Polarization, plasmon, and Debye screening in doped 3D ani-Weyl semimetal

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We compute the polarization function in a doped three-dimensional anisotropic-Weyl semimetal, in which the fermion energy dispersion is linear in two components of the momenta and quadratic in the third. Through detailed calculations, we find that the long wavelength plasmon mode depends on the fermion density $n_e$ in the form $\Omega_{\perp}^2 \propto n_e^{1/2}$ along the third direction. This unique characteristic of the plasmon mode can be probed by various experimental techniques, such as electron energy-loss spectroscopy. The Debye screening at finite chemical potential and finite temperature is also analyzed based on the polarization function.

I. INTRODUCTION

Studying the intriguing properties of various semimetals have been one of the core subjects in condensed matter physics for more than one decade [1–12], starting from the successful fabrication of monolayer graphene [11,12]. The surface state of three-dimensional (3D) topological insulator bears strong similarity to graphene, and both of these two systems are classified as two-dimensional (2D) Dirac semimetal (DSM) [13,14]. It is currently clear that there are a variety of semimetal materials, including 3D DSM [1,2], 3D Weyl semimetal (WSM) [3–8], and nodal line semimetal (NLSM) [9,10]. It is interesting that some of such semimetals provide a nice platform to realize the interesting concepts of chiral anomaly by probing the negative magnetoresistance [13,14]. In addition, there exit 2D semi-DSM [17,18], 3D double-WSM [28–40], 3D triple-WSM [30,34,37,41], and 3D anisotropic-WSM (ani-WSM) [30,44,46].

In this paper, we pay attention to 3D ani-WSM where the Weyl fermion spectrum displays linear dependence on two components of the momenta and quadratic dependence on the third [30,44,46], namely

$$E = \pm \sqrt{v^2 (k_x^2 + k_y^2) + A^2 k_z^2}. \quad (1)$$

Such 3D ani-WSM might be produced at the quantum critical point (QCP) between a normal insulator and a WSM, or at the QCP between a normal insulator and a topological insulator in 3D noncentrosymmetric systems [44]. For 3D ani-WSM, the chirality of the band-touching point is zero [44]. First principle calculations suggested that the topological quantum phase transition (QPT) from normal insulator to topological insulator in a noncentrosymmetric system can be tuned by applying pressure to BiTeI [47]. This theoretical prediction was subsequently confirmed by experiments carried out by means of x-ray powder diffraction and infrared spectroscopy techniques [48].

It is also proposed [30] that a semimetallic state in which the fermion excitations have the dispersion of Eq. (1) can emerge at the QCP between normal insulator and topological 3D DSM, or the QCP between 3D DSM and weak topological insulator or topological crystalline insulator. The analysis made by Yuan et al. [49] showed that the fermion dispersion given by Eq. (1) may be realized in ZrTe$_5$ at the QCP between insulator phase and DSM phase. Remarkably, recent quantum oscillation measurements provided important clue for the existence of such type of fermionic excitations in ZrTe$_5$ under pressure [50].

In an intrinsic semimetal, the chemical potential is exactly zero, and the density of states (DOS) vanishes at the Fermi level. As a result, the dynamically screened Coulomb interaction between the fermion excitations is still long-ranged. Extensive renormalization group (RG) [51] analysis revealed that the long-range Coulomb interaction in various intrinsic semimetals might be marginally irrelevant [12,23,25,32,44,52], relevant [53,54], or irrelevant [45,55], determined by the specific fermion dispersion and the spatial dimension of the system. In the special case of 3D ani-WSM, the Coulomb interaction is found to be irrelevant [45] and thus does not significantly modify the low-energy behaviors of free fermions, which is consistent with the previous work of Abrikosov [56].

In realistic semimetal materials, the chemical potential usually takes a finite value, and its value can be adjusted by changing the gate voltage. For semimetals defined at a finite chemical potential, there are undamped collective modes, namely plasmon. The properties of plasmon is directly related to the energy dispersion of fermion excitations, and can be experimentally investigated by various techniques, such as the electron energy-loss spectroscopy [57,58]. In recent years, there appeared certain amount of theoretic studies for the plasmon mode in several sorts of doped semimetals, including 2D DSM
The bare Coulomb interaction can be written as
\[ V_0(q) = \frac{4\pi e^2}{\kappa |q|^2}. \] (7)

After including the dynamical screening caused by the collective particle-hole excitations, we write down the dressed retarded Coulomb interaction
\[ V_{\text{ret}}(\Omega, q) = \frac{V_0(q)}{\epsilon_r(\Omega, q)}, \] (8)

where \( \epsilon_r(\Omega, q) \) is the retarded polarization function. The collective plasmon mode is determined by the condition
\[ \epsilon_r(\Omega_p, q) = 1 + V_0(q)\Pi_{\text{ret}}(\Omega, q) = 0. \] (9)

In the following, we will calculate the polarization function and then analyze the asymptotic behavior of the plasmon mode in the long wavelength limit.

III. GENERAL EXPRESSION OF POLARIZATION FUNCTION

To the leading order, the polarization function reads
\[ \Pi(i\Omega, q) = -N \frac{1}{\beta} \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ G_0(i\omega_n, k) \right] \times G_0(i(\omega_n + \Omega_m), k + q) \], (11)

where \( \Omega_m = 2m\pi T \) and \( \beta = \frac{1}{2T} \). Utilizing the standard spectral representation
\[ G_0(i\omega_n, k) = -\int_{-\infty}^{+\infty} \frac{d\omega_1}{\pi} \frac{\text{Im} \left[ G_{0}^{\text{ret}}(\omega_1, k) \right]}{i\omega_n - \omega_1}, \] (12)

we obtain
\[ \Pi(i\Omega, q) = -N \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ \int_{-\infty}^{+\infty} \frac{d\omega_1}{\pi} \text{Im} \left[ G_{0}^{\text{ret}}(\omega_1, k) \right] \right] \times \frac{1}{\beta} \sum_{\omega_n} \frac{1}{i\omega_n - \omega_1} \frac{1}{i\Omega_m - \omega_2}. \] (13)

Summing up all frequencies leads to
\[ \frac{1}{\beta} \sum_{\omega_n} \frac{1}{i\omega_n - \omega_1} \frac{1}{i\Omega_m - \omega_2} = \frac{n_F(\omega_1) - n_F(\omega_2)}{\omega_1 - \omega_2 + i\Omega_m}, \] (14)
which then yields

\[
\Pi(i\Omega_m, q) = -N \int \frac{d^3k}{(2\pi)^3} \text{Tr} \left[ \frac{\pi}{\omega_1} \text{Im} \left[ G^\text{ret}_0(\omega_1, k) \right] \right] \\
\times \int_{-\infty}^{+\infty} \frac{d\omega_2}{\pi} \text{Im} \left[ G^\text{ret}_0(\omega_2, k + q) \right] \\
\times \frac{n_F(\omega_1) - n_F(\omega_2)}{\omega_1 - \omega_2 + i\Omega_m}.
\]

\[(15)\]

The imaginary part of the retarded fermion propagator is then given by

\[
\text{Im} \left[ G^\text{ret}_0(\omega, k) \right] = -\pi \text{sgn}(\omega + \mu) \left( \omega + \mu + kv_x \sigma_x + kv_y \sigma_y + Ak_z \sigma_z \right) \\
\times \frac{1}{2E_k} \left[ \delta(\omega + \mu + E_k) + \delta(\omega + \mu - E_k) \right], \quad \text{(16)}
\]

where the energy

\[
E_k = \sqrt{v^2k_z^2 + A^2k_z^2}.
\]

Substituting Eq. \((16)\) into Eq. \((15)\), we find that

\[
\Pi(i\Omega_m, q) = -N \frac{1}{16\pi^2} \sum_{\alpha, \alpha' = \pm 1} \int d^3k \left[ 1 + \alpha \alpha' \frac{F_{k,q}}{E_k E_{k+q}} \right] \\
\times \frac{n_F(\alpha E_k - \mu) - n_F(\alpha' E_{k+q} - \mu)}{\alpha E_k - \alpha' E_{k+q} + i\Omega_m},
\]

where

\[
F_{k,q} = v^2k_x(k_x + q_x) + v^2k_y(k_y + q_y) \\
+ A^2k_z^2(k_z + q_z)^2.
\]

Here, \(n_F(E) = \frac{1}{e^{E/|T|} + 1}\) is the Fermi-Dirac distribution function. We then perform the following analytic continuation:

\[
\frac{1}{x + i\Omega} \rightarrow \frac{1}{x + \Omega + i\eta} = P \frac{1}{x + \Omega} - i\pi \delta(x + \Omega).
\]

Now the imaginary and real parts of the retarded polarization function become

\[
\text{Im} \left[ \Pi^\text{ret}(\Omega, q) \right] = \frac{N}{16\pi^2} \sum_{\alpha, \alpha' = \pm 1} \int d^3k \\
\times \left[ 1 + \alpha \alpha' \frac{F_{k,q}}{E_k E_{k+q}} \right] \\
\times \frac{n_F(\alpha E_k - \mu) - n_F(\alpha' E_{k+q} - \mu)}{\alpha E_k - \alpha' E_{k+q} + \Omega},
\]

\[(21)\]

and

\[
\text{Re} \left[ \Pi^\text{ret}(\Omega, q) \right] = -\frac{N}{16\pi^2} \sum_{\alpha, \alpha' = \pm 1} P \int d^3k \\
\times \left[ 1 + \alpha \alpha' \frac{F_{k,q}}{E_k E_{k+q}} \right] \\
\times \frac{n_F(\alpha E_k - \mu) - n_F(\alpha' E_{k+q} - \mu)}{\alpha E_k - \alpha' E_{k+q} + \Omega}.
\]

\[(22)\]

The derivation for the polarization function in the long wavelength limit is shown in Appendix A and the calculation details for the imaginary and real parts of the retarded polarization function are presented in Appendix C. Since the polarization function is invariant under the transformation \(\mu \rightarrow -\mu\), reflecting the particle-hole symmetry, we will choose \(\mu > 0\) in the subsequent calculations.

### IV. PLASMON MODE

In this section, we analyze the long wavelength plasmon in doped 3D ani-WSM at zero temperature.

As \(T \rightarrow 0\), the function \(n_F(E)\) becomes the step function \(\theta(E)\). In the long wavelength regime with \(\text{max}(vq_\perp, Aq_z^2) \ll \Omega \ll \mu\), it is easy to find that \(\text{Im}\Pi^\text{ret}(\Omega, q) = 0\), which implies the existence of undamped plasmon. According to the values of \(\alpha\) and \(\alpha'\), we can see that the real part of the polarization function is divided into four parts, namely \(\text{Re}\Pi^\text{ret}_++(\Omega, q)\), \(\text{Re}\Pi^\text{ret}_-_+(\Omega, q)\), \(\text{Re}\Pi^\text{ret}_++(\Omega, q)\), and \(\text{Re}\Pi^\text{ret}_-_-(\Omega, q)\). In the regime \(\text{max}(vq_\perp, Aq_z^2) \ll \Omega \ll \mu\), it is easy to verify that \(\text{Re}\Pi^\text{ret}_-(\Omega, q) = 0\) due to the relation

\[
[\theta(-E_k - \mu) - \theta(-E_{k+q} - \mu)] = 0. \quad \text{(23)}
\]

As shown in Appendix A, in the regime \(\text{max}(vq_\perp, Aq_z^2) \ll \Omega \ll \mu\), we have

\[
\text{Re}\Pi^\text{ret}_++(\Omega, q) \approx - \left( C_{++} \frac{q_\perp^2 \mu^2}{\Omega^2} + C_{++}^z \frac{q_z^2 \mu^2}{\Omega^2} \right), \quad \text{(24)}
\]

\[
\text{Re}\Pi^\text{ret}_-_+(\Omega, q) \approx C_{+-} q_\perp^2 + C_{+-}^z q_z^2, \quad \text{(25)}
\]

\[
\text{Re}\Pi^\text{ret}_--(\Omega, q) \approx C_{-+} q_\perp^2 + C_{-+}^z q_z^2. \quad \text{(26)}
\]

where

\[
C_{++} = \frac{N}{5\pi^2 \sqrt{A}},
\]

\[
C_{++}^z = \frac{3N \Gamma \left( \frac{3}{4} \right) \sqrt{A}}{2\pi^2 \Gamma \left( \frac{11}{4} \right) v^2}, \quad \text{(28)}
\]

and

\[
C_{+-} = \frac{N}{20\pi^2 \sqrt{A}} \left( \frac{1}{\Lambda^2} - \frac{1}{\mu^2} \right), \quad \text{(29)}
\]

\[
C_{-+} = \frac{6}{7} - \frac{3\sqrt{\pi} \Gamma \left( \frac{3}{4} \right)}{2\Gamma \left( \frac{11}{4} \right)} \frac{N\sqrt{A}}{32\pi^2} \left( \Lambda^2 - \mu^2 \right). \quad \text{(30)}
\]

Noticing that

\[
|\text{Re}\Pi^\text{ret}_++(\Omega, q)| \gg |\text{Re}\Pi^\text{ret}_-(\Omega, q)|, \quad \text{(31)}
\]

\[
|\text{Re}\Pi^\text{ret}_++(\Omega, q)| \gg |\text{Re}\Pi^\text{ret}_-_+(\Omega, q)|. \quad \text{(32)}
\]
which is valid in the regime \(\max(\vq, Aq_z^2) \ll \Omega \ll \mu\), we finally get

\[
\Re \Pi_{\text{ret}}^\text{is}(\Omega, \vq) \approx \Re \Pi_{\text{ret}}^\text{is}(\Omega, \vq) \\
\approx - \left( C_z^+ + \frac{q_z^2 \mu^e}{\Omega^2} + C_z^- \frac{q_z^2 \mu^o}{\Omega^2} \right). \quad (33)
\]

As mentioned in Sec. [11] the plasmon mode is determined by Eq. [10], which gives rise to

\[
1 - \frac{4\pi^2 v}{q_z} \left( C_z^+ + \frac{q_z^2 \mu^e}{\Omega^2} + C_z^- \frac{q_z^2 \mu^o}{\Omega^2} \right) = 0. \quad (34)
\]

From this equation, we can get the plasmon mode

\[
\Omega_p = C_0 \sqrt{C_z^+ + \mu^e \sin^2(\phi) + C_z^- + \mu^o \cos^2(\phi)}, \quad (35)
\]

where \(C_0 = \sqrt{4\pi^2 v}\) and \(\phi\) is the angle between \(\vq\) and \(z\)-axis. The plasmon mode within the basal plane and the one along the third direction can be respectively written as

\[
\Omega_p^z = \sqrt{4\pi^2 v \kappa C_z^+ + \mu^e \sin^2(\phi) + C_z^- + \mu^o \cos^2(\phi)}, \quad (36)
\]

\[
\Omega_p^z = \sqrt{4\pi^2 v \kappa C_z^+ + \mu^e \sin^2(\phi) + C_z^- + \mu^o \cos^2(\phi)}, \quad (37)
\]

According to calculations presented in Appendix [3], the relation between carrier density \(n_e\) and chemical potential \(\mu\) is

\[
n_e = \frac{1}{3\pi^2 v_e^2 \sqrt{A}} \mu^e. \quad (38)
\]

It is now straightforward to obtain

\[
\Omega_p^z \propto n_e^\frac{1}{2}, \quad (39)
\]

\[
\Omega_p^z \propto n_e^\frac{1}{2}. \quad (40)
\]

which is a unique characteristic of doped 3D ani-WSM, and can be probed experimentally.

We now would like to compare the plasmon mode in doped 3D ani-WSM with other analogous semimetals. The behaviors of plasmon mode obtained in the context of various doped semimetals as well as traditional semimetals are summarized in Table I. It is clear that the plasmon in 3D fermion systems is always gapped, but the one in 2D fermion systems is gapless. The difference arises form the fact that the bare Coulomb interaction has distinct momentum dependence in 2D and 3D. The bare Coulomb interaction is given by Eq. [7] in 3D, and has the form

\[
V_0^{3D}(\vq) = \frac{2\pi^2 v}{|q|}. \quad (41)
\]

in 2D. For a fixed spatial dimension, the behavior of the plasmon mode is closely related to the energy dispersion of the fermionic excitations. More concretely, in the isotropic case, the plasmon is also isotropic and its power takes different value in different semimetals. In semimetals with anisotropic fermion dispersion, the plasmon mode is also anisotropic.

### V. DEBYE SCREENING

Based on the results obtained in Appendix [D], we find that

\[
\Re \Pi_{\text{ret}}(0, 0) = - \frac{N T^2}{2\pi^2 v_e^2 \sqrt{A}} \sum_{\alpha=\pm 1} \int_0^{+\infty} dx \frac{\sqrt{x}}{e^{+\frac{x}{T} + 1}}. \quad (42)
\]

in the limit of \(\Omega = 0\) and \(|q| = 0\), which represents the Debye screening induced by finite chemical potential and finite temperature. At \(T > 0\) and \(\mu = 0\), we have

\[
\Re \Pi_{\text{ret}}(0, 0) = \frac{1}{4}(2 - \sqrt{2}) \zeta(\frac{3}{2}) \frac{N T^2}{\pi^2 v_e^2 \sqrt{A}}. \quad (43)
\]
whereas at $T = 0$ and $\mu > 0$ we find that

$$\text{Re}\Pi^\text{ret}(0, 0) = \frac{N}{3\pi^2 v^2 \sqrt{A}} \mu^3.$$  \hfill (44)

As a consequence of the Debye screening, the dressed Coulomb interaction becomes short-ranged.

VI. SUMMARY

In summary, we calculate the polarization function in the context of a doped 3D anti-WSM, and analyze the behavior of the long wavelength plasmon. We find that the plasmon within the basal plane depends on the fermion density in the form $\Omega^\perp = n_e^\perp$, and that the plasmon along the third direction behaves as $\Omega^z \propto n_e^z$. These behaviors can be experimentally detected and would provide a useful method to verify the existence of 3D anti-Weyl fermions. We expect that such plasmon mode could be confirmed in BiTeI, ZrTe$_5$, and other semimetals that host fermion excitations with dispersion Eq. (1).

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Appendix A: Re$\Pi^\text{ret}(\Omega, q)$ in the limit max($vq_\perp, A q_\perp^2) \ll \Omega \ll \mu$

1. Re$\Pi^\text{ret}_{++}(\Omega, q)$

In the limit max($vq_\perp, A q_\perp^2) \ll \Omega$, we have

$$\begin{align*}
\text{Re}\Pi^\text{ret}_{++}(\Omega, q) &\approx -\frac{N}{16\pi^3} P \int d^3k \\
& \times \left( 1 + \frac{v^2 q_x^2 + v^2 q_y^2 + A^2 q_z^2}{E_k^2} \right) \\
& \times \left( \frac{\delta (\mu - E_k)}{\partial E_k} \frac{\partial E_k}{\partial q_x} q_x + \right. \\
& \left. + \frac{\delta (\mu - E_k)}{\partial E_k} \frac{\partial E_k}{\partial q_y} q_y + \Omega \right) \\
& = -\frac{N}{8\pi^3} P \int d^3k \delta (\mu - E_k) \frac{\partial E_k}{\partial q_x} q_x \\
& - \frac{\partial E_k}{\partial q_y} q_y + 1 \\
& = -\frac{N}{8\pi^3} \int d^3k \delta (\mu - E_k) \left( 1 + \frac{\partial E_k}{\partial q_x} q_x + \frac{\partial E_k}{\partial q_y} q_y \right). \\
\end{align*}$$  \hfill (A1)

Since $E_k = \sqrt{v^2 (k_x^2 + v_y^2) + A^2 k_z^2}$, it is clear that

$$\frac{\partial E_k}{\partial q_x} q_x = \frac{v^2 k_x^2 + v^2 k_y^2 + 2A^2 k_z^2}{E_k \Omega}.$$  \hfill (A2)

Now Re$\Pi^\text{ret}_{++}(\Omega, q)$ can be further written as

$$\begin{align*}
\text{Re}\Pi^\text{ret}_{++}(\Omega, q) &= -\frac{N}{8\pi^3} \int d^2k \delta (\mu - E_k) \\
& \times \left( v^2 k_x^2 q_x + v^2 k_y^2 q_y + 4A^2 k_z^2 q_z \right) \frac{E_k}{\Omega^2} \\
& = -\frac{1}{E_k^4 \Omega^2} \left( [v^4 k_x^2 q_x^2 + v^4 k_y^2 q_y^2 + 4A^4 k_z^2 q_z^2] \right) \\
& \times \frac{1}{E_k^4 \Omega^2} \int d^2k \delta (\mu - E_k) \\
& \times \left( v^2 k_x^2 q_x + v^2 k_y^2 q_y + 4A^2 k_z^2 q_z \right) \frac{E_k}{\Omega^2} \\
& \times \delta (\mu - E_k) \frac{k_0^6}{E_k^4}. \\
\end{align*}$$  \hfill (A3)

We then employ the transformations

$$E = \sqrt{v^2 k_\perp^2 + A^2 k_z^2}, \quad \delta = \frac{Ak_\perp^2}{vk_\perp},$$  \hfill (A4)

which are equivalent to

$$k_\perp = \frac{E}{v \sqrt{1 + \delta^2}}, \quad |k_z| = \frac{\sqrt{\delta} \sqrt{E}}{\sqrt{A (1 + \delta^2)^{\frac{3}{2}}}}.$$  \hfill (A5)

It is easy to verify that

$$\begin{align*}
dk_\perp d|k_z| &= \frac{\partial E}{\partial \delta} \frac{\partial E}{\partial \delta} \frac{dEd\delta}{dEd\delta} \\
& = \frac{\partial k_{\perp1}}{\partial E} \frac{\partial k_{\perp2}}{\partial E} - \frac{\partial k_{\perp1}}{\partial E} \frac{\partial k_{\perp2}}{\partial E} \frac{dEd\delta}{dEd\delta} \\
& = \frac{\sqrt{E}}{2v^2 \A^\delta (1 + \delta^2)^{\frac{3}{2}}} dEd\delta. \\
\end{align*}$$  \hfill (A6)

Utilizing the transformations Eqs. (A3) and (A6), one can get

$$\begin{align*}
\text{Re}\Pi^\text{ret}_{++}(\Omega, q) &= -\frac{N}{4\pi^3} \left[ \frac{1}{2\sqrt{A\Omega^2}} \int_0^{+\infty} dEE^\ast \delta (\mu - E) \\
& \times \int_0^{+\infty} d\delta \frac{1}{\sqrt{\delta (1 + \delta^2)^{\frac{3}{2}}}} \\
& + \frac{4A\Omega^2}{v^4 k_{\perp1}^2} \int_0^{+\infty} dEE^\ast \delta (\mu - E) \\
& \times \int_0^{+\infty} d\delta \frac{1}{\sqrt{\delta (1 + \delta^2)^{\frac{3}{2}}}} \right] \\
& = -\left( C_{++} \frac{q_x^2}{\Omega^2} + C_{++} \frac{q_y^2}{\Omega^2} \right). \\
\end{align*}$$  \hfill (A7)
where

\[ C_{++} = \frac{N}{5 \pi^2 \sqrt{A}}, \quad \text{(A8)} \]
\[ C_{++}^z = \frac{3N \Gamma \left( \frac{3}{2} \right) \sqrt{A}}{2\pi \pi \Gamma \left( \frac{1}{2} \right) v^2}. \quad \text{(A9)} \]

2. Re\( \Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) \)

In the limit max(\(vq_L, Aq_z^2\) \(\ll\) \(\Omega\), Re\( \Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) \) is approximately given by

\[
\text{Re}\Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) \approx \frac{N}{16\pi^2} \int d^3k \left[ 1 - \frac{F_{k,q}}{E_k} \left( \frac{\partial F_{k,q}}{\partial k_i} q_i + E_k \right) \right] \times \frac{\theta (E_k - \mu)}{2E_k + \Omega}.
\]

(A10)

Substituting Eq. (A2) into Eq. (A10), one gets

\[
\text{Re}\Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) = \frac{N}{16\pi^2} \int d^3k \left[ 1 - \frac{A^2 k_i^2 q_i^2}{E_k^2} \right] \times \frac{\theta (E_k - \mu)}{2E_k + \Omega}.
\]

Performing the integration of azimuth angle, we obtain

\[
\text{Re}\Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) = \frac{N}{8\pi^2} \left[ v^4 q_L^4 \int d k_{\perp, L} |k_{\perp}| k_{\perp}^2 \frac{k_{\perp}^2}{E_k^2} \right] \frac{\theta (E_k - \mu)}{2E_k + \Omega} + 2A^2 q_z^2 \int d k_{\perp, \perp} |k_{\perp}| k_{\perp} \times \left( -\frac{k_{\perp}^2}{E_k^2} + \frac{4A^2 k_{\perp}^6}{E_k^4} \right) \frac{\theta (E_k - \mu)}{2E_k + \Omega}.
\]

(A12)

Using the transformations Eqs. (A5) and (A6), and performing the integration of \(\delta\), Re\( \Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) \) can be further written as

\[
\text{Re}\Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) = \frac{N}{8\pi^2} \left[ 4A^2 q_z^2 \int_{-\mu}^{\Lambda} E d E \frac{1}{E + \frac{1}{2E + \Omega}} \right] + \frac{\sqrt{A} q_z^2}{v^2} \int_{-\mu}^{\Lambda} d E E \frac{1}{2E + \Omega} \times \left( \frac{6}{\pi} - \frac{3\sqrt{\pi} \Gamma (\frac{3}{2})}{2\Gamma (\frac{1}{4})} \right).
\]

In the regime \(\Omega \ll \mu\), it can be approximately written as

\[
\text{Re}\Pi_{++}^{\text{ret}} (\Omega, \mathbf{q}) \approx C_{++}^{\perp} q_L^2 + C_{++}^{z} q_z^2.
\]

(A20)
where

\[ C_{-+}^+ = \frac{N}{20\pi^2 \sqrt{A}} \left( \frac{1}{\Lambda^2} - \frac{1}{\mu^2} \right), \quad (A21) \]

\[ C_{-+}^- = \left( \frac{6}{5} - 3\sqrt{\pi} \left( \frac{1}{\Lambda} \right) \right) N \sqrt{A} \frac{1}{2\sqrt{2}} \]

\times \left( \Lambda^2 - \mu^2 \right). \quad (A22) \]

Comparing Eqs. (A14), (A16) with Eqs. (A20)-(A22), we know that

\[ \text{Re} \Pi^R_{\perp} (\Omega, q) \approx \text{Re} \Pi^R_{\perp} (\Omega, q) \quad (A23) \]

in the regime \( \max(vq, Aq^2) \ll \Omega \ll \mu. \)

**Appendix B: The density of fermion**

The density of fermion is given by

\[ n_e = \int \frac{d^3k}{(2\pi)^3} \delta (\mu - E_k) \]

**Appendix C: Further Calculation of polarization function**

1. \( \text{Im} \Pi^R_{\perp} (\Omega, q) \)

Eq. (21) in the main body of the paper can be further written as

\[ \text{Im} \Pi^R_{\perp} (\Omega, q) = I_1 - I_2 + I_3 + I_4 - I_5 - I_6 + I_7 + I_8 - I_9 - I_10. \quad (C1) \]

It can be found that

\[ I_2 = I_1 (\Omega \rightarrow -\Omega), \]

\[ I_4 = I_3 (\mu \rightarrow \mu), \quad I_5 = I_3 (\Omega \rightarrow -\Omega), \quad I_6 = I_3 (\Omega \rightarrow -\Omega, \mu \rightarrow \mu), \]

\[ I_8 = I_7 (\mu \rightarrow \mu), \quad I_9 = I_7 (\Omega \rightarrow -\Omega), \quad I_{10} = I_7 (\Omega \rightarrow -\Omega, \mu \rightarrow \mu). \]

Therefore, we only need to calculate \( I_1, I_3, \) and \( I_7, \) which are given by

\[ I_1 = \frac{N}{16\pi^2} \int dk_{\perp} \int dk_z \int_0^{2\pi} d\varphi \left[ 1 - \frac{F_{k,q}}{E_k E_{k+q}} \right] \delta (E_k + E_{k+q} + \Omega), \quad (C5) \]

\[ I_3 = \frac{N}{16\pi^2} \int dk_{\perp} \int dk_z n_F (E_k - \mu) \int_0^{2\pi} d\varphi \left[ 1 + \frac{F_{k,q}}{E_k E_{k+q}} \right] \delta (E_k - E_{k+q} + \Omega), \quad (C6) \]

\[ I_7 = \frac{N}{16\pi^2} \int dk_{\perp} \int dk_z n_F (E_k - \mu) \int_0^{2\pi} d\varphi \left[ 1 - \frac{F_{k,q}}{E_k E_{k+q}} \right] \delta (E_k + E_{k+q} + \Omega), \quad (C7) \]

where

\[ F_{k,q} = v^2 k_{\perp}^2 + v^2 q_{\perp}^2 \cos(\varphi) + A^2 k_z^2 (k_z + q_z)^2, \quad (C8) \]

\[ E_{k+q} = \sqrt{v^2 (k_{\perp}^2 + q_{\perp}^2 + 2k_{\perp} q_{\perp} \cos(\varphi)) + A^2 (k_z + q_z)^2}. \quad (C9) \]

Suppose that

\[ \delta [F_1 (\varphi)] = \delta (E_k + E_{k+q} + \Omega) = \delta \left( \sqrt{v^2 k_{\perp}^2 + A^2 k_z^2} + \sqrt{v^2 (k_{\perp}^2 + q_{\perp}^2 + 2k_{\perp} q_{\perp} \cos(\varphi)) + A^2 (k_z + q_z)^2} + \Omega \right), \quad (C10) \]
After tedious evaluation, we get
\[
\delta [F_1(\varphi)] = 2 (E_k + \Omega) \left[ (E_k + \Omega)^2 - (E_1(k, q))^2 \right]^{-\frac{1}{2}} \left[ - (E_k + \Omega)^2 + (E_2(k, q))^2 \right]^{-\frac{1}{2}} \left[ \delta (\varphi - \varphi_1^a) + \delta (\varphi - \varphi_2^b) \right]
\times \theta (-E_k - \Omega) \theta (-E_k - \Omega - E_1(k, q)) \theta (E_2(k, q) + E_k + \Omega),
\]
where
\[
E_1(k, q) = \sqrt{v^2 (k_\perp - q_\perp)^2 + A^2 (k_z + q_z)^4},
\]
\[
E_2(k, q) = \sqrt{v^2 (k_\perp + q_\perp)^2 + A^2 (k_z + q_z)^4}.
\]
Here, \(\varphi_1^a\) and \(\varphi_2^b\) are the two angles satisfying \(F_1(\varphi_1^a, \varphi_2^b) = 0\) in the range \((0, 2\pi)\). We then let
\[
\delta [F_2(\varphi)] = \delta (E_k - E_{k+q} + \Omega) = \delta \left( \sqrt{v^2 k_\perp^2 + A^2 k_z^4} - \sqrt{v^2 (k_\perp^2 + q_\perp^2 + 2k_\perp q_\perp \cos(\varphi)) + A^2 (k_z + q_z)^4} + \Omega \right),
\]
We similarly obtain
\[
\delta (F_2(\varphi)) = 2 (E_k + \Omega) \left[ (E_k + \Omega)^2 - (E_1(k, q))^2 \right]^{-\frac{1}{2}} \left[ - (E_k + \Omega)^2 + (E_2(k, q))^2 \right]^{-\frac{1}{2}} \left[ \delta (\varphi - \varphi_1^b) + \delta (\varphi - \varphi_2^b) \right]
\times \theta (E_k + \Omega) \theta (E_k + \Omega - E_1(k, q)) \theta (E_2(k, q) - \Omega),
\]
where \(\varphi_1^b\) and \(\varphi_2^b\) satisfy \(F_2(\varphi_1^b, \varphi_2^b) = 0\) in the range \((0, 2\pi)\). After substituting Eq. \((C11)\) into Eqs. \((C5)\) and \((C7)\), and also substituting Eq. \((C15)\) into Eq. \((C6)\), and then carrying out the integration over \(\varphi\), we can get the simplified expressions of \(I_1\), \(I_2\), and \(I_3\). Making use of the relations Eqs. \((C2)\) \((C4)\), we eventually have
\[
\text{Im\Pi}^{ret}(\Omega, \mathbf{q}) = \sum_{\alpha = \pm 1} \text{sgn}(\Omega) \frac{N}{8 \pi^2} \int dk \int dk \int dk \left[ \delta_{1, \alpha} - n_F(E(k) + \alpha \mu) \right] \frac{1}{E(k)} F_{A_1}(k, q) F_{B_1}(k, q) H_1(k, q)
+ \sum_{\alpha = \pm 1} \text{sgn}(\Omega) \frac{N}{8 \pi^2} \int dk \int dk \int dk n_F(E(k) + \alpha \mu) \frac{1}{E(k)}
\times \{ F_{A_2}(k, q) F_{B_2}(k, q) H_2(k, q) - F_{A_3}(k, q) F_{B_1}(k, q) H_3(k, q) \},
\]
where
\[
F_{A_1}(k, q) = \left[ -(E(k) + |\Omega|)^2 - (E_1(k, q))^2 \right]^{-\frac{1}{2}} \left[ - (E(k) + |\Omega|)^2 + (E_2(k, q))^2 \right]^{-\frac{1}{2}},
\]
\[
F_{A_2}(k, q) = \left[ (E(k) + |\Omega|)^2 - (E_1(k, q))^2 \right]^{-\frac{1}{2}} \left[ - (E(k) + |\Omega|)^2 + (E_2(k, q))^2 \right]^{-\frac{1}{2}},
\]
\[
F_{A_3}(k, q) = \left[ (E(k) - |\Omega|)^2 - (E_1(k, q))^2 \right]^{-\frac{1}{2}} \left[ - (E(k) - |\Omega|)^2 + (E_2(k, q))^2 \right]^{-\frac{1}{2}},
\]
\[
F_{B_1}(k, q) = \left[ 2(E(k) - |\Omega|)^2 - v^2 q_\perp^2 - A^2 (2k_z + q_z)^2 \theta_1 \right],
\]
\[
F_{B_2}(k, q) = \left[ 2(E(k) + |\Omega|)^2 - v^2 q_\perp^2 - A^2 (2k_z + q_z)^2 \theta_2 \right],
\]
\[
H_1(k, q) = \theta (-E(k) + |\Omega|) \theta (-E(k) + |\Omega| - E_1(k, q)) \theta (E_2(k, q) + E(k) - |\Omega|),
\]
\[
H_2(k, q) = \theta (E(k) + |\Omega|) \theta (E(k) + |\Omega| - E_1(k, q)) \theta (E_2(k, q) - E(k) - |\Omega|),
\]
\[
H_3(k, q) = \theta (E(k) - |\Omega|) \theta (E(k) - |\Omega| - E_1(k, q)) \theta (E_2(k, q) - E(k) + |\Omega|).
\]

2. \text{Re\Pi}^{ret}(\Omega, \mathbf{q})

After further calculations, we write Eq. \((C24)\) of the main body of the paper in the form
\[
\text{Re\Pi}^{ret}(\Omega, \mathbf{q}) = J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\]
There exists a number of identities:
\[
J_2 = J_1 (\Omega \rightarrow -\Omega),
\]
\[
J_4 = J_3 (\mu \rightarrow -\mu),
\]
\[
J_5 = J_3 (\Omega \rightarrow -\Omega),
\]
\[
J_6 = J_3 (\mu \rightarrow -\mu, \Omega \rightarrow -\Omega).
\]
It is thus only necessary to calculate $J_1$ and $J_3$, given by

$$J_1 = \frac{N}{8\pi^2} \int dk_L k_L \int dk_z \frac{1}{E_k} M_1,$$

$$J_3 = -\frac{N}{8\pi^3} \int dk_L k_L \int dk_z n_F (E_k - \mu) \frac{1}{E_k} M_1,$$

where

$$M_1 = \mathcal{P} \int_0^{2\pi} d\varphi \frac{E_k (E_k + \Omega) + v^2 k_L^2 + v^2 k_L q_\perp \cos(\varphi) + A^2 k_L^2 (k_z + q_z)^2}{(E_k + \Omega)^2 - (v^2 (k_L^2 + q_\perp^2) + 2k_L q_\perp \cos(\varphi)) + A^2 (k_z + q_z)^4}.$$  \hspace{1cm} (C30)

It is convenient to define $Z = e^{i\varphi}$, which implies that

$$\cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} = \frac{Z + Z^{-1}}{2}, \quad d\varphi = \frac{dZ}{iZ}. \hspace{1cm} (C31)$$

Now $M_1$ can be expressed as

$$M_1 = \frac{1}{i} \mathcal{P} \int_{|Z|=1} dZ F(Z), \hspace{1cm} (C32)$$

where

$$F(Z) = \frac{E_k (E_k + \Omega) Z + v^2 k_L^2 Z + \frac{1}{2}v^2 k_L q_\perp (Z^2 + 1) + A^2 k_L^2 (k_z + q_z)^2 Z}{Z [E_k + \Omega] Z - (v^2 (k_L^2 + q_\perp^2) Z + v^2 k_L q_\perp (Z^2 + 1) + A^2 (k_z + q_z)^4 Z].} \hspace{1cm} (C33)$$

Using the residue theorem, we obtain

$$M_1 = -\pi + \pi \left[ -(3E_k + \Omega) (E_k + \Omega) - v^2 (k_L^2 - q_\perp^2) - A^2 (-2k_L^2 + (k_z + q_z)^2) (k_z + q_z)^2 \right]$$

$$\times \left[ (E_k (k, \mathbf{q}))^2 - (E_k + \Omega)^2 \right] \left[ (E_1 (k, \mathbf{q})) - (E_k + \Omega)^2 \right]^{-\frac{1}{2}}$$

$$\times \theta \left[ (E_2 (k, \mathbf{q}))^2 - (E_k + \Omega)^2 \right] \theta \left[ (E_k + \Omega)^2 - (E_1 (k, \mathbf{q}))^2 \right]. \hspace{1cm} (C34)$$

Substituting Eq. \textit{C34} into Eqs. \textit{C28} and \textit{C29}, it is straightforward to get the expressions of $J_1$ and $J_3$. With help of \textit{C26} and \textit{C27}, we finally find that

$$\text{Re} \Pi^\text{ret}(\Omega, \mathbf{q}) = \frac{N}{4\pi^2} \sum_{\alpha = \pm 1} \int dk_L k_L \int dk_z n_F (E_k + \alpha \mu) \frac{1}{E_k}$$

$$\left[ -(3E_k + \alpha \Omega) (E_k + \alpha \Omega) - v^2 (k_L^2 - q_\perp^2) + A^2 (-2k_L^2 + (k_z + q_z)^2) (k_z + q_z)^2 \right]$$

$$\times \left[ (E_k (k, \mathbf{q}))^2 - (E_k + \alpha \Omega)^2 \right] \left[ (E_1 (k, \mathbf{q}))^2 - (E_k + \alpha \Omega)^2 \right]^{-\frac{1}{2}}$$

$$\times \theta \left[ (E_2 (k, \mathbf{q}))^2 - (E_k + \alpha \Omega)^2 \right] \theta \left[ (E_k + \alpha \Omega)^2 - (E_1 (k, \mathbf{q}))^2 \right]$$

$$\times \theta \left[ (E_k + \alpha \Omega)^2 - (E_2 (k, \mathbf{q}))^2 \right] \theta \left[ (E_k + \alpha \Omega)^2 - (E_2 (k, \mathbf{q}))^2 \right]. \hspace{1cm} (C35)$$

In the derivation, a constant term that is independent of $\Omega, \mathbf{q}, \mu$, and $T$ has been dropped.
Appendix D: Debye Screening

At $\Omega = 0$ and $|q| = 0$, we have

$$ \text{Re} \Pi^{\text{ret}}(0, 0) = \frac{N}{2\pi^2} \sum_{\alpha = \pm 1} \int dk_\perp dk_\parallel |k_\perp n_F (E_k + \alpha \mu) \frac{1}{E_k}. $$

(D1)

Adopting the transformations shown in Eqs. (A5) and (A6) gives rise to

$$ \text{Re} \Pi^{\text{ret}}(0, 0) = \frac{N}{2\pi^2 v^2 \sqrt{A}} \sum_{\alpha = \pm 1} \int_0^{+\infty} dE \sqrt{E} \frac{1}{e^{\frac{E + \mu}{T}} + 1} $$

$$ = \frac{NT_\parallel^+}{2\pi^2 v^2 \sqrt{A}} \sum_{\alpha = \pm 1} \int_0^{+\infty} dx \sqrt{x} \frac{1}{e^{x + \frac{\mu}{T}} + 1}. $$

(D2)
95, 161113(R) (2017).

[41] Q. Liu and A. Zunger, Phys. Rev. X 7, 021019 (2017).

[42] S.-X. Zhang, S.-K. Jian, and H. Yao, arXiv:1610.08973v2.

[43] J.-R. Wang, G.-Z. Liu, and C.-J. Zhang, arXiv:1705.04001.

[44] B.-J. Yang, M. S. Bahramy, R. Arita, H. Isobe, E.-G. Moon, and N. Nagaosa, Phys. Rev. Lett. 110, 086402 (2013).

[45] B.-J. Yang, E.-G. Moon, H. Isobe, and N. Nagaosa, Nat. Phys. 10, 774 (2014).

[46] E.-G. Moon and Y. B. Kim, arXiv:1409.0573v1.

[47] M. S. Bahramy, B.-J. Yang, R. Arita, and N. Nagaosa, Nat. Commun. 3, 679 (2012).

[48] X. Xi., C. Ma, Z. Liu, Z. Chen, W. Ku, H. Berger, C. Martin, D. B. Tanner, and G. L. Carr, Phys. Rev. Lett. 111, 155701 (2013).

[49] X. Yuan, C. Zheng, Y. Liu, A. Narayan, C. Song, S. Shen, X. Sui, J. Xu, H. Yu, Z. An, J. Zhao, S. Sanvito, H. Yan, and F. Xiu, NPG Asia Mater. 8, e325 (2016).

[50] J. L. Zhang, C. Y. Guo, X. D. Zhu, L. Ma, G. L. Zheng, Y. Q. Wang, L. Pi, Y. Chen, H. Q. Yuan, and M. L. Tian, Phys. Rev. Lett. 118, 206601 (2017).

[51] R. Shankar, Rev. Mod. Phys. 66, 129 (1994).

[52] P. Goswami and S. Chakravarty, Phys. Rev. Lett. 107, 196803 (2011).

[53] I. F. Herbut and L. Janssen, Phys. Rev. Lett. 113, 106401 (2014).

[54] L. Janssen and I. F. Herbut, Phys. Rev. B 92, 045117 (2015); L. Janssen and I. F. Herbut, Phys. Rev. B 93, 165109 (2016); L. Janssen and I. F. Herbut, Phys. Rev. B 95, 075101 (2017).

[55] Y. Huh, E.-G. Moon, and Y. B. Kim, Phys. Rev. B 93, 035138 (2016).

[56] A. A. Abrikosov, J. Low Temp. Phys. 8, 315 (1972).

[57] G. F. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid* (Cambridge University Press, Cambridge, 2005).

[58] S. A. Maier, *Plasmonics: Fundamentals and Applications* (Springer, New York, 2007).

[59] B. Wunsch, T. Stauber, F. Sols, and F. Guinea, New J. Phys. 8, 318 (2006).

[60] E. H. Hwang and S. D. Sarma, Phys. Rev. B 75, 205418 (2007).

[61] P. K. Pyatkovskiy, J. Phys.: Condens. Matter 21, 025506 (2009).

[62] S. Das Sarma and E. H. Hwang, Phys. Rev. Lett. 102, 206412 (2009).

[63] M. Lv and S.-C. Zhang, Int. J. Mod. Phys. B 27, 135077 (2013).

[64] I. Panfilov, A. A. Burkov, and D. A. Pesin, Phys. Rev. B 89, 245103 (2014).

[65] J. Zhou, H.-R. Chang, and D. Xiao, Phys. Rev. B 91, 035114 (2015).

[66] Z. Yan, P.-W. Huang, and Z. Wang, Phys. Rev. B 93, 085138 (2016).

[67] J.-W. Rhim and Y. B. Kim, New J. Phys. 180, 043010 (2016).

[68] T. Stauber, J. Phys.: Condens. Matter 26, 123201 (2014).

[69] H.-R. Chang, J. Zhou, H. Zhang, and Y. Yao, Phys. Rev. B 89, 201141(R) (2014).