Sharp estimates for the spreading speeds of the Lotka-Volterra competition-diffusion system: the strong-weak type

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Abstract

We consider the classical two-species Lotka-Volterra competition-diffusion system in the strong-weak competition case. When the corresponding minimal speed of the traveling waves is not linear determined, we establish the precise asymptotic behavior of the solution of the Cauchy problem in two different situations: (i) one species is an invasive one and the other is a native species; (ii) both two species are invasive species.

Key Words: competition-diffusion system, Cauchy problem, long-time behavior, traveling waves

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1 Introduction

We consider the following two-species Lotka-Volterra competition-diffusion system

\[
\begin{align*}
\partial_t u &= u_{xx} + u(1 - u - av), \quad t > 0, \ x \in \mathbb{R}, \\
\partial_t v &= dv_{xx} + rv(1 - v - bu), \quad t > 0, \ x \in \mathbb{R},
\end{align*}
\]

where \( u = u(t, x) \) and \( v = v(t, x) \) represent the population densities of two competing species at the time \( t \) and position \( x \). Here, all parameters are assumed to be positive: \( d \) and \( r \) stand for the diffusion rate and intrinsic growth rate of \( v \), respectively; \( a \) and \( b \) represent the competition coefficient of \( v \) and \( u \), respectively. In present paper, we focus on the strong-weak competition case:

\[(H1) \ 0 < a < 1 < b.\]

The condition \((H1)\) indicates that species \( u \) is the superior species; while \( v \) is an inferior one.

Early in 1937, Fisher [12] and Kolmogorov, Petrovsky, and Piskunov [27] introduced a scalar reaction-diffusion equation

\[w_t = w_{xx} + f(w),\]

with monostable nonlinearity

\[f'(0) > 0, \ f'(1) < 0, \ f(0) = f(1) = 0, \ f(w) > 0 \text{ for all } (0, 1),\]

to model the propagation of dominant gene in a homogeneous environment. With the so-called KPP condition \( f'(0) w \geq f(w) \) for all \( w \in [0, 1] \), they proved that traveling wave solutions of the form

\[w(t, x) = W(x - ct)\]

connecting the states \( 1 \) and \( 0 \) exist if and only if \( c \geq c_{\text{min}} := 2 \sqrt{\frac{f'(0)}{d}} \), where \( c_{\text{min}} \) is called the minimal wave speed. Moreover, they found a mathematical approach to describe the propagation of dominant gene by studying the long-time behavior of the solution to the Fisher-KPP equation:

\[
\begin{align*}
w_t &= w_{xx} + f(w), \quad t > 0, \ x \in \mathbb{R}, \\
w(0, x) &= w_0(x), \ x \in \mathbb{R}
\end{align*}
\]

where \( w_0(x) := H(-x) \), and \( H(x) \) is the Heaviside function.

In the case that \( w_0(x) \neq 0 \) is a nonnegative compactly supported function, Aronson and Weinberger [3] showed that there exists a unique speed \( c_w \) such that the solution of (1.2) satisfies

\[\lim_{t \to \infty} \sup_{|x| \geq ct} w(t, x) = 0 \text{ for all } c > c_w \quad \text{and} \quad \lim_{t \to \infty} \sup_{|x| \leq ct} |1 - w(t, x)| = 0 \text{ for all } c < c_w.\]

Moreover, the spreading speed \( c_w \) coincides with the minimal wave speed \( c_{\text{min}} \). The propagation phenomenon and inside dynamics of the front for more general scalar equation have been discussed widely in the literature. We may refer to, e.g., [6, 7, 11, 14, 17, 30, 40, 42, 43, 45] and references cited therein.

To understand the long-time behavior of solutions of system (1.1), traveling wave solutions play an important role. In the absence of the one species, namely \( u \) or \( v \), system (1.1) can be reduced to the single scale Fisher-KPP equation like (1.2), which admits a unique (up to translations) traveling wave solution \( U_{KPP}(x - ct) \) (resp. \( V_{KPP}(x - ct) \)) with the minimal speed

\[c_u := 2, \quad (\text{resp. } c_v := 2 \sqrt{rd}).\]
Regarding the traveling wave solutions of system (1.1) with (H1), Kon-on [25] showed that there exists a unique speed $c^* \in [2\sqrt{1 - a}, 2]$ such that system (1.1) admits a solution $(c, U, V)$ satisfying

\[
\begin{align*}
U'' + cU' + U(1 - U - aV) &= 0, \\
dV'' + cV' + rV(1 - V - bU) &= 0, \\
(U, V)(-\infty) &= (1, 0), \quad (U, V)(\infty) = (0, 1), \\
U' < 0, \quad V' > 0,
\end{align*}
\]

if and only if $c \geq c^*$. Thus, $c^*$ is called the minimal traveling wave speed of system (1.3).

The linear determinacy of $c^*$ has been widely discussed over several decades to understand the dynamics of diversity for invasive species. It is said that linear determinacy holds if $c^* = 2\sqrt{1 - a}$ since the linearization of (1.3) at the unstable state $(0, 1)$ results in the linear speed $2\sqrt{1 - a}$ (see [28, 46]). In this case, $c^*$ is also said to be linear or be linearly selected. If $c^* > 2\sqrt{1 - a}$, we say that linear determinacy does not hold, nonlinear determinacy holds, or $c^*$ is nonlinear selected. Another terminology comes from Stokes [43]. We may say that $c^* = 2\sqrt{1 - a}$ is "pulled fronts" case since the propagation speed is determined only by the leading edge of the distribution of the population; while $c^* > 2\sqrt{1 - a}$ called "pushed fronts" case since the propagation speed is not determined by the behavior of the leading edge of the population distribution, but by the whole wavefront. We also refer to the work of Roques et al. [41] that introduced another definition of pulled and pushed fronts for system (1.1). In present paper, we mainly focus on the "pushed fronts" case:

(H2) $c^* > \sqrt{1 - a}$.

Linear/nonlinear determinacy of the minimal traveling wave speed of system (1.3) has been investigated in the literature. Among them, Lewis, Li and Weinberger [28] showed that linear determinacy holds when

\[
0 < d < 2 \quad \text{and} \quad r(ab - 1) \leq (2 - d)(1 - a).
\]

(1.4)

An improvement for the sufficient condition for linear determinacy were made by Huang [21]:

\[
\frac{(2 - d)(1 - a) + r}{rb} \geq \max \left\{ a, \frac{d - 2}{2|d - 1|} \right\}.
\]

(1.5)

Note that (1.4) and (1.5) are equivalent when $d \leq 2$. Roques et al. [41] numerically suggested that the parameters region for linear determinacy can still be improved. More recently, Alhasanat and Ou [2] made some improvements.

For $c^*$ being nonlinear selected, Huang and Han [22] constructed examples in which linear determinacy fails to hold under the conditions: $r = d$ and $a$ is sufficiently close to 1. Alhasanat and Ou [2] proved that $c^*$ is nonlinear if

\[
\frac{(d + 2)(1 - a) + r}{rb} < 1 - 2(1 - a).
\]

Therefore, the assumption (H2) is not void. For related discussions, we also refer to, e.g., [1, 16, 18, 19, 20] and the references cited therein.

For the "pulled fronts" case $c^* = 2\sqrt{1 - a}$, the long-time behavior of the solution of system (1.1) is more complicated. We strongly believe that logarithmic phase drift of the location of the wavefront exists as what happens for the scalar monostable equation. This problem will be discussed in our forthcoming paper.
1.1 Main results

The purpose of this paper is to establish the sharp estimate on the long-time behavior of the solution of system (1.1) in the "pushed fronts" case (H2) with two different scenarios for initial data \((u_0, v_0)\):

\[
\begin{align*}
  u(0, x) &= u_0(x) \in C(\mathbb{R}, [0, 1]) \setminus \{0\} : \text{with compact support,} \\
  v(0, x) &= v_0(x) \in C(\mathbb{R}, [0, 1]) : \text{with a positive lower bound},
\end{align*}
\]

or

\[
(u, v)(0, x) = (u_0, v_0)(x) \in \left[C(\mathbb{R}, [0, 1]) \setminus \{0\}\right]^2 : \text{both are with compact support.}
\]

Biologically, (1.6) means that the species \(u\) is the invasive species, while \(v\) is the native species occupying the whole space; (1.7) indicates that both two species are invasive species.

There is a wide variety of literature regarding the traveling wave solution and (asymptotic) spreading speeds for system (1.1). In the weak competition case (i.e., \(a < 1, b < 1\)), Tang and Fife [44] established the existence of the minimal wave speed for traveling waves connecting \((0,0)\) and the coexistence state. For the Cauchy problem, Lin and Li [31] considered system (1.1) with compactly supported initial functions and obtained the spreading speed of the faster species and some estimates on the speed of the slower species. More recently, Liu, Liu and Lam [32, 33] obtained rather complete results.

In the strong (bistable) competition case (i.e., \(a > 1, b > 1\)), the existence of traveling waves connecting \((0,1)\) and \((1,0)\) was established by Gardner [13], Conley and Gardner [9] and Kan-on [24]. For the Cauchy problem, Carrere [8] studied the asymptotic spreading speed of the solution with initial data which are absent on the right half-line \(x > 0\), and the slower species dominates the faster one on the left half-line \(x < 0\). More recently, Peng, Wu and Zhou [38] provided rigorous estimates on the spreading speed and profiles of the solution as \(t \to \infty\).

In the critical competition case (i.e., \(a = b = 1\)), Alfaro and Xiao [4] proved the non-existence of traveling waves with some monotonicity. Moreover, they studied the large time behavior of the solution of the Cauchy problem with compactly supported initial data. More precisely, they not only reveal that the "faster" species excludes the "slower" one, but also found a new bump phenomenon which provides a sharp description of the profile of the solution.

Regarding the strong-weak (monostable) competition case (i.e., (H1)), the asymptotic spreading speed of the Cauchy problem was firstly studied by Lewis, Li and Weinberger [28, 29] with \((u_0, v_0)\) satisfying \(0 \leq u_0(x) \leq 1\) and \(0 \leq 1 - v_0(x) \leq 1\), and both \(u_0\) and \(1 - v_0\) are compactly supported functions. Recently, Girardin and Lam [15] studied the spreading speed of the Cauchy problem with initial data that are null or exponentially decay on the right half line. They obtained a complete understanding of the spreading properties by constructing very technical super-solutions and sub-solutions. Among other things, they also found that a so-called "nonlocal pulling" phenomenon may happen in some cases.

Regarding the study of the spreading property for other reaction-diffusion systems, we refer to [23, 34, 39] for monotone systems; [10, 36] for non-cooperative systems.

Our first result considers the scenario that the initial data \((u_0, v_0)\) satisfies (1.6). The spreading speed has been obtained in [28]. Here we establish the sharp long-time behavior of the solution when linear determinacy does not hold.
**Theorem 1.1** Assume that (H1)-(H2) hold. Then the solution \((u, v)\) of system \((1.1)\) with initial data \((1.6)\) satisfies
\[
\lim_{t \to \infty} \left( \sup_{x \in [0,\infty)} |u(t,x) - U(x - c_1 t - h)| + \sup_{x \in [0,\infty)} |v(t,x) - V(x - c_2 t - h)| \right) = 0,
\]
where \(h\) is a constant and \((c_1^*, U, V)\) is the minimal traveling wave defined as \((1.3)\).

Next, we consider the scenario that the initial function \((u_0, v_0)\) satisfies \((1.7)\). In this case, the spreading property becomes more complicated: the invading speed of the stronger species \(u\) could be nonlocal determined in some cases, as reported in \([15]\). To give a precise illustration, let us recall the auxiliary function given in \([15]\):
\[
f(c) := c - \sqrt{c^2 - 4(1 - a)} + 2\sqrt{a} \text{ and } f^{-1}(c') := \frac{c'}{2} - \sqrt{a} + \frac{2(1 - a)}{c' - 2\sqrt{a}} \tag{1.8}
\]
Note that \(f\) is a decreasing function. If \(2\sqrt{rd} \in (2, f(c^*))\), then we define the accelerated speed
\[
c_{ss} := f^{-1}(2\sqrt{rd}) = \sqrt{rd} - \sqrt{a} + \frac{1 - a}{\sqrt{rd} - \sqrt{a}} \in (c^*, 2).
\]
It has been showed in \([15]\) that, if \(c_v > c_a\), there exist two wavefronts. The fast one moves with the speed \(c_v\). The slow one moves with the speed \(c^*_a\), which satisfies
\[
\begin{cases}
\mathcal{C} = c^* & \text{if } c_v \in [f(c^*), \infty), \\
\mathcal{C} = c_{ss} & \text{if } c_v \in (2, f(c^*)).
\end{cases} \tag{1.9}
\]

In \([15]\), they found this "nonlocal pulling" phenomenon from an observation on the behavior of the solution \((u, v)\) on the leading edge, namely the region where \(u, v \approx 0\). Let us define functions for \(c \geq c^*\) and \(c' \geq \max\{c, f(c)\}\) as follows:
\[
\Lambda(c, c') := \frac{1}{2} \left( c' - \sqrt{c'^2 - 4\lambda(c)(c' - c) - 4} \right) \text{ with } \lambda(c) = \frac{1}{2} \left( c - \sqrt{c^2 - 4(1 - a)} \right).
\]
If \(c_v > c_a\), then we can assume \(v\) invades the uninhabited region \((u, v \approx 0)\) with a speed \(c_1 \geq c_v\) and \(u\) chase \(v\) from behind with a speed \(c_2 \in [c^*, c_a]\). In the region where \(v \approx 1\), the profile of \(u\) converges to the traveling wave solution defined as \((1.3)\) with speed \(c_2\). Therefore, we have \(u(t, x) \approx e^{-\lambda(c_2)(x-c_2 t)}\). Define a new function \(y(t, x) = u(t, x)e^{\lambda(c_2)(x-c_2 t)}\). In the range \(x = \tilde{c} t\) with \(\tilde{c} > c_1\), where \(u, v \approx 0\), it holds
\[
\partial_t y - y_{xx} \approx (1 + \lambda(c_2))(\tilde{c} - c_2)y. \tag{1.10}
\]
Then, by assuming the exponential ansatz \(y(t, x) \approx e^{-\Lambda(x-\tilde{c} t)}\), \((1.10)\) leads to the equation
\[
\Lambda^2 - \tilde{c} \Lambda + (1 + \lambda(c_2)(\tilde{c} - c_2)) = 0.
\]
The minimal root of this equation is equal to \(\Lambda(c_2, \tilde{c})\), which exists if and only if
\[
c_2^2 - 4(\lambda(c_2)(\tilde{c} - c_2) + 1) \geq 0.
\]
This inequality immediately implies that \(\tilde{c}\) has to satisfy \(\tilde{c} \geq f(c_2)\), which implies \(c_1 \geq f(c_2)\).

More precisely, we have the following propagation properties:
Proposition 1.2 (Theorem 1.1 in [15]) Let \((u, v)\) be the solution of system (1.1) with initial data \(u_0 \in C(\mathbb{R}, [0, 1]) \setminus \{0\}\) with support included in a left half-line and \(v_0 \in C(\mathbb{R}, [0, 1]) \setminus \{0\}\) with compact support. Then the following hold:

1. If \(c_1 > c_v\), then it holds
   \[
   \lim_{t \to \infty} \sup_{x \in [0, \infty)} \left( \sup_{x \geq c_1 t} v(t, x) + \sup_{0 \leq x \leq c_2 t} |1 - u(t, x)| \right) = 0
   \]
   for all \(0 < c_2 < c_v < c_1\).

2. If \(c_1 < c_v\), then it holds
   \[
   \lim_{t \to \infty} \sup_{x \geq c_1 t} \left( u(t, x) + v(t, x) \right) = 0 \quad \text{for all} \quad c_v > c_1 > c_3 > \mathcal{C};
   \]
   \[
   \lim_{t \to \infty} \sup_{x \geq c_1 t} \left( |1 - u(t, x)| + v(t, x) \right) = 0 \quad \text{for all} \quad c_4 < \mathcal{C}.
   \]

Here we first establish the convergence of the solution to system (1.1) with initial data (1.7). For \(c_1 > c_v\), in view of statement (1) in Proposition 1.2, we see that \(u\) is the only survival specie, so it can be seen as the fastest species. Therefore, Corollary 4.6 in [38] can be applied to obtain the propagating behavior of \(u\) over \(\{x \geq (c_v + \varepsilon) t\}\) for all \(\varepsilon > 0\) and large \(t\). Thus, combining Proposition 1.2(i) and [38] Corollary 4.6), we immediately conclude that

Proposition 1.3 Assume that (H1) holds. If \(c_1 > c_v\), then the solution \((u, v)\) of system (1.1) with initial data (1.7) satisfies

\[
\lim_{t \to \infty} \left[ \sup_{x \in [0, \infty)} \left| u(t, x) - U_{KPP} \left( x - c_u t + \frac{3}{c_v} \ln t + \omega(t) \right) \right| + \sup_{x \in [0, \infty)} |v(t, x)| \right] = 0,
\]

where \(\omega\) is a bounded function defined on \([0, \infty)\).

For \(c_1 < c_v\), we establish the following result.

Theorem 1.4 Assume that (H1)-(H2) hold. Let \(f\) be the auxiliary function defined in (1.8). If \(c_v \in [f(c^*), \infty)\), then the solution \((u, v)\) of system (1.1) with initial data (1.7) satisfies

\[
\lim_{t \to \infty} \left[ \sup_{x \in [c_0 t, \infty)} \left| v(t, x) - V_{KPP} \left( x - c_v t + \frac{3d}{c_v} \ln t + \omega(t) \right) \right| + \sup_{x \in [c_0 t, \infty)} |u(t, x)| \right] = 0
\]

and

\[
\lim_{t \to \infty} \left[ \sup_{x \in [0, c_0 t]} \left| u(t, x) - U(x - c^* t - \hat{h}) \right| + \sup_{x \in [0, c_0 t]} \left| v(t, x) - V(x - c^* t - \hat{h}) \right| \right] = 0,
\]

where \(c_0 \in (c^*, c^* + \varepsilon)\) and \(\hat{h} \in \mathbb{R}\) are some constants, \(\omega(\cdot)\) is a bounded function defined on \([0, \infty)\), and \((c^*, U, V)\) is the minimal traveling wave defined as (1.3).

Remark 1.5 Theorem 1.4 shows the sharp estimate on the long-time behavior of the solution reported in statement (2) of Proposition 1.2 when the "nonlocal pulling" phenomenon does not occur, i.e., \(\mathcal{C} = c^*\). The case that \(\mathcal{C} = c_{ex}\) is more challenging, and will be discussed in our future work.

Next, we recall some useful known results and establish some asymptotic estimates of the traveling wave for later use.
1.2 Preliminaries

1.2.1 Comparison principle

For the reader’s convenience, we first recall the definitions of super-solution and sub-solution, and the comparison principle. Readers also can see section 2.1 of [15] to find more details. Define the operators as follows:

\[ N_1[u, v](t, x) := u_t - u_{xx} - F(u, v), \quad N_2[u, v](t, x) := v_t - dv_{xx} - G(u, v), \]

where

\[ F(u, v) := u(1 - u - av), \quad G(u, v) := rv(1 - v - bu) \] (1.11)

We say that \((\bar{u}, \bar{v}) \in [C(\Omega) \cap C^2(\Omega)]^2\) is a pair of super-solution (sub-solution) of system (1.1) in \(\Omega := (t_1, t_2) \times (x_1, x_2), 0 \leq t_1 < t_2 \leq \infty, -\infty \leq x_1 < x_2 \leq +\infty\) if \((\bar{u}, \bar{v})\) satisfies \(N_1[\bar{u}, \bar{v}] \geq 0\) and \(N_2[\bar{u}, \bar{v}] \leq 0\) in \(\Omega\).

**Proposition 1.6** (Comparison Principle) Let \((\bar{u}, \bar{v})\) and \((\underline{u}, \underline{v})\) be a super-solution and sub-solution of system (1.1) in \(\Omega\), respectively. If \((\bar{u}, \bar{v})\) and \((\underline{u}, \underline{v})\) satisfy

\[
\begin{cases}
\bar{u}(t, x) \geq \underline{u}(t, x), \quad \underline{v}(t, x) \leq \bar{v}(t, x) & \text{for all } x \in (x_1, x_2),
\bar{u}(t, x_i) \geq \underline{u}(t, x_i), \quad \underline{v}(t, x_i) \leq \bar{v}(t, x_i) & \text{for all } t \in (t_1, t_2) \text{ and } i = 1, 2,
\end{cases}
\] (1.12)

then it holds \(\bar{u} \geq \underline{u}\) and \(\underline{v} \leq \bar{v}\) in \(\Omega\). If \(x_1 = -\infty\) or \(x_2 = +\infty\), the corresponding boundary condition can be omitted.

**Remark 1.7** If both \((\bar{u}_1, \bar{v}_1)\) and \((\bar{u}_2, \bar{v}_2)\) are super-solutions, then Proposition 1.12 still holds if \((\bar{u}, \bar{v})\) is replaced by \((\min\{\bar{u}_1, \bar{u}_2\}, \bar{v})\). If \((\bar{u}, \bar{v})\) and \((\underline{u}, \underline{v})\) are super-solutions, then Proposition 1.12 still holds if \((\bar{u}, \bar{v})\) is replaced by \((\bar{u}, \max\{\underline{u}_1, \underline{u}_2\})\). More details can be found in [15, Section 2].

1.2.2 Asymptotic behavior of the minimal traveling wave near \(\pm \infty\)

The asymptotic behavior of traveling waves \((c, U, V)\) near \(\pm \infty\) for any \(c \geq c^*\) has been reported in [37] (see also [15]). In this subsection, we only recall those for \(c = c^* > 2\sqrt{1 - a}\).

Let \((c, U, V)\) be a solution of system (1.3). To describe the asymptotic behavior of \((U, V)\) near \(+\infty\), we define

\[
\begin{align*}
\lambda_+^+(c) &:= -c \pm \sqrt{c^2 - 4(1 - a)} < 0, \\
\lambda_-^-(c) &:= -c - \sqrt{c^2 + 4rd} / 2d < 0 < \lambda_+^-(c) := -c + \sqrt{c^2 + 4rd} / 2d.
\end{align*}
\]

**Lemma 1.8** ([37]) Let \((c^*, U, V)\) be the minimal traveling wave of system (1.3) with \(c^*\) satisfying (H2). Then there exist positive constants \(l_i (i = 1, 2, 3, 4)\) such that the following hold:

\[
\begin{align*}
\lim_{\xi \to +\infty} & \frac{U(\xi)}{e^{\lambda_u^+(c^*)}\xi} = l_1, \\
\lim_{\xi \to +\infty} & \frac{1 - V(\xi)}{e^{\lambda_u^-(c^*)}\xi} = l_2 \quad \text{if } \lambda_u^-(c^*) < \lambda_u^-(c^*), \\
\lim_{\xi \to +\infty} & \frac{1 - V(\xi)}{e^{\lambda_v^-(c^*)}\xi} = l_3 \quad \text{if } \lambda_u^-(c^*) = \lambda_u^-(c^*), \\
\lim_{\xi \to +\infty} & \frac{1 - V(\xi)}{e^{\lambda_v^+(c^*)}\xi} = l_4 \quad \text{if } \lambda_u^+(c^*) > \lambda_u^+(c^*).
\end{align*}
\]
To describe the asymptotic behavior of \((c^*, U, V)\) near \(-\infty\), we define
\[
\mu_u(c) := \frac{-c - \sqrt{c^2 + 4}}{2} < 0 < \mu_u^+(c) := \frac{-c + \sqrt{c^2 + 4}}{2},
\]
\[
\mu_v(c) := \frac{-c - \sqrt{c^2 + 4rd(b - 1)}}{2d} < 0 < \mu_v^+(c) := \frac{-c + \sqrt{c^2 + 4rd(b - 1)}}{2d}.
\]

**Lemma 1.9** Let \((c^*, U, V)\) be the minimal traveling wave of system (1.3). Then there exist positive constants \(l_i (i = 5, 6, 7, 8)\) such that
\[
\lim_{\xi \to -\infty} \frac{V(\xi)}{e^{\mu_u(c^*)}\xi} = l_5,
\]
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{e^{\mu_u^+(c^*)}\xi} = l_6 \quad \text{if} \quad \mu_u^+(c^*) > \mu_u^+(c^*),
\]
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{[1 + e^{\mu_u(c^*)}]\xi} = l_7 \quad \text{if} \quad \mu_u^+(c^*) = \mu_u^+(c^*),
\]
\[
\lim_{\xi \to -\infty} \frac{1 - U(\xi)}{e^{\mu_u^+(c^*)}\xi} = l_8 \quad \text{if} \quad \mu_u^+(c^*) < \mu_u^+(c^*).
\]

### 1.2.3 Some useful estimates

In this subsection, we provide some estimates for later use. Note that the assumption (H2) is not required in this subsection.

**Lemma 1.10** Let \((u, v)\) be the solution of system (1.1) with initial data satisfying either (1.6) or (1.7). Then there exist \(M > 0\) such that
\[
u(t, x) \leq 1 + Me^{-rt} \quad \text{and} \quad v(t, x) \leq 1 + Me^{-rt} \quad \text{for all} \quad (t, x) \in [0, \infty) \times \mathbb{R}.
\]

**Proof.** Since this can be done by simple comparison with ODEs, the proof is omitted.

**Lemma 1.11** Assume that (H1) holds. Let \((u, v)\) be the solution of system (1.1) with initial data satisfying either (1.6) or (1.7). Then there exist \(c_{uv} > 0\) such that for any \(c \in (0, c_{uv})\), there exist \(C_i > 0, k_i > 0 \ (i = 1, 2)\) and \(T > 0\) such that
\[
u(t, x) \geq 1 - C_1e^{-k_1t} \quad \text{and} \quad v(t, x) \leq C_2e^{-k_2t} \quad \text{for all} \quad (t, x) \in [T, \infty) \times [-ct, ct].
\]

**Proof.** To derive these estimates, we consider the strong-strong (bistable) competition system
\[
\begin{aligned}
\partial_t u^* &= u_{xx}^* + u^*(1 - u^* - a^*v^*), \quad t \geq 0, \ x \in \mathbb{R}, \\
\partial_t v^* &= dv_{xx}^* + rv^*(1 - v^* - b^*u^*), \quad t \geq 0, \ x \in \mathbb{R}, \\
u^*(0, x) &= u(x, T_0), \quad v^*(0, x) = v(x, T_0),
\end{aligned}
\]  \hspace{1cm} (1.13)

where \(a^* > 1 > a, \ b^* = b > 1\) and \(T_0 > 0\) will be determined later. Since \(a^* > a\) and \(b^* = b\), we have \(N_1[u^*, v^*] \leq 0\) and \(N_2[u^*, v^*] \geq 0\) for \(t \geq 0\). By applying the comparison principle, we have
\[
u^* \leq u \quad \text{and} \quad v^* \geq v \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times \mathbb{R}. \] \hspace{1cm} (1.14)

Now we fix the parameters \(b^* = b, \ d\) and \(r\), and let \(a^* > 1\) sufficiently close to 1 such that
\[
\frac{r + d(a^* - 1)}{b^*r} < 3 - 2a^*. \] \hspace{1cm} (1.15)
Due to [13, 24], there exists a unique $c_{uw}$ such that system (1.3) admits a unique traveling wave solution with $c = c_{uw}$, and $a$ is replaced $a^*$. Furthermore, in view of (1.15), Theorem 4.3 in [35] implies that $c_{uw} > 0$.

On the other hand, by the results of [28], we can take $T_0 > 0$ large enough, such that $(u, v)(T_0, \cdot)$ is close to $(1, 0)$ in a sufficiently large interval, and thus the solution $(u^*, v^*)$ of the bistable system (1.13) satisfies that $(u^*(t), v^*(t, x)) \to (1, 0)$ as $t \to \infty$ locally uniformly for $x \in \mathbb{R}$ (see Remark 1.1 in [38]). Together with $c_{uw} > 0$, Lemma 2.6 and Lemma 2.8 in [38] is available to assert that for any $c \in [0, c_{uw})$,

$$u^*(t, x) \geq 1 - C_1 e^{-k_1 t} \quad \text{and} \quad v^*(t, x) \leq C_2 e^{-k_2 t} \quad \text{for all} \quad (t, x) \in [T, \infty) \times [-ct, ct]. \quad (1.16)$$

for some $T > T_0$, $C_i > 0$, $k_i > 0$ ($i = 1, 2$).

Combining (1.14) and (1.16), we immediately obtain the desired result. \qed

Recall $\mathcal{C}$ from (1.9).

Lemma 1.12 Assume that (H1) holds. Let $(u, v)$ be the solution of system (1.1) with initial data (1.7). Further, assume that $c_v > c_u$. Then the following hold:

(i) for any $c > \mathcal{C}$, there exist $C_1, \nu_1, T_1 > 0$ such that

$$\sup_{x \geq ct} u(t, x) \leq C_1 e^{-\nu_1 t} \quad \text{for all} \quad t \geq T_1.$$

(ii) for any $c_1$ and $c_2$ with $\mathcal{C} < c_1 < c_2 < c_v$, there exist $C_2, \nu_2, T_2 > 0$ such that

$$\sup_{c_1 t \leq x \leq c_2 t} v(t, x) \geq 1 - C_2 e^{-\nu_2 t} \quad \text{for all} \quad t \geq T_2.$$

Proof. Let us briefly start with (i). If $c > c_u$, then the conclusion is clear by comparing a supersolution of scalar KPP equation. Since $c_v > c_u$, it thus suffices to consider the case $\mathcal{C} < c < c_v$. In this case, the conclusion is already included in [15, Proposition 1.5] and the proof of [15] Section 3.2.3, Theorem 1.1]. We do not present the full details but only emphasize that a key tool is, for any small $\delta > 0$, the minimal monotone traveling wave of the perturbed system

$$\begin{cases}
U'' + cU' + U(1 + \delta - U - aV) = 0, \\
dV'' + cV' + rV(1 - 2\delta - V - bU) = 0, \\
(U, V)(-\infty) = (1 + \delta, 0), \quad (U, V)(\infty) = (0, 1 - 2\delta), \\
U' < 0, \quad V' > 0.
\end{cases}$$

Let us now turn to (ii) for which the above perturbation argument seems unapplicable. Let $\mathcal{C} < c_1 < c_2 < c_v$ be given. We only deal with $x \geq 0$. From [15, Theorem 1.1] we know

$$\lim_{t \to \infty} \sup_{c_1 t \leq x \leq c_2 t} \left( u(t, x) + |1 - v(t, x)| \right) = 0.$$

From this and (i), we can choose $\varepsilon > 0$ small enough and $T_0 \gg 1$ such that

$$0 < u(t, x) \leq C_1 e^{-\nu_1 t}, \quad v(t, x) > 1 - \varepsilon \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times [c_1 t, c_2 t].$$

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From the \(v\)-equation in system (1.1), we have
\[
v_t \geq dv_{xx} + r(1-\varepsilon)(1-v) - rbC_1v e^{-\nu_1t} \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times [c_1t, c_2t]. \tag{1.17}
\]
Defining
\[
\tilde{v}(t, x) := v(t, x + \tilde{c}t), \quad \tilde{c} := \frac{c_1 + c_2}{2},
\]
it follows from (1.17) that
\[
\tilde{v}_t \geq d\tilde{v}_{xx} + \tilde{c}\tilde{v}_x + r(1-\varepsilon)(1-\tilde{v}) - rbC_1\tilde{v} e^{-\nu_1t} \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times [-c_3t, c_3t],
\]
where \(c_3 := \frac{c_2 - c_1}{2}\).

To estimate \(\tilde{v}\), for any \(T > T_0\), we define
\[
\alpha(t) := 1 + \frac{rbC_1}{1-\varepsilon} e^{-\nu_1(t+T)} \quad \text{for all} \quad t \geq 0.
\]
Up to enlarging \(T > 0\) if necessary, we may assume \(\alpha(0) < \frac{1}{1-\varepsilon}\). Now, let us first consider the auxiliary problem
\[
\begin{align*}
\phi_t &= d\phi_{xx} + \tilde{c}\phi_x + r(1-\varepsilon)[1-\alpha(t)\phi], \quad t > 0, \quad -c_3T < x < c_3T, \\
\phi(t, \pm c_3T) &= 1 - \varepsilon, \quad t > 0, \\
\phi(0, x) &= 1 - \varepsilon, \quad -c_3T \leq x \leq c_3T.
\end{align*} \tag{1.18}
\]
Letting
\[
\Phi(t, x) := e^{Q(t)}[\phi(t, x) - 1 + \varepsilon], \quad Q(t) := r(1-\varepsilon)t - \frac{rbC_1}{\nu_1} e^{-\nu_1(t+T)},
\]
so that \(Q'(t) = r(1-\varepsilon)\alpha(t)\), it follows from (1.18) that
\[
\begin{align*}
\Phi_t &= d\Phi_{xx} + \tilde{c}\Phi_x + r(1-\varepsilon)e^{Q(t)}(1-(1-\varepsilon)\alpha(t)), \quad t > 0, \quad -c_3T < x < c_3T, \\
\Phi(t, \pm c_3T) &= 0, \quad t > 0, \\
\Phi(0, x) &= 0, \quad -c_3T \leq x \leq c_3T. \tag{1.19}
\end{align*}
\]
Up to a rescaling, we may assume \(d = 1\) so that (1.19) is very comparable to [26] problem (3.12)] on which we now rely. Denoting \(G_1(t, x, \cdot)\) the Green function of [26] page 53] (with obvious changes of constants), we obtain the analogous of [26] (3.14)], namely
\[
\Phi(t, x) \geq r(1-\varepsilon) \int_0^t e^{Q(s)}(1-(1-\varepsilon)\alpha(s)) \left( \int_{-c_3T}^{c_3T} G_1(t-s, x, z) dz \right) ds,
\]
for all \(t > 0, -c_3T < x < c_3T\). Next, for any small \(0 < \delta < 1\), we define
\[
D_\delta := \{(t, x) \in \mathbb{R}^2 : 0 < t < \delta^2 c_3T, |x| < (1-\delta)c_3T\}.
\]
From the same process used in [26] pages 54-55], there exist \(C_3, C_4 > 0\) such that the following lower estimate holds
\[
\Phi(t, x) \geq r(1-\varepsilon)(1-(1-\varepsilon)\alpha(0))(1 - C_3 e^{-C_4T}) \int_0^t e^{Q(s)} ds \quad \text{for all} \quad (t, x) \in D_\delta,
\]
resulting in
\[
\phi(t, x) \geq \Psi(t)(1 - C_3 e^{-C_4 t}))(1 - (1 - \varepsilon)\alpha(0)) + 1 - \varepsilon \quad \text{for all} \quad (t, x) \in D_\delta,
\]
where \(\Psi(t) := r(1 - \varepsilon)e^{-Q(t)}\int_0^t e^{Q(s)}ds\). Denoting \(K = \frac{rBC_3}{\nu_1}\), we have
\[
\Psi(t) \geq r(1 - \varepsilon)e^{-Q(t)}\int_0^t e^{r(1-\varepsilon)s}e^{-Ke^{-\nu_1 t}}ds
\]
\[
= e^{-r(1-\varepsilon)t}e^{Ke^{-\nu_1(t+\varepsilon)}}e^{-Ke^{-\nu_1 t}}\int_0^t r(1 - \varepsilon)e^{r(1-\varepsilon)s}ds
\]
\[
= e^{Ke^{-\nu_1(t+\varepsilon)-1}(1 - e^{-r(1-\varepsilon)t})}.
\]
Inserting this into (1.20) and using \(e^y \geq 1 + y\) for all \(y \in \mathbb{R}\), we have, for all \((t, x) \in D_\delta,
\[
\phi(t, x) \geq e^{Ke^{-\nu_1 T}(e^{-\nu_1 t}-1)}(1 - e^{-r(1-\varepsilon)t})(1 - C_3 e^{-C_4 T})(1 - (1 - \varepsilon)\alpha(0)) + 1 - \varepsilon
\]
\[
\geq (1 - Ke^{-\nu_1 T}(1 - e^{-\nu_1 t}))(1 - e^{-r(1-\varepsilon)t})(1 - C_3 e^{-C_4 T})(1 - (1 - \varepsilon)\alpha(0)) + 1 - \varepsilon
\]
\[
\geq (1 - Ke^{-\nu_1 T})(1 - e^{-r(1-\varepsilon)t})(1 - C_3 e^{-C_4 T})(1 - (1 - \varepsilon)\alpha(0)) + 1 - \varepsilon.
\]
Letting
\[
I_1 := 1 - Ke^{-\nu_1 T}, \quad I_2(t) := 1 - e^{-r(1-\varepsilon)t}, \quad I_3 := 1 - C_3 e^{-C_4 T},
\]
we get
\[
\phi(t, x) \geq I_1 I_2 I_3(t) + (1 - \varepsilon)(I_1 I_2 I_3(t)\alpha(0)) \quad \text{for all} \quad (t, x) \in D_\delta.
\]
Now observe that \(I_1 I_2 I_3 \leq I_1 I_2(\delta^2 c_3 T)I_3\). Furthermore some straightforward computations show that, if
\[
r(1 - \varepsilon)\delta^2 c_3 < \nu_1,
\]
then \(I_1 I_2(\delta^2 c_3 T)I_3\alpha(0) \leq 1 \) up to enlarging \(T > 0\) if necessary. As a result, for all \((t, x) \in D_\delta,
\[
\varphi(t, x) \geq I_1 I_2 I_3(t) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-\varepsilon)t},
\]
with some \(K_1, K_2 > 0\). The last inequality holds since we can always choose \(\nu_1 < C_4\). As a conclusion, we have
\[
\phi(t, x) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-\varepsilon)t} \quad \text{for all} \quad (t, x) \in D_\delta,
\]
provided that \(\delta > 0\) is sufficiently small for (1.21) to hold and \(T > 0\) is sufficiently large.
In particular (1.22) implies that, for all \(|x| \leq (1 - \delta)c_3 T,
\[
\phi(\delta^2 c_3 T, x) \geq 1 - K_1 e^{-\nu_1 T} - K_2 e^{-r(1-\varepsilon)\delta^2 c_3 T} \geq 1 - (K_1 + K_2)e^{-r(1-\varepsilon)\delta^2 c_3 T},
\]
in virtue of (1.21). On the other hand, we know from the comparison principle that \(\tilde{\nu}(t + T, x) \geq \phi(t, x)\) for \(t \geq 0\) and \(|x| \leq c_3 T\), which together with (1.23) implies that
\[
\tilde{\nu}(\delta^2 c_3 T + T, x) \geq 1 - (K_1 + K_2)e^{-r(1-\varepsilon)\delta^2 c_3 T} \quad \text{for all} \quad |x| \leq (1 - \delta)c_3 T.
\]
We further take \(t = (\delta^2 c_3 + 1)T\), which yields
\[
\tilde{\nu}(t, x) \geq 1 - C_2 e^{-\nu_2 t} \quad \text{for all large} \ t \ \text{and} \ |x| \leq (1 - \delta)c_3 \tilde{t},
\]
where \(C_2 := K_1 + K_2\) and \(\nu_2 := \frac{\nu_1(1-\varepsilon)\delta^2 c_3}{1 + c_3 \delta^2} > 0\). Recalling that \(\tilde{\nu}(t, x) = \nu(t, x + \tilde{t})\) with \(\tilde{c} = c_3 \delta^2\),
that \(c_3 = \frac{C_2 - \nu_2}{2}\) and since \(\delta > 0\) can be chosen arbitrarily small, the above estimate completes the proof of \((ii)\).
Then by some straightforward computation, we obtain that, for all \( t, x \in \mathbb{R} \),
\[
\begin{align*}
\bar{u}(t, x) &= U(x - c^* t + \zeta(t)) - P(t) \min\{e^{-\alpha_0(x-c^*t+x_0)}, 1\}, \\
\bar{v}(t, x) &= V(x - c^* t + \zeta(t)) + Q(t),
\end{align*}
\] (2.1)

where \( P(t) = p_0 e^{-\mu_0 t}, Q(t) = q_0 e^{-\nu_0 t}, \zeta(t) = \zeta_0 - e^{-\tau t} \), \((c^*, U, V)\) is the minimal traveling wave defined as (1.3). Here, all of the parameters are positive and will be determined in the following proof. For the simplicity, we denote \( \xi = x - c^* t + \zeta(t) \) and \( W(t, x) = e^{-\alpha_0(x-c^*t+x_0)} \).

Clearly, there exists a curve \( \Gamma : [0, \infty) \to \mathbb{R} \) denoted by \( \Gamma(t) := c^* t - x_0 \) such that \( W(t, \Gamma(t)) = 1 \) for all \( t \geq 0 \). Thus, it holds
\[
W(t, x) \leq 1 \quad \text{for} \quad (t, x) \in S_1 := \{(t, x) \mid x \geq \Gamma(t)\},
\]
\[
W(t, x) \geq 1 \quad \text{for} \quad (t, x) \in S_2 := \{(t, x) \mid x \leq \Gamma(t)\}.
\]

Then by some straightforward computation, we obtain that, for \((t, x) \in S_1\), it holds
\[
N_1[u, v] := \partial_t u - u_{xx} - F(u, v) \\
= \zeta' U' + (\mu_0 - c^* \alpha_0 + \alpha_0^2 + 1 - 2U - a(V + Q) + PW)PW + aQU. \tag{2.2}
\]

And for \((t, x) \in S_2\), it holds
\[
N_1[u, v] = \zeta' U' + (\mu_0 + 1 - 2U - a(V + Q) + P)P + aQU.
\]

On the other hand, for \((t, x) \in (0, \infty) \times \mathbb{R}\) it holds
\[
N_2[u, v] := \partial_t v - d\bar{v}_{xx} - G(u, \bar{v}) \\
= \zeta' V' + r(2V + Q + b(U - P \min\{W, 1\}) - 1 - \frac{\mu_0}{r})Q - bvP \min\{W, 1\}. \tag{2.3}
\]

Next, we show that \((u, v)\) is a sub-solution by choosing suitable parameters. In the following discussion, we choose \( M > 0 \) sufficiently large and divide the whole space into three parts:

1. \( \Omega_1 = \{M \leq x - c^* t + \zeta(t)\} \);
2. \( \Omega_2 := \{x - c^* t + \zeta(t) \leq -M\} \);
3. \( \Omega_3 := \{-M \leq x - c^* t + \zeta(t) \leq M\} \).

The rest of this paper is organized as follows. In section 2, we will first study the Cauchy problem of system (1.1) with initial data (1.6) and prove Theorem 1.1. In section 3, we focus on the Cauchy problem of system (1.1) with initial data (1.7) and prove Theorem 1.4.

2 Cauchy problem with Scenario (1.6)

In this section, we shall prove Theorem 1.1. The proof relies on delicate constructions of sub-solution and super-solution, which are presented in subsection 2.1 and subsection 2.2, respectively. The proof of Theorem 1.1 is given in subsection 2.3.

2.1 Construction of sub-solution

We look for a sub-solution \((u, \bar{v})\) in the form of:
\[
\begin{align*}
u(t, x) &= U(x - c^* t + \zeta(t)) - P(t) \min\{e^{-\alpha_0(x-c^*t+x_0)}, 1\}, \\
\bar{v}(t, x) &= V(x - c^* t + \zeta(t)) + Q(t),
\end{align*}
\] (2.1)
Case 1: We first consider \((t, x) \in \Omega_1\). Then, for some small \(\delta > 0\),

\[
\Omega_1 = \{(t, x) \mid 0 \leq U \leq \delta, \ 1 - \delta \leq V \leq 1\}.
\]

Note that, by setting any \(\zeta_0 > 0\) and \(x_0 = \zeta_0 + M\), we have \(\{x = \Gamma(t)\} \subset \Omega_2\) for all \(t \geq 0\), and hence \(\Omega_1 \subset \Omega_2\).

Since \(\delta > 0\) can be chosen arbitrarily small and \(PW \to 0\) uniformly as \(t \to \infty\), by setting

\[
\alpha_0 \in (-\lambda^+ (e^*), -\lambda^- (e^*)),
\]

we have \((\mu_0 - \alpha_0 e^* + \alpha_0^2 + (1 - \alpha + a\delta) + PW)PW \leq -C_0PW\) with some \(\mu_0 > 0\) and \(C_0 > 0\) for all large \(t\). Moreover, by applying Lemma 1.3, there exists \(C_1 > 0\) such that

\[
U' \leq -C_1 U \text{ for all } (t, x) \in \Omega_1.
\]

From (2.2) and (2.5), we can obtain that

\[
\delta > \begin{cases} \Omega_1[\bar{w}, \bar{v}] \leq -C_1 \tau_0 e^{-\tau_0 t} U - C_0 PW + a QU. 
\end{cases}
\]

Therefore, by setting

\[
\tau_0 < \mu_0,
\]

it holds \(N_1[\bar{w}, \bar{v}] \leq 0\) for all \((t, x) \in \Omega_1\) and \(t \geq T\) for some \(T \gg 1\).

Next, we deal with the inequality of \(N_2[\bar{w}, \bar{v}]\). Since \(V' > 0\), \(\zeta' > 0\) and \(\Omega_1 \subset \Omega_2\), (2.3) implies that

\[
N_2[\bar{w}, \bar{v}] \geq r(2V - bP - 1 - \frac{\mu_0}{r})Q - kr PVW.
\]

Since \(V \geq 1 - \delta\) in \(\Omega_1\), by choosing \(\mu_0 < r/2\), it holds that

\[
N_2[\bar{w}, \bar{v}] \geq r \left[1 - 2\delta - b p_0 e^{-\mu_0 t} - \frac{\mu_0}{r}\right] q_0 e^{-\mu_0 t} - b r p_0 e^{-\mu_0 t} W
\]

\[
\geq r \left[\frac{1}{2} \left(1 - \frac{2\mu_0}{r}\right) q_0 - b p_0 W\right] e^{-\mu_0 t}
\]

for all large \(t\), where \(\delta > 0\) is chosen smaller if necessary. Therefore, by setting

\[
\mu_0 < \frac{r}{2} \quad \text{and} \quad \frac{1}{2} \left(1 - \frac{2\mu_0}{r}\right) q_0 > b p_0 e^{-\alpha_0 (M - \zeta_0 + x_0)} = b p_0 e^{-2\alpha_0 M} \quad \text{(since} \ x_0 = \zeta_0 + M),
\]

there exists \(T \gg 1\) such that \(N_2[\bar{w}, \bar{v}] \geq 0\) for all \((t, x) \in \Omega_1\) and \(t \geq T\).

Case 2: We consider \((t, x) \in \Omega_2\). Then, for some small \(\delta > 0\),

\[
\Omega_2 = \{(t, x) \mid 1 \geq U \geq 1 - \delta, \ 1 - \delta \leq V \geq 0\}.
\]

Since \(\zeta' > 0\), \(U' < 0\) and \(P \to 0\) uniformly as \(t \to \infty\), for \((t, x) \in \Omega_2 \cap \Omega_1\), from (2.2), we have

\[
N_1[\bar{w}, \bar{v}] \leq (\mu_0 - \alpha_0 e^* + \alpha_0^2 + 1 - 2U + PW)PW + a QU
\]

\[
\leq (\mu_0 - \alpha_0 e^* + \alpha_0^2 + PW)PW + [1 - 2(1 - \delta)]PW + a Q.
\]

Moreover, for \((t, x) \in \Omega_2 \cap \Omega_1\), we have \(PW \geq P e^{-\alpha_0 (M - \zeta(t) + x_0)} = P e^{-\alpha_0 (t^0)}\). Therefore, by setting \(\alpha_0\) as (2.4) and

\[
\mu_0 < 1 - a \quad \text{and} \quad p_0 > a q_0,
\]

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it holds \( N_1[u, \bar{v}] \leq 0 \) for all \((t, x) \in S_1 \cap \Omega_2 \) and \( t \geq T \) for some \( T \gg 1 \).

On the other hand, for \((t, x) \in \Omega_2 \cap S_2 \), we have \( \min\{W, 1\} = 1 \) and thus

\[
N_1[u, \bar{v}] \leq (\mu_0 + 1 + P - 2U)P + aQU \leq [2 - a + P - 2(1 - \delta)]P + aQ,
\]

where we used \( \mu_0 < 1 - a \). Therefore, by setting

\[
\mu_0 < 1 \quad \text{and} \quad p_0 > q_0,
\]

there exists \( T \gg 1 \) such that \( N_1[u, \bar{v}] \leq 0 \) for all \((t, x) \in S_2 \cap \Omega_2 \) and \( t \geq T \) for some \( T \gg 1 \).

Next, we will deal with the inequality of \( N_2[u, \bar{v}] \). To verify \( N_2[u, \bar{v}] \geq 0 \), we first observe that, from Lemma [1.9] there exists \( C_4 > 0 \) such that \( \zeta'V' \geq C_4\zeta'V \). Then, from (2.4), we have

\[
N_2[u, \bar{v}] \geq C_4\zeta'V + r(bU - 1 - bP - \frac{\mu_0}{r})Q - brPV.
\]

Thus, by setting \( \mu_0 < r(b - 1) \), since \( P \to 0 \) uniformly as \( t \to \infty \), we have

\[
N_2[u, \bar{v}] \geq C_4\zeta'V - brPV.
\]

Therefore, by setting

\[
\tau_0 < \mu_0 < r(b - 1),
\]

it holds \( N_2[u, \bar{v}] \geq 0 \) for all \((t, x) \in \Omega_2 \) and \( t \geq T \) for some \( T \gg 1 \).

**Case 3:** We consider \((t, x) \in \Omega_3 \). Then, for some small \( \delta_i > 0 \) \((i = 1, 2)\),

\[
\Omega_3 = \{(t, x) \mid 1 - \delta_1 \geq U \geq \delta_2, \ 1 - \delta_2 \geq V \geq \delta_1\}.
\]

In this range, there exists \( C_2 > 0 \) such that \( U' \leq -C_2 \), which implies that \( \zeta'U' \leq -C_2\zeta' \). Therefore, we have

\[
N_1[u, \bar{v}] \leq -C_2\zeta' + C_3P + aQU.
\]

Then, for \( \tau_0 \) and \( \mu_0 \) satisfying (2.6), it holds \( N_1[u, \bar{v}] \leq 0 \) for all \((t, x) \in \Omega_3 \) and \( t \geq T \) for some \( T \gg 1 \).

Next, we deal with the inequality of \( N_2[u, \bar{v}] \). Note that, in this range, we have \( V' \geq C_4 > 0 \), which implies that \( \zeta'V' \geq C_5\zeta' \). Therefore, we have

\[
N_2[u, \bar{v}] \geq C_4\zeta' - C_5Q - brP.
\]

Similarly, for \( \tau_0 \) and \( \mu_0 \) satisfying (2.6), it holds \( N_2[u, \bar{v}] \geq 0 \) for all \((t, x) \in \Omega_3 \) and \( t \geq T \) for some \( T \gg 1 \).

By concluding the conditions (2.4), (2.6), (2.7), (2.8), (2.9), (2.10) provided from the discussion above, we get a key lemma as follow:

**Lemma 2.1** For any \( \alpha_0, \mu_0, \tau_0, x_0, \zeta_0 > 0 \) satisfying

1. \( \alpha_0 \in (\lambda_1^+(e^*), -\lambda_2^-(e^*)) \),
2. \( \tau_0 < \mu_0 < \min\{1 - a, r(b - 1), \frac{\zeta_0}{2}\} \),
3. \( x_0 - \zeta_0 > 0 \) sufficiently large,
then there exists \( p_0 > 0, q_0 > 0 \) and \( T \geq 0 \) such that

\[
N_1[u, \bar{v}] \leq 0 \quad \text{and} \quad N_2[u, \bar{v}] \geq 0 \quad \text{in} \quad [T, \infty) \times \mathbb{R},
\]

where \((\bar{u}, \bar{v})\) is defined as \((2.1)\) with \( c = c^* \).

**Remark 2.2** Note that, from the proof of Lemma 2.7, \( T \) always can be chosen independently of all small \( \mu_0 > 0 \). More precisely, if \( \mu_0 > 0 \) becomes smaller, we can choose smaller \( p_0 \) and \( q_0 \) such that the differential inequalities still holds for \( t \geq T \) with the same \( T \).

**Lemma 2.3** Let \((u, \bar{v})\) be defined as \((2.1)\) with \( \alpha_0, \mu_0, \tau_0, \zeta_0, x_0, p_0, q_0 \) satisfying the conditions in Lemma 2.7. Then there exist \( T_0, T^* > 0 \) such that the solution \((u, v)\) with initial data \((1.6)\) satisfies

\[
u(t + T^*, x) \geq u(t, x), \quad v(t + T^*, x) \leq \bar{v}(t, x) \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times [0, \infty).
\]

**Proof.** Let the parameters satisfy the conditions in Lemma 2.7 then we have

\[
N_1[u, \bar{v}] \leq 0 \quad \text{and} \quad N_2[u, \bar{v}] \geq 0 \quad \text{in} \quad [T_0, \infty) \times [0, \infty).
\]

Let us fix this \( T_0 > 0 \). From Lemma 1.11 by setting \( \mu_0 < \min\{k_1, k_2\} \) (see Remark 2.2), there exists \( T_1 \gg 1 \) such that

\[
\begin{align*}
u(t, 0) &\leq 1 - \frac{p_0e^{-\mu_0t} - 1 - C_1e^{-k_1(t+T_1)}}{1 - C_1e^{-k_1(T_1 + T_1)}} \leq u(t + T_1, 0) \quad \text{for all} \quad t \geq T_0, \\
\bar{v}(t, 0) &\geq \frac{q_0e^{\mu_0t} - \frac{1}{e^{-k_1(t+T_1)}}}{1 - C_1e^{-k_1(T_1 + T_1)}} \leq v(t + T_1, 0) \quad \text{for all} \quad t \geq T_0.
\end{align*}
\]

Next, by the definition of \((u, \bar{v})\), condition (1) of Lemma 2.1 and Lemma 1.8 we can choose \( \ell \gg 1 \) such that

\[
\begin{align*}
u(T_0, x) = 0, \quad \bar{v}(T_0, x) &\geq 1 + \frac{q_0}{2}e^{-\mu_0T_0} \quad \text{for all} \quad x \geq \ell.
\end{align*}
\]

Then, by Lemma 1.10 we can take \( T_2 \gg 1 \) such that \( u(T_0 + T_2, x) \geq u(T_0, x) \) and \( v(T_0 + T_2, x) \leq \bar{v}(T_0, x) \) for all \( x \geq \ell \).

For \( x \in [0, \ell] \), we can choose \( T_3 \gg 1 \) such that

\[
\begin{align*}
u(T_0, x) &\leq 1 - P(T_0) \min\{e^{-\alpha_0(x-c^*T_0+x_0)}, 1\} \leq u(t + T_3, x) \quad \text{for all} \quad x \in [0, \ell], \\
v(T_0 + T_3, x) &\leq \min_{x \in [0, \ell]} \bar{v}(T_0, x) \leq \bar{v}(T_0, x) \quad \text{for all} \quad x \in [0, \ell],
\end{align*}
\]

since \((u, v) \to (1, 0)\) as \( t \to \infty \) uniformly for \( x \in [0, \ell] \).

By the above discussion and setting \( T^* = \max\{T_1, T_2, T_3\} \), we can assert that

\[
\begin{align*}
u(T_0 + T^*, x) &\geq u(T_0, x), \quad v(T_0 + T^*, x) \leq \bar{v}(T_0, x) \quad \text{for all} \quad x \in [0, \infty), \\
u(t + T^*, x) &\geq u(t, x), \quad v(t + T^*, x) \leq \bar{v}(t, x) \quad \text{for all} \quad t \geq T_0.
\end{align*}
\]

Therefore, by applying the comparison principle, the proof is complete.

Furthermore, if we consider a sub-solution \((\bar{u}_s, \bar{v}_s)\) defined as

\[
\begin{align*}
\bar{u}_s(t, x) := U(-x - c^*t + \zeta(t)) - P(t) \min\{W(t, -x), 1\}, \\
\bar{v}_s(t, x) := V(-x - c^*t + \zeta(t)) + Q(t),
\end{align*}
\]

then, by repeating the above argument, we can obtain a lemma as follows:

**Lemma 2.4** Let \((\bar{u}_s, \bar{v}_s)\) be defined as \((2.11)\) with \( \alpha_0, \mu_0, \tau_0, \zeta_0, x_0, p_0, q_0 \) satisfying the conditions in Lemma 2.7. Then there exist \( T_0, T^* > 0 \) such that the solution \((u, v)\) with initial data \((1.6)\) satisfies

\[
u(t + T^*, x) \geq \bar{u}_s(t, x), \quad v(t + T^*, x) \leq \bar{v}_s(t, x) \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times (\infty, 0].
\]
2.2 Construction of super-solution

We look for a super-solution \((\bar{u}, \bar{v})\) in the form of

\[
\begin{align*}
\bar{u}(t, x) &:= U(x - c^* t - \zeta(t)) + P(t) \min\{e^{-\alpha_1(x - c^* t + x_1)}, 1\}, \\
\bar{v}(t, x) &:= V(x - c^* t - \zeta(t)) - Q(t),
\end{align*}
\]

(2.12)

where \(\zeta(t) := \zeta_1 - e^{-\tau_1 t}\), \(Q(t) = q_1 e^{-\mu_1 t}\), \(P(t) = p_1 e^{-\mu_1 t}\). The parameters \(\alpha_1, \mu_1, \tau_1, p_1, q_1, \zeta_1\) are positive constants and will be determined later. For the simplicity, we denote \(\zeta := x - c^* t - \zeta(t)\) and \(W(t, x) = e^{-\alpha_1(x - c^* t + x_1)}\). Next, we show that \((\bar{u}, \bar{v})\) is a super-solution by choosing suitable parameters.

Clearly, there exists a curve \(\Gamma(t) = c^* t - x_1 : [0, \infty) \to \mathbb{R}\) such that \(W(t, \Gamma(t)) = 1\). Thus, it holds

\[
W(t, x) \leq 1 \quad \text{for} \quad (t, x) \in S_1 := \{(t, x) \mid x \geq \Gamma(t)\},
\]

\[
W(t, x) \geq 1 \quad \text{for} \quad (t, x) \in S_2 := \{(t, x) \mid x \leq \Gamma(t)\}.
\]

Then by some straightforward computation, we obtain that, for \((t, x) \in S_1\), it holds

\[
N_1[\bar{u}, \bar{v}] = \left( -1 - \mu_1 + \alpha_1 c^* - \alpha_1^2 + 2U + a\bar{u} + PW \right) PW - \zeta'(t)U' - aQU. \tag{2.13}
\]

And for \((t, x) \in S_2\), it holds

\[
N_1[\bar{u}, \bar{v}] = \left( -1 - \mu_1 + 2U + a\bar{v} + P \right) P - \zeta'(t)U' - aQU.
\]

On the other hand, for \((t, x) \in [0, \infty) \times \mathbb{R}\) it holds

\[
N_2[\bar{u}, \bar{v}] = -\zeta V' + r(1 - 2V + Q - b\bar{u} + \frac{\mu_1}{r})Q + brPV \min\{W, 1\}. \tag{2.14}
\]

In the following discussion, we choose \(M > 0\) sufficiently large and divide the whole space into three parts:

1. \(\Omega_1 = \{x \geq c^* t - \zeta(t) + M\}\);
2. \(\Omega_2 := \{c^* t - \zeta(t) - M \geq x \geq 0\}\);
3. \(\Omega_3 := \{c^* t - \zeta(t) + M \geq x \geq c^* t - \zeta(t) - M\}\).

**Case 1:** We consider \((t, x) \in \Omega_1\) with \(M > 0\) sufficiently large. Then, for some small \(\delta > 0\),

\[
\Omega_1 = \{(t, x) \mid 0 \leq U \leq \delta, 1 - \delta \leq V \leq 1\}.
\]

Note that, by setting \(x_1 = \zeta_1 + M\), then for sufficiently large \(t\), we have \(\Gamma(t) \subset \Omega_2\), and hence \(\Omega_1 \subset S_1\).

Since \(\delta\) can be chosen arbitrarily small, by setting \(\alpha_1\) as \((2.4)\), we have

\[
-(\mu_1 - c\alpha_1 + \alpha_1^2 + 1 - 2U - a(V - Q) - PW)PW \geq C_0 PW
\]

with some \(\mu_1 > 0\) and \(C_0 > 0\). Moreover, by applying Lemma \((1.8)\) and Lemma \((1.9)\) there exists \(C_1 > 0\) such that \((2.5)\) holds. From \((2.5)\) and \((2.13)\), we can obtain that

\[
N_1[\bar{u}, \bar{v}] \geq C_1\zeta_1\tau_1 e^{-\tau_1 t}U - C_0 PW - aQU.
\]
Therefore, by setting
\[ \tau_1 < \mu_1, \]  
(2.15)
it holds \( N_1[\bar{u}, \bar{v}] \geq 0 \) for \((t, x) \in \Omega_1 \) and \( t \geq T \) for some \( T \gg 1 \).

Next, we deal with the inequality of \( N_2[\bar{u}, \bar{v}] \). Since \( V' > 0 \) and \( \zeta' > 0 \), from (2.14), we have
\[ N_2[\bar{u}, \bar{v}] \leq r(1 - 2V + Q + \frac{\mu_1}{r})Q + brVP \min\{W, 1\}. \]

Since \( Q \to 0 \) uniformly as \( t \to \infty \) and \( V \geq 1 - \delta \) in \( \Omega_1 \), by choosing \( \mu_1 < \frac{r}{2} \), it holds
\[ N_2[\bar{u}, \bar{v}] \leq -\frac{r}{4}Q + brPW. \]

Then, by setting
\[ \mu_1 < \frac{r}{2} \quad \text{and} \quad \frac{q_1}{4} > bp_1e^{-\alpha_1(M - \zeta_1 + x_1)} = bp_1e^{-2\alpha_1M}, \]
(2.16)
it holds \( N_2[\bar{u}, \bar{v}] \leq 0 \) for \((t, x) \in \Omega_1 \) and \( t \geq T \) for some \( T \gg 1 \).

**Case 2:** We consider \((t, x) \in \Omega_2 \). Then, for some small \( \delta > 0 \), \( \Omega_2 = \{(t, x) \mid 1 \geq U \geq 1 - \delta, \delta \geq V \geq 0\} \). Since \( \zeta' > 0 \), \( U' < 0 \) and \( P \to 0 \) uniformly as \( t \to \infty \), for \((t, x) \in \Omega_2 \cap S_1 \), from (2.13), we have
\[ N_1[\bar{u}, \bar{v}] \geq -(\mu_1 - c\alpha_1 + \alpha_1^2)PW - aQU. \]

Moreover, for \((t, x) \in \Omega_2 \cap S_1 \), we have \( PW \geq p(t)e^{-\alpha_1(x_1 - \zeta_1 - M)} \). Therefore, by setting \( \alpha_1 \) as (2.4) and
\[ \mu_1 < 1 - a \quad \text{and} \quad aq_1 < C_2p_1e^{-\alpha_1(x_1 - \zeta_1 - M)} = C_2p_1, \]
(2.17)
it holds \( N_1[\bar{u}, \bar{v}] \geq 0 \) for \((t, x) \in S_1 \cap \Omega_2 \) and \( t \geq T \) for some \( T \gg 1 \).

On the other hand, for \((t, x) \in \Omega_2 \cap S_2 \), we have
\[ N_1[\bar{u}, \bar{v}] \geq -(\mu_1 + 1 + P - 2U)P - aQU \geq C_3P - aQU. \]

Therefore, by setting
\[ \mu_1 < 1 \quad \text{and} \quad aq_1 < C_3p_1, \]
(2.18)
it holds \( N_1[\bar{u}, \bar{v}] \geq 0 \) for \((t, x) \in S_2 \cap \Omega_2 \) and \( t \geq T \) for some \( T \gg 1 \).

Next, we will deal with the inequality of \( N_2[\bar{u}, \bar{v}] \). To verify \( N_2[\bar{u}, \bar{v}] \leq 0 \), we first observe that, there exists \( C_4 > 0 \) such that \(-\zeta'V' \leq -C_4\zeta'V \). Then, from (2.14), we have
\[ N_2[\bar{u}, \bar{v}] \leq -C_4\zeta'V + r(1 - bU + Q + \frac{\mu_1}{r})Q + brPV. \]

Thus, by choosing \( \mu_1 < r(b - 1) \), since \( P \to 0 \) uniformly as \( t \to \infty \), we have
\[ N_2[\bar{u}, \bar{v}] \leq -C_4\zeta'V + brPV. \]

Therefore, by setting
\[ \tau_1 < \mu_1 < r(b - 1), \]
(2.19)
it holds \( N_2[\bar{u}, \bar{v}] \leq 0 \) for \((t, x) \in \Omega_2 \) and \( t \geq T \) for some \( T \gg 1 \).

**Case 3:** We consider \((t, x) \in \Omega_3 \). Then, for some small \( \delta_1, \delta_2 > 0 \), \( \Omega_3 = \{(t, x) \mid 1 - \delta_1 \geq U \geq \delta_1, 1 - \delta_2 \geq V \geq \delta_2\} \). In this range, there exists \( C_5 > 0 \) such that \( U' \leq -C_5 \), which implies that \(-\zeta'U' \geq C_5\zeta' \). Therefore, we have
\[ N_1[\bar{u}, \bar{v}] \geq C_5\zeta' - C_6P - aQU. \]
Then, for $\tau_1$ and $\mu_1$ satisfying (2.15), it holds $N_1[\bar{u}, \bar{v}] \geq 0$ for $(t, x) \in \Omega_3$ and $t \geq T$ for some $T \gg 1$.

Next, we will deal with the inequality of $N_2[\bar{u}, \bar{v}]$. We observe that, in this range, we have $V' \geq C_\gamma > 0$, which implies that $-\zeta' V' \leq -C_\gamma \zeta'$. Therefore, we have

$$N_2[\bar{u}, \bar{v}] \leq -C_\gamma \zeta' + C_\delta Q + b r P.$$ 

Similarly, for $\tau_1$ and $\mu_1$ satisfying (2.15), it holds $N_2[\bar{u}, \bar{v}] \leq 0$ for $(t, x) \in \Omega_3$ and $t \geq T$ for some $T \gg 1$.

By concluding the conditions (2.14), (2.15), (2.16), (2.17), (2.18), (2.19) provided from the discussion above, we get a key lemma as follow:

**Lemma 2.5** For any $\alpha_1, \mu_1, \tau_1, x_1, \zeta_1 > 0$ satisfying

(1) $\alpha_1 \in (-\lambda_u^+(c^*), -\lambda_u^-(c^*))$,

(2) $\tau_1 < \mu_1 < \min\{1 - a, r(b - 1), \frac{\tau}{\tau_1}\}$,

(3) $x_1 - \zeta_1$ sufficiently large,

then there exists $p_1 > 0, q_1 > 0$ and $T \geq 0$ such that

$$N_1[\bar{u}, \bar{v}] \geq 0$$ 

and

$$N_2[\bar{u}, \bar{v}] \leq 0$$ 

in $[T, \infty) \times \mathbb{R}$,

where $(\bar{u}, \bar{v})$ is defined as (2.12).

**Lemma 2.6** Let $(\bar{u}, \bar{v})$ be defined as (2.12), and $\alpha_1, \mu_1, \tau_1, \zeta_1, x_1, p_1, q_1$ satisfies the conditions in Lemma 2.5. Then there exist $T^* > 0$ such that the solution $(u, v)$ with initial data (1.6) satisfies

$$u(t, x) \leq \bar{u}(t + T^*, x), \quad v(t, x) \geq \bar{v}(t + T^*, x)$$ 

for all $(t, x) \in [0, \infty) \times [0, \infty)$.

**Proof.** First, by setting the parameters as that in Lemma 2.5, we have

$$N_1[\bar{u}, \bar{v}] \geq 0$$ 

and

$$N_2[\bar{u}, \bar{v}] \leq 0$$ 

in $[T^*, \infty) \times [0, \infty)$.

Let us fix this $T^* > 0$.

Then, from Lemma 1.9 and Lemma 1.10, for any $\mu_1 < \min\{1, c^* \mu_u^+(c^*), c^* \mu_u^-(c^*)\}$, there exists $\zeta_1 > 0$ such that for $t \geq 0$, we have

$$\bar{u}(t + T^*, 0) = U(-c^*(t + T^*) - \zeta(t + T^*)) + P(t + T^*) \geq 1 \geq u(t, 0),$$

$$\bar{v}(t + T^*, 0) = V(-c^*(t + T^*) + \zeta(t + T^*)) - Q(t + T^*) \leq 0 \leq v(t, 0).$$

Next, we will deal with the inequality of $N_2$. Then, from the construction of $(\bar{u}, \bar{v})$ and (1.6), up to increasing $\zeta_1$ if it is necessary, we have

$$u(0, x) \leq \bar{u}(T^*, x) \quad \text{and} \quad v(0, x) \geq \bar{v}(T^*, x)$$ 

for all $x \in [0, \infty)$.

Therefore, by applying the comparison principle, the proof is complete.

Furthermore, if we consider a sub-solution $(\bar{u}_s, \bar{v}_s)$ defined as

$$\left\{
\begin{array}{ll}
\bar{u}_s(t, x) := U(-c^*t + \zeta(t)) + P(t) \min\{W(t, -x), 1\}, \\
\bar{v}_s(t, x) := V(-c^*t + \zeta(t)) - Q(t),
\end{array}
\right.$$ 

(2.20)

then, by repeating the above argument, we can obtain a lemma as follow:

**Lemma 2.7** Let $(\bar{u}_s, \bar{v}_s)$ be defined as (2.20) and $\alpha_1, \mu_1, \tau_1, \zeta_1, x_1, p_1, q_1$ satisfy the conditions in Lemma 2.5. Then there exists $T^* > 0$ such that the solution $(u, v)$ with initial data (1.6) satisfies

$$u(t, x) \leq \bar{u}_s(t + T^*, x), \quad v(t, x) \geq \bar{v}_s(t + T^*, x)$$ 

for all $(t, x) \in [0, \infty) \times (-\infty, 0]$. 18
2.3 Proof of Theorem 1.1

Let us set \( \xi := x - c^* t \). Then we can write the solution of system (1.3) as

\[
(\tilde{u}, \tilde{v})(t, \xi) = (u, v)(t, x) = (u, v)(t, \xi + c^* t), \quad t > 0, \; \xi \in \mathbb{R},
\]

which satisfies

\[
\begin{aligned}
\partial_t \tilde{u} &= \tilde{u}_{\xi\xi} + c^* \tilde{u}_\xi + \tilde{u}(1 - \tilde{u} - a \tilde{v}), \quad t > 0, \; \xi \in \mathbb{R}, \\
\partial_t \tilde{v} &= d \tilde{v}_{\xi\xi} + c^* \tilde{v}_\xi + r \tilde{v}(1 - \tilde{v} - b \tilde{u}), \quad t > 0, \; \xi \in \mathbb{R}.
\end{aligned}
\]

Thanks to Lemma 2.3 and Lemma 2.6 we can immediately obtain the following result.

**Lemma 2.8** Let \((c^*, U, V)\) be the minimal traveling wave of system (1.3). Then there exist constants \(p_2, q_2, \alpha_2, \mu_2, x_2, \zeta_2\) and \(K_i (i = 1, 2, 3, 4)\), and \(T > 0\) such that

\[
\begin{aligned}
\tilde{u}(t, \xi) &\geq U \left( \xi + \zeta_2 - K_1 e^{-\frac{-4 \xi}{\alpha_2 t}} \right) - K_2 p_2 e^{-\mu_2 t} \min \{e^{-\alpha_2 (\xi + x_2)}, 1\}, \\
\tilde{u}(t, \xi) &\leq U \left( \xi - \zeta_2 + K_3 e^{-\frac{-4 \xi}{\alpha_2 t}} \right) + K_4 q_2 e^{-\mu_2 t} \min \{e^{-\alpha_2 (\xi + x_2)}, 1\}, \\
\tilde{v}(t, \xi) &\leq V \left( \xi + \zeta_2 - K_1 e^{-\frac{-4 \xi}{\alpha_2 t}} \right) + K_2 q_2 e^{-\mu_2 t}, \\
\tilde{v}(t, \xi) &\geq V \left( \xi - \zeta_2 + K_3 e^{-\frac{-4 \xi}{\alpha_2 t}} \right) - K_4 q_2 e^{-\mu_2 t}.
\end{aligned}
\]

for \(\xi \geq -c^* t\) and \(t \geq T\).

From the construction of sub-solution and super-solution, we actually establish the local stability of traveling waves in the following sense:

**Lemma 2.9** Let \((c^*, U, V)\) be a solution of (1.3). Then there exists a function \(\nu(\varepsilon)\) defined for small \(\varepsilon\) with \(\nu(\varepsilon) \to 0\) as \(\varepsilon \to 0\) satisfying the following property: if

\[
|\tilde{u}(t_*, \xi) - U(\xi - \xi_*)| + |\tilde{v}(t_*, \xi) - V(\xi - \xi_*)| < \varepsilon\quad \text{for all}\quad \xi \in \mathbb{R},
\]

for some \(t_*, \xi_* \in \mathbb{R}\), then

\[
|\tilde{u}(t, \xi) - U(\xi - \xi_*)| + |\tilde{v}(t, \xi) - V(\xi - \xi_*)| < \nu(\varepsilon)\quad \text{for all}\quad (t, \xi) \in [t_*, \infty) \times \mathbb{R}.
\]

**Proof.** In the proof of Lemma 2.1 and Lemma 2.5 we may choose \(q_i = O(\varepsilon), p_i = O(\varepsilon)\) and \(|\xi_i - \xi_*| = O(\varepsilon), i = 0, 1\), such that \((u, v)(t, x)\) can be compared with the sub-solution and super-solution constructed in Lemma 2.1 and Lemma 2.5 in terms of \((U, V)\), respectively, from \(t = t_*\). Therefore, Lemma 2.9 follows from the comparison principle.

Now we are ready to prove Theorem 1.1. Let \((\tilde{u}, \tilde{v})\) defined as (2.21) and \((c^*, U, V)\) be the minimal traveling wave of system (1.3). Let \(\{t_n\}\) be an arbitrary sequence satisfying \(t_n \to \infty\) as \(n \to \infty\). Set

\[
\tilde{u}_n(t, \xi) = \tilde{u}(t + t_n, \xi), \quad \tilde{v}_n(t, \xi) = \tilde{v}(t + t_n, \xi), \quad n \in \mathbb{N}.
\]

By the standard parabolic regularity theory, up to extraction of a subsequence, we have \((\tilde{u}_n, \tilde{v}_n) \to (u^\infty, v^\infty)\) locally uniformly as \(n \to \infty\), and \((u^\infty, v^\infty)\) satisfies

\[
\begin{aligned}
\partial_t u^\infty &= u^\infty_{\xi\xi} + c^* u^\infty_\xi + u^\infty(1 - u^\infty - av^\infty), \quad t \in \mathbb{R}, \; \xi \in \mathbb{R}, \\
\partial_t v^\infty &= dv^\infty_{\xi\xi} + c^* v^\infty_\xi + rv^\infty(1 - v^\infty - bu^\infty), \quad t \in \mathbb{R}, \; \xi \in \mathbb{R}.
\end{aligned}
\]
In addition, by replacing \( t \) by \( t + n \) in the inequalities of Lemma 2.8 we have, for all \((t, \xi) \in \mathbb{R} \times \mathbb{R}\),

\[ U(\xi + \zeta_2) \leq u^\infty(t, \xi) \leq U(\xi - \zeta_2) \quad \text{and} \quad V(\xi - \zeta_2) \leq v^\infty(t, \xi) \leq V(\xi + \zeta_2). \quad (2.23) \]

Note that (2.23) indicates that \((u^\infty, v^\infty)\) is trapped between two shifts of the minimal traveling wave. The following lemma shows that \((u^\infty, v^\infty)\) is exactly the minimal wave with a translation. The proof is based on a sliding method (see [5]).

**Lemma 2.10** There exists \( \bar{\zeta} \in [-\zeta_2, \zeta_2] \) such that

\[ u^\infty(t, \xi) = U(\xi - \bar{\zeta}) \quad \text{and} \quad v^\infty(t, \xi) = V(\xi - \bar{\zeta}) \quad \text{for all} \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}. \]

**Proof.** We choose \( \delta > 0 \) small and let \( A > 0 \) such that

\[ 1 - \delta \leq U(\xi + \zeta_2) \leq 1, \quad 0 \leq V(\xi + \zeta_2) \leq \delta \quad \text{for all} \quad \xi \leq -A. \quad (2.24) \]

For any fixed \( T \in \mathbb{R} \), we denote

\[ w^\sigma_u(t, \xi) = u^\infty(t + T, \xi + \sigma), \quad w^\sigma_v(t, \xi) = v^\infty(t + T, \xi + \sigma) \]

for all \( \sigma \in \mathbb{R} \) and \((t, \xi) \in \mathbb{R} \times \mathbb{R}\). Define now \( \sigma^* = \inf \mathcal{A} \), where

\[ \mathcal{A} := \{ \sigma \in \mathbb{R} \mid w^\sigma_u \leq u^\infty, \ w^\sigma_v \geq v^\infty \text{ in } \mathbb{R} \times \mathbb{R} \text{ for all } \sigma' \geq \sigma \}. \]

Lemma 2.8 implies that \( w^\sigma_u \leq u^\infty \) and \( w^\sigma_v \geq v^\infty \) for \((t, \xi) \in \mathbb{R} \times \mathbb{R} \) and for all \( \sigma \geq 2\zeta_2 \). Thus, \( \mathcal{A} \) is non-empty. Moreover, since \((U, V)(-\infty) = (1, 0)\) and \((U, V)(+\infty) = (0, 1)\), we see that \( \mathcal{A} \) is bounded from below. Thus, \( \sigma^* \) is well defined and is finite. Moreover, by continuity, we have

\[ w^\sigma_u \leq u^\infty, \quad w^\sigma_v \geq v^\infty \quad \text{for all} \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}. \quad (2.25) \]

Define \( E_1 := \{(t, \xi) \in \mathbb{R} \times [-A, \infty)\} \) and \( E_2 := \{(t, \xi) \in \mathbb{R} \times (-\infty, -A]\} \). We now prove the following key result:

**Claim 2.11** There exists no \( \eta_0 > 0 \) such that

\[ w^{\sigma^* - \eta}_u \leq u^\infty \quad \text{and} \quad w^{\sigma^* - \eta}_v \geq v^\infty \quad \text{in} \quad E_1 \quad \text{for all} \quad \eta \in [0, \eta_0]. \]

**Proof.** Assume that such a \( \eta_0 \) exists. We shall show that it would also hold

\[ w^{\sigma^* - \eta}_u \leq u^\infty \quad \text{and} \quad w^{\sigma^* - \eta}_v \geq v^\infty \quad \text{in} \quad E_2 \quad \text{for all} \quad \eta \in [0, \eta_0]. \quad (2.26) \]

Define

\[ \epsilon^*_u = \inf \{ \epsilon > 0 \mid u^\infty + \epsilon \geq w^{\sigma^* - \eta}_u \text{ for all } (t, \xi) \in E_2 \}, \]

\[ \epsilon^*_v = \inf \{ \epsilon > 0 \mid v^\infty - \epsilon \leq w^{\sigma^* - \eta}_v \text{ for all } (t, \xi) \in E_2 \}. \]

Then the real numbers \( \epsilon^*_u \) and \( \epsilon^*_v \) are nonnegative. To show that \( \epsilon^*_u = \epsilon^*_v = 0 \), we first assume \( \epsilon^*_u \geq \epsilon^*_v > 0 \). Since \( w^{\sigma^* - \eta}_u \leq u^\infty \) for \( \xi = -A \), there exist sequences \( \{\xi_n\} \) which converges to \( \xi_\infty \in (-\infty, -A) \cup \{-\infty\} \) and \( \{t_n\} \subset \mathbb{R} \) such that

\[ u^\infty(t_n, \xi_n) + \epsilon^*_u - w^{\sigma^* - \eta}_u(t_n, \xi_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad v^\infty(t_n, \xi_n) - \epsilon^*_v \leq w^{\sigma^* - \eta}_v. \quad (2.27) \]
Since \( U(-\infty) = 1 \) and (2.28), we assert that \( \xi_\infty \neq -\infty \). Set
\[
(u_n^\infty, v_n^\infty)(t, \xi) := (u_0^\infty, v_0^\infty)(t + t_n, \xi), \quad (w_{u,n}^{\sigma^* - \eta}, w_{v,n}^{\sigma^* - \eta})(t, \xi) := (w_{u,n}^{\sigma^* - \eta}, w_{v,n}^{\sigma^* - \eta})(t + t_n, \xi).
\]

Then from the standard parabolic estimates, \((u_n^\infty, v_n^\infty)\) and \((w_{u,n}^{\sigma^* - \eta}, w_{v,n}^{\sigma^* - \eta})\) converge locally uniformly, up to extraction of a subsequence, to a solution \((\bar{u}^\infty, \bar{v}^\infty)\) of (2.22) and \((\bar{w}_u^{\sigma^* - \eta}, \bar{w}_v^{\sigma^* - \eta})\), respectively, such that
\[
z(t, \xi) := \bar{u}^\infty(t, \xi) + \varepsilon_u^* - \bar{w}_u^{\alpha^* - \eta}(t, \xi) \geq 0 \quad \text{for all} \quad (t, x) \in E_2.
\]
Moreover, due to (2.27), we have \( z(0, \xi_\infty) = 0 \) and
\[
\bar{v}^\infty - \varepsilon_v^* \leq \bar{w}_v^{\alpha^* - \eta} \quad \text{for all} \quad (t, \xi) \in E_2. \tag{2.28}
\]

Recall \( F \) from (1.11). Since \( \delta > 0 \) is chosen small enough, it follows that \( F(u, v) \) is decreasing in both \( u \) and \( v \) for \((u, v) \in D := \{ 1 - \delta \leq u \leq 1, 0 \leq v \leq \delta \} \). Also, note that, for \( \xi \leq -A \), we have \((\bar{u}^\infty, \bar{v}^\infty) \in D \). Using \( a < 1 \), it follows from some straightforward computation that
\[
\partial_t z - z_{\xi \xi} - c^* z_\xi = F(\bar{u}^\infty, \bar{v}^\infty) - F(\bar{w}_u^{\sigma^* - \eta}, \bar{w}_v^{\sigma^* - \eta}) \geq F(\bar{u}^\infty + \varepsilon_u^*, \bar{v}^\infty - \varepsilon_v^*) - F(\bar{w}_u^{\sigma^* - \eta}, \bar{w}_v^{\sigma^* - \eta}).
\]

By (2.28), the Lipschitz continuity, and monotonicity of \( F(u, v) \) in \( u \), there exists \( C_\delta > 0 \) such that
\[
\partial_t z - z_{\xi \xi} - c^* z_\xi \geq F(\bar{u}^\infty + \varepsilon_u^*, \bar{w}_v^{\sigma^* - \eta}) - F(\bar{u}_u^{\sigma^* - \eta}, \bar{w}_v^{\sigma^* - \eta}) \geq -C_\delta z
\]
for all \( \xi \leq -A \). Since \( z(0, \xi_\infty) = 0 \), the strong maximum principle implies \( z(t, \xi) = 0 \) for all \( t \leq 0 \) and \( \xi \leq -A \). However, this is contradict to \( z(t, -A) = \varepsilon_u^* > 0 \). Therefore, \( \varepsilon_u^* \geq \varepsilon_v^* > 0 \) is impossible.

If \( \varepsilon_v^* \geq \varepsilon_u^* > 0 \), by repeating the similar argument and considering the function
\[
z(t, \xi) := \bar{w}_u^{\sigma^* - \eta}(t, \xi) - \varepsilon_v^* - \bar{v}^\infty(t, \xi),
\]
we also can prove that \( \varepsilon_v^* \geq \varepsilon_u^* > 0 \) is impossible. Therefore, we assert that \( \varepsilon_u^* = \varepsilon_v^* = 0 \) and (2.26) holds. It follows that \( \sigma^* - \eta_0 \in \mathcal{A} \), which contradicts to the definition of \( \sigma^* \). Therefore, we complete the proof of Claim 2.11.

We next show that \( \sigma^* \leq 0 \). For contradiction, we assume that \( \sigma^* > 0 \). From Claim 2.11 there exist two sequences \( \{\sigma_n\}_{n \in \mathbb{N}} \) in \((0, \sigma^*)\) and \( \{(\tau_n, \xi_n)\}_{n \in \mathbb{N}} \subset \{\xi \geq -A\} \) such that \( \sigma_n \to \sigma^* \) as \( n \to +\infty \), and it holds that
\[
w_{u,n}^{\sigma_n}(\tau, \xi_n) \geq u^{\infty}(\tau, \xi_n) \quad \text{for all} \quad n \in \mathbb{N} \tag{2.29}
\]
or
\[
w_{v,n}^{\sigma_n}(\tau, \xi_n) \leq v^{\infty}(\tau, \xi_n) \quad \text{for all} \quad n \in \mathbb{N}.
\]

**Claim 2.12** \( \{\xi_n\} \) must be bounded.
for all $C$ there exists

Thus, we have

Since $U > 0$ large enough such that, for each $n \geq N$, we have $\sigma^* - \sigma_n < 1$ and $\xi_n \geq A$ and $w^\infty(\tau_n, \xi_n) \in [0, \delta]$ because of (2.23) and (2.24). Moreover, from Lemma 1.8 and (2.23), there exists a constant $C_1 > 0$ such that, for any $t_0 \in \mathbb{R}$ and $\xi_0 \geq -A + 2$, it holds

$$\max_{t_0-1 \leq t \leq t_0, \xi} u^\infty(t, \xi) \leq \min_{t_0-1 \leq t \leq t_0, \xi} C_1 u^\infty(t, \xi).$$

(2.30)

Using (2.29), (2.30) and the standard parabolic estimates, there exists $C_2 > 0$ such that, for $n \geq N$, it holds

$$0 \leq u^\infty(\tau_n, \xi_n) - u^\infty(\tau_n + T, \xi_n + \sigma^*)$$

$$\leq w_{\sigma_n}^\sigma(\tau_n, \xi_n) - w_{\sigma_n}^\sigma(\tau_n, \xi_n)$$

$$\leq C_2(\sigma^* - \sigma_n) \max_{\tau_n-1 \leq t \leq \tau_n, \xi} w_{\sigma_n}^\sigma(t, \xi)$$

$$\leq C_1 C_2(\sigma^* - \sigma_n) w_{\sigma_n}^\sigma(\tau_n, \xi_n)$$

$$\leq C_1 C_2(\sigma^* - \sigma_n) U(\xi_n + \sigma^* - \zeta_1),$$

where the last inequality follows from (2.23).

Now, let us first assume $T > 0$. Then, from the regularity of $u^\infty$, Lemma 1.8 and (2.23), there also exists $C_3 > 0$ such that

$$(u^\infty - w_{\sigma_n}^\sigma)(t - T, \xi - \sigma^*) \leq C_3(u^\infty - w_{\sigma_n}^\sigma)(t, \xi) \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}.$$  

Thus, we have

$$u^\infty(\tau_n - kT, \xi_n - k\sigma^*) - u^\infty(\tau_n - (k - 1)T, \xi_n - (k - 1)\sigma^*) \leq C_1 C_2 C_3^k (\sigma^* - \sigma_n) U(\xi_n + \sigma^* - \zeta_1),$$

for all $k \in \mathbb{N}$ and $n \geq N$, whence

$$u^\infty(\tau_n - kT, \xi_n - k\sigma^*) - u^\infty(\tau_n + T, \xi + \sigma^*) \leq C_1 C_2 \left[ \sum_{i=0}^{k} C_3^i \right] (\sigma^* - \sigma_n) U(\xi_n + \sigma^* - \zeta_1).$$

From (2.23), we have

$$U(\xi_n + \zeta_2 - k\sigma^*) \leq \left[ 1 + C_1 C_2 \sum_{i=0}^{k} C_3^i (\sigma^* - \sigma_n) \right] U(\xi_n + \sigma^* - \zeta_2),$$

(2.31)

for all $k \in \mathbb{N}$ and $n \geq N$. Since $\sigma^* > 0$, we can find $k$ such that $-k\sigma^* < \sigma^* - 2\zeta_2$. From Lemma 1.8 there exists $C_4 > 0$ such that $U(s + 2\zeta_2 - k\sigma^*) \geq (1 + C_4) U(s + \sigma^* - 2\zeta_2)$ for all $s$ sufficiently large. Since $U > 0$ and $\xi_n \to +\infty, \sigma_n \to \sigma^*$ as $n \to +\infty$, (2.31) is impossible to hold for large $n$. For the case $T < 0$, we can get a contradiction by applying the same argument. Therefore, we complete the proof of Claim 2.12.

Due to Claim 2.12 up to extraction of a subsequence we may assume that $\xi_n \to \xi_* \in [-A, +\infty)$ as $n \to \infty$. Consider $(u_n^\infty, v_n^\infty)(t, \xi) := (u^\infty, v^\infty)(t + \tau_n, \xi)$. By standard parabolic estimates, up to extraction of a subsequence, we have that $(u_n^\infty, v_n^\infty)$ converge locally uniformly in $\mathbb{R} \times \mathbb{R}$ to a solution $(\bar{u}^\infty, \bar{v}^\infty)$ of (2.22). Furthermore, in view of (2.25), we have

$$z_u(t, \xi) = \bar{u}^\infty(t, \xi) - \bar{u}^\infty(t + T, \xi + \sigma^*) \geq 0 \text{ in } \mathbb{R} \times \mathbb{R},$$

$$z_v(t, \xi) = \bar{v}^\infty(t, \xi) - \bar{v}^\infty(t + T, \xi + \sigma^*) \leq 0 \text{ in } \mathbb{R} \times \mathbb{R}.$$
Note that \(z_u(0, \xi_*) = 0\). Then the strong maximum principle and uniqueness of solutions to the Cauchy problem for (2.22) imply that \(z_u \equiv 0\) in \(\mathbb{R} \times \mathbb{R}\), whence \(\bar{u}_*(t, \xi) = \bar{u}^\infty(t + T, \xi + \sigma^*)\) in \(\mathbb{R} \times \mathbb{R}\). In particular, \(\bar{u}_*(0, 0) = \bar{u}^\infty(jT, j\sigma^*)\) for all \(j \in \mathbb{Z}\). However, thanks to (2.23), we see that
\[
U(\xi + \zeta_0) \leq \bar{u}(t, \xi) \leq U(\xi - \zeta_0) \quad \text{for all} \quad (t, \xi) \in \mathbb{R} \times \mathbb{R}.
\]
Since \(\sigma^* > 0\) (here we actually use \(\sigma^* \neq 0\)) and \(U(-\infty) = 1 > 0 = U(+\infty)\), we have reached a contradiction.

From the above discussions, we have proved that \(\sigma^* \leq 0\). Thus, for all \(\sigma \geq 0\), we have
\[
\bar{u}_*(t, \xi) \geq w^\sigma_u(t, \xi) = u^\infty(t + T, \xi + \sigma), \quad v^\infty(t, \xi) \leq w^\sigma_v(t, \xi) = u^\infty(t + T, \xi + \sigma),
\]
for all \((t, \xi) \in \mathbb{R} \times \mathbb{R}\). Furthermore, since \(T \neq 0\) can be chosen arbitrarily, it follows that
\[
\bar{u}_*(t, \xi) = \phi_u(\xi), \quad v^\infty(t, \xi) = \phi_v(\xi) \quad \text{for all} \quad (t, \xi) \in \mathbb{R} \times \mathbb{R},
\]
for some nonincreasing function \(\phi_u(\xi)\) and nondecreasing function \(\phi_v(\xi)\). On the other hand, the strong maximum principle implies that it holds either
\[
v^\infty(t, \xi) > u^\infty(t + T, \xi + \sigma) \quad \text{(resp.,} \quad v^\infty(t, \xi) < v^\infty(t + T, \xi + \sigma) \quad \text{)}
\]
or
\[
v^\infty(t, \xi) \equiv u^\infty(t + T, \xi + \sigma) \quad \text{(resp.,} \quad v^\infty(t, \xi) \equiv v^\infty(t + T, \xi + \sigma) \quad \text{)}.
\]
By (2.23), \((\phi_u(\xi), \phi_v(\xi))\) satisfies \((\phi_u(-\infty), \phi_v(-\infty)) = (1, 0)\) and \((\phi_u(+\infty), \phi_v(+\infty)) = (0, 1)\). Thus, (2.33) is impossible; so we assert that both \(\phi_u(\cdot)\) and \(\phi_v(\cdot)\) are strictly monotone functions. Therefore, \((\phi_u(\cdot), \phi_v(\cdot))\) forms a strictly monotone traveling wave solution, and is trapped between two shifts of minimal traveling waves (due to (2.23)). The standard sliding method (see, e.g., [15, Proposition A.7]) yields that for some \(\zeta\),
\[
\phi_u(\xi) = U(\xi - \zeta), \quad \phi_v(\xi) = V(\xi - \zeta) \quad \text{for all} \quad \xi \in \mathbb{R}.
\]
Combining (2.32) and (2.34), we complete the proof of Lemma 2.10

**Remark 2.13** *The uniqueness (up to translations) of traveling wave solutions for (1.1) under (H1) is not completely solved. It was proved in [15, Corollary A.7] that if \(\epsilon > 0\), there exists \(\zeta > 0\) such that (1.1) is equivalent to the equation \(\partial_t u = \partial_x^2 u - u^3\) with initial data \(u_0(x) = \phi(x + \zeta)\) for all \(x \in \mathbb{R}\). In particular, \(u(0, x) = \phi(x + \zeta)\) and \(u(t, x) = \phi(x + \zeta - t)\) for all \(t \geq 0\) and \(x \in \mathbb{R}\). The uniqueness (up to translations) of this family of solutions was established in [15, Proposition A.7].*
3 Cauchy problem with scenario (1.7)

In this section, we shall consider the initial data that satisfies (1.7) and prove Theorem 1.4.

Proof of Theorem 1.4. Let \((u, v)\) be the solution of system (1.1) with initial data \((u_0, v_0)\) satisfying (1.7). We first show that

\[
\lim_{t \to \infty} \left[ \sup_{x \geq c_0 t} |u(t, x) - U(x - c^* t - h_1)| + \sup_{x \in [0, c_0 t]} |v(t, x) - V(x - c^* t - h_1)| \right] = 0, \tag{3.1}
\]

where \(h_1\) is a constant and \(c_0 \in (c^*, c_\nu)\). To do so, let us consider \((\tilde{u}_0, \tilde{v}_0)\) satisfying

\[
\tilde{u}_0(x) = u_0(x), \quad \tilde{v}_0(x) \geq v_0(x) \quad \text{and} \quad \tilde{v}_0(x) \geq \delta > 0 \quad \text{for some} \quad \delta > 0. \tag{3.2}
\]

Let \((\tilde{u}, \tilde{v})\) be the solution of system (1.1) with the initial data \((\tilde{u}_0, \tilde{v}_0)\) satisfying (3.2). Then, by applying comparison principle, we have

\[
\tilde{u}(t, x) \leq u(t, x), \quad \tilde{v}(t, x) \geq v(t, x) \quad \text{for all} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \tag{3.3}
\]

Now, we define a sub-solution \((\bar{u}, \bar{v})\) as (2.11) for \(x \geq 0\). Note that \((\tilde{u}_0, \tilde{v}_0)\) satisfies (1.6). Thus, we can choose suitable parameters in \((\bar{u}, \bar{v})\), and use Lemma 2.3 to conclude that, for some large \(T^*,\)

\[
\tilde{u}(t + T^*, x) \geq \bar{u}(t, x), \quad \tilde{v}(t + T^*, x) \leq \bar{v}(t, x) \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times \mathbb{R}_+. \tag{3.4}
\]

By (3.3) and (3.4), we obtain that

\[
u(t + T^*, x) \geq \bar{u}(t, x), \quad v(t + T^*, x) \leq \bar{v}(t, x) \quad \text{for all} \quad (t, x) \in [T_0, \infty) \times \mathbb{R}_+. \tag{3.5}
\]

Next, let us define a super-solution \((\bar{u}, \bar{v})\) as (2.12) for \(x \geq 0\). We now compare \((u, v)\) with \((\bar{u}, \bar{v})\) over \(\Omega_T(t)\) for some large \(T\) and \(c_0 \in (c^*, c_\nu)\), where

\[
\Omega_T(t) := \{(x, t) | t \geq T, \ 0 \leq x \leq c_0 t\}.
\]

Let us focus on \(\{x = c_0 t\}\) first. From the definition of \(\bar{u}\) and Lemma 1.12(i)

\[
\bar{u}(t, c_0 t) - u(t, c_0 t) \geq U(x - c^* t - \zeta(t)) + P(t)e^{-\alpha_1((c_0 - c^*)t + x_1)} - C_1 e^{-\nu_1 t} \\
\geq p_1 e^{-\mu_1 t} e^{-\alpha_1((c_0 - c^*)t + x_1)} - C_1 e^{-\nu_1 t}
\]

We choose \(\mu_1 > 0\) small enough and \(c_0 > c^*\) sufficiently close to \(c^*\) such that

\[
\mu_1 + \alpha_1(c_0 - c^*) < \nu_1. \tag{3.6}
\]

Note that from the proof of Lemma 2.3 we see that the choice of \(T_0\) is independent on all small \(\mu\). Therefore, there exists some \(T_1 > T_0\) such that

\[
\tilde{u}(t, c_0 t) - u(t, c_0 t) \geq 0 \quad \text{for all} \quad t \geq T_1. \tag{3.7}
\]

Also, from the definition of \(\bar{v}\) and Lemma 1.12(ii), we have

\[
v(t, c_0 t) - \bar{v}(t, c_0 t) \geq 1 - C_2 e^{-\nu_2 t} - V(x - c^* t - \zeta(t)) + Q(t) \geq -C_2 e^{-\nu_2 t} + q_1 e^{-\mu_1 t}.
\]
By setting $\mu_1 < \nu_2$, there exists $T_2 > T_0$ such that
\[
v(t, c_0 t) - v(t, c_0 t) \geq 0 \quad \text{for all} \quad t \geq T_2.
\] (3.8)

Next, we consider the left boundary $\{x = 0\}$. From the definition of $\bar{u}$ and Lemma 1.12(i), we have
\[
\bar{u}(t, 0) - u(t, 0) \geq U(-c^* t - \xi(t)) + P(t)e^{-\alpha_1 (-c^* t + x_1)} - C_1 e^{-\nu t} \geq p_1 e^{-\mu_1 t}\min\{e^{\alpha_1 c^* t - \alpha_1 x_1}, 1\} - C_1 e^{-\nu t}.
\]

For all $t \gg 1$, $\min\{e^{\alpha_1 c^* t - \alpha_1 x_1}, 1\} = 1$. Hence, since (3.6), there exists $T_3 > T_0$ such that
\[
\bar{u}(t, 0) - u(t, 0) \geq 0 \quad \text{for all} \quad t \geq T_3.
\] (3.9)

Similarly, by applying $\mu_1 < \nu_2$, we can assert that for some $T_4 > T_0$,
\[
v(t, 0) - v(t, 0) \geq 0 \quad \text{for all} \quad t \geq T_4.
\] (3.10)

Let us fix $T := \max\{T_1, T_2, T_3, T_4\}$. If necessary, we may shift $(\bar{u}, \bar{v})$ (setting $\xi_0$ sufficiently large does not affect $T_i$ and $T$) such that $\bar{u}(T, \cdot) \geq u(T, \cdot)$ and $\bar{v}(T, \cdot) \leq v(T, \cdot)$. Together with the conclusion of Lemma 2.5 and (3.7), (3.8), (3.9) and (3.10), we can apply the comparison principle to conclude that for some $T^{**}$, it holds
\[
u(t, x) \leq \bar{u}(t + T^{**}, x), \quad v(t, x) \geq \bar{v}(t + T^{**}, x) \quad \text{for all} \quad (t, x) \in [T, \infty) \times [0, c_0 t],
\] (3.11)

for some $c_0 \in (c^*, c_v)$ with sufficiently close to $c^*$. Combining (3.5) and (3.11), we can follow the same line as in the proof of Theorem 1.1 to obtain (3.1).

Finally, by Lemma 1.12(i), we can follow the argument of [38, Section 4.1] that modified the argument of [17] to conclude
\[
\lim_{t \to \infty} \sup_{x \in [c_0 t, \infty)} \left| v(t, x) - V_{KPP}(x - c_v t + \frac{3d}{c_v} \ln t + \omega(t)) \right| + \sup_{x \in [c_0 t, \infty)} |u(t, x)| = 0,
\]
from which the proof of Theorem 1.4 is complete. \hfill \Box

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