Viable hairy black hole solution in Bopp–Podolsky electrodynamics

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Following a recent approach in which the gravitational field equations in curved spacetimes were presented in the Bopp–Podolsky electrodynamics, we obtained a hairy and approximate spherically symmetric black hole solution in this context. The calculations were carried out up to the linear approximation in both the spacetime geometry and the radial electric field. The solution is a hairy black hole because it is not in agreement with the no-hair conjecture, that is to say, the spacetime metric depends on an extra parameter that comes from the Bopp–Podolsky Lagrangian. Moreover, the black hole presented here is viable when its shadow is compared to the Sagittarius A* shadow, recently revealed by the Event Horizon Telescope Collaboration.

Keywords: Black Holes, Electrodynamics, No-Hair Conjecture, Black Hole Shadow

I. INTRODUCTION

The Bopp–Podolsky electrodynamics [1, 2] is an extension of Maxwell’s theory in which a new second-order derivative of the gauge field is present in the Lagrangian, leading then to high-order field equations with fourth-order terms. As is well known, high-order theories in this context contain instabilities (like ghost instabilities). However, the Bopp–Podolsky electrodynamics is able to avoid this situation by means of the concept of Lagrange anchor [3]. Among the reasons to work on the Bopp–Podolsky model is its capability to preserve the linearity of the field equations being the only second-order gauge theory for the $U(1)$ group that accomplishes that [3]. Moreover, in 3+1 dimensions (flat spacetime), the Bopp–Podolsky term in the Lagrangian can also be conceived of as an effective term coming from quantum corrections to the photon action in the low-energy regime [3].

Specific studies on the Bopp–Podolsky context have been done focusing, for example, on self-interaction [4, 7], dark energy [8], massive photons in cosmology [9] and spherical symmetric black holes [10]. In this work, we built an approximate black hole solution with spherical symmetry extending the results of Ref. [10]. But contrary to the mentioned article [10], our approximate black hole solution also violates both the NEC (and the weak energy (WEC) conditions outside the null horizon. Moreover, the spacetime metric brings out the coupling constant $b$, so the black hole solution is not in agreement with the no-hair conjecture/theorem.

The no-hair conjecture/theorem [11, 12] states that black holes are fully described by just three parameters: mass, charge and spin. In this sense, the spacetime metric presented here describes a hairy spacetime, that is to say, the spacetime geometry presents a new parameter other than mass, charge and spin. In this case, a useful task is indicating constraints on the new parameter $b$ unfolded by our approximate metric.

In this regard, we compare the shadow angular diameter or radius of the spacetime metric obtained here with the recent image of the Sagittarius A* (Sgr A*) shadow released by the Event Horizon Telescope (EHT) [14, 15]. Black hole shadows have been a seminal research area since the very first shadow revealed by the EHT, the M87* shadow [16, 17]. Two shadow parameters for constraining general relativity and modified theories of gravity have been adopted recently in the literature, namely the shadow angular diameter $d_{sh}$ [18–23] and the shadow deviation from circularity $\Delta C$ [24–30], which is nonzero for rotating black holes and is zero for nonrotating ones, that is to say, the shadow of spherical black holes is a perfect circle. In particular, for the Sgr A* black hole, due to the more precise measurements of the black hole distance and mass, the EHT [15] used the shadow angular diameter in order to provide a new parameter to be compared to black hole metric candidates, whether in the general relativity context or in modified theories of gravity. From the shadow angular diameter, the new parameter called deviation from the Schwarzschild metric $\delta$ arises from the black hole mass-to-distance ratio and has been adopted, for example, in the pretty complete review of alternative theories of gravity and spherically symmetric black hole metrics carried out by Vagnozzi et al. [20]. According to the authors of the mentioned review, the deviation from the Schwarzschild metric is, for
and conditions, the Bopp–Podolsky Lagrangian in curved spacetimes is described by two new independent equations: (i) invariant under Lorentz transformations, (ii) gauge invariant under the geometrical black hole in the Bopp–Podolsky electrodynamics in curved spacetimes, namely (iii) quadratic in the gauge field and its (iv) dependent on the gauge field and its first two derivatives. As will see, the two new invariant terms in the Lagrangian are either minimally or nonminimally coupled to gravity. Assuming then this four conditions, the Bopp–Podolsky Lagrangian in curved spacetimes is written as

\[ \mathcal{L}_m = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + \frac{(a^2 + 2b^2)}{2} \nabla_\beta F^{\alpha\beta} \nabla_\gamma F_{\alpha\gamma} + b^2 (R_{\alpha\beta} F^{\sigma\alpha} F_{\alpha\beta} + R_{\alpha\beta} F^{\gamma\sigma} F_{\alpha\gamma}) , \] (1)

where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) is the field strength, \( \nabla_\mu \) is the covariant derivative, and \( a \) and \( b \) are coupling constants, \( R_{\mu\nu} \) is the Ricci tensor, and \( R_{\alpha\beta\gamma\delta} \) is the Riemann tensor. By using the so-called Einstein–Hilbert–Podolsky action

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(-R + 4\mathcal{L}_m\right) , \] (2)

in which \( g \) is the metric determinant, \( R \) is the Ricci scalar, and \( \mathcal{L}_m \) is given by Eq. (1), we are able to obtain the gravitational field equations and the equations for the gauge field, also called Podolsky equations. In this regard, by varying the action (2) with respect to the metric field \( g_{\mu\nu} \), one has the corresponding gravitational field equations:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \left(T^M_{\mu\nu} + T^a_{\mu\nu} + T^b_{\mu\nu}\right) , \] (3)

where the components of the energy-momentum tensor are given by

\[ T^M_{\mu\nu} = \frac{1}{4\pi} \left[F_{\mu\sigma} F_{\nu}^{\sigma} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right] , \] (4)

\[ T^a_{\mu\nu} = \frac{a^2}{4\pi} \left[2g_{\mu\nu} F^{\alpha\gamma} \nabla_\alpha K_\beta + \frac{g_{\mu\nu}}{2} K^{\alpha\beta} K_\beta \right. \]

\[ + 2F_{(\mu} \nabla_{\nu)} K_\alpha - 2F_{(\mu} \nabla_\alpha K_\nu) - K_\mu K_\nu \right] , \] (5)

\[ T^b_{\mu\nu} = \frac{b^2}{2\pi} \left[\frac{1}{4} g_{\mu\nu} \nabla_\gamma F^{\alpha\gamma} F_\alpha \gamma + \frac{\nabla_\gamma F_\alpha \gamma}{\mu} (\nabla_\beta F_\alpha \gamma) \right. \]

\[ + F_\gamma (\nabla_\beta \nabla_\gamma F_{\alpha\gamma}) - \nabla_\beta \left( F_\gamma \beta \nabla_\gamma (\nabla_\mu F_\gamma) \right) \big] . \] (6)

The notation (...) indicates symmetrization with respect to the indexes inside the brackets.

On the other hand, by varying the action (2) with respect to the field \( A_\mu \), we have the Podolsky equations for curved spacetimes, namely

\[ \nabla_\nu \left[F^{\mu\nu} - (a^2 + 2b^2) H^{\mu\nu} + 2b^2 S^{\mu\nu}\right] = 0 , \] (7)

where

\[ H^{\mu\nu} \equiv \nabla_\mu K^{\nu} - \nabla_\nu K^\mu , \] (8)

\[ S^{\mu\nu} \equiv F^{\mu\alpha} R_\alpha^{\nu} - F^{\nu\sigma} R_\sigma^{\mu} + 2R_{\sigma^{\mu} \nu} F^{\beta\sigma} , \] (9)

with \( K^\mu \equiv \nabla_\gamma F^{\gamma\mu} \). It is worth pointing out that any valid solution in the Bopp–Podolsky context must be solution of either field equations, whether the gravitational field equations (3) or the Podolsky equations (7).

In Ref. [10], by using the Bekenstein method, even without an explicit form for the black hole metric, it was shown that for \( b = 0 \) the exterior solution of a spherically symmetric black hole is the Reissner–Nordström solution, i.e., the no-hair theorem is fully satisfied. However, the same authors demonstrated that hairy black hole solutions would be possible only if both \( b \neq 0 \) and \( \frac{d^2 a}{dr^2} \geq 0 \), where \( r \) is the radial coordinate. In the next section, using a perturbative approach, we then show explicitly that the \( b \neq 0 \) case generates a hairy black hole solution.

From the energy conservation perspective, the total energy-momentum tensor is conserved, i.e., given \( T_{\mu\nu} = T^M_{\mu\nu} + T^a_{\mu\nu} + T^b_{\mu\nu} \), one has

\[ \nabla_\nu T^{\mu\nu} = 0 , \] (10)

with the aid of the constraint provided by the Podolsky equations (7). In particular, for the spherically symmetric Ansatz that we used in this study (independently of the metric terms) and a radial electric field, every component of Eq. (10) is identically zero, regardless the \( \mu = 1 \) component. Such a component becomes zero by using the \( \mu = 1 \) component of the Podolsky equations.
III. AN APPROXIMATE BLACK HOLE SOLUTION

A. The spacetime metric

When dealing with higher-order theories in which the coupling parameters are very small, the correct perturbative scheme is given by the singular perturbation theory \[11, 12\]. The influence of higher derivatives is important inside a boundary layer, whose size is proportional to the coupling constants coming from the higher-order derivatives, while outside that boundary the solution is asymptotically the nonperturbed solution, and the regular approach for obtaining approximate solutions is enough. In our case, the higher-order derivatives of the electric field are proportional to the parameters or coupling constants \(a\) and \(b\). Since we are interested in the exterior solution, we are dealing with scale distances such that

\[
a, b \ll r_+ \leq r < \infty,
\]

where \(r_+\) is the radius of the external horizon, the usual event horizon radius. This implies that we are able to use only the regular perturbation theory throughout this article.

Having said that, we present here an approximate solution with spherical symmetry in the context indicated in Section [11] by using the regular perturbation theory. Due to the nonlinearity of the field equations (as to the metric components), approximate solutions are an alternative alongside the numerical approach. From the spherically symmetric Ansatz in the \((t, r, \theta, \phi)\) coordinates, i.e.,

\[
ds^2 = A(r)dt^2 - \frac{dr^2}{B(r)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

an approximate solution will be available if we assume

\[
A(r) = A_0(r) + \epsilon A_1(r),
\]

\[
B(r) = B_0(r) + \epsilon B_1(r),
\]

\[
E(r) = E_0(r) + \epsilon E_1(r).
\]

As we are looking for a small deviation from the Reissner–Nordström geometry, then we assume the following input for the metric and the radial electric field, i.e.,

\[
A_0(r) = B_0(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad \text{and} \quad E_0(r) = \frac{Q}{r},
\]

where \(\epsilon = \epsilon(a, b)\) is a small quantity, and we take the perturbations just to the linear order.

Another important input of the model is the field strength \(F_{\mu\nu}\). From the small deviation of the Reissner–Nordström black hole in which we are interested, we assume that

\[
F_{\mu\nu} = E(r) \left[\delta_{\mu}^1 \delta_{\nu}^0 - \delta_{\mu}^0 \delta_{\nu}^1\right].
\]

Thus, the system of equations provided by the gravitational field equations depends on just three functions: \(A(r), B(r)\) and \(E(r)\) or, in the perturbative approach that we adopt, \(A_1(r), B_1(r)\) and \(E_1(r)\).

The field equations \[13\], by using the approximation \[13, 15\] and the field strength \[17\], give us the following system of differential equations:

\[
B'(r) + \left(\frac{Q^2 + F(r)}{r F(r)}\right) B_1(r) - \frac{Q^2}{r F(r)} A_1(r) + \frac{2Q}{r} E_1(r)
\]

\[
- \frac{4b^2 Q^2}{cr^4} \left(7Q^2 + (3r - 10M) r\right) = 0,
\]

\[
A'(r) - \left(\frac{r^2 - F(r)}{r F(r)}\right) A_1(r) + \frac{r}{F(r)} B_1(r) + \frac{2Q}{r} E_1(r)
\]

\[
- \frac{4b^2 Q^2}{cr^4} \left(Q^2 - (3r - 2M) r\right) = 0,
\]

\[
A''(r) + \left(\frac{Q^2 - M r + F(r)}{r F(r)}\right) A_1'(r) + \left(\frac{r - M}{F(r)}\right) B_1'(r)
\]

\[
- \frac{4Q}{r^2} E_1(r) + \left(\frac{2(r - M) (M r - Q^2)}{r F(r)^2}\right) (A_1(r) - B_1(r))
\]

\[
- \frac{16b^2 Q^2}{cr^8} \left(4Q^2 + (3r - 7M) r\right) = 0,
\]

with the operator ' playing the role of the derivative with respect to the radial coordinate, and

\[
F(r) = Q^2 + (r - 2M) r.
\]

An interesting point here is that by assuming \(E_0(r)\), given by \[10\], all terms with \(a\) disappear in the field equations, because all terms coming from the invariant \(\nabla_\beta F_{\alpha\beta} \nabla_\gamma F_{\alpha\gamma}\) are identically zero. On the other hand, only the nonminimal terms will contribute with the perturbed equations, namely terms with \(b\).

As we can see, we have three independent differential equations with three unknown functions: \(A_1(r), B_1(r)\) and \(E_1(r)\). In this regard, the system indicated in Eqs. [13, 20] is solvable, and by assuming that the sought-after geometry is a small deviation from the Reissner–Nordström spacetime, meaning that our solution is asymptotically flat, that is to say, the limits \(\lim_{r \to \infty} A_1(r) = \lim_{r \to \infty} B_1(r) = \lim_{r \to \infty} E_1(r) = 0\) are necessarily true, one has the following solution for the system of differential equations:

\[
A_1(r) = \frac{2b^2 Q^2}{cr^4} \left(1 - \frac{3M}{r} + \frac{9Q^2}{5r^2}\right),
\]

\[
B_1(r) = -b^2 Q^2 \left(2 - \frac{3M}{r} + \frac{6Q^2}{5r^2}\right),
\]

\[
E_1(r) = -\frac{b^2 Q^2}{cr^4} \left(8M - \frac{11Q^2}{r^2}\right).
\]

It is worth pointing out that Eqs. [22-24] are exact solutions of the system [13-20], regardless the value of the radial coordinate \(r\). Therefore, with Eqs. [22-23] substituted into the linear approximation [13-14], we arrive then in the fully calculated geometry, which is
written as

\[ ds^2 = \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{2b^2Q^2}{r^4} - \frac{6Mb^2Q^2}{r^5} \right) dt^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{4b^2Q^2}{r^4} + \frac{6Mb^2Q^2}{r^5} - \frac{12b^2Q^4}{5r^6} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \] (25)

As is clear from the metric (25), by making \( b = 0 \) the Reissner–Nordström geometry is restored. And contrary to the well-known black hole solutions with spherical symmetry in the general relativity context, the black hole obtained here has \( A(r) \neq B(r) \), which is something usual in modified theories of gravity.

The radial electric field, from Eqs. (15) and (24), is now written as

\[ E(r) = \frac{Q}{r^2} \left( 1 - \frac{8Mb^2}{r^3} + \frac{11b^2Q^2}{r^4} \right). \] (26)

And, as expected, turning off \( b \) the usual electric field is obtained. For large values of \( r \), the influence of \( b \) on the electric field is very tiny. In this range of the radial coordinate, the electric field is asymptotically the usual electric field for a point charge: \( E(r) \to E_0(r) \). It is worth pointing out the weirdness of the new term that contains the mass parameter in Eq. (26), for it seems like the radial electric field was truly coupled to the spacetime geometry.

As we said, the metric (25) is an approximate solution with spherical symmetry in the Bopp–Podolsky electrodynamics. We will see that the solution (25) captures the same features pointed out in the analysis carried out by Cuzinatto et al. [10], whose study is made from the Bekenstein method, even without a final expression for the spacetime metric. As we can read from Eq. (25), the spacetime geometry is asymptotically flat, that is, the flatness condition

\[ \lim_{r \to \infty} A(r) = \lim_{r \to \infty} B(r) = 1 \] (27)

is achieved. Also it is worth emphasizing that the metric (25) is solution of the Podolsky equation (11) in its approximate version, in which high-order terms of \( \epsilon \) were also ruled out.

B. The spacetime structure

By spacetime structure we mean the localization of horizons and, of course, type of infinities. As we saw in Eq. (27), the proposed solution is, like the Reissner–Nordström spacetime, asymptotically flat, i.e., it has the same infinite configuration or type of infinities of that well-known solution of Einstein’s equations. But, supposedly, the metric (25) presents a larger number of horizons. The localization of horizons in spacetimes like (12) is given by the equation \( g^{rr} = B(r) = 0 \). Typically, for \( M > Q \), the function \( B(r) \) would have at most three positive roots, which would be horizons: two inner horizons (being the larger one the usual inner horizon of the Reissner–Nordström spacetime, \( r_+ \)) and one outer horizon, which is indicated as \( r_+ \) and plays the role of the event horizon. However, as we said, due to the perturbative approach, we consider here just the interval \( r \geq r_+ \).

Thus, inner horizons are not subject of investigation in the present study.

Fig. 1 shows the dependence of the event horizon on the parameters charge \( Q \) and coupling constant \( b \). As we can read from Fig. 1 for \( M \) and \( Q \) fixed, the larger the value of \( b \), the larger the value of the event horizon radius (this effect is more evident for large values of \( Q \)). With an event horizon, the metric (25) describes then a black hole. And the presence of the parameter \( b \) in the spacetime metric is translated into a hairy black hole. The two conditions for a hairy and spherically symmetric black hole in Bopp–Podolsky electrodynamics, mentioned in Sec. II, namely \( b \neq 0 \) and \( g_{00} \geq 0 \), are fulfilled.

C. Energy conditions

As mentioned in Cuzinatto et al. [10], where an analysis in the context adopted here was made without a final form for the spacetime metric, a black hole solution with spherical symmetry in the Bopp–Podolsky electrodynamics is able to violate energy conditions. In particu-
lar, both the NEC and the WEC are violated outside the event horizon. From the total energy-momentum tensor given by $T_{\mu\nu} = T^M_{\mu\nu} + T^a_{\mu\nu} + T^b_{\mu\nu}$, one has

$$T^a_{\nu} = \begin{pmatrix} \rho (r) & -p_1 (r) & -p_2 (r) & -p_3 (r) \end{pmatrix}. \quad (28)$$

For an energy-momentum tensor written as the diagonal matrix like Eq. (28), the WEC states that $\rho (r) > 0$ and, at the same time, the condition $\rho + p_i \geq 0$ (for every $i = 1, 2, 3$) should also be satisfied. That is not the case for the metric (25). As to the energy density, for the metric (25) in the Bopp–Podolsky electrodynamics, we have

$$\rho (r) = \frac{Q^2}{8\pi r^4} \left( 1 - \frac{12b^2}{r^2} A_0 (r) \right) > 0. \quad (29)$$

From the fact that $A_0 (r) > 0$ for $r > r_+$, being the second term on the right side of Eq. (29) $\ll 1$, we can read that the energy density $\rho (r)$ is positive definite surely for $r \geq r_+$, which is the range of validity of the perturbative approach adopted by us.

On the other hand, as the condition $\rho + p_i \geq 0$ (for every $i = 1, 2, 3$) is a true statement of either energy conditions, thus both the NEC and WEC are not satisfied. In particular, for $i = 1$, this condition reads

$$\rho (r) + p_1 (r) = - \frac{3b^2 Q^2}{2r^6} A_0 (r), \quad (30)$$

which vanishes for $b = 0$ and is negative outside the event horizon, that is to say, the black hole obtained here is not in agreement with the NEC and WEC. Therefore, as a possible interpretation, the coupling constant or the model introduced in Section II could be translated into an exotic matter that does not satisfy energy conditions outside the event horizon.

IV. THE BLACK HOLE SHADOW

A. Radius of the black hole shadow

As the black hole geometry (25) has spherical symmetry, the shadow silhouette is circular. In general, shadows are deformed by the black hole spin, which is not our case here. According to EHT [16, 17], the M87* shadow presents a small deviation from the perfect circle, a deviation from circularity $\Delta C \lesssim 10\%$, indicating then that M87* is spinning. But due to the less precise data from the Sgr A* image (because of intense variability of its surroundings and the sparse coverage of the observations in 2017 [12]), the most precise parameter to be studied is the shadow angular size or angular diameter $d_{\text{sh}}$.

Black hole shadows are generated by the so-called photon sphere, which is a region delimited by unstable photon orbits. Part of photons traveling in these orbits falls into the black hole, and part of them could escape to an observer like us. For supermassive black holes like M87* and Sgr A*, the black hole shadow, a brightness depression, is surrounded by a bright and thick ring of emission. According to EHT [15], under certain circumstances, the size of the emission ring could be an approximate value for the black hole shadow.

The shadow silhouette is given by the photon sphere, and, according to Perlick and Tsupko [33], from an interesting geometric argumentation, metrics like (12) present a photon sphere whose radius is given by the following equation:

$$A(r_{\text{ph}}) - \frac{1}{2} r_{\text{ph}}^2 A'(r_{\text{ph}}) = 0, \quad (31)$$

where $r_{\text{ph}}$ is the photon sphere radius. Following the mentioned article, then the shadow radius $r_{\text{sh}}$ reads

$$r_{\text{sh}} = \frac{r_{\text{ph}}}{\sqrt{A(r_{\text{ph}})}}. \quad (32)$$

As expected, the shadow radius is not equal to the photon sphere radius. The gravitational lensing increases the shadow radius when it is observed from a distance. Due to the high-order polynomial expression for calculating $r_{\text{ph}}$, we will omit the analytic result here. But for our purpose, the main information comes from the shadow radius, where its dependence on the coupling constant is shown in Fig. 2. As we can read from that figure, the coupling constant decreases the shadow radius. This could be a point in favor of our metric, for the EHT image disfavors alternative black holes that generate larger shadows than the Schwarzschild black hole. However, as
the EHT points out [13], the most favorite candidate for the Sgr A* image is the Kerr metric, that is, the observed image size is within $\sim 10\%$ of the Kerr geometry predictions.

It is interesting to point out that the parameter $b$, which generates a hairy black hole, works differently as to the event horizon size and the shadow size. As we said earlier, when compared to the Reissner–Nordström black hole, such a parameter increases the event horizon radius (see Fig. 1), but it decreases the shadow radius.

### B. Constraining the black hole parameters using the Sgr A* shadow

As we said in Introduction, in order to compare the geometry [29] to the Sgr A* black hole, the crucial parameter here is the deviation from the Schwarzschild metric $\delta$. As the Sgr A* spin is still unknown, geometries with spherical symmetry are not ruled out for describing our galactic black hole. A recent estimation for the dimensionless spin parameter for Sgr A* is $a_* \lesssim 0.1$ [32, 37], where $0 < a_* < 1$ for spinning black holes, that is to say, arguably our central black hole is spinning slowly. Thus, describing Sgr A* approximately as a static black hole (like the Schwarzschild black hole or a deviation from the Schwarzschild geometry) is out of question. Therefore, according to EHT [13], the deviation from the Schwarzschild metric is conceived of as

$$\delta = \frac{d_{sh}}{d_{Sch}} - 1,$$

(33)

where $d_{sh} = 48.7 \pm 7.0 \mu\text{as}$ is the shadow angular diameter of Sgr A* (measured by the EHT), $d_{Sch} = 6\sqrt{3}\theta_g$ is the angular diameter for the Schwarzschild black hole in the small angle approximation, and $\theta_g$ is called angular gravitational radius, which is given by $\theta_g = GM/Dc^2$ where $M$ is the black hole mass, and $D$ is its distance from us ($G$ and $c$ are the gravitational constant and the speed of light in vacuum, which were set to 1 in the geometrized unit system adopted in this article). The most precise values for mass and distance of Sgr A*, even adopted by the EHT, were obtained by the Keck observatory and VLTI (Very Large Telescope Interferometer) (see Table 1). Both values provide the mass-to-distance ratio of our central black hole. With those parameters for Sgr A*, the deviation given by Eq. (33) is $\delta = -0.04^{+0.02}_{-0.01}$ for the Keck observatory, and $\delta = -0.08^{+0.03}_{-0.02}$ for the VLTI.

Following Vagnozzi et al. [20], the idea is calculating a valid range for the shadow radius of a black hole like our spacetime [29] using the deviation $\delta$. Spherically symmetric solutions—like the Reissner–Nordström black hole, regular black holes, naked singularities or wormholes and black holes in several contexts like $f(R)$ gravity, brane world, Horndeski model, bumblebee gravity, loop quantum gravity—were constrained by the authors. For that purpose, Vagnozzi et al. [20] obtained a combined value of $\delta$ using the two surveys, Keck and VLTI, considering them independent results. That procedure gives us

$$\delta \simeq -0.060 \pm 0.065,$$

(34)

which leads to the following 1σ constraint for the shadow radius as a deviation from the Schwarzschild metric:

$$3\sqrt{3}(1 - 0.125) \lesssim \frac{r_{sh}}{M} \lesssim 3\sqrt{3}(1 + 0.005),$$

$$4.55M \lesssim r_{sh} \lesssim 5.22M.$$  

(35)

It is worth emphasizing that $r_{sh}$ is the shadow radius, not the angular radius. Having said that, the range above constrains alternative geometries (like ours) allowed by the observations. In this regard, Fig. 2 shows that for small values of the coupling constant $b$ (and charge $Q \lesssim 0.8M$), the geometry built here is viable.

### V. FINAL COMMENTS

In this article, we built an approximate black hole solution with spherical symmetry in Bopp–Podolsky electrodynamics. The final form of the spacetime metric shows a third parameter, alongside the mass and charge parameters, which comes from the action of the model and is conceived of as the nonminimal coupling constant. Therefore, because of this new parameter explicitly written in the metric, the black hole solution presented here is a hairy black hole, that is, it depends on other parameter than mass, charge and spin. When our solution is compared to the Reissner–Nordström black hole, the event horizon radius is larger due to the coupling constant. Also, because of this parameter, the modified radial electric field violates both the NEC and WEC even outside the black hole event horizon.

In the final part of this work, data from the EHT Collaboration and from the Keck and VLTI observatories were adopted in order to compare the black hole obtained here with Sgr A*, the central black hole on the Milky Way galaxy. These data constrain the range of validity of the black hole shadow radius, when Sgr A* is considered as a small deviation from the Schwarzschild black hole. The new parameter in the metric, that which provides a hairy black hole, decreases the radius of the black hole shadow when we compare it to the Reissner–Nordström black hole shadow. But as we pointed out, the metric obtained is viable using the valid range of the shadow radius of Sgr A*.
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