DEFINABILITY IN THE REAL UNIVERSE*

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Logic has its origins in basic questions about the nature of the real world and how we describe it. This article seeks to bring out the physical and epistemological relevance of some of the more recent technical work in logic and computability theory.

"If you are receptive and humble, mathematics will lead you by the hand. Again and again, when I have been at a loss how to proceed, I have just had to wait until I have felt the mathematics lead me by the hand. It has led me along an unexpected path, a path where new vistas open up, a path leading to new territory, where one can set up a base of operations, from which one can survey the surroundings and plan future progress."

– Paul Dirac, 27 November, 1975. In Paul A. M. Dirac Papers, Florida State University Libraries, Tallahassee, Florida, USA, No. 2/29/17.

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1. Introduction

Logic has an impressive history of addressing very basic questions about the nature of the world we live in. At the same time, it has clarified concepts and informal ideas about the world, and gone on to develop sophisticated technical frameworks within which these can be discussed. Much of this work is little known or understood by non-specialists, and the significance of it largely ignored. While notions such as set, proof and consistency have become part of our culture, other very natural abstractions such as that of definability are unfamiliar and disconcerting, even to working mathe-

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maticians. The widespread interest in Gödel’s incompleteness results and their frequent application, often in questionable ways, shows both the potential for logicians to say something important about the world, while at the same time illustrating the limitations of what has been achieved so far. This article seeks to bring out the relevance of some of the more recent technical work in logic and computability theory. Basic questions addressed include: How do scientists represent and establish control over information about the universe? How does the universe itself exercise control over its own development? And more feasibly: How can we reflect that control via our scientific and mathematical representations.

Definability — what we can describe in terms of what we are given in a particular language — is a key notion. As Hans Reichenbach (Hilary Putnam is perhaps his best-known student) found in the 1920s onwards, formalising definability in the real world comes into its own when we need to clarify and better understand the content of a hard-to-grasp description of reality, such as Einstein’s theory of general relativity. Reichenbach’s seminal work on axiomatising relativity has become an ongoing project, carried forward today by Istvan Nemeti, Hajnal Andreka and their co-workers (see, for example, Andréka, Madarász, Németi and Székely). One can think of such work as paralleling the positive developments that models of computation enabled during the early days of computer science, bringing a surer grip on practical computation. But computability theory also gave an overview of what can be computed in principle, with corresponding technical developments apparently unrelated to applications. The real-world relevance of most of this theory remains conjectural.

The capture of natural notions of describability and real-world robustness via the precisely formulated ones of definability and invariance also brings a corresponding development of theory, which can be applied in different mathematical contexts. Such an application does not just bring interesting theorems, which one just adds to the existing body of theory with conjectural relevance. It fills out the explanatory framework to a point where it can be better assessed for power and validity. And it is this which is further sketched out below. The basic ingredients are the notions of definability and invariance, and a mathematical context which best describes the scientific description of familiar causal structure.

2. Computability versus Descriptions

In the modern world, scientists look for theories that enable predictions, and if possible predictions of a computational character. Everyone else lives
with less constrained descriptions of what is happening, and is likely to happen. Albert Einstein⁸ might have expressed the view in 1950 that:

When we say that we understand a group of natural phenomena, we mean that we have found a constructive theory which embraces them.

But in everyday life people commonly use informal language to describe expectations of the real world from which constructive or computational content is not even attempted. And there is a definite mismatch between the scientist’s drive to extend the reach of his or her methodology, and the widespread sense of an intrusion of algorithmic thinking into areas where it is not valid. A recent example is the controversy around Richard Dawkins’ book³² on *The God Delusion*. This dichotomy has some basis in theorems from logic (such as Gödel’s incompleteness theorems): but the basis is more one for argument and confusion than anything more consensual. Things were not always so.

If one goes back before the time of Isaac Newton, before the scientific era, informal *descriptions* of the nature of reality were the common currency of those trying to reason about the world. This might even impinge on mathematics — as when the Pythagoreans wrestled with the ontology of irrational numbers. Calculation had a quite specific and limited role in society.

### 3. Turing’s Model and Incomputability

In 1936, Turing¹⁰¹ modelled what he understood of how a then human “computer” (generally a young woman) might perform calculations — laying down rules that were very restrictive in a practical sense, but which enabled, as he plausibly argued, all that might be achieved with apparently more powerful computational actions. Just as the Turing machine’s primitive actions (observing, moving, writing) were the key to modelling complex computations, so the Turing machine itself provided a route to the modelling of complex natural processes within structures which are discretely (or at least countably) presented. In this sense, it seemed we now had a way of making concrete the Laplacian model of science which had been with us in some form or other ever since the significance of what Newton had done become clear.

But the techniques for presenting a comprehensive range of computing machines gave us the *universal* Turing machine, so detaching computations from their material embodiments: and — a more uncomfortable surprise
— by adding a quantifier to the perfectly down-to-earth description of the
universal machine we get (and Turing proved it) an uncomputable object,
the halting set of the machine. In retrospect, this becomes a vivid indication
of how natural language has both an important real-world role, and quickly
outstrips our computational reach. The need then becomes to track down
material counterpart to the simple mathematical schema which give rise to
incomputability. Success provides a link to a rich body of theory and opens
a Pandora’s box of new perceptions about the failings of science and the
nature of the real universe.

4. The Real Universe as Discipline Problem

The Laplacian model has a deeply ingrained hold on the rational mind. For
a bromeliad-like late flowering of the paradigm we tend to think of Hilbert
and his assertion of very general expectations for axiomatic mathematics.
Or of the state of physics before quantum mechanics. The problem is that
modelling the universe is definitely not an algorithmic process, and that
is why intelligent, educated people can believe very different things, even
in science. Even in mathematics. So for many, the mathematical phase-
transition from computability to incomputability, which a quantifier pro-
vides, is banned from the real world (see for example Cotogno). However
simple the mathematical route to incomputability, when looking out at the
natural world, the trick is to hold the eye-glass to an unseeing eye. The
global aspect of causality so familiar in mathematical structures is denied
a connection with reality, in any shape or form. For a whole community,
the discovery of incomputability made the real universe a real discipline
problem. When Martin Davis says:

The great success of modern computers as all-purpose algorithm-
executing engines embodying Turing’s universal computer in physical
form, makes it extremely plausible that the abstract theory
of computability gives the correct answer to the question What is
a computation?, and, by itself, makes the existence of any more
general form of computation extremely doubtful.

we have been in the habit of agreeing, in a mathematical setting. But in
the context of a general examination of hypercomputational propositions
(whatever the validity of the selected examples) it gives the definite im-
pression of a defensive response to an uncompleted paradigm change. For
convenience, we call this response — that “there is no such discipline as
hypercomputation” — “Davis’ Thesis”.

The universal Turing machine freed us from the need actually *embody* the machines needed to host different computational tasks. The importance of this for building programmable computers was immediately recognised by John von Neuman, and played a key role in the early history of the computer (see Davis\cite{Davis}). The notion of a *virtual machine* is a logical extension of this tradition, which has found widespread favour amongst computer scientists and philosophers of a functionalist turn of mind — for instance, there is the Sloman and Chrisley\cite{Sloman} proposition for releasing consciousness from the philosophical inconvenience of embodiment (see also Torrance, Clowes and Chrisley\cite{Torrance}). Such attempts to tame nature are protected by a dominant paradigm, but there is plenty of dissatisfaction with them based on respect for the complex physicality of what we see.

5. A Dissenting Voice . . .

Back in 1970, Georg Kreisel considered one of the simplest physical situations presenting mathematical predictive problems. Contained within the mathematics one detects uncompleted infinities of the kind necessary for incomputability to have any significance for the real world. In a footnote to Kreisel\cite{Kreisel} he proposed a collision problem related to the 3-body problem, which might result in “an analog computation of a non-recursive function”.

Even though Kreisel’s view was built on many hours of deep thought about extensions of the Church-Turing thesis to the material universe — much of this embodied in Odifreddi’s 20-page discussion of the Church-Turing thesis in his book\cite{Odifreddi} on Classical Recursion Theory — it is not backed up by any proof of the inadequacy of the Turing model built on a precise description of the collision problem.

This failure has become a familiar one, what has been described as a failure to find ‘natural’ examples of incomputability other than those computably equivalent to the halting problem for a universal Turing machine — with even that not considered very natural by the mainstream mathematician. One requirement of a ‘natural’ incomputable set is that it be computably enumerable, like the set of solutions of a diophantine equation, or the set of natural numbers $n$ such that there exists a block of precisely $n$ 7s in the decimal expansion of the real number $\pi$ — or like the halting set of a given Turing machine. The problem is that given a computably enumerable set of numbers, there are essentially two ways of knowing its incomputability. One way is to have designed the set oneself to have complement different to any other set on a standard list of computably enumerable sets. Without working relative to some other incomputable set,
one just gets canonical sets computably equivalent to the halting set of
the universal Turing machine. Otherwise the set one built has no known
robustness, no definable character one can recognise it by once it is built.
The other way of knowing a particular computably enumerable set to be
incomputable is to be able to compute one of the sets built via way one from
the given set. But only the canonical sets have been found so far to work in
this way. So it is known that there is a whole rich universe of computably
inequivalent computably enumerable sets — but the only individual ones
recognisably so are computably equivalent to the halting problem. Kreisel’s
failure is not so significant when one accepts that an arbitrary set picked
from nature in some way is very unlikely to be a mathematically canonical
object. It seems quite feasible that there is a mathematical theorem waiting
to be proved, explaining why there is no accessible procedure for verifying
incomputability in nature.

Since Kreisel’s example, there have been other striking instances of in-
finities in nature with the potential for hosting incomputability. In *Off to
Infinity in Finite Time* Donald Saari and Jeff Xia describe how one can
even derive singularities arising from the behaviour of 5 bodies moving
under the influence of the familiar Newtonian inverse square law.

There is a range of more complex examples which are hard to fit into
the standard Turing model, ones with more real-world relevance. There is
the persistence of problems of predictability in a number of contexts. There
is quantum uncertainty, constrained by computable probabilities, but host-
ing what looks very much like randomness; there are apparently emergent
phenomena in many environments; and chaotic causal environments giving
rise to strange attractors; and one has relativity and singularities (black
holes), whose singular aspects can host incomputability; . Specially inter-
esting is the renewed interest in analog and hybrid computing machines,
leading Jan van Leeuwen and Jiri Wiedermann to observe that “dots the
classical Turing paradigm may no longer be fully appropriate to capture all
features of present-day computing.” And — see later — there is mentality,
consciousness, and the observed shortcomings of the mathematical models
of these.

The disinterested observer of Martin Davis’ efforts to keep nature con-
tained within the Turing/Laplacian model might keep in mind the well-
known comment of Arthur C. Clarke (Clarke’s First Law) that:

> When a distinguished but elderly scientist states that something is
> possible, he is almost certainly right. When he states that some-
> thing is impossible, he is very probably wrong.
In what follows we look in more detail at three key challenges to the attachment of Davis, and of a whole community, to the Turing model in the form of Davis’ thesis.

There is a reason for this. At first sight, it may seem unimportant to know whether we have computational or predictive difficulties due to mere complexity of a real-world computational task, or because of its actual incomputability. And if there is no distinguishable difference between the two possibilities, surely it cannot matter which pertains. Well, no. Attached to two different mathematical characterisations one would expect different mathematical theories. And there is a rich and well-developed theory of incomputability. This mathematics may well constrain and give global form to the real-world which it underlies. And these constraints and structurings may be very significant for our experience and understanding of the universe and our place in it.

6. The Quantum Challenge

In the early days of quantum computing, there was some good news for Davis’ thesis from one of its most prominent supporters. David Deutsch, was one of the originators of the standard model of quantum computation. In his seminal 1985 article “Quantum Theory, the Church-Turing Principle and the Universal Quantum Computer” in the Proceedings of the Royal Society of London, he introduced the notion of a ‘universal quantum computer’, and described how it might exploit quantum parallelism to compute more efficiently than a classical Turing machine. But Deutsch is quite clear that real computers based on this model would not compute anything not computable classically by a Turing machine. And, of course, there are many other instances of successful reductions of “natural examples” of nature-based computational procedures to the Turing model.

But like Martin Davis, Deutsch is keen to take things further — a lot further, attempting a reduction of human mentality to the Turing model in a way even Turing in his most constructive frame of mind might have had misgivings about:

I am sure we will have [conscious computers], I expect they will be purely classical, and I expect that it will be a long time in the future. Significant advances in our philosophical understanding of what consciousness is, will be needed.

Be this as it may, there are aspects of the underlying physics which are not fully used in setting up the standard model for quantum computing. It
is true that measurements do play a role in a quantum computation, but in a tamed guise. This is how Andrew Hodges explains it, in his article What would Alan Turing have done after 1954? in the Teuscher volume.99

Von Neumanns axioms distinguished the U (unitary evolution) and R (reduction) rules of quantum mechanics. Now, quantum computing so far (in the work of Feynman, Deutsch, Shor, etc) is based on the U process and so computable. It has not made serious use of the R process: the unpredictable element that comes in with reduction, measurement, or collapse of the wave function.

The point being that measurements in the quantum context are intrusive, with outcomes governed by computable probabilities, but with the mapping out of what goes on within those probabilities giving the appearance of randomness. There are well-established formalisations of the intuitive notion of randomness, largely coincident, and a large body of mathematical theory built on these (see, for example, Chaitin,15 Downey and Hirschfeldt,36 Nies67). A basic feature of the theory is the fact that randomness implies incomputability (but not the converse). Calude and Svozil14 have extracted a suitable mathematical model of quantum randomness, built upon assumptions generally acceptable to the physicists. Analysing the computability-theoretic properties of the model, they are able to show that quantum randomness does exhibit incomputability. But, interestingly, they are unable as yet to confirm that quantum randomness is mathematically random.

But quantum mechanics does not just present one of the toughest challenges to Davis’ thesis. It also presents the observer with a long-standing challenge to its own realism. Interpretations of the theory generally fail to satisfy everyone, and the currently most widely accepted interpretations contain what must be considered metaphysical assumptions. When we have assembled the key ingredients, we will be in a position to argue that the sort of fundamental thinking needed to rescue the theory from such assumptions is based on some very basic mathematics.

7. Schrödinger’s Lost States, and the Many-Worlds Interpretation

One way of describing the quantum world is via the Schrödinger wave equation. What Hodges refers to above are the processes for change of the wave equation describing the quantum state of a physical system. On the one hand, one has deterministic continuous evolution via Schrödinger’s equation, involving superpositions of basis states. On the other, one has proba-
bistatic non-local discontinuous change due to measurement. With this, one observes a jump to a single basis state. The interpretive question then is:

*Where do the other states go?*

Writing with hindsight: If the physicists knew enough logic, they would have been able to make a good guess. And if the logicians had been focused enough on the foundations of quantum mechanics they might have been able to tell them.

As it is, physics became a little weirder around 1956. The backdrop to this is the sad and strange life-story of Hugh Everett III and his family, through which strode the formidable John Wheeler, Everett’s final thesis advisor, and Bryce DeWitt, who in 1970 coined the term ‘Many-Worlds’ for Everett’s neglected and belittled idea: an idea whose day came too late to help the Everett family, now only survived by the son Mark who relives parts of the tragic story via an autobiography\(^{42}\) and appropriately left field confessional creations as leader of the Eels rock band.

Many-Worlds, with a little reworking, did away with the need to explain the transition from many superposed quantum states to the ‘quasi-classical’ uniqueness we see around us. The multiplicity survives and permeates micro- to macro-reality, via a decohering bushy branching of alternative histories, with us relegated to to our own self-contained branch. Max Tegmark has organised the multiplying variations on the Many-Worlds theme into hierarchical levels of ‘multiverses’, from modest to more radical proposals, with even the underlying mathematics and the consequent laws of physics individuating at Level IV. Of course, if one does not bother anymore to explain why our universe works so interestingly, one needs the ‘anthropic principle’ on which to base our experience of the world — “We’re here because we’re here because we’re here because we’re here . . .”, as they sang during the Great War, marching towards the trenches. The attraction of this picture derives from the drive for a coherent overview, and the lack of a better one. As David Deutsch put it in *The Fabric of Reality* [34, p.48]:

> . . . understanding the multiverse is a precondition for understanding reality as best we can. Nor is this said in a spirit of grim determination to seek the truth no matter how unpalatable it may be . . . It is, on the contrary, because the resulting world-view is so much more integrated, and makes more sense in so many ways, than any previous world-view, and certainly more than the cynical pragmatism which too often nowadays serves as surrogate for a world-view amongst scientists.
Here is a very different view of the multiverse from the distinguished South African mathematician George Ellis [40, p.198], one-time collaborator of Stephen Hawking:

The issue of what is to be regarded as an ensemble of ‘all possible’ universes is unclear, it can be manipulated to produce any result you want . . . The argument that this infinite ensemble actually exists can be claimed to have a certain explanatory economy (Tegmark 1993), although others would claim that Occam’s razor has been completely abandoned in favour of a profligate excess of existential multiplicity, extravagantly hypothesized in order to explain the one universe that we do know exists.

The way out of this foundational crisis, as with previous ones in mathematics and science, is to adopt a more constructive approach. In this way, one can combine the attractions of Tegmark’s Mathematical Universe Hypothesis (MUH) with the discipline one gets from the mathematics of what can be built from very small beginnings.

8. Back in the One World . . .

A constructive approach is not only a key to clarifying the interpretive problem. Eliminating the redundancy of parallel universes, and the reliance on the anthropic principle, also entails the tackling of the unsatisfactory arbitrariness of various aspects of the standard model. The exact values of the constants of nature, subatomic structure, the geometry of space — all confront the standard model of particle physics with a foundational problem. Alan Guth, inventor of the ‘cosmic inflation’ needed to make sense of our picture of the early universe, asks:48

If the creation of the universe can be described as a quantum process, we would be left with one deep mystery of existence: What is it that determined the laws of physics?

And Peter Woit, in his recent book Not Even Wrong — The Failure of String Theory and the Continuing Challenge to Unify the Laws of Physics, comments on the arbitrary constants one needs to give the right values to get the standard model to behave properly:

One way of thinking about what is unsatisfactory about the standard model is that it leaves seventeen non-trivial numbers still to be explained, . . .
Even though the exact number of constants undetermined by theory, but needing special fine-tuning to make the standard model fit with observation, does vary, even one is too many. This dissatisfaction with aspects of the standard model goes back to Einstein. Quoting from Einstein’s *Autobiographical Notes* [39, p.63]:

\[\ldots\] I would like to state a theorem which at present can not be based upon anything more than upon a faith in the simplicity, i.e. intelligibility, of nature \ldots\] nature is so constituted that it is possible logically to lay down such strongly determined laws that within these laws only rationally completely determined constants occur (not constants, therefore, whose numerical value could be changed without destroying the theory) \ldots

What is needed is mathematics which does more than express mechanistic relationships between basic entities. One needs theory expressed in language strong enough to encapsulate not just relations on the material world, but relations on such relations — relations which entail qualifications sophisticated enough to determine all aspects of the our universe, including the laws of nature themselves. Or, as Roger Penrose terms it [70, pp.106-107], we need to capture *Strong Determinism*, whereby:

\[\ldots\] all the complication, variety and apparent randomness that we see all about us, as well as the precise physical laws, are all exact and unambiguous consequences of one single coherent mathematical structure.

The article of Calude, Campbell, Svozil and Stefanescu on *Strong determinism vs. computability* contains a useful discussion of the computability-theoretic ramifications of strong determinism.

In the next section we examine some more approachable phenomena than those at the quantum level. Even though the challenge these present to Davis’ Thesis is less obvious than that of quantum uncertainty, they do point us in the direction of the mathematics needed to make sense of strong determinism.

9. The Challenge from Emergence

The waves on the seashore, the clouds scudding across the sky, the complexity of the Mandelbrot set — observing these, one is made aware of limits on what we can practically compute. The underlying rules governing them are known, but that is not enough. When we talk about the problem of
‘seeing the wood for the trees’ we are approaching the gap between micro- and macro-events from another direction. Either way, there are commonly encountered situations in which either reduction, or seeing the ‘big picture’, entails more than a computation.

Although an interest in such things goes back to Poincaré — we already mentioned the 3-body problem — it was the second half of the twentieth century saw the growth of chaos theory, and a greater of awareness of the generation of informational complexity via simple rules, accompanied by the emergence of new regularities. The most mundane and apparently uncomplicated situations could provide examples — such as Robert Shaw’s strange attractor arising from an appropriately paced dripping tap. And inhospitable as turbulent fluids might appear, there too higher order formations might emerge and be subject to mathematical description, as demonstrated by David Ruelle (see Ruelle86) another early pioneer in the area. Schematic metaphors for such examples are provided by the cellular automaton (CA) model, and famously by John Conway’s Game of Life. Here is the musician Brian Eno41 talking in relation to how his creative work on ‘generative music’ was influenced by ‘Life’:

These are terribly simple rules and you would think it probably couldn’t produce anything very interesting. Conway spent apparently about a year finessing these simple rules. . . . He found that those were all the rules you needed to produce something that appeared life-like. What I have over here, if you can now go to this Mac computer, please. I have a little group of live squares up there. When I hit go I hope they are going to start behaving according to those rules. There they go. I’m sure a lot of you have seen this before. What’s interesting about this is that so much happens. The rules are very, very simple, but this little population here will reconfigure itself, form beautiful patterns, collapse, open up again, do all sorts of things. It will have little pieces that wander around, like this one over here. Little things that never stop blinking, like these ones. What is very interesting is that this is extremely sensitive to the conditions in which you started. If I had drawn it one dot different it would have had a totally different history. This is I think counter-intuitive. One’s intuition doesn’t lead you to believe that something like this would happen.

Margaret Boden and Ernest Edmonds7 make a case for generative art,
emergent from automata-like computer environments, really qualifying as art. While computer pioneer Konrad Zuse was impressed enough by the potentialities of cellular automata to suggest that the physics of the universe might be CA computable.

A specially useful key to a general mathematical understanding of such phenomena is the well-known link between emergent structures in nature, and familiar mathematical objects, such as the Mandelbrot and Julia sets. These mathematical metaphors for real-world complexity and associated patterns have caught the attention of many — such as Stephen Smale and Roger Penrose — as a way of getting a better grip on the computability/complexity of emergent phenomena. Here is Penrose describing his fascination with the Mandelbrot set:

Now we witnessed . . . a certain extraordinarily complicated looking set, namely the Mandelbrot set. Although the rules which provide its definition are surprisingly simple, the set itself exhibits an endless variety of highly elaborate structures.

As a mathematical analogue of emergence in nature, what are the distinctive mathematical characteristics of the Mandelbrot set? It is derived from a simple polynomial formula over the complex numbers, via the addition of a couple of quantifiers. In fact, with a little extra work, the quantifiers can be reduced to just one. This gives the definition the aspect of a familiar object from classical computability theory — namely, a \( \Pi^0_1 \) set. Which is just the level at which we might not be surprised to encounter incomputability. But we have the added complication of working with real (via complex) numbers rather than just the natural numbers. This creates room for a certain amount of controversy around the use of the BSS model of real computation (see Blum, Cucker, Shub and Smale) to show the incomputability of the Mandelbrot set and most Julia sets. The 2009 book by Mark Braverman and Michael Yampolsky on *Computability of Julia Sets* is a reliable guide to recent results in the area, including those using the more mainstream computable analysis model of real computation. The situation is not simple, and the computability of the Mandelbrot set, as of now, is still an open question.

What is useful, in this context, is that these examples both connect with emergence in nature, and share logical form with well-known objects which transcend the standard Turing model. As such, they point to the role of extended language in a real context taking us beyond models which are purely mechanistic. And hence give us a route to mathematically capturing
the origins of emergence in nature, and to extending our understanding of how nature computes. We can now view the halting set of a universal Turing machine as an emergent phenomenon, despite it not being as pretty visually as our Mandelbrot and Julia examples.

One might object that there is no evidence that quantifiers and other globally defined operations have any existence in nature beyond the minds of logicians. But how does nature know anything about any logical construct? The basic logical operations derive their basic status from their association with elementary algorithmic relationships over information. Conjunction signifies an appropriate and very simple merging of information, of the kind commonly occurring in nature. Existential quantification expresses projection, analogous to a natural object throwing a shadow on a bright sunny day. And if a determined supporter of Davis’ Thesis plays at god, and isolates a computational environment with the aim of bringing it within the Turing model, then the result is the delivery of an identity to that environment, the creating a natural entity — like a human being, perhaps — with undeniable naturally emergent global attributes.

There are earlier, less schematic approaches to the mathematics of emergence. Ones which fit well with the picture so far.

It often happens that when one gets interested in a particular aspect of computability, one finds Alan Turing was there before us. Back in the 1950s, Turing proposed a simple reaction-diffusion system describing chemical reactions and diffusion to account for morphogenesis, i.e., the development of form and shape in biological systems. One can find a full account of the background to Turing’s seminal intervention in the field at Jonathan Swinton’s well-documented webpage on Alan Turing and morphogenesis. One of Turing’s main achievements was to come up with mathematical descriptions — differential equations — governing such phenomena as Fibonacci phyllotaxis: the surprising showing of Fibonacci progressions in such things as the criss-crossing spirals of a sunflower head. As Jonathan Swinton describes:

In his reaction-diffusion system [Turing] had the first and one of the most compelling models mathematical biology has devised for the creation process. In his formulation of the Hypothesis of Geometrical Phyllotaxis he expressed simple rules adequate for the appearance of Fibonacci pattern. In his last, unfinished work he was searching for plausible reasons why those rules might hold, and it seems only in this that he did not succeed. It would take many decades before others, unaware of his full progress, would re-
trace his steps and finally pass them in pursuit of a rather beautiful
theory.

Most of Turing’s work in this area was unpublished in his lifetime, only ap-
pearing in 1992 in the *Collected Works*.\(^{104}\) Later work, coming to fruition
just after Turing died, was carried forward by his student Bernard Richards,
appearing in the thesis.\(^{80}\) See Richards\(^{81}\) for an interesting account of
Richards’ time working with Turing.

The field of *synergetics*, founded by the German physicist Hermann
Haken, provides another mathematical approach to emergence. Synergetics
is a multi-disciplinary approach to the study of the origins and evolution of
macroscopic patterns and spacio-temporal structures in interactive systems.
An important feature of synergetics for our purposes is its focus on *self-
organizational* processes in science and the humanities, particularly that of
autopoiesis. An instance of an autopoietic system is a biological cell, and
is distinguished by being sufficiently autonomous and operationally closed,
to recognisably self-reproduce.

A particularly celebrated example of the technical effectiveness of the
theory is Ilya Prigogine’s achievement of the Nobel Prize for Chemistry
in 1977 for his development of dissipative structure theory and its applica-
tion to thermodynamic systems far from equilibrium, with subsequent
consequences for self-organising systems. Nonlinearity and irreversibility
are associated key aspects of the processes modelled in this context.

See Michael Bushev’s comprehensive review of the field in his book\(^{12}\)
*Synergetics – Chaos, Order, Self-Organization*. Klaus Mainzer’s book\(^{61}\)
on *Thinking in Complexity: The Computational Dynamics of Matter, Mind,
and Mankind* puts synergetics in a wider context, and mentions such things as
synergetic computers.

The emphasis of the synergetists on *self-organisation* in relation to the
emergence of order from chaos is important in switching attention from the
*surprise* highlighted by so many accounts of emergence, to the *autonomy*
and *internal organisation* intrinsic to the phenomenon. People like Prig-
gogine found within synergetics, as did Turing for morphogenesis, precise
descriptions of previously mysteriously emergent order.

### 10. A Test for Emergence

There is a problem with the big claims made for emergence in many different
contexts. Which is that, like with ‘life’, nobody has a good definition of it.
Sometimes, this does matter. Apart from which history is littered with
instances of vague concepts clarified by science, with huge benefits to our understanding of the world and to the progress of science and technology. The clarification of what we mean by a *computation*, and the subsequent development of the computer and computer science is a specially relevant example here. Ronald C. Arkin, in his book [3, p.105] *Behaviour-Based Robotics*, summarises the problem as it relates to emergence:

Emergence is often invoked in an almost mystical sense regarding the capabilities of behavior-based systems. Emergent behavior implies a holistic capability where the sum is considerably greater than its parts. It is true that what occurs in a behavior-based system is often a surprise to the system’s designer, but does the surprise come because of a shortcoming of the analysis of the constituent behavioral building blocks and their coordination, or because of something else?

There is a salutary warning from the history of British Emergentists, who had their heyday in the early 1920s — Brian McLaughlin’s book. The notion of emergence has been found to be a useful concept from at least the time of John Stuart Mill, back in the nineteenth century. The emergentists of the 1920s used the concept to explain the irreducibility of the ‘special sciences’, postulating a hierarchy with physics at the bottom, followed by chemistry, biology, social science etc. The emergence was seen, anticipating modern thinking, as being irreversible, imposing the irreducibility of say biology to quantum theory. Of course the British emergentists experienced their heyday before the great quantum discoveries of the late 1920s, and as described in McLaughlin, this was in a sense their undoing. One of the leading figures of the movement was the Cambridge philosopher C. D. Broad, described by Graham Farmelo in his biography of Paul Dirac [43, p.39] as being in 1920 “one of the most talented young philosophers working in Britain”. In many ways a precursor of the current philosophers arguing for the explanatory role of emergence in the philosophy of mind, Charlie Broad was alive to the latest scientific developments, lecturing to the young Paul Dirac on Einstein’s new theory of relativity while they were both at Bristol. But here is Broad writing in 1925 [10, p.59] about the ‘emergence’ of salt crystals:

… the characteristic behaviour of the whole … could not, even in theory, be deduced from the most complete knowledge of the behaviour of its components … This … is what I understand by the ‘Theory of Emergence’. I cannot give a conclusive example of it,
since it is a matter of controversy whether it actually applies to
anything . . . I will merely remark that, so far as I know at present,
the characteristic behaviour of Common Salt cannot be deduced
from the most complete knowledge of the properties of Sodium
in isolation; or of Chlorine in isolation; or of other compounds of
Sodium, . . .

The date 1925 is significant of course, it was in the years following that Dirac
and others developed the quantum mechanics which would explain much
of chemistry in terms of locally described interactions between sub-atomic
particles. The reputation of the emergentists, for whom such examples had
been basic to their argument for the far-reaching relevance of emergence,
ever quite recovered.

For Ronald, Sipper and Capcarrère in 1999, Turing’s approach to pin-
ing down intelligence in machines suggested a test for emergence. Part of
the thinking would have been that emergence, like intelligence, is something
we as observers think we can recognise; while the complexity of what we
are looking for resists observer-independent analysis. The lesson is to po-
lice the observer’s evaluation process, laying down some optimal rules for
a human observer. Of course, the Turing Test is specially appropriate to
its task, our own experience of human intelligence making us well-qualifi-
ced to evaluate the putative machine version. Anyway, the Emergence Test of
Ronald, Sipper and Capcarrère\textsuperscript{84} for emergence being present in a system,
modelled on the Turing Test, had the following three ingredients:

(1) **Design:** The system has been constructed by the designer, by describ-
ing local elementary interactions between components (e.g., artificial
creatures and elements of the environment) in a language $\mathcal{L}_1$.

(2) **Observation:** The observer is fully aware of the design, but describes
global behaviors and properties of the running system, over a period of
time, using a language $\mathcal{L}_2$.

(3) **Surprise:** The language of design $\mathcal{L}_1$ and the language of observation
$\mathcal{L}_2$ are distinct, and the causal link between the elementary interactions
programmed in $\mathcal{L}_1$ and the behaviors observed in $\mathcal{L}_2$ is non-obvious to
the observer — who therefore experiences surprise. In other words,
there is a cognitive dissonance between the observer’s mental image of
the system’s design stated in $\mathcal{L}_1$ and his contemporaneous observation
of the system’s behavior stated in $\mathcal{L}_2$.

Much of what we have here is what one would expect, extracting the
basic elements of the previous discussion, and expressing it from the point
of view of the assumed observer. But an ingredient which should be noted is the formal distinction between the language $\mathcal{L}_1$ of the design and that of the observer, namely $\mathcal{L}_2$. This fits in with our earlier mathematical examples: the halting set of a universal Turing machine, and the Mandelbrot set, where the new language is got by adding a quantifier — far from a minor augmentation of the language, as any logician knows. And it points to the importance of the language used to describe the phenomena, an emphasis underlying the next section.

11. Definability the Key Concept

We have noticed that it is often possible to get descriptions of emergent properties in terms of the elementary actions from which they arise. For example, this is what Turing did for the role of Fibonacci numbers in relation to the sunflower etc. This is not unexpected, it is characteristic of what science does. And in mathematics, it is well-known that complicated descriptions may take us beyond what is computable. This could be seen as a potential source of surprise in emergence.

But one can turn this viewpoint around, and get something more basic. There is an intuition that entities do not just generate descriptions of the rules governing them: they actually exist because of, and according to mathematical laws. And that for entities that we can be aware of, these will be mathematical laws which are susceptible to description. That it is the describability that is key to their observability. But that the existence of such descriptions is not enough to ensure we can access them, even though they have algorithmic content which provides the stuff of observation.

It is hard to for one to say anything new. In this case Leibniz was there before us, essentially with his Principle of Sufficient Reason. According to Leibniz in 1714:

\[
\cdots\text{there can be found no fact that is true or existent, or any true proposition, without there being a sufficient reason for its being so and not otherwise, although we cannot know these reasons in most cases.}
\]

Taking this a little further — natural phenomena not only generate descriptions, but arise and derive form from them. And this connects with a useful abstraction — that of mathematical definability, or, more generally, invariance under the automorphisms of the appropriate structure. So giving precision to our experience of emergence as a potentially non-algorithmic determinant of events.
This is a familiar idea in the mathematical context. The relevance of definability for the real world is implicitly present in Hans Reichenbach’s work\textsuperscript{79} on the axiomatisation of relativity. It was, of course, Alfred Tarski who gave a precise logical form to the notion of definability. Since then logicians have worked within many different mathematical structures, succeeding in showing that different operations and relations are non-trivially definable, or in some cases undefinable, in terms of given features of the structure. Another familiar feature of mathematical structures is the relationship between definability within the structure and the decidability of its theory (see Marker\textsuperscript{62}), giving substance to the intuition that knowledge of the world is so hard to capture, because so much can be observed and described. Tarski’s proof of decidability of the real numbers, contrasting with the undecidability of arithmetic, fits with the fact that one cannot even define the integers in the structure of the real numbers.

Unfortunately, outside of logic, and certainly outside of mathematics, the usefulness of definability remains little understood. And the idea that features of the real world may actually be undefinable is, like that of incomputability, a recent and unassimilated addition to our way of looking at things.

At times, definability or its breakdown comes disguised within quite familiar phenomena. In science, particularly in basic physics, symmetries play an important role. One might be surprised at this, wondering where all these often beautiful and surprising symmetries come from. Maybe designed by some higher power? In the context of a mathematics in which undefinability and nontrivial automorphisms of mathematical structures is a common feature, such symmetries lose their unexpectedness. When Murray Gell-Mann demonstrated the relevance of SU(3) group symmetries to the quark model for classifying of elementary particles, it was based on lapses in definability of the strong nuclear force in relation to quarks of differing flavour. The automorphisms of which such symmetries are an expression give a clear route from fundamental mathematical structures and their automorphism groups to far-reaching macro-symmetries in nature. If one accepts that such basic attributes as position can be subject to failures of definability, one is close to restoring realism to various basic subatomic phenomena.

One further observation: Identifying emergent phenomena with material expressions of definable relations suggests an accompanying robustness of such phenomena. One would expect the mathematical characterisation to strip away much of the mystery which has made emergence so attractive to
theologically inclined philosophers of mind, such as Samuel Alexander [1, p.14]:

The argument is that mind has certain specific characters to which there is or even can be no neural counterpart . . . Mind is, according to our interpretation of the facts, an ‘emergent’ from life, and life an emergent from a lower physico-chemical level of existence.

And further [1, p.428]:

In the hierarchy of qualities the next higher quality to the highest attained is deity. God is the whole universe engaged in process towards the emergence of this new quality, and religion is the sentiment in us that we are drawn towards him, and caught in the movement of the world to a higher level of existence.

In contrast, here is Martin Nowak, Director of the Program for Evolutionary Dynamics at Harvard University, writing in the collection What We Believe But Cannot Prove, describing the sort of robustness we would expect:

I believe the following aspects of evolution to be true, without knowing how to turn them into (respectable) research topics. Important steps in evolution are robust. Multicellularity evolved at least ten times. There are several independent origins of eusociality. There were a number of lineages leading from primates to humans. If our ancestors had not evolved language, somebody else would have.

What is meant by robustness here is that there is mathematical content which enables the process to be captured and moved between different platforms. Though it says nothing about the relevance of embodiment or the viability of virtual machines hostable by canonical machines. We return to this later. On the other hand, it gives us a handle on representability of emergent phenomena, a key aspect of intelligent computation.

12. The Challenge of Modelling Mentality

Probably the toughest environment in which to road-test the general mathematical framework we have associated with emergence is that of human mental activity. What about the surprise ingredient of the Emergence Test?

Mathematical thinking provides an environment in which major ingredients – Turing called them intuition and ingenuity, others might call them
creativity and reason – are easier to clearly separate. A classical source of information and analysis of such thinking is the French mathematician Jacques Hadamard’s *The Psychology of Invention in the Mathematical Field*, based on personal accounts supplied by distinguished informants such as Poincaré, Einstein and Polya. Hadamard was particularly struck by Poincaré’s thinking, including a 1908 address of his to the French Psychological Society in Paris on the topic of *Mathematical Creation*. Hadamard followed Poincaré and Einstein in giving an important role to unconscious thought processes, and their independence of the role of language and mechanical reasoning. This is Hadamard’s account, built on that of Poincaré, of Poincaré’s experience of struggling with a problem:

> At first Poincaré attacked [a problem] vainly for a fortnight, attempting to prove there could not be any such function . . . [quoting Poincaré]:

> “Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me, without anything in my former thoughts seeming to have paved the way for it . . . I did not verify the idea . . . I went on with a conversation already commenced, but I felt a perfect certainty. On my return to Caen, for conscience sake, I verified the result at my leisure.”

This experience will be familiar to most research mathematicians – the period of incubation, the failure of systematic reasoning, and the surprise element in the final discovery of the solution: a surprise that may, over a lifetime, lose some of its bite with repetition and familiarity, but which one is still compelled to recognise as being mysterious and worthy of surprise. Anyway, the important third part of the Emergence Test is satisfied here.

Perhaps even more striking is the fact that Poincaré’s solution had that robustness we looked for earlier: the solution came packaged and mentally represented in a form which enabled it to be carried home and unpacked intact when back home. Poincaré just carried on with his conversation on the bus, his friend presumably unaware of the remarkable thoughts coursing through the mathematicians mind.

Another such incident emphasises the lack of uniqueness and the special character of such incidents – Jacques Hadamard quoting Poincaré again:

> “Then I turned my attention to the study of some arithmetical questions apparently without much success . . . Disgusted with my failure, I went to spend a few days at the seaside and thought of
something else. One morning, walking on the bluff, the idea came to me, with just the same characteristics of brevity, suddenness and immediate certainty, that the arithmetic transformations of indefinite ternary quadratic forms were identical with those of non-Euclidian geometry.

What about the design, and the observer’s awareness of the design? Here we have a large body of work, most notably from neuro-scientists and philosophers, and an increasingly detailed knowledge of the workings of the brain. What remains in question – even accepting the brain as the design (not as simple as we would like!) – is the exact nature of the connection between the design and the emergent level of mental activity. This is an area where the philosophers pay an important role in clarifying problems and solutions, while working through consequences and consistencies.

The key notion, providing a kind of workspace for working through alternatives, is that of supervenience. According to Jaegwon Kim [53, pp.14–15], supervenience:

\[
\text{... represents the idea that mentality is at bottom physically based,}
\]
\[
\text{and that there is no free-floating mentality unanchored in the physical nature of objects and events in which it is manifested.}
\]

There are various formulations. This one is from the online *Stanford Encyclopedia of Philosophy*:

A set of properties $A$ supervenes upon another set $B$ just in case no two things can differ with respect to $A$-properties without also differing with respect to their $B$-properties.

So in this context, it is the mental properties which are thought to supervene on the neuro-physical properties. All we need to know is are the details of how this supervenience takes place. And what throws up difficulties is our own intimate experience of the outcomes of this supervenience.

One of the main problems relating to supervenience is the so-called ‘problem of mental causation’, the old problem which undermined the Cartesian conception of mind-body dualism. The persistent question is: *How can mentality have a causal role in a world that is fundamentally physical?* Another unavoidable problem is that of ‘overdetermination’ – the problem of phenomena having both mental and physical causes. For a pithy expression of the problem, here is Kim\textsuperscript{54} again:

\[
\text{... the problem of mental causation is solvable only if mentality}
\]
is physically reducible; however, phenomenal consciousness resists physical reduction, putting its causal efficacy in peril.

It is not possible here, and not even useful, to go into the intricacies of the philosophical debates which rage on. But it is important to take on board the lesson that a crude mechanical connection between mental activity and the workings of the brain will not do the job. Mathematical modelling is needed to clarify the mess, but has to meet very tough demands.

13. Connectionist Models to the Rescue?

Synaptic interactions are basic to the workings of the brain, and connectionist models based on these are the first hope. And there is optimism about such models from such leading figures in the field as Paul Smolensky, recipient of the 2005 David E. Rumelhart Prize:

There is a reasonable chance that connectionist models will lead to the development of new somewhat-general-purpose self-programming, massively parallel analog computers, and a new theory of analog parallel computation: they may possibly even challenge the strong construal of Church’s Thesis as the claim that the class of well-defined computations is exhausted by those of Turing machines.

And it is true that connectionist models have come a long way since Turing’s 1948 discussion of ‘unorganised machines’, and McCulloch and Pitts’ 1943 early paper on neural nets. (Once again, Turing was there at the beginning, see Teuscher’s book on Turing’s Connectionism.)

But is that all there is? For Steven Pinker “neural networks alone cannot do the job”. And focusing on our elusive higher functionality, and the way in which mental images are recycled and incorporated in new mental processes, he points to a “kind of mental fecundity called recursion”:

We humans can take an entire proposition and give it a role in some larger proposition. Then we can take the larger proposition and embed it in a still-larger one. Not only did the baby eat the slug, but the father saw the baby eat the slug, and I wonder whether the father saw the baby eat the slug, the father knows that I wonder whether he saw the baby eat the slug, and I can guess that the father knows that I wonder whether he saw the baby eat the slug, and so on.
Is this really something new? Neural nets can handle recursions of various kinds. They can exhibit imaging and representational capabilities. They can learn. The problem seems to be with modelling the holistic aspects of brain functionalism. It is hard to envisage a model at the level of neural networks which successfully represent and communicate its own global informational structures. Neural nets do have many of the basic ingredients of what one observes in brain functionality, but the level of developed synergy of the ingredients one finds in the brain does seem to occupy a different world. There seems to be a dependency on an evolved embodiment which goes against the classical universal machine paradigm. We develop these comments in more detail later in this section.

For the mathematician, definability is the key to representation. As previously mentioned, the language functions by representing basic modes of using the informational content of the structure over which the language is being interpreted. Very basic language corresponds to classical computational relationships, and is local in import. If we extend the language, for instance, by allowing quantification, it still conveys information about an algorithmic procedure for accessing information. The new element is that the information accessed may now be emergent, spread across a range of regions of the organism, its representation very much dependent on the material embodiment, and with the information accessed via finitary computational procedures which also depend on the particular embodiment. One can observe this preoccupation with the details of the embodiment in the work of the neuro-scientist Antonio Damasio. One sees this in the following description from Damasio’s book, The Feeling Of What Happens, of the kind of mental recursions Steven Pinker was referring to above [25, p.170]:

As the brain forms images of an object – such as a face, a melody, a toothache, the memory of an event – and as the images of the object affect the state of the organism, yet another level of brain structure creates a swift nonverbal account of the events that are taking place in the varied brain regions activated as a consequence of the object-organism interaction. The mapping of the object-related consequences occurs in first-order neural maps representing the proto-self and object; the account of the causal relationship between object and organism can only be captured in second-order neural maps. . . . one might say that the swift, second-order nonverbal account narrates a story: that of the organism caught in the act of representing its own changing state as it goes about representing something else.
Here we see the pointers to the elements working against the classical independence of the computational content from its material host. We may have a mathematical precision to the presentation of the process. But the presentation of the basic information has to deal with emergence of a possibly incomputable mathematical character, and so has to be dependent on the material instantiation. And the classical computation relative to such information, implicit in the quotations from Pinker and Damasio, will need to work relative to these material instantiations. The mathematics sets up a precise and enabling filing system, telling the brain how to work hierarchically through emergent informational levels, within an architecture evolved over millions of years.

There is some recognition of this scenario in the current interest in the evolution of hardware – see, for example, Hornby, Sekanina and Haddow. We tend to agree with Steven Rose:

Computers are designed, minds have evolved. Deep Blue could beat Kasparov at a game demanding cognitive strategies, but ask it to escape from a predator, find food or a mate, and negotiate the complex interactions of social life outside the chessboard or express emotion when it lost a game, and it couldn’t even leave the launchpad. Yet these are the skills that human survival depends on, the products of 3bn years of trial-and-error evolution.

From a computer scientist’s perspective, we are grappling with the design of a cyber-physical system (CPS). And as Edward Lee from Berkeley describes:

To realize the full potential of CPS, we will have to rebuild computing and networking abstractions. These abstractions will have to embrace physical dynamics and computation in a unified way.

In Lee, he argues for “a new systems science that is jointly physical and computational.”

Within such a context, connectionist models with their close relationship to synaptic interactions, and availability for ad hoc experimentation, do seem to have a useful role. But their are good reasons for looking for a more fundamental mathematical model with which to express the ‘design’ on which to base a definable emergence. The chief reason is the need for a general enough mathematical framework, capable of housing different computationally complex frameworks. Although the human brain is an important example, it is but one part of a rich and heterogeneous compu-
tational universe, reflecting in its workings many elements of that larger context. The history of mathematics has led us to look for abstractions which capture a range of related structures, and which are capable of theoretical development informed by intuitions from different sources, which become applicable in many different situations. And which provide basic understanding to take us beyond the particularities of individual examples.

14. Definability in What Structure?

In looking for the mathematics to express the design, we need to take account of the needs of physics as well as those of mentality or biology. In his *The Trouble With Physics*,\(^9\) Lee Smolin points to a number of deficiencies of the standard model, and also of popular proposals such as those of string theory for filling its gaps. And in successfully modelling the physical universe, Smolin declares [92, p.241]:

\[ \ldots \textit{causality itself is fundamental.} \]

Referring to ‘early champions of the role of causality’ such as Roger Penrose, Rafael Sorkin (the inventor of causal sets), Fay Dowker and Fotini Markopoulou, Smolin goes on to explain [92, p.242]:

It is not only the case that the spacetime geometry determines what the causal relations are. This can be turned around: Causal relations can determine the spacetime geometry \ldots

It’s easy to talk about space or spacetime emerging from something more fundamental, but those who have tried to develop the idea have found it difficult to realize in practice. \ldots We now believe they failed because they ignored the role that causality plays in spacetime. These days, many of us working on quantum gravity believe that causality itself is fundamental – and is thus meaningful even at a level where the notion of space has disappeared.

So, when we have translated ‘causality’ into something meaningful, and the model based on it put in place – the hoped-for prize is a theory in which even the background character of the universe is determined by its own basic structure. In such a scenario, not only would one be able to do away with the need for exotic multiverse proposals, patched with inflationary theories and anthropic metaphysics. But, for instance, one can describe a proper basis for the variation of natural laws near a mathematical singularity, and so provide a mathematical foundation for the reinstatement of the philosophically more satisfying cyclical universe as an alternative to the
inflationary big bang hypothesis – see Paul Steinhardt and Neil Turok’s book\textsuperscript{95} for a well worked out proposal based on superstring theory.

15. The Turing Landscape, Causality and Emergence . . .

If there is one field in which ‘causality’ can be said to be fundamental, it is that of computability. Although the sooner we can translate the term into something more precise, the better. ‘Causality’, despite its everyday usefulness, on closer inspection is fraught with difficulties, as John Earman [37, p.5] nicely points out:

\ldots the most venerable of all the philosophical definitions [of determinism] holds that the world is deterministic just in case every event has a cause. The most immediate objection to this approach is that it seeks to explain a vague concept – determinism – in terms of a truly obscure one – causation.

Historically, one recognised the presence of a causal relationship when a clear mechanical interaction was observed. But Earman’s book makes us aware of the subtleties beyond this at all stages of history. The success of science in revealing such interactions underlying mathematically signalled causality – even for Newton’s gravitational ‘action at a distance’ – has encouraged us to think in terms of mathematical relationships being the essence of causality. Philosophically problematic as this may be in general, there are enough mathematical accompaniments to basic laws of nature to enable us to extract a suitably general mathematical model of physical causality. And to use this to improve our understanding of more complicated (apparent) causal relationships. The classical paradigm is still Isaac Newton’s formulation of a mathematically complete formulation of his laws of motion, sufficient to predict an impressive range of planetary motions.

Schematically, logicians at least have no problem representing Newtonian transitions between mathematically well-defined states of a pair of particles at different times as the Turing reduction of one real to another, via a partial computable (p.c.) functional describing what Newton said would happen to the pair of particles. The functional expresses the computational and continuous nature of the transition. One can successfully use the functional to approximate, to any degree of accuracy, a particular transition.

This type of model, using partial computable functionals extracted from Turing’s\textsuperscript{102} notion of oracle Turing machine, is very generally applicable to basic laws of nature. However, it is well-known that instances of a basic
law can be composed so as to get much more problematic mathematical relationships, relationships which have a claim to be causal. We have mentioned case above – for instance those related to the 3-body problem. Or strange attractors emergent from complex confluences of applications of basic laws. See recent work Beggs, Costa, Loff and Tucker,\(^4\) Beggs and Tucker\(^5\) concerning the modelling of physical interactions as computation relative to oracles, and incomputability from mathematical thought experiments based on Newtonian laws.

The technical details of the extended Turing model, providing a model of computable content of structures based on p.c. functionals over the reals, can be found in Cooper.\(^{19}\) One can also find there details of how Emil Post\(^{75}\) used this model to define the *degrees of unsolvability* – now known as the *Turing degrees* – as a classification of reals in terms of their relative computability. The resulting structure has turned out to be a very rich one, with a high degree of structural pathology. At a time when primarily mathematical motivations dominated the field – known for many years as a branch of mathematical logic called *recursive function theory* – this pathology was something of a disappointment. Subsequently, as we see below, this pathology became the basis of a powerful expressive language, delivering a the sort of richness of definable relations which qualify the structure for an important real-world modelling role.

Dominant as this Turing model is, widely accepted to have a canonical role, there are more general types of relative computation. Classically, allowing non-deterministic Turing computations relative to well-behaved oracles gives one nothing new. But in the real world one often has to cope with data which is imperfect, or provided in real-time, with delivery of computations required in real time. There is an argument that the corresponding generalisation is the ‘real’ relative computability. There are equivalent formalisations – in terms of *enumeration reducibility* between sets of data, due to Friedberg and Rogers,\(^{45}\) or (see Myhill\(^{66}\)), in terms of *relative computability of partial functions* (extending earlier notions of Kleene and Davis). The corresponding extended structure provides an interesting and informative context for the better known Turing degrees – see, for example, Soskova and Cooper.\(^{94}\) The Bulgarian research school, including D. Skordev, I. Soskov, A. Soskova, A. Ditchev, H. Ganchev, M. Soskova and others has played a special role in the development of the research area.

The universe we would like to model is one in which we can describe global relations in terms of local structure – so capturing the emergence of large-scale formations, and giving formal content to the intuition that such
emergent higher structures ‘supervene’ on the computationally more basic local relationships.

Mathematically, there appears to be strong explanatory power in the formal modelling of this scenario as definability over a structure based on reducibilities closely allied to Turing functionals: or more generally, freeing the model from an explicit dependence on language, as Invariance under automorphisms of the Turing structure. In the next section, we focus on the standard Turing model, although the evidence is that similar outcomes would be provided by the related models we have mentioned.

16. An Informational Universe, and Hartley Rogers’ Programme

Back in 1967, the same year that Hartley Rogers’ influential book *Theory of Recursive Functions and Effective Computability* appeared, a paper, based on an earlier talk of Rogers, appeared in the proceedings volume of the 1965 Logic Colloquium in Leicester. This short article initiated a research agenda which has held and increased its interest over a more than 40 year period. Essentially, *Hartley Rogers’ Programme* concerns the fundamental problem of characterising the Turing invariant relations.

The intuition is that these invariant relations are key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure. Where the richness of Turing structure discovered so far becomes the raw material for a multitude of non-trivially definable relations. There is an interesting relationship here between the mathematics and the use of the anthropic principle in physics to explain why the universe is as it is. It is well-known that the development of the complex development we see around us is dependent on a subtle balance of natural laws and associated constants. One would like the mathematics to explain why this balance is more than an accidental feature of one of a multitude, perhaps infinitely many, randomly occurring universes. What the Turing universe delivers is a rich infra-structure of invariant relations, providing a basis for a correspondingly rich material instantiation, complete with emergent laws and constants, a provision of strong determinism, and a globally originating causality equipped with non-localism – though all in a very schematic framework. Of course, echoing Smolin, it is the underlying scheme that is currently missing. We have a lot of detailed information, but the skeleton holding it all together is absent.

However, the computability theorists have their own ‘skeleton in the cupboard’. The modelling potential of the extended Turing model depends
on it giving some explanation of such well-established features as quantum uncertainty, and certain experimentally verified uncertainties relating to human mentality. And there is a widely believed mathematical conjecture which would rob the Turing model of basic credentials for modelling observable uncertainty.

The Bi-Interpretability Conjecture, arising from Leo Harrington’s familiarity with the model theoretic notion of bi-interpretability, can be roughly described as asserting that:

The Turing definable relations are exactly those with information content describable in second-order arithmetic.

Moreover, given any description of information content in second-order arithmetic, one has a way of reading off the computability-theoretic definition in the Turing universe. Actually, a full statement of the conjecture would be in terms of ‘interpreting’ one structure in another, a kind of poor-man’s isomorphism. Seminal work on formalising the global version of the conjecture, and proving partial versions of it complete with key consequences and equivalences, were due to Theodore Slaman and Hugh Woodin. See Slaman’s 1990 International Congress of Mathematicians article for a still-useful introduction to the conjecture and its associated research project.

An unfortunate consequence of the conjecture being confirmed would be the well-known rigidity of the structure second-order arithmetic being carried over to the Turing universe. The breakdown of definability we see in the real world would lose its model. However, work over the years makes this increasingly unlikely.

See Nies, Shore and Slaman for further development of the requisite coding techniques in the local context, with the establishment of a number of local definability results. See Cooper for work in the other direction, both at the global and local levels. What is so promising here is the likelihood of the final establishment of a subtle balance between invariance and non-invariance, with the sort of non-trivial automorphisms needed to deliver a credible basis for the various symmetries, and uncertainties peculiar to mentality and basic physics: along with the provision via partial versions of bi-interpretability of an appropriate model for the emergence of the more reassuring ‘quasi-classical’ world from out of quantum uncertainty, and of other far-reaching consequences bringing such philosophical concepts as epistemological relativism under a better level of control.

To summarise: What we propose is that this most cartesian of research
areas, classical computability theory, regain the real-world significance it was born out of in the 1930s. And that it structure the informational world of science in a radical and revealing way. The main features of this informational world, and its modelling of the basic causal structure of the universe would be:

- A universe described in terms of reals . . .
- With basic natural laws modelled by computable relations between reals.
- Emergence described in terms of definability/invariance over the resulting structure . . .
- With failures of definable information content modelling mental phenomena, quantum ambiguity, etc. . . .
- Which gives rise to new levels of computable structure . . .
- And a familiarly fragmented scientific enterprise.

As an illustration of the explanatory power of the model, we return to the problem of mental causation. Here is William Hasker, writing in The Emergent Self [50, p. 175], and trying to reconcile the autonomy of the different levels:

The “levels” involved are levels of organisation and integration, and the downward influence means that the behavior of “lower” levels – that is, of the components of which the “higher-level” structure consists – is different than it would otherwise be, because of the influence of the new property that emerges in consequence of the higher-level organization.

The mathematical model, making perfect sense of this, treats the brain and its emergent mentality as an organic whole. In so doing, it replaces the simple everyday picture of what a causal relationship is with a more subtle confluence of mathematical relationships. Within this confluence, one may for different purposes or necessities adopt different assessment of what the relevant causal relationships are. For us, thinking about this article, we regard the mentality hosting our thoughts to provide the significant causal structure. Though we know full well that all this mental activity is emergent from an autonomous brain, modelled with some validity via a neural network.

So one might regard causality as a misleading concept in this context. Recognisable causality occurs at different levels of the model, connected by relative definability. And the causality at different levels in the form of
relations with identifiable algorithmic content, this content at higher levels being emergent. The diverse levels form a unity, with the causal structure observed at one level reflected at other levels – with the possibility of non-algorithmic feedback between levels. The incomputability involved in the transition between levels makes the supervenience involved have a non-reductive character.

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