The variety of characters in $\text{PSL}_2(\mathbb{C})$

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Abstract
We study some basic properties of the variety of characters in $\text{PSL}_2(\mathbb{C})$ of a finitely generated group. In particular we give an interpretation of its points as characters of representations. We construct 3-manifolds whose variety of characters has arbitrarily many components that do not lift to $\text{SL}_2(\mathbb{C})$. We also study the singular locus of the variety of characters of a free group.

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1 Introduction
The varieties of representations and characters have many applications in 3-dimensional topology and geometry. The variety of $\text{SL}_2(\mathbb{C})$-characters has been intensively studied since the seminal paper of Culler and Shalen [CS], but for many applications it is more convenient to work with $\text{PSL}_2(\mathbb{C})$ instead of $\text{SL}_2(\mathbb{C})$ (see [BZ] and [BMP] for instance). The purpose of this note is to study some basic properties of the variety of characters in $\text{PSL}_2(\mathbb{C})$. Most of the results of invariant theory that we use can be found in any standard reference (e.g. [KSS], [Kra], [PV]).

Throughout this paper, $\Gamma$ will denote a finitely generated group.

Definition 1.1 The set of all representations of $\Gamma$ in $\text{PSL}_2(\mathbb{C})$ is denoted by $R(\Gamma)$ and it is called the variety of representations of $\Gamma$ in $\text{PSL}_2(\mathbb{C})$.

The variety of representations $R(\Gamma)$ has a natural structure as an affine algebraic set over the complex numbers given as follows: the group $\text{PSL}_2(\mathbb{C})$ is algebraic (see Section 2). Given a presentation $\Gamma = \langle \gamma_1, \ldots, \gamma_s \mid (r_i)_{i \in I} \rangle$ we have a natural embedding:

$$R(\Gamma) \rightarrow \text{PSL}_2(\mathbb{C}) \times \cdots \times \text{PSL}_2(\mathbb{C})$$
$$\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_s))$$
and the defining equations are induced by the relations. This structure can be easily seen to be independent of the presentation. In fact using the isomorphism $\text{PSL}_2(\mathbb{C}) \cong \text{SO}_3(\mathbb{C})$, $R(\Gamma)$ has a structure of an affine set (see Lemma 2.1).

The action of $\text{PSL}_2(\mathbb{C})$ on $R(\Gamma)$ by conjugation is algebraic. The quotient $R(\Gamma)/\text{PSL}_2(\mathbb{C})$ may be not Hausdorff and it is more convenient to consider the algebraic quotient of invariant theory, because $\text{PSL}_2(\mathbb{C})$ is reductive.

**Definition 1.2** The variety of $\text{PSL}_2(\mathbb{C})$-characters $X(\Gamma)$ is the quotient $R(\Gamma)/\text{PSL}_2(\mathbb{C})$ of invariant theory.

This definition means that $X(\Gamma)$ is an affine algebraic set together with a regular map $t: R(\Gamma) \to X(\Gamma)$ which induces an isomorphism

$$t^*: \mathbb{C}[X(\Gamma)] \to \mathbb{C}[R(\Gamma)]^{\text{PSL}_2(\mathbb{C})}$$

(i.e. the regular functions on $X(\Gamma)$ are precisely the regular functions on $R(\Gamma)$ invariant by conjugation). We will use the notation $R(M) = R(\pi_1 M)$ and $X(M) = X(\pi_1 M)$ if $M$ is a path-connected topological space.

In this paper we study the basic properties of $X(\Gamma)$.

First we explain the name “variety of characters”: given a representation $\rho: \Gamma \to \text{PSL}_2(\mathbb{C})$, its character is the map

$$\chi_\rho: \Gamma \to \mathbb{C} \quad \gamma \mapsto \text{tr}^2(\rho(\gamma))$$

**Theorem 1.3** There is a natural bijection between $X(\Gamma)$ and the set of characters of representations $\rho \in R(\Gamma)$. This bijection maps every $t(\rho) \in X(\Gamma)$ to the character $\chi_\rho$.

In many cases the representations of $R(\Gamma)$ lift to $\text{SL}_2(\mathbb{C})$, for instance if $\Gamma$ is a free group. In such a case, $X(\Gamma)$ is just a quotient of the usual variety of characters in $\text{SL}_2(\mathbb{C})$ (See Proposition 1.2). This quotient is the definition already used in [Bur90], [HL1], [HL2] and [Ril84] for 2-bridge knot exteriors. The explicit computation for the figure eight knot exterior is found in [GM].

There are cases where representations do not lift to $\text{SL}_2(\mathbb{C})$, for instance the holonomy representation of an orientable hyperbolic 3-orbifold with 2 torsion. The next result proves that there are manifolds with arbitrarily many components of characters that do not lift.

**Theorem 1.4** For every $n$, there exist a compact irreducible 3-manifold $M$ with $\partial M$ a 2-torus such that $X(M)$ has at least $n$ irreducible one dimensional components whose characters do not lift to $\text{SL}_2(\mathbb{C})$. 

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In Section 2 we prove Theorem 1.3. In Section 3 we study the fiber of the projection \( t: R(\Gamma) \to X(\Gamma) \), introducing the different notions of irreducibility. Section 4 is devoted to the study of lifts of representations and the proof of Theorem 1.4. In the last section we determine the singular set of \( X(\Gamma) \) when \( \Gamma \cong F_n \) is the free group of rank \( n \geq 3 \).

2 Invariants of \( \text{PSL}_2(\mathbb{C}) \)

Before proving Theorem 1.3 we quickly review some basic notions of algebraic geometry and invariant theory (that the reader may prefer to skip and go directly to the proof in Subsection 2.3). For details see [KSS], [Kra] or [PV].

2.1 Basic notions of invariant theory

A closed algebraic subset \( Z \subset \mathbb{C}^N \) is called affine. We denote by \( \mathbb{C}[Z] \) the ring of regular functions on \( Z \). An algebraic group \( G \) that acts algebraically on \( Z \) acts naturally on \( \mathbb{C}[Z] \) via \( g f(z) := f(g^{-1}z) \). We denote by \( \mathbb{C}[Z]^G \) the ring of invariant functions, i.e. functions \( f \in \mathbb{C}[Z] \) for which \( g f = f \) for all \( g \in G \).

The group \( G \) is called reductive if it has the following property: for each finite dimensional rational representation \( \rho: G \to \text{GL}(V) \) and every \( G \)-invariant subspace \( W \subset V \) there exist a complementary \( G \)-invariant subspace \( W' \subset V \), i.e. \( V = W' \oplus W \).

If \( Z \) is affine and \( G \) is reductive, then the ring \( \mathbb{C}[Z]^G \) is finitely generated. The affine set \( Y \) such that \( \mathbb{C}[Y] \cong \mathbb{C}[Z]^G \) is called the algebraic quotient and it is denoted by \( Z//G \).

We shall use the following properties of reductive groups:

– By Maschke’s theorem, finite groups are reductive.

– More generally, let \( G \subset \text{GL}_n(\mathbb{C}) \) be a linear algebraic group. The group \( G \) is reductive if there is a Zariski-dense subgroup \( K \subset G \) which is compact in the classical topology. It follows that \( \text{GL}_n(\mathbb{C}) \), \( \text{SL}_n(\mathbb{C}) \), \( \text{O}_n(\mathbb{C}) \), \( \text{SO}_n(\mathbb{C}) \) and \( \text{Sp}_n(\mathbb{C}) \) are reductive.

– Let \( G \) be a reductive linear algebraic group. Let \( Y \) and \( Z \) be varieties on which \( G \) acts and let \( f: X \to Y \) be a \( G \)-invariant regular map. If \( f^*: \mathbb{C}[Y] \to \mathbb{C}[X] \) is surjective then \( f^*(\mathbb{C}[Y]^G) = \mathbb{C}[X]^G \) holds.
2.2 Algebraic structure of $\text{PSL}_2(\mathbb{C})$

The group $\text{PSL}_2(\mathbb{C})$ is algebraic, it is the quotient of $\text{SL}_2(\mathbb{C})$ by the finite group $\{\pm \text{Id}\}$.

It is useful to recall the isomorphism with $\text{SO}_3(\mathbb{C})$, that we construct next. We denote by

$$\text{Ad}: \text{PSL}_2(\mathbb{C}) \to \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$$

the adjoint action of $\text{PSL}_2(\mathbb{C})$ on its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The Killing form on $\mathfrak{sl}_2(\mathbb{C})$ is a non degenerate symmetric bilinear form over $\mathbb{C}$. For each $A \in \text{PSL}_2(\mathbb{C})$, $\text{Ad}(A)$ preserves the Killing form and $\det(\text{Ad}(A)) = 1$, hence $\text{Ad}(\text{PSL}_2(\mathbb{C})) \subseteq \text{SO}_3(\mathbb{C})$. The following lemma is well known from representation theory (see for instance [FH]):

**Lemma 2.1** The action of $\text{PSL}_2(\mathbb{C})$ on the Lie algebra induces an isomorphism $\text{Ad}: \text{PSL}_2(\mathbb{C}) \to \text{SO}_3(\mathbb{C})$.

In this paper the trace will be abbreviated by $\text{tr}$, and $\text{tr}^2(A)$ stands for $(\text{tr}(A))^2$. By direct computation we obtain the equality

$$\text{tr}(\text{Ad}(A)) = \text{tr}^2(A) - 1 = \text{tr}(A^2) + 1 \quad \text{for all } A \in \text{PSL}_2(\mathbb{C}) \quad (1)$$

that will be used later.

Given $\gamma \in \Gamma$, we have a well defined function

$$\tau_\gamma: R(\Gamma) \to \mathbb{C} \quad \rho \mapsto \text{tr}^2(\rho(\gamma))$$

Since it is invariant by conjugation, it induces a function

$$J_\gamma: X(\Gamma) \to \mathbb{C}.$$

2.3 Proof of Theorem 1.3

Theorem 1.3 is a consequence of:

**Proposition 2.2** The ring of invariant functions $\mathbb{C}[R(\Gamma)]^{\text{PSL}_2(\mathbb{C})}$ is generated by the functions $\tau_\gamma$, with $\gamma \in \Gamma$.

**Proof.** There is a surjection $\psi: F_n \to \Gamma$ where $F_n$ is a free group of rank $n \in \mathbb{N}$. We obtain an inclusion $\psi^*: R(\Gamma) \subset R(F_n)$. This inclusion induces a surjection $\psi_*: \mathbb{C}[R(F_n)] \to \mathbb{C}[R(\Gamma)]$. Now, $\text{PSL}_2(\mathbb{C})$ is reductive and acts...
regularly by conjugation on the representation varieties. Hence we obtain a surjection
\[ \psi_* : \mathbb{C}[R(F_n)]^{\text{PSL}_2(\mathbb{C})} \to \mathbb{C}[R(\Gamma)]^{\text{PSL}_2(\mathbb{C})} \]
and it is sufficient to prove the proposition for \( \Gamma = F_n \) since \( \psi_*(\tau_\gamma) = \tau_{\psi(\gamma)} \).

Using Lemma 2.1 and (1), we have to prove that \( \mathbb{C}[\mathbb{R}(F_n)]^{\text{SO}_3(\mathbb{C})} \) is generated by the trace functions on elements of \( F_n \). Equivalently, we claim that
\[ \mathbb{C}[\mathbb{R}(F_n)]^{\text{SO}_3(\mathbb{C})} \]
\[ \mathbb{C}[\text{SO}_3(\mathbb{C}) \times \cdots \times \text{SO}_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \]
is generated by the traces of products of matrices and their transposes.

Let \( M_3(\mathbb{C}) \) denote the algebra of \( 3 \times 3 \) matrices with complex coefficients. The group \( \text{PSL}_2(\mathbb{C}) \cong \text{SO}_3(\mathbb{C}) \) acts on the product \( M_3(\mathbb{C}) \times \cdots \times M_3(\mathbb{C}) \) diagonally by conjugation. A theorem of Aslaksen, Tan and Zhu (see [ATZ]) states that the algebra of invariant functions
\[ \mathbb{C}[M_3(\mathbb{C}) \times \cdots \times M_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \]
is generated by the traces of products of matrices and their transposes. Thus the proof of the proposition reduces to show that we have a natural surjection
\[ \mathbb{C}[M_3(\mathbb{C}) \times \cdots \times M_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \to \mathbb{C}[\text{SO}_3(\mathbb{C}) \times \cdots \times \text{SO}_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \]
where the proof of the proposition reduces to show that we have a natural surjection
\[ \mathbb{C}[M_3(\mathbb{C}) \times \cdots \times M_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \to \mathbb{C}[\text{SO}_3(\mathbb{C}) \times \cdots \times \text{SO}_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \]
which is of course \( \text{SO}_3(\mathbb{C}) \)-invariant. Using the fact that \( \text{SO}_3(\mathbb{C}) \) is reductive gives the surjection \( \mathbb{C}[M_3(\mathbb{C}) \times \cdots \times M_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \to \mathbb{C}[\text{SO}_3(\mathbb{C}) \times \cdots \times \text{SO}_3(\mathbb{C})]^{\text{SO}_3(\mathbb{C})} \).

Since \( \mathbb{C}[X(\Gamma)] = \mathbb{C}[R(\Gamma)]^{\text{SO}_3(\mathbb{C})} \) is finitely generated, we also obtain:

**Corollary 2.3** There are finitely many elements \( \gamma_1, \ldots, \gamma_N \) in \( \Gamma \) such that \( J_{\gamma_1} \times \cdots \times J_{\gamma_N} : X(M) \to \mathbb{C}^N \) is an embedding and its image is a closed algebraic set.

### 2.4 Other invariant functions

There are other natural functions to consider. Let \( \Gamma^2 \) be the subgroup of \( \Gamma \) generated by the squares \( \gamma^2 \) of all elements \( \gamma \) of \( \Gamma \). It is well known that we have an exact sequence:
\[ 1 \to \Gamma^2 \to \Gamma \to H_1(\Gamma, C_2) \to 1, \]
where $C_2 = \{ \pm 1 \}$ is the group with 2 elements. For instance, if $\Gamma$ is a finite group of odd order, then $\Gamma^2 = \Gamma$. In general, if $\gamma, \mu \in \Gamma$ the commutator $[\gamma, \mu] = \gamma \mu \gamma^{-1} \mu^{-1} = (\mu \gamma)(\mu^{-1} \gamma^{-1})\mu^{-2}$ is in $\Gamma^2$ and hence $\Gamma^2$ contains the commutator group $\Gamma' = [\Gamma, \Gamma]$. Notice that

$$\Gamma^2 = \bigcap_{\epsilon \in H^1(\Gamma, C_2)} \text{Ker}(\epsilon)$$

where $H^1(\Gamma, C_2) = \text{Hom}(\Gamma, C_2)$. Let $R(\Gamma, \text{SL}_2(\mathbb{C}))$ denote the variety of representations of $\Gamma$ in $\text{SL}_2(\mathbb{C})$. The cohomology group $H^1(\Gamma, C_2)$ acts on this variety of representations as follows: an homomorphism $\epsilon: \Gamma \to C_2 = \{ \pm 1 \}$ maps the representation $\rho \in R(\Gamma, \text{SL}_2(\mathbb{C}))$ to the product of representations $\epsilon \cdot \rho$ (which maps $\gamma \in \Gamma$ to $\epsilon(\gamma) \cdot \rho(\gamma)$).

### 2.4.1 Invariant functions for the free group

Let $F$ be a finitely generated free group. For $\gamma \in F^2$ and $\rho \in R(F)$,

$$\text{tr}(\rho(\gamma))$$

is well defined since the representation $\rho: F \to \text{PSL}_2(\mathbb{C})$ lifts to $\tilde{\rho}: F \to \text{SL}_2(\mathbb{C})$ and for $\gamma \in F^2$ the trace $\text{tr}(\tilde{\rho}(\gamma))$ depends only on $\gamma$. Note that two lifts $\tilde{\rho}_1$ and $\tilde{\rho}_2$ of $\rho$ differ by a homomorphism $\epsilon \in H^1(F, C_2)$ and that $F^2 \subset \text{Ker}(\epsilon)$ for each $\epsilon \in H^1(F, C_2)$.

**Proposition 2.4** Let $F$ be a free group. For every $k$-tuple $\gamma_1, \ldots, \gamma_k \in F$ such that the product $\gamma_1 \cdots \gamma_k \in F^2$, the function

$$\sigma_{\gamma_1, \ldots, \gamma_k}: R(F) \to \mathbb{C}$$

$$\rho \mapsto \text{tr}(\tilde{\rho}(\gamma_1)) \cdots \text{tr}(\tilde{\rho}(\gamma_k))$$

is regular (i.e. $\sigma_{\gamma_1, \ldots, \gamma_k} \in \mathbb{C}[R(F)]$). Here, $\tilde{\rho}: F \to \text{SL}_2(\mathbb{C})$ denotes a lift of $\rho$.

In order to prove this proposition we shall use the following:

**Lemma 2.5** Let $F_n$ be the free group of rank $n$. We have a natural isomorphism

$$R(F_n, \text{SL}_2(\mathbb{C}))/H^1(F_n, C_2) \cong R(F_n).$$

**Proof.** Since $R(F_n, \text{SL}_2(\mathbb{C})) \cong \text{SL}_2(\mathbb{C})^n$, $R(F_n) \cong \text{PSL}_2(\mathbb{C})^n$ and $\text{SL}_2(\mathbb{C})/C_2 \cong \text{PSL}_2(\mathbb{C})$, we have the lemma. \hfill $\square$

**Proof of Proposition 2.4.** For a free group $F$ and $\gamma_1, \ldots, \gamma_k \in F$, the function $\tilde{\sigma}: R(F, \text{SL}_2(\mathbb{C})) \to \mathbb{C}$ given by $\tilde{\sigma}(\rho) = \text{tr}(\rho(\gamma_1)) \cdots \text{tr}(\rho(\gamma_k))$ is regular. Moreover, we have $\tilde{\sigma}(\epsilon \cdot \rho) = \epsilon(\gamma_1 \cdots \gamma_k)\tilde{\sigma}(\rho)$. Since the product $\gamma_1 \cdots \gamma_k \in F^2$
we get that \( \tilde{\sigma} \in \mathbb{C}[R(F_n, \text{SL}_2(\mathbb{C}))]^H ) \) is an invariant regular function on the \( \text{SL}_2(\mathbb{C}) \) representation variety. By Lemma 2.5, this function factors through \( R(F) \) and gives the regular function \( \sigma_{\gamma_1, \ldots, \gamma_k} \in \mathbb{C}[R(F)] \).

\[ \square \]

**Example 2.6** Given \( \gamma, \eta \in F \), by Proposition 2.4, \( \sigma_{\gamma, \eta, \gamma \eta} \in \mathbb{C}[R(F)] \), thus by Proposition 2.5, \( \sigma_{\gamma, \eta, \gamma \eta} \) is a polynomial on the functions \( \tau \).

To compute explicitly the polynomial of Example 2.6, we recall some identities of traces in \( \text{SL}_2(\mathbb{C}) \):

\[
\text{tr}(AB) = \text{tr}(BA) \quad \text{and} \quad \text{tr}(A) = \text{tr}(A^{-1}) \quad \forall A, B \in \text{SL}_2(\mathbb{C}).
\]

In addition, we have the fundamental identity:

\[
\text{tr}(AB) + \text{tr}(A^{-1}B) = \text{tr}(A) \text{tr}(B) \quad \forall A, B \in \text{SL}_2(\mathbb{C}).
\]

(2)

This identity can be deduced from \( A^2 - (\text{tr}A)A + \text{Id} = 0 \) multiplying by \( A^{-1}B \) and taking traces. Taking the square of \( \text{tr}(AB^{-1}) = \text{tr}(A) \text{tr}(B) - \text{tr}(AB) \) we deduce:

\[
2\text{tr}(A)\text{tr}(B)\text{tr}(AB) = \text{tr}^2(A)\text{tr}^2(B) + \text{tr}^2(AB) - \text{tr}^2(AB^{-1}).
\]

Thus

\[
\sigma_{\gamma, \eta, \gamma \eta} = \frac{1}{2}(\tau_\gamma \tau_\eta + \tau_\gamma \eta - \tau_{\gamma \eta^{-1}}).
\]

(3)

**Example 2.7** For every \( \gamma, \mu \in F \), the commutator \( [\gamma, \mu] = \gamma \mu \gamma^{-1} \mu^{-1} \) belongs to \( F^2 \) and therefore \( \sigma_{[\gamma, \mu]} \in \mathbb{C}[R(F)] \). Using the same method as for Equation (3) one can find:

\[
\sigma_{[\gamma, \eta]} = \tau_\gamma + \tau_\eta + \frac{1}{2} \tau_{\gamma \eta} + \frac{1}{2} \tau_{\gamma \eta^{-1}} - \frac{1}{2} \tau_\gamma \tau_\eta - 2.
\]

(4)

**2.4.2 Invariant functions for other groups**

Let \( \Gamma \) be a finitely generated group, \( F \) a free group and \( \psi : F \to \Gamma \) a surjection. It induces another surjection \( \psi_* : \mathbb{C}[R(F)] \to \mathbb{C}[R(\Gamma)] \), \( \psi_*f(\rho) = f(\rho \circ \psi) \). Hence we obtain for all \( \eta_1, \ldots, \eta_k \in F \) such that the product \( \eta_1 \cdots \eta_k \in F^2 \) a regular function \( \psi_*\sigma_{\eta_1, \ldots, \eta_k} \in \mathbb{C}[R(\Gamma)] \). Note that the functions \( \psi_*\sigma_{\eta_1} \) and \( \psi_*\sigma_{\eta_2} \) might be different even if \( \psi(\eta_1) = \psi(\eta_2) \) in \( \Gamma \). This reflects the fact that in general not every representation \( \rho : \Gamma \to \text{PSL}_2(\mathbb{C}) \) lifts to \( \text{SL}_2(\mathbb{C}) \).
Example 2.8 Let \( \psi: F \to \Gamma \) be the canonical projection where \( F = \langle x, y \mid \rangle \) and \( \Gamma = \langle x, y \mid [x,y] = 1 \rangle \). We consider the representation \( \rho: \Gamma \to \text{PSL}_2(\mathbb{C}) \) given by \( \rho(x) = \pm A_x \) and \( \rho(y) = \pm A_y \) where

\[
A_x = \begin{pmatrix} \ i & 0 \\ 0 & -i \end{pmatrix} \quad \text{and} \quad A_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

We obtain \( \text{tr}([A_x, A_y]) = -2 \) and hence \( \psi_\ast \sigma_{[x,y]}(\rho) = -2 \). On the other hand, we have \( [x,y] = 1 \) in \( \Gamma \) and \( \psi_\ast \sigma_1 = 2 \) is a constant function.

If the representation \( \rho \in R(\Gamma) \) admits a lift \( \tilde{\rho}: \Gamma \to \text{SL}_2(\mathbb{C}) \) then

\[
\psi_\ast \sigma_{\eta_1,..,\eta_k}(\rho) = \text{tr}(\tilde{\rho}(\psi(\eta_1))) \cdots \text{tr}(\tilde{\rho}(\psi(\eta_k))) \quad (5)
\]

only depends on the elements \( \psi(\eta_1), \ldots, \psi(\eta_k) \in \Gamma \).

3 Irreducibility

To study the fiber of the map \( t: R(\Gamma) \to X(\Gamma) \) we shall consider two different notions of irreducibility for \( \rho \in R(\Gamma) \), the usual one as a representation in \( \text{PSL}_2(\mathbb{C}) \) and the so called Ad-irreducibility, for the three dimensional representation \( \text{Ad} \circ \rho: \Gamma \to \text{SO}_3(\mathbb{C}) \).

3.1 Irreducible representations

Definition 3.1 A representation \( \rho \in R(\Gamma) \) is called reducible if \( \rho(\Gamma) \) preserves a point of \( \mathbb{P}^1(\mathbb{C}) \). Otherwise it is called irreducible. A character \( \chi: \Gamma \to \mathbb{C} \) is called reducible if it is the character of a reducible representation.

Remark 3.2 Up to conjugation, the image of a reducible representation is contained in the set of upper-triangular matrices \((\ast \ast \ast)\).

We shall require the following well known lemma (see [Bea, § 4.3]).

Lemma 3.3 Two non-trivial elements \( g, h \in \text{PSL}_2(\mathbb{C}) \) have a common fixed point in \( \mathbb{P}^1(\mathbb{C}) \) if and only if \( \text{tr}([g, h]) = 2 \). In addition, this fixed point is unique if \([g, h]\) is not the identity.

Irreducibility is a property that can be detected from characters:

Lemma 3.4 A representation \( \rho \in R(\Gamma) \) is reducible iff \( \text{tr}([\rho(\gamma), \rho(\eta)]) = 2 \) for all elements \( \gamma, \eta \) in \( \Gamma \).
Proof. If $\rho$ is reducible then all the $\rho(\gamma)$ have a common fixed point and Lemma 3.3 gives the result.

Assume now that $\text{tr}([\rho(\gamma), \rho(\eta)]) = 2$ for all elements $\gamma, \eta$ in $\Gamma$.

Case 1: There are two elements $\gamma$ and $\eta$ in $\Gamma$ such that $A = [\rho(\gamma), \rho(\eta)]$ is a non-trivial parabolic element in the image of $\Gamma$. For any $\mu \in \Gamma$, either $\rho(\mu)$ commutes with $A$ or $[\rho(\mu), A]$ is non-trivial. The former possibility implies that $\rho(\mu)$ fixes the unique fixed point of $A$, the latter too by Lemma 3.3.

Case 2: The image of $\rho$ is an abelian group. Abelian subgroups of $\text{PSL}_2(\mathbb{C})$ are well-known: either they have a global fixed point in $\mathbb{P}^1(\mathbb{C})$ or they are conjugated to the group with four elements generated by $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Since the commutator of these two generators is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, this possibility does not occur. □

Definition 3.5 A non-cyclic abelian subgroup of $\text{PSL}_2(\mathbb{C})$ with four elements is called Klein’s 4-group. Such a group is realized by rotations about three orthogonal geodesics and it is conjugated to the one generated by $\pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Let $R^\text{red}(\Gamma)$ denote the set of reducible representations and $X^\text{red}(\Gamma) = t(R^\text{red}(\Gamma))$. Let $F$ be a free group and let $\psi : F \to \Gamma$ be surjective. Lemma 3.3 implies that

$$R^\text{red}(\Gamma) = \{ \rho \in R(\Gamma) \mid \psi_\ast \sigma_{[\gamma, \eta]}(\rho) = 2 \quad \forall \gamma, \eta \in F \}$$

is a Zariski closed subset invariant by conjugation. Thus, by invariant theory we have:

Corollary 3.6 The set $X^\text{red}(\Gamma)$ is Zariski closed and $R^\text{red}(\Gamma) = t^{-1}(X^\text{red}(\Gamma))$.

Remark 3.7 Every reducible character $\chi$ is the character of a diagonal representation, because if $\rho(\gamma) = \pm \begin{pmatrix} a_\gamma & b_\gamma \\ 0 & c_\gamma \end{pmatrix}$ is a representation, then $\rho'(\gamma) = \pm \begin{pmatrix} a_\gamma & 0 \\ 0 & c_\gamma \end{pmatrix}$ is also a representation with $\chi_\rho = \chi_{\rho'}$.

3.2 Ad-irreducibility

Definition 3.8 A representation $\rho \in R(\Gamma)$ is Ad-reducible if $\mathfrak{sl}_2(\mathbb{C})$ has a proper invariant subspace by the action of $\text{Ad} \circ \rho$. Otherwise it is Ad-irreducible.

Let $\mathbb{H}^3$ denote the three-dimensional hyperbolic space and $\partial_{\infty} \mathbb{H}^3$ its ideal boundary. We use the isomorphism $\text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}_2(\mathbb{C})$ and the natural identification $\partial_{\infty} \mathbb{H}^3 \cong \mathbb{P}^1(\mathbb{C})$. 

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Lemma 3.9 A representation \( \rho : \Gamma \to \text{PSL}_2(\mathbb{C}) \) is Ad-reducible if and only if \( \rho(\Gamma) \) preserves either a point in \( \partial_\infty \mathbb{H}^3 \) or a geodesic in \( \mathbb{H}^3 \).

Proof. Let \( V \) be a proper subspace of \( \mathfrak{sl}_2(\mathbb{C}) \) invariant by \( \text{Ad} \circ \rho(\Gamma) \). Up to taking \( V^\perp \), we may assume \( \dim V = 1 \), because the Killing form is not degenerate. We have then two possibilities: either the Killing form restricted to \( V \) vanishes or not. In the first case \( V \) consists of parabolic Killing fields, in particular the 1-parameter group \( \{\exp(v) \mid v \in V\} \cong \mathbb{C} \) is parabolic and fixes a unique point at infinity, that has to be fixed also by \( \rho \). In the second case, when the Killing form restricted to \( V \) does not vanish, the 1-parameter group \( \{\exp(v) \mid v \in V\} \cong \mathbb{C}^* \) is a subgroup of index two in the group of isometries which preserve a geodesic in \( \mathbb{H}^3 \). This geodesic has to be preserved by the representation. Conversely, if a representation preserves a point in \( \partial_\infty \mathbb{H}^3 \) or a geodesic, the previous argument shows how to construct an invariant subspace of \( \mathfrak{sl}_2(\mathbb{C}) \).

\[ \square \]

Corollary 3.10 Reducible representations are also Ad-reducible.

Remark 3.11 A representation Ad-reducible but not reducible is a \( C_2 \)-extension of an abelian one that fixes an oriented geodesic. Thus it preserves an unoriented geodesic.

We call a representation \( \rho \in R(\Gamma) \) abelian respectively metabelian if its image is an abelian respectively metabelian subgroup of \( \text{PSL}_2(\mathbb{C}) \).

Lemma 3.12 A representation \( \rho \in R(\Gamma) \) is Ad-reducible iff it is metabelian.

Proof. If \( \rho \) is Ad-reducible then its image is contained in the stabilizer of either a point in \( P^1(\mathbb{C}) \) or a geodesic in \( \mathbb{H}^3 \). Those stabilizers are metabelian, since they are respectively the group of affine transformations of \( \mathbb{C} \) and the semidirect product \( \mathbb{C}^* \rtimes C_2 \).

Now assume that \( \rho(\Gamma) \subset \text{PSL}_2(\mathbb{C}) \) is a metabelian subgroup. We use the fact that an abelian subgroup of \( \text{PSL}_2(\mathbb{C}) \) preserves a unique point of \( P^1(\mathbb{C}) \), a unique geodesic or it is Klein’s 4-group (Def. 3.5). If \( \rho([\Gamma, \Gamma]) \) is trivial then \( \rho \) is Ad-reducible by this fact. If \( \rho([\Gamma, \Gamma]) \) is not trivial, then we look at those unique invariant objects: the unique point in \( P^1(\mathbb{C}) \), the unique geodesic, or the unique three geodesics if it is Klein’s 4-group. Since \( [\Gamma, \Gamma] \) is normal in \( \Gamma \), uniqueness implies that \( \rho(\Gamma) \) preserves the same objects, hence \( \rho \) is Ad-reducible.

\[ \square \]

Lemma 3.13 The set of characters of Ad-reducible representations is Zariski closed.
Proof. Lemma 3.12 gives that the set of Ad-reducible representations is

$$R^{Ad-red} = \{ \rho \in R(\Gamma) \mid \rho(c) = \pm \text{Id} \quad \forall c \in \Gamma'' \}$$

where $\Gamma''$ denotes the second commutator group of $\Gamma$. This is a closed subset of $R(\Gamma)$ invariant under conjugation. Hence we have $X^{Ad-red}(\Gamma) = t(R^{Ad-red})$ is a closed subset of $X(\Gamma)$.

\[\square\]

Remark 3.14 The image of an Ad-reducible representation is elementary, but elementary groups also include groups that fix a point in $\mathbb{H}^3$.

3.3 The fibers of $t$: $R(\Gamma) \to X(\Gamma)$

Lemma 3.15 The fiber of an irreducible character consists of a single closed orbit (i.e. two irreducible representations have the same character iff they are conjugate).

Proof. Let $\rho_1, \rho_2 \in R(\Gamma)$ be two irreducible representations with $\chi_{\rho_1} = \chi_{\rho_2}$.

We assume first that each $\rho_i$ is irreducible but Ad-reducible. Thus each $\rho_i$ preserves a geodesic $l$, that we may assume to be the same after conjugation. The action of $\rho_i(\gamma)$ on $l$ is determined by the value of $\chi_{\rho_i}(\gamma)$, except in the case $\chi_{\rho_i}(\gamma) = 0$, which means that $\rho_i(\gamma)$ is a rotation through angle $\pi$, but it can be either about $\gamma$ or about an axis perpendicular to $\gamma$. Thus if there exists an element $\gamma_0 \in \Gamma$ with $\chi_{\rho_i}(\gamma_0) \neq 4, 0$ (i.e. $\rho_i(\gamma_0)$ is either a loxodromic element or a rotation of angle $\neq \pi$) then $\forall \gamma \in \Gamma$ the action of $\rho_i(\gamma)$ on the geodesic $l$ is determined by $\chi_{\rho_i}(\gamma)$ and $\chi_{\rho_i}(\gamma_0)$. In particular $\rho_i$ is unique up to conjugation. The exceptional case occurs when $\chi_{\rho_i}(\gamma) = 0$ or 4 for every $\gamma \in \Gamma$. In this special case, $\rho_i$ is necessarily a representation into Klein's 4-group. The lemma is also clear in this case.

When $\rho_i$ are Ad-irreducible, we can assume that $\Gamma$ is a free group. Thus we can lift $\rho_i$ to $\tilde{\rho}_1: \Gamma \to \text{SL}_2(\mathbb{C})$. By Example 2.6 for every pair $\gamma, \gamma' \in \Gamma$ we obtain a regular function $\sigma_{\gamma, \gamma', \gamma':} X(\Gamma) \to \mathbb{C}$, given by

$$\sigma_{\gamma, \gamma', \gamma':}(\lambda) = \text{tr} \tilde{\rho}(\gamma \gamma') \text{ tr} \tilde{\rho}(\gamma) \text{ tr} \tilde{\rho}(\gamma')$$

where $\tilde{\rho}: \Gamma \to \text{SL}_2(\mathbb{C})$ is any lift of $\rho$. Thus:

$$\text{tr} \tilde{\rho}_1(\gamma \gamma') \text{ tr} \tilde{\rho}_1(\gamma) \text{ tr} \tilde{\rho}_1(\gamma') = \text{tr} \tilde{\rho}_2(\gamma \gamma') \text{ tr} \tilde{\rho}_2(\gamma) \text{ tr} \tilde{\rho}_2(\gamma'). \quad (6)$$

We define $\epsilon: \Gamma \to C_2 = \{ \pm 1 \}$ by the formula:

$$\text{tr} \tilde{\rho}_1(\gamma) = \epsilon(\gamma) \text{ tr} \tilde{\rho}_2(\gamma), \quad \forall \gamma \in \Gamma \text{ such that } \chi_{\rho_1}(\gamma) \neq 0.$$
When \( \chi_{\rho_1}(\gamma) = 0 \), since we assume that \( \rho_i \) is Ad-irreducible, we can find \( \gamma_0 \in \Gamma \) with \( \chi_{\rho_i}(\gamma_0) \neq 0 \) and \( \chi_{\rho_i}(\gamma\gamma_0) \neq 0 \). In this case we define \( \epsilon(\gamma) = \epsilon(\gamma_0) \cdot \epsilon(\gamma\gamma_0) \).

By (6), \( \epsilon \) is a morphism. Hence \( \tilde{\rho}_1 \) and \( \epsilon \cdot \tilde{\rho}_2 \) are irreducible representations in \( \text{SL}_2(\mathbb{C}) \) with the same character. By [CS] they are conjugate. \( \square \)

**Proposition 3.16**  
(i) A character \( \chi \) is irreducible iff \( \text{PSL}_2(\mathbb{C}) \) acts transitively on the fiber and with finite stabilizer.

(ii) A character is Ad-irreducible iff \( \text{PSL}_2(\mathbb{C}) \) acts faithfully on the fiber.

**Proof.** (i) By Lemma 3.15 if \( \chi \) is irreducible then \( \text{PSL}_2(\mathbb{C}) \) acts transitively on \( t^{-1}(\chi) \). Assume now that the stabilizer is infinite: i.e. there exists non-trivial \( A \in \text{PSL}_2(\mathbb{C}) \) of order \( \geq 3 \) (possibly infinite) and \( \rho \) in the fiber such that \( A \) commutes with \( \rho \). If \( A \) is parabolic, then it has a fixed point in \( P^1(\mathbb{C}) \) and therefore \( \rho \) fixes this point. Otherwise \( A \) has an invariant geodesic; since \( A \) has order \( \geq 3 \), \( \rho \) preserves the oriented geodesic, and therefore \( \rho \) is also reducible.

Assume the character is reducible, then it has a diagonal representation \( \rho \) on the fiber (Rem. 3.7), and therefore the group of diagonal matrices stabilizes it. Thus the stabilizer is infinite.

(ii) Assume \( \text{PSL}_2(\mathbb{C}) \) does not act faithfully on the fiber, i.e. there exists non-trivial \( A \in \text{PSL}_2(\mathbb{C}) \) and \( \rho \) in the fiber such that \( A \) commutes with \( \rho \). If \( A \) is parabolic, then \( \rho \) fixes a point in \( P^1(\mathbb{C}) \) by the previous argument. Otherwise \( A \) has an invariant geodesic, and by commutativity, \( \rho \) must preserve this geodesic. In both cases, \( \rho \) is Ad-reducible.

If the character is irreducible but Ad-reducible, then it preserves a geodesic, and the rotation through angle \( \pi \) about this geodesic commutes with \( \rho \). Hence the stabilizer is nontrivial. \( \square \)

**Remark 3.17** The projection \( t: R(\Gamma) \rightarrow X(\Gamma) \) induces a bijection between irreducible components.

A priori \( R(\Gamma) \) could have more components than \( X(\Gamma) \), but the number of components is the same, because \( \text{PSL}_2(\mathbb{C}) \) is irreducible.

From Corollary 3.6 and Proposition 3.16 we deduce:

**Corollary 3.18** Let \( \rho \in R(\Gamma) \) be an irreducible representation. Let \( R_0 \) denote an irreducible component of \( R(\Gamma) \) that contains \( \rho \) and let \( X_0 \) denote the corresponding irreducible component of \( X(\Gamma) \). Then

\[
\dim R_0 = \dim X_0 + 3.
\]
4 Lifts of representations to $\text{SL}_2(\mathbb{C})$

Let $\overline{R}(\Gamma) \subset R(\Gamma)$ denote the set of representations $\rho \in R(\Gamma)$ that lift to $\text{SL}_2(\mathbb{C})$. According to [Cul] Thm. 4.1 $\overline{R}(\Gamma)$ is a union of connected components of $R(\Gamma)$. In particular $\overline{R}(\Gamma)$ is a Zariski-closed algebraic subset of $R(\Gamma)$, since irreducible complex varieties are connected in the $\mathbb{C}$-topology [Sha, VII, §2]. Moreover, $\overline{R}(\Gamma)$ is invariant under conjugation and hence the algebraic quotient

$$\overline{X}(\Gamma) = \overline{R}(\Gamma)/\text{PSL}_2(\mathbb{C})$$

is a well defined closed subset of $X(\Gamma)$.

In many cases, $\overline{X}(\Gamma) = X(\Gamma)$. For instance this is clear when $\Gamma$ is a free group. It is also true if $H^2(\Gamma, C_2) = 0$ by the following remark (see [GM] or [Cul]).

**Remark 4.1** Let $\rho: \Gamma \to \text{PSL}_2(\mathbb{C})$ be a representation. There is a second Stiefel-Whitney class $w_2(\rho) \in H^2(\Gamma, C_2)$ which is exactly the obstruction for the existence of a lift $\overline{\rho}: \Gamma \to \text{SL}_2(\mathbb{C})$.

4.1 Properties of $\overline{X}(\Gamma)$

Let $R(\Gamma, \text{SL}_2(\mathbb{C}))$ and $X(\Gamma, \text{SL}_2(\mathbb{C}))$ denote the variety of representations and characters in $\text{SL}_2(\mathbb{C})$. The ring $\mathbb{C}[R(\Gamma, \text{SL}_2(\mathbb{C}))]^{\text{SL}_2(\mathbb{C})}$ is generated by the trace functions $\tilde{\tau}_\gamma: R(\Gamma, \text{SL}_2(\mathbb{C})) \to \mathbb{C}$, $\tilde{\tau}_\gamma(\rho) = \text{tr}(\rho(\gamma))$. The function induced by $\tilde{\tau}_\gamma$ is denoted by $I_\gamma: X(\Gamma) \to \mathbb{C}$, therefore $\mathbb{C}[X(\Gamma)]$ is finitely generated by the functions $I_\gamma$, $\gamma \in \Gamma$ [CS].

Elements of the cohomology group $H^1(\Gamma, C_2)$ are homomorphisms $\theta: \Gamma \to C_2 = \{\pm 1\}$ that act on representations by multiplication. The action of $\epsilon \in H^1(\Gamma, C_2)$ on $I_\gamma$ is given by: $\epsilon \cdot I_\gamma = \epsilon(\gamma) I_\gamma$. Since $H^1(\Gamma, C_2)$ is finite, it is reductive and we may take the quotient of invariant theory.

Let $F$ be a finitely generated free group and $\psi: F \to \Gamma$ be a surjection. We fix a $k$-tuple $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that the product $\gamma_1 \cdots \gamma_k \in \Gamma^2$. Moreover, we choose $\eta_i \in F$ such that $\psi(\eta_i) = \gamma_i$ and such that the product $\eta_1 \cdots \eta_k \in F^2$. The function $\psi^* \sigma_{\eta_1, \ldots, \eta_k} \in \mathbb{C}[\overline{R}(\Gamma)]$ is invariant under conjugation and gives us a function $\psi^* \sigma_{\eta_1, \ldots, \eta_k} \in \mathbb{C}[\overline{X}(\Gamma)]$. By Equation (5) we have $\psi^* \sigma_{\eta_1, \ldots, \eta_k}(\chi) = \tilde{\chi}(\gamma_1) \cdots \tilde{\chi}(\gamma_k)$ where $\tilde{\chi} \in X(\Gamma, \text{SL}_2(\mathbb{C}))$ is a character such that $\pi(\tilde{\chi}) = \chi$. Note that $\pi: X(\Gamma, \text{SL}_2(\mathbb{C})) \to \overline{X}(\Gamma)$ is surjective. The function

$$\Sigma_{\gamma_1, \ldots, \gamma_k} := \phi^* \sigma_{\eta_1, \ldots, \eta_k} \in \mathbb{C}[\overline{X}(\Gamma)]$$

(7)

depends only on the elements $\gamma_i \in \Gamma$. 

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Proposition 4.2 There is a natural isomorphism:

\[ \frac{X(\Gamma, \text{SL}_2(\mathbb{C}))}{H^1(\Gamma, C_2)} \cong X(\Gamma). \]

Proof. Composition with the projection \( \text{SL}_2(\mathbb{C}) \to \text{PSL}_2(\mathbb{C}) \) induces a surjection

\[ \pi : X(\Gamma, \text{SL}_2(\mathbb{C})) \to X(\Gamma), \]

which is easily seen to be algebraic and is given by \( \pi(\chi) = \chi^2 \). At the level of function rings it induces an injection

\[ \pi^* : \mathbb{C}[X(\Gamma)] \hookrightarrow \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))]. \]

We have \( \pi^* f(\chi) = f(\chi^2) \) for \( f \in \mathbb{C}[X(\Gamma)] \) and \( \chi \in X(\Gamma, \text{SL}_2(\mathbb{C})) \). The image of \( \pi^* \) is contained in the set of invariant functions:

\[ \text{Im} \pi^* \subseteq \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))]^{H^1(\Gamma, C_2)}. \]

More precisely, we have \( \pi^* f(\epsilon \chi) = f(\epsilon^2 \chi^2) = \pi^* f(\chi) \) for all \( \epsilon \in H^1(\Gamma, C_2) \). It remains to prove that this inclusion is an equality.

Since \( \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))] \) is generated as \( \mathbb{C} \)-algebra by the functions \( I_\gamma \) with \( \gamma \in \Gamma \), the monomials

\[ I_{\gamma_1} I_{\gamma_2} \cdots I_{\gamma_k} \]

generate \( \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))] \) as a \( \mathbb{C} \)-vector space. Taking the average of the action of \( H^1(\Gamma, C_2) \), we deduce that the subspace of invariant functions \( \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))]^{H^1(\Gamma, C_2)} \) is generated by

\[ \frac{1}{2r} \sum_{\epsilon \in H^1(\Gamma, C_2)} \epsilon \cdot I_{\gamma_1} \cdots I_{\gamma_k} = \left( \frac{1}{2r} \sum_{\epsilon \in H^1(\Gamma, C_2)} \epsilon(\gamma_1 \cdots \gamma_k) \right) I_{\gamma_1} \cdots I_{\gamma_k} \]

where \( r \) is the rank of \( H^1(\Gamma, C_2) \) (see [Kra, II.3.6] for instance). Using the fact that

\[ \frac{1}{2r} \sum_{\epsilon \in H^1(\Gamma, C_2)} \epsilon(\gamma) = \begin{cases} 1 & \text{if } \gamma \in \Gamma^2 \\ 0 & \text{otherwise} \end{cases} \]

we deduce that \( \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))]^{H^1(\Gamma, C_2)} \) is generated by the monomials \( I_{\gamma_1} \cdots I_{\gamma_k} \) such that the product \( \gamma_1 \cdots \gamma_k \in \Gamma^2 \).

On the other hand we have for \( \chi \in X(\Gamma, \text{SL}_2(\mathbb{C})) \):

\[ \pi^* \Sigma_{\gamma_1, \ldots, \gamma_k}(\chi) = \Sigma_{\gamma_1, \ldots, \gamma_k}(\chi^2) = \chi(\gamma_1) \cdots \chi(\gamma_k) = I_{\gamma_1} \cdots I_{\gamma_k}(\chi), \]

where \( \Sigma_{\gamma_1, \ldots, \gamma_k} \) is the function defined in (7). This gives that the monomials \( I_{\gamma_1} \cdots I_{\gamma_k} \) such that the product \( \gamma_1 \cdots \gamma_k \in \Gamma^2 \) is in the image of \( \pi^* \) and therefore \( \mathbb{C}[X(\Gamma, \text{SL}_2(\mathbb{C}))]^{H^1(\Gamma, C_2)} = \text{Im} \pi^* \). 

\[ \square \]
Remark 4.3 Let \( p: X(\Gamma, \text{SL}_2(\mathbb{C})) \to \overline{X}(\Gamma) \) denote the projection. If \( \chi \in X(\Gamma) \) is Ad-irreducible, then \( p^{-1}(\chi) \) has \( 2^r \) points where \( r \) is the rank of \( H^1(\Gamma, C_2) \). If \( \chi \) is Ad-reducible then the cardinality of \( p^{-1}(\chi) \) is strictly less than \( 2^r \). Thus \( p \) is a branched covering with branching locus the set of Ad-reducible characters.

Example 4.4 Let \( F_2 \) be the free group of rank 2, with generators \( \alpha \) and \( \beta \). There is an isomorphism:

\[
(I_\alpha, I_\beta, I_{\alpha\beta}): X(F_2, \text{SL}_2(\mathbb{C})) \to \mathbb{C}^3
\]

where \( I_\gamma \) denotes the regular function induced by \( \tilde{\tau}_\gamma \). In particular \( X(F_2, \text{SL}_2(\mathbb{C})) \) is smooth.

Since every representation in \( R(F_2) \) lifts to \( \text{SL}_2(\mathbb{C}) \), we deduce

\[
X(F_2) = X(F_2, \text{SL}_2(\mathbb{C}))/\text{H}(F_2, C_2).
\]

The group \( \text{H}(F_2, C_2) \cong (C_2)^2 \) has four elements, and its action on \( X(F_2, \text{SL}_2(\mathbb{C})) \) is generated by the involutions

\[
(I_\alpha, I_\beta, I_{\alpha\beta}) \mapsto (-I_\alpha, I_\beta, -I_{\alpha\beta})
\]

\[
(I_\alpha, I_\beta, I_{\alpha\beta}) \mapsto (I_\alpha, -I_\beta, -I_{\alpha\beta}).
\]

Thus \( \mathbb{C}[X(F_2), \text{SL}_2(\mathbb{C})]^\text{H}(F_2, C_2) \) is generated by \( X = I^2_\alpha, Y = I^2_\beta, Z = I^2_{\alpha\beta} \) and \( W = I_\alpha I_\beta I_{\alpha\beta} \). Hence

\[
X(F_2) \cong \{(X, Y, Z, W) \in \mathbb{C}^4 \mid W^2 = XYZ\}
\]

(8)

The relationship with Corollary \( \ref{corollary} \) is given by the change of coordinates (cf. Equality \( \ref{equality} \))

\[
\begin{align*}
J_\alpha &= X \\
J_\beta &= Y \\
J_{\alpha\beta} &= Z \\
J_{\alpha\beta^{-1}} &= XY + Z - 2W.
\end{align*}
\]

Remark 4.5 From Equality \( \ref{equality} \) we remark that the singular set of \( X(F_2) \) consists of those points such that two of \( \{X, Y, Z\} \) vanish. This is the same as the set of characters of representations generated by two rotations of angle \( \pi \). This is also the set of Ad-reducible but non-reducible representations.

Example 4.6 If \( M \) is a knot exterior in \( S^3 \), then \( H_2(\pi_1 M) \cong H_2(M) \cong 0 \) and therefore \( X(M) = \overline{X}(M) \). When in addition \( M \) is a 2-bridge knot exterior, explicit methods of how to compute \( X(M) \) are given in [HLM1] and [HLM2], where \( X(M) \) for this particular case was already defined as \( X(M, \text{SL}_2(\mathbb{C}))/C_2 \). The explicit computation for the figure eight knot exterior is found in [GM], for instance.
4.2 Representations that do not lift

Proof of Theorem 1.4. The manifold $M$ is a bundle over $S^1$ with fiber $T^2$, a torus minus a disk. Up to homeomorphism, $M$ is described by the action of the monodromy on $H_1(T^2, \mathbb{Z})$, which is given by the matrix

$$
\begin{pmatrix}
1 & m_2 \\
1 + m_1 m_2 & 1
\end{pmatrix}
$$

with $m_i \in 2\mathbb{Z}$, $m_i > 0$. We shall show that $X(M) - \overline{X}(M)$ has arbitrarily many components by choosing $m_i$ sufficiently large.

To have a presentation of $\pi_1 M$, we use an automorphism $f$ of $\pi_1 T^2$ induced by the monodromy. Since $\pi_1 T^2$ is the free group of rank 2 generated by $\alpha$ and $\beta$,

$$
\pi_1 M = \langle \alpha, \beta, \mu \mid \mu \alpha \mu^{-1} = f(\alpha), \mu \beta \mu^{-1} = f(\beta) \rangle
$$

We choose $f$ such that:

$$
\begin{cases}
\mu \alpha \mu^{-1} = \alpha \beta^{m_2} \\
\mu \beta \mu^{-1} = \beta (\alpha \beta^{m_2})^{m_1}
\end{cases}
$$

We choose odd numbers $p_1, p_2 \in 2\mathbb{Z} + 1$, with $1 \leq p_i \leq m_i/2$ and an arbitrary complex number $z \in \mathbb{C}$. By Example 4.4 there exist matrices $A_z, B_z \in SL_2(\mathbb{C})$ with

$$
\text{tr}(A_z) = 2 \cos \frac{\pi p_1}{m_1}, \quad \text{tr}(B_z) = 2 \cos \frac{\pi p_2}{m_2} \quad \text{and} \quad \text{tr}(A_z B_z) = z.
$$

Those trace equalities imply that $A_z^{m_1} = B_z^{m_2} = -\text{Id}$. In particular

$$
A_z^{m_1} B_z^{m_2} = -A_z, \\
B_z (A_z B_z^{m_2})^{m_1} = -B_z.
$$

Let $\rho_z \in R(\Gamma)$ be the representation that $\rho_z(\alpha) = \pm A_z$, $\rho_z(\beta) = \pm B_z$ and $\rho_z(\mu) = \pm \text{Id}$. Since $m_1$ and $m_2$ are even, this representation does not lift to $SL_2(\mathbb{C})$. In addition, for each value of $p_1$ and $p_2$ we have defined a one parameter family of characters, with parameter $z = \text{tr}(A_z B_z) \in \mathbb{C}$. By [CCGLS, Proposition 2.4] the dimension of each component of $X(M)$ is at most one, hence different values of $p_1$ and $p_2$ give different components.  

\[\square\]

5 The singular set of $X(F_n)$

In this section we compute the singular set of $X(F_n)$, but before we need two preliminary subsections: in Subsection 5.1 we recall some basic facts about the Zariski tangent space and Luna’s slice theorem, and in Subsection 5.2 we compute the cohomology of free groups with twisted coefficients.
5.1 The Zariski tangent space

Given a representation \( \rho \in R(\Gamma) \), we define the space of cocycles

\[
Z^1(\Gamma, \text{Ad} \circ \rho) = \left\{ \theta: \Gamma \to \mathfrak{sl}_2(\mathbb{C}) \mid \theta(\gamma_1 \gamma_2) = \theta(\gamma_1) + \text{Ad}_{\rho(\gamma_1)}(\theta(\gamma_2)), \forall \gamma_1, \gamma_2 \in \Gamma \right\}.
\]

Given a smooth path of representations \( \rho_t \), with \( t \) in a neighborhood of the origin, one can construct a cocycle as follows:

\[
\Gamma \to \mathfrak{sl}_2(\mathbb{C}) \\
\gamma \mapsto \frac{d}{dt}\rho_t(\gamma)\rho_0(\gamma)^{-1}_{|t=0}.
\]

This construction defines an isomorphism, due to Weil [Weil]:

**Theorem 5.1 ([Weil])** The previous construction defines an isomorphism

\[
T^\text{Zar}_\rho(R(\Gamma)) \cong Z^1(\Gamma, \text{Ad} \circ \rho).
\]

Here \( T^\text{Zar}_\rho(R(\Gamma)) \) denotes the Zariski tangent space in the scheme sense (i.e. the defining ideals are not necessary reduced).

We also consider the space of coboundaries

\[
B^1(\Gamma, \text{Ad} \circ \rho) = \left\{ \theta: \Gamma \to \mathbb{R}^2 \mid \text{there exists } a \in \mathfrak{sl}_2(\mathbb{C}) \text{ such that } \theta(\gamma) = \text{Ad}_{\rho(\gamma)}(a) - a, \forall \gamma \in \Gamma \right\}.
\]

The isomorphism of Theorem 5.1 identifies the subspace of the Zariski tangent space corresponding to the orbits by conjugation with \( B^1(\Gamma, \text{Ad} \circ \rho) \).

So it seems natural that in some cases \( T^\text{Zar}_\chi(X(\Gamma)) \) is isomorphic to the cohomology group

\[
H^1(\Gamma, \text{Ad} \circ \rho) = Z^1(\Gamma, \text{Ad} \circ \rho)/B^1(\Gamma, \text{Ad} \circ \rho)
\]

as we will show next.

The stabilizer of a representation \( \rho \in R(\Gamma) \) is denoted by

\[
\text{Stab}_\rho = \{ A \in \text{PSL}_2(\mathbb{C}) \mid A\rho A^{-1} = \rho \}.
\]

In particular, for and Ad-irreducible representation \( \text{Stab}_\rho \) is trivial.

**Proposition 5.2** If \( \rho \) is a smooth point of \( R(\Gamma) \) with closed orbit, then

\[
T^\text{Zar}_\chi(X(\Gamma)) \cong T^\text{Zar}_0(H^1(\Gamma, \text{Ad} \circ \rho)/\text{Stab}_\rho).
\]
Proof. We use the slice theorem of Luna: there exists an algebraic subvariety \( S \subset R(\Gamma) \) that contains \( \rho \) and that is \( \text{Stab}_\rho \)-invariant, such that

\[
Z^1(\Gamma, \text{Ad} \circ \rho) = B^1(\Gamma, \text{Ad} \circ \rho) \oplus T^\text{Zar}_\rho(S)
\]

and the map induced by the projection

\[
S/\text{Stab}_\rho \rightarrow X(\Gamma)
\]
is an \( \acute{e} \)tale isomorphism (in particular their tangent spaces are isomorphic). Since we assume that \( \rho \) is a smooth point, Luna’s theorem shows that \( S/\text{Stab}_\rho \) and \( T^\text{Zar}_\rho(S)/\text{Stab}_\rho \) are \( \acute{e} \)tale equivalent (see [KSS, p. 97]). Since \( T^\text{Zar}_\rho(S) \) and \( H^1(\Gamma, \text{Ad} \circ \rho) \) are isomorphic as \( \text{Stab}_\rho \)-modules (by Equation (9)), the proposition follows. \( \blacksquare \)

5.2 Cohomology of Free groups

We start with irreducible characters:

Lemma 5.3 Let \( \chi_\rho \in X(F_n) \) be an irreducible character. Then

\[
\dim H^1(F_n, \text{Ad} \circ \rho) = 3n - 3.
\]

Proof. Notice first that \( Z^1(F_n, \text{Ad} \circ \rho) \cong \mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^{3n} \). Irreducibility implies that \( \dim B^1(F_n, \text{Ad} \circ \rho) = 3 \), which is maximal (even if Ad-reducible representations have invariant subspaces, irreducibility implies that the eigenvalues are different from 1).

We are interested in computing \( H^1(F_n, \text{Ad} \circ \rho) \) as a \( \text{Stab}_\rho \)-module. If \( \rho \) is Ad-irreducible, then \( \text{Stab}_\rho \) is trivial, and therefore \( H^1(F_n, \text{Ad} \circ \rho) \) is the trivial module \( \mathbb{C}^{3n - 3} \). In the reducible and Ad-reducible cases we need further computations.

Reducible characters. Let \( \chi \in X(F_n) \) be a non trivial reducible character. There exists a representation \( \rho \in R(F_n) \) with character \( \chi \) such that \( \rho \) consists of diagonal matrices, constructed in Remark 3.7.

We decompose the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) = h_0 \oplus h_- \oplus h_+ \), where \( h_0 \), \( h_+ \) and \( h_- \) are the one dimensional \( \mathbb{C} \)-vector spaces generated respectively by \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\).

Lemma 5.4 If \( \rho \) is diagonal then \( \text{Ad} \circ \rho \) preserves the splitting \( \mathfrak{sl}_2(\mathbb{C}) = h_0 \oplus h_- \oplus h_+ \). If in addition \( \rho \) is non-trivial, then \( \text{Stab}_\rho \) preserves the splitting \( \mathfrak{sl}_2(\mathbb{C}) = h_0 \oplus (h_- \oplus h_+ \) (some elements may permute \( h_+ \) and \( h_- \)).
Proof. The first assertion is clear, because diagonal matrices preserve each factor $h_0$ and $h_\pm$.

When the image of $\rho$ has order $\geq 3$, the group $\text{Stab}_\rho$ is precisely the set of diagonal matrices. When the image has order precisely 2, then $\text{Stab}_\rho$ is the group of diagonal and anti-diagonal ones ($0 \, \ast \, 0$). Antidiagonal matrices preserve $h_0$ and permute $h_-$ with $h_+$, hence the second assertion is proved.

\[\blacksquare\]

**Lemma 5.5** Let $\rho \in R(F_n)$ be a non-trivial diagonal representation, then $H^1(F_n, \text{Ad} \circ \rho) \cong h_0^n \oplus (h_+ \oplus h_-)^{n-1}$ as $\text{Stab}_\rho$-modules.

**Proof.** By construction, $Z^1(F_n, \text{Ad} \circ \rho) \cong \mathfrak{sl}_2(\mathbb{C})^n$. We have the splitting

\[H^1(F_n, \text{Ad} \circ \rho) \cong H^1(F_n, h_0) \oplus H^1(F_n, h_+) \oplus H^1(F_n, h_-).\]

A diagonal matrix $\pm \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ acts trivially on $h_0$ and by multiplication by a factor $a^{\pm 2}$ on $h_\pm$. Therefore $B^1(F_n, h_0) \cong 0$ and $B^1(F_n, h_\pm) \cong h_\pm$, and the lemma follows. \[\blacksquare\]

**Ad-reducible but irreducible characters.** Let $\rho \in R(\Gamma)$ be irreducible but Ad-reducible. Up to conjugation the image of $\rho$ is contained in the group of diagonal and anti-diagonal matrices. There are two possibilities for the stabilizer $\text{Stab}_\rho$. If the image of $\rho$ has more than four elements, then $\text{Stab}_\rho$ has two elements: the identity and $\pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Otherwise the image of $\rho$ is Klein’s 4-group (i.e. the group generated by $\pm \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ and $\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$). In this case $\text{Stab}_\rho$ equals the image of $\rho$. With the same argument as in Lemma 5.4, one can prove:

**Lemma 5.6** Let $\rho$ be as above. Then both $\text{Ad} \circ \rho$ and $\text{Stab}_\rho$ preserve the splitting $\mathfrak{sl}_2(\mathbb{C}) = h_0 \oplus (h_+ \oplus h_-)$.

**Lemma 5.7** Let $\rho \in R(F_n)$ be an irreducible but Ad-reducible representation, then $H^1(F_n, \text{Ad} \circ \rho) \cong \mathfrak{sl}_2(\mathbb{C})^{n-1}$ as $\text{Stab}_\rho$-modules.

**Proof.** Again $Z^1(F_n, \text{Ad} \circ \rho) \cong \mathfrak{sl}_2(\mathbb{C})^n$, and we have the decomposition

\[H^1(F_n, \text{Ad} \circ \rho) \cong H^1(F_n, h_0) \oplus H^1(F_n, h_+) \oplus H^1(F_n, h_-).\]

The group $B^1(F_n, h_0)$ has dimension one, because the antidiagonal matrices act on $h_0$ by change of sign. In addition, $\dim(B^1(F_n, h_+ \oplus h_-)) = 2$ is also maximal, because this is the case when we restrict it to diagonal representations (see the proof of Lemma 5.5). \[\blacksquare\]
5.3 Singular locus for free groups

We saw above that $X(F_2, \text{SL}_2(\mathbb{C})) \cong \mathbb{C}^3$ is smooth. We also showed that the singular points of $X(F_2)$ are Ad-reducible but irreducible characters.

**Proposition 5.8** For $n \geq 3$ the singular set of $X(F_n)$ is precisely the set of Ad-reducible characters.

**Proof.** Since $R(F_n) \cong \text{PSL}_2(\mathbb{C})^n$, $X(F_n)$ is irreducible and of dimension $3n - 3$. Thus $\chi \in X(F_n)$ is singular if and only if

$$\dim T^\text{zar}_x X(F_n) > 3n - 3.$$

This dimension is computed by means of Proposition 5.2 if the orbit of $\rho \in t^{-1}(\chi)$ is closed then

$$\dim T^\text{zar}_x X(F_n) = \dim T^\text{zar}_0(H^1(F_n, \text{Ad} \circ \rho) // \text{Stab}_\rho).$$

If $\rho \in R(F_n)$ is irreducible, by Lemma 5.3 $\dim H^1(F_n, \text{Ad} \circ \rho) = 3n - 3$. If in addition $\rho$ is Ad-irreducible, then $\text{Stab}_\rho$ is trivial and therefore $\chi_\rho$ is smooth.

If $\rho$ is irreducible but Ad-reducible, then $H^1(F_n, \text{Ad} \circ \rho) \cong \mathfrak{sl}_2(\mathbb{C})^{n-1}$ as $\text{Stab}_\rho$-modules, by Lemma 5.4. We may assume that the image of $\rho$ has more than 4 elements, because the adherence set of such characters is the whole set of irreducible but Ad-reducible characters, and the singular set is closed. Hence $\text{Stab}_\rho$ is the group generated by the involution $\pm(i 0 0 0)$, that acts trivially on $h_0$ but as a change of sign on $h_+ \oplus h_-$. Thus the action of $\text{Stab}_\rho$ on $H^1(F_n, \text{Ad} \circ \rho)$ is equivalent to the involution on $\mathbb{C}^{3n-3}$ that fixes $(n-1)$ coordinates and changes the sign of the remaining $(2n-2)$ coordinates. The quotient of $\mathbb{C}^{3n-3}$ by this involution is not smooth, hence

$$\dim T^\text{zar}_0(H^1(F_n, \text{Ad} \circ \rho) // \text{Stab}_\rho) > 3n - 3.$$

When $\chi_\rho$ is reducible but non trivial, we may assume that $\rho$ is diagonal and its image has more that three elements (again the adherence set of those characters is the whole set of reducible ones). Thus $\text{Stab}_\rho$ is the group of diagonal matrices, and by Lemma 5.5 $H^1(F_n, \text{Ad} \circ \rho) \cong h_0^n \oplus (h_+ \oplus h_-)^{n-1}$ as $\text{Stab}_\rho$-module. We have an isomorphism $\text{Stab}_\rho \cong \mathbb{C}^*$ and $t \in \mathbb{C}^*$ acts on $h_0$ trivially and on $h_\pm$ by multiplication by $t^{\pm 1}$. An elementary computation shows that $(h_+ \oplus h_-)^{n-1} // \mathbb{C}^*$ has dimension $2n - 3$ and it is not smooth for $n > 2$. \[\square\]

A similar argument yields that for $n \geq 3$ the singular part of $X(F_n, \text{SL}_2(\mathbb{C}))$ is precisely the set of reducible characters.
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