ISOMETRIC EMBEDDABILITY OF SNOWFLAKES

ERIK WALSBERG

To Gabor Elek.

Abstract. We show that a snowflake of a metric space with positive Hausdorff dimension does not admit an isometric embedding into euclidean space.

1. Introduction

Let \((X, d)\) be a metric space. By euclidean space we mean \(\mathbb{R}^k\) equipped with the standard euclidean metric. Given \(\lambda > 0\) a map \(f : X \to \mathbb{R}^k\) gives a \(\lambda\)-bilipschitz embedding of \((X, d)\) into euclidean space if:

\[
\frac{1}{\lambda} d(x, x') \leq ||f(x) - f(x')|| \leq \lambda d(x, x') \quad \text{for all } x, x' \in X.
\]

The map \(f\) is a bilipschitz embedding if it is a \(\lambda\)-bilipschitz embedding for some \(\lambda > 0\). Bilipschitz embeddings are injective, so the term “embedding” is justified. The map \(f\) is an isometric embedding if it is a 1-bilipschitz embedding. Given \(0 < r < 1\), the \(r\)-snowflake of \((X, d)\) is the metric space \((X, d^r)\). We say that \((X, d^r)\) is a snowflake of \((X, d)\). Let \(K > 0\), we say that \((X, d)\) is \(K\)-doubling if every open ball of radius \(t\) contains at most \(K\) pairwise disjoint open balls of radius \(\frac{1}{2}t\). The metric space \((X, d)\) is said to be doubling if it is \(K\)-doubling for some \(K > 0\). It is easy to see that the following facts hold:

- Euclidean space is doubling.
- Any metric space which admits a bilipschitz embedding into euclidean space is doubling.
- A snowflake of a doubling metric space is doubling.

The following marvelous theorem, due to Assouad, gives a kind of converse to the simple facts listed above:

**Theorem 1.1 (Assouad).** Suppose that \((X, d)\) is doubling and \(0 < r < 1\). Then the \(r\)-snowflake of \((X, d)\) admits a bilipschitz embedding into some euclidean space.

It is natural to wonder when snowflakes admit isometric embeddings into euclidean space. In this paper we show that this is generally not the case:

**Theorem 1.2.** Suppose that \((X, d)\) has positive Hausdorff dimension and \(0 < r < 1\). Then the \(r\)-snowflake of \((X, d)\) does not admit an isometric embedding into euclidean space.

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2. Preliminaries

Our proof depends on some basic geometric facts which we gather in this section. We begin with an elementary lemma:

**Lemma 2.1.** Let $x_1, \ldots, x_{l+1} \in \mathbb{R}^l$ be in general position. Then the map $\sigma : \mathbb{R}^l \to \mathbb{R}^{l+1}$ given by

$$\sigma(y) = (\|y - x_1\|, \ldots, \|y - x_{l+1}\|)$$

is injective.

**Proof.** We suppose otherwise towards a contradiction. Suppose that $y, y' \in \mathbb{R}^l$ are such that $y \neq y'$ and $\sigma(y) = \sigma(y')$. Let $H$ be the set of $x \in \mathbb{R}^l$ such that $\|x - y\| = \|x - y'\|$. So $x_1, \ldots, x_{l+1} \in H$. However, as $H$ is a hyperplane of codimension one, this implies that $x_1, \ldots, x_{l+1}$ are not in general position.

We let $D \subseteq \mathbb{R}^{l+1}$ be the image of $\sigma$ and let $\tau : D \to \mathbb{R}^l$ be the compositional inverse of $\sigma$. That is, if $\bar{t} = (t_1, \ldots, t_{l+1}) \in D$ then $\tau(\bar{t})$ is the unique $y \in \mathbb{R}^l$ such that:

$$\|y - x_i\| = t_i \quad \text{for all } 1 \leq i \leq l + 1.$$

We make use of the following:

**Lemma 2.2.** There are smooth submanifolds $D_1, \ldots, D_m \subseteq \mathbb{R}^{l+1}$ such that $D = D_1 \cup \ldots \cup D_m$ and the restriction of $\tau$ to each $D_i$ is smooth.

**Proof.** It is presumably easy to prove Lemma 2.2 in an elementary way. However, the present author is a logician. Therefore, we give a very general proof using semialgebraic geometry. A set $A \subseteq \mathbb{R}^k$ is semialgebraic if it is a finite union of sets of the form

$$\{ \bar{x} \in \mathbb{R}^k : p(\bar{x}) \geq 0 \}$$

for polynomial $p$.

A function $f : A \to B$ between semialgebraic subsets $A, B \subseteq \mathbb{R}^k$ is semialgebraic if its graph is a semialgebraic subset of $\mathbb{R}^k \times \mathbb{R}^k$. We refer to [BCR87] for information about semialgebraic geometry. It is well known that every semialgebraic subset of euclidean space is a finite union of smooth submanifolds of euclidean space and if $A \subseteq \mathbb{R}^k$ and $f : A \to \mathbb{R}^n$ are semialgebraic then $A$ can be written as a finite union of smooth submanifolds of $\mathbb{R}^k$ in such a way that the restriction of $f$ to every submanifold is smooth. It is an immediate consequence of Tarski-Seidenberg quantifier elimination that $D \subseteq \mathbb{R}^{l+1}$ and $\tau : D \to \mathbb{R}^l$ are both semialgebraic. Lemma 2.2 follows.

**Lemma 2.3.** Let $A \subseteq D$. The Hausdorff dimension of $\tau(A)$ is no greater then the Hausdorff dimension of $A$.

Lemma 2.3 is a straightforward consequence of Lemma 2.2 and a few standard facts about Hausdorff dimension which can be found in [Mat95] or other places. We let $\dim$ be the Hausdorff dimension.

**Proof.** Let $D_1, \ldots, D_m$ be as in the statement of Lemma 2.2. Then:

$$\dim(A) = \max\{\dim(D_i \cap A) : 1 \leq i \leq m\}$$

and

$$\dim(\tau(A)) = \max\{\dim(\tau(D_i \cap A)) : 1 \leq i \leq m\}.$$

Smooth maps do not raise Hausdorff dimension, therefore:

$$\dim(\tau(D_i \cap A)) \leq \dim(D_i \cap A) \quad \text{for all } 1 \leq i \leq m.$$
3. Proof

In this section we prove Theorem 1.2. We let \( (X, d) \) be a metric space with positive Hausdorff dimension and \( 0 < r < 1 \). Let \( D \) and \( \tau \) be as in the previous section and let \( \dim \) be the Hausdorff dimension. We suppose toward a contradiction that \( \iota : X \to \mathbb{R}^l \) gives an isometric embedding of \( (X, d') \) into euclidean space. We may suppose that \( \iota(X) \) contains \( l + 1 \) points \( y_1, \ldots, y_{l+1} \) in general position. If this is not the case then \( \iota(X) \) is contained in a hyperplane with positive codimension, and we replace \( \iota \) with an isometric embedding into a euclidean space with smaller dimension. We let \( x_1, \ldots, x_{l+1} \in X \) be such that

\[
\iota(x_i) = y_i \quad \text{for all } 1 \leq i \leq l + 1.
\]

It follows from Lemma 2.1 that for all \( x \in X \), \( \iota(x) \) is the unique \( y \in \mathbb{R}^l \) such that

\[
||y - y_i|| = d(x_i, x)'
\]

for all \( 1 \leq i \leq l + 1 \).

Let \( X' = X \setminus \{x_1, \ldots, x_{l+1}\} \). Let \( U \) be the open subset of \( \mathbb{R}^{l+1} \) consisting of elements with positive coordinates. Let \( f : X' \to U \) be given by

\[
f(x) = (d(x_1, x), \ldots, d(x_{l+1}, x))
\]

and \( g : U \to U \) be given by

\[
g(t_1, \ldots, t_{l+1}) = (t'_1, \ldots, t'_{l+1})
\]

Note that \( g \circ f \) maps \( X' \) into \( D \). The restriction of \( \iota \) to \( X' \) can be factored as the composition

\[
X' \xrightarrow{f} U \xrightarrow{g} U \xrightarrow{\tau} \mathbb{R}^l.
\]

As \( f \) gives a lipschitz map \( (X', d') \to U \) we have \( \dim f(X') \leq \dim(X', d) \). As \( g \) is smooth it does not raise Hausdorff dimension so \( \dim(g \circ f)(X') \leq \dim(X', d) \) as well. It follows from Lemma 2.3 that \( \dim(\tau \circ g \circ f)(X') \leq \dim(X', d) \). Therefore \( \dim \iota(X') \leq \dim(X', d) \). As \( X \setminus X' \) is finite we have \( \dim \iota(X) \leq \dim(X, d) \). As \( \iota(X) \) is isometric to \( (X, d') \) this implies that \( \dim(X, d') \leq \dim(X, d) \). However, it follows immediately from the definition of Hausdorff dimension that \( \dim(X, d') = \frac{1}{r} \dim(X, d) \). This yields a contradiction as \( \frac{1}{r} > 1 \) and \( \dim(X, d) > 0 \).

References

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Institute of Mathematics, The Hebrew University of Jerusalem, Givat Ram, Jerusalem, 9190, Israel.
E-mail address: erikw@math.ucla.edu