Determining the mass of dark matter particles with direct detection experiments

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Abstract. In this paper, I review two data analysis methods for determining the mass (and eventually the spin-independent cross section on nucleons) of weakly interacting massive particles with positive signals from direct dark matter detection experiments: a maximum likelihood analysis with only one experiment and a model-independent method requiring at least two experiments. Uncertainties and caveats of these methods will also be discussed.

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1. Introduction

There is strong evidence that more than 80% of all matter in the Universe is dark (i.e. interacts at most very weakly with electromagnetic radiation and ordinary matter). The dominant component of this cosmological dark matter should be due to some yet to be discovered, non-baryonic particles. Weakly interacting massive particles (WIMPs) $\chi$ arising in several extensions of the standard model of electroweak interactions are one of the leading candidates for dark matter. WIMPs are stable particles with masses roughly between 10 GeV and a few TeV and interact with ordinary matter only weakly (for reviews of WIMPs and some other possible candidates for dark matter, see [1]–[3]).

Currently, the most promising method to detect different WIMP candidates is the direct detection of the recoil energy deposited in a low-background laboratory detector by the elastic scattering of ambient WIMPs on target nuclei [4]–[6]. The basic expression for the differential event rate for elastic WIMP-nucleus scattering is given by [1]:

$$\frac{dR}{dQ} = AF^2(Q) \int_{v_{\text{min}}}^{v_{\text{max}}} \left[ \frac{f_1(v)}{v} \right] dv.$$  (1)

Here $R$ is the direct detection event rate, i.e. the number of events per unit time and unit mass of detector material, $Q$ is the energy deposited in the detector, $F(Q)$ is the elastic nuclear form factor, $f_1(v)$ is the one-dimensional (1D) velocity distribution function of the WIMPs impinging on the detector and $v$ is the absolute value of the WIMP velocity in the laboratory frame. The constant coefficient $A$ is defined as

$$A \equiv \frac{\rho_0 \sigma_0}{2m_\chi m_{r,N}^2},$$  (2)

where $\rho_0$ is the WIMP density near the Earth and $\sigma_0$ is the total cross section ignoring the form factor suppression. The reduced mass $m_{r,N}$ is defined by

$$m_{r,N} \equiv \frac{m_\chi m_N}{m_\chi + m_N},$$  (3)

where $m_\chi$ is the WIMP mass and $m_N$ is that of the target nucleus. Finally, $v_{\text{min}}$ is the minimal incoming velocity of incident WIMPs that can deposit the energy $Q$ in the detector:

$$v_{\text{min}} = \alpha \sqrt{Q},$$  (4)

with

$$\alpha \equiv \sqrt{\frac{m_N}{2m_{r,N}^2}},$$  (5)

and $v_{\text{max}}$ is related to the escape velocity from our Galaxy at the position of the solar system, $v_{\text{esc}}$.

It was found that, by using a time-averaged recoil spectrum $dR/dQ$ and assuming that no directional information exists, the normalized 1D velocity distribution function of incident

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1 Recall that, besides many different candidates for WIMPs, it is also possible that some other particles are (theoretically) candidates for dark matter. For more details about these various possible dark matter particles in many different (exotic) models or scenarios as well as the possible methods to detect them, see e.g. accompanying articles in this Focus issue.

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WIMPs, $f_1(v)$, can be solved from equation (1) directly as \[ f_1(v) = N \left\{ -2Q \frac{d}{dQ} \left[ \frac{1}{F^2(Q)} \left( \frac{dR}{dQ} \right) \right] \right\}_{Q = v^2/\alpha^2}, \] (6)

where the normalization constant $N$ is given by \[ N = \frac{2}{\alpha} \int_0^\infty \frac{1}{\sqrt{Q}} \left[ \frac{1}{F^2(Q)} \left( \frac{dR}{dQ} \right) \right] dQ \right\}^{-1}. \] (7)

Note that, firstly, because $f_1(v)$ in equation (6) is the normalized velocity distribution, the normalization constant $N$ here is independent of the constant coefficient $\mathcal{A}$ defined in equation (2). Secondly, the integral in equation (7) goes over the entire physically allowed range of recoil energies: starting at $Q = 0$, the upper limit of the integral has been written as $\infty$. However, it is usually assumed that the WIMP flux on the Earth is negligible at velocities exceeding the escape velocity $v_{\text{esc}}$. This leads to a kinematic maximum of the recoil energy

\[ Q_{\text{max, kin}} = \frac{v_{\text{esc}}^2}{\alpha^2}. \] (8)

The velocity distribution function of halo WIMPs reconstructed by equation (6) is independent of the local WIMP density $\rho_0$ as well as of the WIMP-nucleus cross section $\sigma_0$. However, not only the overall normalization constant $N$ given in equation (7), but also the shape of the velocity distribution, through the transformation $Q = v^2/\alpha^2$ in equation (6), depends on the WIMP mass $m_\chi$ involved in the coefficient $\alpha$ defined in equation (5). In fact, any (assumed) value of $m_\chi$ will lead to a well-defined, normalized distribution function $f_1(v)$ when one uses equation (6). Hence, $m_\chi$ can be extracted from a single recoil spectrum only if one makes some assumptions about the velocity distribution $f_1(v)$. In contrast, by comparing two (or more) velocity distributions reconstructed from different recoil spectra with different target nuclei, one could avoid using these assumptions and estimate the WIMP mass model-independently.

The remainder of this article is organized as follows. In section 2, I first review a method for determining the WIMP mass with only one direct detection experiment. In section 3, I present a model-independent method for determining $m_\chi$ by combining two experimental data sets. Numerical results based on Monte Carlo simulations of future experiments and uncertainties and caveats of these two methods will also be discussed. I conclude in section 4. Some technical details for the data analysis will be given in an appendix.

2. With one experiment

In this section, I review the method for determining the WIMP mass with only one direct detection experiment based on a maximum likelihood analysis [8]–[11].

2.1. Maximum likelihood analysis

I first describe briefly some (standard) theoretical models/assumptions for fitting the elastic WIMP-nucleus scattering spectrum to experimental data. Then I discuss the determination of the WIMP mass by a maximum likelihood analysis. Note here that only the most commonly used models/assumptions are described as examples to show which information is required for the maximum likelihood analysis; however, it should be understood that other models or assumptions can also be used.
2.1.1. Simple model distributions. The simplest semi-realistic model halo is a Maxwellian halo. The 1D velocity distribution function in the rest frame of our galaxy can be expressed as [1, 6, 7]

\[ f_{1, \text{Gau}}(v) = \begin{cases} \frac{N_{\text{Gau}} v^2}{2} \left( e^{-v^2/v_0^2} - e^{-v_{\text{esc}}^2/v_0^2} \right), & \text{for } v \leq v_{\text{esc}}, \\ 0, & \text{for } v > v_{\text{esc}}. \end{cases} \tag{9} \]

Here \( v_0 \simeq 220 \text{ km s}^{-1} \) is the orbital velocity of the Sun in the galactic frame, and

\[ N_{\text{Gau}} = \left( \frac{\sqrt{\pi} v_0^3}{4} \right) \left( \frac{v_{\text{esc}}}{v_0} \right) \left( \frac{v_0^2}{2} + \frac{v_{\text{esc}}^2}{3} \right) e^{-v_{\text{esc}}^2/v_0^2} \] \tag{10}

is the normalization constant that satisfies

\[ \int_0^{v_{\text{esc}}} f_{1}(v) \, dv = 1. \tag{11} \]

Note that the second term on the right-hand side of equation (9) has been introduced to keep the velocity distribution continuous at \( v = v_{\text{esc}} \). Substituting equation (9) into equation (1), the integral over the velocity distribution function can be calculated as

\[ \int_{v_{\text{min}}}^{v_{\text{esc}}} \left[ \frac{f_{1, \text{Gau}}(v)}{v} \right] \, dv = \frac{N_{\text{Gau}}}{2} \left( \frac{v_0^2}{2} \right) \left( v_{\text{esc}}^2 - v_0^2 \right) \left[ \frac{e^{-\alpha^2 v_0^2}}{v_0^2} - \left( \frac{v_0^2 + v_{\text{esc}}^2 - \alpha^2 Q}{v_0^2} \right) e^{-v_{\text{esc}}^2/v_0^2} \right], \tag{12} \]

where \( v_{\text{min}} = \alpha \sqrt{Q} \) in equation (4) has been used. Note that, in the \( v_{\text{esc}} \to \infty \) limit, \( N_{\text{Gau}} \to 4/\sqrt{\pi} v_0^3 \) and the integral approaches \( (2/\sqrt{\pi} v_0) e^{-\alpha^2 v_0^2/v_0^2} \).

On the other hand, when we take into account the orbital motion of the solar system around the galaxy as well as that of the Earth around the Sun, the velocity distribution function should be modified to [1, 6, 7]

\[ f_{1, \text{sh}}(v) = N_{\text{sh}} v \left[ e^{-v^2/v_0^2} - e^{-v_{\text{esc}}^2/v_0^2} \right] \left[ e^{-v_{\text{esc}}^2/v_0^2} - e^{-v_{\text{esc}}^2/v_0^2} \right], \tag{13} \]

for \( v \leq v_{\text{esc}} \), with the normalization constant

\[ N_{\text{sh}} = \frac{\sqrt{\pi} v_0 v_{\text{esc}}}{2} \left[ \frac{\text{erf} \left( \frac{v_{\text{esc}} + v_e}{v_0} \right) + \text{erf} \left( \frac{v_{\text{esc}} - v_e}{v_0} \right)}{\sqrt{2}} \right] \left[ e^{-v_{\text{esc}}^2/v_0^2} - e^{-v_{\text{esc}}^2/v_0^2} \right] \] \tag{14}

Here \( v_e \) is the Earth’s velocity in the Galactic frame [1, 2, 5]:

\[ v_e(t) = v_0 \left[ 1.05 + 0.07 \cos \left( \frac{2 \pi (t - t_p)}{1 \text{ yr}} \right) \right] ; \tag{15} \]

\( t_p \simeq \) June 2nd is the date on which the velocity of the Earth relative to the WIMP halo is maximal. Consequently, an analytic form of the integral over this velocity distribution can be given as

\[ \int_{v_{\text{min}}}^{v_{\text{esc}}} \left[ \frac{f_{1, \text{sh}}(v)}{v} \right] \, dv = N_{\text{sh}} \left[ \frac{\sqrt{\pi} v_0}{2} \right] \left[ \frac{\text{erf} \left( \frac{\alpha \sqrt{Q} + v_e}{v_0} \right) - \text{erf} \left( \frac{\alpha \sqrt{Q} - v_e}{v_0} \right)}{\sqrt{2}} \right] \left[ e^{-v_{\text{esc}}^2/v_0^2} - e^{-v_{\text{esc}}^2/v_0^2} \right] \right] \left[ e^{-v_{\text{esc}}^2/v_0^2} - e^{-v_{\text{esc}}^2/v_0^2} \right] \] \tag{16}
For practical, numerical uses, an approximate form of the integral over $f_1(v)$ was introduced as [6]

$$
\int_{v_{\text{min}}}^{v_{\text{esc}}} \left[ \frac{f_1(v)}{v} \right] dv = c_0 \left( \frac{2}{\sqrt{\pi} v_0} \right) e^{\frac{-\sigma^2 Q / c_1 v_0^2}{v_0}}.
$$

(17)

where $c_0$ and $c_1$ are two fitting parameters of order unity. Not surprisingly, their values depend on the galactic orbital and escape velocities, the target nucleus, the threshold energy of the experiment, as well as on the mass of incident WIMPs. Note that the characteristic energy

$$
Q_{\text{ch}} \equiv \frac{c_1 v_0^2}{\alpha^2}
$$

and thus the shape of the recoil spectrum depend highly on the WIMP mass: for light WIMPs ($m_\chi \ll m_N$), $Q_{\text{ch}} \propto m_\chi^2$ and the recoil spectrum drops sharply with increasing recoil energy, while for heavy WIMPs ($m_\chi \gg m_N$), $Q_{\text{ch}} \sim \text{const.}$ and the spectrum becomes flatter.

2.1.2. Local WIMP density. Currently, the most commonly used value for the local WIMPs density in equation (2) is given as [1, 2]

$$
\rho_0 \approx 0.3 \text{ GeV cm}^{-3}.
$$

(19)

However, so far it can be estimated only by means of the measurement of the rotational velocity of our Galaxy. Due to our location inside the Milky Way, it is more difficult to measure the accurate rotation curve of our own Galaxy than those of other galaxies. Thus, an uncertainty of around a factor of 2 has been usually adopted [1, 2]:

$$
\rho_0 = 0.2–0.8 \text{ GeV cm}^{-3}.
$$

(20)

2.1.3. Spin-independent (SI) WIMP-nucleus cross section. In most theoretical models, the SI WIMP interaction on a nucleus with an atomic number $A \gtrsim 30$ dominates the spin-dependent (SD) interaction [1, 2]. Additionally, for the lightest supersymmetric neutralino, which is perhaps the best motivated WIMP candidate [1, 2], and for all WIMPs that interact primarily through Higgs exchange, the SI scalar coupling is approximately the same on both protons $p$ and neutrons $n$. The ’pointlike’ cross section $\sigma_0$ in equation (2) can thus be written as

$$
\sigma_0 = A^2 \left( \frac{m_{\chi N}}{m_{\chi p}} \right)^2 \sigma_{\chi p}^{\text{SI}},
$$

(21)

where

$$
\sigma_{\chi p}^{\text{SI}} = \left( \frac{4}{\pi} \right) m_{\chi p}^2 |f_p|^2
$$

(22)

($f_p$ is the effective $\chi \chi pp$ four-point coupling) and $A$ is the atomic number of the target nucleus.

2.1.4. Nuclear form factor. For the SI cross section, an analytic nuclear form factor can be used. The simplest one is the exponential form factor, first introduced by Ahlen et al [12] and Freese et al [5]:

$$
F_{\text{ex}}^2(Q) = e^{-Q / \rho_0}.
$$

(23)
Here \( Q \) is the recoil energy transferred from the incident WIMP to the target nucleus,

\[
Q_0 = \frac{1.5}{m_N R_0^2}
\]

is the nuclear coherence energy and

\[
R_0 = \left[ 0.3 + 0.91 \left( \frac{m_N}{\text{GeV}} \right)^{1/3} \right] \text{fm}
\]

is the radius of the nucleus. The exponential form factor implies a Gaussian form of the radial density profile of the nucleus. This Gaussian density profile is simple, but not very realistic. Engel has therefore suggested a more accurate form factor \[13\], inspired by the Woods–Saxon nuclear density profile \[1, 2\],

\[
F_{\text{WS}}^2(Q) = \left[ \frac{3 j_1(q R_1)}{q R_1} \right]^2 \text{e}^{-(qs)^2}.
\]

Here \( j_1(x) \) is a spherical Bessel function,

\[
q = \sqrt{2m_N Q}
\]

is the transferred three-momentum,

\[
R_1 = \sqrt{R_A^2 - 5s^2}
\]

is the effective nuclear radius\(^2\) with\(^3\) the root-mean-square radius

\[
R_A \simeq 1.2 A^{1/3} \text{ fm}
\]

and

\[
s \simeq 1 \text{ fm}
\]

is the nuclear skin thickness.

\(^2\) In the literature, the form factor given in equation (26) is also known as the ‘Helm’ form factor with \[6, 14\]

\[
R_1 = \sqrt{R_A^2 + \left( \frac{2}{3} \right) \pi^2 r_0^2 - 5s^2},
\]

where

\[
R_A \simeq (1.23 A^{1/3} - 0.6) \text{ fm}, \quad r_0 \simeq 0.52 \text{ fm}, \quad s \simeq 0.9 \text{ fm}.
\]

\(^3\) For \( R_1 \) given by equation (28) with \( s \simeq 1 \text{ fm} \), a more precise approximation for \( R_A \) has also been given \[6, 15\]:

\[
R_A \simeq (1.15 A^{1/3} + 0.39) \text{ fm}.
\]
2.1.5. Extended likelihood function. Now we are ready to put all pieces for predicting the elastic WIMP-nucleus scattering spectrum together and then fit this spectrum to experimental data by maximizing the logarithm of the extended likelihood function [10]:

\[ L = \frac{\lambda^{N_{\text{tot}}} N_{\text{tot}}!}{R} \prod_{a=1}^{N_{\text{tot}}} \left( \frac{dR}{dQ_a} \right) e^{-\lambda N_{\text{tot}}} . \]  

Here

\[ \lambda = \mathcal{E} \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{dR}{dQ} dQ \]  

is the expected event number with the (assumed) exposure of the experiment, \( \mathcal{E} \), \( N_{\text{tot}} \) is the total number of events recorded in one (simulated) experiment, \( Q_a \) are measured recoil energies in the data set between the minimal and maximal cut-off energies, \( Q_{\text{min}} \) and \( Q_{\text{max}} \), and

\[ R = \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{dR}{dQ} dQ \]  

is the total event rate.

Note that, firstly, the definition of \( L \) in equation (34) takes into account the fact that the event number \( N_{\text{tot}} \) and the measured recoil spectrum \( \mathcal{E}(dR/dQ) \) of each (simulated) experiment are not fixed. Secondly, except for \( c_0 \) and \( c_1 \) in equation (17), there are two fitting parameters in the extended likelihood function \( L \), i.e. the WIMP mass \( m_\chi \) (involved in \( \alpha \)) and the SI WIMP-proton cross section \( \sigma_{SI}^\chi p \).

2.2. Numerical results

Here I show some numerical results with 10 000 simulated experiments based on Monte Carlo simulations performed by Green [10, 11]. \(^{76}\text{Ge}\) has been chosen as the target nucleus with a threshold energy of 10 keV. A 3D Maxwellian velocity distribution in the Galactic rest frame for an isotropic isothermal WIMP halo, taking into account the Earth’s motion around the Sun with \( v_0 = 220 \text{ km s}^{-1} \) and \( v_{\text{esc}} = 540 \text{ km s}^{-1} \), and the Helm form factor in equations (26), (27), (29) and (30) have been used. The standard assumption for the local WIMP density of 0.3 GeV cm\(^{-3}\) has been adopted.

Note that the simulations demonstrated here as well as in the next section for the method combining two experimental data sets are based on several simplified assumptions\(^4\). Firstly, the sample to be analyzed contains only signal events, i.e. is free of background. Active background suppression techniques [16]–[18]\(^5\) should make this condition possible. Secondly, all experimental systematic uncertainties as well as the uncertainty on the measurement of the recoil energy have been ignored. The energy resolution of most existing detectors is so good that its error can be neglected when compared to the statistical uncertainty for the foreseeable future.

\(^4\) More realistic modelling with e.g. other WIMP velocity distributions and/or different nuclear form factors could in principle be incorporated into the maximum likelihood analysis.

\(^5\) For more experimental details about current direct detection techniques and the next generation detectors, see accompanying articles in this Focus issue.

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2.2.1. Statistical uncertainty. Figure 1 shows the distributions of the best-fit WIMP mass $m_\chi$ and SI WIMP-proton cross section $\sigma_{SI}^{\chi p}$ on the cross section versus WIMP mass plane. The input WIMP mass and the cross section are 100 GeV and $10^{-7}$ pb, respectively. The exposures have been assumed to be $3 \times 10^3$ (left) and $3 \times 10^4$ (right) kg-day and the corresponding expected event numbers are 78 and 780, respectively. It can be seen that, especially for the smaller exposure, the distribution is asymmetric and there are (significantly) more experiments with best-fit masses and cross sections larger than the input values. Quantitatively, for a WIMP mass of 100 GeV with $\sim 80$ events, the 1σ and 2σ statistical uncertainties are $+40^{-35}$ and $+100^{-50}$ GeV, respectively [10].

Figure 2 shows the 95% (solid) and 68% (dotted) confidence limits on the best-fit WIMP mass as functions of the input WIMP mass. The input SI WIMP-proton cross section has been set here as $10^{-8}$ pb. The assumed exposures are $3 \times 10^3$, $3 \times 10^4$ and $3 \times 10^5$ kg-day, respectively. We see here that since, as mentioned above, the shape of the recoil spectrum varies significantly with the WIMP mass for light WIMP masses ($m_\chi < m_N$), the WIMP mass (and also the cross section) can be fitted with a higher accuracy: the 1σ and 2σ statistical uncertainties for $m_\chi = 25$ GeV are $\pm 4$ GeV and $\pm 8$ GeV, and for $m_\chi = 50$ GeV are $\pm 12$ and $\pm 22$ GeV, respectively [10].

In contrast, the weak dependence of the shape of the recoil spectrum on the WIMP mass for heavy WIMP masses ($m_\chi \gg m_N$) means that it will be more difficult or even impossible to extract the WIMP mass with $\mathcal{O}(100)$ events if WIMPs are (much) heavier than the target nucleus [10]. Note that the dependence of the shape of the recoil spectrum on the WIMP mass as well as on that of the target nucleus suggests that heavy nuclei, e.g. Xe, would be able to measure the mass of heavy WIMPs more accurately; however, the rapid decrease of the nuclear

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Since the event number is directly proportional to the product of the cross section $\sigma_{SI}^{\chi p}$ and the exposure $\mathcal{E}$, we can assume that $\sigma_{SI}^{\chi p} = 10^{-8}$ pb with exposures of $3 \times 10^4$ and $3 \times 10^5$ kg-day.
form factor with increasing recoil energy, which occurs for heavy nuclei, means that, due to less expected events, this is in fact not necessarily the case.

2.2.2. Systematic uncertainties. Different sources of the systematic uncertainties in this model-dependent analysis have been considered [10, 11]. Figure 3 shows the distributions of the best-fit WIMP mass and cross section with different input orbital velocities of the solar system: $v_0 = 200$ (left) and 240 (right) km s$^{-1}$, while the standard value of $v_0 = 220$ km s$^{-1}$ has been used for the data analysis. The exposure assumed here is $3 \times 10^3$ kg-day. The other parameters are as in figures 1 (plots from [10]).
been used for the data analysis. As shown here, for an input WIMP mass of 100 GeV, there could be an $\sim \pm 20$ GeV shift in the best-fit WIMP mass combined with an $\sim \pm 10^{-8}$ pb ($\sim 10\%$) shift in the SI WIMP-proton cross section caused by the $\pm 20$ km s$^{-1}$ difference between the real and the assumed orbital velocities [10]. Moreover, the larger the real orbital velocity, the smaller the expected event number (with a fixed exposure), and thus the larger the statistical uncertainties on both the WIMP mass and the SI WIMP-proton cross section that one could obtain.

More detailed illustrations and discussions about the effects of varying the underlying WIMP mass and cross section, the detector target nucleus, the exposure, the minimal and maximal cut-off energies, the orbital velocity of the solar system, as well as the background event rate and its spectrum can be found in [10, 11, 19].

3. Combining two experiments

In this section, I first review the model-independent method for reconstructing the WIMP mass by using two experimental data sets with different target nuclei\(^7\). Then I also describe an extension of this method for estimating (or at least constraining) the SI WIMP-proton cross section.

3.1. Model-independent determination

As mentioned in the introduction, the normalized 1D velocity distribution function of incident WIMPs can be solved from equation (1) directly and, consequently, its generalized moments can be estimated by [20]

$$\langle v^n \rangle(v(Q_{\text{min}}), v(Q_{\text{max}})) = \int_{v(Q_{\text{min}})}^{v(Q_{\text{max}})} v^n f_1(v) \, dv$$

$$= \alpha^n \left[ \frac{2Q_{\text{min}}^{n+1/2}r(Q_{\text{min}})/F^2(Q_{\text{min}})}{2Q_{\text{min}}^{1/2}r(Q_{\text{min}})/F^2(Q_{\text{min}}) + I_0(Q_{\text{min}}, Q_{\text{max}})} \right].$$

(37)

Here $v(Q) = \alpha \sqrt{Q}$, $Q_{(\text{min,max})}$ are the experimental minimal and maximal cut-off energies,

$$r(Q_{\text{min}}) \equiv \left( \frac{dR}{dQ} \right)_{\text{expt}, \, Q=Q_{\text{min}}}$$

(38)

is an estimated value of the measured recoil spectrum $(dR/dQ)_{\text{expt}}$ (before the normalization by the exposure $\mathcal{E}$) at $Q = Q_{\text{min}}$ and $I_n(Q_{\text{min}}, Q_{\text{max}})$ can be estimated through the sum

$$I_n(Q_{\text{min}}, Q_{\text{max}}) = \sum_a \frac{Q_a^{(n-1)/2}}{F^2(Q_a)},$$

(39)

where the sum runs over all events in the data set that satisfy $Q_a \in [Q_{\text{min}}, Q_{\text{max}}]$. Note that, firstly, by using the second equation (37), $\langle v^n \rangle(v(Q_{\text{min}}), v(Q_{\text{max}}))$ can be determined independently of the local WIMP density $\rho_0$, of the velocity distribution function of incident WIMPs, $f_1(v)$, as well as of the WIMP-nucleus cross section $\sigma_0$. Secondly, as shown later,

\(^7\) In [8], the authors mentioned an attempt for using the maximum likelihood analysis with two (or more) detector materials. However, they found that, since the likelihood contours for different targets are pretty similar when simulating with the same number of events, their results were effectively little different from those obtained with a single experiment.

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3.1.1. Basic expressions for determining $m_X$. By requiring that the values of a given moment of $f_1(v)$ estimated by equation (37) from two detectors with different target nuclei, $X$ and $Y$, agree, $m_X$ appearing in the prefactor $\alpha^n$ on the right-hand side of equation (37) can be solved as [21]

$$m_X^{(\alpha^n)} = \frac{\sqrt{m_X m_Y} - m_X (R_{n,X} / R_{n,Y})}{R_{n,X} / R_{n,Y} - \sqrt{m_X / m_Y}},$$  \tag{40}$$

where

$$R_{n,X} \equiv \left[ \frac{2 Q_{\min,X}^{(n+1)/2} r_X(Q_{\min,X}) / F_X^2(Q_{\min,X}) + (n + 1) I_{n,X}}{2 Q_{\min,X}^{1/2} r_X(Q_{\min,X}) / F_X^2(Q_{\min,X}) + I_{0,X}} \right]^{1/n},$$  \tag{41}$$

and $R_{n,Y}$ can be defined analogously. Here $n \neq 0$, $m_{X,Y}$ and $F_{X,Y}(Q)$ are the masses and the form factors of nuclei $X$ and $Y$, respectively, and $r_{X,Y}(Q_{\min,X,Y})$ refer to the counting rates for detectors $X$ and $Y$ at the respective lowest recoil energies included in the analysis. Note that, first, the general expression (40) can be used either for SI or for SD scattering, one only needs to choose different form factors under different assumptions. Second, the form factors in the estimate of $I_{n,X}$ and $I_{n,Y}$ using equation (39) are also different.

On the other hand, by using the theoretical prediction that the SI WIMP-nucleus cross section given in equation (21) dominates, and the fact that the integral over the 1D WIMP velocity distribution on the right-hand side of equation (1) is the minus-first moment of this distribution, which can be estimated by equation (37) with $n = -1$, one can easily find that [20]

$$\rho_0 |f_p|^2 = \frac{\pi}{4 \sqrt{2}} \left( \frac{m_X + m_N}{E A^2 \sqrt{m_N}} \right) \left[ \frac{2 Q_{\min,X}^{1/2} r_X(Q_{\min,X})}{F_X^2(Q_{\min,X}) + I_{0,X}} \right].$$  \tag{42}$$

Note that the exposure of the experiment, $E$, appears in the denominator. Since the unknown factor $\rho_0 |f_p|^2$ on the left-hand side above is identical for different targets, it leads to a second expression for determining $m_X$ [20],

$$m_X^{(\sigma)} = \frac{(m_X / m_Y)^{3/2} m_Y - m_X (R_{\sigma,X} / R_{\sigma,Y})}{R_{\sigma,X} / R_{\sigma,Y} - (m_X / m_Y)^{3/2}}. \tag{43}$$

Here $m_{X,Y} \propto A_{X,Y}$ has been assumed,

$$R_{\sigma,X} \equiv \frac{1}{E_X} \left[ \frac{2 Q_{\min,X}^{1/2} r_X(Q_{\min,X})}{F_X^2(Q_{\min,X}) + I_{0,X}} \right],$$  \tag{44}$$

and similarly for $R_{\sigma,Y}$.

\footnote{All formulae needed for estimating $r(Q_{\min})$, $I_{\sigma}(Q_{\min}, Q_{\max})$ and their statistical errors are given in the appendix.}
3.1.3. Matching the cut-off energies. In order to yield the best-fit WIMP mass as well as to minimize its statistical error by combining the estimators for different \( n \) in equation (40) with each other and with the estimator in equation (43), a \( \chi^2 \) function has been introduced [20]

\[
\chi^2(m_\chi) = \sum_{i,j} (f_{i,X} - f_{i,Y}) C_{ij}^{-1} (f_{j,X} - f_{j,Y}),
\]

where

\[
f_{i,X} \equiv \alpha_X^i \left[ \frac{2Q_{\min,X}^{(i+1)/2} r_X(Q_{\min}/F_X^2(Q_{\min},x) + (i+1)I_{i,X}]}{2Q_{\min,X}^{1/2} r_X(Q_{\min}/F_X^2(Q_{\min},x) + I_{0,X}]} \right] \left( \frac{1}{300 \text{ km s}^{-1}} \right)^i,
\]

for \( i = -1, 1, 2, \ldots, n_{\text{max}} \), and

\[
f_{n_{\text{max}}+1,X} \equiv \xi_X \left[ \frac{A_X^2}{2Q_{\min,X}^{1/2} r_X(Q_{\min}/F_X^2(Q_{\min},x) + I_{0,X}]} \right] \left( \frac{\sqrt{m_\chi}}{m_\chi + m_X} \right) \frac{\sqrt{m_X}}{m_X + m_X};
\]

the other \( n_{\text{max}} + 2 \) functions \( f_{i,Y} \) can be defined analogously. Here \( n_{\text{max}} \) determines the highest moment of \( f_1(v) \) that is included in the fit. The \( f_i \) are normalized such that they are dimensionless and very roughly of order unity in order to alleviate numerical problems associated with the inversion of their covariance matrix. Note that the first \( n_{\text{max}} + 1 \) fit functions depend on \( m_\chi \) only through the overall factor \( \alpha \) and that \( m_\chi \) in equations (46a) and (46b) is now a fit parameter, which may differ from the true value of the WIMP mass. Finally, \( C \) in equation (45) is the total covariance matrix. Since the \( X \) and \( Y \) quantities are statistically completely independent, \( C \) can be written as a sum of two terms:

\[
C_{ij} = \text{cov}(f_{i,X}, f_{j,X}) + \text{cov}(f_{i,Y}, f_{j,Y}).
\]

3.1.2. \( \chi^2 \) fitting. In order to yield the best-fit WIMP mass as well as to minimize its statistical error by combining the estimators for different \( n \) in equation (40) with each other and with the estimator in equation (43), a \( \chi^2 \) function has been introduced [20]

\[
\chi^2(m_\chi) = \sum_{i,j} (f_{i,X} - f_{i,Y}) C_{ij}^{-1} (f_{j,X} - f_{j,Y}),
\]

where

\[
f_{i,X} \equiv \alpha_X^i \left[ \frac{2Q_{\min,X}^{(i+1)/2} r_X(Q_{\min}/F_X^2(Q_{\min},x) + (i+1)I_{i,X}]}{2Q_{\min,X}^{1/2} r_X(Q_{\min}/F_X^2(Q_{\min},x) + I_{0,X}]} \right] \left( \frac{1}{300 \text{ km s}^{-1}} \right)^i,
\]

for \( i = -1, 1, 2, \ldots, n_{\text{max}} \), and

\[
f_{n_{\text{max}}+1,X} \equiv \xi_X \left[ \frac{A_X^2}{2Q_{\min,X}^{1/2} r_X(Q_{\min}/F_X^2(Q_{\min},x) + I_{0,X}]} \right] \left( \frac{\sqrt{m_\chi}}{m_\chi + m_X} \right) \frac{\sqrt{m_X}}{m_X + m_X};
\]

the other \( n_{\text{max}} + 2 \) functions \( f_{i,Y} \) can be defined analogously. Here \( n_{\text{max}} \) determines the highest moment of \( f_1(v) \) that is included in the fit. The \( f_i \) are normalized such that they are dimensionless and very roughly of order unity in order to alleviate numerical problems associated with the inversion of their covariance matrix. Note that the first \( n_{\text{max}} + 1 \) fit functions depend on \( m_\chi \) only through the overall factor \( \alpha \) and that \( m_\chi \) in equations (46a) and (46b) is now a fit parameter, which may differ from the true value of the WIMP mass. Finally, \( C \) in equation (45) is the total covariance matrix. Since the \( X \) and \( Y \) quantities are statistically completely independent, \( C \) can be written as a sum of two terms:

\[
C_{ij} = \text{cov}(f_{i,X}, f_{j,X}) + \text{cov}(f_{i,Y}, f_{j,Y}).
\]

3.1.3. Matching the cut-off energies. The basic requirement of the expressions for determining \( m_\chi \) given in equations (40) and (43) is that, from two experiments with different target nuclei, the values of a given moment of the WIMP velocity distribution estimated by equation (37) should agree. This means that the upper cuts on \( f_1(v) \) in two data sets should be (approximately) equal\(^{10}\). Since \( v_{\text{cut}} = \alpha \sqrt{Q_{\text{max}}}, \) it requires that [20]

\[
Q_{\text{max},Y} = \left( \frac{\alpha_X}{\alpha_Y} \right)^2 Q_{\text{max},X}.
\]

Note that \( \alpha \) defined in equation (5) is a function of the true WIMP mass. Thus, this relation for matching optimal cut-off energies can be used only if \( m_\chi \) is already known. One possibility to overcome this problem is to fix the cut-off energy of the experiment with the heavier target, minimize the \( \chi^2(m_\chi) \) function defined in equation (45), and estimate the cut-off energy for the lighter nucleus by equation (48) algorithmically [20].

\(^9\) Formulae needed for estimating the entries of \( C \) will be given in the appendix.

\(^{10}\) Here the threshold energies have been assumed to be negligibly small.
3.2. Numerical results

Here I show some numerical results for the reconstructed WIMP mass based on Monte Carlo simulations. The upper and lower bounds on the reconstructed WIMP mass are estimated from the requirement that \( \chi^2 \) exceeds its minimum by 1.\(^{11}\) \(^{28}\)Si and \(^{76}\)Ge have been chosen as two target nuclei. The scattering cross section has been assumed to be dominated by SI interactions. The shifted Maxwellian velocity distribution given in equation (13) (the second term involving \( v_{\text{esc}} \) has been neglected) with \( v_0 = 220 \text{ km s}^{-1} \), \( v_e = 1.05 v_0 \),\(^{12}\) and \( v_{\text{esc}} = 700 \text{ km s}^{-1} \) and the Woods–Saxon form factor in equation (26) have been used. The threshold energies of two experiments were assumed to be negligible and the maximal experimental cut-off energies are set as 100 keV. \( 2 \times 5000 \) experiments have been simulated. In order to avoid large contributions from very few events in the high energy range to higher moments \([7]\), only moments up to \( n_{\text{max}} = 2 \) were included in the \( \chi^2 \) fit.

3.2.1. Statistical uncertainty. In figures 4, the dotted (green) curves show the median reconstructed WIMP mass and its 1\( \sigma \) upper and lower bounds for the case where both \( Q_{\text{max, Si}} \) and \( Q_{\text{max, Ge}} \) have been fixed to 100 keV. As argued earlier, the values of a given moment of the WIMP velocity distribution estimated by equation (37) do not agree when the same maximal cut-off energy for both experimental data sets is used. This causes a systematic underestimate of the reconstructed WIMP mass \([21]\), which can be obviously seen here. The solid (black) curves were obtained by using equation (48) for matching the cut-off energy \( Q_{\text{max, Si}} \) perfectly with \( Q_{\text{max, Ge}} = 100 \text{ keV} \) and the true (input) WIMP mass, whereas the dashed (red) curves show the case where \( Q_{\text{max, Ge}} = 100 \text{ keV} \), and \( Q_{\text{max, Si}} \) has been determined by minimizing \( \chi^2(m_\chi; Q_{\text{max, Si}}) \). As shown here, with only 50 events on average before cuts (upper frame) from each experiment, the algorithmic process seems already to work pretty well for WIMP masses up to \( \sim 500 \text{ GeV} \). For \( m_\chi \lesssim 100 \text{ GeV} \), the median WIMP mass determined in this way overestimates its true value by 15% to 20%; however, the true WIMP mass always lies within the median limits of the 1\( \sigma \) statistical error interval estimated by the algorithmic \( Q_{\text{max}} \) matching procedure up to even \( m_\chi = 1 \text{ TeV} \) \([20]\).

3.2.2. Statistical fluctuation. In order to study the statistical fluctuation of the reconstructed WIMP mass by algorithmic \( Q_{\text{max}} \) matching in the simulated experiments, an estimator \( \delta m \) has been introduced as in \([20]\)

\[
\delta m = \begin{cases} 
1 + \frac{m_{\chi,\text{lo1}} - m_{\chi,\text{in}}}{m_{\chi,\text{lo1}} - m_{\chi,\text{lo2}}}, & \text{if } m_{\chi,\text{in}} \leq m_{\chi,\text{lo1}}; \\
\frac{m_{\chi,\text{rec}} - m_{\chi,\text{in}}}{m_{\chi,\text{rec}} - m_{\chi,\text{lo1}}}, & \text{if } m_{\chi,\text{lo1}} < m_{\chi,\text{in}} < m_{\chi,\text{rec}}; \\
\frac{m_{\chi,\text{rec}} - m_{\chi,\text{in}}}{m_{\chi,\text{rec}} - m_{\chi,\text{hi1}}}, & \text{if } m_{\chi,\text{rec}} < m_{\chi,\text{in}} < m_{\chi,\text{hi1}}; \\
\frac{m_{\chi,\text{hi1}} - m_{\chi,\text{in}}}{m_{\chi,\text{hi1}} - m_{\chi,\text{hi2}}} - 1, & \text{if } m_{\chi,\text{in}} \geq m_{\chi,\text{hi1}}.
\end{cases}
\]

\(^{11}\) The median, rather than the mean, values for the (bounds on the) reconstructed WIMP mass are shown.

\(^{12}\) The time dependence of the Earth’s velocity in the Galactic frame, the second term of \( v_e(t) \) in equation (15), has been ignored.
Figure 4. Results for the reconstructed WIMP mass as well as its 1σ statistical error interval based on the $\chi^2$ fit in equation (45). 50 (upper) and 500 (lower) events on average before cuts from each experiment have been simulated. See the text for further details (plots from [20]).

Here $m_{\chi,\text{in}}$ is the true (input) WIMP mass, $m_{\chi,\text{rec}}$ is its reconstructed value, $m_{\chi,\text{lo1}(2)}$ are the 1 (2) σ lower bounds satisfying $\chi^2(m_{\chi,\text{lo1}(2)}) = \chi^2(m_{\chi,\text{rec}}) + 1$ (4) and $m_{\chi,\text{hi1}(2)}$ are the corresponding 1 (2) σ upper bounds. It has been found that the error intervals of the median reconstructed WIMP mass are quite asymmetric; similarly, the distance between the 2σ and 1σ limits can be quite different from the distance between the 1σ limit and the central value [20].\textsuperscript{13} The asymmetry has also been observed by the maximum likelihood analysis.

\textsuperscript{13} Recall that the same asymmetry has also been observed by the maximum likelihood analysis.
Figure 5. Normalized distribution of the estimator $\delta m$ defined in equation (49) for an input WIMP mass of 50 GeV with 50 events on average (before cuts) in each experiment. The other parameters and notations are as in figures 4 (plot from [20]).

The definition of $\delta m$ in equation (49) takes these differences into account and also keeps track of the sign of the deviation: if the reconstructed WIMP mass is larger (smaller) than the true one, $\delta m$ is positive (negative). Moreover, $|\delta m| \leq 1(2)$ if and only if the true WIMP mass lies between the experimental 1 (2) $\sigma$ limits.

Figure 5 shows the distribution of $\delta m$ calculated from 5000 simulated experiments with 50 events on average before cuts for a rather light WIMP mass of 50 GeV. In this case, simply fixing both $Q_{\text{max}}$ values to 100 keV still works fine (see the upper frame of figures 4). However, the distributions for both fixed $Q_{\text{max}}$ and optimal $Q_{\text{max}}$ matching look somewhat lopsided, since the error interval is already asymmetric, with $m_{\chi,\text{hi}} - m_{\chi,\text{rec}} > m_{\chi,\text{rec}} - m_{\chi,\text{lo}}$. The overestimate of light WIMP masses reconstructed by algorithmic $Q_{\text{max}}$ matching shown in figures 4 is reflected by the dashed (red) histogram here, which has significantly more entries at positive values than at negative values. These distributions also indicate that the statistical uncertainties estimated by minimizing $\chi^2(m_{\chi})$ are indeed overestimated, since nearly 90% of the simulated experiments have $|\delta m| \leq 1$ [20], much more than $\sim 68\%$ of the experiments, that a usual $1\sigma$ error interval should contain.

Unfortunately, as shown in figures 6, when the true (input) WIMP mass increases to 200 GeV and the expected event number (before cuts) increases to 500 (right frame), the situations become less favorable. While optimal $Q_{\text{max}}$ matching seems to approach Gaussian
Figure 6. Normalized distribution of the estimator $\delta m$. Parameters and notations are as in figure 5, except that the input WIMP mass has been increased to 200 GeV. In the right frame, the average event number (before cuts) in each experiment has, in addition, been increased from 50 to 500. Note that the bins at $\delta m = \pm 5$ are overflow bins, i.e. they also contain all experiments with $|\delta m| \geq 5$ (plots from [20]).

very slowly and the overestimated statistical errors become slightly more reliable for larger event numbers [20], the errors estimated by the algorithmic procedure for determining $Q_{\text{max}, Si}$ are not very reliable in the simulations.

More detailed illustrations and discussions about algorithmic $Q_{\text{max}}$ matching with different detector materials or with data sets generated in different halo models, as well as about the statistical fluctuation in the analysis, can be found in [20].

3.3. Estimating the SI WIMP-proton coupling

In the maximum likelihood analysis discussed in section 2, the SI WIMP-proton cross section $\sigma_{\chi p}^{SI}$ is the second fitting parameter that, combined with the WIMP mass $m_{\chi}$, maximizes the extended likelihood function $L$ calculated from an assumed WIMP velocity distribution.

In contrast, as shown above, by combining two experimental data sets, one can estimate the WIMP mass $m_{\chi}$ without knowing the WIMP-nucleus cross section $\sigma_{0}$. Conversely, by means of equation (42), one can also estimate or at least constrain the SI WIMP-proton coupling, $|f_p|^2$, from experimental data directly without knowing the WIMP mass [22].

3.3.1. Making an assumption for the local WIMP density. In equation (42) the WIMP mass $m_{\chi}$ on the right-hand side can be determined by the method described above, $r(Q_{\text{min}})$ and $I_0$ can also be estimated from one of the two data sets used for determining $m_{\chi}$ or from a third experiment. Nevertheless, due to the degeneracy between the local WIMP density $\rho_0$ and the
coupling $|f_p|^2$, one cannot estimate both of them independently. The simplest way is to make an assumption for the local WIMP density $\rho_0$.\(^{14}\)

3.3.2. Numerical results. The left frame of figures 7 shows the reconstructed SI WIMP-proton coupling $|f_p|^2$ as a function of the input WIMP mass $m_{\chi,\text{in}}$. Following simulations for the reconstruction of the WIMP mass, $^{28}\text{Si}$ and $^{76}\text{Ge}$ were chosen as two target nuclei for estimating $m_{\chi}$ in equation (42). In order to avoid complicated calculations of the correlation between the error on the reconstructed $m_{\chi}$ and that on the estimator of $I_0$, a second, independent data set with $^{76}\text{Ge}$ was chosen as the third target for estimating $I_0$. The SI WIMP-proton cross section was set as $10^{-8}$ pb. Each experimental data set has 50 events on average under the common experimental cut-off energy $Q_{\text{max}}$ chosen as 100 GeV.

It can be seen that the reconstructed $|f_p|^2$ values are underestimated for WIMP masses $\gtrsim 100$ GeV. This systematic deviation is caused mainly by the underestimate of $I_0$. However, in spite of this systematic deviation (and in fact due to the fairly large statistical uncertainty),

\(^{14}\) Note that, since the coupling $|f_p|^2$ estimated by equation (42) is inversely proportional to the local density $\rho_0$, whose common value falls at the lower end of the possible range (see equations (19) and (20)), one can therefore at least give an upper bound on this coupling.

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the true value of $|f_p|^2$ always lies within the $1\sigma$ statistical error interval. Moreover, for a WIMP mass of 100 GeV, one could in principle already estimate the SI WIMP-proton coupling with a statistical uncertainty of only $\sim 15\%$ with just 50 events from each experiment. Recall that this is much smaller than the systematic uncertainty of the local dark matter density (of a factor of 2 or even larger).

Combining the estimate for the SI WIMP-proton coupling with the estimate for the WIMP mass, the right frame of figures 7 shows the reconstructed coupling $|f_p|^2_{\text{rec}}$ and the reconstructed WIMP mass $m_{\chi,\text{rec}}$ on the cross section (coupling) versus WIMP mass plane. It is important to note that, as shown here, $|f_p|^2$ and $m_{\chi}$ can be estimated separately and from experimental data directly with neither prior knowledge of each other nor an assumption for the WIMP velocity distribution.

4. Summary and conclusions

In this paper, I have reviewed the methods for the determination(s) of the mass (and eventually the SI cross section on nucleons) of WIMPs with positive signals of their elastic scattering off target nuclei in direct dark matter detection experiments.

With only one experiment, the WIMP mass combined with its SI cross section on nucleons could be estimated by the maximum likelihood analysis using a theoretically predicted scattering spectrum fitted to the measured recoil energies. If WIMPs are light ($m_\chi < m_N$), the shape of the recoil spectrum is sensitive to their mass; then the WIMP mass (and also the cross section) can be estimated with a higher accuracy. However, in the case where WIMPs are (much) heavier than the target nucleus ($m_\chi \gtrsim 200$ GeV), the recoil spectrum becomes nearly independent of $m_\chi$ and it is then more difficult or even impossible to estimate the WIMP mass reasonably with $O(100)$ events.

The maximum likelihood analysis depends on the prior assumption for the velocity distribution of halo WIMPs as well as on the local WIMP density. For a WIMP mass of 100 GeV, an $\sim 10\%$ measurement uncertainty on the orbital velocity of the solar system could cause an $\sim 20\%$ systematic error on the best-fit WIMP mass combined with an $\sim 10\%$ error on the SI WIMP-proton cross section.

In order to determine the WIMP mass without making any assumption for the WIMP velocity distribution, I described a second method based on the reconstruction of (the moments of) the WIMP velocity distribution function from two experiments with different target nuclei. This method can be used without knowing the WIMP-nucleus cross section. The only information needed is the measured recoil energies. By matching the maximal cut-off energies of two experiments, one could in principle estimate the WIMP mass up to $\sim 500$ GeV with $O(50)$ events from each experiment.

Nevertheless, the algorithmic procedure for determining the maximal cut-off energy of the experiment with the lighter target nucleus by minimizing $\chi^2$ could overestimate the WIMP mass by 15–20% if WIMPs are light, or lead to unreliable error estimates if WIMPs are heavy. The latter could become worse with larger event samples. However, the fact that optimal $Q_{\text{max}}$ matching works well in all cases, for both the median reconstructed WIMP mass and its statistical error, gives us hope that a better algorithm for $Q_{\text{max}}$ matching can be found that relies only on the data.

\footnote{Plots shown here were calculated by a different program than that for the Monte Carlo simulations shown in figures 4–6.}
Additionally, by combining two (or three) experimental data sets, one could also estimate the SI WIMP-proton coupling without knowing the WIMP mass. Although, due to the degeneracy between the local WIMP density and the WIMP-nucleus cross section, one needs to adopt the local dark matter density (as the unique assumption), at least an upper bound on this coupling could be given. In fact, for a WIMP mass of 100 GeV, with $O(50)$ events from each experiment, a statistical uncertainty of $\sim 15\%$ could be reached. This is much smaller than the systematic uncertainty on the local dark matter density (of a factor of 2 or even larger).

In summary, by means of currently running and projected experiments using detectors with $10^{-9}$–$10^{-11}$ pb sensitivities [16]–[18] (see footnote 5), we stand a good chance of detecting dark matter particles, if dark matter indeed consists (mainly) of WIMPs. Then the methods presented here can be used to estimate the mass (and eventually the cross section on nucleons) of dark matter particles. This information (perhaps combined with information from indirect detection experiments [19]) will allow us not only to constrain the parameter space in different extensions of the Standard Model of particle physics, but also to identify WIMPs among new particles produced at colliders (hopefully in the near future). Once one is confident of this identification, one can use further collider measurements of the mass and couplings of WIMPs. Together with the reconstruction of the velocity distribution of halo WIMPs [7], this will then yield a new determination of the local WIMP density. On the other hand, knowledge of the WIMP couplings will also permit prediction of the WIMP annihilation cross section. Together with information on the WIMP density, this will allow one to predict the event rate in the indirect dark matter detection [1, 2] as well as to test our understanding of the early Universe.

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Appendix. Formulae needed in section 3

Here I list all formulae needed in the model-independent method described in section 3. Detailed derivations and discussions can be found in [7, 20].

A.1 Estimating $r(Q_{\text{min}})$, $I_n(Q_{\text{min}}, Q_{\text{max}})$ and their statistical errors

First, consider experimental data described by

$$Q_n - \frac{b_n}{2} \leq Q_{n,i} \leq Q_n + \frac{b_n}{2}, \quad i = 1, 2, \ldots, N_n, \quad n = 1, 2, \ldots, B. \quad (A.1)$$

Here the total energy range between $Q_{\text{min}}$ and $Q_{\text{max}}$ has been divided into $B$ bins with central points $Q_n$ and widths $b_n$. In each bin, $N_n$ events will be recorded. Since the recoil spectrum $dR/dQ$ is expected to be approximately exponential, the following ansatz for the spectrum in
the \( n \)th bin has been introduced [7]:
\[
\left( \frac{dR}{dQ} \right)_n \equiv \left( \frac{dR}{dQ} \right)_{Q=Q_n} \equiv r_n e^{k_n(Q_n - Q_{s,n})}. \tag{A.2}
\]
Here \( r_n \) is the standard estimator for \( \frac{dR}{dQ} \) at \( Q = Q_n \):
\[
r_n = \frac{N_n}{b_n}, \tag{A.3}
\]
\( k_n \) is the logarithmic slope of the recoil spectrum in the \( n \)th bin, which can be computed numerically from the average \( Q \) value in the \( n \)th bin:
\[
\frac{Q - Q_n}{b_n} = \left( b_n \right) \coth \left( \frac{k_n b_n}{2} \right) - \frac{1}{k_n}, \tag{A.4}
\]
where
\[
\frac{(Q - Q_n)^2}{b_n} \equiv \frac{1}{N_n} \sum_{i=1}^{N_n} (Q_{n,i} - Q_s)^2. \tag{A.5}
\]
The error on the logarithmic slope \( k_n \) can be computed from equation (A.4) directly:
\[
\sigma^2(k_n) = k_n^4 \left( 1 - \left[ \frac{k_n b_n / 2}{\sinh(k_n b_n / 2)} \right]^2 \right)^{-1} \sigma^2\left( \frac{Q - Q_n}{b_n} \right), \tag{A.6}
\]
with
\[
\sigma^2\left( \frac{Q - Q_n}{b_n} \right) = \frac{1}{N_n - 1} \left[ \frac{(Q - Q_n)^2}{b_n} - \frac{(Q - Q_n)^2}{b_n} \right]. \tag{A.7}
\]
\( Q_{s,n} \) in the ansatz (A.2) is the shifted point at which the leading systematic error due to the ansatz is minimal [7],
\[
Q_{s,n} = Q_n + \frac{1}{k_n} \ln \left[ \frac{\sinh(k_n b_n / 2)}{k_n b_n / 2} \right]. \tag{A.8}
\]
Note that \( Q_{s,n} \) differs from the central point of the \( n \)th bin, \( Q_n \). From the ansatz (A.2), the counting rate at \( Q = Q_{\text{min}} \) can be calculated by
\[
r(Q_{\text{min}}) = r_1 e^{k_1(Q_{\text{min}} - Q_{s,1})}, \tag{A.9}
\]
and its statistical error can be expressed as
\[
\sigma^2(r(Q_{\text{min}})) = r^2(Q_{\text{min}}) \left\{ \frac{1}{N_1} + \frac{1}{k_1} - \left( \frac{b_1}{2} \right) \left( 1 + \coth \left( \frac{b_1 k_1}{2} \right) \right) \right\} \sigma^2(k_1), \tag{A.10}
\]
since
\[
\sigma^2(r_n) = \frac{N_n}{b_n^2}. \tag{A.11}
\]
Finally, since all \( I_n \) are determined from the same data, they are correlated with
\[
\text{cov}(I_n, I_m) = \sum_a \frac{Q_a^{(n+m-2)/2}}{F^a(Q_a)}, \tag{A.12}
\]
where the sum again runs over all events with recoil energy between \( Q_{\text{min}} \) and \( Q_{\text{max}} \). And the correlation between the errors on \( r(Q_{\text{min}}) \), which is calculated entirely from the events in the
first bin, and on $I_n$ is given by
\[
\text{cov}(r(Q_{\text{min}}), I_n) = r(Q_{\text{min}})I_n(Q_{\text{min}}, Q_{\text{min}} + b_1) \left\{ \frac{1}{N_i} + \left[ \frac{1}{k_1} - \left( \frac{b_1}{2} \right) \left( 1 + \coth \left( \frac{b_1 k_1}{2} \right) \right) \right] \right\} \times \frac{I_{n+2}(Q_{\text{min}}, Q_{\text{min}} + b_1)}{I_n(Q_{\text{min}}, Q_{\text{min}} + b_1)} - Q_1 + \frac{1}{k_1} - \left( \frac{b_1}{2} \right) \coth \left( \frac{b_1 k_1}{2} \right) \right] \sigma^2(k_1) \}; \quad \text{(A.13)}
\]

note that the sums $I_n$ here only count in the first bin, which ends at $Q = Q_{\text{min}} + b_1$.

On the other hand, with a functional form of the recoil spectrum (e.g. fitted to experimental data), $(dR/dQ)_{\text{expt}}$, one can use the following integral forms to replace the summations given above. Firstly, the average $Q$ value in the $n$th bin defined in equation (A.5) can be calculated by
\[
(Q - Q_n)^2|_{I_n} = \frac{1}{N_n} \int_{Q_n-b_n/2}^{Q_n+b_n/2} (Q - Q_n)^2 \left( \frac{dR}{dQ} \right)_{\text{expt}} dQ. \quad \text{(A.14)}
\]

For $I_n(Q_{\text{min}}, Q_{\text{max}})$ given in equation (39), we have
\[
I_n(Q_{\text{min}}, Q_{\text{max}}) = \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{Q^{(n-1)/2}}{F^2(Q)} \left( \frac{dR}{dQ} \right)_{\text{expt}} dQ, \quad \text{(A.15)}
\]

and similarly for the covariance matrix for $I_n$ in equation (A.12),
\[
\text{cov}(I_n, I_m) = \int_{Q_{\text{min}}}^{Q_{\text{max}}} \frac{Q^{(n+m-2)/2}}{F^4(Q)} \left( \frac{dR}{dQ} \right)_{\text{expt}} dQ. \quad \text{(A.16)}
\]

Recall that $(dR/dQ)_{\text{expt}}$ is the measured recoil spectrum before the normalization by the exposure. Finally, $I_n(Q_{\text{min}}, Q_{\text{min}} + b_1)$ needed in equation (A.13) can be calculated by
\[
I_n(Q_{\text{min}}, Q_{\text{min}} + b_1) = \int_{Q_{\text{min}}}^{Q_{\text{max}}+b_1} Q^{(n-1)/2} \left[ r_1 \epsilon^{\lambda_1(Q-Q_{\text{min}})} \right] dQ. \quad \text{(A.17)}
\]

Note that $r(Q_{\text{min}})$ and $I_n(Q_{\text{min}}, Q_{\text{min}} + b_1)$ should be estimated by equations (A.9) and (A.17) with $r_1$, $k_1$ and $Q_{n,1}$ estimated by equations (A.3), (A.4) and (A.8) in order to use the other formulae for estimating the (correlations between the) statistical errors without any modification.

A.2 Statistical errors on $m_x$ given in equations (40) and (43)

The expression for $m_x|^{(n)}$ given in equation (40) leads to a lengthy expression for its statistical error:
\[
\sigma(m_x)|^{(n)} = \sqrt{m_x/m_y | m_x - m_y \rangle \langle R_{n,x}/R_{n,y} | R_{n,x}/R_{n,y} - \sqrt{m_x/m_y} |^2} 
\times \left[ \frac{1}{R_{n,x}^2} \sum_{i,j=1}^{3} \left( \frac{\partial R_{n,x}}{\partial c_{i,x}} \right) \left( \frac{\partial R_{n,x}}{\partial c_{j,x}} \right) \text{cov}(c_{i,x}, c_{j,x}) + (X \rightarrow Y) \right]^{1/2}. \quad \text{(A.18)}
\]

Here a short-hand notation for the six quantities on which the estimate of $m_x$ depends has been introduced:
\[
c_{1,x} = I_{n,x}, \quad c_{2,x} = I_{0,x}, \quad c_{3,x} = r_X(Q_{\text{min}}, x), \quad \text{(A.19)}
\]
and similarly for the \( c_{i,Y} \). Estimators for \( \text{cov}(c_i, c_j) \) have been given in equations (A.12) and (A.13). Explicit expressions for the derivatives of \( \mathcal{R}_{n,X} \) with respect to \( c_{i,X} \) are

\[
\frac{\partial \mathcal{R}_{n,X}}{\partial I_{n,X}} = \frac{n+1}{n} \left[ \frac{F_X^2(Q_{\text{min},X})}{2Q_{\text{min},X}^{(n+1)/2}r_X(Q_{\text{min},X}) + (n+1)I_{n,X}F_X^2(Q_{\text{min},X})} \right] \mathcal{R}_{n,X}, \tag{A.20a}
\]

\[
\frac{\partial \mathcal{R}_{n,X}}{\partial I_{0,X}} = -\frac{1}{n} \left[ \frac{F_X^2(Q_{\text{min},X})}{2Q_{\text{min},X}^{1/2}r_X(Q_{\text{min},X}) + I_{0,X}F_X^2(Q_{\text{min},X})} \right] \mathcal{R}_{n,X} \tag{A.20b}
\]

and

\[
\frac{\partial \mathcal{R}_{n,X}}{\partial r_X(Q_{\text{min},X})} = \frac{2}{n} \left[ \frac{Q_{\text{min},X}^{(n+1)/2}I_{0,X} - (n+1)Q_{\text{min},X}^{1/2}I_{n,X}}{2Q_{\text{min},X}^{(n+1)/2}r_X(Q_{\text{min},X}) + (n+1)I_{n,X}F_X^2(Q_{\text{min},X})} \right] \times \left[ \frac{F_X^2(Q_{\text{min},X})}{2Q_{\text{min},X}^{1/2}r_X(Q_{\text{min},X}) + I_{0,X}F_X^2(Q_{\text{min},X})} \right] \mathcal{R}_{n,X}; \tag{A.20c}
\]

explicit expressions for the derivatives of \( \mathcal{R}_{n,Y} \) with respect to \( c_{i,Y} \) can be given analogously. Note that, firstly, factors \( \mathcal{R}_{n,(X,Y)} \) appear in all these expressions, which can practically be cancelled by the prefactors in the bracket in equation (A.18). Secondly, all the \( I_{0,X}, I_{0,Y}, I_{n,X} \) and \( I_{n,Y} \) should be understood to be computed according to equations (39) or (A.15) with integration limits \( Q_{\text{min}} \) and \( Q_{\text{max}} \) specific for that target.

Similar to the analogy between equations (40) and (43), the statistical error on \( m_X \left| \sigma \right. \) given in equation (43) can be expressed as

\[
\sigma (m_X) \mid_\sigma = \frac{(m_X/m_Y)^{5/2} |m_X - m_Y| (\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y})}{[\mathcal{R}_{\sigma,X}/\mathcal{R}_{\sigma,Y} - (m_X/m_Y)^{5/2}]^2} \times \left[ \frac{1}{\mathcal{R}_{\sigma,X}^2} \sum_{i,j=2}^{3} \left( \frac{\partial \mathcal{R}_{\sigma,X}}{\partial c_{i,X}} \right) \left( \frac{\partial \mathcal{R}_{\sigma,X}}{\partial c_{j,X}} \right) \text{cov}(c_{i,X}, c_{j,X}) + (X \rightarrow Y) \right]^{1/2}, \tag{A.21}
\]

where we have again used the short-hand notation in equation (A.19); note that \( c_{1,(X,Y)} = I_{n,(X,Y)} \) do not appear here. Expressions for the derivatives of \( \mathcal{R}_{\sigma,X} \) can be computed from equation (44) as

\[
\frac{\partial \mathcal{R}_{\sigma,X}}{\partial I_{0,X}} = \left[ \frac{F_X^2(Q_{\text{min},X})}{2Q_{\text{min},X}^{1/2}r_X(Q_{\text{min},X}) + I_{0,X}F_X^2(Q_{\text{min},X})} \right] \mathcal{R}_{\sigma,X}, \tag{A.22a}
\]

\[
\frac{\partial \mathcal{R}_{\sigma,X}}{\partial r_X(Q_{\text{min},X})} = \left[ \frac{2Q_{\text{min},X}^{1/2}}{2Q_{\text{min},X}^{1/2}r_X(Q_{\text{min},X}) + I_{0,X}F_X^2(Q_{\text{min},X})} \right] \mathcal{R}_{\sigma,X}, \tag{A.22b}
\]

and similarly for the derivatives of \( \mathcal{R}_{\sigma,Y} \).

A.3 Covariance of \( f_i \), defined in equations (46a) and (46b)

The entries of the \( \mathcal{C} \) matrix in equation (47) involving basically only the moments of the WIMP velocity distribution can be read off equation (82) of [7], with a slight modification due to the
normalization factor in equation (46a):\
\[\text{cov}(f_i, f_j) = N_m^2 \left( f_i f_j \text{ cov}(I_0, I_0) + \tilde{\alpha}^{i+j} (i + 1)(j + 1) \text{ cov}(I_i, I_j) \right)\]
\[\left. - \tilde{\alpha}^i (j + 1) f_i \text{ cov}(I_0, I_j) - \tilde{\alpha}^j (i + 1) f_j \text{ cov}(I_0, I_i) \right) + D_i D_j r^2 (r(Q_{\min})) - (D_i f_i + D_j f_j) \text{ cov}(r(Q_{\min}), I_0)\]
\[+ \tilde{\alpha}^i (j + 1) D_i \text{ cov}(r(Q_{\min}), I_j) + \tilde{\alpha}^j (i + 1) D_j \text{ cov}(r(Q_{\min}), I_i) \right].\]
(A.23)

Here
\[N_m \equiv \frac{1}{2 Q_{\min}^1 r(Q_{\min}) / F^2(Q_{\min}) + I_0},\]
\[\tilde{\alpha} \equiv \frac{\alpha}{300 \text{ km s}^{-1}},\]
and
\[D_i \equiv \frac{1}{N_m} \left( \frac{\partial f_i}{\partial r(Q_{\min})} \right) = \frac{2}{F^2(Q_{\min})} \left( \tilde{\alpha}^{i} Q_{\min}^{(i+1)/2} - Q_{\min}^{1/2} f_i \right),\]
(A.26a)
for \(i = -1, 1, 2, \ldots, n_{\max}\), and
\[D_{n_{\max}+1} = \frac{2}{F^2(Q_{\min})} \left( -Q_{\min}^{1/2} f_{n_{\max}+1} \right).\]
(A.26b)

A.4 Statistical error on \(|f_p|^2\) given in equation (42)

From equation (42), it can easily be found that
\[\sigma(|f_p|^2) = |f_p|^2 \left[ \frac{\sigma^2(m_x)}{(m_x + m_N)^2} + N_m^2 \sigma^2(1/N_m) + \frac{2 N_m \text{ cov}(m_x, 1/N_m)}{(m_x + m_N)} \right]^{1/2},\]
where \(N_m\) is defined in equation (A.24), and
\[\sigma^2(1/N_m) = \left[ \frac{2 Q_{\min}^{1/2}}{F^2(Q_{\min})} \right]^2 \left[ \sigma^2(r(Q_{\min})) + \sigma^2(I_0) + 2 \frac{2 Q_{\min}^{1/2}}{F^2(Q_{\min})} \text{ cov}(r(Q_{\min}), I_0) \right].\]
(A.28)
The correlation between the error on the reconstructed \(m_x\) and that on the estimator of \(1/N_m\), the third term in equation (A.27), can be neglected in case one uses three independent data sets.

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\(^{16}\) Since the last \(f_i\) defined in equation (46b) can be computed from the same basic quantities, i.e. the counting rates at \(Q_{\min}\) and the integrals \(I_0\), it can directly be included in the covariance matrix.
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