Slow rotation of a superfluid trapped Fermi gas

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The moment of inertia, $\Theta$, is one of the possible observables for the experimental determination whether a trapped Fermi system has reached the BCS transition or not. In this article we investigate in detail the temperature dependence of $\Theta$ below the critical temperature $T_c$. Special care is taken to account for the small size of the system, i.e., for the fact that the trapping frequency $\hbar \omega$ is of the same order of magnitude as the gap $\Delta$. It is shown that the usual transport approach, corresponding to the leading order of an expansion in powers of $\hbar$, is not accurate in this case. It turns out that $\Theta$ does not change rapidly if $T$ becomes smaller than $T_c$, but it rather decreases slowly. Qualitatively this behavior can be explained within the two-fluid model, which again corresponds to the leading order in $\hbar$. Quantitatively we find deviations from the two-fluid model due to the small system size.

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I. INTRODUCTION

Since the first observation of Bose-Einstein condensation of magnetically trapped bosonic atoms \cite{arXiv:cond-mat/0211469v1} it has become clear that ultra-cold trapped atomic gases provide an excellent tool to study quantum effects in systems which are almost visible to the naked eye. For example, quantized vortices in the Bose condensate were created by stirring the Bose condensate with the help of a laser beam \cite{4}. Also the quantum pressure related to the Pauli principle could be observed in gases of trapped fermionic atoms \cite{5,6}, which proves that temperatures well below the degeneracy temperature can be reached.

If it was possible to trap two spin states of a fermionic isotope with attractive interaction, and to cool the system below the critical temperature $T_c$, one could study the BCS transition to the superfluid phase. Unlike the transition of a Bose gas to the Bose-Einstein condensate, the BCS transition of a Fermi gas almost does not change the density profile of the atomic cloud \cite{5}. However, there are other observables which may allow to distinguish between the normal-fluid and the superfluid phase. In a preceding paper \cite{8} the moment of inertia was proposed, since it is much smaller in the superfluid phase than in the normal-fluid phase (see also Ref. \cite{9}). Another observable changing from one phase to the other are the frequencies of collective modes \cite{9,10,11}. For example, the frequency of the so-called “scissors mode”, an oscillation of the symmetry axis of the cloud with respect to the symmetry axis of the trap, is closely related to the moment of inertia \cite{9}. Recently one more observable for the detection of the BCS transition was proposed, namely the change of the deformation of the cloud during the expansion of the system when the trapping potential is switched off \cite{12}.

The moment of inertia of a superfluid gas of trapped fermionic atoms at zero temperature was evaluated for the first time in Ref. \cite{13} in close analogy to the calculation of the moment of inertia of superfluid nuclei \cite{13}. This derivation was very similar to the one given by Migdal more than 40 years ago \cite{14}, except that everything was reformulated in phase space in terms of Wigner transforms. In the present article we will generalize the calculation of Ref. \cite{8} to the case of non-zero temperature. In addition, we will give a derivation which further clarifies certain points which in Ref. \cite{8} may have been passed over rather quickly.

In addition to the temperature dependence of the moment of inertia, we will address an interesting question which is relevant already at zero temperature. In nuclear physics it is well known that the moment of inertia of superfluid nuclei is much smaller than the rigid-body value, but still higher than the value corresponding to a purely irrotational motion, and that the currents in rotating nuclei have both rotational and irrotational components \cite{13}. The same behavior is found in trapped Fermi gases at zero temperature \cite{8}. In contrast to this, the ordinary hydrodynamical or transport equations for superfluids at zero temperature, which can be derived from the $\hbar \to 0$ limit of the time-dependent Hartree-Fock-Bogoliubov (TDHFB) equation \cite{15,16,17,18,19} allow only for a purely irrotational motion. We will work out this difference and discuss the limits of validity of the hydrodynamical description.

The article is organized as follows: In Sect. \ref{sect1} we give a brief review of the formalism, mainly in order to recall some definitions and to clarify our notation. In Sect. \ref{sect2} we derive the expression for the density matrix of the slowly rotating system within linear-response theory. This is the generalization of the calculation of Ref. \cite{8} to non-zero temperatures. In Sect. \ref{sect3} we again derive the linear response of the density matrix, but now using the leading order of the $\hbar$ expansion of the TDHFB equation. In Sect. \ref{sect4} we show numerical results for the moment of inertia obtained within both formalisms as a function of temperature and interpret the results and their differences. Finally, in Sect. \ref{sect5} we summarize and draw our conclusions.
II. BRIEF REVIEW OF THE FORMALISM

Before considering the rotating superfluid trapped Fermi gas, we will briefly review the equilibrium case. Our intention is to explain our notation and conventions. Detailed discussions of the subject can be found in many articles [6, 7, 24] and textbooks [22].

In this article we assume for simplicity that equal numbers of atoms with two spin projections \( \sigma = \uparrow, \downarrow \) are trapped in a spin-independent harmonic potential

\[
V_0(r) = \sum_{i=xyz} \frac{m\omega^2}{2} r_i^2.
\]

If the density of the trapped system is very low, the atom-atom interaction can be approximated by a zero-range interaction with a coupling constant \( g \) proportional to the \( s \)-wave scattering length. Due to the Pauli principle only atoms with opposite spin projections can interact in this way. Under these assumptions the Hamiltonian takes the form

\[
H = \int d^3r \left[ \sum_{\sigma=\uparrow,\downarrow} \psi^{\dagger}_{\sigma}(r) \left( -\frac{\hbar^2 \nabla^2}{2m} + V_0(r) \right) \psi_{\sigma}(r) - g \psi_{\uparrow}^{\dagger}(r) \psi_{\uparrow}^{\dagger}(r) \psi_{\downarrow}(r) \psi_{\downarrow}(r) \right].
\]

The mean-field potential corresponding to this interaction reads

\[
V(r) = V_0(r) - g\rho(r, r) = V_0(r) - g\rho(r),
\]

where we have used the following notation for the non-local density matrix:

\[
\rho(r, r') = \langle \psi^{\dagger}_{\uparrow}(r') \psi_{\uparrow}(r) \rangle = \langle \psi^{\dagger}_{\downarrow}(r') \psi_{\downarrow}(r) \rangle.
\]

(Note that with this definition the local part of the density matrix, \( \rho(r) \equiv \rho(r, r) \) corresponds to the density per spin state.) In the presence of pairing correlations, the pairing gap is given by the gap equation

\[
\Delta(r) = g\kappa(r, r)
\]

where the pairing tensor has been defined as

\[
\kappa(r, r') = \langle \psi_{\downarrow}(r') \psi_{\uparrow}(r) \rangle.
\]

It will turn out that the self-consistent solution of Eq. (6) is divergent as a consequence of the zero-range interaction. In the literature several ways how to regularize this divergence can be found [6, 7, 24, 25], but in fact the technical details of the solution of Eq. (6) are not important for our purpose.

In order to write down the Hartree-Fock-Bogoliubov (HFB) equations, which relate the density matrix \( \rho \) and the pairing tensor \( \kappa \) to the potential \( V \) and the gap \( \Delta \), it is useful to expand all quantities in a basis of single-particle wave functions \( \varphi_n(r) \), where \( n \) represents all quantum numbers except spin, i.e., for an arbitrary operator \( A \):

\[
A_{nn'} = \int d^3r d^3r' \varphi^{\dagger}_n(r') \varphi_{n'}(r') A(r, r').
\]

Expressing the field operators \( \psi_\sigma(r) \) and \( \psi^{\dagger}_\sigma(r) \) in terms of annihilation and creation operators \( a_{n\sigma} \) and \( a^{\dagger}_{n\sigma} \), we recover the usual definitions

\[
\rho_{nn'} = \langle a^{\dagger}_{n\uparrow} a_{n\uparrow} \rangle, \quad \kappa_{nn'} = \langle a_{n\downarrow} a_{n\uparrow} \rangle.
\]

The index \( \bar{n}' \) in Eq. (8) denotes the time-reversed state characterized by \( \varphi^{\dagger}_{\bar{n}'}(r) = \varphi^{\dagger}_n(r) \). We need also the matrix elements \( h_{nn'} \) of the grand-canonical (mean-field) single-particle Hamiltonian (i.e., of the single-particle Hamiltonian minus the chemical potential \( \mu \))

\[
h = \frac{\hbar^2}{2m} + V(r) - \mu,
\]

and the matrix elements \( \Delta_{nn'} \) of the gap \( \Delta \). For the more general case that the Hamiltonian is not time-reversal invariant, we introduce the notation \( \tilde{A}_{nn'} = A_{\bar{n}'n} \). If the matrices mentioned above are combined as follows:

\[
\mathcal{R} = \begin{pmatrix} \rho & -\kappa \\ -\kappa & 1 - \tilde{\rho} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} \hbar^2 \Delta & \Delta \\ \Delta & -\hbar \end{pmatrix},
\]

the HFB equations [22, 23] can be written in the form of a \( 2 \times 2 \) matrix equation,

\[
[\mathcal{H}, \mathcal{R}] = 0.
\]

What is relevant for our purpose is the spectrum of the lowest lying quasiparticles, which for a sufficiently small gap can be obtained within the BCS approximation, which is much simpler than the solution of the full HFB equation [13]. We choose a basis in which \( h \) is diagonal, i.e., \( h_{nn'} = h_{0} \delta_{nn'} \). Then, within the BCS approximation, \( \rho \) and \( \kappa \) are diagonal, too, and given by

\[
\rho_n = \frac{1}{2} - \frac{\hbar_n}{2E_n} [1 - 2f(E_n)],
\]

\[
\kappa_n = \frac{\Delta_n}{2E_n} [1 - 2f(E_n)].
\]

The quasiparticle energies \( E_n = \sqrt{\hbar_n^2 + \Delta_n^2} \) and the quasiparticle occupation numbers \( f(E_n) = 1/[\exp(E_n/T) + 1] \) are determined by the diagonal matrix elements \( \Delta_n \equiv \Delta_{nn} \) alone. If we neglect the non-diagonal matrix elements of \( \Delta \), which are irrelevant for the excitation spectrum and, apart from that, much smaller than the diagonal ones, we can rewrite Eqs. (14) and (15) in the compact form

\[
\mathcal{R} = \frac{1}{2} - \frac{\mathcal{H}}{2E} [1 - 2f(E)].
\]
It is evident that the generalized density matrix $\mathcal{R}$ given by Eq. (16) solves the HFB equation (13) if $\hbar$ and $\Delta$ are assumed to be diagonal.

For the spherical case ($\omega_{0x} = \omega_{0y} = \omega_{0z}$) and moderate numbers of particles ($N \leq 10^4$), the self-consistent HFB equation can be solved numerically [20]. However, for the deformed case and large numbers of particles (experimentally numbers of the order $N \approx 10^3 \ldots 10^6$ have been reached), even within the BCS approximation, the self-consistent solution becomes numerically intractable. Therefore it may be indicated to apply semiclassical approximations. Semiclassical methods can become very accurate for large numbers of particles, and in addition they often allow for a very clear interpretation of the results. To that end we will use the Wigner transforms of the density matrix $\rho$, the pairing tensor $\kappa$, the single-particle hamiltonian $h$, etc. The Wigner transform of a single-particle operator $A$ is defined as

$$ A(r, p) = \int d^3s \ e^{-i\mathbf{s}\cdot\mathbf{p}/\hbar} A \left( r + \frac{s}{2}, r - \frac{s}{2} \right). \quad (17) $$

The Wigner transform $h(r, p)$ of the single-particle hamiltonian $h$ is particularly simple: It is just the classical hamiltonian. We also recall the useful relations $[A^\dagger](r, p) = A^\ast(r, p)$ and $[A](r, p) = A(r, -p)$. One advantage of the Wigner transforms in semiclassical calculations is the product rule for the Wigner transform of the product of two operators $A$ and $B$ [21], directly leading to an $h$ expansion:

$$ [AB](r, p) = A(r, p) \exp \left( \frac{i\hbar A}{2} \right) B(r, p), \quad (18) $$

where the symbol $\leftrightarrow$ stands for the Poisson bracket

$$ \leftrightarrow = \sum_{i=x,y,z} \left( \frac{\partial}{\partial r_i} \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial}{\partial r_i} \right). \quad (19) $$

From the definition (17) it is clear that the local density can be written as

$$ \rho(r) = \rho(r, r) = \int \frac{d^3p}{(2\pi\hbar)^3} \rho(r, p). \quad (20) $$

As a very simple case we consider the Thomas-Fermi ($\hbar \to 0$) limit for the density matrix without pairing correlations (i.e., $\Delta = \kappa = 0$) at zero temperature. Quantum-mechanically the density matrix is in this case just given by the Fermi sea filled up to the Fermi energy $\mu$, i.e., $\rho = \theta(-\mu)$. To leading order in $\hbar$ the Wigner transform of this expression gives $\rho(r, p) = \theta[-h(r, p)]$. The corresponding (local) density reads

$$ \rho(r) = \rho_F^3(r) \frac{m}{6\pi^2\hbar^3}, \quad (21) $$

with the local Fermi momentum

$$ p_F(r) = \sqrt{2m[\mu - V(r)]} \theta[\mu - V(r)]. \quad (22) $$

Eq. (21) together with Eq. (3) can easily be solved self-consistently [3, 4]. Since the pairing gaps and temperatures considered in this article are very small compared with the Fermi energy, we will use Eq. (21) also in the presence of pairing correlations and at non-zero temperatures. (The effect of pairing correlations and temperature on the density profile $\rho(r)$ was investigated in Ref. [5].)

In order to include the pairing correlations, one can also use the HFB equation (13) in the limit $\hbar \to 0$:

$$ \left[ \mathcal{H}(r, p), \mathcal{R}(r, p) \right] = 0. \quad (24) $$

This implies that, to leading order in $\hbar$, at each point $r$ the solution $\mathcal{R}(r, p)$ as a function of $p$ is given by the solution for a homogeneous system with the density corresponding to the local density at this point $r$ [Local-Density Approximation (LDA)]:

$$ \mathcal{R}(r, p) = \frac{1}{2} - \frac{\mathcal{H}(r, p)}{2E(r, p)} \left( 1 - 2f[E(r, p)] \right), \quad (25) $$

with the definition $E(r, p) = \sqrt{\hbar^2(r, p) + \Delta^2(r)}$. In terms of the Wigner transform $\kappa(r, p)$, the gap equation [Eq. (3)] can be written as

$$ \Delta(r) = g \int \frac{d^3p}{(2\pi\hbar)^3} \kappa(r, p). \quad (26) $$

Inserting the expression for $\kappa(r, p)$ corresponding to Eq. (21) into Eq. (26), we obtain the following non-linear equation for the gap:

$$ \Delta(r) = g \int \frac{d^3p}{(2\pi\hbar)^3} \Delta(r) \frac{\Delta(r)}{2E(r, p)} \left( 1 - 2f[E(r, p)] \right). \quad (27) $$

As mentioned before, this equation is divergent and needs some regularization (see Refs. [3, 4] for details).

Contrary to the Thomas-Fermi approximation for the unpaired density matrix, Eq. (21), which is valid if the potential can be regarded as constant on a length scale of the inverse Fermi momentum, the local-density approximation in the paired case is valid only if the potential is also constant on a length scale of the coherence length of the Cooper pairs. This latter condition is often not fulfilled. Therefore, in Refs. [3, 22] an alternative semiclassical method for the calculation of the gap has been proposed, which, however, results in an average gap (more precisely: gap averaged over the Fermi surface) of almost the same magnitude as the average gap obtained within the local density approximation.
III. LINEAR RESPONSE TO A SLOW ROTATION

In this section we will describe the formalism used for the calculation of the moment of inertia of a superfluid gas of trapped fermionic atoms. Looking at a system rotating with angular velocity $\Omega$ around the $z$ axis, we can calculate the moment of inertia from

$$\Theta = \frac{\langle L_z \rangle}{\Omega} = \frac{2}{\Omega} \int \frac{d^3p}{(2\pi\hbar)^3} \left( p_x p_y - p_y p_x \right) \rho(r, p),$$

(28)

where $\rho(r, p)$ is the density matrix of the rotating system. Hence the main problem in calculating the moment of inertia is to calculate $\rho(r, p)$, from which also other interesting quantities like the current density (per spin state)

$$j(r) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p}{m} \rho(r, p),$$

(29)

and the velocity field $v(r) = j(r)/\rho(r)$ can be derived.

The system is put into rotation by rotating the external trapping potential around the $z$ axis (of course, for this purpose the trapping potential must not be axially symmetric). In the rotating frame, however, the system is still in a stationary state. In this frame, the Hamiltonian receives the additional term

$$\hat{h}_1 = -\hat{h}_1 = -\Omega L_z,$$

(30)

which for sufficiently small $\Omega$ can be treated as a perturbation. This perturbation induces a change of the density matrix, $\rho_1$, and of the pairing tensor, $\kappa_1$. The mean field potential is not changed to linear order in $\Omega$, since $L_z$ is a time-odd operator. Linearizing Eq. (18), we obtain

$$[\hat{H}, \mathcal{R}_1] = -[\hat{H}_1, \mathcal{R}],$$

(31)

where $\hat{H}$ and $\mathcal{R}$ denote the unperturbed quantities, while $\hat{H}_1$ and $\mathcal{R}_1$ refer to the deviations. Assuming that the unperturbed quantities $\rho$, $\kappa$, $h$, and $\Delta$ are diagonal (BCS approximation), we can solve Eq. (18) for $\rho_1$ and $\kappa_1$. (This is equivalent to solving the linearized Gorkov equations for the normal and anomalous Green’s functions at equal times; see, e.g., Ref. [23]). The solution reads:

$$\rho_{1nn'} = F^{\rho}_{nn'} \rho_{nn'} + F^{\rho\Delta}_{nn'} \Delta_{nn'},$$

(32)

$$\kappa_{1nn'} = F^{\kappa}_{nn'} \kappa_{nn'} + F^{\kappa\Delta}_{nn'} \Delta_{nn'},$$

(33)

where (with the short-hand notation $\rho = \rho_n$, $\rho' = \rho_{n'}$, $h = h_n$, $h' = h_{n'}$, $\kappa = \kappa_n$, $\kappa' = \kappa_{n'}$, etc.)

$$F^{\rho}_{nn'} = \frac{(\rho - \rho')(h + h') - (\kappa - \kappa')(\Delta + \Delta')}{E^2 - E'^2},$$

(34)

$$F^{\rho\Delta}_{nn'} = \frac{(\rho + \rho' - 1)(\Delta + \Delta') + (\kappa + \kappa')(h + h')}{E^2 - E'^2},$$

(35)

$$F^{\kappa}_{nn'} = \frac{(\rho - \rho')(\Delta - \Delta') + (\kappa - \kappa')(h - h')}{E^2 - E'^2},$$

(36)

$$F^{\kappa\Delta}_{nn'} = \frac{(1 - \rho - \rho')(h - h') + (\kappa + \kappa')(\Delta - \Delta')}{E^2 - E'^2},$$

(37)

In practice, Eq. (33) is an integral equation, since the change of the gap, $\Delta_1$, on the r.h.s. is related to the change of the pairing tensor, $\kappa_1$, by the gap equation. In analogy to Eq. (26) the gap equation for the perturbed quantities reads

$$\Delta_1(r) = g \int \frac{d^3p}{(2\pi\hbar)^3} \kappa_1(r, p).$$

(38)

The solution of this integral equation contains some subtleties. For example, the divergence appearing in Eq. (23) as a consequence of the zero-range interaction has to be regularized in the same way as the corresponding divergence of the unperturbed gap equation (21) (see appendix), and the derivations of Eqs. (4.34) in Ref. [3], or the second equation after Eq. (16) in Ref. [14] are not very explicit about this point. However, these problems can be circumvented in the following way [12, 18, 19]: Suppose all single-particle wave functions are multiplied by the same local phase $\exp[i\phi(r)]$. Then the HFB equation (13) can be rewritten in terms of the gauge-transformed matrices

$$\tilde{R} = e^{i\phi} \mathcal{R} e^{-i\phi},$$

(39)

$$\tilde{H} = e^{i\phi} \mathcal{H} e^{-i\phi},$$

(40)

where

$$\Phi = \left( \begin{array}{cc} \phi & 0 \\ 0 & -\phi \end{array} \right).$$

(41)

We will consider $\phi$ as small, i.e., of the order of the perturbation. Then, to linear order in the perturbation, the gauge transformed HFB equation reads

$$[\tilde{H}, \tilde{R}_1] = -[\tilde{H}_1, \tilde{R}],$$

(42)

where

$$\tilde{R}_1 = R_1 + i[\Phi, \mathcal{R}],$$

(43)

$$\tilde{H}_1 = H_1 + i[\Phi, \mathcal{H}],$$

(44)

In the latter expression one has to take into account that $h$ does not commute with $\phi$. Explicitly, for a hamiltonian $h$ of the form (10) and a local gap $\Delta(r)$ one obtains

$$\tilde{h}_1 = \tilde{h}_1 = -\Omega L_z - \frac{\hbar}{2m} \left( p \cdot [\nabla \phi(r)] + [\nabla \phi(r)] \cdot p \right),$$

(45)

$$\tilde{\Delta}_1(r) = \Delta_1(r) + 2i\phi(r)\Delta(r).$$

(46)

Together with the gauge-transformed gap equation

$$\tilde{\Delta}_1(r) = g \int \frac{d^3p}{(2\pi\hbar)^3} \tilde{\kappa}_1(r, p),$$

(47)

Eq. (23) is again a system of integral equations which for an arbitrary function $\phi(r)$ is completely equivalent to the original one, Eqs. (21) and (23). However, since the perturbation $\tilde{h}_1$ is time-odd, the change of the gap,
\( \tilde{\Delta}_1 \), is purely imaginary and therefore can be eliminated by an appropriately chosen function \( \phi(r) \). Physically, this choice of \( \phi(r) \) corresponds to a transformation into the local rest frame of the Cooper pairs \([24]\). In this particular gauge the linearized HFB equation reduces to

\begin{align}
\tilde{\rho}_{1\,nn'} &= F_{nn'}^{ph} \tilde{h}_{1\,nn'}, \\
\tilde{\kappa}_{1\,nn'} &= F_{nn'}^{eq} \tilde{h}_{1\,nn'},
\end{align}

and instead of Eq. (47) we have an equation which determines the phase \( \phi(r) \):

\[ 0 = g \int \frac{d^3p}{(2\pi\hbar)^3} \tilde{\kappa}_1(r, p). \]  

We now proceed to the evaluation of Eq. (18). The unperturbed quantities \( \rho \) and \( \kappa \) entering in \( F_{nn'}^{ph} \) [Eq. (54)] can be rewritten in terms of \( \hbar \) and \( \kappa \) according to the BCS relations \([14]\) and \([15]\). In addition, as in Ref. \([8]\), we replace \( \Delta_0 \) by its average value at the Fermi surface, \( \Delta \), because \( F^{ph} \) and all other relevant quantities are strongly peaked at \( \varepsilon_F \). This allows us to write \( F_{nn'}^{ph} \) as a function of two energies \( \xi = \hbar n \) and \( \xi' = \hbar n' \):

\[ F^{ph}(\xi, \xi') = \frac{[1 - f(E) - f(E')](-4\Delta^2 + \xi^2 + \xi'^2 - 2EE')}{{2EE'}}, \]

where we have introduced the abbreviations \( E = \sqrt{\xi^2 + \Delta^2} \) and \( E' = \sqrt{\xi'^2 + \Delta^2} \). In contrast to Ref. \([8]\), we will not drop the thermal quasiparticle occupation numbers \( f(E) \) and \( f(E') \). As described in detail in Ref. \([3]\), the Wigner transform of an expression like Eq. (18) can be evaluated semiclassically in the following way. First we rewrite Eq. (18) as an operator equation:

\[ \tilde{\rho}_1 = \int d\xi d\xi' F^{ph}(\xi, \xi') \delta(h(\xi) - h(\xi')) \tilde{\rho}_1(\xi, \xi'). \]  

Then we use the Fourier representation for the \( \delta \) functions, i.e., \( \delta(h - \xi) = \int dt/(2\pi\hbar) \exp[(h - \xi)t/\hbar] \), and obtain

\[ \tilde{\rho}_1 = \int \frac{d\xi d\xi' dT dt}{(2\pi\hbar)^2} F^{ph}(\xi, \xi') e^{-i\xi T/\hbar} e^{-i\xi} \tilde{\rho}_1(t) e^{i\xi T/\hbar}, \]  

where we have introduced the notation

\[ \tilde{h}_1(t) = e^{i\hbar T/\hbar} \tilde{h}_1(t) e^{-i\hbar T/\hbar}. \]  

To leading order in \( \hbar \) the Wigner transform of the product of the three operators in the second line of Eq. (53) can be expressed as the product of their Wigner transforms [see Eq. (19)]. Then the integral over \( T \) gives a \( \delta \) function of the form \( \delta(h(r, p) - \tilde{\sigma}) \) and the integral over \( \xi \) becomes trivial.

However, for the operator product in \( \tilde{h}_1(t) \) [Eq. (54)] we will not use the product rule. In this sense we resum certain \( \hbar \) corrections to all orders. One can also say that, since the Wigner transform of Eq. (54) involves the classical trajectories (see below), the long-time information is preserved. On the other hand, developing the Wigner transform of Eq. (54) with the product rule [22] into powers of \( \hbar \) would lead to the Wigner-Kirkwood expansion, which is only valid in the short-time limit (see Ref. [21]). The different treatment of the operator products in Eqs. (53) and (54) is necessary for the following reason: The operator \( h_1 \) connects states with an energy difference of the order \( \hbar \omega \). This is small compared with the Fermi energy, which is the relevant scale for the variable \( \xi \) [since the result \( \tilde{\rho}_1(r, p) \) will be used in integrals over \( p \)], but not necessarily small compared with the gap \( \Delta \), which is the relevant scale for the variable \( \varepsilon \) [this point will become clearer when we investigate the function \( F^{ph}(\xi + \varepsilon/2, \xi - \varepsilon/2) \) explicitly].

In the case of the effective harmonic oscillator potential \([24]\) the Wigner transform of Eq. (54) can be calculated exactly. The result reads

\[ \tilde{h}_1(t)(r, p) = \tilde{h}_1(r^{cl}(r, p; t), p^{cl}(r, p; t)) \]  

where \( r^{cl}(r, p; t) \) and \( p^{cl}(r, p; t) \) are the classical orbits in the potential \([24]\) corresponding to the initial conditions \( r^{cl}(r, p; 0) = r \) and \( p^{cl}(r, p; 0) = p \), which are given by

\[ r^{cl}(r, p; t) = r_i \cos(\omega_t t) + \frac{p_i}{m\omega_t} \sin(\omega_t t), \]

\[ p^{cl}(r, p; t) = p_i \cos(\omega_t t) - m\omega_t r_i \sin(\omega_t t). \]

Putting everything together, we obtain

\[ \tilde{\rho}_1(r, p) = \int d\xi F^{ph}(h(r, p) + \frac{\xi}{2}, h(r, p) - \frac{\xi}{2}) \times \int \frac{dt}{2\pi\hbar} e^{-i\hbar T/\hbar} \tilde{h}_1(r^{cl}(r, p; t), p^{cl}(r, p; t)). \]  

Now we proceed to the calculation of the response of \( \tilde{\rho}_1 \) to the external perturbation \( h_1 \), neglecting for the moment the reaction of the pairing field to the rotation, i.e., the \( p \cdot \nabla \phi \) terms in Eq. (18). In Ref. \([3]\) this contribution was called the “Inglis-Belyaev term” \( \tilde{\rho}_1^{IB} \). In this case the Fourier transform in the second line of Eq. (53) [with \( \tilde{h}_1 \) replaced by \( -\Omega(r_x p_y - r_y p_x) \)] can easily be evaluated with the aid of Eqs. (56) and (57). Inserting the result into Eq. (53) and observing that \( F(\xi, \xi') \) is symmetric under the exchange of its arguments we obtain [the arguments of \( h(r, p) \) will be suppressed for brevity]
with the definition
\[ \omega_\pm = \omega_y \pm \omega_x. \] (60)

To simplify the expression (59) further we note that the distribution function \( \rho(\mathbf{r}, \mathbf{p}) \) is changed only in the vicinity of the Fermi surface, provided the Fermi energy is large compared with \( \hbar \omega_\pm, \Delta, \) and \( T. \) Formally this can be inferred from the fact that \( F^{qh}(\xi + \epsilon/2, \xi - \epsilon/2) \) as a function of \( \xi \) is strongly peaked at \( \xi = 0 \), which leads us to the approximation
\[ F^{qh}(\xi + \epsilon/2, \xi - \epsilon/2) \approx \left[ G\left(\frac{\xi}{2\Delta}\right) - 1 \right] \delta(\xi), \] (61)
with
\[ G(x) = 1 + \int d\xi F^{qh}(\xi + x\Delta, \xi - x\Delta) \] (62)

At zero temperature the integral in Eq. (62) can be evaluated analytically, whereas the terms containing the quasiparticle occupation numbers \( f(E) \) and \( f(E') \) have to be integrated numerically. After some manipulations the function \( G(x) \) can be written as
\[ G(x) = \frac{\text{arsinh}(x)}{x} + \frac{\Delta}{x} \int_0^\infty \frac{d\xi}{\xi} \left( \frac{f(E_+)}{E_+} - \frac{f(E_-)}{E_-} \right), \] (63)
with \( E_\pm = \sqrt{(\xi \pm x\Delta)^2 + \Delta^2}. \) Within the approximation (61) the change of the density matrix corresponding to the Ingles-Belyaev term finally takes the form
\[ \rho_1^{IB}(\mathbf{r}, \mathbf{p}) = \Omega\delta[h(\mathbf{r}, \mathbf{p})]\left[ r_xp_y \left( 1 - \frac{\omega_+ G_- + \omega_- G_+}{\omega_+ + \omega_-} \right) - r_yp_x \left( 1 - \frac{\omega_+ G_- - \omega_- G_+}{\omega_+ - \omega_-} \right) \right], \] (64)
with
\[ G_\pm = G\left(\frac{\hbar \omega_\pm}{2\Delta}\right). \] (65)

Now we will consider also the change of the pairing field \( \Delta, \) i.e., the phase \( \phi(\mathbf{r}). \) As mentioned before, this phase will be determined by Eq. (60), where \( \tilde{\kappa}_1(\mathbf{r}, \mathbf{p}) \) is obtained from the Wigner transform of Eq. (49). Again we replace \( \Delta_\alpha \) and \( \Delta_\alpha' \) entering in \( F_{nn'}^{qh} \) by the average value \( \Delta, \) which allows us to express \( F_{nn'}^{qh} \) as a function of two energies:
\[ F^{qh}(\xi, \xi') = \frac{[1 - f(E) - f(E')]\Delta(\xi - \xi')}{2EE'(E + E')} - \frac{[f(E) - f(E')]\Delta(\xi - \xi')}{2EE'(E - E')}. \] (66)

Then the Wigner transform of Eq. (49) can be calculated semiclassically as given by Eq. (58) with \( \tilde{\kappa}_1 \) and \( F^{qh} \) replaced by \( \tilde{\kappa}_1 \) and \( F^{qh}, \) respectively. As it was the case for \( F^{qh}(\xi + \epsilon/2, \xi - \epsilon/2), \) the function \( F^{qh}(\xi + \epsilon/2, \xi - \epsilon/2) \) is strongly peaked at \( \xi = 0, \) and we approximate it by
\[ F^{qh}(\xi + \epsilon/2, \xi - \epsilon/2) \approx -\frac{\epsilon}{2\Delta} G\left(\frac{\epsilon}{2\Delta}\right) \delta(\xi), \] (67)
with
\[ G(x) = -\frac{1}{x} \int d\xi F^{qh}(\xi + x\Delta, \xi - x\Delta). \] (68)

It turns out that the definitions (62) and (68) indeed define the same function \( G(x), \) which is explicitly given by Eq. (63). Inserting the Wigner transform of Eq. (49) into Eq. (65), we obtain
\[ 0 = -g \int dE \frac{\xi}{2\Delta} G\left(\frac{\xi}{2\Delta}\right) \int \frac{d^3p}{(2\pi\hbar)^3} \delta[h(\mathbf{r}, \mathbf{p})] \times \int \frac{dt}{2\pi\hbar} e^{-ixt/h} \tilde{h}_1[\mathbf{r}^{cl}(\mathbf{r}, \mathbf{p}; t), \mathbf{p}^{cl}(\mathbf{r}, \mathbf{p}; t)]. \] (69)

To solve this equation for the phase \( \phi(\mathbf{r}) \) we make the ansatz
\[ \phi(\mathbf{r}) = \alpha \frac{mr_y r_y}{\hbar}. \] (70)

Then the second line of Eq. (69) is just the Fourier transform of \( \tilde{h}_1 = -\Omega(\mathbf{r}xp_y - \mathbf{r}yp_x) - \alpha(\mathbf{r}xp_y + \mathbf{r}yp_x), \) which is readily evaluated with the aid of Eqs. (55) and (57). Due to the \( \delta \) functions the remaining integrals are trivial, and Eq. (69) finally becomes
\[ 0 = -g \frac{igm^2 p_F(r)x r_y}{8\pi^2 \hbar^2 \Delta} \times [\Omega \delta \omega_+ \omega_- (G_+ + G_-) + \alpha(\omega_+^2 G_+ + \omega_-^2 G_-)], \] (71)
which has the solution
\[ \alpha = -\Omega \frac{\omega_+ \delta \omega_- (G_+ + G_-)}{\omega_+^2 G_+ + \omega_-^2 G_-}. \] (72)

Using this expression we can also calculate the change of the original pairing field, \( \Delta_1: \) Since the change of the gauge-transformed pairing field [Eq. (10)] is zero, the original pairing field is modified according to
\[ \Delta_1(\mathbf{r}) = -2i\Delta_0\phi(\mathbf{r}) = -2i\frac{\Delta_0 m r_y r_y}{\hbar}. \] (73)

Having calculated the phase \( \phi(\mathbf{r}), \) we can now evaluate Eq. (68) with the full \( \tilde{h}_1, \) i.e., including in addition to the Ingles-Belyaev term [Eq. (10)] also the response of the density matrix to the \( \mathbf{p} \cdot \nabla \phi \) terms. This second contribution to \( \tilde{\kappa}_1, \) which we will call \( \tilde{\kappa}_1^{M_1}, \) is obtained in the same way as discussed above for the first one, and the result reads
\[ \tilde{\kappa}_1^{M_1}(\mathbf{r}, \mathbf{p}) = \alpha \delta[h(\mathbf{r}, \mathbf{p})] \left[ r_x p_y \left( 1 - \frac{\omega_+ G_- + \omega_- G_+}{\omega_+ + \omega_-} \right) + r_yp_x \left( 1 - \frac{\omega_+ G_- - \omega_- G_+}{\omega_+ - \omega_-} \right) \right]. \] (74)
However, we are not interested in the change of the gauge-transformed density matrix, $\rho_1$, but of the original density matrix, $\rho_1$. According to Eq. \(3\) the relation between $\rho_1$ and $\tilde{\rho}_1$ is given by

$$\rho_1 = \tilde{\rho}_1 - \delta(\phi, \rho) = \rho_1^{IB} + \rho_1^{M1} + \rho_1^{M2}. \quad (75)$$

Due to the simple $\mathbf{r}$ dependence of $\phi$, the Wigner transform of the commutator $[\phi, \rho]$ is identical to the Poisson bracket of the Wigner transforms of $\phi$ and $\rho$, i.e.

$$\rho_1^{M2}(\mathbf{r}, \mathbf{p}) = \hbar \phi(\mathbf{r}) \Delta \rho(\mathbf{r}, \mathbf{p})$$

$$= \alpha m \left( r_x \frac{\partial}{\partial p_y} + r_y \frac{\partial}{\partial p_x} \right) \rho(\mathbf{r}, \mathbf{p}). \quad (76)$$

As we did before, we will assume that $\Delta$ and $\Theta$ are much smaller than the Fermi energy. Therefore we can write $\rho(\mathbf{r}, \mathbf{p}) \approx \theta(-h(\mathbf{r}, \mathbf{p}))$ and we obtain

$$\rho_1^{M2}(\mathbf{r}, \mathbf{p}) = -\alpha (r_x p_y + r_y p_x) \delta[h(\mathbf{r}, \mathbf{p})]. \quad (77)$$

The total effect of the phase $\phi$, i.e., of the reaction of the pairing field, on the density matrix, which in Ref. \[8\] was called the “Migdal term” $\rho_1^M$, is the sum of the two contributions $\rho_1^{M1}$ and $\rho_1^{M2}$:

$$\rho_1^M(\mathbf{r}, \mathbf{p}) = -\alpha \delta[h(\mathbf{r}, \mathbf{p})] \left( \frac{r_x p_y}{\omega_+ G_+ + \omega_- G_-} \frac{\omega_+ G_+ + \omega_- G_-}{\omega_+ G_+ + \omega_- G_-} \right. \left. + \frac{r_y p_x}{\omega_+ G_+ + \omega_- G_-} \right). \quad (78)$$

Together with the Inglis-Belyaev term, Eq. \(44\), and the explicit expression for $\alpha$, Eq. \(72\), our final result for the change of the density matrix reads

$$\rho_1(\mathbf{r}, \mathbf{p}) = \Omega \delta[h(\mathbf{r}, \mathbf{p})] \left( r_x p_y - r_y p_x \right. \left. - 4 G_+ G_- \frac{\omega_+^2 r_x p_y - \omega_- r_y p_x}{\omega_+^2 G_+ + \omega_-^2 G_-} \right). \quad (79)$$

Given the change of the density matrix, we can immediately calculate the current density $\mathbf{j}(\mathbf{r})$ [Eq. \(29\)] and the velocity field $\mathbf{v}(\mathbf{r})$:

$$\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r}) \mathbf{v}(\mathbf{r}) = \Omega \rho(\mathbf{r}) \left( r_x e_y - r_y e_x \right. \left. - 4 G_+ G_- \frac{\omega_+^2 r_x e_y - \omega_-^2 r_y e_x}{\omega_+^2 G_+ + \omega_-^2 G_-} \right). \quad (80)$$

It is interesting to check explicitly that this current fulfills the continuity equation. In the rotating frame the continuity equation reads

$$\nabla \cdot \mathbf{j}(\mathbf{r}) + \dot{\rho}(\mathbf{r}) - \Omega (e_z \times \mathbf{r}) \cdot \nabla \rho(\mathbf{r}) = 0, \quad (81)$$

where $\dot{\rho}(\mathbf{r}) = 0$ in our case of a stationary rotation. Taking the divergence of Eq. \(80\), we get from the second line a contribution proportional to $[\nabla \rho(\mathbf{r})] \cdot (e_z \times \nabla V(\mathbf{r}))$. This is zero, since the gradient of the density in Thomas-Fermi approximation [Eq. \(23\)], $\nabla \rho(\mathbf{r})$, is parallel to $\nabla V(\mathbf{r})$. Thus, the divergence of the current is equal to the divergence of the first line of Eq. \(80\), which exactly fulfills Eq. \(81\). Note that the contribution of the Migdal term is crucial in order to satisfy the continuity equation. The easiest way to see this is to consider the limit $\Delta \to \infty$. In this limit we have $G_+ \to 1$ and $\rho_1^{IB}(\mathbf{r}, \mathbf{p}) \to 0$, which implies $\mathbf{j}_B(\mathbf{r}) \to 0$. Hence, with the Inglis-Belyaev contribution alone, Eq. \(3\) cannot be satisfied.

As observed in Ref. \[8\], the velocity field $\mathbf{v}(\mathbf{r})$ describes a mixture of rotational motion, corresponding to a velocity field proportional to $e_z \times \mathbf{r}$, and irrotational motion, corresponding to a velocity field proportional to $\nabla \times \mathbf{r}$. The ordinary rigid rotation is realized if $G_+ = G_- = 1$. This is the case if the temperature approaches the critical temperature $T_c$, where the gap vanishes [the temperature dependence of the function $G(x)$ will be discussed in Sect. \[1\]], but it can also happen at zero temperature if $\Delta \ll \hbar \omega_\Delta$, as discussed in Ref. \[8\]. Purely irrotational motion, as it is expected in homogeneous superfluids, is reached if $G_+ = G_- = 1$. This is only possible if the temperature is very low and if $\Delta \gg \hbar \omega_\Delta$.

For completeness let us also discuss the change of the pairing tensor, $\kappa_1(\mathbf{r}, \mathbf{p})$, which can be obtained in a way completely analogous to the calculation of the change of the density matrix, $\rho_1(\mathbf{r}, \mathbf{p})$. The result reads

$$\kappa_1(\mathbf{r}, \mathbf{p}) = \frac{2i \hbar \Omega}{m \Delta} \frac{\omega_+ \omega_- G_+ G_-}{\omega_+^2 G_+ + \omega_-^2 G_-} r_x p_y \delta[h(\mathbf{r}, \mathbf{p})] - 2i \phi(\mathbf{r}) \kappa(\mathbf{r}, \mathbf{p}). \quad (82)$$

Since the last term is of the order $\hbar^{-1}$ [see Eq. \(70\)], it has been argued that a semiclassical description is possible only in the particular gauge where $\Delta + \Delta_1$ is real and where this term vanishes \[15\] \[16\].

IV. SUPERFLUID ROTATION IN TRANSPORT THEORY

The transport or hydrodynamical equations for superfluid systems can be derived by taking the $h \to 0$ limit of the time-dependent Hartree-Fock-Bogoliubov (TDHF) equation \[18\] \[21\] \[23\]

$$i \hbar \dot{R} = [\mathcal{H}, R], \quad (83)$$

i.e., by replacing the Wigner transforms of the commutators in Eq. \(83\) by Poisson brackets of the Wigner transforms \[15\] \[16\] \[17\] \[18\] \[19\]. Due to the transformation into the rotating frame, we are dealing with a static problem, where the TDHF equation \(83\) reduces to the HFB equation \(13\). Again we make use of the gauge transformation and retain only terms of linear order in the perturbation. Then, if $\phi(\mathbf{r})$ is chosen such that $\Delta_1$ vanishes [Eq. \(24\)], the leading order in $h$ of Eq. \(12\) becomes

$$i \hbar \hbar \Delta \tilde{\rho}_1 + 2 \Delta \tilde{\kappa}_1 = -i \hbar \tilde{\Lambda} \Delta \tilde{\rho}, \quad (84)$$

$$i \hbar \Delta \tilde{\Lambda} \tilde{\rho}_1 - 2 \hbar \tilde{\kappa}_1 = i \hbar \tilde{\Lambda} \tilde{\kappa}. \quad (85)$$
In this equation and in the remaining part of this section, $h$, $\rho$, $\kappa$, etc. denote the Wigner transforms of the corresponding operators; the arguments $r$ and $p$ are suppressed for brevity.

Let us first study the zero-temperature limit, $T=0$. In this case the unperturbed quantities are given by $\rho = (1-h/E)/2$ and $\kappa = \Delta/2E$ [see Eq. (23) in the limit $T \to 0$], and it is easy to show that $(\hbar \Lambda \rho)h = (\hbar \Lambda \kappa)\Delta$. Thus, for $\Delta \neq 0$, the solution of Eqs. (84) and (85) reads

$$\tilde{\rho}_1 = 0, \quad \tilde{\kappa}_1 = -\frac{i\hbar}{2\Delta} (\hbar \Lambda \rho) \tag{86}$$

As we will see, the relation $\tilde{\rho}_1 = 0$ implies that the velocity field is completely irrotational independent of the magnitude of $\Delta$, which is a well-known property of homogeneous superfluid systems at $T=0$.

Now we are going to determine the phase $\phi$. To that end we insert Eq. (87) into Eq. (80). If we make again the ansatz (70), we obtain the following equation:

$$0 = -\frac{i\hbar}{2\Delta} \int \frac{d^3p}{(2\pi \hbar)^3} \left( \left((\Omega + \alpha) r_x \frac{\partial \rho}{\partial r_y} - (\Omega - \alpha) r_y \frac{\partial \rho}{\partial r_x} \right) \right). \tag{88}$$

[Note that in this equation $\rho$ still refers to the Wigner transform of the non-local density matrix, $\rho(r,p)$.] It is clear that in general this equation does not have a solution for all $r$, since the ansatz (70) is not general enough. But under certain assumptions it turns out that this ansatz is sufficient. Firstly, we assume that the gap $\Delta(r)$ is either replaced by a constant corresponding to its average value at the Fermi surface (as it was done in the previous section), or that $\Delta(r)$ is calculated within the LDA. In these both cases the function $\Delta(r)$ can formally be written as $\Delta[V(r)]$. Using this, we define the following short-hand notation:

$$\frac{d\rho}{dV} = \frac{d\rho}{dh} \frac{dh}{dV} + \frac{d\rho}{d\Delta} \frac{d\Delta}{dV} = -\frac{\Delta^2}{2E} + \frac{\hbar \Lambda}{2E} \frac{d\Delta}{dV}, \tag{89}$$

which allows us to write $\nabla \rho = (d\rho/dV)^2 \nabla V$. Secondly, as in the previous section, we assume that the potential $V(r)$ is a harmonic oscillator. Then Eq. (88) becomes

$$0 = -\frac{i\hbar m r_x r_y}{2\Delta} \int \frac{d^3p}{(2\pi \hbar)^3} \frac{d\rho}{dV} \times [\Omega(\omega_x^2 - \omega_y^2) + \alpha(\omega_y^2 + \omega_z^2)], \tag{90}$$

with the solution

$$\alpha = \alpha_0 = -\Omega \frac{\omega_y^2 - \omega_z^2}{\omega_y^2 + \omega_z^2}. \tag{91}$$

Not surprisingly, this result is identical to the $h \to 0$ limit of Eq. (23), since for $T = 0$ we have $G(0) = 1$ and consequently $\lim_{h \to 0} G_\pm = 1$.

As in the previous section, the phase $\phi$ implies a change of the density matrix, $\rho_1$, due to the inverse gauge transformation, which to leading order in $h$ reads

$$\rho_1 = \tilde{\rho}_1 + \hbar \phi \nabla \phi \cdot \hat{h}_1 \tag{92}$$

As we have seen, the first term vanishes. Thus, to linear order in the perturbation, Eq. (92) can be rewritten in the following, more suggestive way:

$$\rho(r,p) + \rho_1(r,p) = \rho(r,p + h \nabla \phi(r)). \tag{93}$$

From this equation it follows immediately that the velocity field is given by

$$v(r) = -\frac{h}{m} \nabla \phi(r), \tag{94}$$

which is completely irrotational. Note that this result does not depend on the form of $\phi(r)$ and the approximations made to calculate $\phi(r)$. It also does not at all depend on the magnitude of $\Delta$, as long as $\Delta \neq 0$. It is rather a direct consequence of the vanishing of $\tilde{\rho}_1$, which in turn follows immediately from the $h \to 0$ limit of the linearized HFB equations for time-odd perturbations and zero temperature. However, as we have seen in the previous section, in a small system where $\hbar \omega$ is of the same order of magnitude as $\Delta$, the velocity field is not irrotational. Our conclusion is that one should be careful when applying transport theory to such systems.

So far we have considered only the zero-temperature limit. In the remaining part of this section we are going to consider also the case $T > 0$. In this case it is difficult to solve the coupled Eqs. (84) and (85). However, if we in analogy to the previous section assume that the unperturbed gap $\Delta$ is constant, we find the following solution for $\tilde{\rho}_1$ and $\tilde{\kappa}_1$:

$$\tilde{\rho}_1 = \left( \frac{d\rho}{dh} \frac{\Delta}{\hbar \Lambda} \right), \quad \tilde{\kappa}_1 = \left( \frac{d\rho}{dh} \frac{\Delta}{\hbar \Lambda} \right) \tag{95}$$

If we again make the ansatz (71) and insert Eq. (91) into Eq. (69), we find $\alpha = \alpha_0$ as in the zero-temperature case [see Eq. (61)]. This could have been anticipated from the $h \to 0$ limit of Eq. (72), which does not depend on the actual value of $G(0)$. Finally we are now going to calculate $\rho_1$. To that end we insert Eqs. (53) and (71) with $\alpha = \alpha_0$ into Eq. (72), and we obtain

$$\rho_1 = -\Omega \frac{dG(E)}{dE} (r_x p_y - r_y p_x) - \alpha_0 \left( \frac{dG(E)}{dE} - \frac{d\rho}{dh} \right) (r_x p_y + r_y p_x). \tag{97}$$

Since $dG(E)/dE$ and $d\rho/dh$ are both strongly peaked at the Fermi surface, we can make the same approximation as in the previous section, i.e., we replace the strongly
peaked functions by $\delta$ functions with the appropriate strength. Noting that
\[
\lim_{x \to 0} G(x) = 1 + \int \frac{d^2 \delta}{d^2 \xi} \frac{\Delta}{\rho} \frac{d f(E)}{d^2 \rho} = 1 + \int \frac{d^2 \delta}{d^2 \xi} \frac{f(E)}{d^2 E},
\]
we can write the result as
\[
\rho_1 = \Omega [1 - G(0)] \delta(h)(r_x p_y - r_y p_x) - \alpha_0 G(0) \delta(h)(r_x p_y + r_y p_x),
\]
which is in perfect agreement with the $h \to 0$ limit of Eqs. (32) and (78).

V. RESULTS AND DISCUSSION

Using the results for change of the non-local density matrix $\rho_1(r, p)$ given in the previous sections, we can now calculate the moment of inertia. It should be remembered that an ideal Fermi gas at zero temperature behaves like a rigid body, i.e., the velocity field is given by $v(r) = \Omega \sigma \times r$. Since the critical temperature for the BCS transition is very low, $\Theta$ will approach the rigid-body value $\Theta_{\text{rigid}}$ for $T \to T_c$. Using the Thomas-Fermi density profile (21) with the effective harmonic oscillator potential (22), we can immediately calculate $\Theta_{\text{rigid}}$. The result reads
\[
\Theta_{\text{rigid}} = \frac{\mu^2 (\omega_x^2 + \omega_y^2)}{12 \hbar^2 \omega_x^2 \omega_y^2 \omega_z^2}.\]

In terms of $\Theta_{\text{rigid}}$ the moment of inertia of the superfluid system as obtained from $\rho_1(r, p)$ can be written as
\[
\Theta = \Theta_{\text{rigid}} \left( 1 - \frac{8 \omega_x^2 \omega_y^2 G_+ G_-}{(\omega_x^2 + \omega_y^2) (\omega_x^2 + \omega_z^2) (\omega_y^2 + \omega_z^2)} \right).
\]

In the $h \to 0$ (transport) limit, where $G_\pm \to G(0)$, Eq. (101) reduces to
\[
\Theta = \Theta_{\text{rigid}} \left[ 1 - G(0) + G(0) \left( \frac{\omega_y^2 - \omega_x^2}{\omega_y^2 + \omega_x^2} \right)^2 \right].
\]

In fact, this formula can be understood very easily. The moment of inertia corresponding to the purely irrotational velocity field as it is expected for a large superfluid system at zero temperature, $v(r) = -\alpha_0 \nabla (r_x r_y)$, is given by
\[
\Theta_{\text{irrot}} = \Theta_{\text{rigid}} \left( \frac{\omega_y^2 - \omega_x^2}{\omega_y^2 + \omega_x^2} \right)^2.
\]

Within the two-fluid model a homogeneous system of density $\rho$ is described as a mixture of a superfluid component of density $\rho_s$ and a normal-fluid component of density $\rho_n$, with $\rho_s + \rho_n = \rho$. At $T = 0$ one has $\rho_s = \rho$ and $\rho_n = 0$, whereas at $T \geq T_c$ one has $\rho_s = 0$ and $\rho_n = \rho$. If this model was correct also for finite systems, one would expect that the moment of inertia is given by
\[
\Theta = \frac{\rho_s}{\rho} \Theta_{\text{rigid}} + \frac{\rho_n}{\rho} \Theta_{\text{irrot}}.
\]

This would be exactly Eq. (102), if we could identify $G(0)$ with $\rho_s/\rho$. In fact, the microscopic calculation of $\rho_s$ for a homogeneous system gives [7]
\[
\rho_s = \rho - \frac{1}{6 \pi^2 m \hbar^3} \int_0^\infty dp p^4 \left( -\frac{df(E)}{dE} \right),
\]
with $E = \sqrt{(p^2/2m - \mu)^2 + \Delta^2}$. Noting that the integrand is peaked at $p = p_F$ and remembering $\rho = p_F^3/6\pi^2 \hbar^3$, we rewrite this as
\[
\frac{\rho_s}{\rho} \approx 1 + \int d\xi \frac{f(E)}{E}.
\]
with $\xi = p^2/2m - \mu$. As noted in Sect. IV, the r.h.s. of this equation is identical to $\lim_{x \to 0} G(x)$, so that we are left with
\[
\frac{\rho_s}{\rho} = G(0).
\]

The previous paragraph can be summarized in the statement that the transport approach, corresponding to the leading order of the $\hbar$ expansion, reproduces the two-fluid model for homogeneous systems. It does not give any finite-size corrections, as can be seen from the fact that the result does not depend on the trapping frequencies, except for the purely geometrical dependence contained in $\Theta_{\text{rigid}}$ and $\Theta_{\text{irrot}}$. In contrast to this, the method described in Sect. III is capable to describe the different behavior of the system depending on whether the trapping frequencies (multiplied by $\hbar$) are small or large compared with the gap $\Delta$. This dependence is governed by the $G_\pm$ factors appearing in Eq. (101), resulting from the long-time behavior of the operator $h_1(t)$ [see discussion after Eq. (33)]. In order to reproduce this behavior within the $\hbar$ expansion, one would have to resum a certain class of corrections proportional to $\hbar \omega_\pm/\Delta$ to all orders, in particular if one wants to cover the whole range of possible parameters from $\hbar \omega_\pm \ll \Delta$ to $\hbar \omega_\pm \gg \Delta$.

Let us now proceed to a quantitative analysis. In order to calculate the moment of inertia $\Theta$ as a function of temperature, we need the temperature dependence of the gap $\Delta$. As in Ref. 8, we will assume that it is described by the same universal function relating $\Delta/\Delta_0$ to $T/T_c$ in homogeneous matter, where $\Delta_0$ denotes the gap at $T = 0$ and $T_c = 0.567 \Delta_0$. This universal function is given by the solution of the non-linear equation [20]
\[
-\ln \left( \frac{\Delta}{\Delta_0} \right) = \int d\xi \frac{f(E)}{E}.
\]

For completeness it is displayed in Fig. 4.
temperature dependence of $\Delta$ discussed above, and via
the explicit temperature dependence of the function $G(x)$
due to the thermal quasiparticle occupation numbers as
given by Eq. (3). If only the temperature dependence of
$\Delta$ was included, $G(\varepsilon/2\Delta)$ as a function of $\varepsilon$
would become very strongly peaked at $\varepsilon = 0$ for $T \rightarrow T_c$.
However, due to the explicit temperature dependence of the function
$G(x)$, the peak is suppressed and as a function of $\varepsilon$ the
function $G(\varepsilon/2\Delta)$ even becomes more and more flat with
increasing temperature, as shown in Fig. 2. The decrease of $G(0)$
when $T$ approaches $T_c$ reflects the decrease of the superfluid fraction in the two-fluid model.

Next we have to specify the parameters of the system.
For our comparison we consider, as in Ref. 2, 583000 $^6$Li atoms (i.e., 286500 atoms per spin state) in
a harmonic oscillator potential with average frequency
$\hbar \omega = \hbar ((\omega_x / \omega_y) \omega_z)^{1/3} = 8.21 \text{nK}$.
The corresponding chemical potential is $\mu = 983 \text{nK}$.
In order to simulate the effect of the self-consistent mean-field potential, the
frequency has been chosen slightly higher than the frequency of the external trapping potential ($\hbar \omega_0 = 6.9 \text{nK}$).

In the experiments the traps are generally very elongated, i.e. we have a strong deformation $\sigma = \omega_x / \omega_\perp$, 
where $\omega_\perp = \sqrt{\omega_x \omega_y}$ is the average frequency in the $xy$ plane.
In our examples we choose $\sigma = 1/8$. This results in a rather high value for the average transverse frequency
of $\hbar \omega_\perp = \hbar \omega / \sigma^{1/3} = 16.42 \text{nK}$. In order to rotate the system around the $z$ axis, at least a small deformation in the $xy$ plane is necessary, which we parametrize by
$\delta = \omega_x / \omega_y$. (In practice, the rotating deformation of the potential can be generated by a laser beam.)

The main uncertainty comes from the gap at zero temperature, $\Delta_0$. Note that the coupling constant $g$ does not
appear explicitly. The moment of inertia depends on the interaction only via $\Delta$, which can be written as a function
of $T$ and $\Delta_0$. The value of the critical temperature $T_c = 0.567 \Delta_0$ is still under investigation. In addition, the $s$-wave scattering length $a$ of the atoms, and consequently $g$, $\Delta_0$, and $T_c$ can be tuned in the experiments by a magnetic field due to the presence of Feshbach resonances. Therefore we will treat $\Delta_0$ as a free parameter. As a rough estimate, using the scattering length $a = -216 a_0$, where $a_0$ is the Bohr radius, one obtains that the gap $\Delta$ averaged over the Fermi surface is of the order of magnitude of $15 \text{nK}$, i.e., of the same order of magnitude as the transverse trapping frequency $\omega_\perp$.

In Fig. 3 we display the moment of inertia as a function of the temperature for two different deformations $\delta$. The lower curves correspond to a very small deformation, $\delta = 0.8$. In this case the moment of inertia at $T = 0$
is very small. When $T$ approaches $T_c$, the normal-fluid component becomes more and more important and consequently the moment of inertia increases until it finally reaches the rigid-body value at $T = T_c$. Qualitatively the behavior is similar in the case of a strong deformation in the $xy$ plane (upper curves), except that in this case the whole curve is shifted upwards, mainly due to the much larger value of $\Theta_{\text{rot}}$. The difference between the three curves shown for each deformation will be discussed be-
low.

In order to illustrate the origin of the temperature dependence of \( \Theta \), we show in Fig. 4 the current distributions for the case \( \delta = 0.8 \) for four temperatures between \( T = 0 \) and \( T = T_c \). One can clearly see the continuous transition from the irrotational motion at \( T = 0 \), resulting in a small angular momentum and therefore a small moment of inertia, to the rigid motion at \( T = T_c \).

Now we are going to discuss the differences between the three curves shown in Fig. 3 for each deformation. The short-dashed lines correspond to the results obtained within the \( h \rightarrow 0 \) approach, Eq. (102). The long-dashed and solid lines were obtained from Eq. (103), i.e., they take into account the difference between \( G(x) \) and \( G(0) \), resulting from the long-time behavior of the operator \( \hat{h}_1(t) \), Eq. (24). From the definition [25] it is clear that this difference is less important for large values of \( \Delta \), and indeed the long-dashed lines, corresponding to \( \Delta_0 = 20 \text{ nK} \), are closer to the \( h \rightarrow 0 \) results than the solid lines, corresponding to \( \Delta_0 = 10 \text{ nK} \). More precisely, the criterion for the validity of the \( h \rightarrow 0 \) approach seems to be \( h\omega_\perp \ll \Delta_0 \) rather than \( h\omega_\perp \ll \Delta \), as one might expect. This surprising fact can be understood by looking at Fig. 2. Whatever is the actual value of the temperature \( T \) [i.e., of \( \Delta(T) \)], the value of \( G(\varepsilon/2\Delta) \) can always be replaced by \( G(0) \) if \( \varepsilon/2\Delta_0 \ll 1 \).

To show more clearly the non-trivial dependence of \( \Theta \) on \( \Delta \), we show in Fig. 5 the moment of inertia for zero temperature as a function of \( \Delta \) for the same deformations as in Fig. 3. The irrotational limit, indicated by the dashed lines, is reached for \( \Delta \rightarrow \infty \). If \( \Delta \) is much smaller than \( h\omega_\perp \) (3.67 nK in the case \( \delta = 0.8 \) and 11.61 nK in the case \( \delta = 0.5 \), respectively), the moment of inertia even approaches the rigid body value, and the \( h \) expansion fails completely. For example, in nuclear physics strong deviations from the irrotational value are quite common [13, 14]. Finally let us briefly discuss the question whether the moment of inertia is suitable to detect the superfluidity in experiments. In principle the moment of inertia can be measured directly by measuring the rotational energy

\[
E_{\text{rot}} = \frac{\Theta}{2} \Omega^2. \tag{109}
\]

Since the rotation does not change the potential energy (at least not to linear order in \( \Omega \)), the rotational energy is equal to the difference of the release energies \( E_{\text{rel}} \) of the rotating system and of the non-rotating system. (The release energy \( E_{\text{rel}} \) is the total energy of all particles after the trapping potential has been switched off, i.e., the sum of the kinetic energy \( E_{\text{kin}} \) and of the interaction energy \( E_{\text{int}} \) of the trapped system.) A disadvantage of the direct measurement of \( E_{\text{rot}} \) is that it requires two identical systems, one in rotation and one at rest. As a rough estimate we approximate the release energy \( E_{\text{rel}} \) by the kinetic energy \( E_{\text{kin}} \) of the particles in the “effective” harmonic potential [24],

\[
E_{\text{kin}} = 2 \int \frac{d^3r d^3p}{(2\pi \hbar)^3} \frac{p^2}{2m} \rho(r, p) = \frac{\mu^4}{8\hbar^3 \omega_x \omega_y \omega_z}. \tag{110}
\]

Hence, as a function of the average transverse trapping frequency \( \omega_\perp = \sqrt{\omega_x \omega_y} \) and the deformation \( \delta = \omega_x/\omega_y \) we obtain

\[
\frac{E_{\text{rot}}}{E_{\text{kin}}} = \frac{1 + \delta^2}{3\delta} \frac{\Theta}{\Theta_{\text{rigid}}} \left( \frac{\Omega}{\omega_\perp} \right)^2. \tag{111}
\]

Since we used linear response theory, our results are valid only for slow rotations, \( \Omega \ll \omega_\perp \). In particular the angular velocity must be small enough in order to avoid the

FIG. 4: Current distributions \( j(x, y, 0) \) in the \( xy \) plane (arbitrary units) for \( \omega_x/\omega_y = 0.8 \), \( \Delta_0 = 20 \text{ nK} \), and four different temperatures: \( T/T_c = 0, 0.4, 0.6, \) and 1.
of the gap at $T = T_c$ and $T = 0$ is most pronounced for small deformation (see Fig. 3), we choose $\delta = 0.8$. Using these numbers we find $E_{rot}/E_{kin} \approx 0.1 \times \Theta/\Theta_{rigid}$, i.e., the moment of inertia might indeed be measurable.

VI. SUMMARY AND CONCLUSIONS

In this article we have discussed the temperature dependence of the moment of inertia of a Fermi gas trapped in a slowly rotating trapping potential. The assumption of a slow rotation allowed us to use linear response theory (RPA), but it is clear that in this way certain interesting effects like the creation of vortices could not be considered, since they depend non-linearly on the angular velocity $\Omega$ of the rotation.

In Sect. III we derived the density matrix of the rotating system using a semiclassical method similar to the one described in Ref. [8], but now taking into account the thermal quasiparticle occupation numbers, which were neglected in Ref. [8] and which give rather important contributions. One important point is that the method takes into account that the energy difference $\hbar \omega_{\pm}$ of the states connected by the perturbation hamiltonian (i.e., essentially by $L_z$) is not necessarily negligible in comparison with the gap $\Delta$. This leads to a non-trivial behavior of the density matrix on $\hbar \omega_{\pm}/\Delta$. These effects can also be regarded as finite-size effects, since $\hbar \omega_{\pm}$ vanishes in homogeneous systems.

In Sect. IV we presented an alternative method for the calculation of the density matrix, where only the leading order of the $\hbar$ expansion is retained. This is equivalent to the transport or hydrodynamical approach which is often used in the literature [1-12]. The qualitative difference between the results obtained within the two approaches is that the velocity field obtained in Sect. IV has irrotational and rotational contributions at all temperatures, whereas the transport approach presented in Sect. IV gives a purely irrotational velocity field at zero temperature, as it is the case in homogeneous systems. The dependence on $\hbar \omega_{\pm}/\Delta$ mentioned above is missed within this approach.

In Sect. V we used the density matrices obtained in the preceding sections for the calculation of the moment of inertia. The result can qualitatively be understood within the two-fluid model, which describes the superfluid system as a mixture of a superfluid and a normal-fluid component. The density of the normal-fluid component is zero at $T = 0$ and approaches the total density for $T \rightarrow T_c$. We have shown that the transport approach reproduces this two-fluid model. Somewhat surprisingly, the condition for the transport approach to be valid turns out to be $\hbar \omega \ll \Delta_0$, where $\Delta_0$ is the value of the gap at $T = 0$. This is less restrictive than the condition $\hbar \omega \ll \Delta$, in particular for temperatures near $T_c$.

Within the transport approach, the moment of inertia increases smoothly from the irrotational limit at $T = 0$ to the rigid-body value at $T = T_c$. This is a consequence of the increasing density of the normal-fluid component of the two-fluid model. If the condition $\hbar \omega \ll \Delta$ is not fulfilled, the behavior is qualitatively similar, but the moment of inertia is always larger than it is within the transport approach, because in this case the rotational contributions to the velocity field are always non-zero due to the finite-size effects mentioned above. In both cases, the smoothly increasing moment of inertia as a function of temperature can be obtained only if the thermal quasiparticle occupation numbers are properly included in the calculation. It is not sufficient to perform a zero-temperature calculation and then replace the gap $\Delta$ by the temperature-dependent gap $\Delta(T)$.

Looking at the size of the error made by neglecting the finite-size effects, we conclude that for the trapped fermionic atoms, where $\hbar \omega \lesssim \Delta_0$, the hydrodynamical approach is just at the limit of its applicability. However, we would like to point out that there are other physical situations, where $\hbar \omega \gtrsim \Delta_0$, and where finite-size corrections are crucial. For example, the moments of inertia of rotating superfluid nuclei ($T = 0$) have at least twice the irrotational value [13]. Also for the description of superconducting metallic grains in a weak magnetic field, corresponding to a perturbation $h_1 = (e/mc)\mathbf{p} \cdot \mathbf{A}(\mathbf{r}) = (e/mc)B_zL_z$ (if $\mathbf{B}$ is parallel to the $z$ axis) and therefore being formally equivalent to a slow rotation, these corrections might be important.

The method used in Sect. III for the semiclassical solution of the RPA in superfluid systems can also be extended to the dynamical case, i.e., to time-dependent perturbations. In this way collective excitations of the superfluid system, in particular the change of their frequencies compared with the normal-fluid phase, can be described. So far the collective modes in the superfluid phase have been studied either within the hydrodynamical approach [1, 10] or quantum-mechanically for the case of spherical symmetry and moderate numbers of particles [11].

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APPENDIX: ALTERNATIVE DERIVATION OF THE MIGDAL TERM

In Sect. II we derived the change of the pairing field, $\Delta_1(\mathbf{r})$, via a gauge transformation. Here we will present an alternative method, which is more direct, but also somewhat more difficult. We will solve the original inte-
\( \Delta_1(r) = \Delta_1^B(r) + \Delta_1^M(r) \)

\[
\Delta_1^B(r) = g \int \frac{d^3p}{(2\pi\hbar)^3} \left[ \kappa_1^{IB}(r, p) + \kappa_1^{M}(r, p) \right].
\]  

(A.1) 

with

\[
\kappa_1^{IB} = F_{nn}^h h_{nn'}, \quad \kappa_1^{M} = F_{nn}^h \Delta_{nn'}. \tag{A.2} \]

(A.2)

\[
\kappa_1^{IB} = F_{n'n}^h h_{1n'n'}, \quad \kappa_1^{M} = F_{n'n}^h \Delta_{1nn'}. \tag{A.3} \]

(A.3)

The Wigner transforms of these two contributions to \( \kappa_1 \) can be calculated semiclassically as given by Eq. (58) with \( \tilde{p}_i \) replaced by \( \kappa_1^{IB} \) and \( \kappa_1^{M} \), respectively, and \( F^{\rho h} \) replaced by \( F^{\kappa h} \) and \( F^{\kappa \Delta} \), respectively.

The first term in Eq. (A.1), \( \Delta_1^B \), has already been evaluated in Sect. III. The term proportional to \( \Omega \) in Eq. (71) becomes

\[
\Delta_1^M(r) = \frac{ig^2 p_ho(r) r_x r_y}{8\pi^2 \hbar^2 \Delta} \Omega_{\omega^+ \omega^-} (G_+ + G_-). \tag{A.4} \]

(A.4)

Now we turn to the evaluation of the second term, \( \Delta_1^M \).

The explicit expression for \( F^{\kappa \Delta}(\xi, \xi') \) reads

\[
F^{\kappa \Delta}(\xi, \xi') = \frac{1 - 2f(E)}{4E} - \frac{1 - 2f(E')}{4E'} - \frac{[f(E) - f(E')]\xi^2}{4EE'(E + E')}, \tag{A.5} \]

(A.5)

which in analogy to \( F^{\rho h}(\xi, \xi') \) and \( F^{\kappa h}(\xi, \xi') \) can be approximated by:

\[
F^{\kappa \Delta} \left( \xi + \frac{\varepsilon}{2\Delta}, \xi' - \frac{\varepsilon}{2\Delta} \right) \approx \frac{1 - 2f(\tilde{E})}{2\tilde{E}} - \left( \frac{\varepsilon}{2\Delta} \right)^2 G \left( \frac{\varepsilon}{2\Delta} \right) \delta(\tilde{\xi}), \tag{A.6} \]

(A.6)

with \( \tilde{E} = \sqrt{\xi^2 + \Delta^2} \). Using this approximation we get the arguments of the functions \( h(\mathbf{r}, \mathbf{p}) \) and \( E(\mathbf{r}, \mathbf{p}) \) are omitted for brevity

\[
\Delta_1^M(r) = g \int \frac{d^3p}{(2\pi\hbar)^3} \left[ \frac{1 - 2f(E)}{2E} \Delta_1(r) \right.
\]

\[
- \delta(h) \int d\varepsilon \left( \frac{\varepsilon}{2\Delta} \right)^2 G \left( \frac{\varepsilon}{2\Delta} \right)
\]

\[
\left. \times \int \frac{dt}{2\pi\hbar} e^{-i\varepsilon t/\hbar} \Delta_1[r^{cl}(r, p; t)] \right]. \tag{A.7} \]

(A.7)

At this stage the disadvantage of the present method as compared with the method used in Sect. III becomes obvious, since we encounter a divergent integral over \( d^3p \), whereas in Sect. III all expressions were finite. This divergence is the same one which also appears in the gap equation (27) for the unperturbed gap in local-density approximation. If we assume that this equation is regularized in some way, we can use it to get rid of the divergence in Eq. (A.7), and we obtain

\[
\Delta_1^M(r) = \Delta_1(r) + g \int \frac{d^3p}{(2\pi\hbar)^3} \delta(h) \int d\varepsilon \left( \frac{\varepsilon}{2\Delta} \right)^2 \left( \frac{\varepsilon}{2\Delta} \right)
\]

\[
\times \int \frac{dt}{2\pi\hbar} e^{-i\varepsilon t/\hbar} \Delta_1[r^{cl}(r, p; t)] . \tag{A.8} \]

(A.8)

As we will see, the integral equation (A.1) can be solved by the ansatz (23). With this ansatz the Fourier transform in Eq. (A.8) can easily be evaluated and we obtain

\[
\Delta_1^M(r) = \Delta_1(r) + \frac{ig^2 p_\rho(r) r_x r_y}{8\pi^2 \hbar^2 \Delta} \alpha(\omega^2 G_+ + \omega^2 G_-). \tag{A.9} \]

(A.9)

The coefficient \( \alpha \) can now be determined by inserting Eqs. (A.4) and (A.9) into Eq. (A.1). The solution, of course, coincides with Eq. (72).

However, we have to admit that the above arguments concerning the divergence in Eq. (A.7) are a little bit hand-waving. For example, Eq. (27) (including an appropriate regularization) is valid only in the local-density approximation, and it does not allow for a constant gap \( \Delta \), while we have for simplicity assumed that \( \Delta \) is a constant in order to derive Eq. (A.7). Such inconsistencies do not appear within the formalism presented in Sect. III.

It remains to show that the Migdal term, calculated as the second term of Eq. (22), is consistent with the result given in Eq. (78). This can be done with the aid of the explicit expression for \( F^{\kappa \Delta} \), which turns out to be

\[
F^{\kappa \Delta}(\xi, \xi') = -F^{\kappa h}(\xi, \xi'), \tag{A.10} \]

(A.10)

and the Fourier transform of \( \Delta_1[r^{cl}(r, p; t)] \).

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