The Two-Color Ext Soergel Calculus
or: The Ext-Dihedral Cathedral

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Abstract

We compute Ext groups between Soergel Bimodules associated to the infinite/finite dihedral group for a realization in characteristic 0 and show that they are free right $R$-modules with an explicit basis. We then give a diagrammatic presentation for the corresponding monoidal category of Ext-enhanced Soergel Bimodules.

As applications, we compute $\text{HH}$ of the connect sum of two Hopf links as an $R$-module and show that the Poincare series for the Hochschild homology of Soergel Bimodules of finite dihedral type categorifies Gomi’s trace for finite dihedral groups.

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1 Introduction

1.1 History

Let $(W, S)$ be a Coxeter group and $H_W$ be its associated Hecke Algebra. When $W$ is the Weyl group for a split torus $T$ of a reductive group $G$, Kazhdan and Lusztig [KL80] constructed a categorification of the Hecke algebra, now known as the Hecke category $\mathcal{H}_W$, by considering a certain monoidal subcategory of $D_B^D(G, C)$, the $B \times B$ equivariant derived category of sheaves on $G$ with coefficients in $C$. By categorification, we mean that there is an isomorphism of algebras

$$H_W \cong K_\bullet(\mathcal{H}_W)$$
where $K_a(H_W)$ is the split Grothendieck group of $H_W$. For the case of a general Coxeter group, no such geometric techniques were available at the time for such a categorification.

In the early 1990s, Soergel [Soe92] gave another incarnation of the Hecke category $H_W$ for $W$ a Weyl group that was more "algebraic." Specifically, let $t = \text{Lie}(T)$ and $R = \text{Sym}(t^*)$. Because $H^*_B(S)(\text{pt}) \cong R \otimes C R$, equivariant hypercohomology of any object in $D^b_{B}(G, C)$ is naturally a graded module over $R \otimes C R$. Since $R$ is commutative hypercohomology is therefore a functor

$$H^*_B : H_W \rightarrow R - \text{Bim}. $$

Soergel’s key observation is that $H^*_B$ is fully-faithful and monoidal and as a result we have a monoidal equivalence

$$H_W \cong \text{Bim}_W,$$

where $\text{Bim}$ is the category of Soergel bimodules, a monoidal subcategory of $R - \text{Bim}$ given by the essential image of $H^*_B$. Using this equivalence (or rather a similar equivalence with Soergel modules), Soergel [Soe90] shows how the illustrious Kazhdan-Lusztig conjectures [KL79] (which were proven earlier in [BB81], [BK81] using geometric techniques) on character formulas for irreducible representations of semisimple Lie algebras would follow from a certain desired property of $\text{Bim}_W$ called Soergel’s Conjecture.

One should mention that Soergel gives a completely algebraic definition of $\text{Bim}$ in his work above. Specifically, $W$ acts on $t$ and thus on $R$ via graded algebra automorphisms. Let $R^s \subset R$ be the subalgebra of elements of $R$ invariant under $s \in S$. Define the $R$–bimodule $B_s = R \otimes_{R^s} R(1)$. $\text{Bim}$ is then equivalent to the smallest full additive monoidal Karoubian graded subcategory of $R - \text{Bim}$ containing $B_s$ for each $s \in S$. In [Soe07] Soergel points out that by replacing $t$ with an appropriate (“reflection faithful”) representation of $W$, one can define $\text{Bim}(t, W)$ exactly as in the definition above, where $W$ is now any Coxeter group and shows that $\text{Bim}(t, W)$ categorifies $H_W$. In other words, there is an isomorphism of algebras

$$H_W \cong K_a(\text{Bim}(t, W))$$

In this setting, the statement of Soergel’s conjecture still makes sense and Soergel shows that this would imply a major open question in combinatorics: the Kazhdan-Lusztig positivity conjecture for arbitrary Coxeter groups. However, lacking the Decomposition Theorem in this setting, he was unable to prove his conjecture.

Fortunately $\text{Bim}(t, W)$ is a monoidal category, and thus one can hope to give a presentation by generators modulo local relations as speculated by Rouquier in his ICM talk [Rou06]. Using the language of planar diagrammatics, this was first done by Elias-Khovanov in [EK10] for $W = S_n$, by Elias [Eli16] in the dihedral case and finally by Elias-Williamson [EW16] for any Coxeter group $W$. Although the presentation was found using Soergel bimodules, the resulting monoidal category can be considered independently, thus giving a third incarnation of the Hecke category $D(t, W)$, referred to as the diagrammatic Hecke category. This incarnation has the advantage that many complicated computations can be reduced to algorithmic manipulations of planar diagrams. Using the diagrammatic Hecke category, Elias and Williamson [EW14] were able to prove Soergel’s conjecture for any Coxeter group $W$ thereby proving the Kazhdan-Lusztig positivity conjecture and completing Soergel’s purely algebraic proof of the Kazhdan-Lusztig conjectures. The diagrammatic Hecke category has subsequently led to other breakthroughs in representation theory such as Williamson’s counterexamples [Wil17] to the long-standing Lusztig’s conjecture (the analogue of the Kazhdan-Lusztig conjectures for representations of reductive algebraic groups in characteristic $p$) and the subsequent new character formulas for irreducible and indecomposable tilting modules conjectured in [RW18] and proved in [RW18] and [AMRW19].

In type $A$, the Hecke algebra $H_{S_n}$ also plays an important role in knot theory in the construction of link invariants. Given a link $L$ written as the closure of a braid $\beta_L \in B_n$, Jones [Jon87] shows that the HOMFLY polynomial of $L$ is “essentially equal” to $Tr(\pi(\beta_L))$ where $\pi: B_n \rightarrow H_{S_n}$ and $Tr$ is the Jones-Ocneanu trace on $H_{S_n}$. Khovanov [Kho07] would later categorify this construction to produce a triply graded link homology theory as follows. For a braid $\beta_L \in B_n$, Khovanov associates a complex $F(\beta_L) \in K^b(\text{Bim}(C^n, S_n))$ (categorification of $H_{S_n}$) where $C^n$ is the permutation representation of $S_n$. Applying the functor of Hochschild cohomology (categorification of $Tr$) to $F(\beta_L)$ and taking cohomology gives us the desired link homology $\text{HHH}(\beta_L)$. In recent years there has been a great deal of
In this paper we give a diagrammatic presentation for the monoidal categories $\mathcal{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty})$, $\mathcal{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_m)$ of Ext-enhanced Soergel Bimodules for the infinite and finite dihedral group $W_{\infty}$ and $W_m$ (we will abbreviate both cases using the notation $W_{\infty/m}$ for a realization $\mathfrak{h}$ in characteristic 0 satisfying the usual 2-color assumptions + a linear independence condition. This paper builds upon the $A_1$ case done by Shotaro Makisumi in [Mak22]. We will give some clarifications on the monoidal structure later, but for now note that extending/replacing the graded Hom spaces in $\mathcal{S}\text{Bim}(t, W)$ via

$$\text{Hom}_{R-\text{Bim}}^*(\cdot, \cdot) \rightsquigarrow \text{Ext}^{\cdot,\cdot}_{R-\text{Bim}}(\cdot, \cdot)$$

still results in a category $\mathcal{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$ with composition given by the Yoneda product. This still has a monoidal structure (or rather super-monoidal structure as explained in [Mak22]) and in addition to the Elias-Khovanov generators we have (essentially) three additional one-color generators pictured below using planar diagrammatics

with many new relations (see Section 3). We also have (essentially) one additional two-color generator

with many new relations (see Section 7 and Section 9). These generators and relations can be considered independently of $\mathcal{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$ giving rise to the diagrammatic Ext enhanced Hecke categories $\mathcal{D}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$. To be more precise, as in [EK10] and [EW16] what we actually do is give a presentation for $\mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$, the full subcategory of $\mathcal{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$ with objects the Bott-Samelson bimodules from which $\mathcal{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$ can be recovered by taking the Karoubian envelope. The main result of the paper will then be the following equivalences

$$\mathcal{D}^{\text{Ext}}_{\infty}(\mathfrak{h}, W_{\infty}) \simeq \mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty}), \quad \mathcal{D}^{\text{Ext}}_{m_1}(\mathfrak{h}, W_{m_1}) \simeq \mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{m_1})$$

Note that although the categories $\mathcal{D}^{\text{Ext}}_{\infty/m}(\mathfrak{h}, W_{\infty/m})$ can be considered when $\mathfrak{h}$ has characteristic $p$, it is not clear to the author that the same equivalences above still hold, as there appears to be more generators in $\mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$. In the course of the proof we will obtain a complete description of Hom spaces in $\mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$. Specifically for any two expressions $v, w$, we will explicitly compute

$$\text{Ext}^{\cdot,\cdot}_{R-\text{Bim}}(\mathcal{B}v, \mathcal{B}w)$$

as a right $R$ module (although the results can be equivalently stated with the left $R-$module structure) (see Theorem 4.13, Theorem 8.2). In particular the morphism space above will always be free as a right $R-$ module. When $m = 2, 3, 4, 6$ this is also a special case of [WW11] which uses geometric methods that do not apply when $W_m$ is non-crystallographic.

One thing to note is that throughout the algebraic portion of the paper, diagrammatics are already present, first as a visual aid to help explain the new relations, and then used in proofs of theorems which live in the algebraic category. This is allowed, however, as each time we are either using the equivalence $\mathcal{D}(\mathfrak{h}, W_{\infty/m}) \simeq \mathcal{B}\text{Bim}(\mathfrak{h}, W_{\infty/m})$ established in [Eli16] or a previously established relation in $\mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$ of which we had a diagrammatic interpretation of. When the expression for a relation in $\mathcal{B}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{\infty/m})$ is not too cumbersome, we will write it out explicitly, and accompany it with a diagrammatic description.
1.3 Applications

(a) Because Hochschild cohomology of a $R$–bimodule $M$ is defined to be $HH^k(M) := \text{Ext}^k_{R–\text{Bim}}(R, M)$ one can apply our main results to compute triply graded link homology $HHH^k(\beta_k)$ where $\beta_k$ is a braid on 3 strands by setting $h = C^3(C^2)$ to be the permutation (geometric) realization of $S_3$. An example is done in Appendix B. For general braids one would need a diagrammatic presentation of $\mathbb{S}\text{Bim}^{\text{Ext}}(C^n, S_n)$ which is currently a work in progress by the author.

(b) In [Gom06], Gomi generalizes the two variable Jones-Ocneanu trace on $H_{S_n}$ to a trace on $H_W$ satisfying a "Markov" type condition, where $W$ is any finite Coxeter group as follows. Because $H_W$ is semisimple any trace $\tau:H_W \to \mathbb{Z}[q,q^{-1}][t]$ can be written as

$$\tau = \sum_{\chi \in I(W)} w^\chi \chi_q$$

Gomi then defines $w^\chi:H_W \to \mathbb{Z}[q,q^{-1}][t]$ using entries from Lusztig’s Fourier transform matrix $S$. When $W$ is the Weyl group of a reductive algebraic group $G/F_q$, $S$ is the change of basis matrix between unipotent characters and almost characters for the finite group of Lie type $C^E$ [Lus84]. When $W$ is of dihedral type, $S$ is the Exotic Fourier transform of Lusztig [Lus94] which is similar to above except “unipotent character” is now in quotations. In Appendix C we show that generating function for the Hochschild homology of Soergel Bimodules for $W_m$ coincides with Gomi’s trace defined above. For $m = 2, 3, 4, 6$ this is a special case of [WW11] which uses properties of unipotent character sheaves. This adds more evidence that a suitable category of Soergel Bimodules is the right setting for the study of “spetses" [BMM99].

(c) Soergel Bimodules corresponding to $\mathbb{S}\text{Bim}(C^n, S_n)$ have an alternative pictorial description that is 3D instead of 2D. Instead of depicting morphisms between Bott-Samelson bimodules using planar diagrams, we depict a Bott-Samelson bimodule $B_i \otimes_R \cdots \otimes R B_j$ as a planar diagram, see [Kho07] Figures 2,3. Morphisms can then be realized by linear combinations of foams—decorated two-dimensional CW-complexes embedded in $\mathbb{R}^3$ that arise as cobordisms between the planar diagrams described above. Foams are one of the fundamental objects used in constructing link homology theories. They show up prominently in the doubly graded $sl_n$ theories starting with [Kho04] in $sl_5$ and [MSV09], [QR16] for $sl_4$, and finally in [RW20a] for equivariant $sl_n$.

One major step in the construction of all these link homology theories is to define some variant of foam evaluation for closed foams which is used to associate a vector space $F(\Gamma)$ to a web. In [RW20b], Robert and Wagner show that applying the variant, $F_\infty(\Gamma)$ to a special class of foams recovers $\mathcal{R}(\Gamma)$ the (singular) Soergel Bimodule corresponding to $\Gamma$. Moreover, they show that by closing the planar diagram first and then applying $F_\infty$ recovers $HH_2(\mathcal{R}(\Gamma))$. In [RW20b] they extend this to $HH_2(\mathcal{R}(\Gamma))$ for all $k \geq 0$. As $HH_* (M) \cong HH^*(M)$ for bimodules over polynomial rings, it would be interesting to see how our relations below can be realized in this framework.

(d) A crucial step in the proof of the tilting character formula in [AMRW19] was to relate two different derived categories of sheaves. To do this the authors first establish an equivalence between two different monoidal categories which act on the different categories above. Specifically, using the diagrammatics developed in [EW16], they establish monoidal Koszul duality with modular coefficients for any Kac-Moody group $\mathcal{G}$. For more details there is a nice introduction to these ideas in [Mak19], but the key point right now is that there’s an equivalence of monoidal categories

$$\kappa^{\text{mon}}: \mathcal{D}(h, W, \star) \rightsquigarrow (\text{TiltFM}(h^*, W), \hat{\star})$$

giving rise to the doubly graded equivalence of categories

$$\kappa: K^b(\mathcal{D}(h, W) \otimes_R k) \rightsquigarrow K^b(\mathcal{D}(h^*, W))$$

interchanging indecomposable and tilting objects such that $\kappa \circ (-1)[1] = (1) \circ \kappa$. In [HM20] Hogancamp and Makisumi conjecture that this can be extended to an equivalence of monoidal categories

$$(\kappa^{\text{mon}})^{\text{Ext}}: \mathcal{D}^{\text{Ext}}(h, W, \star) \rightsquigarrow (\text{TiltFM}^{\text{Ext}}(h^*, W), \hat{\star})$$

and hence giving rise to a triply graded equivalence of categories

$$\kappa^{\text{Ext}}: K^b(\mathcal{D}^{\text{Ext}}(h, W) \otimes_R k) \rightsquigarrow K^b(\mathcal{D}^{\text{Ext}}(h^*, W))$$
interchanging indecomposable and tilting objects such that $\kappa^{\text{Ext}} \circ (-1)[1] = (1) \circ \kappa^{\text{Ext}}$ and provide evidence in the $\mathfrak{gl}_2$ realization of $W = S_2$. The main results in this paper can therefore be used to help define $(\kappa^{\text{mon}})^{\text{Ext}}$ to prove such an equivalence. According to Makisumi, "One may speculate that this triply-graded Koszul duality is connected to a duality for character sheaves."

1.4 Organization

- **Background**
  - In Section 2 we review notation and conventions. In particular our notation for Koszul complexes will be important for our computations later on. Throughout the paper we will always assume the reader is familiar with Soergel bimodules and the two-color Soergel diagrammatics such as the light leaf morphisms. A good place to read up on this would be in [EMTW20] Chapters 4, 5, 9, 10, 12.
  - In Section 3 we review the diagrammatics of the $A_1$ case $\mathbb{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, S_2)$. For the computations done in this paper, it turns out that we only need that Theorem 5.2 in [Mak22] holds, i.e. the defining relations of $\mathcal{D}^{\text{Ext}}(\mathfrak{h}, S_2)$ hold in $\mathbb{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, S_2)$.

- **The Algebraic Category**
  - In Section 4 we compute $\text{Ext}^{\bullet, \bullet}_{R^e}(B_r, B_S(\mathfrak{w}))$ as a right $R$–module when $m_{st} = \infty$ which will allow us to define the new generalizing morphism $\Phi^w_{R^e}$ in $\mathbb{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_\infty)$ in Section 5. We should note that $\Phi^w_{R^e}$ doesn’t quite correspond to the new generator in the diagrammatic category (they are off by a rotation+possibly a sign) but it’s much easier to work with $\Phi^w_{R^e}$ in the algebraic category.
  - In Section 6 we give an explicit basis for $\text{Ext}^{\bullet, \bullet}_{R^e}(B_r, B_w)$ as a right $R$ module which shows that $\text{Ext}^{\bullet, \bullet}_{R^e}(B_r, B_w)$ is more or less controlled by $\text{Hom}^{\bullet, \bullet}_{R^e}(B_r, B_w)$.
  - Section 8 proves a lot of the same theorems as above but now in the finite case $m_{st} < \infty$. In most cases, we just check that the argument for $m_{st} = \infty$ still holds when $m_{st} < \infty$.

- **The Diagrammatic Category**
  - In Section 7 and Section 9 we define the Ext-enhanced diagrammatic Hecke categories $\mathcal{D}^{\text{Ext}}_{\infty}$ and $\mathcal{D}^{\text{Ext}}_{m_{st}}$ associated to a (faithful) realization $\mathfrak{h}$ of the infinite and finite dihedral groups $W_\infty$, $W_{m_{st}}$, respectively. We will then establish the equivalences $\mathcal{D}^{\text{Ext}}_{\infty} \cong \mathbb{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_\infty)$ and $\mathcal{D}^{\text{Ext}}_{m_{st}} \cong \mathbb{S}\text{Bim}^{\text{Ext}}(\mathfrak{h}, W_{m_{st}})$. The proof bootstraps the proof of the dihedral equivalence in [Eli16] in that it just suffices to show the isomorphism

  $$\text{Hom}^{\bullet, \bullet}_{\mathcal{D}^{\text{Ext}}_{\infty}}(\mathcal{R}_\varnothing, \mathcal{R}_w) \cong \text{Ext}^{\bullet, \bullet}_{R^e}(B_r, B_w)$$

  and that we have a very explicit description of the rank 2 projectors to the indecomposable summands $B_w$ given by the Jones-Wenzl projectors. Unlike in [Eli16] our proof relies on one of the main theorems in [EW16], namely that double leaves form a right $R$ basis for hom spaces in the diagrammatic Hecke category.

- **Appendices**
  - In Appendix A we define chain lifts for morphisms in $\mathbb{S}\text{Bim}(\mathfrak{h}, W)$. In Appendix B we compute HOMFLY homology of the connect sum of two Hopf links. In Appendix C we show that the Poincare series for Hochschild homology of Soergel Bimodules agrees with Gomi’s trace for $W = W_m$.

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2 Preliminaries and Notation

2.1 Realizations and Gradings

We first recall the definition of a realization of a Coxeter system $(W, S)$ as defined in [EW16] Section 3.

**Definition 2.1.** Let $\mathfrak{k}$ be a commutative ring. A realization of $(W, S)$ over $\mathfrak{k}$ is a triple

$$\mathfrak{h} = (V, \{\alpha_s\}_{s \in S} \subset V^*; \{\alpha_s^\vee\}_{s \in S} \subset V)$$

where $V$ is a free, finite rank $\mathfrak{k}$ module such that

1. $\langle \alpha_s^\vee, \alpha_s \rangle = 2$ for all $s \in S$.
2. The assignment $s(v) = v - \langle v, \alpha_s \rangle \alpha_s^\vee$ for all $v \in V$ yields a representation of $W$.
3. For each $s, t \in S$ let $a_{st} := \langle \alpha_s^\vee, \alpha_t \rangle$ and $m_{st}$ the order of $st$ in $W$. Then $m_{st}a_{st} = |m_{st}a_{ts} = 0$ where $|k|_x$ are the 2–colored quantum numbers defined in Section 3 of [EW16].

For the rest of the paper we will assume the following.

**Assumption 1.** $\mathfrak{h}$ is a balanced realization of rank $n$ that satisfies Demazure surjectivity over an integral domain $\mathfrak{k}$.

All of our algebras and modules will be bigraded with the cohomological grading $[1]$ and Soergel/internal grading $(0, 0)$. We will frequently use the combined shift $[1] := [1]|(-2)$.

**Definition 2.2.** Let $R := \text{Sym}^*(V^*(−2))$ and let $R^e = R \otimes_{\mathfrak{k}} R$. Let $R^{ee} = R \otimes_{\mathfrak{k}} R \otimes_{\mathfrak{k}} R$ and so on with $R^{ee}$, etc.

This paper deals with the case when $W = W_{m_{st}}$ where $W_{m_{st}}$ is the Coxeter group generated by simple reflections $s, t$ with relation $(st)^{m_{st}} = 1$. $W$ will be the infinite dihedral group.

**Definition 2.3.** $\otimes$ will always mean $\otimes_{\mathfrak{k}}$ while $\otimes_{R}$ will mean $\otimes_{R^e}$.

**Definition 2.4.** Given two complexes $A^* = \oplus_p A^p, B^* = \oplus_p B^p$ of $R^e$–graded modules, define the bigraded Hom complex as

$$\text{Hom}_{R^e}(A^*, B^*) = \bigoplus_{i,j} \text{Hom}_{R^e}^i(A^*, B^*) = \bigoplus_{i,j} \text{Hom}_{R^e}(A^p, B^{p+i}(j))$$

with differential $d^n : \text{Hom}_{R^e}^n(A^*, B^*) \to \text{Hom}_{R^e}^{n+1}(A^*, B^*)$ given by

$$d^n(f) = d_B \circ f + (-1)^{n+1} f \circ d_A$$

2.2 Koszul Complexes

For $a \in R$, let $a^e := a \otimes 1 - 1 \otimes a \in R^e$. Given a sequence of homogeneous elements $a_1, \ldots, a_n \in R$ where $|a_i|$ is the internal degree of $a_i$, let $K(a_1^e, \ldots, a_n^e) = K(a_1^e) \otimes_{R^e} \cdots \otimes_{R^e} K(a_n^e)$ be the following graded Koszul complex where $K(a_i^e)$ is the complex

$$K(a_i^e) = [0 \to R^e a_i^e ; R^e 1 \to 0] = \begin{array}{l} [0 \to R^e(-|a_i|) a_i^e \to R^e] \end{array}$$

where the boxed term is in cohomological degree 0 and we have underlined $a_i^e$ and 1 as they are the “exterior” part. Specifically, the graded Koszul complex has the structure of a bigraded dga as $K(a_i^e) = \Lambda^*(k_{a_i^e}[(-|a_i|)]) \otimes R^e$ while $K(a_1^e, \ldots, a_n^e) = \Lambda^*(k_{a_1^e}(-|a_1|) \cdots k_{a_n^e}(-|a_n|)) \otimes R^e$ where the differential is determined by $d(a_i) = 1 \otimes a_i^e$ and the graded Leibniz rule. Here we use the notation $\Lambda^*(V) \otimes R^e := \Lambda^*(V) \otimes R^e$ to distinguish the exterior algebra part from $R^e$. All elements of $\Lambda^*(V)$ will be underlined as well. The $R$–bimodule structure is given by only acting on the $R^e$ part, i.e. $r \cdot (v \otimes f \otimes g) \cdot r' = v \otimes rfr' \otimes g$ and when tensoring two Koszul complexes $\Lambda^*(V_1) \otimes R^e, \Lambda^*(V_2) \otimes R^e$ over $R$ (instead of $R^e$), we tensor the $\Lambda$ and $R^e$ parts separately, i.e

$$\Lambda^*(V_1) \otimes R^e \otimes_R \Lambda^*(V_2) \otimes R^e = \Lambda^*(V_1) \otimes \Lambda^*(V_2) \otimes R^e \otimes_R R^e = \Lambda^*(V_1) \otimes \Lambda^*(V_2) \otimes R^{ee}.$$
Now the $R$–bimodule structure is given by acting on the leftmost and rightmost tensor factors of $R^e$. We will use the shorthand $v^{(1)} := v \otimes 1$ for $v \in V_1$ and likewise $w^{(2)} := 1 \otimes w$ for $w \in V_2$.

Two Koszul complexes will be of great importance to us. As shown in [Mak22] $q_S : K_S \rightarrow R$ and $q_z : K_z \rightarrow B_z$ will be projective resolutions of $R$ and $B_z$ as a $R$–bimodule where

$$K_S := K(e_1^e, \ldots, e_n^e) = \Lambda^*(V^*[1]) \boxtimes R^e$$

$$K_z := K(\rho_1, e_1^e, \ldots, e_n^e)(1) = \Lambda^*(k\rho, s(\rho))[1(-4) \oplus (V^*)^4][1]) \boxtimes R^e$$

where $\rho_s \in V^*$ satisfies $⟨a_s^{\rho_1}, \rho_s⟩ = 1$ which exists by Demazure surjectivity and $\{e_i\}_{i=1}^n$ is a basis for $(V^*)^4$ while $\{e_i\}_{i=1}^n$ is a basis for $V^*$.

Koszul complexes have natural chain maps given by contraction. Specifically, following [Mak22]

**Definition 2.5.** Given $v \in V[-1]$, define $v \prec (-) : \Lambda^*(V^*[1]) \rightarrow \Lambda^*(V^*[1])$ by

$$v \prec (r_1 \wedge \cdots \wedge r_k) = \sum_{i=1}^{k} (-1)^{i+1} r_i(v) r_1 \wedge \cdots \wedge \hat{r}_i \wedge \cdots \wedge r_k$$

One can then check that $v_1 \prec (-)$ anticommutes with $v_2 \prec (-)$ and so there is an induced map $\xi \prec (-) : \Lambda^*(V^*[1]) \rightarrow \Lambda^*(V^*[1])$ for any $\xi \in \Lambda^*(V[-1])$. Similarly, one also obtains a map $w \prec (-) : \Lambda^*(V[-1]) \rightarrow \Lambda^*(V[-1])$ for any $w \in \Lambda^*(V^*[1])$.

Therefore for any $\xi \in \Lambda^*(V[-1])$, by extending $R^e$ linearly, we obtain a map

$$\tilde{\iota}_\xi : K_S \rightarrow K_S$$

and because $\iota_\xi$ is a derivation on $K_S$ it will automatically be a chain map. Let $\iota_\xi$ be the induced map on cohomology. We also have similar contraction maps for the Koszul complex $K_z$ and there is one in particular that will be of great importance, namely,

**Definition 2.6.** Define $\gamma^\xi_S : k\rho_1, s(\rho)_1[1(-4) \oplus (V^*)^4][1] \rightarrow k$ by $\gamma^\xi_S(\rho_1, s(\rho)) = 1$ and $\gamma^\xi_S((V^*)^4) = 0$. Then define the chain map

$$\tilde{\eta}_\xi = -\gamma^\xi_S \prec (-) \in \text{Hom}^{1-4}_R(K_S, K_S)$$

and let $\eta_\xi$ be the induced map on cohomology.

In [Mak22] it was shown that $\alpha^\xi_S \prec (-) : \Lambda^*(V^*[1]) \rightarrow \Lambda^*(V^*[1])$ actually lands in $\Lambda^*((V^*)^4[1])$. As a result, we can define

**Definition 2.7.** Define the chain map

$$\tilde{\eta}_S^{\text{Ext}} = \alpha^\xi_S \prec (-) \in \text{Hom}^{1-3}_R(K_S, K_S)$$

and let $\eta_S^{\text{Ext}}$ be the induced map on cohomology.

**Definition 2.8.** Let $s \in S$. Define the exterior Demazure operator $\partial_s : \Lambda^*(V[-1]) \rightarrow \Lambda^*(V[-1]/k\alpha_S^\gamma)$ as the composition

$$\Lambda^*(V[-1]) \xrightarrow{\alpha^\gamma_S \prec (-)} \Lambda^*(V[-1]) \xrightarrow{\Lambda^*(V[-1]) \rightarrow \Lambda^*(V[-1]/k\alpha_S^\gamma)}$$

**Remark.** Alternatively, as explained in [Mak22] when $k$ doesn’t have characteristic 2, one can "define" $\partial_s$ in a similar fashion as the classical setting $\partial_s : R \rightarrow R(-2)$, i.e.

$$\partial_s(f) := \frac{f - sf}{\alpha^\gamma_S}$$

In order for the two definitions of the exterior Demazure operator to match it is essential that the image is in $\Lambda^*(V[-1]/k\alpha_S^\gamma)$. For example as elements of $\Lambda^*(V[-1])$, we see that

$$\alpha_s \prec (\alpha^\gamma_S \wedge \alpha^\gamma_S) = 2a^\gamma_S - a_{s2}a^\gamma_S$$

$$\partial_s(\alpha^\gamma_S \wedge \alpha^\gamma_S) = \frac{\alpha^\gamma_S \wedge \alpha^\gamma_S - (\alpha^\gamma_S \wedge a_{s2}a^\gamma_S)}{a^\gamma_S} = 2a^\gamma_S$$
3 Review of Main Results of [Mak22]

We first recall the definition of the algebraic category BSBim$^{\text{Ext}}(\mathfrak{h}, W)$.

**Definition 3.1.** Given an expression $\underline{w} = (s_1, \ldots, s_m)$, define the corresponding Bott-Samelson complex to be $K(\underline{w}) = K(s_1, \ldots, s_m) := K_{s_1} \otimes_R \cdots \otimes_R K_{s_m}$

Then BSBim$^{\text{Ext}}(\mathfrak{h}, W)$ is the smallest full additive subcategory of $K^b(\text{Proj}(R^e - \text{gmod}))$ consisting of complexes isomorphic to Bott-Samelson complexes.

**Lemma 3.2.** $q_{\underline{w}} : K(s_1, \ldots, s_m) \xrightarrow{q_1 \otimes_R \cdots \otimes_R q_m} \text{BS}(s_1, \ldots, s_m)$ is a free $R^e$ resolution.

**Proof.** Proceed by induction on $m$. Note that any free $R^e$ resolution will also be a free left or right $R$–module resolution. So by induction we see that

$$H^k(K(s_1, \ldots, s_{m-1}) \otimes_R K_{s_m}) = \text{Tor}_k^R(\text{BS}(s_1, \ldots, s_{m-1})_R, B_{s_m})$$

where BS$(s_1, \ldots, s_{m-1})_R$ is the left $R$ module given by the right action of $R$, aka $R \ast m = m \cdot r$ for $m \in \text{BS}(s_1, \ldots, s_{m-1})_R$.

But since all Bott-Samelson bimodules are free left or right $R$ modules, we see that for $k > 0$ the RHS is 0 and for $k = 0$, we have

$$H^0(K(s_1, \ldots, s_{m-1}) \otimes_R K_{s_m}) = \text{BS}(s_1, \ldots, s_{m-1}) \otimes_R B_{s_m}$$

as sets. But since the morphisms involved were $R^e$–linear it follows that the above is a bimodule isomorphism and so $K(s_1, \ldots, s_m)$ is a free $R^e$ resolution of BS$(s_1, \ldots, s_m)$ as desired.

Thus as $K(s_1, \ldots, s_m)$ is a complex of projective $R^e$ modules that resolves the bimodule BS$(s_1, \ldots, s_m)$ it follows that

$$\text{Hom}_{\text{BSBim}^{\text{Ext}}(\mathfrak{h}, W)}(K(s_1, \ldots, s_m), K(r_1, \ldots, s_k)) \cong \text{Ext}_{R^e}^{*}(\text{BS}(s_1, \ldots, s_m), \text{BS}(r_1, \ldots, r_k))$$

The upshot of working with BSBim$^{\text{Ext}}(\mathfrak{h}, W)$ is that we can easily define the $\otimes$ of morphisms, given by $\otimes_R$ of complexes, and thus there is a clear (super)monoidal structure in BSBim$^{\text{Ext}}(\mathfrak{h}, W)$. As each $K(s_1, \ldots, s_m)$ is a complex of projective $R^e$–modules any morphism in the Soergel category BSBim$(\mathfrak{h}, W)$ will automatically lift to a morphism (unique up to homotopy) in BSBim$^{\text{Ext}}(\mathfrak{h}, W)$ and BSBim$(\mathfrak{h}, W)$ embeds inside BSBim$^{\text{Ext}}(\mathfrak{h}, W)$ fully faithfully as the cohomological degree 0 part.

We now recall the definition of the diagrammatic category $\mathcal{D}^{\text{Ext}}(\mathfrak{h}, S_2)$ as defined in [Mak22] Section 4.

**Definition 3.3.** Let $\mathcal{D}^{\text{Ext}}(\mathfrak{h}, S_2)$ be the strict $k$ linear super monoidal category associated to a realization $\mathfrak{h}$ of $S_2 = \langle s \rangle$ defined as follows.

- **Objects** of $\mathcal{D}^{\text{Ext}}(\mathfrak{h}, S_2)$ are words in $S = \langle s \rangle$, i.e. $\underline{w} = (s_1, \ldots, s_n)$ where $s_i \in S$ where the monoidal structure is given by concatenation.

- **Morphism spaces** in $\mathcal{D}^{\text{Ext}}(\mathfrak{h}, S_2)$ are bigraded $k$ modules. For a morphism $\alpha$ homogeneous of total degree $(\ell, n)$, $\ell$ will be the cohomological degree while $n$ will be the internal or Soergel degree. Let $|\alpha| = \ell$ the cohomological degree. $\mathcal{D}^{\text{Ext}}$ will then be monoidal for the cohomological grading. Specifically, $\otimes$ will satisfy the following super exchange law

$$(h \otimes k) \circ (f \otimes g) = (-1)^{|f||h|} (h \circ f) \otimes (k \circ g)$$

$\text{Hom}_{\mathcal{D}^{\text{Ext}}(\mathfrak{h}, S_2)}(\underline{v}, \underline{w})$ will be the free $k$ module generated by horizontally and vertically concatenating colored graphs built from certain generating morphisms, such that the bottom and top boundaries are $\underline{v}$ and $\underline{w}$. The generating morphisms will be the generating morphisms of the diagrammatic Hecke category $\mathcal{D}(\mathfrak{h}, S_2)$ for $A_1$ plus the additional two "Hochschild" generators

| generator | (bivalent) Hochschild dot | Exterior Box |
|-----------|--------------------------|--------------|
| name      | (1, −4) deg x            |              |
Here, \( x \) is a homogeneous element in \( \Lambda^\ast(V[-1]) \), and \( \deg x \) denotes its bidegree. We also define the following “univalent Hochschild dots” as shorthands:

\[
\begin{align*}
\begin{array}{c}
\text{:=} \\
\end{array}
\end{align*}
\]

\( (3) \)

- **Relations** in \( \mathcal{D}^{\text{Ext}}(h,S_2) \) are as follows. All the defining relations of \( \mathcal{D}(h,S_2) \) in [EK] will be satisfied plus the relations involving the "Hochschild" generators in the subsection below.

### 3.1 1–color Relations

1. Hochschild dot slides past trivalent vertices:

\[
\begin{array}{c}
\text{=} \\
\end{array}
\]

\( (4) \)

2. Hochschild barbell relation:

\[
\begin{array}{c}
\text{=} \\
\end{array}
\]

\( (6) \)

3. Hochschild dot annihilation:

\[
\begin{array}{c}
\text{= 0} \\
\end{array}
\]

\( (7) \)

4. Exterior boxes add and multiply:

\[
\begin{align*}
\begin{array}{c}
\text{=} \\
\end{array}
\end{align*}
\]

\( (8) \)

for \( x, y \in \Lambda^\ast(V[-1]) \).

5. Exterior forcing relation:

\[
\begin{align*}
\begin{array}{c}
\text{=} \\
\end{array}
\end{align*}
\]

\( (9) \)

where \( \partial : \Lambda^\ast(V[-1]) \to \Lambda^\ast(V[-1]/k\alpha^\flat) \) is the exterior Demazure operator defined in **Definition 2.8**.

#### 3.1.1 Further Relations

The following relations follow from the defining relations above.

- 1–color Hochschild Jumping:

\[
\begin{array}{c}
\text{=} \\
\end{array}
\]

\( (10) \)
• 1–color Cohomology:

\[ \langle \xi, \gamma \rangle_s = \begin{array}{c} \text{diag} \\ \phi \end{array} \ 

(11)

• Hochschild coroot annihilation:

\[ \langle 0, \xi \rangle_s = 0 \ 

(12)

3.2 Equivalence

Theorem 3.4 (Main Theorem of [Mak22]). There is a \( k \)-linear monodial equivalence \( \mathcal{F}^\text{Ext} : \mathcal{D}^\text{Ext}(h, S_2) \to \text{BSBim}^\text{Ext}(h, S_2) \)

extending the equivalence \( \mathcal{F} : \mathcal{D}(h, S_2) \to \text{BSBim}(h, S_2) \)

defined on objects by sending \( (s) \to K_s, \emptyset \to K_\emptyset \) and on morphisms by sending

\[ \mathcal{F}^\text{Ext}(\langle \xi \rangle_s) = \eta_s, \quad \mathcal{F}^\text{Ext}(\langle \xi \rangle_s) = \mu_s \quad \text{for} \ \xi \in \Lambda^*([V[-1]]) \]

and sends the generating morphisms of \( \mathcal{D}(h, S_2) \) to their class in cohomology of their chain lifts as defined in Appendix A. Specifically,

\[ \mathcal{F}^\text{Ext}(\langle \rangle_s) = [\eta_s], \quad \mathcal{F}^\text{Ext}(\langle \xi \rangle_s) = [\xi], \quad \mathcal{F}^\text{Ext}(\langle \rangle_s) = [\mu_s], \quad \mathcal{F}^\text{Ext}(\langle \rangle_s) = [\delta_s] \]

Under \( \mathcal{F}^\text{Ext} \) the univalent Hochschild dots defined in Eq. (3) will be mapped to \( \eta_s \circ \phi_s \) and \( \phi_s \circ [\eta_s] \) respectively. In [Mak22] it was shown that \( \phi_s \circ [\eta_s] = \eta_s^\text{Ext} \), in other words

\[ \diamond = \eta_s^\text{Ext} \]

Thus we can and will use the RHS above for our computations involving \( \diamond \) in the algebraic category instead of \( \phi_s \circ [\eta_s] \). It is important to note that BSBim\((h, S_2)\) only sits inside of BSBim\(^\text{Ext}\)(\(h, S_2\)) isomorphically, with objects \( B_s \) replaced by the dga \( K_s \) and morphisms replaced by their cohomology class of their chain lifts. Likewise, throughout the rest of the paper diagrammatics traditionally representing morphisms in BSBim\((h, W)\) will instead refer to their corresponding morphism in BSBim\(^\text{Ext}\)(\(h, W)\).

3.3 Action of Exterior Boxes

Here we explain what diagrams with exterior boxes mean in BSBim\(^\text{Ext}\)(\(h, W)\). Given \( \xi \in \Lambda^*([V[-1]]) \) we have that

\[ \begin{array}{c} \langle \xi \rangle_s \begin{array}{c} \text{diag} \\ \phi \end{array} = \begin{array}{c} \text{diag} \\ \phi \end{array} \end{array} \]

where \( \lambda_s \) is the chain lift of the left unitor for \( B_s \) as defined in [Mak22] (Specifically, \( \lambda_s = K_\emptyset \otimes_R K_s \otimes_R K_\emptyset \otimes_R K_s \) while \( \overline{\tau} \) is the chain lift for the inverse of the left unitor as defined in Appendix A. If \( f \in \text{Hom}^1_{\text{rel}}(K_\emptyset, K_s) \), one can also check that

\[ \begin{array}{c} \langle \xi \rangle_s \begin{array}{c} \phi \end{array} = \begin{array}{c} \phi \end{array} \end{array} \]
as chain maps (each map has a different algebraic interpretation). Note that the relative positioning of the exterior boxes matter because of the Koszul sign rule. For example, suppose \( f \in \text{Hom}_{R\text{-}}^\bullet(K_s, K_s) \) and \( y \in \Lambda^1(V[−1]) \). Then we have that

\[
\begin{pmatrix}
\circ
\
\circ
\end{pmatrix} = -
\begin{pmatrix}
\circ
\
\circ
\end{pmatrix}
\]

4 Computation of \( \text{Ext}_{R\text{-}}^\bullet(B_t, BS(\omega)) \) for \( m_{st} = \infty \)

4.1 Warm-Up computation of \( \text{Ext}_{R\text{-}}^\bullet(R, B_s) \)

Lemma 4.1. Let \( B \) be a \( R\text{-} \)module and suppose there is a subset \( J \subset [n] \) s.t. \( a_j^i = 0 \) in \( B \forall j \in J \), then there is an isomorphism of complexes as bigraded right \( R\text{-} \)modules.

\[
\text{Hom}_{R\text{-}}(K(a^i_1, \ldots, a^i_n), B) \cong \text{Hom}_{R\text{-}}(\oplus_{j \in J} K(a^i_j), B) \otimes_{\mathbb{K}} \Lambda^\bullet(\oplus_{j \in J} ka_j^i[1](−|a_j|))
\]

where the right \( R\text{-} \)module structure is given by \((f \cdot r)(x) = f(x) \cdot r\).

Proof. We have the following chain of isomorphism of complexes

\[
\text{Hom}_{R\text{-}}(\oplus_{j \in J} K(a^i_j) \otimes R^e \otimes_{j \in J} K(a^i_j), B) \equiv \text{Hom}_{R\text{-}}(\oplus_{j \in J} K(a^i_j), \text{Hom}_{R\text{-}}(\oplus_{j \in J} K(a^i_j), B))
\]

\[
\equiv \text{Hom}_{R\text{-}}(\oplus_{j \in J} K(a^i_j), \Lambda^\bullet(\oplus_{j \in J} ka_j^i[1](−|a_j|)) \otimes_{\mathbb{K}} B)
\]

\[
\equiv \text{Hom}_{R\text{-}}(\oplus_{j \in J} K(a^i_j), B) \otimes_{\mathbb{K}} \Lambda^\bullet(\oplus_{j \in J} ka_j^i[1](−|a_j|))
\]

where the isomorphisms arise from the differential of the complex \( \Lambda^\bullet(\oplus_{j \in J} ka_j^i[1](−|a_j|)) \otimes_{\mathbb{K}} B \) being 0 since \( a_j^i = 0 \) in \( B \). It is easy to check that these are all maps of right \( R\text{-} \)modules.

Corollary 4.2. We have an isomorphism of bigraded right \( R\text{-} \) modules

\[
\text{Ext}_{R\text{-}}^\bullet(R, B_s) \cong H^\bullet(\text{Hom}_{R\text{-}}(K(\rho^s), B_s)) \otimes_{\mathbb{K}} \Lambda^\bullet(V[−1]/ka_s^y)
\]

More specifically we have an isomorphism

\[
\text{Ext}_{R\text{-}}^\bullet(R, B_s) \equiv \underset{\mathbb{K}}{\text{d}} R \otimes_{\mathbb{K}} \Lambda^\bullet(V[−1]/ka_s^y) \bigoplus \underset{\mathbb{K}}{\text{d}} R \otimes_{\mathbb{K}} \Lambda^\bullet(V[−1]/ka_s^y)
\]

where \( \mathbb{K} \) is the class of the map in \( \text{Hom}_{R\text{-}}^\bullet(K(\rho^s), B_s) \) sending \( p_s \equiv 1 \otimes 1 \rightarrow 1 \otimes 1 \), and \( \mathbb{K} \equiv 1 \otimes 1 \rightarrow 0 \).

Proof. The LHS above is the cohomology of the complex \( \text{Hom}_{R\text{-}}(K(\rho^s), B_s) \). From the decomposition \( V^* = ka_s \oplus (V^*)^\vee \), we see that we can write \( K(\rho^s) = K(\rho^s, e_s^1, \ldots, e_s^{n−1}) \) in Eq. (1). Here we use the isomorphism \( \Lambda^\bullet((V^*)^\vee[1])^* \equiv \Lambda^\bullet(V[−1]/ka_s^y) \) as explained in [Mak22] Section 2. Now apply Lemma 4.1 as \( e_s^i = 0 \) in \( B_s \) and we see that

\[
\text{Hom}_{R\text{-}}(K(\rho^s), B_s) \equiv \text{Hom}_{R\text{-}}(K(\rho^s), B_s) \otimes_{\mathbb{K}} \Lambda^\bullet(V[−1]/ka_s^y)
\]

(13)

It remains to compute the cohomology of \( \text{Hom}_{R\text{-}}(K(\rho^s), B_s) \) which is the cohomology of the following complex of right \( R\text{-} \) modules

\[
0 \rightarrow \frac{R(1) \oplus R(−1)}{−a_s \ 0 \ 1 \ 0} \rightarrow R(3) \oplus R(1) \rightarrow 0
\]

where we have decomposed \( B_s \equiv R(1) \oplus R(−1) = c_{id} R \oplus c_s R \) as a right \( R\text{-} \)module where

\[
c_{id} = \begin{pmatrix}
\circ
\circ
\end{pmatrix} \quad c_s = \begin{pmatrix}
\circ
\circ
\end{pmatrix}
\]

(14)
Here we are identifying a map in $\text{Hom}_{\mathcal{R}^e}(B_s, B_s)$ with its image after being applied to $1 \otimes_s 1$. This is exactly the $01$-basis for $B_s$, for more information see Section 12.1 of [EMTW20]. The differential $d^0 = \rho_s^*$ will be the bimodule map
\[
\rho_s - \rho_s = |s(\rho_s) - \rho_s + \delta_s(\rho_s)| - \alpha_s + \ldots.
\]
Applying this to the two basis vectors $c_{id}, c_i$ by stacking it above the diagrams in Eq. (14) and simplifying using diagrammatics yields the matrix above. We know $\text{Hom}_{\mathcal{R}^e}(R, B_s) = R_1$ so it remains to compute $\text{Ext}_{\mathcal{R}^e}^1(R, B_s)$. We have
\[
\ker d^1 = R [\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}] \oplus R [\begin{matrix} -\alpha_s \\ 1 \end{matrix}], \quad \text{im} d^0 = R [\begin{matrix} -\alpha_s \\ 1 \end{matrix}]
\]
Applying the invertible matrix to ker $d^1$ below
\[
Q = [\begin{matrix} 1 & -\alpha_s \\ 0 & 1 \end{matrix}]
\]
we see that ker $d^1 \cong Rc_{id} \oplus \text{im} d^0 \implies \text{Ext}_{\mathcal{R}^e}^1(R, B_s) = R_1$ as desired.

Explicitly the isomorphism in Eq. (13) is given on homogenous components by
\[
\begin{align*}
\text{Hom}_{\mathcal{R}^e}^k(K \otimes B_s) & \to \text{Hom}_{\mathcal{R}^e}(K(\rho_s^*), B_s) \otimes \Lambda^* ((V^*)^s[1])^* \\
\psi & \mapsto \sum_{1 \leq i_1 < \ldots < i_k \leq n-1} \left( \psi(e_{i_1} \wedge \ldots \wedge e_{i_k}) \right) \otimes \left( e_{i_1} \wedge \ldots \wedge e_{i_k} \right)^*
\end{align*}
\] (15)
where $v^*$ is the dual basis vector of $v \in \Lambda^* ((V^*)^s[1])$.

**Lemma 4.3.** For any $f \in \text{Hom}_{\mathcal{R}^e}(K(\rho_s^*), B_s)$ and $\xi \in \Lambda^* ((V^*)^s[1])^*$ the following relation holds at chain level
\[
\begin{array}{c c c}
\begin{array}{c}
\text{x}
\end{array} & \begin{array}{c}
\text{y}
\end{array} & \begin{array}{c}
\text{z}
\end{array} \\
\begin{array}{c}
\text{x}
\end{array} & \begin{array}{c}
\text{y}
\end{array} & \begin{array}{c}
\text{z}
\end{array}
\end{array} = f \otimes \xi
\] (16)
where $f \otimes 1^* \in \text{Hom}_{\mathcal{R}^e}(K \otimes B_s)$ under the isomorphism in Eq. (13).

**Proof.** This holds by direct computation.

In other words, the $\Lambda^* ((V^*)^s[1])^*$ part of $\text{Hom}_{\mathcal{R}^e}(K \otimes B_s)$ can always be extracted out as an exterior box leaving just $\text{Hom}_{\mathcal{R}^e}(K(\rho_s^*), B_s)$ which has the advantage of being much easier to work with as it’s a smaller complex. We will sometimes abuse notation and write $f \in \text{Hom}_{\mathcal{R}^e}(K(\rho_s^*), B_s)$ when we really mean $f \otimes 1^* \in \text{Hom}_{\mathcal{R}^e}(K \otimes B_s)$.

**Remark.** Recall that the morphism $\eta^s_{\text{Ext}}$ was defined to be the class of the map $\alpha^s \wedge (-) \in \text{Hom}_{\mathcal{R}^e}^{13}(K \otimes K_s)$. Post-composing the resolution $q_s : K_s \to B_s$ with the isomorphism in Eq. (15), we see that
\[
\begin{align*}
\text{Hom}_{\mathcal{R}^e}^k(K \otimes B_s) & \to \text{Hom}_{\mathcal{R}^e}^k(K \otimes B_s) \otimes \Lambda^* ((V^*)^s[1])^* \\
\alpha^s \wedge (-) & \mapsto \alpha^s \wedge (-) \mid \Lambda^* (V^*[1]) \otimes \mathcal{R}^e \sum_{i=1}^{n-1} \left( \rho_s \otimes 1 \otimes \left( \alpha^s \wedge e_i \right) \otimes R^e 1 \otimes \left( e_i \right)^* \right) + \left( \rho_s \otimes 1 \otimes \left( \alpha^s \wedge \rho_s \right) \otimes R^e 1 \otimes 1^* \right) = \frac{1}{2} \otimes 1^*
\end{align*}
\]
as $\langle \alpha^s, - \rangle$ is 0 on $(V^*)^s$. The first map is an isomorphism after taking cohomology. As $\frac{1}{2}$ was also used to denote $\eta^s_{\text{Ext}}$ in [Mak22], the above calculation and paragraph above this remark justifies our notation.

12
4.2 Main Computation

Let red correspond to the simple reflection $s$ and blue correspond to the simple reflection $t$. Let $w$ be an arbitrary expression in $s$ and $t$. For the rest of the paper we will also assume

Assumption 2. \( \exists \rho_s, \rho_t \in V^* \) such that \( \langle \alpha^{\vee}_s, \rho_s \rangle = 1 = \langle \alpha^{\vee}_t, \rho_t \rangle \) and \( \langle \alpha^{\vee}_s, \rho_t \rangle = 0 = \langle \alpha^{\vee}_t, \rho_s \rangle \).

In particular, this means that \( \rho_s \in (V^*)^t \) and that \( \rho_t \in (V^*)^s \).

Remark. This assumption is satisfied for both \( C^n \), the permutation realization, and the geometric realization for \( W_{m,n} \). This assumption is also satisfied for the Kac-Moody realization for \( W_\infty \), as the coroots \( \{ \alpha^{\vee}_s \} \) of the realization are linearly independent.

Lemma 4.4. Let \( F : \mathcal{A} \to \text{End}(\mathcal{C}) \) be a monoidal functor from \( (\mathcal{A}, \otimes) \) a monoidal category to the category of endofunctors of a category \( \mathcal{C} \). Suppose we have objects \( a_1, a_2 \in \mathcal{A} \) such that \( a_1 \otimes (-) \) is left adjoint to \( a_2 \otimes (-) \) in \( \mathcal{A} \), then

\[
\text{Hom}_\mathcal{C}(F_{a_1}(c), c') \equiv \text{Hom}_\mathcal{C}(c, F_{a_2}(c'))
\]

Proof. We need to give a unit and counit map satisfying the triangle identities. Since \( F \) is monoidal, we see that \( F_{a_2} \circ F_{a_1} = F_{a_2 \otimes a_1} \) and thus the unit map will just come from applying \( F \) to the unit map in \( \mathcal{A} \), \( 1 \to a_2 \otimes a_1 \) and similarly with the counit map since \( F \) is a functor these will satisfy the triangle identities.

Corollary 4.5. Let \( M, N \in \text{BSBim}(\mathfrak{h}, W) \). Then we have a natural isomorphism of right \( R \)-modules,

\[
\text{Ext}^{*}_{R^e}(B_t \otimes_R M, N) \cong \text{Ext}^{*}_{R^e}(M, B_t \otimes_R N)
\]

Proof. Apply the above lemma where \( \mathcal{A} = \text{BSBim}(\mathfrak{h}, W), \mathcal{C} = \text{BSBim}^\text{Ext}(\mathfrak{h}, W) \) and \( F \) is defined on objects as \( F(B_s) = K_s \otimes_R (-) \) and then extended monoidally. On morphisms \( F(f) = \tilde{f} \), where \( \tilde{f} \) is any chain lift of \( f \). Now use that \( B_t \otimes (-) \) is biaadjoint to itself with unit \( u_t \). Then the isomorphism above is given by is given by \( f \mapsto f \circ (u_t \otimes_R id_M) \) which is clearly right \( R \)-linear.

Our goal is to compute \( \text{Ext}^{*}_{R^e}(\text{BS}(w), \text{BS}(w)) \) but as seen above it suffices to compute Hochschild cohomology \( \text{Ext}^{*}_{R^e}(R, \text{BS}(w)) \). However it also suffices to compute \( \text{Ext}^{*}_{R^e}(B_t, \text{BS}(w)) \) for all \( w \) and this turns out to be easier. Because of Assumption 2, \( \rho_s, \rho_t \) will be linearly independent, so we can decompose \( V^* = k\rho_s \otimes k\rho_t \oplus (V^*)^{s,t} \) as \( s, t \) are reflections and thus fix a subspace of codimension 1. Let \( \Lambda_{st} = \Lambda^*((V^*)^{s,t}[1])^* \equiv \Lambda^*(V[-1]/(k\alpha^{\vee}_s \oplus k\alpha^{\vee}_t)) \). As such the resolution of \( B_t \) given in Eq. (2) can be written as

\[
K_i = K(\rho_t t(\rho_1)^e, \rho_s^e, e^e_1, \ldots, e^e_{n-2})(1)
\]

where \( \{e_i\}_{i=1}^{n-2} \) is a basis for \( (V^*)^{s,t} \). As \( e^e_i = 0 \) on \( \text{BS}(w) \) when \( e_i \in (V^*)^{s,t} \) we can use Lemma 4.1 to obtain

Theorem 4.6. We have an isomorphism of bigraded right \( R \) modules,

\[
\text{Ext}^{*}_{R^e}(B_t, \text{BS}(w)) \cong H^*(\text{Hom}_{R^e}(K(\rho_t t(\rho_1)^e, \rho_s^e)(1), \text{BS}(w))) \otimes_{\Lambda_{st}} \Lambda_{st}
\]

As in the previous section, this decomposition shows that elements in \( \Lambda_{st} \) act freely by contraction and so WLOG we will assume \( \Lambda_{st} = \mathbb{k} \) (and thus \( K_i = K(\rho_t t(\rho_1)^e, \rho_s^e)(1) \)) for the remainder of this paper to make notation easier.

To distinguish the terms \( \rho_s^e \) and \( \rho_t t(\rho_1)^e \) in the differential for \( \text{Hom}_{R^e}(K(\rho_t t(\rho_1)^e, \rho_s^e)(1), \text{BS}(w)) \) as \( w \) varies, let

\[
\rho_s^e(w) : \text{BS}(w) \to \text{BS}(w) \quad \rho_t t(\rho_1)^e(w) : \text{BS}(w) \to \text{BS}(w)
\]

Explicitly, \( K(\rho_t t(\rho_1)^e, \rho_s^e)(1) \) is the complex

\[
0 \to \rho_s \wedge \rho_t t(\rho_1)(1) \otimes R^e \xrightarrow{\rho_s} \rho_s(1) \otimes R^e \oplus \rho_t t(\rho_1)(1) \otimes R^e \xrightarrow{\rho_s^e} \rho_t t(\rho_1)^e \oplus \rho_s^e \otimes 1(1) \otimes R^e \to 0
\]
And thus it follows that \( \text{Hom}_{\mathcal{R}}(K(\rho_1 t(\rho_1)^e, \rho_1^e), \text{BS}(w)) \) will be the total complex of the following double complex

\[
\begin{array}{c}
\text{BS}(w) \xrightarrow{\rho_s} \text{BS}(w) \\
\text{BS}(w) \xrightarrow{\rho_s} \text{BS}(w)
\end{array}
\]

\[
\begin{array}{c}
\text{BS}(w) \xrightarrow{\rho_s} \text{BS}(w) \\
\text{BS}(w) \xrightarrow{\rho_s} \text{BS}(w)
\end{array}
\]

\[
\begin{array}{c}
\text{BS}(w) \xrightarrow{\rho_s} \text{BS}(w) \\
\text{BS}(w) \xrightarrow{\rho_s} \text{BS}(w)
\end{array}
\]

where the boxed term is in cohomological degree 0 and on the LHS an element \( b_1 \) for \( b \in \text{BS}(w) \) corresponds to the map sending \( 1 \otimes 1 \in K(\rho_1 t(\rho_1)^e, \rho_1^e) \) to \( b \) and on the RHS we have just replaced this by the corresponding internal degree, etc. It’s clear that the corresponding spectral sequence of the double complex degenerates at the \( E_2 \) page and so taking horizontal cohomology first the \( E_1 \) page will look like

\[
H^0(\text{Hom}_{\mathcal{R}}(K(\rho_1^e), \text{BS}(w)))(3) \quad H^1(\text{Hom}_{\mathcal{R}}(K(\rho_1^e), \text{BS}(w)))(5)
\]

\[
H^0(\text{Hom}_{\mathcal{R}}(K(\rho_1^e), \text{BS}(w)))(-1) \quad H^1(\text{Hom}_{\mathcal{R}}(K(\rho_1^e), \text{BS}(w)))(1)
\]

In fact, we will show in Section 4.2.3 that both arrows in Eq. (19) are 0. The bottom left corner of the \( E_2 \) page will compute \( \text{Ext}^{1,1}_{\mathcal{R}}(B_t, \text{BS}(w)) \) and the top right corner will compute \( \text{Ext}^{2,2}_{\mathcal{R}}(B_t, \text{BS}(w)) \). The diagonal will give us a filtration on \( \text{Ext}^{1,1}_{\mathcal{R}}(B_t, \text{BS}(w)) \). However, we will show that both groups are free right \( R \) modules, and thus the filtration splits and the \( E_2 \) page will also compute \( \text{Ext}^{1,1}_{\mathcal{R}}(B_t, \text{BS}(w)) \) as well. For the remainder of this section we will work in the case \( W = W_\infty \) and will show later that the results will also hold in the finite case \( W = W_{mst} \).

### 4.2.1 Computation of \( H^0(\text{Hom}_{\mathcal{R}}(K(\rho_1^e), \text{BS}(w))) \)

Throughout we will fix the total ordering on \( W_\infty \)

\[
id < s < t < ts < st < \ldots
\]

which refines the Bruhat order. For this section we will make the additional assumption

**Assumption 3.** \( \mathfrak{h} \) is a symmetric realization, i.e. \( a_{st} = a_{ts}, (a_s, a_t) \) is linearly independent and char \( k = 0 \).

Following Section 3 in [Eli16], set \( [2] = q + q^{-1} = -a_t = -a_s \) and let \( [m] = q^m - q^{-m} \) for \( m \in \mathbb{Z}^{\geq 0} \). It was shown in [Eli16] that \( [m] \) is in fact a polynomial in \( [2] \), and thus \( [m] \in k \) is well defined given \( a_{st} \).

Our computations below will use the light leaves basis for \( \text{BS}(w) \) as a free right \( R \)–module, more information can be found in Section 12.4 of [EMTW20]. Here is a brief summary. For any subexpression \( f \subset w \) (A subexpression for \( w = (w_1, \ldots , w_m) \) is a string \( f = (f_1, \ldots , f_m) \) where \( f_i \in \{0,1\} \), let \( r(f) = w^L \in W_\infty \). Then there is a (flipped) light leaf map

\[
\Pi_{w,f} : \text{BS}(r(f)) \to \text{BS}(w)
\]

One technically needs to choose a reduced expression for \( r(f) \in W \) to make sense of above. However this isn’t that big of an issue. For the affine case there is only one reduced expression for any \( x \in W_\infty \), so \( r(f) \) is a well defined expression. For the finite case there any two reduced expressions are related by braid moves and the only effect on the light leaf map will be the addition of various \( 2m_{st} \)–valent morphisms.

Now let \( c_{bat} = c_{id} \otimes_R \cdots \otimes_R c_{id} \in \text{BS}(r(f)) \).

Then the light leaves basis for \( \text{BS}(w) \) is the set

\[
\{ L_{w,f} = \Pi_{w,f}(c_{bat}) \mid f < w \}
\]

**Lemma 4.7.** Arrange the light leaves basis based on \( r(f) \) from least to greatest in the total ordering on \( W \) indicated above (can choose an arbitrary order on those elements with the same \( r(f) \)). In this basis, the matrix for \( \rho_1^e(w) \) as a right \( R \) module will be upper triangular with diagonal entries equal to \( r(w)^{-1}(\rho_s) - \rho_s \).

\(^1\)We will often abuse notation and let \( c_{bat} \) denote the corresponding element \( c_{id} \otimes_R \cdots \) for any choice of \( f \).
Proof. The proof of Proposition 12.26 in [EMTW20] shows
\[ \rho_s \cdot L_{w,e} = L_{w,e} \cdot r(e)^{-1}(\rho_s) + \sum_{w' < r(e)} L_{w',e} a_{f'} \quad a_{f'} \in R \]  
(20)
and thus the lemma follows. \qed

**Corollary 4.8.** When \( m_{st} = \infty \), the image of \( \rho_s^f(w) : BS(w) \to BS(w) \) is a free right \( R \) module and the kernel is also a free right \( R \) module with basis given by \( \{L_{w',e} | r(f) = \text{id or } t\} \).

**Proof.** By Lemma 4.7 we can find a right \( R \) basis for \( BS(w) \) such that \( \rho_s^f(w) \) is upper triangular with diagonal entries equal to \( r(e)^{-1}(\rho_s) - \rho_v \). It is clear that when \( r(e) = \text{id or } t \) that this expression is 0; we claim in all other cases, this will not be zero. One can easily check by induction that
\[ s(ts)^m(\rho_s) = \rho_s - \sum_{k=0}^m (st)^k(\alpha_s) \]  
(21)
\[ (ts)^m(\rho_s) = \rho_s - \sum_{k=0}^{m-1} (tst)^k(\alpha_s) \]  
(22)
By Claim 3.5 in [Eli16], one then computes that
\[ s(ts)^m(\rho_s) - \rho_s = -\sum_{k=0}^m [(2k+1)\alpha_s + [2k]\alpha_t] \]  
(23)
\[ (ts)^m(\rho_s) - \rho_s = -\sum_{k=0}^{m-1} [(2k+1)\alpha_s + [2k+2]\alpha_t] \]  
(24)
To show Eq. (23) is nonzero, as \( \alpha_s, \alpha_t \) are linearly independent, it suffices to show the sums
\[ O_m = \sum_{k=0}^m [2k+1] \quad E_m = \sum_{k=0}^m [2k] \]
don’t simultaneously vanish. Similarly, to show Eq. (24) is nonzero it suffices to show \( O_{m-1} \) and \( E_m \) don’t simultaneously vanish. Depending on the realization there are two cases. First if \( q \neq \pm 1 \), it’s clear that \( O_m, E_m \) is never zero for \( m \geq 0, m \geq 1 \) respectively, in characteristic zero. Now for \( q \neq \pm 1 \), one computes that
\[ O_m = \frac{q + q^3 + \ldots + q^{2m+1} - q^{-1} - q^{-3} - \ldots - q^{-(2m+1)}}{q - q^{-1}} \]
\[ = \frac{1 - q^{2m+2} + 1 - q^{-(2m+2)}}{(1 - q^{-2})(1 - q^2)} = \frac{(q^m - 1)(q^{m+1} - 1)}{(1 - q^{-2})(1 - q^2)} \]
\[ E_m = \frac{q^2 + q^4 + \ldots + q^{2m} - q^{-2} - q^{-4} - \ldots - q^{-(2m)}}{q - q^{-1}} = \frac{q^2 - q^{-2m} + 1 - q^{2m+2}}{(q - q^{-1})(1 - q^2)} \]
\[ = \frac{q^{-2m}(q^{2m+2} - 1) - (q^{2m+2} - 1)}{q^{-1}(1 - q^2)} = \frac{(q^{-2m} - 1)(q^{2m+2} - 1)}{(q - q^{-1})(1 - q^2)} \]
It follows that \( O_m = 0 \iff q^{2m+2} = 1 \) and \( E_m = 0 \iff q^{2m} = 1 \). It follows that
\[ E_k(23) = 0 \iff q^{2m+2} = 1 \quad E_k(24) = 0 \iff q^{2m} = 1 \]  
(25)
But, as noted in [Eli16], a realization for \( W_{\infty} \) will be a realization for \( W_k \iff q^{2k} = 1 \). Thus, for \( q^k \) to be a (faithful) realization for \( W_{\infty} q^m \neq 1 \forall m \in \mathbb{Z}^+ \) and thus Eq. (23) is never 0 for \( m \geq 0 \) and Eq. (24) is never zero for \( m \geq 1 \).

Now note that when \( r(e) = \text{id or } t \) not only does \( r(e)^{-1}(\rho_s) - \rho_s = 0 \), but in fact \( L_{w,e} \in \ker \rho_s^f(w) \). This is clear from the diagramatic picture as \( r(e) \) corresponds to the red or blue lines protruding below \( L_{w,e} \); and it is precisely these lines which we need to slide \( \rho_s \) from left to right over.

Therefore the image of \( \rho_s^f(w) \) will be the span of \( \{\rho_s^f(w) L_{w,e} | r(f) \neq \text{id or } t\} \). As shown above, we can arrange these vectors into a matrix that is upper triangular with nonzero entries on the diagonal. Because \( R \) is a domain this will imply that \( \{\rho_s^f(w) L_{w,e} | r(f) \neq \text{id or } t\} \) is linearly independent over \( R \) and therefore free. It then follows that \( \ker \rho_s^f(w) \) is generated by \( \{L_{w,e} r(f) = \text{id or } t\} \) and thus is a basis. \qed
4.2.2 Computation of \( H^1(\text{Hom}_{R^e}(K(\rho^e_s), \text{BS}(w))) \)

In this section we will use the 01-basis for \( \text{BS}(w) \) as defined in Chapter 12 of [RMTW]. In particular for \( w = (s_1, \ldots, s_m) \), \( c_{\text{bot}} \) as defined in Section 4.2.1 and

\[
c_{\text{top}} = c_{s_1} \otimes_R \cdots \otimes_R c_{s_m} \in \text{BS}(w)
\]

are two special elements in this basis.

**Definition 4.9.** Define \( \text{Tr} : \text{BS}(w) \to R \) by sending any element \( b \) to the coefficient of \( c_{\text{top}} \) when \( b \) is expressed in the 01-basis as a right \( R \)–module.

**Definition 4.10.** The global intersection form on \( \text{BS}(w) \) is the \( R \)–valued pairing

\[
\langle -, - \rangle : \text{BS}(w) \times \text{BS}(w) \to R
\]

defined by

\[
\langle a, b \rangle = \text{Tr}(ab)
\]

The global intersection form on \( \text{BS}(w) \) is non-degenerate as seen in [EMTW20] 18.2.2.2 and gives an isomorphism \( D_w : \text{BS}(w) \cong D(\text{BS}(w)) \) sending \( v \mapsto \langle v, - \rangle \).

Recall that in Corollary 4.8 we found a right \( R \) basis for \( \ker \rho_s(w) \) given by \( \{ L_w, f \mid r(f) = \text{id or } t \} \). For \( f = 00 \ldots \), we clearly have \( r(f) = \text{id} \) and the corresponding light leaves will be all start dots, i.e. the element \( c_{s_1} \otimes_R \cdots \otimes_R c_{s_m} \) which is exactly \( c_{\text{top}} \) in the 01-basis. One can easily check that the dual basis vector under the global intersection form will be \( c_{\text{bot}} \) which we now denote

\[
1(w) := c_{id} \otimes_R \cdots \otimes_R c_{id} \in D(\ker \rho_s^e(w))
\]

**Definition 4.11.** Let \( \mathbb{D} : R^e \text{-gmod} \to R^e \text{-gmod} \) be the functor

\[
\mathbb{D}(N) := \text{Hom}_{R^e}(N, R)
\]

where \( \_R \) means we take right \( R \) module homomorphisms. This has a \( R^e \text{-gmod} \) structure defined as \( (r \cdot f \cdot r')(b) := r \cdot (f(b)) \cdot r' \).

**Theorem 4.12.** We have an isomorphism of right \( R \) modules

\[
H^1(\text{Hom}_{R^e}(K(\rho_s^e), \text{BS}(w))) \cong \mathbb{D}(\ker \rho_s^e(w))
\]

**Proof.** We have a commutative diagram

\[
\begin{array}{ccc}
\text{BS}(w) & \overset{\rho^e_s(w)}{\longrightarrow} & \text{BS}(w) \\
\downarrow{D_w} & & \downarrow{D_w} \\
\mathbb{D}(\text{BS}(w)) & \overset{\mathbb{D}(\rho^e_s(w))}{\longrightarrow} & \mathbb{D}(\text{BS}(w))
\end{array}
\]

(26)

which follows from adjointness of the global intersection form with multiplication by elements of \( R \). As \( D_w \) is an isomorphism it follows that \( H^1(\text{Hom}_{R^e}(K(\rho_s^e), \text{BS}(w))) = \ker \rho_s^e(w) \cong \mathbb{D}(\rho_s^e(w)) \). From Corollary 4.8 \( \text{im} \rho_s^e(w) \) is free and so we have a SES

\[
0 \to \mathbb{D}(\text{im} \rho_s^e(w)) \xrightarrow{\mathbb{D}(\rho^e_s(w))} \mathbb{D}(\text{BS}(w)) \to \mathbb{D}(\ker \rho_s^e(w)) \to 0
\]

and thus it follows that \( \mathbb{D}(\rho^e_s(w)) \cong \mathbb{D}(\ker \rho_s^e(w)) \). \( \square \)
4.2.3 Computation of $H^\ast(\text{Hom}_{R^e}(K(r_1 t(r_t)^e, \rho_w^t), \text{BS}(w)))$

Using our results above, Eq. (19) which is the $E_1$ page of the spectral sequence computing $H^\ast(\text{Hom}_{R^e}(K(r_1 t(r_t)^e, \rho_w^t), \text{BS}(w)))$ as a right $R$–module now reads

$$\begin{array}{c}
\ker \rho_w^t(u)(3) & \overset{D(\ker \rho_w^t(u))(5)}{\longrightarrow} & \text{ker}(\rho_w^t(u)(-1)) \\
\rho_1 t(r_1)^e(u) & \overset{D_2(\rho_1 t(r_1)^e(u))|_{\ker \rho_w^t(u)}}{\longrightarrow} & D(\ker \rho_w^t(u))(1)
\end{array}$$

(27)

We claim that both vertical arrows above are $0$. From Corollary 4.8 we know that $\ker \rho_w^t(u)$ has a basis given by

$$\{L_w f = LL_w f(c_b a_l)| r(f) = id \text{ or } t\}$$

It follows that $\ker \rho_w^t(u) \subseteq \ker \rho_1 t(r_1)^e(u)$ aka $\rho_1 t(r_1)^e(u)|_{\ker \rho_w^t(u)}$ as we can always slide $\rho_1 t(r_1)$ over to the other side.

For the right hand $\rho_1 t(r_1)^e$ note that replacing $\text{BS}(w)$ with $\ker \rho_w^t(u)$ in Eq. (26) still produces a commutative diagram. It follows that for any $v \in \ker \rho_w^t(u)$ we have

$$D_2(\rho_1 t(r_1)^e(u)|_{\ker \rho_w^t(u)}(v, -)) = \langle \rho_1 t(r_1)^e(v), -\rangle|_{\ker \rho_w^t(u)} = \langle v, \rho_1 t(r_1)^e(-)\rangle|_{\ker \rho_w^t(u)} = 0$$

where the last equality follows from the previous paragraph. Thus from Theorem 4.6 we see that

Theorem 4.13. When $m_{\lambda t} = \infty$ and $\Lambda_{\lambda t} = k$ we have an isomorphism of right $R$–modules,

$$\begin{align*}
\text{Ext}^0_{R^e}(B_r, \text{BS}(w)) & \cong \ker \rho_w^t(u)(-1) \cong \ker \rho_w^t(u)(-1) \\
\text{Ext}^1_{R^e}(B_r, \text{BS}(w)) & \cong \ker \rho_w^t(u)(w) \rho_1 t(r_1)(-1) \oplus D(\ker \rho_w^t(u))(1) \\
& \cong \ker \rho_w^t(u)(3) \oplus D(\ker \rho_w^t(u))(1) \\
\text{Ext}^2_{R^e}(B_r, \text{BS}(w)) & \cong D(\ker \rho_w^t(u))(\rho_1 t(r_1)(-1)) \cong D(\ker \rho_w^t(u))(5)
\end{align*}$$

Proof. Follows from the fact that both arrows going up in Eq. (27) are $0$, and that $\ker \rho_w^t(u)$ is a free right $R$–module, so the filtration/SES

$$0 \hookrightarrow \ker \rho_w^t(u)(3) \twoheadrightarrow \text{Ext}^1_{R^e}(B_r, \text{BS}(w)) \twoheadrightarrow \text{Ext}^2_{R^e}(B_r, \text{BS}(w)) \twoheadrightarrow \text{Ext}^3_{R^e}(B_r, \text{BS}(w)) \twoheadrightarrow 0$$

from the spectral sequence splits. □

5 New Generator and Relations in $\text{BSBim}^{\text{Ext}}(f, W_{\infty})$

We will continue to assume that Assumption 1, Assumption 2, and Assumption 3 hold in this section.

5.1 Affine Dimension Calculations

Definition 5.1. An expression is called non repeating if there are no subexpressions of the form $ss \ldots$ or $tt \ldots$.

Definition 5.2. Let $|w|$ be the number of elements in the expression $w$.

Lemma 5.3. Suppose $m_{\lambda t} = \infty$ and $\underline{w}$ is a non repeating expression. If $|w|$ is odd, then the lowest internal degree element in $\text{Hom}_{R^e}(R, \text{BS}(w))$ is of degree 1. If $|w|$ is even, then the lowest internal degree element in $\text{Hom}_{R^e}(R, \text{BS}(w))$ is of degree 2.

Proof. It suffices to show the case when $|w|$ is even as when $|\underline{w}|$ is odd $\underline{w} = (t, \ldots, t)$ or $(s, \ldots, s)$ and we have the following natural isomorphism of graded vector spaces

$$\begin{align*}
\text{Hom}_{R^e}(R, \text{BS}(t, \ldots, t)) & \cong \text{Hom}_{R^e}(B_1, \text{BS}(t, \ldots, t)) \cong \text{Hom}_{R^e}(B_1, B_1 \otimes_R \text{BS}(t, \ldots, t)) \\
& \cong \text{Hom}_{R^e}(B_1, \text{BS}(t, \ldots, t)(1)) \oplus \text{Hom}_{R^e}(B_1, \text{BS}(t, \ldots, t)(-1))
\end{align*}$$
and so the lowest internal degree element in $\text{Hom}_{\mathbb{R}^e}(R, \text{BS}(t, \ldots, t))$ is precisely 1 less than the lowest internal degree element in $\text{Hom}_{\mathbb{R}^e}(B_t, \text{BS}(t, \ldots, t))$.

We now proceed by induction on $|w|$ and use the diagrammatic description of $\text{Hom}_{\mathbb{R}^e}(R, \text{BS}(w))$. As $|w|$ is even the first and last element of $w$ are not the same and so the lowest internal degree element in $\text{Hom}_{\mathbb{R}^e}(R, \text{BS}(w))$ must decompose as

\[
\begin{array}{cc}
\begin{array}{c}
\vspace{1em}
\end{array}
\end{array}
\]

where $a \in \text{Hom}_{\mathbb{R}^e}(R, \text{BS}(w'))$ and $c \in \text{Hom}_{\mathbb{R}^e}(R, \text{BS}(w''))$ such that $(w', w'') = w$ because morphisms are only generated by red and blue trivalent vertices and dots when $m_{st} = \infty$. First assume that both $|w'|$ and $|w''|$ are odd. Notice that $a$ has to be the lowest internal degree element in $\text{Hom}_{\mathbb{R}^e}(R, \text{BS}(w'))$ as otherwise we can replace it with the lowest. Same for $c$. Therefore by induction they both have degree 1 and thus the lowest degree element in $\text{Hom}_{\mathbb{R}^e}(R, \text{BS}(w))$ has degree 2.

Now if $|w'|$ and $|w''|$ are both even, one can show that it can't be the lowest internal degree element as there will exist dots (because $m_{st} = \infty$) which can be turned into trivalent vertices, which lowers the degree.

**Lemma 5.4.** For $m_{st} = \infty$ let $w$ be any expression and let $m(t, w)$ be any non repeating subexpression of $tw$ such that $|m(t, w)|$ is maximal among all non repeating subexpressions of $tw$. If $|m(t, w)|$ is odd, then the lowest internal degree element of $\text{ker} \rho_s^t(w)(-1)$ is $|w| + |m(t, w)|$. If $|m(t, w)|$ is even, then the lowest internal degree element of $\text{ker} \rho_s^t(w)(-1)$ is $1 - |w| + |m(t, w)|$.

**Proof.** As $\text{ker} \rho_s^t(w)(-1) \cong \text{Hom}_{\mathbb{R}^e}(B_t, \text{BS}(w))$ it suffices to show the lowest degree in $\text{Hom}_{\mathbb{R}^e}(B_t, \text{BS}(w))$ is as prescribed above. Note

\[
\text{Hom}_{\mathbb{R}^e}(B_t, \text{BS}(w)) \cong \text{Hom}_{\mathbb{R}^e}(R, R \otimes_R BS(w))
\]

Choose some $m(t, w)$ as defined above. Then the part of $tw$ excluding $m(t, w)$ has to be of the form $ss \ldots$ or $tt \ldots$. As $B_s \otimes_R B_s \cong B_s(1) \oplus B_s(-1)$, we can then simplify $B_t \otimes_R BS(w)$ to $BS(m(t, w))$ at the cost of some grading shifts. As there are exactly $|tw| - |m(t, w)| = 1 + |w| - |m(t, w)|$ places in $tw$ where we need to apply this relation, it follows that

\[
\text{rk} \text{Hom}_{\mathbb{R}^e}(B_t, \text{BS}(w)) = \text{rk} (v + v^{-1})^{1+|w|-|m(t, w)|} \text{Hom}_{\mathbb{R}^e}(R, BS(m(t, w)))
\]

Now apply the previous lemma.

**Proposition 5.5.** For $m_{st} = \infty$, $\text{Ext}_{\mathbb{R}^e}^{1, -|w|+1}(B_t, \text{BS}(w))$ is a 1 dimensional $k$ module when $|m(t, w)| \geq 4$.

**Proof.** Under the decomposition from Theorem 4.13

\[
\text{Ext}_{\mathbb{R}^e}^{1, -|w|+1}(B_t, \text{BS}(w)) \cong \text{ker} \rho_s^t(w)(3) - |w| - 1 \oplus \mathbb{D}(\text{ker} \rho_s^t(w)(1) - |w| - 1)
\]

$\mathbb{D}(\text{ker} \rho_s^t(w)(1) - |w| - 1)$ is a 1 dimensional $k$ module, as $1(w)$ is the lowest degree element in $\mathbb{D}(\text{BS}(w)) \cong \text{BS}(w)$ and we showed above that $1(w) \in \mathbb{D}(\text{ker} \rho_s^t(w))$. Thus it suffices to show the lowest degree element in $\text{ker} \rho_s^t(w)(3)$ is greater than this. By Lemma 5.4, we see that when $|m(t, w)|$ is odd, the lowest degree element in $\text{ker} \rho_s^t(w)(3)$ will be $-|w| + |m(t, w)| - 4$ and thus we want

\[
-|w| + |m(t, w)| - 4 > -|w| - 1 \implies |m(t, w)| > 3
\]

Similarly we find that when $|m(t, w)|$ is even, we want

\[
-|w| + |m(t, w)| - 3 > -|w| - 1 \implies |m(t, w)| > 2
\]

and thus $|m(t, w)| \geq 4$ encompasses both cases above.

**5.2 The New Generator**

We will conti

**Lemma 5.6.** Suppose we have a double complex $C^{**}$ with differentials and terms pictured below
(where \(C^{0,0} = X\) is boxed) such that

1. \(\ker f \subset \ker g\)
2. \(\text{im} g' \subset \text{im} f'
3. \(\ker f' = 0\)

Then \(H^1(\text{Tot}(C^{•,•})) \cong Y / \text{im} f\). Precisely this means that

- Any \(y \in Y\) extends to a unique 1-cocycle \((y, z) \in Y \times Z\) in \(\text{Tot}(C^{•,•})\)
- two 1-cocycles \((y, z)\) and \((y', z')\) define the same class in \(H^1(\text{Tot}(C^{•,•}))\) \iff \(y \equiv y'\) in \(Y / \text{im}(f)\)

**Proof.** Straightforward from assumptions. \(\square\)

**Remark.** Even without the assumptions (1) – (3) above, if \(y \in \text{im}(f)\) and \((y, z)\) is any cocycle, then \((y, z) \equiv (0, 0)\) in \(H^1(\text{Tot}(C^{•,•}))\) and similarly if \(z \in \text{im}(g)\) and \((y, z)\) is any cocycle, then \((y, z) \equiv (0, 0)\) in \(H^1(\text{Tot}(C^{•,•}))\). This follows from the definition of \((y, z)\) being a cocycle.

**Corollary 5.7.** Suppose \(|m(t, w)| \geq 4\). Then for any \(u \in \text{BS}(w) - |w|\), \(\exists ! v \in \text{BS}(w) - |w| + 2\) such that the map \(\psi^u_s : \rho_s(1) \otimes \rho_t(1) \otimes 1 \otimes 1 \to \text{BS}(w)\) defined by

\[
\psi^u_s(\rho_t(1) \otimes 1 \otimes 1) = u
\]

is a cocycle in \(\text{Hom}^{1-|w|+1}(K_t, \text{BS}(w))\) and any relation in \(\text{Ext}^{1-|w|+1}(B_t, \text{BS}(w))\) is determined by evaluating on \(\rho_s \otimes 1 \otimes 1\).

**Proof.** The results will follow from applying Lemma 5.6 to Eq. (18) in internal degree \(-(|w|+1)\) with \(f, f' = \rho_s^w(w)\) and \(g, g' = \rho_t\rho_t^w(w)\). Condition (1) was shown in **Section 4.2.3**, (2) follows from self-duality and (3) follows from the proof of **Proposition 5.5** where it was shown that \(\ker \rho_s^w(w)(3-|w|+1) = 0\) when \(|m(t, w)| \geq 4\). \(\square\)

**Definition 5.8.** Suppose \(|m(t, w)| \geq 4\). Let \(u = 1(w) \in \text{BS}(w) - |w|\) in Lemma 5.6 and define \(\Phi_T^w \in \text{Ext}^{1-|w|+1}(B_t, \text{BS}(w))\) to be

\[
\Phi_T^w = \left[ \psi_s^1\psi_s^w(w) \right] v
\]

where \(v\) is the unique element in \(\text{BS}(w) - |w| + 2\) such that \(\rho_s^v(1) \equiv \rho_t(1) \otimes 1 \otimes 1\) with \(v \in \text{BS}(w) - |w|+2\) such that \(\rho_t^v(1) \equiv \rho_t(1) \otimes 1 \otimes 1\). This follows from the proof of **Proposition 5.5** where it was shown that \(\ker \rho_s^v(1) = 0\) when \(|m(t, w)| \geq 4\).

Except when \(|w|\) is small, we will not give a description for what the corresponding \(v\) should be above, and **Lemma 5.6** essentially tells us don’t need to. We will denote \(\Phi_T^w\) diagrammatically as

\[
\begin{array}{c}
\text{w} \quad \text{or} \\
\end{array}
\]

where the purple lines indicate either \(s\) or \(t\). The red hollow dot is used to indicate that we will typically only know what \(\Phi_T^w\) does on \(\rho_s \otimes 1 \otimes 1\). Likewise for \(\Phi_T^w\) we will color the hollow dot in the middle blue to denote as seen in the left below. In the case when the color of the bottom strand is not stated, we will color the hollow dot in the middle yellow as seen on the right below.
Remark. The isomorphism

\[ \text{Hom}_{K^h(\text{Proj}(R^t)))}(K_t, K(u)) \xrightarrow{d_w} \text{Hom}_{\mathcal{D}^h(R^e)}(K_t, \text{BS}(w)) \]  

(28)

allows us to lift \( \Phi^u \) to a morphism \( \Phi^u_\ast \in \text{Hom}_{\mathcal{D}^h(R^e)}(K_t, K(u)) \) which one needs in order to compute relations in \( \text{BSBi}_{\text{Ext}}^h(h, W_0) \). However like in Appendix A we don’t need to find the entire chain lift and it suffices to work with just \( \Phi^u_0 : K_t^{[-1]} \rightarrow K(u)^0 \) (the notation here means that the map goes from cohomological degree \(-1\) of \( K_t \) to cohomological degree \( 0 \) of \( K(u) \)) which is defined as

\[ \Phi^u_0 : = a \otimes d \cdots \otimes \psi^{-1}_s \otimes \cdots \otimes 1 : \rho_t t(\rho_t) \otimes R^e \otimes \rho_s \otimes R^e \rightarrow 1 \otimes R^0[w]^{d+1} \]

where \( v = a \otimes s_1 \otimes \cdots \otimes s_m d + \cdots \). In other words, we have replaced an expression \( a \otimes s_1 \cdots \otimes s_m d \in \text{BS}(w) \) with \( a \otimes \cdots \otimes d \in R^0[w]^{d+1} \).

Remark. To prevent notation overload, some of the relations in the following subsections are “incorrectly written”. Specifically, on one side of a relation will end up in \( K(u) \) for some \( w = (s_1, \ldots, s_m) \) and the other side will end up in \( K(u) \otimes_R K_S \) or \( K_S \otimes_R K(u) \), etc. This happens when the relation involves the counit \( \epsilon_i : K_t \rightarrow K_S \) for instance. One then needs to apply a chain lift of the right unitor \( \tilde{\epsilon}_s \) or the left unitor \( \tilde{\lambda}_s \) to one side to have actual equality. We will do this in the calculations by replacing \( \text{id}_w \otimes_R \epsilon_i \) with

\[ \text{id}_w \otimes_R \epsilon_i := (\text{id}_{s_1, \ldots, s_m}) \otimes_R (\alpha_{s_m}) \circ (\text{id}_w \otimes_R \epsilon_s) : K(w) \otimes_R K_s \rightarrow K(u) \otimes_R K_S \rightarrow K(u) \]

in the case of \( K(u) \otimes_R K_S \) in our calculations. In fact, we will always end up in the cohomological degree \( 0 \) part of \( K(u) \otimes_R K_S \) after applying \( \text{id}_w \otimes_R \epsilon_i \), and so explicitly we then apply

\[ (\text{id}_{s_1, \ldots, s_m}) \otimes_R (\alpha_{s_m}) (1 \otimes a_1 \otimes \cdots \otimes a_{m+1} \otimes a_{m+2}) = 1 \otimes a_1 \otimes \cdots \otimes a_{m+1} a_{m+2} \]

and similarly with \( K_S \otimes_R K(u) \), etc.

### 5.3 Low Strand Relations

For the expression \( tsts \) we have \( m(tsts) = tsts \) and \( t(tsts) \geq 4 \). Therefore \( \Phi^{(s,t,s)}_t \) is defined.

**Lemma 5.9.** \( -a_t \otimes s \otimes t \otimes s \otimes t \otimes s \) is a cocycle representative for \( \Phi^{(s,t,s)}_t \)

**Proof.** By Lemma 6.6 it suffices to show

\[ \rho^e_s(\tilde{\lambda}_s)[(-a_t \otimes s \otimes t \otimes s 1 + c_t \otimes t 1 \otimes s 1 + c_t 1 \otimes 1 \otimes s 1 + c_t 1 \otimes 1 \otimes t 1 c_s) = \rho^e_t t(\rho^e_t)[(s^t_s)(1 \otimes s 1 \otimes 1 \otimes 1) \]  

which one can check by a brute force calculation. \( \square \)

**Lemma 10.** \( s(\rho_s) + a_t s \rho_t + \psi_s \in (V^\ast)^{s,t} \)

**Proof.** Easy check. \( \square \)

**Proposition 5.11.** \( \Phi^{(s,t,s)}_t \) is signed rotation invariant. Specifically,

\[ (\text{id}_t \otimes_R \text{id}_{s,t} \otimes_R \epsilon_s) \circ (\text{id}_t \otimes_R \text{id}_{s,t} \otimes_R \mu_s) \circ (\text{id}_s \otimes_R \Phi^{(s,t,s)}_t \otimes_R \text{id}_s) \circ (\delta_t \otimes_R \text{id}_s) \circ (\eta_t \otimes_R \text{id}_s) = -\Phi^{(s,t,s)}_t \]

And similarly,

\[ (\epsilon_s \otimes_R \text{id}_{s,t} \otimes_R \text{id}_t) \circ (\mu_s \otimes_R \text{id}_{s,t} \otimes_R \text{id}_t) \circ (\text{id}_s \otimes_R \Phi^{(s,t,s)}_t \otimes_R \text{id}_s) \circ (\text{id}_s \otimes_R \delta_t) \circ (\text{id}_s \otimes_R \eta_t) = -\Phi^{(s,t,s)}_t \]
Proof. We prove the first equality as the second is quite similar (and in fact follows from the first). Diagrammatically we want to show

\[
\begin{array}{c}
\text{From Corollary 5.7 it suffices to show both sides agree on } \rho_t \otimes 1 \otimes 1. \text{ As } (V^*)^I = k \rho_s \otimes (V^*)^{s,t}, \text{ From Lemma 5.10 we have that } \\
p_t + t(p_r) = -a_{st} \rho_s + r \text{ where } r = p_t + t(p_r) + a_{st} \rho_s \in (V^*)^{s,t}. \text{ The first part of the diagram will be the} \\
\text{series of maps} \\
\begin{align*}
K_t & \otimes_R K_t \otimes_R K_s \\
\delta_t^1 & \otimes_{R \text{id}_s} \\
K_t & \otimes_R K_s \\
\eta_t^{-1} & \otimes_{R \text{id}_s} \\
K_{\otimes} & \otimes_R K_s \\
\epsilon_t^1 & \\
K_s & \\
\end{align*}
\end{array}
\]

where we have used the chain lifts defined in Appendix A. Note there is a slight abuse of notation as recall \( \rho_t^{(2)} = 1 \otimes_R \rho_t \), and so we have to apply \( \eta_t^0 \otimes_R \text{id}_s \) to \( \rho_t^{(2)} \otimes 1 \otimes 1 \) when going from the third line to the the second line from the top. At the next step we apply

\[
\text{id}_t \otimes_R a_{st} \rho_s \otimes 1 + c_s \otimes 1 + s - 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1
\]

so only the \( (2) \) terms survive. Because \( r \in (V^*)^{s,t} \), the \( r^{(2)} \) term also disappears. Thus, the next series of maps will be

\[
\begin{align*}
K_t & \otimes_R K(s,t) \\
\text{id}_t & \otimes_{R \text{id}_t \otimes \text{id}_s} \\
K_t & \otimes_R K(s,t) \otimes_R K_s \\
\text{id}_t & \otimes_{R \text{id}_t \otimes \text{id}_s} \\
K_t & \otimes_R K(s,t) \otimes_R K_s \\
\text{id}_t & \otimes_{R \Phi_t^{(s,t,0)}} \\
K_t & \otimes_R K(s,t) \otimes_R K_s \\
\end{align*}
\]

which is exactly equal to \( -\Phi_t^{(s,t,0)} \otimes_R 1 \otimes 1 \) as desired. \( \square \)

**Proposition 5.12.** We have the following reduction identity in \( \text{Ext}_{R^e}^{1,-3}(B_t, B_s, B_t) \).

\[
(\text{id}_s \otimes R \text{id}_t \otimes \text{id}_s) \circ \Phi_t^{(s,t,0)} = (\text{id}_s \otimes \text{id}_t \otimes R \text{id}_s) \circ \tau_t - (\text{id}_s \otimes R \text{id}_t) \circ (\text{id}_s \otimes R \Phi_t) \circ \tau_t
\]

Diagrammatically this will be of the form,

\[
\begin{array}{c}
\text{Diagrammatically this will be of the form,} \\
\begin{array}{c}
\end{array}
\end{array}
\]
Proof. We have to check that both sides agree on both \( \rho_s \boxtimes 1 \otimes 1 \) and \( \rho_t t(\rho_t) \boxtimes 1 \otimes 1 \) as \( \dim_s \text{Ext}^{1,-3}_R(B_t, B_t) \neq 1 \). The RHS will be
\[
\left( \eta_s^{\text{Ext}} \otimes_R \text{id}_t \right) \circ \tilde{\tau}_t^{-1}(\rho_s \boxtimes 1 \otimes 1) = \left( \eta_s^{\text{Ext}} \otimes_R \text{id}_t \right) (\rho_s^{(1)} \boxtimes 1 \otimes 1 + \rho_s^{(2)} \boxtimes 1 \otimes 1) = 1 \boxtimes 1 \otimes 1
\]
and thus the sum agrees with \( (\eta_s^0 \otimes_R \text{id}_t \circ (\text{id}_s \otimes_R \tilde{\phi}_t^{-1}) \circ \tilde{\tau}_t^{-1}(\rho_s \boxtimes 1 \otimes 1) = (\eta_s^0 \otimes_R \text{id}_t) \circ (\text{id}_s \otimes_R \tilde{\phi}_t^{-1})(\rho_s^{(1)} \boxtimes 1 \otimes 1 + \rho_s^{(2)} \boxtimes 1 \otimes 1) = 0 \)
while
\[
(\eta_s^0 \otimes_R \text{id}_t) \circ (\text{id}_s \otimes_R \tilde{\phi}_t^{-1}) \circ \tilde{\tau}_t^{-1}(\rho_t t(\rho_t) \boxtimes 1 \otimes 1) \\
= (\eta_s^0 \otimes_R \text{id}_t) \circ (\text{id}_s \otimes_R \tilde{\phi}_t^{-1})(\rho_t \boxtimes 1 \otimes 1 + t(\rho_t) \boxtimes 1 \otimes 1 + \gamma_t^{(2)} \boxtimes 1 \otimes 1) \\
= (\eta_s^0 \otimes_R \text{id}_t)(-1 \boxtimes 1 \otimes 1) = -1 \boxtimes c_s \otimes 1
\]
On the other hand, we calculate that
\[
(\text{id}_s \otimes_R \text{id}_t \circ \tilde{\phi}_t^{-1}) \circ \tilde{\tau}_t^{-1}(\rho_t t(\rho_t) \boxtimes 1 \otimes 1) \\
= (\text{id}_s \otimes_R \text{id}_t \circ \tilde{\phi}_t^{-1}) \circ \tilde{\tau}_t^{-1}(\rho_t \boxtimes 1 \otimes 1 + t(\rho_t) \boxtimes 1 \otimes 1 + \gamma_t^{(2)} \boxtimes 1 \otimes 1) \\
= 1 \boxtimes -a_{st} \otimes \rho_t \otimes 1 + a_s \boxtimes 1 \otimes 1 + 1 \otimes 1 \otimes a_s - 1 \otimes 1 \otimes c_s
\]
as \( \tilde{\phi}_t^{-1}(c_s) = \rho_s - s(\rho_s) = a_s \). But this is exactly
\[
\left( \eta_s^{\text{Ext}} \otimes_R \text{id}_t \right) \circ \tilde{\tau}_t^{-1}(\rho_t t(\rho_t) \boxtimes 1 \otimes 1) - (\eta_s^0 \otimes_R \text{id}_t) \circ (\text{id}_s \otimes_R \tilde{\phi}_t^{-1}) \circ \tilde{\tau}_t^{-1}(\rho_t t(\rho_t) \boxtimes 1 \otimes 1)
\]
and thus we are done. \( \square \)

**Corollary 5.13.** We have the following reduction identity in \( \text{Ext}^{1,-3}_R(B_t, B_t) \).
\[
(\epsilon_s \otimes_R \text{id}_t \otimes_R \text{id}_s) \circ \Phi^{(s,t,s)}_t = (\text{id}_t \otimes_R \eta_s^{\text{Ext}}) \circ \sigma_t - (\text{id}_t \otimes_R \eta_s) \circ (\phi_t \otimes_R \text{id}_s) \circ \sigma_t
\]
Diagrammatically this will be of the form,

![Diagram](https://example.com/diagram.png)

**Corollary 5.14.** We have the following reduction identity in \( \text{Ext}^{1,-3}_R(B_t, B_s B_s) \).
\[
(\text{id}_s \otimes_R \epsilon_t \otimes_R \text{id}_s) \circ \Phi^{(s,t,s)}_t = -\delta_s \circ \eta_s \circ \epsilon_t^{\text{Ext}} + \delta_s \circ \eta_s^{\text{Ext}} \circ \epsilon_t
\]
Diagrammatically this will be of the form,

![Diagram](https://example.com/diagram.png)
Corollary 5.15. We have the following reduction identity in $\text{Ext}^{1, -3}_{R^w}(R, B_i B_j)$. 
\[
\Phi^{(s,t,i)}_t \circ \eta_i = -(\text{id}_s \otimes R \eta^{\text{Ext}}_t \otimes R \text{id}_s) \circ \delta_s \circ \eta_s + (\text{id}_s \otimes R \eta_i \otimes R \text{id}_s) \circ \delta_s \circ \eta^{\text{Ext}}_s
\]
Diagrammatically this will be of the form,

\[
\begin{array}{c}
\includegraphics[scale=0.5]{diagram1} \\
- \includegraphics[scale=0.5]{diagram2} + \includegraphics[scale=0.5]{diagram3}
\end{array}
\]

The above corollaries all follow by rotating Proposition 5.12 and using Proposition 5.11. Because Proposition 5.11 says that $\Phi^{(s,t,i)}$ is only signed rotation invariant the signs switch in the last two identities.

5.4 High Strand Relations

Definition 5.16. Given an expression $w = (s_1, \ldots, s_m)$, define the expressions
\[
\tilde{w}^i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_m) \\
\tilde{w}^i = (s_1, \ldots, s_i, s_{i+1}, \ldots, s_m)
\]

Lemma 5.17. Let $w = (s_1, \ldots, s_m)$ and suppose that $\ell(m(t\tilde{w}^i)) \geq 4$, then
\[
(\text{id}_{[s_1, \ldots, s_{i-1}]} \otimes R \epsilon_{s_i} \otimes R \text{id}_{(s_{i+1}, \ldots, s_m)}) \circ \Phi^{(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_m)}_t = \Phi^{w^i}_t
\]
Diagrammatically, this will be of the form

\[
\includegraphics[scale=0.5]{diagram4} = \includegraphics[scale=0.5]{diagram5}
\]

Proof. Since $\dim R^{1, -1}(R, BS(\tilde{w}^i)) = 1$ so it suffices to check both sides agree on $\rho_s \otimes 1 \otimes 1$. This will then follow from the fact that $\epsilon_{s_i} \otimes 1 = 1$.

Remark. The lemma above doesn’t apply to $\Phi^{(s,t,i)}$, as seen in Proposition 5.12.

Lemma 5.18. Let $w = (s_1, \ldots, s_m)$ and suppose $|m(t, w)| \geq 4$, then
\[
(\text{id}_{[s_1, \ldots, s_{i-1}]} \otimes R \delta_{s_i} \otimes R \text{id}_{(s_{i+1}, \ldots, s_m)}) \circ \Phi^w_t = \Phi^{w^i}_t
\]
Diagrammatically this will be of the form

\[
\includegraphics[scale=0.5]{diagram6} = \includegraphics[scale=0.5]{diagram7}
\]

Proof. Proceeds similarly to Lemma 5.17.

Remark. The expression $(-1)^{|m(s_m)|-1}$ appearing below is 0 if $s_m = t$ and 1 if $s_m = s$.

Theorem 5.19. $\Phi^w_t$ is signed rotation invariant when $|m(t, w)| \geq 4$. Specifically, letting $w = (s_1, \ldots, s_m)$ we have
\[
(\text{id}_t \otimes R \text{id}_{(s_1, \ldots, s_{m-1})}) \circ (\text{id}_t \otimes R \epsilon_{s_m}) \circ (\text{id}_t \otimes R \text{id}_{(s_1, \ldots, s_{m-1})} \otimes R \mu_{s_m}) \circ (\text{id}_t \otimes R \Phi^w_t \otimes R \text{id}_{s_m}) \circ (\delta_t \otimes R \text{id}_{s_m}) \circ (\eta_t \otimes R \text{id}_{s_m}) = (-1)^{|m(s_m)|-1} \Phi^w_{t(s_1, \ldots, s_{m-1})}
\]
In other words, if $s_m \neq t$, then rotating $\Phi^w_t$ clockwise will pick up a negative sign. And similarly,
\[
(\epsilon_{s_1} \otimes R \text{id}_{(s_2, \ldots, s_m)}) \circ (\mu_{s_1} \otimes R \text{id}_{(s_2, \ldots, s_m) \otimes R \text{id}_t}) \circ (\epsilon_{s_1} \otimes R \Phi^w_t \otimes R \text{id}_t) \circ (\delta_{s_1} \otimes R \eta_{s_1}) \circ (\text{id}_{s_1} \otimes R \Phi^w_t \otimes R \text{id}_t) \circ (\text{id}_{s_1} \otimes R \delta_{s_1}) \circ (\text{id}_{s_1} \otimes R \eta_{s_1}) = (-1)^{|m(s_1)|-1} \Phi^w_{t(s_2, \ldots, s_{m-1})}
\]
In other words, if $s_1 \neq t$, then rotating $\Phi^w_t$ counterclockwise will pick up a negative sign.
Proof. We will prove the first equality as the second is quite similar. First suppose \( s_m = t \). Diagrammatically we want to show

![Diagram](image)

As \( \dim_S \text{Ext}^{1,-(|w|+1)}_{R^e}(B_t, BS(w)) = 1 \) it suffices to check both sides agree on \( \rho_x \boxtimes \id_1 \). The first part of the diagram will be the series of maps

\[
\begin{align*}
K_t \otimes_R K_t \otimes_R K_t & \quad \quad \rho_s^{(1)} \boxtimes (\rho_t \otimes \id_1 \otimes \id_1 - \id_1 \otimes \id_1 \otimes \rho_t) + \rho_s^{(2)} \boxtimes (\rho_t \otimes \id_1 \otimes 1 - \id_1 \otimes \rho_t) + \rho_s^{(3)} \boxtimes (\rho_t \otimes 1 \otimes \id_1 - \rho_t) + \rho_s \boxtimes 1 \\
\delta_t^{-1} \otimes \id_t & \quad \quad \uparrow \\
K_t \otimes_R K_t & \quad \quad \rho_s^{(1)} \boxtimes (\rho_t \otimes \id_1 - \rho_t) + \rho_s \boxtimes (\rho_t \otimes 1 - \rho_t) + \rho_s \boxtimes 1 \\
\eta_t \otimes \id_t & \quad \quad \uparrow \\
K \otimes_R K_t & \quad \quad \rho_s^{(1)} \boxtimes 1 \otimes 1 + \rho_s^{(2)} \boxtimes 1 \otimes 1 \\
\bar{r}_t & \quad \quad \uparrow \\
K_t & \quad \quad \rho_s \boxtimes 1 \otimes 1
\end{align*}
\]

where we have used the chain lifts defined in Appendix A and that \( \rho_s \in (V^*)^t \).

At the next step we apply \( \id \otimes_R \Phi_t^{(0)} \otimes_R \id \) and therefore only the \( \rho_s^{(2)} \) term survives. The next part of the diagram will be the series of maps

\[
\begin{align*}
K_t \otimes_R K(s_1, \ldots, s_{m-1}) & \quad \quad 1 \boxtimes 1 \otimes 1 \otimes 1 \otimes 1 \\
\id_t \otimes \id_{(s_1, \ldots, s_{m-1})} \otimes \id \bar{c}_t & \quad \quad \uparrow \\
K_t \otimes_R K(s_1, \ldots, s_{m-1}) \otimes_R K_t & \quad \quad 1 \boxtimes (0 - 1 \otimes 1 \otimes 1 \otimes 1 \otimes \partial_t(\rho_t) + 1) \\
\id_t \otimes \id_{(s_1, \ldots, s_{m-1})} \otimes \id \bar{u}_t & \quad \quad \uparrow \\
K_t \otimes_R K(w) \otimes R K_t & \quad \quad 1 \boxtimes (\rho_t \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \rho_t) + \rho_s \boxtimes 1 \\
\id_t \otimes \id \bar{v} & \quad \quad \uparrow \\
K_t \otimes_R K_t & \quad \quad \rho_s \boxtimes (\rho_t \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \rho_t) + \rho_s \boxtimes 1
\end{align*}
\]

which is exactly equal to \( \Phi_t^{(1, s_1, \ldots, s_{m-1})} (\rho_x \boxtimes 1 \otimes 1) \).

Now suppose \( s_m = s \). Diagrammatically we want to show

![Diagram](image)

One can check that \( \dim S \text{Ext}^{1,-(|w|+1)}_{R^e}(B_{s_m}, BS(s, s_1, \ldots, s_{m-1})) = 1 \). It follows that

![Diagram](image)
Because $|m(s, t, s_1, \ldots, s_{m-1})| \geq 4$, there is a subexpression of the form $stst$. Therefore there must be a subexpression $st$ in $(s_1, \ldots, s_{m-1})$ say at $(\ldots, s_i, \ldots, s_j, \ldots)$. Now apply

$$\text{id}_t \otimes_R \epsilon_{s_1} \otimes_R \epsilon_{s_2} \otimes_R \ldots \otimes_R \text{id}_{s_i} \otimes_R \ldots \otimes_R \epsilon_{s_{m-1}}$$

to both sides. Using Lemma 5.17 we end up with the equation

$$= c$$

And from Proposition 5.11 it follows that $c = -1$ as desired. □

**Corollary 5.20.** We have the following equality when $|m(t, w)| \geq 4$.

$$(\text{id}_t \otimes_R \Phi_w) \circ (\delta_t) \circ (\eta_t) = (-1)^{|m(s_m)|-1} (\Phi_{s_m} (s_1, \ldots, s_{m-1}) \otimes_R \text{id}_{s_m}) \circ (\delta_{s_m}) \circ (\eta_{s_m})$$

Diagrammatically this will be

$$\cdots = (-1)^{|m(s_m)|-1} \cdots$$

**Proof.** Follows from adding a cup on the bottom right in Theorem 5.19. □

**Corollary 5.21.** Assume that $|m(t, w)| \geq 4$. Then $\Phi_w$ is cyclic. Diagrammatically this means

$$\cdots = \cdots$$

(Note $|m(t, s_1)| - 1 + |m(s_1, s_{i-1})| - 1 = |m(t, s_1, s_{i-1})| - 1$.) Now apply Corollary 5.20, cap off the strands $s_1, \ldots, s_m$ to the left and use isotopy to arrive at

$$\cdots = (-1)^{|m(t, w^{-1})| - 1} \cdots$$

One can then check that in all cases, $|m(t, w^{-1})| - 1$ is always even. □

For more background on cyclicity we refer the reader to [EMTW20, Chapter 7.5] or [Lau10, Section 4.4]. The main upshot of having a cyclic morphism is the following theorem
**Theorem 5.22** (Cockett–Koslowski—Seely [CKS00]). *If all 1-morphisms have biadjoints and all 2-morphisms are cyclic, then diagrams up to true isotopy fixing endpoints unambiguously represent a 2-morphism.*

We should be careful what we mean by fixing endpoints however. Note that

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}
\]

is isotopic “fixing endpoints” to

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}
\]

and these do turn out to be equal in \( \mathcal{D}(t, s) \). However this isn’t a consequence of cyclicity! The discrepancy here arises from the fact that the LHS is a morphism \((s_3) \rightarrow (s_1, s_1, s_2)\) while the RHS is a morphism \((s_1) \rightarrow (s_2, s_3)\) where \(s_1 = s_2 = s_3\) is just a coincidence. Redrawing the diagrams with the corresponding labels,

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}
\]

we see that the LHS is not isotopic fixing endpoints to the RHS. In general if adding a cap on top and a cup on bottom “doesn’t change” the morphism, we will say that is is rotation invariant. Rotation invariance of a morphism greatly simplifies the graphical calculus for a morphism because it implies we don’t need to do this bookkeeping above on how the morphism was constructed, we only need to care about the color(s) of the morphism (See the proof of Lemma 5.26).

**Theorem 5.23.** *Let \( w = (s_1, \ldots, s_m) \) be such that \( |m(t, w)| \geq 4 \) and for a fixed \( i \), consider expressions \( v \) such that \( \ell(m(s_i v)) \geq 4 \). Then the following relation holds in \( \text{Ext}^{2 - |w| - |v| - 2}_{R^e}(B_t, BS(s_1, \ldots, s_{i-1}, v, s_{i+1}, \ldots, s_m)) \)

\[
(id_{s_1} \otimes_R \cdots \otimes_R id_{s_{i-1}} \otimes_R \Phi_{s_i}^v \otimes_R id_{s_{i+1}} \otimes_R \cdots \otimes_R id_{s_m}) \circ \Phi_T^w = 0
\]

WLOG assume that \( s_i = t \). Then diagramatically, this is of the form

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}
\]

\[
= 0
\]

**Proof.** This will be true by degree reasons. Namely from **Theorem 4.13** we know that

\[
\text{Ext}^{2 - |w| - |v| - 2}_{R^e}(B_t, BS(s_1, \ldots, s_{i-1}, v, s_{i+1}, \ldots, s_m)) \cong \mathbb{D}(\ker \rho_t^e(s_1, \ldots, s_{i-1}, v, s_{i+1}, \ldots, s_m)) - |w| - |v| - 2
\]

But notice that the lowest degree element on the RHS is \( 1(s_1, \ldots, s_{i-1}, v, s_{i+1}, \ldots, s_m) \) which has degree \(- |w| - |v|\) and thus the RHS above is \( 0 \) \( \square \)

**Theorem 5.24.** *Let \( w = (s_1, \ldots, s_m) \) be such that \( |m(t, w)| \geq 4 \), then the following relation holds in \( \text{Ext}^{2 - |w| - 4}_{R^e}(B_t, BS(w)) \)

\[
(\phi_{s_1} \otimes_R id_{s_2} \otimes_R \cdots \otimes_R id_{s_m}) \circ \Phi_T^w = (id_{s_1} \otimes_R \phi_{s_2} \otimes_R \cdots \otimes_R id_{s_m}) \circ \Phi_T^w
\]

When \( s_1 = s \) and \( s_2 = t \), diagrammatically this will be of the form

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}
\]

\[
= 0
\]

**Proof.** If \( s_1 = s_2 \) then this follows from 1 color Hochschild jumping. Otherwise wlog assume that \( s_1 = s \) and \( s_2 = t \). Then popping the red Hochschild out and applying **Proposition 5.12** and **Theorem 5.23** it follows that

\[
\begin{array}{c}
\begin{array}{c}
\circ \circ \\
\circ \circ
\end{array}
\end{array}
\]

\[
= 0
\]

\[
(30)
\]

\( \square \)
5.5 Ext Valently Morphisms

Definition 5.25. For any two expressions \( w = (w_1, \ldots, w_m) \) and \( v = (v_1, \ldots, v_j, \ldots, v_n) \) in \( s \) and \( t \) such that \( |m(v^{-1}w)| \geq 4 \) define the morphism \( \Omega^w_{t/v} \) as follows. First choose an anchor, which can be any term \( v_i \) in the expression \( v \) such that \( v_i = t \). Then twist \( \Phi^w_{t}([v_{i-1}, v_i, \ldots, v_{n+1}]) \) using caps and cups until you end up with a morphism in \( \text{Ext}^{1, -(n+m)}(\text{BS}(v), \text{BS}(w)) \). Diagrammatically, we let

\[
\Omega^w_{t/v} = \begin{array}{c}
\Phi^w_{t} \\
\downarrow \\
w \\
\uparrow \\
v
\end{array}
\]

Similarly define \( \Upsilon^w_{t/v} \) by first choosing an anchor, which instead is now any term \( v_j \) in \( v \) such that \( v_j = s \) and then twist \( \Phi^w_{s}([v_{j-1}, v_j, \ldots, v_{n+1}]) \) until you end up with a morphism in \( \text{Ext}^{1, -(n+m)}(\text{BS}(v), \text{BS}(w)) \). Diagrammatically, we let

\[
\Upsilon^w_{t/v} = \begin{array}{c}
\Phi^w_{s} \\
\downarrow \\
w \\
\uparrow \\
v
\end{array}
\]

Lemma 5.26. \( \Omega^w_{t/v} \) and \( \Upsilon^w_{t/v} \) are well defined, i.e. is independent of the choice of the anchor. (We emphasize that the anchor for \( \Omega^w_{t/v} \) has to be blue while the anchor for \( \Upsilon^w_{t/v} \) has to be red.)

Proof. We prove this for \( \Omega^w_{t/v} \) as \( \Upsilon^w_{t/v} \) is quite similar. WLOG (add appropriate caps and cups and use adjunction) assume that \( v = (t, \ldots, t) \) and that our two different anchors are the first and last \( t \) in \( v \) and let \( (\Omega^w_{t/v})_1 \) and \( (\Omega^w_{t/v})_2 \) be the associated morphisms. Since \( \Phi^w_{t} \) is cyclic by Corollary 5.21, we can use Theorem 5.22 to bring \( (\Omega^w_{t/v})_1 \) to the LHS below and \( (\Omega^w_{t/v})_2 \) to the RHS below

\[
(\Omega^w_{t/v})_1 = \begin{array}{c}
w \\
\ldots \\
v \\
\ldots \\
w
\end{array} = (\Omega^w_{t/v})_2
\]

If \( \Phi^w_{t} \) were rotation invariant, then we could conclude that both sides were equal above. But Theorem 5.19 only says that \( \Phi^w_{t} \) is signed rotation invariant so the two sides are the same up to a sign. However, note that there must be an even number of color changes in the expression \( v = (t, \ldots, t) \) and so the sign must be \( +1 \) and thus we have equality as desired.

Remark. \( \Upsilon^w_{t/v} = -\Omega^w_{t/v} \) as a consequence of Theorem 5.19. For the most part we will work with \( \Omega^w_{t/v} \) and occasionally use \( \Upsilon^w_{t/v} \) as needed. Also note that the same argument shows that one can choose an anchor from a spot in the top expression \( w \) as well and \( v \) or \( w \) can be the empty expression as well.

Remark. Up until now we were free to interchange \( t \) with \( s \) in all of our results but as seen in the definition of \( \Omega^w_{t/v} \) and \( \Upsilon^w_{t/v} \) we cannot interchange \( t \) and \( s \). Also note that we have used curved lines with a red dot with a red hollow dot to diagrammatically denote \( \Phi^w_{t} \) and straight lines with a red hollow dot for \( \Omega^w_{t/v} \). However since

\[
\Omega^w_{t/v} = \begin{array}{c}
w \\
\downarrow \\
\Phi^w_{t}
\end{array}
\]

there is no ambiguity in the diagrammatic picture if one cannot determine if a line is straight or not.

Example 1. If \( v_n = t \) and \( v_1 = s \), then one possible presentation of \( \Omega^w_{t/v} \) and \( \Upsilon^w_{t/v} \) is given below to the left and right, respectively.
A huge advantage of working with $\Omega^w_L$ is that

**Lemma 5.27.** $\Omega^w_L$ is rotation invariant. Let $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. Diagrammatically, this means

\[
\begin{align*}
\Omega^w_L &= v_1 w_1 \cdots v_n w_n \\
\end{align*}
\]

**Proof.** This follows from the definition of $\Omega^w_L$ being well defined. Choose some blue strand to be the anchor for $\Omega^w_L$. Then the LHS above can be used as the definition of the RHS above.

**Theorem 5.28** ($\Omega^w_L$ absorbs morphisms). *The following identities hold in BSBim$^\text{Ext}(h, W_\infty)$ whenever both sides of the equalities are defined.*

\[
\begin{align*}
\Omega^w_L &= v_1 w_1 \cdots v_n w_n \\
\end{align*}
\]

**Proof.** Follows from rotation invariance of $\Phi^w_S$ along with Lemma 5.17 and Lemma 5.18.

One particular choice of $w$ and $v$ in $\Omega^w_L$ will be essential to us, namely following [Eli16] we define

**Definition 5.29.** Let $\hat{s} = \hat{s} t \cdots$ where there are $m$ terms in the expression that alternate between $s$ and $t$ starting with $s$. Also let $\hat{m}$ be the corresponding element in $W$. Define $\hat{s}$ and $\hat{m}_s$ similarly.
**Definition 5.30** $(2k-$exvalent). For $k \geq 2$, define the red $2k-$exvalent morphism to be $\Omega^k_{\mathcal{K}}$ or $\Omega^k_{\mathcal{K}} \in \text{Ext}^{1,-2k}_{\mathcal{R}}(\text{BS}(\mathcal{K}), \text{BS}(\mathcal{K}))$ which we will denote diagrammatically by either

\[
\begin{array}{c}
\vdots \\
\otimes \\
\otimes \\
\otimes \\
\vdots
\end{array}
\]

This is well defined as $|m(\hat{k}^{-1}_{s})| = |\hat{k}^{-1}_{s}| = 2k \geq 4$. Also for $k \geq 2$, define the blue $2k-$exvalent morphism to be $\Upsilon^k_{\mathcal{K}}$ or $\Upsilon^k_{\mathcal{K}} \in \text{Ext}^{1,-2k}_{\mathcal{R}}(\text{BS}(\mathcal{K}), \text{BS}(\mathcal{K}))$ which we will denote diagrammatically by

\[
\begin{array}{c}
\vdots \\
\otimes \\
\otimes \\
\otimes \\
\vdots
\end{array}
\]

The reason why the $2k$-exvalent morphisms are essential is because they can be used to generate the rest of the $\Omega^w_{\mathcal{R}}$ morphisms. Namely,

**Lemma 5.31.** $\Omega^w_{\mathcal{R}}$ can be constructed as the composition of $2k$ red exvalent morphisms, along with the generating morphisms of $\text{BSBim}(\mathfrak{h}, W_\infty)$.

**Proof.** Any two adjacent terms in $v$ or $w$ that are the same can be replaced using Theorem 5.28. \[\square\]

**Example 2.** Let us give a partial description for two possible chain maps representing the 4 exvalent morphism $\Omega^{s,t}_{(t,s)} \in \text{Ext}^{1,-4}_{\mathcal{R}}(\mathcal{B}_1 \mathcal{B}_2, \mathcal{B}_2 \mathcal{B}_1)$. The complex $K_t K_s$ in homological degree 1 has a decomposition given by

\[
\rho^{(1)}_t \otimes R^{ee} \oplus \rho_{t}(\rho_{t}^{(1)} \otimes R^{ee} \oplus \rho_{t}^{(2)} \otimes R^{ee} \oplus \rho_{s}(\rho_{s}^{(2)} \otimes R^{ee})
\]

By definition one possible presentation is given by

\[
\Omega^{s,t}_{(t,s)} =
\]

so a possible partial chain lift of $\Omega^{s,t}_{(t,s)}$ is given by $\omega^{s,t}_{(t,s)} : K_t K_s^{[1]} \rightarrow K_s K_t^{[0]}$

\[
\omega^{s,t}_{(t,s)}(\rho^{(1)}_{s} \otimes 1 \otimes h \otimes 1) = 1 \otimes 1 \otimes 1 \otimes \partial_{s}(h)
\]

\[
\omega^{s,t}_{(t,s)}(\rho_{t}(\rho_{t}^{(1)} \otimes 1 \otimes h \otimes 1) = 1 \otimes -a_{ts} \circ \rho_{t} \circ \partial_{s}(h) + c_{s} \circ \partial_{s}(h) + 1 \otimes 1 \otimes (h - a_{s} \circ \partial_{s}(h))
\]

\[
\omega^{s,t}_{(t,s)}(\rho_{t}^{(2)} \otimes 1 \otimes h \otimes 1) = 0
\]

\[
\omega^{s,t}_{(t,s)}(\rho_{s}(\rho_{s}^{(2)} \otimes 1 \otimes h \otimes 1) = 0
\]

On the other hand, we have that $\Omega^{s,t}_{(t,s)} = -\Upsilon^{s,t}_{(t,s)}$ so another presentation is given by

\[
\Omega^{s,t}_{(t,s)} =
\]

so another possible partial chain lift $-\delta^{s,t}_{(t,s)} : K_t K_s^{[1]} \rightarrow K_s K_t^{[0]}$ is given by

\[
-\delta^{s,t}_{(t,s)}(\rho^{(1)}_{s} \otimes 1 \otimes h \otimes 1) = 0
\]

\[
-\delta^{s,t}_{(t,s)}(\rho_{t}(\rho_{t}^{(1)} \otimes 1 \otimes h \otimes 1) = 0
\]

\[
-\delta^{s,t}_{(t,s)}(\rho_{t}^{(2)} \otimes 1 \otimes h \otimes 1) = -1 \otimes \partial_{t}(h) \otimes 1 \otimes 1
\]

\[
-\delta^{s,t}_{(t,s)}(\rho_{s}(\rho_{s}^{(2)} \otimes 1 \otimes h \otimes 1) = -1 \otimes (h - a_{ts} \rho_{s}(h)) \otimes 1 \otimes -\partial_{t}(h) \otimes 1 \partial_{t}(h) \circ c_{t}
\]
5.6 Cohomology Relations

**Theorem 5.32.** Suppose \( k \geq 2 \) and let \( t_i = (\hat{k})_i \) the \( i \)-th spot in the expression \( \hat{k} \) and similarly let \( s_i = (\hat{k})_i \). Then we have the following relation in \( \text{Ext}_{R^e}^{1-2k+2}(\text{BS}(\hat{k}), \text{BS}(\hat{k})) \)

\[
[k] \alpha_s \ldots = - \sum_{i=1}^{k-1} [k-i] \ldots + \sum_{i=1}^{k} [k+1-i] \ldots = s_i
\]

(37)

\[
[k] \alpha_t \ldots = - \sum_{i=1}^{k-1} [k-i] \ldots + \sum_{i=1}^{k} [k+1-i] \ldots = s_i
\]

(38)

where the purple lines are either red or blue and the yellow lines will be the corresponding opposite color.

**Proof.** Suppose \( k \) is odd (\( k \) even will proceed similarly except use the blue extvalent morphism instead) so that the bottom right strand above will be blue. Then we claim Eq. (37) can be obtained by rotating down the left \( k-1 \) strands of the following equation.

\[
\rho_s \cdot \frac{2k-1}{2} - \frac{2k-1}{2} \rho_s = 0
\]

(39)

Specifically, apply polynomial forcing to move the left \( \rho_s \) to the right \( k-1 \) times. The coefficient when we break the \( 2m+1 \)th line starting from the left will be of the form \( \partial_t(s(ts)^m(\rho_s)) \) and the coefficient when we break the \( 2m \)th line starting from the left will be of the form \( \partial_s((ts)^m(\rho_s)) \). Similarly, apply polynomial forcing to move the right \( \rho_s \) to the left \( k \times \) times. The coefficients will be the same as above, except we have negative signs and we start counting from the right. Using Eq. (23) and Eq. (24) we see that

\[
\partial_t(s(ts)^m(\rho_s)) = \frac{s(ts)^m(\rho_s) - (ts)^{m+1}(\rho_s)}{\alpha_t} = [2m+2]
\]

\[
\partial_s((ts)^m(\rho_s)) = \frac{(ts)^m(\rho_s) - s(ts)^m(\rho_s)}{\alpha_s} = [2m+1]
\]

and now rotating the left \( k-1 \) strands of the LHS of Eq. (39) the only terms that are unbroken are of the form

\[
(ts)^{(k-1)/2}(\rho_s) \ldots - s(ts)^{(k-1)/2}(\rho_s) \ldots = [k] \alpha_s \ldots
\]

Rearranging and applying Lemma 5.17, Lemma 5.18 will yield Eq. (39). A similar calculation applies when \( k \) is even aka the bottom right strand will be red.

Now we will show why Eq. (39) is true in \( \text{Ext}_{R^e}^{1-2k+2}(K_t, \text{BS}(2k-1)) \). Because \( \text{BSBim}^{\text{Ext}}(h, W_\infty) \) is a supermonoidal category, we can move both boxed \( \rho_s \) down in Eq. (39), and since \( \rho_s \in (V^*)^t \) we have that

\[
[k] \rho_s - [k] \rho_s = 0
\]

(40)

in \( \text{BSBim}(h, W_\infty) \). By the fully faithful embedding it follows that Eq. (40) also holds in \( \text{BSBim}^{\text{Ext}}(h, W_\infty) \), i.e there are chain homotopies \( K_t \to K_t \) giving rise to Eq. (40).

Eq. (38) can also be obtained from Eq. (39) by instead rotating the right \( k-1 \) strands down and then rotating the diagram by \( 180^\circ \) and applying Lemma 5.27. □
Corollary 5.33. For each $k$, adding a red and blue cap anywhere in Eq. (37) or Eq. (38) will give the cohomology relation for $k - 1$.

Proof. Follows from Eq. (39) and Lemma 5.17.

Example 3. For $k = 2$ one can use Proposition 5.12, Corollary 5.13, and Corollary 5.14 to arrive at the relations

$$\begin{align*}
[2] \alpha_s &= -[2] + [2] + - [2] \\
[2] \alpha_t &= -[2] + [2] - - [2]
\end{align*}
$$

(41)

and the RHS looks similar to the Jones Wenzl projector (see Example 5.10 in Soergel Calculus). Moreover if we cap off appropriately, we can recover the rank 1 cohomology relation (assuming Hochschild jumping and barbell).

$$\begin{align*}
[2] \alpha_s \alpha_t &= -[2] \alpha_s \overset{\alpha_s}{\longrightarrow} -[2] \alpha_s \\
&= [2] \alpha_s \\
&= - [2] \alpha_s + [2] \alpha_s
\end{align*}
$$

(42)

where we have applied Corollary 5.15 on the LHS. We can then cancel $[2] \alpha_t$ from both sides to arrive at the rank 1 cohomology relation.

6 Computation of $\text{Ext}^*_{R_t}(R, B_w)$ for $m_{st} = \infty$

We will continue to assume that Assumption 1, Assumption 2, and Assumption 3 hold in this section.

Theorem 6.1. Assume all quantum numbers are invertible. Then there is an isomorphism of right $R$ modules.

\begin{align*}
\text{Ext}_{R_t}^0(R, B_{s,s}) &= R(-k) \\
\text{Ext}_{R_t}^1(R, B_{s,s}) &= R(4 - k) \oplus R(k) \\
\text{Ext}_{R_t}^2(R, B_{s,s}) &= R(k + 4)
\end{align*}

Proof. Proceed by induction on $k$. $k = 1$ follows from Theorem 4.13. From Theorem 4.13 we know that

\begin{align*}
\text{Ext}_{R_t}^0(R, BS_i(\tilde{k})) &= \ker \rho \tilde{s}_i(\tilde{k} - 1)(-1) \\
\text{Ext}_{R_t}^1(R, BS_i(\tilde{k})) &= \ker \rho \tilde{s}_i(\tilde{k} - 1)(-1)(4) \oplus \text{D}(\ker \rho \tilde{s}_i(\tilde{k} - 1)(-1)) \\
\text{Ext}_{R_t}^2(R, BS_i(\tilde{k})) &= \text{D}(\ker \rho \tilde{s}_i(\tilde{k} - 1)(-1))(4)
\end{align*}

(43)

(44)

In other words, $\text{Ext}_{R_t}^*(R, BS_i(\tilde{k}))$ satisfies the pattern as prescribed by the theorem. Because all quantum numbers are invertible, we have a decomposition

$$BS_i(\tilde{k}) \cong B_{s, s} \bigoplus_{y < k} B_y^{h_y}$$

(45)

By Soergel’s Hom formula we know that $\text{Ext}_{R_t}^0(R, B_{s,s}) \cong R(-k)$. Therefore

$$\ker \rho \tilde{s}_i(\tilde{k} - 1)(-1) = \text{Ext}_{R_t}^0(R, BS_i(\tilde{k})) \cong R(-k) \bigoplus \text{Ext}_{R_t}^0(R, B_y^{h_y})$$

By induction $\text{Ext}_{R_t}^*(R, B_y^{h_y})$ will also satisfy the pattern as prescribed by the theorem and so subtracting $\text{Ext}_{R_t}^*(R, B_y^{h_y})$ from both sides of Eq. (43) and Eq. (44) we complete the induction.

Recall from Theorem 4.13 that we have the isomorphism

$$\text{Ext}_{R_t}^*(B_t, BS(w)) \cong \ker \rho \tilde{s}(w) \rho \tilde{s}(t(\rho \tilde{s}))(-1) \oplus \text{D}(\ker \rho \tilde{s}(w)) \rho \tilde{s}(-1)$$

(46)
Lemma 6.2. The set of all morphisms of the form below
give a right $R$ basis for the $\ker \rho^*_s(w) \rho_t t(\rho_s)(-1)$ part of $\text{Ext}^1_{R^e}(B_t, BS(w))$.

Proof. By definition, the $\ker \rho^*_s(w) \rho_t t(\rho_s)(-1)$ part of $\text{Ext}^1_{R^e}(B_t, BS(w))$ in Theorem 4.13 consists of elements of the form $a_{t_1} \psi_s^0$ where $a \in \ker \rho^*_s(w) \subset BS(w)$ and thus a basis is given by $\{ b_{t_1} \psi_s^0 \}_{b \in \mathcal{B}}$ where $\mathcal{B}$ is a right $R$ basis for $\ker \rho^*_s(w) \subset BS(w)$. Because

and by definition $L_{w,f}$ is the image of $\phi_{bot}$ and so it follows that the morphisms in the lemma are precisely given by $L_{w,f}$ and thus by Corollary 4.8 we are done.

Lemma 6.3. The set of all morphisms of the form below

give a right $R$ basis for the $\ker \rho^*_s(w) \rho_s s(-1)$ part of $\text{Ext}^1_{R^e}(B_t, BS(w))$ as a right $R$ module.

Proof. Similar to Lemma 6.2.

However, note that when $r(f) \in \{ s, t, ts, st, stt \}$ the first/bottom morphism $\Phi_{\cdot \cdot \cdot}^r$ in Lemma 6.3 technically isn't defined! In this case, there is more than one possible choice of $v_f$ such that $v_f \psi_s^0$ is a cocycle. However the proof of Lemma 6.3 only really needs that the bottom/first morphism sends $\rho_s \otimes 1 \otimes 1$ to $1 \otimes 1 \otimes 1$. Thus when $r(f)$ is in the set above we can define

For example,

One can check that this definition is well defined using Corollary 5.13, Corollary 5.14, etc.

Definition 6.4. A pitchfork is a morphism depicted on the left below. A generalized pitchfork is looks like a pitchfork but the middle can have any number of dot morphisms. Such an example is depicted on the right below.

Lemma 6.5. Generalized pitchforks can be written as the sum of morphisms that end with pitchforks somewhere on the top.

Proof. This was shown in the proof of Claim 5.26 in [Eli16].
Theorem 6.6. Assume all quantum numbers are invertible. Then for \( k \geq 1 \) we have an isomorphism of right \( R \)-modules

\[
\text{Ext}^{0\ast}_{R_t}(R, B_t) \cong \text{JW}_t \circ \downarrow \ldots \downarrow R
\]

\[
\text{Ext}^{1\ast}_{R_t}(R, B_t) \cong \text{JW}_t \circ \downarrow \ldots \downarrow R \oplus \text{JW}_t \circ \kappa \downarrow R
\]

\[
\text{Ext}^{2\ast}_{R_t}(R, B_t) \cong \text{JW}_t \circ \kappa \downarrow R
\]

Proof. \( \text{Ext}^{0\ast}_{R_t}(R, B_t) \) was shown in the diagrammatic category in [Eli16] Claim 5.26 and because of the equivalence, it also applies in the algebraic category. Because \( \text{JW}_t \) is the projector for \( B_t \) inside \( \text{BS}(i \hat{k}) \), by the definition of the Karoubian completion we have that

\[
\text{Ext}^{1\ast}_{R_t}(R, B_t) = \text{JW}_t \circ \text{Ext}^{1\ast}_{R_t}(R, \text{BS}(i \hat{k}))
\]

Applying the adjunction

\[
\text{Ext}^{1\ast}_{R_t}(B_t, \text{BS}(i \hat{k})) \cong \text{Ext}^{1\ast}_{R_t}(R, \text{BS}(i \hat{k}))
\]

to Lemma 6.2 shows that the “ker \( \rho^\ast_t(i \hat{k} - 1) \)” part of \( \text{Ext}^{1\ast}_{R_t}(R, \text{BS}(i \hat{k})) \) has a right \( R \)-basis given by

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \downarrow \end{array}
\end{array}
\end{align*}
\text{where } r(f) = \text{id}
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \downarrow \end{array}
\end{array}
\end{align*}
\text{where } r(f) = t
\]

(47)

We want to show that all such morphisms are in the \( R \)-span of \( \downarrow \ldots \downarrow R \) after postcomposing by \( \text{JW}_t \). Theorem 4.13 shows that \( R \) acts freely on \( \text{Ext}^{1\ast}_{R_t}(R, \text{BS}(i \hat{k})) \) and thus it suffices to show that

\[
\text{JW}_t \circ \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \circ \text{a}_1, \quad \text{JW}_t \circ \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \circ \text{a}_1
\]

But applying the 1-color cohomology relation \( \text{a}_1 \downarrow = \xi \downarrow \) we see that all the morphisms above factor as

\[
\text{JW}_t \circ \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \circ \xi_1, \quad r(f) = \text{id}
\]

Because \( \{ \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \}_{r(f) = \text{id}} \) is a right \( R \)-basis for \( \text{Ext}^{0\ast}_{R_t}(R, \text{BS}(i \hat{k})) \), it follows that \( \{ \text{JW}_t \circ \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \}_{r(f) = \text{id}} \) is spanned by \( \text{JW}_t \circ \downarrow \ldots \downarrow R \) as a right \( R \) module from the previous calculation of \( \text{Ext}^{0\ast}_{R_t}(R, B_t) \). Thus we see that the “ker \( \rho^\ast_t(i \hat{k} - 1) \)” part of \( \text{Ext}^{1\ast}_{R_t}(R, \text{BS}(i \hat{k})) \) is spanned by \( \text{JW}_t \circ \downarrow \ldots \downarrow R \) as a right \( R \) module.

Similarly Lemma 6.3 shows that the “\( \text{D}(\text{ker} \rho^\ast_t(i \hat{k} - 1)) \)” part of \( \text{Ext}^{1\ast}_{R_t}(R, \text{BS}(i \hat{k})) \) is generated as a right \( R \) module by elements of the form

\[
\Pi(f) = \begin{cases}
\text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \downarrow \ k - 1 \end{cases} r(f) \text{ can be anything}
\]

(48)

We want to show that all such morphisms are in the right \( R \)-span of \( \kappa \downarrow R \) after postcomposing by \( \text{JW}_t \).

Step 1: First notice that if \( \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \) contains a pitchfork (see [Eli16] Section 5.3.4 for a picture), then \( \text{JW}_t \circ \Pi(f) = 0 \) and we are done. In fact, by Lemma 6.5 the same is true if \( \text{L}_{i \hat{k} - 1}^{\uparrow \downarrow f} \) contains a generalized pitchfork.

Step 2: In the affine case, we claim that all light leaves without generalized pitchforks must be of the form

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \downarrow \end{array}
\end{array}
\end{align*}
\text{or }
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\downarrow \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \diamond \downarrow \end{array}
\end{array}
\end{align*}
\]

(49)
where ••• means the morphism consists of only dot morphisms and ♦♦♦ means the morphism consists of only identity morphisms (aka straight lines). This is because in the light leaf algorithm, in the affine case, we generate generalized pitchforks whenever a decoration is of the form $D0$ or $D1$. As a result, we only need to consider the light leaves that are made up of only dots ($U0$) and lines ($U1$). However, suppose that at step $i$ of our stroll, the corresponding decorations ends in $U1U0$. It will follow the decoration at the next step has to be either $D0$ or $D1$ but this isn’t allowed, or $i = k - 2$ and we end the algorithm at the next step. In other words, we must have all lines after the appearance of the first line or all lines with a dot at the end which is exactly what is depicted in Eq. (49).

**Step 3:** For $L_{\overline{k-1}f}$ in the form in Eq. (49), we claim that $JW_{\hat{k}} \circ \Pi(f)$ is in the right $R$ span of

$$JW_{\hat{k}} \circ \begin{array}{c} \bullet \bullet \bullet \end{array}$$

WLOG we will work with the first expression in Eq. (49). Suppose that the first line in Eq. (49) (The yellow line) is colored blue, then $JW_{\hat{k}} \circ \Pi(f)$ will be equal to

$$JW_{\hat{k}} \circ \begin{array}{c} \bullet \bullet \bullet \end{array} = JW_{\hat{k}} \circ \begin{array}{c} \bullet \bullet \bullet \end{array} = 0$$

where we have used Lemma 5.27 and Theorem 5.28. Now suppose that the first line is colored red. If $L_{\overline{k-1}f}$ starts with a red line we are done. Otherwise $L_{\overline{k-1}f}$ starts with a red dot and so applying fusion and then polynomial forcing and eliminating all diagrams with a generalized pitchforks we see that

$$JW_{\hat{k}} \circ \begin{array}{c} \bullet \bullet \bullet \end{array} - JW_{\hat{k}} \circ \begin{array}{c} \rho_s \bullet \bullet \bullet \end{array} = JW_{\hat{k}} \circ \begin{array}{c} \bullet \bullet \bullet \end{array}$$

But the diagram above is essentially the same as what we started with except we have one more line to the left of •••. We can therefore keep repeating this process, until all the lines ♦♦♦ are to the left of ••• as desired.

**Step 4: Cohomology Reduction** The previous step shows it suffices to prove morphisms of the form

$$\Pi((1, \ldots, 1, 0, \ldots, 0)) = \begin{array}{c} \begin{array}{c} \hat{j} \bullet \end{array} \end{array}$$

are in the right $R$ span of $\overline{k-1}$ after applying $JW_{\hat{k}}$. Note that Eq. (39) will tell us

$$\begin{array}{c} \begin{array}{c} \hat{j} \bullet \end{array} \end{array} = (\text{Morphisms with a pitchfork}) - \begin{array}{c} \begin{array}{c} \hat{j} \bullet \nu_s \end{array} \end{array}$$

in cohomology. As before the morphisms with a pitchfork go to 0 after applying $JW_{\hat{k}}$ and so the LHS of Eq. (52) is in the right $R$ span of $\Pi((1, \ldots, 1, 0, \ldots, 0))$. We can keep repeating this until $j = k - 2$ which means everything is in the right $R$ span of $\overline{k-1}$ as desired.
We have just shown that \( \{ \text{JW}_t \circ \downarrow \ldots \circ \text{JW}_t \circ \downarrow \} \) spans \( \text{Ext}_{R_t}^1(R, B_t) \). However since they also have the correct degrees as specified in Theorem 6.1, they will also be linearly independent over \( R \), and thus give a basis for \( \text{Ext}_{R_t}^1(R, B_t) \).

Similar reasoning works for the computation for \( \text{Ext}_{R_t}^2(R, B_t) \); Theorem 4.13 shows that \( \text{Ext}_{R_t}^2(R, BS(\hat{t})) \) is generated as a right \( R \) module by elements of the form

\[
\Pi(f) = \begin{cases} 
L_{s}^{k-1} f \\

r(f)
\end{cases}
\]

so we can proceed similarly as above.

\[ (35) \]

\section{Ext Dihedral Diagrammatics: \( m_{st} = \infty \)}

For each color \( s \) and \( t \) let \( \mathcal{D}_s^\text{Ext} = \mathcal{D}_s^\text{Ext}(h, S_2 = \langle s \rangle) \) be the strict \( k \)-linear supermonoidal category as defined in Definition 3.3.

\begin{definition}
Let \( \mathcal{D}_s^\text{Ext} := \mathcal{D}_s^\text{Ext}(h, W_\infty) \) be the strict \( k \) linear supermonoidal category associated to a realization \( h \) of the infinite dihedral group \((W_\infty, S)\), where \( S = \{ s, t \} \) is the set of simple reflections, satisfying Assumption 2 and Assumption 3 defined as follows.

- **Objects** of \( \mathcal{D}_s^\text{Ext} \) are expressions \( w = (s_1, \ldots, s_n) \) with \( s_i \in S \) where the monoidal structure is given by concatenation. The color red will correspond to \( s \) and the color blue will correspond to \( t \).

- **Morphism spaces** in \( \mathcal{D}_s^\text{Ext} \) are bigraded \( k \) modules. For a morphism \( \alpha \) homogeneous of total degree \((\ell, n)\), \( \ell \) will be the cohomological degree while \( n \) will be the internal or Soergel degree. \( \mathcal{D}_s^\text{Ext} \) will then be supermonoidal for the cohomological grading. Specifically, \( \otimes \) will satisfy the following super exchange law

\[
(h \otimes k) \circ (f \otimes g) = (-1)^{|f||k|} (h \circ f) \otimes (k \circ g)
\]

\( \text{Hom}_{\mathcal{D}_s^\text{Ext}}(v, w) \) will be the free \( k \) module generated by horizontally and vertically concatenating colored graphs built from certain generating morphisms, such that the bottom and top boundaries are \( v \) and \( w \), respectively. The generating morphisms will be: For the color red we have

| generator | name | bidegree |
|-----------|------|----------|
| \( f \)   | Startdot | \((0, 1)\) |
| \( f \)   | Enddot   | \((0, 1)\) |
| \( f \)   | Merge    | \((0, -1)\) |
| \( f \)   | Split    | \((0, -1)\) |
| \( f \)   | (bivalent) Hochschild dot | \((1, -4)\) |

and likewise for the color blue. In addition we will have the generators

| generator | name       | bidegree |
|-----------|------------|----------|
| \( f \)   | Box        | \((0, \deg f)\) |
| \( f \)   | Exterior Box | \((|\xi|), -2|\xi|)\) |
| \( f \)   | red 2k-exvalent, \( k \geq 2 \) | \((1, -2k)\) |
| \( f \)   | red 2k-exvalent, \( k \geq 2 \) | \((1, -2k)\) |

\(^2\)All generating morphisms will be properly embedded in the planar strip \( \mathbb{R} \times [0, 1] \) meaning each edge ends in a vertex or in the boundary of the strip. The bottom(top) boundary of the morphism will then be the intersection with \( \mathbb{R} \times \{0\} \) (\( \mathbb{R} \times \{1\} \)).
Here \( f \in R \) while \( \xi \in \Lambda^V \) are homogeneous elements. For example elements \( \xi \in V[-1] \) have bidegree \((1, -2)\). The colors in the red \( 2k \)-valent generator alternate between red and blue with \( k \) strands on the bottom and \( k \) strands on top. The circle in the middle will always be red.

• **Relations** in \( \mathcal{E}^{\text{Ext}}_{\infty} \) are as follows. The generating morphisms for the color \( s \) will satisfy the relations of \( \mathcal{E}^{\text{Ext}}_s \) as specified in [Mak22] Section 4, and likewise for the color \( t \) and \( \mathcal{E}^{\text{Ext}}_t \). We will also have \( 2 \)-color relations involving the red \( 2k \)-valent generator which will be given in the subsection below.

### 7.1 \( 2 \)– color Relations

All relations in this subsection will hold with the color of the strands switched.

**Rotation Invariance:**

\[
\begin{align*}
\begin{array}{c}
\text{if } k \text{ is odd} \\
\text{if } k \text{ is even}
\end{array}
\end{align*}
\]

Rotation invariance will imply that the red \( 2k \)-valent morphism is cyclic. Hochschild dots were shown to be cyclic in [Mak22] Section 4, and the other generating morphisms are also cyclic so by Proposition 7.18 in [EMTW20] an isotopy class of diagrams in \( \mathcal{E}^{\text{Ext}}_{\infty} \) unambiguously represents a morphism in \( \mathcal{E}^{\text{Ext}}_{\infty} \). As a result, any diagram isotopic or rotationally equivalent to the remaining relations will also hold in \( \mathcal{E}^{\text{Ext}}_{\infty} \).

**4–Ext Reduction:**

\[
\begin{align*}
\text{(56)}
\end{align*}
\]

\[
\begin{align*}
\text{(57)}
\end{align*}
\]

**Higher Ext Reduction:**

\[
\begin{align*}
\text{(58)}
\end{align*}
\]

**Ext Valant Annihilation:**

\[
\begin{align*}
\text{(59)}
\end{align*}
\]
7.1.1 Further Relations and Morphisms

**Lemma 7.2.** The following relations follow from the defining relations above. More Higher Ext Reduction:

\[ 2^k \cdots = 2^k \cdots , \quad k \geq 3 \quad (60) \]

2–color Hochschild Jumping:

\[ 2^k \cdots = 2^k \cdots \quad (61) \]

2–color Cohomology:

\[ \left[ k \right] \alpha_s 2^k \cdots = - \sum_{i=1}^{k-1} \left[ k - i \right] t_i + \sum_{i=1}^{k} \left[ k + 1 - i \right] s_i 2^{k-2} \cdots \quad (62) \]

\[ \left[ k \right] \alpha_t 2^k \cdots = - \sum_{i=1}^{k-1} \left[ k - i \right] t_i + \sum_{i=1}^{k} \left[ k + 1 - i \right] s_i 2^{k-2} \cdots \quad (63) \]

**Proof.** Eq. (60) clearly follows from Eq. (58). Eq. (61) follows from 4–Ext reduction and Ext Valent Annihilation (see Eq. (30)). Eq. (62) and Eq. (63) follow by using polynomial forcing, i.e. see the proof of Eq. (37) as Eq. (39) also holds in \( D_{\infty} \).

**Definition 7.3.** Given an expression \( w \), let \( mc(w) \) be any non repeating subexpression of \( w \) whose first and last terms are different such that \( \left| mc(w) \right| \) is maximal. If \( w \) consists of only 1 color, we set \( \left| mc(w) \right| = 1 \).

For example, we have that \( mc(tst) = ts \) or \( st \). Note that besides when \( \left| mc(w) \right| = 1, \left| mc(w) \right| \) is always even.

**Definition 7.4.** Given any two expressions \( v \) and \( w \), we will define a morphism in \( \text{Hom}^{\text{Ext}}_\infty (v, w) \) as follows. First suppose that \( \left| mc(v^{-1}w) \right| = 2k \) for some \( k \in \mathbb{Z}_{\geq 2} \). Then define the morphism

\[ w 2k \cdots = \text{add merge, splits to } 2k \cdots \quad \text{and rotate until colors on bottom (top) } \Rightarrow v (w) \text{ respectively} \]

This is independent of how one chooses to rotate or add merge and splits by Proposition 7.17 in [EMTW20]. If \( \left| mc(v^{-1}w) \right| < 4 \), set

\[ 2 := 4 = 4 \quad 1 := 4 \]

and repeat the same definition as above.
Example 4. The RHS of Eq. (58) is the morphism on the left below while the RHS of Eq. (60) is the morphism on the right below

Example 5. Here are possible presentations for two morphisms that we will use later

\[ \begin{align*}
\overline{2k} & \cong \overline{2k} \\
\ldots & = \ldots \\
\overline{2k+1} & \cong \overline{2k} \\
\end{align*} \] (64)

7.2 Equivalence

To distinguish between indecomposable objects in the diagrammatic and bimodule categories, we will let \( B_w \) denote the indecomposable corresponding to \( w \in W_\infty \) in \( \text{Kar}(D_\infty) \) while \( B_w \) will denote the indecomposable Soergel Bimodule corresponding to \( w \). For this section we will also let \( \text{BS}^{\text{Ext}} := \text{BS}^{\text{Ext}}(h, W_\infty) \).

**Theorem 7.5.** Assume all quantum numbers are invertible and that Assumption 1, Assumption 2, and Assumption 3 hold. Define the k-linear functor \( \mathcal{F}_{\infty}^{\text{Ext}} : D_{\infty}^{\text{Ext}} \rightarrow \text{BS}^{\text{Ext}}(h, W_\infty) \) on objects by \( \mathcal{F}_{\infty}^{\text{Ext}}(s) = K_s \), \( \mathcal{F}_{\infty}^{\text{Ext}}(t) = K_t \) and extended monoidally. On morphisms \( \mathcal{F}_{\infty}^{\text{Ext}} \) is defined as in Theorem 3.4 on the subcategories \( D_{\infty}^{\text{Ext}} \) and \( D_{\infty}^{\text{Ext}} \) and additionally sends

\[ \mathcal{F}_{\infty}^{\text{Ext}} \left( \begin{array}{c}
\overline{2k} \\
\overline{2k} \\
\end{array} \right) = \Omega^{\overline{k}}_{\overline{k}} \quad \mathcal{F}_{\infty}^{\text{Ext}} \left( \begin{array}{c}
\overline{2k} \\
\overline{2k} \\
\end{array} \right) = \Omega^{\overline{k}}_{\overline{k}} \forall k \geq 2 \]

Then \( \mathcal{F}_{\infty}^{\text{Ext}} \) is well defined and furthermore will be a monoidal equivalence.

**Proof.** The defining relations of \( D_{\infty}^{\text{Ext}} \) in Section 7.1 all hold in \( \text{BS}^{\text{Ext}}(h, W_\infty) \) as shown in Section 5. Therefore \( \mathcal{F}_{\infty}^{\text{Ext}} \) is well defined and monoidal by construction. Because both \( s \) and \( K_s \) have biadjoints in their respective categories, it suffices to check \( \text{Hom}_{D_{\infty}^{\text{Ext}}}^{\text{Ext}}(\overline{2k}, \overline{2k}) \cong \text{Hom}_{\text{BS}^{\text{Ext}}}(R, \text{BS}(\overline{2k})) \) for all expressions \( \overline{2k} \). As all quantum numbers are invertible the Jones-Wenzl projectors \( J_{\overline{k}} \) are defined for all \( k \) and as shown in [Eli16] Section 5.4.2, the object \( \overline{2k} \) in \( D_{\infty}^{\text{Ext}} \) decomposes in \( \text{Kar}(D_{\infty}) \) into indecomposables exactly as \( BS(\overline{2k}) \) decomposes in \( \text{BS}^{\text{Ext}} \). In other words the decomposition in Eq. (45) holds with \( BS(\overline{k}) \) replaced by \( \overline{2k} \) and \( B_y \) replaced by \( B_w \). It follows that we just need to check the isomorphism on indecomposables, aka for all \( \overline{2k} \in W_\infty \)

\[ \text{Hom}_{D_{\infty}^{\text{Ext}}}^{\text{Ext}}(\overline{2k}, \overline{2k}) \cong \text{Hom}_{\text{BS}^{\text{Ext}}}(K_{\overline{2k}}, K_{\overline{2k}}) = \text{Ext}_{R}^{\text{Ext}}(R, B_{\overline{2k}}) \]

The exterior forcing relation in \( D_{\infty}^{\text{Ext}} \) shows that \( \xi \in \Lambda_{\overline{2k}} \) acts freely on \( \text{Hom}_{D_{\infty}^{\text{Ext}}}^{\text{Ext}}(\overline{2k}, \overline{2k}) \) in agreement with what happens in the bimodule category. Therefore we can assume \( \Lambda_{\overline{2k}} = \overline{k} \) so that Theorem 6.6 gives us a description of the RHS in terms of diagramatics already. WLOG we can assume \( w = \overline{2k} \) (the odd case will proceed exactly the same, but the color of the last strand below will be blue). It follows that we need to show that

\[ \begin{align*}
\text{Hom}_{D_{\infty}^{\text{Ext}}}^{0}(\overline{2k}, \overline{2k}) & = J_{\overline{2k}} \circ \ldots \circ R \\
\text{Hom}_{D_{\infty}^{\text{Ext}}}^{1}(\overline{2k}, \overline{2k}) & = J_{\overline{2k}} \circ \ldots \circ R \oplus J_{\overline{2k}} \circ \overline{2k} \circ R \\
\text{Hom}_{D_{\infty}^{\text{Ext}}}^{2}(\overline{2k}, \overline{2k}) & = J_{\overline{2k}} \circ \overline{2k} \circ R \\
\end{align*} \]
**Step 1:** The case $\text{Hom}^{0,*}_{\mathcal{G}_{\text{Ext}}}(\mathcal{B}_\emptyset, \mathcal{B}_{\xi})$ was already done in [Eli16]. For $\text{Hom}^{1,*}_{\mathcal{G}_{\text{Ext}}}(\mathcal{B}_\emptyset, \mathcal{B}_{\xi})$, notice that the proof of Theorem 6.6 was entirely diagrammatic. It used the equivalence $\mathcal{D}_\infty \rightarrow \text{BSBim}(h, W_\infty)$ so that diagrammatics can be used to prove results in the bimodule category along with a rotated version of the cohomology relations Eq. (37), Eq. (38) which correspond to Eq. (62), Eq. (63) in the diagrammatic category. Therefore the proof of Theorem 6.6 can be applied to show $\text{JW}_\mathcal{N} \circ \mathcal{D}_{\xi} \cdots \mathcal{D}_{\xi}$ and $\text{JW}_\mathcal{N} \circ \mathcal{D}_{\xi}$ span $\text{Hom}^{1,*}_{\mathcal{G}_{\text{Ext}}}(\mathcal{B}_\emptyset, \mathcal{B}_{\xi})$ as a right $R$ module if we can show that any morphism in $\text{Hom}^{1,*}_{\mathcal{G}_{\text{Ext}}}(\mathcal{B}_\emptyset, \mathcal{B}_{\xi})$ can be written as a $R$ linear combination of diagrams of the form

$$
\begin{array}{ll}
\begin{array}{c}
\includegraphics{diagram1} \\
\text{ where } r(f) = \text{id}
\end{array} & 
\begin{array}{c}
\includegraphics{diagram2} \\
\text{ where } r(f) = t
\end{array} & 
\begin{array}{c}
\includegraphics{diagram3} \\
\text{ where } r(f) \text{ is anything}
\end{array}
\end{array}
$$

(65)

using the relations in Section 7.1. Linear independence then follows by applying $\mathcal{G}_{\text{Ext}}$ and noting the corresponding morphisms in the bimodule category forms an $R$ basis.

**Step 2:** By cohomological degree reasons any diagram in $\text{Hom}^{1,*}_{\mathcal{G}_{\text{Ext}}}(\mathcal{B}_\emptyset, \mathcal{B}_{\xi})$ either has exactly one exterior box $\mathcal{D}_{\xi}$ where $\xi \in V$, or a Hochschild dot, or a red $2k$–extvalent map. As we assumed $\Lambda_{st} = k$, $V = k\alpha^t \oplus k\xi_t$. Using the 1 color cohomology relation

$$
\begin{array}{l}
\begin{array}{c}
\includegraphics{diagram11} \\
\text{ where } x_{\alpha^t} \Rightarrow t \alpha^t
\end{array} & 
\begin{array}{c}
\includegraphics{diagram12} \\
\text{ where } x_{\xi_t} \Rightarrow t \alpha^t
\end{array}
\end{array}
$$

and Eq. (8) we can reduce any diagram with an exterior box to a right $R$ linear sum of diagrams with Hochschild dots.

**Step 3:** Given a diagram with a hochschild dot or a red $2k$–extvalent morphism at the bottom, we claim it can be written as a right $R$ linear sum of diagrams in Eq. (65). We first do the case of Hochschild dots. Using Proposition 5.12 any red hochschild dot can be converted to a blue hochschild dot and a $4$–ext valent vertex. Now given a diagram $D$ with a blue hochschild dot at the bottom, the rest of the diagram is some morphism in $\text{Hom}_{\mathcal{G}_{\text{Ext}}}(t, \mathcal{B}_\emptyset)$. From [EW16] we know that double leaves form a right $R$ basis for this. The only possible light leaves for the bottom part of the double leaf map are $\downarrow$ and $\uparrow$. Thus $D$ is a $R$ linear sum of diagrams of the form

$$
\begin{array}{ll}
\begin{array}{c}
\includegraphics{diagram21} \\
\text{ where } r(f) = t
\end{array} & 
\begin{array}{c}
\includegraphics{diagram22} \\
\text{ where } r(f) = \text{id}
\end{array}
\end{array}
$$

(66)

Clearly diagrams in the form on the left above are in the span of diagrams in Eq. (65) while for diagrams in the form on the right above we can use 1 color hochschild jumping Eq. (10) to move the blue hochschild dot onto a blue strand in $L_{\alpha^t}$ and then replacing the barbell with $\alpha^t$ so that the result is again clearly in the the span of diagrams in Eq. (65).

Now, given a diagram with a red $2k$–extvalent morphism at the bottom, the rest of the diagram is some morphism in $\text{Hom}_{\mathcal{G}_{\text{Ext}}}(t, \mathcal{B}_\emptyset)\mathcal{B}_{\xi}$ which has a right $R$ basis given by double leaves. Because of Eq. (58) and Eq. (56), one can show that pitchforks and therefore generalized pitchforks kill the red $2k$–extvalent morphism. Thus, as in Step 2 of the proof of Theorem 6.6 the bottom light leaf of the double leaf map on top of the red $2k$–extvalent consists of only dots and straight lines. But this means that our diagram is exactly in the form of the right most morphism in Eq. (65).

**Step 4:** Now given a diagram with a Hochschild dot, if it’s not at the bottom of the diagram, then it must be trapped by a line, hereafter referred to as the trapping line. If the trapping line has the same color as the Hochschild dot, then we can use 1-color Hochschild Jumping to move the Hochschild dot onto the line. Otherwise if the trapping line is a different color, we can then use 4–ext reduction Eq. (56) to move the Hochschild dot onto the trapping line at the cost of a red $4$–extvalent morphism, as seen below.
Therefore it suffices to consider the case when a red $2k$–extvalent morphism is trapped by a line. First suppose that the red $2k$–extvalent morphism is on the same connected component as the trapping line, as seen below. We can then apply Eq. (58), higher ext reduction so that the red $2k$–extvalent absorbs the trapping line and therefore moves further to the bottom.

On the other hand, if the red $2k$–extvalent morphism is not on the same connected component as the trapping line, we can first apply Eq. (60) to introduce a red and blue dot into the red $2k$–extvalent and then apply fusion to obtain

Because the red $2k$–extvalent is not on the same connected component as the trapping line, the polynomials $\rho_t$ and $t(\rho_t)$ are free to slide all the way to the top of the diagram. As a result, locally we end up with diagrams that exactly look like the LHS of Eq. (67) and so we are done.

**Step 5:** Similarly, for $\text{Hom}^{2\bullet}_{\mathcal{B}_\mathcal{A}^{\mathcal{B}_\mathcal{C}}}(\mathcal{B}_\mathcal{A}, \mathcal{B}_\mathcal{C})$, the proof of Theorem 6.6 can be applied if we can show any possible morphism in $\text{Hom}^{2\bullet}_{\mathcal{B}_\mathcal{A}^{\mathcal{B}_\mathcal{C}}}(\mathcal{B}_\mathcal{A}, \mathcal{B}_\mathcal{C})$ can be written as a $R$ linear combination of diagrams of the form Eq. (53). Again by cohomological degree reasons, any diagram in $\text{Hom}^{2\bullet}_{\mathcal{B}_\mathcal{A}^{\mathcal{B}_\mathcal{C}}}(\mathcal{B}_\mathcal{A}, \mathcal{B}_\mathcal{C})$ has exactly two subdiagrams from the list

Using the diagrammatic reductions above, we can assume both subdiagrams are at the bottom of the diagram. Using fusion and $4$–Ext Reduction Eq. (56) we can assume that both subdiagrams are on the same connected component. Now Hochschild Annihilation Eq. (7) and Ext Valant Annihilation Eq. (59) show that the subdiagrams must be distinct. One of those subdiagrams must be a red $2k$–extvalent morphism, as red and blue Hochschild dots cannot be on the same connected component without the presence of a red $2k$–extvalent morphism. But now we are done, as one can useacolor Hochschild Jumping Eq. (61) to move the Hochschild dot so that our diagram is of the form
and since the top portion of the diagram is a morphism in \( \mathcal{D}_\infty \), we can proceed as in Step 3. \( \square \)

8 The Finite Case: BSBim^{\text{Ext}}(\mathfrak{h}, W_{m_{st}})

In the finite case we will have that \((ts)^{m_{st}} = 1\) and in addition to Assumption 1, Assumption 2, Assumption 3 we will also add the following two assumptions

**Assumption 4.** \( \mathfrak{h} \) is a faithful realization of the finite dihedral group \( W_{m_{st}} \). As noted in [Eli16] Section 1.3, this means that \( q^{2m_{st}} = 1 \) where \( q \) is a primitive \( 2m_{st} \) root of unity and \(|m_{st}| = 0\).

**Assumption 5** (lesser invertibility). For all \( k < m_{st}, |k| \) is invertible in \( k \).

As before, we can take \( \Lambda_{st} = k \). As an aside in [Eli16] Elias assumes

**Assumption 6** (Local non-degeneracy). Whenever \( m_{st} < \infty, 4 - a_{ts}a_{st} \) is invertible in \( k \).

and this will imply Assumption 2, as one can then take in [Eli16] this was denoted \( \omega_s \)

\[
\rho_s = \frac{2a_s - a_{ts}a_t}{4 - a_{ts}a_{st}}
\]

Recall that now there is a \( 2m_{st} \)-valent morphism which we denote by \( v_s^t(m_{st}) \) sending

\[
v_s^t(m_{st}) = \begin{pmatrix}
\cdots & \cdots & \cdots \\
1(s, \overline{m_{st}}) \\
\cdots & \cdots & \cdots \\
1(t, \overline{m_{st}})
\end{pmatrix}
\]

Outside of \( m_{st} = 2 \), this isn’t enough to completely define the \( 2m_{st} \)-valent morphism, but this will suffice for our purposes. Many of the results from the affine case will still hold such as the following analogue of Corollary 4.8

**Lemma 8.1.** When \( m_{st} < \infty \), the image of \( \rho_s^t(w) : \text{BS}(w) \rightarrow \text{BS}(w) \) is a free right \( R \) module and the kernel is also a free right \( R \) module with basis given by \( \{ w_{\mathfrak{f}, \mathfrak{f}} | f \mathfrak{f} = \text{id} \text{ or } t \} \).

**Proof.** Lemma 4.7 holds regardless of what \( m_{st} \) is and so in the proof of Corollary 4.8 we just need to show that Eq. (23) isn’t 0 for \( s(ts)^m \neq \text{id} \) or \( t \) and Eq. (24) isn’t 0 for \( (ts)^m \neq \text{id} \) or \( t \). By Eq. (25) and since \( q \) is a primitive \( 2m_{st} \) root of unity we see that

\[
\text{Eq. (23)} = 0 \iff m = m_{st}\ell - 1 \quad \text{Eq. (24)} = 0 \iff m = m_{st}\ell \quad \ell \in \mathbb{Z}^+
\]

But we have that

\[
s(ts)^{m_{st}\ell - 1} = t(ts)^{m_{st}\ell} = t \quad (ts)^{m_{st}\ell} = \text{id}
\]

and thus our claim is proven. \( \square \)

The rest of Section 4 holds mutatis mutandis and so we also have that

**Theorem 8.2.** When \( m_{st} < \infty \) and \( \Lambda_{st} = k \) we have an isomorphism of right \( R \)-modules,

\[
\begin{align*}
\text{Ext}_R^{0, *}(B_t, \text{BS}(w)) & \cong \ker \rho_s^t(w)(1)(-1) \cong \ker \rho_s^t(w)(1) \\
\text{Ext}_R^{1, *}(B_t, \text{BS}(w)) & \cong \ker \rho_s^t(w)(1) \mathfrak{f} \rho_t(1) - 1 \mathfrak{d}(\ker \rho_s^t(w))(1) \\
& \cong \ker \rho_s^t(w)(3) \mathfrak{d}(\ker \rho_s^t(w))(1) \\
\text{Ext}_R^{2, *}(B_t, \text{BS}(w)) & \cong \mathfrak{d}(\ker \rho_s^t(w)) (1) \rho_s \wedge \rho_t(1) - 1 \mathfrak{d}(\ker \rho_s^t(w))(5)
\end{align*}
\]
Remark. The statements above for the finite case mirror the affine case word for word, but there’s a subtle difference. Namely in the finite case, for a given expression \( w \) and subexpression \( f \), the light leaf morphism \( L_{w,f} \) is possibly different than the affine case. For example, consider \( w = (s,t,s,t,s) \) and \( f = (1,1,1,1,1) \). When \( m_{st} = \infty \) the light leaf algorithm then returns the diagram on the left below

![Diagram](image)

while for \( m_{st} = 3 \) the light leaf algorithm returns the diagram on the right above. In particular \( r((1,1,1,1,1)) = t \) and so by Lemma 8.1 for \( m_{st} = 3 \), \( L_{w,f} \) is possibly different than the affine case. For example, consider \( w = (s,t,s,t,s) \) and \( f = (1,1,1,1,1) \). When \( m_{st} = \infty \)

8.1 Finite Dimension Calculations

Lemma 8.3. Suppose \( m_{st} < \infty \) and suppose \( w \) is a non repeating expression. If \( |w| < 2m_{st} \), then the lowest internal degree element in \( \text{Hom}_{R^e}(R, BS(w)) \) is of degree 1, 2 if \( |w| \) is odd, even respectively. If \( |w| \geq 2m_{st} \) the lowest internal degree element in \( \text{Hom}_{R^e}(R, BS(w)) \) is of degree 2 \( m_{st} - |w| \).

Proof. For \( |w| < 2m_{st} \), \( \text{Hom}_{R^e}(R, BS(w)) \) agrees with the affine case (one can use Soergel’s Hom formula and then use Lemma 3.19 in [EMTW20] and biadjointness of \( b_i \) with the standard form so that one never needs to use the braid relation in the finite Hecke algebra) so this follows from Lemma 5.3. Now, let \( \text{LD}(BS(w), BS(w)) \) be the lowest degree morphism in \( \text{Hom}_{R^e}(BS(w), BS(w)) \). Given any morphism \( \text{Hom}_{R^e}(B_i, BS(w)) \) we can produce a morphism in \( \text{Hom}_{R^e}(R, BS(w)) \) by adding a dot to the bottom of \( B_i \). As a result, we see that

\[
\text{LD}(R, B_i \otimes_R BS(w)) = \text{LD}(B_i, BS(w)) \geq \text{LD}(R, BS(w)) - 1
\]

Therefore each time we tensor with \( B_i \) or \( B_j \), the lowest degree drops by at most 1. It follows that when \( |w| \geq 2m_{st} \), the lowest degree element in \( \text{Hom}_{R^e}(R, BS(w)) \) is at least degree 2 \( m_{st} - |w| \). One can actually show it’s also at most 2 \( m_{st} - |w| \) by explicitly producing a morphism in \( \text{Hom}_{R^e}(R, BS(w)) \) of this degree which we leave to the reader. For a hint, look at the RHS of Eq. (71).

Proposition 8.4. For \( 2 < m_{st} < \infty \), \( \text{Ext}_{R^e}^{1-\lfloor |w|+1 \rfloor}(B_i, BS(w)) \) is a 1 dimensional \( k \) module when \( |m(t,w)| \geq 4 \)

Proof. The proof proceeds similarly to Proposition 5.5 as by Theorem 8.2 we still have the decomposition

\[
\text{Ext}_{R^e}^{1-\lfloor |w|+1 \rfloor}(B_i, BS(w))) \cong \ker \rho^R_s(w)(3-\lfloor |w|+1 \rfloor) \otimes R (\ker \rho^R_s(w)(1)-\lfloor |w|+1 \rfloor)
\]

We then have that an analogue of Lemma 5.4 holds by noting that the proof of Lemma 5.4 still applies for \( m_{st} < \infty \) except we need to replace Lemma 5.3 with Lemma 8.3 in the last step. As a result, when \( 4 \leq |m(t,w)| < 2m_{st} \) we are in the same situation as in Proposition 5.5 while for \( |m(t,w)| \geq 2m_{st} \), one needs to check

\[
-1 - |w| + |m(t,w)| + 2m_{st} - |m(t,w)| - 4 > -|w| - 1
\]

which is true when \( m_{st} > 2 \) as desired.

Corollary 8.5. All the generators/relations in Section 5 for \( m_{st} = \infty \) still exist/hold for \( 2 < m_{st} < \infty \).

We will deal with the case \( m_{st} = 2 \) separately below.
8.2 The Case \( m_{st} = 2 \)

By assumption, \( m_{st} = 0 \) and thus as \( m_{st} = 2 \) we see that \( [2] = a_t = a_s = 0 \). As a result, we see that \( s(a_t) = a_t \) and consequently \( \rho_t t(\rho_t) \in R^{s,t} \) and therefore \( \rho_t t(\rho_t)^{-1} = 0 \). As a result it becomes very easy to compute the cohomology in Eq. (18)

**Theorem 8.6.** When \( m_{st} = 2 \), we have an equality of right \( R \) modules

\[
\begin{align*}
\Ext^0_{R^e}(B_t, BS(w)) &= \ker \rho_t^e(w)(1)(-1) \\
\Ext^1_{R^e}(B_t, BS(w)) &= \ker \rho_t^e(w) \rho_t t(\rho_t)(-1) = \text{BS}(w)/\text{im} \rho_t^e(w) \rho_s(-1) \\
\Ext^2_{R^e}(B_t, BS(w)) &= \text{BS}(w)/\text{im} \rho_t^e(w) \rho_s \wedge \rho_t t(\rho_t)(-1)
\end{align*}
\]

Specifically, the difference between the above theorem and Theorem 8.2 is that \( \psi^u_1 \) are now chain maps and so the \( \Box(\ker \rho_t^e(w)) \rho_s(-1) \) part of \( \Ext^1_{R^e}(B_t, BS(w)) \) can now be described as \( [0,0]^{w,1}_1 \) for \( \psi^u_1 \in \text{BS}(w) \). In other words, blue and red Hochschild dots, along with the generating morphisms of \( \text{BSBim}(h,W_2) \), generate all of the morphisms in \( \text{BSBim}^{\text{Ext}}(h,W_2) \).

**Lemma 8.7.** For \( m_{st} = 2 \), we have the following relation in \( \Ext^{1,4}_{R^e}(B_tB_t,B_s) \)

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\circ \\
\downarrow
\end{array}
\end{array}
\]

\[
(\tilde{t}_1^e(1) \otimes 1 \otimes 1) = 0
\]

\[
(\tilde{t}_1^e(1) \otimes 1 \otimes 1) = 0
\]

**Proof.** Because \( [2] = 0 \), we have that \( \Box(\tilde{t}_1^e(1) \otimes 1 \otimes 1) = 0 \) and one can then compute that the following equalities hold

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\circ \\
\downarrow
\end{array}
\end{array}
\]

and so the lemma follows from rotating the top leftmost red strand down and applying rotation invariance of the 4 valent morphism.

**Corollary 8.8.** We have the following relation in \( \Ext^{2,4}_{R^e}(B_tB_t,B_sB_t) \)

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\circ \\
\downarrow
\end{array}
\end{array}
\]

**Proof.** Follows from adding a dot to the top right red strand of Eq. (69) and applying two-color dot contraction.

8.3 The Case \( m_{st} > 2 \)

Besides the generators and relations from Section 5, the presence of the generator \( v^t_{\{m_{st}\}} \) will result in additional relations in \( \text{BSBim}^{\text{Ext}}(h,W_{m_{st}}) \) not present in \( \text{BSBim}^{\text{Ext}}(h,W_{\infty}) \), namely

**Lemma 8.9.** Suppose that \( m_{st} > 3 \), or when \( m_{st} = 3 \), we have that \( v \) contains at least one \( s \). Then we have the following relation in \( \Ext^{1,4}_{R^e}(B_t,BS(\bar{m}_{m_{st}},v)) \)

\[
(v^t_{\{m_{st}\}} \otimes \text{id}_w) \circ \Phi^{\{\bar{m}_{m_{st}},v\}}_t = \Phi^{\{\bar{m}_{m_{st}},v\}}_t
\]

Diagrammatically this will be of the form

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\circ \\
\downarrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\uparrow \\
\circ \\
\downarrow
\end{array}
\end{array}
\]

43
Proof. The conditions in the lemma will imply that \(|m(t, t, \overline{m_{st}})| \geq 4\) and so \(\dim_{R_{m}} \text{Ext}_{R_{m}}^{1,-(m_{st}+1)}(B_{s}, \text{BS}(t, \overline{m_{st}}, v)) = 1\) by Proposition 8.4. Now the lemma will follow by noting that \((v_{s}^{t}(m_{st}) \otimes_{R} \text{id}_{v})\) sends \(1(s, \overline{m_{st}}, v)\) to \(1(\overline{m_{st}}, v)\). \(\Box\)

Lemma 8.10. For \(m_{st} = 3\), we have the following relation in \(\text{Ext}_{R_{m}}^{1-4}(B_{s}, B_{s}B_{s}R_{1})\).

\[
v_{s}^{t}(3) \circ \Phi^{(s,t,s)}_{t} = (\text{id}_{R} \otimes_{R} \eta_{s}^{\text{Ext}} \otimes_{R} \text{id}_{t}) \circ \delta_{t} - (\text{id}_{R} \otimes_{R} \eta_{s} \otimes_{R} \text{id}_{t}) \circ \delta_{t} \circ \phi_{t}
\]

Proof. Apply \((\text{id}_{\text{Ext}}, \otimes_{R} \eta_{s})\) to the equality

\[
(v_{s}^{t}(3) \otimes_{R} \text{id}_{s}) \circ \Phi^{(s,t,s)}_{t} = \Phi^{(t,s,t,s)}_{t}
\]

from Lemma 8.9 and simplify using Proposition 5.12. \(\Box\)

Another difference between \(\text{BSBim}^{\text{Ext}}(h, W_{m_{st}})\) and \(\text{BSBim}^{\text{Ext}}(h, W_{\infty})\) is that in \(\text{BSBim}^{\text{Ext}}(h, W_{m_{st}})\) for any \(k\), the red/blue \(2k\)–extvalent morphisms can be generated from the \(2m_{st}\)–extvalent morphism and generators of \(\text{BSBim}(h, W_{m_{st}})\). We already know that when \(k < m_{st}\), then the red \(2k\)–extvalent morphism can be obtained from the red \(2m_{st}\)–extvalent morphism by using Lemma 5.17, while for \(k > m_{st}\) we have

Lemma 8.11. \(\Phi^{2k-1}_{t}\) can be expressed as a composition of \(\Phi^{2m_{st}-1}_{t}\), comultiplication, and the \(2m_{st}\)–valent morphism when \(k > m_{st}\).

Proof. We first give an example in the case when \(m_{st} = 3\) and \(2k - 1 = 7\). We claim that

\[
\Phi^{2k-1}_{t}(3) \circ \Phi^{(s,t,s)}_{t} = \Phi^{(t,s,t,s)}_{t}
\]

This follows from the facts that the morphism above the red \(6\)–extvalent on the \(\text{RHS}\) sends \(1(s, t, t, s, s)\) to \(1(s, t, s, t, t, s)\) and \(\dim_{R_{m}} \text{Ext}_{R_{m}}^{1-9}(B_{s}, \text{BS}(s, t, t, s, s, t, t, s)) = 1\) by Proposition 8.4

In general, the red \(2k\)–extvalent can be realized as the postcomposition of the red \(2m_{st}\)–extvalent by an internal degree 0 morphism \(\alpha\) that sends \(1(s, 2m_{st} - 1)\) to \(1(, 2k - 1)\). One can easily generalize the \(\text{RHS}\) of Eq. (71) to produce such an \(\alpha\). \(\Box\)

8.4 Computation of \(\text{Ext}_{R}^{*,*}(R, B_{w})\) for \(m_{st} < \infty\)

Theorem 8.12. Assuming lesser invertibility, for all \(1 \leq k \leq m_{st}\), there is an isomorphism of right \(R\) modules.

\[
\text{Ext}_{R}^{0,*}(R, B_{w}) \cong R(-k) \\
\text{Ext}_{R}^{1,*}(R, B_{w}) \cong R(4-k) \oplus R(k) \\
\text{Ext}_{R}^{2,*}(R, B_{w}) \cong R(k+4)
\]

Proof. Same as Theorem 6.1 where Theorem 4.13 is replaced by Theorem 8.2. \(\Box\)
Theorem 8.13. Assuming lesser invertibility, for all \( 1 \leq k \leq m_{st} \), we have an isomorphism of right \( R \)-modules

\[
\text{Ext}^0_{R^e}(R, B_{\hat{k}}^k) \cong JW_{\hat{k}} \circ \bigoplus \bigoplus R
\]

\[
\text{Ext}^1_{R^e}(R, B_{\hat{k}}^k) \cong JW_{\hat{k}} \circ R \oplus JW_{\hat{k}} \circ \bigoplus \bigoplus R
\]

\[
\text{Ext}^2_{R^e}(R, B_{\hat{k}}^k) \cong JW_{\hat{k}} \circ \bigoplus \bigoplus R
\]

Proof. When \( k \leq m_{st} \), the light leaf morphisms \( L_{\hat{k} - 1, f} \) appearing in the proof in Theorem 6.6 coincide with the light leaf morphisms in affine case and so the proof of Theorem 6.6 still applies with Theorem 6.1 replaced by Theorem 8.12. \( \square \)

9 Ext Dihedral Diagrammatics: \( m_{st} < \infty \)

For this section let \( \mathcal{D}_{m_{st}} := \mathcal{D}_{m_{st}}(\mathfrak{h}, W_{m_{st}}) \) the 2–color diagrammatic Hecke category as defined in [Eli16].

Definition 9.1. Let \( \mathcal{D}_{m_{st}}^{\text{Ext}} := \mathcal{D}_{m_{st}}^{\text{Ext}}(\mathfrak{h}, W_{m_{st}}) \) be the strict \( k \)-linear supermonoidal category associated to a realization \( \mathfrak{h} \) of the finite dihedral group \( (W_{m_{st}}, S) \), where \( S = \{ s, t \} \) is the set of simple reflections, satisfying Assumption 2 and Assumption 3 defined as follows.

- **Objects** are the same as in \( \mathcal{D}_{\infty}^{\text{Ext}} \) as defined in Definition 7.1.
- **(a) Morphism Spaces when** \( m_{st} = 2 \) are again bigraded \( k \)-modules of total degree \((\ell, n)\) where \( \ell \) is the cohomological degree and \( n \) is the internal or Soergel degree. They are generated by

| generator | 4–valent | 4–valent |
|-----------|----------|----------|
| name      | (0,0)    | (0,0)    |
| bidegree  |          |          |

along with the generating morphisms in Definition 7.1 except for the red 2\( k \)–extvalent morphisms.

**Relations in** \( \mathcal{D}_{2}^{\text{Ext}} \) **are as follows.** All relations in \( \mathcal{D}_2 \) will be satisfied (see [Eli16] Section 6.2) and the generating morphisms for the color \( s \) will satisfy the relations of \( \mathcal{D}_s^{\text{Ext}} \) as specified in [Mak22] Section 4, and likewise for the color \( t \) and \( \mathcal{D}_t^{\text{Ext}} \). We will also have the following additional relation.

4–valent Hochschild Sliding:

\[
=x
\]

(b) **Morphism Spaces when** \( m_{st} > 2 \) are bigraded as in the case \( m_{st} = 2 \) above. They are generated by

| generator | \( 2m_{st} \)–valent | \( 2m_{st} \)–valent |
|-----------|---------------------|---------------------|
| name      | (0,0)               | (0,0)               |
| bidegree  |                     |                     |

along with all the generating morphisms in Definition 7.1.

**Relations in** \( \mathcal{D}_{m_{st}}^{\text{Ext}} \) **are as follows.** All relations in \( \mathcal{D}_{m_{st}} \) will be satisfied. The generating morphisms above will satisfy all relations in Section 7 along with the relation.
2m_{st} Absorption (m_{st} > 2)

\[ k > \frac{m_{st} + 1}{2} \]  

(73)

9.1 Equivalence

Theorem 9.2. Assume lesser invertibility. Define the functor \( \mathcal{F}^{\text{Ext}}_{m_{st}} : \mathcal{D}^{\text{Ext}}_{m_{st}} \to \text{BSBim}^{\text{Ext}}(\mathfrak{h}, W_{m_{st}}) \) where

and the rest of the generating morphisms and on objects as in Theorem 7.5. Then \( \mathcal{F}^{\text{Ext}}_{m_{st}} \) will be well defined and furthermore be a monoidal equivalence.

Proof. The defining relations of \( \mathcal{D}^{\text{Ext}}_{m_{st}} \) in Section 7.1 all hold in \( \text{BSBim}^{\text{Ext}}(\mathfrak{h}, W_{m_{st}}) \) as shown in Section 5 and Section 8. Therefore \( \mathcal{F}^{\text{Ext}}_{m_{st}} \) is well defined and as in Theorem 7.5 it suffices to check that for all \( w \in W_{m_{st}} \)

\[ \text{Hom}^{*,*}_{\mathcal{F}^{\text{Ext}}_{m_{st}}} (\mathcal{B}_{\mathfrak{g}}, \mathcal{B}_{w}) \equiv \text{Hom}^{*,*}_{\text{BSBim}^{\text{Ext}}(K_{\mathfrak{g}}, K_{w})} = \text{Ext}^{*,*}_R(R, B_w) \]

WLOG we can assume that \( w = t_{\mathfrak{g}_j}^{\hat{2}} \) and \( 2j \leq m_{st} \) is the longest element in \( W_{m_{st}} \). Then by Theorem 8.13 we need to show

\[ \text{Hom}^{0,*}_{\mathcal{F}^{\text{Ext}}_{m_{st}}} (\mathcal{B}_{\mathfrak{g}}, \mathcal{B}_{t_{\mathfrak{g}_j}^{\hat{2}}}) = \text{JW}^{t_{\mathfrak{g}_j}^{\hat{2}}} \circ \bullet \ldots \circ R \]

\[ \text{Hom}^{1,*}_{\mathcal{F}^{\text{Ext}}_{m_{st}}} (\mathcal{B}_{\mathfrak{g}}, \mathcal{B}_{t_{\mathfrak{g}_j}^{\hat{2}}}) = \text{JW}^{t_{\mathfrak{g}_j}^{\hat{2}}} \circ \bullet \ldots \circ R \oplus \text{JW}^{t_{\mathfrak{g}_j}^{\hat{2}}} \circ \frac{t_{\mathfrak{g}_j}^{\hat{2}}} R \]

\[ \text{Hom}^{2,*}_{\mathcal{F}^{\text{Ext}}_{m_{st}}} (\mathcal{B}_{\mathfrak{g}}, \mathcal{B}_{t_{\mathfrak{g}_j}^{\hat{2}}}) = \text{JW}^{t_{\mathfrak{g}_j}^{\hat{2}}} \circ \frac{t_{\mathfrak{g}_j}^{\hat{2}}} R \]

As in the affine case in Theorem 7.5, we just need to show that any diagram in \( \text{Hom}^{*,*,*}_{\mathcal{F}^{\text{Ext}}_{m_{st}}} (\mathcal{B}_{\mathfrak{g}}, \mathcal{B}_{t_{\mathfrak{g}_j}^{\hat{2}}}) \) with a Hochschild dot or a red 2k–extivalent morphism (for all \( k \geq 2 \)) can be written as a right \( R \) linear combination of diagrams with a Hochschild dot or a red 2k–extalent morphism at the bottom of the diagram. If the diagram if it doesn’t contain the 2m_{st}–valent vertex we are done, as we can just use the proof of Theorem 7.5. Call such diagrams ext–∞ diagrams. It follows that we just need to show that when the red 2k–extalent morphism or Hochschild dot is trapped by the 2m_{st}–valent morphism we can move the Hochschild dot and red 2k–valent morphism further down modulo ext–∞ diagrams.

Suppose we have a red 2k–extalent morphism trapped by a 2m_{st}–valent morphism. We first demonstrate the case when \( m = 3 \). Using Eq. (68) we can assume that the 4–extalent morphism is on the same component as the 6–valent morphism as seen on the left below. Applying fusion, we then see that

\[ \text{Ext}^{*,*,*}_R(R, B_w) \]

(74)
Now in the first diagram on the RHS of Eq. (74), use polynomial forcing to move the $\rho_s$ to the right resulting in

\[
\rho_s^2k\ldots = 2k\ldots - s(\rho_s)^2k\ldots \tag{75}
\]

The first diagram on the RHS of Eq. (75) reduces to $\text{ext}^{-\infty}$ diagrams using two-color dot contraction while the second diagram on the RHS of Eq. (75) is essentially the same as the second diagram on the RHS of Eq. (74) modulo polynomials that are at the top of the diagram. Now apply Eq. (67) and again modulo polynomials both diagrams reduce to

\[
2k\ldots \tag{76}
\]

We are now in same position as the LHS of Eq. (74) except the number of strands of the red $2k-$extvalent morphism connected to the $6-$valent morphism has gone up. We can repeat this argument one more time so that 3 strands of the red $2k-$extvalent morphism are now connected to the $6-$valent morphism, and after another application of Eq. (67) for the first top blue strand we can apply $2m_{st}-\text{Absorption Eq. (73)}$ because the only polynomials that appear are at the top.

For general $m_{st} \geq 3$, the argument is the same, where we keep applying the reductions in Eq. (74) and Eq. (75) until we have $m_{st}$ strands connecting the red $2k-$extvalent morphism and the $2m_{st}$-valent morphism such that the only polynomials that appear are at the top of the diagram. Using $4-\text{ext}$ reductions Eq. (56), Eq. (57), we can also move Hochschild dots past the $2m_{st}-\text{valent morphism}$ at the cost of a red $4-\text{extvalent morphism}$ and so we are done. Note for $m_{st} = 2$, we don't have red $2k-$extvalent morphisms but we can always move Hochschild dots past the $4-\text{valent morphism}$ using Eq. (72).

\section*{Appendix A: Lifting to Chain Level}

We will now define (partial) chain lifts of various bimodule morphisms in $\text{BSBim}(V^*, W)$. Specifically, given a morphism $f$ between Bott-Samelson bimodules, we will define a (partial) morphism between the corresponding Bott-Samelson complexes lifting $f$. As our maps are required to be $R^s$ linear it suffices to define the chain maps on the inner tensor factors of $R^s, R^{se}$, etc. In addition, as all of the bimodule maps we are lifting are of the form $f : R \otimes_{s/t} \ldots \otimes_{s/t} R \rightarrow R \otimes_{s/t} \ldots \otimes_{s/t} R$, we can always define $\tilde{f}^0 : R \otimes \ldots R \rightarrow R \otimes \ldots R$ by the exact same formula as $f$ but with $\otimes_{s/t}$ replaced by $\otimes$.

\textbf{Definition A.1.} Let $\gamma_s = \rho_s \cdot s(\rho_s)$.

\subsection*{A.1 Unit}

The unit map of $B_s$ will be $\eta_s : R \rightarrow B_s$ where $\eta_s(1) = \rho_s \cdot 1 - 1 \otimes s(\rho_s)$. We want to find $\eta_s^{-1}$ making the following diagram commute.
Again since the third equation, we have $\partial_s (r h) = \partial_s (r h) + s(r) \partial_s (h) = r \partial_s (h)$ so the expression on the bottom right above is zero. The second equation will be the same as $s(\gamma_s) = \gamma_s$ and so $\partial_s (k \gamma_s) = \gamma_s \partial_s (k)$. For the third equation, we have

$$
\begin{align*}
\tilde{\eta}_s^0(1) &
\end{align*}
$$

We check the first equation. Note that $\tilde{\eta}_s^0(d(\rho_s \otimes 1 \otimes 1))$ is given by

$$
\begin{align*}
\rho_s \otimes 1 \otimes 1 &
\end{align*}
$$

We want to find $\tilde{\mu}_s^0$ making this diagram commute. Let $W = ((V^*)^d(-2) \otimes k\gamma_s(-4))$.

The multiplication map of $B_s$ is $\mu_s : R \otimes_s R \otimes_s R(2) \rightarrow R \otimes_s R(1)$ where $\mu_s(f \otimes_s g \otimes_s h) = \partial_s(g) f \otimes_s h$. We want to find $\tilde{\mu}_s^0$ making this diagram commute. Let $W = ((V^*)^d(-2) \otimes k\gamma_s(-4))$.

We claim that the following definition will work.

$$
\begin{align*}
\tilde{\eta}_s^0(1) &
\end{align*}
$$

We claim that the following definition will work.

$$
\begin{align*}
\tilde{\eta}_s^0(1) &
\end{align*}
$$

We want to find $\tilde{\mu}_s^0$ making this diagram commute. Let $W = ((V^*)^d(-2) \otimes k\gamma_s(-4))$.

Then $V^* = (V^*)^d \otimes k\rho_s$. We then claim that the following definition will make the diagram commute

$$
\begin{align*}
\tilde{\eta}_s^1(1 \otimes 1 \otimes 1) &= \tilde{\eta}_s^1(1 \otimes 1 \otimes 1 - 1 \otimes s(\rho_s)) \quad \text{if } r \in (V^*)^d \\
\tilde{\eta}_s^1(\rho_s \otimes 1 \otimes 1) &= (\rho_s + s(\rho_s)) \otimes 1 - \gamma_s \otimes 1 \otimes 1
\end{align*}
$$

This is clear if $r \in (V^*)^d$. For $\rho_s$ note that $\tilde{\eta}_s^0(d(\rho_s \otimes 1 \otimes 1))$ is given by

$$
\begin{align*}
\rho_s \otimes 1 \otimes 1 &
\end{align*}
$$

and one quickly computes that $d$ applied to our definition above yields the expression on the bottom right.

### A.2 Multiplication

The multiplication map of $B_s$ is $\mu_s : R \otimes_s R \otimes_s R(2) \rightarrow R \otimes_s R(1)$ where $\mu_s(f \otimes_s g \otimes s h) = \partial_s(g) f \otimes_s h$. We want to find $\tilde{\mu}_s^0$ making this diagram commute. Let $W = ((V^*)^d(-2) \otimes k\gamma_s(-4))$.

We claim that the following definition will work.

$$
\begin{align*}
\tilde{\mu}_s^0(r(1) \otimes 1 \otimes h \otimes 1) &= 0 \quad \text{if } r \in (V^*)^d \\
\tilde{\mu}_s^0(\gamma_s(1) \otimes 1 \otimes h \otimes 1) &= 0 \\
\tilde{\mu}_s^0(r(2) \otimes 1 \otimes h \otimes 1) &= r \otimes \partial_s(h) \otimes 1 \quad \text{if } r \in (V^*)^d \\
\tilde{\mu}_s^0(\gamma_s(2) \otimes 1 \otimes h \otimes 1) &= \gamma_s \otimes \partial_s(h) \otimes 1
\end{align*}
$$

We check the first equation. Note that $\tilde{\mu}_s^0(d(r(1) \otimes 1 \otimes h \otimes 1))$ is given by

$$
\begin{align*}
\mu_s &
\end{align*}
$$

But the twisted Leibniz rule and $r \in (V^*)^d$ we see that $\partial_s(h r) = \partial_s(h) + s(r) \partial_s(h) = r \partial_s(h)$ so the expression on the bottom right above is zero. The second equation will be the same as $s(\gamma_s) = \gamma_s$ and so $\partial_s(k \gamma_s) = \gamma_s \partial_s(k)$. For the third equation, we have

$$
\begin{align*}
\tilde{\eta}_s^0(1) &
\end{align*}
$$

Again since $\partial_s(h r) = \partial_s(h) r$, our choice for $\tilde{\mu}_s^0$ clearly works and the fourth equation will again work because $s(\gamma_s) = \gamma_s$. 

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A.3 Counit

The counit map of $B_s$ will be $\epsilon_s : R \otimes_s R(1) \to R$ where $\epsilon_s(f \otimes_s g) = f g$ so we want to find $\tilde{\epsilon}_s^{-1}$ making the following diagram commute. Again, let $W = ((V^*)^4(-2) \oplus k\gamma_s(-4))$.

![Diagram]

In [Mak22] it was shown that the following definition makes the diagram commute

$$\tilde{\epsilon}_s^{-1}(r \otimes 1 \otimes 1) = r \otimes 1 \otimes 1 \quad \text{if } r \in (V^*)^s$$

$$\tilde{\epsilon}_s^{-1}(\gamma_s \otimes 1 \otimes 1) = \rho_s \otimes s(\rho_s) \otimes 1 + s(\rho_s) \otimes 1 \otimes \rho_s$$

A.4 Comultiplication

The comultiplication map of $B_s$ will be $\lambda_s : R \otimes_s R(1) \to R \otimes_s R \otimes_s R(2)$ where $\lambda_s(f \otimes_s g) = f \otimes_s 1 \otimes_s g$ so we want to find $\tilde{\delta}_s^{-1}$ making the following diagram commute. Again, let $W = ((V^*)^4(-2) \oplus k\gamma_s(-4))$.

![Diagram]

We then claim that the following definition will work.

$$\tilde{\delta}_s^{-1}(r \otimes 1 \otimes 1) = r^{(1)} \otimes 1 \otimes 1 + r^{(2)} \otimes 1 \otimes 1 \quad \text{if } r \in (V^*)^s$$

$$\tilde{\delta}_s^{-1}(\gamma_s \otimes 1 \otimes 1) = \gamma_s^{(1)} \otimes 1 \otimes 1 + \gamma_s^{(2)} \otimes 1 \otimes 1$$

This is an easy check left to the reader.

A.5 Inverse of Left Unitor

The inverse of the left unitor $\lambda_s$ for $B_s$ is the map $\tau_s : B_s \to R \otimes_R B_s$ where $\tau_s(f \otimes_R g) = f \otimes_R 1 \otimes_R g$, so we want to find $\tilde{\tau}_s^{-1}$ making the following diagram commute. Again, let $W = ((V^*)^4(-2) \oplus k\gamma_s(-4))$.

![Diagram]

In [Mak22] it was shown that the following definition will make the diagram commute

$$\tilde{\tau}_s^{-1}(r \otimes 1 \otimes 1) = r^{(1)} \otimes 1 \otimes 1 + r^{(2)} \otimes 1 \otimes 1 \quad \text{if } r \in (V^*)^s$$

$$\tilde{\tau}_s^{-1}(\gamma_s \otimes 1 \otimes 1) = \rho_s^{(1)} \otimes s(\rho_s) \otimes 1 + s(\rho_s) \otimes 1 \otimes \rho_s + \gamma_s^{(2)} \otimes 1 \otimes 1$$

A.6 Inverse of Right Unitor

The inverse of the right unitor $\theta_s$ for $B_s$ is the map $\sigma_s : B_s \to B_s \otimes_R R$ where $\tau_s(f \otimes_R g) = f \otimes_R 1 \otimes_R g$, so we want to find $\tilde{\sigma}_s^{-1}$, $\tilde{\sigma}_s^{-1}$ making the following diagram commute. Again, let $W = ((V^*)^4(-2) \oplus k\gamma_s(-4))$.

![Diagram]
In [Mak22] it was shown that the following definition will make the diagram commute
\[ \tilde{\sigma}_1^{-1}(r \otimes 1 \otimes 1) = \frac{1}{r} \tilde{L}^{(1)} \otimes 1 \otimes 1 + \frac{1}{2} \tilde{L}^{(2)} \otimes 1 \otimes 1 \quad \text{if } r \in (V^*)^8 \]
\[ \tilde{\sigma}_1^{-1}(\gamma \otimes 1 \otimes 1) = \gamma \otimes 1 \otimes 1 + s(\rho) \otimes 1 \otimes s(\rho) \]

**Appendix B: HOMFLY Homology Calculations**

We refer the reader to [EMTW20, Chapter 21.6] for the definition of HOMFLY homology $\text{HHH}$. As shown in [Kho07, Theorem 1], the reduced HOMFLY homology $\text{HHH}$ is computed using the geometric realization $h_{geo}$ of $S_n$. As a result, the complex $F(\beta_L)$ associated to a braid $\beta_L$ lives in $K^{\otimes}(S\text{Bim}(h_{geo}, S_n))$ and $R$ is a polynomial ring in $n−1$ variables instead of $n$. In particular this makes the computation of $\text{HHH}$ easier and so we will be computing reduced HOMFLY homology in this section. The reduced HOMFLY homology categorifies the reduced HOMFLY polynomial $P(a, q)$ defined via the skein relation
\[ \alpha(a, q)(L_+) - \alpha^{-1}(a, q)(L_-) = (q - q^{-1})P(a, q)(L_0) \]
and the normalization $P(a, q)(\text{unknot}) = 1$.

**Definition B.1.** Define $d_s = \frac{1}{2}(a_s \otimes 1 - 1 \otimes a_s) \in R \otimes_R R$.

**Example 6.** Let us review the calculation of the Poincare series for $\text{HHH}$ of the Hopf link $L2a1$. One possible braid presentation is $\sigma_1^-$. By [EMTW20, Eq 19.32] we have that
\[ F_2^1 = B_4(-1) \quad \text{where } B_4(1) \rightarrow R(2) \]
As a result, one can calculate that $\text{HHH}^{A=0}(F_2^3), \text{HHH}^{A=1}(F_2^3)$ will be the cohomology of the following complexes
\[ \text{HHH}^{A=0}(F_2^3) : R(2) \rightarrow R(a) \rightarrow R(2) \]
\[ \text{HHH}^{A=1}(F_2^3) : R(4) \rightarrow R(2) \rightarrow R(4) \]
where $R = \mathbb{k}[a, q]$ as $F_2^1$ is a braid on 2 strands. As a result the Poincare series $P(A, Q, T)(F_2^3)$ for $\text{HHH}(F_2^3)$ will be
\[ P(A, Q, T)(F_2^3) = \left( \frac{Q^2}{1 - Q^2} + Q^{-2}T^2 \right) + \left( A \left( \frac{Q^{-2}}{1 - Q^2} \right) \right) \]
To recover the HOMFLY polynomial $P(a, q)$ up to a unit in $Z[A^\pm1, Q^\pm1, T^\pm1]$ we make the following substitutions
\[ A = \frac{a^2Q^2}{T}, T = -1, \quad a = a^{-1}, Q = q \]
Applying this to $P(A, Q, T)(F_2^3)$ we obtain
\[ P(A, Q, T)(F_2^3)_{\text{sub}} = \frac{a^2 + a^{-2} - 1 - a^{-2}}{1 - q^2} \]
while using the definition of the reduced HOMFLY polynomial above we obtain
\[ \overline{P}(a, q)(\sigma_1^-) = \frac{q}{a} \left( \frac{a^2 + a^{-2} - 1 - a^{-2}}{q^2 - 1} \right) \]
which agrees with $P(A, Q, T)(F_2^3)_{\text{sub}}$ up to the unit $-q/a$. In general, the unit relating $P(A, Q, T)(\beta_L)_{\text{sub}}$ and $P(a, q)(L)$ can be found in [EH19, Appendix A].

Our next example is a braid on 3 strands. Here $h_{geo}$ is the geometric realization of $S_3$ so $R = \mathbb{k}[a_s, \alpha_t]$ where
\[ s = (1, 2), \quad t = (2, 3) \in S_3 \quad \alpha_s = x_1 - x_2, \quad \alpha_t = x_2 - x_3 \quad \alpha_s^* = x_1^* - x_2^*, \quad \alpha_t^* = x_2^* - x_3^* \]
Example 7. A braid representative for the connect sum of two Hopf links $L2a1 \# L2a1$ is given by $\sigma_1^2 \sigma_2^2$. The corresponding Rouquier complex is homotopic to

$$B_tB_t(-2) \xrightarrow{B_tB_t \oplus B_tB_t} B_t(1) \oplus B_t(2) \oplus B_t(1) \xrightarrow{B_t(3) \oplus B_t(3)} R(4).$$

Using Theorem 8.13 and Corollary 4.2 we have the following

**Lemma B.2.**

- $\HHH^0(B_t) = \downarrow R = R(-1)$
- $\HHH^1(B_t) = \downarrow R \oplus \sigma_1 \downarrow \sigma_2 \downarrow, R = R(3) \oplus R(1)$
- $\HHH^2(B_t) = \downarrow \sigma_1 \sigma_2 \downarrow, R = R(5)$

**Lemma B.3.**

- $\HHH^0(R) = \sigma_1 \sigma_2 R = R$
- $\HHH^1(R) = \sigma_1 \sigma_2 \downarrow, R \oplus \sigma_1 \sigma_2 \downarrow, R = R(2) \oplus R(2)$
- $\HHH^2(R) = \sigma_1 \sigma_2 \downarrow, R = R(4)$

The complex for $\HHH^{A=0}$ will then be

$$R(-4) \xrightarrow{0 \ 0 \ 0 \ a_t} R(-2) \oplus R(-2) \xrightarrow{R \oplus R \oplus R} R(2) \oplus R(2) \xrightarrow{R(4)} R(4).$$

As right $R$ modules, the nonzero cohomology will then be

$$\HHH^{A=0} = R(-4), \quad \HHH^{A=0} = R(2) \oplus R(2) \oplus R(4).$$

with corresponding Poincare series $\frac{Q^4(1 - Q^2)}{1 - Q^2} + \frac{2T^2}{1 - Q^2} + Q^{-4}T^4$. The complex for $\HHH^{A=1}$ will be

$$R \oplus R \xrightarrow{(R(2) \oplus R(2)) \oplus (R(2) \oplus R(2))} (R(4) \oplus R(2)) \oplus (R(4) \oplus R(4)) \oplus (R(4) \oplus R(2)) \xrightarrow{(R(6) \oplus R(4)) \oplus (R(6) \oplus R(4)) \oplus (R(6) \oplus R(4)) \oplus (R(6) \oplus R(4)) \oplus (R(6) \oplus R(6))}$$

As right $R$ modules, the nonzero cohomology will then be

$$\HHH^{A=1} = R \oplus R, \quad \HHH^{A=1} = R(4) \oplus R(4) \oplus R(4).$$

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with corresponding Poincare series \( \frac{2}{(1-Q^2)^2} + \frac{2T^2Q^{-4}}{1-Q^2} \). The complex for \( \text{HHH}^{A=2} \) will be

\[
\begin{array}{cccc}
\text{R}(4) & R(6) & R(8) \\
\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\end{array}
\]

As right \( R \) modules, the nonzero cohomology will then be

\[
\left( \text{HHH}^{A=2} \right)^0 = R(4) \quad \text{with corresponding Poincare series } \frac{Q^{-4}}{(1-Q^2)^2}
\]

Putting this all together we obtain

\[
\mathcal{P}(A, Q, T)(F_s^2 F_t^2) = \left( \frac{Q^4}{(1-Q^2)^2} + \frac{2T^2}{1-Q^2 + Q^{-4}} + A \left( \frac{2}{(1-Q^2)^2} + \frac{2T^2Q^{-4}}{1-Q^2} \right) + A^2 \left( \frac{Q^{-4}}{(1-Q^2)^2} \right) \right)
\]

which upon closer inspection is nothing more than \( \mathcal{P}(A, Q, T)(F_s^2 F_t^2) \)! Because

\[
\mathcal{P}(a, q)\delta_1 L_1 \# L_2 = \mathcal{P}(a, q)\delta_1 \mathcal{P}(a, q)\delta_2
\]

it immediately follows from the previous example that

\[
\mathcal{P}(a, q)\delta_1 L_1 \# L_2 \# \delta_2 \# a_1 = \frac{q^2}{a^2} \mathcal{P}(A, Q, T)(F_s^2 F_t^2)\delta_1 \mathcal{P}(a, q)\delta_2
\]

This phenomenon happens in general, i.e.

**Proposition B.4 ([Ras15, Lemma 7.8]).** Given two links \( L_1 \) and \( L_2 \), as graded vector spaces, we have that

\[
\text{HHH}(L_1 \# L_2) = \text{HHH}(L_1) \otimes \text{HHH}(L_2)
\]

### Appendix C: Gomi’s Trace

Before going into Gomi’s trace, we should mention that there are usually two versions/presentations of the Hecke algebra \( H_W \) of a Coxeter group \( W \) in the literature. As originally defined in [KL79], version 1 of \( H_W \) is the associative algebra over \( \mathbb{Z}[q, q^{-1}] \) on generators \( \{ T_s \} \) satisfying the braid relation and the quadratic relation

\[
T_s^2 = (q-1)T_s + q
\]

In [Soe97], Soergel gives version 2 of \( H_W \) as the associative algebra over \( \mathbb{Z}[\nu, \nu^{-1}] \) on generators \( \{ \delta_s \} \) satisfying the braid relation and the quadratic relation

\[
\delta_s^2 = (\nu^{-1} - \nu)\delta_s + 1
\]

and the two presentations are isomorphic under the \( \mathbb{Z} \)-linear map \( \delta_w \rightarrow \nu^{\ell(w)}T_w \) and \( \nu \rightarrow q^{-1/2} \) after tensoring version 1 by \( \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \). Under this isomorphism we also have \( \delta_s^{-1} = \nu^{-1}T_s^{-1} \). For \( w \in W = W_m \), the KL-basis of \( H_{W_m} \) can be explicitly expressed in the standard basis as

\[
b_w = \sum_{y \leq w} \nu^{\ell(y)} \delta_y = \nu^{\ell(w)} \sum_{y \leq w} T_y
\]

[Gom06] uses version 1 of \( H_W \) while [EMTW20] uses version 2 and as such we will be using both presentations interchangeably in what follows.

**Definition C.1.** Define a trace \( \epsilon_t : H_{W_m} \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}][t] \) by first defining \( \epsilon_t \) on the KL-basis via

\[
\epsilon_t(b_w) = \sum_{a, b} \text{dim}_{\mathbb{C}} H_{a,b}(B_w) q^{b/2} t^a
\]

and extend \( \mathbb{Z}[q^{1/2}, q^{-1/2}] \) linearly.
Remark. In order for $\epsilon_t$ to be $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ linear, at the categorical level we should define $q^{1/2}[B_w] := [B_w(1)]$. Consequently this means that we should define $v^{-1}[B_w] := [B_w(q)]$ which is the opposite of what is written in [EMTW20, Section 4.8]. In fact, Soergel’s Hom formula [EMTW20, Theorem 5.27] is only true when using our convention above, rather than the convention in [EMTW20].

Remark. We are using Hochschild homology in the generating function instead of Hochschild cohomology to agree with [WW11]. For $R = k[x_1, \ldots, x_n]$, $|x_i| = 2$ and $M \in R - \text{Bim}$ we have an isomorphism of doubly graded vector spaces

$$HH_{k, \ell}(M) \cong HH^{n-k, \ell-2n}(M)$$

As a result, $(1-q)^2\epsilon_t|_{t=0} = e$, the standard or symmetrizing trace of $H_W$ as a result of Soergel’s Hom Formula. In addition, for $n = 2$ and $W = W_m$ by Theorem 8.12 we have for $1 \leq k \leq m$

$$HH_{0,*}(B, k) \cong R(k)$$

$$HH_{1,*}(B, k) \cong R(k-4) \oplus R(-k)$$

$$HH_{2,*}(B, k) \cong R(-(k+4))$$

and thus for $1 \leq k \leq m - 1$

$$\epsilon_t(b, k) = \epsilon_t(b, k) = \frac{q^{k/2} + (q^{(k-1)/2} + q^{k/2})t + q^{(k+1)/2}t^2}{(1-q^t)^2} \quad (C.2)$$

Lemma C.2. (a) For all $2 \leq k \leq m - 1$

$$(v + v^{-1})\epsilon_t(b, k) = \epsilon_t(b, k) + \epsilon_t(b, k - 1)$$

(b) $\epsilon_t(\delta, k) = \epsilon_t(\delta, k) + \epsilon_t(\delta, k - 1)$.

(c) $\epsilon_t(\delta, k + 1) = (v - 1)\epsilon_t(\delta, k) + \epsilon_t(\delta, k - 1)$ for all $2 \leq k \leq m - 1$.

Proof. (a) follows from direct computation using Eq. (C.2). (b) follows from Eq. (C.2) and induction using Eq. (C.1).

(c) Using Eq. (C.1) to expand the LHS of Eq. (C.3) we obtain $v\epsilon_t(b, k) + v^{-1}\epsilon_t(\delta, k) + \epsilon_t(\delta, k - 1) + \epsilon_t(b, k - 1)$ while the RHS of Eq. (C.3) can be written as $\epsilon_t(\delta, k + 1) + v\epsilon_t(\delta, k) + v\epsilon_t(b, k) + \epsilon_t(b, k - 1)$. Thus it follows that

$$v^{-1}\epsilon_t(\delta, k) + \epsilon_t(\delta, k - 1) = \epsilon_t(\delta, k + 1) + v\epsilon_t(\delta, k)$$

and using part (b) and moving $v\epsilon_t(\delta, k)$ to the LHS gives the result.

For the dihedral group $W_m$ it turns out we do not need to know the entries of Lusztig’s Exotic Fourier transform matrix (although they are explicitly given in [Gom06, Section 4.6]) or even the characters of $H_W$ in order to compute Gomi’s trace $\tau$ by the following theorem proved in [Kih04]

**Theorem C.3.** $\tau : H_{W_m} \to \mathbb{Z}[q^{1/2}, q^{-1/2}]|t|$ is the unique trace\(^3\) function on $H_{W_m}$ satisfying

1. $\tau(1) = 1$
2. $\tau(T_1) = \frac{-(1-q)qt}{1+qt}$
3. $\tau(T_1T_2\ldots T_i T_2^{-1}\ldots T_i^{-1}) = \left(\frac{(1-q)qt}{1+qt}\right)^2$, for all $1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor$

Note, in [Gom06, Theorem 4.5] this was stated using the change of variables $r = qt$.

**Theorem C.4.** For all $x \in H_{W_m}$

$$\frac{(1-q)^2}{(1+qt)^2} \epsilon_t(x) = \tau(x)$$

\(^3\)This means that $\tau$ is a $\mathbb{Z}[q, q^{-1}]$ linear map such that $\tau(ab) = \tau(ba)$ for all $a, b \in H_{W_m}$.
Proof. Because we have an isomorphism of doubly graded vector spaces

$$HH_{a,b}(M \otimes_R N) \cong HH_{a,b}(N \otimes_R M)$$

$\epsilon_I$ the LHS is a trace function on $H_{W_m}$ and by uniqueness we just need to check the LHS satisfies the conditions in Theorem C.3. (1) follows from the well known computation $HH_{*,*}(R) = \text{Sym}(V^* \otimes V/[1])$, where $\dim V = 2$ in this case. (2) is an immediate consequence of Eq. (C.2) and that $T_1 = q^{1/2} b_2 - 1$.

For (3), under the isomorphism between the two versions of the hecke algebra we have that

$$\epsilon_I(T_1 T_2 \ldots T_1^{-1} T_2^{-1}) = v^{-2} \epsilon_I(\delta_1 \delta_2 \ldots \delta_1^{-1} \delta_2^{-1})$$

For $i \geq 1$, let $\varphi_i = \delta_1 \delta_2 \ldots \delta_1^{-1} \delta_2^{-1}$ and $\varphi_i = \delta_2 \delta_1 \ldots \delta_2^{-1} \delta_1^{-1}$. We claim

$$\epsilon_I(\varphi_i) = \epsilon_I(\varphi_i^*)$$

and therefore we only need to compute $\epsilon_I(\varphi_i) = \epsilon_I(\varphi_i^*)$. Note $\delta_2^{-1} = \delta_2 + v - v^{-1}$ and since $\epsilon_I$ is a trace we see that

$$\epsilon_I(\varphi_i) = \epsilon_I((\delta_2 + v - v^{-1}) \delta_1 \varphi_i^*) = \epsilon_I(\delta_2 \delta_1 \varphi_i) + (v - v^{-1}) \epsilon_I(\delta_1 \varphi_i)$$

Now, using the quadratic relation, because $\varphi_i$ alternates between $\delta_1, \delta_2$ at the start, $\delta_1^{-1}, \delta_2^{-1}$ at the end with 2 more $\delta_1, \delta_2$ terms than $\delta_1^{-1}, \delta_2^{-1}$ terms, we can write $\varphi_i = \sum_{j=1}^{m-1} a_j \delta_{_2}^j$ when $i > 1$. As a result, $\delta_2 \delta_1 \delta_2^j = \delta_2^j$ and thus by Lemma C.2 part (c) we have

$$\epsilon_I(\delta_2 \delta_1 \varphi_i) + (v - v^{-1}) \epsilon_I(\delta_1 \varphi_i) = (v^{-1} - v) \epsilon_I(\delta_1 \varphi_i) + \epsilon_I(\varphi_i) + (v - v^{-1}) \epsilon_I(\delta_1 \varphi_i)$$

as desired. Finally, using the above and Eq. (C.2) we compute

$$\frac{(1-q)^2}{(1+qt)^2} \epsilon_I(T_1 T_2 \ldots T_1^{-1} T_2^{-1}) = \frac{(1-q)^2}{(1+qt)^2} \epsilon_I(T_1 T_2) = \frac{(1-q)^2}{(1+qt)^2} \epsilon_I(T_1 T_2)$$

$$= \frac{(1-q)^2}{(1+qt)^2} \left[q \epsilon_I(b_1 b_2) - T_1 - T_2 - 1 \right]$$

$$= \frac{(1-q)^2}{(1+qt)^2} \left[(q-1)qt \right] = \left(\frac{(1-q)qt}{1+qt}\right)^2$$


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