On Enumerating Short Projected Models

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Abstract

Propositional model enumeration, or All-SAT, is the task to record all models of a propositional formula. It is a key task in software and hardware verification, system engineering, and predicate abstraction, to mention a few. It also provides a means to convert a CNF formula into DNF, which is relevant in circuit design. While in some applications enumerating models multiple times causes no harm, in others avoiding repetitions is crucial. We therefore present two model enumeration algorithms, which adopt dual reasoning in order to shorten the found models. The first method enumerates pairwise contradicting models. Repetitions are avoided by the use of so-called blocking clauses, for which we provide a dual encoding. In our second approach we relax the uniqueness constraint. We present an adaptation of the standard conflict-driven clause learning procedure to support model enumeration without blocking clauses. Our procedures are expressed by means of a calculus and proofs of correctness are provided.

1 Introduction

The satisfiability problem of propositional logic (SAT) consists in determining whether for a propositional formula there exists an assignment to its variables which evaluates the formula to true, and which we call satisfying assignment or model. For proving that a formula is satisfiable, it is sufficient to provide one single model. However, sometimes determining satisfiability is not sufficient but all models are required. Propositional model enumeration (All-SAT)\(^1\) is

\(^1\)For the sake of readability, we use the term All-SAT also if not all models are required since in principle such an algorithm could always be extended to determine all models.
the task of enumerating (all) satisfying assignments of a propositional formula. It is a key task in, e.g., bounded and unbounded model checking [1, 2, 3, 4, 5, 6], image computation [7, 8, 9, 10], system engineering [11], predicate abstraction [12], and lazy Satisfiability Modulo Theories [13].

Model enumeration also provides a means to convert a formula in Conjunctive Normal Form (CNF) into a logically equivalent formula in Disjunctive Normal Form (DNF) composed of the models of the CNF formula. This conversion is used in, e.g., circuit design [14] and has also been studied from a computational complexity point of view [15]. If the models found are pairwise contradicting, the resulting DNF is a Disjoint Sum-of-Product (DSOP) formula, which is relevant in circuit design [16, 17], and whose models can be enumerated in polynomial time [18] by simply returning their disjuncts. If the models found are not pairwise contradicting, the resulting formula is still a DNF but does not support polytime model counting. Our model enumeration algorithm basically executes a CNF to DNF conversion, and from this point of view, it can be interpreted as a knowledge compilation algorithm.

The aim of knowledge compilation is to transform a formula into another language\(^2\) on which certain operations can be executed in polynomial time [19, 20]. This can be done, for instance, by recording the trace of an exhaustive search [21, 22, 23], and the target language in these approaches is the deterministic Decomposable Normal Form (d-DNNF),\(^3\) which was applied, for instance, in planning [24]. In contrast, in our work we record the models of the input formula, and the resulting formula is in d-DNNF only if the detected models are pairwise contradicting.

Enumerating models requires to process the search space exhaustively and is therefore a harder task than determining satisfiability. However, since state-of-the-art SAT solvers are successfully applied in industrial applications, it seems natural to use them as a basis for model enumeration. Modern SAT solvers implement conflict-driven clause learning (CDCL) [25, 26, 27] with non-chronological backtracking.\(^4\) If a CDCL-based SAT solver is extended to support model enumeration, adequate measures need be taken to avoid enumerating models multiple times as demonstrated by the following small example.

**Example 1** (Multiple model enumeration). Consider the propositional formula

\[
F = (a \lor c) \land (a \lor \neg c) \land (b \lor d) \land (b \lor \neg d)
\]

\(^2\)A language in this context refers one of the various forms a formula can be expressed in, e.g., CNF and DNF denote the languages we are mostly interested in in this article.

\(^3\)A formula is in d-DNNF, if (1) the sets of variables of the conjuncts of each conjunction are pairwise disjoint, and (2) the disjuncts of each disjunction are pairwise contradicting [20].

\(^4\)Also referred to as backjumping in the literature.
which is defined over the set of variables $V = \{a, b, c, d\}$. Its models are
given by $\text{models}(F) = \{abc\bar{d}, ab\bar{c}\bar{d}, a\bar{b}\bar{c}\bar{d}, ab\bar{c}\bar{d}\}$. These models may
be represented by $ab$, i.e., they are given by all total extensions of $ab$.

Let our model enumerator be based on CDCL with non-chronological
backtracking. Assume we first decide $a$, i.e., assign $a$ the value true, and
then $b$. This (partial) assignment $ab$ is a model of $F$. As in our previous work
on propositional model counting [18], we flip the second decision literal, i.e.,
assign $b$ the value false, in order to explore the second branch, upon which
the literal $d$ is forced to true in order to satisfy clause $C_3$. The resulting
assignment $a\neg b\bar{d}$ now falsifies clause $C_4$, i.e., sets all its literal to false.
Conflict analysis yields the unit clause $C_5 = (\bar{b})$, which is added to $F$. The
enumerator then backtracks to decision level zero, i.e., unassigns $d$, $b$ and $a$,
and propagates $b$ with reason $C_5$. No literal is enforced by the assignment $b$, and a
decision need be taken. If we choose $a$, $F$ is satisfied. The model found
is $ba$, which is the one we had found earlier.

Multiple model enumeration in Example 1 is caused by the fact that after
conflict analysis the same satisfying assignment is repeated, albeit in reverse
order. More generally, the same satisfying assignment might be found again if
the enumerator backtracks past a flipped decision literal. Avoiding enumerating
models multiple times is crucial in, e.g., weighted model counting (WMC)
[28, 29, 30, 31] and Bayesian inference [32], which require enumerating the
models in order to compute their weight or probability. Another example
is weighted model integration (WMI) [33, 34] which generalizes WMC for
hybrid domains. In some applications, repeating models might lead to inefficiency and harm scalability [11]. In the context of model counting but
also relevant in model enumeration, Bayardo and Pehoushek [35] identified
the need for good learning similarly to its learning counterpart in CDCL,
and various measures have therefore been proposed to avoid the multiple
enumeration of models.

One possibility is to rule out a model which was already found by adding
a blocking clause to the formula [4, 36, 37], which in essence is the negation of
the model or the decision literals in the model to be blocked [36]. Whenever
a satisfying assignment is repeated, the clause blocking it is falsified, and
thus this model is not enumerated again. As soon as all models are found
and the relevant blocking clauses added, the formula becomes unsatisfiable.
However, there might be an exponential number of models and adding a
blocking clause for each of them might result in a significant negative impact
on the enumerator performance. In these cases, multiple model enumeration
need be prevented by other measures. Toda and Soh [38] address this issue by
adopting a variant of conflict analysis which is inspired by Gebser et al. [39]
and is exempt from blocking clauses.

The use of blocking clauses can also be avoided by adopting the Davis-
Putnam-Logemann-Loveland (DPLL) algorithm [40]. In DPLL, after a con-
flict or a model the last decision literal is flipped causing the solver to find only pairwise contradicting models. This idea was applied in the context of model counting by Birnbaum and Lozinskii [41] but can readily be adapted to support model enumeration. Chronological backtracking in Grumberg et al. [10] and in part in Gebser et al. [39] ensures that the search space is traversed in a systematic manner similarly to DPLL, and that the use of blocking clauses can be avoided. An apparent drawback of DPLL-based solvers, however, is that they might spend a significant amount of time in regions of the search space having no satisfying assignments, since—unlike CDCL-based solvers—they lack the possibility to escape those regions early.

This last issue can be addressed by the use of chronological CDCL introduced by Nadel and Ryvchin [42, 43]. Chronological CDCL combines the power of conflict-driven clause learning with chronological backtracking. Specifically, after finding a model, the last open (left) decision literal is flipped in order to process neighboring regions of the search space, while in case of a conflict, the solver is able to escape regions without solution early. In our earlier work [18], we developed a calculus for propositional model counting based on chronological CDCL and provided a proof of its correctness. We took a model enumeration approach making our method readily applicable in the context of model enumeration without repetition. However, while finding short models is crucial in, e.g., weighted model integration [33, 34], only total models\(^5\) are detected as is usual in CDCL-based SAT solvers. The reason for this is a simple one.

To detect when a partial assignment\(^6\) is a model of the input formula, the SAT solver would have to carry out satisfiability checks before every decision, as done by Birnbaum and Lozinskii [41]. These satisfiability checks are expensive, and assigning the remaining variables instead is more efficient, computationally. If all variables are assigned and no conflict has occurred, the SAT solver knows to have found a model. This makes sense in SAT solving. Model enumeration, however, is a harder task, and therefore more expensive methods might pay off.

One such method is dual reasoning [37, 44]. Our dual model counter Dualiza\(^7\) takes as input the formula under consideration together with its negation. The basic idea is to execute CDCL on both formulae simultaneously maintaining one single trail. Whenever a conflict in the negated formula occurs, the current (partial) assignment is a model of the formula. Although developed for model counting, its adaptation for model enumeration is straightforward.

Another idea enabling the detection of short models was to check whether all total extensions of the current (partial) assignment evaluate the input

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\(^5\)In total models all variables occur.

\(^6\)In a partial assignment not all variables occur.

\(^7\)https://github.com/arminbiere/dualiza
formula to true before taking a decision, i.e., whether the current assignment logically entails the input formula [45, 46].

Partial assignments evaluating the input formula to true represent sets of total models of the input formula. However, these sets might not be disjoint, as is demonstrated by the following example.

**Example 2** (Short redundant models). Let $F = (a \land b) \lor (a \land c)$ be a propositional formula defined over the set of variables $V = \{a, b, c\}$. Notice that $F$ is not in CNF and significantly differs from the one in our previous example. Its total models are $\text{models}(F) = \{abc, ab\neg c, a\neg bc\}$. These models may also be represented by the two partial models $ab$ and $ac$. The former represents $abc$ and $ab\neg c$, whereas the latter represents $abc$ and $a\neg bc$. Notice that $abc$ occurs twice.

Partial assignments evaluating the input formula to true result in blocking clauses which are shorter than the ones blocking one single total model. Adding short blocking clauses has a twofold effect. First, a larger portion of the search space is ruled out. Second, fewer blocking clauses need be added which mitigates their negative impact on solver performance. Also, short blocking clauses generally propagate more eagerly than long ones. The need for shrinking or minimizing models has been pointed out by Bayardo and Pehoushek [35] and addressed further [6, 47, 48]. Notice that with blocking clauses CDCL can be used as in SAT solving, while in the absence of blocking clauses it need be adapted.

The reason is as follows. If a CDCL-based SAT solver encounters a conflict, it analyzes it and learns a clause in order to prevent the solver from repeating the same assignment which caused the conflict. This clause is determined by traversing the trail in reverse assignment order and resolving the reasons of the literals on the trail, starting with the conflicting clause, until the resolvent contains one single literal at the maximum decision level. If a model is found, the last decision literal is flipped in order to explore another branch of the search space. This leads to issues if this literal is encountered in later conflict analysis and no blocking clause was added, since in this case it is neither a decision literal nor a propagated literal.

To address this issue, Grumberg et al. [10] introduce sub-levels for flipped decision literals treating them similarly to decision literals in future conflict analysis. Similarly to Gebser et al. [39], Toda and Soh [38] limit the level to which the solver is allowed to backtrack. These measures also ensure that enumerating overlapping partial models is avoided. However, in applications where repetitions cause no harm, the power of finding even shorter models representing larger, albeit not disjoint, sets of models, can be exploited. Shorter models are also obtained in the case of model enumeration under *projection*. Workshop on Trends in Information Processing (YSIP2) 2017 [44]

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8We say that a clause is learned if it is added to the formula.
and the 30th International Conference on Tools with Artificial Intelligence (ICTAI) 2018 [37], the 22nd International Conference on Theory and Applications of Satisfiability Testing (SAT) 2019 [43], and the 5th Global Conference on Artificial Intelligence (GCAI) 2019 [18].

Structure of the paper. In Section 2 we give an overview over our contributions, before we introduce our notation and basic concepts needed in Section 3. The definitions of soundness and completeness adopted in SAT and their interpretation for model enumeration are given in Section 4. Dual reasoning is applied for shrinking models in Section 5, and an according dual encoding of blocking clauses is introduced in Section 6. After presenting our algorithm for projected model enumeration without repetition in Section 7 and providing a formalization and correctness proof and a generalization to the detection of partial models in Section 8, we turn our attention to projected model enumeration with repetition. We adapt CDCL for SAT to support conflict analysis in the context of model enumeration without the use of blocking clauses in Section 9 and discuss the changes to our method needed to support multiple model enumeration in Section 10, before we conclude in Section 11.

2 Overview of Contributions

In this section, we give a high-level overview over our contributions.

2.1 Correctness with Respect to Model Enumeration

We recall the definitions of soundness and completeness in the context of SAT solving according to Weidenbach [51], before stating them for All-SAT.

2.2 Model Shrinking

Obtaining short models is our main aim in this work. To this end, in our model enumeration algorithms we adopt a dual method for model shrinking.

Basically, whenever a model is found, a second SAT solver is called incrementally on this model and the negation of the formula. A conflict is obtained and conflict analysis executed to determine the literals involved in the conflict, which constitutes the shortened model.

2.3 Irredundant Model Enumeration Under Projection

We present a CDCL-based algorithm for projected model enumeration without repetition using dual model shrinking and blocking clauses. In a dual setting, models need be blocked not only in the formula but in its negation as well. A model is blocked in the negated formula by disjoining the two.
However, the resulting formula is no longer in CNF. To address this issue, we present a dual encoding for blocking clauses.

The shrunken model is then used to determine the backtrack level, which might be much smaller than the current one. Conflict analysis and unit propagation are executed as in CDCL. The blocking clauses ensure no problems arise in conflict analysis. Our algorithm is expressed by means of a calculus whose rules cover termination, backtracking, unit propagation and decisions.

We identify three invariants and show that they hold in all non-terminal states, that our system always makes progress, and that it eventually terminates. Equivalence of the resulting formula and the input formula projected onto the relevant variables proves soundness and completeness and concludes our proof.

A generalization of our algorithm to the case where partial satisfying assignments are found is discussed. This generalization makes sense since we do not guarantee that our model shrinking method gives us the minimal model.

2.4 Redundant Model Enumeration Under Projection

This method uses dual model shrinking. It is exempt of blocking clauses, and the conflict analysis procedure need be adapted. We propose to annotate the flipped decision literals with the clause which would be added as blocking clause and discuss the relevant changes to our algorithm, calculus, proof, and its generalization to the case where partial models are found.

3 Preliminaries

In this section we provide the concepts and notation on which our presentation relies: propositional satisfiability (SAT) and incremental SAT solving, projection, and the dual representation of a formula, which constitutes the basis for dual reasoning.

3.1 Propositional Satisfiability (SAT)

The set containing the Boolean constants 0 (false) and 1 (true) is denoted with \( \mathbb{B} = \{0, 1\} \). Let \( V \) be a set of propositional (or Boolean) variables. A literal is either a variable \( v \in V \) or its negation \( \neg v \). We write \( \bar{l} \) to denote the complement of \( l \) assuming \( \bar{l} = \neg l \) and \( \neg \neg l = l \). The variable of a literal \( l \) is obtained by \( V(l) \). This notion is extended to formulae, clauses, cubes, and sets of literals.

Most SAT solvers work on formulae in Conjunctive Normal Form (CNF), which are conjunctions of clauses, which are disjunctions of literals. These SAT solvers implement efficient algorithms tailored for CNFs, such as unit propagation, which will be presented below. In contrast, a formula in Disjunctive
Normal Form (DNF) is a disjunction of cubes, which are conjunctions of literals. We interpret formulae as sets of clauses and write $C \in F$ to refer to a clause $C$ occurring in the formula $F$. Accordingly, we interpret clauses and cubes as sets of literals. The empty CNF formula and the empty cube are denoted by $\emptyset$, while the empty DNF formula and the empty clause are represented by $0$.

A total assignment $\sigma: V \mapsto \mathbb{B}$ maps $V$ to the truth values $0$ and $1$. It can be applied to a formula $F$ over a set of variables $V$ to obtain the truth value $\sigma(F) \in \mathbb{B}$, also written $F|_{\sigma}$. The value of $F$ under $\sigma$ is denoted by $\sigma(F)$. A sequence $I = \ell_1, \ldots, \ell_n$ with mutually exclusive variables ($V(\ell_i) \neq V(\ell_j)$ for $i \neq j$) is called a trail. If their variable sets are disjoint, trails and literals may be concatenated, denoted $I = I'I''$ and $I = I'\ell I''$. We treat trails as conjunctions or sets of literals and write $\ell \in I$ if $\ell$ is contained in $I$. Trails can also be interpreted as partial assignments with $I(\ell) = 1$ iff $\ell \in I$. Similarly, $I(\ell) = 0$ if $\neg \ell \in I$, and $I(\ell)$ is undefined iff $V(\ell) \notin V(I)$. The unassigned variables in $V$ are denoted by $V - I$ and the empty trail by $\varepsilon$.

The literal $\ell$ can be either decided or propagated. In the former case, its value is chosen according to some heuristic by a decision, and $\ell$ is called decision literal. In the latter case, there exists a clause $C \in F$ containing $\ell$ in which all literals except $\ell$ evaluate to false under the current (partial) assignment. The literal $\ell$ is called unit literal or unit and $C$ a unit clause. In order to satisfy $C$, and thus $F$, the literal $\ell$ need be assigned the value true. After being propagated, the literal $\ell$ becomes a propagation literal, and $C$ its reason. The corresponding rule is the unit propagation rule. We annotate decision literals on the trail by a superscript, e.g., $\ell^d$, denoting open “left” branches in the sense of DPLL. If a decision literal $\ell$ is flipped, its complement $\overline{\ell}$ opens a “right” branch. Both propagation literals and flipped decision literals are annotated on the trail by their reason, as in $\ell^C$.

The trail is partitioned into blocks, called decision levels, which extend from a decision literal to the last literal preceding the next decision literal. Literals occurring before the first decision are assigned at decision level zero. They are assigned exclusively by unit propagation. The decision level of a variable $v \in V$ is obtained by applying the decision level function $\delta: V \mapsto \mathbb{N} \cup \infty$. If $v$ is unassigned, we have $\delta(V) = \infty$. We extend $\delta$ accordingly to literals $\ell$, non-empty clauses $C$, and non-empty sequences of literals $I$, by defining $\delta(\ell) = \delta(V(\ell)), \delta(C) = \max\{\delta(\ell) \mid \ell \in C\}$, and $\delta(I) = \max\{\delta(\ell) \mid \ell \in I\}$. Accordingly, we define $\delta(L) = \max\{\delta(\ell) \mid \ell \in L\}$ for a set of literals $L \neq \emptyset$. If $v$ is unassigned, we have $\delta(v) = \infty$, and $\delta(\emptyset) = \delta(\varepsilon) = \delta(\emptyset)$ for the empty clause, the empty sequence and the empty set of literals. Whenever a variable is assigned or unassigned, the decision level function $\delta$ is updated. If $V(\ell)$ is assigned at decision level $d$, we write $\delta[\ell \mapsto d]$. If all variables in the set of variables $V$ are unassigned, we write $\delta[V \mapsto \infty]$ or $\delta \equiv \infty$ as a shortcut. Similarly, if all literals occurring in a sequence of literals $I$ are unassigned, we write $\delta[I \mapsto \infty] = \delta[V(I) \mapsto \infty]$. The function $\delta$ is left-associative, i.e.,
δ[I → ∞][ℓ → d] first unassigns all variables on I and then assigns literal ℓ at decision level d.

We call residual of F under I, denoted F|I, the formula I(F) obtained by assigning the variables in F their truth value. If F is in CNF, this amounts to removing from F all clauses containing a literal ℓ ∈ I and removing from the remaining clauses all occurrences of ¬ℓ. If F|I = 1, we say that I satisfies F or that I is a model of F. If all variables are assigned, we call I a total model of F. Following the distinction highlighted by Sebastiani [45], if I is a partial assignment, we say that I evaluates F to 1, written I ⊢ F, if F|I = 1, and that I logically entails F, written I |= F, or that I is a partial model of F, if all total assignments extending I satisfy F. Notice that I ⊢ F implies that I |= F but not viceversa: e.g., it F def = (a ∧ b) ∨ (a ∧ ¬b) and I def = a, then I |= F but I ⊬ F, because F|I = (b ∨ ¬b) ≠ 1. If F is in CNF without valid clauses, i.e., without clauses containing contradicting literals, then I ⊬ F iff F|I = 1. We say that I evaluates F to 0 or that I is a countermodel of F, iff F|I = 0. If F is in CNF, its residual under I contains the empty clause, written 0 ∈ F|I. Similarly, we say that a conflict occurs and call the clause whose literals are set to false under I conflicting clause and its decision level conflict level. In CDCL with non-chronological backtracking this is the current decision level δ(I).

3.2 Conflict-Driven Clause Learning

Suppose the current trail I falsifies the formula F. The basic idea is to determine a clause, let’s say D, containing the negated assignments responsible for the conflict. By adding D to F, this assignment is blocked. Moreover, backtracking to the second highest decision level in D results in D becoming unit, and its literal with highest decision level is propagated.

A main ingredient of the clause learning algorithm is resolution [52]. Given two clauses (A ∨ ℓ) and (B ∨ ¬ℓ), where A and B are disjunctions of literals and ℓ is a literal, their resolvent (A ∨ ℓ) ⊗ℓ (B ∨ ¬ℓ) = (A ∨ B) is obtained by resolving them on ℓ.

The clause D is determined by a sequence of resolution steps, which can be read off either the trail or the implication graph, which is defined as follows. Decision literals are represented as nodes on the left and annotated with their decision level. Propagated literals are internal nodes with one incoming arc originating from each node representing a literal in their reason. A conflict is represented by the special node κ whose incoming arcs are annotated with the conflicting clause.

Example 3 (Trail and implication graph). Consider the formula

\[
F = \left( \neg a \lor b \right) \land \left( \neg c \lor d \right) \land \left( \neg b \lor \neg c \lor \neg d \right)
\]
over the set of variables $V = \{a, b, c, d\}$. Assume we first decide $a$, then propagate $b$ with reason $C_1$ followed by deciding $c$ and propagating $d$ with reason $C_2$. Under this assignment, $C_3$ is falsified. The current trail is given by

$$I = a^d b^{C_1} c^d d^{C_2},$$

and the according implication graph is

During conflict analysis, the conflicting clause is resolved with the reason of one of its literals. This procedure is repeated with the reason of one literal in the resolvent, and so on, until the resolvent contains one single literal at conflict level. In the following example, clause learning based on $I$ and the implication graph is shown.

Example 4 (Conflict-driven clause learning). Consider the situation in Example 3. The conflicting clause is $C_3 = (\neg b \lor \neg c \lor \neg d)$.

In order do determine the sequence of resolution steps in order to learn a clause from $I$, we resolve the conflicting clause $C_3$ with the reason of the last propagated literal on $I$, $C_2$, obtaining $\neg b \lor \neg c$, which already contains only one literal at conflict level 2, namely $\neg c$.

Considering the implication graph, we can resolve $C_3$ with either $C_2$ or $C_1$. If we choose $C_1$, the resolvent $(\neg a \lor \neg c \lor \neg d)$ contains two literals at conflict level, hence resolution with $C_2$ is needed resulting in $(\neg a \lor \neg c)$. Notice that resolving $C_3$ with $C_2$ first, would save one resolution step.

3.3 Incremental SAT Solving

The basic idea of incremental SAT solving is to exploit the progress made during the search process, if similar formulae need be solved. So, instead of discarding the learned clauses they are retained between the single SAT calls.

Hooker [53] presented the idea of incremental in the context of knowledge-based reasoning. Eén and Sörensson [54] introduced the concept of assumptions in the context of incremental SAT solving, which fits our needs best. Assumptions can be viewed as unit clauses added to the formula. They basically represent a (partial) assignment whose literals remain set to true during the solving process. In particular, backtracking does not occur past any assumed literal.
3.4 Projection

We are interested in enumerating the models of a propositional formula projected onto a subset of its variables. To this end we partition the set of variables $V = X \cup Y$ into the set of relevant variables $X$ and the set of irrelevant variables $Y$ and write $F(X \cup Y)$ to express that $F$ depends on the variable set $X \cup Y$. Accordingly, we decompose the assignment $\sigma = \sigma_X \cup \sigma_Y$ into its relevant part $\sigma_X : X \mapsto \mathbb{B}$ and its irrelevant part $\sigma_Y : Y \mapsto \mathbb{B}$ following the convention introduced in our earlier work on dual projected model counting [37]. The main idea of projection onto the relevant variables is to existentially quantify the irrelevant variables. The models of $F(X \cup Y)$ projected onto $X$ are therefore

$$\text{models}(\exists Y. F(X, Y)) = \{ \tau : X \rightarrow \mathbb{B} \mid \text{exists } \sigma : V \rightarrow \mathbb{B} \text{ with } \sigma(F(X,Y)) = 1 \text{ and } \tau = \sigma_X \},$$

and enumerating all models of $F$ without projection is therefore the special case where $Y = \emptyset$. The projection of the trail $I$ onto the set of variables $X$ is denoted by $\pi(I, X)$ and $\pi(F(X, Y), X) \equiv \exists Y [F(X, Y)]$.

**Example 5 (Projected models).** Consider again the formula $F$ in Example 1. Its unprojected models are $\text{models}(F) = \{abcd, abc \neg d, ab \neg c d, a b \neg c \neg d\}$. Its models projected onto $X = \{a, c\}$ are $ac$ and $a \neg c$.

In order to benefit from the efficient methods SAT solvers execute on CNF formulae, we transform an arbitrary formula $F(X, Y)$ into CNF by, e.g., the Tseitin transformation [55]. By this transformation, auxiliary variables, also referred to as Tseitin or internal variables, are introduced. The Tseitin transformation is satisfiability-preserving, i.e., a satisfiable formula is not turned into an unsatisfiable one and, similarly, an unsatisfiable formula is not turned into a satisfiable one. The Tseitin variables, which we denote by $S$, are defined in terms of the variables in $X \cup Y$, which we call input variables. As a consequence, for each total assignment to the variables in $X \cup Y$ there exists one single assignment to the variables in $S$ such that the resulting assignment is a model of $F$, and therefore the model count is preserved. Due to the introduction of the Tseitin variables the resulting formula $P(X, Y, S) = \text{Tseitin}(F(X, Y))$ is not logically equivalent to $F(X, Y)$, i.e., $\text{models}(F) \neq \text{models}(P)$, and the Tseitin transformation is not equivalence-preserving. However, the models of $P(X, Y, S)$ projected onto the input variables are exactly the models of $F(X, Y)$, and

$$\exists S \ [ P(X,Y,S) ] \equiv F(X,Y).$$  \hspace{1cm} (1)
The models of $F$ projected onto $X$ are accordingly given by
\[
\text{models}(\exists Y, S \ [P(X,Y,S)]) = \text{models}(\exists Y \ [F(X,Y)]).
\] (2)

### 3.5 Dual Representation of a Formula

We make use of the dual representation of a formula introduced in our earlier work [37]. Let $F(X,Y)$ and $P(X,Y,S)$ be defined as in Subsection 3.4, and let $N(X,Y,T) = \text{Tseitin}(\neg F(X,Y))$ be a CNF representation of $\neg F$, where $T$ denotes the set of Tseitin variables introduced by the transformation, i.e.,
\[
\exists T \ [N(X,Y,T)] \equiv \neg F(X,Y).
\] (3)

Formulae $P(X,Y,S)$ and $N(X,Y,T)$ are a dual representation of $F(X,Y)$. For the sake of readability, we also may write $F$, $P$, and $N$. Notice that this representation is not unique in general. Besides that, the formulae $P(X,Y,S)$ and $N(X,Y,T)$ share the set of input variables $X \cup Y$, and $S \cap T = \emptyset$. We have
\[
\exists S \ [P(X,Y,S)] \equiv \neg \exists T \ [N(X,Y,T)],
\] (4)
and in an earlier work [37] we showed that during the enumeration process a generalization of the following always holds assuming we first split on variables in $X$ and then on variables in $Y \cup S$ but never on variables in $T$:
\[
(\neg \exists T \ [N(X,Y,T)|_I]) \models (\exists S \ [P(X,Y,S)|_I])
\] (5)
where $I$ is a trail over variables in $X \cup Y \cup S \cup T$. Obviously, also
\[
(\exists S \ [P(X,Y,S)|_I]) \models (\neg \exists T \ [N(X,Y,T)|_I]),
\] (6)
saying that whenever $I$ can be extended to a model of $P$, all extensions of it falsify $N$. This property is a basic ingredient of our dual model shrinking method.

### 4 Soundness and Completeness

In this section, we recall the definitions of soundness and completeness with respect to SAT solving following Weidenbach [51] and present their definition in the context of model enumeration. In his work, Weidenbach refers to total models, but his definitions are readily applicable for partial models as well. Accordingly, and as we are targeting the detection of partial models, our definitions apply to both total and partial models.

**Definition 1** (Soundness in satisfiability). An algorithm for satisfiability is sound, if it is guaranteed that the model it finds is a model of the formula.

---

10Referred to as combined formula pair of $F(X,Y)$ in our previous work [37].
Definition 2 (Completeness in satisfiability). An algorithm for satisfiability is complete, iff it is guaranteed to find a model if the formula is satisfiable.

Definition 3 (Strong completeness in satisfiability). An algorithm for satisfiability is strongly complete iff it is guaranteed that it finds any model of a satisfiable formula.

Definition 1 says that the model returned by a sound satisfiability algorithm is a model of the formula. In the context of model enumeration, the counterpart of a model is a sequence containing all models. Hence, soundness in model enumeration states that the sequence of models returned by the model enumerator contains only models of the formula.

Definition 4 (Soundness in model enumeration). A model enumeration algorithm is sound, iff it is guaranteed that the models in the sequence of all models it enumerates are models of the formula.

Definition 2 ensures that if the input formula is satisfiable, a complete satisfiability algorithm finds a model for it. In the context of model enumeration, we would expect that a complete enumeration algorithm finds a sequence of models, if the formula is satisfiable.

Definition 5 (Completeness in model enumeration). A model enumeration algorithm is complete, iff it is guaranteed to find a sequence containing all models of a satisfiable formula.

According to Definition 3, a strongly complete satisfiability algorithm finds any, i.e., arbitrary, model. For model enumeration this would mean that a strongly complete model enumeration algorithm finds any, i.e., arbitrary, sequence of models.

Definition 6 (Strong completeness in model enumeration). A model enumeration algorithm is strongly complete, iff it is guaranteed to find any arbitrary sequence containing all models of a satisfiable formula.

Note: Strong completeness can be obtained by restarting after the addition of a blocking clause or according to some heuristic if no blocking clauses are used.

5 Dual Reasoning for Model Shrinking

In a previous work, we adopted dual reasoning for obtaining partial models [37]. Basically, we executed CDCL on the formula under consideration and its negation simultaneously exploiting the fact that CDCL is biased towards detecting conflicts. Our experiments showed that dual reasoning detects short models. However, processing two formulae simultaneously turned out to be computationally expensive.
In another work [46] we propose, before taking a decision, to check whether the current (partial) assignment logically entails the formula under consideration. We present four flavors of the entailment check, some of which use a SAT oracle and rely on dual reasoning.

The method introduced in this work, instead, exploits the effectiveness of dual reasoning in detecting short partial models while avoiding both processing two formulae simultaneously and oracle calls, which might be computationally expensive. In essence, we let the enumerator find total models and shrink them by means of dual reasoning.

Assume our task is to determine the models of a formula $F(X,Y)$ over the set of relevant variables $X$ and irrelevant variables $Y$ projected onto $X$, and let $P(X,Y,S)$ and $N(X,Y,T)$ be CNF representations of $F$ and $\neg F$, respectively, as introduced in Section 3. Obviously, Equation 2–Equation 9 hold. Suppose standard CDCL is executed on $P$. We denote with $I$ the trail which ranges over variables in $X \cup Y \cup S \cup T$, where $S$ and $T$ are the Tseitin variables occurring in $P$ and $N$, respectively.

Now assume a total model $I$ of $P$ is found. A second SAT solver is incrementally invoked on $\pi(I,X \cup Y) \land N$. Since $\pi(I,X \cup Y) \models F$ and all variables in $X \cup Y$ are assigned, due to Equation 6 a conflict in $N$ occurs by propagating variables in $T$ only. If conflict analysis is carried out as described in Subsection 3.2, the learned clause $\neg I^*$ contains only negated assumption literals.\(^{11}\) On the one hand, $\neg I^*$ represents a cause for the conflict in $N$. On the other hand, due to Equation 5, its negation $I^*$ represents a (partial) model of $F$. More precisely, $I^*$ represents all total models of $F$ projected onto $X \cup Y$ in which the variables in $X \cup Y$ not occurring in $I^*$ may assume any truth value.

**Example 6** (Model shrinking by dual reasoning). Let $F = (a \lor b) \land (c \lor d)$ be a propositional formula over the set of variables $V = \{a,b,c,d\}$. Without loss of generalization, suppose we want to enumerate the models of $F$ projected onto $V$. Assume a total model $I = a b c d$ has been found. We call a second SAT solver on $N \land I$, where

$$N = \underbrace{\neg t_1 \lor \neg a}_C \land \underbrace{\neg t_1 \lor \neg b}_C \land \underbrace{a \lor b \lor t_1}_C \land \underbrace{\neg t_2 \lor \neg c}_C \land \underbrace{\neg t_2 \lor \neg d}_C \land \underbrace{c \lor d \lor t_2}_C \land \underbrace{t_1 \lor t_2}_C$$

is the Tseitin encoding of $\neg F = (\neg a \land \neg b) \lor (\neg c \land \neg d)$ with Tseitin variables $T = \{t_1, t_2\}$. The clauses $C_1$ to $C_3$ encode $(t_1 \leftrightarrow (\neg a \land \neg b))$, the clauses $C_4$ to $C_6$ encode $(t_2 \leftrightarrow (\neg c \land \neg d))$, and $C_7$ encodes $(t_1 \lor t_2)$.

\(^{11}\)See also the work by Niemetz et al. [57].
The literals on \( I \) are considered assumed variables, annotated by, e.g., \( a^a \), and \( I = a^a b^a c^a d^a \). After propagating \( \neg t_1 \) with reason \( C_1 \) and \( \neg t_2 \) with reason \( C_4 \), the clause \( C_7 \) becomes empty. The trail is \( I' = a^a b^a c^a d^a \neg t_1 C_1 \neg t_2 C_4 \). We resolve \( C_7 \) with \( C_4 \) to obtain the clause \((t_1 \lor \neg c)\), which we then resolve with \( C_1 \). The resolvent is \((\neg c \lor \neg a)\), which contains only assumed literals, which have no reason in \( I' \), and thus can not be resolved further. Below on the left hand side, the implication graph is visualized, and the corresponding resolution steps are depicted on the right hand side.

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\[
\begin{align*}
\text{a} & \quad \text{C}_1 \quad \neg t_1 \quad \text{C}_7 \\
\text{b} & \quad \text{C}_4 \quad \neg t_2 \quad \text{C}_7 \\
\text{c} & \quad \text{c} \quad \text{C}_4 \quad \neg t_2 \quad \text{C}_7 \\
\text{d} & \quad \text{d} \quad \text{d} \\
\end{align*}
\]

The negation of the clause \((\neg c \lor \neg a)\), \( ca \), is a countermodel of \( \neg F \) and hence a model of \( F \). In this case, it is also minimal w.r.t. the number of literals.

Note: The gain obtained by model shrinking is twofold. On the one hand, it enables the (implicit) exploration of multiple models in one pass: e.g., in Example 6, the model \( ca \) represents four total models, namely, \( a b c d \), \( a b c \neg d \), \( a \neg b c d \), and \( a \neg b c \neg d \). On the other hand, short models result in short blocking clauses ruling out a larger part of the search space, as mentioned earlier.

6 Dual Encoding of Blocking Clauses

Recall that we make use of Equation 4. If a blocking clause is added to \( P \) and \( N \) is not updated accordingly, \( P \) and \( N \) do not represent the negation of each other anymore, and Equation 4 ceases to hold. This might lead to multiple model enumerations in the further search. This issue can be remediated by adding the shrunk models disjunctively to \( N \). To retain \( N \) in CNF and ensure Equation 4, we propose the following dual encoding of the blocking clauses.

We denote with \( \text{Tseitin}() \) the function which takes as argument an arbitrary propositional formula and returns its Tseitin transformation. For the sake of readability, we write \( F \), \( P \), and \( N \) as well as their indexed variants instead of \( F(X \cup Y) \), \( P(X \cup Y \cup S) \) and \( N(X \cup Y \cup T) \). We define

\[
P_0 = \text{Tseitin}(F) \\
N_0 = t_0 \land \text{Tseitin}(t_0 \leftrightarrow \neg F).
\]
Let $I_1$ be a trail such that $I_1$ evaluates $F$ to true, i.e., $I_1 \models F$. A second SAT solver SAT is called on $\pi(I_1, X \cup Y) \land N_0$, and a conflict is obtained as argued above. Assume SAT returns the assignment $I^*_1 \subseteq I_1$ such that $SAT(\pi(I^*_1, X \cup Y), N_0) = UNSAT$. Then $\neg \pi(I^*_1, X)$ is added to $P_0$ obtaining $P_1 = P_0 \land \neg \pi(I^*_1, X)$. To ensure Equation 4, we define $N_1 = (t_0 \lor t_1) \land Tseitin(t_0 \leftrightarrow \neg F) \land Tseitin(t_1 \leftrightarrow \pi(I^*_1, X))$, and so on.

At the $n$th step, we have

\[ P_n = P_0 \land \bigwedge_{i=1}^{n} \neg \pi(I^*_i, X) \tag{9} \]

\[ N_n = (t_0 \lor \bigvee_{i=1}^{n} t_i) \land Tseitin(t_0 \leftrightarrow \neg F) \land \bigwedge_{i=1}^{n} Tseitin(t_i \leftrightarrow \pi(I^*_i, X)) \tag{10} \]

where the red parts denote the additions to $P_0$ and $N_0$.

Let $I_{n+1}$ be a trail evaluating $P_n$ to true, i.e., $I_{n+1} \models P_n$. We invoke SAT on $\pi(I_{n+1}, X \cup Y) \land N_n$ leading to a conflict as described above. Assume SAT returns $I^*_n \leq I_{n+1}$, such that $SAT(\pi(I^*_n, X \cup Y), N_n) = UNSAT$. We add $\neg \pi(I^*_n+1, X)$ to $P_n$ and update $N_n$ accordingly. Now we have

\[ P_{n+1} = P_n \land \neg \pi(I^*_n+1, X) \tag{11} \]

\[ N_{n+1} = N_n \setminus \{(t_0 \lor \bigvee_{i=1}^{n} t_i) \land (t_0 \lor \bigvee_{i=1}^{n+1} t_i) \land Tseitin(t_{n+1} \leftrightarrow \pi(I^*_n+1, X)) \} \tag{12} \]

where $I_{i+1} \models P_i$ for $0 \leq i \leq n$ and $I^*_n+1 \leq I_{n+1}$ is s.t. $SAT(\pi(I^*_n+1, X \cup Y), N_i) = UNSAT$.

**Proposition 1.** Let $F(X, Y)$ be an arbitrary propositional formula over the relevant variables $X$ and the irrelevant variables $Y$. Let $F$ and $\neg F$ be encoded into CNFs $P_0$ and $N_0$, respectively, according to Equation 7 and Equation 8. If for all models found blocking clauses are added to $P_0$ and $N_0$ according to Equation 9 and Equation 10, then only pairwise contradicting models are found, i.e., $\pi(I^*_i, X)$ and $\pi(I^*_j, X)$ are pairwise contradicting for every $i \neq j$.

**Proof.** By construction, $N_n \equiv \neg F \lor \bigvee_{i=1}^{n} \pi(I^*_i, X \cup Y)$, and $\pi(I^*, X \cup Y) \land
\( N_n \equiv 0 \). Furthermore, \( \pi(I^*, X \cup Y) \land \neg F \equiv 0 \). We have

\[
0 \equiv \\
\pi(I_{n+1}^*, X \cup Y) \land (-F \lor \bigvee_{i=1}^n \pi(I_i^*, X \cup Y)) = \\
(\pi(I_{n+1}^*, X \cup Y) \land \neg F) \lor (\pi(I_{n+1}^*, X \cup Y) \land \bigvee_{i=1}^n \pi(I_i^*, X \cup Y)) \equiv \\
(\pi(I_{n+1}^*, X) \land \bigvee_{i=1}^n \pi(I_i^*, X)) ,
\]

since \( I_i^* \) contains only relevant variables. Hence, \( \pi(I_{n+1}^*, X) \land \pi(I_i^*, X) \equiv 0 \) for \( i = 1, \ldots, n \).

\( \Box \)

Note: Equation 4 always holds:

\[
\exists S[p_i(X,Y,S)] \equiv \neg \exists T[N_i(X,Y,T)] \quad \text{for all } 0 \leq i \leq n + 1.
\]

Consequently, also Equation 5 and Equation 6 hold:

\[
(\neg \exists T[N_i(X,Y,T)]_{|T}) \vdash (\exists S[p_i(X,Y,S)]_{|T}) \quad \text{for all } 0 \leq i \leq n + 1
\]

\[
(\exists S[p_i(X,Y,S)]_{|T}) \vdash (\neg \exists T[N_i(X,Y,T)]_{|T}) \quad \text{for all } 0 \leq i \leq n + 1
\]

However, for our usage we may use the implication in the forward direction only and write \( t_i \rightarrow \pi(I_i^*, X) \) and \( t_{n+1} \rightarrow \pi(I_{n+1}^*, X) \) in Equation 10 and Equation 12 without compromising correctness for the following reason: The formula \( N_i \) is called always under \( I_{i+1} \) which falsifies all \( I_k^* \) for \( 0 \leq k \leq i \). Hence, \( \pi(I_i^*, X) \rightarrow t_i \) is always true.

**Example 7** (Dual blocking clauses). We clarify the proposed encoding by a small example and show that it prevents multiple model counts. Let our example be \( F = x_1 \lor (x_2 \land \neg x_3) \) and assume we have found the model \( I_1^* = x_1 \).

Then

\[
P_1 = (\neg s_1 \lor x_2) \land (\neg s_1 \lor x_3) \land (s_1 \lor \neg x_2 \lor \neg x_3) \land (x_1 \lor s_1) \land (\neg x_1)
\]

and

\[
N_1 = \frac{\neg t_1 \lor \neg x_1}{C_1} \land \frac{\neg t_1 \lor \neg x_2}{C_2} \land \frac{t_0 \lor x_1 \lor x_2}{C_3} \\
\frac{\neg t_2 \lor \neg x_1}{C_4} \land \frac{\neg t_2 \lor \neg x_3}{C_5} \land \frac{t_2 \lor x_2 \lor x_3}{C_6} \\
\frac{t_1 \lor t_2 \lor t_3}{C_7} \land \frac{\neg t_3 \lor x_1}{C_8} \land \frac{t_3 \lor \neg x_1}{C_9}
\]

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where the blue parts denote the corresponding additions to \( P_0 \) and \( N_0 \). If now we find a total model \( I_2 = \neg x_1 x_2 x_3 \), we obtain a conflict in \( N_1 \) by unit propagating variables \( t_1, t_3, \) and \( t_3 \) only. The conflicting clause is \((t_1 \lor t_2 \lor t_3)\).

The implication graph is depicted below on the left hand side, the corresponding resolution steps for conflict analysis below on the right hand side.

\[
\begin{align*}
\neg x_1 & \quad C_8 \\
x_2 & \quad C_2 \\
x_3 & \quad C_5 \\
\neg t_1 & \quad C_7 \\
\neg t_3 & \quad C_7 \\
\end{align*}
\]

\[
\begin{align*}
(t_1 \lor t_2 \lor t_3) & \quad (~t_2 \lor \neg x_3) \\
(t_1 \lor t_3 \lor \neg x_3) & \quad (~t_1 \lor \neg x_2) \\
(t_3 \lor \neg x_3 \lor \neg x_2) & \quad (~t_3 \lor x_1) \\
(\neg x_3 \lor \neg x_2 \lor x_1) & \quad (~t_1) \\
\end{align*}
\]

Conflict analysis returns the clause \((\neg x_3 \lor \neg x_2 \lor x_1)\), which, after being added to \( P \), blocks the model \( \neg x_1 x_2 x_3 \), which does not overlap with the previously found model \( x_1 \).

7 Projected Model Enumeration Without Repetition

We are given a propositional formula \( F(X,Y) \) over the set of irredundant variables \( X \) and the set of redundant variables \( Y \), and our task is to enumerate its models projected onto the variables in \( X \). Let \( P(X,Y,S) \) and \( N(X,Y,T) \) be a dual representation of \( F \) according to Section 3. Obviously, Equation 2–Equation 6 hold.

In Figure 1, we consider the case with permanent learning of the blocking clauses. Let the first SAT solver execute standard CDCL on \( P \) and let \( I \) denote its trail. Obviously it finds only total models of \( P \). Due to Equation 2, these models satisfy \( F \), too. Now assume a (total) model \( I \) of \( F \) is found. A second SAT solver \( SAT \) is incrementally invoked on \( \pi(I,X \cup Y) \land N \) with the aim to shrink \( I \) obtaining \( I^* \) as described in Section 5.

Let \( b \) denote the decision level of \( \pi(\neg I^*, X) \) and \( \ell \) be the literal in \( \pi(\neg I^*, X) \) with decision level \( b \). If we now add the clause \( \pi(\neg I^*, X) \) to \( P \) and backtrack to decision level \( b - 1 \), it becomes unit and in the next step \( \neg \ell \) is propagated. Notice that \( \pi(\neg I^*, X) \) acts in \( P \) as a blocking clause and must not be deleted anytime which might blow up \( P \) and slow down the first SAT solver. Moreover, the dual encoding of the blocking clause according to Section 6 ensures Equation 4, on which our method relies.

In Subsection 7.1, we present the main function \texttt{EnumerateIrredundant}. Unit propagation (Subsection 7.2) and the schema for conflict analysis (Subsection 7.3) are the same as in CDCL for SAT.
Input: formulae $P(X,Y,S)$ and $N(X,Y,T)$ s.t.
$$\exists S [P(X,Y,S)] \equiv \neg \exists T [N(X,Y,T)] ,$$
set of variables $X \cup Y \cup S \cup T$, trails $I$ and $J$

Output: DNF representation of $\pi(P,X)$

```
EnumerateIrredundant (P, N)  // $P_0 = \text{CNF}(F)$
                           // $N_0 = t_0 \land \text{CNF}(t_0 \leftrightarrow \neg F)$

1 $I := \varepsilon$
2 $\delta[V \mapsto \infty]$
3 $M := 0$
4 $i := 0$
5 forever do
6   $i := i + 1$
7   $C := \text{PropagateUnits}(P, I, \delta)$
8   if $C \neq 0$ then
9     $c := \delta(C)$
10    if $c = 0$ then
11       return $M$
12    else
13       $\text{AnalyzeConflict}(P, I, C, \delta)$
14   else
15     if all variables in $X \cup Y \cup S$ are assigned then
16        // $I$ is total model of $P$ and $F$
17        if $V(\text{decs}(I)) \cap X = \emptyset$ then
18           return $M \lor \pi(I, X)$
19        else
20           $I^* := \text{SAT}(N, \pi(I, X \cup Y))$
21           // $I^*$ is model of $\pi(F, X \cup Y)$ and conflict set of $I$ w. r. t. $N$
22           $P := P \land \neg \pi(\text{decs}(I^*), X)$
23           $N := N \setminus \{(t_0 \lor \bigvee_{j=1}^i t_j)\} \land$
24             $(t_0 \lor \bigvee_{j=1}^i t_j) \land \text{CNF}(t_i \leftrightarrow \pi(\text{decs}(I^*), X))$
25           $M := M \lor \pi(I^*, X)$
26           $b := \delta(\neg \pi(I^*, X))$
27           $\text{Backtrack}(I, b - 1)$
28       else
29          $\text{Decide}(I)$
```

Figure 1: Irredundant model enumeration. The black lines describe CDCL returning a model if one is found and the empty clause otherwise. The blue part represents the extension to model enumeration. A second SAT solver is called incrementally on $N$ assuming the literals on $\pi(I, X \cup Y)$. A conflict occurs by unit propagation only, and $\pi(I^*, X \cup Y)$ is a (partial) model of $F$. The is encoded as a dual blocking clause, and $P$ and $N$ are updated accordingly.
7.1 Main Algorithm

The function \texttt{EnumerateIrredundant} in Figure 1 describes the main algorithm. (Black rows 1–18 and 27–28 represent standard CDCL returning a model if the formula under consideration is satisfiable and the empty clause otherwise, blue rows 19–26 the rest of the algorithm.)

Initially, the trail \( I \) is empty, the target DNF \( M \) is 0, and all variables are unassigned, i.e., assigned decision level \( \infty \). Unit propagation is executed until either a conflict occurs or all variables are assigned a value (line 7).

If a conflict occurs at decision level zero, the search space has been processed exhaustively, and the enumeration terminates (lines 8–11). If a conflict occurs at a decision level higher than zero, conflict analysis is executed (line 13).

If no conflict occurs and all variables are assigned, a total model has been found (line 15). If no relevant decisions are left on the trail \( I \), the search space has been processed exhaustively, the found model is output and the search terminates (lines 17–18). If \( I \) contains a relevant decision, the found model is shrunken (line 20) by means of dual reasoning as described in Section 5. It is blocked, and the last relevant decision literal is flipped (lines 22–26). If no conflict occurs and not all variables are assigned a value, a decision is taken (line 28).

7.2 Unit Propagation

Unit propagation is described by the function \texttt{PropagateUnits} in Figure 2. It takes as input the formula \( F \), the trail \( I \), and the decision level function \( \delta \). If a clause \( C \in F \) is unit under \( I \), its unit literal \( \ell \) is propagated, i.e., \( I \) is extended by \( \ell \) (line 2). Propagated literals are assigned at the current decision level (line 3) as is usual in modern CDCL-based SAT solvers. If the resulting trail falsifies some clause \( D \in F \), this clause is returned (lines 4–5). Otherwise the function returns the empty clause 0 (line 6).

7.3 Conflict Analysis

Conflict analysis is described by the function \texttt{AnalyzeConflict} in Figure 2. It takes as input the formula \( F \), the trail \( I \), the conflicting clause \( C \), and the decision level function \( \delta \). A clause \( D \) is learned as described in Section 9 and added to \( F \) (lines 1–2). The second highest decision level \( j \) in \( D \) is determined (lines 3–4), and the enumerator backtracks (non-chronologically) to decision level \( j \). Backtracking involves unassigning all literals with decision level higher than \( j \) (lines 5–7). After backtracking, the clause \( D \) becomes unit with unit literal \( \ell \), which is propagated and assigned decision level \( j \) (lines 8–9).
PropagateUnits \((F, I, \delta)\)
1. while some \(C \in F\) is unit \((\ell)\) under \(I\) do
2. \(I := I\ell\)
3. \(\delta(\ell) := \delta(I)\)
4. for all clauses \(D \in F\) containing \(-\ell\) do
5. if \(I(D) = 0\) then return \(D\)
6. return \(0\)

AnalyzeConflict \((F, I, C, \delta)\)
1. \(D := \text{Learn}(I, C)\)
2. \(F := F \land D\)
3. \(\ell :=\) literal in \(D\) at decision level \(\delta(I)\)
4. \(j := \delta(D \setminus \{\ell\})\)
5. for all literals \(k \in I\) with decision level \(> j\) do
6. assign \(k\) decision level \(\infty\)
7. remove \(k\) from \(I\)
8. \(I := I\ell\)
9. \(\delta(\ell) := j\)

Figure 2: The function PropagateUnits implements unit propagation in \(F\). The unit literal \(\ell\) is assigned the decision level of \(I\). If some clause \(D \in F\) containing the complement of \(\ell\) becomes falsified, PropagateUnits returns \(D\). Otherwise it returns the empty clause \(0\) indicating that no conflict has occurred. The function AnalyzeConflict is called whenever a clause \(C \in F\) becomes empty under the current assignment. It learns a clause \(D\) starting with the conflicting clause \(C\). The solver then backtracks to the second highest decision level \(j\) in \(D\), upon which \(D\) becomes unit with unit literal \(\ell\), and propagates \(\ell\).
8 Formalizing Projected Irredundant Model Enumeration

In this section, we provide a formalization of our algorithm presented in Section 7. Let $F(X,Y)$ be a formula defined onto the set of relevant (input) variables $X$ and the set of irrelevant (input) variables $Y$, and assume our task is to enumerate its models projected onto $X$.

Our formalization works on a dual representation of $F$, given by $P(X,Y,S)$ and $N(X,Y,T)$ introduced in Subsection 3.5. So, $P(X,Y,S)$ and $N(X,Y,T)$ are defined over the same sets of relevant variables $X$ and irrelevant variables $Y$ as well as the disjoint sets of variables $S$ and $T$, respectively, which are defined in terms of the variables in $X \cup Y$. Recall that Equation 2–Equation 6 hold. We show the working of our calculus by means of an example, before we provide a correctness proof.

8.1 Calculus

We formalize the algorithm presented in Section 7 as a state transition system with transition relation $\sim_{\text{EnumIrred}}$. Non-terminal states are described by tuples $(P,N,M,I,\delta)$. The third element, $M$, is a DSOP formula over variables in $X$. The fourth element, $I$, denotes the trail defined over variables in $X \cup Y \cup S \cup T$, and $\delta$ denotes the decision level function. The initial state is $(P_0,N_0,0,\varepsilon,\delta_0)$, where $P_0$ and $N_0$ denote the initial CNF representations of $F$ and $\neg F$, respectively, $\varepsilon$ denotes the empty trail, and $\delta_0 \equiv \infty$. The terminal state is given by a DSOP formula $M$, which is equivalent to the projection of $P$ onto $X$. The transition relation $\sim_{\text{EnumIrred}}$ is the union of transition relations $\sim_R$, where $R$ is either End1, End0, Unit, Back1, Back0, DecX, or DecYS. The rules are listed in Figure 3.

End1. All variables are assigned and no conflict in $P$ occurred, hence the trail $I$ is a total model of $P$. It contains no relevant decision indicating that the relevant search space has been processed exhaustively. The model projected onto $X$ is added to $M$, and the search terminates. It is sufficient to check for relevant decisions, since flipping an irrelevant one would result in detecting redundant models projected onto $X$. However, due to the addition of blocking clauses, a conflict would occur, and checking for relevant decisions essentially saves work.

End0. A conflict at decision level zero has occurred indicating that the search space has been processed exhaustively. The search terminates leaving $M$ unaltered. We need to make sure no decision is left on the trail, which in particular includes the irrelevant ones. The reason is that after flipping any decision—in particular also irrelevant and internal ones—the resulting trail might be extended to a model of $P$. 

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### Rules for projected model enumeration without repetition

States are represented as tuples \((P, N, M, I, \delta)\). The formulae \(P(X, Y, S)\) and \(N(X, Y, T)\) are a dual representation of the formula \(F(X, Y)\), whose models projected onto \(X\) are sought. These models are recorded in the initially empty DNF \(M\). The last two elements, \(I\) and \(\delta\), denote the current trail and decision level function, respectively. If a model is found or a conflict encountered and the search space has been exhaustively processed, the search terminates (rules **End1** and **End0**). Otherwise, the model is shrunken and a dual blocking clause added (rules **Back1**) or conflict analysis executed followed by non-chronological backtracking (rule **Back0**). If the residual of \(P\) under the current trail \(IJ\) contains a unit literal, this is propagated (rule **Unit**). If none of the mentioned preconditions are met, a decision is taken. Relevant literals are prioritized (rule **DecX**) over irrelevant and internal ones (rule **DecYS**).

---

**End1:** \((P, N, M, I, \delta) \sim_{\text{End1}} (M \vee m)\) if \(P|_I \neq 0\) and \((X \cup Y \cup S) - I = \emptyset\) and \(V(\text{decs}(I)) \cap X = \emptyset\) and \(m \overset{\text{def}}{=} \pi(I, X)\)

**End0:** \((P, N, M, I, \delta) \sim_{\text{End0}} M\) if \(C \in P\) with \(C|_I = 0\) and \(\delta(C) = 0\)

**Unit:** \((P, N, M, I, \delta) \sim_{\text{Unit}} (P, N, M, \ell C, \delta[\ell \mapsto a])\) if \(P|_I \neq 0\) and \(C \in P\) with \(\{\ell\} = C|_I\) and \(a \overset{\text{def}}{=} \delta(I)\)

**Back1:** \((P, N, M, I, \delta) \sim_{\text{Back1}} (P \wedge B, O, M \vee m, J\ell B, \delta[K \mapsto \infty][\ell \mapsto b])\)

- if \((X \cup Y \cup S) - I = \emptyset\) and \(I* \leq \pi(I, X \cup Y)\) with \(JK = I\) such that \(N \wedge I* \vdash_1 0\) and \(m \overset{\text{def}}{=} \pi(I*, X)\) and \(B \overset{\text{def}}{=} \neg \text{decs}(m)\) and \(b + 1 \overset{\text{def}}{=} \delta(B) = \delta(m)\) and \(\ell \in B\) and \(\ell|_K = 0\) and \(b = \delta(B \setminus \{\ell\}) = \delta(J)\) and \(O = \text{Tseitin}(N \lor \neg B)\)

**Back0:** \((P, N, M, I, \delta) \sim_{\text{Back0}} (P \wedge D^r, N, M, J\ell D, \delta[K \mapsto \infty][\ell \mapsto j])\)

- if \(C \in P\) and \(\exists D\) with \(JK = I\) and \(C|_I = 0\) and \(\delta(C) = \delta(D) > 0\) such that \(\ell \in D\) and \(\neg \ell \in \text{decs}(I)\) and \(-\ell|_K = 0\) and \(P \models D\) and \(j \overset{\text{def}}{=} \delta(D \setminus \{\ell\}) = \delta(J)\)

**DecX:** \((P, N, M, I, \delta) \sim_{\text{DecX}} (P, N, M, I\ell, \delta[\ell \mapsto d])\) if \(P|_I \neq 0\) and \(\text{units}(P|_I) = \emptyset\) and \(\delta(\ell) = \infty\) and \(d \overset{\text{def}}{=} \delta(I) + 1\) and \(V(\ell) \in X\)

**DecYS:** \((P, N, M, I, \delta) \sim_{\text{DecYS}} (P, N, M, I\ell, \delta[\ell \mapsto d])\) if \(P|_I = 0\) and \(\text{units}(P|_I) = \emptyset\) and \(\delta(\ell) = \infty\) and \(d \overset{\text{def}}{=} \delta(I) + 1\) and \(V(\ell) \in Y \cup S\) and \(X - I = \emptyset\)
Unit. No conflict in $P$ occurred, and a clause in $P$ is unit under $I$. Its unit literal $\ell$ is propagated and assigned the current decision level.

Back1. All variables are assigned and no conflict in $P$ occurred, hence the trail $I$ is a total model of $P$. It is shrunken as described in Section 5 obtaining $I^*$. The projection of $I^*$ onto $X$, $m$, is added to $M$. The clause $B$ consisting of the negated decision literals of $m$ is added as a blocking clause to $P$. Its negation $\neg B$ is added disjunctively to $N$, which is transformed back into CNF by means of the Tseitin transformation. The solver backtracks to the second highest decision level in $B$ and propagates $\ell$ at the current decision level, i.e., basically flips the relevant decision literal with highest decision level.

Back0. The current trail falsifies a clause in $P$ at a decision level greater than zero indicating that the search space has not yet been processed exhaustively. Conflict analysis returns a clause $D$ implied by $P$, which added to $P$ and marked as redundant. The solver backtracks to the second highest decision level $j$ in $D$. The learned clause $D$ becomes unit, and its unit literal $\ell$ is propagated at decision level $j$. In contrast to End1, any decision literal need be flipped, which particularly applies to irrelevant and internal decision literals.

DecX. No conflict has occurred, the residual of $P$ under $I$ contains no units, and there is an unassigned relevant literal $\ell$. The current decision level is incremented to $d$, $\ell$ is decided and assigned decision level $d$.

DecYS. No conflict has occurred, and the residual of $P$ under $I$ contains no units. All relevant literals are assigned, and there is an unassigned irrelevant or internal literal $\ell$. The current decision level is incremented to $d$, $\ell$ is decided and assigned decision level $d$.

8.2 Example

The working of our calculus is shown by means of an example. Consider again Example 1 and Example 5. We have

$$F = (a \lor c) \land (a \lor \neg c) \land (b \lor d) \land (b \lor \neg d)$$

and assume the set of relevant variables is $X = \{a, c\}$ and the set of irrelevant variables is $Y = \{b, d\}$. The formula $F$ is already in CNF, therefore we define $P_0 = F$ and accordingly $S_0 = \emptyset$. For its negation

$$\neg F = (\neg a \land \neg c) \lor (\neg a \land c) \lor (\neg b \land \neg d) \lor (\neg b \land d)$$
Figure 4: Execution trace for $F = (a \lor c) \land (a \lor \neg c) \land (b \lor d) \land (b \lor \neg d)$ defined over the set of relevant variables $X = \{a, c\}$ and the set of irrelevant variables $Y = \{b, d\}$ (see also Example 1 and Example 5).

we define

$$N_0 = (\neg t_1 \lor \neg a) \land (\neg t_2 \lor \neg a) \land (\neg t_3 \lor \neg b) \land (\neg t_4 \lor \neg b) \land (a \lor c \lor t_1) \land (a \lor c \lor t_2) \land (b \lor d \lor t_3) \land (b \lor d \lor t_4)$$

with the set of internal variables $T_0 = \{t_1, t_2, t_3, t_4\}$. Assume a lexicographic ordering of the input variables, i.e., $a \succ_{lex} b \succ_{lex} c \succ_{lex} d$, and assume we choose the decision variable according to this ordering. The execution steps are depicted in Figure 4.

Step 0: The initial state is given by the empty trail $\varepsilon$, the CNF formulae $P_0$ and $N_0$, and the empty DNF formula 0.

Step 1: The formula $P_0$ contains no units, and there are unassigned relevant variables. The preconditions of rule $\text{DecX}$ are met, and decision $a$ is taken.

Step 2: No conflict occurred, $P_0|_I$ contains no units, and there are unassigned relevant variables. The preconditions of rule $\text{DecX}$ are met, and $c$ is decided.

Step 3: No conflict occurred, $P_0|_I$ contains no units. All relevant variables are assigned, and there are unassigned irrelevant variables. The preconditions of
Step 4: Again, the preconditions of rule $\text{DecYS}$ are met, and decision $b$ is taken. Notice that $I$ already satisfies $P_0$, but the solver is not able to detect this fact.

Step 5: No conflict occurred and all variables are assigned, hence $I$ is a model of $P_0$. It is shrunked following the procedure described in Section 5.

We call a SAT solver incrementally on $N_0 \land I$, i.e., assuming the literals on $I$. A conflict in $N_0$ occurs by propagation of variables in $T_0$ only, and conflict analysis provides us with the shrunk model $a b$ of $F$. Below, the resulting implication graph and trail are depicted:

The conflicting clause is $D_{13}$. For conflict analysis, we resolve $D_{13}$ with $D_{10}$, the resolvent with $D_7$, followed by resolution with $D_4$ and $D_1$. The obtained clause ($\neg b \lor \neg a$) contains only assumed literals. The assumptions $c$ and $d$ do not participate in the conflict and therefore do not occur in the resulting clause. Below, the resolution steps are visualized.

\[
\begin{align*}
(t_1 \lor t_2 \lor t_3 \lor t_4) & \quad (\neg t_4 \lor \neg b) \\
(t_1 \lor t_2 \lor t_3 \lor \neg b) & \quad (\neg t_3 \lor \neg b) \\
(t_1 \lor t_2 \lor \neg b) & \quad (\neg t_2 \lor \neg a) \\
(t_1 \lor \neg b \lor \neg a) & \quad (\neg t_1 \lor \neg a) \\
(\neg b \lor \neg a) & 
\end{align*}
\]

The negation of ($\neg b \lor \neg a$) is $I^* = a b \leq I$ we are looking for. The first model is $m_1 = \pi(I^*, X) = a$ and accordingly $M_1 = M_0 \lor m_1$. Furthermore we have
\[ B_1 = \neg \text{dec}(m_1) = (\neg a), \text{ hence} \]

\[ P_1 = P_0 \land (\neg a) \quad \text{and} \]

\[ N_1 = N_0 \lor (a) \quad \text{where} \]

\[ D_{14} \land D_{15} = (t_5 \leftrightarrow a) \text{ is the Tseitin transformation of } m_1. \]

The clause \( \neg B_1 \) is added disjunctively to \( N_0 \). To retain \( N \) in CNF, \( \neg B_1 \) is encoded as \( (t_5 \leftrightarrow \neg B_1) \), \( t_5 \) is added to \( D_{13} \) resulting in \( D_{16} \), and \( T_1 = T_0 \cup \{ t_5 \} = \{ t_1, t_2, t_3, t_4, t_5 \} \) as described in Section 6. The clause \( B_1 \) acts in \( P \) as blocking clause. The solver backtracks to decision level zero and propagates \( \neg a \) with reason \( B_1 \).

Step 6: The formula \( P_1|_I \) contains two units, \( C_1|_I = (c) \) and \( C_2|_I = (\neg c) \). The literal \( c \) is propagated with reason \( C_1 \).

Step 7: The trail falsifies \( C_2 \), and the current decision level is zero. The preconditions of rule \( \text{End}0 \) are met and the search terminates without altering \( M = a \), which represents exactly the models of \( F \) projected onto \( X \), namely \( ac \) and \( a \neg c \).

8.3 Proofs

Our proofs are based on the ones provided for our work addressing chronological CDCL for model counting [18], which in turn rely on the proof of correctness we provided for chronological CDCL [43]. The method presented here mainly differs from the former in the following aspects: The total models found are shrunken by means of dual reasoning. It adopts non-chronological CDCL instead of chronological CDCL and accordingly makes use of blocking clauses, which affects the ordering or the literals on the trail. In fact, unlike in chronological CDCL, the literals on the trail are ordered in ascending order with respect to their decision level, which simplifies not only the rules but also the proofs. Projection in turn adds complexity to some invariants. In some aspects our proofs are similar to or essentially the same as those in our former proofs [43, 18]. However, they are fully worked out to keep them self-contained.

In order to prove the correctness of our method, we make use of the invariants listed in Figure 5. Invariant InvDualPN in essence is Equation 4.
| Invariant | Description |
|-----------|-------------|
| InvDualPN | $\exists S \ [P(X,Y,S)] \equiv \neg \exists T \ [N(X,Y,T)]$ |
| InvDecs  | $\delta(\text{decs}(I)) = \{1, \ldots, \delta(I)\}$ |
| InvImplIrred | $\forall n \in \mathbb{N}. P \land \text{decs}_\leq n(I) \models I_\leq n$ |
| InvDSOP  | $M$ is a DSOP |

Figure 5: Invariants for projected model enumeration without repetition.

It ensures that the shrunken model is again a model of $P$ projected onto the input variables stating that $P$ and $N$ projected onto the input variables $X \cup Y$ are each other’s negation. Intuitively, Invariant InvDualPN holds because the found models are blocked in $P$ and added to its negation $N$. Invariants InvDecs and InvImplIrred equal Invariants (2) and (3) in our proofs of correctness of chronological CDCL [43] and model counting by means of chronological CDCL [18]. Invariant InvImplIrred differs from the latter in that we need not consider the negation of the DNF $M$ explicitly. The negation of $M$ is exactly the conjunction of the blocking clauses associated with the found models, and these are added to $P$. Invariant InvImplIrred is needed to show that the literal propagated after backtracking is implied by the resulting trail. Its reason is either a blocking clause (rule Back1) or a clause learned by means of conflict analysis (rule Back0).

Our proof is split into several parts. We start by showing that the invariants listed in Figure 5 hold in non-terminal states (Subsubsection 8.3.1). Then we prove that our method always makes progress (Subsubsection 8.3.2), before showing that our procedure terminates (Subsubsection 8.3.3). We conclude the proof by showing that every total model is found exactly once and that all total models are detected, i.e., that upon termination $M \equiv \pi(P,X)$ holds (Subsubsection 8.3.4).

### 8.3.1 Invariants in Non-Terminal States

**Proposition 2.** Invariants InvDualPN, InvDecs, InvDSOP, and InvImplIrred hold in non-terminal states.

**Proof.** The proof is carried out by induction over the number of rule applications. Assuming Invariants InvDualPN to InvImplIrred hold in a non-terminal state $(P, N, M, I, \delta)$, we show that they are met after the transition to another non-terminal state for all rules.

**Unit**

Invariant InvDualPN: Neither $P$ nor $N$ are altered, hence Invariant InvDualPN holds after the application of rule Unit.
**Invariant InvDecs:** The trail \( I \) is extended by a literal \( \ell \). We need to show that \( \ell \) is not a decision literal. Only the case where \( a > 0 \) need be considered, since at decision level zero all literals are propagated. There exists a clause \( C \in P \) s.t. \( C|_I = \{ \ell \} \). Now, \( a = \delta(I) \), i.e., there is already a literal \( k \neq \ell \) on \( I \) with \( \delta(k) = a \). From this it follows that \( \ell \) is not a decision literal. The decisions remain unchanged, and Invariant InvDecs holds after applying rule Unit.

**Invariant InvImplIrred:** Due to \( C|_I = \{ \ell \} \), we have \( P \wedge \text{decs}_{ \leq n}(I) \models \neg(C \setminus \{ \ell \}) \). Since \( C \in P \), also \( P \wedge \text{decs}_{ \leq n}(I) \models C \). Modus ponens gives us \( P \wedge \text{decs}_{ \leq n}(I) \models I_{\leq n} \). Hence, \( P \wedge \text{decs}_{ \leq n}(I_{\ell}) \models I_{\leq n} \), and Invariant InvImplIrred holds after executing rule Unit.

**Invariant InvDSOP:** Due to the premise, \( M \) is a DSOP. It is not altered by rule Unit and after its application is therefore still a DSOP.

**Back1**

**Invariant InvDualPN:** We have \( \exists S \left[ P(X,Y,S) \right] \equiv \exists T \left[ N(X,Y,T) \right] \) and we need to show \( \exists S \left[ (P \wedge B)(X,Y,S) \right] \equiv \exists T \left[ O(X,Y,T) \right] \) where \( B = \neg \text{decs}(m) \) and \( B = T \text{sein}(N \lor \neg B) \) and \( m = \pi(I^*,X) \) is a model of \( P \) projected onto \( X \). Since we have that \( \exists T \left[ O(X,Y,T) \right] \equiv \exists T \left[ (N \lor \neg B)(X,Y,T) \right] \), and furthermore \( \exists T \left[ (N \lor \neg B)(X,Y,T) \right] \equiv \forall T \left[ (\neg N \land B)(X,Y,T) \right] \), we reframe the claim as \( \exists S \left[ (P \wedge B)(X,Y,S) \right] \equiv \forall T \left[ (\neg N \land B)(X,Y,T) \right] \). Together with \( \exists S \left[ P(X,Y,S) \right] \equiv \forall T \left[ \neg N(X,Y,T) \right] \) and observing that \( B \) contains no variable in \( Y \), the claim holds.

**Invariant InvDecs:** We show that the decisions remaining on the trail are unaffected and that no new decision is taken, i.e., \( \ell \) in the post state is not a decision. It is sufficient to consider the case where \( \delta(I) > 0 \). Now, \( J = I_{\leq b} \) by the definition of \( J \), and the decisions on \( J \) are not affected by rule Back1. We have \( \delta(B \setminus \{ \ell \}) = b = \delta(J) \) and \( \delta(B) = b + 1 \). Since relevant decisions are prioritized, also \( B = \neg \text{decs}_{\leq b+1}(\pi(I^*,X)) = \neg \text{decs}_{\leq b+1}(I) \). According to Invariant InvDecs there exists exactly one decision literal for each decision level and in particular in \( B \). Since \( \ell \in B \), we have \( \neg \ell \in \text{decs}(I) \). Precisely, \( \neg \ell \in K \), and \( \neg \ell \) is unassigned upon backtracking. Due to the definition of \( B \) there exists a literal \( k \in B \) where \( k \neq \ell \) such that \( \delta(k) = b \), i.e., \( k \in J \), hence \( k \) precedes \( \ell \) on the resulting trail. By the definition of the blocks on the trail, \( \ell \) is not a decision literal. Since the decisions on \( J \) are unaffected, as argued above, Invariant InvDecs is met.

**Invariant InvImplIrred:** We need to show that \( P \wedge \text{decs}_{\leq n}(J_{\ell}) \models (J_{\ell})_{\leq n} \) for all \( n \). First notice that the decision levels of the literals in \( J \) do not change by applying rule Back1. Only the decision level of the variable of \( \ell \) is decremented from \( b + 1 \) to \( b \). It also stops being a decision. Since \( \delta(J_{\ell}) = b \), we can assume \( n \leq b \). Observe that \( P \wedge \text{decs}_{\leq n}(J_{\ell}) \equiv P \wedge \text{decs}_{\leq n}(J) \), since \( \ell \) is not a decision in \( J_{\ell} \) and \( I_{\leq b} = J \) and thus \( I_{\leq n} = J_{\leq n} \) by definition. Now the induction hypothesis is applied and we get \( P \wedge \text{decs}_{\leq n}(J_{\ell}) \models I_{\leq n} \).
Again using $I_{\leq n} = J_{\leq n}$ this almost closes the proof except that we are left to prove $P \land \text{decs}_{\leq b}(J\ell) \models \ell$ as $\ell$ has decision level $b$ in $J\ell$ after applying the rule and thus $\ell$ disappears in the proof obligation for $n < b$. To see this notice that $P \land \neg B \models I_{\leq b+1}$ using again the induction hypothesis for $n = b + 1$, and recalling that relevant decisions are prioritized, i.e., $I_{\leq b+1}$ contains only relevant decisions, and $\neg B = \text{decs}(\pi(I^*, X)) = \text{decs}_{\leq b+1}(I)$. This gives $P \land \neg \text{decs}_{\leq b}(J) \land \neg \ell \models I_{\leq b+1}$ and thus $P \land \neg \text{decs}_{\leq b}(J) \land \neg I_{\leq b+1} \models \ell$ by conditional contraposition. Therefore, Invariant InvImplIrred holds.

**Invariant InvDSOP:** We assume that $M$ is a DSOP and need to show that $M \lor \ell$ is also a DSOP. Due to the use of the dual blocking clause encoding, Proposition 1 holds, and invariant InvDSOP is met after executing Back1.

**Back0**

**Invariant InvDualPN:** We have $\exists S \{ P(X,Y,S) \} \equiv \neg \exists T \{ N(X,Y,T) \}$, and we need to show that $\exists S \{ (P \land D)(X,Y,S) \} \equiv \neg \exists T \{ N(X,Y,T) \}$. By the premise, $P \models D$, hence $P \land D \equiv P$. Now $\exists S \{ (P \land D)(X,Y,S) \} \equiv \exists S \{ P(X,Y,S) \} \equiv \neg \exists T \{ N(X,Y,T) \}$, and Invariant InvDualPN holds.

**Invariant InvDecs:** We have $J \subseteq I$, hence the decisions on $J$ remain unaltered. Now we show that $\ell$ is not a decision literal. As in the proof for rule Unit, it is sufficient to consider the case where $j > 0$. There exists a clause $D$ where $P \models D$ such that $\delta(D) > 0$ and a literal $\ell \in D$ for which $|\ell|_K = 0$ and $\neg \ell \in K$, hence $\ell$ is unassigned during backtracking. Furthermore, there exists a literal $k \in D$ where $k \neq \ell$ and such that $\delta(k) = j$ which precedes $\ell$ on the trail $J\ell$. Therefore, following the argument in rule Unit, the literal $\ell$ is not a decision literal. Since the decisions remain unchanged, Invariant InvDecs holds after applying rule Back0.

**Invariant InvImplIrred:** Let $n$ be arbitrary but fixed. Before executing Back0, we have $P \land \text{decs}_{\leq n}(I) \models I_{\leq n}$. We need to show that $P \land \text{decs}_{\leq n}(J\ell) \models (J\ell)_{\leq n}$. Now, $I = JK$ and $J < I$, i.e., $P \land \text{decs}_{\leq n}(J) \models J_{\leq n}$. From $j = \delta(D \setminus \{\ell\}) = \delta(J)$ we get $D|J = \{\ell\}$. On the one hand, $P \land \text{decs}_{\leq n}(J) \models (D \setminus \{\ell\})$, and on the other hand $P \land \text{decs}_{\leq n}(J) \models D$. Therefore, by modus ponens, $P \land \text{decs}_{\leq n}(J) \models \ell$. Since $\ell$ is not a decision literal, as shown above, $P \land \text{decs}_{\leq n}(J) \equiv P \land \text{decs}_{\leq n}(J\ell)$ and $P \land \text{decs}_{\leq n}(J\ell) \models J\ell$, and Invariant InvImplIrred holds after applying rule Back0.

**Invariant InvDSOP:** The DSOP $M$ remains unaltered, and InvDSOP still holds after executing rule Back0.

**DecX**

**Invariant InvDualPN:** Both $P$ and $N$ remain unaltered, hence Invariant InvDualPN still holds after executing rule DecX.

**Invariant InvDecs:** The literal $\ell$ is a decision literal by definition. It is assigned decision level $d = \delta(I) + 1$. Since $\ell \in \text{decs}(I\ell)$, we have $\delta(\text{decs}(I\ell)) = \{1, \ldots, d\}$, and Invariant InvDecs holds after applying rule DecX.
Invariant InvImplIrred: Let $n$ be arbitrary but fixed. Since $\ell$ is a decision literal, we have $P \land \text{decs}_{\leq n}(I) \equiv P \land \text{decs}_{\leq n}(I) \land \ell \models I_{\leq n} \land \ell \equiv (I_{\leq n})_{\leq n}$. Hence, Invariant InvImplIrred holds after applying rule DecX.

Invariant InvDSOP: The DSOP $M$ remains unaltered by rule DecX, hence invariant InvDSOP still holds after its application.

DecYS.

The proofs of Invariants InvDualPN, InvDecs, InvDSOP, and InvDSOP are identical to the ones for rule DecX.

\[ \square \]

8.3.2 Progress

Our method can not get caught in an endless loop, as shown next.

Proposition 3. EnumerateIrredundant always makes progress, i.e., in every non-terminal state a rule is applicable.

Proof. The proof is executed by induction over the number of rule applications. We show that in any non-terminal state $(P, N, M, I, \delta)$ a rule is applicable.

Assume all variables are assigned and no conflict has occurred. If no relevant decision is left on the trail $I$, rule End1 can be applied. Otherwise, we execute an incremental SAT call SAT$(N, \pi(I, X \cup Y))$. Since all input variables are assigned, we obtain a conflict by propagating internal variables only. Conflict analysis gives us the subsequence $I^*$ of $\pi(I, X \cup Y)$ consisting of the literals involved in the conflict, which is a model of $F$. Since we are interested in the models of $F$ projected onto $X$, we choose $B = \neg \text{decs}(\pi(I^*, X))$. Now, $\delta(B) = b + 1$, and due to Invariant InvDecs $B$ contains exactly one decision literal $\ell$ such that $\delta(\ell) = b + 1$ and therefore $\delta(B \setminus \{\ell\}) = b$. We choose $J$ and $K$ such that $I = JK$ and $b = \delta(J)$ and in particular $\ell_{|K} = 0$. After backtracking to decision level $b$, we have $I_{\leq b} = J$ where $B|_J = \{\ell\}$. All preconditions of rule Back1 are met.

If instead a conflict has occurred, there exists a clause $C \in P$ such that $C|_I = 0$. If $\delta(C) = 0$, rule End0 is applicable. Otherwise, by Invariant InvImplIrred we have $P \land \text{decs}_{\leq \delta(I)}(I) \equiv P \land \text{decs}_{\leq \delta(I)}(I) \land I_{\leq \delta(I)} \models I_{\leq \delta(I)}$. Since $I(P) \equiv 0$, also $P \land \text{decs}_{\leq \delta(I)}(I) \land I_{\leq \delta(I)} \equiv P \land \text{decs}_{\leq \delta(I)}(I) \equiv 0$. If we choose $D = \neg \text{decs}(I)$ we obtain $P \land \neg D \land I_{\leq \delta(I)} \equiv 0$, thus $P \models D$. Clause $D$ contains only decision literals and $\delta(D) = \delta(I)$. From Invariant InvDecs we know that $D$ contains exactly one decision literal for each decision level in $\{1, \ldots, \delta(I)\}$. We choose $\ell \in D$ such that $\delta(\ell) = \delta(I)$. Then the asserting level is given by $j = \delta(D \setminus \{\ell\})$. Without loss of generalization we assume the trail to be of the form $I = JK$ where $\delta(J) = j$. After backtracking to decision level $j$, the trail is equal to $J$. Since $D|_J = \{\ell\}$, all conditions of rule Back0 hold.
If $P_! \not\in \{0, 1\}$, there are unassigned variables in $X \cup Y \cup S$. If there exists a clause $C \in P$ where $C_! = \{\ell\}$, the preconditions of rule Unit are met. If instead $\text{units}(F_!_!) = \emptyset$, there exists a literal $\ell$ with $V(\ell) \in X \cup Y \cup S$ and $\delta(\ell) = \infty$. If not all relevant variables are assigned, the preconditions of rule DecX are satisfied. Otherwise, rule DecYS is applicable.

All possible cases are covered by this argument. Hence, in every non-terminal state a rule is applicable, i.e., EnumerateIrredundant always makes progress.

8.3.3 Termination

Proposition 4. EnumerateIrredundant terminates.

Proof. In our proof we follow the argument by Nieuwenhuis et al. [58] and Marić and Janičić [59], or more precisely the one by Blanchette et al. [60].

We need to show that from the initial state $(P, N, 0, \varepsilon, \delta_0)$ a final state $M$ is reached in a finite number of steps, i.e., no infinite sequence of rule applications is generated. Otherwise stated, we need to prove that the relation $\sim_{\text{EnumIrred}}$ is well-founded. To this end, we define a well-founded relation $\succ_{\text{EnumIrred}}$ such that any transition $s \sim_{\text{EnumIrred}} s'$ from a state $s$ to a state $s'$ implies $s \succ_{\text{EnumIrred}} s'$.

In accordance with Blanchette et al. [60] but adopting the notation introduced by Fleury [61], we map states to lists. Using the abstract representation of the assignment trail $I$ by Nieuwenhuis et al. [58], we write

$$I = I_0 \ell_1 I_1 \ell_2 I_2 \ldots \ell_m I_m \quad \text{where} \quad \{\ell_1, \ldots, \ell_m\} = \text{decs}(I). \quad (13)$$

The state $(P, N, M, I, \delta)$ is then mapped to

$$[0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0, 2, \ldots, 2]$$

where $V = X \cup Y \cup S$. In this representation, the order of the literals on $I$ is reflected. Propagated literals are denoted by 0, decisions are denoted by 1. Unassigned literals are represented by 2 and are moved to the end. The final state $M$ is represented by $\varepsilon$. The state containing the trail $I$ in Equation 13 is mapped to the list in Equation 14. The first $|I_0|$ entries represent the literals propagated at decision level zero, the 1 at position $|I_0| + 1$ represents the decision literal $\ell_1$, and so on for all decision levels on $I$. The last $|V| - |I|$ entries denote the unassigned variables. Notice that we are not interested in the variable assignment itself, but in its structure, i.e., the number of propagated literals per decision level and the number of unassigned variables. Furthermore, the states are encoded into lists of the same length. This representation induces a lexicographic order $\succ_{\text{lex}}$ on the states. We therefore define $\succ_{\text{EnumIrred}}$ as the restriction of $\succ_{\text{lex}}$ to $\{(v_1, \ldots, v_{|V|}) \mid v_i \in \{0, 1, 2\} \text{ for } 1 \leq i \leq |V|\}$. Accordingly, we have that $s \succ_{\text{EnumIrred}} s'$, if $s \succ_{\text{lex}} s'$. 

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The initial state is depicted above the horizontal rule, the resulting state below. The two end rules lead to the minimal element $\varepsilon$. Rule Unit replaces an unassigned literal (denoted by 2) by a propagated one (denoted by 0) and leaves the rest unchanged. Rules Back1 and Back0 replace a decision literal (denoted by 1) by a propagated one. Finally, the two decision rules replace an unassigned literal by a decision. Clearly, w.r.t. the lexicographic order, the states decrease by a rule application.

In Figure 6, the state transitions for the rules are visualized. In this representation, the unspecified elements occurring prior to the first digit are not altered by the application of the rule. We show that $s \succ_{\text{EnumIrred}} s'$ for each rule.

End1. The state $s'$ is mapped to $\varepsilon$, which is the minimal element with respect to $\succ_{\text{lex}}$, hence $s \succ_{\text{EnumIrred}} s'$ trivially holds. The representation of the state may contain both 0’s and 1’s but no 2’s, since our algorithm detects only total models.\textsuperscript{12} Recall that the associated trail must not contain any relevant decision, which is not reflected in the structure of the trail.

End0. The state $s'$ is mapped to $\varepsilon$, which is the minimal element with respect to $\succ_{\text{lex}}$, hence $s \succ_{\text{EnumIrred}} s'$ trivially holds. The representation of the state may contain both 0’s and 2’s but no 1’s, since any decision need be flipped.

Unit. An unassigned variable is propagated. Its representation changes from 2 to 0, and all elements preceding it remain unaffected. Due to $2 \succ_{\text{lex}} 0$, we also have that $s \succ_{\text{EnumIrred}} s'$.

Back1 / Back0. A decision literal, e.g., $\neg \ell$ is flipped and propagated at a lower decision level, let’s say $d$. The decision level $d$ is extended by $\ell$, which is represented by 0, and replaces the decision literal at decision level $d+1$. All other literals at decision level $d+1$ and higher are unassigned and thus

\textsuperscript{12}This restriction may be weakened in favor of finding partial models. In this section, we refer to the rules introduced in Subsection 8.1 and discuss a generalization of our algorithm enabling the detection of partial models further down.
represented by 2. Therefore, \( s \succeq_{\text{EnumIrred}} s' \). Notice that, although different preconditions of the rules \texttt{Back}1 and \texttt{Back}0 apply and the two rules differ, the structure of their states is the same.

\texttt{DecX} / \texttt{DecYS}. An unassigned variable is decided, i.e., in the representation, the first occurrence of 2 is replaced by 1. The other elements remain unaltered, hence \( s \succeq_{\text{EnumIrred}} s' \). As for the backtracking rules, whether a relevant or irrelevant or internal variable is decided, is irrelevant and not reflected in the mapping of the state, as for rules \texttt{Back}1 and \texttt{Back}0.

We have shown that after any rule application the resulting state is smaller than the preceding one with respect to the lexicographic order on which \( \succeq_{\text{EnumIrred}} \) is based. This argument shows that \( \succeq_{\text{EnumIrred}} \) is well-founded, and that therefore \texttt{EnumerateIrredundant} terminates.

8.3.4 Equivalence

The final state is given by a DSOP \( M \) such that \( M \equiv \pi(F, X) \). The proof is split into several steps. We start by proving that, given a total model \( I \) of \( P \), its subsequence \( I^* \) returned by \texttt{SAT} (line 20 of \texttt{EnumerateIrredundant} in Figure 1) is a (partial) model of \( \pi(P, X) \) and that any total model of \( P \) found during execution either was already found or is found for the first time. Then we show that all models of \( P \) are found and that each model is found exactly once, before concluding by proving that \( M \equiv \pi(F, X) \).

**Proposition 5.** Let \( I \) be a total model of \( P \) and \( I^* = \text{SAT}(N, \pi(I, X \cup Y)) \). Then \( I^* \) is a model of \( \pi(P, X) \).

**Proof.** All variables in \( X \cup Y \cup S \) are assigned, and \( P(X, Y, S) \) and \( N(X, Y, T) \) are a dual representation of \( F(X, Y) \). Invariant \texttt{InvDualPN} holds. In particular it holds for the values of the variables in \( X \cup Y \cup S \) set to their values in \( I \), i.e., we have that \( \exists S [P(X, Y, S)|_I] \equiv \neg \exists T [N(X, Y, T)|_I] \), where only the unassigned variables in \( (X \cup Y) - I \) are universally quantified.

Since \( I \) is a total model of \( P \), Invariant \texttt{InvDualPN} can be rewritten as \( P(X, Y, S)|_I \equiv \neg \exists T [N(X, Y, T)|_I] \), and \( \pi(I, X \cup Y) \) can not be extended to a model of \( N \). Since the variables in \( T \) are defined in terms of variables in \( X \cup Y \), an incremental SAT call on \( N \land I \) yields a conflict in \( N \) exclusively by propagating variables in \( T \).

Exhaustive conflict analysis yields a clause \( D \) consisting of the negations of the (assumed) literals in \( I \) involved in the conflict. Its negation is a countermodel of \( \pi(N, X \cup Y) \), which due to Equation 5 is a model of \( \pi(P, X \cup Y) \). Obviously, the same holds for the projection onto \( X \), and \( \pi(\neg D, X) \models \pi(P, X) \). Since \( I^* = \pi(\neg D, X) \), we have \( I^* \models \pi(P, X) \), and the claim holds.

**Note:** Proposition 5 corresponds to soundness according to Definition 4.
Proposition 6. A total model $I$ of $P$ is either

(i) contained in $M$

(ii) subsumed by a model in $M$

(iii) a model of $P_0 \land \bigwedge_i B_i$ where $B_i$ are the blocking clauses added to $P_0$

Proof. All variables in $X \cup Y \cup S$ are assigned, and $I \models P$. If $I$ was already found earlier, it was shrunken and the resulting model projected onto $X$ to obtain $I^*$, which was then added to $M$ (rule Back1 and line 24 in EnumerateIrredundant in Figure 1). If all assumed variables participated in the conflict and furthermore $Y = S = \emptyset$, $\pi(I^*, X \cup Y) = \pi(I^*, X) = I$, and Item (i) holds. Otherwise, $I^* < I$, and $I^*$ subsumes $I$. Since $I^* \in M$, in this case Item (ii) holds.

Suppose the model $I$ is found for the first time. Since $I \models P$, also $I \models C$ for all clauses $C \in P$. This in particular holds for all blocking clauses which were added to the original formula $P = P_0$ (rule Back1 and line 22 in EnumerateIrredundant), and Item (iii) holds.

Proposition 7. Every model is found.

Proof. According to Proposition 4, EnumerateIrredundant terminates. The final state $M$ is reached if the search space has been processed exhaustively. Hence, all models have been found.

Note: Proposition 7 says that a sequence of all models of the formula is found. It therefore corresponds to completeness according to Definition 5 and to strong completeness according to Definition 6, if restarts are applied.

Proposition 8. Every model is found exactly once.

Proof. We recall Proposition 1 stating that only pairwise contradicting models are detected. In essence, this says that every model is found exactly once.

Theorem 1 (Correctness). If $(P, N, 0, \varepsilon, \delta_0) \leadsto^*_{\text{EnumIrred}} M$, then

(i) $M \equiv \pi(F, X)$

(ii) $C_i \land C_j \equiv 0$ for $C_i, C_j \in M$ and $C_i \neq C_j$

Proof. The cubes in $M$ are exactly the $I^*$ computed from the total models of $P$. These are models of $\pi(P, X)$ (Proposition 5). Since by Proposition 7 all models are found, $M \equiv \pi(P, X)$. By Equation 2 $\text{models}(\exists Y, S . P(X, Y, S)) = \text{models}(\exists Y . F(X, Y))$ holds, i.e. $\text{models}(\pi(P, X)) = \text{models}(\pi(F, X))$. Therefore, $M \equiv \pi(F, X)$, and Item (i) holds.

Due to Proposition 1, the found models are pairwise contradicting, and Item (ii) holds as well. Notice that one could also use Proposition 8, since as its consequence, only pairwise contradicting models are found.
8.4 Generalization to Partial Model Detection

EnumerateIrredundant only finds total models of $P$. In SAT solving, this makes sense from a computational point of view, because checking whether a partial assignment satisfies a formula is more expensive than extending it to a total one. However, model enumeration is computationally more expensive than SAT solving, hence satisfiability checks, e.g., in the form of entailment checks [46], might pay off. Notice that it still might make sense to shrink the models found. In this section, we discuss the changes to be made to our presentation in order to support the detection of partial models.

First, the satisfiability condition need be replaced. This affects “all variables in $X \cup Y \cup S$ are assigned” on line 15 of EnumerateIrredundant (Figure 1) and “$(X \cup Y \cup S) - I = \emptyset$” in rules End1 and Back1 in our calculus (Figure 3). These conditions are replaced by “$I$ is a model of $P$” and “$I(P) = 1$”, respectively.

Second, some proofs need slightly be adapted. In the proof of Proposition 3 we use the assumption that $I$ is total to justify that a conflict in $N|I$ is obtained by propagating only variables in $T$. Now Invariant InvDualPN ensures that a conflict in $N|I$ is obtained also if $I$ is a partial assignment. However, it might be the case that input variables might need be propagated or decided, and we might obtain some $I'$ containing literals which do not occur on $I$. Defining $I^* = \pi(I', V(I))$ solves the issue, and the rest of the proof remains unchanged. Notice that this change is not reflected neither in EnumerateIrredundant nor in the calculus, since the computation of $I^*$ is not specified. Adequate changes need be done in the wording and in the proof of Proposition 5. The rest of the proof remains unaffected.

9 Conflict-Driven Clause Learning for Redundant All-SAT

Let $F$ be a formula in CNF over a set of variables $V$, and suppose the current trail $I$ satisfies $F$. Let the last decision literal on $I$ be $\neg \ell$. It is flipped and the search continues. This corresponds to learning the negation of $I$ and backtracking chronologically, i.e., to the previous decision level, after which $\neg I$ becomes unit and its unit literal $\ell$ is propagated. The clause $\neg I$ acts as a reason for $\ell$. If at a later stage of the search, a conflict occurs, it is analyzed, and a clause blocking this falsifying assignment is learned.

Conflict-driven clause learning (CDCL) is based on the assumption that the reason of every literal which is not a decision literal is contained in the formula. If blocking clauses are added to the input formula, this is indeed the case, and CDCL for model enumeration does not differ from its SAT counterpart. However, this is not the case anymore if no blocking clauses are used. The question now is how $\ell$ should be treated if its reason $\neg I$ was not
added to the formula.

We propose to consider $\ell$ as a propagating literal annotating it on the trail with its reason $\neg I$ but without adding $\neg I$ to $F$. This ensures the working of conflict analysis. Notice that $\neg I$ can not be used for unit propagation. Furthermore, any clause learned in conflict analysis involving $\neg I$ is logically entailed by $F \land \neg I$ but not necessarily by $F$, to which it is added. This is in contrast to CDCL for SAT, where the clauses learned after a conflict are entailed by $F$. The idea is best conveyed by a small example.

Example 8 (Conflict analysis for model enumeration). Consider the formula $F$ over the set of variables $V = \{a, b, c, d, e\}$:

$$F = (a \lor b \lor \neg c) \land (\neg b \lor c) \land (d \lor \neg c \lor e) \land (d \lor \neg c \lor \neg e)$$

Assume $X = \{a, b, c, d\}$ and $Y = \{e\}$. Suppose we decide $a$ and $b$, propagate $c$ with reason $C_2$ and decide $d$ followed by deciding $e$. The resulting trail $I_1 = a^d b^d c^c d^d e^e$ is a model of $F$. This model is blocked by $B_1 = (\neg a \lor \neg b \lor \neg d)$ consisting of the negated relevant decision literals on $I_1$. Considering only the decisions ensures that $B_1$ contains exactly one literal per decision level and that after backtracking chronologically $B_1$ becomes unit. Recall that $B_1$ is not added to $F$. The last decision literal is flipped with reason $B_1$ and $e$ is propagated with reason $C_3$. But now $C_4$ is falsified. The current trail is $I_2 = a^d b^d c^c \neg d^e$, which can also be visualized by the following implication graph:

The nodes on the left hand side having no incoming edge represent decisions and are annotated by their decision level. The other nodes denote propagated literals whose reasons label their incoming edges. The node $\kappa$ represents a conflict, and the conflicting clause is the one labeling its incoming edges.

The resolution steps for determining the clause representing the reason for the conflict can either be read off the trail $I_2$ in reverse assignment order or determined from the incident graph by following the arrows in reverse direction starting with $\kappa$. We first resolve the conflicting clause $C_4$ with the reason of $e$, $C_3$, obtaining the resolvent $(d \lor \neg c)$. Both $d$ and $\neg c$ have the highest decision level 2, and we continue by resolving the resolvent with $B_1$ obtaining $(\neg c \lor \neg a \lor b)$, followed by resolution with $C_2$ resulting in $C_5 = (\neg a \lor \neg b)$, which has only one literal at decision level 2. The resolution process
Back1red: $(P, N, M, I, \delta) \sim_{\text{Back1red}} (P, N, M \lor m, J\ell B, \delta[K \mapsto \infty][\ell \mapsto b])$

if $(X \cup Y \cup S) - I = \emptyset$ and exists $I^* \leq \pi(I, X \cup Y)$ with $JK = I$ such that $N \land I^* \vdash 0$ and $m \overset{\text{def}}{=} \pi(I^*, X)$ and $B \overset{\text{def}}{=} \neg \text{decs}(m)$ and $b + 1 \overset{\text{def}}{=} \delta(B) = \delta(m)$ and $\ell \in B$ and $\ell[K] = 0$ and $b = \delta(B \setminus \{\ell\}) = \delta(J)$

Figure 7: Rule for backtracking after detection of a model in redundant model enumeration. The calculus for redundant projected model enumeration differs from its irredundant counterpart only in the fact that no blocking clauses are used. Hence, all rules in Figure 3 are maintained except for rule Back1, which is replaced by rule Back1red.

stops, and $C_5$ is added to $F$. Notice that learning a clause containing one single literal at conflict level 2 requires resolution with $B_1$.

To lay the focus on the main topic, namely conflict analysis in the absence of blocking clauses, in this section we consider neither model shrinking nor projection. However, the extension of the clause learning algorithm to support the two, is straightforward.

10 Projected Redundant Model Enumeration

Now we turn our attention to the case where enumerating models multiple times is permitted. This allows for refraining from adding blocking clauses to the formula under consideration, since they might significantly slow down the enumerator. This affects both our algorithm and our calculus for irredundant projected model enumeration. Omitting the use of blocking clauses has a minor impact on our algorithm and its formalization. For this reason, in this section we point out the differences between the two methods.

10.1 Algorithm and Calculus

The only difference compared to EnumerateIrredundant consists in the fact that no blocking clauses are added to $P$. However, they are remembered as annotations on the trail in order to enable conflict analysis after finding a model. Our algorithm EnumerateRedundant therefore is exactly the same as EnumerateIrredundant listed in Figure 1 without lines 22–23. The annotation of flipped literals happens in function Backtrack() in line 26.

Accordingly, our formalization consists of all rules of the calculus in Figure 3 but replacing rule Back1 by rule Back1red shown in Figure 7. Rule Back1red differs from rule Back1 only in the fact that both $P$ and $N$ remain unaltered.
10.2 Example

Example 9 (Projected redundant model enumeration). Consider again Example 1 elaborated in detail in Subsection 8.2 for Enumeratelredundant. We have

\[ P = (a \lor c) \land (a \lor \neg c) \land (b \lor d) \land (b \lor \neg d) \]

and

\[ N = (-t_1 \lor \neg a) \land (-t_1 \lor \neg c) \land (a \lor c \lor t_1) \land (-t_2 \lor \neg a) \land (-t_2 \lor c) \land (a \lor \neg c \lor t_2) \land (-t_3 \lor \neg b) \land (-t_3 \lor \neg d) \land (b \lor d \lor t_3) \land (-t_4 \lor \neg b) \land (-t_4 \lor d) \land (b \lor \neg d \lor t_4) \land (t_1 \lor t_2 \lor t_3 \lor t_4) \]

Suppose \( X = \{a, c\} \) and \( Y = \{b, d\} \). The execution trail is depicted in Figure 8.

Assume we decide \( a, b, c, \) and \( d \) (steps 1–4) obtaining the trail \( I_1 = a^d b^d c^d d^d \), which is a model of \( P \). Dual model shrinking occurs as in step 5 in the example elaborated in Subsection 8.2, except that the assumed literals \( b \) and \( c \) occur in a different order, and the same model \( ab \) is obtained. Notice that the clause \( B_1 = (\neg a \lor \neg b) \) is not added to \( P \).

After backtracking, \( P|I_1 = (d) \land (\neg d) \), and after propagating \( d \) (step 6), we obtain a conflict. The current trail is \( I_3 = a^d \neg b^1 d^c^3 \), and \( C_4|I_3 = \) (). Resolution of the reasons on \( I_3 \) in reverse assignment order is executed, starting with the conflicting clause \( C_4 \). We obtain \( C_4 \lor C_3 = (b) = C_5 \), which contains exactly one literal at the maximum decision level, hence no further resolution steps are required. Since \( b \) is unit, the enumerator backtracks to decision level 0 and propagates \( b \) with reason \( C_5 \) (step 7). After deciding \( a, c, \) and \( d \), we find the same model \( b^a c^d d \) as in step 4 (steps 8–10). Obviously, model shrinking provides us with the same model \( ba \), which is added to \( M \), and the last relevant decision is flipped (step 11). Now unit propagation leads to a conflict (step 12), and since there are no decisions on the trail, the procedure stops (step 13). Now the cubes in \( M \), which represent the models of \( P \), are not pairwise disjoint anymore. However, we still have \( M \equiv \pi(P, X) \equiv \pi(F, X) \).
Figure 8: Execution trace for \( F = (a \lor c) \land (a \lor \neg c) \land (b \lor d) \land (b \lor \neg d) \) defined over the set of relevant variables \( X = \{a, b\} \) and the set of irrelevant variables \( Y = \{c, d\} \) (see also Example 1).

### 10.3 Proofs

Invariants \( \text{InvDualPN} \) and \( \text{InvDecs} \) listed in Figure 5 are applicable also for redundant model enumeration, since they involve none of \( P \) and \( N \). Invariant \( \text{InvImplIrred} \) instead need be adapted since no blocking clauses are added to \( P \), and therefore it ceases to hold. Assume a model \( I \) has been found and shrunken to \( m \), and that the last relevant decision literal \( \ell \) has been flipped. Since its reason \( B = \text{decs}(\neg m) \) is not added to \( P \), from \( P \land \text{decs}(I) \) we can not infer \( I \). Recall that instead \( m \) is added to \( M \), hence \( \neg M \) contains the reasons of all decision literals flipped after having found a model. A closer look reveals that this case is analog to the one in our previous work [18]. In this work, we avoided the use of blocking clauses by means of chronological backtracking. However, the basic idea is the same, and we replace Invariant \( \text{InvImplIrred} \) by Invariant \( \text{InvImplRed} \) listed in Figure 9. This is exactly Invariant (3) in our previous work on model counting [18], hence in our proof we use a similar argument. The invariants for redundant model enumeration under projection are given in Figure 9.

#### 10.3.1 Invariants in Non-Terminal States

**Proposition 9** (Invariants in EnumerateRedundant). The Invariants \( \text{InvDualPN}, \text{InvDecs}, \) and \( \text{InvImplRed} \) hold in non-terminal states.
Figure 9: Invariants for projected model enumeration without repetition. Notice that Invariants InvDualPN and InvDecs are the same as for irredundant model enumeration, while due to the lack of blocking clauses, in invariant InvImplRed the models recorded in $M$ need be considered.

**Proof.** The proof is carried out by induction over the number of rule applications. Assuming Invariants InvDualPN, InvDecs, and InvImplRed hold in a non-terminal state $(P, N, M, I, \delta)$, we show that they are met after the transition to another non-terminal state for all rules.

Now rules End1, End0, Back0, DecX, and DecYS are the same as for EnumerateIrredundant (Figure 3). In Subsubsection 8.3.1 we already proved that after the execution of these rules Invariants InvDualPN and InvDecs still hold.

As for invariant InvImplRed, from Item (i) and Item (ii) and observing that $m \leq I$, where $I$ is a total model of $P$ and $m \in M$ its projection onto the relevant variables, we can conclude that invariant InvImplRed holds as well. To see this, remember that in invariant InvImplIrred we consider $P = P_0 \land \bigwedge_i B_i$, where the $B_i$ are the clauses added to $P_0$ blocking the models $m_i$. But $B_i \leq m_i$, hence invariant InvImplRed holds after the application of the rules Unit, Back0, DecX, and DecYS, and we are left to carry out the proof for rule Back1red.

**Back1red**

**Invariant InvDualPN:** Both $P$ and $N$ remain unaltered, therefore Invariant InvDualPN holds after the application of Back1red.

**Invariant InvDecs:** The proof is analogous to the one for rule Back1.

**Invariant InvImplRed:** We need to show $P \land \neg(M \lor m) \land \text{decs}_{\leq n}(J \ell) \models (J \ell)_{\leq n}$ for all $n$. First notice that the decision levels of all the literals in $J$ do not change while applying the rule. Only the decision level of the variable of $\ell$ is decremented from $b + 1$ to $b$. It also stops being a decision. Since $\delta(J \ell) = b$, we can assume $n \leq b$. Observe that $P \land \neg(M \lor m) \land \text{decs}_{\leq n}(J \ell) \equiv \neg m \land (P \land \neg M \land \text{decs}_{\leq n}(I))$, since $\ell$ is not a decision in $J \ell$ and $I_{\leq b} = J$ and $I_{\leq n} = J_{\leq n}$ by definition. Now the induction hypothesis is applied and we get $P \land \neg(M \lor m) \land \text{decs}_{\leq n}(J \ell) \models I_{\leq n}$. Again using $I_{\leq n} = J_{\leq n}$ this almost closes the proof except that we are left to prove $P \land \neg(M \lor m) \land \text{decs}_{\leq n}(J \ell) \models \ell$ as $\ell$ has decision level $b$ in $J \ell$ after applying the rule and thus $\ell$ disappears in the proof obligation for $n < b$. To see this notice that $P \land \neg B \models I_{\leq b+1}$ using again
the induction hypothesis for \( n = b + 1 \) and recalling that \( \neg B = \text{decs}_{\leq b+1}(I) \). This gives \( P \land \neg \text{decs}_{\leq b}(J) \land \neg \ell \models I_{\leq b+1} \) and thus \( P \land \neg \text{decs}_{\leq b}(J) \land \neg I_{\leq b+1} \models \ell \) by conditional contraposition.

\[ \square \]

### 10.3.2 Progress and Termination

The proofs that our method for redundant projected model enumeration always makes progress and eventually terminates are the same as in Subsubsection 8.3.2 and Subsubsection 8.3.3.

### 10.3.3 Equivalence

Some properties proved for the case of irredundant model enumeration cease to hold if we allow enumerating redundant models. Specifically, Proposition 5, and Proposition 7 hold, while Proposition 8 does not. Item (i) and Item (ii) of Proposition 6 hold, while Item (iii) does not. In Theorem 1, Item (i) holds but Item (ii) does not. Their proofs remain the same as for irredundant model enumeration in Subsubsection 8.3.1.

### 10.4 Generalization

The same observations apply as for irredundant model enumeration in Subsection 8.4.

### 11 Conclusion

Model enumeration and projection, with and without repetition, is a key element to several tasks. We have presented two methods for propositional model enumeration under projection. \texttt{EnumerateIrredundant} uses blocking clauses to avoid enumerating models multiple times, while \texttt{EnumerateRedundant} is exempt from blocking clauses and admits repetitions. Our CDCL-based model enumerators detect total models and uses dual reasoning to shrink them.

To ensure correctness of the shrinking mechanism, we developed a dual encoding of the blocking clauses. We provided a formalization and proof of correctness of our blocking-based model enumeration approach and discussed a generalization to the case where partial models are found. These partial models might not be minimal, hence shrinking them still might make sense. Also, there is no guarantee that the shrunken models are minimal as they depend on the order of the variable assignments.

We presented a conflict-driven clause learning mechanism for redundant model enumeration, since standard CDCL might fail in the absence of blocking clauses. Basically, those clauses are remembered on the trail without being added to the input formula. This prevents a blowup of the formula.
but also does not further make use of these potentially short clauses, which in general propagate more eagerly than long clauses.

We discussed the modifications of our blocking-based algorithm and calculus to support redundant model enumeration and provided a correctness proof. Intuitively, shorter partial models representing non-disjoint sets of total models might be found.

Our method does not guarantee that the shrunken model $I^*$ is minimal w. r. t. the decision level $b$ in line `EnumerateIrredundant`. However, finding short DSOPs is important in circuit design [16], and appropriate algorithms have been introduced by, e. g., Minato [62]. While DSOP minimization has been proven to be NP-complete [17], finding a smaller decision level $b$ would already be advantageous, since besides restricting the search space to be explored it generates shorter models. To this end, we plan to adapt our dual shrinking algorithm to exploit the Tseitin encoding as proposed by Iser et al. [63].

In the presence of multiple conflicting clauses, a related interesting question might also be which one to choose as a starting point for conflict analysis with the aim to backtrack as far as possible. This is not obvious unless all conflicts are analyzed.

Determining shortest possible models makes our approach suitable for circuit design. We are convinced that this work provides incentives not only for the hardware-near community but also for the enumeration community.

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References

[1] A. Biere, A. Cimatti, E. M. Clarke, Y. Zhu, Symbolic model checking without BDDs, in: TACAS, Vol. 1579 of Lecture Notes in Computer Science, Springer, 1999, pp. 193–207.

[2] O. Shtrichman, Tuning SAT checkers for bounded model checking, in: CAV, Vol. 1855 of LNCS, Springer, 2000, pp. 480–494.

[3] O. Shtrichman, Pruning techniques for the SAT-based bounded model checking problem, in: CHARME, Vol. 2144 of LNCS, Springer, 2001, pp. 58–70.

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[4] K. L. McMillan, Applying SAT methods in unbounded symbolic model checking, in: CAV, Vol. 2404 of LNCS, Springer, 2002, pp. 250–264.

[5] K. L. McMillan, Interpolation and SAT-based model checking, in: CAV, Vol. 2725 of Lecture Notes in Computer Science, Springer, 2003, pp. 1–13.

[6] H. Jin, H. Han, F. Somenzi, Efficient conflict analysis for finding all satisfying assignments of a Boolean circuit, in: TACAS, Vol. 3440 of Lecture Notes in Computer Science, Springer, 2005, pp. 287–300.

[7] A. Gupta, Z. Yang, P. Ashar, A. Gupta, SAT-based image computation with application in reachability analysis, in: FMCAD, Vol. 1954 of LNCS, Springer, 2000, pp. 354–371.

[8] B. Li, M. S. Hsiao, S. Sheng, A novel SAT all-solutions solver for efficient preimage computation, in: DATE, IEEE Computer Society, 2004, pp. 272–279.

[9] S. Sheng, M. S. Hsiao, Efficient preimage computation using a novel success-driven ATPG, in: DATE, IEEE Computer Society, 2003, pp. 10822–10827.

[10] O. Grumberg, A. Schuster, A. Yadgar, Memory efficient all-solutions SAT solver and its application for reachability analysis, in: FMCAD, Vol. 3312 of LNCS, Springer, 2004, pp. 275–289.

[11] A. Sullivan, D. Marinov, S. Khurshid, Solution enumeration abstraction: A modeling idiom to enhance a lightweight formal method, in: ICFEM, Vol. 11852 of Lecture Notes in Computer Science, Springer, 2019, pp. 336–352.

[12] S. K. Lahiri, R. Nieuwenhuis, A. Oliveras, SMT techniques for fast predicate abstraction, in: CAV, Vol. 4144 of LNCS, Springer, 2006, pp. 424–437.

[13] R. Sebastiani, Lazy satisfiability modulo theories, J. Satisf. Boolean Model. Comput. 3 (3-4) (2007) 141–224.

[14] P. B. Miltersen, J. Radhakrishnan, I. Wegener, On converting CNF to DNF, Theor. Comput. Sci. 347 (1-2) (2005) 325–335.

[15] I. Wegener, The complexity of Boolean functions, Wiley-Teubner, 1987.

[16] S. Minato, G. De Micheli, Finding all simple disjunctive decompositions using irredundant sum-of-products forms, in: ICCAD, ACM / IEEE Computer Society, 1998, pp. 111–117.
[17] A. Bernasconi, V. Ciriani, F. Luccio, L. Pagli, Compact DSOP and partial DSOP forms, Theory Comput. Syst. 53 (4) (2013) 583–608.

[18] S. Möhle, A. Biere, Combining conflict-driven clause learning and chronological backtracking for propositional model counting, in: GCAI, Vol. 65 of EPiC Series in Computing, EasyChair, 2019, pp. 113–126.

[19] M. Cadoli, F. M. Donini, A survey on knowledge compilation, AI Commun. 10 (3-4) (1997) 137–150.

[20] A. Darwiche, P. Marquis, A knowledge compilation map, J. Artif. Intell. Res. 17 (2002) 229–264.

[21] J. Huang, A. Darwiche, The language of search, J. Artif. Intell. Res. 29 (2007) 191–219.

[22] C. J. Muise, S. A. McIlraith, J. C. Beck, E. I. Hsu, Dsharp: Fast d-DNNF compilation with sharpSAT, in: Canadian Conference on AI, Vol. 7310 of Lecture Notes in Computer Science, Springer, 2012, pp. 356–361.

[23] J. Lagniez, P. Marquis, An improved Decision-DNNF compiler, in: IJCAI, ijcai.org, 2017, pp. 667–673.

[24] H. Palacios, B. Bonet, A. Darwiche, H. Geffner, Pruning conformant plans by counting models on compiled d-dnnf representations, in: ICAPS, AAAI, 2005, pp. 141–150.

[25] J. P. M. Silva, K. A. Sakallah, GRASP - a new search algorithm for satisfiability, in: ICCAD, IEEE Computer Society / ACM, 1996, pp. 220–227.

[26] J. P. Marques-Silva, K. A. Sakallah, GRASP: A search algorithm for propositional satisfiability, IEEE Trans. Computers 48 (5) (1999) 506–521.

[27] M. W. Moskewicz, C. F. Madigan, Y. Zhao, L. Zhang, S. Malik, Chaff: Engineering an efficient SAT solver, in: DAC, ACM, 2001, pp. 530–535.

[28] T. Sang, P. Beame, H. A. Kautz, Performing Bayesian inference by weighted model counting, in: AAAI, AAAI Press / The MIT Press, 2005, pp. 475–482.

[29] M. Chavira, A. Darwiche, On probabilistic inference by weighted model counting, Artif. Intell. 172 (6-7) (2008) 772–799.

[30] J. K. Fichte, M. Hecher, S. Woltran, M. Zisser, Weighted model counting on the GPU by exploiting small treewidth, in: ESA, Vol. 112 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018, pp. 28:1–28:16.
[31] J. M. Dudek, V. H. N. Phan, M. Y. Vardi, DPMC: weighted model counting by dynamic programming on project-join trees, in: CP, Vol. 12333 of Lecture Notes in Computer Science, Springer, 2020, pp. 211–230.

[32] F. Bacchus, S. Dalmao, T. Pitassi, DPLL with caching: A new algorithm for #SAT and Bayesian inference, Electron. Colloquium Comput. Complex. 10 (003) (2003).

[33] P. Morettin, A. Passerini, R. Sebastiani, Efficient weighted model integration via SMT-based predicate abstraction, in: IJCAI, ijcai.org, 2017, pp. 720–728.

[34] P. Morettin, A. Passerini, R. Sebastiani, Advanced SMT techniques for weighted model integration, Artif. Intell. 275 (2019) 1–27.

[35] R. Bayardo Jr., J. D. Pehoushek, Counting models using connected components, in: AAAI/IAAI, AAAI Press / The MIT Press, 2000, pp. 157–162.

[36] A. Morgado, J. P. M. Silva, Good learning and implicit model enumeration, in: ICTAI, IEEE Computer Society, 2005, pp. 131–136.

[37] S. Möhle, A. Biere, Dualizing projected model counting, in: ICTAI, IEEE, 2018, pp. 702–709.

[38] T. Toda, T. Soh, Implementing efficient all solutions SAT solvers, ACM Journal of Experimental Algorithmics 21 (1) (2016) 1.12:1–1.12:44.

[39] M. Gebser, B. Kaufmann, T. Schaub, Solution enumeration for projected Boolean search problems, in: CPAIOR, Vol. 5547 of Lecture Notes in Computer Science, Springer, 2009, pp. 71–86.

[40] M. Davis, G. Logemann, D. W. Loveland, A machine program for theorem-proving, Commun. ACM 5 (7) (1962) 394–397.

[41] E. Birnbaum, E. L. Lozinskii, The good old Davis-Putnam procedure helps counting models, J. Artif. Intell. Res. 10 (1999) 457–477.

[42] A. Nadel, V. Ryvchin, Chronological backtracking, in: SAT, Vol. 10929 of LNCS, Springer, 2018, pp. 111–121.

[43] S. Möhle, A. Biere, Backing backtracking, in: SAT, Vol. 11628 of Lecture Notes in Computer Science, Springer, 2019, pp. 250–266.

[44] A. Biere, S. Hölldobler, S. Möhle, An abstract dual propositional model counter, in: YSIP, Vol. 1837 of CEUR Workshop Proceedings, CEUR-WS.org, 2017, pp. 17–26.
[45] R. Sebastiani, Are you satisfied by this partial assignment?, https://arxiv.org/abs/2003.04225 (February 2020).

[46] S. Möhle, R. Sebastiani, A. Biere, Four flavors of entailment, in: SAT, Vol. 12178 of Lecture Notes in Computer Science, Springer, 2020, pp. 62–71.

[47] K. Ravi, F. Somenzi, Minimal assignments for bounded model checking, in: TACAS, Vol. 2988 of LNCS, Springer, 2004, pp. 31–45.

[48] R. A. Aziz, G. Chu, C. J. Muise, P. J. Stuckey, #SAT: Projected model counting, in: SAT, Vol. 9340 of Lecture Notes in Computer Science, Springer, 2015, pp. 121–137.

[49] J. Brauer, A. King, J. Kriener, Existential quantification as incremental SAT, in: CAV, Vol. 6806 of LNCS, Springer, 2011, pp. 191–207.

[50] C. Zengler, W. Küchlin, Boolean quantifier elimination for automotive configuration - A case study, in: FMICS, Vol. 8187 of Lecture Notes in Computer Science, Springer, 2013, pp. 48–62.

[51] C. Weidenbach, Automated reasoning building blocks, in: Correct System Design, Vol. 9360 of Lecture Notes in Computer Science, Springer, 2015, pp. 172–188.

[52] J. A. Robinson, A machine-oriented logic based on the resolution principle, J. ACM 12 (1) (1965) 23–41.

[53] J. N. Hooker, Solving the incremental satisfiability problem, J. Log. Program. 15 (1&2) (1993) 177–186.

[54] N. Eén, N. Sörensson, Temporal induction by incremental SAT solving, Electron. Notes Theor. Comput. Sci. 89 (4) (2003) 543–560.

[55] G. Tseitin, On the complexity of derivation in propositional calculus, Studies in Constructive Mathematics and Mathematical Logic (1968) 115–125.

[56] D. A. Plaisted, S. Greenbaum, A structure-preserving clause form translation, J. Symb. Comput. 2 (3) (1986) 293–304.

[57] A. Niemetz, M. Preiner, A. Biere, Turbo-charging lemmas on demand with don’t care reasoning, in: FMCAD, IEEE, 2014, pp. 179–186.

[58] R. Nieuwenhuis, A. Oliveras, C. Tinelli, Solving SAT and SAT modulo theories: From an abstract Davis–Putnam–Logemann–Loveland procedure to dpll(T), J. ACM 53 (6) (2006) 937–977.
[59] F. Marić, P. Janičić, Formalization of abstract state transition systems for SAT, Log. Methods Comput. Sci. 7 (3) (2011).

[60] J. C. Blanchette, M. Fleury, P. Lammich, C. Weidenbach, A verified SAT solver framework with learn, forget, restart, and incrementality, J. Autom. Reason. 61 (1-4) (2018) 333–365.

[61] M. Fleury, Formalisation of ground inference systems in a proof assistant, Master’s thesis, École normale supérieure de Rennes (2015). URL https://www mpi-inf mpg de/fileadmin/inf/rg1/ Documents/fleury_master_thesis.pdf

[62] S. Minato, Fast generation of prime-irredundant covers from binary decision diagrams, IEICE Trans. Fundamentals E76-A (6) (1993) 967–973.

[63] M. Iser, C. Sinz, M. Taghdiri, Minimizing models for tseitin-encoded SAT instances, in: SAT, Vol. 7962 of Lecture Notes in Computer Science, Springer, 2013, pp. 224–232.