Hamiltonian renormalisation II.
Renormalisation flow of 1+1 dimensional free scalar fields: derivation

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Abstract
In the companion paper Lang et al (2017 arXiv:1711.05685) we motivated a renormalisation flow on Osterwalder–Schrader data (OS-data) consisting of (1.) a Hilbert space, (2.) a cyclic vacuum and (3.) a Hamiltonian annihilating that vacuum. As the name suggests, the motivation was via the OS reconstruction theorem which allows to reconstruct the OS data from an OS measure satisfying (a subset of) the OS axioms, in particular reflection positivity. The guiding principle was to map the usual Wilsonian path integral renormalisation flow onto a flow of the corresponding OS data.

We showed that this induced flow on the OS data has an unwanted feature which disqualifies the associated coarse grained Hamiltonians from being the projections of a continuum Hamiltonian onto vectors in the coarse grained Hilbert space. This motivated the definition of a direct Hamiltonian renormalisation flow which follows the guiding principle but does not suffer from the afore mentioned caveat.

In order to test our proposal, we apply it to the only known completely solvable model, namely the case of free scalar quantum fields. In this paper we focus on the Klein Gordon field in two spacetime dimensions and illustrate the difference between the path integral induced and direct Hamiltonian flow. Generalisations to more general models in higher dimensions will be discussed in our companion papers.

Keywords: constructive QFT, renormalisation, background-independence, canonical formulation
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Notation

In this paper we will deal with quantum fields in the presence of an infrared cut-off $R$ and with smearing functions of finite time support in $[−T, T]$. The spatial ultraviolet cut-off is denoted by $M$ and has the interpretation of the number of lattice vertices in each spatial direction. We will mostly not be interested in an analogous temporal ultraviolet cut-off $N$ but sometimes refer to it for comparison with other approaches. These quantities allow us to define dimensionful cut-offs $\epsilon_{RM} = \frac{R}{M}$, $\delta_{TN} = \frac{T}{N}$. In Fourier space we define analogously $k_R = \frac{2\pi}{R}$, $k_M = \frac{2\pi}{M}$, $k_T = \frac{2\pi}{T}$, $k_N = \frac{2\pi}{N}$.

We will deal with both instantaneous fields, smearing functions and Weyl elements as well as corresponding temporally dependent objects. The instantaneous objects are denoted by lower case letters $\phi_{RM}$, $f_{RM}$, $w_{RM}$, the temporally dependent ones by upper case ones $\Phi_{RM}$, $F_{RM}$, $W_{RM}[F_{RM}]$. As we will see, smearing functions $F_{RM}$ with compact and discrete (sharp) time support will play a more fundamental role for our purposes than those with a smoother dependence.

Osterwalder–Schrader reconstruction concerns the interplay between time translation invariant, time reflection invariant and reflection positive measures (OS measures) $\mu_{RM}$ on the space of history fields $\Phi_{RM}$ and their corresponding Osterwalder–Schrader (OS) data $\mathcal{H}_{RM}, \Omega_{RM}, H_{RM}$ where $\mathcal{H}_{RM}$ is a Hilbert space with cyclic (vacuum) vector $\Omega_{RM}$ annihilated by a self-adjoint Hamiltonian $H_{RM}$. Together, the vector $\Omega_{RM}$ and the scalar product $\langle \cdot \,, \cdot \rangle_{\mathcal{H}_{RM}}$ define a measure $\nu_{RM}$ on the space of instantaneous fields $\phi_{RM}$.

Renormalisation consists in defining a flow or sequence $n \to \mu_{RM}^{(n)}$, $n \in \mathbb{N}_0$ for all $M$ of families of measures $\{\mu_{RM}^{(n)}\}_{M \in \mathbb{N}}$. The flow will be defined in terms of a coarse graining or embedding map $I_{RM} \to \mu_{RM}^{(M')}$ acting on the smearing functions and satisfying certain properties that will grant that (1.) the resulting fixed point family of measures, if it exists, is cylindrically consistent and (2.) the flow stays within the class of OS measures. Fixed point quantities are denoted by an upper case *, e.g. $\mu_{RM}^{*}$. 
1. Introduction

In our companion paper [1] we motivated a renormalisation flow directly on the Osterwalder–Schrader data (OS data) consisting of a Hilbert space \( \mathcal{H} \) supporting a Hamiltonian \( H \) that annihilates a cyclic vacuum vector \( \Omega \) therein. The motivation was through the usual Wilsonian renormalisation flow on OS measures \( \mu \) satisfying at least a subset of the OS axioms [4, 5], specifically time translation invariance, time reflection invariance and time reflection positivity. The OS reconstruction theorem allows to derive OS data directly from an associated OS measure \( \mu \) and thus allows to translate the Wilsonian renormalisation flow on \( \mu \) directly in terms of the OS data.

The reason for the interest in the flow of the OS data rather than of the OS measure stems from applications where the construction of the interacting quantum field theory (QFT) [6–8] has a more natural starting point in its Hamiltonian formulation and where the construction of the corresponding OS measure is much more difficult. This applies in particular to realistic, i.e. Lorentzian signature, quantum gravity in its canonical formulation [9–12] in the presence of matter [13–16] which allows to gauge fix the spacetime diffeomorphism invariance and to define a physical Hamiltonian. Due to the complex, not even polynomial, interaction in this gravitational Hamiltonian and the fact that it does not have a natural (that is, without breaking background independence of Einstein’s theory) split into a free and an interacting piece, the path integral measure corresponding to it cannot be simply constructed using the usual techniques based on Gaussian measures [4] for QFT on Minkowski space and thus presents a major technical challenge. In other words, if one artificially introduces such a split using a background metric, one must make sure that the final result of the Wilsonian flow [17, 18] is independent of that background. It is widely believed that the Gaussian fixed point of the associated flow does not define a renormalisable theory, however, there maybe non-Gaussian fixed points that do. See [19–22] for the state of the art of this so-called ‘Asymptotic Safety Approach’ to quantum gravity which, to the best knowledge of the present authors, is best understood in the Euclidian signature regime.

The renormalisation flow of the path integral measure is defined in terms of a coarse graining map (or block spin transformation in statistical physics jargon) acting on the smearing fields of the spacetime quantum fields in question. This is because the measure can be defined in terms of its generating functional (i.e. the Fourier transform of the measure) which employs (generalised) exponentials of the smeared fields. The smearing fields carry a label \( M \) that specifies its spatial resolution. We restrict to spatial resolution and do not consider temporal resolution because we want to monitor the induced flow of the OS data which requires that time is continuous in order that OS reconstruction applies. When restricting to integrate quantum fields smeared with fields of given resolution \( M \), one obtains a whole family of measures \( \mu_M \). The coarse graining map \( I_{M \rightarrow M'} \) embeds the space of smearing functions with coarse spatial resolution \( M \) into a space of smearing function with finer spatial resolution \( M' > M \). The flow then constructs a sequence \( n \mapsto \{ \mu_M^{(n)} \} \) of measure families where \( \mu_M^{(n+1)} \) is defined in terms of \( \mu_M^{(n)} \cdot M' > M \). We showed that this sequence stays of OS type if the initial family is. By OS reconstruction we then obtain a corresponding OS data family \( (\mathcal{H}_M^{(n)}, \Omega_M^{(n)}, H_M^{(n)}) \).

In [1] we demonstrated that the path integral induced flow of the OS data has a peculiar feature. Namely, the \( H_M^{(n)} \) commute with the projection maps \( P_{M \rightarrow M'}^{(n)} : \mathcal{H}_M^{(n)} \to \mathcal{H}_M^{(n)}, M < M' \) onto the image of \( H_M^{(n+1)} \) in \( \mathcal{H}_M^{(n)} \). This means that the corresponding fixed point objects have the property \( [H_M^{(n)}, P_{M \rightarrow M'}^{(n)}] = 0 \). Hence, the fine resolution Hamiltonian preserves every coarse resolution subspace. This is counter intuitive and contradicts properties of such Hamiltonians in concrete applications, e.g. in lattice gauge theory where \( M \) defines the lattice spacing and.
where the Hamiltonian will have non-vanishing matrix elements between states defined in terms of coarse and fine resolution plaquettes respectively.

In [1] we found that the underlying reason for this peculiarity is that the reconstructed OS data are automatically consistent with the semigroup law of the OS contraction semi-group. This means that the Hamiltonians $H_M^*$ cannot be interpreted as restrictions of the continuum Hamiltonian $H^* = \lim_{M \to \infty} H_M^*$ to $H_M^*$. Thus, the path integral induced flow of the OS data is not what we wished to have. This motivated us to minimally modify the path integral induced flow and define a direct Hamiltonian flow of OS data which does not need the path integral at all. This flow can be considered (1.) as a flow of the canonical Hilbert space measures $\nu^{(n)}_M$ underlying $\mathcal{H}_M^{(n)}$ which carries an imprint of the $\mu^{(n)}_M$ in the sense that $\nu^{(n)}_M$ is the restriction of $\mu^{(n)}_M$ to sharp time zero smearing fields and (2.) of the Hamiltonians $H_M^{(n)}$ which keep track of the flow of vacua $\Omega^{(n)}_M$. The fixed point of that flow, if it exists and if certain mild universality assumptions hold, is an inductive limit of Hilbert spaces $\mathcal{H}$ together with a densely defined, positive quadratic form thereon which in fortunate cases has a Friedrichs extension as a positive self-adjoint operator. The direct and path integral induced flow will thus be very different in general and the interesting question is whether the continuum theories at the fixed points agree with each other.

In this paper we would like to see the formalism in action and apply it to the two-dimensional Klein–Gordon field after having specialised the general theory to free scalar fields in any dimension. The restriction to two dimensions is mainly for illustrative reasons and in order to have a concrete example in mind. The free field case serves as a testing ground for our formalism and allows us to solve all equations analytically. We find explicitly both the path integral induced and direct Hamiltonian flow and can determine their fixed point families. Their flows and fixed point families are very different from each other. Yet, their continuum theories agree. The way this happens is very interesting. At any finite resolution and at the fixed point the path integral induced Hamiltonian is described by an infinite tower of sharp time zero fields species and their conjugate momenta with resolution dependent masses. The Hamiltonian is quadratic in these fields but they are mutually interacting with resolution depending coupling ‘constants’. By contrast, the direct Hamiltonian at any finite resolution and at the fixed point is described by a single field species and its conjugate momentum with resolution dependent mass term. Now, in the continuum limit $M \to \infty$ for the path integral induced Hamiltonian all but a single coupling goes to zero and all but a single corresponding mass goes to infinity. This continuum limit agrees with that of the direct Hamiltonian. Thus, the direct renormalisation flow does find the correct continuum theory, at least in the free field case, and the finite resolution Hamiltonians are the projections to finite resolution subspaces of the continuum Hilbert space, which is what we wanted to achieve. The direct scheme no longer needs any path integral techniques which is what we were after.

The architecture of this paper is as follows:

In section 2 we review the continuum formulation of a free scalar field starting from the classical action. That is, we derive its classical Hamiltonian formulation, the Fock space quantisation, the OS construction—that is, the derivation of the corresponding Wiener measure (Mehler kernel)—and finally the OS reconstruction of the Fock space OS data starting from that Wiener measure.

In section 3 we introduce an initial family of OS data labelled by a finite resolution parameter based on a certain naive discretisation of the continuum theory and its corresponding Fock quantisation. This serves to construct an initial family of Wiener measures. We then compute both the path integral induced flow and the direct flow of these OS data for the concrete model of the 1+1-dimensional Klein Gordon field and compare between them.
In section 4 we summarise our findings. We will continue the investigation of the present model in our companion paper [2] with respect to the aspects of universality of the fixed point (i.e. its dependence on the initial values of the OS data and of the coarse graining map), its convergence behaviour to the fixed point and the decay properties of the contributions of the $n$th order neighbour points of the improved and perfect lattice Laplacian.

The restriction to two dimensions forbids the analysis of the rotational invariance of the renormalisation flow, i.e. the question whether it is possible to show at finite resolution that the flow converges to a rotationally invariant fixed point. This will be remedied in our companion paper [3] where we show that the findings of this paper and [2] easily generalise to higher dimensions. It turns out that it is sufficient to investigate this for the 2+1-dimensional case due to certain factorisation properties. The analysis of rotational invariance is a showcase for the important question how renormalisation restores symmetries of the classical continuum theory when it is broken by the initial choice of discretisations, a question which is especially important in quantum gravity [23–26].

2. Review: OS (re)construction of the free quantum scalar field in the continuum

In this section we review OS reconstruction and OS construction for the free quantum scalar field in the continuum, that is, on Minkowski space without IR cut-off $R$. Experts on the subject can safely skip this section.

2.1. Theory class

We consider the following class of actions

\[
S := \frac{1}{2\kappa} \int_{\mathbb{R}^{D+1}} dt \, d^Dx \left[ \frac{1}{c} \dot{\phi}^2 - c \phi \omega^2 \phi \right]
\]  

(2.1)

where $\omega^2$ is any function of $p, \Delta$ which has dimension of inverse length, for instance

\[
\omega^2 = \frac{1}{p^{2(n-1)}} (-\Delta + p^2)^n, \quad n = 1, 2, ...
\]  

(2.2)

which is not Poincaré invariant unless $n = 1$. Here $x^0 = ct$ has dimension of length and $\hbar \kappa$ has dimension $cm^{D-1}$ if $\phi$ is dimension free. Furthermore, $\Delta = \sum_{a=1}^{D} (\partial / \partial x^a)^2$ is the Laplacian and $p$ is the inverse of the Compton wave length. The Legendre transform of (2.1) results in the Hamiltonian

\[
H := \frac{1}{2} \int_{\mathbb{R}^D} d^Dx \left[ \kappa c \pi^2 + \frac{\omega^2}{\kappa} \phi \right]
\]  

(2.3)

where $\pi = \dot{\phi} / (\kappa c)$ is the momentum conjugate to $\phi$ which has the dimension of an action divided by length$^D$. We will see that our methods work for any choice of $\omega^2$ and in any dimension. Below, for concreteness we consider the simplest case $D = 1$ and the Poincaré invariant choice $n = 1$ in (2.2). The general choices for $D, n$ will be considered in our companion papers [2, 3].
2.2. Canonical quantisation

We choose a Fock representation \( \mathcal{H} \) based on the annihilation operator valued distribution

\[
a = \frac{1}{\sqrt{2\hbar\kappa}} \left[ \sqrt{\omega} \phi - i\kappa \sqrt{\omega} \pi \right]
\]

in terms of which the normal ordered Hamiltonian reads

\[
H = \hbar c \int d^Dx \, a^* \omega a.
\]

The one particle Hilbert space is \( L = L_2(\mathbb{R}^D, d^Dx) \). If we denote the Fock vacuum by \( \Omega \) then

\[
a[f]\Omega = 0 = H\Omega, \quad a[f] := \langle f, a \rangle_L = \int_{\mathbb{R}^D} d^Dx \, \bar{f} \, a
\]

and we verify the canonical commutation relations

\[
[a[f], (a[f'])^*] = \langle f, f' \rangle_L \, 1_{\mathcal{H}}.
\]

The Weyl elements read for real valued smearing functions \( f, g \in S(\mathbb{R}^D) \), which are smooth and of rapid decrease (with \( f \) of inverse length \( D \) dimension)

\[
w[f, g] = e^{i(f, \phi) + (g, \pi)}_L, \quad w[f] := w[f, g = 0].
\]

We compute the generating functional of the Hilbert space measure \( \nu \)

\[
\nu(w[f]) = \langle \Omega, w[f] \Omega \rangle_{\mathcal{H}} = \langle \Omega, e^{i\sqrt{\frac{2\hbar\kappa}{\omega}}(a[\omega^{-1/2}f]^* + a[\omega^{-1/2}f])} \Omega \rangle_{\mathcal{H}} = e^{-\frac{\hbar}{2\omega} (f, \omega^{-1}f)},
\]

which displays \( \nu \) as a Gaussian measure with support on the tempered distributions \( \gamma = S'(\mathbb{R}^D) \) with zero mean and covariance

\[
C = \frac{\hbar \kappa}{2 \omega}^{-1}.
\]

Accordingly, \( \mathcal{H} \cong L_2(\gamma, d\nu) \).

2.3. Wiener measure from Hamiltonian formulation

We compute the corresponding Wiener measure \( \mu \) and anticipate its support \( \Gamma = S'(\mathbb{R}^{D+1}) \).

Let

\[
W[F] := e^{i(F, \Phi)}, \quad F \in S(\mathbb{R}^{D+1}), \quad \Phi \in \Gamma, \quad \langle F, \Phi \rangle = \int_{\mathbb{R}^{D+1}} d\beta \, d^Dx \, F \, \Phi \tag{2.11}
\]

then for \( F \) real valued and in the limit of sharp time support

\[
F(\beta, x) = \sum_{k=1}^N \delta(\beta - \beta_k) f_k(x), \quad \beta_k < \beta_{k+1}, \quad f_k \in S(\mathbb{R}) \tag{2.12}
\]

we have

\[
\mu(W[F]) := \langle \Omega, w[f_N] \, e^{-(\beta_N - \beta_{N-1})H/h} \ldots e^{-(\beta_2 - \beta_1)H/h} \, w[f_1] \, \Omega \rangle_{\mathcal{H}}. \tag{2.13}
\]
To compute (2.13) explicitly we determine the analytically extended Heisenberg field
\[
z(f, \beta) := e^{-i\beta H/H}(f, \phi)e^{i\beta H/H} = \langle \text{ch}(\omega \beta) \cdot f, \phi \rangle - \frac{i}{\omega} \text{sh}(\omega \beta) \cdot f, \pi \rangle,
\]
which can be found by analytic continuation \( t \to -i\beta \) of its unitary evolution with respect to \( H \). It follows that
\[
e^{-i\beta H/H}w[f]e^{i\beta H/H} = e^{i z(f, \beta)}
\]
so that, using the vacuum property and the abbreviation \( K_\beta = e^{-i\beta H/H} \),
\[
\mu(W[F]) = \langle \Omega, w[f_N] K_{\beta_N-\beta_{N-1}} w[f_{N-1}] \ldots w[f_2] K_{\beta_2-\beta_1} w[f_1] \Omega \rangle_H
= \langle \Omega, w[f_N] \rangle \langle K_{\beta_N-\beta_{N-1}} w[f_{N-1}] K_{\beta_{N-1}-\beta_{N-2}} \ldots K_{\beta_2-\beta_1} \rangle \langle K_{\beta_2-\beta_1} w[f_1] \rangle \langle \Omega \rangle_H
= \langle \Omega, e^{i\sqrt{\omega}(f_N, \beta_N - \beta_h)} e^{i\sqrt{\omega}(f_{N-1}, \beta_{N-1} - \beta_h)} \ldots e^{i\sqrt{\omega}(f_1, \beta_1 - \beta_2)} \Omega \rangle_H.
\]
Let \( z_\lambda = z(f_\lambda, \beta_N - \beta_\lambda) \) then by the BCH formula
\[
e^{i z_\lambda} \ldots e^{i z_2} e^{i z_1} = e^{i z_1} \ldots e^{i z_{n-1}} e^{i z_n} = e^{i \sum_{n=0}^{N-1} z_\lambda} e^{-\frac{i}{2} \sum_{n=0}^{N-1} z_\lambda},
\]
and by the CCR with \( \beta_\lambda = \beta_N - \beta_\lambda \)
\[
[z_\lambda, z_{\lambda'}] = -\hbar \langle \text{ch}(\omega \beta_\lambda) \cdot f_\lambda, \frac{\omega}{\omega} \text{sh}(\omega \beta_{\lambda'}) \cdot f_{\lambda'} \rangle + \hbar \langle \text{ch}(\omega \beta_{\lambda'}) \cdot f_{\lambda'}, \frac{\omega}{\omega} \text{sh}(\omega \beta_\lambda) \cdot f_\lambda \rangle.
\]
Next, with the combinations that appear in the single sum exponent of (2.17) we decompose into annihilation and creation operators and apply once more the BCH formula
\[
e^{i \sqrt{\omega} [f, a^*]} = e^{i \sqrt{\omega} [f, a^*] + (\sqrt{\omega} f \cdot a^*)}
= e^{i \sqrt{\omega} f \cdot \sqrt{\omega} a^*} e^{i \sqrt{\omega} f \cdot a^*} = e^{i \sqrt{\omega} f \cdot \sqrt{\omega} a^*} e^{i \frac{1}{2} (\sqrt{\omega} f \cdot \sqrt{\omega} \cdot a^* + \sqrt{\omega} f \cdot \sqrt{\omega} \cdot a^*)} = e^{i \sqrt{\omega} f \cdot \sqrt{\omega} a^*} e^{i \sqrt{\omega} f \cdot a^*} e^{-\frac{i}{2} (\sqrt{\omega} f \cdot \sqrt{\omega} \cdot a^*)} \cdot e^{i \sqrt{\omega} f \cdot a^*}.
\]
The last exponent is explicitly
\[
-\frac{\hbar \omega}{4} \sum_{k=1}^N \langle \text{ch}(\omega \beta_k) - \text{sh}(\omega \beta_k) \rangle \omega^{-1/2} \cdot f_k, [\text{ch}(\omega \beta_k) + \text{sh}(\omega \beta_k) \omega^{-1/2} \cdot f_k]
= -\frac{\hbar \omega}{4} \sum_{k=1}^N \langle e^{-i \omega \beta_k} \cdot f_k, \omega^{-1} e^{i \omega \beta_k} \cdot f_k \rangle.
\]
Now we remember \((2.12)\) and thus may write \((2.21)\) as

\[
- \frac{\hbar \kappa}{4} \int_{\beta_1}^{\beta_N} ds \int_{\beta_1}^{\beta_N} dt \langle e^{-c \omega (\beta_N - t)} \cdot F(s), \omega^{-1} e^{\omega (\beta_N - t)} \cdot F(t) \rangle \\
= - \frac{\hbar \kappa}{4} \int_{\beta_1}^{\beta_N} ds \int_{\beta_1}^{\beta_N} dt \langle e^{\omega (t - s)} \cdot F(s), \omega^{-1} \cdot F(t) \rangle
\]

where in the last step we used that \(e^{-c(\beta_N - s) \omega} \) is a self-adjoint smoothening operator. Likewise, we can write the double sum exponent in \((2.17)\) as, using \((2.18)\)

\[
- \frac{\hbar}{2} \int_{\beta_1}^{\beta_N} ds \int_{\beta_1}^{\beta_N} dt \langle -c (c \omega (\beta_N - s) - F(s), \kappa \omega \cdot c \omega (\beta_N - t) \cdot F(t) \rangle \\
+ \langle c \omega (\beta_N - t) \cdot F(t), \kappa \omega \cdot c \omega (\beta_N - s) \cdot F(s) \rangle
\]

\[
= - \frac{\hbar \kappa}{2} \int_{\beta_1}^{\beta_N} ds \int_{\beta_1}^{\beta_N} dt \langle -c (c \omega (\beta_N - t) \cdot F(s), \omega^{-1} \cdot F(t) \rangle \\
+ \langle \omega^{-1} \cdot F(t), c \omega (\beta_N - t) \cdot F(s) \rangle
\]

\[
= \frac{\hbar \kappa}{2} \int_{\beta_1}^{\beta_N} ds \int_{\beta_1}^{\beta_N} dt \langle c \omega (\beta_N - t) \cdot F(s), \omega^{-1} \cdot F(t) \rangle \\
- \langle c \omega (\beta_N - t) \cdot F(s), \omega^{-1} \cdot F(t) \rangle
\]

\[
= \frac{\hbar \kappa}{2} \int_{\beta_1}^{\beta_N} ds \int_{\beta_1}^{\beta_N} dt \langle c \omega (s - t) \cdot F(s), \omega^{-1} \cdot F(t) \rangle
\]

where in the second step we used that all operators involved are self-adjoint and act on their invariant domain and in the second we used the reality of all functions involved.

We can now put \((2.17)\), \((2.22)\) and \((2.23)\) together and find, dropping the constraints \(\beta_1 \leq s, t \leq \beta_N\) as \(F\) has this compact time support anyway

\[
\mu(W[F]) = \exp(-\frac{\hbar \kappa}{4} \int ds \int_{t \leq s} dt \langle e^{c \omega (t - s)} \cdot F(s), \omega^{-1} F(t) \rangle \\
- 2 \int ds \int_{t \leq s} dt \langle c \omega (s - t) \cdot F(s), \omega^{-1} \cdot F(t) \rangle)
\]

\[
= \exp(-\frac{\hbar \kappa}{4} \int ds \int_{t \geq s} dt \langle e^{c \omega (t - s)} \cdot F(s), \omega^{-1} F(t) \rangle \\
+ \int ds \int_{t \geq s} dt \langle e^{c \omega (t - s)} \cdot F(s), \omega^{-1} F(t) \rangle \\
- \langle e^{c \omega (s - t)} \cdot F(s), \omega^{-1} \cdot F(t) \rangle \\
+ \langle e^{c \omega (s - t)} \cdot F(s), \omega^{-1} \cdot F(t) \rangle)
\]

\[
= \exp(-\frac{\hbar \kappa}{4} \int ds \int_{t \geq s} dt \langle e^{c \omega (t - s)} \cdot F(s), \omega^{-1} F(t) \rangle \\
+ \int ds \int_{t \geq s} dt \langle e^{c \omega (t - s)} \cdot F(s), \omega^{-1} \cdot F(t) \rangle)
\]

\[
= \exp(-\frac{\hbar \kappa}{4} \int ds \int dt \langle e^{c \omega (t - s)} \cdot F(s), \omega^{-1} F(t) \rangle)
\]
where in the second step we split the integration domain of the first term and expressed the hyperbolic sine in terms of exponentials and in the last we combined terms. Note that while $e^{\beta \omega}$, $\beta > 0$ is not defined on Schwarz functions, the actual operator that appears in the above integral is $e^{-|r-t|-\omega}$ which is well defined.

The last step consists in the observation that up to a constant (2.24) just displays the integral kernel of the inverse of the operator

$$-[\partial / \partial x^0]^2 + \omega^2. \quad (2.25)$$

To see this, we note that the Green function of (2.25) is (use translation and reflection invariance)

$$G(x, y) = G(x - y) = G(y - x)$$

$$G(x) = \int \frac{d^{D+1}k}{(2\pi)^D} \frac{e^{i\omega \cdot k}}{k_0^2 + \omega(||k||)^2}. \quad (2.26)$$

We perform the $k_0$ integral by closing the contour over the infinite half circle in the upper/lower complex plane for $x^0 > 0$ and $x^0 < 0$, respectively. We can then apply the residue theorem and pick up the pole contribution at $k_0 = \pm i\omega(||k||)$ with weight $\pm 2\pi i$ of the remaining holomorphic integrand $e^{i\omega \cdot k} / (k_0 \pm i\omega(||k||))$ where

$$G(x) = \frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} e^{-|x|^2/\omega(||k||)} \frac{e^{i\omega \cdot x}}{\omega(||k||)} = \frac{e^{-|x|^2/\omega}}{2\omega} \cdot \delta_\omega.$$ \hspace{1cm} (2.27)

Thus, we conclude that the Wiener measure corresponding to our Hamiltonian parametrised by $\omega$ is a Gaussian measure with zero mean and covariance

$$C := \hbar \kappa (-\partial_0^2 + \omega^2)^{-1}. \quad (2.28)$$

### 2.4. Hamiltonian formulation from measure

This is standard, see e.g. [4]. We sketch some of the key steps for the sake of completeness.

We begin with the Gaussian measure $\mu$ of covariance (2.28) which is reflection and time translation invariant by inspection and analyse the null space of the corresponding reflection positive sesqui-linear and symmetric bilinear form. We consider smearing functions of compact, positive and sharp time support

$$F(t, x) := \sum_{k=1}^N \delta(t, t_k) \delta_\omega F_k(x), \quad G(t, x) := \sum_{l=1}^M \delta(t, s_l) \delta_\omega G_l(x). \quad (2.29)$$

with $0 < t_1 < \ldots < t_N$, $0 < s_1 < \ldots < s_M$ and $F_k, G_l$ are in $S(\mathbb{R}^D)$. We have with the history Hilbert space $H' := L_2(\Gamma, d\mu)$

$$\langle \delta_{\Phi} \Phi, \delta_{\Phi'} \Phi' \rangle_{H'} = \mu(\Phi \Phi' G - \theta F) = e^{-\frac{1}{2} \langle G - \theta F, C(G - \theta F) \rangle_{L_2(\mathbb{R}^{D+1}, d\mu)}}$$

$$= e^{-\frac{1}{2} \langle G, C \Phi \rangle} e^{\frac{1}{2} \langle F, C \Phi \rangle} e^{\frac{1}{2} \langle \Phi, C G \rangle} e^{\langle \Phi, C F \rangle} \cdot \quad (2.30)$$

Now as we can see and by reflection invariance of $C$
\[ \langle \theta \cdot F, C \cdot G \rangle = \langle F, C \cdot \theta \cdot G \rangle = \frac{\hbar \kappa}{2} \int ds \int dt (e^{c \omega(x,t)} \cdot F(s), \omega^{-1} G(-t)) \]

\[ = \frac{\hbar \kappa}{2} \int ds \int dt (e^{c \omega(x,t)} \cdot F(s), \omega^{-1} G(t)) \]

\[ = \frac{\hbar \kappa}{2} \sum_{k,l} (e^{-c \omega t_h} \cdot F_k, \omega^{-1} e^{-c \omega t_h} G_l)_{L_2(\mathbb{R}^d, \delta^d)} \]

\[ = (\theta \cdot F, C \tilde{G}) \] (2.31)

where in the second step we made use of the positive time support of both functions and we have defined

\[ \tilde{G}(t,x) := \delta(t,0) \sum_l e^{-c \omega_p G_l(x)} \] (2.32)

Let

\[ z := e^{-\frac{1}{2} (H \cdot C \cdot G - (\tilde{G} \cdot C \tilde{G})} \] (2.33)

then

\[ \langle e^{\Phi[F]}, e^{\Phi[G]} - ze^{\Phi[\tilde{G}]} \rangle_{\mathcal{H}'} = 0 \] (2.34)

for all \( F \). Thus, we conclude that the function \( e^{\Phi[G]} - ze^{\Phi[\tilde{G}]} \) belongs to the null space provided we can show that the span of the \( e^{\Phi[F]} \) with \( F \) of the form (2.29) is dense in \( \mathcal{H}' \). To see this, we take any \( F \) with positive and compact time support in \( (0, T] \) and consider the approximant

\[ F^N(t) := \sum_{k=1}^{N-1} \delta(t, t_k) F_k^N(x), \quad F_k^N(x) := \int_{t_k - T/(2N)}^{t_k + T/(2N)} dt F(t,x) \] (2.35)

with \( t_k = kT/N, \ k = 1, \ldots, N - 1 \). It follows

\[ ||e^{\Phi[F]} - e^{\Phi[F^N]}||_{\mathcal{H}'}^2 = \mu(e^{\Phi[F - \theta \cdot F^N]} + \mu(e^{\Phi[F^N - \theta \cdot F]})) - \mu(e^{\Phi[F - \theta \cdot F^N]} - \mu(e^{\Phi[F^N - \theta \cdot F]})]. \] (2.36)

We have for instance

\[ \langle F^N, C \cdot F^N \rangle = \frac{\hbar \kappa}{2} \sum_{k,j=1}^{N-1} (F_k^N, e^{-c \omega_{[k,j-\delta], \omega^{-1}} F_j^N)_{L_2(\mathbb{R}^d, \delta^d)} \] (2.37)

which is just a Riemann sum approximation of \( \langle F, C \cdot F \rangle \). The other calculations are similar. Thus, the span of these vectors is dense.

It follows that for any \( G \) the vector \( e^{\Phi[G]} \) can be approximated arbitrarily well by a vector \( e^{\Phi[G]} \) with \( G \) of the form (2.29) and in turn this vector is equivalent to a vector with time zero support up to a constant. We conclude that the Hilbert space of the theory is the completion of the span of vectors with sharp time zero support functions \( F = \delta(t,0) f \) for which we have

\[ \langle e^{\Phi[F]}, e^{\Phi[F']} \rangle_{\mathcal{H}'} = e^{-\frac{\hbar \kappa}{2} \int (f, \omega^{-1} f')} = \langle e^{\Phi[f]}, e^{\Phi[f']} \rangle_{\mathcal{H}} \] (2.38)

displaying \( \mathcal{H} = L_2(S(\mathbb{R}^d), \nu) \) where \( \nu \) is the Gaussian measure of covariance \( \hbar \kappa/(2\omega) \). Here \( e^{\Phi[f]} := [e^{\Phi[F]}] \) denotes the equivalence class of the sharp time zero support vector.

To compute the Hamiltonian, we use the definition \( (F(t) = \delta(t,0) f, \ F'(t) = \delta(t,0) f') \)
where \( \hat{f}' = e^{-i\beta \omega} \cdot f' \) and \( \hat{F}'(t) = \delta(t, 0) \hat{f}' \). As the \( e^{i\phi(f)\Omega} \), \( \Omega := 1 \) span \( \mathcal{H} \) it follows

\[
e^{-\beta H/e^2} e^{i\phi(f)\Omega} = e^{-H/\beta e^2} [(f \omega^{-1}) - (e^{-\beta \omega} f \omega e^{-1} e^{-\beta \omega})] e^{i\phi(e^{-\beta \omega} f)\Omega}. \tag{2.40}
\]

We verify that the contraction (2.40) coincides with the one as obtained from (2.4) and (2.5) if \( H \Omega = 0 \). We have

\[
e^{-\beta H/e^2} a e^{\beta H/e^2} = e^{\beta \omega} \cdot a
\]
\[
e^{-\beta H/e^2} a^* e^{\beta H/e^2} = e^{-\beta \omega} \cdot a^*
\]
\[
e^{-\beta H/e^2} \phi e^{\beta H/e^2} = \sqrt{\frac{\hbar \kappa}{2 \omega}} [e^{\beta \omega} \cdot a + e^{-\beta \omega} \cdot a^*]
\]
\[
= e^{-\beta \omega} \cdot \phi + \sqrt{\frac{2 \hbar \kappa}{\omega}} \text{sh}(\beta \omega) \cdot a
\]
\[
e^{-\beta H/e^2} e^{i\phi(f)\Omega} = e^{i\phi(e^{-\beta \omega} f) + i\sqrt{\frac{\hbar \kappa}{2 \omega}} \text{sh}(\beta \omega) f] \Omega
\]
\[
= e^{i\phi(e^{-\beta \omega} f) + i\sqrt{\frac{\hbar \kappa}{2 \omega}} \text{sh}(\beta \omega) f} \Omega \tag{2.41}
\]

3. Discretised theory and renormalisation

In this section we simulate the typical situation in constructive QFT and pretend not to know what the Hamiltonian or path integral formulation of the given classical theory should be in the continuum. Hence, we will introduce temporal and spatial IR cut-offs \( T, R \), respectively and work on finite lattices with \( M, N \) points in each spatial or temporal direction, respectively. We introduce maybe natural but still ad hoc discretised versions of the Hamiltonian or the corresponding path integral measure and apply the both renormalisation procedures derived in [1]. We will determine explicitly the corresponding fixed point structure and show that the resulting measures or Hamiltonian theories indeed correspond to the continuum theories constructed in section 2.1.

The common starting point for both renormalisation trajectories is a family of either Gaussian, reflection positive measures \( \mu_{R,M} \) or equivalently OS data (\( H_{R,M}^{(0)}, \Omega_{R,M}^{(0)}, H_{R,M}^{(0)} \)) originating from some spatial (lattice) discretisation of the classical continuum theory. In what follows, we construct such a discretisation explicitly using a choice of coarse graining map.

The fields at finite IR cut-off are supposed to obey periodic boundary conditions \( \phi_R(x + R\delta_a) = \phi_R(x), \pi_R(x + R\delta_a) = \pi_R(x) \) for all \( \delta_a \in \mathbb{Z}, a = 1, \ldots, D \). The corresponding one particle Hilbert space is \( L_R := L_2([0, R]^D, d^D x) \). In the presence of an additional UV cut-off we define the one particle Hilbert space as \( L_{R,M} := L_2(\mathbb{R}_M^D) \) with \( \mathbb{Z}_M := \{0, 1, \ldots, M - 1\} \). These are the square summable finite sequences \( f_{RM} \) with norm squared

\[
||f_{RM}||_{L_{RM}}^2 := \epsilon_{RM} \sum_{m \in \mathbb{Z}_M^D} |f_{RM}(m)|^2, \quad \epsilon_{RM} := \frac{R}{M}. \tag{3.1}
\]
Here the prefactor $\rho^{\text{RM}}_R$ is consistent with the interpretation that $f_{\text{RM}}(m) = f_R(m\epsilon_{\text{RM}})$ for some $f_R \in L_R$ so that $\|f_{\text{RM}}\|_{L^1} = \|f_R\|_{L^1}$.

Indeed, we have injections

$$I_{\text{RM}} : L_{\text{RM}} \to L_R; f_{\text{RM}} \mapsto \sum_{m \in \mathbb{Z}^D_{\text{UR}}} f_{\text{RM}}(m) \chi_{m_{\text{UR}}} (x)$$

(3.2)

with

$$\chi_{m_{\text{UR}}} (x) := \prod_{a=1}^D \chi_{m_a e_{\text{UR}}, (m_{a+1}) e_{\text{UR}}} (x).$$

(3.3)

These injections are isometric since the $\chi_{m_{\text{UR}}}$ define a partition of $[0, \infty)$

$$\langle \chi_{m_{\text{UR}}}, \chi_{m'_{\text{UR}}} \rangle_{L_R} = \epsilon^{\text{RM}}_R \delta_{m m'}.$$ 

(3.4)

Likewise, we have evaluation maps (which are densely defined on the continuous elements of $L_R$)

$$E_{\text{RM}} : L_R \to L_{\text{RM}}; f_R \mapsto f_R(m\epsilon_{\text{RM}})$$

(3.5)

which satisfy $E_{\text{RM}} \circ I_{\text{RM}} = \text{id}_{L_{\text{RM}}}$.

We consider the discretised fields

$$\phi_{\text{RM}}(m) := (I_{\text{RM}}^{-1}\phi_R)(m) = \int_{[0, R)^D} d^D x \chi_{m_{\text{UR}}}(x) \phi_R(x)$$

$$\pi_{\text{RM}}(m) := (E_{\text{RM}}\pi_R)(m) := \pi_R(m\epsilon_{\text{RM}}).$$

(3.6)

Notice that (3.6) defines a partial symplectomorphism

$$\{\pi_{\text{RM}}(m), \phi_{\text{RM}}(m')\} = \int d^D x \chi_{m_{\text{UR}}}(x) \{\pi_R(x), \phi_R(m'\epsilon_{\text{RM}})\} = \chi_{m_{\text{UR}}}(m'\epsilon_{\text{RM}}) = \delta_{m m'}.$$ 

(3.7)

Consider the family of Hamiltonians

$$H_{\text{RM}}^{(0)} := \frac{c}{2} \sum_{m \in \mathbb{Z}_{\text{UR}}^D} (\epsilon^2_{\text{RM}}\pi_{\text{RM}}^2(m) + \frac{1}{\epsilon^2_{\text{RM}}} \phi_{\text{RM}}(m)(\omega_{\text{RM}}^{(0)})^2 \cdot \phi_{\text{RM}}(m)).$$

(3.8)

Here we have defined $\omega^{(0)}_{\text{RM}}$ in terms of a suitable, self-adjoint (with respect to $L_{\text{RM}}$) discretisation $\Delta_{\text{RM}}$ of the Laplacian, that is, if the continuum $\omega_R$ is a certain function $G = G(-\Delta_R, p^2)$ of the continuum Laplacian $\Delta_R$ on $[0, R)^D$ then $\omega^{(0)}_{\text{RM}}$ is the function $G(-\Delta_{\text{RM}}, m^2)$. A popular choice is

$$\Delta_{\text{RM}} \cdot f_{\text{RM}}(m) := \frac{1}{\epsilon_{\text{RM}}} [f_{\text{RM}}(m+1) + f_{\text{RM}}(m-1) - 2f_{\text{RM}}(m)].$$

(3.9)

It is not difficult to check that (3.8) converges to

$$H_R := \frac{c}{2} \int_{[0, R)^D} d^D x [\kappa \pi_R^2 + \frac{1}{\kappa} \omega_R^2 \phi]$$

(3.10)

on smooth fields as $M \to \infty$.

The form (3.8) of the Hamiltonian motivates to introduce discrete annihilation operators

$$a_{\text{RM}}^{(0)} := \frac{1}{\sqrt{2\hbar\kappa}} \sqrt{\left(\omega^{(0)}_{\text{RM}} / \epsilon_{\text{RM}} \phi_{\text{RM}} - i\kappa / \epsilon_{\text{RM}} \omega^{(0)}_{\text{RM}} \pi_{\text{RM}}\right]}$$

(3.11)
so that
\[ H^{(0)}_{RM} = \hbar c \sum_{m \in \mathbb{Z}} (a^{(0)}_{RM})^* \omega_{RM}^{(0)} \cdot a^{(0)}_{RM}. \]  
(3.12)

From (3.11) we define a Fock space \( \mathcal{H}^{(0)}_{RM} \) with Fock vacuum \( \Omega^{(0)}_{RM} \) annihilated by (3.12). The Fock space can be presented as the Hilbert space \( \mathcal{H}^{(0)}_{RM} = \mathbb{L}^2(\mathbb{R}^M, d\nu^{(0)}_{RM}) \) where \( \nu^{(0)}_{RM} \) is a Gaussian measure with covariance \( c_{RM} = \frac{1}{2} \hbar c (\omega_{RM}^{(0)})^{-1} \) with vacuum vector \( \Omega_{RM} = 1 \). Hence, we have constructed explicitly a family of OS data \( (\mathcal{H}^{(0)}_{RM}, H^{(0)}_{RM}, \Omega^{(0)}_{RM}) \), which certainly is not a fixed point family.

We sidestep the introduction of a temporal cut-off \( T \) and its corresponding temporal renormalisation and directly construct the Wiener measure \( \mu^{(0)}_{RM} \) on the history spaces \( \Gamma^{(0)}_{RM} \) of fields \( \Phi_{RM} \) corresponding to the OS data constructed above. The construction is entirely identical to the continuum calculation, hence we know that the Wiener measure \( \mu^{(0)}_{RM} \) is described by a Gaussian measure with the covariance \( C^{(0)}_{RM} \) of the operator
\[ -\frac{1}{c^2} \partial_t^2 + |\omega_{RM}|^2. \]  
(3.13)

**Remark.** For completeness sake we sketch how one would proceed if one would also discretise time. The reader not interested in that aspect can skip this remark. If we also discretise time then we would also replace (3.13) by
\[ -\Delta_{TN} + \omega^2_{RM} \]  
(3.14)
where \( \Delta_{TN} \) is a one-dimensional lattice Laplacian defined on \( \mathbb{Z}_N \), e.g.
\[ (\Delta_{TN} F_{TN})(n) = \frac{1}{\delta_{TN}^2} [F_{TN}(n + 1) + F_{TN}(n - 1) - 2F_{TN}(n)]; \quad \delta_{TN} = \frac{T}{N} \]  
(3.15)
where we have introduced a time IR cut-off \( T \) and a time resolution \( N \). In that case we would need coarse graining maps \( I_{TN, RM} \).

The isometric injections are respectively given by
\[ \hat{I}_{RM} : L_T \otimes L_{RM} \to L_T \otimes L_R, \quad I_{TN,RM} : L_{TN} \otimes L_{RM} \to L_T \otimes L_R \]  
(3.16)
where \( L_T = \mathbb{L}^2([0,T], dt) \) and \( L_{TN} \) is the space of square summable finite sequences in \( N \) points with measure \( \delta_{TN} = T/N \), i.e.
\[ \|F_{TN}\|^2_{TN} = \delta_{TN} \sum_{n \in \mathbb{Z}_N} |F_{TN}(n)|^2. \]  
(3.17)
We will take them to be of direct product form
\[ \hat{I}_{RM} = 1_{L_T} \otimes I_{RM}, \quad I_{TN,RM} = I_{TN} \otimes I_{RM} \]  
(3.18)
where \( I_{TN} \) is constructed just as \( I_{RM} \) in (3.1) with \( D = 1 \) and the substitutions \( R \to T, M \to N \). The corresponding evaluation maps are
\[ \hat{E}_{RM} := 1_{L_T} \otimes E_{RM} : L_T \otimes L_R \to L_T \otimes L_{RM}, \quad E_{TN,RM} := E_{TN} \otimes I_{RM} : L_T \otimes L_R \to L_{TN} \otimes L_{RM} \]  
(3.19)
and the coarse graining maps are
\[ I_{RM\rightarrow 2M} = E_{2M} \circ I_{RM} = 1_L \otimes I_{RM}, \quad I_{TN\rightarrow 2N, RM\rightarrow 2M} = E_{2N, 2RM} \circ I_{TN, RM}. \] (3.20)

Fix pointing the corresponding measure family \( \mu_{RM} \) and \( \mu_{TN, RM} \) is equivalent to fix pointing the corresponding covariances which produces the renormalisation sequences
\[ C_{RM}^{(n+1)} = I_{RM\rightarrow 2M} \circ C_{RM}^{(n)} \circ I_{RM\rightarrow 2M}, \quad C_{TN, RM}^{(n+1)} = I_{TN\rightarrow 2N, RM\rightarrow 2M} \circ C_{TN, 2RM}^{(n)} \circ I_{TN\rightarrow 2N, RM\rightarrow 2M} \] (3.21)
which is solved by
\[ C^*_{RM} = I_{RM}^\dagger \circ C^* \circ I_{RM}, \quad C^*_{TN, RM} = I_{TN, RM}^\dagger \circ C^* \circ I_{TN, RM} \] (3.22)
for some covariance \( C^* \) on \( LTR \) for instance
\[ C^* = \left( -\partial^2_f / \epsilon^2 + \omega^2 \right)^{-1}. \] (3.23)

In our companion papers [2, 3] we show that the computations that we perform below for free quantum fields in one spatial dimension can be extended to any such dimensions. Since, as we will also see below, for free quantum fields additional temporal renormalisation can be deduced from the purely spatial renormalisation of a theory in one more spatial dimension, it follows that we also captured the temporal renormalisation scheme which thus leads to the fixed point continuum covariance (3.23).

### 3.1. Path integral induced Hamiltonian renormalisation

We begin with the path integral induced renormalisation flow.

#### 3.1.1. Step 1: computing the path integral flow.

Following the general programme, the first step will be to calculate the flow and the fixed points of the measure family \( \mu_{RM}^{(0)} \). To that end we consider the maps
\[ I_{RM\rightarrow 2M} := E_{2M} \circ I_{RM} : L_{RM} \rightarrow L_{2RM}, \] (3.24)
which have the property \( I_{RM} \circ I_{RM\rightarrow 2M} = I_{RM} \) as we checked in an earlier section. As a consequence of this property and the isometry of the maps \( I_{RM}; (3.24) \) is an isometric injection
\[
\langle I_{RM\rightarrow 2M} \cdot f_{RM} \cdot I_{RM\rightarrow 2M} \cdot \delta_{RM} \rangle_{L_{RM}} = \langle I_{RM} \circ I_{RM\rightarrow 2M} \cdot f_{RM} \rangle_{L_{RM}} \cdot \langle I_{RM\rightarrow 2M} \cdot \delta_{RM} \rangle_{L_{2RM}} = \langle I_{RM} \cdot f_{RM} \cdot I_{RM} \cdot \delta_{RM} \rangle_{L_{2RM}} = \langle f_{RM} \cdot \delta_{RM} \rangle_{L_{2RM}}.
\] (3.25)

Explicitly for \( m \in \mathbb{Z}_{2M}^D \)
\[ [I_{RM\rightarrow 2M} \cdot f_{RM}](m) = \sum_{m' \in \mathbb{Z}_{2M}^D} \chi_{m' \rightarrow m} \cdot f_{RM}(m') = f_{RM}([m/2]) \] (3.26)
where \( [m/2]^a := [m^a/2], \ a = 1, ..., D \) denotes the component wise Gauss bracket.

The path integral flow is defined by
\[ \mu_{RM}^{(n+1)}(e^{i\phi_{RM}} | f_{RM}) := \mu_{RM}^{(n)}(e^{i\phi_{RM}} | f_{RM}) \] (3.27)
and it follows immediately that the flow generates a family of Gaussian measures with covariances \( C_{RM}^{(n)} \), since the initial family is such. Namely, we find
\[ C_{RM}^{(n+1)} = (1_L \otimes I_{RM\rightarrow 2M})^\dagger C_{RM}^{(n)} 1_L \otimes I_{RM\rightarrow 2M} \] (3.28)
where the notation is to indicate, that no temporal renormalisation takes place.

In the continuum the kernel of the covariance is defined as

\[
(F_{R}, C_{R} \cdot F_{R})_{L^{2} \otimes L_{a}} =: \int_{R} ds \int_{R} ds' \int_{[0,R]^{D}} d^{D}x \int_{[0,R]^{D}} d^{D}y \, F_{R}(s,x) \, C_{R}(s,x,(s',y)) \, F_{R}(s',y).
\]

(3.29)

It follows

\[
\langle 1_{L_{z}} \otimes I_{RM} \cdot F_{RM}, C_{R} \cdot 1_{L_{z}} \otimes I_{RM} \cdot F_{RM} \rangle_{L^{2} \otimes L_{a}} = \langle F_{RM}, [I_{L_{z}} \otimes I_{RM}]^{\dagger} C_{R} [I_{L_{z}} \otimes I_{RM}] F_{RM} \rangle_{L^{2} \otimes L_{a}} =: \langle F_{RM}, C_{RM} F_{RM} \rangle_{L^{2} \otimes L_{a}},
\]

(3.30)

which shows that

\[
C_{RM}(s,m),(s',m') = \epsilon^{-2D}_{RM} \langle \chi_{m_{RM}} \cdot C_{R}(s,\cdot),(s',\cdot) \chi_{m'_{RM}} \rangle_{L_{a}}.
\]

(3.31)

Note that the continuum kernel family is automatically a fixed point of (3.28) due to

\[ I_{RM} = I_{RM}^{2} \circ I_{RM} \rightarrow 2M \]. Expression (3.31) tends to \( C_{R}(s,m_{RM}),(s',m'_{RM}) \) as \( M \rightarrow \infty \).

Explicitly, we have in terms of the kernel of the covariance for the flow of the discretised covariance

\[
(F_{RM}, C_{RM}^{(n+1)} F_{RM})_{L^{2} \otimes L_{a,n}} = \epsilon^{2D}_{RM} \sum_{m_{1},m_{2} \in \mathbb{Z}_{2M}^{D}} \int ds \int ds' \, F_{RM}(s,m_{1})F_{RM}(s',m_{2})C_{RM}^{(n)}(s,m_{1},(s',m_{2}))
\]

\[
\times (1_{L_{z}} \otimes I_{RM} \rightarrow 2M \cdot F_{RM})(s,m_{1})(1_{L_{z}} \otimes I_{RM} \rightarrow 2M \cdot F_{RM})(s',m_{2})C_{RM}^{(n)}(s,m_{1},(s',m_{2}))
\]

\[
= \frac{1}{2^{2D}} \epsilon^{2D}_{RM} \sum_{m_{1},m_{2} \in \mathbb{Z}_{2M}^{D}} \int ds \int ds' \, F_{RM}|m_{1}/2||F_{RM}|m_{2}/2|C_{RM}^{(n)}(s,m_{1},(s',m_{2}))
\]

\[
\times F_{RM}(s,m_{1})F_{RM}(s',m_{2}) \sum_{|m_{1}/2|=m_{1}',|m_{2}/2|=m_{2}'} C_{RM}^{(n)}(s,m_{1},(s',m_{2}'))
\]

(3.32)

from which we read off

\[
C_{RM}^{(n+1)}((s,m_{1}),(s',m_{2}')) = 2^{-2D} \sum_{|m_{1}/2|=m_{1}',|m_{2}/2|=m_{2}'} C_{RM}^{(n)}(s,m_{1},(s',m_{2}'))
\]

\[
= 2^{-2D} \sum_{\delta_{1},\delta_{2} \in \{0,1\}^{D}} C_{RM}^{(n)}(s,2m_{1}' + \delta_{1},(s',2m_{2}' + \delta_{2})).
\]

(3.33)

A simplification can be achieved by making use of the translation invariance of the (discrete) Laplacian and thus the corresponding covariances \( C_{RM}((s,m),(s',m')) = C_{RM}(s-s',m-m') \), a property which is preserved by inspection under the block spin transformation (3.33). This suggests to use Fourier transform techniques.

Recall that \( L_{R} \) is equipped with the orthonormal basis \( R^{-D/2} e^{ik_{R} \cdot x}, n \in \mathbb{Z}^{D}, x \in [0,R]^{D} \) where \( k_{R} = \frac{2\pi}{R} \). If we restrict \( x \) to the lattice points \( x = m_{RM}, m \in \mathbb{Z}^{D}_{M} \) then
\[ e^{i k_R m} = e^{i k_R m'}; \quad k_M = \frac{2\pi}{M} \] and we may restrict \( n \) to \( \mathbb{Z}_M \) as well. Indeed, we may define Fourier transform and its inverse on \( L_{RM} \) by \[
abla \text{f}_{RM}(m) := \sum_{n \in \mathbb{Z}_M} \text{f}_{RM}(n) e^{i k_R n m}, \quad \text{\hat{f}}_{RM}(n) := M^{-D} \sum_{m \in \mathbb{Z}_M} \text{f}_{RM}(m) e^{-i k_R n m}. \tag{3.34}
\]

The Fourier transform has the advantage that it diagonalises the Laplacian \( \left[\Delta_{RM} e^{i k_R n m}\right](m) = -\Delta_{RM}(nk_M) e^{i k_R n m} \) and if \( C_{RM}^{(n)} = G(-\partial^2 + \Delta_{RM}) \) then we have
\[
[G \cdot \text{F}_{RM}](s, m) = e^{i k_R n m} \sum_{n' \in \mathbb{Z}_M} \int ds' G(s - s', m - m') \text{f}_{RM}(m')
\]
\[
= \sum_{m' \in \mathbb{Z}_M} \int ds' \sum_{n' \in \mathbb{Z}_M} \frac{dk_0}{2\pi} \hat{F}_{RM}(k_0, n) G(k_0^2 - \Delta_{RM}(nk_M), p^2) e^{i (k_0(s - s') + k_R n (m - m'))}
\]
\[
= \sum_{m' \in \mathbb{Z}_M} \int ds' F_{RM}(s', m') [M^{-D} \sum_{n \in \mathbb{Z}_M} e^{i (k_0(s - s') + k_R n (m - m'))} G(k_0^2 - \Delta_{RM}(nk_M), p^2)]
\tag{3.35}
\]

where
\[
C_{RM}(s - s', m - m') = \sum_{n \in \mathbb{Z}_M} \int [dk_0 e^{i (k_0(s - s') + k_R n (m - m'))} \hat{C}_{RM}(k_0, n)
\]
\[
\hat{C}_{RM}(k_0, n) = R^{-D} G(k_0^2 - \Delta_{RM}(nk_M), p^2)
\tag{3.36}
\]

for the discretised family.

Since for general \( \omega_R \) it is explicitly only possible to study the flow of the covariance in terms of its Fourier transform we translate (3.33) in terms of the Fourier transform
\[
C_{RM}^{(n+1)}((s, m'), (s', m'_{2})) = \sum_{P \in \mathbb{Z}_M^D} \int \frac{dk_0}{2\pi} e^{i (k_0(s - s') + k_R p' (m - m'))} \hat{C}_{RM}^{(n+1)}(k_0, l')
\]
\[
= 2^{-2D} \sum_{P \in \mathbb{Z}_M^D} \int \frac{dk_0}{2\pi} \hat{C}_{RM}^{(n)}(k_0, l') \sum_{\delta_1, \delta_2 \in \{0,1\}^P} e^{i (k_0(s - s') + k_R p' (2(m'_{1} - m'_{2}) + \delta_1 - \delta_2))}
\]
\[
= 2^{-2D} \sum_{P \in \mathbb{Z}_M^D} \int \frac{dk_0}{2\pi} \sum_{\delta_1, \delta_2 \in \{0,1\}^P} \hat{C}_{RM}^{(n)}(k_0, l' + \delta_3 M) e^{i (k_0(s - s') + k_R p' (l' + \delta_3 M) (2(m'_{1} - m'_{2}) + \delta_1 - \delta_2))}
\]
\[
= 2^{-2D} \sum_{P \in \mathbb{Z}_M^D} \hat{e}^{i k_R p' (m'_{1} - m'_{2})} \int \frac{dk_0}{2\pi} \sum_{\delta_1, \delta_2 \in \{0,1\}^P} \hat{C}_{RM}^{(n)}(l' + \delta_3 M) e^{i (k_0(s - s') + k_R p' (l' + \delta_3 M) (\delta_1 - \delta_2))}
\tag{3.37}
\]

where
\[
\hat{C}_{RM}^{(n+1)}(k_0, l') = 2^{-2D} \sum_{\delta_1, \delta_2 \in \{0,1\}^P} \hat{C}_{RM}^{(n)}(k_0, l' + \delta_3 M) e^{i k_R p' (l' + \delta_3 M) (\delta_1 - \delta_2)}.
\tag{3.38}
\]

We will now carry out the details of this procedure for illustrative purposes for the case \( D = 1 \) and the Poincaré invariant choice
\[
\omega_R = -\Delta_R + p^2.
\tag{3.39}
\]

More general models in all dimensions will be discussed in our companion papers. For \( D = 1 \) (3.38) becomes with \( l' \in \mathbb{Z}_M \)
\[ \dot{C}_{RM}^{(n+1)}(k_0, l') = \frac{1}{2} \sum_{\delta_i \in \{0,1\}} \dot{C}_{RM}^{(n)}(k_0, l' + \delta_i M)[1 + \cos(k_{2M}(l' + \delta_i M))] \]
\[ = \frac{1}{2}(\dot{C}_{RM}^{(n)}(k_0, l')[1 + \cos(k_{2M}l')] + \dot{C}_{RM}^{(n)}(k_0, l' + M)[1 - \cos(k_{2M}l')]. \] (3.40)

We start the flow with \( \dot{C}_{RM}^{(0)}(k_0, l') = \dot{C}_{RM}(k_0, l) \) where \( C_{RM}(k_0, l') \) corresponds to the naive discretisation of the Laplacian (3.9). Thus from (3.36) with \( l \in \mathbb{Z}_M \)
\[ \dot{C}_{RM}^{(0)}(k_0, l) = R^{-1} \frac{\hbar c}{2} \left[ \frac{1}{2\epsilon_{RM}^2 [1 - \cos(k_{2M}l)] + k_0^2 + p^2} \right]. \] (3.41)

It is equivalent to study the flow of \( \dot{C}_{RM}(l) := 2R \dot{C}_{RM}(l)/\hbar c \) and it is convenient to introduce the abbreviations \( t := k_{ML}, q := \sqrt{k_0^2 + p^2 \epsilon_{RM}} \). Hence
\[ \dot{C}_{RM}^{(0)}(l) = \frac{\epsilon_{RM}^2}{2[1 - \cos(t)] + q^2}. \] (3.42)

For reasons that will become transparent in a moment we rewrite (3.42) as follows: let
\[ a_0(q) := 1 + q^2/2, \quad b_0(q) := q^3/2, \quad c_0(q) := 0 \] (3.43)
then trivially
\[ \dot{C}_{RM}^{(0)}(l) = \frac{\epsilon_{RM}^2}{q^2} \frac{b_0(q) + c_0(q) \cos(t)}{a_0(q) - \cos(t)}. \] (3.44)

The purpose of doing this trivial rewriting is that it turns out that the parametrisation by three functions \( a_n, b_n, c_n \) of \( q \) in the Ansatz
\[ \dot{C}_{RM}^{(n)}(l) = \frac{\epsilon_{RM}^2}{q^2} \frac{b_n(q) + c_n(q) \cos(t)}{a_n(q) - \cos(t)} \] (3.45)
is invariant under the renormalisation flow. Namely by (3.40) (note \( t = k_{ML} \rightarrow k_{2M}l = t/2, q = \sqrt{k_0^2 + p^2 \epsilon_{RM}} \rightarrow \sqrt{k_0^2 + p^2 \epsilon_{RM}} = q/2 \) and \( \cos(k_{2M}(l + M)) = -\cos(t/2) \))
\[ \dot{C}_{RM}^{(n+1)}(l) = \frac{\epsilon_{RM}^2}{q^2} \frac{b_{n+1}(q) + c_{n+1}(q) \cos(t)}{a_{n+1}(q) - \cos(t)} \]
\[ = \frac{1}{2} \left[ \dot{C}_{RM}^{(n)}(l)[1 + \cos(k_{2M}l)] + \dot{C}_{RM}^{(n)}(l + M)[1 - \cos(k_{2M}l)] \right] \]
\[ = \frac{\epsilon_{RM}^2}{2q^2} \left[ \frac{b_{n}(q/2) + c_{n}(q/2) \cos(t/2)}{a_{n}(q/2) - \cos(t/2)} \right] \frac{1 + \cos(t/2)}{\left[ a_{n}(q/2) - \cos(t/2) \right] [1 + \cos(t/2)]} \]
\[ + \frac{\epsilon_{RM}^2}{q^2} \left[ a_{n}(q/2)^2 - \cos^2(t/2) \right] \frac{\left[ b_{n}(q/2) + c_{n}(q/2) \cos(t/2) \right] \left[ a_{n}(q/2) - \cos(t/2) \right]}{\left[ a_{n}(q/2) - \cos(t/2) \right] [1 + \cos(t/2)]} \]
\[ = \frac{\epsilon_{RM}^2}{q^2} \left[ a_{n}(q/2)^2 - \frac{1}{2} \left[ 1 + \cos(t/2) \right] \right] \frac{\left[ 2a_{n}(q/2)b_{n}(q/2) + 2 \cos^2(t/2)b_{n}(q/2) + c_{n}(q/2) + a_{n}(q/2)c_{n}(q/2) \right]}{\left[ a_{n}(q/2) - \cos(t/2) \right] [1 + \cos(t/2)]} \]
\[ = \frac{\epsilon_{RM}^2}{q^2} \left[ a_{n}(q/2)^2 - \frac{1}{2} \left[ 1 + \cos(t/2) \right] \right] \frac{2a_{n}b_{n} + b_{n} + c_{n} + a_{n}c_{n})q/(2) + 2[b_{n} + c_{n} + a_{n}c_{n}](q/2) \cos(t)}{\left[ a_{n}(q/2) - \cos(t/2) \right] [1 + \cos(t/2)]}. \] (3.46)
We deduce the recursion relations

\[ a_{n+1}(q) = 2a_n(q/2)^2 - 1 \]
\[ b_{n+1}(q) = 2[2a_nb_n + b_n + c_n + a_s c_n](q/2) \]
\[ c_{n+1}(q) = 2[b_n + c_n + a_s c_n](q/2). \] (3.47)

The corresponding fixed point equations become coupled functional equations

\[ a_*(q) = 2a_*(q/2)^2 - 1 \]
\[ b_*(q) = 2[2a_* b_* + b_* + c_* + a_s c_*](q/2) \]
\[ c_*(q) = 2[b_* + c_* + a_s c_*](q/2). \]

The easiest of these three equations is the first one, as it involves only one function and we easily recognise the functional equation of the cosine or hyperbolic cosine. Now, \( a_0(q) > 1 \) for \( q > 0 \) and assuming this to be the case also for \( a_n(q) \) we get \( a_{n+1}(q) = 2a_n(q/2)^2 - 1 > 1 \) for \( q > 0 \). It follows \( a_*(q) > 1 \) for \( q > 0 \), so that

\[ a_*(q) = \text{ch}(q). \] (3.48)

Next we observe

\[ d_*(q) := (b_* + c_*)(q) = 4[1 + a_*(q/2)][b_* + c_*](q/2) \] (3.49)

which is a homogeneous linear functional equation as \( a_* \) is already known. If we define \( [b_* + c_*](q) = q^n P(\text{ch}(q)) \), where \( P \) is a polynomial, then we have a chance to satisfy the fixed point equation since \( q^n \) can take the factor of 4 into account and the rhs depends only on \( \text{ch}(q/2) \) as well as the lhs as \( \text{ch}(q) = 2\text{ch}^2(q/2) - 1 \). Let then \( P = \sum_{k=0}^N \chi_k \text{ch}^k(q) \) then we find in terms of \( x = \text{ch}(q/2) \)

\[ 2^n \sum_{k=0}^N \chi_k [2x^2 - 1]^k = 4(x + 1) \sum_{k=0}^N \chi_k x^k = 4\{z_0 + z_N x^{N+1} + \sum_{k=1}^N [z_k + z_{k-1}] x^k\}. \] (3.50)

We may assume that \( z_N \neq 0 \), otherwise decrease the degree of the polynomial. Then we must have \( 2N = N + 1 \) i.e. \( N = 1 \). It follows

\[ 2^n \{z_0 - z_1 + 2z_1 x^2\} = 4\{z_0 + (z_0 + z_1)x + z_1 x^2\}, \] (3.51)

i.e.

\[ n = 1, z_1 = -z_0 \Rightarrow zq(\text{ch}(q) - 1) =: d_*(q) \] (3.52)

where \( z \) is a constant to be determined later.

Finally, we have

\[ c_*(q) = 2(b_* + c_*)(q/2) + 2a_*(q/2)c_*(q/2) = 2d_*(q/2) + 2a_*(q/2)c_*(q/2) \] (3.53)

which is an inhomogeneous linear functional equation as \( a_*, d_* \) are already known. The general solution will therefore be the linear combination of a special solution \( c_1 \) of the inhomogeneous equation and the general solution \( c_2 \) of the corresponding homogeneous equation. Explicitly

\[ 0 = -c_1(q) + zq(\text{ch}(q/2) - 1) + 2\text{ch}(q/2)c_1(q/2) = -[c_1(q) + zq] + \text{ch}(q/2)[2c_1(q/2) + zq]. \] (3.54)
which is solved by $c_1(q) = -zq$. This leaves us with
\[ c_2(q) = 2\text{ch}(q/2)c_2(q/2), \]
which is the functional equation of $c_2(q) = z'\text{sh}(q)$ where again $z'$ is a constant to be determined later.

To see which values $z, z'$ are chosen by the initial functions of the fixed point equation we notice that $d_0(q) = q^3/2$ and assume $\lim_{q \to 0} 2d_n(q)/q^3 = 1$ up to some $n$ then also
\[ \lim_{q \to 0} \frac{2d_{n+1}(q)}{q^3} = \lim_{q \to 0} \frac{8[qch(q/2) + 1]d_n(q/2)}{q^3} = \lim_{q \to 0} \frac{2d_n(q/2)}{(q/2)^3} = 1. \] (3.56)
Thus also $\lim_{q \to 0} 2d_n(q)/q^3 = 1$ whence $z = 1$. Finally, we have $c_0(q) = 0$ hence $\lim_{q \to 0} c_n(q)/q^3$ is regular. We assume this to be the case up to some $n$, i.e. $c_n(q) = O(q^3)$. Then
\[ c_{n+1}(q)/q^3 = 2d_n(q/2)/q^3 + 2a_n(q/2)c_n(q/2)/q^3 \] (3.57)
is also regular at $q = 0$ hence so must be $c_n(q)$. It follows $z' = 1$.

We summarise: the fixed point equation is uniquely solved by
\[ a_\star(q) = \text{ch}(q) \]
\[ b_\star(q) = q\text{ch}(q) - \text{sh}(q) \]
\[ c_\star(q) = \text{sh}(q) - q. \] (3.58)

We now compare this to the known continuum theory. The continuum theory is described by the covariance $C_R = b e^{(-\partial_\star^2 - \Delta_R + p^2)^{-1}}$ or equivalently $C_R = \frac{2k}{b\pi} C_R = R(-\partial_\star^2 - \Delta_R + p^2)^{-1}$ which can now be directly compared to (3.42). The corresponding cylindrical projection at resolution $M$ is
\[ c_{RM}(s, m, (s', m')) = c_{RM}^{-1}((1L_z \otimes I_{RM})^\dagger c_R(1L_z \otimes I_{RM}))(s, m, (s', m')) = c_{RM}^{-1}\int_{m'\epsilon_{RM}} \int_{m\epsilon_{RM}} dx \int_{m'\epsilon_{RM}} dy c_R((s, x), (s', y)) \] (3.59)
where $c_R(s, y)$ is the continuum kernel. To compute it we employ again Fourier transformation and use the fact that the functions $e_{nk}(x) = e^{ikx}/\sqrt{R}$, $k = 2\pi/R$ form an orthonormal basis on $L_R = L_2([0, R], dx)$. Hence
\[ c_R((s, x), (s', y)) = R(-\partial_\star^2 - \Delta_R + p^2)^{-1}\delta_k(s, s')\delta_R(x, y) = \int \frac{dk_0}{2\pi} \sum_{n \in \mathbb{Z}} e_{-nk_0}(y) R(-\partial_\star^2 - \Delta_R + p^2)^{-1} e_{nk_0}(s') e_{nk_0}(x) \] (3.60)
\[ = \int \frac{dk_0}{2\pi} \sum_{n \in \mathbb{Z}} e^{i(k_0(s-s') + k_0(ny-y))} (nk_0)^2 + k_0^2 + p^2. \]
It follows
\[ c_{RM}(s, m), (s', m') \]
\[ = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} \frac{e^{ik_0(s-s')}}{(nk)^2 + k_0^2 + p^2} \left[ \int_{\mathbb{R}} e^{ik_0x} dx \right] \left[ \int_{\mathbb{R}} e^{ik_0x'} dx' \right] dy e^{ik_0y} \]
\[ = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} \frac{e^{ik_0(s-s')}}{(nk)^2 + k_0^2 + p^2} \left[ \epsilon_{RM}\delta_{n,0} + \frac{1 - \delta_{n,0}}{ik_0} (e^{ik_0(m+1)x'} - e^{ik_0mx'}) \right] \]
\[ \times \left[ \epsilon_{RM}\delta_{n,0} - \frac{1 - \delta_{n,0}}{ik_0} (e^{-ik_0(m+1)x'} - e^{-ik_0mx'}) \right] \]
\[ = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{ik_0(s-s')} \frac{1}{(nk)^2 + k_0^2 + p^2} \left[ \delta_{n,0} + 2 \frac{1 - \delta_{n,0}}{(k_0)^2} (1 - \cos(k_0m)) \right]. \] (3.61)

To compare this expression to \( \hat{c}_{RM}(k_0, l) \), \( l \in \mathbb{Z}_M \) we write \( n = l + NM, N \in \mathbb{Z} \) and split the sum
\[ c_{RM}(s, m), (s', m') \]
\[ = \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} \frac{dk_0}{2\pi} e^{ik_0(s-s')} \frac{1}{(l + NM)(nk)^2 + k_0^2 + p^2} \frac{2(1 - \cos((l + NM)k_0m))}{(l + NM)(k_0)^2} \] (3.62)
where we declare the last fraction to equal unity at \( l = N = 0 \). Comparing with the first line of (3.37) we see that the sum involved in (3.37) coincides with the definition of \( \hat{c}_{RM}(k_0, l) \).

We now carry out the sum over \( N \) by employing the Poisson resummation formula
\[ \sum_{N \in \mathbb{Z}} f(N) = \int_{\mathbb{R}} e^{-2\pi Nx} f(x) \] (3.63)
with
\[ f(x) = \frac{1}{(l + xM)(k_0)^2 + k_0^2 + p^2} \frac{2(1 - \cos((l + xM)k_0m))}{(l + xM)(k_0)^2} \] (3.64)
to which the Poisson resummation may be applied as \( f \) is smooth and decays at infinity as \( 1/x^4 \).

We find with \( q = \sqrt{k_0^2 + p^2} \)
\[ \hat{c}_{RM}(k_0, l) = \sum_{N \in \mathbb{Z}} \frac{1}{(l + NM)(k_0)^2 + k_0^2 + p^2} \frac{2(1 - \cos((l + NM)k_0m))}{(l + NM)(k_0)^2} \]
\[ = \epsilon_{RM} \sum_{N \in \mathbb{Z}} \int dx e^{-2\pi Nx} \frac{1}{(l + xM)(k_0)^2 + q^2} \frac{2(1 - \cos((l + xM)k_0m))}{(l + xM)(k_0)^2} \]
\[ = \epsilon_{RM} \sum_{N \in \mathbb{Z}} \int dx e^{-2\pi Nx} \frac{1}{(k_0l + 2\pi x)^2 + q^2} \frac{2(1 - \cos(k_0l + 2\pi x))}{(k_0l + 2\pi x)^2} \]
\[ = \epsilon_{RM} \sum_{N \in \mathbb{Z}} \int dx \frac{1}{2\pi} e^{-i\pi N} \frac{1}{(k_0l + x)^2 + q^2} \frac{2(1 - \cos(k_0l + x))}{(k_0l + x)^2} \]
\[ = \epsilon_{RM} \sum_{N \in \mathbb{Z}} e^{ik_0Nx} \int dx \frac{2e^{-i\pi N}(1 - \cos(x))}{2\pi x^2 + q^2}. \] (3.65)
We have
\[
\frac{2(1 - \cos(x))e^{-iNx}}{x^2} = \frac{e^{-iNx} - e^{-i(N-1)x}}{x^2} + \frac{e^{-iNx} - e^{-i(N+1)x}}{x^2}. \quad (3.66)
\]
For any \(N\) (3.66) is holomorphic in the entire complex plane and for \(N \geq 1\) decays on the lower infinite half-circle and for \(N \leq -1\) decays on the upper infinite half-circle. It follows that the integrand is holomorphic everywhere in the whole complex plane except at \(x = \pm iq\) and the contour can be closed as described for \(N \neq 0\). Thus we find by the residue theorem
\[
\int \frac{2e^{-iNx}(1 - \cos(x))}{2\pi x^2(x^2 + q^2)} \, dx = \begin{cases} 
\frac{-2\pi}{2\pi} \frac{e^{-iNx}(1-\cos(x))}{x^2(x-iq)} & \text{for } N \geq 1 \\
\frac{2\pi}{2\pi} \frac{e^{-iNx}(1-\cos(x))}{x(x+iq)} & \text{for } N \leq -1 
\end{cases}
\]
\[
= \frac{e^{-N|q|}(\cosh(q) - 1)}{q^3}. \quad (3.67)
\]
For \(N = 0\) the first term in (3.66) decays on the upper while the second decays on the lower infinite half-circle. However, the two terms are not separately holomorphic at \(x = 0\), only their sum is. We thus write the integral as a principal value integral \(\lim_{\delta \to 0}\int_{-\delta}^{\delta} \, dx\left(\frac{1 - e^{-ix}}{x} \right)\) for which we leave out the interval \([-\delta, \delta]\) and then close the contour for the first/second term with a small half-circle of radius \(\delta\) in the upper/lower complex plane, subtract that added contribution and apply the residue theorem. We find
\[
\int \frac{2(1 - \cos(x))}{2\pi x^2(x^2 + q^2)^2} \, dx = \frac{-2\pi}{2\pi} \frac{(1 - e^{-ix})}{x^2(x-iq)} - \lim_{\delta \to 0} \int_{x=\delta e^{i\phi}, \phi \in [-\pi,0]} \frac{dx}{2\pi x} \frac{1 - e^{-ix}}{x(x^2 + q^2)}
\]
\[
+ \frac{2\pi}{2\pi} \frac{(1 - e^{ix})}{x(x+iq)} + \lim_{\delta \to 0} \int_{x=\delta e^{i\phi}, \phi \in [0,\pi]} \frac{dx}{2\pi x} \frac{1 - e^{ix}}{x(x^2 + q^2)}
\]
\[
= q + e^{-q} - 1 \quad (3.68)
\]
It remains to compute the geometric sum in (3.65) with \(k_M = t\)
\[
\frac{\hat{\epsilon}_R(\ell)}{\hat{\epsilon}_R} = \frac{q + e^{-q} - 1}{q^3} + \frac{\cosh(q) - 1}{q^3} \left(-2 + \sum_{N=0}^{\infty} \left\{e^{-N|q+i|} + e^{-N|q-i|}\right\}\right)
\]
\[
= \frac{q + e^{-q} - 1 - \cosh(q) + 1}{q^3} + \frac{\cosh(q) - 1}{q^3} \left(-1 + \frac{1}{1 - e^{-|q+i|}} + \frac{1}{1 - e^{-|q-i|}}\right)
\]
\[
= \frac{q - \sinh(q)}{q^3} + \frac{\cosh(q) - 1}{q^3} \left[2 - 2e^{-q} \cos(t) - \left[1 + e^{-2q} - 2e^{-q} \cos(t)\right]\right]
\]
\[
= \frac{q - \sinh(q)}{q^3} + \frac{\cosh(q) - 1}{q^3} \left[\sinh(q) \cosh(q) - \cos(t)\right]
\]
\[
= \frac{1}{q^3} \frac{1}{\cosh(q) - \cos(t)} \left\{[\cosh(q) - 1] \sinh(q) + [q - \sinh(q)] \cosh(q) - [q - \sinh(q)] \cos(t)\right\}
\]
\[
= \frac{1}{q^3} \frac{1}{\cosh(q) - \cos(t)} \left\{[q \cosh(q) - [q - \cosh(q)] + [\sinh(q) - q] \cos(t)\right\}. \quad (3.69)
\]
Comparing with (3.45) and (3.58) we see that we obtain perfect match! The fixed point equations of the naively discretised covariance of the history field measure have found the precise cylindrical projections of its continuum covariance \([-\partial^2 - \Delta_R + p^2]^{-1}\). It is therefore clear that the fixed point of the path integral induced Hamiltonian renormalisation precisely delivers the continuum OS data via OS reconstruction that we started from and that we artificially discretised. Moreover, it is easy to see that the continuum limit \(\lim_{M \to \infty} c_{RM}(l) = [k_0^2 + m^2 + (lk_R)^2]^{-1}\) coincides with the continuum covariance.

3.1.2. Step 2: computing the path integral induced Hamiltonians. We carry out the explicit OS reconstruction of the measures \(\mu_R^{(n)}\). To do this recall that these are determined by the covariances of their Fourier transform which up to a factor follow the flow \((l \in \mathbb{Z}_M)\), see (3.40)

\[
\hat{c}_{RM}^{(n+1)}(k_0, l) = \frac{1}{2} \left[ c_{R2M}^{(n)}(k_0, l)[1 + \cos(t/2)] + \hat{c}_{RM}^{(n)}(k_0, l + M)[1 - \cos(t/2)] \right],
\]

\[
\hat{c}_{RM}^{(n)}(k_0, l) = \frac{c_{RM}^{(n)}}{2[1 - \cos(t)] + q^2}. \quad t = k_M l, \quad q = \sqrt{p^2 + k_0^2}.
\]

(3.70)

We conclude that while \(c_{RM}^{(n)}(k_0, l)\) displays only one simple pole with respect to \(q^2\), the number of poles gets doubled at each renormalisation step. For instance

\[
\hat{c}_{RM}^{(1)}(k_0, l) = \frac{c_{RM}^{(0)}}{8} \left[ \frac{1 + \cos(t/2)}{2[1 - \cos(t/2)] + q^2/4} + \frac{1 - \cos(t/2)}{2[1 + \cos(t/2)] + q^2/4} \right].
\]

(3.71)

It follows that \(c_{RM}^{(n)}(k_0, l)\) displays \(2^n\) distinct simple poles in \(q^2\). Thus, in the parametrisation

\[
\hat{c}_{RM}^{(n)}(k_0, l) = \frac{c_{RM}^{(n)}}{q^2} b_n(q) \cos(t) + \frac{a_n(q) - \cos(t)}{q^2}.
\]

(3.72)

\(a_n(q)\) must be a polynomial of order \(2^n\) in \(q^2\) whose poles can in principle be read off from the flow equation (3.70). These poles are certain mutually distinct functions of \(t\) as one easily sees from the inductive definition (3.70). The flow equation (3.70) for the covariance at resolution \(M\) is a superposition of covariances at resolution \(2M\) with merely \(t\) dependent, positive coefficients \(1 \pm \cos(t/2)\) (i.e. they do not depend on \(q\)). Since \(c_{RM}^{(0)}(k_0, l)\) has its \(q\) dependence only in the pole and a single positive coefficient, this feature is preserved for the entire flow.

We may therefore display an alternative parametrisation of (3.72) as follows: let \(-\hat{\lambda}_{RM}^{(n)}(t)^2\) with \(N = -2^{n-1} + 1, -2^{n-1} + 2, \ldots, 0, 1, \ldots, 2^{n-1}\) denote the poles of \(c_{RM}^{(n)}(k_0, l)\) for \(n > 0\) with respect to \(q^2\) (from the induction (3.70) it follows that the poles are strictly non-positive) and \(\hat{g}_{RM}^{(n)}(t)c_{RM}^{(n)}\) the corresponding, positive coefficient functions. Then for \(n > 0\)

\[
\hat{c}_{RM}^{(n)}(k_0, l) = \sum_{N = -2^{n-1} + 1}^{2^{n-1}} \frac{\hat{g}_{RM}^{(n)}(t)c_{RM}^{(n)}}{q^2 + \hat{\lambda}_{RM}^{(n)}(t)^2}.
\]

(3.73)

If we trivially extend \(\hat{g}_{RM}^{(n)} \equiv 0\) for \(N > 2^{n-1}, N \leq -2^{n-1}\) we may extend the sum over \(N\) to infinity

\[
\hat{c}_{RM}^{(n)}(k_0, l) = \sum_{N \in \mathbb{Z}} \frac{\hat{g}_{RM}^{(n)}(t)c_{RM}^{(n)}}{q^2 + \hat{\lambda}_{RM}^{(n)}(t)^2}.
\]

(3.74)
which now provides a universal parametrisation for the $c_R^{(n)}(k_0,t)$. The flow is now in terms of the poles and their respective coefficient functions. More and more coefficient functions are switched on from zero to a positive function as the flow number $n$ increases. We even know what the fixed point values of this flow are, if we look at (3.62)

$$[\hat{\Lambda}_{\text{RMN}}^{(n)}(t)]^2 = (t + 2\pi N)^2, \quad \hat{\Lambda}_{\text{RMN}}^{(n)}(t) = 2 \frac{1 - \cos(t)}{(t + 2\pi N)^2}. \quad (3.75)$$

The first question is, what null space of the reflection positive inner product for a covariance of the form (3.74) results. It is convenient to introduce the renormalisation invariant lattice Laplacian

$$(\Delta_{\text{RM}})(m) := f_{\text{RM}}(m + 1) + f_{\text{RM}}(m - 1) - 2 f_{\text{RM}}(m) \quad (3.76)$$
on $L_{\text{RM}}$ which in Fourier space corresponds to multiplication by $2(\cos(k_0 t) - 1) = 2(\cos(t) - 1)$. The functions $\hat{\Lambda}_{\text{RMN}}^{(n)}(t)$, $\hat{\Lambda}_{\text{RMN}}^{(n)}(t)$ can now be considered as the eigenvalues of corresponding operator valued functions of $\Delta_{\text{RM}}$ (using the spectral theorem in the form $t = \pm \arccos(\cos(t))$ for $0 < t < 2\pi$ and noticing that the flow generates a function, which is invariant under $t \rightarrow -t$ so that the sign ambiguity is irrelevant), which we denote by $\Lambda_{\text{RMN}}^{(n)}$, $\Lambda_{\text{RMN}}^{(n)}$, respectively. We also set

$$[\omega_{\text{RMN}}^{(n)}]^2 := \frac{[\Lambda_{\text{RMN}}^{(n)}]^2}{\epsilon_{\text{RM}}} + p^2. \quad (3.77)$$

Then we find for the corresponding reflection positive inner product for functions $F_{\text{RM}}, G_{\text{RM}}$ of positive time support (we drop the label $n$ for a moment)

\[
\langle [e^{i\Phi_{\text{RM}}[F_{\text{RM}}]}]_{\mu_{\text{RM}}}, [e^{i\Phi_{\text{RM}}[G_{\text{RM}}]}]_{\mu_{\text{RM}}} \rangle_{\mu_{\text{RM}}} = \mu_{\text{RM}}(e^{i\Phi_{\text{RM}}[G_{\text{RM}}]} e^{i\Phi_{\text{RM}}[F_{\text{RM}}]}) \quad (3.78)
\]

We now extract representatives of the equivalence classes of the inner product (3.78) corresponding to fields not at a single sharp time zero, but rather a countably infinite set of sharp times. To see how this comes about, we compute, noticing the time support of $G$ and dropping all labels for the sake of the argument

$$\int ds \ e^{-i\omega} G(s,m) = \sum_l \int \frac{dk_0}{2\pi} \int_0^\infty ds \ e^{i(k_0 + k)l} e^{-i\omega(l)} \hat{G}(k_0,l)$$

$$= \sum_l \int \frac{dk_0}{2\pi} \frac{1}{\omega(l) - ik_0} e^{ik_0l} \hat{G}(k_0,l) = - \sum_l e^{ik_0l} \hat{G}(k_0,l) = -i \hat{G}(k_0,l) \quad (3.79)$$
by the residue theorem. Here we used that $G(s) = 0$ for $s < 0$ implies that its Fourier transform $\hat{G}(k_0)$ is holomorphic on the lower complex half plane with at most polynomial growth at infinity. Hence the residue theorem applies. It follows that the value of (3.79) remains unchanged if we replace $\hat{G}(k_0, l)$ by $\hat{G}'(k_0, l) = f(k_0, l) \hat{G}(-i\omega(l), l)$ where $h$ is a fixed function holomorphic in the lower half plane such that $h(-i\omega(l), l) = 1$, e.g. $h \equiv 1$.

If we have a finite number of frequencies $\omega_0 > 0$ labelled by $N \in \mathbb{Z}$ then likewise we consider the functions $\hat{G}_N(l) = \hat{G}(-i\omega_0(l), l)$ and can replace $\hat{G}(k_0, l)$ by

$$\hat{G}'(k_0, l) = \sum_N h_N(k_0, l) \hat{G}(l), \quad h_N(-i\omega_0(l), l) = \delta_{N,N'}.$$  

A possible choice is

$$h_N(k_0, l) := \prod_{N' \neq N} \frac{e^{-i\tau k_0} - e^{-\tau \omega_{N'}(l)}}{e^{-\tau \omega_{N}(l)} - e^{-\tau \omega_{N'}(l)}}$$  

where $\tau > 0$ is any fixed positive real number. Equation (3.81) is well defined because the pole values $\omega_N(l)$ are mutually distinct for different $N$ and equal $l$. It is holomorphic everywhere and a polynomial in $e^{-ik_0\tau}$ where the order coincides with the number of different frequencies $\omega_N$ reduced by one, in our case this number is given by $2^n - 1$. Thus it becomes a constant at the lower half circle in the complex plane of infinite radius and the residue theorem applies. We conclude that the function $G'(s, m)$ itself, at the $n$th renormalisation step, has the form

$$G'(s, l) = \sum_{r=0}^{2^n-1} \delta(s - \tau r) g_r(l), \quad g_r \in L_{RM}$$

i.e. they depend on $2^n$ sharp points of time rather than a single one, except for $n = 0$! It follows that $[e^{i\Phi_{RM}[G_{RM}]}]^{(n)}_{\mu_{RM}}$ can be identified with the representative

$$\hat{\mu}_{RM}^{(n)}(e^{i\Phi_{RM}[G_{RM}]} - \hat{\mu}_{RM}^{(n)}(e^{i\Phi_{RM}[G_{RM}]})$$

or in other words

$$e^{i\Phi_{RM}[G_{RM}]} - \hat{\mu}_{RM}^{(n)}(e^{i\Phi_{RM}[G_{RM}]})$$

is a null vector with respect to the reflection positive inner product defined by $\hat{\mu}_{RM}$. The OS Hilbert space $\mathcal{H}_{RM}^{(n)}$ can thus be thought of as the completion of the finite linear span of the $e^{i\Phi_{RM}[G_{RM}]}$ with $G_{RM}'$ of the form (3.82) and $\Omega_{RM}^{(n)} \equiv 1$.

We compute the corresponding Hamiltonian. This amounts to computing the representative of the equivalence class of $e^{i\Phi_{RM}[F_{RM}]}$ for $F_{RM}$ of the form (3.82). We have

$$(T_{\beta} \cdot F_{RM})(s) = F_{RM}(s - \beta) = \sum_r \delta(s - \beta, \tau r) f_{RM}^r = \sum_r \delta(s, \beta + \tau r) f_{RM}^r$$

whence

$$T_{\beta} \cdot F_{RM}(k_0, l) = \sum_r e^{-ik_0(\beta + \tau r)} f_{RM}^r(l).$$

(3.86)
Thus
\[
T_\beta \cdot F_{RM}^\tau (k_0, l) = \sum_N h_{RMN}(k_0, l) T_\beta \cdot F_{RM}^\tau (-i\omega_{RMN}(l), l)
\]
\[
= \sum_N h_{RMN}(k_0, l) \sum_r e^{-i\omega_{RMN}(l)(\beta + \tau')} \hat{f}_{RM}(l).
\] (3.87)

where \( h_{RMN} \) is defined as in (3.82) with \( \omega_N \) replaced by \( \omega_{RMN} \) and we suppressed the renormalisation step label \( n \) for notational convenience. If we decompose
\[
h_{RMN}(k_0, l) =: \sum_r e^{-ir\tau_k} \hat{h}_{RMN}(l)
\] (3.88)

we obtain
\[
T_\beta \cdot F_{RM}^\tau (k_0, l) = \sum_r e^{-ik_0\tau} \sum_{r'} \sum_N h_{RMN}^r(l) e^{-i\omega_{RMN}(l)(\beta + \tau')} \hat{f}_{RM}(l).
\]

Accordingly, the time evolution is described by the matrix
\[
A_{RM}^{\beta, \tau}(\beta, l) := \sum_N h_{RMN}^\tau(l) e^{-i\omega_{RMN}(l)(\beta + \tau')} \tag{3.89}
\]
or in position space by the corresponding matrix valued operator where \( \omega_{RMN}(l) \) is replaced by the corresponding operator. It follows that we can describe the time translation contraction semigroup on the chosen representatives, reintroducing the renormalisation step label, by
\[
e^{-\beta_{RM}^{(a)}(\omega_{RMN}(l)\tau, l)} = \frac{\mu_{RM}^{(a)}(\omega_{RMN}(l)\tau, l)}{\mu_{RM}^{(a)}(\omega_{RMN}(l)\tau, l)} \cdot \Phi_{RMN}^{(a)}(\beta, l) F_{RM} = \sum_{r'} \delta_{\tau'\tau} \hat{h}_{RM}^{(a)}
\] (3.91)

where \( A_{RM}^{(a)}(\beta) \) is the purely spatial matrix valued operator whose Fourier transform is displayed in (3.90). It is instructive to verify the semigroup law
\[
A_{RM}^{(a)}(\beta_1) \cdot A_{RM}^{(a)}(\beta_2) = A_{RM}^{(a)}(\beta_1 + \beta_2), \tag{3.92}
\]

which rests on the van der Monde identity for polynomials of degree \( d = 2^n - 1 \)
\[
p(x) = \sum_{r=0}^d a_r x^r, \quad p(x_r) = p_r, \quad x_0 < x_1 < \ldots < x_d \quad \Rightarrow \quad p(x) = \sum_{r=0}^d \prod_{r' \neq r} \frac{x - x_r'}{x_r - x_r'}.
\] (3.93)

We prove it by applying (3.80) to \( h_{RMN}^\tau(\omega_{RMN}(l)\tau, l) \) in
\[
\sum_{r', \tau'} A_{RM}^{\beta_{RMN}(\omega_{RMN}(l)\tau, l) \tau, l)} A_{RM}^{\beta_{RMN}(\omega_{RMN}(l)\tau, l) \tau, l)} \]
\[
= \sum_{N, N'} h_{RMN}(l) \sum_{\tau'} e^{-i\omega_{RMN}(l)(\beta_1 - \beta_2)} e^{-i\omega_{RMN}(l)\beta_1 - i\omega_{RMN}(l)\beta_2}
\]
\[
= \sum_{N, N'} h_{RMN}(l) \delta_{N, N'} e^{-i\omega_{RMN}(l)(\beta_1 - \beta_2)} = A_{RM}^{\beta_{RMN}(\omega_{RMN}(l)\tau, l) \tau, l)} (\beta_1 + \beta_2)
\]

(3.94)

As \( n \to \infty \) and for fixed finite \( M \), the Hilbert space can thus no longer be thought of as described by a single sharp time zero field but rather by sharp time fields at an exponentially increasing (with \( n \)) number of sharp times. At the fixed point thus, the number of this sharp points of time is actually infinite. How can this be reconciled with the fact that in the continuum the Hilbert space can be described by a single field at sharp time zero? The answer lies in
the continuum limit $M \to \infty$: if we inspect (3.75) then we see that at fixed $l \in \mathbb{Z}_M$ we obtain for the coupling 'constants' $\hat{g}_{RMN}(l) \to \delta_{l,0}$ as $M \to \infty$. At the same time $\omega_{RMN}(l)$ diverges for all $N$ except $N = 0$ and the time contraction for all modes except for $N = 0$ 'freezes'. Thus in the continuum limit, the theory is described by a single dispersion relation and thus the single sharp time zero description that we are used to applies.

The description using fields at more than one sharp time that we have arrived at means that we cannot express the Hamiltonian in terms of a single time zero field and its conjugate momentum. Thus our discussion suggests to introduce instead an infinite number of sharp time zero field species $\phi_{RMN}$ and their conjugate momenta $\pi_{RMN}$, that is, the non-vanishing commutators are

$$[\pi_{RMN}(m), \phi_{RMN'}(m')] = i\hbar \delta_{m,N}\delta_{m,m'}, \ m, m' \in \mathbb{Z}_M.$$  \hfill (3.95)

At finite $n$ of course we only have $N \in \{-2^{n-1} + 1, \ldots, 2^{n-1}\}$, i.e. we have only $d = 2^n$ field species. Accordingly, instead of $L_{RM} = l_2(M)$ we consider $L_{RM} = l_2(M)^d$ as the one particle Hilbert space and the Hamiltonian

$$H_{RM}' := \frac{1}{2} \sum_{(m,N), (m',N')} [\pi_{RMN}(m)D_{RM}((m,N), (m',N'))\pi_{RMN'}(m')]$$

$$+ \phi_{RMN}(m)E_{RM}((m,N), (m',N'))\phi_{RMN'}(m')] =: \frac{1}{2}\left(\{\phi_{RMN}, D_{RM}\pi_{RMN}\} + \{\phi_{RMN}, E_{RM}\phi_{RMN}\}\right)$$  \hfill (3.96)

for certain operators $D_{RM}, E_{RM}$ on $L_{RM}$. Then we claim that it is possible to choose $D_{RM}, E_{RM}$ such that the Wiener measure corresponding to (3.96) reproduces the path integral measure. To see this, we drop all labels for simplicity

$$H = \frac{1}{2}\left(\langle \pi, D\pi \rangle + \langle \phi, E\phi \rangle\right)$$  \hfill (3.97)

where $D, E$ are self-adjoint, positive and symmetric on $L_{RM}$ and in general not commuting. We define annihilators and frequency

$$a = \frac{1}{\sqrt{2}}[\kappa, \phi] - i(\kappa^{-1}, \pi], : H := \langle a^*, \omega' a \rangle$$  \hfill (3.98)

where normal ordering is with respect to $a$. Note that $\kappa, \omega'$ are operators on $L_{RM}$. This leads to the identities

$$\kappa^\dagger \omega' / \kappa = E, \ (\kappa^{-1})^\dagger \omega' \kappa^{-1} = D$$  \hfill (3.99)

which are solved by

$$\kappa = \kappa^\dagger > 0, \ \kappa = \sqrt{E^{1/2}E^{-1/2}D^{-1/2}E^{-1/2}E^{1/2}}, \ \omega' = (\omega')^\dagger > 0, \ \omega' = \kappa D \kappa.$$  \hfill (3.100)

Now a simple computation similar to the one for the continuum in section 2.2 that generalises the choice $\kappa = \sqrt{\omega}$ shows that the Wiener measure corresponding to (3.98) yields

$$\mu(e^{i\phi|f_i)|} = e^{-1/4 \int ds \int ds' \langle f(s), \frac{\omega(s') - \omega(s)}{\kappa} f(s') \rangle}.$$  \hfill (3.101)

We now pick the sharp time zero Weyl elements to be

$$w_{RM}[f_{RM}] := e^{i\sum_N \phi_{RMN}\langle f'_{RMN}\rangle}, \ f_{RM}' = \{f'_{RMN}\} \in L_{RM}^2$$  \hfill (3.102)
and also $W_{RM}[F_{RM}] = \prod_{N} W_{RMN}[F_{RMN}]$. $W_{RMN}[F_{RMN}] = e^{i\Phi_{RMN}[F_{RMN}]}$. Then the corresponding Wiener measure gives

$$\mu_{RM}^{s}(W_{RM}[F_{RM}]) = \exp(-\frac{1}{2} \sum_{N} \int ds \int ds' \langle F_{RM}(s), \frac{e^{-|s-s'|/2}F_{RM}(s')}{}\rangle).$$

(3.103)

We can use our knowledge from the continuum theory to infer that the Hilbert space corresponding to the reflection positive inner product of $\mu_{RM}^{s}$ is labelled by time zero smearing functions $F_{RM}(s) = \delta(s,0)f_{RM}$ and that the Hamiltonian is defined by

$$e^{-\beta H_{RM}}e^{i\Phi_{RM}[f_{RM}]} = \mu_{RM}^{s}(e^{i\Phi_{RM}[f_{RM}]}|f_{RM}^{s}) = e^{i\Phi_{RM}[e^{-\beta f_{RM}}]}. \tag{3.104}$$

To match this to (3.91) we perform a trivial relabelling between $r, r' \in \{0, \ldots, d - 1\}$ and $N, N' \in \{-2^{n-1} + 1, \ldots, 2^{n-1}\}$ in order to write the matrix elements of $A_{RM}$ in the form $A_{RM}(m, N, (m', N'); \beta)$. Then the semigroup property (3.92) implies that there exists a positive self-adjoint generator $\omega_{RM}$ on $L_{RM}$, such that $A_{RM}(\beta) = e^{-\beta \omega_{RM}}$. Next, for

$$F_{RM} = \sum_{r=0}^{d-1} \delta_{r} f_{RM}^{s} =: \sum_{N=-2^{n-1}+1}^{2^{n-1}} \delta_{(N+2^{n-1}+1)r} f_{RMN} \tag{3.105}$$

we find a positive matrix $B_{RM}$ on $L_{RM}$ such that

$$\mu_{RM}^{s}(e^{i\Phi_{RM}[f_{RM}]}|f_{RM}^{s}) = e^{-\beta \omega_{RM}^{s}}(f_{RM}[f_{RM}^{s}])_{RM}. \tag{3.106}$$

If we now compare (3.91), (3.106) and (3.103), (3.104) we see that we obtain perfect match provided that we pick

$$\omega_{RM}^{s} := \omega_{RM}, \quad \kappa_{RM}^{2} := B_{RM}. \tag{3.107}$$

Accordingly, the path integral induced Hamiltonian theory does have an interpretation in terms of sharp zero-time fields, however, at the price of introducing more and more field species at each renormalisation step. These field species are mutually commuting, however, the Hamiltonian couples them to each other according to the matrices $D_{RM}, E_{RM}$ constructed above.

### 3.2. Direct Hamiltonian renormalisation

We now discuss the direct Hamiltonian renormalisation in terms of the single canonical field species $\phi_{RM}$ at sharp time zero.

#### 3.2.1. Step 1: implementing isotropy

As already remarked before, implementing isotropy of

$$J_{RM}^{(n)} e^{i\phi_{RM}[f_{RM}]} \Psi_{RM}^{(n+1)} := e^{i\phi_{RM}[f_{RM}]f_{RM}} \Psi_{RM}^{(n)} \tag{3.108}$$

is equivalent to studying the flow of the family of Hilbert space measures

$$\nu_{RM}^{(n+1)}(e^{i\phi_{RM}[f_{RM}]}) := \nu_{RM}^{(n)}(e^{i\phi_{RM}[f_{RM}]f_{RM}}) \tag{3.109}$$

Again it is clear that the family stays Gaussian if the original family is. Let $\frac{1}{2\omega_{RM}}$ be the covariance of $\nu_{RM}^{(0)}$. We have the basic identity (in the sense of the spectral theorem)
The fixed point sequence is defined by the matrix element equations

\[
\langle \Omega^{(n+1)}_{\text{FRM}}, \Omega^{(n+1)}_{\text{FRM}} | H^{(n+1)}_{\text{RM}} | \Omega^{(n+1)}_{\text{FRM}} \rangle := \langle \phi^{(n+1)}_{\text{RM}} | H^{(n+1)}_{\text{RM}} | \phi^{(n+1)}_{\text{RM}} \rangle.
\]

Let us define

\[
a^{(n)}_{\text{RM}} := \frac{1}{\sqrt{2}} \left[ \omega^{(n)}_{\text{RM}} \cdot \phi_{\text{RM}} - i \sqrt{\omega^{(n)}_{\text{RM}} \cdot \pi_{\text{RM}}} \right].
\]

Then

\[
\lambda^{(n)}_{\text{RM}} (\phi^{(n)}_{\text{RM}} | f_{\text{RM}}) = e^{-\frac{i}{2} (f_{\text{RM}} - (\omega^{(n)}_{\text{RM}})^{-1} f_{\text{RM}})} = \langle \phi^{(n)}_{\text{RM}} | e^{i \phi^{(n)}_{\text{RM}} | f_{\text{RM}}} | \phi^{(n)}_{\text{RM}} \rangle
\]

is the Fock measure labelled by (3.111) and \(\Omega^{(n)}_{\text{RM}}\) is the Fock vacuum annihilated by (3.113). Then

\[
e^{-i (\phi^{(n)}_{\text{RM}} | f_{\text{RM}} \cdot I_{\text{RM}}^{(n)} + f_{\text{RM}} | f_{\text{RM}}) a^{(n)}_{\text{RM}} (m)} = a^{(n)}_{\text{RM}} (m) - i \sqrt{\omega^{(n)}_{\text{RM}} \cdot \pi_{\text{RM}}} \omega^{(n)}_{\text{RM}} (m)
\]

We now prove by induction that

\[
H^{(n)}_{\text{RM}} = \langle \phi^{(n)}_{\text{RM}} | \omega^{(n)}_{\text{RM}} \cdot a^{(n)}_{\text{RM}} \rangle_{\text{RM}}
\]

which is consistent with \(H^{(n)}_{\text{RM}} | \phi^{(n)}_{\text{RM}} = 0\). By construction, (3.116) holds for \(n = 0\) and all \(M\) and we assume it to hold up to \(n\) and all \(M\). Then
\[
\langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{(n)} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{(n)}
\]

\[
= \frac{\hbar^2 K}{2} \sum_{m, m'} \omega_{RM}^{(n)}(m, m') \langle a_{RM}^{(n)} \rangle_{\Omega_{RM}^{(n)}} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{(n)}
\]

\[
\times (\omega_{RM}^{(n)})^{-1/2} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{(n)}
\]

\[
= \frac{\hbar^2 K}{2} \langle J_{RM} - 2M, f_{RM} \rangle_{HRm} H_{RM}^{(n)}
\]

\[
\times (\omega_{RM}^{(n)})^{-1/2} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{(n)}
\]

\[
= \langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n+1)}} \cdot \langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n+1)}} H_{RM}^{(n+1)}
\]

\[
= \langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n+1)}} H_{RM}^{(n+1)}
\]

\[
\tag{3.117}
\]

where we have made use of isometry of both \(I_{RM} \rightarrow 2M\) and \(J_{RM}^{(n)}\). Thus the matrix elements of \(H_{RM}^{(n)}\) are consistent with (3.116). The fixed point Hamiltonian \(H_{RM}^{*}\) is then simply (3.116) with \(\omega_{RM}^{(n)}\) replaced by \(\omega_{RM}^{*}\).

We claim that

\[
H_{RM}^{*} = J_{RM} H_{Rm} J_{RM}^{\dagger}
\]

\[
\tag{3.118}
\]

where \(H_{Rm}\) is the continuum Hamiltonian and \(J_{RM} : \mathcal{H}_{RM} \rightarrow \mathcal{H}_{RM}^{*}\) the isometric embedding of Fock spaces which is granted to exist due to the equivalence of the fixed point family to an inductive limit Hilbert space family. Indeed in our case this is simply given by

\[
J_{RM} \langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} \Omega_{RM}^{(n)} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}}
\]

\[
\tag{3.119}
\]

This follows because the isometry check and \(J_{RM} : J_{RM} = J_{RM}^{*}\) are equivalent to the corresponding statements for \(I_{RM} \rightarrow 2M\) and to the statement \((\omega_{RM}^{*})^{-1} = I_{RM} \omega_{RM}^{*} I_{RM}^{-1}\) for the fixed point covariances of the Hilbert space measures. To prove (3.118) we compute, using the same steps as in (3.117)

\[
\langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} \langle J_{RM} H_{Rm} J_{RM}^{\dagger} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{*}
\]

\[
= \langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{Rm} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{*}
\]

\[
= \frac{\hbar^2 K}{2} \langle J_{RM} f_{RM} I_{RM} J_{RM}^{\dagger} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{*}
\]

\[
= \frac{\hbar^2 K}{2} \langle f_{RM} I_{RM} f_{RM} I_{RM} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{*}
\]

\[
= \langle e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM} e^{i(\phi_{RM} - f_{RM}) f_{RM}} \rangle_{\Omega_{RM}^{(n)}} H_{RM}^{*}
\]

\[
\tag{3.120}
\]

as claimed. Thus, not only does there exist a consistent family of Hamiltonian quadratic forms but indeed a fixed point Hamiltonian \(H_{Rm}\). This Hamiltonian coincides with the one \(H_{Rm}\) of the
continuum because we checked in the previous subsection that the fixed point covariances \( \omega_{RM}^* \) are obtained from the continuum covariance \( \omega_{RM} \) by 
\[
[\omega_{RM}^*]^{-1} = I_{RM} \omega_R^{-1} I_{RM}.
\]

It is instructive to check that 
\[
\omega_R^{-1} = \lim_{M \to \infty} \left( \omega_{RM}^* \right)^{-1}
\]
by using the explicit presentation (3.69). To do this note that with 
\[
q^2 = (k_0^2 + p^2)\epsilon_{RM}^2
\]
and 
\[
t = k_M l
\]
we have 
\[
\frac{1}{q^3} \left[ q \cosh(q) - \sinh(q) \right] \to 1 \text{ as } M \to \infty
\]
and 
\[
\frac{1}{q^3} \left[ \cosh(q) - q \right] \to 1 \text{ as } M \to \infty
\]
and 
\[
\frac{1}{q^3} \left[ \cosh(q) - \cos(t) \right] \to 1 \text{ as } M \to \infty
\]
and 
\[
k_0^2 + p^2 + (k_M^2 + p^2 + (k_M^2)^2).\]
Note also that (3.62) is an instance of the theorem of Mittag-Leffler applied to (3.69) that allows to write a meromorphic function as a linear combination of simple pole functions and an entire holomorphic function.

3.3. Comparison of the renormalisation flows

The two flows corresponding to the path integral induced Hamiltonian renormalisation and
the direct Hamiltonian renormalisation of the OS data are very different even for free fields:
while the path integral flow generates an infinite number of field species at every finite resolution 
\( M \), even if one starts with a single field species, the direct flow stays in the single field
species regime for every finite resolution \( M \). The corresponding OS data are different and if
one would compute the Wiener measure corresponding to the direct flow OS data we cannot
possibly obtain the finite resolution path integral measure since even the one particle Hilbert
spaces are very different from each other. But this is not what the direct Hamiltonian flow
is supposed to do. Rather, its aim is to construct the continuum Hamiltonian \( H_R \) as the limit
\(
\lim_{M \to \infty} H_{RM}^{*}
\)
where \( \{H_{RM}\}_M \) is the fixed point family as obtained from the direct flow. From
these continuum OS data we can construct the corresponding Wiener measure and then compute
its cylindrical projections. As we have seen, it is those cylindrical projections which do
agree with the finite resolution path integral measures at the fixed point of the path integral
induced flow, at least for the present free field theory that we considered.

4. Summary

In this paper we tested our proposal for a direct Hamiltonian renormalisation flow on the OS
data for the case of a free Klein Gordon field in two spacetime dimensions. Generalisations to
more dimensions and more general models will be supplied in [2, 3]. We find that the flow has
a fixed point whose corresponding continuum Hamiltonian agrees with the continuum Fock
space quantisation of the classical continuum Hamiltonian. That this is not only a fixed point
but that the initial naively discretised family in fact converges to it will be demonstrated in [2].

The direct flow has the advantage that one can sidestep the construction of the Wiener
measures, that is, the corresponding path integrals which has obvious practical implications.
However, as the present paper reveals, there is in fact a more important implication: the direct
Hamiltonian flow is much closer to immediately constructing the finite resolution matrix ele-
ments of the actual continuum Hamiltonian. By contrast, the flow of the OS data induced by
the path integral flow is very far away from that, it leads to finite resolution Hamiltonians with
an ever-increasing number of field species as we increase the number of renormalisation steps
that the continuum theory does not have.

How can these two facts be reconciled? After all, the continuum Hamiltonian, which only
uses a single sharp time zero field species, gives rise to a path integral Wiener measure, which
therefore also uses only a single spacetime field species. The finite resolution cylindrical projections of that continuum measure, which is an OS measure by construction [1], are also OS measures and thus have OS data. Yet, these OS data involve the inflation of field species derived in this paper. The reason lies in the fact that the OS construction enforces that the contraction semigroups of the continuum measure and its cylindrical projections are equivariant with respect to the corresponding Hilbert space embedding. This implies that the finite resolution Hamiltonians must commute with a huge number of subprojection operators. This enhanced symmetry of the finite resolution Hamiltonians cannot be accommodated by a single field species and leads to their inflation.

It is clear that this enhanced symmetry is an unphysical property of the finite resolution matrix elements of the continuum Hamiltonian. In other words, the path integral induced flow of the finite resolution OS data is physically meaningless. The only meaningful OS data that the path integral flow extracts are the OS data of the fixed point continuum measure. On the other hand, the direct renormalisation flow of the OS data runs into a fixed point which does display the finite resolution matrix elements of the continuum Hamiltonian and thus the flow directly generates approximations to those matrix elements that improve their continuum properties eventually with each renormalisation step.

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