OLD AND NEW MORREY SPACES VIA HEAT KERNEL BOUNDS

XUAN THINH DUONG, JIE XIAO AND LIXIN YAN

Abstract. Given $p \in [1, \infty)$ and $\lambda \in (0, n)$, we study Morrey space $L^{p, \lambda}(\mathbb{R}^n)$ of all locally integrable complex-valued functions $f$ on $\mathbb{R}^n$ such that for every open Euclidean ball $B \subset \mathbb{R}^n$ with radius $r_B$ there are numbers $C = C(f)$ (depending on $f$) and $c = c(f, B)$ (relying upon $f$ and $B$) satisfying

$$r_B^{-\lambda} \int_B |f(x) - c|^p dx \leq C$$

and derive old and new, two essentially different cases arising from either choosing $c = f_B = |B|^{-1} \int_B f(y) dy$ or replacing $c$ by $p_t(x, y) = \int_{tB} P_t(x, y) f(y) dy$ where $t_B$ is scaled to $r_B$ and $p_t(\cdot, \cdot)$ is the kernel of the infinitesimal generator $L$ of an analytic semigroup $\{e^{-tL}\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$. Consequently, we are led to simultaneously characterize the old and new Morrey spaces, but also to show that for a suitable operator $L$, the new Morrey space is equivalent to the old one.

1. Introduction

As well-known, a priori estimates mixing $L^p$ and $\text{Lip}_\lambda$ are frequently used in the study of partial differential equations – naturally, the so-called Morrey spaces are brought into play (cf. [24]). A locally integrable complex-valued function $f$ on $\mathbb{R}^n$ is said to be in the Morrey space $L^{p, \lambda}(\mathbb{R}^n)$, $1 \leq p < \infty$ and $\lambda \in (0, n + p)$, if for every Euclidean open ball $B \subset \mathbb{R}^n$ with radius $r_B$ there are numbers $C = C(f)$ (depending on $f$) and $c = c(f, B)$ (relying upon $f$ and $B$) satisfying

$$r_B^{-\lambda} \int_B |f(x) - c|^p dx \leq C.$$ 

The space of $L^{p, \lambda}(\mathbb{R}^n)$-functions was introduced by Morrey [18]. Since then, the space has been studied extensively – see for example [4, 13, 14, 20, 21, 22, 28].

We would like to note that as in [20], for $1 \leq p < \infty$ and $\lambda = n$, the spaces $L^{p,n}(\mathbb{R}^n)$ are variants of the classical BMO (bounded mean oscillation) function space of John-Nirenberg. For $1 \leq p < \infty$ and $\lambda \in (n, n + p)$, the spaces $L^{p, \lambda}(\mathbb{R}^n)$ are variants of the homogeneous Lipschitz spaces Lip($\lambda-n)/p(\mathbb{R}^n)$.

XTD is supported by a grant from Australia Research Council; JX is supported by NSERC of Canada; LXY is partially supported by NSF of China (Grant No. 10371134).

2000 Mathematics Subject Classification. 42B20, 42B35, 47B38.

Key words and phrases. Morrey spaces, semigroup, holomorphic functional calculus, Littlewood-Paley functions.
Clearly, the remaining cases: $1 \leq p < \infty$ and $\lambda \in (0, n)$ are of independent interest, and hence motivate our investigation. The purpose of this paper is twofold. First, we explore some new characterizations of $L^{p,\lambda}(\mathbb{R}^n)$ through the fact that $L^{p,\lambda}(\mathbb{R}^n)$ consists of all locally integrable complex-valued functions $f$ on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^{p,\lambda}} = \sup_{B \subset \mathbb{R}^n} \left[ r_B^{-\lambda} \int_B |f(x) - f_B|^p dx \right]^{1/p} < \infty,$$

where the supremum is taken over all Euclidean open balls $B = B(x_0, r_B)$ with center $x_0$ and radius $r_B$, and $f_B$ stands for the mean value of $f$ over $B$, i.e.,

$$f_B = |B|^{-1} \int_B f(x) dx.$$

The second aim is to use those new characterizations as motives of a continuous study of $\|L^{p,\lambda}\|$ and so to introduce new Morrey spaces $L^{p,\lambda}_L(\mathbb{R}^n)$ associated with operators. Roughly speaking, if $L$ is the infinitesimal generator of an analytic semigroup $\{e^{-tL}\}_{t \geq 0}$ on $L^2(\mathbb{R}^n)$ with kernel $p_t(x, y)$ which decays fast enough, then we can view $P_t f = e^{-tL} f$ as an average version of $f$ at the scale $t$ and use the quantity

$$P_{tB} f(x) = \int_{\mathbb{R}^n} p_{tB}(x, y) f(y) dy$$

to replace the mean value $f_B$ in the equivalent semi-norm (1.1) of the original Morrey space, where $t_B$ is scaled to the radius of the ball $B$. Hence we say that a function $f$ (with appropriate bound on its size $|f|$) belongs to the space $L^{p,\lambda}_L(\mathbb{R}^n)$ (where $1 \leq p < \infty$ and $\lambda \in (0, n)$), provided

$$\|f\|_{L^{p,\lambda}_L} = \sup_{B \subset \mathbb{R}^n} \left[ r_B^{-\lambda} \int_B |f(x) - P_{tB} f(x)|^p dx \right]^{1/p} < \infty$$

where $t_B = r_B^m$ for a fixed constant $m > 0$ – see the forthcoming Sections 2.2 and 3.1.

We pursue a better understanding of (1.1) and (1.2) through the following aspects:

In Section 2, we collect most useful materials on the bounded holomorphic functional calculus.

In Section 3, we study some characterizations of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\lambda}_L(\mathbb{R}^n)$ and give a criterion for $L^{p,\lambda}(\mathbb{R}^n) \subseteq L^{p,\lambda}_L(\mathbb{R}^n)$. The later fact illustrates that $L^{p,\lambda}(\mathbb{R}^n)$ exists as the minimal Morrey space, and consequently induces a concept of the maximal Morrey space.

In Section 4, we establish an identity formula associated with the operator $L$. This formula is a key to handle the quadratic features of the old and new Morrey spaces.

As an immediate continuation of Section 4, Section 5 is devoted to Littlewood-Paley type characterizations of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\lambda}_L(\mathbb{R}^n)$ via the predual of $L^{p,\lambda}(\mathbb{R}^n)$ (cf. [28]) and a number of important estimates for functions in $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\lambda}_L(\mathbb{R}^n)$. Moreover, we show that for a suitable semigroup $\{e^{-tL}\}_{t > 0}$, $L^{p,\lambda}_L(\mathbb{R}^n)$ equals $L^{p,\lambda}(\mathbb{R}^n)$ with equivalent
seminal norms – in particular, if $L$ is either $\triangle$ or $\sqrt{\triangle}$ on $\mathbb{R}^n$, then $L^{p,\lambda}(\mathbb{R}^n)$ coincides with $L^p_{\sqrt{\triangle}}(\mathbb{R}^n)$ and $L^p_{\triangle}(\mathbb{R}^n)$.

Throughout, the letters $c, c_1, c_2, \ldots$ will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

2.1. Holomorphic functional calculi of operators. We start with a review of some definitions of holomorphic functional calculi introduced by McIntosh [17]. Let $0 \leq \omega < \nu < \pi$. We define the closed sector in the complex plane

$$S_\omega = \{ z \in \mathbb{C} : |\arg z| \leq \omega \} \cup \{0\}$$

and denote the interior of $S_\omega$ by $S^0_\omega$.

We employ the following subspaces of the space $H(S^0_\nu)$ of all holomorphic functions on $S^0_\nu$:

$$H_\infty(S^0_\nu) = \{ b \in H(S^0_\nu) : \|b\|_\infty < \infty \},$$

where

$$\|b\|_\infty = \sup\{|b(z)| : z \in S^0_\nu\}$$

and

$$\Psi(S^0_\nu) = \{ \psi \in H(S^0_\nu) : \exists s > 0, \ |\psi(z)| \leq c|z|^s(1 + |z|^{2s})^{-1} \}. $$

Given $0 \leq \omega < \pi$, a closed operator $L$ in $L^2(\mathbb{R}^n)$ is said to be of type $\omega$ if $\sigma(L) \subset S_\omega$, and for each $\nu > \omega$, there exists a constant $c_\nu$ such that

$$\|(L - \lambda\mathcal{I})^{-1}\|_{2,2} = \|(L - \lambda\mathcal{I})^{-1}\|_{L^2 \rightarrow L^2} \leq c_\nu|\lambda|^{-1}, \quad \lambda \notin S_\nu.$$

If $L$ is of type $\omega$ and $\psi \in \Psi(S^0_\nu)$, we define $\psi(L) \in \mathcal{L}(L^2, L^2)$ by

$$(2.1) \quad \psi(L) = \frac{1}{2\pi i} \int_\Gamma (L - \lambda\mathcal{I})^{-1}\psi(\lambda)d\lambda,$$

where $\Gamma$ is the contour $\{ \xi = re^{\pm i\theta} : r \geq 0 \}$ parametrised clockwise around $S_\omega$, and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$ (which is the class of all bounded linear operators on $L^2$), and it is straightforward to show, using Cauchy’s theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu)$. If, in addition, $L$ is one-one and has dense range and if $b \in H_\infty(S^0_\nu)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L) \quad \text{where} \quad \psi(z) = z(1 + z)^{-2}.$$ 

It can be shown that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$. 

We say that $L$ has a bounded $H_\infty$ calculus in $L^2(\mathbb{R}^n)$ provided there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$ and

$$\|b(L)\|_{L^2} = \|b(L)\|_{L^2: L^2} \leq c_{\nu,2} \|b\|_{H_\infty} \quad \forall b \in H_\infty(S^0_\nu).$$

For the conditions and properties of operators which have holomorphic functional calculi, see [17] and [2] which also contain a proof of the following Convergence Lemma.

**Lemma 2.1.** Let $X$ be a complex Banach space. Given $0 \leq \omega < \nu \leq \pi$, let $L$ be an operator of type $\omega$ which is one-to-one with dense domain and range. Suppose $\{f_\alpha\}$ is a uniformly bounded net in $H_\infty(S^0_\nu)$, which converges to $f \in H_\infty(S^0_\nu)$ uniformly on compact subsets of $S^0_\nu$, such that $\{f_\alpha(L)\}$ is a uniformly bounded net in the space $\mathcal{L}(X,X)$ of continuous linear operators on $X$. Then $f(L) \in \mathcal{L}(X,X)$, $f_\alpha(L)u \to f(L)u$ for all $u \in X$ and

$$\|f(L)\| = \|f(L)\|_{X \to X} \leq \sup_\alpha \|f_\alpha(L)\| = \sup_\alpha \|f_\alpha(L)\|_{X \to X}.$$

2.2. **Two more assumptions.** Let $L$ be a linear operator of type $\omega$ on $L^2(\mathbb{R}^n)$ with $\omega < \pi/2$, hence $L$ generates a holomorphic semigroup $e^{-zL}$, $0 \leq |\text{Arg}(z)| < \pi/2 - \omega$. Assume the following two conditions.

**Assumption (a):** The holomorphic semigroup

$$\{e^{-zL}\}_{0 \leq |\text{Arg}(z)| < \pi/2 - \omega}$$

is represented by kernel $p_z(x,y)$ which satisfies an upper bound

$$|p_z(x,y)| \leq c_\theta h_{\|z\|}(x,y) \quad \forall x,y \in \mathbb{R}^n$$

and

$$|\text{Arg}(z)| < \pi/2 - \theta \quad \text{for} \quad \theta > \omega,$$

where $h_t(\cdot, \cdot)$ is determined by

$$h_t(x,y) = t^{-n/m} g\left(\frac{|x-y|}{t^{1/m}}\right),$$

in which $m$ is a positive constant and $g$ is a positive, bounded, decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\epsilon} g(r) = 0 \quad \text{for some} \quad \epsilon > 0.$$  

**Assumption (b):** The operator $L$ has a bounded $H_\infty$-calculus in $L^2(\mathbb{R}^n)$.

Now, we give some consequences of the assumptions (a) and (b) which will be used later.
First, if \( \{ e^{-tL} \}_{t>0} \) is a bounded analytic semigroup on \( L^2(\mathbb{R}^n) \) whose kernel \( p_t(x,y) \) satisfies the estimates (2.2) and (2.3), then for any \( k \in \mathbb{N} \), the time derivatives of \( p_t \) satisfy
\[
\left| t^k \frac{\partial^k p_t(x,y)}{\partial t^k} \right| \leq c \frac{\| x - y \|}{t^{n/m}} g\left( \frac{|x - y|}{t^{1/m}} \right) \quad \text{for all } t > 0 \text{ and almost all } x, y \in \mathbb{R}^n.
\]
(2.4)
For each \( k \in \mathbb{N} \), the function \( g \) might depend on \( k \) but it always satisfies (2.3). See Theorem 6.17 of [19].

Secondly, \( L \) has a bounded \( H_\infty \)-calculus in \( L^2(\mathbb{R}^n) \) if and only if for any non-zero function \( \psi \in \Psi(S_\nu^0) \), \( L \) satisfies the square function estimate and its reverse
\[
c_1 \| f \|_{L^2} \leq \left( \int_0^\infty \| \psi_t(L)f \|_{L^2}^2 \frac{dt}{t} \right)^{1/2} \leq c_2 \| f \|_{L^2}
\]
for some \( 0 < c_1 \leq c_2 < \infty \), where \( \psi_t(\xi) = \psi(t\xi) \). Note that different choices of \( \nu > \omega \) and \( \psi \in \Psi(S_\nu^0) \) lead to equivalent quadratic norms of \( f \).

As noted in [17], positive self-adjoint operators satisfy the quadratic estimate (2.5). So do normal operators with spectra in a sector, and maximal accretive operators. For definitions of these classes of operators, we refer the reader to [27].

The following result, existing as a special case of [6, Theorem 6], tells us the \( L^2 \)-boundedness of a bounded \( H_\infty \)-calculus can be extended to \( L^p \)-boundedness, \( p > 1 \).

**Lemma 2.2.** Under the assumptions (a) and (b), the operator \( L \) has a bounded \( H_\infty \)-calculus in \( L^p(\mathbb{R}^n) \), \( p \in (1, \infty) \), that is, \( b(L) \in \mathcal{L}(L^p, L^p) \) with
\[
\| b(L) \|_{p,p} = \| b(L) \|_{L^p \rightarrow L^p} \leq c_{\nu,p} \| b \|_\infty \quad \forall b \in H_\infty(S_\nu^0).
\]
Moreover, if \( p = 1 \) then \( b(L) \) is of weak type \((1, 1)\).

Thirdly, the Littlewood-Paley function \( G_L(f) \) associated with an operator \( L \) is defined by
\[
G_L(f)(x) = \left( \int_0^\infty |\psi_t(L)f|^2 \frac{dt}{t} \right)^{1/2},
\]
(2.6)
where again \( \psi \in \Psi(S_\nu^0) \), and \( \psi_t(\xi) = \psi(t\xi) \). It follows from Theorem 6 of [3] that the function \( G_L(f) \) is bounded on \( L^p \) for \( 1 < p < \infty \). More specifically, there exist constants \( c_3, c_4 \) such that \( 0 < c_3 \leq c_4 < \infty \) and
\[
c_3 \| f \|_{L^p} \leq \| G_L(f) \|_{L^p} \leq c_4 \| f \|_{L^p}
\]
(2.7)
for all \( f \in L^p, 1 < p < \infty \).

By duality, the operator \( G_{L^*}(f) \) also satisfies the estimate (2.7), where \( L^* \) is the adjoint operator of \( L \).
2.3. **Acting class of semigroup** $\{e^{-tL}\}_{t>0}$. We now define the class of functions that the operators $e^{-tL}$ act upon. Fix $1 \leq p < \infty$. For any $\beta > 0$, a complex-valued function $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ is said to be a function of type $(p; \beta)$ if $f$ satisfies

$$\left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{(1 + |x|)^{n+\beta}} \, dx \right)^{1/p} \leq c < \infty. \tag{2.8}$$

We denote by $\mathcal{M}_{(p;\beta)}$ the collection of all functions of type $(p; \beta)$. If $f \in \mathcal{M}_{(p;\beta)}$, the norm of $f \in \mathcal{M}_{(p;\beta)}$ is defined by

$$\|f\|_{\mathcal{M}_{(p;\beta)}} = \inf\{c \geq 0 : \text{(2.8) holds}\}.$$  

It is not hard to see that $\mathcal{M}_{(p;\beta)}$ is a complex Banach space under $\|f\|_{\mathcal{M}_{(p;\beta)}} < \infty$. For any given operator $L$, let

$$\Theta(L) = \sup \{\epsilon > 0 : \text{(2.3) holds}\} \tag{2.9}$$

and write

$$\mathcal{M}_p = \left\{ \begin{array}{ll}
\mathcal{M}_{(p;\Theta(L))} & \text{if } \Theta(L) < \infty; \\
\bigcup_{\beta: 0 < \beta < \infty} \mathcal{M}_{(p;\beta)} & \text{if } \Theta(L) = \infty.
\end{array} \right.$$  

Note that if $L = \triangle$ or $L = \sqrt{\triangle}$ on $\mathbb{R}^n$, then $\Theta(\triangle) = \infty$ or $\Theta(\sqrt{\triangle}) = 1$.

For any $(x,t) \in \mathbb{R}^n \times (0, +\infty) = \mathbb{R}^{n+1}_+$ and $f \in \mathcal{M}_p$, define

$$P_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^n} p_t(x,y) f(y) \, dy \tag{2.10}$$

and

$$Q_t f(x) = tL e^{-tL} f(x) = \int_{\mathbb{R}^n} -t \left( \frac{d}{dt} p_t(x,y) \right) f(y) \, dy \tag{2.11}.$$  

It follows from the estimate (2.4) that the operators $P_t f$ and $Q_t f$ are well defined. Moreover, the operator $Q_t$ has the following two properties:

(i) For any $t_1, t_2 > 0$ and almost all $x \in \mathbb{R}^n$,

$$Q_{t_1} Q_{t_2} f(x) = t_1 t_2 \left( \frac{d^2 P_t}{dt^2} \bigg|_{t=t_1+t_2} f \right)(x);$$

(ii) The kernel $q_{t^m}(x,y)$ of $Q_{t^m}$ satisfies

$$|q_{t^m}(x,y)| \leq ct^{-n} g \left( \frac{|x - y|}{t} \right) \tag{2.12}$$

where the function $g$ satisfies the condition (2.3).
3. Basic properties

3.1. A comparison of definitions. Assume that $L$ is an operator which generates a semigroup $e^{-tL}$ with the heat kernel bounds (2.2) and (2.3). In what follows, $B(x, t)$ denotes the ball centered at $x$ and of the radius $t$. Given $B = B(x, t)$ and $\lambda > 0$, we will write $\lambda B$ for the $\lambda$-dilate ball, which is the ball with the same center $x$ and with radius $\lambda t$.

Definition 3.1. Let $1 \leq p < \infty$ and $\lambda \in (0, n)$. We say that

(i) $f \in L^p_{loc}(\mathbb{R}^n)$ belongs to $L^p,\lambda(\mathbb{R}^n)$ provided (1.1) holds;
(ii) $f \in M_p$ associated with an operator $L$, is in $L^p,\lambda L(\mathbb{R}^n)$ provided (1.2) holds.

Remark 3.2.
(i) Note first that $(L^p,\lambda(\mathbb{R}^n), \| \cdot \|_{L^p,\lambda})$ and $(L^p,\lambda L(\mathbb{R}^n), \| \cdot \|_{L^p,\lambda L})$ are vector spaces with the seminorms vanishing on constants and

$$K_{L,p} = \left\{ f \in M_p : P_t f(x) = f(x) \text{ for almost all } x \in \mathbb{R}^n \text{ for all } t > 0 \right\},$$

respectively. Of course, the spaces $L^p,\lambda(\mathbb{R}^n)$ and $L^p,\lambda L(\mathbb{R}^n)$ are understood to be modulo constants and $K_{L,p}$, respectively. See Section 6 of [8] for a discussion of the dimensions of $K_{L,2}$ when $L$ is a second order elliptic operator of divergence form or a Schrödinger operator.

(ii) We now give a list of examples of $L^p,\lambda(\mathbb{R}^n)$ in different settings.

(a) Define $P_t$ by putting $p_t(x, y)$ to be the heat kernel or the Poisson kernel:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} \quad \text{or} \quad p_t(x, y) = \frac{c_n t}{(t^2 + |x-y|^2)^{(n+1)/2}} \quad \text{where} \quad c_n = \frac{\Gamma(n+1)}{\pi^{n/2}}.$$

Then we will show that the corresponding space $L^p,\lambda L(\mathbb{R}^n)$ (modulo $K_{L,p}$) coincides with the classical $L^p,\lambda(\mathbb{R}^n)$ (modulo constants).

(b) Consider the Schrödinger operator with a non-negative potential $V(x)$:

$$L = \triangle + V(x).$$

To study singular integral operators associated to $L$ such as functional calculi $f(L)$ or Riesz transform $\nabla L^{-1/2}$, it is useful to choose $P_t$ with kernel $p_t(x, y)$ to be the heat kernel (or Poisson kernel) of $L$. By domination, its kernel $p_t(x, y)$ has a Gaussian upper bound (or a Poisson bound).

The following proposition shows that $L^p,\lambda(\mathbb{R}^n)$ is a subspace of $L^p,\lambda L(\mathbb{R}^n)$ in many cases.
Proposition 3.3. Let \( 1 \leq p < \infty \) and \( \lambda \in (0, n) \). Given an operator \( L \) which generates a semigroup \( e^{-tL} \) with the heat kernel bounds (2.2) and (2.3). A necessary and sufficient condition for the classical space \( L^{p,\lambda}(\mathbb{R}^n) \subseteq L^{p,\lambda}_L(\mathbb{R}^n) \) with
\[
(3.1) \quad \|f\|_{L^{p,\lambda}_L} \leq c\|f\|_{L^{p,\lambda}}
\]
is that for every \( t > 0, \) \( e^{-tL}(1) = 1 \) almost everywhere, that is, \( \int_{\mathbb{R}^n} p_t(x,y)dy = 1 \) for almost all \( x \in \mathbb{R}^n \).

Proof. Clearly, the condition \( e^{-tL}(1) = 1, \text{ a.e.} \) is necessary for \( L^{p,\lambda}(\mathbb{R}^n) \subseteq L^{p,\lambda}_L(\mathbb{R}^n) \).
Indeed, let us take \( f = 1 \). Then, (3.1) implies \( \|1\|_{L^{p,\lambda}_L} = 0 \) and thus for every \( t > 0, \) \( e^{-tL}(1) = 1 \) almost everywhere.

For the sufficiency, we borrow the idea of [16, Proposition 3.1]. To be more specific, suppose \( f \in L^{p,\lambda}(\mathbb{R}^n) \). Then for any Euclidean open ball \( B \) with radius \( r_B \), we compute
\[
\|f - P_t f\|_{L^p(B)} \leq \|f - f_B\|_{L^p(B)} + \|f_B - P_t f\|_{L^p(B)}
\]
\[
\leq \|f\|_{L^{p,\lambda}_B} \left( \int_{B} \left( \int_{\mathbb{R}^n} |f_B - f(y)|^{1/p} \right)^p \right)^{1/p} + \|f|_{L^{p,\lambda}_B} \left( \int_{B} (I(B) + J(B))^p \right)^{1/p},
\]
where
\( I(B) = \int_{2B} |f_B - f(y)| P_t f_B(x,y)dy \)
and
\( J(B) = \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} |f_B - f(y)| P_t f_B(x,y)dy. \)

Next we make further estimates on \( I(B) \) and \( J(B) \). Thanks to (2.2) and (2.3), we have
\[
\|I(B)\|_{L^p(B)} \leq cr_B^{-n} g(0)\|f_B - f\|_{L^1(B)} \leq cr_B^{\lambda/p} \|f\|_{L^{p,\lambda}}.
\]
Again, using (2.2) and (2.3), we derive that for \( x \in B \) and \( y \in 2^{k+1}B \setminus 2^kB, \)
\[
P_t f_B(x,y) \leq cr_B^{-n} g(2^k) \leq cr_B^{-n} 2^{-k(n+\epsilon)}, \quad k = 1, 2, \ldots,
\]
where \( \epsilon > 0 \) is a constant. Consequently,
\[
\|J(B)\|_{L^p(B)} \leq cr_B^{-n} \left( \int_{B} \left( \sum_{k=1}^{\infty} g(2^k) \int_{2^{k+1}B \setminus 2^kB} |f_B - f(y)|dy \right)^p \right)^{1/p}
\]
\[
\leq cr_B^{-n} \left( \sum_{k=1}^{\infty} g(2^k) \left( \int_{2^kB \setminus 2^kB} |f_{2^{k+1}B} - f(y)|dy + (2^k r_B)^n |f_{2^{k+1}B} - f_B| \right) \right)
\leq cr_B^{\lambda/p} \|f\|_{L^{p,\lambda}} \left( \sum_{k=1}^{\infty} 2^{-k(\epsilon + \frac{\lambda}{p})} + \sum_{k=1}^{\infty} k2^{-k} \right).
Putting these inequalities together, we find $f \in \mathcal{L}^{p,\lambda}_L(\mathbb{R}^n)$. \hfill \Box

3.2. **Fundamental characterizations.** In the argument for Proposition 3.3, we have used the following crucial fact that for any $f \in L^{p,\lambda}(\mathbb{R}^n)$ and a constant $K > 1$,

$$|f_B - f_{KB}| \leq cr_B^p \|f\|_{L^{p,\lambda}}.$$

Now, this property can be used to give a characterization of $L^{p,\lambda}(\mathbb{R}^n)$ spaces in terms of the Poisson integral. To this end, we denote the Laplacian by $\Delta = -\sum_{i=1}^n \partial_{x_i}^2$ and $e^{-t\sqrt{\Delta}}$ the Poisson semigroup on $\mathbb{R}^n$. We observe that if

$$f \in \mathcal{M}_{\sqrt{\Delta},p} = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : |f(x)|^p(1 + |x|^{n+1})^{-1} \in L^1(\mathbb{R}^n) \right\},$$

then we can define the operator $e^{-t\sqrt{\Delta}}$ by the Poisson integral as follows:

$$e^{-t\sqrt{\Delta}}f(x) = \int_{\mathbb{R}^n} p_t(x-y)f(y)dy, \quad t > 0,$$

where

$$p_t(x-y) = \frac{c_n t}{(t^2 + |x-y|^2)^{(n+1)/2}}.$$  

**Proposition 3.4.** Let $1 \leq p < \infty$, $\lambda \in (0, n)$ and $f \in \mathcal{M}_{\sqrt{\Delta},p}$. Then $f \in L^{p,\lambda}(\mathbb{R}^n)$ if and only if

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \left( \sup_{(x,t) \in \mathbb{R}^{n+1}_+} t^{n-\lambda} e^{-t\sqrt{\Delta}} \left( |f - e^{-t\sqrt{\Delta}} f(x)|^p \right)(x) \right)^{1/p} < \infty.$$

**Proof.** On the one hand, assume (3.2). Note that $|y - x| < t$ implies

$$\frac{c_n t}{(t^2 + |y - x|^2)^{(n+1)/2}} \geq ct^{-n}.$$

For a fixed ball $B = B(x, r_B)$ centered at $x$, we let $t_B = r_B$. We then have

$$r_B^{-\lambda} \|f - f_B\|_{L^p(B)}^p \leq cr_B^{-\lambda} \|f - e^{-t_B\sqrt{\Delta}} f(x)\|_{L^p(B)}^p \leq cr_B^{-\lambda} \int_B |f(y) - e^{-t_B\sqrt{\Delta}} f(x)|^p \frac{c_n t_B}{(t_B^2 + |y - x|^2)^{(n+1)/2}} dy \leq c \|f\|_{L^{p,\lambda}},$$

whence producing $f \in L^{p,\lambda}(\mathbb{R}^n)$.

On the other hand, suppose $f \in L^{p,\lambda}(\mathbb{R}^n)$. In a similar manner to proving the sufficiency part of Proposition 3.3, we obtain that if $(x,t) \in \mathbb{R}^{n+1}_+$ then

$$e^{-t\sqrt{\Delta}} \left( |f - e^{-t\sqrt{\Delta}} f(x)|^p \right)(x) \leq ct^{-n} \|f\|_{L^{p,\lambda}}^p + c \sum_{j=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} \frac{|f(y) - f_B|^p}{(t^2 + |y - x|^2)^{n+1}} dy \leq ct^{-n} \|f\|_{L^{p,\lambda}}^p,$$

and hence (3.2) holds. \hfill \Box
Remark 3.5. Since a simple computation gives
\[ e^{-t\sqrt{\Delta}}(|f - e^{-t\sqrt{\Delta}} f(x)|^2)(x) \]
\[ = \int_{\mathbb{R}^n} (f(y) - e^{-t\sqrt{\Delta}} f(x))(f(y) - e^{-t\sqrt{\Delta}} f(x)) p_t(x - y) dy \]
\[ = \int_{\mathbb{R}^n} |f(y)|^2 p_t(x - y) dy - e^{-t\sqrt{\Delta}} f(x) \left( \int_{\mathbb{R}^n} f(y) p_t(x - y) dy \right) \]
\[ - e^{-t\sqrt{\Delta}} f(x) \left( \int_{\mathbb{R}^n} f(y) p_t(x - y) dy \right) + |e^{-t\sqrt{\Delta}} f(x)|^2 \]
we have that if \( f \in M_{\sqrt{\Delta},2} \) then \( f \in L^{2,\lambda}(\mathbb{R}^n) \) when and only when
\[ \sup_{(x,t) \in \mathbb{R}^{n+1}_+} t^{n-\lambda} \left( e^{-t\sqrt{\Delta}} |f|^2(x) - |e^{-t\sqrt{\Delta}} f(x)|^2 \right) < \infty \]
which is equivalent to (see also [15] for the BMO-setting, i.e., \( \lambda = n \))
\[ \sup_{(x,t) \in \mathbb{R}^{n+1}_+} t^{n-\lambda} \int_{\mathbb{R}^{n+1}_+} G_{\mathbb{R}^{n+1}}((x,t), (y,s)) |\nabla_{y,s} e^{-s\sqrt{\Delta}} f(y)|^2 dy ds < \infty, \]
where \( G_{\mathbb{R}^{n+1}}((x,t), (y,s)) \) is the Green function of \( \mathbb{R}^{n+1}_+ \) and \( \nabla_{y,s} \) is the gradient operator in the space-time variable \((y,s)\).

To find out an \( L^{p,\lambda}_L(\mathbb{R}^n) \) analog of Proposition 3.3, we take Proposition 2.6 of [7] into account, and establish the following property of the class of operators \( P_t \).

**Lemma 3.6.** Let \( 1 \leq p < \infty \) and \( \lambda \in (0,n) \). Suppose \( f \in L^{p,\lambda}_L(\mathbb{R}^n) \). Then:

(i) For any \( t > 0 \) and \( K > 1 \), there exists a constant \( c > 0 \) independent of \( t \) and \( K \) such that
\[ |P_tf(x) - P_{Kt}f(x)| \leq c t^{\frac{\lambda-n}{pm}} \|f\|_{L^{p,\lambda}_L} \]
for almost all \( x \in \mathbb{R}^n \).

(ii) For any \( \delta > 0 \), there exists \( c(\delta) > 0 \) such that
\[ \int_{\mathbb{R}^n} \frac{t^{\delta/m}}{(t^{1/m} + |x-y|^{n+\delta})} |(I - P_t)f(y)| dy \leq c(\delta) t^{\frac{\lambda-n}{pm}} \|f\|_{L^{p,\lambda}_L} \]
for any \( x \in \mathbb{R}^n \).

**Proof.** (i) For any \( t > 0 \), we choose \( s \) such that \( t/4 \leq s \leq t \). Assume that \( f \in L^{p,\lambda}_L(\mathbb{R}^n) \), where \( 1 \leq p < \infty \) and \( \lambda \in (0,n) \), we estimate the term \( |P_tf(x) - P_{t+s}f(x)| \). Using the commutative property of the semigroup \( \{P_t\}_{t>0} \), we can write
\[ P_tf(x) - P_{t+s}f(x) = P_t(f - P_s f)(x). \]
Since \( f \in L^p_{\lambda} (\mathbb{R}^n) \), one has

\[
|P_t f(x) - P_{t+s} f(x)| \leq \int_{\mathbb{R}^n} |p_t(x, y)| |f(y) - P_s f(y)| dy
\]

\[
\leq \frac{c}{|B(x, t^{1/m})|} \int_{\mathbb{R}^n} \left( 1 + \frac{|x - y|}{t^{1/m}} \right)^{-(n+\epsilon)} |f(y) - P_s f(y)| dy
\]

\[
\leq c \left( \frac{1}{|B(x, s^{1/m})|} \int_{B(x, s^{1/m})} |f(y) - P_s f(y)|^p dy \right)^{1/p}
\]

\[
+ \frac{c}{|B(x, s^{1/m})|} \int_{B(x, s^{1/m})^c} \left( 1 + \frac{|x - y|}{s^{1/m}} \right)^{-(n+\epsilon)} |f(y) - P_s f(y)| dy
\]

\[
\leq c s^{\frac{\lambda-n}{pm}} \|f\|_{L^p_\lambda} + I.
\]

We then decompose \( \mathbb{R}^n \) into a geometrically increasing sequence of concentric balls, and obtain

\[
I = c \sum_{k=0}^{\infty} \frac{1}{|B(x, s^{1/m})|} \int_{B(x, 2^{k+1}s^{1/m}) \setminus B(x, 2^{k}s^{1/m})} \left( 1 + \frac{|x - y|}{s^{1/m}} \right)^{-(n+\epsilon)} |f(y) - P_s f(y)| dy
\]

\[
\leq c \sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \frac{1}{|B(x, s^{1/m})|} \int_{B(x, 2^{k+1}s^{1/m})} |f(y) - P_s f(y)| dy,
\]

since

\[
(1 + s^{-1/m}|x - y|)^{-n-\epsilon} \leq c2^{-k(n+\epsilon)} \quad \forall y \in B(x, 2^{k+1}s^{1/m}) \setminus B(x, 2^{k}s^{1/m}).
\]

For a fixed positive integer \( k \), we consider the ball \( B(x, 2^k s^{1/m}) \). This ball is contained in the cube \( Q[x, 2^k s^{1/m}] \) centered at \( x \) and of the side length \( 2^k s^{1/m} \). We then divide this cube \( Q[x, 2^k s^{1/m}] \) into \( [2k+1 ([\sqrt{n}] + 1)]^n \) small cubes \( \{Q_{x_{ki}}\}_{i=1}^{N_k} \) centered at \( x_{ki} \) and of equal side length \( ([\sqrt{n}] + 1)^{-1} s^{1/m} \), where \( N_k = [2^{k+1}([\sqrt{n}] + 1)]^n \). For any \( i = 1, 2, \ldots, N_k \), each of these small cubes \( Q_{x_{ki}} \) is then contained in the corresponding ball \( B_{k_i} \) with the same center \( x_{k_i} \) and radius \( r = s^{1/m} \). Consequently, for any ball \( B(x, 2^k t), k = 1, 2, \ldots \), there exists a corresponding collection of balls \( B_{k_1}, B_{k_2}, \ldots, B_{k_{N_k}} \) such that

(i) each ball \( B_{k_i} \) is of the radius \( t \);

(ii) \( B(x, 2^k s^{1/m}) \subset \bigcup_{i=1}^{N_k} B_{k_i} \);

(iii) there exists a constant \( c > 0 \) independent of \( k \) such that \( N_k \leq c2^{kn} \);

(iv) each point of \( B(x, 2^k s^{1/m}) \) is contained in at most a finite number \( M \) of the balls \( B_{k_i} \), where \( M \) is independent of \( k \).
Applying the properties (i), (ii), (iii) and (iv) above, we obtain

\[
I \leq c \sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \frac{1}{|B(x,s^{1/m})|} \int_{\bigcup_{i=1}^{N_{k+1}} B_{k_i}} |f(y) - P_t f(y)| dy
\]

\[
\leq c \sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} \sum_{i=1}^{N_{k+1}} \frac{1}{|B_{k_i}|} \int_{B_{k_i}} |f(y) - P_s f(y)| dy
\]

\[
\leq c \sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} N_{k+1} \sup_{i:1\leq i \leq N_{k+1}} \left( \frac{1}{|B_{k_i}|} \int_{B_{k_i}} |f(y) - P_s f(y)|^p dy \right)^{1/p}
\]

\[
\leq c \sum_{k=0}^{\infty} 2^{-k(n+\epsilon)} 2^{kn} s^{\frac{n-\lambda}{m}} \|f\|_{L^p_{\lambda}}
\]

\[
\leq c s^{\frac{n-\lambda}{m}} \|f\|_{L^p_{\lambda}},
\]

which gives (3.3) for the case \( t/4 \leq s \leq t \).

For the case \( 0 < s < t/4 \), we write

\[
P_t f(x) - P_{t+s} f(x) = (P_t f(x) - P_{2t} f(x)) - (P_{t+s} (f - P_{t-s} f)(x)).
\]

Noting that \((t + s)/4 \leq (t - s) < t + s\), we obtain (3.3) by using the same argument as above. In general, for any \( K > 1 \), let \( l \) be the integer satisfying \( 2^l \leq K < 2^{l+1} \), hence \( l \leq 2\log K \). This, together with the fact that \( \lambda \in (0, n) \), imply that there exists a constant \( c > 0 \) independent of \( t \) and \( K \) such that

\[
|P_t f(x) - P_{Kt} f(x)| \leq \sum_{k=0}^{l-1} |P_{2^k t} f(x) - P_{2^{k+1} t} f(x)| + |P_{2^l t} f(x) - P_{Kt} f(x)|
\]

\[
\leq c \sum_{k=0}^{l-1} (2^k t)^{\frac{\lambda-\epsilon}{m}} \|f\|_{L^p_{\lambda}} + c(Kt)^{\frac{\lambda-\epsilon}{m}} \|f\|_{L^p_{\lambda}}
\]

\[
\leq ct^{\frac{\lambda-\epsilon}{m}} \|f\|_{L^p_{\lambda}}
\]

for almost all \( x \in \mathbb{R}^n \).

(ii) Choosing a ball \( B \) centered at \( x \) and of the radius \( r_B = t^{1/m} \), and using (3.3), we have

\[
\left( \frac{1}{|2^k B|} \int_{2^k B} |f(y) - P_t f(y)|^p dy \right)^{1/p}
\]

\[
\leq \left( \frac{1}{|2^k B|} \int_{2^k B} |f(y) - P_{t_{2^k B}} f(y)|^p dy \right)^{1/p} + \sup_{y \in 2^k B} |P_{t_{2^k B}} f(y) - P_t f(y)|
\]

\[
(3.5) \leq ct^{\frac{\lambda-\epsilon}{m}} \|f\|_{L^p_{\lambda}}
\]
for all \( k \). Putting \( 2^{-1}B = \emptyset \), we read off

\[
\int_{\mathbb{R}^n} \frac{t^{\delta/m}}{(t^{1/m} + |x-y|)^{n+\delta}} |(I-P_t)f(y)|dy 
\leq \sum_{k=0}^{\infty} \int_{2^k B \setminus 2^{k-1} B} \frac{t^{\delta/m}}{(t^{1/m} + |x-y|)^{n+\delta}} |(I-P_t)f(y)|dy 
\leq c \sum_{k=0}^{\infty} 2^{kn} 2^{-k(n+\delta)} \left\| f \right\|_{L^p_{\delta}} \left\| f \right\|_{L_{\lambda}^\delta} 
\leq c \sum_{k=0}^{\infty} 2^{-k\delta} t^{\frac{\lambda-n}{pm}} \left\| f \right\|_{L_{\lambda}^p} 
\leq ct^{\frac{\lambda-n}{pm}} \left\| f \right\|_{L_{\lambda}^p}.
\]

The above analysis suggests us to introduce the maximal Morrey space as follows.

**Definition 3.7.** Let \( 1 \leq p < \infty \) and \( \lambda \in (0, n) \). We say that \( f \in M_p \) is in \( L_{p, \lambda}^{\max}(\mathbb{R}^n) \) associated with an operator \( L \), if there exists some constant \( c \) (depending on \( f \)) such that

\[
P_t(|f| - P_t f^p)(x) \leq ct^{\frac{\lambda-n}{pm}} \text{ for almost all } x \in \mathbb{R}^n \text{ and } t > 0.
\]

The smallest bound \( c \) for which (3.6) holds then taken to be the norm of \( f \) in this space, and is denoted by \( \| f \|_{L_{p, \lambda}^{\max}} \).

Using Lemma 3.6, we can derive a characterization in terms of the maximal Morrey space under an extra hypothesis.

**Proposition 3.8.** Let \( 1 \leq p < \infty \) and \( \lambda \in (0, n) \). Given an operator \( L \) which generates a semigroup \( e^{-tL} \) with the heat kernel bounds (2.2) and (2.3). Then \( L_{p, \lambda}(\mathbb{R}^n) \subseteq L_{p, \lambda}^{\max}(\mathbb{R}^n) \). Furthermore, if the kernels \( p_t(x, y) \) of operators \( P_t \) are nonnegative functions when \( t > 0 \), and satisfy the following lower bounds

\[
p_t(x, y) \geq \frac{c}{t^{n/m}}
\]

for some positive constant \( c \) independent of \( t, x \) and \( y \), then, \( L_{p, \lambda}^{\max}(\mathbb{R}^n) = L_{L, \max}^{p, \lambda}(\mathbb{R}^n) \).

**Proof.** Let us first prove \( L_{L, \max}^{p, \lambda}(\mathbb{R}^n) \subseteq L_{p, \lambda}^{\max}(\mathbb{R}^n) \). For any fixed \( t > 0 \) and \( x \in \mathbb{R}^n \), we choose a ball \( B \) centered at \( x \) and of the radius \( r_B = t^{1/m} \). Let \( f \in L_{L, \max}^{p, \lambda}(\mathbb{R}^n) \). To estimate
we use the decay of function \( g \) in (2.3) to get
\[
|P_t(|f - \overline{P_tf}|^p)(x)| \leq \int_{\mathbb{R}^n} |p_t(x, y)||f(y) - P_t f(y)|^p dy
\]
\[
\leq c \sum_{k=0}^{\infty} \frac{1}{|B|} \int_{2^k B \setminus 2^{k-1} B} g \left( \frac{|x - y|}{t^{1/m}} \right) |f(y) - P_t f(y)|^p dy
\]
\[
\leq c \sum_{k=0}^{\infty} 2^k n \int_{2^k B} |f(y) - P_t f(y)|^p dy
\]
\[
\leq c \sum_{k=0}^{\infty} 2^k n \left( 2^{(k-1)} \right) t^{\frac{\lambda - m}{m}} \|f\|^p_{L^p_{L,\lambda}}
\]
\[
\leq ct \frac{\lambda - m}{m} \|f\|^p_{L^p_{L,\lambda}}
\]
This proves \( \|f\|^p_{L^p_{L,\lambda}} \leq c \|f\|^p_{L^p_{L,\lambda}} \).

We now prove \( L^p_{L,\max}(\mathbb{R}^n) \subseteq L^p_{L,\lambda}(\mathbb{R}^n) \) under (3.7). For a fixed ball \( B = B(x, r_B) \) centered at \( x \), we let \( t_B = r_B^m \). For any \( f \in L^p_{L,\max}(\mathbb{R}^n) \), it follows from (3.7) that one has
\[
\frac{1}{|B|} \int_B |f(y) - P_{t_B} f(y)|^p dy \leq c \int_{B(x, r_B^m)} p_{t_B}(x, y)|f(y) - P_{t_B} f(y)|^p dy
\]
\[
\leq c \int_{\mathbb{R}^n} p_{t_B}(x, y)|f(y) - P_{t_B} f(y)|^p dy
\]
\[
\leq ct \frac{\lambda - m}{m} \|f\|^p_{L^p_{L,\max}}
\]
which proves \( \|f\|^p_{L^p_{L,\max}} \leq c \|f\|^p_{L^p_{L,\max}} \). Hence, the proof of Proposition 3.8 is complete. □

4. An identity for the dual pairing

4.1. A dual inequality and a reproducing formula. From now on, we need the following notation. Suppose \( B \) is an open ball centered at \( x_B \) with radius \( r_B \) and \( f \in \mathcal{M}_p \). Given an \( L^q \) function \( g \) supported on a ball \( B \), where \( \frac{1}{q} + \frac{1}{p} = 1 \). For any \( (x, t) \in \mathbb{R}^{n+1}_+ \), let
\[
F(x, t) = Q_{t^m}(I - P_{t^m})f(x) \quad \text{and} \quad G(x, t) = Q_{t^*}^m(I - P_{t^*}^m)g(x),
\]
where \( P^* \) and \( Q^* \) are the adjoint operators of \( P \) and \( Q \), respectively.

**Lemma 4.1.** Assume that \( L \) satisfies the assumptions (a) and (b) of Section 2.2. Suppose \( f, g, F, G, p, q \) are as in (4.4).

(i) If \( f \) also satisfies
\[
\|f\|^p_{L^p_{L,\lambda}} = \sup_{B \subset \mathbb{R}^n} r_B^{\frac{\lambda}{m}} \left\{ \int_0^{r_B} |Q_{t^m}(I - P_{t^m})f(x)|^2 \frac{dt}{t} \right\}^{1/2} \|f\|^p_{L^p(B)} < \infty,
\]
where the supremum is taken over all open ball \( B \subset \mathbb{R}^n \) with radius \( r_B \), then there exists a constant \( c > 0 \) independent of any open ball \( B \) with radius \( r_B \) such that

\[
\int_{\mathbb{R}^{n+1}_+} |F(x,t)G(x,t)| \frac{dxdt}{t} \leq cr_B^{\lambda/p} \| f \|_{L_p^{\lambda}} \| g \|_{L_q}.
\]

(ii) If

\[
h \in L^q(\mathbb{R}^n), \quad b_m = \frac{36m}{5} \quad \text{and} \quad 1 = b_m \int_0^{\infty} t^{2m} e^{-2m} (1 - e^{-t}) \frac{dt}{t},
\]

then

\[
h(x) = b_m \int_0^{\infty} (Q_{tm}^2(I - P_{tm}^*) h(x)) \frac{dt}{t},
\]

where the integral converges strongly in \( L^q(\mathbb{R}^n) \).

Proof. (i) For any ball \( B \subset \mathbb{R}^n \) with radius \( r_B \), we still put

\[
T(B) = \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in B, \ 0 < t < r_B\}.
\]

We then write

\[
\int_{\mathbb{R}^{n+1}_+} |F(x,t)G(x,t)| \frac{dxdt}{t} = \int_{T(4B)} |F(x,t)G(x,t)| \frac{dxdt}{t} + \sum_{k=1}^{\infty} \int_{T(2^{k+1}B) \setminus T(2^kB)} |F(x,t)G(x,t)| \frac{dxdt}{t}
\]

\[
= A_1 + \sum_{k=2}^{\infty} A_k.
\]

Recall that \( q > 1 \) and \( \frac{1}{q} + \frac{1}{p} = 1 \). Using the Hölder inequality, together with (2.7) (here \( \psi(z) = z e^{-z} \)), we obtain

\[
A_1 \leq \left\| \left\{ \int_0^{r_2B} |Q_{tm}(I - P_{tm}) f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p(2B)} \times \left\| \left\{ \int_0^{r_2B} |Q_{tm}^* (I - P_{tm}^*) g(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^q(2B)}
\]

\[
\leq \left\{ \int_0^{r_2B} |Q_{tm}(I - P_{tm}) f(x)|^2 \frac{dt}{t} \right\}^{1/2} \| G_{L^*}((I - P_{tm}^*) g) \|_{L^q}
\]

\[
\leq cr_B^{\lambda/p} \| f \|_{L_p^{\lambda}} \| g \|_{L_q}.
\]
Let us estimate $A_k$ for $k = 2, 3, \ldots$. Note that for $x \in T(2^{k+1}B) \setminus T(2^kB)$ and $y \in B$, we have that $|x - y| \geq 2^k r_B$. Using (2.4) and the commutative property of $\{P_t\}_{t > 0}$, we get

$$|Q_{t_n}^m (I - P_{r_B}^*) g(x)| \leq |Q_{t_n}^m g(x)| + c \left( \frac{t}{t + r_B} \right)^m |Q_{t_n + r_B^*}^m g(x)|$$

$$\leq c \int_B \frac{t^e}{(t + |x - y|)^{n+e}} |g(y)| dy + c \left( \frac{t}{r_B} \right)^m \int_B \frac{r_B^e}{(r_B + |x - y|)^{n+e}} |g(y)| dy$$

$$\leq c \frac{t^e}{(2^k r_B)^{n+e}} \int_B |g(y)| dy \leq c \frac{t^e}{(2^k r_B)^{n+e}} \int_B |g(y)| dy$$

where $e_0 = 2^{-1} \min(m, e)$ and $q = p/(p - 1)$. Consequently,

$$\left\| \int_0^{2^k r_B} |Q_{t_n}^m (I - P_{r_B}^*) g(x)| dx \left( T(2^{k+1}B) \setminus T(2^kB) \right)^{2 \frac{dt}{t}} \right\|_{L^q(2^k B)} \leq c 2^{k(n-\frac{1}{q}-1)} \|g\|_{L^q}.$$ 

Therefore,

$$A_k \leq \left\| \int_0^{2^k r_B} \left| Q_{t_n}^m (I - P_{r_B}^*) f(x) \right|^2 \frac{dt}{t} \right\|_{L^p(2^k B)}^{1/2} \times \left\| \int_0^{2^k r_B} \left| Q_{t_n}^m (I - P_{r_B}^*) g(x) \chi_{T(2^{k+1}B) \setminus T(2^kB)} \right|^2 \frac{dt}{t} \right\|_{L^q(2^k B)}^{1/2}$$

$$\leq c \frac{2^{k(n-\frac{1}{q}/q)}}{r_B} \|f\|_{L^p_r} \|g\|_{L^q_r}$$

Since $\lambda \in (0, n)$, we have

$$\int_{\mathbb{R}^{n+1}} |F(x, t)G(x, t)| \frac{dxdt}{t} \leq cr_B \|f\|_{L^p} \|g\|_{L^q} + c \sum_{k=1}^{\infty} 2^{k(n-\frac{1}{q})/q} r_B \|f\|_{L^p_r} \|g\|_{L^q}$$

as desired.

(ii) From Lemma 2.2 we know that $L$ has a bounded $H_\infty$-calculus in $L^q$ for all $q > 1$. This, together with elementary integration, shows that the set $\{g_{\alpha, \beta}(L^*)\}$ is a uniformly bounded net in $L(L^q, L^q)$, where

$$g_{\alpha, \beta}(L^*) = b_m \int_{\alpha}^{\beta} (Q_{t_n}^m)^2 (I - P_{t_n^*}) \frac{dt}{t}$$

for all $0 < \alpha < \beta < \infty$. 


As a consequence of Lemma 2.1, we have that for any $h \in L^q(\mathbb{R}^n)$,

$$h(x) = b_m \int_0^{\infty} (Q_{t_m}^* t_m^2) (I - P_{t_m}^*) h(x) \frac{dt}{t}$$

where $b_m = \frac{36m^5}{5}$ and the integral is strongly convergent in $L^q(\mathbb{R}^n)$. □

4.2. The desired dual identity. Next, we establish the following dual identity associated with the operator $L$.

**Proposition 4.2.** Assume that $L$ satisfies the assumptions (a) and (b) of Section 2.2. Suppose $B, f, g, F, G, p, q$ are defined as in (4.1). If $\| f \|_{L^p, \lambda} < \infty$ and $b_m = \frac{36m^5}{5}$, then

$$\int_{\mathbb{R}^n} f(x)(I - P_{t_m}^*) g(x) dx = b_m \int_{\mathbb{R}^n+1} F(x, t) G(x, t) \frac{dx dt}{t} .$$

**Proof.** From Lemma 4.1 (i) it turns out that

$$\int_{\mathbb{R}^n+1} \left| F(x, t) G(x, t) \right| dx dt / t < \infty.$$

By the dominated convergence theorem, the following integral converges absolutely and satisfies

$$\int_{\mathbb{R}^n+1} F(x, t) G(x, t) \frac{dx dt}{t} = \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\delta}^{N} \int_{\mathbb{R}^n} F(x, t) G(x, t) \frac{dx dt}{t} .$$

Next, by Fubini’s theorem, together with the commutative property of the semigroup $\{e^{-tL}\}_{t>0}$, we have

$$\int_{\mathbb{R}^n} F(x, t) G(x, t) dx = \int_{\mathbb{R}^n} f(x)(Q_{t_m}^* t_m^2) (I - P_{t_m}^*) (I - P_{t_m}^*) g(x) dx, \quad \forall t > 0.$$ This gives

$$\int_{\mathbb{R}^n+1} F(x, t) G(x, t) \frac{dx dt}{t}$$

$$= \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\delta}^{N} \int_{\mathbb{R}^n} f(x)(Q_{t_m}^* t_m^2) (I - P_{t_m}^*) (I - P_{t_m}^*) g(x) dx \frac{dt}{t}$$

$$= \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\mathbb{R}^n} f(x) \int_{\delta}^{N} (Q_{t_m}^* t_m^2) (I - P_{t_m}^*) (I - P_{t_m}^*) g(x) dx \frac{dt}{t} dx$$

$$= \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\mathbb{R}^n} f_1(x) \left[ \int_{\delta}^{N} (Q_{t_m}^* t_m^2) (I - P_{t_m}^*) (I - P_{t_m}^*) g(x) dx \frac{dt}{t} \right] dx$$

$$+ \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\mathbb{R}^n} f_2(x) \left[ \int_{\delta}^{N} (Q_{t_m}^* t_m^2) (I - P_{t_m}^*) (I - P_{t_m}^*) g(x) dx \frac{dt}{t} \right] dx$$

$$= I + II ,$$

where $f_1 = f \chi_{AB}$, $f_2 = f \chi_{(4B)c}$ and $\chi_E$ stands for the characteristic function of $E \subseteq \mathbb{R}^n$. 

We first consider the term I. Since \( g \in L^q(B) \), where \( q = p/(p - 1) \), we conclude
\[
(I - P_{r_B^m}^\ast)g \in L^q.
\]
By Lemma 4.1 (ii), we obtain
\[
(I - P_{r_B^m}^\ast)g = \lim_{\delta \to 0} \lim_{N \to \infty} b_m \int_\delta^N (Q_{t_m}^\ast)^2 (I - P_{t_m}^\ast)(I - P_{r_B^m}^\ast)(g) \frac{dt}{t}
\]
in \( L^q \). Hence
\[
I = \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\mathbb{R}^n} f_1(x) \left[ \int_\delta^N (Q_{t_m}^\ast)^2 (I - P_{t_m}^\ast)(I - P_{r_B^m}^\ast)(g) \frac{dt}{t} \right] dx
\]
\[
= b_m^{-1} \int_{\mathbb{R}^n} f_1(x)(I - P_{r_B}^\ast)g(x) dx.
\]
In order to estimate the term II, we need to show that for all \( y \not\in 4B \), there exists a constant \( c = c(g, L) \) such that
\[
(4.5) \quad \sup_{\delta > 0, N > 0} \left| \int_\delta^N (Q_{t_m}^\ast)^2 (I - P_{t_m}^\ast)(I - P_{r_B^m}^\ast)(g) \frac{dt}{t} \right| \leq c(1 + |x - x_0|)^{-\epsilon}.
\]
To this end, set
\[
\Psi_{t,s}(L^\ast)h(y) = (2t^m + s^m)^3 \frac{d^3 P_s^\ast}{ds^3} \bigg|_{s=2t^m+s^m}(I - P_{t_m}^\ast)h(y).
\]
Note that
\[
(I - P_{r_B^m}^\ast)g = m \int_0^{r_B} Q_{s_m}^\ast(g) s^{-1} ds.
\]
So, we use (2.3) and (2.4) to deduce
\[
\left| \int_\delta^N (Q_{t_m}^\ast)^2 (I - P_{t_m}^\ast)(I - P_{r_B^m}^\ast)(g) \frac{dt}{t} \right|
\]
\[
= \left| \int_\delta^N \int_0^{r_B} (Q_{t_m}^\ast)^2 Q_{s_m}^\ast(I - P_{t_m}^\ast)g(x) \frac{ds}{s} \frac{dt}{t} \right|
\]
\[
\leq c \int_\delta^N \int_0^{r_B} \frac{t^{2m} s^{m}}{(t^m + s^m)^3} |\Psi_{t,s}(L^\ast)g(x)| \frac{ds}{s} \frac{dt}{t}
\]
\[
\leq c \int_\delta^N \int_0^{r_B} \int_{B(x_0, r_B)} \frac{t^{2m} s^{m}}{(t^m + s^m)^3} (t + s)^\epsilon \left| g(y) \right| \frac{dy ds dt}{s}. \]

Because \( x \not\in 4B \) yields \( |x - y| \geq |x - x_0|/2 \), the inequality
\[
\frac{t^{2m} s^{m}(t + s)^\epsilon}{(t^m + s^m)^3} \leq c \min \left( (ts)^{\epsilon/2}, t^{-\epsilon/2} s^{3\epsilon/2} \right),
\]
together with Hölder’s inequality and elementary integration, produces a positive constant \( c \) independent of \( \delta, N > 0 \) such that for all \( x \not\in 4B \),
\[
\left| \int_\delta^N Q_{t_m}^2 (I - P_{t_m})g(y) \frac{dt}{t} \right| \leq c r_B^\epsilon |x - x_0|^{-\epsilon} \left\| g \right\|_{L^1} \leq c r_B^{\epsilon + \frac{\alpha}{2}} \left\| g \right\|_{L^2} |x - x_0|^{-\epsilon}
\]
Accordingly, (4.5) follows readily.
We now estimate the term II. For \( f \in \mathcal{M}_p \), we derive \( f \in L^p((1 + |x|)^{-(n+\alpha)}dx) \). The estimate (4.5) yields a constant \( c > 0 \) such that

\[
\sup_{\delta > 0, N > 0} \int_{\mathbb{R}^n} \left| f_2(x) \int_\delta^N (Q_{\ell_n}^*)^2(\mathcal{I} - P_{\ell_n}^*)(\mathcal{I} - P_{r_B^*}) (g)(x) \frac{dt}{t} \right| dx \leq c.
\]

This allows us to pass the limit inside the integral of II. Hence

\[
\begin{align*}
II &= \lim_{\delta \to 0} \lim_{N \to \infty} \int_{\mathbb{R}^n} f_2(x) \left( \int_\delta^N (Q_{\ell_n}^*)^2(\mathcal{I} - P_{\ell_n}^*)(\mathcal{I} - P_{r_B^*}) (g)(x) \frac{dt}{t} \right) dx \\
&= \int_{\mathbb{R}^n} f_2(x) \left( \lim_{\delta \to 0} \lim_{N \to \infty} \left[ \int_\delta^N (Q_{\ell_n}^*)^2(\mathcal{I} - P_{\ell_n}^*)(\mathcal{I} - P_{r_B^*}) (g)(x) \frac{dt}{t} \right] \right) dx \\
&= b_m^{-1} \int_{\mathbb{R}^n} f_2(x)(\mathcal{I} - P_{r_B^*})g(x)dx.
\end{align*}
\]

Combining the previous formulas for I and II, we obtain the identity (4.3). \( \square \)

**Remark 4.3.** For a background of Proposition 4.2, see also [8, Proposition 5.1].

5. Description through Littlewood-Paley function

5.1. The space \( L^{p,\lambda}(\mathbb{R}^n) \) as the dual of the atomic space. Following [28], we give the following definition.

**Definition 5.1.** Let \( 1 < p < \infty \), \( q = p/(p - 1) \) and \( \lambda \in (0,n) \). Then

(i) A complex-valued function \( a \) on \( \mathbb{R}^n \) is called a \((q,\lambda)\)-atom provided:

(\( \alpha \)) \( a \) is supported on an open ball \( B \subset \mathbb{R}^n \) with radius \( r_B \):

(\( \beta \)) \( \int_{\mathbb{R}^n} a(x)dx = 0 \);

(\( \gamma \)) \( \|a\|_{L^q} \leq r_B^{-\lambda/p} \).

(ii) \( H^{q,\lambda}(\mathbb{R}^n) \) comprises those linear functionals admitting an atomic decomposition \( f = \sum_{j=1}^{\infty} \eta_j a_j \), where \( a_j \)’s are \((q,\lambda)\)-atoms, and \( \sum_j |\eta_j| < \infty \).

The forthcoming result reveals that \( H^{q,\lambda}(\mathbb{R}^n) \) exists as a predual of \( L^{p,\lambda}(\mathbb{R}^n) \).

**Proposition 5.2.** Let \( 1 < p < \infty \), \( q = p/(p - 1) \) and \( \lambda \in (0,n) \). Then \( L^{p,\lambda}(\mathbb{R}^n) \) is the dual \( (H^{q,\lambda}(\mathbb{R}^n))^* \) of \( H^{q,\lambda}(\mathbb{R}^n) \). More precisely, if \( h = \sum_j \eta_j a_j \in H^{q,\lambda}(\mathbb{R}^n) \) then

\[
\langle h, \ell \rangle = \lim_{k \to \infty} \sum_{j=1}^{k} \eta_j \int_{\mathbb{R}^n} a_j(x)\ell(x)dx
\]

is a well-defined continuous linear functional for each \( \ell \in L^{p,\lambda}(\mathbb{R}^n) \), whose norm is equivalent to \( \|\ell\|_{L^{p,\lambda}} \); moreover, each continuous linear functional on \( H^{q,\lambda}(\mathbb{R}^n) \) has this form.

**Proof.** See [28, Proposition 5] for a proof of Proposition 5.2 \( \square \)
5.2. Characterization of $L^{p,\lambda}(\mathbb{R}^n)$ by means of Littlewood-Paley function. We now state a full characterization of $L^{p,\lambda}(\mathbb{R}^n)$ space for $1 < p < \infty$ and $\lambda \in (0, n)$. For the case $p = 2$, see also [26, Lemma 2.1] as well as [25, Theorem 1 (i)].

**Proposition 5.3.** Let $1 < p < \infty$, $\lambda \in (0, n)$ and $f \in \mathcal{M}_{\sqrt{\Delta}, p}$. Then the following two conditions are equivalent:

(i) $f \in L^{p,\lambda}(\mathbb{R}^n)$;

(ii) $I(f, p) = \sup_{B \subset \mathbb{R}^n} r_B^{-\lambda \frac{1}{p}} \left\{ \int_0^{r_B} \left| \frac{\partial}{\partial t} e^{-t\sqrt{\Delta}} f(x) \right|^2 \frac{dt}{t} \right\}^{1/2} \|f\|_{L^p(B)} < \infty,$

where the supremum is taken over all Euclidean open ball $B \subset \mathbb{R}^n$ with radius $r_B$.

**Proof.** It suffices to verify (ii)⇒(i) for which the reverse implication follows readily from [11, Theorem 2.1]. Suppose (ii) holds. Proposition 5.2 suggests us to show $f \in (H^{p-1,\lambda}(\mathbb{R}^n))^*$ in order to verify (i). Now, let $g$ be a $(\frac{p}{p-1}, \lambda)$-atom and

$$p_t(x) = \frac{c_n t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}.$$

Then for any open ball $B \subset \mathbb{R}^n$ with radius $r_B$ and its tent

$$T(B) = \{(x, t) \in \mathbb{R}^{n+1} : x \in B, t \in (0, r_B)\},$$

we have (cf. [23, p.183])

$$|\langle f, g \rangle| = \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right|$$

$$= 4 \left| \int_{\mathbb{R}^n} \int_0^\infty \left( t \frac{\partial}{\partial t} p_t * f(x) \right) \left( t \frac{\partial}{\partial t} p_t * g(x) \right) t^{-1} dt dx \right|$$

$$\leq 4(I(B) + J(B)).$$

Here,

$$I(B) = \int_{4B} \int_0^{r_{4B}} \left| t \frac{\partial}{\partial t} p_t \ast f(x) \left| t \frac{\partial}{\partial t} p_t \ast g(x) \right| t^{-1} dt dx$$

$$\leq \left( \int_{4B} \left( \int_0^{r_{4B}} \left| t \frac{\partial}{\partial t} p_t \ast f(x) \right|^2 t^{-1} dt \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$$

$$\times \left( \int_{4B} \left( \int_0^{r_{4B}} \left| t \frac{\partial}{\partial t} p_t \ast g(x) \right|^2 t^{-1} dt \right)^{\frac{p}{2(p-1)}} dx \right)^{\frac{p-1}{p}}$$

$$\leq c r_B^{\frac{\lambda}{p}} I(f, p) \|g\|_{L^\frac{p}{p-1}(\mathbb{R}^n)}$$

$$\leq c I(f, p),$$
due to Hölder’s inequality, the $L^{p,q}_{r}$-boundedness of the Littlewood-Paley $G$-function, and $g$ being a $(\frac{p}{p-1}, \lambda)$-atom.

Meanwhile,

$$J(B) = \sum_{k=1}^{\infty} \int_{T(2^{k+1}B) \setminus T(2^kB)} \left| \frac{1}{t} \frac{\partial}{\partial t} p_t f(x) \right| \left| \frac{1}{t} \frac{\partial}{\partial t} g(x) \right| t^{-1} dt dx$$

$$\leq c \sum_{k=1}^{\infty} \left\| \left\{ \int_{0}^{2^{k+1}B} \left| \frac{1}{t} \frac{\partial}{\partial t} p_t f(x) \right|^2 t^{-1} dt \right\}^{\frac{1}{2}} \right\|_{L^p(2^{k+1}B)}$$

$$\times \left\| \left\{ \int_{0}^{2^{k+1}B} \left| \frac{1}{t} \frac{\partial}{\partial t} p_t g(x) \right|^2 t^{-1} dt \right\}^{\frac{1}{2}} \right\|_{L^q(2^{k+1}B)}$$

$$\leq c \sum_{k=1}^{\infty} (2^{k} r_B)^{\frac{1}{2}} I(f, p) 2^{-\frac{k-1}{p}} r_B^{-\lambda}$$

$$\leq c I(f, p),$$

for which we have used the Hölder inequality and the fact that if $|y - x| \geq 2^k r_B$ then

$$\left| \frac{1}{t} \frac{\partial}{\partial t} p_t g(x) \right| \leq c \frac{t^3}{(2^k r_B)^{3+n}} \|g\|_{L^1(B)} \leq c \frac{t^3}{(2^k r_B)^{3+n}} r_B^{-\lambda}$$

for the $(\frac{p}{p-1}, \lambda)$-atom $g$. Accordingly, $f \in L^{p,\lambda}_{L}(\mathbb{R}^n)$.

5.3. **Characterization of $L^{p,\lambda}_{L}(\mathbb{R}^n)$ by means of Littlewood-Paley function.** Of course, it is natural to explore a characterization of $L^{p,\lambda}_{L}(\mathbb{R}^n)$ similar to Proposition 5.3.

**Proposition 5.4.** Let $1 < p < \infty$, $\lambda \in (0, n)$ and $f \in \mathcal{M}_p$. Assume that $L$ satisfies the assumptions (a) and (b) of Section 2.2. Then the following two conditions are equivalent:

(i) $f \in L^{p,\lambda}_{L}(\mathbb{R}^n)$;

(ii) $\left\| f \right\|_{L^{p,\lambda}_{L}} = \sup_{B \subset \mathbb{R}^n} r_B^{-\frac{\lambda}{p}} \left\| \left\{ \int_{0}^{r_B} |Q_{tm}(I - P_m)f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p(B)} < \infty,$

where the supremum is taken over all Euclidean open ball $B \subset \mathbb{R}^n$ with radius $r_B$.

**Proof.** (i)⇒(ii). Suppose $f \in L^{p,\lambda}_{L}(\mathbb{R}^n)$. Note that

$$Q_{tm}(I - P_m) = Q_{tm}(I - P_m)(I - P_{tm}) + Q_{tm}(I - P_m)P_{tm}.$$ 

So, we turn to verify both

$$\left\| \left\{ \int_{0}^{r_B} |Q_{tm}(I - P_m)(I - P_{tm})f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p(B)} \leq c r_B^{-\frac{\lambda}{p}} \|f\|_{L^{p,\lambda}_{L}}$$

and

$$\left\| \left\{ \int_{0}^{r_B} |Q_{tm}(I - P_m)P_{tm}|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p(B)} \leq c r_B^{-\frac{\lambda}{p}} \|f\|_{L^{p,\lambda}_{L}}.$$
and

\begin{equation}
\left\{ \int_{0}^{r_B} |Q_{t_m}(I - P_{t_m})P_{r_B}^m f(x)|^2 \frac{dt}{t} \right\}^{1/2} \leq c r_B^\lambda \|f\|_{L^p_L},
\end{equation}

thereby proving (ii). To do so, we will adapt the argument on pp. 85-86 of [12] to present the situation – see also page 955 of [8].

To prove (5.1), let us consider the square function $G(h)$ given by

\[ G(h)(x) = \left( \int_{0}^{\infty} |Q_{t_m}(I - P_{t_m})h(x)|^2 \frac{dt}{t} \right)^{1/2}. \]

From (2.7), the function $G$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $b = b_1 + b_2$, where $b_1 = (I - P_{r_B}^m)f\chi_{2B}$, and $b_2 = (I - P_{r_B}^m)f\chi_{(2B)^c}$. Using Lemma 3.6, we obtain

\[
\left\{ \int_{0}^{r_B} |Q_{t_m}(I - P_{t_m})b_1(x)|^2 \frac{dt}{t} \right\}^{1/2} \leq c \|G(b_1)\|_{L^p} \leq c \|b_1\|_{L^p} = c \int_{2B} |(I - P_{r_B}^m)f(x)|^p dx \right\}^{1/p} + cr_B^{n/p} \sup_{x \in 2B} |P_{r_B}^m f(x) - P_{r_B}^m f(x)|^p \leq c r_B^\lambda \|f\|_{L^p_L}.
\]

(5.3)

On the other hand, for any $x \in B$ and $y \in (2B)^c$, one has $|x - y| \geq r_B$. From Proposition 3.5, we obtain

\[
|Q_{t_m}(I - P_{t_m})b_2(x)| \leq c \int_{\mathbb{R}^n \setminus 2B} \frac{t^\epsilon}{(t + |x - y|)^{n+\epsilon}} |(I - P_{r_B}^m)f(y)| dy \leq c \left( \frac{t}{r_B} \right)^\epsilon \int_{\mathbb{R}^n} \frac{r_B^\epsilon}{(r_B + |x - y|)^{n+\epsilon}} |(I - P_{r_B}^m)f(y)| dy \leq c \left( \frac{t}{r_B} \right)^\epsilon r_B^{\frac{n}{p}} \|f\|_{L^p_L},
\]

which implies

\[
\left\{ \int_{0}^{r_B} |Q_{t_m}(I - P_{t_m})b_2(x)|^2 \frac{dt}{t} \right\}^{1/2} \leq c r_B^\lambda \|f\|_{L^p_L}.
\]

This, together with (5.3), gives (5.1).

Next, let us check (5.2). This time, we have $0 < t < r_B$, whence getting from Lemma 3.6 that for any $x \in \mathbb{R}^n$,

\[
|P_{r_B}^m f(x) - P_{t_m} f(x)| \leq c r_B^{\frac{\lambda-n}{p}} \|f\|_{L^p_L}.
\]
By (2.4), the kernel $K_{t,r_B}(x,y)$ of the operator

$$Q_{tm}P_{\frac{r_B}{2}} = \frac{t^m}{r_B}Q_{tm+\frac{r_B^m}{2}}$$

satisfies

$$|K_{t,r_B}(x,y)| \leq c\left(\frac{t}{r_B}\right)^m \frac{r_B^m}{(r_B + |x-y|)^{n+\epsilon}}.$$

Using the commutative property of the semigroup $\{e^{-tB}\}_{t>0}$ and the estimate (2.4), we deduce

$$|Q_{tm}(I - P_{tm})P_{\frac{r_B}{2}}f(x)| = |Q_{tm}P_{\frac{r_B}{2}}(P_{\frac{r_B}{2}} - P_{(tm+\frac{r_B^m}{2})})f(x)| \leq c\left(\frac{t}{r_B}\right)^m \int_{\mathbb{R}^n} \frac{r_B^m}{(r_B + |x-y|)^{n+\epsilon}} |(P_{\frac{r_B}{2}} - P_{(tm+\frac{r_B^m}{2})})f(y)| dy \leq c\left(\frac{t}{r_B}\right)^m r_B^\epsilon \|f\|_{L^p,B},$$

whence deriving

$$\left\| \left\{ \int_0^{r_B} |Q_{tm}(I - P_{tm})P_{\frac{r_B}{2}}f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p(B)} \leq cr_B^\epsilon \|f\|_{L^p,B}.$$ 

This gives (5.2) and consequently (ii).

(ii) $\Rightarrow$ (i). Suppose (ii) holds. The duality argument for $L^p$ shows that for any open ball $B \subset \mathbb{R}^n$ with radius $r_B$,

$$\left( \int_B |f(x) - P_{r_B}f(x)|^p \right)^{1/p} = \sup_{\|g\|_{L^p(B)} \leq 1} \int_B (I - P_{r_B}f(x))g(x)dx \geq c \int_{\mathbb{R}^n} |Q_{tm}(I - P_{tm})f(x)| \left| Q_{tm}^\epsilon(I - P_{r_B}^\epsilon)g(x)dx \right| \frac{dxdt}{t} \leq c \|f\|_{L^p,B} \|g\|_{L^q}.$$

Using the identity (4.3), the estimate (4.2) and the Hölder inequality, we have

$$\left| \int_{\mathbb{R}^n} f(x)(I - P_{r_B}^\epsilon)g(x)dx \right| \leq c \int_{\mathbb{R}^{n+1}}|Q_{tm}(I - P_{tm})f(x)| Q_{tm}^\epsilon(I - P_{r_B}^\epsilon)g(x)| \frac{dxdt}{t} \leq c \|f\|_{L^p,B} \|g\|_{L^q}.$$ 

Substituting (5.5) back to (5.4), by Definition 3.1 we find a constant $c > 0$ such that

$$\|f\|_{L^p,B} \leq c \|f\|_{L^p,B} < \infty.$$

This just proves $f \in L^p_{r_B}((\mathbb{R}^n)$, thereby yielding (i).

\[\square\]

**Remark 5.5.** In the case of $p = 2$, we can interpret Proposition 5.4 as a measure-theoretic characterization, namely, $f \in L^2_{r_B}((\mathbb{R}^n)$ when and only when

$$d\mu_f(x,t) = |Q_{tm}(I - P_{tm})f(x)|^2 \frac{dxdt}{t}$$

where

$$d\mu_f(x,t) = \sum_{n=1}^{\infty} |Q_{tm}(I - P_{tm})f(x)|^2 \frac{dxdt}{t}.$$
is a $\lambda$-Carleson measure on $\mathbb{R}^{n+1}_+$. According to [10] Lemma 4.1, we find further that $f \in L^2_{\mathcal{L}}(\mathbb{R}^n)$ is equivalent to

$$\sup_{(y,s) \in \mathbb{R}^{n+1}_+} \int_{\mathbb{R}^n} \left( \frac{s}{(|x-y|^2 + (t+s)^2)^{\frac{n+1}{2}}} \right)^\lambda d\mu_f(x,t) < \infty.$$ 

5.4. A sufficient condition for $L_{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$. In what follows, we assume that $L$ is a linear operator of type $\omega$ on $L^2(\mathbb{R}^n)$ with $\omega < \pi/2$ -- hence $L$ generates an analytic semigroup $e^{-tL}, 0 \leq |\text{Arg}(z)| < \pi/2 - \omega$. We also assume that for each $t > 0$, the kernel $p_t(x,y)$ of $e^{-tL}$ is Hölder continuous in both variables $x, y$ and there exist positive constants $m, \beta > 0$ and $0 < \gamma \leq 1$ such that for all $t > 0$, and $x, y, h \in \mathbb{R}^n$,

$$|p_t(x,y)| \leq c \frac{t^{\beta/m}}{(t^{1/m} + |x-y|)^{n+\beta}} \quad \forall \ t > 0, \ x, y \in \mathbb{R}^n,$$

$$|p_t(x+h,y) - p_t(x,y)| + |p_t(x,y+h) - p_t(x,y)| \leq c|h|^\gamma \frac{t^{\beta/m}}{(t^{1/m} + |x-y|)^{n+\beta+\gamma}} \quad \forall h \in \mathbb{R}^n \text{ with } 2|h| \leq t^{1/m} + |x-y|,$$

and

$$\int_{\mathbb{R}^n} p_t(x,y)dx = \int_{\mathbb{R}^n} p_t(x,y)dy = 1 \quad \forall t > 0.$$

**Proposition 5.6.** Let $1 < p < \infty$ and $\lambda \in (0, n)$. Given an operator $L$ which generates a semigroup $e^{-tL}$ with the heat kernel bounds (2.2) and (2.3). Assume that $L$ satisfies the conditions (3.6), (3.7) and (3.8). Then $L_{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\lambda}(\mathbb{R}^n)$ coincide, and their norms are equivalent.

**Proof.** Since Proposition 3.3 tells us that $L^{p,\lambda}(\mathbb{R}^n) \subseteq L_{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$ under the above-given conditions, we only need to check $L_{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n) \subseteq L^{p,\lambda}(\mathbb{R}^n)$. Note that $L^{p,\lambda}(\mathbb{R}^n)$ is the dual of $\mathcal{H}^{q,\lambda}(\mathbb{R}^n)$, $q = p/(p-1)$. It reduces to prove that if $f \in L_{\mathcal{L}}^{p,\lambda}(\mathbb{R}^n)$, then $f \in (\mathcal{H}^{q,\lambda}(\mathbb{R}^n))^\ast$. Let $g$ be a $(q, \lambda)$-atom. Using the conditions (5.6), (5.7) and (5.8) of the operator $L$, together with the properties of of $(q, \lambda)$-atom of $g$, we can follow the argument for Lemma 4.1 (ii) to verify

$$\int_{\mathbb{R}^n} f(x) g(x) dx = b_m \int_{\mathbb{R}^{n+1}_+} Q_{tm}(\mathcal{I} - P_{tm}) f(x) Q^*_tm g(x) \frac{dx dt}{t} \quad \text{where} \quad b_m = \frac{36m}{5}.$$
Consequently,

\[ |⟨f, g⟩| = \left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| = \left| \int_{\mathbb{R}^n_+} Q_{t^m}(I - P_{t^m})f(x)Q_{t^m}^*g(x) \frac{dxdt}{t} \right| \]
\[ \leq \int_{T(4B)} |Q_{t^m}(I - P_{t^m})f(x)Q_{t^m}^*g(x)| \frac{dxdt}{t} \]
\[ + \sum_{k=1}^{\infty} \int_{(2^{k+1}B)\setminus (2^kB)} |Q_{t^m}(I - P_{t^m})f(x)Q_{t^m}^*g(x)| \frac{dxdt}{t} \]
\[ = D_1 + \sum_{k=2}^{\infty} D_k. \]

Define the Littlewood-Paley function \( G^* \) by

\[ G^*(h)(x) = \left[ \int_0^{\infty} |Q_{t^m}^* h(x)|^2 \frac{dt}{t} \right]^{1/2}. \]

By (2.7), \( G^* \) is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

Following the proof of Lemma 4.1 (i), together with the property \((\gamma)\) of \((q, \lambda)\)-atom \( g \), we derive

\[ D_1 \leq \left\| \left\{ \int_{0}^{r^2B} |Q_{t^m}(I - P_{t^m})f(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^p(2B)} \left\| \left\{ \int_{0}^{r^2B} |Q_{t^m}^* g(x)|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^q(2B)} \]
\[ \leq c r^\frac{\lambda}{2} \left\| f \right\|_{L^{p,\lambda}} \left\| g \right\|_{L^q} \leq c \| f \|_{L^{p,\lambda}}. \]

On the other hand, we note that for \( x \in T(2^{k+1}B)\setminus T(2^kB) \) and \( y \in B \), we have that \( |x - y| \geq 2^kr_B \). Using the estimate (2.4) and the properties \((\alpha)\) and \((\gamma)\) of \((q, \lambda)\)-atom \( g \), we obtain

\[ |Q_{t^m}^* g(x)| \leq c \int_{B} \frac{t^k}{(t + |x - y|)^{n+\epsilon}} |g(y)|dy \]
\[ \leq c \frac{t^k}{(2^kr_B)^{n+\epsilon}} \int_{B} |g(y)|dy \]
\[ \leq c \frac{t^k}{(2^kr_B)^{n+\epsilon}} \frac{r_B^\lambda}{r_B^\lambda}, \]

which implies

\[ \left\| \left\{ \int_{0}^{r^2B} |Q_{t^m}^* g(x)\chi_{T(2^{k+1}B)\setminus T(2^kB)}|^2 \frac{dt}{t} \right\}^{1/2} \right\|_{L^q(2^kB)} \leq c 2^{k(n(\frac{1}{p} - 1))r_B^\lambda}. \]
Therefore,

\[
D_k \leq \left\{ \int_0^{2k r_B} \left| Q_{t^m} (I - P_{t^m}) f(x) \right|^2 \frac{dt}{t} \right\}^{1/2} \left\| \int_0^{2k r_B} \left| Q_{t^m} g(x) \chi_{T(2^{k+1} B)} f(x) \right|^2 \frac{dt}{t} \right\}^{1/2} \leq c (2^k r_B)^{\frac{\lambda}{p} + 2^k r_B^{n-1} - \frac{\lambda}{p}} \left\| f \right\|_{L^p_{L,\lambda}}
\]

Since \( \lambda \in (0, n) \), we have

\[
|\langle f, g \rangle| \leq c \left\| f \right\|_{L^p_{L,\lambda}} + c \sum_{k=1}^{\infty} 2^{k(\lambda-n)} \left\| f \right\|_{L^p_{L,\lambda}} \leq c \left\| f \right\|_{L^p_{L,\lambda}}.
\]

This, together with Proposition 5.2, implies \( f \in \left( \mathcal{H}^{q,\lambda} (\mathbb{R}^n) \right)^* = L^{p,\lambda} (\mathbb{R}^n) \). \( \square \)

**References**

[1] D.R. Adams and J. Xiao, Nonlinear potential analysis on Morrey spaces and their capacities. *Indiana Univ. Math. J.* 53 (2004), 1629-1663.

[2] D. Albrecht, X.T. Duong and A. McIntosh, Operator Theory and Harmonic Analysis. *Workshop in Analysis and Geometry 1995*. Proceedings of the Centre for Mathematics and its Applications, ANU 34 (1996), 77-136.

[3] P. Auscher, X.T. Duong and A. McIntosh, Boundedness of Banach space valued singular integral operators and Hardy spaces. Preprint (2005).

[4] S. Campanato, Proprietà di una famiglia di spazi funzionali. *Ann Scuola Norm. Sup. Pisa* (3). 18 (1964), 137-160.

[5] D.G. Deng, X.T. Duong and L.X. Yan, A characterization of the Morrey-Campanato spaces. *Math. Z.* 250 (2005), 641-655.

[6] X.T. Duong and A. McIntosh, Singular integral operators with non-smooth kernels on irregular domains. *Rev. Mat. Iberoamerican* 15 (1999), 233-265.

[7] X.T. Duong and L.X. Yan, New function spaces of BMO type, the John-Nirenberg inequality, interpolation and applications. *Comm. Pure Appl. Math.* 58 (2005), 1375-1420.

[8] X.T. Duong, L.X. Yan, Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. *J. Amer. Math. Soc.* 18 (2005), 943-973.

[9] X.T. Duong and L.X. Yan, New Morrey-Campanato spaces associated with operators and applications. Preprint (2005).

[10] M. Essén, S. Janson, L. Peng and J. Xiao, Q Spaces of several real variables. *Indiana Univ. Math. J.* 49(2000), 575-615.

[11] E.B. Fabes, R.L. Johnson and U. Neri, Spaces of harmonic functions representable by Poisson integrals of functions in BMO and \( L^p_{\alpha,\lambda} \). *Indiana Univ. Math. J.* 25(1976), 159-170.

[12] J.L. Journé, *Calderón-Zygmund operators, pseudo-differential operators and the Cauchy integral of Calderón*. Lecture Notes in Math. 994. Springer, Berlin-New York, 1983.

[13] F. John and L. Nirenberg, On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* 14(1961), 415–426.

[14] S. Janson, M.H. Taibleson and G. Weiss, Elementary characterizations of the Morrey-Campanato spaces. *Lecture Notes in Math.* 992 (1983), 101-114.

[15] H. Leutwiler, BMO on harmonic spaces. *Univ. of Joensuu Pub. Sci.* 14(1989), 71-78.
[16] J.M. Martell, Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications. *Studia Math.* **161** (2004), 113-145.

[17] A. McIntosh, Operators which have an $L^\infty$ functional calculus. *Miniconference on operator theory and partial differential equations 1986*. Proceedings of the Centre for Mathematical Analysis, ANU **14** (1986), 210-231.

[18] C.B. Morrey, *Multiple integral problems in the calculus of variations and related topics*. Univ. of California Publ. Math. (N.S.) **1** (1943), 1-130.

[19] E.M. Ouhabaz, *Analysis of heat equations on domains*. London Math. Soc. Mono. **31**, Princeton Univ. Press, (2004).

[20] J. Peetre, On the theory of $L^{p,\lambda}$ spaces. *J. Funct. Anal.* **4** (1969), 71-87.

[21] S. Spanne, Some function spaces defined by using the mean oscillation over cubes. *Ann Scuola Norm. Sup. Pisa* **19** (1965), 593-608.

[22] G. Stampacchia, $L^{p,\lambda}$ spaces and interpolation. *Comm. Pure Appl. Math.* **17** (1964), 293-306.

[23] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*. Princeton Univ. Press, Princeton, NJ, 1993.

[24] M.E. Taylor, Analysis of Morrey spaces and applications to Navier-Stokes and other evolution equations. *Commun. P.D.E.* **17**, 1407-1456 (1992).

[25] Z.J. Wu and C.P. Xie, $Q$ spaces and Morrey spaces. *J. Funct. Anal.* **201** (2003), 282-297.

[26] J. Xiao, Towards $Q_0(\mathbb{R}^n)$ extension of $\text{BMO}(\mathbb{R}^n)$ by quadratic Campanato-Morrey space and incompressible Navier-Stokes system. Preprint, (2006).

[27] K. Yosida, *Functional Analysis* (Fifth edition). Spring-Verlag, Berlin, 1978.

[28] C.T. Zorko, Morrey space. *Proc. Amer. Math. Soc.* **98** (1986), 586-592.

DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2109, AUSTRALIA
E-mail address: duong@ics.mq.edu.au

DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN’S, NL, A1C 5S7, CANADA
E-mail address: jxiao@math.mun.ca

DEPARTMENT OF MATHEMATICS, ZHONGSHAN UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA
E-mail address: mcsylx@mail.sysu.edu.cn