Firefighting on Trees

Pierre Coupechoux\textsuperscript{a}, Marc Demange\textsuperscript{b}, David Ellison\textsuperscript{b}, Bertrand Jouve\textsuperscript{c}

\textsuperscript{a} LAAS-CNRS, Université de Toulouse, CNRS, Toulouse, France
\textsuperscript{b} RMIT University, School of Science, Melbourne, Australia
\textsuperscript{c} LISST, CNRS, Université de Toulouse, Toulouse, France

pierre.coupechoux@laas.fr, marc.demange@rmit.edu.au, david.ellison2@rmit.edu.au, bertrand.jouve@cnrs.fr

Abstract

In the Firefighter problem, introduced by Hartnell in 1995, a fire spreads through a graph while a player chooses which vertices to protect in order to contain it. In this paper, we focus on the case of trees and we consider as well the Fractional Firefighter game where the amount of protection allocated to a vertex lies between 0 and 1. While most of the work in this area deals with a constant amount of firefighters available at each turn, we consider three research questions which arise when including the sequence of firefighters as part of the instance. We first introduce the online version of both Firefighter and Fractional Firefighter, in which the number of firefighters available at each turn is revealed over time. We show that a greedy algorithm on finite trees is 1/2-competitive for both online versions, which generalises a result previously known for special cases of Firefighter. We also show that the optimal competitive ratio of online Firefighter ranges between 1/2 and the inverse of the golden ratio. Next, given two firefighter sequences, we discuss sufficient conditions for the existence of an infinite tree that separates them, in the sense that the fire can be contained with one sequence but not with the other. To this aim, we study a new purely numerical game called targeting game. Finally, we give sufficient conditions for the fire to be contained, expressed as the asymptotic comparison of the number of firefighters and the size of the tree levels.

1 Introduction and Definitions

1.1 Context

Since it was formally introduced by B. Hartnell in 1995 (\cite{18}, cited in \cite{13}) the firefighting problem - Firefighter - has raised the interest of many researchers. While this game started as a very simple model for fire spread and containment problems for wildfires, it can also represent any kind of threat able to spread sequentially in a network (diseases, viruses, rumours, flood . . .).

It is a deterministic discrete-time one-player game defined on a graph. In the beginning, a fire breaks out on a vertex and at each step, if not blocked, the fire spreads to all adjacent vertices. In order to contain the fire, the player is given a number $f_i$ of firefighters at each turn $i$ and can use them to protect vertices which are neither burning nor already protected. The game terminates when the fire cannot spread any further. In the case of finite graphs the aim is to save as many vertices as possible, while in the infinite case, the player wins if the game finishes, which means that the fire is contained.

This problem and its variants give rise to a generous literature; the reader is referred to \cite{13} for a broad presentation of the main research directions. A significant amount of theoretical work...
deals with its complexity and approximability behaviour in various classes of graphs [6, 8, 12, 15] and its parametrised complexity (e.g. [5, 8]). In particular, when one firefighter is available at each turn it is known to be polynomially solvable in some classes of graphs, which include graphs of maximum degree 3 if the fire breaks out on a vertex of degree at most 2 [12], interval graphs, permutation graphs and split graphs [15]. However it is known to be very hard, even in some restrictive cases. In particular, the case of trees was revealed to be very rich and a lot of research focuses on it. The problem, with the same number of firefighters at each turn, is NP-hard on finite trees of maximum degree 3 [12], as well as in even more restricted cases [6]; the reader is also referred to [9] for further complexity results. Regarding approximation results on trees, a greedy strategy was first shown to be a $\frac{1}{2}$-approximation algorithm [17] if a fixed number of firefighters is available at each turn. For a single firefighter, a $(1 - \frac{1}{e})$-approximation algorithm is proposed in [8] for the problem in trees. This ratio was improved in [19] for ternary trees and, very recently, a polynomial time approximation scheme was obtained in trees [1]; which essentially closes the question of approximating the firefighter problem in trees with one firefighter and motivates considering some generalisations. The problem is hard to approximate within $n^{1-\epsilon}$ on general graphs and with a single firefighter [3].

Most papers on this subject deal with a constant firefighter sequence. In fact, the problem was originally defined with one firefighter per turn. The case of infinite grids is of particular interest and has led to the model being extended by varying the available resources per turn. The change was motivated by the fact that a fire of any size on a 2-dimensional infinite grid can be contained with two firefighters per turn but not with one [14, 24]. In order to refine these results, M.-E. Messinger started considering periodic firefighter sequences [23] while more general sequences are considered in [11]. A related research direction investigates integer linear programming models for the problem, especially on trees [1, 16, 22]. This line of research makes very natural a relaxed version where the amount of firefighters available at each turn is any non-negative number and the amount allocated to vertices lies between 0 and 1. A vertex with a protection less than 1 is partially protected and its unprotected part can burn partially and transmit only its fraction of fire to the adjacent vertices. Thus, the $f_i$ may take any non-negative value. This defines a variant game called Fractional Firefighter which was introduced in [14].

1.2 Our contribution

The main thread of this paper is the focus on general firefighter sequences, which raises three specific research questions. We address these questions when a single fire spreads throughout a rooted tree.

First, we introduce an online version of both Firefighter and Fractional firefighter where the sequence of firefighters is revealed over time (online) while the graph (a tree in our case) is known from the start. To our knowledge, this is the first attempt at analysing online firefighter problems. Although our motivation is mainly theoretical, this paradigm is particularly natural in emergency management where one has to make quick decisions despite lack of information. Any progress in this direction tells us how lack of information impacts the quality of the solution. Note that a version of the game introduced in [7] also models a lack of information. In that version, rather than the firefighting resources, the missing information is where the fire will spread. Also, they propose randomised analyses to maximise the expected number of saved vertices while we use worst case analyses expressed in terms of competitive ratios.

A second question, the separating problem, deals specifically with infinite trees. Separating two given firefighter sequences means finding an infinite tree on which the fire can be contained with
one sequence but not the other.

The third question deals with criteria for the fire to be contained based on the asymptotic behaviours of the firefighter sequence and the size of the levels in the tree. Unlike the first two questions, it has already been investigated in other papers (e.g., [10, 20]) for Firefighter with firefighter sequences of the form \((\lambda^n)\).

The paper is organised as follows: in Section 2 we define formally Firefighter and Fractional Firefighter as well as their online versions. Section 3 deals with competitive analysis when the fire spreads in a finite tree and the firefighter sequence is revealed online. We first generalise an analysis of a greedy algorithm known only in special cases of Firefighter to Fractional Firefighter. For the offline case, it answers an open question proposed in [13, 16]. Then we propose improved competitive algorithms for online Firefighter with a small total number of firefighters while establishing that the greedy approach is optimal in the general case. The last two sections (Section 4 and Section 5) both deal with the infinite case. Section 4 deals with our second question. Considering the class of spherically symmetric trees where all vertices at the same level have the same degree, we express the separation problem as a purely numerical one-player game, which we call the targeting game. We propose two sufficient conditions for the existence of a winning strategy. Section 5 deals with our third question. We establish sufficient conditions for containing the fire expressed as asymptotic comparisons of the number of available firefighters and the size of the levels in the tree. In the online case, for a particular class of trees the level size of which grows linearly, we also give a sufficient condition to contain the fire.

### 1.3 Some notations

Given a tree \(T\) rooted in \(r\), \(V(T)\) and \(E(T)\) will denote the vertex set and the edge set of \(T\), respectively. Given two vertices \(v\) and \(v'\), \(v \prec v'\) denotes that \(v\) is an ancestor of \(v'\) (or \(v'\) is a descendant of \(v\)) and \(v \preceq v'\) denotes that either \(v = v'\) or \(v \prec v'\). For any vertex \(v\), let \(T[v]\) denote the sub-tree induced by \(v\) and its descendants. Let \(T_i\) denote the \(i\)-th level of \(T\) rooted in \(r\), where \(\{r\} = T_0\). For a finite tree \(T\) rooted in \(r\), the height \(h(T)\) is the maximum length of a path from \(r\) to a leaf. If \(i > h(T)\), we have \(T_i = \emptyset\). The weight \(w(v)\) of a vertex \(v\) is the number of vertices of \(T[v]\). When no ambiguity may occur, we will simply write \(w_v = w(v)\).

We denote by \(B(T)\) the tree obtained from \(T\) by contracting all vertices from levels 0 and 1 into a new root vertex \(r_B\): for all \(u_1 \in T_1\) and \(u_2 \in T_2\), every edge \(ru_1\) is contracted and every edge \(u_1u_2 \in E(T)\) gives rise to an edge \(ru_2 \in E(B(T))\). For \(k \leq h(T)\), \(B^k(T)\) will denote the \(k\)-th iteration of \(B\) applied to \(T\): all vertices from levels 0 to \(k\) are contracted into a single vertex denoted by \(r_B\) which becomes the new root.

Given a predicate \(P\), we denote by \(\mathbb{1}_P\) the associated characteristic function so that \(\mathbb{1}_{P(x)} = 1\) if \(P(x)\) is true and 0 otherwise.

### 2 Problems and preliminary results

#### 2.1 Firefighter and Fractional Firefighter

An instance of the Fractional Firefighter is defined by a triple \((G, r, (f_i))\), where \(G = (V(G), E(G))\) is a finite graph, \(r \in V(G)\) is the vertex where the fire breaks out and \((f_i)_{i \geq 1}\) is the non-negative firefighter sequence. \((f_i\) indicates the amount of protection that can be placed at turn \(i\)). Note that the game could be extended by allowing negative values for \(f_i\), however, we will
exclude pyromaniac firefighters from this paper, with one exception in Section 4.2 for the purpose of simplifying a proof. Let \( S_i \) denote the cumulative amount of firefighters received: \( S_i = \sum_{j=1}^{i} f_j \).

Turn \( i = 0 \) is the initial state where \( r \) is burning and all other vertices are unprotected, and \( i \geq 1 \) corresponds to the different rounds of the game. At each turn \( i \geq 1 \) and for every vertex \( v \), the player decides which amount \( p_i(v) \) of protection to add to \( v \), with \( 0 \leq p_i(v) \leq 1 \). Throughout the game, for every vertex \( v \) the part of \( v \) which is burning at turn \( i \geq 0 \) is denoted by \( b_i(v) \), with \( 0 \leq b_i(v) \leq 1 \), \( b_0(r) = 1 \) and \( b_0(v) = 0 \) for all \( v \neq r \). Similarly the cumulative protection received by vertex \( v \) is \( p_i^c(v) \) with \( p_i^c(v) = 0 \) for all \( v \). Both \((b_i(v))_{i \geq 0} \) and \((p_i^c(v))_{i \geq 0} \) are non-decreasing sequences with \( b_i(v) + p_i^c(v) \leq 1 \) for all \( i \) and \( v \). At each turn \( i \), the player’s choice of \( p_i(v) \) is subject to the constraints \( p_i(v) \geq 0 \), \( b_{i-1}(v) + p_{i-1}^c(v) + p_i(v) \leq 1 \) and \( \sum_{v \in V(G)} p_i(v) \leq f_i \). The new protection of \( v \) is \( p_i^c(v) = p_i^c(v) + p_i(v) \). The fire then spreads following the rule

\[
b_i(v) = \max \{ \max_{v' \in N(v)} b_{i-1}(v') - p_i^c(v'), b_{i-1}(v) \},
\]

where \( N(v) \) denotes the open neighbourhood of \( v \). The game finishes when the fire stops spreading (i.e. \( b_i(v) = b_{i-1}(v) \) for all \( v \)).

The standard Firefighter problem is similar to Fractional Firefighter, but with the additional constraint that the \( p_i(v) \) are all binary variables. It follows that \( p_i^c(v) \) and \( b_i(v) \) are also binary. In this case, we require that the firefighter sequence has integral values.

We will now show that both versions of the game always terminate on a finite graph \( G \). Let \( L_r(G) \) denote the maximum length of an induced path in \( G \) with extremity \( r \), we have:

**Proposition 1.** The maximum number of turns before a game of Firefighter or Fractional Firefighter on a finite graph \( G \) will terminate is \( L_r(G) \).

**Proof.** First, we show by induction on \( i \) that:

For all \( i \geq 1 \), for all vertex \( v \), if \( b_i(v) > b_{i-1}(v) \), then there is an induced path \( P_{i,v} = (u_0 = r, u_1, \ldots, u_i = v) \) of length \( i \) such that \( b_j(u_j) \) is non-increasing along the path.

For \( i = 1 \) and \( v \in V(G) \), if \( b_1(v) > b_0(v) \) then \( v \) is a neighbour of \( r \) and \( b_0(r) = 1 \geq b_1(v) \). The path \( P_{1,v} = (r, v) \) is of length 1.

Suppose the property holds at turn \( i \) and that \( b_{i+1}(v) > b_i(v) \) for some \( v \). Necessarily, \( v \) receives the additional amount of fire from a neighbour \( w \):

\[
b_{i+1}(v) = b_i(w) - p_{i+1}^c(v) > b_i(v).
\]

We necessarily have \( b_i(w) > b_{i-1}(w) \) since otherwise \( b_i(w) = b_{i-1}(w) \) and we would have the following contradiction:

\[
b_i(v) \geq b_{i-1}(w) - p_i^c(v) = b_i(w) - p_i^c(v) \geq b_i(w) - p_{i+1}^c(v) > b_i(v).
\]

Applying to \( w \) the induction hypothesis, there is an induced path \( P_{i,w} = (u_0 = r, u_1, \ldots, u_i = w) \) such that \( (b_j(u_j)) \) is non-increasing. Since, \( b_{i+1}(v) = b_i(w) - p_{i+1}^c(v) \), we have \( b_{i+1}(v) \leq b_i(w) \).

Also, for all \( j < i \), \( u_j \) and \( v \) are not adjacent since otherwise, we would have the following contradiction:

\[
b_i(v) \geq b_{j+1}(v) \geq b_j(u_j) - p_{j+1}^c(v) \geq b_i(w) - p_{i+1}^c(v) > b_i(v).
\]
It follows that the path $P_{i+1,v}$ obtained by adding the edge $uv$ to $P_{i,w}$ is an induced path which satisfies the required property. Thus, if $b_{i+1}(v) > b_i(v)$ for some $v$, we have $i + 1 \leq L_r(G)$. Consequently, the fire can no longer spread at turn $L_r(G) + 1$.

Conversely, let $P$ be an induced path with extremity $r$ of length $L_r(G)$. If $(f_i) = (|G| - L_r(G) - 1, 0, 0, 0, \cdots)$ and if the complement of $P$ is protected during the first turn, the game will terminate in exactly $L_r(G)$ turns.

Example 1. On a perfect binary tree $(B_n, r)$ of height $n$, given one firefighter per turn, the length of (Fractional) Firefighter is exactly $n = L_r(B_n)$, whatever the player’s strategy (this will be an immediate consequence of Proposition 5).

2.2 Simplification for Trees

In this paper, we focus on the case of trees. Given an instance $(T, r, (f_i))$, where $T$ is a tree, $T$ will be considered rooted in $r$. In order to remove trivial cases, we will exclude algorithms which place at turn $i$ more protection on a vertex $v$ than the part of $v$ that would burn if no protection were placed starting from turn $i$. If $T$ is finite, Proposition 1 implies that the game will end in at most $h(T)$ turns. We consider that it has exactly $h(T)$ turns, eventually with empty turns where no firefighters are allocated towards the end of the game.

Solutions on trees have a very specific structure. Indeed, it immediately follows from Equation (1) that, when playing on a tree, at each turn $i$, the amounts of fire $b_i(v)$ are non-increasing along any path from the root, which means that the fire will only spread outwards from the root. Also, for every vertex $v$ in $T_j$ for some $j$, the amount of fire $b_j(v)$ can no longer increase after turn $j$. Hence, no protection is placed in $T_j$ at turn $i > j$. Note also that for any solution which allocates a positive amount of protection at turn $i$ to a vertex $v \in T_k$, $k > i$, allocating the same amount of protection to the parent of $v$ instead strictly improves the performance. Indeed, if vertex $v$ can still burn, so can its parent. So we may consider only algorithms that play in $T_i$ at turn $i$. For an optimal algorithm, this property was emphasised in [17].

This holds for both Firefighter and Fractional Firefighter on trees. For such an algorithm, $f_i^+(v) = f_i(v)$ and the values of $p_i(v)$ and $b_i(v)$ will not change after turn $i$ for $v \in T_i$. Hence, the index $i$ may be dropped by denoting $p(v) = p_i(v)$ and $b(v) = b_i(v)$ for $v \in T_i$. A solution $p$ is then characterised by the values $p(v), v \in V(T)$. For any solution $p$, while $p(v)$ represents the amount of protection received directly, vertex $v$ also receives protection through its ancestors, the amount of which is denoted by $P_p(v) = \sum_{v' \prec v} p(v')$ (used in Section 3.1). Since we only consider algorithms that play in $T_i$ at turn $i$ and do not place extraneous protection, for any vertex $v$, $p(v) + P_p(v) \leq 1$. Also, for any vertex $v \in T_i$, we have $b_{i-1}(v) = 0$ and by summing Equation (i) from 1 to $i$, we deduce that $p(v) + P_p(v) + b(v) = 1$.

Any solution $p$ for Firefighter or Fractional Firefighter will satisfy the constraints:

$$\begin{align*}
[C] \begin{cases} 
\sum_{v \in T_i} p(v) \leq f_i & (i) \\
\forall v, p(v) + P_p(v) \leq 1 & (ii)
\end{cases}
\end{align*}$$

In [22], a specific boolean linear model has been proposed for solving Firefighter on a tree $T$ involving these constraints. Solving Fractional Firefighter on $T$ corresponds to solving the relaxed version of this linear program.
2.3 Online version

Online optimisation \[4\] is a generalisation of approximation theory which represents situations where the information is revealed over time and one needs to make irrevocable decisions. Following the definitions of \[2\], we now introduce online versions of Firefighter and Fractional Firefighter on trees. The graph and starting point of the fire are known from the start by the player, but the firefighter sequence \((f_i)_{i \geq 1}\) is revealed over time, i.e., at step \(i\) the player does not know any request \(f_j\) with \(j > i\). This set-up can be seen as a game between the online player (or algorithm) and an oblivious adversary. At each turn \(i\), the oblivious adversary reveals \(f_i\) and then the player chooses where to allocate this resource. We refer to the usual case, where \((f_i)_{i \geq 1}\) is known in advance by the player, as offline.

Let us consider an online algorithm \(OA\) for one of the two problems and let us play the game on a finite tree \(T\) until the fire stops spreading. The value \(\lambda_{OA}\) achieved by the algorithm, defined as the amount of saved vertices, is measured against the best value performed by an algorithm which knows in advance the sequence \((f_i)\). In the present case, it is simply the optimal value of the offline instance, referred to as the offline optimal value, denoted by \(\beta\) when considering the online Firefighter (I stands for “Integral”) and \(\beta_F\) for the online Fractional Firefighter. We will call Bob such an algorithm, able to see the future and guaranteeing the value \(\beta\) or \(\beta_F\) for online Firefighter and Fractional Firefighter.

Algorithm \(OA\) is said to be \(\gamma\)-competitive, \(\gamma \in [0, 1]\), for the online Firefighter (resp. Fractional Firefighter) if for every instance, \(\frac{\lambda_{OA}}{\lambda_{Bob}} \geq \gamma\) (resp. \(\frac{\lambda_{OA}}{\lambda_{Bob}} \geq \gamma\)); \(\gamma\) is also called a competitive ratio guaranteed by \(OA\). An online algorithm will be called optimal if it guarantees the best possible competitive ratio. It should be noted that to calculate the competitive ratio, it is necessary to evaluate the worst case of all the adversary’s possible choices. This is actually like considering a malicious adversary instead of an ignorant one.

The arguments given in Section 2.2 to justify considering only algorithms that play in \(T_i\) at turn \(i\), are still valid for online algorithms; thus we will only consider such online algorithms. Let us start with a reduction:

**Proposition 2.** We can reduce online (Fractional) Firefighter on trees to instances where \(f_1 > 0\).

**Proof.** If \(f_i = 0\) for all \(i\) such that \(1 \leq i \leq k\), then the instance \((T, r, (f_i))\) is equivalent to the instance \((B^k(T), r_{B^k}, (f_{i+k})).\)

In the infinite case we do not define competitive ratios, but only ask whether the fire can be contained by an online algorithm. Sections 3.1 and 3.2 deal with the finite case while Section 5.2 deals with a class of infinite trees.

3 Online Firefighting on Finite Trees

3.1 Competitive analysis of a Greedy algorithm

Greedy algorithms are usually very good candidates for online algorithms, sometimes the only known approach. Mainly two different greedy algorithms have been considered in the literature for Firefighter on a tree \[13\] and they are both possible online strategies in our set-up. The degree greedy strategy prioritises saving vertices of large degree; it has been shown in \[6\] that it cannot
guarantee any approximation ratio on trees, even for a constant firefighter sequence. A second greedy algorithm, introduced in [17] for an integral sequence \((f_i)\), maximises at each turn the total weight of the newly protected vertices. We generalise it to any firefighter sequence for both the integral and the fractional problems. Let \(\text{GR}\) denote the greedy algorithm that selects at each turn \(i\) an optimal solution of the linear program \(P_i\) with variables \(x(v), v \in T_i\) and constraints \([C]\):

\[
P_i : \begin{cases}
\max \sum_{v \in T_i} x(v)w(v) \\
\sum_{v \in T_i} x(v) \leq f_i \\
\forall v \in T_i, x(v) + P_x(v) \leq 1 
\end{cases}
\]

An optimal solution of \(P_i\) is obtained by ordering vertices \(\{v_1, \ldots, v_{|T_i|}\}\) of level \(i\) by non-increasing weight and taking them one by one in this order and greedily assigning to vertex \(v_j\) the value \(x(v_j) = \min(f_i - \sum_{k < j} x(v_k), 1 - P_x(v_j))\). Note that \(\text{GR}\) is valid for both \textsc{Firefighter} and \textsc{Fractional Firefighter}.

It was shown in [17] that the greedy algorithm on trees gives a \(\frac{1}{2}\)-approximation of the restriction of \textsc{Firefighter} when a single firefighter is available at each turn. They claim that this approximation ratio remains valid for a fixed number \(D \in \mathbb{N}\) of firefighters at each turn. We extend this result to any firefighter sequence \((f_i)_{i \geq 1}\), integral or not. Since \(\text{GR}\) is an online algorithm, the performance can also be seen as a competitive ratio for the online version.

**Theorem 1.** The greedy algorithm \(\text{GR}\) is \(\frac{1}{2}\)-competitive for both online \textsc{Firefighter} and \textsc{Fractional Firefighter} on finite trees.

**Proof.** Let us first consider the fractional case with an online instance \((T, r, (f_i))\) of \textsc{Fractional Firefighter} on a tree.

Let \(x(v)\) and \(y(v)\) be the amounts of firefighters placed on vertex \(v\) by \(\text{GR}\) and \(\text{Bob}\), respectively. We have \(\lambda_{\text{GR}} = \sum_v x(v)w(v)\) and \(\beta_P = \sum_v y(v)w(v)\).

Recall that \(P_x(v) = \sum_{v' < v} x(v')\) and \(P_y(v) = \sum_{v' < v} y(v')\). We split \(y(v)\) into two non-negative quantities, \(y(v) = g(v) + h(v)\), where \(g(v)\) is the part of \(y(v)\) already protected by \(\text{GR}\) through the ancestors of \(v\), while \(h(v)\) is the part of \(y(v)\) which, when added on top of \(P_y(v)\), exceeds \(P_x(v)\). So, if \(P_y(v) < P_x(v) < y + P_y(v)\), we have \(h(v) = y(v) - P_x(v)\). The general formula is:

\[
g(v) = \min\{g(v), \max\{0, P_x(v) - P_y(v)\}\} \text{ and } h(v) = \max\{0, y(v) - \min\{0, P_y(v) - P_x(v)\}\}.
\]

We now claim that \(\forall v' \in T, \sum_{v \subseteq v'} g(v) \leq P_x(v')\) and prove it by induction. Since \(g(r) = 0\), it holds for the root \(r\). Assuming that the inequality holds for a vertex \(v'\), let \(v''\) be a child of \(v'\). If \(P_x(v'') - P_y(v'') \geq 0\), then we directly have:

\[
\sum_{v \subseteq v''} g(v) = \sum_{v \subseteq v''} g(v) + g(v'') \leq \sum_{v \subseteq v''} y(v) + (P_x(v'') - P_y(v'')) = P_x(v'').
\]

Else, \(g(v'') = 0\) and using \(\sum_{v \subseteq v'} g(v) \leq P_x(v')\) and \(P_x(v'') \geq P_x(v')\), the inequality holds for \(v''\); which completes the proof of the claim. Thus:

\[
\sum_{v' \in |T|} \sum_{v \subseteq v'} g(v) \leq \sum_{v' \in |T|} P_x(v') = \sum_{v' \in |T|} \sum_{v \subseteq v'} x(v) \leq \sum_{v' \in |T|} \sum_{v \subseteq v'} x(v).
\]

Since \(w(v) = \sum_{v \subseteq v'} 1\), by changing the order of summation on both sides, we obtain:

\[
\sum_v g(v)w(v) \leq \sum_v x(v)w(v) = \lambda_{\text{GR}}.
\]
Let us now consider the coefficients \( h(v) \). We claim that the coefficients \( h(v) \) with \( v \in T \), satisfy the constraints (i) and (ii) of \( P_i \); indeed for (i), we have \( h(v) \leq y(v) \) and \( y \) satisfies constraint (i). For (ii) note that \( h(v) + P_x(v) = \max\{P_x(v), y(v) + \min\{P_x(v), P_y(v)\}\} \leq \max\{P_x(v), y(v) + P_y(v)\} \leq 1.

Hence, \( \forall v, \sum_{v \in T_t} h(v)w(v) \leq \sum_{v \in T} x(v)w(v) \) and therefore:

\[
\sum_{v \in T} h(v)w(v) \leq \sum_{v \in T} x(v)w(v) = \lambda_{\text{GR}}. \tag{3}
\]

Finally, since \( g(v) + h(v) = y(v) \), we conclude from Equations (2) and (3) that \( \beta_F \leq 2\lambda_{\text{GR}}. \) Hence the Greedy algorithm is \( \frac{1}{2} \)-competitive for the online Fractional Firefighter problem. Since the greedy algorithm gives an integral solution if \((f_i)\) has integral values and since \( \beta_F \geq \beta_I \), it is also \( \frac{1}{2} \)-competitive for the Firefighter problem. This concludes the proof of Theorem 1. \( \square \)

Conjecture 2.3 in [16] (which is also Conjecture 3.5 in [13]) claims that there is a constant \( \rho \) such that the optimal value of Fractional Firefighter on trees is at most \( \rho \) times the optimal value of Firefighter. It was supported by extensive experimental tests [16], but finding such a constant and proving the ratio is one of the open problems proposed in [13] (Problem 7). Theorem 1 can be expressed as \( \lambda_{\text{GR}} \leq \beta_I \leq \beta_F \leq 2\lambda_{\text{GR}} \), which shows that \( \rho = 2 \) is such a constant:

**Corollary 1.** In Fractional Firefighter on trees, the amount of vertices saved is at most twice the maximum number of vertices saved in Firefighter.

### 3.2 Improved Competitive Algorithm for Firefighter

In this section, we investigate possible improvements for online strategies for Firefighter on finite trees. Let \( \varphi = \frac{\sqrt{5} + 1}{2} \) denote the golden ratio, satisfying \( \varphi^2 = \varphi + 1 \).

For any integer \( k \geq 2 \), we denote by \( \alpha_{I,k} \) the best possible competitive ratio for online Firefighter on finite trees if at most \( k \) firefighters are available in the entire game. We have:

\[
\alpha_{I,k} = \inf_{T \in \mathcal{T}} \max_{\mathcal{OA} \in \mathcal{A}_k} \min_{(f_i) \in \mathbb{N}^* \sum_i f_i \leq k} \frac{\lambda_{\text{OA}}}{\beta_I},
\]

where \( \mathcal{T} \) denotes the set of finite rooted trees, \( \mathcal{A}_k \) the set of online algorithms for Firefighter on finite trees and \( \mathbb{N}^* \) denotes the set of positive integers.

Note that in the definition of \( \alpha_{I,k} \), \( \lambda_{\text{OA}} \) and \( \beta_I \) depend on \( T \). Also, the maximum and the minimum are well defined since on a finite tree \( T \), the set of possible ratios is finite. An online algorithm, choosing for any fixed \( T \) a strategy which achieves this maximum, will be \( \alpha_{I,k} \)-competitive for instances with at most \( k \) firefighters. Such an algorithm is optimal for these instances.

The sequence \( (\alpha_{I,k}) \) is non-increasing. We define \( \alpha_I = \lim_{k \to \infty} \alpha_{I,k} \); again, the index \( I \) stands for Integral and refers to the Firefighter problem.

**Remark 1.** The limit \( \alpha_I \) is the greatest competitive ratio that can be reached on any tree. Indeed, given a finite tree \( T \), it suffices to consider the instances with at most \( |V(T)| \) firefighters.

In this section, we give an online algorithm for instances of Firefighter on a finite tree that is optimal (i.e., \( \alpha_{I,2} \)-competitive) if at most two firefighters are presented. Based on Proposition 2
Lemma 1. Let \( a \) and \( b \) be two vertices of maximum weights in \( T_1 \). If \( \sum_i f_i \leq 2 \), there is an optimal offline algorithm for Firefighter which places the first firefighter on either \( a \) or \( b \).

Proof. If the first firefighter is placed on \( v \in T_1 \setminus \{a, b\} \) by an optimal offline algorithm, since at most two firefighters are available, \( \exists u \in \{a, b\}, T[u] \) burns completely. Hence, replacing \( v \) by \( u \) when assigning the first firefighter would produce another optimal solution (necessarily \( w_a = w_b \)).

We suppose Bob has this property. However, even if \( w_a > w_b \), he will not necessarily choose \( a \); as illustrated by the graph \( W_{1, 10, 20} \) (Figure 2) where if the firefighter sequence is \( (1, 0, 1, 0, 0, 0, \ldots) \), then Bob's needs to protect \( x \) during the first turn. Note also that, when the root is of degree at least 3, the second firefighter is not necessarily in \( V(T[a]) \cup V(T[b]) \).

We now consider Algorithm 1 and assume that the adversary will reveal at most two firefighters. The algorithm works on an updated version \( \tilde{T} \) of the tree: if one vertex is protected, then the corresponding sub-tree is removed and all the burnt vertices are contracted into the new root \( \tilde{r} \) so that the algorithm always considers vertices of level 1 in \( \tilde{T} \). Before starting the online process, the algorithm computes the weights of all vertices. The weights of the unburnt vertices do not change when updating \( \tilde{T} \). The value of \( h(T) \), required in line 9, can be computed during the initial calculation of weights and easily updated with \( \tilde{T} \). For the sake of clarity, we do not detail all updates in the algorithm.

In this section only, for any vertex \( v \in T_i \) and any \( i \leq j \leq i + h(T[v]) \), we denote by \( v_j \) a vertex of maximum weight \( w_{v_j} \) in \( T_j \cap V(T[v]) \), i.e. among the descendants of \( v \) which are in level \( j \) (or \( v \) itself if \( i = j \)). We also define \( \tilde{w}_{v_j} \) for all \( j \) via:

\[
\tilde{w}_{v_j} = w_{v_j} \text{ if } j \in [i; i + h(T[v])] \text{ and } 0 \text{ otherwise.}
\]

Theorem 2. Algorithm 1 is a \( \frac{1}{2} \)-competitive online algorithm for online Firefighter with at most two firefighters available. It is optimal for this case.

Proof. While Algorithm 1 runs feasibly on any instance, we limit the analysis to the case where at most two firefighters are available. If the adversary does not present any firefighter before the turn \( h(T) \), both Algorithm 1 and Bob cannot save any vertex and, by convention, the competitive ratio is 1.

Let us suppose that at least one firefighter is presented at some turn \( k \leq h(T) \); the tree still has at least one unburnt vertex. During the first \( (k - 1) \) turns, the instance is updated to \((B^{k-1}(T), r_{B^{k-1-1}}, (f_{i+k-1}))\). In the updated instance, at least one firefighter is presented during the first turn and the root has at least one child. Proposition 2 ensures that it is equivalent to the original instance.

If the root \( r_{B^{k-1}} \) has only one child, line 8 gives \( a = b \) and Algorithm 1 selects \( a \) at line 10. In the updated instance, all vertices are saved; so the competitive ratio is equal to 1.
Algorithm 1

**Require:** A finite tree $T$ with root $r$ - An online adversary.

1. $(T, \check{r}) \leftarrow (T, r)$; Compute $w_v, \forall v \in V(T)$
2. \texttt{FirstFirefighter} \leftarrow TRUE;
3. \{Start of the online process\}
4. At each turn, after the fire spreads, $\check{T}$ is updated - burnt vertices are contracted to $\check{r}$;
5. If several firefighters are presented at the same time, we consider them one by one in the following lines;
6. \textbf{if} a new Firefighter is presented \textbf{and} $\check{r}$ has at least one child \textbf{then}
   7. \textbf{if} \texttt{FirstFirefighter} \textbf{then}
      8. Let $a$ and $b$ denote two children of $\check{r}$ with maximum weight $w_a, w_b$ and $w_a \geq w_b$ ($a = b$ if $\check{r}$ has only one child);
      9. if $\min_{2 \leq i \leq 1 + h(\check{T})} \frac{w_a + \check{w}_a}{w_b + \check{w}_b} \geq \frac{1}{\varphi}$ then
         10. Place the first firefighter on $a$;
      11. else
         12. Place the first firefighter on $b$;
      13. \texttt{FirstFirefighter} \leftarrow FALSE;
      14. else
         15. Place the firefighter on a child $v$ of $\check{r}$ of maximum weight

Else, we have $a \neq b$ with $w_a \geq w_b$ (line $\spadesuit$). If the adversary presents a single firefighter for the whole game, then Algorithm $\clubsuit$ protects either $a$ or $b$. Meanwhile, Bob will protect $a$, saving $w_a$ vertices. If $w_a \geq \varphi w_b$, then we have:

\[
\forall i, 2 \leq i \leq 1 + h(T), \frac{w_a + \check{w}_a}{w_b + \check{w}_b} \geq \frac{w_a}{w_b + \check{w}_a} \geq \frac{\varphi w_b}{w_b + \check{w}_a} = \frac{1}{\varphi}. \tag{4}
\]

So Algorithm $\clubsuit$ protects $a$ (line $\spadesuit$), guaranteeing a competitive ratio of 1. Otherwise, if $w_b > \frac{1}{\varphi} w_a$, even placing the firefighter on $b$ guarantees a ratio of at least $\frac{1}{\varphi}$.

Suppose now that the adversary presents two firefighters. We consider two cases.

Case (i): If Algorithm $\clubsuit$ places the first firefighter on $a$ at line $\spadesuit$ and if the adversary presents the second firefighter at turn $i \geq k$, then the algorithm will save $w_a + \check{w}_x$, for some $x \in T_k \setminus \{a\}$ such that $\check{w}_x = \max_{u \in T_k \setminus \{a\}} \check{w}_u$. For the same instance, Bob will save $w_a + \check{w}_y$, for some $v \in \{a, b\}$ and $y \in T_k \setminus \{v\}$. If the two values are different (the optimal one is strictly better), then necessarily $v = b$ and $y = a$. In this case the criterion of line $\spadesuit$ ensures that the related competitive ratio is at least $\frac{1}{\varphi}$.

Case (ii): Suppose now that Algorithm $\clubsuit$ places the first firefighter on $b$ at line $\spadesuit$ and say the adversary presents the second firefighter at turn $j \geq k$. Lines $\spadesuit$ and $\spadesuit$ ensure that:

\[
\exists i, 2 \leq i \leq 1 + h(T), \frac{w_a + \check{w}_a}{w_b + \check{w}_b} < \frac{1}{\varphi}. \tag{5}
\]

Hence, we have $w_a < \varphi w_b$, since in the opposite case, Equation $\clubsuit$ would hold. Algorithm $\clubsuit$ now saves $w_b + \check{w}_x$, for $x \in T_k \setminus \{b\}$ such that $\check{w}_x = \max_{u \in T_k \setminus \{b\}} \check{w}_u$. Meanwhile, Bob selects $v \in \{a, b\}$ and, if it exists, $y_j$ for some $y \in T_k \setminus \{v\}$, for a total of $w_v + \check{w}_y$ vertices saved. If $y \neq b$, then
$\bar{w}_y \leq \bar{w}_x$, by definition of $x$, and thus:

$$\frac{w_b + \bar{w}_x}{w_a + \bar{w}_y} \geq \frac{w_b + \bar{w}_x}{w_a + \bar{w}_y} \geq \frac{w_b}{w_a} > \frac{1}{\varphi}. \quad (6)$$

Finally, if $y = b$, then $v = a$ and the competitive ratio to evaluate is $\frac{w_b + \bar{w}_x}{w_a + \bar{w}_y}$. We claim that the following holds:

$$\frac{w_a + \bar{w}_b}{w_b + \bar{w}_a} \times \frac{w_b + \bar{w}_x}{w_a + \bar{w}_y} \geq \frac{w_a + \bar{w}_b}{w_b + \bar{w}_a} \geq \frac{w_a}{w_b + \bar{w}_a} \geq \frac{1}{\varphi^2}. \quad (7)$$

If $i \geq j$, then $\bar{w}_{ai} \leq \bar{w}_{aj}$ and since $a \neq b$, $\bar{w}_{aj} \leq \bar{w}_{xj}$. Hence: $\frac{w_b + \bar{w}_x}{w_b + \bar{w}_a} \geq \frac{w_b + \bar{w}_x}{w_b + \bar{w}_a} \geq \frac{1}{\varphi^2}$. Together with Equation (6), this concludes case (ii) and shows that Algorithm 1 is $\frac{2}{\varphi}$-competitive.

Even though complexity analyses are not usually proposed for online algorithms, it is worth noting that line 9 only requires the weights of vertices in $V(T[a]) \cup V(T[b])$ and the maximum weight per level in $T[a]$ and $T[b]$. Hence, Algorithm 1 requires $O(|V(T[a])| + |V(T[b])|)$ to choose the position of the first firefighter and $O(|V(T)|)$ altogether.

We conclude this section with a hardness result justifying that the greedy algorithm GR is optimal and that Algorithm 1 is optimal if at most two firefighters are available. These hardness results will all be derived from the graphs $W_{k,l,m}$ (Figure 1).

![Figure 1: Graph $W_{k,l,m}$](image_url)
Proposition 3. For all \( k \geq 2 \), \( \frac{1}{2} \leq \alpha_{I,k} \leq \frac{1}{\phi} \), more precisely:

(i) \( \alpha_{I} = \frac{1}{2} \), which means that the greedy algorithm is optimal for Firefighter in finite trees;

(ii) \( \alpha_{I,2} = \frac{1}{\phi} \), which means that Algorithm \( \text{GR} \) is optimal if at most two firefighters are available;

(iii) \( \alpha_{I,4} < \frac{1}{\phi} \).

Proof. Theorem 1 shows that \( \alpha_{I} \geq \frac{1}{2} \). Given integers \( l, m, k \) such that \( k|m - 1 \), we define the graph \( W_{k,l,m} \) as shown in Figure 1. We will assume that \( m > k^{2} \).

(i) Let us consider an online algorithm for \( W_{k,l,m} \). As established in Section 2, we can assume that the online algorithm plays in \( T_{1} \) at turn \( i \). If \( f_{1} = 1 \), the algorithm will protect either \( x \) or \( y \). If \( x \) is selected and the firefighter sequence is \( (1, 0, 1, 0, 0, \ldots) \), our online algorithm protects the branch of \( x \) and one of the \( k \) chains, while the optimal offline algorithm protects \( y \) and the star.

Its performance is then \( k + m - 1 \). If, however, \( y \) is protected instead during the first turn and if the firefighter sequence is \( (1, 0, 1, 1, 1, \ldots) \), the online algorithm protects the branch of \( y \) and one vertex of the star whilst the optimal algorithm protects the branch of \( x \) as well as the \( k \) chains, minus \( \frac{k(k+1)}{2} \) vertices. If \( l = m - 1 = k^{4} \), for large values of \( k \), the online algorithm which protects \( x \) is more performant and its competitive ratio is \( \frac{1+\phi}{\phi} \). Having \( k \rightarrow +\infty \) shows that \( \alpha \leq \frac{1}{\phi} \). Since the greedy algorithm \( \text{GR} \) guarantees \( \alpha_{I} \geq \frac{1}{2} \), we have \( \alpha_{I} = \frac{1}{\phi} \).

(ii) Consider the graphs \( W_{1,l,|x|} \). If the online algorithm protects \( x \), the adversary selects the sequence \( (1, 0, 0, 0, \ldots) \), whereas if the online algorithm protects \( y \), \( (1, 0, 1, 0, 0, 0, \ldots) \) is selected. In both cases, the performance tends to \( \frac{1}{\phi} \) when \( l \rightarrow +\infty \).

(iii) If at most 4 firefighters are available, the graph \( W_{4,901,1001} \) gives an example where \( \frac{1}{\phi} \) cannot be reached. Indeed, if \( f_{1} = 1 \) and the online algorithm protects \( x \), then the adversary will select the sequence \( (1, 1, 0, 0, 0, \ldots) \), as in the proof of (i), for a performance of \( \frac{1002}{1645} \). If the online algorithm protects \( y \), since firefighters are limited to 4, the adversary will select \( (1, 0, 1, 1, 0, 0, 0, \ldots) \), for a performance of \( \frac{1002}{1645} \). This second choice is slightly better; however, \( \frac{1002}{1645} < \frac{1}{\phi} \).  

We have also proved that there is a \( \frac{1}{\phi} \)-competitive algorithm if three firefighters are presented (i.e., \( \alpha_{I,3} = \frac{1}{\phi} \)). This algorithm is similar to Algorithm 1 in that it places the first firefighter on one of the three largest branches and greedily places each of the other two on the largest branch available at the time they are presented. However, our proof involves a much more technical case-by-case analysis, and will not be detailed here.

4 Separating Firefighter Sequences

4.1 Definitions

We now consider the fractional firefighter problem on infinite graphs. We say that a sequence of firefighters \( (f_{i}) \) is weaker than \( (f'_{i}) \) (or \( (f'_{i}) \) is stronger than \( (f_{i}) \)) if \( \forall k, S_{k} \leq S'_{k} = \sum_{i=1}^{k} f'_{i} \), and we write \( (f_{i}) \leq (f'_{i}) \). If we also have \( \exists k : S_{k} < S'_{k} \), \( (f_{i}) \) is said to be strictly weaker than \( (f'_{i}) \) and we write \( (f_{i}) < (f'_{i}) \).

Lemma 2. If the fire can be contained in the instance \((G, r, (f_{i}))\) and if \( (f_{i}) \leq (f'_{i}) \), then the fire can also be contained in \((G, r, (f'_{i}))\) by an online algorithm that knows \( (f_{i}) \) in advance.
Proof. Given a winning strategy in the instance \((G, r, (f_i))\), if \((f'_i)\) firefighters are available, we contain the fire by protecting the same vertices, possibly earlier than in the initial strategy.

However, if \((f_i) \prec (f'_i)\), for Fractional Firefighter, it is not always the case that there is an infinite graph \(G\) such that the fire can be contained in \((G, r, (f'_i))\) but not in \((G, r, (f_i))\) (see Example 2). We call such a \(G\) a separating graph for \((f_i)\) and \((f'_i)\), and we say that \(G\) separates \((f_i)\) and \((f'_i)\) in \(N\) turns if the fire can be contained in \(N\) turns for \((f'_i)\) but not for \((f_i)\). In this section, we give sufficient conditions for the existence of a separating graph.

Example 2. Let \(f_1 = 1, f'_1 = 1.5\) and \(\forall i \geq 2, f_i = f'_i = 0\). Although \((f_i) \prec (f'_i)\), no graph separates those two sequences.

Note that for Firefighter, the problem is trivial, as shown in Corollary 2.

4.2 Spherically Symmetric Trees

Given a sequence \((a_i) \in (\mathbb{N}^*)^{\mathbb{N}^+}\), the spherically symmetric tree \(T((a_i))\) is the tree rooted in \(r\) where every vertex of level \(i - 1\) has \(a_i\) children [21]. Note that if \(T = T((a_i))\), we have \(|T_i| = \prod_{j=1}^{i} a_j\).

The total amount of fire at level \(i\) is the sum of the amounts of fire on all vertices of level \(i\).

Proposition 4. In the instance \((T((a_i)), r, (f_i))\), the total amount of fire that spreads to level \(i\) is \(\max\{0, F_i\}\), where \(F_0 = 1\) and \(F_i = a_iF_{i-1} - f_i\) for all \(i\).

Proof. At turn \(i\), the player only protects vertices on level \(i\). If no protection is placed, the total amount of fire is multiplied by \(a_i\). Hence, if \(f_i \geq a_iF_{i-1}\), the fire is contained; else, the total amount of fire spreading to level \(i\) is \(a_iF_{i-1} - f_i\), regardless of how the protection is distributed among the vertices of level \(i\).

Corollary 2. Let \((f_i)\) and \((f'_i)\) be two distinct integral valued sequences. There is a spherically symmetrical tree which separates \((f_i)\) and \((f'_i)\).

Proof. Let \(k\) be the first rank where \(f_k \neq f'_k\). We may assume that \(f_k < f'_k\). It follows from Proposition 4 that in the instance \((T((f_i+1)), r, (f_i))\), the amount of fire that spreads to each level is equal to 1. Yet, in \((T((f_i+1)), r, (f'_i))\), the fire is contained at turn \(k\).

For the purpose of the following technical lemma, we define a new firefighter sequence, which may include a negative term. Given a firefighter sequence \((f_i)\) and two non-zero integers \(k\) and \(\epsilon\), we define the firefighter sequence \((f_i^{(k, \epsilon)})\) via:

\[
\begin{align*}
f_i^{(k, \epsilon)} &= \begin{cases} 
    f_k + \epsilon & \text{if } i = k \\
    f_{k+1} - \epsilon & \text{if } i = k + 1 \\
    f_i & \text{otherwise}
\end{cases}
\end{align*}
\]

Note that there is a possibility that \(f_{k+1}^{(k, \epsilon)}\) might be negative; however, this does not impact the reasoning. We also define the sequence \((F_i^{(k, \epsilon)})\) via \(F_0^{(k, \epsilon)} = 1\) and \(F_i^{(k, \epsilon)} = a_iF_{i-1}^{(k, \epsilon)} - f_i^{(k, \epsilon)}\). It follows from Proposition 4 that the amount of fire which spreads to level \(i\) in the instance \((T((a_i)), r, (F_i^{(k, \epsilon)}))\) is \(\max\{0, F_i^{(k, \epsilon)}\}\).
Lemma 3. The spherically symmetric tree $T = T((a_i))$ separates $(f_i)$ and $(f_i^{(k,\epsilon)})$ if and only if there is a rank $N$ such that: $A \leq \sum_{i=k+2}^{N} \frac{f_i}{\prod_{j=k+2}^{i} a_j} < B$, where $A = F^{(k,\epsilon)}_{k+1}$ and $B = F_{k+1}$.

Proof. It follows from Proposition 4 that $F_n = \prod_{j=1}^{n} a_j - \sum_{i=1}^{n} f_i \prod_{j=i+1}^{n} a_j$. So, $F_n = |T_n|(1 - \sum_{i=1}^{n} \frac{f_i}{|T_i|})$. The condition for $T((a_i))$ to separate $(f_i)$ and $(f^{(k,\epsilon)})$ can be stated as follows: there is a rank $N$ such that $F^N_N \leq 0 < F_N$. Hence, there is an $N$ such that

$$|T_N|(1 - \sum_{i=1}^{N} \frac{f_i^{(k,\epsilon)}}{|T_i|}) \leq 0 < |T_N|(1 - \sum_{i=1}^{N} \frac{f_i}{|T_i|}).$$

Therefore

$$1 - \sum_{i=1}^{k+1} \frac{f_i^{(k,\epsilon)}}{|T_i|} \leq \sum_{i=k+2}^{N} \frac{f_i}{|T_i|} < 1 - \sum_{i=1}^{k+1} \frac{f_i}{|T_i|}. $$

And finally,

$$A \leq \sum_{i=k+2}^{N} \frac{f_i}{\prod_{j=k+2}^{i} a_j} < B,$$

with $A = F^{(k,\epsilon)}_{k+1} = |T_{k+1}|(1 - \sum_{i=k+1}^{k+1} \frac{f_i^{(k,\epsilon)}}{|T_{k+1}|})$ and $B = F_{k+1} = |T_{k+1}|(1 - \sum_{i=1}^{k+1} \frac{f_i}{|T_{k+1}|})$.

Proposition 5. Given two sequences $(f_i)$ and $(f_i')$ such that $(f_i) \leq (f_i')$, let $k$ be the smallest integer such that $f_k \neq f_k'$ and let $\epsilon = f_k' - f_k$. The spherically symmetric tree $T = T((a_i))$ separates $(f_i)$ and $(f_i')$ if and only if there is a rank $N$ such that: $A \leq \sum_{i=k+2}^{N} \frac{f_i}{\prod_{j=k+2}^{i} a_j} < B$,

where $A = F^{(k,\epsilon)}_{k+1}$ and $B = F_{k+1}$.

Proof. We have $(f_i^{(k,\epsilon)}) \leq (f_i')$, as indeed, $(f_i^{(k,\epsilon)})$ is the weakest sequence in $\{(g_i) \in \mathbb{R}^N \mid \forall i < k, g_i = f_i, g_k = f_k + \epsilon$ and $(f_i) \leq (g_i)\}$. Hence, any tree separating $(f_i)$ and $(f_i^{(k,\epsilon)})$ also separates $(f_i)$ and $(f_i')$. We conclude using Lemma 3.

4.3 Targeting Game

Given the form of the condition in Proposition 5, we can view this as a special case of a purely numerical problem, which we will call the targeting game. The instance of the problem is given by two positive real numbers, $A < B$, and a sequence of non-negative real numbers $(f_i)$ which represents the movements towards the target $[A, B]$. The player starts at position $u_0 = 0$ with an initial step size of 1. We denote by $\delta_i$ the step size at turn $i$, so $\delta_0 = 1$. At each turn $i > 0$, the player chooses a positive integer $a_i$ by which he will divide the previous step size, that is to say $\delta_i = \frac{\delta_{i-1}}{a_i} = \prod_{j=1}^{i} a_j^{-1}$. Then, the position of the player is updated with the rule $u_i = u_{i-1} + f_i \delta_i$. If there is an integer $N$ such that $u_N \in [A, B]$, then the player wins with the strategy $(a_i)$.

The targeting game can be summarised as follows: Given $0 < A < B$ and a sequence $(f_i)$, is there an $N$ and a sequence $(a_i)$ such that $A \leq \sum_{i=1}^{N} \frac{f_i}{\prod_{j=1}^{i} a_j} < B$?
We give two sufficient conditions on the data to ensure the existence of a winning strategy for the player.

**Theorem 3.** If there is an N such that $\sum_{i=1}^{N} f_i \geq A \left[ \frac{A}{B - A} \right]$, then there exists a sequence $(a_i)$ with $a_i = 1, \forall i \geq 2$ such that the player wins the targeting game at turn $N$ by selecting $(a_i)$.

**Proof.** First, note that if $m \geq \frac{A}{B - A}$, then $(m + 1)A \leq mB$. It follows that

$$[A \left[ \frac{A}{B - A} \right], +\infty] \subseteq \bigcup_{k \in \mathbb{N}^*} [kA, kB].$$

Hence, there is a $k$ such that $kA \leq \sum_{i=1}^{N} f_i < kB$. So, $A \leq \sum_{i=1}^{N} \frac{f_i}{k} < B$. Therefore, if the player chooses $a_1 = k$ and $a_i = 1$, for $i \geq 2$, he will have reached the target at turn $N$. \(\square\)

**Theorem 4.** If $|\{i : f_i \geq B\}| \geq \log_2 \left( \frac{B}{B - A} \right)$, then the player wins the targeting game by choosing at each turn $i$ the smallest positive integer $a_i$ such that $u_i < B$.

**Proof.** Consider a turn $i$ such that $a_i > 1$. Given that the player chooses the minimum $a_i$, it follows that $B \leq u_{i-1} + \frac{\delta_{i-1}}{a_i - 1} f_i$. By definition of $u_i$ and $\delta_i$, we have $\delta_{i-1} f_i = a_i (u_i - u_{i-1})$, so $B \leq u_{i-1} + \frac{a_i}{a_i - 1} (u_i - u_{i-1})$. Then $B(a_i - 1) \leq u_{i-1} (a_i - 1) + a_i (u_i - u_{i-1})$ and

$$a_i (B - u_i) \leq B - u_{i-1}. \tag{8}$$

Now consider the sequence $x_i = \frac{B - u_i}{\delta_i}$. By dividing Equation (8) by $\delta_{i-1}$, we see that $x_i \leq x_{i-1}$ when $a_i > 1$. When $a_i = 1$, we also have $x_i = x_{i-1} - f_i \leq x_{i-1}$. Thus $(x_i)$ is non-increasing, and $\forall i, x_i \leq x_0 = B$.

At any turn $i$ where $f_i \geq B$, we have $f_i \geq x_{i-1}$ and

$$\delta_{i-1} f_i \geq \delta_{i-1} x_{i-1} = B - u_{i-1} > u_i - u_{i-1} = \delta_i f_i.$$

So $\delta_{i-1} > \delta_i$, and $a_i > 1$. It then follows from Equation (8) that $B - u_i \leq \frac{B - u_{i-1}}{2^{|\{i \leq N : f_i \geq B\}|}}$.

Note also that, since $(x_i)$ and $(\delta_i)$ are non-increasing, $(B - u_i)$ is also non-increasing. Hence, for all $N$, we have:

$$B - u_N \leq \frac{B - u_0}{2^{|\{i \leq N : f_i \geq B\}|}} = \frac{B}{2^{|\{i \leq N : f_i \geq B\}|}}.$$

Finally, choosing $N$ such that $|\{i \leq N : f_i \geq B\}| \geq \log_2 \left( \frac{B}{B - A} \right)$, we have $A \leq u_N < B$. \(\square\)

**Proposition 6.** Given $(f_i) < (f'_i)$, let $k$ be the smallest integer such that $f_k \neq f'_k$ and let $\epsilon = f'_k - f_k$. If there is an $N$ such that $\sum_{k+2}^{N} f_i \geq 2 \left\lfloor \frac{2}{\epsilon} \right\rfloor$ or $|\{k+2 \leq i \leq N : f_i \geq 2\}| > 1 - \log_2 \epsilon$, then there is a spherically symmetric tree which separates $(f_i)$ and $(f'_i)$ in $N$ turns.
Proof. For \( i \leq k \), we choose the smallest \( a_i \) such that \( F_i > 0 \); i.e., \( a_i = \left\lceil \frac{f_i}{F_i} \right\rceil + 1 \). We then choose \( a_{k+1} = \max\{2, \left\lceil \frac{f_{k+1}}{F_{k+1}} \right\rceil + 1\} \).

Using Proposition 5, it is sufficient to have a rank \( N \) such that:

\[
A \leq \sum_{i=k+2}^{N} \frac{f_i}{\prod_{j=k+2}^{i} a_j} < B,
\]

where \( A = |T_{k+1}|(1 - \sum_{i=1}^{k+1} \frac{f_i^{(k, \epsilon)}}{|T_i|}) \) and \( B = |T_{k+1}|(1 - \sum_{i=1}^{k+1} \frac{f_i}{|T_i|}) \).

It follows from the choice of \( a_i \) that for \( i \leq k \), \((a_i - 1)F_{i-1} - f_i \leq 0\). Hence, \( a_iF_{i-1} - f_i \leq F_{i-1} \), and \( F_i \leq F_{i-1} \). If \( a_{k+1} = \left\lceil \frac{f_{k+1}}{F_{k+1}} \right\rceil + 1 \), then \( F_{k+1} \leq F_k \). Otherwise, if \( a_{k+1} = 2 \), \( F_{k+1} \leq 2F_k \).

Finally, we have \( B = F_{k+1} \leq 2F_0 = 2 \).

Also, \( B - A = \sum_{i=k+2}^{k+1} (f_i - f_i) \prod_{j=k+1}^{i+1} a_j = (f_{k+1} - f_k)a_{k+1} + (f_{k+1} - f_{k+1}) = (a_{k+1} - 1) \epsilon \).

Having chosen \( a_{k+1} \geq 2 \), we have \( B - A \geq \epsilon \).

Thus, we have an \( N \) such that \( \sum_{i=k+2}^{N} f_i \geq 2 \left\lceil \frac{2}{\epsilon} \right\rceil \geq A \left\lceil \frac{A}{B - A} \right\rceil \) or \( |\{k+2 \leq i \leq N; f_i \geq 2\}| > 1 - \log_2 \epsilon \geq \log_2 \left( \frac{B}{B - A} \right) \). The result follows by applying Theorem 3 or Theorem 4.

\[ \square \]

Remark 2. In the case where \( |\{k+2 \leq i \leq N; f_i \geq 2\}| > 1 - \log_2 \epsilon \), the sequence \( (a_i) \) is entirely created by a greedy algorithm which selects the minimum value of \( a_i \) such that \( F_i > 0 \) (and \( a_{k+1} \geq 2 \)). The value of \( a_i \) is therefore a function of \( F_{i-1} \) and \( f_i \).

5 Firefighting Sequence vs. Level Growth

5.1 Infinite Offline Instances

On infinite trees, the objective is to contain the fire. We will consider only locally finite trees, i.e., trees where each vertex has finite degree. Given a locally finite rooted tree \( T \) with at least one infinite branch, we consider \( T^* \) the leafless sub-tree obtained from \( T \) by pruning finite branches. Formally, \( T^* \) is the union of all leafless sub-trees of \( T \) with the same root, where the union \( T_1 \cup T_2 \) of two such sub-trees of \( T \) is the sub-tree induced by \( V(T_1) \cup V(T_2) \). Since \( T \) is locally finite, the fire is contained on \( T \) if and only if it is contained on \( T^* \). Hence, without loss of generality, we may restrict the infinite case to leafless trees. Note that if \( T \) is leafless, then \( (|T_i|) \) is non-decreasing.

Intuitively, it seems that when the firefighter sequence grows faster, in some sense, than the number of vertices per level, the firefighter should be able to contain the fire. Following this line of reasoning, Proposition 7 and Theorem 5 give criteria for infinite instances to be winning based on the asymptotic behaviours of those two sequences.

Proposition 7. Let \((T, r, (f_i))\) be an instance of Fractional Firefighter where \( T \) is a tree of infinite height. If \( \sum_{i=1}^{\infty} \frac{f_i}{|T_i|} > 1 \), then the instance is winning.

Proof. The firefighter wins by spreading at each turn \( n \) the amount of protection evenly among all vertices of \( T_n \). The amount of fire that reaches \( v \in T_n \) is max\(\{0, 1 - \sum_{i \leq n} \frac{f_i}{|T_i|}\}\). Hence, the fire is contained after a finite number of turns. \[ \square \]
Unfortunately, we need a more complex criterion to obtain a sufficient condition to win in both Firefighter and Fractional Firefighter.

**Theorem 5.** Let \((T, r, (f_i))\) be an instance of Firefighter or Fractional Firefighter where \(T\) is a leafless tree. If \(S_i \to +\infty\) and \(\frac{S_i}{|T_i|} \to 0\), then the instance \((T, r, (f_i))\) is winning for the firefighter.

The proof of Theorem 5 will require the following lemma, which is probably well-known:

**Lemma 4.** If \((u_n)\) is a positive sequence that increases towards \(+\infty\), then \(\sum u_n \to \infty\) diverges.

**Proof.** Let \(v_n = \frac{u_n - u_n - 1}{u_n}\). If \(v_n \to 0\), then \(\sum v_n\) diverges. Let us assume that \(v_n \to 0\). Since \(\frac{u_n - 1}{u_n} = 1 - v_n\), we have \(\ln\frac{u_n}{u_n} = \ln\prod_{i=1}^n 1 - v_i = \sum_{i=1}^n \ln(1 - v_i)\). Since \(u_n \to +\infty\), \(\ln\frac{u_n}{u_n} \to -\infty\), so \(\sum \ln(1 - v_i) \to -\infty\), and since \(v_n \to 0\), \(\sum v_n \to +\infty\). \(\square\)

We may now prove Theorem 5.

**Proof.** Since \(T\) is leafless, \(|T_i|\) is non-decreasing and since \(\frac{S_i}{|T_i|} \to 0\), there is a positive constant \(C\) and an increasing injection \(\sigma: \mathbb{N} \to \mathbb{N}\) such that

\[
\forall i, |T_{\sigma(i)}| \leq C S_{\sigma(i)} < \infty.
\]

Let \(a: V(T) \to [0, 1]\) denote the amount of protection we will place on each vertex. In order to describe the amount of each vertex that remains unprotected at the end of each turn, we use a sequence of labellings \(l_i : V(T) \to [0, 1]\). Initially, all vertices are unprotected, so \(l_0 = 1\). At turn \(i\), protection is placed on vertices of \(T_i\), so \(\forall v \in V(T), l_i(v) = l_{i-1}(v) - \sum_{v' \in T_i} a(v')\mathbb{1}_{v' \leq v}\). For any \(W \subset V(T)\) and any labelling \(l\), we define \(l(W) = \sum_{v \in W} l(v)\).

For all \(i\) and \(h \in \mathbb{N}\) with \(h > i\), for all \(v \in T_i\), let \(w_h(v) = |\{v' \in T_h, v < v'\}|\). Thus, \(\sum_{v \in T_i} w_h(v) = |T_h|\) and for all \(j < i\),

\[
\sum_{v \in T_i} w_h(v) l_j(v) = l_j(T_h).
\]

It follows from Lemma 4 that \(\sum \frac{S_{\sigma(i)} - S_{\sigma(i-1)}}{S_{\sigma(i)}}\) diverges. Hence \(\prod (1 + \frac{S_{\sigma(i)} - S_{\sigma(i-1)}}{C S_{\sigma(i)}})\) also diverges. Let \(N\) be such that \(\prod_{i=1}^N (1 + \frac{S_{\sigma(i)} - S_{\sigma(i-1)}}{C S_{\sigma(i)}}) > 2C\) and let \(h\) be such that \(S_{\sigma(h)} > 2 S_{\sigma(N)}\).

We consider the following strategy. At each turn \(i\), we protect the vertices which have the most descendants in level \(\sigma(h)\), i.e., \(a(v), v \in T_i\) is an optimal solution of the following linear program:

\[
\begin{align*}
\max \sum_{v \in T_i} a(v) w_{\sigma(h)}(v) \\
a(v) &\leq l_{i-1}(v) ; \ v \in T_i \ \\
\sum_{v \in T_i} a(v) &\leq f_i
\end{align*}
\]

If \(f_i \geq l_{i-1}(T_i)\) then we can protect the whole level \(T_i\), thus \(T_{\sigma(h)}\), and the fire is contained. So we assume \(f_i < l_{i-1}(T_i)\) for all \(i < \sigma(h)\). Then, \(f_i \frac{l_{i-1}(v)}{l_{i-1}(T_i)}, v \in T_i\) is a solution of the linear program with \(\sum_{v \in T_i} f_i \frac{l_{i-1}(v)}{l_{i-1}(T_i)} = f_i\). It follows, by optimality and using Equation 4, that:

\[
\sum_{v \in T_i} a(v) w_{\sigma(h)}(v) \geq \sum_{v \in T_i} f_i \frac{l_{i-1}(v)}{l_{i-1}(T_i)} w_{\sigma(h)}(v) = \frac{f_i l_{i-1}(T_{\sigma(h)})}{l_{i-1}(T_i)}
\]

17
Note that for \( j \leq i \), \( l_{j-1}(T_j) \leq |T_j| \leq |T_{\sigma(i)}| \). Hence,

\[
l_{\sigma(i-1)}(T_{\sigma(h)}) - l_{\sigma(i)}(T_{\sigma(h)}) = \sum_{j=\sigma(i-1)+1}^{\sigma(i)} \sum_{v \in T_j} a(v)w_{\sigma(h)}(v)
\]

\[
\geq \sum_{j=\sigma(i-1)+1}^{\sigma(i)} \frac{f_j l_{j-1}(T_{\sigma(h)})}{l_{j-1}(T_j)}
\]

\[
\geq \sum_{j=\sigma(i-1)+1}^{\sigma(i)} \frac{f_j l_{\sigma(i)}(T_{\sigma(h)})}{|T_{\sigma(i)}|}
\]

\[
\geq \frac{S_{\sigma(i)} - S_{\sigma(i-1)}}{CS_{\sigma(i)}} l_{\sigma(i)}(T_{\sigma(h)}) \quad \text{(since } |T_{\sigma(i)}| \leq CS_{\sigma(i)} \text{).}
\]

So

\[
l_{\sigma(i-1)}(T_{\sigma(h)}) \geq (1 + \frac{S_{\sigma(i)} - S_{\sigma(i-1)}}{CS_{\sigma(i)}})l_{\sigma(i)}(T_{\sigma(h)}).
\]

Therefore,

\[
|T_{\sigma(h)}| \geq l_{\sigma(0)}(T_{\sigma(h)}) \geq \prod_{i=1}^{N} (1 + \frac{S_{\sigma(i)} - S_{\sigma(i-1)}}{CS_{\sigma(i)}})l_{\sigma(N)}(T_{\sigma(h)}) > 2Cl_{\sigma(N)}(T_{\sigma(h)}).
\]

And consequently,

\[
l_{\sigma(N)}(T_{\sigma(h)}) \leq \frac{|T_{\sigma(h)}|}{2C} \leq \frac{1}{2} S_{\sigma(h)} \leq S_{\sigma(h)} - S_{\sigma(N)}.
\]

This means that the firefighters available between turns \( \sigma(N) \) and \( \sigma(h) \) outnumber the unprotected vertices on level \( \sigma(h) \). Hence, the strategy will win in at most \( \sigma(h) \) turns.

Conversely, asymptotic behaviours cannot guarantee that an instance will be losing. Indeed, if \( f_1 \geq |T_1| \), the instance is winning regardless of asymptotic behaviours. However, having selected asymptotic behaviours where the levels of the tree grow faster than the firefighter sequence, Theorem 6 guarantees that some instances with those asymptotic behaviours will be losing.

**Theorem 6.** Let \((t_i) \in \mathbb{N}^+\mathbb{N}\) and \((f_i) \in \mathbb{R}^+\mathbb{N}\) be such that \((t_i)\) is non-decreasing and tends towards \(+\infty\). Then, \(\sum \frac{f_i}{t_i}\) converges if and only if there exists a spherically symmetric tree \(T\) rooted in \(r\) such that:

- \(\exists N : \forall i \geq N, \frac{f_i}{2} \leq |T_i| \leq t_i\)
- \(\text{the instance } (T, r, (f_i)) \text{ is losing for (FRACTIONAL) FIREFIGHTER.}\)

**Proof.** 1) Suppose that \(\sum \frac{f_i}{t_i}\) converges. Let \(M\) be such that \(\sum_{i=M+1}^{+\infty} \frac{f_i}{t_i} < \frac{1}{4}\) and let \(N > M\) be such that \(t_N > 4S_M\). We choose \(a_1 = t_N\), \(a_i = 1\) for \(2 \leq i \leq N\), and \(a_i = \frac{t_i}{\prod_{j=1}^{i-1} a_j}\) for \(i > N\). We will show that \(T = T((a_i))\) is a solution.
Let us show by induction that $\forall i \geq N, \frac{t_i}{2} \leq |T_i| \leq t_i$. Note that

$$|T_N| = \prod_{j=1}^{N} a_j = t_N.$$ 

Assume that the result holds for $i - 1$ where $i > N$:

$$\frac{t_{i-1}}{2} \leq \prod_{j=1}^{i-1} a_j \leq t_{i-1}.$$ 

Since $t_i \geq t_{i-1}$, we have $a_i \geq 1$. Hence,

$$a_i \leq \frac{t_i}{\prod_{j=1}^{i-1} a_j} \leq a_i + 1 \leq 2a_i.$$ 

So, $\frac{t_i}{2} \leq |T_i| \leq t_i$, and the result holds for all $i \geq N$.

Since $T$ is spherically symmetric, the amount of fire that spreads to level $n$ is $\max(0, F_n)$, where $F_n = |T_n|(1 - \sum_{i=1}^{n} \frac{f_i}{|T_i|})$. For $n > N$, we have:

$$\sum_{i=1}^{n} \frac{f_i}{|T_i|} = \frac{S_N}{t_N} + \sum_{i=N+1}^{n} \frac{f_i}{|T_i|} \leq \frac{S_M}{t_N} + \frac{1}{t_N} \sum_{i=M+1}^{N} f_i + 2 \sum_{i=N+1}^{n} \frac{f_i}{t_i} < \frac{1}{4} + \frac{1}{4} + \frac{2}{4} = 1.$$ 

Hence, $F_n > 0$ for all $n$, and therefore the fire cannot be contained.

2) Conversely, if $\sum \frac{f_i}{|T_i|}$ diverges and $T = T((a_i))$ is such that $\exists N : \forall i \geq N, \frac{t_i}{2} \leq |T_i| \leq t_i$, then $\sum \frac{f_i}{|T_i|}$ also diverges. It follows that $F_n = |T_n|(1 - \sum_{i=1}^{n} \frac{f_i}{|T_i|})$ is negative above a certain rank. Hence, the fire is contained. \[\square\]

**Corollary 3.** Let $(t_i) \in \mathbb{N}^{\mathbb{N}}$ and $(f_i) \in \mathbb{R}^{+\mathbb{N}}$ be such that $(t_i)$ is non-decreasing and tends towards $+\infty$. Let $S_i = \sum_{1 \leq k \leq i} f_k$. If $S_i \to +\infty$ and $\frac{S_i}{t_i} \to 0$, then $\sum \frac{f_i}{t_i}$ diverges.

**Proof.** If $\sum \frac{f_i}{t_i}$ were convergent, it follows from Theorem 6 that there would be a spherically symmetric tree $T$ such that:

- $\exists N : \forall i \geq N, \frac{t_i}{2} \leq |T_i| \leq t_i$
- the instance $(T, r, (f_i))$ is losing for (Fractional) Firefighter.

It then follows from Theorem 5 that $S_i \to +\infty$ or $\frac{S_i}{t_i} \to 0$. Hence $S_i \to +\infty$ or $\frac{S_i}{t_i} \to 0$. \[\square\]

It follows that for Fractional Firefighter, Theorem 6 is weaker than Proposition 7. Theorem 5 remains interesting for Firefighter and it gives an alternative winning method for Fractional Firefighter.
Remark 3. Under the hypotheses of Theorem 5, if \( \sum \frac{t_i}{|T_i|} \) converges, we can create a losing instance \((T', r', (f_i))\) with \( \forall i, |T_i| = t_i \) by adding \( t_i - |T_i| \) leaves to level \( i \) for all \( i \). We will have \( |T_n| = t_n \) without adding leaves if and only if there exists a spherically symmetric tree with \( t_i \) vertices on level \( i \) for all \( i \).

Remark 4. Remark 1.12 in [11] gives a sufficient condition for an instance to be losing for Firefighter in a general graph satisfying some growth condition, using a similar criteria to the convergence of \( \sum \frac{t_i}{|T_i|} \). In general, both results cannot be compared. In our set-up however, their result can be seen as the particular case of Firefighter where \( (t_i) = (\lambda^i) \) for some \( \lambda \).

5.2 Online Firefighting on Trees with Linear Level Growth

In the previous section, Proposition 7 gives a winning strategy for online Fractional Firefighter in cases where \( \sum \frac{t_i}{|T_i|} > 1 \). However, Theorem 5 is limited to the offline case, as the winning strategy requires the player to be able to compute \( \sigma(h) \) from the start. In this section, we give a result which works for online Firefighter in the case of rooted trees \((T, r)\) where the number of vertices per level increases linearly, i.e. \( |T_i| = O(i) \). We say that such a tree has linear level growth.

Remark 5. The linear level growth property of \( T \) remains if we choose a different root \( r' \). Indeed, if \( d \) is the distance between \( r \) and \( r' \), the set of vertices at distance \( i \) from \( r' \) is included in \( \bigcup_{j=1}^{i+d} T_j \), the cardinal of which is \( O(i) \).

Theorem 7. Let \( \mathcal{I} \) be the set of instances \((T, r, (f_i))\) of Firefighter where \( T \) has linear level growth and there exists a non-zero periodic sequence which is weaker than \( (f_i) \). There is an online algorithm which contains the fire for every instance in \( \mathcal{I} \).

The proof of Theorem 7 will use the following lemma:

Lemma 5. For any real number \( a > 1 \), \( \lim_{n \to +\infty} \prod_{j=1}^{n} \frac{j^{a-1}}{j^{a}} = 0 \).

Proof. We have \( \ln \prod_{j=1}^{n} \frac{j^{a-1}}{j^{a}} = \sum_{j=1}^{n} \ln(1 - \frac{1}{j^{a}}) \) and since \( \sum_{j=1}^{n} \frac{1}{j^{a}} \to +\infty \), we have \( \sum_{j=1}^{n} \ln(1 - \frac{1}{j^{a}}) \to -\infty \). Hence, \( \prod_{j=1}^{n} \frac{j^{a-1}}{j^{a}} \to 0 \).

We can now prove Theorem 7.

Proof. Since \( T \) has linear level growth, let \( C \) be such that \( \forall i, |T_i| \leq Ci \). Without loss of generality, we assume \( C > 1 \). That a non-zero periodic sequence is weaker than \( (f_i) \) means that \( (1_{n|j}) \preceq (f_i) \) for all \( n \) greater than some \( m \). First, we will give an offline strategy to contain the fire with one firefighter every \( n \) turns. Then, we will show that online instances with \((1_{n|j}) \preceq (f_i) \) for an \( n \) known to the player are winning. Finally, we will describe an online winning strategy when such a \((1_{n|j})\) is unknown.

Given an integer \( n \), let us first consider the instance \((T, r, (1_{n|j}))\). It follows from Lemma 5 that there exists an integer \( N \) such that \( \prod_{j=1}^{N} \frac{C_{nj}-1}{C_{nj}} < \frac{1}{2CN} \). Let \( h(n) = 2nN \). A winning strategy for this offline instance is obtained by protecting at turn \( nj \) the unprotected vertex of \( T_{nj} \) with the highest number of descendants in level \( h(n) \). Since \( |T_{nj}| \leq C_{nj} \), the remaining number of unprotected vertices in \( T_{h(n)} \) is reduced by at least \( \frac{1}{C_{nj}} \) of its previous value. So the number of unprotected vertices of \( T_{h(n)} \) remaining after \( nN \) turns is less than \( |T_{h(n)}| \prod_{i=1}^{N} \frac{C_{nj}-1}{C_{nj}} \leq \frac{|T_{h(n)}|}{2CN} \leq N \).
Since $N$ firefighters remain to be placed between turns $N$ and $h(n)$, the strategy is winning in at most $h(n)$ turns.

If the player knows in advance that $(1_{n|i}) \preceq (f_i)$ for a given $n$, the above strategy can be adapted using Lemma 2.

In the general case, assume that $(1_{n|i}) \preceq (f_i)$ for some $n$, but the player does not know which $n$. The online strategy proceeds as follows: we initially play as though under the assumption that $(1_{n_0|i}) \preceq (f_i)$ with $n_0 = 100$. If the fire is not contained by turn $h(n_0)$, or later on by turn $h(n_k)$, we choose $n_{k+1} = h(n_k) \left( \lceil S(n_k) \rceil + 1 \right)$. We now assume that $(1_{n_k|\ell}) \preceq (f_i)$. It follows that after cancelling the first $h(n_k)$ terms of $(f_i)$, i.e., replacing $f_\ell$ with 0 for $\ell \leq h(n_k)$, the resulting sequence is stronger than $(1_{n_{k+1}|\ell})$. So we can consider that the first $h(n_k)$ turns were wasted and follow the strategy for $n_{k+1}$ until turn $h(n_{k+1})$. Eventually, this strategy will win when $n_k$ is large enough.

6 Conclusion

The main thread of this paper is to consider a general sequence of number of firefighters available at each turn in (Fractional) Firefighter, whereas most of the existing work on this topic considers constant sequences. We give first results, in the case of trees, for three independent research questions that arise when including such a sequence as part of the instance.

We introduce the online version of (Fractional) Firefighter on trees and provide initial results for the finite case. So far, our results outline the potential of this approach and suggest many open questions. To our knowledge, Theorem 1 is the first non-trivial competitive (and also approximation) analysis for Fractional Firefighter and a first question would be to investigate whether a better competitive ratio can be obtained for Fractional Firefighter in finite trees. Although the case of trees is already challenging, the main open question will be to study online (Fractional) Firefighter problem in other classes of finite graphs.

As far as we know, the second question has never been considered yet. The existence of a separating tree for any two given firefighter sequences seems very hard in general. Spherically symmetric trees provide convenient examples of separating trees since they allow us to ignore the playing strategy. This allowed us to express the problem in terms of the targeting game, which completely hides the structure of the tree. An interesting question will be to investigate whether the existence of a separating tree implies that of a spherically symmetric separating tree. So far, we only considered the case where one of the sequences is weaker than the other. The general case remains fully open.

We have shown that some conditions on the asymptotic behaviours of the firefighter sequence vs. the tree growth guarantee that the instance is winning. Yet, other conditions guarantee the existence of losing instances. We conjecture that all Firefighter instances where $\sum \frac{1}{\ell^{1}}$ diverges are winning.

Finally, note that the question of approximating (Fractional) Firefighter in finite trees for a general firefighter sequence is also an important research direction that, to our knowledge, remains uninvestigated.
Acknowledgements

We are grateful to the anonymous reviewers for their helpful comments and suggestions, especially for highlighting Proposition 7. We also acknowledge the support of GEO-SAFE, H2020-MSCA-RISE-2015 project # 691161.

References

[1] David Adjiashvili, Andrea Baggio, and Rico Zenklusen. Firefighting on trees beyond integrality gaps. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, pages 2364–2383, 2017.

[2] Susanne Albers. Online algorithms: a survey. Mathematical Programming Ser. B, Ser. B(97):3–26, 2003.

[3] Elliot Anshelevich, Deeparnab Chakrabarty, Ameya Hate, and Chaitanya Swamy. Approximability of the firefighter problem - computing cuts over time. Algorithmica, 62(1-2):520–536, 2012.

[4] Giorgio Ausiello and Luca Becchetti. On-line algorithms. In V. Th. Paschos, editor, Paradigms of Combinatorial Optimization: Problems and New Approaches, Vol. 2, chapter 15, pages 473–509. ISTE - WILEY, London - Hoboken, 2010.

[5] Cristina Bazgan, Morgan Chopin, Marek Cygan, Michael R. Fellows, Fedor V. Fomin, and Erik Jan van Leeuwen. Parameterized complexity of firefighting. Journal of Computer and System Sciences, 80(7):1285–1297, 2014.

[6] Cristina Bazgan, Morgan Chopin, and Bernard Ries. The firefighter problem with more than one firefighter on trees. Discrete Applied Mathematics, 161(1-2):301–306, 2013.

[7] Anthony Bonato, Margaret-Ellen Messinger, and Pawe Pralat. Fighting constrained fires in graphs. Theoretical Compututer Science, 434:11–22, 2012.

[8] Leizhen Cai, Elad Verbin, and Lin Yang. Firefighting on trees: (1-1/e)-approximation, fixed parameter tractability and a subexponential algorithm. In Algorithms and Computation, 19th International Symposium, ISAAC 2008, Gold Coast, Australia, December 15-17, 2008. Proceedings, pages 258–269, 2008.

[9] Janka Chlebiková and Morgan Chopin. The firefighter problem: Further steps in understanding its complexity. Theoretical Compututer Science, 676:42–51, 2017.

[10] Danny Dyer, Eduardo Martínez-Pedroza, and Brandon Thorne. The coarse geometry of Hartnell’s firefighter problem on infinite graphs. Discrete Mathematics, 340(5):935–950, 2017.

[11] Ohad N. Feldheim and Rani Hod. 3/2 firefighters are not enough. Discrete Applied Mathematics, 161(1-2):301–306, 2013.

[12] Stephen Finbow, Andrew King, Gary Macgillivray, and Romeo Rizzi. The firefighter problem for graphs of maximum degree three. Discrete Mathematics, 307(16):2094–2105, 2007.
[13] Stephen Finbow and Gary MacGillivray. The firefighter problem: a survey of results, directions and questions. Australasian Journal of Combinatorics, 43(6):57–77, 2009.

[14] Patricia Fogarty. Catching the Fire on Grids, PhD thesis. University of Vermont, 2003.

[15] Fedor V. Fomin, Pinar Heggernes, and Erik Jan van Leeuwen. The firefighter problem on graph classes. Theoretical Computer Science, 613(C):38–50, February 2016.

[16] Stephen G. Hartke. Attempting to narrow the integrality gap for the firefighter problem on trees. In Discrete Methods in Epidemiology, pages 225–232, 2004.

[17] Bert Hartnell and Qiyan Li. Firefighting on trees: How bad is the greedy algorithm? Congressus Numerantium, pages 187–192, 2000.

[18] Bert Hartnell. Firefighter! An application of domination. 1995. presented at the 10th Conference on Numerical Mathematics and Computing, University of Manitoba in Winnipeg, Canada.

[19] Yutaka Iwaikawa, Naoyuki Kamiyama, and Tomomi Matsui. Improved approximation algorithms for firefighter problem on trees. IEICE Transactions on Information and Systems, E94.D(2):196–199, 2011.

[20] Florian Lehner. Firefighting on trees and Cayley graphs. ArXiv e-prints, (arXiv:1707.01224v1 [math.CO]), July 2017.

[21] Russell Lyons and Yuval Peres. Probability on Trees and Networks. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.

[22] Gary MacGillivray and Ping Wang. On the firefighter problem. Journal of Combinatorial Mathematics and Combinatorial Computing, 47:83–96, 2003.

[23] Margaret-Ellen Messinger. Average firefighting on infinite grids. The Australasian Journal of Combinatorics, 41:15–28, 2008.

[24] Ping Wang and Stephanie A. Moeller. Fire control on graphs. Journal of Combinatorial Mathematics and Combinatorial Computing, 41:19–34, 2002.