu} \omega_{\mu \lambda} \in V^*_c \otimes V^*_c,$$  \hspace{1cm} (1.16)

such that

$$\mathbb{C} \otimes \mathfrak{sp}(V_3(\mathbb{K}), \omega) \simeq S^{2,0}.$$ 

The complex structures of Proposition 3 are $I = \theta \in \mathfrak{u}(1)$ and $J, K \in \text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2})$:

$$I_{ab} = \theta_{ab}, \quad J_{ab} = \omega_{\alpha\beta} + \bar{\omega}_{\alpha\beta}, \quad K_{ab} = -i \omega_{\alpha\beta} + i \bar{\omega}_{\alpha\beta},$$

and $\omega$ itself satisfies $\omega_{\alpha\mu} \bar{\omega}^{\mu \beta} = -h_{\beta\alpha}$. These three generate the $\mathfrak{sp}(1)$.

We can now start proving the Proposition.

**Proof of points 1 & 2 of Proposition 4**

1. In the first family, the group $G(\mathbb{K}, \mathbb{C})$ is simply $\text{Aut}\mathfrak{h}_3\mathbb{K}$ acting in the natural way on $V_1(\mathbb{K}) = \mathfrak{h}_3\mathbb{K}$. Now, the automorphisms of $\mathfrak{h}_3\mathbb{K} = \mathbb{R} \oplus \mathfrak{h}_3\mathbb{K}$ automatically preserve the trace, the latter decomposition and the scalar product on both $\mathfrak{h}_3\mathbb{K}$ and $\mathfrak{h}_3\mathbb{K}$. They moreover act faithfully on the latter space. It thus follows that they are the isotropy group of the cubic form $X \mapsto \langle X, X \circ X \rangle$ on $\mathfrak{h}_3\mathbb{K}$, or – equivalently – of the cubic form $X \mapsto \langle X, X \times X \rangle$ on $\mathfrak{h}_3\mathbb{K}$. But the latter is, up to the constant $c_2 \neq 0$, the tensor $\Upsilon$.

2. In the second family, we consider a rescaled linear extension of the determinant,

$$\Lambda_{\alpha\beta\gamma} \in S^{3,0}_{abc}, \quad \Lambda(Z, Z, Z) = c_2 z^3 \text{det} X$$

and its complex conjugate, the antilinear cubic

$$\bar{\Lambda}_{\alpha\beta\gamma} \in S^{0,3}_{abc}, \quad \bar{\Lambda}(Z, Z, Z) = c_2 z^3 \text{det} X.$$
for $Z = z \otimes X \in V_2(K) = C \otimes h_3K$. Our tensor $\Xi$ is then
\[
\Xi = \Lambda \cdot \bar{\Lambda} \in \text{Re} S^{3,3},
\]
projecting under the isomorphism (1.15) onto a multiple of
\[
\Lambda_{\alpha\beta\gamma} \bar{\Lambda}_{\delta\epsilon\phi} \in S_{\text{abcd}}^{3,0} \otimes S_{\text{def}}^{0,3}
\]
by a combinatorial factor. Let now $E \in \mathfrak{o}(V_2(K))$ be given by
\[
E = A + \bar{A} + B, \quad A \in \Lambda^{2,0}, \quad B \in \text{Re}\Lambda^{1,1}.
\]
Then
\[
(A + \bar{A})(\Xi) \in S^{4,2} \oplus S^{2,4},
\]
\[
B(\Xi) \in S^{3,3}
\]
and as such must both vanish independently if $E$ is to preserve $\Xi$.

The projection of $(A + \bar{A})(\Xi)$ onto $S_{\text{abcd}}^{3,0} \otimes S_{\text{def}}^{0,2}$ is proportional to
\[
\Lambda_{(\alpha\beta\gamma} A_{\delta)} \bar{\Lambda}_{\mu\nu\phi},
\]
vanishing iff $A = 0$, since
\[
\Lambda(X, \cdot, \cdot, \cdot) = 0 \iff L^*_X = 0 \iff X = 0.
\]
The projection of $B(\Xi)$ onto $S^{3,0} \otimes S^{0,3}$ is proportional to
\[
B(\Lambda) \otimes \bar{\Lambda} + \Lambda \otimes \overline{B(\Lambda)},
\]
vanishing iff $B(\Lambda) = \lambda \Lambda$ with $\lambda \in i\mathbb{R}$. Thus, if $E$ preserves $\Xi$, then
\[
B^0_{ab} = B_{ab} - i\lambda(h_{\beta\delta} - h_{\alpha\beta}) = B_{ab} - \lambda\theta_{ab}
\]
must preserve $\Lambda$. As the latter is the linear extension of the determinant, it follows that $B^0 \in \text{C} \otimes \text{str}_0(h_3K)$, and finally
\[
B = B^0 + \lambda\theta \in G'_{2}(K) \oplus u(1).
\]
Thus $E$ preserves $\Xi$ iff $A = 0$ and $B \in g(K, \mathbb{H})$, which proves point 2 of the Proposition.

Proving the last point is considerably more involved. We consider a rescaled linear extension of the quartic on $\mathcal{F}(h_3K)$,
\[
q_{\alpha\beta\gamma\delta} \in S_{\text{abcd}}^{4,0} \quad q(Z, Z, Z, Z) = c_3z^4Q(X, X, X, X)
\]
and its complex conjugate, the antilinear quartic
\[
\bar{q}_{\alpha\beta\gamma\delta} \in S_{\text{abcd}}^{0,4} \quad \bar{q}(Z, Z, Z, Z) = c_3\bar{z}^4\overline{Q}(X, X, X, X)
\]
for $Z = z \otimes X \in V_3(K) = \mathbb{C} \otimes h_3K$.

As the tensor $\bar{U}$ has been defined in a rather obscure form, we need some more convenient formula. Note first, that, being a symmetric rank eight tensor, invariant with respect to $I$, it must be in $S^{4,4}$ (invariance follows from Lemma 17). We now have the following lemma, to be proved in the next section:
1.4. CONSTRUCTING SYMMETRIC INVARIANTS

Lemma 18. The projection of $\mathcal{U} \in S^{4,4}$ onto $S^{4,0} \otimes S^{0,4}$ is

$$\mathcal{U}|_{S^{4,0} \otimes S^{0,4}} = \frac{256}{70} (1 \otimes J^*) P_{44}(q \otimes q),$$

where $P_{44} : S^{4,0} \otimes S^{4,0} \rightarrow S^{4,0} \otimes S^{4,0}$ is the projection

$$(P_{44})_{\alpha \beta \gamma \delta \epsilon \phi \kappa \lambda}^{\xi \eta \zeta \tau} = \delta_{[\alpha}^{\xi} \delta_{\beta]}^{\eta} \delta_{[\gamma}^{\zeta} \delta_{\delta]}^{\tau} \delta_{[\epsilon}^{\phi} \delta_{\phi]}^{\kappa} \delta_{[\delta}^{\lambda} \delta_{\lambda]}^{\tau}$$
on the subspace corresponding to the Young diagram with two rows of four boxes each, and $J^*$ denotes the map taking $t_{\alpha \beta \gamma \delta} \in S^{4,0}_{abcd}$ to $\bar{\omega}^{\mu} \bar{\omega}^{\nu} \bar{\omega}^{\rho} \bar{\omega}^{\sigma} \chi_{\mu \nu \rho \sigma} \in S^{0,4}_{ijkl}.$

From now on, we define

$$\kappa = \dim \mathcal{K} \quad \text{and} \quad c_3 = 24 \sqrt{\frac{1}{\kappa + 3}}$$
to claim the following:

Lemma 19.

1. Introducing $\chi = \frac{\kappa + 2}{\sqrt{\kappa + 3}}$ and $N = \dim V_c = 6\kappa + 8$, the tensor $q$ satisfies the following identities:

$$\begin{align*}
q_{\alpha \beta \mu \nu} \bar{\omega}^{\mu \nu} &= 0 \\
q_{\mu \nu \rho \sigma} \bar{\omega}^{\mu} \bar{\omega}^{\nu} \bar{\omega}^{\rho} \bar{\omega}^{\sigma} &= \tilde{q}^{\alpha \beta \gamma \delta} \\
q_{\alpha \mu \nu \rho} q^{3 \mu \nu \rho} &= \frac{N + 1}{2} \delta_{\alpha}^{\beta} \\
q_{\alpha \mu \nu} q_{\beta \gamma \delta} &= \frac{1}{2} [\delta^{\gamma \delta}_{\alpha \beta} + \delta^{\delta \gamma}_{\alpha \beta}] + \chi \ q_{\alpha \beta \mu \nu} \bar{\omega}^{\mu} \bar{\omega}^{\nu}.
\end{align*}$$

2. Let us introduce a map

$$D_q : S^{2,0} \rightarrow S^{2,0}$$

$S^{2,0}_{ab} \ni b_{\alpha \beta} \mapsto D_q(b)_{\alpha \beta} = b_{\mu \nu} \tilde{q}^{\mu \nu \rho \sigma} \omega_{\rho \sigma} \bar{\omega}^{\mu} \bar{\omega}^{\nu} \in S^{2,0}_{ab}.$

Then, with respect to the decomposition

$$S^{2,0} \simeq \mathbb{C} \otimes \mathfrak{sp}(V_3(\mathbb{K}), \omega) = \mathbb{C} \otimes [G_3'(\mathbb{K}) \oplus G_4'(\mathbb{K})],$$

the map $D_q$ is given by:

$$\begin{align*}
D_q|_{G_3'(\mathbb{K})} &= -\sqrt{\kappa + 3} \\
D_q|_{G_4'(\mathbb{K})} &= \sqrt{\frac{1}{\kappa + 3}}.
\end{align*}$$

It also satisfies:

$$D_q^2 = 1 - \chi D_q.$$
Note that the term \( P_{44}(q \otimes q) \) in our expression for the \( S^{4,0} \otimes S^{0,4} \) component of \( \mathcal{U} \) is itself an element of \( S^{4,0} \otimes S^{4,0} \), i.e. it is completely linear. It is then convenient to consider only the End\( V_c \) component of elements of \( u(V_3(K)) \), so that a map \( F_{\alpha\beta} = f^{\alpha\beta} + \bar{f}^{\alpha\beta} \in u(V_3(K))^\alpha \beta \) is represented by \( f^{\alpha\beta} \in u(V_c)^{\alpha\beta} \subset (\text{End}V_c)^{\alpha\beta} \). (the isomorphism \( V_3(K) \simeq V_c \) is implicit).
The algebra \( u(V_c) \) decomposes into the symplectic subalgebra and its complement, such that
\[
\text{sp}_{c}(V_c, \omega) \oplus \text{sp}_{c}'(V_c, \omega) \simeq S^{2,0} \oplus \Lambda^{2,0},
\]
where the isomorphism is given by \( \omega_{\alpha\beta} \) as in (1.16). In the following we will omit indicating the complexifications of algebras explicitly – it is clear that the final stabilizer is an intersection of a complexified one with the real algebra \( \text{so}(V_3(K)) \).

**Lemma 20.** The stabilizer algebra of \( \mathcal{U} \) in \( \text{su}(V_c) \) is \( \mathcal{G}'_3(K) \).

**Proof.** Let \( E = A + B + C \) with \( A \in \mathcal{G}'_3(K) \), \( B \in \mathcal{G}'_3^{-}(K) \subset \text{sp}(V_c) \) and \( C \in \text{sp}^{+}(V_c) \subset \text{su}(V_c) \). Clearly, \([A, J] = [B, J] = 0\) and \( CJ + JC = 0 \). Now, according to the previous lemma, we have
\[
E(\mathcal{U}) = 0 \iff E((1 \otimes J^*)P_{44}(q \otimes q)) = 0.
\]
Using the properties of \( A, B \) and \( C \), we have
\[
(1 \otimes J^*) \circ [(A + B + C) \otimes 1 + 1 \otimes (A + B - C)] \circ P_{44}(q \otimes q) = 0,
\]
which under the action of \( 1 \otimes J^{-1} \) yields
\[
[(A + B + C) \otimes 1 + 1 \otimes (A + B - C)]P_{44}(q \otimes q) = 0.
\]
However terms involving \( A \) shall obviously vanish, it is instructive to keep them for the sake of verification. We shall contract the latter expression with two copies of \( q \):
\[
[(A + B + C) \otimes 1 + 1 \otimes (A + B - C)]P_{44}(q \otimes q)_{\alpha\beta\gamma\delta\mu \nu \rho \sigma} q^\ell_{\beta\gamma\delta} q^\ell_{\mu \nu \rho \sigma} = 0.
\]
In order to simplify bookkeeping, we introduce the following maps:
\[
\Phi, \Psi : \text{su}(V_c) \to \text{su}(V_c)
\]
\[
\Phi(F)_{\xi} \alpha = \{(F \otimes 1)P_{44}(q \otimes q)\}_{\alpha\beta\gamma\delta\mu \nu \rho \sigma} q^\ell_{\beta\gamma\delta} q^\ell_{\mu \nu \rho \sigma}
\]
\[
\Psi(F)_{\xi} \alpha = \{(1 \otimes F)P_{44}(q \otimes q)\}_{\alpha\beta\gamma\delta\mu \nu \rho \sigma} q^\ell_{\beta\gamma\delta} q^\ell_{\mu \nu \rho \sigma},
\]
so that our condition becomes \( \Phi(A + B + C) + \Psi(A + B - C) = 0 \).

Expanding the projection \( P_{44} \), and utilising the symmetry of \( q \otimes q \), we have (for some irrelevant combinatorial constant \( c' \neq 0 \)):
\[
c' \Phi(F)_{\xi} \alpha = 2 q_{(\alpha\beta\gamma\delta)q_{\mu \nu \rho \sigma}} q^\ell_{\beta\gamma\delta} q^\ell_{\mu \nu \rho \sigma}
- 8 \left[ \frac{3}{4} q_{(\mu\alpha\beta\gamma\delta)q_{\rho \sigma}} + \frac{1}{4} q_{(\mu\alpha\gamma\beta\delta)q_{\rho \sigma}} q_{\mu \nu \rho \sigma} \right] q^\ell_{\beta\gamma\delta} q^\ell_{\mu \nu \rho \sigma}
+ 6 q_{(\mu\nu\alpha\beta\gamma\delta)q_{\rho \sigma}} q^\ell_{\beta\gamma\delta} q^\ell_{\mu \nu \rho \sigma}
\]
We will show that both $\Phi(F)$ and $\Psi(F)$ are linear combinations of $F$ and $D_q(F)$, where the domain of $D_q$ is trivially extended onto entire $\mathfrak{su}(V_c)$ (so that $\mathfrak{sp}^+(V_c)$ is its kernel). To this end we first get rid of the symmetrizers:

$$
c' \Phi(F)^\xi_\alpha = 2 \left[ \frac{3}{4} \eta_{\mu\alpha\beta\gamma} F^\eta_{\delta\mu\rho\sigma} + \frac{1}{4} \eta_{\mu\beta\gamma\alpha} F^\eta_{\delta\mu\rho\sigma} \right] \xi^{\beta\gamma\delta} q^{\mu\rho\sigma},
$$

and then perform contractions:

$$
c' \Phi(F)^\xi_\alpha = \frac{N(N+1)}{2} \left[ 3 \varphi(F)_\alpha + \frac{N+1}{2} F^\xi_\alpha \right] - \frac{3N+1}{2} \varphi(F)^\xi_\alpha - \frac{3N+1}{2} \varphi(F)^\xi_\alpha + \frac{3}{2}(1+\chi^2) \left[ \mathcal{D}_q(F)^\xi_\alpha + \chi \mathcal{D}_q(F)^\xi_\alpha \right] + \frac{3}{2}(1+\chi^2) \frac{N+1}{2} F^\xi_\alpha + 3 \chi \left[ \frac{1}{2} + \chi^2 \right] D_q(F)^\xi_\alpha + \frac{\chi}{2} (F^\eta_\alpha + F^\mu_{\mu\delta} \delta^\xi_\alpha),
$$

where in the last lines we used the following identities:

$$
q_{\alpha\beta\gamma\delta} q^{\mu\nu\rho\sigma} q_{\rho\gamma\delta} = (1+\chi^2) q_{\alpha\beta\gamma\delta} + \frac{\chi}{2} (\omega_{\alpha\gamma} \omega_{\beta\delta} + \omega_{\alpha\delta} \omega_{\beta\gamma})
$$

and we have introduced

$$
\varphi(F)^\xi_\alpha = F^\eta_\nu q_{\alpha\mu\rho\sigma} q^{\xi\nu\rho\sigma} = \frac{1}{2} F^\xi_\alpha + \frac{1}{2} F^{\mu}_{\mu\delta} \delta^\xi_\alpha + \chi q_{\alpha\mu\rho\sigma} \omega^\xi_\alpha \omega_{\mu\nu} F^\eta_{\nu},
$$

so that $\varphi(F) = \frac{1}{2} F + \chi \mathcal{D}_q(F)$, since we assume $F$ to be traceless (note also that the last term is indeed zero for $F \in \mathfrak{sp}^+(V_c)$). We perform the same operations...
for $\Psi$: expand symmetrizers,

$$c' \Psi(F)_{\alpha}^{\xi} = 2 q_{\rho\sigma\gamma\delta} F_{\rho}^{\sigma} q_{\alpha\beta\gamma\delta} q_{\mu\nu\rho\sigma} d_{\mu\nu\rho\sigma}$$

$$- 8 \left[ \frac{3}{4} q_{\rho\sigma\gamma\delta} F_{\sigma}^{\rho} q_{\alpha\beta\gamma\delta} + \frac{3}{4} q_{\rho\sigma\gamma\delta} F_{\sigma}^{\rho} q_{\alpha\beta\gamma\delta} \right]$$

$$+ \frac{1}{4} \left[ \frac{3}{4} q_{\rho\sigma\gamma\delta} F_{\sigma}^{\rho} q_{\alpha\beta\gamma\delta} + \frac{3}{4} q_{\rho\sigma\gamma\delta} F_{\sigma}^{\rho} q_{\alpha\beta\gamma\delta} \right]$$

$$+ 6 \left[ q_{\rho\sigma\gamma\delta} F_{\sigma}^{\rho} q_{\alpha\beta\gamma\delta} + \frac{1}{2} q_{\rho\sigma\gamma\delta} F_{\sigma}^{\rho} q_{\alpha\beta\gamma\delta} \right]$$

and contract $q$:

$$c' \Psi(F)_{\alpha}^{\xi} = 2 \left( \frac{N + 1}{2} \right)^2 F_{\mu}^{\sigma} d_{\mu}^{\sigma}$$

$$- 3 \left( \frac{N + 1}{2} \right)^2 F_{\xi}^{\alpha} - 3 \frac{N + 1}{2} \varphi(F)_{\alpha}^{\xi}$$

$$+ 3 \left[ (1 + \chi^2) \varphi(F)_{\alpha}^{\xi} - \frac{1}{2} D_q(F)_{\alpha}^{\xi} \right]$$

$$+ 3 \varphi \left[ \left( \frac{N + 1}{2} \right) D_q(F)_{\alpha}^{\xi} - \frac{1}{2} \varphi(F)_{\alpha}^{\xi} \right]$$

$$+ 3 (1 + \chi^2) D_q(F)_{\alpha}^{\xi} - \frac{1}{2} \varphi(F)_{\alpha}^{\xi}$$

where in the last line we used one more identity:

$$q_{\gamma\alpha\lambda\omega} q_{\beta\mu\nu} q_{\mu\nu\delta\xi} q_{\beta\delta\gamma\xi} = \left( \frac{1}{2} - \chi^2 \right) q_{\gamma\alpha\lambda\omega} q_{\beta\mu\nu} q_{\mu\nu\delta\xi} q_{\beta\delta\gamma\xi}$$

and we have introduced a map

$$\sigma(F) = J^{-1} F J,$$

so that $\sigma|_{sp(V_c)} = id$ and $\sigma|_{sp^\perp(V_c)} = -id$. The square of $\varphi$, appearing in the second line, reads

$$\varphi^2(F) = \left( \frac{1}{4} + \chi^2 \right) F + \chi(1 - \chi^2) D_q(F).$$

We thus have:

$$c' \Phi(F) = (1 - N + N^2 + 3\chi^2) \left\{ \frac{N + 1}{4} F + \frac{3\chi}{2} D_q(F) \right\}$$

$$c' \Psi(F) = \frac{8 - 2N - N^2 - 3\chi^2 (\sigma + 2)}{8} F + \frac{3\chi}{2} (3\chi^2 - N - 1) D_q(F)$$

Using $D_q(A) = \lambda_+, D_q(B) = \lambda_-, D_q(C) = 0$ and $\sigma(A) = A$, $\sigma(B) = B$, $\sigma(C) = -C$, and substituting $N, \lambda_+, \chi$ in

$$\Phi(A + B + C) + \Psi(A + B - C) = 0,$$
we obtain

\[ f_1(\kappa)B + f_2(\kappa)C = 0, \]

where \( f_{1,2}(\kappa) \) are nonzero for \( \kappa = 1, 2, 4, 8 \) (note that terms involving \( A \) vanish as expected). Thus

\[ E(\mathcal{U}) = 0 \Rightarrow E \in \mathcal{G}'_3(K). \]

The converse is obviously true from the definition of \( \mathcal{U} \).

\[ \square \]

**Lemma 21.** The subspace of \( u^\perp(V_3(K)) \subset \mathfrak{so}(V_3(K)) \) stabilizing \( \mathcal{U} \) is spanned by \( J \) and \( K \).

**Proof.** We shall repeat the procedure applied in the previous proof. Let

\[ E(\mathcal{U}) = 0 = \Rightarrow E \in \mathcal{G}'_3(K). \]

The converse is obviously true from the definition of \( \mathcal{U} \).

\[ \square \]
and performing contractions gives:

\[
0 = \frac{2}{5} N(N + 1) F^\delta_\zeta + \frac{8}{5} \left( \frac{N + 1}{2} \right)^2 F^\delta_\zeta \\
- \frac{6}{5} \left( \frac{N + 1}{2} \right)^2 F^\delta_\zeta - \frac{6}{5} N + 1 \varphi(F)^\delta_\zeta \\
- \frac{18}{5} \left[ \frac{1}{2} N + 1 \right] F^\delta_\zeta + \frac{1}{2} \varphi(F)^\delta_\zeta \\
- \frac{18}{5} \left( \frac{1}{2} - \chi^2 \right) D_\eta(F)^\delta_\zeta - \frac{\chi}{2} (\sigma(F)^\delta_\zeta - F^\mu_\mu \delta^\delta_\zeta) \\
- \frac{2}{5} \left( \frac{N + 1}{2} \right)^2 F^\delta_\zeta - \frac{2}{5} N + 1 \varphi(F)^\delta_\zeta \\
+ \frac{6}{5} (1 + \chi^2) N + 1 \varphi(F)^\delta_\zeta + \frac{12}{5} \varphi^2(F)^\delta_\zeta \\
+ \frac{12}{5} \varphi(F)^\delta_\zeta + \frac{12}{5} \chi \left[ \left( - \frac{1}{2} + \chi^2 \right) D_\eta(F)^\delta_\zeta + \frac{\chi}{2} (F^\delta_\zeta + F^\mu_\mu \delta^\delta_\zeta) \right],
\]

where we used the identities and maps introduced in the previous proof, the latter understood as mapping \(\text{End}(V_\zeta)\) to itself, namely:

\[
D_\eta(F)^\mu_\nu = F^\xi_\eta \mu [\xi, \lambda] \omega^{\nu \lambda} \\
\varphi(F)^\mu_\nu = \frac{1}{2} F^\mu_\nu + \frac{1}{2} F^\alpha_\alpha \delta^\mu_\nu + \chi D_\eta(F)^\mu_\nu \\
\sigma(F)^\mu_\nu = F^\xi_\eta \omega^{\mu \eta} \omega^\nu_\zeta.
\]

Collecting all terms gives:

\[
0 = \left[ 2N^2 + (6\chi^2 - 7)N + 6\chi(3\chi + 1)(1 + \sigma) \right] F \\
+ 12\chi(N + 1 - 3\chi^2) D_\eta(F) \\
- (8N + 5 + 6\chi^2) (\text{tr}F) \text{id}.
\]

Recalling the eigenvalues of the (commuting) operators \(D_\eta\) and \(\sigma\), one finds that the latter equation is satisfied only for

\[
F^\mu_\nu = \lambda \delta^\mu_\nu, \quad \lambda \in \mathbb{C}.
\]

Hence \(E\) is a linear combination of \(J\) and \(K\).

\[\square\]

At last, we arrive at the final step:

*Proof of point 3 of Proposition* Let us consider \(E \in \mathfrak{so}(\mathcal{V}_3(\mathbb{K}))\). Then \(E = A + B\) with \(A \in \mathfrak{u}(\mathcal{V}_3(\mathbb{K}))\) and \(B \in \mathfrak{u}^+(\mathcal{V}_3(\mathbb{K}))\), and the components map \(\mathfrak{u}\) into independent subspaces:

\[
A(\mathfrak{u}) \in S^{4,4} \quad \text{and} \quad B(\mathfrak{u}) \in S^{5,3} \oplus S^{3,5}.
\]

Thus \(E(\mathfrak{u}) = 0 \iff A(\mathfrak{u}) = 0 \& B(\mathfrak{u}) = 0\). Let now \(A = A_0 + aI\) with \(A_0 \in \mathfrak{su}(\mathcal{V}_3(\mathbb{K}))\) and \(a \in \mathbb{R}\). Clearly, \(I(\mathfrak{u}) = 0\), so that \(A(\mathfrak{u}) = 0 \iff A_0(\mathfrak{u}) = 0\). We can now apply the lemmas to \(A_0\) and \(B\) to find that

\[
A_0 \in G^{2}_3(\mathbb{K}) \quad \text{and} \quad B \in \mathfrak{sp}(1) \cap \mathfrak{u}^+(\mathcal{V}_3(\mathbb{K})),
\]

and finally \(E \in \mathfrak{g}(\mathbb{K}, \mathfrak{O})\) as claimed.

\[\square\]
1.5. Important identities and proofs of Lemmas 18 & 19

The invariants of $G_{\sigma}(K)$ are essentially derived from certain product maps: the traceless product $\times$, Freudenthal product $\bullet$ and $\tau$, the triple product on a FTS. As usually, finite dimension of the spaces the maps operate on leads to some characteristic equations; these in turn are translated to certain identities the tensors $\Upsilon$, $\Lambda$ and $q$ satisfy under contraction (the constants $c_1$, $c_2$ and $c_3$ of Proposition 4 have been introduced in order to simplify these identities).

1.5.1 First family

Lemma 22. Let $L^X_X : Y \mapsto X \times Y$ denote the (left) multiplication map for $X, Y \in \mathfrak{sh}_3 K$. Then:

\begin{align*}
\text{tr } L^X_X &= 0 \quad (1.17) \\
\text{tr } L^X_X L^X_X &= \frac{3 \dim K + 4}{12} \langle X, X \rangle \quad (1.18) \\
\text{tr } L^X_X L^X_X L^X_X &= -\frac{\dim K}{8} \langle X, X \times X \rangle \quad (1.19) \\
\langle X \times X, X \times X \rangle &= \frac{1}{6} \langle X, X \rangle^2 \quad (1.20)
\end{align*}

for $X \in \mathfrak{sh}_3 K$.

Proof. Identities (1.17) and (1.18) follow from irreducibility of $\mathfrak{sh}_3 K$ as a representation of $\text{Aut}(\mathfrak{sh}_3 K)$, where in the second case we employ Schur’s lemma and check the formula for some simple $X$ to find the proportionality constant.

To prove (1.20) we recall Lemma 10, substituting $T(X), S(X, X)$ and $N(X, X, X)$:

\begin{equation*}
X^3 - \frac{1}{2} (\text{tr} X^2) X - \frac{1}{3} \text{tr} X^3 = 0,
\end{equation*}

where we used $\text{tr} X = 0$. Taking the scalar product of this expression with $X$ and using symmetry of $L^X_X$ yields

\begin{equation*}
\langle X^2, X^2 \rangle = \frac{1}{2} \langle X, X \rangle^2.
\end{equation*}

Now,

\begin{align*}
\langle X \times X, X \times X \rangle &= \langle X^2 - \frac{1}{3} \text{tr} X^2, X^2 - \frac{1}{3} \text{tr} X^2 \rangle \\
&= \langle X^2, X^2 \rangle - \frac{1}{3} \langle X, X \rangle^2 \\
&= \frac{1}{6} \langle X, X \rangle^2.
\end{align*}

Using $i, j, k, \ldots$ to index $V_1(K) \simeq V^*_1(K)$, with the identification given by $(\cdot, \cdot)$, we introduce the structure constants of $\times$:

\begin{equation*}
(X \times Y)_i = f_{ijk} X^j Y^k.
\end{equation*}
With \( g_{ij}X^iY^j = \langle X, Y \rangle \), identities (1.17, 1.18, 1.20) read:

\[
\begin{align*}
&f_{imm} = 0, \quad f_{imn}f_{jmn} = \frac{3 \dim K + 4}{12}g_{ij}, \quad f_{m(ij}f_{kl)m} = \frac{1}{6}g_{(ij}g_{kl)}.
\end{align*}
\]

Contraction of the latter one with \( c_{kln} \) yields:

\[
\begin{align*}
&\frac{3 \dim K + 4}{24}f_{ijn} + f_{ipq}f_{jqr}f_{nrp} = \frac{1}{6}f_{ijn}, \\
&f_{ipq}f_{jqr}f_{nrp} = -\frac{3}{6\kappa + 8}f_{ijn},
\end{align*}
\]

proving (1.19).

**Corollary 4.** Let us set \( c_1 = \sqrt{\frac{12}{\kappa + 1}} \), where \( \kappa = \det K \). Using \( i, j, k, \ldots \) to index \( V_1(K) \simeq V_1^*(K) \), with the identification given by \( \langle \cdot, \cdot \rangle \), we have:

\[
\begin{align*}
&\Upsilon^{m}{}_{i}{}^{m} = 0, \\
&\Upsilon^{m}{}_{i}{}^{m}\Upsilon^{m}{}_{j}{}^{m} = g_{ij}, \\
&\Upsilon^{i}{}^{p}{}^{r}\Upsilon^{q}{}^{r}{}^{p}{}^{q}{}^{r} = -\frac{3\kappa}{6\kappa + 8}\Upsilon^{i}{}^{j}{}^{k}, \\
&\Upsilon^{(i}{}^{j}{}^{m}{}^{r}{}^{p}{}^{m}{}^{r}{}^{q}{}^{m}{}^{n} = \frac{10}{3\kappa + 4}g_{(ij}g_{kl)},
\end{align*}
\]

where \( g(X, Y) = \langle X, Y \rangle \).

Notably, the last identity is, up to a proportionality constant, the one used by Nurowski in [2] to define his cubic invariant (without referring to all the Jordan machinery).

**Lemma 23.** Let us introduce a map

\[
\mathcal{D}_\Upsilon : \Lambda^2 V_1(K) \to \Lambda^2 V_1(K)
\]

\[
(\Lambda^2 V_1(K))_{ij} \ni E_{ij} \mapsto E_{pq}\Upsilon_{iqm}\Upsilon_{jpm} \in (\Lambda^2 V_1(K))_{ij}.
\]

Then, with respect to the decomposition

\[
\Lambda^2 V_1(K) \simeq \mathfrak{so}(V_1(K)) = \mathcal{G}'_1(K) \oplus \mathcal{G}'_1(K)^\perp,
\]

the map \( \mathcal{D}_\Upsilon \) is given by:

\[
\begin{align*}
\mathcal{D}_\Upsilon|_{\mathcal{G}'_1(K)} &= -\frac{1}{2}, \\
\mathcal{D}_\Upsilon|_{\mathcal{G}'_1(K)^\perp} &= \frac{3}{3\kappa + 4}.
\end{align*}
\]

**Proof.** Let us first consider the familiar map into derivations (Lemma 4):

\[
\Lambda^2 V_1(K) \ni X \wedge Y \mapsto \mathcal{D}_{X,Y} = [L_X, L_Y] \in \mathcal{G}'_1(K) \subset \Lambda^2 V_1(K).
\]

We have

\[
(X^p Y^q - Y^p X^q)\Upsilon_{iqm}\Upsilon_{jpm} = -c_1^2[L_X^i, L_Y^j].
\]

(1.21)
and

\[ [L_X^*, L_Y^*] = [L_X, L_Y] - \frac{1}{3} X \wedge Y. \]

Since the map (1.21) is symmetric and its image is \( G'_1(\mathbb{K}) = \text{der}_{\text{sh}}^3 \mathbb{K} \), it follows that it vanishes on \( \mathcal{G}'_1(\mathbb{K})^\perp \). It then follows from equivariance, that its action on \( \mathcal{G}'_1(\mathbb{K}) \) is a multiple of identity. We thus have:

\[ \mathcal{D} \Upsilon |_{\mathcal{G}'_1(\mathbb{K})^\perp} = \frac{c_2}{13}, \quad \mathcal{D} \Upsilon |_{\mathcal{G}'_1(\mathbb{K})} = \alpha \]

for some constant \( \alpha \). Then, demanding

\[ \text{tr} \mathcal{D} \Upsilon = \Upsilon \sum_{m} \sum_{p} \delta_{[p]}^{[m]} \frac{N}{2} \]

and substituting the dimensions of \( \text{der}_{\text{sh}}^3 \mathbb{K} \), one finds \( \alpha = -\frac{c_2}{13} (3 \kappa + 4) \), and the lemma follows.

1.5.2 Second family

It is convenient to consider the following general result, which shall prove useful in both second and third family. The idea is to use some abstract real spaces \( W, \tilde{W} \simeq h^3 \mathbb{K} \) instead of \( V_c, \bar{V}_c \) (recall Remark [1]):

**Lemma 24.** Let us introduce two spaces \( W, \tilde{W} \simeq h^3 \mathbb{K} \) and maps

\[ l_X^* : W \to \tilde{W} \quad \text{for} \quad X \in W \]

\[ \tilde{l}_X^* : \tilde{W} \to W \quad \text{for} \quad \tilde{X} \in \tilde{W} \]

such that \( l_X^* \) and \( \tilde{l}_X^* \) become respectively \( L_X^* \) and \( L_X^* \) under the isomorphisms \( W, \tilde{W} \simeq h^3 \mathbb{K} \). Taking the traces over \( W \oplus \tilde{W} \), we have:

\[ \text{tr} l_X^* = 0 \quad \text{(1.22)} \]

\[ \text{tr} l_X^* l_T^* = \frac{\dim \mathbb{K} + 2}{4} \langle \tilde{X}, Y \rangle \quad \text{(1.23)} \]

\[ \text{tr} l_X^* \tilde{l}_Y^* = 0 \quad \text{(1.24)} \]

\[ \text{tr} l_X^* l_T^* = \frac{\dim \mathbb{K} + 2}{32} \left[ \langle \tilde{X}, Y \rangle \langle \tilde{Z}, T \rangle + \langle \tilde{X}, T \rangle \langle \tilde{Z}, Y \rangle \right] - \frac{\dim \mathbb{K}}{8} (l_Y^* l_Z^* T) \quad \text{(1.25)} \]

for \( Y, T \in W \) and \( \tilde{X}, \tilde{Z} \in \tilde{W} \).

**Proof.** Identities (1.22) and (1.24) are obvious, as we trace maps which swap the (sub)spaces \( W \) and \( \tilde{W} \). Formula (1.23) follows, up to the constant, from Schur’s lemma: indeed, let us complexify \( W, \tilde{W} \) and identify them complex-linearly with \( V_c, \bar{V}_c \) respectively. Then (1.23) defines a sesquilinear form on \( V_2(\mathbb{K}) \) – as the latter is an irreducible representation of \( \mathcal{G}_2(\mathbb{K}) \), the form must be a multiple of the hermitian inner product. The factor can be found by checking for some simple \( X \) and \( Y \).
To prove \(1.24\), we introduce some extra notation: we shall use \(i, j, \ldots\) to index \(W\) and \(\bar{i}, \bar{j}, \ldots\) to index \(\bar{W}\), and the same for their duals. We introduce a tensor \(\gamma(X, Y) = \langle \bar{X}, \bar{Y} \rangle\), so that \(\gamma_{ij} \in \bar{W}^* \otimes W^*\), with its inverse satisfying
\[
\gamma_{ij} \gamma^{ji} = \dim \mathfrak{h}_3 \mathbb{K} = 3 \dim \mathbb{K} + 3.
\]
The determinant defines tensors \(N_{ijk}\) and \(\bar{N}_{\bar{i}\bar{j}\bar{k}}\) such that
\[
\det X = N_{ijk} X^i X^j X^k, \quad \det \bar{X} = \bar{N}_{\bar{i}\bar{j}\bar{k}} \bar{X}^\bar{i} \bar{X}^\bar{j} \bar{X}^\bar{k}.
\]
We will use \(\gamma\) and its inverse to lower and raise indices, so that the maps \(l^*\) and \(\bar{l}^*\) are simply given by
\[
(l^*_X Y)^i = 3 \bar{N}^i_{\bar{j}k} X^j Y^k, \quad (\bar{l}^*_X \bar{Y})^i = 3 \bar{N}^i_{\bar{j}k} \bar{X}^\bar{j} \bar{Y}^\bar{k}.
\]
Equation \(1.23\) is expressed as:
\[
9 \bar{N}^m_{\bar{i}in} N^n_{jjm} = \frac{\dim \mathbb{K} + 2}{4} \gamma_{ij}.
\]
We now recall that \(X^H = (\det X) X\) for \(X \in \mathfrak{h}_3 \mathbb{K}\). Utilising the isomorphisms \(W, \bar{W} \simeq \mathfrak{h}_3 \mathbb{K}\), we can write this identity as follows:
\[
9 [N_{imp} N_{kqn} \bar{N}_{\bar{j}pq} + N_{mkp} N_{nqn} \bar{N}_{\bar{j}pq}] + N_{skp} N_{mqn} \bar{N}_{\bar{j}pq}] = \frac{1}{4} [N_{imk} \gamma_{jn} + N_{imk} \gamma_{jm} + N_{imn} \gamma_{jk} + N_{imn} \gamma_{jl}].
\]
Contraction with \(\bar{N}_{\bar{i}^jmn}\) yields:
\[
18 N_{ip} \bar{N}_{\bar{j}q} \bar{N}_{\bar{m}}^p N_{kn} \bar{N}_{\bar{o}n}^q \bar{N}_{\bar{l}o}^m + \frac{\dim \mathbb{K} + 2}{4} N_{skp} \bar{N}_{\bar{j}^l}^p = \frac{1}{2} N_{imk} \bar{N}_{\bar{j}^l}^m + \frac{1}{36} \frac{\dim \mathbb{K} + 2}{4} [\gamma_{ij} \gamma_{jk} + \gamma_{ij} \gamma_{lk}],
\]
where we used \(1.23\). Thus
\[
81 N_{ip} \bar{N}_{\bar{j}q} \bar{N}_{\bar{m}}^p N_{kn} \bar{N}_{\bar{o}n}^q \bar{N}_{\bar{l}o}^m = \frac{\dim \mathbb{K}}{8} 9 N_{skp} \bar{N}_{\bar{j}^l}^p + \frac{\dim \mathbb{K}}{32} [\gamma_{ij} \gamma_{jk} + \gamma_{ij} \gamma_{lk}],
\]
which is equivalent to \(1.24\). 

For \(C \otimes W = V_c\) and \(C \otimes \bar{W} = V_{\bar{c}}\), where \(V_c, V_{\bar{c}}\) are the duals of the (complex) spaces of linear and antilinear complex forms on \(V_2(\mathbb{K})\), as in \(1.14\), we readily have the following

**Corollary 5.** Let us set \(c_2 = \sqrt{\frac{1}{\kappa} - 2}\), where \(\kappa = \det \mathbb{K}\). Recalling the conventions introduced in Remark \(7\), we have:
\[
\Lambda_{\alpha \mu \nu} \Lambda_{\beta}^\mu = h_{\beta \alpha},
\]
\[
\Lambda_{\alpha \mu \nu} \Lambda_{\beta}^\mu \Lambda_{\gamma \rho} \Lambda_{\delta}^\gamma = \frac{1}{2\kappa + 4} (h_{\beta \alpha} h_{\gamma \nu} + h_{\beta \alpha} h_{\delta \nu} - \frac{\kappa}{2\kappa + 4} \Lambda_{\alpha \gamma} \Lambda_{\delta}^\mu).
\]
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We shall further make use of the following:

Lemma 25. Let us introduce a map

\[ D_\Lambda : \Lambda^{1,1} \rightarrow \Lambda^{1,1} \]

\[ \Lambda_{\alpha \beta}^{1,1} \mapsto b_{\alpha \beta} - b_{\beta \alpha} \mapsto b_{\mu \nu} \Lambda_{\mu \nu}^{\beta \alpha} - b_{\mu \nu} \Lambda_{\mu \nu}^{\alpha \beta} \in \Lambda_{\alpha \beta}^{1,1}. \]

Then, with respect to the decomposition

\[ \text{Re}\Lambda^{1,1} \simeq u(V_2(\mathbb{K})) = \theta R \oplus G'_2(\mathbb{K}) \oplus G'_2(\mathbb{K})^\perp, \]

where \( G'_2(\mathbb{K})^\perp \) is the orthogonal complement of \( G'_2(\mathbb{K}) \) in \( su(V_2(\mathbb{K})) \), the map \( D_\Lambda \) is given by:

\[
D_\Lambda|_{\theta} = 1 \quad \quad D_\Lambda|_{G'_2(\mathbb{K})} = -\frac{1}{2} \quad \quad D_\Lambda|_{G'_2(\mathbb{K})^\perp} = \frac{1}{\kappa + 2}.
\]

It also satisfies:

\[
D_\Lambda^2|_{su(V_2(\mathbb{K}))} = \frac{1}{2\kappa + 4}[1 - \kappa D_\Lambda|_{su(V_2(\mathbb{K}))}].
\]

Proof. It is easy to check that \( D_\Lambda(\theta) = \theta \). Now, let

\[ B_{\alpha \beta} = b_{\alpha \beta} - b_{\beta \alpha} \in su(V_2(\mathbb{K}))_{\alpha \beta}. \]

Then \( D_\Lambda(B)_{\alpha \beta} = b_{\mu \nu} \Lambda_{\mu \nu}^{\beta \alpha} \Lambda_{\rho \sigma}^{\alpha \beta} - b_{\mu \nu} \Lambda_{\mu \nu}^{\sigma \alpha} \Lambda_{\rho \beta}^{\alpha \beta} \) (still in \( su(V_2(\mathbb{K}))_{\alpha \beta} \), due to hermiticity of \( D_\Lambda \)) and

\[
D_\Lambda^2(B)_{\alpha \beta} = b_{\mu \nu} \Lambda_{\mu \nu}^{\beta \alpha} \Lambda_{\rho \sigma}^{\alpha \beta} - b_{\mu \nu} \Lambda_{\mu \nu}^{\sigma \alpha} \Lambda_{\rho \beta}^{\alpha \beta}
\]

\[
= \frac{1}{2\kappa + 4} [b_{\alpha \beta} + b_{\mu \nu} \theta_{\alpha \beta} - \kappa b_{\mu \nu} \Lambda_{\mu \nu}^{\beta \alpha} \Lambda_{\rho \beta}^{\alpha \beta}]
\]

\[
- \frac{1}{2\kappa + 4} [b_{\beta \alpha} + b_{\mu \nu} \theta_{\beta \alpha} - \kappa b_{\mu \nu} \Lambda_{\mu \nu}^{\alpha \beta} \Lambda_{\rho \alpha}^{\beta \alpha}],
\]

where we used Corollary 35 and \( b_{\mu \nu} = 0 \). Thus, when restricted to \( su(V_2(\mathbb{K})) \),

\[
D_\Lambda^2 = \frac{1}{2\kappa + 4}[1 - \kappa D_\Lambda].
\]

Then, since \( D_\Lambda \) is hermitian, it splits the space \( su(V_2(\mathbb{K})) \) orthogonally into \( su^+(V_2(\mathbb{K})) \oplus su^-(V_2(\mathbb{K})) \) with

\[
D_\Lambda|_{su^+(V_2(\mathbb{K}))} = \lambda^+ \text{id}, \quad \lambda^\pm = -\kappa \pm (\kappa + 4) \quad \quad \frac{1}{4\kappa + 8}.
\]

It follows that

\[ \lambda^+ \dim su^+(V_2(\mathbb{K})) + \lambda^- \dim su^-(V_2(\mathbb{K})) = \text{tr} D_\Lambda = \Lambda_{\mu \nu} \Lambda^{\mu \nu} = 3\kappa + 3. \]

Using \( \dim su(V_2(\mathbb{K})) = N^2 - 1 \) we can compute the dimensions of \( su^+(V_2(\mathbb{K})) \) and identify \( su^-(V_2(\mathbb{K})) \) as \( G'_2(\mathbb{K}) \).
1.5.3 Third family

Lemma 26. Let $L_{X,Y} : Z \mapsto \tau(X,Y,Z)$ be the left multiplication map for $X,Y,Z \in \mathcal{F}(\mathfrak{h}_3\mathbb{K})$. Then:

$$\operatorname{tr} L_{X,Y} = 0$$ (1.26)

$$\operatorname{tr} L_{X,Y}^T L_{Z,T} = \dim \mathbb{K} + 3 \cdot \frac{\omega(X,Z)\omega(Y,T) + \omega(X,T)\omega(Y,Z)}{2^23^2}$$ (1.27)

Moreover, let $J_0 : \mathcal{F}(\mathfrak{h}_3\mathbb{K}) \rightarrow \mathcal{F}(\mathfrak{h}_3\mathbb{K})$ be the map given by $\langle J_0X,Y \rangle = \omega(X,Y)$.

Then

$$Q(J_0X,J_0X,J_0X,J_0X) = Q(X,X,X,X).$$ (1.28)

Proof. The equation (1.26) follows from symmetry of $Q$ and antisymmetry of $\omega$, and (1.28) is easy to check directly. On the other hand, proving (1.27) will require considerably more effort, and some extra notation.

To use the results of Lemma (24), we introduce spaces $W, \tilde{W} \cong \mathfrak{h}_3\mathbb{K}$, $L, \tilde{L} \cong \mathbb{R}$ such that

$$\mathcal{F}(\mathfrak{h}_3\mathbb{K}) = (\mathbb{R} \oplus \mathfrak{h}_3\mathbb{K}) \otimes \mathbb{R}^2 = (L \oplus W) \otimes (1,0) \oplus (\tilde{L} \oplus \tilde{W}) \otimes (0,1).$$

In the same manner as in the proof of Lemma (24), the scalar product on $\mathfrak{h}_3\mathbb{K}$ is extended to a tensor

$$\gamma \in \tilde{W}^* \otimes W^*, \quad \gamma(\tilde{X},Y) = \langle \tilde{X},Y \rangle,$$

and we introduce for later convenience the normalised forms on the real lines $L, \tilde{L}$, denoted

$$\xi \in L^*, \tilde{\xi} \in \tilde{L}^*, \quad \xi(x) = x, \quad \tilde{\xi}(\tilde{x}) = \tilde{x}.$$

We shall now use $i,j,\ldots$ to index $L \oplus W$ and $\tilde{i},\tilde{j},\ldots$ to index $\tilde{L} \oplus \tilde{W}$, and the same for their duals. Indices are lowered and raised with help of the tensor

$$\gamma_{ij} + \tilde{\xi}^i\tilde{\xi}^j$$

and its inverse, $\gamma^{ij} + \tilde{\xi}^i\tilde{\xi}^j$, where

$$\gamma_{ij}\gamma^{ij} = \dim \mathfrak{h}_3\mathbb{K} = 3 \dim \mathbb{K} + 3, \quad \tilde{\xi}^i\tilde{\xi}^i = 1 = \tilde{\xi}^j\tilde{\xi}^j$$

and

$$\gamma_{ij}\tilde{\xi}^i = 0, \quad \gamma^{ij}\xi^i = 0, \quad \gamma_{ij}\xi^j = 0, \quad \gamma^{ij}\tilde{\xi}^j = 0.$$

The determinant defines tensors $N_{ijk}$ and $\tilde{N}_{\tilde{ij}k}$ such that

$$\det X = N_{ijk}X^iX^jX^k, \quad \det \tilde{X} = \tilde{N}_{\tilde{ijk}}\tilde{X}^i\tilde{X}^j\tilde{X}^k$$

for $X \in W$ and $\tilde{X} \in \tilde{W}$, vanishing on $L$ and $\tilde{L}$, so that

$$N_{ijk}\xi^k = 0, \quad \tilde{N}_{\tilde{ij}k}\tilde{\xi}^k = 0.$$
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The maps \( l^* \) and \( \tilde{l}^* \) are given by

\[
(l_X^* Y)^i = 3 N_{ijk} X^j Y^k, \quad (\tilde{l}_X^* \tilde{Y})^i = 3 \tilde{N}_{ijk} \tilde{X}^j \tilde{Y}^k
\]

for \( X, Y \in W \) and \( \tilde{X}, \tilde{Y} \) in \( \tilde{W} \), and identities of Lemma 24 are expressed as:

\[
9 \tilde{N}_{ipq} R_{kpq} N_{mjq} = \operatorname{dim} K + 2 \frac{\gamma_{ij}}{4},
\]

\[
81 \tilde{N}_{ipq} R_{kpq} N_{mjq} = \operatorname{dim} K + 2 \frac{[\gamma_{ij} + \gamma_{il} \gamma_{kj}]}{32} - \frac{9 \operatorname{dim} K}{8} \tilde{N}_{mij} N_{njq}.
\]

Using corresponding uppercase letters to index full \( \mathcal{F}(\mathfrak{h}_3 K) \), we have

\[
\mathcal{F}(\mathfrak{h}_3 K)^I = (L \oplus W)^I \oplus (\tilde{L} \oplus \tilde{W})^I,
\]

etc., and the scalar product and symplectic form on this space is represented by

\[
k^{PQ} = \gamma \bar{p}^q + \bar{x} \xi \bar{u} + \gamma \bar{q}^p + \bar{x} \xi \bar{u}
\]

\[
\omega^{PQ} = \gamma \bar{p}^q + \bar{x} \xi \bar{u} - \gamma \bar{q}^p - \bar{x} \xi \bar{u}.
\]

The quartic defining the triple product is \( Q_{IJKL} \), and, recalling that \( \tau \) is given in terms of \( Q_{IJKLM} \omega^{ML} \), the expression we wish to compute is

\[
Q_{IJKL}^2 = Q_{IJPQ} Q_{KLR} \omega^{PR} \omega^{QS}.
\]

Projecting onto subspaces of \( \mathcal{F}(\mathfrak{h}_3 K) \), the components of the symmetric tensor \( Q \) are given by

\[
Q_{IJPQ} = Q_{ijpq} + Q_{ijqp} + Q_{ijpq} + Q_{ijpq} + Q_{ijpq} + Q_{ijpq} + Q_{ijpq} + Q_{ijpq}.
\]

with

\[
Q_{ijkl} = \frac{1}{4} \left[ \xi_i N_{jkl} + \xi_j N_{ikl} + \xi_k N_{ijl} + \xi_l N_{ijk} \right]
\]

\[
6Q_{ijkl} = 9 N_{mij} \tilde{N}_{mkl} - \frac{1}{4} \xi_i \xi_j \tilde{\xi}_k \tilde{\xi}_l
\]

\[
+ \frac{1}{8} \left[ -\gamma_{kl} \gamma_{ij} - \gamma_{ij} \gamma_{kl} + \gamma_{ik} \tilde{\xi}_l \xi_l + \gamma_{ij} \tilde{\xi}_l \xi_l + \gamma_{ik} \tilde{\xi}_l \xi_l + \gamma_{ij} \tilde{\xi}_l \xi_l \right].
\]

Recalling that we contract \( Q_{IJPQ} \) with

\[
Q_{KLR} \omega^{PR} \omega^{QS} = Q_{klpq} + Q_{klpq} + Q_{klpq} + Q_{klpq} - Q_{klpq} - Q_{klpq} - Q_{klpq},
\]

we find that the following expressions need to be computed:

\[
24 Q_{ijpq} Q_{klpq} = \left[ 2 \xi_{(ij)} N_{pq} + 2 \xi_{(pq)} N_{ij} \right] \left[ 9 N_{mkl} \tilde{N}_{mqp} - \frac{1}{4} \xi_k \xi_l \tilde{\xi}_p \tilde{\xi}_q \right]
\]

\[
+ \frac{1}{8} \left[ -\delta^p_k \delta^q_l - \delta^q_k \delta^p_l + \delta^p_k \xi_l \tilde{\xi}_q + \delta^q_k \xi_l \tilde{\xi}_p + \delta^q_k \xi_l \tilde{\xi}_p + \delta^p_k \xi_l \tilde{\xi}_q \right]
\]

\[
= \left( \frac{\operatorname{dim} K}{4} + \frac{1}{2} \right) \xi_{(ij)} N_{kl} + \frac{1}{2} N_{ij(k \xi_l)}.
\]
Using these we evaluate the components of $Q^{pq}_{ij}$:

$$16Q^{pq}_{ijpq}Q^{pq}_{kl} = [2\xi(pN_{ijpq} + \xi_q N_{pq})] [\tilde{\xi}^p N^q_{kl} + \tilde{\xi}^q N^p_{kl} + 2\tilde{\xi}_k N^p_{ijpq}]$$

$$= 2N_{mij}\tilde{N}^m_{kl} + \frac{1}{4} \dim K + 2 \frac{\gamma_{kk}}{4} [\gamma_{kl}\tilde{\xi}_{kl} + \gamma_{ij}\tilde{\xi}_{ij} + \gamma_{ij}\tilde{\xi}_{kl} + \gamma_{ij}\tilde{\xi}_{ik}]$$

$$36Q^q_{ijpq}Q^{pq}_{kl} = [9N_{mij}\tilde{\xi}^m_{kl} - \frac{1}{4} \gamma_{ij}\tilde{\xi}^q_{kl} - \frac{1}{4} \gamma_{ij}\tilde{\xi}^q_{kl} + \frac{1}{4} (\gamma_{ij}\tilde{\xi}_{kl} + \gamma_{kl}\tilde{\xi}_{ij})]$$

$$\times [9N_{mik}\tilde{\xi}^m_{ij} - \frac{1}{4} \gamma_{ij}\tilde{\xi}^p_{ij} - \frac{1}{4} \gamma_{ij}\tilde{\xi}^p_{ij} + \frac{1}{4} (\gamma_{ij}\tilde{\xi}_{ij} + \gamma_{ij}\tilde{\xi}_{ij})]$$

$$= 9 \left( \frac{\dim K + 2}{4} - \frac{1}{4} \right) N_{mij}\tilde{N}_{mkl} + \frac{1}{32} (\gamma_{kl}\gamma_{ij} + \gamma_{ij}\gamma_{kl}) + \frac{1}{16} \gamma_{kl}\tilde{\xi}_{kl} + \frac{1}{16} \gamma_{kl}\tilde{\xi}_{kl} + \gamma_{kl}\tilde{\xi}_{kl} + \gamma_{kl}\tilde{\xi}_{kl})$$

$$36Q^q_{ipjq}Q^{q}_{kl} = [9N_{mij}\tilde{\xi}^m_{pq} - \frac{1}{8} (\gamma_{ij}\tilde{\xi}_{pq} + \gamma_{ij}\tilde{\xi}_{pq}) - \frac{1}{4} \gamma_{ij}\tilde{\xi}_{pq}\tilde{\xi}_{pq}]$$

$$\times [9N_{mik}\tilde{\xi}^m_{pq} - \frac{1}{8} (\gamma_{ij}\tilde{\xi}_{pq} + \gamma_{ij}\tilde{\xi}_{pq}) - \frac{1}{4} \gamma_{ij}\tilde{\xi}_{pq}\tilde{\xi}_{pq}]$$

$$= \frac{32}{\dim K + 2} \left( \frac{\dim K + 2}{4} \right) [\gamma_{ij}\gamma_{pq} + \gamma_{ij}\gamma_{pq}] - \frac{9}{8} \frac{\dim K + 2}{4} \tilde{N}_{mij}\tilde{N}_{mkl}$$

$$+ \frac{1}{64} \left( 3 \frac{\dim K + 2}{4} - \frac{1}{8} \right) [\gamma_{ij}\gamma_{pq} + \gamma_{ij}\gamma_{pq}] + \frac{1}{16} \gamma_{ij}\tilde{\xi}_{pq}\tilde{\xi}_{pq}$$

$$+ \frac{1}{64} \left( 3 \frac{\dim K + 2}{4} - \frac{1}{8} \right) [\gamma_{ij}\gamma_{pq} + \gamma_{ij}\gamma_{pq}] + \frac{1}{16} \gamma_{ij}\tilde{\xi}_{pq}\tilde{\xi}_{pq}$$

$$+ \frac{1}{64} \left( 3 \frac{\dim K + 2}{4} - \frac{1}{8} \right) [\gamma_{ij}\gamma_{pq} + \gamma_{ij}\gamma_{pq}]$$

Using these we evaluate the components of $Q^{ijkl}_{ijkl}$:

$$Q^{ijkl}_{ijkl} = Q^{ijkl}_{ijkl} + Q^{ijkl}_{ijkl}$$

$$= \frac{\dim K + 2}{48} [\gamma_{ijkl} + \gamma_{ijkl}]$$

$$= \frac{\dim K + 2}{24} Q_{ijkl}^{ijkl}.$$
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\[ Q_{ijkl}^2 = -2Q_{ijpq}Q_{k\ell q} \]

\[ = \frac{\dim K + 2}{16} N_{mk}N_{j\ell} + \ldots \]

\[ = \frac{\dim K + 2}{24} Q_{ijkl} + \frac{\dim K + 3}{3^2 2^7} [2\xi_i\xi_k\xi_j\xi_l + \gamma_{ki}\gamma_{lj} + \gamma_{ij}\gamma_{kj} + \gamma_{il}\gamma_{kj} + \gamma_{il}\gamma_{kj}]. \]

Finally, it follows that

\[ Q_{ijkl}^2 = \dim K + 2 \frac{2}{24} Q_{ijkl} + \dim K + 3 \frac{3}{3^2 2^7} (\omega_{ij}\omega_{k\ell} + \omega_{I\ell}\omega_{KJ}), \]

which is equivalent to (1.27). □

Passing to the complexification \( V_3(K) = C \otimes \mathcal{F}(h_3K) \), we readily have the following

**Corollary 6.** Let us set \( c_3 = 24\sqrt{\frac{1}{\kappa + 3}} \), where \( \kappa = \dim K \). Recalling the conventions introduced in Remark 1, we have

\[ q_{\alpha\beta\mu\nu} = 0 \]  \hspace{1cm} (1.29)

\[ q_{\alpha\beta\mu\nu} \bar{q}^{\gamma\delta\mu\nu} = \frac{1}{2} [\delta_\beta^\gamma \delta_\alpha^\delta + \delta_\alpha^\gamma \delta_\beta^\delta] + \chi q_{\alpha\beta\mu\nu} \bar{q}^{\mu\gamma\rho\sigma} \omega^\rho\gamma\omega^\sigma. \]  \hspace{1cm} (1.30)

Moreover, \( J^* q = \bar{q} \), i.e.

\[ q_{\mu\nu\rho\sigma} \bar{q}^{\alpha\beta} \omega^\rho\beta \omega^\sigma\gamma \omega^\alpha = q^{\alpha\beta}. \]

These lead us directly to the missing proof:

**Proof of Lemma 19.**

1. The only formula absent in Corollary 6 is \( q_{\alpha\mu\nu\rho} \bar{q}^{\delta\mu\nu\rho} = \frac{N+1}{2} \delta_\alpha^\delta \) with \( N = 6\kappa + 8 \) being the complex dimension of \( V_3(K) \). But this follows by contraction from (1.30).

2. Let \( b \in S^{2,0} \). Then \( D_q(b)_{\alpha\beta} = b_{\mu\nu} \bar{q}^{\mu\nu\rho\sigma} \omega_{\rho\alpha} \omega_{\sigma\beta} \) and

\[ D_q^2(b)^{\alpha\beta} = b^\mu \nu q_{\mu\nu\rho\sigma} \omega_{\rho\alpha} \omega_{\sigma\beta} q_{\delta\theta} \xi_{\theta\delta} \omega^{\xi\rho\sigma}. \]

\[ = -b^\mu \nu q_{\mu\nu\rho\sigma} \bar{q}^{\delta\theta\rho\sigma} \omega_{\omega\beta} \omega_{\rho\sigma}. \]

\[ = b^\alpha \beta - \chi b^\mu \nu q_{\mu\nu\beta\alpha} \omega^{\delta\theta\rho\sigma} \omega_{\rho\sigma}, \]

so that \( D_q^2 = 1 - \chi D_q \). Then, since \( D_q \) is hermitian, it splits the space \( \text{sp}(V_3(K), \omega) \) orthogonally into \( \text{sp}^+(V_3(K), \omega) \oplus \text{sp}^-(V_3(K), \omega) \) with

\[ D_q|_{\text{sp}^\pm(V_3(K), \omega)} = \lambda^\pm \text{id}, \quad \lambda^\pm = -\frac{\chi \pm \sqrt{\chi^2 + 4}}{2}. \]

Now, using (1.29), one easily checks that \( \text{tr} D_q = 0 \). It thus follows that

\[ \lambda^+ \dim \text{sp}^+(V_3(K), \omega) + \lambda^- \dim \text{sp}^-(V_3(K), \omega) = 0. \]
CHAPTER 1. ALGEBRAIC PART

Using \( \dim \mathfrak{sp}(V_3(\mathbb{K}),\omega) = \frac{N(N+1)}{2} \) we can compute the dimensions of \( \mathfrak{sp}(V_3(\mathbb{K}),\omega) \) and identify \( \mathfrak{sp}(V_3(\mathbb{K}),\omega) \) as \( \mathcal{G}_3(\mathbb{K}) \). Finally, it turns out that one can further simplify \( \lambda^\pm \) and \( \chi \), so that
\[
\lambda_+ = -\sqrt{\kappa + 3}, \quad \lambda_- = -\frac{1}{\lambda_+}.
\]

1.5.4 Explicit formula for \( \mathcal{U} \)

We now wish to prove Lemma 18, which gave us an explicit expression for the projection of \( \mathcal{U} \) onto \( S^4.0 \otimes S^0.4 \). We first state a simpler result, which essentially expands the result of applying the construction of Lemma 17 to a one-form:

**Lemma 27.** Let \( F : V_3(\mathbb{K}) \rightarrow \mathbb{R} \) be a real linear map \( F_a = f_\alpha + \bar{f}_\alpha \). Let \( \Phi : S^2 V_3(\mathbb{K}) \rightarrow \mathbb{R} \) be the quadratic map given by
\[
\Phi(X,X) = F(X)^2 + F(I(X))^2 + F(J(X))^2 + F(K(X))^2.
\]

Then \( \Phi \in S^{1,1} \) and
\[
\frac{1}{2}\Phi = f \otimes \bar{f} + J^* \bar{f} \otimes J^* f + J^* f \otimes J^* \bar{f} + \bar{f} \otimes f + J^* \bar{f} \otimes J^* f + J^* f \otimes J^* \bar{f} + J^* \bar{f} \otimes J^* f + J^* f \otimes J^* \bar{f} + J^* \bar{f} \otimes J^* f + J^* f \otimes J^* \bar{f} + J^* \bar{f} \otimes J^* f,
\]
reducing to the former expression.

With indices present, the formula reads
\[
\Phi_{ab} = 2f_\mu \bar{f}_\nu [\delta^\mu_\alpha \delta^\nu_\ell - \omega_\alpha^\beta \omega_\ell^\beta] + \delta^\mu_\alpha \delta^\nu_\ell - \omega_\alpha^\beta \omega_{\ell \beta}.
\]

It is now easy to extend this result to multilinear maps. This leads us to the following

**Proof of Lemma 18.** Let us introduce on \( V_3(\mathbb{K}) \) a representation of an orthonormal basis in \( \mathbb{H} : L_1 = 1, L_2 = i, L_3 = J, L_4 = K \). Recalling the definition \( \mathcal{U}(Z,\ldots,Z) = c_3^2 \| Q_L(Z, Z, Z) \|^2 \), we expand it as
\[
\mathcal{U}(Z) = c_3^2 \sum_{ABCD} \tilde{Q}(L_A Z, L_B Z, L_C Z, L_D Z)^2,
\]
with \( A, B, C, D \) ranging from 1 to 4, where \( \tilde{Q} \) is defined as in Lemma 17
\[
\tilde{Q}(Z, Z, Z, Z) = |z|^4 Q(X, X, X, X)
\]
for \( Z = z \otimes X \in \mathbb{C} \otimes \mathcal{F}(\mathfrak{h}_3 \mathbb{K}) \).
Performing the sums over $A, B, C, D$ separately, we can apply the former lemma to obtain

$$
\mathcal{U}_{abcdefkl} = \frac{1}{70} q_{\mu\nu\rho\sigma} \bar{q}_{\xi\eta\zeta\tau} \left\{ \begin{array}{c}
[2\delta^\rho_\delta \delta^\xi_\xi + 2\omega^\xi_\alpha \omega^\mu_\epsilon] [2\delta^\eta_\beta \delta^\alpha_\beta + 2\omega^\eta_\beta \omega^\nu_\gamma] \\
[2\delta^\xi_\delta \delta^\eta_\gamma + 2\omega^\eta_\gamma \omega^\rho_\kappa] [2\delta^\xi_\delta \delta^\gamma_\nu + 2\omega^\gamma_\nu \omega^\sigma_\chi] + \text{sym.} \end{array} \right\}
$$

where symmetrization in $\alpha \ldots l$ is indicated, producing $\binom{8}{4} = 70$ terms on the right hand side. Projecting onto $S^4 \otimes S^0$, we obtain

$$
\frac{70}{16} \mathcal{U}_{\alpha\beta\gamma\delta\epsilon\zeta\lambda} = q_{\alpha\beta\gamma\delta} \bar{q}_{\epsilon\zeta\lambda} \\
+ 4 \, q_{\mu(\alpha\beta\gamma\omega^\delta)\bar{q}^\mu(\epsilon\bar{q}^\delta\zeta\lambda)} \xi \\
+ 6 \, q_{\mu\nu(\alpha\beta\omega^\delta\gamma\omega^\rho_\delta)\bar{q}^\mu(\epsilon\bar{q}^\nu\bar{q}^\rho_\delta)} \xi \eta \\
+ 4 \, q_{\mu\nu\rho(\alpha\omega^\delta\beta\omega^\gamma\delta)\omega^\mu_\epsilon(\epsilon\bar{q}^\nu\bar{q}^\rho_\delta)} \xi \eta \zeta \\
+ q_{\mu\nu\rho\sigma\omega^\delta(\alpha\omega^\beta\gamma\omega^\gamma\nu)\omega^\rho_\epsilon(\epsilon\bar{q}^\nu\bar{q}^\rho_\delta)} \xi \eta \zeta \tau.
$$

Now, using $\bar{q} = J^* q$ and $\bar{q}_\mu \mu^\beta = -\delta^\beta_\beta$, we find that

$$
\frac{70}{16} \omega^\xi_\mu \omega^\delta_\nu \omega^\lambda_\rho \omega^\sigma_\sigma \mathcal{U}_{\alpha\beta\gamma\delta\epsilon\zeta\lambda} \propto q_{\alpha\beta\gamma\delta} q_{\mu\nu\rho\sigma} \\
- 4 \, q_{\mu(\alpha\beta\gamma\eta)\nu\rho\sigma} \\
+ 6 \, q_{\mu\nu(\alpha\beta\gamma\delta)\rho\sigma} \\
- 4 \, q_{(\mu\nu\rho)(\alpha\beta\gamma\delta)\sigma} \\
+ q_{\mu\nu\rho\sigma} q_{\alpha\beta\gamma\delta},
$$

where $(\alpha\beta\gamma\delta)$ and $(\mu\nu\rho\sigma)$ are symmetrized separately. It is now easy to see that the expression on the right hand side is $16P_{44} (q \otimes q)$. Inverting the omegas on the left, we obtain the lemma. \qed
Chapter 2

Geometric part

2.1 Summary of the algebraic results

We wish to keep this part of our work possibly independent of the Jordan-algebra-related theory developed in the previous chapter. We will now recall the results we need, in a form which claims the existence of certain tensors subject to a number of identities.

Still, the symbols $\mathbb{K}$ and $\mathbb{K}'$ denote, respectively, one of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and one of $\mathbb{C}, \mathbb{H}, \mathbb{O}$. Pairs $(\mathbb{K}, \mathbb{K}')$ enumerate compact Riemannian symmetric spaces collected in the following table:

| $\mathbb{K}'$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
|---------------|-------------|-------------|-------------|
| $\mathbb{R}$  | $SU(3)$    | $Sp(1)$    | $E_6$       |
| $\mathbb{C}$  | $SU(3) \times SU(3)$ | $SU(6)$    | $Sp(3) \times Sp(1)$ |
| $\mathbb{H}$  | $SU(6)$    | $SO(12)$   | $E_{7,1}$   |
| $\mathbb{O}$  | $E_8^*$     | $E_7 \times O(1)$ | $E_7 \times Sp(1)$ |

Let $M_S(\mathbb{K}, \mathbb{K}')$ denote the corresponding symmetric space and $G(\mathbb{K}, \mathbb{K}')$ its underlying isotropy group, with a Lie algebra $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$. The isotropy representation is denoted $V(\mathbb{K}, \mathbb{K}')$.

Abstract index notation, including the conventions introduced in Remark 1, is assumed. Moreover, for given $\mathbb{K}$ and $\mathbb{K}'$, the following symbols are defined:

$$\kappa = \dim \mathbb{K}, \quad N = \begin{cases} \dim M_S(\mathbb{K}, \mathbb{C}) = 3\kappa + 2 & \text{for } \mathbb{K}' = \mathbb{C} \\ \frac{1}{2} \dim M_S(\mathbb{K}, \mathbb{H}) = 3\kappa + 3 & \text{for } \mathbb{K}' = \mathbb{H} \\ \frac{1}{2} \dim M_S(\mathbb{K}, \mathbb{O}) = 6\kappa + 8 & \text{for } \mathbb{K}' = \mathbb{O} \end{cases}$$

and $\chi = \frac{\kappa + 2}{\sqrt{\kappa + 3}}$ for $\mathbb{K}' = \mathbb{O}$.

The following statements are reformulations or simple corollaries of: Corollary [9], Proposition [9], Lemmas [16], [19], [23], [25], and Corollaries [9, 9].

2.1.1 First family

Set $\mathbb{K}' = \mathbb{C}$ and choose $\mathbb{K}$. Let $V_1(\mathbb{K})$ be a real vector space of dimension $N$ equipped with a positive definite scalar product $g$ identifying $V_1(\mathbb{K}) \simeq V_1(\mathbb{K})^*$. 53
There exists a tensor $\Upsilon \in S^3 V_1(\mathbb{K})$ reducing the group $GL(N)$ to $G(\mathbb{K}, \mathbb{C})$ acting on $V_1(\mathbb{K})$ in the isotropy representation of $M_S(\mathbb{K}, \mathbb{C})$, with $g$ as the preserved scalar product.

This tensor moreover satisfies the following identities:
\[
\begin{align*}
\Upsilon_{i m m} &= 0 \\
\Upsilon_{i m n} \Upsilon_{j m n} &= g_{i j} \\
\Upsilon_{i r p} \Upsilon_{j p q} \Upsilon_{k q r} &= -\frac{3\kappa}{6\kappa + 8} \Upsilon_{i j k} \\
\Upsilon_{(i j m} \Upsilon_{k l) m} &= \frac{10}{3\kappa + 4} g_{(i j k l)}.
\end{align*}
\]

The latter shows that, up to rescaling, $\Upsilon$ is the same tensor as the one used by Nurowski in [1].

An operator $D_\Upsilon : \Lambda^2 V_1(\mathbb{K}) \to \Lambda^2 V_1(\mathbb{K})$
\[
D_\Upsilon(E)_{i j} = E_{pq} \Upsilon_{i q m} \Upsilon_{j p n}
\]
acts as
\[
D_\Upsilon = -\frac{1}{2} pr_g + \frac{3}{3\kappa + 4} pr_{\perp}
\]
where $pr_g$ and $pr_{\perp}$ are projections corresponding to the orthogonal decomposition
\[
\Lambda^2 V_1(\mathbb{K}) = g(\mathbb{K}, \mathbb{C}) \oplus \perp,
\]
with $g(\mathbb{K}, \mathbb{C}) \subset so(V_1(\mathbb{K}), g) \simeq \Lambda^2 V_1(\mathbb{K})$
being the isotropy algebra of $\Upsilon$.

### 2.1.2 Second family

Set $\mathbb{K}' = \mathbb{H}$ and choose $\mathbb{K}$. Let $V_2(\mathbb{K})$ be a complex vector space of (complex) dimension $N$ equipped with a hermitian inner product $h$, giving rise to a positive definite real scalar product $g$ on (the realification of) $V_2(\mathbb{K})$.

Let $S^{p,q}$ and $\Lambda^{p,q}$ denote the spaces of $p$-linear $q$-antilinear, respectively symmetric and antisymmetric, complex-valued forms on $V_2(\mathbb{K})$.

There exists a real tensor $\Xi \in \text{Re} S^{3,3}$ reducing the group $O(V_2(\mathbb{K}), g)$ to a subgroup whose connected component is $G(\mathbb{K}, \mathbb{H})$, acting on $V_2(\mathbb{K})$ in the isotropy representation of $M_S(\mathbb{K}, \mathbb{H})$, with $g$ as the preserved scalar product.

Recall that $h$ gives rise to an identification
\[
u(V_2(\mathbb{K}), h) \simeq \text{Re} \Lambda^{1,1} = \theta \mathbb{R} \oplus su(V_2(\mathbb{K}), h),
\]
where $\theta_{ab} = i h_{ab} - i h_{ba}$. In particular, the orthogonal isotropy algebra of $\Xi$ is a subalgebra in $u(V_2(\mathbb{K}), h)$.

There moreover exists a tensor $\Lambda \in S^{3,0}$ such that $\Xi = \Lambda \cdot \bar{\Lambda}$, satisfying the following identities:
\[
\begin{align*}
\Lambda_{\alpha \mu \nu} \bar{\Lambda}_{\beta}^{\mu \nu} &= h_{\beta \alpha} \\
\Lambda_{\alpha \mu \nu} \bar{\Lambda}_{\beta}^{\nu \rho} \Lambda_{\gamma \rho \sigma} \bar{\Lambda}_{\delta}^{\sigma \mu} &= \frac{1}{2\kappa + 4} (h_{\beta \alpha} h_{\delta \gamma} + h_{\beta \sigma} h_{\delta \alpha}) \\
&- \frac{\kappa}{2\kappa + 4} \Lambda_{\alpha \gamma \mu} \Lambda_{\beta}^{\gamma \mu}.
\end{align*}
\]
The algebra \( g(\mathbb{K}, \mathbb{H}) \) is a direct sum
\[
g(\mathbb{K}, \mathbb{H}) = G'_{2}(\mathbb{K}) \oplus u(1)
\]
of the orthogonal stabilizer algebra of \( \Lambda \) and a \( u(1) \) spanned by multiplication by \( i \). The latter is the centre of \( g(\mathbb{K}, \mathbb{H}) \).

There is a \( G(\mathbb{K}, \mathbb{H}) \)-invariant operator
\[
D_{\Lambda} : \Lambda^{1,1} \to \Lambda^{1,1}
\]
for \( B_{ab} = b_{\alpha\beta} - b_{\beta\alpha}, \) acting as
\[
D_{\Lambda} = p_{r} - \frac{1}{2}p_{r'v'} + \frac{1}{\kappa + 2}p_{r'},
\]
where \( p_{r}, p_{r'}, \) and \( p_{r'} \) are projections corresponding to the orthogonal decomposition into \( G(\mathbb{K}, \mathbb{H}) \)-invariant spaces:
\[
\Lambda^{1,1} = \mathbb{C} \otimes u(1) \oplus \mathbb{C} \otimes G'_{2}(\mathbb{K}) \oplus \perp.
\]

### 2.1.3 Third family

Set \( \mathbb{K}' = 0 \) and choose \( \mathbb{K} \). Let \( V_{3}(\mathbb{K}) \) be a real vector space of dimension \( 2N \) equipped with a positive definite scalar product \( g \) and a quaternion-hermitian structure, i.e. three hermitian structures \( I, J, K \) subject to
\[
I^{2} = J^{2} = K^{2} = IJK = -\text{id}.
\]

There exists a tensor \( \mathcal{O} \in S^{8}V_{3}(\mathbb{K}) \) reducing the group \( O(V_{3}(\mathbb{K}), g) \) to a subgroup whose connected component is \( G(\mathbb{K}, \mathbb{O}) \), acting on \( V_{3}(\mathbb{K}) \) in the isotropy representation of \( M_{3}(\mathbb{K}, \mathbb{O}) \), with \( g \) as the preserved scalar product.

Consider now \( V_{3}(\mathbb{K}) \) as a complex vector space of dimension \( N \), with one of the three hermitian structures, for concreteness \( I \), as the complex structure. Let \( S^{p,q} \) and \( \Lambda^{p,q} \) denote the spaces of \( p \)-linear \( q \)-antilinear, respectively symmetric and antisymmetric, complex-valued forms. By polarisation, the scalar product \( g \) gives rise to a hermitian inner product \( h \in S^{1,4} \). Moreover, by the identification
\[
\mathfrak{so}(V_{3}(\mathbb{K}, g)) \simeq \Lambda^{2}V_{3}(\mathbb{K}, g) = \text{Re}(\Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}),
\]
one has a symplectic form \( \omega_{\alpha\beta} \in \Lambda^{2,0} \) such that
\[
J \in \text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2}), \quad J^{\alpha}_{\beta} = \omega^{\alpha}_{\beta} + \omega^{\beta}_{\beta}.
\]

In this context, \( \mathcal{U} \in \text{Re}S^{1,4} \). There moreover exists a tensor \( q \in S^{4,0} \) such that the projection of \( \mathcal{U} \) in \( S^{4,4} \) onto \( S^{4,0} \oplus S^{0,4} \) is
\[
(\mathcal{U}|_{S^{4,0} \oplus S^{0,4}})_{\alpha\gamma\delta\epsilon\phi\chi\lambda} = \frac{256}{70} \delta^{[\mu}_{\alpha} \omega^{\xi]} (\epsilon^{[\nu}_{\beta} \omega^{\eta]} \chi^{[\rho}_{\gamma} \omega^{\delta]} \phi^{\lambda]}_{\delta} \sigma)_{\chi} q_{\mu\nu\rho\sigma\eta\xi\gamma\lambda},
\]
where barred and unbarred indices are symmetrized separately. Following identities are satisfied:
\[
\begin{align*}
q_{\alpha\beta\mu\nu} & = 0 \quad (2.1) \\
q_{\alpha\mu\nu\rho} q^{\beta\nu\mu\rho} & = \frac{N + 1}{2} \delta_{\beta}^{\alpha} \quad (2.2) \\
q_{\alpha\beta\mu\nu} q^{\gamma\delta\mu\nu} & = \frac{1}{2} [\delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} + \delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma}] + \chi q_{\alpha\beta\mu\nu} \omega^{\mu\gamma} \omega^{\nu\delta} \\
q_{\mu\nu\rho\sigma} & = \chi \omega^{\mu\nu}\omega^{\rho\sigma} \quad (2.3)
\end{align*}
\]
Let $\text{Sp}(\mathcal{V}_3(K), \omega)$ denote the symplectic group preserving the quaternionic structure, and $\text{Sp}(1)$ the group generated by $I, J, K$. Let $\mathfrak{sp}(\mathcal{V}_3(K), \omega)$ and $\mathfrak{sp}(1)$ be the corresponding Lie algebras. From the complex viewpoint, with $I$ the distinguished complex structure, one has

$$\mathfrak{sp}(\mathcal{V}_3(K), \omega) \subset u(\mathcal{V}_3(K), h) \simeq \text{Re}\Lambda^{1,1}.$$

Moreover, $\omega$ gives rise to an identification

$$\Lambda^{1,1} \supset C \otimes \mathfrak{sp}(\mathcal{V}_3(K), \omega) \simeq S^{2,0}$$

via

$$\Lambda^{1,1}_{ab} \ni b_{\alpha\beta} - b_{\beta\alpha} \mapsto b^\mu_{\alpha\beta} \omega_{\mu\rho\sigma} \in \Lambda^{1,0}_a \otimes \Lambda^{1,0}_b.$$ 

The orthogonal isotropy algebra of $U$ can be decomposed as

$$\mathfrak{g}(K, \Omega) = \mathcal{G}'_3(K) \oplus \mathfrak{sp}(1)$$

where $\mathcal{G}'_3(K)$ is the orthogonal stabilizer of $q$.

An operator

$$D_q : S^{2,0} \to S^{2,0}$$

acts as

$$D_q(b)_{\alpha\beta} = b_{\mu\rho} q^{\nu\rho\sigma} \omega_{\mu\rho\sigma} \omega_{\sigma\beta},$$

where $\text{pr}_g$ and $\text{pr}_\perp$ are projections corresponding to the decomposition

$$S^{2,0} \simeq C \otimes \mathcal{G}'_3(K) \oplus \perp.$$

2.2 $G-$structures and intrinsic torsion

$G$-structures provide a covariant description of additional structure introduced on a manifold (be it Riemannian, complex, CR etc.). While they are defined as reductions of the frame bundle, or simply principal bundles equipped with a soldering form, related objects are pulled back to the manifold and its tangent bundle by means of local sections, i.e. adapted frames.

One can also introduce somewhat weaker structure directly on the tangent bundle, specifying endomorphisms of the latter corresponding to the Lie algebra $\mathfrak{g}$ of the original structure group $G$. Under certain technical assumption, the latter can be then reconstructed up to connected components, so that in general one ends up, locally, with several associated $G$-structures (although these may fail to be globally defined on a manifold which is not simply connected).

Finally, we shall discuss structures defined in terms of invariant tensors. However, the groups $G(K, K')$ we are interested in are in general defined only as identity components of the isotropy groups. Selecting a $G(K, K')$-structure requires thus an extra choice of a frame in a single point of each connected component of the manifold, and may fail due to global problems even if the tensor is globally defined.
2.2. \textit{G}–\textit{structures and the intrinsic torsion}

Recall, that a \textit{G}–\textit{structure} \( Q \) on an \( m \)–dimensional Riemannian manifold \((M,g)\), for \( G \) being a subgroup of \( \text{O}(m) \), is a reduction

\[ i : Q \hookrightarrow P \]

of the orthonormal frame bundle \( \pi : P \to M \) to a subbundle \( Q \) with structure group \( G \), acting on \( Q \) by restriction of the action of \( \text{O}(m) \) on \( P \). Its local sections will be called \textit{adapted frames}.

A connection on the bundle \( P \) is said to be compatible with \( Q \) iff the associated horizontal distribution \( \text{Hor} \subset TP \) is tangent to \( Q \):

\[ \text{Hor}_q \subset T_qQ \quad \text{for each} \ q \in Q. \]

In terms of a connection form \( \omega \in \Omega^1(P) \otimes \mathfrak{so}(m) \), the compatibility condition reads simply\footnote{Indeed, consider a vector \( X \in T_qQ \subset T_qP \). Then \( X - \omega(X)_q \) is horizontal, where \( \dot{A} \in \mathcal{X}(P) \) denotes the vertical vector field associated to \( A \in \mathfrak{so}(m) \) by the structure group action. Demanding \( \text{Hor}_q \subset T_qQ \) implies \( \omega(X)_q \in T_qQ \), so that \( \omega(X) \in \mathfrak{g} \).}

\[ i^* \omega \in \Omega^1(Q) \otimes \mathfrak{g}, \]

where \( \mathfrak{g} \subset \mathfrak{so}(m) \) is the Lie algebra of \( G \). If it is satisfied, \( i^* \omega \) becomes a connection on \( Q \) (since \( G \) acts on \( Q \) by restriction of its action on \( P \)).

\textbf{Definition 2.} A \textit{G}–\textit{structure} is said to be \textit{integrable} iff it admits a torsion-free compatible connection.

Following Agricola \cite{13}, we consider the Levi-Civita connection \( \omega^{LC} \in \Omega^1(P) \otimes \mathfrak{so}(m) \) and decompose its restriction to \( Q \) orthogonally:

\[ \mathfrak{so}(m) = \mathfrak{g} \oplus \mathfrak{t}, \quad [\mathfrak{g}, \mathfrak{t}] \subset \mathfrak{t} \]

\[ i^* \omega^{LC} = \omega^{\mathfrak{g}} - \alpha^1 \] \quad (2.5)

so that \( \omega^{\mathfrak{g}} \in \Omega^1(Q) \otimes \mathfrak{g} \) and \( \alpha^1 \in \Omega^1(Q) \otimes \mathfrak{t} \).

\textbf{Lemma 28} (cf. \cite{13}),

1. \( \omega^{\mathfrak{g}} \) is a connection on \( Q \).
2. \( \alpha^1 \) is a horizontal one-form of type \( \text{Ad} \) on \( Q \).

The usual name for the \( \alpha^1 \) is the \textit{intrinsic torsion} of \( Q \) (in fact, this notion can be defined also for a general \( G \)–\textit{structure}, not necessarily Riemannian, although it becomes less explicit \cite{34}).

The form \( \omega^{\mathfrak{g}} \) can be extended by covariance to a connection on entire \( P \), with values in the orthogonal orbit of \( \mathfrak{g} \):

\[ \tilde{\omega}^{\mathfrak{g}} \in \Omega^1(P) \otimes \text{Ad}_{\text{O}(m)} \mathfrak{g}. \]

Let \( \tilde{D}^{\mathfrak{g}} \) be the associated covariant derivative:

\[ \tilde{D}^{\mathfrak{g}} : \Omega^p_{\text{hor}}(P) \otimes H \to \Omega^{p+1}_{\text{hor}}(P) \otimes H. \]
\[ \tilde{D}^\theta \phi = d\phi + \tilde{\omega}^\theta \wedge \phi \]

for any \( O(m) \)-module \( H \). Recall that a point \( p \) in the frame bundle \( P \) is an orthogonal isomorphism \( p : \mathbb{R}^m \to T_{\pi(p)} M \), where \( \mathbb{R}^m \) is equipped with some fixed positive definite scalar product. Introducing the soldering form

\[ \theta \in \Omega^1(P) \otimes \mathbb{R}^m, \quad \theta_p = p^{-1} \circ T_p \pi \quad \text{for each} \quad p \in P, \]

we have the torsion of \( \omega^\theta \):

\[ \Theta^\theta = \tilde{\Phi}^\theta \theta = d\theta + \tilde{\omega}^\theta \wedge \theta \in \Omega^2(P) \otimes \mathbb{R}^m. \]

Recalling that the Levi-Civita connection is torsion-free and using (2.5), one finds

\[ i^* \Theta^\theta = i^* d\theta + \omega^\theta \wedge i^* \theta = i^* (d\theta + \omega^{LC} \wedge \theta) + \alpha^t \wedge i^* \theta = \alpha^t \wedge i^* \theta. \]

It then follows that

\[ \Theta^\theta(X,Y) = \alpha^t(X)(\theta(Y)) - \alpha^t(Y)(\theta(X)) \]

for \( X,Y \in T_q Q \). Conversely, it turns out that \( \Theta^\theta \) determines the horizontal form \( \alpha^t \) (it needn’t be true in the non-Riemannian case, cf. [13]):

\[ 2 \langle \alpha^t(X)(\theta(Y)), \theta(Z) \rangle = \langle \Theta^\theta(X,Y), \theta(Z) \rangle - \langle \Theta^\theta(Y,Z), \theta(X) \rangle + \langle \Theta^\theta(Z,X), \theta(Y) \rangle \]

for \( X,Y,Z \in T_q Q \), where \( \langle \cdot, \cdot \rangle \) denotes the scalar product on \( \mathbb{R}^m \).

Note that a \( Q \)-compatible connection is necessarily metric. Thus a torsion-free \( Q \)-compatible connection, provided it exists, necessarily coincides with the Levi-Civita one. This leads us to the following obvious

**Proposition 5** (cf. [13,34]). Let \( Q \) be a \( G \)-structure on an \( m \)-dimensional Riemannian manifold \((M,g)\), with \( G \subset O(m) \) The following are equivalent:

1. \( Q \) is integrable.
2. The intrinsic torsion of \( Q \) is trivial.
3. The Levi-Civita connection on \( M \) is compatible with \( Q \).

Each of these implies reduction of the Riemannian holonomy group \( \text{Hol}(g) \) to a subgroup of \( G \).

Particular examples one should have in mind are the almost-hermitian and almost-quaternion-hermitian structures, with structure groups \( U(m/2) \) and \( \text{Sp}(m/2)\text{Sp}(1) \) respectively (their integrable cases being featured in the celebrated theorem of Berger). The geometries modelled on the second and third family of symmetric spaces described so far fall into these categories.
2.2.2 Tangent bundle view

Choosing a local section $e : M \supset U \to Q$, i.e. an adapted frame, one can pull $\omega^g$ and $\alpha^1$ back to $M$, so that $\omega^g$ gives rise to a connection in the tangent bundle, and $\alpha^1$ becomes a well-defined tensorial one-form.

It is however instructive to consider the structure which appears on the tangent bundle independently of any section. In what follows, we use $g$ to identify $TM \cong T^*M$. Observe first that, having fixed a point $x \in M$, the image of $g \subset \Lambda^2 \mathbb{R}^n$ under (the extension of) $g : \mathbb{R}^m \to T_xM$, where $q \in Q$ and $\pi(q) = x$, does not depend on the choice of $q$ from the fibre of $Q$ over $x$. Indeed, for every two frames $q, q' \in \pi|_Q^{-1}(x)$ there exists $g \in G$ such that $q' = g \circ q$; on the other hand, $Ad_q(g) = g$, so that $q(g) = q'(g)$. We shall denote this subspace as $g_M(x) \subset \Lambda^2 T_xM$.

It thus follows, that a $G$–structure equips $M$ with an orthogonal splitting

$$\Lambda^2 TM = g_M \oplus t_M$$

(2.6)

into subbundles such that at each point $x \in M$ the subspace

$$g_M(x) \subset \Lambda^2 T_xM \cong \mathfrak{so}(T_xM, g_x)$$

is a Lie subalgebra isomorphic to $g$.

A natural question is to what extent can one reconstruct the original $G$–structure from the data given by (2.6). In what follows we shall assume that:

1. $G$ is connected

2. $t$ contains no one-dimensional $G$–invariant subspace

(these are true for $G(K, K')$, as one can check in Section 2.6). Point 2 is needed in particular for the following result to hold:

**Lemma 29.** The set of all orthonormal frames mapping $g$ to $g_M$ forms a principal bundle with a structure group whose identity component is $G$.

**Proof.** Consider an orthonormal frame $e_x : \mathbb{R}^m \to T_xM$ at a fixed point $x \in M$, such that $e_x(g) = g_M(x)$. Every other frame at $x$ is of the form $e'_x = e_x \circ a$ for some $a \in O(m)$, so that $e'_x$ maps $g$ to $g_M(x)$ iff $Ad_a(g) = g$. It thus follows that all such frames at $x$ form a fibre of a principal bundle with structure group

$$\tilde{G} = \{a \in O(m) \mid Ad_a(g) = g\}.$$

The corresponding Lie algebra is

$$\tilde{\mathfrak{g}} = \{A \in \mathfrak{so}(m) \mid [A, g] \subset g\}.$$

Now, every $A \in \mathfrak{so}(m)$ is of the form $A_1 + A_2$ with $A_1 \in g$ and $A_2 \in t$, so that $[A, g] = [A_1, g] + [A_2, g]$, where automatically $[A_1, g] \subset g$ and $[A_2, g] \subset t$, since $[g, t] \subset t$. Thus $[A, g] \subset g$ iff $[A_2, g] = 0$. Then the assumption we have made on $t$ implies that every such $A_2$ is zero, so that finally $\tilde{\mathfrak{g}} = g$, and $\tilde{G}$ has $G$ as its component of identity. 

One moreover obtains a characterization of compatible connections:
Lemma 30. A metric connection $\nabla$ in the tangent bundle, lifted to a connection on the frame bundle, is compatible with the $G$–structure iff it preserves the splitting \[(2.6):\]

$$f \in \Omega^0(M, g_M) \implies \nabla f \in \Omega^1(M, g_M).$$

Proof. Choose (locally) an adapted frame, i.e. a (local) section $e : M \to Q$. A compatibility condition for a connection form $\omega$ on the frame bundle $P$ is $i^* \omega \in \Omega^1(Q) \otimes g$. Consider now the corresponding connection $\nabla^\omega$ in the frame bundle, whose local connection form in the adapted frame $e$ is

$$\Gamma^\omega \in \Omega^1(M, \Lambda^2 TM), \quad \Gamma^\omega(X) = e_x(e^* \omega(X)) \text{ for } X \in T_x M.$$

Then the compatibility condition is equivalent to $\Gamma^\omega \in \Omega^1(M, g_M)$. On the other hand, $\nabla^\omega$ preserves $g_M$ iff for each $X \in T_x M$

$$[\Gamma^\omega(X), g_M(x)] \subset g_M(x).$$

From the assumption on $t$, via an argument given in the proof of Lemma 29 it follows that $\nabla^\omega$ preserves $g_M$ iff $\Gamma^\omega \in \Omega^1(M, g_M)$. Hence the lemma.

Corollary 7. The $G$–structure is integrable iff the Levi-Civita derivative of each section of $g_M$ is $g_M$–valued.

Recall now, that a difference of two connections in the tangent bundle is a well-defined tensor

$$\nabla_X Y - \nabla'_X Y = A(X)(Y), \quad A \in \Omega^1(M, \text{End } TM),$$

where in particular $A \in \Omega^1(M, \Lambda^2 M)$ for metric connections. We have the following

Lemma 31. There exists a unique metric connection $\nabla^g$ preserving $g_M$ such that the difference tensor of $\nabla^g$ and the Levi-Civita connection $\nabla^{LC}$ is $t_M$–valued, i.e.

$$\exists A^1 \in \Omega^1(M, t_M) \forall X, Y \in \mathcal{X}(M) : \nabla^g_X Y - \nabla^{LC}_X Y = A^1(X)(Y).$$

Proof. Choose (locally) an adapted frame $e$. Then the local connection form of $\nabla^{LC}$ is uniquely decomposed with respect to

$$\Lambda^2 TM = g_M \oplus t_M$$

$$\Gamma^{LC} = \Gamma^g - A^1,$$

where $\Gamma^g \in \Omega^1(M, g_M)$ and $A^1 \in \Omega^1(M, t_M)$. One easily checks that $\Gamma^g$ transforms as a connection form corresponding to some connection $\nabla^g$ in the tangent bundle, while $A^1$ is a well defined tensor. It further follows that, irrespectively of the frame,

$$\nabla^g_X Y - \Gamma^{LC}_X Y = A^1(X)(Y).$$

\[2\] By $A(X)(Y)$ we mean the element of $\text{End } TM$, obtained by evaluation of $A$ on $X$, acting on $Y$.\[\square\]
2.2. \textit{G–STRUCTURES AND INTRINSIC TORSION}

It is not difficult to check, that $A^i \in \Omega^1(M, \mathfrak{t}_M)$ coincides with a pull-back by an arbitrary section $e : M \to Q$ of the intrinsic torsion $\alpha^i$ introduced in the previous subsection:

$$A^i(X) = e_x(e^*\alpha^i(X)) \quad \text{for } X \in T_xM$$

As the reader may expect, $A^i$ is equivalent to the torsion $T^g$ of $\nabla^g$:

$$T^g(X, Y) = A^i(X)(Y) - A^i(Y)(X),$$

$$2g(A^i(X)(Y), Z) = g(T^g(X, Y), Z) - g(T^g(Y, Z), X) + g(T^g(Z, X), Y).$$

2.2.3 Invariant tensors

Finally, we consider the following structure: let $\tilde{G} \subset O(m)$ be the orthogonal stabilizer group of a tensor $Y \in \otimes^p \mathbb{R}^m$, with $G$ its identity component and $\mathfrak{g}$ the Lie algebra. Then a $\tilde{G}$–structure on an $m$–dimensional manifold $(M, g)$ can be defined by a tensor

$$Y_M \in \otimes^p TM$$

such that at each point $x \in M$ there exists an orthonormal frame $e_x : \mathbb{R}^m \to T_xM$ such that $e_x(Y) = Y_M(x)$. Clearly, the set of all such frames at $x$ forms a fibre of a principal bundle with structure group $\tilde{G}$, a reduction $i : \tilde{Q} \to P$ of the orthonormal frame bundle.

In a typical situation, we will however be interested in having a $G$–structure $Q$, where $G$ is the identity component of possibly not connected $\tilde{G}$. A reduction of $\tilde{Q}$ to $Q$ is always possible locally (i.e. on $\pi^{-1}(U)$ for $U$ a neighbourhood of a point on $M$), by choosing a single point $q \in \tilde{Q}$, declaring it to be a member of $Q$ and identifying the latter by continuity. It may however fail to yield a $G$–structure over entire $M$, even if $Y_M$, and thus $\tilde{Q}$, are globally defined. Such a situation is illustrated by the following:

\textbf{Example 1.} A simple flat geometry related locally to the symmetric space $Sp(3)/U(3)$. Consider first the (real) space $M_0 = \mathbb{C}^6$ whose tangent spaces carry a natural complex structure and a fixed complex-linear identification with $V_2(\mathbb{R}) \simeq \mathbb{C}^6$, so that they become equipped with a metric $g_{M_0}$ and a parallel tensor $\Xi_{M_0}$ being a real section of $S^{3,3}TM_0$. The latter defines a trivial $\tilde{G}$–structure on $M_0$, and restricting to holomorphic frames gives a reduction to a trivial $G$–structure, $G = U(3)$.

Introduce now the natural complex coordinates $z^1, \ldots, z^6 : \mathbb{C}^6 \to \mathbb{C}$, such that $g_{M_0} = \sum_{i=1}^6 dz^i \overline{dz^i}$, and consider a (real) manifold $M$ obtained by identifying points subject to the equivalence relation

$$(z^1, z^2, \ldots, z^6) \sim (\bar{z}^1 + 2\pi, \bar{z}^2, \ldots, \bar{z}^6).$$

Considering the projection $p : M_0 \to M$ one finds that $M$ inherits a metric and orientation (complex conjugation on a space of even complex dimension has determinant one). The tensor $\Xi_{M_0}$ is also uniquely pushed forward to $\Xi_M$ on $M$, since $\Xi = \Xi_M$.

We thus have a $\tilde{G}$–structure on $M$, where $\tilde{G}$ is the full orthogonal isotropy group of $\Xi$, with $U(3)$ as its identity component. However, as we have just noted, $G$ also contains an antiunitary component, in particular the complex conjugation
CHAPTER 2. GEOMETRIC PART

map, $C : \mathbb{C}^6 \to \mathbb{C}^6$. A local $U(3)$-structure can be defined by picking a frame at a single point, say the one obtained by projecting the natural holomorphic frame at the origin of $M_0 = \mathbb{C}^6$ to

$$e_{p(0)} : \mathbb{C}^6 \to T_{p(0)}M.$$ 

However, parallel transporting $e_{p(0)}$ along a closed loop, given by the projection of a curve joining $(0, \ldots, 0)$ with $(2\pi, \ldots, 0)$ in $M_0$, one ends with $e_{p(0)} \circ C$, which clearly belongs to a different connected component of the fibre than the original frame. One thus fails to define a global $U(3)$ structure (which is already implied by the fact that the complex structure of $M_0$ does not descend to $M$). Note that this problem is not resolved by fixing an orientation.

Nevertheless, ignoring global questions and considering any of the local $G$-structures defined by $\mathcal{Y}_M$, we can recover most of the local information about the former from geometric data given by the latter (i.e. $\mathcal{Y}_M$ and its Levi-Civita derivatives). Note first, that in an adapted frame $e$ we have $e_x^{-1} \mathcal{Y}_M(x) = \mathcal{Y}$, a constant $\otimes \mathbb{R}^m$-valued function, so that

$$\nabla_X \mathcal{Y}_M = \Gamma(X)(\mathcal{Y}_M)$$

for a connection $\nabla$ with local connection form $\Gamma \in \Omega^1(M, \Lambda^2 T\mathcal{M})$. This leads to the following

**Lemma 32.** A connection $\nabla$ in the tangent bundle is compatible with a local $G$-structure defined by $\mathcal{Y}_M$ iff $\nabla \mathcal{Y}_M = 0$.

**Proof.** Choose an adapted frame $e$. Then for $X \in T_xM$

$$\nabla_X \mathcal{Y}_M = \Gamma(X)(\mathcal{Y}_M) = 0 \iff \Gamma(X) \in \mathfrak{g}_M.$$

The latter is the compatibility condition for $\nabla$. \hfill $\Box$

**Corollary 8.** A local $G$-structure defined by $\mathcal{Y}_M$ is integrable iff $\nabla^{LC} \mathcal{Y}_M = 0$.

While $\mathcal{Y}_M$ being parallel implies vanishing of the intrinsic torsion, it turns out that entire $A^t$ can be reconstructed from $\nabla^{LC} \mathcal{Y}_M$. Since $\mathfrak{t}$ is the complement of the isotropy algebra of $\mathcal{Y}$, we have the following obvious

**Lemma 33.** The kernel of

$$\mathfrak{t} \ni E \mapsto E(\mathcal{Y}) \in \otimes \mathbb{R}^m$$

is trivial.

**Corollary 9.** There exists a $G$-equivariant map $\varphi : \otimes \mathbb{R}^m \to \mathfrak{t}$ such that, independently of the choice of an adapted frame $e$,

$$A^t(X) = -(e_x \circ \varphi \circ e_x^{-1})\nabla_X^{LC} \mathcal{Y}_M \quad \text{for each } X \in T_xM.$$

**Proof.** Set $\varphi$ to be the left inverse of $E \mapsto E(\mathcal{Y})$, which exists due to the Lemma. Then, in an adapted frame $e$, we have

$$(e_x \circ \varphi \circ e_x^{-1})\nabla_X^{LC} \mathcal{Y}_M = (e_x \circ \varphi)(e_x^{-1} \circ A^t(X) \circ e_x(\mathcal{Y})) = e_x(e_x^{-1} \circ A^t(X) \circ e_x) = A^t(X).$$ \hfill $\Box$
Remark 2. We can derive a more direct result for a symmetric tensor
\[ \mathcal{Y} \in S^p(\mathbb{R}^m), \quad p \geq 3, \]
assuming \( \mathbb{R}^m \) to be \( G \)-irreducible. Since \( \mathcal{Y} \) is \( G \)-invariant, irreducibility implies
\[ \mathcal{Y}_{im_2...m_p} \mathcal{Y}_{jm_2...m_p} = \lambda_0 g_{ij} \]
for some \( \lambda_0 \in \mathbb{R} \), where \( g_{ij}X^iY^j = \langle X, Y \rangle \). Similarly, the map
\[ \mathcal{D}_{\mathcal{Y}} : \Lambda^2 \mathbb{R}^m \to \Lambda^2 \mathbb{R}^m \]
\[ \mathcal{D}_{\mathcal{Y}}(E)_{ij} = E_{kl} \mathcal{Y}_{km_2...m_p} \mathcal{Y}_{lm_2...m_p} \]
restricted to \( t \) is given by
\[ \mathcal{D}_{\mathcal{Y}}|_t = \lambda_1 p_{r_1} + \cdots + \lambda_r p_{r_t}, \]
for some \( \lambda_1, \ldots, \lambda_r \in \mathbb{R} \), where \( p_{r_1}, \ldots, p_{r_t} \) are projections onto \( G \)-irreducible subspaces of \( t : \)
\[ t = t_1 \oplus \cdots \oplus t_r. \]
We finally introduce the map
\[ \mathcal{Y}^t : t \to \mathcal{Y}^t \]
\[ \mathcal{Y}^t(E)_{ij} = E_{kl} \mathcal{Y}_{km_2...m_p} \mathcal{Y}_{lm_2...m_p} \]
and, performing the contractions, easily find that \( \mathcal{Y}^t = \lambda_0 \text{id} + (p-1)\mathcal{D}_{\mathcal{Y}} \).
If we thus calculate \( \lambda_0, \lambda_1, \ldots, \lambda_r \) and check that \( \lambda_0 + (p-1)\lambda_k \neq 0 \) for \( k = 1, \ldots, r \), we can express the intrinsic torsion as in Corollary 9 with
\[ \varphi(t)_{ij} = \sum_{k=1}^r \frac{1}{\lambda_0 + (p-1)\lambda_k} (p_{r_k})^{k l}_{ij} t_{km_2...m_p} \mathcal{Y}_{lm_2...m_p} \]
for \( t \in S^p \mathbb{R}^m \). As one can find in Section 2.6 the space \( t \) in case of \( G(\mathbb{K}, \mathbb{K}') \)-structures decomposes into at most two subspaces. Thus the projections \( p_{r_k} \) with \( k = 1, 2 \) can be expressed as combinations of \( \mathcal{D}_{\mathcal{Y}} \) and the identity map (provided \( \lambda_1 \neq \lambda_2 \)). This will provide us with explicit expressions for \( A^t \) in terms of \( \mathcal{Y}_M \) and \( \nabla^{KC} \mathcal{Y}_M \).

2.3 \( g(\mathbb{K}, \mathbb{K}') \)-geometries and their torsion

We will now finally define the geometries we are to investigate. Our choice is to consider Riemannian manifolds equipped with the sole invariant tensor, disregarding global existence of a \( G(\mathbb{K}, \mathbb{K}') \)-structure and its local choice. When referring to a local \( G(\mathbb{K}, \mathbb{K}') \)-structure associated with the \( g(\mathbb{K}, \mathbb{K}') \)-geometry, we mean any of the possible ones (i.e. any connected component of the full bundle of frames defined on some region of the manifold and preserving the special form of the tensor).3

Recall the conventions summarized at the beginning of this chapter. In particular, \( \mathbb{K} \) is any of \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \), and \( \mathbb{K}' \) is set to \( \mathbb{C}, \mathbb{H}, \mathbb{O} \) in, respectively, first, second and third family. We have defined \( \kappa = \dim \mathbb{K} \). To avoid index clutter, we abandon the convention of writing \( \mathcal{Y}_M \) in favour of \( \mathcal{Y} \), provided \( M \) is clear from context.

3 In particular, the tensor gives global \( G(\mathbb{K}, \mathbb{K}') \)-structures on a simply connected manifold. This can be seen noting, that every loop lifted to the full bundle of adapted frames (i.e. the one whose structure group is the full isotropy group of the tensor) must necessarily start and end in the same connected component of a fibre.
2.3.1 First family

The following definition essentially coincides with that of the geometries studied by Nurowski [1]:

**Definition 3.** A \( g(K, C) \)-geometry is a \((3\kappa + 2)\)-dimensional Riemannian manifold \( M \) equipped with a tensor

\[
\Upsilon \in \Omega^0(M, S^3TM)
\]

such that at each \( x \in M \) there exists an orthonormal frame \( e_x : V_1(K) \to T_xM \) such that \( \Upsilon = e_x(\Upsilon) \).

Let thus \( M \) be a \( g(K, C) \)-geometry. Since in the first family the groups \( G(K, C) \) are the precise isotropy groups of \( \Upsilon \), it follows that the global principal bundle of all frames mapping \( \Upsilon \) to \( \Upsilon \) has \( G(K, C) \) as its structure group:

**Proposition 6.** A \( g(K, C) \)-geometry possesses a natural global \( G(K, C) \)-structure.

In what follows, we shall use \( i, j, \ldots \) to index \( TM \simeq T^*M \), when referring to tensors and tensor-valued forms, so that we have e.g.

\[
\Upsilon_{ijk} \in \Omega^0(M, (S^3TM)_{ijk}) \quad (A^i)_{ij} \in \Omega^1(M, (\Lambda^2TM)_{ij}) \quad (\nabla^{LC}\Upsilon_M)_{ijk} \in \Omega^1(M, (S^3TM)_{ijk}).
\]

**Proposition 7.** The intrinsic torsion of the \( G(K, C) \)-structure associated with a \( g(K, C) \)-geometry \( (M, \Upsilon) \) is given by

\[
(A^i)_{ij} = \frac{3\kappa + 4}{3\kappa + 10} (\nabla^{LC}\Upsilon)_{imn}\Upsilon_{jmn}.
\]

**Proof.** The proof is by application of the procedure described in Remark [2]. The decompositions of \( \Lambda^2V_1(K) \) given in Section 2.6 show that \( t = g(K, C)^\perp \) is always irreducible; one has

\[
D_{\Upsilon} = \frac{3}{3\kappa + 4},
\]

according to the summary presented in the first section of this chapter. One thus uses the formula (2.7) with \( \lambda_0 = 1, \lambda_1 = \frac{1}{3\kappa + 4} \) and \( p = 3 \), while the projection is simply identity. Having the map \( \varphi : S^3V_1(K) \to t \) expressed using \( \Upsilon \), the Proposition follows from Corollary [6].

2.3.2 Second family

**Definition 4.** A \( g(K, \mathbb{H}) \)-geometry is a \((6\kappa + 6)\)-dimensional Riemannian manifold \( M \) equipped with a tensor

\[
\Xi \in \Omega^0(M, S^6TM)
\]

such that at each \( x \in M \) there exists an orthonormal frame \( e_x : V_2(K) \to T_xM \) such that \( \Xi = e_x(\Xi) \), where \( V_2(K) \) is considered as a real vector space.
2.3. $\mathfrak{g}(\mathbb{K}, \mathbb{H})$-GEOMETRIES AND THEIR TORSION

Let thus $M$ be a $\mathfrak{g}(\mathbb{K}, \mathbb{H})$-geometry. As it has been demonstrated in Example 1, there may be in general no globally defined $G(\mathbb{K}, \mathbb{H})$-structure associated to $\Xi$. However, one has a global split

$$\Lambda^2 TM = \mathfrak{g}_M \oplus \mathfrak{t}_M$$

with $\mathfrak{g}_M(x) \simeq \mathfrak{g}(\mathbb{K}, \mathbb{H})$ as a Lie subalgebra of $\Lambda^2 T_x M \simeq \mathfrak{so}(V_2(\mathbb{K}))$. Recall that the algebra $\mathfrak{g}(\mathbb{K}, \mathbb{H})$ has a one-dimensional centre spanned by the hermitian complex structure of $V_2(\mathbb{K})$. The same is true of each $\mathfrak{g}_M(x)$, so that defining a bundle

$$u(1)_M \subset \mathfrak{g}_M, \quad u(1)_M(x) = \mathcal{Z}(\mathfrak{g}_M(x)),$$

with $\mathcal{Z}$ defining the centre, we have the following

**Proposition 8.** A $\mathfrak{g}(\mathbb{K}, \mathbb{H})$-geometry $(M, \Xi)$ is naturally equipped with a one-dimensional subbundle $u(1)_M \subset \Lambda^2 TM$ spanned locally by an almost hermitian structure.

Each fibre $u(1)_M(x)$ contains exactly two complex structures on $T_x M$ (singled out by normalization), and there is a priori no way to distinguish between these. A particular choice of one complex structure at a single point of $M$ can be extended by continuity to a neighbourhood, but may fail to yield a globally defined almost hermitian structure on $M$ (cf. Example 1).

Before we proceed, an algebraic consideration is necessary:

**Lemma 34.** Reintroducing $N = 3\kappa + 3$, we have:

1. $\Xi_{abcdefkl}^\Xi = \Xi_{abcdefkl} = \frac{N}{30}\delta^a_b$.

2. Let us introduce a map

$$D_\Xi : \Lambda^2 V_2(\mathbb{K}) \to \Lambda^2 V_2(\mathbb{K})$$

$$D_\Xi(E)_{ab} = E_{cd} \Xi_a^{defkl} \Xi_b^{efkl}.$$ 

Then, with respect to the decomposition

$$\Lambda^2 V_2(\mathbb{K}) = u(1) \oplus G^r_2(\mathbb{K}) \oplus (t \cap \Lambda^{1,1}) \oplus \text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2})$$

the map $D_\Xi$ is given by

$$200 D_\Xi|_{u(1)} = 2N + 3$$

$$200 D_\Xi|_{G^r_2(\mathbb{K})} = -N$$

$$200 D_\Xi|_{t \cap \Lambda^{1,1}} = \frac{2N}{\kappa + 2}$$

$$200 D_\Xi|_{\text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2})} = 3.$$ 

The proof is given in Section 2.8. We are now ready to express the intrinsic torsion in terms of $\Xi$ and its derivative. In the following we use $a, b, \ldots$ to index $TM \simeq T^* M$ :

**Proposition 9.** The intrinsic torsion of a local $G(\mathbb{K}, \mathbb{H})$-structure associated with a $\mathfrak{g}(\mathbb{K}, \mathbb{H})$-geometry $(M, \Xi)$ is given by

$$(A^I)_{ab} = \frac{40}{3\kappa^2 + 15\kappa + 12} \left[ \frac{\kappa^2 + 6\kappa + 6}{\kappa + 2} \delta^d_e \delta^d_h - \frac{200}{3} \Xi_a^{defkl} \Xi_b^{efkl} \right] \times (\nabla^L C_\Xi)^{cdefkl} \Xi_d^{mnrst}.$$
Proof. The proof is by application of the procedure described in Remark 2. The decompositions of \( \Lambda^2 V_2(K) \) given in Section 2.6 show that \( t = g(K, C)^{12} \) decomposes into two irreducible subspaces:

\[
t = (t \cap \Lambda^{1,1}) \oplus \text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2}).
\]

The map \( \mathcal{D}_T|_t \) and the contraction \( \Xi_{adefkl} \Xi^{efkl} \) is found in Lemma 34 so that one uses the formula (2.7) with \( \lambda_0 = \frac{N}{20}, \lambda_1 = \frac{1}{100} \), \( \lambda_2 = \frac{1}{200} \) and \( p = 6 \), while the projections are (having checked that \( \lambda_1 \neq \lambda_2 \) for \( \kappa = 1, 2, 4, 8 \))

\[
\begin{align*}
\text{pr}_{t \cap \Lambda^{1,1}} &= -\frac{\kappa + 2}{\kappa} \text{id} + \frac{\kappa + 2}{3\kappa} 200 D \Xi \\
\text{pr}_{\text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2})} &= \frac{2\kappa + 2}{\kappa} \text{id} - \frac{\kappa + 2}{3\kappa} 200 D \Xi.
\end{align*}
\]

Having the map \( \varphi : S^6 V_2(K) \rightarrow t \) expressed using \( \Xi \), the Proposition follows from Corollary 9.

Recall that, by Proposition 8, normalized local sections of \( u(1)_M \) give local almost hermitian structures on \( M \). These are integrable iff \( u(1)_M \) is parallel, i.e. iff

\[
f \in \Omega^0(M, u(1)_M) \implies \nabla^{\text{LC}} f \in \Omega^1(M, u(1)_M).
\]

Since \( \nabla^g \) has this property, it follows that in this case the intrinsic torsion must satisfy

\[
[A^i(X), u(1)_M(x)] \subset u(1)_M(x)
\]

for each \( X \in T_xM \). Expressed in an adapted frame \( e_x : V_2(K) \rightarrow T_xM \), the condition reads

\[
A^i(X) \in e_x(\text{Re}\Lambda^{1,1})
\]

i.e. the antiunitary component, obtained by projecting onto Re(\( \Lambda^{2,0} \oplus \Lambda^{0,2} \)), must vanish. Applying the projection (2.8) to the intrinsic torsion expressed in Proposition 9 the following formula is readily proved:

**Proposition 10.** A \( g(K, \mathbb{H}) \)-geometry \( (M, \Xi) \) possesses a natural local Kähler structure, given by a unit local section of \( u(1)_M \), iff

\[
\left[ \Xi_{adefkl} \Xi^{efkl}, \frac{3}{100} \frac{\kappa + 1}{\kappa} \frac{1}{2} \delta_a^c \delta_b^d \right] (\nabla^{\text{LC}} \Xi)_{nmrst} \Xi_{mnmrst} = 0.
\]

### 2.3.3 Third family

**Definition 5.** A \( g(K, \mathcal{O}) \)-geometry is a \((12\kappa + 16)\)-dimensional Riemannian manifold \( M \) equipped with a tensor

\[
\mathcal{U} \in \Omega^0(M, \mathcal{H}^{8}TM)
\]

such that at each \( x \in M \) there exists an orthonormal frame \( e_x : V_3(K) \rightarrow T_xM \) such that \( \mathcal{U} = e_x(\mathcal{U}) \).

Let thus \( M \) be a \( g(K, \mathcal{O}) \)-geometry. There exists a global split

\[
\Lambda^2 TM = g_M \oplus t_M
\]
such that \( g_M(x) \simeq g(K, \mathcal{O}) \) at each \( x \in M \). Recall that \( g(K, \mathcal{O}) \) decomposes under its adjoint representation into two a direct sum
\[
g(K, \mathcal{O}) = G_3^*(K) \oplus \mathfrak{sp}(1)
\]
with \( \mathfrak{sp}(1) \) generated by a quaternionic structure on \( V_3(K) \). The same is true of each \( g_M(x) \), so that defining a bundle
\[
\mathfrak{sp}(1)_M \subset g_M, \quad [g_M, \mathfrak{sp}(1)_M] = \mathfrak{sp}(1)_M, \quad \dim \mathfrak{sp}(1)_M(x) = 3
\]
we have the following

**Proposition 11.** A \( g(K, \mathcal{O}) \)-geometry \( (M, \mathcal{U}) \) is naturally equipped with a three-dimensional subbundle \( \mathfrak{sp}(1)_M \subset \Lambda^2 TM \) spanned locally by three almost hermitian structures subject to the algebra of imaginary quaternions.

Note that the full orthogonal isotropy group of \( \mathcal{U} \) necessarily preserves the structure constants of \( \mathfrak{sp}(1) \subset g(K, \mathcal{O}) \), and thus an orientation on this space, so that it acts on it as \( SO(3) \simeq \text{Aut} \mathbb{H} \). This in turn implies that the images of \( I, J, K \in \Lambda^2 V_3(K) \) under different adapted frames are related by automorphisms of the unique quaternionic structure. Hence the following

**Corollary 10.** A \( g(K, \mathcal{O}) \)-geometry \( (M, \mathcal{U}) \) is naturally equipped with a unique global almost quaternion-hermitian structure defined by
\[
\Omega \in \Omega^4(M), \quad \Omega(x) = e_x(I \wedge I + J \wedge J + K \wedge K)
\]
in an arbitrary adapted frame \( e_x : V_3(K) \to T_x M \).

Once again some algebra is needed before we can express the intrinsic torsion:

**Lemma 35.** Reintroducing \( N = 3\kappa + 4 \), we have:
1. \( L_{x_m m}^{j_{m_2} \ldots m_s} \delta_{j_{m_2} \ldots m_s} = \frac{64}{35} N + 1 \sqrt{25(N - 1) + 12\chi^2} \delta_{x} \)
2. Let us introduce a map
\[
D_{\mathcal{U}} : \Lambda^2 V_3(K) \to \Lambda^2 V_3(K)
\]
\[
D_{\mathcal{U}}(E)_{ab} = E_{cd} \delta_{a} \delta_{b} \delta_{d} \delta_{m} \delta_{s} \delta_{m} \delta_{s} 
\]
Then, with respect to the decomposition
\[
\Lambda^2 V_3(K) = \mathfrak{sp}(1) \oplus \mathfrak{sp}^*(3) \oplus [t \cap \mathfrak{sp}(V_3(K, \omega))] \oplus \bot,
\]
the map \( D_{\mathcal{U}} \) is given by
\[
D_{\mathcal{U}}|_{\mathfrak{sp}(1)} = \frac{32}{245} (30N^2 - 9N - 21 - 636\chi^2)
\]
\[
D_{\mathcal{U}}|_{\mathfrak{sp}^*(3)} = \frac{32}{245} (51N^2 + 13N - 83 - 18\chi^2 - \sqrt{\kappa + 3})
\]
\[
D_{\mathcal{U}}|_{t \cap \mathfrak{sp}(V_3(K, \omega))} = \frac{32}{245} (51N^2 + 13N - 83 - 18\chi^2 + \frac{1}{\sqrt{\kappa + 3}})
\]
\[
D_{\mathcal{U}}|_{\bot} = \frac{32}{245} (15N^2 + 53N + 47 - 386\chi^2)
\]

There are many equivalent ways of introducing (almost) quaternion-hermitian structures on a manifold — e.g. by means of a 4-form, or a 2-sphere bundle in \( \text{End} TM \). For reference, see \([34,35,36]\).
The proof can be found in Section 2.8. We are now able to read the intrinsic torsion from \( \hat{\mathcal{U}} \) and its derivative:

**Proposition 12.** The intrinsic torsion of a local \( G(\mathcal{K}, \mathcal{O}) \)-structure associated with a \( g(\mathcal{K}, \mathcal{O}) \)-geometry \((M, \hat{\mathcal{U}})\) is given by

\[
(A^t)_{ab} = \gamma^{-1} [\alpha \hat{\mathcal{U}}_a \delta_{b_{m_3 \ldots m_8}} \hat{\mathcal{U}}_b + \beta \delta_d^c \delta_e^d] \\
\times (\nabla^{LC})_{c_{n_3 \ldots n_8}} \hat{\mathcal{U}}_d^{k_{n_2 \ldots n_8}}
\]

where

\[
\alpha = \frac{1}{\lambda_0 + 7\lambda_1 - \lambda_0 + 7\lambda_2} \\
\beta = -\frac{\lambda_2}{\lambda_0 + 7\lambda_1} + \frac{\lambda_1}{\lambda_0 + 7\lambda_2} \\
\gamma = \frac{32}{245} \left( 36N^2 + 40N - 130 + 368\chi^2 + \frac{1}{\sqrt{\kappa + 3}} \right) \\
\lambda_0 = \frac{64}{35} N + 1 - \frac{1}{2} [25(N - 1) + 12\chi^2] \\
\lambda_1 = \frac{32}{245} \left( 51N^2 + 13N - 83 - 18\chi^2 + \frac{1}{\sqrt{\kappa + 3}} \right) \\
\lambda_2 = \frac{32}{245} (15N^2 + 53N + 47 - 386\chi^2).
\]

Before we give the proof, we must sadly admit that even after \( N, \chi \) and \( \kappa \) have been substituted by numbers for given \( \mathcal{K} \), the constants remain unreasonably complicated.

**Proof.** The proof is by application of the procedure described in Remark 2. The decompositions of \( \wedge^2 V_3(\mathcal{K}) \) given in Section 2.6 show that \( t = g(\mathcal{K}, \mathcal{O})^\perp \) decomposes into two irreducible subspaces:

\[
t = [t \cap \text{sp}(V_3, \omega)] \oplus [\text{sp}(V_3, \omega) + \text{sp}(1)]^\perp.
\]

The map \( D_\hat{\mathcal{U}}|_t \) and the contraction \( \hat{\mathcal{U}}_{a m_2 \ldots m_8} \hat{\mathcal{U}}^{b m_2 \ldots n} \) is found in Lemma 35, so that one uses the formula (2.7) with

\[
\lambda_0 = \frac{64}{35} N + 1 - \frac{1}{2} [25(N - 1) + 12\chi^2] \\
\lambda_1 = \frac{32}{245} \left( 51N^2 + 13N - 83 - 18\chi^2 + \frac{1}{\sqrt{\kappa + 3}} \right) \\
\lambda_2 = \frac{32}{245} (15N^2 + 53N + 47 - 386\chi^2).
\]

and \( p = 8 \), while the projections are (having checked that \( \lambda_1 \neq \lambda_2 \) for \( \kappa = 1, 2, 4, 8 \))

\[
\text{pr}_{t \cap \text{sp}(V_3(\mathcal{K}), \omega)} = \frac{1}{\lambda_1 - \lambda_2} (D_\hat{\mathcal{U}} - \lambda_2 \text{id})
\]

\[
\text{pr}_{[\text{sp}(V_3(\mathcal{K}), \omega) + \text{sp}(1)]^\perp} = \frac{1}{\lambda_2 - \lambda_1} (D_\hat{\mathcal{U}} - \lambda_1 \text{id}).
\]

Having the map \( \varphi : S^8 V_3(\mathcal{K}) \rightarrow t \) expressed using \( \hat{\mathcal{U}} \), the Proposition follows from Corollary 9. \( \square \)
2.4. G-STRUCTURES WITH CHARACTERISTIC TORSION

The condition for integrability of the almost-quaternion-hermitian structure described in Corollary 10 is that the Levi-Civita parallel transport preserve a bundle of unit 2-spheres in \( \mathfrak{sp}(1) \):

\[ S \subset \mathfrak{sp}(1)_M, \quad S_x = \{ E \in \mathfrak{sp}(1)_M(x) \mid E^2 = -\text{id} \}, \]

which under this condition becomes the natural twistor bundle of the quaternion-Kähler structure (cf. [34]). It is equivalent to

\[ f \in \Omega^0(M, \mathfrak{sp}(1)_M) \implies \nabla^{\text{LC}} f \in \Omega^1(M, \mathfrak{sp}(1)_M). \]

Since \( \nabla^g \) has this property, it follows that in this case the intrinsic torsion must satisfy

\[ [A^i(X), \mathfrak{sp}(1)_M(x)] \subset \mathfrak{sp}(1)_M(x) \]

for each \( X \in T_xM \). Expressed in an adapted frame \( e_x : V_3(\mathbb{K}) \to T_xM \), the condition reads

\[ A^i(X) \in e_x(\mathfrak{sp}(V_3(\mathbb{K}), \omega)) \]

i.e. the component in \( \mathfrak{sp}(V_3(\mathbb{K}), \omega)^\perp \) must vanish. Applying the projection \( (2.9) \) to the intrinsic torsion expressed in Proposition 12 the following formula is readily proved:

**Proposition 13.** A \( \mathfrak{g}(\mathbb{K}, \mathcal{O}) \)-geometry \( (M, \mathcal{O}) \) possesses a natural quaternion-Kähler structure, given by the 2-sphere bundle

\[ S \subset \mathfrak{sp}(1)_M, \quad S_x = \{ E \in \mathfrak{sp}(1)_M(x) \mid E^2 = -\text{id} \}, \]

iff

\[ \sum_{b}^{d} \epsilon_{bms_{1}} \sum_{c}^{e} \epsilon_{c_{n}s_{2}} - \lambda_1 \delta_{a}^{c} \delta_{b}^{d} (\nabla^{LC} \mathcal{O})_{c_{n}s_{2}} \sum_{d}^{a} \epsilon_{d_{n}s_{3}} = 0 \]

where

\[ \lambda_1 = \frac{32}{245} \left( 51N^2 + 13N - 83 - 18\chi^2 + \frac{1}{\sqrt{\kappa} + 3} \right). \]

2.4 G-structures with characteristic torsion

In the previous section we have at last defined the geometries modelled after the Magic Square symmetric spaces, and expressed their intrinsic torsion in terms of geometric data, i.e. the defining tensorial invariant and its Levi-Civita derivative. It is of course natural to investigate at first the integrable case, that is, manifolds with parallel \( \mathfrak{I}, \mathfrak{X} \) or \( \mathfrak{O} \). However, we immediately have the following

**Proposition 14** (Corollary of Berger’s theorem). Let \( (M, g, \mathcal{Y}_M) \) be a \( \mathfrak{g}(\mathbb{K}, \mathcal{K}') \)-geometry, whose underlying local \( G(\mathbb{K}, \mathbb{K}') \)-structures are integrable. Then \( (M, g) \) is either a locally symmetric space or a product of Riemannian manifolds of lower dimension, with the product metric tensor.

**Proof (sketch).** Integrability implies that the connected holonomy group \( \text{Hol}(g) \), considered via some adapted frame as a subgroup of \( GL(V_n(\mathbb{K})) \), must be contained in \( G(\mathbb{K}, \mathbb{K}') \). Simple dimension count shows that none of the irreducible Riemannian holonomy groups listed in Berger’s theorem meet this requirement. Thus, \( (M, g) \) cannot be irreducible and not locally symmetric. \( \square \)
A larger variety of geometries is available once one relaxes the integrability condition to some extent. While integrability is equivalent to existence of a compatible connection with trivial torsion, we can consider a milder condition of vanishing all but one of the torsion’s irreducible components. The torsion tensor of a connection on a Riemannian manifold \((M,g)\) is a section of the bundle \(\Lambda^2 TM \otimes TM\), which decomposes under the action of the orthogonal group into three (for \(\dim M > 3\)) irreducible components:

\[
\Lambda^2 TM \otimes TM \simeq \Lambda^3 TM \oplus TM \oplus T.
\]

Particularly interesting classes of connections are those with completely skew \((\Lambda^3 TM)\) and vectorial \((TM)\) torsion, where the associated projections act on \(T_{ijk} \in (\Lambda^2 TM)_{ij} \otimes (TM)_k\) as:

\[
(pr_{\text{skew}} T)_{ijk} = T_{[ijk]}, \quad (pr_{\text{vec}} T)_{ijk} = \frac{1}{\dim M} T_{[i j] k}.
\]

In what follows, we shall restrict our attention to the skew case (information about the vectorial one can be found e.g. in \([37]\), mostly following the exposition given in \([13]\). A geometric characterisation of this class is given by the following

**Proposition 15** (cf. \([13]\)). A metric connection \(\nabla\) in the tangent bundle of a Riemannian manifold \((M,g)\) has completely skew torsion iff its unparametrised geodesics coincide with those of the Levi-Civita connection on \(M\).

Let us first introduce a simple

**Lemma 36.** Let \(\nabla\) be a metric connection in the tangent bundle of a Riemannian manifold \((M,g)\) and \(A \in \Omega^1(M, \Lambda^2 TM)\) its difference tensor with the Levi-Civita connection:

\[
A(X)(Y) = \nabla_X Y - \nabla^LC_X Y.
\]

Then \(\nabla\) has completely skew torsion \(T\) iff \(A\) is completely skew, i.e.

\[
A(X)(Y) = -A(Y)(X).
\]

Moreover, in such case \(A = \frac{1}{2} T\) as a section of \(\Lambda^3 TM\).

**Proof.** The torsion of \(\nabla\) is

\[
T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \nabla^LC_X Y - \nabla^LC_Y X - [X,Y] + A(X)(Y) - A(Y)(X)
\]

and

\[
g(T(X,Y), Z) + g(T(X,Z), Y) = g(A(X)(Y), Z) - g(A(Y)(X), Z) + g(A(X)(Z), Y) - g(A(Z)(X), Y) = g(A(Y)(Z), X) + g(A(Z)(Y), X).
\]

Vanishing of l.h.s. is equivalent to complete antisymmetry of \(T\), while vanishing of r.h.s. is equivalent to complete antisymmetry of \(A\).
2.4. $G$-STRUCTURES WITH CHARACTERISTIC TORSION

Proof of Proposition 15. Let $\gamma : \epsilon, \epsilon \rightarrow M$ be a curve in $M$ and $X$ a vector field defined on some neighbourhood of $\gamma$ such that $X \circ \gamma$ is tangent to $\gamma$. Then $\gamma$ is (unparametrised) geodesic of $\nabla$ iff

$$(\nabla_X X) \circ \gamma = (\nabla^{LC}_X X + A(X)(X)) \circ \gamma = f X \circ \gamma$$

for some function $f$. The latter is equivalent to $\gamma$ being a Levi-Civita geodesic iff $A(X)(X)$ is a multiple of $X$. However, since $A(X)(X)$ is skew, $A(X)(X)$ must be simply zero. Demanding it for every curve $\gamma$ is equivalent to $A(X)(X) = 0$ for each $X \in TM$, i.e. $A$ completely skew. This in turn, via the previous Lemma, is equivalent to complete antisymmetry of the torsion of $\nabla$. 

2.4.1 $G$-structures with skew torsion

Let us now consider a general $G$-structure $Q$ on an $m$-dimensional Riemannian manifold $(M,g)$ as described in subsection 2.2.1. The basic result is the following:

Proposition 16 (cf. [13]). A $G$-structure $Q$ on $M$ admits a compatible connection with completely skew torsion iff there exists a function $H \in \Omega^1(Q) \otimes \Lambda^3 \mathbb{R}^m$ (of the natural type) such that for each $X \in TQ$

$$\alpha^t(X) = \text{pr}_1 H(\theta(X))$$

where $\alpha^t$ is the intrinsic torsion of $Q$, and $\theta$ the soldering form.

We first give a simple Lemma 37.

A connection $\omega$ compatible with a $G$-structure $Q$ has skew torsion iff

$$[\omega(X) - \omega^{LC}(X)](\theta(Y)) = -[\omega(Y) - \omega^{LC}(Y)](\theta(X)).$$

Proof. The torsion of $\omega$ is

$$\Theta = d\theta + \omega \wedge \theta = (d\theta + \omega^{LC} \wedge \theta) + (\omega - \omega^{LC}) \wedge \theta = \alpha \wedge \theta,$$

where we introduced $\alpha = \omega - \omega^{LC}$, so that $\alpha|_Q \in \Omega^1(Q) \otimes g$ is horizontal of type Ad. We now have

$$\langle \Theta(X,Y), \theta(Z) \rangle + \langle \Theta(X,Z), \theta(Y) \rangle = \alpha(X)(\theta(Y),\theta(Z) - \alpha(Y)(\theta(X),\theta(Z)) + \alpha(X)(\theta(Z),\theta(Y) - \alpha(Z)(\theta(X),\theta(Y)) = \alpha(Y)(\theta(Z),\theta(X)) + \alpha(Z)(\theta(Y),\theta(X)),$$

where vanishing of l.h.s is equivalent to complete antisymmetry of the torsion.  

Proof of Proposition 16. Assume that indeed $\alpha^t$ satisfies this condition for some $H$. Then

$$\omega^{LC}|_Q = \omega^g - \alpha^t = \omega^g|_Q - H(\theta(\cdot)) + \beta$$
where \( \beta \in \Omega^1_{tor}(Q) \otimes g \). One easily checks that \( \omega^s = \omega^g + \beta \) defines a connection on \( Q \). Now, we have \( \omega^s - \omega^{LC} \big|_Q = H(\theta(\cdot)) \), and by Lemma 37 the torsion of \( \omega^s \) is skew.

Conversely, the same lemma implies that the difference of a connection \( \omega^s \) with skew torsion and the Levi-Civita connection is a completely skew tensor, and the function \( H \) is simply given by

\[
H_q(\theta(X), \theta(Y), \theta(Z)) = \langle (\omega(X) - \omega^{LC}(X))\theta(Y), \theta(Z) \rangle
\]

for \( X, Y, Z \in T_qQ \). We have

\[
\alpha^t(X) = H(\theta(X)) + \omega^g(X) - \omega^s(X),
\]

and projecting on \( t \) proves the Proposition.

Having stated a condition for existence of a compatible connection with skew torsion, it is natural to ask how many such connections can be found. A particularly interesting case occurs when the connection is unique. There is a well known, purely algebraic condition:

**Proposition 17** (cf. [13]). A compatible connection with skew torsion on a \( G \)-structure \( Q \) is unique, provided it exists, iff

\[
(\mathbb{R}^m \otimes g) \cap \Lambda^3 \mathbb{R}^m = 0. \tag{2.10}
\]

**Proof.** Assume \( \omega^s \) and \( \omega^\ast \) are two such connections. Then, by Lemma 37 the function \( C \in \Omega^0(Q) \otimes (\mathbb{R}^m \otimes g) \) defined by

\[
C_q(\theta(X), \theta(Y), \theta(Z)) = \langle (\omega^s(X) - \omega^\ast(X))\theta(Y), \theta(Z) \rangle
\]

\[
= \langle (\omega^s(X) - \omega^{LC}(X))\theta(Y), \theta(Z) \rangle
\]

\[
- \langle (\omega^\ast(X) - \omega^{LC}(X))\theta(Y), \theta(Z) \rangle
\]

for \( X, Y, Z \in T_qQ \) is completely skew. Thus, if the intersection of \( \mathbb{R}^m \otimes g \) and \( \Lambda^3 \mathbb{R}^m \) is trivial, one has \( \omega^s = \omega^\ast \).

Conversely, let \( \omega^s \) be the unique compatible connection with skew torsion and \( C \in \Omega^0(Q) \otimes [(\mathbb{R}^m \otimes g) \cap \Lambda^3 \mathbb{R}^m] \) a function of the natural type. Then

\[
\omega^\ast(X) = \omega^s(X) - C(\theta(X))
\]

for \( X \in TQ \) defines a connection on \( Q \) and by Lemma 37 its torsion is skew. Now, if the intersection of \( \mathbb{R}^m \otimes g \) and \( \Lambda^3 \mathbb{R}^m \) was nontrivial, \( \omega^\ast(X) \neq \omega^s(X) \) and there would exist different compatible connections with skew torsion. As it contradicts the uniqueness of \( \omega^s \), the intersection must be trivial.

If such a connection is indeed unique, it is called the **characteristic connection** of the \( G \)-structure, and its torsion tensor – the characteristic torsion. It is not very difficult to check the intersections (2.10) for classical irreducible holonomy groups, and the result is that a skew-torsion connection is unique in all those cases [13]. Moreover, there is a recent important result of Nagy, which solves the problem of computing the l.h.s. of (2.10) completely:

**Proposition 18** (Nagy [5]). Let \( g \subset \mathfrak{so}(m) \) be proper and act irreducibly on \( \mathbb{R}^m \). Then the intersection (2.10) is trivial, unless \( g \) compact simple and \( \mathbb{R}^m \simeq g \) as a \( g \)-module. In the latter case, it is one dimensional and spanned by the structure constants of \( g \).
Considering the case with nontrivial intersection (i.e. no characteristic connection), we have:

**Corollary 11.** Let $G$ be simple, with $\mathfrak{g}$ proper in $\mathfrak{so}(m)$ and $\mathbb{R}^m \simeq \mathfrak{g}$ as a $G$-module. Then a $G$-structure on $M$ admits either no compatible connections with skew torsion, or precisely a one-parameter family thereof. In the latter case, the torsion tensors of connections in this family differ by a section of a one-dimensional invariant subbundle $T_0 \subset \Lambda^3 TM$. Choosing an adapted frame $e$, an intertwiner $\psi : \mathfrak{g} \rightarrow \mathbb{R}^m$, and defining $f = e \circ \psi$, we have a fibre of $T_0$:

$$T_0(x) = \text{Span}\{C\}, \quad C(X, Y, Z) = \langle e_x^{-1}X, \psi[f_x^{-1}Y, f_x^{-1}Z] \rangle$$

for $X, Y, Z \in T_x M$.

**Proof.** Let $Q$ be the $G$-structure. Every connection $\omega \in \Omega^1(Q) \otimes \mathfrak{g}$ on $Q$ with skew torsion $\Theta \in \Omega^2(Q) \otimes \mathbb{R}^m$ is uniquely defined by the latter, since (cf. Lemma 36)

$$\langle \omega(X) - \omega^{LC}(X)\{\theta(Y), \theta(Z)\} = \frac{1}{2}\{\Theta(X, Y), \theta(Z)\}.$$  

We can introduce a function $H^\omega \in \Omega^0(Q) \otimes \Lambda^3 \mathbb{R}^m$ such that

$$\langle \Theta(X, Y), \theta(Z) \rangle = H^\omega(q)(\theta(X), \theta(Y), \theta(Z))$$

for each $X, Y, Z \in T_q Q$.

On the other hand, two such connections $\omega, \cong$ differ by a horizontal one-form of type $\text{Ad}$:

$$\omega - \cong \in \Omega^1(Q) \otimes \mathfrak{g},$$

so that

$$H^\omega(q)(\theta(X)) - H^\cong(q)(\theta(X)) \in \mathfrak{g}$$

for each $X \in T_q Q$, and thus

$$H^\omega(q) - H^\cong(q) \in \Lambda^3 \mathbb{R}^m \cap (\mathbb{R}^m \otimes \mathfrak{g}).$$

Assume now that there exists a skew-torsion connection $\omega_0$ on $Q$. It thus follows, that if we identify each skew-torsion connection $\omega$ on $Q$ with the corresponding function $H^\omega$, the space of all such connections at a point $q \in Q$ is the affine space

$$H^\omega(q) + [\Lambda^3 \mathbb{R}^m \cap (\mathbb{R}^m \otimes \mathfrak{g})].$$

We can now apply Proposition 18 to find that the intersection is spanned by the map

$$c : \Lambda^3 \mathbb{R}^m \rightarrow \mathbb{R}, \quad c(X, Y, Z) = \langle X, \psi^{-1}Y, \psi^{-1}Z \rangle,$$

for $X, Y, Z \in \mathbb{R}^m$, where $\psi : \mathbb{R}^m \rightarrow \mathfrak{g}$ is an intertwiner. Pulling everything back to $M$ via some adapted frame, i.e. a (local) section $e : M \rightarrow Q$, we arrive at the Corollary.

We shall refer to such a family as the one-parameter family of skew-torsion connections. Their torsion can be considered to be ‘characteristic modulo $T_0$'.
CHAPTER 2. GEOMETRIC PART

2.4.2 Invariant tensors and nearly-integrability

We shall finally focus on structures defined on a Riemannian manifold \((M, g)\) by a symmetric tensor \(Y \in \Omega^0(M, S^p TM)\), as in Subsection 2.2.3. The first important fact is a necessary condition for the existence of a compatible connection with skew torsion, first discussed by Nurowski [1]:

**Proposition 19.** Let a local \(G\)-structure defined by a symmetric tensor \(Y\) admit a compatible connection with skew torsion. Then the symmetrized Levi-Civita derivative of \(Y\) vanishes:

\[
(\nabla_{LC}^X Y)(X, \ldots, X) = 0 \quad \text{for each } X \in TM.
\]  

(2.11)

**Proof.** Let \(\nabla\) be such a connection. Clearly, \(\nabla Y = 0\). Then, recalling Lemma 36,

\[
\nabla_{LC}^X Y = \nabla_X Y - \frac{1}{2} T(X)(Y) = -\frac{1}{2} T(X)(Y)
\]

where \(T\) is the torsion of \(\nabla\), a section of \(\Lambda^3 TM \subset TM \otimes \Lambda^2 TM\). Indexing \(TM \simeq T^* M\) with \(i, j, k, \ldots\), we have

\[
\nabla_{LC}^{(i_1 Y_{j_1 \ldots j_p})} = \frac{p}{2} T_{(i_1 m jou \ldots j_p)} = 0
\]

due to the antisymmetry of \(T\).

The next natural step is to ask when is (2.11) sufficient. After [1], we give an algebraic condition on the tensor \(Y\), mapped to \(Y^\prime\) in adapted frames:

**Lemma 38.** Let us introduce a map

\[
Y^\prime : \mathbb{R}^m \otimes \Lambda^2 \mathbb{R}^m \to S^{p+1} \mathbb{R}^m
\]

\[
Y^\prime(A)_{i_0 \ldots j_p} = A_{(i_0 \ldots j_p = 0)}.
\]

Then (2.11) implies existence of a compatible connection with skew torsion iff

\[
\ker Y^\prime = \mathbb{R}^m \otimes g + \Lambda^3 \mathbb{R}^m.
\]

**Proof.** Choose locally an adapted frame \(e\). We have, at each \(x \in M\),

\[
[e_x^{-1}(\nabla_{LC}^X Y)(x)]_{(i_0 \ldots j_p)} = [e_x^{-1}\Gamma_{LC}^X(x)]_{(i_0 \ldots j_p = 0)}
\]

Thus, (\nabla_{LC}^X Y)(X, \ldots, X) = 0 for each \(X \in TM\) iff \(e_x^{-1}\Gamma_{LC}^X(x) \in \ker Y^\prime\) for each \(x \in M\).

Now, if (2.11) implies existence of a compatible connection with skew torsion, then \(\ker Y^\prime\) must be contained in \(\mathbb{R}^m \otimes g + \Lambda^3 \mathbb{R}^m\), as the Levi-Civita connection can be expressed as a sum of the compatible connection and half of its torsion. However, Proposition 19 implies that \(\mathbb{R}^m \otimes g + \Lambda^3 \mathbb{R}^m \subset \ker Y^\prime\). Thus these must be equal.

Conversely, if the kernel of \(Y^\prime\) is \(\mathbb{R}^m \otimes g + \Lambda^3 \mathbb{R}^m\), then (2.11) implies that \(\Gamma_{LC}\) can be decomposed (not necessarily in a unique way) into a \(g\)-valued local connection form and a skew difference tensor. Lemma 36 completes the proof. \(\square\)
The condition (2.11) will be referred to as nearly-integrability of the geometric structure defined by $\mathcal{Y}$ (or of the tensor itself). There exists an interesting geometric interpretation:

**Lemma 39.** A $G$-structure defined by $\mathcal{Y}$ is nearly-integrably iff for each parametrised geodesic $\gamma : \mathbb{R} \supset I \rightarrow M$ the value of $\mathcal{Y}$ evaluated on $\dot{\gamma}$ is constant along the geodesic.

**Proof.** Let $\gamma$ be such a geodesic, $\nabla^L_{\dot{\gamma}} \dot{\gamma} = 0$. We have

$$\frac{d}{dt} \mathcal{Y}(\dot{\gamma}(t), \ldots, \dot{\gamma}(t)) = \nabla^L_{\dot{\gamma}} [\mathcal{Y}(\dot{\gamma}, \ldots, \dot{\gamma})](t) = (\nabla^L_{\dot{\gamma}} \mathcal{Y})(\dot{\gamma}, \ldots, \dot{\gamma})(t).$$

Now, $\mathcal{Y}$ is nearly integrable iff the latter expression vanishes for every parametrised geodesic, at every point. This in turn is equivalent to the evaluation of $\mathcal{Y}$ on $\dot{\gamma}$ being constant. \qed

It follows that for a nearly-integrable $G$-structure there is a well-defined notion of geodesics which are null with respect to the tensor. The spaces of such geodesics seem to be interesting in their own right.

### 2.5 $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$-geometries with characteristic torsion

We return now to the $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$-geometries defined in Section 2.3, and collect results on skew-torsion connections compatible with related $G(\mathbb{K}, \mathbb{K}')$-structures.

The problem of uniqueness of a skew-torsion connection for the first family has been already investigated by Nurowski [1], while uniqueness for the other two families can be readily established once one knows an analogous result for almost-Kähler and almost-quaternion-Kähler structures. Currently however, we can present it as a simple corollary of Nagy’s general result:

**Proposition 20 (Corollary of Proposition [18]).** Let $(M, g, \mathcal{Y})$ be a $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$-geometry admitting a skew-torsion compatible connection. The such a connection is unique, unless $\mathbb{K} = \mathbb{C}$ and $\mathbb{K}' = \mathbb{C}$. In the latter case, there is a one-parameter family of such connections, whose torsion differ by a section of a one-dimensional $G(\mathbb{C}, \mathbb{C})$-invariant section of $\Lambda^3 TM$.

**Proof (sketch).** We apply Proposition [18] to $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$ as subalgebras of $\text{End}V(\mathbb{K}, \mathbb{K}')$. These algebras are clearly proper and act irreducibly. The only case when $V(\mathbb{K}, \mathbb{K}') \simeq \mathfrak{g}(\mathbb{K}, \mathbb{K}')$ is the geometry modelled after $\frac{SU(3) \times SU(3)}{SU(3)}$, i.e. $\mathbb{K} = \mathbb{C}$ and $\mathbb{K}' = \mathbb{C}$ (it suffices to notice that there is a single 8-dimensional irreducible representation of $SU(3)$, and thus both the isotropy and adjoint representations are equivalent). The, we apply Corollary [11] and the Proposition follows. \qed

As we have mentioned, such unique connection is called the characteristic connection, and its torsion tensor – the characteristic torsion. Indeed, one may consider the latter as characterising the geometry in a manner more convenient than the intrinsic torsion (however the two can be clearly obtained from each other).
CHAPTER 2. GEOMETRIC PART

Of course, these notions make sense only if the connection exists. Bobienski and Nurowski [2] proposed nearly integrability as a candidate for an existence condition, and checked that it is one indeed for the geometry they considered – i.e. the one modelled after \( \text{SU}(3)/\text{SO}(3) \). It then turned out [3] that it also works for the next two geometries in the first column, failing however in case of the last one, i.e. \( E_6/F_4 \).

Proposition 21 (Nurowski [3]). Let \((M, g, \Xi)\) be a \( g(K, \mathbb{C}) \)-geometry with \( K \neq \mathbb{C} \). Then \( M \) admits a skew-torsion compatible connection iff \( \Xi \) is nearly integrable.

In the following, we extend the equivalence between nearly-integrability and existence of a characteristic connection onto the second family.

Theorem 1. Let \((M, g, \Xi)\) be a \( g(K, \mathbb{H}) \)-geometry. Then \( M \) admits a characteristic connection iff \( \Xi \) is nearly integrable.

Proof. That nearly-integrability is implied by existence of the characteristic connection follows immediately from Proposition [19]. To prove the converse, we will apply Lemma [38] to the map

\[
\Xi' : \mathcal{V}_2(K) \otimes \Lambda^2 \mathcal{V}_2(K) \to S^2 \mathcal{V}_2(K)
\]

\[
\Xi'(A)_{a_0a_1...a_6} = A_{a_0} m_a \Xi_{a_2...a_6} m_a.
\]

It is clear that its kernel contains \( \mathcal{V}_2(K) \otimes g(K, \mathbb{H}) + \Lambda^3 \mathcal{V}_2(K) \). In order to check that these are actually equal, we extend \( \Xi' \) by complex linearity and consider a decomposition of a generic element \( A \in \mathcal{V}_2(K) \otimes \Lambda^2 \mathcal{V}_2(K) \) :

\[
A = O + B + C + \hat{O} + \hat{B} + \hat{C}
\]

\[
O \in \Lambda^{1,0} \otimes \Lambda^{2,0}, \quad B \in \Lambda^{0,1} \otimes \Lambda^{2,0}, \quad C \in \Lambda^{1,0} \otimes \Lambda^{1,1}
\]

where \( c_{\alpha\beta\gamma} = c_{\alpha\gamma\beta} - c_{\alpha\beta\gamma} \). The action of extended \( \Xi' \) gives then

\[
\Xi'(O) \in \mathcal{S}^{5,2}, \quad \Xi'(B + C) \in \mathcal{S}^{4,3}
\]

\[
\Xi'((\hat{O}) \in \mathcal{S}^{2,5}, \quad \Xi'(\hat{B} + \hat{C}) \in \mathcal{S}^{3,4}
\]

and \( \Xi'(A) = 0 \) iff each of these vanishes separately. Applying the isomorphisms [1,15], we find \( \Xi'(O) = 0 \) and \( \Xi'(B + C) = 0 \) to be equivalent to

\[
\Lambda_{(\alpha\beta\gamma)O} \delta_{\rho} \epsilon^\rho \Lambda_{\delta\kappa\ell} = 0
\]

\[
\Lambda_{(\mu\nu\kappa)B} \rho (\alpha\Lambda_{\beta\gamma\delta}) + \Lambda_{(\mu\nu\gamma\delta)C} \rho (\beta\Lambda_{\alpha\beta\gamma}) - \Lambda_{(\alpha\beta\gamma)\epsilon(\rho\Lambda_{\delta\kappa\ell})} = 0.
\]

Contracting the first equation with \( \Lambda_{\delta\kappa\ell}^\mu \) and the second one with \( \Lambda^\alpha_{\beta\gamma} \), we obtain

\[
\Lambda_{(\alpha\beta\gamma)O} \delta_{\rho} = 0 \quad (2.12)
\]

\[
X_{\delta\rho}^\mu (\epsilon \Lambda_{\delta\kappa\ell}) + \xi_\delta \Lambda_{\epsilon\delta\kappa} = 0 \quad (2.13)
\]

where

\[
\xi_\delta = \frac{1}{N + 3} [c_{\delta\mu}^\rho + c_{\rho\mu}^\delta + 2c_{\alpha\mu}^\rho \Lambda^\alpha_{\beta\gamma} \Lambda_{\gamma\mu\delta}]
\]
2.6. DECOMPOSITIONS OF TWO- AND THREE-FORMS

\[ X_{\delta \mu \nu} = B_{\delta \mu \nu} - c_{\delta \mu \nu}. \]

Equation (2.12) gives simply \( O(\alpha^\rho_{\beta \gamma}) = 0 \), so that \( O \in \Lambda^{3,0} \). Writing \( X \) in the form

\[ X_{\delta \mu \nu} = -\xi_{\delta \mu \nu} + X'_{\delta \mu \nu} \]

equation (2.13) simplifies to \( X'_{\delta \mu \nu} \in \Lambda^{1,0} \otimes G'_2(\mathbb{K})_{mn} \), so that

\[ X'_{\delta \mu \nu} - X'_{\delta \nu \mu} \in \Lambda^{1,0} \otimes g(\mathbb{K}, \mathbb{H})_{mn}. \]

Now, using \( X \) and \( B \) to eliminate \( c \), we get

\[ A_{abc} = O_{\alpha \beta \gamma} + B_{\alpha \beta \gamma} + B_{\beta \gamma \alpha} + B_{\gamma \alpha \beta} + X_{\alpha \beta \gamma} - X_{\alpha \gamma \beta} + \]
where the terms involving \( B \), due to \( B_{\gamma \alpha \beta} = -B_{\gamma \beta \alpha} \), give an element of \( \Lambda^{2,1} \oplus \Lambda^{1,2} \). We thus finally obtain

\[ A \in \text{Re}[\Lambda^{3,0} \oplus \Lambda^{2,1} \oplus \Lambda^{1,2} \oplus \Lambda^{0,3} \oplus \Lambda^{1,0} \otimes g(\mathbb{K}, \mathbb{H}) \oplus \Lambda^{0,1} \otimes g(\mathbb{K}, \mathbb{H}) \], \]

so that \( \ker \Xi' = V_2(\mathbb{K}) \otimes g(\mathbb{K}, \mathbb{H}) + \Lambda^3 V_2(\mathbb{K}), \) and the theorem follows by Lemma 38.

It is instructive to note that the latter proof relies on the complex structure of \( V_2(\mathbb{K}) \) : indeed, in spite of complete symmetrization in the definition of \( \Xi' \), using the projections (1.15) splits linear and antilinear indices and allows for separate contraction. That did not occur in the first family, and thus Nurowski had to resort to explicit calculations using computer algebra. Unfortunately, this method fails also for the third family, since none of the complex structures is invariant. The question whether nearly-integrability guarantees existence of a characteristic connection for \( g(\mathbb{K}, \mathbb{O}) \)-geometries remains open.

2.6 Decompositions of two- and three-forms

In order to classify possible \( g(\mathbb{K}, \mathbb{K}') \)-geometries with skew torsion, we need to decompose the space of three-forms into \( G(\mathbb{K}, \mathbb{K}') \)-irreducibles. This is easily done using the computer algebra package LiE by Marc van Leeuwen et al. 38.

One needs to note that we are ultimately interested in real representations, while the package we use operates on complex ones. Thus in the first and third family, where the representations are respectively real and quaternionic, we consider complexifications of \( V_1(\mathbb{K}) \) and \( V_3(\mathbb{K}) \), and, after decomposing, take the real section (recalling that \( \text{Re}(V_n \oplus \overline{V}_n) \), with \( V_n \) being some complex irrep, does not decompose). On the other hand, the representations in the second column are already unitary, and can be dealt with directly.

We only provide here the dimensions of the irreducible subspaces. The invariants can be used to construct suitable projection operators.
2.6.1 First family

The following was already done by Nurowski [1].

**Proposition 22.** With $W_n$ denoting an $n$–dimensional real representation, we have the following decompositions into:

1. **SO(3) irreps:**
   \[
   \Lambda^2 V_1(\mathbb{R}) \cong \mathfrak{so}(3) \oplus W_7 \\
   \Lambda^3 V_1(\mathbb{R}) \cong \mathfrak{so}(3) \oplus W_7
   \]

2. **SU(3) irreps:**
   \[
   \Lambda^2 V_1(\mathbb{C}) \cong \mathfrak{su}(3) \oplus W_{20} \\
   \Lambda^3 V_1(\mathbb{C}) \cong \mathfrak{su}(3) \oplus W_{20} \oplus W_{27} \oplus \mathbb{R}
   \]

3. **Sp(3) irreps:**
   \[
   \Lambda^2 V_1(\mathbb{H}) \cong \mathfrak{sp}(3) \oplus W_{70} \\
   \Lambda^3 V_1(\mathbb{H}) \cong \mathfrak{sp}(3) \oplus W_{70} \oplus W_{84} \oplus W_{189}
   \]

4. **F_4 irreps:**
   \[
   \Lambda^2 V_1(\mathbb{O}) \cong f_4 \oplus W_{273} \\
   \Lambda^3 V_1(\mathbb{O}) \cong W_{273} \oplus W_{1274} \oplus W_{1053}
   \]

The **SU(3) case is special**, since the isotropy representation is equivalent to the adjoint one, $V_1(\mathbb{C}) \cong \mathfrak{su}(3)$. In particular, there exists an invariant three-form (spanning the $\mathbb{R} \subset \Lambda^3 V_1(\mathbb{C})$ subspace) corresponding to the structure constants of $\mathfrak{su}(3)$.

2.6.2 Second family

Recall first, that the spaces of two- and three-forms on the second family spaces decompose under $U(V_2(\mathbb{K}))$ into

\[
\Lambda^2 V_2(\mathbb{K}) \cong \text{Re}\Lambda^{1,1} \oplus \text{Re}(\Lambda^{2,0} \oplus \Lambda^{0,2}) = [\mathbb{R} \oplus \mathfrak{su}(V_2(\mathbb{K}))] \oplus u^+(V_2(\mathbb{K})),
\]

\[
\Lambda^3 V_3(\mathbb{K}) \cong \text{Re}(\Lambda^{2,1} \oplus \Lambda^{1,2}) \oplus \text{Re}(\Lambda^{3,0} \oplus \Lambda^{0,3}) = [\mathcal{U}_3 + \mathcal{U}_4](V_2(\mathbb{K})) \oplus \mathcal{U}_4(V_2(\mathbb{K})),
\]

where $\mathcal{U}_1$, $\mathcal{U}_4$ and $\mathcal{U}_4$ are completely skew analogues of the usual spaces introduced in the Gray-Hervella classification of almost-Hermitian structures [39]. These are further decomposed when the unitary group is reduced to one of our groupus $G(\mathbb{K}, \mathbb{H})$.

**Proposition 23.** With $V_n$ denoting (a realification of) an $n$–dimensional complex representation, we have the following decompositions into:
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1. U(3) irreps:
   \[ \text{su}(V_2(\mathbb{R})) \simeq \text{su}(3) \oplus V_{27} \]
   \[ u^+(V_2(\mathbb{R})) \simeq V_{15} \]
   \[ U_1(V_2(\mathbb{R})) \simeq V_{10} \oplus \bar{V}_{10} \]
   \[ U_3(V_2(\mathbb{R})) \simeq V_3 \oplus V_{15} \oplus V_{24} \oplus V_{42} \]

2. S(U(3) × U(3)) irreps:
   \[ \text{su}(V_2(\mathbb{C})) \simeq \{\text{su}(3) \oplus \text{su}(3)\} \oplus V_{64} \]
   \[ u^+(V_2(\mathbb{C})) \simeq \{V_{18} \oplus V_{18}\} \]
   \[ U_1(V_2(\mathbb{C})) \simeq \{V_{10} \oplus V_{10}\} \oplus V_{64} \]
   \[ U_3(V_2(\mathbb{C})) \simeq V_3(\mathbb{C}) \oplus \{V_{18} \oplus V_{18}\} \oplus \{V_{45} \oplus V_{45}\} \oplus \{V_{90} \oplus V_{90}\} \]

   where in \( \{V_n \oplus V_n\} \) the first (resp. second) copy of \( V_n \), being a SU(3) irrep, is affected only by the first (resp. second) SU(3) in SU(3) × SU(3).

3. U(6) irreps:
   \[ \text{su}(V_2(\mathbb{H})) \simeq \text{su}(6) \oplus V_{189} \]
   \[ u^+(V_2(\mathbb{H})) \simeq V_{105} \]
   \[ U_1(V_2(\mathbb{H})) \simeq V_{175} \oplus V_{280} \]
   \[ U_3(V_2(\mathbb{H})) \simeq V_{21} \oplus V_{105} \oplus V_{384} \oplus V_{1050} \]

4. E_6 · U(1) irreps:
   \[ \text{su}(V_2(\mathbb{O})) \simeq \mathfrak{e}_6 \oplus V_{650} \]
   \[ u^+(V_2(\mathbb{O})) \simeq V_{351} \]
   \[ U_1(V_2(\mathbb{O})) \simeq V_{2925} \]
   \[ U_3(V_2(\mathbb{O})) \simeq V_{351} \oplus V_{1728} \oplus V_{7371} \]

The space \( U_4(V_2(\mathbb{K})) = V_2(\mathbb{K}) \land \theta \), where \( \theta_{ab} = i h_{\alpha \beta} - i h_{\beta \alpha} \), is already isomorphic to \( V_2(\mathbb{K}) \) and thus irreducible as a \( G(\mathbb{K}, \mathbb{H}) \)-module.

2.6.3 Third family

Proposition 24. With \( W_n \) denoting an \( n \)-dimensional real representation, we have the following decompositions into:

1. Sp(3)Sp(1) irreps:
   \[ \Lambda^2 V_3(\mathbb{R}) \simeq \text{sp}(3) \oplus \text{sp}(1) \oplus W_{84} \oplus W_{270} \]
   \[ \Lambda^3 V_3(\mathbb{R}) \simeq V_3(\mathbb{R}) \oplus W_{56} \oplus W_{128} \oplus W_{432} \oplus W_{1232} \oplus W_{1400} \]

2. SU(6)Sp(1) irreps:
   \[ \Lambda^2 V_3(\mathbb{C}) \simeq \text{su}(6) \oplus \text{sp}(1) \oplus W_{175} \oplus W_{576} \]
   \[ \Lambda^3 V_3(\mathbb{C}) \simeq V_3(\mathbb{C}) \oplus W_{80} \oplus W_{280} \oplus W_{1080} \oplus W_{3920} \oplus W_{4480} \]
3. SO(12)Sp(1) irreps:
\[ \Lambda^2 V_3(\mathbb{H}) \simeq \mathfrak{so}(12) \oplus \mathfrak{sp}(1) \oplus W_{1463} \oplus W_{1485} \]
\[ \Lambda^3 V_3(\mathbb{H}) \simeq V_3(\mathbb{H}) \oplus W_{128} \oplus W_{704} \oplus W_{3456} \oplus W_{17600} \oplus W_{19712} \]

4. E_7Sp(1) irreps:
\[ \Lambda^2 V_3(\mathbb{O}) \simeq e_7 \oplus \mathfrak{sp}(1) \oplus W_{1463} \oplus W_{14617} \]
\[ \Lambda^3 V_3(\mathbb{O}) \simeq V_3(\mathbb{O}) \oplus W_{224} \oplus W_{1824} \oplus W_{12960} \oplus W_{102144} \oplus W_{110656} \]

Note that one always has \( V_3(\mathbb{K}) \subset \Lambda^3 V_3(\mathbb{K}) \) with the intertwiner given by the natural \( \text{Sp}(V_3(\mathbb{K})) \}-\text{invariant 4-form} \( I \wedge I + J \wedge J + K \wedge K \).

2.6.4 A classification
Let \( M \) be a \( g(\mathbb{K}, \mathbb{K}') \)-geometry. Then the decomposition of \( \Lambda^3 \mathcal{V}_n(\mathbb{K}) \) into \( G(\mathbb{K}, \mathbb{K}') \)-irreducibles
\[ \Lambda^3 \mathcal{V}_n(\mathbb{K}) = \bigoplus_{r=1}^s W^{(r)} \]
(where \( n = 1, 2, 3 \) for \( \mathbb{K}' \) being, respectively, \( \mathbb{C}, \mathbb{H}, \mathbb{O} \)) gives rise to a decomposition of the bundle of three-forms
\[ \Lambda^3 T M = \bigoplus_{r=1}^s T^{(r)} \]
into subbundles such that in an adapted frame \( e_x : \mathcal{V}_n(\mathbb{K}) \to T_x M \) one has \( e_x(W^{(r)}) = T^{(r)}_x \). As the spaces \( W^{(r)} \) are \( G(\mathbb{K}, \mathbb{K}') \)-invariant, the latter decomposition does not depend on the choice of a frame.

Assume now that \( M \) admits a characteristic connection, with a characteristic torsion \( T^c \in \Omega^0(M, \Lambda^3 T M) \). It follows that \( M \) falls into one of 2s classes, numbered by
\[ t(M) = \sum_{r=1}^s 2^{r-1} t_r(M), \quad t_r(M) = \begin{cases} 0 & \text{pr}_{T^{(r)}, T^c} = 0 \\ 1 & \text{otherwise} \end{cases} \]
This way we obtain:
- 4 classes of \( g(\mathbb{R}, \mathbb{C}) \)-geometries with characteristic torsion.
- 16 classes of \( g(\mathbb{H}, \mathbb{C}) \)-geometries with characteristic torsion.
- 8 classes of \( g(\mathbb{O}, \mathbb{C}) \)-geometries with characteristic torsion.
- 128 classes of \( g(\mathbb{K}, \mathbb{H}) \)-geometries with characteristic torsion, where \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H} \).
- 32 classes of \( g(\mathbb{O}, \mathbb{H}) \)-geometries with characteristic torsion.
- 64 classes of \( g(\mathbb{K}, \mathbb{O}) \)-geometries with characteristic torsion.
A connection with skew torsion is not unique for a \( g(C,C) \)-geometry, due to the one-dimensional subspace \( \mathbb{R} \subset \Lambda^3 \mathcal{V}_1(C) \) which is also contained in \( \mathcal{V}_1(C) \otimes g(C,C) \). There is thus no genuine characteristic torsion – instead, one has a one-parameter family of connections whose torsion differs by a section of the one-dimensional \( G(C,C) \)-invariant subbundle \( T^{(4)} \subset \Lambda^3 TM \). (Nurowski \cite{nurowski2000d} introduces the notion of restricted nearly-integrability to rule this subbundle out.) Anyway, we can still perform analogous classification, projecting the skew torsion of any compatible connection onto the complement of \( T^{(4)} \). We thus have in addition:

- 8 classes of \( g(C,C) \)-geometries admitting a skew-torsion compatible connection.

### 2.7 Locally reductive \( g(K,K') \)-geometries

While symmetric spaces provide integrable models for \( g(K,K') \)-geometries, it is the reductive spaces that give homogeneous examples with nontrivial characteristic torsion (cf. \cite{knapp1996representation,trautman1970}). We will first show how certain locally reductive spaces become equipped with a \( G(K,K') \)-structure, and how such reductive spaces can be obtained from the symmetric models. Then, we shall perform a construction of such spaces at the Lie-algebraic level, having previously shown how to extend results derived for a single pair \((K,K')\) onto all the ‘larger’ cases.

#### 2.7.1 The existence theorem

In what follows, we construct a manifold equipped with a \( G \)-structure from adequate Lie-algebraic data.

**Lemma 40.** Let \( G \) be a subgroup of \( O(m) \) with \( g \subset \mathfrak{so}(m) \) its Lie algebra. Let \( \mathfrak{k} \) be a Lie algebra admitting a reductive, non-symmetric decomposition

\[
\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{l}
\]

\([\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{l}] \subset \mathfrak{l}, \quad \text{pr}_l [\mathfrak{l}, \mathfrak{l}] \neq 0\)

such that \( \mathfrak{l} \) possesses a \( \mathfrak{k} \)-invariant scalar product. Let there moreover exist a Lie algebra monomorphism \( \varphi : \mathfrak{h} \rightarrow \mathfrak{g} \) and an orthogonal isomorphism \( \psi : \mathfrak{l} \rightarrow \mathbb{R}^m \) satisfying

\[
\varphi(A) \circ \psi = \psi \circ \text{ad}_A
\]

for each \( A \in \mathfrak{h} \).

Then there exists an \( m \)-dimensional manifold equipped with a \( G \)-structure admitting a compatible connection with nontrivial skew torsion.

**Proof.** Let \( K \) be a Lie group with Lie algebra \( \mathfrak{k} \) (for example the simply-connected one). While one is tempted to produce a subgroup from \( \mathfrak{h} \subset \mathfrak{k} \) and form a reductive homogeneous space by taking a quotient, it is not trivial to guarantee that the subgroup is closed. Instead, we shall perform a local construction.

Let \( \mathfrak{h}^L \subset TK \) be the left-invariant distribution such that \( \mathfrak{h}^L(g) = (T_{eL(g)} \mathfrak{h}) \) for each \( g \in K \). Since \( \mathfrak{h} \) is a subalgebra in \( \mathfrak{k} \), it follows that \( \mathfrak{h}^L \) is integrable and there exists a neighbourhood \( Q_0 \subset K \) of identity, such that \( \mathfrak{h}^L \) gives rise to a
Possibly choosing smaller \( Q_0 \), consider a global section \( \sigma_0 : M \rightarrow Q_0 \subset K \), i.e. a map satisfying \( \pi_0 \circ \sigma_0 = \text{id}_M \). The Maurer-Cartan form \( \vartheta_{MC} \in \Omega^1(K) \otimes \mathfrak{k} \) pulls back to
\[
\vartheta = \sigma_0^* \vartheta_{MC} \in \Omega^1(M) \otimes \mathfrak{k},
\]
and the latter is further decomposed into
\[
\vartheta \in \Omega^1(M) \otimes \mathbb{R}^m, \quad \varpi \in \Omega^1(M) \otimes \mathfrak{g}
\]
\[
\vartheta = \psi \circ \text{pr}_1 \vartheta, \quad \varpi = \varphi \circ \text{pr}_2 \vartheta.
\]
Moreover \( \vartheta(\pi_0(e)) : T_{\pi_0(e)}M \rightarrow \mathbb{R}^m \) is an isomorphism and extends to a coframe on entire \( M \). The structure equation \( d\vartheta_{MC} + \frac{1}{2}[\vartheta_{MC}, \vartheta_{MC}] = 0 \) pulls back to
\[
\Omega(X,Y) = - (\varphi \circ \text{pr}_2)[\psi^{-1}(\vartheta(X), \psi^{-1}(\vartheta(Y))]
\]
\[
\Theta(X,Y) = - (\psi \circ \text{pr}_1)[\psi^{-1}(\vartheta(X), \psi^{-1}(\vartheta(Y))]
\]
for \( X, Y \in T_xM \), where \( \Omega = d\varpi + \varpi \wedge \varpi \) and \( \Theta = d\vartheta + \vartheta \wedge \vartheta \). Since \( \psi \) is orthogonal, it follows that
\[
(\Theta(X,Y), Z) = -(\Theta(X,Z), Y). \tag{2.14}
\]
Considering \( \vartheta \) as an adapted orthogonal coframe on \( M \), we equip the latter with a compatible metric, and a \( G \)-structure \( \pi : Q \rightarrow M \). The fibre over \( x \) of the latter consists of all frames \( e_x : \mathbb{R}^m \rightarrow T_xM \) such that there exists \( g \in G \) such that \( e_x^{-1}(g\vartheta(X)) = X \) for all \( X \in T_xM \).

The frame dual to \( \vartheta \) is a section \( \sigma : M \rightarrow Q \) with \( \sigma^* \vartheta = \vartheta \), where \( \vartheta \) is the soldering form on \( Q \). Considering
\[
\Gamma^\omega \in \Omega^1(M, \mathfrak{g}_M), \quad \Gamma^\sigma(X) = \sigma_x(\varpi(X)) \quad \text{for } X \in T_xM
\]
as a connection form relative to the frame \( \sigma \), we obtain a \( Q \)-compatible connection \( \nabla^\sigma \) in the tangent bundle, whose torsion tensor
\[
T^\omega \in \Omega^2(M, TM), \quad T^\omega(X,Y) = \sigma_x(\Theta(X,Y)) \quad \text{for } X, Y \in T_xM
\]
is completely skew due to (2.14). Moreover, it is nontrivial, since \( \text{pr}_1[l,l] \neq 0 \). \( \square \)

The latter lemma reduces the problem of producing locally reductive \( \mathfrak{g}([\mathbb{K}, \mathbb{K}']) \)-geometries with characteristic torsion to an algebraic one. This in turn can be first dealt with on a Lie-algebraic level. Recall first the notation introduced when decomposing the Magic Square algebras into symmetric pairs, Subsection 1.1.5 of the previous Chapter. The idea is, being given an original symmetric decomposition \( \mathfrak{m}(\mathbb{K}, \mathbb{K}') = \mathfrak{g}(\mathbb{K}, \mathbb{K}') \oplus \mathfrak{v}(\mathbb{K}, \mathbb{K}') \), to produce a reductive pair by reducing \( \mathfrak{g}(\mathbb{K}, \mathbb{K}') \) to a subalgebra \( \mathfrak{h} \) and twisting the \([\mathfrak{v}(\mathbb{K}, \mathbb{K}'), \mathfrak{v}(\mathbb{K}, \mathbb{K}')] \) bracket.

The following proposition performs such a construction whenever the decomposition of \( \mathfrak{v}(\mathbb{K}, \mathbb{K}') \) into \( \mathfrak{h} \)-irreducibles includes \( \mathfrak{h} \) itself.
Lemma 41. Let \( \mathfrak{M} = \mathfrak{g} \oplus \mathfrak{V} \) be a symmetric pair, with \( \mathfrak{M} \) semisimple. Let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \) such that there exists an orthogonal (with respect to the Killing form) map \( B : \mathfrak{h} \to \mathfrak{V} \) equivariant under the adjoint action of \( \mathfrak{h} \).

Let us equip the space \( \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{V} \) with the following bracket:

\[
\begin{align*}
[A, A']_\mathfrak{k} & = [A, A'] \\
[A, X]_\mathfrak{k} & = [A, X] \\
[X, Y]_\mathfrak{k} & = \text{pr}_\mathfrak{h}[X, Y] + B(\text{pr}_\mathfrak{h}[X, Y]) + [B^*(X), Y] - [B^*(Y), X] - B([B^*(X), B^*(Y)])
\end{align*}
\]

for \( A, A' \in \mathfrak{h} \) and \( X, Y \in \mathfrak{V} \), where the commutators on the r.h.s are those in \( \mathfrak{M} \) and \( B^* \) is the adjoint of \( B \) with respect to the Killing form.

Then \( \mathfrak{k} \) becomes a Lie algebra, and the decomposition \( \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{V} \) is reductive pair. Moreover, \( \mathfrak{k} \) possesses an invariant nondegenerate quadratic form which, restricted to \( \mathfrak{V} \), coincides with the Killing form of \( \mathfrak{M} \).

This way we obtained a reductive pair \( \mathfrak{h} \oplus \mathfrak{V} = \mathfrak{k} \) such that the action on \( \mathfrak{h} \) on \( \mathfrak{V} \) in \( \mathfrak{k} \) is the same as in \( \mathfrak{M} \). Moreover, \( \mathfrak{k} \) is constructed in such a way that the action on \( \mathfrak{h} \) on \( \mathfrak{V} \) is nontrivial.

Then, since the action on \( \mathfrak{h} \) on \( \mathfrak{V} \) is nontrivial, there exist such \( X, Y \in \mathfrak{W} \) that \( \text{pr}_\mathfrak{V}[X, Y] \neq 0 \). It thus follows that the reductive pair we have obtained is not symmetric.

The proof is by an explicit verification of the Jacobi identities, and can be found in Section 2.8. We finally arrive at the following result, which we shall use in the next section.

Theorem 2. Let \( \mathfrak{h} \subset \mathfrak{g}(\mathbb{K}, \mathbb{K}') \) be a simple subalgebra and \( \mathfrak{W} \subset \mathfrak{V}(\mathbb{K}, \mathbb{K}') \) a \( \mathfrak{h} \)-submodule equivalent to \( \mathfrak{h} \). Let moreover the action of \( \mathfrak{h} \) on the orthogonal complement of \( \mathfrak{W} \) be nontrivial.

Then there exists a \( \mathfrak{g}(\mathbb{K}, \mathbb{K}') \)-geometry admitting a compatible connection with nonvanishing skew torsion.

Proof. We first apply Lemma 41 to the symmetric decomposition \( \mathfrak{M}(\mathbb{K}, \mathbb{K}') = \mathfrak{g}(\mathbb{K}, \mathbb{K}') \oplus \mathfrak{V}(\mathbb{K}, \mathbb{K}') \) and the subalgebra \( \mathfrak{h} \) with the map \( B \) being the intertwiner between \( \mathfrak{h} \) and \( \mathfrak{W} \) (due to semisimplicity both \( \mathfrak{h} \) and \( \mathfrak{W} \) are irreducible, and thus \( B \) is automatically orthogonal).

This way we obtained a reductive pair \( \mathfrak{h} \oplus \mathfrak{V}(\mathbb{K}, \mathbb{K}') = \mathfrak{k} \) such that the action on \( \mathfrak{h} \) on \( \mathfrak{W} \) in \( \mathfrak{k} \) is the same as in \( \mathfrak{M}(\mathbb{K}, \mathbb{K}') \). Moreover, for \( X, Y \in \mathfrak{W}^\perp \) (the orthogonal complement of \( \mathfrak{W} \)), we have

\[
\text{pr}_\mathfrak{V}(\mathbb{K}, \mathbb{K}')[X, Y]_\mathfrak{k} = [B \circ \text{pr}_\mathfrak{h}[X, Y]].
\]

Then, since the action on \( \mathfrak{h} \) on \( \mathfrak{W}^\perp \) is nontrivial, there exist such \( X, Y \in \mathfrak{W}^\perp \) that \( \text{pr}_\mathfrak{V}(\mathbb{K}, \mathbb{K}')[X, Y]_\mathfrak{k} \neq 0 \). It thus follows that the reductive pair we have obtained is not symmetric.

We can finally apply Lemma 40 to obtain a Riemannian manifold \((\mathfrak{M}, \mathcal{G})\) equipped with a \( \mathfrak{G}(\mathbb{K}, \mathbb{K}') \)-structure admitting a compatible connection with nontrivial skew torsion. Then, choosing an adapted frame \( e \), we can equip \( \mathfrak{M} \) with a corresponding tensor \( \mathcal{Y}(x) = e_x(\mathcal{Y}) \), where \( \mathcal{Y} \) is one of \( \Upsilon, \Xi, \Upsilon, \) depending on \( \mathbb{K}' \). Clearly, \( \mathcal{Y} \) does not depend on the choice of a frame. \( \square \)
2.7.2 A generalizing proposition

We are now going to give several examples of subalgebras \( h \subset g(\mathbb{K}, \mathbb{K}') \) satisfying the conditions of Theorem 2. While we tried to keep previous sections of the present chapter possibly independent of the Jordan-algebraic constructions, here we will need the structures introduced in Sections 1.1 and 1.2 of the previous chapter.

It useful to note that an example for a pair \((\mathbb{K}, \mathbb{K}')\) gives rise to examples for all ‘further’ such pairs, with the ordering given by inclusions:

**Proposition 25.** Let \( \tilde{\mathbb{K}} \supset \mathbb{K} \) and \( \tilde{\mathbb{K}}' \supset \mathbb{K}' \) where \( \mathbb{K}, \tilde{\mathbb{K}} \) are chosen from \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) and \( \mathbb{K}', \tilde{\mathbb{K}}' \) from \( \mathbb{C}, \mathbb{H}, \mathbb{O} \). Assume there is a subalgebra \( h \subset g(\mathbb{K}, \mathbb{K}') \) and a subspace \( W \subset V(\mathbb{K}, \mathbb{K}') \) satisfying the conditions of Theorem 2.

Then there exist natural inclusions \( h \hookrightarrow g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \) and \( W \hookrightarrow V(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \) such that their images satisfy the conditions of Theorem 2 for \( g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \)-geometries.

We need the following Lemma, deriving form the properties of the Magic Square:

**Lemma 42.**

1. Let \( \mathbb{K}, \tilde{\mathbb{K}} \) and \( \mathbb{K}', \tilde{\mathbb{K}}' \) be as in Proposition 25. Then there exist natural monomorphisms (of Lie algebras and of vector spaces):
   \[
   i_h : g(\mathbb{K}, \mathbb{K}') \rightarrow g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \\
   i_V : V(\mathbb{K}, \mathbb{K}') \rightarrow V(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}')
   \]
   such that
   \[
   [i_h(E), i_V(X)] = i_V[E, X]
   \]
   for \( E \in g(\mathbb{K}, \mathbb{K}') \) and \( X \in V(\mathbb{K}, \mathbb{K}') \), with the l.h.s bracket taken in \( \mathfrak{M}(\mathbb{K}, \mathbb{K}') \) and the r.h.s. one in \( \mathfrak{M}(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \).

2. Let \( \tilde{\mathbb{K}} \) and \( \mathbb{K}', \tilde{\mathbb{K}}' \) be as in Proposition 25. Then there exist natural monomorphisms (of Lie algebras and of vector spaces):
   \[
   j_h : g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \rightarrow g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \\
   j_V : V(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \rightarrow V(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}')
   \]
   such that
   \[
   [j_h(E), j_V(X)] = j_V[E, X]
   \]
   for \( E \in g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \) and \( X \in V(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \), with the l.h.s bracket taken in \( \mathfrak{M}(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \) and the r.h.s. one in \( \mathfrak{M}(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') \).

**Proof.**

1. Recall that
   \[
   g(\mathbb{K}, \mathbb{K}') = \text{der} \mathfrak{h}_3 \mathbb{K} \oplus \text{der}_0 \mathfrak{K}' \oplus \mathfrak{K}'^0 \oplus \text{sh}_3 \mathbb{K} \\
   V(\mathbb{K}, \mathbb{K}') = \text{der}_1 \mathfrak{K}' \oplus \mathfrak{K}'_1 \oplus \text{sh}_3 \mathbb{K},
   \]
   while
   \[
   g(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') = \text{der} \mathfrak{h}_3 \tilde{\mathbb{K}} \oplus \text{der}_0 \mathfrak{K}' \oplus \mathfrak{K}'^0 \oplus \text{sh}_3 \tilde{\mathbb{K}} \\
   V(\tilde{\mathbb{K}}, \tilde{\mathbb{K}}') = \text{der}_1 \mathfrak{K}' \oplus \mathfrak{K}'_1 \oplus \text{sh}_3 \tilde{\mathbb{K}}.
One easily checks that the inclusion $\mathbb{K} \hookrightarrow \tilde{\mathbb{K}}$ induces a Jordan algebra monomorphism

$$\mu : \mathfrak{h}_3 \mathbb{K} \hookrightarrow \mathfrak{h}_3 \tilde{\mathbb{K}}.$$ 

Recalling that the derivations of $\mathfrak{h}_3 \mathbb{K}$ are defined as an image of the der $\mathfrak{h}_3 \mathbb{K}$--equivariant map $\mathcal{D}$ defined in Lemma 4:

$$\text{der} \mathfrak{h}_3 \mathbb{K} = \{D_{X,Y} | X, Y \in \mathfrak{sh}_3 \mathbb{K}\},$$

we see that $\mu$ induces a map

$$\text{der} \mathfrak{h}_3 \mathbb{K} \ni D_{X,Y} \mapsto D_{\mu(X),\mu(Y)} \in \text{der} \mathfrak{h}_3 \tilde{\mathbb{K}} \quad (2.15)$$

such that

$$D_{\mu(X),\mu(Y)}(\mu(Z)) = \mu(D_{X,Y}(Z)) \quad (2.16)$$

for $X, Y \in \mathfrak{sh}_3 \mathbb{K}$ and $Z \in \mathfrak{h}_3 \mathbb{K}$. Due to equivariance of $\mathcal{D}$, the map (2.15) gives a monomorphism of Lie algebras,

$$\gamma : \text{der} \mathfrak{h}_3 \mathbb{K} \rightarrow \text{der} \mathfrak{h}_3 \tilde{\mathbb{K}},$$

and (2.16) translates to $\gamma(D)(\mu(X)) = \mu(D(X))$ for $D \in \text{der} \mathfrak{h}_3 \mathbb{K}$ and $X \in \mathfrak{sh}_3 \mathbb{K}$.

It then follows that the maps

$$i_\phi = \gamma \oplus \text{id} \oplus \text{id} \otimes \mu|_{\mathfrak{sh}_3 \mathbb{K}}$$

$$i_\nu = \text{id} \oplus \text{id} \otimes \mu|_{\mathfrak{sh}_3 \mathbb{K}}$$

are as such claimed by the Lemma.

2. Recalling Proposition 3, we have the following identifications:

$$V(\tilde{\mathbb{K}}, \mathbb{C}) \simeq V_1(\tilde{\mathbb{K}}) = \mathfrak{sh}_3 \tilde{\mathbb{K}}$$

$$V(\tilde{\mathbb{K}}, \mathbb{H}) \simeq V_2(\tilde{\mathbb{K}}) = \mathbb{C} \otimes \mathfrak{h}_3 \tilde{\mathbb{K}}$$

$$V(\tilde{\mathbb{K}}, \mathbb{O}) \simeq V_3(\tilde{\mathbb{K}}) = \mathbb{C} \otimes (\mathbb{R} \oplus \mathfrak{h}_3 \tilde{\mathbb{K}})$$

and the algebras $\mathfrak{g}(\tilde{\mathbb{K}}, \mathbb{K}')$ seen as endomorphisms of the latter spaces are

$$\mathfrak{g}(\tilde{\mathbb{K}}, \mathbb{C}) = \text{der} \mathfrak{h}_3 \tilde{\mathbb{K}}$$

$$\mathfrak{g}(\tilde{\mathbb{K}}, \mathbb{H}) = \text{der} \mathfrak{h}_3 \tilde{\mathbb{K}} \oplus iL_{\mathfrak{h}_3 \tilde{\mathbb{K}}}$$

$$\mathfrak{g}(\tilde{\mathbb{K}}, \mathbb{O}) = \mathcal{H}_0(\text{der} \mathfrak{h}_3 \tilde{\mathbb{K}} \oplus i\mathfrak{h}_3 \tilde{\mathbb{K}})$$

$$\oplus \quad \mathcal{H}_1(1 \otimes \mathfrak{h}_3 \tilde{\mathbb{K}} \oplus i \otimes \mathfrak{h}_3 \tilde{\mathbb{K}}) \oplus \mathfrak{sp}(1),$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the maps defined in Lemma 12 and extended by complex linearity, and $\mathfrak{sp}(1)$ denotes the additional algebra related to a quaternion-hermitian structure, as described in Proposition 3. In $\mathfrak{g}(\tilde{\mathbb{K}}, \mathbb{I})$ we have included the additional $\mathfrak{u}(1)$ as spanned by $iL_1 \in iL_{\mathfrak{h}_3 \tilde{\mathbb{K}}}$.

Let us now consider monomorphisms

$$\nu_1(\tilde{\mathbb{K}}) \xrightarrow{\mu} \nu_2(\tilde{\mathbb{K}}) \xrightarrow{\nu} \nu_3(\tilde{\mathbb{K}})$$

$$\alpha(\tilde{\mathbb{K}}, \mathbb{C}) \xrightarrow{\beta} \beta(\tilde{\mathbb{K}}, \mathbb{H}) \xrightarrow{\beta} \beta(\tilde{\mathbb{K}}, \mathbb{O})$$
given by
\[ \mu(X) = 1 \otimes X, \quad \nu(z \otimes Y) = (0, Y) \otimes (z, 0) \]
for \( X \in \mathfrak{h}_3 \overline{\mathbb{K}}, \ Y \in \mathfrak{h}_3 \overline{\mathbb{K}} \) and \( z \in \mathbb{C} \), and
\[ \alpha(D) = D, \quad \beta(D + iL_X) = \mathcal{H}_0(D + iX) \]
for \( D \in \text{der} \mathfrak{h}_3 \overline{\mathbb{K}} \) and \( X \in \mathfrak{h}_3 \overline{\mathbb{K}} \).

It is then readily checked that \( \alpha(D)(\mu(X)) = \mu(D(X)) \) for \( D \in \text{der} \mathfrak{h}_3 \overline{\mathbb{K}} \) and \( X \in \mathfrak{h}_3 \overline{\mathbb{K}} \).

Comparing with the formula for \( \mathcal{H}_0 \) given in Lemma 12, one also easily verifies \( \beta(E)(\nu(Z)) = \nu(E(Z)) \) for \( E \in \mathfrak{g}(\overline{\mathbb{K}}, \mathbb{H}) \) and \( Z \in \mathcal{V}_2(\overline{\mathbb{K}}) \).

Finally, the map \( j_\mathfrak{g} \) is given by either id, \( \alpha \), \( \beta \) or \( \beta \circ \alpha \) (depending on \( \mathbb{K}' \) and \( \overline{\mathbb{K}}' \)), while \( j_\mathcal{V} \) is respectively id, \( \mu \), \( \nu \) or \( \nu \circ \mu \). It follows from the previous paragraph, that \( j_\mathfrak{g} \) and \( j_\mathcal{V} \) are such as claimed by the Lemma.

By composing \( m_\mathfrak{g} = j_\mathfrak{g} \circ i_\mathfrak{g} \) and \( m_\mathcal{V} = j_\mathcal{V} \circ i_\mathcal{V} \) we have the obvious

**Corollary 12.** Let \( \mathbb{K}, \overline{\mathbb{K}}, \mathbb{K}', \overline{\mathbb{K}}' \) be as in Proposition 25. Then there exist natural monomorphisms (of Lie algebras and of vector spaces):
\[ m_\mathfrak{g} : \mathfrak{g}(\mathbb{K}, \mathbb{K}') \rightarrow \mathfrak{g}(\overline{\mathbb{K}}, \overline{\mathbb{K}}') \]
\[ m_\mathcal{V} : \mathcal{V}(\mathbb{K}, \mathbb{K}') \rightarrow \mathcal{V}(\overline{\mathbb{K}}, \overline{\mathbb{K}}') \]
such that
\[ [m_\mathfrak{g}(E), m_\mathcal{V}(X)] = m_\mathcal{V}[E, X] \quad (2.17) \]
for \( E \in \mathfrak{g}(\mathbb{K}, \mathbb{K}') \) and \( X \in \mathcal{V}(\mathbb{K}, \mathbb{K}') \), with the l.h.s bracket taken in \( \mathfrak{M}(\mathbb{K}, \mathbb{K}') \) and the r.h.s. one in \( \mathfrak{M}(\overline{\mathbb{K}}, \overline{\mathbb{K}}') \).

This finally leads us to a proof of the Proposition:

**Proof of Proposition 25.** The inclusions are simply the restrictions \( m_\mathfrak{g}|_\mathfrak{h} \) and \( m_\mathcal{V}|_\mathcal{W} \) of the maps described by Corollary 12. That \( m_\mathcal{V}(W) \simeq m_\mathfrak{g}(\mathfrak{h}) \) as a \( m_\mathfrak{g}(\mathfrak{h}) \)-module is guaranteed by the intertwining property (2.17) of \( m_\mathcal{V} \). Finally, \( m_\mathfrak{g}(\mathfrak{h}) \) acts nontrivially at least on \( m_\mathcal{V}(W^\perp) \subset m_\mathcal{V}(W)^\perp \).

**2.7.3 Examples**

We will now give a couple of examples of subalgebras satisfying the conditions of Theorem 2. As the examples of \( \mathfrak{g}(\mathbb{K}, \mathbb{C}) \)-geometries have been given by Nurowski in [1], we concentrate on constructions applicable to \( \mathfrak{g}(\mathbb{K}, \mathbb{H}) \)- and \( \mathfrak{g}(\mathbb{K}, \mathbb{O}) \)-geometries.

**Example 2** (Subalgebra of type \( \mathfrak{su}(2) \)). Set
\[ \mathfrak{g} = \mathfrak{g}(\mathbb{R}, \mathbb{H}) = \text{der} \mathfrak{h}_3 \mathbb{R} \oplus iL_{\mathfrak{h}_3 \mathbb{R}}, \quad V = V(\mathbb{R}, \mathbb{H}) = \mathbb{C} \otimes \mathfrak{h}_3 \mathbb{R}. \]

Define \( X, Y \in \mathfrak{h}_3 \mathbb{R} : \)
\[
X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
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and $Z = X^2 = Y^2$. We have:

$$Z \circ X = X, \quad Z \circ Y = Y, \quad X \circ Y = 0.$$ 

Let

$$\mathfrak{h} = \text{Span}\{D, iL_X, -iL_Y\} \subset \mathfrak{g},$$

where $D = [L_X, L_Y]$. One easily checks that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$. Let

$$W = \text{Span}\{1 \otimes X, 1 \otimes Y, i \otimes Z\} \subset V.$$ 

One easily checks that $\mathfrak{h}$ preserves $W$ (and $W^\perp$) and that the map

$$B : \mathfrak{h} \to W, \quad B(izL_X - iyL_Y + zD) = xY + yX + izZ$$

satisfies $B([A', A]) = A'(B(A))$ for $A, A' \in \mathfrak{h}$. The action of $\mathfrak{h}$ on $W^\perp$ is evidently nontrivial.

Theorem 2 for $\mathfrak{h}$ and $W$, extended by Proposition 25 yields an example of a $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$-geometry with characteristic torsion for all $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\mathbb{K}' = \mathbb{H}, \mathbb{O}$.

**Example 3** (Subalgebra of type $\mathfrak{su}(3)$). Set

$$\mathfrak{g} = \mathfrak{g}(\mathbb{C}, \mathbb{H}) = \text{der} \mathfrak{h}_3 \mathbb{C} \oplus iL_{\mathfrak{h}_3} \mathbb{C}, \quad V = V(\mathbb{C}, \mathbb{H}) = \mathbb{C} \otimes \mathfrak{h}_3 \mathbb{C}.$$ 

Let

$$\mathfrak{h} = \text{der} \mathfrak{h}_3 \mathbb{C} \subset \mathfrak{g},$$

being of course isomorphic to $\mathfrak{su}(3)$. Let

$$W = 1 \otimes \mathfrak{h}_3 \mathbb{C} \subset V.$$ 

One easily checks, that $\mathfrak{h}$ preserves $W$ (and $W^\perp$) and the map

$$B : \mathfrak{h} \to W, \quad B([L_X, L_Y]) = 1 \otimes i(XY - YX)$$

is bijective and satisfies $B([A', A]) = A'(B(A))$ for $A, A' \in \mathfrak{h}$. The action of $\mathfrak{h}$ on $W^\perp$ is evidently nontrivial.

Theorem 2 for $\mathfrak{h}$ and $W$, extended by Proposition 25 yields an example of a $\mathfrak{g}(\mathbb{K}, \mathbb{K}')$-geometry with characteristic torsion for all $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\mathbb{K}' = \mathbb{H}, \mathbb{O}$.

**Example 4** (Subalgebra of type $\mathfrak{so}(8)$). Set

$$\mathfrak{g} = \mathfrak{g}(\mathbb{O}, \mathbb{O}) \simeq \mathfrak{e}_7 \oplus \mathfrak{sp}(1), \quad V = V(\mathbb{O}, \mathbb{O}) = \mathbb{C} \otimes \mathfrak{F}(\mathfrak{h}_3 \mathbb{O}).$$

The compact algebra $\mathfrak{e}_7$ has $\mathfrak{so}(8)$ among its maximal subalgebras, and one can check (e.g. using LiE [38]) that the complex 56-dimensional module $V$ decomposes under the action of $\mathfrak{so}(8) \subset \mathfrak{e}_7$ into

$$V \simeq \Lambda^2 \mathbb{C}^8 \oplus \Lambda^2 \mathbb{C}^8,$$

where $\mathbb{C}^8$ is the defining representation of $\mathfrak{so}(8)$.

Regarding now the usual inclusion $\mathfrak{so}(8) \subset \mathfrak{su}(8)$ (given by some generic nondegenerate quadratic form on $\mathbb{C}^8$), we find that $V$ decomposes under $\mathfrak{so}(8) \subset \mathfrak{e}_7$ into two copies of the complexified adjoint representation:

$$V \simeq \mathfrak{so}(8, \mathbb{C}) \oplus \mathfrak{so}(8, \mathbb{C}).$$

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5 Since $\mathfrak{so}(8, \mathbb{C}) \simeq \Lambda^2 \mathbb{C}^8 \simeq \mathbb{C}^8 \otimes \mathbb{C}^8$ as $\mathfrak{so}(8)$-modules.
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The explicit form of the intertwiner is rather complicated. Nevertheless, taking
\[ h = so(8) \subset su(8) \subset e_7 \subset g \]
and
\[ W = so(8) \subset so(8, \mathbb{C}) \subset V, \]
the real part of one of the two copies of \( so(8, \mathbb{C}) \), we can apply Theorem 2 to obtain an example of a \( g(\mathbb{O}, \mathbb{O}) \)-geometry with characteristic torsion.

2.8 Proofs of Lemmas 34, 35 & 41

Proof of Lemma 34. To find correct combinatorial factors, note that
\[ \Xi_{abcdef} = \frac{1}{20} (\Lambda_{\alpha\beta\gamma\bar{\Lambda}_{\delta\epsilon\phi\kappa} + \Lambda_{\beta\gamma\epsilon\bar{\Lambda}_{\alpha\delta\phi\kappa} + \ldots} ) \]
with \( \binom{6}{3} = 20 \) terms on the r.h.s., so that we have
\[ 20^2 \Xi_{abcdefkl} \Xi_{\beta defkl} = 20 \Lambda_{\alpha\epsilon\delta\epsilon\phi\kappa} \Lambda_{\delta\epsilon\phi\kappa} + 10 \delta_{\alpha}^{\beta} \phi \]
so that \( \Xi_{abcdefkl} \Xi_{bdefkl} = \frac{1}{40} \delta_{\alpha}^{\beta} \phi \). Moreover,
\[ 20^2 \Xi_{\delta ef k l} \Xi_{\gamma \beta e f k l} = 6 \Lambda_{\alpha\beta\gamma\delta\epsilon\phi\kappa} \Lambda_{\delta\epsilon\phi\kappa} + 6 \delta_{\alpha}^{\beta} \gamma \]
\[ 20^2 \Xi_{\delta ef kl} \Xi_{\gamma \beta ef kl} = 4 \Lambda_{\alpha\beta\gamma\delta\epsilon\phi\kappa} \Lambda_{\delta\epsilon\phi\kappa} + 4 \Lambda_{\delta\epsilon\phi\kappa} \Lambda_{\beta\epsilon\phi\kappa} \]
\[ 20^2 \Xi_{\epsilon \delta ef kl} \Xi_{\gamma \beta ef kl} = 6 \Lambda_{\alpha\beta\gamma\delta\epsilon\phi\kappa} \Lambda_{\delta\epsilon\phi\kappa} + 6 \Lambda_{\delta\epsilon\phi\kappa} \Lambda_{\beta\epsilon\phi\kappa} \]
\[ \Xi_{\delta ef kl} \Xi_{\gamma \beta ef kl} = 0, \]
so that
\[ \mathcal{D}_{\Xi} |_{\text{Re}(\Lambda^2 \oplus \Lambda^0 \oplus \Lambda^2)} = \frac{3}{200}, \]
\[ \mathcal{D}_{\Xi} |_{\text{Re}(\Lambda^1 \oplus \Lambda^1 \oplus \Lambda^1 \oplus \Lambda^1)} = \frac{N}{100} \mathcal{D}_{\Lambda} + \frac{3}{200} \text{pr}_0, \]
where \( \text{pr}_0 \) is the orthonormal projection onto \( u(1) \). The Lemma is then proved recalling the formula for \( \mathcal{D}_{\Lambda} \) given at the beginning of the present chapter.

Proof of Lemma 35. Note first that the map \( \mathcal{D}_{\Omega} \), being by construction \( G(\mathbb{K}, \mathbb{O}) \)-equivariant, is given by a multiple of identity when restricted to each of the irreducible subspaces of \( \Lambda^2 \mathcal{V}_{\mathbb{K}}(\mathbb{K}) \), namely:
\[ \Lambda^2 \mathcal{V}_{\mathbb{O}}(\mathbb{K}) = sp(1) \oplus G'_3(\mathbb{K}) \oplus [t \cap sp(\mathcal{V}_3(\mathbb{K}, \omega))] \oplus \perp. \]
Distinguishing as usually one of the complex structures of $\mathfrak{sp}(1)$, by convention $I$, to consider $V_3(\mathbb{K})$ as a complex vector space, one notes that each of the former subspaces has a nonzero intersection with $\Lambda^{1,1} \cong u(V_3(\mathbb{K}))$. It is therefore sufficient to check the eigenvalue of $D_3$ on each of the following:

$$\text{Re}\Lambda^{1,1} = u(1) \oplus G'_3(\mathbb{K}) \oplus \{t \cap \mathfrak{sp}(V_3(\mathbb{K}), \omega)\} \oplus \mathfrak{sp}_t^\perp(V_3(\mathbb{K}), \omega),$$

where $u(1)$ is generated by the distinguished complex structure and $\mathfrak{sp}_t^\perp(V_3(\mathbb{K}), \omega)$ is the complement of $\mathfrak{sp}(V_3(\mathbb{K}), \omega)$ in $\mathfrak{su}(V_3(\mathbb{K}))$.

Recall now the formula for the orthogonal projection of $\bar{u} \in S^{4,4}$ onto $S^{4,0} \otimes S^{0,4}$:

$$\bar{u}_{\mu_1\ldots\mu_4\bar{\mu}_5\ldots\bar{\mu}_8} = \frac{256}{70} P_{44}(q \otimes q)_{\mu_1\ldots\mu_4\sigma_1\ldots\sigma_4} \omega^\sigma_{\bar{\mu}_5} \cdots \omega^\sigma_{\bar{\mu}_8},$$

where the operator $P_{44} : S^{4,0} \otimes S^{1,0} \to S^{4,0} \otimes S^{4,0}$ is such that the image in $S^{4,0}_{\mu_1\ldots\mu_4} \otimes S^{4,0}_{\nu_1\ldots\nu_4}$ of a tensor $t_{\tau_1\ldots\tau_4\sigma_1\ldots\sigma_4} \in S^{4,0}_{\tau_1\ldots\tau_4} \times S^{4,0}_{\sigma_1\ldots\sigma_4}$ is:

$$P_{44}(t)_{\mu_1\ldots\mu_4\nu_1\ldots\nu_4} = \delta_{\mu_1}^{\tau_1} \delta_{\sigma_1}^{\nu_1} \delta_{\mu_2}^{\tau_2} \delta_{\nu_2}^{\sigma_2} \delta_{\mu_3}^{\tau_3} \delta_{\nu_3}^{\sigma_3} \delta_{\mu_4}^{\tau_4} \delta_{\nu_4}^{\sigma_4},$$

where the index sets $\mu_1\ldots\mu_4$ and $\nu_1\ldots\nu_4$ are symmetrized separately. Note that, introducing $c = \frac{256}{70}$ to avoid clutter,

$$c^{-1} \bar{u}_{\mu_1\ldots\mu_4\bar{\mu}_5\ldots\bar{\mu}_8} = P_{44}(q \otimes q)_{\mu_1\mu_2\mu_3\mu_4\sigma_1\sigma_2\sigma_3\sigma_4} \omega^\sigma_{\bar{\mu}_5} \omega^\sigma_{\bar{\mu}_6} \omega^\sigma_{\bar{\mu}_7} \omega^\sigma_{\bar{\mu}_8} + \ldots$$

with $C(\bar{\alpha}) = 70$ terms on the r.h.s., so that we have

$$c^{-2} \bar{u}_{\nu_1\ldots\nu_4} \bar{u}^{\nu_1\ldots\nu_4} = 35 P_{44}(q \otimes q)_{\alpha\mu_2\ldots\mu_8} P_{44}(q \otimes q)^{\beta\mu_2\ldots\mu_8}.$$
where the contractions of \( q, q, \bar{q}, \bar{q} \) are:

\[
(K_0)_\alpha^\beta = q_{\alpha\phi\phi} q_{\mu\nu\rho} q^{\beta\mu\nu\rho} q^{\phi\alpha\lambda} = \left(1 + \chi^2 \right) \frac{N + 1}{2} \delta_\alpha^\beta
\]

\[
(K_1)_\alpha^\beta = q_{\alpha\xi\phi\phi} q_{\lambda\mu\nu\rho} q^{\beta\xi\mu\nu\rho} q^{\phi\alpha\lambda} = \left(1 + \chi^2 \right) \frac{N + 1}{2} \delta_\alpha^\beta
\]

\[
(K_2)_\alpha^\beta = q_{\alpha\xi\eta\phi} q_{\lambda\mu\nu\rho} q^{\beta\xi\eta\mu\nu\rho} q^{\phi\alpha\lambda} = \left(1 + \chi^2 \right) \frac{N + 1}{2} \delta_\alpha^\beta
\]

\[
(K_3)_\alpha^\beta = q_{\alpha\xi\eta\phi} q_{\phi\lambda\mu\nu} q^{\beta\xi\phi\lambda\mu\nu} q^{\eta\alpha\lambda} = N \left(1 + \chi^2 \right) \frac{N + 1}{2} \delta_\alpha^\beta
\]

\[
(H_1)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\mu\nu\rho} q_{\beta\xi\eta\phi} q^{\gamma\mu\nu\rho} q^{\delta\xi\eta\phi} = \frac{N + 1}{2} \delta_\alpha^\gamma \delta_\beta^\delta
\]

\[
(H_2)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\beta\mu\nu} q_{\beta\xi\eta\phi} q^{\gamma\delta\mu\nu} q^{\phi\xi\eta\phi} = \frac{N(N + 1)}{2} \left[ \frac{1}{2} (\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma) + \chi q_{\alpha\beta}^{\gamma\delta} \right]
\]

\[
(H_3)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\beta\mu\nu} q_{\beta\xi\eta\phi} q^{\gamma\delta\mu\nu} q^{\phi\xi\eta\phi} = \frac{N + 1}{2} \left[ \frac{1}{2} (\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma) + \chi q_{\alpha\beta}^{\gamma\delta} \right]
\]

\[
(H_4)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\beta\mu\nu} q_{\beta\xi\phi\phi} q^{\gamma\delta\mu\nu} q^{\phi\xi\phi\phi} = \frac{N + 1}{2} \left[ \frac{1}{2} (\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma) + \chi q_{\alpha\beta}^{\gamma\delta} \right]
\]

\[
(H_5)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\beta\mu\nu} q_{\beta\xi\phi\phi} q^{\gamma\delta\phi\phi} q^{\phi\xi\phi\phi} = \left(1 + \chi^2 \right) \left[ \frac{1}{2} (\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma) + \chi q_{\alpha\beta}^{\gamma\delta} \right] - \frac{\chi}{2} q_{\alpha\beta}^{\gamma\delta}
\]

\[
(H_6)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\beta\mu\nu} q_{\beta\xi\phi\phi} q^{\gamma\delta\phi\phi} q^{\phi\xi\phi\phi} = \left(1 + \chi^2 \right) \left[ \frac{1}{2} (\delta_\alpha^\gamma \delta_\beta^\delta + \delta_\alpha^\delta \delta_\beta^\gamma) + \chi q_{\alpha\beta}^{\gamma\delta} \right] + \chi q_{\alpha\beta}^{\gamma\delta}
\]

\[
(H_7)_{\alpha\beta}^{\gamma\delta} = q_{\alpha\beta\mu\nu} q_{\beta\xi\phi\phi} q^{\gamma\delta\mu\nu} q^{\phi\xi\phi\phi} = \frac{N + 2}{4} \delta_\alpha^\gamma \delta_\beta^\delta + \frac{1 + 2 \chi^2}{4} \delta_\alpha^\gamma \delta_\beta^\delta - \frac{\chi}{2} \omega_{\alpha\beta}^{\gamma\delta} + \chi (1 - \chi^2) q_{\alpha\beta}^{\gamma\delta}
\]

and by slight abuse of notation \( q_{\alpha\beta}^{\gamma\delta} := q_{\alpha\beta\mu\nu} \bar{\omega}^{\gamma\delta} \). Collecting terms yields

\[
16^2 P_{44}(q \otimes \bar{q})_{\alpha\mu_1\beta\mu_2\nu_1 \ldots \nu_4} P_{44}(\bar{q} \otimes q)^{\beta\mu_1\mu_2\nu_1 \ldots \nu_4} = \frac{N + 1}{2} [25(N - 1) + 12 \chi^2] \delta_\alpha^\beta
\]
and
\[16^2 P_{44}(q \otimes q)_{\alpha \beta \mu_1 \mu_2 \nu_1 \ldots \nu_4} P_{44}(q \otimes q)_{\gamma \delta \mu_1 \mu_2 \nu_1 \ldots \nu_4} = \frac{11N(N + 1) - 6 + 10\chi^2}{2} \delta_\gamma^\gamma \delta_\delta^\delta \]
\[+ \frac{N(11N - 5) - 22(1 + \chi^2)}{2} \delta_\gamma^\gamma \delta_\delta^\delta \]
\[+ 16\chi^2 \omega_{\alpha \beta} \bar{\omega}^{\gamma \delta} + [N(11N - 5) - 40 + 42\chi^2] \chi_{\alpha \beta} \gamma \delta \]
and
\[16^2 P_{44}(q \otimes q)_{\alpha \mu_1 \mu_2 \nu_1 \nu_2 \beta} P_{44}(q \otimes q)_{\gamma \delta \mu_1 \mu_2 \nu_1 \nu_2 \delta} \]
\[= \frac{N(18N + 47) + 2 + 84\chi^2}{8} \delta_\alpha^\alpha \delta_\beta^\beta \]
\[+ \frac{2N(9N - 10) - 65 + 138\chi^2}{8} \delta_\alpha^\alpha \delta_\gamma^\gamma \]
\[+ \frac{111\chi^2}{4} \omega_{\alpha \beta} \bar{\omega}^{\gamma \delta} + \frac{9}{2} [N^2 - 4 + 3\chi^2] \chi_{\alpha \beta} \gamma \delta .\]

Thus applying $\Omega_{\alpha \mu \nu \mu_3 \ldots m_3 ... m_4}$ to $\text{Re} A_{\alpha \beta} := F_{\alpha \beta} = f_{\alpha \beta} + f_{\beta \alpha}$, with $f_{\alpha \beta} = -f_{\beta \alpha}$, we have $D_{\Omega}(F)_{ab} = f'_{\alpha \beta} + \bar{f}'_{\beta \alpha}$ where
\[
f'_{\alpha \beta} = \bar{\U}_\alpha \delta_{\mu_3 ... m_3} \U_\beta_{\mu_1 \mu_2 \nu_1 ... \nu_4} f_{\gamma \delta} + \bar{\U}_\alpha \delta_{\mu_3 ... m_3} \U_\beta_{\mu_1 \mu_2 \nu_1 ... \nu_4} \bar{f}_{\gamma \delta} \]
\[= \left\{ \frac{c^2}{16^2} f_{\gamma \delta} \left\{ 15 \left[ \frac{11N(N + 1) - 6 + 10\chi^2}{2} \delta_\gamma^\gamma \delta_\delta^\delta \right. \right. \right. \]
\[+ \left. \frac{N(11N - 5) - 22(1 + \chi^2)}{2} h_{\beta \alpha} h_{\gamma \delta} \right. \]
\[+ 16\chi^2 \omega_{\alpha} \bar{\omega}^{\gamma} \bar{\omega}^{\delta} + (N(11N - 5) - 40 + 42\chi^2) \chi_{\alpha \beta} \gamma \delta \right] \]
\[- 20 \left[ \frac{N(18N + 47) + 2 + 84\chi^2}{8} h_{\beta \alpha} h_{\gamma \delta} \right. \]
\[+ \left. \frac{2N(9N - 10) - 65 + 138\chi^2}{8} \omega_{\alpha} \bar{\omega}^{\gamma} \bar{\omega}^{\delta} \right. \]
\[+ \left. \frac{111\chi^2}{4} \delta_\alpha^\alpha \delta_\beta^\beta - \frac{9}{2} (N^2 - 4 + 3\chi^2) \chi_{\alpha \beta} \gamma \delta \right\} \right. \right. \}
\].

Again by abuse of notation we have introduced $q_{\alpha \beta} \gamma \delta := q_{\mu \nu} \delta \bar{\omega}_{\mu} \bar{\omega}_{\bar{\nu}}$.

Recollcet terms we finally have
\[
\frac{2}{5} 16^2 c^{-2} D_{\Omega} \mid_\nu = 3(11N^2 + 1N - 6 - 84\chi^2) \text{id} \]
\[+ (15N^2 - 62N - 68 - 150\chi^2) \text{pr}_0 \]
\[+ (18N^2 - 20N - 65 + 234\chi^2) \sigma \]
\[+ 6(17N^2 - 52N - 64 + 60\chi^2) D_q \]
where $D_q$ is trivially extended to the unitary complement of $\mathfrak{sp}(V_3(K))$, $\omega$ and $\sigma(E) = J^{-1} E J$, so that it is 1 on $\mathfrak{sp}(V_3(K))$ and -1 on its unitary complement.
Recalling the eigenvalues of $D_q$ on $G_3^q(K)$ and its symplectic complement, the result is

$$D_{\Omega}|_{u(1)} = 5c_2 \frac{512}{512} \left(30N^2 - 9N - 21 - 636\chi^2\right)$$

$$D_{\Omega}|_{G_3^q(K)} = 5c_2 \frac{512}{512} \left(51N^2 + 13N - 83 - 18\chi^2 + \frac{1}{\sqrt{\kappa + 3}}\right)$$

$$D_{\Omega}|_{t^\perp sp(V_3(K),\omega)} = 5c_2 \frac{512}{512} \left(51N^2 + 13N - 83 - 18\chi^2 + 1\sqrt{\kappa + 3}\right)$$

$$D_{\Omega}|_{sp^\perp(\nu_2(K),\omega)} = 5c_2 \frac{512}{512} \left(15N^2 + 53N + 47 - 386\chi^2\right).$$

We also have

$$D_{am_2...m_8} D^{bm_2...m_8} = 35c_2 \frac{256}{256} N + 1 \left[25(N - 1) + 12\chi^2\right] \delta^b_a.$$ 

Substituting $c = \frac{256}{10}$, and using equivariance of $D_{\Omega}$ and irreducibility of the subspaces the Lemma refers to, the proof is complete.

**Proof of Lemma 41.** We use the Killing form $m : M \to \mathfrak{M}^*$ to identify $\mathfrak{M}$ with $\mathfrak{M}^*$ in what follows. We will moreover, independently of the previous indexing conventions, use $a, b, c, \ldots$ to index $\mathfrak{g}$, $i, j, k, \ldots$ to index $V$ and $\alpha, \beta, \gamma, \ldots$ to index $\mathfrak{h}$ in such a way that greek letters denote subalgebras of the algebras indexed by corresponding latin ones:

$$\mathfrak{h}_\alpha \subset \mathfrak{g}_a, \quad \mathfrak{h}_\beta \subset \mathfrak{g}_\beta, \quad \ldots$$

Let the symbols $\epsilon$ and $\rho$ express the bracket on $\mathfrak{M}$:

$$[E, F]_\epsilon = \epsilon_{abc} E^a F^b$$

$$[E, X]_\rho = \rho_{aij} E^a X^i$$

$$[X, Y]_\rho = -\rho_{aij} X^i Y^j$$

for $E, F \in \mathfrak{g}$ and $X, Y \in V$. The bracket on $\mathfrak{k}$ is then given by:

$$[E, F]_\epsilon = \epsilon_{\alpha\beta\gamma} E^\alpha F^\beta$$

$$[E, X]_\rho = \rho_{aij} E^a X^i$$

$$[X, Y]_\epsilon = \epsilon_{aij} X^i Y^j$$

$$[X, Y]_\rho = c_{ijk} X^i Y^j$$

where $\rho_{aij}$ denotes the restriction of $\rho_{aij} \in \mathfrak{g}_a \otimes (\Lambda^2 V)_{ij}$ to $\mathfrak{h}_\alpha \otimes (\Lambda^2 V)_{ij}$ and

$$c_{aij} = \rho_{aij}$$

$$c_{ijk} = 3B_\alpha^a B_\beta^b B_\gamma^c - \epsilon_{\alpha\beta\gamma} B_\alpha^a B_\beta^b B_\gamma^c$$

and $B(E)_i = B_\alpha^a E^a$ satisfies

$$B_\alpha^a B_\beta^b = n^{\alpha\beta}$$

$$\epsilon_{\alpha\beta\gamma} B_\gamma^\alpha - \rho_{aij} B_\beta^a B_\gamma^b = 0.$$ 

We now investigate the Jacobi identities for $\mathfrak{k}$:
2.8. PROOFS OF LEMMAS 34, 35 & 41

1. \([\mathfrak{b}, \mathfrak{b}], \mathfrak{b}] : \epsilon^{\alpha}_{\beta\gamma}\epsilon^{\delta}_{\alpha_\beta} = 0\), satisfied since \(\mathfrak{b}\) is a subalgebra in \(\mathfrak{g}\).

2. \([\mathfrak{b}, \mathfrak{b}], V\] : \(\varepsilon_\alpha\beta\gamma\varepsilon_{\rho k}^\beta + 2\rho_{m[\rho_{k}]^m}^\beta = 0\), satisfied since \(\rho\) is equivariant.

3. \([V, V], \mathfrak{b}]\):

\[
\sum_{ijk} c^\beta_{ij} \varepsilon_{\beta\gamma\delta} + 2\rho_{m[\rho_{k}]^m}^\beta = 0 \quad \text{i.e.} \quad c^\alpha_{ij} \text{ equivariant}
\]

\[
c_m^c \rho_{k}^\beta = 0 \quad \text{i.e.} \quad c_{ijk} \text{ equivariant}
\]

4. \([V, V], V\) :

\[
c_m^c = 0
\]

\[
c_m^c c_{kl} = \rho_{ijkl}^0 = 0.
\]

Equivariance of \(B\), \(\rho\) and \(\epsilon\) automatically ensures satisfaction of all the equations except for the last one, namely:

\[
c_m^c c_{kl} = \rho_{ijkl}^0 = 0.
\]

We shall compute the first term. In the following formulas the indices \(ijk\) are implicitly antisymmetrized (the parentheses have been supressed for the sake of readability):

\[
c_{mij} c_{mkl} = (B_{m}^\alpha \rho_{ij}^\alpha + 2B_{m}^\alpha \rho_{jm}^\alpha - \varepsilon_\alpha\beta\gamma B_{m}^\alpha B_{i}^\beta B_{j}^\gamma)\]
\[
\times (B_{m}^\lambda \rho_{kl}^\lambda + B_{k}^\lambda \rho_{lm}^\lambda - B_{l}^\lambda \rho_{km}^\lambda - \varepsilon_{\alpha\mu\nu} B_{m}^\alpha B_{k}^\mu B_{l}^\nu).
\]

Calculating each term (using the antisymmetry in \(ijk\)) yields:

\[
B_{m}^\alpha B_{m}^\beta \rho_{ij}^\alpha \rho_{kl}^\beta = \rho_{ij}^\alpha \rho_{kl}^\beta,
\]
\[
2B_{m}^\alpha B_{k}^\beta \rho_{jm}^\alpha =
\]
\[
-2B_{m}^\alpha B_{k}^\beta \rho_{jm}^\alpha = \epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma,
\]
\[
-2\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = 2\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
-\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = -\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = \epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = \epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
-\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = -\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
-\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = -\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
\epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma = \epsilon_{\alpha\beta\gamma} B_{m}^\alpha B_{k}^\beta B_{j}^\gamma
\]
\[
2B_{m}^\alpha B_{k}^\beta \rho_{jm}^\alpha = 0.
\]

Summing the expressions on the r.h.s. we obtain \(\rho_{ijkl}^0\).
Thus the Jacobi equations are all satisfied and \( \mathfrak{h} \) is a Lie algebra. Then \( \mathfrak{h} \oplus V \) is a reductive pair by construction. Finally we note that the following quadratic form \( k \) is invariant:

\[
k(A, B) = -m(A, B), \quad k(A, X) = 0, \quad k(X, Y) = m(X, Y)
\]

for \( A, B \in \mathfrak{h} \) and \( X, Y \in V \). It clearly coincides with \( m \) on \( V \). \qed
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