CP-violation due to Majorana phase in two flavour neutrino oscillations

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(Dated: August 10, 2022)

We study the conditions under which the Majorana phase of the two flavour neutrino mixing matrix appears in the oscillation probabilities and causes CP-violation. We find that the Majorana phase remains in the neutrino evolution equation if the neutrino decay eigenstates are not aligned with the mass eigenstates. We show that, in general, two kinds of CP-violation are possible: one due to the Majorana phase and the other due to the phase of the off-diagonal element of the neutrino decay matrix. We find that the CP violating terms in the oscillation probabilities are also sensitive to neutrino mass ordering.

I. Introduction

Neutrinos are the most intriguing particles in nature. They are the only known elementary neutral fermions. Even their fundamental nature, whether they are Dirac or Majorana fermions, is an open question. In the Standard Model (SM), neutrinos are massless. But the discovery of neutrino oscillations showed that different neutrino flavours mix to form mass eigenstates and these states have tiny, non-degenerate masses, which are more than a million times smaller than the electron mass. The mechanism which gives rise to such tiny masses is also an open problem in particle physics.

Since neutrinos are neutral, it is possible for them to be their own anti-particle, i.e., they can be Majorana fermions. The mass term for Majorana neutrinos has a very different form compared to that of Dirac neutrinos and can be naturally made small via see-saw mechanism. Majorana masses violate lepton number by two units and lead to interesting signals such as neutrinoless double beta decay or same sign lepton pairs at colliders. They are the only known elementary neutral fermions.

There are, however, some neutrino evolution equations for which the Majorana phases appear in neutrino oscillation probabilities. A new form of neutrino decoherence, with an off-diagonal term in the decoherence matrix was considered in Ref. [5]. It was shown that the neutrino oscillation probabilities depend on Majorana phases in such a case. It was also shown that these probabilities are CP-violating [6]. This leads us to the question, “what are the other possibilities under which the Majorana phases appear in neutrino oscillation probabilities and lead to CP-violation?”

In this paper we address the above question for the case of two flavour oscillations. Extension of the discussion to three flavour oscillations is straight forward. We consider the most general neutrino evolution Hamiltonian including decay terms. Then we identify the terms in this Hamiltonian which lead to the appearance of Majorana phase in oscillation probabilities and discuss their CP and CPT properties.

II. Vacuum neutrino oscillations

In this section we briefly discuss the dynamics of two-flavour neutrino oscillations in vacuum. In general, neutrino mass eigenstates $\nu_i$ mix via a unitary matrix to introduce flavour states $\nu_\alpha$ of neutrinos

$$\nu_\alpha = U \nu_i = O U_{ph} \nu_i,$$

where $\nu_\alpha = (\nu_e \nu_\mu)^T$ and $\nu_i = (\nu_1 \nu_2)^T$ and

$$O = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad U_{ph} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}. \quad (1)$$

The mixing matrix $U$ is parameterized in terms of the mixing angle $\theta$ and the Majorana phase $\phi$. We assume that two other phases are pulled out on the left and are absorbed in the flavour states. In the case of Dirac neutrinos, the phase $\phi$ gets absorbed in the neutrino mass eigenstates through rephasing and we are left with the orthogonal mixing matrix. Such rephasing cannot be done for Majorana neutrinos.

Let us now consider the traditional diagonal Hamiltonian in mass-basis that governs the time evolution of
neutrino mass eigenstates

\[ \mathcal{H} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \]

\[ = \begin{pmatrix} (a_1 + a_2) & (a_2 - a_1) \\ (a_2 - a_1) & (a_1 + a_2) \end{pmatrix} \]

where \( a_1 = m_1^2/2E \) and \( a_2 = m_2^2/2E \) where \( m_i \) are the mass eigenvalues and \( E \) is the energy of the neutrinos. Evolution equations in mass eigenbasis is

\[ i \frac{d}{dt} \nu_i(t) = \begin{bmatrix} (a_1 + a_2)^2 \sigma_0 - (a_2 - a_1) \sigma_z \end{bmatrix} \nu_i(t), \]

(2)

where, \( \sigma_0 \) is the 2 \( \times \) 2 identity matrix and \( \sigma_z \) is the diagonal Pauli matrix. In the flavour basis, eq. (2) has the form

\[ \begin{bmatrix} \sigma_0 & \sigma_z \end{bmatrix} U \mathcal{H} U^\dagger \begin{bmatrix} \sigma_0 & \sigma_z \end{bmatrix} = \begin{bmatrix} (a_1 + a_2)^2 \sigma_0 - (a_2 - a_1) \sigma_z \end{bmatrix} \]

and the phase \( \phi \) disappears from the evolution equation. The vacuum neutrino flavour transition probability is obtained as

\[ P(\nu_e \rightarrow \nu_\mu) = \sin^2 2\theta \sin^2 \left( \frac{(a_2 - a_1) t}{2} \right) \equiv P_{e\mu}^{\text{vac}}. \]

(4)

The term proportional to \( \sigma_0 \) makes no distinction between the flavours and hence is absent from the probabilities. Since, \( U_{ph} \) and \( \sigma_z \) are diagonal matrices they commute and \( \sigma_z \) term simplifies to \( \mathcal{O} \sigma_z \mathcal{O}^T \) and the phase \( \phi \) disappears from the evolution equation. The vacuum neutrino flavour transition probability is obtained as

\[ P_{e\mu}^{\text{vac}} = \sin^2 2\theta \sin^2 \left( \frac{\Delta m^2 L}{4E} \right). \]

(5)

The relations between different probabilities, \( P_{ee}^{\text{vac}} = 1 - P_{e\mu}^{\text{vac}} = P_{\mu\mu}^{\text{vac}} \) and \( P_{ee}^{\text{vac}} = P_{\mu\mu}^{\text{vac}} \), follow trivially. Expressing \( a_i \) as \( m_i^2/2E \) we get the standard expression for transition probability.

where, \( L \) is the distance traveled by neutrinos.

### III. Oscillations with general decay-Hamiltonian

We now consider the case of the most general Hamiltonian in neutrino mass basis including the decay terms

\[ \mathcal{H} = M - i\Gamma/2, \]

(6)

where the hermitian matrices \( M \) and \( \Gamma/2 \) have the form

\[ \mathcal{M} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \Gamma/2 = \begin{pmatrix} b_1 & i\eta e^{i\xi} \\ -i\eta e^{-i\xi} & b_2 \end{pmatrix}. \]

(7)

In Eq. (7), the parameters \( a_1, a_2, b_1, b_2, \eta, \xi \) are real with \( a_2 - a_1 = \Delta m^2/2E \) depicting the frequency of neutrino oscillations and rest causing decay of neutrino mass eigenstates. The matrix \( \Gamma \) needs to be positive semi-definite, i.e., non negative which leads to the following constraints: \( b_1, b_2 \geq 0 \) and \( \eta^2 \leq 4b_1b_2 \).

Neutrino evolution through this Hamiltonian describes both neutrino oscillation as well as neutrino-decay. If \( \Gamma \) is diagonal (\( \eta = 0 \)), i.e., the decay eigenbasis is the same as the mass eigenbasis. In such a case it is straightforward to show that the Majorana phase \( \phi \) disappears from neutrino evolution equations through a discussion similar to that of the previous section. We now consider the case where \( \Gamma \) is non-diagonal (\( \eta \neq 0 \)), i.e., the mass eigenstates are not decay eigenstates. In this case, the evolution equation in the mass eigenbasis takes the form

\[ \begin{bmatrix} \sigma_0 \sigma_z \end{bmatrix} U \mathcal{H} U^\dagger \begin{bmatrix} \sigma_0 \sigma_z \end{bmatrix} = \begin{bmatrix} (a_1 + a_2)^2 \sigma_0 - (a_2 - a_1) \sigma_z \end{bmatrix} \]

and the phase \( \phi \) remains in the evolution equation.

\begin{align*}
\int \frac{d}{dt} \nu_i(t) &= \begin{bmatrix} (a_1 + a_2)^2 \sigma_0 - (a_2 - a_1) \sigma_z \end{bmatrix} \nu_i(t), \\
\int \frac{d}{dt} \nu_\alpha(t) &= \begin{bmatrix} (a_1 + a_2)^2 \sigma_0 - (a_2 - a_1) \sigma_z \end{bmatrix} \nu_\alpha(t).
\end{align*}

(8)

Since \( \sigma_x \) and \( \sigma_y \) do not commute with \( U_{ph} \) matrix, the phase \( \phi \) remains in the evolution equation.

\[ \frac{d}{dt} \nu_i(t) = \begin{bmatrix} (a_1 + a_2)^2 \sigma_0 - (a_2 - a_1) \sigma_z \end{bmatrix} \nu_i(t), \]

(9)

The time evolution operator for neutrinos in the mass eigenbasis is \( \mathcal{U} = e^{-i\mathcal{H}t} \). This matrix can be ex-
panded in the basis spanned by $\sigma_0$ and Pauli matrices $\gamma$. This expansion is parameterized by a complex 4-vector $n_\mu \equiv (n_0, \vec{n})$, whose components are given by $n_\mu = Tr[(-i\mathcal{H})_\mu]/2$. Explicitly, they are expressed in terms of the parameters of $\mathcal{H}$ as

$$n_0 = -\frac{i}{2}(a_1 + a_2)t - \frac{1}{2}(b_1 + b_2)t,$$

$$n_x = \frac{1}{2}(\eta \cos \xi)t,$$

$$n_y = \frac{1}{2}(\eta \sin \xi)t,$$

$$n_z = \frac{i}{2}(a_2 - a_1)t + \frac{1}{2}(b_2 - b_1)t.$$

(10)

In terms of these components the evolution matrix $U$ is

$$U = e^{i n_0} \left[ \begin{array}{c} \cosh n \sigma_0 + \frac{\vec{n} \cdot \vec{\sigma}}{n} \sinh n \end{array} \right].$$

(11)

where

$$n = \sqrt{n_0^2 + n_x^2 + n_y^2 + n_z^2} = \frac{t}{2} \sqrt{\eta^2 - (a_2 - a_1 - i(b_2 - b_1))}.$$

(12)

The evolution matrix in flavour basis can be obtained through the transformation $U_f = U U_d U_i^{-1}$, where $U$ is defined in Eq. (1). Oscillation probabilities can be obtained as

$$P_{\alpha\beta} = |(U_f)_{\alpha\beta}|^2.$$

The general probability expressions with all the decay parameters non-zero are quite complicated. In this article, we are interested in how the probabilities depend on the Majorana phase $\phi$. To illustrate this, consider the probability expressions in the limit $b_1 = b = b_2$ and $\eta \ll |a_2 - a_1|$. For convenience we define

$$A = \left( \frac{\sin(2\theta) \sin((a_2 - a_1) t)}{(a_2 - a_1)} \right) \quad \text{and} \quad B = \left( \frac{\sin(2\theta) \sin^2 \left( \frac{1}{2}t(a_2 - a_1) \right)}{(a_2 - a_1)} \right).$$

(13)

Neglecting terms of $O(\eta^2)$ and higher order, we get the survival probabilities as

$$P_{ee} = e^{-2bt} \left( P_{ee}^{\text{vac}} - \eta \cos(\xi - \phi)A \right) \quad \text{and} \quad P_{\bar{\mu}\bar{\mu}} = e^{-2bt} \left( P_{\bar{\mu}\bar{\mu}}^{\text{vac}} + \eta \cos(\xi - \phi)A \right)$$

(14)

and the oscillation probabilities as

$$P_{e\mu} = e^{-2bt} \left( P_{e\mu}^{\text{vac}} + 2\eta \sin(\xi - \phi)B \right) \quad \text{and} \quad P_{\mu e} = e^{-2bt} \left( P_{\mu e}^{\text{vac}} - 2\eta \sin(\xi - \phi)B \right).$$

(15)

Hence, we see that the Majorana phase $\phi$ appears in the probability expressions if the neutrino evolution equation contains the off-diagonal term of the decay matrix $\Gamma_{12} \propto \eta$. The presence of this term also violates the equalities $P_{\mu\mu} = P_{ee}$ and $P_{\mu e} = P_{e\mu}$ that we see in the case of two flavour vacuum oscillations. In addition, we note that the terms with $B$, present in oscillation probabilities, have opposite signs for the two cases $a_2 > a_1$ ($m_2 > m_1$) and $a_2 < a_1$ ($m_2 < m_1$). That is the oscillation probability is sensitive to the mass hierarchy.

We now consider the oscillations of antineutrinos. We assume CPT-conservation which implies the following relations for the mass and decay matrices $\tilde{\mathcal{M}} = M$ and $\tilde{\Gamma} = \Gamma^*$. (16)

Hence, antineutrino probabilities expressions can be obtained by making the substitutions $\phi \to -\phi$ and $\xi \to -\xi$ in the neutrino probability expressions. Explicitly these expressions are

$$P_{ee} = e^{-2bt} \left( P_{ee}^{\text{vac}} - \eta \cos(\xi - \phi)A \right)$$

$$P_{\bar{\mu}\bar{\mu}} = e^{-2bt} \left( P_{\bar{\mu}\bar{\mu}}^{\text{vac}} + \eta \cos(\xi - \phi)A \right)$$

and

$$P_{e\bar{\mu}} = e^{-2bt} \left( P_{e\bar{\mu}}^{\text{vac}} - 2\eta \sin(\xi - \phi)B \right)$$

$$P_{\bar{\mu}e} = e^{-2bt} \left( P_{\bar{\mu}e}^{\text{vac}} + 2\eta \sin(\xi - \phi)B \right).$$

(17)

(18)

Since we assumed CPT invariance we find $P_{e\bar{\mu}} = P_{ee}$, $P_{\bar{\mu}\bar{\mu}} = P_{\bar{\mu}\bar{\mu}}$ and $P_{e\bar{\mu}} = P_{\bar{\mu}e}$. However, there is CPT-violation ($P_{e\bar{\mu}} \neq P_{e\mu}$) and T-violation ($P_{\mu e} \neq P_{e\mu}$).

IV. Results and Discussions

In this section we discuss our results. In transforming the evolution equation from the mass eigenbasis to the flavour eigenbasis we get the matrix product $U_{ph} \mathcal{H} U_{ph}^\dagger$, where $\mathcal{H}$ is the Hamiltonian in mass eigenbasis. The diagonal phase matrix $U_{ph}$ commutes with $\mathcal{H}$ whenever $\mathcal{H}$ is diagonal. This is true for the usual neutrino oscillations and for the case where the mass eigenstates are also the decay eigenstates. In such situations, $U_{ph}$ matrix drops out from neutrino evolution equation which in turn leads to the oscillation probabilities being independent of the Majorana phase $\phi$. But when the decay eigenstates are not aligned with the mass eigenstates, there is an off-diagonal term in the decay matrix $\Gamma$ of $\mathcal{H}$. Since $\mathcal{H}$ is no longer diagonal, it does not commute with $U_{ph}$ thus leading to the presence of $\phi$ in the evolution equation and in the probabilities. In addition to the off-diagonal dissipator discussed in [5], the off-diagonal decay matrix is another possible source for the appearance of Majorana phases in the oscillation probabilities and the corresponding CPT-violation.

The off-diagonal term of the decay matrix $\Gamma$ has the general form $\eta e^{i\xi}$ and the CPT violating term in the oscillation probabilities is proportional to $\eta \sin(\xi - \phi)$. We distinguish different forms of CPT-violation based on the values of the phases $\xi$ and $\phi$.

- For $\eta \neq 0$ and $\xi = 0$ the decay matrix $\Gamma$ is real and is CPT-conserving. In such a case we need $\phi \neq 0$ for CPT-violation. We call this CPT-violation in mass because $\phi$ arises due to the diagonalization of the complex mass matrix.
• If $\phi = 0$, it is still possible to have $CP$-violation if $\eta \neq 0$ and $\xi \neq 0$. We call this $CP$-violation in decay because the $CP$ violating phase $\xi$ comes from the decay matrix.

• The most general possibility is $\eta \neq 0$, $\xi \neq 0$ and $\phi \neq 0$. In this case, we have $CP$-violation due to both mass and decay provided $\phi \neq \xi$.

We see that $\eta \neq 0$ in all the above three cases. However, a non-zero value of $\eta$ is a necessary condition for $CP$-violation but not a sufficient condition. For the two special cases, (a) $\phi = 0 = \xi$ and (b) $\phi = \xi$, there is no $CP$-violation even when $\eta \neq 0$. In these two cases the $CP$ violating terms vanish and the flavour conversion probabilities are the same as the vacuum probabilities multiplied by the decay term. However, the presence of a non-zero value of $\eta$ is discernible in the survival probabilities.

V. Summary & Conclusions

In this article, we point out scenarios in which the Majorana phase can appear in neutrino oscillation probabilities that also causes $CP$-violation. We did this analysis for two flavour oscillations but the extension of this work for three flavour oscillations is straightforward. $CP$-violation in neutrino oscillations requires complex values of neutrino mixing matrix. In the case of standard two flavour oscillations the phases of this matrix, including the Majorana phase, drop out of the evolution equation and there is no $CP$-violation. In this work we have shown that the Majorana phase in two flavour mixing remains in the evolution equation and causes $CP$-violation provided the neutrinos decay and the decay eigenstates are not the same as mass eigenstates. This requires the decay matrix $\Gamma$ to have an off-diagonal term $\Gamma_{12}$. We have shown that two types of $CP$-violation are possible: (a) that due to the Majorana phase $\phi$ which we call $CP$-violation in mass and (b) that due to the phase $\xi$ of $\Gamma_{12}$ which we call $CP$-violation in decay. The $CP$ violating term in the oscillation probability is also sensitive to the neutrino mass ordering. In the two special cases, when $\phi$ and $\xi$ are equal to each other or when both are zero, there is no $CP$-violation even if the decay eigenstates are different from the mass eigenstates. In such a situation, the flavour conversion probabilities are insensitive to off-diagonal elements of $\Gamma$ but the flavour survival probabilities do depend on them.

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