CANONICAL TRANSFER-FUNCTION REALIZATION FOR SCHUR MULTIPLIERS ON THE DRURY-ARVESON SPACE AND MODELS FOR COMMUTING ROW CONTRACTIONS

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Abstract. We develop a $d$-variable analog of the two-component de Branges-Rovnyak reproducing kernel Hilbert space associated with a Schur-class function on the unit disk. In this generalization, the unit disk is replaced by the unit ball in $d$-dimensional complex Euclidean space, and the Schur class becomes the class of contractive multipliers on the Drury-Arveson space over the ball. We also develop some results on a model theory for commutative row contractions which are not necessarily completely noncoisometric (the case considered in earlier work of Bhattacharyya, Eschmeier and Sarkar).

1. Introduction

For $\mathcal{U}$ and $\mathcal{Y}$ two Hilbert spaces we let $\mathcal{L}(\mathcal{U}, \mathcal{Y})$ be the space of bounded linear operators mapping $\mathcal{U}$ into $\mathcal{Y}$, abbreviated to $\mathcal{L}(\mathcal{U})$ in case $\mathcal{U} = \mathcal{Y}$. The operator-valued version of the classical Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ is defined to be the set of all holomorphic, contractive $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions on the unit disk $D$. With any such function $S : D \to \mathcal{L}(\mathcal{U}, \mathcal{Y})$, one can associate the following three operator-valued kernels

$$K_S(z, \zeta) = \frac{I_Y - S(z)S(\zeta)^*}{1 - z\zeta}, \quad K_S^*(z, \zeta) = \frac{I_U - S(z)^*S(\zeta)}{1 - z\zeta}, \quad (1.1)$$

$$\hat{K}(z, \zeta) = \begin{bmatrix} K_S(z, \zeta) & \frac{S(z) - S(\zeta)}{z - \zeta} \\ \frac{Z(z)^* - S(\zeta)^*}{\zeta - \zeta} & K_S^*(z, \zeta) \end{bmatrix}. \quad (1.2)$$

The Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y})$ can be characterized as the set of all $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function on $D$ for which any (and therefore every) of the above three kernels is positive on $D \times D$. Furthermore, for every function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ there exists an auxiliary Hilbert space $\mathcal{X}$ and a unitary connecting operator (or colligation)

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \quad (1.3)$$

so that $S(z)$ can be expressed as

$$S(z) = D + zC(I - zA)^{-1}B \quad \text{for all} \quad z \in D. \quad (1.4)$$

On the other hand, if $U$ of the form (1.3) is a contraction, then the function $S$ of the form (1.4) belongs to $\mathcal{S}(\mathcal{U}, \mathcal{Y})$. The formula (1.4) is called a realization of the function $S$ which in turn, is called the characteristic function of the colligation (1.3).

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The realization is called unitary, isometric, coisometric or contractive if the connecting operator \( U \) is respectively, unitary, isometric, coisometric or contractive. It is seen from (1.4) that for any realization \( U \) of \( S \), the entry \( D \) is uniquely determined and equals \( S(0) \). As was shown in [19], [20], [21] for unitary, isometric or coisometric realizations, the state space \( X \) and the operators \( A, B \) and \( C \) can be chosen in a certain canonical way (specific for each type) and these realizations are unique up to unitary equivalence under certain minimality conditions which we now recall. With a colligation (1.3) we associate the observability subspace \( H_{C,A}^O \) and the controllability subspace \( H_{A,B}^C \) by

\[
H_{C,A}^O := \bigvee_{n \geq 0} \text{Ran} A^n C^*, \quad H_{A,B}^C := \bigvee_{n \geq 0} \text{Ran} A^n B,
\]

where \( \bigvee \) denotes the closed linear span.

**Definition 1.1.** The colligation \( U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \to Y \) is called observable, controllable or closely connected if respectively, \( H_{C,A}^O = X \), \( H_{A,B}^C = X \) or \( H_{C,A}^O \bigvee H_{A,B}^C = X \).

Furthermore, \( U \) is called unitarily equivalent to a colligation \( \tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} : \tilde{X} \to \tilde{Y} \) if there exists a unitary operator \( U : X \to \tilde{X} \) such that \( UA = \tilde{A}U \), \( UB = \tilde{B} \) and \( C = \tilde{C}U \).

We remark that some authors attribute the notions of observability and controllability to the pairs \((C, A)\) and \((A, B)\) rather than to the whole colligation \( U \), and call the colligation with an observable output pair \((C, A)\) and/or with a controllable input pair \((A, B)\) respectively closely outer-connected and/or closely inner-connected.

Recall that given an \( S \in S(U, Y) \), the three associated kernels in (1.1) and (1.2) are positive and give rise to the respective reproducing kernel Hilbert spaces \( \mathcal{H}(K_S) \), \( \mathcal{H}(\tilde{K}_S) \) and \( \mathcal{H}(\hat{K}_S) \) (called de Branges-Rovnyak reproducing kernel Hilbert spaces). Observe that the kernel \( K_S(z, \zeta) \) is analytic in \( z, \zeta \) and therefore, all functions in the associated space \( \mathcal{H}(K_S) \) are analytic on \( \mathbb{D} \). The kernel \( \tilde{K}_S \) is analytic in \( \overline{\mathbb{D}} \) and \( \zeta \) and the associated space \( \mathcal{H}(\tilde{K}_S) \) consists of conjugate-analytic functions. Similarly, the elements of \( \mathcal{H}(\hat{K}_S) \) are the functions of the form \( f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix} \) where \( f_+ \) is analytic and \( f_- \) is conjugate-analytic. The following theorem summarizes realization results from [19], [20], [21].

**Theorem 1.2.** Let \( S \in S(U, Y) \) and let \( D := S(0) \).

(1) Let \( X = \mathcal{H}(K_S) \) and let

\[
A : f(z) \mapsto \frac{f(z) - f(0)}{z}, \quad B : u \mapsto \frac{S(z) - S(0)}{z} u, \quad C : f(z) \mapsto f(0),
\]

Then \( U \) (1.3) is an observable coisometric colligation with its characteristic function equal to \( S \). Any observable coisometric colligation \( \tilde{U} \) (1.7) with its characteristic function equal to \( S \) is unitarily equivalent to \( U \).
(2) Let \( X = \mathcal{H}(\tilde{K}_S) \) and let \( B: u \mapsto (I_d - S(z)S(0))u, \)
\[
A^* : f(z) \mapsto \frac{f(z) - f(0)}{z}, \quad C^* : y \mapsto \frac{S(z)^* - S(0)^*}{z} y.
\]
Then \( U \) is a controllable isometric colligation with its characteristic function equal to \( S \). Any controllable isometric colligation \( \tilde{U} \) with its characteristic function equal to \( S \) is unitarily equivalent to \( U \).

(3) Let \( X = \mathcal{H}(\tilde{K}_S) \) and let \( A \) and \( B \) be as in (2). Then \( U \) is a closely connected unitary colligation with its characteristic function equal to \( S \). Any closely connected unitary colligation with its characteristic function equal to \( S \) is unitarily equivalent to \( U \).

We mention that such de Branges-Rovnyak reproducing kernel Hilbert spaces can be used as canonical functional model Hilbert spaces for contraction operators of various classes (namely, completely noncoisometric, completely nonisometric, and completely nonunitary)—see [27, 12].

The objective of this paper is to extend these realization results to the following multivariable setting. We denote by \( \mathbb{B} = \{ z = (z_1, \ldots, z_d) \in \mathbb{C}^d : |z| < 1 \} \) the unit ball of the Euclidean space \( \mathbb{C}^d \) with the standard inner product \( \langle z, \zeta \rangle = \sum_{j=1}^d z_j\bar{\zeta}_j \). The kernel \( k_d(z, \zeta) = \frac{1}{1 - \langle z, \zeta \rangle} \) is positive on \( \mathbb{B} \times \mathbb{B} \) and we denote by \( \mathcal{H}(k_d) \) the associated reproducing kernel Hilbert space (the Drury-Arveson space) which for \( d = 1 \) is the usual Hardy space \( H^2 \) of the unit disk. For a Hilbert space \( \mathcal{Y} \), we use notation \( \mathcal{H}_\mathcal{Y}(k_d) \) for the Drury-Arveson space of \( \mathcal{Y} \)-valued functions which can be characterized in terms of power series as follows:
\[
\mathcal{H}_\mathcal{Y}(k_d) = \left\{ f(z) = \sum_{n \in \mathbb{Z}^d_+} f_n z^n : \|f\|^2 = \sum_{n \in \mathbb{Z}^d_+} \frac{n!}{|n|!} \|f_n\|_{\mathcal{Y}}^2 < \infty \right\}.
\]
Here and in what follows, we use standard multivariable notations: for multi-integers \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d_+ \) and points \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \) we set
\[
|n| = n_1 + n_2 + \ldots + n_d, \quad n! = n_1!n_2! \ldots n_d!, \quad z^n = z_1^{n_1}z_2^{n_2} \ldots z_d^{n_d}.
\]
Given two Hilbert spaces \( \mathcal{U} \) and \( \mathcal{Y} \), we denote by \( \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) the class of \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued functions \( S \) on \( \mathbb{B}^d \) such that the multiplication operator \( M_S : f \mapsto S \cdot f \) defines a contraction from \( \mathcal{H}_\mathcal{U}(k_d) \) into \( \mathcal{H}_\mathcal{Y}(k_d) \) or equivalently, such that the kernel
\[
K_S(z, \zeta) = \frac{I_\mathcal{Y} - S(z)S(\zeta)^*}{1 - \langle z, \zeta \rangle}
\]
is positive on \( \mathbb{B} \times \mathbb{B} \). It is readily seen that the class \( S_1(\mathcal{U}, \mathcal{Y}) \) is the Schur class introduced above. In general, it follows from positivity of \( K_S \) that \( S \) is holomorphic and takes contractive values on \( \mathbb{B}^d \). However, for \( d > 1 \) there are analytic contractive-valued functions on \( \mathbb{B}^d \) not in \( \mathcal{S}_d \). The class \( \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) can be characterized in various ways similarly to the one-variable situation. Here we recall the one from [13] given in terms of norm-constrained realizations.
In what follows we use notation \( Z_{\text{row}}(z) = [z_1 \ldots z_d] \) and for a Hilbert space \( \mathcal{X} \) we let
\[
Z_{\mathcal{X}}(z) := Z_{\text{row}}(z) \otimes I_{\mathcal{X}} = [z_1 I_{\mathcal{X}} \ldots z_d I_{\mathcal{X}}].
\] (1.14)

**Theorem 1.3.** If a function \( S : \mathbb{B}^d \to \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) belongs to \( \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \), then there is an auxiliary Hilbert space \( \mathcal{X} \) and a unitary connecting operator (or colligation)
\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{bmatrix} : [\mathcal{X}] \to [\mathcal{X}^d]
\] (1.15)
so that \( S(z) \) can be realized as
\[
S(z) = D + C(I - z_1 A_1 - \cdots - z_d A_d)^{-1}(z_1 B_1 + \cdots + z_d B_d)
= D + C(I_{\mathcal{X}} - Z_{\mathcal{X}}(z) A)^{-1} Z_{\mathcal{X}}(z) B \quad (z \in \mathbb{B}^d).
\] (1.16)

Conversely, if \( U \) of the form (1.15) is a contraction, then the function \( S \) of the form (1.16) belongs to \( \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \).

As in the univariate case, \( S \) of the form (1.16) will be referred to as the characteristic function of the colligation (1.15). The main goal of the present paper is to establish the analog of Theorem 1.2 for the present multivariable setting, that is to obtain coisometric, isometric and unitary functional-model realizations of a given \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) in a certain canonical way and to show that these types of realizations are unique up to unitary equivalence under suitable minimality conditions. As an application of our functional model spaces for this multivariable ball setting, we show how the model theory of Bhattacharyya-Eschmeier-Sarkar [17, 18] for commutative row-contractive operator \( d \)-tuples can be extended beyond the completely noncoisometric case, and we relate these results with those of the first author and Vinnikov [15] established for general (possibly noncommutative) completely nonunitary row-contractive operator \( d \)-tuples.

We now introduce the minimality conditions which will play a key role in the sequel. We denote by \( I_i : \mathcal{X} \to \mathcal{X}^d \) the inclusion map of the space \( \mathcal{X} \) into the \( i \)-th slot in the direct-sum space \( \mathcal{X}^d = \bigoplus_{k=1}^d \mathcal{X} \); the adjoint then is the orthogonal projection of \( \mathcal{X}^d \) down to the \( i \)-th coordinate:
\[
I_i : x_i \mapsto \begin{bmatrix} 0 \\ \vdots \\ x_i \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad I_i^* : \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_d \end{bmatrix} \mapsto x_i.
\] (1.17)

With a structured colligation (1.15) we associate the observability subspace \( \mathcal{H}_{\mathcal{C},A}^O \) and the controllability subspace \( \mathcal{H}_{\mathcal{A},B}^O \) as follows:
\[
\mathcal{H}_{\mathcal{C},A}^O := \bigvee \{ (I_{\mathcal{X}} - A^* Z_{\mathcal{X}}(z)^* )^{-1} C^* y : z \in \mathbb{B}^d, y \in \mathcal{Y} \},
\] (1.18)
\[
\mathcal{H}_{\mathcal{A},B}^O := \bigvee_{j=1}^d \{ I_j^* (I_{\mathcal{X}^d} - A Z_{\mathcal{X}}(z))^{-1} B u : z \in \mathbb{B}^d, u \in \mathcal{U} \},
\] (1.19)
where $Z_X$ and $T^*_X$ are given in (1.14) and (1.17). Observe, that in case $d = 1$ these definitions are equivalent to those in (1.5). Similarity becomes more transparent if one writes definitions (1.18), (1.19) in terms of powers of the state space operators $A_1, \ldots, A_d$; however, at this point we try to avoid power notation which requires some more explanations and notation in case the state space operators do not commute. With the spaces (1.18), (1.19) in hand, the multivariable extension of Definition 1.1 is now immediate.

**Definition 1.4.** The structured colligation $\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} X \\ U \end{bmatrix} \to \begin{bmatrix} X^d \\ Y \end{bmatrix}$ as in (1.15) is called observable, controllable or closely connected if respectively,

$$H_{O,C,A} = X, \quad H_{A,B} = X \quad \text{or} \quad H_{C,A} \lor H_{A,B} = X.$$  

Furthermore, $\mathbf{U}$ is called unitarily equivalent to a colligation

$$\tilde{\mathbf{U}} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} : \begin{bmatrix} X \\ U \end{bmatrix} \to \begin{bmatrix} X^d \\ Y \end{bmatrix}$$  

if there exists a unitary operator $U : X \to \tilde{X}$ such that

$$(\oplus_{i=1}^d U) A = \tilde{A} U, \quad (\oplus_{i=1}^d U) B = \tilde{B} \quad \text{and} \quad C = \tilde{C} U.$$  

(1.21)

It is readily seen that equalities (1.21) is what we need to guarantee (as in the univariate case) that the characteristic functions of $\mathbf{U}$ and of $\tilde{\mathbf{U}}$ are equal.

As was pointed out on many occasions, a more useful analog of coisometric (isometric, unitary) realizations appearing in the classical univariate case is not that the whole connecting operator $\mathbf{U}$ be coisometric (isometric, unitary), but rather that $\mathbf{U}$ and/or $\mathbf{U}^*$ be contractive and isometric on certain canonical subspaces closely related to the subspaces (1.18) and (1.19).

**Definition 1.5.** A contractive colligation $\mathbf{U}$ of the form (1.15) is called

1. **weakly isometric** if $\mathbf{U}$ is isometric on the subspace $\mathcal{D}_{A,B} \oplus U$ where

$$\mathcal{D}_{A,B} := \bigvee_{\zeta \in \mathbb{B}^d, u \in U} Z_X(\zeta)(I - AZ_X(\zeta))^{-1} Bu \subset X;$$

2. **weakly coisometric** if the adjoint $\mathbf{U}^* : X^d \oplus Y \to X \oplus U$ is isometric on the subspace $\mathcal{D}_{C,A} \oplus Y$ where

$$\mathcal{D}_{C,A} := \bigvee_{\zeta \in \mathbb{B}^d, y \in Y} Z_X(\zeta)^*(I - A^* Z_X(\zeta)^*)^{-1} C^* y \subset X^d;$$

(1.22)

3. **weakly unitary** if it is weakly isometric and weakly coisometric.

We remark that the above weak notions do not appear in the single-variable case for a simple reason that if the pair $(C, A)$ is observable than $\mathcal{D}_{C,A} = X$ so that a weakly coisometric colligation is automatically coisometric and similarly, if the pair $(A, B)$ is controllable, then $\mathcal{D}_{A,B} = X$ so that a weakly isometric colligation is automatically isometric.

As was shown in [9], a function $S \in S_d(U, Y)$ may not admit an observable coisometric realization. In contrast, observable weakly-coisometric realizations always exist and up to unitary equivalence, all these realizations are canonical functional...
model \((c.f.m.)\) realizations (see Definition 2.1 below) with the state space equal to the de Branges-Rovnyak space \(\mathcal{H}(K_S)\) with reproducing kernel \((1.13)\), with the output operator \(C\) equal to evaluation at zero on \(\mathcal{H}(K_S)\), and with operators \(A\) and \(B\) whose adjoints are uniquely determined on the subspace \(\mathcal{D}_{C,A} \subset \mathcal{X}^d\) given in \((1.22)\). Realizations of this type were studied in [9], [10], [8] and will be briefly reviewed in Section 2. Section 3 is a brief sketch of the dual canonical functional model \((d.c.f.m.)\) colligations which provide a canonical functional model for controllable weakly isometric realizations of \(S\) (the analog of part (2) of Theorem 1.2); this section is kept quite short as the proofs of the results can be seen as special cases of the more general manipulations carried out in Section 3. Section 4 is the core of the paper where the theory of the two-component canonical functional model \((t.c.f.m.)\) colligations (the analog of part (3) of Theorem 1.2) is carried out; these form the precise class of canonical models which provide weakly unitary closely connected realizations for the contractive Drury-Arveson-space multiplier \(S\). Section 5 gives the application to the model theory for row-contractive operator \(d\)-tuples, i.e., how these \(t.c.f.m.\) colligations can be used to extend at least partially the role of the de Branges-Rovnyak two-component spaces \(\mathcal{H}(\hat{K}_S)\) as the model space for completely nonunitary contractions to the setting of commutative row-contractive operator \(d\)-tuples \(T = (T_1, \ldots, T_d)\) where each \(T_k\) \((k = 1, \ldots, d)\) is an operator on a fixed Hilbert space \(\mathcal{X}\) and the block row-matrix \(T = [T_1 \cdots T_d]\) is a contraction operator from \(\mathcal{X}^d\) to \(\mathcal{X}\). Here we also give a simple example (specifically, a point on the boundary of the unit ball in \(\mathbb{C}^2\) viewed as a 2-tuple of operators on \(\mathbb{C}\)) for which the completely noncoisometric version of the model theory gives no information but for which our added invariant gives complete information. The final Section 6 sketches on the \(t.c.f.m.\) approach to operator-model theory leads to more definitive results for unitary classification of not necessarily commutative row-contractive operator-tuples; here we draw on results from [15] and [14].

We close the Introduction with a short discussion of the Gleason problem to give the reader some orientation for the multivariable formalisms to follow. The reader will note that the difference-quotient transformation

\[ f(z) \mapsto \frac{f(z) - f(0)}{z} \]

(where \(f\) is a function which is holomorphic at 0) plays a key role in the definition of the model operators \(A, B, C^*\) in Theorem 1.2. For some time now it has been recognized that the multivariable analog of the difference-quotient transformation is any solution of the so-called Gleason problem for a space of holomorphic functions \(\mathcal{H}\) (see [26, 2, 3, 4]). Given a space \(\mathcal{H}\) of holomorphic functions \(h\) which are holomorphic in a neighborhood of the origin in \(d\)-dimensional complex Euclidean space \(\mathbb{C}^d\), we say that the operators \(R_1, \ldots, R_d\) mapping \(\mathcal{H}\) into itself solve the Gleason problem for \(\mathcal{H}\) if every function \(h \in \mathcal{H}\) has a decomposition (not necessarily unique) of the form

\[ h(z) = h(0) + \sum_{k=1}^{d} z_k[R_k h](z). \]

We shall see that more structured variations on this idea appear in the definition of the model operators for the various canonical functional-model spaces over the ball \(\mathbb{B}^d\) to appear in the sequel.
2. Weakly coisometric canonical realizations

For any $S \in \mathcal{S}_d(U, Y)$, the associated kernel $K_S$ is positive on $\mathbb{B}^d \times \mathbb{B}^d$ so we can associate with $S$ the de Branges-Rovnyak reproducing kernel Hilbert space $\mathcal{H}(K_S)$. In parallel to the univariate case, $\mathcal{H}(K_S)$ is the state space of certain canonical functional-model realization for $S$.

**Definition 2.1.** We say that the contractive operator-block matrix

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K_S) \\ U \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}(K_S)^d \\ Y \end{bmatrix}$$

(2.1)

is a canonical functional-model (abbreviated to c.f.m. in what follows) colligation for the given function $S \in \mathcal{S}_d(U, Y)$ if

1. The operator $A = \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}$ solves the Gleason problem for $\mathcal{H}(K_S)$, i.e.,

$$f(z) - f(0) = \sum_{j=1}^d z_j (A_j f)(z) \quad \text{for all } f \in \mathcal{H}(K_S).$$

(2.2)

2. The operator $B = \begin{bmatrix} B_1 \\ \vdots \\ B_d \end{bmatrix}$ solves the Gleason problem for $S$:

$$S(z) u - S(0) u = \sum_{j=1}^d z_j (B_j u)(z) \quad \text{for all } u \in U.$$ 

(2.3)

3. The operators $C : \mathcal{H}(K_S) \rightarrow Y$ and $D : U \rightarrow Y$ are given by

$$C : f \mapsto f(0), \quad D : u \mapsto S(0) u.$$ 

(2.4)

We next rearrange equality (1.13) as follows

$$\sum_{j=1}^d z_j \bar{\zeta}_j K_S(z, \zeta) + I_Y = K_S(z, \zeta) + S(z) S(\zeta)^*$$

(2.5)

and write (2.5) in the inner product form as

$$\left\langle \begin{bmatrix} \text{Row}(z)^* \otimes K_S(\cdot, z) y \\ y \end{bmatrix}, \begin{bmatrix} \text{Row}(z)^* \otimes K_S(\cdot, z) y' \\ y' \end{bmatrix} \right\rangle_{\mathcal{H}(K_S)^d \oplus Y}$$

$$= \left\langle \begin{bmatrix} K_S(\cdot, \zeta) y \\ S(\zeta)^* y \end{bmatrix}, \begin{bmatrix} K_S(\cdot, z) y' \\ S(z)^* y' \end{bmatrix} \right\rangle_{\mathcal{H}(K_S) \oplus U}.$$ 

It now follows that the map

$$V : \begin{bmatrix} \text{Row}(z)^* K_S(\cdot, \zeta) y \\ y \end{bmatrix} \rightarrow \begin{bmatrix} K_S(\cdot, \zeta) y \\ S(\zeta)^* y \end{bmatrix}$$

(2.6)

extends by linearity and continuity to an isometry with initial space

$$\mathcal{D}_V = \bigvee_{\zeta \in \mathbb{B}^d, y \in Y} \begin{bmatrix} \text{Row}(z)^* K_S(\cdot, \zeta) y \\ y \end{bmatrix}.$$ 

The following result can be found in [8].
Theorem 2.2. Given a function \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \), let \( V \) be the isometric operator defined in (2.6). Then

1. A block-operator matrix \( U \) of the form (2.4) is a c.f.m. colligation for \( S \) if and only if \( U^* \) is a contractive extension of \( V \) to all of \( \mathcal{H}(K_S)^d \oplus \mathcal{Y} \), i.e.,

\[
U^*|_{\mathcal{D}_V} = V \quad \text{and} \quad ||U^*|| \leq 1. \tag{2.7}
\]

In particular, a c.f.m. colligation for \( S \) exists.

2. Every c.f.m. colligation \( U \) for \( S \) is weakly coisometric and observable and furthermore, \( S(z) = D + C(I - Z_{\mathcal{H}(K_S)}(z)A)^{-1}Z_{\mathcal{H}(K_S)}(z)B \).

3. Any observable weakly coisometric colligation \( \tilde{U} \) of the form (1.20) with the characteristic function equal \( S \) is unitarily equivalent to some c.f.m. colligation \( U \) for \( S \).

Remark 2.3. Since \( Z_{\mathcal{H}(K_S)}(0)(0) = 0 \), the space \( \mathcal{D}_V \) contains all vectors of the form \([0 \ y] \) and therefore it splits into the direct sum \( \mathcal{D}_V = \mathcal{D} \oplus \mathcal{Y} \) where

\[
\mathcal{D} = \bigvee_{\zeta \in \mathbb{B}^d, y \in \mathcal{Y}} Z_{\mathcal{Y}}(\zeta)^* K_S(\cdot, \zeta)y \subset \mathcal{H}(K_S)^d.
\]

It is readily checked that the orthogonal complement \( \mathcal{D}^\perp = \mathcal{H}(K_S)^d \ominus \mathcal{D} \) of \( \mathcal{D} \) is given by

\[
\mathcal{D}^\perp = \{ h \in \mathcal{H}(K_S)^d : Z_{\mathcal{Y}}(z)h(z) \equiv 0 \}. \tag{2.8}
\]

We now see from (2.7) that nonuniqueness of c.f.m. colligations for \( S \) is achieved by different choices of \( A^*|_{\mathcal{D}^\perp} \) and of \( B^*|_{\mathcal{D}^\perp} \).

Remark 2.4. Observe also that in the univariate case \( d = 1 \), the operators \( A \) and \( B \) are uniquely recovered from (2.2), (2.3) and one then arrives at the operators \( A \) and \( B \) exactly as in (1.8). Also, the space \( \mathcal{D} \) equals \( \mathcal{H}(K_S) \) so that \( U^* = V \) and now it is seen that for \( d = 1 \) Theorem 2.2 collapses to part (1) in Theorem 1.2.

Definition 2.1 does not require \( U \) to be a realization for \( S \): representation (1.16) is automatic once the operators \( A, B, C \) and \( D \) are of the required form. Theorem 2.2 below (see Theorem 2.10 in [8] for the proof) characterizes which operators \( A \) and which operators \( B \) can arise in a c.f.m. colligation for \( S \). Let us say that \( A: \mathcal{H}(K_S) \to \mathcal{H}(K_S)^d \) is a contractive solution of the Gleason problem for \( \mathcal{H}(K_S) \) if in addition to (2.2), the inequality

\[
\sum_{k=1}^{d} ||A_kf||_{\mathcal{H}(K_S)}^2 \leq ||f||_{\mathcal{H}(K_S)}^2 - ||f(0)||_{\mathcal{Y}}^2
\]

holds for every \( f \in \mathcal{H}(K_S) \). An equivalent operator form of this inequality is \( A^*A + C^*C \leq I \) where the operator \( C: \mathcal{H}(K_S) \to \mathcal{Y} \) is given in (2.4). It therefore follows from Definition 2.1 that for every c.f.m. colligation \( U = [A \ B] \) for \( S \), the operator \( A \) is a contractive solution of the Gleason problem (2.2).

Theorem 2.5. Let \( S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y}) \) be given and let us assume that \( C, D \) are given by formulas (2.4). Then

1. For every contractive solution \( A \) of the Gleason problem (2.2), there exists an operator \( B: \mathcal{U} \to \mathcal{H}(K_S) \) such that \( U = [A \ B] \) is contractive and \( S \) is realized as in (1.16).

2. Every such \( B \) solves the \( \mathcal{H}(K_S) \)-Gleason problem (2.3) so that \( U \) is a c.f.m. colligation.
An object of independent interest is the class of Schur-class functions admitting contractive commutative realizations of the form (1.16) where the state space operators $A_1, \ldots, A_d$ commute with each other. A key role here is played by the backward shift on the Drury-Arveson space $\mathcal{H}(k_d)$, the commuting $d$-tuple $M^*_z := (M^*_z, \ldots, M^*_z)$ consisting of the adjoints (in metric of $\mathcal{H}(k_d)$) of operators $M_z$'s of multiplication by the coordinate functions of $\mathbb{C}^d$. It was shown in [10] that any Schur-class function $S$ with associated de Branges-Rovnyak space $\mathcal{H}(K_S)$ finite-dimensional and not $M^*_z$-invariant does not admit a contractive commutative realization. The following theorem also can be found in [10].

**Theorem 2.6.** A Schur-class function $S \in S_d(U, Y)$ admits a commutative weakly coisometric realization if and only if the following conditions hold:

1. The associated de Branges-Rovnyak space $\mathcal{H}(K_S)$ is $M^*_z$-invariant, and
2. the inequality
   \[
   \sum_{j=1}^d \|M^*_zf\|^2_{\mathcal{H}(K_S)} \leq \|f\|^2_{\mathcal{H}(K_S)} - \|f(0)\|^2_Y
   \]
   holds for all $f \in \mathcal{H}(K_S)$. (2.9)

Furthermore, if conditions (1) and (2) are satisfied, then there exists a commutative c.f.m. colligation for $S$. Moreover, the state-space operators tuple is equal to the Drury-Arveson backward shift restricted to $\mathcal{H}(K_S)$: $A_j = M^*_z|_{\mathcal{H}(K_S)}$ for $j = 1, \ldots, d$.

3. **Weakly isometric realizations**

In the univariate case, the state space of the functional-model isometric realization for a Schur-class function $S$ can be taken to be equal to the reproducing kernel Hilbert space $\mathcal{H}(\tilde{K}_S)$ with reproducing kernel $\tilde{K}_S(z, \zeta)$ as in (1.1). A natural multivariable counterpart of this kernel would be the kernel

\[
\tilde{K}_S(z, \zeta) = \frac{I_U - S(z)^*S(\zeta)}{1 - \langle \zeta, z \rangle}.
\]

However, if $S \in S_d(U, Y)$ for $d > 1$, this kernel is not positive in general. Instead, we have the following Agler-type decomposition result (see [13, Theorem 2.4] for the proof).

**Theorem 3.1.** A function $S: \mathbb{B}^d \to \mathcal{L}(U, Y)$ belongs to $S_d(U, Y)$ if and only if there exists a positive kernel

\[
\Phi = \begin{bmatrix}
\Phi_{11} & \cdots & \Phi_{1d} \\
\vdots & \ddots & \vdots \\
\Phi_{d1} & \cdots & \Phi_{dd}
\end{bmatrix} : \mathbb{B}^d \times \mathbb{B}^d \to \mathcal{L}(\mathbb{U}^d) \tag{3.1}
\]

so that for every $z, \zeta \in \mathbb{B}^d$,

\[
I_U - S(z)^*S(\zeta) = \sum_{j=1}^d \Phi_{jj}(z, \zeta) - \sum_{i, \ell=1}^d z_i\zeta^\ell \Phi_{i\ell}(z, \zeta). \tag{3.2}
\]

The kernel $\Phi$ in Theorem 3.1 is not determined from $S$ uniquely. With each such kernel, one can associate weakly-isometric functional-model colligations as follows. Given a decomposition (3.2) with a positive kernel $\Phi$, let $\mathcal{H}(\Phi)$ be the reproducing kernel Hilbert space with reproducing kernel $\Phi$. Clearly, the elements of $\mathcal{H}(\Phi)$ are
We next rearrange the block columns $\Phi_{\bullet k}$ of $\Phi$ to produce the kernel
\[
T(z, \zeta) := \begin{bmatrix} \Phi_{\bullet 1}(z, \zeta) \\ \vdots \\ \Phi_{\bullet d}(z, \zeta) \end{bmatrix}, \quad \text{where} \quad \Phi_{\bullet k}(z, \zeta) = \begin{bmatrix} \Phi_{1k}(z, \zeta) \\ \vdots \\ \Phi_{dk}(z, \zeta) \end{bmatrix}, \quad (3.3)
\]
and we then introduce the subspace
\[
\tilde{D} = \sqrt{\bigoplus_{j=1}^{d} \zeta_j \Phi_{\bullet j}(, \zeta) u : \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{B}^d, u \in \mathcal{U}} \subset \mathcal{H}(\Phi). \quad (3.4)
\]
Decomposition (3.2) then can be written in the inner product form as
\[
\left\langle \left[ \sum_{j=1}^{d} \zeta_j \Phi_{\bullet j}(, \zeta) u \right], u' \right\rangle_{\mathcal{H}(\Phi)\oplus \mathcal{U}} = \left\langle \left[ \sum_{j=1}^{d} \zeta_j \Phi_{\bullet j}(, \zeta) u \right], u' \right\rangle_{\mathcal{H}(\Phi)\oplus \mathcal{Y}}
\]
so that the linear map $\tilde{V}$ given by formula
\[
\tilde{V} : \left[ \sum_{j=1}^{d} \zeta_j \Phi_{\bullet j}(, \zeta) u \right] \rightarrow \left[ \begin{array}{c} T(, \zeta) u \\ S(\zeta) u \end{array} \right]
\]
extends by continuity to define the isometry $\tilde{V} : \mathcal{D}_{\tilde{V}} \rightarrow \mathcal{R}_{\tilde{V}}$ where
\[
\mathcal{D}_{\tilde{V}} = \tilde{D} \oplus \mathcal{U} \quad \text{and} \quad \mathcal{R}_{\tilde{V}} = \bigvee_{\zeta \in \mathbb{B}^d, u \in \mathcal{U}} \left[ \begin{array}{c} T(, \zeta) y \\ S(\zeta) u \end{array} \right] \subset \left[ \mathcal{H}(\Phi)\oplus \mathcal{Y} \right].
\]

**Definition 3.2.** Given a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$, we shall say that the contractive block-operator matrix
\[
\tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} : \left[ \begin{array}{c} \mathcal{H}(\Phi) \\ \mathcal{U} \end{array} \right] \rightarrow \left[ \begin{array}{c} \mathcal{H}(\Phi)\oplus \mathcal{Y} \end{array} \right]
\]

is a dual canonical functional-model (abbreviated to d.c.f.m. in what follows) coligation associated with decomposition (3.2) for $S$ if:

1. The restrictions of operators $A$ and $C$ to the subspace $\tilde{D} \subset \mathcal{H}(\Phi)^d$ defined in (3.3) have the following action on special kernel functions:
   \[
   \tilde{A}|_{\tilde{D}} : \sum_{j=1}^{d} \zeta_j \Phi_{\bullet j}(, \zeta) u \rightarrow T(, \zeta) u - T(, 0) u,
   \]
   \[
   \tilde{C}|_{\tilde{D}} : \sum_{j=1}^{d} \zeta_j \Phi_{\bullet j}(, \zeta) u \rightarrow S(\zeta) u - S(0) u.
   \]

2. The operators $\tilde{B} : \mathcal{U} \rightarrow \mathcal{H}(\mathbb{K}_R)^q$ and $\tilde{D} : \mathcal{U} \rightarrow \mathcal{Y}$ are given by
   \[
   \tilde{B} : u \mapsto T(, 0) u, \quad \tilde{D} : u \mapsto S(0) u.
   \]

The following theorem is parallel to Theorem 2.2; note that when $d = 1$ this theorem amounts to part (2) of Theorem 1.2.
Theorem 3.3. Given a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ with a a fixed decomposition (3.2), let $\tilde{V}$ be the isometric operator defined in (3.5). Then

(1) A block-operator matrix $\mathbf{U}$ of the form (3.9) is a d.c.f.m. colligation for $S$ if and only if $\mathbf{U}$ is a contractive extension of $\tilde{V}$ to all of $\mathcal{H}(\Phi)^d \oplus \mathcal{Y}$. In particular, a d.c.f.m. colligation for $S$ exists.

(2) Every d.c.f.m. colligation $\mathbf{U}$ for $S$ is weakly isometric and controllable and furthermore, $S(z) = D + C(I - Z_\mathcal{H}(\Phi)(z)A)^{-1}Z_\mathcal{H}(\Phi)(z)B$.

(3) Any controllable weakly isometric colligation $\tilde{U}$ (1.20) with its characteristic function equal $S$ is unitarily equivalent to some d.c.f.m. colligation for $S$ based on the Agler decomposition (3.2) with $\Phi_{\tilde{d}}(z, \zeta)$ given by

$$\Phi_{\tilde{d}}(z, \zeta) = \tilde{B}^*(I - Z_X(z)^*A^*)^{-1}\mathbf{I}_d(\mathcal{Z}_z)$$

The proof can be extracted from the proof of Theorem 4.11 in the next section and will be omitted.

4. Weakly unitary realizations

In this section we will construct functional model weakly-unitary realizations for functions $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. The state space for these realizations will be the reproducing kernel Hilbert space with reproducing kernel $\mathcal{K}$ and will be omitted.

Theorem 4.1. Let $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ and let $M(z)$ and $N_j(z)$ be defined as in (4.1). The kernel $K_S$ defined in (1.13) can be extended to the positive kernel

$$K(z, \zeta) = \begin{bmatrix} K_S(z, \zeta) & \Psi_1(z, \zeta) & \cdots & \Psi_d(z, \zeta) \\ \Psi_1^*(z, \zeta) & \Phi_{11}(z, \zeta) & \cdots & \Phi_{1d}(z, \zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_d^*(z, \zeta) & \Phi_{d1}(z, \zeta) & \cdots & \Phi_{dd}(z, \zeta) \end{bmatrix} : \mathbb{B}^d \times \mathbb{B}^d \rightarrow \mathcal{L}(\mathcal{Y} \oplus \mathcal{U}^d)$$

subject to identity

$$M(z)^*K(z, \zeta)M(\zeta) = \sum_{j=1}^{d} N_j(z)^*K(z, \zeta)N_j(\zeta).$$

Proof. By Theorem 1.3 we know that any $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ can be realized as in (1.16) with $\mathbf{U}$ as in (1.15) unitary. It is then a straightforward calculation to show that the kernel

$$\mathcal{K}(z, \zeta) = G(z)G(\zeta)^*$$

with

$$G(z) := \begin{bmatrix} C(I - Z_X(z)A)^{-1} \\ B^*(I - Z_X(z)^*A^*)^{-1}\mathbf{I}_d \\ \vdots \\ B^*(I - Z_X(z)^*A^*)^{-1}\mathbf{I}_d \end{bmatrix} : \mathbb{B}^d \rightarrow \mathcal{L}(\mathcal{X}, \mathcal{Y} \oplus \mathcal{U}^d)$$
provides a positive-kernel solution of the identity (4.3). Note that \( K(z, \zeta) \) in (4.4) has the form (4.2) with

\[
\Psi_k(z, \zeta) = \left(C(I - Z_X(z)A)^{-1}I_k(I - AZ_X(\zeta))^{-1}B, \right. \\
\Phi_{ij}(z, \zeta) = \left(B^*(I - Z_X(z)^*A^*)^{-1}I_iI_j^*(I - AZ_X(z))^{-1}B. \right)
\]

Identity (4.3) (as well as the kernel \( K \) itself) will be called an Agler decomposition for \( S \). Equating the diagonal block entries in (4.3) one gets (2.5) and (3.2); equality of nondiagonal blocks gives

\[
S(z) - S(\zeta) = \sum_{j=1}^{d} (z_j - \zeta_j) \Psi_j(z, \zeta). 
\]

We let \( \mathcal{H}(K) \) be the reproducing kernel Hilbert space associated with the kernel \( K \) and remark that the elements of \( \mathcal{H}(K) \) are the \( \mathcal{Y} \oplus \mathcal{U}^d \)-valued functions of the form

\[
f = \begin{bmatrix} f_+ \\ f_- \end{bmatrix} : \mathbb{B}^d \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U}^d \end{bmatrix}, \quad f_- = \bigoplus_{i=1}^{d} f_{-,i},
\]

where \( f_+ \) is analytic and \( f_- \) is conjugate-analytic on \( \mathbb{B}^d \). For functions \( g \in \mathcal{H}(K)^d \), we will use the following representation and notation

\[
g = \bigoplus_{i=1}^{d} g_i := \begin{bmatrix} g_1 \\ \vdots \\ g_d \end{bmatrix} : \quad g_i = \begin{bmatrix} g_{i,+} \\ g_{i,-} \end{bmatrix} \in \mathcal{H}(K), \quad g_{i,-} = \bigoplus_{j=1}^{d} g_{i,-,j}. 
\]

We next observe that Agler decomposition (4.3) can be written in the inner product form as the identity

\[
\langle y + S(\zeta)u, y' + S(z)u' \rangle_{\mathcal{Y}} - \langle S(\zeta)^*y + u, S(z)^*y' + u' \rangle_{\mathcal{U}} \\
= \left\langle K(\cdot, \cdot)M(\zeta) \begin{bmatrix} y \\ u \end{bmatrix}, K(\cdot, z)M(z) \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{H}(K)} \\
- \left\langle \bigoplus_{j=1}^{d} K(\cdot, \zeta)N_j(\zeta) \begin{bmatrix} y \\ u \end{bmatrix}, \bigoplus_{j=1}^{d} K(\cdot, z)N_j(z) \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{H}(K)^d}. 
\]

holding for all \( z, \zeta \in \mathbb{B}^d, y, y' \in \mathcal{Y} \) and \( u, u' \in \mathcal{U} \). We next denote by \( K_0, K_1, \ldots, K_d \) the block columns of the kernel (4.2):

\[
K_0(z, \zeta) = \begin{bmatrix} K_s(z, \zeta) \\ \Psi_1(z, \zeta)^* \\ \vdots \\ \Psi_d(z, \zeta)^* \end{bmatrix}, \quad \Psi_j(z, \zeta) = \begin{bmatrix} \Psi_j(z, \zeta) \\ \Phi_{1j}(z, \zeta) \\ \vdots \\ \Phi_{dj}(z, \zeta) \end{bmatrix} \quad (j = 1, \ldots, d)
\]

and use them to define a new kernel

\[
T(z, \zeta) := \begin{bmatrix} K_1(z, \zeta) \\ \vdots \\ K_d(z, \zeta) \end{bmatrix} : \mathbb{B}^d \times \mathbb{B}^d \to \mathcal{L}(\mathcal{U}, (\mathcal{Y} \oplus \mathcal{U}^d)^d).
\]
The relations
\[
K(\cdot, \zeta)M(\zeta) \begin{bmatrix} y \\ u \end{bmatrix} = K_0(\cdot, \zeta)y + \sum_{j=1}^{d} \zeta_j K_j(\cdot, \zeta)u,
\]
\[
K(\cdot, \zeta)N_j(\cdot) \begin{bmatrix} y \\ u \end{bmatrix} = \bar{\zeta}_j K_0(\cdot, \zeta)y + K_j(\cdot, \zeta)u,
\]
follow immediately from (4.11) and (4.12) and allow us to rewrite (4.10) as
\[
\begin{bmatrix} Z_{row}(\cdot)^* \otimes K_0(\cdot, \zeta)y + T(\cdot, \zeta)u \\ y + S(\zeta)u \end{bmatrix}, \begin{bmatrix} Z_{row}(\cdot)^* \otimes K_0(\cdot, z)y' + T(\cdot, z)u' \\ y' + S(z)u' \end{bmatrix}
\]
and
\[
\begin{bmatrix} K_0(\cdot, \zeta)y + \sum_{j=1}^{d} \zeta_j K_j(\cdot, \zeta)u \\ S(\zeta)^* y + u \end{bmatrix}, \begin{bmatrix} K_0(\cdot, z)y' + \sum_{j=1}^{d} \zeta_j K_j(\cdot, z)u' \\ S(z)^* y' + u' \end{bmatrix}
\].
(4.13)

**Lemma 4.2.** Let $K$ be a fixed Agler decomposition for a function $S \in S_d(U, Y)$ and let $K_j$ and $T$ be given by (4.11), (4.12). Then the map
\[
V : \begin{bmatrix} Z_{row}(\cdot)^* \otimes K_0(\cdot, \zeta)y + T(\cdot, \zeta)u \\ y + S(\zeta)u \end{bmatrix} \rightarrow \begin{bmatrix} K_0(\cdot, \zeta)y + \sum_{j=1}^{d} \zeta_j K_j(\cdot, \zeta)u \\ S(\zeta)^* y + u \end{bmatrix}
\]
(4.14)
extends by linearity and continuity to an isometry from
\[
D_V = D \oplus Y \quad \text{onto} \quad R_V = R \oplus U
\]
(4.15)
where the subspaces $D \subset H(K)^d$ and $R \subset H(K)$ are given by
\[
D = \bigvee \left\{ Z_{row}(\cdot)^* \otimes K_0(\cdot, \zeta)y, T(\cdot, \zeta)u : \zeta \in \mathbb{B}^d, y \in Y, u \in U \right\}, \quad (4.16)
\]
\[
R = \bigvee \left\{ K_0(\cdot, \zeta)y, \sum_{j=1}^{d} \zeta_j K_j(\cdot, \zeta)u : \zeta \in \mathbb{B}^d, y \in Y, u \in U \right\}. \quad (4.17)
\]

**Proof.** It follows from (4.13) that $V$ defined as in (4.14) extends by linearity and continuity to an isometry from
\[
D_V = \bigvee \left\{ Z_{row}(\cdot)^* \otimes K_0(\cdot, \zeta)y, T(\cdot, \zeta)u : \zeta \in \mathbb{B}^d, y \in Y, u \in U \right\}
\]
on to
\[
R_V = \bigvee \left\{ T(\cdot, \zeta)y, \sum_{j=1}^{d} \zeta_j K_j(\cdot, \zeta)u : \zeta \in \mathbb{B}^d, y \in Y, u \in U \right\}.
\]
It is readily seen that $D_V$ and $R_V$ contain respectively all vectors of the form $\begin{bmatrix} 0 \\ y \end{bmatrix}$ and $\begin{bmatrix} y \\ 0 \end{bmatrix}$ and therefore they split into the direct sums (4.15).

A straightforward verification shows that the orthogonal complements $D^\perp = H(K)^d \ominus D$ and $R^\perp = H(K) \ominus R$ (the defect spaces of the isometry $V$) can be described as
\[
D^\perp = \left\{ g \in H(K)^d : \sum_{i=1}^{d} z_i g_{i,+}(z) \equiv 0 \quad \text{and} \quad \sum_{i=1}^{d} g_{i,-,i}(z) \equiv 0 \right\}, \quad (4.18)
\]
\[
R^\perp = \left\{ f \in H(K) : f_{+}(z) \equiv 0 \quad \text{and} \quad \sum_{i=1}^{d} f_{-,i}(z) \equiv 0 \right\} \quad (4.19)
\]
where we have used notation (4.18) and (4.19). We next use the same notation to define two linear maps $s : \mathcal{H}(K) \to \mathcal{H}(K_S)$ and $\overline{s} : \mathcal{H}(K)^d \to \mathcal{H}(\Phi)$ by

$$s : f \mapsto f_+, \quad \overline{s} : g = \bigoplus_{i=1}^d g_i \mapsto \sum_{i=1}^d g_{i,-i}, \quad (4.20)$$

and observe the equalities

$$\langle f, \mathbb{K}_0(\cdot, \zeta) y \rangle_{\mathcal{H}(K)} = \langle (sf)(\zeta), y \rangle_{\mathcal{Y}}, \quad \langle g, \overline{T}(\cdot, \zeta) u \rangle_{\mathcal{H}(K)^*} = \langle (\overline{s}g)(\zeta), u \rangle_{\mathcal{U}} \quad (4.21)$$

holding for all $f \in \mathcal{H}(K)$, $g \in \mathcal{H}(K)^d$, $\zeta \in \mathbb{B}^d$, $y \in \mathcal{Y}$ and $u \in \mathcal{U}$. Indeed, for a function $f$ in $\mathcal{H}(K)$, we have from (4.11) by the reproducing kernel property

$$\langle f, \mathbb{K}_0(\cdot, \zeta) y \rangle_{\mathcal{H}(K)} = \left< f, \mathbb{K}(\cdot, \zeta) \begin{bmatrix} y \\ 0 \end{bmatrix} \right>_{\mathcal{H}(K)} = \langle f_+(\zeta), y \rangle_{\mathcal{Y}} = \langle (sf)(\zeta), y \rangle_{\mathcal{Y}}$$

which proves the first equality in (4.21). The proof of the second is much the same.

**Definition 4.3.** A contractive colligation

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K)^d \\ \mathcal{Y} \end{bmatrix} \quad (4.22)$$

will be called a two-component canonical functional-model (abbreviated to t.c.f.m. in what follows) colligation associated with a fixed Agler decomposition (4.3) of a given $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ if

1. The state space operator $A = \text{Col}_{1 \leq k \leq d} A_k$ solves the structured Gleason problem

$$\left( sf \right)(z) - \left( sf \right)(0) = \sum_{k=1}^d z_k \left( A_k f \right)_+(z) \quad \text{for all} \quad f \in \mathcal{H}(K), \quad (4.23)$$

whereas the adjoint operator $A^*$ solves the dual structured Gleason problem

$$\left( \overline{s}g \right)(z) - \left( \overline{s}g \right)(0) = \sum_{k=1}^q z_k \left( A^* g \right)_{-k}(z) \quad \text{for all} \quad g \in \mathcal{H}(K)^d. \quad (4.24)$$

2. The operators $C : \mathcal{H}(K) \to \mathcal{Y}$, $B^* : \mathcal{H}(K)^d \to \mathcal{U}$ and $D : \mathcal{U} \to \mathcal{Y}$ are of the form

$$C : f \to (sf)(0), \quad B^* : g \to (\overline{s}g)(0) \quad \text{and} \quad D : u \to S(0)u. \quad (4.25)$$

**Proposition 4.4.** Relations (4.22), (4.24) and (4.25) are equivalent respectively to equalities

$$A^* \left( Z_{\text{row}}(\zeta)^* \otimes \mathbb{K}_0(\cdot, \zeta) y \right) = \mathbb{K}_0(\cdot, \zeta) y - \mathbb{K}_0(\cdot, 0) y, \quad (4.26)$$

$$A \left( \sum_{j=1}^d \zeta_j K_j(\cdot, \zeta) u \right) = \overline{T}(\cdot, \zeta) u - \overline{T}(\cdot, 0) u, \quad (4.27)$$

$$C^* y = \mathbb{K}_0(\cdot, 0) y, \quad Bu = \overline{T}(\cdot, 0) u, \quad \text{and} \quad D^* y = S(0)^* y \quad (4.28)$$

holding for every $\zeta \in \mathbb{B}^d$, $y \in \mathcal{Y}$ and $u \in \mathcal{U}$.
Proof. It follows from the first equality in (4.21) that
\[ \langle (sf)(z) - (sf)(0), y \rangle_{\mathcal{Y}} = \langle f, \mathbb{K}_0(\cdot, z)y - \mathbb{K}_0(\cdot, 0)y \rangle_{\mathcal{H}(\mathbb{K})} \]
and on the other hand,
\[ \left\langle \sum_{k=1}^{d} (A_k f)_+(z), y \right\rangle_{\mathcal{Y}} = \sum_{k=1}^{d} \left\langle A_k f, \mathbb{K}_0(\cdot, z)y \right\rangle_{\mathcal{H}(\mathbb{K})} = \left\langle Af, Z_{\text{row}}(z)^* \mathbb{K}_0(\cdot, z)y \right\rangle_{\mathcal{H}(\mathbb{K})^d} = \left\langle f, A^* (Z_{\text{row}}(z)^* \mathbb{K}_0(\cdot, z)y) \right\rangle_{\mathcal{H}(\mathbb{K})^d}. \]
Since the two latter equalities hold for every \( f \in \mathcal{H}(\mathbb{K}) \) and \( y \in \mathcal{Y} \), the equivalence (4.23) \( \Leftrightarrow \) (4.26) follows. The equivalence (4.21) \( \Leftrightarrow \) (4.27) follows from (4.21) in much the same way; the formula for \( C^* \) in (4.27) follows from
\[ \langle f, C^* y \rangle = \langle Cf, y \rangle = \langle (sf)(0), y \rangle = \langle f, \mathbb{K}_0(\cdot, 0)y \rangle \]
and the formula for \( D^* \) is a consequence of a similar computation. The formula for \( D^* \) is self-evident. \( \square \)

**Proposition 4.5.** Let \( B, C \) and \( D \) be the operators defined in (4.25). Then
\[ CC^* + DD^* = I_{\mathcal{Y}} \quad \text{and} \quad B^* B + D^* D = I_{\mathcal{Y}}. \]
Furthermore,
\[ B^*: Z_{\text{row}}(\zeta)^* \mathbb{K}_0(\cdot, \zeta)y \rightarrow S(\zeta)^* y - S(0)^* y, \quad (4.30) \]
\[ B^*: T(\cdot, \zeta)u \rightarrow u - S(0)^* S(\zeta)u \]
for all \( \zeta \in \mathbb{H}^d, y \in \mathcal{Y} \) and \( u \in \mathcal{U} \), where \( \mathbb{K}_0 \) and \( T \) are defined in (4.11), (4.12).

**Proof.** Upon letting \( f = \mathbb{K}_0(\cdot, \zeta)y \) and \( g = T(\cdot, \zeta)u \) in formulas (4.21) and making use of (4.11) we get
\[ \langle \mathbb{K}_0(\cdot, \zeta)y, \mathbb{K}_0(\cdot, z)y \rangle_{\mathcal{H}(\mathbb{K})^p} = \langle K_S(z, \zeta)y, y \rangle_{\mathcal{Y}}, \]
\[ \langle T(\cdot, \zeta)u, T(\cdot, z)u \rangle_{\mathcal{H}(\mathbb{K})^d} = \sum_{j=1}^{d} \langle \Phi_{jj}(z, \zeta)u, u \rangle_{\mathcal{U}}. \]
We then have
\[ ||C^* y||^2 = ||\mathbb{K}_0(\cdot, 0)y||^2 = \langle K_S(0, 0)y, y \rangle = \langle (I - S(0)^* S(0))y, y \rangle, \]
\[ ||Bu||^2 = ||T(\cdot, 0)u||^2 = \left( \sum_{k=1}^{d} \Phi_{kk}(0, 0)u, u \right) = \langle (I - S(0)^* S(0))u, u \rangle, \]
where the first equalities follow from formulas (4.28) for \( B \) and \( C^* \), the second equalities follow upon letting \( z = \zeta = 0 \) in (4.32), (4.33), and finally, the third equalities follow from the representation formulas (4.22) and (4.3) evaluated at \( z = \zeta = 0 \). Taking into account formulas (4.25) and (4.28) for \( D \) and \( D^* \), we then have equalities
\[ ||C^* y||^2 = ||y||^2 - ||S(0)^* y||^2 = ||y||^2 - ||D^* y||^2, \quad (4.34) \]
\[ ||Bu||^2 = ||u||^2 - ||S(0)u||^2 = ||u||^2 - ||Du||^2 \]
holding for all \( y \in \mathcal{Y} \) and \( u \in \mathcal{U} \) which are equivalent to operator equalities (4.29).
By definitions \((4.25)\) of \(B^*\) and \((4.11), (4.12)\) of \(\mathbb{K}_j\) and \(T\),

\[
B^* (Z_{row}(\cdot, \cdot)^* \otimes \mathbb{K}_0)(\cdot, \cdot) y = \mathbb{s} (Z_{row}(\cdot, \cdot)^* \otimes \mathbb{K}_0)(\cdot, \cdot) y(0) = \sum_{j=1}^{d} \zeta_j \Psi_j(\cdot, 0)^* y, \quad (4.35)
\]

\[
B^* T(\cdot, \cdot) u = \mathbb{s} (T(\cdot, \cdot) u)(0) = \sum_{j=1}^{d} \Phi_{jj}(0, \cdot) u. \quad (4.36)
\]

Upon letting \(z = 0\) in \((4.7)\) and \((3.2)\) we get

\[
S(\cdot, 0)^* - S(0)^* = \sum_{j=1}^{d} \zeta_j \Psi_j(\cdot, 0)^* \quad \text{and} \quad I_{\mathcal{H}} - S(0)^* S(\cdot) = \sum_{j=1}^{d} \Phi_{jj}(0, \cdot) \quad (4.37)
\]

which being combined with \((4.35)\) and \((4.36)\) give \((4.30)\) and \((4.31)\).

Formulas \((4.30), (4.31)\) describing the action of the operator \(B^*\) on elementary kernels of the subspace \(D\) defined in \((4.16)\) were easily obtained from the general formula \((4.25)\) for \(B^*\). Although the operator \(A^*\) is not defined in Definition \((4.3)\) on the whole space \(\mathcal{H}(\mathbb{K})^d\), it turns out that its action on elementary kernels of \(D\) is completely determined by conditions \((4.23)\) and \((4.24)\). Formula \((4.26)\) (which is equivalent to \((4.23)\)) does half of the job; the next proposition takes care of the other half.

**Proposition 4.6.** Let \(\mathcal{U} = \{A \in \mathbb{C}^{d \times d} \} \) be a t.c.f.m. colligation associated with the Agler decomposition \((4.3)\) of a given \(S \in S_d(\mathcal{U}, \mathcal{Y})\) and let \(T\) be given by \((4.12)\). Then

\[
A^* T(\cdot, \cdot) u = \sum_{j=1}^{d} \zeta_j \mathbb{K}_j(\cdot, \cdot) u - \mathbb{K}_0(\cdot, 0) S(\cdot) u \quad (\zeta \in \mathbb{B}^d, \ y \in \mathcal{Y}, \ u \in \mathcal{U}). \quad (4.38)
\]

**Proof.** We have to show that formula \((4.38)\) follows from conditions in Definition \((4.3)\)

To this end, we first verify the equality

\[
\| h_{\zeta, u} \|^2_{\mathcal{H}(\mathbb{K})} - \| A h_{\zeta, u} \|^2_{\mathcal{H}(\mathbb{K})^d} = \| C h_{\zeta, u} \|^2_{\mathcal{U}}, \quad \text{where} \quad h_{\zeta, u} = \sum_{j=1}^{d} \zeta_j \mathbb{K}_j(\cdot, \cdot) u. \quad (4.39)
\]

Indeed, it follows from the explicit formula \((4.25)\) for \(C\) that

\[
Ch_{\zeta, u} = \mathbb{s} \left( \sum_{j=1}^{d} \zeta_j \mathbb{K}_j(\cdot, \cdot) u \right)(0) = \sum_{j=1}^{d} \zeta_j \Psi_j(0, \cdot) u = S(\cdot) u - S(0) u \quad (4.40)
\]

where the last equality is a consequence of \((4.7)\). By the reproducing kernel property,

\[
(\mathbb{K}_i(\cdot, \cdot) u, \mathbb{K}_l(\cdot, \cdot) v)_{\mathcal{H}(\mathbb{K})} = (\Phi_{il}(\zeta, \zeta) u, v)_{\mathcal{U}}
\]

for \(i, l = 1, \ldots, d\), and therefore,

\[
\| h_{\zeta, u} \|^2_{\mathcal{H}(\mathbb{K})} = \left\| \sum_{j=1}^{d} \zeta_j \mathbb{K}_j(\cdot, \cdot) u \right\|^2_{\mathcal{H}(\mathbb{K})} = \sum_{i, l=1}^{d} \zeta_i \zeta_l \Phi_{il}(\zeta, \zeta). \quad (4.41)
\]
Making use of (1.27) (which holds by Proposition 1.4) and of (4.33) we have
\[
\|Ah\zeta,u\|_{\mathcal{H}(\mathbb{K})^d}^2 = \|T(\cdot,\zeta)u - T(\cdot,0)u\|_{\mathcal{H}(\mathbb{K})^d}^2
= \sum_{j=1}^d \langle (\Phi_{jj}(\zeta,\zeta) - \Phi_{jj}(\zeta,0) - \Phi_{jj}(0,\zeta) + \Phi_{jj}(0,0)) u, u \rangle_{\mathcal{U}}. \tag{4.42}
\]

Observe that by (3.22),
\[
\sum_{j=1}^d (\Phi_{jj}(\zeta,\zeta) - \Phi_{jj}(\zeta,0) - \Phi_{jj}(0,\zeta) + \Phi_{jj}(0,0)) - \sum_{i,t=1}^d \zeta_i\Phi_{it}\zeta_t
= I_\mathcal{U} - S(\zeta)^*S(\zeta) - (I_\mathcal{U} - S(\zeta)^*S(0)) - (I_\mathcal{U} - S(0)^*S(\zeta)) + I_\mathcal{U} - S(0)^*S(0)
= -(S(\zeta)^* - S(0)^*)(S(\zeta) - S(0)).
\]

Subtracting (4.42) from (4.41) and taking into account the last identity we get
\[
\|h_{\zeta,u}\|^2 - \|Ah_{\zeta,u}\|^2 = \|S(\zeta)u - S(0)u\|_{\mathcal{Y}}^2
\]
which proves (4.39), due to (4.40). Writing (4.39) in the form
\[
\langle (I - A^*A - C^*C)h_{\zeta,u}, h_{\zeta,u} \rangle_{\mathcal{H}(\mathbb{K})^d} = 0
\]
and observing that the operator $I - A^*A - C^*C$ is positive semidefinite (since $\mathcal{U}$ is contractive by Definition 4.3), we conclude that
\[
(I - A^*A - C^*C)h_{\zeta,u} \equiv 0 \quad \text{for all } \zeta \in \mathbb{B}^d, u \in \mathcal{U}. \tag{4.43}
\]

Applying the operator $C^*$ to both parts of (4.40) we get
\[
C^*Ch_{\zeta,u} = \mathcal{H}_0(\cdot,0)(S(\zeta) - S(0))u \tag{4.44}
\]
by the explicit formula (4.28) for $C^*$. From the same formula and the formula (1.26) for $D$ we get
\[
C^*Du = C^*S(0)^*u = \mathcal{H}_0(\cdot,0)S(0)u. \tag{4.45}
\]
We next apply the operator $A^*$ to both parts of equality (1.27) to get
\[
A^*Ah_{\zeta,u} = A^*T(\cdot,\zeta)u - A^*T(\cdot,0)u.
\]
Due to the second formula in (4.28) (which holds by Proposition 1.4) the latter equality can be written as
\[
A^*T(\cdot,\zeta)u = A^*Ah_{\zeta,u} + A^*Bu. \tag{4.46}
\]

Since $\mathcal{U}$ is contractive (by Definition 4.3) and since $B$ and $D$ satisfy the second equality in (4.29), it then follows that $A^*B + C^*D = 0$. Thus,
\[
A^*Bu = -C^*Du = -C^*S(0)^*u = -\mathcal{H}_0(\cdot,0)S(0)u.
\]
Taking the latter equality into account and making subsequent use of (4.43)–(4.45) we then get from (4.46)
\[
A^*T(\cdot,\zeta)u = (I - C^*C)h_{\zeta,u} - C^*Du
\]
\[
= h_{\zeta,u} - \mathcal{H}_0(\cdot,0)(S(\zeta) - S(0))u - \mathcal{H}_0(\cdot,0)S(0)u
\]
\[
= \sum_{j=1}^d \zeta_jK_j(\cdot,\zeta)u - \mathcal{H}_0(\cdot,0)S(\zeta)u
\]
which completes the proof of (4.38). □
Remark 4.7. Since any t.c.f.m. colligation is contractive, we have in particular that $AA^* + BB^* \leq I$. Therefore, formulas (4.30), (4.31) and (4.38), (4.26) defining the action of operators $B^*$ and $A^*$ on elementary kernels of the space $D$ (see (4.16)) can be extended by continuity to define these operators on the whole space $D$.

Proposition 4.8. Any t.c.f.m. colligation $U = [A \ B]_\lambda$ associated with a fixed Agler decomposition (4.3) of a given $S \in S_d(U, Y)$ is weakly unitary and closely connected. Furthermore,

$$S(z) = D + C(I - Z_{\mathcal{H}(K)}(z)A)^{-1}Z_{\mathcal{H}(K)}(z)B.$$  

(4.47)

Proof. Let $U = [A \ B]_\lambda$ be a t.c.f.m. colligation of $S$ associated with a fixed Agler decomposition (4.3). Then equalities (4.20)–(4.28) hold by Proposition 4.3. Upon representing the left hand side expressions in (4.26), (4.28) as $A^*Z_{\mathcal{H}(K)}(\zeta)^*\mathbb{K}_0(\cdot, \zeta)y$ and $AZ_{\mathcal{H}(K)}(\zeta)\mathcal{T}(\cdot, \zeta)u$ respectively and replacing $\zeta$ by $z$, we then solve the system (4.26)–(4.28) for $\mathcal{T}(\cdot, z)y$ and $\mathcal{T}(\cdot, z)u$ as follows:

$$\mathbb{K}_0(\cdot, z)y = (I - A^*Z_{\mathcal{H}(K)}(z)^*)^{-1}\mathbb{K}_0(\cdot, 0)y = (I - A^*Z_{\mathcal{H}(K)}(z)^*)^{-1}C^*y,$$  

(4.48)

$$\mathcal{T}(\cdot, z)u = (I - AZ_{\mathcal{H}(K)}(z))^{-1}\mathcal{T}(\cdot, 0)u = (I - AZ_{\mathcal{H}(K)}(z))^{-1}Bu.$$  

(4.49)

From (4.48) and (4.30) we conclude that equalities

$$(D^* + B^*Z_{\mathcal{H}(K)}(z)^*)(I - A^*Z_{\mathcal{H}(K)}(z)^*)^{-1}C^*y$$

$$= S(0)^*y + B^*Z_{\mathcal{H}(K)}(z)^*\mathbb{K}_0(\cdot, z)y$$

$$= S(0)^*y + S(z)^*y - S(0)^*y = S(z)^*y$$

(4.50)

hold for every $z \in \mathbb{B}^d$ and $y \in \mathcal{Y}$, which proves representation (4.47). Furthermore, in view of (4.11) and (4.12),

$$\mathcal{H}_{C, A}^\mathcal{C} := \bigvee \{ (I - A^*Z_{\mathcal{H}(K)}(z)^*)^{-1}C^*y : z \in \mathbb{B}^d, \ y \in \mathcal{Y} \}$$

$$= \bigvee \{ \mathbb{K}_0(\cdot, z)y : z \in \mathbb{B}^d, \ y \in \mathcal{Y} \},$$

$$\mathcal{H}_{A, B}^\mathcal{C} := \bigvee \{ \mathcal{T}_j^*(I - AZ_{\mathcal{H}(K)}(z))^{-1}Bu : z \in \mathbb{B}^d, \ u \in \mathcal{U}, \ j = 1, \ldots, d \}$$

$$= \bigvee \{ \mathcal{T}_j^*\mathcal{T}(\cdot, z)u : z \in \mathbb{B}^d, \ u \in \mathcal{U}, \ j = 1, \ldots, d \}$$

$$= \bigvee \{ \mathbb{K}_j(\cdot, z)u_j : z \in \mathbb{B}^d, \ u_j \in \mathcal{U}, \ j = 1, \ldots, d \},$$

and therefore,

$$\mathcal{H}_{C, A}^\mathcal{C} \bigwedge \mathcal{H}_{A, B}^\mathcal{C} = \bigvee \{ \mathbb{K}_0(\cdot, z)y, \mathbb{K}_j(\cdot, z)u_j : z \in \mathbb{B}^d, \ y \in \mathcal{Y}, \ u_j \in \mathcal{U}, \ j = 1, \ldots, d \}$$

$$= \bigvee \{ \mathbb{K}(\cdot, z) \begin{bmatrix} y \\ u \end{bmatrix} : z \in \mathbb{B}^d, \begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{Y} \oplus \mathcal{U}^d \} = \mathcal{H}(\mathcal{K})$$

where the last equality follows by the very construction of the reproducing kernel Hilbert space. The colligation $U = [A \ B]_\lambda$ is closely connected by Definition 1.4. To show that $U$ is weakly unitary, we let $u = u' = 0$, $y = y'$ and $z = \zeta$ in (4.13) to get

$$\left\| \begin{bmatrix} Z_{\mathcal{H}(K)}(\zeta)^*\mathbb{K}_0(\cdot, \zeta)y \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} \mathbb{K}_0(\cdot, \zeta)y \\ S(\zeta)^*y \end{bmatrix} \right\|$$

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which on account of (4.48) can be written as
\[
\left\| \begin{bmatrix} Z_{H(\mathbb{K})}(\zeta^*)(I - A^*Z_{H(\mathbb{K})}(\zeta^*))^{-1}C^*y \\ y \end{bmatrix} \right\| = \left\| \begin{bmatrix} (I - A^*Z_{H(\mathbb{K})}(\zeta^*))^{-1}C^*y \\ S(\zeta)^*y \end{bmatrix} \right\|.
\]  
(4.51)

Since
\[
\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} Z_{H(\mathbb{K})}(\zeta^*)(I - A^*Z_{H(\mathbb{K})}(\zeta^*))^{-1}C^*y \\ y \end{bmatrix} = \begin{bmatrix} (I - A^*Z_{H(\mathbb{K})}(\zeta^*))^{-1}C^*y \\ S(\zeta)^*y \end{bmatrix}
\]
the top components in the latter formula are equal automatically whereas the bottom components are equal due to (4.47), equality (4.51) tells us that \( U \) is weakly coisometric by Definition 1.4. Similarly letting \( u = u' \) and \( y = y' = 0 \) in (4.13) we get
\[
\left\| \begin{bmatrix} T(\cdot, \zeta)u \\ S(\zeta)u \end{bmatrix} \right\| = \left\| \begin{bmatrix} Z_{H(\mathbb{K})}(\zeta)T(\cdot, \zeta)u \\ u \end{bmatrix} \right\|
\]
which in view of (4.49) can be written as
\[
\left\| \begin{bmatrix} (I - AZ_{H(\mathbb{K})}(\zeta))^{-1}Bu \\ S(\zeta)u \end{bmatrix} \right\| = \left\| \begin{bmatrix} Z_{H(\mathbb{K})}(\zeta)(I - AZ_{H(\mathbb{K})}(\zeta))^{-1}Bu \\ u \end{bmatrix} \right\|
\]
and since
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z_{H(\mathbb{K})}(\zeta)(I - AZ_{H(\mathbb{K})}(\zeta))^{-1}Bu \\ u \end{bmatrix} = \begin{bmatrix} (I - AZ_{H(\mathbb{K})}(\zeta))^{-1}Bu \\ S(\zeta)u \end{bmatrix}
\]
(again, the top components are equal automatically and the bottom components are equal due to (4.47)), the colligation \( U \) is weakly isometric by Definition 1.4. \( \square \)

Proposition 4.8 establishes common features of t.c.f.m. colligations leaving the question about the existence of at least one such colligation open.

Lemma 4.9. Given an Agler decomposition \( \mathbb{K} \) for a function \( S \in S_d(\mathcal{U}, \mathcal{Y}) \), let \( V \) be the isometric operator associated with this decomposition as in (4.14). A block-operator matrix \( U = [A \ B] \) of the form (4.22) is a t.c.f.m. colligation associated with \( \mathbb{K} \) if and only if
\[
\|U^*\| \leq 1, \quad U^*|_{\mathcal{D} \oplus \mathcal{Y}} = V \quad \text{and} \quad B^*|_{\mathcal{D}^\perp} = 0,
\]
that is, \( U^* \) is a contractive extension of \( V \) from \( \mathcal{D} \oplus \mathcal{Y} \) to all of \( \mathcal{H}(\mathbb{K})^d \oplus \mathcal{Y} \) subject to the condition \( B^*|_{\mathcal{D}^\perp} = 0 \).

Proof. Let us write the isometry \( V \) from (4.14) in the form
\[
V = \begin{bmatrix} A_V & B_V \\ C_V & D_V \end{bmatrix} : \begin{bmatrix} \mathcal{D} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \mathcal{R} \\ \mathcal{U} \end{bmatrix}.
\]
(4.53)
Then we get from (4.13) the following relations for the block entries \( A_V, B_V, C_V, \) \( D_V \):
\[
A_V (Z_{row}(\zeta)^* \otimes \mathbb{K}_0(\cdot, \zeta)) y + B_V y = \mathbb{K}_0(\cdot, \zeta) y, \tag{4.54}
\]
\[
A_V T(\cdot, \zeta) u + B_V S(\zeta) u = \sum_{j=1}^d \zeta_j \mathbb{K}_j(\cdot, \zeta) u, \tag{4.55}
\]
\[
C_V (Z_{row}(\zeta)^* \otimes \mathbb{K}_0(\cdot, \zeta)) y + D_V y = S(\zeta)^* y, \tag{4.56}
\]
\[
C_V T(\cdot, \zeta) u + D_V S(\zeta) u = u. \tag{4.57}
\]
Indeed, equalities (4.54) and (4.55) are obtained upon equating the top components in (4.14) for the respective special cases \( u = 0 \) and \( y = 0 \). Equalities (4.56) and (4.57) are obtained similarly upon equating the bottom components in (4.14). Letting \( \zeta = 0 \) in (4.54) and (4.56) gives

\[
B_V y = K_0(\cdot,0)y \quad \text{and} \quad D_V y = S(0)^*y.
\]

(4.58)

Substituting the first and the second formula in (4.58) respectively into (4.54), (4.55) and into (4.56) and (4.57) results in equalities

\[
A_V : Z_{row}(\zeta)^* \otimes K_0(\cdot,\zeta)y \rightarrow K_0(\cdot,\zeta)y - K_0(\cdot,0)y,
\]

(4.59)

\[
A_V : T(\cdot,\zeta)u \rightarrow \sum_{j=1}^{d} \zeta_j K_j(\cdot,\zeta)u - K_0(\cdot,0)S(\zeta)u,
\]

(4.60)

\[
C_V : Z_{row}(\zeta)^* \otimes K_0(\cdot,\zeta)y \rightarrow S(\zeta)^*y - S(0)^*y,
\]

(4.61)

\[
C_V : T(\cdot,\zeta)u \rightarrow u - S(0)^*S(\zeta)u
\]

(4.62)

holding for all \( \zeta \in B^d \), \( u \in U \) and \( y \in \mathcal{Y} \) and completely defining the operators \( A_V \) and \( C_V \) on the whole space \( \mathcal{D} \).

Let \( U = [A \ B] \) be a t.c.f.m. colligation associated with \( \mathbb{K} \). Then \( U \) is contractive by definition and relations (4.26)–(4.28) and (4.38) hold by Propositions 4.4 and 4.6. Comparing (4.20) and (4.38) with (4.59), (4.60) we see that \( A^*|D = A_V \). Comparing (4.30), (4.31) with (4.61), (4.62) we conclude that \( B^*|D = C_V \). Also, it follows from (4.28) and (4.58) that \( C^* = B_V \) and \( D^* = D_V \). Finally, the second of formulas (4.25) combined with the characterization (4.18) of \( D^+ \) enables us to see that \( B^*f = \overline{s}f = 0 \)

for every \( f \in D^+ \). The last equality in (4.62) now follows.

Conversely, let us assume that a colligation \( U = [A \ B] \) meets all the conditions in (4.52). From the second relation in (4.52) we conclude the equalities (4.58)–(4.62) hold with operators \( A_V, B_V, C_V \) and \( D_V \) replaced by \( A^*, C^* \). \( B^* \) and \( D^* \) respectively. In other words, we conclude from (4.58) that \( C^* \) and \( D^* \) are defined exactly as in (4.28) which means (by Proposition 4.4) that they are already of the requisite form. Equalities (4.61), (4.62) tell us that the operator \( B^* \) satisfies formulas (4.30), (4.31). As we have seen in the proof of Proposition 4.6 these formulas agree with the second formula in (4.25) and then define \( B^* \) on the whole space \( H(\mathbb{K})^d \). From the third condition in (4.52) we now conclude that \( B^* \) is defined by formula (4.26) on the whole space \( H(\mathbb{K})^d \) and therefore, \( B \) is also of the requisite form. The formula (4.59) (with \( A^* \) instead of \( A_V \)) leads us to (4.26) which means that \( A \) solves the Gleason problem (4.24).

To complete the proof, it remains to show that \( A^* \) solves the dual Gleason problem (4.24) or equivalently, that (4.27) holds. Rather than (4.27), what we know is the equality (4.55) (with \( A^* \) and \( C^* \) instead of \( A_V \) and \( B_V \) respectively):

\[
A^* T(\cdot,\zeta)u = \sum_{j=1}^{d} \zeta_j K_j(\cdot,\zeta)u - C^* S(\zeta)u.
\]

(4.63)

We use (4.63) to show that equality

\[
\|T(\cdot,\zeta)u\|_{\mathcal{H}(\mathbb{K})^d}^2 - \|A^* T(\cdot,\zeta)u\|_{\mathcal{H}(\mathbb{K})}^2 = \|B^* T(\cdot,\zeta)u\|_{U}^2
\]

(4.64)
holds for every $\zeta \in \mathbb{B}^d$ and $u \in \mathcal{U}$. Indeed, it follows from (4.63) that

$$\|T(\cdot, \zeta)u\|^2 - \|A^*T(\cdot, \zeta)u\|^2 = \|T(\cdot, \zeta)u\|^2 - \sum_{j=1}^d \|\zeta_j K_j(\cdot, \zeta)u - C^*S(\zeta)u\|^2$$

$$= \|T(\cdot, \zeta)u\|^2 - \sum_{j=1}^d \|\zeta_j K_j(\cdot, \zeta)u\|^2 - \|C^*S(\zeta)u\|^2$$

$$- 2\Re \left\langle C \left( \sum_{j=1}^d \zeta_j K_j(\cdot, \zeta)u \right), S(\zeta)u \right\rangle. \quad (4.65)$$

We next express all the terms on the right of (4.65) in terms of the function $S$: $\|T(\cdot, \zeta)u\|^2$ and $\sum_{j=1}^d \|\zeta_j K_j(\cdot, \zeta)u\|^2$ follow from (4.33), (4.41) and (3.2); the second equality is a consequence of (4.40); the third equality is obtained upon letting $y = S(\zeta)u$ in (4.33). We now substitute the three last equalities into (4.65) to get

$$\|T(\cdot, \zeta)u\|^2 - \|A^*T(\cdot, \zeta)u\|^2 = \|R(\zeta)u, u\|_{\mathcal{H}(\mathcal{K})}$$

where

$$R(\zeta) = I_{\mathcal{U}} - S(\zeta)^*S(\zeta) + S(\zeta)^*(S(\zeta) - S(0))$$

$$+ (S(\zeta)^* - S(0)^*)S(\zeta) - S(\zeta)^*S(\zeta) + S(\zeta)^*S(0)S(0)^*S(\zeta)$$

$$= I_{\mathcal{U}} - S(\zeta)^*S(0) - S(0)^*S(\zeta) + S(\zeta)^*S(0)S(0)^*S(\zeta)$$

$$= (I_{\mathcal{U}} - S(\zeta)^*S(0))(I_{\mathcal{U}} - S(0)^*S(\zeta)).$$

By (4.31) we have

$$B^*T(\cdot, \zeta)u = u - S(0)^*S(\zeta)u$$

and therefore

$$\|B^*T(\cdot, \zeta)u\|^2_{\mathcal{U}(\mathcal{K})} = \|u - S(0)^*S(\zeta)u\|^2_{\mathcal{U}(\mathcal{K})} = \langle R(\zeta)u, u\rangle_{\mathcal{H}(\mathcal{K})},$$

which together with (4.66) completes the proof of (4.64). Writing (4.64) as

$$\langle (I - AA^* - BB^*)T(\cdot, \zeta)u, T(\cdot, \zeta)u\rangle = 0$$

and observing that the operator $I - AA^* - BB^*$ is positive semidefinite (since $U = [A \ B]$ is a contraction), we conclude that

$$(I - AA^* - BB^*)T(\cdot, \zeta)u = 0 \quad \text{for all} \quad \zeta \in \mathbb{B}^d, \ u \in \mathcal{U}. \quad (4.68)$$

Since the operators $C$ and $D$ satisfy the first equality (4.29) and since $U = [A \ B]$ is a contraction, necessarily we have $AC^* + BD^* = 0$. We now combine this latter
equality with (4.67) and formula (4.28) for $D^*$ to get
\[
\mathcal{T}(\cdot, 0)u = Bu = B(B^*\mathcal{T}(\cdot, \zeta)u + S(0)^*S(\zeta)u) \\
= BB^*\mathcal{T}(\cdot, \zeta)u + BD^*S(\zeta)u \\
= BB^*\mathcal{T}(\cdot, \zeta)u - AC^*S(\zeta)u.
\] (4.69)

We now apply the operator $A$ to both parts of (4.63),
\[
AA^*\mathcal{T}(\cdot, \zeta)u = A \sum_{j=1}^d \zeta_j \mathcal{K}_j(\cdot, \zeta)u - AC^*S(\zeta)u
\]
and combine the obtained identity with (4.68) and formula (4.28) for
\[
A \left( \sum_{j=1}^d \zeta_j \mathcal{K}_j(\cdot, \zeta)u \right) = AA^*\mathcal{T}(\cdot, \zeta)u + AC^*S(\zeta)u \\
= \mathcal{T}(\cdot, \zeta)u - BB^*\mathcal{T}(\cdot, \zeta)u - BD^*S(\zeta)u \\
= \mathcal{T}(\cdot, \zeta)u - \mathcal{T}(\cdot, 0)u.
\]
This completes the proof of (4.27). \qed

As a consequence of Lemma 4.10 we get a description of all t.c.f.m. colligations associated with a given Agler decomposition of a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$.

**Lemma 4.10.** Let $\mathbb{K}$ be a fixed Agler decomposition of a function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$. Let $V$ be the associated isometry defined in (4.14) with the defect spaces $D^\perp$ and $R^\perp$ defined in (4.18), (4.19). Then all t.c.f.m. colligations associated with $\mathbb{K}$ are of the form
\[
U^* = \begin{bmatrix} X & 0 \\ 0 & V \end{bmatrix} : \begin{bmatrix} D^\perp \\ \mathcal{D} \oplus \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} R^\perp \\ \mathcal{R} \oplus \mathcal{U} \end{bmatrix}
\] (4.70)
where we have identified $[\mathcal{H}(\mathbb{K})]^d_{\mathcal{Y}}$ with $[\mathcal{D}^\perp]_{\mathcal{D} \oplus \mathcal{Y}}$ and $[\mathcal{H}(\mathbb{K})]_{\mathcal{U}}$ with $[\mathcal{R}^\perp]_{\mathcal{R} \oplus \mathcal{U}}$ and where $X$ is an arbitrary contraction from $D^\perp$ into $R^\perp$. The colligation $U^*$ is isometric (coisometric, unitary) if and only if $X$ is coisometric (isometric, unitary).

For the proof, it is enough to recall that $V$ is unitary as an operator from $\mathcal{D}_V = \mathcal{D} \oplus \mathcal{Y}$ onto $\mathcal{R}_V = \mathcal{R} \oplus \mathcal{U}$ and then to refer to Lemma 4.9. The meaning of description (4.70) is clear: the operators $B^*$, $C^*$, $D^*$ and the restriction of $A^*$ to the subspace $\mathcal{D}$ in the operator colligation $U^*$ are prescribed. The objective is to guarantee $U^*$ be contractive by suitably defining $A^*$ on $D^\perp$. Lemma 4.10 states that $X = A^*|_{D^\perp}$ must be a contraction with range contained in $\mathcal{R}^\perp$.

We now are ready to formulate the main result of this section.

**Theorem 4.11.** Let $S$ be a function in $\mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ with given Agler decomposition $\mathbb{K}$. Then

1. There exists a t.c.f.m. colligation $U = [A \; B]_{\mathbb{K}}$ associated with $\mathbb{K}$.
2. Every t.c.f.m. colligation $U$ associated with $\mathbb{K}$ is weakly unitary and closely connected and furthermore, $S(z) = D + C(I - Z_{\mathcal{H}(\mathbb{K})}(z)A)^{-1}Z_{\mathcal{H}(\mathbb{K})}(z)B$.
3. Any weakly unitary closely connected colligation $\tilde{U}$ of the form (4.20) with the characteristic function equal $S$ is unitarily equivalent to a t.c.f.m. colligation
$U$ associated with associated Agler decomposition $K_U$ for $S$ given by (1.2) with
$\Psi_k$ and $\Phi_{ij}$ given as in (1.5) and (1.6):

$$
\Psi_k(z, \zeta) = \tilde{C}(I - Z_X(z)\tilde{A})^{-1}I_k^\ast(I - \tilde{A}Z_X(\zeta))^{-1}\tilde{B}
$$

(4.71)

$$
\Phi_{ij}(z, \zeta) = \tilde{B}^\ast(I - Z_X(z)^\ast\tilde{A}^\ast)^{-1}I_iI_j^\ast(I - \tilde{A}Z_X(\zeta))^{-1}\tilde{B}
$$

(4.72)

where the inclusion operators $I_j$ are as in (1.17).

**Proof.** Part (1) is contained in Lemma 4.10. Part (2) was proved in Proposition 4.8.

To prove part (3) we assume that $\tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} : [\mathcal{Y}] \rightarrow [\mathcal{X}^d]$ is a closely connected weakly unitary colligation with the state space $X$ and such that

$$S(z) = D + \tilde{C}(I - Z_X(z)\tilde{A})^{-1}Z_X(z)\tilde{B}. 
$$

(4.73)

The proof of unitary equivalence of $\tilde{U}$ to a $t.c.f.m.$ colligation for $S$ associated with

the Agler decomposition as in (1.17), (172) will be broken into three steps below. Let $G(z)$ be the operator-valued function

$$G(z) := 
\begin{bmatrix}
\tilde{C}(I - Z_X(z)\tilde{A})^{-1} & \tilde{B}^\ast(I - Z_X(z)^\ast\tilde{A}^\ast)^{-1}I_1

\vdots

\tilde{B}^\ast(I - Z_X(z)^\ast\tilde{A}^\ast)^{-1}I_d
\end{bmatrix} : \mathbb{B}^d \rightarrow \mathcal{L}(X, \mathcal{Y} \oplus \mathcal{U}^d),
$$

(4.74)

with the operators $I_j$ as in (1.17).

**Step 1:** The Agler decomposition (1.3) holds for the kernel $K$ given by

$$K(z, \zeta) = G(z)G(\zeta)^\ast 
$$

(4.75)

**Proof of Step 1:** It follows by straightforward calculations (see e.g., [13]) that for

the characteristic function $S$ (4.73) of the colligation $\tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$, \n

$$
I_S(z)S(\zeta)^\ast = (1 - \langle z, \zeta \rangle)\tilde{C}(I - Z_X(z)\tilde{A})^{-1}(I - \tilde{A}^\ast Z_X(\zeta)^\ast)^{-1}\tilde{C}^\ast 
+ F(z) \left( I - \tilde{U}^\ast\tilde{U} \right) F(\zeta)^\ast 
$$

where $F(z) = \begin{bmatrix} \tilde{C}(I - Z_X(z)\tilde{A})^{-1}Z_X(z) & I \end{bmatrix}$, and

$$
I - S(z)^\ast S(\zeta) = \tilde{B}^\ast(I - Z_X(z)^\ast\tilde{A}^\ast)^{-1}(I - Z_X(z)^\ast Z_X(\zeta))(I - \tilde{A}Z_X(\zeta))^{-1}\tilde{B} 
+ \tilde{F}(z) \left( I - \tilde{U}^\ast\tilde{U} \right) \tilde{F}(\zeta)^\ast 
$$

where $\tilde{F}(z) = \begin{bmatrix} \tilde{B}^\ast(I - Z_X(z)^\ast\tilde{A}^\ast)^{-1}Z_X(z)^\ast & I \end{bmatrix}$, from which it is clear that weak-

coisometric and weak-isometric properties of $\tilde{U}$ (see Definition 1.4) are exactly what is needed for the respective identities

$$
K_S(z, \zeta) = \tilde{C}(I - Z_X(z)\tilde{A})^{-1}(I - \tilde{A}^\ast Z_X(\zeta)^\ast)^{-1}\tilde{C}^\ast,
$$

(4.76)

$$
I - S(z)^\ast S(\zeta) = \tilde{B}^\ast(I - Z_X(z)^\ast\tilde{A}^\ast)^{-1}(I - Z_X(z)^\ast Z_X(\zeta))(I - \tilde{A}Z_X(\zeta))^{-1}\tilde{B}.
$$

(4.77)
Since $\tilde{U}$ is weakly unitary by assumption, the two latter identities hold. Also we observe that for $S$ of the form (4.73),

$$S(z) - S(\zeta) = \tilde{C}(I - Z X(z) \tilde{A})^{-1} Z X(z) \tilde{B} - \tilde{C} Z X(\zeta) (I - \tilde{A} Z X(\zeta))^{-1} \tilde{B}$$

$$= \tilde{C}(I - Z X(z) \tilde{A})^{-1} (Z X(z) - Z X(\zeta)) (I - \tilde{A} Z X(\zeta))^{-1} \tilde{B}. \quad (4.78)$$

We now conclude from (4.74), (4.75) and (4.76)–(4.78) that the kernel $K$ indeed extends $K_\chi$ and is of the form (4.22) with $\Psi_k(z, \zeta)$ and $\Phi_{ij}(z, \zeta)$ given by (4.71) and (4.72) for $k, i, j = 1, \ldots, d$. It follows from (4.71), (4.74) and (4.78) that

$$\sum_{k=1}^{d}(z_k - \zeta_k) \Psi_k(z, \zeta) = \tilde{C}(I - Z X(z) \tilde{A})^{-1} \left( \sum_{k=1}^{d}(z_k - \zeta_k) T_k^* \right) (I - \tilde{A} Z X(\zeta))^{-1} \tilde{B}$$

$$= \tilde{C}(I - Z X(z) \tilde{A})^{-1} (Z X(z) - Z X(\zeta)) (I - \tilde{A} Z X(\zeta))^{-1} \tilde{B}$$

$$= S(z) - S(\zeta)$$

so that equality (4.7) holds. Equality (3.2) follows in much the same way from (4.72) and (4.74). Thus, the identity (4.3) holds which completes the proof.

**Step 2:** The linear map $U: \mathcal{X} \to \mathcal{H}(\mathbb{K})$ defined by the formula

$$U: x \mapsto G(z)x \quad (4.79)$$

is unitary.

**Proof of Step 2:** Due to factorization (4.75), the reproducing kernel Hilbert space $\mathcal{H}(\mathbb{K})$ can be characterized as the range space

$$\mathcal{H}(\mathbb{K}) = \{ f(z) = G(z)x : x \in \mathcal{X} \}$$

with the lifted norm $\|Gx\|_{\mathcal{H}(\mathbb{K})} = \|(I - \pi)x\|_{\mathcal{X}}$ where $\pi$ is the orthogonal projection onto the subspace $\mathcal{X}^\circ = \{ x \in \mathcal{X} : \tilde{G}x = 0 \}$. For every vector $x \in \mathcal{X}^\circ$ we have by (4.74),

$$\tilde{C}(I - Z X(z) \tilde{A})^{-1} x = 0 \quad \text{and} \quad \tilde{B}^*(I - Z X(z)^* \tilde{A}^*)^{-1} T_j^* x = 0$$

for all $j = 1, \ldots, d$. Then $x$ is orthogonal to the spaces $\mathcal{H}_{C, \tilde{A}}^\circ$ and $\mathcal{H}_{A, \tilde{B}}^\circ$ (see Definition 1.3) and since the colligation $\tilde{U}$ is closely connected, it follows that $x = 0$. Thus, $\mathcal{X}^\circ$ is trivial and $\|Gx\|_{\mathcal{H}(\mathbb{K})} = \|x\|_{\mathcal{X}}$ which means that the operator $U: x \mapsto G(z)x$ is a unitary operator from $\mathcal{X}$ to $\mathcal{H}(\mathbb{K})$.

**Step 3:** Define the operators $A: \mathcal{H}(\mathbb{K}) \to \mathcal{H}(\mathbb{K})^d$, $B: \mathcal{U} \to \mathcal{H}(\mathbb{K})^d$ and $C: \mathcal{H}(\mathbb{K}) \to \mathcal{Y}$ by

$$AU = \bigoplus_{j=1}^{d} U_j \tilde{A}, \quad B = \bigoplus_{j=1}^{d} U_j \tilde{B} \quad \text{and} \quad CU = \tilde{C} \quad (4.80)$$

where $U: \mathcal{X} \to \mathcal{H}(\mathbb{K})$ is defined in (4.79). The colligation $U = [A \ B]$ is a t.c.f.m. colligation associated with the Agler decomposition $\mathbb{K}$ for $S$.

**Proof of Step 3:** We first observe that the colligation $U = [A \ B]$ is a contraction since it is unitarily equivalent to a weakly unitary colligation $\tilde{U}$. It remains to show that $A$ solves the Gleason problems (4.23), (4.24) and that $C$ and $B^*$ are of the form (4.25).

Take the generic element $f$ of $\mathcal{H}(\mathbb{K})$ in the form

$$f(z) = G(z)x, \quad x \in \mathcal{X}$$
so that \( f = Ux \) by (4.79), or equivalently, \( x = U^*f \), since \( U \) is unitary. By definitions (4.20) and (4.74) we have

\[
(sf)(z) = \bar{C}(I - Z_X(z)\bar{A})^{-1}x.
\]

(4.81)

Upon evaluating the latter equality at \( z = 0 \) we get for the operator \( C \) from (4.80)

\[
Cf = \bar{C}U^*f = \bar{C}x = (sf)(0)
\]

so that the formula (4.25) for \( C \) holds. We also have from (4.81)

\[
(sf)(z) - (sf)(0) = \bar{C}(I - Z_X(z)\bar{A})^{-1}x - \bar{C}x = \bar{C}(I - Z_X(z)\bar{A})^{-1}Z_X(z)\bar{A}x.
\]

(4.82)

On the other hand, for the operator \( A \) defined in (4.80), we have

\[
Z_{H(\mathbb{K})}(z)Af = Z_{H(\mathbb{K})}(z)AUx = Z_{H(\mathbb{K})}(\oplus_{j=1}^d U)\bar{A}x = UZ_X(z)\bar{A}x
\]

and therefore, by formula (4.81) applied to \( Z_X(z)\bar{A}x \) rather than to \( x \) we get

\[
s(Z_{H(\mathbb{K})}(z)Af)(z) = \bar{C}(I - Z_X(z)\bar{A})^{-1}Z_X(z)\bar{A}x
\]

which together with (4.82) implies (4.23).

We now take the generic element \( g \) of \( H(\mathbb{K})^d \) in the form

\[
g(z) = \bigoplus_{j=1}^d G(z)x_j \quad \text{and let} \quad x := \bigoplus_{j=1}^d x_j \in X^d,
\]

so that \( x = (\oplus_{j=1}^d U^*)g \). By definitions (4.20) and (4.74) we have

\[
(\tilde{s}g)(z) = \sum_{j=1}^d \bar{B}^*(I - Z_X(z)\bar{A}^*)^{-1}Z_jx_k = \bar{B}^*(I - Z_X(z)\bar{A}^*)^{-1}x.
\]

(4.83)

Upon evaluating the latter equality at \( z = 0 \) we get for the operator \( B^* \) from (4.80)

\[
B^*g = \bar{B}^*(\oplus_{j=1}^d U^*)g = \bar{B}^*x = (\tilde{s}g)(0)
\]

so that the formula (4.25) for \( B^* \) holds. We also have from (4.83)

\[
(\tilde{s}g)(z) - (\tilde{s}g)(0) = \bar{B}^*(I - Z_X(z)\bar{A}^*)^{-1}Z_X(z)^*\bar{A}^*x.
\]

On the other hand, for the operator \( A \) defined in (4.80), we have

\[
Z_{row}(z)^* \otimes A^*g = Z_{row}(z)^* \otimes (A^*(\oplus_{j=1}^d U)x)
\]

\[
= Z_{row}(z)^* \otimes U(\bar{A}^*x) = (\oplus_{j=1}^d U)Z_X(z)^*\bar{A}^*x
\]

and therefore, by formula (4.83) applied to \( Z_X(z)^*\bar{A}^*x \) instead of \( x \) we get

\[
\tilde{s}(Z_{H(\mathbb{K})}(z)^*A^*g)(z) = \bar{B}^*(I - Z_X(z)^*\bar{A}^*)^{-1}Z_X(z)^*\bar{A}^*x
\]

which together with (4.82) implies (4.24). This completes the proof of Step 3.

To complete the proof of the theorem, it suffices to observe that the colligation \( \tilde{U} \) is unitarily equivalent to a \textbf{t.c.f.m.} colligation \( U \) by construction (4.80) and definition (1.21). \( \Box \)
5. Characteristic functions of commutative row contractions: unitary equivalence and coincidence

The Sz.-Nagy-Foias characteristic function of a Hilbert space contraction $T \in \mathcal{L}(\mathcal{X})$ is defined as

$$\theta_T(z) = (-T + zD_T^*(I_X - zT^*)^{-1}D_T)\big|_{D_T}$$

where $D_T$ and $D_{T^*}$ are the defect operators and $D_T, D_{T^*}$ are the defect spaces recalled in (5.3) below. The function $\theta_T$ belongs to the Schur class $\mathcal{S}(D_T, D_{T^*})$ and is pure in the sense that

$$\|\theta_T(0)u\| = \|u\| \quad \text{for some} \quad u \in \mathcal{U} \implies u = 0. \quad (5.1)$$

The Schur-class membership and pureness are the properties which characterize characteristic functions. A classical result of B. Sz.-Nagy and C. Foias (see [27]) is: If $T$ is completely nonunitary (c.n.u.) contraction (i.e., $T$ is a contraction and there is no nontrivial reducing subspace $\mathcal{M}$ for $T$ so that the $T|_{\mathcal{M}}$ is unitary), then the characteristic function $\theta_T$ is a complete unitary invariant of $T$. More precisely, if $T \in \mathcal{L}(\mathcal{X})$ and $R \in \mathcal{L}(\mathcal{X})$ are two c.n.u. contractions, then they are unitarily equivalent if and only if their characteristic functions $\theta_T$ and $\theta_R$ coincide; by definition two operator-valued functions $S$ and $\tilde{S}$ with the same domain of definition coincide if $S(z) = \alpha S(z)\beta$ for some unitary transformations $\alpha$ and $\beta$. Moreover, if one starts with a pure Schur-class function $S$, one can associate a c.n.u. contraction $T(S)$ defined on the associated Sz.-Nagy-Foias canonical model Hilbert space $\mathcal{K}(S)$. We mention that there is a dictionary between the Sz.-Nagy–Foias model $(T(S), \mathcal{K}(S))$ and the de Branges-Rovnyak model space $(T_{dBR}(S), \mathcal{H}(\tilde{K}_S))$ where $\tilde{K}_S$ is the positive kernel given by (1.2) and where $T_{dBR}(S) = A^*$ where $A$ is the operator on $\mathcal{H}(\tilde{K}_S)$ given in part (3) of Theorem 1.2 (see e.g. [12]). It is easy to see that if $T$ and $R$ are unitarily equivalent, then the associated model contraction operators $T(\theta_T)$ and $R(\theta_R)$ (or $T_{dBR}(\theta_T)$ and $T_{dBR}(\theta_R)$) are unitarily equivalent. The result mentioned above can be further elaborated as follows: If $T \in \mathcal{L}(\mathcal{X})$ is a c.n.u. contraction operator, then $T$ is unitarily equivalent to its functional model contraction operator $T(\theta_T)$ on $\mathcal{K}(\theta_T)$ or $T_{dBR}(\theta_T)$ on $\mathcal{H}(\tilde{K}_{\theta_T})$. We should also mention that if $T$ is completely noncoisometric (c.n.c.—see the discussion below for precise definitions), then it suffices to take $T_{dBR}(S) = A^*$ where $A$ is the backward shift operator acting on $\mathcal{H}(\tilde{K}_S)$ as in (1.3). We mention the paper of Foias-Sarkar [25] as a very recent application of the Sz.-Nagy-Foias model theory for a single contraction operator. Let us also mention that there is a second approach to unitary classification of Hilbert space operators based on the curvature invariant of Cowen and Douglas [22]; for a recent comparison between these two approaches, we refer to [24].

In this section we discuss extensions of this Sz.-Nagy–Foias model theory to the context of row contractions, that is to $d$-tuples of operators $T = (T_1, \ldots, T_d)$ on a Hilbert space $\mathcal{X}$ for which the associated block-row matrix is contractive:

$$\|T\| \leq 1 \quad \text{where} \quad T = [T_1 \cdots T_d] : \mathcal{X}^d \to \mathcal{X}. \quad (5.2)$$

For such a row-contraction, let

$$D_T = (I_{\mathcal{X}^d} - T^*T)^{1/2}, \quad D_T = \overline{\text{Ran}} D_T \subset \mathcal{X}^d, \quad D_{T^*} = (I_{\mathcal{X}^d} - TT^*)^{1/2}, \quad D_{T^*} = \overline{\text{Ran}} D_{T^*} \subset \mathcal{X}. \quad (5.3)$$
The characteristic function \( \theta_{T,nc}(z) \) of a row contraction has been introduced in [29] (in slightly different terms) as a formal power series in \( d \) noncommuting indeterminates \( z_1, \ldots, z_d \) which can be written in a compact realization form as

\[
\theta_{T,nc}(z) = (-T + DT^*) (I_X - Z_X(z)T^*)^{-1} Z_X(z)DT \big|_{D_T} : D_T \to D_T^*,
\]

where \( Z_X(z) \) is of the form (1.14) (but with the noncommuting indeterminates \( z_1, \ldots, z_d \) replacing the commuting variables \( z_1, \ldots, z_d \)). To write this expression out more explicitly, we need the following notation connected with formal power series in noncommuting indeterminates. Let \( F_d \) consists of all words \( v = i_N \cdots i_1 \) with letters \( i_j \) coming from the alphabet \( \{ 1, \ldots, d \} \). Then the operator of concatenation \( v \cdot v' = v'' \) where

\[
v'' = j_N \cdots j_1 i_N \cdots i_1 \quad \text{if} \quad v' = j_N \cdots j_1 \quad \text{and} \quad v = i_N \cdots i_1
\]

makes \( F_d \) a free semigroup; here we include the empty word, denoted as \( \emptyset \), as an element of \( F_d \) which serves as the identity element of the semigroup. For \( \{ z_1, \ldots, z_d \} \) a \( d \)-tuple of freely noncommuting indeterminates and for \( v = i_N \cdots i_1 \) an element of \( F_d \), we let \( z^v \) denote the noncommutative monomial \( z_{i_N} \cdots z_{i_1} \). For the case \( v = \emptyset \), we set \( z^\emptyset = 1 \). We extend this noncommutative functional calculus to a \( d \)-tuple of operators \( T = (T_1, \ldots, T_d) \) on a Hilbert space \( \mathcal{X} \):

\[
T^v = T_{i_N} \cdots T_{i_1} \quad \text{if} \quad v = i_N \cdots i_1 \in F_d \ \setminus \ \{ \emptyset \}; \quad T^\emptyset = I_X.
\]

Similarly, for \( T^* = (T_1^*, \ldots, T_d^*) \) equal to a \( d \)-tuple of (not necessarily commuting) operators, we use the notation \( T^{*v} \) to indicate the product \( T^{*v} = T_{i_N}^* \cdots T_{i_1}^* \), with \( T^{*\emptyset} \) equal to the identity operator \( I \). We wish to point out that the expression (5.4) for the characteristic function can also be written more explicitly as the noncommutative formal power series

\[
\theta_{T,nc}(z) = \sum_{v \in F_d} [\theta_{T,nc}]_v z^v
\]

where the power series coefficients \([\theta_{T,nc}]_v : D_T \to D_T^*\) are given by

\[
[\theta_{T,nc}]_v = \begin{cases} 
-T|_{D_T} & \text{if } v = \emptyset, \\
DT^*T^{*v}I^*_vDT & \text{if } v \neq \emptyset \quad \text{has the form } v = v' \cdot j 
\end{cases}
\]

(where \( I^*_v : \mathcal{H}^d \to \mathcal{H} \) is as in (1.14)). Thus we see that knowledge of the characteristic function amounts to knowledge of all the moment operators (5.7). It is readily seen that if \( T = [T_1 \ldots T_d] \) and \( R = [R_1 \ldots R_d] \) are two unitarily equivalent row contractions (i.e., \( UT^*_iU^* = R_i \) for \( i = 1, \ldots, d \) and some unitary operator \( U \)), then the formal power series \( \theta_{T,nc}(z) \) and \( \theta_{R,nc}(z) \) coincide (the coincidence for noncommutative formal power series is defined in much the same way as for usual functions), or equivalently, the set of moments (5.7) associated with \( T \) coincide with those associated with \( R \). The converse was proved in [29] under the assumption that \( T \) and \( R \) are completely non-coisometric (c.n.c.). Recall that a row contraction \( T \) as in (5.2) is called completely non-coisometric (c.n.c.) if there is no nontrivial subspace \( \mathcal{M} \subset \mathcal{X} \) invariant under \( T^*_i \) for \( i = 1, \ldots, d \) so that the operator

\[
P_M [T_1|_\mathcal{M} \ldots T_d|_\mathcal{M}] : \mathcal{M}^d \to \mathcal{M}
\]

is a coisometry. An equivalent formulation is that

\[
\mathcal{X} = \sqrt{\{ \text{Ran } T^{*k}D_T^* : v \in F_d, k = 1, \ldots, d \}}.
\]
Thus, the result from [29] states that if \( T \) is a c.n.c. row contraction, then the characteristic function \( \theta_{T,nc} \) is a complete unitary invariant for \( T \). In view of the explicit formula (5.6) for \( \theta_{T,nc} \), we see that the latter result can be rephrased as saying that the set of moments

\[
-T^* , D_T , T^v \Lambda^* D_T : D_T \to D_T ,
\]

(5.10)

(where \( v \) runs over all words in \( F_d \) and \( j \) runs over all indices in \( \{1 , \ldots , d\} \)) form a complete set of unitary invariants for a row contraction in the c.n.c. case.

The more general class of completely nonunitary (c.n.u.) row contractions as defined in [15] consists of those \( T \) for which there is no nontrivial subspace \( \mathcal{M} \) reducing for each \( T_1 , \ldots , T_d \) on which the operator block-row matrix (5.8) is unitary. An equivalent formulation (see page 89 in [15]) is that

\[
X = \sqrt{\{ \text{Ran } T^v D_T , \text{Ran } T^v \Lambda^* \Lambda^* D_T : \alpha , \beta \in F_d , k = 1 , \ldots , d \}}.
\]

(5.11)

The characteristic function \( \theta_{T,nc} \) does not recover \( T \) up to unitary equivalence. However, it was shown in [15] that there is an operator \( L_T \) so that the pair \( (\theta_{T,nc} , L_T) \) is a complete unitary invariant for \( T \). A more concrete version of the result from [15] (see the discussion around equations (5.3.6) there) is that a complete set of invariants (up to coincidence) for the c.n.u. case is given by the expanded set of moments

\[
-T , D_T , T^v \Lambda^* \Lambda^* D_T : D_T \to D_T , \quad D_T \Lambda , T^v \Lambda^* \Lambda^* D_T : D_T \to D_T
\]

(5.12)

where \( v \) and \( v' \) run over all words in \( F_d \) and where \( k \) and \( j \) run over the set of indices \( \{1 , \ldots , d\} \). This work also makes explicit the construction of a model contraction operator acting on a Sz.-Nagy–Foias canonical model space (see [29] for the c.n.c. case and [15] for the c.n.u. case).

It is not difficult to see that any such characteristic function \( \theta_{T,nc} \) defines a contractive multiplier on the Fock space which commutes with the right creation operators (see [15]). Formal power series for which the associated multiplication operator is bounded on the Fock space are called multianalytic functions in [29]. Conversely, any contractive multianalytic function \( S(\mathbf{z}) = \sum_{v \in F_d} S_v \mathbf{z}^v \) is a characteristic function for some c.n.u. row contraction \( T \) if and only if \( S \) is also pure in the sense that \( \|S_u\|_Y = \|u\|_U \) only when \( u = 0 \) (see page 89 in [15]). Contractive multipliers \( S(\mathbf{z}) = \sum_{v \in F_d} S_v \mathbf{z}^v \) equal to the characteristic function of a c.n.c. row contraction \( T \) are characterized by having the additional property that \( I - S(\mathbf{z})^* S(\mathbf{z}) \geq G(\mathbf{z})^* G(\mathbf{z}) \) for some multianalytic \( G \) forces \( G = 0 \) (see Remark 5.3.5 in [15]). Thus one can say that c.n.c. row contractions are parametrized by (equivalence classes up to coincidence of) pure contractive multianalytic functions \( S(\mathbf{z}) \) for which the defect \( I - S(\mathbf{z}) \) has zero maximal factorable minorant, while c.n.u. row contractions are parametrized by equivalence classes of pure contractive multianalytic functions combined with the second invariant \( L_T \), the details of which need not concern us here.

If we replace the noncommutative indeterminates \( z_1 , \ldots , z_d \) with commuting variables \( z_1 , \ldots , z_d \) in formula (5.12), then we get a function \( \theta_T(\mathbf{z}) \) analytic on \( \mathbb{B}^d \) and certainly depending on \( T \) only. Moreover, this function is the characteristic function of the colligation

\[
U_T = D_T , T^* : \mathcal{X} \to \mathcal{X} , \quad D_T , T^* : \mathcal{Y} \to \mathcal{Y}
\]

(5.13)

which is the Halmos unitary dilation of \( T^* \); therefore, \( \theta_T \) belongs to \( \mathcal{S}_d(\mathcal{D}_T , \mathcal{D}_T^* \) by Theorem 1.3. It is not hard to show that an \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued function \( S \) coincides with
a function \( \theta_T \) of the form (5.4) for some Hilbert-space row contraction \( T \) of the form (5.2) if and only if \( S \) belongs to the Schur class \( S_d(U, \mathcal{Y}) \) and is pure in the sense of (5.1). Thus, the commutative (analytic) version of formula (5.4) perfectly fits the framework of the present paper. However, this version is meaningful only in case (5.1). Thus, the commutative (analytic) version of formula (5.4) perfectly fits the characteristic function \( \theta \). Thus, the convenient (analytic) version of formula (5.4) perfectly fits the characteristic function \( \theta \) of more general operator-tuples associated with a general “positive regular freely holomorphic function” \( f \) (see [32]). We give here an alternative direct proof of the result based on the results from Section 2 which suggests extensions to the cases beyond the c.n.c. setting.

For the case of commutative row-contractions, note that the conditions (5.9) and (5.11) simplify. Thus the commutative row contraction \( T \) is c.n.c. exactly when

\[
\mathcal{X} = \bigvee \{ \text{Ran } T^n D_T^*: n \in \mathbb{Z}_+^d, k = 1, \ldots, d \},
\]

where we use the standard multivariable notation

\[
T^n = T_1^{n_1} \cdots T_d^{n_d} \quad \text{for} \quad n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d
\]

and for \( T = (T_1, \ldots, T_d) \) a commutative operator tuple. Similarly, the commutative row-contraction \( T \) of the form (5.2) is c.n.c. exactly when

\[
\mathcal{X} = \bigvee \{ \text{Ran } T^n D_T^*, \text{Ran } T^n T^m D_T: n, m \in \mathbb{Z}_+^d, k = 1, \ldots, d \}.
\]

**Remark 5.1.** Before proceeding further, let us first observe that any commutative row-unitary (or even just row-isometric) tuple \( U \) as in (5.2) is trivial if \( d > 1 \). Indeed, suppose that \( d > 1 \) and \( U = (U_1, \ldots, U_d) \) is a row-isometric tuple. This means that each \( U_j \) is an isometry and the ranges of \( U_1, U_2, \ldots, U_d \) have pairwise orthogonal ranges (spanning the whole space \( \mathcal{X} \) in case \( U \) is row-unitary). In particular, \( \text{Ran } U_j \perp \text{Ran } U_k \) for \( j \neq k \). But then we also have

\[
\text{Ran } U_j U_k = \text{Ran } U_k U_j \subset \text{Ran } U_j \cap \text{Ran } U_k = \{0\} \quad \text{for} \quad j \neq k.
\]

As \( U_k U_j \) is also an isometry, it follows that the ambient Hilbert space \( \mathcal{X} \) is the zero space. As a consequence of this observation, it follows that any commutative row contraction \( T \) is c.n.c.. Indeed, there can be no nonzero reducing subspace for \( T \) on which \( T \) is row-unitary, since then necessarily the restriction of \( T \) to such a subspace would have to be simultaneously commutative and non-commutative. We conclude that any commutative row-contraction \( T \) as in (5.2) is unitarily equivalent to a noncommutative Sz.-Nagy-Foias functional model as in [15] based on its n.c.-characteristic function \( \theta_{T, nc} \). The drawback of this model of course is that it does not display prominently the additional structure that \( T \) is commutative.

The following result was first obtained in [18]; it is also possible to give a unified proof which includes the noncommutative and commutative setting in one formalism (see [16, 30, 31]) and there is now an extension of the general theory to the setting of more general operator-tuples associated with a general “positive regular freely holomorphic function” \( f \) (see [32]). We give here an alternative direct proof of the result based on the results from Section 2 which suggests extensions to the cases beyond the c.n.c. setting.
Theorem 5.2. Two commutative c.n.c. row contractions \( T = [T_1 \ldots T_d] \) and \( R = [R_1 \ldots R_d] \) are unitarily equivalent if and only if their characteristic functions \( \theta_T \) and \( \theta_R \) coincide.

Proof. We prove the nontrivial “if” part. We first observe that a commutative row contraction \( T \) is completely non-coisometric if and only if the colligation \( (5.13) \) is observable, i.e.,

\[
D_T \cdot (I_X - Z_X(z)T^*)^{-1}x = 0 \quad \implies \quad x = 0.
\]

Indeed, the latter implication can be equivalently written as

\[
D_T \cdot T^{*n}x = 0 \quad \text{for all} \quad n \in \mathbb{Z}_+^d \quad \implies \quad x = 0,
\]

as can be seen from the expansion

\[
D_T \cdot (I_X - Z_X(z)T^*)^{-1} = D_T \cdot \left(I_X - \sum_{j=1}^d z_j T_j^*\right)^{-1} = \sum_{n \in \mathbb{Z}_+^d} \frac{|n|!}{n!} D_T \cdot T^{*n}z^n,
\]

where we have used notation \((1.12)\) and \((5.15)\). On the other hand,

\[
\mathcal{M} := \{ x \in \mathcal{X} : D_T \cdot T^{*n}x = 0 \quad \text{for all} \quad n \in \mathbb{Z}_+^d \}
\]

is the maximal \( T^* \)-invariant subspace of \( \mathcal{X} \) such that the operator \((5.8)\) is a coisometry. Combining this with observability characterization \((5.17)\) and the definition of a c.n.c. tuple, we get the desired equivalence.

Let us assume that \( \theta_T \) and \( \theta_R \) coincide, i.e., that

\[
\theta_T(z) = \alpha \theta_R(z) \beta^*
\]

where \( \alpha : D_{R^*} \to D_{T^*} \) and \( \beta : D_R \to D_T \) are unitary operators. Thus,

\[
\theta_T(z) = (-T + D_T \cdot (I_X - Z_X(z)T^*)^{-1}Z_X(z)D_T) \big|_{D_R} = (-\alpha R \beta^* + \alpha D_{R^*} (I_X - Z_X(z)T^*)^{-1}Z_X(z)D_R \beta^*) \big|_{D_R}
\]

and we have two commutative unitary colligations

\[
U_1 = \begin{bmatrix} T^* & D_T \\ D_{T^*} & -T \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} R^* & D_R \beta^* \\ \alpha D_{R^*} & -\alpha R \beta^* \end{bmatrix}
\]

with the same input and output spaces and with the same characteristic function \( \theta_T \).

Since \( T \) and \( R \) are completely non-coisometric, these colligations are both observable. As the lower diagonal entries in \( U_1 \) and \( U_2 \) are equal (evaluate \((5.18)\) at \( z = 0 \)), Corollary 3.7 from \([10]\) implies that \( T^* \) is unitarily equivalent to \( R^* \).

We now discuss how our t.c.f.m. colligations can be used to study unitary equivalence and unitary invariants for row contractions more general that c.n.c. Before proceeding further, let us recall that any unitary realization \( \tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \) of an \( S \in \mathcal{S}_d(U, \mathcal{Y}) \) induces an Agler decomposition \((4.3)\) for \( S \) via formulas \((4.71)\), \((4.72)\). If \( T \) is a row contraction, then the unitary realization \( U_T \) \((5.13)\) for \( \theta_T \) induces the Agler decomposition

\[
K_T(z, \zeta) = G_T(z)G_T(\zeta)^*, \quad \text{where} \quad G_T(z) = \begin{bmatrix} D_T \cdot (I_X - Z_X(z)T^*)^{-1} \\ D_T \cdot (I_X T^* - Z_X(z)T)^{-1} \mathbb{I}_1 \\ \vdots \\ D_T \cdot (I_X T^* - Z_X(z)T)^{-1} \mathbb{I}_d \end{bmatrix}.
\]
Moreover, in case the Halmos-dilation colligation \[5.13\] is closely connected, Theorem 4.11 tells us that \(U_T\) is unitarily equivalent to some t.c.f.m. colligation associated with the Agler decomposition \(K_T\) \[5.20\] for \(\theta_T\). Let us write \(D(T)\) and \(R(T)\) for the domain and range of the isometry \(V\) given by \[4.14\] for the case where \(K = K_T\); we also write \(V_T\) rather than \(V\) for this case. Then any t.c.f.m. colligation associated with \(K_T\) has the form

\[
U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H}(K) \\ D_T \end{bmatrix} \to \begin{bmatrix} \mathcal{H}(K)^d \\ D_T^* \end{bmatrix}
\]

where, upon the identifications

\[
\begin{bmatrix} \mathcal{H}(K) \\ D_T \end{bmatrix} \sim \begin{bmatrix} R(T)^\perp \\ R(T) \oplus D_T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathcal{H}(K)^d \\ D_T^* \end{bmatrix} \sim \begin{bmatrix} D(T)^\perp \\ D(T) \oplus D_T^* \end{bmatrix}
\]

as in Lemma 4.10. \(U^*\) has the form

\[
U^* = \begin{bmatrix} X_T & 0 \\ 0 & V_T \end{bmatrix} : \begin{bmatrix} D(T)^\perp \\ D(T) \oplus D_T^* \end{bmatrix} \to \begin{bmatrix} R(T)^\perp \\ R(T) \oplus D_T \end{bmatrix}.
\]

The fact that \(U_T\) is unitary implies that \(U\) is unitary and hence also \(\dim D(T)^\perp = \dim R(T)^\perp\) and \(X_T : D^\perp \to R^\perp\) is unitary. We conclude:

**Proposition 5.3.** Given a row-contraction \(T\) such that the the Halmos-dilation colligation \[5.13\] is closely connected and given an Agler decomposition \(K_T\) for \(\theta_T\), there is a choice of unitary \(X_T\) from \(D(T)^\perp\) to \(R(T)^\perp\) so that \(T\) is unitarily equivalent to \(A(T)^*\), where \(A(T)\) is determined just from \((\theta_T, K_T, X_T)\) via the decomposition \[5.21\] for \(U\) defined by \[5.22\].

This suggests that we define a new invariant consisting of a triple of objects \((S, K, X)\) defined as follows. We let \(S\) be any pure Schur-class function in \(S_d(U, Y)\). We then let \(K\) be any Agler decomposition for \(S\). The remaining ingredient to form a t.c.f.m. colligation associated with the Agler decomposition \(K\) for \(S\) is a choice of contraction operator \(X : D^\perp \to R^\perp\). Part of our admissibility requirements on \((S, K, X)\) is that

1. it turns out that \(\dim D^\perp = \dim R^\perp\).

In this case there exists a unitary operator \(X : D^\perp \to R^\perp\) which then defines completely a t.c.f.m. associated with \(K\) and \(S\) which gives a unitary realization of \(S\). As the final admissibility requirement, we demand that

2. the choice of unitary \(X : D^\perp \to R^\perp\) is such that the \(d\)-tuple of operators \((A_1, \ldots, A_d)\) constructed from the decomposition \[5.21\] for \(U\) is commutative.

Let us call any such triple \((S, K, X)\) (consisting of a pure Schur-class function \(S\), an Agler decomposition \(K\) for \(S\), and a unitary operator \(X : D^\perp \to R^\perp\) such that the admissibility requirements (1) and (2) are also satisfied) an admissible triple. In the discussion above we explained how to attach a particular admissible triple \((\theta_T, K_T, X_T)\) (the characteristic admissible triple of \(T\)) to any commutative row contraction \(T\) for which \(U_T\) is closely connected.

It is not difficult to characterize when the colligation \(U_T\) is closely connected directly in terms of \(T\). Toward this end, given a commutative row contraction \(T\), introduce the subspace

\[
M_T^{(1)} = \{ x \in X : D_T (I_X - Z_X(z)T^*)^{-1} x \equiv 0 \} \quad \text{and} \quad D_T (I_X - Z_X(z)^* T)^{-1} X \equiv 0 \quad \text{for} \quad i = 1, \ldots, d.
\]

(5.23)
The space $\mathcal{M}_T^{(1)}$ is the orthogonal complement in $\mathcal{X}$ of the space $\mathcal{H}_{D_T, T}^c \bigvee \mathcal{H}_{R, T}^c$ (see definitions (1.19)); thus $\mathcal{M}_T^{(1)} = \{0\}$ if and only if the colligation is closely connected. For the case $d = 1$, the condition $\mathcal{M}_T^{(1)} = \{0\}$ simply means that $T$ is completely nonunitary. With this as motivation, we make the following definition.

**Definition 5.4.** We say that the commutative row-contraction $T = [T_1 \ldots T_d]$ is **closely connected** (c.c.) if $\mathcal{M}_T^{(1)} = \{0\}$ where $\mathcal{M}_T^{(1)}$ is as in (5.23). Equivalently, $T$ is c.c. if and only if

$$\mathcal{X} = \{\text{Ran } T^n D_{T^*}, \text{ Ran } I_n^k X_n D_T : n \in \mathbb{Z}_+^d, k = 1, \ldots, d\}$$

where $X_n$ is given by

$$\left(I_{X^d} - T^* Z_{\mathcal{X}}(z)\right)^{-1} = \sum_{n \in \mathbb{Z}_+^d} X_n z^n. \tag{5.25}$$

We note that the $X_n$'s in (5.25) are difficult to compute explicitly in general. Nevertheless it is clear that

$$\bigvee \{\text{Ran } T^n D_{T^*} : n \in \mathbb{Z}_+^d\} \subset \bigvee \{\text{Ran } T^n D_{T^*}, \text{Ran } I_n^k X_n D_T : n \in \mathbb{Z}_+^d, k = 1, \ldots, d\} \subset \bigvee \{\text{Ran } T^n D_{T^*}, \text{Ran } T^n T^{*m} I_k D_T : n, m \in \mathbb{Z}_+^d, k = 1, \ldots, d\} = \mathcal{X}, \tag{5.26}$$

from which it follows that c.n.c. $\Rightarrow$ c.c. $\Rightarrow$ c.n.u. (since c.n.u. holds for any commutative row contraction by Remark 5.1). Henceforth, unless otherwise stipulated, we assume that $T$ is a **c.c. commutative row contraction**.

We next observe that if two c.c. commutative row contractions $T$ and $R$ are unitarily equivalent, then the associated Agler decompositions $\mathbb{K}_T$ and $\mathbb{K}_R$ together with the characteristic functions $\theta_T$ and $\theta_R$ defined as in (5.20) jointly coincide in the sense that

$$\theta_T(z) = \alpha \theta_R(z) \beta^*, \quad \mathbb{K}_T(z, \zeta) = \begin{bmatrix} \alpha & 0 \\ 0 & \bigoplus_{1}^{d} \beta \end{bmatrix} \mathbb{K}_R(z, \zeta) \begin{bmatrix} \alpha^* & 0 \\ 0 & \bigoplus_{1}^{d} \beta^* \end{bmatrix} \tag{5.27}$$

for some unitary $\alpha : D_{R^*} \rightarrow D_{T^*}$ and $\beta : D_{R} \rightarrow D_{T}$. Indeed, if $T_i = U R U^*$ for a unitary $U : \mathcal{X} \rightarrow \mathcal{X}$, then (5.27) holds with $\alpha = U|_{D_{T^*}}$ and $\beta = \bigoplus_{1}^{d} U|_{D_{T}}$. Moreover, it is easy to see that the unitary operators $X_T : D(T)^\perp \rightarrow \mathcal{R}(T)^\perp$ and $X_R : D(R)^\perp \rightarrow \mathcal{R}(R)^\perp$ are unitarily equivalent. This suggests that we define an equivalence relation on admissible triples: we say that the two admissible triples $(\mathcal{S}, \mathbb{K}, \mathcal{X})$ and $(\mathcal{S}', \mathbb{K}', \mathcal{X}')$ are equivalent if

(i) $(\mathcal{S}, \mathbb{K})$ and $(\mathcal{S}', \mathbb{K}')$ jointly coincide, and

(ii) $X : D^\perp \rightarrow \mathcal{R}^\perp$ and $X' : D'^\perp \rightarrow \mathcal{R}'^\perp$ are unitarily equivalent.

The discussion above shows that the equivalence class of $(\theta_T, \mathbb{K}_T, X_T)$ is a unitary invariant for any c.c. commutative row contraction $T$. The next result gives the converse.

**Theorem 5.5.** Suppose that $T$ and $R$ are two c.c. commutative row contractions such that the associated characteristic triples $(\theta_T, \mathbb{K}_T, X_T)$ and $(\theta_R, \mathbb{K}_R, X_R)$ are equivalent as admissible triples. Then $T$ and $R$ are unitarily equivalent.
Proof. We have seen that $T$ is unitarily equivalent to $A(T)^*$ and $R$ is unitarily equivalent to $A(R)^*$ where $A(T)$ (respectively $A(R)$) appears in the t.c.f.m. colligation $U(T)$ (respectively $U(R)$) \(5.21\) and \(5.22\) associated with \((\theta_T, K_T, X_T)\) (respectively \((\theta_R, K_R, X_R)\)). By applying unitary changes of bases on the input and output spaces coming from the joint coincidence of \((\theta_T, K_T)\) and \((\theta_R, K_R)\) in the input and output spaces, we may even assume that $\theta_T = \theta_R$ and $K_T = K_R$. It then follows that $V_T = V_R$ (as in \(5.22\)), and the assumption that $X_T$ is unitarily equivalent to $X_R$ then implies that $A(T)$ is unitarily equivalent to $A(R)$. As $Y$ is unitarily equivalent to $A(T)^*$ and $R$ is unitarily equivalent to $A(R)^*$, it now follows that $T$ and $R$ are unitarily equivalent to each other.

\[\Box\]

Remark 5.6. Note that if \((S, K, X)\) is an admissible triple with associated t.c.f.m. colligation \([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]\), then $A^* = [A_1^* \cdots A_d^*]$ is a c.c. commutative row contraction. Moreover, if \((S', K', X')\) is another admissible triple, that $A^*$ and $A'^*$ are unitarily equivalent if and only if the associated admissible triples \((S, K, X)\) and \((S', K', X')\) are equivalent as triples. Thus admissible triples can be viewed as providing a parametrization of c.c. commutative row contractions. This parametrization is somewhat crude, however, since there is no explicit way (1) to write down all the Agler decompositions $\mathcal{K}$ associated with $S$, and (2) pick out among them which $\mathcal{K}$ lead to reproducing kernel spaces $\mathcal{H}(\mathcal{K})$ so that (a) the dimension criterion $\dim D^\perp = \dim R^\perp$ holds, and (b) there exist a unitary $X$; $D^\perp \rightarrow R^\perp$ giving rise to a commutative $A$ in the associated t.c.f.m. colligation.

In the c.n.c. case as developed in \cite{17-18}, the complete invariant for a c.n.c. commutative row contraction $T$ is just the characteristic function $\theta_T$ and there is a version of the Sz.-Nagy-Foias functional model space. Even in this case, there remains the issue of characterizing which pure Schur-class functions $S$ coincide with the characteristic function $\theta_T$ of a c.n.c. commutative row contraction; the following partial result on this issue follows by combining Theorem \ref{thm:characterization} above with Proposition 6.1 in \cite{10}.

Theorem 5.7. If the function $S \in \mathcal{S}_d(\mathcal{U}, \mathcal{Y})$ coincides with a characteristic function $\theta_T$ of a commutative c.n.c. row contraction $T$, then $S$ is pure, the space $\mathcal{H}(\mathcal{K}_S)$ is $M^*_S$-invariant, inequality \(2.9\) holds, and finally,

\[
\begin{equation}
\dim \ker A^*|_{D^\perp} = \dim \mathcal{U}_S^0,
\end{equation}
\]

where

\[
A = \begin{bmatrix}
M^*_1|_{\mathcal{H}(K_S)} \\
\vdots \\
M^*_d|_{\mathcal{H}(K_S)}
\end{bmatrix}, \quad \mathcal{U}_S^0 = \{u \in \mathcal{U} : S(z)u \equiv 0\}.
\]

In general the construction of Agler decompositions $\mathcal{K}$ from a given Schur-class function $S$ is mysterious\footnote{However, one case where a multitude of Agler decompositions can be written down for a given function $S = 0$ is presented as Example \ref{ex:agler-decomposition} below.}. In particular, we do not know any characterization of when the Agler decomposition is unique. If it were the case that there is a unique Agler decomposition in case $S = \theta_T$ with $T$ c.n.c., then one could see Theorem 5.2 as a corollary to Theorem 5.5.
In case \( \dim D(T) = \dim R(T) = 0 \), then the third object in the admissible triple \( X_T \) is trivial and can be ignored. To analyze this situation, let us introduce another subspace associated with a c.c.c. commutative row contraction \( T \), namely

\[
\mathcal{M}_T^{(2)} = \{ x \in X : D_T, (I_X - Z_X(z)T^*)^{-1}x = 0 \text{ and } D_T(I_X - Z_X(z)T^*)^{-1}Z_X(z)^*x = 0 \}.
\] (5.29)

It is readily seen that \( \mathcal{M}_T^{(1)} \subset \mathcal{M}_T^{(2)} \). Indeed, if \( D_T(I - Z(z)^*)^{-1}I^*_z x = 0 \) for \( i = 1, \ldots, d \), then also

\[
0 \equiv \sum_{i=1}^{d} z_i D_T(I_X - Z_X(z)^*)^{-1}I^*_z x = D_T(I_X - Z_X(z)^*)^{-1} \sum_{i=1}^{d} z_i I^*_z x = D_T(I_X - Z_X(z)^*)^{-1}Z_X(z)^*x.
\]

In case \( d = 1 \), we have \( \mathcal{M}_T^{(1)} = \mathcal{M}_T^{(2)} \) and either space is the maximal reducing space for \( T \) on which \( T \) is unitary. Hence \( \mathcal{M}_T^{(1)} = \{ 0 \} \) and \( \mathcal{M}_T^{(2)} = \{ 0 \} \) are both equivalent to \( T \) being c.n.u. for the single-variable \( d = 1 \) case. For the multivariable setting, as we have already taken \( \mathcal{M}_T^{(1)} = 0 \) as the definition of \( T \) being c.c., we make the following definition.

**Definition 5.8.** Given a commutative row contraction \( T \) we say that \( T \) is strongly closely connected (strongly c.c.) if \( \mathcal{M}_T^{(2)} = \{ 0 \} \) with \( \mathcal{M}_T^{(2)} \) as in (5.29). Equivalently,

\[
X = \bigvee \left\{ \text{Ran } T^n D_T^*, \text{Ran } \left( \sum_{k=1}^{d} I_k^* X_n - e_k D_T \right) : n \in \mathbb{Z}_+^d \right\}.
\] (5.30)

where \( X_n \) is given in (5.25) and where \( e_k \) stands for the element in \( \mathbb{Z}_+^d \) with one in the \( k \)-th slot and zeros in all other slots.

From the chain of containments (5.26) combined with the observation made above that \( \mathcal{M}_T^{(1)} \subset \mathcal{M}_T^{(2)} \), we get

\[
\bigvee \{ \text{Ran } T^n D_T^* : n \in \mathbb{Z}_+^d \} \subset (\mathcal{M}_T^{(2)})^+ = \left\{ \text{Ran } T^n D_T^*, \text{Ran } \left( \sum_{k=1}^{d} I_k^* X_n - e_k D_T \right) : n \in \mathbb{Z}_+^d \right\}
\]

\[
\subset (\mathcal{M}_T^{(1)})^+ = \bigvee \{ \text{Ran } T^n D_T^*, \text{Ran } I_k^* X_n D_T : n \in \mathbb{Z}_+^d, k = 1, \ldots, d \}
\]

\[
\subset \bigvee \{ \text{Ran } T^n D_T^*, \text{Ran } T^n T^m I_k^* D_T^* : n, m \in \mathbb{Z}_+^d, k = 1, \ldots, d \} = X,
\] (5.31)

from which we see that c.n.c. \( \Rightarrow \) strongly c.c. \( \Rightarrow \) c.c. \( \Rightarrow \) c.n.u. for a commutative row contraction \( T \) (where the last property c.n.u. holds for any commutative row contraction).

One can check that \( \mathcal{M}_T^{(2)} = \{ 0 \} \) amounts to the condition that \( \mathcal{R}(T) = \{ 0 \} \) (here \( \mathcal{R}(T) \) is as in (5.22)). Thus the characteristic triple \( (\theta_T, K_T, X_T) \) for a strongly c.c. commutative row contraction collapses to \( (\theta_T, K_T, 0) \). Given two strongly c.c. commutative row contractions \( T \) and \( R \), equivalence of the characteristic triples \( (\theta_T, K_T, 0), (\theta_R, K_R, 0) \) collapses to joint coincidence of the characteristic function/Agler decomposition pairs \( (\theta_T, K_T), (\theta_R, K_R) \). Thus the following result is an immediate special case of Theorem 5.6.
Theorem 5.9. Let $T$ and $R$ be two strongly c.c. commutative row contractions and let us assume that the associated characteristic function/Agler decomposition pairs $(\theta_T, \mathbb{K}_T)$ and $(\theta_R, \mathbb{K}_R)$ jointly coincide (i.e., let us assume that (5.27) holds). Then $T$ and $R$ are unitarily equivalent.

Remark 5.10. In Theorem 5.9 it is enough to assume that $T$ is strongly c.c. with $R$ only c.c. or vice versa.

We have seen that the model-theory results are the best for the case where the commutative row contraction is c.n.c. It essentially follows from the definitions that any commutative row-contractive $d$-tuple $T = (T_1, \ldots, T_d)$ can be decomposed as

$$T = \begin{bmatrix} T_{cne} & \Gamma \\ 0 & T_c \end{bmatrix} = \begin{bmatrix} T_{cne,1} & \Gamma_1 \\ 0 & T_c \end{bmatrix} \cdots \begin{bmatrix} T_{cne,d} & \Gamma_d \\ 0 & T_c \end{bmatrix}$$

where $T_{cne}$ is c.n.c. while $T_c$ is coisometric, i.e., the operator block-row matrix $T_c = [T_{c,1} \cdots T_{c,d}]$ is coisometric as an operator from $\mathcal{X}^d$ to $\mathcal{X}_d$.

$$T_{c,1}T_{c,1}^* + \cdots + T_{c,d}T_{c,d}^* = I_{\mathcal{X}_c}.$$ It is known (see [3]) that any column isometry such as $T_c^*$ (sometimes also called a spherical isometry) is jointly subnormal and extends to a spherical unitary, i.e., a commutative $d$-tuple $N = (N_1, \ldots, N_d)$ with joint spectral measure supported on the unit sphere $\partial \mathbb{B}^d$. Thus there is rather complete unitary-equivalence classification theory (in terms of the absolutely-continuous equivalence class of a spectral measure supported on $\partial \mathbb{B}^d$ together with specification of a multiplicity function) for spherical-unitary $d$-tuples. Nevertheless it makes sense to apply our t.c.f.m.-model theory to spherical-unitary tuples. In this case the characteristic function $\theta_T$ has values in $\mathcal{L}(\mathcal{X}_d, \{0\})$ and is thus trivial. Thus this case separates out the extra invariant (i.e., the Agler-decomposition kernel $\mathbb{K}_T$) as the only object of interest. Since the space $\mathcal{X}$ is trivial in this case, the two-component Agler-decomposition then collapses to the single-component Agler decomposition occurring for the weakly isometric case as sketched in Section [3]. To make all objects explicitly computable, in the following example, we specialize even further to the simplest case where $T$ is just a pair of complex numbers $(\lambda_1, \lambda_2)$ on the boundary of the unit ball in $\mathbb{C}^2$.

Example 5.11. Let

$$T = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $|\lambda_1|^2 + |\lambda_2|^2 = 1$. (5.32)

Thus, $\lambda = (\lambda_1, \lambda_2)$ is fixed point on the boundary of the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$ and we view $T$ as a commutative row-contraction on the Hilbert space $\mathcal{X} = \mathbb{C}$, to which our model theory applies. Our goal is to compute explicitly the model characteristic-function/Agler-decomposition pair $(\theta_T, \mathbb{K}_T)$ for this case.

Since $TT^* = 1$, it follows that $D_T = 0$ (as an operator on $\mathbb{C}$) and that $D_T$ is the orthogonal projection onto the orthogonal complement of $\text{Ran} T^*$. Thus, $D_T$ is spanned by the vector $[\begin{smallmatrix} -\lambda_1 \\ \lambda_1 \end{smallmatrix}]$, and if we write the characteristic colligation $U_T = \begin{bmatrix} T^* & D_T \\ D_T & -I \end{bmatrix}$, as a matrix with respect to the choice of basis $[\begin{smallmatrix} -\lambda_1 \\ \lambda_1 \end{smallmatrix}]$ for $D_T$, we arrive at

$$U_T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \\ 0 & 0 \end{bmatrix} : [\mathbb{C}] \to [\mathbb{C}^2].$$
Then the characteristic function \( \theta_T(z) = D + C(I - Z_{row}(z)A)^{-1}Z_{row}(z)B \) is trivial since it lands in the zero-dimensional space. The Agler-decomposition kernel \( \mathbb{K}_T(z, \zeta) \) given by (5.20) collapses to the lower diagonal block \( \Phi(z, \zeta) = [\Phi_{ij}(z, \zeta)] (i, j = 1, 2) \), again since \( \mathcal{Y} = \mathcal{D}_T^* = \{0\} \). We then compute

\[
B^*(I - Z_{row}(z)^*A^*)^{-1} = \begin{bmatrix} -\overline{\lambda}_2 & \overline{\lambda}_1 \\ \overline{\lambda}_2 & \overline{\lambda}_1 \end{bmatrix} \begin{bmatrix} 1 - \overline{\tau}_1\lambda_1 & -\overline{\tau}_1\lambda_2 \\ -\overline{\tau}_2\lambda_1 & 1 - \overline{\tau}_2\lambda_2 \end{bmatrix}^{-1}
\]

\[
= \frac{1}{d(z, \lambda)} \begin{bmatrix} -\overline{\lambda}_2 & \overline{\lambda}_1 \\ \overline{\lambda}_2 & \overline{\lambda}_1 \end{bmatrix} \begin{bmatrix} 1 - \overline{\tau}_2\lambda_2 & \overline{\tau}_1\lambda_2 \\ \overline{\tau}_2\lambda_1 & 1 - \overline{\tau}_1\lambda_1 \end{bmatrix}
\]

where we made use of the assumed identity \( |\lambda_1|^2 + |\lambda_2|^2 = 1 \) and where we have set

\[
d(z, \lambda) = \det(I - Z_{row}(z)^*A^*) = 1 - A^*Z_{row}(z)^* = 1 - \overline{\tau}_1\lambda_1 - \overline{\tau}_2\lambda_2.
\]

Thus \( \mathbb{G}_\lambda(z) \) in (5.20) (with \( \lambda \) in place of \( T \) and with respect to our choice of basis for \( \mathcal{D}_T \)) becomes

\[
\mathbb{G}_T(z) = \frac{1}{d(z, \lambda)} \begin{bmatrix} -\overline{\tau}_2 - \overline{\lambda}_2 \\ -\zeta_2 \end{bmatrix} \begin{bmatrix} \zeta_2 - \lambda_2 & -(\zeta_1 - \lambda_1) \end{bmatrix}
\]

and hence

\[
\mathbb{K}_\lambda(z, \zeta) = \mathbb{G}_\lambda(z)\mathbb{G}_\lambda(\zeta)^* = \frac{1}{d(z, \lambda)d(\zeta, \lambda)} \begin{bmatrix} -\overline{\tau}_2 - \overline{\lambda}_2 \\ -\zeta_2 \end{bmatrix} \begin{bmatrix} \zeta_2 - \lambda_2 & -(\zeta_1 - \lambda_1) \end{bmatrix}. \quad (5.33)
\]

The expected Agler decomposition

\[
I - \theta_T(z)^*\theta_T(\zeta) = \sum_{k=1}^{2} \Phi_{kk}(z, \zeta) - \sum_{i,j=1}^{2} \zeta_i\zeta_j\Phi_{ij}(z, \zeta)
\]

can be expressed as

\[
(1 - \overline{\tau}_1\lambda_1 - \overline{\tau}_2\lambda_2) (1 - \zeta_1\overline{\lambda}_1 - \zeta_2\overline{\lambda}_2) = (\overline{\tau}_2 - \overline{\lambda}_2)(\zeta_2 - \lambda_2) + (\overline{\tau}_1 - \overline{\lambda}_1)(\zeta_1 - \lambda_1)
\]

\[
- \overline{\tau}_1\zeta_1(\overline{\tau}_2 - \overline{\lambda}_2)(\zeta_2 - \lambda_2) + \overline{\tau}_1\zeta_2(\overline{\tau}_2 - \overline{\lambda}_2)(\zeta_1 - \lambda_1)
\]

\[
+ \overline{\tau}_2\zeta_1(\overline{\tau}_1 - \overline{\lambda}_1)(\zeta_2 - \lambda_2) - \overline{\tau}_2\zeta_2(\overline{\tau}_1 - \overline{\lambda}_1)(\zeta_1 - \lambda_1).
\]

This identity in turn can be checked directly by a routine but tedious calculation (or as an exercise for a software package such as MATHEMATICA). For this simple example it is easily checked that we are in the strongly c.c. case. Thus by Theorem 5.9 the characteristic function/Agler decomposition pair \( (0, \mathbb{K}_\lambda) \) is a complete unitary invariant for \( \lambda = [\lambda_1 \quad \lambda_2] \) within the class of commutative strongly c.c. row-contractive operator 2-tuples. Since the matrix entries of \( \mathbb{K}_\lambda \) are scalar, it is clear that two such kernels \( \mathbb{K}_\lambda \) and \( \mathbb{K}_{\lambda'} \) coincide if and only if they are identical. It is also elementary that two such \( \lambda \)'s are unitarily equivalent as operator tuples if and only if they are identical. We conclude as a consequence of Theorem 5.9 that, given two points \( \lambda \) and \( \lambda' \) on \( \partial B^2 \), then \( \mathbb{K}_\lambda = \mathbb{K}_{\lambda'} \) if and only if \( \lambda = \lambda' \)—a point which of course can also be verified directly from the formula (5.33). In summary, for this case we have used a more complicated object \( \mathbb{K}_\lambda \) to classify a much simpler object \( \lambda = (\lambda_1, \lambda_2) \); presumably there are other examples \( T = (T_1, \ldots, T_d) \) where the characteristic pair \( (\theta_T, \mathbb{K}_T) \)
is a simpler object than $T$ and for which the **t.c.f.m.** associated with $(\theta_T, K_T)$ sheds some light on the structure of $T$.

Note that in this example we have arrived at a whole family $K_\lambda(z, \zeta)$ of essentially different Agler decompositions for the fixed Schur-class function $S(z) = 0: \mathbb{C} \to \{0\}$, indexed by a point $\lambda \in \partial \mathbb{B}^2$. In general, identification of a family of row-contractive operator tuples $\mathbf{T}_\lambda$ all having the same characteristic function $\theta_{\mathbf{T}_\lambda} = S$ leads to the construction of a whole family $\{K_{\mathbf{T}_\lambda}\}$ of Agler decompositions for the fixed Schur-class function $S$. This illustrates the non-uniqueness of Agler decompositions for a given $S$ and may lead to other examples where a whole family of distinct Agler decompositions can be exhibited explicitly.

## 6. Noncommutative Agler decompositions

It is possible also to study noncommutative Agler decompositions for the noncommutative characteristic function $\theta_{T, nc}(z)$ of a (possibly noncommutative) row contraction $T$ as follows. We first need to review some basic facts concerning noncommutative kernels; a systematic treatment can be found in [14].

A noncommutative kernel (with operator coefficients) is a formal power series in two sets of noncommuting indeterminates $z = (z_1, \ldots, z_d)$ and $\zeta = (\zeta_1, \ldots, \zeta_d)$ of the form

$$K(z, \zeta) = \sum_{\alpha, \beta \in \mathcal{F}_d} K_{\alpha, \beta} z^\alpha \zeta^\beta.$$ 

While we assume that the $z_1, \ldots, z_d$ do not commute with each other and similarly for $\zeta_1, \ldots, \zeta_d$, it is convenient to assume that the $z$’s commute with the $\zeta$’s: $z_i \zeta_j = \zeta_j z_i$ for all $i, j = 1, \ldots, d$. Let us say that the noncommutative kernel $K(z, \zeta)$ is positive if it has a Kolmogorov decomposition of the form

$$K(z, \zeta) = H(z)H(\zeta)^*.$$ 

Here $H(\zeta)^* = \sum_{v \in \mathcal{F}_d} H_v \zeta^v^T$ if $H(z) = \sum_{v \in \mathcal{F}_d} H_v z^v$. Note that here we follow the conventions of [14, 15] and avoid introduction of formal conjugate variables $\overline{\zeta}_1, \ldots, \overline{\zeta}_d$: we define $(\zeta^v)^* = \zeta^v^T$ where $v^T = i_1 \ldots i_N$ is the transpose of $v = i_N \ldots i_1 \in \mathcal{F}_d$.

If the formal power series in noncommuting indeterminates $z = (z_1, \ldots, z_d)$ has a realization of the form

$$S(z) = D + C(I - Z_X(z)A)^{-1}Z_X(z)B,$$

with colligation matrix $U = \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_d \\ C \\ B_d \\ D \end{bmatrix} : \begin{bmatrix} x \end{bmatrix} \to \begin{bmatrix} x^d \\ y \end{bmatrix}$, then the noncommutative kernel

$$K(z, \zeta) = \begin{bmatrix} C(I_x - Z_X(z)A)^{-1} \\ B^*(I_x - Z_X(z)^* A^*)^{-1} \end{bmatrix} \begin{bmatrix} (I_x - A^* Z_X(\zeta)^* A)^{-1}C^* \\ (I_x^d - AZ_X(\zeta))^{-1}B \end{bmatrix}$$

is positive and has a decomposition of the form

$$K(z, \zeta) = \begin{bmatrix} K(z, \zeta) & \Psi_1(z, \zeta) & \cdots & \Psi_d(z, \zeta) \\ \Psi_1^*(\zeta, z) & \Phi_{11}(z, \zeta) & \cdots & \Phi_{1d}(z, \zeta) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_d^*(\zeta, z) & \Phi_{d1}(z, \zeta) & \cdots & \Phi_{dd}(z, \zeta) \end{bmatrix}$$

(6.2)
with
\[
K_S(z, \zeta) = C(I - Z_X(z)A)^{-1}(I - A^*Z_X(\zeta)^*)^{-1}C^*,
\]
\[
\Psi_k(z, \zeta) = C(I - Z_X(z)A)^{-1}I_k^\dagger(I - AZ_X(\zeta))^{-1}B,
\]
\[
\Phi_{ij}(z, \zeta) = B^*(I - Z_X(z)^*A^*)^{-1}I_j^\dagger(I - AZ(\zeta))^{-1}B.
\]

Furthermore, by making use of the assumed unitary property of the colligation matrix \(U\), one can check that the block matrix entries of \(K\) in (6.2) satisfy the additional identities
\[
K_S(z, \zeta) = k_{S,z}(z, \zeta)I_Y - S(z)(k_{S,z}(z, \zeta)I_d)S(\zeta)^*,
\]
\[
S(z) - S(\zeta) = \sum_{k=1}^d [\Psi_k(z, \zeta)z_k - \zeta_k\Psi_k(z, \zeta)],
\]
\[
I - S(z)^*S(\zeta) = \sum_{k=1}^d \Phi_{kk}(z, \zeta) - \sum_{i,j=1}^d \zeta_j\Phi_{ij}(z, \zeta)z_i.
\]

Here we use the noncommutative formal Szegő kernel \(k_{S,z}(z, \zeta)\) given by
\[
k_{S,z}(z, \zeta) = \sum_{\alpha \in F_d} z^\alpha \zeta^{\alpha^T}.
\]

Conversely, given a formal power series \(S(z) = \sum_{\alpha \in F_d} S_\alpha z^\alpha\) with coefficients \(S_\alpha \in \mathcal{L}(U, Y)\), we say that a kernel \(K\) of the form (6.2) is an Agler decomposition for \(S\) if the relations (6.3) all hold true. In case the colligation matrix \(U\) has the Halmos-dilation form \(U = \begin{bmatrix} T^\dagger & D_T \\ D_T^\dagger & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X}^d \\ \mathcal{Y} \end{bmatrix}\) for a row-contractive operator \(T = [T_1, \ldots, T_d]\) from \(\mathcal{X}^d\) to \(\mathcal{X}\), then the pair \((S = \theta_{T,nc}, K_{T,nc})\) (where \(K_{T,nc}\) is as in (6.1)) is a unitary invariant for \(T\). Several questions arise: (1) to what extent is \((\theta_{T,nc}, K_{T,nc})\) a complete unitary invariant for \(T\), and (2) to what extent can we use a given noncommutative Schur-class multiplier/noncommutative Agler-decomposition pair \((S, K)\) to construct a row contraction \(T\) such that \((S, K) = (\theta_{T,nc}, K_{T,nc})\)?

The first question is more elementary than the second and can be resolved as follows. From the formula (9.1) we see that \(K_{T,nc}(z, \zeta)\) is given by
\[
K_{T,nc}(z, \zeta) = \begin{bmatrix} D_T(I_X - Z_X(z)T^*)^{-1}I_{Yd} - T^*Z_X(\zeta) & -1D_T \end{bmatrix} \cdot \begin{bmatrix} I_X - T^*Z_X(\zeta)^{-1}D_T \end{bmatrix}.
\]

From the formula (6.2) and the noncommutative Agler decomposition formulas (6.3), we see that the upper diagonal block and the off-diagonal blocks are already uniquely determined by \(S(z) = \theta_{T,nc}(z)\). The lower diagonal block has the form \(\Phi(z, \zeta) = [\Phi_{ij}(z, \zeta)]_{i,j=1}^{\text{dom}}\) where
\[
\Phi_{ij}(z, \zeta) = D_T(I_{Yd} - Z_X(z)^*T)^{-1}I_j^\dagger(I_{Xd} - T^*Z_X(\zeta))^{-1}D_T.
\]
It then follows that
\[
\sum_{i,j=1}^{d} \zeta_j \Phi_{ij}(z, \zeta) z_i \\
= D_T (I_X - Z_X(z)^*)^{-1} Z_X(z)^* Z_X(\zeta)(I_X - Z_X(\zeta))^{-1} D_T \\
= D_T Z_X(z)^* (I_X - T Z_X(z))^*^{-1} (I_X - Z_X(\zeta)T^*)^{-1} Z_X(\zeta) D_T \\
= \sum_{i,j=1}^{d} z_i D_T I_i (I_X - T Z_X(z))^* (I_X - Z_X(\zeta)T^*)^{-1} I_j^* D_T \zeta_j.
\]

In the present noncommutative setting, any collection of nonzero formal power series of the form
\[
\{z_i G_{ij}(z, \zeta) \zeta_j : i, j = 1, \ldots, d\}
\]
is linearly independent and it follows that knowledge of \(\Phi_{ij}(z, \zeta)\) uniquely determines the modified kernels
\[
\widehat{\Phi}_{ij}(z, \zeta) := D_T I_i (I_X - T Z_X(z))^*^{-1} (I_X - Z_X(\zeta)T^*)^{-1} I_j^* D_T, \quad i, j = 1, \ldots, d
\]
as well. Using the formal power series expansion
\[
(I_X - T Z_X(z))^*^{-1} = \sum_{\alpha \in \mathcal{F}_d} T^{\alpha} z^{\alpha},
\]
by looking at the coefficient of \(z^{\alpha} \zeta^{\beta}\) in the expansion for \(\widehat{\Phi}_{ij}(z, \zeta)\) we see that the \(\widehat{\Phi}_{ij}\)'s determine uniquely the moments
\[
D_T I_i T^{\alpha} T^{*\beta} I_j^* D_T, \quad \alpha, \beta \in \mathcal{F}_d \text{ and } i, j = 1, \ldots, d.
\]
Combining these moments with the moments
\[
- T, \quad D_T T^{*\alpha} I_j^* D_T : D_T \to D_T^*, \quad \alpha \in \mathcal{F}_d \text{ and } j = 1, \ldots, d
\]
determined by the characteristic function \(\theta_{T,nc}\) gives us the list \([5, 12]\). By the result from \([15]\) we conclude that \((\theta_{T,nc}, K_{T,nc})\) is a complete unitary invariant for the general \(\text{c.n.u.}\) row contraction \(T\) (in particular, for commutative such \(T\)).

We conclude that the two-component Agler-decomposition approach to operator-model theory (i.e., using the two-component Agler decomposition in addition to the characteristic function as a unitary invariant) has mixed results. In the commutative case, some additional information is added and the characteristic-function/Agler-decomposition pair is definitive in some special cases which go beyond the \(\text{c.n.c.}\) case for which the characteristic function \(\theta_T\) alone is definitive. On the other hand, for the noncommutative setting, the results from \([15]\) can be reinterpreted to say that the characteristic-function/Agler-decomposition pair is a complete unitary invariant for the general \(\text{c.n.u.}\) row contraction \(T\). The following table summarizes our results on complete unitary invariants for various classes of Hilbert-space row-contraction operator \(d\)-tuples (note that the class on each line is a subclass of the class on the
As for the second question posed above (construction of a canonical model for a given noncommutative Schur-class function/Agler decomposition pair $(S, K_S)$), we can say the following. By using the analysis in [15], from such a pair $(S, K_S)$ one can construct a characteristic pair $(S, L)$ in the sense of [15] from which one can construct a noncommutative Sz.-Nagy-Foias functional-model space on which there is a canonical choice of c.n.u. row-contractive operator $d$-tuple $T = T(S, L)$. It should also be possible to construct a noncommutative de Branges-Rovnyak model space directly from the noncommutative Schur-class function/Agler decomposition pair $(S, K_S)$; some machinery in this direction has already been developed in [11], but we leave the fleshing out of the complete details for another occasion.

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