Self-consistent Nonlinear Force-Free Field Reconstruction from Weighted Boundary Conditions

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Abstract Photospheric vector magnetogram data are often used as boundary conditions for force-free coronal magnetic field extrapolations. In general, however, vector magnetogram data are not consistent with the force-free assumption. In this article, we demonstrate a way to deal with inconsistent boundary data, by generalizing the “self-consistency procedure” of Wheatland and Régnier (Astrophys. J. Lett. 700, L88, 2009). In that procedure, the inconsistency is resolved by an iterative process of constructing two solutions based on the values of the force-free parameter $\alpha$ on the two polarities of the field in the boundary (the P and N polarities), and taking uncertainty-weighted averages of the boundary $\alpha$ values in the P and N solutions. When the $\alpha$ values in the P and N regions are very different, the self-consistent solution may lose high $\alpha$ values from the boundary conditions. We show how, by altering the weighting of the uncertainties in the P or N boundary conditions, we can preserve high $\alpha$ values in the self-consistent solution. The weighted self-consistent extrapolation method is demonstrated on an analytic bipole field and applied to vector magnetogram data taken by the Helioseismic and Magnetic Imager (HMI) instrument on board the Solar Dynamics Observatory (SDO) spacecraft for NOAA active region AR 12017 on 2014 March 29.

Keywords Active regions, magnetic fields · Magnetic fields, corona · Magnetic fields, models

1. Introduction

The magnetic field vector in the solar corona cannot be measured directly. The vector magnetic field at the photosphere, however, can be inferred from spectropolarimetric measurements of lines in the low atmosphere. The photospheric measurements can then, in principle,
be used as boundary conditions to extrapolate the coronal magnetic field. A common simplifying assumption is that the coronal field is force-free, i.e., the Lorentz force is zero. The problem then reduces to solving a boundary value problem, given by two vector partial differential equations for the magnetic field $B$, namely the physical requirement that the magnetic field is divergence-free

$$\nabla \cdot B = 0,$$

and the force-free condition

$$\nabla \times B = \alpha B,$$

where $\alpha$ is the scalar force-free parameter (which in general is a function of position). Taking the divergence of Equation 2 allows us to recast it into

$$B \cdot \nabla \alpha = 0,$$

i.e., $\alpha$ is constant along field lines. Equations 1–3 define a nonlinear force-free field (NLFFF).

There are several methods available for extrapolating an NLFFF in the solar corona from photospheric boundary conditions [for summaries and comparisons, see, e.g., Schrijver et al. (2006, 2008), De Rosa et al. (2009), Wiegelmann and Sakurai (2012), Régnier (2013), and Wiegelmann, Petrie, and Riley (2017)]. One class of methods uses the Grad–Rubin approach (Grad and Rubin, 1958; Amari, Boulmezaoud, and Mikic, 1999; Wheatland, 2006), which solves for $B$ and $\alpha$ via an iterative procedure, and uses $\alpha$ and the normal component of $B$ at the photosphere as boundary conditions. The boundary conditions on the lateral sides and the top of the computational box also need to be defined. The Grad–Rubin method discussed and modified in this article uses Fourier transforms of the magnetic vector potential (Wheatland, 2006, 2007), leading to periodic boundary conditions at the sides of the box. The top of the box can be closed (i.e., $B = 0$ at the top) or open (i.e., $B \to 0$ at infinity). In the analytic test case discussed in Section 3 the top of the box is closed, in the application to active region AR 12017 discussed in Section 4 the top of the box is open.

For the lower boundary condition, Grad–Rubin methods only require the value of $\alpha$ over one magnetic polarity, for field lines connected to the lower boundary at both ends. If the values of $B$ and $\alpha$ are available over an entire solar active region, the boundary value problem is over-specified. One needs to make a choice as to which set of $\alpha$ values to take as boundary condition — those from the positive magnetic polarity region or the negative region. The results of the two possible reconstructions are referred to as the P solution and the N solution, respectively. In principle, if the boundary conditions are consistent with a force-free field, this over-specification of the problem is not an issue and the two solutions are equal. However, in practice the P and N solutions can differ significantly, both qualitatively and quantitatively: they may have substantially different energies, and one solution may show field structures and field line connectivities that the other lacks. This occurs because the magnetic field on the photosphere, which is used to derive the boundary conditions for NLFFF reconstructions, is not force-free (Gary, 2001). One approach to deal with the inconsistency of the boundary conditions with the force-free model, developed by Wiegelmann, Inhester, and Sakurai (2006), is “preprocessing,” which alters the magnetogram data to meet necessary conditions for the existence of a force-free solution.

A different approach is to take the data as given, but allow the boundary values of $B$ and $\alpha$ to change during the NLFFF extrapolation process (Wiegelmann and Sakurai, 2012). This procedure is used in the modified Grad–Rubin code (XTRAPOL) of Amari and Aly.
(2010) and the “self-consistency” procedure of Wheatland and Régnier (2009), which uses the Grad–Rubin current-field iteration code CFIT of Wheatland (2007). The Wheatland and Régnier (2009) method calculates P and N solutions from the two sets of boundary conditions on $\alpha$, and then averages the values of $\alpha$ for the two solutions at every point on the boundary, taking into account the uncertainties in $\alpha$ (calculated from the uncertainties in the measurements of the magnetic field components). The resulting map of $\alpha$ allows two new P and N solutions to be constructed, and then the cycle is iterated until a self-consistent set of values is obtained, i.e., the P and N solutions are the same. Note that Wheatland and Régnier (2009) did not actually use measurement uncertainties (which were not available for their data) in their test calculation. They assumed equal uncertainties in the $\alpha$ values at all points of the boundary.

The self-consistency procedure uses the uncertainties in the values of $\alpha$ on the boundary (calculated from $B$-measurement uncertainties) as weightings when calculating the average value of $\alpha$ at the end of each cycle. When applied with the observational uncertainties, the procedure can lead to a loss of information regarding areas with high values of $\alpha$, resulting in a self-consistent solution with a smooth distribution of boundary values of $\alpha$ compared to the observed boundary conditions. This is because, when the uncertainties in $\alpha$ on the two polarities are approximately equal, each self-consistency cycle takes a simple average of the values of $\alpha$ from the P and N polarities. For example, Kleint et al. (2018) presented reconstructions using the self-consistency procedure of the coronal field associated with active region (AR) 12017 on 2014 March 29. They found the P and N solutions to be quite different: the P solution included a highly twisted flux rope along the magnetic polarity inversion line, and this was absent in the N solution. In this particular case, the positive polarity field has regions with high values of $\alpha$, which do not have counterparts in the negative polarity regions (Kleint et al., 2018). In successive self-consistency cycles, the large values of $\alpha$ at certain boundary points in the P solution are averaged with small values of $\alpha$, leading to a decrease in $\alpha$ at those locations. Therefore, while the procedure constructs a final NLFFF that is consistent with the boundary conditions on both polarities, that field may be close to potential, and have magnetic field lines which lack resemblance to structures observed in extreme ultraviolet (EUV) observations.

In this article, we present a modification to the self-consistency procedure that can generate a self-consistent NLFFF that agrees better with observations by weighting the uncertainties in the values of $\alpha$ in the boundary conditions towards areas which are considered significant. The article is structured as follows. In Section 2 we describe our modification to the self-consistency procedure as implemented by the CFIT method. In Section 3, we demonstrate the method using an analytic bipolar field. In Section 4 we apply our method to the NLFFF extrapolation of active region AR 12017 using the data obtained by the Helioseismic and Magnetic Imager (HMI) instrument aboard the Solar Dynamics Observatory (SDO) spacecraft. We compare our NLFFF reconstructions to the EUV observations taken by the Atmospheric Imaging Assembly (AIA) instrument aboard SDO. In Section 5 we summarize our results.

2. Data and Methods

For the solar coronal field reconstructions in this article, we use the SDO/HMI vector magnetogram product called Space-Weather HMI Active Region Patches (SHARPs) (Sun, 2013; Hoeksema et al., 2014). The NLFFF code used here (CFIT) works in a Cartesian geometry. In application to the SHARP data, which are in a cylindrical equal-area projection, we
assume the magnetic vector field values are on a Cartesian grid. We take the photosphere to be in the \(x\)-\(y\) plane and the positive \(z\)-direction is the radial direction away from the Sun, and we assume \(B_x = B_\phi\), \(B_y = -B_\theta\), and \(B_z = B_r\), where \(B_r\), \(B_\theta\), and \(B_\phi\) are the field components in the SHARP data. The force-free parameter \(\alpha_0\) at \(z = 0\), which we require as boundary condition, is obtained from

\[
\alpha_0 = \frac{1}{B_z} \left( \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right) \bigg|_{z=0},
\]

where the derivatives are approximated using centered differencing. The uncertainties in \(\alpha_0\), which we designate as \(\sigma_0\), are derived from the uncertainties in the observed \(B\) field values. These uncertainties arise from instrumental errors, as well as uncertainties in the inversion process (Hoeksema et al., 2014), which are propagated to assign uncertainty values for each quantity at every pixel in the SHARP magnetogram (Bobra et al., 2014).

The self-consistency procedure is as follows (Wheatland and Régnier, 2009). The CFIT method is first applied to the boundary value problem given by the values of \(B_z\) and \(\alpha_0\) on the P region (the region where \(B_z > 0\)) until the NLFFF “P solution” is obtained. Since the force-free parameter is constant along force-free field lines, the values of \(\alpha_0\) from the P region are mapped via this solution to the conjugate magnetic foot points in the N region (the region where \(B_z < 0\)). These mappings define a new set of values \(\alpha_1\) in N. CFIT is applied again using the values of \(\alpha_0\) on the N region until a second NLFFF result is obtained (the “N solution”), and the values of \(\alpha_0\) from the N region map via this solution to new values \(\alpha_1\) on the P region. We now have two NLFFF solutions and two complete sets of \(\alpha\) values (defined over the entire region, both P and N): \(\alpha_0\) and \(\alpha_1\). The uncertainties for the \(\alpha_1\) values are obtained by mapping the values of \(\sigma_0\) from the P (N) region to the conjugate magnetic foot points in the N (P) region. At each point on the region, we now have two possible sets of values \((\alpha_0, \sigma_0)\) and \((\alpha_1, \sigma_1)\).

Bayes’ theorem can be used to decide on the most probable value of \(\alpha\), given some weighting factor (Wheatland and Régnier, 2009). When working with real data, the weighting factor is taken to be the uncertainties \(\sigma\) in the values of \(\alpha\). The most probable value of \(\alpha\) is

\[
\alpha_2 = \frac{\alpha_0/\sigma_0^2 + \alpha_1/\sigma_1^2}{1/\sigma_0^2 + 1/\sigma_1^2},
\]

with the corresponding uncertainty

\[
\sigma_2 = \left( \frac{1/\sigma_0^2 + 1/\sigma_1^2}{2} \right)^{-1/2}.
\]

This process defines one “self-consistency cycle” (Wheatland and Régnier, 2009).\(^1\) The resulting values of \(\alpha_2\) may still be inconsistent with a force-free field, but they are expected to be closer to consistency. The self-consistency cycle is repeated using the force-free parameter values \(\alpha_2\) to obtain a new set of values \(\alpha_3\), and so on, until the P and N solutions converge to an identical NLFFF solution.

If the boundary values of \(\alpha\) on the P and N regions are highly inconsistent with the force-free model, the final self-consistent values of \(\alpha\) may be altogether different again. The

\(^1\)Wheatland and Régnier (2009) derived Equations 5 and 6 assuming Gaussian errors in \(\alpha\). In this article, we proceed with this tacit assumption. We do not expect our results to be substantially different for some range of different probability distributions.
field lines of the self-consistent field may disagree with structures seen in EUV, and the self-consistent field may have less free energy than that implied by the explosive events produced by the region. These problems were seen in the study of AR 12017 by Kleint et al. (2018). In such cases, we can consider one set of boundary conditions (i.e. P or N) to be favored over the other, because the corresponding solution reproduces structures of interest. This suggests the idea of weighting the self-consistent solution towards the favored boundary condition. We can weight the solution by decreasing the uncertainty values of $\alpha$ in the P or N region (that is, $\sigma_0$ or $\sigma_1$, respectively) by some factor. When $\alpha_2$ is calculated using Equation 5, reducing $\sigma_0$ (or $\sigma_1$) results in $\alpha_2$ being closer to $\alpha_0$ (or $\alpha_1$).

3. Application to an Analytic Bipole

In this section we demonstrate the weighting method, applied to an artificially constructed analytic bipole field.

Figure 1 shows the boundary conditions for our calculations. The boundary field $B_z(x, y, z = 0)$ is constructed in the region $0 \leq x \leq 1, 0 \leq y \leq 1$. In dimensionless units, $B_z$ at the bottom boundary of the volume is

$$B_z(x, y, z = 0) = B_{\text{max}} \exp[-c_1(x - 0.5)^2 - c_1(y - 0.6)^2] - B_{\text{max}} \exp[-c_2(x - 0.5)^2 - c_2(y - 0.4)^2],$$

where we set $B_{\text{max}}$ such that the maximum absolute field strength is 1, and $c_1 = c_2 = 200$. The boundary values of the force-free parameter $\alpha$ (again in dimensionless units) are chosen to be

$$\alpha(x, y, z = 0) = \begin{cases} 20, & \text{if } B_z(x, y, z = 0) > 0.9, \\ -20, & \text{if } B_z(x, y, z = 0) < -0.9. \end{cases}$$

We define $\alpha$ through this piecewise function for the sake of convenience. Other distributions of $\alpha$ can also be used, e.g. a ring distribution of the vertical current (Régnier, 2012). We note that the discontinuity in $\alpha$ has no effect on the convergence of the code. These boundary values of $\alpha$ are clearly inconsistent with a closed force-free field. A field line traced from the positive polarity to the negative polarity cannot have the same values of $\alpha$ at the two foot points, contradicting Equation 3.

We can apply the self-consistency procedure (Wheatland and Régnier, 2009) to find a single NLFFF solution. However, to do this we need to assign uncertainties $\sigma$ to the values of $\alpha$. A nominal choice is to choose equal uncertainties, i.e. $\sigma = 1$ everywhere. We call this the “equally-weighted” solution. We also consider choosing $\sigma$ to be constant in each polarity, but with a different value. Solutions constructed in this way are weighted towards the polarity with the smaller value of $\sigma$. Note that, strictly speaking, $\sigma$ in this case has no physical meaning, it is simply a weighting factor assigned to $\alpha$.

We use CFIT to calculate the P, N, and self-consistent NLFFF solution in the region $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 0.75$, with the calculation performed on a $400 \times 400 \times 300$ grid. The P (N) solution is calculated using the $\alpha$ value from the positive (negative) $B_z$ region. The self-consistent solution is calculated using the self-consistency procedure described in Section 2, for different choices of the uncertainties $\sigma$.

Figure 2 shows the magnetic field lines of the solutions calculated by CFIT. Blue (red) field lines are those traced from the positive (negative) poles, and yellow field lines are...
selected field lines that pass through $0.45 \leq x \leq 0.55$, $y = 0.5$, $z = 0.04$. Figure 2a shows the P solution, while Figure 2b shows the N solution. Because $\alpha$ is positive in the positive pole and negative in the negative pole, as given by Equation 8, the P and N solutions are antisymmetric, i.e. the field lines twist in opposite senses. We see in Figure 2c that the equally-weighted ($\sigma = 1$ everywhere) self-consistent solution is approximately a potential bipole field, and the self-consistent solution is very different from both the P and N solutions.

Figure 2d shows the self-consistent CFIT solution obtained by setting $\sigma = 0.8$ in the P region, and $\sigma = 1$ in the N region. The field lines show some twist in the sense of the P solution (Figure 2a). Figure 2e shows the solution with $\sigma = 0.4$ in the P region, and $\sigma = 1$ in the N region. The field lines begin to approximate those of the P solution (Figure 2a). Figures 2d and 2e therefore show how we can make the self-consistent solution more similar to the P solution by reducing $\sigma$ in the P region.

Figure 3 shows the self-consistent values of $\alpha$ at the bottom boundary. Figure 3a shows the P solution and Figure 3b shows the N solution. In the P (N) solution, field lines traced between the poles are assigned $\alpha = 20$ ($-20$), i.e. the value of $\alpha$ in the P (N) region. Figure 3c shows the self-consistent values of $\alpha$ at the bottom boundary for the equally-weighted solution. The self-consistent values of $\alpha$, calculated by successive averaging of the boundary values of the P and N solutions, have absolute values much smaller than 20 (the boundary value of $|\alpha|$ before the self-consistency procedure) everywhere, with positive and negative values of $\alpha$ dispersed in the boundary (note that, for clarity, the color scale in Figure 3c is different from the other panels of Figure 3). Figure 3d shows the values of $\alpha$ with $\sigma = 0.8$ in the P region, and $\sigma = 1$ in the N region. The values and distribution of $\alpha$ are now closer to the P solution, although the values are much smaller. Figure 3e shows the values of $\alpha$ with $\sigma = 0.4$ in the P region, and $\sigma = 1$ in the N region. The values and distribution of $\alpha$ approximate the P solution, with some small negative values of $\alpha$ remaining.
Figure 2. Nonlinear force-free field extrapolations of the analytic bipole generated by CFIT, with the boundary conditions given by Equations 7 and 8. (a) P solution, (b) N solution. (c) Self-consistent solution where the uncertainty value is uniform. (d) Self-consistent solution where the uncertainty value in the P region is 0.8 and the uncertainty in the N region is 1. (e) Self-consistent solution where the uncertainty in the P region is 0.4 and the uncertainty in the N region is 1.
Figure 3  The values of $\alpha$ at the lower boundary for the CFIT solutions generated using the boundary conditions given by Equations 7 and 8.  
(a) P solution.  
(b) N solution.  
(c) Self-consistent solution where the uncertainty value is uniform.  
(d) Self-consistent solution where the uncertainty value in the P region is 0.8 and the uncertainty in the N region is 1.  
(e) Self-consistent solution where the uncertainty in the P region is 0.4 and the uncertainty in the N region is 1.
4. Application to AR 12017

In this section, we apply our weighting method to the NLFFF reconstruction of the coronal field of active region AR 12017 for 2014 March 29, 17:36 UT.

On 2014 March 29, 17:48 UT, AR 12017 produced an X1 flare which was simultaneously observed and recorded by a number of instruments and observatories, such as the Interferometric BIdimensional Spectropolarimeter (IBIS) instrument at the Dunn Solar Telescope, the Hinode spacecraft, and the RHESSI spacecraft (Kleint et al., 2015).

Yang, Guo, and Ding (2016) generated NLFFF reconstructions of AR 12017 using the optimization method (Wheatland, Sturrock, and Roumeliotis, 2000; Wiegelmann, 2004), and Woods et al. (2018) reconstructed the field using the magnetohydrodynamic relaxation method of Inoue et al. (2014) and Inoue (2016). Kleint et al. (2018) presented a reconstruction of the field using CFIT, with the self-consistency procedure. The results of the optimization and relaxation methods, as well as SDO/AIA EUV observations, clearly show a filament present near the polarity inversion line which erupts during the flare. However, the self-consistent CFIT NLFFF solution does not contain a flux rope structure at the polarity inversion line (Kleint et al., 2018). The P solution does show a highly twisted flux rope, which is absent in the N solution, and correspondingly, is absent in the self-consistent solution. Hence we consider applying the weighted self-consistent reconstruction procedure, with the boundary conditions on $\alpha$ weighted towards the positive polarity.

The relevant vector magnetogram boundary data for our NLFFF extrapolation are the SHARP data for AR 12017, at 17:36 UT on 2014 March 29. The SHARP patch is $642 \times 504$ pixels, where each pixel is $0.502 \times 0.502$ arcsec. We approximate the SHARP magnetic field data, given in Lambert equal-area projection, as field values on a Cartesian grid as explained in Section 2, and solve for the magnetic field in a $642 \times 504 \times 200$ Cartesian box. The vertical size is chosen to capture as much large-scale structure as possible without unduly slowing down the calculation. For convenience, we work in dimensionless quantities. We use the width of the solution box as the length scale, such that the solution box covers the region $0 \leq x \leq 1$, $0 \leq y \leq 0.785$, and $0 \leq z \leq 0.31$. We normalize the strength of each component of the vector magnetic field ($B_x$, $B_y$, and $B_z$) to the maximum of the absolute value of $B_z$ at $z = 0$ (in this case, $|B_z|_{\text{max}} = 2.649 \times 10^3$ G). The boundary values of $\alpha$ are calculated using Equation 4, and the associated uncertainties are propagated from the uncertainties of $B$ given in the SHARP data (Bobra et al., 2014; Hoeksema et al., 2014), to give a map of the uncertainties $\sigma$ in $\alpha$ that we use in the self-consistency cycles. The values of $\alpha$ and $\sigma$ are non-dimensionalized by multiplying by the length scale. To reduce the influence of unphysical currents, we set the current to zero in pixels where the signal-to-noise ratio of the current is less than one or where $|B_z| < 0.05$. Figure 4 shows the maps of the boundary values of $B_z$ in units of gauss, $\alpha$ in dimensionless units, and $\sigma$ in dimensionless units (saturated to 250 for clarity), from the data. Weak-field areas, where $|B_z| < 0.05|B_z|_{\text{max}}$, are assigned $\alpha = 0$ and a maximum value of $\sigma$.

Figure 6 shows the results of the CFIT NLFFF extrapolations. The P solution, Figure 6a, shows a highly twisted flux rope, associated with high values of $\alpha$ near the polarity inversion line, which coincides with the filament structure seen in the AIA image. On the other hand, the N solution, Figure 6b, does not show such a structure. The self-consistent solution, Figure 6c, has lost the high values of $\alpha$ present in the P boundary condition, and appears as a sheared rather than a twisted field. These results are consistent with Kleint et al. (2018).

Because we want to generate a self-consistent solution which preserves the high values of $\alpha$ in the P solution (in particular associated with the twisted flux rope), we weight the boundary condition towards the P solution by multiplying $\sigma$ in the P region at locations...
Figure 4  Maps of the vertical magnetic field values ($B_z$), force-free parameter ($\alpha$, in dimensionless units), and the absolute value of the uncertainty $\sigma$ in $\alpha$ on the plane tangent to the solar surface. Panel (a) shows $B_z$, (b) shows the parameter $\alpha$ from the data (before the self-consistency procedure), and (c) shows the absolute value of the uncertainty $\sigma$ in $\alpha$ from the data.

where $50 < |\alpha| < 150$ (in dimensionless units) by a factor of 0.1. We arrive at these upper and lower limits on $|\alpha|$ by visually comparing the P solution to the AIA images. We inspect the foot points of the flux rope in the P solution and we conclude that they are associated with $|\alpha| \gtrsim 50$ (or $|\alpha| \gtrsim 2.14 \times 10^{-7}$ m\(^{-1}\) in dimensional units). In addition, because we want to limit the effects of high, unphysical values of $\alpha$ (which can prevent CFIT from converging to a self-consistent solution), we set an upper limit of $|\alpha| < 150$ (or $|\alpha| < 6.42 \times 10^{-7}$ m\(^{-1}\)). Figure 5 shows maps of the boundary condition on $B_z$, $\alpha$, and $\sigma$ (saturated to 250 for clarity) with the weighted regions outlined.\(^2\) More systematic ways of identifying $\alpha$ values are available, such as that proposed by Malanushenko, Longcope, and McKenzie (2009). For our purposes, visual inspection and comparison suffice. We present the result in Figure 6d. The new self-consistent solution clearly preserves the flux rope in the P solution (Figure 6a), which is absent in the unweighted self-consistent solution (Figure 6c).

We can calculate the free energy of our solutions, defined as the difference between total energy $E$ of the extrapolated NLFFF and the energy $E_0$ of the potential component of the NLFFF. The free energy of the P solution is $8.1 \times 10^{31}$ erg (corresponding to $E/E_0 = 1.18$), the free energy of the N solution is $2.0 \times 10^{31}$ erg ($E/E_0 = 1.05$), the free energy of the

\(^2\) Examples of similarly highly twisted flux ropes in other active region extrapolations can be found in, e.g. Sharykin et al. (2017), Yan et al. (2017), and Inoue et al. (2018).
Figure 5  Maps of the vertical magnetic field values ($B_z$), force-free parameter ($\alpha$, in dimensionless units), and the absolute value of the uncertainty $\sigma$ in $\alpha$ on the plane tangent to the solar surface, with the weighted regions indicated. Panel (a) shows $B_z$, (b) shows the parameter $\alpha$ from the data (before the self-consistency procedure), and (c) shows the absolute value of the uncertainty $\sigma$ in $\alpha$ from the data. In panels (a)–(c), the contours enclose the regions where we reduce $\sigma$.

The unweighted self-consistent solution is $5.4 \times 10^{30}$ erg ($E/E_0 = 1.015$), and the free energy of the weighted self-consistent solution is $3.4 \times 10^{31}$ erg ($E/E_0 = 1.09$). The unweighted self-consistent solution has the least free energy of these four solutions, even less than the free energy of the N solution. The weighted solution has more free energy than the N solution, but less free energy than the P solution.

We can also characterize the fields using relative helicity $H_r$ (Berger and Field, 1984; Finn and Antonsen, 1985). Using the definition given by, e.g., Démoulin and Berger (2003), we separate the magnetic field $B$ in the computational volume $V$ into the potential field $B_p$ and the current-carrying field $B_j$

$$B = B_p + B_j,$$

where

$$B \cdot \hat{n} |_{\partial V} = B_p \cdot \hat{n} |_{\partial V},$$

where $\partial V$ is the boundary surface and $\hat{n}$ is the unit vector normal to $\partial V$, and we write

$$H_r = \int_V (A + A_p) \cdot (B - B_p) dV.$$
For the NLFFF presented here, \( H_r = 8.72 \times 10^{44} \text{ Mx}^2 \) for the P solution, \( H_r = 8.73 \times 10^{44} \text{ Mx}^2 \) for the N solution, \( H_r = 9.11 \times 10^{44} \text{ Mx}^2 \) for the unweighted self-consistent solution, and \( H_r = 9.91 \times 10^{44} \text{ Mx}^2 \) for the weighted solution. The weighted solution has larger currents than the unweighted solution and consequently higher relative helicity. It is, however, interesting that the values of \( H_r \) for the P and N solutions are similar, despite the P solution having larger currents. To see the reason, we express \( H_r \) as a sum of the self helicity of the current-carrying part \( H_j \) and the mutual helicity between the potential and current-carrying fields \( H_{pj} \) (Berger, 1999), where

\[
H_j = \int_V (A - A_p) \cdot (B - B_p) dV
\]  

(12)

and

\[
H_{pj} = 2 \int_V A_p \cdot (B - B_p) dV.
\]  

(13)

For our NLFFF, we find \( H_{pj} = 7.29 \times 10^{44} \text{ Mx}^2 \) for the P solution, \( H_{pj} = 8.41 \times 10^{44} \text{ Mx}^2 \) for the N solution, \( H_j = 1.43 \times 10^{44} \text{ Mx}^2 \) for the P solution, and \( H_j = 3.14 \times 10^{43} \text{ Mx}^2 \) for the N solution. While the value of \( H_j \) for the P solution is higher than that for the N solution, the value of \( H_j \) is dominated by the mutual helicity \( H_{pj} \) which is similar for both the P and N solutions. In this case, the presence of large currents in the P solution is obscured in the relative magnetic helicity \( H_r \) as defined by Equation 11.

CFIT calculates a potential field solution using the Fourier transform method (Alissandrakis, 1981; Wheatland, 2006, 2007). This potential field \( B_0 \) satisfies periodic lateral boundary conditions and does not satisfy Equation 10 at the lateral boundaries. The field \( B_0 \) has a lower energy than the field \( B_p \), but cannot be chosen as the reference field directly, since its lateral boundary conditions lead to a gauge-dependent result for relative helicity. To make use of \( B_0 \), a new definition for relative magnetic helicity was proposed by Yang, Wheatland, and Gilchrist (2020). The field \( B \) is decomposed as

\[
B = B_0 + B_c,
\]  

(14)

where

\[
B \cdot \hat{n} = B_0 \cdot \hat{n}
\]  

(15)

at the top and bottom boundaries, and \( B_0 \) is periodic at the lateral boundaries. Note that \( B_c \cdot \hat{n} \) no longer vanishes at the lateral boundaries, unlike \( B_j \cdot \hat{n} \). The field \( B_c \) is further decomposed as (Berger and Field, 1984)

\[
B_c = B_{c1} + B_{p1},
\]  

(16)

where \( B_{p1} \) is the solution of Laplace’s equation that satisfies

\[
B_c \cdot \hat{n}|_{\partial V} = B_{p1} \cdot \hat{n}|_{\partial V}.
\]  

(17)

A gauge-independent relative magnetic helicity \( H_{cr} \) for the field \( B_c \) can then be defined as (Yang, Wheatland, and Gilchrist, 2020)

\[
H_{cr} = \int_V (A_c + A_{p1}) \cdot (B_c - B_{p1}) dV.
\]  

(18)
Figure 6  The nonlinear force-free magnetic field solutions obtained from the CFIT code and overlaid onto the contemporaneous 171 Å AIA image. The field extrapolations are derived from the 2014 March 29, 17:36 UT, SHARP vector magnetogram data for AR 12017. The panels show (a) the P solution, (b) the N solution, (c) the (unweighted) self-consistent solution, and (d) the self-consistent solution from a boundary condition weighted towards the P solution.

For the NLFFF presented here, \( H_{cr} = 1.48 \times 10^{44} \text{ Mx}^2 \) for the P solution, \( H_{cr} = 4.04 \times 10^{43} \text{ Mx}^2 \) for the N solution, \( H_{cr} = 5.70 \times 10^{41} \text{ Mx}^2 \) for the unweighted self-consistent solution, and \( H_{cr} = 6.38 \times 10^{43} \text{ Mx}^2 \) for the weighted solution. We note that the value of \( H_{cr} \) for the P solution is higher than that for the N solution as expected on the basis of the currents in the two fields. These results suggest that \( H_{cr} \) may be a more suitable measure of the relative magnetic helicity for these solutions.

Figure 7 shows maps the current density \( J_z(x, y, z = 0) \) in units of mA m\(^{-2}\) and a selection of extrapolated NLFFF field lines. Figure 7a and Figure 7b show the values of the boundary data and the self-consistent vertical current densities \( J_z \), respectively, with the weighted areas outlined. We see that, while the distribution of the current has been smoothed out, the weighting procedure preserves areas of strong current density. We show a plot of
Figure 7  Maps of the vertical magnetic field values ($B_z$) and vertical current density ($J_z$) on the plane tangent to the solar surface. Panel (a) shows the initial boundary conditions on $J_z$ (before the self-consistency procedure), (b) shows $J_z$ corresponding to the weighted self-consistent NLFFF solution, and (c) shows selected field lines of the weighted self-consistent NLFFF solution, superimposed on the $B_z$ map. In panels (a)–(b), the contours enclose the regions where we reduce $\sigma$.

As Figure 6 shows, the weighted self-consistency procedure results in a significantly better match to SDO/AIA observations than the unweighted self-consistency procedure of Wheatland and Régnier (2009) as implemented by Kleint et al. (2018). Comparison can also be made with the NLFFF extrapolated by Woods et al. (2018) using the magnetohydrodynamical relaxation method of Inoue et al. (2014) and Inoue (2016). Figure 2d of Woods et al. (2018) shows their result for the 17:36 UT time step. The red field lines in our Figure 6d and Figure 7c are similar to the blue field lines in Figure 2d of Woods et al. (2018), and display similar connectivities. Our result does not show any structures similar to the gold flux rope of Woods et al. (2018). This flux rope may be associated with $\alpha$ values outside our selected weighting threshold. More work is required to see if we can reproduce the gold flux rope using the weighted self-consistent Grad–Rubin method. Direct quantitative comparison be-
tween our method and others, however, is not possible without a detailed test involving a known 3D NLFFF.

5. Conclusions

In this article, we present an improvement to the self-consistency procedure implemented using the Grad–Rubin code CFIT (Wheatland and Régnier, 2009). It is desirable to arrive at a self-consistent NLFFF solution for coronal magnetic fields using the boundary data given by vector magnetogram observations. In general, the values of $\alpha$ in the P and N magnetic polarities are not consistent with a force-free field. The self-consistency procedure of Wheatland and Régnier (2009) solves this problem by constructing P and N solutions and taking uncertainty-weighted averages of the boundary values of $\alpha$ from the solutions, iteratively, to arrive at a self-consistent solution. We demonstrate this with an analytic bipole, which has inconsistent boundary values of $\alpha$ on the P and N poles by construction. The unweighted self-consistent solution using these boundary conditions is different from both the P and N solutions, and is close to a potential field. We show that by reducing the uncertainties $\sigma$ in the values of $\alpha$ in the P region, we can skew the self-consistent solution towards the P solution.

We apply this procedure to active region AR 12017. The P and N solutions obtained by CFIT from the 2014 March 29, 17:36 UT time step are inconsistent with each other. When the P and N boundary conditions differ significantly, the resulting self-consistent NLFFF solution may, like the simple bipole case, be close to a potential field, and not provide an accurate model for the true coronal field, based e.g. on comparison of field lines with structures observed in EUV images. The P and N solutions for this time step demonstrate this. Because the P solution has field lines which are more consistent with the observations, and contains greater free energy, we weight the boundary conditions towards the P region. To achieve this, we identify the foot points of the flux ropes in the P solution and multiply the values of $\sigma$ at the foot points by a factor of 0.1. We calculate the new self-consistent solution and, like the P solution, it shows the flux rope evident in EUV.

In this article, we show how a simple modification to the Grad–Rubin CFIT code can achieve self-consistent NLFFF solutions using boundary conditions derived from readily available SDO/HMI photospheric vector magnetograms, taking into account the errors in the tangential field strength measurements. We show how visible field structures can be used as an additional constraint in a Grad–Rubin NLFFF extrapolation. In the example presented in this article, the choice of which pixels to be weighted is based on visual inspection. In future work, we will investigate an automated approach, similar to those proposed by Malanushenko, Longcope, and McKenzie (2009) and Malanushenko et al. (2014).

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References

Alissandrakis, C.E.: 1981, On the computation of constant alpha force-free magnetic field. *Astron. Astrophys.* 100(1), 197. DOI.

Amari, T., Aly, J.-J.: 2010, Observational constraints on well-posed reconstruction methods and the optimization-Grad–Rubin method. *Astron. Astrophys.* 522, A52. DOI.

Amari, T., Boulmezaoud, T.Z., Mikic, Z.: 1999, An iterative method for the reconstruction of the solar coronal magnetic field. I. Method for regular solutions. *Astron. Astrophys.* 350, 1051. ADS.

Berger, M.A.: 1999, Introduction to magnetic helicity. *Plasma Phys. Control. Fusion* 41(12B), B167. DOI.

Berger, M.A., Field, G.B.: 1984, The topological properties of magnetic helicity. *J. Fluid Mech.* 147, 133. DOI.

Bobra, M.G., Sun, X., Hoeksema, J.T., Turmon, M., Liu, Y., Hayashi, K., Barnes, G., Leka, K.D.: 2014, The Helioseismic and Magnetic Imager (HMI) vector magnetic field pipeline: SHARPs – Space-Weather HMI Active Region Patches. *Solar Phys.* 289(9), 3549. DOI.

De Rosa, M.L., Schrijver, C.J., Barnes, G., Leka, K.D., Lites, B.W., Aschwanden, M.J., Amari, T., Canou, A., McTiernan, J.M., Régnier, S., Thalmann, J.K., Valori, G., Wheatland, M.S., Wiegelmann, T., Cheung, M.C.M., Conlon, P.A., Fuhrmann, M., Inhester, B., Tadesse, T.: 2009, A critical assessment of nonlinear force-free field modeling of the solar corona for active region 10953. *Astrophys. J.* 696(2), 1780. DOI.

Démoulin, P., Berger, M.A.: 2003, Magnetic energy and helicity fluxes at the photospheric level. *Solar Phys.* 215(2), 203. DOI.

Finn, J.M., Antonsen, T.M.J.: 1985, Magnetic helicity: what is it and what is it good for. *Comments Plasma Phys. Control. Fusion* 9, 111. http://inis.iaea.org/search/search.aspx?orig_q=RN:16069504.

Gary, G.A.: 2001, Plasma beta above a solar active region: rethinking the paradigm. *Solar Phys.* 203(1), 71. DOI.

Grad, H., Rubin, H.: 1958, Hydromagnetic equilibria and force-free fields. In: *Proc. 2nd Int. Conf. on Peaceful Uses of Atomic Energy* 31, 190.

Hoeksema, J.T., Liu, Y., Hayashi, K., Sun, X., Schou, J., Couvidat, S., Norton, A., Bobra, M., Centeno, R., Leka, K.D., Barnes, G., Turmon, M.: 2014, The Helioseismic and Magnetic Imager (HMI) vector magnetic field pipeline: overview and performance. *Solar Phys.* 289(9), 3483. DOI.

Inoue, S.: 2016, Magnetohydrodynamics modeling of coronal magnetic field and solar eruptions based on the photospheric magnetic field. *Prog. Earth Planet. Sci.* 3(1), 19. DOI.

Inoue, S., Magara, T., Pandey, V.S., Shiota, D., Kusano, K., Choe, G.S., Kim, K.S.: 2014, Nonlinear force-free extrapolation of the coronal magnetic field based on the magnetohydrodynamic relaxation method. *Astrophys. J.* 780(1), 101. DOI.

Inoue, S., Shiota, D., Bamba, Y., Park, S.-H.: 2018, Magnetohydrodynamic modeling of a solar eruption associated with an X9.3 flare observed in the active region 12673. *Astrophys. J.* 867(1), 83. DOI.

Klein, L., Battaglia, M., Reardon, K., Sainz Dalda, A., Young, P.R., Krucker, S.: 2013, The fast filament eruption leading to the X-flare on 2014 March 29. *Astrophys. J.* 806(1), 9. DOI.

Klein, L., Wheatland, M.S., Mastrano, A., McCauley, P.I.: 2018, Nonlinear force-free modeling of flare-related magnetic field changes at the photosphere and chromosphere. *Astrophys. J.* 865(2), 146. DOI.

Malanushenko, A., Longcope, D.W., McKenzie, D.E.: 2009, Reconstructing the local twist of coronal magnetic fields and the three-dimensional shape of the field lines from coronal loops in extreme-ultraviolet and X-ray images. *Astrophys. J.* 707(2), 1044. DOI.

Malanushenko, A., Schrijver, C.J., DeRosa, M.L., Wheatland, M.S.: 2014, Using coronal loops to reconstruct the magnetic field of an active region before and after a major flare. *Astrophys. J.* 783(2), 102. DOI.

Régnier, S.: 2012, Magnetic energy storage and current density distributions for different force-free models. *Solar Phys.* 277(1), 131. DOI.

Régnier, S.: 2013, Magnetic field extrapolations into the corona: success and future improvements. *Solar Phys.* 288(2), 481. DOI.

Schrijver, C.J., DeRosa, M.L., Metcalf, T.R., Liu, Y., McTiernan, J., Régnier, S., Valori, G., Wheatland, M.S., Wiegelmann, T.: 2006, Nonlinear force-free modeling of coronal magnetic fields part I: a quantitative comparison of methods. *Solar Phys.* 235(1-2), 161. DOI.

Schrijver, C.J., DeRosa, M.L., Metcalf, T., Barnes, G., Lites, B., Tarbell, T., McTiernan, J., Valori, G., Wiegelmann, T., Wheatland, M.S., Amari, T., Aulanier, G., Démoulin, P., Fuhrmann, M., Kusano, K., Régnier, S., Thalmann, J.K.: 2008, Nonlinear force-free field modeling of a solar active region around the time of a major flare and coronal mass ejection. *Astrophys. J.* 675(2), 1637. DOI.
Sharykin, I.N., Sadykov, V.M., Kosovichev, A.G., Vargas-Dominguez, S., Zimovets, I.V.: 2017, Flare energy release in the lower solar atmosphere near the magnetic field polarity inversion line. Astrophys. J. 840(2), 84. DOI. ADS.

Sun, X.: 2013, On the coordinate system of Space-Weather HMI Active Region Patches (SHARPs): a technical note. arXiv e-prints, arXiv. ADS.

Wheatland, M.S.: 2006, A fast current-field iteration method for calculating nonlinear force-free fields. Solar Phys. 238, 29. DOI. ADS.

Wheatland, M.S.: 2007, Calculating and testing nonlinear force-free fields. Solar Phys. 245, 251. DOI. ADS.

Wheatland, M.S., Régnier, S.: 2009, A self-consistent nonlinear force-free solution for a solar active region magnetic field. Astrophys. J. Lett. 700, L88. DOI. ADS.

Wheatland, M.S., Sturrock, P.A., Roumeliotis, G.: 2000, An optimization approach to reconstructing force-free fields. Astrophys. J. 540, 1150. DOI. ADS.

Wiegelmann, T.: 2004, Optimization code with weighting function for the reconstruction of coronal magnetic fields. Solar Phys. 219(1), 87. DOI. ADS.

Wiegelmann, T., Inhester, B., Sakurai, T.: 2006, Preprocessing of vector magnetograph data for a nonlinear force-free magnetic field reconstruction. Solar Phys. 233, 215. DOI.

Wiegelmann, T., Petrie, G.J.D., Riley, P.: 2017, Coronal magnetic field models. Space Sci. Rev. 210(1-4), 249. DOI. ADS.

Wiegelmann, T., Sakurai, T.: 2012, Solar force-free magnetic fields. Living Rev. Solar Phys. 9(1), 5. DOI. ADS.

Woods, M.M., Inoue, S., Harra, L.K., Matthews, S.A., Kusano, K., Kalmoni, N.M.E.: 2018, The triggering of the 2014 March 29 filament eruption. Astrophys. J. 860(2), 163. DOI. ADS.

Yan, X.L., Jiang, C.W., Xue, Z.K., Wang, J.C., Priest, E.R., Yang, L.H., Kong, D.F., Cao, W.D., Ji, H.S.: 2017, The eruption of a small-scale emerging flux rope as the driver of an M-class flare and of a coronal mass ejection. Astrophys. J. 845(1), 18. DOI. ADS.

Yang, K., Guo, Y., Ding, M.D.: 2016, Quantifying the topology and evolution of a magnetic flux rope associated with multi-flare activities. Astrophys. J. 824(2), 148. DOI. ADS.

Yang, K.E., Wheatland, M.S., Gilchrist, S.A.: 2020, Relative magnetic helicity based on a periodic potential field. Astrophys. J. 894(2), 151. DOI. ADS.