Deforming maps between

\[ sl(n), \ sp(2n) \]

and

\[ U_q(sl(n)), U_q(sp(2n)) \]

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Abstract

Using crystal basis, in the space of symmetric irreducible representations, we explicitly write invertible deforming functionals between the \( q \)-deformed universal enveloping algebra and the Lie algebras \( sl(n) \) and \( sp(2n) \). For \( sl(2) \) we obtain the Curtright-Zachos map.
1 Introduction

Quantum algebras $G_q$ or $U_q(G)$, i.e. the $q$-deformed universal enveloping algebra of a semi-simple Lie algebra $G$ [1], [2] admit irreducible highest weight representations (IR) which are for $q$ generic, i.e. not a root of unity, in one-to-one correspondence with the IR of the Lie algebra $G$ [3], [4]. This property strongly suggests the existence of a correspondence between the generators of $G$ and $U_q(G)$. Indeed Curtright and Zachos [5] have found an invertible deforming functional $Q$ which allows the connection $sl(2) \iff sl_q(2)$. Denoting by small (resp. capital) letter $j_{\pm,0}$ ($J_{\pm,0}$) the generator of $sl(2)$ ($sl_q(2)$) it is possible to write ($q$ real number)

$$
J_+ = Q(j_{\pm,0}) j_+ \quad J_- = j_- Q(j_{\pm,0}) \quad J_0 = Q_0(j_0) j_0 = j_0
$$

where

$$
Q = \sqrt{\frac{[J_0 + J_q][J_0 - J_q - 1]}{(j_0 + j)(j_0 - j - 1)}}
$$

and the operator $j$ ($J$) is defined by the Casimir operator of $sl(2)$ ($sl_q(2)$)

$$
C = j(j + 1) \quad (C_q = [J]_q [J + 1]_q)
$$

In [5] it has been argued that the construction of the invertible functional can be generalized for any $sl(n)$, but, at our knowledge, no proof has yet been given. It is the aim of this paper to show that indeed such a functional can explicitly constructed for $sl(n + 1)$ and $sp(2n)$ ($n \geq 1$) in the space of the symmetric IRs, i.e. the IRs labelled by the Dynkin labels $a_1 > 0$, $a_i = 0$ ($i > 1$). In order to prove our statement we use the crystal basis, whose existence has been proven for the classical Lie algebra by Kashiwara [6]. In Sec. 2, in order to make the paper self-contained and to fix the notation, we recall the basic definitions of deformed Lie algebra $G_q$ in the Cartan-Chevalley basis and of crystal basis. In Sec. 3 and Sec. 4 we give our main results. A few remarks are given in Sec. 5.

2 Reminder of deformed algebras and crystal basis

Let us recall the definition of $G_q$ associated with a simple Lie algebra $G$ of rank $r$ defined by the Cartan matrix $(a_{ij})$ in the Chevalley basis. $G_q$ is generated by $3r$ elements $e_i^+, f_i = e_i^-$ and $h_i$ which satisfy ($i, j = 1, \ldots, r$)
\[ [e_i^+, e_j^-] = \delta_{ij} [h_i]_{q_i} \quad [h_i, h_j] = 0 \]

\[ [h_i, e_j^+] = a_{ij} e_j^+ \quad [h_i, e_j^-] = -a_{ij} e_j^- \]  

where

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \]

and \( q_i = q^{d_i} \), \( d_i \) being non-zero integers with greatest common divisor equal to one such that \( d_i a_{ij} = d_j a_{ji} \). For simply laced Lie algebra \( d_i = 1 \), for \( sp_q(2n) d_n = 2 \). Further the generators have to satisfy the Serre relations:

\[ \sum_{0 \leq n \leq 1 - a_{ij}} (-1)^n \left[ \frac{1 - a_{ij}}{n} \right]_q (e_i^\pm)^{1-a_{ij}-n} e_j^\pm (e_i^\pm)^n = 0 \]  

where

\[ \left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[m-n]_q! [n]_q} \]

\[ [n]_q! = [1]_q [2]_q \ldots [n]_q \]

In the following the deformation parameter \( q \) will be assumed different from the roots of the unity. The algebra \( G_q \) is also endowed with a Hopf algebra structure, but we shall not discuss here this aspect, although very relevant.

Let us also recall the definition of \( gl(n)_q, (k, j = 1, 2, \ldots, n-1; i = 1, 2, \ldots, n-1, n) \):

\[ [e_k^+, e_j^-] = \delta_{kj} [N_k - N_{k+1}]_q \quad [N_i, N_k] = 0 \]

\[ [N_i, e_j^+] = \pm (\delta_{i,j} - \delta_{i-1,j}) e_j^+ \]  

The Serre relations are computed using \( a_{i,j} = -(\delta_{i-1,j} + \delta_{i,j-1}) \). So \( gl(n)_q \) can be considered as \( sl(n)_q \oplus N_n \) with \( h_i = N_i - N_{i+1} \).

It has been shown in [6] that for any \( q \)-deformed universal enveloping algebra of a classical Lie algebra \( G \) of rang \( r \), in the limit \( q \to 0 \), a canonical basis exists, called crystal basis, such that, in any highest weight \( \tilde{\lambda} \) IR of \( G \): \( (i, j = 1, 2 \ldots r) \)

\[ \tilde{h}_i \psi(\tilde{\lambda}; \tilde{\lambda} = \{\lambda_i\}) = \lambda_i \psi(\tilde{\lambda}; \tilde{\lambda} = \{\lambda_i\}) \]

\[ \tilde{e}_i^+ \psi(\tilde{\lambda}; \tilde{\lambda} = \{\lambda_i\}) = \tilde{e}_i^- \psi(\tilde{\lambda}; \tilde{\lambda} = \{\lambda_i\}) = 0 \]

\[ \tilde{e}_i^\pm \psi(\tilde{\lambda}; \tilde{\lambda} = \{\lambda_i\}) = \psi(\tilde{\lambda}; \tilde{\lambda} = \{\lambda_i \pm a_i\}) \]

\[ \tilde{\lambda} = \{\lambda_j \pm a_{ij}\} \]
where we have denoted by $\hat{e}^\pm_i$, $\hat{h}_i = h_i$ the generators of $G_{q \rightarrow 0}$, by $\psi(\bar{\Lambda}; \bar{\lambda})$ a generic weight $\bar{\lambda}$ state, in the space of the IR with highest weight $\bar{\Lambda}$, and by $\psi(\bar{\Lambda}; \pm \bar{\lambda}_i)$ a state annihilated by $\hat{e}^\pm_i$. In [3] the relation between $\hat{e}^\pm_i$ and $\hat{e}^\pm_i$ is given. The previous equations imply

$$\hat{e}_i^+ \hat{e}_j^- \psi(\bar{\Lambda}; \bar{\lambda}) = \hat{e}_j^- \hat{e}_i^+ \psi(\bar{\Lambda}; \bar{\lambda}) = \psi(\bar{\Lambda}; \bar{\lambda} + \bar{a}_i - \bar{a}_j)$$

(13)

which for $i = j$ can be written

$$\hat{e}_i^+ \hat{e}_i^- \psi(\bar{\Lambda}; \bar{\lambda}) = 1 \psi(\bar{\Lambda}; \bar{\lambda}) \quad (\bar{\lambda} \neq -\bar{\lambda}_i)$$

$$\hat{e}_i^- \hat{e}_i^+ \psi(\bar{\Lambda}; \bar{\lambda}) = 1 \psi(\bar{\Lambda}; \bar{\lambda}) \quad (\bar{\lambda} \neq \bar{\lambda}_i)$$

(14)

Note that from eq.(13) the Serre relations for $G$ are trivially satisfied by $\{\hat{e}^\pm_i\}$.

### 3 Relation between $sl_q(n)$ and $sl(n)$

In this section we derive explicitly the invertible functionals which connect $sl_q(n)$ and $sl(n)$. Let us recall, e.g. see [4], that a state of an IR of $sl(n)$ is identified by a Young tableaux, which is a pattern of $n$ rows, the $i$-th row containing $l_i$ boxes, where $l_i$ are not negative integer satisfying, for the highest weight state, $l_k \geq l_{k+1}$ ($k = 1, 2, \ldots, n - 1$). Strictly speaking one should consider class of Young tableaux as two tableaux differing for a block of $(p \times n)$ boxes i.e. by the first $p$ columns $(1 \leq p \leq l_n)$, identify the same IR. The relation between the $n - 1$ Dynkin label $a_k$ and the $n$ labels $l_i$ is

$$a_k = l_k - l_{k+1}$$

(15)

The integer $l_i$ is the eigenvalue of the operator $N_i$ on the state identified by the corresponding Young tableaux and $\lambda_i = l_i - l_{i+1}$. From eq.(8) it follows

$$[N_i, \hat{e}^\pm_k] = \pm(\delta_{i,k} - \delta_{i-1,k}) \hat{e}^\pm_k$$

(16)

As first step we prove the following proposition

**Prop. 1** - The generators $(i = 1, 2, \ldots, n - 1)$

$$E_i^+ = \hat{e}_i^+ \sqrt{(N_i + 1) N_{i+1}} \quad E_i^- = \hat{e}_i^- \sqrt{(N_i + 1) N_{i+1}} \quad H_i = N_i - N_{i+1}$$

(17)

where $\{\hat{e}_i^\pm, N_i, N_n\}$ are the generators in the crystal basis of $gl(n)_{q \rightarrow 0}$, satisfy the defining relations, in the Cartan-Chevalley basis, of $sl(n)$ in the space of the symmetric IRs labelled by the Young tableaux $l_1 = \Lambda, \ l_i = 0$. 

3
Proof: On the highest weight state we have

\[ [E_i^+, E_i^-] \psi(\vec{\Lambda}; \vec{\Lambda}) = E_i^+ E_i^- \psi(\vec{\Lambda}; \vec{\Lambda}) = N_i (N_{i+1} + 1) \psi(\vec{\Lambda}; \vec{\Lambda}) = (l_i l_{i+1} + l_i) \psi(\vec{\Lambda}; \vec{\Lambda}) \]  

(18)

The eigenvalue of the r.h.s. of eq.(18) has to identified with the eigenvalue of \( H_i \), i.e. \( l_i - l_{i+1} \), which implies \( l_1 = \Lambda, l_i = 0 \) \( (i \geq 1) \). Then we consider the states \( \{ \psi(\vec{\Lambda}; -\vec{\Lambda}_i) \} \) which are eigenstates with vanishing eigenvalue of \( N_i \)

\[ N_i \psi(\vec{\Lambda}; -\vec{\Lambda}_i) = 0 \]  

(19)

We have, from eq.(17):

\[ [E_i^+, E_i^-] \psi(\vec{\Lambda}; -\vec{\Lambda}_i) = -E_i^- E_i^+ \psi(\vec{\Lambda}; -\vec{\Lambda}_i) = -(N_i + 1) N_{i+1} \psi(\vec{\Lambda}; -\vec{\Lambda}_i) = -N_{i+1} \psi(\vec{\Lambda}; -\vec{\Lambda}_i) = H_i \psi(\vec{\Lambda}; -\vec{\Lambda}_i) \]  

(20)

On the state \( \psi(\vec{\Lambda}; \vec{\lambda}) \) \( (\vec{\lambda} \neq -\vec{\Lambda}_i) \), from eqs.(11)-(14) we have

\[ [H_i, E_j^\pm] \psi(\vec{\Lambda}; \vec{\lambda}) = \pm a_{ji} E_j^\pm \psi(\vec{\Lambda}; \vec{\lambda}) \]  

(21)

\[ [E_i^+, E_j^-] \psi(\vec{\Lambda}; \vec{\lambda}) = \delta_{ij} H_i \psi(\vec{\Lambda}; \vec{\lambda}) \]  

(22)

So we can write

\[ [H_i, E_j^\pm]_\psi = \pm a_{ji} E_j^\pm \]  

(23)

\[ [E_i^+, E_j^-]_\psi = \delta_{ij} H_i \]  

(24)

where the lower label \( \psi \) in the commutator reminds that the relations hold when applied on a state of a symmetric IR and not as general algebraic expressions. Finally we have to prove that generators \( E_i^\pm \) satisfy the Serre relation for \( sl(n) \):

\[ \sum_{0 \leq n \leq 1-a_{ij}} (-1)^n \binom{1-a_{ij}}{n} (E_i^\pm)^{1-a_{ij}-n} E_j^\pm (E_i^\pm)^n = 0 \]  

(25)

This is an immediate consequence of eq.(17) and of the following identity \( (a \geq 1, a \in \mathbb{Z}_+, z \in \mathbb{C}) \), evaluated (for \( E_i^+ \)) for \( N = N_{i+1} + 1 \), \( q = 1 \), and \( a = z = 1 \)

\[ \sum_{0 \leq n \leq (1+a)} (-1)^n \left[ \frac{1+a}{n} \right]_q [N-nz]_q = 0 \]  

(26)

Then we easily get
Prop. 2 - The generators \((i = 1, 2, \ldots, n - 1)\)
\[
e_i^+ = \tilde{e}_i^+ \sqrt{[N_i + 1]_q [N_{i+1}]_q} \quad e_i^- = \sqrt{[N_i + 1]_q [N_{i+1}]_q} \quad \tilde{e}_i^- \quad h_i = H_i
\]

define \(sl_q(n)\) in the Cartan-Chevalley basis in the space of the symmetric IRs.

Proof: The second defining relation in eq.(4) is immediately proven while the first one follows by the identity
\[
[N_i]_q [N_{i+1} + 1]_q - [N_i + 1]_q [N_{i+1}]_q = [N_i - N_{i+1}]_q
\]

The deformed Serre relations eq.(6) are proven using eq.(26).

From Prop.1 and Prop.2, in the symmetric basis, it follows the relation between \(sl(n)\) and \(sl_q(n)\)
\[
e_i^+ = E_i^+ \sqrt{[N_i + 1]_q [N_{i+1}]_q} \quad e_i^- = \sqrt{[N_i + 1]_q [N_{i+1}]_q} E_i^- \]

Let us discuss in more detail the case \(sl(2)\). In this case all the IRs are of the type we have called symmetric and the eq.(29) reads
\[
j^+ = J^+ \sqrt{[N_1 + 1]_q [N_2]_q} \quad j^- = \sqrt{[N_1 + 1]_q [N_2]_q} J^- \]

From the following relation between the operators \(j\) and \(j_0\) and our operators
\[
j = \frac{N_1 + N_2}{2} \quad j_0 = \frac{N_1 - N_2}{2}
\]
we see that eq.(31) is equivalent to the Curtright-Zachos formula eq.(2). Moreover, in this case, the deforming map can be turned into a general algebraic relation. As a first consequence of eq.(29), we can give a new construction of \(so_q(3)\) in terms of standard bosons. Let us recall that Van der Jeugt [8] has shown that the following generators satisfy the commutation of \(so_q(3)\)
\[
L_+ = q^{N_1 - N_0/2} q^{N_1} b_0^{-1} b_0 + b_0^2 q^{-N_1} q^{-N_0/2} \sqrt{q^{N_1} + q^{-N_1}}
\]
\[
L_- = b_0^2 b_1 q^{N_1} q^{-N_0/2} \sqrt{q^{N_1} + q^{-N_1}} + q^{N_1} q^{-N_0/2} \sqrt{q^{N_1} + q^{-N_1}} b_0^{-1} b_0
\]
\[
L_0 = N_1 - N_1
\]

5
where $b_{\pm}, b_{\pm,0}$ are the Biedenharn-MacFarlane $[^9], [^10]$ $q$-bosons. From our result it follows that the following generators satisfy, on the states of any IR, the commutation relations of $so_q(3)$

\[
L_+ = \sqrt{2} \left( \tilde{b}_0^+ \tilde{b}_0 - \tilde{b}_1^+ \tilde{b}_-1 \right) \sqrt{\frac{[N_1 + 1]_q [N_2]_q}{(N_1 + 1) N_2}}
\]

\[
L_- = \sqrt{2} \left( \tilde{b}_1^+ \tilde{b}_0^+ + \tilde{b}_0^+ \tilde{b}_-1 \right) \sqrt{\frac{[N_1 + 1]_q [N_2]_q}{(N_1 + 1) N_2}}
\]

\[
N_1 = 2 \tilde{b}_1^+ \tilde{b}_1 + \tilde{b}_0^+ \tilde{b}_0
\]

\[
N_2 = 2 \tilde{b}_-1^+ \tilde{b}_-1 + \tilde{b}_0^+ \tilde{b}_0
\]

\[
L_0 = (N_1 - N_2)/2 = \tilde{b}_1^+ \tilde{b}_1 - \tilde{b}_-1^+ \tilde{b}_-1
\]

where $\tilde{b}_{\pm,0}, \tilde{b}_{\pm,0}$ are standard bosonic operators.

4 Relation between $sp_q(2n)$ and $sp(2n)$

In this section we derive explicitly the invertible functionals which connect $sp_q(2n)$ and $sp(2n)$. Let us recall, e.g. see $[^3]$, that an IR of $sp(2n)$ can be identified by a $n$-rows Young tableaux, the $i$-th row containing $l_i$ boxes, $l_k \geq l_{k+1}$ ($k = 1, 2, \ldots, n-1$).

The relation between the $n$ Dynkin label $a_k$ and the $n$ labels $l_i$ is

\[
a_k = l_k - l_{k+1} \quad a_n = l_n
\]

The integer $l_i$ is the eigenvalue of the operator $N_i$ on the state identified by the corresponding Young tableaux. Now we define

\[
[N_n, \hat{e}_j^\pm] = \pm (2 \delta_{n,j} - \delta_{n-1,j}) \hat{e}_j^\pm
\]

while the action of for $N_j, j \neq n$, is given by eq.(16). In complete analogy with the previous case, using eq.(19), we get

Prop. 3 - The generators ($k = 1, 2, \ldots, n-1$)

\[
E^+_k = \hat{e}_k^+ \sqrt{(N_k + 1) N_{k+1}} \quad E^-_k = \sqrt{(N_k + 1) N_{k+1}} \hat{e}_k^-
\]

\[
E^+_n = \hat{e}_n^+ \sqrt{(N_n + 1) (N_n - 2)} \quad E^-_n = \sqrt{(N_n + 1) (N_n - 2)} \hat{e}_n^-
\]

\[
H_k = N_k - N_{k+1} \quad H_n = N_n + \frac{1}{2}
\]

define, in the Cartan-Chevalley basis, $sp(2n)$ in the spaces of symmetric IRs labelled by the Young tableaux with $l_1 = \Lambda, l_i = 0$ ($i \geq 1$).

6
To prove the Serre relations between $E_{n-1}^+$ and $E_n^+$ one has to use eq.(26) evaluated for $N = N_n$, $q = 1$, $z = -2$ and $a = 2$, for $a_{n,n-1}$, and $a = 3$, for $a_{n-1,n}$.

Prop. 4 - The generators $(k = 1, 2, \ldots, n - 1)$

\[
e_k^+ = \hat{e}_k^+ \sqrt{([N_k + 1]_q [N_{k+1}]_q)}, \quad E_k^- = \sqrt{([N_k + 1]_q [N_{k+1}]_q)}, \quad E_n^- = \frac{1}{q^{n+1}} \sqrt{([N_n + 1]_q [-N_n - 2])_q} e_n^-
\]

\[
h_k = H_k, \quad h_n = H_n
\]

(37)

define $sp_q(2n)$ in the Cartan-Chevalley basis in the space of the symmetric IRs.

The first defining relation in eq.(4) is proven using the identity

\[
[N - 1]_q [-N]_q - [N + 1]_q [-N - 2]_q = [2N + 1]_q \times (q + q^{-1})
\]

(38)

From Prop.3 and Prop.4 it follows the relation between $sp(2n)$ and $sp_q(2n)$

\[
e_k^+ = E_k^+ \sqrt{([N_k + 1]_q [N_{k+1}]_q)} \quad E_k^- = \sqrt{([N_k + 1]_q [N_{k+1}]_q)}, \quad E_n^- = \frac{2}{q^{n+1}} \sqrt{([N_n + 1]_q [-N_n - 2])_q} E_n^-
\]

(39)

5 Conclusions

We have found explicit invertible maps between $U_q(sl(n))$, $U_q(sp(2n))$ and $sl(n)$, $sp(2n)$ which hold on the states of the IRs labelled by the Dynkin label $a_1$, that is the symmetric representations. Let us remark that the Cartan sub-algebra is left undeformed and it can also be identified with the set of diagonal generators in the crystal basis. As a byproduct result we have obtained:

- in the spaces of the symmetric IRs, explicit expressions of the generators of $U_q(sl(n))$ and $U_q(sp(2n))$ in terms of the generators of $sl(n)$ and $sp(2n)$. Note however that the correspondence $E_i^\pm \leftrightarrow \hat{e}_i^\pm$ has to be handled with some precautions as the factor $(N_i + 1) N_{i+1}$ appearing in Prop.1 and in Prop.3 is vanishing on some states. This is an interesting result as the relations given in eq.8 between $\hat{e}_i^\pm$ and $e_i^\pm$ are rather cumbersone for Lie algebra $G$ of rank larger than one.

- from eq.(37)-(27) (resp. eqs.(38)-(37)) an expression of the action of the generators of $sl(2)$ (resp. $sl_q(2)$) embedded in $sl(n)$ and $sp(2n)$ (resp. $sl_q(n)$ and $sl_q(2n)$ on the states of the symmetric IRs.
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