Equilibrium configuration of perfect fluid orbiting around black holes in some classes of alternative gravity theories

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Abstract

The hydrodynamic behavior of perfect fluid orbiting around black holes in spherically symmetric spacetime for various alternative gravity theories has been investigated. For this purpose we have assumed a uniform distribution for the angular momentum density of the rotating perfect fluid. The contours of equipotential surfaces are illustrated in order to obtain the nature of inflow and outflow of matter. It has been noticed that the marginally stable circular orbits originating from decreasing angular momentum density lead to closed equipotential surfaces along with cusps, allowing the existence of accretion disks. On the other hand, the growing part of the angular momentum density exhibits central rings for which stable configurations are possible. However, inflow of matter is prohibited. Among the solutions discussed in this work, the charged $F(R)$ gravity and Einstein–Maxwell–Gauss–Bonnet solutions exhibit inflow and outflow of matter with central rings present. These varied accretion disk structures of perfect fluid attribute astrophysical importance to these spacetimes. The effect of higher curvature terms predominantly arises from the region near the black hole horizon. Hence the structural difference of the accretion disk in modified gravity theories in comparison to general relativity may act as an experimental probe for these alternative gravity theories.

Keywords: alternative gravity theory, accretion disc, black holes

(Some figures may appear in colour only in the online journal)

1. Introduction

General relativity (GR) is a very successful theory and is the best contender so far to describe the geometrical properties of spacetime. It has passed through all the experimental and
observational tests so far, ranging from local gravity tests like perihelion precession and bending of light to precision tests using pulsars [1–5]. In spite of these outstanding successes for GR there are still unresolved issues. These include the problem of dark energy and the problem of inflation. Phrased in a different way, the reason for the accelerated expansion of the Universe at both very small scale and very large scale is unknown [6–8]. These issues are generally dealt with by assuming the existence of additional matter components in the Universe, like the cosmological constant, fluid with complicated equations of state, scalar fields, etc. However, all these models are plagued by several issues, e.g., coupling with usual matter, consistency with elementary particle theories and consistency of formulation [9–11]. These results prompted research that took a new direction by modifying the Einstein–Hilbert (EH) action for GR itself.

This idea of modifying the EH action stems from the belief that GR is just a low energy approximation of some underlying fundamental theory [12, 13]. In this spirit the classical generalizations of EH action should explain both early-time inflation and late-time cosmic acceleration without ever introducing additional matter components in the energy momentum tensor. There exist large number of ways in which the standard EH action can be modified by introducing non-linear terms. However the criterion that the field equations should remain second order in the dynamical variable (otherwise some ghost fields would appear) uniquely fixes the action to be the Lanczos–Lovelock action [14–16]. The Lanczos–Lovelock Lagrangian has the special property that though the Lagrangian contains higher curvature terms the field equation turns out to be of second order [17, 18]. One of the important models is the second order correction term in the gravity action in addition to the EH term, called the Gauss–Bonnet (GB) Lagrangian.

Another such model explaining the above mentioned problems is obtained by replacing $R$, the scalar curvature in the EH action, by some arbitrary function of the scalar curvature $F(R)$. This alternative theory for describing gravitational interaction is very interesting in its own right, for it can provide an explanation to a large number of experimental phenomena. These include: late time cosmic acceleration [19], early power law inflation [20], and the problem of singularity in the strong gravity regime [21], along with possible detection of gravitational waves [22]. Compatibility with Newtonian and post-Newtonian approximations as well as estimations of cosmological parameters are shown in [8, 23]. $F(R)$ theory can also bypass a long known instability problem, called ‘Ostrogradski’ instability [24], and is capable of explaining recent graviton mass bound in the LHC, with possible constraints on the model itself [25].

There also exist quite a few gravity models which have come into existence via some low energy string inspired theories. For example, the GB Lagrangian discussed earlier is actually a low energy realization of a string inspired model [26]. In this context, we should mention that there exists a natural generalization of the Reissner–Nordström (RN) solution in GR to that in string inspired models with dilaton coupling. The dilaton field couples with the electromagnetic field tensor $F_{\mu\nu}$ in such a fashion that every solution with non-zero $F_{\mu\nu}$ couples with the dilaton field [27–30].

In this work our main aim is to obtain the equilibrium configuration of perfect fluid rotating around black holes in these alternative gravity theories. This in turn will allow us to address the new features that come out in comparison to their general relativistic counterpart. For this purpose we have first studied the equilibrium configuration of fluid in a general static spherically symmetric spacetime. Having derived various quantities of interest in this general
context, we apply them to the black hole solutions in various alternative gravity theories. This can be achieved by substituting the metric elements describing spherically symmetric spacetime in these alternative theories into our general result. Throughout this work we have mainly concentrated on three alternative theories, the \( F(R) \) gravity theory, dilaton induced gravity theory and finally the Einstein–Maxwell–Gauss–Bonnet (EMGB) gravity.

After deriving the results for a general static spherically symmetric spacetime, as a warm up we applied our results to the RN solution in Einstein gravity. Then we applied these results to a charged black hole solution in \( F(R) \) gravity theory. It turns out that the equipotential contours of the perfect fluid rotating around a black hole in this gravity model are quite different from those of its GR counterpart. Identical features are shared by the equilibrium configuration of the perfect fluid rotating around a black hole in the EMGB theory as well. However the equipotential contours of perfect fluids rotating around black holes in dilaton gravity are different from those found in the previous two situations. Still the equipotential contours exhibit features distinct from those of the respective GR solution. All these results suggest that the equilibrium configurations of perfect fluid rotating around black holes in alternative gravity theories have structures different from those found in GR. By comparing these alternative theories ‘under one roof’, as presented in this work, we have been able to understand the structure of accretion disks and in-fall of matter to them in some detail. This might help to understand the accretion disk structure around other black hole solutions by direct comparison with the results presented in this work.

Since the accretion disk is intimately connected with various astrophysical processes, it is natural to ask for some observational consequences of our result. We should stress at this point that though the accretion disk structure gets modified with the outflow and inflow of associated matter due to the introduction of alternative theories, this is unlikely to be detected in an observational test. These results have to do with the fact that the accretion disk structure across a black hole in these alternative theories is determined by the dimensionless parameter \( y = \Lambda M^2/12 \) in geometrized units, where \( \Lambda \) is the cosmological constant and \( M \) is the black hole mass. In order to have the stable circular orbits necessary for the existence of the accretion disc, the cosmological parameter has a restriction \( y < \chi \sim 4.8 \times 10^{-5} \) as we will show later. Along with this, we can use cosmological tests using the magnitude-redshift relation for supernovae with the measurements of cosmic microwave background fluctuation [42, 43] implying \( \Lambda \sim 10^{-50} \text{ cm}^{-2} \). This leads to very low values of \( y \) for realistic black holes. For example, in the case of an extremely massive black hole as seen in quasar TON 618, with mass \( \sim 6.6 \times 10^{10} M_\odot \) [44] leads to \( y \sim 4.1 \times 10^{-25} \). This suggests that the parameter space for observations in astrophysical scenarios has the value of cosmological parameter \( y \sim 10^{-30} \). This is quite small compared to the value \( y \sim 10^{-7} \) presented in the text for being of astrophysical importance [31–39]. Thus for normal astrophysical systems it is difficult to detect any departure from the result predicted by GR. Even for quiet massive systems, observational accuracy has not been possible to date, in order to detect departure from GR\(^1\). Such small values are out of the parameter space for present day astrophysical observations, making the test for these alternative theories difficult.

However, there is another window to look for signatures of these alternative theories. For primordial black holes in the early Universe the effective cosmological constant was higher [45] and consequently \( y \) would have had a much larger value comparable to \( \chi \). Thus in the

\(^1\) One such system where the effect can be observed is a pulsar–white dwarf system. Such a recent system, PSR J0348+0432, has a huge mass for a neutron star, making sensitive tests for the strong gravity regime possible. To date the orbital period decay of this system is in agreement with GR [40]. Another such system, PSR J1738+0333, also shows results consistent with GR [41].
early Universe the accretion disk structure around primordial black holes may possess the features discussed in this work. But this is also far from being observed experimentally. In spite of these difficulties in relating to observational results, the work stands on its own, for it describes the accretion structure in alternative theories and its departure from the GR results. For some choice of the parameter space the structure in alternative theories is much richer and differs significantly from that in GR. This analysis brings out the fact that introduction of higher curvature terms in the EH action alters the accretion disk structure of perfect fluid orbiting the black hole in a non-trivial manner with more structures included.

The paper is organized as follows. In section 2 we have illustrated Boyer’s condition [46–48] in order to have an equilibrium configuration of the perfect fluid. Then in section 3 we have derived various physical quantities for a general static spherically symmetric spacetime, which we have subsequently applied to determine equipotential surfaces of fluid moving around various black hole solutions in different gravity theories. Among these alternative theories we have discussed the RN solution, topologically charged black holes in $F(R)$ theory, charged black holes in dilaton gravity and finally black holes in EMGB theory in 4. At the end we conclude with a discussion of our results.

2. Equilibrium configuration of rotating perfect fluid

In this section we briefly summarize the well known results regarding the general theory of equipotential surfaces inside any relativistic, differentially rotating, perfect fluid body [46, 47]. This has also been applied to configurations of perfect fluid rotating in the stationary and axi-symmetric spacetimes [35–37]. In standard coordinate systems the spacetime is described by the following line element,

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt d\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

where the metric elements depend neither on time coordinate $t$ nor on azimuthal coordinate $\phi$. Hence energy and angular momenta are two conserved quantities in this spacetime implying the existence of two killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$. We shall consider perfect fluid rotating in the $\phi$ direction. Then its four velocity field $U^\mu$ has only two non-zero components,

$$U^\mu = \left(U^t, \ U^\phi, \ 0, \ 0 \right)$$

which can be functions of the coordinates $(r, \theta)$. The stress energy tensor of the perfect fluid is,

$$T^\mu_{\nu} = (\rho + \epsilon)U^\mu U^\nu + p\delta^\mu_{\nu}$$

where $\epsilon$ and $\rho$ denote the total energy density and the pressure of the fluid. The rotating fluid can be characterized by the vector fields of the angular velocity $\Omega$, and the angular momentum per unit mass (angular momentum density) $\ell$, defined by

$$\Omega = \frac{U^\phi}{U^t}; \quad \ell = -\frac{U_\phi}{U_t}.$$ 

These vector fields are related to the metric elements by the following result

$$\Omega = \frac{g_{tt} + \ell g_{\theta\theta}}{g_{\phi\phi} + \ell g_{\theta\phi}}$$

4
In static spacetimes \((g_{\phi} = 0)\), the above relation reduces to the simple formula
\[
\frac{\Omega}{\ell} = -\frac{g_{tt}}{g_{\phi\phi}}.
\] (6)

The surfaces of constant \(t\) and \(\Omega\) are called von Zeipel’s cylinders. These surfaces do not depend on the rotation law for fluids in static spacetimes but do depend on the rotation law for stationary spacetimes [35].

Projecting the stress energy tensor conservation law \(V^\mu T_{\mu}^\nu = 0\) onto the hypersurface orthogonal to the four velocity \(U^\mu\) by the projective tensor \(h_{\mu\nu} = g_{\mu\nu} + U^\mu U_\nu\), we obtain the relativistic Euler equation in the form,
\[
\frac{\partial_p \rho}{p + \epsilon} = -\partial_p (\ln U_t) + \frac{\Omega \rho \ell}{1 - \Omega \ell}
\]
where
\[
(U_t)^2 = \frac{g_{\phi\phi} - g_{tt} g_{\phi\phi}}{g_{\phi\phi} + 2\ell g_{\phi\phi} + \ell^2 g_{tt}}.
\] (8)

The solution to the relativistic Euler equation given in equation (7) can be obtained by defining the potential \(W(r, \theta)\) whose ‘equipotential surfaces’ are surfaces of constant pressure through the following relation [36, 48, 49]:
\[
\int_0^\rho \frac{dp}{p + \epsilon} = W_{in} - W
\]
which leads to
\[
W_{in} - W = \ln (U_t)_{in} - \ln (U_t) + \int_{\ell_{in}}^{\ell} \frac{\Omega \ell}{1 - \Omega \ell}.
\] (10)

The subscript ‘in’ in the above equations refers to the inner edge of the disk. For an alternative definition and thus derivation of Boyer’s conditions see [36, 50, 51]. The equipotential surfaces are determined by the condition, \(W(r, \theta) = \) constant and in a given spacetime \(W\) can be found from equation (10), if the rotation law \(\Omega = \Omega(\ell)\) is known. The surfaces of constant pressure are being determined by equation (9). The structure of thick accretion disks can also be found using the accurate Newtonian framework called the pseudo-Newtonian potential method [52, 53].

3. Equilibrium configuration of a perfect fluid for a general spherically symmetric spacetime

In this section we shall start with a general metric ansatz. This is very appropriate for describing various static spherically symmetric solutions in both Einstein gravity and other alternative gravity theories [54]. The metric ansatz is simple enough that it can be easily generalized to an arbitrary number of spacetime dimensions. The general metric ansatz is taken as,
\[
dx^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2.
\] (11)

Note that the characteristic property of equipotential surfaces is retained in this simplest choice, which is uniform distribution of angular momentum density \(\ell\) [37]. The importance of the above statement is enhanced considerably by the fact that marginally stable configurations
come into existence under the condition \[ \ell(r, \theta) = \text{const.} \] (12)

This holds for any rotating fluid irrespective of its rotation law for static spacetime (however for stationary spacetime the above condition depends on the rotation law [35]). Using this we can obtain the following expression for equipotential surfaces from equation (10) as

\[ W(r, \theta) = \ln U_i(r, \theta). \] (13)

Note that the quantity \( U_i(r, \theta) \) depends only on the metric and is being completely determined by equation (12). The equations for these equipotential surfaces are to be given by the following relation \( \theta = \theta(r) \), which can be obtained by solving the following differential equation

\[ \frac{d\theta}{dr} = -\frac{\partial p / \partial r}{\partial p / \partial \theta}. \] (14)

For the \( \ell = \text{constant} \) configurations the above expression leads to,

\[ \frac{d\theta}{dr} = -\frac{\partial U_i / \partial r}{\partial U_i / \partial \theta}. \] (15)

Having obtained all these basic tools we now proceed to determine the equilibrium configuration of perfect fluid orbiting around a black hole in a spacetime with a metric ansatz given by equation (11). Thus the potential turns out to be

\[ W(r, \theta) = \ln \left[ \frac{r \sin \theta \sqrt{f(r)}}{\sqrt{r^2 \sin^2 \theta - \ell^2 f(r)}} \right] \] (16)

and the differential equation satisfied by \( \theta \) leads to

\[ \frac{d\theta}{dr} = \frac{r^3 \sin^2 \theta f'(r) - 2\ell^2 f'(r)^2}{2r \ell^2 f'(r)^2} \tan \theta \] (17)

where ‘prime’ denotes derivative with respect to the radial co-ordinate \( r \). Insights about the \( \ell = \text{constant} \) surfaces are gained by examination of the potential \( W(r, \theta) \) in the equatorial plane \( \theta = \pi/2 \). From physical principles the potential should be real and that eventually leads to the following conditions

\[ f(r) \geq 0 \] (18)

\[ r^2 - \ell^2 f(r) \geq 0. \] (19)

The first condition would lead to static regions outside the black hole horizon where the perfect fluid orbiting the black hole can have equilibrium configurations. The second condition can be expressed in an elegant form as:

\[ \ell^2 \leq \ell^2_{\text{ph}} \equiv \frac{r^2}{f(r)} \] (20)

where the function defined as \( \ell^2_{\text{ph}} \) is the effective potential of the photon geodesic, with impact parameter \( \ell = U_p / U_i \) [56]. Note that the extremization of the potential \( W(r, \theta = \pi/2) \) is identical with the condition that the pressure gradient should vanish i.e. \( \partial U_i / \partial r = 0 \) and \( \partial U_i / \partial \theta = 0 \). However since we are concerned with the equatorial plane \( (\theta = \pi/2) \), the criterion \((\partial W / \partial r) = 0\) is sufficient for maximization of \( W(r, \theta = \pi/2) \) leading to
\[
\frac{\partial U(r, \theta = \pi/2)}{\partial r} = \frac{r f'(r) - 2f(r)^2}{2 \sqrt{f(r^2 - \ell^2 r(f(r))^3/2}}. \quad (21)
\]

It is well known that the extrema of the potential on the equatorial plane \(W(r, \theta = \pi/2)\) correspond to motion along circular geodesics. Thus the angular momentum distribution \(\ell_K^2\) obtained from the condition \(\partial_r U_r = 0\) represents the angular momentum density for circular orbits. From equation (21) the following estimation of angular momentum density \(\ell_K^2\) becomes possible

\[
\ell_K^2 = \frac{r f'(r)}{2f(r)^2}. \quad (22)
\]

With this identification of \(\ell_K^2\), the extrema for the potential \(W(r, \theta = \pi/2)\) can be obtained as

\[
W_{\text{extrema}}(r, \theta = \pi/2) = \ln E_c
\]

which determines

\[
E_c(r) = \frac{\sqrt{2f(r)}}{\sqrt{2f(r) - rf'(r)}} \quad (23)
\]

to be the specific energy attributed to motion along circular geodesics. Thus in order to summarize our results, we have discussed that one of the most important properties of the potential \(W(r, \theta)\) is that its behavior at equatorial plane \(\theta = \pi/2\) completely determines all the parameters under interest through \(\ell_{ph}^2\) and \(\ell_K^2\). From these two angular momentum densities we will be able to introduce more parameters in order to explain the behavior of the fluid configuration in a compact and understandable fashion.

For that purpose there exist two equations of utmost importance. One of them comes from the fact that the minima of \(\ell_{ph}^2\) gives the photon radius \(r_{ph}\) and is determined by solving the equation,

\[
r f'(r) = 2f(r). \quad (25)
\]

Also note that \(\ell_K^2\) diverges at the horizon given by \(f(r) = 0\). Since we have

\[
\frac{\partial \ell_K^2}{\partial r} = \frac{2f(r)^2 \left[3r^2 f'(r) + r f''(r) \right] - 4r^2 f'(r) f''(r) r^2}{4f(r)^4} \quad (26)
\]

the local extrema of the angular momentum density \(\ell_K^2\) are being determined by the following equation,

\[
4r f'(r)^2 = 2f(r) \left[3r^2 f'(r) + r f''(r) \right]. \quad (27)
\]

Hence all the relevant quantities necessary to uniquely characterize the fluid motion are derived for a general metric ansatz. We will henceforth use these results frequently throughout the latter part of this work.

**4. Equipotential surfaces of marginally stable configurations orbiting around black holes in various gravity theories**

In this section we will apply the results derived in the previous section to spherically symmetric and static spacetimes in both GR and alternative gravity theories. We will start with the RN solution in general relativity and then subsequently will generalize to alternative gravity theories. Among these alternative theories, we will consider the black hole solution in \(F(R)\)
gravity first, then Einstein–Maxwell gravity with dilaton field, finally specializing to EMGB
gridy (similar aspects within the GR framework have been considered in [57, 58]).

4.1. Reissner–Nordström black hole

Before proceeding to alternative theories we shall first consider a solution in GR itself. This will help the reader to understand various physical quantities of interest in the other theories better. The line element for the RN black hole has the following expression

$$\text{ds}^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\text{dr}^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^3}\right)^{-1}\text{dr}^2 + r^2\text{d\Omega}^2$$  \hspace{1cm} (28)

where $Q$ is the charge of the black hole and $M$ is its mass. The potential $W(r, \theta)$, introduced by equations (9) and (16) for the above metric takes the following form

$$W(r, \theta) = \frac{1}{2} \ln \left[\frac{r^2 \sin^2 \theta (r^2 - 2Mr + Q^2)}{r^4 \sin^2 \theta - (r^2 - 2Mr + Q^2)\ell^2}\right]$$  \hspace{1cm} (29)

Hence from the reality criteria of the potential we arrive at the following constraints

$$\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \geq 0$$  \hspace{1cm} (30)

$$\ell^2 \leq \ell_{\text{ph}}^2 = \frac{r^4}{r^2 - 2Mr + Q^2}$$  \hspace{1cm} (31)

where $\ell_{\text{ph}}$ corresponds to angular momentum associated with the photon orbit. Also we have the following result for $(d\theta/dr)$ from equation (17) as

$$\frac{d\theta}{dr} = \frac{r^4(Mr - Q^2)\sin^2 \theta - \ell^2(r^2 - 2Mr + Q^2)^2}{r\ell^2(r^2 - 2Mr + Q^2)^2/\tan \theta}.$$  \hspace{1cm} (32)

Now there are two horizons of this spacetime: one is the regular event horizon, located at $r = r_{\text{eh}}$ while the other, known as the Cauchy horizon, is located at $r = r_{\text{ch}}$. These have the following expressions

$$r_{\text{eh}} = M + \sqrt{M^2 - Q^2}$$  \hspace{1cm} (33)

$$r_{\text{ch}} = M - \sqrt{M^2 - Q^2}.$$  \hspace{1cm} (34)

Another important radius is the photon circular orbit radius, whose location in this spacetime is given by the following expression:

$$r_{\text{ph}} = \frac{3M \pm \sqrt{9M^2 - 8Q^2}}{2}.$$  \hspace{1cm} (35)

The extrema of the potential $W(r, \theta)$ on the equatorial plane are being determined by an angular momentum density $\ell_\kappa$ from equation (22) with the following expression
\[ \ell^2_k = \frac{r^4(Mr - Q^2)}{(r^2 - 2Mr + Q^2)^2}. \] (36)

Then we also have the following expression for derivative of \( \ell_k^2 \):

\[ \frac{d\ell^2_k}{dr} = \frac{(r^2 - 2Mr + Q^2)}{(r^2 - 2r + Q^2)^3} \left( 5Mr^4 - 4r^3Q^2 - 4r^4(M - Q^2) \right). \] (37)

The minima of the photon angular momentum can be obtained by plugging equation (35) into equation (31) which leads to:

\[ \ell_{\text{ph}}(\text{min})^2 = \frac{(3M + \sqrt{9M^2 - 8Q^2})^4}{8(3M^2 - 2Q^2 + M\sqrt{9M^2 - 8Q^2})}. \] (38)

Thus we have presented all these results for a well known solution in GR to facilitate easy comparison with other gravity theories. This also brings out the key parameters involved in these computations which will be helpful as we go along discussing various alternative gravity theories.

4.2. Charged black hole in F(R) gravity

In this section we will restrict ourselves to four dimensional spacetime. The action in this four dimensional spacetime with \( F(R) \) gravity in the presence of a matter field can be presented as:

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ F(R) + \mathcal{L}_{\text{matter}} \right]. \] (39)

In the above expression for action \( F(R) \) is an arbitrary function of the Ricci scalar \( R \) and \( \mathcal{L}_{\text{matter}} \) is the Lagrangian for matter fields. Then variation of the above action with respect to the metric leads to the field equation \[9, 59\]:

\[ \nabla^a \nabla_b F_R - \nabla_a \nabla_b F_R + \left( \nabla_R \nabla - \frac{1}{2} F(R) \right) g_{ab} \equiv T_{ab}^{\text{matter}} \] (40)

where \( R_{ab} \) is the Ricci tensor, \( F_R \equiv dF(R)/dR \) and \( T_{ab}^{\text{matter}} \) is the standard matter stress-energy tensor, derived from the matter part of the Lagrangian \( \mathcal{L}_{\text{matter}} \) as given in action (39). Also the trace of the above equation connects the trace of the stress-energy tensor to the scalar curvature. We are interested in determining a spherically symmetric solution to the above field equation. Following the analysis presented in \[60\] we arrive at a charged solution in this \( F(R) \) gravity model with \( F(R) = R - \lambda \exp(-\xi R) \). It should be noted as emphasized in section 1 that these modifications in EH action should pass all tests starting from clustering of galaxies down to Solar System tests. For this special exponential correction factor it has been shown \[61\] that there is no contradiction with Solar System tests. Also the solutions of this model are indistinguishable from the standard EH solution except for a possible change in Newton’s constant \[62\]. The solution to the above field equation presented in equation (40) is topologically charged and can be presented following equation (11) as

\[ f(r) = 1 - \frac{\Lambda}{3} r^2 - \frac{M}{r} + \frac{Q^2}{r^2}. \] (41)
where $\Lambda = \lambda \left(4 + 2\xi R\right)/\left(8e^{\xi R}\right)$. In order for this metric ansatz to satisfy equation (40) we should have some constraint among the parameters appearing in our theory, which are the following [60]:

$$1 + \frac{\lambda \xi}{\exp(\xi R)} = 0$$

$$\frac{\lambda}{\exp(\xi R)} + \frac{R}{2} \left( \frac{\lambda \xi}{\exp(\xi R)} - 1 \right) = 0. \quad (43)$$

Although this solution looks similar to the charged black hole, its higher dimensional behavior is different. The charge term in the above expression goes as $r^{-(d-2)}$ in $d$ dimension, while it goes as $r^{-2(d-3)}$ in standard charged higher dimensional black hole spacetime. Therefore it is interesting to ask whether this term is originating from the scalar-tensor representation of $F(R)$ gravity. In order to obtain the representation the standard way is to start with conformal transformations on the metric elements. However in this case no new physical interpretation can be obtained except its relation to the scale of the problem [10, 60, 63].

The above result can also be interpreted in a slightly different manner: uncharged solutions in $F(R) = R + f(R)$ theory show exact similarity with Einstein gravity in the presence of a cosmological constant. Thus in those situations $f(R)$ plays the role of cosmological constant. As shown in [60] this charged solution is equivalent to Einstein gravity coupled with a conformally invariant Maxwell field, such that the $f(R)$ term in this situation has the role of electromagnetic field. This acts as an origin of the charge term.

Next we will consider the equilibrium configurations of perfect fluid rotating around this black hole. For notational simplicity we will assume $M = 2$, $Q = 1$ and $\Lambda = \Lambda/3$. Then the line element for this black hole spacetime reduces to the following form

$$\mathrm{d}s^2 = -\left(1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{\left(1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}\right)} + r^2\mathrm{d}\Omega^2. \quad (44)$$

We must stress that static region exists in this spacetime with sub-critical values of cosmological parameter $y$ as

$$y < y_c = \frac{1}{16}. \quad (45)$$

Thus equilibrium configurations are possible in this spacetime provided cosmological parameter $y$ satisfies the constraint given by equation (45). The equipotential surfaces are being given by

$$W(r, \theta) = \ln \left[\frac{r \sin \theta \sqrt{\left(1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}\right)}}{\sqrt{r^2 \sin^2 \theta - \ell^2 \left(1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}\right)}}\right] \quad (46)$$

and we also have,

$$\frac{d\theta}{dr} = \frac{\left(2r - 2 - 2yr^4\right) \sin^2 \theta - 2\ell^2 \left(1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}\right)^2}{2\ell^2 \left(1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}\right)^2} \tan \theta. \quad (47)$$

Note that for $y = 0$ it reduces to the RN scenario discussed earlier. The best idea about the nature of an $\ell = \text{constant}$ surface can be extracted from the behavior of the potential $W(r, \theta)$
in equatorial plane ($\theta = \pi/2$). There we have two reality conditions imposed on the potential,

$$\left( 1 - yr^2 - \frac{2}{r} + \frac{1}{r^2} \right) \geq 0$$

(48)

$$\ell^2 \leq \ell^2_{\text{ph}} \equiv \frac{r^2}{\left( 1 - yr^2 - \frac{2}{r} + \frac{1}{r^2} \right)}.$$  

(49)

The function $\ell_{\text{ph}}$ represents the angular momentum of the photon’s orbit. Further extremizing the potential $W(r, \theta = \pi/2)$ we arrive at the particular expression for angular momentum density

$$\ell^2 = \ell^2_{K}(r, y) \equiv \frac{(2r - 2 - 2 yr^4)}{2 \left( 1 - yr^2 - \frac{2}{r} + \frac{1}{r^2} \right)^2}.$$  

(50)

The specific energy of circular geodesics, corresponding to local extrema of the effective potential, can be expressed as

$$E_c(r, y) = \frac{1 - yr^2 - \frac{2}{r} + \frac{1}{r^2}}{\sqrt{1 - \frac{2}{r} + \frac{1}{r^2}}}.$$  

(51)

Now we turn back to the horizons in this spacetime. For $y > 0$, the photon angular momentum $\ell^2_{\text{ph}}(r, y)$ diverges at the black hole horizon $r_h$ and cosmological horizon $r_c$ determined by the equality in equation (48). This leads to the following expression for horizons,

$$r_h = \frac{1}{2} \left( \frac{1}{\sqrt{y}} - \frac{\sqrt{1 - 4\sqrt{y}}}{\sqrt{y}} \right)$$

(52)

$$r_c = \frac{1}{2} \left( \frac{1}{\sqrt{y}} + \frac{\sqrt{1 - 4\sqrt{y}}}{\sqrt{y}} \right).$$  

(53)

Also the photon circular orbit radius can be obtained from the extrema of $\ell^2_{K}$. This has the numerical value: $r_{\text{ph}} = 3.56155$. For this circular orbit the photon angular momentum density could be given by,

$$\ell^2_{\text{ph}(c)} \equiv \ell^2_{\text{ph}}(r_{\text{ph}}, y) = \frac{12.6846}{0.359611 - 12.9846y}.$$  

(54)

Additionally the zero point of $\ell^2_{K}$ leads to a radius called static radius and in this black hole spacetime is expressed parametrically as,

$$y(r_s) = \frac{r_s - 1}{r_s^4}.$$  

(55)

Note that $\ell^2_{K}$ is not well defined in the region $r > r_s$, being negative there. At this static radius black hole attraction is compensated for cosmological repulsion. Then local extrema of $\ell^2_{K}$ corresponds to the following expression for the cosmological parameter as,
This determines the marginally stable circular geodesics. The local maxima of this function $y_{ms}$ give the critical value for the cosmological parameter $y$ admitting stable circular orbits,

$$y_{ms} = 0.000692.$$ (57)

For $y < y_{ms}$, there exists an inner or outer marginally stable circular geodesic at $r_{ms(i)}$ or $r_{ms(o)}$. There exists another special value of $y$, which corresponds to the situation where the minimum value of $\ell_{pl}$ equals the maximum of $\ell_{K}$. This value is denoted by $y_{e}$ and has the following numerical value,

$$y_{e} \sim 4.8 \times 10^{-5}.$$ (58)

We can now separate out four different classes depending on the behavior of the functions $\ell_{pl}$ and $\ell_{K}$. These four classes are defined according to the cosmological parameter in the following way,

(i) $0 < y < y_{e}$

(ii) $y = y_{e}$

(iii) $y_{e} < y < y_{ms}$

(iv) $y_{ms} < y < y_{e}$.

For all these classes the variation of the two angular momentum quantities is illustrated in figure 1. In all these figures the descending parts of the curve $\ell_{K}^{2}(r, y)$ connect to unstable circular geodesics, along with the growing part (if it exists) determining stable circular geodesics. The extrema of $\ell_{K}^{2}(r, y)$ have an important significance: the minima determines $\ell_{ms(i)}$ at $r_{ms(i)}$, the inner marginally stable circular orbit and the maxima determines $\ell_{ms(o)}$ at $r_{ms(o)}$, which corresponds to the outer marginally stable circular orbit.

The equipotential surfaces are also astrophysically quite important. Their properties can be established by studying the properties of the potential $W(r, \theta)$ in the equatorial plane. Note that the potential $W(r, \theta = \pi/2, y)$ has closely related properties with the effective potential of geodesic motion. It is worthwhile to mention that in the limit $r \to r_{h}$ or $r \to r_{c}$, $W(r, \theta = -\pi/2, y) \to -\infty$ hence the topological properties of the equipotential surfaces are directly inferred from the behavior of the potential $W(r, \theta = \pi/2, y)$. As pointed out earlier, local extrema of the potential are determined by the condition,

$$\ell^{2} = \ell_{K}^{2}(r, y).$$ (59)

Hence the decreasing part of $\ell_{K}^{2}$ determines the maxima of the potential $W(r, \theta = \pi/2, y)$. These correspond to cusps such that at these radii matter moves along an unstable geodesic orbit. While the rising part in $\ell_{K}^{2}$ determines the minima of the potential. These correspond to central rings of the equilibrium configurations along which matter moves in stable geodesics.

Next we provide a complete study of the behavior of equipotential surfaces along with the potential $W(r, \theta = \pi/2, y)$. We begin with the situations of astrophysical importance.

(i) $0 < y < y_{e}$. This is the situation illustrated in figure 1(a). Here we discuss eight different configurations according to the values of $\ell = \text{constant}$ satisfying $\ell > 0$.

(a) $\ell < \ell_{ms(i)}$. Open surfaces are the only ones in existence. No disks are possible while surfaces with outer cusps exist (figure 2(a1–2)).

(b) $\ell = \ell_{ms(i)}$. An infinitesimally thin, unstable ring located at $r_{ms(i)}$ exists. Open surfaces with outer cusps also remain in the picture (figure 2(b1–2)).
Closed surfaces come into existence. Many equilibrium configurations even without any cusps are now possible. We also have surfaces with inner and outer cusps (figure 2(c1–2)).

Here also equilibrium configurations without cusps are possible. A surface comes into existence with both inner and outer cusp. Along with inflow into the black hole due to mechanical non-equilibrium, an outflow from the disk also occurs. This comes due to the repulsive nature of the cosmological parameter. This is an astrophysically important scenario (figure 2(d1–2)).

Equilibrium configurations are possible but accretion is not possible since we have no closed surface with inner cusp. The surface with an inner cusp is an open equipotential surface, while that with an outer cusp is a closed equipotential surface. This makes outflow from the disk possible (figure 2(e1–2)).

The potential $W(r, \theta = \pi/2, y)$ diverges at the photon circular orbit. Thus the inner cusp completely disappears. Closed equipotential surfaces with outer cusps still exist enabling outflow from the disk (figure 2(f1–2)).

An infinitesimally thin, unstable ring located at $r_{m(o)}$ exists with coalescing outer and central cusp (figure 2(g1–2)).

Only open equipotential surfaces exist with no cusp (figure 2(h1–2)).
Figure 2. The figures (a)–(k) show the variation of potential $W(r, \theta = g/2, \gamma)$ with $\log r$ (the ‘1’ part) and the contour plot of $W$ with $\log r \sin \theta$ and $\log r \cos \theta$ (the ‘2’ part). These actually determine the equipotential surfaces i.e. meridional sections of marginally stable configurations with $\ell = \text{constant}$ for fluid motion around a black hole in this $F(R)$ theory. All possible variations for different values of cosmological parameter $\gamma$ and angular momentum $\ell$ are shown. The cusps are originating from local maxima of the potential while the central rings are from local minima. All these situations are discussed in more detail in the text.
For this special $y$ value the variation of $\ell_{K}^{2}$ and $\ell_{ph}^{2}$ has been illustrated in figure 1(b). For this case also we obtain all the hypersurfaces with similar nature. All of them fall into a single class, given by

(a) $\ell = \ell_{ph} = \ell_{usi}$. Here there exists no inner cusp, only an outer cusp mixing with the center exists (figure 2(i1–2)).

(ii) $y = y_{c}$. For this special $y$ value the variation of $\ell_{K}^{2}$ and $\ell_{ph}^{2}$ has been illustrated in figure 1(b). For this case also we obtain all the hypersurfaces with similar nature. All of them fall into a single class, given by

(a) $\ell = \ell_{ph} = \ell_{usi}$. Here there exists no inner cusp, only an outer cusp mixing with the center exists (figure 2(i1–2)).
This situation is illustrated by figure 1(c). However in this situation many other choices come into existence. We classify them below:

(a) $\ell_{mb} < \ell < \ell_{ms(o)}$. This condition is equivalent to the condition (1e) illustrated above.

(b) $\ell = \ell_{ms(o)}$. This situation is slightly different. Here there exists an inner cusp of an open equipotential surface, while the outer cusp gets mixed with the center corresponding to a thin ring at the radius $r_{ms(o)}$ (figure 2(j1–2)).

(c) $\ell_{ms(o)} < \ell < \ell_{ph(c)}$. This condition has only open equipotential surfaces; among them, one has an inner cusp (figure 2(k1–2)).

(iv) $y_{ms} < y < y_{\nu}$. For this choice of cosmological parameter the variations of $\ell_K^2$ and $\ell_{ph}^2$ are shown in figure 1(d). Note that in this situation $\ell_K^2$ has only a declining part. Thus we can have only maxima of the potential and existence of open equipotential surfaces. This leads to the fact that in these spacetimes stable circular geodesics do not exist. In this case there are only two physically meaningful intervals.

(a) $\ell < \ell_{ph(c)}$. This situation has equipotential surfaces identical to the situation illustrated in (1a).

(b) $\ell \geq \ell_{ph(c)}$ This is similar to the situation (2a) discussed earlier.
We should also mention in this connection that the following numerical values are obtai

ted for angular momentum densities, which are important for the study of fluid orbiting around a black hole. The angular momentum density for an inner marginally stable orbit is \( \ell_{\text{in}} = 3.03965 \) ms, while that for an outer marginally stable orbit is given by \( \ell_{\text{out}} = 4.9 \) for the cosmological parameter being \( y = 10^{-6} \). Two other important angular momentum densities are that of the marginally bound orbit, with \( \ell_{\text{mb}} = 3.28852 \), and photon circular orbit, \( \ell_{\text{phc}} = 4.27392 \). These expressions are of importance for studying the structure of accretion disk.

### 4.3. Charged black hole in dilaton gravity

Equilibrium configurations originating from test particle fluid, which is rotating in a given spacetime, are determined by the equipotential surfaces, where the gravitational and inertial forces are being compensated by the pressure gradient. (For an axially symmetric spacetime the rotation axis of the equilibrium configuration coincides with the axis of symmetry for the spacetime, while for a spherically symmetric case this axis can be any radial line; see for example [48]).

The influence of a non-zero dilaton charge coming from an effective string theory on the character of the equipotential surfaces of marginally stable orbits has been studied for configuration rotating around such black holes. Static uncharged black holes in general relativity are described by the well known and well studied Schwarzschild solution. However even for large mass black holes (compared to Planck mass) the Schwarzschild solution is a good approximation to describe uncharged black holes in string theory except for regions near the singularity. This situation is completely different for the Einstein–Maxwell solution in string inspired theory due to dilaton coupling.

The dilaton couples with \( F^2 \) with the implication that every solution having non zero \( F_{\mu\nu} \) couples with the dilaton. Thus the charged black hole solution in general relativity (the RN solution) appears in string theory with the presence of the dilaton. Thus the effective four dimensional low energy action is given by,

\[
S = \int d^4x \sqrt{-g} \left[ -R + e^{-2\Phi} F^2 + 2(\nabla\Phi)^2 \right]
\]  

where \( F_{\mu\nu} \) is the Maxwell field and we have set other gauge and antisymmetric tensor fields, e.g. the Kalb–Ramond field \( H_{\mu
u\rho} \) to zero in order to focus on the dilaton field \( \Phi \) [27–30, 64]. Extremizing the above action with respect to the \( U(1) \) potential \( A_\mu \), dilaton field \( \Phi \) and metric \( g_{\mu\nu} \) lead to the following field equations,

\[
\nabla_\mu \left( e^{-2\Phi} F^{\mu\nu} \right) = 0
\]  

\[
\nabla^2 \Phi + \frac{1}{2} e^{-2\Phi} F^2 = 0
\]  

\[
R_{\mu\nu} = 2 \nabla_\mu \Phi \nabla_\nu \Phi + 2 e^{-2\Phi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} g_{\mu\nu} e^{-2\Phi} F^2.
\]

The static spherically symmetric solution to the above field equations yields the line element as [27]:
\[ ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 + r\left(r - e^{2\Phi_0}Q^2/M\right)d\Omega^2 \]  

(64)

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \). Due to isometry we can confine our motion in the equatorial plane such that \( d\Omega^2 = d\phi^2 \) [i.e. \( \theta = \pi/2 \)]. Here \( \Phi_0 \) gives the asymptotic value of the dilaton field and \( Q \) represents the black hole charge. This solution is almost identical to the Schwarzschild metric with the difference being that areas of the two spheres depend on \( Q \).

The surface \( r = \frac{Q^2e^{\Phi_0}}{M} \) is singular and \( r = 2M \) is the regular event horizon. The solution of the scalar field \( \Phi \) as a function of \( r \) in terms of its asymptotic value \( \Phi_0 \) can be obtained from equation (62) leading to the following result [27]:

\[ e^{-2\Phi} = e^{-2\Phi_0} - \frac{Q^2}{Mr}. \]  

(65)

From the above result note that we have the following limit, \( r \to \infty \) leading to \( \Phi \to \Phi_0 \). Thus the dilaton charge can be defined as,

\[ D = \frac{1}{4\pi} \int d^2\sigma^t V_\sigma^r \Phi \]  

(66)

where the integral is over the two spheres located at spatial infinity and \( \sigma^t \) is the normal to the corresponding two spheres at spatial infinity. For the charged black hole we are considering this leads to,

\[ D = -\frac{Q^2e^{2\Phi_0}}{2M}. \]  

(67)

Note that dilaton charge \( D \) depends on the asymptotic value of dilaton field \( \Phi_0 \), which is determined solely by \( M \) and \( Q \) and is always negative.

One point that must be mentioned while passing is that this charge originates from the coupling of the electromagnetic field with a scalar field, and the actual charge present in the system is the electromagnetic charge \( Q \). If there were no electromagnetic charge the dilaton charge \( D \) defined through equation (67) would have been zero. Hence this dilaton charge is not a scalar charge, it is a coupling between the electromagnetic charge and the scalar field at asymptotic infinity. This charge is also responsible for long range attractive force between black holes [27].

Note that the actual dependence on the dilaton field depends on the Planck scale described by \( e^{-\Phi/M_p} \). Since we have worked in the unit \( M_{pl} \sim 1 \) the term modified to \( e^{-\Phi} \). As \( \Phi \to \Phi_0 \sim M_{pl} \), this term becomes significant. For notational convenience we shall define a quantity \( q = -D \) and by virtue of the above discussion \( q \) is always positive. Hence our solution is parameterized by the variable \( q \) and choosing the dimensionless expression such that \( M = 1 \) we get the line element as

\[ ds^2 = -\left(1 - \frac{2}{r}\right)dt^2 + \left(1 - \frac{2}{r}\right)^{-1}dr^2 + r(r - 2q)d\Omega^2. \]  

(68)

Then we have two horizons in our system given by \( r_1 = 2 \) and \( r_2 = 2q \). We shall take the choice in which \( r_1 > r_2 \) or equivalently \( q \leq 1 \). Thus equilibrium configurations are possible only for \( r > 2 \). Now the equipotential surfaces are possible only in these spacetimes. Now, the equipotential surfaces are determined by the result,
\[ W(r, \theta) = \frac{1}{2} \ln \left[ \frac{r(r-2)(r-2q) \sin^2 \theta}{r^2(r-2q) \sin^2 \theta - (r-2)\ell^2} \right] \] (69)

and

\[ \frac{d\theta}{dr} = \frac{r^2(r-2q)^2 \sin^2 \theta - \ell^2(r-2)^2(r-q)}{\ell^2(r-2)^2(r-2q)} \tan \theta. \] (70)

For \( q = 0 \) the above results reduce to the well known Schwarzschild result [37]. The best insight into the nature of the \( \ell = \text{constant} \) fluid configurations is obtained by examining the behavior of \( W(r, \theta) \) in the equatorial plane \( (\theta = \pi/2) \). There are two reality conditions on \( W(r, \theta = \pi/2) \):

\[ r > 2; \; r > 2q \] (71)

and

\[ r^2(r-2q) > r^2\ell^2. \] (72)

The first condition is just the statement that the fluid should remain outside the black hole horizons and we shall calculate its property in that non singular region of spacetime. While the second condition implies,

\[ \ell^2 \leq \ell_{ph}^2(r, q) \equiv \frac{r^2(r-2q)}{r^2}. \] (73)

This function \( \ell_{ph}^2(r, q) \) can be thought of as an effective potential of the photon geodesic motion, also note that \( \ell = \frac{U_\theta}{U_r} \) has a close correspondence with the impact parameter for photon geodesic motion [56]. Further, the condition to obtain local extrema of the potential \( W(r, \theta = \pi/2) \) is identical to the condition for vanishing of the pressure gradient \( (\partial U_r/\partial r = 0, \partial U_\theta/\partial \theta = 0) \). Since at the equatorial plane we have \( \partial U_t/\partial \theta = 0 \) independently along with the criteria \( \ell = \text{constant} \), and the following relation:

\[ \frac{\partial U_t}{\partial r} = \frac{r^2(r-2q)^2 - \ell^2(r-2)^2(r-q)}{[r^2(r-2q) - \ell^2(r-2)]^{3/2} [r(r-2)(r-2q)]^{1/2}}. \] (74)

From which we arrive at the particular expression for angular momentum density,

\[ \ell^2 = \ell_{ph}^2 (r, q) \equiv \frac{r^2(r-2q)^2}{(r-q)(r-2)^2}. \] (75)

The extrema for \( W(r, \theta = \pi/2) \) correspond to the spacetime points, where fluid moves along a circular geodesics, since \( \ell_{ph}^2 (r, q) \) relates to the distribution of the angular momentum density of circular geodesic orbits. Clearly this leads to,

\[ W_{ext}(r, \theta = \pi/2; q) = \ln E_r(r, y) \] (76)

where

\[ E_r(r, q) = \frac{(r-2)\sqrt{(r-q)}}{\sqrt{r(q-r-2q)}} \] (77)

is defined as the specific energy along these circular geodesics. Important properties of the potential \( W(r, \theta) \) are determined by its behavior at the equatorial plane, and especially by the properties of the functions \( \ell_{ph}^2 (r, q) \) and \( \ell_{ph}^2 (r, q) \). Discussion of these properties enables us to
classify the dilaton gravity spacetime according to the properties of equipotential surfaces of test particle fluid. For pure Schwarzschild spacetime \( q = 0 \) the analysis can be found in [35].

From the analytical form of \( \ell_{ph}^2(r, q) \) it is evident that the quantity diverges at the black hole horizon \( r = 2 \). However it shows no such divergence for the inner horizon given by \( r = 2q \). The local minimum of the function \( \ell_{ph}^2(r, q) \) corresponds to the radius of the photon circular orbit and has the following expression,

\[
r_{ph} = \frac{3 + q \pm \sqrt{(q + 3)^2 - 16q}}{2}
\]

(78)

with the impact parameter,

\[
\ell_{ph}^2 = \ell_{ph}^2(\text{min}) \equiv r^2(r - q).
\]

(79)

The function \( \ell_K^2(r, q) \), determining the Keplerian (geodesic) circular orbits, has a zero point at the so-called static radius given by,

\[
r_s = 2q
\]

(80)

which coincides with the inner horizon, also note that the function has divergent nature at \( r = 2 \) and as well as at \( r = q \). The function \( \ell_K^2(r, q) \) diverges at the black hole horizon i.e. \( \ell_K^2(r \to 2) \to +\infty \). Since,

\[
\frac{d\ell_K^2}{dr} = \frac{4r(r - 2q)}{(r - 2)^2} - \frac{r^2(r - 2q)^2(3r - 2 - 2q)}{(r - q)^2(r - 2)^3}
\]

(81)

the local extrema of \( \ell_K^2(r, q) \) are given by the condition \( q_{ms} = 1 \) determining the marginally stable circular orbits. For \( q < q_{ms} \), there exists an inner (outer) marginally stable circular geodesic at \( r_{ms(i)}(r_{ms(o)}) \).

Other special values of \( q \) correspond to the situation where the value of the minimum of \( \ell_{ph}^2(r, q) \) equals the maximum of \( \ell_K^2(r, q) \). We denote this value by \( q_e \) and it is a solution of the algebraic equation,

\[
q_e^5 - 22q_e^4 + 80q_e^3 - 64q_e^2 - 32 = 0.
\]

(82)

However this situation is less important as the above equation has no physical solution. Thus we will not have any inner cusp present in the system. Hence this has less attractive features, still interesting, which we will present now.

In this situation the behavior of \( \ell_{ph}^2 \) and \( \ell_K^2 \) is similar to that of the RN scenario, while we have only four available situations. As illustrated in figure 3 we have a minimum for \( \ell_K^2 \) having value \( \ell_{ms} = 3.5739 \). We also have a minimum in \( \ell_{ph}^2 \) which corresponds to the value \( \ell_{ph(c)} = 4.2786 \). Thus all together we have four cases in this dilaton gravity model from the behavior of the potential \( W(r, \theta = \pi/2, q) \). These cases are given for the following intervals of \( \ell \):

(i) \( \ell < \ell_{ms} \). We have only open equipotential surfaces in this angular momentum range (see figure 4(a1–2)).

(ii) \( \ell = \ell_{ms} \). An infinitesimally thin and unstable ring comes into existence at the marginally stable radius (see figure 4(b1–2)).

(iii) \( \ell_{ms} < \ell \leq \ell_{ph(c)} \). Closed equipotential surfaces come into existence, with one such surface having a cusp, allowing inflow to the black hole (see figure 4(c1–2)).
Closed equipotential surfaces exist, however cusps do not exist. Thus in the near horizon region equipotential surfaces cannot cross the equatorial plane (see figure 4(d1–2)).

Thus we have obtained the structure of equipotential surfaces in dilaton gravity. We have established how the inflow occurs in this spacetime along with necessary criteria on the angular momentum density; also the behavior that determines the angular momentum density which admits no minima. Hence this can be thought of as having less structure than the previous one while still having quite interesting features.

4.4. Einstein–Maxwell–Gauss–Bonnet gravity

In high energy physics, assuming spacetime to have more than four dimensions is a common practice. In these scenarios, the spacetime is assumed to be a four dimensional brane, embedded on a higher dimensional bulk. Though ordinary matter fields are confined to these branes, gravity can interact via the bulk as well. The EH action needs to be modified too and the most natural choice would be to include higher order terms in the action. A famous second order term is the GB term, under whose presence the modified action looks like,

$$S = \int dx^5 \sqrt{-g} \left[ R + \alpha \left( R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \right) + F_{ab} F^{ab} \right]$$

where $R$, $R_{ab}$ and $R_{abcd}$ are the Ricci scalar, Ricci tensor and Riemann tensor respectively. $F_{ab}$ is the electromagnetic tensor field and $\alpha$ is the GB coupling coefficient with dimension of length squared. In order to get equations of motion we need to vary the action with respect to the metric $g_{ab}$ and electromagnetic field $F_{ab}$ respectively. This leads to [65],

$$R_{ab} = \frac{1}{2} g_{ab} R - \alpha \left[ \frac{1}{2} g_{ab} \left( R_{pqrs} R^{pqrs} - 4 R_{mu} R^{mu} + R^2 \right) - 2 R_{ab} R^{2} + 4 R_{am} R_{b}^{m} \right]$$

$$+ 4 R_{mn} R_{abmn} - 2 R_{a}^{pq} R_{bq} - T_{ab}$$

Figure 3. The figures show the behavior of $f_\phi^2(r, q)$ and $f_\phi^2(r, q)$ with radial variable for different values of dilaton charge $q$. All of them are in units of $M$. Here we have all the figures in the range $q < 1$. The curve for $f_\phi^2(r, y)$ has only growing part showing only the existence of the inner ring. Thus no inner cusp exists in this spacetime.
Figure 4. Figures (a)–(d) show the variation of potential $W(r, \theta = \pi/2, q)$ with $\log r$ in the ‘1’ part and the contour plot of $W$ with $\log (r \sin \theta)$ and $\log (r \cos \theta)$ in the ‘2’ part. Each figure has a dilaton charge value of $q < 1$. Here only central rings exist, while no inner cusp comes into existence.
where $T_{ab}$ is the usual stress tensor for the electromagnetic field. The important thing to notice is that the field equation only contains second order derivatives of the metric, and no higher derivatives are present. This is expected since GB gravity is a subclass of Lovelock gravity, which does not contain higher derivative terms of the Riemann tensor.

Now it is possible to obtain a static spherically symmetric solution to this field equation having the form of equation (11). It turns out that those solutions are asymptotically de Sitter or anti-de Sitter. Thus this situation would be identical to that in section 4.2. This immediately tells us that only a correspondence between the parameters in this theory with those of the charged $F(R)$ scenario discussed in an earlier section is required.

To this end we must mention that the solution obtained above corresponds to a five dimensional spacetime, i.e. the line element takes the form: $ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2_5$, where $d\Omega^2_5 = d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2 d\theta_3^2)$. However, as we have been emphasizing, this solution can be interpreted from a brane world point of view, such that the visible brane is characterized by $\theta = $ constant hypersurface. Hence the solution reduces to four dimensions with $(t, r, \theta_1, \theta_2)$ as a set of coordinates. Thus on the visible brane the spherically symmetric solution can be treated on the same footing as those discussed earlier. Such spherically symmetric solutions are obtained in [65] and have the particular form with reference to equation (11) as

$$f(r) = K + \frac{r^2}{4a} \left[ 1 \pm \sqrt{1 + \frac{8a(m + 2a|K|)}{r^4} - \frac{8aq^2}{3r^6}} \right]$$

where $K$ determines the scalar curvature of the spacetime, to which we attribute a positive value. Then from Solar System tests and neutrino oscillation experiments [66, 67] it turns out that $\alpha^{-1}$ has stringent bounds. Also in all astrophysical scenarios the distance from black hole $r$ is quite large, thus for our study we can make a power series expansion of the terms inside the square root and arrive at the following form:

$$f(r) = 1 + \frac{r^2}{3a} + \frac{m + 2a}{r^2} - \frac{q^2}{3r^4}. \quad (87)$$

Then we readily identify this with equation (44) leading to the following mapping, $\alpha = -1/(2\gamma)$. The possibility that $\alpha$ may be negative has been discussed in detail by [68] and [69]. With this identification we can run all our machinery described in previous sections and obtain all the relevant physical quantities.

All the numerical values for cosmological parameters discussed earlier are directly mapped to those of $\alpha$. We can also conclude that it has identical structure to equipotential surfaces, with the existence of inner and outer cusps. Thus matter outflow can also occur in these topological dS or AdS black holes along with the possibility of inflow to it.

Some numerical estimates can be made for this model following previous discussions. The angular momentum densities for inner and outer marginally stable orbits are 3.04 and 4.9 respectively with $\alpha = 5 \times 10^3$. Also marginally bound and photon circular orbits have similar expressions.

Thus for various alternative gravity theories we have obtained the equipotential surfaces for rotating perfect fluid. For some of them we have observed dynamical accretion disks requiring inflow, outflow and cusps, while in some cases the structure was simpler and did not have such dynamical behavior. Thus we may conclude that black holes in different alternative theories affect accretion disk structure differently.
5. Conclusions

The effect of additional higher curvature correction terms in the EH action on the structure of equipotential surfaces for perfect fluid rotating around black holes in these modified gravity theories has been investigated. For that purpose we have performed the whole analysis for an arbitrary static spherically symmetric spacetime. Then we have applied the results obtained to different classes of alternative theories, with higher curvature terms. These theories include: charged $F(R)$ theory, dilaton induced gravity theory and finally the EMGB gravity. In the $F(R)$ theory and EMGB theory the black hole solutions asymptotically behave as dS or AdS. The equilibrium structure of perfect fluid orbiting around these black holes in the above mentioned alternative gravity theories leads to modifications of the equipotential contours.

Having provided the basic features let us now summarize the results:

- For the situation $\ell = 0$, we always have an open equipotential surface. The potential is determined by the relation $W = \ln \sqrt{f(r)}$. Thus different spherically symmetric solutions lead to different equipotential surfaces.
- Existence of an outer cusp does not facilitate accretion onto the black hole. We need to have an inner cusp for the equipotential surfaces as well, in order to have inflow to the black hole. Note that the outer cusp is also necessary for accretion to occur.
- Closed equipotential surfaces are possible if and only if the angular momentum lies in the range $\ell \in (\ell_{mi}, \ell_{mo})$. The quantities $\ell_{mi}$ and $\ell_{mo}$ represents the local minima and maxima of the angular momentum density respectively. For the charged $F(R)$ theory closed surfaces come into existence provided the cosmological parameter $\gamma$ satisfies $\gamma < \gamma_{mi} = 0.000692$, while for the EMGB gravity we obtain $\alpha = -782.47$. Note that these closed surfaces are necessary for the existence of the toroidal accretion disc.
- The accretion by the Paczyński mechanism becomes possible if angular momentum density lies in the range $\ell \in (\ell_{mi}, \ell_{mb})$, where $\ell_{mb}$ is the corresponding value of angular momentum density for marginally bound circular geodesics. In this scenario, the outflow from the accretion disc becomes possible through both the inner and outer cusps. The existence of the inner cusp leads to accretion flow directed towards the black hole.
- For marginally stable configurations, due to overfilling of marginally closed equipotential surfaces, efficient inflow and outflow occurs. However very near the horizon, the behavior may change significantly.
- Another physical situation of interest corresponds to the angular momentum range: $\ell \in (\ell_{mb}, \ell_{mo})$. In this situation the flow down to the black hole becomes non-existent since open self-crossing surfaces come into existence.
- Finally, for the situation where $\ell > \ell_{mo}$, the toroidal structure cannot exist. Equipotential surfaces exist, however they are always open. Also the surfaces become narrower as it approaches the static radius which may have a significant effect on collimation of jets from the black hole.

These effects were most prominent for accretion of perfect fluid to the black holes in $F(R)$ theory and the EMGB theory, both of which are asymptotically de Sitter for certain choices of parameter space. Richness in the structure of the black hole accretion disk for asymptotically de Sitter solutions in GR were pointed out earlier in [48]. Thus it is evident from this work that their generalization to alternative theories also comes up with varied structure of the accretion disk. There are also asymptotically flat solutions like the dilaton black hole, which have new features compared to the GR scenario. However in this spacetime the accretion disk does not possess all the structures which were present in the other two cases.
Thus we observe that in Einstein gravity as well as in alternative theories, spacetime with an analogue of repulsive cosmological constant (for instance, $y$ in the $F(R)$ model, $1/\alpha$ in the EMGB theory) shows a wide variety of phenomena as the accretion disk structure is considered. Though the structure of the accretion disk changes considerably due to the presence of higher curvature correction terms in the EH action it is very difficult to detect their signature astrophysically. This is mainly because the parameter space of these alternative theories for being of astrophysical interest is quite large compared to present day estimates of those parameters. For example, the black holes fuelling active galactic nuclei have typical masses in the range $\sim 10^8 M_\odot - 10^9 M_\odot$ which leads to an estimation of the cosmological parameter as $y < 10^{-10}$, which is very small and thus will not result in significant changes in the accretion disk structure.

However the primordial black holes have much larger value cosmological parameters. Thus their accretion structure could be best hoped to test these alternative theories. However this is beyond the scope of present day observations. Even though the departure of the equipotential contours in alternative theories from their general relativistic counterparts due to the presence of higher curvature correction terms is an important effect it is unlikely to be observed in the near future.

To summarize, in this work we have discussed various possible candidates for alternative theories and the accretion disk structure in them. We have also left some questions unanswered. Among these, the runaway instability criterion for toroidal accretion disks, i.e. stability with time, has not been addressed, and nor has the self-gravitation influence of the disk been considered. These topics we leave for future investigations.

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