Analysis of an epidemic model with latent infection

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Abstract. An infectious disease model with latent infection is studied, and the basic reproduction number of the model is obtained. When the basic reproduction number is less than 1, the model has only one disease-free equilibrium. When the basic reproduction number is greater than 1, the disease-free equilibrium is unstable. The model also has a unique positive equilibrium. The global asymptotic stability of the two equilibria is analyzed by linearization method, Lyapunov function method and geometric method.

1. Introduction
Herpesvirus is a kind of enveloped DNA virus, which can infect human and animals and has certain infectivity. Herpes virus has a common characteristic, that is, once infected, it is difficult to remove from the body, and the virus can be latent for several years or for a lifetime. The virus in the latent state will not cause clinical symptoms, but when the body's resistance is reduced due to the stimulation of harsh environment, the virus in the latent state will be activated into an infectious virus, thus causing animal disease [1] Infected by parasites such as protozoa and helminth, or pathogens such as bacteria and bacteria, diseases that can spread among related populations are called infectious diseases [2]. Infectious diseases have always threatened human health. In history, outbreaks of infectious diseases have brought great disasters to human beings [3-5].

In this paper, we mainly study this kind of infectious diseases caused by herpes virus among human or animals. The infection characteristics of this kind of infectious diseases are latent infection, which refers to the latent infection of virus established by naturally infected animals or experimental infected animals after the initial infection and recovery [6].

2. Modeling
We divide the population into three chambers: susceptible population, susceptible population, latent period population, and the proportion of the total population in the total population is recorded as \( S(t), I(t), L(t) \). Assuming that the birth rate of pigs is \( b \), \( \beta \) is the incidence rate, \( d \) means the natural mortality rate of pigs, \( m \) is the rate of killing, \( p \) means vertical transmission rate, \( \gamma \) is the conversion coefficient from pig to infectious period, \( \eta \) means the recurrence of pigs in incubation period, have: \( S(t) + I(t) + L(t) = 1 \).
According to the above assumptions, the following model is obtained
\[
\begin{align*}
\frac{dS}{dt} &= b(1 - pI) - \beta SI - (d + m)S, \\
\frac{dI}{dt} &= \beta SI - (d + m + \gamma)I + \eta L + bpI, \\
\frac{dL}{dt} &= \gamma I - (\eta + d + m)L.
\end{align*}
\] (1)

the disease-free equilibrium point is obtained
\[
E_0(S^0, I^0, L^0) = \left(\frac{b}{d + m}, 0, 0\right).
\]

From the third part of reference [7], the basic reproduction number of model (1) is
\[
R_0 = \frac{(\beta S^0 + bp)(\eta + d + m)}{(d + m + \gamma)(\eta + d + m) - \eta \gamma}.
\]

When \(R_0 > 1\), there is also a unique positive equilibrium point in model (1) is \(E^*(S^*, I^*, L^*)\),
\[
S^* = \frac{(\eta + d + m)(d + m - bp) + \gamma(d + m)}{\beta(\eta + d + m)},
I^* = \frac{\gamma}{\eta + d + m}I^*,
L^* = \frac{b[(\beta S^0 + bp)(\eta + d + m) - (d + m + \gamma)(\eta + d + m) + \eta \gamma]}{\beta S^0(d + m)(\eta + d + m) + \beta b \gamma}.
\]

3. Global stability of disease free equilibrium

**Theorem 1** when \(R_0 < 1\), disease free equilibrium \(E_0(S^0, I^0, L^0)\) is global stability; when \(R_0 > 1\), disease free equilibrium \(E_0\) is instability.

**Prove:** The model (1) is linearized at \(E_0\). The characteristic equation at \(E_0\) is
\[
(\lambda + d + m + \delta)(\lambda^2 + a\lambda + c) = 0.
\] (2)

Inside
\[
a = \eta + d + m - (\beta S^0 + bp) + d + m + \gamma, \\
c = (\eta + d + m)[d + m + \gamma - (\beta S^0 + bp)] - \eta \gamma,
\]

When \(R_0 < 1\), have \((\beta S^0 + bp)(\eta + d + m) < (d + m + \gamma)(\eta + d + m)\).

So there is \(\beta S^0 + bp < d + m + \gamma\), so \(a > 0\), easy to know that \(R_0 < 1\) equivalent to \(c > 0\).

According to Weida's theorem, the roots of characteristic equation (2) have negative real parts, disease free equilibrium \(E_0(S^0, I^0, L^0)\) is local stability.

When \(R_0 < 1\), there is a sufficiently small \(\varepsilon > 0\), such that
\[
M_\varepsilon = (\eta + d + m)[\beta(S^0 + \varepsilon) + bp - (d + m + \gamma)] + \eta \gamma < 0.
\]

It can be seen from the first equation of model (1) \(S' \leq b - (d + m)S\),

For the above reasons \(\varepsilon > 0\), exist \(T > 0\), when \(t \geq T\), \(S \leq \frac{b}{d + m} + \varepsilon = S^0 + \varepsilon\).

Construct Liapunov function \(V = (\eta + d + m)I + \eta L\).

when \(t \geq T\), receive \(V' \leq (\eta + d + m)[\beta(S^0 + \varepsilon) + bp - (d + m + \gamma)]I + \eta \gamma I = M_\varepsilon I \leq 0\).

Order \(\Gamma = \{(S, I, L) \in \Omega | V'(t) = 0\} = \{I = 0\}\), from the model (1), the maximum invariant set of
\[ \Gamma = \{ E_0 \} \text{.} \] from the principle of LaSalle invariant set\(^8\), disease free equilibrium \( E_0 \) is global stability. when \( R_0 > 1 \), have \( c < 0 \), The characteristic equation has positive roots, so disease free equilibrium \( E_0 \) is instability.

4. Global asymptotic stability of positive equilibrium

**Theorem 2** when \( R_0 > 1 \), positive equilibrium \( E^* = (S^*, I^*, L^*) \) is Global asymptotic stability.

**Prove:** The model (1) is linearized at \( E^* \), The characteristic equation at \( E^* \) is

\[
\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0, \tag{3}
\]

Inside \( c_1 = m_3 + m_4 - m_2 + m_5 + 2m_1 \),

\[
c_2 = \left[ m_4 (m_3 - m_2) - \eta \gamma \right] + (m_1 + m_3)(m_3 + m_4 - m_2) + m_1 (m_2 + m_4 + bp),
\]

\[
c_3 = (m_1 + m_3) \left[ m_4 (m_5 - m_2) - \eta \gamma \right] + m_4 (m_2 + bp) + \gamma m_1
\]

\[
m_1 = \beta I^* > 0, m_2 = \beta S^* > 0, m_3 = d + m + \gamma - bp > d + m - bp \geq b(1 - p) > 0,
\]

\[
m_4 = \eta + d + m > 0, m_5 = d + m > 0.
\]

equilibrium \( E^* \) satisfy equation

\[
\begin{align*}
\beta S^* I^* - (d + m + \gamma - bp) I^* + \eta L^* &= 0, \\
\gamma I^* - (\eta + d + m) L^* &= 0.
\end{align*}
\]

Therefore \( m_3 - m_2 = \frac{\eta L^*}{I^*} = \frac{\eta \gamma}{m_4} > 0 \), so \( c_1 > m_3 - m_2 > 0 \),

\[
c_2 = (m_1 + m_3)(m_3 + m_4 - m_2) + m_1 (m_2 + m_4 + bp) > 0, c_3 = m_1 m_4 (m_2 + bp) + \gamma m_1 > 0,
\]

\[
c_1 c_2 - c_3 = (m_1 + m_3)(m_5 - m_2)^2 + m_4^2 + m_1 \eta \gamma > 0 .
\]

Know from the Hurwitz criterion\(^9\) that when \( R_0 > 1 \), positive equilibrium \( E^* \) is local stability.

from the theorem 2.2 in \([10]\): model (1) has a compact set of attractors \( \bar{E} \) in \( \Omega \). So the lemma in\([10]\) is satisfied. we only need to prove to prove positive equilibrium \( E^* \) is global stability \( q < 0 \).

The Jacobian matrix and the second additive composite matrix of model (1) are

\[
J = \begin{pmatrix}
-\beta I - (d + m) & -bp - \beta S & 0 \\
\beta I & \beta S - (d + m + \gamma - bp) & \eta \\
0 & \gamma & -(\eta + d + m)
\end{pmatrix},
\]

\[
J[2] = \begin{pmatrix}
\beta S - \beta I - m_3 - m_5 & \eta & 0 \\
\gamma & -\beta I - m_4 - m_5 & -bp - \beta S \\
0 & \beta I & \beta S - m_5 - m_4
\end{pmatrix}.
\]

choose \( P(x) = P(S, I, L) = diag(1, \frac{I}{L}, \frac{I}{L}) \), get

\[
P P^{-1} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{I'}{I} - \frac{L'}{L} & 0 \\
0 & 0 & \frac{I'}{I} - \frac{L'}{L}
\end{pmatrix}.
\]
\[
P_{I}^{[3]} P^{-1} = \begin{pmatrix}
\beta S - \beta I - m_3 - m_5 & \eta L / I & 0 \\
\gamma L / I & -\beta I - m_4 - m_5 & -bp - \beta S \\
0 & \beta I & \beta S - m_3 - m_4
\end{pmatrix}.
\]

So \( B = P_{I} P^{-1} + P_{I}^{[2]} P^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \). Inside \( B_{11} = \beta S - \beta I - m_3 - m_5 \), \( B_{12} = \frac{\eta L / I}{0} \), \( B_{21} = \left( \frac{\gamma L / I}{0} \right), B_{22} = \begin{pmatrix}
\frac{I'}{I} - \frac{L'}{L} - \beta I - m_4 - m_5 & -bp - \beta S \\
\beta I & \frac{I'}{I} - \frac{L'}{L} + \beta S - m_3 - m_4
\end{pmatrix}.
\]

by using the valuation method in reference [11] \( \mu(B) \leq \max \{ g_1, g_2 \} \),
inside \( g_1 = \mu_1(B_{11}) + |B_{12}|, g_2 = |B_{21}| + \mu_2(B_{22}) \). \( B_{21} \) and \( |B_{21}| \) are matrix norms corresponding to \( L \) vector norms, \( \mu_i \) is Lozinski \( i \) measure corresponding to \( L \) vector norm.

So \( \mu_1(B_{11}) = \beta S - \beta I - m_3 - m_5, |B_{12}| = \max \left\{ \frac{\eta L / I}{0} \right\} = \frac{\eta L}{I}, |B_{21}| = \frac{\gamma L}{I} \)
\( \mu_2(B_{22}) = \frac{I'}{I} - \frac{L'}{L} - m_4 - (d + m) \)

So \( g_1 = \frac{I'}{I} - \beta I - (d + m); \ g_2 = \frac{I'}{I} - (d + m + \gamma) \).
\( \mu(B) \leq \max \{ g_1, g_2 \} = \frac{I'}{I} - (d + m + \gamma) \).

So \( q = \lim_{t \to \infty} \sup_{x \in \Omega} \frac{1}{t} \int_{0}^{t} \mu(B) dS \leq -\frac{d + m + \gamma}{2} < 0. \)

Therefore, when \( R_0 > 1 \), positive equilibrium \( E^* \) is global stability in \( \Omega \).

5. Concluding remarks
In this paper, the dynamic behavior of an infectious disease model with latent infection is discussed. The global stability of the disease-free equilibrium of the model is obtained by using the eigenvalue method and Liapunov function method. The global asymptotic stability of the positive equilibrium is obtained by linearization and geometric methods.

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