Abstract. The Yamabe problem in compact closed Riemannian manifolds is concerned with finding a metric with constant scalar curvature in the conformal class of a given metric. This problem was solved by the combined work of Yamabe, Trudinger, Aubin, and Schoen. In particular, Aubin solved the case when the Riemannian manifold is compact, is nonlocally conformally flat and has a dimension equal to or greater than 6. In 2015, Case considered a Yamabe-type problem in the setting of smooth measure space in manifolds and for a parameter $m$, which generalizes the original Yamabe problem when $m = 0$. Additionally, Case solved this problem when the parameter $m$ is a natural number. In the context of the Yamabe-type problem, we generalize Aubin’s result for nonlocally conformally flat manifolds, with dimension equal and greater than 6 and parameter $m$ close to nonnegative integers.

1. Introduction

Let $(M^n, g)$ be an $n$-dimensional compact Riemannian manifold and $R_g$ be the scalar curvature associated with the metric $g$. The Yamabe problem is concerned with finding a metric of constant scalar curvature in the conformal class of $g$. It is well known that the Yamabe problem was solved by the combined work of Yamabe [11], Trudinger [10], Aubin [1], and Schoen [9]; for a presentation of this topic, see [6]. In particular, we mention that Aubin in [1] solved the problem under the hypothesis that the Riemannian manifold is compact, nonlocally conformally flat and with dimension $n \geq 6$.

In [2] and [3], Case considered some geometric invariants that he called the weighted Yamabe constants, which constitute a one-parameter family and interpolate between the Yamabe constant and Perelman’s $\nu$-entropy when the parameter is zero and infinity, respectively. The Yamabe constant is the curved analogue of the sharp Sobolev inequality, and the $\nu$-entropy is the curved analogue of the sharp logarithmic Sobolev inequality. The weighted Yamabe constants are thus curved analogues of a family of sharp Gagliardo-Nirenberg-Sobolev inequalities.

Before we explain Case’s results, we introduce some terminology. Let us denote by $dV_g$ the volume form induced by the metric $g$. Fix a function $\phi \in C^\infty(M)$ and a dimensional
parameter \( m \in [0, \infty] \). A smooth metric measure space is a four-tuple \((M^n, g, e^{-\phi} dV_g, m)\). Let us denote by \( \Delta_g \) and \( \nabla_g \) the Laplacian and the gradient associated to the metric \( g \), respectively. The weighted scalar curvature \( R^m_\phi \) of a smooth metric measure space for \( m = 0 \) is \( R^m_\phi = R_g \) and for \( m \neq 0 \) is the function \( R^m_\phi := R_g + 2\Delta_g \phi - \frac{m+1}{m} |\nabla_g \phi|^2 \). The weighted Yamabe quotient is the functional \( Q : C^\infty(M) \to \mathbb{R} \) defined by

\[
Q(w) = \frac{\int_M (|\nabla_g w|^2 + \frac{m+n-2}{4(m+n-1)} R^m_\phi w^2) e^{-\phi} dV_g}{\left( \int_M |w|^{\frac{2(m+n)}{m+n-2}} e^{-\frac{(m-1)\phi}{m}} dV_g \right)^{2m}}.
\]  

The weighted Yamabe constant is the number

\[
\Lambda[M^n, g, e^{-\phi} dV_g, m] = \inf \{ Q(w) : w \in C^\infty(M), w \neq 0 \}.
\]

For \( m = \infty \), Case defined the weighted Yamabe quotient as the limit of (1) when \( m \) goes to infinity and the weighted Yamabe constant as (2). Note that when \( m = 0 \), the weighted Yamabe constant coincides with the Yamabe constant, and when \( m = \infty \), if the weighted Yamabe constant is positive, this is equivalent to Perelman’s entropy (see [8]). By the Gagliardo-Nirenberg inequality, it is possible to consider the functional defined in (1) and the constant defined in (2) for \((\mathbb{R}^n, dx^2, dV, m)\), where \( dx^2 \) is the usual metric in \( \mathbb{R}^2 \) and \( dV \) is the volume form induced by this metric. We denote by \( \Lambda_{m,n} \) the weighted Yamabe constant \( \Lambda(\mathbb{R}^n, dx^2, dV, m) \) (see Theorem 3 below).

Two smooth measure metric spaces \((M^n, g, e^{-\phi} dV_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}} d\hat{V}_g, m)\) are pointwise conformally equivalent if there is a function \( \sigma \in C^\infty(M) \) such that \( \hat{g} = e^{a(m+n-2) \sigma} g \) and \( \hat{\phi} = \frac{m \sigma}{m+n-2} + \phi \). We say that \((M^n, g, e^{-\phi} dV_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}} d\hat{V}_g, m)\) are conformally equivalent if there is a diffeomorphism \( F : \hat{M} \to M \) such that \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}} d\hat{V}_g, m)\) is pointwise conformally equivalent to \((F^{-1}(M), F^*g, F^*(e^{-\phi} dV_g), m)\).

The weighted Yamabe problem is to find a function that minimizes the weighted Yamabe quotient. In [2], Case proved an Aubin-type criterion for the existence of a minimizer of the weighted Yamabe quotient. The exact statement is:

**Theorem 1** ([2]). Let \((M^n, g, e^{-\phi} dV_g, m)\) be a compact smooth metric measure space such that \( m \geq 0 \). Then,

\[
\Lambda[M^n, g, e^{-\phi} dV_g, m] \leq \Lambda_{m,n}.
\]

Furthermore, if the inequality (3) is strict, then there exists a smooth positive function \( w \) such that

\[
Q(w) = \Lambda[M^n, g, e^{-\phi} dV_g, m].
\]

Additionally, in [2], Case proved the strict inequality in (3) when \( m \in \mathbb{N} \cup \{0\} \), together with a characterization for the equality in (3).
Theorem 2 ([2]). Let \((M^n, g, e^{-\phi}dV_g, m)\) be a compact smooth metric measure space such that \(m \in \mathbb{N} \cup \{0\}\). The following equality holds
\[
\Lambda[M^n, g, e^{-\phi}dV_g, m] = \Lambda_{m,n}
\]
if and only if \((M^n, g, e^{-\phi}dV_g, m)\) is conformally equivalent to \((S^n, g_0, dV_{g_0}, m)\), where \((S^n, g_0)\) is the n-dimensional sphere with a metric of constant sectional curvature and \(m \in \{0, 1\}\). Therefore, there exists a smooth positive function \(w\) such that
\[
Q(w) = \Lambda[M^n, g, e^{-\phi}dV_g, m].
\]

In this paper, we prove for nonlocally conformally flat manifolds with dimension \(n \geq 6\) and \(m \in \bigcup_{i \in \mathbb{N} \cup \{0\}} [i, i + \delta)\) for some \(\delta > 0\) that inequality (3) is strict. By Theorem [1], the existence of a minimizer of the weighted Yamabe problem follows. This result is a generalization of the Aubin existence theorem and a generalization of the Case existence result for \(m\) close to the integers.

Theorem A. Let \((M^n, g)\) be a compact Riemannian manifold, \(\phi \in C^\infty(M)\), and \(n \geq 6\). If \((M, g)\) is nonlocally conformally flat, there exist \(0 < \delta \leq 1\) such that for
\[
m \in \bigcup_{i \in \mathbb{N} \cup \{0\}} [i, i + \delta)
\]
we obtain
\[
\Lambda[M^n, g, e^{-\phi}dV_g, m] < \Lambda_{m,n}.
\]
Therefore, there exists a smooth positive minimizer of the weighted Yamabe quotient.

We use arguments similar to those Aubin used in [1] to prove Theorem A. These arguments involve test functions in the Yamabe quotient with support in a neighborhood of a point where the Weyl tensor is nonzero. However, when we restrict to the case \(m = 0\), we use different test functions from the ones used in [1]. For this reason, our proof is different than Aubin’s Theorem for \(n \geq 7\).

This paper is organized as follows. In sections [2] and [3] we present some basic concepts about smooth metric measure spaces and the Yamabe-type problem on these spaces, respectively. In section [4] we prove Theorem A. We complement this with an appendix devoted to some calculus lemmas that we use in the proof of Theorem A.

2. Smooth metric measure space and the conformal Laplacian

Our approach in this section is based on [2] and [3]. The first step is to introduce the definition of a smooth metric measure space.
Definition 1. Let \( (M^n, g) \) be a Riemannian manifold, and let us denote by \( dV_g \) the volume form induced by \( g \) in \( M \). Fix a function \( \phi \in C^\infty(M) \) and a dimensional parameter \( m \in [0, \infty] \). When \( m = 0 \), we require that \( \phi = 0 \). A smooth metric measure space is the four-tuple \((M^n, g, e^{-\phi}dV_g, m)\).

As in [2], we sometimes denote by the three-tuple \((M^n, g, v^m dV_g)\) a smooth metric measure space where \( v \) and \( \phi \) are related by \( v^m = e^{-\phi} \). We denote by \( R_g, \text{Ric}_g, T_g, \text{and } W_g \) the scalar curvature, the Ricci tensor, traceless Ricci tensor, and the Weyl tensor of \((M, g)\), respectively. In some cases, we omit the reference to the metric \( g \) and write \( R, \text{Ric}, T, \text{and } W \). In the following definitions, we consider the case of \( m = \infty \) as the limit case of the parameter \( m \).

Definition 2. Given a smooth metric measure space \((M^n, g, e^{-\phi}dV_g, m)\), the weighted scalar curvature \( R^m_\phi \) and the Bakry-Émery Ricci curvature \( \text{Ric}^m_\phi \) are the tensors

\[
R^m_\phi := R_g + \frac{m+1}{m} |\nabla_g \phi|^2_g
\]

and

\[
\text{Ric}^m_\phi := \text{Ric} + \text{Hess}_g \phi - \frac{1}{m} d\phi \otimes d\phi,
\]

where \( \Delta_g \) is the usual Laplacian, \( \nabla_g \) is the gradient, \( |\cdot|_g \) is the tensor norm, and \( \text{Hess}_g \) is the Hessian, all of which are calculated in the metric \( g \).

Since \( \phi = -m \ln v \), equality (6) takes the form of

\[
R^m_\phi = R_g + \frac{m(1-m)}{v^2} |\nabla_g v|^2_g - \frac{2m}{v} \Delta_g v.
\]

Definition 3. Let \((M^n, g, e^{-\phi}dV_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}d\hat{V}_g, m)\) be smooth metric measure spaces. We say that they are pointwise conformally equivalent if there is a function \( \sigma \in C^\infty(M) \) such that

\[
\hat{g} = e^{m\sigma/2n-2} g \quad \text{and} \quad \hat{\phi} = \frac{-m\sigma}{m+n-2} + \phi.
\]

\((M^n, g, e^{-\phi}dV_g, m)\) and \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}d\hat{V}_g, m)\) are conformally equivalent if there is a diffeomorphism \( F : \hat{M} \to M \) such that \((F^{-1}(M), F^*g, F^*(e^{-\phi}dV_g), m)\) is pointwise conformally equivalent to \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}d\hat{V}_g, m)\).

Remark 1. We will denote with a hat all quantities computed with respect to the smooth metric measure space \((\hat{M}^n, \hat{g}, e^{-\hat{\phi}}d\hat{V}_g, m)\). On the other hand, equalities in (9) imply

\[
e^{-\phi}dV_g = e^{m\sigma/2}\nabla g dV_g.
\]
Definition 4. Given a compact smooth metric measure space \((M^n, g, e^{-\phi}dV_g, m)\), the weighted Laplacian \(\Delta_\phi : C^\infty(M) \to C^\infty(M)\) is an operator defined by
\[
\Delta_\phi u = \Delta_g u - \nabla_g u \cdot \nabla_g \phi,
\]
where \(u \in C^\infty(M)\).

Definition 5. Let \((M^n, g, e^{-\phi}dV_g, m)\) be a smooth metric measure space. The weighted conformal Laplacian \(L^m_\phi\) is given by the operator
\[
L^m_\phi = -\Delta_\phi + \frac{m+n-2}{4(m+n-1)} R^m_\phi.
\]

Proposition 1. Let \((M^n, g, e^{-\phi}dV_g, m)\) and \((M^n, \hat{g}, e^{-\hat{\phi}}dV_{\hat{g}}, m)\) be two pointwise conformally equivalent smooth metric measure spaces such that \(\hat{g} = e^{\frac{2\phi}{m+n-2}}g\) and \(\hat{\phi} = \frac{m\sigma}{m+n-2} + \phi\). Let us denote by \(\hat{L}^m_\phi\) and \(\hat{L}^m_\phi\) their respective weighted conformal Laplacians. Then, we have \(\hat{v} = e^{\frac{m\sigma}{m+n-2}}v\), and the following transformation rules
\[
\begin{align*}
\hat{L}^m_\phi(w) &= e^{-\frac{m+n+2}{2(m+n-2)}\sigma} L^m_\phi(e^{\frac{\sigma}{2}}w), \\
\nabla_{\hat{g}} \hat{v} &= e^{-\frac{\sigma}{m+n-2}} \left( \nabla_g v + \frac{v}{m+n-2} \nabla_g \sigma \right), \\
R_{\hat{g}} &= e^{-\frac{2\phi}{m+n-2}} (R_g + \frac{2(n-1)}{m+n-2} \Delta_g \sigma - \frac{(n-1)(n-2)}{(m+n-2)^2} |\nabla_g \sigma|_g^2), \\
\hat{R}_{ij}(p) &= R_{ij} - \frac{n-2}{m+n-2} \sigma_{ij} + \frac{n-2}{(m+n-2)^2} \sigma_i \sigma_j + \left( \frac{\Delta_g \sigma}{m+n-2} - \frac{n-2}{(m+n-2)^2} |\nabla_g \sigma|_g^2 \right) g_{ij}.
\end{align*}
\]

Proof. We mention that the identity (10) appears in [2]. Equality (12) follows from (10) when \(m = 0\) and \(w \equiv 1\). The identity \(\hat{v} = e^{\frac{m\sigma}{m+n-2}}v\) follows from the relations \(\hat{v}^m = e^{-\phi}, v^m = e^{-\phi}\), and \(\hat{\phi} = \frac{m\sigma}{m+n-2} + \phi\). The equation (11) follows from equalities \(\hat{v} = e^{\frac{m\sigma}{m+n-2}}v\) and \(\nabla_{\hat{g}} = e^{-\frac{2\phi}{m+n-2}} \nabla_g\). For the equation (13), see (1.1) in [5].

We denote by \((w, \varphi)_M = \int_M w\varphi v^m dV_g\) the inner product in \(L^2(M, v^m dV_g)\). Additionally, we denote by \(||\cdot||_{2,M}\) the norm in the space \(L^2(M, v^m dV_g)\); in some cases, we use the notation |||| for this norm. \(H^1(M, v^m dV_g)\) denotes the closure of \(C^\infty(M)\) with respect to the norm \(\int_M |\nabla w|_g^2 + |w|_g^2\). Here and subsequently, the integrals are computed using the measure \(v^m dV_g\).
3. YAMABE-TYPE PROBLEM

In this section, we recall some concepts necessary to study the Yamabe-type problem in a smooth measure space \((M^n, g, v^m dV_g)\). In particular, we consider the weighted Yamabe quotient, which generalizes the Sobolev quotient in the case of \(m = 0\) and a suitable \(W\)-functional. These definitions are taken from [2]. Following the presentation in [2], we also consider the energies of these functionals and some of their properties.

3.1. The weighted Yamabe quotient. We start with the definition of the weighted Yamabe quotient.

**Definition 6.** The weighted Yamabe quotient \(Q[M^n, g, v^m dV_g] : C^\infty(M) \to \mathbb{R}\) of a compact smooth metric measure space \((M^n, g, e^{-\phi} dV_g, m)\) is, by definition, the functional

\[
Q[M^n, g, v^m dV_g](w) = \frac{(L^m_\phi w, w)_M (\int_M |w|^{2(m+n-1) \over (m+n-2)} v^{-1})^{2m \over n}}{(\int_M |w|^{2(m+n) \over m+n-2} v^{-1})^{2m+n-2 \over n} - 1}.
\]

The weighted Yamabe constant \(\Lambda[M^n, g, v^m dV_g] \in \mathbb{R}\) of \((M^n, g, v^m dV_g)\) is defined by

\[
\Lambda[M^n, g, v^m dV_g] = \inf \{Q[M^n, g, v^m dV_g](w) : w \in H^1(M, v^m dV_g) \setminus \{0\}\}.
\]

The weighted Yamabe problem is to find a function that minimizes the weighted Yamabe quotient.

**Remark 2.** In some cases, when the context is clear, we will not write the dependence of the smooth metric measure space: for example, we write \(Q\) and \(\Lambda\) instead of \(Q[M^n, g, v^m dV_g]\) and \(\Lambda[M^n, g, v^m dV_g]\), respectively. We note that since \(C^\infty(M)\) is dense in \(H^1(M, v^m dV_g)\) and \(Q(|w|) = Q(w)\), it is sufficient to consider the weighted Yamabe constant by minimizing over the space of nonnegative smooth functions on \(M\), and we will subsequently make this assumption without further comment.

Note that the weighted Yamabe quotient is conformally invariant in the sense of definition 3 (see Proposition 3.3 in [2]). To simplify the computations and to avoid the trivial noncompactness of the weighted Yamabe problem, as in [2], we consider the next definition:

**Definition 7.** Let \((M^n, g, v^m dV_g)\) be a smooth metric measure space. A smooth positive function \(w\) is volume-normalized if

\[
\int_M w^{2(m+n) \over m+n-2} = 1.
\]
3.2. $\mathcal{W}$-functional. Let us start with the definition of the $\mathcal{W}$-functional considered by Case in [2].

**Definition 8.** The functional $\mathcal{W} : C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$ of a compact smooth metric measure space $(M^n, g, v^m dV_g)$ is defined by

$$\mathcal{W}(w, \tau) = \tau \frac{m}{m+n}(L^m \phi w, w) + m \int_M \left( \tau \frac{m}{2(m+n)} w \frac{2m+n-1}{m+n-2} v^{-1} - \frac{2m+2}{m+n-2} \right)$$

when $m \in [0, \infty)$. We will refer to this functional as the $\mathcal{W}$-functional.

Additionally, $\mathcal{W}$ satisfies the following conformal property; see Proposition 3.10 in [2].

**Proposition 2** ([2]). Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space. In the first component, the $\mathcal{W}$-functional is conformally invariant:

$$(14) \quad \mathcal{W}[M^n, e^{2\sigma} g, e^{(m+n)\sigma} v^m dV_g](w, \tau) = \mathcal{W}[M^n, g, v^m dV_g](e^{(m+n)\sigma/2} w, \tau)$$

for all $\tau > 0$, $\sigma$, and $w \in C^\infty(M)$.

Since we are interested in minimizing the weighted Yamabe quotient, it is natural to define the following energies as infima of the $\mathcal{W}$-functional. It is also natural to relate one of these energies with the weighted Yamabe constant.

**Definition 9.** Given a compact smooth metric measure space $(M^n, g, v^m dV_g)$ and $\tau > 0$, the $\tau$-energy, denoted by $\nu[M^n, g, v^m dV_g](\tau) \in \mathbb{R}$, is defined to be

$$\nu[M^n, g, v^m dV_g](\tau) = \inf \left\{ \mathcal{W}(w, \tau) : w \in H^1(M, v^m dV_g), \int_M w^{2(m+n)/(m+n-2)} = 1 \right\}.$$

Define the energy $\nu[M^n, g, v^m dV_g] \in \mathbb{R} \cup \{-\infty\}$, where

$$\nu[M^n, g, v^m dV_g] = \inf_{\tau > 0} \nu[M^n, g, v^m dV_g](\tau).$$

The conformal invariance property in the $\mathcal{W}$-functional is transferred to the energies.

**Proposition 3** ([2]). Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space. Then

$$\nu[M^n, g, v^m dV_g](\tau) = \nu[M^n, ce^{2\sigma} g, e^{(m+n)\sigma} v^m dV_{cg}](c\tau),$$

$$\nu[M^n, g, v^m dV_g] = \nu[M^n, ce^{2\sigma} g, e^{(m+n)\sigma} v^m dV_{cg}]$$

for all $c > 0$, and for all $\sigma \in C^\infty(M)$. 

The following proposition shows that it is equivalent to considering the energy instead of the weighted Yamabe constant when the latter is nonnegative.

**Proposition 4** ([2]). Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space, and let $\nu$ and $\Lambda$ denote the energy and the weighted Yamabe constant, respectively. Then

(a) $\nu = -\infty$ if and only if $\Lambda \in \left(-\infty, 0\right)$;

(b) $\nu = -m$ if and only if $\Lambda = 0$; and

(c) $\nu > -m$ if and only if $\Lambda > 0$. Moreover, in this case we obtain

$$
\nu = \frac{2m + n}{2} \left[ \frac{2\Lambda}{n} \right]^{\frac{n}{2m+n}} - m
$$

and if $w$ is a volume-normalized function, we have that $(w, \tau)$ is a minimizer of $\nu$ with

$$
\tau = \left[ \frac{n \int_M w^{2(m+n-1)/(m+n-2)} v^{-1}}{2(L^m_\phi w, w)} \right]^{\frac{2(m+n)}{2m+n}}.
$$

if and only if $w$ is a minimizer of $\Lambda$.

Next, we consider the Euler-Lagrange equation of the $W$-functional.

**Proposition 5** ([2]). Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space $(M^n, g, v^m dV_g)$. Fix $\tau > 0$ and consider the map $\xi \rightarrow \mathcal{W}(\xi, \tau)$, where every $\xi$ is a volume-normalized function in $H^1(M, v^m dV_g)$. Suppose that $w \in H^1(M, v^m dV_g)$ is a critical point of this map. Then, $w$ is a weak solution of

$$
\tau \frac{m}{m+n} L^m_\phi w + \frac{m(m+n-1)}{m+n-2} \tau - \frac{n}{2(m+n)} w^{m+n-2} v^{-1} = cw^{m+n+2},
$$

for some constant $c$. Furthermore, if $(w, \tau)$ minimizes the $\nu$-energy, then

$$
c = \frac{(2m + n - 2)(m + n)}{(2m + n)(m + n - 2)(\nu + m)}.\n$$

3.3. **Euclidean space as the model space for the weighted Yamabe problem.** In this subsection, we consider a family of functions together with some of its properties, which are fundamental in our proof of the Aubin-type existence result for minimizers of the Yamabe quotient.

We start by mentioning Del Pino and Dolbeault’s result in [4] regarding the sharp Gagliardo-Nirenberg-Sobolev inequalities in the manner as Case presented it in [2].
Theorem 3 (\[4\]). Fix $m \in [0, \infty)$. For all $w \in H^1(\mathbb{R}^n) \cap L^{\frac{2(m+n-1)}{m+n-2}}(\mathbb{R}^n)$, it holds that

$$\Lambda_{m,n} \left( \int_{\mathbb{R}^n} |w|^{\frac{2(m+n)}{m+n-2}} \right)^{\frac{2m}{n}} \leq \left( \int_{\mathbb{R}^n} |\nabla w|^2 \right) \left( \int_{\mathbb{R}^n} |w|^{\frac{2(m+n-1)}{m+n-2}} \right)^{\frac{2m}{n}} ,$$

where the constant $\Lambda_{m,n}$ is given by

$$\Lambda_{m,n} = \frac{n \pi (m + n - 2)^2}{(2m + n - 2)} \left( \frac{2(m + n - 1)}{(2m + n - 2)} \right)^{\frac{2m}{n}} \left( \frac{\Gamma(\frac{2m+n}{2})}{\Gamma(m+n)} \right)^{\frac{2}{n}} .$$

Moreover, the equality holds in (16) if and only if $w$ is a constant multiple of the function $w_{\epsilon,x_0}$ defined on $\mathbb{R}^n$ by

$$w_{\epsilon,x_0}(x) := \left( \frac{2 \epsilon}{\epsilon^2 + |x - x_0|^2} \right)^{\frac{m+n-2}{2}} ,$$

where $\epsilon > 0$, and $x_0 \in \mathbb{R}^n$.

Fix $n \geq 3$ and $m \geq 0$. Next, we consider a particular case of function in (17) defined by

$$\varphi_{x_0,\tau}(x) = \tau^{-\frac{n(m+n-2)}{4(m+n)}} \left( 1 + \frac{c(m,n)}{\tau} |x - x_0|^2 \right)^{\frac{(m+n-2)}{2}} ,$$

where $c(m,n) = \frac{m+n-1}{(m+n-2)}$, $x_0 \in \mathbb{R}^n$ and $\tau > 0$.

We denote the normalization of $\varphi_{x_0,\tau}$ by

$$\tilde{\varphi}_{x_0,\tau} = V^{-\frac{m+n-2}{2(2m+n)}} \varphi_{x_0,\tau} ,$$

where

$$V = \int_{\mathbb{R}^n} \varphi_{0,1}^{\frac{2(m+n)}{m+n-2}} dx = \int_{\mathbb{R}^n} \varphi_{x_0,\tau}^{\frac{2(m+n)}{m+n-2}} dx ;$$

we used the change of variables in the second equality. On the other hand, a computation shows

$$-\tau^{\frac{m}{m+n}} \Delta \varphi_{x_0,\tau} + \frac{m(m+n-1)}{m+n-2} \varphi_{x_0,\tau}^{\frac{m+n-2}{m+n}} = \frac{(m+n)(m+n-1)}{m+n-2} \varphi_{x_0,\tau}^{\frac{m+n-2}{m+n}} .$$

The definition of $\varphi_{x_0,\tau}$, the definition of $V$, and identity (20) correspond to (5.1), (5.2), and (5.3) in \[2\], respectively. Since $\tilde{\varphi}_{x_0,\tau}$ is a volume-normalized function and attains the
infimum of the weighted Yamabe quotient (see Theorem 3), by Proposition 4 there is \( \tilde{\tau} > 0 \) such that

\[
\nu(\mathbb{R}^n, dx^2, 1^m dV_g) + m = \mathcal{W}(\mathbb{R}^n, dx^2, 1^m dV_g)(\tilde{\varphi}_{x_0, \tilde{\tau}}) + m
\]

\[
= \frac{2m}{m+n} \int_{\mathbb{R}^n} |\nabla \varphi_{x_0, \tau}|^2 + \frac{m^2-n}{m+n} \int_{\mathbb{R}^n} \varphi_{x_0, \tau}^2.
\]

It follows that \( \tilde{\tau} = \tau V^{-\frac{2}{2m+n}} \), since \( \tilde{\varphi}_{x_0, \tau} \) satisfies the equation (15) and \( \varphi_{x_0, \tau} \) satisfies the equation (20), respectively.

The next result, which corresponds to Theorem 7.1 in [2], links the weighted Yamabe constants of \((M^n, g, v^m dV_g)\) and \((M^n, g, v^m+1 dV_g)\) with the weighted Yamabe constants for the Euclidean space with parameters \( m \) and \( m + 1 \). This result allows us to prove the existence of a minimizer for the weighted Yamabe constant in an inductive argument for the parameter \( m \).

**Theorem 4 ([2])**. Let \((M^n, g, v^m dV_g)\) be a compact smooth metric measure space with a nonnegative weighted Yamabe constant, and suppose that the weighted Yamabe constant was minimized by a smooth positive function. Then

\[
\Lambda[M^n, g, v^m dV_g] \leq \frac{\Lambda[M^n, g, v^m dV_g]}{\Lambda[\mathbb{R}^n, dx^2, dV, m+1]} \frac{\Lambda[M^n, g, v^m+1 dV_g]}{\Lambda[\mathbb{R}^n, dx^2, dV, m]}.
\]

4. An Aubin-type existence theorem

In this section, we are dedicated to proving Theorem A. Briefly, the proof consists of performing some computation with a family of test functions in the \( \mathcal{W} \)-functional around a point \( p \) where the Weyl tensor is nonzero. To simplify these computations, we consider a smooth metric measure space conformal to the original so that \( T(p) = 0 \) and the new density \( v \) has suitable properties. The family of functions are supported in a small neighborhood of the point \( p \). Such functions are of the form of a standard cutoff function times the family of functions \( \varphi_{x_0, \tau} \) defined by (18). Then, taking the limit when the parameter \( \tau \) goes to zero, we prove that the entropy is less than the entropy of Euclidean space when \( m < \delta \), for some \( \delta > 0 \). By Proposition 4 and Theorem 1, we obtain the result for \( m < \delta \). Finally, using Theorem 4 in an inductive argument, we obtain the result for \( m \in \bigcup_{i \in \mathbb{N} \cup \{0\}} [i, i+\delta] \).

In this section, on the right-hand side of every equality or inequality that involves the terms \( R^m_\phi, v^k, R_g, R_{ij}, T_{ij} R_{ijkl} \) or \( W_{ijkl} \), we will be calculated in the point \( p \) and we will omit this point from notation. We will denote \( \Delta^2 f \) instead \( \Delta(\Delta f) \) where \( f \in C^\infty(M) \). The big O notation is considered when the variables involved are going to zero. Additionally, in this section, \( C \) is a positive constant that depends only on the smooth metric measure.
(\(M^n, g, v^m dV_g\)) and possibly changes from line to line or within the same line. Finally, we use the Einstein summation convention when an index variable appears twice in a single term. In some few cases, we will write the summation notation when the indexes appear more than twice or when the index is not over all of the values.

4.1. **Suitable smooth measure space.** In this subsection, we consider a suitable conformally smooth measure space to begin the proof of Theorem A.

**Lemma 1.** Let \((M^n, g, v^m dV_g)\) be a compact smooth metric measure space. There exists a conformal smooth metric measure space \((\hat{M}, \hat{g}, \hat{v}^m dV_{\hat{g}})\) such that for \(p \in M\) we have \(\hat{v}(p) = 1\), \(\nabla_{\hat{g}} \hat{v}(p) = 0\), and \(\text{Ric}_{\hat{g}}(p) = 0\).

**Proof.** If \(\sigma \in C^\infty(M)\), \(\hat{g} = e^{\frac{2\sigma}{n-2}}g\), by equations (11) and (12) we can take \(\sigma\) and \(\hat{v} = e^{\frac{\sigma}{n-2}}v\), such that we have \(\hat{v}(p) = 1\), \(\nabla_{\hat{g}} \hat{v}(p) = 0\) and \(\text{Ric}_{\hat{g}}(p) = 0\). Then, we change our original smooth metric measure space by \((M^n, \tilde{g}, \tilde{v}^m dV_{\tilde{g}})\). We denote with tilde all quantities computed with respect to the smooth metric measure space \((M^n, \tilde{g}, \tilde{v}^m dV_{\tilde{g}})\).

Let \(\tilde{g} = e^{\frac{2\tilde{\sigma}}{n-2}}\tilde{g}\) and \(\tilde{v} = e^{\frac{\tilde{\sigma}}{n-2}}\tilde{v}\) be such that in the point \(p\), \(\tilde{\sigma}\) satisfies that \(\tilde{\sigma} = 0\), \(\nabla_{\tilde{g}} \tilde{\sigma} = 0\), and \(\tilde{\sigma}_{ij} = \frac{m+n-2}{n-2}\tilde{T}_{ij} = \frac{m+n-2}{n-2}\tilde{R}_{ij}\). Since \(\tilde{T}_{ij}\) is trace-free, we obtain in the point \(p\) that \(\Delta_{\tilde{g}} \tilde{\sigma} = 0\). We denote by a hat all quantities computed with respect to the metric \(\tilde{g}\). In \(p\), the relation (13) for the Ricci tensor in the metrics \(\hat{g}\) and \(\tilde{g}\) yields \(\tilde{R}_{ij}(p) = 0\). Given that in \(p\) we have \(\nabla_{\tilde{g}} \tilde{v} = 0\), \(\nabla_{\tilde{g}} \tilde{\sigma} = 0\), and transformation rule (11) yields \(\nabla_{\tilde{g}} \tilde{v}(p) = 0\). The smooth metric measure space requested is \((M^n, \hat{g}, \hat{v}^m dV_{\hat{g}})\). \(\square\)

In the following lemma, we present some computation for a smooth metric measure space which satisfies the thesis of Lemma 1.

**Lemma 2.** Let \((M^n, g, v^m dV_g)\) be a compact smooth metric measure space such that for \(p \in M\) we have \(v(p) = 1\), \(\nabla_g v(p) = 0\), and \(\text{Ric}_g(p) = 0\). Then, for \(k \in \mathbb{R}\), we have

\[
\Delta_g v^k(p) = k \Delta_g v(p),
\]

(22)

\[
\Delta_g^2 v^k(p) = (v^k)_{ijij} = 2k(k-1)|\text{Hess}_g v|^2_g + k(k-1)(\Delta v)^2 + k \Delta_g^2 v + k(k-1)(\Delta_g v)^2,
\]

(23)

and

\[
\Delta_g R^{m}_{\phi}(p) = \Delta_g R_g + 2m(1-m)|\text{Hess}_g v|^2_g - 2m \Delta^2 v + 2m(\Delta v)^2.
\]

(24)

**Proof.** We will consider normal coordinates around \(p\) to compute the covariant derivatives. By hypothesis \(v(p) = 1\) and \(\nabla_g v(p) = 0\), we obtain
\[ \Delta_g v^k(p) = k \Delta_g v + k(k - 1) |\nabla_g v|^2 = k \Delta_g v. \]

The Ricci formula implies

\[ (v^k)_{iij}(p) = (v^k)_{jii} = (v^k)_{jij} + R^s_{ijij}(v^k)_s = (v^k)_{jii} + R_{sj}(v^k)_s = (v^k)_{jij} \]

and

\[ (v^k)_{iijj}(p) = ((v^k)_{iij})_j = ((v^k)_{iij} + R^s_{iijj}(v^k)_s)_j = (v^k)_{iijj} + R^s_{iijj}(v^k)_s + R^s_{iijj}(v^k)_{sj} = \Delta^2 v^k. \]

To compute the second equality in \((23)\), we will use \(v(p) = 1, \nabla_g v(p) = 0\), and \((25)\) to obtain

\[ \Delta^2_g v^k(p) = k(k-1)(k-2)(k-3) |\nabla_g v|^4 + 2k(k-1)(k-2) |\nabla_g v|^2 \Delta_g v + 4k(k-1)(k-2) \text{Hess}_v(\nabla_g v, \nabla_g v) + k(k-1)(\Delta_g v)^2 \]

\[ + 4k(k-1) |\Delta g_v| + 2k(k-1) \text{Ric}_g(\nabla_g v, \nabla_g v) \]

\[ = 2k(k-1) |\nabla_g v|^2 + k \Delta^2_g v + k(k-1)(\Delta_g v)^2. \]

Finally, by hypothesis, Ricci formula, and formula \((8)\) we obtain

\[ \Delta_g R^m_\phi(p) = \Delta_g (R_g + \frac{m(1-m)}{v^2} |\nabla_g v|^2 - \frac{2m}{v} \Delta_g v) \]

\[ = \Delta_g R_g + 2m(m-3) |\nabla_g v|^2 \Delta g_v + 2m(\Delta_g v)^2 - 2m(m-3) |\nabla_g v, \nabla_g \Delta g_v| - 2m \Delta^2_g v + 6m(1-m) |\nabla_g v|^4 \]

\[ + 8m(m-1) \text{Hess}_v(\nabla_g v, \nabla_g v) + 2m(1-m) \text{Hess}_v |^2 + 2m(1-m) \text{Ric}_g(\nabla_g v, \nabla_g v) \]

\[ = \Delta_g R_g + 2m(1-m) \text{Hess}_v |^2 - 2m \Delta^2_g v + 2m(\Delta_g v)^2. \]

\[ \square \]

Next, we write some formulae needed in the proof of Theorem A. In \(g\)-normal coordinates, it holds that

\[ dV_g = (1 - \frac{1}{6} R_{ij} x_i x_j - \frac{1}{12} R_{ijk} x_i x_j x_k - \frac{1}{720} R_{ijkl} x_i x_j x_k x_l + O(|x|^5)) dx \]

and

\[ g^{ij}(x) = \delta_{ij} - \frac{1}{3} R_{iklj} x_k x_l - \frac{1}{6} R_{iklj,s} x_k x_l x_s - (\frac{1}{20} R_{iklj,su} - \frac{3}{45} R_{iklh} R_{jlsur} x_k x_l x_s x_u + O(|x|^5), \]

where \(1 \leq i,j,k,l,r,s,u \leq n\), we recall that the coefficients are computed in \(p\). For formula \((28)\), see Lemma 5.5 in \([6]\), and for formula \((29)\), see formula (5.4) in \([6]\) and an argument akin to Lemma 2.3 in \([7]\).

We recall the following identities

\[ T_{ij} = R_{ij} - \frac{R}{n} g_{ij}, \]
4.2. Some estimates. The underlying idea of the proof of Theorem A is to improve the upper bound estimated in Proposition 6.3 in [2]. For this purpose, we fix a point \( p \in M \) and let \( \{x_i\} \) be normal coordinates in some fixed neighborhood \( U \), centered at \( p := (0, \ldots, 0) \). Let \( 0 < \epsilon < 1 \) be such that \( B_{2\epsilon}(p) \subset U \), where \( B_{2\epsilon}(p) \) is the geodesic ball of radius \( 2\epsilon \) around \( p \). Let \( \eta : M \to [0, 1] \) be a cutoff function such that \( \eta \equiv 1 \) on \( B_\epsilon \), \( \text{supp}(\eta) \subset B_{2\epsilon} \), and \( |\nabla \eta| < C\epsilon^{-1} \) in \( A_\epsilon := B_{2\epsilon} \setminus B_\epsilon \). For each \( 0 < \tau < 1 \), define \( f_\tau : M \to \mathbb{R} \) by

\[
\tau(x_1, \ldots, x_n) = \eta(x_1, \ldots, x_n),
\]

and set

\[
\tilde{f}_\tau = V_\tau^{-\frac{m+n-2}{2(m+n)}} f_\tau\]

where

\[
V_\tau = \int_M f_\tau^{\frac{2(m+n)}{m+n}} v^m dV_g.
\]

Taking \( \tilde{\tau} = \tau V^{-\frac{2}{2m+n}} \), where \( V \) is defined by (19), the definition of \( W \) and equality (14) in Proposition 2 yield

\[
W[\mathcal{M}, g, v^m dV_g](\tilde{f}_\tau, \tilde{\tau}) + m = \frac{\tilde{\tau}^{\frac{m}{m+n}}}{V_\tau^{\frac{m+n-2}{m+n}}} \left( \int_{B_{2\epsilon}} |\nabla_g f_\tau|^2 v^m + \frac{m+n-2}{4(m+n-1)} R_{\phi} f_\tau^2 v^m dV_g \right)
\]

\[
+ m \tilde{\tau}^{\frac{n}{m+n}} \int_{B_{2\epsilon}} f_\tau^{\frac{2(m+n-1)}{m+n}} v^{m-1} dV_g.
\]

Next, we establish lemmas to estimate each term on the right-hand side of the equality (35). First, we estimate on the right-hand side of (35) the term with the Bakry-Émery scalar curvature \( R_{\phi} \).

**Lemma 3.** Let \((\mathcal{M}, g, v^m dV_g)\) be a compact smooth metric measure space such that for \( p \in M \), we have \( v(p) = 1 \), \( \nabla_g v(p) = 0 \), and \( \text{Ric}_g(p) = 0 \). If \( f_\tau \) is defined by (32), then,

\[
\int_{B_{2\epsilon}} R_{\phi} f_\tau^2 v^m dV_g = -2m \Delta_g v \frac{m+n}{c(m,n)} I_1 + (\Delta_g R_{\phi} - 2m^2 (\Delta_g v)^2) \frac{\tau^{\frac{n}{m+n}+\frac{1}{2}}}{2nc(m,n)\tau^{\frac{n}{m+n}+\frac{m+n}{2}}} I_2
\]

\[
+ O(\tau^{\frac{n}{m+n}+\frac{1}{2}}) + O(\epsilon^{4-2m-n} \tau^{\frac{n}{m+n}+m+n-4}),
\]
where

\[(37)\quad I_1 = \int_{\mathbb{R}^n} (1 + |y|^2)^{-(m+n-2)}dy\]

and

\[(38)\quad I_2 = \int_{\mathbb{R}^n} |y|^2(1 + |y|^2)^{-(m+n-2)}dy.\]

**Proof.** First, we estimate the term with the Bakry-Émery scalar curvature $R^m_\phi$ in the region $A_\epsilon$. Given that $dV_g = (1 + \epsilon C)dx$ around $p$, we obtain

\[
\int_{A_\epsilon} R^m_\phi f^2 v^m dV_g \leq C(1 + \epsilon C) \int_{A_\epsilon} \frac{\varphi^2_0}{\varphi_0} dx
\]

\[
= C(1 + \epsilon C) \int_{A_\epsilon} (1 + \frac{c(m, n)}{\tau} |x|^2)^{-(m+n-2)} dx
\]

\[
= C(1 + \epsilon C) \int_{\mathbb{R}^n} \frac{1}{\sqrt{\varphi_0}} \left(1 + \frac{c(m, n)}{\tau} \right)^{\frac{1}{2}}
\]

\[
\int_{A_\epsilon} \frac{1}{\sqrt{\varphi_0}} \left(1 + \frac{c(m, n)}{\tau} \right)^{\frac{1}{2}} (1 + r^2)^{-(m+n-2)} r^{n-1} dr
\]

\[
\leq C(1 + \epsilon C) \epsilon^{4-2m-n} \frac{n-1}{m+n+\frac{n+1}{2}}.
\]

Next, we estimate on the right-hand side of (33) the term with the Bakry-Émery curvature $R^m_\phi$ in the region $B_\epsilon$. For this purpose, we will use the Taylor expansion around $p$ of $R^m_\phi$ and $v^m$. Since $v(p) = 1$, $\nabla_g v(p) = 0$, $R_{ij}(p) = 0$, and $R_g(p) = 0$, it follows by formula (8) that $R^m_\phi(p) = -2m \Delta_g v$, and the Taylor expansion of $R^m_\phi$ and $v^m$ takes the form of

\[
R^m_\phi(x) = -2m \Delta_g v + (R^m_\phi)_{ii}x_i + \frac{1}{2}(R^m_\phi)_{ij} x_j + O(|x|^3),
\]

\[
v^m(x) = 1 + \frac{(v^m)_{ij} x_i x_j}{2} + \frac{(v^m)_{ij} x_i x_j x_l}{6} + \frac{(v^m)_{ij} x_i x_j x_l x_k}{24} + O(|x|^5),
\]

we recall that the coefficients were computed in $p$. Since $R_{ij}(p) = 0$, formula (28) takes the form

\[
dV_g = (1 - \frac{1}{12} R_{ij,k} x_i x_j x_k - \frac{1}{180} R_{ijkl} x_i x_j x_k + O(|x|^5)) dx.
\]
Using equalities (40), (41), and (42) and the symmetries in the ball, we have

\[
\int_{B_\varepsilon} R^m f^2 v^m dV_g = -2m \Delta g v \int_{B_\varepsilon} \varphi^2_{0, \tau} dx + \frac{1}{2} \left( (R^m_{\phi})_{ij} - 2m(v^m)_{ij} \Delta g v \right) \int_{B_\varepsilon} \varphi^2_{0, \tau} x_i x_j dx
\]

\[
+ \int_{B_\varepsilon} \varphi^2_{0, \tau} O(|x|^4) dx
\]

\[
= -2m \Delta g v \int_{B_\varepsilon} \varphi^2_{0, \tau} dx + \frac{1}{2n} (\Delta g R^m_{\phi} - 2m(\Delta g v^m) \Delta g v) \int_{B_\varepsilon} \varphi^2_{0, \tau} |x|^2 dx
\]

\[
+ \int_{B_\varepsilon} \varphi^2_{0, \tau} O(|x|^4) dx.
\]

Then,

\[
\int_{B_\varepsilon} \varphi^2_{0, \tau} dx = \tau \frac{-n(m+n-2)}{2(m+n)} \int_{B_\varepsilon} \left( 1 + \frac{c(m,n)}{\tau} |x|^2 \right)^{(m+n-2)} dx
\]

\[
= \tau \frac{-n(m+n-2)}{2(m+n)} + \frac{a}{2} \int_{B_{\sqrt{c(m,n)}}} (1 + |y|^2)^{(m+n-2)} dy
\]

\[
= \tau \frac{-n(m+n-2)}{2(m+n)} \left( I_1 - \int_{R^n \setminus B_{\sqrt{c(m,n)}}} (1 + |y|^2)^{(m+n-2)} dy \right)
\]

\[
= \tau \frac{-n(m+n-2)}{2(m+n)} I_1 + O(c^{4-2m-n} \tau^{\frac{n}{m+n} + m + \frac{n-4}{4}})
\]

and similarly,

\[
\int_{B_\varepsilon} \varphi^2_{0, \tau} |x|^2 dx = \tau \frac{-n(m+n-2)}{2(m+n)} \int_{B_\varepsilon} \left( 1 + \frac{c(m,n)}{\tau} |x|^2 \right)^{(m+n-2)} |x|^2 dx
\]

\[
= \tau \frac{-n(m+n-2)}{2(m+n)} I_2 + O(c^{6-2m-n} \tau^{\frac{n}{m+n} + m + \frac{n-4}{4}}).
\]
Now, taking \( q < \min\{2m+n-6,1\} \) and \( 0 < \epsilon < 1, \) then for \( |x| \leq \epsilon \) we obtain \( |x|^q > |x|^2 \) and
\[
\int_{B_{\tau}} \varphi_{0,\tau}^2 |x|^4 \, dx \leq \int_{B_{\tau}} \varphi_{0,\tau}^2 |x|^{2+q} \, dx \\
\leq \tau^{-\frac{n(m+n-2)}{2(m+n)}} \int_{B_{\tau}} \frac{|x|^{2+q}}{(1 + c(m,n)|x|^2)^{m+n-2}} \, dx \\
\leq C \tau^{-\frac{n}{m+n}+1+\frac{q}{2}} \int_{B_{\sqrt{c(m,n)}}} \frac{|y|^{2+q}}{(1 + |y|^2)^{(m+n-2)}} \, dy \\
\leq C \tau^{-\frac{n}{m+n}+1+\frac{q}{2}} (C + C \int_{1}^{\infty} r^{6-2m-n+q-1} \, dr) \\
\leq C \tau^{-\frac{n}{m+n}+1+\frac{q}{2}}.
\]
Equality (22), together with estimates (39), (43), (44), and (46), leads to equality (36). \( \square \)

The following lemma estimates the integral with the gradient term in (35).

**Lemma 4.** Let \( (M^n, g, v^m dV_g) \) be a compact smooth metric measure space such that for \( p \in M, \) we have \( v(p) = 1, \nabla_g v(p) = 0, \) and \( Ric_g(p) = 0. \) If \( f_\tau \) is defined by (32), then
\[
\int_{B_{2\tau}} |\nabla g f_\tau|_g^2 v^m dV_g = \int_{B_{\tau}} |\nabla \varphi_{0,\tau}|^2 \, dx + \tau^{-\frac{n}{m+n}} \frac{m(m+n-2)^2}{2n c(m,n)^{\frac{n}{2}}} \Delta_g I_3 \\
- \frac{\tau^{-\frac{n}{m+n}+1}(m+n-2)^2}{2c(m,n)^{\frac{n}{2}} n(n+2)} I_4 A_1 + O(\tau^{-\frac{n}{m+n}+1+\frac{q}{2}} \epsilon^{2-2m-n}) + O(\tau^{-\frac{n}{m+n}+1+\frac{q}{2}}),
\]
where \(| \cdot | \) is the Euclidean norm,
\[
I_3 = \int_{\mathbb{R}^n} |y|^4(1 + |y|^2)^{-(m+n)} \, dy,
\]
\[
I_4 = \int_{\mathbb{R}^n} |y|^6(1 + |y|^2)^{-(m+n)} \, dy,
\]
and
\[
A_1 = -\frac{m(m-1)}{2} |\text{Hess} v|_g^2 - \frac{m(m-1)}{4} (\Delta_g v)^2 - \frac{m}{4} \Delta^2_g v + \frac{\Delta R}{10} + \frac{|W|_g^2}{60}.
\]

**Proof.** To estimate the gradient term in \( A_4, \) note that, in \( g \)-normal coordinates, the term \(|\nabla f_\tau|_g^2 \) in \( A_4 \) satisfies the following inequality
\[
|\nabla f_\tau|_g^2 \leq C |\nabla f_\tau|^2 \leq C (\eta^2 |\nabla \varphi_{0,\tau}|^2 + |\nabla \eta|^2 \varphi_{0,\tau}^2) \leq C (|\nabla \varphi_{0,\tau}|^2 + \epsilon^2 \varphi_{0,\tau}^2).
\]
Additionally, we obtain

\[
\int_{A_\epsilon} \epsilon^{-2} \varphi_{0,\tau}^2 dV_g \leq (1 + \epsilon C) \epsilon^{-2} \int_{A_\epsilon} \varphi_{0,\tau}^2 dx \leq (1 + \epsilon C) \epsilon^{2-n-2m} \frac{n}{m+n} + m + \frac{n+4}{2}
\]

and

\[
\int_{A_\epsilon} |\nabla \varphi_{0,\tau}|^2 dV_g \leq (1 + \epsilon C) \int_{A_\epsilon} |\nabla \varphi_{0,\tau}|^2 dx \leq (1 + \epsilon C) \epsilon^{2-n-2m} \frac{n}{m+n} + m + \frac{n+4}{2}.
\]

Then,

\[
\int_{A_\epsilon} |\nabla f_{\tau}|^2_g dx \, dt = O(\epsilon^{2-n-2m} \frac{n}{m+n} + m + \frac{n+4}{2}).
\]

Next, we estimate the integral with the gradient term in \(B_\epsilon\) in equality (35). Then, the symmetries of the ball, equality (29), equality (42), Taylor expansion (41), Taylor expansion (40), and \(\nabla g v^m(p) = 0\) imply that

\[
\int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 v^m_g dV_g = \int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 dx + \frac{\Delta_g v^m}{2n} \int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 |x|^2 dx - \frac{1}{3} R_{ijkl} \int_{B_\epsilon} (\partial_i \varphi_{0,\tau})(\partial_j \varphi_{0,\tau}) x_k x_l dx - \frac{1}{6} R_{ijkl}(v^m)_{su} \int_{B_\epsilon} (\partial_i \varphi_{0,\tau})(\partial_j \varphi_{0,\tau}) x_k x_s x_u x_l dx
\]

\[
= \frac{1}{20} R_{ijkl,us} - \frac{3}{45} R_{iklr,rsu} \int_{B_\epsilon} (\partial_i \varphi_{0,\tau})(\partial_j \varphi_{0,\tau}) x_k x_j x_s x_u x_l dx + \frac{1}{24} (v^m)_{ijkl} \int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 x_i x_j x_k x_l dx - \frac{1}{40} R_{ijkl} + \frac{1}{180} R_{ijkl} R_{kl} \int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 x_i x_j x_k x_l dx + \int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 O(|x|^6) dx.
\]

For the second integral in (55), we have

\[
\int_{B_\epsilon} |\nabla \varphi_{0,\tau}|^2 |x|^2 dx = \tau \frac{n}{m+n} \frac{(m+n-2)^2}{2} \int_{B_{\sqrt{(m,n)}}} |y|^4 (1 + |y|^2)^{-(m+n)} dy = \tau \frac{n}{m+n} \frac{(m+n-2)^2}{c(m,n)} I_3 + O(\epsilon^{4-2m-n} \frac{n}{m+n} + m + \frac{n+4}{2}).
\]
Using the symmetries of the ball and of the Riemann curvature tensor, we obtain

\[
\int_{B_r} \frac{\partial_i \varphi_0(x, \tau)}{\partial_j \varphi_0(x, \tau)} R_{ijkl}(0) x_k x_l \, dx = \int_{B_r} \frac{R_{ijkl} x_i x_j x_k x_l}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx
\]

and

\[
\sum_{i=1}^{n} R_{iiii} x_i^4 + \sum_{i \neq j} (R_{ijij} + R_{jiji} + R_{ijji}) x_i^2 x_j^2 \frac{1}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx
\]

\[
= 0.
\]

Again, the symmetries of the ball and of the Riemann curvature tensor yield

\[
R_{ijkl}(0)(v^m)_{su}(0) \int_{B_r} \frac{\partial_i \varphi_0(x, \tau)}{\partial_j \varphi_0(x, \tau)} x_k x_l x_s x_u \, dx = 0
\]

and

\[
\frac{1}{40} R_{ijkl, su}(0) - \frac{3}{30} R_{iklr}(0) R_{rsuj}(0) \int_{B_r} \frac{\partial_i \varphi_0(x, \tau)}{\partial_j \varphi_0(x, \tau)} x_k x_l x_s x_u \, dx = 0.
\]

In order to compute the sixth integral on the right-hand side of (55), we obtain

\[
(v^m)_{ijkl}(0) \int_{B_r} |\nabla \varphi_0(x, \tau)|^2 x_i x_j x_k x_l \, dx = (v^m)_{ijkl} \int_{B_r} \frac{|x|^2 x_i x_j x_k x_l}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx.
\]

The symmetries of the ball, Lemma 7 in the appendix, and \((v^m)_{ijji} = (v^m)_{jiji}\) imply

\[
\int_{B_r} \frac{(v^m)_{ijkl}(0) x_i x_j x_k x_l}{|x|^{-2}(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx = \int_{B_r} \frac{\sum_{i=1}^{n} (v^m)_{iiii} x_i^4 + \sum_{i \neq j} ((v^m)_{ijij} + (v^m)_{ijji} + (v^m)_{ijji}) x_i^2 x_j^2}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}|x|^{-2}} \, dx
\]

\[
= \frac{1}{3} ((v^m)_{ijij} + (v^m)_{ijji} + (v^m)_{ijji}) \int_{B_r} \frac{|x|^2 x_i^4}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx
\]

\[
= \frac{((v^m)_{ijij} + (v^m)_{ijji} + (v^m)_{ijji})}{n(n + 2)} \int_{B_r} \frac{|x|^6}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx
\]

\[
= \frac{((v^m)_{ijij} + (v^m)_{ijji} + (v^m)_{ijji})}{n(n + 2)} \int_{B_r} \frac{|x|^6}{(1 + \frac{c(m, n)}{r}|x|^2)^{m+n}} \, dx.
\]

Formula (23) for \(k = m\) implies

\[
(v^m)_{ijkl}(0) \int_{B_r} |\nabla \varphi_0(x, \tau)|^2 x_i x_j x_k x_l \, dx = \frac{3 \tau^{\frac{n}{m+n} + 1}(m + n - 2)^2 \Delta^2 v^m}{n(n + 2)c(m, n)^{\frac{m}{2}}} I_4
\]

\[
+ O(\epsilon^{6-2m-n} \tau^{\frac{m}{m+n} + \frac{n}{m+n} - \frac{1}{2}}).
\]
A GENERALIZATION OF AUBIN’S RESULT

Using a similar argument to obtain equality (62) and the contraction of Bianchi’s identity \( R_{g_{ij}} = 2 R_{ij} \), we have

\[
R_{ij,kl} \int_{B_\epsilon} |\nabla \phi_{0,\tau}|^2 x_i x_j x_k x_l dx = \frac{\tau^{n+1} (R_{iij,jj} + R_{ij,ij} + R_{ij,jj})}{n(n+2)c(m,n)^{n+2}} \left( m + n - 2 \right)^{-2} I_4 \\
+ O(\epsilon^{6-2m-n} \frac{n}{m+n} \frac{n+n}{2}) \\
= \frac{2\tau^{n+1} (m + n - 2)^2 \Delta R_g m}{n(n+2)c(m,n)^{n+2}} I_4 \\
+ O(\epsilon^{6-2m-n} \frac{n}{m+n} \frac{n+n}{2}).
\]

The identities \( R_{ijkl} R_{ijkl} = 2 R_{ijkl} R_{iklj} \) and \( \text{Ric}_g(p) = 0 \) yield

\[
R_{i,j,s}(0) R_{r,k,l}(0) \int_{B_\epsilon} |\nabla \phi_{0,\tau}|^2 x_i x_j x_k x_l dx = \frac{\tau^{n+1} (R_{r,s} R_{r,s} + R_{r,s} R_{r,s} + R_{r,s} R_{r,s})}{n(n+2)c(m,n)^{n+2}} \left( m + n - 2 \right)^{-2} I_4 \\
+ O(\epsilon^{6-2m-n} \frac{n}{m+n} \frac{n+n}{2}) \\
= \frac{3\tau^{n+1} (m + n - 2)^2 \Delta R_g m}{2n(n+2)c(m,n)^{n+2}} I_4 \\
+ O(\epsilon^{6-2m-n} \frac{n}{m+n} \frac{n+n}{2}).
\]

For the last integral on the right-hand side of (55), taking \( q < \min \{2m + n - 6, 1\} \) and \( \epsilon < 1 \) as in (46), we obtain

\[
\int_{B_\epsilon} |\nabla \phi_{0,\tau}|^2 |x|^6 dx \leq \int_{B_\epsilon} |\nabla \phi_{0,\tau}|^2 |x|^{4+q} dx \leq C \tau^{n+1} \frac{n}{m+n} \frac{n+\frac{n}{2}}{2}.
\]

Equalities (22) for \( k = m \), (54), (55), (56), (57), (58), (59), (62), (63), and (64) and inequality (65) lead to (47) where

\[
A_1 = -\Delta^2 + \frac{\Delta R_g}{10} + \frac{R_{ijkl} R_{iklj}}{60}.
\]

By equality (23) for \( k = m \), we obtain

\[
A_1 = -\frac{m(m-1)|\text{Hess} v|^2}{2} - \frac{m(m-1)(\Delta g)^2}{4} - \frac{m\Delta^2 g}{4} + \frac{\Delta R_g}{10} + \frac{R_{ijkl} R_{iklj}}{60}.
\]

Finally, by hypothesis \( \text{Ric}_g(p) = 0 \), and formulas (30) and (31), we conclude that \( A_1 \) takes the form of (50).

□

Next, we analyze the last integral on the right-hand side of (35).
Lemma 5. Let \((M^n, g, v^m dV_g)\) be a compact smooth metric measure space such that for \(p \in M\) we have \(v(p) = 1\), \(\nabla_g v(p) = 0\), \(\text{Ric}_g(p) = 0\). If \(f_\tau\) is defined by \((32)\), then

\[
\int_{B_{2\epsilon}} f_{\tau}^{2(m+n-1)} v^{m-1} dV_g = \int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} dx + \frac{\tau^{n+1} (m-1) \Delta_g v}{2nc(m, n)^2} I_5
\]

\[(68)\]

\[
= -\frac{\tau^n n + 2 + 2}{2n(n + 2) c(m, n)^{n+1}} A_2
\]

\[
+ O(\tau^{2(n+2)m} + m + n - 2 \epsilon^{4-2m-n}) + O(\tau^{n+2+\frac{q}{2}})
\]

where

\[(69)\]

\[I_5 = \int_{\mathbb{R}^n} |y|^2 (1 + |y|^2)^{- (m+n-1)} dy,\]

\[(70)\]

\[I_6 = \int_{\mathbb{R}^n} |y|^4 (1 + |y|^2)^{- (m+n-1)} dy,\]

and

\[(71)\]

\[A_2 = \frac{(m-1)(m-2)}{2} |\text{Hess} v|^2 - \frac{(m-1)(m-2)}{4} (\Delta g v)^2 - \frac{(m-1)}{4} \Delta_g v + \frac{\Delta R_g}{10} + \frac{|W|^2}{60}.\]

Proof. In the region \(A_{\epsilon}\), we have

\[
\int_{A_{\epsilon}} f_{\tau}^{2(m+n-1)} v^{m-1} dV_g \leq C(1 + \epsilon C) \int_{A_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} dx
\]

\[(72)\]

\[\leq C(1 + \epsilon C) \tau^{\frac{n}{2(m+n)}} + m + n - 2 \epsilon^{2-2m-n}.\]

In order to estimate the integral on the left-hand size of \((68)\) in the region \(B_{\epsilon}\), we use equality \((28)\), Taylor expansion around \(p\) for \(v^{m-1}\), Taylor expansion \((40)\), \(\nabla v(p) = 0\), equality \((22)\) for \(k = m - 1\), and the symmetries in the ball to obtain

\[
\int_{B_{\epsilon}} f_{\tau}^{2(m+n-1)} v^{m-1} dV_g = \int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} dx + \frac{\Delta_g v^{m-1}}{2n} \int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} |x|^2 dx
\]

\[\quad + \frac{1}{24} (v^{m-1})_{ijkl} \int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} x_i x_j x_k x_l dx
\]

\[\quad - \frac{1}{40} R_{ijkl} + \frac{1}{180} R_{ijkl} R_{klm} \int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} x_i x_j x_k x_l dx
\]

\[\quad + \int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} O(|x|^6) dx.
\]

Now, the second term on the right-hand side of \((73)\) takes the form

\[
\int_{B_{\epsilon}} \varphi_{0, \tau}^{2(m+n-1)} |x|^2 dx = \frac{\tau^{n+1}}{c(m, n)^{n+2}} I_5 + O(\tau^{\frac{n}{2(m+n)}} + m + n - 2 \epsilon^{4-2m-n}).
\]

\[(74)\]
For the third term on the right-hand side of (73), a similar argument as in (60) to (62) using formula (23) for \( k = m - 1 \) yields

\[
(v^{m-1})_{ijkl}(0) \int_{B_{\epsilon}} \varphi_{0,\epsilon}^{2(m+n-1)} x_i x_j x_k x_l dx = \frac{3\tau^{\frac{n}{2(m+n)}} + 2 \Delta^2 \nu^{m-1}}{n(n+2)c(m,n)} \frac{\tau^2}{n+2} I_6 \\
+ O(\epsilon^{2m-n} \tau^{\frac{n}{2(m+n)}+m+n+2}).
\]

Using the contraction of Bianchi’s identity \( R_{ij} i = 2R_{ij} \), we obtain

\[
R_{ijkl}(0) \int_{B_{\epsilon}} \varphi_{0,\epsilon}^{2(m+n-1)} x_i x_j x_k x_l dx = \frac{2\tau^{\frac{n}{2(m+n)}} + 2 \Delta R_{ij} i}{n(n+2)c(m,n)} \frac{\tau^2}{n+2} I_6 \\
+ O(\epsilon^{2m-n} \tau^{\frac{n}{2(m+n)}+m+n+2}).
\]

The identity \( R_{ijkl} R^{ijkl} = \frac{1}{2} R_{ijkl} R^{iklj} \) and \( Ric(p) = 0 \) imply

\[
R_{ijkl}(0) R_{ijkl}(0) \int_{B_{\epsilon}} \varphi_{0,\epsilon}^{2(m+n-1)} x_i x_j x_k x_l dx = \frac{3\tau^{\frac{n}{2(m+n)}} + 2 \Delta R_{ijkl} R_{ijkl} i}{2n(n+2)c(m,n)} \frac{\tau^2}{n+2} I_6 \\
+ O(\epsilon^{2m-n} \tau^{\frac{n}{2(m+n)}+m+n+2}).
\]

On the other hand, since \( \epsilon < 1 \) and we choose \( 0 < q < \min\{2m + n - 6, 1\} \), the last term on the right-hand side of (73) is estimated as follows

\[
\int_{B_{\epsilon}} \varphi_{0,\epsilon}^{\frac{2(m+n-1)}{m+n-2}} |x|^6 dx \leq \tau^{\frac{n(2(m+n-1)}{2(m+n)}} \int_{B_{\epsilon}} |x|^{4+q} (1 + \frac{c(m,n)}{\tau^{\frac{n}{2}}} |x|^{2-(m+n-1)}) dx \\
\leq C\tau^{\frac{n}{2(m+n)}} + \frac{2+q}{2}.
\]

Equality (22) and estimates (72) to (78) allow us to conclude formula (68), where

\[
A_2 = -\frac{\Delta^2 \nu^{m-1}}{4} + \frac{\Delta R_{ijkl} R_{ijkl} i}{10} + \frac{R_{ijkl} R_{ijkl} i}{60}.
\]

By hypothesis \( Ric(p) = 0 \), equality (31), and equality (23) for \( k = m - 1 \), it follows that \( A_2 \) satisfies equality (71). \( \square \)

Now, we analyze the behavior of \( V_{\tau}, V_{\tau}^{-\frac{m+n-2}{m+n}} \), and \( V_{\tau}^{-\frac{m+n-1}{m+n}} \) when \( \tau \) is near zero.

**Lemma 6.** Let \( (M^n, g, v^m dV_g) \) be a compact smooth metric measure space such that for \( p \in M \) we have \( \nu(p) = 1, \nabla_g v(p) = 0, \) and \( Ric(p) = 0 \). If \( V, v, \) and \( V_{\tau} \) are defined by (19), (32), and (33), respectively; then,

\[
V - V_{\tau} = -\frac{\tau^{m} \Delta_g v}{2nc(m,n)^{\frac{1}{2}}} I_7 + \tau^{2} I_3 A_1 + O(\tau^{m+\frac{4}{2}} \epsilon^{-2m-n}) + O(\tau^{2+\frac{q}{2}}),
\]
where $A_1$ is defined by (50) and

$$I_7 = \int_{\mathbb{R}^n} |y|^2(1 + |y|^2)^{-(m+n)}dy.$$  

Additionally, we obtain the following estimates

$$V_\tau \frac{m+n-2}{m+n} = V_\tau \frac{m+n-2}{m+n} - \frac{\tau m(m+n-2)I_7\Delta_g v}{2n(m+n)c(m,n)\frac{n+2}{m+n}} V_\tau \frac{2m+2n-2}{m+n}$$

$$+ \frac{\tau^2(m+n-2)I_3A_1}{2n(n+2)(m+n)c(m,n)\frac{n+4}{m+n}} V_\tau \frac{2m+2n-2}{m+n}$$

$$+ \frac{\tau^2m^2(m+n-2)(m+n-1)(I_7)^2(\Delta_g v)^2}{4n^2(m+n)^2c(m,n)n^2V_\tau \frac{2m+2n-2}{m+n}}$$

$$+ O(\tau^m \frac{\eta}{\tau} e^{-2m-n}) + O(\tau^{2+\frac{m}{2}})$$

and

$$V_\tau \frac{m+n-1}{m+n} = V_\tau \frac{m+n-1}{m+n} - \frac{\tau m(m+n-1)\Delta_g v}{2n(m+n)c(m,n)\frac{n+2}{m+n}} I_7$$

$$+ \frac{\tau^2(m+n-1)I_3A_1}{2n(n+2)(m+n)c(m,n)\frac{n+4}{m+n}} V_\tau \frac{2m+2n-1}{m+n}$$

$$+ \frac{\tau^2m^2(m+n-1)(2m+2n-1)(\Delta_g v)^2(I_7)^2}{8n^2(m+n)^2c(m,n)n^2V_\tau \frac{2m+2n-1}{m+n}}$$

$$+ O(\tau^m \frac{\eta}{\tau} e^{-2m-n}) + O(\tau^{2+\frac{m}{2}}).$$

**Proof.** By the definitions of $V$ and $V_\tau$, we obtain

$$V - V_\tau = \int_{\mathbb{R}^n \setminus B_{2\varepsilon}} \frac{2(m+n)}{m+n} \varphi_{0,\tau} \frac{2(m+n)}{m+n} dx + (\int_{A_{\varepsilon}} \frac{2(m+n)}{m+n} \varphi_{0,\tau} dx - \int_{A_{\varepsilon}} \frac{2(m+n)}{m+n} v^m dV_g)$$

$$+(\int_{B_{\varepsilon}} \frac{2(m+n)}{m+n} \varphi_{0,\tau} dx - \int_{B_{\varepsilon}} \frac{2(m+n)}{m+n} v^m dV_g).$$

For the first integral on the right-hand side of (84) we have

$$\int_{\mathbb{R}^n \setminus B_{2\varepsilon}} \frac{2(m+n)}{m+n} \varphi_{0,\tau} \frac{2(m+n)}{m+n} dx \leq C\varepsilon^{-n-2m} \tau^{m+\frac{m}{2}}.$$ 

Using the expansion for the volume form (28), and the fact that $v$ is bounded, we have in the second integral on the right-hand side of (84) that

$$\left| \int_{A_{\varepsilon}} \frac{2(m+n)}{m+n} \varphi_{0,\tau} dx - \int_{A_{\varepsilon}} \frac{2(m+n)}{m+n} v^m dV_g \right| \leq C(1 + \epsilon C) \int_{A_{\varepsilon}} \frac{2(m+n)}{m+n} \varphi_{0,\tau} dx$$

$$\leq C(1 + \epsilon C)\varepsilon^{-n-2m} \tau^{m+\frac{m}{2}}.$$
By the expansion for the volume form (28), the Taylor expansion around $p$ for $v^m$ and the symmetries of the ball in the third integral on the right-hand side of (84), we obtain
\[\int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} \, dx - \int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} v^m \, dV_g = -\frac{\Delta_g v^m}{2n} \int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} |x|^2 \, dx - \frac{1}{24} (v^m)_{ijkl} \int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} x_i x_j x_k x_l \, dx + \frac{1}{180} R_{ijkl} R_{rkl} \int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} x_i x_j x_k x_l \, dx + \frac{1}{40} R_{ij,kl} + \frac{1}{180} R_{rj}\, R_{rkl} \int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} x_i x_j x_k x_l \, dx + \int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} O(|x|^6) \, dx.\]

To analyze (87), we consider the first integral on its right-hand side
\[\int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} |x|^2 \, dx = \frac{\tau}{c(m,n)} I_7 + O(\epsilon^{2-2m-n} \tau^{m+n+\frac{4}{7}}).\]

Additionally, we have
\[\int_{B_\varepsilon} \varphi_{0,\tau}^{2(m+n)} |x|^4 \, dx = \frac{\tau^2}{c(m,n)} I_3 + O(\epsilon^{4-2m-n} \tau^{m+n+\frac{4}{7}}).\]

For the last integral on the right-hand side of (87), recalling that $q < \min\{2m-n-4, 1\}$ and $\epsilon < 1$, we obtain
\[\int_{B_\varepsilon} \varphi_{0,\tau}^{m+n} |x|^6 \, dx \leq C \tau^{2+\frac{q}{2}}.\]

Equalities (22) for $k = m$, (84), (87), (88), and (89); inequalities (85), (86), and (90); and similar arguments like we used in (75) to (77) lead to formula (80).

It follows that the terms $V_\tau$ are uniformly bounded away from zero. Using estimate (80) and Taylor expansion around $V$ for the functions $x^{\frac{m+n}{m+n-2}}$ and $x^{-\frac{m+n-1}{m+n}}$, we obtain formulas (82) and (83), respectively. \qed

4.3. Proof of Theorem A. In this subsection, we prove Theorem A.

**Proof of Theorem A.** Let $(M^n, g, v^m dV_g)$ be a compact smooth metric measure space, where $(M, g)$ is nonlocally conformally flat. Then, there exist $p \in M$ such that the Weyl tensor in $p$ is nonzero, i.e. $|W|_g(p) \neq 0$. Since $|W|_g$ is conformally invariant, and with the equality (14) and Lemma 11 we can assume without loss of generality that $v(p) = 1$, $\nabla_g v(p) = 0$, $\Delta_g v(p) = 0$, and $\text{Ric}_g(p) = 0$. Let $\tau > 0$ and consider $f_\tau$, $\tilde{f}_\tau$, and $V_\tau$, defined by (32), (33), and (34), respectively.
Using the lemmas in subsection 4.2 we estimate $W[M^n, g, v^m dV_g] (\tilde{f}_\tau, \tilde{\tau})$ where $\tilde{\tau} = \tau V^\frac{-2}{m+n}$. First, we obtain

$$
\int_{B_{2c}} |\nabla \varphi_{0, \tau}|^2 dx = \frac{\tau m}{c(m,n)} I_7 + O(e^{2-2m-n} \tau^{\frac{n}{m+n} + \frac{n}{2}}).
$$

Estimate (47) in Lemma 4, estimate (82) in Lemma 6, $\tilde{\tau} = \tau V^\frac{-2}{m+n}$, and equality (91) implies that

$$
\int_{B_{2c}} |\nabla g f_{\tau}^m v^m dV_g = \frac{\tau m}{V_{\tau}} \int_{B_{2c}} |\nabla \varphi_{0, \tau}|^2 dx
$$

where $\tilde{\tau} = \tau V^\frac{-2}{m+n}$, estimate (36) in Lemma 3, and equality (82) in Lemma 6 yield

$$
\frac{\tau m}{V_{\tau}} \int_{B_{2c}} F^m f_{\tau}^2 v^m dV_g = - \frac{2 \tau m \Delta_g v I_1}{c(m,n)^2 V^{(m+n)/(2m+n)} + \frac{m+n-2}{m+n}} + \frac{\tau^2 m^2 (m + n - 2)(\Delta_g v)^2 I_1 I_7}{n (m + n) c(m,n)^{n+1} V^{2m/(m+n) + \frac{2m+2n-2}{m+n}}} + \frac{\tau^2 (\Delta_g R_{\phi}^m - 2 m^2 (\Delta_g v)^2) I_2}{2 n c(m,n)^{m+2} V^{2m/(m+n) + \frac{2m+2n-2}{m+n}}} + O(\tau^{m+2} e^{-2m-n}) + O(\tau^{2+\frac{2}{n}}).
$$

Equality $\tilde{\tau} = \tau V^\frac{-2}{m+n}$, estimate (36) in Lemma 3 and estimate (82) in Lemma 6 yield
Now, we obtain
\begin{equation}
\int_{B_{\tau}} \varphi_{0,\tau}^{2(m+n-1)} dx = \frac{\tau^{2(m+n)}}{c(m,n)^{2}} I_8 + O(\epsilon^{2-2m-n} \tau^{2(m+n)+m+\frac{n-2}{2}})
\end{equation}
where
\begin{equation}
I_8 = \int_{\mathbb{R}^n} (1 + |y|^{2})^{-(m+n-1)} dy.
\end{equation}

Equality \( \tilde{\tau} = \tau V^{\frac{1}{2m+n}} \), estaminative \((83)\) in Lemma \(6\) and equality \((94)\) imply that
\begin{align}
m \int_{B_{\tau}} f_{\tau}^{2(m+n-1)} v^{m-1} dV_g &= \frac{\tau^{2(m+n)}}{2(m+n)} V^{\frac{n}{m+n}} \\
&- \frac{\tau m^2(m+n-1)I_7 I_8 \Delta_g v}{2n(m+n)c(m,n)^{n+1} V^{\frac{n}{(m+n)(2m+n)}} + 2n + 2(n+1)} \\
&+ \frac{\tau^2 m(m+n-1)}{2n(n+2)(m+n)c(m,n)^{n+2} V^{\frac{n}{(m+n)(2m+n)}} + 2n + 2(n+1)} \\
&+ \frac{\tau^2 m^3(m+n-1)(2m+2n-1)(I_7)^2 I_8 (\Delta_g v)^2}{8n^2(m+n)^2 c(m,n) V^{\frac{n}{(m+n)(2m+n)}} + 2n + 2(n+1)} \\
&- \frac{\tau m(m+n)}{2nc(m,n)^{\frac{n}{2}} V^{\frac{n}{(m+n)(2m+n)}} + m+n+1} \\
&- \frac{\tau^2 m^2(m+n-1)(m+n-1)}{4n^2(m+n)c(m,n)^{n+2} V^{\frac{n}{(m+n)(2m+n)}} + 2n + 2(n+1)} \\
&- \frac{\tau^2 m^2 I_6 A_2}{2n(n+2)c(m,n)^{\frac{n+4}{2}} V^{\frac{n}{(m+n)(2m+n)}} + m+n+1} \\
&+ O(\tau^{m+\frac{n-2}{2}} \epsilon^{2m-n}) + O(\tau^{2+\frac{n}{2}}).
\end{align}

The equality \((21)\) and the estimates \((92), (93),\) and \((96)\) imply that \((35)\) takes the following form
\begin{equation}
|W[M^n, g, v^n dV_g]|(\tilde{f}_\tau, \tilde{\tau}) + m = \nu(\mathbb{R}^n, dx^2, dV, m) + m + \tau A_3 + \tau^2 A_4 \\
+ O(\tau^{m+\frac{n-2}{2}} \epsilon^{2m-n}) + O(\tau^{2+\frac{n}{2}})
\end{equation}
where

\[(98)\]

\[
A_3 := \frac{mV - (m+n)(2m+n)}{2c(m,n)^2} \frac{m+n-2}{m+n} \Delta_g v \left( \frac{(m+n-2)^2}{n} I_3 + \frac{m-1}{nc(m,n)} I_5 - \frac{m+n-2}{(m+n-1)} I_1 - \frac{m(m+n-1)I_7 I_8}{n(m+n)c(m,n)} \frac{2+m}{2+m} V - \frac{(m+n-2)^3}{n(m+n)c(m,n)^2} V \right)
\]

and

\[(99)\]

\[
A_4 := \frac{V}{(m+n)(2m+n)} \frac{m+n-1}{m+n} \frac{m+n-2}{m+n} \left( \frac{m+n-2}{2n(n+2)(m+n)c(m,n)^2} V \right)
\]

Using the comparisons for integrals given in Lemma (see appendix below) and the equality \(c(m,n) = \frac{m+n-1}{(m+n-2)^2}\), it follows that

\[(100)\]

\[
A_3 = \frac{m(m+n-2)^2 \Delta_g v I_7}{2c(m,n)^2 V} \frac{2m}{(m+n)(2m+n)} + \frac{m+n-2}{m+n} \left( \frac{n+2}{n(2m+n-4)} + \frac{2(m-1)}{n(2m+n-4)} - \frac{m+n-2}{n(2m+n-4)} \right)
\]
Next, we analyze $A_4$. For this purpose, note that

$$\frac{m-5}{n(2m+n-4)(2m+n-6)} = \frac{(m+n-2)}{2(m+n)(2m+n-2)(2m+n-4)} \frac{(n+4)}{2n(2m+n-4)(2m+n-6)} + \frac{m(m+n-1)}{n(m+n)(2m+n-2)(2m+n-4)}. \tag{101}$$

By equality (101) and Lemma 8 we obtain

$$A_4 = \frac{I_7(m+n-2)^2}{c(m,n)^{n+2} V(\frac{m}{m+n})^{\frac{n}{m+n}} - m+1} \left( \frac{(m-5)A_1}{n(2m+n-4)(2m+n-6)} \right.$$

$$\left. + \frac{m^2(m+n-2)(m+n-1)(\Delta_g v)^2}{4(m+n)^2(2m+n-2)^2} \right.$$

$$\left. - \frac{m^2(m+n-2)(n+2)(\Delta_g v)^2}{4n(m+n)(2m+n-2)(2m+n-4)} \right.$$

$$\left. + \frac{m^2(m+n-2)(\Delta_g v)^2}{n(m+n)(2m+n-2)(2m+n-4)} \right.$$ (102)

$$\left. + \frac{\Delta_g R_{\phi}^m - 2m^2(\Delta_g v)^2}{2n(2m+n-4)(2m+n-6)} \right.$$

$$\left. + \frac{m^3(m+n-1)(2m+2n-1)(\Delta_g v)^2}{4n(m+n)^2(2m+n-2)^2} \right.$$ (103)

$$\left. - \frac{m^2(m-1)(m+n-1)(\Delta_g v)^2}{2n(m+n)(2m+n-2)(2m+n-4)} \right.$$ (104)

$$\left. - \frac{mA_2}{n(2m+n-4)(2m+n-6)} \right).$$

The definitions of $A_1$ and $A_2$ in (50) and (71), respectively, and equality (24) imply that $A_4$ takes the form

$$A_4 = \frac{(-\frac{1}{6} |W|^2 g + m(m-1) |\text{Hess } v|^2 g}{2n(2m+n-4)(2m+n-6)} + A_5 (\Delta_g v)^2 \tag{103}$$

where

$$A_5 = \frac{m(1-\frac{1}{m})}{4n(2m+n-4)(2m+n-6)} + \frac{m^2(m+n-1)(m+n-2)}{4(m+n)^2(2m+n-2)^2} - \frac{m^2(m+n-2)(n+2)}{4n(m+n)(2m+n-2)(2m+n-4)}$$

$$+ \frac{m^2(m+n-2)}{n(m+n)(2m+n-2)(2m+n-4)} + \frac{m^3(m+n-1)(2m+2n-1)}{4n(m+n)^2(2m+n-2)^2} - \frac{m^2(m-1)(m+n-1)}{2n(m+n)(2m+n-2)(2m+n-4)}.$$ (104)

Since the problem coincides with the Yamabe problem when $m = 0$, we analyze the term on the right-hand side of (103) for $m > 0$. Note that $m(m-1)|\text{Hess } v|^2 g$ is nonpositive.
if $0 < m \leq 1$. For $n \geq 7$, we have $\lim_{m \to 0} A_5 = 0$, and for $n = 6$, the term $A_5$ is bounded and

$$\lim_{m \to 0} \frac{-\frac{1}{6} W_5^2(p)}{2n(2m + n - 4)(2m + n - 6)} = -\infty.$$  

Then, for $n \geq 6$ there exists $0 < \delta \leq 1$ such that $A_4 < 0$ for $m < \delta$.

For $0 < m \leq \delta$, using $A_3 = 0$ and $A_4 < 0$, taking $\epsilon$ as small and fixed, and after choosing a sufficiently small $\tau$, equality (97) yields

$$\nu \left[ M^n, g, v^n dV_g \right] < \nu \left[ R^n, dx^2, dV, m \right].$$

Proposition 4 implies

$$\Lambda \left[ M^n, g, v^n dV_g \right] < \Lambda \left[ R^n, dx^2, dV, m \right].$$

Theorem 1 concludes the proof for $0 < m \leq \delta$. Finally, Theorem 4 and an inductive argument imply that

$$\Lambda \left[ M^n, g, v^{m+1} dV_g \right] \leq \Lambda \left[ M^n, g, v^n dV_g \right] \Lambda \left[ R^n, dx^2, dV, m \right] \Lambda \left[ R^n, dx^2, dV, m + 1 \right]$$

$$< \Lambda \left[ R^n, dx^2, dV, m + 1 \right],$$

which leads to our result for every $m \in \bigcup_{i \in \mathbb{N} \cup \{0\}} [i, i + \delta)$. \qed

**Remark 3.** Note that the proof works if $A_5 \leq 0$, which is false for a general $m > 0$.

5. **APPENDIX**

In this section, we show some calculus lemmas that we used in the proof of Theorem A.

**Lemma 7.** Letting $1 \leq i, j \leq n$ with $i \neq j$, then

$$\int_{B_\rho} \frac{x_i^4 |x|^l}{(1 + \frac{c(m,n)}{\tau} |x|^2)^k} dx = 3 \int_{B_\rho} \frac{x_i^2 x_j^2 |x|^l}{(1 + \frac{c(m,n)}{\tau} |x|^2)^k} dx$$

$$= \frac{3}{n(n+2)} \int_{B_\rho} \frac{|x|^{4+l}}{(1 + \frac{c(m,n)}{\tau} |x|^2)^k} dx.$$  

**Proof.** We will use the formula

$$\int_{S^{n-1}_\rho} q dS_\rho = \frac{\rho^2}{d(d+n-2)} \int_{S^{n-1}_\rho} \Delta q dS_\rho,$$

where $S^{n-1}_\rho$ is the sphere of radius $\rho$ and $q$ is a homogeneous polynomial of degree $d$. Then

$$\int_{S^{n-1}_\rho} x_i^4 dS_\rho = 3 \int_{S^{n-1}_\rho} x_i^2 x_j^2 dS_\rho = \frac{3}{n(n+2)} \rho^4 \int_{S^{n-1}_\rho} dS_\rho.$$
Using the last equality and polar coordinates, we obtain the result.

Next, we compare the integrals $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$ and $V$ considered in the above section. This kind of comparison appeared, for example, in [1] and [5].

**Lemma 8.** We obtain the following equalities

\[ I_1 = \frac{4(m + n - 1)(m + n - 2)}{n(2m + n - 4)}I_7, \quad I_2 = \frac{4(m + n - 1)(m + n - 2)}{(2m + n - 4)(2m + n - 6)}I_7, \]

\[ I_3 = \frac{n + 2}{2m + n - 4}I_7, \quad I_4 = \frac{(n + 2)(n + 4)}{(2m + n - 4)(2m + n - 6)}I_7, \]

\[ I_5 = \frac{2(m + n - 1)}{2m + n - 4}I_7, \quad I_6 = \frac{2(m + n - 1)(n + 2)}{(2m + n - 4)(2m + n - 6)}I_7, \]

\[ I_8 = \frac{2(m + n - 1)}{n}I_7, \quad \text{and} \quad \frac{I_7}{V} = \frac{nc(m, n)^\frac{n}{2}}{2m + n - 2}. \]

**Proof.** Using polar coordinates, we obtain

\[ I_1 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^{m+n-2}} dr, \]

\[ I_2 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n-2}} dr, \]

\[ I_3 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^{m+n}} dr, \]

\[ I_4 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n+5}}{(1 + r^2)^{m+n}} dr, \]

\[ I_5 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n-1}} dr, \]

\[ I_6 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^{m+n-1}} dr, \]

\[ I_7 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n}} dr, \]

\[ I_8 = \text{vol}(S^{n-1}) \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^{m+n-1}} dr, \]
and

\begin{equation}
V = \frac{\text{vol}(S^{n-1})}{c(m, n)^{\frac{n}{2}}} \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^m \, dr}.
\end{equation}

Integrating by parts, we obtain for every \( k > 1, \ l > 1, \) and \( k > l \)

\begin{equation}
\int_0^\infty \frac{r^{l+1}}{(1 + r^2)^k \, dr} = \frac{l}{2(k - 1)} \int_0^\infty \frac{r^{l-1}}{(1 + r^2)^{k-1} \, dr},
\end{equation}

which implies that \( I_7 = \frac{n}{2(m+n-1)} I_8, \ I_3 = \frac{n+2}{2(m+n-1)} I_5 \) and \( I_4 = \frac{n+4}{2(m+n-1)} I_6. \) To compare \( I_5 \) with \( I_7, \) we write

\begin{equation}
\int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n-1} \, dr} = \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^m \, dr} + \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^{m+n} \, dr}.
\end{equation}

Using equality \( \text{(118)} \) in \( \text{(119)} \) yields

\begin{equation}
\frac{2m + n - 4}{2(m + n - 1)} \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n-1} \, dr} = \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^m \, dr}.
\end{equation}

Hence, \( I_5 = \frac{2(m+n-1)}{2m+n-4} I_7 \) and \( I_3 = \frac{n+2}{2m+n-4} I_7. \) Similarly

\begin{equation}
\int_0^\infty \frac{r^{n+3}}{(1 + r^2)^{m+n-1} \, dr} = \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^m \, dr} + \int_0^\infty \frac{r^{n+5}}{(1 + r^2)^{m+n} \, dr}.
\end{equation}

Using equality \( \text{(118)} \) in \( \text{(121)} \) yields

\begin{equation}
\frac{2m + n - 6}{2(m + n - 1)} \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^{m+n-1} \, dr} = \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^m \, dr}.
\end{equation}

Then, \( I_6 = \frac{2(m+n-1)}{2m+n-6} I_7 = \frac{2(m+n-1)(n+2)}{(2m+n-4)(2m+n-6)} I_7 \) and \( I_4 = \frac{(n+2)(n+4)}{(2m+n-4)(2m+n-6)} I_7. \)

To compare \( I_7 \) with \( V, \) we write

\begin{equation}
\int_0^\infty \frac{r^{n-1}}{(1 + r^2)^{m+n-1} \, dr} = \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^m \, dr} + \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n} \, dr}.
\end{equation}

Using equality \( \text{(118)} \) in equality above, we obtain

\begin{equation}
\frac{2m + n - 2}{n} \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n} \, dr} = \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^m \, dr}.
\end{equation}

Therefore,

\begin{equation}
\frac{I_7}{V} = \frac{nc(m, n)^{\frac{n}{2}}}{2m + n - 2}.
\end{equation}

Now, we compare \( I_1 \) with \( I_7; \) for this purpose, observe that

\begin{equation}
\int_0^\infty \frac{r^{n-1}}{(1 + r^2)^{m+n-2} \, dr} = \int_0^\infty \frac{r^{n-1}}{(1 + r^2)^{m+n-1} \, dr} + \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n} \, dr}.
\end{equation}
Hence, \( I_1 = I_8 + I_5 \). Therefore, \( I_1 = \frac{4(m+n-1)(m+n-2)}{n(2m+n-4)} I_7 \). It remains to compare \( I_2 \) with \( I_7 \). We have

\[
(126) \quad \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n-2}} \, dr = \int_0^\infty \frac{r^{n+1}}{(1 + r^2)^{m+n-1}} \, dr + \int_0^\infty \frac{r^{n+3}}{(1 + r^2)^{m+n-1}} \, dr.
\]

It follows from equalities (126) and (118) that \( I_2 = \frac{n}{2(m+n-2)} I_5 + I_6 \). As a consequence, \( I_2 = \frac{4(m+n-1)(m+n-2)}{(2m+n-4)(2m+n-6)} I_7 \). □

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**References**

[1] Aubin, Thierry. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures Appl. (9) 55, no. 3, pp. 269-296. (1976)

[2] Case, Jeffrey S. A Yamabe-type problem on smooth metric measure spaces, J. Differential Geom. 101 no. 3, pp 467-505. (2015)

[3] Case, Jeffrey S. Conformal invariants measuring the best constants for Gagliardo-Nirenberg- Sobolev inequalities. Calc. Var. Partial Differential Equations, 48(3-4), pp 507-526. (2013)

[4] Del Pino M. and Dolbeault, J. Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. (9), 81(9) pp 847-875(2002)

[5] Escobar, J. F., Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. Ann. of Math. (2) 136, no. 1, pp 1-50, (1992).

[6] J. M. Lee and T. H. Parker. The Yamabe problem. Bull. Amer. Math. Soc. (N.S.) 17 no. 1, pp 37-91, (1987).

[7] Marques, F. C. Existence results for the Yamabe problem on manifolds with boundary. Indiana Univ. Math. J.54, no. 6, pp 1599-1620, (1992).

[8] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. arXiv:0211159, preprint.

[9] Schoen, Richard. Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 no. 2, pp 479-495 (1984)

[10] Trudinger, Neil S. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa (3) 22, pp 265-274, (1968)

[11] Yamabe, Hiddeiko. On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. no. 12, pp 21-37 (1960)
1 Department of Mathematics, Universidad del Valle, Calle 13#100 – 00, Cali, Colombia
E-mail address: jhovanny.munoz@correounivalle.edu.co

2 Instituto de Matemáticas Puras e Aplicadas, Estrada Dona Castorina 110, Rio de Janeiro, Brasil.
E-mail address: jhovamu@impa.br