THE GROUPOID $C^*$-ALGEBRA OF A RATIONAL MAP

KLAUS THOMSEN

Abstract. This paper contains a quite detailed description of the $C^*$-algebra arising from the transformation groupoid of a rational map of degree at least two on the Riemann sphere. The algebra is decomposed stepwise via extensions of familiar $C^*$-algebras whose nature depend on the structure of the Julia set and the stable regions in the Fatou set, as well as on the behaviour of the critical points.

1. Introduction and presentation of results

In non-commutative geometry it is a basic principle, referred to as Connes’ dictum in [Kh], that a quotient space should be replaced with a non-commutative algebra, preferably a $C^*$-algebra, in the cases where the topology of the quotient is ill-behaved. Following this dictum the procedure should go via an intermediate step which first produces a groupoid, and the non-commutative algebra should then arise as the convolution algebra of the groupoid. The structure of the resulting non-commutative algebra offers to compensate for the poor topology which the quotient very often is equipped with, and at the same time it encodes the equivalence relation defining the quotient space which is otherwise lost.

A standard example of the construction is the classical crossed product arising from a group acting on a locally compact Hausdorff space which in this picture is a non-commutative substitute for the quotient of the space under the orbit equivalence relation given by the action. There are several other examples of this type of construction arising from dynamical systems. The $C^*$-algebras introduced by D. Ruelle in [Ru] are non-commutative algebras representing the quotient space under the homoclinic equivalence relation arising from a hyperbolic homeomorphism, while the extension of Ruelle’s approach by I. Putnam ([Pu]) also allows to consider the quotient by the heteroclinic equivalence relations of the same type of dynamics. The full orbit equivalence relation arising from a non-invertible continuous self map can also serve as input when the map is locally injective. For local homeomorphisms the construction was developed in stages by J. Renault ([Re]), V. Deaconu ([De]) and C. Anantharaman-Delaroche ([An]), while the extension to locally injective maps was carried out in [Th1]. In all cases, including the work of Ruelle and Putnam, a major problem is to equip the natural groupoid with a sufficiently nice topology which allows the construction of the convolution $C^*$-algebra. The best one can hope for is to turn the groupoid into a locally compact Hausdorff groupoid in such a way that the range map becomes a local homeomorphism. In this case the groupoid is said to be étale. These crucial properties come relatively cheap for the transformation groupoid of Renault, Deaconu and Anantharaman-Delaroche, while it is harder to obtain them for the groupoids in Putnam’s construction ([PS]). In a recent work
(Th2) it was shown that it is possible to formulate the definition of the transformation groupoid of a homeomorphism or a local homeomorphism in such a way that it not only makes sense for a larger class of continuous maps, but also retains the structure of a locally compact second countable Hausdorff groupoid with the important étale property. This class of continuous maps includes the non-constant holomorphic self-maps of a Riemann surface and for such maps it was shown in [Th2] that the convolution $C^*$-algebra of the transformation groupoid obtained in this way is equipped with a one-parameter group of automorphisms for which the KMS-states correspond to the conformal measures introduced in complex dynamics by D. Sullivan. Applied to a particular class of quadratic maps the result was systems for which the KMS states exhibit phase transition with spontaneous symmetry breaking in the sense of Bost and Connes. This illustrates one application of Connes’ dictum, obtaining new models in quantum statistical mechanics. In [Th2] the focus was on the one-parameter action with its KMS states, and the structure of the $C^*$-algebras carrying the action was not investigated. It is the purpose of the present paper to present a relatively detailed description of these $C^*$-algebras $C^*_r(R)$ when they arise from a rational map $R$ of degree at least two acting on the Riemann sphere $\overline{\mathbb{C}}$. It is well-known that the dynamics of such a map is highly complicated, exhibiting features that are both beautiful and fascinating. As we try to show here, the structure of the associated $C^*$-algebra is no less fascinating, although the appreciation of it may require a somewhat more specialized background of the observer than what is needed to admire the colorful pictures used to depict the dynamics of the maps.

The dynamics of a rational map is partitioned by two totally invariant subsets; the Julia set on which the map behaves chaotically under iteration and the Fatou set on which its iterates form an equi-continuous family. As one would expect from familiarity with crossed products this division gives rise to a decomposition of $C^*_r(R)$ as an extension where the Fatou set, as the open subset, gives rise to an ideal $C^*_r(F_R)$ and the Julia set, as the closed subset, represents the corresponding quotient $C^*_r(J_R)$. Thus the first decomposition of $C^*_r(R)$ is given by an extension

$$
0 \longrightarrow C^*_r(F_R) \longrightarrow C^*_r(R) \longrightarrow C^*_r(J_R) \longrightarrow 0
$$

which reflects the partitioning of the $\overline{\mathbb{C}}$ by the Julia and Fatou sets. The two $C^*$-algebras $C^*_r(J_R)$ and $C^*_r(F_R)$ in this extension are of very different nature. The $C^*$-algebra $C^*_r(J_R)$ of the Julia set is always purely infinite, nuclear and satisfies the universal coefficient theorem of Rosenberg and Schochet ([RS]), and it is often, but not always simple. Ideals in $C^*_r(J_R)$ arise from the possible presence of finite subsets of the Julia set invariant under the equivalence relation represented by the transformation groupoid which produces the $C^*$-algebra $C^*_r(R)$; we call this relation 'restricted orbit equivalence' and it is a relation which is slightly stronger than orbit equivalence. The finite subsets of the Julia set invariant under restricted orbit equivalence comprise the finite subsets considered by Makarov and Smirnov in their work on phase transition in the thermodynamic formalism associated to the dynamics ([MS1], [MS2]), and they are closely related to, but not identical with the subsets introduced in [GPRR] in connection with work on exceptional rational maps. The possible presence of such subsets of the Julia set implies that in general...
the structure of $C^*_r(J_R)$ must be decoded from an extension of the form

\[
0 \longrightarrow C^*_r(J_R \setminus E_R) \longrightarrow C^*_r(J_R) \longrightarrow B \longrightarrow 0 \quad (1.1)
\]

where $C^*_r(J_R \setminus E_R)$ is purely infinite and simple, while $B$ is a finite direct sum of algebras of the form $M_n(\mathbb{C})$ for some $n \leq 3$ or $C(\mathbb{T}) \otimes M_n(\mathbb{C})$ for some $n \leq 4$.

In contrast to $C^*_r(J_R)$ the $C^*$-algebra $C^*_r(F_R)$ of the Fatou set is finite, and its ideal structure is typically much more complex than that of $C^*_r(J_R)$. This is partly due to the fact that the Fatou set is partitioned into classes of connected components, the so-called stable regions which are termed super-attracting, attracting, parabolic, Siegel and Herman regions according to the asymptotic behaviour of their elements under iteration. This division of $F_R$ results in a direct sum decomposition of the ideal $C^*_r(F_R)$ where each direct summand is further decomposed as an extension where the structure of the ideal depends on the type of the stable region and where the nature of the quotient is governed by the presence or absence of critical and periodic points in the region. Specifically,

\[
C^*_r(F_R) = \bigoplus_{i=1}^{N} C^*_r(\Omega_i),
\]

where $C^*_r(\Omega_i)$ is the $C^*$-algebra obtained by restricting attention to the stable region $\Omega_i \subseteq F_R$. It turns out that the nature of the algebra $C^*_r(\Omega_i)$ varies with the type of the stable region:

When $\Omega_i$ is super-attractive there is an extension

\[
0 \longrightarrow K \otimes MT_d \longrightarrow C^*_r(\Omega_i) \longrightarrow B \longrightarrow 0
\]

where $B$ is a finite direct sum of algebras stably isomorphic to either $\mathbb{C}$ or the continuous functions on the Cantor set. The algebra $MT_d$ is the mapping torus of an endomorphism on a Bunce-Deddens algebra of type $d^\infty$ where $d$ is the product of the valencies of the elements in the critical orbit.

When $\Omega_i$ is attractive there is a an extension

\[
0 \longrightarrow K \otimes C(\mathbb{T}^2) \longrightarrow C^*_r(\Omega_i) \longrightarrow B \longrightarrow 0
\]

where $B$ is a finite direct sum of algebras stably isomorphic to either $\mathbb{C}$ or the continuous functions on the circle $\mathbb{T}$.

When $\Omega_i$ is parabolic there is a an extension

\[
0 \longrightarrow K \otimes C(\mathbb{T}) \otimes C_0(\mathbb{R}) \longrightarrow C^*_r(\Omega_i) \longrightarrow B \longrightarrow 0
\]

where $B$ is a finite direct sum of algebras stably isomorphic to $\mathbb{C}$.

When $\Omega_i$ is of Siegel type there is a an extension

\[
0 \longrightarrow K \otimes C_0(\mathbb{R}) \otimes A_\theta \longrightarrow C^*_r(\Omega_i) \longrightarrow B \longrightarrow 0 \quad (1.2)
\]

where $B$ is a finite direct sum of algebras stably isomorphic to either $\mathbb{C}$ or the continuous functions on the circle $\mathbb{T}$, and $A_\theta$ is the irrational rotation algebra corresponding to the angle of rotation in the Siegel domain inside $\Omega_i$.

Finally, when $\Omega_i$ is of Herman type there is a an extension quite similar to (1.2). The only difference is that while the quotient algebra $B$ must contain a summand stably isomorphic to $C(\mathbb{T})$ in the Siegel case, in the Herman case all summands are stably isomorphic to $\mathbb{C}$.

It almost goes without saying that the entire structure in the decomposition of $C^*_r(R)$ described above reflects easily identified structures of the dynamics of $R$. For
example, the difference between the structure of the summands in $C^*_r(F_R)$ coming from a Siegel and Herman region is due to the periodic point in a Siegel domain which is absent in a Herman ring.

From the point of view of operator algebra theory a study of a non-simple $C^*$-algebra often begins with a description of the primitive ideals and the corresponding irreducible quotients. In [CT] T. Carlsen and the author identified the primitive and maximal ideals of the $C^*$-algebras arising from the transformation groupoid of a locally injective surjection on a finite dimensional compact metric space. In the final section of the present paper the method from [CT] is carried over to the groupoid $C^*$-algebras of rational maps and we obtain in this way a description of the primitive ideals and primitive quotients. In particular, it is shown that the primitive ideal space of $C^*_r(R)$ is only Hausdorff in the hull-kernel topology when it has to be, i.e. when $C^*_r(R)$ is simple. This occurs only when $J_R = \mathbb{C}$ and there are no finite sets invariant under restricted orbit equivalence. In all other cases the primitive ideal space is not even $T_0$.

While there is often a rich variety of primitive quotients, there are always very few types of simple quotients. The finite invariant subsets under restricted orbit equivalence give rise to maximal ideals, but the corresponding simple quotients are matrix algebras of size no more than 4. In most cases $C^*_r(J_R)$ is also a simple quotient, but only when there are no finite subsets of $J_R$ invariant under restricted orbit equivalence. There are no other simple quotients. In particular, when there are finite subsets of $J_R$ invariant under restricted orbit equivalence the only simple quotients of $C^*_r(R)$ are matrix algebras of size not exceeding 4.

There are other ways to associate a $C^*$-algebra to a rational map, and we refer to [DM] and [KW] for these. It would be interesting to find the precise relationship between the algebras investigated here and those of Kajiwara and Watatani. Presently it is only clear that they are generally very different.

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2. Étale groupoids and $C^*$-algebras from dynamical systems

Let $G$ be an étale second countable locally compact Hausdorff groupoid with unit space $G^{(0)}$. Let $r : G \to G^{(0)}$ and $s : G \to G^{(0)}$ be the range and source maps, respectively. For $x \in G^{(0)}$ put $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$ and $I_s_x = s^{-1}(x) \cap r^{-1}(x)$. Note that $I_s_x$ is a group, the isotropy group at $x$. The space $C_c(G)$ of continuous compactly supported functions is a $*$-algebra when the product is defined by

$$(f_1f_2)(g) = \sum_{h \in G^r(s)} f_1(h)f_2(h^{-1}g)$$

and the involution by $f^*(g) = \overline{f(g^{-1})}$. Let $x \in G^{(0)}$. There is a representation $\pi_x$ of $C_c(G)$ on the Hilbert space $l^2(G_x)$ of square-summable functions on $G_x$ given by

$$\pi_x(f)\psi(g) = \sum_{h \in G^r(s)} f(h)\psi(h^{-1}g).$$
The reduced groupoid $C^*$-algebra $G^*_r(G)$ is the completion of $C_c(G)$ with respect to the norm
\[ \|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\|. \]

2.0.1. Stability of $C^*_r(G)$. We shall need the following sufficient condition for stability of $C^*_r(G)$. Recall that a $C^*$-algebra $A$ is stable when $A \otimes K \simeq A$ where $K$ denotes the $C^*$-algebra of compact operators on an infinite dimensional separable Hilbert space. Recall also that a bi-section in $G$ is an open subset $U \subseteq G$ such that $r : U \to G^{(0)}$ and $s : U \to G^{(0)}$ both are injective.

**Lemma 2.1.** Let $G$ be a locally compact second countable étale groupoid. Assume that for every compact subset $K \subseteq G^{(0)}$ there is a finite collection \( \{U_i\}_{i=1}^N \) of bi-sections in $G$ such that
\begin{enumerate} [i)]  \item $K \subseteq \bigcup_{i=1}^N s(U_i)$,  \item $r(U_i) \cap r(U_j) = \emptyset$ when $i \neq j$, and  \item $K \cap \bigcup_{i=1}^N r(U_i) = \emptyset$. \end{enumerate}
It follows that $C^*_r(G)$ is a stable $C^*$-algebra.

**Proof.** By Theorem 2.1 and b) of Proposition 2.1 in [HR] it suffices to consider a positive element $a \in C^*_r(G)$ and an $\epsilon > 0$ and show that there is an element $v \in C^*_r(G)$ such that $\|v^*v - a\| \leq \epsilon$ and $v^2 = 0$. Write $a = a_0^\varphi a_0$ for some $a_0 \in C^*_r(G)$. By approximating $a_0$ with an element $h \in C_c(G)$ and taking $b = h^*h$ we obtain an element $b \in C_c(G)$ which is positive in $C^*_r(G)$ and satisfies that $\|a - b\| \leq \epsilon$. Let $\text{supp} b$ be the support of $b$ in $G$ and set $K = r(\text{supp} b)$. By assumption there is a finite collection \( \{U_i\}_{i=1}^N \) of bi-sections such that i)-iii) hold. Let \( \{h_i\} \subseteq C_c(G^{(0)}) \) be a partition of unity on $K$ subordinate to \( \{s(U_i)\}_{i=1}^N \). For each $i$ let $f_i \in C_c(G)$ be supported in $U_i$ and satisfy that $f_i = 1$ on $s^{-1}(\text{supp} h_i) \cap U_i$. Set $w = \sum_{i=1}^N f_i \sqrt{h_i} \in C_c(G)$ and note that $w^*wb = \sum_i h_i b = b$ while $bw = 0$. Set $v = w\sqrt{b}$. \qed

2.0.2. $G$-orbits and reductions. When $W$ is a subset of $G^{(0)}$ we set
\[ G_W = \{ g \in G : r(g), s(g) \in W \}, \]
which is a subgroupoid of $G$, called a reduction of $G$. When $W$ is an open subset of $G$ the reduction $G_W$ will be an étale groupoid in the relative topology inherited from $G$ and there is an embedding $C^*_r(G_W) \subseteq C^*_r(G)$, cf. e.g. Proposition 1.9 in [Ph]. In fact, $C^*_r(G_W)$ is a hereditary $C^*$-subalgebra of $C^*_r(G)$. When $x \in G^{(0)}$, the set $Gx = \{ r(g) : g \in G_x \}$ will be called the $G$-orbit of $x$. We say that $W \subseteq G^{(0)}$ is $G$-invariant when $x \in W \Rightarrow Gx \subseteq W$. When $W$ is $G$-invariant and locally compact in the topology inherited from $G^{(0)}$, the reduction $G_W$ is an étale locally compact groupoid in the topology inherited from $G$. If $W$ is both open and $G$-invariant $C^*_r(G_W)$ is an ideal in $C^*_r(G)$. Similarly, when $F$ is a closed subset of $G^{(0)}$ which is also $G$-invariant, $C^*_r(G_F)$ is a quotient of $C^*_r(G)$. It is known that under a suitable amenability condition the kernel of the quotient map
\[ \pi_F : C^*_r(G) \to C^*_r(G_F) \]
is $C^*_r(G_{G^{(0)} \setminus F})$. We shall avoid the amenability issue here and prove this equality directly in the cases we are interested in. See Section $3$.

We shall need the following fact which follows straightforwardly from the definitions.
Lemma 2.2. Assume that there is a finite partition $G^{(0)} = \sqcup_{i=1}^{n} W_i$ such that each $W_i$ is open and $G$-invariant. It follows that

$$C^*_r(G) \simeq \oplus_{i=1}^{n} C^*_r(G_{W_i}).$$

The following result was obtained by Muhly, Renault and Williams in [MRW].

Theorem 2.3. Let $G$ be an étale second countable locally compact Hausdorff groupoid and $W \subseteq G^{(0)}$ an open subset such that $Gx \cap W \neq \emptyset$ for all $x \in G^{(0)}$. It follows that $C^*_r(G)$ is stably isomorphic to $C^*_r(G_W)$.

Proof. The set $\bigcup_{x \in W} G_x$ is a $(G, G_W)$-equivalence in the sense of [MRW] and hence Theorem 2.8 of [MRW] applies. \hfill $\square$

2.0.3. Pure infiniteness of $C^*_r(G)$. Following [An] we say that an étale groupoid $G$ is essentially free when the points $x$ of the unit space $G^{(0)}$ for which the isotropy group $\text{Is}_x$ is trivial (i.e. only consists of $\{x\}$) is dense in $G^{(0)}$. In the same vein we say that $G$ is locally contracting when every open non-empty subset of $G^{(0)}$ contains an open subset $V$ with the property that there is an open bisection $S$ in $G$ such that $\overline{V} \subseteq s(S)$ and $\alpha_S^{-1}(\overline{V}) \subseteq \overline{V}$, when $\alpha_S : r(S) \to s(S)$ is the homeomorphism defined such that $\alpha_S(x) = s(g)$ where $g \in S$ is the unique element with $r(g) = x$, cf. Definition 2.1 of [An] (but note that the source map is denoted by $d$ in [An]).

We say that a $C^*$-algebra is purely infinite when every non-zero hereditary $C^*$-subalgebra of $A$ contains an infinite projection. Proposition 2.4 of [An] then says the following.

Theorem 2.4. Let $G$ be an étale second countable locally compact Hausdorff groupoid. Assume that $G$ is essentially free and locally contracting. Then $C^*_r(G)$ is purely infinite.

2.1. The transformation groupoid of a local homeomorphism. In this section we describe the construction of an étale second countable locally compact Hausdorff groupoid from a local homeomorphism of a locally compact Hausdorff space which was introduced in increasing generality by J. Renault [Re], V. Deaconu [De] and Anantharaman-Delaroche [An].

Let $X$ be a second countable locally compact Hausdorff space and $\varphi : X \to X$ a local homeomorphism. Thus we assume that $\varphi$ is open and locally injective, but not necessarily surjective. Set

$$G_{\varphi} = \{(x, k, y) \in X \times \mathbb{Z} \times X : \exists n, m \in \mathbb{N}, k = n - m, \varphi^n(x) = \varphi^m(y)\}.$$\nonumber

This is a groupoid with the set of composable pairs being

$$G_{\varphi}^{(2)} = \{((x, k, y), (x', k', y')) \in G_{\varphi} \times G_{\varphi} : y = x'\}.$$\nonumber

The multiplication and inversion are given by

$$(x, k, y)(y', k', y') = (x, k + k', y') \text{ and } (x, k, y)^{-1} = (y, -k, x).$$

Note that the unit space of $G_{\varphi}$ can be identified with $X$ via the map $x \mapsto (x, 0, x)$. Under this identification the range map $r : G_{\varphi} \to X$ is the projection $r(x, k, y) = x$ and the source map the projection $s(x, k, y) = y$.

To turn $G_{\varphi}$ into a locally compact topological groupoid, fix $k \in \mathbb{Z}$. For each $n \in \mathbb{N}$ such that $n + k \geq 0$, set

$$G_{\varphi}(k, n) = \{(x, l, y) \in X \times \mathbb{Z} \times X : l = k, \varphi^{k+n}(x) = \varphi^n(y)\}.$$
This is a closed subset of the topological product \( X \times \mathbb{Z} \times X \) and hence a locally compact Hausdorff space in the relative topology. Since \( \varphi \) is locally injective \( G_\varphi(k, n) \) is an open subset of \( G_\varphi(k, n + 1) \) and hence the union

\[
G_\varphi(k) = \bigcup_{n \geq -k} G_\varphi(k, n)
\]

is a locally compact Hausdorff space in the inductive limit topology. The disjoint union

\[
G_\varphi = \bigcup_{k \in \mathbb{Z}} G_\varphi(k)
\]

is then a locally compact Hausdorff space in the topology where each \( G_\varphi(k) \) is an open and closed set. In fact, as is easily verified, \( G_\varphi \) is an étale groupoid, i.e. the range and source maps are local homeomorphisms. The groupoid \( G_\varphi \) will be called the transformation groupoid of \( \varphi \). To simplify notation we denote in the following the corresponding \( C^* \)-algebra by \( C^*_r(\varphi) \); i.e. we set

\[
C^*_r(G_\varphi) = C^*_r(\varphi).
\]

Note that the \( G_\varphi \)-orbit \( G_\varphi x \) of a point \( x \in X \) is the orbit of \( x \) under \( \varphi \), i.e.

\[
G_\varphi x = \bigcup_{n, m \in \mathbb{N}} \varphi^{-n}(\varphi^m(x)) ;
\]

by some authors called the full or grand orbit to distinguish it from \( \{\varphi^n(x) : n \in \mathbb{N}\} \), which we will call the forward orbit.

When the local homeomorphism \( \varphi \) is proper, in the sense that inverse images of compact sets are compact, the \( C^* \)-algebra \( C^*_r(\varphi) \) can be realised as the crossed product by an endomorphism, cf. \cite{De, An}, in the following way: The subset

\[
R_\varphi = G_\varphi(0) \simeq \{(x, y) \in X \times X : \varphi^n(x) = \varphi^n(y) \text{ for some } n \in \mathbb{N}\}
\]

is an open subgroupoid of \( G_\varphi \) and an étale groupoid in itself. The reduced groupoid \( C^* \)-algebra \( C^*_r(R_\varphi) \) is a \( C^* \)-subalgebra of \( C^*_r(\varphi) \). When \( \varphi \) is proper there is an endomorphism \( \widehat{\varphi} : C^*_r(R_\varphi) \to C^*_r(R_\varphi) \) defined such that

\[
\widehat{\varphi}(f)(x, y) = (\#\varphi^{-1}(\varphi(x))\#\varphi^{-1}(\varphi(y)))^{-1/2} f(\varphi(x), \varphi(y))
\]

when \( f \in C^*_c(R_\varphi) \). We will refer to this endomorphism as the Deaconu endomorphism. As shown in \cite{An} there is an isomorphism

\[
C^*_r(\varphi) = C^*_r(R_\varphi) \rtimes_{\varphi} \mathbb{N},
\]

where the crossed product is a crossed product by an endomorphism both in the sense of Paschke \cite{P} and the sense of Stacey \cite{S}.

2.2. The transformation groupoid of a rational map. In this section we describe an étale groupoid coming from a non-constant holomorphic map on a Riemann surface by a construction introduced in \cite{Th2}. But since we shall focus on the Riemann sphere in this paper we restrict the presentation accordingly.

Let \( \overline{\mathbb{C}} \) be the Riemann sphere and \( R : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) a rational map of degree at least 2. Consider a subset \( X \subseteq \overline{\mathbb{C}} \) which is locally compact in the topology inherited from \( \overline{\mathbb{C}} \), without isolated points and \emph{totally \( R \)-invariant} in the sense that \( R^{-1}(X) = X \).

Let \( P \) be the pseudo-group on \( X \) of local homeomorphisms \( \xi : U \to V \) between open subsets of \( X \) with the property that there are open subsets \( U_1, V_1 \) in \( \overline{\mathbb{C}} \) and a bi-holomorphic map \( \xi_1 : U_1 \to V_1 \) such that \( U_1 \cap X = U, \ V_1 \cap X = V \) and \( \xi = \xi_1 \)
on $U$. For each $k \in \mathbb{Z}$ we denote by $\mathcal{T}_k(X)$ the elements $\eta : U \to V$ of $\mathcal{P}$ with the property that there are natural numbers $n, m$ such that $k = n - m$ and

$$R^n(z) = R^m(\eta(z)) \quad \forall z \in U. \quad (2.1)$$

The elements of $\mathcal{T} = \bigcup_{k \in \mathbb{Z}} \mathcal{T}_k(X)$ will be called local transfers for $R|_X$. We denote by $[\eta]_x$ the germ at a point $x \in X$ of an element $\eta \in \mathcal{T}$. Set

$$G_X = \{(x, k, \eta, y) \in X \times \mathbb{Z} \times \mathcal{P} \times X : \eta \in \mathcal{T}_k(X), \eta(x) = y\}.$$

We define an equivalence relation $\sim$ in $G_X$ such that $(x, k, \eta, y) \sim (x', k', \eta', y')$ when $x = x'$, $y = y'$, $k = k'$ and $[\eta]_x = [\eta']_x$. Let $[x, k, \eta, y]$ denote the equivalence class of $(x, k, \eta, y) \in G_X$. The quotient space $G_X = G_X/\sim$ is a groupoid where two elements $[x, k, \eta, y]$ and $[x', k', \eta', y']$ are composable when $y = x'$ and their product is

$$[x, k, \eta, y] \circ [x', k', \eta', y'] = [x, k + k', \eta' \circ \eta, y'] .$$

The inversion in $G_X$ is defined such that $[x, k, \eta, y]^{-1} = [y, -k, \eta^{-1}, x]$. The unit space of $G_X$ can be identified with $X$ via the map $x \mapsto [x, 0, \text{id}, x]$, where $\text{id}$ is the identity map on $X$. When $\eta \in \mathcal{T}_k(X)$ and $U$ is an open subset of the domain of $\eta$ we set

$$U(\eta) = \{[z, k, \eta, \eta(z)] : z \in U\} . \quad (2.2)$$

It is straightforward to verify that by varying $k$, $\eta$ and $U$, the sets (2.2) constitute a base for a topology on $G_X$. A crucial point is that the topology is also Hausdorff since the local transfers are holomorphic. It follows that $G_X$ is an étale second countable locally compact Hausdorff groupoid, cf. [Th2].

It is the possible presence of critical points of $R$ in $X$ which prevents us from using the procedure of Section 2.1 to get an étale groupoid. The additional feature, the local transfers, which is introduced to obtain a well-behaved étale groupoid is therefore not necessary when there are no critical points in $X$, and it is therefore reassuring that the two constructions coincide in the absence of critical points in $X$.

**Proposition 2.5.** Assume that $Y \subseteq X$ is an open subset of $X$ which does not contain any critical point of $R$ and is $R$-invariant in the sense that $R(Y) \subseteq Y$. Then the reduction $(G_X)_Y$ is isomorphic, as a topological groupoid, to the transformation groupoid of the local homeomorphism $R|_Y : Y \to Y$.

**Proof.** Define $\mu : (G_X)_Y \to G_{R|_Y}$ such that $\mu[x, k, \eta, y] = (x, k, y)$.

$\mu$ is surjective: If $(x, k, y) \in G_{R|_Y}$ there are $n, m \in \mathbb{N}$ such that $k = n - m$ and $R^n(x) = R^m(y)$. Note that $(R^n)'(x) \neq 0$ and $(R^m)'(y) \neq 0$ since $Y$ does not contain critical points. It follows that there are open neighbourhoods $U$ and $V$ of $x$ and $y$ in $\mathbb{C}$, respectively, such that $R^n : U \to R^n(U)$ and $R^m : V \to R^m(V)$ are univalent holomorphic maps. Let $R^{-m} : V \to R^{-m}(V)$ be the inverse of $R^m : V \to R^m(V)$. Then $\eta_0 = R^{-m} \circ R^n : V \to R^{-m}(R^m(V)) \cap V$ is bi-holomorphic. Set $\eta = \eta_0|_X$ and note that $[x, k, \eta, y] \in (G_X)_Y$.

$\mu$ is injective: Assume that $\mu[x, k, \eta, y] = \mu[x', k', \eta', y']$. Then $(x, k, y) = (x', k', y')$. Since $(R^n)'(y) \neq 0$ it follows that $R^n$ is injective in a neighborhood of $y$. Since $R^n(\eta(z)) = R^n(z) = R^n(\eta'(z))$ for all $z$ close to $x$ it follows that $\eta = \eta'$ in a neighborhood of $x$, i.e. $[x, k, \eta, y] = [x', k', \eta', y']$.

$\mu$ is continuous: Let $\mu[x, k, \eta, y] = (x, k, y)$ and consider an open neighborhood $\Omega$ of $(x, k, y)$ in $G_{R|_Y}$. There is then an $m \in \mathbb{N}$ and a local transfer $\eta \in \mathcal{T}_k(X)$ such that $k + m \geq 0$, $\eta(x) = y$ and $R^{k+m}(z) = R^m(\eta(z))$ for all $z$ in a neighborhood of $x$. 
By definition of the topology of $G_{R|_Y}$, the set $\Omega \cap G_{R|_Y}(k, m)$ is open in $G_{R|_Y}(k, m)$ when this set has the relative topology inherited from $X \times Z \times X$. From this fact it follows that there is an open neighborhood $W$ of $x$ in the domain of $\eta$ such that $(z, k, \eta(z)) \in \Omega \cap G_{R|_Y}(k, m)$ for all $z \in W$. Since $Y$ is open in $X$ we can shrink $W$ to ensure that $W \subseteq Y$ and $\eta(W) \subseteq Y$. Then $\{[z, k, \eta, \eta(z)] : z \in W\}$ is an open neighborhood of $[x, k, \eta, \eta]$ in $(G_X)_Y$ such that $\mu([z, k, \eta, \eta(z)] : z \in W) \subseteq \Omega$.

$\mu$ is open: Let $\eta \in T_k(X)$ be a local transfer. Let $U$ be an open subset of the domain of $\eta$ such that $U \subseteq Y$ and $\eta(U) \subseteq Y$. It suffices to show that 

$$\eta(\{[z, k, \eta, \eta(z)] : z \in U\})$$

is open in $G_{R|_Y}$. To this end note that there is an $m \in \mathbb{N}$ such that $k + m \geq 0$ and $R^{k+m}(z) = R^m(\eta(z))$ for all $z \in U$. Consider a point $(z_1, k, z_2) \in \eta(\{[z, k, \eta, \eta(z)] : z \in U\})$. Then $(z_1, k, z_2) \in G_{R|_Y}(k, m)$. Let $W$ be an open neighborhood of $z_2$ in $X$ such that $W \subseteq \eta(U)$ and $R^m$ is injective on $W$. If $(z'_1, k, z'_2) \in (\eta^{-1}(W) \times \{k\} \times W) \cap G_{R|_Y}(k, m)$ we have that $R^m(z'_1) = R^{k+m}(z'_2) = R^m(\eta(z'_1))$ which implies that $z'_2 = \eta(z'_1)$. This shows that 

$$(\eta^{-1}(W) \times \{k\} \times W) \cap G_{R|_Y}(k, m) \subseteq \eta(\{[z, k, \eta, \eta(z)] : z \in U\})$$.

\[\square\]

When we take the set $X$ in the construction of $G_X$ to be the whole Riemann sphere $\mathbb{C}$ we obtain the groupoid $G_\mathbb{C}$ and the corresponding $C^*$-algebra $C^*_r(G_\mathbb{C})$. There is no good reason to emphasize $\mathbb{C}$ so we will denote the groupoid $G_\mathbb{C}$ by $G_R$ instead and $C^*_r(G_\mathbb{C})$ by $C^*_r(R)$. We call $G_R$ the transformation groupoid of $R$. It follows from Lemma 4.1 in [Th2] that two elements $x, y \in \mathbb{C}$ are in the same $G_R$-orbit if and only if there are natural numbers $n, m \in \mathbb{N}$ such that $R^n(x) = R^m(y)$ and $\text{val}(R^n, x) = \text{val}(R^m, y)$, where $\text{val}(R^n, x)$ is the valency of $R^n$ at the point $x$. In particular, the $G_R$-orbit of $x$, which we will denote by $\text{RO}(x)$ in the following, is a subset of the orbit of $x$ conditioned by an equality of valencies. Specifically,

$$\text{RO}(x) = \{y \in \mathbb{C} : R^n(y) = R^m(x) \text{ and } \text{val}(R^n, y) = \text{val}(R^m, x) \text{ for some } n, m \in \mathbb{N}\}.$$ 

We call $\text{RO}(x)$ the restricted orbit of $x$. In the following a subset $Y$ of $\mathbb{C}$ will be called restricted orbit invariant, or $\text{RO}$-invariant, when $y \in Y \Rightarrow \text{RO}(y) \subseteq Y$. Note that a totally $R$-invariant subset is $\text{RO}$-invariant, but the reverse can fail when there are isolated points in $Y$.

We shall work with many different reductions of $G_R$. Recall that for any subset $Y \subseteq \mathbb{C}$ the reduction of $G_R$ to $Y$ is the set 

$$G_Y = \{[x, k, \eta, y] \in G_R : x, y \in Y\}.$$ 

This is always a subgroupoid of $G_R$. If $X$ is both locally compact in the topology inherited from $\mathbb{C}$ and $\text{RO}$-invariant, $G_X$ will be a locally compact Hausdorff étale groupoid in the topology inherited from $G_R$. When $X$ is also totally $R$-invariant and has no isolated points the reduction is equal to the groupoid $G_X$ described above, showing that our notation is consistent. Furthermore, when $X$ is open in the relative topology of a subset $Y \subseteq \mathbb{C}$ which is $\text{RO}$-invariant and locally compact in the topology inherited from $\mathbb{C}$, the reduction $G_X$ is again a locally compact Hausdorff étale groupoid in the topology inherited from $G_R$. The reduction $G_X$ of $G_R$ by such
a set is the most general type of reduction we shall need in this paper. To simplify notation we set
\[ C_r^*(X) = C_r^*(G_X). \]

3. The Julia-Fatou extension

Let \( X \) be an RO-invariant subset of \( \overline{C} \) which is locally compact in the relative topology, or at least an open subset of such a set. The \( C^* \)-algebra \( C_r^*(X) \) carries a natural action \( \beta \) by the circle group \( \mathbb{T} \) defined such that
\[ \beta(f)(x, k, \eta, y) = \mu^k [x, k, \eta, y] \]
when \( f \in C_c(G_X) \) and \( \mu \in \mathbb{T} \). To identify the fixed point algebra of \( \beta \) set
\[ G_0^X = \{ [x, k, \eta, y] \in G_X : k = 0 \}. \]

Since \( G_0^X \) is an open subgroupoid of \( G_X \) there is an inclusion \( C_c(G_0^X) \subseteq C_c(G_X) \) which extends to an embedding \( C_r^*(G_0^X) \subseteq C_r^*(X) \) of \( C^* \)-algebras.

**Lemma 3.1.** \( C_r^*(G_0^X) \) is the fixed point algebra \( C_r^*(X)^\beta \) of the gauge action.

**Proof.** Let \( a \in C_r^*(X)^\beta \). For any \( \epsilon > 0 \) there is a function \( f \in C_c(G_X) \) such that \( \|a - f\| \leq \epsilon \). We can then write \( f \) as a finite sum \( f = \sum_{k \in \mathbb{Z}} f_k \) where \( f_k \) is supported in
\[ \{ [x, l, \eta, y] \in G_X : l = k \}. \]

Since \( f_k \) is \( \beta \)-invariant and \( \|a - f\| \leq \epsilon \), we deduce that \( \|a - f_k\| \leq \epsilon \). This shows that \( C_r^*(X)^\beta \subseteq C_r^*(G_0^X) \), and the reversed inclusion is trivial. □

Just as for local homeomorphisms, [An], [De], there is an inductive limit decomposition of \( C_r^*(G_0^X) \) which throws some light on its structure. Let \( n \in \mathbb{N} \). Set
\[ G_0^X(n) = \{ [x, 0, \eta, y] \in G_X : R^n(z) = R^n(\eta(z)) \} \]
in a neighbourhood of \( x \).

Each \( G_0^X(n) \) is an open subgroupoid of \( G_0^X \), \( G_0^X(n) \subseteq G_0^X(n + 1) \) for all \( n \) and
\[ G_0^X = \bigcup_n G_0^X(n). \]

It follows that \( C_r^*(G_0^X(n)) \subseteq C_r^*(G_0^X(n + 1)) \subseteq C_r^*(G_0^X) \) for all \( n \), and
\[ C_r^*(G_0^X) = \bigcup_n C_r^*(G_0^X(n)). \] (3.2)

Assume now that \( X \) is an RO-invariant subset of \( \overline{C} \) which is locally compact in the relative topology, and not just an open subset of such a set. Let \( Y \) be a closed RO-invariant subset of \( X \). Then \( X \setminus Y \) is open in \( X \) and RO-invariant. Since \( Y \) and \( X \setminus Y \) are locally compact in the topology inherited from \( \overline{C} \) we can consider the reduced groupoid \( C^* \)-algebras \( C_r^*(Y) \) and \( C_r^*(X \setminus Y) \). Furthermore, we have a surjective *-homomorphism
\[ \pi_Y : C_r^*(X) \rightarrow C_r^*(Y) \]
because \( Y \) is closed and RO-invariant in \( X \).

**Lemma 3.2.** Let \( X \) be a RO-invariant subset of \( \overline{C} \) which is locally compact in the relative topology. Let \( Y \) be a closed RO-invariant subset of \( X \). The sequence
\[ 0 \rightarrow C_r^*(X \setminus Y) \rightarrow C_r^*(X) \xrightarrow{\pi_Y} C_r^*(Y) \rightarrow 0 \]
is exact.
Proof. It is clear that \( C^*_r(X \setminus Y) \subseteq \ker \pi_Y \). To establish the reverse inclusion, let \( a \in \ker \pi_Y \) and let \( \epsilon > 0 \). Note first that the formula (3.1) also defines an action by \( \mathbb{T} \) on \( C^*_r(Y) \) which makes \( \pi_Y \) equivariant. It follows that \( \ker \pi_Y \) is left globally invariant by the gauge action, and there is therefore an approximate unit in \( \ker \pi_Y \) consisting of elements fixed by the gauge action. It follows then from Lemma 3.1 that there is an element \( u \in C^*_r(G^0_X) \cap \ker \pi_Y \) such that \( \|ua - a\| \leq \epsilon \). It follows from (3.2) that

\[
C^*_r(G^0_X) \cap \ker \pi_Y = \bigcup_n C^*_r(G^0_X(n)) \cap \ker \pi_Y,
\]

and there is therefore an \( n \in \mathbb{N} \) and an element \( v \in C^*_r(G^0_X(n)) \cap \ker \pi_Y \) such that \( \|ua - va\| \leq \epsilon \). Note now that \( \#R^{-n}(x) \leq (\deg R)^n \) for all \( x \in \mathbb{C} \) when \( \deg R \) is the degree of \( R \). By definition of the norm on \( C^*_r(G^0_X(n)) \) this gives us the estimate

\[
\|f\| \leq (\deg R)^n \sup_{\gamma \in G^0_X(n)} |f(\gamma)|
\]

for all \( f \in C_c(G^0_X(n)) \). Set \( \delta = \epsilon((2(\deg R)^n + 1)(\|a\| + 1))^{-1} \) and choose \( g \in C_c(G^0_X(n)) \) such that \( \|g - v\| \leq \delta \). Then \( \|\pi_Y(g)\| \leq \delta \) and hence

\[
\sup_{\gamma \in G^0_X(n)s^{-1}(Y)} |g(\gamma)| \leq \delta,
\]

by Proposition II 4.2 of [Re]. We can therefore write \( g = g_1 + g_2 \) where \( g_1 \in C_c(G^0_X(n)) \) has support in \( s^{-1}(X \setminus Y) \) and \( \sup_{\gamma \in G^0_X(n)} |g_2(\gamma)| \leq 2\delta \). It follows then from (3.3) that \( \|g_1a - va\| \leq \delta \|a\| + \|g_2a\| \leq \epsilon \), and hence that \( \|g_1a - a\| \leq 3\epsilon \). Since \( g_1a \in C^*_r(X \setminus Y) \) it follows that \( a \in C^*_r(X \setminus Y) \).

Since the Fatou set \( F_R \) and the Julia set \( J_R \) are totally \( R \)-invariant and hence also RO-invariant we get the following.

**Corollary 3.3.** The sequence

\[
0 \longrightarrow C^*_r(F_R) \longrightarrow C^*_r(R) \xrightarrow{\pi J_R} C^*_r(J_R) \longrightarrow 0.
\]

is exact.

4. THE STRUCTURE OF \( C^*_r(J_R) \)

4.1. Pure infiniteness of \( C^*_r(J_R) \).

**Proposition 4.1.** \( G_{J_R} \) is essentially free and locally contractive, and \( C^*_r(J_R) \) is purely infinite.

For the proof of Proposition 4.1 we need a couple of lemmas.

**Lemma 4.2.** Assume that \( (R^n)'(x) \neq 0 \). Then \( R^n(x) \in RO(x) \).

**Proof.** When \( x \) is not critical for \( R^n \) there is an open neighbourhood \( U \) of \( x \) such that \( R^n : U \to R^n(U) \) is a local transfer and \( [x, n, R^n|_U, R^n(x)] \in G_R \). \( \square \)

In the following proof and in the rest of the paper \( \text{Crit} \) will denote the set of critical points of \( R \).
Lemma 4.3. Let $Y$ be a closed RO-invariant subset of $\mathbb{C}$. If $Y$ does not contain an isolated point which is periodic or critical, it follows that the elements of $Y$ that are neither pre-periodic nor pre-critical are dense in $Y$.

Proof. Assume that there is a non-empty open subset $U \subseteq Y$ consisting entirely of pre-critical and pre-periodic point. Let $\text{Per}_n R$ denote the set of $n$-periodic points of $R$. Since

$$ U \subseteq \bigcup_{n,j} R^{-j} (\text{Per}_n R \cup \text{Crit}) $$

by assumption it follows from the Baire category theorem that there are $n, j \in \mathbb{N}$ and a non-empty open subset $W$ of $Y$ such that

$$ W \subseteq U \cap R^{-j} (\text{Per}_n R \cup \text{Crit}). $$

Since $R$ is neither periodic nor constant it follows that $R^{-j} (\text{Per}_n R \cup \text{Crit})$ is finite. Hence $W$ must contain a point $z_0$ which is isolated in $Y$. If $z_0 \notin \text{Crit}$ we conclude from Lemma 4.2 that $R(z_0) \in Y$ since $Y$ is RO-invariant. By repeating this argument we either reach an $l < j$ such that $R^l(z_0) \in \text{Crit} \cap Y$ or conclude that $R^j(z_0) \in Y$. In the first case $R^l$ is a local transfer in an open neighbourhood of $z_0$ and in the second $R^j$ is. Hence $R^j(z_0)$ is isolated in $Y$ in the first case, and $R^j(z_0)$ in the second. In any case we conclude that $Y$ contains an isolated point which is either critical or periodic. This contradicts our assumption on $Y$. \hfill \Box

We can then give the proof of Proposition 4.1.

Proof. By Proposition 4.4 of Th2 the elements of $J_R$ with trivial isotropy group in $G_{J_R}$ are the points that are neither pre-periodic nor pre-critical. It is well-known that $J_R$ is closed, totally $R$-invariant and without isolated points. It follows therefore from Lemma 4.3 that $G_{J_R}$ is essentially free.

To prove that $G_{J_R}$ is also locally contracting we use that the repelling periodic points are dense in $J_R$ by ii) of Theorem 14.1 in Mi. The argument is then essentially the same used in the proof of Lemma 4.2 in Th3: Let $U \subseteq J_R$ be an open non-empty set. There is then a repelling periodic point $z_0 \in U \cap \mathbb{C}$, and there is an $n \in \mathbb{N}$, a positive number $\kappa > 1$ and an open neighbourhood $W \subseteq U \cap \mathbb{C}$ of $z_0$ such that $R^n(z_0) = z_0$, $R^n$ is injective on $W$ and

$$ |R^n(y) - z_0| \geq \kappa |y - z_0| $$

for all $y \in W$. Let $\delta_0 > 0$ be so small that

$$ \{y \in \mathbb{C} : |y - z_0| \leq \delta_0\} \subseteq R^n(W) \cap W. $$

Since $z_0$ is not isolated in $J_R$ there is an element $z_1 \in J_R \cap \mathbb{C}$ such that $0 < |z_1 - z_0| < \delta_0$. Choose $\delta$ strictly between $|z_1 - z_0|$ and $\delta_0$ such that

$$ \kappa |z_1 - z_0| > \delta. $$

(4.3)

Set $V = \{y \in J_R \cap \mathbb{C} : |y - z_0| < \delta\}$. Then

$$ V \subseteq R^n(V). $$

Indeed, if $|y - z_0| \leq \delta$ (4.2) implies that there is a $y' \in W$ such that $R^n(y') = y$ and then (4.1) implies that $|y' - z_0| < \delta$. Since $y' \in J_R$ because $R^{-1}(J_R) = J_R$, it...
follows that $\overline{V} \subseteq R^n(V)$. On the other hand, it follows from (4.3) and (4.1) that $R^n(z_1) \notin \overline{V}$. This shows that (4.4) holds. Then

$$S = \{[z,n,R^n|_V,R^n(z)] \in G_{J_R} : z \in V\}$$

is an open bisection in $G_{J_R}$ such that $\overline{V} \subseteq s(S)$ and

$$\alpha_{S^{-1}}(\overline{V}) \subseteq V,$$

where $\alpha_{S^{-1}} : s(S) \rightarrow r(S)$ is the homeomorphism defined by $S$, cf. Section 2.0.3. This shows that $G_{J_R}$ is locally contracting and it follows then from Theorem 2.4 that $C^*_r(J_R)$ is purely infinite. \[\square\]

4.2. Exposed points and finite quotients of $C^*_r(J_R)$.

**Lemma 4.4.** Let $X$ be a totally $R$-invariant set which is locally compact in the topology inherited from $\overline{X}$. Let $Y \subseteq X$ be a closed RO-invariant subset of $X$ and let $Y_0$ be the subset of $Y$ obtained by deleting the isolated points of $Y$. Then $Y_0$ is totally $R$-invariant, i.e $R^{-1}(Y_0) = Y_0$.

**Proof.** Let $y \in Y_0$. Let $n, m \in \mathbb{N}$ and consider a point $x \in X$ a such that $R^n(x) = R^m(y)$. We must show that $x \in Y_0$. Since $R^{-n}(R^n(x))$ is a finite set there is an open neighborhood $U_0$ of $x$ such that

$$R^{-n}(R^n(y)) \cap \overline{U_0} = \{x\}.$$  \hspace{1cm} (4.5)

Because $R^n$ and $R^m$ are both open maps we can also arrange that there is an open neighborhood $V_0$ of $y$ such that $R^n(U_0) = R^m(V_0)$. Set $U = U_0 \cap X$ and $V = V_0 \cap X$ and note that $R^n(U) = R^m(V)$. Since $\bigcup_{j=0}^n R^{-j}(\text{Crit})$ and $R^{-m}\left(\bigcup_{j=0}^n R^j(\text{Crit})\right)$ are both finite sets and $y$ is not isolated in $Y$, there is a sequence $\{y_k\}$ of mutually distinct elements in

$$Y \cap V \setminus \left(R^{-m}\left(\bigcup_{j=0}^n R^j(\text{Crit})\right) \cup \bigcup_{j=0}^m R^{-j}(\text{Crit})\right)$$

converging to $y$. Since $\#R^{-m}(z) \leq (\deg R)^m$ for all $z$ we can prune the sequence $\{y_k\}$ to arrange that $k \neq l \Rightarrow R^m(y_k) \neq R^m(y_l)$. Let $\{x_k\} \subseteq U$ be points such that $R^n(x_k) = R^m(y_k)$ for all $k$. Then $x_k \in U \setminus \bigcup_{j=0}^n R^{-j}(\text{Crit})$ for all $k$. Passing to a subsequence we can arrange that $\{x_k\}$ converges in $X$, necessarily to $x$ because of (4.3). Note that $\text{val}(R^m,y_k) = \text{val}(R^n,x_k) = 1$ for all $k$ since $x_k \notin \bigcup_{j=0}^n R^{-j}(\text{Crit})$ and $y_k \notin \bigcup_{j=0}^m R^j(\text{Crit})$. It follows therefore, either from Proposition 4.1 in [Th2] or from Lemma 4.2 above, that $x_k \in \text{RO}(y_k)$. Hence $x_k \in Y$ because $y_k \in Y$ and $Y$ is RO-invariant. It follows that $x \in Y$ because $Y$ is closed. Furthermore, since the $x_k$’s are distinct, $x \in Y_0$. This shows that $Y_0$ is totally $R$-invariant. \[\square\]

**Lemma 4.5.** Let $L$ be a closed RO-invariant subset of $J_R$. Then $L$ is either finite or equal to $J_R$.

**Proof.** It follows from Lemma 4.4 that we can write $L = L_0 \cup L_1$ where $L_0$ is closed and totally $R$-invariant while $L_1$ is discrete. If $L_0 \neq \emptyset$ it follows from Corollary 4.13 of [Mi] that $L_0 = J_R = L$. If $L_0 = \emptyset$ the compactness of $L$ implies it is finite. \[\square\]
4.2.1. Exposed points. The preceding lemma forces us to look for points in $J_R$, or more generally in $\overline{\mathbb{C}}$, whose restricted orbits are finite. In the following we say that a point $x \in \overline{\mathbb{C}}$ is exposed when the restricted orbit $RO(x)$ of $x$ is finite. A non-empty subset $A \subseteq \overline{\mathbb{C}}$ is exposed when it is finite and RO-invariant. Clearly the so-called exceptional points, those with finite orbit, are exposed. There are at most two of them and they are always elements of the Fatou set. See e.g. §4.1 in [B]. In contrast exposed points can occur in the Julia set as well.

**Lemma 4.6.** Let $A \subseteq \overline{\mathbb{C}}$ be a finite subset with the property that
\[
R^{-1}(A) \setminus \mathrm{Crit} \subseteq A. 
\]  
Then $A$ contains at most 4 elements, and at most 3 if it contains a critical point.

**Proof.** The proof is a repetition of the proof of Lemma 1 in [GPRR]. Set $d = \deg R$ and let $\text{val}(R, x)$ denote the valency of $R$ at $x$. Then
\[
d(A) = \sum_{x \in R^{-1}(A)} \text{val}(R, x) = \sum_{x \in R^{-1}(y)} \text{val}(R, x) = d 
\]

\[
= \#R^{-1}(A) + \sum_{x \in R^{-1}(A)} (\text{val}(R, x) - 1) 
\]

\[
\leq \#A + \# \mathrm{Crit} + \sum_{x \in R^{-1}(A)} (\text{val}(R, x) - 1) 
\]

\[
= \#A + \# \mathrm{Crit} + 2d - 2 
\]

\[
\leq \#A + 4(d - 1) 
\]

It follows that $\#A \leq 4$. When $A$ contains a critical point the first inequality above is strict and hence $\#A \leq 3$.

It follows from Lemma 4.2 that an exposed subset satisfies (4.6). This gives us the following.

**Corollary 4.7.** An exposed subset does not contain more than 4 elements. If it contains a critical point it contains at most 3 elements.

The upper bound on the number of elements in an exposed set can be improved when $R$ is a polynomial since the number of critical points for a polynomial is at most $\deg R$, and $\infty$ is always one of them. Specifically, applied to a polynomial the proof of Lemma 4.6 yields the following.

**Lemma 4.8.** Assume that $R$ is a polynomial of degree at least 2 and $A$ an exposed subset of $\mathbb{C}$. Then $\#A \leq 2$, and $\#A \leq 1$ when $A$ contains a critical point.

We say that an exposed subset is of type 1 when it does not contain a critical point, of type 2 when it contains a pre-periodic critical point and type 3 when it contains a critical point, but no pre-periodic critical point.

**Lemma 4.9.** Let $A$ be a finite subset of $\overline{\mathbb{C}}$. Then $A$ is an exposed subset of type 1 if and only if
\[
R^{-1}(A) \setminus \mathrm{Crit} = A. 
\]  

**Proof.** Assume first that $A$ is an exposed subset of type 1. Since $A$ does not contain a critical point it follows from Lemma 4.2 that $R(A) \subseteq A$, i.e. $A \subseteq R^{-1}(A) \setminus \mathrm{Crit}$. Combined with (4.6) this shows that (4.7) holds.
Conversely, it follows from (4.7) that $R(A) \subseteq A$ and that $A \cap \text{Crit} = \emptyset$. Hence \( \text{val}(R^n, x) = 1 \) for all \( x \in A \) and all \( n \in \mathbb{N} \). Thus if \( x \in A \) and \( y \in RO(x) \) we have both that \( R^m(y) = R^n(x) \in A \) for some \( n, m \), and that \( \text{val}(R^m, y) = \text{val}(R^n, x) = 1 \). But then \( R^j(y) \notin \text{Crit} \) for all \( 0 \leq j \leq m - 1 \), and repeated use of (4.7) then shows that \( y \in A \), i.e. \( A \) is RO-invariant.

Finite subsets of the Julia set satisfying (4.7) were considered by Makarov and Smirnov in [MS1] and their work can be used to find examples of polynomials with exposed subsets of type 1 in the Julia set. In [MS2] a rational map with an exposed subset of type 1 was called exceptional. This notion was extended in [GPRR] where a rational map was called exceptional when the Julia set contains a finite forward invariant subset satisfying (4.6). There are exceptional rational maps, in the sense of [GPRR], without any exposed subsets, and conversely, there are non-exceptional rational maps with exposed subsets in the Julia set. Thus although there is of course a relationship between exposed subsets and the subsets used to define the exceptional maps in [GPRR], the two notions are not the same. Note that it follows from Corollary 4.7 that the total number of exposed points is at most 4.

4.2.2. Finite quotients of \( C_r^* (J_R) \). Let \( E_R \) denote the union of the exposed subsets in \( \mathbb{C} \); a set with at most 4 elements. When \( E_R \cap J_R \neq \emptyset \) the purely infinite \( C^* \)-algebra \( C_r^* (J_R) \) will have \( C_r^* (E_R \cap J_R) \) as a quotient. The corresponding ideal, however, is simple.

**Proposition 4.10.** \( C_r^* (J_R \setminus E_R) \) is simple.

**Proof.** It follows from Lemma 4.3 that there are many points in \( J_R \setminus E_R \) whose isotropy group in \( G_{J_R \setminus E_R} \) is trivial. By Corollary 2.18 of [Th1] it suffices therefore to show that there are no non-trivial (relatively) closed RO-invariant subset in \( J_R \setminus E_R \). Assume that \( L \) is a non-empty RO-invariant subset of \( J_R \setminus E_R \) which is closed in \( J_R \setminus E_R \). It follows first that \( L \cup (E_R \cap J_R) \) is closed and RO-invariant in \( J_R \) and then from Lemma 4.5 that \( L = J_R \setminus E_R \), which is the desired conclusion. \( \square \)

In order to obtain a description of the quotient

\[
C_r^* (J_R \cap E_R) \simeq C_r^* (J_R) / \bigcap C_r^* (J_R \setminus E_R)
\]

we consider a more general situation because the result can then be used when we examine \( C_r^* (F_R) \).

**Lemma 4.11.** Let \( x \in \mathbb{C} \) and assume that the restricted orbit \( RO(x) \) of \( x \) is discrete in \( \mathbb{C} \). There is an isomorphism

\[
C_r^* (RO(x)) \simeq C^* (\text{Is}_x) \otimes \mathbb{K} (\ell^2 (RO(x)))
\]

where \( \mathbb{K} (\ell^2 (RO(x))) \) denotes the \( C^* \)-algebra of compact operators on \( \ell^2 (RO(x)) \) and \( \text{Is}_x \) the isotropy group \( \text{Is}_x = \{ g \in G_R : s(g) = r(g) = x \} \).

**Proof.** Note that the reduction \( G_{RO(x)} \) is discrete in \( G_R \) and that \( C_r^* (RO(x)) \) is generated by \( 1_z \), \( z \in G_{RO(x)} \), when \( 1_z \) denotes the characteristic function of the set \( \{ z \} \). For every \( y \in RO(x) \) choose an element \( \eta_y \in r^{-1}(y) \cap s^{-1}(x) \). For every \( g \in \text{Is}_x \) set

\[
u_g = \sum_{y \in RO(x)} 1_{\eta_y g \eta_y^{-1}}.
\]
The sum converges in the strict topology and defines a unitary in the multiplier algebra of $C^*_r(RO(x))$. Note that $u_g u_h = u_{gh}$ so that $u$ is a unitary representation of Is$_x$ as multipliers of $C^*_r(RO(x))$. The elements $1_{\eta_n^{-1}}$, $u, v \in RO(x)$, constitute a system of matrix units generating a copy of $\mathbb{K}(l^2(RO(x)))$ inside $C^*_r(RO(x))$. Since each $u_g$ commutes with $1_{\eta_n^{-1}}$ for all $u, v \in RO(x)$ we get a $*$-homomorphism $C^*_r(\text{Is}_x) \otimes \mathbb{K}(l^2(RO(x))) \to C^*_r(RO(x))$ sending $1_g \otimes 1_{\eta_n^{-1}}$ to $u_g 1_{\eta_n^{-1}}$. It is easy to see that this is an isomorphism. □

**Lemma 4.12.** Let $A$ be a finite RO-orbit in $J_R$. Set $n = \#A$.

a) Assume that $A$ is of type 1. Then $n \leq 4$ and

$$C^*_r(A) \simeq C(\mathbb{T}) \otimes M_n(\mathbb{C}).$$

b) Assume that $A$ is of type 2. Then $n \leq 3$ and

$$C^*_r(A) \simeq M_n(\mathbb{C}) \otimes C(\mathbb{T}) \otimes \mathbb{C}^d$$

where $d = \lim_{k \to \infty} \text{val}(R^k, x)$ for any critical point $x \in A$.

c) Assume that $A$ is of type 3. Then $n \leq 3$ and

$$C^*_r(A) \simeq M_n(\mathbb{C}) \otimes \mathbb{C}^d$$

where $d = \lim_{k \to \infty} \text{val}(R^k, x)$ for any critical point $x \in A$.

**Proof.** a): It follows from Corollary 4.7 that $n \leq 4$. It follows from Lemma 4.9 that $A = RO(x)$ for some point $x \in J_R$ which is periodic and whose forward orbit is contained in $A$. Since $A$ contains no critical points Proposition 4.4 b) in [Th2] tells us that Is$_x \simeq \mathbb{Z}$. Then the conclusion follows from Lemma 4.11.

b) When $A$ contains a critical point the last assertion in Corollary 4.7 says that $\#A \leq 3$. When $x$ is a pre-periodic critical point in $A$ we can determine the isotropy group Is$_x$ from Proposition 4.4 in [Th2]. Since $J_R$ does not contain any periodic critical orbit we are in situation d2) of that proposition and obtain therefore the stated conclusion from Lemma 4.11.

c) follows in the same way as b). The only difference is that Is$_x$ is determined by use of c) in Proposition 4.4 of [Th2]. □

Given a point $z \in \overline{\mathbb{C}}$ we call the limit $\lim_{k \to \infty} \text{val}(R^k, z)$ occurring in Lemma 4.12 the asymptotic valency of $z$. It can be infinite, but only when $z$ is pre-periodic to a critical periodic orbit.

**Theorem 4.13.** There is an extension

$$0 \longrightarrow C^*_r(J_R \setminus E_R) \longrightarrow C^*_r(J_R) \longrightarrow \oplus_{A} C^*_r(A) \longrightarrow 0, \tag{4.8}$$

where the direct sum $\oplus_{A}$ is over the (possibly empty) collection of finite RO-orbits $A$ in $J_R$. Furthermore, $C^*_r(J_R \setminus E_R)$ is separable, purely infinite, nuclear, simple and satisfies the universal coefficient theorem of Rosenberg and Schochet, [RS]. If non-zero, the quotient $\oplus_{A} C^*_r(A)$ is isomorphic to a finite direct sum of matrix algebras $M_n(\mathbb{C})$ with $n \leq 3$ and circle algebras $C(\mathbb{T}) \otimes M_n(\mathbb{C})$ with $n \leq 4$.

**Proof.** The extension (4.8) is a special case of the extension from Lemma 3.2. The direct sum decomposition of the quotient follows from Lemma 2.2 and the description of it from Lemma 4.12. The pure infiniteness of the ideal follows from Proposition 4.10 because pure infiniteness is inherited by ideals. It is simple by Proposition 4.10.
That $C^*_r(J_R \setminus E_R)$ is nuclear and satisfies the UCT will be shown in Section 4.3 below by making a connection to the work of Katsura. [Ka] □

In view of Theorem 4.13 it seems appropriate to point out that the Julia set can contain exposed orbits of all three types. For an example of type 1 observe that for the Chebyshev polynomial

$$R(z) = z^2 - 2$$

the set $E_R \cap J_R$ consists of the points $\{-2, 2\}$ which is a finite RO-orbit of type 1 in the Julia set $\{-2, 2\}$. Hence

$$\oplus \mathbb{A} C^*_r(A) \simeq C(T) \otimes M_2(\mathbb{C}).$$

No other polynomial in the quadratic family $z^2 + c$ has an exposed point in the Julia set.

For an example of a finite RO-orbit of type 2 consider the rational map

$$R(z) = \left(1 - \frac{2}{z}\right)^2$$

The Julia set is the entire sphere in this case, cf. §11.9 in [B], and the maximal exposed subset consists of the points $\{0, \infty, 1\}$ which is the union of the finite RO-orbit $\{1, \infty\}$ of type 1 and the finite RO-orbit $\{0\}$ which is of type 2. The asymptotic valency $\lim_{n \to \infty} \text{val}(R^n, 0)$ is 2 and hence

$$\oplus \mathbb{A} C^*_r(A) \simeq C(T) \oplus C(T) \oplus (C(T) \otimes M_2(\mathbb{C})).$$

To give examples of finite RO-orbits of type 3 in the Julia set we use the work of M. Rees. She shows in Theorem 2 of [R] that for 'many' $\lambda \in \mathbb{C} \setminus \{0\}$ the rational map

$$R(z) = \lambda \left(1 - \frac{2}{z}\right)^2$$

has a dense critical forward orbit. In particular, the Julia set $J_R$ is the entire sphere. The critical points are 0 and 2, and $R^{-1}(0) = \{2\}$. Since the forward orbit of 0 is dense it follows that $\{0\}$ is a finite RO-orbit of type 2. There are no other exposed points, i.e. $E_R = \{0\}$. Hence

$$\oplus \mathbb{A} C^*_r(A) \simeq \mathbb{C} \oplus \mathbb{C}$$

in this case because the asymptotic valency of 0 is 2.

4.3. Amenability and the UCT. Set

$$J'_R = J_R \setminus \left(E_R \cup \bigcup_{j=0}^{3} R^{-j}(\text{Crit})\right)$$

and consider

$$\Gamma = \{(x, k, \eta, y) \in G_{J'R} : k = 1\}$$

which is an open subset of $G_{J'R}$. Let $X_\Gamma$ be the closure of $C_\Gamma(\Gamma)$ in $C^*_r(J'_R)$. Since $X_\Gamma X_\Gamma \subseteq C^*_r\left(G_{J'R}^0\right)$ we can consider $X_\Gamma$ as a Hilbert $C^*_r\left(G_{J'R}^0\right)$-module with the 'inner product' $\langle a, b \rangle = a^*b$. Furthermore, since

$$C^*_r\left(G_{J'R}^0\right) X_\Gamma \subseteq X_\Gamma$$

we can consider any $a \in C^*_r\left(G_{J'R}^0\right)$ as an adjointable operator $\varphi(a)$ on $X_\Gamma$. Then the pair $(\varphi, X_\Gamma)$ is a $C^*$-correspondence in the sense of Katsura, [Ka], and we aim to show that the $C^*$-algebra $O_{X_\Gamma}$ introduced in [Ka] is a copy of $C^*_r(J'_R)$. 
Lemma 4.14. \( \varphi \) is injective.

Proof. Assume that \( \varphi(a) = 0 \). To show that \( a = 0 \) consider the continuous function \( j(a) \) on \( G_{J_R^0}^0 \), defined by \( a \), cf. Proposition 4.2 on page 99 in [Re]. Consider an element \( \gamma \in G_{J_R^0}^0 \) such that \( s(\gamma) \notin R^{-4}(\text{Crit}) \). It follows from Lemma 1.2 that there is a function \( f \in C_c(\Gamma) \) supported in \([z, 1, R|U, R(z)] : z \in U \) for some open neighborhood \( U \) of \( s(\gamma) \) such that \( ff^* \in C_c(J_R^0) \) and \( f\bar{f}^* = 1 \). Since \( af\bar{f}^* = 0 \) it follows that \( j(a)(\gamma) = 0 \), i.e. \( j(a) = 0 \) on the set

\[
\{ \gamma \in G_{J_R^0}^0 : s(\gamma) \notin R^{-4}(\text{Crit}) \}.
\]

Since this set is dense in \( G_{J_R^0}^0 \) it follows first that \( j(a) = 0 \), and then that \( a = 0 \) because \( j \) is injective. \( \square \)

Note that we can consider the inclusions \( C_r^* \left( G_{J_R^0}^0 \right) \subseteq C_r^* (J_R^0) \) and \( X_\Gamma \subseteq C_r^* (J_R^0) \) as an injective \(*\)-homomorphism \( \psi_t : \mathcal{K}(X_\Gamma) \to C_r^* (J_R^0) \) such that

\[
\psi_t (\Theta_{ab}) = ab^*,
\]

cf. Definition 2.3 and Lemma 2.4 in [Ka]. Note that the range of \( \psi_t \) is the ideal

\[
X_\Gamma X_\Gamma^* = \text{Span} \{ab^* : a, b \in X_\Gamma\}
\]

in \( C_r^* \left( G_{J_R^0}^0 \right) \). We are here and in the following lemma borrowing notation from [Ka].

Lemma 4.15. The ideal \( \{a \in C_r^* \left( G_{J_R^0}^0 \right) : \varphi(a) \in \mathcal{K}(X_\Gamma) \} \) is equal to \( X_\Gamma X_\Gamma^* \), and \( a = \psi_t(\varphi(a)) \) for all \( a \in J_{X_\Gamma} \).

Proof. The inclusion

\[
X_\Gamma X_\Gamma^* \subseteq \left\{ a \in C_r^* \left( G_{J_R^0}^0 \right) : \varphi(a) \in \mathcal{K}(X_\Gamma) \right\}
\]

is trivial so we focus on the inverse. Let therefore \( a \in C_r^* \left( G_{J_R^0}^0 \right) \) be an element such that \( \varphi(a) \in \mathcal{K}(X_\Gamma) \). There is then a sequence

\[
\sum_{n=1}^{N_n} \Theta_{a^n_i, b^n_i}, \quad n = 1, 2, 3, \ldots,
\]

where \( a^n_i, b^n_i \in X_\Gamma \) for all \( i, n \), which converges to \( \varphi(a) \) in \( \mathcal{K}(X_\Gamma) \). In particular,

\[
af = \lim_{n \to \infty} \sum_{i=1}^{N_n} \Theta_{a^n_i, b^n_i} f = \lim_{n \to \infty} \sum_{i=1}^{N_n} a^n_i b^n_i^* f \quad (4.9)
\]

for all \( f \in X_\Gamma \). By continuity of \( \psi_t \) it follows that the sequence \( \psi_t \left( \sum_{i=1}^{N_n} \Theta_{a^n_i, b^n_i} \right) = \sum_{i=1}^{N_n} a^n_i b^n_i^* \) converges in \( X_\Gamma X_\Gamma^* \) to \( \psi_t(\varphi(a)) \). It follows from (4.9) that \( af = \psi_t(\varphi(a))f \) for all \( f \in X_\Gamma \) and as in the proof of Lemma 4.14 we deduce first that \( j(a - \psi_t(\varphi(a))) = 0 \) and then that \( a = \psi_t(\varphi(a)) \). \( \square \)

It follows from Lemma 4.15 that the representation of the \( C^* \)-correspondence \( X_\Gamma \) given by the inclusions \( C_r^* \left( G_{J_R^0}^0 \right) \subseteq C_r^* (J_R^0) \) and \( X_\Gamma \subseteq C_r^* (J_R^0) \) is covariant in the sense of Katsura, [Ka]. Combined with the presence of the gauge action on \( C_r^* \left( G_{J_R^0}^0 \right) \)
this allows us now to use Theorem 6.4 from [Ka] to conclude that the $C^*$-algebra $O_{X_\Gamma}$ defined from the $C^*$-correspondence $X_\Gamma$ is isomorphic to the $C^*$-subalgebra of $C^*_r(J'_R)$ generated by $C^*_r\left(G_{J'_R}^0\right)$ and $X_{\Gamma}$. It remains to show that this is all of $C^*_r(J'_R)$.

**Lemma 4.16.** For all $x \in J'_R$ there is an element $\gamma \in \Gamma$ such that $s(\gamma) = x$.

**Proof.** For any $z \in J_R \setminus E_R$ and $n \geq 1$, set

$$D_n(z) = \{ y \in R^{-n}(z) : y, R(y), \ldots, R^{n-1}(y) \notin \text{Crit} \}.$$ Let $x \in J'_R$. It follows from Lemma 4.2 that $[y, 1, \eta, x] \in \Gamma$ for some local transfer $\eta$. Assume therefore that $R^{-1}(x) \subseteq \text{Crit}$. Then $R(x) \neq x$ since otherwise $\text{RO}(x) = \{x\}$, contradicting that $x \notin E_R$. If $D_2(R(x)) \neq \emptyset$, choose an element $y \in D_2(R(x))$ and note that $[y, 1, \eta, x] \in \Gamma$ for some local transfer $\eta$. Assume therefore now that $D_2(R(x)) = R^{-1}(x) \setminus \text{Crit} = \emptyset$. Then $R^2(x) \notin \{x, R(x)\}$, since otherwise $\text{RO}(x) \subseteq \{x, R(x)\} \cup D_1(R(x))$, contradicting that $x \notin E_R$. If $D_3(R^2(x)) \neq \emptyset$, choose $y \in D_3(R^2(x))$ and note that $[y, 1, \eta, x] \in \Gamma$ for some local transfer $\eta$. Assume therefore that $D_3(R^2(x)) = R^2(x) = R^{-1}(x) \setminus \text{Crit} = \emptyset$. Then $R^3(x) \notin \{x, R(x), R^2(x)\}$, since otherwise $\text{RO}(x) \subseteq \{x, R(x), R^2(x)\} \cup D_1(R(x)) \cup D_1(R^2(x)) \cup D_2(R^2(x))$, contradicting that $x \notin E_R$. We claim that $D_4(R^3(x))$ cannot be empty. Indeed, if it is empty we have either $R^4(x) \in \{x, R(x), R^2(x), R^3(x)\}$, and then

$$\text{RO}(x) \subseteq \{x, R(x), R^2(x), R^3(x)\} \cup D_1(R(x)) \cup D_1(R^2(x)) \cup D_2(R^2(x)) \cup D_3(R^3(x)),$$

which is impossible since $x \in E_R$, or that $R^4(x) \notin \{x, R(x), R^2(x), R^3(x)\}$ in which case

$$A = \{x, R(x), R^2(x), R^3(x)\} \cup D_1(R(x)) \cup D_1(R^2(x)) \cup D_2(R^2(x)) \cup D_1(R^3(x)) \cup D_2(R^3(x)) \cup D_3(R^3(x)),$$

is a set with more than 4 elements for which (4.6) holds, contradicting Lemma 4.6. Thus $D_4(R^3(x))$ is not empty and we choose $y \in D_4(R^3(x))$ and note that $[y, 1, \eta, x] \in \Gamma$ for some local transfer $\eta$. 

**Proposition 4.17.** $O_{X_\Gamma} \simeq C^*_r(J'_R)$, and $C^*_r(J_R \setminus E_R)$ is stably isomorphic to $O_{X_\Gamma}$.

**Proof.** By the remarks preceding Lemma 4.16 to establish the isomorphism $O_{X_\Gamma} \simeq C^*_r(J'_R)$ it suffices to show that $C^*_r(J'_R)$ is generated by $C^*_r\left(G_{J'_R}^0\right)$ and $X_{\Gamma}$. Let $k \geq 0$. Consider a bisection in $G_{J'_R}$ of the form

$$S = \{ [z, k, \eta, \eta(z) ] : z \in U \}$$

for some local transfer $\eta$ and an open subset $U \subseteq J'_R$ in the domain of $\eta$ such that $\eta(U) \subseteq J'_R$. By varying $k$, $\eta$ and $U$ functions $f \in C_c\left(G_{J'_R}^0\right)$ with support in a set of the form $S$ generate $C^*_r(J'_R)$ as a $C^*$-algebra. It suffices therefore to show that such functions are elements of the $\ast$-algebra generated by $C^*_r\left(G_{J'_R}^0\right)$ and $X_{\Gamma}$. We will prove this by induction in $k$, starting with the observation that the assertion is trivial when $k = 0$ and $k = 1$. Assume that the assertion is established for $k - 1$ and consider $f \in C_c\left(G_{J'_R}^0\right)$ supported in $S$. Let $x \in U$. It follows from Lemma 4.16
that we can find $\gamma \in \Gamma$ such that $s(\gamma) = \eta(x)$. This means that there is a bisection in $\Gamma$ of the form $\{ [z, 1, \xi, \xi(z)] : z \in V \}$ such that $\xi(y) = \eta(x)$ for some $y \in V$. Then

$$[z, k, \eta, \eta(z)] = \left[ z, k - 1, \xi^{-1} \circ \eta, \xi^{-1}(\eta(z)) \right] \left[ \xi^{-1}(\eta(z)) \right], 1, \xi, \eta(z)$$

for all $z$ in a neighborhood $U_x \subseteq U$ of $x$. It follows that there are functions $h, g \in C_c(G_{J_R^0})$ supported in

$$\{ [z, k - 1, \xi^{-1} \circ \eta, \xi^{-1}(\eta(z))] : z \in U_x \}$$

and

$$\{ [\xi^{-1}(\eta(z)), 1, \xi, \eta(z)] : z \in U_x \},$$

respectively, such that $f = hg$ in a neighborhood of $[x, k, \eta, \eta(x)]$. Note that $h$ is then in the $*$-algebra generated by $C_r^* \left( G_{J_R^0}^0 \right)$ and $X_\Gamma$ by induction hypothesis and that $g \in X_\Gamma$. We choose then functions $\psi_i \in C_c(J_R^0)$ forming a finite partition of unity $\{ \psi_i \}$ on $r(\text{supp } f)$ such that

$$f = \sum_i \psi_i h_i g_i$$

where $g_i \in X_\Gamma$ and $h_i$ is in the $*$-algebra generated by $C_r^* \left( G_{J_R^0}^0 \right)$ and $X_\Gamma$. Since $\psi_i \in C_r^* \left( G_{J_R^0}^0 \right)$ for each $i$, this completes the induction step and hence the proof of the isomorphism $O_{X_r} \simeq C_r^*(J_R^0)$.

Since $G_{J_R^0}$ is the reduction to the open subset $J_R'$ of $J_R \setminus E_R$ the algebra $C_r^*(J_R')$ is a hereditary $C^*$-subalgebra of $C_r^*(J_R \setminus E_R)$. The latter algebra is simple by Proposition 4.10 and it follows therefore from [Br] that the two algebras are stably isomorphic.

\[ \square \]

**Corollary 4.18.** $C_r^*(J_R \setminus E_R)$ is a nuclear $C^*$-algebra which satisfies the universal coefficient theorem of Rosenberg and Schochet, [RS].

**Proof.** Thanks to Proposition 4.17 and Lemma 4.15 the assertions follow from Corollary 7.4 and Proposition 8.8 in [Ka] provided we show that both $C_r^* \left( G_{J_R}^0 \right)$ and the ideal $X_\Gamma X_\Gamma^*$ are nuclear and satisfy the UCT. To this end we use the inductive limit decomposition

$$C_r^* \left( G_{J_R}^0 \right) = \bigcup_n C_r^* \left( G_{J_R}^0(n) \right), \quad (4.10)$$

cf. (3.2). We claim that $C_r^* \left( G_{J_R}^0(n) \right)$ is liminary for each $n$. In fact we will show that every irreducible representation of $C_r^* \left( G_{J_R}^0(n) \right)$ is finite dimensional for each $n$. By Theorem 6.2.3 in [Pe] this will show that $C_r^* \left( G_{J_R}^0(n) \right)$ is a type I $C^*$-algebra and it follows then from (4.10) that $C_r^* \left( G_{J_R}^0 \right)$ is nuclear and satisfies the UCT. Furthermore, since we also have the inductive limit decomposition

$$X_\Gamma X_\Gamma^* = \bigcup_n C_r^* \left( G_{J_R}^0(n) \right) \cap X_\Gamma X_\Gamma^*, \quad (4.11)$$
and since we know that every irreducible representation of $C^*_r\left(G^0_{fr}(n)\right) \cap X_\Gamma X_\Gamma^*$ is finite dimensional when this holds for $C^*_r\left(G^0_{fr}(n)\right)$, we can at the same time conclude that also $X_\Gamma X_\Gamma^*$ is nuclear and satisfies the UCT.

To show that $C^*_r\left(G^0_{fr}(n)\right)$ is liminary consider an irreducible representation $\pi$ of $C^*_r\left(G^0_{fr}(n)\right)$. Every function $f \in C(\overline{G})$ defines a multiplier $\psi(f)$ of $C^*_r\left(G^0_{fr}(n)\right)$ such that

$$\psi(f)g[x,0,\eta,y] = f(R^\eta(x))g[x,0,\eta,y]$$

when $g \in C_c\left(G^0_{fr}(n)\right)$. Note that $\psi(f)$ is central in the multiplier algebra and that $\psi$ is a $*$-homomorphism. Since $\pi$ is irreducible there is a point $z \in \overline{G}$ such that $\pi(\psi(f)g) = f(z)\pi(g)$ for all $f \in C(\overline{G})$ and all $g \in C^*_r\left(G^0_{fr}(n)\right)$. Consequently $\pi(g) = 0$ for every $g \in C_c\left(G^0_{fr}(n)\right)$ whose support does not contain elements from $F = r^{-1}(R^\eta(z))$. Since all the isotropy groups in $G^0_{fr}(n)$ are finite by Lemma 4.2 in [Th2] it follows that $F$ is a finite set. Therefore $\pi\left(C_c\left(G^0_{fr}(n)\right)\right)$ must be finite dimensional and the same is then necessarily true for $\pi$.  

Note that the $C^*$-correspondance $X_\Gamma$ represents an element

$$[X_\Gamma] \in KK\left(X_\Gamma X_\Gamma^*, C^*_r\left(G^0_{fr}(n)\right)\right).$$

This element defines a homomorphism

$$[X_\Gamma]_* : K_*\left(X_\Gamma X_\Gamma^*\right) \to K_*\left(C^*_r\left(G^0_{fr}(n)\right)\right)$$

which fits into the following six terms exact sequence, cf. Theorem 8.6 in [K4].

**Corollary 4.19.** There is an exact sequence

$$\begin{array}{cccc}
K_0(X_\Gamma X_\Gamma^*) & \xrightarrow{id-[X_\Gamma]_*} & K_0(C^*_r(G^0_{fr}(n))) & \xrightarrow{\iota_*} & K_0(C^*_r(J_R\setminus E_R)) \\
& \uparrow & & \downarrow & \\
K_1(C^*_r(J_R\setminus E_R)) & \xleftarrow{\iota_*} & K_1(C^*_r(G^0_{fr}(n))) & \xleftarrow{id-[X_\Gamma]_*} & K_1(X_\Gamma X_\Gamma^*)
\end{array}$$

(4.12)

where $\iota : C^*_r\left(G^0_{fr}(n)\right) \to C^*_r\left(J_R\setminus E_R\right)$ is the inclusion $C^*_r\left(G^0_{fr}(n)\right) \subseteq C^*_r\left(J_R\right)$ followed by the stable isomorphism $C^*_r\left(J_R\right) \simeq C^*_r\left(J_R\setminus E_R\right)$.

It may seem possible to calculate the $K$-theory of $C^*_r\left(J_R\setminus E_R\right)$ from Corollary 4.19 and the inductive limit decompositions (4.10) and (4.11). In practice, however, the task is very difficult, not only because the six terms exact sequence of Corollary 4.19 is less helpful than the one which is available for local homeomorphs, [DM], [Th3], and which can be applied here when there are no critical points in the Julia set, but also because the topology of $J_R$ is poorly understood in general.
5. The structure of \( C^*_r(F_R) \)

It is well-known and not difficult to see that \( R \) takes a connected component \( W \) of \( F_R \) onto another connected component \( R(W) \) of \( F_R \). It follows that we can define an equivalence relation \( \sim \) in \( F_R \) such that \( x \sim y \) if and only if there are \( n, m \in \mathbb{N} \) such that \( R^n(x) \) and \( R^m(y) \) are contained in the same connected component of \( F_R \). By Sullivan’s no-wandering-domain theorem, Theorem 16.4 in [Mi], and a result of Shishikura, [Sh], the set of equivalence classes \( F_R/\sim \) is finite, and in fact can not have more than \( 2 \deg R - 2 \) elements. We can therefore write

\[
F_R = \bigsqcup_{i=1}^{N} \Omega_i,
\]

where \( N \leq 2 \deg R - 2 \), such that each \( \Omega_i \) is open, \( R^{-1}(\Omega_i) = \Omega_i \) and \( \Omega_i \cap \Omega_j = \emptyset \) when \( i \neq j \). The \( \Omega_i \)'s will be called the stable regions of \( R \). Since they are RO-invariant it follows from Lemma 2.2 that

\[
C^*_r(F_R) = \bigoplus_{i=1}^{N} C^*_r(\Omega_i).
\] (5.1)

The stable regions are divided into different types reflecting the faith of their elements under iteration.

Definition 5.1. Let \( U \) be an open subset of \( F_R \) and \( p \in \mathbb{N} \). We say that

a) \( U \) is a super-attracting domain of period \( p \) when \( R^p(U) \subseteq U \), \( R^i(U) \cap U = \emptyset, 1 \leq i \leq p-1 \), and there are a natural number \( d \geq 2 \), an \( r \in [0, 1] \) and a conformal conjugacy \( \psi: U \to D_r = \{ z \in \mathbb{C} : |z| < r \} \) such that

\[
\begin{array}{ccc}
U & \xrightarrow{R^p} & U \\
\psi \downarrow & & \psi \\
D_r & \xrightarrow{z \to zd} & D_r
\end{array}
\]

commutes;

b) \( U \) is an attracting domain of period \( p \) when \( R^p(U) \subseteq U \), \( R^i(U) \cap U = \emptyset, 1 \leq i \leq p-1 \), and there are a \( \lambda \in \mathbb{C}, |\lambda| < 1 \), an \( r > 0 \) and a conformal conjugacy \( \psi: U \to D_r = \{ z \in \mathbb{C} : |z| < r \} \) such that

\[
\begin{array}{ccc}
U & \xrightarrow{R^p} & U \\
\psi \downarrow & & \psi \\
D_r & \xrightarrow{z \to \lambda z} & D_r
\end{array}
\]

commutes;

c) \( U \) is a parabolic domain of period \( p \) when \( R^p(U) \subseteq U \), \( R^i(U) \cap U = \emptyset, 1 \leq i \leq p-1 \), and there is a conformal conjugacy \( \alpha: U \to \mathbb{H} = \{ z \in \mathbb{C} : \Re z > 0 \} \) such that

\[
\begin{array}{ccc}
U & \xrightarrow{R^p} & U \\
\alpha \downarrow & & \alpha \\
\mathbb{H} & \xrightarrow{z \to z+1} & \mathbb{H}
\end{array}
\]

commutes;
d) $U$ is a Siegel disk of period $p$ when $R^p(U) = U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p - 1$, and there are a $t \in \mathbb{R} \setminus \mathbb{Q}$ and a conformal conjugacy $\psi : U \to D_1 = \{z \in \mathbb{C} : |z| < 1\}$ such that

\[
\begin{array}{ccc}
D_1 & \xrightarrow{\psi} & U \\
\downarrow \scriptstyle{z \mapsto e^{2\pi i t}z} & & \downarrow \scriptstyle{\psi} \\
D_1 & \xrightarrow{\psi} & U
\end{array}
\]

commutes;

e) $U$ is a Herman ring of period $p$ when $R^p(U) = U$, $R^i(U) \cap U = \emptyset$, $1 \leq i \leq p - 1$, and there are a $t \in \mathbb{R} \setminus \mathbb{Q}$ and a conformal conjugacy $\psi : U \to \mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < 2\}$ such that

\[
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\psi} & U \\
\downarrow \scriptstyle{z \mapsto e^{2\pi i t}z} & & \downarrow \scriptstyle{\psi} \\
\mathbb{A} & \xrightarrow{\psi} & U
\end{array}
\]

commutes.

It follows from the classification of the periodic Fatou components, [Mi], that a stable region $\Omega$ contains a domain $U$ of one of the types described in a)-e) of Definition 5.1 with the property that

\[\Omega = \bigcup_{n=0}^{\infty} R^{-n}(U).\]  

(5.2)

We will say that $\Omega$ is a super-attractive, attractive, parabolic, Siegel or Herman region in accordance with the nature of the domain $U$ which we will refer to as a core of $\Omega$.

**Lemma 5.2.** Let $x \in F_R$ be periodic. Then $\text{RO}(x)$ is closed and discrete in $F_R$.

**Proof.** Let $W$ be a connected component of $F_R$ and $K \subseteq W$ a compact subset. We must show that $K \cap \text{RO}(x)$ is finite. Assume therefore that $W \cap \text{RO}(x) \neq \emptyset$. Let $\Omega$ be the stable region containing $W$ and $U$ a core for $\Omega$. It follows from [5,2] and the compactness of $K$ that there is an $l \in \mathbb{N}$ such that $R^l(K) \subseteq U$. When $y \in K \cap \text{RO}(x)$ there is then a $k > l$ such that $R^k(y) = x$. Thus $R^l(y)$ is a pre-periodic element of $U$. By inspecting the possible types of $U$ we see that $U$ contains at most one point pre-periodic under $R$. Thus $R^l(y) = x$, proving that $K \cap \text{RO}(x) \subseteq R^{-l}(x)$ which is a finite set. \hfill $\square$

**Lemma 5.3.** $\bigcup_{c \in \text{Crit}} \text{RO}(c) = \bigcup_{n=0}^{\infty} R^{-n}(\text{Crit})$.

**Proof.** Assume that $x \in \bigcup_{c \in \text{Crit}} \text{RO}(c)$. There is then a critical point $c \in \text{Crit}$ and $n,m \in \mathbb{N}$ such that $R^n(x) = R^m(c)$ and $\text{val}(R^n, x) = \text{val}(R^m, c)$. Since $\text{val}(R^m, c) \geq 2$ this implies that $\text{val}(R^n, x) = \text{val}(R, R^{m-1}(x)) \text{val}(R, R^{m-2}(x)) \cdots \text{val}(R, x) \geq 2$ and hence that $x \in \bigcup_{j=0}^{n-1} R^{-j}(\text{Crit})$. Conversely, if $x \in \bigcup_{n=0}^{\infty} R^{-n}(\text{Crit})$ let $j \in \mathbb{N}$ be the least natural number such that $R^j(x) \in \text{Crit}$. It follows from Lemma 4.2 that $x \in \text{RO}(R^j(x)) \subseteq \bigcup_{c \in \text{Crit}} \text{RO}(c)$. \hfill $\square$

**Lemma 5.4.** Let $x$ be a critical point in $F_R$. Then $\text{RO}(x)$ is closed and discrete in $F_R$. 

Proof. As in the proof of Lemma 5.2 we take a connected component \( W \) of \( F_R \) and show that \( K \cap \text{RO}(x) \) is finite for any compact subset \( K \subseteq W \). Let \( \Omega \) be the stable region containing \( W \) and \( U \) a core for \( \Omega \) of period \( p \). There is an \( l \in \mathbb{N} \) such that \( R^l(K) \subseteq U \). By inspecting the possible types of \( U \) we see that there is at most one element of \( U \) which is pre-critical under \( R \), and that element is already critical for \( R^p \) when it exists. It follows therefore from Lemma 5.3 that \( K \cap \text{RO}(x) \subseteq \bigcup_{n=0}^{l+p} R^{-n}(\text{Crit}) \) which is a finite set.

Since there are only finitely many periodic and critical points in \( F_R \) the union of their restricted orbits, which we will denote by \( I \), is closed and discrete in \( F_R \) by Lemma 5.2 and Lemma 5.4. It follows therefore from Lemma 3.2 that for each stable region \( \Omega \) in \( F_R \) there is an extension

\[
0 \longrightarrow C^*_r(\Omega|\mathcal{I}) \longrightarrow C^*_r(\Omega) \xrightarrow{\pi_x} C^*_r(\Omega \cap \mathcal{I}) \longrightarrow 0. \tag{5.3}
\]

To study the ideal \( C^*_r(\Omega|\mathcal{I}) \) we need the following lemma which seems to be folklore for mathematicians working with rational maps. We sketch a proof for the benefit of the operator algebraists.

Lemma 5.5. Let \( U \subseteq \overline{\mathbb{C}} \) be an open simply connected subset such that

\[
U \cap \bigcup_{n=1}^{\infty} R^n(\text{Crit}) = \emptyset.
\]

Let \( d = \deg R \) be the degree of \( R \). For each \( n \in \mathbb{N} \) there are \( d^n \) open connected subsets \( W_1^n, W_2^n, \ldots, W_{d^n} \) and \( d^n \) holomorphic maps \( \chi_i^n : U \to W_i^n \), \( i = 1, 2, \ldots, d^n \), such that

\[
\begin{align*}
&i) \ R^n(\chi_i^n(z)) = z, \ z \in U, \\
&ii) \ \chi_i^n(U) = W_i^n, \ i = 1, 2, \ldots, d^n, \\
&iii) \ W_i^n \cap W_j^n = \emptyset, \ i \neq j, \ \text{and} \\
&iv) \ R^{-n}(U) = \bigcup_{i=1}^{d^n} W_i^n.
\end{align*}
\]

Proof. Note that \( R^n \) is a \( d^n \)-fold covering of \( U \) by \( R^{-n}(U) \) since \( U \cap \bigcup_{j=1}^{d^n} R^j(\text{Crit}) = \emptyset \). Let \( W_i, \ i = 1, 2, \ldots, N \), be the connected components of \( R^{-n}(U) \). We claim that \( R^n(W_i) = U \). To see this let \( x \in W_i \) and let \( y \in U \). Since \( R^n : R^{-n}(U) \to U \) has the path-lifting property we can lift a path in \( U \) connecting \( R^n(x) \) to \( y \) in \( U \) to a path starting in \( x \). This path must end in a point in \( W_i \) which maps to \( y \) under \( R^n \), proving the claim. Then \( R^n : W_i \to U \) is also a covering and since \( U \) is simply connected it must be a homeomorphism. Let \( \chi_i^n : U \to W_i \) be its inverse and note that \( \chi_i^n \) is holomorphic since \( R^n \) is. It follows also that \( N = d^n \) since \( \#R^{-n}(y) = d^n \) for all \( y \in U \). The proof is complete.

\[
\square
\]

Lemma 5.6. Let \( \Omega \) be a stable region for \( R \) and \( U \subseteq \Omega \) a core for \( \Omega \). Then

\[
C^*_r(\Omega|\mathcal{I}) \simeq C^*_r(U|\mathcal{I}) \otimes \mathbb{K}.
\]

Proof. Let \( x \in \Omega|\mathcal{I} \). It follows from (5.2) that there is a \( k \in \mathbb{N} \) such that \( R^k(x) \in U \). Since \( x \notin \mathcal{I} \) it follows from Lemma 5.3 and Lemma 4.2 that \( R^k(x) \in \text{RO}(x) \). This shows that \( \text{RO}(x) \cap (U|\mathcal{I}) \neq \emptyset \) and it follows then from Theorem 2.3 that \( C^*_r(\Omega|\mathcal{I}) \) is stably isomorphic to \( C^*_r(U|\mathcal{I}) \). It suffices therefore now to show that \( C^*_r(\Omega|\mathcal{I}) \) is stable. To this end we use Lemma 2.1 and consider therefore a compact subset \( K \)
of $\Omega \setminus \mathcal{I}$. The construction of the required bi-sections will be performed differently for the different core domains.

Assume first that $U$ is a Siegel disk or a Herman ring. It follows from Theorem 16.1 in [M] that $U$ is a connected component of $F_R$ in these cases. There is an $l \in \mathbb{N}$ such that $R^l(K) \subseteq U$. Note that when $U$ is a Siegel disk the periodic point at the center of $U$ is not in $R^l(K)$. For each $z \in \text{Crit} \cap \Omega$, let $\chi(z)$ be the first element from $U$ in the forward orbit of $z$, i.e. $\chi(z) \in U$ is the element determined by the condition that $R^m(z) = \chi(z)$ while $R^i(z) \notin U$ for $i = 0, 1, 2, \ldots, m - 1$. Then $\chi(\text{Crit})$ is a finite (possibly empty) set. Since the rotation in the core is irrational and $K \cap \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit}) = \emptyset$, there is for each point $x$ in $K$ an open neighbourhood $V_x$ of $x$ such that $R^i$ is injective on $V_x$ for all $i \in \mathbb{N}$ and an $n_x \in \mathbb{N}$ with the property that $R^{n_x+l}(V_x) \cap \chi(\text{Crit}) = \emptyset$. We can also arrange that $R^{n_x+l}(V_x)$ is simply connected (e.g. a small disc). By compactness of $K$ there is a finite collection $V_{x_i}, i = 1, 2, \ldots, N$, such that $K \subseteq \bigcup_{i=1}^{N} V_{x_i}$. Let $U_{-1}, U_{-2}, U_{-3}, \ldots$ be a sequence of Fatou components such that $U_{-1} \notin \{U, R(U), R^2(U), \ldots, R^{p-1}(U)\}$ and $R(U_{-i}) = U_{-i+1}, i \geq 2$. Such a sequence exists since $R^{-p}(U) \notin U$. Note that $U_{-i} \cap U_{-j} = \emptyset$ when $i \neq j$. Since $K$ is compact there is an $m \in \mathbb{N}$ such that

$$K \cap U_{-j} = \emptyset, \quad j \geq m.$$  

Let $i \in \{1, 2, \ldots, N\}$. By using Lemma 2.3 we choose a connected subset $V_i' \subseteq U_{-i-m}$ such that $R^{i+m}$ is a homeomorphism from $V_i'$ onto $R^{n_x+l}(V_{x_i})$. Let $R^{-i-m} : R^{n_x+l}(V_{x_i}) \to V_i'$ denote its inverse. Then

$$S_i = \left\{ \left( R^{-i-m} \circ R^{n_x+l}(z), i + m - n_x - l, (R^{-i-m} \circ R^{n_x+l}|_{V_{x_i}})^{-1}, z \right) : z \in V_i \right\},$$  

$i = 1, 2, \ldots, N$, is a collection of open bi-sections in $G_{\Omega \setminus \mathcal{I}}$ meeting the requirements in Lemma 2.1.

Assume instead that $U$ is attracting or parabolic. We choose first $L$ such that $R^L(K) \subseteq U$, and then a finite open and relatively compact cover $V_1, V_2, \ldots, V_{N}$ of $K$ in $\Omega \setminus \mathcal{I}$ such that $R^L : V_i \to R^L(V_i) \subseteq U$ is injective for each $i$. Subsequently we choose $n_1, n_2, \ldots, n_{N} \in \mathbb{N}$ such that $R^{n_i}(R^L(V_i)) \cap R^{n_j}(R^L(V_j)) = \emptyset$ when $i \neq j$ and

$$R^{n_i+l}(V_i) \cap K = \emptyset$$  

for all $i$. Then

$$S_i = \left\{ \left( R^{n_i+l}(z), -n_i - L, (R^{n_i+l}|_{V_i})^{-1}, z \right) : z \in V_i \right\},$$  

$i = 1, 2, \ldots, N$, is a collection of open bi-sections in $G_{\Omega \setminus \mathcal{I}}$ meeting the requirements in Lemma 2.1.

Finally, in the super-attractive case choose $L$ such that $R^L(K) \subseteq U$ and $K \cap R^L(K) = \emptyset, i \geq L$. Let $U_0$ be an open subset of $U$ such that $R^L(K) \subseteq U_0$ and $\overline{U_0}$ is a compact subset of $U \setminus \{x\}$, where $x$ is the critical point in $U$. Set

$$Y = R^{-L} \left( \bigcup_{n=1}^{\infty} R^n(\text{Crit}) \right) \cap K$$  

and note that $Y$ is a finite set. For every $z \in Y$ choose a small neighbourhood $V_z$ of $z$ and a natural number $n_z$ such that $R^{n_z+l}$ is injective on $V_z$, $R^{n_z+l}(V_z) \subseteq U \setminus \{x\}$ and

$$R^{n_z+l}(V_z) \cap K = \emptyset$$  

for all $z \in Y$. The construction of the required bi-sections will be performed differently for the different core domains.
for all \( z \), and \( R^{n_z+L}(V_z) \cap R^{n_{z'}+L}(V_{z'}) = \emptyset \) when \( z \neq z' \). This is possible because \( K \cap I = \emptyset \); cf. Lemma 5.3. Then \( R^L(K \setminus \bigcup_{z \in Y} V_z) \cap \bigcup_{n=1}^{\infty} R^n(\text{Crit}) = \emptyset \). We can therefore cover \( K \setminus \bigcup_{z \in Y} V_z \) by a finite collection \( W_i \), \( i = 1, 2, \ldots, N \), of open sets such that \( R^L(W_i) \) is an open simply connected subset of \( U_0 \setminus \bigcup_{n=1}^{\infty} R^n(\text{Crit}) \). It follows then from Lemma 5.5 that for any collection \( n_i, i = 1, 2, \ldots, N \), of natural numbers we can find univalent holomorphic maps \( \chi_i : R^L(W_i) \to R^{-n_i}(U_0) \) such that \( R^{n_i} \circ \chi_i(z) = z \) for all \( z \in R^L(W_i) \). Set \( U_1 = U_0 \cup \bigcup_{z \in Y} R^{n_z+L}(V_z) \) and note that \( U_1 \) is a relatively compact subset of \( U \setminus \{ x \} \). There is therefore an \( N_1 \in \mathbb{N} \) such that \( R^s(U_1) \cap U_1 = \emptyset \), \( i \geq N_1 \). Thus, if we arrange that \( N_1 + L \leq n_1 \) and \( n_i \geq n_{i-1} + N_1, i = 2, 3, \ldots, N \), we will have that the sets

\[
\chi_i \circ R^L(W_i), \quad i = 1, 2, \ldots, N,
\]

are mutually disjoint, and also disjoint from \( K \cup U_1 \). For each \( z \in Y \) the set

\[
T_z = \left\{ \left( R^{n_z+L}(z), -n_z - L, (R^{n_z+L}|_{V_z})^{-1}, z \right) : \ z \in V_z \right\}
\]

is an open bi-section in \( G_{\Omega \setminus I} \). The same is true for

\[
S_i = \left\{ \left( \chi_i \circ R^L(z), n_i - L, (\chi_i \circ R^L|_{W_i})^{-1}, z \right) : \ z \in W_i \right\},
\]

\( i = 1, 2, \ldots, N \). Taken together we have a collection of bi-sections in \( G_{\Omega \setminus I} \) with the properties required in Lemma 2.1.

\[
\square
\]

5.1. Super-attractive stable regions. In this section we study the \( C^* \)-algebra \( C^*_r(\Omega \setminus I) \) in the case where \( \Omega \) is a super-attractive stable region. Let \( U \) be a core domain for \( \Omega \). Then \( I \cup U \) consists only of the super-attracting periodic point \( x \) at the center of \( U \). It follows from Lemma 5.10 and Proposition 2.9 that \( C^*_r(\Omega \setminus I) \simeq C^*_r(\psi) \otimes \mathbb{K} \) where \( \psi : D_r \setminus \{ 0 \} \to D_r \setminus \{ 0 \} \), for some \( r \in [0,1] \), is the local homeomorphism \( \psi(z) = z^d \). Note that \( d = \text{val}(R^s, x) \geq 2 \) where \( p \) is the period of \( x \). Let \( D = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \). Define \( \alpha : D \to D \) such that \( \alpha(z) = z^d \). Since \( D = \bigcup_j \alpha^{-j}(D \setminus \{ 0 \}) \) it follows from Theorem 2.3 that \( C^*_r(\psi) \) is stably isomorphic to \( C^*_r(\alpha) \). Thus \( C^*_r(\Omega \setminus I) \simeq C^*_r(\alpha) \otimes \mathbb{K} \) since \( C^*_r(\Omega \setminus I) \) is stable by Lemma 5.6 and in this section we identify the stable isomorphism class of \( C^*_r(\alpha) \).

First identify \( D \) with \( ]0, 1[ \times \mathbb{T} \) via that map \( (t, \lambda) \mapsto t \lambda \). In this picture

\[
\alpha(t, \lambda) = (t^d, \lambda^d).
\]

Map \( ]0, 1[ \times \mathbb{T} \) to \( \mathbb{R} \times \mathbb{T} \) using the map

\[
(t, \lambda) \mapsto \left( \frac{\log (-\log t)}{\log d}, \lambda \right).
\]

This gives us a conjugacy between \( (D, \alpha) \) and \( (\mathbb{R} \times \mathbb{T}, \tau \times \beta) \) where \( \tau(t) = t + 1 \) and \( \beta(\lambda) = \lambda^d \). It follows that \( C^*_r(\alpha) \simeq C^*_r(\tau \times \beta) \). Let \( S^1 \) be the one-point compactification of \( \mathbb{R} \) and let \( \tau^+ \) be the continuous extension of \( \tau \) to \( S^1 \). To simplify notation set \( \varphi = \tau^+ \times \beta \). It follows from Proposition 4.6 in [CT] that there is an extension

\[
0 \longrightarrow C^*_r(\tau \times \beta) \longrightarrow C^*_r(\varphi) \longrightarrow C^*_r(\beta) \longrightarrow 0. \tag{5.4}
\]

In the notation of Section 2 observe that \( R_\varphi = S^1 \times R_\beta \) and that this decomposition gives rise to an isomorphism

\[
C^*_r(R_\varphi) \simeq C(\left( S^1 \right) \otimes C^*_r(R_\beta). \tag{5.5}
\]
Under this identification the Deaconu endomorphism \( \hat{\varphi} \) of \( C^*_r(R) \) becomes the tensor product \( \hat{\tau} \otimes \hat{\beta} \) where \( \hat{\beta} : C^*_r(R) \to C^*_r(R) \) is the Deaconu endomorphism of \( C^*_r(\beta) \) and \( \hat{\tau} : C(S^1) \to C(S^1) \) is given by

\[
\hat{\tau}(f)(x) = f(\tau^+(x)).
\]

Let \( B_\beta \) be the inductive limit of the sequence

\[
C^*_r(R) \xrightarrow{\hat{\tau}} C^*_r(R) \xrightarrow{\hat{\beta}} C^*_r(R) \xrightarrow{\hat{\beta}} C^*_r(R) \xrightarrow{\hat{\beta}} \cdots \tag{5.6}
\]

and let \( \hat{\beta}_\infty \) be the automorphism of \( B_\beta \) induced by letting \( \hat{\beta} \) act on all copies of \( C^*_r(R) \) in the sequence \( (5.6) \). Similarly, we can consider the inductive limit \( B_\varphi \) of the sequence

\[
C^*_r(R) \xrightarrow{\hat{\varphi}} C^*_r(R) \xrightarrow{\hat{\varphi}} C^*_r(R) \xrightarrow{\hat{\varphi}} C^*_r(R) \xrightarrow{\hat{\varphi}} \cdots .
\]

Using \( (5.5) \) and the tensor product decomposition \( \hat{\varphi} = \hat{\tau} \otimes \hat{\beta} \) it follows that \( B_\varphi \cong C(S^1) \otimes B_\beta \) under an isomorphism which turns \( \hat{\varphi}_\infty \) into \( \hat{\tau} \otimes \hat{\beta}_\infty \). It follows in this way from Theorem 4.8 in [Th1] that there are embeddings of \( C^*_r(\tau^+ \times \beta) \) and \( C^*_r(\beta) \) into full corners of \( (C(S^1) \otimes B_\beta) \times_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z} \) and \( B_\beta \times_{\hat{\beta}_\infty} \mathbb{Z} \), respectively. Together with the extension \( (5.4) \) this gives us a commuting diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C^*_r(\tau \times \beta) & \longrightarrow & C^*_r(\tau^+ \times \beta) & \longrightarrow & C^*_r(\beta) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (C_0(\mathbb{R}) \otimes B_\beta) \times_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z} & \longrightarrow & (C(S^1) \otimes B_\beta) \times_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z} & \longrightarrow & B_\beta \times_{\hat{\beta}_\infty} \mathbb{Z} & \longrightarrow & 0
\end{array}
\]

with exact rows. Consequently the range of the embedding

\[
C^*_r(\tau \times \beta) \to (C_0(\mathbb{R}) \otimes B_\beta) \times_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z}
\]

is a full corner, and we conclude therefore from [Br] that \( C^*_r(\tau \times \beta) \) is stably isomorphic to \( (C_0(\mathbb{R}) \otimes B_\beta) \times_{\hat{\tau} \otimes \hat{\beta}_\infty} \mathbb{Z} \).

Recall that the mapping torus \( MT_\gamma \) of an endomorphism \( \gamma : B \to B \) of a \( C^* \)-algebra \( B \) is the \( C^* \)-algebra

\[
MT_\gamma = \{ f \in C[0, 1] \otimes B : \; \gamma(f(0)) = f(1) \}.
\]

We need the following lemma.

**Lemma 5.7.** Let \( A \) be a \( C^* \)-algebra and \( \alpha : A \to A \) an automorphism. It follows that the crossed product \( (C_0(\mathbb{R}) \otimes A) \times_{\hat{\tau} \otimes \alpha} \mathbb{Z} \) is isomorphic to the mapping torus \( MT_{id_\mathbb{K} \otimes \alpha} \) of the automorphism \( id_\mathbb{K} \otimes \alpha : \mathbb{K} \otimes A \to \mathbb{K} \otimes A \).

**Proof.** For \( f \in C_0(\mathbb{R}) \) and \( n \in \mathbb{Z} \), let \( f_n \in C[0, 1] \) be the function \( f_n(t) = f(n + t) \). Let \( e_{ij} \) be the standard matrix units in \( \mathbb{K} = \mathbb{K}(l^2(\mathbb{Z})) \). Define a \( * \)-homomorphism \( \pi : C_0(\mathbb{R}) \otimes A \to C[0, 1] \otimes \mathbb{K} \otimes A \) such that

\[
\pi(f \otimes a) = \sum_{n \in \mathbb{Z}} f_n \otimes e_{nn} \otimes \alpha^{-n}(a).
\]

Let \( S \) be the two-sided shift on \( l^2(\mathbb{K}) \) such that \( Se_{nn}S^* = e_{n-1,n-1} \). Then \( \pi \) maps into the mapping torus of \( (Ad S) \otimes \alpha \) and \( \pi \circ (\hat{\tau} \otimes \alpha) = Ad T \circ \pi \) where \( T = \)
\(1_{C^\star_{[0,1]} \otimes S \otimes 1_A}\). It follows from the universal property of the crossed product that we get a \(*\)-homomorphism

\[
\Phi : (C_0(\mathbb{R}) \otimes A) \rtimes_{\tau \otimes \alpha} \mathbb{Z} \to C^\star_{[0,1]} \otimes \mathbb{K} \otimes A
\]

which is injective since its restriction to \(C_0(\mathbb{R}) \otimes A\) clearly is. Its range is generated by elements in \(C^\star_{[0,1]} \otimes \mathbb{K} \otimes A\) of the form \(T\pi(f \otimes a)\) and is therefore contained in the mapping torus of \((\text{Ad } S) \otimes \alpha\). To see that

\[
B = \Phi ((C_0(\mathbb{R}) \otimes A) \rtimes_{\tau \otimes \alpha} \mathbb{Z})
\]

actually is equal to this mapping torus, let \(\text{ev}_t : C^\star_{[0,1]} \otimes \mathbb{K} \otimes A \to \mathbb{K} \otimes A\) denote evaluation at \(t \in [0,1]\). Then \(\text{ev}_t(B)\) is generated by elements of the form \(e_{n-1,n} \otimes a\) for some \(n \in \mathbb{Z}\) and some \(a \in A\), and it is easy to see that this is all of \(\mathbb{K} \otimes A\). Consider then a continuous function \(g : [0,1] \to \mathbb{K} \otimes A\) which is an element of \(MT_{(\text{Ad } S) \otimes \alpha}\), i.e. has the property that

\[
(\text{Ad } S) \otimes \alpha (g(0)) = g(1).
\]

Let \(\epsilon > 0\). For each \(t \in [0,1]\) there is then an element \(f_t \in B\) such that \(g(t) = f_t(t)\).

We can therefore choose intervals \(I_j = \left[\frac{i-1}{M}, \frac{i}{M}\right]\) and elements \(f_j \in B\) such that \(\|g(t) - f_j(t)\| < \epsilon, t \in I_j\), for all \(j\), and such that

\[
(\text{Ad } S) \circ \alpha (f_0(0)) = f_{M-1}(1).
\]

Choose a partition of unity \(h_j \in C^\star_{[0,1]}, j = 0,1,2,\ldots, M-1\), such that \(h_0(0) = h_{M-1}(1) = 1\) and \(\text{supp } h_j \subseteq I_j\) for all \(j\). Then \(f = \sum_{j=0}^{M-1} h_j f_j \in B\) because \(B\) is a module over \(\{ f \in C^\star_{[0,1]} : f(0) = f(1)\}\). Since \(\|f - g\| < \epsilon\) this shows that \(B\) is equal to the entire mapping torus of \((\text{Ad } S) \otimes \alpha\). This mapping torus is isomorphic to that of \(\text{id } \mathbb{K} \otimes \alpha\) because the automorphism group of \(\mathbb{K}\) is connected, cf. Proposition 10.5.1 in [13].

It follows from Lemma 5.7 and the preceding considerations that \(C^\star_{\tau}(\alpha)\) is stably isomorphic to the mapping torus of \(\hat{\beta}_\infty : B_\beta \to B_\beta\).

It is known that \(C^\star_{\tau}(R_\beta)\) is isomorphic to the Bunce-Deddens algebra \(\text{BD}(d^\infty)\) of type \(d^\infty\), cf. Example 3 in [14]. Thus \(\text{BD}(d^\infty)\) is the unique simple unital \(\text{AT}\)-algebra with a unique trace state such that \(K_1(\text{BD}(d^\infty)) \simeq \mathbb{Z}\) and \(K_0(\text{BD}(d^\infty)) \simeq \mathbb{Z}[1/d]\) of \(d\)-adic rationals when the latter has the order inherited from \(\mathbb{R}\) and the order unit 1. As shown in Example 3 of [14] the map \(\hat{\beta}_\tau : K_1(\text{BD}(d^\infty)) \to K_1(\text{BD}(d^\infty))\) is the identity while \(\hat{\beta}_\tau : K_0(\text{BD}(d^\infty)) \to K_0(\text{BD}(d^\infty))\) is multiplication by \(\frac{1}{d}\) on \(\mathbb{Z}[1/d]\). To emphasise the number \(d\), which is the determining input for the construction, we will denote the mapping torus of \(\hat{\beta}\) by \(MT_d\) in the following.

**Lemma 5.8.** The mapping torus \(MT_{\hat{\beta}_\infty}\) of \(\hat{\beta}_\infty\) is stably isomorphic to

\[
MT_d = \left\{ f \in C^\star_{[0,1]} : C^\star_{\tau}(\beta) : \hat{\beta}(f(0)) = f(1) \right\};
\]

the mapping torus of the Deaconu endomorphism of \(C^\star_{\tau}(R_\beta)\).

**Proof.** Note that the mapping torus of \(\hat{\beta}_\infty\) is isomorphic to the inductive limit

\[
MT_d \xrightarrow{\text{id}_{C^\star_{[0,1]} \otimes \hat{\beta}}} MT_d \xrightarrow{\text{id}_{C^\star_{[0,1]} \otimes \hat{\beta}}} MT_d \xrightarrow{\text{id}_{C^\star_{[0,1]} \otimes \hat{\beta}}} \cdots \cdots \hspace{1cm} (5.7)
\]
Let $\rho_{\infty,1} : C^*_r(R_\beta) \to B_\beta$ be the canonical homomorphism out of the first copy of $C^*_r(R_\beta)$ in the sequence (5.6). As observed in [An] the isometry $V \in C^*_r(\beta)$ which implements the Deaconu endomorphism, in the sense that $\hat{\beta}(a) = VaV^*$, has the property that $V^*C^*_r(R_\beta) V \subseteq C^*_r(R_\beta)$. It follows that $\hat{\beta} (C^*_r(R_\beta)) = VV^*C^*_r(R_\beta)VV^*$ and that

$$\rho_{\infty,1}(C^*_r(R_\beta)) = qB_\beta q$$

where $q = \rho_{\infty,1}(1)$. It follows then from the commuting diagram

that the image in the mapping torus $MT_{\beta_\infty}$ of the first copy of $MT_d$ from the sequence (5.7) is equal to

$$\left\{ f \in C[0,1] \otimes B_\beta : \hat{\beta}_{\infty}(f(0)) = f(1), \; qf(t) = f(t)q = f(t) \forall t \right\}, \quad (5.8)$$

which is visibly a hereditary $C^*$-subalgebra of the mapping cone of $\hat{\beta}_{\infty}$. Since $B_\beta$ is simple (because BD($d^{\infty}$) is) it follows that an ideal in $MT_{\beta_\infty}$ which contains (5.8) must have full fiber over every $t \in [0,1]$. Then a standard partition of unity argument shows, much as in the proof of Lemma 5.7, that such an ideal must be all of $MT_{\beta_\infty}$, i.e. (5.8) is both hereditary and full in $MT_{\beta_\infty}$. The desired conclusion follows then from Corollary 2.6 of [Br].

We can now summarise with the following.

**Proposition 5.9.** Let $\Omega$ be a super-attractive stable region. Then $C^*_r(\Omega \cap I)$ is isomorphic to $\mathbb{K} \otimes MT_d$ where $MT_d$ is the mapping torus of the Deaconu endomorphism on BD($d^{\infty}$).

To describe the quotient $C^*_r(\Omega \cap I)$ in (5.3) we need to determine the restricted orbits in $I$ and find the isotropy groups of their elements. Note that every periodic point in $\Omega$ is RO-equivalent to a critical point in the critical periodic orbit. Hence every RO-equivalence class in $\Omega \cap I$ is represented by a critical point $z$ in $\Omega$. When $z$ is eventually periodic it follows from Proposition 4.4 of [Tu2] that $\text{Is}_z$ is an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$ and hence $C^*_r(\text{Is}_z) \simeq C \left( \hat{\text{Is}}_z \right) \simeq C(K)$ where $K$ is the Cantor set. When $z$ is not pre-periodic it follows from Proposition 4.4 of [Tu2] that $\text{Is}_z = \mathbb{Z}v$ where $v$ is the asymptotic valency of $z$. By using Lemma 2.2 and Lemma 4.11 we get in this way a complete description of $C^*_r(\Omega \cap I)$ and we can then put the information we have obtained into (5.3). To summarise our findings we introduce the notation $\mathbb{K}_x$ for the $C^*$-algebra of compact operators on the Hilbert space $l^2(\text{RO}(x))$. Thus

$$\mathbb{K}_x = \begin{cases} \mathbb{K} & \text{when } x \text{ is not exposed,} \\ M_n(\mathbb{C}) & \text{where } n = \# \text{RO}(x) \leq 4 \text{ when } x \text{ is exposed.} \end{cases}$$
Theorem 5.10. Let \( \Omega \) be a super-attractive stable region and \( c_1, c_2, \ldots, c_{n+m} \) critical points in \( \Omega \) such that \( \Omega \cap \mathcal{I} = \bigcup_{i=1}^{n+m} \text{RO}(c_i) \), and \( c_1, c_2, \ldots, c_n \) are pre-periodic while \( c_{n+1}, c_{n+2}, \ldots, c_{n+m} \) are not. Let \( \nu_i \) be the asymptotic valency of \( c_i \), \( n + 1 \leq i \leq n + m \). There is an extension

\[
0 \longrightarrow K \otimes \text{MT}_d \longrightarrow C_r^\ast(\Omega) \longrightarrow (\oplus_{i=1}^n C(K) \otimes K_{c_i}) \oplus (\oplus_{i=n+1}^{n+m} C^{\nu_i} \otimes K_{c_i}) \longrightarrow 0,
\]

where \( K \) is the Cantor set and \( \text{MT}_d \) is the mapping torus of the Deaconu endomorphism on the Bunce-Deddens algebra of type \( d^\infty \).

5.2. Attractive stable regions. Let now \( \Omega \) be an attractive stable region. Let \( q \) be an element of the periodic orbit in \( \Omega \). The number \( \lambda \) given by Proposition 4.4 of [Th2]. This leads to the following description of \( \lambda \) where \( \theta \in [0, 1[ \times \mathbb{T} \) given by

\[
(t, \mu) \mapsto (|t|, \mu e^{2\pi i \theta}).
\]

Via the map \((t, \mu) \mapsto \left( \frac{\log t}{\log |t|}, \mu \right)\) we see that \( \alpha \) is conjugate to the map \((t, \mu) \mapsto (t + 1, \mu e^{2\pi i \theta})\) on \( \mathbb{R}_+ \times \mathbb{T} \). The transformation groupoid of the last map is a reduction of the transformation groupoid of the homeomorphism \((t, \lambda) \mapsto (t + 1, \lambda e^{2\pi i \theta})\) on \( \mathbb{R} \times \mathbb{T} \). Hence \( C_r^\ast(\alpha) \) is stably isomorphic to the corresponding crossed product \( C_0(\mathbb{R} \times \mathbb{T}) \rtimes \mathbb{Z} \) by Theorem 2.3. It follows from Lemma 5.7 that the latter crossed product is isomorphic to \( K \otimes C(\mathbb{T}^2) \). In this way we obtain the following.

Proposition 5.11. Let \( \Omega \) be an attractive stable region. Then

\[
C_r^\ast(\Omega \setminus \mathcal{I}) \simeq K \otimes C(\mathbb{T}^2).
\]

It is also straightforward to adopt the methods from the preceding section to obtain a description of \( C_r^\ast(\Omega \cap \mathcal{I}) \). The periodic points lie in the same restricted orbit and the isotropy group of any of its members is a copy of \( \mathbb{Z} \) by Proposition 4.4 of [Th2]. The restricted orbits of the critical points are divided according to whether or not they are pre-periodic. Since the periodic orbit is not critical the isotropy group of a critical pre-periodic point is now \( \mathbb{Z} \oplus \mathbb{Z}^d \) where \( d \) is the asymptotic valency by Proposition 4.4 of [Th2]. This leads to the following description of \( C_r^\ast(\Omega) \).

Theorem 5.12. Let \( \Omega \) be an attractive stable region and \( q \) a periodic point in \( \Omega \). Let \( c_1, c_2, \ldots, c_{n+m} \) be critical points in \( \Omega \) such that

\[
\Omega \cap \mathcal{I} = \text{RO}(q) \cup \bigcup_{i=1}^{n+m} \text{RO}(c_i),
\]

and \( c_1, c_2, \ldots, c_n \) are pre-periodic while \( c_i, i \geq n + 1 \), are not. Let \( \nu_i \) be the asymptotic valency of \( c_i \). There is an extension

\[
0 \longrightarrow K \otimes C(\mathbb{T}^2) \longrightarrow C_r^\ast(\Omega) \longrightarrow A \longrightarrow 0
\]

where

\[
A = (C(\mathbb{T}) \otimes K_q) \oplus (\oplus_{i=1}^n C^{\nu_i} \otimes C(\mathbb{T}) \otimes K_{c_i}) \oplus (\oplus_{i=n+1}^{n+m} C^{\nu_i} \otimes K_{c_i}).
\]
5.3. **Parabolic stable regions.** The remaining cases, corresponding to stable regions of parabolic, Siegel or Herman type can be handled by similar methods. Since the considerations are simpler than those involved in the attractive cases, we merely state the results.

**Theorem 5.13.** Let \( \Omega \) be parabolic stable region. Let \( c_i, i = 1, 2, \ldots, N \), be representatives for the restricted orbits of the critical points in \( \Omega \) and let \( v_i \) be the asymptotic valency of \( c_i \). There is an extension

\[
0 \longrightarrow K \otimes C(T) \otimes C_0(\mathbb{R}) \longrightarrow C_r^*(\Omega) \longrightarrow \bigoplus_{i=1}^N C^{v_i} \otimes K_{c_i} \longrightarrow 0.
\]

5.4. **Stable regions of Siegel type.** Let \( \theta \in [0, 1) \setminus \mathbb{Q} \). The corresponding *irrational rotation algebra* is the universal \( C^* \)-algebra generated by two unitaries \( U, V \) satisfying the relation \( UV = e^{2\pi i \theta} VU \). See [EE] for more on its structure.

**Theorem 5.14.** Let \( \Omega \) be a stable region of Siegel type. Let \( q \) be a periodic point in \( \Omega \). Let \( c_1, c_2, \ldots, c_{n+m} \) be critical points in \( \Omega \) such that

\[
\Omega \cap I = \text{RO}(q) \sqcup \bigcup_{i=1}^{n+m} \text{RO}(c_i),
\]

and \( c_1, c_2, \ldots, c_n \) are pre-periodic while \( c_i, \ i \geq n+1 \), are not. Let \( v_i \) be the asymptotic valency of \( c_i \). There is an extension

\[
0 \longrightarrow K \otimes C_0(\mathbb{R}) \otimes A_\theta \longrightarrow C_r^*(\Omega) \longrightarrow B \longrightarrow 0
\]

where \( A_\theta \) is the irrational rotation algebra, corresponding to the rotation by the angle \( 2\pi \theta \) in the core domain, and

\[
B = (C(T) \otimes K_q) \oplus \left( \bigoplus_{i=1}^n C^{v_i} \otimes C(T) \otimes K_{c_i} \right) \oplus \left( \bigoplus_{i=n+1}^{n+m} C^{v_i} \otimes K_{c_i} \right).
\]

5.5. **Stable regions of Herman type.**

**Theorem 5.15.** Let \( \Omega \) be a stable region of Herman type. Let \( c_i, i = 1, 2, \ldots, N \), be representatives for the restricted orbits of the critical points in \( \Omega \) and let \( v_i \) be the asymptotic valency of \( c_i \). There is an extension

\[
0 \longrightarrow K \otimes C_0(\mathbb{R}) \otimes A_\theta \longrightarrow C_r^*(\Omega) \longrightarrow \bigoplus_{i=1}^N C^{v_i} \otimes K_{c_i} \longrightarrow 0
\]

where \( A_\theta \) is the irrational rotation algebra corresponding to the rotation by the angle \( 2\pi \theta \) in the core domain.

5.6. **A square of six extensions.** It is possible to combine the extensions from the last sections into an exact square of 6 extensions in the following way. Let \( I_p \) be the union of the RO-orbits containing a non-critical periodic orbit in \( F_R \) and \( I_c = I \setminus I_p \) its complement in \( I \). Several applications of Lemma 3.2 gives us the following commuting diagram with exact rows and columns.
The algebras in the corners, $C^* (F_R \setminus \mathcal{I})$, $C^* (\mathcal{I}_p)$, $C^* (\mathcal{I}_c)$ and $C^* (J_R)$, can all be identified from the preceding sections. Specifically, $C^* (J_R)$ is either nuclear, simple and purely infinite, or an extension of such an algebra by a finite direct sum of circle and matrix algebras, cf. Theorem 4.13. $C^* (F_R \setminus \mathcal{I})$ is a finite direct sum of algebras each of which is the stabilization of $MT_d$, $C(T)^2$, $C(T) \otimes C_0(\mathbb{R})$ or $C_0(\mathbb{R}) \otimes A_\theta$. Which of the four types are present depends on the nature of the stable regions in $F_R$. The algebra $C^* (\mathcal{I}_p)$ is a finite direct sum of algebras stably isomorphic to $C$, $C(T)$ or $C(K)$. Which summands occur depends on the behaviour under iteration of the critical points in $F_R$.

It should be noted that the decomposition of $C^* (R)$ depicted in (5.9) is not the only possible. In fact there is a commuting square of the form (5.9) for any RO-invariant partitioning of $I$; not just for the partition $I = \mathcal{I}_p \cup \mathcal{I}_p$ chosen above.

6. Primitive ideals and primitive quotients

In the following an ideal in a $C^*$-algebra is a closed two-sided and proper ideal. Recall that an ideal $I$ is primitive when it is the kernel of an irreducible non-zero representation, and prime when it has the property that $I_1 I_2 \subseteq I \Rightarrow I_1 \subseteq I$ or $I_2 \subseteq I$ when $I_1$ and $I_2$ are also ideals. Since we shall only deal with separable $C^*$-algebras the primitive ideals will be the same as the prime ideals, cf. e.g. [RW].

6.1. The primitive ideals. When $I$ is an ideal in $C^*_r (R)$ we set

$$\rho(I) = \{ x \in \overline{\mathbb{C}} : \ f(x) = 0 \ \forall f \in C(\overline{\mathbb{C}}) \cap I \} .$$

We call $\rho(I)$ the co-support of $I$.

**Lemma 6.1.** $\rho(I)$ is a closed non-empty RO-invariant subset of $\overline{\mathbb{C}}$.

**Proof.** See Lemma 4.5 in [CT]. \qed

**Lemma 6.2.** Let $I$ be an ideal in $C^*_r (R)$ and let $A$ be a closed RO-invariant subset of $\overline{\mathbb{C}}$. If $\rho(I) \subseteq A$, then $\ker \pi_A \subseteq I$.

**Proof.** See Lemma 4.8 in [CT]. \qed
When $I \subseteq C^*_\tau(R)$ is an ideal we let $q_I : C^*_\tau(R) \to C^*_\tau(R)/I$ denote the corresponding quotient map. Note that it follows from Lemma 6.2 that $q_I$ factorises through $C^*_\tau(\rho(I))$, i.e. there is an $*$-homomorphism $C^*_\tau(\rho(I)) \to C^*_\tau(R)/I$ such that

$$
\begin{array}{ccc}
C^*_\tau(R) & \xrightarrow{q_I} & C^*_\tau(R)/I \\
\pi_{\rho(I)} & & \\
& & C^*_\tau(\rho(I))
\end{array}
$$

(6.1)

commutes.

A non-empty closed RO-invariant subset $A \subseteq \mathbb{T}$ is prime when the implication

$$
A \subseteq B \cup C \Rightarrow A \subseteq B \text{ or } A \subseteq C
$$

holds for all closed RO-invariant subsets $B$ and $C$ of $\mathbb{T}$.

**Lemma 6.3.** Assume that $I$ is a primitive ideal in $C^*_\tau(R)$. It follows that $\rho(I)$ is prime.

**Proof.** See Proposition 4.10 in [CT]. \qed

**Lemma 6.4.** Let $Y$ be a prime subset of $\mathbb{T}$. Assume that $x \in Y$ is isolated in $Y$. Then all elements of RO$(x)$ are isolated in $Y$ and $Y = \overline{\text{RO}(x)}$.

**Proof.** It is clear that all elements of RO$(x)$ are isolated in $Y$ since $x$ is. Set

$$
B = \{z \in Y : z \notin \text{RO}(x)\}.
$$

Since $Y \subseteq \overline{\text{RO}(x)} \cup B$ the primeness of $Y$ implies $Y \subseteq \overline{\text{RO}(x)}$ or $Y \subseteq B$. Note that $x \notin B$ since $x$ is isolated in $Y$. It follows that $Y \subseteq \overline{\text{RO}(x)}$. \qed

In the following we denote by Orb$(x)$ the (full) orbit of $x$, i.e.

$$
\text{Orb}(x) = \{y \in \mathbb{T} : R^n(x) = R^m(y) \text{ for some } n,m \in \mathbb{N}\}.
$$

**Lemma 6.5.** Let $Y$ be a prime subset of $\mathbb{T}$. Assume $Y$ has no isolated points. It follows that there is a point $x \in Y \setminus \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$ such that $Y = \overline{\text{RO}(x)} = \overline{\text{Orb}(x)}$.

**Proof.** The proof is largely the same as the proof of Proposition 4.9 in [CT], but with a few crucial modifications. It follows from Lemma 6.4 that $Y$ is totally $R$-invariant and hence in particular that $\overline{\text{RO}(x)} \subseteq \overline{\text{Orb}(x)} \subseteq Y$ for all $x \in Y$. It suffices therefore to find an $x \in Y \setminus \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$ such that $Y \subseteq \overline{\text{RO}(x)}$. Let $\{U_k\}_{k=1}^{\infty}$ be a basis for the topology of $Y$. We will by induction construct compact sets $\{C_k\}_{k=0}^{\infty}$ and $\{C'_k\}_{k=0}^{\infty}$ with non-empty interiors in $Y$ and positive integers $(n_k)_{k=0}^{\infty}$ and $(n'_k)_{k=0}^{\infty}$ such that

i) $C_k \subseteq U_k$,

ii) $C'_k \subseteq R^{n_{k-1}}(C_{k-1}) \cap R^{n'_{k-1}}(C'_{k-1})$ when $k \geq 1$ and

iii) $C_k \cap \left(\bigcup_{j=0}^{n'_k-1} R^j(\text{Crit}) \cup \bigcup_{j=0}^{n_k-1} R^j(\text{Crit})\right) = \emptyset$ when $k \geq 1$.

Let $C_0 = C'_0$ be any compact subset of $Y$ with non-empty interior in $Y$. Assume that $k \geq 1$ and that $C_1, \ldots, C_k, C'_1, \ldots, C'_k, n_0, \ldots, n_{k-1}$ and $n'_0, \ldots, n'_{k-1}$ satisfying the conditions above have been chosen. Choose non-empty open subsets $V_k \subseteq C_k$ and $V'_k \subseteq C'_k$. Then

$$
\bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V_k)) \text{ and } \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V'_k))
$$
are non-empty open and totally $R$-invariant subsets of $Y$, and hence

\[ Y \setminus \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V_k)) \text{ and } Y \setminus \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V'_k)) \quad (6.2) \]

are closed and totally $R$-invariant subsets of $Y$. Since $Y$ is prime and not contained in either of the sets from (6.2), it is also not contained in their union. That is,

\[ \left( \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V_k)) \right) \cap \left( \bigcup_{l,m=0}^{\infty} R^{-l}(R^m(V'_k)) \right) \neq \emptyset. \]

It follows that there are positive integers $n_k$ and $n'_k$ such that $R^{n_k}(V_k) \cap R^{n'_k}(V'_k)$ is non-empty. We can therefore choose a non-empty compact set $C_{k+1}$ with non-empty interior such that $C_{k+1} \subseteq U_{k+1}$ and a non-empty compact set $C'_{k+1}$ with non-empty interior such that $C'_{k+1} \subseteq R^{n_k}(V_k) \cap R^{n'_k}(V'_k)$. Since

\[ \bigcup_{j=0}^{n'_k+n_k-1} R^j(C) \cup \bigcup_{j=0}^{n_k} R^j(C) \]

is a finite set and $Y$ contains no isolated points, we can arrange that

\[ C'_{k+1} \cap \left( \bigcup_{j=0}^{n'_k+n_k-1} R^j(C) \cup \bigcup_{j=0}^{n_k} R^j(C) \right) = \emptyset. \]

This completes the induction step.

It follows from ii) that

\[ C'_k = R^{n'_0+n'_1+\cdots+n'_k-1}(C'_0 \cap R^{-n'_0}(C'_1) \cap \cdots \cap R^{-(n'_0+n'_1+\cdots+n'_k-1)}(C'_k)) \]

for all $k$ and hence that

\[ C'_0 \cap R^{-n'_0}(C'_1) \cap \cdots \cap R^{-(n'_0+n'_1+\cdots+n'_k-1)}(C'_{k+1}), \quad k = 0, 1, \ldots \]

is a decreasing sequence of non-empty compact sets. Let

\[ x \in \bigcap_{k=0}^{\infty} R^{-n'_0+n'_1+\cdots+n'_k}(C'_{k+1}) \cap C'_0. \]

By construction there is for each $k$ an element $u_k \in U_k$ such that $R^{n'_0+\cdots+n'_k}(x) = R^{n_k}(u_k)$ and

\[ \text{val} \left( R^{n'_0+\cdots+n'_k}, x \right) = \text{val} (R^{n_k}, u_k) = 1. \]

Since this implies that $u_k \in \text{RO}(x)$ we conclude that $\text{RO}(x)$ is dense in $Y$. Furthermore, it implies also that $\text{val} (R, R^j(x)) = 1$ for all $j$, i.e. $x \not\in \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit})$.

\[ \square \]

**Corollary 6.6.** Let $Y \subseteq \overline{C}$ be a closed RO-invariant subset. Then $Y$ is prime if and only if there is a point $x \in \overline{C}$ such that $Y = \text{RO}(x)$.

**Proof.** If $Y$ is prime it follows from Lemma 6.4 and Lemma 6.5 that there is an element $x \in \overline{C}$ such that $Y = \text{RO}(x)$. This proves the necessity of the condition. Sufficiency follows immediately from the definitions. \[ \square \]
Let $\mathcal{M}$ be the set of prime subsets of $\overline{\mathbb{C}}$. Let $\mathcal{M}_{\text{ex}}$ denote the collection of elements $Y \in \mathcal{M}$ with the property that $Y$ contains an isolated point which is either periodic or critical.

**Lemma 6.7.** Let $Y \in \mathcal{M} \setminus \mathcal{M}_{\text{ex}}$. It follows that $\ker \pi_Y$ is the only ideal $I$ in $C^*_r(R)$ with $\rho(I) = Y$, and that $\ker \pi_Y$ is a primitive ideal in $C^*_r(R)$.

**Proof.** Let $I$ be an ideal in $C^*_r(R)$ with $\rho(I) = Y$. Then $\ker \pi_Y \subseteq I$ by Lemma 6.2. To conclude that $I = \ker \pi_Y$ it suffices therefore to show that $\pi_Y(I) = \{0\}$ in $C^*_r(Y)$.

To this end note first that $\pi_Y(I) \cap C(Y) = \{0\}$. Indeed, if $h \in \pi_Y(I) \cap C(Y)$, let $g \in C(\overline{\mathbb{C}})$ be a function such that $g|_Y = h$ and let $a \in I$ be an element such that $\pi_Y(a) = h$. Then $\pi_Y(a - g) = 0$ and hence $a - g \in \ker \pi_Y \subseteq I$. It follows that $g = a - (a - g) \in I \cap C(\overline{\mathbb{C}})$ and hence that $g(y) = 0$ for all $y \in \rho(I) = Y$. Thus $h = 0$, proving that $\pi_Y(I) \cap C(Y) = \{0\}$. To conclude from this that $\pi_Y(I) = 0$ note first that the elements of $Y$ with non-trivial isotropy in $G_Y$ are dense in $Y$. This follows from Lemma 4.3 because a point $y \in Y$ with non-trivial isotropy in $G_Y$ must be pre-periodic or pre-critical for $R$ by Proposition 4.4 a) in [Th2]. It follows then from Lemma 2.15 of [Th1] that $P(\pi_Y(Y)) = \{0\}$ when $P : C^*_r(Y) \rightarrow C(Y)$ denotes the conditional expectation. Since $P$ is faithful this shows that $\pi_Y(I) = 0$ and hence that $I = \ker \pi_Y$.

To show that $\ker \pi_Y$ is primitive we may as well show that $C^*_r(Y)$ is a prime $C^*$-algebra. Consider therefore two ideals $I_j \subseteq C^*_r(Y)$, $j = 1, 2$, such that $I_1I_2 = \{0\}$. Then

$$\{y \in Y : f(y) = 0 \ \forall f \in I_1 \cap C(Y)\} \cup \{y \in Y : f(y) = 0 \ \forall f \in I_2 \cap C(Y)\} = Y.$$  

By Corollary 6.6 there is an element $x \in Y$ such that $Y = \text{RO}(x)$. Then $x$ must be in $\{y \in Y : f(y) = 0 \ \forall f \in I_j \cap C_0(Y)\}$ for either $j = 1$ or $j = 2$. Assume without loss of generality that $x \in \{y \in Y : f(y) = 0 \ \forall f \in I_1 \cap C(Y)\}$. The latter set is both closed and $\text{RO}$-invariant so we conclude that

$$Y = \text{RO}(x) = \{y \in Y : f(y) = 0 \ \forall f \in I_1 \cap C(Y)\},$$

i.e $I_1 \cap C(Y) = \{0\}$. As above we conclude from this that $I_1 = \{0\}$, thanks to Lemma 2.15 of [Th1].

Let $Y \in \mathcal{M}_{\text{ex}}$ and let $y \in Y$ be an isolated point which is either periodic or critical. By Proposition 4.4 of [Th2] the isotropy group $I_{sy}$ is abelian and in fact either $\mathbb{Z}$, a non-zero subgroup of $\mathbb{Q}/\mathbb{Z}$ or isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_d$ for some $d \in \mathbb{N}$. Let $\widehat{I_{sy}}$ be its Pontryagin dual group. Since $y$ is isolated in $Y$ every element $\xi \in \widehat{I_{sy}}$ is isolated in $G_Y$ and hence the characteristic function $1_\xi$ of the set $\{\xi\}$ is an element of $C_c(G_Y) \subseteq C^*_r(Y)$. For each $\omega \in \widehat{I_{sy}}$ set

$$I(y, \omega) = \pi_Y^{-1}(I_0(y, \omega)),$$

where $I_0(y, \omega)$ is the ideal in $C^*_r(Y)$ generated by the elements

$$1_{[y,0,\text{id},y]} - \omega(\xi)1_\xi, \ \xi \in \widehat{I_{sy}}.$$  

By adopting the proof of Proposition 4.15 from [CT] in a straightforward way we obtain the following.

**Lemma 6.8.** Let $Y \in \mathcal{M}_{\text{ex}}$ and let $y \in Y$ be an isolated point. Then the map $\widehat{I_{sy}} \ni \omega \mapsto I(y, \omega)$ is a bijection from $\widehat{I_{sy}}$ onto the collection of primitive ideals $I$ in $C^*_r(R)$ with $\rho(I) = Y$. 

In particular, it follows from Lemma 6.8 and Lemma 6.7 that every prime subset of \( \mathbb{C} \) is the co-support of a primitive ideal in \( C^*_r(R) \). By combining Lemma 6.8 with Lemma 6.7 we get the following.

**Lemma 6.9.** For each \( A \in \mathcal{M}_{\text{ex}} \) choose an isolated point \( y_A \) in \( A \) which is either periodic or critical. Then the set of primitive ideals in \( C^*_r(R) \) is the disjoint union

\[
\ker \pi_B : B \in \mathcal{M}\setminus \mathcal{M}_{\text{ex}} \cup \bigcup_{A \in \mathcal{M}_{\text{ex}}} \left\{ I(y_A, \omega) : \omega \in \mathcal{I}_{y_A} \right\}.
\]

**Lemma 6.10.** Let \( A \in \mathcal{M}_{\text{ex}} \). There is either an exposed point \( x \in E_R \) such that \( A = \text{RO}(x) \), or a critical or periodic point \( x \in F_R \setminus E_R \) such that \( A = \text{RO}(x) \cup J_F \).

**Proof.** Let \( x \) be periodic or critical point such that \( x \) is isolated in \( A \) and \( A = \text{RO}(x) \). If \( x \in J_R \) it follows from Lemma 4.5 that \( \text{RO}(x) \) is finite since \( J_R \) has no isolated points, i.e. \( x \) is exposed. Assume \( x \in F_R \). It follows from Lemma 5.2 and Lemma 5.4 that \( \text{RO}(x) \setminus \text{RO}(x) \subseteq J_R \). Since \( \text{RO}(x) \setminus \text{RO}(x) \) is closed, \( \text{RO} \)-invariant and has no isolated points, it follows then from Lemma 4.5 that \( \text{RO}(x) \setminus \text{RO}(x) = \emptyset \) or \( \text{RO}(x) \setminus \text{RO}(x) = J_R \). In the first case \( x \) is exposed and in the second we have that \( A = \text{RO}(x) \cup J_F \).

**Lemma 6.11.** Let \( A \in \mathcal{M}\setminus \mathcal{M}_{\text{ex}} \). Then either \( A = J_R \) or \( A = \text{RO}(x) \) for some \( x \in F_R \setminus \mathcal{I} \). In the last case, \( J_R \subseteq A \).

**Proof.** Let \( x \in A \) such that \( \text{RO}(x) = A \). If \( x \in J_R \) it follows from Lemma 4.5 that \( A = J_R \) or \( A \) is finite. In the last case \( A = \text{RO}(x) \) is an exposed \( \text{RO} \)-orbit which must contain either a periodic point or a critical point, cf. Section 4.2. Since this is impossible when \( A \notin \mathcal{M}_{\text{ex}} \) we must have that \( A = J_R \).

Assume that \( x \in F_R \). If \( A \) contains an isolated point \( y \) it follows form Lemma 6.3 that \( A = \text{RO}(y) \). Note that \( y \in F_R \) because \( y \in \text{RO}(x) \) and \( F_R \) is totally \( R \)-invariant. It follows that \( y \notin \mathcal{I} \) since \( A \notin \mathcal{M}_{\text{ex}} \). To prove that \( J_R \subseteq A \) note that \( R^n(y) \in \text{RO}(y) \) for all \( n \in \mathbb{N} \) since \( y \notin \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit}) \). Thus \( y \) can not be pre-periodic since this would contradict that \( A \notin \mathcal{M}_{\text{ex}} \). Now an argument from the proof of Lemma 7.3 in [1] shows that there is an \( n \in \mathbb{N} \) such that the backward orbit of \( R^n(y) \) contains no points. Then the backward orbit of \( R^n(y) \) is contained in \( \text{RO}(y) \), and since \( R^n(y) \) is not exceptional, it follows therefore from Theorem 4.2.5 in [13] that \( J_R \subseteq \text{RO}(x) \).

If \( A \) has no isolated points it follows from Lemma 6.5 that there is a point \( y \in A \setminus \bigcup_{j=0}^{\infty} R^{-j}(\text{Crit}) \) such that \( A = \text{RO}(y) \). Note that \( y \) can not be pre-periodic because \( A \notin \mathcal{M}_{\text{ex}} \). Since the asymptotic valency of \( y \) is 1 it follows that \( y \notin \mathcal{I} \), i.e. \( y \in F_R \setminus \mathcal{I} \). As above it follows that there is an \( n \in \mathbb{N} \) such that the backward orbit of \( R^n(y) \) is in \( \text{RO}(y) \) and it follows again that \( J_R \subseteq A \).

We can now show that the primitive ideal space of \( C^*_r(R) \) is not Hausdorff, or even \( T_0 \), in the hull-kernel topology unless \( J_R = \mathbb{C} \) and there are no exposed points. Indeed, if \( J_R \neq \mathbb{C} \) it follows from Lemma 4.3 that \( F_R \) contains a point \( y \) which is neither pre-critical nor pre-periodic or exposed. Then Lemma 6.10 combined with Lemma 6.3 and Lemma 5.2 shows that \( A = \text{RO}(y) \in \mathcal{M}\setminus \mathcal{M}_{\text{ex}} \). Furthermore, \( J_R \subseteq A \) by Lemma 6.11. When \( J_R = \mathbb{C} \) and there is an exposed point, its restricted orbit will be an element \( B \in \mathcal{M}_{\text{ex}} \) such that \( B \subseteq J_R \). In the first case it follows from Lemma...
that \( \ker \pi_A \subseteq \ker \pi_{JR} \) so that \( \ker \pi_{JR} \) is in the closure of \( \{ \ker \pi_A \} \) with respect to the hull-kernel topology. In the second case \( \{0\} = \ker \pi_{JR} \subseteq \ker \pi_B \subseteq I(y_B, \omega) \) for any \( y_B \in B \) and any \( \omega \in \hat{K}_{y_B} \), and then \( I(y_B, \omega) \) is a primitive ideal in the closure of \( \{ \ker \pi_{JR} \} \). In both cases we conclude that the primitive ideal spectrum is not \( T_0 \). Note that when \( JR = \overline{C} \) and there are no exposed points, \( C^*_r(R) \) is simple by Proposition \( 4.10 \) and the primitive ideal spectrum reduces to one point.

6.2. The primitive quotients. It follows from Lemma \( 6.10 \) and Lemma \( 6.11 \) that we can divide the primitive ideals \( I \) of \( C^*_r(R) \) into four types, according to the nature of their co-supports:

i) \( \rho(I) = JR \),

ii) \( \rho(I) = RO(x) \) for some exposed point \( x \),

iii) \( \rho(I) = RO(x) \sqcup JR \) for some \( x \in I \cap FR \setminus ER \), and

iv) \( \rho(I) = RO(x) \) for some \( x \in FR \setminus I \).

When \( \rho(I) = JR \) the quotient \( C^*_r(R)/I \) is \( C^*_r(JR) \) whose structure was elucidated in Section \( \mathbf{4} \). When \( \rho(I) = RO(x) \) for some exposed point it follows from \( (6.1) \), Lemma \( 4.11 \) and Corollary \( 4.7 \) that \( C^*_r(R)/I \simeq M_n(\mathbb{C}) \) for some \( n \leq 4 \). In case iii) it follows first from Lemma \( 3.2 \) and Lemma \( 4.11 \) that there is an extension

\[
0 \rightarrow C^*(Is_x) \otimes \mathbb{K} \rightarrow C^*_r(\rho(I)) \rightarrow C^*_r(JR) \rightarrow 0
\]

and then from \( (6.1) \) and Lemma \( 6.8 \) that there is an extension

\[
0 \rightarrow \mathbb{K} \rightarrow C^*_r(R)/I \rightarrow C^*_r(JR) \rightarrow 0.
\]

It remains to describe the primitive quotient \( C^*_r(R)/I \) in case iv). The result depends very much on which stable region the point \( x \in FR \setminus I \) which generates \( \rho(I) \) comes from. We consider the different possibilities in the following subsections.

6.2.1. The super-attractive and attractive stable regions. Assume \( x \) is contained in a super-attracting stable region \( \Omega \). It follows from Lemma \( (6.1) \) that \( C^*_r(R)/I \simeq C^*_r(\rho(I)) \). Since \( I \cap C^*_r(\Omega \setminus I) \) is a primitive ideal in \( C^*_r(\Omega \setminus I) \) it follows that

\[
C^*_r(\rho(I) \cap \Omega \setminus I) = C^*_r(\Omega \setminus I)/I
\]

is a primitive quotient of \( C^*_r(\Omega \setminus I) \) and hence isomorphic to the stabilised Bunce-Deddens algebra \( \mathbb{K} \otimes BD(d^\infty) \) by Proposition \( 5.9 \). When we apply the method from Section \( 5.6 \) to \( C^*_r(\rho(I)) \) rather than \( C^*_r(R) \), we obtain therefore the following commuting diagram with exact rows and columns because there are no periodic
non-critical orbits.

$$\begin{array}{ccc}
0 & \longrightarrow & C^*_r (\rho(I) \cap F_R) \\
& \downarrow & \\
0 & \longrightarrow & C^*_r (J_R) \\
0 & \longrightarrow & A \\
& \downarrow & \\
0 & \longrightarrow & C^*_r (J_R \cup (I_c \cap \rho(I))) \\
& \downarrow & \\
0 & \longrightarrow & 0 \\
\end{array}$$

Here

$$A = C^*_r (I_c \cap \rho(I)) = (\oplus_{i=1}^{n} C(K) \otimes K_{c_i}) \oplus (\oplus_{i=n+1}^{n+m} C_v \otimes K_{c_i})$$

where $c_1, c_2, \ldots, c_{n+m}$ are critical points in $\Omega$ such that $\rho(I) \cap I_c = \sqcup_{i=1}^{n+m} \text{RO}(c_i)$, and $c_1, \ldots, c_n$ are pre-periodic while $c_{n+1}, \ldots, c_{n+m}$ are not. As usual $v_i$ is the asymptotic valency of $c_i$ and $K$ is the Cantor set.

When $\Omega$ is attractive with periodic point $p$ we get by the same reasoning the diagram

$$\begin{array}{ccc}
0 & \longrightarrow & \mathbb{K} \\
& \downarrow & \\
0 & \longrightarrow & C^*_r (\text{RO}(p)) \\
& \downarrow & \\
0 & \longrightarrow & C^*_r (J_R \cup \text{RO}(p)) \\
0 & \longrightarrow & A \\
& \downarrow & \\
0 & \longrightarrow & C^*_r (J_R) \\
\end{array}$$

Here

$$A = C^*_r (I_c \cap \rho(I)) = (\oplus_{i=1}^{n} C(T) \otimes C_v \otimes K_{c_i}) \oplus (\oplus_{i=n+1}^{n+m} C_v \otimes K_{c_i})$$

where $c_1, c_2, \ldots, c_{n+m}$ are critical points in $\Omega$ such that $\rho(I) \cap I_c = \sqcup_{i=1}^{n+m} \text{RO}(c_i)$, and $c_1, \ldots, c_n$ are pre-periodic while $c_{n+1}, \ldots, c_{n+m}$ are not.

6.2.2. Parabolic stable regions. Assume now that $x$ is contained in a parabolic stable region $\Omega$. In this case there is no periodic point in $\Omega$ and we get the following
diagram.

\[
\begin{array}{cccc}
0 & 0 & & \\
& \uparrow & & \\
\mathbb{K} & = & \mathbb{K} & \\
& \downarrow & & \\
0 & \longrightarrow & C^*_{\tau}(\rho(I) \cap F_R) & \longrightarrow \mathbb{C}^* \left( \left( J_R \right) / I \right) & \longrightarrow C^*_{\tau}(J_R) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{i=1}^N C^*_{c_i} \otimes \mathbb{K}_{c_i} & \longrightarrow C^*_{\tau}(J_R \cup (\mathcal{I}_C \cap \rho(I))) & \longrightarrow C^*_{\tau}(J_R) & \longrightarrow 0 \\
& 0 & 0 & & \\
\end{array}
\]

(6.3)

where \( c_i, i = 1, 2, \ldots, N \), are critical points in \( \Omega \) such that \( \mathcal{I}_C \cap \rho(I) = \bigsqcup_{i=1}^N \text{RO}(c_i) \).

6.2.3. Stable regions of Siegel or Herman type. Assume now that \( x \) is contained in a stable region \( \Omega \) of Siegel type. In this case there is a periodic point in \( \Omega \) with a non-critical orbit, but since \( x \notin \mathcal{I} \) this orbit is not in \( \rho(I) \). Therefore the picture is the same as in the case of a Herman type stable region and we get in both cases a diagram similar to the parabolic case. The only difference is that the algebra \( \mathbb{K} \) in (6.3) is exchanged with the stabilized irrational rotation algebra \( A_{2\pi\theta} \) corresponding to the rotation by the angle \( 2\pi\theta \) in the core of \( \Omega \).

This completes the list of primitive quotients of \( C^*_{\tau}(R) \). Note that only very few of the primitive quotients are simple. In fact, the simple quotients of \( C^*_{\tau}(R) \) are all matrix algebras \( M_n(\mathbb{C}) \) with \( n \leq 4 \), together with \( C^*_{\tau}(J_R) \) when there are no exposed point in \( J_R \).

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E-mail address: matkt@imf.au.dk

INSTITUT FOR MATHEMATISKE FAG, NY MUNKEGADE, 8000 AARHUS C, DENMARK