TORSION PAIRS IN A TRIANGULATED CATEGORY
GENERATED BY A SPHERICAL OBJECT

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Abstract. We extend Ng’s characterisation of torsion pairs in the 2-Calabi-Yau
triangulated category generated by a 2-spherical object to the characterisation of tor-
sion pairs in the $w$-Calabi-Yau triangulated category, $T_w$, generated by a $w$-spherical
object for any $w \in \mathbb{Z}$. Inspired by the combinatorics of $T_w$, we also characterise the
torsion pairs in a certain $w$-Calabi-Yau orbit category of the bounded derived cate-
gory of the path algebra of Dynkin type $A$.

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Introduction

Calabi-Yau (CY) triangulated categories are triangulated categories that satisfy an
important duality. They are becoming increasingly important throughout mathemat-
ics and physics, for example as 3-CY categories arising from Calabi-Yau threefolds
in algebraic geometry and string theory, to 3-CY categories arising in representation
theory coming from quivers with potential. Of particular importance in representation
theory are (2-)cluster categories, which provide categorifications of important aspects
of the theory of cluster algebras. There are higher analogues, so-called $w$-cluster cate-
gories for $w \geq 2$, which are $w$-CY. These give rise to an important family of categories
of positive CY dimension which satisfy many interesting and important homological
and combinatorial properties.

Let $k$ be an algebraically closed field, $w \in \mathbb{Z}$, and $T_w$ a $k$-linear triangulated category
that is idempotent complete and generated by a $w$-spherical object. These categories
provide a family of categories which are $w$-CY and whose structure is sufficiently simple
to allow concrete computation. As such, they provide a nice ‘toolbox’ of examples
with which to explore the properties of CY triangulated categories, as witnessed by
the intense recent interest in these categories; see [12, 14, 19, 25, 27]. Indeed, for
$w \geq 2$, $T_w$ occurs naturally as a $w$-cluster category of type $A_\infty$.

Owing to their importance and ubiquity, much work has been carried out on under-
standing triangulated categories of positive CY-dimension. However, very little
work has been carried out in understanding the properties of triangulated categories
of negative CY-dimension, although there is the beginning of a theory emerging in
[9, 11, 12, 23]. In [23], it was shown that for \( w \geq 1 \), the category \( \mathcal{T}_w \) has one family
of bounded t-structures and no bounded co-t-structures, whilst for \( w \leq 0 \) the
opposite is true. This shows that there are important homological differences between
triangulated categories of positive and negative CY dimension.

Both t-structures and co-t-structures are examples of torsion pairs in triangulated
categories [24]. Torsion pairs have long been studied in representation theory in the
context of tilting theory to provide important structural information about module
categories of finite-dimensional algebras and a means of comparing different cate-
gories. In the context of cluster-tilting theory, they can be seen as a generalisation of
cluster-tilting objects; furthermore, they admit a mutation theory [33]. Thus charac-
terising and understanding torsion pairs is central to understanding the structure of
triangulated categories.

In the context of cluster theory, geometric/combinatorial models are a useful tool,
first arising in [8, 30]. Combinatorial models for the \( \mathcal{T}_w \) were obtained by Holm and
Jørgensen in [18, 19] and employed by Ng in [28] to characterise torsion pairs in \( \mathcal{T}_2 \).
Building on Ng’s ideas, characterisations of torsion pairs have since been given in
various settings, see [3, 20, 21, 22], and used for detailed studies of their mutation
theories [15, 33].

The combinatorial models involve setting up a correspondence between indecompos-
able objects of the category and certain ‘admissible’ arcs or diagonals of some geometric
object. For \( \mathcal{T}_w \) with \( w \neq 1 \), the combinatorial model consists of \((w - 1)\)-admissible’
arcs of the \( \infty \)-gon; see Section 4 for precise details. It is well-known consequence of
the 2-Calabi-Yau property of cluster categories that the crossing of arcs corresponds
to the existence of a non-trivial extension between the corresponding indecomposable
objects. Given two crossing arcs, the admissible Ptolemy arcs are defined to be the
admissible arcs connecting the endpoints of the two crossing arcs.

We extend Ng’s characterisation of torsion pairs for \( \mathcal{T}_2 \) to the entire family:

**Theorem A.** Let \( X \) be a full additive subcategory of \( \mathcal{T}_w \) for \( w \neq 1 \) and \( X \) be the
corresponding set of arcs in the appropriate combinatorial model of \( \mathcal{T}_w \). Then \((X, X^\perp)\)
is a torsion pair in \( \mathcal{T}_w \) if and only if

1. for \( w \geq 2 \), any so-called ‘right fountain’ in \( X \) is a so-called ‘left fountain’ and
   \( X \) is closed under taking admissible Ptolemy arcs.
2. for \( w \leq 0 \), any left fountain in \( X \) is a right fountain and \( X \) is closed under
taking admissible Ptolemy arcs and ‘modified Ptolemy’ arcs.

See Sections 5 and 6 for precise statements. The statement for \( w \geq 2 \) is somewhat
expected, although it is not a completely straightforward generalisation of Ng’s charac-
terisation for \( w = 2 \) in [28] because crossings of arcs instead correspond to the existence
of some higher extension instead of simply extensions, which requires a substantially
different approach from [28]. The case \( w = 1 \) is degenerate and does not admit such
a combinatorial model; it is treated in the short Section 7. However, surprisingly, for
\( w \leq 0 \) there is a pleasant combinatorial description using this combinatorial model.

As observed in [12], it turns out that the combinatorial model for \( \mathcal{T}_w \) when \( w < 0 \)
duces a combinatorial model on another important \( w \)-CY category, namely the
following orbit category: \( \mathcal{C}_w(Q) := \mathcal{D}^b(kQ)/\Sigma^{1-w}\tau \) for \( w \leq -1 \). When \( w = -1 \) and
\( Q = A_n \), the maximal rigid objects of this category are classified in [11] using a different
combinatorial model. With the combinatorial model of [11], the characterisation of
torsion pairs in \( C_{-1}(A_n) \) proved intractible. However, with the induced combinatorial model, the characterisation is tractible and gives us our second main result.

**Theorem B.** Let \( X \) be a full additive subcategory of \( C_w(A_n) \) for \( w \leq -1 \) and \( X \) be the corresponding set of arcs in the combinatorial model for \( C_w(A_n) \). Then \((X, X^\perp)\) is a torsion pair in \( C_w(A_n) \) if and only if \( X \) is closed under taking admissible Ptolemy arcs and ‘modified’ Ptolemy arcs.

It is our viewpoint that for \( w \leq -1 \) the categories \( C_w(Q) \) are naturally \( w \)-CY, i.e. natural examples of triangulated categories having negative CY dimension. However, even in the case \( w = -1 \) there is some debate on the CY dimension of these categories. For example, in [13], Dugas takes the CY dimension to be defined as the least positive integer \( d \) such that \( \Sigma^d \) is (isomorphic to) the Serre functor. According to this definition, the CY dimension of \( C_{-1}(A_n) \) is \( 2n - 1 \); [13, Theorem 6.1]. Note, however, that in \( C_{-1}(A_n) \), the inverse suspension \( \Sigma^{-1} \) is also isomorphic to the Serre functor of \( C_{-1}(A_n) \).

In contrast, by [23, Proposition 2.8], \( T_w \) is unambiguously \( w \)-CY. It was argued in [12] that for \( C_w(A_n) \), with \( w \leq -1 \), the ‘correct’ CY dimension should be \( w \), owing to similarities in the combinatorics of so-called \( w \)-Hom-configurations in the categories \( C_w(A_n) \) and \( T_w \). We believe the similarities in the combinatorics of torsion pairs in Theorems A and B provide further support for this viewpoint. Moreover, we believe that this means triangulated categories of negative CY dimension are more widespread than previously believed, and warrant further, systematic, study. This article should be considered as a step in this direction.

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1. Torsion pairs, extension closure and functorial finiteness

Let \( T \) be a triangulated category. Throughout this paper all subcategories will be considered to be full and additive.

A *torsion pair* in \( T \) consists of a pair of full subcategories \((X, Y)\), which are closed under direct summands, and satisfy \( \text{Hom}_T(X, Y) = 0 \) and \( X \ast Y = T \), where \( X \ast Y := \{ t \in T \mid \exists x \to t \to y \to \Sigma x \text{ with } x \in X \text{ and } y \in Y \} \). The torsion pair is called a *\( t \)-structure* when \( \Sigma X \subseteq X (\Leftrightarrow \Sigma^{-1} Y \subseteq Y) \); see [1]. It is called a *\( co-t \)-structure* (or *weight structure*) when \( \Sigma^{-1} X \subseteq X (\Leftrightarrow \Sigma Y \subseteq Y) \); see [5, 29]. If \( T \) is Krull-Schmidt, a torsion pair \((X, Y)\) is called *split* if for any \( t \in \text{ind}(T) \) we have either \( t \in X \) or \( t \in Y \).

A subcategory \( X \) of \( T \) is *closed under extensions* or *extension-closed* if given any distinguished triangle \( x' \to x \to x'' \to \Sigma x' \) in \( T \) with \( x', x'' \in X \) then \( x \in X \). The object \( x \) will be called the *middle term* of the extension. We denote by \( \langle X \rangle \) the smallest extension-closed subcategory of \( T \) containing \( X \).

Let \( C \) be any category and \( A \) be a subcategory. A morphism \( f : a \to c \) is called a *right \( A \)-approximation* of \( c \) if the induced map \( \text{Hom}_C(a', f) : \text{Hom}_C(a', a) \to \text{Hom}_C(a', c) \) is surjective. In the case that any object of \( C \) admits a right \( A \)-approximation we say that \( A \) is a *contravariantly finite subcategory* of \( C \). There are dual notions of *left \( A \)-approximation* and *covariantly finite*. If \( A \) is both contra- and covariantly finite,
A is called functorially finite. Right (resp. left) \( A \)-approximations are often called \( A \)-precovers (resp. \( A \)-preenvelopes).

These concepts are linked by the following proposition.

**Proposition 1.1** ([24, Proposition 2.3]). Let \( T \) be a triangulated category. The following conditions are equivalent:

1. \((X, Y)\) is a torsion pair;
2. \( X \) is an extension-closed contravariantly finite subcategory of \( T \) and \( Y = X^\perp \);  
3. \( Y \) is an extension-closed covariantly finite subcategory of \( T \) and \( X = {^\perp}Y \).

A triangulated category \( T \) is called Krull-Schmidt if every object \( t \) admits a direct sum decomposition \( t = t_1 \oplus \cdots \oplus t_n \) into indecomposables, which is unique up to reordering and isomorphism. In a Krull-Schmidt triangulated category, we shall denote the collection of (isomorphism classes of) indecomposable objects by \( \text{ind}(T) \).

2. **Triangulated categories generated by \( w \)-spherical objects**

Let \( k \) be an algebraically closed field, \( w \in \mathbb{Z} \) and \( T_w \) be a \( k \)-linear algebraic triangulated category that is idempotent complete and generated by a \( w \)-spherical object. By [27, Theorem 2.1], \( T_w \) is unique up to triangle equivalence. We recall only the definitions of spherical object and generator; the others are explained succinctly in [23, Section 1.1] and are not explicitly required in this article.

The notion of a \( w \)-spherical object is originally due to Seidel and Thomas [31]. An object \( s \) in a \( k \)-linear triangulated category \( T \) is called \( w \)-spherical if

(S1) it is a \( w \)-spherelike object [17], i.e.

\[
\text{Hom}_T(s, \Sigma^i s) = \begin{cases} 
k & \text{if } i = 0, w; 
0 & \text{otherwise}
\end{cases}
\]

(S2) if it is a \( w \)-Calabi-Yau object (\( w \)-CY, for short), i.e. there is a functorial isomorphism \( \text{Hom}_T(s, t) \cong D \text{Hom}_T(t, \Sigma^w s) \), where \( D(-) := \text{Hom}_k(-, k) \) is the usual vector space duality.

An object \( s \) generates \( T \) if \( \text{thick}_T(s) = T \), i.e. the smallest triangulated subcategory containing \( s \) which is also closed under direct summands is \( T \).

The categories \( T_w \) satisfy many nice properties:

- \( T_w \) is \( \text{Hom} \)-finite and Krull-Schmidt;
- \( T_w \) has a Serre functor \( \mathcal{S} \), i.e. a functor \( \mathcal{S}: T_w \to T_w \) satisfying a functorial isomorphism \( \text{Hom}_{T_w}(x, y) \cong D \text{Hom}_{T_w}(y, \mathcal{S}x) \) for all \( x, y \in T_w \). Moreover, \( \mathcal{S} = \Sigma \tau \), where \( \tau \) is the Auslander–Reiten translate in \( T_w \).
- \( T_w \) is \( w \)-CY, i.e. \( \mathcal{S} \simeq \Sigma^w \) and all objects \( s, t \in T_w \) satisfy (S2).

In general \( w \) may not be the unique integer such that \( \mathcal{S} \simeq \Sigma^w \). However, in the case of the categories \( T_w \), \( w \) is the unique integer such that \( \mathcal{S} \simeq \Sigma^w \).

2.1. **The AR quiver of \( T_w \).** For background on Auslander–Reiten (AR) theory we direct the reader to [1], [2] and, in the triangulated setting [16].

The structure of the AR quiver of \( T_w \) was described in [23] by using a model of \( T_w \) as a thick subcategory of the derived category of a certain differential graded algebra. The indecomposable objects of \( T_w \) and the form of the AR quiver of \( T_w \) was determined for \( w \geq 2 \) in [23, Theorem 8.13] and for general \( w \) in [14, Section 3.3]. We summarise this below using the notation from [23].

**Proposition 2.1.** The indecomposable objects of \( T_w \) are precisely the (co)suspensions of a family of objects \( X_r \) for \( r \geq 0 \). If \( w \neq 1 \) then the AR quiver of \( T_w \) consists of
Hom-hammocks in $T_w$ for $w \neq 1$. To describe the Hom-hammocks in $T_w$ conveniently, we need to introduce some notation regarding rays and corays, which is borrowed from \cite{6}.

Consider the object $\Sigma^i X_j$ and make the following definitions:

$$ \text{ray}_+(\Sigma^i X_j) := \{ \Sigma^{i-n} X_{j+n} \mid n \geq 0 \}, \quad \text{coray}_+(\Sigma^i X_j) := \{ \Sigma^i X_k \mid k \leq j \}; $$

$$ \text{ray}_-(\Sigma^i X_j) := \{ \Sigma^{i+n} X_{j-n} \mid 0 \leq n \leq j \}, \quad \text{coray}_-(\Sigma^i X_j) := \{ \Sigma^i X_k \mid k \geq j \}. $$

For an object $x \in \text{ind}(T_w)$ define $L(x) \in \text{ray}_-(x)$ to be the unique object lying on the mouth of the component. Analogously, define $R(x) \in \text{coray}_+(x)$ to be the unique object lying on the mouth. Thus, if $x$ itself lies on the mouth, then $x = L(x) = R(x)$.

Given two indecomposable objects $a, b \in \text{ind}(T_w)$ that lie on the same ray or coray in the AR quiver of $T_w$, then the finite set consisting of these two objects and all indecomposables lying between them on the (co)ray is denoted by $\overline{\mathbf{ab}}$. In an abuse of notation, we identify $\text{ray}_+(a) \cap \text{coray}_-(b)$ with its indecomposable additive generator.

Following the usage prevalent in algebraic geometry, for objects $a, b \in T_w$, we set $\text{hom}_{T_w}(a, b) := \dim_k \text{Hom}_{T_w}(a, b)$. For $a \in \text{ind}(T_w)$, define the forward Hom-hammock and the backward Hom-hammock of $a$ as, respectively,

$$ H^+(a) := \text{ray}_+(aR(a)) \text{ and } H^-(a) := \text{coray}_-(L(a)a). $$

**Proposition 2.2 (\cite{23} Propositions 3.2 and 3.3).** Let $a, b \in \text{ind}(T_w)$ for $w \neq 1$. Then

(i) If $w \neq 0$, then

$$ \text{hom}_{T_w}(a, b) = \begin{cases} 1 & \text{if } b \in H^+(a) \cup H^-(Sa), \\ 0 & \text{otherwise}. \end{cases} $$

(ii) If $w = 0$, then

$$ \text{hom}_{T_w}(a, b) = \begin{cases} 1 & \text{if } b \in H^+(a) \cup H^-(Sa) \setminus \{a\}, \\ 2 & \text{if } b = a, \\ 0 & \text{otherwise}. \end{cases} $$
2.3. **Factorisation properties.** Later, it will be important to know how morphisms between indecomposable objects of $T_w$ factor. This is dealt with in the following proposition, which generalises the statements of [19 Propositions 2.1 and 2.2].

**Proposition 2.3.** Suppose $a, b$ and $c$ are in $\text{ind}(T_w)$ for $w \in \mathbb{Z} \setminus \{0, 1\}$. We have:

(i) Suppose that $c \in H^+(a) \cap H^+(b)$ and $b \in H^+(a)$. If $g: a \to b$ is a nonzero morphism, then each morphism $f: a \to c$ factors through $g$ as $a \twoheadrightarrow b \rightarrowtail c$. Similarly for any nonzero morphism $h: b \to c$.

(ii) Suppose that $a \in H^+(S^{-1}b) \cap H^+(S^{-1}c)$ and $c \in H^+(b)$. If $g: b \to c$ is a nonzero morphism, then each morphism $f: a \to c$ factors through $g$ as $a \twoheadrightarrow b \rightarrowtail c$.

**Proof.** For $w \geq 2$ these statements are [19 Propositions 2.1 and 2.2] and their duals. The proofs in [19] uses only one-dimensionality of the Hom-spaces and Serre duality. Thus, the same arguments apply to $T_w$ when $w \leq -1$. □

Note that the obvious dual statements to (i) and (ii) above also hold.

When $w = 0$, each $a \in \text{ind}(T_0)$ has a two-dimensional endomorphism space and we tweak the result for this case.

**Proposition 2.4.** Suppose $a, b$ and $c$ are indecomposable objects in $T_0$. We have:

(i) If $a, b$ and $c$ are pairwise non-isomorphic, then the statements in Proposition 2.3 hold without modification.

(ii) Suppose $a \neq b$ and $b \in H^+(a)$. Then for any non-isomorphism $f \in \text{Hom}_{T_0}(a, a)$ there exist nonzero maps $g: a \to b$, $h: b \to a$ and $\lambda \in k$ such that $f = hg + \lambda \text{id}_a$.

**Proof.** For statement (i) use one-dimensionality of the Hom-spaces and [19] as above.

For $b \in H^+(a)$ there are nonzero morphisms $g: a \to b$ and $h: b \to a$, which are unique up to scalars by Proposition 2.2. Moreover, $g = g_1 \cdots g_n$ is a composition of irreducible maps, by the $w = 0$ analogue of [19 Proposition 2.1], which works again by one-dimensionality of the Hom-spaces.

First, we claim that the composition $hg$ is nonzero. It is sufficient to show that the induced map $\text{Hom}_{T_0}(g, a): \text{Hom}_{T_0}(b, a) \to \text{Hom}_{T_0}(a, a)$ is injective. By Serre duality and $k$-linearity of $T_0$ this is equivalent to $\text{Hom}_{T_0}(S^{-1}a, g): \text{Hom}_{T_0}(S^{-1}a, a) \to \text{Hom}_{T_0}(S^{-1}a, b)$ being surjective. Noting again that $S^{-1}a \simeq a$, so that the vector space on the right hand side is one-dimensional, and the image of $\text{id}_a$ is $g$, gives surjectivity.

We now show that $hg$ is not an isomorphism. Suppose that $hg$ is an isomorphism with inverse $h'$. Writing $g' = h'hg_1 \cdots g_{n-1}$ gives $\text{id}_a = g'g_n$, which says that $g_n$ is a split monomorphism, contradicting the irreducibility of $g_n$. Thus $hg$ is a non-isomorphism. The fact that $\text{hom}_{T_0}(a, a) = 2$ then gives the claim. □

**Remark 2.5.** Let $a, b \in \text{ind}(T_0)$ and $b \in H^+(a)$. Let $g \in \text{Hom}_{T_0}(a, b)$ and $h \in \text{Hom}_{T_0}(b, a)$ be nonzero maps. Then $\{\text{id}_a, hg\}$ forms a basis of $\text{Hom}_{T_0}(a, a)$.

3. **Extensions with indecomposable outer terms in $T_w$ for $w \neq 1$**

In this section, we describe how to compute the middle terms of extensions in $T_w$ for which the outer terms are indecomposable.

3.1. **A necessary condition.** In this subsection we assume only that $T$ is a Krull-Schmidt triangulated category. Given a triangle $a \to e \to b \to \Sigma a$ we give necessary conditions that the object $e$ must satisfy with respect to $a$ and $b$. The material is well-known to experts, but we give brief proofs for the convenience of the reader.

In a distinguished triangle $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{} \Sigma x$, the object $z$ is called the cone of $f$ and written $\text{cone}(f)$, and the object $x$ is called the cocone of $g$ and written $\text{cocone}(g)$.
Lemma 3.1. Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and $f: a \to e_1$ and $g: e_2 \to b$ be morphisms in $\mathcal{T}$.

(i) Consider the map $[f_0]: a \to e_1 \oplus e_2$. We have $\text{cone}([f_0]) = \text{cone}(f) \oplus e_2$.

(ii) Consider the map $[g_0]: e_1 \oplus e_2 \to b$. We have $\text{cocone}([g_0]) = \text{cocone}(g) \oplus e_2$.

Proof. Apply the octahedral axiom to the triangle $a \xrightarrow{f} e_1 \xrightarrow{\text{cone}(f)} \Sigma a$ and the split triangle $e_1 \xrightarrow{0} e_1 \oplus e_2 \xrightarrow{e_2} 0 \xrightarrow{\Sigma e_1}$; see Figure 2. The conclusion is now read off from the third column. Similarly for the statement regarding cocones. □

An analogue of the following lemma is contained in the proof of [7, Proposition 8.3].

Lemma 3.2. Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and suppose that

$$
\begin{array}{c}
\vdots \\
[x] \\
\oplus \\
\vdots \\
[f_1] \\
\vdots \\
[y] \\
\oplus \\
\vdots \\
[g_1 \cdots g_n] \\
y \to \Sigma x \\
\end{array}
$$

is a non-split distinguished triangle for some $x, y \in \text{ind}(\mathcal{T})$. Then:

1. The $f_i$ and $g_i$ are each nonzero.

2. If additionally, $\text{hom}(x, y) \leq 1$ for each $x, y \in \text{ind}(\mathcal{T})$ and $\text{Ext}^1(x, x) = 0$ for all $x \in \text{ind}(\mathcal{T})$, then $\text{Ext}^1(e_i, x) = 0$ and $\text{Ext}^1(y, e_i) = 0$.

Proof. The statement about non-vanishing homomorphisms follows immediately from Lemma 3.1. To get the statements about extensions, apply the functors $\text{Hom}(-, x)$ and $\text{Hom}(y, -)$ to the distinguished triangle and use one-dimensionality of the Hom-spaces and the vanishing self-extension property. □

3.2. Ext-hammocks. As with Hom-spaces, we write $\text{ext}^1_{T_w}(b, a) := \dim_k \text{Ext}^1_{T_w}(b, a)$ for $a, b \in \text{ind}(T_w)$. The Ext-hammocks for $a$ can be obtained by combining Proposition 2.2 with Serre duality. The forward and backward Ext-hammocks of $a$ are, respectively,

$$
\begin{align*}
E^+(a) &:= H^+(\tau^{-1}a) \\
E^-(a) &:= H^-(\Sigma a).
\end{align*}
$$

Proposition 3.3. Suppose that $w \notin \mathbb{Z} \setminus \{1\}$ and $a, b \in \text{ind}(T_w)$. Then

(i) If $w \neq 0$, then

$$
\text{ext}^1_{T_w}(b, a) = \begin{cases} 
1 & \text{if } b \in E^+(a) \cup E^-(a), \\
0 & \text{otherwise}.
\end{cases}
$$
(ii) If \( w = 0 \), then

\[
\text{ext}^1_{T_w}(b, a) = \begin{cases} 
1 & \text{if } b \in E^+(a) \cup E^-(a), \\
2 & \text{if } b = \Sigma a, \\
0 & \text{otherwise.}
\end{cases}
\]

Consider the object \( X_r \in \text{ind}(T_w) \) for \( r \geq 0 \). The Ext-hammocks of \( X_r \) are given by

\[
E^+(X_r) = \bigcup_{i=0}^{r} \text{ray}_+(\Sigma^{-d}X_{r-i}) \quad \text{and} \quad E^-(X_r) = \bigcup_{i=0}^{r} \text{coray}_-(\Sigma^{i+1}X_{r-i}).
\]

These are indicated graphically in Figure 3.

3.3. Cohomology of the middle terms. In this section we compute the cohomology of the middle terms of extensions in \( T_w \) for \( w \notin \{0, 1\} \) whose outer terms are indecomposable. Since the action of \( \Sigma \) and \( \tau \) is transitive on the AR quiver of \( T_w \), without loss of generality we may restrict our attention to the objects \( X_r \) for \( r \geq 0 \).

We first deal with the Ext-hammock \( E^+(X_r) \). Note that the non-trivial extensions occurring in this Ext-hammock have the form

\[
(1) \quad X_r \rightarrow E \rightarrow \Sigma^{-sd}X_{r+s-i} \xrightarrow{f} \Sigma X_r \quad \text{for } s \geq 1, \text{ and } 1 \leq i \leq r + 1.
\]

Lemma 3.4. Consider a triangle of the form (1) above. Then:

\[
H^n(E) = \begin{cases} 
k & \text{for } n = d, 2d, \ldots, sd, \\
k^2 & \text{for } n = 0, -d, \ldots, -(r-i)d, \\
k & \text{for } n = -(r-i+1)d, -(r-i+2)d, \ldots, -rd, \\
0 & \text{otherwise,}
\end{cases}
\]

where when \( i = r + 1 \), we take the second condition to be empty.

Proof. Suppose \(|d| > 1\). Applying the functor \( H^n(\cdot) \) to the distinguished triangle (1) and using the fact that, by the proofs of [23, Propositions 3.2–3.4],

\[
H^n(X_i) = \begin{cases} 
k & \text{if } n = 0, -d, -2d, \ldots, -td, \\
0 & \text{otherwise.}
\end{cases}
\]

to read off the cohomology from the following (short) exact sequences when \( 0 \leq i \leq r \):

\[
H^{(s-j)d}(E) \xrightarrow{\sim} H^{-jd}(X_{r+s-i}) \quad \text{for } 0 \leq j < s;
\]

\[
(2) \quad H^{-jd}(X_r) \xrightarrow{\sim} H^{-jd}(E) \xrightarrow{H^{-(j+s)d}} H^{-jd}(X_{r+s-i}) \quad \text{for } 0 \leq j \leq r - i;
\]

\[
H^{-jd}(X_r) \xrightarrow{\sim} H^{-jd}(E) \quad \text{for } r - i + 1 \leq j \leq r.
\]

When \( i = r + 1 \), the sequence (2) degenerates into the isomorphism on the third line.

Now suppose \( d = 1 \). Then the short exact sequences (2) above are connected into the long exact sequence

\[
H^{-(r-i)}(X_r) \xrightarrow{\sim} H^{-(r-i)}(E) \xrightarrow{H^{-(r+s-i)}} H^{-(r-i)}(X_{r+s-i}) \xrightarrow{H^{-(r-i)}(f)} H^{-(r-i-1)}(X_r) \rightarrow \cdots
\]

\[
\cdots \xrightarrow{H^{-(s-i)}(f)} H^0(X_r) \xrightarrow{H^0(E)} H^0(X_{r+s-i}).
\]

For \( i = r \) and \( i = r + 1 \) there is nothing to prove: in the first case, there is already only one short exact sequence, and in the second, (2) degenerates into an isomorphism. Thus, we need only consider the cases \( 1 \leq i < r \). There are nonzero maps \( g : \Sigma^{-s}X_{r+s-i} \rightarrow \Sigma^{r+2-i}X_{i-1} \) and \( h : \Sigma^{r+2-i}X_{i-1} \rightarrow \Sigma X_r \) by Proposition 2.2. Thus
the map $f$ in triangle (1) factors as $f = hg$ by Proposition 2.3. It follows that $H(f) = H(h)H(g)$. Now for $0 \leq j \leq r - i$ we have

$$H^{-j}(\Sigma^{r+2-i}X_{i-1}) = H^{r+2-i-j}(X_{i-1}) = 0$$

because $r + 2 - i - j > 0$ and $H^n(X_i) = 0$ for any $n > 0$ and $t \geq 0$ in the case $d > 0$. Thus it follows that the connecting maps $H^{-j}(f)$ for $0 \leq j \leq r - i$ in the long exact sequence above are zero, which thus decomposes back into the short exact sequences (2) allowing us to again read off the cohomology of $E$.

We now deal with the Ext-hammock $E^-(X_r)$. Note that the non-trivial extensions occurring in this Ext-hammock have the form

(3) \quad $X_r \to E \to \Sigma^{1+id}X_{r+s-i} \xrightarrow{f} \Sigma X_r$ for $s \geq 0$ and $0 \leq i \leq r$.

Lemma 3.5. Consider a triangle of the form (3) above. Then:

$$H^n(E) = \begin{cases} k & \text{for } n = 0, d, \ldots, (i - 1)d, \\ k & \text{for } n = (r + i)d - 1, (r + i + 1)d - 1, \ldots, (r + s + i)d - 1, \\ 0 & \text{otherwise}, \end{cases}$$

where when $i = 0$ we assume the first condition to be empty.

Proof. Apply the functor $H^n(-)$ to the triangle (3) and note that the long exact cohomology sequence decomposes into exact sequences

$$H^{-jd-1}(E) \to H^{-jd}(E) \xrightarrow{(i-j)d} H^{-jd}(X_r) \to H^{-jd}(E).$$

Since $H^{-jd-1}(E) = 0$ for $0 \leq j < i$ and $i + r \leq j \leq i + r + s$, we have $H^{-jd}(X_r) \simeq H^{-jd}(E)$ for $0 \leq j < i$ and $H^{-jd-1}(E) \simeq H^{-jd}(E)$ for $i + r < j \leq i + r + s$.

The map $f: \Sigma^{1+id}X_{r+s-i} \to \Sigma X_r$, factors as $\Sigma^{1+id}X_{r+s-i} \xrightarrow{g} \Sigma X_r$ by Proposition 2.3. The map $g$ is induced from an inclusion map of the underlying DG modules; see [23, Section 2] for precise details. As such $H^{-jd-1}(g): H^{-jd}(X_{r+i}) \to H^{-jd}(X_r)$ is nonzero and thus an isomorphism (by one-dimensionality) for $i \leq j \leq i + r$, and zero otherwise. Similarly the induced map $H^{-jd}(h): H^{-jd}(X_{r+i}) \to H^{-jd}(X_{r+i})$ is an isomorphism for $i \leq j \leq i + r$, and zero otherwise. Since $H(f) = H(g)H(h)$, it follows that $H^{-jd-1}(f)$ is an isomorphism for $i \leq j \leq i + r$. Now one can read off the cohomology of $E$ from the sequences above.

Lemma 3.6. In the case $w = 0$, the statements of Lemmas 3.4 and 3.5 also hold, with the modification that there are, up to equivalence, two extensions,

$$X_r \to E \to \Sigma X_r \to \Sigma X_r,$$

one whose middle term has cohomology as in Lemma 3.4, and one whose middle term has trivial cohomology.

Proof. In the case that $w = 0$, $d = -1$ and so the AR quiver of $T_0$ consists of only one $\mathbb{Z}A_{\infty}$ component. However, the extensions with indecomposable outer terms are formulated exactly as in Lemmas 3.4 and 3.5. The only difference occurs because the Ext-hammocks $E^+(X_r)$ and $E^-(X_r)$ have non-trivial intersection $E^+(X_r) \cap E^-(X_r) = \{ \Sigma X_r \}$. The two-dimensional Ext-space, $\text{Ext}^1(\Sigma X_r, X_r) = \text{Hom}(\Sigma X_r, \Sigma X_r)$ (see Proposition 2.2), has a basis $(1d, f)$ where $f$ can be chosen to be a non-isomorphism factoring through any indecomposable object in $H^+(\Sigma X_r)$; see Proposition 2.4 and Remark 2.5. The corresponding extensions are:

$$X_r \to E \to \Sigma X_r \xrightarrow{f} \Sigma X_r \quad \text{and} \quad X_r \to 0 \to \Sigma X_r \xrightarrow{id} \Sigma X_r.$$
Figure 3. Top: Middle terms of extensions whose outer terms lie in the same component. Bottom: Middle terms of extensions whose outer terms lie in different components. Shaded regions are the Ext-hammocks $E^+(X_r)$ and $E^-(X_r)$. The black square denotes $e_1$, the black diamond $e_2$ and the triangle $\tau^{-1}X_r$.

The first triangle is (equivalent to) the AR triangle, thus its cohomology is known. However, one can also argue exactly as in the case $d = 1$ in the proof of Lemma 3.4.

It is clear that the middle term of the second triangle has trivial cohomology. □

3.4. Graphical calculus. The main technical result of this section is the following computation of the middle terms of extensions whose outer terms are indecomposable. It is analogous to the graphical calculus in [7, Corollary 8.5]. The strategy of our proof is inspired by [25, Section 8].

Theorem 3.7. Let $a, b \in \text{ind}(T_w)$ for $w \neq 0, 1$. Suppose $\text{Ext}^1_0(b, a) \neq 0$. Let $a \to e \to b \to \Sigma a$ be the unique non-split extension of $b$ by $a$. Then $e = e_1 \oplus e_2$, with $e_i$ either indecomposable or zero for $i = 1, 2$, which can be computed as follows:

(i) If $b \in E^+(a)$ then $e_1 = \text{ray}_+(a) \cap \text{coray}_-(b)$ and $e_2 = \text{coray}_+(a) \cap \text{ray}_-(b)$.
(ii) If $b \in E^-(a)$ then $e_1 = \text{coray}_-(L(Sa)) \cap \text{ray}_-(b)$ and $e_2 = \text{coray}_+(a) \cap \text{ray}_+(R(S^{-1}b))$.

If any of the intersections in parts (i) and (ii) are empty, then we interpret the corresponding object as being the zero object. See Figure 3 for an illustration.

Proof. Without loss of generality, we may assume that $a = X_r$ for some $r \geq 0$. Firstly consider triangle (1):

$$X_r \to E \to \Sigma^{-sd}X_{r+s-i} \to \Sigma X_r$$

Suppose $E = \bigoplus_{i=1}^n E_i^{m(i)}$ with $E_i \in \text{ind}(T_w)$ and $m(i) \geq 0$. Note that when $w \neq 0, 1$, Lemma 3.2 and the one-dimensionality of the Hom spaces mean that the $m(i) \leq 1$. Moreover, the only indecomposable objects satisfying the necessary conditions of Lemma 3.2 are \{$X_{r-i}, \Sigma^{-sd}X_{r+s}$\}. Call these objects candidates. Note that when $i = r + 1$, the candidate is simply $\Sigma^{-sd}X_{r+s}$. We now use Lemma 3.4 to identify
that $E$ one summand of $E = E_1 \oplus E_2$ with $E_1$ indecomposable.

Suppose that $d < -1$. In this case the cohomology of the indecomposable objects $X_t$ is concentrated in non-negative degrees. By Lemma 3.4, the lowest degree in which $E$ has non-trivial cohomology is $sd$. Thus, $\Sigma^{-sd}X_{r+s}$ must be a direct summand of $E$. Set $E_1 = \Sigma^{-sd}X_{r+s}$, and note that $E_1$ has one-dimensional cohomology in degrees $sd, (s-1)d, \ldots, -rd$. This leaves $E_2$ with one-dimensional cohomology in degrees $0, -d, \ldots, -(r-i)d$. The only candidate object with cohomology in these degrees is $X_{r-i}$, giving the unique non-split triangle as

$$X_r \to \Sigma^{-sd}X_{r+s} \oplus X_{r-i} \to \Sigma^{-sd}X_{r+s-i} \to \Sigma X_r.$$ 

Inspecting the AR quiver gives:

$$\text{ray}_+(X_r) \cap \text{coray}_-(\Sigma^{-sd}X_{r+s-i}) = \{\Sigma^{-sd}X_{r+s}\},$$
$$\text{coray}_+(X_r) \cap \text{ray}_-(\Sigma^{-sd}X_{r+s-i}) = \{X_{r-i}\}.$$

The case $d > 0$ is analogous, taking into account that the indecomposables $X_t$ now have non-trivial cohomology only in non-positive degrees, giving statement (i). An analogous argument applied to the triangle \((3)\) using Lemma 3.5 gives (ii). \hfill \Box

**Proposition 3.8.** If $w = 0$ and $a, b \in \text{ind}(T_0)$ with $b \neq \Sigma a$, then the statement of Theorem 3.7 also holds. If $b = \Sigma a$, then there are two non-split triangles

$$a \to e_1 \oplus e_2 \to \Sigma a \xrightarrow{f} \Sigma a \text{ and } a \to 0 \to \Sigma a \xrightarrow{id} \Sigma a,$$

where $f$ is a non-isomorphism. The first is computed as in Theorem 3.7(i), the second corresponds to Theorem 3.7(ii).

**Proof.** Here we have $d = -1$ and we cannot apply Lemma 3.2(ii) because each indecomposable has two-dimensional endomorphism spaces and self-extensions. However, we can apply Lemma 3.2(i) and brute force using Lemmas 3.4 and 3.5.

Consider the triangle \((1)\) corresponding to the Ext-hammock $E^+(X_r)$. Again write $E = E_1 \oplus E_2$ with $E_1$ indecomposable. First observe that Lemma 3.4 implies that one summand of $E$ is $\Sigma^t X_t$ for some $t \geq 0$. Lemma 3.2(i) and Proposition 2.2 mean that $\Sigma^t X_{r+s}$ is the only possibility, which we take to be $E_1$. This means that $E_2$ has cohomology in degrees $0, 1, \ldots, r-i$. This means that there is precisely one summand of $E_2$ that consists of an unsuspended $X_t$ for some $0 \leq t \leq r-i$. Note that if $t > r-i$ then $X_t$ has cohomology in too many degrees. Inspecting the AR quiver and using Proposition 2.2 again, we see that if $t < r-i$, there is no map $X_t \to \Sigma^{-s}X_{r+s-i}$, and thus by Lemma 3.2(i) such $X_t$ cannot be summands of $E_2$. This leaves only $X_{r-i}$ itself, giving $E = \Sigma^{-s}X_{r+s} \oplus X_{r-i}$ again, as claimed.

The argument for the triangle \((3)\) is analogous, however, one must deal with the case $X_r \to E \to \Sigma X_r \to \Sigma X_r$ separately. As remarked in Lemma 3.6, the two triangles are the standard triangle $X_r \to 0 \to \Sigma X_r \xrightarrow{id} \Sigma X_r$ and the AR triangle. The AR triangle puts us in case (i) of the theorem, and the second triangle puts us in case (ii) of the theorem with empty intersections, and therefore zero middle term. \hfill \Box

## 4. THE COMBINATORIAL MODEL AND CONTRAVARIANT-FINITENESS

Here we recall the combinatorial model for $T_w$ from [19] in the case $w \geq 2$ and its natural extension to $w \leq 0$ in terms of ‘arcs/diagonals of the $\infty$-gon’. Namely, we regard each pair of integers $(t, u)$ as an arc connecting the integers $t$ and $u$.

For $w \in \mathbb{Z} \setminus \{1\}$ set $d = w - 1$. A pair of integers $(t, u)$ is called a $d$-admissible arc if (i) for $w \geq 2$, one has $u - t \geq w$ and $u - t \equiv 1 \mod d$;
The length functor in this model are given by

described.

Below, we state a characterisation of contravariantly finite subcategories of $\mathbb{T}_w$ when $w \neq 1$. We note that the action of the suspension, AR translate and Serre functor in this model are given by

$$\Sigma(t, u) = (t - 1, u - 1) \quad \tau(t, u) = (t - d, u - d) \quad S(t, u) = (t - w, u - w).$$

Let $A$ be a collection of $d$-admissible arcs for $d = w - 1$. Then an integer $t$ is called a left fountain of $A$ if $A$ contains infinitely many arcs of the form $(s, t)$. Dually, one defines a right fountain of $A$; a fountain of $A$ is both a left fountain and a right fountain.

By Proposition 4.1 to obtain a characterisation of torsion pairs in $\mathbb{T}_w$ we need to characterise extension-closed contravariantly finite subcategories $X$ of $\mathbb{T}_w$. The computations of Section 3 will be used to characterise the extension-closed subcategories. Below, we state a characterisation of contravariantly finite subcategories of $\mathbb{T}_w$.

**Proposition 4.1.** Suppose $w \in \mathbb{Z} \setminus \{1\}$. Let $X$ be a full subcategory of $\mathbb{T}_w$ and $X$ be the set of admissible arcs corresponding to $\text{ind}(X)$. Then

1. If $w \geq 2$ then $X$ is contravariantly finite in $\mathbb{T}_w$ if and only if every right fountain in $X$ is also a left fountain.
2. If $w \leq 0$ then $X$ is contravariantly finite in $\mathbb{T}_w$ if and only if every left fountain in $X$ is also a right fountain.

**Proof.** When $w = 2$ this is [28] Theorem 2.2 and for $w > 2$ this is [19] Proposition 2.4. For $w = 2$, a right fountain at a given integer corresponds to having infinitely many objects from a ray, $\text{ray}_+(\Sigma^iX_0)$, in the subcategory $X$. The corresponding left fountain consists of infinitely many objects from $\text{coray}_-(\Sigma^iX_0)$. For $w \leq 0$ the situation is dual: a left fountain at a given integer corresponds to having infinitely many objects from a ray, $\text{ray}_-(\Sigma^iX_0)$ in $X$, and the corresponding right fountain consists of infinitely many objects from $\text{coray}_+(\Sigma^iX_0)$. After making the appropriate modifications to take into account this difference, the proof of [19] Proposition 2.4 also works for $w \leq -1$.

For $w = 0$, the reverse implication works exactly as in [28] Theorem 2.2 in the case $w = 2$. For the forward implication, we need a new argument, which is given in Lemma 4.3. The forward implication in [28] Theorem 2.2 contains an unexplained step; for the convenience of the reader we include a proof of this as Lemma 4.3. □

**Figure 4.** A component of type $\mathbb{Z}A_\infty$ with the endpoints of the $d$-admissible arcs described.

(ii) for $w = 0$, a $(-1)$-admissible arc $(t, u)$ is one with $u - t \leq 0$; and

(iii) for $w \leq -1$, one has $u - t \leq w$ and $u - t \equiv 1 \mod d$.

The length of the arc $(t, u)$ is $|u - t|$.

When $d$ is clear from context we refer to $d$-admissible arcs simply as admissible arcs. Figure 4 shows how admissible arcs correspond to the indecomposable objects of $\mathbb{T}_w$ when $w \neq 1$. We note that the action of the suspension, AR translate and Serre functor in this model are given by

$$\Sigma(t, u) = (t - 1, u - 1) \quad \tau(t, u) = (t - d, u - d) \quad S(t, u) = (t - w, u - w).$$

Let $A$ be a collection of $d$-admissible arcs for $d = w - 1$. Then an integer $t$ is called a left fountain of $A$ if $A$ contains infinitely many arcs of the form $(s, t)$. Dually, one defines a right fountain of $A$; a fountain of $A$ is both a left fountain and a right fountain.

By Proposition 4.1 to obtain a characterisation of torsion pairs in $\mathbb{T}_w$ we need to characterise extension-closed contravariantly finite subcategories $X$ of $\mathbb{T}_w$. The computations of Section 3 will be used to characterise the extension-closed subcategories. Below, we state a characterisation of contravariantly finite subcategories of $\mathbb{T}_w$.

**Proposition 4.1.** Suppose $w \in \mathbb{Z} \setminus \{1\}$. Let $X$ be a full subcategory of $\mathbb{T}_w$ and $X$ be the set of admissible arcs corresponding to $\text{ind}(X)$. Then

1. If $w \geq 2$ then $X$ is contravariantly finite in $\mathbb{T}_w$ if and only if every right fountain in $X$ is also a left fountain.
2. If $w \leq 0$ then $X$ is contravariantly finite in $\mathbb{T}_w$ if and only if every left fountain in $X$ is also a right fountain.

**Proof.** When $w = 2$ this is [28] Theorem 2.2 and for $w > 2$ this is [19] Proposition 2.4. For $w \geq 2$, a right fountain at a given integer corresponds to having infinitely many objects from a ray, $\text{ray}_+(\Sigma^iX_0)$, in the subcategory $X$. The corresponding left fountain consists of infinitely many objects from $\text{coray}_-(\Sigma^iX_0)$. For $w \leq 0$ the situation is dual: a left fountain at a given integer corresponds to having infinitely many objects from a ray, $\text{ray}_-(\Sigma^iX_0)$ in $X$, and the corresponding right fountain consists of infinitely many objects from $\text{coray}_+(\Sigma^iX_0)$. After making the appropriate modifications to take into account this difference, the proof of [19] Proposition 2.4 also works for $w \leq -1$.

For $w = 0$, the reverse implication works exactly as in [28] Theorem 2.2 in the case $w = 2$. For the forward implication, we need a new argument, which is given in Lemma 4.3. The forward implication in [28] Theorem 2.2 contains an unexplained step; for the convenience of the reader we include a proof of this as Lemma 4.3. □
Lemma 4.2. Let $X$ be a full subcategory of $T_0$ and $\mathbf{x}$ be the corresponding set of arcs. If $X$ is contravariantly finite, then every left fountain of $\mathbf{x}$ is also a right fountain.

Proof. Without loss of generality, we may assume that the left fountain corresponds to infinitely many objects $s_i \in \text{ray}_+(X_0)$ occurring as objects in $X$. We claim that this implies there are infinitely many objects $c \in \text{coray}_-(X_0)$ occurring in $X$, recalling that $\mathcal{S}X_0 = X_0$. This will be the right fountain. The diagrams in [28] may be useful to help understand our arguments.

Since $X$ is contravariantly finite, there is a right $X$-approximation $x \to c$, where $x = x_1 \oplus \cdots \oplus x_n$. We may assume that the map $x \to c$ is nonzero from each summand. Since $c \in H^-(\mathcal{S}s_i) = H^-(s_i)$ for each $i \in \mathbb{N}$, the map $s_i \to c$ factors as $s_i \to x \to c$. In particular, there is a summand, $x_i$ say, of $x$ such that the map $s_i \xrightarrow{f} x_i \xrightarrow{g} c$ is nonzero. Now inspecting the Hom-hammocks shows that $x_i$ either lies on $\text{coray}_-(X_0)$ or lies in the region of the AR quiver of $T_0$ bounded by the following:

$$\text{coray}_-(X_0), \text{coray}_-(R(s_i)), \text{and}, \text{ray}_+(L(c)), \text{ray}_+(X_0).$$

If $x_i \in \text{coray}_-(X_0)$ there is nothing to show, so suppose that $x_i \notin \text{coray}_-(X_0)$.

Now, by Proposition [2.4] the map $gf$ factors as $s_i \xrightarrow{f} x_i \xrightarrow{h} s_i \xrightarrow{g'} c$, where the map $hf$ is a non-isomorphism and $g'$ is unique up to scalars. Again, by Proposition [2.4] $g'$ factors through $R(s_i)$ as $g': s_i \xrightarrow{a} R(s_i) \xrightarrow{b} c$. Note that unless $x_i \simeq s_i$ we have $R(s_i) \notin H^+(x_i) \cup H^-(x_i)$, whence the composite $hg' = 0$, giving $gf = 0$ and a contradiction. Now, since $x$ contains only finitely many indecomposable summands, only finitely many of the $s_i$ may occur as summands of $x$. Applying the above argument to an $s_j$ that is not a summand of $x$ thus yields the required contradiction. This shows that each of the $x_i$ must lie on $\text{coray}_-(X_0)$ above $c$. Repeating this argument indefinitely for $c \in \text{coray}_-(X_0)$ further and further from the mouth then gives infinitely many objects of $\text{coray}_-(X_0)$ in $X$, as claimed. \hfill \Box

Lemma 4.3. Let $X$ be a full subcategory of $T_2$ and $\mathbf{x}$ be the corresponding set of arcs. If $X$ is contravariantly finite, then every right fountain of $\mathbf{x}$ is also a left fountain.

Proof. The first part of the proof is as Lemma [4.3] with the roles of left and right fountains interchanged. The coray corresponding to the required left fountain is $\text{coray}_-(\Sigma^2 X_0)$, recalling that $\mathcal{S}X_0 = \Sigma^2 X_0$. Suppose there is a $c \in X$ such that $c \in \text{coray}_-(\Sigma^2 X_0)$. Again, we suppose $x = x_1 \oplus \cdots \oplus x_n \to c$ is a right $X$-approximation and find that the $x_i$ either lie on $\text{coray}_-(\Sigma^2 X_0)$ or in the region of the AR quiver of $T_2$ bounded by the following:

$$\text{coray}_+(S^{-1}c), \text{coray}_-(\mathcal{S}s_i), \text{and}, \text{ray}_+(S^{-1}c), \text{ray}_-(\mathcal{S}s_i).$$

We assume that a summand $x_i$ of the approximation lies in this region and assume that the factorisation $s_i \xrightarrow{f} x_i \xrightarrow{g} c$ is nonzero. By Proposition [2.3] the map $g$ factors through $s_i$ as $g': x_i \xrightarrow{h} s_i \xrightarrow{g'} c$. By [21 Proposition 2.1], the map $h$ is a composition of irreducible maps, $h = h_n \cdots h_1$ say. Thus, the map $gf$ factors as

$$s_i \xrightarrow{f} x_i \xrightarrow{h} s_i \xrightarrow{g'} c.$$

Hence, we have a nonzero map $hf: s_i \to s_i$. We argue as in Proposition [2.4] By Proposition [2.2] $\text{hom}_{T_2}(s_i, s_i) = 1$, in which case $hf = \lambda \text{id}_{s_i}$, with $\lambda \neq 0$. Without loss of generality, we assume $\lambda = 1$. It follows that $h_{n-1} \cdots h_1 f$ is a right inverse for $h_n$, i.e. $h_n$ is a split epimorphism. This contradicts the irreducibility of $h_n$, whence the original factorisation $gf$ must have been zero. Thus, the $x_i$ lie on $\text{coray}_-(\Sigma^2 X_0)$ above
Repeating this argument indefinitely for $c \in \text{coray}_{\Sigma^2 X_0}$ further and further from the mouth then gives infinitely many objects of $\text{coray}(\Sigma^2 X_0)$ in $X$. $\square$

5. Torsion pairs and the Ptolemy condition in $T_w$ for $w \neq 0, 1$

Throughout this section $w \in \mathbb{Z} \setminus \{0, 1\}$. We will give a combinatorial description of the extension closure of a subcategory of $T_w$ using the combinatorial model of the previous section. This description is in terms of Ptolemy diagrams of different classes, which include those defined in [23].

For $a \in \text{ind}(T_w)$ denote the corresponding admissible arc by $a$. If $a = (t, u)$ then the starting point is $s(a) = t$ and the ending point is $t(a) = u$.

When $w \leq -1$, the first coordinate of an admissible arc is strictly bigger than the second coordinate, and when $w \geq 2$ the opposite holds.

**Notation.** Whenever the value of $w$ is not specified, the arc incident with the distinct integers $t$ and $u$ is denoted by $\{t, u\}$. In other words, assuming $t > u$,

$\{t, u\} = (t, u)$ when $w \leq -1$ and $\{t, u\} = (u, t)$ when $w \geq 2$.

**Definitions.** Let $a, b$ be two admissible arcs of $T_w$.

(1) If $a$ and $b$ are crossing arcs, then the Ptolemy arcs of class I associated to $a$ and $b$ are the remaining four arcs connecting the vertices incident with $a$ or $b$, i.e. the set of Ptolemy arcs is $\{(x, y) \mid x, y \in \{s(a), t(a), s(b), t(b)\}, x \neq y, \{x, y\} \neq a, b\}$.

(2) The distance between $a = \{t, u\}$ and $b = \{v, w\}$ is defined as $d(a, b) := \min\{|t - v|, |u - w|, |t - w|, |u - v|\}$.

(3) The arcs $a$ and $b$ are neighbouring if they do not cross and $d(a, b) = 1$.

(4) If $a$ and $b$ are neighbouring arcs and $d(a, b)$ is given by the distance between vertices $x$ and $x - 1$, then the corresponding Ptolemy arc of class II is the arc connecting the vertices incident with $a$ or $b$ which are not $x$ and $x - 1$.

Figures 5 and 6 illustrate these concepts. For brevity we shall sometimes refer to Ptolemy arcs of both classes simply as Ptolemy arcs.

![Figure 5. The Ptolemy arcs of class I.](image)

Recall that for a full subcategory $X$ of $T_w$, $\langle X \rangle$ denotes the smallest extension-closed subcategory of $T_w$ containing $X$. In this section, we prove the following main result.

**Theorem 5.1.** Let $X$ be a subcategory of $T_w$ and let $X$ be the arcs corresponding to the objects of $\text{ind}(X)$. Then the objects of $\text{ind}(\langle X \rangle)$ correspond to the arcs of

1. for $w \geq 2$, the closure of $X$ under admissible Ptolemy arcs of class I,
2. for $w \leq -1$, the closure of $X$ under admissible Ptolemy arcs of classes I and II.

Putting this together with Propositions 1.1 and 4.1 gives us part of Theorem A.

**Corollary 5.2.** Let $w \in \mathbb{Z} \setminus \{0, 1\}$, $X$ be a subcategory of $T_w$ and $X$ be the corresponding set of arcs. Then $(X, X^\perp)$ is a torsion pair in $T_w$ if and only if
Figure 6. The Ptolemy arcs of class II.

(1) for \( w \geq 2 \), any right fountain in \( X \) is also a left fountain and \( X \) is closed under taking admissible Ptolemy arcs of class I.

(2) for \( w \leq -1 \), any left fountain in \( X \) is also a right fountain and \( X \) is closed under taking admissible Ptolemy arcs of classes I and II.

5.1. Combinatorial description of the Ext-hammocks. We first need to introduce some notation regarding partial fountains, which is borrowed from [12].

**Notation.** Let \( v > t > u \) be integers such that \( \{ t, u \} \) and \( \{ t, v \} \) are \( d \)-admissible arcs. Define the partial right fountain at \( t \) starting at \( v \) and the partial left fountain at \( t \) starting at \( u \) by

\[
RF(t; v) := \{ \{ t, x \} \text{ d-admissible} | x \geq v \};
\]

\[
LF(t; u) := \{ \{ t, y \} \text{ d-admissible} | y \leq u \}.
\]

Let \( V \subseteq \mathbb{Z} \) such that \( \{ t, v \} \) is a \( d \)-admissible arc for each \( v \in V \). Write

\[
RF(V; t) := \bigcup_{v \in V} RF(v; t) \quad \text{and} \quad LF(V; t) := \bigcup_{v \in V} LF(v; t).
\]

Below is a description of the Ext-hammocks in terms of partial fountains.

**Lemma 5.3.** Let \( a, b \in \text{ind}(T_w) \), and \( V_a = \{ s(a) + id | i = 1, \ldots, k \} \), where \( k \geq 1 \) is such that \( t(a) - s(a) = kd + 1 \).

(1) If \( w \geq 2 \), then \( \text{Ext}_{T_w}^1(b, a) \neq 0 \) if and only if \( b \in LF(V_a; s(a) - 1) \cup RF(V_a; t(a) + d) \). In particular,

\[
b \in E^+(a) \iff b \in RF(V_a; t(a) + d) \quad \text{and} \quad b \in E^-(a) \iff b \in LF(V_a; s(a) - 1).
\]

(2) If \( w \leq 0 \), then \( \text{Ext}_{T_w}^1(b, a) \neq 0 \) if and only if \( b \in RF(V_a; s(a) - 1) \cup LF(V_a; t(a) + d) \). In particular,

\[
b \in E^+(a) \iff b \in LF(V_a; t(a) + d) \quad \text{and} \quad b \in E^-(a) \iff b \in RF(V_a; s(a) - 1).
\]

**Proof.** The case when \( w \leq -1 \) is [12] Remark 2.2. In the case when \( w = 0 \), the first coordinate of an admissible arc is greater than or equal to the second coordinate, and so the result is the same as in \( w \leq -1 \). When \( w \geq 2 \), the admissible arcs are ordered from left to right, and so we swap \( RF(-; -) \) and \( LF(-; -) \). \( \square \)

The following two propositions are direct consequences of the previous lemma.

**Proposition 5.4.** Let \( V_a \) be as in Lemma 5.3. If \( w \geq 2 \), and \( a, b \in \text{ind}(T_w) \), then \( \text{Ext}_{T_w}^1(b, a) \neq 0 \) if and only if \( a \) and \( b \) cross and \( b \) is incident with a vertex in \( V_a \).
Proposition 5.5. Let $V_a$ be as in Lemma 5.3. If $w \leq -1$, and $a, b \in \text{ind}(T_w)$, then $\text{Ext}^1_{T_w}(b, a) \neq 0$ if and only if we are in one of the following situations:

1. $a$ and $b$ cross and $b$ is incident with a vertex in $V_a \setminus \{t(a) - 1\}$;
2. $a$ and $b$ are neighbouring arcs such that $b$ is incident with $s(a) - 1$ or $t(a) - 1$;
3. $b = (s(a) - 1, t(a) - 1)$, i.e. $b = \Sigma a$.

5.2. The middle terms of extensions correspond to admissible Ptolemy arcs.

We now define some arcs associated with $a, b \in \text{ind}(T_w)$ for which $\text{Ext}^1_{T_w}(b, a) \neq 0$. A priori these arcs need not be admissible, but when they are, they will correspond to the indecomposable summands of the middle term of the extension.

Definition. Let $a, b \in \text{ind}(T_w)$ and suppose $\text{Ext}^1_{T_w}(b, a) \neq 0$.

1. If $b \in \text{Ext}^+(a)$ then $e_1 := (s(a), t(b))$ and $e_2 := (s(b), t(a))$.
2. If $b \in \text{Ext}^-(a)$ then $e_1 := (s(b), s(a))$ and $e_2 := (t(b), t(a))$.

Proposition 5.6. Let $a, b \in \text{ind}(T_w)$ be such that $\text{Ext}^1_{T_w}(b, a) \neq 0$, and $e_1$ and $e_2$ be as in Theorem 2.7.

1. Suppose $b \in \text{Ext}^+(a)$. The arc $e_1$ is always admissible. The arc $e_2$ is admissible if and only if $s(b) \neq t(a) - 1$.
2. Suppose $b \in \text{Ext}^-(a)$. For $w \geq 2$, the arc $e_1$ is admissible if and only if $s(b) < s(a) - 1$. For $w \leq -1$, $e_1$ is admissible if and only if $w \leq -1, s(b) > s(a)$. Moreover, $e_i$ is nonzero if and only if the corresponding arc $e_i$ is admissible.

Proof. We will first see when $e_1$ and $e_2$ are admissible. Let $k$ and $V_a$ be as in Lemma 5.3 and $k' \geq 1$ be such that $t(b) - s(b) = k'd + 1$.

Suppose $b \in \text{Ext}^+(a)$. Then $s(b) \in V_a$, and so we have:

- $t(e_1) - s(e_1) = (k' + i)d + 1$, for some $i \in \{1, \ldots, k\}$; and
- $t(e_2) - s(e_2) = (k - i)d + 1$.

Hence, $e_1$ is always admissible, and $e_2$ is admissible if and only if $k - i \geq 1$, i.e. $s(b) \neq t(a) - 1$.

Suppose now that $b \in \text{Ext}^-(a)$. Then $t(b) \in V_a$, and so we have:

- $t(e_1) - s(e_1) = (k' - i)d + 1$, for some $i \in \{1, \ldots, k\}$; and
- $t(e_2) - s(e_2) = (k - i)d + 1$.

Hence, $e_1$ is admissible if and only if $k' - i \geq 1$, i.e. $s(b) < s(a) - 1$ if $w \geq 2$ and $s(b) > s(a)$ if $w \leq -1$. On the other hand, $e_2$ is admissible if and only if $k - i \geq 1$, i.e. $t(b) \neq t(a) - 1$.

The last statement of the proposition follows from applying the following facts:

- For $x \in \text{ind}(T_w)$, the arc corresponding to $L(Sx)$ is $(s(x) - w, s(x))$ and the arc corresponding to $R(S^{-1}x)$ is $(t(x), t(x) + w)$.
- Two indecomposable objects lie in the same ray (resp. coray) if and only if the first (resp. second) coordinate of the corresponding arcs coincides.

Corollary 5.7. If $w \leq -1$ and $a$ and $b$ are neighbouring admissible arcs, then the corresponding Ptolemy arcs of class II are always admissible.

Definition. Let $a, b \in \text{ind}(T_w)$. Write $a \bowtie b := \text{add} a \bowtie \text{add} b$, and $E(a, b) := \text{ind}(a \bowtie b \cup b \bowtie a) \setminus \{a, b\}$. Denote the set of arcs corresponding to the objects in $E(a, b)$ by $E(a, b)$.

Note that $E(a, b)$ consists of the indecomposable objects occurring as middle terms of both the extensions of $b$ by $a$ and $a$ by $b$. With this definition, we have the following remark and corollary of Proposition 5.6.
Remark 5.8. Note that the only case in Proposition 5.5 where $a$ and $b$ are neither crossing nor neighbouring arcs is when $w = -1$, $a$ lies on the mouth of the AR quiver and $b = \Sigma a$. In this case, $a \notin E^+ (\Sigma a) \cup E^- (\Sigma a)$, and so $\text{Ext}^1_{T_w} (a, b) = 0$. On the other hand, the extension of $b$ by $a$ is $a \to 0 \to b \to \Sigma a$. Therefore $E(a, b) = \emptyset$.

Corollary 5.9. Let $a, b \in \text{ind}(T_w)$ and $a, b$ the corresponding admissible arcs. Then $E(a, b) \subseteq \{ \text{admissible Ptolemy arcs incident with the endpoints of } a \text{ and } b \}$.

Proof. We have $E(a, b) = \{ e_1, e_2, f_1, f_2 \}$, where the $e_i$’s and $f_i$’s are such that the extensions are $a \to e_1 \oplus e_2 \to b \to \Sigma a$ and $b \to f_1 \oplus f_2 \to a \to \Sigma b$. Note that some of these objects may be zero. We will only check that $e_1$ and $e_2$ correspond to admissible Ptolemy arcs when nonzero, as the proof is analogous for $f_1$ and $f_2$.

Suppose $e_i \neq 0$. By Proposition 5.6, $e_1$ is admissible. It remains to check that $e_1$ is a Ptolemy arc. By definition, the Ptolemy arcs and the arc $e_1$ connect endpoints of $a$ and $b$. Ptolemy arcs of class I cover all the possibilities for this connection, so the only non-trivial case that we need to consider is when $a$ and $b$ are neighbours. The problem here lies in the fact that the Ptolemy arc of class II might not be the only admissible arc connecting the endpoints of $a$ and $b$. Namely, when $w = -1$, the arc $(x, x - 1)$ connecting the two closest endpoints of $a$ and $b$ is also admissible, and it is not in general a Ptolemy arc of class II.

Since $\text{Ext}^1(b, a) \neq 0$, we have $b$ incident with either $s(a) - 1$ or $t(a) - 1$ (see Proposition 5.3 (2)). Hence, the arc in question is:

- $(s(a), s(b))$, if $b$ is incident with $s(a) - 1$,
- $(t(a), s(b))$ or $(t(a), t(b))$, if $b$ is incident with $t(a) - 1$.

Note that, by definition, $e_1$ is not any of these arcs, and therefore $e_1$ is the Ptolemy arc of class II. \qed

Given a full subcategory $X$ of $T_w$, it follows that the arcs corresponding to objects of $\text{ind}((X))$ are a subset of the closure of $X$ under admissible Ptolemy arcs. We now need to show that the inclusion in Corollary 5.9 is in fact an equality.

5.3. The extension closure. Let $a, b \in \text{ind}(T_w)$. Combinatorially, Ptolemy arcs of class I and II arise out of the following situations, respectively:

- $w \in \mathbb{Z} \setminus \{0, 1\}$ and $a$ and $b$ are crossing arcs,
- $w \leq -1$ and $a$ and $b$ are neighbouring arcs.

We shall show that in each of the two cases above $E(a, b)$ is precisely the set of all the admissible Ptolemy arcs of the appropriate class associated to $a$ and $b$. First, let us consider the case when $a$ and $b$ are neighbours.

Proposition 5.10. Let $w \leq -1$ and $a, b \in \text{ind}(T_w)$ be such that $a$ and $b$ are neighbouring arcs. Then $\text{Ext}^1_{T_w}(a, b) \neq 0$ or $\text{Ext}^1_{T_w}(b, a) \neq 0$, and $E(a, b)$ is the set of admissible Ptolemy arcs of class II associated to $a$ and $b$.

Proof. If $b$ is incident with $s(a) - 1$ or $t(a) - 1$, then $\text{Ext}^1_{T_w}(b, a) \neq 0$, by Proposition 5.6. Dually, if $b$ is incident with $s(a) + 1$ or $t(a) + 1$, then $\text{Ext}^1_{T_w}(a, b) \neq 0$. In both cases, the middle term of the extension must be nonzero, since $b \neq \Sigma a, \Sigma^{-1} a$. Therefore, $E(a, b) \neq \emptyset$, and so in particular, its cardinality is greater than or equal to one.

Note that the set of (admissible) Ptolemy arcs of class II associated to $a$ and $b$ has cardinality one or two. If the cardinality is one, then by Corollary 5.9, $E(a, b)$ must be equal to the set of admissible Ptolemy arcs associated to $a$ and $b$. Now, suppose there are two admissible Ptolemy arcs associated to $a$ and $b$. Then these must be of the form $(x, y)$ and $(x - 1, y + 1)$, with $x \geq y + 3$, for $w = -1$. In this case we have
extensions in both directions, giving rise to the Ptolemy arcs \((x, x - 1)\) and \((y, y - 1)\). Hence, we also have equality in this case. \(\Box\)

We now turn our attention to the case when \(a\) and \(b\) cross each other.

**Proposition 5.11.** Let \(a, b \in \text{ind}(T_w)\). If \(\text{Ext}_{T_w}^1(b, a) \neq 0\) and \(a, b\) are crossing arcs, then the set of admissible Ptolemy arcs of class I associated to \(a\) and \(b\) is contained in \(E(a, b)\).

**Proof.** Firstly, let us consider the case when \(w \geq 2\).

**Case 1:** \(b \in E^+(a)\). We need to check whether \((s(a), s(b))\) and \((t(a), t(b))\) are admissible. Note that these two arcs are Ptolemy arcs associated to \(a\) and \(b\), but they do not correspond to the middle term of the extension \(a \to e \to b \to \Sigma a\).

We have \(s(b) = s(a) + id\), for some \(i = 1, \ldots, k\) because \(s(b) \in V_a\). Hence, \(s(b) \neq s(a) = id\) and \(t(b) - t(a) = (i + k' + k)d\). Therefore, the arcs \((s(a), s(b))\) and \((t(a), t(b))\) are admissible if and only if \(w = 2\). Indeed, \(\text{Ext}_{T_w}^1(b, a) \simeq D \text{Ext}_{T_w}^1(b, a)\) since \(T_2 = 2\text{-CY}\). Since \(a \in E^-(b)\), \((s(a), s(b))\) and \((t(a), t(b))\) are the arcs corresponding to the indecomposable summands of the middle term of the extension \(b \to e \to a \to \Sigma b\), by Proposition 5.6.

**Case 2:** \(b \in E^-(a)\). We need to check whether \((s(b), t(a))\) and \((s(a), t(b))\) are admissible. We have \(t(b) = t(a) + id\), for some \(i = 1, \ldots, k\) and so, \(t(b) - s(a) = id\) and \(t(a) - s(b) = (k + k' + i)d + 2\). Therefore, \((s(b), t(a))\) and \((s(a), t(b))\) are admissible if and only if \(w = 2\), as in case 1. As we have seen above, when \(w = 2\), \(\text{Ext}_{T_w}^1(a, b) \simeq \text{Ext}_{T_w}^1(b, a) \neq 0\), and here we have \(a \in E^+(b)\). Hence, by Proposition 5.6, \((s(b), t(a))\) and \((s(a), t(b))\) are the arcs corresponding to the indecomposable summands of the middle term of the triangle \(b \to e \to a \to \Sigma b\).

Now, let \(w \leq -1\). We need to consider the following three cases.

**Case 1:** \(b\) crosses \(a\), it is to the left of \(a\), and \((s(b), s(a)) \in V_a\). As we have seen in case 1 for \(w \geq 2\), we have \(s(b) - s(a) = id\) and \(t(b) - t(a) = (i + k + k')d\). However, \(d \leq -2\) since \(w \leq -1\), and so the arcs \((s(a), s(b))\) and \((t(a), t(b))\) are never admissible.

**Case 2:** \(b\) crosses \(a\), it is to the right of \(a\), and \((t(b), t(a)) \in V_a\). As in case 2 for \(w \geq 2\), we have \(t(b) - s(a) = id\) and \(t(a) - s(b) = (k + k' + i)d + 2\). But again, since \(d \leq -2\), there is no \(l \geq 1\) such that \(t(b) - s(a) = ld + 1\) or \(t(a) - s(b) = ld + 1\). Therefore, \((s(b), t(a))\) and \((s(a), t(b))\) are never admissible.

**Case 3:** \(b = \Sigma a\), i.e. \(b = (s(a) - 1, t(a) - 1)\), and \(a\) does not lie on the mouth if \(w = -1\). It is easy to check that \((s(a), t(b))\) and \((s(b), t(a))\) are never admissible. On the other hand, \((s(a), s(b))\) and \((t(a), t(b))\) are admissible if and only if \(w = -1\). From now on let \(w = -1\). We have \(\text{Ext}_{T_w}^1(a, \Sigma a) \simeq D \text{Hom}_{T_w}^1(a, \tau \Sigma a) \neq 0\) if and only if \(\tau \Sigma a \in H^-(\Sigma a)\), which always holds unless \(a\) lies on the mouth of the AR quiver.

Since, by hypothesis, \(a\) does not lie on the mouth, we have \(\text{Ext}_{T_w}^1(a, \Sigma a) \neq 0\) and \(\Sigma a \in E^-(a)\). So \((s(a), s(a) - 1)\) and \((t(a), t(a) - 1)\), which are the only admissible Ptolemy arcs of class I associated to \(a\) and \(\Sigma a\), are the arcs corresponding to the indecomposable summands of the middle term of the triangle \(\Sigma a \to e \to a \to \Sigma^2 a\). \(\Box\)

In contrast to the case \(w = 2\), for \(w \in \mathbb{Z} \setminus \{0, 1, 2\}\), there are crossing arcs whose corresponding indecomposable objects do not have extensions between them. We must check that these yield no admissible Ptolemy arcs.

**Lemma 5.12.** Let \(w \neq 0, 1, 2\) and \(a, b \in \text{ind}(T)\) be such that \(\text{Ext}_{T_w}^1(a, b) = 0 = \text{Ext}_{T_w}^1(b, a)\) and \(a\) and \(b\) cross each other. Then none of the associated Ptolemy arcs of class I is admissible.
Proof. Let \( w \geq 3 \) and suppose \( a \) and \( b \) cross each other in such a way that \( s(a) < s(b) < t(a) < t(b) \). Since there are no extensions between \( a \) and \( b \), we cannot have \( s(b) = s(a) + id, \) for some \( i \geq 1 \), nor can we have \( t(a) = s(b) + jd, \) for some \( j \geq 1 \).

Since \( a \) and \( b \) are admissible arcs, we have \( t(a) - s(a) = kd + 1 \) and \( t(b) - s(b) = k'd + 1 \), for some \( k, k' \geq 1 \). We also have \( s(b) = s(a) + x \) for some \( x \geq 1 \), and \( t(a) = s(b) + y, \) for some \( y \geq 1 \).

Since \( t(b) - s(a) = t(b) - s(b) + s(b) - s(a) = k'd + x + 1, \) we conclude that the Ptolemy arc \((s(a), t(b))\) is admissible if and only if \( k'd + x + 1 = ld + 1, \) for some \( l \geq 1 \). This is equivalent to \( x = (l - k')d, \) and \( l - k' \geq 1 \) since \( s(a) < s(b) \). However, this contradicts the hypothesis, and therefore this Ptolemy arc is not admissible.

The arc \((s(b), t(a))\) is admissible if and only if \( t(a) - s(b) = kd + x + 1 = ld + 1, \) for some \( l \geq 1 \), i.e. \( x = (k - l)d \). Since \( x, d \geq 1 \), we must also have \( k - l \geq 1 \), but this contradicts the hypothesis.

Using the fact that \( t(a) = s(b) + y, \) for some \( y \geq 1 \), we can similarly show that \((t(a), t(b))\) is not admissible.

Finally, \((s(a), s(b))\) is admissible if and only if \( s(b) - s(a) = x = ld + 1, \) for some \( l \geq 1 \). Suppose, for a contradiction, that this holds, i.e. \( s(b) = s(a) + ld + 1, \) for some \( l \geq 1 \). Then we have \( t(a) = s(b) + (k - l)d, \) with \( k - l \geq 1 \), a contradiction.

The proof for the case when \( w \leq -1 \) is analogous. \( \square \)

Corollary 5.13. Let \( a, b \in \text{ind}(T_w) \) be such that \( a \) and \( b \) are crossing arcs. Then \( E(a, b) \) is the set of admissible Ptolemy arcs of class I associated to \( a \) and \( b \).

Proof. This is an immediate consequence of Proposition 5.11 and Lemma 5.12. \( \square \)

Putting these together with Corollary 5.9 yields the following corollary, which in turn gives Theorem 5.1.

Corollary 5.14. Let \( a, b \in \text{ind}(T_w) \) and \( a, b \) be the corresponding admissible arcs. Then

\[
E(a, b) = \{ \text{admissible Ptolemy arcs incident with the endpoints of } a \text{ and } b \}.
\]

6. Torsion pairs and the Ptolemy condition in \( T_0 \)

In this section we complete the proof of Theorem A. Throughout this section \( w = 0 \). The AR quiver of \( T_0 \) has only one component since \( |d| = 1 \). Recall that \( T_0 \) is 0-CY, i.e. for \( a \in T_0 \) we have \( \Sigma a = \tau^{-1} a, \) and \( \delta a = a. \) In this case, the admissible arcs are pairs \((t, u)\) with \( t \geq u. \) In particular, for \( a \in \text{ind}(T_0) \) lying on the mouth of the AR quiver, the corresponding arc \( a \) is a loop, i.e. a pair of integers \((x, x)\).

To classify extension closed subcategories of \( T_0 \) we need to introduce a new class of Ptolemy arcs.

Definition. Let \( a \) and \( b \) be \((-1)\)-admissible arcs which are not loops. Note that we can have \( a = b. \) We say \( a \) and \( b \) are adjacent if they are incident with a common vertex \( x. \) In this case, the Ptolemy arcs of class III associated to \( a \) and \( b \) are the loop at \( x \) and the arc connecting the other two vertices. See Figure 7 for an illustration.

Note that the notion of crossing arcs only makes sense for non-loops. However, we admit loops in the notion of neighbouring arcs. Note also that Ptolemy arcs in \( T_0 \) are always admissible.

The aim of this section is to prove the following theorem.

Theorem 6.1. Let \( X \) be a subcategory of \( T_0 \) and \( X \) be the arcs corresponding to the objects of \( \text{ind}(X) \). Then the objects of \( \text{ind}(\langle X \rangle) \) correspond to the arcs of the closure of \( X \) under Ptolemy arcs of classes I, II, and III.
Figure 7. Ptolemy arcs of class III.

Putting this together with Propositions 1.1 and 4.1 completes Theorem A.

Corollary 6.2. Let $X$ be a subcategory of $T_0$ and $X$ be the corresponding set of arcs. Then $(X, X^⊥)$ is a torsion pair in $T_w$ if and only if any left fountain in $X$ is also a right fountain and $X$ is closed under taking admissible Ptolemy arcs of classes I, II and III.

6.1. Combinatorial description of the Ext-hammocks. The following propositions are direct consequences of Lemma 5.3(2) and give us a combinatorial description of the arcs $b$ for which there is an extension of $b$ by $a$.

Proposition 6.3. Let $a, b \in \text{ind}(T_0)$, and assume $a$ has length greater than or equal to one. We have $\text{Ext}^1_{T_0}(b, a) \neq 0$ if and only if $b$ satisfies one of the following conditions:

1. $b$ crosses $a$,
2. $b$ is a neighbour of $a$ incident with $t(a) - 1$ or $s(a) - 1$,
3. $s(b) = t(a)$ and $s(b) \leq t(a) - 1$,
4. $t(b) = t(a)$ and $s(b) \geq s(a) - 1$,
5. $s(b) = s(a)$ and $t(a) - 1 \leq t(b) \leq s(a) - 1$.

Proposition 6.4. If $a$ is a loop, and $b \in \text{ind}(T_0)$, then $\text{Ext}^1_{T_0}(b, a) \neq 0$ if and only if $b$ satisfies condition (2) or (5) of Proposition 6.3.

6.2. The middle terms of extensions correspond to admissible Ptolemy arcs. Recall the definitions of $E^+(a)$ and $E^-(a)$ given in Section 5. The next proposition shows that, like the case $w \neq 0, 1$, these arcs, when admissible, correspond to the indecomposable summands of the middle term of the extension.

Proposition 6.5. Let $a, b \in \text{ind}(T_0)$ be such that $\text{Ext}^1_{T_0}(b, a) \neq 0$, and let $e_1$ and $e_2$ be as in Theorem 3.7.

1. Suppose $b \in E^+(a)$. The arc $e_1$ is always admissible. The arc $e_2$ is admissible if and only if $s(b) \geq t(a)$.
2. Suppose $b \in E^-(a)$. The arc $e_1$ is admissible if and only if $s(b) \geq s(a)$. The arc $e_2$ is admissible if and only if $t(b) \leq t(a)$.

Moreover, $e_i$ is nonzero if and only if the corresponding $e_i$ is admissible.

Proof. Since $w = 0$, the arcs $e_1$ and $e_2$ are admissible if and only if their first component is greater than or equal to the second component. The proof of the last statement is the same as in Proposition 5.6.

Remark 6.6. The only cases in Propositions 6.3 and 6.4 where $a$ and $b$ are neither crossing, neighbouring nor adjacent arcs are when:
6.3. The extension closure. Let \( a, b \in \text{ind}(T_0) \) and \( a, b \) be the corresponding admissible arcs. Then

\[ E(a, b) \subseteq \{ \text{admissible Ptolemy arcs incident with the endpoints of } a \text{ and } b \} . \]

**Proof.** The proof is similar to that of Corollary 5.9 when \( a \) and \( b \) are crossing or neighbouring arcs. When \( a \) and \( b \) are adjacent, any arc connecting endpoints of \( a \) and \( b \) are Ptolemy arcs of class III, so in particular, \( e_1 \) is a Ptolemy arc. \( \square \)

Corollary 6.7. Let \( a, b \in \text{ind}(T_0) \) and \( a, b \) be the corresponding admissible arcs. Then

\[ E(a, b) \subseteq \{ \text{admissible Ptolemy arcs incident with the endpoints of } a \text{ and } b \} . \]

**Proof.** The proof is similar to that of Corollary 5.9 when \( a \) and \( b \) are crossing or neighbouring arcs. When \( a \) and \( b \) are adjacent, any arc connecting endpoints of \( a \) and \( b \) are Ptolemy arcs of class III, so in particular, \( e_1 \) is a Ptolemy arc. \( \square \)

6.3. The extension closure. Let \( a, b \in \text{ind}(T_0) \). Combinatorially, Ptolemy arcs of class I, II and III arise out of the following situations, respectively:

- \( a \) and \( b \) are crossing arcs,
- \( a \) and \( b \) are neighbouring arcs,
- \( a \) and \( b \) are adjacent arcs.

We shall show that in each of the three cases above \( E(a, b) \) is precisely the set of all the admissible Ptolemy arcs of the appropriate class associated to \( a \) and \( b \). We consider each situation in turn.

**Proposition 6.8.** If \( a \) and \( b \) cross each other then \( \text{Ext}_{T_0}^1(b, a) \neq 0 \), \( \text{Ext}_{T_0}^1(a, b) \neq 0 \) and \( E(a, b) \) is the set of the Ptolemy arcs of class I associated to \( a \) and \( b \).

**Proof.** The fact that there are extensions in both directions is an immediate consequence of Proposition 6.3(1).

We can assume that \( b \) crosses \( a \) to the right, as the other case is dual. We must show that the four Ptolemy arcs associated to \( a \) and \( b \) lie in \( E(a, b) \).

On one hand, \( b \in E^-(a) \) and so the admissible Ptolemy arcs \((s(b), s(a))\) and \((t(b), t(a))\) lie in \( E(a, b) \), by Proposition 6.5(2). On the other hand, \( a \in E^+(a) \), and so the other two admissible Ptolemy arcs, namely \((s(b), t(a))\) and \((s(a), t(b))\), also lie in \( E(a, b) \), by Proposition 6.5(1). \( \square \)

**Proposition 6.9.** If \( a \) and \( b \) are neighbouring arcs, then \( \text{Ext}_{T_0}^1(b, a) \neq 0 \) or \( \text{Ext}_{T_0}^1(a, b) \neq 0 \), and \( E(a, b) \) is the set of Ptolemy arcs of class II associated to \( a \) and \( b \).

**Proof.** The proof is similar to that of Proposition 6.10. The only difference is that \( b \) can be \( \Sigma a \) or \( \Sigma^{-1}a \), namely when \( a \) is a loop. In these cases, the extension has dimension two, and the middle term of the extension whose middle term is nonzero corresponds to the Ptolemy arc of associated to \( a \) and \( b \). \( \square \)

**Proposition 6.10.** If \( a \) and \( b \) are adjacent arcs, then \( \text{Ext}_{T_0}^1(b, a) \neq 0 \) or \( \text{Ext}_{T_0}^1(a, b) \neq 0 \), and \( E(a, b) \) contains the set of Ptolemy arcs of class III associated to \( a \) and \( b \).

**Proof.** Let us fix an arc \( a \) and check the possibilities for \( b \).

- **Case 1:** \( b \) and \( a \) are adjacent at \( t(a) \) and \( t(b) < t(a) \). We have \( \text{Ext}_{T_0}^1(b, a) \neq 0 \) and \( b \in E^+(a) \). So, by Proposition 6.5(1), the two Ptolemy arcs associated to \( a \) and \( b \) lie in \( E(a, b) \).

- **Case 2:** \( b \) and \( a \) are adjacent at \( t(a) \) and \( s(b) \geq s(a) \). Then \( \text{Ext}_{T_0}^1(b, a) \neq 0 \) and \( b \in E^-(a) \). So, by Proposition 6.5(2), the middle term of the extension has two indecomposable summands, which correspond to the Ptolemy arcs associated to \( a \) and \( b \).
Case 3: \( b \) and \( a \) are adjacent at \( s(a) \) and \( t(a) \leq t(b) < s(a) \). In this case we also have \( \text{Ext}^1_{T_{a}}(b, a) \neq 0 \) and \( b \in E^-(a) \). By Proposition 6.5(2), the two Ptolemy arcs of class III associated to \( a \) and \( b \) lie in \( E(a, b) \).

The remaining three cases are dual. \( \square \)

Putting these together with Corollary 6.7 yields the following corollary, which in turn gives Theorem 6.11.

**Corollary 6.11.** Let \( a, b \in \text{ind}(T_0) \) and \( a, b \) be the corresponding admissible arcs. Then

\[
E(a, b) = \{ \text{admissible Ptolemy arcs incident with the endpoints of } a \text{ and } b \}.
\]

7. Torsion pairs and extensions in \( T_1 \)

There is no combinatorial model of \( T_1 \) in terms of admissible arcs of the infinity-gon, and thus no characterisation of torsion pairs in terms of Ptolemy diagrams. However, the classification of torsion pairs in \( T_1 \) is quite simple, see Theorem 7.1 below.

Recall that the AR quiver of \( T_1 \) consists of \( \mathbb{Z} \) copies of the homogenous tube below:

\[
\begin{array}{c}
X_0 \xrightarrow{1} X_1 \xrightarrow{1} X_2 \xrightarrow{1} X_3 \xrightarrow{1} \cdots \\
\end{array}
\]

Let \( T_0 \) be the additive (even abelian) category generated by this tube and \( T_n := \Sigma^n T_0 \).

There is a \( \mathbb{Z} \)-indexed family of split t-structures in \( T_1 \), \( (X_n, Y_n) \) given by

\[
X_n := \text{add} \bigcup_{i \geq n} T_i \quad \text{and} \quad Y_n := \text{add} \bigcup_{i < n} T_i,
\]

whose heart is \( T_n \), which is a hereditary abelian category. The short exact sequences in \( T_n \) correspond precisely to distinguished triangles \( a' \to a \to a'' \to \Sigma a' \) with \( a', a, a'' \in T_n \); see [3].

7.1. Torsion pairs in \( T_1 \). The main result of this section is the following.

**Theorem 7.1.** The only torsion pairs in \( T_1 \) are the (de)suspensions of the standard t-structure, i.e. \( (X_n, Y_n) \) for \( n \in \mathbb{Z} \), and the trivial torsion pairs \( (T_1, 0) \) and \( (0, T_1) \).

Before proving Theorem 7.1 we need two preliminary results. The first one describes the Hom- and Ext-hammocks of \( T_1 \).

**Proposition 7.2 ([23 Proposition 3.4]).** Consider the indecomposable object \( X_r \) in \( T_1 \) and let \( b \in \text{ind}(T_1) \) be any other indecomposable. Then

\[
\text{hom}_{T_1}(X_r, b) = \begin{cases} 
\min\{r, s\} + 1 & \text{if } b = X_S \text{ or } b = \Sigma X_s, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that since \( T_1 \) is 1-Calabi-Yau, the Hom-hammocks and Ext-hammocks coincide, i.e. \( H^+(X_r) = E^+(X_r) \) and \( H^-(\Sigma X_r) = E^-(X_r) \).

**Lemma 7.3.** All torsion pairs in \( T_1 \) are split.

**Proof.** Suppose \( (X, Y) \) is a non-split torsion pair in \( T_1 \) and that \( t \in \text{ind}(T_1) \) does not belong to either \( X \) or \( Y \). Therefore, there is a non-trivial approximation triangle \( x \to t \to y \to \Sigma x \) with \( x \in X \) and \( y \in Y \). The object \( t \) lies in some homogeneous tube, \( T_k \) say, and thus \( t = \Sigma^k X_u \) for some \( u \geq 0 \). By Proposition 7.2 there are maps to \( t \) only from the tubes \( T_{k-1} \) and \( T_k \). Suppose \( x \) contains a summand \( \Sigma^{k-1} X_r \in T_{k-1} \). By the properties of homogeneous tubes (see for example [32 Chapter X] and combine it with the properties of derived categories of hereditary categories [16]), the maps \( \Sigma^{k-1} X_r \to t \) factor through \( \Sigma^{k-1} X_s \) for all \( s > r \). Hence, no finite sum of indecomposable objects
from $\mathcal{T}_{k-1}$ can be a summand of $x$. The only possibility remaining is that $x$ contains a summand from $\mathcal{T}_k$. But since $x$ is extension-closed, we get $\mathcal{T}_k \subseteq X$, whence $t \in X$; a contradiction. Thus, any torsion pair $(X, Y)$ is split.

**Proof of Theorem 7.7** Let $(X, Y)$ be a torsion pair in $T_1$ and suppose $\mathcal{T}_k \subseteq X$. By Proposition 7.2 $\text{Hom}_{T_1}(\mathcal{T}_k, \mathcal{T}_{k+1}) \neq 0$, whence $\mathcal{T}_{k+1} \subseteq X$ since $(X, Y)$ is split by Lemma 7.3. Thus, if $\mathcal{T}_k \subseteq X$ then $\mathcal{T}_j \subseteq X$ for each $j \geq k$. There is either a minimal such $k$, in which case $(X, Y) = (X_k, Y_k)$, or there is not, in which case $(X, Y) = (T_1, 0)$. Arguing from the point of view of $Y$ gives the other trivial torsion pair $(0, T_1)$.

**Remark 7.4.** In [23] it was shown that $T_0$ has only one family of non-trivial (bounded) t-structures, namely the $(X_n, Y_n)$. Here we have shown that this family of bounded t-structures are the only non-trivial torsion pairs in $T_0$.

### 7.2. Extensions with indecomposable outer terms in $T_1$.

Whilst this is not needed in the classification of torsion pairs in $T_1$, for the sake of completeness, we include a brief description.

Without loss of generality we may consider extensions starting at $X_r$ for some $r \geq 0$. Using Proposition 7.2 and the 1-Calabi-Yau property, the extensions whose outer terms are indecomposable and first term is $X_r$ have the following form for $s \geq 0$:

$$X_r \to E \to X_s \to \Sigma X_r \text{ and } X_r \to F \to \Sigma X_s \to \Sigma X_r.$$

The first extension $X_r \to E \to X_s \to \Sigma X_r$ has all three objects lying in the heart $\mathcal{T}_0$. Thus, it is enough to compute the extensions in the abelian category $\mathcal{A}_0$, which is a special case of [3, Lemma 5.1].

The middle term of the second extension is isomorphic to $\text{cone}(f) = (\Sigma \ker f) \oplus \text{coker} f$, where $f : X_s \to X_r$, where we use the fact $\mathcal{T}_0$ is hereditary. We can now compute the cones of these morphisms in $T_1$ using, for example, [32, Chapter X].

We summarise these considerations below.

**Proposition 7.5.** Consider a non-trivial extension $X_r \to E \to B \to \Sigma X_r$ in $T_1$, where $B$ is either $X_s$ or $\Sigma X_s$ for some $s \geq 0$. We interpret $X_{-1}$ as the zero object.

(i) If $B = X_s$, write $n = \min\{r, s\}$ and $m = \max\{r, s\}$. Then the $n + 1$ extensions are $X_r \to X_{m+i} \oplus X_{n-i} \to X_r \to \Sigma X_r$ for $1 \leq i \leq n + 1$.

(ii) If $B = \Sigma X_s$ for $s \geq r$ then the $r + 1$ extensions are $X_r \to \Sigma X_{s-r} \oplus X_{r-1} \to \Sigma X_s \to \Sigma X_r$ for $1 \leq i \leq r + 1$.

(iii) If $B = \Sigma X_s$ for $s < r$ then the $s + 1$ extensions are $X_r \to X_{r-s} \oplus \Sigma X_{r-2} \to \Sigma X_s \to \Sigma X_r$ for $1 \leq i \leq s + 1$.

### 8. Torsion pairs in $C_w(A_n)$

Throughout this section $w \leq -1$ and $m = -w + 1$ and we shall consider the orbit category

$$C_m : = C_m(A_n) = D^b(kA_n)/\Sigma^m \tau,$$

where $\tau$ denotes the AR translate of $D^b(kA_n)$. For more detailed background on these categories we refer the reader to the papers [9, 11, 12]. These categories are triangulated by Keller’s Theorem [20], and satisfy $\Sigma^w \simeq S$, where $S = \Sigma \tau$ is the Serre functor. In particular, they can be considered to be $w$-CY.

#### 8.1. The combinatorial model for $C_m$.

Given two indecomposable objects $a$ and $b$ in $T_w$, we say that $a$ is an innerarc of $b$ if one has $t(b) < t(a) < s(a) < s(b)$.

It was shown in [12] that the combinatorial model in $T_w$ induces a combinatorial model in $C_m$ as follows: $C_m$ is equivalent to the full subcategory $\mathcal{C}_w$ of $T_w$ whose set...
of indecomposable objects correspond to the admissible innerarcs of an admissible arc a of length \(|(n+1)(-w-1)+1|\).

We briefly recall the explicit combinatorial model. Let \(P_{n,m}\) be the regular \(N\)-gon, where \(N = m(n+1) - 2\), with vertices numbered clockwise from 1 to \(N\). All operations on vertices of \(P_{n,m}\) will be done modulo \(N\), with representatives \(1, \ldots, N\). An \(m\)-diagonal of \(P_{n,m}\) is a diagonal that divides \(P_{n,m}\) into two polygons each of whose number of vertices is divisible by \(m\).

The AR quiver of \(C_m\) is equivalent to the stable translation quiver \(\Gamma(n, m)\) whose vertices are the \(m\)-diagonals of \(P_{n,m}\). We denote a vertex of \(\Gamma(n, m)\) by \(\{i, j\}\), where \(i\) and \(j\) are vertices of \(P_{n,m}\). The arrows of \(\Gamma(n, m)\) are obtained in the following way: given two \(m\)-diagonals \(D\) and \(D'\) with a vertex \(i\) in common, let \(j\) and \(j'\) be the other vertices of \(D\) and \(D'\) respectively. Then, there is an arrow from \(D\) to \(D'\) in \(\Gamma(n, m)\) if and only if \(D'\) can be obtained from \(D\) by rotating clockwise \(m\) steps around \(i\). The translation automorphism \(\tau : \Gamma(n, m) \to \Gamma(n, m)\) sends an \(m\)-diagonal \(\{i, j\}\) to \(\tau(\{i, j\}) := \{i - m, j - m\}\). Note that \(\Sigma\{i, j\} = \{i + 1, j + 1\}\). Figure 8 shows an example of this stable translation quiver.

![Figure 8](image)

**Figure 8.** The AR quiver of \(C_2(A_3)\) is equivalent to \(\Gamma(3, 2)\).

Define Ptolemy diagonals of class I, neighbouring \(m\)-diagonals and Ptolemy diagonals of class II in the same manner as in \(T_w\). The main result of this section is:

**Theorem 8.1.** Let \(X\) be a subcategory of \(C_m\) and \(X\) the set of \(m\)-diagonals corresponding to the objects of \(\text{ind}(X)\). Then the objects of \(\text{ind}(|X|)\) correspond to the \(m\)-diagonals of the closure of \(X\) under Ptolemy \(m\)-diagonals of classes I and II.

Since \(C_m\) has finitely many indecomposable objects up to isomorphism, any subcategory of \(C_m\) is contravariantly finite. Thus, Theorem 8.1 is an immediate corollary of Theorem 8.1 and Proposition 1.1.

**Corollary 8.2.** Let \(X\) be a subcategory of \(C_m\) and \(X\) the corresponding set of \(m\)-diagonals. Then \((X, X^\perp)\) is a torsion pair in \(C_m\) if and only if \(X\) is closed under Ptolemy \(m\)-diagonals of class I and II.

**Remark 8.3.** The characterisation of torsion pairs in \(C_m\) does not follow immediately from that in \(T_w\): \(C_w\) is not a triangulated subcategory of \(T_w\); see [12, Theorem 5.1]. Hence, if there is an extension in \(T_w\) between two objects of \(C_w\) then there is an extension in \(C_m\) between their images, but the converse is not true. As an example, consider \(m = 2, n = 3\) and \(a = (7, 0)\) to be the arc that defines the equivalence. The admissible arcs \((2, 1)\) and \((6, 5)\) are objects in \(C_w\) and the corresponding images \(\{1, 2\}\) and \(\{5, 6\}\) are 2-diagonals of a hexagon. It is easy to check that \(\text{Ext}^1_{\text{ind}}((1, 2), (5, 6)) = 0\) but \(\text{Ext}^1_{\text{ind}}(\{1, 2\}, \{5, 6\}) \neq 0\).

8.2. **Combinatorial description of the Ext-hammocks.** We require the following notation to describe the Hom- and Ext-hammocks in \(C_m\).
Notation. Given the vertices $i_1, i_2, \ldots, i_k$ of $\mathcal{P}_{n,m}$, we write $C(i_1, i_2, \ldots, i_k)$ to mean that $i_1, i_2, \ldots, i_k, i_1$ follow each other under the clockwise circular order on the boundary of $\mathcal{P}_{n,m}$.

Lemma 8.4. Let $a, b \in \text{ind}(C_m)$ and $a = \{a_1, a_2\} (a_1 < a_2)$, $b = \{b_1, b_2\}$ be the corresponding $m$-diagonals of $\mathcal{P}_{n,m}$. We have $\text{Hom}_{C_m}(a, b) \neq 0$ if and only if $b$ satisfies the following condition:

$$b_1 = a_2 + im, b_2 = a_1 + jm, \text{ for some } i, j \geq 0, \text{ and } C(a_2, b_1, a_1, b_2).$$

Proof. Explicit computation using the combinatorial model of $C_m$. □

Using the Auslander–Reiten formula, we obtain the Ext-hammocks:

Corollary 8.5. We have $\text{Ext}^1_{C_m}(b, a) \neq 0$ if and only if the following condition holds:

$$b_1 = a_2 + im, b_2 = a_1 + jm, \text{ for some } i, j \geq 1, \text{ and } C(a_2 + m, b_1, a_1 + m, b_2).$$

Corollary 8.5 can be unpacked into the following more readable statement.

Corollary 8.6. Let $a, b \in \text{ind}(C_m)$ and $a = \{a_1, a_2\}$, with $a_1 < a_2$, be the $m$-diagonal corresponding to $a$. Then $\text{Ext}^1_{C_m}(b, a) \neq 0$ if and only if $b$ satisfies one of the following:

1. $b$ is a neighbour of $a$ incident with $a_2 + 1$,
2. $b$ is a neighbour of $a$ incident with $a_1 + 1$,
3. $b$ crosses $a$ in such a way that $b$ is obtained by adding multiples of $m$ to the endpoints of $a$,
4. $b = \{a_1 + 1, a_2 + 1\}$, i.e. $b = \Sigma a$.

Figure 9 shows where the indecomposable objects corresponding to the arcs that satisfy the conditions in Corollary 8.6 lie in the AR quiver.

8.3. Graphical calculus. To prove Theorem 8.1 we need a graphical calculus analogous to Theorem 3.7. Before doing this, we briefly recall some useful notation and facts about $C_m$.

Write $\text{inj}(kA_n)$ for the subcategory of $\text{mod}(kA_n)$ containing the injective $kA_n$-modules. We have the following fundamental domain for $C_m$ in $\text{D}^b(kA_n)$:

$$F = \text{ind}(\bigcup_{i=1}^m \Sigma^{i-1} \text{mod}(kA_n) \cup \Sigma^m (\text{mod}(kA_n) \setminus \text{inj}(kA_n))).$$

From now on, we identify objects in $\text{ind}(C_m)$ with their representatives in $F$. Given $X \in \text{ind}(\text{D}^b(kA_n))$, we denote by $d(X)$ the degree of $X$, i.e. the integer such that $X = \Sigma^{d(X)} \overline{X}$ for $\overline{X} \in \text{ind}(\text{mod}(kA_n))$.

Lemma 8.7. Let $A, B \in F$, $P$ be a projective $kA_n$-module and $C \in \text{ind}(\text{D}^b(kA_n))$. 

\[\text{Ext}^1_{C_m}(-, a) \neq 0.\]
(1) $\text{Hom}_{D^b(kA_n)}(A, \tau^k \Sigma^m B) = 0$, for every $k \neq 0, 1$.

(2) $\text{Hom}_{D^b(kA_n)}(A, \tau^k \Sigma^m B) \neq 0$ for at most one value of $k$.

(3) If $\text{Hom}_{D^b(kA_n)}(P, C) \neq 0$, then $C \in \text{ind}(\text{mod}(kA_n))$.

Proof. Statements (1) and (2) are proved for $m = 1$ in [9] Proposition 2.1; the proof for $m \geq 2$ is similar. Statement (3) is trivial using the hereditary property. \hfill \Box

For $a \in \text{ind}(C_m)$ the starting and ending frames of $a$ are:

$$F_s(a) := \{ b \in \text{ind}(C_m) \mid \text{Hom}_{C_m}(a, b) \neq 0, \text{Ext}^1_{C_m}(b, a) = 0 \};$$

$$F_e(a) := \{ b \in \text{ind}(C_m) \mid \text{Hom}_{C_m}(a, b) \neq 0, \text{Ext}^1_{C_m}(a, b) = 0 \},$$

cf. Lemma 3.2(ii), ray$_+$ $(a) \cup$ coray$_+$ $(a)$ and ray$_-$ $(a) \cup$ coray$_-$ $(a)$ in Theorem 6.7.

We are now ready to state the graphical calculus result. For $w = -1$ this was done in [10] Chapter 5.

Proposition 8.8. Let $a, b \in \text{ind}(C_m)$ be such that $\text{Ext}^1_{C_m}(a, b) \neq 0$. Then $\text{ext}^1_{C_m}(a, b) = 1$ and the unique (up to equivalence) non-split extension $b \rightarrow e \rightarrow a \rightarrow \Sigma b$ has middle term $e$ whose summands are given by $F_s(b) \cap F_e(a)$. If this intersection is empty, then we interpret $e$ to be the zero object.

Proof. We can assume, without loss of generality, that the quiver of type $A_n$ has the linear orientation $n \rightarrow n - 1 \rightarrow \cdots \rightarrow 1$. Recall that we see $a$ and $b$ as sitting inside the fundamental domain $F$. Suppose $\text{Ext}^1_{C_m}(a, b) \neq 0$. By taking a suitable AR translate of $a$ and $b$, we may assume that $b$ is a projective $kA_n$-module.

Case 1: Assume $a$ is a non-projective $kA_n$-module. Then $\tau a \in \text{mod}(kA_n) \subseteq F$ and so

$$\text{Ext}^1_{C_m}(a, b) \simeq D \text{Hom}_{C_m}(b, \tau a) = D \text{Hom}_{D^b(kA_n)}(b, \tau a) \oplus D \text{Hom}_{D^b(kA_n)}(b, \tau^2 \Sigma^m a).$$

Since $\tau a$ is a module, $d(\tau^2 \Sigma^m a) = m - 1$ or $m$, so $d(\tau^2 \Sigma^m a) \geq 1$. Hence, the second summand is zero, since $b$ is a projective module. Therefore $\text{Ext}^1_{C_m}(a, b) \simeq \text{Hom}_{kA_n}(b, \tau a)$. Hence, $\text{ext}^1_{C_m}(a, b) = 1$, since Hom spaces in type $A$ are either zero or one dimensional. The result then follows from [7] Corollary 8.5.

Case 2: Assume $a$ is a projective indecomposable module. Then $\tau a = \Sigma^{-1} I$ for some indecomposable injective $I$. In $C_m$ we have $\tau a \simeq \Sigma^{m-1} \tau I$, which lies in $F$. Hence, $\text{Ext}^1_{C_m}(a, b) \simeq D \text{Hom}_{C_m}(b, \tau a) = D \text{Hom}_{D^b(kA_n)}(b, \Sigma^m \tau I) \oplus D \text{Hom}_{D^b(kA_n)}(b, \tau^2 \Sigma^{2m-1} I).$

The second summand is always zero since $d(\tau^2 \Sigma^{2m-1} I) \geq 1$ and $b$ is projective. We have $d(\tau \Sigma^m I) = m - 2$ or $m - 1$. For any $m \geq 2$, we have $m - 1 \geq 1$. Thus, if $d(\tau \Sigma^{m-1} I) = m - 1$, the first summand is also zero giving a contradiction. Analogously if $d(\tau \Sigma^{m-1} I) = m - 2$ and $m > 2$. To avoid a contradiction, we must have $m = 2$ and $d(\tau \Sigma I) = 0$ and $I \cong P(n) \cong I(1)$ the indecomposable projective-injective, whence $a \cong P(1)$. Now,

$$0 \neq \text{Ext}^1_{C_2}(a, b) \simeq D \text{Hom}_{D^b(kA_n)}(b, \Sigma \tau P(n)) = D \text{Hom}_{kA_n}(b, I(n)),$$

where one-dimensionality follows from being in type $A$. Since $I(n) = S(n)$, i.e. the simple at vertex $n$, and $b$ is projective, we get $b = P(n)$. In $C_2$ we have $\Sigma P(n) \cong P(1)$, whence the non-split triangle is $P(n) \rightarrow 0 \rightarrow P(1) \rightarrow \Sigma P(n)$. However, $F_s(b) \cap F_e(a) = \emptyset$, giving the claim in this case.

Case 3: $a = \Sigma^i \overline{A}$, for some indecomposable non-injective module $\overline{A}$ and $0 < i < m$. If $i = 1$, assume also that $\overline{A}$ is also non-projective. Then $\tau a$ lies in $F$. Therefore,

$$\text{Ext}^1_{C_m}(a, b) \simeq D \text{Hom}_{C_m}(b, \tau a) = D \text{Hom}_{D^b(kA_n)}(b, \tau a) \oplus D \text{Hom}_{D^b(kA_n)}(b, \Sigma^m \tau^2 a).$$
Both summands are zero since \( d(\Sigma^m \tau^2 a) \geq 2 \) and \( d(\tau a) \geq 1 \), the latter since \( \overline{A} \) is non-projective when \( i = 1 \); a contradiction, so this case provides no extensions.

**Case 4:** \( a = \Sigma \overline{A} \), where \( \overline{A} \) is projective. Then \( \tau a \) is an injective module, and thus lies in \( F \). Hence

\[
\text{Ext}^1_{C_m}(a, b) \cong D \text{Hom}_{C_m}(b, \tau a) = \text{Hom}_{D^b(kA_n)}(b, \tau a) \oplus D \text{Hom}_{D^b(kA_n)}(b, \Sigma^m \tau^2 a).
\]

The second summand is zero because \( d(\Sigma^m \tau^2 a) \geq 1 \). Therefore \( \text{ext}^1_{C_m}(a, b) = 1 \), since \( \text{Ext}^1_{C_m}(a, b) \neq 0 \) by assumption. We construct a non-split triangle with first term \( b \) and last term \( a \). Let \( b = P(i) \) and \( a = \Sigma P(j) \).

For \( i < j \) we have \( \text{Hom}_{D^b(kA_n)}(b, \tau a) = \text{Hom}_{D^b(kA_n)}(P(i), I(j)) = 0 \), giving a contradiction. Thus \( i \geq j \). For \( i > j \), there is a short exact sequence \( 0 \to P(j) \to P(i) \to E \to 0 \), which induces a triangle \( P(j) \to P(i) \to E \to \Sigma P(j) \). Shifting this triangle gives the desired triangle. It is easy to check that the cokernel \( E \) is the unique object in \( F_e(\Sigma P(i)) \cap F_e(\Sigma P(j)) \). For \( i = j \), we get the standard triangle \( P(i) \to 0 \to \Sigma P(i) \to \Sigma P(i) \) in which the last map is an isomorphism. Moreover, \( F_e(\Sigma P(i)) = \emptyset \), corresponding to the zero middle term. \( \square \)

Using Proposition 8.8, one can check that the \( m \)-diagonals corresponding to the middle term are as in Figure 10.

**8.4. Extension closure.** Let \( a \) and \( b \) be \( m \)-diagonals and recall the notion of \( E(a, b) \) from Section 5.

**Remark 8.9.** Let \( m = 2 \), \( a = \{i, i+1\} \) and \( b = \Sigma a \). Note that \( a \) and \( b \) are incident with the vertex \( i + 1 \), and this is the only case when extensions occur between noncrossing and non-neighbouring \( m \)-diagonals. In this case, the middle term of the extension of \( b \) by \( a \) is zero, and \( \text{Ext}^1_{C_2}(a, b) \cong D \text{Hom}_{C_2}(a, \tau^{-1} a) = 0 \). Therefore, \( E(a, b) = \emptyset \).

Figure 10 shows us that \( E(a, b) \) is contained in the set of Ptolemy \( m \)-diagonals associated to \( a \) and \( b \). Our aim is to prove that these two sets are in fact equal, giving Theorem 8.11. This follows from the following three propositions.

**Proposition 8.10.** Let \( a, b \in \text{ind}(C_m) \) be such that \( a \) and \( b \) are neighbouring \( m \)-diagonals. Then \( E(a, b) \) contains all the Ptolemy \( m \)-diagonals of class II associated to \( a \) and \( b \).

**Proof.** Similar to the proof of Proposition 5.10 \( \square \)

**Proposition 8.11.** Let \( a, b \in \text{ind}(C_m) \) be such that \( a \) and \( b \) are crossing \( m \)-diagonals and \( \text{Ext}^1_{C_m}(b, a) \neq 0 \). Then \( E(a, b) \) contains all the Ptolemy \( m \)-diagonals of class I associated to \( a \) and \( b \).

**Proof.** It follows immediately from the fact that the Ptolemy diagonals of class I associated to \( a \) and \( b \) other than \( e_1 \) and \( e_2 \) (see Figure 10) are not \( m \)-diagonals. \( \square \)

**Proposition 8.12.** Let \( a, b \in \text{ind}(C_m) \) be such that \( a \) and \( b \) are crossing \( m \)-diagonals and \( \text{Ext}^1_{C_m}(a, b) = 0 = \text{Ext}^1_{C_m}(b, a) \). Then none of the corresponding Ptolemy diagonals is a \( m \)-diagonal.

**Proof.** By Remark 8.3, there is no extension between the corresponding \( d = (w - 1) \)-admissible arcs in \( \mathbb{T}_w \). Hence, by Lemma 5.12 the corresponding Ptolemy diagonals of class I are not \( m \)-diagonals. \( \square \)
Figure 10. The middle term of the extension of $b$ by $a$. The arrows in the third case mean that the distance between the corresponding endpoints is a multiple of $m$.

References

[1] I. Assem, D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras. 1: Techniques of Representation Theory*, London Math. Soc. Stud. Texts, vol 65, Cambridge University Press (2006).

[2] M. Auslander, I. Reiten, S.O. Smalø, *Representation Theory of Artin Algebras*, Cambridge University Press (1997).

[3] K. Baur, A. B. Buan, R. Marsh, *Torsion pairs and rigid objects in tubes*, arXiv:1112.6132

[4] A.A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque 100 (1982).

[5] M. V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences and complexes (for motives and in general); J. K-theory* 6 (2010), 387–504, also arXiv:0704.4003

[6] N. Broomhead, D. Pauksztello, D. Ploog, *Discrete derived categories I: Homomorphisms, autoequivalences and t-structures*, arXiv:1312.5203
[7] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), 572–618, also [arXiv:math/0402054](http://arxiv.org/abs/math/0402054).

[8] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from clusters (A_n case)*; Trans. Amer. Math. Soc. **358** (2006), 1347–1364, also [arXiv:math/0401316](http://arxiv.org/abs/math/0401316).

[9] R. Coelho Simões, *Hom-configurations and noncrossing partitions*; J. Algebraic Combin. **35** (2012), 313–343, also [arXiv:1012.1278](http://arxiv.org/abs/1012.1278).

[10] R. Coelho Simões, *On a triangulated category which models positive noncrossing partitions*; PhD thesis, University of Leeds, March 2012.

[11] R. Coelho Simões, *Maximal rigid objects as noncrossing bipartite graphs*; Algebr. Represent. Theory **16** (2013), 1243–1272, also [arXiv:1111.2306](http://arxiv.org/abs/1111.2306).

[12] R. Coelho Simões, *Hom-configurations in triangulated categories generated by spherical objects*; arXiv:1312.4769.

[13] A. Dugas, *Resolutions of mesh algebras: periodicity and Calabi-Yau dimensions*; Math. Z. **271** (2012), 1151–1184, also [arXiv:1003.4960](http://arxiv.org/abs/1003.4960).

[14] C. Fu, D. Yang, *The Ringel-Hall Lie algebra of a spherical object*; J. London Math. Soc. **85** (2012), 511–533, also [arXiv:1103.1241](http://arxiv.org/abs/1103.1241).

[15] S. Gratz, *Mutation of torsion pairs in cluster categories of Dynkin type D*; arXiv:1311.4804.

[16] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Mathematical Society Lecture Notes Series **119**, Cambridge University Press (1988).

[17] A. Hochenegger, M. Kalck, D. Ploog, *Spherical subcategories: algebraic geometry*, arXiv:1208.4046.

[18] T. Holm, P. Jørgensen, *On a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon*, Math. Z. **270** (2012), 277–295, also [arXiv:0907.3195](http://arxiv.org/abs/0907.3195).

[19] T. Holm, P. Jørgensen, *Cluster tilting vs. weak cluster tilting in Dynkin type A infinity*, to appear in Forum Math, also [arXiv:0902.4125](http://arxiv.org/abs/0902.4125).

[20] T. Holm, P. Jørgensen, M. Rubey, *Torsion pairs in cluster tubes*; arXiv:1207.3206.

[21] T. Holm, P. Jørgensen, M. Rubey, *Ptolemy diagrams and torsion pairs in the cluster category of Dynkin type D*; Adv. in Appl. Math. **51** (2013), 583–605, also [arXiv:1303.1684](http://arxiv.org/abs/1303.1684).

[22] T. Holm, P. Jørgensen and D. Yang, *Sparseness of t-structures and negative Calabi-Yau dimension in triangulated categories generated by a spherical objects*; Bull. London Math. Soc. **45** (2013), 120–130, also [arXiv:1108.2195](http://arxiv.org/abs/1108.2195).

[23] O. Iyama, Y. Yoshino, *Mutation in triangulated categories and rigid Cohen-Macaulay modules*; Invent. math. **172** (2008), 117–168, also [arXiv:math/0607736](http://arxiv.org/abs/math/0607736).

[24] P. Jørgensen, *Auslander-Reiten theory over topological spaces*; Comment. Math. Helv. **79** (2004), 160–182, also [arXiv:math/0304080](http://arxiv.org/abs/math/0304080).

[25] B. Keller, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581, also [arXiv:math/0503240](http://arxiv.org/abs/math/0503240).

[26] B. Keller, D. Yang, G. Zhou, *The Hall algebra of a spherical object*; J. London Math. Soc. **80** (2009), 771–784, also [arXiv:0810.5546](http://arxiv.org/abs/0810.5546).

[27] P. Ng, *A characterization of torsion theories in the cluster category of type $A_\infty$*; [arXiv:1005.4364](http://arxiv.org/abs/1005.4364).

[28] D. Pauksztello, *Compact corigid objects in triangulated categories and co-t-structures*; Cent. Eur. J. Math. **6** (2008), 25–42, also [arXiv:0705.0102](http://arxiv.org/abs/0705.0102).

[29] R. Schiffler, *A geometric model for cluster categories of type $D_n$*; J. Algebraic Combin. **27** (2008), 1–21, also [arXiv:math/0607264](http://arxiv.org/abs/math/0607264).

[30] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*; Duke Math. J. **108** (2001), 37–108, also [arXiv:math/0001043](http://arxiv.org/abs/math/0001043).

[31] D. Simson, A. Skowroński, *Elements of the Representation Theory of Associative Algebras. 2: Tubes and Concealed Algebras of Euclidean type*, London Math. Soc. Stud Texts, vol 71, Cambridge University Press (2007).

[32] Y. Zhou, B. Zhou, *Mutation of torsion pairs in triangulated categories and its geometric realisation*; [arXiv:1105.3521](http://arxiv.org/abs/1105.3521)

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