Dependence Properties of Multivariate Max-Stable Distributions

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Abstract

For an $m$-dimensional multivariate extreme value distribution there exist $2^m - 1$ exponent measures which are linked and completely characterise the dependence of the distribution and all of its lower dimensional margins. In this paper we generalise the inequalities of Schlather and Tawn (2002) for the sets of extremal coefficients and construct bounds that higher order exponent measures need to satisfy to be consistent with lower order exponent measures. Subsequently we construct nonparametric estimators of the exponent measures which impose, through a likelihood-based procedure, the new dependence constraints and provide an improvement on the unconstrained estimators.

Keywords: max-stable distributions; multivariate extremes; exponent measure; inequalities; constrained estimators

1 Introduction

Max-stable distributions arise naturally from the study of limiting distributions of appropriately scaled componentwise maxima of independent and identically distributed random variables. Here and throughout the vector algebra is to be interpreted as componentwise. A vector random variable $Y = (Y_1, \ldots, Y_m)$ with unit Fréchet margins, i.e., $G_i(y) := P(Y_i < y) = \exp(-1/y)$, $y > 0$, $i \in M_m = \{1, \ldots, m\}$, is called max-stable if its distribution function is max-stable, i.e., if

$$G_{M_m}(y_{M_m}) := P(Y < y_{M_m}) = \exp\left\{-\int_{S_m} \max_{i \in M_m} \left(\frac{w_i}{y_i}\right) dH(w_1, \ldots, w_m)\right\}, \quad (1)$$

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where \( y_{M_m} = (y_1, \ldots, y_m) \in \mathbb{R}_+^m \), \( S_m = \{(w_1, \ldots, w_m) \in \mathbb{R}_+^m : \sum_{i=1}^m w_i = 1\} \) is the \((m-1)\)-dimensional unit simplex and \( H \) is an arbitrary finite measure that satisfies
\[
\int_{S_m} w_i dH(w_1, \ldots, w_m) = 1 \quad \text{for any} \quad i \in M_m.
\]
The last condition is necessary for \( G_{M_m} \) to have unit Fréchet margins and representation (1) is due to [Pickands (1981)]. There is no loss of generality in assuming unit Fréchet margins since our focus is placed on the dependence structure of max-stable distributions, i.e., we are interested in the copula function [Nelsen (1999)] which is invariant to strictly monotone marginal transformations and in practice we can standardise random variables to unit Fréchet margins.

The dependence properties of max-stable distributions have received attention in the multivariate extreme value literature. Dating back to [Sibuya (1960)] and [Tiago de Oliveira (1962/63)], it has been known that max-stable distributions are necessarily positively quadrant dependent, i.e.,
\[
G_{M_m}(y_{M_m}) \geq \prod_{i=1}^m G_i(y_i) \quad y_{M_m} \in \mathbb{R}_+^m,
\]
which implies that no pair of random variables can be negatively dependent. Additionally, max-stable distributions satisfy even stronger forms of dependence. [Marshall and Olkin (1983)] show that \( \text{Cov}\{g(Y), h(Y)\} \geq 0 \) for every pair of non-decreasing real functions \( g \) and \( h \) on \( \mathbb{R}^m \), i.e., they are associated. For a review of the dependence properties of max-stable distributions we refer the reader to [Beirlant et al. (2004)] and the references therein.

Although all of the aforementioned properties exhibit characteristics for the dependence structure of the class of max-stable distributions, they are far too general to be either tested or implemented in practice. In this paper, we introduce additional constraints for the dependence structure that can be incorporated, through a likelihood-based procedure, into the estimation of max-stable distributions from observed componentwise maxima. The new constraints are in essence the generalisation of the [Schlather and Tawn (2002) 2003] inequalities for the extremal coefficients which correspond to the dependence properties of max-stable distributions for the special case of \( G_{M_m}(y, \ldots, y), y > 0 \). As such, our notation and strategy are influenced by the
work of Schlather and Tawn (2002, 2003). The new inequalities presented in this paper are related to the general case of $G_M(y_m), y_m \in \mathbb{R}_+^m$.

In Section 2 we introduce the class of max-stable distributions along with the Schlather and Tawn (2002) inequalities for the extremal coefficients. Subsequently, we present the general result of the paper that gives rise to inequalities for the exponent measures. In Section 3 we consider the Hall and Tajvidi (2000) nonparametric estimator for the exponent measure and extend it, through a likelihood-based procedure, to satisfy the new inequalities. Finally, in Section 4 a simulation study is conducted to assess the performance of the constrained estimator.

2 Dependence Properties

2.1 Background

The class of max-stable distributions arises naturally from the study of appropriately scaled component-wise maxima of random variables. Consider a set of independent and identically distributed random vectors $X^j = (X^j_1, \ldots, X^j_m), j = 1, \ldots, n$, with unit Fréchet margins. Under weak conditions (Resnick, 1987) it follows that

$$\lim_{n \to \infty} \mathbb{P}\left( \bigcap_{i=1}^{m} \left\{ \max_{j=1}^{n} X^j_i / n < y_i \right\} \right) = G_M(y_{M_m}), \quad y_{M_m} \in \mathbb{R}_+^m. \quad (3)$$

The distribution function $G_{M_m}$ can be completely characterised by the following representations

$$V_{M_m}(y_{M_m}) = -\log G_{M_m}(y) = \int_{S_m} \max_{i \in M_m} \left( \frac{w_i}{y_i} \right) dH(w_1, \ldots, w_m), \quad (4)$$

$$= \left\{ \sum_{i=1}^{m} 1/y_i \right\} A_{M_m} \left( \frac{1/y_1}{\sum_{i=1}^{m} 1/y_i}, \ldots, \frac{1/y_m}{\sum_{i=1}^{m} 1/y_i} \right) \quad (5)$$

where the function $V_{M_m}$ is known as the exponent measure of the multivariate extreme value distribution $G_{M_m}$ and $A_{M_m}$, called the Pickands’ dependence function, is a convex function that satisfies $\max\{w_1, \ldots, w_m\} \leq A_{M_m}(w_1, \ldots, w_m) \leq 1, (w_1, \ldots, w_m) \in S_m$. This condition implies that $A_{M_m}(e_j) = 1, j \in M_m$, where $e_j$ is the $j$-th unit vector in $\mathbb{R}^m$. 3
Let $C_m = 2^m \setminus \{\emptyset\}$ and denote also by $y_B = \{y_i : i \in B\}$ for $B \in C_m$. Then we can define $2^m - 1$ exponent measures for an $m$-dimensional max-stable random vector $Y$, where each one characterises completely the distribution function of a marginal random variable $Y_B$ of $Y$, i.e.,

$$V_B(y_B) = -\log \{P(Y_B < y_B)\} = -\log \left( \lim_{y_{Mm} \setminus B \rightarrow \infty} G_{Mm}(y_{Mm}) \right), \quad B \in C_m.$$ 

The set of exponent measures $\{V_B : B \in C_m\}$ describes completely the dependence structure of a max-stable distribution given by equation (1) and all of its lower dimensional margins. It is also trivial to see that with each exponent measure $V_B$ there is an associated Pickands’ dependence function $A_B$. Additionally, $V_B, B \in C_m$, is homogeneous of order $-1$, i.e., $V_B(y, \ldots, y) = y^{-1}V_B(1, \ldots, 1), \ y > 0$.

The importance of the homogeneity property is mostly illustrated through one widely used measure of extremal dependence for the variables indexed by a set $B \in C_m$. More specifically, the quantity defined by

$$\theta_B = V_B(1, \ldots, 1) = \int_{S_m} \max_{i \in B} w_i dH(w_1, \ldots, w_m), \quad 1 \leq \theta_B \leq |B|,$$ 

(6)

describes the effective number of independent variables in the set $B$ and arises naturally from the distribution of the maximum of all the variables indexed by the set $B$, i.e.,

$$P\left\{\max_{i \in B} Y_i < y\right\} = P\{Y_i < y\}^{\theta_B}, \quad y > 0.$$ 

(7)

The measure $\theta_B$ is termed the extremal coefficient and complete dependence and independence corresponds to $\theta_B = 1$ and $\theta_B = |B|$ respectively. Also, from expression (7) it follows trivially that $\theta_B = 1$ for any $B \in C_m$ with $|B| = 1$. Due to its simple interpretation, the set of extremal coefficients $\{\theta_B : B \in C\}$ has been used as a dependence measure in various applications (Tawn, 1990; Schlather and Tawn, 2003).

2.2 Schlather and Tawn (2002, 2003) inequalities for the extremal coefficients

Schlather and Tawn (2002, 2003) constructed bounds for the set of extremal coefficients $\{\theta_B :
$B \in C_m$) of max-stable distributions that characterise the dependence structure for the special case of $G_{M_m}(y, \ldots, y)$, $y > 0$. Here we use the terminology of Schlather and Tawn (2002) and for non-empty distinct subsets $B_1, \ldots, B_s$ of $M_m$, $s \in \mathbb{N}$, we refer to the set of extremal coefficients \{\theta_{B_1}, \ldots, \theta_{B_s}\} as complete and consistent if $s = 2^m - 1$ and $\theta_{B_i}$ is given by expression (6), respectively. Their main result is given in the following theorem.

**Theorem 1** (Schlather and Tawn (2002) Corollary 5). A complete set of extremal coefficients \{\theta_B : B \in C_m\}, where $M_m$ is a finite set of indices, is consistent if and only if

$$\sum_{B \in C_m, B \supseteq M_m \setminus L} (-1)^{|B \cap L| + 1} \theta_B \geq 0, \text{ for all } L \in C_m. \quad (8)$$

Theorem 1 yields bounds that higher order extremal coefficients need to satisfy to be consistent with lower order extremal coefficients. For example consider the inequalities (8) for the cases $m = 2$ and $m = 3$ and let for ease of notation $\theta_{\{i,j\}}$ and $\theta_{\{i,j,k\}}$ be $\theta_{ij}$ and $\theta_{ijk}$, for $i, j, k \in M_m$ and $i \neq j \neq k$. These are respectively

$$1 \leq \theta_{12}, \theta_{13}, \theta_{23} \leq 2$$

and

$$\max\{\theta_{12}, \theta_{13}, \theta_{23}, \theta_{12} + \theta_{13} + \theta_{23} - 3\} \leq \theta_{123} \leq \min\{\theta_{12} + \theta_{13} - 1, \theta_{12} + \theta_{23} - 1, \theta_{13} + \theta_{23} - 1\}.$$

The first set of inequalities represents the well known bounds of the extremal coefficients that come from the positively quadrant dependence property of max-stable distributions. However, the second set of inequalities gives tighter bounds for the higher order extremal coefficient $\theta_{123}$. This can be seen easily since the combined inequalities for the cases $m = 2$ and $m = 3$ reduce to $1 \leq \theta_{123} \leq 3$.

**2.3 Inequalities for the exponent measures of max-stable distributions**

It transpires that similar inequalities as with those in expression (8) can be obtained for the exponent measures \{\nu_B : B \in C_m\} of max-stable distributions. Analogously with the terminology for the extremal coefficients in Section 2.2 we introduce the following definition.

**Definition 1.** Let $s$ be an integer, for $i = 1, \ldots, s$ $B_i$ are distinct non-empty subsets of $M_m =$
\( \{1, \ldots, m\} \) and \( y_{M_m} = (y_1, \ldots, y_m) \in \mathbb{R}_+^m \). An ensemble \( \{V_{B_1}(y_{B_1}), \ldots, V_{B_s}(y_{B_s})\} \) of exponent measures, where \( y_{B_i} = \{y_j : j \in B_i\} \), is called consistent if

\[
V_{B_i}(y_{B_i}) = \int_{S_m} \max_{j \in B_i} \left( \frac{w_i}{y_j} \right) dH(w_1, \ldots, w_m),
\]

for \( i = 1, \ldots, s \) and \( H \) is an arbitrary finite measure that satisfies \( \int_{S_m} w_i dH(w_1, \ldots, w_m) = 1 \) for any \( i \in M_m \).

If \( s = 2^m - 1 \) then the set of exponent measures is called complete. The following theorem provides a new representation of the exponent measures of multivariate extreme-value distributions in terms of non-negative and uniquely defined real functions.

**Theorem 2.** Let \( \{V_B : B \in C_m\} \) be a complete and consistent set of exponent measures. Then, there exist \( 2^m - 1 \) non-negative functions \( d_L : \mathbb{R}_+^m \rightarrow \mathbb{R}_+ \), \( L \in C_m \), such that, for any \( B \in C_m \)

\[
V_B(y_B) = \sum_{L \in M_m, L \cap B \neq \emptyset} d_L(y_{M_m}), \quad (9)
\]

and the functions \( d_L \) are uniquely given by

\[
d_L(y_{M_m}) = \sum_{B \in C_m, B \supseteq M_m \setminus L} (-1)^{|B \cap L| + 1} \int_{S_m} \max_{j \in B} \left( \frac{w_j}{y_j} \right) dH(w_1, \ldots, w_m). \quad (10)
\]

**Proof**

The proof of equation \( (9) \) of Theorem 2 follows along the lines of Schlather and Tawn (2002) proof of Theorem 5 for the simpler case of the extremal coefficients by replacing the constants \( \alpha_k(n) \) of Deheuvels (1983) representation of max-stable distributions with \( \alpha_k(n)/y_i, i \in M_m, k \in \mathbb{Z} \). Equation \( (10) \) is the Möbius inversion of equation \( (9) \). \( \square \)

The characterisation of a consistent set of exponent measures is obtained from the following corollary.

**Corollary 1.** A complete set of exponent measures \( \{V_B : B \in C_m\} \) is consistent if and only if

\[
\sum_{B \in C_m, B \supseteq M_m \setminus L} (-1)^{|B \cap L| + 1} \int_{S_m} \max_{j \in B} \left( \frac{w_j}{y_j} \right) dH(w_1, \ldots, w_m) \geq 0, \quad (11)
\]

for all \( y_{M_m} \in \mathbb{R}_+^m \) and \( L \in C_m \).
3 Inference

3.1 The Hall and Tajvidi (2000) estimator of the exponent measure

The fundamental premise in all statistical extreme value modelling is that the observed extremes of a stochastic process are well modelled by the limiting theoretical extreme-value distributions. Let for example $X^j = (X^j_1, \ldots, X^j_m)$, $j = 1, \ldots, N$, be a set of independent and identically distributed $m$-dimensional random vectors with unit Fréchet margins. Here and throughout we assume that the normalised componentwise block maxima

$$Y^j := \left( \max_{r=(j-1)d+1}^{jd} X^r \right), \quad j = 1, \ldots, n,$$

where $nd = N$, follow exactly the law $G_{M_m}$ of the limiting expression (3).

Let now $w_B \in S_{|B|} = \left\{ w_B \in \mathbb{R}^{B} : \sum_{i \in B} w_{B,i} = 1 \right\}$, $B \in C_m$, and define $Z^j_B = w_B Y^j$, for $j = 1, \ldots, n$. It then follows that the cumulative distribution function of $\max_{i \in B} Z^j_i$ is Fréchet with scale parameter equal to the Pickands’ dependence function $A_B(w_B)$ of $G_B$, i.e.,

$$P \left\{ \max_{i \in B} Z^j_i < y \right\} = \exp \left\{ - \frac{A_B(w_B)}{y} \right\}, \quad y > 0.$$

A natural consistent estimator of $A_B$ then is the Hall and Tajvidi (2000) corrected version of Pickands’ estimator (Pickands, 1981) which maximises the likelihood

$$\ell_B \{ A_B(w_B) \} = n \log \{ A_B(w_B) \} - 2 \sum_{j=1}^{n} \log W^j_B - A_B(w_B) \sum_{j=1}^{n} \frac{1}{W^j_B},$$

where $W^j_B = \max_{i \in B} \left\{ w_{B,i} Y^j_i \sum_{j=1}^{n} (1/Y^j_i)/n \right\}$, $j = 1, \ldots, n$, is the Hall and Tajvidi (2000) correction which ensures that $\max w_B \leq \hat{A}_B(w_B)$, for all $w_B \in S_{|B|}$, as well as $\hat{A}_B(e_j) = 1$, for any $j \in M_m$, where $e_j$ is the $j$-th unit vector in $\mathbb{R}^m$. The maximum likelihood estimator is given by $\hat{A}_B(w_B) = \left\{ n^{-1} \sum_{j=1}^{n} (1/W^j_B) \right\}^{-1}$ which is subsequently corrected by

$$\tilde{A}_B(w_B) = \min \left\{ \hat{A}_B(w_B), 1 \right\}$$

to satisfy $\tilde{A}_B(w_B) \leq 1$, for all $w_B \in S_{|B|}$. On combining the estimator $\tilde{A}_B$ with equation (5),
the following consistent estimator of the exponent measure $V_B$ is obtained,

$$
\tilde{V}_B(y_B) = \left\{ \sum_{i \in B} 1/y_i \right\} \tilde{A}_B \left( \frac{1/y_B}{\sum_{i \in B} 1/y_i} \right), \quad y_B \in \mathbb{R}_+^{|B|}, \quad B \in C_m.
$$

(13)

Other types of estimators exist in the literature such as the non-parametric estimators proposed by Deheuvels (1991) and Capéraà et al. (1997) for the bivariate case. Zhang et al. (2008) gives a detailed overview of the existing estimators and extends them to the multivariate case. In this paper though we use the Hall and Tajvidi (2000) estimator since it arises as the maximum of a log-likelihood function based on which the new inequalities of Section 2.3 can be imposed.

### 3.2 Constrained estimators

It transpires that the aforementioned nonparametric estimators of the exponent measures do not necessarily ensure that the resulting estimated set of exponent measures satisfy inequalities [11]. The focus here is placed on incorporating these additional constraints in the estimation procedure so that the resulting complete set of estimated exponent measures $V_B(y_B), B \in C_m$, is consistent in the sense of Definition 1 for fixed $y_{M_m} \in \mathbb{R}_+^m$. To incorporate the inequalities we construct similarly with Schlather and Tawn (2003) a joint log-likelihood function $\ell$ of $\{A_B; B \in C_m\}$ by falsely assuming independence between the observations for all different $B$ to give the pseudo-log-likelihood

$$
\ell\left( \left\{ A_B \left( \frac{1/y_B}{\sum_{i \in B} 1/y_i} \right) : B \in C_m \right\} \right) = \sum_{B \in C_m, |B| \geq 2} \ell_B \left\{ A_B \left( \frac{1/y_B}{\sum_{i \in B} 1/y_i} \right) \right\}.
$$

(14)

The maximum pseudo-likelihood estimators are consistent (Liang and Self, 1996) and the constrained estimators are obtained by maximising the pseudo-log-likelihood (14) subject to

$$
\sum_{B \in C_m, B \supseteq M_m \setminus L} (-1)^{|B \cap L| + 1} \left\{ \sum_{i \in B} 1/y_i \right\} A_B \left( \frac{1/y_B}{\sum_{i \in B} 1/y_i} \right) \geq 0, \quad \text{for all } L \in C_m
$$

and

$$
A_B \left( \frac{1/y_B}{\sum_{i \in B} 1/y_i} \right) \leq 1, \quad \text{for all } B \in C_m.
$$

The resulting constrained estimators are denoted by $\tilde{A}_B^c$ which in turn yield the estimators $\tilde{V}_B^c$. 

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as in equation (13). The joint estimation of the exponent measures ensures that all estimators are self-consistent. Note that the resulting estimates of lower order exponent measures are affected by higher order measures, i.e., estimates of $V_{B_0}(y_{B_0})$ are affected by estimates of $V_{B_1}(y_{B_1})$, where $B_0 \subset B_1$. The major benefit of this feature is that this guarantees the existence of higher order measures which are self-consistent with the lower order measures.

An alternative way of obtaining a set of estimated exponent measures is via sequential estimation, i.e., the lower order exponent measures are estimated firstly and then are used as constraints in the estimation of the higher order exponent measures, see also Schlather and Tawn (2003). Although this method is faster than the joint optimization problem described by equation (14), it does not have the desirable feature described above.

4 Simulation Study

4.1 Design

We illustrate the impact of constraining the Hall and Tajvidi (2000) estimators to satisfy the new inequalities (11) over the unconstrained estimators of the set of exponent measures using simulated data from a 3-dimensional max-stable distribution, i.e., the extreme value logistic distribution with dependence parameter $\alpha \in (0, 1]$ and set of exponent measures given by

$$\left\{ V_B(y_B) = \left( \sum_{i \in B} y_i^{-1/\alpha} \right)^\alpha : B \in C_3 \right\}, \quad y_B \in \mathbb{R}_{+}^{\left| B \right|}. $$

(15)

The values $\alpha = 1$ and $\alpha = 0$, taken as $\alpha \to 0$, correspond to independence and complete dependence, respectively.

All comparisons are based on the root mean square error (RMSE) performance of the exponent measure estimators for a range of dependence parameters $\alpha$ and a cube grid of values, say $L^3 \subseteq \mathbb{R}_{+}^3$, for $y_M$. Specifically, the values chosen for the dependence parameter and the sample size are $\alpha \in \{0.2, 0.5, 0.8\}$ and $n = 50$, respectively. Results from larger sample sizes are not reported in the paper since they are unrealistic for applications and also, the efficiency of the estimators $\tilde{V}_B$ and $\tilde{V}_B^c$ is similar, a fact that comes from the consistency property of the estimators $\tilde{V}_B$ and $\tilde{V}_B^c$. The Hall and Tajvidi (2000) estimators are compared to the new inequalities (11) over the unconstrained estimators for a range of dependence parameters $\alpha$ and a cube grid of values, say $L^3 \subseteq \mathbb{R}_{+}^3$, for $y_M$. Specifically, the values chosen for the dependence parameter and the sample size are $\alpha \in \{0.2, 0.5, 0.8\}$ and $n = 50$, respectively. Results from larger sample sizes are not reported in the paper since they are unrealistic for applications and also, the efficiency of the estimators $\tilde{V}_B$ and $\tilde{V}_B^c$ is similar, a fact that comes from the consistency property of the estimators $\tilde{V}_B$ and $\tilde{V}_B^c$. The Hall and Tajvidi (2000) estimators are compared to the new inequalities (11) over the unconstrained estimators for a range of dependence parameters $\alpha$ and a cube grid of values, say $L^3 \subseteq \mathbb{R}_{+}^3$, for $y_M$. Specifically, the values chosen for the dependence parameter and the sample size are $\alpha \in \{0.2, 0.5, 0.8\}$ and $n = 50$, respectively. Results from larger sample sizes are not reported in the paper since they are unrealistic for applications and also, the efficiency of the estimators $\tilde{V}_B$ and $\tilde{V}_B^c$ is similar, a fact that comes from the consistency property of the estimators $\tilde{V}_B$ and $\tilde{V}_B^c$.
and Tajvidi (2000) estimator. The set \( L \) was chosen to be the discrete set \( \{ x_{p_1}, \ldots, x_{p_7} \} \) with \( x_p \) denoting the \( p \)-th quantile of the unit Fréchet distribution. We chose \( p_1 = 0.05 \), \( p_7 = 0.95 \) and step size \( p_j - p_{j-1} = 0.15 \). The Monte Carlo size used to compute estimates of the RMSE is 500.

To obtain an aggregated measure of performance, we also report the Monte Carlo estimates of the integrated square deviation of the estimators from the theoretical function, i.e.,

\[
\check{T}_B = \int_{C(L|B|)} \left( \hat{V}_B(y_B) - V_B(y_B) \right)^2 dy_B, \quad \text{for all } B \in C_3,
\]

where \( C(L|B|) \) is the smallest \(|B|\)-hypercube that contains the set \( L|B| \). The integral in expression (16) is approximated in each Monte Carlo iteration by the quadrature mid-point numerical integration technique on the grid \( L|B| \) and the measure \( \check{T}_B \) is defined analogously by replacing \( \hat{V}_B \) in expression (16) with \( \hat{V}_B^c \).

### 4.2 Results

Figure 1 shows the histograms of the ratio of RMSEs between \( \hat{V}_B^c \) and \( \hat{V}_B \), \( B \in C_3 \), for all grid points in \( \mathbb{R}^3_+ \) for the extreme value logistic distribution with \( \alpha = 0.2, 0.5 \) and 0.8. The figures indicate either similar or better performance of the constrained estimators.

In particular, for the \( \alpha = 0.8 \) case, the constrained estimators are more efficient than the unconstrained estimators especially for the higher order exponent measure \( V_{123} \) and improvement in RMSE, although lower in magnitude, can be also seen in the bivariate exponent measures \( V_B, B \in C_3 \setminus M_3 \). Also, the percentage of Monte Carlo samples where the constrained estimates changed with respect to the Hall and Tajvidi (2000) estimates is 62%. Regarding the \( \alpha = 0.5 \) case, we found better performance of the constrained estimators for \( V_{123} \), although lower in magnitude than the \( \alpha = 0.8 \) case, and similar performance for the bivariate exponent measures.

This feature is also supported by the smaller percentage of change in estimates which is 30%. For the case of strong dependence, i.e., \( \alpha = 0.2 \), the percentage of change in estimates is very low and equal to 6% which results in similar efficiency of the estimators for all exponent measures as is also shown from Figure 1.

Table 1 shows the Monte Carlo estimates of the integrated square deviation of the estimators from the theoretical function. For the case of strong dependence there is no practical benefit
Figure 1: Histograms of the ratio of Monte Carlo estimates of RMSEs between the constrained and unconstrained estimators of the exponent measures $V_{12}, V_{13}, V_{23}$ and $V_{123}$ (top to bottom) for the extreme value logistic distribution with $\alpha = 0.2$ (left) $\alpha = 0.5$ (centre) and $\alpha = 0.8$ (right).

Table 1: Monte Carlo estimates of $\tilde{T}_B$ and $\tilde{T}_B^c$, $B \in C_3$, for the extreme value logistic case with $\alpha = 0.2, 0.5$ and 0.8.

| $B$ | $\alpha = 0.2$ | $\alpha = 0.5$ | $\alpha = 0.8$ |
|-----|----------------|-----------------|----------------|
|     | $\tilde{T}_B$  | $\tilde{T}_B^c$| $\tilde{T}_B$  |
| $\{1, 2\}$ | 0.02 0.02 | 0.36 0.35 | 0.76 0.72 |
| $\{1, 3\}$ | 0.02 0.02 | 0.34 0.34 | 0.76 0.74 |
| $\{2, 3\}$ | 0.02 0.02 | 0.37 0.36 | 0.74 0.72 |
| $\{1, 2, 3\}$ | 0.66 0.66 | 11.15 10.63 | 26.80 22.14 |
of \( \tilde{V}_B^C \) over \( \tilde{V}_B \). However, in all other cases the constrained estimators are more efficient than the unconstrained estimators. This shows that not only does the imposition of the constraints improve the performance of the estimators for the higher order exponent measures, but so does for the bivariate level of dependence.

To conclude, the performance of the estimators \( \tilde{V}_B \) and \( \tilde{V}_B^C \) is similar as the dependence increases and becomes identical in the limiting case of \( \alpha \to 0 \). This feature is explained by the increase in performance of the \textbf{Hall and Tajvidi (2000)} estimators \( \tilde{V}_B \) as dependence increases which yields a consistent set of estimated exponent measures. Overall, we found the imposition of the new constraints to be beneficial for the simplest max-stable distribution, i.e., the extreme value logistic, and superior in efficiency, especially for the case of moderate or weak dependence. The largest improvement is observed for higher order exponent measures which is promising for implementations in higher than 3 dimensions.

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