HARBINGERS OF ARTIN’S RECIPROCITY LAW.
I. THE CONTINUING STORY OF AUXILIARY PRIMES

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Abstract. In this article we present the history of auxiliary primes used in proofs of reciprocity laws from the quadratic to Artin’s reciprocity law. We also show that the gap in Legendre’s proof can be closed with a simple application of Gauss’s genus theory.

Introduction

Artin’s reciprocity law, the central result in class field theory, is an isomorphism between some ideal class group and a Galois group, and sends the class of a prime ideal \( p \) to the Frobenius automorphism of \( p \). In this series of articles we would like to discuss results that, in the long run, turned out to be related to certain pieces of the quite involved proof of Artin’s reciprocity law.

In this article we sketch the history of auxiliary primes. These are prime numbers whose existence was needed in various proofs of the quadratic as well as the higher reciprocity laws from Kummer to Artin. Subsequent articles will discuss the irreducibility of the cyclotomic equation, Gauss’s Lemma and the transfer map, and finally Bernstein’s reciprocity law, which is the special case of Artin’s reciprocity law for unramified abelian extensions stated by Bernstein long before Artin.

Auxiliary primes in connection with proofs of the quadratic reciprocity law have a long history. Legendre’s attempt at proving the reciprocity law failed (see [20, Ch. 1]) because he could not guarantee the existence of certain primes. Gauss’s first proof via induction used auxiliary primes whose existence was secured with the help of an ingenious elementary argument. After Kummer had proved the \( p \)-th power reciprocity law in the fields of \( p \)-th roots of unity, where \( p \) is a regular prime, he gave a proof of the quadratic reciprocity law based on analogous principles which also required auxiliary primes; their existence, Kummer claimed, would follow easily from the analytic techniques of Dirichlet.

1. Legendre

Legendre attempted to prove the quadratic reciprocity law with the help of his theorem on the solvability of the equation \( ax^2 + by^2 = cz^2 \), whose proof relied heavily on the technique of reduction provided by Lagrange:

**Theorem 1.1.** Assume that the integers \( a, b, c \) are positive, pairwise coprime, and squarefree. Then the diophantine equation

\[
ax^2 + by^2 + cz^2 = 0
\]

(1.1)

has non-trivial solutions in \( \mathbb{Z} \) if the following three conditions are satisfied: \( bc, ac \) and \( -ab \) are squares modulo \( a, b \) and \( c \), respectively.
Let us now briefly explain Legendre’s idea for proving the quadratic reciprocity law
\[
\left( \frac{b}{a} \right) = (-1)^{\frac{a-1}{2} \cdot \frac{b-1}{2}} \left( \frac{a}{b} \right)
\]
for distinct odd prime numbers \( p \). Like Gauss, Legendre distinguished eight cases; he denoted primes \( \equiv 1 \mod 4 \) by \( a \) and \( A \), and primes \( \equiv 3 \mod 4 \) by \( b \) and \( B \). The first of Legendre’s eight cases then states that

\[
\left( \frac{b}{a} \right) = 1 \quad \implies \quad \left( \frac{a}{b} \right) = 1.
\]

Legendre observes that the equation \( x^2 + ay^2 = bz^2 \) is impossible modulo 4; thus the conditions in Thm. 1.1 cannot all be satisfied. But there are only two conditions, one of which, namely \( \left( \frac{b}{a} \right) = +1 \), holds by assumption. Thus the second condition \( \left( \frac{-a}{b} \right) = +1 \) must fail; but then \( \left( \frac{-a}{b} \right) = -1 \) implies \( \left( \frac{a}{b} \right) = +1 \) as desired.

Next consider the equation \( Ax^2 + ay^2 = bz^2 \); it is easily seen to be impossible modulo 4, hence at least one of the three conditions in Thm. 1.1, \( \left( \frac{-A}{b} \right) = +1 \), \( \left( \frac{A}{b} \right) = +1 \), and \( \left( \frac{a}{b} \right) = +1 \) must fail. Now choose an auxiliary prime \( b \) such that \( \left( \frac{b}{a} \right) = 1 \) and \( \left( \frac{A}{b} \right) = -1 \). By what Legendre has already proved, this implies \( \left( \frac{A}{a} \right) = 1 \) and \( \left( \frac{a}{A} \right) = -1 \). Then the first condition \( \left( \frac{-A}{a} \right) = +1 \) always holds, and the last two conditions are equivalent to \( \left( \frac{A}{a} \right) = +1 \) and \( \left( \frac{a}{A} \right) = -1 \). Since these cannot hold simultaneously, Legendre concludes that

\[
\left( \frac{a}{A} \right) = 1 \quad \implies \quad \left( \frac{A}{a} \right) = 1,
\]

as well as

\[
\left( \frac{A}{a} \right) = -1 \quad \implies \quad \left( \frac{a}{A} \right) = -1.
\]

This result is proved under the assumption that the following lemma holds:

**Lemma 1.2.** Given distinct primes \( a \equiv A \equiv 1 \mod 4 \), there exists a prime \( b \) satisfying the conditions \( b \equiv 3 \mod 4 \), \( \left( \frac{b}{a} \right) = 1 \) and \( \left( \frac{A}{b} \right) = -1 \).

As for the existence of these primes \( b \), Legendre remarks

On peut s’assurer qu’il y en a une infinité; mais voici une démonstration directe qui écarte toute difficulté.

He then gives a second proof of this claim, and this time admits

Dans cette démonstration, nous avons supposé seulement qu’il y
avois un nombre premier \( b \) de la form \( 4n - 1 \), qui pouvait diviser la
formule \( x^2 + Ay^2 \).

In fact, an equation of the form \( x^2 + Ay^2 = abz^2 \) is impossible in nonzero integers since \( x^2 + y^2 \equiv -z^2 \mod 4 \) implies that \( x, y \) and \( z \) must be even. Thus the conditions in Legendre’s Theorem cannot be satisfied. If we choose \( b \) prime such that \( \left( \frac{b}{a} \right) = -1 \), then by what we have already proved we know that \( \left( \frac{b}{A} \right) = -1 \). Since \(-A\) is a square modulo \( a \) and \( b \), the last condition that \( ab \) be a square modulo \( A \) cannot hold; thus \( \left( \frac{A}{a} \right) \left( \frac{a}{A} \right) = -1 \), and this implies the claim \( \left( \frac{A}{a} \right) = +1 \).

For future reference, let us state Legendre’s assumption explicitly as

\[1\]It is possible to make sure that there are infinitely many such primes; but here is a direct proof that avoids any difficulty.

\[2\]In this proof we have only assumed that there be a prime number \( b \) of the form \( 4n - 1 \) which divides the form \( x^2 + Ay^2 \).
Lemma 1.3. For each prime \( p \equiv 1 \mod 4 \) there exists a prime \( q \equiv 3 \mod 4 \) such that \( (p/q) = -1 \).

Finally, for proving that
\[
\left( \frac{b}{B} \right) = 1 \quad \implies \quad \left( \frac{B}{b} \right) = -1,
\]
Legendre assumes the following

Lemma 1.4. For primes \( q \equiv r \equiv 3 \mod 4 \) there exists a prime \( p \equiv 1 \mod 4 \) such that \( (p/q) = (p/r) = -1 \).

Legendre returns to the problem of the existence of his auxiliary primes at the end of his memoir [18]: on p. 552, he writes

"Il seroit peut-être nécessaire de démontrer rigoureusement une chose que nous avons supposée dans plusieurs endroits de cet article, savoir, qu’il y a une infinité de nombres premiers compris dans toute progression arithmétique, dont le premier terme & la raison sont premier entre eux . . ."

He then sketches an approach to this result which he says is too long to be given in detail.

This last remark is problematic in more than one way: Legendre’s Lemma 1.4 certainly follows from the theorem on primes in arithmetic progressions: take a quadratic nonresidues \( m \mod q \) and \( n \mod r \), and use the Chinese Remainder Theorem to find an integer \( c \) with \( c \equiv 3 \mod 4 \), \( c \equiv m \mod q \) and \( c \equiv n \mod r \). Then any prime in the progression \( c + 4qrk \) will satisfy Lemma 1.4.

On the other hand it is not clear at all whether Lemma 1.2 and Lemma 1.3 can be deduced from the theorem on primes in arithmetic progression without assuming the quadratic reciprocity law: this was already observed by Gauss in [11, p. 449].

In the second version of his proof in [19], Legendre managed to do with the existence result in Lemma 1.3 alone.

2. Gauss

Gauss’s first proof of the quadratic reciprocity law used auxiliary primes that were quite similar to those used by Legendre. Below we will first sketch the main idea behind Gauss’s first proof and then use the techniques from his second proof (genus theory of binary quadratic forms) to prove Legendre’s Lemma 1.3.

Gauss’s First Proof. The first proof of the quadratic reciprocity law found by Gauss was based on induction (or descent, as Fermat would have said). Assuming that the reciprocity law holds for small primes, Gauss assumes that \( (\frac{r}{q}) = +1 \), where \( r \) and \( q \equiv 1 \mod 4 \) are prime; then \( x^2 \equiv r \mod q \) shows that there is an equation of the form \( e^2 = r + fq \). Choosing \( e \) and \( f \) carefully and reducing the equation modulo \( p \), Gauss is able to prove that \( (q/r) = +1 \).

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It is perhaps necessary to give a rigorous proof of something that we have assumed in several places of this article, namely, that there is an infinite number of primes contained in every arithmetic progression whose first term and ratio are coprime . . .

In Legendre’s times, this result was credited to Bachet; the name Chinese Remainder Theorem became common only much later.
This procedure fails if \((r/q) = -1\) because in this case we do not have an equation like \(e^2 = r + fq\) to work with. To get around this problem, Gauss used an auxiliary prime: he found an elementary, but highly ingenious proof of the following

**Lemma 2.1.** For every prime \(q \equiv 1\) mod 4 there is a prime \(p < q\) with \((q/p) = -1\).

By what he already proved, Gauss concluded that \((p/q) = -1\) as well, hence \(pq\) is a quadratic residue modulo \(q\): by carefully studying the equation \(e^2 = pr +fq\) and reducing it modulo various primes, Gauss finally proves that \((q/r) = -1\).

The proof of Lemma 2.1 is trivial if one is allowed to use quadratic reciprocity: let \(a < q\) be an odd quadratic nonresidue modulo \(q\); then there must be a prime number \(p \mid a\) with \((p/q) = -1\). Quadratic reciprocity takes care of inverting the symbol. If \(p \equiv 1\) mod 8, then 2 is a quadratic residue modulo \(p\), and proving the existence of \(a\) is easy (in this case, \(a\) actually can always be chosen < \(\sqrt{q}\)). If \(p \equiv 5\) mod 8, then \(a = \frac{q+1}{2}\) will work.

**A Proof of Legendre's Lemma.** Let us now give a proof of Legendre’s Lemma [1,3] based on Gauss’s genus theory. We start by recalling the basic results. The equivalence classes of primitive binary quadratic forms with nonsquare discriminant \(\Delta\) form a group called the class group, which, for fundamental discriminants (these are discriminants of quadratic number fields) is isomorphic to the class group in the strict sense of \(Q(\sqrt{\Delta})\). Every form \(Q = (A, B, C) = Ax^2 + Bxy + Cy^2\) represents integers \(a\) coprime to \(\Delta\), i.e., there exist integers \(x, y\) such that \(Q(x, y) = a\). Fundamental discriminants \(\Delta\) can be written uniquely as a product \(\Delta = \Delta_1 \cdots \Delta_t\) of prime discriminants \(\Delta_j\); the value \(\chi_j(a) = (\Delta_j/a)\) is well defined and does not depend on the choice of \(a\) or on the representative of the equivalence class of \(Q\). This allows us to define \(\chi_j(Q) = \chi_j(a)\), where \(a\) is any integer coprime to \(\Delta\) represented by \(Q\). The forms for which all characters \(\chi_j\) are trivial form a subgroup of the class group called the principal genus, which consists of all square classes. The main result in genus theory is

**Theorem 2.2.** Given any set of integers \(c_1, \ldots, c_t\), there exists a primitive form \(Q\) with discriminant \(\Delta = \Delta_1 \cdots \Delta_t\) such that

\[
\chi_1(Q) = (-1)^{c_1}, \ldots, \chi_t(Q) = (-1)^{c_t}
\]

if and only if \(c_1 + \ldots + c_t\) is even.

Given a prime \(p \equiv 1\) mod 4 we have to find a prime \(q \equiv 3\) mod 4 such that \((p/q) = -1\). The last condition is equivalent to \((-p/q) = +1\).

If \(p \equiv 1\) mod 4, then \(Q = (2, 2, \frac{p+1}{2})\) is primitive and has discriminant \(\Delta = -4p\).

If \(p \equiv 5\) mod 8, then \(\frac{p+1}{2} \equiv 3\) mod 4, and the form \(Q\) represents integers \(\equiv 3\) mod 4 since e.g. \(Q(0,1) = \frac{p+1}{2}\). This implies that \(Q(0,1)\) must have a prime divisor \(q \equiv 3\) mod 4. Now we observe

**Lemma 2.3.** Let \(Q\) be a primitive binary quadratic form with discriminant \(\Delta\). If a prime number \(q \nmid \Delta\) divides \(Q(x, y)\) for some coprime integers \(x, y\), then \((\frac{\Delta}{q}) = +1\).

**Proof.** Assume that \(Q = (A, B, C)\) is a primitive form with discriminant \(\Delta = B^2 - 4AC\); if \(Ax^2 + Bxy + Cy^2 \equiv 0\) mod \(q\), then

\[
4A(Ax^2 + Bxy + Cy^2) \equiv (2Ax + By)^2 - \Delta x^2 \equiv 0\mod q.
\]

If \(q \mid x\), then \(q \mid C\) since gcd\((x, y) = 1\); but then \(\Delta \equiv B^2 - 4AC \equiv B^2 \mod C\). □
If \( p \equiv 1 \mod 8 \), then \( Q \) only represents integers that are even or \( \equiv 1 \mod 4 \). But in this case, the class number of forms with discriminant \( -p \) is divisible by 4, and it is easy to see that \( [Q] = 2[Q_1] \) for some form \( Q_1 \). If \( Q_1 \) represents an integer \( \equiv 3 \mod 4 \), then we are done. If the odd integers represented by \( Q_1 \) are all \( \equiv 1 \mod 4 \), then \( Q_1 \) is in the principal genus:

**Lemma 2.4.** Let \( p \equiv 1 \mod 4 \) be a prime number and let \( Q \) be a primitive binary quadratic form with discriminant \( -4p \) that represents only integers that are even or \( \equiv 1 \mod 4 \). Then \( Q \) is in the principal genus.

**Proof.** Since \( \Delta = -4 \cdot p \) is a product of two prime discriminants, there are only two genus characters, namely \( \chi_4(a) = \left( \frac{-1}{a} \right) \) and \( \chi_p(a) = \left( \frac{a}{p} \right) \). If an integer \( a \) coprime to \( \Delta \) is represented by \( Q \), then \( \chi_4(a) = \chi_p(a) \), and \( Q \) is in the principal genus if and only if \( \chi_1(a) = +1 \).

Thus if the odd integers represented by \( Q \) are all \( \equiv 1 \mod 4 \), the form \( Q \) must lie in the principal genus. \( \square \)

Continuing in this way we see that some form \( Q_j \) with discriminant \( -4p \) must represent an integer \( \equiv 3 \mod 4 \) (unless the 2-class group of primitive forms with discriminant \( -4p \) has an infinitely divisible element – but this would contradict the finiteness of the class number). This integer must have a prime factor \( q \equiv 3 \mod 4 \), and since \( q \) is represented by some form with discriminant \( -4p \), we must have \((-p/q) = +1 \). Since \((-1/q) = -1 \), Lemma 2.3 follows.

The proof of Legendre’s Lemma given above closely follows the way it was discovered. Actually we can give a much shorter proof: let \( Q \) be a primitive form with discriminant \( \Delta = -4p \) and with genus characters \( \chi_1(Q) = \chi_2(Q) = -1 \). Then any odd integer represented by \( Q \) is \( \equiv 3 \mod 4 \), and now our claim follows immediately.

It is clear that this proof can be generalized considerably. I have meanwhile found that Lubelski \([26]\) used similar methods for proving the following result:

**Proposition 2.5.** Every polynomial \( f(x) = ax^2 + bx + c \in \mathbb{Z}[x] \), where \( d = b^2 - 4ac \) is not a negative square, has infinitely many prime divisors \( q \equiv -1 \mod 4 \).

3. **Dirichlet**

Dirichlet, who had studied mathematics in Paris, was not only familiar with the work of Euler and Gauss, but also knew the results due to Lagrange and Legendre\(^5\). In particular he knew Legendre’s conjecture about primes in arithmetic progressions, and also was aware of the fact that the proof Legendre had given was incomplete; in \([6]\), he wrote:

Legendre bases the theorem he wants to prove on the problem of finding the largest number of consecutive terms of an arithmetic progression that are divisible by one of a set of given primes; however, he solves this problem only by induction. If one tries to prove the solution of the maximum problem found by him, whose form is most remarkable because of its simplicity, then one comes across some big difficulties which I did not succeed in surmounting.

\(^5\)Dirichlet also was aware of Lambert’s article on the irrationality of \( \pi \); Lambert’s proof was streamlined by Legendre and later led to dramatic progress under the hands of Hermite. In Germany, Lambert’s work was apparently less appreciated.
Dirichlet therefore looked for different tools and found inspiration in the work of Euler, who had proved the infinitude of primes using the divergence of the zeta function at $s = 1$. Using Euler’s method Dirichlet was able to prove the following

**Theorem 3.1.** Let $a$ and $m$ be coprime natural numbers. Then there exist infinitely many primes $p \equiv a \mod m$.

For proving the result Dirichlet introduced characters $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ and their associated L-series $L(s, \chi) = \sum_{n \geq 1} \chi(n) n^{-s}$. For real $s > 1$, these L-series can be written as an Euler product: $L(s, \chi) = \prod_{p}(1 - \chi(p)p^{-s})^{-1}$ if one carefully defines the values of $\chi(p)$ for primes $p | m$. Theorem 3.1 is a direct consequence of the following result:

**Theorem 3.2.** If $\chi$ is a nontrivial Dirichlet character modulo $m$, then $\lim_{s \to 1} L(s, \chi) \neq 0$.

Dirichlet never worked with the value $L(1, \chi)$; it was Mertens who first proved that $L(s, \chi)$ converges at $s = 1$ for all nontrivial Dirichlet characters $\chi$. Below we will use $L(1, \chi)$ as an abbreviation of $\lim_{s \to 1} L(s, \chi)$.

For prime values $p$, Dirichlet could express $L(1, \chi)$ as a finite sum and directly prove that it does not vanish. For composite values of $m$, on the other hand, he had a rather technical proof that he abandoned in favor of a more conceptual approach. It is rather easy to show that $L(1, \chi) \neq 0$ if $\chi$ assumes nonreal values, so the problem is showing $L(1, \chi) \neq 0$ for all real Dirichlet characters. Dirichlet found that the values $L(1, \chi)$ were connected to arithmetic properties of quadratic number fields. For making the connection, Dirichlet had to show

**Lemma 3.3 (Dirichlet’s Lemma).** If $\chi$ is a primitive quadratic Dirichlet character defined modulo $d$, then $\chi(a) = (d/a)$ for all positive integers $a$ coprime to $D$.

Dirichlet’s Lemma is essentially Euler’s version of quadratic reciprocity: since $\chi$ is defined modulo $d$, Dirichlet’s Lemma implies that $(d/p) = (d/q)$ for positive prime numbers $p \equiv q \mod d$.

We have already remarked that it was Mertens who first succeeded in proving Dirichlet’s Theorem without using Dirichlet’s Lemma; there are reasons to believe that Mertens’ proof is similar to the one that Dirichlet abandoned since it is in fact based on Dirichlet’s results on the asymptotic behaviour of the divisor function.

We have also observed that Dirichlet’s Theorem 3.1 implies Legendre’s Lemma but that it cannot be used for deriving Lemma 1.3. The last lemma would follow from a result that Dirichlet announced but whose proof he only sketched in a very special case:

**Theorem 3.4.** Let $Q = (A, B, C)$ be a primitive quadratic form with discriminant $\Delta = B^2 - 4AC$. Then the set $S_Q$ of primes represented by $Q$ has Dirichlet density

$$\delta(S_Q) = \begin{cases} \frac{1}{h} & \text{if } Q \not\sim (A, -B, C), \\ \frac{1}{2h} & \text{if } Q \sim (A, -B, C), \end{cases}$$

where $h$ is the class number of forms of discriminant $\Delta$.

Dirichlet’s claims were slightly different, since he worked with forms $(A, 2B, C)$ whose middle coefficients are even. If $Q = (1, 0, 1)$, then $h = 1$, hence Thm. 3.4 tells us that primes represented by $Q$ have Dirichlet density $\frac{1}{4}$. 
If $\Delta = -23$, then $h = 3$, and the primes represented by the principal form $(1, 1, 6)$ have density $\frac{1}{6}$, whereas the primes represented by $(2, 1, 3)$ have density $\frac{1}{3}$. The forms $(2, 3, 4)$ and $(2, -1, 3)$ clearly represent the same primes, and primes represented by any form with discriminant $\Delta$ have density $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$: these are the primes $p$ satisfying $(-23/p) = +1$. In general, Dirichlet’s Theorem 3.4 has the following

**Corollary 3.5.** Let $\Delta$ be a quadratic discriminant. Then the prime numbers $p$ with $(\Delta/p) = +1$ and those with $(\Delta/p) = -1$ each have Dirichlet density $\frac{1}{2}$.

This corollary, by the way, can be proved directly without having to assume the quadratic reciprocity law; in fact, Dirichlet’s Lemma has to be invoked only for transforming a quadratic Dirichlet character $\chi$ modulo $\Delta$ into a Kronecker character $(\Delta/\cdot)$, and the character in Cor. 3.5 is already a Kronecker character.

We now apply Cor. 3.5 to forms with discriminant $\Delta = -4p$, where $p \equiv 1 \mod 4$ is a prime number. In this case there are exactly two forms with discriminant $\Delta = -4p$ satisfying $(A, B, C) \sim (A, -B, C)$, namely the principal form $Q_0 = (1, 0, p)$ and the ambiguous form $Q_1 = (2, 2, p+1)$.

If $p \equiv 5 \mod 8$, then the number $h_1 = h/2$ of classes in the principal genus is odd, and the primes represented by forms in the principal genus have density $\frac{1}{2} + \frac{h_1}{2h} = \frac{1}{2}$.

If $p \equiv 1 \mod 8$, then the number $h_1 = h/2$ of classes in the principal genus is even, and the primes represented by forms in the principal genus have density $\frac{1}{2} + \frac{h_1}{2h} = \frac{1}{4}$. In particular, the primes represented by some form not in the principal genus have density $\frac{1}{4}$, and in particular, we find the following strengthening of Legendre’s Lemma 1.3 above: the primes $q$ with $(\frac{-1}{q}) = (\frac{p}{q}) = -1$ have Dirichlet density $\frac{1}{4}$.

4. Kummer and Hilbert

Starting with Kummer, auxiliary primes became an indispensable tool for proving higher reciprocity laws, and variants of the corresponding existence results can be found in the work of Hilbert, Furtwängler and Takagi on class field theory.

Kummer’s Proof of Quadratic Reciprocity. In [17], Kummer gave two new proofs of the $p$-th power reciprocity law for regular primes; in the introduction, he proves the quadratic reciprocity law by similar methods. The first proof is similar to Legendre’s; the main difference is that Kummer replaces Legendre’s equation $ax^2 + by^2 = cz^2$ by the Pell equation. In the course of his proof, Kummer assumes the following existence results:

**Lemma 4.1.** Let $p \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$ be distinct primes numbers. Then there exist primes $p' \equiv 3 \mod 4$ such that $(p'/p) = (p'/q) = -1$.

This clearly follows from Dirichlet’s theorem on primes in arithmetic progression.

**Lemma 4.2.** Let $p \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$ be distinct primes numbers. Then there exist primes $p' \equiv 3 \mod 4$ such that $(p/p') = (q/p') = -1$.

This lemma can be proved in the same way as Legendre’s Lemma 1.3 above by exploiting the genus theory for forms with discriminant $\Delta = -4pq$, and also follows from Dirichlet’s Theorem 3.4.
**Lemma 4.3.** Given primes $q \equiv q' \equiv 1 \mod 4$ be distinct prime numbers. Then there exist primes $p \equiv p' \equiv 3 \mod 4$ such that $(p/q) = -1$, $(p/q') = +1$, $(p'/q) = +1$ and $(p'/q') = -1$.

This is again a consequence of Dirichlet’s theorem on primes in arithmetic progression.

Kummer does not give proofs for these existence results and simply remarks that they can be derived using Dirichlet’s methods. It must be observed, however, that Dirichlet’s proof of the theorem on primes in arithmetic progression used the quadratic reciprocity law for showing that every real primitive Dirichlet character $\chi$ modulo $D$ has the form $\chi(p) = (D/p)$ for some quadratic discriminant $D$; thus things are not as easy as Kummer pretends. Dirichlet actually gave a proof that there exist infinitely many primes in coprime residue classes modulo $p$, where $p$ is a prime number, that did not depend on quadratic reciprocity. With a little effort, this proof can be extended to residue classes modulo $4pq$, which would cover the existence results that Kummer had used.

**Kummer’s Reciprocity Law.** After Gauss and Jacobi had stated and proved the first “higher” reciprocity laws for fourth and third powers, it became clear that even for stating any $p$-th power reciprocity law one had to work in the field of $p$-th roots of unity. The problems coming from nonunique factorization were overcome by Kummer’s invention of ideal numbers. Eisenstein used Kummer’s ideal numbers for proving “Eisenstein’s reciprocity law”, which turned out to be as indispensible for proving higher reciprocity laws as the existence of auxiliary primes; only Artin finally succeeded in eliminating Eisenstein’s reciprocity law by replacing it with the technique of abelian twists used by Chebotarev.

Kummer’s existence result was stated and proved in [16, p. 138]. In its original form, it reads as follows:

**Theorem 4.4.** Let $F(\alpha), F_1(\alpha), F_2(\alpha), \ldots, F_{n-1}(\alpha)$ denote real complex numbers satisfying the condition that the product

$$F(\alpha)^m F_1(\alpha)^{m_1} F_2(\alpha)^{m_2} \cdots F_{n-1}(\alpha)^{m_{n-1}}$$

for integral values of the exponents becomes a $\lambda$-th power if and only if all these exponents are $\equiv 0 \mod \lambda$. Then there exist infinitely many prime numbers $\phi(\alpha)$ with respect to which the indices of the complex numbers $F(\alpha), F_1(\alpha), \ldots, F_{n-1}(\alpha)$ are proportional modulo $\lambda$ to arbitrarily given numbers.

In Kummer’s terminology, $\alpha$ is a primitive $\lambda$-th root of unity. Kummer distinguished between ideal complex numbers (roughly corresponding to Dedekind’s ideals; see [21] for a more precise explanation of ideal numbers) and real complex numbers (“wirkliche komplexe Zahlen”, that is, elements of the ring $\mathbb{Z}[\alpha]$), which he denoted by $F(\alpha)$ even if the numbers were ideal. The index of $F(\alpha)$ with respect to $\phi(\alpha)$ is the integer $c$ for which

$$\left( \frac{F(\alpha)}{\phi(\alpha)} \right) = \alpha^c,$$

where the symbol on the left is the $\lambda$-th power residue symbol in $\mathbb{Z}[\alpha]$ and $c$ is determined modulo $\lambda$.

Translated into modern terms, Kummer’s theorem becomes
Theorem 4.5. Let $\zeta$ be a primitive $p$-th root of unity. If $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}[\zeta]$ are independent modulo $p$-th powers, then for any set of integers $c_1, \ldots, c_r$ there is an integer $m$ not divisible by $p$ and infinitely many prime ideals $\mathfrak{p}$ in $\mathbb{Z}[\zeta]$ such that
\[
\left( \frac{\alpha_1}{\mathfrak{p}} \right)^m = \zeta^{c_1}, \ldots, \left( \frac{\alpha_r}{\mathfrak{p}} \right)^m = \zeta^{c_r}.
\]

The main idea behind Kummer’s result is applying Dirichlet’s method to the $L$-series $L(s, \chi)$, where the $\chi$ are $p$-th power residue characters. The nonvanishing of $L(1, \chi)$ comes from the analytic class number formula (which essentially shows that $\lim_{s \to 1} (s-1) \zeta_K(s)$ is finite and nonzero) and the factorization $\zeta_K(s) = \prod_j L(s, \chi_j)$ of “Dedekind’s” zeta function as a product of $L$-series.

Hilbert. Hilbert proved the nonvanishing of the $L$-series needed for proving Theorem 4.5 in [14, Hs. 27] and Kummer’s theorem in [14, Satz 152] of his Zahlbericht. Later, Hilbert worked out the theory of quadratic extensions of number fields with odd class number by closely following Kummer’s work on Kummer extensions of the fields of $p$-th roots of unity for regular primes $p$. In particular, Hilbert needed the following existence theorem, which can be found in [15, Satz 18]:

Theorem 4.6. Let $K$ be a number field, and assume that $\alpha_1, \ldots, \alpha_s \in K^\times$ are independent modulo squares. Then for any choice of $c_1, \ldots, c_s \in \{\pm 1\}$, there exist infinitely many prime ideals $\mathfrak{p}$ in $K$ with
\[
\left( \frac{\alpha_1}{\mathfrak{p}} \right) = c_1, \ldots, \left( \frac{\alpha_s}{\mathfrak{p}} \right) = c_s.
\]

The nonvanishing of the corresponding $L$-series is an immediate consequence of Dedekind’s class number formula, according to which $\lim_{s \to 1} \zeta_K(s)$ is finite and nonzero. The special case in which $K$ is a multiquadratic extension of $\mathbb{Q}$ is also the result that Kummer alluded to when he stated that the existence of his auxiliary primes can be verified using Dirichlet’s methods. One should, however, bear in mind that Kummer did not have the theory of ideals (or ideal numbers) in quadratic number fields at his disposal; this was worked out later by Dedekind.

In the special case $K = \mathbb{Q}$, $\alpha_1 = -1$ and $\alpha_2 = p$, Thm. 4.6 is exactly Legendre’s Lemma [13].

Similar results were used by Furtwängler and Takagi in their proofs of the main results of class field theory.

5. **Kronecker, Frobenius, Chebotarev**

The simplest case of Kummer’s existence theorem claims that in Kummer extensions $L/K$ of $K = \mathbb{Q}(\zeta_p)$, there are infinitely many prime ideals that split completely and infinitely many prime ideals that remain inert. A more general result (although initially restricted to the case where the base field is $\mathbb{Q}$) was stated by Kronecker:

**Theorem 5.1.** Let $L$ be a number field. Then there exist infinitely many prime ideals of inertia degree 1 in $L$. More exactly, the Dirichlet density of such primes is $1/(N: \mathbb{Q})$, where $N$ is the normal closure of $L/\mathbb{Q}$.

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6In [10, Satz 17], Furtwängler sketches the proof, in most other places he simply refers to Hilbert’s Bericht.

7See in particular [28, § 3].
Applied to quadratic extensions $L = \mathbb{Q}(\sqrt{m})$, this implies the existence of infinitely many primes $p$ with $(\frac{m}{p}) = +1$, and in fact half the primes have this property; consequently, there also must be infinitely many primes with $(\frac{m}{p}) = -1$.

Similarly, applied to $L = \mathbb{Q}(\zeta_m)$, Kronecker’s density theorem predicts the existence of infinitely many primes $p \equiv 1 \mod m$. Both of these special cases can actually be proved by elementary means.

Kronecker’s density theorem was generalized immediately by Frobenius [9]. The following special case of Frobenius’ result can be stated in the same language as Kronecker’s:

**Theorem 5.2** (Frobenius Density Theorem for Abelian Extensions). Let $L/K$ be an abelian extension, and $F$ an intermediate field such that $L/F$ is cyclic. Then the set $S_F$ of prime ideals with decomposition field $F$ has Dirichlet density $\delta(S_F) = \frac{\phi(L:F)}{(L:K)}$.

The main idea behind this special case is again Dedekind’s density theorem. Applied to the biquadratic number field $L = \mathbb{Q}(\sqrt{-1}, \sqrt{p})$ and the subextension $F = \mathbb{Q}(\sqrt{-p})$, Thm. [5.2] guarantees the existence of infinitely many primes $q$ with $(\frac{q}{p}) = +1$ and $(\frac{-1}{q}) = (\frac{p}{q}) = -1$. In particular, it implies Legendre’s Lemma [1.3].

For generalizing his result to general normal extensions, Frobenius had to introduce the notion of a division: the division $\text{Div}(\phi)$ of an element $\phi \in G$ is the set of all $\sigma \in G$ with the property that $\sigma = \tau^{-1}\phi^k\tau$ for some $\tau \in G$ and an exponent $k$ coprime to the order of $\sigma$. Modulo some group theoretical preliminaries, the following result is then rather easily proved:

**Theorem 5.3** (Frobenius Density Theorem). Let $L/K$ be a normal extension, and let $D$ be a division in $G = \text{Gal}(L/K)$. Let $S$ denote the set of unramified prime ideals $p$ in $K$ with the property that the prime ideals $\mathfrak{p}$ above $p$ in $L$ satisfy $[L/K]_{\mathfrak{p}} \in D$. Then $S$ has Dirichlet density $\delta(S) = \frac{\#D}{\#G}$.

The Frobenius density theorem is not the best we can hope for: in fact it does not even contain Dirichlet’s theorem on primes in arithmetic progression. Since the Frobenius automorphism of a prime ideal $p$ is determined up to conjugacy, the best possible density result would predict the infinitude of prime ideals whose Frobenius automorphism lies in some conjugacy class (in fact, divisions are unions of conjugacy classes).

Frobenius’ density theorem with divisions replaced by conjugacy classes contains Dirichlet’s Theorem [3.1] as a special case and was stated as a conjecture by Frobenius at the end of his article [9]. For stating it, let $[\sigma]$ denote the conjugacy class of $\sigma \in G$:

**Theorem 5.4** (Chebotarev’s Density Theorem). Let $K/\mathbb{Q}$ be a normal extension, and fix a $\sigma \in G = \text{Gal}(K/\mathbb{Q})$. Let $S$ denote the set of unramified primes $p$ with the property that the prime ideals $\mathfrak{p}$ above $p$ in $K$ satisfy $[K/Q]_{\mathfrak{p}} \in [\sigma]$. Then $S$ has Dirichlet density $\delta(S) = \frac{\#[\sigma]}{\#G}$.
The problem that Frobenius was unable to solve was that of resolving the divisions into conjugacy classes. This was accomplished by Chebotarev [3] using the technique of abelian twists. Already Hilbert, in his proof of the theorem of Kronecker and Weber, had introduced this technique: given a cyclic extension \( K/\mathbb{Q} \), there is a cyclotomic extension \( L/\mathbb{Q} \) such that the compositum \( KL \) contains a cyclic extension \( M/\mathbb{Q} \) with a Galois group isomorphic to \( K/\mathbb{Q} \) but with less ramification.

**Chebotarev and Artin.** Artin, who had conjectured his reciprocity law in 1923, had faced a problem similar to that of Frobenius. When he saw Chebotarev’s article [3], Artin immediately suspected that this paper would hold the key for a proof of his reciprocity law, and it did. In order to convince the reader of the deep connection between the proofs, let us compare the proofs of Chebotarev’s density theorem as given by Ribenboim [27, § 25.3] with the exposition of Artin’s reciprocity law as given by Childress [5]:

| Step | Chebotarev | Artin |
|------|------------|-------|
| 1.   | The result holds if \( K = \mathbb{Q} \) and \( L/\mathbb{Q} \) is a cyclotomic extension. This case follows from Dirichlet’s Theorem 3.1 (Chebotarev) and the irreducibility of the cyclotomic equation (Dedekind; see the subsequent article [22]). | [27, p. 554] | [5, p. 112] |
| 2.   | The result holds for general base fields \( K \) if \( L = K(\zeta) \) is a cyclotomic extension. | [27, p. 554] | [5, p. 113] |
| 3.   | The result holds for general base fields \( K \) if \( L \subseteq K(\zeta) \) is a subextension of a cyclotomic extension. | [27, p. 558] | [5, Ex. 5.6] |
| 4.   | The result holds for arbitrary cyclic extensions \( L/K \). This step uses the technique of abelian twisting. | [27, p. 558] | [5, p. 114 ff] |
| 5.   | The result holds for general normal (Chebotarev) resp. abelian (Artin) extensions. | [27, p. 561] | [5, p. 114] |

**Table 1.** Chebotarev’s Density Theorem and Artin’s Reciprocity Law

For proving step 4, Artin needed, in addition to Chebotarev’s ideas, auxiliary primes. These will be discussed in the next two sections.

In [1], where Artin conjectured his reciprocity law, he was able to prove it for cyclic extensions of prime degree using the known reciprocity laws due to Kummer, Furtwängler and Takagi. Even this proof already has a structure similar to the one above:

1. The Artin map sends the principal class (and only this class) to the trivial automorphism.
2. If the reciprocity law holds for \( K/k \), then it holds for every subextension \( F/k \).
3. If the reciprocity law holds for \( K_1/k \) and \( K_2/k \), then it holds for the compositum \( K_1K_2/k \).
4. The reciprocity law holds for cyclotomic extensions \( K = k(\zeta) \).
5. If \( k \) contains the \( \ell^n \)-th roots of unity, then the reciprocity law holds for cyclic extensions \( K/k \) of degree \( \ell^n \). Artin derives this result from a Takagi’s
general reciprocity law, which was known to hold only for extensions of prime degree \( \ell \), i.e., for \( n = 1 \).

(6) Let \( K/k \) be a cyclic extension of prime power degree \( \ell^n \), \( \zeta \) a primitive \( \ell^n \)-th root of unity, and let \( k' = k(\zeta) \) and \( K' = K(\zeta) \). If the reciprocity law holds for \( K'/k' \), then it also holds for \( K/k \).

6. Artin’s Reciprocity Law

The existence of the auxiliary primes necessary for Step 4 of Artin’s proof is secured by the following lemma (Hilfssatz 1 in [2]):

**Lemma 6.1.** Let \( f \) be a positive integer, and let \( p_1 \) and \( p_2 \) be primes. Then there exist infinitely many primes \( q \) with the following property: the group \( (\mathbb{Z}/q\mathbb{Z})^\times \) has a subgroup \( H \) such that \( p_1 H = p_2 H \), and the coset \( p_1 H \) has order divisible by \( f \).

Artin actually proved something stronger, namely that, in many cases, the subgroup \( H \) can be taken to be the group of \( f \)-th power residues modulo \( q \). Below we only state this stronger result for \( f = 2 \) (I have modified the proof slightly in order to allow for the possibility \( p_2 = -1 \)); observe that if \( H \) is the group of squares modulo \( q \), then \( pH \) has order 2 if and only if \( (p/q) = -1 \):

**Lemma 6.2.** Let \( p_1 \) be a positive odd prime, and \( p_2 \neq p_1 \) a prime or \( p_2 = -1 \). Then there exist infinitely many primes \( q \) with \( (p_1/q) = (p_2/q) = -1 \).

**Proof.** Consider the quadratic extension \( F = \mathbb{Q}(\sqrt{p_1 p_2}) \), and let \( K = F(\sqrt{p_1}) \). The Takagi group \( T_{K/F} \) has index 2 in the group \( D \) of ideals coprime to the conductor of \( K/F \), which divides \( 2p_1p_2 \). The coset \( D/T_{K/F} \) different from \( T_{K/F} \) contains infinitely many prime ideals of degree 1; let \( q \) be such a prime ideal coprime to \( 2p_1p_2 \), and let \( q \) be its norm.

Since \( q \) splits in \( F/\mathbb{Q} \), we must have \( (p_1/q) = (p_2/q) \). Since the prime ideals in \( F \) above \( q \) remain inert in \( K/F \), we must have \( (p_1/q) = -1 \). \( \square \)

Observe that Legendre’s Lemma [1,3] is exactly the case \( p_1 \equiv 1 \mod 4 \) and \( p_2 = -1 \) of this special case of Artin’s Lemma. Perhaps the fact that an incarnation of Legendre’s Lemma comes up as a special case in the proof of Artin’s reciprocity law shows that Legendre’s work is much more than a failed attempt of proving the quadratic reciprocity law.

7. Arithmetization

When Hasse [13] later provided a proof of Artin’s reciprocity law via the theory of algebras (see [24, 8]), he also had to prove the existence of certain auxiliary primes; his set of conditions is slightly different from Artin’s:

**Lemma 7.1.** If \( p_1, \ldots, p_r \) are distinct prime numbers and \( k_1, \ldots, k_r \) given natural numbers, then there is a modulus \( m \) and a subgroup \( U \) of the group \( R = (\mathbb{Z}/m\mathbb{Z})^\times \) such that

(1) \( R/U \) is cyclic;

(2) the order of residue class \( p_j + m\mathbb{Z} \) in \( R/U \) is a multiple of \( k_j \) for all \( 1 \leq j \leq r \);

(3) the coset \( -1 + m\mathbb{Z} \) has order 2 in \( R/U \).
Hasse remarked that the Frobenius density theorem guarantees the existence of such a modulus $m$, and that it even can be chosen to be prime. He also mentioned that he expected that this result can be proved with elementary means by allowing $m$ to be composite, and that Artin meanwhile found a proof of his reciprocity law by using the special case $r = 1$ of Hasse’s Lemma

There is a certain analogy with the following classical result in algebraic number theory: Dirichlet’s analytic methods and class field theory show that in a number field $K$, each ideal class in $\text{Cl}(K)$ contains a prime ideal with degree 1. Kummer, on the other hand, observed that there is an algebraic proof of the slightly weaker fact that each ideal class contains an ideal whose prime ideal factors all have degree 1 (strictly speaking, the result in this generality is due to Hilbert [14 Satz 89]; Kummer only had the general theory of ideal numbers in cyclotomic fields). For a similar approach to special cases of the Chebotarev density theorem see Lenstra & Stevenhagen [27].

Chevalley [4] succeeded in giving a non-analytic proof of the main theorems of class field theory; in particular, he proved the special case $r = 1$ of Hasse’s Lemma 7.1 which Hasse presented (with a simplified proof) in his lectures [12 Satz 139] on class field theory:

**Theorem 7.2.** Let $a > 1$, $k$ and $n$ be given integers. Then there exists a modulus $m$ coprime to $k$ for which $R = (\mathbb{Z}/m\mathbb{Z})^\times$ contains a subgroup $U$ with the following properties:

1. $R/U$ is cyclic;
2. the order of the coset $aU$ is divisible by $n$.

If $a$ is an odd prime, and if we demand that $m$ be prime and that $U$ is the subgroup of squares modulo $m$, then the case $n = 2$ of this theorem predicts the existence of a prime $m$ such that the coset $a + m\mathbb{Z}$ has order 2 modulo squares, i.e., that $(\frac{a}{m}) = -1$. Showing the existence of primes in certain classes seems to require analytic techniques in most cases; Chevalley’s success in giving an arithmetic proof of class field theory is due in part to Hasse’s insight that Artin’s proof can be modified in such a way that one can do with nonprime moduli $m$. An elementary proof of Lemma 7.1 for general $r$ was given by van der Waerden [29].

As we have seen, an important step in the arithmetization of class field theory was the realization that the role of auxiliary primes could be played by composite numbers with suitable properties, whose existence could be proved without Dirichlet’s analytic techniques. This brings up the question whether Legendre’s proof of the quadratic reciprocity law or Gauss’s first proof can be modified in a similar way.

**References**

[1] E. Artin, Über eine neue Art von L-Reihen, Abh. Math. Sem. Hamburg (1923), 89–108; Coll. Papers, 105–124
[2] E. Artin, Beweis des allgemeinen Reziprozitätsgesetzes, Abh. Math. Sem. Hamburg 5 (1927), 353–363; Coll. Papers, 131–141
[3] N. Chebotarev, Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören, Math. Ann. 95 (1926), 191–228
[4] C. Chevalley, Sur la théorie du corps de classes dans le corps finis at les corps locaux, J. Fac. Sci. Univ. Tokyo 2 (1933), 365–476
[5] N. Childress, Class Field Theory, Springer-Verlag 2009
[6] P.G.L. Dirichlet, *Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Faktor sind, unendlich viele Primzahlen enthält*, Abh. Preuss. Akad. Wiss. (1837), 45–81; Werke I (1889), 313–342

[7] P.G.L. Dirichlet, *Über eine Eigenschaft der quadratischen Formen*, Ber. Königl. Preuss. Akad. Wiss. (1840), 49–52; Werke II, 499–502

[8] G. Frei, P. Roquette, *Emil Artin und Helmut Hasse. Die Korrespondenz 1923–1934*, Universitätsverlag Göttingen 2008

[9] G. Frobenius, *Über Beziehungen zwischen den Primidealen eines algebraischen Zahlkörpers und den Substitutionen seiner Gruppe*, Ber. Berl. Akad. Wiss. 1896

[10] Ph. Furtwängler, *Über die Reziprozitätsgesetze zwischen den Potenzresten in algebraischen Zahlkörpern, wenn l eine ungerade Primzahl bedeutet*, Math. Ann. 58 (1904), 1–50

[11] C.-F. Gauss, *Disquisitiones Arithmeticae*, 1801; German Transl. *Arithmetische Untersuchungen*, (H. Maser. ed.), Berlin 1889

[12] H. Hasse, *Klassenkörpertheorie*, Marburg Lectures 1933; Physica Verlag 1967

[13] H. Hasse, *Die Struktur der R. Brauerschen Algebrenklassengruppe*, Math. Ann. 107 (1933), 731–760

[14] D. Hilbert, *Die Theorie der algebraischen Zahlen* (Zahlbericht), Jahresber. DMV 4 (1897), 175–546; French transl. Toulouse Ann. (3) 1 (1905), 257–328; Engl. transl. Springer-Verlag 1998; Romanian transl. Bukarest 1998

[15] D. Hilbert, *Über die Theorie des relativquadratischen Zahlkörpers*, Math. Ann. 51 (1899), 1–127; Gesammelte Werke I, 370–482

[16] E.E. Kummer, *Über die allgemeinen Reziprozitätsgesetze der Potenzreste*, Berliner Akad. Ber. (1858), 158–171; Coll. Papers I, 673–687

[17] E.E. Kummer, *Zwei neue Beweise der allgemeinen Reziprozitätsgesetze unter den Resten und Nichtresten der Potenzen, deren Grad eine Primzahl ist*, Berliner Akad. Abh. 1861; J. Reine Angew. Math. 100 (1887), 10–50; Coll. Papers I, 842–882

[18] A.M. Legendre, *Recherches d’analyse indéterminée*, Hist. de l’ac. Royale des sciences 1785

[19] A.-M. Legendre, *Essai sur la théorie des nombres*, Deprat, Paris (1798)

[20] F. Lemmermeyer, *Reciprocity Laws. From Euler to Eisenstein*, Springer-Verlag 2000

[21] F. Lemmermeyer, *Jacobi and Kummer’s ideal numbers*, Abh. Math. Sem. Hamburg 79 (2009), 165–187

[22] F. Lemmermeyer, *Harbingers of Artin’s Reciprocity Law, II. The Irreducibility of Cyclotomic Equation*, in preparation

[23] F. Lemmermeyer, *The Quadratic Reciprocity Law*, with an appendix by O. Baumgart, to appear

[24] F. Lemmermeyer, P. Roquette (eds.), *Correspondence Helmut Hasse - Emmy Noether*, Gött. Univ.-Verlag 2006

[25] H.W. Lenstra, P. Stevenhagen, *Primes of degree one and algebraic cases of Cebotarev’s theorem*, L’Ens. Math. 37 (1991), 17–30

[26] S. Lubelski, *Zur Reduzibilität von Polynomen in der Kongruenztheorie*, Acta Arith. 1 (1936), 169–183; see also Prace mat.-fiz. 43, 207–221

[27] P. Ribenboim, *Classical Theory of Algebraic Numbers*, Springer-Verlag 2001

[28] T. Takagi, *Über eine Theorie des relativ Abel’schen Zahlkörpers*, J. Coll. Sci. Univ. Tokyo 41 (1920), 1–133

[29] B. van der Waerden, *Elementarer Beweis eines zahlentheoretischen Existenztheorems*, J. Reine Angew. Math. 171 (1934), 1–3

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