Mean Field Games with Major and Minor Agents and Conditional Distribution Dependent FBSDEs

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Abstract

In this paper, we consider mean field games (MFGs) with a major and \( N \) minor agents. We first consider the limiting problem and allow the state coefficients to vary with the conditional distribution in a nonlinear way. We use the sufficient Pontryagin principle for optimality to transform the limiting control problem into a system of two coupled conditional distribution dependent forward–backward stochastic differential equations (FBSDEs), and prove the existence and uniqueness of solutions of the FBSDEs when the dependence between major agent and minor agents is sufficiently weak and the convexity parameter of the running cost of minor agents on the control is sufficiently large. A weak monotonicity property is required for minor agents’ cost functions and the proof is based on a continuation method in coefficients. We then consider the equilibrium property of MFG with major and minor agents and use the solution of the limit problem to construct an \( O(N^{-\frac{1}{2}}) \)-Nash equilibrium for MFG with a major and \( N \) minor agents.

Keywords: mean field games, major and minor agents, Nash equilibrium, McKean-Vlasov, FBSDE.

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1 Introduction

Stochastic games are widely used in economic, engineering and social science applications, and the notion of Nash equilibrium is one of the most prevalent notions of equilibrium used in their analyses. Searching for a Nash equilibrium for a \( N \)-player game is known to be intractable when \( N \) is large due to the high dimensionality. However, in view of the theory of McKean–Vlasov limits and propagation of chaos for uncontrolled weakly interacting particle systems [23], one may expect to obtain a convergence result by assuming independence of the random noise in the players’ state processes, symmetry of the cost functions and a mean-field interaction, and taking \( N \to \infty \). This limiting system is more tractable and one can use its solution to approximate Nash equilibrium of the finite-player games. With this heuristic in mind, Lasry and Lions [17, 18, 19] initiated the theory of mean field games (MFGs) for a type of games in which all the players are statistically identical, and only interact through their empirical distributions. They successfully identify the limiting problem and reduced it to solving a fully-coupled system of forward-backward partial differential equations. MFGs were proposed independently by Huang, Caines and Malhamé [12], under the different name of Nash Certainty Equivalence. The literature in this area is huge. See [3].
for a summary of a series of Lions’ lectures given at the Collège de France. Carmona and Delarue [4, 5, 6] approached the MFG problem from a probabilistic point of view. There are rigorous results about construction of \( \varepsilon \)-Nash equilibria for \( N \)-player games. See for example [5, 7, 13, 15, 16]. For some results about MFGs with common noises, we refer to [12, 14].

An important class of MFGs is that of dynamic games with one major agent and a large number of minor agents, which was first introduced by Huang [11]. He considered a linear-quadratic infinite-horizon model in which the influence of major agent will not fade away when the number of minor agents tends to infinity. Nourian and Caines [20] generalized this model to the nonlinear case using dynamic programming principle. Carmona and Zhu [8] developed a systematic scheme to find approximate Nash equilibria for the finite-player games using a probabilistic approach. There are two major issues of interest in MFGs with major and minor agents: (I) solvability of the limiting problem of MFG with major and minor agents, (II) convergence result for approximate Nash equilibria for \((N + 1)\)-player game as \( N \to \infty \). The goal of this paper is to answer both questions (I) and (II) for our model.

First we discuss issue (I). Nourian and Caines [20] studied MFGs with major and minor agents assuming that the diffusion of the major agent’s system is independent of all the states of the major and \( N \) minor agents, and the noise of the major agent does not appear in the state dynamic of minor agents. Carmona and Zhu [8] used the stochastic maximum principle to reduce the solution of MFG with major and minor agents to a forward–backward system of stochastic differential equations (FBSDEs) of the conditional McKean–Vlasov type. However, they only proved the solvability of the linear quadratic setting, in which the diffusions of all the agents are assumed to be constants. In this paper, we allow the state coefficients to vary with the conditional distribution in a nonlinear way and allow the noise of the major agent to appear in the state dynamics of all minor agents. We allow the diffusion of an agent’s state dynamic to depend on its own state, control, and the conditional distribution of all minor agents’ state. We use the sufficient Pontryagin principle for optimality to transform the limiting control problem into a system of two coupled conditional distribution dependent FBSDEs (see Theorem 3.3). Our main result is the existence and uniqueness of solutions of the FBSDEs when the dependence between major agent and minor agents is sufficiently weak, and the convexity parameter of the running cost of minor agents on the control is sufficiently large or the dependence of minor agents’ state coefficients on the conditional distribution is sufficiently weak (see Theorem 4.2). A weak monotonicity property is required for minor agents’ cost functions. The proof of the existence and uniqueness of solutions of the FBSDEs is based on a continuation method in coefficients, which was developed by Hu and Peng [10]. In this line of studies, Carmona and Delarue [4] proved the solvability only for a linear case where the state distribution appears as an expectation. Ahuja et al. [2] showed the uniquely sovability for a linear-convex setting with a common noise where the state dynamic is independent of the distribution of state. Huang and Tang [14] proved the uniquely sovability of the FBSDE with state coefficients varying with the conditional distribution in a nonlinear way under convexity and weak monotonicity assumptions.

Next we discuss issue (II). Nourian and Caines [20] constructed a \( \mathcal{O}(N^{-\frac{1}{2}}) \)-Nash equilibrium for MFG with a major and \( N \) minor agents for the case that controls of minor agents are adapted to the natural filtration of the major agent. Carmona et al. [12, 13] proved that the optimal control of the McKean-Vlasov dynamics forms an \( \mathcal{O}(N^{-\frac{1}{2}}) \)-Nash equilibrium using results of propagation of chaos, where \( d \) is the dimension of state space. In this paper, we show that the solution of the limit problem can provide an \( \mathcal{O}(N^{-\frac{1}{2}}) \)-Nash equilibrium for MFG with a major and \( N \) minor agents.
where control of the $i$-th agent is adapted to the natural filtration generated by his/her own noise and major agent’s noise (see Theorem 3.2).

The paper is organized as follows. In Section 2, we introduce our model and state the main two problems. In Section 3 we use the sufficient condition of the Pontryagin principle for optimality to transform the limiting control problem into a system of two coupled conditional distribution dependent FBSDEs. The existence and uniqueness result of the coupled conditional distribution dependent FBSDEs is proved in Sections 4. Finally, in Section 5 we use the solution of the limit problem to construct an $O(N^{-\frac{1}{2}})$-Nash equilibrium for MFG with a major and $N$ minor agents.

1.1 Notations

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, 0 \leq t \leq T\}, \mathbb{P})$ denote a complete filtered probability space augmented by all the $\mathbb{P}$-null sets. Let $\mathcal{F}^0$ be a subfiltration of $\mathcal{F}$. $\mathcal{L}(\cdot|\mathcal{F}_t^0)$ is the law conditioned at $\mathcal{F}_t^0$ for $t \in [0, T]$. Let $\mathcal{L}^2_{\mathcal{F}_t}$ denote the set of all $\mathcal{F}_t$-measurable square-integrable $\mathbb{R}$-valued random variables. Let $\mathcal{L}^2_{\mathcal{F}_t}(0, T)$ denote the set of all $\mathcal{F}_t$-progressively-measurable $\mathbb{R}$-valued processes $\alpha = \{\alpha_t, 0 \leq t \leq T\}$ such that $\mathbb{E}[\int_0^T |\alpha_t|^2 dt] < +\infty$. Let $\mathcal{S}^2_{\mathcal{F}_t}(0, T)$ denote the set of all $\mathcal{F}_t$-progressively-measurable $\mathbb{R}$-valued processes $\alpha = \{\alpha_t, 0 \leq t \leq T\}$ such that $\mathbb{E}[\sup_{0 \leq t \leq T} |\alpha_t|^2] < +\infty$. Let $\mathcal{P}(\mathbb{R})$ denote the space of all Borel probability measures on $\mathbb{R}$, and $\mathcal{P}_2(\mathbb{R})$ the space of all probability measures $m \in \mathcal{P}(\mathbb{R})$ such that

$$|m|_2 := \left( \int_{\mathbb{R}} x^2 dm(x) \right)^{\frac{1}{2}} < \infty.$$ 

The Wasserstein distance is defined on $\mathcal{P}_2(\mathbb{R})$ by

$$W_2(m_1, m_2) = \left( \inf_{\gamma \in \Gamma(m_1, m_2)} \int_{\mathbb{R}^2} |x(\omega_1) - x(\omega_2)|^2 d\gamma(\omega_1, \omega_2) \right)^{\frac{1}{2}}, \quad m_1, m_2 \in \mathcal{P}_2(\mathbb{R}),$$

where $\Gamma(m_1, m_2)$ denotes the collection of all probability measures on $\mathbb{R}^2$ with marginals $m_1$ and $m_2$. The space $(\mathcal{P}_2(\mathbb{R}), W_2)$ is a complete separable metric space. Let $\mathcal{P}^2_{\mathcal{F}_0}(0, T)$ denote the set of all $\mathcal{F}^0_t$-progressively-measurable stochastic flow of probability measures $m = \{m_t, 0 \leq t \leq T\}$ such that $\mathbb{E}[\sup_{0 \leq t \leq T} |m_t|_2^2] < +\infty$.

2 Problem Formulation

Let $T > 0$ be a fixed terminal time, $W^i = \{W^i_t, 0 \leq t \leq T\}$, $i = 0, 1, 2, \ldots, N$ are one-dimensional independent Brownian motions defined on a complete filtered probability space $(\Omega, \mathbb{P})$ satisfying the usual conditions. Consider a stochastic dynamic game with a major agent and $N$ minor agents. The major agent regulates his/her own state process $X^0_t$ governed by

$$X^0_t = \xi^0 + \int_0^t b^0(s, X^0_s, u^0_s, m^N_s) ds + \int_0^t \sigma^0(s, X^0_s, u^0_s, m^N_s) dW^0_s, \quad t \in [0, T],$$

via the control process $u^0 \in \mathcal{L}^2_{\mathcal{F}_0}(0, T)$, and the $i$-th minor agent regulates his/her own state process $X^i_t$ governed by

$$X^i_t = \xi^i + \int_0^t b(s, X^i_s, u^i_s, m^N_s) ds + \int_0^t \sigma(s, X^i_s, u^i_s, m^N_s) dW^i_s + \int_0^t \tilde{\sigma}(s, X^i_s, u^i_s, m^N_s) dW^0_s, \quad t \in [0, T],$$

where $\tilde{\sigma}(s, X^i_s, u^i_s, m^N_s) = \sigma(s, X^i_s, u^i_s, m^N_s) - \sigma^0(s, X^0_s, u^0_s, m^N_s)$ is the control drift of the minor agent with respect to the major agent.

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via the control process \( u^i \in \mathcal{L}^2_{\mathcal{F}^i}(0,T) \), where \((b^0, \sigma^0, b, \sigma, \bar{\sigma}) : [0,T] \times \mathbb{R} \times \mathcal{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}^5\), \( \mathcal{F}^0 \) is the natural filtration of \((\xi^0, W^0)\) and \( \mathcal{F}^i \) is the natural filtration of \((\xi^0, \xi^i, W^0, W^i)\) for \(1 \leq i \leq N\), and \( m^N_t \) is the empirical distribution of \( \{X^i_t, 1 \leq i \leq N\} \), i.e.,

\[
m^N_t = \frac{1}{N} \sum_{i=1}^N \delta_{X^i_t}(dx), \quad t \in [0,T].
\]

We assume that \(\{\xi^i, 0 \leq i \leq N\}\) be square integrable, independent, identically distributed and independent of all Brownian motions. Given minor agents’ strategies, the major agent selects a control \(u^0 \in \mathcal{L}^2_{\mathcal{F}^0}(0,T)\) to minimize his/her expected total cost given by

\[
J_0(u^0|u^{-0}) := \mathbb{E}\left[ \int_0^T f^0(t, X^0_t, u^0, m^N_t)dt + g^0(X^0_T, m^N_T) \right],
\]

where \(f^0 : [0,T] \times \mathbb{R} \times \mathcal{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}\) and \(g^0 : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}\). Given major agent’s strategy and the other minor agents’ strategies, the \(i\)-th agent selects a control \(u^i \in \mathcal{L}^2_{\mathcal{F}^i}(0,T)\) to minimize his/her expected total cost given by

\[
J_i(u^i|u^{-i}) := \mathbb{E}\left[ \int_0^T f(t, X^i_t, u^i, m^N_t, X^0_t)dt + g(X^i_T, m^N_T, X^0_T) \right],
\]

where \(u^{-i}\) dotes a strategy profile of other agents excluding the \(i\)-th agent, and \(f : [0,T] \times \mathbb{R} \times \mathcal{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}\), \(g : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}\) are assumed to be indentical for all minor agents. We are seeking a type of equilibrium solution widely used in game theory setting called Nash equilibrium.

**Definition 2.1.** We call a set of admissible strategies \(\{u^i, 0 \leq i \leq N\}\) a Nash equilibrium for \((N+1)\) agents, if \(u^i\) is optimal for the \(i\)-th agent given the other agents’ strategies \(\{u^j, j \neq i\}\). In other words,

\[
J_i(u^i|u^{-i}) = \min_{u \in \mathcal{L}^2_{\mathcal{F}^i}(0,T)} J_i(u|u^{-i}), \quad 0 \leq i \leq N.
\]

Given \(\epsilon > 0\), we call a set of admissible strategies \(\{u^i, 0 \leq i \leq N\}\) an \(\epsilon\)-Nash equilibrium, if

\[
J_i(u^i; u^{-i}) - \epsilon \leq \inf_{u \in \mathcal{L}^2_{\mathcal{F}^i}(0,T)} J_i(u; u^{-i}) \quad 0 \leq i \leq N.
\]

One of the main problems of our paper is stated as follows.

**Problem 2.1.** Find an \(\epsilon\)-Nash equilibrium for MFGs with a major and \(N\) minor agents.

Solving for a Nash equilibrium of an \((N+1)\)-player game is impractical when \(N\) is large due to high dimensionality. So we formally take the limit as \(N \to \infty\) and consider the limiting problem instead. We assume that each minor agent adopts the same strategy. Therefore, minor agents’ distribution can be represented by a law of a single representative minor agent conditional on \(\mathcal{F}^0\). Let \(W = \{W_t, 0 \leq t \leq T\}\) be a one-dimensional independent Brownian motion and \(\xi\) a square integrable random variable, which are independent of \(\mathcal{F}^0\). Let \(\mathcal{F}\) be the natural filtration of \((\xi^0, \xi, W^0, W)\) augmented by \(\mathbb{P}\)-null sets. The limit problem of MFGs with major and minor agents is defined as follows.
Problem 2.2. Find an optimal control \((\hat{u}^0, \hat{u}) \in \mathcal{L}_F^2(0, T) \times \mathcal{L}_F^2(0, T)\) for the stochastic control problem

\[
\begin{aligned}
\hat{u}^0 \in \argmin_{u^0 \in \mathcal{L}_F^2(0, T)} J_0(u^0|m) := \mathbb{E}\left[ \int_0^T f^0(t, X_t^0, u_t^0, m_t) dt + g^0(X_T^0, m_T) \right], \\
\hat{u} \in \argmin_{u \in \mathcal{L}_F^2(0, T)} J(u|\hat{u}^0, m) := \mathbb{E}\left[ \int_0^T f(t, X_t^u, u_t, m_t, X_t^{u^0}) dt + g(X_T^u, m_T, X_T^{u^0}) \right]; \\
X_t^{u^0} = \xi^0 + \int_0^t b^0(s, X_s^{u^0}, u_s^0, m_s) ds + \int_0^t \sigma^0(s, X_s^{u^0}, u_s^0, m_s) dW_s^0, \quad t \in [0, T], \\
X_t^u = \xi + \int_0^t b(s, X_s^u, u_s, m_s) ds + \int_0^t \sigma(s, X_s^u, u_s, m_s) dW_s^0 + \int_0^t \tilde{\sigma}(s, X_s^u, u_s, m_s) dW^0_s, \quad t \in [0, T]; \\
m_t = \mathcal{L}(X_t^\hat{u}|\mathcal{F}_t^0), \quad t \in [0, T].
\end{aligned}
\]

3 Stochastic Maximum Principle

In this section, we discuss the stochastic maximum principle for Problem 2.2. The stochastic maximum principle gives optimality conditions satisfied by an optimal control. It gives sufficient and necessary conditions for the existence of an optimal control in terms of solvability of the adjoint process as a backward stochastic differential equation (BSDE). For more details about stochastic maximum principle, we refer to [22] or [24].

We first state our assumptions of this section. For notational convenience, we use the same constant \(L \geq 1\) for all the conditions below, and assume that \(\max\{L_m, l_m, l_{x^0}\} \leq L\).

(H1) The functions \((b^0, \sigma^0, b, \sigma, \tilde{\sigma})\) are linear in \(x, u\) and \(t\). That is,

\[
\phi(t, x, u, m) = \phi_0(t, m) + \phi_1(t) x + \phi_2(t) u, \quad \phi = (b^0, \sigma^0, b, \sigma, \tilde{\sigma}), \quad \phi_i = (b_i^0, \sigma_i^0, b_i, \sigma_i, \tilde{\sigma}_i),
\]

with \(\phi_0\) growing linearly in \(m\) and \((\phi_1, \phi_2)\) being bounded by \(L\). The functions \((b_0, \sigma_0, \tilde{\sigma}_0)\) are \(L_m\)-Lipschitz continuous in \(m\), and the functions \((b_0^0, \sigma_0^0)\) are \(L_m\)-Lipschitz continuous in \(m\).

(H2) The functions \((f^0, g^0)(t, 0, 0, m)\) satisfy a quadratic growth condition in \(m\). The functions \((f^0, g^0)\) are differentiable in \((x^0, u^0)\), with the derivatives growing linearly in \((x^0, u^0)\), being \(L\)-Lipschitz continuous in \((x^0, u^0)\) and being \(l_m\)-Lipschitz continuous in \(m\). The functions \((f, g)(t, 0, 0, m, x^0)\) satisfy a quadratic growth condition in \((m, x^0)\). The function \(f\) is of the form

\[
f(t, x, u, m, x^0) = f^1(t, x, u, x^0) + f^2(t, x, m, x^0), \quad (t, x, u, m, x^0) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}.
\]

The functions \((f^1, f^2, g)\) are differentiable in \((x, u)\), with the derivatives growing linearly in \((x, u, m, x^0)\), being \(L\)-Lipschitz continuous in \((x, u, m)\) and being \(l_{x^0}\)-Lipschitz continuous in \(x^0\).

(H3) The function \(g^0\) is convex in \(x^0\) and the function \(f^0\) is jointly convex in \((x^0, u^0)\) with a strict convexity in \(u^0\), in such a way that, for some \(C_{f^0} \geq L^{-1}\),

\[
f^0(t, x_2^0, u_2^0, m) - f^0(t, x_1^0, u_1^0, m) - (f^0_x(t, x_1^0, u_1^0, m)) (x_2^0 - x_1^0, u_2^0 - u_1^0) \geq C_{f^0} |u_2^0 - u_1^0|^2,
\]

\((t, m) \in [0, T] \times \mathcal{P}_2(\mathbb{R}), \quad x_1^0, x_2^0, u_1^0, u_2^0 \in \mathbb{R}.
\]

The functions \((f^2, g)\) are convex in \(x\) and the function \(f_1\) is jointly convex in \((x, u)\) with a strict convexity in \(u\), in such a way that, for some \(C_f \geq L^{-1}\),

\[
f_1(t, x_2, u_2, x^0) - f_1(t, x_1, u_1, x^0) - (f^1_x(t, x_1, u_1, x^0)) (x_2 - x_1, u_2 - u_1) \geq C_f |u_2 - u_1|^2,
\]

\((t, x^0) \in [0, T] \times \mathbb{R}, \quad x_1, x_2, u_1, u_2 \in \mathbb{R}.
\]
We begin with discussing the maximum principle of stochastic control problems $\mathbf{P}^m$ and $\mathbf{P}^{X_0,m}$.

Given $m \in \mathcal{P}_2(0, T)$, $\mathbf{P}^m$ is defined as

$$
\begin{cases}
\hat{u}^0 \in \arg\min_{u^0 \in \mathcal{L}_2^{\mathcal{F}_0}(0,T)} \mathbb{E} \int_0^T f^0(t, X^0_t, u^0_t, m_t)dt + g^0(X^0_T, u^0_T, m_T), \\
X^0_t, u^0_t = \xi^0 + \int_0^t b^0(s, X^0_s, u^0_s, m_s)ds + \int_0^t \sigma^0(s, X^0_s, u^0_s, m_s)dW^0_s, \quad t \in [0, T].
\end{cases}
$$

Given $(m, X^0) \in \mathcal{P}_2^2(0, T) \times \mathcal{S}_2(0, T)$, $\mathbf{P}^{X_0,m}$ is defined as

$$
\begin{cases}
\hat{u} \in \arg\min_{u \in \mathcal{L}_2^{\mathcal{F}_0}(0,T)} \mathbb{E} \int_0^T f(t, X^u_t, u_t, m_t, X^0_t)dt + g(X^u_T, m_T, X^0_T); \\
X^u_t = \xi + \int_0^t b(s, X^u_s, u_s, m_s)ds + \int_0^t \sigma(s, X^u_s, u_s, m_s)dW_s + \int_0^t \hat{\sigma}(s, X^u_s, u_s, m_s)dW^0_s, \quad t \in [0, T].
\end{cases}
$$

We define the generalized Hamiltonians

$$
\begin{align*}
H^0(t, x^0, p^0, q^0, u^0, m) &= b^0(t, x^0, u^0, m)p^0 + \sigma^0(t, x^0, u^0, m)q^0 + f^0(t, x^0, u^0, m), \\
H(t, x, p, q, \tilde{q}, u, m, x^0) &= b(t, x, u, m)p + \sigma(t, x, u, m)q + \tilde{\sigma}(t, x, u, m)\tilde{q} + f(t, x, u, m, x^0),
\end{align*}
$$

and the minimizing control functions

$$
\begin{align*}
\hat{u}^0(t, x^0, p^0, q^0, m) := \arg\min_{u^0 \in \mathbb{R}} H^0(t, x^0, p^0, q^0, u^0, m), \\
(t, x^0, p^0, q^0, m) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}), \\
\hat{u}(t, x, p, q, \tilde{q}, u, m, x^0) := \arg\min_{u \in \mathbb{R}} H(t, x, p, q, \tilde{q}, u, m, x^0), \\
(t, x, p, q, \tilde{q}, u, m, x^0) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}.
\end{align*}
$$

Assumptions (H1) and (H3) ensure that the Hamiltonians $H^0$ and $H$ are strictly convex in $u^0$ and $u$, respectively, so that there is a unique minimizer in the feedback form. The separability condition of $f$ in Assumption (H2) ensures that $\hat{u}$ is independent of $m$. The following result is borrowed from [5] Lemma 1.

**Lemma 3.1.** Let Assumptions (H1)-(H3) be satisfied. Then, the function $\hat{u}^0$ is measurable, locally bounded, $L(2C_{l^0})^{-1}$-Lipschitz continuous in $(x^0, p^0, q^0)$ and $l_{m}(2C_{l^0})^{-1}$-Lipschitz continuous in $m$; and the function $\hat{u}$ is measurable, locally bounded, $L(2C_f)^{-1}$-Lipschitz-continuous in $(x, p, q, \tilde{q})$ and $l_{x^0}(2C_f)^{-1}$-Lipschitz continuous in $x^0$.

We have the following solvability of $\mathbf{P}^m$ and $\mathbf{P}^{X_0,m}$.

**Lemma 3.2.** Let Assumptions (H1)-(H3) be satisfied. Given $m \in \mathcal{P}_2^2(0, T)$, $\hat{u}^0 = \{\hat{u}^0(t, X^0_t, p^0_t, q^0_t, m_t), 0 \leq t \leq T\}$ is the unique optimal control of $\mathbf{P}^m$, where $(X^0, p^0, q^0) \in \mathcal{S}_2^2(0, T) \times \mathcal{L}_2^2(0, T)$ is the solution of the following FBSDE

$$
\begin{align*}
&dX^0_t = b^0(t, X^0_t, \hat{u}^0(t, X^0_t, p^0_t, q^0_t, m_t), m_t)dt + \sigma^0(t, X^0_t, \hat{u}^0(t, X^0_t, p^0_t, q^0_t, m_t), m_t)dW^0_t, \quad t \in (0, T]; \\
dp^0_t = -H^0_0(t, X^0_t, p^0_t, q^0_t, \hat{u}^0(t, X^0_t, p^0_t, q^0_t, m_t), m_t)dt + q^0_tdW^0_t, \quad t \in [0, T); \\
X^0_0 = \xi^0, \quad p^0_T = \beta^0_0(X^0_T, m_T).
\end{align*}
$$

(1)
Given \((m, X^0) \in \mathcal{P}_T^2(0, T) \times \mathcal{S}_T^2(0, T)\), \(\hat{u} = \{\hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), 0 \leq t \leq T\}\) is the unique optimal control of \(\mathbf{P}^{X^0, m}\), where \((X, p, q, \hat{q}) \in \mathcal{S}_T^2(0, T) \times \mathcal{S}_T^2(0, T) \times \mathcal{L}_T^2(0, T) \times \mathcal{L}_T^2(0, T)\) is the solution of the following FBSDE
\[
\begin{aligned}
 dX_t &= b(t, X_t, \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), m_t)dt + \sigma(t, X_t, \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), m_t)dW_t \\
 &\quad + \hat{\sigma}(t, X_t, \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), m_t)dW_t^0, \quad t \in (0, T]; \\
 dp_t &= -H_x(t, X_t, p_t, q_t, \hat{q}_t, \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), m_t, X_t^0)dt + q_t dW_t + \hat{q}_t dW_t^0, \quad t \in [0, T]; \\
 X_0 &= \xi, \quad p_T = g_x(X_T, m_T, X_T^0).
\end{aligned}
\]

\textit{Proof.} The stochastic maximum principle for \(\mathbf{P}^m\) and \(\mathbf{P}^{X^0, m}\) is a direct consequence of \([14, \text{Theorem 3.1}]\), and the existence and uniqueness of solutions of FBSDEs \([11\text{ and } 12]\) is an immediate consequence of \([21, \text{Theorem 2.3}]\). \(\square\)

Now we turn to Problem \[2.2\]. Given \(m \in \mathcal{P}_T^2(0, T)\), we denote by \(\hat{u}^0\) the optimal control of \(\mathbf{P}^m\) and \(\hat{X}^0\) the corresponding state process. We denote by \(\hat{u}\) the optimal control of \(\mathbf{P}^{\hat{X}^0, m}\) and \(\hat{X}\) the corresponding state process. The definition of Problem \[2.2\] states that, \((\hat{u}^0, \hat{u})\) is an optimal control of Problem \[2.2\] if the following consistency condition is satisfied
\[
m_t = \mathcal{L}(\hat{X}_t|\mathcal{F}_t^0), \quad t \in [0, T].
\]

Therefore, we have the following stochastic maximum principle for Problem \[2.2\].

\textbf{Theorem 3.3.} Let Assumptions (H1)-(H3) be satisfied. If the following FBSDE
\[
\begin{aligned}
 dX^0_t &= b^0(t, X^0_t, \hat{u}^0(t, X^0_t, p^0_t, q^0_t, \mathcal{L}(X_t|\mathcal{F}_t^0)), \mathcal{L}(X_t|\mathcal{F}_t^0))dt \\
 &\quad + \sigma^0(t, X^0_t, \hat{u}^0(t, X^0_t, p^0_t, q^0_t, \mathcal{L}(X_t|\mathcal{F}_t^0)), \mathcal{L}(X_t|\mathcal{F}_t^0))dW_t^0, \quad t \in (0, T]; \\
 dp^0_t &= -H^0_x(t, X^0_t, p^0_t, q^0_t, \hat{u}^0(t, X^0_t, p^0_t, q^0_t, \mathcal{L}(X_t|\mathcal{F}_t^0)), \mathcal{L}(X_t|\mathcal{F}_t^0))dt + q^0_t dW_t^0, \quad t \in [0, T]; \\
 dX_t &= b(t, X_t, \hat{u}^0(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), \mathcal{L}(X_t|\mathcal{F}_t^0))dt \\
 &\quad + \sigma(t, X_t, \hat{u}^0(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), \mathcal{L}(X_t|\mathcal{F}_t^0))dW_t \\
 &\quad + \hat{\sigma}(t, X_t, \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), \mathcal{L}(X_t|\mathcal{F}_t^0))dW_t^0, \quad t \in (0, T]; \\
 dp_t &= -H_x(t, X_t, p_t, q_t, \hat{q}_t, \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), \mathcal{L}(X_t|\mathcal{F}_t^0), X_t^0)dt \\
 &\quad + q_t dW_t + \hat{q}_t dW_t^0, \quad t \in [0, T]; \\
 X^0_0 &= \xi^0, \quad p^0_T = g_x(X_T, \mathcal{L}(X_T|\mathcal{F}_T^0), X_T^0), \quad X_0 = \xi, \quad p_T = g_x(X_T, \mathcal{L}(X_T|\mathcal{F}_T^0), X_T^0)
\end{aligned}
\]
has an solution \((X^0, p^0, q^0, X, p, q, \hat{q}) \in \mathcal{S}_T^2(0, T) \times \mathcal{S}_T^2(0, T) \times \mathcal{L}_T^2(0, T) \times \mathcal{S}_T^2(0, T) \times \mathcal{L}_T^2(0, T) \times \mathcal{S}_T^2(0, T) \times \mathcal{L}_T^2(0, T) \times \mathcal{L}_T^2(0, T)\), then, \((\hat{u}^0, \hat{u}) = \{(\hat{u}^0(t, X^0_t, p^0_t, q^0_t, \mathcal{L}(X_t|\mathcal{F}_t^0)), \hat{u}(t, X_t, p_t, q_t, \hat{q}_t, X_t^0), 0 \leq t \leq T\}\) is an optimal control of Problem \[2.2\].

4 Solvability of FBSDEs \([3]\)

In this section, we prove the existence and uniqueness of the solutions of FBSDE \([3]\) by the method of continuation in coefficients. We always suppose that assumptions (H1)-(H3) hold true. Moreover, we need the following weak monotonicity assumption:
(H4) For any $\gamma \in \mathcal{P}_2(\mathbb{R}^2)$ with marginals $m, m'$,

$$
\int_{\mathbb{R}^2} [(f_2^2(t, x, m, x^0) - f_2^2(t, y, m', x^0)) (x - y)] \gamma (dx, dy) \geq 0, \quad (t, x^0) \in [0, T], \ m, m' \in \mathcal{P}_2(\mathbb{R});
$$

$$
\int_{\mathbb{R}^2} [(g_x(x, m, x^0) - g_x(y, m', x^0)) (x - y)] \gamma (dx, dy) \geq 0, \quad m, m' \in \mathcal{P}_2(\mathbb{R}).
$$

Equivalently, for any square-integrable random variables $\xi$ and $\xi'$ on the same probability space,

$$
\mathbb{E} [(f_2^2(t, \xi', \mathcal{L}(\xi'), x^0) - f_2^2(t, \xi, \mathcal{L}(\xi), x^0))(\xi' - \xi)] \geq 0, \quad (t, x^0) \in [0, T] \times \mathbb{R};
$$

$$
\mathbb{E} [(g_x(\xi', \mathcal{L}(\xi'), x^0) - g_x(\xi, \mathcal{L}(\xi), x^0))(\xi' - \xi)] \geq 0, \quad x^0 \in \mathbb{R}.
$$

For notational convenience, we denote by $\theta^0$ a process $(X^0, u^0) \in \mathcal{S}^2_{\mathcal{F}_0}(0, T) \times \mathcal{L}^2_{\mathcal{F}_0}(0, T)$ and $\theta$ a process $\{(X_t, u_t, \mathcal{L}(X_t|\mathcal{F}_0^0))\}, 0 \leq t \leq T \} \in \mathcal{S}_{\mathcal{F}_T}(0, T) \times \mathcal{L}_{\mathcal{F}_T}(0, T) \times \mathcal{P}_{\mathcal{F}_T}(0, T)$. We denote by $\Theta^0$ a process $(\theta^0, p^0, q^0)$ with $(p^0, q^0) \in \mathcal{S}^2(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T)$, and $\Theta$ a process $(\theta, p, q, \tilde{q})$ with $(p, q, \tilde{q}) \in \mathcal{S}_T(0, T) \times \mathcal{L}_{\mathcal{F}_T}(0, T) \times \mathcal{L}_{\mathcal{F}_T}(0, T)$. We then denote by $\mathcal{S}$ the space of processes $(\Theta^0, \Theta)$ and

$$
\| (\Theta^0, \Theta) \|_\mathcal{S} := (\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(X^0_t, p^0_t, X_t, u_t)|^2 + \int_0^T |(u^0_t, q^0_t, u_t, q_t, \tilde{q}_t)|^2 dt \right])^{1/2} < +\infty.
$$

The strategy to prove the solvability of FBSDE [3] is to prove that the solvability are kept preserved when coefficients are slightly perturbed. We denote by $(\mathcal{I}^0, \mathcal{I})$ an input for FBSDE [3] with

$$
\mathcal{I}^0 = (\mathcal{I}^{\theta^0}, \mathcal{I}^{\sigma^0}, \mathcal{I}^0, \mathcal{I}^0_T) \in \mathcal{L}^2_{\mathcal{F}_0}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T),
$$

$$
\mathcal{I} = (\mathcal{I}^{\theta}, \mathcal{I}^{\sigma}, \mathcal{I}^1, \mathcal{I}^1_T) \in \mathcal{L}^2_{\mathcal{F}_0}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T) \times \mathcal{L}^2_{\mathcal{F}_T},
$$

and $\mathcal{I}$ the space of all inputs, endowed with the norm

$$
\| (\mathcal{I}^0, \mathcal{I}) \|_1 = (\mathbb{E} \left[ |(\mathcal{I}^{\theta^0}_T, \mathcal{I}^{\sigma^0}_T)|^2 + \int_0^T |(\mathcal{I}^{\theta}_T, \mathcal{I}^{\sigma}_T, \mathcal{I}^{\theta^0}_T, \mathcal{I}^{\sigma^0}_T)|^2 dt \right])^{1/2}.
$$

For any $(\gamma, \xi^0, \xi, \mathcal{I}^0, \mathcal{I}) \in [0, 1] \times \mathcal{L}^2_{\mathcal{F}_0} \times \mathcal{L}^2_{\mathcal{F}_T} \times \mathbb{I}$, we denote by $\mathcal{E}(\gamma, \xi^0, \xi, \mathcal{I}^0, \mathcal{I})$ the FBSDE

$$
\begin{align*}
\begin{cases}
    dX^0_t = (\gamma b^0(t, \theta^0_t, m_t) + \mathcal{I}^{\theta^0}_t) dt + (\gamma \sigma^0(t, \theta^0_t, m_t) + \mathcal{I}^{\sigma^0}_t) dW^0_t, & t \in (0, T]; \\
    dp^0_t = -(\gamma H^0_x(t, \theta^0_t, \mathcal{I}^{\theta^0}_t, m_t) + \mathcal{I}^{\sigma^0}_t) dt + q^0_t dW^0_t, & t \in [0, T); \\
    dX_t = (\gamma b(t, \theta_t) + \mathcal{I}^1_t) dt + (\gamma \sigma(t, \theta_t) + \mathcal{I}^2_t) dW_t + (\gamma \bar{\sigma}(t, \bar{\theta}_t) + \mathcal{I}^2_t) dW_t, & t \in (0, T]; \\
    dp_t = -(\gamma H_x(t, \theta_t, X^0_t) + \mathcal{I}^1_t) dt + q_t dW_t + \tilde{q}_t dW_t, & t \in [0, T), \\
    X^0_0 = \xi^0, \quad X_0 = \xi, \quad p^0_T = \gamma g_{x^0}(X^0_T, m^0_T) + \mathcal{I}^{\theta^0}_T, \quad p_T = \gamma g_x(X_T, m_T, X^0_T) + \mathcal{I}^{\theta}_T, \\
    m_t = \mathcal{L}(X_t|\mathcal{F}^0_t), \quad u^0_t = \gamma u^0(t, X^0_t, p^0_t, \xi, m_t), \quad u_t = \gamma u(t, X_t, p_t, \xi, \tilde{q}_t, X^0_t), \quad t \in [0, T].
\end{cases}
\end{align*}
$$

Note that $\mathcal{E}(1, 0^0, 0, 0)$ coincides with FBSDE [3]. For $\gamma \in [0, 1]$, we say that property $(S_\gamma)$ holds true if, for any $(\xi^0, \xi, \mathcal{I}^0, \mathcal{I}) \in \mathcal{L}^2_{\mathcal{F}_0} \times \mathcal{L}^2_{\mathcal{F}_T} \times \mathbb{I}$, $\mathcal{E}(\gamma, \xi^0, \xi, \mathcal{I}^0, \mathcal{I})$ has a unique solution in $\mathcal{S}$. Our aim is to show $(S_1)$ holds true. We first the following lemma which is crucial in the proof of our main result Theorem 4.2.

**Lemma 4.1.** Let Assumptions (H1)-(H4) be satisfied and $\gamma \in [0, 1)$ such that $(S_\gamma)$ holds true. Then, there exist positive constants $(\delta, \eta_0)$ depending only on $(L, T)$, such that $(S_{\gamma + \eta})$ holds true for any $\eta \in (0, \eta_0 \wedge (1 - \gamma)]$ when $L_m C^{-1}_f \vee I_m \leq \delta$. 

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Theorem 4.2. Let Assumptions (H1)-(H4) be satisfied. Then, there exist positive constants \((\delta, C)\) depending only on \((L, T)\), such that for any \((\Theta_1^0, \Theta_1)\), \((\Theta_2^0, \Theta_2)\) \(\in \mathbb{S}\),

\[
\|\Phi(\Theta_2^0, \Theta_2) - \Phi(\Theta_1^0, \Theta_1)\|_\mathbb{S} \leq C\|\tilde{\Theta}_1^0 - \tilde{\Theta}_2^0, \tilde{\Theta}_2 - \tilde{\Theta}_1\|_1 \leq C\eta\|\Theta_2^0 - \Theta_1^0, \Theta_2 - \Theta_1\|_\mathbb{S},
\]

when \(L_mC_f^{-1} \vee l_{v1}l_m \leq \delta\). So when \(\eta\) is small enough, \(\Phi\) is a contraction.

Our main result of this section is stated as follows.

Theorem 4.2. Let Assumptions (H1)-(H4) be satisfied. Then, there exist \(\delta > 0\) depending only on \((L, T)\), such that FBSDE \((\text{3})\) is uniquely solvable when \(L_mC_f^{-1} \vee l_{v1}l_m \leq \delta\).

Proof. In view of Lemma 4.1, we only need to prove that \(\mathcal{S}_0\) holds true, which is obviously true. □

To complete the previous proof, it remains to state and prove the following lemma.

Lemma 4.3. Let Assumptions (H1)-(H4) be satisfied and \(\gamma \in [0, 1]\) such that \(\mathcal{S}_\gamma\) holds true. Then, there exist positive constants \((\delta, C)\) depending only on \((L, T)\), such that for any \((\xi_1^0, \xi_1, \mathcal{I}_1^0, \mathcal{I}_1)\), \((\xi_2^0, \xi_2, \mathcal{I}_2^0, \mathcal{I}_2) \in \mathcal{L}_{\mathcal{F}_0}^2 \times \mathcal{L}_{\mathcal{F}_0}^2 \times \mathbb{I}\), the solutions \((\Theta_1^0, \Theta_1)\) and \((\Theta_2^0, \Theta_2)\) of \(\mathcal{E}(\gamma, \xi_1^0, \xi_1, \mathcal{I}_1^0, \mathcal{I}_1)\) and \(\mathcal{E}(\gamma, \xi_2^0, \xi_2, \mathcal{I}_2^0, \mathcal{I}_2)\) satisfy

\[
\|\Theta_1^0 - \Theta_2^0, \Theta_1 - \Theta_2\|_\mathbb{S} \leq C(\mathbb{E}[|\xi_1^0 - \xi_2^0|^2 + |\xi_1 - \xi_2|^2] + ||\mathcal{I}_1^0 - \mathcal{I}_2^0, \mathcal{I}_1 - \mathcal{I}_2||_2^2)
\]

when \(L_mC_f^{-1} \vee l_{v1}l_m \leq \delta\).

Proof. We set \((\Delta \Theta_0, \Delta \Theta) = (\Theta_2^0 - \Theta_1^0, \Theta_2 - \Theta_1)\) and \((\Delta \mathcal{I}_0, \Delta \mathcal{I}) = (\mathcal{I}_2^0 - \mathcal{I}_1^0, \mathcal{I}_2 - \mathcal{I}_1)\). Then
From Assumption (H2), we have

From Assumptions (H1)-(H2) and standard estimates for SDEs and BSDEs, there exists

We note the fact that for

\( \Delta \Theta \) satisfy the following FBSDE

\[
\begin{aligned}
&d\Delta X^i_t = [\gamma((b^i_0(t,m^i_t) - b^i_0(t,m^i_1)) + b^i_2(t)\Delta X^i_t + b^i_2(t)\Delta u^i_0) + \Delta T^0_t]dt \\
&\quad + [\gamma(\sigma^i_0(t,m^i_t) - \sigma^i_0(t,m^i_1)) + \sigma^i_2(t)\Delta X^i_t + \sigma^i_2(t)\Delta u^i_0)]dW^i_t, \quad t \in (0,T]; \\
&d\Delta p^i_t = -[\gamma(b^i_1(t)\Delta p^i_t + \sigma^i_1(t)\Delta q^i_t + f^i_2(t,X^i_t, u^i_t, m^i_t, X^{i,0,2}_t) - f^i_2(t,X^i_t, u^i_t, m^i_t)) \\
&\quad + \Delta T^0_t]dt + \Delta q^i_t dW^i_t, \quad t \in [0,T); \\
&d\Delta X_t = [\gamma((b_0(t,m^i_t) - b_0(t,m^i_1)) + b_2(t)\Delta X_t + \Delta T^0_t]dt \\
&\quad + [\gamma(\sigma_0(t,m^i_t) - \sigma_0(t,m^i_1)) + \sigma_2(t)\Delta X_t + \Delta T^0_t]dW_t \\
&\quad + [\gamma(\sigma_0(t,m^i_t) - \sigma_0(t,m^i_1)) + \sigma_2(t)\Delta X_t + \Delta T^0_t]dW^i_t, \quad t \in (0,T]; \\
&d\Delta p_t = -[\gamma(b_1(t)\Delta p_t + \sigma_1(t)\Delta q_t + \sigma_1(t)\Delta q_t + f_2(t,X_t, u_t, m_t, X^{i,0,2}_t) \\
&\quad - f_2(t,X_t, u_t, m_t, X^{i,0,2}_t) + \Delta T^0_t]dt + \Delta q_t dW_t + \Delta q_t dW^i_t, \quad t \in [0,T); \\
&\Delta X^0_t = \Delta \xi, \quad \Delta p^0_t = \gamma(g_0(X^0_T,m^0_T) - g_0(X^0_T, m^0_T)) + \Delta T^0_t, \\
&\Delta X_0 = \Delta \xi, \quad \Delta p_T = \gamma(g(x,X^0_T,m^0_T) - g(x,X^0_T, m^0_T)) + \Delta T^0_T
\end{aligned}
\]

with \( m^i_t = \mathcal{L}(X^i_t, \mathcal{F}^i_t) \) for \( i = 1,2 \), and the condition

\[
\begin{aligned}
&b^i_2(t)\Delta p^i_t + \sigma^i_2(t)\Delta q^i_t + f^i_2(t,X^i_t, u^i_t, m^i_t) - f^i_0(t,X^i_t, u^i_t, m^i_t) = 0, \\
&b_2(t)\Delta p_t + \sigma_2(t)\Delta q_t + \sigma_2(t)\Delta q_t + f_2(t,X_t, u_t, m_t) - f_0(t,X_t, u_t, m_t) = 0.
\end{aligned}
\]

(5)

We note the fact that for \( t \in [0,T] \),

\[
\mathbb{E}[W_2(m^1_t, m^2_t)^2] \leq \mathbb{E}[\mathbb{E}[|\Delta X_t|^2|\mathcal{F}^i_t]] = \mathbb{E}[|\Delta X_t|^2].
\]

From Assumptions (H1)-(H2) and standard estimates for SDEs and BSDEs, there exists \( C_1 > 0 \) depending only on \( (L,T) \), such that

\[
\begin{aligned}
&\mathbb{E}[\sup_{0 \leq t \leq T} |(\Delta X^0_t, \Delta p^0_t)|^2 + \int_0^T |\Delta q^0_t|^2 dt] \\
&\leq C_1 \left( \mathbb{E}[|\Delta \xi|^2] + \mathbb{E}[\gamma^2 \sup_{0 \leq t \leq T} |\Delta X_t|^2 + \gamma \int_0^T |\Delta u_t|^2 dt] + \|\Delta T^0\|^2 \right).
\end{aligned}
\]

(6)

Applying Itô’s lemma on \( \Delta p^0_t \Delta X^0_t \) and taking expectation, we have

\[
\begin{aligned}
&\mathbb{E}[\Delta p^0_t \Delta X^0_t] - \mathbb{E}[\Delta p^0_0 \Delta \xi^0] \\
&= \mathbb{E}\left[ \int_0^T \gamma[(b^0_0(t,m^0_t) - b^0_0(t,m^0_1))\Delta p^0_t + (\sigma^0_0(t,m^0_t) - \sigma^0_0(t,m^0_1))\Delta q^0_t] \\
&\quad + \gamma[(b^0_2(t)\Delta p^0_t + \sigma^0_2(t)\Delta q^0_t)\Delta u^0_t - (f^0_0(t,X^0_t, u^0_t, m^0_t) - f^0_0(t,X^0_t, u^0_t, m^0_t))\Delta X^0_t] \\
&\quad + (\Delta p^0_t \Delta T^0_t + \Delta q^0_t \Delta T^0_t - \Delta X^0_t \Delta T^0_t)dt \right].
\end{aligned}
\]

(7)

From Assumption (H2), we have

\[
\begin{aligned}
&\mathbb{E}\left[ \int_0^T \gamma[(b^0_0(t,m^0_t) - b^0_0(t,m^0_1))\Delta p^0_t + (\sigma^0_0(t,m^0_t) - \sigma^0_0(t,m^0_1))\Delta q^0_t]dt \right] \\
&\leq \mathbb{E}\left[ \int_0^T \gamma [W_2(m^1_t, m^2_t)(|\Delta p^0_t| + |\Delta q^0_t|)dt] \right].
\end{aligned}
\]

(8)
From (8) and Assumptions (H2)-(H3), we have

\[
(b_2^0(t)\Delta p_t^0 + \sigma_0^2(t)\Delta q_t^0)\Delta u_t^0 - (f^0(x_t^{0,2}, u_t^0, m_t^2) - f^0(x_t^{0,1}, u_t^{0,1}, m_t^1))\Delta X_t^0 \\
= -[(f^0, f^0)(t, X_t^{0,2}, u_t^0, m_t^2) - (f^0, f^0)(t, X_t^{0,1}, u_t^{0,1}, m_t^1)] \cdot (\Delta u_t^0, \Delta X_t^0) \\
+ [(f^0, f^0)(t, X_t^{0,1}, u_t^{0,1}, m_t^1) - (f^0, f^0)(t, X_t^{0,1}, u_t^{0,1}, m_t^1)] \cdot (\Delta u_t^0, \Delta X_t^0) \\
\leq -2C_{f^0}|\Delta u_t^0|^2 + l_m W_2(m_t^1, m_t^2)(|\Delta u_t^0| + |\Delta X_t^0|).
\]

From Assumptions (H2)-(H3), we have

\[
\mathbb{E}[\Delta p_T^0 \Delta X_T^0] = \gamma \mathbb{E}[(g^0(x_T^{0,2}, m_T^2) - g^0(x_T^{0,1}, m_T^1))\Delta X_T^0] \\
+ \mathbb{E}[\gamma (g^0(x_T^{0,1}, m_T^1))\Delta X_T^0 + \Delta T_T^0 \Delta X_T^0] \\
\geq -\mathbb{E}[\gamma l_m W_2(m_t^1, m_t^2)|\Delta X_T^0| + |\Delta T_T^0||\Delta X_T^0|].
\]

Plugging (8)-(10) into (7), we have for any \(\epsilon \in (0, 1),\)

\[
2\gamma C_{f^0}\mathbb{E}\left[ \int_0^T |\Delta u_t^0|^2 dt \right] \\
\leq \mathbb{E}\left[ \gamma l_m W_2(m_t^1, m_t^2)|\Delta X_T^0| + |\Delta T_T^0||\Delta X_T^0| + |\Delta p_0^0||\Delta \xi_0| \\
+ \gamma \int_0^T l_m W_2(m_t^1, m_t^2)(|\Delta X_t^0| + |\Delta p_t^0| + |\Delta q_t^0| + |\Delta u_t^0|)dt \\
+ \int_0^T (\Delta p_t^0 \Delta T_t^0 + \Delta q_t^0 \Delta T_t^0 - \Delta X_t^0 \Delta T_t^0)dt \right] \\
\leq \epsilon \mathbb{E}\left[ (T + 1) \sup_{0 \leq t \leq T} |(\Delta X_t^0, \Delta p_t^0)|^2 + \int_0^T |\Delta q_t^0|^2 + \mathbb{E}[|\Delta u_t^0|^2]dt \right] \\
+ \frac{1}{\epsilon} \mathbb{E}\left[ \gamma l_m^2(T + 1) \sup_{0 \leq t \leq T} |\Delta X_t|^2 + |\Delta \xi_0|^2 \right] + \frac{1}{\epsilon} \mathbb{E}[|\Delta \xi_0|^2].
\]

Plugging (9) into above, we have

\[
2\gamma C_{f^0}\mathbb{E}\left[ \int_0^T |\Delta u_t^0|^2 dt \right] \\
\leq \epsilon \gamma (T + 1)(C_1 + 1)\mathbb{E}\left[ \int_0^T |\Delta u_t^0|^2 dt \right] + \gamma l_m^2(T + 1)(\frac{1}{\epsilon} + C_1)\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Delta X_t|^2 \right] \\
+ \left( \frac{1}{\epsilon} + (T + 1)C_1 \right) (\|\Delta \xi_0\|^2 + \mathbb{E}[|\Delta \xi_0|^2]).
\]

We choose \(\epsilon = C_{f^0}(T + 1)^{-1}(C_1 + 1)^{-1},\) then we have

\[
\gamma \mathbb{E}\left[ \int_0^T |\Delta u_t^0|^2 dt \right] \\
\leq \gamma l_m^2 L^2(T + 1)^2(2C_1 + 1)\mathbb{E}\left[ \sup_{0 \leq t \leq T} |\Delta X_t|^2 \right] \\
+ L^2(T + 1)(2C_1 + 1)(\|\Delta T_0\|^2 + \mathbb{E}[|\Delta \xi_0|^2]).
\]

Plugging (11) into (6), we have

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |(\Delta X_t^0, \Delta p_t^0)|^2 + \int_0^T |\Delta q_t^0|^2 dt \right]
\]
\[
\leq 2\gamma l^2_m L^2 (T + 1)^2 C_1 (C_1 + 1) \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta X_t|^2 \right] + C(L, T) (\|\Delta T\|^2 + \mathbb{E}[|\Delta \xi|^2]),
\]

where \(C(L, T)\) is a constant depending only on \((L, T)\). We know from Lemma \(3.1\) that

\[
|\Delta u_t^0| \leq \frac{L^2}{2} (|\Delta X_t^0| + |\Delta p_t^0| + |\Delta q_t^0|) + \frac{L^2 m}{2} W_2(m_1, m_2),
\]

so we eventually have

\[
\|\Delta \Theta^0\|^2 \leq 6 l^2_m L^6 (T + 1)^3 (C_1 + 1)^2 \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Delta X_t|^2 \right] + C(L, T) (\|\Delta T\|^2 + \mathbb{E}[|\Delta \xi|^2]). \tag{12}
\]

Now we give the estimates of \(\Delta \Theta\). From Assumptions (H1)-(H2) and standard estimates for SDEs and BSDEs, there exists \(C_2 > 0\) depending only on \((L, T)\), such that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |(\Delta X_t, \Delta p_t)|^2 + \int_0^T |(\Delta q_t, \Delta \tilde{q}_t)|^2 \, dt \right]
\leq C_2 \left( \mathbb{E}[|\Delta \xi|^2] + \gamma l^2_m \sup_{0 \leq t \leq T} |\Delta X_t^0|^2 + \gamma \int_0^T |\Delta u_t|^2 \right) \tag{13}
+ \|\Delta T\|^2.
\]

Applying Itô’s lemma on \(\Delta p_t \Delta X_t\) and taking expectation, we have

\[
\mathbb{E}[\Delta p_T \Delta X_T] = \mathbb{E}[\Delta p_0 \Delta \xi]
= \mathbb{E} \left[ \int_0^T \gamma ((b_0(t, m_1^2) - b_0(t, m_1^1)) \Delta p_t + (\sigma_0(t, m_1^2) - \sigma_0(t, m_1^1)) \Delta q_t
+ (\tilde{\sigma}_0(t, m_1^2) - \tilde{\sigma}_0(t, m_1^1)) \Delta \tilde{q}_t + (b_2(t) \Delta p_t + \sigma_2(t) \Delta q_t + \tilde{\sigma}_2(t) \Delta \tilde{q}_t) \Delta u_t
- (f_x(t, X_t^1, u_t^1, m_1^2, X_t^{01}) - f_x(t, X_t^1, u_t^1, m_1^1, X_t^{01})) \Delta X_t
+ (\Delta p_t \Delta T_t^b + \Delta q_t \Delta T_t^\theta + \Delta \tilde{q}_t \Delta T_t^\tilde{\theta} - \Delta X_t \Delta T_t^f) \, dt \right]. \tag{14}
\]

From Assumption (H2), we have

\[
\mathbb{E} \left[ \int_0^T \gamma ((b_0(t, m_1^2) - b_0(t, m_1^1)) \Delta p_t + (\sigma_0(t, m_1^2) - \sigma_0(t, m_1^1)) \Delta q_t
+ (\tilde{\sigma}_0(t, m_1^2) - \tilde{\sigma}_0(t, m_1^1)) \Delta \tilde{q}_t) \, dt \right]
\leq \mathbb{E} \left[ \int_0^T \gamma M W_2(m_1^1, m_1^2) (|\Delta p_t| + |\Delta q_t| + |\Delta \tilde{q}_t|) \, dt \right]
\leq \gamma M (T + 1) \mathbb{E} \left[ \sup_{0 \leq t \leq T} |(\Delta X_t, \Delta p_t)|^2 + \int_0^T |(\Delta q_t, \Delta \tilde{q}_t)|^2 \, dt \right]. \tag{15}
\]

From \(5\) and Assumptions (H2)-(H3), we have

\[
(b_2(t) \Delta p_t + \sigma_2(t) \Delta q_t + \tilde{\sigma}(t) \Delta \tilde{q}_t) \Delta u_t - (f_x^1(t, X_t^1, u_t^1, X_t^{01}) - f_x^1(t, X_t^1, u_t^1, X_t^{01})) \Delta X_t
\leq -2C_f |\Delta u_t|^2 + l_{\omega^0} |\Delta X_t^0|(|\Delta u_t| + |\Delta X_t|). \tag{16}
\]
From Assumptions (H2) and (H4), we have
\[
\mathbb{E}[(f^2_x(t, X_t^2, m_t^2, X_t^0) - f^2_x(t, X_t^1, m_t^1, X_t^0))\Delta X_t] \\
= \mathbb{E}[\mathbb{E}[(f^2_x(t, X_t^2, m_t^2, X_t^0) - f^2_x(t, X_t^1, m_t^1, X_t^0))\Delta X_t|\mathcal{F}_T]] \\
+ \mathbb{E}[(f^2_x(t, X_t^1, m_t^1, X_t^0) - f^2_x(t, X_t^1, m_t^1, X_t^0))\Delta X_t] \\
\geq -\ell_{x0}\mathbb{E}[\|\Delta X_t^0\|\|\Delta X_t\|] \\
\mathbb{E}[\Delta p_T|\Delta X_T] \\
= \mathbb{E}[\mathbb{E}[\gamma(g_x(X_T^2, m_T^2, X_T^0) - g_x(X_T^1, m_T^1, X_T^0))\Delta X_T|\mathcal{F}_T]] \\
+ \mathbb{E}[\gamma(g_x(X_T^1, m_T^1, X_T^0) - g_x(X_T^1, m_T^1, X_T^0))\Delta X_T + \Delta X_T\Delta I_T^b] \\
\geq -\gamma\ell_{x0}\mathbb{E}[\|\Delta X_T^0\|\|\Delta X_T\| + \mathbb{E}[\Delta X_T\Delta I_T^b]].
\]
Plugging (15)-(17) into (14), we have for any \(\epsilon \in (0, 1)\),
\[
2\gamma C_f\mathbb{E}[\int_0^T |\Delta u_t|^2 dt] \\
\leq \gamma L_m(T + 1)\mathbb{E}[\sup_{0 \leq t \leq T} |(\Delta X_t, \Delta p_t)|^2 + \int_0^T |(\Delta q_t, \Delta \tilde{q}_t)|^2 dt] \\
+ \mathbb{E}[\gamma\ell_{x0}\|\Delta X_T^0\|\|\Delta X_T\| + |\Delta X_T\||\Delta I_T^b| + |\Delta p_0||\Delta \xi|] \\
+ \gamma\ell_{x0}\int_0^T |\Delta X_T^0|(|\Delta u_t| + 2|\Delta X_t|)dt \\
+ \int_0^T |\Delta p_t||\Delta I_T^b| + |\Delta q_t||\Delta I_T^b| + |\Delta \tilde{q}_t||\Delta I_T^b| + |\Delta X_t||\Delta I_T^b||dt] \\
\leq (L_m + \epsilon)(T + 1)\mathbb{E}[\sup_{0 \leq t \leq T} |(\Delta X_t, \Delta p_t)|^2 + \int_0^T |(\Delta q_t, \Delta \tilde{q}_t)|^2 dt] \\
+ \epsilon\mathbb{E}[\int_0^T |\Delta u_t|^2 dt] + \frac{1}{\epsilon}\mathbb{E}[3\ell_{x0}^2(T + 1) \sup_{0 \leq t \leq T} |\Delta X_T^0|^2 + |\Delta \xi|^2] + \frac{1}{\epsilon}\|\Delta I_T|^2.
\]
Plugging (13) into above, we have
\[
2\gamma C_f\mathbb{E}[\int_0^T |\Delta u_t|^2 dt] \\
\leq (L_m + \epsilon)\gamma(T + 1)(C_2 + 1)\mathbb{E}[\int_0^T |\Delta u_t|^2] \\
+ \ell_{x0}^2(T + 1)\left(\frac{3}{\epsilon} + (L + 1)C_2\right)\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_T^0|^2] \\
+ \left(\frac{1}{\epsilon} + (L + 1)(T + 1)C_2\right)(\mathbb{E}[|\Delta \xi|^2] + \|\Delta I_T\|^2).
\]
We now set \(\delta_1 := 2^{-1}(T + 1)^{-1}(C_2 + 1)^{-1}\), which depends only on \((L, T)\). When \(L_mC_f^{-1} \leq \delta_1\), we choose \(\epsilon = 2^{-1}(T + 1)^{-1}(C_2 + 1)^{-1} C_f\), then we have
\[
\gamma\mathbb{E}[\int_0^T |\Delta u_t|^2 dt] \\
\leq 6\ell_{x0}^2(L + 1)^2(T + 1)^2(2C_2 + 1)\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_T^0|^2] \\
+ 2(L + 1)^2(T + 1)(2C_2 + 1)\mathbb{E}[|\Delta \xi|^2] + \|\Delta I_T\|^2) \\
\](18).
We know from Lemma 3.1 that
\[ |\Delta u_t| \leq \frac{L^2}{2} (|\Delta X_t| + |\Delta p_t| + |\Delta q_t| + |\Delta \tilde{q}_t|) + \frac{l_{x_0} L}{2} |\Delta X_0^0|, \]
so we eventually have
\[ \|\Delta \Theta\|_3^2 \leq 24 l_{x_0}^2 (L + 1)^3 (T + 1)^3 (C_2 + 1)^2 \mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t^0|^2] + C(L, T)(\mathbb{E}[|\Delta \xi|^2] + \|\Delta T\|^2). \]  

(19)

In view of (12) and (19), we have
\[ \|(\Delta \Theta^0, \Delta \Theta)\|_3^2 \leq 144 l_{x_0}^2 l_m^2 (L + 1)^4 (T + 1)^6 (C_1 + 1)^2 (C_2 + 1)^2 \mathbb{E}[|\Delta \Theta^0|^2 + |\Delta \Theta|^2] \]
\[ + C(L, T)(\mathbb{E}[|\Delta \xi|^2] + \|\Delta T^0, \Delta T\|^2_2). \]

Now we set \( \delta_2 := 24^{-1}(L + 1)^{-5}(T + 1)^{-3}(C_1 + 1)^{-1}(C_2 + 1)^{-1} \), which depends only on \( (L, T) \). When \( l_{x_0} l_m \leq \delta_2 \), we have
\[ \|(\Delta \Theta^0, \Delta \Theta)\|_3^2 \leq C(L, T)(\mathbb{E}[|\Delta \xi^0|^2 + |\Delta \xi|^2] + \|\Delta T^0, \Delta T\|^2_2). \]

\[ \square \]

5 \( \epsilon \)-Nash Equilibrium Property

In this section, we show how the solution of Problem 2.2 can provide an \( O(\frac{1}{\sqrt{N}}) \)-Nash equilibrium for Problem 2.1. We always suppose that Assumptions (H1)-(H4) hold true. Moreover, we make the following assumption:

\( (H5) \) The functions \((b_0^0, \sigma_0^0, b_0, \sigma_0, \tilde{b}_0, f^0, g^0, f, g)\) depend on \( m \in \mathcal{P}(\mathbb{R}) \) in a scalar form: for \( \phi = b_0^0, \sigma_0^0, b_0, \sigma_0, \tilde{b}_0, f^0, g^0, f, g, \)
\[ \phi(t, x, u, m, x^0) = \int_{\mathbb{R}} \phi'(t, x, u, y, x^0) m(dy), \quad (t, x, u, m, x) \in [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{R}, \]
with \( \phi' \) being \( L \)-Lipschitz continuous in \( y \in \mathbb{R} \). For notation convenience, we still denote \( \phi' \) by \( \phi \).

We let \((\tilde{u}^0, \tilde{X}^0, \tilde{u}^i, \tilde{X}^i)\) be the solution of FBSDE (3) with the Brownian motion \( W \) replaced by \( W^i \) for \( 1 \leq i \leq N \). Since all the minor agents are statistically identical, we know that \( \mathcal{L}(\tilde{X}_t^i|\tilde{F}_t^0) = \mathcal{L}(\tilde{X}_t^j|\tilde{F}_t^0) \) for \( 1 \leq i, j \leq N \) and \( t \in [0, T] \), which is denoted by \( m_t \). From Theorem 3.3 we know that, \((\tilde{u}^0, \tilde{u}^i)\) is an optimal solution of Problem 2.2 with the Brownian motion \( W \) replaced by \( W^i \) for \( 1 \leq i \leq N \). Applying \((\tilde{u}^0, \ldots, \tilde{u}^N)\) into the \((N + 1)\)-agent game, the dynamics of agents are as
We set follows:

\[
\tilde{X}_t^{0,N} = \xi^0 + \int_0^t \frac{1}{N} \sum_{j=1}^N b_0^0(s, \tilde{X}_s^{j,N}) + b_1^0(s) \tilde{X}_s^{0,N} + b_2^0(s) u_s^0 ds \\
+ \int_0^t \frac{1}{N} \sum_{j=1}^N \sigma_0^0(s, \tilde{X}_s^{j,N}) + \sigma_1^0(s) \tilde{X}_s^{0,N} + \sigma_2^0(s) \tilde{u}_s^0 dW_s^0, \quad t \in [0, T],
\]

\[
\tilde{X}_t^{i,N} = \xi^i + \int_0^t \frac{1}{N} \sum_{j=1}^N b_0(s, \tilde{X}_s^{j,N}) + b_1(s) \tilde{X}_s^{i,N} + b_2(s) u_s^i ds \\
+ \int_0^t \frac{1}{N} \sum_{j=1}^N \sigma_0(s, \tilde{X}_s^{j,N}) + \sigma_1(s) \tilde{X}_s^{i,N} + \sigma_2(s) \tilde{u}_s^i dW_s^i \\
+ \int_0^t \frac{1}{N} \sum_{j=1}^N \tilde{\sigma}_0(s, \tilde{X}_s^{j,N}) + \tilde{\sigma}_1(s) \tilde{X}_s^{i,N} + \tilde{\sigma}_2(s) \tilde{u}_s^i dW_s^0, \quad t \in [0, T], \quad 1 \leq i \leq N.
\]

The following lemma gives estimate of the distance between \( \tilde{X}^i \) and \( \tilde{X}^{i,N} \).

**Lemma 5.1.** Let Assumptions (H1)-(H5) be satisfied. Then, we have

\[
\sup_{0 \leq i \leq N} \sup_{0 \leq t \leq T} \mathbb{E}[|\tilde{X}_t^{i,N} - \tilde{X}_t^i|^2] = O\left(\frac{1}{N}\right).
\]

**Proof.** We set \( \Delta X_t^i = \tilde{X}_t^{i,N} - \tilde{X}_t^i \) for \( 0 \leq i \leq N \), then we have

\[
d\Delta X_t^0 = \left[\left(\frac{1}{N} \sum_{j=1}^N b_0^0(t, \tilde{X}_t^{j,N}) - b_0^0(t, m_t)\right) + b_1(t) \Delta X_t^0\right] dt \\
+ \left[\left(\frac{1}{N} \sum_{j=1}^N \sigma_0^0(t, \tilde{X}_t^{j,N}) - \sigma_0^0(t, m_t)\right) + \sigma_1(t) \Delta X_t^0\right] dW_t^0, \quad t \in (0, T),
\]

\[
d\Delta X_t^i = \left[\left(\frac{1}{N} \sum_{j=1}^N b_0(t, \tilde{X}_t^{j,N}) - b_0(t, m_t)\right) + b_1(t) \Delta X_t^i\right] dt \\
+ \left[\left(\frac{1}{N} \sum_{j=1}^N \sigma_0(t, \tilde{X}_t^{j,N}) - \sigma_0(t, m_t)\right) + \sigma_1(t) \Delta X_t^i\right] dW_t^i \\
+ \left[\left(\frac{1}{N} \sum_{j=1}^N \tilde{\sigma}_0(t, \tilde{X}_t^{j,N}) - \tilde{\sigma}_0(t, m_t)\right) + \tilde{\sigma}_1(t) \Delta X_t^i\right] dW_t^0, \quad t \in (0, T), \quad 1 \leq i \leq N,
\]

where \( m_t = \mathcal{L}(\tilde{X}_t^i | \mathcal{F}_t^0) \) for all \( i = 1, 2, \ldots, N \). From Assumption (H5), we have

\[
\left|\frac{1}{N} \sum_{j=1}^N b_0^0(t, \tilde{X}_t^{j,N}) - b_0^0(t, m_t)\right| \leq \frac{L}{N} \sum_{j=1}^N |\Delta X_t^j| + \left|\frac{1}{N} \sum_{j=1}^N b_0^0(t, \tilde{X}_t^{j}) - b_0^0(t, m_t)\right|.
\]

We set \( Y_t^j := b_0^0(t, \tilde{X}_t^j) - b_0^0(t, m_t) \), then, we know from the definition of \( m_t \) that \( Y_t^j = b_0^0(t, \tilde{X}_t^j) - \mathbb{E}[b_0^0(t, \tilde{X}_t^j) | \mathcal{F}_t^0] \) and \( \mathbb{E}[Y_t^j | \mathcal{F}_t^0] = 0 \). Recall that \( W^j \) and \( W^{j'} \) are independent when \( 1 \leq j \neq j' \leq N \),
so we have  
\[
\mathbb{E}[Y_t^j Y_t^{j'} | \mathcal{F}_t^0] = \mathbb{E}[Y_t^j | \mathcal{F}_t^0] \mathbb{E}[Y_t^{j'} | \mathcal{F}_t^0] = 0,
\]
and then  
\[
\mathbb{E}[Y_t^j Y_t^{j'}] = \mathbb{E}[\mathbb{E}[Y_t^j Y_t^{j'} | \mathcal{F}_t^0]] = 0.
\]
Since \(Y_t^j, 1 \leq j \leq N\) are identically distributed, we have
\[
\mathbb{E}[\frac{1}{N} \sum_{j=1}^N b_0^j(t, X_t^j) - b_0^0(t, m_t)]^2 = \frac{1}{N^2} \sum_{j=1}^N \mathbb{E}[|Y_t^j|^2] + \frac{1}{N^2} \sum_{1 \leq j \neq j' \leq N} \mathbb{E}[Y_t^j Y_t^{j'}]
\]
\[
= \frac{1}{N} \sum_{j=1}^N \mathbb{E}[|Y_t^j|^2] = \mathcal{O}(\frac{1}{N}).
\]

Therefore, from standard estimates for SDEs, we have
\[
\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t^0|^2] \leq C(L, T) \mathbb{E}[\frac{1}{N} \sum_{1 \leq j \leq N} \int_0^T |\Delta X_t^j|^2 dt] + \mathcal{O}(\frac{1}{N}),
\]
where \(C(L, T)\) is a constant depending only on \((L, T)\). Similarly, we have
\[
\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t^j|^2] \leq C(L, T) \mathbb{E}[\frac{1}{N} \sum_{1 \leq j \leq N} \int_0^T |\Delta X_t^j|^2 dt] + \mathcal{O}(\frac{1}{N}).
\]
Combine them together, we have
\[
\mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t^0|^2] + \frac{1}{N} \sum_{1 \leq j \leq N} \mathbb{E}[\sup_{0 \leq t \leq T} |\Delta X_t^j|^2]
\]
\[
= C(L, T) \mathbb{E}[\frac{1}{N} \sum_{1 \leq j \leq N} \int_0^T |\Delta X_t^j|^2 dt] + \mathcal{O}(\frac{1}{N}).
\]

From Gronwall’s inequality, we have \([21]\).

Our main result of this section is stated as follows.

**Theorem 5.2.** Let Assumptions (H1)-(H5) be satisfied. Then, \((\bar{u}^0, \ldots, \bar{u}^N)\) is an \(\mathcal{O}(\frac{1}{\sqrt{N}})\)-Nash equilibrium of the \((N + 1)\)-agent game.

**Proof. Case 1 (strategy change of the major agent):**

While the minor agents are using the responses \((\bar{u}^1, \ldots, \bar{u}^N)\), a strategy change from \(\bar{u}^0\) to some \(u^0 \in \mathcal{L}^2_{\mathcal{F}_0}(0, T)\) of the major agent yields
\[
X_t^0 = \xi^0 + \int_0^t \frac{1}{N} \sum_{j=1}^N b_0^j(s, X_s^j, N) + b_1^0(s) X_s^0 + b_2^0(s) u_s^0 ds
\]
\[
+ \int_0^t \frac{1}{N} \sum_{j=1}^N \sigma_0^0(s, X_s^j, N) + \sigma_1^0(s) X_s^0 + \sigma_2^0(s) u_s^0 dW_s^0, \quad t \in [0, T].
\]

Note that the \(i\)-th minor agent’s state is still \(X_{i,N}\) for \(1 \leq i \leq N\). We define
\[
\hat{X}_t^0 := \xi^0 + \int_0^t b_0^0(s, m_s) + b_1^0(s) \hat{X}_s^0 + b_2^0(s) u_s^0 ds
\]
\[
+ \int_0^t \sigma_0^0(s, m_s) + \sigma_1^0(s) \hat{X}_s^0 + \sigma_2^0(s) u_s^0 dW_s^0, \quad t \in [0, T].
\]
Similar as Lemma 5.1, we have
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X^0_t - \hat{X}^0_t|^2 \right] = O\left(\frac{1}{N}\right). \]

So we know from above and Lemma 5.1 that
\[ J_0(u^0; \hat{u}^{-0}) \geq \mathbb{E} \left[ \int_0^T \frac{1}{N} \sum_{j=1}^N f^0(t, \hat{X}^0_t, u^0_t, \hat{X}^j_t) dt + \frac{1}{N} \sum_{j=1}^N g^0(\hat{X}^0_T, \hat{X}^j_T) \right] - O\left(\frac{1}{\sqrt{N}}\right). \]

Similar as the proof of Lemma 5.1, we have
\[ \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^N \left( f^0(t, \hat{X}^0_t, u^0_t, \hat{X}^j_t) - f^0(t, \hat{X}^0_t, u^0_t, m_t) \right)^2 + \frac{1}{N} \sum_{j=1}^N \left( g^0(\hat{X}^0_T, \hat{X}^j_T) - g^0(\hat{X}^0_T, m_T) \right)^2 \right] = O\left(\frac{1}{N}\right). \]

Therefore,
\[ J_0(u^0; \hat{u}^{-0}) \geq \mathbb{E} \left[ \int_0^T f^0(t, \hat{X}^0_t, u^0_t, m_t) dt + g^0(\hat{X}^0_T, m_T) \right] - O\left(\frac{1}{\sqrt{N}}\right). \] (22)

In a similar way, we have
\[ J_0(u^0; \hat{u}^{-0}) \leq \mathbb{E} \left[ \int_0^T f^0(t, \hat{X}^0_t, \hat{u}^0_t, m_t) dt + g^0(\hat{X}^0_T, m_T) \right] + O\left(\frac{1}{\sqrt{N}}\right). \] (23)

From Theorem 3.3 we know that
\[ \mathbb{E} \left[ \int_0^T f^0(t, \hat{X}^0_t, u^0_t, m_t) dt + g^0(\hat{X}^0_T, m_T) \right] \geq \mathbb{E} \left[ \int_0^T f^0(t, \hat{X}^0_t, \hat{u}^0_t, m_t) dt + g^0(\hat{X}^0_T, m_T) \right]. \] (24)

Therefore, we know from (22)-(24) that
\[ J_0(u^0; \hat{u}^{-0}) - O\left(\frac{1}{\sqrt{N}}\right) \leq J_0(u^0; \hat{u}^{-0}). \]

**Case 2 (strategy change of an minor agent):**
Without loss of generality, we assume that the 1-th minor agent changes his/her best response control strategy \( \hat{u}^1 \) to \( u \in \mathcal{L}^2_{\mathbb{P}_1}(0, T) \). This leads to
\[
X^0_t = \xi^0 + \int_0^t \frac{1}{N} \sum_{j=1}^N b^0_0(s, X^j_s) + b^0_1(s)X^0_s + b^0_2(s)\hat{u}^0_s ds \\
+ \int_0^t \frac{1}{N} \sum_{j=1}^N \sigma^0_0(t, X^j_s) + \sigma^0_1(s)X^0_s + \sigma^0_2(s)u^0_s dW^0_s, \quad t \in [0, T],
\]
\[
X^1_t = \xi^1 + \int_0^t \frac{1}{N} \sum_{j=1}^N b_0(s, X^j_s) + b_1(s)X^1_s + b_2(s)u_ds \\
+ \int_0^t \frac{1}{N} \sum_{j=1}^N \sigma_0(s, X^j_s) + \sigma_1(t)X^1_s + \sigma_2(s)u_ds dW^1_s.
\]
Similar as Lemma 5.1, we have

\[ X_t^i = \xi^i + \int_0^t \frac{1}{N} \sum_{j=1}^N b_0(s, X_s^j) + b_1(s)X_s^i + b_2(s)\bar{u}_s^i)ds + \int_0^t \frac{1}{N} \sum_{j=1}^N \sigma_0(s, X_s^j) + \sigma_1(s)X_s^i + \sigma_2(s)\bar{u}_s^i dW_s^i + \int_0^t \frac{1}{N} \sum_{j=1}^N \bar{\sigma}_0(s, X_s^j) + \bar{\sigma}_1(s)\bar{X}_s^i + \bar{\sigma}_2(s)\bar{u}_s^i dW_s^0, \quad t \in [0, T], \quad 2 \leq i \leq N. \]

From standard estimates for SDEs, we have

\[ \sup_{j=0,2,3,...,N} \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^j - \bar{X}_t^{j,N}|^2] = \mathcal{O}(\frac{1}{N}). \]

From above and Lemma 5.1, we know that

\[ \sup_{j=0,2,3,...,N} \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^j - \bar{X}_t^j|^2] = \mathcal{O}(\frac{1}{N}). \] (25)

We define

\[ \bar{X}_t^{1,N} := \xi^1 + \int_0^t \frac{1}{N} (b_0(s, \bar{X}_s^{1,N}) + \sum_{j=2}^N b_0(s, \bar{X}_s^j)) + b_1(s)\bar{X}_s^{1,N} + b_2(s)u_s ds + \int_0^t \frac{1}{N} (\sigma_0(s, \bar{X}_s^{1,N}) + \sum_{j=2}^N \sigma_0(s, \bar{X}_s^j)) + \sigma_1(s)\bar{X}_s^{1,N} + \sigma_2(s)u_s dW_s^1 + \int_0^t \frac{1}{N} (\bar{\sigma}_0(s, \bar{X}_s^{1,N}) + \sum_{j=2}^N \bar{\sigma}_0(s, \bar{X}_s^j)) + \bar{\sigma}_1(s)\bar{X}_s^{1,N} + \bar{\sigma}_2(s)u_s dW_s^0, \quad t \in [0, T]. \]

From (25) and standard estimates for SDEs, we have

\[ \mathbb{E}[\sup_{0 \leq t \leq T} |X_t^1 - \bar{X}_t^{1,N}|^2] = \mathcal{O}(\frac{1}{N}). \] (26)

We define

\[ \bar{X}_t^1 := \xi^1 + \int_0^t b_0(s, m_s) + b_1(s)\bar{X}_s^1 + b_2(s)u_s ds + \int_0^t \sigma_0(s, m_s) + \sigma_1(s)\bar{X}_s^1 + \sigma_2(s)u_s dW_s^1 + \int_0^t \bar{\sigma}_0(s, m_s) + \bar{\sigma}_1(s)\bar{X}_s^1 + \bar{\sigma}_2(s)u_s dW_s^0, \quad t \in [0, T]. \]

Similar as Lemma 5.1, we have

\[ \mathbb{E}[\sup_{0 \leq t \leq T} \bar{X}_t^{1,N} - \bar{X}_t^1|^2] = \mathcal{O}(\frac{1}{N}). \]
Similarly, from above and (26), we have
\[ \mathbb{E}[ \sup_{0 \leq t \leq T} |X_t^1 - \hat{X}_t^1|^2] = O\left(\frac{1}{N}\right). \quad (27) \]

From (25) and (27), we have
\[ J_1(u; \bar{u}^{-1}) \geq \mathbb{E}\left[ \int_0^T \frac{1}{N} \sum_{j=1}^N f(t, \hat{X}_t^1, u_t, \hat{X}_t^j, \bar{X}_T^0) dt + \frac{1}{N} \sum_{j=1}^N g(\hat{X}_T^1, \hat{X}_T^j, \bar{X}_T^0) \right] - O\left(\frac{1}{\sqrt{N}}\right). \quad (28) \]

We set \( Y_t^j := f(t, \hat{X}_t^1, u_t, \bar{X}_t^j, \bar{X}_T^0) - f(t, \hat{X}_t^1, u_t, m_t, \bar{X}_T^0) \) for \( 2 \leq j \leq N \), then, from the definition of \( m_t \) and the fact that \( \mathcal{F}_t^j \) is independent from \( \{W^1, \xi^1\} \), we have \( \mathbb{E}[Y_t^j | \mathcal{F}_t^1] = 0 \). For \( 2 \leq j \neq j' \leq N \), since \( \{W^1, W^j, W^{j'}\} \) are independent, we have \( \mathbb{E}[Y_t^j Y_t^{j'} | \mathcal{F}_t^1] = \mathbb{E}[Y_t^j | \mathcal{F}_t^1] \mathbb{E}[Y_t^{j'} | \mathcal{F}_t^1] = 0 \), and then \( \mathbb{E}[Y_t^j Y_t^{j'}] = 0 \). Since \( \{Y_t^j, 2 \leq j \leq N\} \) are identically distributed, we have
\[
\mathbb{E}\left[ \left\| \frac{1}{N} \sum_{j=1}^N (f(t, \hat{X}_t^1, u_t, \bar{X}_t^j, \bar{X}_T^0) - f(t, \hat{X}_t^1, u_t, m_t, \bar{X}_T^0)) \right\|^2 \right] \\
= \frac{1}{N^2} \sum_{j=1}^N \mathbb{E}[|Y_t^j|^2] + \frac{1}{N^2} \sum_{1 \leq j \neq j' \leq N} \mathbb{E}[Y_t^j Y_t^{j'}] \\
= \frac{1}{N^2} \sum_{j=1}^N \mathbb{E}[|Y_t^j|^2] + \frac{2}{N^2} \sum_{1 < j \leq N} \mathbb{E}[Y_t^j Y_t^j] = O\left(\frac{1}{N}\right).
\]

Similarly,
\[
\mathbb{E}\left[ \left\| \frac{1}{N} \sum_{j=1}^N (g(\hat{X}_T^1, \hat{X}_T^j, \bar{X}_T^0) - g(\hat{X}_T^1, m_T, \bar{X}_T^0)) \right\|^2 \right] = O\left(\frac{1}{N}\right).
\]

Therefore, we know that
\[
\mathbb{E}\left[ \int_0^T \frac{1}{N} \sum_{j=1}^N f(t, \hat{X}_t^1, u_t, \bar{X}_t^j, \bar{X}_T^0) dt + \frac{1}{N} \sum_{j=1}^N g(\hat{X}_T^1, \hat{X}_T^j, \bar{X}_T^0) \right] \\
\geq \mathbb{E}\left[ \int_0^T f(t, \hat{X}_t^1, u_t, m_t, \bar{X}_T^0) dt + g(\hat{X}_T^1, m_T, \bar{X}_T^0) \right] - O\left(\frac{1}{\sqrt{N}}\right). \quad (29) \]

From (28) and (29), we have
\[ J_1(u; \bar{u}^{-1}) \geq \mathbb{E}\left[ \int_0^T f(t, \hat{X}_t^1, u_t, m_t, \bar{X}_T^0) dt + g(\bar{X}_T^1, m_T, \bar{X}_T^0) \right] - O\left(\frac{1}{\sqrt{N}}\right). \quad (30) \]

In a similar way, we have
\[ J_1(\bar{u}^1; \bar{u}^{-1}) \leq \mathbb{E}\left[ \int_0^T f(t, \hat{X}_t^1, \bar{u}_t^1, m_t, \bar{X}_T^0) dt + g(\bar{X}_T^1, m_T, \bar{X}_T^0) \right] + O\left(\frac{1}{\sqrt{N}}\right). \quad (31) \]

From Theorem 3.3 we know that
\[
\mathbb{E}\left[ \int_0^T f(t, \hat{X}_t^1, u_t, m_t, \bar{X}_T^0) dt + g(\hat{X}_T^1, m_T, \bar{X}_T^0) \right] \\
\geq \mathbb{E}\left[ \int_0^T f(t, \hat{X}_t^1, \bar{u}_t^1, m_t, \bar{X}_T^0) dt + g(\bar{X}_T^1, m_T, \bar{X}_T^0) \right]. \quad (32) \]
Therefore, we know from (30)-(32) that
\[ J_1(\bar{u}^1; \bar{u}^{-1}) - \mathcal{O}(\frac{1}{\sqrt{N}}) \leq J_1(u; \bar{u}^{-1}). \]

\[ \square \]

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