BIQUADRATIC FIELDS HAVING A NON-PRINCIPAL EUCLIDEAN IDEAL CLASS

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Abstract. H. W. Lenstra [7] introduced the notion of an Euclidean ideal class, which is a generalization of norm-Euclidean ideals in number fields. Later, families of number fields of small degree were obtained with an Euclidean ideal class (for instance, in [3] and [6]). In this paper, we construct certain new families of biquadratic number fields having a non-principal Euclidean ideal class and this extends the previously known families given by H. Graves [3] and C. Hsu [6].

1. Introduction

Let $K$ be an algebraic number field and $\mathcal{O}_K$ be its ring of integers. Following the standard notations, let $d_K$, $\text{Cl}_K$, $h_K$ and $\mathcal{O}_K^*$ be the discriminant, the ideal class group, the class number and the multiplicative group of units of $K$, respectively. For an integral ideal $I$ of $\mathcal{O}_K$, $\text{Nm}(I)$ denotes its norm and $[I]$ stands for its ideal class. The class group $\text{Cl}_K$ and its size $h_K$, which are invariants associated with $K$, play a central role in number theory. As a consequence, we value results that shed light on $\text{Cl}_K$ and $h_K$.

An integral domain $R$ is said to be an Euclidean domain if there exists a function $\phi : R \to \mathbb{N} \cup \{0\}$ such that for every $\alpha, \beta \neq 0 \in R$, there exists $\gamma \in R$ such that $\phi(\alpha - \beta \gamma) < \phi(\alpha)$. One of the classical problems in algebraic number theory is to classify all number fields which are Euclidean with respect to the absolute value of the standard norm map. By multiplicatively extending the norm map $\text{Nm}$ from $\mathcal{O}_K$ to $K \setminus \{0\}$, the above definition can be written as follows: for every $x \in K \setminus \{0\}$ there exists $\alpha \in \mathcal{O}_K$ such that $\text{N}(x - \alpha) < 1$. In 1979, H.W. Lenstra [7] made a crucial observation and generalized the aforementioned definition. He defined an integral ideal $C$ to be norm-Euclidean if for every $x \in K \setminus \{0\}$ there exists $\alpha \in C$ such that $\text{N}(x - \alpha) < 1$.

Further, he generalized the notion of norm-Euclidean ideals as follows.

**Definition 1.1.** [7] Let $R$ be a Dedekind domain and $\mathbb{I}$ be the set of all fractional ideals containing $R$. A fractional ideal $C$ of $R$ is said to be an Euclidean ideal if there exists a function $\psi : \mathbb{I} \to W$, for some well-ordered set $W$, such that for all $I \in \mathbb{I}$ and all $x \in I^{-1}C \setminus C$, there exists some $y \in C$ such that

$$\psi((x - y)I^{-1}C) < \psi(I).$$

2010 Mathematics Subject Classification. 11A05.

Key words and phrases. Euclidean ideal class, Hilbert class field, cyclic class group, quartic field.

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Lenstra proved that if \( C \) is an Euclidean ideal, then every ideal in the ideal class \([C]\) is also Euclidean. Thus we can unambiguously define Euclidean ideal classes.

It is a well known fact that if \( \mathcal{O}_K \) is Euclidean, then its ideal class group \( \text{Cl}_K \) is trivial \([8]\). Likewise, if \( \mathcal{O}_K \) has an Euclidean ideal class then \( \text{Cl}_K \) is cyclic (Theorem 1.6, \([7]\)). More precisely, if \( \mathcal{O}_K \) has an Euclidean ideal class \([C]\) then \( \text{Cl}_K \) is a cyclic group and \([C]\) generates \( \text{Cl}_K \). However, the converse is not always true, for \( \mathbb{Q}(\sqrt{-d}) \) for \( d = 19, 23, 24, 31, 35, 39, 40, 43, 47 \) has no Euclidean ideal class even though \( \text{Cl}_K \) is cyclic (Proposition 2.1, \([7]\)).

Thus to classify all number fields in which the converse holds true is an intriguing problem and was addressed by Lenstra \([7]\). He proved under the assumption of the generalized Riemann Hypothesis (GRH), that all number fields with cyclic class group, except for the imaginary quadratic fields, have an Euclidean ideal class. In other words, if \( \text{rank}(\mathcal{O}_K^*) \geq 1 \), then \( \mathcal{O}_K \) has an Euclidean ideal class if and only if \( \text{Cl}_K \) is cyclic, assuming the GRH for number fields.

H. Graves proved a useful growth result for certain algebraic number fields, which removed the appeal to GRH \([1]\). We state the precise statement below, which will be necessary in the course our proof.

**Theorem 1.2.** \([4]\) Suppose that \( K \) is a number field such that \( |\mathcal{O}_K^*| = \infty \), and that \( C \) is a non-zero ideal of \( \mathcal{O}_K \). If \([C]\) generates the class group of \( K \) and

\[
\left|\{\text{prime ideal } \mathfrak{p} \subset \mathcal{O}_K : N(\mathfrak{p}) \leq x, [\mathfrak{p}] = [C], \pi_\mathfrak{p} \text{ is onto }\}\right| \gg \frac{x}{(\log x)^2},
\]

where \( \pi_\mathfrak{p} \) is the canonical map from \( \mathcal{O}_K^* \) to \( (\mathcal{O}_K/\mathfrak{p})^* \), then \([C]\) is an Euclidean ideal class.

Using this growth result Ram Murty and Graves removed GRH for large class of number fields \([5]\). The precise statement is as follows.

**Theorem 1.3.** Let \( K \) be a number field that is Galois over \( \mathbb{Q} \) and has cyclic class group. If its Hilbert class field \( H(K) \) has abelian Galois group over \( \mathbb{Q} \) and if \( \text{rank}(\mathcal{O}_K^*) \geq 4 \), then \( \text{Cl}_K = \langle [C] \rangle \) if and only if \([C]\) is an Euclidean ideal class.

Using her growth result, Graves \([3]\) constructed an explicit number field, viz. \( \mathbb{Q}(\sqrt{2}, \sqrt{35}) \), having a non-principal Euclidean ideal class. In this paper, we generalize this work and construct a new family of quartic fields of the form \( \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2}) \) which has a non-principal Euclidean ideal class. The precise statement is given at the end of this section. It is worthwhile to note that the quartic field of the form \( \mathbb{Q}(\sqrt{2}, \sqrt{p_1 p_2}) \) has unit rank 3.

In \([6]\), Hsu explicitly constructed a family of biquadratic and cyclic quartic fields with unit rank 3 having an Euclidean ideal class, described below.

**Theorem 1.4.** \([6]\) Suppose \( K \) is a quartic field of the form \( \mathbb{Q}(\sqrt{q}, \sqrt{kr}) \). Then \( K \) has a non-principal Euclidean ideal class whenever \( h_K = 2 \). Here the integers \( q, k, r \) are all primes \( \geq 29 \) and are all congruent to 1 (mod 4).
In this article we shall provide new families of number fields each having an Euclidean ideal class. This produces a new class of number fields other than the ones constructed by H. Graves [3] and C. Hsu [6]. Our main theorems are stated as follows.

**Theorem 1.5.** Let $K_1 = \mathbb{Q}(\sqrt{q}, \sqrt{kr})$, where $q \equiv 3 \pmod{4}$ and $k, r \equiv 1 \pmod{4}$ are prime numbers. Suppose that $h_{K_1} = 2$. Then $K_1$ has an Euclidean ideal class.

**Theorem 1.6.** Let $p, q$ be two prime numbers with $p, q \equiv 1 \pmod{4}$. Then the biquadratic field $K_2 = \mathbb{Q}(\sqrt{2}, \sqrt{pq})$ with $h_{K_2} = 2$ has an Euclidean ideal class.

It is efficacious to note at this point that combining Hsu’s result with the Theorem 1.5 above, we get a much larger class of biquadratic fields having an Euclidean ideal class. More precisely,

**Theorem 1.7.** Let $K = \mathbb{Q}(\sqrt{p}, \sqrt{qr})$, where $p$ is any rational prime and $q, r \equiv 1 \pmod{4}$ are prime numbers. If $h_K = 2$, then $K$ has an Euclidean ideal class.

This article is organized as follows. In section 2, we recall some standard preliminary results. Section 3 contains the computation of conductor and Hilbert class fields of certain biquadratic fields. The proof of the main theorems are presented in section 4. Finally, we list the class numbers of some of the biquadratic fields of our interest in section 5 and the table provides numerous examples of such fields with class number two.

### 2. Preliminaries

Let $K/\mathbb{Q}$ be a finite Galois extension with Galois group $G$. If $G$ is abelian, then by the famous Kronecker-Weber theorem, $K \subseteq \mathbb{Q}(\zeta_m)$ for some positive integer $m$. The smallest such integer $m$ is called the **conductor** of $K$. The following well known Proposition, which will be occasionally used in the next section, provides the conductor of real quadratic fields.

**Proposition 2.1.** The real quadratic field $\mathbb{Q}(\sqrt{m})$ has conductor $m$ if $m \equiv 1 \pmod{4}$ and $4m$ if $m \equiv 2, 3 \pmod{4}$.

For our purposes of this article, we recall a couple of standard results from class field theory. For that, we start with the following proposition.

**Proposition 2.2.** Let $L/K$ be a Galois extension of number fields with Galois group $G$. Let $\mathfrak{p}$ be a non-zero prime ideal in $\mathcal{O}_K$ which is unramified in $L$ and let $\mathfrak{P}$ be a prime ideal in $\mathcal{O}_L$ lying above $\mathfrak{p}$. Then there is a unique element $\sigma \in G$ such that for all $\alpha \in \mathcal{O}_L$, we have

$$\sigma(\alpha) \equiv \alpha^{[\mathcal{O}_K : \mathfrak{p}]} \pmod{\mathfrak{P}}.$$
**Definition 2.3.** The unique element $\sigma$ in Proposition 2.2 is called the Artin symbol and is often denoted by $(\frac{L/K}{\mathfrak{p}})$.

**Remark:** Suppose $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_g$ be all the primes in $\mathcal{O}_L$ lying above $\mathfrak{p}$. For $\tau \in G$, we have,

$$\left(\frac{L/K}{\tau(\mathfrak{p}_i)}\right) = \tau \left(\frac{L/K}{\mathfrak{p}_i}\right) \tau^{-1} \text{ for all } i.$$ 

Since $G$ acts transitively on $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_g\}$, the set $\left\{ \left(\frac{L/K}{\mathfrak{p}_i}\right) : i = 1, 2, \ldots, g \right\}$ is a conjugacy class in $G$. Thus by $(\frac{L/K}{\mathfrak{p}})$, we unambiguously mean this conjugacy class.

Next, we define the Dirichlet density of a set of prime ideals as follows.

**Definition 2.4.** Let $K$ be a number field and let $\mathcal{S}$ be a set of prime ideals in $\mathcal{O}_K$. The Dirichlet density of the set $\mathcal{S}$ is defined to be

$$\delta(\mathcal{S}) = \lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in \mathcal{S}} N(\mathfrak{p})^{-s}}{-\log(s - 1)},$$

provided the limit exists. Here, $N(\mathfrak{p})$ stands for the quantity $|\mathcal{O}_K/\mathfrak{p}|$.

Now, we are in a position to state the Chebotarev density theorem, that goes as follows.

**Theorem 2.5. (Chebotarev density theorem)** Let $L/K$ be a finite Galois extension of number fields with Galois group $G$ and let $C$ be a conjugacy class in $G$. Then the Dirichlet density of the set

$$\left\{ \text{prime ideal } \mathfrak{p} \text{ in } \mathcal{O}_K : \mathfrak{p} \text{ is unramified in } L \text{ and } \left(\frac{\mathfrak{p}}{L/K}\right) = C \right\}$$

exists and equals $\frac{|C|}{[L:K]}$, where $\left(\frac{\mathfrak{p}}{L/K}\right)$ is the Artin symbol of $\mathfrak{p}$.

Finally, we state the following consequence of the Artin reciprocity law.

**Lemma 2.6.** Let $K$ be a number field and let $H(K)$ be its Hilbert Class field i.e., $H(K)$ is the unique maximal, unramified, abelian extension of $K$. Then the Galois group of $H(K)$ over $K$ is isomorphic to $\text{Cl}_K$.

Alternatively, the Hilbert class field $H(K)$ of $K$ is the unique maximal, abelian extension of $K$ such that the principal prime ideals of $K$ split completely in $H(K)$. 
3. Computations for conductors and Hilbert class fields

The construction of the Hilbert class field of a number field is almost a century old classical problem. We are interested in the explicit construction of Hilbert class fields for certain family of biquadratic fields as these computations are essential to the proof of our main theorems. As mentioned, the conductors for biquadratic fields of the form $K = \mathbb{Q}(\sqrt{q}, \sqrt{kr})$, where $q, k$ and $r$ are primes, can be obtained by the following elementary arguments.

**Lemma 3.1.** Let $L/K$ be an extension of number fields. Assume that $L/\mathbb{Q}$ is a Galois extension and it is abelian. Then $f(K)$ divides $f(L)$, where $f(K), f(L)$ stands for the conductors of $K$ and $L$, respectively.

**Proof.** Let $G$ be the Galois group of the extension $L/\mathbb{Q}$. As $G$ is abelian, the fundamental theorem of Galois theory implies that $K/\mathbb{Q}$ is also an abelian extension. Thus the conductor of $K$ is well-defined.

Since $K \subseteq L \subseteq \mathbb{Q}(\zeta_{f(L)})$ and $f(K)$ is the smallest positive integer such that $K \subseteq \mathbb{Q}(\zeta_{f(K)})$, we have

$$K \subseteq \mathbb{Q}(\zeta_{f(K)}) \subseteq \mathbb{Q}(\zeta_{f(L)}).$$

Since $\zeta_{f(L)}$ is a primitive root of unity and $\mathbb{Q}(\zeta_{f(K)}) \subseteq \mathbb{Q}(\zeta_{f(L)})$, we get that $f(K)$ divides $f(L)$.

□

The next lemma, which will often be helpful to us, provides the conductor of the compositum of two number fields in terms of those of the constituent fields.

**Lemma 3.2.** Let $L$ be an abelian number field and let $K_1, K_2$ be two subfields of $L$ such that $L$ is the compositum of $K_1$ and $K_2$. Then $f(L) = \text{lcm}(f(K_1), f(K_2))$, where $f(K_1), f(K_2)$ and $f(L)$ are the conductors of $K_1, K_2$ and $L$, respectively.

**Proof.** Consider the following tower of number field extensions.

$$Q(\zeta_{f(L)})$$

$$\downarrow$$

$$Q(\zeta_{f(K_1)})$$

$$\downarrow$$

$L$

$$\downarrow$$

$$Q(\zeta_{f(K_2)})$$

$$\downarrow$$

$K_1$

$$\downarrow$$

$K_2$

$$\downarrow$$

$\mathbb{Q}$

(Fig. 3.1)

Since $K_1$ and $K_2$ are subfields of an abelian extension $L/\mathbb{Q}$, by Lemma 3.1 $f(K_1)$ divides $f(L)$ and $f(K_2)$ divides $f(L)$. Therefore, $\text{lcm}(f(K_1), f(K_2))$ divides $f(L)$.

On the other hand, using the following facts that

$$L = K_1 K_2 \subseteq \mathbb{Q}(\zeta_{f(K_1)})\mathbb{Q}(\zeta_{f(K_2)}) = \mathbb{Q}(\zeta_{\text{lcm}(f(K_1), f(K_2))})$$
along with the minimality of the conductor implies that \( \mathbb{Q}(\zeta_{f(L)}) \subseteq \mathbb{Q}(\zeta_{\text{lcm}(f(K_1), f(K_2))}) \). Thus \( f(L) \) divides \( \text{lcm}(f(K_1), f(K_2)) \) and hence \( f(L) = \text{lcm}(f(K_1), f(K_2)) \).

\[ \square \]

**Lemma 3.3.** The conductor \( f(K) \) of \( K = \mathbb{Q}(\sqrt{q}, \sqrt{kr}) \) is \( 4qkr \), where \( q, k, r \) are primes such that \( q \equiv 3 \pmod{4} \) and \( k, r \equiv 1 \pmod{4} \).

**Proof.** Let \( K_1 = \mathbb{Q}(\sqrt{q}) \) and \( K_2 = \mathbb{Q}(\sqrt{kr}) \) be two quadratic fields. By Proposition 2.1 we have \( f(K_1) = 4q \) and \( f(K_2) = kr \). As \( K = K_1K_2 \), we get by Lemma 3.2

\[ f(K) = \text{lcm}(f(K_1), f(K_2)) = 4qkr. \]

\[ \square \]

We now use Lemma 3.3 to find the Hilbert class field of the biquadratic fields considered above.

**Lemma 3.4.** Let \( K = \mathbb{Q}(\sqrt{q}, \sqrt{kr}) \), where \( q, k, r \) are prime numbers satisfying \( q \equiv 3 \pmod{4} \) and \( k, r \equiv 1 \pmod{4} \). If \( K \) has class number two, then the Hilbert class field \( H(K) \) is \( \mathbb{Q}(\sqrt{q}, \sqrt{k}, \sqrt{r}) \).

**Proof.** Let \( K' = \mathbb{Q}(\sqrt{q}, \sqrt{k}, \sqrt{r}) \) be a number field, where \( q, k, r \) are defined as above. By applying Lemma 3.3 twice, we get \( f(K') = \text{lcm}(f(K_1K_2), f(K_3)) = \text{lcm}(4qr, r) = 4qr \).

Now, we are ready to compute the Hilbert class field of \( K \). Artin reciprocity law gives (by Lemma 2.6), \( \text{Gal}(H(K)/K) \simeq Cl_K \).

By our assumption on the has class number of \( K \), the Hilbert class field \( H(K) \) is a quadratic extension of \( K \). Therefore, it is sufficient to prove that \( K' \) is an unramified extension of \( K \).

From the tower of number fields, we have

\[ \mathbb{Q} \subseteq K \subseteq K' \subseteq \mathbb{Q}(\zeta_{4qkr}). \]

Thus, the prime ideals in \( K \) lying above 2, \( q, k \) and \( r \) may ramify \( K' \). In fact, we prove that all such prime ideals are unramified for \( K' \) over \( K \).

We first prove this for the rational prime 2. Let \( L_1 = \mathbb{Q}(\sqrt{k}, \sqrt{r}) \) and let \( p_1, p_2, p_3 \) and \( p_4 \) be primes in \( L_1, K, K' \) and \( \mathbb{Q}(\zeta_{4qkr}) \) respectively, all lying above 2 and the containments are indicated in the diagram below.

Since \( k, r \equiv 1 \pmod{4} \), the prime 2 is unramified in \( \mathbb{Q}(\sqrt{k}) \) and \( \mathbb{Q}(\sqrt{r}) \) and so in the compositum \( L_1 \) i.e., \( e(p_1|2) = 1 \). And, since \( q \equiv 3 \pmod{4} \), 2 is ramified in \( \mathbb{Q}(\sqrt{q}) \). As \( \mathbb{Q}(\sqrt{q}) \subseteq K \subseteq K' \), we have \( e(p_2|2) > 1 \) and \( e(p_3|2) > 1 \).

From the multiplicativity of the ramification indices, we get,

\[ 1 < e(p_3|2) = e(p_3|p_1)e(p_1|2) \leq 2 \cdot 1. \]

Thus, \( e(p_3|2) = 2 \) and \( e(p_2|2) = 2 \) and hence \( e(p_3|p_2) = \frac{e(p_3|2)}{e(p_2|2)} = 1. \) That is, \( p_3 \) is unramified over \( p_2 \).
By a close examination of the arguments given above, we immediately see that just by replacing 2 with \( q \), we get the unramifiedness for the prime \( q \).

For the other two primes, viz, \( k \) and \( r \), the proof goes through ad verbatim. That is, we simply need to replace \( L_1 \) by \( L_2 = \mathbb{Q}(\sqrt{q}, \sqrt{r}) \) for the prime \( k \) and \( L_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p}) \) for the prime \( r \). Thus all the prime ideals in \( K \) lying above the rational primes \( q, k \) and \( r \) are unramified in \( K' \). Hence, the Hilbert class field \( H(K) = K' = \mathbb{Q}(\sqrt{q}, \sqrt{k}, \sqrt{r}) \). □

Next, we turn our attention to compute the conductor and the Hilbert class field of a class of biquadratic fields which is slightly different from the one we discussed above. The proof of the following lemmas are similar, but still we present it again to make it complete.

**Lemma 3.5.** Let \( K_2 = \mathbb{Q}(\sqrt{2}, \sqrt{pq}) \), where \( p, q \) are rational primes with \( p, q \equiv 1 \pmod{4} \). Then the conductor \( f(K_2) \) of \( K_2 \) is \( 8pq \).

**Proof.** It is an application of Lemma 3.3. □

**Lemma 3.6.** Let \( K_2 = \mathbb{Q}(\sqrt{2}, \sqrt{pq}) \), be the biquadratic field as in the above lemma. If the class number \( h_{K_2} \) of \( K_2 \) is 2, then the Hilbert class field \( H(K_2) \) of \( K_2 \) is \( \mathbb{Q}(\sqrt{q}, \sqrt{k}, \sqrt{r}) \).

**Proof.** By close examination of the extension of number fields, the proof is similar to the proof of the Lemma 3.4. But, it is crucial to note that the prime 2 is ramified in \( K_2 \). Therefore, we only sketch a proof of unramification of primes in \( K_2 \) lying above 2. Let \( K'' = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}) \). Using our standard arguments, we can prove that \( K'' \) has conductor \( 8pq \). Since \( K_2 \) has class number 2 and \( K''/K_2 \) is a quadratic extension, the Hilbert class field of \( K_2 \) is equal to \( K'' \) provided that \( K''/K_2 \) is an unramified extension.

As \( K \subseteq K'' \subseteq \mathbb{Q}(\zeta_{8})\mathbb{Q}(\zeta_{p})\mathbb{Q}(\zeta_{q}) = \mathbb{Q}(\zeta_{8pq}) \), we only need to take care of the ramification of the primes 2, \( p \) and \( q \). Since the argument is similar for all the primes exactly like in the previous Lemma 3.5, we just outline it for the prime 2.

The prime 2 is unramified in \( \mathbb{Q}(\sqrt{p}, \sqrt{q}) \) and it is ramified in \( \mathbb{Q}(\sqrt{2}) \). In other words, \( e(p_2|2) = 1 \) and \( e(p_1|2) = 2 \). Therefore,

\[
e(p_2|2) = e(p_2|p_2) \cdot e(p_2|2) = e(p_2|p_2) \leq 2
\]

and hence \( e(p_2|2) = 1 \). Finally, \( e(p_1|p_3) = \frac{e(p_1|2)}{e(p_3|2)} = 1 \), that is, 2 is unramified in \( K'' \) over \( K_2 \).
4. Proof of the main theorems

4.1. Proof of Theorem 1.5. The main strategy of our proof is to find an arithmetic progression modulo the conductor, suitable enough to apply the growth result as in the Theorem 1.2 by Graves. That is to show that the number of prime ideals \( \wp \) belonging to the non-principal ideal class with norm less than or equal to \( x \) and a unit generating the group \((\mathcal{O}_K/\wp)^* \) is \( \gg x/\log^2 x \).

We first recall the following result as pointed in [3], which is crucial to fulfill our goal.

**Theorem 4.1.** [3] Let \( K \) be a totally real number field with conductor \( f(K) \) and let \( \{e_1, e_2, e_3\} \) be a multiplicatively independent set contained in \( \mathcal{O}_K^* \). If \( l = \text{lcm}(16, f(K)) \), and if \( \gcd(u, l) = \gcd(u - 1, l) = 1 \) for some integer \( u \), then

\[
\left| \{ \text{primes } \in \mathcal{O}_K : N(\wp) \equiv u \pmod{l}, N(\wp) \leq x, \langle -1, e_i \rangle \rightarrow (\mathcal{O}_K/\wp)^* \} \right| \gg \frac{x}{(\log x)^2},
\]

for at least one \( i \).

Since we confine ourselves to totally real quartic fields, by the previous theorem, we have enough number of prime ideals in the field satisfying the growth result [3], but we need them to be in the non-principal ideal class. Here is the place where the construction of the Hilbert class field is crucial.

**Proof of Theorem 1.5:** Let \( K_1 = \mathbb{Q}(\sqrt{q}, \sqrt{kr}) \) be a biquadratic field as in the Lemma 3.3 which is totally real with signature \((4,0)\). The field \( K_1 \) has the unit rank 3, and therefore, there exists a multiplicatively independent set consisting of 3 elements.

We first claim the following:

There exists an integer \( u \) such that whenever \( p \equiv u \pmod{4qkr} \), \( f(p|p) = 1 \) and \( f(\wp|p) = 2 \) hold true. (2)

Here, \( f(p|p) \) and \( f(\wp|p) \) are the residual degrees of the primes \( p \) and \( \wp \) in \( K_1 \) and \( H(K_1) \) respectively, both lying above \( p \).

The proof of our claim goes as follows. Let \( \left( \frac{p}{K_1/\mathbb{Q}} \right) \) and \( \left( \frac{p}{H(K_1)/\mathbb{Q}} \right) \) be the Artin symbols of the prime \( p \) in the number field \( K_1 \) and its Hilbert class field \( H(K_1) \), respectively.

Consider the following two sets of prime numbers in \( \mathbb{Z} \), namely,

\[ X_{K_1} = \left\{ p : p \text{ is prime and } \left( \frac{p}{K_1/\mathbb{Q}} \right) = 1 \right\}, \]

and

\[ X_{H(K_1)} = \left\{ p : p \text{ is prime and } \left( \frac{p}{H(K_1)/\mathbb{Q}} \right) = 1 \right\}. \]
The containment of the sets $K_1 \subseteq H(K_1)$, implies that a prime $p$ splits completely in $K_1$ whenever it splits completely in $H(K_1)$. Therefore, $X_{H(K_1)} \subseteq X_{K_1}$.

Since both the groups $Gal(K_1/Q)$ and $Gal(H(K_1)/Q)$ are abelian, the identity element itself constitutes a conjugacy class. So, by Theorem (2.5), the Dirichlet densities of the sets $X_{K_1}$ and $X_{H(K_1)}$ are $\frac{1}{4}$ and $\frac{1}{8}$, respectively. Since $X_{H(K_1)} \subseteq X_{K_1}$, the Dirichlet density of the difference $X_{K_1} \setminus X_{H(K_1)}$ is $\frac{1}{4} - \frac{1}{8} = \frac{1}{8}$. In particular, the set $X_{K_1} \setminus X_{H(K_1)}$ is infinite and thus any of its element can serve our purpose. Understanding the set $X_{K_1} \setminus X_{H(K_1)}$ more closely will be convenient for us. We do this now.

First, we recall a result about splitting of primes in a number field from [8].

**Theorem 4.2.** [8] Let $K$ be a number field and let $L$ and $M$ be two extensions of $K$. Let $p$ be a prime ideal in the ring of integers of $K$. If $p$ splits completely in both $L$ and $M$, then $p$ splits completely in the compositum field $LM$.

Using the above theorem, we characterize all the prime numbers in the set $X_{K_1} \setminus X_{H(K_1)}$.

We note that, from Theorem 4.2, a prime $p$ splits completely in $K_1$ if and only if it splits completely in $\mathbb{Q}(\sqrt{q})$ as well as in $\mathbb{Q}(\sqrt{kr})$. By the theory of decomposition of primes in quadratic fields, we have:

A prime $p$ splits completely in $\mathbb{Q}(\sqrt{q})$ and in $\mathbb{Q}(\sqrt{kr})$ iff
\[
\left(\frac{q}{p}\right) = 1 \quad \text{and} \quad \left(\frac{kr}{p}\right) = 1. \tag{3}
\]

Similarly, $p \not\in X_{H(K_1)}$ if and only if $p$ does not split completely in at least one of the fields $\mathbb{Q}(\sqrt{q})$, $\mathbb{Q}(\sqrt{k})$ or $\mathbb{Q}(\sqrt{r})$. That is
\[
\left(\frac{q}{p}\right) = -1 \quad \text{or} \quad \left(\frac{k}{p}\right) = -1 \quad \text{or} \quad \left(\frac{r}{p}\right) = -1. \tag{4}
\]

Thus, combining the equations (3) and (4), we get
\[
X_{K_1} \setminus X_{H(K_1)} = \left\{ p : p \text{ is prime and } \left(\frac{q}{p}\right) = 1 \text{ and } \left(\frac{k}{p}\right) = \left(\frac{r}{p}\right) = -1 \right\} \tag{5}
\]

We proceed for the proof now.

Let $l = \text{lcm}(16, f(K_1)) = \text{lcm}(16, 4qr) = 16qr$. In order to apply Theorem (1.1), we are required to find an integer $u$ satisfying the condition (2) along with the following:

\[
(i) \quad \gcd(u, 16qr) = 1, \tag{6}
\]
\[
(ii) \quad \gcd\left(\frac{u - 1}{2}, 16qr\right) = 1. \tag{7}
\]
We note that the condition \((ii)\) above is equivalent to the following simultaneous congruences:

\[
\begin{align*}
    u &\not\equiv 1 \pmod{q}; \\
    u &\not\equiv 1 \pmod{r}; \\
    u &\not\equiv 1 \pmod{k}; \\
    u &\not\equiv 1 \pmod{4}.
\end{align*}
\]

In other words, it suffices to find a prime number \(w \in X_{K_1} \setminus X_{H(K_1)}\) satisfying \((5), (6)\) and \((7)\).

Here, we quote the following result due to P. Pollack \([10]\) which will be of our help to prove the existence of \(w\).

**Theorem 4.3.** \([10]\) *Let* \(p \geq 5\) *be a prime number. Then there exists a prime number* \(q < p\) *such that* \(\left(\frac{q}{p}\right) = -1\) *and* \(q \equiv 3 \pmod{4}\).*

Now we are ready to find our desired \(w\). By the aforementioned Theorem \((4.3)\), there exist prime numbers \(p_1, p_2, p_3\) with \(p_1 < q, p_2 < k, p_3 < r\) and satisfying

\[
\left(\frac{p_1}{q}\right) = \left(\frac{p_2}{k}\right) = \left(\frac{p_3}{r}\right) = -1 \text{ and } p_1, p_2, p_3 \equiv 3 \pmod{4}.
\]

\[\tag{8}\]

Now, we look for a solution to the following system of simultaneous congruences.

\[
\begin{align*}
x &\equiv p_1 \pmod{q}; \\
x &\equiv p_2 \pmod{k}; \\
x &\equiv p_3 \pmod{r}; \\
x &\equiv 3 \pmod{4}.
\end{align*}
\]

\[\tag{9}\]

The Chinese remainder theorem guarantees a unique solution modulo \(4qkr\) to the above system congruences, say \(x_0\).

Since \(gcd(x_0, 4qkr) = 1\), by Dirichlet’s Theorem for primes in arithmetic progression, there exist infinitely many prime numbers \(l'\) satisfying \(l' \equiv x_0 \pmod{4qkr}\). We pick such a prime number \(l'\) and call it \(w\). We note that \(w\) satisfies conditions \((i)\) and \((ii)\) above. It remains to check that \(w\) satisfies \((5)\).

For that, using the law of quadratic reciprocity along with the facts \(q \equiv 3 \pmod{4}, k, r \equiv 1 \pmod{4}\) and \(w \equiv 3 \pmod{4}\), we get

\[
\left(\frac{q}{w}\right) = \left(\frac{w}{q}\right) (-1)^{\frac{w-1}{2}} = \left(\frac{w}{q}\right) = -\left(\frac{p_1}{q}\right) = 1.
\]

Also,

\[
\left(\frac{k}{w}\right) = \left(\frac{w}{k}\right) = \left(\frac{p_2}{k}\right) = -1
\]
and
\[
\left( \frac{r}{w} \right) = \left( \frac{w}{r} \right) = \left( \frac{p_3}{r} \right) = -1.
\]
Thus, the prime number \( w \) satisfies the conditions (5), (6) and (7). Hence, for any prime \( p \) satisfying \( p \equiv w \pmod{4qkr} \), we have that \( f(\varphi(p)) = 2 \). That is, \( p \) does not split completely in \( H(K_1) \). Therefore, using lemma 3.4, we can say that \( p \) is not a principal ideal and hence it will be a generator of \( Cl_{K_1} \). This will prove that \( K_1 \) has an Euclidean ideal class.

**Proof of Theorem (1.6)**

The proof is almost similar to that of the previous one. By following the argument given above, (5) reads the following.

\[
X_{K_2} \setminus X_{H(K_2)} = \left\{ s : s \text{ is prime and } \left( \frac{\varphi}{s} \right) = 1 \text{ and } \left( \frac{p}{s} \right) = \left( \frac{q}{s} \right) = -1 \right\} \tag{10}
\]

In this case we have, \( l = \text{lcm}(16, f(K_2)) = \text{lcm}(16, 8pq) = 16pq \). So, we need to find an integer \( u \) such that the following two conditions hold: (i) \( \gcd(u, 16pq) = 1 \) and (ii) \( \gcd\left(\frac{u-1}{2}, 16pq\right) = 1 \). Again, condition (ii) can be replaced by the following equivalent conditions: \( u \not\equiv 1 \pmod{p} \); \( u \not\equiv 1 \pmod{q} \); and \( u \not\equiv 1 \pmod{4} \).

Again using the Theorem (4.3), we can choose prime numbers \( p_1 \equiv 3 \pmod{4} \) and \( p_2 \equiv 3 \pmod{4} \) such that

\[
p_1 < p, p_2 < q \text{ and } \left( \frac{p_1}{p} \right) = \left( \frac{p_2}{q} \right) = -1.
\]

Let \( x_0 \) be a unique solution \( \pmod{8pq} \) to the system of congruences below.

\[
\begin{cases} 
  x \equiv p_1 \pmod{p} \\
  x \equiv p_2 \pmod{q} \\
  x \equiv 7 \pmod{8}
\end{cases} \tag{11}
\]

By Dirichlet’s theorem for primes in an arithmetic progression, there exist infinitely many primes \( w \) satisfying \( w \equiv x_0 \pmod{8pq} \). We choose such a prime number \( w \). Then

\[
\left( \frac{2}{w} \right) = 1 \text{ since } w \equiv 7 \pmod{8}.
\]

Also, by the quadratic reciprocity law, we get

\[
\left( \frac{p}{w} \right) = \left( \frac{w}{p} \right) = \left( \frac{x_0}{p} \right) = \left( \frac{p_1}{p} \right) = -1.
\]

and

\[
\left( \frac{q}{w} \right) = \left( \frac{w}{q} \right) = \left( \frac{x_0}{q} \right) = \left( \frac{p_2}{p} \right) = -1.
\]

We take \( u = w \) and clearly it fulfills all our requirements. This completes the proof. \( \square \)
There are numerous examples of quartic fields of the form $K = \mathbb{Q}(\sqrt{q}, \sqrt{kr})$ with class number 2. We list some quartic fields of our interest and its class number. The authors have computed the class number of such fields using Sage.

Now, we list the class number of biquadratic fields of the form $\mathbb{Q}(\sqrt{2}, \sqrt{pq})$. 



| $(q,k,r)$ | $h_K$ | $(q,k,r)$ | $h_K$ | $(q,k,r)$ | $h_K$ | $(q,k,r)$ | $h_K$ | $(q,k,r)$ | $h_K$ |
|-----------|-------|-----------|-------|-----------|-------|-----------|-------|-----------|-------|
| $(3, 5, 13)$ | 2 | $(3, 5, 17)$ | 2 | $(3, 5, 29)$ | 8 | $(3, 5, 37)$ | 2 | $(3, 5, 41)$ | 4 |
| $(3, 5, 53)$ | 4 | $(3, 5, 61)$ | 4 | $(3, 5, 73)$ | 4 | $(3, 5, 89)$ | 4 | $(3, 5, 97)$ | 4 |
| $(3, 5, 101)$ | 8 | $(3, 5, 109)$ | 4 | $(3, 5, 113)$ | 2 | $(3, 5, 137)$ | 6 | $(3, 5, 149)$ | 4 |
| $(3, 5, 157)$ | 6 | $(3, 5, 173)$ | 4 | $(3, 5, 181)$ | 8 | $(3, 5, 193)$ | 4 | $(3, 5, 197)$ | 12 |
| $(3, 5, 229)$ | 4 | $(3, 13, 5)$ | 2 | $(3, 13, 17)$ | 4 | $(3, 13, 29)$ | 4 | $(3, 13, 37)$ | 4 |
| $(3, 13, 41)$ | 6 | $(3, 13, 53)$ | 8 | $(3, 13, 61)$ | 16 | $(3, 13, 73)$ | 4 | $(3, 13, 89)$ | 2 |
| $(3, 13, 97)$ | 4 | $(3, 13, 101)$ | 4 | $(3, 13, 109)$ | 8 | $(3, 13, 113)$ | 4 | $(3, 13, 137)$ | 2 |
| $(3, 13, 149)$ | 6 | $(3, 13, 157)$ | 8 | $(3, 13, 173)$ | 4 | $(3, 13, 181)$ | 8 | $(3, 13, 193)$ | 4 |
| $(3, 13, 197)$ | 2 | $(3, 13, 229)$ | 8 | $(3, 17, 5)$ | 2 | $(3, 17, 13)$ | 4 | $(3, 17, 29)$ | 2 |
| $(3, 17, 37)$ | 2 | $(3, 17, 41)$ | 12 | $(3, 17, 53)$ | 12 | $(3, 17, 61)$ | 2 | $(3, 17, 73)$ | 4 |
| $(3, 17, 89)$ | 4 | $(3, 17, 97)$ | 4 | $(3, 17, 101)$ | 4 | $(3, 17, 109)$ | 2 | $(3, 17, 113)$ | 4 |
| $(3, 17, 137)$ | 4 | $(3, 17, 149)$ | 8 | $(3, 17, 157)$ | 8 | $(3, 17, 173)$ | 6 | $(3, 17, 181)$ | 2 |
| $(3, 17, 193)$ | 12 | $(3, 17, 197)$ | 2 | $(3, 17, 229)$ | 4 | $(3, 29, 5)$ | 4 | $(3, 29, 13)$ | 2 |
| $(3, 29, 73)$ | 4 | $(3, 29, 89)$ | 6 | $(3, 29, 97)$ | 4 | $(3, 29, 101)$ | 4 | $(3, 29, 109)$ | 4 |
| $(3, 29, 113)$ | 2 | $(3, 29, 137)$ | 6 | $(3, 29, 149)$ | 20 | $(3, 29, 157)$ | 2 | $(3, 29, 173)$ | 6 |

| $(2,k,r)$ | $h_K$ | $(2,k,r)$ | $h_K$ | $(2,k,r)$ | $h_K$ | $(2,k,r)$ | $h_K$ | $(2,k,r)$ | $h_K$ |
|-----------|-------|-----------|-------|-----------|-------|-----------|-------|-----------|-------|
| $(2, 5, 13)$ | 4 | $(2, 5, 17)$ | 2 | $(2, 5, 29)$ | 4 | $(2, 5, 37)$ | 2 | $(2, 5, 41)$ | 4 |
| $(2, 5, 53)$ | 4 | $(2, 5, 61)$ | 2 | $(2, 5, 73)$ | 6 | $(2, 5, 89)$ | 4 | $(2, 5, 97)$ | 2 |
| $(2, 5, 101)$ | 4 | $(2, 5, 109)$ | 6 | $(2, 5, 113)$ | 4 | $(2, 5, 137)$ | 4 | $(2, 5, 149)$ | 2 |
| $(2, 5, 157)$ | 6 | $(2, 5, 173)$ | 2 | $(2, 5, 181)$ | 4 | $(2, 5, 193)$ | 2 | $(2, 5, 197)$ | 12 |
| $(2, 5, 229)$ | 4 | $(2, 13, 5)$ | 4 | $(2, 13, 17)$ | 4 | $(2, 13, 29)$ | 2 | $(2, 13, 37)$ | 2 |
| $(2, 13, 41)$ | 4 | $(2, 13, 53)$ | 4 | $(2, 13, 61)$ | 4 | $(2, 13, 73)$ | 2 | $(2, 13, 89)$ | 2 |
| $(2, 13, 97)$ | 2 | $(2, 13, 101)$ | 8 | $(2, 13, 109)$ | 2 | $(2, 13, 113)$ | 4 | $(2, 13, 137)$ | 4 |
| $(2, 13, 149)$ | 6 | $(2, 13, 157)$ | 2 | $(2, 13, 173)$ | 4 | $(2, 13, 181)$ | 6 | $(2, 13, 193)$ | 2 |
| $(2, 13, 197)$ | 2 | $(2, 13, 229)$ | 12 | $(2, 17, 5)$ | 2 | $(2, 17, 13)$ | 4 | $(2, 17, 29)$ | 2 |
| $(2, 17, 37)$ | 2 | $(2, 17, 41)$ | 12 | $(2, 17, 53)$ | 4 | $(2, 17, 61)$ | 2 | $(2, 17, 73)$ | 4 |
| $(2, 17, 89)$ | 8 | $(2, 17, 97)$ | 8 | $(2, 17, 101)$ | 4 | $(2, 17, 109)$ | 6 | $(2, 17, 113)$ | 4 |
| $(2, 17, 137)$ | 8 | $(2, 17, 149)$ | 8 | $(2, 17, 157)$ | 8 | $(2, 17, 173)$ | 6 | $(2, 17, 181)$ | 2 |
| $(2, 17, 193)$ | 24 | $(2, 17, 197)$ | 2 | $(2, 17, 229)$ | 16 | $(2, 29, 5)$ | 4 | $(2, 29, 13)$ | 2 |
### Acknowledgements

We express our heartfelt gratitude towards Prof. R. Thangadurai for his encouragement throughout this project and valuable comments to improve the quality of the paper. We sincerely thank Dr. Hester Graves for going through the manuscript meticulously several times and her suggestions which greatly improved the presentation of the paper. We are also thankful to Prof. K. Srinivas and Prof. T.R. Ramadas for their careful reading of an earlier version of this manuscript and for their useful comments. The second author would like to thank the Roman Number Theory Association and Prof. Francesco Pappalardi for their financial support to visit the Università Roma Tre where this work was initiated. We also thank Mr. R. Dixit for his valuable suggestions regarding the computational part. We are grateful to the Dept. of Atomic Energy, Govt. of India and Harish-Chandra Research Institute for providing financial support to carry out this research.

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