WILLMORE SURFACES IN SPHERES VIA LOOP GROUPS III: ON MINIMAL SURFACES IN SPACE FORMS

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Abstract

This paper aims to provide a characterization of the conformal Gauss map of minimal surfaces in Riemannian space forms among Willmore surfaces by using normalized potentials. We first show that, if a strongly conformally harmonic map \( F \) from a Riemann surface into \( G_{1,3}(\mathbb{R}^{1,n+3}) = SO^+(1, n+3)/SO^+(1, 3) \times SO(n) \) contains a constant light-like vector (looking on the images \( F \) as 4-dimensional Lorentzian subspaces of \( \mathbb{R}^{1,n+3} \)), then its normalized potential must have a special form (taking values in one special nilpotent Lie sub-algebra, see Theorem 2.1). Moreover, the normalized potentials of minimal surfaces in other space forms are also shown to have a special form. The third main result shows that if the normalized potential of a strongly conformally harmonic map \( F \) has such special form, then it contains a constant lightlike vector. As a consequence, it is either the conformal Gauss map of a minimal surface in \( \mathbb{R}^{n+2} \) or it can not be the conformal Gauss map of any Willmore surface. The basic methods used here are applications of Wu’s formula to derive potentials and performing an Iwasawa decomposition for the meromorphic frame of this type normalized potential.

Keywords: Willmore surfaces; harmonic maps of finite uniton type; normalized potential; minimal surfaces; Iwasawa decompositions.

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1. Introduction

This is the third paper of a series of papers concerning the global geometry of Willmore surfaces in term of loop group theory, aiming to derive a criterion picking up the normalized potentials of all strongly conformally harmonic maps corresponding to minimal surfaces in \( \mathbb{R}^{n+2} \) or no Willmore surfaces. On the one hand, this provides a characterization of minimal surfaces in \( \mathbb{R}^{n+2} \), as well as this special class of strongly conformally harmonic maps. On the other hand, it also allows us to derive Willmore surfaces different from minimal surfaces in \( \mathbb{R}^{n+2} \) by excluding this special type of normalized potentials, which is important for the application of the main results of [14] to generic Willmore surfaces in \( S^{n+2} \).
It is well-known that minimal surfaces in Riemannian space forms provide standard examples of Willmore surfaces \[3, 4, 28\]. To pick up all minimal surfaces in space forms among Willmore surfaces, we provide a description of them via potentials.

To be concrete, let \(F\) be a harmonic map \(F\) from a Riemann surface \(M\) into \(Gr_{1,3}(\mathbb{R}^{1,n+3})\) (\(= SO^+(1,n+3)/SO^+(1,3) \times SO(n)\)), with a lift \(F : M \rightarrow SO^+(1,n+3)\) and M-C form
\[
\alpha = F^{-1}dF = \left(\begin{array}{c}
A_1 \\
-B_1^t I_{1,3} \\
A_2
\end{array}\right)dz + \left(\begin{array}{c}
\hat{A}_1 \\
-B_1^t I_{1,3} \\
\hat{A}_2
\end{array}\right)d\hat{z}.
\]

Here \(A_1 \in Mat(4 \times 4, \mathbb{C}), A_2 \in Mat(n \times n, \mathbb{C}), B_1 \in Mat(4 \times n, \mathbb{C})\), \(I_{1,3} = diag(-1,1,1,1)\). \(F\) is called strongly conformal if \(B_1\) satisfies \(B_1^t I_{1,3} B_1 = 0\). Note that this condition is independent of the choice of \(F\) \[14\]. We also recall that it is shown that the conformal Gauss map of a Willmore surface is a strongly conformally harmonic map \[14\]. Conversely, by Theorem 3.10 of \[14\], there are two different kinds of strongly conformally harmonic maps:

those which contain a constant lightlike vector and those which do not contain a constant lightlike vector.

Moreover from Theorem 3.10 of \[14\], we see that if a strongly conformally harmonic map \(F\) does not contain a lightlike vector, \(f\) will always be the conformal Gauss map of some Willmore map. Especially, this kind of Willmore maps corresponds exactly to all the Willmore maps which are not Möbius equivalent to any minimal surface in \(\mathbb{R}^{n+2}\), since minimal surfaces in \(\mathbb{R}^{n+2}\) can be characterized as Willmore surfaces with their conformal Gauss map containing a constant lightlike vector \[19, 30, 21\] (See Lemma 1.2 below, see also \[3, 17, 23, 22\] and \[5\]). Since minimal surfaces in \(\mathbb{R}^{n+2}\) can be constructed by a straightforward way, one will be mainly interested in Willmore surfaces not Möbius equivalent to minimal surfaces in \(\mathbb{R}^{n+2}\). It is therefore vital to derive a criterion to determine whether a strongly conformally harmonic map \(f\) contains a lightlike vector or not. Note that this will also yields an interesting description of minimal surfaces in \(\mathbb{R}^{n+2}\). Applying Wu’s formula, one will obtain the following description of the normalized potential of \(F\) when it contains a constant light-like vector.

**Theorem 1.1.** Let \(\mathbb{D}\) denote the Riemann surface \(S^2\), \(\mathbb{C}\) or the unit disk of \(\mathbb{C}\). Let \(F : \mathbb{D} \rightarrow SO^+(1,n+3)/SO^+(1,3) \times SO(n)\) be a strongly conformally harmonic map which contains a constant light-like vector. Assume that \(f(p) = I_{n+4} \cdot K\) w.r.t some base point \(p \in \mathbb{D}\) and \(z\) is a local coordinate with \(z(p) = 0\). Then the normalized potential of \(f\) with reference point \(p\) is of the form

\[
\eta = \lambda^{-1} \begin{pmatrix}
0 & \hat{B}_1 \\
-B_1^t I_{1,3} & 0
\end{pmatrix} dz, \quad \text{where} \quad \hat{B}_1 = \begin{pmatrix}
\hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1n} \\
-\hat{f}_{11} & -\hat{f}_{12} & \cdots & -\hat{f}_{1n} \\
\hat{f}_{31} & \hat{f}_{32} & \cdots & \hat{f}_{3n} \\
i\hat{f}_{31} & i\hat{f}_{32} & \cdots & i\hat{f}_{3n}
\end{pmatrix}.
\]

Here \(f_{ij}\) are meromorphic functions on \(\mathbb{D}\).
It is well-known that minimal surfaces in Riemannian space forms can be characterized by the following lemma (The statements and proofs can be found in [19], see also [21] for a proof of Case (1)).

**Lemma 1.2.** [19], [30] Let \( y : M \to S^{n+2} \) be a Willmore surface, with \( F \) as its conformal Gauss map. We say that \( F \) contains a constant vector \( a \in \mathbb{R}^{1,n+3} \) if for any \( p \in M \), \( a \) is in the \( 4\)-dim Lorentzian subspace \( F(p) \). Then

1. \( y \) is Möbius equivalent to a minimal surface in \( \mathbb{R}^{n+2} \) if and only if \( F \) contains a non-zero constant lightlike vector.
2. \( y \) is Möbius equivalent to a minimal surface in some \( S^{n+2}(c) \) if and only if \( F \) contains a non-zero constant timelike vector.
3. \( y \) is Möbius equivalent to a minimal surface in \( \mathbb{H}^{n+2}(c) \) if and only if \( F \) contains a non-zero constant spacelike vector.

Applying this lemma and Wu’s formula, one obtains the following descriptions of minimal surfaces in space forms.

**Theorem 1.3.** Let \( F : M \to SO^+(1,n+3)/SO(1,3) \times SO(n) \) be a strongly conformal harmonic map. Let

\[
\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix} dz, \quad \hat{B}_1 = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix},
\]

be the normalized potential of \( F \) with respect to some base point \( z_0 \). Then, up to a conjugation by some \( T \in O^+(1,3) \times O(n) \),

1. \( F \) contains a constant lightlike vector, if and only if every \( v_j \) has the form

\[
(1.3) \quad v_j = f_j ( f_{j1} \quad -f_{j3} \quad f_{j3} \quad if_{j3})^t, \quad \text{with } f_{jl} \text{ meromorphic.}
\]
2. \( F \) contains a constant timelike vector, if and only if every \( v_j \) has the form

\[
(1.4) \quad v_j = g_j ( 0 \quad 2g_0 \quad 1 - g_0^2 \quad i(1 + g_0^2))^t, \quad \text{with } g_j, g_0 \text{ meromorphic.}
\]
3. \( F \) contains a constant spacelike vector, if and only if every \( v_j \) has the form

\[
(1.5) \quad v_j = h_j ( 2ih_0 \quad 0 \quad 1 - h_0^2 \quad i(1 + h_0^2))^t, \quad \text{with } h_j, h_0 \text{ meromorphic.}
\]

Potentials of above form will be named as the canonical potentials of the corresponding minimal surfaces in space forms.

The proof of Case (1) of this theorem needs more techniques and this case has a more general background. The classical theorem of Bryant [3] tells us that every Willmore two-sphere in \( S^3 \) is Möbius equivalent to some minimal surface with embedded planar ends in \( \mathbb{R}^3 \). However, when the co-dimension adds, this does not hold true and there are more Willmore two-spheres in \( S^4 \) different from minimal surfaces in \( \mathbb{R}^4 \) [17]. In [27], by using loop group methods developed in [14], [15], we provide a classification of Willmore two-spheres in \( S^{n+2} \) via the normalized potentials of their harmonic conformal Gauss maps. The basic idea is to deform the work of Burstall and Guest [6], [16] into the DPW version [15] and to characterize harmonic conformal Gauss maps of Willmore surfaces by
describing their normalized potentials, which take values in nilpotent Lie sub-algebras by the results of [6] and [15]. In [27], the corresponding nilpotent Lie sub-algebras are shown to be \((m - 2)\) kinds when \(n + 4 = 2m\). The possible forms of normalized potentials are also listed explicitly.

To obtain the geometric properties, moreover, to derive concrete expressions of Willmore surfaces, one needs to perform Iwasawa decompositions for the meromorphic frames of the given normalized potentials. Since the potentials take values in some nilpotent Lie sub-algebra, the meromorphic frames, i.e., the integrations of normalized potentials, are Laurent polynomials in \(\lambda \in S^1\) [6], [16], [15]. A theoretical procedure of Iwasawa decompositions of such algebra elements in a loop group has been presented in [8]. While for the first type normalized potentials in the classification theorem of [27], the Iwasawa decomposition has an easier and more straightforward way.

To be concrete, assume that the normalized potential is of the form (See Section 3 of [14], Section 2 of [27] for the definitions and notations)

\[
\eta = \lambda^{-1}\eta_{-1}dz
\]

with

\[
(1.7) \quad \eta_{-1} = \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,3} & 0 \end{pmatrix}, \quad \text{and} \quad \hat{B}_1 = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{m-1,1} & f_{m-1,2} \\ f_{11} & f_{12} & \cdots & f_{m-2,1} & f_{m-2,2} \\ f_{13} & f_{14} & \cdots & f_{m-2,3} & f_{m-2,4} \\ i f_{13} & i f_{14} & \cdots & i f_{m-2,3} & i f_{m-2,4} \end{pmatrix},
\]

where \(f_{ij}\) are meromorphic functions on the Riemann surface \(\tilde{M}\). Note that this \(\eta\) is conjugate to the one in (1.1) by \(\hat{T} = \text{diag}(1, -1, 1, \cdots, 1)\). So the corresponding harmonic maps are Möbius equivalent to each other. Moreover, we have the following theorem, which is in fact part of (1) of Theorem 1.3.

**Theorem 1.4.** Let \(\mathcal{F} : \mathbb{D} \to SO^+(1, 2m - 1)/SO^+(1, 3) \times SO(2m - 4)\) be a strongly conformally harmonic map with its normalized potential being of the form in (1.7). Then \(\mathcal{F}\) contains a constant light-like vector.

Moreover, if \(\mathcal{F}\) is the conformal Gauss map of a strong Willmore map \(y : \mathbb{D} \to S^{2m-2}\), then rank(\(\hat{B}_1\)) \(\leq 1\) and \(y\) is Möbius equivalent to a minimal surface in \(\mathbb{R}^{2m-2}\).

It is easy to verify that such \(\mathcal{F}\) is of finite uniton type. Moreover, \(\mathcal{F}\) actually belongs to one of the simplest cases, called \(S^1 - \text{invariant}\) (See [6], [11], [27]). For such harmonic maps, by using a straightforward and lengthy computation, one can derive the harmonic map explicitly and then read off all needed information, which will provide a proof of Theorem 1.4.

**Corollary 1.5.** Let \(\mathcal{F} : \mathbb{D} \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n)\) be a strongly conformally harmonic map with its normalized potential \(\eta\) of the form (1.1) and of maximal rank(\(\hat{B}_1\)) = 2. Then \(\mathcal{F}\) can not be the conformal Gauss map of a Willmore surface. In particular, there exist conformally harmonic maps which are not related to any Willmore map.
As a consequence, now we have a complete knowledge on the strongly conformally harmonic maps which produce either minimal surfaces in \( \mathbb{R}^{n+2} \) or no Willmore surfaces at all, which make our theory workable for the study of non-Euclidean-minimal Willmore surfaces. This corollary shows that the characterization theorems here do make sense for the theory in [14] to deal with global Willmore surfaces different from minimal surfaces in \( \mathbb{R}^{n+2} \).

This paper is organized as follows. Section 2 provides the forms of potentials of strongly conformally harmonic maps containing a constant lightlike vector. The proofs of case (2) and (3) of Theorem 1.3 are also derived in this section. Section 3 contains the characterizations of minimal surfaces in \( \mathbb{R}^{n+2} \) in terms of potentials and several technical lemmas providing a proof of our main theorem. The proofs of these technical lemmas are derived in Section 4.

For simplicity we will always retain the notions and results in [14] and [27] in this paper.

2. Potentials of strongly conformally harmonic maps containing a constant lightlike vector

This section is to derive the forms of the normalized potentials of strongly conformally harmonic maps containing a non-zero constant real vector. The basic idea is to characterize the Maurer-Cartan form of such a strongly conformally harmonic map \( F : \mathbb{D} \to SO^+(1, n + 3)/SO^+(1, 3) \times SO(n) \).

2.1. Proof of Theorem 1.1. The proof of Theorem 1.1 relies on the following technical lemma.

**Lemma 2.1.** Let

\[
A_1 = \begin{pmatrix}
0 & 0 & a_{13} & a_{14} \\
0 & 0 & -a_{13} & -a_{14} \\
a_{13} & a_{14} & 0 & a_{34} \\
a_{14} & a_{13} & -a_{34} & 0
\end{pmatrix}
\]

be a holomorphic matrix function on a contractible open Riemann surface \( U \). Let \( F_{01} \) be a solution to the equation

\[
F_{01}^{-1} dF_{01} = A_1 dz, \quad F_{01}|_{z=0} = I_4.
\]

Then

\[
F_{01} = \begin{pmatrix}
1 + \frac{1}{2}(b_{13}^2 + b_{14}^2) & \frac{1}{2}(b_{13}^2 + b_{14}^2) & b_{13} & b_{14} \\
-\frac{1}{2}(b_{13}^2 + b_{14}^2) & 1 - \frac{1}{2}(b_{13}^2 + b_{14}^2) & -b_{13} & -b_{14} \\
b_{13} & b_{14} & 1 & 0 \\
b_{14} & b_{13} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
cos \varphi \\
-sin \varphi
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
-cos \varphi \\
-sin \varphi
\end{pmatrix}
\]

with

\[
\varphi = \int_0^z a_{34} \, dw
\]
and

\[ b_{13} = \int_0^z (a_{13} \cos \varphi + a_{14} \sin \varphi) dz, \quad b_{14} = \int_0^z (-a_{13} \sin \varphi + a_{14} \cos \varphi) dz. \]

**Proof.** Set

\[
\tilde{F}_{01} = \begin{pmatrix} 1 & 1 \\ \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix},
\]

and

\[
\hat{F}_{01} = \begin{pmatrix} 1 + \frac{1}{2} (b_{13}^2 + b_{14}^2) & \frac{1}{2} (b_{13}^2 + b_{14}^2) & b_{13} & b_{14} \\ -\frac{1}{2} (b_{13}^2 + b_{14}^2) & 1 - \frac{1}{2} (b_{13}^2 + b_{14}^2) & -b_{13} & -b_{14} \\ b_{13} & b_{13} & 1 & 0 \\ b_{14} & b_{14} & 0 & 1 \end{pmatrix}.
\]

Straightforward computations yield

\[
\tilde{F}_{01}^{-1} d\tilde{F}_{01} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{34} & 0 \\ 0 & 0 & -a_{34} & 0 \end{pmatrix} dz,
\]

and

\[
\hat{F}_{01}^{-1} d\hat{F}_{01} = \begin{pmatrix} 0 & 0 & b_{13}' & b_{14}' \\ 0 & 0 & -b_{13}' & -b_{14}' \\ b_{13}' & b_{13}' & 0 & 0 \\ b_{14}' & b_{14}' & 0 & 0 \end{pmatrix} dz,
\]

with

\[ b_{13}' = a_{13} \cos \varphi + a_{14} \sin \varphi, \quad b_{14}' = -a_{13} \sin \varphi + a_{14} \cos \varphi. \]

Moreover, one obtains

\[
\tilde{F}_{01}^{-1} \left( \hat{F}_{01}^{-1} d\hat{F}_{01} \right) \tilde{F}_{01} = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & -a_{13} & -a_{14} \\ a_{13} & a_{13} & 0 & 0 \\ a_{14} & a_{14} & 0 & 0 \end{pmatrix}.
\]

Since \( F_0 = \tilde{F}_{01} \hat{F}_{01} \), one derives

\[
F_0^{-1} F_{01z} = \tilde{F}_{01}^{-1} \left( \hat{F}_{01}^{-1} d\hat{F}_{01} \right) \tilde{F}_{01} + \tilde{F}_{01}^{-1} \tilde{F}_{01z} = A_1.
\]

\[ \square \]

**Proof of Theorem 1.1:**

Let \( F(z, \bar{z}, \lambda) = (\phi_1, \phi_2, \phi_3, \phi_4, \psi_1, \ldots, \psi_n) \) be a frame of \( f \) with the initial condition \( F(0,0,\lambda) = I_{n+4} \). W.l.o.g., we may assume that

\[ Y_0 = \phi_1 - \phi_2 \]
is the constant lightlike vector contained in $f$. As a consequence, we derive

$$\phi_{1z} = \phi_{2z} = a_{13}\phi_3 + a_{14}\phi_4 + \sqrt{2}\sum_{j=1}^{n} \beta_j\psi_j.$$  

That is

$$\begin{pmatrix} \phi_{1z}, \phi_{2z} \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} & \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \end{pmatrix}. F^t$$

Comparing with

$$F^{-1} F_z = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,3} & A_2 \end{pmatrix},$$

we obtain

$$A_1 = \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & -a_{13} & -a_{14} \\ a_{13} & a_{14} & 0 & a_{34} \\ a_{14} & a_{13} & -a_{34} & 0 \end{pmatrix},$$

and

$$B_1 = \begin{pmatrix} \sqrt{2}\beta_1 & \cdots & \sqrt{2}\beta_n \\ -\sqrt{2}\beta_1 & \cdots & -\sqrt{2}\beta_n \\ -\hat{k}_1 & \cdots & -\hat{k}_n \\ -\hat{k}_n & \cdots & -\hat{k}_n \end{pmatrix}.$$

Since $B_1^t I_{1,3} B_1 = 0$,

$$\hat{k}_1 = ik_1, \cdots, \hat{k}_n = ik_n, \text{ or } \hat{k}_1 = -ik_1, \cdots, \hat{k}_n = -ik_n.$$

Similar to the discussion in Lemma 3.8 of Section 3 of [14], without loss of generality, we assume that on $\tilde{M}$, $\hat{k}_1 = ik_1, \cdots, \hat{k}_n = ik_n$.

For the computation of the normalized potential, we will apply Wu’s formula (Theorem 4.23, Section 4.3 of [14] (See also [29]). Let $\delta_1 = (\hat{a}_{ij})$ denote the “holomorphic part” of $A_1$ with respect to the base point $z = 0$, i.e., the part of the Taylor expansion of $A_1$ which is independent of $\bar{z}$. Let $F_{01}$ be a solution to the equation

$$F_{01}^{-1} dF_{01} = \delta_1 dz, \quad F_{01}|_{z=0} = I_4.$$

By Lemma 2.1 $F_{01}$ is equal to

$$\begin{pmatrix} 1 + \frac{1}{2}(b_{13}^2 + \hat{a}_{14}^2) & \frac{1}{2}(b_{13}^2 + \hat{a}_{14}^2) & -b_{13}\cos\varphi - b_{14}\sin\varphi & b_{13}\sin\varphi + b_{14}\cos\varphi \\ -\frac{1}{2}(b_{13}^2 + \hat{a}_{14}^2) & 1 - \frac{1}{2}(b_{13}^2 + \hat{a}_{14}^2) & -b_{13}\cos\varphi + b_{14}\sin\varphi & -(b_{13}\sin\varphi + b_{14}\cos\varphi) \\ b_{13} & b_{13} & \cos\varphi & \sin\varphi \\ b_{14} & b_{14} & -\sin\varphi & \cos\varphi \end{pmatrix}$$

with

$$\varphi = \int_0^z \hat{a}_{34} \, dz$$

and

$$b_{13} = \int_0^z (\hat{a}_{13}\cos\varphi + \hat{a}_{14}\sin\varphi) \, dz, \quad b_{14} = \int_0^z (-\hat{a}_{13}\sin\varphi + \hat{a}_{14}\cos\varphi) \, dz.$$
Let $\delta_2$ denote the “holomorphic part” of $A_2$, with respect to the base point $z = 0$, and let $F_{02}$ be a solution to the equation

$$F_{02}^{-1} dF_{02} = \delta_2 dz, \quad F_{02}|_{z=0} = I_n.$$ 

Let $\tilde{B}_1$ denote the holomorphic part of $B_1$. By Wu’s formula (Theorem 4.23 of [14]), the normalized potential can be represented in the form

$$\eta = \lambda^{-1} \left( \begin{array}{cc} F_{01} & 0 \\ 0 & F_{02} \end{array} \right) \left( \begin{array}{cc} 0 & \tilde{B}_1 \\ -\tilde{B}_1^t I_{1,3} & 0 \end{array} \right) \left( \begin{array}{cc} F_{01} & 0 \\ 0 & F_{02} \end{array} \right)^{-1} d\bar{z},$$

with

$$\tilde{B}_1 = F_{01} \tilde{B}_1 F_{02}^{-1}.$$

$$= F_{01} \cdot \left( \begin{array}{cccc} \hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1n} \\ -\hat{f}_{11} & -\hat{f}_{12} & \cdots & -\hat{f}_{1n} \\ -\hat{f}_{31} & -\hat{f}_{32} & \cdots & -\hat{f}_{3n} \\ -i\hat{f}_{31} & -i\hat{f}_{32} & \cdots & -i\hat{f}_{3n} \end{array} \right) + F_{02}^{-1}.$$

$$= \left( \begin{array}{cccc} \hat{f}_{11} & \hat{f}_{12} & \cdots & \hat{f}_{1n} \\ -\hat{f}_{11} & -\hat{f}_{12} & \cdots & -\hat{f}_{1n} \\ -\hat{f}_{31} & -\hat{f}_{32} & \cdots & -\hat{f}_{3n} \\ -i\hat{f}_{31} & -i\hat{f}_{32} & \cdots & -i\hat{f}_{3n} \end{array} \right),$$

$$\square$$

2.2. Proof of Theorem 1.3

Proof of Theorem 1.3:

Case (1) comes from Theorem 1.1 and Theorem 1.4.

Now we consider Case (2). Since $F$ contains a constant timelike vector $e_0$. We can assume $|e_0| = 1$ and it is time forward. Then there exists a transformation $T \in SO(1, n + 3)$ transforming $e_0$ into $(1, 0, \cdots, 0)^t$ and transforming $\mathcal{F}$ into $T\mathcal{F}$. So without lose of generality, we assume $e_0 = (1, 0, \cdots, 0)^t$. Let $F = (e_0, \hat{e}_0, e_1, e_2, \psi_1, \cdots, \psi_n)$ be a lift of $\mathcal{F}$. As a consequence, every entry of the first column and the first row of $\alpha = F^{-1} dF$ is zero. So does $F_\lambda$ when introducing loops for $F$. Then using Wu’s formula [29] (see also Theorem 4.23, and Theorem 4.24 of [14] for the Willmore case), we see that every entry of the first column and the first row of the normalized potential stays zero. Moreover, by Theorem 4.24, $\tilde{B}_1$ also satisfies $\tilde{B}_1^t I_{1,3} \tilde{B}_1 = 0$, which yields

$$v_j^t I_{1,3} v_l = 0, \quad \text{for all } j, l = 1, \cdots, n.$$ 

Formula (1.4) follows by a simple computation like deriving Weierstrass representation of minimal surfaces in $\mathbb{R}^3$.

The converse part is straightforward. In fact, integrating $\eta$, we see that all the entries of the first column and the first row of $F_-$ are 0, except the $(1, 1)$—entry being 1. Then
performing Iwasawa decompositions one see that $F$ inherits the same property, that is, the harmonic map $F$ contains $e_0 = (1, 0, \cdots, 0)^t$ at every point.

Case (3) holds by similar discussions.

\[\square\]

### 3. Minimal surfaces in $\mathbb{R}^n$ as Willmore surfaces

This and the next section aim to give a concrete description of all Willmore surfaces corresponding to the first type of nilpotent Lie subalgebras in [27], by direct computations. To this end, since there are many lengthy and elementary computations, we will reduce the proofs into several technical lemmas, which will be stated in this section. The proofs of these lemma will be left to Section 4.

The basic idea in our computations is to express the normalized potentials by some strictly upper triangular matrices-valued 1-forms, since the computations will be essentially simplified. For this purpose, we need to transform the original group into a different one such that the matrices coefficients of the potentials in (1.7) will be upper triangular matrices-valued. So we will first recall the Lie group isometry in Section 3.1. Then we state the five technical lemmas, which combine the proof of our main results in Section 3.2.

#### 3.1. Preliminary

To begin with, we first recall some basic notations and results. The detailed descriptions and proofs can be found in Section 3 of [27]. We will retain the notations in [27].

Recall that $SO^+(1, n + 3) = SO(1, n + 3)_0$ is the connected subgroup of

$$SO(1, n + 3) := \{ A \in Mat(n + 4, \mathbb{R}) \mid A^t I_{1,n+3} A = I_{1,n+3}, \det A = 1 \},$$

with

$$I_{1,n+3} = \text{diag}\{-1, 1, \cdots, 1\}.$$

The subgroup $K = SO^+(1, 3) \times SO(n)$ is defined by the involution (Here $D = \text{diag}\{-I_4, I_n\}$)

$$\sigma : SO^+(1, n + 3) \to SO^+(1, n + 3) \quad A \mapsto DAD^{-1}.$$  

(3.1)

For simplicity, we assume that $n$ is even and $n + 4 = 2m$. We also have

$$G(n + 4, \mathbb{C}) := \{ A \in Mat(n + 4, \mathbb{C}) \mid A^t J_{n+4} A = J_{n+4}, \det A = 1 \},$$

with

$$J_{n+4} = (j_{k,l})_{(n+4) \times (n+4)}, \quad j_{k,l} = \delta_{k+l,n+5} \text{ for all } 1 \leq k, l \leq n + 4,$$

By Lemma 3.1 of [27], one obtains a Lie group isometry from $SO^+(1, 2m - 1, \mathbb{C})$ into $G(2m, \mathbb{C})$, defined by the following map

$$P : \quad SO^+(1, 2m - 1, \mathbb{C}) \to G(2m, \mathbb{C}) \quad A \mapsto \tilde{P}^{-1} A \tilde{P}.$$  

(3.3)
Under this isometry, we have that $\mathcal{P}(SO^+(1, 2m - 1))$ is equal to the connected component of $\{F \in G(2m, \mathbb{C}) \mid F = S_{2m}^{-1}FS_{2m}\}$ containing $I_{2m}$. Here

$$S_{2m} = \begin{pmatrix} 1 & & & & \\ & J_{2m-2} & & & \\ & & & & \\ & & & J_{2m-2} & \\ & & & & 1 \end{pmatrix}.$$  

Moreover, this induces an involution of the loop group $\Lambda G(2m, \mathbb{C})$:

$$\hat{\tau} : \Lambda G(2m, \mathbb{C}) \to \Lambda G(2m, \mathbb{C})$$

$$F \mapsto S_{2m}^{-1}FS_{2m}$$

with $\mathcal{P}(\Lambda SO^+(1, 2m - 1)) = \{F \in \Lambda G(2m, \mathbb{C}) \mid \hat{\tau}(F) = F\}$ as its fixed point set. We also have that the image of $SO^+(1, 3) \times SO(2m - 4)$ under $\mathcal{P}$ is of the form

$$\mathcal{P}((SO^+(1, 3) \times SO(2m - 4))^C) = \{F \in G(2m, \mathbb{C}) \mid F = D_0^{-1}FD_0\}$$

with

$$D_0 = \tilde{P}^{-1}D\tilde{P} = \text{diag}\{-1, -1, I_{2m-4}, -1, -1\} = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & & & \\ & & & I_{2m-4} & \\ & & & & -1 \end{pmatrix}.$$  

### 3.2. Technical Lemmas

With the notations as above, we are able to state the following lemmas:

**Lemma 3.1.** Let $\eta \in \Lambda^{-\mathfrak{s}\mathfrak{o}}(1, n+3)$ be the normalized potential defined on $\mathbb{D}$ in Theorem 1.4. Then

$$\mathcal{P}(\eta) = \lambda^{-1} \begin{pmatrix} 0 & \tilde{f} & 0 \\ 0 & 0 & -\tilde{f}^2 \\ 0 & 0 & 0 \end{pmatrix} dz, \text{ with } \tilde{f} \in \text{Mat}(2 \times (2m - 4), \mathbb{C}), \tilde{f}^2 := J_{2m-4}\tilde{f}J_2.$$  

**Lemma 3.2.** Let $\eta$ be as in Lemma 3.1. Then $H = I_{2m} + \lambda^{-1}H_1 + \lambda^{-2}H_2$ is the solution to

$$H^{-1}dH = \mathcal{P}(\eta), \quad H|_{z=0} = I_{2m}.$$
with

\[ H_1 = \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & -f^2 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ f = \int_0^z \tilde{f} \, dz, \quad g = -\int_0^z (f \tilde{f}^2) \, dz. \]

Note that if \( \eta \) is derived from some strong Willmore map, then \( \eta \) is meromorphic and also \( H \), the integration of \( \mathcal{P}(\eta) \), is meromorphic. If in Lemma 3.1 we start from some normalized potential and want to construct a strong Willmore map defined on \( \mathbb{D} \), then we need to assume that \( \eta \) is meromorphic and also that \( H \) is meromorphic.

**Lemma 3.3.** Retaining the assumptions and the notation of the previous lemmas, assume that \( \mathcal{P}(\eta) \) is the normalized potential of some harmonic map, we obtain:

The Iwasawa decomposition of \( H \) is

\[ H = \tilde{F} \tilde{F}_+, \quad \text{with} \quad \tilde{F} \in \mathcal{P}(\Lambda SO^+(1,2m-1)_\sigma) \subset \Lambda G(2m,\mathbb{C})_\sigma, \quad \tilde{F}_+ \in \Lambda^+ G(2m,\mathbb{C})_\sigma. \]

And \( \tilde{F} \) is given by (see also (3.7) of \[27\])

\[ \tilde{F} = H \hat{\tau}(W) L_0^{-1}. \]

Here \( W, W_0 \) and \( L_0 \) are the solutions to the matrix equations

\[ \hat{\tau}(H)^{-1} H = WW_0 \hat{\tau}(W)^{-1}, \quad W_0 = \hat{\tau}(L_0)^{-1} L_0 \]

with

\[ W = I_{2m} + \lambda^{-1} W_1 + \lambda^{-2} W_2, \quad W_1 = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & -u^* \\ 0 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & \hat{g} \\ 0 & 0 & 0 \end{pmatrix}, \]

and

\[ W_0 = \begin{pmatrix} a & 0 & b \\ 0 & q & 0 \\ 0 & 0 & d \end{pmatrix}, \quad L_0 = \begin{pmatrix} l_1 & 0 & l_2 \\ 0 & l_0 & 0 \\ 0 & 0 & l_4 \end{pmatrix}, \quad \text{with} \ l_1, \ l_4 \text{ upper triangular}. \]

Moreover, we have

\[ d = I_2 + E_4 \tilde{f}^2 f^* + E_4 \hat{g}^t E_1 g, \]
\[ u^* d = f^* - \tilde{f}^t E_1 g, \]
\[ q + u^* d E_4 \tilde{u} f^* = I_{2m-4} + f^t E_1 f, \]
\[ a + uq^* E_1 + g E_4 \tilde{g}^t \hat{g} f E_1 = I_2, \]
\[ b + uq^* E_2 + g E_4 \tilde{g}^t \hat{g} f E_2 = E_2 \tilde{f}^* f^* + E_2 \hat{g}^t E_1 g, \]
\[ uq - g E_4 \tilde{u} f^* = f. \]
Here $E_1$, $E_2$, $E_3$ and $E_4$ are defined as

$$E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ E_2 = E_3^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ E_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

**Remark 3.4.**

1. Since in Lemma 3.3 the matrices $f$ and $g$, whence also $f^\sharp$, are given, equation (3.13a) determines $d$, where $d$ is invertible (certainly true for small $\varepsilon$ close to $\varepsilon = 0$). Then equation (3.13b) determines $u^\sharp$, hence $u$. Inserting this into (3.13c) results in determining $q$. Inserting what we have so far into (3.13d) determines $a$ and similarly from (3.13e) we obtain $b$. The last equation, (3.13f), is a consequence of the previous equations. Therefore, the only condition for the solvability of the system of equations is the invertibility of $d$.

2. If $f$ and $g$ are rational functions of $\varepsilon$, the invertibility of $d$ is satisfied locally, whence on an open dense subset due to the rational expression in $\varepsilon$, $\bar{\varepsilon}$.

**Lemma 3.5.** Retaining the assumptions and the notation of the previous lemmas, the Maurer-Cartan form of $\tilde{F}$ in (3.12) is of the form

$$\tilde{\alpha}'_p = \lambda^{-1} \begin{pmatrix} 0 & \bar{b} & 0 \\ 0 & 0 & -\bar{b}^* \\ 0 & 0 & 0 \end{pmatrix} \, dz, \quad \tilde{\alpha}'_t = \begin{pmatrix} a_1 & 0 & a_2 \\ 0 & a_0 & 0 \\ 0 & 0 & a_4 \end{pmatrix} \, dz.$$

**Lemma 3.6.** Let $\mathcal{F} : M \to SO^+(1, 2m-1)/SO^+(1, 3) \times SO(2m-4)$ be a strongly conformally harmonic map with an extended frame $F$. If the Maurer-Cartan form of $\tilde{F} = \mathcal{P}(F)$ is of the form (3.15) in $\mathfrak{g}(2m, \mathbb{C})$, then $\mathcal{F}$ contains a constant light-like vector. Therefore, if $\mathcal{F}$ is the conformal Gauss map of some Willmore map $y$, $y$ is Möbius equivalent to a minimal surface in $\mathbb{R}^{2m}$.

Lemma 3.1 and Lemma 3.2 can be verified by straightforward computations. And the other lemmas will be proven in the following section.

**Proof of Theorem 1.4:**

Combination of the above five lemmas provides the proof of Theorem 1.4. \qed

Note that Corollary 1.5 also already comes from above lemmas. The fact that $\eta_{-1}$ takes values in a nilpotent Lie subalgebra of rank 2 comes from Theorem 2.6 and Lemma 3.5 of [27].

**Remark 3.7.**

1. For a general procedure for the computation of Iwasawa decompositions for algebraic loops, or more generally for rational loops, see §I.2 of [8], where they provided a somewhat constructive method to do the decompositions for such loops. One may also compare our treatment here with the ones for CMC surfaces in [12].

2. For the theoretical descriptions of loop groups of non-compact Lie group, we refer to [1] and [20]. Another concrete discussions of $\Lambda SU(1, 1)$ can be found in [2].

3. Recently there are several publications concerning harmonic maps into compact Lie groups and compact symmetric spaces, under different methods from us, see [18], [9], [25].
and reference therein. Most of their treatments basically followed the theories developed of Uhlenbeck [26] and Segal [24]. We also note that in [25], an inverse of this procedure is used also for the computations for the Iwasawa decompositions of algebraic loop group \( \lambda_{alg} U(n)^C \), which provides another way to do the concrete Iwasawa decompositions for algebraic loops.

4. Iwasawa Decompositions

In this section we first provide the proof of Lemma 3.3, which corresponds to the Iwasawa decompositions. Then, we derive the M-C forms by the information from the explicit Iwasawa decompositions, which yields the geometric descriptions of the corresponding harmonic maps.

4.1. Iwasawa decompositions and Lemma 3.3.

Proof of Lemma 3.3:

The first question is the existence of the Iwasawa decomposition for \( H \). This is guaranteed by the existence of an Iwasawa decomposition on an open subset containing the identity and \( H|_{z=0} = I \) (see Theorem 4.1 of [14], Theorem 2.3 of [15], also [20]). So certainly near \( z = 0 \) we have \( H = \tilde{F} \tilde{F}_+ \). Since \( \hat{\tau}(\tilde{F}) = \tilde{F} \), the maximal and minimal powers of \( \lambda \) of \( \tilde{F} \) are \( \lambda^2 \) and \( \lambda^{-2} \) respectively. Hence in \( \tilde{F} \) the maximal power of \( \lambda \) is at most 2.

Moreover, from the definition of \( G(n+4, \mathbb{C}) \) we infer that also \( (\tilde{F}_+)^{-1} \) only contains the powers of \( \lambda \) from \(-2\) to \(2\). Moreover,

\[
I_{n+4} = \hat{\tau}(\tilde{F})^{-1} \tilde{F} = \left( \hat{\tau}(\tilde{F}_+^{-1}) \right)^{-1} \hat{\tau}(H)^{-1} H \tilde{F}_+^{-1}
\]

implies that

\[
\hat{\tau}(H)^{-1} H = \hat{\tau}(\tilde{F}_+^{-1}) \tilde{F}_+. \]

Let’s write

\[
\hat{\tau}(\tilde{F}_+^{-1}) = W \hat{\tau}(L_0)^{-1}
\]

with

\[
W = I_{2m} + \lambda^{-1} W_1 + \lambda^{-2} W_2,
\]

where

\[
W_1 = \begin{pmatrix} 0 & u & 0 \\ -v^* & 0 & -u^* \\ 0 & v & 0 \end{pmatrix}.
\]

Set \( W_0 = \hat{\tau}(L_0)^{-1} L_0 \) with

\[
W_0 = \begin{pmatrix} a & 0 & b \\ 0 & q & 0 \\ c & 0 & d \end{pmatrix}, \quad W_0^{-1} = \begin{pmatrix} \hat{a} & 0 & \hat{b} \\ 0 & q^{-1} & 0 \\ \hat{c} & 0 & \hat{d} \end{pmatrix}, \text{ and } L_0 = \begin{pmatrix} l_1 & 0 & l_2 \\ 0 & l_0 & 0 \\ 0 & 0 & l_4 \end{pmatrix}.
\]

With these notations, we obtain

\[
\hat{\tau}(H)^{-1} H = WW_0 \hat{\tau}(W)^{-1}.
\]
For explicit computations we recall from (3.6) (See also (3.7) of [27]) that for any \( F \in G(2m, \mathbb{C}) \) we have

\[
(4.1) \quad \hat{\tau}(F)^{-1} = \hat{J}_{2m} F^t \hat{J}_{2m},
\]

where

\[
(4.2) \quad \hat{J}_{2m} = S_{2m} J_{2m} = \begin{pmatrix} E_1 & 0 & E_2 \\ 0 & I_{2m-4} & 0 \\ E_3 & 0 & E_4 \end{pmatrix}.
\]

Recall that (3.14)

\[
E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_2 = E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

As a consequence, we derive

\[
W_2 W_0 = H_2, \quad W_1 W_0 + W_2 W_0 (\hat{J}_{2m} \hat{W}_1^t \hat{J}_{2m}) = H_1 + (\hat{J}_{2m} \hat{H}_1^t \hat{J}_{2m}) H_2, \quad \hat{W} = I_{2m} + (\hat{J}_{2m} \hat{H}_1^t \hat{J}_{2m}) H_1 + (\hat{J}_{2m} \hat{H}_2^t \hat{J}_{2m}) H_2.
\]

Here

\[
\hat{W} = W_0 + W_1 W_0 (\hat{J}_{2m} \hat{W}_1^t \hat{J}_{2m}) + W_2 W_0 (\hat{J}_{2m} \hat{W}_2^t \hat{J}_{2m}).
\]

The first equation of (4.3) yields

\[
W_2 = H_2 W_0^{-1} = \begin{pmatrix} g \hat{c} & 0 & g \hat{d} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Next we evaluate the second matrix equation of (4.3). We compute

\[
H_1 + (\hat{J}_{2m} \hat{H}_1^t \hat{J}_{2m}) H_2 = \begin{pmatrix} 0 & f & 0 \\ 0 & 0 & -f^t + \hat{f}^t E_1 g \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\hat{J}_{2m} \hat{W}_1^t \hat{J}_{2m} = \begin{pmatrix} 0 & \hat{u}^t E_1 + \hat{v}^t E_3 & -E_1 \hat{v}^t - E_2 \hat{u}^t \\ \hat{u}^t E_1 + \hat{v}^t E_3 & 0 & \hat{u}^t E_2 + \hat{v}^t E_4 \\ -E_1 \hat{v}^t - E_2 \hat{u}^t & \hat{u}^t E_2 + \hat{v}^t E_4 & 0 \end{pmatrix},
\]

\[
H_2 (\hat{J}_{2m} \hat{W}_1^t \hat{J}_{2m}) = \begin{pmatrix} 0 & -g E_3 \hat{v}^t - g E_4 \hat{u}^t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
W_1 W_0 = \begin{pmatrix} 0 & uq & 0 \\ -v^t a - u^t c & 0 & -v^t b - u^t d \\ 0 & vq & 0 \end{pmatrix}.
\]

As a consequence we read off the equations

\[
vq = 0, -v^t a - u^t c = 0, \quad uq - g E_3 \hat{v}^t - g E_4 \hat{u}^t = f, \quad -v^t b - u^t d = -f^t + \hat{f}^t E_1 g.
\]
Since $q$ is invertible, $v = 0$. Therefore these equations reduce to the following ones:

$$v = 0, \ u^*c = 0, \ uq - gE_4\bar{u}^t = f, \ u^*d = f^* - \bar{f}^tE_1g.$$  

Similarly, for the third equation of (4.3), we first compute matrix expressions for the summands:

$$(\hat{J}\hat{H}^1_1\hat{J})H_1 = \begin{pmatrix} 0 & 0 & E_2\hat{f}^t f^* \\ 0 & \hat{f}^t E_1 f & 0 \\ 0 & 0 & E_4\hat{f}^t f^* \end{pmatrix},$$

$$(\hat{J}\hat{H}^2_2\hat{J})H_2 = \begin{pmatrix} 0 & 0 & E_2\bar{g}^t E_1g \\ 0 & 0 & 0 \\ 0 & E_4\bar{g}^t E_1g \end{pmatrix},$$

$$W_1W_0(\hat{J}\hat{W}^1_1\hat{J}) = \begin{pmatrix} uq\bar{u}^t E_1 & 0 & uq\bar{u}^t E_2 \\ 0 & u^*dE_4\bar{u}^t & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{J}\hat{W}^2_2\hat{J} = \begin{pmatrix} E_1\hat{c}^t\hat{g}^t E_1 + E_2\hat{d}^t\hat{g}^t E_1 & 0 & E_1\hat{c}^t\hat{g}^t E_2 + E_2\hat{d}^t\hat{g}^t E_2 \\ 0 & 0 & 0 \\ E_3\hat{c}^t\hat{g}^t E_1 + E_4\hat{d}^t\hat{g}^t E_1 & 0 & E_3\hat{c}^t\hat{g}^t E_2 + E_4\hat{d}^t\hat{g}^t E_2 \end{pmatrix},$$

$$W_2W_0(\hat{J}\hat{W}^2_2\hat{J}) = \begin{pmatrix} E_3\hat{c}^t\hat{g}^t E_1 + gE_4\hat{d}^t\hat{g}^t E_1 & 0 & gE_3\hat{c}^t\hat{g}^t E_2 + gE_4\hat{d}^t\hat{g}^t E_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Substituting these expressions into the third matrix equation of (4.3), we derive that $c = 0$. Therefore we have

$$\begin{cases} 
    c = 0, \ a + uq\bar{u}^t E_1 + gE_4\hat{d}^t\hat{g}^t E_1 = I_2, \\
    b + uq\bar{u}^t E_2 + gE_4\hat{d}^t\hat{g}^t E_2 = E_2\hat{f}^t f^* + E_2\hat{g}^t E_1g, \\
    q + u^*dE_4\bar{u}^t = I_{2m-4} + \hat{f}^t E_1 f, \ d = I_2 + E_4\hat{f}^t f^* + E_4\hat{g}^t E_1g.
\end{cases}$$

Summing up, we obtain (3.13).

The only thing left to prove Lemma 3.3 is the statement of the form of $L_0$. To derive this, we first consider $W_0$. Recall that

$$W_0 = W^{-1}\hat{r}(H)^{-1}H\hat{r}(W) = \hat{r}(W_0)^{-1} = \hat{J}_{2m}\hat{W}^t_0\hat{J}_{2m},$$

and $W_0 \in G(2m, \mathbb{C})$. Hence, in particular, we also have

$$W^t_0\hat{J}_{2m}W_0 = J_{2m}.$$  

A direct computation using these equations shows

$$\hat{W}_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & \bar{a}_{12} \\ 0 & 0 & a_{33} & \bar{a}_{13} \\ 0 & 0 & 0 & \bar{a}_{11} \end{pmatrix}.$$
with

\[ |a_{11}|^2 = 1, \ a_{22}a_{33} = 1, \ \bar{a}_{13} = -\frac{a_{12}a_{11}}{a_{22}}, \ \bar{a}_{14} = \frac{|a_{12}|^2}{a_{22}}. \]

Set

\[ \tilde{l}_0 = \begin{pmatrix} l_1 & l_3 \\ 0 & l_4 \end{pmatrix} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & l_{14} \\ 0 & l_{22} & 0 & l_{24} \\ 0 & 0 & l_{33} & l_{34} \\ 0 & 0 & 0 & l_{44} \end{pmatrix}, \]

with \( l_{ij} \) satisfying

\[ a_{11} = l_{11}^2, \ a_{12} = l_{11}l_{12} + \bar{l}_{24}l_{22}, \ a_{22} = |l_{22}|^2, \]

and

\[ l_{14} = -\frac{l_{12}l_{13}}{l_{11}}, \ l_{24} = -\frac{l_{13}l_{22}}{l_{11}}, \ l_{33} = \frac{1}{l_{22}}, \ l_{34} = -\frac{l_{12}}{l_{11}l_{22}}, \ l_{44} = \frac{1}{l_{11}}. \]

Moreover, let \( l_0 \) be a solution to

\[ q = \tilde{l}_0 l_0. \]

Applying (4.1), it is a straightforward computation to verify that

\[ \hat{\tau}(L_0)^{-1}L_0 = \tilde{J}_{2m}L_0^t\tilde{J}_{2m}L_0 = W_0. \]

This finishes the proof of Lemma 3.3.

\[ \blacksquare \]

**Remark 4.1.** In above proof the splitting \( q = \tilde{l}_0 l_0 \) implies a strong restriction on \( q \). This condition will always be satisfied near the identity. While in general it may happen that there need some middle term in the splitting \( q = \tilde{l}_0 l_0 \) due to non-globality of the Iwasawa splitting in our case. Actually the situation is much more complicated (for a similar situation, see [2]). Since we have two open Iwasawa cells by Section 6 [14], Theorem 6.7, it may happen that \( q \) starts at \( I \) in the first open Iwasawa cell \( I_1 \) for some \( z_0 \) and move forwards the boundary between the two open Iwasawa cells \( I_1 \) and \( I_2 \). It could touch the boundary and return to \( I_1 \) or it could cross into \( I_2 \). What this means geometrically has been investigated only to a very small extent so far, but would seem to be a highly interesting project. But there are certainly cases where everything works just fine. See the example below.

**Example 4.2.** Let \( 2m = 6 \) and assume

\[ f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}, \ g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}, \]

with \( f_j, g_j, j = 1, \cdots, 4, \) meromorphic functions. It is easy to derive that

\[ g_1 + g_4 = -f_1f_4 - f_2f_3, \ g_2 = -f_1f_2, \ g_3 = -f_3f_4 \]

holds. From the last equation in (3.13a), and from

\[ \tilde{f}^2f^2 = J\tilde{f}fJ = \begin{pmatrix} |f_3|^2 + |f_4|^2 & \tilde{f}_3f_1 + \tilde{f}_4f_2 \\ \tilde{f}_1f_3 + \tilde{f}_2f_4 & |f_1|^2 + |f_2|^2 \end{pmatrix}, \]

holds.
and
\[ \tilde{g}'E_1 \tilde{g} = \begin{pmatrix} \frac{|g_3|^2}{\bar{g}_4 g_3} & \frac{\bar{g}_3 g_4}{|g_4|^2} \\ \bar{g}_4 g_3 & |g_4|^2 \end{pmatrix}, \]
we obtain
\[ d = \begin{pmatrix} (1 + |f_3|^2)(1 + |f_4|^2) & \bar{f}_3 f_1 + \bar{f}_4 f_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_1 & d_2 \\ 0 & 1 \end{pmatrix}. \]
Therefore
\[ a_1 = 1, \ a_5^{-1} = (1 + |f_3|^2)(1 + |f_4|^2), \ a_2 = -\frac{f_1 \bar{f}_3 + f_2 \bar{f}_4 + \bar{g}_3 g_4}{1 + |f_3|^2 + |f_4|^2 + |g_3|^2}. \]
Moreover, we have
\[ d^{-1} = \begin{pmatrix} d_1^{-1} & -d_1^{-1} d_2 \\ 0 & 1 \end{pmatrix}. \]
From (3.13b) we obtain
\[ u^\sharp = \begin{pmatrix} f_4 - \bar{f}_3 g_3 & f_2 - \bar{f}_4 g_3 \\ f_3 - \bar{f}_4 g_3 & f_1 - \bar{f}_3 g_4 \end{pmatrix}, \quad d^{-1} = \begin{pmatrix} u_4 & u_2 \\ u_3 & u_1 \end{pmatrix} \]
with
\[ u_1 = \frac{f_1 - f_2 \bar{f}_3 g_4 - \bar{f}_4 g_4}{1 + |f_3|^2}, \quad u_2 = \frac{f_2 - f_1 \bar{f}_3 g_4 - \bar{f}_3 g_4}{1 + |f_4|^2}, \quad u_3 = \frac{f_3}{1 + |f_3|^2}, \quad u_4 = \frac{f_4}{1 + |f_4|^2}. \]
Moreover, since
\[ \tilde{f}'E_1 f = \begin{pmatrix} \bar{f}_3 f_3 & \bar{f}_3 f_4 \\ \bar{f}_4 f_3 & \bar{f}_4 f_4 \end{pmatrix}, \]
\[ u^\sharp d E_4 \tilde{u}^\sharp = \begin{pmatrix} |u_4|^2 & u_4 \bar{u}_3 d_1 \\ u_3 \bar{u}_4 d_1 & |u_3|^2 d_1 \end{pmatrix} = \begin{pmatrix} \frac{|f_4|^2(1 + |f_3|^2)}{1 + |f_3|^2} & \frac{\bar{f}_3 f_4}{\bar{f}_4 f_3} \\ \frac{|f_3|^2(1 + |f_4|^2)}{1 + |f_4|^2} & \frac{\bar{f}_3 f_4}{\bar{f}_4 f_3} \end{pmatrix}, \]
from (3.13c), we have
\[ q = \begin{pmatrix} 1 + |f_3|^2 & 0 \\ 0 & 1 + |f_3|^2 \end{pmatrix}. \]
Clearly, such expression can be written in the form \( q = \bar{q} q_1 \), and the above computations provides a global solution of equation (3.13). This shows that for the case \( 2m = 6 \) there exists a global Iwasawa splitting for harmonic maps with normalized potential of the form in Theorem 1.4.

4.2. The Maurer-Cartan form of \( \tilde{F} \).

**Proof of Lemma 3.5:**

To take a look at the Maurer-Cartan form of \( \tilde{F} \), we first recall that
\[ H^{-1} H_z = \lambda^{-1} \eta_{-1} = \lambda^{-1} \left( \begin{array}{ccc} 0 & \tilde{f} & 0 \\ 0 & 0 & -\tilde{f}^2 \\ 0 & 0 & 0 \end{array} \right). \]
Since
\[
\hat{F}^{-1}\hat{F}_z = \lambda^{-1}L_0\hat{\tau}(W)^{-1}\eta_{-1}\hat{\tau}(W)L_0^{-1} + L_0\hat{\tau}(W)^{-1}\hat{\tau}(W)zL_0^{-1} + L_0(L_0^{-1})z,
\]
and
\[
\hat{\tau}(W) = I_{2m} + \lambda\hat{\tau}(W_1) + \lambda^2\hat{\tau}(W_2),
\]
we may assume that
\[
\hat{F}^{-1}\hat{F}_zd\zeta = \lambda^{-1}\tilde{a}_{-1} + \tilde{a}_0 + \lambda\tilde{a}_1 + \lambda^2\tilde{a}_2 + \lambda^3\tilde{a}_3 + \lambda^4\tilde{a}_4, \quad \tilde{a}_{-1} = L_0\eta_{-1}L_0^{-1}d\zeta.
\]
On the other hand, the reality condition yields
\[
\hat{\tau}(\hat{F}^{-1}d\hat{F}) = \hat{F}^{-1}d\hat{F},
\]
and
\[
\hat{F}^{-1}\hat{F}_zd\zeta = \lambda^{-1}\tilde{a}_{-1} + \tilde{a}_0 + \lambda\tilde{a}_1
\]
with
\[
\tilde{a}_1 = \hat{\tau}(\tilde{a}_{-1}), \quad \text{and} \quad \tilde{a}_0 = \hat{\tau}(\tilde{a}_0).
\]
Moreover, a straightforward computation yields
\[
\tilde{a}_0' = L_0[\eta_{-1}, \hat{\tau}(W_1)]L_0^{-1} + L_0(L_0^{-1})d\zeta.
\]
Since $L_0$ is an upper triangular block matrix, $\tilde{a}'$ is of the desired form stated in (3.15).

\[\Box\]

**Proof of Lemma 3.6:**

By Lemma 3.5, there exists a frame $\hat{F}$ such that $\tilde{a}'$ is of the form stated in (3.15). By (3.3), (3.5) and (3.7), we obtain
\[
\alpha' = \mathcal{P}^{-1}(\tilde{a}') = \begin{pmatrix} A_1 & \lambda^{-1}B_1 \\ -\lambda^{-1}B_1^tI_{1,3} & A_2 \end{pmatrix} dz,
\]
with
\[
A_1 = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{12} & 0 & a_{13} & a_{14} \\ a_{13} & -a_{13} & 0 & a_{34} \\ a_{14} & -a_{14} & -a_{34} & 0 \end{pmatrix},
\]
and
\[
B_1 = \begin{pmatrix} h_{11} & \hat{h}_{11} & h_{12} & \hat{h}_{12} & \cdots & h_{1,m-2} & \hat{h}_{1,m-2} \\ h_{11} & \hat{h}_{11} & h_{12} & \hat{h}_{12} & \cdots & h_{1,m-2} & \hat{h}_{1,m-2} \\ h_{31} & \hat{h}_{31} & h_{32} & \hat{h}_{32} & \cdots & h_{3,m-2} & \hat{h}_{3,m-2} \\ ih_{31} & \hat{i}h_{31} & ih_{32} & \hat{i}h_{32} & \cdots & ih_{3,m-2} & \hat{i}h_{3,m-2} \end{pmatrix}.
\]
Set
\[
F = \mathcal{P}^{-1}(\hat{F}) = (\phi_1, \phi_2, \phi_3, \phi_4, \psi_1, \ldots, \psi_{2m-4}).
\]
We have now
\[
\begin{cases}
\phi_{1z} = a_{12}\phi_2 + a_{13}\phi_3 + a_{14}\phi_4 + h_{11}\psi_1 + \cdots + \hat{h}_{1,m-2}\psi_{2m-4}, \\
\phi_{2z} = a_{12}\phi_1 - a_{13}\phi_3 - a_{14}\phi_4 - h_{11}\psi_1 - \cdots - \hat{h}_{1,m-2}\psi_{2m-4}.
\end{cases}
\]
This indicates
\[(\phi_1 + \phi_2)_z = a_{12}(\phi_1 + \phi_2).\]
Since \(\phi_1 + \phi_2\) is a non-zero real vector-valued function, let \(\phi_{01}\) be the first coordinate of \(\phi_1 + \phi_2\), it is straightforward to verify that \(\frac{1}{\phi_{01}}(\phi_1 + \phi_2)\) is a non-zero constant lightlike vector. As a consequence, a well-known fact states that if \(\mathcal{F}\) is the conformal Gauss map of a Willmore surface \(y\), \(y\) is Möbius equivalent to some minimal surface in \(\mathbb{R}^{2m-2}\). **[19]**

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