ASYMMETRY MODELS BASED ON LOGIT TRANSFORMATIONS FOR SQUARE CONTINGENCY TABLES WITH ORDINAL CATEGORIES

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ABSTRACT

Many observations tend to concentrate in the main diagonal cells when analyzing square contingency tables with ordered categories. Although many statisticians have proposed a variety of symmetry and asymmetry models, constraints on the main diagonal cells are not considered. This implies that the observed frequencies on the main diagonal cells are not utilized. Herein we propose three models that indicate an asymmetric structure for the log odds ratio for cell probabilities. These models constrain the main diagonal cells such that the information in the main diagonal cells can be utilized. Then we decompose the symmetry model using the proposed models.

1. Introduction

In a square contingency table with ordered row and column classifications, many observations tend to concentrate in the main diagonal cells. Our research is interested in determining whether the structure in a contingency table is symmetric or asymmetric since independence between the row and column classifications is unlikely to hold. For example, consider Table 1, which is a square contingency table using the data from Stuart (1953). These data are constructed from the unaided distance vision of 7477 women aged 30 to 39 employed in Royal Ordnance factories from 1943 to 1946. Various types of models (symmetric and asymmetric) have been proposed such as those by Bowker (1948), McCullagh (1978), Goodman (1979), and Agresti (1983). Many of these models do not constrain the main diagonal cells. Consequently, the parameters on the main diagonal cells are saturated.

Many observations tend to fall along the main diagonal cells. For the data in Table 1, 5296 (=1520+1512+1772+492) observations fall on the main diagonal cells, corresponding to approximately 71% of the sample size. In other words, the main diagonal cells contain much information. Because we are interested in utilizing the information on the main diagonal cells, we consider three models where the parameters on the main diagonal cells are not saturated. These models provide an asymmetric structure for cell probabilities. Agresti (1984) considered the marginal cumulative logistic model, which indicates one marginal distribution is a location shift of the other marginal distribution on a logistic-scale. Also, see Miyamoto, Niibe and Tomizawa (2005). Under the model, the parameters on the main diagonal cells are not saturated. Therefore, we propose a new model by using the logit transformation. This may be useful to see the structure that one cell probability is a location shift of the symmetric cell probability on a logistic-scale.

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Table 1: Unaided distance vision of 7477 women aged 30-39 (Stuart, 1953)

| Right eye | Highest (1) | Second (2) | Third (3) | Lowest (4) | Total |
|-----------|-------------|------------|-----------|------------|-------|
| Highest (1) | 1520        | 266        | 124       | 66         | 1976  |
|           | 1520.004    | 268.453    | 129.557   | 54.860     | 1972.874 |
|           | 1520.012    | 263.163    | 133.402   | 59.207     | 1975.784 |
|           | 1520.004    | 269.193    | 121.391   | 66.000     | 1976.588 |
| Second (2) | 234         | 1512       | 432       | 78         | 2256  |
|           | 231.561     | 1512.004   | 425.673   | 86.048     | 2255.285 |
|           | 236.831     | 1512.012   | 417.443   | 88.614     | 2254.900 |
|           | 230.825     | 1512.004   | 426.825   | 80.595     | 2250.249 |
| Third (3)  | 117         | 362        | 1772      | 205        | 2456  |
|           | 111.457     | 368.278    | 1772.004  | 206.283    | 2458.022 |
|           | 107.633     | 376.479    | 1772.014  | 202.190    | 2458.316 |
|           | 119.609     | 367.134    | 1772.005  | 206.855    | 2465.602 |
| Lowest (4) | 36          | 82         | 179       | 492        | 789   |
|           | 47.129      | 73.966     | 177.723   | 492.001    | 790.819 |
|           | 42.779      | 71.412     | 181.805   | 492.004    | 788.000 |
|           | 36.000      | 79.406     | 177.154   | 492.001    | 784.561 |
| Total     | 1907        | 2222       | 2507      | 841        | 7477  |
|           | 1910.150    | 2222.700   | 2504.957  | 839.192    | 7479.492 |
|           | 1907.254    | 2223.066   | 2504.664  | 842.015    | 7483.948 |
|           | 1906.438    | 2227.736   | 2497.374  | 845.452    | 7499.986 |

Notes: 
- "a" Estimated expected frequencies from the LoCS model.
- "b" Estimated expected frequencies from the LoLDPS model.
- "c" Estimated expected frequencies from the LoDPS model.

Caussinus (1965) provided the decomposition of the symmetry model. Namely, the symmetry model holds if and only if both the quasi-symmetry model and the marginal homogeneity model hold. Hence, the symmetry of the cell probability can be separated into the symmetry of the odds ratio and the homogeneity of the marginal distributions. This result may be useful to elucidate the origin of the poor fit when the symmetry model does not fit the given real dataset. [See Kateri and Papaioannou (1997) and Tomizawa and Tahata (2007)]. Therefore, we are also interested in considering the decomposition of symmetry using proposed models.

This paper is organized as follows. Section 2 describes the proposed models. Section 3 decomposes the symmetry model. Section 4 evaluates the goodness-of-fit of the proposed models. Section 5 gives a numerical example, Section 6 discusses relationship between related models, and Section 7 concludes this paper.

2. Models

Consider an $r \times r$ square contingency table with the same row and column classifications. Let $\pi_{ij}$ denote the probability that an observation will fall in the $i$th row and the $j$th column of the table ($i = 1, \ldots, r; j = 1, \ldots, r$). Bowker (1948) proposed the symmetry (S) model,
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which is defined as

$$\pi_{ij} = \psi_{ij} \quad (i = 1, \ldots, r; j = 1, \ldots, r),$$

where $$\psi_{ij} = \psi_{ji}$$. Note that $$\{\psi_{ij}\}$$ are unknown parameters. This indicates that the cell probabilities have a symmetric structure. The S model suggests that the right and left eye visions are balanced using the data in Table 1.

When the S model fits a given dataset poorly, an asymmetric model may be more appropriate. For the square contingency tables with same row and column categories, consider the following asymmetric structure

$$\pi_{ij} = \begin{cases} \delta_{ji} \psi_{ij} & (i < j), \\ \psi_{ij} & (i \geq j), \end{cases}$$

(1)

where $$\psi_{ij} = \psi_{ji}$$. Various models have been proposed by changing the structure of $$\{\delta_{ji}\}$$. McCullagh (1978) proposed the conditional symmetry (CS) model in which equation (1) is replaced by $$\{\delta_{ji} = \delta\}$. For the data in Table 1, the CS model shows that the probability when a women’s right eye grade is $$i$$ and her left eye grade is $$j$$ ($$> i$$) is $$\delta$$ times higher than the probability when a woman’s right eye grade is $$j$$ and her left eye grade is $$i$$. Goodman (1979) proposed the diagonals-parameter symmetry (DPS) model in which equation (1) is replaced by $$\{\delta_{ji} = \delta_{j-i}\}$. Agresti (1983) proposed the linear diagonals-parameter symmetry (LDPS) model in which equation (1) is replaced by $$\{\delta_{ji} = \delta_{j-i}\}$. The DPS model indicates that the ratios of the symmetric cells are $$\delta_{j-i}$$, while the LDPS model indicates that the ratios of symmetric cells increase or decrease exponentially for the difference $$j-i$$. A special case of this model with $$\{\delta_{ji} = 1\}$$ is the S model. Under these models, the parameters on the main diagonal cells are saturated. Also, see Tahata and Tomizawa (2015).

Let $$X$$ and $$Y$$ denote the row and column variables, respectively. Let

$$L_{ij} = \log \frac{\pi_{ij}}{1 - \pi_{ij}} = \log \frac{P(X = i, Y = j)}{1 - P(X = i, Y = j)} \quad (i = 1, \ldots, r; j = 1, \ldots, r).$$

Namely, $$L_{ij}$$ is the logit transformation. The log odds that $$(X, Y) = (i, j)$$ instead of $$(X, Y) \neq (i, j)$$ is $$L_{ij}$$, and the log odds that $$(X, Y) = (j, i)$$ instead of $$(X, Y) \neq (j, i)$$ is $$L_{ji}$$. We propose that the asymmetric structure for cell probabilities is

$$L_{ij} = \Delta_{ji} + L_{ji} \quad (i < j).$$

(2)

Note that this model with $$\{\Delta_{ji} = 0\}$$ is the S model. This model shows the asymmetric structure of the log odds. Under equation (2),

$$\pi_{ij} = \frac{\exp(\Delta_{ji} + L_{ji})}{1 + \exp(\Delta_{ji} + L_{ji})}, \quad \pi_{ji} = \frac{\exp(L_{ji})}{1 + \exp(L_{ji})} \quad (i < j).$$

Namely, this model indicates that one cell probability is a location shift of the symmetric cell probability on a logistic-scale. We see that (i) if $$\Delta_{ji} > 0$$, then $$\pi_{ij} > \pi_{ji}$$, (ii) if $$\Delta_{ji} < 0$$, then $$\pi_{ij} < \pi_{ji}$$, and (iii) if $$\Delta_{ji} = 0$$, then $$\pi_{ij} = \pi_{ji}$$. Therefore, parameters $$\{\Delta_{ji}\}$$ may be useful to visualize the magnitude relation of the cell probabilities in symmetric positions. We note that (i) $$\{\delta_{ji}\}$$ in equation (1) indicate the ratio between two symmetric cells and (ii) $$\{\Delta_{ji}\}$$ in equation (2) indicate the log odds ratio between two symmetric cells. Namely, although the equations (1) and (2) are very similar, these models indicate a different structure of asymmetry (also see Section 6).
We consider various types of special cases by changing the structure of \( \{ \Delta_{ji} \} \). The logit conditional symmetry (LoCS) model refers to the model where equation (2) is replaced by \( \{ \Delta_{ji} = \Delta \} \), that is,

\[
L_{ij} = \Delta + L_{ji} \quad (i < j).
\]

The LoCS model indicates that the log odds ratios are constant for \( i < j \).

Similarly, the logit diagonals-parameter symmetry (LoDPS) model refers to the model where equation (2) is replaced by \( \{ \Delta_{ji} = \Delta_{j-i} \} \), namely,

\[
L_{ij} = \Delta_{j-i} + L_{ji} \quad (i < j).
\]

The LoDPS model implies that the log odds ratio depends on only the difference \( j - i \) between the value of \( X \) and the value of \( Y \).

The logit linear diagonals-parameter symmetry (LoLDPS) model can be expressed as

\[
L_{ij} = (j - i)\Delta + L_{ji} \quad (i < j).
\]

This model is equivalent to replacing equation (2) with \( \{ \Delta_{ji} = (j - i)\Delta \} \). The LoLDPS model indicates that the log odds ratio is proportional to the difference \( j - i \) between the value of \( X \) and the value of \( Y \).

The parameters on the main diagonal cells are saturated under the CS, DPS, and LDPS models since these models have the asymmetric structures for the ratios of cell probabilities. By contrast, the parameters on the main diagonal cells are not saturated under the LoCS, LoDPS, and LoLDPS models since these models have asymmetric structures for the log odds ratios of cell probabilities.

3. Decomposition of the symmetry model

Many statisticians have shown the decomposition of the S model to analyze a square contingency table. Caussinus (1965) gave the theorem that the S model holds if and only if both the quasi-symmetry model and the marginal homogeneity model hold. See also Read (1977) and Tomizawa and Tahata (2007).

Although the proof is omitted here, we can obtain the following lemma:

**Lemma 1** Equation (2) can be expressed as

\[
\pi_{ij} - \pi_{ji} = \pi_{ji}(1 - \pi_{ij})(\exp(\Delta_{ji}) - 1) \quad (i < j).
\]

Consider the global symmetry (GS) model defined by

\[
\delta_U = \delta_L,
\]

where \( \delta_U = \sum \sum_{i<j} \pi_{ij} \) and \( \delta_L = \sum \sum_{i<j} \pi_{ji} \) (Read, 1977). We can obtain the following theorem.

**Theorem 1** The S model holds if and only if both the LoCS model and the GS model hold.

**Proof.** If the S model holds, then \( \sum \sum_{i<j} \pi_{ij} = \sum \sum_{i<j} \pi_{ji} \). Namely, the GS model holds. Since the LoCS model with \( \Delta = 0 \) is the S model, the necessity is proved. Next, we show the sufficiency. Assume that both the LoCS model and the GS model hold. We can see that

\[
\pi_{ij} - \pi_{ji} = \pi_{ji}(1 - \pi_{ij})(\exp(\Delta) - 1) \quad (i < j),
\]
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from Lemma 1. From the GS model, we can obtain

\[(1 - \exp(\Delta)) \sum_{i<j} (\pi_{ji}(1 - \pi_{ij})) = 0.\]

Therefore, we can show \(\Delta = 0\). The proof is completed.

**Theorem 2** The S model holds if and only if both the LoLDPS model and the GS model hold.

**Proof.** If the S model holds, then \(\sum \sum_{i<j} \pi_{ij} = \sum \sum_{i<j} \pi_{ji}\). Namely, the GS model holds. Since the LoLDPS model with \(\Delta = 0\) is the S model, the necessity is proved. Next, we prove the sufficiency. Assume that both the LoLDPS model and the GS model hold. We can see that

\[\pi_{ij} - \pi_{ji} = \pi_{ji}(1 - \pi_{ij})(\exp((j - i)\Delta) - 1) \quad (i < j),\]

from Lemma 1. From the GS model, we can obtain

\[\sum_{i<j} \pi_{ji}(1 - \pi_{ij})(\exp((j - i)\Delta) - 1) = 0.\]

The left-hand side of this equation with \(\Delta > 0\) is positive since the difference \(j - i\) is positive. The left-hand side of this equation with \(\Delta < 0\) is negative for the same reason. Namely, this equation with \(\Delta = 0\) only holds. The proof is completed.

Moreover, we consider the following model.

\[\delta^{(k)}_U = \delta^{(k)}_L \quad (k = 1, \ldots, r - 1),\]

where \(\delta^{(k)}_U = \sum \sum_{j-i=k} \pi_{ij}\) and \(\delta^{(k)}_L = \sum \sum_{j-i=k} \pi_{ji}\). We refer to this model as the diagonals-global symmetry (DGS) model. We can obtain the decomposition of the S model.

**Theorem 3** The S model holds if and only if both the LoDPS model and the DGS model hold.

**Proof.** If the S model holds, then \(\sum \sum_{j-i=k} \pi_{ij} = \sum \sum_{j-i=k} \pi_{ji}\). Namely, the DGS model holds. Since the LoDPS model with \(\Delta_k = 0\) \((k = 1, \ldots, r - 1)\) is the S model, the necessity is proved. Next, assuming that both the LoDPS model and the DGS model hold, we can see that for \(k = 1, \ldots, r - 1,\)

\[\pi_{ij} - \pi_{ji} = \pi_{ji}(1 - \pi_{ij})(\exp(\Delta_k) - 1) \quad (j - i = k),\]

from Lemma 1. From the DGS model, we can obtain that for \(k = 1, \ldots, r - 1,\)

\[(1 - \exp(\Delta_k)) \sum_{j-i=k} \pi_{ji}(1 - \pi_{ij}) = 0.\]

Thus, \(\Delta_k = 0\) \((k = 1, \ldots, r - 1)\). The LoDPS model with \(\Delta_k = 0\) for all \(k\) is the S model. The proof is completed.
4. Goodness-of-fit test

For $r \times r$ contingency tables, let $n_{ij}$ denote the observed frequency in the $(i, j)$th cell of the table and $m_{ij}$ denote the corresponding expected frequency with $n = \sum \sum n_{ij}$ ($i = 1, \ldots, r; j = 1, \ldots, r$). Assuming that $\{n_{ij}\}$ has a multinomial distribution, $\hat{m}$ denotes the maximum likelihood estimate (MLE) of $m$ under a model. The MLE of $m$ under a model is obtained by applying the Newton-Raphson method to the log-likelihood equations. For example, we consider the LoCS and GS models. For the GS model, please see Read (1977) for the detail. We show the log-likelihood equations for the LoCS model. To obtain the MLEs under the LoCS model, we must maximize the Lagrangian

$$L = \sum \sum n_{ij} \log \pi_{ij} + \lambda (\sum \sum \pi_{ij} - 1) + \sum \sum \lambda_{ij} \left( \log \pi_{ij} - \log \sum \sum \pi_{kl} - \Delta - \log \pi_{ji} + \log \sum \sum \pi_{lk} \right),$$

with respect to $\{\pi_{ij}\}, \lambda, \{\lambda_{ij}\}$, and $\Delta$. Setting the partial derivatives of $L$ equal to zero, we obtain the equations

$$\begin{cases} \frac{n_{st}}{\pi_{st}} + \lambda + \sum \sum \lambda_{ij} \left( \frac{1}{\sum \sum \pi_{kl}} - \frac{1}{\sum \sum \pi_{lk}} \right) = 0 & (s < t), \\ \frac{n_{ss}}{\pi_{ss}} + \lambda - \sum \sum \lambda_{ij} \left( \frac{1}{\sum \sum \pi_{kl}} - \frac{1}{\sum \sum \pi_{lk}} \right) = 0 & (s = 1, \ldots, r), \\ \frac{n_{ts}}{\pi_{ts}} + \lambda - \sum \sum \lambda_{ij} \left( \frac{1}{\sum \sum \pi_{kl}} - \frac{1}{\sum \sum \pi_{lk}} \right) = 0 & (s < t), \end{cases}$$

(3)

as well as

$$\sum \sum \lambda_{ij} = 0, \quad \sum \sum \pi_{ij} = 1,$$

and

$$\log \pi_{ij} - \log \sum \sum \pi_{kl} - \Delta - \log \pi_{ji} + \log \sum \sum \pi_{lk} = 0 \quad (i < j).$$

We can solve the equations by using the Newton-Raphson method.

The likelihood ratio chi-squared statistic for the goodness-of-fit of the model is defined by

$$G^2 = 2 \sum_{i=1}^{r} \sum_{j=1}^{r} n_{ij} \log \left( \frac{n_{ij}}{\hat{m}_{ij}} \right).$$

The numbers of degrees of freedom for testing the goodness-of-fit of LoCS, LoLDPS, LoDPS, and DGS are $(r + 1)(r - 2)/2$, $(r + 1)(r - 2)/2$, $(r - 1)(r - 2)/2$, and $r - 1$, respectively. For example, we consider the case of the LoCS model. For $i = 1$ and $j = 2$, 

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\[
\log \frac{\pi_{12}(1 - \pi_{21})}{\pi_{21}(1 - \pi_{12})} = \Delta.
\]

Then we substitute \(\log(\pi_{12}(1 - \pi_{21})/(\pi_{21}(1 - \pi_{12})))\) for \(\Delta\) in the equation (2). Namely

\[
\frac{\pi_{ij}(1 - \pi_{ji})}{\pi_{ji}(1 - \pi_{ij})} = \frac{\pi_{12}(1 - \pi_{21})}{\pi_{21}(1 - \pi_{12})} \quad (i < j),
\]

where \((i, j) \neq (1, 2)\). Therefore, the LoCS model has \((r + 1)(r - 2)/2\) restrictions. This is equal to the degree of freedom of the LoCS model for the goodness-of-fit test.

We note that the MLE of \(m_{ii}\) under the CS, DPS, and LDPS models is \(n_{ii}\) \((i = 1, \ldots, r)\). This implies that the observed frequencies on the main diagonal cells do not affect the likelihood ratio chi-squared statistic. For more details, see Section 7.

5. Example

The models described herein are used to analyze the data in Table 1. Table 2 shows the value of \(G^2\) for each model applied to the data in Table 1. The S model fits poorly. We infer that the cell probabilities do not have a symmetric structure. Consequently, we applied our proposed asymmetric models. The LoCS, LoLDPS, and LoDPS models fit well. The MLE of \(\Delta\) is \(\hat{\Delta} = 0.15\) in the LoCS model. Therefore, the result in the LoCS model infers that the log odds ratio \(L_{ij} - L_{ji}\) is 0.15 for all \(i < j\). A woman’s right eye vision tends to be better than her left eye vision because of \(\hat{\Delta} > 0\). Similarly, the MLE of \(\Delta\) is \(\hat{\Delta} = 0.11\) under the LoLDPS model, and the MLEs under the LoDPS model of \(\Delta_1, \Delta_2,\) and \(\Delta_3\) are \(\hat{\Delta}_1 = 0.16, \hat{\Delta}_2 = 0.02,\) and \(\hat{\Delta}_3 = 0.61,\) respectively. Thus, the same results are obtained in the LoCS model.

On the other hand, one may be interested in considering the structure of ratios between two symmetric cells. In this case, we shall apply the CS, LDPS, and DPS models. These models fit well from Table 2. The MLE of \(\delta\) is \(\hat{\delta} = 1.16\) in the CS model, that of \(\delta\) is \(\hat{\delta} = 1.11\) in the LDPS model, and those of \(\delta_1, \delta_2,\) and \(\delta_3\) are \(\hat{\delta}_1 = 1.17, \hat{\delta}_2 = 1.02,\) and \(\hat{\delta}_3 = 1.83\) in the DPS model. Thus, we obtain similar results: the values of MLE are greater than 1 for all of the models.

Finally, we are interested in deducing the reason for the poor fit of the S model. The S model, for instance, can be separated into the LoDPS model and the DGS model from Theorem 3. The LoDPS model fits very well, but the DGS model fits very poorly. Therefore,

| Model | Degree of freedom | \(G^2\) |
|-------|------------------|---------|
| S     | 6                | 19.25*  |
| LoCS  | 5                | 7.39    |
| LoLDPS| 5                | 7.39    |
| LoDPS | 3                | 0.53    |
| CS    | 5                | 7.35    |
| LDPS  | 5                | 7.28    |
| DPS   | 3                | 0.50    |
| GS    | 1                | 11.90*  |
| DGS   | 3                | 18.75*  |

Notes: *Significant at the 0.05 level
we infer that the lack of structure of the DGS model is responsible for the poor fit of the S model.

6. Discussions

We consider the artificial data in Table 3. Tables 3(a), 3(b), 3(c), and 3(d) have the same frequencies in the off diagonal cells and the sum of frequencies in the main diagonal cells are different. Namely, the sample log odds ratios \( \log \frac{n_{ij}(n-n_{ji})}{n_{ji}(n-n_{ij})} \) are not the same, but the sample ratios \( \frac{n_{ij}}{n_{ji}} \) are the same for Tables 3(a), 3(b), 3(c), and 3(d). This implies concentration of information in the main diagonal cells. From Table 4, we can see that \( G^2 \) for the CS, LDPS, and DPS models are the same for Tables 3(a), 3(b), 3(c), and 3(d), because the parameters on the main diagonal cells are saturated. Also see Section 7. On the other hand, the \( G^2 \) for the proposed models are not the same for Tables 3(a), 3(b), 3(c), and 3(d). As the sum of frequencies on the main diagonal cells increases, the difference between \( G^2 \) for the existing model and the proposed model decreases. Therefore, the proposed models may be useful to see the asymmetric structure by utilizing the information on the main diagonal cells.

Table 3: Artificial data

|       | (a) n = 199   |       | (b) n = 235   |
|-------|---------------|-------|---------------|
|       | 1 30 20 20    | 10 30 20 20   |
|       | 15 1 30 10    | 15 10 30 10   |
|       | 10 25 1 15    | 10 25 10 15   |
|       | 5 5 10 1      | 5 5 10 1      |

|       | (c) n = 595   |       | (d) n = 4195  |
|-------|---------------|-------|---------------|
|       | 100 30 20 20  | 1000 30 20 20 |
|       | 15 100 30 10  | 15 1000 30 10 |
|       | 10 25 100 15  | 10 25 1000 15 |
|       | 5 5 10 100    | 5 5 10 1000   |

Table 4: Likelihood ratio chi-square values \( G^2 \) for models applied to the data in Table 3

|       | CS      | LoCS    | LDPS    | LoLDPS   | DPS     | LoDPS   |
|-------|---------|---------|---------|----------|---------|---------|
| (a)   | 5.57    | 5.16    | 1.78    | 1.73     | 1.53    | 1.46    |
| (b)   | 5.57    | 5.22    | 1.78    | 1.74     | 1.53    | 1.47    |
| (c)   | 5.57    | 5.44    | 1.78    | 1.76     | 1.53    | 1.50    |
| (d)   | 5.57    | 5.55    | 1.78    | 1.78     | 1.53    | 1.52    |

7. Concluding remarks

We propose models based on the logit transformation. These models indicate that a cell’s probability is a location shift of the symmetric cell’s probability on a logistic-scale. Thus, when we want to see the asymmetric structure of two symmetric cells on a logistic-
scale, we should apply the proposed models rather than the model given by equation (1). (The model given by equation (1) may be useful for seeing the asymmetric structure of the ratio between two symmetric cells.)

We propose models with constraints on the main diagonal cells since many observations tend to concentrate in the main diagonal cells. These models have features where the observed frequencies on the main diagonal cells affect the likelihood ratio chi-squared statistic. That is, these models utilize information on the main diagonal cells. For example, we consider the CS and LoCS models. Under the CS model, the MLEs of expected frequencies \( m_{ij} \) are given by

\[
\hat{m}_{ij} = \begin{cases} 
\frac{n_U(n_{ij} + n_{ji})}{n_U + n_L} & (i < j), \\
\frac{n_{ii}}{n_U + n_L} & (i = j), \\
\frac{n_L(n_{ij} + n_{ji})}{n_U + n_L} & (i > j), 
\end{cases}
\]

where \( n_U = \sum_{i<j} n_{ij} \) and \( n_L = \sum_{i<j} n_{ji} \) (Read, 1977). Therefore, we can see that

\[
\sum_{i=1}^{r} n_{ii} \log \left( \frac{n_{ii}}{\hat{m}_{ii}} \right) = 0.
\]

That is, observations on the main diagonal cells do not contribute to \( G^2 \). On the other hand, the MLEs of expected frequencies \( m_{ij} \) are given by the solution of equation (3) under the LoCS model. These are not equal to \( n_{ii} \) for \( i = 1, \ldots, r \). So, observations on the main diagonal cells contribute to \( G^2 \). In this sense, we can utilize the information in the main diagonal cells.

We partially decompose the S model using the proposed models. When the S model fits poorly, the decomposition may be useful to determine the reason for the poor fit of the S model.

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