Abstract

From the realization of $q$–oscillator algebra in terms of generalized derivative, we compute the matrix elements from deformed exponential functions and deduce generating functions associated with Rogers-Szegő polynomials as well as their relevant properties. We also compute the matrix elements associated to the $(p, q)$–oscillator algebra (a generalization of the $q$–one) and perform the Fourier-Gauss transform of a generalization of the deformed exponential functions.

Key-words: Deformed algebra, matrix elements, Fourier-Gauss transform.

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1 Introduction

The deformation of quantum algebras, and consequently quantum $q$-deformations of Lie algebras and related properties still continue to be of great relevance in mathematics and physics due to their important applications in the quantum field theories and quantum groups [1].

Lie group theory and representations spawned new ideas and results providing a unifying framework for discussing special functions. The latter appear as solutions of differential equations describing specific physical problems and satisfy suitable properties such as orthogonality giving rise to generalized Fourier analysis. Specifically, the development of the theory of group representations has made it possible to comprehend the theory of the most important classes of special functions from a single point of view. Indeed, the appraisal of the importance of individual classes of special functions has greatly changed during the last hundred years. In particular, the class of special functions associated with the hypergeometric function and its various special and degenerate cases, the functions of Bessel and Legendre, the orthogonal polynomials of Jacobi, Tchebychev, Laguerre, Hermite, etc. which play a big role in different branches of mathematics and its numerous applications to astronomy and mathematical physics, lends itself to a group-theoretical treatment.

The connection between special functions and group representation was first discovered by Cartan [2] in 1929. However, a connection between the theory of special functions and the theory of invariants, which is one of the aspects of the theory of group representations, was established even earlier. The application of the theory of representations to quantum mechanics played a significant part in the investigation of these connections. Further development in this field was stimulated by the works of Gel’fand and Naimark and their students and collaborators in the field of infinite-dimensional group representations. They linked the theory of group representations with the automorphic functions, and developed the theory of special functions over finite fields, investigated the special functions in homogeneous domains, and so on. A more detailed statement of the above, including a systematic account of the theory of special functions from the group theoretical point of view, is found in form of a nice compilation in the book by Vilenkin [3], the most used classics in the field.

In the same vein, the discovery of quantum groups has in turn prompted the undertaking of a systematic investigation of the algebraic properties of the $q-$analogs of special functions. The matrix elements in representation spaces, yielding relevant properties of $q-$special functions, are defined from deformed exponential functions of the generators associated with a given deformed algebra. Intense research activities in such an area as $q$-special functions are mainly motivated by their importance in quantum theory. This work highlights some relevant properties of Rogers-Szegő polynomials from $(q, \mu)$—
exponential functions and provides with matrix elements and Fourier-Gauss transform of \((p,q,\mu,\nu)\)-exponential functions.

The paper is organized as follows. In section 2, we give some well-known connection between quantum algebra representations and \(q\)-polynomials. In section 3, we compute the matrix elements of a known deformed exponentials related to the generators of standard \(q\)-oscillator \([4]\), which we use to deduce some properties of Rogers-Szegő polynomials. We then provide a generalization of the introduced deformed exponential functions and study the associated matrix elements and the Fourier-Gauss transform in section 4.

2 Overview of known results

In this section, let us briefly recall the most popular matrix elements issued from irreducible quantum algebra representations generating \(q\)-polynomials.

(1) Little \(q\)-Jacobi and \(q\)-Legendre polynomials. They arise from the noncommutative algebra \(U_q(sl_2(\mathbb{C}))\) generated by \(I\) and \(X_{\pm}\), \(q^{\pm \frac{H}{2}}\) with the relations \([5]\)

\[
q^{\frac{H}{2}} X_{\pm} q^{-\frac{H}{2}} = q^{\pm} X_{\pm} \quad [X_{+}, X_{-}] = \frac{q^{H} - q^{-H}}{q - q^{-1}} \tag{2.1}
\]

and \(V_\epsilon(\lambda)\), the irreducible highest weight \(U_\epsilon(sl_2(\mathbb{C}))\)-module with highest weight \(\lambda \in \mathbb{N}\). Indeed, as quoted in \([6]\), the matrix elements \(C^\lambda_{\nu,\mu}\) of the irreducible \(U_\epsilon(sl_2(\mathbb{C}))\)-module \(V_\epsilon(\lambda)\), with respect to an orthonormal basis are related to little \(q\)-Jacobi polynomials

\[
p_n(z; \alpha, \beta; q) = \binom{q^{-n}, q^{n+1}\alpha\beta}{\alpha q} \binom{q, qz}{q^2} \tag{2.2}
\]

and defined by the following statement.

**Proposition 2.1** For different values of \(\mu\) and \(\nu\), the matrix elements \(C^\lambda_{\nu,\mu}\) of the irreducible \(U_\epsilon(sl_2(\mathbb{C}))\)-module of highest weight \(\lambda\) are given as follows:

(i) if \(\mu + \nu \geq 0\) and \(\mu \geq \nu\),

\[
C^\lambda_{\nu,\mu} = a^{(\mu+\nu)/2}b^{(\mu-\nu)/2}c^{(\lambda-\mu)(\nu-\mu)/4} \left[ \begin{array}{c} \frac{1}{2}(\lambda - \nu) \\ \frac{1}{2}(\mu - \nu) \end{array} \right] q^{\mu+\nu} \left[ \begin{array}{c} \frac{1}{2}(\lambda + \mu) \\ \frac{1}{2}(\mu - \nu) \end{array} \right] q^2 \right]^{1/2} \times \left[ \begin{array}{c} \frac{1}{2}(\lambda + \mu) \\ \frac{1}{2}(\mu - \nu) \end{array} \right] q^2 \right]^{1/2} \times \left[ \begin{array}{c} \frac{1}{2}(\lambda - \nu) \\ -q^{-1}bc; q^{\mu-\nu}, q^{\mu+\nu}; q^2 \end{array} \right] \tag{2.3}
\]
(ii) if \( \mu + \nu \geq 0 \) and \( \mu \leq \nu \),

\[
C_{\nu,\mu}^\lambda = a^{(\mu+\nu)/2} b^{(\nu-\mu)/2} q^{(\lambda-\nu)(\mu-\nu)/4} \left( \left[ \frac{1}{2} (\lambda + \mu) \right]_q \right)^{1/2} \\
\times \left( \left[ \frac{1}{2} (\lambda - \mu) \right]_q \right)^{1/2} q_{\frac{1}{2}(\lambda-\nu)} (-q^{-1} bc; q^{\nu-\mu}, q^{\mu+\nu}; q^{2}) \tag{2.4}
\]

(iii) if \( \mu + \nu \leq 0 \) and \( \mu \leq \nu \),

\[
C_{\nu,\mu}^\lambda = q^{-(\lambda+\mu)(\nu-\mu)/4} \left( \left[ \frac{1}{2} (\lambda + \nu) \right]_q \right)^{1/2} \left( \left[ \frac{1}{2} (\lambda - \mu) \right]_q \right)^{1/2} \\
\times q_{\frac{1}{2}(\lambda+\nu)} (-q^{-1} bc; q^{\mu-\nu}, q^{\mu-\nu}; q^{2}) b^{(\nu-\mu)/2} d^{(\mu+\nu)/2} \tag{2.5}
\]

(iv) if \( \mu + \nu \leq 0 \) and \( \mu \geq \nu \),

\[
C_{\nu,\mu}^\lambda = q^{-(\lambda+\nu)(\mu-\nu)/4} \left( \left[ \frac{1}{2} (\lambda - \nu) \right]_q \right)^{1/2} \left( \left[ \frac{1}{2} (\lambda + \mu) \right]_q \right)^{1/2} \\
\times q_{\frac{1}{2}(\lambda+\nu)} (-q^{-1} bc; q^{\mu-\nu}, q^{\mu-\nu}; q^{2}) b^{(\mu-\nu)/2} d^{(\mu+\nu)/2} \tag{2.6}
\]

where

\[
\begin{align*}
\left[ \begin{array}{c} m \\ n \end{array} \right]_q^P &= [n]_q^{-1} [m]_q^P ! \\
[n]_q^P &= [n]_q^P [n-1]_q^P \ldots [2]_q^P [1]_q^P \\
[n]_q^m &= q^n - q^{-n} \\
\end{align*}
\tag{2.7}
\]

(v) For \( \lambda \) even and \( \mu = \nu = 0 \). one obtains

\[
C_{0,0}^\lambda = p_{\lambda/2} (-q^{-1} bc; 1, 1; q^{2}) \tag{2.8}
\]

which is a little \( q \)-Legendre polynomial with argument \( -q^{-1} bc \).

The \( q \)-hypergeometric series \( r \phi_s \) are defined by

\[
\begin{align*}
\phi_s (a_1, a_2, \ldots, a_r) ; q, x) \\
&= \sum_{k=0}^{+\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(q; q)_k (b_1; q)_k \cdots (b_s; q)_k} (-1)^k q^{k(k-1)/2} \sum_{s-r} x^k \tag{2.10}
\end{align*}
\]

where \((a; q)_k = (1-a)(1-aq) \ldots (1-aq^{k-1})\).
(2) Hahn-Exton $q-$Bessel functions. They can be generated considering the quantized universal enveloping, $U_\epsilon(e_2)$, of $E(2)$ through the following statements.

**Theorem 2.2** Every irreducible unitarizable representation of $U_\epsilon(e_2)$ on which the $*$-structure $L$ acts semi-simply with finite-dimensional eigenspaces is equivalent to the representation $\rho_{\pm}^{\lambda,\mu}$ on the Hilbert space $l^2(\mathbb{Z})$ given by

\[
\begin{align*}
\rho_{\pm}^{\lambda,\mu}(L)(e_k) &= \pm \mu e_k \\
\rho_{\pm}^{\lambda,\mu}(X^+)(e_k) &= \lambda \mu e_{k+1} \\
\rho_{\pm}^{\lambda,\mu}(X^-)(e_k) &= \pm \lambda \mu^{-1} e_{k-1}
\end{align*}
\]  

(2.11)

in terms of the standard orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$ of $l^2(\mathbb{Z})$, where $L^\pm, X^\pm$ are the generators of $U_\epsilon(e_2)$; $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$ and $1 \leq |\mu| < \epsilon$.

The matrix elements of $\rho_+^{\lambda,1}$ and $\rho_+^{\lambda,\epsilon/2}$ involve the Hahn-Exton $q-$Bessel functions

\[
J_n(z; q) = z^n \frac{(q^{n+1}; q)_\infty}{(q; q)_\infty} \phi_1 \left( \frac{0}{q^{n+1}} ; q, qz^2 \right).
\]  

(2.12)

**Proposition 2.3** The following formulas give the matrix elements of the representations $\rho_+^{\lambda,1}$ and $\rho_+^{\lambda,\epsilon/2}$:

(i) for $j \geq k$,

\[
\begin{align*}
(\rho_+^{\lambda,1})_{jk} &= \left( \frac{\lambda(1-q^2)}{q^{k+1}} \right)^{j-k} \frac{q^{j+k}b_j^{j-k}}{(q^2; q^2)_{j-k}} \\
&\times_1 \phi_1 \left( \frac{0}{q^{2(j-k+1)}} ; q^2, -(1-q^2)^2 \lambda^2 q^{-2k-1}bc \right)
\end{align*}
\]

(ii) for $j \leq k$,

\[
\begin{align*}
(\rho_+^{\lambda,1})_{jk} &= \left( \frac{\lambda(1-q^2)}{q^{k+1}} \right)^{k-j} \frac{q^{j+k}b_j^{k-j}}{(q^2; q^2)_{k-j}} \\
&\times_1 \phi_1 \left( \frac{0}{q^{2(k-j+1)}} ; q^2, -(1-q^2)^2 \lambda^2 q^{-2j-1}bc \right)
\end{align*}
\]  

(2.13)
(ρ_+^{λe^{1/2}})_{jk} = \left( \frac{\lambda(1-q^2)}{q^{j+\frac{1}{2}}} \right)^{k-j} \frac{q^{j+k+1}c^{k-j}}{(q^2;q^2)_{k-j}^{}} \times_1 \phi_1 \left( \begin{array}{c} 0 \\ q^{2(k-j+1)} \\ q^{2} \end{array} ; q^2, -(1-q^2)^2\lambda^2q^{-2j-2bc} \right). (2.14)

(3) Big $q$–Jacobi polynomials. They can be built using the connection between big $q$–Jacobi polynomials and quantum spheres.

**Proposition 2.4** The quantized algebra $\mathcal{F}_e(S^2)$ of functions on the 2–spheres is isomorphic to the associative algebra over $\mathbb{C}$ with generators $x, y, z$ satisfying the relations

\begin{align*}
    xz &= \epsilon zx \quad yz = \epsilon^2zy \\
    xy &= -\epsilon^{-1}z(1-\epsilon^{-2}z) \quad yx = -\epsilon z(1-z)
\end{align*}

(2.15) (2.16)

and the $\ast$–structure

\begin{align*}
    x^\ast &= -\epsilon^{-1}y \quad y^\ast = -\epsilon x \quad z^\ast = z.
\end{align*}

(2.17)

The monomials $y^jz^kx^l$ with $j, k, l \in \mathbb{N}$, are basis of $\mathcal{F}_e(S^2)$ over $\mathbb{C}$. Moreover, $\mathcal{F}_e(S^2)$ is a quantum $sl_2(\mathbb{C})$–space.

The algebra of polynomial functions on the classical 2–spheres, as representation of $SO_3$, contains each irreducible representation exactly once [3]. The same result holds in the quantum case [7]:

**Proposition 2.5** As an $\mathcal{F}_e(S^2)$–comodule, $\mathcal{F}_e(S^2)$ decomposes into irreducibles as follows:

\[ \mathcal{F}_e(S^2) \cong \bigoplus_{\lambda \in 2\mathbb{N}} V_\lambda(\lambda) \]  

(2.18)

where $V_\lambda(\lambda)$ is the irreducible $\mathcal{F}_e(S^2)$–comodule of dimension $\lambda + 1$.

From this proposition, there exist a unique basis $\{ S^\lambda_\mu \}$ of $\mathcal{F}_e(S^2)$, where $\mu = \lambda, \lambda - 2, \ldots, -\lambda, \lambda \in 2\mathbb{N}$ such that $S^\lambda_0 = 1$ and

\[ \Delta_{\mathcal{F}_e}(S^\lambda_\mu) = \sum_\nu C^\lambda_{\mu,\nu} \otimes S^\lambda_\nu \]  

(2.19)

where $\Delta_{\mathcal{F}_e}$ is the comultiplication defined by

\[ \Delta_{\mathcal{F}_e} : \mathcal{F}_e(S^2) \rightarrow \mathcal{F}_e(S^2) \otimes \mathcal{F}_e(S^2). \]  

(2.20)
The $S^\lambda_{\mu}$ are $q$–analogue of spherical functions. They can be derived in terms of big $q$–Jacobi polynomials

$$P_n(z; \alpha, \beta; q) = \frac{3\phi_2}{\varphi_2(q^n; q, q)}.$$ (2.21)

by the following statement [9]

**Proposition 2.6** The $q$–spherical functions $S^\lambda_{\mu}$ are given by the following formulas:

(i) if $\mu \geq 0$,

$$S^\lambda_{\mu} = (-1)^{(\lambda-\mu)/2} q^{-(\lambda-\mu)(\lambda+3\mu+6)/8} \left[ \left[ \frac{\lambda}{2} (\lambda - \mu) \right]_{q^2}^P \right]^{-1/2}$$

$$\times \left[ \frac{\lambda}{2} (\lambda + \mu) \right]_{q^2}^P \times y^{\mu/2} P_{\frac{1}{2}(\lambda-\mu)}(z; q^\mu, q^\mu; q^2).$$ (2.22)

(ii) if $\mu \leq 0$,

$$S^\lambda_{\mu} = (-1)^{(\lambda+\mu)/2} q^{-(\lambda+\mu)(\lambda-3\mu+6)/8} \left[ \left[ \frac{\lambda}{2} (\lambda + \mu) \right]_{q^2}^P \right]^{-1/2}$$

$$\times \left[ \frac{\lambda}{2} (\lambda - \mu) \right]_{q^2}^P \times x^{\mu/2} P_{\frac{1}{2}(\lambda+\mu)}(z; q^{-\mu}, q^{-\mu}; q^2).$$ (2.23)

(4) $q$–analogs of Bessel functions. Floreanini [10] showed that the matrix elements of the two dimensional quantum Euclidean algebra $U_q(E(2))$ representations are given by

$$U_{k,n}(\alpha, \beta) = q^{(k-n)/2} \left( -\frac{\alpha}{\beta} \right)^{(k-n)/2} J^{(2)}_{k-n} \left( 2w(-\frac{\alpha\beta}{q})^{1/2}; q \right)^{1/2}$$ (2.24)

where $n, k \in \mathbb{Z}$ and $J^{(2)}_nu(x; q)$ are $q$–analogs of Bessel functions defined by

$$J^{(2)}_{nu}(x; q) = \sum_{n=0}^{\infty} q^{n(n+\nu)} \frac{(-1)^n}{(q;q)_n(q;q)_{n+\nu}} \left( \frac{x}{2} \right)^{2n+\nu}. (2.25)$$

The authors used this model to entail a $q$–analog of Graf’s addition formula for Bessel functions and $q$–analog of the Fourier-Gegenbauer expansion.

In the same vein, a series of papers [11]-[23] should be quoted for their relevance to $q$–orthogonal polynomials and $q$–special functions deduced from matrix elements or basis vectors of quantum algebra representations.

In the next section, we aim at using this formalism to investigate some properties of Rogers-Szegő polynomials which play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson polynomials [24, 25, 26].
3 \textit{\((q, \mu)\)-exponential functions: matrix elements}

The \(q\)--oscillator algebra is generated by three elements \(A_-, A_+\) and \(N\) obeying the relations \([4]\)

\[
\begin{align*}
   A_- A_+ - A_+ A_- &= q^N \quad A_- A_+ - q A_+ A_- = I \\
   [N, A_-] &= -A_- \quad [N, A_+] = A_+.
\end{align*}
\]

(3.1) \quad (3.2)

In general, the parameter \(q\) may be real or a phase factor. Throughout, we suppose that it is real and positive.

The algebra (3.1)-(3.2) admit a class of irreducible representations defined on the vector space spanned by the basis vectors \(\theta_n, n = 0, 1, 2, \ldots\) such that \([27]\)

\[
\begin{align*}
   A_+ \theta_n &= \theta_{n+1} \\
   A_- \theta_n &= \frac{1 - q^n}{1 - q} \theta_{n-1}.
\end{align*}
\]

(3.3) \quad (3.4)

One can easily verify that these definitions are compatible with the relations (3.1)-(3.2).

3.1 \textit{Matrix elements}

In order to make a link between the algebra (3.1)-(3.2) and \(q\)--polynomials, one can note that the representation (3.3)-(3.4) is given in terms of the Rogers-Szegő polynomials

\[
\theta_n(y) \equiv H_n(y|q) = \sum_{k=0}^{n} \binom{n}{k}_q y^k
\]

(3.5)

with the \(q\)-binomial coefficients given by

\[
\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.
\]

(3.6)

Indeed, by taking \(A_+, A_-\) to be the following operators, defined in terms of the \(q\)--Jackson derivative acting on the space of analytic functions

\[
\begin{align*}
   A_- f(y) &:= q D_y f(y) \\
   A_+ f(y) &:= (1 + y) f(y) - (1 - q) y_q D_y f(y)
\end{align*}
\]

(3.7) \quad (3.8)

where

\[
q D_y f(y) = \frac{f(y) - f(qy)}{(1 - q)y}
\]

(3.9)
one proves that relations (3.3)-(3.4) are verified.

In order to reproduce the exponential mapping, necessary to pass from Lie algebras to Lie groups, we consider the following \((q, \mu)\)-exponential function [28]

\[
E_q^{(\mu)}(z) = \sum_{n=0}^{+\infty} \frac{q^{\mu n^2}}{(q; q)_n} z^n \quad \mu \geq 0 \quad 0 < q < 1. \tag{3.10}
\]

In the limit \(q \to 1\), \(E_q^{(\mu)}((1 - q)z)\) tend to the ordinary exponential: \(\lim_{q \to 1} E_q^{(\mu)}((1 - q)z) = \exp(z)\). We also note that for some specific values of \(\mu\), they correspond to standard \(q\)-exponentials. Indeed, for \(\mu = 0\) and \(\mu = 1/2\) one has [29]

\[
e_q(z) \equiv E_q^{(0)}(z) = \sum_{n=0}^{+\infty} \frac{1}{(q; q)_n} z^n = \frac{1}{(z; q)_\infty} \tag{3.11}
\]

\[
E^{(1/2)}(z) = E_q(q^{1/2}z) = (-q^{1/2}z; q)_\infty. \tag{3.12}
\]

For an algebraic interpretation of special class of \(q\)-hypergeometric functions, let us introduce the following operators

\[
\mathcal{U}^{(\mu, \nu)}(\alpha, \beta) = E_q^{(\mu)}(\alpha(1 - q)A_+)E_q^{(\nu)}(\beta(1 - q)A_-). \tag{3.13}
\]

Then, in the limit \(q \to 1\), they go into the Lie group element \(\exp(\alpha A_+) \exp(\beta A_-)\). Their matrix elements, in the representation space spanned by the vectors \(\theta_n\), are defined by

\[
\mathcal{U}^{(\mu, \nu)}(\alpha, \beta) \theta_n = \sum_{m=0}^{+\infty} U_{m, n}^{\mu, \nu}(\alpha, \beta) \theta_m \tag{3.14}
\]

and, when evaluated, are found to involve generalized hypergeometric functions. By using (3.10) and various identities for the \(q\)-shifted factorials, one explicitly finds

\[
U_{m, n}^{\mu, \nu}(\alpha, \beta) = \beta^{n-m} \left[ \begin{array}{c} n \\ m \end{array} \right]_q q^{\nu(n-m)^2} \times Q^{(\mu, \nu)}(\alpha, \beta) \text{ if } n \geq m \tag{3.15}
\]

\[
U_{m, n}^{\mu, \nu}(\alpha, \beta) = \frac{[(1 - q)\alpha]^{m-n} q^{\mu(m-n)^2}}{(q; q)_{m-n}} \times Q^{(\nu, \mu)}(\alpha, \beta) \text{ if } m \geq n \tag{3.16}
\]

where \(Q^{(\mu, \nu)}(x; q^n|q)\) is the polynomial given by

\[
Q_n^{(\mu, \nu)}(x; q^n|q) = \sum_{k=0}^{n} \frac{q^{k^2(\mu+\nu)+(2\nu+\gamma+n)k} (q^{-n}; q)_k (q^\gamma+1; q)_k}{(q; q)_k (q^\gamma+1; q)_k} q^{-k(k-1)/2} x^k. \tag{3.17}
\]
Note that in passing from expression (3.15) to (3.16) for $U_{m,n}^{\mu,\nu}(\alpha,\beta)$ or vice-versa, $m$ and $n$ as well as $\mu$ and $\nu$ are exchanged in the polynomials $Q_n^{(\mu,\nu)}(x;q^\gamma|q)$.

The connection with standard $q$-polynomials is observed for particular values of $\mu$ and $\nu$.

For instance, for $\mu = \nu = 0$

$$Q_n^{(0,0)}(x;q^\gamma|q) = \phi_1 \left( \begin{array}{c} q^{-n}, 0, 0 \\ q^{\gamma+1} \\ q^{\gamma+1} \end{array} ; q, -xq^n \right)$$

for $\mu = 0, \nu = \frac{1}{2}$

$$Q_n^{(0,1/2)}(x;q^\gamma|q) = \phi_1 \left( \begin{array}{c} q^{-n}, 0 \\ q^{\gamma+1} \\ q^{\gamma+1} \end{array} ; q, xq^{\gamma+n+\frac{1}{2}} \right) = p_n \left( xq^{\gamma+n+\frac{1}{2}}, q^\gamma, 0|q \right)$$

where $p_n(z;\alpha,\beta|q)$ is the little $q$–Jacobi polynomials, for $\mu = \nu = \frac{1}{2}$

$$Q_n^{(1/2,1/2)}(x;q^\gamma|q) = \phi_1 \left( \begin{array}{c} q^{-n} \\ q^{\gamma+1} \\ q^{\gamma+1} \end{array} ; q, xq^{\gamma+n+1} \right) = \frac{(q;q)_n}{(q^{\gamma+1};q)_n} L_n^{(\gamma)}(x)$$

where $L_n^{(\gamma)}(x)$ are the $q$–Laguerre polynomials [29].

### 3.2 Main properties of Rogers-Szegő polynomials

Let us turn back to the relations (3.3)-(3.4) and make use of the matrix elements to derive some properties of the Rogers-Szegő polynomials.

**Theorem 3.1** The Rogers-Szegő polynomials possess the generating functions

i) $$S_q(\alpha; y) \equiv \frac{1}{(\alpha; q)_\infty (\alpha y; q)_\infty} = \sum_{m=0}^{+\infty} \frac{\alpha^m}{(q; q)_m} H_m(y|q)$$

ii) $$\phi_1 \left( \begin{array}{c} 0 \\ q^{1/2} \end{array} ; q, ty \right) (-tq^{1/2};q)_\infty = \sum_{m=0}^{+\infty} \frac{t^m q^{m(m-1)/2}}{(q; q)_m} H_m(y|q).$$
Proof.

From the definition of the matrix elements and (3.16) one can write

\[ U^{(\mu,0)}(\alpha/(1-q),0) \cdot 1 = \sum_{m=0}^{+\infty} \alpha q^m y^m. \]  

(3.24)

This relation indicates that if \( U^{(\mu,0)}(\alpha/(1-q),0) \cdot 1 \) can be written in a closed form then, this would be generating functions for the Rogers-Szegő polynomials. Let us consider two particular cases.

i) For \( \mu = 0 \), (3.24) becomes

\[ E^{(0)}(\alpha A_+) \cdot 1 = \sum_{k,m=0}^{+\infty} \alpha^m (q;q)_{m-k} (q;q)_k y^k. \]  

(3.25)

By introducing the new summation index \( l = m - k \) on the right-hand side of (3.25), one obtains

\[ E^{(0)}(\alpha A_+) \cdot 1 = e_q(\alpha) e_q(\alpha y). \]  

(3.26)

Hence, the following generating relation

\[ \frac{1}{(\alpha; q)_{\infty}(\alpha y; q)_{\infty}} = \sum_{m=0}^{+\infty} \alpha^m (q;q)_m \mathcal{H}_m(y/q). \]  

(3.27)

ii) For \( \mu = 1/2 \), (3.24) can be rewritten as

\[ U^{(1/2,0)}(\alpha/(1-q),0) \cdot 1 = \sum_{m,k=0}^{+\infty} \frac{q^m q^{(m-1)/2}}{(q;q)_m (q;q)_k} (\alpha q^{1/2})^m y^k. \]  

(3.28)

The two sums are reorganized by using \( l = m - k \) instead of \( m \) as summation index. This allows us to perform the sum over \( l \) thanks to the explicit expansion for (3.12) and the Heine’s binomial theorem which states that \[ 29 \]

\[ \sum_{n=0}^{+\infty} \frac{(a;q)_n}{(q;q)_n} z^n = (az; q)_{\infty}. \]  

(3.29)

One obtains

\[ U^{(1/2,0)}(\alpha/(1-q),0) \cdot 1 = {}_1\phi_1 \left( \begin{array}{c} 0 \\ \alpha q^{1/2} y \end{array} ; q, -\alpha q \right) (\alpha q; q)_{\infty}. \]  

(3.30)

Finally, we set \( t = \alpha q^{1/2} \) on the right-hand sides of (3.28) and (3.30) to find the generating function identity (3.24). □

It is worth mentioning that the formula (3.22) which was here constructively derived using algebraic methods, coincide with the one given in [30].
The generating function \((3.22)\) can be used to determine another realization of Rogers-Szegö polynomials. Indeed, by introducing the \(q\)-dilatation operator \(qT_y f(y) = f(qy)\) and recalling that the \(q\)-exponential \((3.11)\) obeys the difference rules
\[
qD_\alpha e_q(\alpha y) = \frac{y}{1-q} e_q(\alpha y)
\]
(3.31)
one can derive for \(S_q(\alpha; y)\) the following relation
\[
qD_\alpha S_q(\alpha; y) = \frac{1}{1-q} \left( 1 + yq T_y^{-1} qT_\alpha \right) S_q(\alpha; y).
\]
(3.32)
Also, by applying the operator \(qD_\alpha\) on the right-hand side of \((3.22)\), we arrive at the relation
\[
\mathcal{H}_{n+1}(y|q) = \mathcal{H}_n(y|q) + y q^n qT_y^{-1} \mathcal{H}_n(y|q).
\]
(3.33)
which can be converted into \([27]\)
\[
\mathcal{H}_{n+1}(y|q) - (1+y) \mathcal{H}_n(y|q) + y(1-q^n) \mathcal{H}_{n-1}(y|q) = 0.
\]
(3.34)
\((3.33)\) yields \(\mathcal{H}_{n+1}(y|q) = (I + y q^n qT_y^{-1}) \mathcal{H}_n(y|q)\). Therefore, we state the following.

**Theorem 3.2** Let
\[
S_+ := I + y q^N qT_y^{-1} \quad S_- := qD_y
\]
(3.35)
be raising and lowering operators, respectively, and
\[
N_q := S_+ S_-
\]
(3.36)
Then, their realizations are performed as:

i) \(S_+ \mathcal{H}_n(y|q) = \mathcal{H}_{n+1}(y|q)\)
ii) \(S_- \mathcal{H}_n(y|q) = [n]_q M^N \mathcal{H}_{n-1}(y|q) \quad [n]_q^M = \frac{1-q^n}{1-q}\)
iii) \(N_q \mathcal{H}_n(y|q) = [n]_q M^N \mathcal{H}_n(y|q)\).
(3.37)

**Corollary 3.3** The following commutation relations hold:

i) \([S_-, S_+] = q^N\) \quad ii) \([N_q, S_+] = S_+ q^N\)
iii) \([N, S_-] = -S_-\) \quad iv) \([N_q, S_-] = -q^N S_-\).
(3.40)

Now, coming back to \((3.39)\) and using the explicit realization of the raising and lowering operators, i.e. \(S_+\) and \(S_-\), we can write
\[
N_q \mathcal{H}_n(y|q) = \left( qD_y + y q^n qT_y^{-1} qD_y \right) \mathcal{H}_n(y|q) = [n]_q^M \mathcal{H}_n(y|q)
\]
(3.42)
from which, after using the identity \(qT_y^{-1} qD_y = qD_y qT_y^{-1}\), we get a \(q\)-difference equation obeyed by the Rogers-Szegö polynomials
\[
\left( qD_y + y q^{n+1} qD_y qT_y^{-1} - [n]_q^M \right) \mathcal{H}_n(y|q) = 0.
\]
(3.43)
4 \[(p, q, \mu, \nu)\text{–exponential function: matrix elements and Fourier-Gauss transform}\]

As a straightforward generalization of the \((q, \mu)\text{–exponential function}\), let us define the following \((p, q, \mu, \nu)\text{–exponential}\)

\[
E_{p, q}^{\mu, \nu}(z) = \sum_{n=0}^{+\infty} \left( \frac{q^n}{p^n} \right)^2 \frac{z^n}{[p, q; p, q]_n} \quad q^{2\mu} p^{1-2\nu} < 1 \quad 0 < pq < 1 
\]

(4.1)

where

\[
[p^\rho, q^\delta; p, q]_n = \left( \frac{1}{p^\rho} - q^\delta \right) \left( \frac{1}{p^{\rho+1}} - q^{\delta+1} \right) \cdots \left( \frac{1}{p^{\rho+n-1}} - q^{\delta+n-1} \right) 
\]

(4.2)

with \(\rho = \delta = 1\) here. In terms of the \(q\text{–shifted factorial}\)

\[
[p^\rho, q^\delta; p, q]_n = p^{-(n(n-1)/2+\rho n)} (p^\rho q^\delta; pq)_n.
\]

(4.3)

In the limit \((p, q) \to (1, 1)\), once \(z\) has been rescaled by \((p^{-1} - q)\), all these functions tend to the ordinary exponential:

\[
\lim_{(p, q) \to (1, 1)} E_{p, q}^{\mu, \nu}((p^{-1} - q)z) = \exp(z).
\]

(4.4)

It is worth noticing that for \(p = 1\), (4.1) yields (3.10). For some specific values of \(\mu\) and \(\nu\), standard \((p, q)\text{–exponentials}\) are recovered. Indeed, for \(\mu = \nu = 1/2\) and \(\mu = \nu = 0\) one has, respectively,

\[
E_{p, q}^{1/2, 1/2}(z) = E_{p, q} \left( \left( \frac{q}{p} \right)^{1/2} z \right)
\]

(4.5)

\[
E_{p, q}^{0, 0}(z) \equiv e_{p, q}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{[p, q; p, q]_n}
\]

(4.6)

where \(E_{p, q}(z)\) is the \((p, q)\text{–exponential}\) defined by Vinet et al [31]:

\[
E_{p, q}(z) = \sum_{n=0}^{+\infty} \left( \frac{q}{p} \right)^{n(n-1)/2} \frac{z^n}{[p, q; p, q]_n}. 
\]

(4.7)

4.1 \((p, q, \mu, \nu)\text{–exponential function: matrix elements}\)

Next, we deal with the computation of the matrix elements associated with a generalization of the \((q, \mu)\text{–exponential function}\).

The \((p, q)\text{–oscillator algebra}\) is generated by three elements \(A_-, A_+\) and \(N\) obeying the relations [32]

\[
A_- A_+ - p A_+ A_- = q^{-N} \quad A_- A_+ - q^{-1} A_+ A_- = p^N \quad [N, A_-] = -A_- \quad [N, A_+] = A_+.
\]

(4.8)
In general, the two parameters $q$ and $p$ may be real or a phase factor. Throughout, we 
suppose that they are real and positive.

The connection between quantum algebras and special functions depends not only on 
the particular algebra which is considered but also on the special realization which is 
used. The representation may be useful to construct models for bibasic special functions.

We consider a realization of (4.8) and (4.9) in terms of operators acting on the space of 
analytic functions as follows:

\[
A_- f(z) := \frac{1}{z(q^{-1} - p)} \left[ f \left( (pq)^{1/2} z \right) - f \left( (pq)^{-1/2} z \right) \right] \tag{4.10}
\]

\[
A_+ f(z) := -z(p/q)^{1/2} f \left( (p/q)^{1/2} z \right) \tag{4.11}
\]

\[
N f(z) := z \frac{d}{dz} f(z) \tag{4.12}
\]

and the basis \{ $\zeta_n = z^n, n \in \mathbb{N}$ \} such that

\[
A_+ \zeta_n = - \left( \frac{q}{p} \right)^{(n+1)/2} \zeta_{n+1}, \tag{4.13}
\]

\[
A_- \zeta_n = \left( \frac{q}{p} \right)^{1+n/2} \left( \frac{p^n - q^{-n}}{p^{-1} - q} \right) \zeta_{n-1}. \tag{4.14}
\]

For an algebraic interpretation of special class of bibasic functions, let us introduce the 
following operators

\[
U^{(\mu, \nu)}(\alpha, \beta) = E_p^{\mu, \nu}(\alpha(p^{-1} - q)A_+) E_p^{\mu, \nu}(\beta p^{-1} - q)A_-). \tag{4.15}
\]

Then, in the limit $(p, q) \to (1, 1)$, they go into the Lie group element $\exp(\alpha A_+) \exp(\beta A_-)$. 
Their matrix elements, in the representation space spanned by the vectors $\zeta_n$, are defined 
by

\[
U^{(\mu, \nu)}(\alpha, \beta)\zeta_n = \sum_{m=0}^{+\infty} U_{m,n}^{(\mu, \nu)}(\alpha, \beta)\zeta_m \tag{4.16}
\]

and, when evaluated, are found to involve generalized bibasic hypergeometric functions.

Explicitly, after a straightforward computation, one obtains

\[
U_{m,n}^{(\mu, \nu)}(\alpha, \beta) = (-\beta)^{n-m} \left[ \frac{n}{m} \right]_{p,q} \left( \frac{q^{\mu-1/4}}{p^{\nu-1/4}} \right)^{(m-n)^2} \left( \frac{q}{p} \right)^{-(n-m)(1+2m)/4}
\times L_m^{(m-n; \mu, \nu)}(-\alpha \beta; p, q) \quad \text{if} \quad m \leq n \tag{4.17}
\]

\[
U_{m,n}^{(\mu, \nu)}(\alpha, \beta) = \frac{(-\alpha(p^{-1} - q))^{m-n} \left( \frac{q^{\mu-1/4}}{p^{\nu-1/4}} \right)^{(m-n)^2} \left( \frac{q}{p} \right)^{-(m-n)(1+2n)/4}}{\prod_{l=1}^{m-n} (p^{-l} - q^l)} \times L_n^{(m-n; \mu, \nu)}(-\alpha \beta; p, q) \quad \text{if} \quad m \geq n \tag{4.18}
\]
where \( \mathcal{L}_n^{(\gamma;\mu,\nu)}(x;p,q) \) are the polynomials given by

\[
\mathcal{L}_n^{(\gamma;\mu,\nu)}(x;p,q) = \sum_{k=0}^{n} \binom{q^\mu \gamma}{p^\nu} \frac{((pq)^{-n};pq)_k}{(pq;pp)_k((pq)^{\gamma+1};pq)_k} \\
\times p^{(k+1)/2} \left[(1 - pq)p^{\gamma+n}\right]^k.
\] (4.19)

Note that in passing from expression (4.17) to (4.18) for \( U_{m,n}^{(\mu,\nu)}(\alpha,\beta) \) or vice versa, \( m \) and \( n \) are exchanged in the polynomials \( \mathcal{L}_n^{(\gamma;\mu,\nu)}(x;p,q) \).

The connection with standard \((p,q)-\)bibasic hypergeometric functions is observed for particular values of \( \mu \) and \( \nu \).

For \( \mu = \nu = 0 \)

\[
\mathcal{L}_n^{(\gamma;0,0)}(x;p,q) = \Phi \left[ \begin{array}{c}
(pq)^{-n}, 0 : - \\
(pq)^{\gamma+1}, - : 0
\end{array} \right] \]

(4.20)

for \( \nu = \mu = 1/4 \)

\[
\mathcal{L}_n^{(\gamma;1/4,1/4)}(x;p,q) = \Phi \left[ \begin{array}{c}
(pq)^{-n} : 0 \\
(pq)^{\gamma+1} : - \end{array} \right] \]

(4.21)

where

\[
\Phi \left[ \begin{array}{c}
a : c \\
b : d
\end{array} \right] = \sum_{l=0}^{+\infty} \frac{(a; q)_l (c; p)_l}{(q; q)_l (b; q)_l (d; p)_l} \left[ (-1)^l q^{(l+1)} \right] z^{l+m-n} \\
\times \left[ (-1)^l p^{(l+1)} \right] z^n
\] (4.22)

is the well-known bibasic hypergeometric series [33]. In (4.22)

\[
a = (a_1, \ldots, a_n) \quad c = (c_1, \ldots, c_r) \\
b = (b_1, \ldots, b_m) \quad d = (d_1, \ldots, d_s)
\] (4.23)

\[
(a; q)_l = (a_1; q)_l \cdots (a_n; q)_l.
\] (4.24)

### 4.2 \((p,q,\zeta)-\)exponential function: Fourier-Gauss transform

**Proposition 4.1** Let \( \mu = \nu = \zeta/2 \). Then, we obtain the \((p,q,\zeta)-\)exponential function

\[
E_{p,q}^{\zeta/2,(\zeta/2)}(z) \equiv E_{p,q}^{(\zeta)}(z) = \sum_{n=0}^{+\infty} \binom{q^\zeta}{p} \frac{\zeta^{n/2}}{(pq;p,q)_n} z^n.
\] (4.25)
to which correspond the following two types of Fourier-Gauss transforms

\[
E_{p,q}^{(\zeta+1/2)} \left( t e^{-kx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} E_{p,q}^{(\zeta)}(te^{iky})dy
\]  
(4.26)

and

\[
E_{p,q}^{(\zeta-1/2)} \left( t e^{ikx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} E_{p,q}^{(\zeta)}(te^{ky})dy
\]  
(4.27)

where \( q = p \exp(-2k^2) \).

**Proof.**

To evaluate the right-hand side of (4.26) (resp. (4.27)) one only needs to use the definition (4.25) with \( z = te^{iky} \) (resp. \( z = te^{ky} \)) and to integrate the sums termwise by the Fourier transform

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2}dy = e^{-x^2/2}
\]  
(4.28)

for the Gauss exponential function \( \exp(-x^2/2) \). □

Important particular cases of (4.26) are

\[
\epsilon_{p,q} \left( t e^{-kx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} \epsilon_{p,q}(te^{iky})dy
\]  
(4.29)

and

\[
E_{p,q} \left( \left( \frac{q}{p} \right)^{1/2} t e^{-kx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} \epsilon_{p,q}(te^{iky})dy
\]  
(4.30)

where

\[
\epsilon_{p,q}(z) = E_{p,q}^{(1/2)}(z).
\]  
(4.31)

These case correspond to the values 0 and 1/2 of the parameter \( \zeta \), respectively.

When \( \zeta = 1/2 \), from (4.27) follows the inverse Fourier transformation with respect to (4.29)

\[
\epsilon_{p,q} \left( t e^{ikx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} \epsilon_{p,q}(te^{ky})dy
\]  
(4.32)

whereas the value \( \zeta = 1 \) yields the inverse to (4.30), i.e.

\[
\epsilon_{p,q} \left( t e^{ikx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} E_{p,q} \left( \left( \frac{q}{p} \right)^{1/2} t e^{ky} \right) dy.
\]  
(4.33)

Actually, the Fourier-Gauss transforms (4.26) and (4.27) may be written in the unified form

\[
E_{p,q}^{(\zeta+\epsilon^2/2)} \left( t e^{-\phi kx} \right) e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixy-y^2/2} E_{p,q}^{(\zeta)}(te^{\phi kx})dy.
\]  
(4.34)
This is easy to prove in exactly the same way as (4.26) and (4.27). It is worth noticing that, for $\zeta = 0$ and $\varrho = \sqrt{2}$, the Fourier-Gauss transform (4.34) gives the following relation

$$E_{p,q}\left(\left(\frac{q}{p}\right)^{1/2} te^{i\varrho k x}\right)e^{-x^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\varrho y - y^2/2} e_{p,q}(te^{i\varrho y})dy \tag{4.35}$$

which is a particular version of the $(p, q)$–Ramanujan’s integral with respect to a complex parameter $\zeta$.

The formalism used here provides with a straightforward algorithm for characterizing $q$–polynomials. In this paper, we have shown that Rogers-Szeg\'o polynomials also could be studied along the same basis. The key of all these investigations undoubtedly remains the expression of conveniently chosen deformed exponential function. On this basis, using the relation (4.11) giving a generalized $(p, q, \mu, \nu)$–exponential function, this formalism may also turn out to be useful for achieving a global definition as well as a better understanding of $(p, q)$–analogs of special functions. These aspects are now under consideration.

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