RANGES OF UNITARY DIVISOR FUNCTIONS

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Abstract. For any real $t$, the unitary divisor function $\sigma^*_t$ is the multiplicative arithmetic function defined by $\sigma^*_t(p^\alpha) = 1 + p^{\alpha t}$ for all primes $p$ and positive integers $\alpha$. Let $\overline{\sigma^*_t(\mathbb{N})}$ denote the topological closure of the range $\sigma^*_t$. We calculate an explicit constant $\eta^* \approx 1.9742550$ and show that $\overline{\sigma^*_t(N)}$ is connected if and only if $r \in (0, \eta^*)$. We end with an open problem.

Keywords: Dense, divisor function, unitary divisor, connected

2010 Mathematics Subject Classification: Primary 11B05; Secondary 11A25.

1. Introduction

For any $c \in \mathbb{C}$, the divisor function $\sigma_c$ is defined by $\sigma_c(n) = \sum_{d|n} d^c$. Divisor functions, especially $\sigma_1$, $\sigma_0$, and $\sigma_{-1}$, are among the most extensively-studied arithmetic functions [2, 9, 11]. For example, two very classical number-theoretic topics are the study of perfect numbers and the study of friendly numbers. A positive integer $n$ is said to be perfect if $\sigma_{-1}(n) = 2$, and $n$ is said to be friendly if there exists $m \neq n$ with $\sigma_{-1}(m) = \sigma_{-1}(n)$ [12]. Motivated by the very difficult problems related to perfect and friendly numbers, Laatsch studied $\sigma_{-1}(\mathbb{N})$, the range of $\sigma_{-1}$. He showed that $\sigma_{-1}(\mathbb{N})$ is a dense subset of the interval $[1, \infty)$ and asked if $\sigma_{-1}(\mathbb{N})$ is in fact equal to the set $\mathbb{Q} \cap [1, \infty)$ [10]. Weiner answered this question in the negative, showing that $(\mathbb{Q} \cap [1, \infty)) \setminus \sigma_{-1}(\mathbb{N})$ is also dense in $[1, \infty)$ [14].

The author has studied ranges of divisor functions in a variety of contexts [4, 5, 6, 7]. In this paper, we study the close relatives of the divisor functions known as unitary divisor functions. A unitary divisor of an integer $n$ is a divisor $d$ of $n$ such that $\gcd(d,n/d) = 1$. The unitary divisor function $\sigma^*_c$ is defined by [1, 3, 8]

$$\sigma^*_c(n) = \sum_{\substack{d|n \\ \gcd(d,n/d)=1}} d^c.$$ 

The function $\sigma^*_c$ is multiplicative and satisfies $\sigma^*_c(p^\alpha) = 1 + p^{\alpha c}$ for all primes $p$ and positive integers $\alpha$.

If $t \in [-1, 0)$, then one may use the same argument that Laatsch employed in [10] in order to show that $\overline{\sigma^*_t(\mathbb{N})} = [1, \infty)$. Here, the overline denotes the topological closure. In particular, $\overline{\sigma^*_t(\mathbb{N})}$ is connected if $t \in [-1, 0)$. On the other hand, $\overline{\sigma^*_t(\mathbb{N})}$ is a discrete disconnected set if $t \geq 0$ (this follows from a simple modification of the proof of Theorem
2.2 in [1]). The purpose of this paper is to prove the following theorem. Let $\zeta$ denote the Riemann zeta function.

**Theorem 1.1.** Let $\eta^*$ be the unique number in the interval $(1, 2]$ that satisfies the equation

$$
\frac{2^{\eta^*} + 1}{2^{\eta^*}} \cdot \frac{(3^{\eta^*} + 1)^2}{3^{2^{\eta^*}} + 1} = \frac{\zeta(\eta^*)}{\zeta(2^{\eta^*})}.
$$

If $r \in \mathbb{R}$, then $\sigma_{-r}(N)$ is connected if and only if $r \in (0, \eta^*]$.

**Remark 1.1.** In the process of proving Theorem 1.1, we will show that there is indeed a unique solution to the equation (1) in the interval $(1, 2]$.

In all that follows, we assume $r > 1$ and study $\sigma_{-r}(N)$. We first observe that $\sigma_{-r}(N) \subseteq [1, \zeta(r)/\zeta(2r))$. This is because if $q_1^{\beta_1} \cdots q_v^{\beta_v}$ is the prime factorization of some positive integer, then

$$
\sigma_{-r}(q_1^{\beta_1} \cdots q_v^{\beta_v}) = \prod_{i=1}^{v} \sigma_{-r}(q_i^{\beta_i}) = \prod_{i=1}^{v} \left(1 + q_i^{-\beta_i r}\right) < \prod_{i=1}^{v} \left(1 + q_i^{-r}\right) = \prod_{i=1}^{v} \left(1 + p_i^{-r}\right) = \frac{\zeta(r)}{\zeta(2r)}.
$$

It is straightforward to show that 1 and $\zeta(r)$ are elements of $\sigma_{-r}(N)$. Therefore, Theorem 1.1 tells us that $\sigma_{-r}(N) = [1, \zeta(r)/\zeta(2r)]$ if and only if $r \in (0, \eta^*]$.

2. Proofs

In what follows, let $p_i$ denote the $i^{th}$ prime number. Let $\nu_p(x)$ denote the exponent of the prime $p$ appearing in the prime factorization of the integer $x$.

To start, we need the following technical yet simple lemma.

**Lemma 2.1.** If $s, m \in \mathbb{N}$ and $s \leq m$, then

$$
\frac{p_s^{2r} + 1}{p_s^{2r} + p_r^{2r}} \leq \frac{p_m^{2r} + 1}{p_m^{2r} + p_r^{2r}}\text{ for all } r > 1.
$$

**Proof.** Fix some $r > 1$, and write $h(x) = \frac{x^{2r} + 1}{x^{2r} + x^r}$. Then

$$
h'(x) = \frac{r x^r (x^{2r} + x^r) - (x^{2r} + 1) x^r}{(x^{2r} + 1)^2} = \frac{r x^r - 1}{x^{2r} + 1}.
$$

We see that $h(x)$ is increasing when $x \geq 3$. Hence, in order to complete the proof, it suffices to show that $h(2) \leq h(3)$. For $r > 1$, we have $2^{2r}3^r + 3^{2r} + 3^r < 2^{2r}3^r + 2^{2r} + 2^r$ (in fact, equality occurs if we set $r = 1$), so $(2^{2r} + 1)(3^{2r} + 3^r) < (2^{2r} + 2^r)(3^{2r} + 1)$. This shows that

$$
\frac{2^{2r} + 1}{2^{2r} + 2^r} < \frac{3^{2r} + 1}{3^{2r} + 3^r},
$$

which completes the proof. \qed
The following theorem replaces the question of whether or not \( \sigma^*_r(\mathbb{N}) \) is connected with a question concerning infinitely many inequalities. The advantage in doing this is that we will further reduce this problem to the consideration of a finite list of inequalities in Theorem 2.2 Recall from the introduction that \( \sigma^*_r(\mathbb{N}) \) is connected if and only if it is equal to the interval \([1, \zeta(r)/\zeta(2r)]\).

**Theorem 2.1.** If \( r > 1 \), then \( \sigma^*_r(\mathbb{N}) = [1, \zeta(r)/\zeta(2r)] \) if and only if

\[
\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \leq \prod_{i=m+1}^{\infty} \left( 1 + \frac{1}{p_i^r} \right)
\]

for all positive integers \( m \).

**Proof.** First, suppose that

\[
\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \leq \prod_{i=m+1}^{\infty} \left( 1 + \frac{1}{p_i^r} \right)
\]

for all positive integers \( m \). We will show that the range of \( \log \sigma^*_r \) is dense in \([0, \log (\zeta(r)/\zeta(2r))]\), which will then imply that the range of \( \sigma^*_r \) is dense in \([1, \zeta(r)/\zeta(2r)]\). Fix some \( x \in (0, \log (\zeta(r)/\zeta(2r))) \). We will construct a sequence \((C_i)_{i=1}^{\infty}\) of elements of the range of \( \log \sigma^*_r \) that converges to \( x \). First, let \( C_0 = 0 \). For each positive integer \( n \), if \( C_{n-1} < x \), let

\[
C_n = C_{n-1} + \log \left( 1 + p_n^{-\alpha_n r} \right)
\]

where \( \alpha_n \) is the smallest positive integer that satisfies

\[
C_{n-1} + \log \left( 1 + p_n^{-\alpha_n r} \right) \leq x.
\]

If \( C_{n-1} = x \), simply set \( C_n = C_{n-1} = x \). For each \( n \in \mathbb{N} \), \( C_n \in \log \sigma^*_r(\mathbb{N}) \). Indeed, if \( C_n \neq C_{n-1} \), then

\[
C_n = \sum_{i=1}^{n} \log \left( 1 + p_i^{-\alpha_i r} \right) = \log \left( \prod_{i=1}^{n} \left( 1 + p_i^{-\alpha_i r} \right) \right) = \log \sigma^*_r \left( \prod_{i=1}^{n} p_i^{\alpha_i} \right).
\]

If, however, \( C_n = C_{n-1} = x \), then we may let \( l \) be the smallest positive integer such that \( C_l = x \) and show, in the same manner as above, that

\[
C_n = C_l = \log \sigma^*_r \left( \prod_{i=1}^{l} p_i^{\alpha_i} \right).
\]

Let us write \( \gamma = \lim_{n \to \infty} C_n \). Note that \( \gamma \) exists and that \( \gamma \leq x \) because the sequence \((C_i)_{i=1}^{\infty}\) is nondecreasing and bounded above by \( x \). If we can show that \( \gamma = x \), then we will be done. Therefore, let us assume instead that \( \gamma < x \).

We have \( C_n = C_{n-1} + \log(1 + p_n^{-\alpha_n r}) \) for all positive integers \( n \). Write \( D_n = \log(1 + p_n^{-r}) - \log(1 + p_n^{-\alpha_n r}) \) and \( E_n = \sum_{i=1}^{n} D_i \). As

\[
x + \lim_{n \to \infty} E_n > \gamma + \lim_{n \to \infty} E_n = \lim_{n \to \infty} (C_n + E_n) = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \log \left( 1 + p_i^{-\alpha_i r} \right) + \sum_{i=1}^{n} D_i \right)
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} \log \left( 1 + p_i^{-r} \right) = \log \left( \zeta(r)/\zeta(2r) \right),
\]

we have \( \lim_{n \to \infty} E_n > \log \left( \zeta(r)/\zeta(2r) \right) - x \). Therefore, we may let \( m \) be the smallest positive integer such that \( E_m > \log \left( \zeta(r)/\zeta(2r) \right) - x \). If \( \alpha_m = 1 \) and \( m > 1 \), then \( D_m = 0 \). This forces

\[
E_{m-1} = E_m > \log \left( \zeta(r)/\zeta(2r) \right) - x,
\]

contradicting the minimality of \( m \). If \( \alpha_m = 1 \) and \( m = 1 \), then \( 0 = E_m > \log \left( \zeta(r)/\zeta(2r) \right) - x \), which is also a contradiction since we originally
chose \( x < \log(\zeta(r)/\zeta(2r)) \). Therefore, \( \alpha_m > 1 \). Due to the way we defined \( C_m \) and \( \alpha_m \), we have \( C_m - \log (1 + p_{n-1}(\alpha_m-1)r) > x \). Hence,
\[
\log (1 + p_n^{-(\alpha_m-1)r}) - \log (1 + p_{n-1}(\alpha_m-1)r) > x - C_m.
\]
Using our original assumption that \( \frac{p_{m+1}^2 + p_m^r}{p_m^2 + 1} < \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right) \), we have
\[
\log \left(\frac{p_{m+1}^2 + p_m^r}{p_m^2 + 1}\right) \leq \sum_{i=m+1}^{\infty} \log \left(1 + \frac{1}{p_i^r}\right) = \log \left(\frac{\zeta(r)}{\zeta(2r)}\right) - E_m - C_m
\]
\[
< x - C_m < \log (1 + p_n^{-(\alpha_m-1)r}) - \log (1 + p_{n-1}(\alpha_m-1)r) = \log \left(\frac{p_m^{\alpha_m r} + p_m^r}{p_m^{\alpha_m r} + 1}\right).
\]
Thus,
\[
\frac{p_{m+1}^2 + p_m^r}{p_m^2 + 1} < \frac{p_m^{\alpha_m r} + p_m^r}{p_m^{\alpha_m r} + 1}.
\]
Rewriting this inequality, we get \( \frac{p_{m+1}^2 + p_m^{(\alpha_m+1)r}}{p_m^2 + p_m^r} < \frac{p_m^{(2-\alpha_m)r} + p_m^r}{1 + p_m^{(3-\alpha_m)r}} \). Now, dividing through by \( p_m^{\alpha_m} \) yields \( p_m^{(2-\alpha_m)r} + p_m^r < 1 + p_m^{(3-\alpha_m)r} \), which is impossible since \( \alpha_m \geq 2 \). This contradiction proves that \( \gamma = x \), so \( \sigma_{\gamma}^*(\bar{N}) = [1, \zeta(r)/\zeta(2r)] \).

To prove the converse, suppose there exists some positive integer \( m \) such that
\[
\frac{p_{m+1}^2 + p_m^r}{p_m^2 + 1} > \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right).
\]
We may write this inequality as
\[
\frac{p_{m+1}^2 + 1}{p_m^2 + p_m^r} < \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)^{-1}.
\]
Fix a positive integer \( N \). If \( \nu_{p_s}(N) = 1 \) for all \( s \in \{1, 2, \ldots, m\} \), then
\[
\sigma_{\gamma}^*(N) \geq \prod_{s=1}^{m} \left(1 + \frac{1}{p_s^r}\right) = \frac{\zeta(r)}{\zeta(2r)} \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_i^r}\right)^{-1}.
\]
On the other hand, if \( \nu_{p_s}(N) \neq 1 \) for some \( s \in \{1, 2, \ldots, m\} \), then \( \sigma_{\gamma}^*(p_s) \leq 1 + \frac{1}{p_s^{2r}} \). This implies that
\[
\sigma_{\gamma}^*(N) \leq \left(1 + \frac{1}{p_s^{2r}}\right) \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i^r}\right) = \frac{\zeta(r)}{\zeta(2r)} \frac{1 + p_s^{-2r}}{1 + p_s^{-r}} = \frac{\zeta(r)}{\zeta(2r)} \frac{p_s^{2r} + 1}{p_s^{2r} + p_s^r}
\]
in this case. Using Lemma 2.1, we have
\[
\sigma_{\gamma}^*(N) \leq \frac{\zeta(r)}{\zeta(2r)} \frac{p_m^{2r} + 1}{p_m^2 + p_m^r}.
\]
As \( N \) was arbitrary, we have shown that there is no element of the range of \( \sigma^* - r \) in the interval
\[
\left( \frac{\zeta(r)}{\zeta(2r)} \frac{p_m^{2r} + 1}{p_m^{2r}} \right) \prod_{i=m+1}^{\infty} \left( 1 + \frac{1}{p_i^r} \right)^{-1}. \]
This interval is a gap in the range of \( \sigma^* - r \) because of the inequality (2). □

As mentioned above, we wish to reduce the task of checking the infinite collection of inequalities given in Theorem 2.1 to that of checking finitely many inequalities. We do so in Theorem 2.2, the proof of which requires the following lemma.

**Lemma 2.2.** If \( j \in \mathbb{N} \setminus \{1, 2, 3, 4, 6, 9\} \), then \( \frac{p_{j+1}}{p_j} < \sqrt{2} \).

**Proof.** A simple manipulation of the corollary to Theorem 3 in [13] shows that
\[
\frac{p_{j+1}}{p_j} < \frac{(j+1)(\log(j+1) + \log \log(j+1))}{j \log j}
\]
for all integers \( j \geq 6 \). It is easy to verify that \( \frac{(j+1)(\log(j+1) + \log \log(j+1))}{j \log j} < \sqrt{2} \) for all \( j \geq 3100 \). Therefore, the desired result holds for \( j \geq 3100 \). A quick search through the values of \( \frac{p_{j+1}}{p_j} \) for \( j < 3100 \) yields the desired result. □

**Theorem 2.2.** If \( r \in (1, 3] \), then \( \sigma^*_r(\mathbb{N}) = [1, \zeta(r)/\zeta(2r)] \) if and only if
\[
\frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \leq \prod_{i=m+1}^{\infty} \left( 1 + \frac{1}{p_i^r} \right)
\]
for all \( m \in \{1, 2, 3, 4, 6, 9\} \).

**Proof.** Let
\[
F(m, r) = \frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \prod_{i=1}^{m} \left( 1 + \frac{1}{p_i^r} \right)
\]
so that the inequality \( \frac{p_m^{2r} + p_m^r}{p_m^{2r} + 1} \leq \prod_{i=m+1}^{\infty} \left( 1 + \frac{1}{p_i^r} \right) \) is equivalent to \( F(m, r) \leq \frac{\zeta(r)}{\zeta(2r)} \). Let \( r \in (1, 3] \). By Theorem 2.1 it suffices to show that if \( F(m, r) \leq \frac{\zeta(r)}{\zeta(2r)} \) for all \( m \in \{1, 2, 3, 4, 6, 9\} \), then \( F(m, r) \leq \frac{\zeta(r)}{\zeta(2r)} \) for all \( m \in \mathbb{N} \). Therefore, assume that \( r \) is such that \( F(m, r) \leq \frac{\zeta(r)}{\zeta(2r)} \) for all \( m \in \{1, 2, 3, 4, 6, 9\} \).

We will show that \( F(m+1, r) > F(m, r) \) for all \( m \in \mathbb{N} \setminus \{1, 2, 3, 4, 6, 9\} \). This will show that \( (F(m, r))_{m=10}^{\infty} \) is an increasing sequence. As \( \lim_{m \to \infty} F(m, r) = \frac{\zeta(r)}{\zeta(2r)} \), it will then follow that
$F(m, r) < \frac{\zeta(r)}{\zeta(2r)}$ for all integers $m \geq 10$. Furthermore, we will see that $F(5, r) < F(6, r) \leq \frac{\zeta(r)}{\zeta(2r)}$ and $F(7, r) < F(8, r) < F(9, r) \leq \frac{\zeta(r)}{\zeta(2r)}$, which will complete the proof.

Let $m \in \mathbb{N}\{1, 2, 3, 4, 6, 9\}$. By Lemma 2.2, $\frac{p_{m+1}}{p_m} < \sqrt[2]{2} \leq \sqrt[3]{2}$. This shows that $p_{m+1} < 2p_m$, implying that $2p_{m+1} > p_mp_{m+1}$. Therefore,

$$2p_{m+1}^r > p_mp_{m+1} + \frac{p_{m+1}^r}{p_{m+1}^r} - p_{m+1}^r - 1 = \frac{(p_{m+1}^r - 1)(p_{m+1}^r + 1)}{p_{m+1}^r}.
$$

Multiplying each side of this inequality by $\frac{p_{m+1}^r}{(p_{m+1}^r + 1)(p_{m+1}^r + 1)}$ and adding 1 to each side, we get

$$1 + \frac{2p_{m+1}^r}{p_{m+1}^r + 1} > 1 + \frac{p_{m+1}^r - 1}{p_{m+1}^r + 1},$$

which we may write as

$$\frac{(p_{m+1}^r + 1)^2}{p_{m+1}^r + 1} > \frac{p_{m+1}^r + p_{m+1}^r}{p_{m+1}^r + 1}.$$

Finally, we get

$$F(m + 1, r) = \frac{p_{m+1}^r + p_{m+1}^r}{p_{m+1}^r + 1} \prod_{i=1}^{m+1} \left(1 + \frac{1}{p_{i}^r}\right) = \frac{(p_{m+1}^r + 1)^2}{p_{m+1}^r + 1} \prod_{i=1}^{m+1} \left(1 + \frac{1}{p_{i}^r}\right) = F(m, r). \quad \Box$$

Now, let

$$V_m(r) = \log \left(\frac{p_{m+1}^r + p_{m+1}^r}{p_{m+1}^r + 1}\right) - \sum_{i=m+1}^{\infty} \log \left(1 + \frac{1}{p_{i}^r}\right).$$

Equivalently, $V_m(r) = \log(F(m, r)) - \log\left(\frac{\zeta(r)}{\zeta(2r)}\right)$, where $F$ is the function defined in the proof of Theorem 2.2. Observe that

$$\frac{p_{m+1}^r + p_{m+1}^r}{p_{m+1}^r + 1} \leq \prod_{i=m+1}^{\infty} \left(1 + \frac{1}{p_{i}^r}\right)$$

if and only if $V_m(r) \leq 0$. If we let $J_m(r) = \sum_{i=m+1}^{m+6} \frac{1}{p_{i}^r + 1} - \frac{p_{m+1}^r - 2p_{m+1}^r}{(p_{m+1}^r + 1)(p_{m+1}^r + 1)}$, then we have

$$\frac{\partial}{\partial r} J_m(r) = \frac{p_r^r((p_{m+1}^r - 1)^4 - 12p_{m+1}^r) \log p_m}{(p_{m+1}^r + 1)^2(p_{m+1}^r + 1)^2} - \sum_{i=m+1}^{m+6} \frac{p_{r}^r \log p_{i}^r}{(p_{i}^r + 1)^2}.$$
It is not difficult to verify that \( \frac{p_r^m((p_r^m - 1)^4 - 12p_r^m) \log p_m}{(p_r^m + 1)^2(p_r^{2m} + 1)^2} \geq -1 \) for all \( r \in [1, 2] \) and \( m \in \{1, 2, 3, 4, 6, 9\} \). Therefore, when \( r \in [1, 2] \) and \( m \in \{1, 2, 3, 4, 6, 9\} \), we have

\[
\frac{\partial}{\partial r} J_m(r) \geq -1 - \sum_{i=m+1}^{m+6} \frac{p_r^i \log p_i}{(p_r^i + 1)^2} \geq -1 - \sum_{i=m+1}^{m+6} \frac{\log p_i}{p_r^i} > -7.
\]

Numerical calculations show that \( J_m(r) > \frac{1}{400} \) for all \( m \in \{1, 2, 3, 4, 6, 9\} \) and \( r \in \left\{ 1 + \frac{n}{2800} : n \in \{0, 1, 2, \ldots, 2800\} \right\} \).

Because each function \( J_m \) is continuous in \( r \) for \( r \in [1, 2] \), we see that

\[
J_m(r) > \frac{1}{400} - 7 \left( \frac{1}{2800} \right) = 0
\]

for all \( r \in [1, 2] \) and \( m \in \{1, 2, 3, 4, 6, 9\} \).

We introduced the functions \( J_m \) so that we could write

\[
\frac{\partial}{\partial r} V_m(r) = \sum_{i=m+1}^{\infty} \frac{\log p_i}{p_r^i + 1} - \frac{(p_r^{2m} - 2p_r^m - 1) \log p_m}{(p_r^m + 1)(p_r^{2m} + 1)} > (\log p_m) J_m(r) > 0
\]

for all \( m \in \{1, 2, 3, 4, 6, 9\} \) and \( r \in [1, 2] \). A quick numerical calculation shows that \( V_2(1.5) < 0 < V_2(2) \), so the function \( V_2 \) has exactly one root, which we will call \( \eta^* \), in the interval \((1, 2]\). Further calculations show that \( V_m(2) < 0 \) for all \( m \in \{1, 3, 4, 6, 9\} \). Hence, \( V_m(r) \leq 0 \) for all \( m \in \{1, 2, 3, 4, 6, 9\} \) and \( r \in (1, \eta^*[\mathbb{N}]) \). By Theorem 2.2, this means that if \( r \in (1, 2] \), then \( \sigma^*_r(\mathbb{N}) \notin [1, \zeta(r)/\zeta(2r)] \) if and only if \( r \leq \eta^* \).

Next, note that

\[
\frac{\partial}{\partial r} V_2(r) = \sum_{i=3}^{\infty} \frac{\log p_i}{p_r^i + 1} - \frac{(3^{2r} - 2 \cdot 3^r - 1) \log 3}{(3^{2r} + 1)(3^r + 1)} > -\frac{(3^{2r} - 2 \cdot 3^r - 1) \log 3}{(3^{2r} + 1)(3^r + 1)}
\]

\[
> -\frac{(3^{2r} + 1) \log 3}{(3^{2r} + 1)(3^r + 1)} \geq \frac{\log 3}{3^2 + 1} > -1.1
\]

for all \( r \in [2, 3] \). Let \( A = \left\{ 2 + \frac{n}{400} : n \in \{0, 1, 2, \ldots, 400\} \right\} \). With a computer program, one may verify that \( V_2(r) > 0.003 \) for all \( r \in A \). Because \( V_2 \) is continuous, this shows that \( V_2(r) > 0.003 - 1.1 \left( \frac{1}{400} \right) > 0 \) for all \( r \in [2, 3] \). Consequently, \( \sigma^*_r(\mathbb{N}) \notin [1, \zeta(r)/\zeta(2r)] \) if \( r \in [2, 3] \).

We are now in a position to prove Theorem 1.1. Note that the equation defining \( \eta^* \) in the statement of this theorem is simply a rearrangement of the equation \( V_2(\eta^*) = 0 \). Therefore, we have shown that the theorem is true for \( r \in (1, 3] \). In order to prove the theorem for
$r > 3$, it suffices (by Theorem 2.2) to show that $F(1, r) > \frac{\zeta(r)}{\zeta(2r)}$ for all $r > 3$. If $r > 3$, then
\[
F(1, r) = \frac{(2^r + 1)^2}{2^{2r} + 1} = \frac{2^r + 2^{r+1} + 1}{2^{2r} + 1} > \frac{2^r + 2^r + \frac{2^{r+1}}{r-1}}{2^{2r} + 1} = 1 + \frac{1}{2^r} + \frac{1}{(r-1)2^{2r}}
\]
\[
> \frac{1}{2^r} + \frac{1}{(r-1)2^{2r}} = 1 + \frac{1}{2^r} + \int_\infty^\infty x^{-r}dx > \frac{\zeta(r)}{\zeta(2r)}.
\]

3. An Open Problem

Let $\mathcal{N}(S)$ denote the number of connected components of a set $S \subseteq \mathbb{R}$, and let $\mathcal{E}_k^* = \{t \in \mathbb{R} : \mathcal{N}(\sigma_t^*(\mathbb{N})) = k\}$. Theorem 1.1 tells us that $\mathcal{E}_1^* = [-\eta^*, 0)$. What can be said about $\mathcal{E}_k^*$ for $k \geq 2$? Are these sets all half-open intervals? What is the growth rate of the sequence $(-\inf_{k=1}^\infty \mathcal{E}_k^*)$?

4. Acknowledgements

This work was supported by National Science Foundation grant no. 1262930.

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