CLASSIFICATION OF GRAPH FRACTALOIDS

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Abstract. In this paper, we observe graph fractaloids, which are the graph groupoids with fractal property. In particular, we classify them in terms of the spectral data of certain Hilbert space operators, called the radial operators. Based on these information, we can define the pair of two numbers \((N_0, N^0)\), for a given graph fractaloid \(G\), called the fractal pair of \(G\). The graph fractaloids are classified by such pairs.

1. Introduction

The word “fractaloids” in the title may at first seem a bit puzzling; in any case, calling for an explanation. We have chosen the terminology in order to call attention to a certain feature in the study of analysis and spectral theory on countable directed graphs. The idea is sketched briefly below, and then taken up systematically again in Section 2, with precise definitions.

While the area of spectral theory of countable directed graphs usually refers to weighted graphs and then the spectrum of a suitable associated graph Laplacian, our approach here builds instead on two different tools: Starting with a given graph \(G\) (vertices and edges) we build a groupoid (morphisms from edges, etc. in the usual way), and we then construct an associated von Neumann algebra \(M_G\). The von Neumann algebra construction is reminiscent of von Neumann’s original way of generating a ring operators (alias von Neumann algebra) from a free group on a finite number of generators. The feature the two constructions have in common is a set of branching rules, and indeed these branching rules capture an essential feature of fractals.

In fact, a given graph \(G\) can be turned a “symbol space” for the kind of fractals that are built from iterated function systems (IFSs). Here, we use IFSs in the sense of Hutchinson (See [50]). i.e., spaces and measures built from a repeated application of a given finite set of maps in an ambient space, and a subsequent limit construction. (Two popular examples of IFSs in very special cases are the familiar middle-third Cantor set, and the Sierpinsky gasket.) Spectral analysis on \(G\) then carries over to the IFS fractal under consideration. Even for the familiar IFSs under current study, spectral theory is not yet fully developed, and our use of the von Neumann algebra \(M_G\) adds some global invariants to the study of fractals.
The distinction between local and global is relevant when analysis or spectral theory is considered for graphs, or more generally for infinite systems, such as arise in statistical mechanics and thermodynamics. Erwin Schrödinger, in his little book [51], illustrated this point with reference to macroscopic laws vs. microscopic in the physics of diffusion. While bulk diffusion as predicted by the heat equation is deterministic, it results by contrast from taking limits of microscopic components (at the quantum level), i.e., molecular movements. Hence, as Einstein noted (1905), in explaining Brownian motions, the local theory may be modeled by (purely statistical) random walk on discrete configurations, later to be widely studied in the form of statistical graph models.

A second kind of “fractal” amenable to our von Neumann algebra approach derives from a different family of iteration systems, again a von Neumann construction, but now the objects are automata; i.e., the study of abstract machines and problems they are able to solve. The study of automata is related to formal language theory, understood as classes of formal languages they can recognize. More specifically, an automaton is a mathematical model for a finite state machine (FSM). Roughly, an input-output machine that, given an input of symbols, then “jumps” through a series of states according to a transition function (expressed as a table).

Again, an automaton $A$ arises as an iteration limit $L(A)$, and we will study $L(A)$ with the use of our von Neumann algebra $M_G$.

In both applications of $M_G$, we are taking advantage of a certain atomic decomposition (developed in our paper) of $M_G$, and we explore its use in the study of fractals in the two senses outlined above.

Our subject is at the crossroads of operator algebra and analysis of graphs and fractals. As a result, in Section 2 below, we develop the basic tools we will need from both subjects. This section includes careful definitions of the concepts from both subjects. To make the paper more accessible, we take the liberty of explaining and motivating the fundamental tools we need from operator algebras so they make sense to researchers working on analysis of countable directed graphs, and vice versa.

The main purpose of this paper is to introduce a new algebraic structures having certain fractal property, which is, sometimes, called fractality. In particular, we are interested in the groupoidal version of fractal groups. In [16], [19] and [20], we constructed (graph) groupoids with fractal property, called fractaloids. And we considered the spectral data of fractaloids in operator theoretical point of view. In this paper, we observe the classification of graph fractaloids.

In [10] through [15], and [17] through [22], we introduced graph groupoids induced by countable directed graphs. A graph groupoid is a categorical groupoid having its base, the set of all vertices, i.e., we can regard all vertices as (multi-)units. Every groupoid having only one base element is a group. So, if $G$ is a finite directed graph with its graph groupoid $G$, and if the vertex set $V(G) \subset G$ consists of only one element, then the graph groupoid $G$ is a group. For example, if $G$ is the one-vertex-$n$-loop-edge graph, then the graph groupoid $G$ of $G$ is group-isomorphic to the free group $F_n$, with $n$-generators (See [10] and [11]). Notice that the free group $F_n$ is a fractal group (See [1]), for all $n \in \mathbb{N}$. Remark that every graph groupoid is a groupoid, but the converse does not hold in general. So, our fractaloids may be partially understood in groupoid theory. Therefore, to avoid the confusion, different from [19] and [20], we call fractaloids (in the sense of [19] and [20]), graph
fractaloids, like in [16]. In [19], we conjectured that the only “connected,” “finite” directed graphs, generating graph fractaloids, are graph-isomorphic to (i) the one-vertex-multi-loop-edge graphs, or (ii) the one-flow circulant graphs, or the certain connection of the previous kind of graphs. And, in [16], this conjecture is solved. And the conclusion of the conjecture in [16] shows that there are sufficiently many fractaloids, since we can find sufficiently many “finite” directed graphs, generating graph fractaloids. i.e., we have rich fractality on (graph) groupoids. We can have that the connected finite directed graphs, generating graph fractaloids, are

(i) the one-vertex-multi-loop-edge graphs, or
(ii) the regularized graphs of the one-flow circulant graphs or the shadowed graphs of them, or
(iii) the regularized graphs of the complete graphs or the shadowed graphs of them, or
(iv) the regularized graphs of the vertex-fixed iterated glued graphs $G \#^v O_n$, where $G$ are one of the forms in (i) through (iv).

Again, the above conclusion shows that, even though we restrict our interests to the case where we only consider graph fractaloids, generated by a connected “finite” directed graphs, we have the rich fractoidal structures to handle. i.e., there are more connected finite directed graphs what we expected in [19], which means good for the richness of fractaloids.

To detect the fractality of graph groupoids, we used automata theory in [16], [19] and [20]: We found the “automata-theoretical,” and “algebra” characterization of graph fractaloids. In this paper, we avoid to use the automata theory. However, our construction is completely based on automata theory. Based on the characterizations of graph fractaloids in [19], we can find the “graph-theoretical” characterization of graph fractaloids in [16], and it leads us to define graph fractaloids without using automata theory. Recall that, in [16], we show that: the graph groupoid $G$ of a connected locally finite directed graph $G$ is a graph fractaloid, if and only if the out-degrees and the in-degrees of all vertices of $G$ are identical from each other. So, by using this characterization, we can re-define graph fractaloids as in Section 3, below.

As in [10] through [15], we construct a von Neumann algebra $M_G$, generated by the graph groupoid $G$ of $G$, as a groupoid $W^*$-algebra $vN(L(G))$ generated by the graph groupoid $G$ in $B(H_G)$, where $(H_G, L)$ is the canonical (left) representation of $G$, consisting of a suitable Hilbert space $H_G$, and the groupoid action $L$ of $G$, acting on $H_G$. We call $M_G$, the (left) graph von Neumann algebra of $G$. In [16], [19], and [20], we use the right graph von Neumann algebra $M_G = vN(R(G))$ of $G$ in $B(H_G)$, where $(H_G, R)$ is the canonical “right” representation of $G$, where $R$ is the right action of $G$, acting on $H_G$. The right graph von Neumann algebras $M_G$ are the opposite $W^*$-algebras $M_G^{op}$ of the graph von Neumann algebras $M_G$. Thus the right graph von Neumann algebras $M_G$ and the graph von Neumann algebras $M_G$ are anti-$\ast$-isomorphic from each other. The only difference is the choice of actions of a graph groupoid. In this paper, we will use right graph von Neumann algebras, as in [19] and [20].

Let $G$ be a given connected locally finite directed graph with its graph groupoid $G$, and let $M_G$ the right graph von Neumann algebras of $G$. Then the graph groupoid $G$ induces a certain Hilbert space operator $T_G$ in $M_G$, called the labeling operator of $G$ (See [19] and [20]). It is self-adjoint in $M_G$. It is known that the
free distributional data, represented by the $D_G$-valued (amalgamated or operator-valued) free moments $\{E(T_G^n)\}_{n=1}^{\infty}$ of $T_G$, contain the spectral information of $T_G$, where $D_G$ is the diagonal subalgebra of $M_G$.

In [20], we found the general computations of $D_G$-valued free moments of $T_G$, and in [19], the spectral information of $T_G$ of graph fractaloids $G$ is completely characterized by computing the $D_G$-valued free moments: The computations are based on the observation of the cardinalities of lattice paths with axis property (See [40]).

In this paper, under our new setting, we re-define the same operator $T_G$, called the radial operators of the graph groupoid $G$, as an element of the right graph von Neumann algebra $M_G$ (See Section 4). By definition, we can realize that the labeling operators in the sense of [19] and [20], and our radial operators are equivalent. i.e., if a graph $G$ is fixed, then the labeling operator and the radial operator are identically distributed over $D_G$ in $B(H_G)$.

A graph is a set of objects called vertices (or points or nodes) connected by links called edges (or lines). In a directed graph, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), jointed by curves (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots jointed by arrowed curves, where the arrows point the direction of the directed edges.

In this paper, we consider direct graph $G$ as a combinatorial pair $(V(G), E(G))$, where $V(G)$ is the vertex set of $G$ and $E(G)$ is the edge set of $G$. As we assumed at the beginning of the paper, throughout this paper, every graph is a locally finite countably directed graph. Equivalently, the degree of $v \in V(G)$ is finite, for all $v \in V(G)$. Notice that, since $G$ is directed (or oriented on $E(G)$), each edge $e$ has its initial vertex $v_1$ and its terminal vertex $v_2$. i.e., $e$ connects from $v_1$ to $v_2$. Remark that the vertices $v_1$ and $v_2$ are not necessarily distinct, in general; for instance, if $e$ is a loop edge, then $v_1 = v_2$.

Recall that the degree $\deg(v)$ of a vertex $v$ is defined to be the sum of the out-degree $\deg_{\text{out}}(v)$ and the in-degree $\deg_{\text{in}}(v)$, dependent upon the direction on $G$. i.e.,

$$\deg(v) \overset{\text{def}}{=} \deg_{\text{out}}(v) + \deg_{\text{in}}(v)$$

where

$$\deg_{\text{out}}(v) \overset{\text{def}}{=} |\{e \in E(G) : e \text{ has its initial vertex } v\}|$$

and

$$\deg_{\text{in}}(v) \overset{\text{def}}{=} |\{e \in E(G) : e \text{ has its terminal vertex } v\}|.$$

Define now the number $N$ by

$$N \overset{\text{def}}{=} \max \{\deg_{\text{out}}(v) : v \in V(\hat{G}) = V(G)\}.$$

Notice that, by the locally finiteness of $G$, $N < \infty$ in $\mathbb{N}$.

The main purpose of this paper is to classify the graph fractaloids, in terms of their spectral information. In [19], we showed that the free distribution of the labeling operators (and hence, that of the radial operators) of graph fractaloids are scalar-valued:

$$E(T_G^n) = \gamma_n \cdot 1_{D_G}, \text{ for all } n \in \mathbb{N},$$
where $T_G$ is the labeling operator of a graph fractaloid $G$ in the right graph von Neumann algebra $M_G$, and where $\gamma_n$ is the cardinality of a certain subset of the collection of all lattice paths induced by $N$-lattices.

The above free-moment computations show that if two connected locally finite directed graphs $G_1$ and $G_2$ have graph-isomorphic shadowed graphs, then the corresponding graph groupoids $G_1$ and $G_2$ are groupoid-isomorphic; and if $G_k$ are graph fractaloids, for $k = 1, 2$, then the radial operators $T_{G_1}$ and $T_{G_2}$ are identically distributed over $D_G$ in $M_G$.

Therefore we can determine the classification of graph fractaloids in terms of their spectral property, with respect to the fractal pairs, consisting of the certain numbers.

Let

$$n = \max\{\deg_{\text{out}}(v) : v \in V(G)\} \in \mathbb{N},$$

and

$$m = |V(G)| \in \mathbb{N}_\infty \overset{\text{def}}{=} \mathbb{N} \cup \{\infty\}.$$

Then the pair $(n, m)$ is well-determined, whenever we have a connected “locally finite” directed graph $G$. If $G$ generates a graph fractaloid $G$, then this pair $(n, m)$ is called the fractal pair of $G$. Our main result of this paper is that the given two graph fractaloids $G_1$ and $G_2$ have the same fractal pair $(N_0, \mathbb{N}^0)$, then the radial operators $T_{G_1}$ and $T_{G_2}$ of them are identically free distributed over $\mathbb{C} \oplus \mathbb{N}_0$. In particular,

$$E(T_{G_k}^n) = |\mathcal{L}_{N_0}^n(n)| \cdot 1_{\mathbb{C} \oplus \mathbb{N}_0},$$

for all $n \in \mathbb{N}$,

where $\mathcal{L}_{N_0}^n(n)$ is the set consisting of all length-$N_0$ lattice paths in $\mathbb{R}^2$, starting at $(0, 0)$, and ending on the horizontal axis. These fractal pairs on the set $\mathcal{F}_{\text{fractal}}$ of all graph fractaloids make us classify the set $\mathcal{F}_{\text{fractal}}$, as follows:

$$\mathcal{F}_{\text{fractal}} = \bigcup_{(n, m) \in \mathbb{N} \times \mathbb{N}_\infty} \left(\left[(n, m)\right]\right),$$

where $\left[(n, m)\right]$ is an equivalence class in $\mathcal{F}_{\text{fractal}}$.

2. Background and Definitions

Recently, countable directed graphs have been studied in Pure and Applied Mathematics, because not only that they are involved by a certain noncommutative structures but also that they visualize such structures. Furthermore, the visualization has nice matricial expressions, (sometimes, the operator-valued matricial expressions dependent on) adjacency matrices or incidence matrices of the given graph. In particular, partial isometries on a Hilbert space can be expressed and visualized by directed graphs: in [10] through [15], [17], and [23], we have seen that each edge (resp. each vertex) of a graph corresponds to a partial isometry (resp. a projection) on a Hilbert space. In [18], [21], and [22], we showed that any finite partial isometries (and the initial and final projections induced by these partial isometries) on an arbitrary separable infinite Hilbert space induces a (locally finite) directed graph. This shows that there are close relations between Hilbert space operators and directed graphs.

Also, in this paper, we consider the property of fractaloids in terms of the spectral property of certain operators on Hilbert spaces (Also, see [19]). To do that, in this section, we introduce the concepts we will use in the rest of the context.
2.1. Graph Groupoids and Representations. Let $G$ be a directed graph with its vertex set $V(G)$ and its edge set $E(G)$. Let $e \in E(G)$ be an edge connecting a vertex $v_1$ to a vertex $v_2$. Then we write $e = v_1 \overset{e}{\longrightarrow} v_2$, for emphasizing the initial vertex $v_1$ of $e$ and the terminal vertex $v_2$ of $e$. For a graph $G$, we can define the oppositely directed graph $G^{-1}$, with $V(G^{-1}) = V(G)$ and $E(G^{-1}) = \{ e^{-1} : e \in E(G) \}$, where each element $e^{-1}$ satisfies that $e = v_1 \overset{e}{\longrightarrow} v_2$ in $E(G)$, with $v_1, v_2 \in V(G)$, if and only if $e^{-1} = v_2 \overset{e}{\longrightarrow} v_1$, in $E(G^{-1})$. This opposite directed edge $e^{-1}$ is called the shadow of $e$. Also, this new graph $G^{-1}$, induced by $G$, is said to be the shadow of $G$. It is clear that $(G^{-1})^{-1} = G$.

Define the shadowed graph $\tilde{G}$ of $G$ by a directed graph with its vertex set $V(\tilde{G}) = V(G) = V(G^{-1})$ and its edge set $E(\tilde{G}) = E(G) \cup E(G^{-1})$, where $G^{-1}$ is the shadow of $G$. We say that two edges $e_1 = v_1 \overset{e_1}{\longrightarrow} v_1'$ and $e_2 = v_2 \overset{e_2}{\longrightarrow} v_2'$ are admissible, if $v_1' = v_2$, equivalently, the finite path $e_1 e_2$ is well-defined on $\tilde{G}$. Similarly, if $w_1$ and $w_2$ are finite paths on $G$, then we say $w_1$ and $w_2$ are admissible, if $w_1 w_2$ is a well-defined finite path on $G$, too. Similar to the edge case, if a finite path $w$ has its initial vertex $v$ and its terminal vertex $v'$, then we write $w = v \overset{e}{\longrightarrow} v'$. Notice that every admissible finite path is a word in $E(\tilde{G})$. Denote the set of all finite path by $FP(\tilde{G})$. Then $FP(\tilde{G})$ is the subset of $E(\tilde{G})^*$, consisting of all words in $E(\tilde{G})$.

We can construct the free semigroupoid $F^+(\tilde{G})$ of the shadowed graph $\tilde{G}$, as the union of all vertices in $V(\tilde{G}) = V(G) = V(G^{-1})$ and admissible words in $FP(\tilde{G})$, with its binary operation, the admissibility. Naturally, we assume that $F^+(\tilde{G})$ contains the empty word $\emptyset$. Remark that some free semigroupoid $F^+(\tilde{G})$ of $\tilde{G}$ does not contain the empty word; for instance, if a graph $G$ is a one-vertex-multi-edge graph, then the shadowed graph $\tilde{G}$ of $G$ is also a one-vertex-multi-edge graph, and it induces the free semigroupoid $F^+(\tilde{G})$, which does not have the empty word. However, in general, if $|V(G)| > 1$, then $F^+(\tilde{G})$ always contain the empty word. Thus, if there is no confusion, we always assume the empty word $\emptyset$ is contained in the free semigroupoid $F^+(\tilde{G})$ of $\tilde{G}$.

By defining the reduction (RR) on $F^+(\tilde{G})$, we can construct the graph groupoid $\mathbb{G}$, i.e., the graph groupoid $\mathbb{G}$ is a set of all “reduced” words in $E(\tilde{G})$, with the inherited admissibility on $F^+(\tilde{G})$, where the reduction (RR) on $\mathbb{G}$ is

\[
(w \overset{v}{\longrightarrow} v')^{-1} = v \overset{w^{-1}}{\longrightarrow} v',
\]

for all $w = v \overset{v'}{\longrightarrow} v' \in \mathbb{G}$, with $v, v' \in V(\tilde{G})$. In fact, this graph groupoid $\mathbb{G}$ is indeed a categorial groupoid with its base $V(\tilde{G})$ (See Section 2.2).

Construct the canonical representation of the given graph groupoid $\mathbb{G}$. Let

\[
H_\mathbb{G} \overset{\text{def}}{=} \bigoplus_{w \in FP_r(\tilde{G})} (\mathbb{C} \cdot \xi_w)
\]

be the Hilbert space with its Hilbert basis $\{ \xi_w : w \in FP_r(\tilde{G}) \}$, where

\[
FP_r(\tilde{G}) \overset{\text{def}}{=} \mathbb{G} \setminus \big( V(\tilde{G}) \cup \{ \emptyset \} \big).
\]

We will call $H_\mathbb{G}$, the graph Hilbert space induced by the graph $G$. Notice that the basis elements $\xi_w$’s satisfy the multiplication rule;

\[
\xi_{w_1} \xi_{w_2} = \xi_{w_1 w_2}, \text{ for all } w_1, w_2 \in FP_r(\tilde{G}),
\]

with $\xi_\emptyset \overset{\text{def}}{=} 0_{H_\mathbb{G}}$ in $H_\mathbb{G}$. Also, we have, for any $w \in FP_r(\tilde{G})$,

\[
\xi_w \xi_{w^{-1}} = \xi_{w w^{-1}}, \text{ and } \xi_{w^{-1}} \xi_w = \xi_{w^{-1} w}.
\]
By the reduction (RR), $ww^{-1}$ and $w^{-1}w$ are vertices in $V(\hat{G})$. This shows that naturally, we can determine the Hilbert space elements $\xi_w$, for all $w \in G$.

Define now the groupoid action of $G$, acting on $H_G$,

$$ R : G \rightarrow B(H_G) $$

by

$$ R(w) \overset{\text{def}}{=} R_w, \text{ for all } w \in G, $$

where

$$ R_w \xi_{w'} \overset{\text{def}}{=} \xi_{w'w}, \text{ for all } w, w' \in G. $$

i.e., the operator $R_w$ is the “right” multiplication operator with its symbol $\xi_w$ on $H_G$, for all $w \in G$.

**Definition 2.1.** Let $H_G$ be the graph Hilbert space induced by a given graph $G$, and let $R$ be the right action of the groupoid $G$ of $G$, acting on $H_G$, defined as above. Then the pair $(H_G, R)$ is called the (canonical) right representation of $G$.

### 2.2. Categorial Groupoids and Groupoid Actions.

We say an algebraic structure $(\mathcal{X}, \mathcal{Y}, s, r)$ is a (categorial) groupoid if it satisfies that (i) $\mathcal{Y} \subset \mathcal{X}$, (ii) for all $x_1, x_2 \in \mathcal{X}$, there exists a partially-defined binary operation $(x_1, x_2) \mapsto x_1 x_2$, for all $x_1, x_2 \in \mathcal{X}$, depending on the source map $s$ and the range map $r$ satisfying the followings:

(ii-1) $x_1 x_2$ is well-determined, whenever $r(x_1) = s(x_2)$ and in this case, $s(x_1 x_2) = s(x_1)$ and $r(x_1 x_2) = r(x_2)$, for $x_1, x_2 \in \mathcal{X}$.

(ii-2) $(x_1 x_2) x_3 = x_1 (x_2 x_3)$, if they are well-determined in the sense of (ii-1), for $x_1, x_2, x_3 \in \mathcal{X}$.

(ii-3) if $x \in \mathcal{X}$, then there exist $y, y' \in \mathcal{Y}$ such that $s(x) = y$ and $r(x) = y'$, satisfying $x = y x y'$ (Here, the elements $y$ and $y'$ are not necessarily distinct).

(ii-4) if $x \in \mathcal{X}$, then there exists a unique element $x^{-1}$ for $x$ satisfying $x x^{-1} = s(x)$ and $x^{-1} x = r(x)$.

Thus, every group is a groupoid $(\mathcal{X}, \mathcal{Y}, s, r)$ with $|\mathcal{Y}| = 1$ (and hence $s = r$ on $\mathcal{X}$). This subset $\mathcal{Y}$ of $\mathcal{X}$ is said to be the base of $\mathcal{X}$. Remark that we can naturally assume that there exists the empty element $\emptyset$ in a groupoid $\mathcal{X}$. The empty element $\emptyset$ means the products $x_1 x_2$ are not well-defined, for some $x_1, x_2 \in \mathcal{X}$. Notice that if $|\mathcal{Y}| = 1$ (equivalently, if $\mathcal{X}$ is a group), then the empty word $\emptyset$ is not contained in the groupoid $\mathcal{X}$. However, in general, whenever $|\mathcal{Y}| \geq 2$, a groupoid $\mathcal{X}$ always contain the empty word. So, if there is no confusion, we will naturally assume that the empty element $\emptyset$ is contained in $\mathcal{X}$.

It is easily checked that our graph groupoid $G$ of a countable directed graph $G$ is indeed a groupoid with its base $V(\hat{G})$. I.e., every graph groupoid $G$ of a countable directed graph $G$ is a groupoid $(G, V(\hat{G}), s, r)$, where $s(w) = s(v w) = v$ and $r(w) = r(w v') = v'$, for all $w = v w v' \in G$ with $v, v' \in V(\hat{G})$. I.e., the vertex set $V(\hat{G}) = V(G)$ is a base of $G$.

Let $\mathcal{X}_k = (\mathcal{X}_k, \mathcal{Y}_k, s_k, r_k)$ be groupoids, for $k = 1, 2$. We say that a map $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is a groupoid morphism if (i) $f$ is a function, (ii) $f(\mathcal{Y}_1) \subseteq \mathcal{Y}_2$, (iii) $s_2(f(x)) = f(s_1(x))$, for all $x \in \mathcal{X}_1$, and (iv) $r_2(f(x)) = f(r_1(x))$, for all $x \in \mathcal{X}_1$. If a groupoid morphism $f$ is bijective, then we say that $f$ is a groupoid-isomorphism, and the groupoids $\mathcal{X}_1$ and $\mathcal{X}_2$ are said to be groupoid-isomorphic.
Notice that, if two countable directed graphs $G_1$ and $G_2$ are graph-isomorphic, via a graph-isomorphism $g : G_1 \rightarrow G_2$, in the sense that (i) $g$ is bijective from $V(G_1)$ onto $V(G_2)$, (ii) $g$ is bijective from $E(G_1)$ onto $E(G_2)$, (iii) $g(e) = g(v_1 e v_2) = g(v_1) g(e) g(v_2)$ in $E(G_2)$, for all $e = v_1 e v_2 \in E(G_1)$, with $v_1, v_2 \in V(G_1)$, then the graph groupoids $G_1$ and $G_2$ are groupoid-isomorphic. More generally, if two graphs $G_1$ and $G_2$ have graph-isomorphic shadowed graphs $\hat{G}_1$ and $\hat{G}_2$, then $\hat{G}_1$ and $\hat{G}_2$ are groupoid-isomorphic (See [10] and [11]).

Let $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ be a groupoid. We say that this groupoid $\mathcal{X}$ acts on a set $Y$ if there exists a groupoid action $\pi$ of $\mathcal{X}$ such that $\pi(x) : Y \rightarrow Y$ is a well-determined function, for all $x \in \mathcal{X}$. Sometimes, we call the set $Y$, a $\mathcal{X}$-set.

Let $\mathcal{X}_1 \subset \mathcal{X}_2$ be a subset, where $\mathcal{X}_2 = (\mathcal{X}_2, \mathcal{Y}_2, s, r)$ is a groupoid, and assume that $\mathcal{X}_1 = (\mathcal{X}_1, \mathcal{Y}_1, s, r)$, itself, is a groupoid, where $\mathcal{Y}_1 = \mathcal{X}_2 \cap \mathcal{Y}_2$. Then we say that the groupoid $\mathcal{X}_1$ is a subgroupoid of $\mathcal{X}_2$.

Recall that we say that a countable directed graph $G_1$ is a full-subgraph of a countable directed graph $G_2$, if

$$E(G_1) \subseteq E(G_2)$$

and

$$V(G_1) = \{ v \in V(G_1) : e = v e \text{ or } e = e v, \forall e \in E(G_1) \}.$$ 

Remark the difference between full-subgraphs and subgraphs: We say that $G_1'$ is a subgraph of $G_2$, if

$$V(G_1') \subseteq V(G_2)$$

and

$$E(G_1') = \{ e \in E(G_2) : e = v_1 e v_2, \text{ for } v_1, v_2 \in V(G_1') \}.$$ 

We can see that the graph groupoid $G_1$ of $G_1$ is a subgroupoid of the graph groupoid $G_2$ of $G_2$, whenever $G_1$ is a full-subgraph of $G_2$.

2.3. **Right Graph von Neumann Algebras.** In this section, we briefly introduce right graph von Neumann algebras of the graphs. Frankly speaking, we will not consider such operator algebraic structures in detail, here. However, to study the spectral property of our fractaloids, we need the frameworks where the corresponding labeling operators of fractaloids work. For more about groupoid topological algebras, see [19], [20], [22], [24] and [26]. And, for more about free probability, see [5], [10], [11], and [28].

**Definition 2.2.** Let $G$ be a graph with its graph groupoid $\mathcal{G}$, and let $(H_G, R)$ be the right representation of $\mathcal{G}$, in the sense of Section 2.1. Under the representation $(H_G, R)$, define the groupoid $W^*$-algebra $M_G = C[\hat{R}(\mathcal{G})]$ in $B(H_G)$, as a $W^*$-subalgebra. This groupoid $W^*$-algebra $M_G$ is called the right graph von Neumann algebra of $G$. Define a $W^*$-subalgebra $D_G$ of $M_G$ by

$$D_G \overset{def}{=} \bigoplus_{v \in V(\hat{G})} (\mathcal{C} \cdot R_v).$$

It is called the diagonal subalgebra of $M_G$.

**Remark 2.1.** In [10] through [14], we observed the (left) multiplication operators $L_w$'s, for all $w \in \mathcal{G}$, instead of using right multiplication operators $R_w$'s. Then we can define the (left) graph von Neumann algebra $M_G^{op} = C[L(\mathcal{G})]^{op}$ in $B(H_G)$, where $L : \mathcal{G} \rightarrow B(H_G)$ is the left groupoid action of $\mathcal{G}$, acting on $H_G$, i.e., $L_w \xi_{w'} \overset{def}{=} \xi_{w w'}$, for all $w, w' \in \mathcal{G}$. Notice that $M_G^{op}$ and $M_G$ are anti-*-isomorphic. Thus
they share the fundamental properties (See [19]). Indeed, the von Neumann algebra $M^G$ is the opposite $*$-algebra of our right graph von Neumann algebra $M_G$ of $G$.

Notice that, every element $x$ in the right graph von Neumann algebra $M_G$ of $G$ has its expression,

$$x = \sum_{w \in G} t_w R_w, \quad \text{with} \quad t_w \in \mathbb{C}.$$ 

Let $D_G$ be the diagonal subalgebra of $M_G$. Define the canonical conditional expectation $E : M_G \to D_G$ by

$$E \left( \sum_{w \in G} t_w R_w \right) \triangleq \sum_{v \in V(G)} t_v R_v,$$

for all $\sum_{w \in G} t_w R_w \in M_G$. Then the pair $(M_G, E)$ is a $D_G$-valued $W^*$-probability space over $D_G$, in the sense of Voiculescu (See [5] and [28]).

**Definition 2.3.** The $D_G$-valued $W^*$-probability space $(M_G, E)$ is called the graph $W^*$-probability space induced by the given graph $G$.

By [10], [11], [19], and [20], we have the following two theorems.

**Theorem 2.1.** (See [10] and [11]) Let $M_G$ be the right graph von Neumann algebra of $G$. Then it is $*$-isomorphic to the $D_G$-valued reduced free product algebra $\bigstar_{e \in E(G)}^* M_e$, where $M_e \triangleq vN(G_e, D_G)$ in $B(H_G)$, where $G_e$ are the subgroupoid of $G$, induced by $\{e, e^{-1}\}$, for all $e \in E(G)$. □

**Theorem 2.2.** (See [11]) Let $M_G$ be the right graph von Neumann algebra of $G$, and let $\bigstar_{e \in E(G)}^* M_e$ be the $D_G$-valued reduced free product algebra of $M_e$'s, which is $*$-isomorphic to $M_G$, in $B(H_G)$.

1. If $e$ is a loop edge, then the corresponding $D_G$-free block $M_e$ is $*$-isomorphic to the group von Neumann algebra $L(\mathbb{Z})$, generated by the infinite cyclic abelian group $\mathbb{Z}$, which is also $*$-isomorphic to the $L^\infty$-algebra $L^\infty(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$.

2. If $e$ is a non-loop edge, then $M_e$ is $*$-isomorphic to the matricial algebra $M_2(\mathbb{C})$, consisting of all $(2 \times 2)$-matrices. □

Also, we can have the following classification theorem, in terms of graph theory.

**Theorem 2.3.** (See [11]) Let $G_1$ and $G_2$ be directed graphs and assume that the shadowed graphs $\hat{G}_1$ and $\hat{G}_2$ are graph-isomorphic. Then the graph von Neumann algebras $M_{G_1}$ and $M_{G_2}$ are $*$-isomorphic. □

Unfortunately, the converse of the previous theorem is unknown (e.g., [10], [11], and [49]).

3. **Graph Trees and Graph Fractaloids**

In this section, we define the fractality on graph groupoids. Our “new” definition of graph fractaloids is based on the original automata theoretical definition of graph fractaloids in the sense of [19]. In Section 3.1, we briefly introduce the automata.
theoretical definition of graph fractaloids. And Section 3.2, we re-define graph fractaloids.

3.1. Fractality on Graph Groupoids. Automata theory is the study of abstract machines, and we are using it in the formulation given by von Neumann. It is related to the theory of formal languages. In fact, automata may be thought of as the class of formal languages they are able to recognize. In von Neumann’s version, an automaton is a finite state machine (FSM). i.e., a machine with input of symbols, transitions through a series of states according to a transition function (often expressed as a table). The transition function tells the automata which state to go to next, given a current state and a current symbol. The input is read sequentially, symbol by symbol, for example as a tape with a word written on it, registered by the head of the automaton; the head moves forward over the tape one symbol at a time. Once the input is depleted, the automaton stops. Depending on the state in which the automaton stops, it is said that the automaton either accepts or rejects the input. The set of all the words accepted by the automaton is called the language of the automaton. For the benefit for the readers, we offer the following references for the relevant part of Automata Theory: [1], [33], [34], [35], [48] and [49].

Let the quadruple $A = (D, Q, \varphi, \psi)$ be given, where $D$ and $Q$ are sets and

$$\varphi : D \times Q \rightarrow Q \quad \text{and} \quad \psi : D \times Q \rightarrow D$$

are maps. We say that $D$ and $Q$ are the (finite) alphabet and the state set of $A$, respectively and we say that $\varphi$ and $\psi$ are the output function and the state transition function, respectively. In this case, the quadruple $A$ is called an automaton. If the map $\psi(\bullet, q)$ is bijective on $D$, for any fixed $q \in Q$, then we say that the automaton $A$ is invertible. Similarly, if the map $\varphi(x, \bullet)$ is bijective on $Q$, for any fixed $x \in D$, then we say that the automaton $A$ is reversible. If the automaton $A$ is both invertible and reversible, then $A$ is said to be bi-reversible.

To help visualize the use of automata, a few concrete examples may help. With some oversimplification, they may be drawn from the analysis and synthesis of input / output models in Engineering, often referred to as black box diagram: excitation variables, response variables, and intermediate variables (e.g., see [52] and [53]).

Roughly speaking, a “undirected” tree is a connected simplicial graph without loop finite paths. Recall that a (undirected) graph is simplicial, if the graph has neither loop-edges nor multi-edges connecting distinct two vertices. A directed tree is a connected simplicial graph without loop finite paths. In particular, we say that a directed tree $T_n$ is a $n$-regular tree, if $T_n$ is rooted, one-flowed, infinite directed tree, having the same out-degrees $n$ for all vertices (See Section 3.2, for details). For example, the 2-regular tree $T_2$ can be depicted by
Let \( \mathcal{A} = (D, Q, \varphi, \psi) \) be an automaton with \( |D| = n \). Then, we can construct automata actions \( \{ \mathcal{A}_q : q \in Q \} \) of \( \mathcal{A} \), acting on \( T_n \). Let’s fix \( q \in Q \). Then the action of \( \mathcal{A}_q \) is defined on the finite words \( D^* \) of \( D \), by

\[
\mathcal{A}_q (x) \overset{\text{def}}{=} \varphi(x, q), \text{ for all } x \in D,
\]

and recursively,

\[
\mathcal{A}_q ((x_1, x_2, ..., x_m)) = \varphi(x_1, \mathcal{A}_q(x_2, ..., x_m)),
\]

for all \( (x_1, ..., x_m) \in D^* \), where

\[
D^* \overset{\text{def}}{=} \bigcup_{m=1}^\infty \left\{ (x_1, ..., x_m) \in D^m \mid x_k \in D, \text{ for all } k = 1, ..., m \right\}.
\]

Then the automata actions \( \mathcal{A}_q \)'s are acting on the \( n \)-regular tree \( T_n \). In other words, all images of automata actions are regarded as an elements in the free semigroupoid \( \mathbb{F}^+(T_n) \) of the \( n \)-regular tree. i.e.,

\[
V(T_n) \supseteq D^*
\]

and its edge set

\[
E(T_n) \supseteq \{ \mathcal{A}_q(x) : x \in D, q \in Q \}.
\]

This makes us to illustrate how the automata actions work.

Let \( \mathcal{C} = \{ \mathcal{A}_q : q \in Q \} \) be the collection of automata actions of the given automaton \( \mathcal{A} = < D, Q, \varphi, \psi > \). Then we can create a group \( G(\mathcal{A}) \) generated by the collection \( \mathcal{C} \). This group \( G(\mathcal{A}) \) is called the automata group generated by \( \mathcal{A} \). The generator set \( \mathcal{C} \) of \( G(\mathcal{A}) \) acts fully on the \( |D| \)-regular tree \( T_{|D|} \), we say that this group \( G(\mathcal{A}) \) is a fractal group. There are many ways to define fractal groups, but we define them in the sense of automata groups. (See [1] and [35]. In fact, in [35], Bartholdi, Grigorchuk and Nekrashevych did not define the term “fractal”, but they provide the fractal properties.)

Now, we will define a fractal group more precisely (Also see [1]). Let \( \mathcal{A} \) be an automaton and let \( \Gamma = G(\mathcal{A}) \) be the automata group generated by the automata actions acting on the \( n \)-regular tree \( T_n \), where \( n \) is the cardinality of the alphabet of \( \mathcal{A} \). By \( St_l(k) \), denote the subgroup of \( \Gamma = G(\mathcal{A}) \), consisting of those elements of \( \Gamma \), acting trivially on the \( k \)-th level of \( T_n \), for all \( k \in \mathbb{N} \cup \{0\} \).
 Analogously, for a vertex $u$ in $T_n$, define $St_T(u)$ by the subgroup of $\Gamma$, consisting of those elements of $\Gamma$, acting trivially on $u$. Then

$$St_T(k) = \bigcap_{u: \text{vertices of the } k\text{-th level of } T_n} (St_T(u)).$$

For any vertex $u$ of $T_n$, we can define the algebraic projection $p_u : St_T(u) \to \Gamma$.

**Definition 3.1.** Let $\Gamma = G(A)$ be the automata group given as above. We say that this group $\Gamma$ is a fractal group if, for any vertex $u$ of $T_n$, the image of the projection $p_u (St_T(u))$ is group-isomorphic to $\Gamma$, after the identification of the tree $T_n$ with its subtree $T_u$ with the root $u$.

For instance, if $u$ is a vertex of the 2-regular tree $T_2$, then we can construct a subtree $T_u$, as follows:

As we can check, the graphs $T_2$ and $T_u$ are graph-isomorphic. So, the above definition shows that if the automata actions $A_q$’s of $A$ are acting fully on $T_n$, then the automata group $G(A)$ is a fractal group.

The original definition of (graph) fractaloids in [19] are based on that of fractal groups (Also, see [20] and [22]). To detect the fractality on a connected locally finite directed graph $G$ (or the graph groupoid $G$ of $G$), we define the corresponding automaton

$$A_G = (\pm X, E(G), \varphi, \psi),$$

induced by $G$. To do that we put the weight on $G$ (or the labeling on $G$) by the labeling set $X$. And then consider the automata actions $\{A_w : w \in F^+(\hat{G})\}$: if the actions act fully on the $2N$-regular tree $T_{2N}$, then the groupoid $G$ has fractality, like fractal groups, where
\( N = \max\{\deg_{\text{out}}(v) : v \in V(G)\} \in \mathbb{N}, \) in \( G. \)

Let \( G \) be a given connected locally finite directed graph with the number \( N \), the maximum of the out-degrees of all vertices of \( G \). As usual, we understand the real plane \( \mathbb{R}^2 \) as a 2-dimensional space generated by the horizontal axis (or the \( x \)-axis) and the vertical axis (or the \( y \)-axis), which are homeomorphic to \( \mathbb{R} \). For the given number \( N \), define the lattices \( l_1, ..., l_N \) in \( \mathbb{R}^2 \) by

\[
l_k \overset{\text{def}}{=} (1, e^k) \text{, for all } k = 1, ..., N,\]

where \((t_1, t_2)\) means the vector connecting the origin \((0, 0)\) to the point \((t_1, t_2)\), for \( t_1, t_2 \in \mathbb{R} \). We call \( l_1, ..., l_N \), the upward lattices for \( N \). Define the set \( X \) by the collection of all upward lattices for \( N \), i.e., \( X = \{l_1, ..., l_N\} \). With respect to the upward lattices \( l_1, ..., l_N \), define the downward lattices \( l_{-1}, ..., l_{-N} \) for \( N \), by

\[
l_{-k} \overset{\text{def}}{=} (1, -e^k) \text{, for all } k = 1, ..., N.\]

Define the set \(-X\) by the collection of all downward lattices for \( N \), i.e., \(-X = \{-l_{-1}, ..., -l_{-N}\}\). Define the set \( \pm X \) by the union of \( X \) and \(-X\), i.e., \( \pm X = X \cup -X \). Then the set \( \pm X \) is called the labeling set of \( G \) (or \( \mathcal{G} \)).

For the given lattices in \( \pm X \), we can construct the lattice paths in \( \mathbb{R}^2 \) by the following rules:

\[
l_i l_j = \begin{cases} \text{the vector sum of } l_i \text{ and } l_j, \\ \text{by identifying the ending point } (1, e^i) \text{ of } l_i \text{ to the starting point } (0, 0) \text{ of } l_j, \end{cases}\]

for all \( i, j \in \{\pm 1, ..., \pm N\} \); inductively, we can construct the lattice paths \( l_i, l_{i_2}, ..., l_{i_n} \), for all \( n \in \mathbb{N} \), where \( i_1, ..., i_n \in \{\pm 1, ..., \pm N\} \). Define the lattice path set \( \mathcal{L}_N \) generated by \( \pm X \) by the collection of all lattice paths defined as above. Let \( l = l_{i_1} ... l_{i_n} \in \mathcal{L}_N \). Then the length \( |l| \) of \( l \) is defined to be the number \( n \), the cardinality of the lattices generating the lattice path \( l \). So, the lattice path set \( \mathcal{L}_N \) is decomposed by

\[
\mathcal{L}_N = \bigcup_{k=1}^{\infty} \mathcal{L}_N(k),
\]

where

\[
\mathcal{L}_N(k) \overset{\text{def}}{=} \{l \in \mathcal{L}_N : |l| = k\}, \text{ for all } k \in \mathbb{N}.\]

Clearly, \( \pm X = \mathcal{L}_N(1) \), by definition.

Now, put the weights on edges of the shadowed graphs \( \widehat{G} \) of \( G \). The weighting process on \( G \) is as follows:

(3.1.1) If \( v \in V(G) \) and assume that \( \deg_{\text{out}}(v) = k \in \mathbb{N} \), then

\[
0 \leq k \leq N, \text{ in } \mathbb{N}.\]

Indeed, by the definition of \( N \), the out-degree \( k \leq N \) in \( \mathbb{N} \). Now, let \( e_1, ..., e_k \) be the edges in \( E(G) \), having their initial vertex \( v \), i.e., \( e_j = v e_j \), for all \( j = 1, ..., k \). Then, by the suitable re-arrange of these edges, we can give the lattice weights \( l_j \in X \) to the edges \( e_j \), for all \( j = 1, ..., k \). Let’s denote the weights of \( e_j \)'s by \( \varpi(e_j) \), then, under our setting, \( \varpi(e_j) = l_j \), for all \( j = 1, ..., k \). Do this process for all \( v \in V(G) \). The graph \( G \) with the weighting process \( \varpi \) is called the canonical labeled graph, denoted by \( (G, \varpi) \).

(3.1.2) For the shadowed graph \( G^{-1} \) of \( G \), we do the similar process like (3.1.1). But, in this time, we use the set \(-X\) instead of \( X \). More precisely, if \( e \in E(G) \) with \( \varpi(e) = l_j \in X \), for \( j \in \{1, ..., k\} \), then take the weight \( \varpi(e^{-1}) \) of the shadow \( e^{-1} \)
of $e$ by $l_e \in -X$. The pair $(G^{-1}, \varpi)$ is said to be the canonical labeled shadow of $(G, \varpi)$.

(3.1.3) The shadowed graph $\hat{G}$ of $G$ can have the weighting process based on (3.1.1) and (3.1.2), i.e., if $e \in E(\hat{G})$, then the weight $\varpi(e)$ of $e$ is determined by (3.1.1), whenever $e \in E(G)$, and it is determined by (3.1.2), whenever $e \in E(G^{-1})$. Recall that $E(\hat{G}) = E(G) \cup E(G^{-1})$. The pair $(\hat{G}, \varpi)$ is called the canonical labeled shadowed graph of $(G, \varpi)$.

In the rest of this section, all connected locally finite directed graphs (resp., their shadowed graphs) are canonically labeled by the labeling set $X$ (resp. $\pm X$), as in (3.1.1), (resp., (3.1.2), and (3.1.3)).

Let’s denote the empty lattice by $\emptyset_X$. i.e.,

$$\emptyset_X = (0, 0) \in \mathbb{R}^2.$$ 

Define the sets $\pm X_0$ and $\pm X_0^*$ by

$$\pm X_0 \overset{\text{def}}{=} \{ \emptyset \} \cup X \cup (-X),$$

and

$$\pm X_0^* \overset{\text{def}}{=} \{ \emptyset_X \} \cup \mathcal{L}_N.$$ 

Define the subset $E(\hat{G})_0$ of the free semigroupoid $\mathbb{F}^+(\hat{G})$ of $\hat{G}$ by

$$E(\hat{G})_0 \overset{\text{def}}{=} E(\hat{G}) \cup \{ \emptyset \}.$$ 

Now, for the given graph $G$, define the corresponding automaton $A_G$ by

$$A_G = (\pm X_0, E(\hat{G})_0, \varphi, \psi),$$

satisfying that

$$\varphi(l, e) \overset{\text{def}}{=} \begin{cases} l & \text{if } \exists e_o \in E(\hat{G}), \text{ s.t., } \varpi(e_o) = l \\ \emptyset_X & \text{otherwise.} \end{cases}$$

and

$$\psi(l, e) \overset{\text{def}}{=} \begin{cases} e_o & \text{if } \varphi(l, e) = l \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $l \in \pm X_0$ and $e \in E(\hat{G})_0$, with

$$\varphi(\emptyset_X, e) = \emptyset_X, \text{ for all } e \in E(\hat{G})_0$$

and

$$\psi(l, \emptyset) = \emptyset, \text{ for all } l \in \pm X_0.$$ 

Such an automaton $A_G$ is called the graph-automaton induced by $G$ (or, in short, the $G$-automaton). Then we can construct the automata actions \{ $A_w : w \in \mathbb{F}^+(\hat{G})$ \}, acting on $\pm X_0^*$, and it is easy to check that they act on the $2N$-regular tree $T_{2N}$, because all elements of $\pm X_0^*$ can be embedded in $T_{2N}$, in the natural manner. Assume that $T^G$ is a full-subgraph of $T_{2N}$, where the automata actions of $A_G$ act “fully” on. Then this full-subgraph $T^G$ is called the automata tree of $A_G$ (or $A_{G'}$-tree).

For any nonempty $\varphi(l, w)$, for $l \in \mathcal{L}_N$, and $w \in FP_r(\hat{G})$, we can define the tree $T_w$, where the automata actions

$$\{ A_w' : w' = w w'' , \text{ for } w'' \in \mathbb{F}^+(\hat{G}) \}$$

are acting on. We call the trees $T_w$ the $w$-parts of $T_{2N}$, for all $w \in FP_r(\hat{G})$. The reason why we call $T_w$’s the $w$-parts is that they are full-subgraph of the automata.
tree $T^G$ of $T_{2N}$. Similar to the definition of fractality on groups, we can define the fractality on graph groupoids as follows.

**Definition 3.2.** Let $G$ be a connected locally finite canonical labeled graph and let $A_G$ be the $G$-automaton. Then the graph groupoid $G$ of $G$ is said to be a graph fractaloid, if all $w$-parts $T_w$ are graph-isomorphic to the $A_G$-tree $T^G$, for all $w \in FP_r(G)$.

The above definition is a natural extension of fractality on groups to that on graph fractaloids. Actually the above definition can be extended to define the fractality on groupoids with fractality. So, in [19], instead of using the term “graph fractaloids,” we simply use the term “fractaloids.” However, we prefer to use the term graph fractaloids, because all graph groupoids are groupoids, but the converse does not hold. In [19], we found the following two characterizations of graph fractaloids.

The following theorem is the automata-theoretical characterization of graph fractaloids.

**Theorem 3.1.** (See [19]) Let $G$ be a canonical labeled graph with

$$N = \max\{\deg_{out}(v) : v \in V(G)\} \in \mathbb{N}.$$ 

and let $A_G$ be the $G$-automaton. Then the graph groupoid $G$ of $G$ is a graph fractaloid, if and only if the automata actions $\{A_w : w \in F^+(\hat{G})\}$ act fully on the $2N$-regular tree $T_{2N}$. □

The following theorem is the algebraic characterization of graph fractaloids.

**Theorem 3.2.** (See [19]) Let $G$ be given as in the previous theorem. Then the graph groupoid $G$ of $G$ is a graph fractaloid, if and only if the $A_G$-tree $T^G$ is graph-isomorphic to the $2N$-regular tree $T_{2N}$. □

The above theorems in fact show the difference between fractaloids (groupoids with fractality) and graph fractaloids. Motivated by the previous theorems, without using the automata theory, we can re-define graph fractaloids in Section 3.2. In the rest of this section, we introduce several examples for graph fractaloids. For more interesting examples, see [22].

**Example 3.1.** (1) Let $O_N$ be the one-vertex-$N$-loop-edge graph, for $N \in \mathbb{N}$. Then the graph groupoid $G_O$ of $O_N$ is a graph fractaloid. Recall that, in fact, $O_N$ is a group, which is group-isomorphic to the free group $F_N$ with $N$-generators. And the free groups are fractal groups (See [1]).

(2) Let $K_N$ be the one-flow circulant graph with $N$-vertices with

$$V(K_N) = \{v_1, \ldots, v_N\},$$

and

$$E(K_N) = \{e_j = v_j e_j v_{j+1} : j = 1, \ldots, N, \text{ with } v_{N+1} \overset{\text{def}}{=} v_1\}.$$ 

Then the graph groupoid $G_K$ of $K_N$ is a graph fractaloid.

(3) Let $L_\infty$ be the infinite linear graph, graph-isomorphic to

$$\cdots \to \bullet \to \bullet \to \bullet \to \cdots$$

Then the graph groupoid $G_L$ is a graph fractaloid.

(4) Let $C_N$ be the complete graph with $N$-vertices. Recall that we say that a graph $G$ is complete, if, for any pair $(v_1, v_2)$ of a distinct vertices, there always
exists an edge $e \in E(G)$, such that $e = v_1 e v_2$. Then the graph groupoid $G(C_N)$ of $C_N$ is a graph fractaloid.

In [16], we obtain the following graph-theoretical characterization of graph fractaloids, induced by connected locally finite (finite or infinite) directed graphs.

**Theorem 3.3.** (See [16]) Let $G$ be a connected locally finite directed graph with its graph groupoid $G$. Then $G$ is a graph fractaloid, if and only if the out-degrees and the in-degrees of all vertices are identical in $G$. i.e., a graph $G$ generates a graph fractaloid, if and only if

$$\deg_{\text{out}}(v) = N = \deg_{\text{in}}(v), \text{ in } G,$$

for all $v \in V(G)$, where

$$N = \max\{\deg_{\text{out}}(v) : v \in V(G)\}.$$

□

By the previous theorem, without using automata theory, we can define the graph fractaloids in the following section. However, we want to emphasize that the above theorem is proven in [16], thanks to the automata-theoretical and algebraic characterization of graph fractaloids obtained in [19], based on the automata-theoretical setting on graph groupoids.

### 3.2. Graph Fractaloids

In this section, we construct the graph tree $T_G$ induced by a given connected locally finite directed graph $G$. Throughout this section, all graphs are automatically assumed to be connected, and locally finite. Recall that a directed graph, having neither multi-edges nor loop finite paths, is called a directed tree. If a directed tree $G$ has at least one vertex $v$, satisfying that $\deg_{\text{in}}(v) = 0$, is said to be a directed tree with root(s). The vertices with 0 in-degree are called the roots of $G$. Suppose we have a directed tree $G$ with roots, and assume that we fix one root $v_0$. Then $G$ is called a rooted tree with its root $v_0$. Now, let $G$ be a rooted tree with its root $v_0$, and assume that the direction of $G$ is one-flowed from the root $v_0$ (equivalently, $v_0$ is the only root of $G$). Then $G$ is a one-flow rooted tree. An one-flow rooted tree is infinite, then it is said to be a growing rooted tree. Assume that a growing rooted tree $G$ satisfies that, for any $v \in V(G)$, the out-degree $\deg_{\text{out}}(v)$ are all identical. Then $G$ is a regular tree. In particular, if $\deg_{\text{out}}(v) = N$, for all $v \in V(G)$, then this regular tree $G$ is called the $N$-regular tree. To emphasize the regularity of this tree $G$, we denote this $N$-regular tree $G$ by $T_N$. For instance, the 2-regular tree $T_2$ is as follows:
Let $G$ be a graph, and let $N = \max\{\deg_{\text{out}}(v) : v \in V(G)\} < \infty$ in $\mathbb{N}$.

Consider the shadowed graph $\hat{G}$ of $G$. Define the subsets $E^v_{v'}$ of $E(\hat{G})$ by

$$E^v_{v'} \overset{\text{def}}{=} \{ e \in E(\hat{G}) : e = v v' \},$$

for all $(v, v') \in V(\hat{G})^2$. Remark that $v$ and $v'$ are not necessarily distinct in $V(\hat{G})$. It is possible that there exists a pair $(v_1, v_2)$ of vertices such that $E^{v_2}_{v_1}$ is empty. By definition,

$$E(\hat{G}) = \bigcup_{(v, v')} E^v_{v'}.$$

Then construct the graph tree $T_G$ of $G$, by re-arranging the elements $V(\hat{G}) \cup E(\hat{G})$, up to the admissibility on the free semigroupoid $\mathbb{F}^+(\hat{G})$, as follows. First fix any arbitrary vertex $v_0 \in V(\hat{G}) = V(G)$. Then arrange $e \in \bigcup_{v \in V(\hat{G})} E^v_{v_0}$, by attaching them to $v_0$, preserving the direction on $G$, i.e.,

$$(*)(**)$$

Then we can have the above finite rooted tree with its root $v_0$. Of course, if the set $\bigcup_{v \in V(\hat{G})} E^v_{v_0}$ is empty, then we only have the trivial tree $G_{v_0}$, with $V(G_{v_0}) = \{v_0\}$, and $E(G_{v_0}) = \emptyset$. The edges in the column $(*)$ is induced by the re-arrangement of the elements in $\bigcup_{v \in V(\hat{G})} E^v_{v_0}$, and the vertices in the column $(**)$ means the re-arrangement of the “terminal” vertices of the edges in $\bigcup_{v \in V(\hat{G})} E^v_{v_0}$.

Now, let $v_1 \in V(\hat{G})$ be an arbitrary chosen vertex of the shadowed graph $\hat{G}$ of $G$, re-arranged in $(**)$. Then we can do the same process for $v_1$, i.e., arrange the
edges in \( \bigcup_{v \in V(\hat{G})} E_v \) (if it is not empty), by attaching them to \( v_1 \), preserving the direction on \( G \). i.e., we can construct

\[
\begin{array}{c}
\bullet \\
v_0 \quad \rightarrow \quad \bullet \\
\rightarrow & \quad \rightarrow & \bullet \\
\downarrow & \quad \downarrow & \bullet \\
\end{array}
\]

Here, the column ($) is induced by the re-arrangement of the edges in \( \bigcup_{v \in V(\hat{G})} E_v \), and the vertices in the column ($$) means the re-arrangement of the terminal vertices of the edges in \( \bigcup_{v \in V(\hat{G})} E_v \). We can do the same processes for all vertices in (**). Now, notice that it is possible that one of the vertices in the columns (** or ($$) can be \( v_0 \). For instance, if \( E_{v_0} \) is not empty (equivalently, if \( v_0 \) has an incident loop-edge), then \( v_0 \) is located in (**). Similarly, \( v_0 \) can be located in ($$). For instance, if \( v_0 \) has its incident length-2 loop finite path in \( F^+(\hat{G}) \), then \( v_0 \) is in ($$).

We admit such cases. i.e., a same vertex of \( V(\hat{G}) \) can appear several times in this rooted-tree-making process. Do this process inductively. If \( G \) is infinite, then do this process infinitely. The one-flow rooted tree, induced by this process, with its root \( v_0 \) is denoted by \( T_{v_0} \).

**Definition 3.3.** Let \( G \) be a connected locally finite directed graph with its shadowed graph \( \hat{G} \). And let \( T_{v_0} \) be a rooted tree with its root \( v_0 \), induced by \( G \). We say that this process is the graph-tree making of \( G \). And the tree \( T_{v_0} \) is called the \( v_0 \)-tree (or a vertex-fixed graph tree) of \( G \).

By definition, every connected locally finite directed graph \( G \) has \( |V(\hat{G})| \)-many vertex-trees of \( G \). Notice that the vertex-trees of \( G \) are determined by the vertices and edges in the “shadowed” graph \( \hat{G} \) of \( G \). The following proposition is easily proven by the definition of the vertex-trees of a given graph, and by the connectedness of our graphs.

**Proposition 3.4.** Let \( G \) be a connected locally finite directed graph with its shadowed graph \( \hat{G} \). Let \( F^+(\hat{G}) \) be the free semigroupoid of \( \hat{G} \). Then all elements in \( F^+(\hat{G}) \) are embedded in the \( v \)-tree \( T_v \) of \( G \), for all \( v \in V(\hat{G}) = V(G) \).

Observe now several examples for the construction of vertex-graphs of a given graph.

**Example 3.2.** Let \( O_1 \) be a one-vertex-1-loop-edge graph with

\[
V(O_1) = \{ v \} \quad \text{and} \quad E(O_1) = \{ e = v \ e \ v \}. \]
Then the shadowed graph $\hat{O}_1$ of $O_1$ has its vertex set $V(\hat{O}_1)$, identical to $V(O_1)$, and its edge set

$$E(\hat{O}_1) = \{v\}, \text{ and } E(\hat{O}_1) = \{e, e^{-1}\}.$$  

Then we can construct the $v$-graph of $O_1$ by

$$T_v = v \xrightarrow{e} v_2 \xrightarrow{e^{-1}} v \xrightarrow{v_1} v \cdots$$

We can realize that the $v$-graph $T_v$ is graph-isomorphic to the 2-regular graph $T_2$.

**Example 3.3.** Let $G_e$ be the two-vertices-one-edge graph with

$$V(G_e) = \{v_1, v_2\} \text{ and } E(G_e) = \{e = v_1 e v_2\}.$$  

Then the shadowed graph $\hat{G}_e$ is a directed graph with

$$V(\hat{G}_e) = \{v_1, v_2\} \text{ and } E(\hat{G}_e) = \{e, e^{-1}\}.$$  

So, we can have the $v_1$-tree $T_{v_1}$ of $G$,

$$T_{v_1} = v_1 \xrightarrow{e} v_2 \xrightarrow{e^{-1}} v_1 \xrightarrow{e} v_2 \cdots,$$

and the $v_2$-tree $T_{v_2}$ of $G$,

$$T_{v_2} = v_2 \xrightarrow{e^{-1}} v_1 \xrightarrow{e} v_2 \xrightarrow{e^{-1}} v_1 \cdots.$$  

Therefore, both $T_{v_1}$ and $T_{v_2}$ are graph-isomorphic to the 1-regular tree $T_1$.

**Example 3.4.** Let $T_{2,1}$ be the finite tree with

$$V(T_{2,1}) = \{v_1, v_2, v_3\}$$

and

$$E(T_{2,1}) = \{e_1 = v_1 e_1 v_2, e_2 = v_1 e_2 v_3\}.$$  

i.e.,
Then, after finding, the shadowed graph $\hat{T}_{2,1}$ of $T_{2,1}$, we can have the $v_1$-tree $T_{v_1}$ of $T_{2,1}$,

and the $v_2$-tree $T_{v_2}$ of $T_{2,1}$,

and the $v_3$-graph of $T_{2,1}$,
We can check that $T_{v_2}$ and $T_{v_3}$ are graph-isomorphic, but neither of them is graph-isomorphic to $T_{v_1}$.

**Example 3.5.** Let $K_2$ be the one-flow circulant graph with

$$V(K_2) = \{v_1, v_2\},$$

and

$$E(K_2) = \{e_1 = v_1 v_1, e_2 = v_2 e_2 v_1\}.$$  

Then the shadowed graph $\hat{K}_2$ of $K_2$ has

$$V(\hat{K}_2) = \{v_1, v_2\},$$

and

$$E(\hat{K}_2) = \{e_1^{\pm 1}, e_2^{\pm 1}\}.$$  

By using the tree-making process, we obtain that the $v_1$-tree $T_{v_1}$ and the $v_2$-tree $T_{v_2}$ are graph-isomorphic to the 2-regular tree $T_2$. In general, every one-flow circulant graph $K_n$ has its vertex-trees graph-isomorphic to the 2-regular tree $T_2$.

As we have seen in the previous examples, sometimes, the vertex-trees of a given graph are graph-isomorphic from each other, or not. In general, the vertex-trees of a graph $G$ are not graph-isomorphic from each other.

**Definition 3.4.** Let $G$ be a connected locally finite directed graph and $\{T_v : v \in V(G)\}$, the collection of all vertex-trees of $G$. Also, let

$$N = \max \{\deg_{out}(v) : v \in V(G)\}$$

(“not” in $\hat{G}$). If every $v$-tree $T_v$ of $G$ is graph-isomorphic to the $2N$-regular tree $T_{2N}$, for all $v \in V(\hat{G})$, then the graph groupoid $\mathbb{G}$ of $G$ is called the graph fractaloid induced by $G$. And the graph $G$ is said to be a fractal graph.

Under the above (new) definition of the fractal graphs and graph fractaloids, we can re-obtain the graph-theoretical characterization of graph fractaloids of [16]. In fact, the above new definition for graph fractaloids (and fractal graphs) is based on the re-expression of automata trees in the sense of [19]. The vertex-trees $T_v$’s of a given graph $G$ can be understood as the re-expression of the automata-trees without using the automata-theoretical labeling process on $G$ (or on $\hat{G}$). Depending on the new definition for graph fractaloids, we can get the graph-theoretical characterization of graph fractaloids as follows:

**Theorem 3.5.** (See [16]) Let $G$ be a connected locally finite directed graph. The graph $G$ is a fractal graph, if and only if
\[
\text{deg}_{\text{out}}(v) = \text{deg}_{\text{in}}(v), \text{ in } G,
\]
for all \( v \in V(G) \). \( \square \)

Thus, without loss of generality, we can re-define the fractal graphs and graph fractaloids as follows: A connected locally finite directed graph \( G \) is a fractal graph, if the out-degrees and the in-degrees of all vertices of \( G \), in \( G \), are identical from each other. And, if a graph \( G \) is a fractal graph, then the graph groupoid \( \mathcal{G} \) of \( G \) is said to be a graph fractaloid.

**Example 3.6.** (1) The one-vertex-\( n \)-loop-edge graph \( O_n \) is a fractal graph, for all \( n \in \mathbb{N} \), since

\[
\text{deg}_{\text{out}}(v) = n = \text{deg}_{\text{in}}(v), \text{ in } O_n
\]
where \( v \) is the only vertex of \( O_n \), for all \( n \in \mathbb{N} \).

(2) The one-flow circulant graph \( K_n \) is a fractal graph, for all \( n \in \mathbb{N} \setminus \{1\} \), since

\[
\text{deg}_{\text{out}}(v) = 1 = \text{deg}_{\text{in}}(v), \text{ in } K_n,
\]
for all \( v \in V(K_n) \).

(3) Let \( C_n \) be the complete graph with \( n \)-vertices, for \( n \in \mathbb{N} \setminus \{1\} \). i.e., it is a graph with

\[
V(C_n) = \{ v_1, ..., v_n \},
\]
and

\[
E(C_n) = \{ e_{ij} \mid i \neq j \in \{1, ..., n\} \},
\]
where \( e_{ij} \) means the edge connecting the vertex \( v_i \) to the vertex \( v_j \). Then the graph \( C_n \) is a fractal graph, for all \( n \in \mathbb{N} \setminus \{1\} \), since

\[
\text{deg}_{\text{out}}(v_j) = n - 1 = \text{deg}_{\text{in}}(v_j), \text{ in } C_n,
\]
for all \( j = 1, ..., n \).

(4) Let \( L \) be the infinite linear graph, graph-isomorphic to

\[
\cdots \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots
\]

Then it is a fractal graph, since

\[
\text{deg}_{\text{out}}(v) = 1 = \text{deg}_{\text{in}}(v), \text{ in } L,
\]
for all \( v \in V(L) \).

(5) Let \( T_k \) be the \( k \)-regular graph, for \( k \in \mathbb{N} \). Then it is not a fractal graph. Assume that \( v_0 \) is a root of \( T_k \). Then

\[
\text{deg}_{\text{out}}(v_0) = k \neq 0 = \text{deg}_{\text{in}}(v), \text{ in } T_k.
\]

Therefore, the regular trees are not fractal.

More generally, we define graph fractaloids, without connectedness condition.

**Definition 3.5.** Let \( G \) be a locally finite directed graph with its connected components \( G_1, ..., G_t \), for \( t \in \mathbb{N} \). Let \( \mathcal{G} \) be the graph groupoid of \( G \). We say that \( \mathcal{G} \) is a graph fractaloid, if each \( G_j \) generates a graph fractaloid \( \mathcal{G}_j \), in the above sense, for all \( j = 1, ..., t \). In this case, the graph \( G \) is called the “disconnected” fractal graph.

However, in the rest of this paper, all our graphs are connected.

Let \( G \) be a connected locally finite graph with its graph groupoid \( \mathcal{G} \). Let \((v_1, v_2)\) be the pair of vertices of \( G \) (Remark that \( v_1 \) and \( v_2 \) are not necessarily distinct), and assume that there exists an edge \( e = v_1 \rightarrow v_2 \). Let’s replace this edge \( e \) to the \( k \)-multi-edges \( e_1, ..., e_k \), satisfying \( e_j = v_1 \rightarrow e_j \rightarrow v_2 \). Do this process for all pair \((v, v')\) of the vertices of \( G \), whenever there exists at least one edge connecting \( v \) to
Clearly, if there is no edge connecting \( v \) to \( v' \), then we do not need to do this process. Then we can create a new connected locally finite graph \( G' \), satisfying that

\[
V(G') = V(G).
\]

**Definition 3.6.** The new connected locally finite graph \( G' \) induced by a given connected locally finite graph \( G \), in the previous paragraph, is called the regularized graph of \( G \), denoted by \( R_k(G) \), where \( k \) is the cardinality of the multi-edges in \( G' \) replaced by the edges in \( G \), for all \( k \in \mathbb{N} \).

In [16], we showed that:

**Theorem 3.6.** (Also, see [16]) Let \( G \) be a fractal graph. Then the \( k \)-regularized graph \( R_k(G) \) is a fractal graph, too.

**Proof.** Indeed, assume that \( G \) is a fractal graph, satisfying that

\[
\deg_{out}(v) = N = \deg_{in}(v), \text{ in } G,
\]

for all \( v \in V(G) \). Then the \( k \)-regularized graph \( R_k(G) \) satisfies that

\[
\deg_{out}(x) = kN = \deg_{in}(x), \text{ in } R_k(G),
\]

for all \( x \in V(R_k(G)) = V(G) \), for all \( k \in \mathbb{N} \). Therefore, by the graph-theoretical characterization, the graph \( R_k(G) \) is again a fractal graph. \( \blacksquare \)

In the previous example, we showed that the graphs \( O_N, K_n, C_n \) and \( L \) are fractal graphs, for \( N \in \mathbb{N}, n \in \mathbb{N} \setminus \{1\} \). By the previous theorem, we can conclude that the \( k \)-regularized graphs \( R_k(O_N), R_k(K_n), R_k(C_n) \), and \( R_k(L) \) are fractal graphs, too, for all \( k \in \mathbb{N} \).

Let \( G_1 \) and \( G_2 \) be connected locally finite graphs. Define the unioned graph \( G = G_1 \cup G_2 \) of \( G_1 \) and \( G_2 \) by a new directed graph with

\[
V(G) = V(G_1) \cup V(G_2),
\]

and

\[
E(G) = E(G_1) \cup E(G_2).
\]

So, every “disjoint” unioned graph \( G \), satisfying

\[
V(G) = V(G_1) \sqcup V(G_2),
\]

and

\[
E(G) = E(G_1) \sqcup E(G_2),
\]

is a unioned graph. But, notice that not all unioned graphs are disjoint unioned graphs! For instance, if \( G_1 \) and \( G_2 \) are full-subgraphs of a connected locally finite graph \( K \), then it is possible that

\[
V(G_1) \cap V(G_2) \neq \emptyset,
\]

or

\[
E(G_1) \cap E(G_2) \neq \emptyset.
\]

Also, our shadowed graphs are unioned graphs which are not disjoint unioned graphs, i.e., \( \hat{G} = G \sqcup G^{-1} \), where \( G^{-1} \) is the shadow of \( G \). Furthermore, we are not interested in the disjoint unioned graphs, because the disjoint union of graphs generates the “disconnected” graphs.
Now, let \( G_k \) be connected locally finite graphs, and let \( v_k \in V(G_k) \) be the fixed vertices, for \( k = 1, 2 \). Then, by identifying the chosen vertices \( v_1 \) and \( v_2 \), we can create a new graph \( G \), denoted by
\[
G_1 \overset{v_1}{\#} v_2^{v_2} G_2.
\]
The identified vertex of \( v_1 \) and \( v_2 \) is called the **glued vertex of** \( v_1 \in V(G_1) \) and \( v_2 \in V(G_2) \). Denote it by \( v_\times \). Then the graph \( G = G_1 \overset{v_1}{\#} v_2^{v_2} G_2 \) is the graph with
\[
V(G) = \{v_\times\} \cup (V(G_1) \setminus \{v_1\}) \cup (V(G_2) \setminus \{v_2\}),
\]
and
\[
E(G) = E(G_1) \cup E(G_2),
\]
under the identification rule: if \( e \in E(G_k) \) satisfies either \( e = v_k \) or \( e = v_\times \), then this edge \( e \) is identified with the edge, also denoted by \( e \), satisfying that \( e = v_\times \), respectively, \( e = v_\times \).

This new graph \( G = G_1 \overset{v_1}{\#} v_2^{v_2} G_2 \), by identifying the vertices \( v_1 \in V(G_1) \) and \( v_2 \in V(G_2) \), is called the **glued graph of** \( G_1 \) and \( G_2 \), with the **glued vertex of** \( v_1 \) and \( v_2 \). Again, let \( v_k \in V(G_k) \) be the fixed vertices, for \( k = 1, 2 \). Define the new connected locally finite graphs
\[
G_1 \overset{v_1}{\#} v_2^{v_2} G_2 \overset{\text{def}}{=} \bigcup_{v \in V(G_1)} (G_1 \overset{v_1}{\#} v^{v_2} G_2)
\]
and
\[
G_1 \overset{v_1}{\#} G_2 \overset{\text{def}}{=} \bigcup_{v \in V(G_2)} (G_1 \overset{v_1}{\#} v G_2).
\]
Then the graph \( G_1 \overset{v_2}{\#} G_2 \) (resp., \( G_1 \overset{v_1}{\#} G_2 \)) is called the **iterated glued graph with the fixed vertex** \( v_2 \in V(G_2) \) (resp., \( v_1 \in V(G_1) \)). In [16], we showed that:

**Theorem 3.7.** (Also, see [16]) Let \( G \) be a fractal graph and let \( O_n \) be the one-vertex-\( n \)-loop-edge graph, for \( n \in \mathbb{N} \). If \( v \) is the unique vertex of \( O_n \), then the iterated glued graph \( G \overset{v}{\#} O_n \) is a fractal graph, too, for all \( n \in \mathbb{N} \).

**Proof.** Roughly speaking the iterated glued graph \( G \overset{v}{\#} O_n \) is the graph gotten by gluing the unique vertex \( v \) of \( O_n \) to every vertex of \( G \), recursively. Assume that \( G \) is a fractal graph, and assume that
\[
\deg_{\text{out}}(v) = N = \deg_{\text{in}}(v), \quad \text{in } G,
\]
for all \( v \in V(G) \), and for some \( N \in \mathbb{N} \). Notice that, by the construction of the iterated glued graph \( G \overset{v}{\#} O_n \),
\[
V(G) = V(G \overset{v}{\#} O_n),
\]
since \( O_n \) has only one vertex \( v \), for all \( n \in \mathbb{N} \). So, we can check that
\[
\deg_{\text{out}}(v) = N + n = \deg_{\text{in}}(v), \quad \text{in } G \overset{v}{\#} O_n,
\]
for all \( v \in V(G \overset{v}{\#} O_n) = V(G) \), for all \( n \in \mathbb{N} \). Thus, by the graph-theoretical characterization of graph fractaloids, the graph \( G \overset{v}{\#} O_n \) is a fractal graph, too. \( \blacksquare \)

Again, by the previous example, we can conclude that \( O_n \overset{v}{\#} O_n \overset{\text{Gr}}{=} O_{2n}, K_n \overset{v}{\#} O_n, C_n \overset{v}{\#} O_n \), and \( L \overset{v}{\#} O_n \) are fractal graphs, too. For example, \( L \overset{v}{\#} O_1 \) is a graph,
\[
\ldots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \ldots
\]
and hence
\[ \deg_{out}(v) = 2 = \deg_{in}(v), \text{ in } L \#^v O_1, \]
for all \( v \in V(L \#^v O_1) = V(L) \). Thus \( L \#^v O_1 \) is a fractal graph.

4. Radial Operators of Graph Fractaloids

In this section, we provide a tool to study the property of graph fractaloids, in operator theory. Let \( G \) be a graph fractaloids, induced by a connected locally finite directed graph \( G \). Then, as in [19], we define a suitable Hilbert space operator \( T_G \), induced by \( G \), and we observe the spectral property of \( T_G \), by considering the operator-valued free distributional data of \( \Gamma \). Then such spectral information of \( T_G \) explains how the fractality of \( G \) acts on Hilbert spaces. In [16] and [19], we define \( T_G \), in terms of the labelings on the given fractaloid \( G \), on the graph Hilbert space \( H_G \). Recall that the labelings are determined by the graph automaton \( A_G \), induced by the graph \( G \), and we called \( T_G \), the labeling operator of \( G \) on the graph Hilbert space \( H_G \). In this paper, we define graph fractaloids without using automata theory. Hence, we need to define \( T_G \), differently.

Let \( \Gamma \) be an arbitrary group and let \( L(\Gamma) = \mathbb{C}[\theta(\Gamma)]^w \) be the group von Neumann algebra, generated by the group \( \Gamma \), acting on the group Hilbert space \( H_\Gamma = l^2(\Gamma) \), where \((H_\Gamma, \theta)\) is the (left) unitary regular representation of \( \Gamma \), i.e., \( \theta: \Gamma \to B(H_\Gamma) \) is a group action, acting on \( H_\Gamma \),

\[ \theta(g) \xi_{g'} = u_g \xi_{gg'}, \text{ for all } \xi_{g'} \in H_\Gamma, \]
where \( u_g \) is the unitary with its adjoint \( u_g^* = u_{g^{-1}} \), where \( g^{-1} \) is the group inverse of \( g \) in \( \Gamma \). Clearly, we can define the “right” group action \( \pi: \Gamma \to B(H_\Gamma) \) by

\[ \pi(g) \xi_{g'} = u_{g^\op} \xi_{g'g}, \text{ for all } g' \in \Gamma. \]

Then the right group von Neumann algebra \( R(\Gamma) = \mathbb{C}[\pi(\Gamma)]^w \) is well-defined in \( B(H_\Gamma) \), and it is the opposite \( W^* \)-algebra \( L(\Gamma)^{op} \) of \( L(\Gamma) \), and hence they are anti-*-isomorphic from each other. Now, fix a group \( \Gamma \) and its right (or left) group von Neumann algebra \( R(\Gamma) \) (resp., \( L(\Gamma) \)). Then the radial operator \( T_\Gamma \) of \( \Gamma \) in \( R(\Gamma) \) (resp., \( L(\Gamma) \)) is defined by

\[ T_\Gamma \overset{def}{=} \sum_{g \in X} \pi(g) = \sum_{g \in \Gamma} u_{g^\op}, \]

(See [9]), where \( X \subset \Gamma \) is the set of all generators of \( \Gamma \). Instead of using automata theory, like the radial operators of groups, we will re-define the labeling operators of graph fractaloids (in the sense of [20] and [19]) without the labeling process, and we call them, the radial operators of graph fractaloids.

4.1. Radial Operators. Let \( G \) be a connected locally finite directed graph with its graph groupoid \( G \), and let

\[ N \overset{def}{=} \max \{ \deg_{out}(v) : v \in V(G) \}. \]

Let \( M_G \) be the right graph von Neumann algebra of \( G \), which is the groupoid \( W^* \)-algebra \( \mathbb{C}[R(G)]^w \) in \( B(H_G) \), where \( (H_G, R) \) is the canonical right representation of \( G \) (See Section 2.4).
Definition 4.1. Let $M_G$ be the right graph von Neumann algebra of $G$. The radial operator $T_G$ of the graph groupoid $\mathcal{G}$ of $G$ is defined by an element

$$T_G = \sum_{e \in E(\hat{G})} R_e = \sum_{e \in E(G)} (R_e + R_{e^{-1}}),$$

in $M_G$, where $\hat{G}$ is the shadowed graph of $G$.

The radial operator $T_G$ of $\mathcal{G}$ is similarly defined like the Hecke-type operators or the Ruelle operators of groups. It is easy to check that:

Lemma 4.1. The radial operator $T_G$ of the graph groupoid $\mathcal{G}$ of $G$ is self-adjoint. □

Notice that the radial operator $T_G$ of $\mathcal{G}$ is defined by the edges in the “shadowed” graph $\hat{G}$ of $G$. Therefore, for any summand $R_e$, we can find its adjoint $R_e^* = R_{e^{-1}}$, as a summand of $T_G$, i.e., we can re-define $T_G$ by

$$T_G = \sum_{e \in E(G)} (R_e + R_e^*).$$

Therefore, indeed, the operator $T_G$ is self-adjoint in $M_G$. By the self-adjointness of $T_G$, the free distributional data of $T_G$, in $M_G$, represents the spectral property of $T_G$ on $H_G$.

4.2. Spectral Property of Graph Fractaloids. By Voiculescu, the spectral property of a “self-adjoint” operator $x$ in a von Neumann algebra $\mathcal{M}$ is represented by the free distributional data of $x$ over a $W^*$-subalgebra $\mathcal{N}$ of $\mathcal{M}$, if there is a suitable conditional expectation $E : \mathcal{M} \to \mathcal{N}$ (See [5] and [28]). So, to consider the spectral property of our radial operator $T_G$ of $\mathcal{G}$ (on $H_G$), we can compute the free moments $\{E(T_G^n)\}_{n=1}^{\infty}$, where $E : M_G \to D_G$ is the canonical conditional expectation in the sense of [10] and [11], where

$$D_G \overset{\text{def}}{=} \bigoplus_{v \in V(\hat{G})} (\mathbb{C} \cdot R_v)$$

is the diagonal subalgebra of $M_G$. Since every element $a \in M_G$ has its expression

$$a = \sum_{w \in \hat{G}} t_w R_w, \text{ with } t_w \in \mathbb{C},$$

we can define $E$ by

$$E(a) \overset{\text{def}}{=} \sum_{v \in V(\hat{G})} t_v R_v,$$

for all $a \in M_G$. Then the pair $(M_G, E)$ is a $D_G$-valued $W^*$-probability space in the sense of Voiculescu. The $D_G$-valued moments of $a$ is defined by the sequence $\{E(a^n)\}_{n=1}^{\infty}$, and the $D_G$-valued $*$-moments of $a$ is defined by the collection,

$$\{E(a^{i_1} a^{i_2} \ldots a^{i_n}) : (i_1, \ldots, i_n) \in \{1, \ast\}^n, \forall n \in \mathbb{N}\}.$$

Then these $D_G$-valued $*$-moments of $a$ contain the free distributional data of $a$. Clearly, if $a$ is self-adjoint, then the sequence $\{E(a^n)\}_{n=1}^{\infty}$ contains the same data (See [5] and [28]).

Let $\mathbb{R}^2$ be the 2-dimensional $\mathbb{R}$-vector space. Without loss of generality, we assume $\mathbb{R}^2$ is the $\mathbb{R}$-plane, generated by the horizontal axis and the vertical axis. When we denote the point $P$ in $\mathbb{R}^2$, we will use the pair notation $(a, b)$, as usual. When we regard the point $P$ in $\mathbb{R}^2$ as a vector connecting the origin $(0, 0)$ to $P$, we use the vector notation $(\overline{a}, \overline{b})$. Fix $N \in \mathbb{N}$, and define the lattices $l_1, \ldots, l_N$, embedded in $\mathbb{R}^2$, by the vectors,
\[ l_k \overset{\text{def}}{=} (1, e^k), \text{ for all } k = 1, \ldots, N, \]

where \( e \) is the natural exponential number in \( \mathbb{R} \). Define the corresponding downward lattices \( l_{-1}, l_{-2}, \ldots, l_{-N} \) by

\[ l_{-k} \overset{\text{def}}{=} (1, -e^k), \text{ for all } k = 1, \ldots, N. \]

Now, construct lattice paths, by attaching \( l_{\pm 1}, \ldots, l_{\pm N} \) as follows; \( l_i \) \( l_j \) is a lattice path, by transforming the starting point of the lattice \( l_j \) to the end point of \( l_i \).

i.e., we identify the starting point \((0, 0)\) of \( l_j \) to the ending point \((1, \varepsilon_i e^i)\) of \( l_i \), where

\[ \varepsilon_i = \begin{cases} 1 & \text{if } i > 0 \\ -1 & \text{if } i < 0 \end{cases}, \]

for all \( i \in \{ \pm 1, \ldots, \pm N \} \). Inductively, we can determine the lattice paths

\[ l_{i_1} l_{i_2} \ldots l_{i_n}, \]

for all \( (i_1, \ldots, i_n) \in \{ \pm 1, \ldots, \pm N \}^n \), for all \( n \in \mathbb{N} \).

By \( \mathcal{L}_N \), we denote the collection of all lattice paths induced by the lattices \( l_{\pm 1}, \ldots, l_{\pm N} \). Now, let \( l_{i_1} \ldots l_{i_n} \in \mathcal{L}_N \). Then we define the length \( |l_{i_1} \ldots l_{i_n}| \) of the given lattice path by the cardinality \( n \) of the lattices, generating the lattice path i.e.,

\[ |l_{i_1} \ldots l_{i_n}| = n. \]

Define the subset \( \mathcal{L}_N(n) \) of \( \mathcal{L}_N \) by the collection of all length-\( n \) lattice paths,

\[ \mathcal{L}_N(n) \overset{\text{def}}{=} \{ l \in \mathcal{L}_N : |l| = n \}, \text{ for all } n \in \mathbb{N}. \]

Then \( \mathcal{L}_N \) is decomposed by \( \mathcal{L}_N(n)'s:\n
\[ \mathcal{L}_N = \bigsqcup_{n=1}^{\infty} (\mathcal{L}_N(n)). \]

Let \( l \in \mathcal{L}_N(n) \), for some \( n \in \mathbb{N} \). And assume that \( l \) starts at \((0, 0)\), and \( l \) end on the horizontal axis. Such a lattice path \( l \) is said to be a lattice path satisfying the axis property. Define the subset \( \mathcal{L}_N^o(n) \) of \( \mathcal{L}_N(n) \) by

\[ \mathcal{L}_N^o(n) \overset{\text{def}}{=} \{ l \in \mathcal{L}_N(n) : l \text{ satisfies the axis property} \}, \]

for all \( n \in \mathbb{N} \). By definition, \( \mathcal{L}_N^o(n) \) is empty, whenever \( n \) is odd.

In [19], we found the following \( D_G \)-valued moments of the radial operator \( T_G \) of \( \mathbb{G} \), where \( \mathbb{G} \) is a graph fractaloid.

**Theorem 4.2.** Let \( T_G \) be the radial operator of a graph fractaloid \( \mathbb{G} \), induced by a connected locally finite directed graph \( G \), and let

\[ N \overset{\text{def}}{=} \max\{ \deg_{\text{out}}(v) : v \in V(G) \}. \]

Then

\[ E(T_G^n) = |\mathcal{L}_N^o(n)| \cdot 1_{D_G}, \text{ for all } n \in \mathbb{N}. \]

More precisely,

\[ E(T_G^n) = \begin{cases} |\mathcal{L}_N^o(n)| \cdot 1_{D_G} & \text{if } n \text{ is even} \\ 0_{D_G} & \text{if } n \text{ is odd}. \end{cases} \]

\[ \square \]

In [40], the first author and his undergraduate students computed the cardinalities \( |\mathcal{L}_N^o(n)| \) of \( \mathcal{L}_N^o(n) \). In particular, they could find:

**Example 4.1.** (See [40]) For any \( n \in \mathbb{N} \),

\[ |\mathcal{L}_1(2n)| = 2nC_n, \]

and
\[ |\mathcal{L}_2(2n)| = 2^n C_n \left( \sum_{j=0}^{2n} (\alpha C_j)^2 \right), \]

where \( mC_k \overset{\text{def}}{=} \frac{m!}{m(m-n)!} \), for all \( n \leq m \) in \( \mathbb{N} \).

More precisely, we have the following computation.

**Proposition 4.3.** (See [19] and [40]) Let \( N \in \mathbb{N} \), and \( \mathcal{L}_N(n) \), the collection of all length-\( n \) lattice paths induced by the lattices \( l_{\pm 1}, \ldots, l_{\pm N} \). If \( \mathcal{L}_N(2n) \) is the subset of \( \mathcal{L}_N(2n) \), consisting of all length-2n lattice paths satisfying the axis property, for all \( n \in \mathbb{N} \), then

\[ |\mathcal{L}_N(2n)| = \sum_{(j_1, \ldots, j_{2n}) \in \mathcal{C}_{2n}} c_{j_1, \ldots, j_{2n}}, \]

where

\[ \mathcal{C}_{2n} \overset{\text{def}}{=} \left\{ (j_1, \ldots, j_{2n}) \mid (j_1, \ldots, j_{2n}) \in \{\pm 1, \ldots, \pm N\}^{2n}, \right. \]

\[ j_1 \leq j_2 \leq \cdots \leq j_{2n} \]

\[ \sum_{k=1}^{2n} j_k = 0 \]

and where the numbers \( c_{j_1, \ldots, j_{2n}} \) satisfies the recurrence relation:

\[ c_{j_1, \ldots, j_{2n}} = c_{j_1, \ldots, j_{2n-m}, j_0, j_0, \ldots, j_0, j_0} = c_{j_1, \ldots, j_{2n-m}, 2nC_m}, \]

with

\[ c_j, j = 1, \text{ for all } j \in \{\pm 1, \ldots, \pm N\}, \]

for all \( 1 \leq m \leq 2n \), in \( \mathbb{N} \). Here, \( \alpha C_k \) means \( \frac{m!}{m(m-n)!} \), for all \( k \leq n \) in \( \mathbb{N} \).

For instance, assume that we have \( c_{-3, -2, -2, -1, 1, 2, 3} \), as a summand of \( |\mathcal{L}_N(6)| \), for \( N \geq 3 \). Indeed, it is a summand of \( |\mathcal{L}_N(6)| \), since

\[ (-3, -2, -2, -1, 1, 2, 3) \in \mathcal{C}_6. \]

Then, by the recurrence relation, we can compute it by

\[ c_{-3, -2, -2, -1, 1, 2, 3} = c_{-3, -2, -2, -1, 1, 2} \]

\[ = c_{-3, -2, -2, -1} \cdot 5C_2 \cdot 6C_1 \]

\[ = c_{-3, -2, -2} \cdot 4C_1 \cdot 5C_2 \cdot 6C_1 \]

\[ = c_{-3} \cdot 3C_2 \cdot 4C_1 \cdot 5C_2 \cdot 6C_1 \]

\[ = 1 \cdot 3C_2 \cdot 4C_1 \cdot 5C_2 \cdot 6C_1. \]

5. Classification of Graph Fractaloids

Let \( A_1 \) and \( A_2 \) be von Neumann algebras and let \( a_k \in A_k \) be arbitrary fixed “self-adjoint” operators, for \( k = 1, 2 \). Assume that \( A_1 \) and \( A_2 \) contains their \( W^\ast \)-subalgebras \( B_1 \) and \( B_2 \), which are \( * \)-isomorphic to a von Neumann algebra \( B \). Without loss of generality, assume that \( A_1 \) and \( A_2 \) contain their common \( W^\ast \)-subalgebra \( B \). Let \( E_k : A_k \to B \) be a conditional expectation, for \( k = 1, 2 \). We say that the self-adjoint operators \( a_1 \) and \( a_2 \) are identically free distributed over \( B \), if

\[ E(a_1^n) = E(a_2^n) \text{ in } B, \text{ for all } n \in \mathbb{N}. \]

If two connected locally finite directed graphs \( G_1 \) and \( G_2 \) are fractal graphs, and if

\[ \text{deg}_{out}(v_1) = N = \text{deg}_{out}(v_2), \]

for \( v_1 \in V(G_1) \) and \( v_2 \in V(G_2) \), then the corresponding graph fractaloids \( \mathbb{G}_1 \) and \( \mathbb{G}_2 \) have the identically free distributed spectral properties “up to identity elements.” Indeed, if \( N \in \mathbb{N} \) is given as above, then
Radial operators

\[ E(T_{G_1}^n) = |L^e_n(n)| \cdot 1_{D_{G_1}} \]
and
\[ E(T_{G_2}^n) = |L^e_n(n)| \cdot 1_{D_{G_2}}, \]

where \( L^e_n(n) \) is the collection of all length-\( n \) lattice paths, induced by the lattices \( l_{\pm 1}, \ldots, l_{\pm N} \), satisfying the axis property, for \( n \in \mathbb{N} \). For instance, let \( G_1 = K_n \) be the one-flow circulant graph with \( n \)-vertices, for \( n \in \mathbb{N} \), and \( G_2 = L \), where \( L \) is the infinite linear graph with

\[ V(L) = \{ \ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, v_3, \ldots \} \]
and
\[ E(L) = \{ e_j = v_j, v_{j+1} : j \in \mathbb{Z} \}. \]

Then the corresponding graph groupoids \( G_1 \) and \( G_2 \) are graph fractaloids, since

\[ \deg_{\text{out}}^{(G_1)}(v) = 1 = \deg_{\text{in}}^{(G_1)}(v), \text{ in } G_1, \]
and
\[ \deg_{\text{out}}^{(G_2)}(x) = 1 = \deg_{\text{in}}^{(G_2)}(x), \text{ in } G_2, \]
for all \( v \in V(G_1) \), and \( x \in V(G_2) \). So, the radial operators \( T_{G_1} \) and \( T_{G_2} \) have their amalgamated free moments,

\[ E(T_{G_1}^k) = |L^e_1(k)| \cdot 1_{C^\otimes n}, \]
and
\[ E(T_{G_2}^k) = |L^e_1(k)| \cdot 1_{C^\otimes \infty}, \]

because

\[ D_{G_1} \overset{\ast}{=} \text{isomorphic } C^\otimes n, \text{ in } M_{G_1}, \]
and
\[ D_{G_1} \overset{\ast}{=} \text{isomorphic } C^\otimes \infty, \text{ in } M_{G_2}. \]

**5.1. Fractal Pairs of Graph Fractaloids.** Motivated by the example, in the previous paragraph, we can obtain the following proposition.

**Proposition 5.1.** Let \( G_1 \) and \( G_2 \) be connected locally finite directed graphs with

\[ \max\{ \deg_{\text{out}}(v_1) : v_1 \in V(G_1) \} = N_0 = \max\{ \deg_{\text{out}}(v_2) : v_2 \in V(G_2) \}, \]
in \( \mathbb{N} \), and

\[ |V(G_1)| = N^0 = |V(G_2)|, \]

where \( N_0 \in \mathbb{N} \cup \{ \infty \} \). If \( G_1 \) and \( G_2 \) are fractal graphs, then the radial operators \( T_{G_1} \) and \( T_{G_2} \) are identically free distributed over \( C^\otimes N^0 \).

**Proof.** Since \( |V(G_1)| = N^0 = |V(G_2)| \) in \( \mathbb{N} \cup \{ \infty \} \), the diagonal subalgebras \( D_{G_1} \) and \( D_{G_2} \) are \( \ast \)-isomorphic to \( C^\otimes N^0 \). Let’s denote \( C^\otimes N^0 \) by \( B \). By the fractality of the graphs \( G_1 \) and \( G_2 \), we have

\[ E(T_{G_1}^n) = |L^e_{N_0}(n)| \cdot 1_B = E(T_{G_2}^n), \]
for all \( n \in \mathbb{N} \), where \( 1_B \) is the identity \((N^0 \times N^0)\)-matrix in \( B \). Therefore, the radial operators \( T_{G_1} \) and \( T_{G_2} \) are identically free distributed over \( B \). □

The above proposition shows that the pair \((N_0, N^0)\) of the numbers \( N_0 \in \mathbb{N} \), and \( N^0 \in \mathbb{N} \cup \{ \infty \} \), can explain the fractality (characterized by the spectral information of the radial operator \( T_G \)) of a graph fractaloid \( \mathbb{G} \), via the sequence
\[
\left(\left|\mathcal{L}_{N_0}^n(n)\right|\right)_{n=1}^\infty \text{ over } \mathbb{C}^{\oplus N^0},
\]
where
\[
N_0 = \text{deg}_\text{out}(v) = \text{deg}_\text{in}(v), \text{ in } G,
\]
for all \(v \in V(G)\), and
\[
N^0 = \left|V(G)\right|.
\]

**Definition 5.1.** Let \(G\) be a connected locally finite directed graph with its graph groupoid \(G\), and assume that \(G\) is a graph fractaloid. Then the pair \((N_0, N^0)\), where
\[
N_0 = \max\{\text{deg}_\text{out}(v) : v \in V(G)\} \in \mathbb{N}
\]
and
\[
N^0 = \left|V(G)\right| \in \mathbb{N} \cup \{\infty\} \text{ denote } \mathbb{N}_\infty,
\]
is called the fractal pair of \(G\). Denote the fractal pair \((N_0, N^0)\) of a graph fractaloid \(G\) by \(fp(G)\).

By the previous proposition, we can get the following theorem.

**Theorem 5.2.** Let \(G_k\) be connected locally finite fractal graphs with their graph fractaloids \(G_k\), for \(k = 1, 2\). If the fractal pairs of \(G_k\) are identical to \((N_0, N^0)\), for \(N_0 \in \mathbb{N}\), and \(N^0 \in \mathbb{N}_\infty\), then the radial operators \(T_{G_k}\) of \(G_k\) are identically free distributed over \(\mathbb{C}^{\oplus N^0}\). Moreover,
\[
E(T_{G_k}^n) = \left|\mathcal{L}_{N_0}^n(n)\right| \cdot 1_{\mathbb{C}^{\oplus N^0}},
\]
for all \(n \in \mathbb{N}\). \(\square\)

The fractal pairs of graph fractaloids give the classification of graph fractaloids in terms of the spectral information of the corresponding radial operators.

### 5.2. Equivalence Classes of Graph Fractaloids

Now, let’s collect all graph fractaloids induced by “connected” fractal graphs, and denote this collection by \(\mathcal{F}_{\text{fractal}}\), i.e.,
\[
\mathcal{F}_{\text{fractal}} \overset{\text{def}}{=} \left\{ G \mid \begin{array}{c}
G \text{ is a graph fractaloid of } G, \\
\text{where } G \text{ is a connected locally finite fractal graph.}
\end{array} \right\}.
\]

Define an equivalence relation \(\mathcal{R}\) on the set \(\mathcal{F}_{\text{fractal}}\) by
\[
G_1 \mathcal{R} G_2 \overset{\text{def}}{\iff} fp(G_1) = fp(G_2) \text{ in } \mathbb{N} \times \mathbb{N}_\infty.
\]

Then the relation \(\mathcal{R}\) on \(\mathcal{F}_{\text{fractal}}\) is indeed an equivalence relation:
\begin{align*}
(5.1) & \quad G \mathcal{RG}_1 \text{ for all } G \in \mathcal{F}_{\text{fractal}}, \\
(5.2) & \quad G_1 \mathcal{RG}_2 \implies G_2 \mathcal{RG}_1, \text{ and} \\
(5.3) & \quad G_1 \mathcal{RG}_2 \text{ and } G_2 \mathcal{RG}_3 \implies G_1 \mathcal{RG}_3,
\end{align*}
for all \(G, G_1, G_2, G_3 \in \mathcal{F}_{\text{fractal}}\). By (5.1), (5.2), and (5.3), the relation \(\mathcal{R}\) is an equivalence relation on the set \(\mathcal{F}_{\text{fractal}}\).

**Definition 5.2.** Let \(\mathcal{F}_{\text{fractal}}\) be the collection of all graph fractaloids, induced by connected locally finite directed graphs, and let \(\mathcal{R}\) be an equivalence relation on \(\mathcal{F}_{\text{fractal}}\), defined as above. Then we call \(\mathcal{R}\), the spectral (equivalence) relation on \(\mathcal{F}_{\text{fractal}}\). And the equivalence classes of \(\mathcal{R}\) are said to be spectral classes of \(\mathcal{F}_{\text{fractal}}\). Denote each spectral class by \([(N_0, N^0)]\), for all \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}_\infty\). i.e.,
\[
[(N_0, N^0)] \overset{\text{def}}{=} \{ G \in \mathcal{F}_{\text{fractal}} : fp(G) = (N_0, N^0)\}.
\]
Now, fix \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}_\infty\). Then we can always choose at least one element \(G\) in \(\mathcal{F}_{\text{fractal}}\), equivalently, for any \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}_\infty\), we can always have a “nonempty” graph fractaloid \(G \in \mathcal{F}_{\text{fractal}}\), such that \(fp(G) = (N_0, N^0)\). The following lemma is proven by construction.

**Lemma 5.3.** Let \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}_\infty\). Then there exists at least one \(G \in \mathcal{F}_{\text{fractal}}\), such that \(G \in \{(N_0, N^0)\}\), where \(\{(N_0, N^0)\}\) is the spectral class. In other words, each spectral class is nonempty.

**Proof.** Fix \((N_0, 1) \in \mathbb{N} \times \{1\}\). Then we can construct the one-vertex-\(N_0\)-loop-edge graph \(O_{N_0}\), generating the graph fractaloid \(\mathcal{G}_{N_0}\), which is the graph groupoid of \(O_{N_0}\). Now, take \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}\). Then we can construct the \(N_0\)-regularized graph \(R_{N_0}(K_{N^0})\), in the sense of Section 3.2, of the one-flow circulant graph \(K_{N^0}\) with \(N^0\)-vertices, generating the graph fractaloid \(\mathcal{G}(R_{N_0}(K_{N^0})) \in \mathcal{F}_{\text{fractal}}\). Assume that \((N_0, \infty) \in \mathbb{N} \times \{\infty\}\). Construct the infinite linear graph \(L\), graph-isomorphic to

\[
\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots
\]

Then the graph groupoid \(\mathbb{L}\) of \(L\) is in \(\mathcal{F}_{\text{fractal}}\), with its fractal pair \(fp(\mathbb{L}) = (1, \infty)\). If we construct the \(N_0\)-regularized graph \(R_{N_0}(L)\), then the fractal pair of the graph fractaloid \(\mathcal{G}(R_{N_0}(L))\) is identical to \((N_0, \infty)\).

Therefore, for any \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}_\infty\), there always exists at least one graph fractaloid \(G\) in \(\mathcal{F}_{\text{fractal}}\), such that \(fp(G) = (N_0, N^0)\). □

By the previous lemma, we can get the following theorem of this paper. Actually the following theorem is the summary of all our main results of this paper.

**Theorem 5.4.** Let \((N_0, N^0) \in \mathbb{N} \times \mathbb{N}_\infty\). Then the corresponding spectral class \(\{(N_0, N^0)\}\) of \(\mathcal{F}_{\text{fractal}}\) is nonempty. Moreover, if \(G \in \{(N_0, N^0)\}\), and if \(T_G\) is the radial operator of \(G\) in the right graph von Neumann algebra \(M_G\), then the \(D_G\)-valued moments \(E(T_G^n)\) satisfy

\[
E(T_G^n) = |C_{N_0}^\mathcal{G}(n)| \cdot 1_{C^\mathcal{G} \cap N^0}, \text{ for all } n \in \mathbb{N}.
\]

□

The following corollary is the direct consequence of the above theorem.

**Corollary 5.5.** Let \(\mathcal{F}_{\text{fractal}}\) be the set of all graph fractaloids induced by connected locally finite directed graphs. Then \(\mathcal{F}_{\text{fractal}}\) is classified by the spectral classes \(\{(n, m)\}\), for all \((n, m) \in \mathbb{N} \times \mathbb{N}_\infty\). i.e.,

\[
\mathcal{F}_{\text{fractal}} = \bigcup_{(n, m) \in \mathbb{N} \times \mathbb{N}_\infty} \{(n, m)\},
\]

where \(\bigcup\) means the disjoint union. □

Notice that our spectral classes in \(\mathcal{F}_{\text{fractal}}\) are determined by the spectral information of graph fractaloids, not by the graph-theoretical (and algebraic) information of fractal graphs (resp., graph fractaloids).

**Proposition 5.6.** Let \(G\) be a connected locally finite directed graph with its graph groupoid \(\mathcal{G}\), and assume that \(\mathcal{G}\) is a graph fractaloid in \(\mathcal{F}_{\text{fractal}}\). Let \(G'\) be a directed graph. If \(G\) and \(G'\) are graph-isomorphic, then the graph groupoid \(\mathcal{G}'\) of \(G'\) is a graph fractaloid in \(\mathcal{F}_{\text{fractal}}\), too. Moreover, the graph fractaloids \(\mathcal{G}\) and \(\mathcal{G}'\) are contained in the same spectral class \(\{(n, m)\}\), for some \(n \in \mathbb{N}\), and \(m \in \mathbb{N}_\infty\).
Proof. Since the graphs $G$ and $G'$ are graph-isomorphic, the graph groupoids $G$ of $G$ and $G'$ of $G'$ are groupoid-isomorphic. And since $G \in F_{\text{fractal}}$, the graph groupoid $G' \in F_{\text{fractal}}$, too. Assume now that $fp(G) = (n, m) \in \mathbb{N} \times \mathbb{N}_\infty$. Then, since

$$V(G) = V(\hat{G}) = V(\hat{G'}) = V(G'),$$

we have

$$|V(G)| = m = |V(G')|.$$ 

Also, since $\hat{G}$ and $\hat{G'}$ are graph-isomorphic,

$$\max\{\deg_{\text{out}}(v) : v \in V(G')\} = n,$$

too, because

$$n = \max\{\deg_{\text{out}}(v) : v \in V(G)\}.$$ 

So, we obtain that

$$fp(G') = (n, m),$$

too. Therefore, the graph fractaloids $G$ and $G'$ are contained in the same spectral class $[(n, m)]$ in $F_{\text{fractal}}$. 

How about the converse of the previous proposition? The following example shows that the converse does not hold true.

Example 5.1. Let $K_3$ be the one-flow circulant graph with 3-vertices, and let $G_1$ be the 2-regularized graph $R_2(K_3)$ of $K_3$. Then $G_1$ is a fractal graph, and hence the graph groupoid $G_1$ is a graph fractaloid in $F_{\text{fractal}}$, since

$$\deg_{\text{out}}(G_1)(v) = 2 = \deg_{\text{in}}(G_1)(v), \text{ in } G_1 = R_2(K_3),$$

for all $v \in V(G_1)$. Now, let $G_2$ be the complete graph $C_3$ with 3-vertices. Then $\deg_{\text{out}}(G_2)(x) = 2 = \deg_{\text{in}}(G_2)(x)$, in $G_2 = C_3$, for all $x \in V(G_2)$. Thus, the graph groupoid $G_2$ of $G_2$ is a graph fractaloid in $F_{\text{fractal}}$, too. Moreover, both $G_1$ and $G_2$ have the same fractal pair $(2, 3)$, and hence

$$G_1, G_2 \in [(2, 3)] \in F_{\text{fractal}}.$$

But the graphs $R_2(K_3)$ and $C_3$ are not graph-isomorphic. This shows that the converse of the previous proposition does not hold true, in general.

In the previous example, we showed that even though two graphs are not graph-isomorphic, the corresponding graph groupoids can be fractaloids contained in the same spectral class in the set $F_{\text{fractal}}$. More general to the previous proposition, we can obtain the following generalized result.

Theorem 5.7. Let $G_1$ and $G_2$ be connected locally finite graphs, and assume that the corresponding shadowed graphs $\bar{G}_1$ and $\bar{G}_2$ are graph-isomorphic. If $G_1$ is a fractal graph, then $G_2$ is a fractal graph, too. Moreover, the graph fractaloids $G_k$ of $G_k$ are contained in the same spectral class in $F_{\text{fractal}}$.

Proof. Let $G_1$ be a fractal graph with its graph fractaloid $G_1$, and let $G_1 \in [(n, m)] \in F_{\text{fractal}}$, for some $n \in \mathbb{N}$, and $m \in \mathbb{N}_\infty$. And assume that the shadowed graphs $\bar{G}_1$ and $\bar{G}_2$ are graph-isomorphic. Then the graph groupoids $G_1$ and $G_2$ are groupoid-isomorphic. So, the radial operators $T_{G_k}$ of $G_k$ are identically distributed over $\mathbb{C}^{\oplus m}$, and hence $fp(G_2) = (n, m)$, too. Therefore, $G_2 \in [(n, m)]$ in $F_{\text{fractal}}$. 

How about the converse of the above theorem? Unfortunately, the converse does not hold, in general, either.

**Example 5.2.** Let $G_1$ be the 2-regularized graph $R_2(K_3)$, and let $G_2$ be the iterated glued graph $K_3 \#^v O_1$, where $v$ is the unique vertex of the one-vertex-1-loop-edge graph $O_1$. As we observed in Section 3.2, since the one-flow circulant graph $K_3$ is a fractal graph, its 2-regularized graph $R_2(K_3)$, and the iterated glued graph $K_3 \#^v O_1$ are fractal graphs, too. i.e., the graph groupoids $G_k$ of $G_k$ are graph fractaloids, for $k = 1, 2$. Moreover, we have that

$$|V(G_1)| = 3 = |V(K_3)| = |V(G_2)|,$$

and

$$\deg_{out}^{(G_1)}(v) = 2 = \deg_{in}^{(G_1)}(v), \text{ in } G_1,$$

for all $v \in V(G_1)$, and

$$\deg_{in}^{(G_2)}(x) = 2 = \deg_{in}^{(G_2)}(x), \text{ in } G_2,$$

for all $x \in V(G_2)$. i.e., the fractal pairs $fp(G_k)$ of the graph fractaloids $G_k$ are

$$fp(G_1) = (2, 3) = fp(G_2).$$

Therefore, the graph fractaloids $G_1$ and $G_2$ are contained in the spectral class $[(2, 3)]$ in $F_{\text{fractal}}$. However, it is easy to check that the shadowed graphs $\hat{G}_1$ and $\hat{G}_2$ are not graph-isomorphic. Indeed, the shadowed graphs $\hat{G}_2$ contain loop edges, but $\hat{G}_1$ does not contain any loop edges. This example shows that even though the shadowed graphs $\hat{G}_1$ and $\hat{G}_2$ are not graph-isomorphic, it is possible that the graph fractaloids $G_1$ and $G_2$ are contained in the same spectral class in $F_{\text{fractal}}$.

By the previous example, the converse of the previous theorem does not hold true, in general.

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