A Computer Library for Ray Tracing in Analytical Media

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Abstract. Ray tracing technique is an important tool not only for forward but also for inverse problems in Geophysics, which most of the seismic processing steps depends on. However, implementing ray tracing codes can be very time consuming. This article presents a computer library to trace rays in 2.5D media composed by stack of layers. The velocity profile inside each layer is such that the eikonal equation can be analytically solved. Therefore, the ray tracing within such profile is made fast and accurately. The great advantage of an analytical ray tracing library is the numerical precision of the quantities computed and the fast execution of the implemented codes. Although ray tracing programs already exist for a long time, for example the seis package by Červený, with a numerical approach to compute the ray. Regardless of the fact that numerical methods can solve more general problems, the analytical ones could be part of a more sophisticated simulation process, where the ray tracing time is completely relevant. We demonstrate the feasibility of our codes using numerical examples.

1. The Computer Library art

We have developed a free C-library called art, standing for analytical ray tracing. The 2.5D synthetic medium for ray tracing considered by the library is presented in Figure 1. The main goal of the library is the determination of a ray within the profile π of Figure 1, starting a source point (usually at the surface, identified by the symbol *) and arriving at a receiver or at an interface (identified by ∇). Since the medium is composed by a stack of layers, where each layer has one of the analytical velocities presented in the previous section, the first step in tracing a ray through a profile, is the computation of incidence points along a sequence of layers. Computationally, the union of incidence points determines completely the raypath, because we know exactly (or analytically) the behavior of the curve between such extreme points. The sequence of layers is given a priori, and is known as the ray code. In a synthetic profile like the described above, the library allows the following two basic operations: one-way ray tracing, given a point source and an initial slowness (inverse of velocity) direction, find the ray through the profile, following a ray code sequence and two-point ray tracing, find the ray through the profile that starts at a given point source and finish at a given receiver, obeying a ray code sequence. Others operations are also allowed in the library, but are nothing more than applications of this two basic procedures.

Figure 1. Seismic exploration through plane π.
As other free libraries, like gsl\textsuperscript{1}, our library provide many ‘ray tracing functions’, so the user can be free to implement his/her own codes; which is basically the idea of art. On the other hand, we also provide programs that already contain some standard routines in seismic ray tracing, like two-point ray tracing, one-way ray tracing, common-shot experiments, among others. These functions are contained within art-fun. Also, the library depends on other free softwares, the libConfuse\textsuperscript{2} and the gengetopt\textsuperscript{3}, as shown in the fluxogram (at the left). The library is in continuous expansion and in a near future we will present an expanded article with the whole library, examples, and more sophisticated applications. This paper has the purpose to present the library, comparing with the well-known software seis.

2. Seismic Ray Tracing

The seismic wave propagating through an elastic medium obeys the elastodynamic wave equation \[ \rho \frac{\partial^2 u}{\partial t^2} = \mathcal{E}(x,u,\nabla \cdot u, \nabla \times u) \], where \( \mathcal{E} \) is an operator over the displacement vector \( u \) and \( \rho \) is the density of the isotropic medium. In acoustic media, it is easy to show that elastic waves are a superposition of two other waves, the compressional scalar wave \( \theta = \nabla \cdot u \) and the shear vectorial wave \( \Omega = \nabla \times u \), also called P and S waves respectively. For a non-acoustic medium, the splitting is no longer possible, although in a high-frequency sense, this is approximately true. Hence, the ray-theory, assume that the zeroth-order approximation \( u(x,t) = U(x)F(t - T(x)) \) is a high-frequency solution of the above PDE, where \( U \) is a complex-valued vectorial amplitude function, \( F \) is a high-frequency signal and \( T \) is the real-valued traveltime function. So, to estimate the zeroth-order solution, we need to compute the traveltime and amplitude functions. After several operations with the ansatz \( u \) and the elastodynamic equation, it can be shown that the traveltime obeys the so-called eikonal equation \( p^T p = \|p\|^2 = \frac{1}{\Omega(x)^T}, \) where \( p = \nabla T \) is the slowness vector and \( v \) is the wave velocity.

Usually, we had to solve this equation for P and S waves, so we denote \( v = \alpha \) for the first case and \( v = \beta \) for the second. Functions \( \alpha, \beta \) are related to the parameters of the medium. For pressure waves the displacement vector is perpendicular to the wavefront and thus, we can write \( U(x) = \alpha(x)A(x)p \). Since shear waves propagate in the parallel wavefront direction, we write \( U(x) = B(x)e_1(x) + C(x)e_2(x) \). In both cases, we need to compute the scalar amplitude functions \( A, B \) and \( C \). In the case of P waves, the function \( A \) obeys the transport equation \( 2p^T \nabla \left( \sqrt{\rho \alpha^2 A} \right) + \sqrt{\rho \alpha^2 A} \nabla^2 T = 0 \). Also, the same transport equation applies for S waves, with \( v = \beta \).

The eikonal equation is associated to a Hamiltonian operator,
which is related to an index \( n \in \mathbb{N} \) - which we describe as the hamiltonian index. After using the method of characteristics, which is fully described in [4] and [3], we arrive at a system of ordinary equations. In our case, these equations are given by

\[
\begin{align*}
\begin{cases}
\frac{d u}{du} x &= v(x)^2 n p, \\
\frac{d u}{du} T &= \frac{1}{n} \nabla \left( \frac{1}{v(x)^n} \right)
\end{cases}
\end{align*}
\]

for \( n \neq 0 \) and \( n = 0 \) respectively, and with \( d_u(\cdot) = (\cdot)/du \). These are the so-called ray equations. The curve \( x: \mathbb{R} \to \mathbb{R}^3 \) has the slowness vector \( p \) as a tangent vector and the integration parameter \( u \) varying over the domain \( u \geq u_0 \). Usually, we will adopt the convention \( u_0 = 0 \). For the case \( n = 0 \), the integration parameter is the traveltime itself.

The kinematic part of a ray is mathematically represented by the characteristic curve \((x, p, T)\) and the raypath is determined by \( x \). Along this ray, the amplitude function \( A \) can be computed by \( A(u) = \sqrt{\frac{\rho(u)v(u)J(u)}{\rho(u)v(u)J(u)}} \), where \( v(u) = v(x(u_0)) \) and \( J \) is the ray Jacobian related to a usual orthonomic ray system parameterization, see [4]. The dynamical ray tracing provides a theory mainly concerned with the computation of \( J \) through a new system of ordinary differential equation that can be decoupled into the so called in-plane and out-of-plane systems.

### 3. Analytical Medium

Since a ray is determined by the solution of (1), either for \( n \neq 0 \) or \( n = 0 \), we are inspired to choose velocity fields that enable analytical solutions. In this sense, we select particular values for the index \( n \) in order to make easier the ray equations. In our case, these values are \( n \in \{-1, 0, 1, 2\} \). In a general sense, we consider that velocities can be written in the form \( \Phi(v(x)) = x^T A x + a^T x + c \) where \( \Phi \) assume the value of one of the functions \( v, v^{-2}, \ln v \), as illustrated by the following Table. By a suitable choice of the parameters \( A, a \) and \( c \) we can always ensure that \( \Phi(v(x)) > 0 \).

To distinguish the velocities, we established the following identification: vass standing for affine square of slowness, vqss for quadratic square of slowness, vcgl for constant gradient of logarithmic velocity, vaff for affine velocity and vconst for constant velocity. Since we are mainly interested in rays that are confined in a vertical plane along the seismic profile, we can assume that parameters in equation \( \Phi(v(x)) \) are two-dimensional, i.e, \( A \in \mathbb{R}^{2 \times 2} \) and \( a \in \mathbb{R}^2 \) where the position vector \( x \) has first component related to distance and second component related to depth. Figure 7 shows a profile with nonconstant velocity. We note that the solution of the kinematic ray tracing system (1) can always be written by \( x(u) = x_0 + d_n(u) \) for the position vector, \( T(u) = T_0 + t_n(u) \) for the traveltime and \( p(u) = p_0 + s_n(u) \) for the slowness vector; where \( \{x_0, T_0, p_0\} \) are initial conditions for the ray. The analytical solution varies according to \( n \) for each velocity model. Functions \( \{t_n, s_n, p_n\} \) can be found explicitly in [1] and [5], also in [3].
4. Numerical Examples

The ray tracing with art is presented in Figures 9.a to 9.e. Each figure has depth (in Km) as the vertical axis and distance (in Km) as the horizontal axis. In figure (a), we present a typical common-shot experiment (CS) with source placed at \(x = 3\) and receivers equally distributed at the surface; in this case, each reflector branch has dipping angle \(\theta = 11, 30^\circ\). The seismogram obtained for this example is presented in Figure 8, using seismic traces as the one depicted in Figure 5. Figure (b) and (c) presents the behaviour of a ray family (including Fermat rays), usually needed for the computation of the Kirchhoff integral [3], the first one being at a synthetic anticlinal reflector and the second at a synclinal reflector; both using affine velocity \((n = -1)\) within the first layer (also, source at \(x = 1\) and receiver at \(x = 3.5\)). Figure (d) presents a ray-diagram for a common-shot experiment at a profile with affine velocity \((n = -1)\) within second, third and fourth layers, as shown in Figure 7. The associated Kirchhoff seismic section to this velocity model is presented in Figure 10. It is shown in Figures (e) and (f) a comparison between the rays obtained at a common-shot example, with package art and seis, respectively.

Figure 8. Seismogram obtained with art, from rays at Figure 9.a.

Figure 9. Ray tracing using art. See text for more details.

Figure 10. Seismic section for rays from Figure 9.d and velocity model from Figure 7, using art.

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