A DERIVED CATEGORY APPROACH TO KEMPF’S VANISHING THEOREM

ALEXANDER SAMOKHIN

ABSTRACT. We give a proof of the Andersen–Haboush identity (cf. [1], [6]) that implies Kempf’s vanishing theorem. Our argument is based on the structure of derived categories of coherent sheaves on flag varieties over \( \mathbb{Z} \).

1. Introduction

Let \( G \) be a split semisimple simply connected algebraic group over a perfect field \( k \) of characteristic \( p \). The weight \( -\rho = -\sum \omega_i \), where \( \omega_i \) are the fundamental weights of \( G \), is known to play a fundamental rôle in representation theory of \( G \). For \( q = p^n, n \geq 1 \), the Steinberg weight \((q - 1)\rho\) is equally important in representation theory of semisimple groups in defining characteristic. In particular, there is a remarkable property that the corresponding line bundle \( L_{(q - 1)\rho} \) on the flag variety \( G/B \) enjoys: its pushforward under the \( n \)-th iteration of Frobenius morphism is a trivial vector bundle whose space of global sections is canonically identified with the Steinberg representation \( St_q \):

\[
F_n^* L_{(q - 1)\rho} = St_q \otimes O_{G/B}. \tag{1}
\]

This was proven independently and at around the same time by Andersen in [1] and by Haboush in [6]. Back to the weight \(-\rho\), isomorphism of vector bundles (1) is equivalent to saying that the line bundle \( L_{-\rho} \) is an "eigenvector" with respect to Frobenius morphism, i.e. \( F_q^* L_{-\rho} = St_q \otimes L_{-\rho} \). This fact has many important consequences for representation theory of algebraic groups in characteristic \( p \): in particular, the Kempf vanishing theorem [9] easily follows from it (see [1] and [6]). The proofs of \textit{loc.cit.} were essentially representation–theoretic. The goal of this note is to prove isomorphism (1) using the structure of the derived category of coherent sheaves on the flag variety \( G/B \). In a nutshell, the idea is as follows. Given a smooth algebraic variety \( X \) and a semiorthogonal decomposition \( \langle D_0, D_1 \rangle \) of the derived category \( D^b(X) \) (see Section 2 for the details), any object of \( D^b(X) \) – in particular, any vector bundle \( F \) on \( X \), can be decomposed with respect to \( D_0 \) and \( D_1 \). Thus, if \( F \) is right orthogonal to \( D_1 \), i.e. \( \text{Hom}_{D^b(X)}(D_1, F) = 0 \), it automatically belongs to \( D_0 \).

It turns out that for a semiorthogonal decomposition of the derived category \( D^b(G/B) \) into two pieces, one of which is the admissible subcategory \( \langle L_{-\rho} \rangle \) generated by the single line bundle \( L_{-\rho} \) and the other one being its left orthogonal \( \perp \langle L_{-\rho} \rangle \), the bundle \( F_q^* L_{-\rho} \) is right orthogonal to \( \perp \langle L_{-\rho} \rangle \). Therefore, it should belong to the subcategory \( \langle L_{-\rho} \rangle \). Being generated by a single exceptional bundle, the latter subcategory is equivalent to the derived category of vector spaces over \( k \); thus, one has \( F_q^* L_{-\rho} = L_{-\rho} \otimes V \) for some graded vector space \( V \). Since the left hand side of this isomorphism is a vector bundle, i.e. a pure object of \( D^b(G/B) \), the graded vector space \( V \) should only have a
non-trivial zero degree part, which is a vector space of dimension $q^{\dim(G/B)}$. Tensoring the both sides with $L_\rho$ and taking the cohomology, one obtains an isomorphism $V = H^0(G/B, L_{(q-1)\rho}) = St_q$, and hence isomorphism (1).

Unfolding this argument takes the rest of the note. The key step consists of proving a special property of the semiorthogonal decomposition described above that allows to easily check the orthogonality properties for the bundle $F_\ast L_{\rho}$. This is done in Section 3. Theorem 4.1, which is equivalent to isomorphism (1), immediately follows from it.

The present note was initially motivated by the author’s computations of Frobenius pushforwards of homogeneous vector bundles on flag varieties. The derived localization theorem of [2] implies, in particular, that for a regular weight $\chi$ (that is, for a weight having trivial stabilizer with respect to the dot–action of the (affine) Weyl group) the bundle $F_\ast L_{\chi}$ is a generator in the derived category $D^b(G/B)$; in other words, there are sufficiently many indecomposable summands of $F_\ast L_{\chi}$ to generate the whole derived category $D^b(G/B)$. Knowing indecomposable summands of these bundles (e.g., for $p$–restricted weights) may clarify, in particular, cohomology vanishing patterns of line bundles on $G/B$. On the contrary, the weight $-\rho$ being the most singular, the thick category generated by the bundle $F_\ast L_{-\rho}$ ”collapses” to the subcategory generated by the single line bundle $L_{-\rho}$, which is encoded in isomorphism (1).

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Notation. Given a split semisimple simply connected algebraic group $G$ over a perfect field $k$, let $T$ denote a maximal torus of $G$, and let $T \subset B$ be a Borel subgroup containing $T$. The flag variety of Borel subgroups in $G$ is denoted $G/B$. Denote $X(T)$ the weight lattice, and let $R$ and $R^\vee$ denote the root and coroot lattices, respectively. The Weyl group $W = N(T)/T$ acts on $X(T)$ via the dot–action: if $w \in W$, and $\lambda \in X(T)$, then $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho$ is the sum of fundamental weights. Let $S$ be the set of simple roots relative to the choice of a Borel subgroup than contains $T$. A parabolic subgroup of $G$ is usually denoted by $P$; in particular, for a simple root $\alpha \in S$, denote $P_\alpha$ the minimal parabolic subgroup of $G$ associated to $\alpha$. Given a weight $\lambda \in X(T)$, denote $L_{\lambda}$ the corresponding line bundle on $G/B$. Given a morphism $f : X \to Y$ between two schemes, we write $f_*, f^*$ for the corresponding derived functors of push–forwards and pull–backs.

2. Some preliminaries

2.1. Flag varieties of Chevalley groups over $\mathbb{Z}$. Let $G \to \mathbb{Z}$ be a semisimple Chevalley group scheme (a smooth affine group scheme over $\text{Spec}(\mathbb{Z})$ whose geometric fibres are connected semisimple algebraic groups), and $G/B \to \mathbb{Z}$ be the corresponding Chevalley flag scheme (resp., the corresponding parabolic subgroup scheme $G/P \to \mathbb{Z}$ for a standard parabolic subgroup scheme $P \subset G$ over $\mathbb{Z}$). Then $G/P \to \text{Spec}(\mathbb{Z})$ is flat and any line bundle $L$ on $G/P$ also comes from a line bundle
2.2. Cohomology of line bundles on flag varieties. We recall first the classical Bott’s theorem (see [5]). Let $G \to Z$ be a semisimple Chevalley group scheme as above. Assume given a weight $\chi \in X(T)$, and let $L_\chi$ be the corresponding line bundle on $G/B$. The weight $\chi$ is called singular, if it lies on a wall of some Weyl chamber defined by $\langle - , \alpha^\vee \rangle = 0$ for some coroot $\alpha^\vee \in R^\vee$. Weights, which are not singular, are called regular. A weight $\chi$ such that $\langle \chi, \alpha^\vee \rangle \geq 0$ for all simple coroots $\alpha^\vee$ is called dominant. Let $k$ be a field of characteristic zero, and $G/B \to \text{Spec}(k)$ the corresponding flag variety over $k$. The weight $\chi \in X(T)$ defines a line bundle $L_\chi$ on $G/B$.

Theorem 2.1. [5] Theorem 2]

(a) If $\chi + \rho$ is singular, then $H^i(G/B, L_\chi) = 0$ for all $i$.

(b) If $\chi + \rho$ is regular and dominant, then $H^i(G/B, L_\chi) = 0$ for $i > 0$.

(c) If $\chi + \rho$ is regular, then $H^i(G/B, L_\chi) \neq 0$ for the unique degree $i$, which is equal to $l(w)$. Here $l(w)$ is the length of an element $w$ of the Weyl group that takes $\chi$ to the dominant chamber, i.e. $w \cdot \chi \in X_+(T)$. The cohomology group $H^{l(w)}(G/B, L_\chi)$ is the irreducible $G$–module of highest weight $w \cdot \chi$.

Remark 2.1. Some bits of Theorem 2.1 are still true over $\mathbb{Z}$: if a weight $\chi$ is such that $\langle \chi + \rho, \alpha^\vee \rangle = 0$ for some simple root $\alpha$, then the corresponding line bundle is acyclic. Indeed, Lemma from [5] Section 2] holds over fields of arbitrary characteristic. Besides this, however, very little of Theorem 2.1 holds over $\mathbb{Z}$ (see [8] Part II, Chapter 5)).

From now on, unless specified otherwise, the base field $k$ is assumed to be a perfect field of characteristic $p > 0$.

2.3. Kempf’s vanishing theorem. Kempf’s vanishing theorem, originally proven by Kempf in [9], and subsequently by Andersen [1] and Haboush [6] with shorter representation–theoretic proofs (see also [8] Part II, Chapter 4]), states that given a dominant weight $\chi \in X(T)$, the cohomology groups $H^i(G/B, L_\chi)$ vanish in positive degrees, i.e. $H^i(G/B, L_\chi) = 0$ for $i > 0$. This theorem is ubiquitous in representation theory of algebraic groups in characteristic $p$. For convenience of the reader, we briefly recall how it can be obtained from the main isomorphism $F_q^n L_{q^i-1} = St_q \otimes O_{G/B}$ (recall that $q = p^n$ for $n \in \mathbb{N}$). From $\langle \chi, \alpha^\vee \rangle \geq 0$ one obtains $\langle \chi + \rho, \alpha^\vee \rangle > 0$ for all simple coroots $\alpha^\vee$. By [8] Part II, Proposition 4.4], the line bundle $L_{\chi + \rho}$ is ample on $G/B$. Consider the weight $q(\chi + \rho) - \rho = q\chi + (q - 1)\rho$. Since $L_{\chi + \rho}$ is ample, one can choose $n \in \mathbb{N}$ large enough so that the line bundle $L_{q(\chi + \rho)}$ be very ample. From the well–known properties of the Frobenius morphism it then follows

$$H^i(G/B, L_{q(\chi + (q - 1)\rho)}) = H^i(G/B, L_{\chi} \otimes F_q^n L_{(q - 1)\rho}) = H^i(G/B, L_{\chi}) \otimes St_q.$$
Now the left hand side group vanishes for \(i > 0\) by Serre’s vanishing, the line bundle \(\mathcal{L}_{q(\chi + \rho)}\) being very ample. Hence, \(H^i(G/B, \mathcal{L}_\chi) = 0\) for \(i > 0\) as well.

2.4. Derived categories of coherent sheaves. The content of this section can be found, e.g., in [7, Section 1.2, 1.4].

Let \(k\) be a field. Assume given a \(k\)-linear triangulated category \(D\), equipped with a shift functor \([1]\): \(D \to D\). For two objects \(A, B \in D\) let \(\text{Hom}_D^\bullet(A, B)\) be the graded \(k\)-vector space \(\oplus_{i \in \mathbb{Z}} \text{Hom}_D(A, B[i])\). Let \(A \subset D\) be a full triangulated subcategory, that is a full subcategory of \(D\) which is closed under shifts and forming distinguished triangles.

**Definition 2.1.** The right orthogonal \(A^\perp \subset D\) is defined to be the full subcategory

\[
A^\perp = \{ B \in D : \text{Hom}_D(A, B) = 0 \}
\]

for all \(A \in A\). The left orthogonal \(\perp A\) is defined similarly.

**Definition 2.2.** A full triangulated subcategory \(A\) of \(D\) is called right admissible if the inclusion functor \(A \hookrightarrow D\) has a right adjoint. Similarly, \(A\) is called left admissible if the inclusion functor has a left adjoint. Finally, \(A\) is admissible if it is both right and left admissible.

If a full triangulated category \(A \subset D\) is right admissible then every object \(X \in D\) fits into a distinguished triangle

\[
\cdots \to Y \to X \to Z \to Y[1] \to \cdots
\]

with \(Y \in A\) and \(Z \in A^\perp\). One then says that there is a semiorthogonal decomposition of \(D\) into the subcategories \((A^\perp, A)\). More generally, assume given a sequence of full triangulated subcategories \(A_1, \ldots, A_n \subset D\). Denote \((A_1, \ldots, A_n)\) the triangulated subcategory of \(D\) generated by \(A_1, \ldots, A_n\).

**Definition 2.3.** A sequence \((A_1, \ldots, A_n)\) of admissible subcategories of \(D\) is called semiorthogonal if \(A_i \subset A_j^\perp\) for \(1 \leq i < j \leq n\), and \(A_i \subset A_j^\perp\) for \(1 \leq j < i \leq n\). The sequence \((A_1, \ldots, A_n)\) is called a semiorthogonal decomposition of \(D\) if \(\langle A_1, \ldots, A_n \rangle^\perp = 0\), that is \(D = \langle A_1, \ldots, A_n \rangle\).

**Lemma 2.1.** For a semi–orthogonal decomposition \(D = \langle A, B \rangle\), the subcategory \(A\) is left admissible and the subcategory \(B\) is right admissible. Conversely, if \(A \subset D\) is left (resp. right) admissible, then there is a semi–orthogonal decomposition \(D = \langle A, B \rangle\) (resp. \(D = \langle A, A \rangle\)).

**Definition 2.4.** An object \(E \in D\) of a \(k\)-linear triangulated category \(D\) is said to be exceptional if there is an isomorphism of graded \(k\)-algebras

\[
\text{Hom}_D^\bullet(E, E) = k.
\]

A collection of exceptional objects \((E_0, \ldots, E_n)\) in \(D\) is called exceptional if for \(1 \leq i < j \leq n\) one has
Denote by \( \langle E_0, \ldots, E_n \rangle \subset D \) the full triangulated subcategory generated by the exceptional objects \( E_0, \ldots, E_n \). One proves [7] Lemma 1.58 that such a category is admissible.

Given a smooth algebraic variety \( X \) over a field \( k \), denote \( D^b(X) \) the bounded derived category of coherent sheaves, and let \( D(\text{QCoh}(X)) \) denote the unbounded derived category of quasi-coherent sheaves. These are \( k \)-linear triangulated categories. Let \( \mathcal{E} \) be a vector bundle of rank \( r \) on \( X \), and consider the associated projective bundle \( \pi : \mathbb{P}(\mathcal{E}) \to X \). Denote \( \mathcal{O}_\pi(-1) \) the line bundle on \( \mathbb{P}(\mathcal{E}) \) of relative degree \(-1\), such that \( \pi_*\mathcal{O}_\pi(1) = \mathcal{E}^* \). One has [7] Corollary 8.36:

**Theorem 2.2.** The category \( D^b(\mathbb{P}(\mathcal{E})) \) has a semiorthogonal decomposition:

\[
D^b(\mathbb{P}(\mathcal{E})) = \langle \pi^* D^b(X) \otimes \mathcal{O}_\pi(-r+1), \ldots, \pi^* D^b(X) \otimes \mathcal{O}_\pi(-1), \pi^* D^b(X) \rangle.
\]

We also need some basic facts about generators in triangulated categories (see [10]).

**Definition 2.5.** Let \( D \) be a \( k \)-linear triangulated category. An object \( C \) of \( D \) is called compact if for any coproduct of objects one has \( \text{Hom}_D(C, \coprod_{\lambda \in \Lambda} X_\lambda) = \prod_{\lambda \in \Lambda} \text{Hom}_D(C, X_\lambda) \).

**Definition 2.6.** A \( k \)-linear triangulated category \( D \) is called compactly generated if \( D \) contains small coproducts, and there exists a small set \( \mathcal{T} \) of compact objects of \( D \), such that \( \text{Hom}_D(\mathcal{T}, X) = 0 \) implies \( X = 0 \). In other words, if \( X \) is an object of \( D \), and for every \( T \in \mathcal{T} \) one has \( \text{Hom}_D(T, X) = 0 \), then \( X \) must be the zero object.

**Definition 2.7.** Let \( D \) be a compactly generated triangulated category. A set \( \mathcal{T} \) of compact objects of \( D \) is called a generating set if \( \text{Hom}_D(\mathcal{T}, X) = 0 \) implies \( X = 0 \) and \( \mathcal{T} \) is closed under the shift functor, i.e. \( \mathcal{T} = \mathcal{T}[1] \).

**Definition 2.8.** Let \( X \) be a quasi-compact, separated scheme. An object \( C \in D(\text{QCoh}(X)) \) is called perfect if, locally on \( X \), it is isomorphic to a bounded complex of locally free sheaves of finite type.

**Proposition 2.1.** [10] Example 1.10) Let \( X \) be a quasi-compact, separated scheme, and \( \mathcal{L} \) be an ample line bundle on \( X \). Then the set \( \langle \mathcal{L}^\otimes m[n] \rangle, m, n \in \mathbb{Z} \) is a generating set for \( D(\text{QCoh}(X)) \).

Finally, recall that given two smooth varieties \( X \) and \( Y \) over \( k \), an object \( \mathcal{P} \in D^b(X \times Y) \) gives rise to an integral transform \( \Phi_{\mathcal{P}}(-) := \pi_Y^* (\pi_X^*(-) \otimes \mathcal{P}) \) between \( D^b(X) \) and \( D^b(Y) \), where \( \pi_X, \pi_Y \) are the projections of \( X \times Y \) onto corresponding factors.

**Proposition 2.2.** [7] Proposition 5.1) Let \( X, Y, \) and \( Z \) be smooth projective varieties over a field \( k \). Consider objects \( \mathcal{P} \in D^b(X \times Y) \) and \( \mathcal{Q} \in D^b(Y \times Z) \). Define the object \( \mathcal{R} \in D^b(X \times Z) \) by the formula \( \pi_{XZ}^* (\pi_{XY}^* \mathcal{P} \otimes \pi_{YZ}^* \mathcal{Q}) \), where \( \pi_{XZ}, \pi_{XY}, \) and \( \pi_{YZ} \) are the projections from \( X \times Y \times Z \) to \( X \times Z \) (resp., to \( X \times Y \), resp., to \( Y \times Z \)). Then the composition \( \Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} : D^b(X) \to D^b(Z) \) is isomorphic to the integral transform \( \Phi_{\mathcal{R}} \).
3. Semiorthogonal decompositions for flag varieties

In order to prove Lemma 3.1 below, the key statement of this section, we need an auxiliary proposition which is a derived category counterpart of the main theorem of [1].

**Proposition 3.1.** Let \( \pi : G/B \to \text{Spec}(k) \) be the structure morphism, and for a simple root \( \alpha_i \) denote \( \pi_{\alpha_i} : G/B \to G/P_{\alpha_i} \), the projection, a \( \mathbb{P}^1 \)-bundle over \( G/P_{\alpha_i} \). Let \( w_0 \) be the longest element of \( \mathcal{W} \), and let \( s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N} \) be a reduced expression of \( w_0 \). Then there is an isomorphism of functors:

\[
\pi^* \pi_* = \pi_{\alpha_N}^* \pi_{\alpha_{N-1}}^* \pi_{\alpha_{N-2}}^* \cdots \pi_{\alpha_1}^* \pi_0^*.
\]  

**Proof.** Denote \( Z \) the fibered product \( G/B \times_{G/P_{\alpha_1}} G/B \times \cdots \times_{G/P_{\alpha_N}} G/B \), and let \( p : Z \to G/B \times G/B \) denote the projection onto the two extreme factors. Then, by Proposition 2.2, the functor in the right hand side of (8) is given by an integral transform whose kernel is isomorphic to the direct image \( p_* O_Z \in D^b(G/B \times G/B) \). Observe that \( Z \) is isomorphic to \( G \times B Z_{w_0} \), where \( Z_{w_0} := P_{\alpha_1} \times \cdots \times P_{\alpha_N}/B^N \) is the Demazure variety corresponding to the reduced expression of \( w_0 \) as above. Indeed, by definition of these varieties [3, Definition 2.2.1] and [Diagram (D), p.66] of loc. cit., one has an isomorphism \( (G \times B Z_{w_0 s_{\alpha_N}}) \times G/P_{\alpha_N} G/B = G \times B (Z_{w_0 s_{\alpha_N} s_{\alpha_N}} G/B) = G \times B Z_{w_0} \).

The projection \( p \) maps \( Z = G \times B Z_{w_0} \) onto \( G/B \times G/B \). Consider the base change of \( p \) along the quotient morphism \( q : G \times B \to G/B \times G/B \). One obtains the projection \( q' : G \times Z_{w_0} \to G \times G/B \) that factors as \( q = q' \circ p \). Applying \( q_* \) to \( q'^* p_* O_Z \) and using the projection formula, one arrives at an isomorphism \( q_* (q'^* p_* O_Z) = p_* O_Z \otimes q_* O_{G \times G/B} = q_* O_{G \times G/B} \). It follows that \( p_* O_Z \) is an invertible sheaf on \( G/B \times G/B \) isomorphic to \( L \otimes O_{G/B} \) for some line bundle \( L \) on \( G/B \). Applying the integral transform \( \Phi_{w_0} = \Phi_{w_0}^{-1} = \Phi_{p_* O_Z} \) to \( O_{G/B} \), one obtains \( \Phi_{w_0}^{-1}(O_{G/B}) = \pi_{\alpha_1}^* \pi_{\alpha_2}^* \pi_{\alpha_3}^* \cdots \pi_{\alpha_N}^* \pi_0^*(O_{G/B}) = O_{G/B} = \Phi_{O_{G/B} \otimes L}(O_{G/B}) = \pi_* O_{G/B} \otimes L = \mathbb{H}(G/B, O_{G/B}) \otimes L = L \), where the last isomorphism follows from Corollary 3.1 below. Therefore, \( p_* O_Z = O_{G/B \times G/B} \). Finally, by flat base change for the morphism \( \pi : G/B \to \text{Spec}(k) \) along itself, the integral transform \( \Phi_{O_{G/B \times G/B}} \) is isomorphic to \( \pi^* \pi_* \). \( \square \)

**Proposition 3.2.** Let \( d : Z_{w_0} \to G/B \) be the projection map as above. Then \( d_* O_{Z_{w_0}} = O_{G/B} \).

**Proof.** Given an element \( w \in \mathcal{W} \), denote the corresponding Schubert variety by \( X_w \), and let \( d_w : Z_w = P_{\alpha_1} \times \cdots \times P_{\alpha_n}/B^n \to X_w \) denote the Demazure desingularization. Then \( d = d_{w_0} : Z_{w_0} \to G/B \) is a birational morphism onto \( G/B \), since \( X_{w_0} = G/B \). The flag variety being smooth, hence normal, by Zariski’s main theorem one has \( R^0 d_* O_{Z_{w_0}} = O_{G/B} \). To prove the vanishing of higher direct images \( R^i d_* O_{Z_{w_0}} \) for \( i > 0 \), one can argue as in the [3, Theorem 3.3.4, (b)]. More specifically, one argues by induction on the length \( l(w) \) of an element \( w \in \mathcal{W} \) to prove that \( R^i d_{w-*} O_{Z_w} = 0 \)
for \( i \geq 1 \); if \( l(w) = 1 \) then \( d_w \) is an isomorphism. Given a reduced expression \( s_{\alpha_1} \ldots s_{\alpha_n} \) of \( w \), set \( v = s_{\alpha_2} \ldots s_{\alpha_n} \), and consider the factorization of \( d_w \) as \( d_{\alpha_1} \circ Z_w = P_{\alpha_1} \times B Z_v \rightarrow P_{\alpha_1} \times B X_v \) followed by the product morphism \( f : P_{\alpha_1} \times B X_v \rightarrow X_w \). Then, by induction one obtains \( R^i d_{\alpha_1} \circ Z_{w} = 0 \) for \( i \geq 1 \) which implies \( R^i d_{\alpha_1} \circ Z_{w} = 0 \) for \( i \geq 1 \) as well. Finally, \( [\mathfrak{H} \text{ Proposition 3.2.1}, (b)] \) implies that for the product morphism \( f : P \times B X_v \rightarrow P X_v \), where \( P \) is the minimal parabolic subgroup corresponding to a simple reflection, the higher direct images \( R^i f_* \mathcal{O}_{P \times B X_v} \) are trivial for \( i \geq 1 \). Hence, for the composed morphism \( d_w \) the higher direct images \( R^i d_w \circ \mathcal{O}_{Z_w} = 0 \) are trivial for \( i \geq 1 \).

**Corollary 3.1.** One has \( H^i(G/B, \mathcal{O}_{G/B}) = 0 \) for \( i > 0 \). Given a line bundle \( \mathcal{L} \) on \( G/B \), the triangulated subcategory \( \langle \mathcal{L} \rangle \) of \( D^b(G/B) \) generated by \( \mathcal{L} \) is admissible.

**Proof.** By Proposition \[3.2\] one has \( d_* \mathcal{O}_{X_{w_0}} = \mathcal{O}_{G/B} \). On the other hand, by its construction, the variety \( X_{w_0} \) is isomorphic to an iterated sequence of \( \mathbb{P}^1 \)–bundles over a point; hence, \( H^i(X_{w_0}, \mathcal{O}_{X_{w_0}}) = 0 \) for \( i > 0 \). Therefore, \( H^i(G/B, \mathcal{O}_{G/B}) = H^i(G/B, d_* \mathcal{O}_X) = H^i(X_{w_0}, \mathcal{O}_{X_{w_0}}) = 0 \) for \( i > 0 \). It follows from Section \[2.4\] that the category \( \langle \mathcal{L} \rangle \) is admissible once the bundle \( \mathcal{L} \) is exceptional, i.e. \( \text{Hom}^*_{G/B}(\mathcal{L}, \mathcal{L}) = k \). The latter condition is equivalent to \( H^i(G/B, \mathcal{O}_{G/B}) = 0 \) for \( i > 0 \).

**Lemma 3.1.** Consider the semiorthogonal decomposition of \( D^b(G/B) = \langle \mathcal{L}_{\chi} \rangle \). Then the subcategory \( \langle \mathcal{O}_{G/B} \rangle \) is generated, as an admissible triangulated subcategory of \( D^b(G/B) \), by acyclic line bundles \( \mathcal{L}_{\chi} \) with the following property: there exists a simple coroot \( \alpha^\vee \in R^\vee \), such that \( \langle \chi + \rho, \alpha^\vee \rangle = 0 \).

**Remark 3.1.** The generating set of the subcategory \( \langle \mathcal{O}_{G/B} \rangle \) in Lemma \[3.1\] is not at all minimal.

**Proof.** By Corollary \[3.1\] the category \( \langle \mathcal{O}_{G/B} \rangle \) is admissible, hence its right orthogonal is an admissible subcategory of \( D^b(G/B) \).

Given a simple root \( \alpha \), consider the corresponding minimal parabolic subgroup \( P_\alpha \), and let \( \pi_\alpha : G/B \rightarrow G/P_\alpha \) denote the projection. Observe first that \( \langle \mathcal{O}_{G/B} \rangle \) contains the subcategory generated by \( \pi_\alpha^* \mathcal{F} \otimes \mathcal{L}_{-\rho} \), where \( \mathcal{F} \in D^b(G/B) \). Indeed,

\[
\text{Hom}_{G/B}^*(\mathcal{O}_{G/B}, \pi_\alpha^* \mathcal{F} \otimes \mathcal{L}_{-\rho}) = H^*(G/P_\alpha, \pi_\alpha^* \mathcal{F} \otimes \mathcal{L}_{-\rho}) = 0,
\]

as \( \pi_\alpha^* \mathcal{L}_{-\rho} = 0 \). Let \( \mathcal{C} \subset D^b(G/B) \) be the full triangulated category generated by \( \pi_\alpha^* \mathcal{F} \otimes \mathcal{L}_{-\rho} \), where \( \mathcal{F} \in D^b(G/P_\alpha) \) and \( \alpha \) runs over the set of all the simple roots. Observe next that \( \mathcal{C} \) coincides with the triangulated subcategory generated by line bundles satisfying the condition of Lemma \[3.1\] one the one hand, given a simple root \( \alpha \), any line bundle \( \mathcal{L}_{\chi} \) on \( G/P_\alpha \) satisfies \( \langle \chi, \alpha^\vee \rangle = 0 \), and the projection functor \( \pi_\alpha^* : D^b(G/B) \rightarrow D^b(G/P_\alpha) \) is surjective. Hence, \( \mathcal{C} \supset \langle \mathcal{L}_{\chi} \rangle \) with \( \langle \chi + \rho, \alpha^\vee \rangle = 0 \) for a simple coroot \( \alpha^\vee \). On the other hand, upon choosing an ample line bundle \( \mathcal{L} \) on \( G/P_\alpha \), the category \( D^b(G/P_\alpha) \) is generated by the set \( \langle \mathcal{L}^\otimes m[n] \rangle, m, n \in \mathbb{Z} \) in virtue of Proposition \[2.1\] and one obtains a converse inclusion \( \mathcal{C} \subset \langle \mathcal{L}_{\chi} \rangle \) with \( \chi \) as in the statement of the lemma.
Consider its left orthogonal $\perp C \subset D^b(G/B)$. By Lemma 2.1, it is an admissible subcategory of $D^b(G/B)$. The same lemma implies that the statement of Lemma 3.1 is equivalent to saying that the category $\perp C$ is equivalent to $\langle O_{G/B} \rangle$. To this end, observe that any object $G$ of $\perp C$ belongs to $\pi^*_\alpha D^b(G/P_\alpha)$ for each simple root $\alpha$; in other words, $\perp C \subset \bigcap_\alpha \pi^*_\alpha D^b(G/P_\alpha)$. Indeed, by Serre duality

$$\text{Hom}^\bullet_{G/B}(G, \pi^*_\alpha \pi_{\alpha*} F \otimes L_{-\rho}) = \text{Hom}^\bullet_{G/B}(\pi_{\alpha*} F, \pi_{\alpha*}(G \otimes L_{-\rho}[\dim(G/B)])^*) = 0;$$

since $F$ is arbitrary and functor $\pi_{\alpha*}$ is surjective, it follows that $\pi_{\alpha*}(G \otimes L_{-\rho}) = 0$. By Theorem 2.2 this implies $G \in \pi^*_\alpha D^b(G/P_\alpha)$, the line bundle $L_{-\rho}$ having degree $-1$ along $\pi_\alpha$.

Let $M$ be an object $\perp C$. By the above, $M = \pi^*_\alpha \pi_{\alpha*} M$ for any simple root $\alpha$. By Proposition 3.1 one has an isomorphism $\pi^* \pi_{\alpha*} M = \pi^* \pi_{\alpha*} \pi_{\alpha N} \pi_{\alpha_{N-1}} \pi_{\alpha_{N-1}} \ldots \pi_{\alpha_1} \pi_{\alpha_1} M = M$, hence $M \in \langle O_{G/B} \rangle$.

**Corollary 3.2.** Consider the semiorthogonal decomposition of $D^b(G/B) = \langle \langle L_{-\rho} \rangle, \perp \langle L_{-\rho} \rangle \rangle$. Then the category $\perp \langle L_{-\rho} \rangle$ is generated, as an admissible triangulated subcategory of $D^b(G/B)$, by the set of line bundles $L_\chi$, where $\langle \chi, \alpha^\vee \rangle = 0$ for some simple coroot $\alpha^\vee \in \mathbb{R}^\vee$.

**Proof.** Let $E \in \perp \langle L_{-\rho} \rangle$, then by Serre duality

$$\text{Hom}^\bullet_{G/B}(E, L_{-\rho}) = \text{Hom}^\bullet_{G/B}(L_{-\rho}, E \otimes L_{-2\rho}[\dim(G/B)])^* = \text{Hom}^\bullet_{G/B}(O_{G/B}, E \otimes L_{-\rho}[\dim(G/B)])^* = 0.$$

Therefore, up to a twist by the line bundle $L_\rho$, the category $\perp \langle L_{-\rho} \rangle$ is equivalent to the subcategory $\langle O_{G/B} \rangle$. Lemma 3.1 implies the statement.

4. The Steinberg line bundle

Consider the admissible subcategory $\langle L_{-\rho} \rangle$ of $D^b(G/B)$. It follows from the above that the isomorphism $F^*_u L_{-\rho} = St_q \otimes L_{-\rho}$ is equivalent to the following statement:

**Theorem 4.1.** One has $F^*_u L_{-\rho} \in \langle L_{-\rho} \rangle$.

**Proof.** By Corollary 3.2 the fact that the bundle $F^*_u L_{-\rho}$ belongs to the subcategory $\langle L_{-\rho} \rangle \subset D^b(G/B)$ is equivalent to saying that $F^*_u L_{-\rho}$ is right orthogonal to the subcategory $\langle L_\chi \rangle$ generated by all $L_\chi$, where $\langle \chi, \alpha^\vee \rangle = 0$ for some simple coroot $\alpha^\vee \in \mathbb{R}^\vee$. In other words, one has to ensure that

$$\text{Hom}^\bullet_{G/B}(L_\chi, F^*_u L_{-\rho}) = \mathbb{H}^*(G/B, L_{-p^\alpha \chi - \rho}) = 0.$$

By Remark 2.1 the line bundle $L_\mu$ is acyclic if $\langle \mu + \rho, \alpha^\vee \rangle = 0$ for some simple coroot $\alpha^\vee$. Taking $\mu = -p^\alpha \chi - \rho$, one obtains $\langle \mu + \rho, \alpha^\vee \rangle = \langle -p^\alpha \chi - \rho + \rho, \alpha^\vee \rangle = -p^\alpha \langle \chi, \alpha^\vee \rangle = 0$. Hence, the bundle $L_{-p^\alpha \chi - \rho}$ is acyclic, and (12) holds. \qed
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Math. Institut, Heinrich-Heine-Universität, D-40204 Düsseldorf, Germany

and

Institute for Information Transmission Problems, Moscow, Russia

E-mail address: alexander.samokhin@gmail.com