Polynomials with $r$-Lah coefficient and hyperharmonic numbers

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Abstract

In this paper, we take advantage of the Mellin type derivative to produce some new families of polynomials whose coefficients involve $r$-Lah numbers. One of these polynomials leads to rediscover many of the identities of $r$-Lah numbers. We show that some of these polynomials and hyperharmonic numbers are closely related. Taking into account of these connections, we reach several identities for harmonic and hyperharmonic numbers.

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1 Introduction

Operator theory is one of the attractive way used in number theory as in other branches of mathematics. One of the often used operators is the operator $(xD) = x^2d/dx$, called Mellin derivative [7]. For a $n$-times differentiable function $f$, we have [5]

$$(xD)^n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k f^{(k)}(x),$$

where $\binom{n}{k}$ are the Stirling numbers of the second kind. This operator has a long mathematical history and firstly used by Euler as a tool in one of his study [23]. Depending on the choice of function $f$ in (1) appear exponential, geometric, higher-order geometric or harmonic geometric polynomials [5, 19]. These polynomials are closely related to Bernoulli and Euler numbers and some of their generalizations [6, 31–33], which numbers play important roles in number theory. Further, the operator $(xD)$ has been used in the evaluation of some power series and integrals [5, 12, 19, 32, 34, 36]. Numerous generalizations of the Mellin derivative have also been studied according to the generalizations of the Stirling numbers of the second kind [5, 20, 30, 32].

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The $r$-Lah numbers $\left\lfloor \frac{n}{k} \right\rfloor_r$ can be defined by following generating function [38]

$$\frac{1}{k!} \left( \frac{t}{1-t} \right)^k \left( \frac{1}{1-t} \right)^{2r} = \sum_{n=k}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor_r \frac{t^n}{n!},$$

which are also mentioned and studied with different names in [3, 15, 37]. The Lah numbers $\left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{n}{k} \right\rfloor_0$ (rarely called Stirling numbers of the third kind [40]) have many interesting applications in analysis and combinatorics [1, 2, 10, 26, 27].

The hyperharmonic numbers $h^{(r)}_n$ are defined by [16]

$$h^{(r)}_n = \sum_{k=1}^{n} h^{(r-1)}_k, \ h^{(0)}_k = \frac{1}{k}$$

and have the generating function [4]

$$\sum_{n=0}^{\infty} h^{(r)}_n t^n = -\frac{\ln(1-t)}{(1-t)^r}.$$ (3)

The hyperharmonic numbers are related to the harmonic numbers $H_n$ by $h^{(1)}_n = H_n$ and

$$h^{(r)}_n = \binom{n+r-1}{n} (H_{n+r-1} - H_{r-1})$$

(cf. [4, 16]). The harmonic numbers satisfy the binomial harmonic identity [11]

$$H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k}$$

and symmetric formula

$$\frac{H_n}{n+1} = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{H_k}{k+1}.$$ (5)

The binomial harmonic identity was generalized to binomial hyperharmonic identity [35, Corollary 3.1]

$$h^{(r)}_n = \sum_{k=0}^{n} \binom{n}{k} \alpha (k, r),$$ (6)

with the help of Euler-Siedel matrix method. Here

$$\alpha (k, r) = \begin{cases} h^{(r-k)}_k, & 0 \leq k < r \\ (-1)^{k+h} (r-1)!/k! \delta_r, & k \geq r \end{cases}$$

$\delta_r = 0$ or 1, according to $r$ is even or odd, and $(x)^\underline{m}$ is the falling factorial function (see Section 2).
In this paper, we capitalize the operator

\[(xD + 2r)^n = (xD + 2r)(xD + 2r + 1) \cdots (xD + 2r + n - 1),\]

which is the key tool of this study. When it is applied to an appropriate function \(f\), the coefficients of the resulting function are the \(r\)-Lah numbers, namely,

\[(xD + 2r)^n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k f^{(k)}(x). \quad (7)\]

Some families of polynomials appear due to the choice of \(f\) in (7). One of these polynomials is the exponential \(r\)-Lah polynomials (see Section 3). With the help of these polynomials, we rediscover many of the identities given by Nyul and Rácz [38] for \(r\)-Lah numbers. In addition, some new relations are presented for these numbers. The another arising polynomial is the geometric \(r\)-Lah polynomial (see Section 4). We reach a relationship between geometric \(r\)-Lah polynomials and hyperharmonic numbers. This connection gives rise to a new formulation for the binomial hyperharmonic identity (6):

\[h^{(r)}_{n+1} = \sum_{k=0}^{n} \binom{n+r}{k+r} (-1)^k \frac{k!}{k+1},\]

(see Theorem 3). Additionally, we show that Bernoulli polynomials can be described as a sum of the products of hyperharmonic numbers and \(r\)-Stirling numbers of the second kind (see Theorem 5). This formula leads a generalization of some identities, obtained by Pascal matrix method, presented in [14]. Finally, we examine a polynomial which we call harmonic geometric \(r\)-Lah polynomial (see Section 5). These polynomials are also related to hyperharmonic numbers. Considering this, we infer a generating function for the hyperharmonic numbers with respect to the upper index. We then deduce a closed-form evaluation formula for the Euler-type sum

\[\sum_{m=0}^{\infty} \frac{h_n^{(m+1)}}{(m+1)^r} x^m\]

(see Theorem 9). It should be noted that the sum in question is over the upper index, however, the studies on the Euler-type sums containing hyperharmonic numbers depend on the lower index [18,21,29]. Furthermore, we produce some generalizations of the symmetric formula (5) (see Theorems 10 and 11), for instance, one of the generalizations is

\[h^{(r)}_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{H_k}{(k+1)^r},\]

where \((x)^r\) is the rising factorial function (see Section 2). Moreover, we give formulas for the sum of the products of hyperharmonic numbers (see Theorem 13). Particular cases give rise to some interesting results, for instance,

\[2 \sum_{k=1}^{n} \frac{H_k}{k+1} = \sum_{k=1}^{n} \frac{H_k}{n+1-k} = (H_{n+1})^2 - H_{n+1}^{(2)}\]
where \( H^{(2)}_{n} = 1 + \frac{1}{2^n} + \cdots + \frac{1}{n^n} \).

2 Preliminaries

Let \((x)^{\pi}\) and \((x)^{\mu}\) denote the rising and falling factorial functions defined by

\[
(x)^{\pi} = x(x+1) \cdots (x+n-1), \quad \text{with} \quad (x)^{\pi}_0 = 1,
\]

\[
(x)^{\mu} = x(x-1) \cdots (x-n+1), \quad \text{with} \quad (x)^{\mu}_0 = 1.
\]

The \( r \)-Stirling numbers of the first kind \([n]_r \), the second kind \( \{n\}_r \) and the \( r \)-Lah numbers \( \lfloor n \rfloor_r \) can be defined by [13, 38]

\[
(x + r)^{\pi} = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_r x^k,
\]

\[
(x + r)^{\mu} = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r (x)^k,
\]

and

\[
(x + 2r)^{\pi} = \sum_{k=0}^{n} \lfloor n \rfloor_r (x)^k.
\]

Note that \( \left[ \begin{array}{c} n \\ k \end{array} \right]_0 = \left[ \begin{array}{c} n \\ k \end{array} \right] \) and \( \left\{ \begin{array}{c} n \\ k \end{array} \right\}_0 = \left\{ \begin{array}{c} n \\ k \end{array} \right\} \) are the Stirling numbers of the first and second kind.

Replace \( x \) by \((xD)\) in

\[
(x)^{\mu} = \sum_{k=0}^{n} (-1)^{n-k} \left[ \begin{array}{c} n \\ k \end{array} \right] x^k
\]

and then apply it to a \( n \)-times differentiable function \( f \). Utilizing (1) and the identity

\[
\sum_{j=k}^{n} (-1)^{j-k} \left[ \begin{array}{c} n \\ j \end{array} \right] \left\{ \begin{array}{c} j \\ k \end{array} \right\} = \left\{ \begin{array}{l} 1, \quad n = k \\
0, \quad n \neq k
\right. \]

we obtain that

\[
(xD)^k f(x) = x^k f^{(k)}(x).
\]

Therefore, applying the operator \((xD + 2r)^{\pi}\) to a \( n \)-times differentiable function \( f \) and using (10) and (11), we obtain (7), i.e.,

\[
(xD + 2r)^{\pi} f(x) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_r x^k f^{(k)}(x).
\]

By the similar fashion, it follows from (9) and (11) that

\[
(xD + r)^{\mu} f(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_r x^k f^{(k)}(x).
\]
We finally want to recall the $r$-Stirling transform which will be useful in the next sections:

\[ a_n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] r b_k \quad (n \geq 0) \] if and only if 
\[ b_n = \sum_{k=0}^{n} (-1)^{n-k} \left[ \frac{n}{k} \right] r a_k \quad (n \geq 0), \]

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3 Exponential $r$-Lah polynomials

We first deal with (7) for $f(x) = e^x$ and present some identities for arising polynomials. We then rediscover many of the identities recorded in [38] and present some new relations for $r$-Lah numbers.

It is seen from (7) that

\[ (xD + 2r)^n e^x = e^x \sum_{k=0}^{n} \left[ \frac{n}{k} \right] r x^k = e^x L_{n,r} (x), \] (13)

where

\[ L_{n,r} (x) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] r x^k, \] (14)

which we call exponential $r$-Lah polynomials. In particular,

\[ (xD)^n e^x = e^x L_{n,0} (x) = e^x L_n (x). \] (15)

The notation $L_n (x)$ for this polynomial was firstly used by Guo and Qi in [26] and their related papers.

On the other hand, from (8) and (12), we have

\[ (xD + r + s)^n e^x = \sum_{j=0}^{n} \left[ \frac{n}{j} \right] r (xD + s)^j e^x \]
\[ = e^x \sum_{k=0}^{n} \sum_{j=k}^{n} \left[ \frac{n}{j} \right] r \left[ \frac{j}{k} \right] s e^x, \] (16)

Thus, we find from (13) and (16) that

\[ \left[ \frac{n}{k} \right] r^+_{s+t} = \sum_{j=k}^{n} \left[ \frac{n}{j} \right] r \left[ \frac{j}{k} \right] s, \]

which is Theorem 3.11(d) of [38].

**Proposition 1** For all non-negative integer $n$,

\[ L_{n+2s, r} (x) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] r^+_{s+t} x^k L_{2s, r+t} (x). \] (17)
Proof. Appealing to (13) and noting that \( a^{m+n} = a^m (a + m)^n \), we see that
\[
e^x \mathcal{L}_{n+2s} (x) = (xD + 2r)^{2s} (xD + 2r + 2s) e^x
\]
\[
= (xD + 2r)^{2s} e^x \mathcal{L}_{n+2s} (x)
\]
\[
\stackrel{(14)}{=} \sum_{k=0}^{n} \left\lfloor \frac{n}{k} \right\rfloor \frac{2s}{r+s} (xD + 2r)^{\frac{s}{r+s}} (x^k e^x) .
\]
Using the Taylor expression of \( e^x \) in (13) and considering that \((xD + 2r)^{2s} x^k = (k + 2r)^{2s} x^k\), we obtain
\[
(xD + 2r)^{2s} (x^k e^x) = x^k e^x \mathcal{L}_{n,r+2s} (x),
\]
which completes the proof. ■

Comparing the coefficients of \( x^k \) in (17), with the use of (14), yields to the following relation:

Corollary 1
\[
\left\lfloor \frac{n+2s}{m} \right\rfloor = \sum_{k=0}^{n} \left\lfloor \frac{n}{k} \right\rfloor \frac{2s}{m-k} .
\]
Multiplying both sides of (14) with \( t^n/n! \) and then summing over \( n \) give the following generating function for \( \mathcal{L}_{n,r} (x) \):
\[
\frac{1}{(1-t)^{2s}} e^{tx} = \sum_{n=0}^{\infty} \mathcal{L}_{n,r} (x) \frac{t^n}{n!} .
\]
It is easily seen from (19) that
\[
\mathcal{L}_{n,r+s} (x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{L}_{k,r} (x) \mathcal{L}_{n-k,s} (y),
\]
\[
\mathcal{L}_{n,r+s} (x) = \sum_{k=0}^{n} \binom{n}{k} (2s)^{n-k} \mathcal{L}_{k,r} (x) .
\]
As a consequence of (20), we have (cf. [38, Theorem 3.4])
\[
\left\lfloor \frac{n}{l} \right\rfloor = \sum_{k=l}^{n} \binom{n}{k} \frac{2r}{l} (2r+2s)^{n-k}
\]
\[
= \sum_{k=l}^{n} \binom{n}{k} \frac{2s}{l} (2r)^{n-k} .
\]
Let us continue by differentiating both sides of (19) with respect to \( x \). Taking \( m \) times derivative, we see that
\[
\frac{d^m}{dx^m} \mathcal{L}_{n,r} (x) = n^{m} \mathcal{L}_{n-m,r+2s} (x) .
\]
On the other hand, we have
\[ \sum_{n=0}^{\infty} \frac{d}{dx} L_{n,r}(x) \frac{t^n}{n!} = t \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} L_{n,r}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{L_{k,r}(x)}{k!} t^n \]
and then, by (21),
\[ \frac{1}{n!} L_{n,r+\frac{1}{2}}(x) = \sum_{k=0}^{n} \frac{L_{k,r}(x)}{k!}. \]

Therefore, (21) implies that
\[ \frac{1}{n!} n^m L_{n-m,r+\frac{m+1}{2}}(x) = \sum_{k=0}^{n-m} \frac{(k+m)^m}{(k+m)!} L_{k,r+\frac{m}{2}}(x), \]
which can be stated as
\[ \frac{1}{n!} L_{n+r+\frac{m}{2}}(x) = \sum_{k=0}^{n} \frac{1}{k!} L_{k,r+\frac{m}{2}}(x) \] (22)
by substituting \( n \to n + m \). We now compare the coefficients of \( x \) to deduce
\[ \binom{n+m}{n} \binom{n}{k} \frac{n}{k+r+\frac{m}{2}} = \binom{n+m}{k+m} \frac{n}{k+m+\frac{m}{2}} \] (23)
and
\[ \frac{1}{n!} \binom{n}{l} \frac{n}{l+r+\frac{m}{2}} = \frac{1}{k!} \binom{k}{l} \frac{n}{l+m+\frac{m}{2}}, \]
from (21) and (22), respectively. Thus, we arrive at the following:

**Proposition 2** For all non-negative integers \( n \) and \( m \),
\[ \frac{1}{(n+m+1)!} \binom{n+m+1}{l+m+1} r = \sum_{k=l}^{n} \frac{1}{(k+m)!} \binom{k+m}{l+m+1} r. \] (24)

**Corollary 2** We have
\[ \sum_{k=l}^{p} \frac{k}{(k+m)!} \binom{k+m}{l+m} r = \frac{p+1}{(p+m+1)!} \binom{p+m+1}{l+m+1} r - \frac{1}{(p+m+2)!} \binom{p+m+2}{l+m+2} r \] (25)
and
\[ \sum_{k=l}^{p} \frac{k^2}{(k+m)!} \binom{k+m}{l+m} r = \frac{(p+1)^2}{(p+m+1)!} \binom{p+m+1}{l+m+1} r - \frac{2p+3}{(p+m+2)!} \binom{p+m+2}{l+m+2} r \]
\[ + \frac{1}{(p+m+3)!} \binom{p+m+3}{l+m+3} r. \]
Proof. Summing both sides of (24) over \( n \), we find that
\[
\frac{1}{(p + m + 2)!} \left[ \begin{array}{c} p + m + 2 \\ l + m + 2 \end{array} \right]_r = \frac{p}{(p + m + 2)!} \sum_{k=l}^{p} \frac{1}{(k + m)!} \left[ \begin{array}{c} k + m \\ l + m \end{array} \right]_r
\]
which is (25). The second relation follows from (25) by summing over \( p \). □

The following proposition offers formula that is analogue to a familiar Leibniz rule for higher derivatives of the product of two functions.

**Proposition 3** For \( n \)-times differentiable functions \( f \) and \( g \),
\[
(xD + 2r)^n [f(x)g(x)] = \sum_{k=0}^{n} \binom{n}{k} \left[ (xD)^k f(x) \right] \left[ (xD + 2r + k)^{n-k} g(x) \right].
\]

**Proof.** It follows from (7) and (23) that
\[
(xD + 2r)^n [f(x)g(x)] = \sum_{l=0}^{n} \binom{n}{l} x^l \left[ f(x)g(x) \right]^{(l)}
\]
\[
= \sum_{k=0}^{n} x^k f^{(k)}(x) \sum_{l=0}^{n-k} \binom{n-k}{l+k} \left[ (l+k) \right]_r x^l g^{(l)}(x)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} x^k f^{(k)}(x) \sum_{l=0}^{n-k} \binom{n-k}{l} \left[ (l) \right]_{r+k/2} x^l g^{(l)}(x).
\]

Hence, (7) and (11) give (26). □

Taking \( f(x) = x^m \) and \( g(x) = e^x \) in (26), it is seen that
\[
(xD + 2r)^n [x^m e^x] = \sum_{k=0}^{n} \binom{n}{k} \left( m \right)_{n-k} x^m e^x L_{k,r+\frac{m}{2}}(x)
\]
and from (18) (with substitution \( m = 2s \))
\[
L_{n,r+s}(x) = \sum_{k=0}^{n} \binom{n}{k} (2s)_{n-k} L_{k,r+\frac{2s}{2}}(x).
\]
Comparing the coefficients of \( x^k \) in (27), with the use of (14), we arrive at
\[
\binom{n}{l} = \sum_{k=l}^{n} \binom{n}{k} \binom{k}{l} (2s)^{n-k}. \quad (28)
\]
It emerges that the relation (28) is equivalent to [38, Theorem 3.5] by (23):
\[
\binom{n}{l} = \sum_{k=l}^{n} \binom{n}{k} \binom{k}{l} (2s)^{k-l}. \quad (28)
\]
If we take $f(x) = e^x$ and $g(x) = x^m$ in (26), then we have

$$x^m e^x \mathcal{L}_{n,r} \left( \frac{w}{x} \right) \overset{(18)}{=} (xD + 2r) \left[ e^x x^m \right]$$

$$
\overset{(26)}{=} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \left[ (xD)^k e^x \right] \left[ (xD + 2r + k)^{n-k} x^m \right] \\
= \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) x^k e^x (m + 2r + k)^{n-k} x^m,
$$
or (by substitution $m = 2s$)

$$\mathcal{L}_{n,r+s} (x) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (2r + 2s + k)^{n-k} x^k.$$ 

This and (14) yield that

$$\left[ \begin{array}{c} n \\ l \end{array} \right]_r = \left( \begin{array}{c} n \\ k \end{array} \right) (2r + k)^{n-k} = n! \left( \frac{n + 2r - 1}{k + 2r - 1} \right). \quad (29)$$

This is nothing but Theorem 3.7 of [38].

The exponential $r$-Lah polynomials satisfy the following three-term recurrence relation:

**Theorem 1** We have

$$\mathcal{L}_{n+1,r} (x) = (2n + 2r + x) \mathcal{L}_{n,r} (x) - n (2r + n - 1) \mathcal{L}_{n-1,r} (x). \quad (30)$$

**Proof.** Differentiating both sides of (19) with respect to $t$ gives

$$\sum_{n=0}^{\infty} \mathcal{L}_{n+1,r} (x) \frac{t^n}{n!} = \frac{2r}{(1-t)^{2r+1}} e^x \frac{t^1}{1!} + \frac{x}{(1-t)^{2r+2}} e^x \frac{t^1}{1!}$$

$$= \frac{2r (1-t) + x}{(1-t)^2} \sum_{n=0}^{\infty} \mathcal{L}_{n,r} (x) \frac{t^n}{n!}.$$ 

We multiply both sides with $(1-t)^2$ and arrange to find

$$\sum_{n=0}^{\infty} \mathcal{L}_{n+1,r} (x) \frac{t^n}{n!} \frac{n!}{n!} - \sum_{n=1}^{\infty} 2n \mathcal{L}_{n,r} (x) \frac{t^n}{n!} + \sum_{n=2}^{\infty} n (n-1) \mathcal{L}_{n-1,r} (x) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} (2r + x) \mathcal{L}_{n,r} (x) \frac{t^n}{n!} - \sum_{n=1}^{\infty} 2rn \mathcal{L}_{n-1,r} (x) \frac{t^n}{n!}.$$ 

This gives the desired result. ■

Note that (30) with (14) and (29) yields the following recurrence relation [38, Theorem 3.1]

$$\left[ \begin{array}{c} n + 1 \\ k \end{array} \right]_r = \left[ \begin{array}{c} n \\ k-1 \end{array} \right]_r + (n + k + 2r) \left[ \frac{n}{k} \right]_r, \quad 1 \leq k \leq n. \quad (31)$$
4 Geometric $r$-Lah polynomials

In this section, we shall present several identities involving hyperharmonic numbers. These identities follow from the connection between hyperharmonic numbers and polynomials that appear in (7) for $f(x) = (1 - x)^{-1}$.

It is seen from (7) that

$$(xD + 2r)^n \left( \frac{1}{1-x} \right) = \frac{1}{1-x} \sum_{k=0}^{n} \binom{n}{k}_r k! \left( \frac{x}{1-x} \right)^k.$$ 

We define

$$L_n, r(x) = \sum_{k=0}^{n} \binom{n}{k}_r k! x^k,$$  \hspace{1cm} (32)

and call these polynomials geometric $r$-Lah polynomials. Using the fact $(xD + 2r)^n x^m = (m + 2r)^n x^m$, we see that

$$(xD + 2r)^n \left( \frac{1}{1-x} \right) = \sum_{m=0}^{\infty} (m + 2r)^n x^m = \frac{1}{1-x} L_n, r \left( \frac{x}{1-x} \right),$$  \hspace{1cm} (33)

It follows from (29) and (32) that $L_{n,1/2}(x) = n!(1 + x)^n$, in which case (33) reduce to the generating function of rising factorial.

The geometric $r$-Lah polynomials have the following generating function, which leads to investigate some properties of these polynomials.

**Theorem 2** We have

$$\sum_{n=0}^{\infty} L_n, r (x-1) \frac{t^n}{n!} = \frac{1}{(1-t)^{2r-1}} \frac{1}{1-xt}.$$  \hspace{1cm} (34)

**Proof.** From (32) and (2), we have

$$\sum_{n=0}^{\infty} L_n, r (x-1) \frac{t^n}{n!} \equiv \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k}_r k! x^k
= \sum_{k=0}^{\infty} x^k \left( \sum_{n=k}^{\infty} \binom{n}{k}_r t^n k! \right)
= \left( \frac{1}{1-t} \right)^{2r-1} \sum_{k=0}^{\infty} \left( \frac{xt}{1-t} \right)^k
= \left( \frac{1}{1-t} \right)^{2r-1} \frac{1}{1-xt - t}, \quad \left| \frac{xt}{1-t} \right| < 1.$$

This completes the proof. \hspace{1cm} $\blacksquare$

In particular, for $x = 0$ in (34), we have

$$L_n, r (-1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}_r k! = (2r-1)^n, \quad n \geq 0, r \geq 1.$$  \hspace{1cm} (35)
Appealing [38, Theorem 3.12] gives
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2r - 1)^k = n!. \]

As an application of (34), we present the binomial hyperharmonic identity with a different formulation from (6).

**Theorem 3** For all non-negative integers \( n \) and \( r \),
\[ h_{n+1}^{(r)} = \sum_{k=0}^{n} \binom{n+r}{k+r} (-1)^k \frac{1}{k+1}. \]

**Proof.** Let \( 2r - 1 \geq 0 \) be an integer. Integrating both sides of (34) with respect to \( x \) from 0 to 1, we have
\[ \sum_{n=0}^{\infty} t^n \frac{1}{n!} \left[ \int_{0}^{1} \mathcal{H}_{n,r} (x-1) \, dx \right] = \frac{1 - \ln (1-t)}{t (1-t)^{2r-1}}. \]

Using the generating function of hyperharmonic numbers (3), we have
\[ \sum_{n=0}^{\infty} t^n \frac{1}{n!} \left[ \int_{0}^{1} \mathcal{H}_{n,r} (x-1) \, dx \right] = \sum_{n=0}^{\infty} t^n h_{n+1}^{(2r-1)}. \]

Comparing the coefficients of \( t^n \) gives
\[ \int_{0}^{1} \mathcal{H}_{n,r} (x-1) \, dx = nH_{n+1}^{(2r-1)}. \] (36)

Hence, (32) and (29) complete the proof. \( \blacksquare \)

We now write (34) in the form
\[ \sum_{n=0}^{\infty} \mathcal{H}_{n,r} (x-1) \frac{(1-e^{-t})^n}{n!} e^{-tm} = e^{-t(m-2r+1)} \frac{1}{1 + x (e^{-t} - 1)} \]
by setting \( t \to 1 - e^{-t} \) and then multiplying both sides by \( e^{-tm} \). We recall the \( r \)-geometric polynomials defined by the generating function [20]
\[ \sum_{n=0}^{\infty} w_{n,r} (x) \frac{t^n}{n!} = \frac{1}{1 - x (e^{t} - 1)} e^{rt}. \] (37)

Utilizing (37) and the generating function of \( r \)-Stirling numbers of the second kind
\[ \sum_{n=k}^{\infty} \left\{ \binom{n}{k} \frac{t^n}{n!} \right\} = \frac{(e^{t} - 1)^k}{k!} e^{rt}, \] (38)
we see that
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k \binom{n}{k}_{m} \mathcal{L}_{k,r} (x - 1) \left( -\frac{t}{n!} \right)^n = \sum_{n=0}^{\infty} w_{n,m-2r+1} (-x) \left( -\frac{t}{n!} \right)^n,
\]
which relate the \( r \)-geometric polynomials and geometric \( r \)-Lah polynomials as in the following:

**Theorem 4** For all integers \( n \geq 1 \) and \( m+1 \geq 2r \geq 1 \), we have
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \mathcal{L}_{k,r} (x - 1) = w_{n,m+1-2r} (-x). \tag{39}
\]
It should be noted that for \( r = 1/2 \) and \( x = 1/2 \), (39) becomes the well known identity
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k!}{2^k} = E_n (m),
\]
upon the use of \( \mathcal{L}_{k,1/2} (-1/2) = k!/2^k \) and \( w_{n,m} (-1/2) = E_n (m) \). Here, \( E_n (x) \) is the \( n \)th Euler polynomial [40, p. 529].

Next theorem shows that the Bernoulli polynomials can be described as a sum of the products of hyperharmonic numbers and \( r \)-Stirling numbers of the second kind.

**Theorem 5** For all non-negative integers \( n, r, m \), we have
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} m! h_{k+1}^{(r)} = B_n (m-r) \tag{40}
\]
and
\[
\sum_{k=0}^{n} \binom{n}{k} m! B_k (r) = n! h_{n+1}^{(r+m-1)}. \tag{41}
\]

**Proof.** Integrating both sides of (39) with respect to \( x \) from 0 to 1, and using (36) we see that
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} m! h_{k+1}^{(2r-1)} = \int_{0}^{1} \int_{0}^{1} w_{n,m+1-2r} (-x) \, dx. \tag{42}
\]
We now integrate (37) with respect to \( x \) from 0 to 1 and use the generating function of Bernoulli polynomials [40, p. 529]
\[
\sum_{n=0}^{\infty} B_n (x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt},
\]

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to deduce that
\[
\int_0^1 w_{n,r} (-x) \, dx = B_n (r). \tag{43}
\]
Thus, (40) follows from (42) and (43).

To obtain (41) we apply \( r \)-Stirling transform to (40) and then use the well-known formula \( B_k (1 - x) = (-1)^k B_k (x) \). \[\blacksquare\]

**Remark 1**

- Note that (43) is a natural extension of Keller’s identity [33]

\[
\int_0^1 w_n (-x) \, dx = B_n,
\]

where \( B_n = B_n (0) \) is the \( n \)th Bernoulli number.

- Since \( h_n^{(0)} = 1/n \), we have the well-known formula for Bernoulli polynomials

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{m+1} k! = B_n (m) \cdot
\]

- (41) specializes some formulas in [14, p. 129].

We want to finalize this section giving a connection between the exponential \( r \)-Lah polynomials and geometric \( r \)-Lah polynomials, namely,

\[
\mathfrak{L}_{n,r} (x) = \int_0^\infty e^{-\lambda} \mathfrak{L}_{n,r} (x\lambda) \, d\lambda. \tag{44}
\]

This connection follows from (14), (32) and the well-known identity

\[
\int_0^\infty z^k e^{-z} \, dz = k!, \quad k \in \mathbb{N}.
\]

Then, with the use of (36), we see that this connection leads some identities for the hyperharmonic numbers:

**Theorem 6** We have

\[
(n + 1) h_{n+1}^{(r)} = (n + r) h_n^{(r)} + \frac{r^n}{n!},
\]

\[
h_{n+1}^{(r+s)} = \sum_{k=0}^{n} \binom{n-k+s}{s} h_{k+1}^{(r-1)},
\]

and

\[
h_{n+1}^{(r+s)} = \sum_{k=0}^{\min(n,s)} \binom{s}{k} h_{n-k+1}^{(r+k)}.
\]
Proof. To prove the first identity, we replace $x$ by $x\lambda$ in (30) and multiply both sides by $e^{-\lambda}$. We then integrate with respect to $\lambda$ from 0 to $\infty$, with the use of (44), and obtain that

$$
\mathcal{L}_{n+1,r}(x) = (2n + 2r) \mathcal{L}_{n,r}(x) - n(n + 2r - 1) \mathcal{L}_{n-1,r}(x) + \int_0^\infty x\lambda \mathcal{L}_{n,r}(x\lambda) e^{-\lambda} d\lambda. \quad (45)
$$

It is clear from (14) that

$$
\int_0^\infty x\lambda \mathcal{L}_{n,r}(x\lambda) e^{-\lambda} d\lambda = \sum_{k=0}^n \binom{n}{k} \frac{n+k+1}{r+1} (k+1)!.
$$

We now integrate both sides of (45) with respect to $x$ from $-1$ to 0 and use (36) to deduce that

$$
(n+1)!h_{n+2}^{(2r-1)} = (2n + 2r) n!h_{n+1}^{(2r-1)} - (n + 2r - 1) n!h_{n+1}^{(2r-1)}
$$

$$
+ \sum_{k=0}^n \binom{n}{k} \frac{(-1)^{k+1}}{k+2} (k+1)!. \quad (46)
$$

Now utilizing (31), (36) and (35), we find that

$$
\sum_{k=0}^n \binom{n}{k} \frac{(-1)^{k+1}}{k+2} (k+1)! = \sum_{k=1}^{n+1} \frac{n+1}{k} \left[ \frac{(-1)^k}{k+1} k! - \sum_{k=1}^n \frac{n}{k} \frac{(-1)^k}{k+1} k! \right] (n + 2r + k) \frac{(-1)^k}{k+1} k!
$$

$$
= (n+1)!h_{n+2}^{(2r-1)} - \sum_{k=1}^{n+1} \frac{n+1}{k} \frac{n}{r} (2r - 1)\pi
$$

$$
+ (n + 2r) \frac{n}{0+2} \frac{n}{r} (2r - 1)\pi
$$

$$
= (n+1)!h_{n+2}^{(2r-1)} - (n + 2r - 1) n!h_{n+1}^{(2r-1)} - (2r - 1)\pi.
$$

Hence, (46) completes the proof of the first identity.

Proofs of the second and the third identities are similar, but for this time we use (20) and (27) instead of (30), respectively.

It is worth noting that the first and second identities are proved in the recent paper [22] with a different method.

5 Harmonic geometric $r$-Lah polynomials

We continue to present identities for hyperharmonic numbers, such as generating function with respect to upper index, generalizations of the symmetric formula (5), formulas for the sum of the products of hyperharmonic numbers.
Setting \( f(x) = -\ln(1-x)/(1-x) \) in (7) and using [19, Eq. (27)]

\[
\frac{d^k}{dx^k} \left( -\ln(1-x) \right) = k! \frac{H_k - \ln(1-x)}{(1-x)^{k+1}}.
\]

we deduce that

\[
(xD + 2r)^n \left( -\ln(1-x) \right) = \frac{1}{1-x} \sum_{k=0}^{n} \binom{n}{k} H_k k! \left( \frac{x}{1-x} \right)^k - \frac{\ln(1-x)}{1-x} \mathcal{L}_{n,r} \left( \frac{x}{1-x} \right).
\]

Let \( \mathcal{L}_{n,r}(x) \) denote the sum in the right-hand side of the above equation, i.e.,

\[
\mathcal{L}_{n,r}(x) = \sum_{k=0}^{n} \binom{n}{k} H_k k! x^k,
\]

which we call harmonic geometric \( r \)-Lah polynomials. Then, considering the generating function of Harmonic numbers (3), we arrive at a closed form evaluation formula for power series involving harmonic numbers.

**Theorem 7** For all non-negative integers \( n, r \)

\[
\sum_{m=0}^{\infty} \binom{m+n}{m} H_m x^m = \frac{1}{1-x} H_{n,r} \left( \frac{x}{1-x} \right) - \frac{\ln(1-x)}{1-x} \mathcal{L}_{n,r} \left( \frac{x}{1-x} \right).
\]

For \( r = 1/2 \), the above relation can be written as

\[
\sum_{m=0}^{\infty} \binom{m+n}{n} H_m x^m = \frac{1}{1-x} \sum_{k=0}^{n} \binom{n}{k} H_k x^k - \frac{\ln(1-x)}{1-x} \mathcal{L}_{n,r} \left( \frac{x}{1-x} \right).
\]

We use (3) and (4) to see that

\[
\sum_{m=0}^{\infty} \binom{m+n}{n} H_m x^m = \frac{1}{1-x} \sum_{k=0}^{n} \binom{n}{k} H_k x^k + \sum_{m=0}^{\infty} \binom{m+n}{n} H_{m+n} x^m - \sum_{m=0}^{\infty} \binom{m+n}{n} H_n x^m.
\]

Considering this formula as

\[
\sum_{m=0}^{\infty} \binom{m+n}{m} (H_{n+m} - H_m) x^m = H_n \sum_{m=0}^{\infty} \binom{m+n}{n} x^m - \frac{1}{1-x} \sum_{k=0}^{n} \binom{n}{k} H_k x^k,
\]

then using (4) and

\[
(1-t)^{-\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{t^n}{n!},
\]

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we have  
\[
\sum_{m=0}^{\infty} h_n^{(m+1)} x^m = \frac{H_n}{(1 - x)^{n+1}} - \frac{1}{1-x} \sum_{k=0}^{n} \binom{n}{k} H_k \left( \frac{x}{1-x} \right)^k.
\]  
(49)

Using [8, Corollary 8]  
\[
\sum_{k=0}^{n} \binom{n}{k} H_k \lambda^k = (1 + \lambda)^n H_n - \sum_{j=1}^{n} \frac{1}{j} (1 + \lambda)^{n-j}
\]
in the last part of (49) for \( \lambda = x/(1 - x) \), we obtain a generating function for hyperharmonic numbers with respect to upper index:

**Theorem 8** We have  
\[
\sum_{m=0}^{\infty} h_n^{(m+1)} x^m = \sum_{j=1}^{n-1} \frac{1}{n-j} \left( \frac{1}{1-x} \right)^{j+1}.
\]

It is good to note that using (48) gives  
\[
\sum_{m=0}^{\infty} h_n^{(m+1)} x^m = \sum_{j=0}^{n-1} \frac{1}{n-j} \sum_{m=0}^{\infty} \frac{(j+1)^\mu}{m!} x^m.
\]
(50)

Comparing the coefficients of \( x^m \) in the above equation, we deduce  
\[
h_n^{(m+1)} = \sum_{j=1}^{n} \binom{n + m - j}{m} \frac{1}{j},
\]
which was proved in [4,17] by different methods.

Moreover, integrate both sides of (50) with respect to \( x \) from 0 to \( x \) and multiply it by \( 1/x \). Repeat this procedure for \( q \) times to obtain  
\[
\sum_{m=0}^{\infty} h_n^{(m+1)} \frac{x^m}{(m+1)^q} = \sum_{j=0}^{n-1} \frac{1}{n-j} \sum_{m=0}^{\infty} \frac{(j+1)^\mu}{m!} \frac{x^m}{(m+1)^q}.
\]
Then we have obtained the following closed-form evaluation formula for a Euler-type sum:

**Theorem 9**  
\[
\sum_{m=0}^{\infty} h_n^{(m+1)} \frac{x^m}{(m+1)^q} = \sum_{j=0}^{n-1} \frac{1}{n-j} \Phi_{j+1}^* (x, q, 1),
\]

where  
\[
\Phi_{\mu}^* (z, s, a) = \sum_{m=0}^{\infty} \frac{(\mu)^m}{m!} \frac{z^m}{(m+a)^s}
\]
is a generalization of the Hurwitz–Lerch Zeta function [28].
One of the generalizations of the symmetric formula \((5)\) is as follows:

**Theorem 10** For all integers \(n \geq 0\) and \(r \geq 1\),

\[
h_n^{(r)} = \sum_{k=0}^{n} (-1)^{k+1} \binom{n + r}{k + r} H_k. \tag{51}
\]

**Proof.** From \((47)\), \((2)\) and \((3)\) it is seen that

\[
\sum_{n=0}^{\infty} H_n^{(-1)} x^n \frac{t^n}{n!} = \left( \frac{1}{1-t} \right)^{2r-1} \ln \left( \frac{1}{1-xt} \right). \tag{52}
\]

Comparing \((3)\) and \((52)\) with \(x = 0\), we reach that

\[
H_n^{(-1)} = -n! h_n^{(2r-1)}. \]

Utilizing \((47)\) and \((29)\) give \((51)\). \(\blacksquare\)

In the following theorem, we give a generalization both of \((51)\) and the symmetric formula \((5)\).

**Theorem 11**

\[
\sum_{k=0}^{n} (-1)^{k+m+1} \binom{n + m + r}{k + m + r} \binom{k + m}{m} H_{k+m} = h_n^{(r)} - \binom{n + r - 1}{n} H_m. \tag{53}
\]

In particular,

\[
\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{k+m+1}}{k + m + 1} H_{k+m} = \frac{1}{n + m + 1} \binom{n + m}{m}^{-1} (H_n - H_m)
\]

and

\[
\sum_{k=0}^{n} (-1)^{k+r+1} \binom{n}{k} \frac{n}{r+1} (k + r)! H_{k+r} = (n + r)! (H_{n+r} - 2H_r).
\]
Proof. By induction on \( m \), it can be shown that
\[
\sum_{n=m}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} H^n_{L_n,r} (x-1) = (-1)^m \ln \left( \frac{1-t}{1-x t} \right) + H_m t^m.
\]

Thus, we have
\[
\left. \sum_{n=0}^{\infty} \frac{t^n}{(n+m)!} \frac{d^m}{dx^m} H^m_{L_{n+m},r} (x-1) \right|_{x=0} = (-1)^m m! \ln \left( \frac{1-t}{1-x t} \right) + H_m t^m.
\]

Using (3) and (48) in the above equation and then comparing the coefficients, we obtain
\[
\left. \frac{d^m}{dx^m} H^m_{L_{n+m},r} (x-1) \right|_{x=0} = (-1)^{m+1} m! \left( h_n^{(2r-1)} - H_m \left( \frac{2r-1}{n!} \right)^n \right).
\]

From (47) and (29), we have (53).

To investigate the relation between the harmonic geometric \( r \)-Lah polynomials and some other well-known numbers or polynomials, we recall the harmonic \( r \)-geometric polynomials, defined by
\[
H^w_{n,r} (x) = \sum_{k=0}^{n} \binom{n}{k} r^k \frac{H_k x^k}{k!}.
\]

The harmonic \( r \)-geometric polynomials have the following generating function and a relation with Bernoulli polynomials.

**Lemma 1** The harmonic \( r \)-geometric polynomials have the following identities
\[
\sum_{n=0}^{\infty} H^w_{n,r} (x) \frac{t^n}{n!} = -\ln \left( \frac{1-x (e^t-1)}{1-x (e^t-1)} \right) e^{rt},
\]
\[
\int_{0}^{1} H^w_{n,r} (-x) \, dx = -\frac{n}{2} B_{n-1} (r), \quad n \geq 1, \quad r \geq 0.
\]

**Proof.** From (54), we have
\[
\sum_{n=0}^{\infty} H^w_{n,r} (x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} r^k \frac{H_k x^k}{n!} = \sum_{k=0}^{\infty} k! H_k x^k \sum_{n=k}^{\infty} \frac{n!}{k!} \binom{n}{k} \frac{t^n}{n!} = e^{rt} \sum_{k=0}^{\infty} H_k \left( \frac{e^t-1}{1-x} \right)^k = \frac{1}{1-x (e^t-1)} e^{rt}.
\]

From (3) and (53), we have (55).
The proof of (56) is similar to that of (43) and is omitted. Remark that the relation (56) is a generalization of the second identity given in [33, Theorem 1.3].

The following theorem present a relationship between the harmonic geometric r-Lah, r-geometric and harmonic r-geometric polynomials. The proof is similar to that of Theorem 4, so we omit it.

**Theorem 12** For all integers \( n \geq 1 \) and \( m + 1 \geq 2r \geq 1 \)

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k}_m H_{k,r}(x-1) = n w_{n-1,m-2r+1}(-x) + H_{w,n,m-2r+1}(-x). \quad (57)
\]

Now, we want to deal with some applications of this theorem. Since \( w_{n,r}(0) = r^n \), \( H_{w,n,r}(0) = 0 \) and \( H_{r,n,r}(1) = -n!h_n^{(2r-1)} \), taking \( x = 0 \) in (57) gives (58) in the following corollary. Moreover, applying r-Stirling transform to (58) yields (59) which is also given in [14] by different means.

**Corollary 3** We have

\[
\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k}_m k! h_k^{(r)} = n (m - r)^{n-1}, \quad (58)
\]

\[
\sum_{k=1}^{n} k \binom{n}{k}_m r^{k-1} = n! h_n^{(r+m)}. \quad (59)
\]

**Theorem 13** For all integers \( n, r, s \geq 0 \)

\[
\sum_{k=1}^{n} h_k^{(r)} h_k^{(s)} = 2 \sum_{k=1}^{n} \binom{n + r + s}{k + r + s} \frac{(-1)^{k+1}}{k+1} H_k \quad (60)
\]

and

\[
\sum_{k=0}^{n} h_k^{(r-1)} h_k^{(r)} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k}_m k B_{k-1} (m - 2r + 1) \quad (61)
\]

**Proof.** Integrating both sides of (52) with respect to \( x \) from 0 to 1, we have

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^1 H_{n,r+s}(x-1) \, dx = - \frac{1}{2t} \frac{\ln^2(1-t)}{(1-t)^{2r+2s-1}}.
\]

Using (3) in the above equation yields

\[
\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} \int_0^1 H_{n,r+s}(x-1) \, dx = - \frac{1}{2} \sum_{n=0}^{\infty} t^n h_n^{(2s)} \sum_{k=0}^{\infty} t^k h_k^{(2r-1)}
\]

\[
= - \frac{1}{2} \sum_{n=0}^{\infty} t^n \left[ \sum_{k=0}^{n} h_k^{(2r-1)} h_k^{(2s)} \right].
\]
Comparing the coefficients of \( t^n \) gives
\[
\int_0^1 H_{n-1,r+s} (x - 1) \, dx = \frac{(n-1)!}{2} \sum_{k=1}^{n-1} h_k^{(2r-1)} h_{n-k}^{(2s)}.
\] (62)

On the other hand, it is seen from (47) that
\[
\int_0^1 H_{n-1,r+s} (x - 1) \, dx = -\sum_{k=0}^{n-1} \left( \begin{array}{c} n-1 \\ k \end{array} \right) \frac{(-1)^k}{k+1} H_k k!.
\]

Hence, (60) follows from (29).

To prove (61) we first integrate both sides of (57) with respect to \( x \) from 0 to 1 and use (43), (56) and (62). Then we have
\[
\sum_{k=1}^n (-1)^{k+1} \left( \frac{n}{k} \right) k! m \sum_{l=1}^{k} (-1)^{k-1} h_l^{(r)} h_{k+1-l} = n B_{n-1} (m - 2r + 1).
\]

Hence, \( r \)-Stirling transform implies (61). \( \blacksquare \)

Particular cases \( r = 1 \) in (61) and \( r = 1, s = 0 \) in (60) give
\[
\sum_{k=1}^n H_k = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} \left( \begin{array}{c} n \\ k \end{array} \right) k! B_{k-1} (m - 1) = 2 \sum_{k=1}^{n} \left( \frac{n+1}{k+1} \right) (-1)^{k+1} H_k,
\]

respectively. However, the last formula can be evaluated as follows:

**Proposition 4** For all integers \( n \geq 1 \),
\[
\sum_{k=1}^n \left( \frac{n+1}{k+1} \right) \frac{(-1)^{k+1}}{k+1} H_k = \frac{1}{2} \left\{ (H_{n+1})^2 - H_{n+1}^{(2)} \right\}.
\]

**Proof.** We have
\[
\sum_{k=1}^n \left( \frac{n+1}{k+1} \right) \frac{(-1)^{k+1}}{k+1} H_k
= \sum_{k=1}^{n-1} \left\{ \left( \frac{n}{k+1} \right) + \left( \frac{n}{k} \right) \right\} \frac{(-1)^{k+1}}{k+1} H_k + \frac{(-1)^{n+1}}{n+1} H_n
= n \sum_{k=1}^{n-1} \left( \frac{n-1}{k} \right) \frac{(-1)^{k+1}}{(k+1)^2} H_k + n \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{(-1)^{k+1}}{k+1} H_k.
\]

If we set
\[
A (n) = \sum_{k=1}^n \left( \frac{n+1}{k+1} \right) \frac{(-1)^{k+1}}{k+1} H_k = (n+1) \sum_{k=1}^{n} \left( \frac{n}{k} \right) \frac{(-1)^{k+1}}{(k+1)^2} H_k
\]

...
and use (5), then (64) becomes

\[ A(n) = \frac{H_n}{n+1} + A(n-1) = \sum_{k=1}^{n} \frac{H_k}{k+1} \]

\[ = \sum_{k=1}^{n} \frac{1}{k+1} \left( H_{k+1} - \frac{1}{k+1} \right) \]

\[ = \frac{1}{2} \left\{ (H_{n+1})^2 + H_{n+1}^{(2)} \right\} - H_{n+1}^{(2)}, \]

where we have used that

\[ \sum_{k=1}^{n} \frac{1}{k} H_k = \sum_{j=1}^{n} \frac{1}{j} \sum_{k=j}^{n} \frac{1}{k} = \sum_{j=1}^{n} \frac{1}{j} \left( H_n - H_j + \frac{1}{j} \right) \]

\[ = (H_n)^2 + H_n^{(2)} - \sum_{j=1}^{n} \frac{1}{j} H_j. \]

This completes the proof. □

Combining (63) and (65), we conclude that

\[ 2 \sum_{k=1}^{n} \frac{H_k}{k+1} = \sum_{k=1}^{n} \frac{H_k}{n+1-k} = (H_{n+1})^2 - H_{n+1}^{(2)} \]

and

\[ \frac{1}{n!} \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} k B_{k-1} (m-1) = (H_{n+1})^2 - H_{n+1}^{(2)}, \]

and then, from the r-Stirling transform

\[ \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} m! \left( (H_{k+1})^2 - H_{k+1}^{(2)} \right) = n B_{n-1} (m-1). \]

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