ON CONTINUOUS CHOICE OF RETRACTIONS ON NONCONVEX SUBSETS

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Abstract. For a Banach space \( B \) and for a class \( A \) of its bounded closed retracts, endowed with the Hausdorff metric, we prove that retractions on elements \( A \in A \) can be chosen to depend continuously on \( A \), whenever nonconvexity of each \( A \in A \) is less than \( \frac{1}{2} \). The key geometric argument is that the set of all uniform retractions onto an \( \alpha \)-paraconvex set (in the spirit of E. Michael) is \( \alpha^{1-\alpha} \)-paraconvex subset in the space of continuous mappings of \( B \) into itself. For a Hilbert space \( H \) the estimate \( \frac{1}{2-\alpha} \) can be improved to \( \frac{\alpha(1+\alpha^2)}{1-\alpha^2} \) and the constant \( \frac{1}{2} \) can be reduced to the root of the equation \( \alpha + \alpha^2 + a^3 = 1 \).

0. Introduction

The initial source of the present paper was two-fold. Probably it was Bing [12] who first asked whether there exists a continuous function which selects a point from each arc of the Euclidean plane. Hamström and Dyer [3] observed that this problem reduces to the problem of continuous choice of retractions onto arcs. In fact, it suffices to consider the images of a chosen point with respect to continuously chosen retractions. A simple construction based, for example, on the \( \sin(\frac{1}{x}) \)–curve shows that in general there are no continuously chosen retractions for the family of arcs topologized by the Hausdorff metric.

Therefore a stronger topology is needed for an affirmative answer. In fact, for any homeomorphic compact subsets \( A_1 \) and \( A_2 \) of a metric space \( B \) one can consider the so-called \( h \)–metric \( d_h(A_1, A_2) \) defined by:

\[
d_h(A_1, A_2) = \sup \{ \text{dist}(x, h(x)) : h \text{ runs over all homeomorphisms of } A_1 \text{ onto } A_2 \}
\]

and consider the completely regular topology on the family of all subarcs, generated by such a metric. With respect to this topology, Pixley [12] affirmatively resolved the problem of continuous choice for retractions onto subarcs in an arbitrary separable metric space.

By returning to the more standard Hausdorff topology in the subspace \( \text{exp}_{\text{AR}}(B) \) of all compact absolute retracts in \( B \) one can try to search for a degree of nonconvexity of such a retracts. In the simplest situation, for convex exponent \( \text{exp}_{\text{conv}}(B) \)

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consisting of all compact convex subsets of $B$, continuous choice of retraction is a direct corollary of the following Michael theorem [8]:

**Convex-Valued Selection Theorem.** Any multivalued mapping $F : X \rightarrow Y$ admits a continuous singlevalued selection $f : X \rightarrow Y$, $f(x) \in F(x)$, provided that:

1. $X$ is a paracompact space;
2. $Y$ is a Banach space;
3. $F$ is a lower semicontinuous (LSC) mapping;
4. For every $x \in X$, $F(x)$ is a nonempty convex subset of $Y$; and
5. For every $x \in X$, $F(x)$ is a closed subset of $Y$.

In fact, let $X = \exp_{\text{conv}}(B)$, let $Y$ be the space $\mathcal{C}_b(B, B)$ of all continuous bounded mappings of $B$ into itself and suppose that $F : X \rightarrow Y$ associates to each $A \in X$ the nonempty set of all retractions of $B$ onto $A$. Then all hypotheses (1)–(5) can be verified and the conclusion of the theorem gives the desired continuously chosen retractions.

However, what can one say about nonconvex absolute retracts? In general, there exists an entire branch of mathematics dedicated to various generalizations and versions of the convexity. In our opinion, even if one simply lists the titles of "generalized convexities" one will find as a minimum, nearly 20 different notions. Among them are Menger’s metric convexity [7], Levy’s abstract convexity [5], Michael’s convex structures [9], Prodanov’s algebraic convexity [13], Mägel’s paved spaces [6], van de Vel’s topological convexity [21], decomposable sets [1], Belyawski’s simplicial convexity [2], Horvath’s structures [4], Saveliev’s convexity [18], and many others.

Typically, a creation of "generalized convexities", is usually related to an extraction of several principal properties of the classical convexity which are used in one of the key mathematical theorems or theories and, consequently deals with analysis and generalization of these properties in maximally possible general settings. Based on the ingenious idea of Michael who proposed in [10] the notion of a paraconvex set, the authors of [14-17, 19] systematically studied another approach to weakening or controlled omission of convexity on a set of principal theorems of multivalued analysis and topology. Roughly speaking, to each closed subset $P \subset B$ of a Banach space we have associated a numerical function, say $\alpha_P : (0, +\infty) \rightarrow [0, 2)$, the so-called function of nonconvexity of $P$. The identity $\alpha_P \equiv 0$ is equivalent to the convexity of $P$ and the more $\alpha_P$ differs from zero the "less convex" is the set $P$.

Such classical results about multivalued mappings as the Michael selection theorem, the Cellina approximation theorem, the Kakutani-Glicksberg fixed point theorem, the von Neumann - Sion minimax theorem, etc. are valid with the replacement of the convexity assumption for values $F(x)$, $x \in X$ of a mapping $F$ by some appropriate control of their functions of nonconvexity.

In comparison with usual ideas of "generalized convexities", we never define in this approach, for example, a "generalized segment" joining $x \in P$ and $y \in P$. We look only for the distances between points $z$ of the classical segment $[x, y]$ and the set $P$ and look for the ratio of these distances and the size of the segment. So the following natural question immediately arises: Does paraconvexity of a set with respect to the classical convexity structure coincide with convexity under some generalized convexity structure? Corollaries 2.5 and 2.6, based on continuous choice of a retraction, in particular provide an affirmative answer.
1. Preliminaries

Below we denote by $D(c, r)$ the open ball centered at the point $c$ with the radius $r$ and denote by $D_r$ an arbitrary open ball with the radius $r$ in a metric space. So for a nonempty subset $P \subset Y$ of a normed space $Y$, and for an open $r$-ball $D_r \subset Y$ we define the relative precision of an approximation of $P$ by elements of $D_r$ as follows:

$$\delta(P, D_r) = \sup \left\{ \frac{\text{dist}(q, P)}{r} : q \in \text{conv}(P \cap D_r) \right\}.$$

For a nonempty subset $P \subset Y$ of a normed space $Y$ the function $\alpha_P(\cdot)$ of nonconvexity of $P$ associates to each positive number $r$ the following nonnegative number

$$\alpha_P(r) = \sup \{ \delta(P, D_r) : D_r \text{ is an open } r\text{-ball} \}.$$

Clearly, the identity $\alpha_P(\cdot) \equiv 0$ is equivalent to the convexity of the closed set $P$.

**Definition 1.1.** For a nonnegative number $\alpha$ the nonempty closed set $P$ is said to be $\alpha$-paraconvex, whenever $\alpha$ majorates the function $\alpha_P(\cdot)$ of nonconvexity of the set, i.e.

$$\text{dist}(q, P) \leq \alpha \cdot r, \quad \forall q \in \text{conv}(P \cap D_r).$$

The nonempty closed set $P$ is said to be paraconvex if it is $\alpha$-paraconvex for some $\alpha < 1$.

Recall, that a multivalued mapping $F : X \to Y$ between topological spaces is called lower semicontinuous (LSC for shortness) if for each open $U \subset Y$, its full preimage, i.e. the set

$$F^{-1}(U) = \{ x \in X | F(x) \cap U \neq \emptyset \}$$

is open in $X$. Recall also that a singlevalued mapping $f : X \to Y$ is called a selection (resp. an $\varepsilon$-selection) of a multivalued mapping $F : X \to Y$ if $f(x) \in F(x)$ (resp. $\text{dist}(f(x), F(x)) < \varepsilon$), for all $x \in X$. Michael [9] proved the following selection theorem:

**Paraconvex-Valued Selection Theorem.** For each number $0 \leq \alpha < 1$ any multivalued mapping $F : X \to Y$ admits a continuous singlevalued selection whenever:

1. $X$ is a paracompact space;
2. $Y$ is a Banach space;
3. $F$ is a lower semicontinuous (LSC) mapping; and
4. all values $F(x)$, $x \in X$ are $\alpha$-paraconvex.

As a corollary, every $\alpha$-paraconvex set, $\alpha < 1$, is contractible and moreover, it is an absolute extensor (AE) with respect to the class of all paracompact spaces. Hence, it is an absolute retract (AR). Moreover by [17], metric $\varepsilon$-neighborhood of a paraconvex set in any uniformly convex Banach space $Y$, is also a paraconvex set, and hence is also an AR.

For each number $0 \leq \alpha < 1$ we denote by $\exp_{\alpha}(B)$ the family of all $\alpha$-paraconvex compact subsets and by $b\exp_{\alpha}(B)$ the family of all $\alpha$-paraconvex bounded subsets of a Banach space $B$ endowed with the Hausdorff metric. Recall that the Hausdorff
distance between two bounded sets is defined as the infimum of the set of all \( \varepsilon > 0 \) such that each of the sets is a subset of an open \( \varepsilon \)-neighborhood of the other set.

For each retract \( A \subseteq B \) we denote by \( \text{Retr}(A) \) the set of all continuous retraction of \( B \) onto \( A \). So the multivalued mapping \( \text{Retr} \) associates to each retract \( A \subseteq B \) the set of all retractions of \( B \) onto \( A \). For checking of the lower semicontinuity of a mappings into the spaces of retractions and for proving paraconvexity of these spaces we also need the notion of an uniform retraction (in terminology of [11]), or a regular retraction (in terminology of [20]). Recall that a continuous retraction \( R : B \to A \) is said to be uniform (with respect to \( A \)) if

\[
\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in B : \quad \text{dist}(x, A) < \delta \Rightarrow \text{dist}(x, R(x)) < \varepsilon.
\]

We emphasize that a uniform retraction in general is not a uniform mapping in the classical metric sense. Clearly, each continuous retraction onto a compact subset is uniform with respect to the set. So we denote by \( U\text{Retr}(A) \) the set of all continuous retractions of \( B \) onto \( A \) which are uniform with respect to \( A \).

2. The Banach space case

**Theorem 2.0.** Let \( 0 \leq \alpha < \frac{1}{2} \) and \( F : X \to \text{exp}_\alpha(B) \) be a continuous multivalued mapping of a paracompact space \( X \) into a Banach space \( B \). Then there exists a continuous singlevalued mapping \( \delta : X \to C_\beta(B, B) \) such that for every \( x \in X \) the mapping \( \delta_x : B \to B \) is a continuous retraction of \( B \) onto the value \( F(x) \) of \( F \).

**Sketch of proof of Theorem 2.0.** Proposition 2.4 below is a corollary of the Paraconvex-valued selection theorem due to Propositions 2.1-2.3 and the fact that \( 0 \leq \frac{\alpha}{1 - \alpha} < 1 \Leftrightarrow 0 \leq \alpha < \frac{1}{2} \). In turn, Theorem 2.0. follows directly from Proposition 2.4, it suffices to put \( \delta = \mathcal{R} \circ F \).

**Proposition 2.1.** For every \( 0 \leq \alpha < 1 \) and for each bounded \( \alpha \)-paraconvex subset \( P \) the set \( U\text{Retr}(P) \) is a nonempty closed subset of \( C_\beta(B, B) \).

**Proposition 2.2.** For every \( 0 \leq \alpha < 1 \) and for every bounded \( \alpha \)-paraconvex subset \( P \subset B \) the set \( U\text{Retr}(P) \) is an \( \frac{\alpha}{1 - \alpha} \)-paraconvex subset of \( C_\beta(B, B) \).

**Proposition 2.3.** For every \( 0 \leq \alpha < 1 \) the restriction \( U\text{Retr}|_{\text{exp}_\alpha(B)} : P \to U\text{Retr}(P) \) is lower semicontinuous.

**Proposition 2.4.** For every \( 0 \leq \alpha < \frac{1}{2} \) the restriction \( U\text{Retr}|_{\text{exp}_\alpha(B)} : P \to U\text{Retr}(P) \) admits a singlevalued continuous selection, say

\[
\mathcal{R} : \text{exp}_\alpha(B) \to C_\beta(B, B), \quad \mathcal{R}_P \in U\text{Retr}(P).
\]

**Proof of Proposition 2.1.** Clearly for each bounded closed retract \( A \) the sets \( \text{Retr}(A) \) and \( U\text{Retr}(A) \) are closed in the Banach space \( C_\beta(B, B) \). To obtain the nonemptiness of \( \text{Retr}(P) \) for the \( \alpha \)-paraconvex set \( P \) it suffices to apply the Paraconvex-valued selection theorem to the mapping \( F : B \to B \) defined by setting \( F(x) = P \) for \( x \in B \setminus P \) and \( F(x) = \{x\} \) for \( x \in P \). To construct a uniform retraction \( R : B \to P \) one must study more in detail the idea of the proof of the Paraconvex-valued selection theorem.

Let us denote by \( d(x) \) the distance between a point \( x \in B \) and a fixed \( \alpha \)-paraconvex subset \( P \subset B \). For every \( x \in B \setminus P \) first consider the intersection of the set
by setting $C$ for the constant mappings $x, x \in P$ once again due to the $\alpha$-construction.

Theorem guarantees the existence of a continuous singlevalued selection, say $h_1 : B \setminus P \to B$, $h_1(x) \in H_1(x)$.

For an arbitrary $\alpha < \beta < 1$ the $\alpha$-paraconvexity of $P$ implies the inequalities

$$
\text{dist}(h_1(x), P) < \beta \cdot 2d(x), \quad \text{dist}(x, h_1(x)) \leq 2d(x), \quad x \in B \setminus P.
$$

Similarly, define the convex-valued and closed-valued LSC mapping $H_2 : B \setminus P \to B$ by setting

$$
H_2(x) = \overline{\text{conv}}\{P \cap D(h_1(x), \beta \cdot 2d(x))\}, \quad x \in B \setminus P.
$$

For its continuous singlevalued selection $h_2 : B \setminus P \to B$, $h_2(x) \in H_2(x)$ we see that for every $x \in B \setminus P$,

$$
\text{dist}(h_2(x), P) \leq \alpha \cdot \beta \cdot 2d(x) < \beta^2 \cdot 2d(x), \quad \text{dist}(h_2(x), h_1(x)) \leq \beta \cdot 2d(x),
$$

once again due to the $\alpha$-paraconvexity of $P$.

One can inductively construct a sequence $\{h_n\}_{n=1}^\infty$ of continuous singlevalued mappings $h_n : B \setminus P \to B$ such that for every $x \in B \setminus P$,

$$
\text{dist}(h_{n+1}(x), P) < \beta^{n+1} \cdot 2d(x), \quad \text{dist}(h_{n+1}(x), h_n(x)) \leq \beta^n \cdot 2d(x).
$$

So the sequence $\{h_n\}_{n=1}^\infty$ is locally uniformly convergent and hence its pointwise limit $h(x) = \lim_{n \to \infty} h_n(x)$ is well-defined and continuous. Moreover, $h(x) \in P$, $x \in B \setminus P$, due to the closedness of $P$ and convergency of $\{h_n\}_{n=1}^\infty$.

Hence the mapping $R : B \to P$ defined by $R(x) = h(x)$, $x \in B \setminus P$ and $R(x) = x$, $x \in P$ is a retraction of $B$ onto $P$ which is continuous over the set $B \setminus P$ by construction.

To finish the proof we estimate that for every $x \in B \setminus P$:

$$
\text{dist}(x, h(x)) \leq \text{dist}(x, h_1(x)) + \sum_{n=1}^\infty \text{dist}(h_n(x), h_{n+1}(x)) \leq \text{dist}(x, h_1(x)) + \sum_{n=1}^\infty \beta^n \cdot 2d(x) \leq 2d(x)(1 + \beta + \beta^2 + \beta^3 + \ldots) = C \cdot d(x),
$$

for the constant $C = \frac{1}{1-\beta}$. So for $x_0 \in P$ and for $x \in B \setminus P$ we have

$$
\text{dist}(R(x_0), R(x)) = \text{dist}(x_0, h(x)) \leq \text{dist}(x_0, x) + \text{dist}(x, h(x)) \leq \text{dist}(x_0, x) + C \cdot d(x) \leq (1 + C)\text{dist}(x_0, x).
$$

The continuity of the retraction $R : B \to P$ over the closed subset $P \subset B$ and its uniformity clearly follow from the last inequality. \qed
Proof of Proposition 2.2. Pick an open ball $D(h,r)$ with the radius $r$ in the space $C_b(B,B)$ centered at the mapping $h \in C_b(B,B)$ which intersects with the closed set $U\text{Retr}(P)$. Let $R_1, R_2, ..., R_n$ be elements of the intersection $D(h,r) \cap U\text{Retr}(P)$ and let $Q$ be a convex combination of $R_1, R_2, ..., R_n$. We want to estimate the distance between $Q$ and $U\text{Retr}(P)$.

Pick a point $x \in B$. Passing from the mappings $h, Q, R_1, R_2, ..., R_n \in C_b(B,B)$ to their values at $x$ we find the open ball $D(h(x),r)$ with the radius $r$ in the space $B$ centered at $h(x) \in B$, the finite set $\{R_1(x), R_2(x), ..., R_n(x)\}$ of elements from the intersection $D(h(x),r) \cap P$ and the point $Q(x) \in \text{conv}(D(h(x),r) \cap P)$. Having all fixed continuous mappings $h, Q, R_1, R_2, ..., R_n \in C_b(B,B)$ we see that all points $h(x), Q(x), R_1(x), R_2(x), ..., R_n(x) \in B$ continuously depend on $x \in B$.

Let $r(x)$ be the Chebyshev radius of the compact convex finite-dimensional set

$$
\Delta(x) = \text{conv}\{R_1(x), ..., R_n(x)\},
$$

i.e. the infimum (in fact, the minimum), of the set of radii of all closed balls containing $\Delta(x)$. Clearly $r(x) < r, x \in X$. Moreover $r(x)$ continuously depends on $x$ and for any positive $\gamma > 0$ the entire set $\Delta(x)$ lies in the open ball $D(C(x), r(x) + \gamma)$ for some suitable point $C(x) \in \Delta(x)$.

Henceforth, the $\alpha$–paraconvexity of $P$ implies that for an arbitrary $\alpha < \beta$ the inequality

$$
\text{dist}(Q(x), P) < \beta \cdot g(x), \quad g(x) = r(x) + \gamma
$$

holds. So, the multivalued mapping

$$
F_1(x) = \text{conv}\{P \cap D(Q(x), \beta \cdot g(x))\}.
$$

is a LSC mapping with nonempty convex and closed values. Note that for each $x \in P$ all points $R_1(x), R_2(x), ..., R_n(x), Q(x)$ coincide with $x$ because all $R_1, ..., R_n$ are retractions onto $P$. So the identity mapping $id|_{P}$ is a continuous selection of $F_1|_{P}$. Therefore the mapping $G_1$ which is identity on $P \subset B$ and otherwise coincides with $F_1$ admits a continuous singlevalued selection, say $Q_1 : B \rightarrow B, Q_1(x) \in G_1(x)$. The $\alpha$–paraconvexity of $P$ and the construction imply that

$$
\text{dist}(Q_1(x), P) < \beta^2 \cdot g(x), \quad \text{dist}(Q_1(x), Q(x)) \leq \beta \cdot g(x), \quad Q_1|_{P} = id|_{P}.
$$

Similarly, the multivalued mapping defined by setting

$$
F_2(x) = \text{conv}\{P \cap D(Q_1(x), \beta^2 \cdot g(x))\}
$$

admits a continuous singlevalued selection, say $Q_2 : B \rightarrow B$ such that

$$
\text{dist}(Q_2(x), P) < \beta^3 \cdot g(x), \quad \text{dist}(Q_2(x), Q_1(x)) \leq \beta^2 \cdot g(x), \quad Q_2|_{P} = id|_{P}.
$$

Inductively we obtain a sequence $\{Q_n\}_{n=1}^{\infty}$ of continuous singlevalued mappings $Q_n : B \rightarrow B$ with the properties that $Q_n|_{P} = id|_{P}$ and

$$
\text{dist}(Q_{n+1}(x), P) < \beta^{n+2} \cdot g(x), \quad \text{dist}(Q_{n+1}(x), Q_n(x)) \leq \beta^{n+1} \cdot g(x).
$$
Clearly the pointwise limit $R$ of the sequence $\{Q_n\}_{n=1}^{\infty}$ is continuous retraction of $B$ onto $P$ and, moreover,

\[
dist(Q(x), R(x)) \leq dist(Q(x), Q_1(x)) + \sum_{n=1}^{\infty} dist(Q_n(x), Q_{n+1}(x)) \leq \\
= \beta \cdot (1 + \beta + \beta^2 + \beta^3 + \ldots) \cdot g(x) = \frac{\beta}{1-\beta} \cdot g(x).
\]

Hence,

\[
dist(Q, Retr(P)) \leq \frac{\beta}{1-\beta} \cdot g(x) = \frac{\beta}{1-\beta} \cdot (r(x) + \gamma) < \frac{\beta}{1-\beta} \cdot (r + \gamma).
\]

Passing to $\beta \to \alpha + 0$ and to $\gamma \to 0 + 0$ we conclude $dist(Q, Retr(P)) \leq \frac{\alpha}{1-\alpha} \cdot r$.

To finish the proof we must check that the retractions $R(x) = \lim_{n \to \infty} Q_n(x)$, $x \in X$ onto $P$ constructed above are uniform with respect to $P$. To this end, using uniformity of all retractions $R_1, ..., R_n$, for an arbitrary $\varepsilon > 0$ choose $\delta > 0$ such that

\[
dist(x, P) < \delta \Rightarrow dist(x, R_i(x)) < \varepsilon.
\]

In particular, for every point $x$ with $dist(x, P) < \delta$ all values $R_1(x), ..., R_n(x), Q(x)$ are in the open ball $D(x, \varepsilon)$. Hence $r(x) < \varepsilon$ and this is why

\[
dist(x, R(x)) \leq dist(x, Q(x)) + dist(Q(x), R(x)) < \varepsilon + \frac{\beta}{1-\beta} \cdot \gamma(x) < \frac{1}{1-\beta} \cdot (\varepsilon + \gamma)
\]

Therefore $R \in URetr(P)$ and $dist(Q, URetr(P)) \leq \frac{\alpha}{1-\alpha} \cdot r$. So $URetr(P)$ is $\frac{\alpha}{1-\alpha}$-paraconvex. □

Proof of Proposition 2.3. Pick $P \in bexp_{\alpha}(B)$, an uniform retraction $R \in URetr(P)$ and a number $\varepsilon > 0$. So let $\delta > 0$ be such that $\delta < (1 - \alpha) \cdot \varepsilon$ and

\[
dist(x, P) < \delta \Rightarrow dist(x, R(x)) < (1 - \alpha) \cdot \varepsilon.
\]

Consider any $P' \in bexp_{\alpha}(B)$ which is $\delta$—close to $P$ with respect to the Hausdorff distance. We must find a uniform retraction $R' \in URetr(P')$ such that $dist(R, R') < \varepsilon$.

The multivalued mapping $F' : B \to B$ such that $F'(x) = \{x\}$, $x \in P'$ and $F'(x) = P'$ otherwise is a LSC mapping with $\alpha$—paraconvex values. Any selection of $F'$ is a retraction onto $P'$. So let us check that $R$ is almost selection of $F'$ and hence, is almost a retraction onto $P'$.

For every $x \in B \setminus P'$ we have

\[
dist(R(x), F'(x)) = dist(R(x), P') < \delta < (1 - \alpha)\varepsilon
\]

because $R(x) \in P$ and the set $P$ lies in the $\delta$—neighborhood of the set $P'$. If $x \in P'$ then

\[
dist(R(x), F'(x)) = dist(R(x), x) < (1 - \alpha)\varepsilon
\]
because the set $P'$ lies in the $\delta$-neighborhood of the set $P$ and due to the choice of the number $\delta$. Hence, the retraction $R$ of $B$ onto the set $P$ is a continuous singlevalued $\varepsilon'$-selection of the mapping $F'$, $\varepsilon' = (1-\alpha)\varepsilon$.

Following the proofs of Propositions 2.1 and 2.2 we can improve the $\varepsilon'$-selection $R$ of $F'$ to a selection $R'$ of $F'$ such that $dist(R, R') < \frac{\varepsilon'}{1-\alpha} = \varepsilon$. So $R'$ is a continuous retraction onto $P'$ and the checking of uniformity of $R'$ can be performed by repeating the arguments on Chebyshev radii from the proof of Proposition 2.2. □

Observe the proof of Theorems 2.0 for the case of compact paraconvex sets is much more easier, because for any compact retract $A \subset B$ each continuous retraction $B \rightarrow A$ automatically will be uniform with respect to $A$. So, one can uses directly $\text{Retr}(A)$ instead of $\text{URetr}(A)$.

**Corollary 2.5.** Under the assumptions of Theorem 2.0 if in addition all values $F(x)$, $x \in X$, are pairwise disjoint then the metric subspace $Y = \bigcup_{x \in X} F(x) \subset B$ admits a convex metric structure $\sigma$ (in the sense of [9]) such that each value $F(x)$ is convex with respect to $\sigma$.

*Proof.* By Theorem 2.0, let $R(x) : B \rightarrow F(x)$, $x \in X$, be a continuous family of uniform continuous retractions onto the values $F(x)$. One can define a convex metric structure $\sigma$ on $Y = \bigcup_{x \in X} F(x)$ by setting that $\sigma$-convex combinations are defined only for finite subsets $\{y_1, y_2, ..., y_n\}$ which are entirely displaced in a value $F(x)$ and

$$\sigma - \text{conv}_{F(x)} \{y_1, y_2, ..., y_n\} = R(x)(\text{conv}_B \{y_1, y_2, ..., y_n\}).$$

□

**Corollary 2.6.** Let $f : Y \rightarrow X$ be a continuous singlevalued surjection and let all point-inverses $f^{-1}(x)$, $x \in X$, are $\alpha$-paraconvex subcompacta of $Y$ with $\alpha < \frac{1}{2}$. Then $Y$ admits a convexity metric structure such that each point-inverse is convex with respect to this structure.

### 3. The Hilbert space case

Hilbert spaces have a many of advantages inside the class of all Banach spaces. In this chapter we demonstrate such a advantage related to paraconvexity. Briefly we prove the estimate $\alpha(1+\alpha^2)$ for paraconvexity of the set $\text{Retr}(P)$ onto $\alpha$-paraconvex set $P$ can be improved with

$$\frac{\alpha(1+\alpha^2)}{1-\alpha^2} = \frac{\alpha}{1-\alpha} \cdot \frac{1+\alpha^2}{1+\alpha} < \frac{\alpha}{1-\alpha}.$$  

Hence in Theorem 2.0 one can substitute the root of the equation $\alpha + \alpha^2 + \alpha^3 = 1$ instead of $\frac{1}{2}$. In fact, a generalization of such type can be performed for any uniformly convex Banach spaces but for Hilbert space the proofs differ only in technical details.

**Theorem 3.0.** Let $H$ be a Hilbert space and $F : X \rightarrow \text{bexp}_\alpha(H)$ be a continuous mapping of a paracompact space $X$, where $\alpha + \alpha^2 + \alpha^3 < 1$. Then there exists a continuous singlevalued mapping $\tilde{\mathbf{F}} : X \rightarrow C_b(H, H)$ such that for every $x \in X$ the mapping $\tilde{\mathbf{F}}_x : H \rightarrow H$ is a continuous retraction of $H$ onto the value $F(x)$ of $F$.

So we repeat the original definition of $\alpha$-paraconvexity of $P$ but with the appropriate estimate for distances between points of simplices and points of $P$ inside open balls.
Definition 3.1. Let $0 \leq \alpha < 1$. A nonempty closed subset $P \subset B$ of a Banach space $B$ is said to be strongly $\alpha$-paraconvex if for every open ball $D \subset B$ with radius $r$ and for every $q \in \text{conv}(P \cap D)$ the distance $\text{dist}(q, P \cap D)$ is less than or equal to $\alpha \cdot r$.

Clearly, strong $\alpha$–paraconvexity of a set implies its $\alpha$–paraconvexity. In a Hilbert space the converse is almost true: for some $1 > \beta > \alpha$, $\alpha$–paraconvexity implies strong $\beta$–paraconvexity for some suitable $\beta$.

Proposition 3.2. Any $\alpha$–paraconvex subset $P$ of a Hilbert space is its strong $\varphi(\alpha)$–paraconvex subset, where $\varphi(\alpha) = \sqrt{1 - (1 - \alpha)^2} = \sqrt{2\alpha - \alpha^2}$.

Proposition 3.2 is an immediate corollary of the following purely geometrical lemma:

Lemma 3.3. Let $D = D_r$ be an open ball with the radius $r$ in the Hilbert space $H$. Let $z$ be a point of the convex hull $\text{conv}(P \cap D)$ of the intersection $D$ with a set $P$ and let $\text{dist}(z, P) \leq \alpha \cdot r$. Then $\text{dist}(z, P \cap D) \leq \varphi(\alpha) \cdot r$.

Proof of Lemma 3.3. Pick an arbitrary $\alpha < \gamma < 1$ and let $c$ be the center of the open ball $D = D(c, r)$.

If $\text{dist}(c, z) \leq (1 - \gamma) \cdot r$ then the whole open ball $D(z, \gamma \cdot r)$ lies inside of $D$. Hence, a point $p \in P$ which is $(\gamma \cdot r)$–close to $z$ automatically will be in $D$. So

$$\text{dist}(z, P \cap D) \leq \text{dist}(z, P) \leq \gamma \cdot r \leq \varphi(\gamma) \cdot r.$$  

Let us look for the opposite case when $z$ is "close" to the boundary of the ball $D$, i.e. when $(1 - \gamma) \cdot r < \text{dist}(c, z) < r$. Draw the hyperplane $\Pi$ supporting to the ball $D(c, \text{dist}(c, z))$ at the point $z$. Such the hyperplane $\Pi$ divides the ball $D$ into two open convex parts: the center $c$ belongs to the "larger" part $D_+$ whereas the point $z$ belongs to the the boundary of "smaller" part $D_-$. Clearly, $\text{Clos}(D_-)$ contains a point $p \in P$ (if, to the contrary, $P \cap D$ is subset of $D_+$ then $z \in \text{conv}(P \cap D) \subset D_+$). Hence, the distance $\text{dist}(z, P \cap D)$ majorates by

$$\text{dist}(z, P \cap D) \leq \max\{\text{dist}(z, u) : u \in \text{Clos}(D_-)\} = \varphi\left(\frac{\text{dist}(c, z)}{r}\right) \cdot r < \varphi(\gamma) \cdot r.$$  

So in both cases $\text{dist}(z, P \cap D) \leq \varphi(\gamma) \cdot r$ and passing to $\gamma \to \alpha + 0$ we see that $\text{dist}(z, P \cap D) \leq \varphi(\alpha) \cdot r$. $\Box$

Recall that for a multivalued mapping $F : X \to Y$ and for a numerical function $\varepsilon : X \to (0, +\infty)$ a singlevalued mapping $f : X \to Y$ is said to be an $\varepsilon$–selection of $F$ if $\text{dist}(f(x), F(x)) < \varepsilon(x), \ x \in X$.

Proposition 3.4. Let $0 \leq \alpha < 1$ and let $F : X \to H$ be an $\alpha$–paraconvex valued LSC mapping from a paracompact domain into a Hilbert space. Then

1. for each constant $C > \frac{1}{1 + \alpha^2}$, for every continuous function $\varepsilon : X \to (0, +\infty)$ and for every continuous $\varepsilon$–selection $f_\varepsilon : X \to H$ of the mapping $F$ there exists a continuous selection $f : X \to H$ of $F$ such that

$$\text{dist}(f_\varepsilon(x), f(x)) < C \cdot \varepsilon(x), \quad x \in X;$$  

2. $F$ admits a continuous selection $f$.  

Proof. Clearly (1) implies (2): the mapping \( x \mapsto [1 + \text{dist}(0, F(x)), +\infty) \), \( x \in X \), is a LSC mapping with nonempty closed and convex values and therefore admits a continuous selection, say \( \varepsilon : X \to (0, +\infty) \). Therefore \( f_\varepsilon \equiv 0 \) is an \( \varepsilon \)-selection of \( F \).

To prove (1) let \( \varphi(t) = \frac{\sqrt{2t - t^2}}{2} \), \( 0 < t < 1 \), choose any \( \gamma \in (0, 1) \) and denote by \( D(x) = D(f_\varepsilon(x), \varepsilon(x)) \). As above, the multivalued mapping
\[
F_1(x) = \overline{\text{conv}}\{F(x) \cap D(x)\}, \quad x \in X
\]
admits a single valued continuous selection, say \( f_1 : X \to H \).

For each \( x \in X \) the point \( f_1(x) \) belongs to the convex hull \( \overline{\text{conv}}\{F(x) \cap D(x)\} \) and \( \text{dist}(f_1(x), F(x)) \leq \alpha \cdot \varepsilon(x) \) due to the \( \alpha \)-paraconvexity of the value \( F(x) \). Lemma 3.3 implies that
\[
\text{dist}(f_1(x), F(x) \cap D(x)) \leq \varphi(\alpha) \cdot \varepsilon(x) < \varphi(\gamma) \cdot \varepsilon(x).
\]

Therefore, the multivalued mapping \( F_2 : X \to H \) defined by
\[
F_2(x) = \overline{\text{conv}}\{F(x) \cap D(x) \cap D(f_1(x), \varphi(\gamma) \cdot \varepsilon(x))\}, \quad x \in X
\]
is a LSC mapping with nonempty closed and convex values. Hence there exists a selection of \( F_2 \), say \( f_2 : X \to H \).

For each \( x \in X \) the point \( f_2(x) \) belongs to the convex hull \( \overline{\text{conv}}\{F(x) \cap D(x)\} \) and \( \text{dist}(f_2(x), F(x)) \leq \alpha \cdot \varepsilon(x) \) due to the \( \alpha \)-paraconvexity of the value \( F(x) \) and because \( f_2(x) \in \overline{\text{conv}}\{F(x) \cap D(f_1(x), \varphi(\gamma) \cdot \varepsilon(x))\} \).

Lemma 3.3 implies that
\[
\text{dist}(f_2(x), F(x)) \leq \varphi(\alpha \cdot \varphi(\gamma)) \cdot \varepsilon(x) < \varphi(\gamma \cdot \varphi(\gamma)) \cdot \varepsilon(x), \quad x \in X.
\]

Put
\[
F_3(x) = \overline{\text{conv}}\{F(x) \cap D(f_2(x), \varepsilon(x)) \cap D(f_2(x), \varphi(\gamma \cdot \varphi(\gamma)) \cdot \varepsilon(x))\}, \quad x \in X
\]
and so on. Hence we have constructed a sequence \( \{f_n : X \to H\}_{n=1}^{\infty} \) of continuous single valued mappings such that
\[
\text{dist}(f_n(x), f_n(x)) \leq \varepsilon(x), \quad \text{dist}(f_n(x), F(x)) < \gamma_n \cdot \varepsilon(x)
\]
where \( \gamma_1 = \gamma \) and \( \gamma_{n+1} = \gamma \cdot \varphi(\gamma_n) \).

The sequence \( \{\gamma_n\} \) is monotone, decreasing and converges to the (positive!) root of equation \( t = \gamma \cdot \varphi(t) \), i.e. to the number \( t = \frac{2\gamma^2}{1 + \gamma^2} > \frac{2\alpha^2}{1 + \alpha^2} \). Therefore we can choose numbers \( N \in \mathbb{N} \) and \( \lambda \) such that
\[
1 > 1 - \frac{1}{C} > \lambda > \gamma_N > \frac{2\gamma^2}{1 + \gamma^2} > \frac{2\alpha^2}{1 + \alpha^2}.
\]
Hence, the mapping \( g_1 = f_N \) is a \( (\lambda \cdot \varepsilon) \)-selection of \( F \) and
\[
\text{dist}(f_n(x), g_1(x)) \leq \varepsilon(x).
\]

Starting with \( g_1 \) one can find \( \lambda^2 \cdot \varepsilon \)-selection \( g_2 \) of \( F \) such that
\[
\text{dist}(g_1(x), g_2(x)) \leq \lambda \cdot \varepsilon(x).
\]

Continuation of this construction produces a continuous selection \( f = \lim_{n \to \infty} g_n \) of \( F \) such that
\[
\text{dist}(f_n(x), f(x)) \leq \varepsilon(x) \cdot (1 + \lambda + \lambda^2 + ...) = \frac{1}{1 - \lambda} \cdot \varepsilon(x) < C \cdot \varepsilon(x), \quad x \in X.
\]
\[\square\]

Proposition 3.4 implies the following analog of Proposition 2.2:
**Corollary 3.5.** For every $0 \leq \alpha < 1$ and for every bounded $\alpha$-paraconvex subset $P \subset H$ the set $\text{Retr}(P)$ is an $\alpha \left( \frac{1}{1+\alpha^2} \right)$-paraconvex subset of $C_b(H,H)$.

**Proof of Theorem 3.0.** It suffices to repeat the proof of Theorem 2.0 but we use Corollary 3.5 instead of Proposition 2.2. □

4. Concluding remarks

Roughly speaking, we have proved that $\alpha$-paraconvexity of a set implies $\beta$-paraconvexity of a set of all retractions onto this set with $\beta = \beta(\alpha) = \frac{\alpha}{1-\alpha}$.

Such an estimate for $\beta = \beta(\alpha)$ naturally appears as a result of the usual geometric progression procedure. However, it is unclear to us whether the constant $\frac{\alpha}{1-\alpha}$ is the best possible?

Some examples in the Euclidean plane show that in some particular cases (for some curves) the constant $\beta = \beta(\alpha)$ admits more precise estimates. Unfortunately, calculations in these examples are based on geometric properties of concrete $\alpha$-paraconvex curves which are in general false for arbitrary $\alpha$-paraconvex subsets of the plane.

Hence the question about continuous choice of retractions onto bounded $\alpha$-paraconvex sets with $\frac{1}{2} \leq \alpha < 1$ remains open. Even the case of subsets of the Euclidean plane presents an evident interest. The main obstructions for various counterexamples are related to problems of constructing retractions with some prescribed constraints.

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