MODERATE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH HARDY POTENTIAL UNDER MINIMAL RESTRICTIONS ON THE POTENTIAL

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Abstract. We study semilinear elliptic equations with Hardy potential

\[(E) \quad -\mathcal{L}_\mu u + u^q = 0\]

in a bounded smooth domain \(\Omega \subset \mathbb{R}^N\). Here \(q > 1\), \(\mathcal{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega}\), and \(\delta_\Omega(x) = \text{dist}(x, \partial\Omega)\). Assuming that \(0 \leq \mu < C_H(\Omega)\), boundary value problems with measure data and discrete boundary singularities for positive solutions of \((E)\) have been studied in [10]. In the case of convex domains \(C_H(\Omega) = 1/4\). In this case similar problems have been studied in [8]. In the present paper we study these problems, in arbitrary domains, assuming only \(-\infty < \mu < 1/4\), even if \(C_H(\Omega) < 1/4\). We recall that \(C_H(\Omega) \leq 1/4\) and, in general, strict inequality holds. The key to our study is the fact that, if \(\mu < 1/4\) then in smooth domains there exist local \(\mathcal{L}_\mu\)-superharmonic functions in a neighborhood of \(\partial\Omega\) (even if \(C_H(\Omega) < 1/4\)). Using this fact we extend the notion of normalized boundary trace introduced in [10], to arbitrary domains, provided that \(\mu < 1/4\). Further we study the b.v.p. with normalized boundary trace \(\nu\) in the space of positive finite measures on \(\partial\Omega\). We show that existence depends on two critical values of the exponent \(q\) and discuss the question of uniqueness. Part of the paper is devoted to the study of the linear operator: properties of local \(\mathcal{L}_\mu\)-subharmonic and superharmonic functions and the related notion of moderate solutions. Here we extend and/or improve results of [5] and [10] which are later used in the study of the nonlinear problem.

1. Introduction and main results

1.1. Introduction. On bounded smooth domains \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\) we study semilinear elliptic equations with Hardy potential of the form,

\[(P_\mu) \quad -\Delta u - \frac{\mu}{\delta_\Omega^2} u + |u|^{q-1} u = 0 \quad \text{in } \Omega,\]

where \(q > 1\), \(-\infty < \mu < 1/4\) and

\(\delta_\Omega(x) := \text{dist}(x, \partial\Omega)\).

Equations \((P_0)\) had been extensively studied in the past two decades and by now the structure of the set of positive solution of such equations is well understood, see [11] and further references therein. Equation \((P_\mu)\) with Hardy potential, i.e. with \(\mu \neq 0\), had been first considered in [5], where a

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classification of positive solutions had been introduced and conditions for the existence and nonexistence of large solutions for $(P_{\mu})$ had been derived.

The study and classification of positive solutions of equation $(P_{\mu})$ relies on the properties of the associated linear equation

$$-\mathcal{L}_\mu h = 0 \text{ in } \Omega,$$

where

$$\mathcal{L}_\mu := \Delta + \frac{\mu}{\delta^2_{\Omega}}.$$

Denote

$$\alpha_\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}$$

and note that $\alpha_+ + \alpha_- = 1$. For $\rho > 0$ and $\varepsilon \in (0, \rho)$ we use the notation

$$\Omega_\rho := \{x \in \Omega : \delta(x) < \rho\}, \quad \Omega_{\varepsilon, \rho} := \{x \in \Omega : \varepsilon < \delta(x) < \rho\},$$

$$D_\rho := \{x \in \Omega : \delta(x) > \rho\}, \quad \Sigma_\rho := \{x \in \Omega : \delta(x) = \rho\}.$$

A function $w \in L^1_{\text{loc}}(G)$ is a $\mathcal{L}_\mu$-subharmonic in $\Omega$ if $\mathcal{L}_\mu w \leq 0$ in the distribution sense, i.e.,

$$\int_G w(-\Delta \varphi) \, dx - \int_G \frac{\mu}{\delta^2_{\Omega}} w \varphi \, dx \leq 0 \quad \forall 0 \leq \varphi \in C^\infty_c(\Omega).$$

We say that $w$ is a local $\mathcal{L}_\mu$-subharmonic function if there exists $\rho > 0$ such that $w \in L^1_{\text{loc}}(\Omega_\rho)$ is subharmonic in $\Omega_\rho$. Similarly, (local) $\mathcal{L}_\mu$-superharmonic functions are defined with “$\geq$” in the above inequality.

1.2. The role of the Hardy constant. The existence and properties of positive $\mathcal{L}_\mu$-harmonic and superharmonic functions in $\Omega$ are controlled by the Hardy constant of the domain, defined as

$$C_H(\Omega) := \inf_{C^\infty_c(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 \, dx}{\int_{\Omega} \frac{u^2}{\delta^2_{\Omega}} \, dx}.$$

For a bounded Lipschitz domain it is known that $C_H(\Omega) \in (0, 1/4]$. If $\Omega$ is convex then $C_H(\Omega) = 1/4$. In general, $C_H(\Omega)$ varies with the domain and could be arbitrary small (see, e.g. Theorem I and Section 4) for a discussion and examples).

Denote the local Hardy constant in $\Omega_\rho$ relative to $\partial \Omega$ by

$$C_H^{\partial \Omega}(\Omega_\rho) := \inf_{C^\infty_c(\Omega_\rho) \setminus \{0\}} \frac{\int_{\Omega_\rho} |\nabla u|^2 \, dx}{\int_{\Omega_\rho} \frac{u^2}{\delta^2_{\Omega_\rho}} \, dx}.$$

Note the difference between $C_H^{\partial \Omega}(\Omega_\rho)$ and $C_H(\Omega_\rho)$: the distance involved in the first one is $\delta_{\Omega_\rho}(x) = \text{dist} (x, \partial \Omega)$ while in the second it is $\delta_{\Omega_\rho}(x) = \text{dist} (x, \partial \Omega_\rho)$. Obviously $C_H^{\partial \Omega}(\Omega_\rho) \geq C_H(\Omega_\rho)$.

The following lemma shows that in contrast to the ”global” Hardy constant $C_H(\Omega)$ the value of the ”local” Hardy constant $C_H^{\partial \Omega}(\Omega_\rho)$ does not depend on the shape of $\Omega$, provided that $\rho$ is sufficiently small.

Lemma 1.1. (Local Hardy Inequality) There exists $\bar{\rho} = \bar{\rho}(\Omega) > 0$ such that for every $\rho \in (0, \bar{\rho})$ one has $C_H^{\partial \Omega}(\Omega_\rho) = C_H(\Omega_\rho) = 1/4$. 
The fact that $C_{\partial\Omega}(\Omega, \rho) = 1/4$ is due to [9, p.3246], while $C_{H}(\Omega, \rho) = 1/4$ follows from [6, Lemma 1.2].

Lemma 1.2. Equation (1.1) admits a positive $L_{\mu}$-superharmonic function in $\Omega$ if and only if $\mu \leq C_{H}(\Omega)$.

Equation (1.1) admits a positive $L_{\mu}$-superharmonic in $\Omega_{\rho}$ with $\rho \in (0, \bar{\rho})$ if and only if $\mu \leq 1/4$.

Thus, according to Lemma 1.1, if $C_{H}(\Omega) < 1/4$ then, for $\mu \in (C_{H}(\Omega), 1/4)$, there exist local positive $L_{\mu}$-superharmonic functions but no “global” positive $L_{\mu}$-superharmonic functions in $\Omega$.

1.3. Moderate solutions and normalised boundary trace. In this work we study moderate positive solutions of nonlinear equation ($P_{\mu}$) in the range $\mu < 1/4$, including negative values of $\mu$. Recall that in the classical theory of equations ($P_{\mu}$) with $\mu = 0$, moderate solution is a solution which is dominated by a positive harmonic function, cf. [11, pp.66-69]. This concept had been extended to equations ($P_{\mu}$) with $0 \leq \mu < C_{H}(\Omega)$ in [10], where $L_{\mu}$-moderate solution is defined as a solution dominated by a positive $L_{\mu}$-harmonic function. This definition is not applicable in the range $\mu \in (C_{H}(\Omega), 1/4)$, when the set of positive $L_{\mu}$-harmonic function is empty. Therefore we modify it as follows:

Definition 1.3. A solution $u \in L^{1}_{\text{loc}}(\Omega)$ of equation ($P_{\mu}$) is $L_{\mu}$-moderate if there exists a local positive $L_{\mu}$-harmonic function $h$ such that $|u| \leq h$ in $\Omega_{\rho}$ for some $\rho \in (0, \bar{\rho})$.

We are going to show that equation ($P_{\mu}$) admits $L_{\mu}$-moderate solutions, with prescribed (normalized) boundary data, in the entire domain $\Omega$ for every $\mu < 1/4$, even when $C_{H}(\Omega) < 1/4$. The existence of a certain class of positive solutions was observed in [5, Lemma 4.15].

More specifically, we study the generalised boundary trace problem

($P_{\mu}$)
\[
\begin{align*}
-\mathcal{L}_{\mu}u + |u|^{q-1}u &= 0 \quad \text{in } \Omega, \\
\text{tr}_{\partial\Omega}^{*}(u) &= \nu,
\end{align*}
\]

where $\mu < 1/4$, $q > 1$, $\nu \in \mathcal{M}^{+}(\partial\Omega)$ and $\text{tr}_{\partial\Omega}^{*}(u)$ denotes the normalized boundary trace of a positive Borel function $u$ on $\partial\Omega$. A function $u \in L^{q}_{\text{loc}}(\Omega)$ is a solution of ($P_{\mu}$) if it satisfies the equation in the distribution sense and attains the indicated boundary data.

The concept of normalized boundary trace was introduced in [10] in order to classify positive moderate solutions of ($P_{\mu}$) in terms of their behaviour at the boundary, when $0 < \mu < C_{H}(\Omega)$. It is defined as follows.

A nonnegative Borel function $u : \Omega \to \mathbb{R}$ possesses a normalized boundary trace $\nu \in \mathcal{M}^{+}(\partial\Omega)$ if,

1Actually, the assumption $\mu > 0$ was introduced in [10] only for simplicity: the normalised boundary trace is well-defined and the related results remain valid for any $\mu < C_{H}(\Omega)$.
\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |u - K^\Omega_\mu [\nu]| dS = 0 \]

where \( K^\Omega_\mu \) is the Martin kernel of \( L_\mu \) in \( \Omega \). If, for a given \( u \) there exists a measure \( \nu \) as above then it is unique.

By Ancona [2], if \( \mu < C_H(\Omega) \) there is a (1-1) correspondence between the set of positive \( L_\mu \)-harmonic functions in \( \Omega \) and \( \mathcal{M}^+(\partial \Omega) \); the \( L_\mu \)-harmonic function \( v \) corresponding to a measure \( \nu \) has the representation \( v = K^\Omega_\mu [\nu] \).

(For details and notation see Subsection 2.1 below.)

We point out that, except in the case \( \mu = 0 \), \( \text{tr}^*_\Omega (u) \) is not the standard measure boundary trace of \( u \). In fact, when \( \mu > 0 \), the measure boundary trace of any \( L_\mu \)-harmonic function is zero.

In order to extend the definition of normalised boundary trace to arbitrary \( \mu < 1/4 \) we pick \( \rho \in (0, \bar{\rho}) \) (with \( \bar{\rho} \) as in Lemma 1.1) and employ (1.4) with \( K^\Omega_{\rho \mu} \) instead of \( K^\Omega_\mu \). Since \( C_H(\Omega_\rho) = 1/4 \), \( K^\Omega_{\rho \mu} \) is well defined for every \( \mu < 1/4 \).

We show that if, for some \( \rho \) as above, there exists \( \nu \in \mathcal{M}_+(\partial \Omega) \) such that

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |u - K^\Omega_{\rho \mu} [\nu]| dS = 0 \]

then (1.5) holds for every \( \rho \in (0, \bar{\rho}) \) and the measure \( \nu \) is independent of \( \rho \).

In addition we show that a positive solution of equation \( (P_\mu) \) possesses a normalised boundary trace if and only if it is a moderate solution.

1.4. Main results. We start with a few results about the linear operator.

**Theorem 1.4.** Let \( \mu < 1/4 \). Suppose that \( u \) is positive and \( L_\mu \)-subharmonic in \( \Omega^\rho \). Then \( u \) has a normalized boundary trace on \( \partial \Omega^\rho \) if and only if \( u \) is dominated in \( \Omega_\rho \) (for some \( \rho \in (0, \bar{\rho}) \)) by an \( L_\mu \)-harmonic function.

**Theorem 1.5.** Let \( \mu < 1/4 \). Suppose that \( u \) is a non-negative, \( L_\mu \)-subharmonic function in \( \Omega^\rho \). In addition assume that, for some \( \rho \in (0, \bar{\rho}) \) \( u \) is dominated in \( \Omega_\rho \) by an \( L_\mu \)-harmonic function. Then, either

(i) \( \text{tr}^*_\Omega u = 0 \), in which case, for every \( \beta \in (0, \rho) \) there exists a constant \( c_\beta > 0 \) such that

\[ u(\mathbf{x}) \leq c_\beta \delta(\mathbf{x})^{\alpha_+} \quad \text{in} \quad \Omega_\beta \]

or

(ii) \( \text{tr}^*_\Omega u > 0 \), in which case, for every \( \beta \) as above,

\[ \frac{1}{c_\beta} \beta^\alpha - \leq \int_{\Sigma_\beta} u dS \leq c_\beta \beta^\alpha - \quad \text{in} \quad \Omega_\beta. \]

**Theorem 1.6.** Let \( \mu < 1/4 \). Suppose that \( u \) is positive and \( L_\mu \)-superharmonic in \( \Omega^\rho \). If \( \text{tr}^*_\Omega u \neq 0 \) then (1.7) holds.

**Corollary 1.7.** Suppose that \( u \) is non-negative and \( L_\mu \)-subharmonic in \( \Omega^\rho \). Then either (1.6) holds or

\[ 0 < \limsup_{\beta \to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} u dS. \]
Remark 1.8. The corollary is an improved version of [5, Thm. 2.9]. Since we do not assume that \( u \) is dominated by an \( \mathscr{L}_\mu \)-harmonic function the alternative to (1.6) is not necessarily (1.7) but only (1.8) which is nothing more than the negation of the statement \( \text{tr}_{\partial \Omega}^\ast u = 0 \).

Clearly every positive subsolution of the nonlinear equation \( (P_\mu) \) is \( \mathscr{L}_\mu \)-subharmonic so that the above results apply to it.

We turn to the nonlinear problem.

**Theorem 1.9.** Let \( \mu < 1/4 \) and \( \nu \in \mathfrak{M}^+ (\partial \Omega) \setminus \{0\} \). Assume that \( K_{\mathcal{O}^\mu}^{\Omega^\mu} [\nu] \in L^q_\mu (\Omega_\rho) \) for some \( \rho \in (0, \bar{\rho}] \). Then the boundary value problem \( (P_\mu^\nu) \) admits a positive solution \( u \).

We emphasise that if \( \overline{C}_H (\Omega) < 1/4 \) then for \( \mu \in (\overline{C}_H (\Omega), 1/4) \) an \( \mathscr{L}_\mu \)-harmonic extension of \( \nu \) exists only locally in a strip \( \Omega_\rho \). Nevertheless, problem \( (P_\nu^\mu) \) has a positive solution in \( \Omega \), for any \( \mu < 1/4 \).

When \( \mu < \overline{C}_H (\Omega) \) problem \( (P_\nu^\mu) \) admits at most one solution for every \( \nu \in \mathfrak{M}^+ (\partial \Omega) \) [10]. However, if \( \overline{C}_H (\Omega) < \mu < 1/4 \) then uniqueness fails. Indeed, it was proved in [5, Theorem 5.3] that in the latter case there exists a positive solution of \( (P_\nu^\mu) \) with \( \nu = 0 \). An alternative, more direct proof, of this result is presented in Appendix A.

**Theorem 1.10.** Let \( u \) be a positive solution of \( (P_\mu) \). Then,

(i) \( u \) has a normalized boundary trace if and only if \( u \in L^q (\Omega; \delta^{\alpha+}) \).

(ii) If \( u \) has normalized boundary trace \( \nu \) then

\[
(1.9) \quad \lim_{x \to y} \frac{u(x)}{K_{\mathcal{O}^\mu}^{\Omega^\mu} [\nu](x)} = 1 \quad \text{non-tangentially, for } \nu\text{-a.e. } y \in \partial \Omega.
\]

**Theorem 1.11.** Let \( \nu \in \mathfrak{M}^+ (\partial \Omega) \). If \( K_{\mathcal{O}^\mu}^{\Omega^\mu} [\nu] \in L^q (\Omega; \delta^{\alpha+}) \) then \( (P_\nu^\mu) \) has a solution.

In general, the existence of a solution of \( (P_\mu) \) does not imply that \( K_{\mathcal{O}^\mu}^{\Omega^\mu} [\nu] \in L^q (\Omega; \delta^{\alpha+}) \). In fact, for any \( \mu > 0 \) and \( q > 1 \), one can construct functions \( f \in L^1 (\partial \Omega) \) such that \( K_{\mathcal{O}^\mu}^{\Omega^\mu} [f] \notin L^q (\Omega; \delta^{\alpha+}) \) while \( (P_\mu^\nu) \) has a solution whenever \( \nu = f \in L^1 (\partial \Omega) \).

Let

\[
(1.10) \quad q_{\mu, c} := \frac{N + \alpha_+}{N - 1 - \alpha_-} \quad \forall \mu < 1/4.
\]

The next result has been obtained in [10, Theorems E and F] for \( \mu \in (0, \overline{C}_H (\Omega)) \). A similar result is presented in [8, Theorems D and E], under the assumption that \( \Omega \) is a convex domain, in which case it is known that \( \overline{C}_H (\Omega) = 1/4 \).

**Proposition 1.12.** Let \( \mu < 1/4 \). If \( 1 < q < q_{\mu, c} \) then the boundary value problem \( (P_\mu^\nu) \) has a solution for every Borel measure \( \nu \in \mathfrak{M}^+ (\partial \Omega) \). Moreover, if \( q \geq q_{\mu, c} \) then problem \( (P_\mu^\nu) \) has no solution when \( \nu \) is the Dirac measure.

In the next proposition, the existence statement is a consequence of Theorem 1.11. The non-existence part is more subtle.
Proposition 1.13. (i) For every $\mu < 1/4$ put
\[ q^*_\mu = \begin{cases} \infty & \text{if } \mu \geq 0 \\ 1 - \frac{2}{\alpha_-} & \text{if } \mu < 0. \end{cases} \]
If $1 < q < q^*_\mu$ then problem $\left( P^*_\mu \right)$ has a solution for every measure $\nu = fdS$, $f \in L^1(\partial\Omega)$.
(ii) If $q \geq q^*_\mu$ then problem $\left( P^*_\mu \right)$ has no solution for any $\nu \in \mathcal{M}_+(\partial\Omega) \backslash \{0\}$.

Remark 1.14. If $\mu < 0$ then $\alpha_- < 0$ so that $q^*_\mu > 1$ and $q_{\mu,c} < q^*_\mu$.

The paper is organised as follows. In Section 2 we study the linear problem. We derive estimates of the Green and Martin kernels of $L^\mu$ in $\Omega$ and discuss the boundary behavior of local positive $L^\mu$-sub and superharmonic functions in terms of the normalized trace.

In Section 3 these results are applied to the study of the nonlinear boundary value problem $\left( P^*_\mu \right)$.

2. Linear equation and normalised boundary trace

2.1. The local behavior of Green and Martin kernels. We recall some results concerning Schrödinger equations, that are needed in what follows. The results are due to Ancona [2]. Let $D$ be a bounded Lipschitz domain and consider the Schrödinger operator $L^V = \Delta + V$ where $V \in C(D)$ is a potential such that, for some constant $a > 0$, $|V(x)| \leq a \text{dist}(x, \partial D)^{-2}$ and $L^V$ possesses a positive supersolution. (If $V \leq 0$ there is always a supersolution namely, $u = 1$.) Then $L^V$ has a Green function $G^V$ and Martin kernel $K^V$ in $D$. The Martin boundary coincides with $\partial D$ and the following holds,

Theorem 2.1 (Representation Theorem). For every $\nu \in \mathcal{M}^+ (\partial D)$ the function
\[ K^V[\nu](x) := \int_{\partial D} K^V(x, y) d\nu(y), \quad x \in D, \]
is $L^V$-harmonic in $D$. Conversely, if $u$ is a positive $L^V$-harmonic function in $D$ then there exists a unique measure $\nu \in \mathcal{M}^+ (\partial D)$ such that $u = K^V[\nu]$.

In order to state the boundary Harnack principle we need additional notation. Let $y \in \partial D$ and let $\xi = \xi^y$ be a local set of coordinates centered at $y$ such that the $\xi_1$-axis is in the direction of an interior pseudo normal $n_y$. (If $D$ is a $C^1$ domain we may take $n_y$ to be the interior unit normal.) Denote $T_y(r, \rho) = \{\xi = (\xi_1, \xi') : |\xi_1| < \rho, |\xi'| < r\}$. Assume that $r$ and $\rho$ are so chosen that
\[ \omega_y := T_y(r, \rho) \cap D = \{\xi : F_y(\xi') < \xi_1 < \rho, |\xi'| < r\} \]
where $F_y$ is a Lipschitz function in $\mathbb{R}^{N-1}$, with Lipschitz constant $\Lambda$, $F_y(0) = 0$ and $12\Lambda < \rho/r$. Since $D$ is a bounded Lipschitz domain $\Lambda, r, \rho$ can be chosen independently of $y \in \partial D$. 

Let $A \in T(r, \rho)$ be the point such that $\xi(A) = (\rho/2, 0)$. Then the boundary Harnack principle reads as follows: If $u, v$ are positive $\mathcal{L}_\mu$-harmonic functions in $\omega_y$ vanishing continuously on $\partial \Omega \cap T_y(r, \rho)$ then

$$ (2.1) \quad C^{-1} \frac{u(A)}{v(A)} \leq \frac{u(\xi)}{v(\xi)} \leq C \frac{u(A)}{v(A)} \quad \forall \xi \in T_y(r/2, \rho/2) \cap D, $$

where the constant $C$ depends only on $N, M, \rho/r$ and the Lipschitz constant of $F_y$, say $\Lambda$. ($\Lambda$ may be taken to be independent of $y \in \partial D$.)

We also need the following consequence of the boundary Harnack principle (c.f. Ancona [1, Lemma 3.5]): there exist positive numbers $c, t_0$ such that

$$ (2.2) \quad c^{-1}|x - y|^{2-N} \leq K^V(x, y)G^V(x, x_0) \leq c|x - y|^{2-N} $$

for every $y \in \partial \Omega'$ and $x$ on the interior pseudo normal at $y$ such that $|x - y| \leq t_0$.

Recall that if $V(x) = \mu \text{dist}(x, \partial D)^{-2}$ and $\mu < C_H(D)$ then $\mathcal{L}^V$ has a positive supersolution. In particular, if $D = \bar{\Omega} = \partial D$ then $C_H(D) = 1/4$. Therefore, in this case, the above results apply to the operator $\mathcal{L}_\mu = \Delta + \frac{\mu}{\delta_D}$ for every $\mu < 1/4$.

**Notation.** Let $D$ be a subdomain of $\Omega$ and denote

$$ \mathcal{L}_{\mu, D} = \Delta + \frac{\mu}{\delta_D} \quad \text{where} \quad \delta_D(x) = \text{dist}(x, \partial D). $$

Assume that $\mu < C_H(D)$ and let $D'$ be a subdomain of $D$. Obviously $C_H(D') \geq C_H(D)$. Denote the Green kernel (resp. the Martin kernel) of $\mathcal{L}_\mu$ in $D$ by $G_{\mu, D}^D$ (resp. $K_{\mu, D}^D$). Denote the Green kernel (resp. the Martin kernel) of $\mathcal{L}_{\mu, D}$ in $D'$ by $G_{\mu, D'}^D$ (resp. $K_{\mu, D'}^D$).

**Lemma 2.2.** Assume that $\mu < 1/4$. Let $\bar{\rho}$ be as in Lemma 1.1 and $t \in (0, \bar{\rho})$.

Put $U = \Omega_{\bar{\rho}} = [\delta(x) < \bar{\rho}], \Omega_t = [\delta(x) < t], U_t = [\bar{\rho} > \delta(x) > t]$. Then,

$$ (2.3) \quad G_{\mu, \Omega_t}^{\Omega_t}(x, y) \leq C(t) \inf(|x - y|^{2-N}, \delta(x)\alpha_+ \delta(y)\alpha_+ |x - y|^{2\alpha_- - N}) \quad \forall x, y \in \Omega_t/2 $$

**Proof.** Note that $\mathcal{L}_\mu = \mathcal{L}_{\mu, U}$ in $\Omega_t/2$. Hence

$$ G_{\mu, \Omega_t}^{\Omega_t} = G_{\mu, U}^{\Omega_t} $$

It is well-known that the Green function is monotone with respect to the domain. Therefore $G_{\mu, U}^{\Omega_t} < G_{\mu, U}^{\Omega_t}$ which implies

$$ (2.4) \quad G_{\mu, \Omega_t}^{\Omega_t}(x, y) \leq cG_{\mu, U}^{\Omega_t}(x, y) \quad \forall x, y \in \Omega_t/2. $$

By (2.1) and the estimate of the Green function of $\mathcal{L}_{\mu, U}$ (see [7] and [10] (2.6)),

$$ (2.5) \quad G_{\mu, \Omega_t}^{\Omega_t}(x, y) \leq cG_{\mu, U}^{\Omega_t}(x, y) \leq cG_{\mu, U}^{U}(x, y) 
\sim \inf(|x - y|^{2-N}, \delta(x)\alpha_+ \delta(y)\alpha_+ |x - y|^{2\alpha_- - N}) $$

for every $x, y \in \Omega_t/2$. This implies (2.3). $\square$
Theorem 2.3. Assume that $\mu < 1/4$, let $\bar{\rho}$ be as in Lemma 1.1 and let $t \in (0, \bar{\rho}/2)$. Using the notations of the previous lemma, pick $x_t \in U_t$ and $x'_t \in \Omega_t$ such that $\delta(x_t) = (t + \bar{\rho})/2$ and $\delta(x'_t) = t/2$. As usual $G^U_{\mu}$ denotes the Green function for $-\Delta$ in $U$. A similar notation is employed for the corresponding Martin kernels. Then,

\[ \begin{align*}
(2.6) & \quad c_1(t)^{-1} G^U_{\mu}(x, x_t) \leq G^U_{\mu}(x, x_t) \leq c_1(t) G^U_{\mu}(x, x_t) \quad \forall x \in \Omega_t \\
(2.7) & \quad c_2(t)^{-1} G^U_0(x, x'_t) \leq G^U_0(x, x'_t) \leq c_2(t) G^U_0(x, x'_t) \quad \forall x \in U_t,
\end{align*} \]

and

\[ \begin{align*}
(2.8) & \quad c_3(t)^{-1} K^U_{\mu}(x, y) \leq K^U_{\mu}(x, y) \leq c_3(t) K^U_{\mu}(x, y) \quad \forall (x, y) \in \Omega_t \times \partial \Omega, \\
(2.9) & \quad c_4(t)^{-1} K^U_0(x, y) \leq K^U_0(x, y) \leq c_4(t) K^U_0(x, y) \quad \forall (x, y) \in U_t \times \Omega.
\end{align*} \]

Proof. Note that $\mathcal{L}_\mu = \mathcal{L}_{\mu, U}$ in $\Omega_{\bar{\rho}/2}$. Hence both $G^U_{\mu}(\cdot, x_t)$ and $G^U_{\mu, U}(\cdot, x_t)$ are $\mathcal{L}_{\mu}$-harmonic in $\Omega_t$ and vanish on $\partial \Omega$. Therefore, by the boundary Harnack principle they are equivalent in a strip $S$ along $\partial \Omega$. In addition they are continuous and bounded away from zero in $\Omega_t \setminus S$. This implies the first inequality in (2.6). For the second inequality: $G^U_{\mu}(\cdot, x'_t)$ is $\mathcal{L}_{\mu}$-harmonic in $U_t$, $G^U_0(\cdot, x'_t)$ is $\Delta$ harmonic in $U_t$ and $\mathcal{L}_{\mu} - \Delta = \mu/\delta(x)^2$ is bounded in $U_t$. Therefore, since they both vanish on $\bar{\Omega}$, we can still apply the boundary Harnack principle (c.f. Ancona [4]) to deduce that they are equivalent in the strip $U_t$. This implies the second inequality in (2.6).

Recall that, $G^U_{\mu, U}(x, x_t) \sim \delta_U(x)^{\alpha+}$ in $\Omega_t$ for $t \in (0, \rho)$. (Of course the constants involved in this relation depend on $t$.) Since $\delta_\Omega \sim \delta_U$ in $\Omega_t$, this fact and (2.6) imply,

\[ G^U_{\mu}(x, x_t) \sim \delta_\Omega(x)^{\alpha+} \quad \forall x \in \Omega_t. \]

In what follows we use the notation introduced for the statement of the boundary Harnack principle. Let $y \in \partial \Omega$ and let $\xi = \xi_y$ be a local set of coordinates at $y$ relative to $U$. Thus

\[ \omega_y = T_y(r, \rho) \cap U = \{ \xi : F_y(\xi') < \xi_1 < \rho, |\xi'| < r \}. \]

We assume that $\gamma = \rho/r > 12\Lambda$.

Since $K^U_{\mu}(\cdot, y)$ and $G^U_{\mu}(\cdot, x_t)$ satisfy the (classical) Harnack inequality (2.2) remains valid in $\mathcal{C}_y(b) \cap T_y(r, \rho)$. Therefore, assuming that $\rho < t < \bar{\rho}$,

\[ K^U_{\mu}(\xi, y)G^U_{\mu}(\xi, x_t) \sim K^U_{\mu}(\xi_1, 0, y)G^U_{\mu}(\xi_1, 0, x_t) \sim |\xi|^{2-N} \]

for every $\xi \in \mathcal{C}_y(b) \cap T_y(r, \rho)$. By (2.8) and (2.9),

\[ K^U_{\mu}(\xi, y) \sim |\xi|^{2-N}\delta(\xi)^{-\alpha+} \quad \forall \xi \in \mathcal{C}_y(b) \cap T_y(r, \rho). \]

Let $\eta$ be a point in $\mathbb{R}^{N-1}$ such that $0 < |\eta| < r/2$ and denote by $P$ the point $(F_y(\eta), \eta)$ in the local coordinates $\xi_y$. Then $P \in \partial \Omega$ and $\xi_P := \xi_y - P$ is a standard set of local coordinates at $P$. Choose $r_P, \rho_P$ such that $r_P = |\eta|/2$ and $\rho_P/r_P = \gamma$. Then,

\[ |x - y| = |\xi_y| \sim |\xi'_y| \sim r_P \quad \forall x \in \Omega \cap T_P(r_P, \rho_P). \]
Let $A_P = (\rho_P/2, 0)$ in $\xi_P$ coordinates, i.e., $A_P = (F_y(\eta) + \gamma \tau P/2, \eta)$ in $\xi_y$ coordinates. Pick $\rho$ such that $0 < \rho < 2\Lambda$. Then

$$F_y(\eta) + \rho_P/2 \geq -\Lambda|\eta| - \gamma \tau P/2 = |\eta|(-\Lambda + \gamma/4) > 2\Lambda|\epsilon\eta|.$$  
Consequently, $F_y(\eta) < b|\eta| < F_y(\eta) + \rho_P/2$, which implies

$$A_P \in \mathcal{C}(b) = \{\xi_y = (\xi_1, \xi') : \xi_1 > b|\xi'|\}.$$  
Observe that

$$\delta \Omega(A_P) \sim \rho_P/2, \quad |\xi_y(A_P)| = |A_P - y| \sim (\rho_P^2 + \rho_P^2)^{1/2} \sim \rho_P.$$  
Therefore, by (2.10),

$$K^U_\mu(A_P, y) \sim \rho_P^{2-N-\alpha_+}.$$  
In fact,

$$|x - y| = |\xi_y| \sim \rho_P \quad \forall x \in \Omega \cap T_P(r_P, \rho_P).$$  
Therefore applying (2.1) in $\Omega \cap T_P(r_P, \rho_P)$ with $u(x) = K^U_\mu(x, y)$ we obtain,

(2.11)

$$K^U_\mu(x, y) \sim K^U_\mu(A_P, y) \frac{G^U_\mu(x, x_t)}{G^U_\mu(A_P, x_t)} \sim \rho_P^{2-N-\alpha_+} (\delta(x)/\rho_P)^{\alpha_+} \sim |x - y|^{2-N-\alpha_+} \delta(x)^{\alpha_+} = \delta(x)^{\alpha_+} |x - y|^{2\alpha_+ - N}$$
for every $x \in \Omega \cap T_P(r_P/2, \rho_P/2)$. Combining (2.10) and (2.11), we obtain,

(2.12)

$$K^U_\mu(x, y) \sim |x - y|^{2-N-\alpha_+} (\delta(x)/|x - y|)^{\alpha_+} = \delta(x)^{\alpha_+} |x - y|^{2\alpha_+ - N}$$
for every $x \in T_y(r_P/2, \rho_P/2)$. As (2.12) holds uniformly with respect to $y \in \partial \Omega$ we conclude that there exists $\rho' > 0$ such that this relation holds for every $(x, y) \in \Omega_{\rho'} \times \partial \Omega$. Consequently, for every $t \in (0, \rho')$,

(2.13)

$$K^U_\mu(x, y) \sim |x - y|^{2-N-\alpha_+} (\delta(x)/|x - y|)^{\alpha_+} = \delta(x)^{\alpha_+} |x - y|^{2\alpha_+ - N}$$
for every $(x, y) \in \Omega_t \times \partial \Omega$ with similarity constants depending on $t$. Since $K^U_\mu$ behaves precisely in the same way (see [10, Sec. 2.2]) we obtain the first inequality in (2.7). The second inequality is proved in a similar way. \qed

We state below two key results concerning the operator $\mathcal{L}_\mu$ in $\Omega = \Omega_{\rho'}$. These have been recently proved in [10], with respect to the operator $\mathcal{L}_\mu$ in $\Omega$ under the assumption that $0 < \mu < C_H(\Omega)$. (In fact, the condition $\mu > 0$ is redundant and does not affect the proofs.) Since $C_H(\Omega_{\rho'}) = 1/4$, the results apply to the operator $\mathcal{L}_{\mu, \Omega_{\rho'}}$ for every $\mu < 1/4$. In view of the relation between the Martin kernels and Green functions of $\mathcal{L}_{\mu, \Omega_{\rho'}}$ and $\mathcal{L}_\mu$ in $\Omega_{\rho'}$, these results also apply to the operator $\mathcal{L}_\mu$ in $\Omega_{\rho'}$.

**Theorem 2.4.** (i) If $\nu_0 \in \mathfrak{M}^+(\partial \Omega) \setminus \{0\}$ then there exist positive numbers $c$ and $\rho_0 < \rho$ such that,

(2.14)

$$c^{-1} \|\nu_0\| \leq \frac{1}{\epsilon_0} \int_{\Sigma_\rho} \mathcal{K}_\nu \nu_0 dS \leq c \|\nu_0\|, \quad \epsilon \in (0, \rho_0).$$  
(ii) Let $\rho \in (0, \rho)$ and let $\tau$ be a Radon measure in $\Omega_{\rho}$. Denote

$$G^\Omega_{\mu}(\tau) := \int_{\Omega_{\rho}} G^\mu_{\rho}(x, y)d\tau(y), \quad x \in \Omega_{\rho}.$$
If \( \tau \in \mathfrak{M}_\rho^+ (\Omega_\rho) \) then for every \( 0 < \varepsilon < \rho' < \rho \),

\[
(2.15) \quad \frac{1}{\varepsilon^{\alpha^+}} \int_{\Sigma_{\varepsilon}} G_{\mu^+}^\Omega [\tau] dS_x \leq c \int_{\Omega_\rho} \delta^{\alpha^+} d\tau,
\]

where \( c \) is a constant depending on \( \mu, \rho' \), but not on \( \varepsilon \). Moreover,

\[
(2.16) \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha^+}} \int_{\Sigma_{\varepsilon}} G_{\mu^+}^\Omega [\tau] dS = 0.
\]

Remark 2.5. If \( G_{\mu^+}^\Omega [\tau](x') < \infty \) for some point \( x' \in \Omega_\rho \) then \( \tau \in \mathfrak{M}_\rho^+ (\Omega_\rho) \) and \( G_{\mu^+}^\Omega [\tau](x) < \infty \) for every \( x \in \Omega_\rho \). This follows from the fact that there exists \( c > 0 \) such that for every fixed \( x \in \Omega_\rho \),

\[
\frac{1}{c} \delta(y)^{\alpha^+} \leq G_{\mu^+}^\Omega (x, y) \leq c\delta(y)^{\alpha^+} \quad \forall y \in \Omega_{\delta(x)/2}.
\]

Proof. In view of (2.13), inequality (2.14) follows from [10, Corollary 2.11].

The proof of (2.15) and (2.16) is similar to that of [10, Proposition 2.12]. However several modifications are needed; therefore we provide the proof of these statements in detail.

We may assume that \( \tau > 0 \). Denote \( v := G_{\mu^+}^\Omega [\tau] \). We start with the proof of (2.15).

By Fubini’s theorem and (2.6),

\[
\int_{\Sigma_{\beta}} v dS_x \leq c \left( \int_{\Omega} \int_{\Sigma_{\beta} \cap B_{\beta}(y)} |x - y|^{2-N} dS_x d\tau(y) + \beta^{\alpha^+} \int_{\Omega} \int_{\Sigma_{\beta} \cap B_{\beta}(y)} |x - y|^{2\alpha^+} dS_x \delta^{\alpha^+}(y) d\tau(y) \right) = I_1(\beta) + I_2(\beta).
\]

Note that, if \( x \in \Sigma_{\beta} \) and \( |x - y| \leq \beta/2 \) then \( \beta/2 \leq \delta(y) \leq 3\beta/2 \). Therefore

\[
I_1(\beta) \leq c_1 \beta^{\alpha^+} \int_{\Sigma_{\beta} \cap B_{\beta}(y)} |x - y|^{2-N} dS_x \int_{\Omega_\rho} \delta(y)^{\alpha^+} d\tau(y)
\]

\[
\leq c_1' \beta^{1-\alpha^+} \int_{\Omega_\rho} \delta(y)^{\alpha^+} d\tau(y) = c_1' \beta^{\alpha^+} \int_{\Omega_\rho} \delta(y)^{\alpha^+} d\tau(y)
\]

and

\[
I_2(\beta) \leq c_2 \beta^{\alpha^+} \int_{\beta/4}^{\infty} \int_{\Omega_\rho} \delta(y)^{\alpha^+} d\tau(y) \leq c_2' \beta^{\alpha^+} \int_{\Omega_\rho} \delta(y)^{\alpha^+} d\tau.
\]

This implies (2.15).

Given \( \ell \in (0, \|\tau\|_{\mathfrak{M}_\rho^+ (\Omega_\rho)}) \) and \( \beta_1 \in (0, \beta_0) \) put \( \tau_1 = \tau \chi_{B_{\beta_1}} \) and \( \tau_2 = \tau - \tau_1 \).

Pick \( \beta_1 = \beta_1(\ell) \) such that

\[
(2.17) \quad \int_{\Omega_{\beta_1}} \delta(y)^{\alpha^+} d\tau \leq \ell.
\]

Thus the choice of \( \beta_1 \) depends on the rate at which \( \int_{\Omega_\beta} \delta_x^{\alpha^+} d\tau \) tends to zero as \( \beta \to 0 \).
Put $v_i = G_{\Omega_{\mu}}[\nu_i]$. Then, for $0 < \beta < \beta_1/2$,
$$\int_{\Sigma_{\beta}} v_1 dS_x \leq c_3 \beta^\alpha + \beta_1^{2\alpha - N} \int_{\Omega_{\rho}} \delta^\alpha(y) d\tau_1(y).$$
Thus,
$$\lim_{\beta \to 0} \frac{1}{\beta^\alpha} \int_{\Sigma_{\beta}} v_1 dS_x = 0. \quad (2.18)$$
On the other hand, by (2.15) (replacing $\Omega_\rho$ by $\Omega_{\beta_0}$) and (2.17),
$$\frac{1}{\beta^\alpha} \int_{\Sigma_{\beta}} v_2 dS_x \leq c_\ell \forall \beta < \beta_1. \quad (2.19)$$
This proves (2.16). \hfill \square

**Corollary 2.6.** Let $\rho \in (0, \bar{\rho}]$ and assume that $h$ is a nonnegative $L_{\mu}$-harmonic function in $\Omega_{\rho}$ such that
$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^\alpha} \int_{\Sigma_{\epsilon}} h dS = 0. \quad (2.20)$$
Then: (i) $h = K_{\Omega_{\mu}}[\nu]$ for some measure $\nu \in M(\Sigma_{\rho})$ and (ii) For $t \in (0, \tilde{\rho})$,
$$h \sim \delta^{\alpha_+}_{\Omega_t} \text{ in } \Omega_t, \quad (2.21)$$
with the similarity constant depending on $t$.

**Proof.** (i) By the Representation Theorem, $h = K_{\Omega_{\mu}}[\nu]$ for some $\nu \in M(\partial \Omega_{\rho})$.
By (2.14) and (2.20), $\nu_0 := \nu|_{\Sigma_{\rho}} = 0$. Thus $\nu = \nu_{\rho} := \nu|_{\Sigma_{\rho}}$.
(ii) This is a consequence of (i) and (2.13). \hfill \square

**Corollary 2.7.** If $\tau \in M_{\rho, +}(\Omega_{\rho}) \setminus \{0\}$ then there exists a positive constant $c = c(\tau)$ such that
$$G_{\mu \rho}^{\Omega_{\rho}}[\tau](x) \geq c \delta(x)^{\alpha_+} \forall x \in \Omega_{\rho}, \quad (2.22)$$
and
$$\liminf_{x \to \partial \Omega} \frac{G_{\mu \rho}^{\Omega_{\rho}}[\tau](x)}{\delta(x)^{\alpha_-}} < \infty. \quad (2.23)$$

**Proof.** Let $t \in (0, \rho)$ be a number such that $\tau(\Omega_{\rho} \setminus \Omega_t) > 0$. Let $\tau' \in M_{\rho}(\Omega_{\rho})$ be defined by: $\tau' = \tau$ in $\Omega_{\rho} \setminus \Omega_t$ and $\tau' = 0$ in $\Omega_t$. Then
$$G_{\mu \rho}^{\Omega_{\rho}}[\tau] \geq G_{\mu \rho}^{\Omega_{\rho}}[\tau'] := h.$$ 

Since $h$ is $L_{\mu}$-harmonic in $\Omega_t$, (2.22) is a consequence of (2.21).
Inequality (2.23) follows from (2.15). \hfill \square

The next result was proved in [10] for $L_{\mu}$ in a domain $\Omega$ such that $\mu < C_H(\Omega)$. 

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Theorem 2.8. Let \( w \) be a nonnegative \( \mathcal{L}_\mu \)-subharmonic function in \( \Omega_\rho \). If \( w \) is dominated by an \( \mathcal{L}_\mu \)-superharmonic function in \( \Omega_\rho \), then \( \mathcal{L}_\mu w = \lambda \in \mathcal{M}_\rho^+[\Omega_\rho] \) and there exists \( \nu \in \mathcal{M}_\rho^+(\partial \Omega_\rho) \) such that
\[
(2.24) \quad w = \mathbb{K}^{\Omega_\rho}_\mu[\nu] - \mathbb{G}^{\Omega_\rho}_\mu[\lambda].
\]

Proof. There exists a nonnegative Radon measure \( \lambda \) in \( \Omega_\rho \), such that \( -\mathcal{L}_\mu w = -\lambda \) in \( \Omega_\rho \). Since \( w \) is dominated by an \( \mathcal{L}_\mu \)-superharmonic function in \( \Omega_\rho \), one shows, as in the proof of [10, Proposition 2.14], that \( \lambda \in \mathcal{M}_\rho^+(\Omega_\rho) \). Then \( v := w + \mathbb{G}^{\Omega_\rho}_\mu[\lambda] \) is a nonnegative \( \mathcal{L}_\mu \)-harmonic function in \( \Omega_\rho \). By the Representation Theorem, \( v = \mathbb{K}^{\Omega_\rho}_\mu[\nu] \) for some \( \nu \in \mathcal{M}_\rho^+(\partial \Omega_\rho) \). \( \square \)

Definition 2.9. A Borel function \( u : \Omega \to \mathbb{R} \) possesses a normalised boundary trace \( \nu_0 \in \mathcal{M}_\rho^+(\partial \Omega) \) if, for some \( \rho \in (0, \bar{\rho}] \),
\[
(2.25) \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} \int_{\Sigma_\varepsilon} |u - \mathbb{K}^{\Omega_\rho}_\mu[\nu_0]| dS = 0.
\]

The normalised boundary trace on \( \partial \Omega \) will be denoted by \( \text{tr}^*_{\partial \Omega}(u) \).

Remark. Since \( u \) is a Borel function \( u|_{\Sigma_\rho} \) is well defined and (2.25) implies that this function is in \( L^1(\Sigma_\rho) \) for all sufficiently small \( \varepsilon \).

We say that \( u \) has a measure boundary trace on \( \Sigma_\rho \) if there exists \( \nu_1 \in \mathcal{M}_\rho^+(\Sigma_\rho) \) such that
\[
\lim_{\alpha \to 0} \int_{\Sigma_\alpha} u \phi dS = \int_{\Sigma_\rho} \phi d\nu_1 \quad \forall \phi \in C_0(\bar{\Omega}_\rho).
\]

This trace is denoted by \( \text{tr}_{\Sigma_\rho}(u) \). If both \( \text{tr}_{\Sigma_\rho}(u) \) and \( \text{tr}_{\partial \Omega}(u) \) exist then the measure \( \nu \in \mathcal{M}_\rho^+(\partial \Omega_\rho) \) given by \( \nu |_{\Omega_\rho} = \text{tr}_{\partial \Omega}(u) \) and \( \nu |_{\Sigma_\rho} = \text{tr}_{\Sigma_\rho}(u) \) is denoted by \( \text{tr}_{\partial \Omega}(u) \).

Lemma 2.10. The normalised boundary trace \( \nu_0 \) is uniquely defined, independently of \( \rho \).

Proof. First we note that (2.20) remains valid if \( \nu_0 \) is replaced by any measure \( \nu \in \mathcal{M}_\rho^+(\partial \Omega_\rho) \) such that \( \nu_0 = \nu |_{\partial \Omega} \). This follows from the fact that, for every measure \( \nu_\rho \in \mathcal{M}_\rho^+(\Sigma_\rho) \),
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} \int_{\Sigma_\rho} |\mathbb{K}^{\Omega_\rho}_\mu[\nu_\rho]| dS = 0.
\]

This implies that if (2.20) holds with respect to some \( \rho \in (0, \bar{\rho}] \) then it is valid for any \( \rho' < \bar{\rho} \) in this range. Suppose for instance that \( \rho < \rho' < \bar{\rho} \) and put \( v = \mathbb{K}^{\Omega_\rho}_\mu[\nu_\rho] \). Let \( \nu \in \mathcal{M}_\rho^+(\partial \Omega_\rho) \) be the measure equal to \( \nu_0 \) on \( \partial \Omega \) and to \( h = v |_{\Sigma_\rho} d\omega_\rho \) on \( \Omega_\rho \). (Here \( \omega_\rho \) is the \( \mathcal{L}_\mu \)-harmonic measure on \( \Sigma_\rho \) relative to \( \Omega_\rho \). Since \( \Sigma_\rho \) is ‘smooth’, \( \omega_\rho \) is absolutely continuous with respect to surface measure.) Then \( v = \mathbb{K}^{\Omega_\rho}_\mu[\nu] \) in \( \Omega_\rho \) and
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\alpha} \int_{\Sigma_\rho} |\mathbb{K}^{\Omega_\rho}_\mu[\nu] - \mathbb{K}^{\Omega_\rho}_\mu[\nu_0]| dS = 0.
\]

It remains to verify that, if (2.25) holds then \( \nu_0 \) is uniquely determined by \( u \) in a fixed domain \( \Omega_\rho \).
Suppose, by negation, that there exist $\nu_1, \nu_2 \in \mathcal{M}_+(\partial \Omega)$ such that (2.25) holds for both $v_1 = K^{\Omega^+}_{\mu^+}[\nu_1]$ and $v_2 = K^{\Omega^+}_{\mu^+}[\nu_2]$. Then $w := |v_1 - v_2|$ is $\mathcal{L}_{\mu}$-subharmonic and $\text{tr}^*_{\partial \Omega}(w) = 0$.

Clearly $w$ is dominated by the $\mathcal{L}_{\mu}$-superharmonic function $v_1 + v_2$. Therefore, by Theorem 2.8 there exist $\lambda \in \mathcal{M}_+^{\delta \alpha + (\Omega^+_{\rho})}$ and $\chi \in \mathcal{M}_+^{\partial \Omega_{\rho})}$ such that,

$$w = K^{\Omega^+}_{\mu^+}[\chi] - G^{\Omega^+}_{\mu^+}[\lambda].$$

Thus $w + G^{\Omega^+}_{\mu^+}[\lambda]$ is $\mathcal{L}_{\mu}$-harmonic. By (2.16) and the fact that $\text{tr}^*_{\partial \Omega}(w) = 0$ we have $\text{tr}^*_{\partial \Omega}(w + G^{\Omega^+}_{\mu^+}[\lambda]) = 0$. Hence $w = 0$ and therefore $\nu_1 = \nu_2$. \qed

**Theorem 2.11.** Let $w$ be a nonnegative $\mathcal{L}_{\mu}$-subharmonic function in $\Omega_{\rho}$ dominated by an $\mathcal{L}_{\mu}$-superharmonic function in this domain. Then the boundary trace $\nu = \text{tr}^*_{\partial \Omega_{\rho}}(w)$ is well-defined and

$$w \leq K_{\mu^+}[\nu].$$

If $\nu_0 := \nu_{1, \rho}$ then,

$$\lim_{x \to \partial \Omega} \frac{w(x)}{K_{\mu^+}[\nu_0](x)} = 1 \quad \text{non-tangentially, } \nu_0\text{-a.e. on } \partial \Omega.$$

If $\nu_0 = 0$ then,

$$\limsup_{x \to \partial \Omega} \frac{w(x)}{\delta^{-\alpha}(x)} < \infty.$$

**Proof.** The first statement (2.26) follows from (2.24) and Theorem 2.4 (ii).

The second statement (2.27) follows from (2.24) and the fact that $G^{\Omega^+}_{\mu^+}[\lambda]$ is an $\mathcal{L}_{\mu}$-potential (i.e. a positive superharmonic function that does not dominate any positive $\mathcal{L}_{\mu}$-harmonic function). This fact implies (see, e.g. [3]):

$$\lim_{x \to \partial \Omega} \frac{G^{\Omega^+}_{\mu^+}[\lambda](x)}{K_{\mu^+}[\nu](x)} = 0 \quad \nu\text{-a.e. on } \partial \Omega.$$

By Fatou’s limit theorem

$$\lim_{x \to \partial \Omega} \frac{K_{\mu^+}[\nu_0](x)}{K_{\mu^+}[\nu](x)} = 1 \quad \nu\text{-a.e. on } \partial \Omega.$$

Therefore (2.24) implies (2.27).

The third statement (2.28) follows from (2.26) and Corollary 2.4. \qed

**Corollary 2.12.** Let $w$ be a nonnegative $\mathcal{L}_{\mu}$-subharmonic function in $\Omega_{\rho}$ for some $\rho \in (0, \bar{\rho})$. Then $w$ possesses a normalised boundary trace in $\mathcal{M}_+^{\partial \Omega}$ if and only if $w$ is dominated by a positive $\mathcal{L}_{\mu}$-superharmonic function $\nu$ in a strip around $\partial \Omega$.

**Proof.** If $w$ is dominated by a positive $\mathcal{L}_{\mu}$-superharmonic function in $\Omega_{\rho}$ then the existence of $\text{tr}^*_{\partial \Omega}(w)$ follows from (2.16) and Theorem 2.8.
Next suppose that $w$ has a normalized boundary trace $\nu_0 \in \mathcal{M}^+ (\partial \Omega)$. Without loss of generality we may assume that it also has a measure boundary trace $\nu_\beta$ on $\Sigma_\beta$. Since $u$ is $\mathcal{L}_\mu$-subharmonic, there exists a positive Radon measure $\tau$ in $\Omega$ such that

$$-\mathcal{L}_\mu u = -\tau.$$  

Let $\tau_\beta := \tau 1_{D_\beta \setminus \bar{D}_\rho}$, $w = \mathbb{K}_{\mu\nu}^\beta [\nu_0 + \nu_\beta]$ and $\nu_\beta = w [\Sigma_\beta]$.

Let $u_\beta$ be the solution of the boundary value problem,

$$-\mathcal{L}_\mu v = -\tau_\beta \text{ in } D_\beta \setminus \bar{D}_\rho, \quad v = \nu_\rho \text{ on } \Sigma_\rho, \quad v = \nu_\beta \text{ on } \Sigma_\beta.$$

Then

$$u_\beta + \mathbb{G}_{\mu\nu}^{D_\beta \setminus \bar{D}_\rho} [\tau_\beta] = w.$$

It follows that

$$G_{\mu\nu}^\Omega [\tau] = \lim_{\beta \to 0} \mathbb{G}_{\mu\nu}^{D_\beta \setminus \bar{D}_\rho} [\tau_\beta] < \infty,$$

which in turn implies that $\tau \in \mathcal{M}^+ (\Omega; \delta^{\alpha+})$ and finally

$$u + \mathbb{G}_{\mu\nu}^\Omega [\tau] = w.$$

In particular,

$$u \leq w = \mathbb{K}_{\mu\nu}^\beta [\nu_0 + \nu_\beta]. \quad \Box$$

**Corollary 2.13.** (i) Suppose that $u$ is positive and $\mathcal{L}_\mu$-subharmonic in $\Omega_\rho$. Then $\text{tr}^*_{\partial \Omega} = 0$ if and only if, for every $\rho \in (0, \rho)$, there exists a constant $c_\rho$ such that

$$(\text{2.30}) \quad u(x) \leq c_\rho \delta(x)\alpha^+ \quad \forall x \in \Omega_\rho.$$

(ii) Suppose that $u$ is positive and $\mathcal{L}_\mu$-superharmonic in $\Omega_\rho$. Then $u$ has a normalized boundary trace $\nu \in \mathcal{M}^+ (\partial \Omega)$ and consequently there exists $c_\rho$ such that

$$(\text{2.31}) \quad \int_{\Sigma_\beta} udS \leq c_\rho \beta^\alpha \quad \forall \beta \in (0, \rho).$$

**Proof.** (i) Obviously (2.30) implies that $\text{tr}^*_{\partial \Omega} (u) = 0$. Conversely assume that $\text{tr}^*_{\partial \Omega} (u) = 0$.

By the previous corollary $u$ is dominated by an $\mathcal{L}_\mu$-harmonic function. Therefore, by Theorem 2.8 there exist $\lambda \in \mathcal{M}^+ (\delta^{\alpha+})$ and $\nu \in \mathcal{M}^+ (\partial \Omega)$ such that $u = \mathbb{K}_{\mu\nu}^\Omega [\nu] - \mathbb{G}_{\mu\nu}^\Omega [\lambda]$. Since $\text{tr}^*_{\partial \Omega} (u) = 0$, $\nu_0 = \nu 1_{\Sigma_\rho} = 0$. Hence $u < \mathbb{K}_{\mu\nu}^\Omega [\nu_\rho]$ where $\nu_\rho = \nu 1_{\Sigma_\rho}$. Therefore the result follows from Corollary 2.7.

(ii) By the Riesz decomposition theorem (see [3]), $u = u_p + u_h$ where $u_p$ is an $\mathcal{L}_\mu$-potential and $u_h$ is a nonnegative $\mathcal{L}_\mu$-harmonic function in $\Omega_\rho$. It is known that every $\mathcal{L}_\mu$-potential is the Green potential of a positive measure. Thus there exists $\tau \in \mathcal{M}^+ (\Omega; \delta^{\alpha+})$ such that $u_p = \mathbb{G}_{\mu\nu}^\Omega [\tau]$. By the Representation Theorem $u_h = \mathbb{K}_{\mu\nu}^\Omega [\nu_\rho]$ for some $\nu \in \mathcal{M}^+ (\partial \Omega_\rho)$. Thus

$$u = \mathbb{G}_{\mu\nu}^\Omega [\tau] + \mathbb{K}_{\mu\nu}^\Omega [\nu].$$
The required result follows from Theorem 2.4.

3. \(L_\mu\)-MODERATE SOLUTIONS OF NONLINEAR EQUATION

In this section we study the nonlinear equation

\[ (P_\mu) \quad -L_\mu u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \]

where \(\Omega \subset \mathbb{R}^N\) is a bounded smooth domain, \(\mu < 1/4\) and \(q > 1\).

3.1. Preliminaries. Suppose that \(u \in L^q_{\text{loc}}(\Omega)\) is either a subsolution or a supersolution of \((P_\mu)\), in the distribution sense. Then, \(u \in W^{1,p}_{\text{loc}}(\Omega)\) for \(1 \leq p < N/(N-1)\). If, in addition, \(u\) is a distributional solution of \((P_\mu)\) then it is also a classical solution.

Consequently, if \(u \in L^q_{\text{loc}}(\Omega)\) is a distributional subsolution in \(\Omega\) then

\[ \int_\Omega \nabla u \cdot \nabla \varphi \, dx - \int_\Omega \frac{\mu}{\delta^2} u \varphi \, dx + \int_\Omega |u|^{q-1} u \varphi \, dx \leq 0 \quad \forall \varphi \in C^\infty_c(\Omega). \]

If, in addition, \(u \in H^1_{\text{loc}}(\Omega)\) then (3.1) holds for every \(\varphi \in H^1_c(\Omega)\).

A similar statement holds for supersolutions, in which case the inequality sign in (3.1) is inversed. Of course these statements remain valid for local subsolutions and supersolutions (in a subdomain \(G \subset \Omega\)).

We state below two results from [5] that will be used in the sequel.

Lemma 3.1. (Comparison principle [5, Lemma 3.2])

(i) Let \(G\) be open with \(G \subset \Omega\). Let \(0 \leq u, \overline{u} \in H^1_{\text{loc}}(G) \cap C(G)\) be a pair of sub and supersolutions to \((P_\mu)\) in \(G\) such that

\[ \limsup_{x \to \partial G} [u(x) - \overline{u}(x)] < 0. \]

Then \(u \leq \overline{u}\) in \(G\).

(ii) Let \(G\) be open with \(G \subset \Omega\). Let \(u, \overline{u} \in H^1(G) \cap C(\overline{G})\) be a pair of sub and supersolutions to \((P_\mu)\) in \(G\) and \(u \leq \overline{u}\) on \(\partial G\). Then \(u \leq \overline{u}\) in \(G\).

Lemma 3.2. ([5, Lemma 4.10]) Assume that \((P_\mu)\) admits a subsolution \(u\) and a supersolution \(\overline{u}\) in \(\Omega\) so that \(0 \leq u \leq \overline{u}\) in \(\Omega\). Then \((P_\mu)\) has a solution \(U\) in \(\Omega\) such that \(u \leq U \leq \overline{u}\) in \(\Omega\).

In [5, Proposition 3.5] the Keller–Osserman estimate has been extended to equation \((P_\mu)\). Specifically it was proved that every subsolution \(u\) of \((P_\mu)\) in \(\Omega\) satisfies,

\[ u(x) \leq \gamma_* \delta^{-\frac{2}{q-1}}(x) \quad \text{in } \Omega, \]

where \(\gamma_*\) is a constant independent of \(u\). In addition it was shown that, if \(u\) is a local subsolution in \(\Omega_{\rho}\), continuous at \(\Sigma_{\rho}\), then \(u\) satisfies (3.2) in \(\Omega_{\rho}\), but \(\gamma_*\) may depend on \(u\). We prove below a stronger version that is used later on.

Lemma 3.3. (Keller–Osserman estimate) If \(u\) is a subsolution of \((P_\mu)\) in \(\Omega\) then it satisfies (3.2) with a constant depending only on \(q, N, \mu\). If \(u\) is a subsolution of \((P_\mu)\) in \(\Omega_{\rho}\) then (3.2) holds with a constant depending only on \(q, N, \mu, \rho\) and \(\delta(x)\) replaced by \(\delta_\rho(x) := \text{dist}(x, \partial \Omega_{\rho})\).
Proof. Without loss of generality we may assume that $u \geq 0$ because $u_+$ is a subsolution. If $\mu \leq 0$ then $u$ is also a subsolution of the equation $-\Delta u + u^q = 0$. Therefore in this case (3.2) is a direct consequence of the classical Keller–Osserman inequality.

Now assume that $\mu > 0$. Let $y \in \Omega$ and $R = \delta(y)/2$. Then,

$$-\Delta u - \frac{\mu}{R^2} u + u^q \leq 0 \quad \text{in } B_R(y).$$

Therefore in $B_R(y)$ either $u \leq (8\mu/R^2)^{\frac{1}{q-1}}$ or $-\Delta u + u^q/2 \leq 0$. Hence, by Kato’s inequality, the function $v := (u - (8\mu/R^2)^{\frac{1}{q-1}})_+$ satisfies

$$-\Delta v + v^q/2 \leq 0 \quad \text{in } B_R(y).$$

By the classical Keller–Osserman inequality,

$$v(y) \leq c(q, N) R^{-\frac{2}{q-1}}.$$ 

Since $u(y) \leq v(y) + (8\mu/R^2)^{\frac{1}{q-1}}$ we conclude that

(3.3) $$u(y) \leq c(\mu, q, N) \delta\Omega(y)^{-\frac{1}{q-1}} \quad \forall y \in \Omega.$$ 

Next, let $u$ be a subsolution in $\Omega_\rho$. As before we may assume that $u \geq 0$ and that $\mu > 0$. By the first part of the proof, (3.3) holds in $\Omega_{3\rho/4}$. Further,

$$-\Delta u - (4\mu/\rho^2) u + u^q \leq 0 \quad \text{in } \Omega'_{\rho} = \{ x : \rho/2 \leq \delta(x) < \rho \}.$$ 

Therefore, either $u \leq (8\mu/\rho^2)^{\frac{1}{q-1}}$ or $-\Delta u + u^q/2 \leq 0$. By the same argument as before, the function $v := (u - (8\mu/\rho^2)^{\frac{1}{q-1}})_+$ satisfies

$$v(x) \leq c(q, N) \text{dist} \ (x, \Sigma_{\rho})^{-\frac{1}{q-1}} \quad \forall x : 3\rho/4 \leq \delta(x) < \rho.$$ 

Consequently,

(3.4) $$u(x) \leq c(\mu, q, N, \rho) \text{dist} \ (x, \partial\Omega_{\rho})^{-\frac{1}{q-1}} \quad \forall x \in \Omega_{\rho}. \quad \square$$

3.2. Moderate solutions. We study the generalised boundary trace problem $\{ \mu \partial^{\nu}/D \}$ where $\mu < 1/4$, $q > 1$ and $\nu \in \mathcal{M}^+(\partial\Omega)$. First we prove,

Lemma 3.4. Let $D$ be a $C^2$ domain such that $D \Subset \Omega$. If $0 \leq f \in C(\partial D)$ then there exists a unique solution of the problem

(3.5) \[ \begin{cases} \mathcal{L}_{\mu} u + u^q = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases} \]

Proof. For $u \in H^1(D)$, let

$$J_D(u) = \int_D \left( \frac{1}{2} |\nabla u|^2 - \frac{\mu}{2\delta\Omega} u^2 + \frac{1}{q+1} |u|^{q+1} \right) dx.$$ 

Since $\mu\delta_{\Omega}^{-2} \in L^\infty(D)$, it is standard to see that $J_D$ is coercive and weakly l.s.c. on

$$H^1_D(D) = \{ u \in H^1(D) : u = f \text{ on } \partial D \}.$$ 

Therefore there exists a minimizer $u_f \in H^1_D(D)$. We may assume that $u_f > 0$ because $|u_f|$ too is a minimizer. The minimizer is a solution of (3.5). The uniqueness is a consequence of the comparison principle. \quad \square
Next consider the problem,

$$(P_{\mu}^{\nu}(\rho)) \begin{cases} -\mathcal{L}_{\mu}u + u^{q} = 0 \quad \text{in } \Omega_{\rho}, \\ \text{tr}_{\nu,\Omega}^{\ast}(u) = \nu_{0}, \\ \text{tr}_{\nu,\Sigma}^{\ast}(u) = \nu_{\rho}. \end{cases}$$

where $\mu < 1/4$, $q > 1$, $\nu \in \mathcal{M}^{+}(\partial \Omega_{\rho})$ and $\rho \in (0, \bar{\rho}]$.

The following result is an adaptation of [10] Theorem C to problem $P_{\mu}^{\nu}(\rho)$. Since $C_{\mu}(\Omega_{\rho}) = 1/4$ the result applies to every $\mu < 1/4$. The proof follows the argument in [10]; for the convenience of the reader it is presented below.

**Proposition 3.5.** Let $\nu \in \mathcal{M}^{+}(\partial \Omega_{\rho})$ and assume that $\mathbb{K}_{\mu}^{\Omega}[\nu] \in L_{g^{+}}^{q}(\Omega_{\rho})$ for some $\rho \in (0, \bar{\rho})$. Then $(P_{\mu}^{\nu}(\rho))$ admits a unique solution $U_{\nu}$.

**Proof.** Let $\{D_{n}\}$ be a sequence of $C^{2}$ domains such that $\bar{D}_{n} \subset D_{n+1}$ and $D_{n} \uparrow \Omega_{\rho}$. Let $u_{n}$ be the solution of (3.5) with $D = D_{n}$ and $f = f_{n} := \mathbb{K}_{\mu}^{\Omega}[\nu]|_{\partial D_{n}}$. Since $\mathbb{K}_{\mu}^{\Omega}[\nu]$ is a supersolution of the equation $\mathcal{L}_{\mu}v + v^{q} = 0$ in $\Omega_{\rho}$ it follows that $u_{n}$ decreases and $u = \lim u_{n}$ is a solution of this equation. We claim that $u$ is a solution of $(P_{\mu}^{\nu}(\rho))$. Indeed,

$$(3.6) \quad u_{n} + G_{\mu}^{D_{n}}[u_{n}] = \mathbb{P}_{\mu}^{D_{n}}[f_{n}] = \mathbb{K}_{\mu}^{\Omega}[\nu] \quad \text{in } D_{n},$$

where $\mathbb{P}_{\mu}^{D_{n}}$ denotes the Poisson kernel of $\mathcal{L}_{\mu}$ in $D_{n}$.

Since $u_{n} \leq \mathbb{K}_{\mu}^{\Omega}[\nu] \in L_{g^{+}}^{q}(\Omega)$ it follows that

$$G_{\mu}^{D_{n}}[u_{n}] \to \mathbb{K}_{\mu}^{\Omega}[u^{q}].$$

Hence, by (3.6),

$$u + \mathbb{K}_{\mu}^{\Omega}[u^{q}] = \mathbb{K}_{\mu}^{\Omega}[\nu] \quad \text{in } \Omega_{\rho}.$$

By Theorem 2.4 $\text{tr}_{\nu,\Omega}^{\ast}(u) = \nu_{0}$ and (by (2.7)) $\text{tr}_{\nu,\Sigma}^{\ast}(u) = \nu_{\rho}$. \hfill $\square$

The next result is an adaptation of [10] Theorem D. We omit the proof which – except for obvious modifications – is the same as in [10].

**Proposition 3.6.** Assume that $u$ is a positive solution of $(P_{\mu}^{\nu}(\rho))$. Then

$$(3.7) \quad \lim_{x \to \Omega} \frac{u(x)}{\mathbb{K}_{\mu}^{\Omega}[\nu_{0}](x)} = 1 \quad \text{non-tangentially, } \nu-a.e. \text{ on } \partial \Omega,$$

where $\nu_{0} = \nu_{1,\rho}$.

**Theorem 3.7.** Let $\nu \in \mathcal{M}^{+}(\partial \Omega)$ and $\rho \in (0, \bar{\rho})$. Let $\nu' \in \mathcal{M}^{+}(\partial \Omega_{\rho})$ be defined by $\nu' = \nu$ on $\partial \Omega$ and $\nu' = 0$ on $\Sigma$. Assume that, for some $\rho$ as above, $\mathbb{K}_{\mu}^{\Omega}[\nu'] \in L_{g^{+}}^{q}(\Omega_{\rho})$. Then the boundary value problem $(P_{\mu}^{\nu}(\rho))$ admits a solution in $\Omega$.

**Proof.** By Proposition 3.5 there exists a (unique) solution $U_{\nu,0}$ of the problem $(P_{\mu}^{\nu}(\rho))$. For every $k \geq 0$, let $\nu_{k} \in \mathcal{M}^{+}(\partial \Omega_{\rho})$ be the measure given by, $\nu_{k}1_{\Omega_{\rho}} = \nu$ and $\nu_{k}1_{\Sigma_{\rho}} = k \text{d}S_{\Sigma_{\rho}}$. By the same proposition there exists a (unique) solution $U_{\nu,k}$ of $(P_{\mu}^{\nu}(\rho))$. Put

$$U_{\nu,\infty} = \lim_{k \to \infty} U_{\nu,k}.$$
Let $R \in (0, \rho)$. By Lemma 3.4 there exists a unique solution $v_R$ of (3.5) in $D_R$ with $f = U_{\nu,0}|\Sigma_R$. By the comparison principle,

$$U_{\nu,0} \leq v_R \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho \cap D_R.$$  

By Proposition 3.3 the family $\{v_R : 0 < R < \rho\}$ is bounded in compact subsets of $\Omega$. Therefore there exists a sequence $\{R_j\}$ converging to zero such that $v_{R_j}$ converges to a solution $v$ of the nonlinear equation in $\Omega$. By construction,

$$U_{\nu,0} \leq v \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho.$$  

Therefore $\text{tr}^*_{\partial\Omega}(v) = \nu$. □

Remark 3.8. If $\mu < C_H(\Omega)$ then problem $(P_{\nu,\mu})$ has at most one solution, [10, Theorem B]. However uniqueness fails when $C_H(\Omega) < \mu < 1/4$. It was proved in [5, Theorem 5.3] that in this case there exists a positive solution of $(P_{\nu,\mu})$ with $\nu = 0$. An alternative, more direct proof, is presented in Appendix A.

**Proposition 3.9.** Assume that $u \in L^q_{\text{loc}}(\Omega)$ is a positive solution of $(P_{\mu})$.

Then the following assertions are equivalent:

(i) $u$ has a normalized boundary trace,

(ii) $u$ is a moderate solution in the sense of Definition 1.3,

(iii) $u \in L^q(\Omega; \delta^{\alpha+})$.

**Proof.** The assumption implies that $\mathcal{L}_{\mu} u \leq 0$ in $\Omega$. If $\rho \in (0, \bar{\rho})$ then, by Lemma 2.12 (i) holds if and only if $u$ is dominated by an $\mathcal{L}_{\mu}$-superharmonic function in $\Omega_\rho$. Consequently, by Lemma 3.2 (i) holds if and only if $u$ is dominated by an $\mathcal{L}_{\mu}$-harmonic function in $\Omega_\rho$. Thus (i) and (ii) are equivalent.

If (iii) holds then $v := u + G_{\Omega_\rho}^{\mu}[u^q]$ is $\mathcal{L}_{\mu}$-harmonic. By the representation theorem there exists $v \in \mathcal{M}(\partial \Omega)$ such that $v = G_{\Omega_\rho}^{\mu}[v]$. Since $\text{tr}^*_{\partial\Omega} G_{\Omega_\rho}^{\mu}[u^q] = 0$ it follows that $\nu 1_{\partial\Omega}$ is the normalized boundary trace of $u$. Conversely if (ii) holds then by Theorem 2.8 $\mathcal{L}_{\mu} u = u^q \in \mathcal{M}^{\mu+}(\Omega_\rho)$ which is the same as (iii). □

### 3.3. Critical exponents.

The next result provides necessary and sufficient conditions in order that a positive measures $\nu \in \mathcal{M}^+(\partial \Omega)$ satisfy

$$\mathcal{K}_{\mu}^{\Omega_\rho} [\nu] \in L^q_{\mu_{\nu}^{-}}(\Omega_\rho) \quad \text{for some } \rho > 0.$$  

Let $\Gamma_a(x - y) = |x - y|^{-(N - a)}$ denote the Riesz kernel of order $0 < a < N$ in $\mathbb{R}^N$.

**Proposition 3.10.** Let $\nu \in \mathcal{M}^+(\partial \Omega)$.

(i) If $\Gamma_1 * \nu \in L^q_{\mu_{\nu_{\mu}^{-}}(\Omega)}$ then $\nu$ satisfies (3.8).

(ii) Assume $\mu \geq 0$. If $\nu$ satisfies (3.8) then $F_0^{\Omega} [\nu] \in L^q_{\mu_{\nu_{\mu}^{-}}(\Omega)}$.

Here $F_0^{\Omega}$ is the Poisson kernel of $-\Delta$ in $\Omega$: $P_0^{\Omega}(x, y) = \delta(x)|x - y|^{-N}$.
Proof. By (2.13),
\begin{equation}
K_\mu^\Omega(x, y) \sim \frac{\delta(x)^{\alpha_+}}{|x - y|^{N - 2\alpha_+}} \sim \delta(x)^{\alpha} - P_0^\Omega(x, y) |x - y|/\delta(x))^{2\alpha_-} \sim \delta(x)^{\alpha - \Gamma_1(x - y) |x - y|/\delta(x))^{-1 + 2\alpha_-},
\end{equation}
for every \((x, y) \in \Omega_{\rho/2} \times \partial \Omega\).

For every \(\mu < 1/4\) we have \(-1 + 2\alpha_- < 0\). Consequently,
\begin{equation}
K_\mu^\Omega(x, y) \leq c\delta(x)^{\alpha} - \Gamma_1(x - y) \quad \forall (x, y) \in \Omega_{\rho/2} \times \partial \Omega.
\end{equation}
Hence,
\[
\|F^\Omega_{\mu}\|_{L_{q, \alpha_+}^q(\Omega_{\rho/2})} \leq c \int_{\Omega_{\rho/2}} \left( \int_{\partial \Omega} \Gamma_1(x - y)d\nu(y) \right)^q \delta(x)^{q\alpha + \alpha_+} dx.
\]
This proves (i).

If \(\mu \geq 0\), so that \(\alpha_- \geq 0\) then, by (3.9),
\begin{equation}
K_\mu^\Omega(x, y) \geq c\delta(x)^{\alpha} - P_0^\Omega(x, y) \quad \forall (x, y) \in \Omega_{\rho/2} \times \partial \Omega.
\end{equation}
Therefore,
\[
\|F^\Omega_{\mu}[^{\mu}]\|_{L_{q, \alpha_+}^q(\Omega_{\rho/2})} \geq c \int_{\Omega_{\rho/2}} \left( \int_{\partial \Omega} P_0^\Omega(x, y)d\nu(y) \right)^q \delta(x)^{q\alpha + \alpha_+} dx.
\]
This proves (ii). □

Using this result we provide a necessary and sufficient condition for the existence of positive moderate solutions of \((P_\mu)\).

**Proposition 3.11.** Let \(\nu \in \mathcal{M}^+(\partial \Omega)\).

(i) If \(\alpha_- > -\frac{2}{q - 1}\) then the boundary value problem \((P_\mu^\nu)\) has a solution for every measure \(\nu = f dS_{\partial \Omega}\) such that \(f \in L^1(\partial \Omega)\).

(ii) If \(\alpha_- \leq -\frac{2}{q - 1}\) then, for every \(\nu \geq 0\), \((P_\mu^\nu)\) has no solution.

**Remark.** When \(\mu > 0\) and consequently \(\alpha_- > 0\), the condition in (i) holds for every \(q > 1\).

**Proof.** Let \(\nu = f dS_{\partial \Omega}\) and \(f \in L^\infty(\partial \Omega)^+\). Let \(x \in \Omega_{\delta_0}\) and pick \(x' \in \partial \Omega\) such that \(|x' - x| = \delta(x)|x\). Then,\n\begin{equation}
\int_{\partial \Omega} |x - y|^{1-N} f(y)dS(y) \leq c\|f\|_{L^\infty} \left( \int_{y \in \partial \Omega |x' - y| \geq \delta(x)} |x' - y|^{1-N}dS(y) + 1 \right)
\leq c\|f\|_{L^\infty} (1 + |\ln \delta(x)|) \leq c'\|f\|_{L^\infty} |\ln \delta(x)|,
\end{equation}
where \(c'\) is independent of \(x\). Therefore, if \((q - 1)\alpha_- + 1 \geq 1\) then \(\Gamma_1 * \nu \in L_{q, \alpha_+}^q(\Omega)\). Consequently, by Proposition 3.10 (i) and Theorem 3.7, problem \((P_\mu^\nu)\) has a solution.

Next, let \(f \in L^1(\partial \Omega)^+\) and \(\nu = f dS_{\partial \Omega}\). If \(\nu_n = \min(f, n)dS_{\partial \Omega}\) then problem \((P_\mu^{\nu_n})\) has a solution \(u_n\) and the sequence \(\{u_n\}\) is non-decreasing. In view of the Keller–Osserman estimate (3.2), \(\{u_n\}\) converges to a solution \(u\) of \((P_\mu^\nu)\). This proves (i).
We turn to part (ii). Suppose that $\alpha_- \leq -\frac{2}{q-1}$ and that there exists $\nu \in \mathcal{M}^+(\partial \Omega) \setminus \{0\}$ such that problem (P$_\nu$) has a solution $u$. Then, there exists $c > 0$ such that
\[
c^\beta - \frac{2}{q-1} \leq c^{\alpha_-} \leq \int_{\Sigma_\beta} \| K^\Omega_{\mu}[\nu] \| dS \quad \forall \beta \in (0, \beta_0).
\]
Since $u = -G_\mu[w^q] + K_\mu[\nu]$ and $\text{tr}_{\partial \Omega}(G_\mu[w^q]) = 0$ it follows that, for sufficiently small $\beta_1$,
\[
c^{\beta_0} \leq \int_{\Sigma_{\beta_1}} u dS \quad \forall \beta \in (0, \beta_1).
\]
But, by the Keller-Osserman estimate, $u(x) \leq c_1 \delta(x)^{-\frac{2}{q-1}}$ so that
\[
c^{\beta_0} \leq \int_{\Sigma_{\beta_1}} u dS \leq c_2 \beta^{-\frac{2}{q-1}} \quad \forall \beta \in (0, \beta_1).
\]
If $\alpha_- < -2/(q-1)$ we reached a contradiction. If $\alpha_- = -2/(q-1)$ then, in view of the Keller-Osserman estimate (3.2) we conclude that $u(x) \sim \delta(x)^{-\frac{2}{q-1}}$. This implies that $u \sim U_{\max} (= \text{the maximal solution of } -L_\mu v + v^q = 0)$. Thus $\sup U_{\max} u := c < \infty$. Now $cu$ is a supersolution and, if $v$ is the largest solution dominated by $cu$ then $\text{tr}^*(v) = c \text{tr}^*(u) = cu$. It follows that $U_{\max} \leq v$ which is impossible. \hfill \square

**Remark 3.12.** When $\mu > 0$ and consequently $\alpha_- > 0$ – the condition in (i) holds trivially for every $q > 1$. However, if $\mu < 0$ and
\[
q \geq q^*_\mu := 1 - \frac{2}{\alpha_-}
\]
then equation (P$_\nu$) has no moderate solution except for the trivial solution.

**Lemma 3.13.** Let $\mu < C_H(\Omega)$ and put
\[
q_{\mu,c} = \frac{N + 1 - \alpha_-}{N - 1 - \alpha_-}.
\]
Then, for $y \in \partial \Omega$,
\[
K^\Omega_{\mu}(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+}) \iff q < q_{\mu,c}.
\]
For every $q \in (1, q_{\mu,c})$ there exists a number $c = c(q, N, \mu)$ such that
\[
\| K^\Omega_{\mu}[\nu] \|_{L^{\frac{N + \alpha_+}{N - 1 - \alpha_-}}(\Omega, \delta^{\alpha_+})} \leq c \| \nu \| \quad \forall \nu \in \mathfrak{M}(\partial \Omega).
\]

**Proof.** Recall that
\[
K^\Omega_{\mu}(x, y) \sim |x - y|^{2-N-\alpha_+} (\delta(x)/|x - y|)^{\alpha_+} = \delta(x)^{\alpha_+} |x - y|^{2\alpha_- - N},
\]
(see [10, Section 2.2]). Therefore,
\[
c'(\delta(x)/|x - y|)^{\alpha_+} |x - y|^{1+\alpha_- - N} \leq K_{\mu}(x, y) \leq c |x - y|^{1+\alpha_- - N}.
\]
It follows that $K_{\mu}(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+})$ if and only if
\[
I := \int_0^1 t^{q(1+\alpha_- - N)} t^{\alpha_+} t^{N-1} dt < \infty
\]
and
\[ \| K_\mu (\cdot, y) \|_{L^q(\Omega, \delta^\alpha)} \sim I. \]
A simple computation shows that \( I < \infty \) if and only if
\[ q < q_{\mu,c} = \frac{N + 1 - \alpha_+}{N - 1 - \alpha_+}. \]

Finally,
\[ \| K_\mu^\Omega [\nu] \|_{L^q(\Omega, \delta^\alpha)} \leq \int_{\partial \Omega} \| K_\mu (\cdot, y) \|_{L^q(\Omega, \delta^\alpha)} d|\nu|(y) \leq c\| \nu \|. \]

Corollary 3.14. Let \( \mu < 1/4 \). If \( 1 < q < q_{\mu,c} \) then the boundary value problem \((P_{\nu}^\mu)\) has a solution for every Borel measure \( \nu \). Moreover, if \( q \geq q_{\mu,c} \) then problem \((P_{\nu}^\mu)\) has no solution when \( \nu \) is the Dirac measure.

Proof. In view of Lemma 3.13, the first assertion follows from Theorem 3.7. The second assertion follows from Proposition 3.6.

Appendix A. Non-uniqueness for \( C_H(\Omega) < \mu < 1/4 \)

We are going to show that for \( C_H(\Omega) < \mu < 1/4 \) the problem
\[
(P^0_{\mu}) \begin{cases} 
-L_\mu u + u^q = 0 & \text{in } \Omega, \\
\text{tr}^*_{\mu}(u) = 0,
\end{cases}
\]
admits a nontrivial solution. This was proved in [5, Theorem 5.3]. Here we provide a more direct argument.

Recall that if \( C_H(\Omega) < 1/4 \) then the operator \(-L_{C_H(\Omega)}\) admits a positive ground state solution \( \phi_H \in H^1_0(\Omega) \) such that \(-L_{C_H(\Omega)}\phi_H = 0 \) in \( \Omega \), see [9].

Proposition A.1. Assume that \( C_H(\Omega) < \mu < 1/4 \) and \( q > 1 \). Then \((P^0_{\mu})\) admits a positive solution \( U_0 \) such that
\[ \liminf_{x \to \partial \Omega} \frac{U_0(x)}{\phi_H(x)} > 0. \]

Proof. Since \(-L_{C_H(\Omega)}\phi_H = 0 \) in \( \Omega \), for a small \( \tau > 0 \) we obtain
\[ -L_\mu (\tau \phi_H) + (\tau \phi_H)^q = -\frac{\mu - C_H(\Omega)}{\delta^2} (\tau \phi_H) + (\tau \phi_H)^q \leq 0 \] in \( \Omega \), so that \( \tau \phi_H \) is a subsolution for \((P^0_{\mu})\) in \( \Omega \).

Fix \( \rho \in (0, \bar{\rho}] \). Similarly to the proof of Theorem 3.7, for every \( k \geq 0 \) denote \( \nu_{p,k} = k dS_{\Sigma_\rho} \) and let \( \nu \in \mathcal{M}^+(\partial \Omega_{\rho}) \) be the measure such that \( \nu 1_{\partial \Omega} = 0 \) and \( \nu 1_{\Sigma_\rho} = \nu_{p,k} \). By Proposition 3.5 there exists a (unique) solution of \((P_{\nu}^\mu(\rho))\) with this boundary data. Denote this solution by \( U_{0,k} \) and put
\[ U_{0,\infty} = \lim_{k \to \infty} U_{0,k}. \]
Let \( R \in (0, \rho) \). By Lemma 3.4 there exists a unique solution \( v_R \) of (3.5) in \( D_R \) with \( f = 2U_{0,\infty} \) on \( \Sigma_R \). We define,
\[ \varpi := \min\{U_{0,\infty}, u_R\} \] in \( D_R \cap \Omega^c_R \).
Then $u$ is a supersolution of $(P_\mu)$ in $D_R \cap \Omega_\rho$, $\Omega = U_{0,\infty}$ in $D_R \cap \Omega_\rho'$ for some $\rho' \in (R, \rho)$ and $u = u_R$ in $D_{R'} \cap \Omega_\rho$ for some $R' \in (R, \rho)$. Therefore setting $\bar{u} = u_R$ in $\Omega \setminus \Omega_\rho$ and $\bar{u} = U_{0,\infty}$ in $\Omega \setminus D_R$ provides an extension (still denoted by $\bar{u}$) that is a supersolution of $(P_\mu)$ in $\Omega$. As $\bar{u} = U_{0,\infty}$ in a neighborhood of $\partial \Omega$ it follows that $\bar{u} \sim \delta^{a_+}$ in such a neighborhood. On the other hand $\phi_H \sim \delta^{a_+}$ where $a_+ := \frac{1}{2} + \sqrt{\frac{1}{4} - C_H(\Omega)}$. As $C_H(\Omega) < \mu$ it follows that $\alpha_+ < a_+$ so that $\delta^{\alpha_+} > \delta^{a_+}$. Therefore $\tau \phi_H < \bar{u}$ near $\partial \Omega$ and therefore, by Lemma 3.1 everywhere in $\Omega$. Finally by Lemma 3.2 we conclude that there exists a solution $U_0$ of $(P_\mu)$ in $\Omega$ such that $\tau \phi_H < U_0 < \bar{u}$. Thus $U_0$ is a positive solution such that $\text{tr}^*(U_0) = 0$. □

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References

[1] A. Ancona, Comparaison des mesures harmoniques et des fonctions de Green pour des opérateurs elliptiques sur un domaine lipschitzien, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 15, 505–508 (French, with English summary).
[2] , Negatively curved manifolds, elliptic operators, and the Martin boundary, Ann. of Math. (2) 125 (1987), no. 3, 495–536.
[3] , Théorie du potentiel sur les graphes et les variétés, École d’été de Probabilités de Saint-Flour XVIII—1988, Lecture Notes in Math., vol. 1427, Springer, Berlin, 1990, pp. 1–112 (French).
[4] , First eigenvalues and comparison of Green’s functions for elliptic operators on manifolds or domains, J. Anal. Math. 72 (1997), 45–92.
[5] C. Bandle, V. Moroz, and W. Reichel, ‘Boundary blowup’ type sub-solutions to semilinear elliptic equations with Hardy potential, J. Lond. Math. Soc. (2) 77 (2008), no. 2, 503–523.
[6] H. Brezis and M. Marcus, Hardy’s inequalities revisited, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 1–2, 217–237 (1998).
[7] S. Filippas, L. Moschini, and A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains, Comm. Math. Phys. 273 (2007), no. 1, 237–281.
[8] K. T. Gkikas and L. Véron, Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials, Nonlinear Anal. 121 (2015), 469–540.
[9] M. Marcus, V. J. Mizel, and Y. Pinchover, On the best constant for Hardy’s inequality in $\mathbb{R}^n$, Trans. Amer. Math. Soc. 350 (1998), no. 8, 3237–3255.
[10] M. Marcus and P.-T. Nguen, Moderate solutions of semilinear elliptic equations with Hardy potential, Ann. Inst. H. Poincaré Anal. Non Linéaire (2015) doi: 10.1016/j.anihpc.2015.10.001 (available online).
[11] M. Marcus and L. Véron, Nonlinear second order elliptic equations involving measures, De Gruyter Series in Nonlinear Analysis and Applications, vol. 21, De Gruyter, Berlin, 2014.
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