Cognitive Hierarchies in Multi-Stage Games of Incomplete Information

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Abstract

We explore the dynamic cognitive hierarchy (CH) theory proposed by Lin and Palfrey (2022) in the setting of multi-stage games of incomplete information. In such an environment, players will learn other players’ payoff-relevant types and levels of sophistication at the same time as the history unfolds. As we apply the dynamic CH solution to a class of two-person dirty faces games, we find that lower-level players will figure out their face types in later periods than higher-level players, which is in sharp contrast with the equilibrium. Finally, we re-analyze the dirty faces game experimental data from Bayer and Chan (2007) and demonstrate the dynamic CH solution can better explain the data than the static CH solution.

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"The natural way of looking at game situations...is not based on circular concepts, but rather on a step-by-step reasoning procedure."

—Reinhard Selten (1998)

1 Introduction

In many situations, people interact with others over time, in a multi-stage environment with incomplete information, such as social learning, sequential bargaining, signaling, reputation building, or cheap talk, etc. The standard method to analyzing these environments is to model them as extensive form games and solve for the sequential equilibrium (or the perfect Bayesian equilibrium). The equilibration is demanding which requires players to form consistent beliefs at every information set, and best respond to such belief everywhere.

When there is incomplete information, no matter how simple the rules are, the number of possible information sets could be tremendously large. For instance, Johanson (2013) finds that in a two-person Texas Hold’em game, the number of information sets is around $10^{162}$, which is $10^{82}$ times larger than the number of atoms in the observable universe (around $10^{80}$, estimated by Villanueva (2009)). Therefore, it seems impractical to expect people to behave as predicted by sequential equilibrium in such games. Even in much simpler games that can be played in laboratory experiments, it is well-documented that people systematically violate the predictions of sequential equilibrium (see, for example, Camerer (2003)).

To accommodate these anomalous findings, Lin and Palfrey (2022) develop the concept of dynamic cognitive hierarchy (CH) that extends the CH framework proposed by Camerer et al. (2004)—which is considerably successful in explaining the behavior in normal form games—to general extensive form games. In this framework, players are heterogeneous with respect to levels of sophistication. The iterative process starts from level 0 players who will uniformly randomize at every information set. For any $k > 0$, a level $k$ player believes all other players have lower levels distributed from 0 to $k - 1$. The idea of dynamic CH is that players will update their beliefs about other players’ levels as history unfolds.

In this paper, we apply the dynamic CH framework to analyze people’s behavior in extensive form games of incomplete information. In such an environment, players will learn about other players’ levels and the basic elements of the game structure, such as other players’ private information at the same time. To study this learning process in a tractable way, we consider the framework of “multi-stage games with observed actions” proposed by Fudenberg and Levine (1983) and Fudenberg and Tirole (1991b). In this framework, players will have two pieces of private information: a payoff-relevant type and a level of sophistication, and they will update their joint beliefs of types and levels at every information set.

In a multi-stage game with observed actions, the dynamic CH solution is a level-dependent assessment consisting of the level-dependent behavioral strategy profile and posterior beliefs. In games of perfect information, the posterior belief is only about other players’ levels of sophistication. However, when the information sets are non-singleton, as pointed out by Lin and Palfrey (2022), the posterior beliefs are typically correlated across histories at some information sets in such an environment. In other words, the posterior beliefs of types and
levels are generally correlated even if they are independently drawn at the beginning.

Despite the fact that the posterior beliefs of types and levels are correlated, we can still characterize how the beliefs will evolve along the history. In Proposition 1, we first show that players will never eliminate the possibility of any type profile at every information set, while they will gradually rule out the possibility of levels at later stages. This result holds for general prior distributions of types. Moreover, when assuming the types are independently drawn across players, Proposition 2 establishes that every level of player’s posterior belief is independent across players at every information set. Finally, if the prior distribution of types is not independent across players, Proposition 3 points out the original game can be transformed to another game with independent types and the dynamic CH solutions are invariant in both games.

At a more conceptual level, the dynamic CH solution is in the same spirit of the standard equilibrium model—the only difference is that dynamic CH solution replaces the requirement of mutual consistency of the belief system with level-dependent beliefs. There are several distinct advantages of dynamic CH approach. First, since every level \( k > 0 \) of players always believes all others are possibly level 0 players who will uniformly randomize everywhere, they believe every information set is possible. Namely, there are no off-path information sets in the dynamic CH solution, and thus, the level-dependent belief is well-defined everywhere.

Second, the dynamic CH solution is attainable without the assumption of common knowledge of rationality, which is critical to equilibration (see, for instance, Bernheim (1984), Pearce (1984) and Aumann (1995)). Instead, the dynamic CH framework imposes a partial consistency requirement connecting the level-dependent prior beliefs to the true distribution of levels. Specifically, level \( k \) players’ prior beliefs of levels are specified as the normalized true distribution of levels, from level 0 to \( k - 1 \). That is, players have “truncated rational expectation.” One important feature of this property is that more sophisticated players have beliefs that are closer to the true distribution of levels. For level infinity players, their prior beliefs of levels coincide with the true underlying distribution. In short, compared to the equilibrium model, the dynamic CH framework replaces the assumption of common knowledge of rationality with the assumption of truncated rational expectation. Under this assumption, we find some general properties of the dynamic CH model closely mirror the standard equilibrium model. Proposition 2 and 3 are indeed two of the general properties of perfect Bayesian equilibrium characterized in Fudenberg and Tirole (1991b).

To further investigate the relation between the truncated rational expectation and the common knowledge of rationality, in the second half of the paper, we apply the dynamic CH solution to a class of dirty faces games first introduced by (Littlewood (1953), pp.3): “Three ladies, A,B,C, in a railway carriage all have dirty faces and are all laughing. It suddenly flashes on A: why doesn’t B realize C is laughing at her? Heavens! I must be laughable.” In fact, A’s epiphany consists of a subtle chain of reasoning starting from a missing piece of information: a public laughter will laugh whenever there is at least one lady having a dirty face.\(^1\) Given this common knowledge, if B sees that A has a clean face, then B should realize that C is laughing at herself; otherwise, C would have blushed. Next, if A doesn’t see B

\(^1\)This public announcement makes the event of that there is at least one dirty face a common knowledge to all players. Yet, the public announcement will not reveal the identity of the lady whose face is dirty.
blushing, she will realize that her face is dirty.

The reason why the dirty faces game is an important application is threefold. First, it is theoretically interesting because the equilibrium does not depend on the structure of the game, such as the payoffs, or the probability of having a dirty face, etc. Therefore, the predictions of different behavioral models are in sharp contrast with the equilibrium. Second, the dirty faces game is also important from the behavioral perspective that players will behave as sophisticatedly as they can. Notice that the equilibrium outcome is Pareto efficient because all players are better off if they can figure out their face types earlier. In this case, when deviating from the equilibrium, players will make others unable to make inferences from their actions, which makes them unable to make inferences from others’ actions. Therefore, players have no incentives to mimic lower-level players’ behavior—which yields a reliable estimate of the average level of sophistication among the population. The third reason is a practical concern. In this pandemic era, it is extremely challenging to conduct new experiments. Thus, it is ideal to study dirty faces games since there is an existing experimental data set that hasn’t been fully analyzed. In summary, the dirty faces game is an ideal application for dynamic CH analysis because of the theoretical, behavioral and practical concerns.

As we examine the equilibrium argument of the dirty faces games carefully, we can find that the deductive process relies on the requirements of sequential rationality and common knowledge of rationality. For A to realize that her face is dirty, A must believe B and C are rational enough to draw inferences from what they have observed. However, previous dirty faces game experiments by Weber (2001) and Bayer and Chan (2007) have demonstrated that people generally fail to perform such an iterative reasoning argument. In fact, both experiments find that around half of the subjects cannot even make two steps of reasoning, which again suggests the empirical fragility of common knowledge of rationality.

To this end, since the dynamic CH solution does not require common knowledge of rationality, it is possible for dynamic CH to generate empirically plausible predictions. For a class of two-person dirty-faces games, Proposition 4 fully characterizes the dynamic CH solution, finding that when both players’ faces are dirty, different levels of players are heterogeneous with respect to how soon they can realize their faces are dirty. Higher-level players tend to figure out their face types sooner than lower-level players. This characterization contrasts with the equilibrium which predicts a degenerated distribution of terminal periods.

Since dirty faces games are extensive form games, if we incorrectly apply the static CH model to this class of games, we in fact ignore the information contained in the observed history. Moreover, in the static CH solution, players have to make decisions based on hypothetical events rather than statistical inferences. As experimentally studied by Esponda and Vespa (2014), they are two behaviorally different learning processes. In Proposition 5,

\[2\]

\[3\]

See Section 5 for details of the game specification.

Because the static CH solution is defined on one-shot games, to apply the static CH model to dirty faces games, we need to transform the game to its corresponding reduced normal form.
we solve for the static CH solution, and compare the two CH solutions in Proposition 6. We find that players do not necessarily behave closer to the equilibrium in either of which CH solution. When players are impatient, dynamic CH predicts players tend to behave closer to the equilibrium prediction, and vice versa.

Finally, to see how the dynamic CH solution can bridge the gap between the theory and experiments, we revisit the experimental data of Bayer and Chan (2007) with the dynamic CH solution. We fit the dynamic CH solution to the data of the two-person and three-person dirty faces games experiments, finding that dynamic CH can explain the data significantly better than the static CH solution. In addition, we compare the fitness of the dynamic CH with the agent quantal response equilibrium (AQRE) developed by McKelvey and Palfrey (1998), and we find that the fitness of two models is not significantly different. Conceptually speaking, AQRE is a solution concept that relaxes the requirements of sequential equilibrium, which drops the requirement of sequential rationality while maintaining the consistency of the belief system. Similar to dynamic CH, AQRE also attempts to generalize the sequential equilibrium, but from a different angle. Therefore, the insignificance suggests that these two approaches are empirically comparable ways to relaxing the standard equilibrium model.

The paper is organized as follows. We discuss the related literature in section 2. Section 3 sets up the model. Section 4 establishes general properties of the belief updating process. In section 5, we analyze the two-person dirty faces games and discuss the theoretical implications of the dynamic CH solution. We revisit the experimental data of Bayer and Chan (2007) in section 6. Finally, section 7 concludes the paper.

2 Related Literature

As discussed in Section 1, this paper is closely related to the extensive literature of limited depth of reasoning in strategic environments. Over the past thirty years, this idea has been studied by a variety of theoretical researches (see, for instance, Binmore (1987, 1988), Selten (1991, 1998), Aumann (1992), Stahl (1993), and Alaoui and Penta (2016, 2018)). Beyond theoretical work, Nagel (1995) conducts the first experiment to study people’s iterative reasoning process, using the “beauty contest” game. In this game, each player simultaneously chooses an integer between 0 and 100. The winner is the player whose choice is closest to the average of all numbers multiplied by a parameter \( p \in (0, 1) \). The unique equilibrium is that all players should choose 0, while empirically, there is almost no player choosing the equilibrium action. Instead, players seem to behave as if performing iterative best response.\(^4\)

To explain the data, Nagel (1995) proposes the “level-\( k \) model,” which assumes each player has a level of reasoning. Level 0 players will uniformly randomize in their action sets. For every \( k \geq 1 \), level \( k \) players have the (degenerated) beliefs that they are one level of reasoning deeper than the rest and best respond to such beliefs. Level-\( k \) model has been applied to a range of different environments, such as matrix games (Costa-Gomes et al., 2001; Crawford and Iriberri, 2007a), two-person guessing games (Costa-Gomes and Crawford, 2001; Crawford and Iriberri, 2007a), two-person guessing games (Costa-Gomes and Crawford, 2001; Crawford and Iriberri, 2007a),

\(^4\)This empirical pattern can be robustly replicated in different environments. For instance, Ho et al. (1998) and Bosch-Domenech et al. (2002) find similar results in both the laboratory and field experiments.
2006), auctions (Crawford and Iriberri, 2007b), and sender-receiver games (Cai and Wang, 2006; Wang et al., 2010). Although the level-\(k\) model has been considerably successful in explaining the data, the specification is disentangled with the equilibrium model.

To this end, Stahl and Wilson (1995) are the first to propose a specific mixture model where each level of player best responds to a mixture between lower levels and equilibrium players. Later, Camerer et al. (2004) develop the CH framework where level \(k\) players best respond to a mixture of lower levels, from level 0 to \(k - 1\). Moreover, players have correct beliefs about the relative proportions of the lower levels. The specification of truncated rational expectation connects the perspective of a behavioral model to the equilibrium theory. Yet, these theories are only built for normal form games. Finally, Lin and Palfrey (2022) extend the CH approach from normal form games to general extensive form games. In the dynamic CH model, each level of player also has a level of sophistication and has a correct prior belief about the relative proportions of the lower levels. However, the difference is that in the dynamic CH model, players will update their beliefs about the levels and best respond to the posterior beliefs at every subgame.

This paper is also related to other behavioral models in game theory. First, in most laboratory experiments in economics and game theory, subjects play the same game with multiple repetitions, in order to gain experience and facilitate convergence to equilibrium behavior. Ho and Su (2013) and Ho et al. (2021) propose a modification of CH that allows for learning across repeated plays of the same sequential game, in a different way than in Stahl (1996), but in the same spirit. In their setting, players repeatedly play the same sequential game and update their beliefs about the distribution of levels after observing past outcomes of earlier games, while holding the fixed beliefs during each play of the game. Moreover, players endogenously choose new levels of sophistication for the next iteration of the game. This is different from the framework of dynamic CH where players update their beliefs about other players’ levels after each move within a single game. Furthermore, players are strategic learners in dynamic CH as they can correctly anticipate the evolution of posterior beliefs in later subgames—which leads to a much different learning dynamic compared with naive adaptive learning models.

Furthermore, dynamic CH is also conceptually related to the agent quantal response equilibrium (AQRE) (McKelvey and Palfrey, 1998), and the cursed equilibrium (Eyster and Rabin, 2005). Dynamic CH is connected to AQRE since both solutions attempt to relax the requirements of equilibrium. AQRE maintains the requirement of belief consistency while allowing players to make “better” responses instead of best responses. Yet, dynamic CH replaces the mutually consistent belief system with the level-dependent beliefs but assuming all strategic players to make best responses. On the other hand, although the cursed equilibrium is not defined on extensive form games, it is related to dynamic CH in the sense that every level of player is somehow “cursed”—level 1 players are fully cursed since they believe all other players are level and their actions are unrelated to their private information. Higher-level players are partially cursed as they believe others may be strategic and their actions will reveal their private information. The differences are that players in the cursed equilibrium do not update other players’ cursedness as the history unfolds and the cursed equilibrium maintains the consistency of the belief system.

Last but not least, this paper also relates to the literature of dirty faces games. The dirty
faces game is first introduced by Littlewood (1953) to demonstrate how common knowledge is transmitted. Binmore and Brandeburger (1988) are the first to theoretically study the dirty faces games with the knowledge operator. In addition, Liu (2008) shows that in theory, if players are unaware of other players’ faces, they might wrongly claim their face types, and hence influence the knowledge transmission among the players.

On the experimental studies, Weber (2001) and Bayer and Chan (2007) conduct the first two dirty faces game experiments, finding that there is a significant portion of subjects who are not able perform such iterative reasoning. More recent experiments have shown that the failure of iterative reasoning is still observed when playing against fully rational robot players (Grehl and Tutić, 2015), and is correlated with cognitive abilities (Devetag and Warglien, 2003; Bayer and Renou, 2016a,b), while the deviations from the equilibrium will significantly decrease when the participants are selected through a market mechanism (Choo and Zhou, 2022). In summary, these experiment findings support the existence of level 0 players, who are not sequentially rational, and the heterogeneity with respect to strategic sophistication among the population.

3 The Model

In this paper, we will focus on a restricted class of extensive form games: the multi-stage games with observed actions introduced by Fudenberg and Levine (1983) and Fudenberg and Tirole (1991b). This framework provides a tractable framework to study players’ strategic behavior when they are uncertain about other players’ types and levels of sophistication at the same time. This section defines multi-stage games with observed actions and the dynamic CH solution for this family of games.

3.1 Multi-Stage Games with Observed Actions

Let \( N = \{1, \ldots, n\} \) be a finite set of players. Each player \( i \in N \) has a type \( \theta_i \) drawn from a finite set \( \Theta_i \). Let \( \theta \in \Theta \equiv \times_{i=1}^{n} \Theta_i \) be the type profile and \( \theta_{-i} \) be the type profile without player \( i \). All players have the common (full support) prior distribution \( F(\cdot): \Theta \rightarrow [0, 1] \). Therefore, for every player \( i \), the belief of other players’ types conditional on his own type is

\[
 F(\theta_{-i}|\theta_i) = \frac{F(\theta_{-i}, \theta_i)}{\sum_{\theta'_{-i} \in \Theta_{-i}} F(\theta'_{-i}, \theta_i)}.
\]

If the types are independent across players, then for each player \( i \), his belief of other players’ types is \( F_{-i}(\theta_{-i}) = \Pi_{j \neq i} F_j(\theta_j) \) where \( F_j(\theta_j) \) is the marginal distribution of player \( j \)’s type. At the beginning of the game, players are told their own types, but they are not informed anything about other players’ types. That is, each player’s type is his own (payoff-relevant) private information.

The game is played in periods \( t = 1, 2, \ldots, T \).\(^5\) In each period, players simultaneously

\(^5\)In this paper, we focus on the class of finite horizon games although \( T \) can be arbitrarily large. In other words, we consider the games that the horizon is long but well foreseen.
choose their actions, which will be revealed at the end of the period. We allow the feasible set of actions varies with histories, so games with alternating moves are also included. Let $\mathcal{H}_{t-1}$ be the set of all available histories at period $t$, where $\mathcal{H}_0 = \{h_0\}$ and $\mathcal{H}^T$ is the set of terminal histories. Let $\mathcal{H} = \bigcup_{t=0}^{T} \mathcal{H}_t$ be the set of all available histories of the game, and let $\mathcal{H}\setminus\mathcal{H}^T$ be the set of non-terminal histories.

For every player $i$, the available information at period $t$ is in $\Theta_i \times \mathcal{H}_{t-1}$. Therefore, player $i$’s information sets can be specified as $\mathcal{I}_i \in \Pi_i = \{(\theta_i, h) : \theta_i \in \Theta_i, h \in \mathcal{H}\setminus\mathcal{H}^T\}$. For the sake of simplicity, we assume that the feasible set of actions for every player is independent of their types.\footnote{The consequence of this assumption is that players cannot signal their own type by choosing some action that is not available to some other types.} We use $A_i(h_{t-1})$ to denote the feasible set of actions for player $i$ at history $h_{t-1}$. Let $A_i = \bigcup_{h \in \mathcal{H}\setminus\mathcal{H}^T} A_i(h)$ be the set of player $i$’s all feasible actions in the game. We assume that all players in the game have perfect recall (see Kreps and Wilson (1982b) for the definition). In addition, we assume $A_i$ is finite for all $i \in N$ and $|A_i(h)| \geq 1$ for all $i \in N$ and any $h \in \mathcal{H}\setminus\mathcal{H}^T$.

A behavioral strategy for player $i$ is a function $\sigma_i : \Pi_i \rightarrow \Delta(A_i)$ satisfying $\sigma_i(\theta_i, h_{t-1}) \in \Delta(A_i(h_{t-1}))$. Furthermore, we use $\sigma_i(a_t^{i} | \theta_i, h_{t-1})$ to denote the probability for player $i$ to choose $a_t^{i} \in A_i(h_{t-1})$. We use $a^{t} = (a_1^{t}, \ldots, a_n^{t}) \in \times_{i=1}^{n} A_i(h_{t-1})$ to denote the action profile at period $t$ and $a^{t}_{i-1}$ be the action profile at period $t$ without player $i$. If $a^{t}$ is the action profile realized in period $t$, then $h^{t} = (h_{t-1}, a^{t})$. Finally, each player $i$ has a payoff function (in von Neumann-Morgenstern utilities) $u_i : \mathcal{H}^T \times \Theta \rightarrow \mathbb{R}$, and we let $u = (u_1, \ldots, u_n)$ be the profile of utility functions.\footnote{Notice that each player’s payoff depends on the whole type profile. For player $i$, if $\theta_i$ does not directly kick in the utility function, we say $\theta_i$ has “informational value.”}

A multi-stage game with observed actions, $\Gamma$, is defined by the tuple $\Gamma = (N, \mathcal{H}, \Theta, \mathcal{F}, u)$.

### 3.2 Dynamic Cognitive Hierarchy Solution

Each player $i$ is endowed with a level of sophistication $\tau_i \in \mathbb{N}_0$. Let $\tau = (\tau_1, \ldots, \tau_n)$ be the level profile and $\tau_{-i}$ be the level profile without player $i$. The level profile is drawn from a distribution $P : \mathbb{N}_0^n \rightarrow [0, 1]$. Following Lin and Palfrey (2022), we assume $P$ has full support and it is independent across players. That is, $P(\tau) = \prod_{i=1}^{n} P_i(\tau_i)$ where $P_i$ is the marginal distribution of player $i$’s level of sophistication. Each level of each player has different prior beliefs about other players’ levels while the prior beliefs satisfy truncated rational expectations. That is, for each $i$, $j \neq i$, and $k$, let $\hat{P}^k_{ij}(\tau_j)$ be level $k$ player $i$’s prior belief about player $j$’s level, and $\hat{P}^k_{ij}(\tau_j)$ satisfies:

$$\hat{P}^k_{ij}(\tau_j) = \begin{cases} \frac{P_j(\tau_j)}{\sum_{m=0}^{k-1} P_j(m)} & \text{if } \tau_j < k \\ 0 & \text{if } \tau_j \geq k. \end{cases} \tag{1}$$

The intuition behind (1) is that all players have correct beliefs about the relative proportions of players who are less sophisticated than they are, while they incorrectly believe there does not exist any player who is more sophisticated than they are.
Therefore, in a multi-stage game with observed actions, every player has two pieces of private information—the payoff-relevant “type” and the “level of sophistication,” which are drawn from $\mathcal{F}$ and $P$, respectively. Moreover, we assume that every player’s type and level of sophistication are drawn independently.

**Assumption 1.** $\mathcal{F}$ and $P$ are independent distributions.

The dynamic cognitive hierarchy solution requires every level of players to best respond to their beliefs at every continuation game starting in each period $t$ after every possible history $h^{t-1}$. Since these continuation games themselves are not proper subgames, we still need to specify the beliefs at the beginning of each continuation game. Let $\mu_{i}^{k}(\theta_{-i}, \tau_{-i}|\theta_{i}, h^{t-1})$ be player $i$’s belief of other players’ types and levels of sophistication at history $h^{t-1}$, conditional on being type $\theta_{i}$ and level $k$. In addition, we use $\mu_{i}^{k}(\theta_{-i}|\theta_{i}, h^{t-1})$ and $\mu_{i}^{k}(\tau_{-i}|\theta_{i}, h^{t-1})$ to denote player $i$’s marginal beliefs of other players’ types and levels at history $h^{t-1}$ (conditional on $\theta_{i}$ and $k$), respectively. Also, for any $j \neq i$, we use $\mu_{ij}^{k}(\theta_{j}, \tau_{j}|\theta_{i}, h^{t-1})$ to denote player $i$’s belief about player $j$’s type and level at history $h^{t-1}$ conditional on being type $\theta_{i}$ and level $k$.

In the dynamic cognitive hierarchy solution, a strategy profile is a level-dependent profile of behavioral strategy of each level of each player. Let $\sigma_{i}^{k}$ be level $k$ player $i$’s behavioral strategy, where level 0 players uniformly randomize at every information set.\(^8\) That is, for every $i \in N$, $\theta_{i} \in \Theta_{i}$, $h \in \mathcal{H}\backslash\mathcal{H}^{T}$, and for all $a \in A_{i}(h)$,

$$\sigma_{i}^{0}(a|\theta_{i}, h) = \frac{1}{|A_{i}(h)|}.$$  

In the following, we may interchangeably call level 0 players non-strategic players and level $k \geq 1$ players strategic players.

Each player $i$ with level $k \geq 1$ and type $\theta_{i}$ will update their beliefs about all other players’ types and levels at every history.\(^9\) Their posterior beliefs at history $h^{t-1}$ depend on the level-dependent strategy profile and the prior beliefs. To formally characterize the belief updating process, we need to introduce some additional notations. Let $\sigma_{j}^{k} = (\sigma_{0}^{k}, \ldots, \sigma_{j-1}^{k})$ be the profile of strategies adopted by the levels below $k$ of player $j$. Furthermore, let $\sigma_{-i}^{k} = (\sigma_{k}^{k}, \ldots, \sigma_{i}^{k}, \sigma_{i+1}^{-k}, \ldots, \sigma_{n}^{-k})$ be the profile of behavioral strategies of the levels below $k$ of all players other than player $i$. Notice that all strategic players believe every history is possible since $\tau_{-i} = (0, \ldots, 0)$ is always possible. Therefore, every level of players can use Bayes’ rule to derive the posterior belief about other players’ types and levels. Specifically, for any $i \in N$, $k \geq 1$ and $\theta_{i} \in \Theta_{i}$, a level-dependent strategy profile will induce the posterior belief $\mu_{i}^{k}(\theta_{-i}, \tau_{-i}|\theta_{i}, h)$ at every $h \in \mathcal{H}\backslash\mathcal{H}^{T}$.

In the dynamic CH solution, players correctly anticipate how they will update their posterior beliefs at all future histories of the game. Therefore, for any $k \geq 1$, $i \in N$ and

\(^8\)This is a placeholder assumption for level 0 players’ behavior. Dynamic CH solution will be well-defined as long as level 0’s behavioral strategy is full support at every information. An alternative model of level 0 is to assume level 0 players are more likely to choose “salient” actions that will not lead to the outcome with the minimum payoff (see Chong et al. (2016) for details).

\(^9\)Level 1 players will always believe other players are non-strategic players, so they don’t update their beliefs about other players’ levels. However, they may update their beliefs about other players’ types.
\( \theta_i \in \Theta_i \), given any level-dependent strategy-profile \( \sigma_{-i}^k \), level \( k \) type \( \theta_i \) player \( i \) believes the probability of \( a_{-i}^k \in A_{-i}(h^{t-1}) \) being chosen is

\[
\tilde{\sigma}_{-i}^k(a_{-i}^k|\theta_i, h^{t-1}) = \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}; \tau_j < k \forall j \neq i\}} \mu_{i}^{k}(\theta_{-i}, \tau_{-i}|\theta_i, h^{t-1}) \prod_{j \neq i} \sigma_j^{T_i}(a_j^i|\theta_j, h^{t-1}).
\]

Furthermore, for every level of every player, given lower-level players’ strategies, they can compute the probability of any outcome being realized at any non-terminal history. Specifically, for any \( i \in N, \tau_i > 0, \theta \in \Theta, \sigma, \) and \( \tau_{-i} \) such that \( \tau_j < \tau_i \) for any \( j \neq i \), let \( P_{i}^{\tau_i}(h^{T}|h^{t-1}, \theta, \tau_{-i}, \sigma_{-i}^{-\tau_i}, \sigma_{i}^{\tau_i}) \) be level \( \tau_i \) player \( i \)'s belief about the conditional realization probability of \( h^{T} \in \mathcal{H}^T \) at history \( h^{t-1} \in \mathcal{H}\setminus\mathcal{H}^T \) if the type profile is \( \theta \), the level profile is \( \tau \), and player \( i \) uses \( \sigma_{i}^{\tau_i} \).

Finally, level \( \tau_i \) player \( i \) will use Bayes’ rule to derive the posterior belief in every information set. Thus, level \( \tau_i \) player \( i \)'s expected payoff at any \( h^{T} \in \mathcal{H}\setminus\mathcal{H}^T \) is given by:

\[
\mathbb{E}u_{i}^{\tau_i}(\sigma|\theta_i, h^{T}) = \sum_{h^{T} \in \mathcal{H}^{T}} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}; \tau_j < k \forall j \neq i\}} \mu_{i}^{\tau_i}(\theta_{-i}, \tau_{-i}|\theta_i, h^{T}) P_{i}^{\tau_i}(h^{T}|h^{t-1}, \theta, \tau_{-i}, \sigma_{-i}^{-\tau_i}, \sigma_{i}^{\tau_i}) u_{i}(h^{T}, \theta_i, \theta_{-i}).
\]

The dynamic CH solution is defined as the level-dependent assessment \((\sigma^{*}, \mu^{*})^{10}\), such that for any level \( k \) player \( i \), \( \mu_{i}^{k*} \) is computed by Bayes’ rule as other players are using \( \sigma_{-i}^{-k*} \), and for every \( i, k, \) and \( h^{T}, \sigma_{i}^{k*} \) maximizes player \( i \)'s expected payoff:

\[
\mathbb{E}u_{i}^{k*}(\sigma^{*}|\theta_i, h^{T}) = \sum_{h^{T} \in \mathcal{H}^{T}} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}; \tau_j < k \forall j \neq i\}} \mu_{i}^{k*}(\theta_{-i}, \tau_{-i}|\theta_i, h^{T}) P_{i}^{k*}(h^{T}|h^{t-1}, \theta, \tau_{-i}, \sigma_{-i}^{-k*}, \sigma_{i}^{k*}) u_{i}(h^{T}, \theta_i, \theta_{-i}).
\]

### 4 General Properties of the Belief Updating Process

In this section, we characterize some general properties of the belief updating process of the dynamic CH solution. Assuming the prior distributions of types and levels of sophistication are independent, we can first characterize the (posterior) belief at every information set recursively in Lemma 1.

**Lemma 1.** Consider any multi-stage game with observed actions \( \Gamma \), any \( i \in N, \theta_i \in \Theta_i \), \( h \in \mathcal{H}\setminus\mathcal{H}^T \), and every level \( k \in \mathbb{N} \). For every information set \( \mathcal{I}_i = (\theta_i, h) \), level \( k \) player \( i \)'s belief at \( \mathcal{I}_i \) can be characterized as follows.

1. Level \( k \) player \( i \)'s prior belief about other players’ types and levels are independent. That is, \( \mu_{i}^{k}(\theta_{-i}, \tau_{-i}|\theta_i, h_0) = \mathcal{F}(\theta_{-i}|\theta_i) \prod_{j \neq i} \hat{P}_{ij}^{k}(\tau_j) \).

---

10When players are indifferent, we follow Lin and Palfrey (2022), assuming they will uniformly randomize over optimal actions. This is a typical assumption in level \( k \) models, and it is convenient because it ensures a unique solution. In addition, although the dynamic CH solution is defined as a fixed point, it can be solved for recursively, starting with the lowest level and iteratively working up to higher levels.
Proposition 1. Consider any multi-stage game with observed actions $\tau$, any history $h \in H^t$, level $k$ player $i$’s belief at information set $(\theta_i, h^t)$ is

$$
\mu_i^k(\theta_{-i}, \tau_{-i}|\theta_i, h^t) = \frac{\mathcal{F}(\theta_{-i}|\theta_i) \prod_{j \neq i} \{\hat{P}_{ij}^k(\tau_j) \prod_{l=1}^{t} \sigma_j^r(a_l^j|\theta_j, h^{l-1})\}}{\sum_{\theta'_{-i}} \sum_{\{\tau'_{-i}: \tau_j < k \forall j \neq i\}} \mathcal{F}(\theta'_{-i}|\theta_i) \prod_{j \neq i} \{\hat{P}_{ij}^k(\tau'_j) \prod_{l=1}^{t} \sigma_j^r(a_l^j|\theta'_j, h^{l-1})\}}.
$$

Proof. See Appendix A. \hfill \square

One property of the dynamic CH solution is that in the later histories, the support of the posterior beliefs is (weakly) shrinking (Lin and Palfrey (2022), Proposition 2). In dynamic games of incomplete information, players will learn other players’ types and levels in every subgame. Proposition 1 shows that the marginal belief about other players’ types always have full support. The intuition of this result is that when history gets longer, players will rule out the possibility of level profiles but not type profiles. To state the proposition, we first define the support of the marginal beliefs.

Definition 1 (Support). For any multi-stage game with observed actions $\Gamma$, any $i \in N$, any $\tau_i \in \mathbb{N}$, any $\theta_i \in \Theta_i$, and any history $h \in H^t$, let $\text{supp}_i(\theta_{-i}|\tau_i, \theta_i, h)$ and $\text{supp}_i(\tau_{-i}|\tau_i, \theta_i, h)$ be the support of level $\tau_i$ player $i$’s marginal belief about other players’ types and levels at information set $(\theta_i, h)$, respectively. In other words, for any $\theta_{-i}$ and $\tau_{-i}$,

$$
\theta_{-i} \in \text{supp}_i(\theta_{-i}|\tau_i, \theta_i, h) \iff \sum_{\{\tau_{-i}: \tau_j < \tau_i \forall j \neq i\}} \mu_i^r(\theta_{-i}, \tau_{-i}|\theta_i, h) > 0,
$$

$$
\tau_{-i} \in \text{supp}_i(\tau_{-i}|\tau_i, \theta_i, h) \iff \sum_{\theta_{-i} \in \Theta_{-i}} \mu_i^r(\theta_{-i}, \tau_{-i}|\theta_i, h) > 0.
$$

Proposition 1. Consider any multi-stage game with observed actions $\Gamma$, any $i \in N$, any $\tau_i \in \mathbb{N}$, and any $\theta_i \in \Theta_i$. The following two statements hold.

1. For any $h^t = (h^{t-1}, a^t) \in H^t \setminus H^T$, $\text{supp}_i(\tau_{-i}|\tau_i, \theta_i, h^t) \subseteq \text{supp}_i(\tau_{-i}|\tau_i, \theta_i, h^{l-1})$.

2. For any $h \in H \setminus H^T$. $\text{supp}_i(\theta_{-i}|\tau_i, \theta_i, h) = \Theta_{-i}$.

Proof. See Appendix A. \hfill \square

The intuition of Proposition 1 is that since it is always possible for other players to be level 0, players can always rationalize any type profile by assuming all other players are level 0. Notice that this argument relies on two underlying assumptions. First, the action sets are independent of types. If this assumption doesn’t hold, it is possible to find some history that can only be created by some specific type profile. Consequently, players would rule out the possibility of some other type profiles when such history is realized. Second, the horizons are finite. For infinite horizon games, consider a history that can be realized only if some type of player chooses infinitely many dominated actions. At such history, that type will be eliminated from the support since the possibility of level 0 converges to 0 when the length of history is infinite.
In the following, we further assume every player’s type is independently drawn. That is, \( F(\theta) = \prod_{i \in N} F_i(\theta_i) \). In this case, the belief updating process will satisfy a particular independence property. Proposition 2 establishes that at every information set, the posterior beliefs are independent across players.

**Proposition 2.** For any multi-stage game with observed actions \( \Gamma \), any \( h \in \mathcal{H} \setminus \mathcal{H}^T \), any \( i \in N \), \( \theta_i \in \Theta_i \), and for any \( k \in \mathbb{N} \), if the prior distribution of types is independent across players, i.e., \( F(\theta) = \prod_{i=1}^n F_i(\theta_i) \), then level \( k \) player \( i \)'s posterior belief about other players’ types and levels at \( h \) is independent across players. That is,

\[
\mu^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h) = \prod_{j \neq i} \mu^k_{ij}(\theta_j, \tau_j|\theta_i, h).
\]

**Proof.** See Appendix A. \( \square \)

Proposition 2 is an analogous property of the “no-signaling-what-you-don’t-know” condition of perfect Bayesian equilibrium (see Fudenberg and Tirole (1991b)). Since the prior distribution is independent, and the past history is public information at the beginning of each period, each player’s belief will remain independent across players. In other words, when the types are drawn independently, each player’s action does not convey any information about other players’ private information. Proposition 2 demonstrates the independence does not only hold in equilibrium, but also hold in the dynamic CH solution. This property does not rely on Assumption 1—it would hold as long as the priors of types and levels are both independent across players. Finally, it is worth noticing that it is a useful property for solving the dynamic CH solution when the game structure is really complex or when there are a lot of players. In these cases, it is easier to compute the posterior beliefs by each player rather than by each stage.

Although every level of players’ posterior beliefs about others are independent across players, the belief of any other player’s type and level is generally correlated. As players start observing the histories, they will learn the types and levels at the same time, causing these two dimensions to become correlated. In Section 5, we will discuss how the beliefs of types and levels are correlated in details in the context of dirty faces games.

To conclude this section, we analyze the case where every player’s type is not drawn independently. When players’ types are correlated, their actions may signal not only their own types but also those of players whose types are correlated with them. Similar to the observations of Myerson (1985) and Fudenberg and Tirole (1991b), to deal with correlated types, we can simply transform the original game (correlated types) into one game with independent types. After solving the transformed games (independent types), we then map the solution back to the original game.

Proposition 3 shows that the dynamic CH solution is invariant in the transformed and the original game. The insight of this result is that the independence assumption of the types is without loss of generality. Moreover, since the types and the levels are drawn independently, the transformation is in fact level-independent. To this end, the dynamic CH solution closely mirrors the equilibrium model.
Specifically, for any multi-stage game with observed actions $\Gamma$, we can consider a corresponding transformed game $\hat{\Gamma}$ where the prior distribution of types is the product of independent uniform marginal distributions. Namely,

$$\hat{F}(\theta) = \frac{1}{\prod_{i=1}^{n} |\Theta_i|} \quad \forall \theta \in \Theta.$$ 

In addition, we can transform the utility functions to be

$$\hat{u}_i(h^T, \theta_i, \theta_{-i}) = \hat{F}(\theta_{-i}|\theta_i)u_i(h^T, \theta_i, \theta_{-i}).$$

**Proposition 3.** The level-dependent assessment $(\hat{\sigma}, \hat{\mu})$ is the dynamic CH solution of the transformed (independent types) game if and only if the level-dependent assessment $(\sigma, \mu)$ is the dynamic CH solution of the original (correlated types) game where $\sigma = \hat{\sigma}$ and for any $i \in N, \theta_i \in \Theta_i, k > 0$, and $h^t \in H \setminus H^T$,

$$\mu^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t) = \frac{\hat{F}(\theta_{-i}|\theta_i)\hat{\mu}^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t)}{\sum_{\theta'_i} \sum_{\{\tau'_{-i}, \tau'_j < k \forall j \neq i\}} \hat{F}(\theta'_{-i}|\theta_i)\hat{\mu}^k_i(\theta'_{-i}, \tau'_{-i}|\theta_i, h^t)}.$$

*Proof.* See Appendix A.

## 5 Dirty Faces Games: Theory

The dirty faces game is first described by Littlewood (1953) to study the relationship between common knowledge and actions. In an earlier version of the puzzle, Littlewood describes an incident where three ladies—all with dirty faces yet unable to see their own faces—are sitting in a railway carriage and laugh at each other. These ladies will laugh at anyone with a dirty face, but stop laughing when they realize their own faces are dirty. As each lady realizes other ladies do not stop laughing, they suddenly infer that their own faces must be dirty. The reason is that, as argued by (Littlewood (1953), pp. 4): “If I, A, am not laughable, B will be arguing: if I, B, am not laughable, C has nothing to laugh at. Since B does not so argue, I, A, must be laughable.”

Notice that this logic is extremely bold that does not rely on any structural assumption. The argument is independent of the payoffs, the timing, and the (prior) probability of having a dirty face. Moreover, it can be easily extended to $n$-lady scenarios. The only piece of information required is that there is a public laughter who will laugh whenever there is at least one lady having a dirty face.

To analyze the dirty faces game by the standard game theory approach, we slightly reframe the game into the environment introduced by Fudenberg and Tirole (1991a), which is also been experimentally studied by Bayer and Chan (2007). Let $N = \{1, 2, \ldots, n\}$ be the

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11The dirty faces game has also been reframed as the “cheating wives puzzle” (Gamow and Stern, 1958), the “cheating husbands puzzle” (Moses et al., 1986), the “muddy children puzzle” (Barwise, 1981) and (Halpern and Moses, 1990), and the “red hat puzzle” (Hardin and Taylor, 2008).
set of players. For each $i \in N$, let $x_i \in \{O, X\}$ represent whether the player has a clean face ($O$) or a dirty face ($X$). Each player’s face type is independently and identically determined by a commonly known probability $p = \Pr(x_i = X) = 1 - \Pr(x_i = O)$.\footnote{Notice that how the face types are drawn does not play any role in Littlewood’s argument. Therefore, the equilibrium analysis remains the same if players’ face types are correlated.} Each player $i$ can observe other players’ faces $x_{-i}$ but not their own faces.\footnote{To fit into the framework, each player’s “type” (their own private information) can be specified as “other players’ faces.” That is, $\theta_i = x_{-i}$.} If there is at least one player having a dirty face, there will be a public announcement to all players at the beginning of the game. Let $a \in \{0, 1\}$ denote the event of whether there is an announcement. If there is an announcement ($a = 1$), all players are informed there is at least one dirty face but not the identities.

There are up to $T \geq 2$ periods. In each period, every player $i$ simultaneously chooses $s_i \in \{U, D\}$. The game ends after any period where any player chooses $D$. Furthermore, their actions are revealed at the end of each period (so this is a multi-stage game with observed actions). Finally, as the game proceeds to period $t \leq T$, the payoffs depend on their own face types and actions. As some player chooses $D$, he will get $\alpha > 0$ if he has a dirty face while receive $-1$ if he has a clean face. We assume that

$$p\alpha - (1 - p) < 0 \iff 0 < \alpha < (1 - p)/p, \quad (2)$$

where $p\alpha - (1 - p)$ is the expected payoff of $D$ when the belief of having a dirty face is $p$. Thus, Assumption (2) guarantees it is strictly dominated to choose $D$ in period 1 when observing at least one dirty face. In other words, players will be rewarded when correctly inferring the dirty face but penalized when wrongly claiming the dirty face. Besides, the payoffs will be discounted with a common discount factor $\delta \in (0, 1)$. To summarize, conditional on reaching period $t$, each player’s payoff function (which depends on their own faces) can be written as:

$$u_i(s_i|t, X) = \begin{cases} \delta^{t-1}\alpha & \text{if } s_i = D \\ 0 & \text{if } s_i = U \end{cases} \quad \text{and} \quad u_i(s_i|t, O) = \begin{cases} -\delta^{t-1} & \text{if } s_i = D \\ 0 & \text{if } s_i = U. \end{cases}$$

Therefore, a dirty faces game is defined by a tuple $\langle N, p, T, \alpha, \delta \rangle$.

To better understand the dirty faces game, Figure 1 plots the game tree of a two-person two-period dirty faces game. At the beginning, Nature will randomly determine each player $i$’s face type with probability $\Pr(x_i = X) = p$. There are four possible realizations, and we use $(x_1, x_2)$ to denote the profile of face types. We ignore the case where $(x_1, x_2) = OO$ because in this case, it is common knowledge to both players that their faces are clean. From this game tree, we can see how complicated the dirty faces games are—even in the simplest case (two-person two-period games), there are multiple non-singleton information sets where players need to form correct posterior beliefs. Despite the complexity of the game structure, there is a unique equilibrium.
Figure 1: The game tree of a two-person two-period dirty faces game. Here we omit the case of OO since it is common knowledge to all players that both of their faces are clean.
To solve for the equilibrium, we assume from now on that there is a public announcement. Otherwise, it is common knowledge to all players that their faces are clean. In this case, let
\[ 0 \leq k_i \leq n - 1 \]
be the number of dirty faces observed by player \( i \). Then for any \( i \in N \) and any \( k_i \), the unique Nash equilibrium (and hence perfect Bayesian equilibrium) is that player \( i \) will choose \( U \) in periods \( t < k_i + 1 \), and \( D \) in periods \( t \geq k_i + 1 \).

To see this, we can proceed by induction on the number of observed dirty faces. If the player doesn’t observe any dirty face, he knows his face is dirty at the beginning. Hence, he will choose \( D \) from the first period (because of discounting). Since all players know that their opponents know the game structure, all players know that a player will choose \( D \) whenever one doesn’t observe any dirty face. Therefore, if the game proceeds to period 2, it is common knowledge to all players that there are at least two dirty faces. Continuing the same argument, if the game proceeds to period \( k \), then it is common knowledge to all players that there are at least \( k \) dirty faces. Therefore, if player \( i \) observes \( k \) dirty faces, he knows the game will end at period \( k \) if his face is clean. Yet, if the game proceeds to period \( k + 1 \), he will realize that his face is dirty for sure, and choose \( D \) from period \( k + 1 \).

Notice that the common knowledge of rationality is critical for reaching the equilibrium. Without common knowledge of rationality, the failure of choosing \( D \) is not necessarily caused by observing that many dirty faces. Instead, it is possibly because of lack of rationality or because they don’t believe other players are not rational. In fact, as documented in previous dirty faces game experiments, common knowledge of rationality is an empirically implausible assumption. To bridge the gap between the theory and experimental results, we apply the dynamic cognitive hierarchies to the dirty faces games. In this section, to avoid the intuition from being blurred by the algebra, we focus on the analysis of two-person dirty faces games. We defer the analysis of three-person games to Appendix C.

5.1 Dynamic CH Solution for Two-Person Dirty Faces Game

In two-person dirty faces games, let \( N = \{1, 2\} \) be the set of players. Hence, any two-person dirty faces game can be described by the four-tuple \( \langle p, T, \alpha, \delta \rangle \) where \( p, \delta \in (0, 1) \), \( T \geq 2 \), and Assumption (2) is satisfied. We use \( D_2 \) to denote the set of two-person dirty faces games.

In a two-person dirty faces game, given there is a public announcement, each player’s information sets can be described by the period and the other player’s face type since the game can proceed to the next period only if all players choose \( U \) in previous periods. As a result, each player \( i \)’s behavioral strategy can be represented by:

\[
\sigma_i : \{1, \ldots, T\} \times \{O, X\} \rightarrow [0, 1],
\]

which is a mapping from the period and the observed face to the probability of choosing \( D \). As analyzed above, the unique equilibrium is that players will choose \( D \) in period 1 when \( x_{-i} = O \), while choose \( U \) in period 1 and \( D \) in period 2 when \( x_{-i} = X \).

In the dynamic CH solution, let player \( i \)’s level of sophistication be independently and identically drawn from the distribution \( p = (p_k)_{k=0}^\infty \) where \( p_k > 0 \) for all \( k \). We maintain the assumption that the distribution of face types and the distribution of levels are independent. Each player’s optimal behavioral strategy is level-dependent. We denote level \( k \) player \( i \)’s
strategy as $\sigma^k_i$. Following previous notations, we let $\mu^k_i(x_i, \tau_i|t, x_i)$ be level $k$ player $i$’s belief about their own face and the other player’s level of sophistication conditional on observing $x_{-i}$ and being at period $t$. Level 0 players will uniformly randomize everywhere, so $\sigma^0_i(t, x_{-i}) = 1/2$ for all $t, x_{-i}$.

Proposition 4 fully characterizes the dynamic CH solution. The intuition is straightforward. When a player observes a clean face, one can immediately figure out the face type. Therefore, the prediction of dynamic CH coincides with the equilibrium when $x_{-i} = O$. However, when a player observes a dirty face and $U$ in period 1, he cannot tell whether he has a dirty face or not for sure. Instead, he will believe that he is more likely to have a dirty face as the game continues. As a result, conditional on observing a dirty face, players will claim having a dirty face as long as the reward $\alpha$ is high enough or the discount rate $\delta$ is sufficiently low. Otherwise, they will wait for more evidence.

**Proposition 4.** For any two-person dirty faces game, the level-dependent strategy profile of the dynamic CH solution can be characterized as following. For any $i \in N$,

1. $\sigma_i^k(t, O) = 1$ for any $k \geq 1$ and $1 \leq t \leq T$.

2. $\sigma_i^1(t, X) = 0$ for any $1 \leq t \leq T$. Moreover, for any $k \geq 2$,

   (1) $\sigma_i^k(1, X) = 0$,

   (2) for any $2 \leq t \leq T - 1$, $\sigma_i^k(t, X) = 1$ if and only if

   $$\alpha \geq \left(1 - \frac{p}{p_0}\right) \left(\frac{\left(\frac{1}{2}\right)^{t-1} - \left(\frac{1}{2}\right)^{t} \delta}{p_0 + (1 - \delta) \sum_{j=1}^{k-1} p_j}\right),$$

   (3) $\sigma_i^k(T, X) = 1$ if and only if

   $$\alpha \geq \left(1 - \frac{p}{p_0}\right) \left(\frac{\left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-1} p_0 + \sum_{j=1}^{k-1} p_j}\right).$$

**Proof.** See Appendix B.

To better understand this result, we can focus on the analysis of level 2 players who will best respond to a mixture of level 0 and level 1 players. Level 1 players believe the other player is non-strategic. The only useful information for level 1 players is the announcement and what they have observed—the other player’s action doesn’t convey any information about the face type. Namely, level 1 players would view the decision at every period as the same problem as in period 1, and make the same choice as in period 1. As a result, level 1 players will always choose $U$ when $x_{-i} = X$, and always choose $D$ when $x_{-i} = O$.

When observing a clean face, level 2 players will know their face type immediately, and they know level 1 players would know when observing a clean face. On the other hand, when observing a dirty face, level 2 players will use their prior beliefs to make inferences at the
first period, and choose $U$. As the game proceeds to period 2, level 2 players will know it is impossible that the other player is level 1 and observes a dirty face. Specifically, the posterior belief $\mu^2_i(x_i, \tau_{-i}|2, X)$ is

\[
\mu^2_i(X, 0|2, X) = \frac{\left(\frac{1}{2}\right)p p_0}{\left(\frac{1}{2}\right) p_0 + p p_1}, \quad \mu^2_i(O, 0|2, X) = \left(\frac{1}{2}\right) (1 - p) p_0 \left(\frac{1}{2}\right) p_0 + p p_1,
\]

\[
\mu^2_i(X, 1|2, X) = \frac{p p_1}{\left(\frac{1}{2}\right) p_0 + p p_1}, \quad \mu^2_i(O, 1|2, X) = 0.
\]

At period 2, level 2 players would believe there is only 50% chance that the game continues to this period if the other player is level 0. In addition, they believe the game will end in period 1 if $(x_i, \tau_{-i}) = (O, 1)$ while the game will continue to period 2 if $(x_i, \tau_{-i}) = (X, 1)$. Therefore, the marginal probability of having a dirty face is

\[
\mu^2_i(X|2, X) = \mu^2_i(X, 0|2, X) + \mu^2_i(X, 1|2, X) = \frac{p \left[\left(\frac{1}{2}\right) p_0 + p_1\right]}{\left(\frac{1}{2}\right) p_0 + p p_1} > p,
\]

suggesting that level 2 players would think their face is more likely to be dirty.

Since the impossibility of $(x_i, \tau_{-i}) = (O, 1)$ has been updated in period 2, the only information to learn in later periods is that the other player is less likely to be level 0 if the game continues. For any period $2 \leq t \leq T$, the marginal probability of having a dirty face is

\[
\mu^2_i(X|t, X) = \frac{p \left[\left(\frac{1}{2}\right)^{t-1} p_0 + p_1\right]}{\left(\frac{1}{2}\right)^{t-1} p_0 + p p_1},
\]

which is an increasing function of $t$. This suggests that in later periods, level 2 players are more certain about having a dirty face. In other words, level 2 players can benefit from waiting in order to get more information. Yet, the risks of choosing $U$ are that the other player may (randomly) end the game and the utility will be discounted. Hence, this problem becomes a sequential sampling problem similar to Wald (1947) where level 2 players decide when to stop sampling and claim the dirty face.

To solve the problem, we need to calculate level 2 player’s expected payoff. For any $2 \leq t \leq T$, level 2 player $i$’s expected payoff of choosing $D$ at period $t$ is

\[
\mathbb{E}u^2_i(D|t) := \delta^{t-1} \left[\alpha \mu^2_i(X|t, X) - \mu^2_i(O|t, X)\right].
\]

Note that since period $T$ is the last period, it is optimal for level 2 players to choose $D$ if and only if

\[
\mathbb{E}u^2_i(D|T) \geq 0 \iff \alpha \geq \frac{\mu^2_i(O|T, X)}{\mu^2_i(X|T, X)} = \left(\frac{1 - p}{p}\right) \left(\frac{\left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-1} p_0 + p_1}\right).
\]

For other periods between 2 and $T - 1$, it is optimal to choose $D$ at some period $t$ only if

\[
\mathbb{E}u^2_i(D|t) \geq \Pr(t + 1|t, X)\mathbb{E}u^2_i(D|t + 1),
\]

17
where \( \Pr(t+1|t, X) \) is level 2 player 2’s belief of the probability that the other player would choose \( U \) in period \( t \). As we rearrange the inequality, we can obtain the condition stated in Proposition 4. The proof in Appendix B shows the condition is not only necessary but sufficient to make level 2 players to choose \( D \) in period \( t \). Moreover, we prove the statement by induction on \( k \), demonstrating that every level \( k > 2 \) player is in fact facing with a similar sequential sampling problem as level 2 players.

Proposition 4 characterizes the level-dependent behavioral strategies. Alternatively, we can characterize the solution by computing the level-dependent stopping period for any \( x_{-i} \).

**Definition 2 (Stopping Period).** For any two-person dirty faces games and its dynamic CH solution, let \( \hat{\sigma}^k_i(x_{-i}) \) be level \( k \) player \( i \)'s earliest period to choose \( D \) conditional on observing \( x_{-i} \) for any \( k \geq 1 \) and \( i \in N \). Specifically,

\[
\hat{\sigma}^k_i(x_{-i}) = \begin{cases} 
\arg\min_t \{ \sigma^k_i(t, x_{-i}) = 1 \}, & \text{if } \exists t \text{ s.t. } \sigma^k_i(t, x_{-i}) = 1 \\
T + 1, & \text{otherwise.}
\end{cases}
\]

With Proposition 4, Corollary 1 follows directly. If \( x_{-i} = O \), every strategic level of players would always choose \( D \). Therefore, \( \hat{\sigma}^k_i(O) = 1 \) for every \( k \geq 1 \) as \( \sigma^k_i(t, O) = 1 \) for every \( 1 \leq t \leq T \) and \( k \geq 1 \). Focusing on the case where \( x_{-i} = X \), Corollary 1 shows that the optimal stopping period is decreasing in \( k \). In other words, high-level players are more rational in the sense that they can figure out their face type in fewer stages.

**Corollary 1.** For any two-person dirty faces game, the level-dependent strategy profile of the dynamic CH solution can be equivalently characterized by level-dependent optimal stopping periods. For any \( i \in N \),

1. \( \hat{\sigma}^k_i(O) = 1 \);

2. \( \hat{\sigma}^1_i(X) = T + 1 \) and \( \hat{\sigma}^k_i(X) \geq 2 \) for all \( k \geq 2 \). Moreover, level \( k \geq 2 \) players’ optimal stopping periods are weakly decreasing in \( k \).

**Proof.** By Proposition 4, we know \( \sigma^k_i(t, 0) = 1 \) for all \( t \) and \( k \geq 1 \), and \( \sigma^1_i(t, X) = 0 \) for all \( t \). Then by Definition 2, we can obtain that \( \hat{\sigma}^k_i(O) = 1 \) for every \( k \geq 1 \), and \( \hat{\sigma}^1_i(X) = T + 1 \). In addition, since \( \sigma^k_i(1, X) = 0 \) for all \( k \geq 2 \), \( \hat{\sigma}^k_i(X) \neq 1 \). Moreover, we can characterize the dynamic CH solution by the optimal stopping period because for any \( t \geq 2 \) and \( k \geq 2 \),

\[
\hat{\sigma}^k_i(X) = t \iff \sigma^k_i(t - 1, X) = 0 \quad \text{and} \quad \sigma^k_i(t, X) = 1,
\]

\[
\hat{\sigma}^k_i(X) = T + 1 \iff \sigma^k_i(t', X) = 0 \quad \text{for any } 1 \leq t' \leq T.
\]

\[\text{Specifically, } \Pr(t + 1|t, X) \text{ can be calculated from level 2 player’s posterior belief where} \]

\[
\Pr(t + 1|t, X) = \frac{1}{2} \mu^2_2(0|t, X) + \mu^2_2(1|t, X) = \frac{(1/2)^t p_0 + p_1}{(1/2)^t p_0 + p_1}.
\]
Finally, to show the monotonicity, it suffices to show that for any \( k' > k \geq 2 \) and any \( 2 \leq t \leq T \), if \( \sigma^k_i(t, X) = 1 \), then \( \sigma^{k'}_i(t, X) = 1 \). We can separate the discussion into two cases. First, if \( t = T \), then by Proposition 4, we know \( \sigma^k_i(T, X) = 1 \) suggests

\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} \right)^{T-1} p_0}{\left( \frac{1}{2} \right)^{T-1} p_0 + \sum_{j=1}^{k-1} p_j} \right) > \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} \right)^{T-1} p_0}{\left( \frac{1}{2} \right)^{T-1} p_0 + \sum_{j=1}^{k'-1} p_j} \right),
\]

which implies \( \sigma^{k'}_i(T, X) = 1 \). Second, for any \( 2 \leq t \leq T - 1 \), we can obtain from Proposition 4 that \( \sigma^k_i(t, X) = 1 \) suggests

\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta}{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta} \right) p_0 \left( \frac{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta}{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta} \right) p_0 + (1 - \delta) \sum_{j=1}^{k-1} p_j > \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta}{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta} \right) p_0 \left( \frac{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta}{\left( \frac{1}{2} \right)^{t-1} - \left( \frac{1}{2} \right)^t \delta} \right) p_0 + (1 - \delta) \sum_{j=1}^{k'-1} p_j,
\]

implying that \( \sigma^{k'}_i(t, X) = 1 \). This completes the proof. \( \square \)

### Visualization

To summarize the characterization of the dynamic CH solution, we illustrate the optimal stopping periods for level 2 and level infinity players when \( x_{-i} = X \). For illustrative purposes, we assume \( p = 0.5 \) and \( T = 5 \) so that the set of two-person dirty faces games \( \mathcal{D}_2 \) can simply be described by two parameters \((\delta, \alpha)\). When \( p = 0.5 \), Assumption (2) is equivalent to \( 0 < \alpha < 1 \). Therefore, \( \mathcal{D}_2 \) is the unit square on the \((\delta, \alpha)\)-plane.

In addition, we assume the distribution of levels follows Poisson\((1.5)\), which is a “good omnibus guess” of prior according to Camerer et al. \(2004\). Once the distribution of levels is specified, we can solve for the dynamic CH solution (and the level-dependent optimal stopping periods) by Proposition 4. Figure 2 plots the level-dependent optimal stopping periods for level 2 and level infinity players in the left and right panel, respectively. From this figure, we can find that the optimal stopping periods form a partition of the set of dirty faces games. For instance, level 2 players would choose \( D \) at period 2 if and only if

\[
\alpha \geq \left( \frac{1 - p}{p} \right) \frac{\left( \frac{1}{2} - \frac{1}{4} \delta \right) p_0}{\left( \frac{1}{2} - \frac{1}{4} \delta \right) p_0 + (1 - \delta) p_1} = \frac{\left( \frac{1}{2} - \frac{1}{4} \delta \right) e^{-1.5}}{\left( \frac{1}{2} - \frac{1}{4} \delta \right) e^{-1.5} + (1 - \delta) 1.5 e^{-1.5}} = \frac{2 - \delta}{8 - 7\delta},
\]

which corresponds to the red area in the left panel.

Focusing on level 2 players (left panel), we can find that dynamic CH model predicts that it is possible for them to choose any stopping period in \( \{2, 3, 4, 5, 6\} \), depending on the parameters \( \alpha \) and \( \delta \). This result contrasts to the equilibrium which predicts they will choose 2 in any dirty faces game. More surprisingly, dynamic CH model makes similar predictions even for level infinity players (right panel)—no matter how sophisticated the players are,
their behavior is dependent with the parameters. This is because they always believe it is possible that the other player’s action is due to randomness. Therefore, whenever the reward is not high enough, every level of players would have the incentive to strategically delay in order to get more information.

Figure 2: Optimal stopping periods for level 2 (left) and level ∞ players (right) as \( x - i = X \) where \( p = 0.5, T = 5 \), and the levels are drawn from Poisson(1.5).

5.2 Static CH Solution for Two-Person Dirty Faces Game

Since the static CH solution is defined on simultaneous move games, to solve the static CH for dirty faces games, we need to transform the original game into its corresponding reduced normal form. That is, the dirty faces game is alternatively specified as a static Bayesian game where players simultaneously decide the earliest period to choose \( D \) conditional on \( x - i \). Specifically, each player has \( T + 1 \) actions, corresponding to the earliest period to choose \( D \) or never \( D \). Hence, the set of actions is \( S = \{1, 2, \ldots, T, T + 1\} \) where \( T + 1 \) corresponds to never \( D \). In any strategic form two-person dirty faces game, a mixed strategy for player \( i \) is a function from the other player’s face type to a probability distribution over the action space, i.e.,

\[
\tilde{\sigma}_i : \{O, X\} \rightarrow \Delta(S).
\]

In the static CH solution, players’ strategies are also level-dependent. We use \( \tilde{\sigma}^k_i(x - i) \) to denote level \( k \) player \( i \)'s strategy. Following the analysis of the dynamic CH solution, we let the level of sophistication of each player be independently and identically drawn from the distribution \( p = (p_k)_{k=0}^\infty \) where \( p_k > 0 \) for all \( k \) and keep assuming the levels and face types are independent. Level 0 players will uniformly randomize at every state, so \( \tilde{\sigma}_i^0(x - i) = \frac{1}{T+1} \) for all \( i, x - i \). In the static CH solution, level \( k \geq 1 \) players will generically choose pure
strategies. Therefore, in the following, we slightly abuse the notation to use \( \tilde{\sigma}_k^i(x_{-i}) \) to denote the pure strategies.\(^{15}\)

Here we emphasize the two main differences between the dynamic and static CH solutions. First, level 0 players are mixing on different sets of actions. The static CH solution is defined on strategic form games, and hence level 0 players will randomize on the set of contingent strategies. On the contrary, level 0 players in the dynamic CH solution will randomize at every information set. Although level 0 players uniformly randomize in both CH solutions, this difference will make level 0 players to have generically different choice probabilities conditional on reaching each information set.\(^{16}\)

Second, the spirits of “learning” are different in the dynamic and static CH solutions. In the dynamic CH solution, players can observe other players’ past actions, and make statistical inferences on other players’ levels of sophistication and basic game structures. On the other hand, players in the static CH cannot observe past actions (since it is defined on simultaneous games), they have to make decisions conditional on hypothetical events. As experimentally examined by Esponda and Vespa (2014), making decisions conditional on hypothetical events and extracting information from opponents’ strategies are behaviorally different types of learning.

In the following, we solve for the static CH solution of the two-person dirty faces games. Notice that the equilibrium analysis of the strategic form is essentially the same as the analysis of the extensive form. When observing a clean face, players can learn that they have a dirty face immediately, and hence choose 1. On the other hand, when observing a dirty face, the unique equilibrium predicts players would choose 2, i.e., choosing \( U \) in period 1 and \( D \) in period 2.

Proposition 5 is parallel to Proposition 4 that characterizes the static CH solution. The intuition is also similar to the dynamic CH solution. When observing a clean face, players can figure out their face types immediately. Hence, they will choose the strictly dominant strategy \( \tilde{\sigma}_1^i(O) = 1 \) for all \( k \geq 1 \). On the other hand, when observing a dirty face, players have to make similar inferences as before, but the only available information now is the prior belief and what they observe at the beginning. Players cannot update the other player’s level of sophistication from the history—they can only make inferences hypothetically.

**Proposition 5.** For any two-person dirty faces game, the static CH solution can be characterized as following. For any \( i \in N \),

1. \( \tilde{\sigma}_1^i(O) = 1 \) for any \( k \geq 1 \).
2. \( \tilde{\sigma}_1^i(X) = T + 1 \). Moreover, for any \( k \geq 2 \),

\[^{15}\text{Specifically, for any } t \in \{1, 2, \ldots, T, T + 1\}, \text{ we use } \tilde{\sigma}_k^i(x_{-i}) = t \text{ to denote the degenerated distribution: } \tilde{\sigma}_k^i(x_{-i})(t) = 1, \text{ and } \tilde{\sigma}_k^i(x_{-i})(t') = 0 \forall t' \neq t.\]

\[^{16}\text{For instance, in the first period of the dirty faces game, the dynamic CH solution predicts level 0 players will choose } D \text{ with probability } 1/2, \text{ while the static CH solution predicts the probability of level 0 players choosing } D \text{ is } 1/(T + 1).\]
(1) $\tilde{\sigma}_i^k(X) \geq 2$,

(2) for any $2 \leq t \leq T - 1$, $\tilde{\sigma}_i^k(X) \leq t$ if and only if

$$\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{T + 2 - t}{T + 1} - \frac{T + 1 - t}{T + 1} \delta}{1 - \frac{T + 1}{T + 1} \delta} p_0 + (1 - \delta) \sum_{j=1}^{k-1} p_j \right),$$

(3) $\tilde{\sigma}_i^k(X) \leq T$ if and only if

$$\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{2}{T + 1} p_0}{\frac{2}{T + 1} p_0 + \sum_{j=1}^{k-1} p_j} \right).$$

\textbf{Proof.} See Appendix B. \qed

The result is similar to Proposition 4 in the sense that when $x_{-i} = X$, strategic players will not choose 1; instead, they will choose to claim a dirty face at period $t$ if and only if the reward $\alpha$ is sufficiently high or the player is impatient enough. However, the critical level of $\alpha$ is different. In the static CH solution, the critical level of $\alpha$ depends on the horizon $T$ while it is independent of $T$ in the dynamic CH solution. We will discuss this contrast later.

Although the dynamic CH solution and the static CH solution are quantitatively different, we can still obtain the similar qualitative result as Corollary 1—higher-level players claim having a dirty face earlier than lower-level players.

\textbf{Corollary 2.} For any strategic form two-person dirty faces game, any $k \geq 2$ and any $i \in N$, when $x_{-i} = X$, the static CH solution $\tilde{\sigma}_i^k(X) \geq 2$ and is weakly decreasing in $k$.

\textbf{Proof.} It suffices to prove the monotonicity by showing for all $k' > k \geq 2$, if $\tilde{\sigma}_i^k(X) \leq t$, then $\tilde{\sigma}_i^{k'}(X) \leq t$ for any $2 \leq t \leq T$. We can separate the analysis into two cases. First, if $t = T$, then by Proposition 5, we know $\tilde{\sigma}_i^k(X) \leq T$ suggests

$$\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{2}{T + 1} p_0}{\frac{2}{T + 1} p_0 + \sum_{j=1}^{k-1} p_j} \right) > \left( \frac{1 - p}{p} \right) \left( \frac{\frac{2}{T + 1} p_0}{\frac{2}{T + 1} p_0 + \sum_{j=1}^{k'-1} p_j} \right),$$

which implies $\tilde{\sigma}_i^{k'}(X) \leq T$. Second, for any $2 \leq t \leq T - 1$, we can obtain from Proposition 5 that $\tilde{\sigma}_i^k(X) \leq t$ suggests

$$\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{T + 2 - t}{T + 1} - \frac{T + 1 - t}{T + 1} \delta}{1 - \frac{T + 1}{T + 1} \delta} p_0 + (1 - \delta) \sum_{j=1}^{k-1} p_j \right) > \left( \frac{1 - p}{p} \right) \left( \frac{\frac{T + 2 - t}{T + 1} - \frac{T + 1 - t}{T + 1} \delta}{1 - \frac{T + 1}{T + 1} \delta} p_0 + (1 - \delta) \sum_{j=1}^{k'-1} p_j \right),$$

implying $\tilde{\sigma}_i^{k'}(X) \leq t$ by Proposition 5. This completes the proof. \qed
Visualization

Similar to the analysis of the dynamic CH solution, we illustrate the static CH optimal strategies (stopping periods) for level 2 and level infinity players when $x_{-i} = X$. For illustrative purposes, we again assume $p = 0.5$ and $T = 5$ so that $D_2$ is simply described by two parameters $(\delta, \alpha)$. When $p = 0.5$, Assumption (2) is equivalent to $0 < \alpha < 1$, so $D_2$ is the unit square on the $(\delta, \alpha)$-plane. Moreover, to compare the static and dynamic CH solutions, we assume the distribution of levels follows Poisson(1.5).

From Figure 3, we can observe that the static CH solution is similar to the dynamic solution. Both solutions predict players’ optimal stopping periods depend on the reward $\alpha$ and their patience $\delta$. If $\alpha$ is sufficiently low, players will never claim a dirty face, no matter how high the players’ levels are. In the next section, we will compare both solutions and characterize how players will behave differently in extensive form and strategic form games.

![Static CH Solution (Level 2)](image1)

![Static CH Solution (Level Infinity)](image2)

Figure 3: Static CH solution for level 2 (left) and level $\infty$ players (right) as $x_{-i} = X$ where $p = 0.5$, $T = 5$, and the levels are drawn from Poisson(1.5).

5.3 Representation Effect

As discussed in the previous section, besides level 0 players’ behavior, the concepts of learning are completely different in the dynamic and static CH solutions. Therefore, if the game is in extensive form, static CH is essentially a misspecified model. The comparison between the dynamic and static CH is then quantifying how much the prediction will be distorted in the wrong model. However, if we view both models as correctly specified models, the difference between the dynamic and static CH models is the behavioral difference in different representations of the game.

The representation effect is still an unsolved debate in experimental economics. As surveyed by Brandts and Charness (2011), there is no definitive answer to whether the direct-
response method (extensive form representation) is behaviorally equivalent to the strategy method (strategic form representation). In addition to experimental work, there is some theoretical work attempting to provide insights from a different angle. For instance, Lin and Palfrey (2022) applies the dynamic CH model to centipede games and show that players tend to terminate the game earlier in the extensive form representation than the strategic form, which agrees with the empirical pattern found in García-Pola et al. (2020).

To this end, we here analyze the representation effect of the two-person dirty faces game with the CH models. To formally compare the two solutions, we define the following two ways to partition the set of dirty faces games \( D_2 \). Because strategic players will choose \( D \) immediately when observing a clean face no matter in which representation, we focus on the situation where \( x_i = X \). First, we can partition the set of dirty faces games (with extensive form representation) based on the dynamic CH solution. For any \( t \geq 2 \) and \( k \geq 1 \), let \( E^k_t \) be the set of dirty faces games where \( \hat{\sigma}^k_i(X) \leq t \). For level 1 players, since \( \hat{\sigma}^1_i(X) = T + 1 \) by Corollary 1, we know \( E^1_t = \emptyset \) for all \( t = 2, \ldots, T \), and \( E^1_{T+1} = D_2 \). For higher-level players, \( E^k_t \) can also be visualized in Figure 2. For instance, \( E^2_2 \) corresponds to the “2 (EQ)” area in the left panel.\(^{17}\)

Second, we can also partition the set \( D_2 \) (with strategic form representation) based on the static CH solution. Namely, for any \( t \geq 2 \) and \( k \geq 1 \), we can define \( S^k_t \) as the set of dirty faces games where \( \tilde{\sigma}^k_i(X) \leq t \). This partition is illustrated in Figure 3. Proposition 6 compares the dynamic and static CH solutions by the set inclusions of \( E^k_t \) and \( S^k_t \).

**Proposition 6.** Consider any \( T \geq 2 \) and the set of two-person dirty faces games. For any level \( k \geq 2 \), the following relationships hold.

1. \( S^k_t \subset E^k_t \).

2. \( S^k_t \subset E^k_t \) for any \( [\ln(T+1)/\ln 2] \leq t < T - 1 \).

3. There is no set inclusion relationship between \( S^k_t \) and \( E^k_t \) for \( 2 \leq t < [\ln(T+1)/\ln 2] \). Moreover, for any \( i \in N \), there exists \( \tilde{\delta}(T,t) \in (0,1) \) such that \( t = \hat{\sigma}^k_i(X) \leq \tilde{\sigma}^k_i(X) \) if \( \delta \leq \tilde{\delta}(T,t) \) and \( \hat{\sigma}^k_i(X) \geq \tilde{\sigma}^k_i(X) = t \) if \( \delta > \tilde{\delta}(T,t) \). Specifically,

\[
\tilde{\delta}(T,t) = \frac{(2^t - 2)(T + 1) - (t - 1)2^t}{(2^t - 1)(T + 1) - t2^t}.
\]

**Proof.** See Appendix B. \( \square \)

Proposition 6 formally compares the dynamic and the static CH solutions, demonstrating how different representations would affect players’ behavior. We can first observe that for any level \( k \geq 2 \), when both players’ faces are dirty, they are more likely to learn their face

\(^{17}\)Formally, when \( p = 0.5, T = 5 \), and the distribution of levels is Poisson(1.5), \( E^2_2 \) is characterized by:

\[
(\delta, \alpha) \in E^2_2 \iff \frac{2 - \delta}{8 - 7\delta} \leq \alpha < 1 \text{ and } 0 < \delta < 1.
\]
type before the game ends in extensive form than in strategic form as $S^k_T \subset E^k_T$. However, more likely to learn their face type eventually does not imply players would learn their face type earlier. The second and third results show that when the horizon is long enough and the players are sufficiently patient, i.e., $\delta > \delta(T, t)$, they will claim having a dirty later in the extensive form. That is, dynamic CH model predicts players do not always behave closer to the equilibrium in the extensive form than in the strategic form. More surprisingly, the cutoff $\delta(T, t)$ is independent of the level of sophistication. To some extend, the representation of the game has the same impact on each level of players’ behavior.

![Dynamic CH vs. Static CH Solution (Level 2)](image1)

![Dynamic CH vs. Static CH Solution (Level Infinity)](image2)

Figure 4: Representation effect of level 2 (left) and level $\infty$ players (right) when $x_{-i} = X$ where $p = 0.5$, $T = 5$, and the levels are drawn from Poisson(1.5).

To illustrate the representation effect, we keep focusing on the running example where $p = 0.5$, $T = 5$, and the distribution of levels follows Poisson(1.5). By Proposition 6, we can find that $S^k_T \subset E^k_T$ for any $k \geq 2$ and $3 \leq t \leq 5$. Yet, since there is no set inclusion relationship between $S^k_T$ and $E^k_T$, we plot $S^k_T$ and $E^k_T$ for $k = 2$ and $\infty$ in Figure 4. Specifically, by Proposition 4 and Proposition 5, we know for any $\delta \in (0, 1)$, the boundaries of $E^2_T$ and $S^2_T$ can be characterized by:

\[
\begin{align*}
(\delta, \alpha) \in E^2_T & \iff \alpha \geq \left(\frac{1}{2} - \frac{1}{4}\delta\right) e^{-1.5} + \left(1 - \delta\right)1.5 e^{-1.5} = \frac{2 - \delta}{8 - 7\delta} \\
(\delta, \alpha) \in S^2_T & \iff \alpha \geq \left(\frac{5}{6} - \frac{2}{3}\delta\right) e^{-1.5} + \left(1 - \delta\right)1.5 e^{-1.5} = \frac{5 - 4\delta}{14 - 13\delta}.
\end{align*}
\]

Hence, the boundaries intersect at $\delta = 0.8$. This suggests that when $\delta \leq 0.8$, $(\delta, \alpha) \in S^2_T$ implies $(\delta, \alpha) \in E^2_T$, and vice versa. By similar calculation, we can find that the boundaries of $E^\infty_T$ and $S^\infty_T$ also intersect at $\delta = 0.8$. This illustrates the third result of Proposition 6—the cutoff $\delta(5, 2)$ is the same for every level.

This result complements to the finding in Lin and Palfrey (2022) where they find players will always behave closer to the equilibrium in the centipede game when the game is played
in extensive form. The intuition behind the difference is that when there is incomplete information, players do not only learn the other players’ levels but also the payoff-relevant types at the same time. Since each subgame in this environment is more informative, it is more valuable for players to strategically deviate from the equilibrium to gain more information. In the dirty faces game, when players are patient enough, once the players are able to observe the history, they tend to strategically wait for more evidence.

Finally, we analyze the representation effect when there are almost infinitely many periods, i.e., $T \to \infty$. Besides the observability of past actions, another difference between the extensive form and the strategic form is that level 0 players have more available actions to randomize in strategic form. In extensive form, level 0 players can only randomize between two actions at every information set. Yet, they can randomize across $T + 1$ actions in strategic form. As a result, when $T \to \infty$, for any period $t \geq 2$ and level $k \geq 2$, $S_k^t$ and $E_k^t$ do not have set inclusion relationship, suggesting neither the dynamic nor the static CH solution is definitely closer to the equilibrium in the limit. The result is formally stated in Corollary 3.

Corollary 3. Consider the set of two-person dirty faces games. When $T \to \infty$, for any $t \geq 2$ and $k \geq 2$, there is no set inclusion relationship between $S_k^t$ and $E_k^t$. Specifically, $t = \hat{\sigma}_i^k (X) \leq \bar{\sigma}_i^k (X)$ if $\delta \leq \bar{\delta}^t (t)$ and $\hat{\sigma}_i^k (X) \geq \bar{\sigma}_i^t (X) = t$ if $\delta > \bar{\delta}^t (t)$ where

$$\bar{\delta}^t (t) = \frac{[2^t - 2]}{[2^t - 1]}.$$  

Proof. By Proposition 6, we know for any $k \geq 2$, there is no set inclusion relationship between $S_k^t$ and $E_k^t$ if $2 \leq t < \frac{\ln(T + 1)}{\ln(2)}$. When $T \to \infty$, this condition holds for any $t \geq 2$. Moreover, from Proposition 6, we can obtain that

$$\bar{\delta}^t (t) = \lim_{T \to \infty} \delta (T, t) = \lim_{T \to \infty} \frac{(2^t - 2)(T + 1) - (t - 1)2^t}{(2^t - 1)(T + 1) - t2^t} = \frac{2^t - 2}{2^t - 1}.$$  

This completes the proof.  

5.4 Three-Person Games

Thus far, we have focused on two-person games. When there are more than two players, it is challenging to characterize the dynamic CH solution analytically. In the dynamic CH framework, no matter how sophisticated the players are and which information set is reached, players always need to make statistical inferences as it is always possible that others are level 0 players. Therefore, when there are more players, the number of information sets and possible level profiles is too large for us to solve the dynamic CH solution analytically.

To provide some insights of how the dynamic CH solution would look like when there are more players, we consider a special case, the three-person three-period dirty faces game, which is also one of the treatments in Bayer and Chan (2007). We defer the general characterization of the dynamic CH solution to Appendix C. Here we numerically illustrate level 3 players’ behavior. Notice that level 3 players are the least sophisticated players than would possibly choose $D$ when observing two dirty faces.
For illustrative purposes, we assume $p = 0.5$ and the distribution of levels follows Poisson(1.5). Therefore, similar to the analysis of two-person games, the set of dirty faces games becomes the unit square on the $((\delta, \alpha))$-plane. When observing two clean faces, level 3 players will know their faces are dirty immediately. Hence, they will choose $D$ at period 1.

When observing one ($x_i = OX$) or two dirty faces ($x_i = XX$), level 3 players cannot tell their faces for sure in period 1, and thus they will choose $U$. If $x_i = OX$ and the game proceeds beyond the first period, level 3 players will know it is impossible that his face is clean and the player with a dirty face is strategic at the same time. As a result, level 3 players’ belief of having a dirty face at period $t \in \{2, 3\}$ is:

$$
\mu_3^t(X|t, OX) = \sum_{\tau_{-i}} \mu_3^t(X, \tau_{-i}|t, OX) = p \left( \frac{1}{2} \right)^{t-1} \frac{p_0 + p_1 + p_2}{p_0 + p_1 + p_2}.
$$

Therefore, at period 3, the last period of the game, level 3 players will choose $D$ if and only if the expected payoff is non-negative. That is,

$$
\sigma_3^3(3, OX) = 1 \iff \alpha \geq \frac{1}{4}p_0 = \frac{2}{23}. 
$$

(3)

At period 2, level 3 players would think the probability that all others choose $U$ is

$$
\left[ \frac{1}{2}p_0 + p_1 + p_2 \right] \left[ \frac{1}{2}p_0 + p(p_1 + p_2) \right] = \gamma_3 \left[ \frac{1}{2}p_0 + p(p_1 + p_2) \right]
$$

where the first (second) term corresponds to the probability of the player with a clean (dirty) face choosing $U$ at period 2. As a result, level 3 players would choose $D$ at period 2 if and only if

$$
\alpha \geq \frac{1}{4} \frac{p_0}{\gamma_3} = \frac{100 - 46\delta}{625 - 529\delta}. 
$$

(4)

Finally, when observing two dirty faces, level 3 players cannot tell their face types in the first two periods. This is because level 1 and 2 players will choose $U$ in period 1 no matter which face types they are. At period 3, level 3 players will choose $D$ if and only if (1) level 2 players would choose $D$ at period 2 when seeing only one dirty face,\(^\text{18}\) and (2) the expected payoff of $D$ is non-negative. Therefore, it is optimal to choose $D$ at period 3 if and only if

$$
\alpha \geq \max \left\{ \frac{16 - 7\delta}{64 - 49\delta}, \left( \frac{14}{23} \right)^2 \right\}.
$$

(5)

The dynamic CH solution for level 3 players is visualized in Figure 5. Here we naturally extend the definition of optimal stopping periods to three-person games. We plot the optimal

\(^{18}\)Otherwise, if level 1 and 2 players will both choose $U$ at period 2 when seeing only one dirty face, then level 3 players can still not tell their face types even when the game proceeds to period 3.
stopping periods in the left and right panels of Figure 5, respectively. From this figure, we can observe two features that are different from the two-person games. First, when \( x_\rightarrow = OX \) and \( \delta \to 1 \), the level 3 players would choose \( D \) at period 2 when \( \alpha \geq 9/16 \). However, in two-person games, when \( \delta \to 1 \), players will always wait till the last period. This is because when there are more players, the game is more likely to be randomly terminated, causing the players to choose \( D \) earlier even if the payoff is not discounted.

Second, from the right panel, we can see that when \( x_\rightarrow = XX \), players’ behavior at period 3 depends on \( \delta \) even if it is the last period. The reason is that level 3 players’ beliefs at period 3 depend on level 2 players’ behavior at period 2 which is related to the discount factor. Finally, it is worth remarking that for general dirty faces games, since there are more information sets, the boundaries may be much kinkier, which makes the analytic characterization more challenging. In Appendix C, we also solve for the static CH solution and find a stronger representation effect in three-person three-period games.

![Dynamic CH Solution (Level 3, OX)](image)

![Dynamic CH Solution (Level 3, XX)](image)

Figure 5: Optimal stopping periods for level 3 players when \( x_\rightarrow = OX \) (left) and \( x_\rightarrow = XX \) (right) where \( p = 0.5 \) and the levels are drawn from Poisson(1.5).

### 6 Dirty Faces Games: Experimental Evidence

#### 6.1 Experimental Environment and Data Description

In this section, we revisit the dirty faces game experiment conducted by Bayer and Chan (2007) with the dynamic CH solution. The experiment consists of two treatments: two-person two-period games (Treatment 1) and three-person three-period games (Treatment 2). In both treatments, the prior probability of having a dirty face is 2/3, the discount factor \( \delta \)
is 0.8, and the reward $\alpha$ is $1/4$.\(^{19}\) In Weber (2001)’s experiment, there is no discount factor, i.e., $\delta = 1$, and there are fewer observations than Bayer and Chan (2007).\(^{20}\) Therefore, we will focus on Bayer and Chan (2007)’s experiment in the following analysis.

There are two sessions in each treatment. In the experiment, there are 42 and 48 subjects in Treatment 1 and 2, respectively. At the beginning of each session, the computer randomly matches two (Treatment 1) or three (Treatment 2) subjects into a group, and the matching groups are fixed for all rounds in a session. This is commonly known to all subjects. There are 14 consecutive rounds in both treatments and the face types are independently drawn in each round according to the prior probabilities. In each group, an announcement is made on the screen if there is at least one person having type $X$.

After observing the face types and the announcement, subjects enter the first period and are asked to choose either $U$ or $D$ simultaneously. When every subject in the group has made the decision, the action profile of this period is revealed to everyone in the same group. In the next period, players are asked to choose either $U$ or $D$ again. The game continues to the next period unless someone has chosen $D$ or the game has reached the horizon. At the end of each round, the subjects are told their own payoffs in that round but they are never told their own face types. The subjects are paid with the sum of the earnings of all 14 rounds. See Bayer and Chan (2007) for the instructions and screen shots of the experimental program.

In our analysis, we will exclude the data from the situation where there is no public announcement\(^{21}\) because it is common knowledge to all players that their faces are clean. Moreover, to gain the most statistical power, we pool the data from all 14 rounds, which yields 1,611 observations (690 in Treatment 1 and 921 in Treatment 2).\(^{22}\)

Table 1 reports the raw data at each information set of the game. Notice the information sets can be represented by the period and what the player has observed, which is $(t, x_{−i})$. Each entry in the table states the number of observations and the percentage of the choices that follow the equilibrium predictions. For instance, in the information set $(t, x_{−i}) = (2, X)$, there are 170 choices and 62 percent of the choices are $D$, which is the action predicted by the equilibrium.

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\(^{19}\)To be more specific, in their experiment, the payoff of correctly claiming a dirty face is 100 ECU (experimental currency unit) and the penalty of wrongly claiming a dirty face is $−400$ ECU. As we normalize the payoff, we can obtain that the relative reward of correctly claiming a dirty face is $1/4$, which corresponds to the parameter $\alpha$ in the analysis of Section 5.

\(^{20}\)Weber (2001)’s dataset consists of two experiments where experiment 2 is comparable with Bayer and Chan (2007)’s design. In this experiment, there are 10 groups in two-person games and 9 groups in three-person games, and the experiment lasts for 9 repetitions. Hence, there are much fewer observations in Weber (2001) than in Bayer and Chan (2007).

\(^{21}\)We exclude 144 observations from Treatment 1 and 63 observations from Treatment 2. When there is no public announcement, players should always choose $U$, while among these excluded data, there are eight choices of $D$ in total (four from Treatment 1 and four from Treatment 2).

\(^{22}\)Since the matching groups are fixed through the experiment, players’ behavior may be affected by the experience from previous rounds. In Appendix D, we separate the data by the first and last seven rounds, finding that the estimation results are similar in two subsamples. Therefore, we assume the learning effect across rounds is mild.
Table 1: Experimental Data from Bayer and Chan (2007)

| Number of Players | 2       | 3       |
|-------------------|---------|---------|
| \( x_{-i} \)     | \( O \) | \( X \) | \( OO \) | \( OX \) | \( XX \) |
| EQ                | \( D \) | \( UD \) | \( D \) | \( UD \) | \( UUD \) |

| Period | Number of Obs (EQ %) |
|--------|----------------------|
| 1      | 123 (0.94) 391 (0.79) 48 (0.92) 280 (0.61) 320 (0.76) |
| 2      | 6 (0.50) 170 (0.62) 2 (0.50) 60 (0.58) 145 (0.79) |
| 3      | — — — 10 (0.20) 56 (0.36) |

Note: In Treatment 1, there are 21 groups of subjects (42 subjects in total), and in Treatment 2, there are 16 groups of subjects (48 subjects in total). Because each group plays 14 rounds, the data set consists of \((21 + 16) \times 14 = 518\) games.

From Table 1, we can observe that when players do not observe any dirty face, the behavior is highly aligned with the equilibrium prediction. In this situation \((x_{-i} = O \text{ or } OO)\), players know their face type is \(X\), and they should choose \(D\) in period 1. The corresponding frequencies for both treatments are 94% and 92%. Moreover, the behavior becomes less consistent with the equilibrium as the problem gets more complicated. When they observe only one dirty face \((x_{-i} = X \text{ or } OX)\), players should realize their face type is \(X\) as the game proceeds to period 2. Yet, the frequencies that players claim to have a dirty face at period 2 are 62% and 58% for Treatment 1 and 2, respectively. Similarly, when observing two dirty faces \((x_{-i} = XX)\), only 30% of the players claim to have a dirty face in period 3 when observing two dirty faces.

These findings suggest equilibrium fails to explain a significant portion of the data. In the following, we will compare the fitness of the dynamic CH model with the static CH model and the agent quantal response equilibrium (AQRE) proposed by McKelvey and Palfrey (1998). By comparing the dynamic and static CH, we can quantify the improvement of incorporating the learning from past actions into the CH model. On the other hand, AQRE is an equilibrium model for extensive form games where players make stochastic choices and assume other players do so as well. The comparison between the dynamic CH and AQRE would demonstrate how hierarchical thinking models can generate statistically comparable predictions as equilibrium-based models.

### 6.2 Likelihood Functions

In this section, we derive the likelihood functions. For the cognitive hierarchy theories, we follow Camerer et al. (2004) and Chong et al. (2016) to assume the prior distribution of levels follows Poisson distribution. Therefore, for both of the dynamic CH and static CH, there is one parameter to be estimated—the average number of levels, \(\tau\). For AQRE, we follow McKelvey and Palfrey (1998) to estimate the logit-AQRE which has a single parameter \(\lambda\). In addition, in order to capture the learning from repeated play, we consider another
specification that incorporate the time trend into the likelihood functions.

Poisson-CH Models

The assumption of Poisson distribution has some distinct advantages, especially the interpretability of the parameter $\tau$. Specifically, the Poisson CH model assumes each player’s level of sophistication is identically and independently drawn from $(p_k)_{k=0}^{\infty}$ where

$$p_k \equiv \frac{e^{-\tau} \tau^k}{k!}, \quad \text{for all } k = 0, 1, 2, \ldots$$

and $\tau > 0$. Because $\tau$ is the mean and variance of the Poisson distribution, the economic meaning of $\tau$ is the average level of sophistication among the population.

Moreover, another theoretic property of the Poisson CH model in dirty faces games is when $\tau \to \infty$, the prediction of Poisson CH would converge to the equilibrium, in the sense of aggregate choice frequencies. This convergence property provides the second interpretation of the parameter $\tau$—the higher of $\tau$, the closer to the equilibrium. Notice that this convergence property does not hold generically (see Camerer et al. (2004)). Here we formally discuss the convergence property of the Poisson DCH in two-person games. A similar argument holds for three-person three-period games.

For any two-person dirty faces game, conditional on there is an announcement, there are two possible states: one dirty face or two dirty faces, which are denoted as $\Omega = \{OX, XX\}$. For each $\omega \in \Omega$, equilibrium predicts a deterministic terminal period. We use $F^*_\omega(t)$ to express the (degenerated) distribution of terminal periods at the equilibrium.\footnote{For two-person dirty-faces games, the equilibrium predicts that players will choose $D$ in period 1 when observing $O$, and choose $D$ in period 2 when seeing $X$. Therefore, when $\omega = OX$, the game will end in period 1, and when $\omega = XX$, the game will be terminated at period 2. In other words,

$$F^*_{OX}(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}, \quad \text{and} \quad F^*_{XX}(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1 & \text{if } t \geq 2 \end{cases}.$$}

On the other hand, given any $\tau > 0$ and $\omega \in \Omega$, the dynamic CH solution predicts a non-degenerated distribution over all possible terminal periods. We use $F^D_\omega(t|\tau)$ to denote the distribution predicted by the dynamic CH solution. Proposition 7 states that when $\tau \to \infty$, the max norm between $F^D_\omega(t|\tau)$ and $F^*_\omega(t)$ will converge to 0 for any $\omega \in \Omega$.

**Proposition 7.** Consider any extensive form two-person dirty faces games. When the prior distribution of levels follows Poisson($\tau$), then for any $\omega \in \Omega$,

$$\lim_{\tau \to \infty} \|F^*_\omega(t) - F^D_\omega(t|\tau)\|_\infty = 0.$$

**Proof.** See Appendix B. \qed
After discussing the theoretical properties of Poisson CH, we now construct the likelihood functions. We first build the likelihood function for dynamic CH. For each subject \( i \), we use \( \Pi_i \) to denote the set of information sets that subject \( i \) has encountered in the game. Notice that every information set can be described by the period \( t \) and the observed face types \( x_{-i} \). We use \( \mathcal{I}_i \) to denote a generic information set. At any information set \( \mathcal{I}_i \), subject \( i \) can choose \( c_i \in \{U, D\} \). Let \( P_k(c_i|\mathcal{I}_i, \tau) \) be the probability of level \( k \) players choosing \( c_i \) at information set \( \mathcal{I}_i \). Moreover, let \( f(k|\mathcal{I}_i, \tau) \) be the posterior distribution of levels at information set \( \mathcal{I}_i \).

At period 1, \( f(k|\mathcal{I}_i, \tau) = e^{-\tau}\tau^k/k! \). For later periods, we can analytically solve \( f(k|\mathcal{I}_i, \tau) \) given any \( \tau \) by Proposition 4 (two-person games) and Proposition 8 (three-person games) in Appendix C. Finally, the predicted choice probability for \( c_i \) at information set \( \mathcal{I}_i \) is simply the aggregation of best responses from all levels weighted by the proportion \( f(k|\mathcal{I}_i, \tau) \):

\[
D(c_i|\mathcal{I}_i, \tau) = \sum_{k=0}^{\infty} f(k|\mathcal{I}_i, \tau) P_k(c_i|\mathcal{I}_i, \tau).
\]  

(6)

We then aggregate over all subjects \( i \), actions \( c_i \) and information sets \( \mathcal{I}_i \) to form the log-likelihood function for the dynamic CH model:

\[
\ln L^D(\tau) = \sum_i \sum_{\mathcal{I}_i \in \Pi_i} \sum_{c_i \in \{U, D\}} 1\{c_i, \mathcal{I}_i\} \ln [D(c_i|\mathcal{I}_i, \tau)],
\]

(7)

where \( 1\{c_i, \mathcal{I}_i\} \) is the indicator function which is 1 when subject \( i \) chooses \( c_i \) at information set \( \mathcal{I}_i \).

Second, the log-likelihood function for the static CH model can be constructed in the similar way. Given any \( \tau \), the static CH model predicts a probability distribution over \( \{1, \ldots, T, T+1\} \) (earliest period to choose \( D \) or never \( D \)) for each level of players conditional on the announcement and other players’ faces. Following previous notations, the probability of level \( k \) subject \( i \) choosing \( t \) conditional on \( x_{-i} \) is denoted by \( \tilde{\sigma}_i^k(t|x_{-i}) \), which can be analytically solved by Proposition 5 (two-person games) and Proposition 9 (three-person games) in Appendix C. Therefore, subject \( i \)'s predicted choice probability for \( t \in \{1, \ldots, T, T+1\} \) conditional on \( x_{-i} \) is the aggregation of choice frequencies of all levels weighted by Poisson(\( \tau \)):

\[
\tilde{S}(t|x_{-i}, \tau) = \sum_{k=0}^{\infty} \frac{e^{-\tau}\tau^k}{k!} \tilde{\sigma}_i^k(t|x_{-i}).
\]  

(8)

Notice that since \( \tilde{\sigma}_i^0(t|x_{-i}) = \frac{1}{T+1} \) for all \( t \), \( \tilde{S}(t|x_{-i}, \tau) > 0 \) for all \( t \). In addition, we can compute the conditional probability to choose \( D \) or \( U \) at information set \( \mathcal{I}_i \). Specifically, the predicted conditional probability to choose \( c_i \in \{U, D\} \) at \((t, x_{-i})\) is

\[
S(D|\mathcal{I}_i, \tau) = \frac{\tilde{S}(t|x_{-i}, \tau)}{\sum_{t' \geq t} \tilde{S}(t'|x_{-i}, \tau)} \quad \text{and} \quad S(U|\mathcal{I}_i, \tau) = 1 - S(D|\mathcal{I}_i, \tau),
\]

where \( \mathcal{I}_i = (t, x_{-i}) \). Finally, we can construct the log-likelihood function for the static CH model by aggregating over all subjects \( i \), actions \( c_i \), and information sets \( \mathcal{I}_i \):

\[
\ln L^S(\tau) = \sum_i \sum_{\mathcal{I}_i \in \Pi_i} \sum_{c_i \in \{U, D\}} 1\{c_i, \mathcal{I}_i\} \ln [S(c_i|\mathcal{I}_i, \tau)].
\]  

(9)

\(^{24}\)When pooling the data from all rounds, we implicitly assume that players’ levels of sophistication are randomly drawn in every round.
Logit-AQRE Model

Let $Q(c_i|I_i, \lambda)$ be the probability of subject $i$ choosing $c_i$ at information set $I_i$ predicted by the logit-AQRE. We describe the details of the model as it applies to the two-person two-period dirty faces games. The AQRE of three-person three-period games can be found in Appendix C. In this game, each player’s strategy is defined by a four-tuple $(q_1, q_2, r_1, r_2)$ which corresponds to $Q(D|1, O, \lambda)$, $Q(D|2, O, \lambda)$, $Q(D|1, X, \lambda)$, and $Q(D|2, X, \lambda)$, respectively. At information set $(t, x_{-i}) = (1, O)$, players would estimate the payoff of $D$ and $U$ by

$$U_{1,O}(D) = \alpha + \epsilon_{1,O,D}$$
$$U_{1,O}(U) = \delta\alpha(1 - r_1)q_2 + \epsilon_{1,O,U},$$

where $\epsilon_{1,O,D}$ and $\epsilon_{1,O,U}$ are independent random variables with a Weibull distribution with the precision parameter $\lambda$. Then the logit formula suggests

$$q_1 = \frac{1}{1 + \exp\{\lambda[\delta\alpha(1 - r_1)q_2 - \alpha]\}}.$$

Similarly, we can express $q_2$ by:

$$q_2 = \frac{1}{1 + \exp\{-\delta\lambda\}}.$$

On the other hand, when observing a dirty face and the game proceeds to period 2, players’ posterior beliefs become:

$$\mu \equiv \Pr(X|2, X) = \frac{p(1 - r_1)}{p(1 - r_1) + (1 - p)(1 - q_1)} = \frac{1}{1 + \left(\frac{1 - p}{p}\right)\left(\frac{1 - q_1}{1 - r_1}\right)},$$

and hence the expected payoff to choose $D$ at information set $(2, X)$ is:

$$\delta [\alpha \mu - (1 - \mu)] = \delta [(1 + \alpha)\mu - 1].$$

As a result, we can obtain that $r_2$ satisfies that

$$r_2 = \frac{1}{1 + \exp\{\lambda\delta[1 - (1 + \alpha)\mu]\}}.$$

Finally, the expected payoff of choosing $D$ at information set $(1, X)$ is $\alpha p - (1 - p)$, while the expected payoff of $U$ is

$$\frac{[p(1 - r_1) + (1 - p)(1 - q_1)]r_2\delta [(1 + \alpha)\mu - 1]}{\text{prob. to reach period 2}} \equiv A,$$

and therefore, $r_1$ can be expressed by:

$$r_1 = \frac{1}{1 + \exp\{\lambda[A + (1 - p) - \alpha p]\}}.$$

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As plugging \( p = 2/3, \delta = 4/5 \) and \( \alpha = 2/3 \) into the choice probabilities, we can obtain that

\[
\begin{align*}
\text{r}_1 &= \frac{1}{1 + \exp \left\{ \lambda \left[ \frac{2}{15} (1 - \text{r}_1) \text{r}_2 - \frac{4}{15} (1 - \text{q}_1) \text{r}_2 + \frac{1}{6} \right] \right\}} \quad (10) \\
\text{r}_2 &= \frac{1}{1 + \exp \left\{ \lambda \left[ \frac{4}{5} - \frac{2 - 2\text{r}_1}{3 - 2\text{r}_1 - \text{q}_1} \right] \right\}} \quad (11) \\
\text{q}_1 &= \frac{1}{1 + \exp \left\{ \lambda \left[ \frac{1}{5} (1 - \text{r}_1) \text{q}_2 - \frac{1}{4} \right] \right\}} \quad (12) \\
\text{q}_2 &= \frac{1}{1 + \exp \left\{ -\frac{1}{5} \lambda \right\}}. \quad (13)
\end{align*}
\]

Given each \( \lambda \), the system of four equations with four unknowns can be solved uniquely. In addition, for each information set \( \mathcal{I}_i \), we can compute \( Q(U|\mathcal{I}_i, \lambda) = 1 - Q(D|\mathcal{I}_i, \lambda) \). Finally, we form the log-likelihood function by aggregating over all subjects \( i \), actions \( c_i \), and information sets \( \mathcal{I}_i \):

\[
\ln L^Q(\lambda) = \sum_i \sum_{\mathcal{I}_i \in \Pi_i} \sum_{c_i \in \{U,D\}} \mathbb{1}\{c_i, \mathcal{I}_i\} \ln [Q(c_i|\mathcal{I}_i, \lambda)].
\]

### 6.3 Estimation Results

Table 2: Estimation Results for Treatment 1 and Treatment 2 Data

| Parameters | Two-Person Games | Three-Person Games |
|------------|------------------|--------------------|
|            | Dynamic CH | Static CH | AQRE | Dynamic CH | Static CH | AQRE |
| \( \tau \) | 1.269     | 1.161     | —     | 0.370     | 0.140     | —     |
| S.E. | (0.090) | (0.095) | —     | (0.043) | (0.039) | —     |
| \( \lambda \) | —       | —       | 7.663 | —       | —       | 5.278 |
| S.E. | —       | —       | (0.493) | —       | —       | (0.404) |
| Fitness | LL -360.75 | -381.46 | -368.38 | LL -575.30 | -608.45 | -565.05 |
|        | AIC 723.50 | 764.91 | 738.76 | AIC 1152.61 | 1218.89 | 1132.11 |
|        | BIC 728.04 | 769.45 | 743.29 | BIC 1157.43 | 1223.72 | 1136.93 |
| Vuong Test | 6.517 | 1.463 | — | 3.535 | -1.330 | — |
| p-value | < 0.001 | 0.144 | — | < 0.001 | 0.184 | — |

Note: There are 294 games (rounds \times groups) in Treatment 1 and 224 games in Treatment 2.

Table 2 reports the estimation results of dynamic CH, static CH and AQRE on Treatment 1 and Treatment 2 data. The table shows the estimated parameters and the fitness of each model. As we compare the fitness of the models, we can observe that no matter in which data set, static CH has the lowest log-likelihood, while dynamic CH and AQRE have quantitatively
comparable log-likelihood. Since these three models are non-nested, we test the difference of the log-likelihood between these models by Vuong test (Vuong, 1989). The result shows dynamic CH can fit the data significantly better than static CH in both treatments (Vuong Test p-value < 0.001). However, the difference between dynamic CH and AQRE is not statistically significant (Treatment 1: p-value = 0.144; Treatment 2: p-value = 0.184).

Finding 1. Dynamic CH can explain the data significantly better than static CH while the difference between dynamic CH and AQRE is not statistically significant.

As we compare the estimation results of Treatment 1 and Treatment 2, we can find that there is more randomness in three-person three-period games than two-person two-period games—no matter from the perspective of dynamic CH, static CH or AQRE. Dynamic CH estimates in two-person games, players can think 1.269 steps (95% C.I. = [1.093, 1.445]) on average but they can only think 0.370 steps (95% C.I. = [0.286, 0.454]). On the other hand, AQRE finds when the game changes from two-person games to three-person games, the precision becomes significantly smaller (from 7.663 to 5.278), suggesting players are less likely to make best responses in three-person games.

Finding 2. Players perform significantly more sophisticatedly in two-person games than three-person games.

In addition, as we compare the estimates of dynamic CH and static CH, we can observe that static CH systematically under-estimates the average level of sophistication, especially in three-person three-period games. In three-person games, dynamic CH estimates $\hat{\tau} = 0.37$ with 95% C.I. = [0.286, 0.454] while static CH estimates $\hat{\tau} = 0.14$ with 95% C.I. = [0.006, 0.216]. The driving force of this result is that the learning process is not captured by static CH. When players choose to claim their dirty faces later than the equilibrium prediction, dynamic CH is more likely to think this is a result of strategic delaying rather than randomness. Therefore, the difference between the estimates of dynamic CH and static CH tends to be larger when the length of the game is longer.

Finding 3. Compared with dynamic CH, static CH tends to under-estimates the average level of sophistication.

To analyze the differences between the models in details, we compare the choice probabilities predicted by each model. Figure 6 and 7 plot the choice probabilities of two-person games and three-person games, respectively. Comparing the dynamic and the static CH models, we find that the static CH generally under-estimates the probability to choose $D$.

\[ V = \frac{N^{-1} \sum_{i=1}^{N} [l_{i1}(\hat{\theta}_1) - l_{i2}(\hat{\theta}_2)]}{\{N^{-1} \sum_{i=1}^{N} [l_{i1}(\hat{\theta}_1) - l_{i2}(\hat{\theta}_2)]^2\}^{1/2}/\sqrt{N}} \xrightarrow{\text{d}} \mathcal{N}(0,1) \]
in period 1. In two-person games, the empirical frequencies of $D$ at information sets $(1, O)$ and $(1, X)$ are 0.943 and 0.210, respectively, while the predictions of static CH are 0.791 and 0.104. Besides, in three-person games, the empirical frequencies of $D$ at information sets $(1, OO)$, $(1, OX)$ and $(1, XX)$ are 0.917, 0.393 and 0.241 but the estimates of static CH are 0.348, 0.217 and 0.217.

This phenomenon is caused by the difference of level 0 players’ behavior. In two-person games, static CH assumes level 0 players uniformly randomizes across $\{1, 2, 3\}$, causing the static CH predicts the probability of level 0 players choosing $D$ at period 1 is $1/3$. Yet, level 0 players in the dynamic CH would uniformly randomize everywhere, yielding them to choose $D$ with probability $1/2$. Similarly, in three-person games, level 0 players uniformly randomize among $\{1, 2, 3, 4\}$, and hence the probability for them to choose $D$ at period 1 is $1/4$, rather than $1/3$ in two-person games. Yet, from the perspective of dynamic CH, level 0 players’ behavior is exactly the same in both two-person and three-person games.

**Finding 4.** *Static CH systematically under-estimates the probabilities of $D$ at period 1.*
To better understand the key difference between CH approaches and AQRE, we can focus on the off-equilibrium-path information sets. Conceptually speaking, the key difference between CH approaches and AQRE is the reason why the game could proceed to the off-equilibrium-path information sets. From the perspective of AQRE, the off-equilibrium-path information sets are reached by mistakes. As a result, AQRE predicts a high probability of choosing $D$ at these off-equilibrium-path information sets because the expected payoff of choosing $D$ is much larger than $U$ at these information sets. On the other hand, in the framework of CH, the off-equilibrium-path information sets are reached because the players are not sophisticated enough. For instance, when observing no dirty face, players should choose $D$ immediately since this is a dominant strategy. If some one doesn’t choose $D$, he is definitely a level 0 player from the perspective of CH.

Figure 7: The choice probabilities in three-person games at different information sets. Each panel plots the empirical choice frequencies and the predictions of different models at one information set. The gray panels represent the off-equilibrium-path information sets.

From the choice probabilities, we can find that dynamic CH has the most accurate pre-
dictions at off-path information sets no matter in two-person games or three-person games. At information sets \((2, O)\) and \((2, OO)\), the empirical choice probabilities of \(D\) are 0.5, which are correctly predicted by dynamic CH. Furthermore, at information set \((3, OX)\), the empirical choice probability of \(D\) is 0.2, and the prediction of dynamic CH is 0.291 while the predictions of static CH and AQRE are 0.385 and 0.624, respectively.

**Finding 5.** Dynamic CH has the most accurate predictions at the off-equilibrium-path information sets.

Notice that dynamic and static CH models predict only level 0 players would possibly choose strictly dominated strategies, no matter what private information the players have received. In three-person games, when \(x_{-i} = OX\) or \(XX\), it is strictly dominated to choose \(D\) at period 1. Thus, both CH models predict the choice probabilities will be the same at \((1, OX)\) and \((1, XX)\). However, the empirical frequency of \(D\) at \((1, OX)\) is 0.393 which is significantly higher than 0.241, the probability to choose \(D\) at \((1, XX)\) (Mann-Whitney test: p-value < 0.001). This empirical pattern is not captured by the current CH models. One possible explanation is that players are not fully aware of the observed faces (Liu, 2008). When \(x_{-i} = OX\) and players are unaware of the dirty face, they may therefore choose \(D\) at period 1. This finding suggests incorporating psychological biases to the model is a potential way to improve the predictivity. We leave this extension for future research.

**Table 3: Estimation Results for Pooled Data**

|          | Dynamic CH | Static CH | AQRE |
|----------|------------|-----------|------|
| Parameters | \(\tau\)   | 1.030     | 0.241| —   |
|          | S.E.       | (0.060)   | (0.033)| —   |
|          | \(\lambda\) | —         | —   | 6.235|
|          | S.E.       | —         | —   | (0.302)|
| Fitness  | LL         | -956.92   | -1047.12| -940.65|
|          | AIC        | 1915.84   | 2096.23| 1883.30|
|          | BIC        | 1921.22   | 2101.62| 1888.69|
| Vuong Test |           | 7.513     | -1.363|      |
| p-value  |            | < 0.001   | 0.173|      |
| LR Test  | \(\chi^2\) | 41.74     | 114.42| 14.44|
| p-value  |            | < 0.001   | < 0.001| < 0.001|

Note: The likelihood ratio test is testing if the log-likelihood of two-parameter models (Treatment 1 and 2) is significantly higher than the log-likelihood of one-parameter models.

In addition, we estimate the three models on the pooled data, and the results are reported in Table 3. Similar to the results of two-person games and three-person games, we again
find that dynamic CH can explain the data significantly better than static CH (Vuong test: p-value < 0.001) while the difference between dynamic CH and AQRE is not statistically significant (Vuong test p-value = 0.173). Moreover, we perform a likelihood ratio test on all three models to see whether allowing two-person and three-person games to have different parameter can significantly improve the fitness. The result shows heterogeneous models are significantly better than homogeneous models. Coupled with the second finding, we conclude that either the level of sophistication or the precision varies with the complexity of the games.

Finding 6. The null hypothesis that two-person and three-person games share the same $\tau$ and $\lambda$ is rejected.

This result is in line with the prediction of Alaoui and Penta (2016). In the theory of endogenous depth of reasoning, players will behave as if following a cost-benefit analysis and endogenously determine the steps of reasoning to perform. One of the predictions is that when the complexity increases, players tend to be less sophisticated. This is exactly the pattern we observe in this experiment. Moreover, the novelty of the dynamic CH approach is that it allows us to precisely pin down the shift of sophistication when the complexity of some extensive form game has changed.

To summarize, since static CH is a misspecified model for extensive form games, it’s not surprising that dynamic CH can better explain the data. However, what is surprising is as we correctly specify the Poisson-CH model for extensive form games, we can obtain $\hat{\tau} = 1.030$, which is a regular $\tau$ (between 1 and 2) predicted by Camerer et al. (2004). Besides, it’s interesting that dynamic CH and AQRE can generate quantitatively and statistically similar fits even though the spirits are completely different. In Appendix D, we separate the data by the first and the last seven rounds to control for the learning effect. We find none of the estimates are significantly different in the first and second half, suggesting the robustness of our results.

7 Discussion and Conclusion

Multi-stage games of incomplete information include a variety of applications in economics and political science. The standard approach to analyzing such an environment is to solve for the sequential equilibrium or the perfect Bayesian equilibrium, which require the belief system to be mutually consistent everywhere, no matter on or off the equilibrium path. Yet, as documented in the literature, mutual consistency is an empirically fragile requirement.

To this end, this paper considers an alternative solution concept, the dynamic cognitive hierarchy solution proposed by Lin and Palfrey (2022), to the class of multi-stage games of incomplete information. One appealing feature of this approach is that the dynamic CH solution can be attained without mutual consistency on the belief system. Instead, players are heterogeneous with respect to the depth of reasoning, and each level of players will best respond a mixture of lower levels with truncated rational expectations. In a multi-stage game of incomplete information, players will learn about other players’ payoff-dependent types and levels of sophistication at the same time as the history unfolds.
Moreover, we characterize some general properties of the belief updating process. First, Proposition 1 demonstrates that the posterior belief of other players’ types always has full support, while players will rule out the possibility of levels as the game proceeds. Secondly, when the types are drawn independently across players, Proposition 2 establishes that every level of player’s posterior belief is independent across players at every information set. Thirdly, if the types are correlated across players, Proposition 3 proves that we can transform the game into another game with independent types, suggesting the independence result holds without loss of generality.

As pointed out by Lin and Palfrey (2022), when information sets are non-singleton, the beliefs could be correlated across histories. In this class of games, players’ posterior beliefs of others’ types and levels are generally corrected at some information sets. To further understand how the beliefs of types and levels are correlated, in the second half of the paper, we analyze the dynamic CH solution in a class of two-person dirty faces games. As characterized by Proposition 4, dynamic CH solution predicts lower-level players will figure out their face types later than higher-level players. This contrasts with the equilibrium which predicts a degenerate distribution of terminal periods given each status of the face types.

Besides, we compare the dynamic and the static CH solutions of the dirty faces games, finding that the representation would affect players’ behavior. Proposition 6 demonstrates whether players behave closer to the equilibrium in the extensive form or the strategic form depends on the horizon $T$ and the patience $\delta$. This result complements to the representation effect characterized by Lin and Palfrey (2022) that in a class of centipede games, players will always behave closer to the equilibrium in the extensive form than in the strategic form.

Finally, to quantitatively calibrate the dynamic CH model and compare with other solutions, we re-visit the dirty faces game experimental data by Bayer and Chan (2007). As assuming the prior distribution of levels follows a Poisson distribution, dynamic CH estimates players can think 1.03 steps on average, while the average level of sophistication significantly varies with the complexity of the game. Furthermore, when comparing dynamic CH with other models, we find that dynamic CH can significantly explain the data better than the static CH while its fitness is not significantly different from agent quantal response equilibrium. Although the overall fitness is similar to AQRE, dynamic CH has the most accurate predictions at off-path information sets, which is not expected a priori.

For future research, we suggest to apply the dynamic CH solution to other applications where common knowledge of rationality and mutual consistency are critical to the equilibrations. Besides, hybridizing the dynamic CH solution with other behavioral models is also an interesting and important direction to pursue. Here are some potential applications where dynamic CH might provide some new insights.

- **Social learning**: In social learning games with repeated actions, players will make inferences about the true state based on their private signals and publicly observed actions (see Bala and Goyal (1998) and Harel et al. (2021)). The dynamic CH solution posits that players do not commonly believe others are able to make correct inferences. Specifically, level 0 players’ actions do not convey any information about the true states, while level 1 players will always obey their private signals. For higher-level players, they will constantly update their beliefs about the true state and other players’ levels.
• **Sequential bargaining:** The equilibrium of a sequential bargaining game is first characterized by Rubinstein (1982). To reach the perfect equilibrium, players are required to choose the optimal proposal among a continuum of choices at every subgame, and believe the other player to optimally respond to each proposal. This could be an empirically fragile requirement. Later, McKelvey and Palfrey (1993, 1995) consider a two-person multi-stage bargaining game where each player has a private payoff-relevant type and makes a binary decide (whether to give in or hold out) in every period. The game continues until at least one of the players gives in. In this game, it is strictly dominant for the strong type of players to hold out forever, but not for the weak type. In the dynamic CH solution, players will update their beliefs about the opponent’s types and levels at the same time.

• **Signaling:** In a multi-stage signaling game, an informed player will have a persistent type and interact with an uninformed player repeatedly. Kaya (2009) analyzes such an environment, finding that the set of equilibrium signal sequences includes a large class of possibly complex signal sequences. By contrast, in the dynamic CH solution, the uninformed player will learn about the informed player’s true type and level when observing a new signal. Besides, the informed player will also learn about the uninformed player’s level at each stage. The analysis has a variety of applications. One of the potential applications is the “reputation building” game by Kreps and Wilson (1982a). In such an environment, each entrant will update their beliefs about the monopolist’s type (“tough” or “benevolent”) and level after every decision of the monopolist.

• **Sequential voting:** There is a large class of voting rules that include multiple rounds, such as sequential voting over agendas (Baron and Ferejohn, 1989) or elections based on repeated ballots and elimination of one candidate in each round (Bag et al., 2009). To reach the Condorcet consistent outcomes, players are required to behave strategically. When voters are not strategic or when they believe others are not strategic, we can apply the dynamic CH model to the context of sequential voting. In this case, voters will update their beliefs about others’ preferences and levels of sophistication simultaneously, and vote according to their posterior beliefs in each round.

• **Behavioral models:** In various environments, it is commonly found that players’ decisions will be substantially affected by their behavioral characteristics, such as their other-regarding preferences (Fehr and Schmidt, 1999), degrees of loss aversion (Tversky and Kahneman, 1991; Brown et al., 2021), or risk aversion (Holt and Laury, 2002), etc. When players interact with others repeatedly, especially in a lab setting, players may learn about other players’ behavioral traits and levels of sophistication along the gameplay history. To analyze this dynamics, we can potentially specify the behavioral traits as the payoff-relevant types and apply the dynamic CH model.

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For instance, Lin et al. (2020) recently documents that in ultimatum game experiments, there is a significant proportion of the players who strongly prefer the equal-sharing allocation. Furthermore, Camerer et al. (2019, 2022) find that in an unstructured bargaining game with asymmetric information, players’ preferences about the efficiency or equality would significantly affect the equilibrium selection.
References

Alaoui, Larbi and Antonio Penta, “Endogenous depth of reasoning,” *The Review of Economic Studies*, 2016, 83 (4), 1297–1333.

_ and _, “Cost-benefit analysis in reasoning,” *Working Paper*, 2018.

Aumann, Robert J, “Irrationality in game theory,” *Economic analysis of markets and games*, 1992, pp. 214–227.

_ , “Backward induction and common knowledge of rationality,” *Games and Economic Behavior*, 1995, 8 (1), 6–19.

Bag, Parimal Kanti, Hamid Sabourian, and Eyal Winter, “Multi-stage voting, sequential elimination and Condorcet consistency,” *Journal of Economic Theory*, 2009, 144 (3), 1278–1299.

Bala, Venkatesh and Sanjeev Goyal, “Learning from neighbours,” *The review of economic studies*, 1998, 65 (3), 595–621.

Baron, David P and John A Ferejohn, “Bargaining in legislatures,” *American political science review*, 1989, 83 (4), 1181–1206.

Barwise, Jon, “Scenes and other situations,” *The journal of Philosophy*, 1981, 78 (7), 369–397.

Bayer, R-C and Ludovic Renou, “Logical abilities and behavior in strategic-form games,” *Journal of Economic Psychology*, 2016, 56, 39–59.

Bayer, Ralph C and Ludovic Renou, “Logical omniscience at the laboratory,” *Journal of Behavioral and Experimental Economics*, 2016, 64, 41–49.

_ and Mickey Chan, “The dirty faces game revisited,” Technical Report, University of Adelaide, School of Economics 2007.

Bernheim, B Douglas, “Rationalizable strategic behavior,” *Econometrica: Journal of the Econometric Society*, 1984, pp. 1007–1028.

Binmore, Ken, “Modeling rational players: Part I,” *Economics & Philosophy*, 1987, 3 (2), 179–214.

_ , “Modeling rational players: Part II,” *Economics & Philosophy*, 1988, 4 (1), 9–55.

_ and Adam Brandeburger, “Common knowledge and game theory,” *MichU DeptE CenREST W89-06*, 1988.

Bosch-Domenech, Antoni, Jose G Montalvo, Rosemarie Nagel, and Albert Satorra, “One, two,(three), infinity,...: Newspaper and lab beauty-contest experiments,” *American Economic Review*, 2002, 92 (5), 1687–1701.

42
Brandts, Jordi and Gary Charness, “The strategy versus the direct-response method: a first survey of experimental comparisons,” *Experimental Economics*, 2011, 14 (3), 375–398.

Brown, Alexander L, Taisuke Imai, Ferdinand Vieider, and Colin Camerer, “Meta-analysis of empirical estimates of loss-aversion,” *Available at SSRN 3772089*, 2021.

Cai, Hongbin and Joseph Tao-Yi Wang, “Overcommunication in strategic information transmission games,” *Games and Economic Behavior*, 2006, 56 (1), 7–36.

Camerer, Colin F. *Behavioral game theory: Experiments in strategic interaction*, Princeton university press, 2003.

, Gideon Nave, and Alec Smith, “Dynamic unstructured bargaining with private information: theory, experiment, and outcome prediction via machine learning,” *Management Science*, 2019, 65 (4), 1867–1890.

, Hung-Ni Chen, Po-Hsuan Lin, Gideon Nave, Alec Smith, and Joseph Tao yi Wang, “Using Machine Learning to Understand Bargaining Experiments,” in “Bargaining,” Springer, 2022, pp. 407–431.

, Teck-Hua Ho, and Juin-Kuan Chong, “A cognitive hierarchy model of games,” *The Quarterly Journal of Economics*, 2004, 119 (3), 861–898.

Chong, Juin-Kuan, Teck-Hua Ho, and Colin Camerer, “A generalized cognitive hierarchy model of games,” *Games and Economic Behavior*, 2016, 99, 257–274.

Choo, Lawrence and Xiaoyu Zhou, “Can market selection reduce anomalous behaviour in games?,” *European Economic Review*, 2022, 141, 103958.

Costa-Gomes, Miguel A and Vincent P Crawford, “Cognition and behavior in two-person guessing games: An experimental study,” *American economic review*, 2006, 96 (5), 1737–1768.

Costa-Gomes, Miguel, Vincent P Crawford, and Bruno Broseta, “Cognition and behavior in normal-form games: An experimental study,” *Econometrica*, 2001, 69 (5), 1193–1235.

Crawford, Vincent P and Nagore Iriberri, “Fatal attraction: Salience, naivety, and sophistication in experimental” hide-and-seek” games,” *American Economic Review*, 2007, 97 (5), 1731–1750.

and _, “Level-k auctions: Can a nonequilibrium model of strategic thinking explain the winner’s curse and overbidding in private-value auctions?,” *Econometrica*, 2007, 75 (6), 1721–1770.

Devetag, Giovanna and Massimo Warglien, “Games and phone numbers: Do short-term memory bounds affect strategic behavior?,” *Journal of Economic Psychology*, 2003, 24 (2), 189–202.
Esponda, Ignacio and Emanuel Vespa, “Hypothetical thinking and information extraction in the laboratory,” *American Economic Journal: Microeconomics*, 2014, 6 (4), 180–202.

Eyster, Erik and Matthew Rabin, “Cursed equilibrium,” *Econometrica*, 2005, 73 (5), 1623–1672.

Fehr, Ernst and Klaus M Schmidt, “A theory of fairness, competition, and cooperation,” *The quarterly journal of economics*, 1999, 114 (3), 817–868.

Fudenberg, Drew and David Levine, “Subgame-perfect equilibria of finite- and infinite-horizon games,” *Journal of Economic Theory*, 1983, 31 (2), 251–268.

_ and Jean Tirole, *Game theory*, MIT press, 1991.

_ and _, “Perfect Bayesian equilibrium and sequential equilibrium,” *Journal of Economic Theory*, 1991, 53 (2), 236–260.

Gamow, G and M Stern, “Forty unfaithful wives,” *Puzzle math*, 1958, pp. 20–23.

García-Pola, Bernardo, Nagore Iriberri, and Jaromír Kovářík, “Hot versus cold behavior in centipede games,” *Journal of the Economic Science Association*, 2020, 6 (2), 226–238.

Grehl, Sascha and Andreas Tutić, “Experimental Evidence on Iterated Reasoning in Games,” *PloS one*, 2015, 10 (8), e0136524.

Halpern, Joseph Y and Yoram Moses, “Knowledge and common knowledge in a distributed environment,” *Journal of the ACM (JACM)*, 1990, 37 (3), 549–587.

Hardin, Christopher S and Alan D Taylor, “An introduction to infinite hat problems,” *The Mathematical Intelligencer*, 2008, 30 (4), 20–25.

Harel, Matan, Elchanan Mossel, Philipp Strack, and Omer Tamuz, “Rational groupthink,” *The Quarterly Journal of Economics*, 2021, 136 (1), 621–668.

Ho, Teck-Hua and Xuanming Su, “A dynamic level-k model in sequential games,” *Management Science*, 2013, 59 (2), 452–469.

_ , Colin Camerer, and Keith Weigelt, “Iterated dominance and iterated best response in experimental” p-beauty contests”,” *The American Economic Review*, 1998, 88 (4), 947–969.

_ , So-Eun Park, and Xuanming Su, “A Bayesian Level-k Model in n-Person Games,” *Management Science*, 2021, 67 (3), 1622–1638.

Holt, Charles A and Susan K Laury, “Risk aversion and incentive effects,” *American economic review*, 2002, 92 (5), 1644–1655.
Johanson, Michael, “Measuring the size of large no-limit poker games,” *arXiv preprint arXiv:1302.7008*, 2013.

Kaya, Ayça, “Repeated signaling games,” *Games and Economic Behavior*, 2009, 66 (2), 841–854.

Kreps, David M and Robert Wilson, “Reputation and imperfect information,” *Journal of economic theory*, 1982, 27 (2), 253–279.

and _, “Sequential Equilibria,” *Econometrica: Journal of the Econometric Society*, 1982, 50 (4), 863–894.

Lin, Po-Hsuan, Alexander L Brown, Taisuke Imai, Joseph Tao yi Wang, Stephanie W Wang, and Colin F Camerer, “Evidence of general economic principles of bargaining and trade from 2,000 classroom experiments,” *Nature Human Behaviour*, 2020, 4 (9), 917–927.

and Thomas R Palfrey, “Cognitive Hierarchies in Extensive Form Games,” *Caltech Social Science Working Paper*, 2022.

Littlewood, John Edensor, *A Mathematician’s Miscellany*, London, England: Meuthen & Co. Ltd., 1953.

Liu, Zhen, “The dirty face problem with unawareness,” *The BE Journal of Theoretical Economics*, 2008, 8 (1), 1313–1326.

McKelvey, Richard D and Thomas R Palfrey, “Engodeneity of Alternating Offers in a Bargaining Game,” *Caltech Social Science Working Paper*, 1993.

and _, “The holdout game: An experimental study of an infinitely repeated game with,” in “Social choice, welfare, and ethics: Proceedings of the Eighth International Symposium in Economic Theory and Econometrics,” Vol. 8 Cambridge University Press 1995, p. 321.

and _, “Quantal response equilibria for extensive form games,” *Experimental economics*, 1998, 1 (1), 9–41.

Moses, Yoram, Danny Dolev, and Joseph Y Halpern, “Cheating husbands and other stories: a case study of knowledge, action, and communication,” *Distributed computing*, 1986, 1 (3), 167–176.

Myerson, Roger B, “Bayesian equilibrium and incentive-compatibility: An introduction,” *Social goals and social organization: Essays in memory of Elisha Pazner*, 1985, pp. 229–260.

Nagel, Rosemarie, “Unraveling in guessing games: An experimental study,” *The American Economic Review*, 1995, 85 (5), 1313–1326.

Pearce, David G, “Rationalizable strategic behavior and the problem of perfection,” *Econometrica: Journal of the Econometric Society*, 1984, pp. 1029–1050.
Rubinstein, Ariel, “Perfect equilibrium in a bargaining model,” *Econometrica: Journal of the Econometric Society*, 1982, pp. 97–109.

Selten, Reinhard, “Anticipatory learning in two-person games,” in “Game equilibrium models I,” Springer, 1991, pp. 98–154.

_ , “Features of experimentally observed bounded rationality,” *European Economic Review*, 1998, *42* (3-5), 413–436.

Stahl, Dale O, “Evolution of smartn players,” *Games and Economic Behavior*, 1993, *5* (4), 604–617.

_ , “Boundedly rational rule learning in a guessing game,” *Games and Economic Behavior*, 1996, *16* (2), 303–330.

_ and Paul W Wilson, “On players’ models of other players: Theory and experimental evidence,” *Games and Economic Behavior*, 1995, *10* (1), 218–254.

Tversky, Amos and Daniel Kahneman, “Loss aversion in riskless choice: A reference-dependent model,” *The quarterly journal of economics*, 1991, *106* (4), 1039–1061.

Villanueva, John Carl, “How many atoms are there in the universe,” *Universe Today*, 2009, 30.

Vuong, Quang H, “Likelihood ratio tests for model selection and non-nested hypotheses,” *Econometrica: Journal of the Econometric Society*, 1989, pp. 307–333.

Wald, Abraham, *Sequential analysis*, John Wiley and Sons, New York, 1947.

Wang, Joseph Tao-Yi, Michael Spezio, and Colin F Camerer, “Pinocchio’s pupil: using eyetracking and pupil dilation to understand truth telling and deception in sender-receiver games,” *American Economic Review*, 2010, *100* (3), 984–1007.

Weber, Roberto A, “Behavior and learning in the “dirty faces” game,” *Experimental Economics*, 2001, *4* (3), 229–242.
A Omitted Proofs for General Properties

Proof of Lemma 1

1. At the beginning of the game, the only information available to player $i$ is his own type $\theta_i$ and his level of sophistication $\tau_i = k$. Therefore, the prior belief is the probability of the opponents’ types and levels conditional on $\theta_i$ and $\tau_i$, which is

$$\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h_0) = \Pr(\theta_{-i}, \tau_{-i} | \theta_i, \tau_i = k) = \Pr(\theta_{-i} | \theta_i) \Pr(\tau_{-i} | \tau_i = k) = \mathcal{F}(\theta_{-i} | \theta_i) \prod_{j \neq i} \hat{P}^k_{ij}(\tau_j).$$

The second equality holds because the types and levels are independently drawn.

2. We can prove this statement by induction on $t$. First, consider any available history at period 2, $h^1 \in \mathcal{H}^1$. Level $k$ player $i$’s belief at information set $(\theta_i, h^1)$ is

$$\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h^1) = \frac{\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h_0) \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h_0)}{\sum_{\theta_{-i}} \sum_{\tau_{-i} : \tau_{-i}^j < k \forall j \neq i} \mu^k_i(\theta_{-i}, \tau_{-i}^j | \theta_i, h_0) \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h_0) \mathcal{F}(\theta_{-i} | \theta_i) \prod_{j \neq i} \left\{ \hat{P}^k_{ij}(\tau_j) \sigma_j^\tau_{ij}(a^1_j | \theta_j, h_0) \right\}}. \quad (A.1)$$

By step 1, we know $\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h^0) = \mathcal{F}(\theta_{-i} | \theta_i) \prod_{j \neq i} \hat{P}^k_{ij}(\tau_j)$. Plugging in Equation (A.1), we can obtain that

$$\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h^1) = \frac{\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h_0) \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h_0)}{\sum_{\theta_{-i}} \sum_{\tau_{-i} : \tau_{-i}^j < k \forall j \neq i} \mu^k_i(\theta_{-i}, \tau_{-i}^j | \theta_i, h_0) \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h_0) \mathcal{F}(\theta_{-i} | \theta_i) \prod_{j \neq i} \left\{ \hat{P}^k_{ij}(\tau_j) \sigma_j^\tau_{ij}(a^1_j | \theta_j, h_0) \right\}}. \quad (A.2)$$

Next, suppose there is $t'$ such that the statement holds for every period $t = 2, \ldots, t'$. Consider period $t' + 1$ and any history available at period $t' + 1$, $h^{t'} \in \mathcal{H}^{t'}$. Then level $k$ player $i$’s belief at information set $(\theta_i, h^{t'})$ is

$$\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h^{t'}) = \frac{\mu^k_i(\theta_{-i}, \tau_{-i} | \theta_i, h^{t'-1}) \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h^{t'-1})}{\sum_{\theta_{-i}} \sum_{\tau_{-i} : \tau_{-i}^j < k \forall j \neq i} \mu^k_i(\theta_{-i}, \tau_{-i}^j | \theta_i, h^{t'-1}) \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h^{t'-1}) \mathcal{F}(\theta_{-i} | \theta_i) \prod_{j \neq i} \left\{ \hat{P}^k_{ij}(\tau_j) \prod_{l=1}^{t'-1} \sigma_j^\tau_{ij}(a^0_l | \theta_j, h^{l-1}) \right\} \prod_{j \neq i} \sigma_j^\tau_{ij}(a^1_j | \theta_j, h^{t'-1})}.$$

The second equality holds because of the induction hypothesis. This completes the proof. □
Proof of Proposition 1

The proof for the first statement can be found in Proposition 2 of Lin and Palfrey (2022). For the second statement, we can prove by induction on \( t \).

**Base Case:** Consider \( t = 1 \). For any \( i \in N, \tau_i \in \mathbb{N} \) and \( \theta_i \in \Theta_i \), by Lemma 1, we know the belief about other players’ types and levels is \( \mu_i^\tau_i(\theta_i, \tau_i|\theta_i, h_0) = F(\theta_i|\theta_i) \prod_{j \neq i} \hat{P}_{ij}^\tau_j(\tau_j) \). Since \( F \) has full support, for any \( \theta_i \in \Theta_i \),

\[
\sum_{\{\tau_i: \tau_j < \tau_i, \forall j \neq i\}} \mu_i^\tau_i(\theta_i, \tau_i|\theta_i, h_0) = \sum_{\{\tau_i: \tau_j < \tau_i, \forall j \neq i\}} F(\theta_i|\theta_i) \prod_{j \neq i} \hat{P}_{ij}^\tau_j(\tau_j) = F(\theta_i|\theta_i) > 0.
\]

Hence, the statement is true at period 1.

**Induction Step:** Next, suppose there is \( t' > 1 \) such that the result holds for all \( t = 1, \ldots, t' \). We want to show the statement holds at period \( t' + 1 \) where \( h^{t'} = (h^{t'-1}, a^{t'}) \). Therefore, player \( i \)'s posterior belief at \( h^{t'} \) is

\[
\mu_i^{\tau_i}(\theta_i, \tau_i|\theta_i, h^{t'}) = \frac{\mu_i^{\tau_i}(\theta_i, \tau_i|\theta_i, h^{t'-1}) \prod_{j \neq i} \sigma_j^{\tau_j}(a_j^{t'}|\theta_j, h^{t'-1})}{\sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}: \tau_{-j} < \tau_i, \forall j \neq i\}} \mu_i^{\tau_i}(\theta_i, \tau_i|\theta_i, h^{t'-1}) \prod_{j \neq i} \sigma_j^{\tau_j}(a_j^{t'}|\theta_j, h^{t'-1})},
\]

which is well-defined because level 0 players are always in the support and \( \sigma_j^0(a_j^{t'}|\theta_j, h^{t'-1}) = \frac{1}{|A_j(h^{t'-1})|} > 0 \) for all \( j \). By induction hypothesis, we know \( supp_i(\theta_i|\tau_i, \theta_i, h^{t'-1}) = \Theta_i \). Therefore, as we fix any \( \theta_i \in \Theta_i \), we know \( \mu_i^{\tau_i}(\theta_i, (0, \ldots, 0)|\theta_i, h^{t'-1}) > 0 \), suggesting that \( \theta_i \in supp_i(\theta_i|\tau_i, \theta_i, h^{t'}) \) because

\[
\mu_i^{\tau_i}(\theta_i|\theta_i, h^{t'}) = \frac{\sum_{\{\tau_{-i}: \tau_{-j} < \tau_i, \forall j \neq i\}} \mu_i^{\tau_i}(\theta_i, \tau_i|\theta_i, h^{t'-1}) \prod_{j \neq i} \sigma_j^{\tau_j}(a_j^{t'}|\theta_j, h^{t'-1})}{\sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}: \tau_{-j} < \tau_i, \forall j \neq i\}} \mu_i^{\tau_i}(\theta_i, \tau_i|\theta_i, h^{t'-1}) \prod_{j \neq i} \sigma_j^{\tau_j}(a_j^{t'}|\theta_j, h^{t'-1})} > 0.
\]

This completes the proof of the proposition. \( \square \)

**Proof of Proposition 2**

We prove this by induction on \( t \). Let \( \sigma \) be any level-dependent strategy profile and \( F \) and \( P \) be any distributions of types and levels. First, consider \( t = 1 \). By Lemma 1, we know

\[
\mu_i^k(\theta_i, \tau_i|\theta_i, h_0) = F(\theta_i|\theta_i) \prod_{j \neq i} \hat{P}_{ij}^k(\tau_j).
\]

As the prior distribution of types is independent
across players, we can obtain that
\[
\mu_i^k(\theta, h_{-i}, h_0) = F(\theta_{-i} | \theta_i, h) \prod_{j \neq i} \tilde{P}^k_{ij}(\tau_j)
\]
\[
= \prod_{j \neq i} F_j(\theta_j) \prod_{j \neq i} \tilde{P}^k(\tau_j)
\]
\[
= \prod_{j \neq i} \left[ F_j(\theta_j) \tilde{P}^k(\tau_j) \right] = \prod_{j \neq i} \mu_i^k(\theta_j, \tau_j | \theta_i, h_0).
\]

Therefore, we know the result is true at \( t = 1 \). Next, suppose there is \( t' > 1 \) such that the result holds for all \( t = 1, \ldots, t' \). We want to show that the result holds at period \( t' + 1 \). Let \( h' \in H' \) be any available history in period \( t' + 1 \) with \( h' = (h'^{-1}, a') \). Therefore, player \( i \)'s posterior belief at history \( h' \) is
\[
\mu_i^k(\theta, h') = \frac{\mu_i^k(\theta_{-i}, h'^{-1}) \prod_{j \neq i} \sigma_j^l(a_j' | \theta_j, h'^{-1})}{\sum_{\theta'_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i} | \tau_j' < k \} \forall j \neq i} \mu_i^k(\theta'_{-i}, h'^{-1}) \prod_{j \neq i} \sigma_j^l(a_j' | \theta_j, h'^{-1})}.
\]

By induction hypothesis, we know
\[
\mu_i^k(\theta, h') = \prod_{j \neq i} \mu_i^k(\theta_j, \tau_j | \theta_i, h').
\]

Therefore, as we rearrange the posterior belief \( \mu_i^k(\theta_{-i}, h') \), we can obtain that
\[
\mu_i^k(\theta_{-i}, h') = \frac{\mu_i^k(\theta_{-i}, h'^{-1}) \prod_{j \neq i} \sigma_j^l(a_j' | \theta_j, h'^{-1})}{\sum_{\theta'_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i} | \tau_j' < k \} \forall j \neq i} \mu_i^k(\theta'_{-i}, h'^{-1}) \prod_{j \neq i} \sigma_j^l(a_j' | \theta_j, h'^{-1})}.
\]

As a result, we can conclude that
\[
\mu_i^k(\theta, h') = \prod_{j \neq i} \left[ \frac{\mu_i^k(\theta_j, \tau_j | \theta_i, h') \sigma_j^l(a_j' | \theta_j, h'^{-1})}{\sum_{\theta' | \in \Theta_j} \sum_{\tau_j' < k} \mu_i^k(\theta_j, \tau_j' | \theta_i, h') \sigma_j^l(a_j' | \theta_j, h'^{-1})} \right] = \prod_{j \neq i} \mu_i^k(\theta_j, \tau_j | \theta_i, h').
\]

This completes the proof of the proposition.
Proof of Proposition 3

By Lemma 1, we know that in the transformed (independent types) game \( \Gamma \), level \( k \) player \( i \)'s belief at \( h^t \in \mathcal{H}^t \) is

\[
\hat{\mu}^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t) = \frac{\mathcal{F}(\theta_{-i}|\theta_i) \prod_{j \neq i} \{ \hat{P}^k_{ij}(\tau_j) \prod_{l=1}^{t} \sigma_j^T(a_j'|\theta_j, h^{l-1}) \}}{\sum_{\theta'_{-i}} \sum_{\{\tau'_{-i}, \tau'_{j} < k \forall j \neq i\}} \mathcal{F}(\theta'_{-i}|\theta_i) \prod_{j \neq i} \{ \hat{P}^k_{ij}(\tau'_j) \prod_{l=1}^{t} \sigma_j^T(a_j'|\theta'_j, h^{l-1}) \}}
\]

Therefore, we can obtain that

\[
\mu^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t) = \frac{\mathcal{F}(\theta_{-i}|\theta_i) \prod_{j \neq i} \{ \hat{P}^k_{ij}(\tau_j) \prod_{l=1}^{t} \sigma_j^T(a_j'|\theta_j, h^{l-1}) \}}{\sum_{\theta'_{-i}} \sum_{\{\tau'_{-i}, \tau'_{j} < k \forall j \neq i\}} \mathcal{F}(\theta'_{-i}|\theta_i) \prod_{j \neq i} \{ \hat{P}^k_{ij}(\tau'_j) \prod_{l=1}^{t} \sigma_j^T(a_j'|\theta'_j, h^{l-1}) \}}
\]

To complete the proof, it suffices to show that for each level \( k \) player \( i \) and every \( h^t \in \mathcal{H} \setminus \mathcal{H}^T \), maximizing \( \mathbb{E}u^k_i \) given belief \( \hat{\mu}^k_i \) and \( \sigma^{-k} \) is equivalent to maximizing \( \mathbb{E}u^k_i \) given belief \( \hat{\mu}^k_i \) and \( \hat{\sigma}^{-k} = \sigma^{-k} \). This is true because the expected payoff in the original (correlated types) game is:

\[
\mathbb{E}u^k_i(\sigma|\theta_i, h^t) = \sum_{h^T \in \mathcal{H}^T} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}, \tau_{j} < k \forall j \neq i\}} \mu^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t) P_1^k(h^T|h^t, \theta, \tau_{-i}, \sigma^{-k}, \sigma^k) u_i(h^T, \theta_i, \theta_{-i}),
\]

which is proportional to

\[
\mathbb{E}\hat{u}^k_i(\sigma|\theta_i, h^t) = \sum_{h^T \in \mathcal{H}^T} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}, \tau_{j} < k \forall j \neq i\}} \mathcal{F}(\theta_{-i}|\theta_i) \hat{\mu}^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t) P_1^k(h^T|h^t, \theta, \tau_{-i}, \sigma^{-k}, \sigma^k) u_i(h^T, \theta_i, \theta_{-i})
\]

\[
= \sum_{h^T \in \mathcal{H}^T} \sum_{\theta_{-i} \in \Theta_{-i}} \sum_{\{\tau_{-i}, \tau_{j} < k \forall j \neq i\}} \hat{\mu}^k_i(\theta_{-i}, \tau_{-i}|\theta_i, h^t) P_1^k(h^T|h^t, \theta, \tau_{-i}, \sigma^{-k}, \sigma^k) \hat{u}_i(h^T, \theta_i, \theta_{-i}).
\]

This completes the proof of the proposition. \( \square \)
B Omitted Proofs for Two-Person Dirty-Faces Games

Proof of Proposition 4

Step 1: Consider any $i \in N$. If $x_{-i} = O$, then player $i$ knows his face is dirty immediately. Therefore, $D$ is a dominant strategy, suggesting $\sigma^1_i(t, O) = 1$ for all $k \geq 1$ and $1 \leq t \leq T$. If $x_{-i} = X$, player $i$'s belief of having a dirty face at period 1 is $p$. Hence, the expected payoff of choosing $D$ at period 1 is $pa - (1 - p) < 0$, implying $\sigma^k_i(1, X) = 0$ for all $k \geq 1$. Finally, since level 1 players believe the other player’s actions don’t convey any information about their own face types, the expected payoff of $D$ at each period is $pa - (1 - p) < 0$, implying $\sigma^1_i(t, X) = 0$ for any $1 \leq t \leq T$.

Step 2: Consider any level $k \geq 2$, and period $2 \leq t \leq T$. In this step, we characterize the posterior beliefs of the dynamic CH solution when $x_{-i} = X$. When the game proceeds to period $t$, the posterior belief of $(x_i, \tau_{-i}) = (f, l)$ for any $f \in \{O, X\}$ and $0 \leq l \leq k - 1$ is:

$$
\mu^k_i(f, l|t, X) = \frac{[\prod_{t' = 1}^{t-1}(1 - \sigma^l_{-i}(t', f))] \sum_{x \in \{O, X\}} \sum_{j=0}^{k-1} \prod_{t' = 1}^{t-1}(1 - \sigma^j_{-i}(t', x))}{\sum_{x \in \{O, X\}} \sum_{j=0}^{k-1} \prod_{t' = 1}^{t-1}(1 - \sigma^j_{-i}(t', x))}. \tag{A.2}
$$

Notice that by step 1, when observing a dirty face, strategic players will choose $D$ in period 1. That is, $\sigma^l_{-i}(t', O) = 1$ for all $1 \leq t' \leq t - 1$. Therefore, as the game proceeds to period $t$, level $k$ players would update that it is impossible for the other player to observe a dirty face and have a positive level of sophistication at the same time. Furthermore, let $S^k_i(t)$ be the support of level $k$ player’s marginal belief of $\tau_{-i}$ at period $t$. For any $0 \leq l \leq k - 1$,

$$
l \in S^k_i(t) \iff \sum_{x_i \in \{O, X\}} \prod_{t' = 1}^{t-1}(1 - \sigma^l_{-i}(t', x_i)) > 0,
$$

and we let $S^k_{i+}(t) \equiv S^k_i(t) \setminus \{0\}$. Therefore, we can obtain from (A.2) that

$$
\mu^k_i(X, 0|t, X) = \frac{\left(\frac{1}{2}\right)^{t-1} \sum_{j \in S^k_{i+}(t)} p_j}{\left(\frac{1}{2}\right)^{t-1} p_0 + \sum_{j \in S^k_{i+}(t)} p_j}, \quad \mu^k_i(O, 0|t, X) = \frac{\left(\frac{1}{2}\right)^{t-1} (1 - p) p_0}{\left(\frac{1}{2}\right)^{t-1} p_0 + \sum_{j \in S^k_{i+}(t)} p_j}.
$$

Moreover, for any $1 \leq k' \leq k - 1$, $\mu^{k'}_i(O, k'|t, X) = 0$, and for any $l \in S^k_{i+}(t)$,

$$
\mu^k_i(X, l|t, X) = \frac{\sum_{j \in S^k_{i+}(t)} p_j}{\left(\frac{1}{2}\right)^{t-1} p_0 + \sum_{j \in S^k_{i+}(t)} p_j}.
$$

Under the posterior beliefs, the marginal belief of having a dirty face at period $2 \leq t \leq T$ is:

$$
\mu^k_i(X|t, X) = \sum_{j=0}^{k-1} \mu^k_i(X, j|t, X) = \frac{p \left[\left(\frac{1}{2}\right)^{t-1} p_0 + \sum_{j \in S^k_{i+}(t)} p_j\right]}{\left(\frac{1}{2}\right)^{t-1} p_0 + \sum_{j \in S^k_{i+}(t)} p_j}.
$$
Therefore, the expected payoff of choosing $D$ at period $t$ is $\delta^{t-1} [(1 + \alpha)\mu^k_t(X|t, X) - 1]$, which equals to $\mathbb{E}u^k_t(D|t, X) = \frac{\delta^{t-1}}{(\frac{1}{2})^{t-1}p_0 + p \sum_{j \in S^k_{t+1}(t)} p_j} \left\{ p\alpha \left[ \left( \frac{1}{2} \right)^{t-1}p_0 + \sum_{j \in S^k_{t+1}(t)} p_j \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^{t-1}p_0 \right] \right\}. \tag{A.3}

Finally, at period $t$, level $k$ players believe the other player would choose $U$ with probability

$$\frac{1}{2} \mu^k_t(0|t, X) + \sum_{j \in S^k_{t+1}(t+1)} \mu^k_j(t|X) = \frac{(\frac{1}{2})^t p_0 + \sum_{j \in S^k_{t+1}(t+1)} p_j}{(\frac{1}{2})^{t-1} p_0 + \sum_{j \in S^k_{t+1}(t)} p_j} \tag{A.4}$$

**Step 3:** This step proves a monotonicity result—if $\sigma^k_t(t, X) = 1$, then $\sigma^{k+1}_t(t, X) = 1$ for any $k \geq 2$ and $2 \leq t \leq T$. The proof consists of two cases. We first consider period $T$. From (A.3), we know $\sigma^k_t(T, X) = 1$ if and only if

$$\frac{\delta^T}{(\frac{1}{2})^T p_0 + p \sum_{j \in S^k_{t+1}(T)} p_j} \left\{ p\alpha \left[ \left( \frac{1}{2} \right)^T p_0 + \sum_{j \in S^k_{t+1}(T)} p_j \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^T p_0 \right] \right\} \geq 0$$

$$\iff \alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{2})^T p_0}{(\frac{1}{2})^{T-1} p_0 + \sum_{j \in S^k_{t+1}(T)} p_j} \right).$$

Because $S^k_i(T) \subseteq S^{k+1}_i(T)$, we can find that

$$\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{(\frac{1}{2})^T p_0}{(\frac{1}{2})^{T-1} p_0 + \sum_{j \in S^k_{t+1}(T)} p_j} \right) \geq \left( \frac{1 - p}{p} \right) \left( \frac{(\frac{1}{2})^T p_0}{(\frac{1}{2})^{T-1} p_0 + \sum_{j \in S^{k+1}_{t+1}(T)} p_j} \right),$$

implying it is also optimal for level $k + 1$ players to choose $D$ at period $T$.

Second, we consider any period $2 \leq t \leq T - 1$. Notice that since level $k$ players would choose $D$ at period $t$, $k \notin S^{k+1}_i(t+1)$, suggesting $S^k_i(t') = S^{k+1}_i(t')$ for any $t + 1 \leq t' \leq T$. Therefore, as the game proceeds beyond period $t$, level $k$ and level $k + 1$ players will have the same value. As letting $V^k_{t+1}$ be level $k$ player’s value as period $t$, we can obtain that $V^k_{t+1} = V^{k+1}_{t+1}$. Coupled with that $S^k_i(t) \subseteq S^{k+1}_i(t)$, we can find level $k + 1$ player’s expected payoff of choosing $U$ at period $t$ satisfies

$$\frac{(\frac{1}{2})^t p_0 + \sum_{j \in S^{k+1}_{t+1}(t+1)} p_j}{(\frac{1}{2})^{t-1} p_0 + \sum_{j \in S^{k+1}_{t+1}(t)} p_j} V^{k+1}_{t+1} \leq \frac{(\frac{1}{2})^t p_0 + \sum_{j \in S^k_{t+1}(t+1)} p_j}{(\frac{1}{2})^{t-1} p_0 + \sum_{j \in S^k_{t+1}(t)} p_j} V^k_{t+1},$$

where the RHS is level $k$ player’s expected payoff of choosing $U$ at period $t$. The inequality shows level $k$ player’s expected payoff of choosing $U$ is weakly higher than level $k + 1$ player’s expected payoff of choosing $U$. It suffices to complete the proof by arguing that level $k + 1$
player’s expected payoff of \( D \) at period \( t \) is higher than level \( k \) player’s expected payoff of \( D \). This is true because \( S^2_{t+1}(t) \subseteq S^1_{t+1}(t) \) implies \( \mu^{1+1}_i(X|t, X) \geq \mu^k_i(X|t, X) \), and hence,

\[
\delta^{t-1} [(1 + \alpha)\mu^{k+1}_i(X|t, X) - 1] \geq \delta^{t-1} [(1 + \alpha)\mu^k_i(X|t, X) - 1].
\]

**Step 4:** We prove the proposition by induction on \( k \). In this step, we show the statement holds for level 2 players, which is the base case of the induction argument. Notice that from step 1, we know \( \sigma^1_i(t, X) = 0 \) for all \( 1 \leq t \leq T \), so \( S^2_{t+1}(t) = \{1\} \) for all \( 1 \leq t \leq T \). Therefore, from equation (A.3), we can obtain that the expected payoff of choosing \( D \) at period \( T \) is

\[
\mathbb{E}u^2(D|T, X) = \frac{\delta^{T-1}}{(\frac{1}{2})^{T-1} p_0 + pp_1} \left\{ p\alpha \left[ \left( \frac{1}{2} \right)^{T-1} p_0 + p_1 \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^{T-1} p_0 \right] \right\},
\]

suggesting \( D \) is optimal at period \( T \) if and only if

\[
\mathbb{E}u^2(D|T, X) \geq 0 \iff \alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} \right)^{T-1} p_0}{\left( \frac{1}{2} \right)^{T-1} p_0 + p_1} \right).
\]

For any period \( 2 \leq t \leq T - 1 \), we first prove the direction of necessity. If the player chooses \( U \) at period \( t \), the game can proceed to period \( t + 1 \) only if the other player chooses \( U \) at period \( t \) as well. From equation (A.4), we know level 2 players believe the other player would choose \( U \) at period \( t \) with probability

\[
\frac{1}{2} \mu^2_i(0|t, X) + \mu^2_i(1|t, X) = \frac{(\frac{1}{2})^t p_0 + pp_1}{(\frac{1}{2})^{t-1} p_0 + pp_1}.
\]

Thus, choosing \( U \) at period \( t \) can gain at least

\[
\frac{\delta^t}{(\frac{1}{2})^{t-1} p_0 + pp_1} \left\{ p\alpha \left[ \left( \frac{1}{2} \right)^t p_0 + p_1 \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^t p_0 \right] \right\}.
\]

Since \( U \) is always available, \( D \) is strictly dominated at period \( t \) for level 2 players if

\[
\mathbb{E}u^2(D|t, X) < \left[ \left( \frac{1}{2} \right)^t p_0 + pp_1 \right] \mathbb{E}u^2(D|t + 1, X) \iff \alpha < \left( \frac{1 - p}{p} \right) \left( \frac{\left[ \left( \frac{1}{2} \right)^t - (\frac{1}{2})^t \delta \right] p_0}{\left[ \left( \frac{1}{2} \right)^{t-1} - (\frac{1}{2})^t \delta \right] p_0 + (1 - \delta)p_1} \right).
\]

This proves the direction of necessity.

Second, we prove the sufficiency by induction on the periods. Namely, we show the sufficiency holds for any period \( T - t' \) where \( 1 \leq t' \leq T - 2 \). We now prove the statement is
true at period $T - 1$. Because

$$\alpha \geq \left(\frac{1 - p}{p}\right) \left(\frac{\left[\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta\right] p_0 \left(\frac{1}{2}\right)^{T-1} \frac{p_0}{p_0 + p_1}\right)}{\left[\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta\right] p_0 + (1 - \delta)p_1}\right) > \left(\frac{1 - p}{p}\right) \left(\frac{\left(\frac{1}{2}\right)^{T-1} p_0}{p_0 + p_1}\right),$$

we know level 2 players will choose $D$ in period $T$. Therefore, it is optimal to choose $D$ at period $T - 1$ if

$$\mathbb{E}u_i^2(D|T - 1, X) \geq \left(\frac{1}{2}\right)^{T-1} \frac{p_0 + pp_1}{p_0 + pp_1} \mathbb{E}u_i^2(D|T, X) \iff \alpha \geq \left(\frac{1 - p}{p}\right) \left(\frac{\left[\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta\right] p_0}{\left[\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta\right] p_0 + (1 - \delta)p_1}\right).$$

Now, suppose there is $t' \leq T - 2$ such that the statement holds at any period $T - t$ where $1 \leq t \leq t' - 1$. We want to show that the sufficiency holds at period $T - t'$. Because

$$\alpha \geq \left(\frac{1 - p}{p}\right) \left(\frac{\left[\left(\frac{1}{2}\right)^{T-t'} - \left(\frac{1}{2}\right)^{T-t'} \delta\right] p_0}{\left[\left(\frac{1}{2}\right)^{T-t'} - \left(\frac{1}{2}\right)^{T-t'} \delta\right] p_0 + (1 - \delta)p_1}\right) > \left(\frac{1 - p}{p}\right) \left(\frac{\left[\left(\frac{1}{2}\right)^{T-t'} - \left(\frac{1}{2}\right)^{T-t'} + 1 \delta\right] p_0}{\left[\left(\frac{1}{2}\right)^{T-t'} - \left(\frac{1}{2}\right)^{T-t'} + 1 \delta\right] p_0 + (1 - \delta)p_1}\right),$$

we know level 2 players will choose $D$ in period $T - t' + 1$ by induction hypothesis. Therefore, it is optimal to choose $D$ at period $T - t'$ if

$$\mathbb{E}u_i^2(D|T - t', X) \geq \left(\frac{1}{2}\right)^{T-t'} \frac{p_0 + pp_1}{p_0 + pp_1} \mathbb{E}u_i^2(D|T - t' + 1, X) \iff \alpha \geq \left(\frac{1 - p}{p}\right) \left(\frac{\left[\left(\frac{1}{2}\right)^{T-t'} - \left(\frac{1}{2}\right)^{T-t'} \delta\right] p_0}{\left[\left(\frac{1}{2}\right)^{T-t'} - \left(\frac{1}{2}\right)^{T-t'} \delta\right] p_0 + (1 - \delta)p_1}\right).$$

This completes the proof of sufficiency.

**Step 5:** Step 4 establishes the base case where $k = 2$. Now, suppose there is $K > 2$ such that the statement holds for all $2 \leq k \leq K$. We want to show the statement holds for level $K + 1$ players. The proof for period $T$ is straightforward. From step 3, we know if $\sigma_i^K(T, X) = 1$, then $\sigma_i^{K+1}(T, X) = 1$. Hence, it suffices to consider the case where

$$\alpha < \left(\frac{1 - p}{p}\right) \left(\frac{\left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-1} p_0 + \sum_{j=1}^{K-1} p_j}\right).$$
By induction hypothesis, we know \( \sigma^l_i(t, X) = 0 \) for all \( 1 \leq l \leq K \) and for all \( 1 \leq t \leq T \). Therefore, \( \sigma^K_i(T, X) = 1 \) if and only if \( \mathbb{E} u^K_i(D|T, X) \geq 0 \), which is equivalent to

\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{(\frac{1}{2})^{T-1} p_0}{(\frac{1}{2})^{T-1} p_0 + \sum_{j=1}^{K} p_j} \right).
\]

For any period \( 2 \leq t \leq T - 1 \), we first prove the direction of necessity. If

\[
\alpha < \left( \frac{1 - p}{p} \right) \left( \frac{(\frac{1}{2})^{t-1} p_0}{(\frac{1}{2})^{t-1} p_0 + \sum_{j=1}^{K} p_j} \right),
\]

then by induction hypothesis, we know \( \sigma^l_i(t', X) = 0 \) for all \( 1 \leq l \leq K \) and \( 1 \leq t' \leq t \). Hence, \( S^K_i(t) = \{1, \ldots, K\} \), and we can obtain from equation (A.3) that the expected payoff of \( D \) at period \( t \) is

\[
\frac{\delta^{t-1}}{(\frac{1}{2})^{t-1} p_0 + p \sum_{j=1}^{K} p_j} \left\{ p \alpha \left[ \left( \frac{1}{2} \right)^{t-1} p_0 + \sum_{j=1}^{K} p_j \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^{t-1} p_0 \right] \right\}.
\]

Furthermore, equation (A.4) suggests level \( K + 1 \) players believe the other player would choose \( U \) at period \( t \) with probability

\[
\frac{1}{2} \mu^K_i(0|t, X) + \sum_{j=1}^{K} \mu^K_i(l|t, X) = \frac{(\frac{1}{2})^t p_0 + p \sum_{j=1}^{K} p_j}{(\frac{1}{2})^{t-1} p_0 + p \sum_{j=1}^{K} p_j}.
\]

Therefore, by similar calculation as in step 4, choosing \( D \) is strictly dominated if

\[
\frac{\delta^{t-1}}{(\frac{1}{2})^{t-1} p_0 + p \sum_{j=1}^{K} p_j} \left\{ p \alpha \left[ \left( \frac{1}{2} \right)^{t-1} p_0 + \sum_{j=1}^{K} p_j \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^{t-1} p_0 \right] \right\} < \frac{\delta^t}{(\frac{1}{2})^t p_0 + p \sum_{j=1}^{K} p_j} \left\{ p \alpha \left[ \left( \frac{1}{2} \right)^t p_0 + \sum_{j=1}^{K} p_j \right] - (1 - p) \left[ \left( \frac{1}{2} \right)^t p_0 \right] \right\},
\]

which is implied by

\[
\alpha < \left( \frac{1 - p}{p} \right) \left( \frac{(\frac{1}{2})^{t-1} p_0 - (\frac{1}{2})^t \delta}{(\frac{1}{2})^{t-1} p_0 + (1 - \delta) \sum_{j=1}^{K} p_j} \right).
\]

This proves the direction of necessity.

Second, we prove the sufficiency by induction on the periods. Namely, we show the sufficiency holds for any period \( T - t' \) where \( 1 \leq t' \leq T - 2 \). We first prove the statement is true at period \( T - 1 \). By step 3, if \( \sigma^K_i(T - 1, X) = 1 \), then \( \sigma^K_i(T - 1, X) = 1 \). Therefore,
it suffices to consider the case where
\[
\left(\frac{1 - p}{p}\right) \left[ \frac{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta} \right] p_0 + (1 - \delta) \sum_{j=1}^{K} p_j \right) \leq \alpha
\]
\[
< \left(\frac{1 - p}{p}\right) \left[ \frac{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta} \right] p_0 + (1 - \delta) \sum_{j=1}^{K-1} p_j \right).
\]

By induction hypothesis, we know \(\sigma_i^l(t, X) = 0\) for all \(1 \leq t \leq T - 1\) and \(1 \leq l \leq K\). Moreover, we know \(\sigma_i^{K+1}(T, X) = 1\) because
\[
\left[ \frac{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta} \right] p_0 > \frac{(1/2)^{T-1} p_0}{(1/2)^{T-2} + \sum_{j=1}^{K} p_j}.
\]

Therefore, by similar calculation as in step 4, we can find that it is optimal for level \(K + 1\) players to choose \(D\) at period \(T - 1\) if
\[
\mathbb{E}u_i^{K+1}(D|T - 1, X) \geq \left[ \frac{(1/2)^{T-1} p_0 + \sum_{j=1}^{K} p_j}{(1/2)^{T-2} + \sum_{j=1}^{K} p_j} \right] \mathbb{E}u_i^{K+1}(D|T, X)
\]
\[
\Leftrightarrow \alpha \geq \left(\frac{1 - p}{p}\right) \left[ \frac{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} p_0}{\left(\frac{1}{2}\right)^{T-2} - \left(\frac{1}{2}\right)^{T-1} \delta} \right] p_0 + (1 - \delta) \sum_{j=1}^{K} p_j \right).
\]

Now, suppose there is \(t' \leq T - 2\) such that the statement holds for any period \(T - t\) where \(1 \leq t \leq t' - 1\). We want to show that the sufficiency at period \(T - t'\). By step 3, if \(\sigma_i^K(T - t', X) = 1\), then \(\sigma_i^{K+1}(T - t', X) = 1\). Therefore, it suffices to consider the case:
\[
\left(\frac{1 - p}{p}\right) \left[ \frac{\left(\frac{1}{2}\right)^{T-t' - 1} - \left(\frac{1}{2}\right)^{T-t'} \delta}{\left(\frac{1}{2}\right)^{T-t' - 1} - \left(\frac{1}{2}\right)^{T-t'} \delta} \right] p_0 + (1 - \delta) \sum_{j=1}^{K} p_j \right) \leq \alpha
\]
\[
< \left(\frac{1 - p}{p}\right) \left[ \frac{\left(\frac{1}{2}\right)^{T-t' - 1} - \left(\frac{1}{2}\right)^{T-t'} \delta}{\left(\frac{1}{2}\right)^{T-t' - 1} - \left(\frac{1}{2}\right)^{T-t'} \delta} \right] p_0 + (1 - \delta) \sum_{j=1}^{K-1} p_j \right).
\]

By induction hypothesis, we know \(\sigma_i^l(t, X) = 0\) for all \(1 \leq t \leq T - t'\) and \(1 \leq l \leq K\), and \(\sigma_i^{K+1}(T - t' + 1, X) = 1\). Hence, by similar calculation as in step 4, we can find that it is optimal for level \(K + 1\) players to choose \(D\) at period \(T - t'\) if
\[
\alpha \geq \left(\frac{1 - p}{p}\right) \left[ \frac{\left(\frac{1}{2}\right)^{T-t' - 1} - \left(\frac{1}{2}\right)^{T-t'} \delta}{\left(\frac{1}{2}\right)^{T-t' - 1} - \left(\frac{1}{2}\right)^{T-t'} \delta} \right] p_0 + (1 - \delta) \sum_{j=1}^{K} p_j \right).
\]

This completes the proof of the proposition. \(\square\)
Proof of Proposition 5

**Step 1:** Consider any \(i \in N\). If \(x_{-i} = O\), player \(i\) knows his face is dirty immediately, suggesting 1 is a dominant strategy and \(\tilde{\sigma}_i^k(O) = 1\) for any \(k \geq 1\). If \(x_{-i} = X\), the expected payoff of 1 is \(p \alpha - (1 - p) < 0\), implying \(\tilde{\sigma}_i^k(X) \geq 2\) for any \(k \geq 1\). Moreover, level 1 players believe the other player is level 0, so when observing \(X\), suggesting 1 is a dominant strategy and \(\tilde{\sigma}_i^1(X) = 1\).

**Step 2:** In this step, we claim for any \(i\) implying \(\tilde{\sigma}_i^1(X) = 1\) is a dominant strategy and \(\tilde{\sigma}_i^0(X) \geq 2\). Step 1.

**Step 3:** We prove the proposition by induction on \(K\). In this step, we show the statement holds for level 2 players, which is the base case of the induction argument. For any \(2 \leq j \leq T\),
the expected payoff of choosing \( j \) is \( \mathbb{E}u_i^2(j|X) = p \left[ \left( \frac{T + 2 - j}{T + 1} \delta^j \right) \frac{p_0}{p_0 + p_1} + \left( \frac{p_1}{p_0 + p_1} \right) \right] - (1 - p) \left[ \left( \frac{T + 2 - j}{T + 1} \delta^j \right) \frac{p_0}{p_0 + p_1} \right]. \)

For level 2 players and any \( 2 \leq j \leq T - 1 \), let \( \Delta_j^2 = \mathbb{E}u_i^2(j|X) - \mathbb{E}u_i^2(j+1|X) \) be the difference of expected payoffs between \( j \) and \( j + 1 \). That is,

\[
\Delta_j^2 = \delta^j - \delta^{j+1} p \alpha \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} + \left( \frac{p_1}{p_0 + p_1} \right) \right] - \delta^j \left( 1 - p \right) \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} \right],
\]

suggesting that \( j \) dominates \( j + 1 \) if and only if

\[
\Delta_j^2 \geq 0 \iff \alpha \geq \left( \frac{1 - p}{p} \right) \left( \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} \right] - \left( \frac{p_1}{p_0 + p_1} \right) \right). \]

Because the RHS is a decreasing function in \( j \), \( \Delta_j^2 \geq 0 \) implies \( \Delta_{j+1}^2 \geq 0 \). Moreover, since

\[
\mathbb{E}u_i^2(j|X) \geq 0 \iff \alpha \geq \left( \frac{1 - p}{p} \right) \left( \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} \right] - \left( \frac{p_1}{p_0 + p_1} \right) \right),
\]

\( \Delta_j^2 \geq 0 \) implies \( \mathbb{E}u_i^2(j|X) \geq 0 \) because

\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} \right] - \left( \frac{p_1}{p_0 + p_1} \right) \right) > \left( \frac{1 - p}{p} \right) \left( \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} \right] - \left( \frac{p_1}{p_0 + p_1} \right) \right).
\]

As a result, \( \tilde{\sigma}_i^2(X) \leq T \) if and only if \( \mathbb{E}u_i^2(T|X) \geq 0 \), which is equivalent to

\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{T + 2 - j}{T + 1} \frac{p_0}{p_0 + p_1} \right),
\]

and for any other \( 2 \leq t \leq T - 1 \), \( \tilde{\sigma}_i^2(X) \leq t \) if and only if

\[
\Delta_t^2 \geq 0 \iff \alpha \geq \left( \frac{1 - p}{p} \right) \left( \left[ \left( \frac{T + 2 - j}{T + 1} \delta \right) \frac{p_0}{p_0 + p_1} \right] - \left( \frac{p_1}{p_0 + p_1} \right) \right). \]

**Step 4:** Step 3 establishes the base case where \( k = 2 \). Now suppose there is \( K > 2 \) such that the statement holds for any \( 2 \leq k \leq K \). We want to show that the statement also holds for level \( K + 1 \) players. We can first obtain from step 1 that \( \tilde{\sigma}_i^{K+1}(X) \geq 2 \).

Besides, notice that for any \( 1 \leq t \leq T \) and \( 1 \leq l \leq K \), if \( \tilde{\sigma}_i^l(X) > t \), then level \( K + 1 \) player \( i \)’s expected payoff of choosing \( 2 \leq j \leq t + 1 \) is \( \mathbb{E}u_i^{K+1}(j|X) = \delta^{j-1} \left[ \alpha \left( \frac{T + 2 - j}{T + 1} \frac{p_0}{\sum_{j=0}^{K} p_j} + \sum_{j=0}^{K} \frac{p_j}{\sum_{j=0}^{K} p_j} \right) - \left( 1 - p \right) \left( \frac{T + 2 - j}{T + 1} \frac{p_0}{\sum_{j=0}^{K} p_j} \right) \right]. \)
Similar to step 3, we define $\Delta_{t'}^{K+1}$ for any $2 \leq t' \leq t$ where $\Delta_{t'}^{K+1}$ is the difference of expected payoff between choosing $t'$ and $t' + 1$. That is,

$$
\Delta_{t'}^{K+1} = \delta^{t'-1} p \alpha \left[ \left( \frac{T + 2 - t'}{T + 1} - \frac{T + 1 - t'}{T + 1} \delta \right) - \frac{p_0}{\sum_{j=0}^{K} p_j} \right] + \left( 1 - \delta \right) \frac{\sum_{j=1}^{K} p_j}{\sum_{j=0}^{K} p_j}
$$

By the same argument as in step 3, $\Delta_{t'}^{K+1} < 0$ implies $\Delta_{t-1}^{K+1} < 0$. Therefore, we can find that if $\tilde{\sigma}^{l_{i}}(X) > t$ for any $1 \leq l \leq K$, then it is strictly dominated for level $K + 1$ players to choose $t'$ (and all strategies $s < t'$) where $2 \leq t' \leq t$ if

$$
\alpha < \left( \frac{1 - p}{p} \right) \left( \frac{\sum_{j=0}^{K-1} p_j}{\sum_{j=1}^{K} p_j} \right)
$$

By induction hypothesis, $\tilde{\sigma}^{l_{i}}(X) = T + 1$ for all $1 \leq l \leq K$, so $\tilde{\sigma}_{t}^{K+1}(X) \leq T$ if $\tilde{\sigma}_{t}^{K}(X) \leq T$. Thus, it suffices to consider the case where

$$
\alpha < \left( \frac{1 - p}{p} \right) \left( \frac{2}{T+1} \frac{p_0}{\sum_{j=1}^{K} p_j} \right)
$$

On the other hand, we consider any $2 \leq t \leq T - 1$. By induction hypothesis and step 2, if $\tilde{\sigma}_{t}^{K}(X) \leq t$, then $\tilde{\sigma}_{t}^{K+1}(X) \leq t$. Hence, it suffices to complete the proof by considering

$$
\alpha < \left( \frac{1 - p}{p} \right) \left( \frac{\sum_{j=0}^{K-1} p_j}{\sum_{j=1}^{K} p_j} \right)
$$

In this case, $t < \tilde{\sigma}^{l+1}_{t}(X) \leq \tilde{\sigma}^{l}_{t}(X)$ for all $1 \leq l \leq K - 1$. Therefore, from equation (A.5), we can obtain that $\tilde{\sigma}_{t}^{K+1}(X) \leq t$ if and only if

$$
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\sum_{j=0}^{K-1} p_j}{\sum_{j=1}^{K} p_j} \right)
$$

This completes the proof of this proposition. □
Proof of Proposition 6

First of all, for any \( k \geq 2 \), it suffices to prove \( S_t^k \subset \mathcal{E}_t^k \) by showing if \( \hat{\sigma}_t^k(X) \leq T \), then \( \hat{\sigma}_t^k(X) \leq T \). This is true because

\[
\left( \frac{1 - p}{p} \right) \left( \frac{\frac{2}{T+1} p_0}{\frac{2}{T+1} p_0 + \sum_{j=1}^{k-1} p_j} \right) > \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{2} T^{-1} p_0}{\frac{1}{2} T^{-1} p_0 + \sum_{j=1}^{k-1} p_j} \right).
\]

Similarly, for other \( 2 \leq t \leq T - 1 \), it suffices to prove \( S_t^k \subset \mathcal{E}_t^k \) by showing

\[
\left( \frac{1 - p}{p} \right) \left( \frac{\frac{T+2-t}{T+1} - \frac{T+1-t}{T+1} \delta}{\frac{T+2-t}{T+1} - \frac{T+1-t}{T+1} \delta} p_0 + (1 - \delta) \sum_{j=1}^{k-1} p_j \right) \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{2} t^{-1} - \frac{1}{2} \delta}{\frac{1}{2} t^{-1} - \frac{1}{2} \delta} \sum_{j=1}^{k-1} p_j \right) \iff \frac{T + 2 - t}{T + 1} - \frac{T + 1 - t}{T + 1} \delta \geq \frac{1}{2} t^{-1} - \frac{1}{2} \delta \iff \delta \leq \frac{(2^t - 2) (T + 1) - (t - 1) 2^t}{(2^t - 1) (T + 1) - t 2^t} \equiv \delta(T, t).
\]

Notice that since \((2^t - 2)(T + 1) - (t - 1) 2^t \geq 2(T + 1) - 4 > 0\) and \((2^t - 1)(T + 1) - t 2^t \geq 3(T + 1) - 8 > 0\), we know \( \delta(T, t) > 0 \). Moreover, if \( \delta(T, t) > 1 \), then we know the inequality holds for any \( \delta \in (0, 1) \), and hence \( S_t^k \subset \mathcal{E}_t^k \). Otherwise, if \( \delta(T, t) < 1 \), the inequality does not hold for all \( \delta \). Therefore, there is no set inclusion relationship between \( S_t^k \) and \( \mathcal{E}_t^k \). In addition, we can find that \( \hat{\sigma}_t^k(X) \leq \hat{\sigma}_t^k(X) \) if \( \delta \leq \delta(T, t) \) and \( \hat{\sigma}_t^k(X) \geq \hat{\sigma}_t^k(X) \) if \( \delta > \delta(T, t) \).

Finally, as we rearrange the inequality, we can obtain that

\[
\delta(T, t) < 1 \iff \frac{(2^t - 2) (T + 1) - (t - 1) 2^t}{(2^t - 1) (T + 1) - t 2^t} < 1 \iff t < \frac{\ln(T + 1)}{\ln(2)}.
\]

This completes the proof of this proposition. \( \square \)

Proof of Proposition 7

When \( \omega = OX \), the player who observes a clean face will know his face is dirty at the beginning and choose \( D \) at period 1. Therefore,

\[
F_{OX}^D(1|\tau) = 1 - \left( \frac{1}{2} e^{-\tau} \right) \left( 1 - \frac{1}{2} e^{-\tau} \right).
\]

To show \( \| F_{OX}^*(t) - F_{OX}^D(t|\tau) \|_{\infty} \to 0 \), it suffices to prove that \( F_{OX}^D(1|\tau) \to 1 \) as \( \tau \to \infty \), which is true because

\[
\lim_{\tau \to \infty} F_{OX}^D(1|\tau) = \lim_{\tau \to \infty} 1 - \left( \frac{1}{2} e^{-\tau} \right) \left( 1 - \frac{1}{2} e^{-\tau} \right) = 1.
\]
When \( \omega = XX \), we need to show \( F^D_{XX}(1|\tau) \to 0 \) and \( F^D_{XX}(2|\tau) \to 1 \) as \( \tau \to \infty \) in order to prove the convergence. Since every level \( k \geq 1 \) would choose \( U \) in period 1 when observing a dirty face, \( F^D_{XX}(1|\tau) = 1 - [1 - (1/2)e^{-\tau}]^2 \), implying that
\[
\lim_{\tau \to \infty} F^D_{XX}(1|\tau) = \lim_{\tau \to \infty} 1 - \left[ 1 - \frac{1}{2}e^{-\tau} \right]^2 = 0.
\]

On the other hand, we need to introduce one additional piece of notation to show \( F^D_{XX}(2|\tau) \) converges to 1. Let \( K^*(\tau) \) be the lowest level of players to choose \( D \) at period 2 when observing a dirty face and the prior distribution of levels is Poisson(\( \tau \)). By Proposition 4, we know \( K^*(\tau) \) is weakly decreasing in \( \tau \), and \( K^*(\tau) \to 2 \) as \( \tau \to \infty \). Hence, \( F^D_{XX}(2|\tau) = 1 - \left[ (1/4)e^{-\tau} + \sum_{j=1}^{K^*(\tau)-1} e^{-\tau j} / j! \right]^2 \), suggesting the limit is
\[
\lim_{\tau \to \infty} F^D_{XX}(2|\tau) = \lim_{\tau \to \infty} 1 - \left[ \frac{1}{4}e^{-\tau} + \sum_{j=1}^{K^*(\tau)-1} \frac{e^{-\tau j}}{j!} \right]^2 = \lim_{\tau \to \infty} 1 - \left[ \frac{1}{4}e^{-\tau} + \tau e^{-\tau} \right]^2 = 1.
\]

This completes the proof of this proposition. \( \square \)
C Three-Person Three-Period Dirty Faces Games

C.1 Dynamic CH Solution

In this section, we will analyze a specific class of dirty faces games: the three-person three-period games. Following previous notations, we let \( N = \{1, 2, 3\} \) be the set of players. For each player \( i \), \( x_i \in \{O, X\} \) represents the player’s face type. We maintain the assumption that each player’s face type is independently and identically determined by a commonly known probability \( p = \Pr(x_i = X) \). Each player \( i \) can observe other two players’ faces \( x_{-i} \) but not their own face. If there is at least one player having a dirty face, there will be a public announcement to every player at the beginning. The announcement will tell all players whether there is a dirty face but not the identity of the players with dirty faces. In any period \( t \in \{1, 2, 3\} \), all players simultaneously choose \( U \) or \( D \). The game ends after any period where there is at least one player choosing \( D \).

We will focus on the case where there is an announcement. Otherwise, it is commonly known to all players that everyone’s face is clean. A behavioral strategy for player \( i \) is

\[
\sigma_i : \{1, 2, 3\} \times \{OO, OX, XX\} \rightarrow [0, 1],
\]

which is a mapping from the period and what player \( i \) observes to the probability of choosing \( D \). Proposition 8 characterizes the dynamic CH solution of three-person three-period dirty faces games, which predicts heterogeneous stopping periods for different levels of players.

**Proposition 8.** For any three-person three-period dirty faces game, the dynamic CH solution can be characterized as following. For any \( i \in N \),

1. \( \sigma_i^k(t, OO) = 1 \) for all \( k \geq 1 \) and \( 1 \leq t \leq 3 \).
2. \( \sigma_i^1(t, OX) = 0 \) for any \( 1 \leq t \leq 3 \). Moreover, for any \( k \geq 2 \),
   \[
   (1) \quad \sigma_i^k(1, OX) = 0,
   (2) \quad \sigma_i^k(2, OX) = 1 \text{ if and only if } \alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{2} - \frac{1}{4} \gamma_k \delta}{\frac{1}{2} - \frac{1}{4} \gamma_k \delta} \right) p_0 + (1 - \gamma_k \delta) \sum_{j=1}^{k-1} p_j
   \]
   where \( \gamma_k \equiv \left[ \frac{1}{2} p_0 + \sum_{j=1}^{k-1} p_j \right] / \left[ \frac{1}{2} p_0 + \sum_{j=1}^{k-1} p_j \right] \),
   \[
   (3) \quad \sigma_i^k(3, OX) = 1 \text{ if and only if } \alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{2} p_0}{\frac{1}{4} p_0 + \sum_{j=1}^{k-1} p_j} \right),
   \]
3. \( \sigma_i^1(t, XX) = \sigma_i^2(t, XX) = 0 \) for any \( 1 \leq t \leq 3 \). Moreover, for any \( k \geq 3 \),
(1) $\sigma^k_l(1, XX) = \sigma^k_l(2, XX) = 0$,

(2) $\sigma^k_l(3, XX) = 1$ if and only if there exists $2 \leq l \leq k-1$ such that $\sigma^l_l(2, OX) = 1$ where we denote $L^*_k = \arg \min_j \{ \sigma^l_l(2, OX) = 1 \}$, and

$$\alpha \geq \max \left\{ \left( \frac{1-p}{p} \right) \left( \frac{1}{2} - \gamma_k \delta \right) \frac{p_0}{p} \right\},$$

$$\frac{\left( 1 - \frac{1}{4} \gamma \delta \right) p_0 + \left( 1 - \gamma \delta \right) \sum_{j=1}^{k-1} p_j}{\left( 1 - \frac{1}{4} \gamma \delta \right) p_0 + \left( 1 - \gamma \delta \right) \sum_{j=1}^{l-1} p_j} \geq \frac{\left( 1 - \frac{1}{4} \gamma \delta \right) p_0 + \left( 1 - \gamma \delta \right) \sum_{j=1}^{l-1} p_j}{\left( 1 - \frac{1}{4} \gamma \delta \right) p_0 + \left( 1 - \gamma \delta \right) \sum_{j=1}^{l-1} p_j} \quad \Leftrightarrow \quad \left[ \frac{1}{4} \gamma \delta \sum_{j=1}^{l-1} p_j + (1 - \gamma \delta) \left( 1 - \frac{1}{4} \gamma \delta \right) p_l \right] \geq 0$$

Notice that the LHS of the inequality is decreasing in $\delta$ since

$$\frac{d}{d\delta} \left[ \left( -\frac{1}{4} \gamma \delta + \frac{1}{4} \gamma \delta \sum_{j=1}^{l-1} p_j + (1 - \gamma \delta) \left( \frac{1}{2} - \frac{1}{4} \gamma \delta \right) p_l \right] \right]$$

$$\leq \left( -\frac{1}{4} \gamma \delta + \frac{1}{4} \gamma \delta \sum_{j=1}^{l-1} p_j + (1 - \gamma \delta) \left( \frac{1}{2} - \frac{1}{4} \gamma \delta \right) p_l \right) < 0.$$
Therefore, we can complete the proof by showing
\[
\left(-\frac{1}{4}\gamma_{l+1} + \frac{1}{4}\gamma_l\right)\sum_{j=1}^{l-1} p_j + (1 - \gamma_{l+1}) \left(\frac{1}{2} - \frac{1}{4}\gamma_l\right) p_l \geq 0,
\]
which holds because the inequality is equivalent to
\[
\frac{p_l}{\sum_{j=1}^{l-1} p_j} \geq \frac{1}{3} \left(\gamma_{l+1} - \gamma_l\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_l\right) = \frac{3}{4} p_0 + \sum_{j=1}^{l-1} p_j.
\]

**Step 3:** We analyze level \(k\) player's behavior when \(x_{-i} = OX\) for all \(k \geq 2\). We prove this case by induction on \(k\). First, we show the statement holds for \(k = 2\). At period 3, level 2 player's belief of having a dirty face is
\[
\mu^2_i(X|3, OX) = \sum_{\tau_{-i}} \mu^2_i(X, \tau_{-i}|3, OX) = \frac{p\left(\frac{1}{4}p_0 + p_1\right)}{\frac{1}{4}p_0 + p_1}.
\]
Therefore, it is optimal to choose \(D\) at period 3 if and only if
\[
\mu^2_i(X|3, OX)\alpha - (1 - \mu^2_i(X|3, OX)) \geq 0 \iff \alpha \geq \left(\frac{1 - p}{p}\right) \left(\frac{\frac{1}{4}p_0}{\frac{1}{4}p_0 + p_1}\right).
\]
Second, we show the statement is also true for period 2. Notice that at period 2, level 2 player’s belief of having a dirty face is
\[
\mu^2_i(X|2, OX) = \sum_{\tau_{-i}} \mu^2_i(X, \tau_{-i}|2, OX) = \frac{p\left(\frac{1}{2}p_0 + p_1\right)}{\frac{1}{2}p_0 + p_1},
\]
and the belief of that two other players choose \(U\) at period 2 is
\[
\left(\frac{1}{2}p_0 + p_1\right)\left(\frac{1}{2}p_0 + pp_1\right) = \gamma_2 \left(\frac{1}{2}p_0 + pp_1\right).
\]
Conditional on reaching period 3, the expected payoff of choosing \(U\) is 0, and the expected payoff of \(D\) is
\[
\frac{\delta^2}{\frac{1}{4}p_0 + pp_1} \left[p\alpha \left(\frac{1}{4}p_0 + p_1\right) - (1 - p) \left(\frac{1}{4}p_0\right)\right].
\]
Therefore, it is optimal to choose \(D\) at period 2 if and only if
\[
\frac{\delta}{\frac{1}{4}p_0 + pp_1} \left[p\alpha \left(\frac{1}{4}p_0 + p_1\right) - (1 - p) \left(\frac{1}{4}p_0\right)\right] \geq \max \left\{\gamma_2 \left(\frac{1}{2}p_0 + pp_1\right) \frac{\delta^2}{\frac{1}{4}p_0 + pp_1} \left[p\alpha \left(\frac{1}{4}p_0 + p_1\right) - (1 - p) \left(\frac{1}{4}p_0\right)\right], 0\right\}
\]
\[
\iff \alpha \geq \max \left\{\left(\frac{1 - p}{p}\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta\right) \frac{\delta^2}{\frac{1}{2} - \frac{1}{4}\gamma_2 \delta}, \left(\frac{1 - p}{p}\right) \left(\frac{\frac{1}{4}p_0}{\frac{1}{2} - \frac{1}{4}\gamma_2 \delta}\right)\right\}.
\]
Furthermore, because for any $\delta \in (0, 1)$,
\[
\left(1 - \frac{p}{p}\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 + (1 - \gamma_2 \delta)p_1 \right) > \left(1 - \frac{p}{p}\right) \left(\frac{1}{2}p_0 + \frac{1}{2}p_0 + p_1 \right),
\]
$D$ is optimal for level 2 players at period 2 if and only if
\[
\alpha \geq \left(1 - \frac{p}{p}\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 + (1 - \gamma_2 \delta)p_1 \right).
\]
This completes the proof for level 2 players.

Now suppose there is $K > 2$ such that the statement holds for any level 2 $\leq k \leq K$. We want to show the statement holds for level $K + 1$ players. By the same argument as in the proof of Proposition 4, level $K + 1$ players would choose $D$ when it is already optimal for level $K$ players to choose $D$. Therefore, for period 3, it suffices to consider the case where
\[
\alpha < \left(1 - \frac{p}{p}\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 + (1 - \gamma_2 \delta)p_1 \right).
\]
By induction hypothesis, we know for every level 1 $\leq k \leq K$ player, they will choose $U$ in three periods when observing one dirty face. Therefore, level $K + 1$ players' beliefs of having a dirty face at period 3 when $x_{-i} = OX$ are
\[
\mu_{i}^{K+1}(X|3, OX) = \sum_{\tau_{-i}} \mu_{i}^{K+1}(X, \tau_{-i}|3, OX) = \frac{p \left(\frac{1}{4}p_0 + \sum_{j=1}^{K} p_j \right)}{\frac{1}{4}p_0 + \sum_{j=1}^{K} p_j}.
\]
Consequently, level $K + 1$ players would choose $D$ at period 3 if and only if
\[
\mu_{i}^{K+1}(X|3, OX)\alpha - (1 - \mu_{i}^{K+1}(X|3, OX)) \geq 0 \iff \alpha \geq \left(1 - \frac{p}{p}\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 + (1 - \gamma_2 \delta)p_1 \right).
\]
For period 2, because of step 2 and the induction hypothesis, it suffices to consider
\[
\alpha < \left(1 - \frac{p}{p}\right) \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 \left(\frac{1}{2} - \frac{1}{4}\gamma_2 \delta \right) p_0 + (1 - \gamma_2 \delta)p_1 \right);
\]
otherwise, level $K$ players would choose $D$ at period 2 and so do level $K + 1$ players. By similar argument, we can obtain that level $K + 1$ player would choose $D$ at period 2 if and only if
\[
\frac{\delta}{\frac{1}{2}p_0 + p \sum_{j=1}^{K} p_j} \left[ p\alpha \left(\frac{1}{2}p_0 + \sum_{j=1}^{K} p_j \right) - (1 - p) \left(\frac{1}{2}p_0 \right) \right] \geq \max \left\{ \gamma_{K+1} \left(\frac{\delta^2}{\frac{1}{2}p_0 + p \sum_{j=1}^{K} p_j} \right) \left[ p\alpha \left(\frac{1}{4}p_0 + \sum_{j=1}^{K} p_j \right) - (1 - p) \left(\frac{1}{4}p_0 \right) \right] , 0 \right\},
\]
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which is equivalent to
\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} - \frac{1}{4} \gamma K \delta \right) p_0}{\left( \frac{1}{2} - \frac{1}{4} \gamma K \delta \right) p_0 + (1 - \gamma K \delta) \sum_{j=1}^{K-1} p_j} \right).
\]

**Step 4:** We analyze level \( k \) player’s behavior when \( x_{-i} = XX \) for all \( k \geq 3 \). Consider any level \( k \geq 3 \). For level \( k \) players, they can update their beliefs about having a dirty face at period 3 only if there is some lower level of players that would choose \( D \) at period 2 when observing one dirty face. That is, \( \sigma^k_i(3, XX) = 1 \) only if there is \( 2 \leq l \leq k - 1 \) such that
\[
\alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} - \frac{1}{4} \gamma l \delta \right) p_0}{\left( \frac{1}{2} - \frac{1}{4} \gamma l \delta \right) p_0 + (1 - \gamma l \delta) \sum_{j=1}^{l-1} p_j} \right).
\]
If there exists such level of players, we denote \( L^*_k \) as the lowest level that would choose \( D \) at period 2 when observing one dirty face. In this case, level \( k \) players’ beliefs of having a dirty face at period 3 are
\[
\mu^k_i(X|3, XX) = \frac{p \left( \frac{1}{4} p_0 + \sum_{j=1}^{k-1} p_j \right)^2}{p \left( \frac{1}{4} p_0 + \sum_{j=1}^{k-1} p_j \right)^2 + (1 - p) \left( \frac{1}{4} p_0 + \sum_{j=1}^{L^*_k-1} p_j \right)^2},
\]
and expected payoff of \( D \) is greater than the expected payoff of \( U \) if and only if
\[
\mu^k_i(X|3, XX) \alpha - (1 - \mu^k_i(X|3, XX)) \geq 0 \iff \alpha \geq \left( \frac{1 - p}{p} \right) \left[ \frac{1}{4} p_0 + \sum_{j=1}^{L^*_k-1} p_j \right]^2.
\]
Therefore, we can conclude that \( \sigma^k_i(3, XX) = 1 \) if and only if
\[
\alpha \geq \max \left\{ \left( \frac{1 - p}{p} \right) \left( \frac{\left( \frac{1}{2} - \frac{1}{4} \gamma L^*_k \delta \right) p_0}{\left( \frac{1}{2} - \frac{1}{4} \gamma L^*_k \delta \right) p_0 + (1 - \gamma L^*_k \delta) \sum_{j=1}^{L^*_k-1} p_j} \right), \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{4} p_0 + \sum_{j=1}^{L^*_k-1} p_j}{\frac{1}{4} p_0 + \sum_{j=1}^{k-1} p_j} \right)^2 \right\}.
\]
This completes the proof of step 4 and this proposition. \( \square \)

**C.2 Static CH Solution**

Since the static CH solution is defined on the reduced normal form, to solve for the static CH solution, we need to transform the extensive form into its reduced normal form, which is a static Bayesian game. In the reduced normal form, players determine the earliest period to choose \( D \) given the observed \( x_{-i} \) (and hearing the public announcement). Specifically, given
each \( x_{-i} \), player \( i \)'s action set is \( S \equiv \{1, 2, 3, 4\} \) which corresponds to the stopping period or never \( D \). A strategy of player \( i \) is a function from what \( i \) observes to a distribution over the action set. That is,

\[
\tilde{\sigma}_i : \{OO, OX, XX\} \to \Delta(S).
\]

The equilibrium analysis for the strategic form is essentially the same as the extensive form. However, Proposition 9 characterizes the static CH solution which is different from the dynamic CH solution.

**Proposition 9.** For any three-person three-period dirty faces games, the static CH solution can be characterized as following. For any \( i \in N \),

1. \( \tilde{\sigma}_i^k(OO) = 1 \) for all \( k \geq 1 \).
2. \( \tilde{\sigma}_i^1(OX) = 4 \). Moreover, for any \( k \geq 2 \), \( \tilde{\sigma}_i^k(OX) > 1 \) and

   \[
   (1) \quad \tilde{\sigma}_i^k(OX) = 2 \text{ if and only if } \alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{3}{4} p \delta}{\frac{3}{4} p_0 + \sum_{j=1}^{k-1} p_j} \right),
   \]

   \[
   (2) \quad \tilde{\sigma}_i^k(OX) \leq 3 \text{ if and only if } \alpha \geq \left( \frac{1 - p}{p} \right) \left( \frac{\frac{1}{2} p_0}{\frac{1}{2} p_0 + \sum_{j=1}^{k-1} p_j} \right),
   \]

   \[
   (3) \quad \tilde{\sigma}_i^1(XX) = \tilde{\sigma}_i^2(XX) = 4. \text{ Moreover, for any } k \geq 3, \tilde{\sigma}_i^k(XX) > 2, \text{ and } \tilde{\sigma}_i^k(XX) = 3 \text{ if and only if there exists } 2 \leq l \leq k - 1 \text{ such that } \tilde{\sigma}_i^l(OX) = 2 \text{ where we denote } L_k^* = \arg \min_j \{ \tilde{\sigma}_i^j(OX) = 2 \}, \text{ and }
   \]

\[
\alpha \geq \max \left\{ \left( \frac{1 - p}{p} \right) \left( \frac{\frac{3}{4} p_0 \left( \frac{3}{4} p_0 + \sum_{j=1}^{L_{k-1}^* - 1} p_j \right) - \delta \left( \frac{1}{2} p_0 + \sum_{j=1}^{L_{k-1}^* - 1} p_j \right) \right)}{\left( \frac{3}{4} p_0 + \sum_{j=1}^{L_{k-1}^* - 1} p_j \right)^2 - \delta \left( \frac{1}{2} p_0 + \sum_{j=1}^{L_{k-1}^* - 1} p_j \right)^2} \right\} .
\]

**Proof.** **Step 1:** Consider any \( i \in N \). If \( x_{-i} = OO \), player \( i \) knows his face is dirty immediately, suggesting 1 is a dominant strategy and \( \tilde{\sigma}_i^k(OO) = 1 \) for any \( k \geq 1 \). If \( x_{-i} = OX \) or \( XX \), the expected payoff of 1 is \( p\alpha - (1 - p) < 0 \), implying \( \tilde{\sigma}_i^k(OX) \geq 2 \) and \( \tilde{\sigma}_i^k(XX) \geq 2 \) for any \( k \geq 1 \). Moreover, level 1 players believe all other players are level 0, so when observing \( OX \) or \( XX \), the expected payoff of \( t \in \{2, 3\} \) is

\[
p \left[ \sigma^{t-1} \alpha \left( \frac{5 - t}{4} \right)^2 \right] + (1 - p) \left[ -\sigma^2 \left( \frac{5 - t}{4} \right)^2 \right] = \sigma^{t-1} \left( \frac{5 - t}{4} \right)^2 \left[ p\alpha - (1 - p) \right] < 0,
\]

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implying \( \hat{\sigma}^1_t(OX) = \hat{\sigma}^1_t(XX) = 4 \).

Finally, we claim \( \hat{\sigma}^k_t(XX) \geq 3 \) for all \( k \geq 1 \), which can be proven by induction on \( k \). From previous calculation, we know \( \hat{\sigma}^1_t(XX) = 4 \), which establishes the base case. Now, suppose \( \hat{\sigma}^k_t(XX) \geq 3 \) for all \( 1 \leq i \leq K \) for some \( K > 1 \). We want to show \( \hat{\sigma}^{K+1}_t(XX) \geq 3 \). It suffices to show 2 is a strictly dominated strategy for level \( K+1 \) players, which holds as

\[
p \left[ \hat{\sigma} \left( \frac{3}{4} \frac{p_0}{\sum_{j=0}^{K-1} p_j} + \frac{\sum_{j=1}^{K-1} p_j}{\sum_{j=0}^{K-1} p_j} \right)^2 \right] + (1 - p) \left[ -\hat{\sigma} \left( \frac{3}{4} \frac{p_0}{\sum_{j=0}^{K-1} p_j} + \frac{\sum_{j=1}^{K-1} p_j}{\sum_{j=0}^{K-1} p_j} \right)^2 \right]
\]

\[
= \delta \left( \frac{3}{4} \frac{p_0}{\sum_{j=0}^{K-1} p_j} + \frac{\sum_{j=1}^{K} p_j}{\sum_{j=0}^{K} p_j} \right)^2 \left[ p\alpha - (1 - p) \right] < 0.
\]

**Step 2:** In this step, we claim for any \( K > 1 \), if \( \hat{\sigma}^{i+1}_t(OX) \leq \hat{\sigma}^i_t(OX) \) for all \( 1 \leq i \leq K-1 \), then \( \hat{\sigma}^{K+1}_t(OX) \leq \hat{\sigma}^K_t(OX) \). Notice that if \( \hat{\sigma}^{K}_t(OX) = 4 \), then there is nothing to prove. Now suppose \( \hat{\sigma}^{i+1}_t(OX) \leq \hat{\sigma}^i_t(OX) \) for all \( 1 \leq i \leq K-1 \). If \( \hat{\sigma}^{K}_t(OX) = 3 \), then it is necessary that level \( K \) player’s expected payoff of choosing 3 is non-negative. Namely,

\[
\delta^2 \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{K-1} p_j} + \frac{\sum_{j=1}^{K-1} p_j}{\sum_{j=0}^{K-1} p_j} \right) \left[ p\alpha \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{K-1} p_j} + \frac{\sum_{j=1}^{K-1} p_j}{\sum_{j=0}^{K-1} p_j} \right) - (1 - p) \left( \frac{1}{2} \sum_{j=0}^{K-1} p_j \right) \right] \geq 0,
\]

which implies:

\[
\delta^2 \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{K} p_j} + \frac{\sum_{j=1}^{K} p_j}{\sum_{j=0}^{K} p_j} \right) \left[ p\alpha \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{K} p_j} + \frac{\sum_{j=1}^{K} p_j}{\sum_{j=0}^{K} p_j} \right) - (1 - p) \left( \frac{1}{2} \sum_{j=0}^{K} p_j \right) \right] > 0.
\]

Therefore, we know \( \hat{\sigma}^{K+1}_t(OX) \leq 3 \). If \( \hat{\sigma}^{K}_t(OX) = 2 \), we want to show \( \hat{\sigma}^{K+1}_t(OX) = 2 \) as well. Notice that if \( \hat{\sigma}^{K}_t(OX) = 2 \), then it is necessary that for level \( K \) players, 2 dominates 3 and 4. Let \( M \) be the lowest level of players that would choose 2 when observing \( OX \). Then level \( K \) player’s expected payoff of choosing 2 would satisfy that

\[
\delta \left( \frac{3}{4} \frac{p_0}{\sum_{j=1}^{K} p_j} \right) \left[ p\alpha \left( \frac{3}{4} \frac{p_0}{\sum_{j=1}^{K} p_j} \right) - (1 - p) \left( \frac{3}{4} \frac{p_0}{\sum_{j=1}^{K} p_j} \right) \right]
\]

\[
\geq \max \left\{ \delta^2 \left( \frac{1}{2} \frac{p_0}{\sum_{j=1}^{M} p_j} \right) \left[ p\alpha \left( \frac{1}{2} \frac{p_0}{\sum_{j=1}^{M} p_j} \right) - (1 - p) \left( \frac{1}{2} \frac{p_0}{\sum_{j=1}^{M} p_j} \right) \right], 0 \right\},
\]

which suggests:

\[
\delta \left( \frac{3}{4} \frac{p_0}{\sum_{j=1}^{K} p_j} \right) \left[ p\alpha \left( \frac{3}{4} \frac{p_0}{\sum_{j=1}^{K} p_j} \right) - (1 - p) \left( \frac{3}{4} \frac{p_0}{\sum_{j=1}^{K} p_j} \right) \right]
\]

\[
\geq \max \left\{ \delta^2 \left( \frac{1}{2} \frac{p_0}{\sum_{j=1}^{M} p_j} \right) \left[ p\alpha \left( \frac{1}{2} \frac{p_0}{\sum_{j=1}^{M} p_j} \right) - (1 - p) \left( \frac{1}{2} \frac{p_0}{\sum_{j=1}^{M} p_j} \right) \right], 0 \right\}.
\]
Hence, we can obtain that $\tilde{\sigma}^{K+1}_i(OX) = 2$.

**Step 3:** We analyze level $k$ player’s behavior as $x_{-i} = OX$ for all $k \geq 2$. The statement can be proven by induction on $k$. Level 2 players’ the expected payoff of choosing $t \in \{2, 3\}$ is

$$\delta^{t-1} \left( \frac{5 - t}{4} \frac{p_0}{p_0 + p_1} + \frac{p_1}{p_0 + p_1} \right) \left[ p\alpha \left( \frac{5 - t}{4} \frac{p_0}{p_0 + p_1} + \frac{p_1}{p_0 + p_1} \right) - (1 - p) \left( \frac{5 - t}{4} \frac{p_0}{p_0 + p_1} \right) \right] \text{ increasing in } t$$

Therefore, $\tilde{\sigma}^2_i(OX) \leq 3$ if and only if

$$p\alpha \left( \frac{1}{2}p_0 + p_1 \right) - (1 - p) \left( \frac{1}{2}p_0 \right) \geq 0 \iff \alpha \geq \frac{\left( \frac{1}{2}p_0 \right)}{p},$$

and $\tilde{\sigma}^2_i(OX) = 2$ if and only if

$$\delta \left( \frac{3}{4}p_0 + p_1 \right) \left[ p\alpha \left( \frac{3}{4}p_0 + p_1 \right) - (1 - p) \left( \frac{3}{4}p_0 \right) \right] \geq \max \left\{ \delta^2 \left( \frac{1}{2}p_0 + p_1 \right) \left[ p\alpha \left( \frac{1}{2}p_0 + p_1 \right) - (1 - p) \left( \frac{1}{2}p_0 \right) \right], 0 \right\} \iff \alpha \geq \max \left\{ \frac{\left( \frac{1}{2}p_0 \right)}{p} \left[ \left( \frac{3}{4}p_0 \right) \left( \frac{3}{4}p_0 + p_1 \right) - \delta \left( \frac{1}{2}p_0 \right) \left( \frac{1}{2}p_0 + p_1 \right) \right], \left( \frac{1}{2}p_0 \right) \left( \frac{3}{4}p_0 \right) \right\}.$$
suggesting 4 is a dominated strategy if and only if
\[ \alpha \geq \left( \frac{1-p}{p} \right) \left( \frac{1}{2} p_0 + \frac{\sum_{j=1}^{K} p_j}{\sum_{j=0}^{k-1} p_j} \right) . \]

If \( \tilde{\sigma}_i^K(OX) = 2 \), then \( \tilde{\sigma}_i^{K+1}(OX) = 2 \) by step 2. Thus, it suffices to consider the case where \( \tilde{\sigma}_i^l(OX) \geq 3 \) for all \( 1 \leq l \leq K \). In this case, \( \tilde{\sigma}_i^{K+1}(OX) = 2 \) if and only if
\[
\delta \left( \frac{3}{4} p_0 + \sum_{j=1}^{K} p_j \right) \left[ p \alpha \left( \frac{3}{4} p_0 + \sum_{j=1}^{K} p_j \right) - (1-p) \left( \frac{3}{4} p_0 \right) \right] \geq \max \left\{ \delta^2 \left( \frac{1}{2} p_0 + \sum_{j=1}^{K} p_j \right) \left[ p \alpha \left( \frac{1}{2} p_0 + \sum_{j=1}^{K} p_j \right) - (1-p) \left( \frac{1}{2} p_0 + \sum_{j=1}^{K} p_j \right) \right], 0 \right\}
\]
\[ \iff \alpha \geq \left( \frac{1-p}{p} \right) \left[ \left( \frac{3}{4} p_0 \right) \left( \frac{3}{4} p_0 + \sum_{j=1}^{K} p_j \right) - (1-p) \left( \frac{3}{4} p_0 + \sum_{j=1}^{K} p_j \right) \right] . \]

**Step 4:** We analyze level \( k \) player's behavior when \( x_{-i} = XX \) for level \( k \geq 3 \). Consider any level \( K \geq 3 \). For level \( k \) players, they would choose 3 only if there is some level \( 2 \leq l \leq k-1 \) such that \( \tilde{\sigma}_i^l(OX) = 2 \). Let \( L_k^i \) be the lowest level of player that would choose 2 when observing \( OX \). Then level \( k \) player’s expected payoff of 3 when observing \( XX \) is
\[
\delta^2 \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{k-1} p_j} + \sum_{j=0}^{k-1} p_j \right) \left[ p \alpha \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{k-1} p_j} + \sum_{j=0}^{k-1} p_j \right) - (1-p) \left( \frac{1}{2} \frac{p_0}{\sum_{j=0}^{k-1} p_j} + \sum_{j=0}^{k-1} p_j \right) \right] ,
\]
which dominates 4 if and only if
\[ \alpha \geq \left( \frac{1-p}{p} \right) \left( \frac{1}{2} p_0 + \frac{\sum_{j=0}^{k-1} p_j}{\sum_{j=0}^{k-1} p_j} \right) . \]

Coupled with the existence of \( L_k^i \), we can find that \( \tilde{\sigma}_i^k(XX) = 3 \) if and only if
\[
\alpha \geq \max \left\{ \left( \frac{1-p}{p} \right) \left( \frac{3}{4} p_0 + \sum_{j=1}^{L_k^i-1} p_j \right) - \delta \left( \frac{1}{2} p_0 + \sum_{j=1}^{L_k^i-1} p_j \right) , \left( \frac{1-p}{p} \right) \left( \frac{1}{2} p_0 + \sum_{j=1}^{L_k^i-1} p_j \right) \right\} .
\]

This completes the proof of Proposition 9.
To better understand the dynamic and static CH solutions for three-person three-period games, we illustrate the optimal stopping periods for level 3 players when $x_{-i} = OX$ or $XX$ in Figure 8. We focus on level 3 players because they are the least sophisticated players that would possibly choose $D$ when observing two dirty faces. The intuition of this analysis applies to higher-level players.

![Figure 8: Dynamic CH (left) and static CH (right) solutions for level 3 players.](image)

Similar to the illustration for two-person games, we assume $p = 0.5$ and the level of distribution follows Poisson(1.5). Therefore, the set of three-person three-period dirty faces games is the unit square on the $(\delta, \alpha)$-plane. The dynamic and static CH solutions are plotted in the left and right columns, respectively.

Focusing on the top row, we can compare the predictions of two solutions when observing one dirty face ($x_{-i} = OX$). In this case, players tend to behave closer to the equilibrium in the dynamic CH solution than the static CH solution, suggesting that players are closer to the
equilibrium in the extensive form representation than in the strategic form representation. Besides, for both solutions, the boundary of the red area does not converge to one when $\delta$ is close to one, suggesting that players tend to stop searching earlier even if the payoff is not discounted. The intuition is that when there is an addition player, it is less likely that the game can proceed to the next period because others could be level 0 players and end the game randomly.

When observing two dirty faces, we know the earliest period that players could possibly tell their face type is period 3. Although this is the last period of the game, we can observe that the behavior still depends on the discount factor $\delta$. This is because players can tell their face type in period 3 only if other players would choose $D$ in period 2 when observing only one dirty face, which depends on $\delta$. In addition, from the bottom row of the figure, we can observe a sharp difference between the dynamic and static CH solutions when $x_{-i} = XX$. Our theory predicts the set of games where level 3 players would choose $D$ when observing $x_{-i} = XX$ in the extensive form is much larger than in the strategic form.\(^{28}\)

To summarize, the analysis of three-person three-period games demonstrates how the predictions of CH solutions would change when there are more players. Equilibrium theory predicts the optimal stopping period only depends on the number of observed dirty faces, not the number of players in the game. Yet, both CH solutions predict the behavior would be affected by the number of players. The intuition is that when there are more players, the game is more likely to be randomly terminated by level 0 players, and hence strategic players tend to be closer to the equilibrium.

### C.3 Logit-AQRE

In the three-person three-period dirty faces game implemented by Bayer and Chan (2007), each player’s strategy is defined by a nine-tuple $\{(q_t, r_t, s_t)\}_{t=1}^3$ where $q_t \equiv Q(D|t, OO)$, $r_t \equiv Q(D|t, OX)$, and $s_t \equiv Q(D|t, XX)$ for each $1 \leq t \leq 3$. At period 1 with observing two clean faces, players would estimate the payoff of $D$ and $U$ by

\[
U_{1,OO}(D) = \alpha + \epsilon_{1,OO,D} \\
U_{1,OO}(U) = \delta\alpha(1 - r_1)^2q_2 + \delta^2\alpha(1 - r_1)^2(1 - r_2)^2(1 - q_2)q_3 + \epsilon_{1,OO,U}.
\]

\(^{28}\)Specifically, level 3 players would choose $D$ in period 3 when observing two dirty faces if and only if level 2 players choose $D$ when observing one dirty face, and choosing $D$ at period 3 yields a non-negative expected payoff. Therefore, the dynamic CH predicts level 3 players would choose $D$ if and only if

$$\alpha \geq \max \left\{ \frac{16 - 7\delta}{64 - 49\delta}, \left(\frac{14}{23}\right)^2 \right\}.$$  

Similarly, the static CH predicts players would choose 3, i.e., $D$ in period 3, if and only if

$$\alpha \geq \max \left\{ \frac{27 - 16\delta}{81 - 64\delta}, \frac{16}{25} \right\} > \max \left\{ \frac{16 - 7\delta}{64 - 49\delta}, \left(\frac{14}{23}\right)^2 \right\}.$$
where $\epsilon_{1,OO,D}$ and $\epsilon_{1,OO,U}$ are independent random variables with a Weibull distribution with parameter $\lambda$. Then the logit formula suggests

$$q_1 = \frac{1}{1 + \exp \{ \lambda [\delta \alpha (1 - r_1)^2 q_2 + \delta^2 \alpha (1 - r_2)^2 (1 - q_2) q_3 - \alpha] \}}.$$  

Similarly, we can express $q_2$ and $q_3$ by:

$$q_2 = \frac{1}{1 + \exp \{ \lambda [\delta^2 \alpha (1 - r_2)^2 q_3 - \delta \alpha] \}} \quad (A.6)$$

$$q_3 = \frac{1}{1 + \exp \{ -\lambda \delta^2 \alpha \}} \quad (A.8)$$

Plugging $p = 2/3$, $\delta = 4/5$ and $\alpha = 1/4$, we can obtain that

$$q_1 = \frac{1}{1 + \exp \{ \lambda \left[ \frac{1}{5} (1 - r_1)^2 q_2 + \frac{4}{25} (1 - r_1)^2 (1 - r_2)^2 (1 - q_2) q_3 - \frac{1}{4} \right] \}} \quad (A.6)$$

$$q_2 = \frac{1}{1 + \exp \{ \lambda \left[ \frac{4}{25} (1 - r_2)^2 q_3 - \frac{1}{5} \right] \}} \quad (A.7)$$

$$q_3 = \frac{1}{1 + \exp \{ -\frac{4}{25} \lambda \}} \quad (A.8)$$

When observing at least one dirty face, players cannot tell their face type for sure. That is, at every information set, they will form posterior beliefs about their face types. We use $\mu_i(X|I_i)$ to denote player $i$’s belief (under AQRE) about having a dirty face at information set $I_i$. Therefore, at the AQRE, we can solve for the system of posterior beliefs. When observing one dirty face, we can find $\mu_i(X|1,OX) = p$, and

$$\pi_2 \equiv \mu_i(X|2,OX) = \frac{1}{1 + \left( \frac{1 - p}{p} \right) \left( \frac{1 - q_1}{1 - s_1} \right)}$$

$$\pi_3 \equiv \mu_i(X|3,OX) = \frac{1}{1 + \left( \frac{1 - s_2}{\pi_2} \right) \left( \frac{1 - q_2}{1 - s_2} \right)} = \frac{1}{1 + \left( \frac{1 - p}{p} \right) \left( \frac{(1 - q_1)(1 - q_2)}{(1 - s_1)(1 - s_2)} \right)}.$$  

Similarly, when observing two dirty faces, the beliefs would be $\mu_i(X|1,XX) = p$, and

$$\nu_2 \equiv \mu_i(X|2,XX) = \frac{1}{1 + \left( \frac{1 - p}{p} \right) \left( \frac{1 - r_1}{1 - s_1} \right)^2}$$

$$\nu_3 \equiv \mu_i(X|3,XX) = \frac{1}{1 + \left( \frac{1 - r_2}{\nu_2} \right) \left( \frac{1 - r_2}{1 - s_2} \right)^2} = \frac{1}{1 + \left( \frac{1 - p}{p} \right) \left( \frac{(1 - r_1)(1 - r_2)}{(1 - s_1)(1 - s_2)} \right)^2}.$$  

Hence, when observing one dirty face, the expected payoff to choose $D$ at period 3 is

$$\delta^2 [\alpha \pi_3 - (1 - \pi_3)] = \delta^2 [(1 + \alpha) \pi_3 - 1].$$
At period 2, the expected payoff of choosing $D$ is
\[ \delta[\alpha \pi_2 - (1 - \pi_2)] = \delta[(1 + \alpha)\pi_2 - 1], \]
while the expected payoff of choosing $U$ is
\[ \left[ \frac{\pi_2(1 - r_2)(1 - s_2) + (1 - \pi_2)(1 - r_2)(1 - q_2)}{\text{prob. to reach period 3}} \right] r_3 \delta^2[(1 + \alpha)\pi_3 - 1] = (1 - r_2) r_3 \left[ \frac{\pi_2(1 - s_2) + (1 - \pi_2)(1 - q_2)}{\text{prob. to reach period 3}} \right] \delta^2[(1 + \alpha)\pi_3 - 1] \equiv B. \]

Similarly, at period 1, the expected payoff of choosing $D$ is $p\alpha - (1 - p)$, and the expected payoff of choosing $U$ is
\[ \left[ p(1 - r_1)(1 - s_1) + (1 - p)(1 - r_1)(1 - q_1) \right] \{ r_2 \delta[(1 + \alpha)\pi_2 - 1] + (1 - r_2) B \} \equiv A. \]

Plugging $p = 2/3$, $\delta = 4/5$ and $\alpha = 1/4$, we can obtain that
\[ A = (1 - r_1) \left\{ r_2 \left[ \frac{2}{15}(1 - s_1) - \frac{4}{15}(1 - q_1) \right] + (1 - r_2) \left[ \frac{2}{3}(1 - s_1) + \frac{1}{3}(1 - q_1) \right] B \right\}, \]
\[ B = (1 - r_2) \left[ \frac{4}{25}\pi_2(1 - s_2) - \frac{16}{25}(1 - \pi_2)(1 - q_2) \right] r_3, \]
and hence the choice probabilities ($r_1, r_2, r_3$) can be expressed as
\[ r_1 = \frac{1}{1 + \text{exp} \left\{ \lambda \left[ A + \frac{1}{6} \right] \right\}} \quad \text{(A.9)} \]
\[ r_2 = \frac{1}{1 + \text{exp} \left\{ \lambda \left[ B + \frac{4}{5} - \pi_2 \right] \right\}} \quad \text{(A.10)} \]
\[ r_3 = \frac{1}{1 + \text{exp} \left\{ \lambda \left[ \frac{16}{25} - \frac{4}{5}\pi_3 \right] \right\}}. \quad \text{(A.11)} \]

Finally, when observing two dirty faces, the expected payoff to choose $D$ at period 3 is
\[ \delta^2[\alpha \nu_3 - (1 - \nu_3)] = \delta^2[(1 + \alpha)\nu_3 - 1]. \]

At period 2, the expected payoff of choosing $D$ is
\[ \delta[\alpha \nu_2 - (1 - \nu_2)] = \delta[(1 + \alpha)\nu_2 - 1], \]
while the expected payoff of choosing $U$ is
\[ \left[ \nu_2(1 - s_2)^2 + (1 - \nu_2)(1 - r_2)^2 \right] s_3 \delta^2[(1 + \alpha)\nu_3 - 1] \equiv D. \]
At period 1, the expected payoff of choosing $D$ is $p\alpha - (1 - p)$, and the expected payoff of choosing $U$ is

$$\left[ p(1 - s_1)^2 + (1 - p)(1 - r_1)^2 \right] \{ s_2\delta[(1 + \alpha)\nu_2 - 1] + (1 - s_2)D \} \equiv C.$$ 

Plugging $p = 2/3$, $\delta = 4/5$ and $\alpha = 1/4$, we can obtain that

$$C = s_2 \left[ \frac{2}{15}(1 - s_1)^2 - \frac{4}{15}(1 - r_1)^2 \right] + (1 - s_2) \left[ \frac{2}{3}(1 - s_1)^2 + \frac{1}{3}(1 - r_1)^2 \right] D,$$

$$D = s_3 \left[ \frac{4}{25}\nu_2(1 - s_2)^2 - \frac{16}{25}(1 - \nu_2)(1 - r_2)^2 \right],$$

and therefore the choice probabilities $(s_1, s_2, s_3)$ can be expressed as

$$s_1 = \frac{1}{1 + \exp\{\lambda [C + \frac{1}{6}]\}} \quad \text{(A.12)}$$

$$s_2 = \frac{1}{1 + \exp\{\lambda [D + \frac{4}{5} - \nu_2]\}} \quad \text{(A.13)}$$

$$s_3 = \frac{1}{1 + \exp\{\lambda [\frac{16}{25} - \frac{4}{5}\nu_3]\}} \quad \text{(A.14)}$$

Given each $\lambda$, the system of nine equations (from (A.6) to (A.14)) with nine unknowns $(q_t, r_t, s_t$ where $t \in \{1, 2, 3\}$) can be solved uniquely.
D Robustness of Estimation Results

To see whether players will learn from repeated play, we separate the data by the first and the last seven rounds and compare the results of both subsamples. Table A.1 to Table A.3 report the results for round 1 to round 7 data, and Table A.4 to Table A.6 show the results for round 8 to round 14 data.

Table A.1: Estimation Results for Two-Person Dirty Faces Games (Round 1 to 7)

| $(t, x_{-i})$ | $N$ | $\sigma^*_i(t, x_{-i})$ | $\hat{\sigma}_i(t, x_{-i})$ | Dynamic CH | Static CH | AQRE |
|--------------|-----|-----------------|----------------|----------|----------|------|
| $(1, O)$     | 68  | 1.000           | 0.926          | 0.846    | 0.773    | 0.709 |
| $(2, O)$     | 4   | 1.000           | 0.750          | 0.500    | 0.500    | 0.815 |
| $(1, X)$     | 194 | 0.000           | 0.211          | 0.154    | 0.114    | 0.204 |
| $(2, X)$     | 84  | 1.000           | 0.560          | 0.480    | 0.458    | 0.584 |

Parameters

| $	au$ | S.E.   | $\lambda$ | S.E. | $\lambda$ |
|--------|--------|------------|------|------------|
|        | (0.122)|          |      | (0.691)    |

Fitness

| LL     | AIC    | BIC    |
|--------|--------|--------|
| -183.64| 369.28 | 373.13 |
| -193.41| 388.82 | 392.68 |
| -187.92| 377.85 | 381.71 |

Vuong Test

| $p$-value |
|-----------|
| $< 0.001$ |
| 0.236     |

Note: There are 147 games (rounds × groups) in total.
Table A.2: Estimation Results for Three-Person Dirty Faces Games (Round 1 to 7)

| $(t, x_{-i})$ | $N$ | $\sigma_i(t, x_{-i})$ | $\hat{\sigma}_i(t, x_{-i})$ | Dynamic CH | Static CH | AQRE |
|---------------|-----|------------------------|-----------------|------------|-----------|-------|
| $(1, OO)$     | 27  | 1.000                  | 0.889            | 0.660      | 0.361     | 0.704 |
| $(2, OO)$     | 2   | 1.000                  | 0.500            | 0.500      | 0.333     | 0.699 |
| $(3, OO)$     | 0   | 1.000                  | —               | 0.500      | 0.500     | 0.686 |
| $(1, OX)$     | 142 | 0.000                  | 0.437            | 0.340      | 0.213     | 0.300 |
| $(2, OX)$     | 22  | 1.000                  | 0.727            | 0.257      | 0.271     | 0.503 |
| $(3, OX)$     | 2   | 1.000                  | 0.500            | 0.291      | 0.371     | 0.606 |
| $(1, XX)$     | 155 | 0.000                  | 0.245            | 0.340      | 0.213     | 0.333 |
| $(2, XX)$     | 63  | 0.000                  | 0.175            | 0.257      | 0.271     | 0.324 |
| $(3, XX)$     | 28  | 1.000                  | 0.179            | 0.173      | 0.371     | 0.471 |

| Parameters   | $\tau$ | 0.386 | 0.160 | —       |
|--------------|---------|-------|-------|---------|
| S.E.         | (0.063) | (0.057) | —     |         |
| $\lambda$    | —       | —     | 4.884 |         |
| S.E.         | —       | —     | (0.569) |        |

| Fitness   | LL | -272.95 | -299.80 | -273.76 |
| AIC       | 547.90 | 601.61 | 549.51 |
| BIC       | 551.99 | 605.70 | 553.60 |

| Vuong Test | 4.102 | 0.174 |
| p-value    | < 0.001 | 0.862 |

Note: There are 112 games (rounds × groups) in total.

Table A.3: Results for Pooled Data (Round 1 to 7)

| Parameters   | $\tau$ | 1.000 | 0.284 | —       |
|--------------|---------|-------|-------|---------|
| S.E.         | (0.078) | (0.048) | —     |         |
| $\lambda$    | —       | —     | 5.910 |         |
| S.E.         | —       | —     | (0.424) |        |

| Fitness   | LL | -469.01 | -519.36 | -465.83 |
| AIC       | 940.02 | 1040.72 | 933.67 |
| BIC       | 944.69 | 1045.39 | 938.34 |

| Vuong Test | 6.134 | -0.367 |
| p-value    | < 0.001 | 0.714 |

Note: There are 259 games (rounds × groups) in total.
Table A.4: Estimation Results for Two-Person Dirty Faces Games (Round 8 to 14)

| $(t, x_{-i})$ | $N$ | $\sigma_i(t, x_{-i})$ | $\hat{\sigma}_i(t, x_{-i})$ | Dynamic CH | Static CH | AQRE |
|---------------|-----|------------------------|-------------------------------|-----------|----------|------|
| $\sigma_i(t, x_{-i})$ |     |                        |                               |           |          |      |
| $(1, O)$      | 55  | 1.000                  | 0.964                         | 0.873     | 0.810    | 0.713 |
| $(2, O)$      | 2   | 1.000                  | 0.000                         | 0.500     | 0.500    | 0.829 |
| $(1, X)$      | 197 | 0.000                  | 0.208                         | 0.127     | 0.095    | 0.187 |
| $(2, X)$      | 86  | 1.000                  | 0.674                         | 0.528     | 0.500    | 0.597 |
| Parameters   |     |                        |                               |           |          |      |
| $\tau$       |     |                        |                               | 1.369     | 1.257    | —    |
| S.E.         |     |                        |                               | (0.132)   | (0.140)  | —    |
| $\lambda$    |     |                        |                               | —         | —        | 7.903 |
| S.E.         |     |                        |                               | —         | —        | (0.703) |
| Fitness      |     |                        |                               |           |          |      |
| LL           |     | -176.54                | -187.58                       | -180.34   |          |      |
| AIC          |     | 355.09                 | 377.17                        | 362.67    |          |      |
| BIC          |     | 358.91                 | 381.00                        | 366.50    |          |      |
| Vuong Test   |     |                        |                               |           |          |      |
| p-value      |     |                        |                               |           |          |      |

Note: There are 147 games (rounds × groups) in total.
Table A.5: Estimation Results for Three-Person Dirty Faces Games (Round 8 to 14)

| $(t, x_{-i})$ | $N$ | $\sigma_i^*(t, x_{-i})$ | $\hat{\sigma}_i(t, x_{-i})$ | Dynamic CH | Static CH | AQRE |
|---------------|-----|-------------------------|-----------------------------|------------|-----------|-------|
| $(1, OO)$     | 21  | 1.000                   | 0.952                       | 0.649      | 0.334     | 0.722 |
| $(2, OO)$     | 0   | 1.000                   | —                           | 0.500      | 0.500     | 0.713 |
| $(3, OO)$     | 0   | 1.000                   | —                           | 0.500      | 0.500     | 0.713 |
| $(1, OX)$     | 138 | 0.000                   | 0.348                       | 0.351      | 0.222     | 0.267 |
| $(2, OX)$     | 38  | 1.000                   | 0.500                       | 0.270      | 0.285     | 0.518 |
| $(3, OX)$     | 8   | 1.000                   | 0.125                       | 0.290      | 0.399     | 0.643 |
| $(1, XX)$     | 165 | 0.000                   | 0.236                       | 0.351      | 0.222     | 0.307 |
| $(2, XX)$     | 82  | 0.000                   | 0.244                       | 0.270      | 0.285     | 0.290 |
| $(3, XX)$     | 28  | 1.000                   | 0.536                       | 0.185      | 0.399     | 0.495 |

Parameters

| Parameter | $\tau$ | S.E. | $\lambda$ | S.E. | $\lambda$ | S.E. |
|-----------|--------|------|------------|------|------------|------|
| $\tau$   | 0.355  | (0.059) |  —        | —    |  —         | —    |
| S.E.     | 0.119  | (0.054) |  —        | —    |  —         | —    |
| $\lambda$ |  —    | —    |  5.689    | (0.571) |  —         | —    |

Fitness

| Metric | $LL$ | $AIC$ | $BIC$ |
|--------|------|-------|-------|
| $LL$   | -302.29 | -308.51 | -290.80 |
| $AIC$  | 606.58 | 619.01 | 583.60 |
| $BIC$  | 610.75 | 623.18 | 587.77 |

Vuong Test

| Metric | $\Delta LL$ | $p$-value |
|--------|--------------|-----------|
| $\Delta LL$ | 0.926 | 0.354 |
| $p$-value  | -1.831 | 0.067 |

Note: There are 112 games (rounds $\times$ groups) in total.

Table A.6: Results for Pooled Data (Round 8 to 14)

| Metric | Dynamic CH | Static CH | AQRE |
|--------|------------|-----------|------|
| $LL$   | -487.73    | -526.90   | -474.21 |
| $AIC$  | 977.46     | 1055.80   | 950.43 |
| $BIC$  | 982.17     | 1060.51   | 955.14 |

Vuong Test

| Metric | $\Delta LL$ | $p$-value |
|--------|--------------|-----------|
| $\Delta LL$ | 4.332 | < 0.001 |
| $p$-value  | -1.550 | 0.121 |

Note: There are 259 games (rounds $\times$ groups) in total.