Generalized theta functions, projectively flat vector bundles and noncommutative tori

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Abstract

In this paper, the well known relationship between theta functions and Heisenberg group actions thereon is resumed by merging complex algebraic and noncommutative geometry: in essence, we describe Hermitian-Einstein vector bundles on 2-tori via representations of noncommutative tori, thereby reconstructing Matsushima’s setup [11] and making the ensuing Fourier-Mukai-Nahm (FMN) aspects transparent. We prove the existence of noncommutative torus actions on the space of smooth sections of Hermitian-Einstein vector bundles on 2-tori preserving the eigenspaces of a natural Laplace operator. Motivated by the Coherent State Transform approach to theta functions ([7, 22]), we extend the latter to vector valued thetas and develop an additional algebraic reinterpretation of Matsushima’s theory making FMN-duality manifest again.

Keywords: Hermitian-Einstein vector bundles; generalized theta functions; Heisenberg groups; Fourier-Mukai transform; noncommutative tori.

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1 Introduction

In this paper we address the well-known intriguing and multifaceted relationship between theta functions and representations of Heisenberg groups (both infinite and finite [11, 13, 15]), from a blended complex differential geometric
and noncommutative geometric viewpoint, possibly bringing in novel insights and, in particular, improving the treatment given in [20]. Specifically, in Section 3, via a series of Propositions, we prove the existence of a representation of the noncommutative torus $\mathbb{A}_{1/\theta}$ ($\theta = q/r$, $q$ and $r$ being coprime positive integers) on the space of sections $\Gamma(\mathcal{E}_{r,q})$ of a projectively flat Hermitian-Einstein holomorphic vector bundle (or HE-vector bundle for short) $\mathcal{E}_{r,q}$ of rank $r$ and degree $q$ on a two-dimensional torus. This representation will actually preserve the eigenspaces (“Landau levels”) of a natural Laplace operator (essentially, a quantum harmonic oscillator), hence, in particular, its holomorphic sections, thereby recovering the classical algebraic-geometric portrait. The vector bundle $\mathcal{E}_{r,q}$ itself, in turn, can be manufactured from a representation of $\mathbb{A}_\theta$ on its the typical fibre. Another representation of $\mathbb{A}_\theta$ on $\Gamma(\mathcal{E}_{r,q})$, commuting with the representation of $\mathbb{A}_{1/\theta}$, is produced out of the parallel transport pertaining to the Chern-Bott connection on $\mathcal{E}_{r,q}$. The above developments bring in a vivid portrait of the Fourier-Mukai-Nahm (FMN, [12, 14]) duality between $\mathcal{E}_{r,q}$ and $\mathcal{E}_{q,r}$ together with their respective Chern-Bott connections. Actually, all objects, representations and bundles, will come in (torus-) families (moduli). In Section 4, upon resorting to the well-known heat equation interpretation of theta functions (described via the so-called Coherent State Transform (CST) of [7]) and further insisting on a noncommutative torus perspective, we present a “matrix” description of Matsushima’s theory making again the above duality manifest. Finally, we prove that, as pre-$C^*$-algebras (and for the unique $C^*$-tensor product involved), $\mathbb{A}_{q/r} \otimes \mathbb{A}_{r/q}$ and $\mathbb{A}_{1/rq}$ are isomorphic. This will be a byproduct of a “categorical” reinterpretation of Gauss sums identities also shedding light on Fourier-Mukai-Nahm transform issues. Moreover, a vector analogue of the CST will be set up. The layout of the paper is completed by Section 2 – gathering together background material from different areas in order to fix notation and to pave the ground for the successive developments in Sections 3 and 4 – and by Section 5, pointing out possible applications and further research directions.

2 Preliminary tools

In this section we establish our notation and collect several miscellaneous technical tools for the benefit of a wider readership.

2.1 $k$-level theta functions and the Coherent State Transform

We begin by providing minimal background on $k$-level theta functions and on their relationship with the heat equation closely following the exposition of [7] (see also [22])—up to slight notational changes—and referring to it for a complete treatment. We restrict to the genus one case, namely to an Abelian torus $(M, \tau)$. 

2
Let us start from the following (tempered) distributions on $S^1$

$$\theta^\ell_0(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i (\ell + kn)x}$$

with $\ell = 0, 1, \ldots, k - 1$. They are mapped, via the so-called Coherent State Transform (CST):

$$\text{CST}(\theta^\ell_0)(z) = \partial_\ell(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (\ell + kn)(\tau/k)(\ell + kn)} e^{2\pi i (\ell + kn)z}$$

to the $k$-level theta functions. These, in turn, are interpreted as holomorphic sections of the $k$-th power of the so called Theta line bundle, and yield a basis thereof, as a consequence of the Riemann-Roch theorem. A far reaching generalization for HE-vector bundles has been developed by Matsushima [11] and his theory will be retrieved and elaborated on in what follows.

2.2 Review of Matsushima’s theory

In this subsection we outline Matsushima’s theory [11], tailoring the exposition to our purposes and referring to [10], especially Ch.IV-7 and to [20], Section 3.2, for background material. Here we just recall that an irreducible holomorphic vector bundle (i.e. without proper holomorphic direct summands) admits a HE-metric if and only if it is stable in the algebraic-geometric sense: this is the celebrated Kobayashi-Hitchin correspondence, fully established for compact Kähler manifolds in [23]. In particular, projectively flat holomorphic vector bundles (i.e. equipped with a Hermitian metric whose corresponding canonical (Chern-Bott) connection has constant curvature) are stable. This will be the case for all bundles dealt with in the present work.

Let $r$ and $q$ be coprime positive integers, i.e. $\gcd(r, q) = 1$. Let $V$ a one-dimensional complex vector space and let us consider a complex torus $V/L$ where $L \cong \mathbb{Z}^2$ is a lattice. Let $L' \subset L$ be a complete sublattice of $L$. Specifically, if $L = \langle \omega_1, \omega_2 \rangle$ is the lattice generated by a (real) basis $\{\omega_j\}_{j=1,2}$ of $V$, let $L' = \langle r\omega_1, \omega_2 \rangle$ and

$$K := L/L' \cong \mathbb{Z}_r,$$

thus we have an $r$-covering of complex tori

$$\varphi : V/L' \rightarrow V/L.$$

Let $A$ be the $\mathbb{Q}$-valued form defined by $A(\omega_1, \omega_2) = q/r$, and $A' = rA$ the $\mathbb{Z}$-valued form fulfilling $A'(\omega_1, \omega_2) = q$ (the pull-back of $A$ via $\varphi$). The form $A$ gives rise to a HE-vector bundle $E_{r,q} \rightarrow V/L$ —of rank $r$ and degree $q$— i.e. such that its canonical (Chern-Bott) connection has constant curvature $\Omega = -2\pi i A$. Correspondingly one has a HE-line bundle $E_{1,q} \rightarrow V/L'$, the $q$-level theta line bundle over $V/L'$, related to the form $A'$. Let us denote, as usual, by $H^0(E)$ the space of holomorphic sections of a holomorphic vector bundle $E$, of dimension $h^0(E)$. It is known (by Riemann-Roch) that $h^0(E_{r,q}) = h^0(E_{1,q}) = q$, thus the
corresponding section spaces are isomorphic. Now, given an orthonormal basis of \( H^0(E_{1,q}) \) made up by \( q \)-level theta functions \( \{ \vartheta_m \}_{m=0}^{q-1} \), one has, according to Matsushima, a splitting of \( \varphi^*E_{r,q} \rightarrow V/L' \) as

\[
\varphi^*E_{r,q} \cong \bigoplus_{\sigma \in K} (E_{1,q})_{\sigma}
\]

where \( (E_{1,q})_{\sigma} \) is a translate of \( E_{1,q} \) and any two different translates being non isomorphic as holomorphic line bundles.

Therefore, one has an injective correspondence

\[
\mathcal{M} : H^0(E_{r,q}) \rightarrow \bigoplus_{\sigma \in \mathbb{Z}} H^0(E_{1,q})
\]
given by (picking an orthonormal basis \( \{ s_m \}, m = 0, 1, \ldots, q-1 \))

\[
\mathcal{M} : s_m \mapsto \text{vec}(\vartheta_m) := \left[ (\sigma \cdot \vartheta_m)_{\sigma \in \mathbb{Z}} \right]
\]

where the map \( \text{vec} \) arranges the translates of \( \vartheta_m \in H^0(E_{1,q}) \) into a column vector.

We do not spell out the action of \( \sigma \) in the original Matsushima picture in detail, since we shall essentially recover it anew in what follows, see Section 4.2.

### 2.3 The noncommutative torus

General references for the present subsection are, among others, [1,3]. Let \( \theta \in \mathbb{R} \). The noncommutative torus is the pre-C*-algebra \( \mathcal{A}_\theta \) consisting of rapidly decaying series

\[
a = \sum_{n,m=-\infty}^{\infty} a_{nm} u^n v^m, \quad a_{n,m} \in \mathbb{C}
\]

where \( u, v \) are unitary operators in a Hilbert space \( \mathcal{H} \) satisfying the relation

\[
v u = e^{2\pi i \theta} u v.
\]

We have a natural smooth structure on \( \mathcal{A}_\theta \) given by the noncommutative integral

\[
\tau(a) = a_{00}, \quad a \in \mathcal{A}_\theta,
\]

and noncommutative derivatives

\[
\partial_1(u^n v^m) = imu^n v^m, \quad \partial_2(u^n v^m) = imu^n v^m.
\]

In the sequel we shall take \( \theta \in \mathbb{Q} \), \( \theta > 0 \) and, ultimately, we shall deal with \( \theta = q/r \), with \( q \) and \( r \) positive and coprime.
2.4 The Canonical Commutation Relations and the quantum harmonic oscillator

In this subsection we assemble basic information on the quantum harmonic oscillator and its relationship to the Canonical Commutation Relations and the associated Stone-von Neumann theorem [24], referring to the comprehensive survey [17] for elucidation of their modern ramifications.

Let us consider a representation of the Canonical (or Weyl-Heisenberg) Commutation Relations (CCR) on a (necessarily infinite dimensional) separable Hilbert space $H$,

$$[Q, P] = i1$$

(one degree of freedom), with $Q$ and $P$ (“position” and “momentum” operators, respectively) unbounded self-adjoint operators on a suitable domain. In order to avoid problems arising from the latter issue (see however [16, Section X.6], for amplification and further use, together with [6]) the CCR are reformulated (Weyl) in integral form:

$$U(a) V(b) = e^{iab} V(b) U(a), \quad a, b \in \mathbb{R}$$

with $Q$ and $P$ becoming the infinitesimal generators of the one parameter unitary groups $U(\cdot)$ and $V(\cdot)$, respectively.

The quantum harmonic oscillator Hamiltonian reads

$$H = \frac{1}{2}(P^2 + Q^2) = A^\dagger A + \frac{1}{2} = \frac{1}{2}(A^\dagger A + AA^\dagger)$$

in terms of annihilation and creation operators

$$A = \frac{1}{\sqrt{2}} (Q + iP) \quad A^\dagger = \frac{1}{\sqrt{2}} (Q - iP)$$

subject to the commutation relation

$$AA^\dagger - A^\dagger A = 1.$$

In the irreducible case the spectrum of $H$ only consists of simple eigenvalues \{\(n + \frac{1}{2}\)\}_{n=0}^\infty and the $n$-th eigenspace $H_n$ is generated by $\phi_n = (1/\sqrt{n!})(A^\dagger)^n \phi_0$, with the ground state $\phi_0$ fulfilling $A\phi_0 = 0$. The operator $A^\dagger A$, namely, the Hamiltonian without constant term (“zero-point energy”) is called number operator.

In general the multiplicity of a representation of the CCR (phrased into Weyl’s integral form) is given by $k = \dim H_0$: this is a version of the Stone-von Neumann uniqueness theorem (see e.g. [24]).

2.5 Gauss sums

First of all, let us recall the celebrated Gauss sums (see [8], Section 5.6): 

$$S(\mu, r) := \sum_{0 \leq \ell \leq r-1} e^{2\pi i \ell^2 \mu r},$$
for integers $\mu$ and $r$, the latter different from zero, together with the well-known multiplicative formula

$$S(\mu q, r)S(\mu r, q) = S(\mu, rq)$$

valid for coprime integers $r$ and $q$ and any integer $\mu$ (cf. [8, Theorem 64]).

Here is a quick outline of the proof. The r.h.s. reads

$$\sum_{k=0}^{r-1} e^{2\pi i \frac{k^2}{rq}}.$$ 

Now, upon exploiting the group isomorphism

$$\mathbb{Z}_r \times \mathbb{Z}_q \cong \mathbb{Z}_{rq}$$

stemming from the equation

$$q \cdot [\ell]_r + r \cdot [m]_q = [k]_{rq}$$

which, given a residue class $[k]$ modulo $rq$, yields unique residue classes $[\ell]$ modulo $r$ and $[m]$ modulo $q$ (the converse being clear), we see that, setting $k = \ell q + m r$ (no dependence on representatives), the r.h.s. splits into the product appearing in the l.h.s. Explicitly:

$$\frac{k^2}{rq} = \frac{(\ell q + mr)^2}{rq} = \frac{\ell^2 q}{r} + \frac{m^2 r}{q} + 2\ell m$$

and the last term in the r.h.s. exponentiates to 1. Notice that the problem of finding $[k]_{rq}$ such that $[k]_r = [\ell]_r$ and $[k]_q = [m]_q$, with given classes $[\ell]_r$ and $[m]_q$ is solved via the Chinese Remainder Theorem: if $a$ and $b$ are integers such that $ar + bq = 1$, then $k = qb\ell + ram$, see again [8, Theorem 121].

## 3 Representations of noncommutative tori and HE-vector bundles

In this Section we reinterpret the Matsushima construction of holomorphic HE-vector bundles over a two-dimensional torus $\mathbb{C}/\Lambda$ — with lattice $\Lambda = \langle 1, \tau \rangle$ and $\operatorname{Im} \tau > 0$ — via representations of the two-dimensional noncommutative torus, see also [20]. This will be unfolded through the following series of propositions.

**Proposition 3.1.** An irreducible representation of $\mathcal{A}_\theta$ on a finite-dimensional Hilbert space $\mathcal{H}$ produces a HE-vector bundle $E_\theta \to \mathbb{C}/\Lambda$ over a two-dimensional torus $\mathbb{C}/\Lambda$, where $\Lambda = \langle 1, \tau \rangle$, $\operatorname{Im}(\tau) > 0$ with degree $\theta \dim(\mathcal{H})$ and rank $\dim(\mathcal{H})$.

**Proof.** Let $v, u$ be unitary operators on a finite-dimensional Hilbert space $\mathcal{H}$ satisfying (1), then if $\gamma = n + \tau m \in \Lambda$ the function (theta multiplier)

$$J_{\gamma}(z) = e^{\frac{iz}{\sqrt{\tau}}(v + \frac{1}{2}\gamma)^2} e^{\pi i \theta nm} u^{-n} v^{-m}, \quad z = x + \tau y \in \mathbb{C}, \gamma \in \Lambda,$$

where $\gamma = n + \tau m \in \Lambda$. This will be unfolded through the following series of propositions.

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**Proof.** Let $v, u$ be unitary operators on a finite-dimensional Hilbert space $\mathcal{H}$ satisfying (1), then if $\gamma = n + \tau m \in \Lambda$ the function (theta multiplier)

$$J_{\gamma}(z) = e^{\frac{iz}{\sqrt{\tau}}(v + \frac{1}{2}\gamma)^2} e^{\pi i \theta nm} u^{-n} v^{-m}, \quad z = x + \tau y \in \mathbb{C}, \gamma \in \Lambda,$$
satisfying
\[ J_{\gamma + \delta}(z) = J_\gamma(z + \delta)J_\delta(z), \]
defines a holomorphic vector bundle \( E \) with generic fibre \( \mathcal{H} \) over the torus \( \mathbb{C}/\Lambda \) given as the quotient
\[(\mathbb{C} \times \mathcal{H})/\sim \]
where
\[(z + \gamma, v) \sim (z, J_\gamma(z)v).\]
Its (smooth) sections \( s: \mathbb{C} \to \mathcal{H} \) (collectively denoted by \( \Gamma(E) \)) are then characterized by the following boundary conditions:
\[ s(z + 1) = e^{i\frac{\theta}{\text{Im}(\tau)}}u(s(z)), \quad s(z + \tau) = e^{i\frac{\theta}{\text{Im}(\tau)}|\tau|^2}v^*(s(z)). \] (3)

On \( \Gamma(E) \) we have a Hermitian structure \((\cdot | \cdot)\) given by
\[ (s|s')(z) = \langle s(z), s'(z) \rangle_{\mathcal{H}}h(z) \]
with Chern-Bott connection (the unique connection compatible with the Hermitian and the holomorphic structure)
\[ \nabla = \left( d - \frac{\theta \pi}{\text{Im}(\tau)} \bar{z}dz \right) \otimes 1_{\mathcal{H}} \]
having constant curvature and Chern class
\[ \frac{i}{2\pi} \nabla^2 = \theta \omega_1_{\mathcal{H}}, \quad c_1(E) = \theta \dim(\mathcal{H})\omega \]
with
\[ \omega = \frac{i}{2\text{Im}(\tau)}dz \wedge d\bar{z} \otimes 1. \]
Indeed, a short computation shows that, if \( Q = i\nabla_{\frac{\phi}{\phi}}, \quad P = i\nabla_{\frac{\bar{\phi}}{\phi}} \), we have
\[ \frac{1}{2\pi i}[Q, P] = \theta 1. \]
Moreover, it is clear that the rank of \( E \) is \( c_0(E) = \dim(\mathcal{H}) \). This vector bundle will be our \( E_\theta \).

**Proposition 3.2.** The correspondence that assigns to each representation \( \pi \) of \( A_\theta \) a holomorphic vector bundle \( E_\pi \) over the torus \( \mathbb{C}/\Lambda \) is functorial.

**Proof.** Let \( \pi : A_\theta \to \mathcal{B}(\mathcal{H}_\pi) \) and \( \sigma : A_\theta \to \mathcal{B}(\mathcal{H}_\sigma) \) be two such representations and let \( U : \mathcal{H}_\pi \to \mathcal{H}_\sigma \) be an intertwining unitary map. Then the map on sections
\[ \psi_U : \Gamma(E_\pi) \to \Gamma(E_\sigma) \]
given by
\[ (\psi_U(s))(z) = U(s(z)) \]
is an isomorphism of $C(\mathbb{T})$-modules. The above map is indeed well defined, i.e.
that it maps sections to sections:

$$(\psi_U(s))(z + n + \tau n) = e^{\frac{\pi i \theta}{r} (z + \frac{1}{r}|\tau|^2)} e^{\pi i \theta n m} U \pi(u)^{-n} \pi(v)^{-m} s(z)$$

$$= e^{\frac{\pi i \theta}{r} (z + \frac{1}{r}|\tau|^2)} e^{\pi i \theta n m} \sigma(u)^{-n} \sigma(v)^{-m} U s(z)$$

$$= e^{\frac{\pi i \theta}{r} (z + \frac{1}{r}|\tau|^2)} e^{\pi i \theta n m} \sigma(u)^{-n} \sigma(v)^{-m} (\psi_U s)(z).$$

$$\square$$

**Corollary 3.1.** Since $A_\theta$ is Morita equivalent to the classical torus, its repre-
sentations are indexed by points in $\mathbb{T}^2$, they all produce, topologically the same
vector bundle.

**Proof.** Let $u, v$ denote a representation of the noncommutative torus $A_\theta$, then
any other representation is given by $\rho(z, 0) u, \rho(0, w) v$ where $\rho$ is a 1-dimensional
representation of $\mathbb{T}^2$. Then the map $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ given by

$$F(m, z) = (m, z \rho(m))$$

maps one vector bundle to the other (given by the distinct theta characters $J$).  
$$\square$$

**Proposition 3.3.** Let $\pi : A_{q/r} \to B(\mathcal{H})$ be an irreducible finite dimen-
sional representation of the noncommutative torus $A_{q/r}$ where $r$ and $q$ are positive and
coprime. Then, the dimension of $\mathcal{H}$ is $r$ and $\pi(u) = U$, $\pi(v) = V$ fulfil

$$U^r = a \mathbf{1} \quad \text{and} \quad V^r = b \mathbf{1}, \quad (4)$$

for some $a, b \in S^1$.

**Proof.** Let $V, U$ define a finite-dimensional representation on $\mathcal{H}$ of dimension $d$.
Let us write $\theta = q/r$ with $\gcd(r, q) = 1$. Taking the determinant of $vu = e^{2\pi i \theta} uv$ we see that
$\theta d \in \mathbb{Z}$. Also observe that, since $U^r$ and $V^r$ commute with the representation
by Schur’s lemma, there is a constant $a \in S^1$ such that $U^r = a \mathbf{1}$; therefore, the minimal polynomial
of $u$, call it $P$, has to divide $x^r - a$ and has degree at most $r$. Moreover, $P$ satisfies $VP(U)V^* = P(U e^{2\pi i \theta})$ = 0 and
the polynomial $Q(x) = P(x e^{2\pi i \theta})$ must also satisfy $Q(U) = 0$; thus, given a root $\lambda$ of $P$ we see that $e^{2\pi i \theta} \lambda$ is a different root of it, whence the polynomial
$P(x) := (x - \lambda)(x - e^{2\pi i/\lambda}) \cdots (x - e^{2\pi i(r-1)/r} \lambda)$ divides $P$ and, having the same
degree as $P$, coincides with it. In particular $P(x) = x^r - \lambda^r$. Let $\varphi \neq 0$ be
an eigenvector for $U$, then $\{U^n V^m \varphi\}_{n, m = 0}^{r-1}$ generate the whole Hilbert space,
one checks that $U^n V^m \varphi = \lambda^n e^{-2\pi i n m d} V^m \varphi$ so the dimension $d$ of the Hilbert
space $\mathcal{H}$ is at most $r$, and therefore equal to $r$.  
$$\square$$

Note that if $s, t \in S^1$ and $s^r \neq 1, t^r \neq 1$ then

$$\pi'(u) = s U, \quad \pi'(V) = t V,$$
defines a second irreducible representation since the minimal polynomial of \( U' = \pi'(u) \) is now given by \( U'' - s'a \mathbf{1} \) (notational abuse) and the minimal polynomial is intrinsic to the representation \( \pi \not\cong \pi' \).

**Proposition 3.4.** Let \( \theta = \frac{q}{r} \), with \( q \) and \( r \) positive and coprime. Let \( \pi \) be an irreducible representation of \( \mathcal{A}_\theta \), inducing as above a HE-vector bundle \( E_\theta \to \mathbb{C}/\Lambda \). Then, on the space \( \mathcal{H} \) consisting of the \( L^2 \)-sections of \( E_\theta \) and in particular on the holomorphic ones (i.e. the Matsushima generalised (vector) theta functions) we have a natural representation of \( \mathcal{A}_{1/\theta} \) given by the operators

\[
(\hat{v}s)(z) = e^{\frac{\theta}{2\pi i}(\frac{1}{2} \frac{1}{z - \frac{1}{\theta}})^2} s \left( z - \frac{1}{\theta} \right) \tag{5}
\]

\[
(\hat{u}s)(z) = e^{\frac{\theta}{2\pi i}(\frac{1}{2} \frac{1}{z - \frac{1}{\theta}})^2} s \left( z - \frac{1}{\theta} \right). \tag{6}
\]

That is, \( \hat{u}, \hat{v} \) are unitary operators satisfying \( \hat{u}\hat{v} = e^{2\pi i \frac{1}{\theta}} \hat{u}\hat{v} \).

**Proof.** By virtue of Proposition 3.3, \( \dim \mathcal{H} = r \), so we shall denote \( E_\theta \) also as \( E_{r,q} \). Then notice that \( Q = i\nabla_{\frac{1}{\theta}x}, P = i\nabla_{\frac{1}{\theta}y} \) are symmetric and essentially self-adjoint on \( \Gamma(E_{r,q}) \) since for all \( s, s' \in \Gamma(E_{r,q}) \)

\[
\int_{\mathbb{C}/\Lambda} \left[ \left( \nabla_{\frac{1}{\theta}x} s | s' \right) + \left( s | \nabla_{\frac{1}{\theta}y} s' \right) \right] = \int_{\mathbb{C}/\Lambda} \frac{\partial}{\partial x} (s | s') = 0
\]

and similarly for \( \nabla_{\frac{1}{\theta}y} \). Essential self-adjointness ultimately follows from Nelson’s analytic vector theorem, see e.g. [16], X.6, together with Example 2.

Then, on the same domain, we have

\[
\frac{1}{2\pi i} [Q, P] = \theta \cdot 1_{\mathcal{H}}
\]

and we shall check below that \( P \) and \( Q \) and hence \( \nabla_{\frac{1}{\theta}y} = \frac{1}{2\text{Im}(\tau)} Q - \frac{1}{2\text{Im}(\tau)} P \) commute with \( \hat{u}, \hat{v} \). So in the space \( \mathcal{H} \) we have operators satisfying

\[
\hat{v}\hat{u} = e^{2\pi i \frac{1}{\theta}} \hat{u}\hat{v}
\]

\[
V(b)U(a) = e^{2\pi i \theta ab} U(a) V(b), \quad a, b \in \mathbb{R}
\]

where \( U(a) = e^{iaQ}, V(b) = e^{ibP} \) (parallel transport operators along the fundamental directions). In particular, setting \( \hat{u} := U(1) \) and \( \hat{v} := V(1) \) we get

\[
\hat{v}\hat{u} = e^{2\pi i \theta} \hat{u}\hat{v}
\]

Rephrasing a bit, we have then both a representation of the Heisenberg group \( \mathbb{H} = \mathbb{C} \times S^1 \) with parameter \( \theta \) and a representation of the discrete Heisenberg group \( \mathbb{H}_r = \Lambda \times S^1 \) with parameter \( r \):

\[
W(z, t) = e^{2\pi i \theta} e^{i\pi \theta xy} U(x) V(y), \quad z = x + \tau y \in \mathbb{C}
\]
\[ \hat{w}(n, t) = t^n e^{i \pi n_1 n_2} \tilde{w}_n, \quad n = n_1 + \tau n_2 \in \Lambda \]

where the group structures are given, respectively, by

\[(z, t)(z', t') = (z + z', tt'e^{i\pi(x'y'-x')}), \quad z, z' \in \mathbb{C}, \quad t, t' \in S^1\]

and

\[(n, t)(n', t') = (n + n', tt'e^{i\pi n_1 n_2 - n_1 n_2'}), \quad n, n' \in \Lambda.\]

Now we claim that

\[ W(z, t) \hat{w}(n, s) = \hat{w}(n, s) W(z, t) \]

for all \(z, t, n, s\). To see this, it is enough to show that, on \(\Gamma(E_{r,q})\),

\[ [Q, \hat{u}] = 0 = [Q, \hat{v}] \]

and that the same relation holds for \(P\). Let us start with the proof for \(Q\). Let

\[ \alpha = \frac{\theta \pi}{\text{Im}(\tau)}. \]

We have

\[ -iQ\hat{u}s(z) = \frac{\partial}{\partial x} \hat{u}s(z) - \alpha z \hat{u}s(z) \]

\[ -i\hat{u}Qs(z) = \hat{u} \frac{\partial s}{\partial x}(z) - z - \frac{1}{\theta} \hat{u}s(z) \]

so if \((\hat{u}s)(z) = e^{\beta(z)}s(z - 1/\theta),\)

\[ [-iQ, \hat{u}]s(z) = \left[ \frac{\partial}{\partial x}, \hat{u} \right] s(z) - \alpha \frac{\partial s}{\partial x} = e^{\beta(z)}s(z - 1/\theta) - \beta s(z - 1/\theta) \]

Now, repeating the computation for \(P\) we find:

\[ -iP\hat{u}s(z) = \frac{\partial}{\partial y} \hat{u}s(z) - \alpha \tau z \hat{u}s(z) \]

\[ -i\hat{u}Ps(z) = \hat{u} \frac{\partial s}{\partial y}(z) - \alpha z - \frac{1}{\theta} \hat{u}s(z) \]

Thus

\[ [-iP, \hat{u}]s(z) = \left[ \frac{\partial}{\partial y}, \hat{u} \right] s(z) - \tau \frac{\partial \hat{u}s(z)}{\partial y} = \frac{\partial \beta}{\partial y} e^{\beta s(z - 1/\theta)} - \tau \frac{\partial s(z - 1/\theta)}{\partial y} \]

In the same vein, if \(\hat{v}s(z) = e^{\gamma s(z - \tau/\theta)},\)

\[ [-iQ, \hat{v}]s(z) = \left[ \frac{\partial}{\partial x}, \hat{v} \right] s(z) - \alpha \frac{\partial s(z - \tau/\theta)}{\partial x} = \left( \frac{\partial \gamma}{\partial \theta} - \gamma \frac{1}{\theta} \right) \]

\[ [-iP, \hat{v}]s(z) = \left[ \frac{\partial}{\partial y}, \hat{v} \right] s(z) - \alpha \frac{\partial s(z - \tau/\theta)}{\partial y} = \left( \frac{\partial \gamma}{\partial y} - \gamma \frac{1}{\theta} \right) \]

yielding the conclusion.
We also notice the following consequence.

**Proposition 3.5.** We have $\hat{u}^q = \mu \cdot 1, \hat{v}^q = \nu \cdot 1$, where $\mu, \nu \in S^1$ are constants given by $u' = \mu, v' = \nu$.

**Proof.** We find, successively

\[
(\hat{u}^q)(z) = e^{\frac{a}{\mu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{a}{\mu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{a}{\mu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{a}{\mu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{a}{\mu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

where we used (2) in the second to last equality. Similarly,

\[
(\hat{v}^q)(z) = e^{\frac{b}{\nu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{b}{\nu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{b}{\nu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{b}{\nu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

\[
= e^{\frac{b}{\nu} r (q (z^{q-1}) - \frac{q-1}{q})} \left( z - \frac{q}{\theta} \right)
\]

Note that, in accordance with the Stone-von Neumann theorem, the representation $W$ is not irreducible: indeed, by Riemann-Roch, its multiplicity is precisely $\theta \dim(H) = q/r$, also cf. [18][20].

**Corollary 3.2.** Each $k$-level theta line bundle over the two-dimensional torus produces an irreducible finite dimensional representation of $A_\theta$ with $\theta = 1/k$.

In view of the preceding discussion (Subsection 2.4) on the harmonic oscillator we have (with obvious and inessential notational changes), the following result.

**Proposition 3.6.** Let $\Delta = (\nabla_{\theta x})^* \nabla_{\theta x} \equiv \Delta^* A$ be the Laplacian on $E_\theta$ (the "number operator"). Then its spectrum only consists of eigenvalues, whose eigenspaces are finite-dimensional with the same dimension $q$ and each carrying a representation of $A_{1/\theta} = A_{r/q}$ (with generators $\hat{u}, \hat{v}$).
Remark 3.1. The above proposition, together with the preceding developments, reformulate and possibly improve (in the classical case) the celebrated results establishing the action of finite Heisenberg groups on spaces of theta functions (viewed as holomorphic sections of line bundles on complex tori), see e.g. [13] (esp. vol. III) and [15] (cf. in particular the remark in Section 3.1).

Proposition 3.7. (Bimodule structure). The HE-vector bundle $E_{r,q} \to \mathbb{C}/\Lambda$ (actually, its space of smooth sections $\Gamma(E_{r,q})$) comes equipped with a $A_\theta - A_{-1/\theta}$ bimodule structure, where $A_\theta$ acts on the left by $\hat{u} \hat{v}$, and $A_{1/\theta}$ acts on the right by $\check{u}$ and $\check{v}$.

Proof. This is clear in view of the preceding discussion. The minus sign comes from regarding $\hat{u}$ and $\check{v}$ as acting on the right.

Remark 3.2. It is known that above algebras are actually strongly Morita equivalent, see e.g. the remarks in [20] and [19]. It would be interesting to explicitly compare the two algebra-valued Hermitian structures involved.

4 Gauss sums, vector theta functions and the FMN-transform

The above construction can be interpreted in terms of the so-called Fourier Mukai-Nahm (FMN*) transform (plus dualization) as in [20] (see in particular Section 4.3). For background on the FMN-transform see e.g. —in addition to the original sources [12, 14]— the article [21] and the textbook [2].

Specifically, in view of Proposition 3.3, an irreducible representation $\pi'$ of $A_{1/\theta}$ on a finite dimensional Hilbert space $\mathcal{H}'$ yields in turn, à la Matsushima, a HE-vector bundle $E'_{1/\theta} \to \mathbb{C}/\Lambda$ with rank $q = \dim H^0(E_\theta) = \dim \mathcal{H}'$ and degree $r$ (FMN* dual to $E_\theta \to \mathbb{C}/\Lambda$) equipped with a Chern-Bott connection $\nabla'$ having constant curvature and Chern class given respectively by

$$\frac{i}{2\pi} \nabla'^2 = (1/\theta) \omega 1_{\mathcal{H}'}, \quad c_1(E'_{1/\theta}) = (1/\theta) \dim(\mathcal{H}') \omega.$$

The above connection can be also readily computed via noncommutative geometric tools as in [20]. In general, the moduli dependence is governed by Proposition 3.5.

In the following sections we shall reformulate the Matsushina approach by a further enhancement of a noncommutative torus perspective and by enforcing FMN* from the outset via a matrix portrait and by building upon the Coherent State Transform of [7].

4.1 $\delta$-description of vector theta functions and Gauss sums

Let us consider coprime positive integers $r$ and $q$, set, for $x \in S^1$,

$$\delta^{(q)}_{\ell}(x) := \delta[(x - \ell/r)q], \quad \ell = 0, 1, \ldots, r - 1.$$
If $q = 1$ we simply write $\delta_\ell$ instead of $\delta^{(q)}_\ell$. From the (distributional) Fourier expansion (involving a $q$-covering $S^1 \to S^1$ and, dually, the subgroup $q\mathbb{Z} \subset \mathbb{Z}$)

$$\delta^{(q)}_\ell(x) = \delta(qx) = \sum_{n \in \mathbb{Z}} e^{2\pi inqx} \equiv \mathcal{F}\delta^{(q)} \equiv \theta^{(q)}_0$$

one gets

$$\delta^{(q)}_\ell(x) = \sum_{0 \leq \ell' \leq r-1} e^{-2\pi i\ell' \ell} \sum_{n \in \mathbb{Z}} e^{2\pi i(\ell' + rn)qx} = \sum_{0 \leq \ell' \leq r-1} e^{-2\pi i\ell' \ell} \theta^{(q)}_{\ell'} \equiv a_\ell' \theta^{(q)}_{\ell'}$$

via the introduction of the (invertible) $r \times r$ matrix (cf. [7])

$$A := \left(\begin{array}{c} a_\ell' \\ \end{array} \right) = \left(\begin{array}{c} e^{-2\pi i\ell' \ell} \end{array} \right)$$

(Einstein’s convention is employed) relating the $\delta$ and (boundary) theta descriptions; thus

$$\text{Tr} A = \sum_{0 \leq \ell \leq r-1} e^{-2\pi i\ell^2 \ell} = S(q,r)$$

i.e. a Gauss sum. Similarly (obvious notation, with $y \in S^1$), one has

$$\delta^{(r)}_m(y) := \delta((y - m/q) r) = \sum_{0 \leq m' \leq q-1} e^{-2\pi im'm} \sum_{n \in \mathbb{Z}} e^{2\pi i(m' + qn)ry} \equiv \theta^{(r)}_{m'}$$

with a corresponding matrix

$$B := \left(\begin{array}{c} \theta^{(r)}_{m'} \\ \end{array} \right) = \left(\begin{array}{c} e^{-2\pi i(m' \ell)} \end{array} \right)$$

with

$$\text{Tr} B = \sum_{0 \leq m \leq q-1} e^{-2\pi im^2 \ell} = S(r,q).$$

Then consider the tensor product distributions

$$\delta^{(q)}_\ell(x) \delta^{(r)}_m(y), \quad x, y \in S^1.$$ 

Upon fixing $m \in \{0, 1, \ldots, q - 1\}$, one has an obvious $r$-component column vector, representing a model for the $m$-th Matsushima holomorphic section for the vector bundle $\mathcal{E}_{r,q} \to V/L$. More precisely, we have (with a natural abridged notation), upon suitably reinterpreting Matsushima’s construction ([11], Section 8, and our previous discussion on the CST in Section 2.1):

$$\vec{\delta}_{m,} := (\delta_{0,m}, \delta_{1,m}, \ldots, \delta_{r-1,m})^T \leftrightarrow \delta_{0,m}.$$ 

The $q$ columns thus obtained yield a basis for a $q$-dimensional Hilbert space $H^q \cong H^0(\mathcal{E}_{r,q})$. Similarly, fixing $\ell \in \{0, 1, \ldots, r - 1\}$, we get a row vector, giving rise to a model for the $\ell$-th holomorphic section of the (FMN*) dual vector bundle $\mathcal{E}_{q,r} \to V/L$, namely:

$$\vec{\delta}_{\ell,} := (\delta_{\ell,0}, \delta_{\ell,1}, \ldots, \delta_{\ell,q-1}) \leftrightarrow \delta_{\ell,0}.$$
and the ensuing $r$ rows yield a basis for an $r$-dimensional Hilbert space $\mathcal{H}^r \cong H^0(\mathcal{E}_{q,r})$.

Also, in view of the previous considerations (Section 2.5), we can naturally establish a bijective correspondence

$$\delta^{(q)}_k(x)\delta^{(r)}_m(y) \leftrightarrow \delta_k(z) = \delta(z - k/rq), \quad z \in S^1$$

($k \in \{0, 1, \ldots, rq - 1\}$), with the $rq$-level thetas, viewed as holomorphic sections of the complex line bundle $\mathcal{E}_{1,rq} \to V/L'$.

Therefore, one finds a third matrix

$$C := (c_{kk'} = e^{-2\pi ik'k\frac{1}{rq}})$$

with

$$\text{Tr } C = \sum_{0 \leq k \leq rq-1} e^{-2\pi ik^2 \frac{1}{rq}} = S(1,rq).$$

The above matrix is related to the former ones in the following way. Let us consider the following $rq$-dimensional Hilbert spaces: $\mathcal{H}^{rq}$, generated by the orthonormal basis $\delta^{(q)}_k(x)\delta^{(r)}_m(y)$, and $\mathbb{H}^{rq}$, generated by the orthonormal basis $\delta_k(z)$; we have then a natural unitary transformation $U : \mathcal{H}^{rq} \to \mathbb{H}^{rq}$

$$U(\delta^{(q)}_k \delta^{(r)}_m) := \delta_k$$

whereby

$$U(\theta^{(q)}_k \theta^{(r)}_m) = \theta_k$$

as well (shorthand notation), this easily leading to

$$C = U (A \otimes B) \ U^{-1}.$$ 

Therefore, from

$$\text{Tr } C = \text{Tr } [U (A \otimes B) \ U^{-1}] = \text{Tr } (A \otimes B) = \text{Tr } A \cdot \text{Tr } B$$

we get a special case of the above multiplicative formula for Gauss sums with $\mu = 1$.

Actually, the general formula is also obtained via the same technique, after introducing from the outset another $\mu$-covering $S^1 \to S^1$, resulting in an extra factor $\mu$ in the numerators of all arguments of the exponentials.

This may be viewed as a sort of categorification of Gauss sums in the sense that, as numerical objects, they come from the multiplicativity of tensor product traces.

A variant of the above procedure consists in exploiting the (algebra) isomorphism $M_{rq}(\mathbb{C}) \cong M_r(\mathbb{C}) \otimes M_q(\mathbb{C})$ via the elementary matrix bases $E_{ij}$ (i.e. the matrices whose $(i,j)$-entry is 1 and all others are zero):

$$E_{kk'} \leftrightarrow E_{\ell\ell'} \otimes E_{mm'}$$

(abbreviated notation: $k, k'$ and so on are taken modulo the size of the respective matrices).
Remark 4.1. A few words about the heuristics behind the above discussion are maybe in order: upon formally multiplying the deltas labelled by \( \ell \) and \( m \) after taking the same argument \( x = y \) (this is an ill-defined object!), one has, for the product of their Fourier series, after an obvious index relabelling, the (meaningless) expression

\[
\sum_{n,N \in \mathbb{Z}} e^{2\pi i (\ell q + mr + rqN)x}
\]  

which, upon discarding the sum in \( n \), yields the distributional Fourier series expressing \( \delta(z - k/rq) \) —after changing \( x \) to \( z \)— with \( k/rq \) obeying the above equation.

4.2 Noncommutative torus aspects of the \( \delta \)-formulation

Set (abridged notation) \( \delta_{\ell m} := \delta_{\ell} \delta_{m} \) and define, in \( \mathcal{H}^{q} \), for \( \mu, \nu, \tilde{\mu}, \tilde{\nu} \in S^{1} \)

\[
U^{q} := \mu^{q} 1_{\mathcal{H}^{q}} \quad V^{q} := \nu^{q} 1_{\mathcal{H}^{q}}, \quad \tilde{U}^{r} := \tilde{\mu}^{r} 1_{\mathcal{H}^{r}}, \quad \tilde{V}^{r} := \tilde{\nu}^{r} 1_{\mathcal{H}^{r}}
\]

and

\[
U_{\delta_{\ell m}} := \mu_{\delta_{\ell,m-1}} \quad V_{\delta_{\ell m}} := \nu e^{-2\pi i \frac{\ell}{q} \delta_{\ell m}}
\]

and

\[
\tilde{U}_{\delta_{\ell m}} := \tilde{\mu}_{\delta_{\ell-1,m}} \quad \tilde{V}_{\delta_{\ell m}} := \tilde{\nu} e^{-2\pi i \frac{\ell}{r} \delta_{\ell m}}
\]

(cyclic ordering understood: for instance, \( U_{\delta_{\ell,0}} := \mu_{\delta_{\ell,q}} \) et cetera). One has, upon restriction to the spaces indicated, \( U_{\delta_{\ell m}}^{q} = \mu_{\delta_{\ell,m-1}}^{q} \quad V_{\delta_{\ell m}}^{q} = \nu^{q} 1_{\mathcal{H}^{q}} \quad \tilde{U}_{\delta_{\ell m}}^{r} = \tilde{\mu}^{r} 1_{\mathcal{H}^{r}} \quad \tilde{V}_{\delta_{\ell m}}^{r} = \tilde{\nu}^{r} 1_{\mathcal{H}^{r}} \) and subsequently

\[
UV = e^{-2\pi i \frac{\ell}{q}} VU, \quad \tilde{U} \tilde{V} = e^{-2\pi i \frac{\ell}{r}} \tilde{V} \tilde{U}
\]

Then define

\[
U_{\delta_{\ell m}} := \mu_{\delta_{\ell,m-1}} = U_{\delta_{\ell m}}, \quad V_{\delta_{\ell m}} := \nu^{r} e^{-2\pi i \frac{\ell}{q} \delta_{\ell m}} = V^{r} \delta_{\ell m},
\]

yielding

\[
U V = e^{-2\pi i \frac{\ell}{q}} V U
\]

(cyclic ordering again understood), \( U_{\delta_{\ell m}}^{q} = \mu^{q} 1_{\mathcal{H}^{q}} \quad V_{\delta_{\ell m}}^{q} = \nu^{q} 1_{\mathcal{H}^{q}} \). Similarly, upon setting

\[
\tilde{U}_{\delta_{\ell m}} := \tilde{\mu}_{\delta_{\ell-1,m}} = \tilde{U}_{\delta_{\ell m}}, \quad \tilde{V}_{\delta_{\ell m}} := \tilde{\nu}^{q} e^{-2\pi i \frac{\ell}{r} \delta_{\ell m}} = \tilde{V}^{q} \delta_{\ell m},
\]

we get

\[
\tilde{U} \tilde{V} = e^{-2\pi i \frac{\ell}{q}} \tilde{V} \tilde{U}
\]

together with \( \tilde{U}^{r} = \tilde{\mu}^{r} 1_{\mathcal{H}^{r}} \quad \tilde{V}^{r} = \tilde{\nu}^{r} 1_{\mathcal{H}^{r}} \). These two representations mutually commute (since they do not mix first and second subscripts) and they are exchanged upon application of the FMN*-transform.

The action of the various operators involved can be cast in a more compact way as follows: in \( \mathcal{H}^{q} \) one has

\[
U_{\delta_{.,m}} = \mu_{\delta_{-,m-1}} \quad V_{\delta_{.,m}} = \nu^{r} e^{-2\pi i \frac{\ell}{q} \delta_{-,m}}, \quad m = 0, 1, \ldots, q - 1
\]
with the “tilded” operators acting as the identity:

\[ \tilde{\bar{\delta}}_{j,m} = \bar{\delta}_{j,m}, \quad \tilde{\bar{\delta}}_{j,m} = \bar{\delta}_{j,m}, \quad m = 0, 1, \ldots, q - 1. \]

A similar portrait, \textit{mutatis mutandis}, holds in \( H^r \).

**Remark 4.2.** Geometrically, the above “toric” families of representations correspond to tensoring the initial holomorphic bundle \( E_\theta \to \mathbb{C}/\Lambda \) with the flat line bundle \( \mathcal{P}_x \to \mathbb{C}/\Lambda \), the restriction of the Poincaré bundle to \( \mathbb{C}/\Lambda \times \{\xi\} \cong \mathbb{C}/\Lambda \), where \( \xi = (\mu, \nu) \) and similarly for \( E_{1/\theta} \to \mathbb{C}/\Lambda \). Also notice that the torus also classifies holonomies of the different Chern-Bott connections, see also [20].

We recover the standard Matsushima correspondence involving the holomorphic vector bundle \( E_{r,q} \to V/L \) and the \( q \)-level theta line bundle \( E_{1,q} \to V/L' \) via the Coherent State Transform \( CST \) through the following steps (obvious abridged notation), also setting \( \mu = \tilde{\mu} = \nu = \tilde{\nu} = 1 \) for simplicity. Indeed:

\[ \vartheta_{j,m} = CST(\delta_{j,m}) = CST(\tilde{\bar{U}} \delta_{0,m}) = CST(\tilde{\bar{U}} \tilde{\delta}_{0,m}), \quad j = 0, 1, \ldots, r - 1 \]

whence

\[ \bar{\vartheta}_{j,m} := (\vartheta_{j,m})_{j=0}^{r-1} =: CST(\tilde{\bar{\delta}}_{m}) \leftrightarrow \vartheta_{0,m}. \]

Actually, in this manner we have defined a \textit{vector} version of the Coherent State Transform:

\[ CST = M^{-1} \circ \text{vec} \circ CST \circ F \]

mapping \( \delta_{0,m} \mapsto s_m \) and spelled out as follows

\[ \delta_{0,m} \mapsto \vartheta_{0,m}(x) \mapsto \vartheta_{0,m}(z) \overset{\text{vec}}{\rightarrow} \bar{\vartheta}_{0,m}(z) \mapsto M^{-1}s_m. \]

The notation \( M^{-1} \) is justified since \( M \) is injective and \( \text{Im} (\text{vec}) \subset \text{Im} M \). We may recap the previous discussion via the isomorphism

\[ H^0(\mathcal{E}_{r,q}) \otimes H^0(\mathcal{E}_{q,r}) \cong H^0(\mathcal{E}_{1,rq}) \]

induced by the correspondence

\[ s_m \otimes s_l \leftrightarrow s_k \]

where again

\[ [k]_{rq} = q[l]_r + r[m]_q \]

and by further noticing the following “categorical” result:

**Proposition 4.1.** Under the above assumptions, we have

\[ \mathcal{A}_{1/rq} \cong \mathcal{A}_{q/r} \otimes \mathcal{A}_{r/q}. \]
Proof. First observe that noncommutative tori are nuclear \( C^* \)-algebras, so their \( C^* \)-tensor product appearing in the r.h.s. is uniquely determined ([1], IV.3.5.3, p. 392). Then, starting from

\[
U V = e^{-2\pi i \frac{q}{r}} V U
\]

and

\[
\tilde{U} \tilde{V} = e^{-2\pi i \frac{q}{r}} \tilde{V} \tilde{U}
\]

(all tilded operators commuting with untilded ones), define (same notation as before: \( k = \ell q + mr \) and so on):

\[
\mathcal{U}^k := U^m \tilde{U}^\ell, \quad \forall k' := V^{m'} \tilde{V}^{\ell'}.
\]

A straightforward computation then yields:

\[
\mathcal{U}^k \mathcal{V}^{k'} = e^{-2\pi i \frac{k k'}{r}} \mathcal{V}^{k'} \mathcal{U}^k.
\]

The above reasoning is clearly invertible, achieving the sought for conclusion.

\[\square\]

Remark 4.3. Upon further requiring that \( U^q = V^q = 1 \) and \( \tilde{U}^r = \tilde{V}^r = 1 \), we get \( U^{rq} = V^{rq} = 1 \) (with 1 the identity in the respective algebras).

Also notice that, at the vector bundle level, this reflects an operation (denoted by \( * \))

\[
\mathcal{E}_{r,q} * \mathcal{E}_{q,r} = \mathcal{E}_{1,rq}.
\]

casting light on FMN-duality via a “Gauss” perspective.

5 Conclusions and outlook

In this paper we used complex algebraic-geometric and noncommutative geometric techniques in order to understand and possibly enhance, at least in the classical case (i.e. on the complex field) the fascinating relationship between theta functions and Heisenberg groups. Our research is strongly motivated by condensed matter physics as well, which however was not explicitly addressed herein. Nevertheless, our results may be possibly applied to Quantum Hall Effect issues: for instance, the above developments illustrate the symmetry \( \theta \leftrightarrow 1/\theta \) ultimately leading, in applications to the integral Quantum Hall Effect, to the duality occurring between Hofstadter’s and Harper’s regimes, see e.g. [4]. Also, they may provide a clear-cut mathematical formulation of the important Laughlin gauge principle for a toral configuration, see e.g. the comprehensive review [9]. Finally, instances of the vector bundles dealt with in the present paper also appear in the works [4][5], devoted to a far reaching generalization of the TKNN equations. These questions will be possibly tackled elsewhere.
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