Multi-criteria decision making for a large heating system with heat loss

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Abstract. A large heat supply system, which always has a multi-level hierarchical structure, is considered. There are levels of trunk and distribution heating networks with intermediate levels of regulation. Planning and selection of a mode that provides the required heat loads are associated in such a system with a large number of different factors that complicate the decision-making process. The present study investigates heat transfer in a flat anisotropic insulating plate with uniformly distributed internal heat sinks subjected to heat flows. An analytical solution is obtained that allowed us to study and build temperature distribution fields for various orientations of principal axes of thermal conductivity tensor in an insulating plate. The thermal field localization regime is shown to be possible in the presence of thermal energy sinks/sources inside the plate wherein heat propagation is bound with limiting isotherms.

1. Introduction
The problem of multi-level thermal-hydraulic adjustment was considered in [1]. In [2], this problem was solved for the relationship between control variables and the states of a hierarchical system that is controlled by several criteria. The task was considered within the framework of a decision-making model in a three-level hierarchical system. In [3], a study of thermal processes in inhomogeneous materials was carried out and a mathematical model of heat transfer was proposed in regions with multidimensional discontinuities of thermophysical and geometric characteristics. In [4], a study of surface heat and mass transfer under intense heat loading was made. In [5], the heating of anisotropic half-space by a point source of heat was studied. The problem of heat conduction in an orthotropic body by a variational method was solved in [6], the problem of heat conduction in an anisotropic plate with heat fluxes set at the boundaries was addressed in [7], and the problem of conjugate heat transfer in anisotropic bodies was solved in [8]. The theory of solving inverse coefficient problems of nonlinear heat conduction in bodies with arbitrary anisotropy of properties was described in [9].

In the global aspect, safety, reliability, and efficiency of a complex system as a whole during operation must be ensured. Therefore, the evaluation of the optimality criteria for such a system is carried out by several indicators at once. Such a complex system analysis always implies the existence of various kinds of uncertain and random factors.

The choice of the optimal mode of operation of a complex heat supply system is made on the basis of an iterative coordination process, which should converge to a consistent solution of local optimization problems. The decision is based on the principle of guaranteed results. The lack of correlation between random and uncertain factors allows improving this result by changing the sequence of averaging operations by chance and optimizing operations on uncertainties.
In the event of faults, deviations in the system can only be compensated in interactive real-time mode. The decision maker launches a new recalculation according to a predetermined algorithm of parameters and modes of operation of the system.

The main technical problem of transporting the coolant is the reduction of heat loss through the insulation of pipelines of heat networks. Insulating coatings from anisotropic materials today are widespread. This refers to heat-shielding coatings and thermal insulation used in engineering, including thermally and mechanically loaded structures of heat supply systems. These are various composites of film, laminated and fibrous materials, textolites, glass, asbestos, carbon, phenol-formaldehyde plastics, graphite-containing materials and so on.

This paper aims to derive and study a new solution to the problem of thermal conductivity in an anisotropic thermal insulation layer with internal sink heat sources proportional to the current temperature, where heat fluxes are applied to both boundaries of the thermal insulation material. The problem is solved analytically for the first time using the method of source function applied to a computational area with anisotropy of properties. Based on the obtained solution, the distributions of non-stationary temperature fields inside the heat-insulating plate are analyzed for varying model parameters such as different values of the orientation angles of the axes and components of the thermal conductivity tensor.

2. Problem formulation

Computational domain is represented by an infinitely long strip of anisotropic material of thickness \( l \) shown in figure 1. The main axis of thermal conductivity tensor is rotated with angle \( \varphi \) with respect to axis \( Ox \) of Cartesian frame \( Oxy \). Let us consider the following unsteady heat conduction problem with boundary conditions of second kind.

\[
c p \frac{\partial T}{\partial t} = \lambda_{xx} \frac{\partial^2 T}{\partial x^2} + \lambda_{xy} \frac{\partial^2 T}{\partial x \partial y} + \lambda_{yy} \frac{\partial^2 T}{\partial y^2} - gT, \quad -\infty < x < \infty, \quad 0 < y < l, \quad t > 0; \tag{1}
\]

\[
\lambda_{xy} \frac{\partial T}{\partial x} + \lambda_{yy} \frac{\partial T}{\partial y} = -q_1(x,t), \quad -\infty < x < \infty, \quad y = 0, \quad t > 0; \tag{2}
\]

\[
\lambda_{yy} \frac{\partial T}{\partial x} + \lambda_{yy} \frac{\partial T}{\partial y} = q_2(x,t), \quad -\infty < x < \infty, \quad y = l, \quad t > 0; \tag{3}
\]

\[
T(x,y,0) = f(x,y), \quad -\infty < x < \infty, \quad 0 \leq y \leq l, \quad t = 0, \tag{4}
\]

where \( x, \ y \) are the dimension variables, \( m; \ t \) is the time, sec; \( c \) is the thermal conductivity of the material, \( J/(kg\cdot K) \); \( \rho \) is the density, \( kg/m^3 \); \( T \) is the temperature, K; \( q_1, q_2 \) are the densities of heat fluxes, \( W/m^2 \); and \( \lambda_{xx}, \lambda_{xy}, \lambda_{yy}, \lambda_{yy} \) are the components of thermal conductivity tensor of anisotropic material, \( W/(m\cdot K) \). Note that there is the source term \( gT \) in equation (1), which is proportional to temperature and \( g = Const \) is the absorption coefficient, \( W/(m^3\cdot K) \). It is required to find the distribution of temperature field \( T(x,y,t) \) in the plate material.

The mathematical model proposed above corresponds to a real possible physical process. In the statement (1)–(4), the direct problem of anisotropic thermal conductivity is formulated [7], which...
contains the sink term in the main equation (1) proportional to the current temperature. The following significant assumptions have been made of the considered two-dimensional idealization: a) a strip of infinite length and limited height, which is consistent with a physically real heat-insulating plate, the thickness of which is significantly less than the length and width; b) a uniform distribution of thermal energy sinks throughout the calculation area, which is also acceptable, for example, in fibrous composites during physicochemical transformations inside the plate; c) specification of sources/sinks at the boundaries of the computational area. The last of these assumptions is explained by the focus of the study, where the influence of the boundary heat fluxes and thermal energy sinks inside the plate is of interest for showing that the distribution of temperature fields is bounded in the vicinity of the points of application of these flows. Thus, the system of equations (1)–(4) is admissible; it is adequate to the real conditions of heat transfer.

3. Solution method

To solve problem (1)–(4) let us exclude source term using the following substitution:

\[ T(x, y, t) = \exp(-gt/cp) \cdot \theta(x, y, t) , \]  

where \( \theta(x, y, t) \) is a new function to be found. Then we use source functions method, according to which we obtain the following conductivity problem with respect to \( G(x, y, \zeta, \psi, t) \):

\[ \frac{\partial G}{\partial t} = a_{xx} \frac{\partial^2 G}{\partial x^2} + 2a_{xy} \frac{\partial^2 G}{\partial x \partial y} + a_{yy} \frac{\partial^2 G}{\partial y^2}, \quad -\infty < x < \infty, \quad 0 < y < l, \quad t > 0; \]

\[ a_{xx} \frac{\partial G}{\partial x} + a_{yy} \frac{\partial G}{\partial y} = 0, \quad -\infty < x < \infty, \quad y = 0, \quad t > 0; \]

\[ a_{xx} \frac{\partial G}{\partial x} + a_{yy} \frac{\partial G}{\partial y} = 0, \quad -\infty < x < \infty, \quad y = l, \quad t > 0; \]

\[ G(x, y, \zeta, \psi, 0) = \delta(x - \zeta) \cdot \delta(y - \psi), \quad -\infty < x < \infty, \quad 0 < y \leq l, \]

where \( \alpha = \lambda_y / \lambda_x, \quad \{a_{xx}, a_{xy}, a_{yy}\} = \{\lambda_x, \lambda_y, \lambda_y\} / cp \). Equation (6) has mixed partial derivatives, hence let us apply Fourier transform \( G_\omega(y, \zeta, \psi, t) \) for longitudinal variable \( x \) to the problem (6)–(9) and get the following single-dimension problem with respect to \( y \) axis:

\[ \frac{\partial G_\omega}{\partial t} = -a_{xx} \omega^2 G_\omega + 2a_{xy} i\omega \frac{\partial G_\omega}{\partial y} + a_{yy} \frac{\partial^2 G_\omega}{\partial y^2}, \quad 0 < y < l, \quad t > 0; \]

\[ \left. \frac{\partial G_\omega}{\partial y} \right|_{y=0} + i\omega \alpha G_\omega(0, \zeta, \psi, t) = 0; \quad \left. \frac{\partial G_\omega}{\partial y} \right|_{y=l} + i\omega \alpha G_\omega(l, \zeta, \psi, t) = 0; \]

\[ G_\omega(y, \zeta, \psi, 0) = \exp(-i\omega \zeta) \delta(y - \psi), \quad 0 \leq y \leq l, \quad t = 0. \]

This problem can be simplified with substitution

\[ G_\omega(y, \zeta, \psi, t) = \exp(-i\omega \gamma + i\beta \gamma'^{-1}) G_0(y, \zeta, \psi, t), \]

where \( \beta = (\lambda_y / \lambda_x - \lambda_y^2) / \lambda_y^2; \quad \gamma = c \rho \gamma'^{-1} \). We obtain a problem with initial and boundary conditions for heat equation for \( y \) axis with respect to a function \( G_0(y, \zeta, \psi, t) \):

\[ \frac{\partial G_0}{\partial t} = \frac{1}{\gamma} \frac{\partial^2 G_0}{\partial y^2}, \quad 0 < y < l, \quad t > 0; \]

\[ \left. \frac{\partial G_0}{\partial y} \right|_{y=0} = \left. \frac{\partial G_0}{\partial y} \right|_{y=l} = 0, \quad t > 0; \]

\[ G_0(y, \zeta, \psi, 0) = \exp(-i\omega \zeta \exp(i\omega \gamma) \delta(y - \psi), \quad 0 < y < l. \]

Solution to the last problem can easily be found by separation of variables:

\[ G_0(y, \zeta, \psi, t) = \sum_{n=0}^\infty F_n(\zeta, \psi) \cdot \exp(-\pi^2 n^2 \gamma'^{-1}) \cdot \cos(n \pi l^{-1} y), \]

where \( F_n(\zeta, \psi) = 2l^{-1} \exp(-i\omega \zeta - i\omega \gamma) \cos(n \pi l^{-1} \psi) \). Substituting (11) into (10) we obtain Fourier transformant. Applying the inverse Fourier transform we find the solution to the problem (6)–(9),
which is the source function:

\[
G(x, y, \zeta, \psi, t) = \sqrt{\pi \beta t} \exp \left\{ -\left( \frac{\zeta - x + \alpha(y - \psi)}{\sqrt{4 \beta t}} \right)^2 \right\} \times \\
\sum_{n=0}^{\infty} \left\{ \cos(\pi n l^{-1} \psi) \cos(\pi n l^{-1} y) \exp \left[ -\pi^2 n^2 l^{-2} \beta^{-1} t \right] \right\}.
\]

Then according to the general theory of source functions we find

\[
\theta(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, \zeta, \psi, t) f(\zeta, \psi) d\zeta d\psi + \frac{1}{\gamma_{yy}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, \zeta, l, t - \tau) q_1(\zeta, \tau) \exp \{ g[l \rho]^{-1} \} d\zeta + \\
\frac{1}{\gamma_{yy}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, \zeta, l, t - \tau) q_2(\zeta, \tau) \exp \{ g[l \rho]^{-1} \} d\zeta.
\]

Thus we obtain analytic solution of the initially stated problem – distribution of temperature field \(T(x, y, t)\), determined by (5).

4. Results analysis

Simulation of temperature field distribution (5), (12) in anisotropic plate of thickness \(l\) is conducted using the following sets of parameters: \(l = 0.01\) m; \(\lambda_f = 3.0\) W·(m·K)\(^{-1}\); \(\lambda_\eta = 0.1\) W·(m·K)\(^{-1}\); \(\varphi = \pi/12\) rad; \(c = 0.4\) kJ·(kg·K)\(^{-1}\); \(\rho = 2500\) kg/m\(^3\); \(T(x, y, 0) = f(x, y) = 0\) K; \(q_1(x, t) = A_1 \exp(-x^2 l_1^{-2})\); \(\gamma_{yy} = 7.0 \cdot 10^5\) W·m\(^{-2}\); \(l_1 = 2l/3\); \(q_2(x, t) = A_2 \cdot \eta(|l_2| - x)\). \(A_2 = 5.0 \cdot 10^5\) W·m\(^{-2}\), \(l_2 = l/2\); \(g = 1.0 \cdot 10^5\) W·(m\(^3\)·K)\(^{-1}\); \(\eta(x)\) – unit step function.

![Temperature field in anisotropic plate, \(\varphi = \pi/12\) rad](image1)

(a) – \(g = 0\) W·(m\(^3\)·K)\(^{-1}\), \(t = 5.0\) s; (b) – \(g = 0\) W·(m\(^3\)·K)\(^{-1}\), \(t = 10.0\) s; (c) – \(g = 1.0 \cdot 10^5\) W·(m\(^3\)·K)\(^{-1}\), \(t = 5.0\) s; (d) – \(g = 1.0 \cdot 10^5\) W·(m\(^3\)·K)\(^{-1}\), \(t = 10.0\) s.
Figure 3. Temperature changes is middle section of the plate, $\varphi = \pi / 12$ rad:
(a) – absence of absorption $g = 0$ W·(m$^3$·K)$^{-1}$; (b) – presence of absorption $g = 1.0 \cdot 10^5$ W·(m$^3$·K)$^{-1}$.

Figure 2 shows that the heating of the strip is oriented along the main axis $O\xi$ with a large value $\xi\lambda$ of the thermal conductivity tensor. The elliptic isotherm, located in the central part of the strip, degenerates to a point with time. This point is the saddle point of the separatrixes’ intersection, which separate temperature flows from the flows applied to opposite boundaries.

Figure 2a, 2b show that with increasing time, in the absence of absorption, the temperature field is not localized, but continues to grow along the coordinate axes. But with the presence of absorption, the situation shown in figure 2c, 2d, where $g = 1.0 \cdot 10^5$ W·(m$^3$·K)$^{-1}$, is possible when the temperature field is localized in the central part of the band, that is, does not grow along the axes, $Ox$, $Oy$.

With a fixed value of the absorption coefficient and an increase in the orientation angle $\varphi$ of the principal axes of the thermal conductivity tensor, the picture does not change qualitatively. As we tend to orthotropy ($\varphi \rightarrow \pi/2$ rad), the temperature distribution becomes more and more symmetrical along the axis $Oy$. Figure 3 shows temperature versus time at some points along the middle of the strip cut along $Ox$ ($y = l/2$). In the absence of absorption (figure 3a), temperatures monotonously increase, whereas with the presence and at the expense of absorption (figure 3b) they reach their maximum value, and then decrease and have their own limit temperature of natural conditions, in this case zero: $T \rightarrow 0$, $t \rightarrow \infty$.

5. Validation of model
In section 1, the verification of the model representing the mathematical formulation of the problem is carried out. The proposed model is a mathematical description of the physical heat exchange system. The adequacy check (validation) of the model was carried out in the following areas.
1. The basic values of the input parameters of the simulation are physically real, they are consistent with the works [5-9]. The initial temperature, for the convenience of perceiving its increments, is conventionally taken as 0 K. Numerical experiments were carried out in various ranges and with various combinations of parameters. The results have shown the correctness and stability of the model in terms of input parameters. A qualitatively correct change in isotherms was observed at different values of components and orientation angles of the axes of the thermal conductivity tensor. In particular, an increase in the orientation angle of the principal axes of the thermal conductivity tensor...
extends the temperature field to the larger component of the thermal conductivity tensor and in the direction perpendicular to it. At zero angle ($\varphi = 0$ rad), the obtained orientations of the principal axes are close to those of [6].

2. The presence in the main equation (1) of the sink term ($-gT$) physically characterizes the localization of temperature fields in the vicinity of the given boundary conditions. In the absence of thermal energy sinks ($g = 0$ W·(m$^3$·K)$^{-1}$), the results of the numerical simulation are in good agreement with the study [7].

3. The load from heat sources applied to the plate boundaries corresponds to intense heating: the values used are in agreement with the works [3-5, 7]. For various distributions of heat fluxes and powers of sources (including zero power of one of the sources), a qualitatively and quantitatively correct distribution of temperature fields was observed [5].

4. In the numerical implementation of the obtained analytical solution (5), (12), the well-known quadrature formulas for numerical integration are used. For improper integrals in (5) with infinite limits, we take finite limits in such a way that the perturbation from the boundary conditions over the entire width of the plate is close to zero.

Thus, the mathematical model proposed in the work and the obtained analytical solution are valid; they meet the conditions of real heat transfer in the anisotropic plate.

6. Conclusion

New analytical solution of conductivity problem has been obtained for anisotropic material in the flat plate form with account for thermal energy absorption and heat transfer at boundaries. Calculations of thermal fields on the basis of that solution reveal localization of temperature fields in the vicinity of heat fluxes application zones. The mentioned result bears evidence of principal possibility (by means of anisotropic material coatings) of heating medium transport conditions with minimal thermal energy loss. In addition, it is indicated that each plate point temperature reaches maximum at given time and then (under condition of energy absorption via internal endothermic physicochemical processes) tends to initial value. The obtained results may be used in designing thermal protection with local zones of temperature gradient with ambient space keeping initial temperature.

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