Quantization of scalar fields in curved background, deformed Hopf algebra and entanglement

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Abstract. A suitable deformation of the Hopf algebra of the creation and annihilation operators for a complex scalar field, initially quantized in Minkowski space–time, induces the canonical quantization of the same field in a generic gravitational background. The deformation parameter $q$ turns out to be related to the gravitational field. The entanglement of the quantum vacuum appears to be robust against interaction with the environment.

1. Introduction

We shortly report on two main results of some recent works [1, 2] on the quantization of a scalar field in curved background: i) a suitable deformation of the Hopf algebra for a complex scalar operator field, initially quantized in Minkowski space–time, induces the canonical quantization of the same field in a generic gravitational background. The deformation parameter $q$ thus turns out to be related to the gravitational field. ii) The entanglement of the quantum vacuum appears to be robust against interaction with the environment.

Thermal properties of quantum field theory (QFT) in curved space–time can be derived in this deformed algebra setting. On the other hand, it is well known the intimate relationship between space–times with an event horizon and thermal properties [3, 4]. In particular, it has been shown [4] that global thermal equilibrium over the whole space–time implies the presence of horizons in this space–time. We find that the doubling of the degrees of freedom implied by the coproduct map of the deformed Hopf algebra turns out to be most appropriate for the description of the modes on both sides of the horizon. The entanglement between inner and outer particles with respect to the event horizon appears to be rooted in the background curvature and it is therefore robust against interaction with the environment.

2. Quantization and deformed Hopf algebra

We consider a complex scalar operator field $\phi(x)$, initially quantized in Minkowski space–time. To study the quantization procedure in curved space–time, we treat the gravitational field as a classical background. We start with few notions on the deformation of the Hopf algebra [5, 6]. We shall focus on the case of bosons for simplicity.

The coproduct is a homomorphism which duplicates the algebra, $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$. The operational meaning of the coproduct is that it provides the prescription for operating on two modes. Associated to that, there is the doubling of the degrees of freedom of the system. Our finding is that in the presence of a single event horizon such a doubling perfectly describes the modes on the two sides of the horizon [1, 2] (see also [7]).
The bosonic Hopf algebra for a single mode (the case of modes labelled by the momentum is straightforward), also called $h(1)$, is generated by the set of operators \{a, a^\dagger, H, N\} with commutation relations:
\begin{align}
[a, a^\dagger] &= 2H,
[N, a] &= -a,
[N, a^\dagger] &= a^\dagger,
[H, \bullet] &= 0,
\end{align}
where $H$ is a central operator, constant in each representation. The Casimir operator is given by $C = 2NH - a^\dagger a$. In $h(1)$ the coproduct is defined by $\Delta \mathcal{O} = \mathcal{O} \otimes 1 + 1 \otimes \mathcal{O} \equiv \mathcal{O}^{(+)} + \mathcal{O}^{(-)}$, where $\mathcal{O}$ stands for $a, a^\dagger, H$ and $N$. The $q$-deformation of $h(1)$ is the Hopf algebra $h_q(1)$:
\begin{align}
[a_q, a_q^\dagger] &= [2H]_q,
[N, a_q] &= -a_q,
[N, a_q^\dagger] &= a_q^\dagger,
[H, \bullet] &= 0,
\end{align}
where $N_q \equiv N$, $H_q \equiv H$ and $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$. The Casimir operator is given by $C_q = N[2H]_q - a_q^\dagger a_q$. The coproduct stays the same for $H$ and $N$, while for $a_q$ and $a_q^\dagger$ now it changes. In the fundamental representation, obtained by setting $H = 1/2, C = 0$, it is written as
\begin{align}
\Delta a_q &= a_q \otimes q^{1/2} + q^{-1/2} \otimes a_q = a^{(+)} q^{1/2} + q^{-1/2} a^{(-)},
\Delta a_q^\dagger &= a_q^\dagger \otimes q^{1/2} + q^{-1/2} \otimes a_q^\dagger = a^{(+)}^\dagger q^{1/2} + q^{-1/2} a^{(-)}^\dagger,
\end{align}
where self-adjointness requires that $q$ can only be real or of modulus one. In this representation $h(1)$ and $h_q(1)$ coincide. The differences appear in the coproduct. Note that $[a^{(\sigma)}, a^{(\sigma')^\dagger}] = 0$, $\sigma \neq \sigma'$ with $\sigma \equiv \pm$. Now the key point is that, by setting $q = q(\epsilon) \equiv e^{2\epsilon(p)}$, suitable linear combinations of the deformed coproduct operation (5) (where the momentum label is introduced) give (6):
\begin{align}
d_p^{(\sigma)}(\epsilon) &= d_p^{(\sigma)} \cosh \epsilon(p) + \bar{d}_p^{(-\sigma)^\dagger} \sinh \epsilon(p),
\bar{d}_p^{(-\sigma)^\dagger}(\epsilon) &= d_p^{(\sigma)} \sinh \epsilon(p) + \bar{d}_p^{(-\sigma)^\dagger} \cosh \epsilon(p),
\end{align}
where $d_p^{(\sigma)} \equiv \sum_k F(k,p) a_k^{(\sigma)}$, $\bar{d}_p^{(-\sigma)^\dagger} \equiv \sum_k F(k,p) \bar{a}_k^{(\sigma)}$ and $\{F(k,p)\}$ is a complete orthonormal set of functions, $p \in \mathbb{Z}^{n-1}$, as for $k = (k_1, k)$, and $p = (\Omega, \mathbf{p})$, $\bar{p} = (\Omega, -\mathbf{p})$. We use $q(p) = q(\bar{p})$. In general $k \neq p$. $a_k^{(\sigma)}$ and $\bar{a}_k^{(\sigma)}$ are the two (annihilation) operator modes of the complex scalar field $\phi(x)$ (for each of the sides $\pm$ of the horizon). Eqs. (6) are recognized to be the Bogolubov transformations obtained in the quantization procedure in the gravitational background in the semiclassical approximation (8). We thus see that use of the deformed coproducts is equivalent to such a quantization procedure.

The generators of (8) is $g(\epsilon) = \sum_p \sum_\sigma \epsilon(p) [d_p^{(\sigma)} \bar{d}_p^{(-\sigma)^\dagger} - \bar{d}_p^{(-\sigma)^\dagger} d_p^{(\sigma)}]$ and $G(\epsilon) \equiv \exp g(\epsilon)$ is a unitary operator at finite volume. The Hilbert–Fock space $\mathcal{H}$ associated to the Minkowski space is built by repeated action of $(d_p^{(\sigma)^\dagger}, \bar{d}_p^{(-\sigma)^\dagger})$ on the vacuum state $|0_M\rangle$. The generator $G(\epsilon)$ maps vectors of $\mathcal{H}$ to vectors of another Hilbert space $\mathcal{H}_c$: $\mathcal{H} \rightarrow \mathcal{H}_c$. In particular,
\begin{align}
|0(\epsilon)\rangle = G(\epsilon) |0_M\rangle,
\end{align}
where $|0(\epsilon)\rangle$ is the vacuum state of the Hilbert space $\mathcal{H}_c$ annihilated by the new operators $(d_p^{(\sigma)^\dagger}(\epsilon), \bar{d}_p^{(-\sigma)^\dagger}(\epsilon))$. We use the short-hand notation for the Hilbert spaces ($\mathcal{H}$ stands for $\mathcal{H} \otimes \mathcal{H}$), as well as for the states (for instance $|0_M\rangle$ stands for $|0_M\rangle \otimes |0_M\rangle$). The group underlying this construction is $SU(1,1)$. By inverting Eq. (5), $|0_M\rangle$ can be expressed as a $SU(1,1)$ generalized coherent state (8) of Cooper-like pairs
\begin{align}
|0_M\rangle = \frac{1}{Z} \exp \left[ \sum_\sigma \sum_p \tanh \epsilon(p)d_p^{(\sigma)^\dagger}(\epsilon)\bar{d}_p^{(-\sigma)^\dagger}(\epsilon) \right] |0(\epsilon)\rangle,
\end{align}
where $|0(\epsilon)\rangle$ is the vacuum state of the Hilbert space $\mathcal{H}_c$.
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where \( Z = \prod_p \cosh^2 \epsilon(p) \). Moreover, \( \langle 0(\epsilon) | 0(\epsilon) \rangle = 1, \forall \epsilon, \) and \( \langle 0(\epsilon) | 0_M \rangle \rightarrow 0 \) and \( \langle 0(\epsilon) | 0(\epsilon) \rangle \rightarrow 0 \) as \( V \rightarrow \infty, \) \( \forall \epsilon, \epsilon', \epsilon \neq \epsilon' \), i.e. \( \mathcal{H} \) and \( \mathcal{H}_\epsilon \) become unitarily inequivalent in the infinite-volume limit. In this limit \( \epsilon \) labels the set \( \{ \mathcal{H}_\epsilon, \forall \epsilon \} \) of the infinitely many unitarily inequivalent representations of the canonical commutation relations [6, 10, 11].

The physical meaning of having two distinct momenta \( k \) and \( p \) for states in the Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}_\epsilon \), respectively, is the occurrence of two different reference frames: the \( M \)-frame (Minkowski) and the \( M_\epsilon \)-frame. To explore the physics in the \( M_\epsilon \)-frame, one has to construct a diagonal operator \( H_{\epsilon} \), which plays the role of the Hamiltonian in the \( M_\epsilon \)-frame. In order to do that one has to use the generator of the boosts. Thus one finds [1]

\[
H_{\epsilon} = G(\epsilon) \mathcal{M}_{10} G^{-1}(\epsilon) = \sum_{\sigma} \sum_p \sigma \Omega [d_p(\sigma) \dagger (\epsilon) d_p(\sigma) (\epsilon) + \bar{d}_p(\sigma) (\epsilon) \bar{d}_p(\sigma) \dagger (\epsilon)] \\
= H^{(+)}(\epsilon) - H^{(-)}(\epsilon).
\]

Here \( \mathcal{M}_{10} \) denotes the deformed generator of the boosts. Eq. (7) gives the wanted Hamiltonian in the \( M_\epsilon \)-frame, as also suggested by the customary results of QFT in curved space-time [8].

3. Entropy and entanglement

The condensate structure of the vacuum (6) suggests to consider the thermal properties of the system. The entropy operator is \( S^{(\sigma)}(\epsilon) = S^{(\sigma)}(\epsilon) + S^{(\sigma)}(\epsilon) \) with \( S^{(\sigma)}(\epsilon) \) given by (\( \sigma \equiv \pm \))

\[
S^{(\sigma)}(\epsilon) = - \sum_p [d_p(\sigma) \dagger (\epsilon) d_p(\sigma) (\epsilon) \ln \sinh^2 \epsilon(p) - d_p(\sigma) (\epsilon) d_p(\sigma) \dagger (\epsilon) \ln \cosh^2 \epsilon(p)].
\]

\( S^{(\sigma)}(\epsilon) \) has a similar form (with \( d_p \rightarrow \bar{d}_p \)). The total entropy operator is \( S_\epsilon = S^{(+)}(\epsilon) - S^{(-)}(\epsilon) \) and it is invariant under the Bogoliubov transformations. Similarly one may introduce the free energy as [13, 12]

\[
\mathcal{F}^{(+)}(\epsilon) \equiv \langle 0_M | H^{(+)}(\epsilon) - \frac{1}{\beta} S^{(+)}(\epsilon) | 0_M \rangle.
\]

with \( \beta \equiv T^{-1} \). Stationarity of \( \mathcal{F}^{(+)}(\epsilon) \) gives

\[
\mathcal{N}^{(+)}_{d(\epsilon)} = \sinh^2 \epsilon(p) = \frac{1}{e^{\beta \Omega} - 1}, \tag{10}
\]

and similarly for \( \mathcal{N}^{(+)}_{\bar{d}(\epsilon)} \). Eq. (10) shows that for vanishing \( T \) the deformation parameter \( \epsilon \) vanishes too. In that limit thermal properties as well as the event horizon are lost, and \( M_\epsilon \)-frame \( \rightarrow M \)-frame. Moreover, \( i) \) \( \beta \) is related to the event horizons, and being \( \beta \) constant in the \( M \), space–time is static and stationary; \( ii) \) the gravitational field itself vanishes as \( \epsilon \rightarrow 0 \). The vanishing of the gravitational field occurs either if the \( M \)-frame is far from the gravitational source where space-time is flat, or if there exists a reference frame locally flat, i.e. the \( M \)-frame is a free–falling reference frame. This clearly is a realization of the equivalence principle, which manifests itself when "\( \epsilon \)-effects" are shielded.

We now consider the entanglement. The expansion of \( |0_M \rangle \) in (6) contains terms such as

\[
\sum_p \tanh \epsilon(p) \left( |1_p^{(+)} , \bar{0} \rangle \otimes |0, 1_p^{(-)} \rangle + |0, 1_p^{(+) \dagger} \rangle \otimes |1_p^{(-)}, \bar{0} \rangle \right) + \ldots, \tag{11}
\]

where, we denote by \( |n_p^{(\sigma)}, \bar{m}_p^{(\sigma)} \rangle \) a state of \( n \) particles and \( m \) "antiparticles" in whichever sector \( (\sigma) \). For the generic \( n^{th} \) term, it is \( |n_p^{(\sigma)}, \bar{0} \rangle \equiv |1_p^{(\sigma)}, \ldots, 1_p^{(\sigma)}, \bar{0} \rangle \), and similarly for antiparticles. By introducing a well known notation, \( \uparrow \) for a particle, and \( \downarrow \) for an antiparticle, the two-particle state in (11) can be written as

\[
| \uparrow^{(+)} \rangle \otimes | \downarrow^{(-)} \rangle + | \downarrow^{(+)} \rangle \otimes | \uparrow^{(-)} \rangle,
\]

(12)
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which is an entangled state of particle and antiparticle living in the two sectors \((\pm)\). The generic \(n\)th term in (11) shares exactly the same property as the two-particle state, but this time the \(\uparrow\) describes a set of \(n\) particles, and \(\downarrow\) a set of \(n\) antiparticles. The mechanism of the entanglement, induced by the q-deformation, takes place at all orders in the expansion, always by grouping particles and antiparticles into two sets. Thus the whole vacuum \(|0_M\rangle\) is an infinite superposition of entangled states (a similar structure also arises in the temperature-dependent vacuum of Thermo-Field Dynamics [13] (see also [14])):

\[
|0_M\rangle = \sum_{n=0}^{\infty} \sqrt{W_n} |\text{Entangled}\rangle_n, \quad W_n = \prod_p \frac{\sinh^{2n_p} \epsilon(p)}{\cosh^{2(n_p+1)} \epsilon(p)},
\]

with \(0 < W_n < 1\) and \(\sum_{n=0}^{\infty} W_n = 1\). The probability of having entanglement of two sets of \(n\) particles and \(n\) antiparticles is \(W_n\). At finite volume, being \(W_n\) a decreasing monotonic function of \(n\), the entanglement is suppressed for large \(n\). It appears then that only a finite number of entangled terms in the expansion (13) is relevant. Nonetheless this is only true at finite volume (the quantum mechanics limit), while the interesting case occurs in the infinite volume limit, which one has to perform in a QFT setting.

The entanglement is generated by \(G(\epsilon)\), where the field modes in one sector \((\sigma)\) are coupled to the modes in the other sector \((-\sigma)\) via the deformation parameter \(q(\epsilon)\). Since the deformation parameter describes the background gravitational field (environment), it appears that the origin of the entanglement is the environment, in contrast with the usual quantum mechanics view, which attributes to the environment the loss of the entanglement. In the present treatment such an origin for the entanglement makes it quite robust. One further reason for the robustness is that this entanglement is realized in the limit to the infinite volume once and for all since then there is no unitary evolution to disentangle the vacuum: at infinite volume one cannot ”unknot the knots”. Such a non-unitarity is only realized when all the terms in the series (13) are summed up, which indeed happens in the \(V \to \infty\) limit [2].

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