A THRESHOLD-BASED RISK PROCESS WITH A WAITING PERIOD TO PAY DIVIDENDS

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ABSTRACT. In this paper, a modified dividend strategy is proposed by delaying dividend payments until the insurer’s surplus level remains at or above a threshold level $b$ for a predetermined period of time $h$. We consider two cases depending on whether the period of time sustained at or above level $b$ is counted either consecutively or accumulatively (referred to as standard or cumulative waiting period). In both cases, we develop a recursive computational procedure to calculate the expected total discounted dividend payments made prior to ruin for a discrete-time Sparre Andersen renewal risk process. By varying the values of $b$ and $h$, a numerical study of the trade-off effects between finite-time ruin probabilities and expected total discounted dividend payments is investigated under a variety of scenarios. Finally, a generalized threshold-based strategy with a delayed dividend payment rule is studied under the compound binomial model.

1. Introduction and preliminaries. In recent years, risk processes perturbed by dividend payment strategies have received considerable attention in the actuarial literature on ruin theory. Strategies involving both single and multiple dividend barriers/thresholds have been studied by various authors under a multitude of distributional risk models (for an excellent survey of the work that has been carried out in this regard, see [4]). The central focus has predominantly involved the characterization of the Gerber-Shiu discounted penalty function (e.g., see [20]) and its

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connection to the expected value of the total discounted dividends paid prior to ruin.

In this paper, we consider a variation of the threshold-based risk model in which dividends only get paid provided that a constant threshold level \( b > 0 \) is reached and some period of “good health” is sustained at or above this threshold. A similar variation in the study of barrier options is the so-called Parisian option, as introduced by [7]. The owner of this option has the right to exercise it depending on whether the underlying asset price stays above or below a certain level for a predetermined time before the maturity date. Since the trigger event does not depend on a single crossing of this level, it is difficult to manipulate the underlying asset price over a short period of time. Pricing of cumulative Parisian options is related to the concept of occupation times. There is a vast literature on occupation times of stochastic processes such as Brownian motion (e.g., see [1], [13], and [30]).

The motivation for our study is two-fold. On the one hand, for many of the threshold-based risk models in the literature, it is typically the case that the dividends modification involves the introduction of a fixed level \( b > 0 \), so that whenever the surplus process achieves a level of at least \( b \), the insurer immediately begins paying dividends to shareholders at some specified rate. The same essential principle continually applies as the surplus process moves up and down over the course of time, potentially crossing level \( b \) several times along the way. A particular shortcoming of this type of model lies in the fact that the company is presumed to be healthy enough to begin paying dividends the very moment level \( b \) is achieved, instead of perhaps trying to assess whether the company can sustain a period of good health by remaining at or above level \( b \) for a certain length of time. On the other hand, this concept of Parisian delay was first considered by [14] who studied the optimal barrier dividend strategy in the classical Poisson risk process with exponentially distributed claims. By applying martingale techniques, they derived some explicit expressions for the expected present value of dividend payments. While it is known that a barrier strategy is optimal only when the dividend rate is unbounded, it may be worthwhile to investigate a threshold-based strategy in which the dividend rate is bounded (which should be more realistic in practice). To be precise, when the surplus level has remained at or above level \( b \) for a certain amount of time, it is not the lump-sum exceeding level \( b \) that is paid out (as in the original type of Parisian option in [7] or the barrier strategy with Parisian delay in [14], but a particular bounded dividend rate per period is applied instead. In the first part of this paper, we assume that the dividend rate is bounded by the premium rate (i.e., the classical threshold strategy). Later on, we extend the analysis to a generalized threshold-based strategy in which the dividend rate is now bounded by the lump-sum exceeding level \( b \) at the time a dividend is to be paid. In this scenario, it is clear that the dividend rate can be greater than the premium rate (although this is impossible in the zero delay case).

In other related work, [10] and [28] applied the idea of Parisian delay to the recognition of ruin problem (i.e., ruin is only declared when the surplus level is negative for more than a fixed period of time) in a spectrally negative Lévy risk process. Also, within the same process, [24] studied the Parisian ruin problem using stochastic implementation delays in conjunction with the concept of Erlangization (e.g., see [2]) to provide nice approximations for the model with deterministic implementation delays. In very recent, [25] generalized the results in [28] to a standard Lévy risk process with adaptive premium rate. Although most of these studies have
been performed under a continuous-time setting, [12] did analyze a discrete-time counterpart of Parisian ruin for a compound binomial risk model.

It is clear that recent research in this area has mainly focused on the Parisian ruin problem, and the risk processes under consideration have been restricted to a certain group of Lévy processes. No corresponding investigation seems to have been conducted with respect to general renewal risk processes. Although the model with a fixed period of delay, either in terms of dividend payments or ruin recognition, can be analyzed within the framework of such Lévy processes, it is difficult to derive analytic results for a general renewal risk process. Moreover, while several papers have addressed the dividend problem with Parisian delay (e.g., see [14], [11], [36], [8]), we are unaware of any research that studies the dual relationship between the expected total discounted dividend payments and the ruin probability under a threshold-based Parisian-type framework. In [15], the so-called firm value is formulated as the present value of all future dividend payments. In line with the idea of de Finetti’s model, most studies related to dividend strategies in risk models focus on maximizing the expected total discounted dividend payments until ruin. However, it has been shown that reducing the risk of insolvency can also increase firm value by improving the likelihood of extending the firm’s life (e.g., see [34]). Therefore, we take into account both of these trade-off criteria when determining a divided payment strategy which addresses both the firm’s value and risk. In our model, the expected total discounted dividend payments and the probability of ruin are regarded as the firm’s value and risk, respectively. The effect of pairs of threshold level and period of excursion time above the threshold on the aforementioned quantities of interest are investigated numerically. Although explicit solutions for optimal pairs are not attainable, we are still able to provide useful information for implementing a modified form of dividend strategy with control of dividend payment delay and threshold level.

As in the case of [7] who distinguished between standard and cumulative Parisian options, we also consider two situations where the excursion time at or above level $b$ is calculated either consecutively or in total. In what follows, we use the term *standard waiting period* for dividend payments which start when the surplus process reaches level $b$ and remains consecutively at or above $b$ for a period of time longer than some (predetermined) waiting time $h$ prior to ruin. On the other hand, the term *cumulative waiting period* pertains to dividend payments made when the total amount of time spent by the surplus process at or above level $b$ is at least $h$ in advance of ruin occurring. A related but different modified dividend strategy – namely, a randomized dividend strategy – has been studied previously by [33], [5], and [23] in the case of a compound binomial risk model, by [18] for a discrete-time delayed Sparre Andersen model, and by [22] within a Markovian-type risk environment.

In this paper, we consider an insurer’s business in discrete time. Specifically, we define the insurer’s basic surplus process $\{U_t, t \in \mathbb{N}\}$ as follows:

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i,$$

where $u \in \mathbb{N}$ represents the initial capital and premiums are collected at rate $c \in \mathbb{Z}^+$ per unit time. Throughout this paper, $\mathbb{Z}^+$ denotes the set of positive integers and $\mathbb{N} = \{0\} \cup \mathbb{Z}^+$. We remark that $U_t, t \in \mathbb{Z}^+$, represents the amount of surplus at the end of the time interval $(t-1, t]$, where we assume premiums are received at $(t-1)^+$
and any claims are paid out at \( t^− \) (in general, \( t^+ \) and \( t^− \) represent the infinitesimal time instants just after and before time \( t \), respectively). The claim number process \( \{N_t, t \in \mathbb{N}\} \) is assumed to be a discrete-time renewal process, having independent and identically distributed (iid) positive interclaim times \( \{W_i\}_{i=1}^{\infty} \) with common probability mass function (pmf) \( a_w = \Pr\{W_i = w\} \) for \( w = 1, 2, \ldots, n \), and survival function \( \overline{A}(w) = \Pr\{W_i > w\} = 1 - \sum_{j=1}^{w} a_j \). We remark that the interclaim time distribution can either be of finite or infinite support (i.e., \( n \leq \infty \)). The claim sizes \( \{Y_i\}_{i=1}^{\infty} \) are iid positive random variables with common pmf \( \alpha_y = \Pr\{Y_i = y\} \) for \( y \in \mathbb{Z}^+ \). Let an arbitrary \( W_i \) and \( Y_i \) be denoted by \( W \) and \( Y \), respectively. We assume that the positive loading condition \( cE[W] > E[Y] \) holds true.

The above process, referred to as the ordinary Sparre Andersen renewal risk process in discrete time, has been studied previously in papers by [26], [27], [9], [3], [38] and [37], the latter of which dealt with a delayed version of the risk process (i.e., \( W_1 \) has a different distribution from \( \{W_i\}_{i=2}^{\infty} \)). In addition, [29] analyzed a discrete-time stationary risk model in which \( W_1 \) has the equilibrium distribution of \( \{W_i\}_{i=2}^{\infty} \). As a special case, when interclaim times are geometrically distributed, the compound binomial risk model (which is a discrete analogue of the classical Poisson risk model) is obtained and has been studied extensively in the works of [19], [31], [35], [16] and [6].

We now formally define the minimum healthy period, denoted by \( h \in \mathbb{N} \), in the following manner. If the surplus process has remained at or above a threshold level \( b \in \mathbb{Z}^+ \) over the period of \( h \) time units, then the insurer pays a dividend \( d \in \{0, 1, \ldots, c\} \) per unit time. To be precise, it is in fact the premium rate which is modified and becomes \( c_1 = c - d \) in this case. In order to analyze the associated surplus process, we must keep track of either the consecutive time units at or above level \( b \) under the standard waiting period model or the total time units at or above level \( b \) under the cumulative waiting period model. Let \( L_t \in \{0, 1, \ldots, h + 1\} \) represent this tracking variable at time \( t \). Under the standard waiting period model, we assume that \( L_t = 0 \) if the surplus process at time \( t \) is below level \( b \). On the other hand, if the surplus process drops below level \( b \) at time \( t \), then \( L_t \) retains its previous value (i.e., \( L_t = L_{t-1} \)) under the cumulative waiting period model (however, this excludes the case when \( L_{t-1} = h+1 \), which we elaborate on momentarily). In either model, \( L_t \) counts up to \( h + 1 \) while the process remains at or above level \( b \). It is known that the surplus process has been at or above level \( b \) for \( h \) time units when \( L_t = h + 1 \), and thus the insurer would pay a dividend at \( t^+ \). Finally, under the cumulative waiting period model, if \( \tau^+ \) represents a dividend payment time point (i.e., \( L_{\tau^+} = h+1 \)) and a claim happens to be applied at \( (\tau^+)^− \), then \( L_{\tau^+ + 1} = 0 \) if the resulting claim drops the surplus process below level \( b \) or \( L_{\tau^+ + 1} = h + 1 \) otherwise. Sample paths of the surplus process under both the standard and cumulative waiting period models are given in Figures 1 and 2.

In what follows, it is convenient to introduce a function, denoted by \( z_{u,b} \), which represents the earliest possible time point when the surplus process reaches level \( b \) starting from an initial capital of \( u \). It is defined as \( z_{u,b} = 0 \) for \( u \geq b \) and \( z_{u,b} = \min\{i \in \mathbb{Z}^+ | u + ci \geq b\} \) for \( u < b \). Equivalently, this last expression can be written succinctly as

\[
z_{u,b} = \left\lfloor \frac{b - u - 1}{c} \right\rfloor + 1 \text{ for } u < b,
\]

where, in general, \( \lfloor x \rfloor \) denotes the floor function of \( x \) (i.e., the largest integer less than or equal to \( x \)). Furthermore, if \( L_0 = 0 \), then \( z_{u,b}(h) = z_{u,b} + h \) represents the earliest possible time point when a dividend payment can be made.
Let us suppose that $W_1 = k$. Assuming that $U_0 = u$ and $L_0 = i$, the value of the surplus process at time $k$ (prior to the claim being applied) is given by

$$s_{u,k}(i) = u + c \times \min\{r_{u,i}, k\} + c_1 \times \max\{k - r_{u,i}, 0\},$$

where $r_{u,i} = z_{u,b}(h) - \max\{i - 1, 0\}$, or equivalently,

$$r_{u,i} = \begin{cases} z_{u,b}(h) & \text{for } u < b \\ h - i + 1 & \text{for } u \geq b \end{cases}.$$  

Clearly, when $i = h + 1$ (only possible when the surplus level is at or above level $b$), we have $r_{u,i} = 0$ and $s_{u,k}(i) = u + c_1 k$. For simplicity, in the case when $L_0 = 0$, we suppress the $i$ notation and write $s_{u,k}$ and $r_u$ for $s_{u,k}(i)$ and $r_{u,i}$.  

**Figure 1.** Illustration of the threshold-based dividend strategy: Standard waiting period

**Figure 2.** Illustration of the threshold-based dividend strategy: Cumulative waiting period
respectively. Moreover, it is obvious that for $u < b$, $s_{u,k} = u + ck$ if $k \leq z_{u,b}(h)$ and $s_{u,k} = u + c_{u,b}(h) + c_1[k - z_{u,b}(h)]$ if $k \geq z_{u,b}(h) + 1$. Note that the notation used here (i.e., $z_{u,b}(h)$, $s_{u,k}(f)$, and $r_{u,i}$) pertains specifically to the standard waiting period model. In Section 2.2, we will add a superscript “*” to the notation corresponding to the cumulative waiting period model. Throughout the paper, we adopt the usual conventions that an empty sum is equal to zero and a set of the form $\{x, x+1, \ldots, y\}$ is the empty set if $x > y$.

The rest of the paper is as follows. In the next section, for both the standard and cumulative waiting period models, we derive recursive computational formulas for the expected total discounted dividend payments made prior to ruin. In Section 3, we focus our attention on the expected total discounted dividend payments and finite-time ruin probabilities under the standard waiting period model and examine their behavior in several numerical examples. In Section 4, we consider a compound binomial risk model and introduce a generalized threshold-based dividend strategy which is an extension of the barrier-type strategies with implementation delay studied by [14]. Finally, we provide some concluding remarks in Section 5.

2. Dividend analysis. In this section, our focus lies in the determination of the expected total discounted dividend payments made prior to ruin. Similar to the notation used by [17], let $V(u) = E[D_u]$ represent the expected total discounted (i.e., to time 0 according to discount factor $v \in (0, 1)$ per unit time) dividend payments made prior to ruin starting with an initial capital of $u$. Similarly, let $V(u, m) = E[D_{u,m}]$ be the expected total discounted dividend payments before ruin occurs or before time $m \in \mathbb{Z}^+$, whichever happens first. By construction, we have that $\lim_{m \to \infty} V(u, m) = V(u)$. In order to determine $V(u, m)$ in our risk model, we introduce the related function

$$V_i(u, m) = E[D_{u,m}|L_0 = i] \text{ for } i = 0, 1, \ldots, h + 1,$$

(3)

where $L_i$ is the tracking time at time $t$ as introduced in Section 1. As $L_0$ denotes how long initially the process has remained at or above level $b$, it is clear that $V(u, m) = V_0(u, m)$ if $u < b$ and $V(u, m) = V_1(u, m)$ if $u \geq b$. A similar idea was used to calculate $E[D_{u,m}]$ via (3) in the model considered by [21].

For convenience, we define $V_i(u, m) = 0$ for $m \in \{0, -1, -2, \ldots\}$. We proceed to find recursive computational expressions for $V_i(u, m)$ under both the standard and cumulative waiting period models.

2.1. Standard waiting period model. Assume that $u < b$, so that $L_0 = 0$. Conditioning on $W_1 = k$, we immediately get

$$V_0(u, m) = \sum_{k = 1}^{\min\{m - 1, n_u\}} a_k E[D_{u,m}|L_0 = 0, W_1 = k] + \sum_{k = m}^{n_u} a_k E[D_{u,m}|L_0 = 0, W_1 = k].$$

For $k = m, m + 1, \ldots, n_u$, the only dividend payments to be taken into account are potentially those paid before time $m$, which happens only when $z_{u,b}(h) = z_{u,b} + h < m$ (otherwise, nothing is paid out). This amount is given by

$$E[D_{u,m}|L_0 = 0, W_1 = k] = \sum_{\ell = z_{u,b}(h)}^{m - 1} dv^\ell = \frac{d(v^{\min\{z_{u,b}(h), m\}} - v^m)}{1 - v}.$$

For $k = 1, 2, \ldots, \min\{m - 1, n_u\}$, an expected dividend payment of amount $d$ would occur at times $z_{u,b}(h)^+, (z_{u,b}(h) + 1)^+, \ldots, (k - 1)^+$, followed by possible future
dividend payments (starting from time \( k \) which is a renewal point) once the initial claim is applied. Thus, we obtain

\[
E[D_{u,m}|L_0 = 0, W_1 = k] = \sum_{\ell = u,b}^{k-1} dv^\ell + \sum_{j=1}^{s_{u,k}} \alpha_j E[D_{u,m}|L_0 = 0, W_1 = k, Y_1 = j]
\]

\[
= \frac{d(v^\min(z_{u,b}(h,k)) - v^k)}{1 - v} + v^k \left\{ \sum_{j=1}^{s_{u,k} - b} \alpha_j V_{\min(h,k-z_{u,b})+1}(s_{u,k} - j, m-k) \right\}
\]

\[
+ \sum_{j=1+\max\{s_{u,k} - b,0\}}^{s_{u,k}} \alpha_j V_0(s_{u,k} - j, m-k) \right\}.
\]

Note that the possible future dividend payments in (4) have two cases. If the surplus process does not reach level \( b \) prior to the first claim (i.e., \( k < z_{u,b} \)), then \( L_t \) for \( t \in \{1, 2, \ldots, k\} \) is not increased and simply remains equal to 0. On the other hand, if \( k \geq z_{u,b} \), then we must consider separate cases depending on the size of the first claim (i.e., whether it brings the value of the surplus process below level \( b \) or not).

Next, for \( u \geq b \), we derive an expression for \( V_i(u, m) \) for \( i = 1, 2, \ldots, h+1 \). Conditioning once more on \( W_1 = k \), we have

\[
V_i(u, m) = \sum_{k=1}^{\min\{m-1,n_a\}} a_k E[D_{u,m}|L_0 = i, W_1 = k] + \sum_{k=m}^{n_a} a_k E[D_{u,m}|L_0 = i, W_1 = k]. \tag{5}
\]

Once again, for \( k = m, m+1, \ldots, n_a \), the only dividend payments to be taken into account are potentially those paid before time \( m \), but which occur only when \( h-i+1 < m \). That is,

\[
E[D_{u,m}|L_0 = i, W_1 = k] = \sum_{\ell = h-i+1}^{k-1} dv^\ell = \frac{d(v^\min(h-i+1,k) - v^k)}{1 - v}.
\]

For \( k = 1, 2, \ldots, \min\{m-1,n_a\} \), we know that the only situation which results in dividend payments being made is when the first claim occurs after time \( h-i+2 \) (i.e., \( L_{k-1} = h+1 \)), and in this case, dividends are payable at rate \( d \) at times \( (h-i+1)^+, (h-i+2)^+, \ldots, (k-1)^+ \). This leads to

\[
E[D_{u,m}|L_0 = i, W_1 = k] = \frac{d(v^\min(h-i+1,k) - v^k)}{1 - v} + v^k \left\{ \sum_{j=1}^{s_{u,k}(i) - b} \alpha_j V_{\min(h,i+k-1)+1}(s_{u,k}(i) - j, m-k) \right\}
\]

\[
+ \sum_{j=1+\max\{s_{u,k}(i) - b,0\}}^{s_{u,k}(i)} \alpha_j V_0(s_{u,k}(i) - j, m-k) \right\}. \tag{6}
\]

The following proposition provides a unified expression for \( V_i(u, m) \) which combines the above cases for \( u < b \) and \( u \geq b \).
Proposition 1. Under the standard waiting period model, the recursive computational formula for \( V_i(u, m) \) in (3) is given by

\[
V_i(u, m) = \sum_{k=1}^{n_a} \left\{ \frac{d(u, \min\{r_{u,k}, m\})}{1-v} + v^k \sigma_k(u, m, i) \right\} a_k \text{ for } u \in \mathbb{N},
\]

where

\[
\sigma_k(u, m, i) = \sum_{j=1}^{s_{u,k}(i)-b} \alpha_j V_{g_{u,k}(i)}(s_{u,k}(i)-j, m-k) + \sum_{j=1+\max\{s_{u,k}(i)-b, 0\}}^{s_{u,k}(i)} \alpha_j V_0(s_{u,k}(i)-j, m-k),
\]

with \( g_{u,k}(i) = h + 1 + \min\{0, k - r_{u,i}\} \) and \( r_{u,i} \) given by (2).

First of all, we remark that (8) accounts for the possible future dividend payments (i.e., after the first claim) with a remaining observation period of length \( m - k \) and “new” starting capital which is obtained via the subtraction of the first claim amount \( j \) from the surplus process level \( (s_{u,k}(i)) \) at time \( k \). The first term of (8) deals with the situation when the first claim amount is less than or equal to \( s_{u,k}(i) - b > 0 \) (i.e., the process remains at or above level \( b \) following the claim), resulting in the continued consecutive tracking of the surplus process at or above level \( b \). The second term in (8) handles the case when the tracking variable is either reset to 0 (if the non-ruin causing claim is large enough to drop the surplus process below level \( b \)) or remains at 0 (if \( s_{u,k}(i) < b \)). Secondly, when \( u \geq b \) and we set \( i = h + 1 \) (implying that the process has remained at or above level \( b \) for at least \( h \) consecutive periods), we have that \( s_{u,k}(h+1) = u + c_1 k \) from (1) since \( r_{u,h+1} = 0 \) from (2), and this yields \( g_{u,k}(h+1) = h+1 \). Hence, expressions (7) and (8) simplify nicely to become

\[
V_{h+1}(u, m) = \sum_{k=1}^{n_a} \left\{ \frac{d(u, \min\{k, m\})}{1-v} + v^k \sigma_k(u, m, h+1) \right\} a_k \text{ for } u \geq b,
\]

where

\[
\sigma_k(u, m, h+1) = \sum_{j=1}^{u+c_1 k-b} \alpha_j V_{h+1}(u+c_1 k-j, m-k) + \sum_{j=1+\max\{u+c_1 k-b, 0\}}^{u+c_1 k} \alpha_j V_0(u+c_1 k-j, m-k).
\]

According to Proposition 1, \( V_i(u, m) \) satisfies a 3-dimensional recursive formula.
denote the Kronecker delta function (which has value 0 if $x \neq y$ or 1 if $x = y$), we have that $V_i(u, 1) = d\delta_{i,h+1}$ for $u \geq b$ and $V_i(u, 1) = 0$ for $u < b$. In other words, values of $d$ and 0 only appear in the bottom layer.

As an application of Proposition 1, we consider some different dividend strategies in the following examples.

**Example 1.** (Randomized dividend threshold strategy) As in [18], the major difference between this strategy and the one considered in the present paper is that the modified premium rate is assumed to be a random variable and not simply a constant. In this case, $h = 0$ (i.e., $z_{u,b}(h) = z_{u,b}$) and $V_i(u, m)$ in (3) is independent of $i$. Since a random dividend is immediately paid once the surplus process reaches level $b$, the premium rate $c$ changes to a (random) rate denoted by $X \in \{c_1, c_1+1, \ldots, c_2\}$ where $0 \leq c_1 \leq c_2 \leq c$. Let $S_k$ be the total amount of randomized premiums collected by time $k$. Using (7) and (8), along with the additional conditioning based on $S_k = s$ so that $s_{u,k}$ in (1) is expressed as $s_{u,k} = u + c \times \min\{z_{u,b}, k\} + s$, we ultimately obtain

\[
V(u, m) = \sum_{k=1}^{n_a} \left\{ \sum_{\ell = z_{u,b}}^{\min\{k,m\}-1} \{c - E(X)\} v^\ell \right. \\
+ v^k \left. \sum_{s = c_1 \times \max\{k-z_{u,b},0\}}^{\max\{k-z_{u,b},0\}} \sum_{j=1}^{\gamma_s} \alpha_j \gamma_s \right. \\
\left. \left\{ \frac{\gamma_s}{a_k} \sum_{j=1}^{\alpha_j} V(s_{u,k} - j, m-k) \right\} a_k \right. \\
\text{for } u \in \mathbb{N}, \quad (9)
\]

where $\gamma_s = \Pr\{S_k = s\}$ denotes the pmf of $S_k$. We remark that (9) is in agreement with the iterative formula given in p.745 of [18].

**Example 2.** (Classical dividend threshold strategy) It is a common assumption among many dividend-based models that an insurance company immediately pays dividends at rate $d$ whenever the surplus level reaches level $b$. In other words, from the previous example, we have that $X = c_1$ with probability 1 and hence $\Pr\{S_k = c_1 k\} = 1$. Thus, $s_{u,k} = u + c \times \min\{z_{u,b}, k\} + c_1 \times \max\{k-z_{u,b}, 0\}$ and
we immediately obtain the following representation for $V(u, m)$:

$$V(u, m) = \sum_{k=1}^{m_u} \left\{ \sum_{t=2u,b}^{\min\{k,m\}-1} dv^t + v^k \sum_{j=1}^{s_{u,k}} \alpha_j V(s_{u,k} - j, m - k) \right\} a_k \text{ for } u \in \mathbb{N}.
$$

2.2. Cumulative waiting period model. Under this model, we recall that if the surplus process happens to drop below level $b$ due to a claim at time $t$ in which $L_{t-1} \in \{1, 2, \ldots, h\}$, then $L_t$ retains the same value as $L_{t-1}$ until the surplus process next achieves level $b$. However, if a dividend payment was made prior to the claim at time $t$ (i.e., $L_{t-1} = h + 1$) which resulted in the surplus process dropping below level $b$, then $L_t$ is reset to 0. Otherwise, the dividend payment strategy does not make sense with a surplus value below level $b$ and $L_t = h + 1$ (which would indicate a dividend payment to be made). As a result, we need to modify the formulas for $V_i(u, m)$ obtained in the previous subsection appropriately. Certainly, under this model, we can expect dividend payments to occur more frequently in comparison to the standard waiting period model.

We begin by modifying the values of $s_{u,k}(i)$ and $r_{u,i}$ given by (1) and (2), respectively. Assuming that $u < b$ and $L_0 = i$ ($\neq h + 1$), the earliest time point when a dividend is payable is given by $z_{u,b}^*(i) = z_{u,b} + h - i$. It is obvious that when $i = 0$, $z_{u,b}(0)$ reduces to give $z_{u,b}(h)$. Therefore, with the aid of $z_{u,b}^*(i)$, the total amount of premiums collected prior to applying the first claim at time $k$ is given by

$$s_{u,k}^*(i) = u + c \times \min\{r_{u,i}^*, k\} + c_1 \times \max\{k - r_{u,i}^*, 0\} \text{ for } i = 0, 1, \ldots, h + 1,$$

where $r_{u,i}^* = z_{u,b}^*(i) - \max\{i - 1, 0\}$, or equivalently,

$$r_{u,i}^* = \begin{cases} z_{u,b}^*(i) & \text{for } u < b \\ h - i + 1 & \text{for } u \geq b. \end{cases}
$$

In what follows, it is convenient to introduce a function, denoted by $l_{u,k}(i)$, which indicates the value of $L_k$. For $u < b$, it is defined as

$$l_{u,k}(i) = \begin{cases} i & \text{for } k < z_{u,b} \\ (1 - \delta_{k+1, \min\{h,k - z_{u,b} + i\} + 1})(\min\{h,k - z_{u,b} + i\} + 1) & \text{for } k \geq z_{u,b}. \end{cases}
$$

To explain how (11) is obtained, we remark that if the surplus process does not reach level $b$ before the first claim occurs (i.e., $k < z_{u,b}$ which implies that $s_{u,k}^*(i) < b$), then $L_k = L_0 = i$. In the case when $k > z_{u,b}$, we need to ascertain whether the surplus process drops below level $b$ after the first claim is applied. It does so if the claim size lies between $s_{u,k}^*(i) - b + 1$ and $s_{u,k}^*(i)$. In this situation, $L_k$ either takes on the value of $L_{k-1}$ (provided that $L_{k-1} \neq h + 1$) or is reset to 0 if $L_{k-1} = h + 1$. The use of the Kronecker delta function in (11) achieves this result. We omit the details, but a similar line of reasoning ultimately yields

$$l_{u,k}(i) = (1 - \delta_{k+1, \min\{h,i+k-1\} + 1})(\min\{h,i+k-1\} + 1) \text{ for } u \geq b.
$$

By once again conditioning on $W_1 = k$, it is clear that $V_i(u, m)$ for $u < b$ and $i = 0, 1, \ldots, h$ is given by (5). In the case when $k = m, m+1, \ldots, n_u$ and $z_{u,b}(i) < m$, we note that dividend payments are made at times $z_{u,b}^*(i)^+, (z_{u,b}^*(i) + 1)^+, \ldots, (m-1)^+$, resulting in

$$E[D_{u,m}|L_0 = i, W_1 = k] = \sum_{t=z_{u,b}^*(i)}^{m-1} dv^t = \frac{d(u^{\min\{z_{u,b}^*(i),m\}} - u^m)}{1 - v}.
$$
For $k = 1, 2, \ldots, \min\{m - 1, n_a\}$, dividends would now be payable at times $z_{u,b}^*(i)^+$, $(z_{u,b}^*(i) + 1)^+$, \ldots, $(k - 1)^+$, along with any future dividends starting from time $k$. Hence, we ultimately obtain

$$E[D_{u,m}|L_0 = i, W_1 = k]$$

$$= \frac{d(v^\min(z_{u,b}^*(i),k) - v^k)}{1 - v} + v^k \left\{ \sum_{j=1}^{s_{u,k}^*(i)-b} \alpha_j V_{\min(h,k,z_{u,b}^*(i))} + (s_{u,k}^*(i) - j, m - k) + \sum_{j=1+\max\{s_{u,k}^*(i)-b,0\}}^{s_{u,k}^*(i)} \alpha_j V_{l_{u,k}^*(i)}(s_{u,k}^*(i) - j, m - k) \right\},$$

where $l_{u,k}^*(i)$ is given by (11).

For $u \geq b$, we recognize that $V_i(u, m)$, $i = 1, 2, \ldots, h + 1$, has a similar form to its counterpart in the standard waiting period model, but the potential future dividend payments, equivalent to the second and third terms on the right-hand side of (6), now become

$$v^k \left\{ \sum_{j=1}^{s_{u,k}^*(i)-b} \alpha_j V_{\min(h,i+k-1) + 1}(s_{u,k}^*(i) - j, m - k) + \sum_{j=1+\max\{s_{u,k}^*(i)-b,0\}}^{s_{u,k}^*(i)} \alpha_j V_{l_{u,k}^*(i)}(s_{u,k}^*(i) - j, m - k) \right\},$$

where $l_{u,k}^*(i)$ is given by (12). Combining the above two cases, a unified expression for $V_i(u, m)$ is provided in the following result.

**Proposition 2.** Under the cumulative waiting period model, the recursive computational formula for $V_i(u, m)$ in (3) is given by

$$V_i(u, m) = \sum_{k=1}^{n_a} \left\{ \frac{d(v^\min(r_{u,b,k}^*,k,m) - v^k)}{1 - v} + v^k \sigma_k^*(u, m, i) \right\} a_k$$

where

$$\sigma_k^*(u, m, i) = \sum_{j=1}^{s_{u,k}^*(i)} \alpha_j V_{g_{u,k}^*(i)}(s_{u,k}^*(i) - j, m - k) + \sum_{j=1+\max\{s_{u,k}^*(i)-b,0\}}^{s_{u,k}^*(i)} \alpha_j V_{l_{u,k}^*(i)}(s_{u,k}^*(i) - j, i - k),$$

with $g_{u,k}^*(i) = h + 1 + \min\{0, k - r_{u,i}^a\}$, $r_{u,i}^a$ given by (10), and

$$l_{u,k}^*(i) = \begin{cases} i \\ (1 - \delta_{h+1, g_{u,k}^*(i)}) g_{u,k}^*(i) \end{cases}$$

for $k < z_{u,b}$, $k \geq z_{u,b}$.

We emphasize that the main results given in Propositions 1 and 2 are expressed as recursive computational procedures. We remark that further simplification of these results are generally not available. In fact, we have discovered that it is difficult (and not all that meaningful) to derive simpler forms even for a specific case such as geometric interclaim times and/or deterministic claim sizes.
3. Numerical examples. In this section, we consider several numerical scenarios and calculate the expected total discounted dividend payments made prior to ruin in the case of the standard waiting period model for our modified threshold-based dividend strategy. In addition, we compute associated finite-time ruin probabilities. More precisely, our aim is to compute \( V(u, m) \) for large values of \( m \) (so as to closely approximate \( V(u) = E[D_u] \)) and the quantity \( \psi(u, m) = \Pr \{ T \leq m \mid U_0 = u \} \), where \( T \) is the time of ruin defined as \( T = \min \{ t \in \mathbb{Z}^+ : U_t < 0 \} \) with \( T = \infty \) if \( U_t \geq 0 \) for all \( t \). 

In order to calculate \( \psi(u, m) \), we work with the corresponding survival probability \( \phi(u, m) = \Pr \{ T > m \mid U_0 = u \} = 1 - \psi(u, m) \). In a manner similar to the derivation of \( V(u, m) \), we first obtain an expression for \( \phi_i(u, m) \) defined as

\[
\phi_i(u, m) = \Pr \{ T > m \mid U_0 = u, L_0 = i \} \text{ for } i = 0, 1, \ldots, h + 1.
\]

We assume that \( \phi_i(u, m) = 1 \) for \( m \in \{0, -1, -2, \ldots\} \). Also, if \( u < b \), then \( \phi_i(u, m) \) is defined only for \( i = 0 \). Likewise, if \( u \geq b \), then \( \phi_i(u, m) \) is defined only for \( i \in \{1, 2, \ldots, h + 1\} \). Clearly, \( \phi(u, m) = \phi_0(u, m) \) if \( u < b \) and \( \phi(u, m) = \phi_1(u, m) \) if \( u \geq b \).

Conditioning on the time and amount of the first claim ultimately yields the following recursive computational formula for \( \phi_i(u, m) \):

\[
\phi_i(u, m) = \min \{m, n_u\} \sum_{k=1}^{\min \{m, n_u\}} \left\{ \sum_{j=1}^{s_{u,k}(i) - b} \alpha_j \phi_{g_{u,k}(i)}(s_{u,k}(i) - j, m - k) \right. \\
+ \left. \sum_{j=1}^{s_{u,k}(i)} \alpha_j \phi_0(s_{u,k}(i) - j, m - k) \right\} a_k + \overline{A}(m),
\]

where \( g_{u,k}(i) = h + 1 + \min\{0, k - r_{u,i}\} \) and \( r_{u,i} \) is given by (2). In particular, when \( u < b \) and \( m = 1 \), (13) reduces to give

\[
\phi_0(u, 1) = \overline{A}(1) + a_1 \sum_{j=1}^{u+e} \alpha_j,
\]

which is the correct result and initial value to use in the recursive formula. Moreover, in the case when \( b \rightarrow \infty \) (i.e., no dividend strategy is in place), (13) simplifies to become

\[
\phi_0(u, m) = \sum_{k=1}^{\min \{m, n_u\}} \left\{ \sum_{j=1}^{u + ek} \alpha_j \phi_0(u + ck - j, m - k) \right\} a_k + \overline{A}(m),
\]

which agrees with the survival probability given in [18] p.744 when no dividends are paid (i.e., \( c_1 = c_2 = c \)).

In the following examples, we consider an ordinary Sparre Andersen risk model with \( u = 10 \), \( c = 5 \), \( c_1 = 2 \) (so that \( d = 3 \)), and \( v = 0.99 \). Claim amounts are assumed to follow a discretized Pareto distribution with mean 10 and pmf of the form

\[
\alpha_j \left( 1 + \frac{j - 1}{30} \right)^{-4} - \left( 1 + \frac{j}{30} \right)^{-4} \text{ for } j \in \mathbb{Z}^+.
\]

We wish to investigate the impact that the parameters \( h \) and \( b \) have on the expected total discounted dividend payments and associated finite-time ruin probabilities. Moreover, to illustrate the effect that the choice of interclaim time distribution has on these two performance measures, we consider three different interclaim time distributions, each having the same mean of 5.5 but different standard deviations. Specifically, we examine how the coefficient of variation (cv) of the interclaim time
distribution influences our two performance measures. The three distributions we consider are:

- **Distribution 1**: Zero-truncated binomial distribution ($cv = 0.374$)
  \[ a_w = \frac{\binom{25}{w}(0.22)^w(0.78)^{25-w}}{1 - (0.78)^{25}} \text{ for } w = 1, 2, \ldots, 25, \]

- **Distribution 2**: Zero-truncated geometric distribution ($cv = 0.877$)
  \[ a_w = \begin{cases} 
    \frac{9}{11}w^{-1} & \text{for } w = 1, 2, \ldots, 24 \\ 
    \left(\frac{9}{11}\right)^{24} & \text{for } w = 25 
  \end{cases} \]

- **Distribution 3**: Zero-truncated mixture of two geometric distributions ($cv = 1.466$)
  \[ a_w = \begin{cases} 
    0.645(1/2)^w + 0.355(11/12)^{w-1}(1/12) & \text{for } w = 1, 2, \ldots, 14 \\ 
    0.645(1/2)^{14} + 0.355(11/12)^{14}(1/12) & \text{for } w = 15 \\ 
    0.355(11/12)^{w-1}(1/12) & \text{for } w = 16, 17, \ldots, 49 \\ 
    0.355(11/12)^{49} & \text{for } w = 50 
  \end{cases} \]

Generally speaking, we would expect that if there is a greater likelihood of experiencing a shorter interclaim time, then this should translate to lower dividend payments and a higher ruin probability. We consider the following three examples:

**Example 3.** We calculate $V(10, 80)$ and $\psi(10, 80)$ for $h = 0, 1, \ldots, 6$ with different threshold levels of $b = 10, 20, 40$ under each of the three interclaim time distributions. The results are illustrated in Figures 4-6.

**Example 4.** Assuming that the interclaim time distribution follows that of Distribution 2, we calculate $V(10, 80)$ and $\psi(10, 80)$ for the pair $(h, b)$ satisfying $h \in \{0, 1, \ldots, 4\}$ and $b \in \{15, 16, \ldots, 40\}$. The results are illustrated in Figures 7 and 8.

**Example 5.** To demonstrate the convergence of $V(u, m)$ to $V(u)$ numerically, we construct Figure 9 which shows the convergence trend of $V(10, m)$ for $h = 0, 2, 4$ under the assumption that the interclaim time distribution follows that of Distribution 2.

Based on these numerical examples, we make the following observations:
(i) Figures 4(a)-6(a) show that the influence of the delay period $h$ on the expected total discounted dividend payments depends on the threshold level $b$. For example, when $b = 40$, the expected total discounted dividend payments appear to decrease linearly as $h$ increases, but when $b$ is small, it behaves...
as a convex function of $h$. In particular, when $b = 10$, the expected total discounted dividend payments first increase with $h$ for small values of $h$ (i.e., $h < 5$). However, the effect of $h$ on the finite-time ruin probability is always monotone and looks to be more stable with respect to $b$. As is evident in Figures 4(b)-6(b), $\psi(10, 80)$ decreases as $h$ increases (since less frequent dividend payments results in the process sustaining above level 0 longer). We also note that the effect of $h$ on the ruin probabilities seems to be much less than that on the expected total discounted dividend payments. This is reasonable since ruin probabilities are mainly driven by the frequency of claim arrivals and the amount of each claim, whereas delay of dividend payment only has a limited influence on the ruin probabilities. In addition, although different interclaim time distributions result in a different range of values for the expected total discounted dividend payments and finite-time ruin probabilities, the patterns with respect to $(h, b)$ remain the same.
(ii) Figure 7 reports the convexity of the expected total discounted dividend payments concerning the threshold level \( b \) for each fixed \( h \). From the 3-dimensional plot, we can identify an approximate straight line parallel to the \( x\)-\( y \)-plane (i.e., the darkest red part) which maximizes the expected total discounted dividend payments. Then, under the criterion of maximizing the expected total discounted dividend payments, we can choose different pairs of \((h, b)\) (located on the straight line) to achieve the same optimization point, which means that the introduction of a delay period \( h \) will improve the flexibility of the decision-making process. In addition, if we wish to consider both the expected total discounted dividend payments and finite-time ruin probabilities as our optimization criteria, then Figure 8 can indicate some suggestions for decision-making purposes. Each curve represents a pair of \( h \) and \( b \) yielding various sets of values of expected total discounted dividend payments (value) and finite-time ruin probability (risk). Efficient choices of \( h \) and \( b \) then lie on the curve with a positive slope, and the maximum of the dividend payments is located at the most right. We can define this as an optimal frontier for \( h \in \{0, 1, \ldots, 4\} \) and \( b \in \{15, 16, \ldots, 40\} \). Note that for most of the points on the optimal frontier, we have multiple choices of \( h \) and \( b \), although some points with a unique pair of \((h, b)\) could still exist.

(iii) As one might anticipate, Figure 9 illustrates that the influence of the delay period \( h \) dissipates as \( m \) tends to \( \infty \).

4. Extension to a generalized threshold-based dividend strategy. In this section, we assume that dividend payments are still made following a predetermined waiting period of length \( h \) (according to our standard waiting period definition), but that the (constant) dividend rate \( d \) can now exceed the premium rate \( c \) so long as it is bounded by the lump-sum which is the difference between the surplus level at a dividend paying moment and level \( b \). As a result, the barrier strategy with implementation delay (e.g., see [14]) in a discrete-time analogue can be recovered as a special case. A sample path of the surplus process under this generalized threshold-based dividend strategy is given in Figure 10.

From (1), \( s_{u,r_u,i}(i) = u + cr_{u,i} \) represents the surplus level at the earliest dividend payment time point if no claim occurs beforehand. Under our generalized threshold-based strategy, it is possible for the dividend rate \( d \) to take on any value between \( c \) and \( s_{u,r_u,i}(i) - b \), assuming that a dividend is paid before the collection of a premium. We remark that this assumption is necessary in the current strategy so that it can be consistent with respect to the barrier strategy with implementation delay. Let \( s_{u,t}^+(i) = s_{u,t}(i) - d\delta_{h+1,L_t} \) be the surplus level at time \( t^+ \), just after a possible dividend payment but prior to the collection of the premium. In addition, let \( r_{u,i}^* \) be the last possible dividend payment time point prior to the occurrence of the first claim, where \( r_{u,i}^* \) denotes the maximum period of dividend payments starting from surplus level \( s_{u,r_u,i}(i) \geq b \). The function \( r_{u,i}^* \) is given by

\[
\tau_{u,i}^* = \begin{cases} 
\left\lceil \frac{s_{u,r_u,i}(i) - b - d}{d - c} \right\rceil & \text{for } c < d \leq s_{u,r_u,i}(i) - b \\
\infty & \text{for } 0 \leq d \leq \min\{c, s_{u,r_u,i}(i) - b\} 
\end{cases},
\]

(14)

where, in general, \( \lceil x \rceil \) denotes the ceiling function of \( x \) (i.e., the smallest integer greater than or equal to \( x \)). To be specific, when \( d \leq \min\{c, s_{u,r_u,i}(i) - b\} \) (i.e., the classical threshold strategy), the surplus process can only jump downward to
level \( b \) at claim instants and, in this case, \( \tau_{u,i}^* = \infty \). When \( d > c \), after starting to pay dividends, the surplus process will decrease steadily at rate \( d - c \). Thus, \( \tau_{u,i}^* \) represents the maximum period of dividend payments without the occurrence of a claim.

Under this generalized threshold-based strategy, we note that the surplus process does not necessarily possess the renewal property at \( \tau_{u,i}^* \), which is a critical time point to be compared with the arrival time of the first claim when we try to implement a similar analysis performed previously. Therefore, a geometric inter-claim time distribution (which possesses the memoryless property) is assumed in the subsequent analysis (i.e., \( \lambda_w = \rho^{w-1}(1 - \rho) \) for \( w \in \mathbb{Z}^+ \)). For other more general risk models beyond the compound binomial model, different approaches should be adopted.

For \( u < b \), we determine \( V_0(u,m) \) by conditioning on \( W_1 = k \) and the size of the initial claim \( Y_1 = j \). First of all, if \( k \geq m \), then dividends are paid starting from \( z_{u,b}(h) \) and going up to either \( (m - 1) + (z_{u,b}(h) + \tau_{u,0}) \), whichever is earlier. Then the expected discounted dividend payment for this period is

\[
\min\{z_{u,b}(h) + \tau_{u,0}, m-1\}
\]

\[
\sum_{\ell = z_{u,b}(h)}(d - c) - v^\ell \min\{z_{u,b}(h) + \tau_{u,0}, m-1\}
\]

In particular, if \( m - 1 > z_{u,b}(h) + \tau_{u,0} \), then dividends will be paid up to time \( z_{u,b}(h) + \tau_{u,0} \), resulting in a surplus level of \( s_{u,z_{u,b}(h) + \tau_{u,0}}(0) \). However, recognizing that \( z_{u,b}(h) + \tau_{u,0} \) happens to be a dividend paying time point, we must have \( s_{u,z_{u,b}(h) + \tau_{u,0}}(0) = s_{u,z_{u,b}(h) + \tau_{u,0}}(0) - d \). Thus, the future contribution to the expected total discounted dividend payments is given by

\[
v^{\tau_{u,0}} \mathcal{A}(s_{u,z_{u,b}(h) + \tau_{u,0}}(0), m - (z_{u,b}(h) + \tau_{u,0}))
\]

where, for convenience, we define \( \mathcal{A} = \delta_{s_{u,z_{u,b}(h) + \tau_{u,0}}(0),b} \).
On the contrary, if the first claim occurs at or before the maximum period of dividend payments \( \tau_{\ast,0} \) is achieved or before \( m - 1 \) (i.e., \( z_{u,b}(h) < k \leq \min\{m - 1, z_{u,b}(h) + \tau_{\ast,0}^\ast\} \)), then the contribution to the expected total discounted dividend payments from \( z_{u,b}(h) \) to \( k \) is

\[
\sum_{\ell = z_{u,b}(h)}^{k-1} dv^\ell = \frac{d(v^{z_{u,b}(h)} - v^k)}{1 - v}.
\]

As for future dividend contributions beyond this point, they can be divided into two cases: (i) if \( s_{u,k}(0) - j \geq b \), then dividends will continue to be paid at rate \( d \) and the future contribution is given by \( v^k V_{h+1}(s_{u,k}(0) - j, m - k) \), or (ii) if \( s_{u,k}(0) - j < b \), then the future expected discounted dividend payments will simply be given by \( v^k V_{0}(s_{u,k}(0) - j, m - k) \). Finally, if \( k \leq z_{u,b}(h) \), the same method which we used in Section 2.1 can be applied again to handle this case. Putting all the various pieces together eventually yields

\[
V_0(u, m) = v(1 - \rho) \sum_{k=1}^{z_{u,b}(h)} (vp)^{k-1} \sigma_k(u, m, 0) \\
+ \min\{z_{u,b}(h)+\tau_{\ast,0}^\ast+1, m-1\} \sum_{k = z_{u,b}(h)+1}^{\min\{z_{u,b}(h)+\tau_{\ast,0}^\ast+1, m\}} \rho^{k-1}(1 - \rho) \left\{ \frac{d(v^{z_{u,b}(h)} - v^k)}{1 - v} + v^k \sigma_k(u, m, 0) \right\} \\
+ \sum_{k=\min\{z_{u,b}(h)+\tau_{\ast,0}^\ast+1, m\}}^{\infty} \rho^{k-1}(1 - \rho) \left\{ \frac{d(v^{z_{u,b}(h)} - v^{\min\{z_{u,b}(h)+\tau_{\ast,0}^\ast+1, m\}})}{1 - v} \right\} \\
+ v^{z_{u,b}(h)+\tau_{\ast,0}^\ast+1} V_A(s_{u,z_{u,b}(h)+\tau_{\ast,0}^\ast+m} - (z_{u,b}(h) + \tau_{\ast,0}^\ast)) .
\]

where \( \sigma_k(u, m, 0) \) is evaluated via (8).

In a similar fashion, we can derive an expression for \( V_i(u, m) \) for \( u \geq b \) and \( i = 1, 2, \ldots, h + 1 \). Omitting the details, we ultimately find

\[
V_i(u, m) = v(1 - \rho) \sum_{k=1}^{h-i+1} (vp)^{k-1} \sigma_k(u, m, i) \\
+ \min\{h-i+\tau_{\ast,i}^\ast+1, m-1\} \sum_{k = h-i+2}^{\min\{h-i+\tau_{\ast,i}^\ast+1, m\}} \rho^{k-1}(1 - \rho) \left\{ \frac{d(v^{h-i+1} - v^k)}{1 - v} + v^k \sigma_k(u, m, i) \right\} \\
+ \sum_{k=\min\{h-i+\tau_{\ast,i}^\ast+1, m\}}^{\infty} \rho^{k-1}(1 - \rho) \left\{ \frac{d(v^{h-i+1} - v^{\min\{h-i+\tau_{\ast,i}^\ast+1, m\}})}{1 - v} \right\} \\
+ v^{h-i+\tau_{\ast,i}^\ast+1} V_A(s_{u,h-i+\tau_{\ast,i}^\ast+1}^+ - (h-i + \tau_{\ast,i}^\ast+1)) .
\]

Combining the above two cases, we can obtain a general recursive computational formula for \( V_i(u, m) \) under our generalized threshold-based strategy, as stated in the following result.

**Proposition 3.** For the generalized threshold-based dividend strategy operating under the standard waiting period model and geometric interclaim time distribution
\[ a_w = \rho^{w-1}(1 - \rho) \text{ for } w \in \mathbb{Z}^+, \]  

the recursive computational formula for \( V_i(u, m) \) is given by

\[
V_i(u, m) = \sum_{k=1}^{\infty} \left\{ \frac{d(\min(r_{u,i}, k, m) - \min(r_{u,i} + \tau^*_u, k, m))}{1 - v} + \min(r_{u,i} + \tau^*_u, k, m) \eta_k(u, m, i) \right\} \rho^{k-1}(1 - \rho)
\]

for \( u \in \mathbb{N}, \quad (15) \)

where

\[
\eta_k(u, m, i) = \sigma_k(u, m, i) I\{k \leq r_{u,i} + \tau^*_u, i\}
\]

\[ + \ V_A \left( s^*_{u,r_{u,i}+\tau^*_u, i}(i), m - (r_{u,i} + \tau^*_u, i) \right) I\{k > r_{u,i} + \tau^*_u, i\}, \]

\( I\{\cdot\} \) denotes the indicator function, and \( r_{u,i} \) and \( \sigma_k(u, m, i) \) are given by (2) and (8), respectively.

We remark that (15) exhibits a similar form to our earlier formulas given in Propositions 1 and 2, where the first term within the braces represents all dividend payments up to the first claim or the downward hitting time of level \( b \), while the second term involving \( \eta_k(u, m, i) \) accounts for future dividend payments. When \( i = h + 1 \) (i.e., \( r_{u,h+1} = 0 \)), (14) becomes

\[
\tau_{u,h+1}^* = \begin{cases} \left\lceil \frac{u - b - d}{d - c} \right\rceil & \text{for } c < d \leq u - b \\ \infty & \text{for } 0 \leq d \leq \min\{c, u - b\} \end{cases}
\]

In this case, (15) simplifies to give

\[
V_{h+1}(u, m) = \sum_{k=1}^{\infty} \left\{ \frac{d(1 - \min(\tau_{u,h+1,k,m}^*, m))}{1 - v} \right\} \rho^{k-1}(1 - \rho)
\]

for \( u \geq b, \quad (16) \)

where

\[
\eta_k(u, m, h + 1) = \sigma_k(u, m, h + 1) I\{k \leq \tau_{u,h+1}^*\}
\]

\[ + \ V_A \left( u + (c - d)\tau_{u,h+1,k,m}^* - \tau_{u,h+1}^* \right) I\{k > \tau_{u,h+1}^*\}. \]

In addition, when \( d \leq \min\{c, s_{u,r_{u,i}}(i) - b\} \), we have \( \tau_{u,i}^* = \infty \), which implies that \( \eta_k(u, m, i) = \sigma_k(u, m, i) \) and (15) reduces to (7) under a geometric interclaim time distribution. When \( d \) takes on its largest possible value (so that \( \tau_{u,i}^* = 0 \)), we can capture from Proposition 3 the corresponding recursive computational formula for the classical barrier strategy with implementation delay. Specifically, when \( \tau_{u,i}^* = 0 \), (15) becomes

\[
V_i(u, m) = \sum_{k=1}^{\infty} \left\{ \frac{d(\min(r_{u,i}, k, m) - \min(r_{u,i} + \tau^*_u, k, m))}{1 - v} + \min(r_{u,i} + \tau^*_u, k, m) \eta_k(u, m, i) \right\} \rho^{k-1}(1 - \rho),
\]

(17)
where $\eta_{k}(u, m, i) = \sigma_{k}(u, m, i)\mathbb{P}\{k \leq r_{u,i}\} + V_{1}(b, m - r_{u,i})\mathbb{P}\{k > r_{u,i}\}$. Note that when $r_{u,i} < \min(k, m)$, the terms within the braces in (17) simply become $\psi^{*}, \{d + V_{1}(b, m - r_{u,i})\}$ with $d = s_{u,r_{u,i}}(i) - b$. This ultimately leads to

$$V_{i}(u, m) = v(1 - \rho) \sum_{k=1}^{\min\{m-1, r_{u,i}\}} (v\rho)^{k-1} \sigma_{k}(u, m, i) + w^{*}(1 - \rho) \sum_{k=r_{u,i} + 1}^{\infty} \rho^{k-1} \left\{ (s_{u,r_{u,i}}(i) - b) + V_{1}(b, m - r_{u,i}) \right\}.$$ 

Moreover, the finite-time survival probability for our generalized threshold-based dividend strategy can be expressed as

$$\phi_{i}(u, m) = \left(\rho^{\min\{m, r_{u,i} + \tau_{u,i}^{*}\}} - \rho^{m}\right) \phi_{0}(s_{u,r_{u,i}}(i), m - (r_{u,i} + \tau_{u,i}^{*})) + \rho^{m} \sum_{k=1}^{\min\{m, r_{u,i} + \tau_{u,i}^{*}\}} \left\{ \sum_{j=1}^{s_{u,k}(i)-b} \alpha_{j} \phi_{0}(s_{u,k}(i) - j, m-k) \right\} \rho^{k-1}(1 - \rho),$$

where $g_{u,k}(i) = h + 1 + \min\{0, k - r_{u,i}\}$. When $\tau_{u,i}^{*} = \infty$, it is a straightforward exercise to verify that (18) reduces to (13) under a geometric interclaim time distribution. Moreover, when $d$ assumes its largest possible value (so that $\tau_{u,i}^{*} = 0$), the corresponding expression in the case of the classical barrier strategy with implementation delay becomes

$$\phi_{i}(u, m) = \left(\rho^{\min\{m, r_{u,i}\}} - \rho^{m}\right) \phi_{1}(b, m - r_{u,i}) + \rho^{m} \sum_{k=1}^{\min\{m, r_{u,i}\}} \left\{ \sum_{j=1}^{s_{u,k}(i)-b} \alpha_{j} \phi_{0}(s_{u,k}(i) - j, m-k) \right\} \rho^{k-1}(1 - \rho).$$

To demonstrate the use of these results, we consider the following example:

**Example 6.** Letting $m = 100$, we calculate $V(u, m)$ and $\psi(u, m)$ for $h \in \{1, 2, \ldots, 6\}$ and $b \in \{10, 11, \ldots, 50\}$ under the generalized threshold-based dividend strategy with $d = 7$, $c = 5$, $u = 10$, $v = 0.99$, and $\rho = 9/11$.

Figure 11 displays the values of the expected total discounted dividend payments prior to ruin by varying the delay period $h$ and generalized threshold level $b$. In comparison to Figure 7 in Section 3, we can conclude that, under the generalized threshold-based strategy, the expected total discounted dividend payments prior to ruin is still convex with respect to $b$. In addition, the maximum dividend payment amount is also convex with respect to the delay period $h$, as in this particular example, we can identify that the maximum value of $V(10, 100)$ first increases (as $h$ increases to around 3 or 4) and then subsequently decreases. In conjunction with the corresponding results for the finite-time ruin probability in Figure 12, we can conjecture two conclusions. First of all, with the implementation delay and bounded
dividend rate, the optimal strategy belongs to the set of generalized threshold-based rules. Secondly, the decision-making process can be made more flexible for different risk and profit targets through the introduction of the delay period $h$.

FIGURE 11. Plot of $V(10,100)$ against $(h,b)$ under the generalized threshold-based dividend strategy

FIGURE 12. Plot of $\psi(10,100)$ against $(h,b)$ under the generalized threshold-based dividend strategy

5. **Concluding remarks.** We considered a modified threshold-based risk model by incorporating some form of waiting period into the dividend payments (i.e., a similar concept to a Parisian option). With different pairs of waiting period and threshold level chosen, we investigated the trade-off effect of firm’s value and risk in terms of expected total discounted dividend payments and finite-time ruin probability. As revealed in empirical studies, policyholders are reluctant to purchase products
from a less stable insurer (e.g., see [32]). Hence, risk aspect, which is quantified here as the finite-time ruin probability, was also taken into account to determine a dividend-paying strategy under the current threshold-based risk model.

Furthermore, previous studies in the literature of risk theory have mostly focused on the Parisian ruin probability problem in the Lévy risk model (rather than the dividend problem). This lead us to consider the discrete-time Sparre Andersen renewal risk model as an approximate way to study the dividend problem in the continuous-time risk model with Parisian-type delay. From our numerical results, it is natural to observe that enhancement of expected dividend payments comes with higher finite-time ruin probabilities and those values vary depending on the threshold level and waiting period, as well as distributional assumptions on the claim counting process. Given a constraint on the risk tolerance for a specified period or reward portion to shareholders, different dividend strategies can be determined. Finally, a generalized threshold-based strategy with a delayed dividend payment rule enabled us to choose extended ranges for the dividend rate (which was not necessarily bounded by the premium rate as in the classical threshold strategy). Although this strategy is analyzed only for the compound binomial risk model, it is interesting to note that this model contains various threshold and barrier strategies as special cases.

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