Conversion of gravitational and electromagnetic waves without any external background field

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Abstract
Applying a simple harmonic map method to the cylindrically symmetric Einstein-Maxwell system, we obtain exact solutions representing strong nonlinear interaction between gravitational waves and electromagnetic waves in the case without any background field. As an interesting fact, we can show that with adjusted parameters the solution represents occurrences of large conversion phenomena in the intense region of fields near the cylindrically symmetric axis.

Keywords Cylindrical symmetry · Exact solution · Gravitational wave

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1 Introduction
The first direct confirmation of the existence of gravitational waves [1] has inspired a lot of fundamental study of gravitational waves from various perspective. As the
observation accuracy improves in the future, it will become more and more important to know the fundamental aspects of gravitational waves themselves. In particular, clarification of the nonlinear behavior of gravitational waves in the regions with intense fields will continue to be one of the important clues to explore new gravitational physics. For this purpose, we have robust numerical techniques which have already being used vigorously. On the one hand, we have a long history of studying non-linear aspects of gravity using exact solutions and have an accumulated heritage [2–5]. The latter way will be still useful to add new insights matching this new trend if the appropriate ingenuity is introduced.

From a more realistic point of view, the study of gravitational waves combined with other fields will be needed more in the future. Because of the universality of the gravitational interaction with matters, it may be natural to expect that as nonlinear phenomena, various conversions between gravitational waves and matter fields occur in regions with intense fields. Hence, as a simple and important example, we consider the Einstein-Maxwell system, that is, a coupled system with gravitational and electromagnetic waves. In fact, by using linear perturbation methods, so far several studies have been done to treat the occurrence of conversion phenomena based on the existence of external background fields, such as electromagnetic fields around black holes [6–13]. On the other hand, within the scope of linear perturbative method, such conversions cannot be expected in the limit of neutral vacuums, because the mixing tendency of the waves corresponding to the perturbed Einstein-Maxwell system becomes weak and eventually disappears in such cases [7–9, 14–17]. From this, as a guess, the conversions from gravitational fields to electromagnetic fields seem to be not so easy phenomena to occur in comparison with the reverse conversions. We therefore have taken interest in whether remarkable conversions occur by full nonlinearity or not between the gravitational waves and the electromagnetic waves even in the absence of a background field. For this purpose, we here perform the analysis on the base of a suitable exact solution here.

It should be noticed that possibility and importance to analyze the conversion phenomena by using the exact solutions have been already pointed out by Alekseev [18], and also the possibility of using cylindrical symmetric solutions has been suggested by us [19].

The plan of the paper is the following. Using the simple and convenient method summarized in Sect. 2, we first construct simple exact solutions which represent the waves of the Einstein-Maxwell system under cylindrical symmetry in Sect. 3. Then, in Sect. 4 we use the solutions to reveal the novel and interesting aspects caused by strong nonlinearities in the Einstein-Maxwell system; especially, we clarify some interesting features of the conversion phenomena, as mentioned above. Finally, Summary and discussion are given in Sect. 5.

2 Summary of construction method

Under the adoption of the geometric unit system: $c = G = 1$, we assume the following useful line element [20, 21] (see [22] for general treatments of cylindrical symmetric systems) and gauge potential $A$:
\[
\begin{align*}
\text{ds}^2 &= e^{2\psi}(dz - w d\phi)^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2(\gamma - \psi)} (-dt^2 + d\rho^2), \\
A &= A_\phi d\phi + A_z dz,
\end{align*}
\]

where the metric functions \(\psi, \gamma\) and gauge potentials \(A_\phi, A_z\) depend only on the coordinates \(t\) and \(\rho\). The basic equations are derived from the following Einstein-Maxwell equations:

\[
\begin{align*}
R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R &= 8\pi T^\mu_\nu, \\
T^\mu_\nu &= \frac{1}{4\pi} (F^\mu_\rho F^\nu_\delta - \frac{1}{4} \delta^\mu_\nu F^\rho_F^\delta), \\
\nabla_\mu F^\mu_\nu &= 0.
\end{align*}
\]

If we introduce the following two complex potentials:

\[
E := e^{2\psi} + |F|^2 - i \Phi, \quad F := A_z + i \chi,
\]

the Einstein-Maxwell equations (3) and (4) are reduced to the following compact forms, which correspond to the cylindrically symmetric version of Ernst equations [23],

\[
\begin{align*}
(\text{Re}[E] - F \bar{F}) \nabla^2 E &= (\nabla E - 2 \bar{F} \nabla F) \cdot \nabla E, \\
(\text{Re}[E] - F \bar{F}) \nabla^2 F &= (\nabla E - 2 \bar{F} \nabla F) \cdot \nabla F,
\end{align*}
\]

where the dot (\(\cdot\)) means scalar product defined with the three dimensional Minkowski metric:

\[
ds^2 = h_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\rho^2 + \rho^2 d\phi^2.
\]

Here, a twist potential \(\Phi\) and a ‘magnetic’ potential \(\chi\) corresponding to \(w\) and \(A_\phi\) are introduced by \(d\chi := -i \xi \wedge d\xi\) and \(d\Phi := \Phi^*(\xi \wedge d\xi) - 2(A_z d\chi - \chi dA_z)\), where \(\xi\) corresponds to a Killing vector \(\partial/\partial z\) and \(F\) is an electromagnetic field. The explicit expressions in the cylindrical symmetric case are given, as follows

\[
(\partial_t \chi, \partial_\rho \chi) = 1 \rho e^{2\psi} (\partial_t A_\phi + w \partial_\rho A_z), \quad 1 \rho e^{2\psi} (\partial_t A_\phi + w \partial_\rho A_z),
\]

\[
(\partial_t \Phi, \partial_\rho \Phi) = 1 \rho e^{4\psi} \partial_\rho w - 2(A_z \partial_t \chi - \chi \partial_t A_z), \quad 1 \rho e^{4\psi} \partial_t w - 2(A_z \partial_\rho \chi - \chi \partial_\rho A_z).
\]

For further convenience, we adopt another type of the Ernst equation in the following [23, 24]:

\[
\begin{align*}
(\xi \bar{\xi} + \eta \bar{\eta} - 1) \nabla^2 \xi &= 2(\bar{\xi} \nabla \xi + \bar{\eta} \nabla \eta) \cdot \nabla \xi, \\
(\xi \bar{\xi} + \eta \bar{\eta} - 1) \nabla^2 \eta &= 2(\bar{\xi} \nabla \xi + \bar{\eta} \nabla \eta) \cdot \nabla \eta,
\end{align*}
\]
where the complex potentials $\xi$ and $\eta$ are related to the potentials $E$ and $F$, as follows (for example, see [5], because the mathematical structure of the target space is exactly the same as the case of colliding plane waves)

$$
\xi = \frac{E - 1}{E + 1}, \quad \eta = \frac{2F}{E + 1}.
$$

(13)

and $\nabla$ is the gradient defined on the three dimensional Minkowski space $M^{(1,2)}$.

When the solutions of Eqs. (11) and (12) are given, through the above relations (5) and (13), the quantities $e^{2\psi}$, $\Phi$, $\chi$ and $A_z$ can be derived algebraically. Then, the metric function $w$ and gauge field component $A_\phi$ are obtained after some quadrature using Eqs. (9) and (10). Once the above quantities determined, the metric function $\gamma$ can be also given by the following equations, which are derived from the constraints of the Einstein equation:

$$
\begin{align*}
\partial_\rho \gamma &= \rho \left[ (\partial_t \psi)^2 + (\partial_\rho \psi)^2 \right] + \frac{1}{4\rho} e^{4\psi} \left[ (\partial_t w)^2 + (\partial_\rho w)^2 \right] \\
&+ \rho e^{-2\psi} \left[ (\partial_t A_z)^2 + (\partial_\rho A_z)^2 \right] + \frac{1}{\rho} e^{2\psi} \left[ (\partial_t A_\phi + w\partial_t A_z)^2 + (\partial_\rho A_\phi + w\partial_\rho A_z)^2 \right].
\end{align*}
$$

(14)

$$
\begin{align*}
\partial_t \gamma &= 2\rho \partial_t \psi \partial_\rho \psi + \frac{1}{2\rho} e^{4\psi} \partial_t w \partial_\rho w \\
&+ 2\rho e^{-2\psi} \partial_t A_z \partial_\rho A_z + \frac{2}{\rho} e^{2\psi} \left( \partial_t A_\phi + w\partial_t A_z \right) \left( \partial_\rho A_\phi + w\partial_\rho A_z \right).
\end{align*}
$$

(15)

To construct a new solution, it is convenient to regard the basic equations (11) and (12) as a harmonic map equation (more precisely, a wave map equation). So the solutions can be realized as harmonic maps from the base space $M^{(1,2)}$ to the final target space which corresponds to the ball model of complex two dimensional hyperbolic space $H_2^C$ (hereafter $H_2^C$ means the ball model) [25, 26]. The field variables $\xi$ and $\eta$ of Eqs. (11) and (12) correspond to the two complex coordinates which are used to describe $H_2^C$. As a simple method, we adopt here the composite harmonic method [27]. In short, the procedure of the method is the following: first construct a harmonic map transforming the base space to an intermediate target space (here we adopt the Poincare disc model of complex one dimensional hyperbolic space $H_1^C$), next find out an appropriate totally geodesic embedding map of $H_1^C$ into the $H_2^C$, and finally combine these maps.

### 3 A solution and its related useful quantities

First, note that the harmonic map in the first step of the procedure mentioned above is nothing but a solution to the vacuum Ernst equation (i.e. Equation (11) for $\eta = 0$). So, we adopt the following simple and convenient harmonic map $\xi_v$ given in the previous work [19]:

$\xi_v$ Springer
\[
\xi_v = \frac{1 - e^{-2\tau} + iA}{1 + e^{-2\tau} - iA}.
\] 

(16)

Here the constant \( A \) is real. The real parameter \( \tau \) is replaced with a cylindrically symmetric wave function which satisfies the linear wave equation defined on the three dimensional Minkowski space \( M^{(1,2)} \). In a sense, the wave function \( \tau \) can be considered as a seed function. The field variable \( \xi_v \) corresponds to the complex coordinate of the \( H^1_C \). Next, as a simple totally geodesic embeddings of \( H^1_C \), we use the subspace of \( H^2_C \) defined by \( (\xi, \eta) = (\cos 2\theta z, \sin 2\theta) \), where the parameter \( z \) is considered as a complex coordinate of the \( H^1_C \). In fact, the similar ansatz has been already used by Halilsoy for a different situation to treat colliding plane waves [28] (see also [5]). Finally we obtain the desired harmonic map by replacing the parameter \( z \) of the subspace introduced above with the function \( \xi_v \), defined by Eq. (16), as follows

\[
(\xi, \eta) = (\cos 2\theta \xi_v, \sin 2\theta \xi_v).
\]

(17)

From this harmonic map, the expressions of the parts of the Ernst potentials are given after some algebraic calculation, as below:

\[
e^{2\psi} = \frac{1}{A^2 e^{2\tau} \cos^4 \theta + e^{-2\tau} (\cos^2 \theta + e^{2\tau} \sin^2 \theta)^2},
\]

\[
\Phi = -A e^{2\tau} \cos 2\theta e^{2\psi}, \quad \chi = \frac{1}{2} A e^{2\tau} \sin 2\theta e^{2\psi},
\]

\[
A_z = -\frac{1}{2} [A^2 e^{2\tau} \cos^2 \theta + (e^{-2\tau} - 1)(\cos^2 \theta + e^{2\tau} \sin^2 \theta)] \sin 2\theta e^{2\psi}.
\]

(18)

Then, the metric functions \( w, \gamma \) and gauge field component \( A_\phi \) are obtained after some calculation, as follows

\[
w = -4A \cos^4 \theta \int \left\{ 2\rho \partial_{\rho} \tau \partial_t dt + \rho (\partial_t \tau) dt + \rho \left[ (\partial_t \tau)^2 + (\partial_{\rho} \tau)^2 \right] d\rho \right\},
\]

(19)

\[
\gamma = \int \left\{ 2\rho \partial_{\rho} \tau \partial_t dt + \rho \left[ (\partial_t \tau)^2 + (\partial_{\rho} \tau)^2 \right] d\rho \right\},
\]

(20)

\[
A_\phi = -w A_z - \tan \theta w + \text{const}.
\]

(21)

The parameters \( A \) and \( \theta \) correspond to the gravitational \( x \)-mode and the electromagnetic modes, respectively.

To clarify the nonlinear aspects of the fields using these expressions, let us use the metric function \( \gamma \), so-called the C-energy [29], which is extended to cylindrically symmetric Einstein-Maxwell system. The corresponding C-energy density \( \mathcal{E} \) is given by the derivative of the function \( \gamma \) with respect to \( \rho \). The useful formula is deduced from the Einstein equation, as below:

\[
\mathcal{E} := \partial_\rho \gamma = \frac{\rho}{8} \left( A_c^2 + B_\phi^2 + A_\phi^2 + B_\phi^2 + A_x^2 + B_x^2 + A_z^2 + B_z^2 \right).
\]

(22)
Here, according to the previous work by Piran, Safier, and Stark [30], the quantities, called ‘amplitudes’, are introduced with an extension to the Einstein-Maxwell system:

\[
\begin{align*}
A_+ &:= 2 \partial_v \psi, \quad A_\times := \frac{1}{\rho} e^{2\psi} \partial_v w = e^{-2\psi} \left[ \partial_v \Phi + 2(A_z \partial_v \chi - \chi \partial_v A_z) \right], \\
A_z &:= 2e^{-\psi} \partial_v A_z, \quad A_\phi := \frac{2}{\rho} e^\psi \left( \partial_v A_\phi + w \partial_v A_z \right) = 2 e^{-\psi} \partial_v \chi, \\
B_+ &:= 2 \partial_u \psi, \quad B_\times := \frac{1}{\rho} e^{2\psi} \partial_u w = -e^{-2\psi} \left[ \partial_u \Phi + 2(A_z \partial_u \chi - \chi \partial_u A_z) \right], \\
B_z &:= 2e^{-\psi} \partial_u A_z, \quad B_\phi := \frac{2}{\rho} e^\psi \left( \partial_u A_\phi + w \partial_u A_z \right) = -2 e^{-\psi} \partial_u \chi, 
\end{align*}
\]

where the symbols \(A_\cdot\) and \(B_\cdot\) mean ‘ingoing’ and ‘outgoing’, respectively, and the subscripts indicate which mode the amplitude corresponds to, and also the null coordinates \(u = (t - \rho)/2\) and \(v = (t + \rho)/2\) are introduced. Let us divide the C-energy density \(\mathcal{E}\) given by Eq. (22) into the two parts, the first four terms and its rest, which are referred to as \(\mathcal{E}_g\) and \(\mathcal{E}_{em}\) in the following, respectively. \(\mathcal{E}_g\) can be considered a gravitational part, according to the work [30]. On the other hand \(\mathcal{E}_{em}\) can be regarded as an electromagnetic part, since \(\mathcal{E}_{em}\) essentially corresponds to the \(tt\)-component of electromagnetic energy momentum tensor.

For the next step, let us introduce the occupancy ratios \(R_g\) and \(R_{em}\) \((R_g + R_{em} = 1)\), which correspond to the gravitational and electromagnetic parts, respectively, as follows,

\[
\begin{align*}
\mathcal{E}_g &= R_g \mathcal{E}, \quad R_g := \frac{A^2 e^{4\tau} \cos^4 \theta + (\cos^2 \theta - e^{2\tau} \sin^2 \theta)^2}{A^2 e^{4\tau} \cos^4 \theta + (\cos^2 \theta + e^{2\tau} \sin^2 \theta)^2}, \\
\mathcal{E}_{em} &= R_{em} \mathcal{E}, \quad R_{em} := \frac{e^{2\tau} \sin^2(2\theta)}{A^2 e^{4\tau} \cos^4 \theta + (\cos^2 \theta + e^{2\tau} \sin^2 \theta)^2}, \\
\mathcal{E} &= \rho \left[ (\partial_t \tau)^2 + (\partial_\rho \tau)^2 \right],
\end{align*}
\]

where the explicit forms of \(R_g, R_{em}\), and \(\mathcal{E}\) are derived from Eq. (18). It should be noticed that the total C-energy density \(\mathcal{E}\) itself depends on neither of \(A\) nor \(\theta\) because of the invariance of \(\mathcal{E}\) under the isometry of the target space. So, the non-trivial effects of non-linear interaction between gravitational modes and electromagnetic modes will appear only through the occupancy ratios \(R_g\) and \(R_{em}\). These quantities show how the seed function \(\tau\) determines the local spacetime dependence of the occupancy ratio of each contribution, once the parameters \((A, \theta)\) are fixed.

### 4 Analysis of conversion phenomena

To consider the conversion phenomena, let us introduce occupancy diagrams by using the formulas (24) and (25), as depicted in Fig. 1. From the occupancy diagrams, we can qualitatively predict how the conversion phenomenon will occur according to the
behavior of the seed function $\tau$. Each set of the parameters $(A, \theta)$ gives one occupancy diagram, which corresponds to one of various patterns of the conversion phenomenon. The first diagram, corresponding to the parameters $(A, \theta) = (1/6, \pi/20)$, shows the typical behavior of occupancy diagrams. Some conversion usually can be expected. In particular, if the parameters $(A, \theta)$ are appropriately adjusted, the graphs of the occupancy ratio of the electromagnetic part show a large peak at some value of $\tau$. This makes us expect that the electromagnetic part of the C-energy can become overwhelmingly dominant over the gravitational part at some time. For example, in the graphs of Fig. 1, such peaks appear at $\tau = 0$ for the second case, and near $\tau = -1.6$ for the third case.

Here let us note the following facts: (i) when a regular wave packet like the Weber-Wheeler-Bonnor solution (WWB) [31, 32] is adopted as the seed function, in general at null infinity the seed function becomes zero and near the cylindrical symmetric axis $\rho = 0$ the seed function can have any large values, (ii) Most of the C-energy is generally concentrated around the peak of the wave. From these, we can immediately deduce the following consequences. First, let us assume that the seed function is a wave packet like the WWB solution, whose peak is initially put on the symmetric axis and take a value of about $\tau = 5$ as an example. Then, from the second figure of Fig. 1, it is naturally expected that the initial wave composed mostly of gravitational modes rapidly changes to a wave composed mainly of electromagnetic modes, as the wave spreads. Similarly, from the third figure in Fig. 1, under the assumption that the initial peak of the seed takes a value of about $-1.6$ on the axis, it can be expected that the corresponding initial configuration of the wave is mainly composed of electromagnetic parts, and then the wave is converted to the gravitational wave after leaving the axis.

To proceed the analysis further, we should give the seed function $\tau$ explicitly. Hence in the rest, we adopt the following expression of WWB solution as a seed function $\tau$ (for details, see [19, 33]),

$$
\tau(t, \rho) = \frac{c}{\sqrt{2}} \left[ \sqrt{4a^2t^2 + (a^2 + \rho^2 - t^2)^2 + a^2 + \rho^2 - t^2} \right]^{1/2},
$$

(27)

two shapes of which, for examples, are given in Figs. 2 and 3 for two different sets of parameters $(a, c)$. It is a useful fact that at $t = 0$, the height of the peak and the half width of the wave are given by $c/a$ and $a$, respectively.
Fig. 2 The left and right figures display the snapshots of the seed function $\tau$, which correspond to $t = 0$ and $t = 1$, respectively. The parameters $(a, c)$ set to $(1/20, 1/4)$

Fig. 3 The left and right figures display the snapshots of the seed function $\tau$, which correspond to $t = 0$ and $t = 1$, respectively. The parameters $(a, c)$ set to $(1/18, -2/15)$

Fig. 4 The left and right figures show the time dependence of $T_{em}$ for the cases: $(A, \theta, a, c) = (0, \pi/4, 1/20, 1/4)$ or $(A, \theta, a, c) = (1/6, 11\pi/25, 1/18, -2/15)$. The Red line segments on both the figures mean rates of electromagnetic contributions.

Using the explicit form of the seed given above, let us give the total conversion ratio $T_{em}$, which is useful to show the time dependence of total electromagnetic contributions to the C-energy, as follows

$$T_{em}(t) := \frac{\gamma_{em}(t, \rho = \infty)}{\gamma(t, \rho = \infty)}, \quad \left( \gamma(t, \rho) := \int_0^\rho E(t, r) \, dr \right).$$

Here the quantity $\gamma(t, \rho = \infty)$ takes a time-independent value $(c/2a)^2$. Figure 4 shows, for example, the time dependence of the $T_{em}$ from $t = 0$ to $t = 2$ for $(A, \theta, a, c) = (0, \pi/4, 1/20, 1/4)$ or $(A, \theta, a, c) = (1/6, 11\pi/25, 1/18, -2/15)$, respectively. Here, to evaluate the integral numerically we have changed the upper limit of the integration range from infinity ($\rho = \infty$) to a sufficiently large finite value (here $\rho = 1000$). The left figure in Fig. 4 shows the ratio $T_{em}$ increases rapidly, while the right figure shows the ratio $T_{em}$ decreases promptly. After sufficiently long time (e.g., from $t = 0$ to $t = 200$), for the case of the left-hand side, the electromagnetic contribution is amplified times about 9.8 (i.e., $\gamma_{em}(t = 200, \rho = 1000)/\gamma_{em}(t = 0, \rho = 1000) \approx 9.8$), whereas, for the right-hand case, the gravitational contribution is amplified times about 3.4 (i.e., $\gamma_g(t = 200, \rho = 1000)/\gamma_g(t = 0, \rho = 1000) \approx 3.4$).
5 Summary and discussions

To summarize, we have clarified the occurrence of large conversion phenomena between gravitational and electromagnetic waves as a novel aspect of full nonlinear interaction of Einstein-Maxwell system. There is already a similar attempt to treat the conversion based on the numerical analysis [34]. However, their attempt is interesting but somewhat limited because of the research that deals only with the conversion from electromagnetic waves to gravitational waves. As mentioned above, one of our main concerns is about how large an initially small electromagnetic wave can be amplified being supplied with the gravitational wave energy. In the case of the above electromagnetic wave amplification, the amplification factor is about 10 times. Though its factor can be considered large enough, we expect, however, that any larger amplification factor can be achieved. From the diagrams of Fig. 1, we easily know that the electromagnetic contribution decreases sharply as the magnitude of the wave function $|\tau|$ becomes larger. Therefore, considering that the size of the wave function $|\tau|$ is controlled by the parameter $c$ of the WWB solution (27), we may expect that the amplification factor of the electromagnetic wave will increase without limit as the size of the parameter $c$ becomes larger. In fact, for the case of $(A, \theta, a, c) = (0, \pi/4, 1/20, 3)$, we can show that the electromagnetic amplification factor $\gamma_{em}(t = 200, \rho = 1000)/\gamma_{em}(t = 0, \rho = 1000)$ exceeds 1000 numerically.

In the rest, we shall give a few comments related to the spacetime structure, the detailed analysis of which will be given elsewhere. First, it should be noticed that as long as the WWB solution is used, the space-time structure may be expected to be regular because the behaviors of the metric functions including the C-energy are essentially the same as the vacuum solutions treated previously [19]. Next, when the solution shows a significant conversion, we easily know that the corresponding deficit angle behaves like $\Delta\phi/2\pi = O(1)$ at the spacelike infinity, from the formula [19]:

$$\Delta\phi = 2\pi \left[1 - e^{-(c^2/2a)^2}\right].$$

(28)

For further discussion, let us recover the Newton constant $G$ and the speed of light $c$. It should be noticed that the observer at the infinity considers the initial concentration of the waves around the axis as an infinitely long stringy object packed with energy. Then, according to the formula given in the work [36], the deficit angle evaluated at the infinity may be given, as follows

$$\Delta\phi = \frac{8\pi G c^4}{\mu},$$

(29)

where the quantity $\mu$ means the linear energy density along with the $z$-axis. If $\Delta\phi/2\pi = O(1)$, from Eqs. (28) and (29) the corresponding linear energy density $\mu$ may be roughly $c^4/2G$. When some mass scale $M$ (e.g. solar mass $M_\odot$, etc.) is introduced here due to the scale invariance of the Einstein-Maxwell system, $\mu$ is represented, as follows

$$\mu \approx \frac{c^4}{2G} = \frac{Mc^4}{2GM} = \frac{Mc^2}{rg},$$

(30)
where \( r_g = \frac{2MG}{c^2} \) (i.e., gravitational radius). Though in the cylindrical symmetric spacetime no black holes can be formed, if we dare to relate this linear energy density to “black holes”, the linear energy density may correspond to that of an array of an infinite number of black holes with mass \( M \) aligned equally at distance \( r_g \). Now let us consider an elongated and localized energy distributed around the finite region on the \( z \) axis which corresponds to the more realistic waves, and approximate it with the cylindrically symmetric waves dealt with above, assuming that the effects at both ends are negligible. According to the so-called hoop conjecture [35], which is applied here to the elongated and finite objects, it is expected that a black hole may be formed in the case of \( \text{LINEAR ENERGY DENSITY} \geq \frac{Mc^2}{r_g} \). Hence, when we consider the conversions of more realistic gravitational and electromagnetic waves which emanate from elongated finite energy distributions, the black hole formation may prevent the spread of the waves. As a speculation, the corresponding conversions may be suppressed in comparison with the cylindrically symmetric solutions of the same linear energy density because of the confinement of the conversion region inside the black hole.

It would be, however, fair to comment on the opposite possibility. From the formula (28), in fact, we can see that the upper limit of the deficit angle evaluated at the infinity is \( 2\pi \), and the deficit angle can only approach \( 2\pi \) no matter how large value the C energy takes. This means that the cylindrically symmetric wave approximation cannot handle the formation of black holes itself at least within the solutions treated above. However, the occurrence of large conversions of the cylindrically symmetric waves may suggest that remarkably large conversion occurs just before the black hole formation as a critical phenomenon. Therefore, it may be an interesting problem to determine which will occur for more realistic conversion phenomena, though non-cylindrically symmetric approaches are needed.

In this paper, we have proceeded with the analysis using only the C-energy and the related quantities. As other useful quantities, there are the cylindrically symmetric analogues of the Bondi mass/news and Bel-Robinson superenergy [37, 38]. According to the discussion in [37], time dependence of the analogous Bondi mass/news is described with the temporal derivative of Thorn’s C energy with respect to retarded time. Hence, we may consider that our results obtained by using directly the C energy density is essentially the same as the treatment with news functions. On the other hand, for the analogue of the Bel-Robinson superenergy, it seems to be basically different from the C energy (see, for example, [38]). Therefore, the analysis based on the superenergy is interesting, but is somewhat complicated, so we would like to leave it as a future problem.

As a further study, it would be interesting and important to see whether the basic conclusions of the paper hold in other cases when including other type of seed solutions (for example, [19, 39]).

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