A dynamic scheme for generating number squeezing in Bose-Einstein condensates through nonlinear interactions

Simon A. Haine$^{1,2}$ and Mattias T. Johnsson$^{1,3}$

$^1$Australian Research Council Centre for Excellence for Quantum-Atom Optics
$^2$School of Physical Sciences, University of Queensland, Brisbane, 4072, Australia
$^3$Australian National University, Canberra, 0200, Australia

We develop a scheme to generate number squeezing in a Bose-Einstein condensate by utilizing interference between two hyperfine levels and nonlinear atomic interactions. We describe the scheme using a multimode quantum field model and find agreement with a simple analytic model in certain regimes. We demonstrate that the scheme gives strong squeezing for realistic choices of parameters and atomic species. The number squeezing can result in noise well below the quantum limit, even if the initial noise on the system is classical and much greater than that of a poisson distribution.

Introduction. — The experimental realization of Bose-Einstein condensates (BECs) has allowed the creation of macroscopic quantum systems that are highly controllable, and hence provide an excellent system to test predictions of many body quantum dynamics. The generation of nonclassical states in BECs, such as number squeezed states, would allow for measurements of particle number statistics that differ from classical predictions. However, while the motional state of the atoms is reasonably simple to manipulate, the quantum statistics referring to the number of particles present is difficult to control, with BECs typically produced with a large (5%) shot to shot variation in the number. The generation of nonclassical states in BECs is currently of great interest, as it could potentially enhance the sensitivity of atomic interferometers used to measure electric, magnetic, gravitational fields, accelerations, and atomic interactions. The generation of nonclassical states also provides a method for testing the fidelity of recent quantum state transfer schemes, and an atom laser produced from a number squeezed BEC will have a reduced linewidth.

The generation of nonclassical states via self-interaction in samples of cold atoms has been considered before. The schemes proposed in [8, 9, 10] describe the generation of quadrature squeezing, which requires the use of a well-defined phase reference in order to be observed. A well-defined phase reference is difficult to obtain in atom optical systems, especially in the presence of strong nonlinearities, which are required to produce the quadrature squeezing, thus making the schemes somewhat unrealistic. In addition, both of these schemes assume the BEC is initially in a coherent state, rather than a more realistic statistical mixture of coherent states with random phases. [8, 9, 10] have demonstrated that the atomic nonlinearity can be used to generate number difference squeezing, creating a state with angular momentum projection below the standard quantum limit (spin squeezing). These schemes assume that the total number of particles is initially well defined. Recently, Esteve et al. [2] have directly observed squeezing in the number difference between two adjacent lattice sites.

In this Letter we describe a scheme that allows the creation of absolute number squeezing in a BEC (as opposed to number difference squeezing), which is experimentally realistic, utilizes only the relatively simple experimental technique of Ramsey interferometry, and does not require manipulation of the scattering length via a Feshbach resonance or a coherent phase reference for the atoms. We show that our scheme achieves number squeezing below the quantum limit even if there is initially considerable classical noise on the system, and that this result holds even when we realistically assume that the initial state of the BEC is a statistical mixture of states with random phase.

We consider a BEC with two internal states confined to an optical trap, with all the atoms initially in one state. A short state-dependent coupling is applied, transferring a small fraction of the population to another state. The system is then left to evolve for some time, allowing nonlinear interactions and interference between the two states, before the coupling is applied for a second time, transferring some of the population back to the initial state. Provided the s-wave scattering lengths of the atoms in the different internal states are not all identical, and by choosing appropriate coupling strengths, hold times and trap geometry, it is possible to generate number squeezing. An important difference between our scheme and the one demonstrated by Esteve et al. [2] is that our scheme is based on dynamic interference between the two modes to obtain absolute number squeezing in one of the modes, whereas Esteve et al. obtain their squeezing by adiabatically changing the potential to one where the ground state of the system exhibits number difference squeezing.

Scheme. — Our proposed scheme is based on a Ramsey interference experiment between two hyperfine states of sodium, namely $|F = 1, m_F = +1\rangle \equiv |1\rangle$ and $|F = 2, m_F = 0\rangle \equiv |2\rangle$. The timing for our scheme is outlined in Figure 1. We begin with all the atoms in a BEC in the $|1\rangle$ state in an optical trap. At $t = t_0$, the microwave coupling is turned on for...
a brief duration of time $\Delta t_1 = t_1 - t_0$. During this time, a fraction of the atoms are then transferred to state $|2\rangle$. The coupling is switched off, and the system is left to evolve for an amount of time $t_{\text{hold}} = t_2 - t_1$, before we interfere the two modes with a second microwave coupling pulse for a duration $\Delta t_2 = t_3 - t_2$. We assume that this pulse is phase locked to the first, with an adjustable phase shift $\phi$. This phase shift is an important ingredient, as it allows us to ensure that the two modes interfere in such a way as to produce number squeezing, independently of the phase shift required to produce number squeezing, and thus optimize our squeezing depth. After $t_3$, we separate the two modes with a magnetic field, and count the number of atoms in mode $|2\rangle$ to determine the number statistics. The Hamiltonian for the system is $\hat{H} = \hat{H}_0 + \hat{H}_c(t)$, with

$$\hat{H}_0 = \sum_{j=1,2} \int \bar{\psi}_{j1}(r) H_j \psi_{j1}(r) \, d^3r + \sum_{i,j=1,2} U_{ij} \int \bar{\psi}_{i1}(r) \bar{\psi}_{j1}(r) \psi_{i1}(r) \psi_{j1}(r) \, d^3r, \quad (1)$$

and

$$\hat{H}_c = \int \left( \hbar \Omega(t) e^{i\phi} \bar{\psi}_{j1}(r) \psi_{j1}(r) + \text{h.c.} \right) \, d^3r, \quad (2)$$

where $\psi_{j1}(r)$ represents the annihilation operator for state $|j\rangle$, $H_j = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{opt}}(r) + (j-1)\hbar\delta$ is the single particle Hamiltonian, and $V_{\text{opt}}(r)$ is the optical dipole potential. $\Omega(t)$ represents the microwave coupling field, which is switched on and off to control the coupling between the two hyperfine levels. The phase of this RF field can also be tuned between each pulse. We will assume that the coupling is on resonance, such that $\Omega(t) = \Omega_0 e^{-i\delta t}$, with $\hbar\delta$ the hyperfine splitting between $|1\rangle$ and $|2\rangle$, and $\Omega_0$ is the Rabi frequency.

**Analytic model.** — We first consider a two mode model, which demonstrates how atomic nonlinearities can be used to generate number squeezing. A two-mode model can be derived from Eq. (1) and Eq. (2) by assuming that the atoms remain in the ground motional state of the optical trap. With this assumption, the modified Hamiltonian for the system is $\hat{H} = \hat{H}_0 + \hat{H}_c$, with

$$\hat{H}_0 = \hbar \delta \hat{a}_{12} \hat{a}_{2} + \sum_{i,j=1,2} \hbar \chi_{ij} \hat{a}_{i} \hat{a}_{i} \hat{a}_{j} \hat{a}_{j} \quad (3)$$

$$\hat{H}_c = \hbar \left( \Omega(t) e^{i\phi} \hat{a}_{1} \hat{a}_{1} + \text{h.c.} \right) \quad (4)$$

where $\hat{a}_{1}$ ($\hat{a}_{2}$) annihilates an atom from state $|1\rangle$ ($|2\rangle$), and $\chi_{ij} = \frac{U_{ij}}{2} \int |\psi_{0}(r)|^4 \, d^3r$, where $\psi_{0}(r)$ is the ground state wavefunction of the optical potential.

Beginning with the coupling initially switched off ($\Omega_0 = 0$), and assuming that our initial state is a Poissonian mixture of number states for mode $|1\rangle$, (or, equivalently, a mixture of coherent states with random phase) and vacuum for mode $|2\rangle$ we notice that the evolution is trivial, as the density operator commutes with $\hat{H}_0$. At $t_0$, the coupling is turned on for a duration $\Delta t_1$ coupling a fraction of the atoms $(\sin^2(\Omega_0\Delta t_1))$ into mode $|2\rangle$, creating a relative phase between the two modes. This relative phase is then ‘sheared’ during $t_{\text{hold}}$ by evolution under $\hat{H}_0$ due to the Kerr effect. We note that if $\chi_{11} = \chi_{22} = \chi_{12}$, there is no phase shearing due to this effect, and our scheme does not work. We chose sodium as our atomic species, as it has a relatively large difference between the scattering lengths of $|F = 1, m_F = +1\rangle$ and $|F = 2, m_F = 0\rangle$. If we assume that $\Delta t_1$ is sufficiently short that we can ignore the effect of $\hat{H}_0$ during the coupling time, the density matrix for the system after $t_{\text{hold}}$ is

$$\rho(t_2) = \sum_{n_1, n_2, m_1 = 0}^{\infty} A_{n_1, n_2, m_1} e^{-i\Phi_{n_1, n_2, m_1} t_{\text{hold}}} \times \langle n_1, n_2 | \{ n_1, n_1 + n_2 - m_1 \} |, \quad (5)$$

with $\Phi_{n_1, n_2, m_1} = \chi_{11} (n_1 (n_1 - 1) - m_1 (m_1 - 1)) + \chi_{22} ((m_1 - n_1)(2n_2 + n_1 - n_1 - 1) + 2\chi_{12} (n_1 n_2 - m_1 (n_1 + n_2 - m_1))$ and $A_{n_1, n_2, m_1} = e^{-|\alpha(t_1)|^2} \alpha(t_1)^n (|\beta(t_1)|^2)^{n_2-m_1} \frac{n!}{n_1! n_2!} \sqrt{n_1! n_1! n_2!} \xi(n_1 + n_2 - m_1)$ with $\alpha(t_1) = \alpha_0 \cos \theta_1$, $\beta(t_1) = -i \alpha_0 \sin \theta_1$, where $\theta_1 = \Omega_0 \Delta t_1$ and $|\alpha_0|^2 = N_0$, the mean number of atoms initially in state $|1\rangle$. We note that at this point, both modes still contain a Poissonian number distribution. Finally, we describe the dynamics caused by the second microwave pulse in the Heisenberg picture, by noting that the Heisenberg operators after the second pulse are $\hat{a}_{1H} = \hat{a}_{1}(0) \cos \theta_2 - i \hat{a}_{2}(0) \sin \theta_2 e^{i\phi}$, and $\hat{a}_{2H} = \hat{a}_{2}(0) \cos \theta_2 - i \hat{a}_{1}(0) \sin \theta_2 e^{-i\phi}$, with $\theta_2 = \Omega_0 \Delta t_2$, where $\Delta t_2$ is the duration of the second microwave pulse, and $\phi$ is the phase of the microwave field relative to the first pulse. Again, we have assumed that the duration
of the pulse is sufficiently short that we can ignore the evolution due to $H_0$. As we are only interested in the number statistics, we can neglect the rest of the evolution after the second microwave pulse, as the number operators for both modes commute with $H_0$. Assuming we can distinguish state [2] atoms from state [1] atoms, we define the normalized number variance for state [2] atoms as $\langle N_2^2 \rangle = \langle N_2 \rangle^2 - \langle N_2 \rangle^2$. Figure 2 shows the $v(N_2)$ as a function of $t_{\text{hold}}$ and $\phi$, using the scattering properties of sodium ($a_{11} = a_{12} = 2.8$ nm, $a_{22} = 3.0$ nm) in a 500 Hz spherical harmonic trap. For some values of $t_{\text{hold}}$ and $\phi$, $v(N_2)$ dips below 0.01, as compared to the quantum limit $v(N_2) = 1$ associated with a coherent state, indicating significant number squeezing. The parameter space for this model is quite large, as we can adjust the length of the first and second coupling pulses, the hold time, and the phase of the second coupling pulse. If we did not have the ability to adjust the phase of the second coupling pulse, we would be constrained to a vertical line in figure 2 and not necessarily be able to access the optimum value of the squeezing. We found that the best number squeezing was obtained when the first coupling pulse was quite weak (approximately 8% of the atoms transferred). We also found we could still get a good level of squeezing when we began with an initial state which had number fluctuations 150 times larger than a Poissonian distribution (about 5% shot to shot fluctuations in the number) (see Figure 3). When starting with a such an initial condition, the best squeezing was found when the first beam splitter was relatively weak. This is due to the fact that the addition of vacuum to a super-Poissonian number distribution drives it to a Poissonian number distribution.

**Multimode model.** — To investigate if the approximations we made in the previous section were valid, we performed a 1D multimode simulation of the system using a stochastic phase space method. Specifically, we utilize a Truncated Wigner (TW) approach [13]. We reduce Eqs. 1 and 2 to one dimension by integrating out the dynamics in the $y$ and $z$ dimensions. A Fokker Plank equation (FPE) is then found from the master equation for the system using the Wigner representation. This equation can then be converted into a set of stochastic partial differential equations (SPDEs), which can be solved numerically. By averaging over many trajectories with different noises, expectation values of quantities corresponding to operators in the full quantum field theory can be extracted. When converting our FPE to a SPDEs, we ignore third and higher order derivatives in the FPE, as these terms do not have a simple mapping to the stochastic PDEs, and can be assumed to be negligible when the field has a high occupation number [13]. This truncated Wigner approximation will eventually fail, as it can not describe negative components of the Wigner function, which eventually occur when evolving under a Hamiltonian such as Eq. (1). However, we have checked our simulations in limits where the multimode dynamics can be neglected and they agree with the results obtained from our two-mode analytical model. In addition, over our simulation times no anomalous results such as significant negative densities were seen, indicating that the truncation of third order derivatives was a valid approximation. The SPDEs describing the one-dimensional system are

$$i\hbar \dot{\psi}_1(x) = \mathcal{L}_1 \psi_1(x) + \hbar \Omega(t) \psi_2(x)$$

$$i\hbar \dot{\psi}_2(x) = \mathcal{L}_2 \psi_2(x) + \hbar \Omega^*(t) \psi_1(x),$$

with

$$\mathcal{L}_j = H_j + U_{jj}(|\psi_j|^2 - 1/dx) + U_{jj}(|\psi_1|^2 - 1/2dx),$$

where $dx$ is the grid spacing of the numerical simulations. The terms inversely proportional to $dx$ compensate for the mean field of the vacuum, which is nonzero in the Wigner approach. The noise on the initial conditions for each trajectory of the evolution of these equations was chosen such that they corresponded to the specific initial state of interest.

Figure 3 shows the results from the TW simulation for two different parameter regimes. The left column shows a case where there is excellent agreement between the multimode model and the two-mode analytic model. However, the right column shows a case where the comparison between the two models is poor, due to significant multimode dynamics preventing the system acting as a two mode system. The multimode dynamics is a consequence of the unequal scattering lengths, meaning
that when atoms are created in state $|2\rangle$), they are no longer in a motional eigenstate of the system. These dynamics are relatively insignificant in first case, when only 0.25\% of the atoms are transferred in the first coupling pulse. However, in the second case, \sim 9\% of the atoms are transferred in the first coupling pulse, and the perturbation to the dynamics during the hold time is significant, even though the system is left to evolve for a much shorter time. Discussion. — Detection of atoms with high quantum efficiency will be required in order to observe the squeezing. This is experimentally challenging, but has been demonstrated before \cite{Esteve2008}. Furthermore, as the parameter space for the experiment is large, a good knowledge of the parameters such as trapping frequency and Rabi frequency is required such that theoretical modeling can predict roughly where to look for the squeezing. Apart from that, the scheme relies on having good control of optical dipole traps and microwave spectroscopy, and should be straightforward to implement. An addition effect that may degrade the squeezing is atomic loss due to inelastic collisions. Using the three-body recombination rates recently measured in \cite{Bourdel2008}, we estimate that roughly 10\% of the atoms from state $|1\rangle$ (the state that we are not looking for squeezing) are lost during the 16 ms hold time. However, a more important concern is the two-body inelastic collision rate, as it scales as $\int |v_0(r)|^4 \, d^3r$, that is, the same way as $\chi_{ij}$, the nonlinear interaction parameter, so reducing the atomic density will not help, as the lifetime due to collisions and the time taken to achieve squeezing scale identically. It was observed in \cite{Bourdel2008} that with mixtures of different hyperfine states decayed on timescales of order several milliseconds. However, if the maximal stretched combination of states was used \cite{Bourdel2008} (for example $|F = 1, m_F = 1\rangle$, $|F = 2, m_F = 2\rangle$), lifetimes of several seconds were observed. Using this particular combination of states would allow for squeezing under our scheme, as it has $a_{11} \neq a_{12}$.

It may be possible to create an intensity squeezed atom laser via a similar technique to that discussed in this Letter. By outcoupling two co-propagating hyperfine states, and interfering them at particular distance from the condensate, it may be possible to create intensity squeezing in one of the modes. However, this would require a species of atom with two magnetic field insensitive states, with scattering lengths $a_{11} + a_{22} \neq 2a_{12}$. This requirement could be avoided by using a separated beam path interferometer, as the effective $\chi_{ij}$ goes to zero. However, it may be difficult to achieve the required mode matching in this case.

The authors would like to acknowledge useful discussions with Andy Ferris, and support from the NCI supercomputing facility.

* Electronic address: haine@physics.uq.edu.au

\begin{thebibliography}{32}
\bibitem{Scully1997} M. O. Scully and M. S. Zubairy, ‘\textit{Quantum Optics},’ Cambridge University press, (1997).
\bibitem{Esteve2008} J. Esteve, \textit{et al.}, Nature, 01, 10, (2008).
\bibitem{Dowling1998} J. P. Dowling, Phys. Rev. A, \textbf{57} 4736 (1998).
\bibitem{Olsen2008} M. K. Olsen, \textit{et al.}, The European Physical Journal Special Topics, \textbf{160}, 331 (2008).
\bibitem{Johnsson2007} M. T. Johnsson and J. J. Hope, Phys. Rev. A, \textbf{75}, 043619 (2007).
\bibitem{Wuester2008} S. Wüster, \textit{et al.}, Phys. Rev. A, \textbf{77}, 023619 (2008).
\bibitem{Johnsson2007} M. T. Johnsson \textit{et al.}, Phys. Rev. Lett. \textbf{99}, 010401 (2007).
\bibitem{Sorensen2001} A. Sørensen, \textit{et al.}, Nature \textbf{409} 63 (2001).
\bibitem{Thankvanthri2007} S. Thankvanthri \textit{et al.}, Phys. Rev. A \textbf{75}, 023618 (2007).
\bibitem{Li2008} Y. Li, \textit{et al.}, Phys. Rev. Lett. 100, 210401 (2008).
\bibitem{Nandi2008} G. Nandi, \textit{et al.}, Phys. Rev. A, \textbf{78} 013605 (2008).
\bibitem{Sammels2000} C. Sammels, \textit{et al.}, Phys. Rev. A, \textbf{63}, 012710 (2000).
\bibitem{Norrie2006} A. A. Norrie \textit{et al.}, Phys. Rev. A \textbf{73} 043617 (2006).
\bibitem{Bourdel2006} T. Bourdel, \textit{et al.}, Phys. Rev. A \textbf{73} 043602 (2006).
\bibitem{Gorlitz2003} A. Görlitz, \textit{et al.}, Phys. Rev. Lett. \textbf{90} 090401 (2003).
\bibitem{Ferris2008} We were unaware of the favorable inelastic loss rates for the $|F = 1, m_F = 1\rangle$, $|F = 2, m_F = 2\rangle$ collisions at the time when we performed the calculations.
\end{thebibliography}