GLOBAL TORELLI THEOREM FOR PROJECTIVE MANIFOLDS OF CALABI–YAU TYPE

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Abstract. We prove that the period map from the Teichmüller space of polarized and marked Calabi–Yau type manifolds to the period domain of polarized Hodge structures is injective. The proof is based on the constructions of holomorphic affine structure on the Teichmüller space and the construction of the completion of the Teichmüller space with respect to the Hodge metric.

1. Introduction

The basic object we consider in this paper is the Calabi–Yau type manifold. A compact simply connected projective manifold $M$ of complex dimension $n$ is called a Calabi–Yau type manifold in this paper, if it satisfies the following: (i) there exists some $[n/2] < s \leq n$, such that $h^{s,n-s}(M) = 1$ and $h^{s,n-s'}(M) = 0$ for any $s' > s$; (ii) $H^{s,n-s}(M) = H^{s,n-s}(pr)$; (iii) for any generator $[\Omega] \in H^{s,n-s}(M)$, the contraction map $\cdot \mapsto \cdot \Omega$ is an isomorphism. A polarized and marked Calabi–Yau type manifold is a triple consisting of a Calabi–Yau type manifold $M$, an ample line bundle $L$ over $M$, and a basis of the integral middle homology group modulo torsion, $H_n(M,\mathbb{Z})/\text{Tor}$.

The Teichmüller space is the deformation of the complex structure on the polarized and marked Calabi–Yau type manifold $M$, which will be denoted by $T$. Let $Z_m$ be the moduli space of polarized Calabi–Yau type manifolds with level $m$ structure, which can be viewed analogously as that of Calabi–Yau manifolds constructed by Popp, Viehweg, and Szendrői, for example in Section 2 of [23]. Assume that $Z_m$ is a connected and smooth quasi-projective moduli space and there exists a versal family $X_{Z_m} \rightarrow Z_m$ with polarized Calabi–Yau type manifolds as fibers. We will show that the Teichmüller space is precisely the universal cover of the smooth moduli space $Z_m$ with the universal covering map denoted by $\pi_m : T \rightarrow Z_m$. Therefore there exists the pull-back versal family $\mathcal{U} \rightarrow T$ of the polarized and marked Calabi–Yau type manifolds via $\pi_m$ of the versal family $\mathcal{X}_{Z_m} \rightarrow Z_m$. Moreover, we have that $T$ is a connected and simply connected smooth complex manifold with $\dim_{\mathbb{C}} T = \dim_{\mathbb{C}} H^{s-1,n-s+1}_{pr}(pr) = N$.

Let $D$ be the period domain of the Hodge structures of polarized and marked Calabi–Yau type manifolds of weight $n$ and let $\Phi : T \rightarrow D$ be the period map from the Teichmüller space of polarized and marked Calabi–Yau type manifolds to the period domain. Denote the period map on the smooth moduli space by $\Phi_{Z_m} : Z_m \rightarrow D/\Gamma$, where $\Gamma$ is the global monodromy group which acts properly and discontinuously on $D$. Then $\Phi : T \rightarrow D$ is the lifting of $\Phi_{Z_m} \circ \pi_m$. Since $\Phi$ is locally injective (see Proposition [2.11]), so is $\Phi_{Z_m}$. It is studied in [6] that there is a complete homogeneous Hodge metric $h$ on $D$. Then the pull-back of $h$ on $Z_m$ and $T$ via $\Phi_{Z_m}$ and $\Phi$ respectively are both well-defined Kähler...
metrics, as \( \Phi_{Z_m} \) and \( \Phi \) are both locally injective. By abuse of notation, they are still called Hodge metrics in this paper. The main theorem of this paper is the following.

**Theorem 1.1** (Global Torelli Theorem). The period map \( \Phi : \mathcal{T} \to D \) is injective.

The proof of this theorem is outlined as follows. First, we construct a holomorphic affine structure on the Teichmüller space. We first fix a base point \( p \in \mathcal{T} \) with the Hodge structure \( \{H^{k,n-k}_p\}_{k=0}^n \) as the reference Hodge structure. With this fixed base point \( \Phi(p) \in D \), we identify the unipotent subgroup \( N_+ \) with its orbit in \( \bar{D} \) (see Section 3.1 and Remark 3.1) and define \( \bar{T} = \Phi^{-1}(N_+) \subseteq \mathcal{T} \). We first show that \( \Phi : \bar{T} \to N_+ \cap D \) is a bounded map with respect to the Euclidean metric on \( N_+ \), and that \( \mathcal{T} \setminus \mathcal{T} \) is an analytic subvariety with \( \text{codim}_C(\mathcal{T} \setminus \mathcal{T}) \geq 1 \). Then by applying Riemann extension theorem, we conclude that \( \Phi(\mathcal{T}) \subseteq N_+ \cap D \). Using this property, we then show \( \Phi \) induces a global holomorphic map \( \tau : \mathcal{T} \to \mathbb{C}^N \), which is non-degenerate at each point in \( \mathcal{T} \). Thus \( \tau : \mathcal{T} \to \mathbb{C}^N \) induces a global holomorphic affine structure on \( \mathcal{T} \). In fact we prove that \( \tau = P \circ \Phi : \mathcal{T} \to \mathbb{C}^N \) is a composition map with \( P : N_+ \to \mathbb{C}^N \simeq H^{s-1,n-s+1}(M_p) \) a natural projection map into a subspace, where \( N_+ \simeq \mathbb{C}^d \) with the fixed base point \( p \in \mathcal{T} \).

Secondly, suppose \( Z^H_m \) is the metric completion of the smooth moduli space \( Z_m \) with respect to the Hodge metric. It is easy to see that \( Z^H_m \) is a connected and complete smooth complex manifold. Consider the universal cover \( \mathcal{U}^H_m \) of \( Z^H_m \) with the universal covering map \( \pi^H_m : \mathcal{U}^H_m \to Z^H_m \). Then \( \mathcal{U}^H_m \) is a connected and simply connected complete smooth complex manifold. Hence, we get the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_m} & \mathcal{U}^H_m \\
\pi_m \downarrow & & \downarrow \pi^H_m \\
Z^H_m & \xrightarrow{i_m} & D/\Gamma,
\end{array}
\]

with \( \Phi^H_m \) the continuous extension map of \( \Phi Z_m : Z_m \to D/\Gamma, i \) the inclusion map, \( i_m \) the lifting of \( i \circ \pi_m \), and \( \Phi^H_m \) the lifting of \( \Phi^H \circ \pi_m \). We remark that there is a choice of \( i_m \) and \( \Phi^H_m \) such that \( \Phi = \Phi^H_m \circ i_m \). If we denote \( \mathcal{T}_m := i_m(\mathcal{T}) \) and \( \Phi_m := \Phi^H_m \mid_{\mathcal{T}_m} \), then \( \Phi = i_m \circ \Phi_m \). We first show that the map \( \Phi_m \) is a bounded holomorphic map from \( \mathcal{T}_m \) to \( N_+ \cap D \) with respect to the Euclidean metric on \( N_+ \). Thus we can define \( \tau_m^H : \mathcal{T}_m \to \mathbb{C}^N \) analogously as the definition of \( \tau \). The restriction of \( \tau_m \) on \( \mathcal{T}_m \) is denoted by \( \tau_m \). Then we conclude that \( \tau_m \) defines a holomorphic affine structure on \( \mathcal{T}_m \). This property implies that \( \tau_m \) also defines a holomorphic affine structure on \( \mathcal{T}_m^H \). Here the property that \( \tau : \mathcal{T} \to \mathbb{C}^N \) is a global coordinate map on \( \mathcal{T} \) comes in play substantially.

Thirdly, we use the affineness, completeness of \( \mathcal{T}_m^H \), the affineness and local injectivity of \( \tau_m^H \) to show that \( \tau_m^H : \mathcal{T}_m^H \to \mathbb{C}^N \) is injective for any \( m \geq 3 \). This induces that \( \Phi^H_m \) is injective as well.

Finally, by showing that \( \mathcal{T}_m^H \) and \( \mathcal{T}_m^H \) are biholomorphic for any \( m, m' \geq 3 \), we define the complete complex manifold \( \mathcal{T}^H \) by \( \mathcal{T}^H = \mathcal{T}_m^H \), the holomorphic maps \( i_T : \mathcal{T} \to \mathcal{T}^H \) by \( i_T = i_m \) and \( \Phi^H : \mathcal{T}^H \to D \) by \( \Phi^H = \Phi^H_m \) for any \( m \geq 3 \). By these definitions, it is easy to see \( \mathcal{T}^H \) is a complex affine manifold, which is also a bounded domain in \( \mathbb{C}^N \); and the extended period map \( \Phi^H : \mathcal{T}^H \to \mathbb{C}^N \) is a holomorphic injection. We then show that \( \mathcal{T}^H \) is the Hodge metric completion space of \( \mathcal{T} \).
an embedding. More importantly, we conclude that $\Phi = \Phi^H \circ i_T : \mathcal{T} \rightarrow D$ is injective as both $\Phi^H$ and $i_T$ are injective. This gives Theorem 1.1.

Furthermore, we assume that the moduli space $\mathcal{M}$ of polarized Calabi–Yau type manifolds is smooth. Then consider the period map $\Phi_M : \mathcal{M} \rightarrow D/\Gamma$, where $\Gamma$ denotes the global monodromy group which acts properly and discontinuously on the period domain $D$. For simplicity we also assume that the action of $\Gamma$ is free so that $D/\Gamma$ is a smooth analytic manifold. The following theorem of this paper proves that the period map $\Phi_M$ is a covering map onto its image. Moreover, we know that generic Torelli theorem on the moduli space is a weaker property than global Torelli theorem on the moduli space in general. However, as a consequence of this theorem, we show that if the generic Torelli theorem on the moduli space of polarized Calabi–Yau type manifolds is known, then we also get the global Torelli theorem on the moduli space.

**Theorem 1.2.** Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau type manifolds. If $\mathcal{M}$ is smooth, and $\Gamma$ acts on $D$ freely, then the period map on the moduli space $\Phi_M : \mathcal{M} \rightarrow D/\Gamma$ is a covering map from $\mathcal{M}$ onto its image in $D/\Gamma$. As a consequence, if the period map $\Phi_M$ on the moduli space is generically injective, then it is globally injective.

Note that in the most interesting cases it is always possible to find a subgroup $\Gamma_0$ of $\Gamma$, which is of finite index in $\Gamma$ such that $\Gamma_0$ acts on freely and thus $D/\Gamma_0$ is smooth. In such cases we consider the lifting $\Phi_{M_0} : M_0 \rightarrow D/\Gamma_0$ of the period map $\Phi_M$ with $M_0$ a finite cover of $\mathcal{M}$. Then our arguments can be applied to prove that $\Phi_{M_0}$ is actually a covering map onto its image for polarized Calabi–Yau type manifolds with smooth moduli spaces.

This paper is organized as follows: in Section 2 we give the definition of Calabi–Yau type manifolds and briefly review the definition of period domain of polarized Hodge structures. Then we construct the Teichmüller space of Calabi–Yau type manifolds and show its basic properties. We then introduce the period map from the Teichmüller space to the period domain. In Section 3, we construct the holomorphic affine structure on the Teichmüller space $\mathcal{T}$. In Section 4, we introduce the completion space $\mathcal{T}_m^H$ and show that $\mathcal{T}_m^H$ is a complex affine manifold. In Section 5, we give the proof of the global Torelli theorem on the Teichmüller space. We first show that $\Phi^H : \mathcal{T}_m^H \rightarrow D$ is injective. Then we define the completion space $\mathcal{T}^H$ and the extended period map $\Phi^H$ and show that the map $\Phi^H : \mathcal{T}^H \rightarrow D$ is also injective. Moreover, we conclude that $\Phi = \Phi^H \circ i_T : \mathcal{T} \rightarrow D$ is injective by showing that the map $i_T : \mathcal{T}^H \rightarrow \mathcal{T}$ is an embedding. In Section 6, we make smoothness assumptions on both the moduli space $\mathcal{M}$ and on the quotient space $D/\Gamma$. Then using the global Torelli theorem on the Teichmüller space of polarized and marked Calabi–Yau type manifolds, we show that the period map $\Phi_M : \mathcal{M} \rightarrow D/\Gamma$ is a covering map onto its image. As a consequence, we conclude that generic Torelli implies global Torelli on the moduli space of Calabi–Yau type manifolds. We remark that the first five sections follow closely from our paper [2], in which the case of Calabi–Yau manifolds are proved.

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2. The period map on the Teichmüller space

In Section 2.1, we introduce the definition of polarized and marked Calabi–Yau type manifolds and some basic properties. In Section 2.2, we review the construction of the period domain of polarized Hodge structures. In Section 2.3, we briefly describe the definitions of the moduli space of polarized Calabi–Yau type manifolds and the Teichmüller space of polarized and marked Calabi–Yau type manifolds. We then make basic assumptions on the Teichmüller space and show the simply connectedness of it. In Section 2.4, we introduce the period map from the Teichmüller space to the period domain.

2.1. Calabi–Yau type manifolds. We generalize the definition of Calabi–Yau type manifolds in [12] as follows.

Definition 2.1. Let $M$ be a simply connected compact complex projective manifold of $\dim \mathbb{C}M = n$ and $L$ be an ample line bundle over $M$ with $c_1(L)$ the Kähler class on $M$. We call $M$ a manifold of Calabi–Yau type if it satisfies the following conditions:

1. There exists some $[n/2] < s \leq n$ and $\ h^s,n-s(M) = 1$, and $\ h^{s',n-s'}(M) = 0$ for $s' > s$;

2. $H^{s,n-s}(M) = H^{pr,n-s}(M)$;

3. For any generator $[\Omega] \in H^{pr,n-s}(M)$, the contraction map $\iota : H^{0,1}(M, T^{1,0}M) \to H^{s-1,n-s+1}(M)$, $[\varphi] \mapsto [\varphi \iota \Omega]$ is an isomorphism, where $\Omega$ is a $\overline{\partial}$-closed $(s, n-s)$-form;

where $H^{pr,n-s}(M)$ and $H^{s-1,n-s+1}(M)$ are the primitive cohomology group of corresponding type, which will be defined in the following.

We call a triple $(M, L, \{\gamma_1, \cdots, \gamma_h\})$ a polarized and marked Calabi–Yau type manifold with a Calabi–Yau type manifold $M$ of $\dim \mathbb{C}M = n$, $L$ an ample line bundle over $M$ and $\{\gamma_1, \cdots, \gamma_h\}$ a basis of the integral homology group modulo torsion, $H_n(M, \mathbb{Z})/\text{Tor}$. For a polarized and marked Calabi–Yau type manifold $M$ with the background smooth manifold $X$, we identify the basis of $H_n(M, \mathbb{Z})/\text{Tor}$ to a lattice $\Lambda$ as in [23]. This gives us canonical identification $H^n(M, \mathbb{C}) = H^n(X, \mathbb{C})$. In the rest of the paper we will use the Chern form $\omega \in c_1(L)$ of the line bundle $L$ as the Kähler form on $M$. The polarization $L$, which is an integer class, defines a map

$$L : H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q}), \quad A \mapsto L \wedge A.$$ 

Then $H^{pr}_n(X) = \ker(L)$ is called the primitive cohomology groups, where the coefficient ring can be $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Moreover, let $H^{k,n-k}_{pr}(M) = H^{k,n-k}(M) \cap H^n_{pr}(M, \mathbb{C})$ denote $\dim H^{k,n-k}_{pr}(M, \mathbb{C}) = h^{k,n-k}$. We then have the Hodge decomposition

$$H^n_{pr}(M, \mathbb{C}) = H^{n,0}_{pr}(M) \oplus \cdots \oplus H^{0,n}_{pr}(M).$$

One may refer to Chapter 7.1 in [25] for more details of polarized Hodge structure on primitive cohomology group of Kähler manifolds. One also notices that the middle dimensional Hodge numbers of a Calabi–Yau type manifold are similar to that of a Calabi–Yau manifold. Actually, a Calabi–Yau manifold, for example as considered in [23] or in [2], satisfies all the conditions in Definition 2.1. Therefore, a Calabi–Yau manifold is of course...
a Calabi–Yau type manifold. Hence, the results presented in this paper can be applied to
Calabi–Yau manifolds. In the rest of this paper, one may consider Calabi–Yau manifolds
as our special cases. We remark that one may refer to [12] for many interesting examples
of Calabi–Yau type manifolds of Fano type.

2.2. Period domain of polarized Hodge structures. In this section, we will review
the construction of period domain of polarized Hodge structures. We remark that most
of the discussions in this section is based on Section 3 in [21].

For a polarized and marked Calabi–Yau type manifold \((M, L)\) with background smooth
manifold \(X\), we identify the basis of \(H_n(M, \mathbb{Z})/\text{Tor}\) to a lattice \(\Lambda\) as in [23]. This gives us
a canonical identification

\[ H^n(M) \simeq H^n(X), \]

where the coefficient ring can be \(\mathbb{Q}, \mathbb{R}\) or \(\mathbb{C}\). The polarization \(L\), which is an integer class,
defines a map

\[ L : H^n(X, \mathbb{Q}) \to H^{n+2}(X, \mathbb{Q}), \quad A \mapsto L \wedge A. \]

We denote by \(\ker(L) = H^npr(M)\) the primitive cohomology. Moreover, let \(H^{k,n-k}_pr(M) = H^{k,n-k}(M, \mathbb{C}) \cap H^npr(M, \mathbb{C})\) and its \(\dim H^{k,n-k}_pr(M, \mathbb{C}) = h^{k,n-k}.\) We then have the Hodge
decomposition \(H^{n}_pr(M, \mathbb{C}) = H^{0,n}_pr(M) \oplus \cdots \oplus H^{n,n}_pr(M).\) The Poincaré bilinear form \(Q\) on
\(H^npr(X, \mathbb{Q})\) is defined by

\[ Q(u, v) = (-1)^{n(n-1)/2} \int_X u \wedge v \]

for any \(d\)-closed \(n\)-forms \(u, v\) on \(X\). The bilinear form \(Q\) is symmetric if \(n\) is even and
is skew-symmetric if \(n\) is odd. Furthermore, \(Q\) is nondegenerate and can be extended to
\(H^npr(X, \mathbb{C})\) bilinearly, and it satisfies the Hodge-Riemann relations

\[ Q(H^{k,n-k}_pr(M), H^{l,n-l}_pr(M)) = 0 \quad \text{unless} \quad k + l = n, \quad \text{and} \]

\[ (\sqrt{-1})^{2k-n} Q(v, \bar{v}) > 0 \quad \text{for} \quad v \in H^{k,n-k}_pr(M) \setminus \{0\}. \]

Let \(f^k = \sum_{i=k}^n h^{i,n-i}, f^0 = m,\) and \(F^k = F^k(M) = H^{n,0}_pr(M) \oplus \cdots \oplus H^{k,n-k}_pr(M)\) from
which we have the decreasing filtration \(H^npr(M, \mathbb{C}) = F^0 \supset \cdots \supset F^n.\) We know that

\[ \dim_{\mathbb{C}} F^k = f^k, \]

\[ H^{n}_pr(X, \mathbb{C}) = F^k \oplus F^{n-k+1}, \quad \text{and} \quad H^{k,n-k}_pr(M) = F^k \cap F^{n-k}. \]

In terms of the Hodge filtration \(F^n \subset \cdots \subset F^0 = H^npr(M, \mathbb{C}),\) the Hodge-Riemann relations [2] and [3] can be written as

\[ Q(F^k, F^{n-k+1}) = 0, \quad \text{and} \]

\[ (\sqrt{-1})^{2k-1} Q(v, \bar{v}) > 0, \quad \text{for} \quad 0 \neq v \in H^{k,n-k}_pr(M). \]

The period domain \(D\) for polarized Hodge structures and its compact dual \(\hat{D}\) are then
defined as follows,

\[ D = \{ F^n \subset \cdots \subset F^0 = H^npr(X, \mathbb{C}) \mid [4], [5] \text{ and } [7] \text{ hold} \}, \]

\[ \hat{D} = \{ F^n \subset \cdots \subset F^0 = H^npr(X, \mathbb{C}) \mid [4] \text{ and } [6] \text{ hold} \}, \]

where the the period domain \(D\) is an open subset of \(\hat{D}\). From the definition of period
domain, we naturally get the Hodge bundles over \(\hat{D}\) by associating to each point in \(\hat{D}\) the
vector spaces \( \{ F^k \}_{k=0}^n \) in the Hodge filtration of that point. Without confusion we will also denote by \( F^k \) the bundle with \( F^k \) as the fiber, for each \( 0 \leq k \leq n \).

Given a complex manifold \( S \), a variation of Hodge structures is a holomorphic map
\[
\Phi : S \to D,
\]
such that the tangent map satisfies the Griffiths transversality, that is,
\[
\Phi_* : T^{1,0}S \to \bigoplus_{k=0}^n \text{Hom}(F^k/F^{k+1}, F^{k-1}/F^k).
\]
The map \( \Phi \) is called a period map on \( S \). We denote by \( \Phi(p) = \{ F^n_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p \} \) for any \( p \in S \).

**Remark 2.2.** We remark the notation change for the primitive cohomology groups. Since we mainly consider \( \Phi(q) = \{ F^n_q \subseteq \cdots \subseteq F^0_q = H^{n}_{pr}(M, \mathbb{C}) \} \), which is defined using the primitive cohomology, by abuse of notation, we will simply use \( H^n(M, \mathbb{C}) \) and \( H^{k,n-k}(M) \) to denote the primitive cohomology groups \( H^n_{pr}(M, \mathbb{C}) \) and \( H^{k,n-k}_{pr}(M) \) respectively. Moreover, we will use cohomology to mean primitive cohomology in the rest of the paper.

Recall that we can identify a point \( \Phi(p) = \{ F^n_p \subseteq F^{n-1}_p \subseteq \cdots \subseteq F^0_p \} \in D \) with its Hodge decomposition \( \bigoplus_{k=0}^n H^{k,n-k}_p \); and thus with any fixed basis for the Hodge decomposition. In particular, with the fixed adapted basis for the Hodge decomposition of the base point, we have matrix representations of elements in the above Lie groups and Lie algebras.

### 2.3. Moduli space and Teichmüller space

The **moduli space** \( \mathcal{M} \) of polarized complex structures on a given differential manifold \( X \) is a complex analytic space consisting of biholomorphically equivalent pairs \( (M, L) \) of complex structures and ample line bundles. Let us denote by \( [M, L] \) the point in \( \mathcal{M} \) corresponding to a pair \( (M, L) \), where \( M \) is a complex manifold diffeomorphic to \( X \) and \( L \) is an ample line bundle on \( M \). If there is a biholomorphic map \( f \) between \( M \) and \( M' \) with \( f^*L' = L \), then \( [M, L] = [M', L'] \in \mathcal{M} \).

Before we make assumptions on the moduli space of Calabi–Yau type manifolds, let us recall that a family of compact complex manifolds \( \pi : \mathcal{U} \to \mathcal{T} \) is **versal** at a point \( p \in \mathcal{T} \) if it satisfies the following conditions:

1. If given a complex analytic family \( \iota : \mathcal{V} \to \mathcal{S} \) of compact complex manifolds with a point \( s \in \mathcal{S} \) and a biholomorphic map \( f_0 : V = \iota^{-1}(s) \to U = \pi^{-1}(p) \), then there exists a holomorphic map \( g \) from a neighbourhood \( \mathcal{N} \subseteq \mathcal{S} \) of the point \( s \) to \( \mathcal{T} \) and a holomorphic map \( f : \iota^{-1}(\mathcal{N}) \to \mathcal{U} \) with \( \iota^{-1}(\mathcal{N}) \subseteq \mathcal{V} \) such that they satisfy that \( g(s) = p \) and \( f|_{\iota^{-1}(s)} = f_0 \), with the following commutative diagram

\[
\begin{array}{ccc}
\iota^{-1}(\mathcal{N}) & \xrightarrow{f} & \mathcal{U} \\
\downarrow{\iota} & & \downarrow{\pi} \\
\mathcal{N} & \xrightarrow{g} & \mathcal{T}.
\end{array}
\]

2. For all \( g \) satisfying the above condition, the tangent map \( (dg)_s \) is uniquely determined.

If a family \( \pi : \mathcal{U} \to \mathcal{T} \) is versal at every point \( p \in \mathcal{T} \), then it is a **versal family** on \( \mathcal{T} \). In this case, if \( \pi : \mathcal{U} \to \mathcal{T} \) is a versal family, then the family \( \iota^{-1}(\mathcal{N}) \to \mathcal{N} \) is complex...
analytically isomorphic to the pull-back of the family \( \pi : U \to T \) by the holomorphic map \( g \). If a complex analytic family satisfies the above condition (1), then the family is called complete at \( p \). If a complex analytic family \( \pi : U \to T \) of compact complex manifolds is complete at each point of \( T \) and versal at the point \( p \in T \), then the family \( \pi : W \to T \) is called the Kuranishi family of the complex manifold \( V = \pi^{-1}(p) \). If the family is complete at each point in a neighbourhood of \( p \in T \) and versal at \( p \), then the family is called a local Kuranishi family at \( p \in T \). We refer the reader to page 8-10 in [24], page 94 in [17] or page 19 in [21] for more details about versal families and local Kuranishi families.

Let \((M, L)\) be a polarized Calabi–Yau type manifold. For any integer \( m \geq 3 \), we call a basis of the quotient space \( (H_n(M, \mathbb{Z})/\text{Tor})/m(H_n(M, \mathbb{Z})/\text{Tor}) \) a level \( m \) structure on the polarized Calabi–Yau type manifold. Let \( Z_m \) be the moduli space of polarized Calabi–Yau type manifolds with level \( m \) structure. In this paper, we assume that \( Z_m \) is a connected quasi-projective smooth complex manifold with a versal family of Calabi–Yau type manifolds with level \( m \) structures.

\[
X_{Z_m} \to Z_m, \tag{8}
\]
containing \( M \) as a fiber and polarized by an ample line bundle \( \mathcal{L}_{Z_m} \) on \( X_{Z_m} \). We remark that these conditions hold for the smooth moduli \( Z_m \) of Calabi–Yau manifold with level \( m \) structure according to the work of Szendrői in [23] and Viehweg in [24].

We now define the Teichmüller space \( T \) to be a complex analytic space consisting of biholomorphically equivalent triples of \((M, L, \{\gamma_1, \cdots, \gamma_h^m\})\), where \( \{\gamma_1, \cdots, \gamma_h^m\} \) is a basis of \( H_n(M, \mathbb{Z})/\text{Tor} \). To be more precise, for two triples \((M, L, \{\gamma_1, \cdots, \gamma_h^m\})\) and \((M', L', \{\gamma'_1, \cdots, \gamma'^h_{n}\})\), if there exists a biholomorphic map \( f : M \to M' \) with

\[
f^*L' = L, \quad f^*\gamma_i' = \gamma_i \quad \text{for} \quad 1 \leq i \leq h^n,
\]
then \([M, L, \{\gamma_1, \cdots, \gamma_h^m\}] = [M', L', \{\gamma'_1, \cdots, \gamma'^h_{n}\}] \in T\). For simplicity we use \([M, L, \gamma]\) to denote the triple \([M, L, \{\gamma_1, \cdots, \gamma_h^m\}]\). By this definition, we know that the Teichmüller space \( T \) is a covering space of \( Z_m \) with the covering map denoted by \( \pi_m : T \to Z_m \). We then have the pull-back family \( \pi : U \to T \) of \( X_{Z_m} \to Z_m \).

**Proposition 2.3.** The Teichmüller space \( T \) is a smooth and connected complex manifold and the family

\[
\pi : U \to T, \tag{9}
\]
containing \( M \) as a fiber, is local Kuranishi at each point of \( T \).

**Proof.** One notices that there is a natural covering map \( \pi_m : T \to Z_m \) for any \( m \geq 3 \) according to the definition of \( T \). Therefore \( T \) is a smooth and connected complex manifold, as \( Z_m \) is a connected smooth complex manifold. Since the family \( \text{(8)} \) is a versal family at each point of \( Z_m \) and that \( \pi_m \) is locally biholomorphic, the pull-back family via \( \pi_m \) is also versal at each point of \( T \). By the definition of local Kuranishi family, we get that \( U \to T \) is local Kuranishi at each point of \( T \). \( \Box \)

**Remark 2.4.** As a consequence, since the versal family \( \pi : U \to T \) is local Kuranishi at any point \( p \in T \), the Kodaira-Spencer map

\[
\kappa : T^{1,0}_p T^{-1,0}_p \to H^{0,1}(M_p, T^{1,0}_p M_p), \quad \text{for any} \quad p \in T
\]
is an isomorphism. Therefore, one may conclude that $\dim T = \dim H^{0,1}(M_p, T^{1,0}(M_p)) = \dim H^{s-1,n-s+1}_p = N$. We refer the reader to Chapter 4 in [14] for more details about versal family, deformation of complex structures and the Kodaira-Spencer map.

For a polarized Calabi–Yau type manifold $(M, L)$, let

$$\text{Aut}(M, L) = \{\alpha : M \to M | \alpha^* L = L\}$$

be the group of biholomorphic maps on $M$ preserving the polarization $L$. We have a natural representation of $\text{Aut}(M, L)$,

$$\sigma : \text{Aut}(M, L) \to \text{Aut}(H_n(M, \mathbb{Z})/\text{Tor}), \quad \alpha \mapsto \alpha^*.$$

Moreover, as $H^n(M, \mathbb{C})$ is the dual space of $H_n(M, \mathbb{C})$, we may view $\sigma$ as an automorphism of $H^n(M, \mathbb{C})$ via the duality.

**Theorem 2.5.** The Teichmüller space $T$ of polarized and marked Calabi–Yau type manifolds is simply connected.

**Proof of Theorem 2.5.** Suppose towards a contradiction that $T$ is not simply connected. Assume that $\tilde{T}$ is the universal cover of $T$ with a covering map $\pi : \tilde{T} \to T$. Then for each point $p = [M, L, \gamma] \in T$, the preimage $\pi^{-1}(p) = \{p_i | i \in I\} \subseteq \tilde{T}$ satisfies that $|I| > 1$. Let $\tilde{T} : \tilde{U} \to \tilde{T}$ be the pull back family of $\pi : U \to T$ with the following commutative diagram,

$$\xymatrix{ \tilde{U} \ar[r] \ar[d]^{\tilde{\pi}} & U \ar[d]^{\pi} \\
\tilde{T} \ar[r]_{\pi} & T. }$$

Then $\tilde{T} : \tilde{U} \to \tilde{T}$ is also a versal family of polarized and marked Calabi–Yau type manifolds. On the other hand, for any $i \neq j \in I$, there exists a deck transformation $\alpha : \tilde{T} \to \tilde{T}$ of the covering map $\pi : \tilde{T} \to T$ such that $\alpha(p_i) = p_j$. Because $\pi^{-1}(p_i) = \pi^{-1}(p_j) = [M, L, \gamma]$, this $\alpha$ can be viewed as a biholomorphic map on $M$ which preserves the polarization $L$ and the marking $\gamma$. Therefore $\alpha \in \ker(\sigma)$. Let us now first show the following lemma. We remark that in the first part of the proof of this lemma, we will mainly adopt the analogous arguments of Lemma 2.6 in [23].

**Lemma 2.6.** Let $[M, L, \gamma]$ be a fiber of the versal family $\pi : \tilde{U} \to \tilde{T}$. Then for any $\alpha \in \ker \sigma$, there is an extension $\tilde{\alpha}$ on $\tilde{U}$ leaving the base space $\tilde{T}$ fixed and also the polarization fixed on each fiber of the family.

**Proof of Lemma 2.6.** Let $p \in \tilde{T}$ with $\pi^{-1}(p) = [M, L, \gamma]$. Since the family $\pi : \tilde{U} \to \tilde{T}$ is local Kuranishi at any point, there exists a neighborhood $U_p$ of $p \in \tilde{T}$ with holomorphic morphisms $\tilde{\alpha} : \pi^{-1}(U_p) \to \tilde{U}$ and $f : U_p \to \tilde{T}$ such that the following diagram is commutative,

$$\xymatrix{ \pi^{-1}(U_p) \ar[r]^{\tilde{\alpha}} \ar[d]^{\pi} & \tilde{U} \ar[d]^{\tilde{\pi}} \\
U_p \ar[r]^{f} & \tilde{T}. }$$
To see $\tilde{\alpha}$ leaves the base space $\tilde{\mathcal{T}}$ fixed, it is sufficient to show that $f$ is the identity map on $U_p$. Notice that $f(p) = p$ from the definition of $f$. Suppose towards a contradiction that $f$ were not the identity map on $U_p$. Then the tangent map $f_* : T_pU_p \to T_p\tilde{\mathcal{T}}$ is not identity either. By the analogous discussion in Remark 2.4, we know that the Kodaira-Spencer map $\kappa : T_pU_p = T_p^{1,0}\mathcal{T} \to H^{0,1}(M_p, T^{1,0}M_p)$ is an isomorphism. Moreover, since the contraction map $\sigma : H^{0,1}(M_p, T^{1,0}M) \to H^{s-1,n-s+1}(M_p)$ is also an isomorphism, we have the following commutative diagram,

\[
\begin{array}{ccc}
T_p^{1,0}\mathcal{T} & \xrightarrow{f_*^{-1}} & T_p^{1,0}U_p \\
\downarrow & & \downarrow \\
H^{0,1}(M, T^{1,0}M) & \xrightarrow{\sigma} & H^{s-1,n-s+1}(M). \\
\end{array}
\]

But $\alpha \in \ker(\sigma)$ implies the map $\alpha^* : H^{s-1,n-s+1}(M) \to H^{s-1,n-s+1}(M)$ is the identity map. This contradicts the assumption that $f_*$ is not identity.

It is not hard to show that for each $q \in U_p$ the biholomorphic map $\tilde{\alpha}_q$ on the fiber $M_q$ preserves the polarization $L$. Indeed, because $H^2(M, \mathbb{Z})$ is a discrete group, we have

\[c_1(\tilde{\alpha}_qL) = c_1(\tilde{\alpha}_pL) = c_1(L), \quad \text{for any} \quad q \in U_p.\]

Since $M$ is simply connected, holomorphic line bundles on $M$ are uniquely determined by the first Chern class. Therefore we conclude that $\tilde{\alpha}_q^*(L) = L$ for any $q \in U_q$.

Let us define a sheaf $\mathcal{Z}$ on the base space $\tilde{\mathcal{T}}$ as follows: for any open set $U \subset \tilde{\mathcal{T}}$, we assign the group $\mathcal{Z}(U)$ to be all the biholomorphic maps $\alpha_U : \tilde{\pi}^{-1}(U) \to \tilde{\pi}^{-1}(U)$ which leaves the open set $U$ fixed and preserving the polarization on each fiber. In other words, for any $\alpha_U \in \mathcal{Z}(U)$, we have the following commutative diagram,

\[
\begin{array}{ccc}
\tilde{\pi}^{-1}(U) & \xrightarrow{\alpha_U} & \tilde{\pi}^{-1}(U) \\
\downarrow & & \downarrow \\
U & \xrightarrow{id} & U; \\
\end{array}
\]

and the restriction of $\alpha_U$ on the fiber $M_q = \tilde{\pi}^{-1}(q)$ preserves the polarization $L$ over $M_q$ for any $q \in U$. If $V \subseteq U$ is open, then the restriction map of the sheaf is given by

\[\text{res} : \mathcal{Z}(U) \to \mathcal{Z}(V), \quad \alpha_U \to \alpha_U|_{\tilde{\pi}^{-1}(V)}.\]

From the local extension result discussed above, we have that for any point $p \in \tilde{\mathcal{T}}$, there exists a neighborhood $U_p \subset \tilde{\mathcal{T}}$ such that any $\alpha \in \ker(\sigma)$ on $M_p$ can be extended to the family $\tilde{\pi}^{-1}(U_p)$. This means the restriction map

\[\text{res} : \mathcal{Z}(U_p) \to \ker(\sigma), \quad \alpha_U \mapsto \alpha_U|_{M_p}\]

is an isomorphism. Therefore the sheaf $\mathcal{Z}$ is a locally constant sheaf. Using the fact that $\tilde{\mathcal{T}}$ is simply connected and Proposition 3.9 in [26], we have $\mathcal{Z}$ is a constant sheaf. This means $\mathcal{Z}(\tilde{\mathcal{T}}) = \ker(\sigma)$. Therefore, for each point $p \in \tilde{\mathcal{T}}$ and $\alpha \in \ker(\sigma)$, there is a global
section \( \tilde{\alpha} \in \mathcal{S}(\tilde{T}) \) such that \( \tilde{\alpha}|_{M_p} = \alpha \). By the definition of the sheaf \( \mathcal{S} \), we have the commutative diagram,

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\tilde{\alpha}} & \tilde{U} \\
\downarrow{\tilde{\pi}} & & \downarrow{\tilde{\pi}} \\
\tilde{T} & \xrightarrow{id} & \tilde{T}.
\end{array}
\]

And restricted to each fiber \( M_q \) the morphism \( \tilde{\alpha}|_{M_q} \) preserves the polarization \( L \).

\[\square\]

**Corollary 2.7.** The action of \( \ker(\sigma) \) on \( \tilde{T} \) is trivial.

**Proof of Corollary 2.7.** For each element \( \alpha \in \ker(\sigma) \), we have a global extension \( \tilde{\alpha} \) acts on the family \( \tilde{U} \) with the commutative diagram

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\tilde{\alpha}} & \tilde{U} \\
\downarrow{\pi} & & \downarrow{\pi} \\
\tilde{T} & \xrightarrow{\tilde{f}} & \tilde{T},
\end{array}
\]

where \( \alpha \) acts on \( \tilde{T} \) by the holomorphic map \( \tilde{f} : \tilde{T} \to \tilde{T} \). But Lemma 2.6 implies that \( \tilde{f} \) is the identity map. Therefore the action of \( \alpha \) on \( \tilde{T} \) is trivial. \[\square\]

**Proof of Theorem 2.5 (continued).** Recall that the deck transformation \( \alpha : \tilde{T} \to \tilde{T} \) satisfies \( \alpha(p_i) = p_j \neq p_i \) can also be viewed as a biholomorphic map on \( M \) with \( \alpha \in \ker(\sigma) \). However, Corollary 2.7 shows that the action of \( \ker(\sigma) \) on \( \tilde{T} \) is trivial. Thus \( \alpha = \text{Id} : \tilde{T} \to \tilde{T} \), which contradicts with the assumption that \( \alpha(p_i) = p_j \neq p_i \). \[\square\]

**Corollary 2.8.** The covering map \( \pi_m : T \to Z_m \) is a universal covering map.

**Remark 2.9.** By Corollary 2.8 we know that the Teichmüller space \( T \) is the universal cover of the smooth moduli space \( Z_m \) for any \( m \geq 3 \). Therefore, we can alternatively define the Teichmüller space of Calabi–Yau type manifolds to be the universal cover of the smooth moduli \( Z_m \) with any \( m \geq 3 \). And this definition is well-defined, as in fact, let \( m_1 \) and \( m_2 \) be two different integers, and \( U_1 \to T_1 \) and \( U_2 \to T_2 \) be two versal families constructed via level \( m_1 \) and level \( m_2 \) structures respectively as above, and both of which contain \( M \) as a fiber. By using the properties that \( T_1 \) and \( T_2 \) are simply connected and the definition of versal family, we have a biholomorphic map \( f : T_1 \to T_2 \), such that the versal family \( U_1 \to T_1 \) is the pull back of the versal family \( U_2 \to T_2 \) by \( f \). Thus these two families are isomorphic to each other.

2.4. The period map on the Teichmüller space and induced Hodge metric.

**Remark 2.10.** To start this section, we make an important remark of notation change of primitive cohomology groups. Since primitive cohomology groups are the major type of cohomology groups we are considering in this paper, we will simply use \( H^n(M, \mathbb{C}) \) and \( H^{k,n-k}(M) \) to denote \( H^n_{pr}(M, \mathbb{C}) \) and \( H^{pr,k,n-k}(M) \) respectively. Using these notations, we will denote \( h^{k,n-k} = \dim H^{k,n-k}(M) \), \( f^k = \dim F^k \), and \( m = f^{n-s} = \sum_{k=n-s} h^{k,n-k} \).

For any point \( p \in T \), let \( M_p \) be the fiber of the family \( \pi : U \to T \) at \( p \), which is a polarized and marked Calabi–Yau type manifold. Since the Teichmüller space is
simply connected and we have fixed a basis of the middle cohomology group modulo torsions. We identify the basis of \( H_n(M, \mathbb{Z})/\text{Tor} \) to a lattice \( \Lambda \) as in [23]. This gives us a canonical identification of the middle dimensional de Rham cohomology of \( M \) to that of the background manifold \( X \), that is, \( H^n(M, \mathbb{C}) \cong H^n(X, \mathbb{C}) \). Therefore, we can use this to identify \( H^n(M_p, \mathbb{C}) \) for all fibers on \( T \), and thus get a canonical trivial bundle \( H^n(M_p, \mathbb{C}) \times T \). The Hodge numbers \( \{h^{s,n-s}, h^{s-1,n-s+1}, \ldots, h^{n-s,s}\} \) of \( M_p \) are independent to the choice of \( p \in T \). Therefore we have corresponding data of \( H^n(M, \mathbb{C}), H^n(M, \mathbb{Z}), \{h^k,n-k\} \) and the intersection form on \( H^n(M, \mathbb{C}) \) on a given polarized and marked Calabi–Yau type manifolds.

Let \( D \) be the period domain of the Hodge structures of the \( n \)-th primitive cohomology of the Calabi–Yau type manifold as defined as in Section [22]. Then the period map is defined by assigning to each point \( p \in T \) the Hodge structure on \( H^n(M_p, \mathbb{C}) \) by the definition of Calabi–Yau type manifolds. For this reason, we will denote \( \Phi(p) = \{F^s(M_p) \subseteq \cdots \subseteq F^0(M_p)\} \). We denote \( F^k(M_p) \) by \( F_p^k \) for simplicity. Recall that we have described general period maps in Section [22]. We remark that the period maps have several nice properties, the reader may refer to Chapter 10 in [25] for details. Among these properties, the one we are most interested in is the following Griffiths transversality of the period map, that is, the tangent map satisfies,

\[
\Phi_s(v) \in \bigoplus_{k=n-s}^s \text{Hom}(F_p^k/F_p^{k+1}, F_p^{k-1}/F_p^k), \quad \text{for any } p \in T \text{ and any } v \in T^{1,0} T
\]

where \( F^{s+1} = 0 \), or equivalently as in page 29 of [5] that \( \frac{\partial H^{p,q}}{\partial \tau} \subseteq H^{p,q}_\tau \oplus H^{p-1,q+1}_\tau \), where \( \tau \) is a local coordinate function on \( T \). We have an immediate property of tangent map of the period map as follows.

**Proposition 2.11.** For any \( p \in T \) and any generator \( [\Omega_p] \) of \( F_p^s \), the map

\[
P_p^s \circ \Phi_s : T^{1,0}_p T \simeq H^{0,1}(M_p, T^{1,0} M_p) \to \text{Hom}(F_p^s, F_p^{s-1}/F_p^s) \simeq H^{s-1,n-s+1}_p
\]

is an isomorphism, where \( \Phi_s \) is the tangent map of \( \Phi \).

**Proof.** The first isomorphism \( T^{1,0}_p T \simeq H^{0,1}(M_p, T^{1,0} M_p) \) follows from the property that the Kodaira-Spencer map is an isomorphism. The second isomorphism

\[
\text{Hom}(F_p^s, F_p^{s-1}/F_p^s) \simeq H^{s-1,n-s+1}_p
\]

follows from the condition on Calabi–Yau type manifold that \( \dim F_p^s = 1 \). This isomorphism depends on the choice of the generator \( [\Omega_p] \). Now it is clear that the map

\[
P_p^s \circ \Phi_s : H^{0,1}(M_p, T^{1,0} M_p) \to H^{s-1,n-s+1}_p
\]

is given by contraction \( P_p^s \circ \Phi_s(v) = [\kappa(v) \cdot \Omega_p] \). This contraction map is an isomorphism by the third condition in the definition of Calabi–Yau type manifolds.

From this proposition, one notices that the tangent map of the period map \( \Phi : T \to D \) is a nondegenerate map for any \( p \in T \). In particular, the period map \( \Phi : T \to D \) is locally injective. Let us denote the period map on the moduli space \( \Phi_{Z_m} : Z_m \to D/\Gamma \), where \( \Gamma \) denotes the global monodromy group which acts properly and discontinuously.
on the period domain $D$. Then $\Phi: \mathcal{T} \to D$ is the lifting of $\Phi_{\mathcal{Z}_m} \circ \pi_m$ with $\pi_m: \mathcal{T} \to \mathcal{Z}_m$ the universal covering map. Therefore, $\Phi_{\mathcal{Z}_m}$ is also locally injective.

In [6], Griffiths and Schmid studied the so-called Hodge metric on the period domain $D$, which is a complete homogeneous metric. We denote it by $h$. Since both $\Phi$ and $\Phi_{\mathcal{Z}_m}$ are locally injective, the pull-backs of $h$ by $\Phi_{\mathcal{Z}_m}$ and $\Phi$ on $\mathcal{Z}_m$ and $\mathcal{T}$ respectively are both well-defined Kähler metrics. By abuse of notation, we still call these pull-back metrics the Hodge metrics, and they are still denoted by $h$.

3. Holomorphic affine structure on the Teichmüller space

In Section 3.1 we review some properties of the period domain from Lie group and Lie algebra point of view. In Section 3.2 we fix a base point $p \in \mathcal{T}$ and introduce the unipotent space $N_+ \subseteq D$, which is biholomorphic to $\mathbb{C}^d$. Then we show that the image $\Phi(\mathcal{T})$ is bounded in $N_+ \cap D$ with respect to the Euclidean metric on $N_+$. In Section 3.3 using the property that $\Phi(\mathcal{T}) \subseteq N_+$, we define a holomorphic map $\tau: \mathcal{T} \to \mathbb{C}^N$. Then Proposition 2.11 implies that $\tau$ defines a local coordinate chart around each point in $\mathcal{T}$, and this shows that $\tau: \mathcal{T} \to \mathbb{C}^N$ defines a holomorphic affine structure on $\mathcal{T}$.

3.1. Preliminary. Let us briefly recall some properties of the period domain from Lie group and Lie algebra point of view. All of the results in this section is well-known to the experts in the subject. The purpose to give details is to fix notations. One may either skip this section or refer to [6] and [21] for most of the details. One may also find this preliminary in [2].

Let us simply use the notations $H_C := H_{pr}^n(M, \mathbb{C})$ and $H_R := H_{pr}^n(M, \mathbb{C})$. The orthogonal group of the bilinear form $Q$ is a linear algebraic group which is defined over $\mathbb{Q}$. The group $G_C$ of the $\mathbb{C}$-rational points and the group $G_R$ of real points in $G_C$ are the following:

$$G_C = \{ g \in GL(H_C) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_C \}.$$  

$$G_R = \{ g \in GL(H_R) | Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_R \}.$$  

Then $G_C$ acts on $\tilde{D}$ transitively and $G_R$ acts transitively on $D$.

Consider the period map $\Phi: S \to D$. Fix a point $p \in S$ with the image $o := \Phi(p) = \{ F_p^1 \subseteq \cdots \subseteq F_p^n \} \in D$. The points $p \in S$ and $o \in D$ may be referred as the base points or reference points. Notice that a linear transformation $g \in G_C$ preserves the base point precisely when $gF_p^k = F_p^k$. Thus we have the identification

$$\tilde{D} \simeq G_C/B, \quad \text{with} \quad B = \{ g \in G_C | gF_p^k = F_p^k, \text{ for any } k \}.$$  

Similarly, one obtains an analogous identification $D \simeq G_R/V$ with $V = G_R \cap B$, and the embedding corresponding to the inclusion $G_R/V = G_R/G_R \cap B \subseteq G_C/B$.

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_C$ can be described as

$$\mathfrak{g} = \{ X \in \text{End}(H_C) | Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_C \}.$$  

It is a simple complex Lie algebra, which contains $\mathfrak{g}_0 = \{ X \in \mathfrak{g} | XH_R \subseteq H_R \}$ as a real form, that is, $\mathfrak{g} = \mathfrak{g}_0 \oplus i \mathfrak{g}_0$. With the inclusion $G_R \subseteq G_C$, $\mathfrak{g}_0$ becomes Lie algebra of $G_R$. One observes that the reference Hodge structure $\{ H_p^{k,n-k} \}_{k=0}^n$ induces a Hodge structure
of weight zero on \( \text{End}(H^n_{pr}(M, \mathbb{C})) \), namely,

\[
g = \bigoplus_{k \in \mathbb{Z}} g^{k,-k}, \quad \text{with } g^{k,-k} = \{ X \in g | X H^{r,n-r}_p \subseteq H^{r+k,n-r-k}_p \}.
\]

Since the Lie algebra \( b \) of \( B \) consists of those \( X \in g \) that preserves the reference Hodge filtration \( \{ F^l_p \subseteq \cdots \subseteq F^0_p \} \), one has

\[
b = \bigoplus_{k \geq 0} g^{k,-k}.
\]

The Lie algebra \( v \) of \( V \) is thus \( v = g_0 \cap b = g_0 \cap b \cap \bar{b} = g_0 \cap g^{0,0} \). With the above isomorphisms, one gets that the holomorphic tangent space of \( \tilde{D} \) at the base point \( o \) is naturally isomorphic to \( g/b \).

Let us consider the nilpotent Lie subalgebra \( n_+ := \bigoplus_{k \geq 1} g^{k,k} \). Then one gets the holomorphic isomorphism \( g/b \simeq n_+ \). We take the unipotent group \( N_+ = \exp(n_+) \).

As \( \text{Ad}(g)(g^{k,-k}) \) is in \( \bigoplus_{i \geq 1} g^{i,-i} \) for each \( g \in B \), the sub-Lie algebra \( b \oplus g^{-1,1}/b \subseteq g/b \) defines an \( \text{Ad}(B) \)-invariant subspace. By left translation via \( G_{\mathbb{C}} \), \( b \oplus g^{-1,1}/b \) gives rise to a \( G_{\mathbb{C}} \)-invariant holomorphic subbundle of the holomorphic tangent bundle at the base point. It will be denoted by \( T^{1,0}_{o,h}\tilde{D} \), and will be referred to as the holomorphic horizontal tangent bundle at the base point. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle at the base point \( o \), restricted to \( D \), determines a subbundle \( T^{1,0}_o D \) of the holomorphic tangent bundle \( T^1_o D \) of \( D \) at the base point. The \( G_{\mathbb{C}} \)-invariance of \( T^{1,0}_o\tilde{D} \) implies the \( G_{\mathbb{C}} \)-invariance of \( T^{1,0}_o D \). As another interpretation of this holomorphic horizontal bundle at the base point, one has

\[
T^{1,0}_o\tilde{D} \simeq T^{1,0}_o D \cap \bigoplus_{k=1}^n \text{Hom}(F^k_p/F^{k+1}_p, F^{k-1}_p/F^k_p).
\]

In [21], Schmid call a holomorphic mapping \( \Psi : M \to \tilde{D} \) of a complex manifold \( M \) into \( \tilde{D} \) horizontal if at each point of \( M \), the induced map between the holomorphic tangent spaces takes values in the appropriate fibre \( T^{1,0}\tilde{D} \). It is easy to see that the period map \( \Phi : \mathcal{T} \to D \) is horizontal since \( \Phi_* (T^{1,0}_p \mathcal{T}) \subseteq T^{1,0}_o D \) for any \( p \in \mathcal{T} \). Since \( D \) is an open set in \( \tilde{D} \), we have the following relation:

\[
T^{1,0}_o D = T^{1,0}_o \tilde{D} \cong b \oplus g^{-1,1}/b \hookrightarrow g/b \cong n_+.
\]

Remark 3.1. With a fixed base point, we can view \( N_+ = N_+( \text{base point} ) \cong N_+ B/B \subseteq \tilde{D} \), or we identify an element \( c \in N_+ \) with \([c] = cB \) in \( \tilde{D} \). Thus we can identify \( N_+ \) with its unipotent orbit of the base point in \( \tilde{D} \). In particular, when the base point \( o \) is in \( D \), we have \( N_+ \cap D \subseteq D \).

Let us introduce the notion of an adapted basis for the given Hodge decomposition or the Hodge filtration. For any \( p \in \mathcal{T} \) and \( f^k = \dim F^k_p \) for any \( n-s \leq k \leq s \), we call a basis

\[
\xi = \{ \xi_0, \xi_1, \ldots, \xi_N, \ldots, \xi_{f^{k+1}}, \ldots, \xi_{f^{k-1}}, \ldots, \xi_{f^2}, \ldots, \xi_{f^1}, \xi_{f^0} \}
\]

of \( H^n(M, \mathbb{C}) \) an adapted basis for the given Hodge decomposition

\[
H^n(M, \mathbb{C}) = H^{s,n-s}_p \oplus H^{s-1,n-s+1}_p \oplus \cdots \oplus H^{n-s+1,s-1}_p \oplus H^{n-s,s}_p.
\]
if it satisfies $H_p^{k,n-k} = \text{Span}_C \{\xi_{f+1}, \ldots, \xi_{f+k} \}$ with $\dim H_p^{k,n-k} = f^k - f^{k+1}$. We call a basis
\[ \zeta = \{\zeta_0, \zeta_1, \ldots, \zeta_N, \ldots, \zeta_{f+1}, \ldots, \zeta_{f+k}, \ldots, \zeta_{f^2}, \ldots, \zeta_{f^3-0}, \zeta_{f^3-1} \} \]
of $H^n(M_p, C)$ an adapted basis for the given filtration
\[ F^s \subseteq F^{s-1} \subseteq \cdots \subseteq F^{n-s} \]
if it satisfies $F^k = \text{Span}_C \{\zeta_0, \ldots, \zeta_{f^k-1} \}$ with $\dim F^k = f^k$. Moreover, unless otherwise pointed out, the matrices in this paper are $m \times m$ matrices, where $m = f^{n-s} = f^0$. The blocks of the $m \times m$ matrix $T$ is set as follows: for each $0 \leq \alpha, \beta \leq 2s - n$, the $(\alpha, \beta)$-th block $T^{\alpha,\beta}$ is
\[ T^{\alpha,\beta} = [T_{ij}(\tau)]_{f^{-\alpha + s+1} \leq i \leq f^{-\alpha + s-1}, f^{-\beta + s+1} \leq j \leq f^{-\beta + s-1}} \]
where $T_{ij}$ is the entries of the matrix $T$, and $f^{n+1}$ is defined to be zero. In particular, $T = [T^{\alpha,\beta}]$ is called a block lower triangular matrix if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

**Remark 3.2.** We remark that by fixing a base point, we can identify the above quotient Lie groups or Lie algebras with their orbits in the corresponding quotient Lie algebras or Lie groups. For example, $N_+ \cong g/b$, $g^{-1,1} \cong b \oplus g^{-1,1}/b$, and $N_+ \cong N_+B/B \subseteq D$. We can also identify a point $\Phi(p) = \{ F_p^s \subseteq F_p^{s-1} \subseteq \cdots \subseteq F_p^{n-s} \} \in D$ with its Hodge decomposition $\bigoplus_{k=0}^n H_p^{k,n-k}$, and thus with any fixed adapted basis of the corresponding Hodge decomposition for the base point, we have matrix representations of elements in the above Lie groups and Lie algebras. For example, elements in $N_+$ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in $B$ can be realized as nonsingular block upper triangular matrices.

We shall review and collect some facts about the structure of simple Lie algebra $g$ in our case. Again one may refer to [6] and [21] or [2] for more details. Let $\theta : g \to g$ be the Weil operator, which is defined by
\[ \theta(X) = (-1)^p X \quad \text{for} \quad X \in g^{p,-p}. \]
Then $\theta$ is an involutive automorphism of $g$, and is defined over $\mathbb{R}$. The $(+1)$ and $(-1)$ eigenspaces of $\theta$ will be denoted by $\mathfrak{e}$ and $\mathfrak{p}$ respectively. Moreover, set
\[ \mathfrak{e}_0 = \mathfrak{e} \cap g_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap g_0. \]
The fact that $\theta$ is an involutive automorphism implies
\[ g = \mathfrak{e} \oplus \mathfrak{p}, \quad g_0 = \mathfrak{e}_0 \oplus \mathfrak{p}_0, \quad [\mathfrak{e}, \mathfrak{e}] \subseteq \mathfrak{e}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{p}, \quad [\mathfrak{e}, \mathfrak{p}] \subseteq \mathfrak{p}. \]
Let us consider $g_\mathbb{C} = \mathfrak{e}_0 \oplus \sqrt{-1} \mathfrak{p}_0$. Then $g_\mathbb{C}$ is a real form for $g$. Recall that the killing form $B(\cdot, \cdot)$ on $g$ is defined by
\[ B(X, Y) = \text{Trace}(\text{ad}(X) \circ \text{ad}(Y)) \quad \text{for} \quad X, Y \in g. \]
A semisimple Lie algebra is compact if and only if the Killing form is negative definite. Thus it is not hard to check that $g_\mathbb{C}$ is actually a compact real form of $g$, while $g_0$ is a non-compact real form. Recall that $G_\mathbb{R} \subseteq G_\mathbb{C}$ is the subgroup which corresponds to the subalgebra $g_0 \subseteq g$. Let us denote the connected subgroup $G_\mathbb{C} \subseteq G_\mathbb{C}$ which corresponds to the subalgebra $g_\mathbb{C} \subseteq g$. Let us denote the complex conjugation of $g$ with respect to the
compact real form by $\tau_c$, and the complex conjugation of $\mathfrak{g}$ with respect to the compact real form by $\tau_0$.

The intersection $K = G_c \cap G_\mathbb{R}$ is then a compact subgroup of $G_\mathbb{R}$, whose Lie algebra is $\mathfrak{k}_0 = \mathfrak{g}_\mathbb{R} \cap \mathfrak{g}_c$. With the above notations, Schmid showed in [21] that $K$ is a maximal compact subgroup of $G_\mathbb{R}$, and it meets every connected component of $G_\mathbb{R}$. Moreover, $V = G_\mathbb{R} \cap B \subseteq K$.

As remarked in §1 in [6] of Griffiths and Schmid, one gets the following important proposition about the Cartan subalgebra of $\mathfrak{g}_0$.

**Proposition 3.3.** There exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{t}_0$.

Proposition 3.3 implies that the simple Lie algebra $\mathfrak{g}_0$ in our case is a simple Lie algebra of first category as defined in §4 in [19]. In the upcoming part, we will briefly derive the result of a simple Lie algebra of first category in Lemma 3 in [20]. One may also refer to [28] Lemma 2.2.12 at pp. 141-142 for the same result.

Let us still use the above notations of the Lie algebras we consider. By Proposition 4, we can take $\mathfrak{h}_0$ to be a Cartan subalgebra of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{t}_0$. Let us denote $\mathfrak{h}$ to be the complexification of $\mathfrak{h}_0$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Write $\mathfrak{h}_0^* = \text{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_k^* = \sqrt{-1}\mathfrak{h}_0^*$. Then $\mathfrak{h}_k^*$ can be identified with $\mathfrak{h}_k := \sqrt{-1}\mathfrak{h}_0$ by duality using the restriction of the Killing form $\mathcal{B}$ of $\mathfrak{g}$ to $\mathfrak{h}_k$. Let $\rho \in \mathfrak{h}_k^* \simeq \mathfrak{h}_k$, one can define the following subspace of $\mathfrak{g}$

$$\mathfrak{g}^\rho = \{ x \in \mathfrak{g} | [h, x] = \rho(h)x \text{ for all } h \in \mathfrak{h} \}.$$  

An element $\varphi \in \mathfrak{h}_k^* \simeq \mathfrak{h}_k$ is called a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\mathfrak{g}^\varphi \neq \{0\}$.

Let $\Delta \subseteq \mathfrak{h}_k^* \simeq \mathfrak{h}_k$ denote the space of nonzero $\mathfrak{h}$-roots. Then each root space

$$\mathfrak{g}^\varphi = \{ x \in \mathfrak{g} | [h, x] = \varphi(h)x \text{ for all } h \in \mathfrak{h} \}$$

belongs to some $\varphi \in \Delta$ is one-dimensional over $\mathbb{C}$, generated by a root vector $e_\varphi$.

Since the involution $\theta$ is a Lie-algebra automorphism fixing $\mathfrak{k}$, we have $[h, \theta(e_\varphi)] = \varphi(h)\theta(e_\varphi)$ for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$. Thus $\theta(e_\varphi)$ is also a root vector belonging to the root $\varphi$, so $e_\varphi$ must be an eigenvector of $\theta$. It follows that there is a decomposition of the roots $\Delta$ into $\Delta_\mathfrak{r} \cup \Delta_\mathfrak{p}$ of compact roots and non-compact roots with root spaces $\mathbb{C}e_\varphi \subseteq \mathfrak{r}$ and $\mathfrak{p}$ respectively. The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ determines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^\varphi.$$  

There also exists a Weyl base $\{h_i, 1 \leq i \leq l; e_\varphi, \text{ for any } \varphi \in \Delta\}$ with $l = \text{rank}(\mathfrak{g})$ such that $\text{Span}_\mathbb{C}\{h_1, \cdots, h_l\} = \mathfrak{h}$, $\text{Span}_\mathbb{C}\{e_\varphi\} = \mathfrak{g}^\varphi$ for each $\varphi \in \Delta$, and

$$\tau_c(h_i) = \tau_0(h_i) = -h_i, \quad \text{for any } 1 \leq i \leq l;$$

$$\tau_c(e_\varphi) = \tau_0(e_\varphi) = -e_{-\varphi} \quad \text{for any } \varphi \in \Delta_\mathfrak{r}; \quad \tau_0(e_\varphi) = -\tau_c(e_\varphi) = e_\varphi \quad \text{for any } \varphi \in \Delta_\mathfrak{p}.$$
With respect to this Weyl base, we have
\[ t_0 = h_0 + \sum_{\varphi \in \Delta_f} \mathbb{R}(e_\varphi - e_{-\varphi}) + \sum_{\varphi \in \Delta_f} \mathbb{R}\sqrt{-1}(e_\varphi + e_{-\varphi}); \]
\[ p_0 = \sum_{\varphi \in \Delta_p} \mathbb{R}(e_\varphi + e_{-\varphi}) + \sum_{\varphi \in \Delta_p} \mathbb{R}\sqrt{-1}(e_\varphi - e_{-\varphi}). \]

**Lemma 3.4.** Let \( \Delta \) be the set of \( \mathfrak{h} \)-roots as above. Then for each root \( \varphi \in \Delta \), there is an integer \(-n \leq k \leq n\) such that \( e_\varphi \in \mathfrak{g}^{k,-k} \). In particular, if \( e_\varphi \in \mathfrak{g}^{k,-k} \), then \( \tau_0(e_\varphi) \in \mathfrak{g}^{k,-k} \) for any \(-n \leq k \leq n\).

**Proof.** Let \( \varphi \) be a root, and \( e_\varphi \) be the generator of the root space \( \mathfrak{g}^\varphi \), then \( e_\varphi = \sum_{k=-n}^n e_{-k,k} \), where \( e_{-k,k} \in \mathfrak{g}^{k,-k} \). Because \( \mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0,0} \), the Lie bracket \( [e_{-k,k}, h] \in \mathfrak{g}^{k,-k} \) for each \( k \).

Then the condition \( [e_\varphi, h] = \varphi(h)e_\varphi \) implies that
\[ \sum_{k=-n}^n [e_{-k,k}, h] = \sum_{k=-n}^n \varphi(h) e_{-k,k} \quad \text{for each } h \in \mathfrak{h}. \]

By comparing the type, we get
\[ [e_{-k,k}, h] = \varphi(h) e_{-k,k} \quad \text{for each } h \in \mathfrak{h}. \]

Therefore \( e_{-k,k} \in \mathfrak{g}^\varphi \) for each \( k \). As \( \{ e_{-k,k} \}_{k=-n}^n \) forms a linear independent set, but \( \mathfrak{g}^\varphi \) is one dimensional, thus there is only one \(-n \leq k \leq n\) with \( e_{-k,k} \neq 0 \).

Let us now introduce a lexicographic order (cf. pp.41 in [28] or pp.416 in [19]) in the real vector space \( \mathfrak{h}_\mathbb{R} \) as follows: we fix an ordered basis \( e_1, \ldots, e_l \) for \( \mathfrak{h}_\mathbb{R} \). Then for any \( h = \sum_{i=1}^l \lambda_i e_i \in \mathfrak{h}_\mathbb{R} \), we call \( h > 0 \) if the first nonzero coefficient is positive, that is, \( \lambda_1 = \cdots = \lambda_k = 0, \lambda_{k+1} > 0 \) for some \( 1 \leq k < l \). For any \( h, h' \in \mathfrak{h}_\mathbb{R} \), we say \( h > h' \) if \( h - h' > 0 \), \( h < h' \) if \( h - h' < 0 \) and \( h = h' \) if \( h - h' = 0 \). In particular, let us identify the dual spaces \( \mathfrak{h}_\mathbb{R}^* \) and \( \mathfrak{h}_\mathbb{R} \), thus \( \Delta \subseteq \mathfrak{h}_\mathbb{R} \). Let us choose a maximal linearly independent subset \( \{ s_1, \ldots, s_k \} \) of \( \Delta_f \), then a linearly independent subset \( \{ e_1, \ldots, e_{l-k} \} \) of \( \Delta_p \) such that \( \{ e_1, \ldots, e_{l-k}, s_1, \ldots, s_k \} \) forms an ordered basis for \( \mathfrak{h}_\mathbb{R}^* \). Then define the above lexicographic order in \( \mathfrak{h}_\mathbb{R} \) using the ordered basis \( \{ e_1, \ldots, e_{l-k}, s_1, \ldots, s_k \} \). In this way, we can also define
\[ \Delta^+ = \{ \varphi > 0 : \varphi \in \Delta \}; \quad \Delta^+_p = \Delta^+ \cap \Delta_p. \]

Similarly we can define \( \Delta^-, \Delta^-_p, \Delta^+_f \), and \( \Delta^-_f \). Then one can conclude the following lemma from Lemma 2.2.10 and Lemma 2.2.11 at pp.141 in [28],

**Lemma 3.5.** Using the above notation, we have
\[ (\Delta_f + \Delta^+_p) \cap \Delta \subseteq \Delta^+_p; \quad (\Delta^-_p + \Delta^+_p) \cap \Delta = \emptyset. \]

If one defines
\[ p^\pm = \sum_{\varphi \in \Delta^\pm_p} \mathfrak{g}^\varphi \subseteq \mathfrak{p}, \]
then \( \mathfrak{p} = p^+ \oplus p^- \) and \( [p^+, p^+] = 0, [p^+, p^-] \subseteq \mathfrak{t}, [\mathfrak{t}, p^\pm] \subseteq p^\pm \).

**Definition 3.6.** Two different roots \( \varphi, \psi \in \Delta \) are said to be strongly orthogonal if and only if \( \varphi \pm \psi \notin \Delta \cup \{0\} \), which is denoted by \( \varphi \perp \psi \).
For the real simple Lie algebra \( g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \) which has a Cartan subalgebra \( \mathfrak{h}_0 \) in \( \mathfrak{k}_0 \), the maximal abelian subspace of \( \mathfrak{p}_0 \) can be described as in the following lemma, which is a slight extension of a lemma of Harish-Chandra in [7]. One may refer to Lemma 3 in [20] or Lemma 2.2.12 at pp. 141–142 in [28] for more details. One may also refer to [2] for a detailed proof of this lemma.

**Lemma 3.7.** There exists a set of strongly orthogonal noncompact positive roots \( \Lambda = \{ \varphi_1, \cdots, \varphi_r \} \subseteq \Delta^+_0 \) such that

\[
a_0 = \sum_{i=1}^{r} \mathbb{R} (e_{\varphi_i} + e_{-\varphi_i})
\]
is a maximal abelian subspace in \( \mathfrak{p}_0 \).

For further use, we also state a proposition about the maximal abelian subspaces of \( \mathfrak{p}_0 \) according to Ch V in [8],

**Proposition 3.8.** Let \( \mathfrak{a}'_0 \) be an arbitrary maximal abelian subspaces of \( \mathfrak{p}_0 \), then there exists an element \( k \in K \) such that \( k \cdot \mathfrak{a}_0 = \mathfrak{a}'_0 \). Moreover, we have

\[
\mathfrak{p}_0 = \bigcup_{k \in K} \text{Ad}(k) \cdot \mathfrak{a}_0,
\]
where \( \text{Ad} \) denotes the adjoint action of \( K \) on \( \mathfrak{a}_0 \).

### 3.2. Boundedness of the period map.

Now let us fix the base point \( p \in \mathcal{T} \) with \( \Phi(p) \in D \). Then according to the above remark, \( N_+ \) can be viewed as a subset in \( \tilde{D} \) by identifying it with its orbit in \( \tilde{D} \) with base point \( \Phi(p) \). Let us also fix an adapted basis \( (\eta_0, \cdots, \eta_{m-1}) \) for the Hodge decomposition of the base point \( \Phi(p) \in D \). Then we can identify elements in \( N_+ \) with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. We define

\[
\tilde{\mathcal{T}} = \Phi^{-1}(N_+).
\]

At the base point \( \Phi(p) = o \in N_+ \cap D \), the tangent space \( T_o^{1,0}N_+ = T_o^{1,0}D \simeq \mathfrak{n}_+ \simeq N_+ \), then the Hodge metric on \( T_o^{1,0}D \) induces an Euclidean metric on \( N_+ \). In the proof of the following lemma, we require all the root vectors to be unit vectors with respect to this Euclidean metric.

Because the period map is a horizontal map, and the geometry of horizontal slices of the period domain \( D \) is similar to Hermitian symmetric space as discussed in detail in [6], the proof the following theorem is basically an analogue of the proof of the Harish-Chandra embedding theorem for Hermitian symmetric spaces, see for example [11]. We remark that one may also find an analogous theorem in [2].

**Theorem 3.9.** The restriction of the period map \( \Phi : \mathcal{T} \to N_+ \) is bounded in \( N_+ \) with respect to the Euclidean metric on \( N_+ \).

**Proof.** We need to show that there exists \( 0 \leq C < \infty \) such that for any \( q \in \tilde{\mathcal{T}} \),

\[
d_E(\Phi(p), \Phi(q)) \leq C,
\]
where \( d_E \) is the Euclidean distance on \( N_+ \).

For any \( q \in \tilde{\mathcal{T}} \), there exists a vector \( X^+ \in \mathfrak{g}^{-1,1}_0 \subseteq \mathfrak{n}_+ \) and a real number \( T_0 \) such that \( \beta(t) = \exp(tX^+) \) defines a geodesic \( \beta : [0, T_0] \to N_+ \subseteq G_C \) from \( \beta(0) = I \) to \( \beta(T_0) \) with \( \pi_1(\beta(T_0)) = \Phi(q) \), where \( \pi_1 : N_+ \to N_+B/B \subseteq D \) is the projection map with the fixed
base point $\Phi(p) = o \in D$. In this proof, we will not distinguish $N_+ \subseteq G_C$ from its orbit $N_+B/B \subseteq D$ with fixed base point $\Phi(p) = o$.

Consider $X^- = \tau_0(X^+) \in \mathfrak{g}^{-1,1}$, then $X = X^+ + X^- \in (\mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1}) \cap \mathfrak{g}_0$. For any $q \in T$, there exists $T_1$ such that $\gamma = \exp(tX) : [0, T_1] \to G_{\mathbb{R}}$ defines a geodesic from $\gamma(0) = I$ to $\gamma(T_1) \in G_{\mathbb{R}}$ such that $\pi_2(\gamma(T_1)) = \Phi(q) \in D$, where $\pi_2 : G_{\mathbb{R}} \to G_{\mathbb{R}}/V \simeq D$ denotes the projection map with the fixed base point $\Phi(p) = o$.

Let $\Lambda = \{\varphi_1, \ldots, \varphi_r\} \subseteq \Delta_+^\ast$ be a set of strongly orthogonal roots given in Proposition 3.8. We denote $x_{\varphi_i} = e_{\varphi_i} + e_{-\varphi_i}$ and $y_{\varphi_i} = \sqrt{-1}(e_{\varphi_i} - e_{-\varphi_i})$ for any $\varphi_i \in \Lambda$. Then

$$a_0 = \mathbb{R}x_{\varphi_1} \oplus \cdots \oplus \mathbb{R}x_{\varphi_r}, \quad \text{and} \quad a_c = \mathbb{R}y_{\varphi_1} \oplus \cdots \oplus \mathbb{R}y_{\varphi_r},$$

are maximal abelian spaces in $p_0$ and $\sqrt{-1}p_0$ respectively.

Since $X \in \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{1,-1} \subseteq p_0$, by Proposition 3.8 there exists $k \in K$ such that $X \in Ad(k) \cdot a_0$. As the adjoint action of $K$ on $p_0$ is unitary action and we are considering the length in this proof, we may simply assume that $X \in a_0$ up to a unitary transformation. With this assumption, there exists $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq r$ such that

$$X = \lambda_1 x_{\varphi_1} + \lambda_2 x_{\varphi_2} + \cdots + \lambda_r x_{\varphi_r}.$$ 

Since $a_0$ is commutative, we have

$$\exp(tX) = \prod_{i=1}^r \exp(t\lambda_i x_{\varphi_i}).$$

Now for each $\varphi_i \in \Lambda$, we have $\text{Span}_\mathbb{C}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ with

$$h_{\varphi_i} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\varphi_i} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

and $\text{Span}_\mathbb{R}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{R})$ with

$$\sqrt{-1}h_{\varphi_i} \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad x_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y_{-\varphi_i} \mapsto \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$ 

Since $\Lambda = \{\varphi_1, \ldots, \varphi_r\}$ is a set of strongly orthogonal roots, we have that

$$\mathfrak{g}_C(\Lambda) = \text{Span}_\mathbb{C}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{C}))^r,$$

and $\mathfrak{g}_R(\Lambda) = \text{Span}_\mathbb{R}\{x_{\varphi_i}, y_{-\varphi_i}, \sqrt{-1}h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{R}))^r$.

In fact, we know that for any $\varphi, \psi \in \Lambda$ with $\varphi \neq \psi$, $[e_{\pm \varphi}, e_{\pm \psi}] = 0$ since $\varphi$ is strongly orthogonal to $\psi$; $[h_{\varphi}, h_{\psi}] = 0$, since $\mathfrak{h}$ is abelian; $[h_{\varphi}, e_{\pm \varphi}] = \sqrt{-1}[e_{\pm \varphi}, e_{\pm \varphi}] = 0$, $[e_{\varphi}, e_{\pm \psi}] = 0$.

Then we denote $G_C(\Lambda) = \exp(\mathfrak{g}_C(\Lambda)) \simeq (SL_2(\mathbb{C}))^r$ and $G_R(\Lambda) = \exp(\mathfrak{g}_R(\Lambda)) = (SL_2(\mathbb{R}))^r$, which are subgroups of $G_C$ and $G_R$ respectively. With the fixed reference point $o = \Phi(p)$, we denote $D(\Lambda) = G_R(\Lambda)(o)$ and $S(\Lambda) = G_C(\Lambda)(o)$ to be the corresponding orbits of these two subgroups, respectively. Then we have the following isomorphisms,

$$D(\Lambda) = G_R(\Lambda) \cdot B/B \simeq G_R(\Lambda)/G_R(\Lambda) \cap V;$$

$$S(\Lambda) \cap (N_+B/B) = (G_C(\Lambda) \cap N_+) \cdot B/B \simeq (G_C(\Lambda) \cap N_+)/(G_C(\Lambda) \cap N_+ \cap B).$$

With the above notations, we will show that (i). $D(\Lambda) \subseteq S(\Lambda) \cap (N_+B/B) \subseteq \tilde{D}$; (ii). $D(\Lambda)$ is bounded inside $S(\Lambda) \cap (N_+B/B)$. 

Notice that since $X \in g^{-1,1} \oplus g^{1,1}$. By Proposition 3.4, we know that for each pair of roots $\{e_\varphi, e_{-\varphi}\}$, either $e_\varphi \in g^{-1,1} \subseteq n_+$ and $e_{-\varphi} \in g^{1,1}$, or $e_\varphi \in g^{1,1}$ and $e_{-\varphi} \in g^{-1,1} \subseteq n_+$. For notation simplicity, for each pair of root vectors $\{e_\varphi, e_{-\varphi}\}$, we may assume the one in $g^{-1,1} \subseteq n_+$ to be $e_\varphi$ and denote the one in $g^{1,1}$ by $e_{-\varphi}$. In this way, one can check that $\{\varphi_1, \cdots, \varphi_r\}$ may not be a set in $\Delta_0^+$, but it is a set of strongly orthogonal roots in $\Delta_0$. Therefore, we have the following description of the above groups,

$$
G_R(\Lambda) = \exp(g_R(\Lambda)) = \exp(\text{Span}_R\{x_{\varphi_1}, y_{\varphi_1}, \sqrt{-1}h_{\varphi_1}, \cdots, x_{\varphi_r}, y_{-\varphi_r}, \sqrt{-1}h_{-\varphi_r}\})
$$

$$
G_R(\Lambda) \cap V = \exp(g_R(\Lambda) \cap v_0) = \exp(\text{Span}_R\{\sqrt{-1}h_{\varphi_1}, \cdots, \sqrt{-1}h_{-\varphi_r}\})
$$

$$
G_C(\Lambda) \cap N_+ = \exp(g_C(\Lambda) \cap n_+) = \exp(\text{Span}_C\{e_{\varphi_1}, e_{-\varphi_2}, \cdots, e_{-\varphi_r}\});
$$

$$
G_C(\Lambda) \cap B = \exp(g_C(\Lambda) \cap B) = \exp(\text{Span}_C(h_{\varphi_1}, e_{-\varphi_2}, \cdots, h_{-\varphi_r}, e_{-\varphi_r}))
$$

Thus by the isomorphisms in (15) and (16), we have

$$
D(\Lambda) \simeq \prod_{i=1}^r \exp(\text{Span}_R\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) / \exp(\text{Span}_R\{\sqrt{-1}h_{\varphi_i}\}),
$$

$$
S(\Lambda) \cap (N_+B/B) \simeq \prod_{i=1}^r \exp(\text{Span}_C\{e_{\varphi_i}\}).
$$

Let us denote $G_C(\varphi_i) = \exp(\text{Span}_C\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}) \simeq SL_2(\mathbb{C})$, $S(\varphi_i) = G_C(\varphi_i)(o)$, and $G_R(\varphi_i) = \exp(\text{Span}_R\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) \simeq SL_2(\mathbb{R})$, $D(\varphi_i) = G_R(\varphi_i)(o)$. On one hand, each point in $S(\varphi_i) \cap (N_+B/B)$ can be represented by

$$
\exp(ze_{\varphi_i}) = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} \quad \text{for some } z \in \mathbb{C}.
$$

Thus $S(\varphi_i) \cap (N_+B/B) \simeq \mathbb{C}$. One the other hand, denote $z = a + bi$ for some $a, b \in \mathbb{R}$, then

$$
\exp(ax_{\varphi_i} + by_{\varphi_i}) = \begin{bmatrix} \cosh |z| & \frac{b}{|z|} \sinh |z| \\ \frac{b}{|z|} \sinh |z| & \cosh |z| \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 \\ \frac{b}{|z|} \tanh |z| & 1 \end{bmatrix} \begin{bmatrix} \cosh |z| & 0 \\ 0 & (\cosh |z|)^{-1} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{|z|} \tanh |z| \\ 0 & 1 \end{bmatrix}
$$

$$
= \exp \left[ \left( \frac{b}{|z|} \tanh |z| \right) e_{\varphi_i} \right] \exp \left[ \left( - \log \cosh |z| \right) h_{\varphi_i} \right] \exp \left[ \left( \frac{b}{|z|} \tanh |z| \right) e_{-\varphi_i} \right].
$$

So elements in $D(\varphi_i)$ can be represented by $\exp[(z/|z|)(\tanh |z|)]e_{\varphi_i}$, i.e. the lower triangular matrix

$$
\begin{bmatrix} 1 & 0 \\ \frac{b}{|z|} \tanh |z| & 1 \end{bmatrix},
$$

in which $\frac{b}{|z|} \tanh |z|$ is a point in the unit disc $\mathfrak{D}$ of the complex plane. Therefore the $D(\varphi_i)$ is a unit disc $\mathfrak{D}$ in the complex plane $S(\varphi_i) \cap (N_+B/B)$. Therefore

$$
D(\Lambda) \simeq \mathfrak{D}^r \quad \text{and} \quad S(\Lambda) \cap N_+ \simeq \mathbb{C}^r.
$$
So we have obtained both (i) and (ii). As a consequence, we get that for any \( q \in \mathcal{T} \), \( \Phi(q) \in D(\Lambda) \). This implies

\[
d_E(\Phi(p), \Phi(q)) \leq \sqrt{r}
\]

where \( d_E \) is the Euclidean distance on \( S(\Lambda) \cap (N_+ B/B) \).

To complete the proof, we only need to show that \( S(\Lambda) \cap (N_+ B/B) \) is totally geodesic in \( N_+ B/B \). In fact, the tangent space of \( N_+ \) at the base point is \( n_+ \) and the tangent space of \( S(\Lambda) \cap N_+ B/B \) at the base point is \( \text{Span}_\mathbb{C}\{e_{v_1}, e_{v_2}, \ldots, e_{v_r}\} \). Since \( \text{Span}_\mathbb{C}\{e_{v_1}, e_{v_2}, \ldots, e_{v_r}\} \) is a sub-Lie algebra of \( n_+ \), the corresponding orbit \( S(\Lambda) \cap N_+ B/B \) is totally geodesic in \( N_+ B/B \). Here the basis \( \{e_{v_1}, e_{v_2}, \ldots, e_{v_r}\} \) is an orthonormal basis with respect to the pull-back Euclidean metric. \( \square \)

In order to prove Corollary \[3.12\] we first show that \( \mathcal{T} \setminus \mathcal{\hat{T}} \) is an analytic subvariety of \( \mathcal{T} \) with \( \text{codim}_\mathbb{C}(\mathcal{T} \setminus \mathcal{\hat{T}}) \geq 1 \).

**Lemma 3.10.** Let \( p \in \mathcal{T} \) be the base point with \( \Phi(p) = \{F_p^s \subseteq F_p^{s-1} \subseteq \cdots \subseteq F_p^{n-s}\} \). Let \( q \in \mathcal{T} \) be any point with \( \Phi(q) = \{F_q^s \subseteq F_q^{s-1} \subseteq \cdots \subseteq F_q^{n-s}\} \), then \( \Phi(q) \in N_+ \) if and only if \( F_q^k \) is isomorphic to \( F_p^k \) for all \( n - s \leq k \leq s \).

**Proof.** For any \( q \in \mathcal{T} \), we choose an arbitrary adapted basis \( \{\zeta_0, \ldots, \zeta_{m-1}\} \) for the given Hodge filtration \( \{F_q^s \subseteq F_q^{s-1} \subseteq \cdots \subseteq F_q^{n-s}\} \). Recall that \( \{\eta_0, \ldots, \eta_{m-1}\} \) is the adapted basis for the Hodge filtration \( \{F_p^s \subseteq F_p^{s-1} \subseteq \cdots \subseteq F_p^{n-s}\} \) for the base point \( p \). Let \( [A^{i,j}(q)]_{0 \leq i,j \leq 2n-s} \) be the transition matrix between the basis \( \{\eta_0, \ldots, \eta_{m-1}\} \) and \( \{\zeta_0, \ldots, \zeta_{m-1}\} \) for the same vector space \( H^n(M, \mathbb{C}) \), where \( A^{i,j}(q) \) are the corresponding blocks. Recall that elements in \( N_+ \) and \( B \) have matrix representations with the fixed adapted basis at the base point: elements in \( N_+ \) can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in \( B \) can be realized as nonsingular block upper triangular matrices. Therefore \( \Phi(q) \in N_+ = N_+ B/B \subseteq \hat{D} \) if and only if its matrix representation \( [A^{i,j}(q)]_{0 \leq i,j \leq 2n-s} \) can be decomposed as \( \hat{L}(q) \cdot \hat{U}(q) \), where \( \hat{L}(q) \) is a nonsingular block lower triangular matrix with identities in the diagonal blocks, and \( \hat{U}(q) \) is a nonsingular block upper triangular matrix. By basic linear algebra, we know that \( [A^{i,j}(q)] \) has such decomposition if and only if \( \det[A^{i,j}(q)]_{0 \leq i,j \leq n-k} \neq 0 \) for any \( n - s \leq k \leq s \).

In particular, we know that \( [A(q)^{i,j}]_{0 \leq i,j \leq s-k} \) is the transition map between the bases of \( F_p^k \) and \( F_q^k \). Therefore, \( \det([A(q)^{i,j}]_{0 \leq i,j \leq s-k}) \neq 0 \) if and only if \( F_q^k \) is isomorphic to \( F_p^k \). \( \square \)

**Lemma 3.11.** The subset \( \mathcal{\hat{T}} \) is an open dense submanifold in \( \mathcal{T} \), and \( \mathcal{T} \setminus \mathcal{\hat{T}} \) is an analytic subvariety of \( \mathcal{T} \) with \( \text{codim}_\mathbb{C}(\mathcal{T} \setminus \mathcal{\hat{T}}) \geq 1 \).

**Proof.** From Lemma \[3.10\] one can see that \( \hat{D} \setminus N_+ \subseteq \hat{D} \) is defined as an analytic subvariety by equations

\[
\det[A^{i,j}(q)]_{0 \leq i,j \leq s-k} = 0 \quad \text{for each } n - s \leq k \leq s.
\]

Therefore \( N_+ \) is dense in \( \hat{D} \), and that \( \hat{D} \setminus N_+ \) is an analytic subvariety, which is close in \( \hat{D} \) and \( \text{codim}_\mathbb{C}(\hat{D} \setminus N_+) \geq 1 \). We consider the period map \( \Phi : \mathcal{T} \to \hat{D} \) as a holomorphic map to \( \hat{D} \), then \( \mathcal{T} \setminus \mathcal{\hat{T}} = \Phi^{-1}(\hat{D} \setminus N_+) \) is the pre-image of the holomorphic map \( \Phi \). So \( \mathcal{T} \setminus \mathcal{\hat{T}} \) is also an analytic subvariety and a close set in \( \mathcal{T} \). Because \( \mathcal{T} \) is smooth and connected, if \( \dim(\mathcal{T} \setminus \mathcal{\hat{T}}) = \dim \mathcal{T} \), then \( \mathcal{T} \setminus \mathcal{\hat{T}} = \mathcal{T} \) and \( \mathcal{\hat{T}} = \emptyset \). But this contradicts to the fact...
that the reference point $p \in \mathcal{T}$. So we have $\dim(\mathcal{T} \setminus \mathcal{T}) < \dim \mathcal{T}$, and consequently $\text{codim}_C(\mathcal{T} \setminus \mathcal{T}) \geq 1$. □

**Corollary 3.12.** The image of $\Phi : \mathcal{T} \to D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on $N_+$.

**Proof.** According to Lemma 3.11, $\mathcal{T} \setminus \mathcal{T}$ is an analytic subvariety of $\mathcal{T}$ and the complex codimension of $\mathcal{T} \setminus \mathcal{T}$ is at least one; by Theorem 3.9, the holomorphic map $\Phi : ˇ\mathcal{T} \to N_+ \cap D$ is bounded in $N_+$ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\Phi' : \mathcal{T} \to N_+ \cap D$ such that $\Phi'|_\mathcal{T} = \Phi|_\mathcal{T}$. Since as holomorphic maps, $\Phi'$ and $\Phi$ agree on the open subset $\mathcal{T}$, they must be the same on the entire $\mathcal{T}$. Therefore, the image of $\Phi$ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on $N_+$. As a consequence, we also get $\mathcal{T} = ˇ\mathcal{T} = \Phi^{-1}(N_+)$. □

### 3.3. Holomorphic affine structure on the Teichmüller space

We first review the definition of complex affine manifolds. One may refer to page 215 of [15] for more details.

**Definition 3.13.** Let $M$ be a complex manifold of complex dimension $n$. If there is a coordinate cover $\{(U_i, \varphi_i); i \in I\}$ of $M$ such that $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ is a holomorphic affine transformation on $\mathbb{C}^n$ whenever $U_i \cap U_k$ is not empty, then $\{(U_i, \varphi_i); i \in I\}$ is called a complex affine coordinate cover on $M$ and it defines a holomorphic affine structure on $M$.

Let us still fix an adapted basis $(\eta_0, \cdots, \eta_{m-1})$ for the Hodge decomposition of the base point $\Phi(p) \in D$. Recall that we can identify elements in $N_+$ with nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix, and element in $B$ with nonsingular block upper triangular matrices. Therefore $N_+ \cap B = Id$. By Corollary 3.12, we know that $\mathcal{T} = \Phi^{-1}(N_+)$. Thus we get that each $\Phi(q)$ can be uniquely determined by a matrix, which we will still denote by $\Phi(q) = [\Phi_{ij}(q)]_{0 \leq i, j \leq m-1}$ of the form of nonsingular block lower triangular matrices whose diagonal blocks are all identity submatrix. Thus we can define a holomorphic map

$$\tau : \mathcal{T} \to \mathbb{C}^N \cong H^{s,n-s}_{p} \quad q \mapsto (1,0)\text{-block of the matrix } \Phi(q) \in N_+,$$

that is

$$\tau(q) = (\tau_1(q), \tau_2(q), \cdots, \tau_N(q)) = (\Phi_{10}(q), \Phi_{20}(q), \cdots, \Phi_{N0}(q)).$$

**Remark 3.14.** If we define the following projection map with respect to the base point and the its pre-fixed adapted basis to the Hodge decomposition,

$$F : N_+ \cap D \to H^{s,n-s}_{p} \cong \mathbb{C}^N, \quad F(q) = (\eta_1, \cdots, \eta_N)F^{(1,0)} = F_{10}\eta_1 + \cdots + F_{N0}\eta_N,$$

where $F^{(1,0)}$ is the $(1,0)$-block of the unipotent matrix $F$, according to our convention in (14), then $\tau = P \circ \Phi : \mathcal{T} \to \mathbb{C}^N$.

**Proposition 3.15.** The holomorphic map $\tau = (\tau_1, \cdots, \tau_N) : \mathcal{T} \to \mathbb{C}^N$ defines a coordinate chart around each point $q \in \mathcal{T}$.

**Proof.** We have that the generator $\{\eta_0\} \subseteq H^{n-s}(M_p, \Omega^s(M_p))$, the generators $\{\eta_1, \cdots, \eta_N\} \subseteq H^{n-s-1}(M_p, \Omega^{s-1}(M_p))$, and the generators $\{\eta_{N+1}, \cdots, \eta_{m-1}\} \subset \bigoplus_{k \geq n-s+2} H^k(M_p, \Omega^{n-k}(M_p))$.

On one hand, the 0-th column of the matrix $\Phi(q) \in N_+$ for each $q \in \mathcal{T}$ gives us the following data:

$$\Omega : \mathcal{T} \to F^s; \quad \Omega(q) = (\eta_0, \cdots, \eta_{m-1})(\Phi_{00}(q), \Phi_{10}(q), \cdots, \Phi_{N0}(q), \cdots)^T$$

$$= \eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \cdots + \tau_N(q)\eta_N + \eta_0(q) \in F_q^n \cong H^{n-s}(M_q, \Omega^s(M_q)),$$
where \(g_0(q) \in \bigoplus_{k \geq n-s+2} H^k(M_p, \Omega^{n-k}(M_p))\).

The 1-st to \(N\)-th columns of \(\Phi(q) \in \mathbb{N}_+\) give us the following data:

\[
\theta_1(q) = \eta_1 + g_1(q), \quad \ldots, \quad \theta_N(q) = \eta_N + g_N(q) \in F_q^{s-1},
\]

where \(g_k(q) \in \bigoplus_{k \geq n-s+2} H^k(M_p, \Omega^{n-k}(M_p))\), such that \(\{\Omega(q), \theta_1(q), \ldots, \theta_N(q)\}\) forms a basis for \(F_q^{s-1}\) for each \(q \in \mathcal{T}\).

On the other hand, by Proposition 2.11 we know that for any holomorphic coordinate \(\{\sigma_1, \ldots, \sigma_N\}\) around \(q\), \(\{\Omega(q), \partial\Omega(q)/\partial \sigma_1, \ldots, \partial\Omega(q)/\partial \sigma_N\}\) forms a basis of \(F_q^{s-1}\).

As both \(\{\Omega(q), \theta_1(q), \ldots, \theta_N(q)\}\) and \(\{\Omega(q), \partial\Omega(q)/\partial \sigma_1, \ldots, \partial\Omega(q)/\partial \sigma_N\}\) are bases for \(F_q^{s-1}\), there exists \(\{X_1, \ldots, X_N\}\) such that \(X_k = \sum_{i=1}^N a_{ik} \partial\Omega(q)/\partial \sigma_i\) for each \(1 \leq k \leq N\) such that

\[
(18) \quad \theta_k = X_k(\Omega(q)) + \lambda_k \Omega(q) \quad \text{for} \quad 1 \leq k \leq N.
\]

Note that we have \(X_k(\Omega(q)) = X_k(\tau_1(q))\eta_1 + \cdots + X_k(\tau_N(q))\eta_N + X_k(g_0(q))\) and \(\theta_k = \eta_k + g_k(q)\), where \(X_k(g_0(q)), g_k(q) \in \bigoplus_{k \geq n-s+2} H^k(M_q, \Omega^{n-k}(M_q))\). By comparing the types of classes in (18), we get

\[
(19) \quad \lambda_k = 0, \quad \text{and} \quad X_k(\Omega(q)) = \theta_k(q) = \eta_k + g_k(q) \quad \text{for each} \quad 1 \leq k \leq N.
\]

Since \(\{\theta_1(q), \ldots, \theta_N(q)\}\) are linearly independent set for \(F_q^{s-1}(M_q)\), we know that \(\{X_1, \ldots, X_N\}\) are also linearly independent in \(T_q^{1,0}(\mathcal{T})\). Therefore \(\{X_1, \ldots, X_N\}\) forms a basis for \(T_q^{1,0}(\mathcal{T})\). Without loss of generality, we may assume \(X_k = \partial\Omega(q)/\partial \sigma_k\) for each \(1 \leq k \leq N\). Thus by (19), we have

\[
\frac{\partial\Omega(q)}{\partial \sigma_k} = \theta_k = \eta_k + g_k(q) \quad \text{for any} \quad 1 \leq k \leq N.
\]

Since we also have

\[
\frac{\partial\Omega(q)}{\partial \sigma_k} = \frac{\partial}{\partial \sigma_k}(\eta_0 + \tau_1(q)\eta_1 + \tau_2(q)\eta_2 + \cdots + \tau_N(q)\eta_N + g_0(q)),
\]

we get \(\left[\frac{\partial \tau_i(q)}{\partial \sigma_j}\right]_{1 \leq i,j \leq N} = I_N\). This shows that \(\tau_q : T_q^{1,0}(\mathcal{T}) \to T_{\tau(q)}(\mathbb{C}^N)\) is an isomorphism for each \(q \in \mathcal{T}\), as \(\left\{\frac{\partial}{\partial \sigma_1}, \ldots, \frac{\partial}{\partial \sigma_N}\right\}\) is a basis for \(T_q^{1,0}(\mathcal{T})\).

Thus the holomorphic map \(\tau : \mathcal{T} \to \mathbb{C}^N\) defines local coordinate map around each point \(q \in \mathcal{T}\). In particular, the map \(\tau\) itself gives a global holomorphic coordinate for \(\mathcal{T}\). Thus the transition maps are all identity maps. Therefore,

**Theorem 3.16.** The global holomorphic coordinate map \(\tau : \mathcal{T} \to \mathbb{C}^N\) defines a holomorphic affine structure on \(\mathcal{T}\).

**Remark 3.17.** This affine structure on \(\mathcal{T}\) depends on the choice of the base point \(p\). Affine structures on \(\mathcal{T}\) defined in this ways by fixing different base point may not be compatible with each other.
4. Hodge metric completion space with level structure

In this section, we introduce the Hodge metric completion space with level $m$ structure $\mathcal{T}_m^H$, which is the universal cover of $\mathcal{Z}_m^H$, with $\mathcal{Z}_m^H$ the completion space of the smooth moduli space $\mathcal{Z}_m$ with respect to the Hodge metric on $\mathcal{Z}_m$. We fix the lifting maps $i_m : \mathcal{T} \to \mathcal{T}_m^H$ and $\Phi_m^H : \mathcal{T}_m^H \to D$ for each $m \geq 3$ (see diagram (20)) such that $\Phi = \Phi_m^H \circ i_m$. Then we prove that there is a complex affine structure on $\mathcal{T}_m^H$ by substantially using Theorem 3.16.

Recall that in Section 2.3, we used the smooth moduli space $\mathcal{Z}_m$, which denotes the smooth moduli space of polarized Calabi–Yau type manifolds with level $m$ structure. We then proved that the Teichmüller space $\mathcal{T}$ is the universal cover of $\mathcal{Z}_m$ for any $m \geq 3$ with the universal covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$.

We have assumed that $\mathcal{Z}_m$ is quasi-projective. Then by the work of Viehweg in [21], we can find a smooth projective compactification $\overline{\mathcal{Z}_m}$ such that $\mathcal{Z}_m$ is open in $\overline{\mathcal{Z}_m}$ and the complement $\overline{\mathcal{Z}_m} \setminus \mathcal{Z}_m$ is a divisor of normal crossing. Therefore, $\mathcal{Z}_m$ is dense and open in $\overline{\mathcal{Z}_m}$ with the complex codimension of the complement $\overline{\mathcal{Z}_m} \setminus \mathcal{Z}_m$ at least one. Moreover, as $\overline{\mathcal{Z}_m}$ a compact space, it is a complete space. Let us now take $\mathcal{Z}_m^H$ to be completion space of $\mathcal{Z}_m$ with respect to the Hodge metric on $\mathcal{Z}_m$. Then $\mathcal{Z}_m^H$ is the smallest complete space with respect to the Hodge metric that contains $\mathcal{Z}_m$. In particular, $\mathcal{Z}_m^H \subseteq \overline{\mathcal{Z}_m}$. Therefore, the complex codimension of the complement $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is at least one. We remark that any two points in $\mathcal{Z}_m^H$ are of Hodge finite distance. Moreover, we have the following properties of the completion space $\mathcal{Z}_m^H$; the detailed proof of which is referred to Lemma 4.1 in [2].

**Lemma 4.1.** The Hodge metric completion $\mathcal{Z}_m^H$ is a dense and open smooth submanifold in $\overline{\mathcal{Z}_m}$, and the complex codimension of $\mathcal{Z}_m^H \setminus \mathcal{Z}_m$ is at least one.

Let $\mathcal{T}_m^H$ be the universal cover of $\mathcal{Z}_m^H$ with the universal covering map $\pi_m^H : \mathcal{T}_m^H \to \mathcal{Z}_m^H$. Thus $\mathcal{T}_m^H$ is a connected and simply connected complete smooth complex manifold with respect to the Hodge metric. The complete manifold $\mathcal{T}_m^H$ is called the Hodge metric completion space with level $m$ structure, or simply the Hodge metric completion space. Since $\mathcal{Z}_m^H$ is the Hodge metric completion of $\mathcal{Z}_m$, we have the continuous extension map $\Phi_m^H : \mathcal{Z}_m^H \to D/\Gamma$. In particular, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_m} & \mathcal{T}_m^H & \xrightarrow{\Phi_m^H} & D \\
\downarrow{\pi_m} & & \downarrow{\pi_m^H} & & \downarrow{\pi_D} \\
\mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H & \xrightarrow{\Phi_m^H} & D/\Gamma,
\end{array}
\]

with $i$ the inclusion map, $i_m$ a lifting map of $i \circ \pi_m$, $\pi_D$ the covering map and $\Phi_m^H$ a lifting map of $\Phi_m^H \circ \pi_m^H$. We remark that there exists a suitable choice of $i_m$ and $\Phi_m^H$ such that $\Phi = \Phi_m^H \circ i_m$. One may refer to Lemma A.1 in Appendix A in [2] for the proof. We will fix such choice of $i_m$ and $\Phi_m^H$ in the rest of the paper. Moreover, we have the following result and the proof is referred to Proposition 4.3 in [2].

**Proposition 4.2.** The image $\mathcal{T}_m := i_m(\mathcal{T})$ equals to the preimage $(\pi_m^H)^{-1}(\mathcal{Z}_m)$. 

Since \( \mathcal{Z}_m \) is an open submanifold in \( \mathcal{Z}^H \) and \( \pi^H_m : \mathcal{T}^H_m \to \mathcal{Z}^H \) is a holomorphic cover map, the preimage \( \mathcal{T}_m \) is then a connected open submanifold in \( \mathcal{T}^H_m \). In particular, since the complex codimension of \( \mathcal{Z}^H \setminus \mathcal{Z}_m \) is at least one, the complex codimension of \( \mathcal{T}^H \setminus \mathcal{T}_m \) is at least one in \( \mathcal{T}^H_m \) as well. On the other hand, it is easy to check that \( \Phi_m \) is a holomorphic map from \( \mathcal{T}_m \) to \( N_+ \cap D \), where \( N_+ \) is identified with its unipotent orbit in \( \dot{D} \) by fixing the base point \( \Phi(p) \in D \) with \( p \in \mathcal{T} \). In fact, according to the commutativity of the diagram (20), we have \( i_m : \mathcal{T} \to \mathcal{T}_m \) is the lifting of \( i \circ \pi_m, i_m \) is locally invertible. Since \( \pi^H_m \mid \mathcal{T}_m : \mathcal{T}_m \to \mathcal{Z}_m \) is holomorphic covering map, \( i_m \) is also holomorphic. As \( \Phi = \Phi_m \circ i_m \) as \( i_m(T) = \mathcal{T}_m \) with \( \Phi \) and \( i_m \) both holomorphic and \( i_m \) locally invertible, \( \Phi_m \) is holomorphic. In particular, since \( \Phi_m(\mathcal{T}_m) = \Phi_m(i_m(T)) = \Phi(T) \subseteq N_+ \cap D \). Thus \( \Phi_m \) is a holomorphic map from \( \mathcal{T}_m \) to \( N_+ \cap D \). Therefore, as \( \Phi_m \) is locally bounded, we can apply the Riemann extension theorem to \( \Phi_m : \mathcal{T}_m \to N_+ \cap D \) to first conclude that there exists a holomorphic map \( \Phi'_m : \mathcal{T}_m^H \to N_+ \cap D \) such that \( \Phi'_m \mid \mathcal{T}_m = \Phi_m \). Then since \( \Phi'_m \) and \( \Phi^H_m \) are both continuous extension of \( \Phi_m \) and they agree on the dense subset \( \mathcal{T}_m \), we conclude that \( \Phi^H_m \) and \( \Phi'_m \) must agree on the whole set of \( \mathcal{T}_m^H \). Moreover, since \( \Phi = i_m \cdot \Phi_m \) is bounded, so is \( \Phi_m \). Therefore \( \Phi^H_m \) is also bounded. Thus we get the following proposition.

**Proposition 4.3.** The map \( \Phi^H_m \) is a bounded holomorphic map from \( \mathcal{T}_m^H \) to \( N_+ \cap D \) with respect to the Hodge metric on \( N_+ \).

Let \( P \) be the projection map given by (17) with the same fixed base point \( \Phi(p) \in D \) and \( p \in \mathcal{T} \) and the fixed adapted basis \( (\eta_0, \cdots, \eta_{m-1}) \) for the Hodge decomposition of \( \Phi(p) \). Then based on the above proposition, we can define the following holomorphic map

\[
\tau^H_m = P \circ \Phi^H_m : \mathcal{T}_m^H \to \mathbb{C}^N.
\]

Moreover, we also have \( \tau = P \circ \Phi = P \circ \Phi^H_m \circ i_m = \tau^H_m \circ i_m \). Let us denote the restriction map \( \tau_m = \tau^H_m \mid \mathcal{T}_m : \mathcal{T}_m \to \mathbb{C}^N \) in the following context. Then \( \tau^H_m \) is the continuous extension of \( \tau_m \) and \( \tau = \Psi_m \circ i_m \). By the definition of \( \tau^H_m \), we can easily conclude the following lemma.

**Lemma 4.4.** If the holomorphic map \( \tau^H_m \) is injective, then \( \Phi^H_m : \mathcal{T}_m^H \to N_+ \cap D \) is also injective.

In the rest of this section, we will present affineness results about the space \( \mathcal{T}_m \) and \( \mathcal{T}_m^H \) for any \( m \geq 3 \) by adopting the same arguments of analogous results in [2].

**Lemma 4.5.** The restriction map \( \tau_m : \mathcal{T}_m \to \mathbb{C}^N \cong H^s_{p,n-s} \) is a local embedding. In particular, \( \tau_m \) defines a holomorphic affine structure on \( \mathcal{T}_m \).

**Proof.** We know that \( i : \mathcal{Z}_m \to \mathcal{Z}^H \) is the natural inclusion map, \( \pi_m, \pi^H_m \) are both universal covering map. Since \( i \circ \pi_m = \pi^H_m \circ i_m \), we have that \( i_m \) is locally biholomorphic. On the other hand, we showed in Proposition 3.15 that \( \tau \) is also a local embedding. Consider an open cover \( \{ U_{\alpha} \}_{\alpha \in \Lambda} \) of \( \mathcal{T}_m \). We may assume that for each \( U_{\alpha} \subseteq \mathcal{T}_m, i_m \) is biholomorphic on \( U_{\alpha} \) and thus the inverse \( (i_m)^{-1} \) is also an embedding on \( U_{\alpha} \). We may also assume that \( \tau \) is an embedding on \( (i_m)^{-1}(U_{\alpha}) \). In particular, the relation \( \tau = \tau_m \circ i_m \) implies that \( \tau_m \mid U_{\alpha} = \tau \circ (i_m)^{-1} \mid U_{\alpha} \) is also an embedding on \( U_{\alpha} \). This shows that \( \tau_m \) is a local embedding on \( \mathcal{T}_m \). In particular, \( \tau_m : \mathcal{T}_m \to \mathbb{C}^N \) defines a local coordinate map around each point of \( \mathcal{T}_m \). Thus \( \tau_m \) gives a holomorphic coordinate cover on \( \mathcal{T}_m \) whose transition maps are
all identity maps. In conclude, we get that \( \tau_m \) defines a holomorphic affine structure on \( \mathcal{T}_m \).

As a corollary the above lemma, we conclude the following property of the map \( \tau_m^H \).

One may refer to Lemma 4.7 in [2] for a detailed proof.

**Corollary 4.6.** The map \( \tau_m^H : \mathcal{T}_m^H \to \mathbb{C}^N \cong H_{p,n-s}^s \) is a local embedding.

By applying analogous arguments in the proof of Lemma 4.5 we conclude the following.

**Theorem 4.7.** The holomorphic map \( \tau_m^H : \mathcal{T}_m^H \to \mathbb{C}^N \) is a local embedding, and it defines a holomorphic affine structure on \( \mathcal{T}_m^H \).

5. Global Torelli Theorem on the Teichmüller space

In this section, we prove the global Torelli theorem on the Teichmüller space of Calabi–Yau type manifolds.

**Theorem 5.1 (Global Torelli).** The period map \( \Phi : \mathcal{T} \to D \) is injective.

This theorem follows from three steps. The first step is to show Proposition 5.2 which states that for any \( m \geq 3 \), the map \( \Phi_m^H : \mathcal{T}_m^H \to N_+ \cap D \) is injective. The second step is using the injectivity of \( \Phi_m^H \) for any \( m \geq 3 \) to show Proposition 5.4 which asserts that \( \mathcal{T}_m^H \) is biholomorphic to \( \mathcal{T}_{m'}^H \) for any \( m, m' \geq 3 \). Therefore, this allows us to define the completion space \( \mathcal{T}^H \) by \( \mathcal{T}^H = \mathcal{T}_m^H \), the map \( i_T : \mathcal{T} \to \mathcal{T}^H \) by \( i_T = i_m \), and the extended period map \( \Phi^H : \mathcal{T}^H \to D \) by \( \Phi^H = \Phi_m^H \) for any \( m \geq 3 \). Then \( \mathcal{T}^H \) is a complete complex affine manifold, and Proposition 5.2 implies that \( \Phi^H : \mathcal{T}^H \to N_+ \cap D \) is a holomorphic injective map. The third step is to show Proposition 5.4 which says that \( i_T : \mathcal{T} \to \mathcal{T}^H \) is an embedding. As a consequence, the period map \( \Phi : \mathcal{T} \to D \) is an injective map as \( \Phi = \Phi^H \circ i_T \) with both \( \Phi^H : \mathcal{T}^H \to D \) and \( i_T : \mathcal{T} \to \mathcal{T}^H \) injective, as needed.

**Proposition 5.2.** For any \( m \geq 3 \), the holomorphic map \( \Phi_m^H : \mathcal{T}_m^H \to D \) is injective.

**Proof.** By Lemma 4.4 to show \( \Phi_m^H \) is injective, it is enough to prove \( \tau_m^H \) is injective, which is given by Lemma 5.3.

**Lemma 5.3.** The holomorphic map \( \tau_m^H : \mathcal{T}_m^H \to \mathbb{C}^N \) is injective.

**Proof.** Note that as \( \mathcal{T}_m^H \) is a complex affine manifold, we have the notion of straight lines in it with respect to the affine structure. We first claim that for any two points in \( \mathcal{T}_m^H \), there is a straight line in \( \mathcal{T}_m^H \) connecting them. Let us take an arbitrary point \( p \in \mathcal{T}_m^H \), and \( S \subseteq \mathcal{T} \) be the set so that for any \( q \in S \), there exists a straight line in \( \mathcal{T}_m^H \) that connects \( p \) and \( q \). Then to show the claim, we need to show \( S = \mathcal{T}_m^H \), and this is enough to show that \( S \) is an open and closed set in \( \mathcal{T}_m^H \).

We first show that \( S \) is a closed set. Let \( \{q_i\}_{i=1}^\infty \subseteq S \) be a Cauchy sequence with respect to the Hodge metric. Then for each \( i \) we have the straight line \( l_i \) connecting \( p \) and \( q_i \) such that \( l_i(0) = p \), \( l_i(T_i) = q_i \) for some \( T_i \geq 0 \) and \( v_i := \frac{\partial}{\partial t} l_i(0) \) a unit vector with respect to the Euclidean metric on \( n_+ \). We can view these straight lines \( l_i : [0, T_i] \to \mathcal{T}_m^H \) as the solutions of the affine geodesic equations \( \frac{\partial}{\partial t} l_i(t) = 0 \) with initial conditions \( v_i := \frac{\partial}{\partial t} l_i(0) \) and \( l_i(0) = p \) in particular \( T_i = d_E(p, q_i) \) is the Euclidean distance between \( p \) and \( q_i \). It is
well-known that solutions of these geodesic equations analytically depend on their initial data.

Proposition 4.3 showed that \( \Phi^H_m : T^H_m \to N_+ \cap D \) is a bounded map, which implies that the image of \( \Phi^H_m \) is bounded with respect to the Euclidean metric on \( N_+ \). Because a linear projection will map a bounded set to a bounded set, we have that the image of \( \tau^H_m = P \circ \Phi^H_m \) is also bounded in \( \mathbb{C}^N \) with respect to the Euclidean metric on \( \mathbb{C}^N \).

Passing to a subsequence, we may therefore assume that \( \{T_i\} \) and \( \{v_i\} \) converge, with \( \lim_{i \to \infty} T_i = T_\infty \) and \( \lim_{i \to \infty} v_i = v_\infty \), respectively. Let \( l_\infty(t) \) be the local solution of the affine geodesic equation with initial conditions \( \frac{\partial}{\partial t} l_\infty(0) = v_\infty \) and \( l_\infty(0) = p \). We claim that the solution \( l_\infty(t) \) exists for \( t \in [0, T_\infty] \). Consider the set

\[
E_\infty := \{ a : l_\infty(t) \text{ exists for } t \in [0, a) \}.
\]

If \( E_\infty \) is unbounded above, then the conclusion is obvious. Otherwise, let \( a_\infty = \sup E_\infty \), and then we need to show \( a_\infty > T_\infty \). Suppose towards a contradiction that \( a_\infty \leq T_\infty \). One defines the sequence \( \{a_k\}_{k=1}^\infty \) so that \( a_k/T_k = a_\infty/T_\infty \). We have \( a_k \leq T_k \) and \( \lim_{k \to \infty} a_k = a_\infty \). Then the continuous dependence of solutions of the geodesic equation on initial data implies that the sequence \( \{l_k(a_k)\}_{k=1}^\infty \) is a Cauchy sequence. Note that \( T^H_m \) is a completion space, thus the sequence \( \{l_k(a_k)\}_{k=1}^\infty \) converges to some \( q' \in T^H_m \). Define \( l_\infty(a_\infty) := q' \). Then the solution \( l_\infty(t) \) exists for \( t \in [0, a_\infty] \). On the other hand, \( q' \) is an inner point of \( T^H_m \) since \( T^H_m \) is a smooth manifold. Thus the affine geodesic equation has a local solution at \( q' \) that extends the geodesic \( l_\infty \). That is, there exists \( \epsilon > 0 \) such that the solution \( l_\infty(t) \) exists for \( t \in [0, a_\infty + \epsilon] \). This contradicts the fact that \( a_\infty \) is an upper bound of \( E_\infty \). We have therefore proven that \( l_\infty(t) \) exists for \( t \in [0, T_\infty] \). Then again since the continuous dependence of solutions of the affine geodesic equations on the initial data, we conclude that \( l_\infty(T_\infty) = \lim_{k \to \infty} l_k(T_k) = \lim_{k \to \infty} q_k = q_\infty \). This means the limit point \( q_\infty \in S \), and hence \( S \) is a closed set.

We now show that \( S \) is an open set. For any point \( q \in S \), there exists a straight line \( l \) connecting \( p \) and \( q \). Then for each point \( x \in l \) there exists an open neighborhood \( U_x \subset T^H_m \) with diameter \( 2r_x \). Therefore the collection \( \{U_x\}_{x \in l} \) forms an open cover of \( l \). Since \( l \) is a compact set, there is a finite subcover \( \{U_{x_i}\}_{i=1}^K \) of \( l \). Let \( r = \min \{r_{x_i} : 1 \leq i \leq K \} \), then the straight line \( l \) is covered by a cylinder \( C_r \) of radius \( r \) in \( T^H_m \). As \( C_r \) is a convex set, each point in \( C_r \) can be connected to \( p \) by a straight line. Therefore we have found an open neighborhood \( C_r \subseteq S \) of \( q \in S \). This implies that \( S \) is an open set.

Therefore, we have proved the claim that any two points in \( T^H_m \) can by connected by a straight line.

Let \( p, q \in T^H_m \) be two different points. Suppose towards a contradiction that \( \tau^H_m(p) = \tau^H_m(q) \in \mathbb{C}^N \). On one hand, we have showed there is a straight line \( l \subset T^H_m \) connecting \( p \) and \( q \). Since \( \tau^H_m : T^H_m \to \mathbb{C}^N \) is affine, we have that \( \tau^H_m|_l \) is a linear map. Thus the restriction of \( \tau^H_m \) to the straight line \( l \) is a constant map. On the other hand, Corollary 4.6 shows that \( \tau^H_m : T^H_m \to \mathbb{C}^N \) is locally injective. Therefore, let us take \( U_p \) to be a neighborhood of \( p \) in \( T^H_m \) such that \( \tau^H_m : U_p \to \mathbb{C}^N \) is injective. Then the intersection of \( U_p \) and \( l \) contains infinitely many points, but the restriction of \( \tau^H_m \) to \( U_p \cap l \) is a constant map. Therefore, we get the contradiction that \( \tau^H_m \) is both injective and constant on \( U_p \cap l \). Thus \( \tau^H_m(p) \neq \tau^H_m(q) \) for any different \( p, q \in T^H_m \). \( \square \)
For any two integers $m, m' \geq 3$, let $\mathcal{Z}_m$ and $\mathcal{Z}_{m'}$ be smooth moduli space of Calabi–Yau type manifolds with level structure and $\mathcal{T}_m^H$ and $\mathcal{T}_{m'}^H$ be the universal cover spaces of $\mathcal{Z}_m^H$ and $\mathcal{Z}_{m'}^H$ respectively. Then we have the following proposition.

**Proposition 5.4.** The complete complex manifolds $\mathcal{T}_m^H$ and $\mathcal{T}_{m'}^H$ are biholomorphic to each other.

**Proof.** First, Proposition 5.2 shows that $\Phi_m$ and $\Phi_{m'}$ are embeddings for any $m, m' \geq 3$. Therefore, we have the isomorphisms $\mathcal{T}_m \cong \Phi_m(\mathcal{T}_m)$ and $\mathcal{T}_{m'} \cong \Phi_{m'}(\mathcal{T}_{m'})$. Secondly, since $\Phi = \Phi_m \circ i_m = \Phi_{m'} \circ i_{m'}$, and $\Phi$ is independent of $m$ and $m'$, we have $\Phi_m(\mathcal{T}_m) = \Phi_{m'}(\mathcal{T}_{m'}) = \Phi(\mathcal{T})$. Therefore $\mathcal{T}_m \cong \mathcal{T}_{m'} \cong \Phi(\mathcal{T})$ biholomorphically. Finally, since $\mathcal{T}_m^H$ is the completion space of $\mathcal{T}_m$, $\mathcal{T}_{m'}^H$ is the completion space of $\mathcal{T}_{m'}$, and that the metric completions space is unique up to biholomorphism, we conclude that $\mathcal{T}_m^H$ is biholomorphic to $\mathcal{T}_{m'}^H$. \qed

Proposition 5.4 shows that $\mathcal{T}_m^H$ is independent of the choice of the level $m$ structure up to biholomorphisms. Therefore, it allows us to give the following definition.

**Definition 5.5.** We define the completion space $\mathcal{T}^H = \mathcal{T}_m^H$, the holomorphic map $i_T : \mathcal{T} \to \mathcal{T}^H$ by $i_T = i_m$, and the holomorphic map $\Phi^H : \mathcal{T}_m^H \to D$ by $\Phi^H = \Phi_m^H$ for any $m \geq 3$.

Notice that with these new notations, we obtain the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{i_T} & \mathcal{T}^H \\
\pi_m & & \Phi^H \\
\mathcal{Z}_m & \xrightarrow{i} & \mathcal{Z}_m^H \\
\end{array}
\]

which satisfies $\Phi = \Phi^H \circ i_T$. Moreover, by the above definitions, we can also combine Theorem 4.7 and Proposition 5.2 to conclude the following corollary.

**Corollary 5.6.** The complete manifold $\mathcal{T}_m^H$ is a smooth holomorphic affine manifold and the holomorphic map $\Phi^H : \mathcal{T}_m^H \to N_+ \cap D$ is a holomorphic injection.

The third step of the proof is mainly to show the following proposition. The same proof can be found from the proof of Lemma 5.4 in [2]. We repeat the proof here for reader’s convenience.

**Proposition 5.7.** The map $i_T : \mathcal{T} \to \mathcal{T}^H$ is an embedding.

**Proof.** Let us define $\mathcal{T}_0$ to be $\mathcal{T}_0 = \mathcal{T}_m$ for any $m \geq 3$ (as $\mathcal{T}_m$ is independent of choice of $m$ as well). Since $\mathcal{T}_0 = (\pi_m^H)^{-1}(\mathcal{Z}_m)$, the map $\pi_m^H : \mathcal{T}_0 \to \mathcal{Z}_m$ is a covering map. Thus the fundamental groups satisfies $\pi_1(\mathcal{T}_0) \subseteq \pi_1(\mathcal{Z}_m)$ for any $m \geq 3$. Therefore, the universal property of the universal covering map $\pi_m : \mathcal{T} \to \mathcal{Z}_m$ and that $i \circ \pi_m = \pi_m^H |_{\mathcal{T}_0} \circ i_T$ implies that $i_T : \mathcal{T} \to \mathcal{T}_0$ is a covering map (where $\mathcal{T}_0 = i_T(\mathcal{T})$).

We now claim that $\pi_1(\mathcal{T}_0)$ is a trivial group. In fact, let $\{m_1, m_2, \cdots, m_k, \cdots\}$ be a sequence of positive integers satisfying $m_k | m_{k+1}$ and $m_k < m_{k+1}$ for each $k \geq 1$. Because each point in $\mathbb{Z}_{m_{k+1}}$ is a polarized Calabi–Yau type manifold with a basis $\gamma_{m_{k+1}}$ for the space $(H_n(M, \mathbb{Z})/\text{Tor})/m_{k+1}(H_n(M, \mathbb{Z})/\text{Tor})$ and $m_k | m_{k+1}$, then the basis $\gamma_{m_{k+1}}$ induces a
basis for the space \( (H_n(M, \mathbb{Z})/\text{Tor})/m_k (H_n(M, \mathbb{Z})/\text{Tor}) \). Therefore we get a well-defined map \( Z_{m_{k+1}} \rightarrow Z_{m_k} \) by assigning to a polarized Calabi–Yau type manifold with the basis \( \gamma_{m_{k+1}} \) the same polarized Calabi–Yau type manifold with the basis \( (\gamma_{m_{k+1}} \mod m_k) \in (H_n(M, \mathbb{Z})/\text{Tor})/m_k (H_n(M, \mathbb{Z})/\text{Tor}) \). Moreover, using the versal properties of both the families \( X_{m_{k+1}} \rightarrow Z_{m_{k+1}} \) and \( X_{m_k} \rightarrow Z_{m_k} \), we can conclude that the map \( Z_{m_{k+1}} \rightarrow Z_{m_k} \) is locally biholomorphic. Therefore we get a natural covering \( Z_{m_{k+1}} \rightarrow Z_{m_k} \).

Thus the fundamental group \( \pi_1(Z_{m_k}) \) is a subgroup of \( \pi_1(Z_{m_{k+1}}) \) for each \( k \). Hence, the inverse system of fundamental groups

\[
\pi_1(Z_{m_1}) \leftarrow \pi_1(Z_{m_2}) \leftarrow \cdots \leftarrow \pi_1(Z_{m_k}) \leftarrow \cdots
\]

has an inverse limit, and this limit is actually the fundamental group of \( T \). Since \( \pi_1(T_0) \subseteq \pi_1(Z_{m_k}) \) for each \( k \), the fundamental group of \( T_0 \) is also a subgroup of \( \pi_1(T) \). However, simply-connectedness of \( T \) implies that \( \pi_1(T) \) is a trivial group. Therefore \( \pi_1(T_0) \) is also a trivial group. Thus the covering map \( i_T : T \rightarrow T_0 \) is a one-to-one covering and therefore \( i_T : T \rightarrow T^H \) is an embedding. \( \square \)

Corollary 5.8. The complete complex manifold \( T^H \) is the completion space of \( T \) with respect to the Hodge metric.

Proof of Theorem 5.1. According to Corollary 5.6 and Proposition 5.7, both \( \Phi^H \) and \( i_T \) are injective. Since \( \Phi = \Phi^H \circ i_T \), we conclude that \( \Phi \) is also injective. \( \square \)

6. Applications

In this section, we make two assumptions: the moduli space \( M \) of polarized Calabi–Yau type manifolds is smooth and that the global monodromy group acts on the period domain freely. Then we use the global Torelli theorem on Teichmüller space of polarized and marked Calabi–Yau type manifolds to show that the period map on the moduli space of polarized Calabi–Yau type manifolds is a covering map onto its image. As a consequence, we derive that the generic Torelli theorem on the moduli space of polarized Calabi–Yau type manifolds implies the global Torelli theorem on the moduli space.

Let \( \Gamma \) denote the global monodromy group which acts properly and discontinuously on \( D \). Then for the smooth moduli space \( M \), we consider the period map \( \Phi_M : M \rightarrow D/\Gamma \). Thus for the two period maps \( \Phi \) and \( \Phi_M \), we have the following commutative diagram,

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\Phi} & D \\
\pi_T \downarrow & & \pi_D \\
\mathcal{M} & \xrightarrow{\Phi_M} & D/\Gamma,
\end{array}
\]

where \( \pi_T : T \rightarrow M \) is a covering map and \( \Phi = \Phi_D \circ \Phi_M \). The image of the period map \( \Phi_M \) is an analytic subvariety of \( D/\Gamma \). We refer the reader to page 156 of [4] for details of the analyticity of the image of the period mapping.

Global Torelli problem on the moduli space \( M \) asks when \( \Phi_M \) is injective, and generic Torelli problem asks when there exists an open dense subset \( U \subseteq M \) such that \( \Phi_M|_U \) is injective. In both cases, we need to understand the global Torelli property of the period map \( \Phi_M \) on the moduli space.
Furthermore, we assume $\Gamma$ acts on $D$ freely, thus the quotient space $D/\Gamma$ is a smooth analytic variety. Therefore the quotient map $\pi_D : D \to D/\Gamma$ is a covering map. Then we will show the following theorem.

**Theorem 6.1.** Let $\mathcal{M}$ be the moduli space of polarized Calabi–Yau type manifolds. If $\mathcal{M}$ is smooth and the global monodromy group $\Gamma$ acts on $D$ freely, then the period map $\Phi_\mathcal{M} : \mathcal{M} \to D/\Gamma$ is a covering map from $\mathcal{M}$ to its image in $D/\Gamma$. As a consequence, if the period map $\Phi_\mathcal{M}$ is generically injective, then it is globally injective.

**Proof of Theorem 6.1.** First, we show that $\Phi_\mathcal{M}$ is a covering map from $\mathcal{M}$ to its image in $D/\Gamma$. This follows from the following lemma,

**Lemma 6.2.** Let $\tilde{\Phi} : \mathcal{T} \to D \to D/\Gamma$ be the composition of $\Phi$ and the covering map $\pi_D : D \to D/\Gamma$. If $p_1, p_2 \in \mathcal{T}$ are distinct points such that $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$, then there exist $V_1$ and $V_2$, which are neighborhoods of $p_1$ and $p_2$ respectively, such that $\tilde{\Phi}(V_1) = \tilde{\Phi}(V_2)$, $V_1 \cap V_2 = \emptyset$, and the map $\tilde{\Phi} : V_1 \to \tilde{\Phi}(V_i)$ is biholomorphic for each $i = 1, 2$.

**Proof of Lemma 6.2.** By the argument at the beginning of Section 2.2, we can identify a point $\Phi(p) = \{F^s_p \subset F^s_p \subset \cdots \subset F^{s-n}_p\} \in D$ with its Hodge decomposition $\Phi(p) = \{H^{k,n-k}_p\}_{k=n-s}$. Therefore, let $\Phi(p_1) = \{H^{k,n-k}_{p_1}\}_{k=n-s}$ and $\Phi(p_2) = \{H^{k,n-k}_{p_2}\}_{k=n-s}$ be the corresponding Hodge decompositions and thus a fixed adapted basis of the Hodge decomposition. Then the condition $\tilde{\Phi}(p_1) = \tilde{\Phi}(p_2)$ implies that there exists some $\alpha \in \Gamma \subseteq \text{Aut}(H^n(M, \mathbb{Z}))$ such that $\alpha \cdot \Phi(p_1) = \Phi(p_2)$, where “$\cdot$” means $\alpha$ acts on a fixed adapted basis of the Hodge decomposition $\{H^{k,n-k}_p\}_{k=n-s}$.

We fix an adapted basis $\{\eta^{(1)}_0, \eta^{(1)}_1, \cdots, \eta^{(1)}_N, \cdots, \eta^{(1)}_{m-1}\}$ for the Hodge decomposition of $\Phi(p_1)$. Let $(\eta^{(2)}_0, \eta^{(2)}_1, \cdots, \eta^{(2)}_N, \cdots, \eta^{(2)}_{m-1}) = (\alpha \cdot \eta^{(1)}_0, \alpha \cdot \eta^{(1)}_1, \cdots, \alpha \cdot \eta^{(1)}_N, \cdots, \alpha \cdot \eta^{(1)}_{m-1})$. Then $(\zeta_0, \zeta_1, \cdots, \zeta_N, \cdots, \zeta_{m-1})$ forms an adapted basis for the Hodge decomposition of $\Phi(p_2)$.

Let $(U_{p_1}, \{\tau^{(1)}_1, \cdots, \tau^{(1)}_N\})$ be the local Kuranishi coordinate chart associated to the basis $\{\eta^{(i)}_0, \eta^{(i)}_1, \cdots, \eta^{(i)}_N\}$. Let $\rho_1 : U_{p_1} \to \mathbb{C}^N \cong H^{s-1,n-s+1}_{p_1}$ and $\rho_2 : U_{p_2} \to \mathbb{C}^N \cong H^{s-1,n-s+1}_{p_2}$ for $i = 1, 2$ respectively, which are defined in Section ???. Then Proposition ?? shows that we have the following embeddings,

$$\rho_1 : U_{p_1} \to \mathbb{C}^N \cong H^{s-1,n-s+1}_{p_1}, \quad \rho_2 : U_{p_2} \to \mathbb{C}^N \cong H^{s-1,n-s+1}_{p_2},$$

with $\rho_i(p^{(i)}) = \sum_{j=1}^N \tau^{(i)}_j (p^{(i)}) \eta^{(i)}_j$ for any $p^{(i)} \in U_{p_i}$ and $i = 1, 2$.

Because $\dim_c U_{p_i} = \dim_c H^{s-1,n-s+1}_{p_i} = \dim_c \mathcal{T}$ and that $\rho_i$ is an embedding, the image $\rho_i(U_{p_i})$ is open in $H^{s-1,n-s+1}_{p_i}$ for each $i = 1, 2$. This implies that $\alpha \cdot \rho_1(U_{p_1})$ and $(\alpha \cdot \rho_1(U_{p_1})) \cap \rho_2(U_{p_2})$ are also open in $H^{s-1,n-s+1}_{p_2}$. Together with the fact that $\alpha \cdot \rho_1(p_1) = \rho_2(p_2) \in (\alpha \cdot \rho_1(U_{p_1})) \cap \rho_2(U_{p_2}) \neq \emptyset$, we get that there exists a neighborhood $W$ of $\rho_2(p_2)$ in $H^{s-1,n-s+1}_{p_2}$, such that $W \subseteq (\alpha \cdot \rho_1(U_{p_1})) \cap \rho_2(U_{p_2})$.

Let $V_1 = \rho_1^{-1}(\alpha^{-1} \cdot W) \subseteq U_{p_1}$ and $V_2 = \rho_2^{-1}(W) \subseteq U_{p_2}$, then the restriction maps

$$\rho_1|_{V_1} : V_1 \to W, \quad \rho_2|_{V_2} : V_2 \to W,$$
are biholomorphic maps since the $\rho_1$ and $\rho_2$ are embeddings. Then from the commutative diagram (??) in Section ??, we get the following composition maps,

$$\alpha \cdot \Phi_1|_{V_1} : V_1 \simeq W \hookrightarrow \mathbb{C}^N \cong H_{p_2}^{s-1,n-s+1}, \quad \Phi_2|_{V_2} : V_2 \simeq W \hookrightarrow \mathbb{C}^N \cong H_{p_2}^{s-1,n-s+1}$$

Notice that the composition maps from $W$ to $H_{p_2}^{s-1,n-s+1}$ in the above two maps are the same. Together with the isomorphisms between $V_1 \simeq W$ and $V_2 \simeq W$ as defined in (23), we now can conclude that $\alpha \cdot \Phi(V_1) = \Phi(V_2)$, which implies $\widetilde{\Phi}(V_1) = \widetilde{\Phi}(V_2) = \pi_D(W)$. By shrinking $W$ properly, we can make $V_1$ and $V_2$ disjoint, and also the map

$$\widetilde{\Phi} : V_i \xrightarrow{\Phi} W \xrightarrow{\pi_D} \widetilde{\Phi}(V_i)$$

for each $i = 1, 2$ is biholomorphic. \hfill \Box

**Proof of Theorem 6.1 (continued).** Notice that in the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\Phi} & D \\
\downarrow{\pi_T} & & \downarrow{\pi_D} \\
\mathcal{M} & \xrightarrow{\Phi_M} & D/\Gamma,
\end{array}
$$

since $\mathcal{T}$ and $\mathcal{M}$ are both smooth, the map $\pi_T$ is a covering map. This implies that $\pi_T$ is locally biholomorphic.

For any Hodge structure $\{H^{k,n-k}\}_{k=0}^{s} \in D/\Gamma$, if the preimage $\Phi_M^{-1}(\{H^{k,n-k}\}_{k=0}^{s}) = \{p_i^j | i \in I\}$ is not empty, then take $\{p_j | j \in J\} = \pi_T^{-1}(\{p_i^j | i \in I\})$. By Lemma 6.2, for each $j \in J$, we get $V_j \subseteq \mathcal{T}$ which is a neighbourhood around $p_j$ such that

$$\widetilde{\Phi} : V_j \to \widetilde{\Phi}(V_j)$$

is biholomorphic, $\cup_j V_j$ is a disjoint union, and all the $\widetilde{\Phi}(V_j)$ are the same for any $j \in J$. Now take $\{V_k' \subseteq \mathcal{M} | k \in K\} = \{\pi_T(V_j) | j \in J\}$. By shrinking the set $W$ properly, we may assume that $\cup_k V_k'$ is a disjoint union and that the images $\Phi_M(V_k') = \pi_D(W)$ are still all the same for any $k$. Moreover, since $\pi_T$ is covering map, the map

$$\Phi_M : V_k' \to \Phi_M(V_k')$$

is still biholomorphic for each $k \in K$. This shows that the holomorphic map $\Phi_M : \mathcal{M} \to \Phi_M(\mathcal{M})$ is a covering map. In particular, if the map $\Phi_M : \mathcal{M} \to \Phi_M(\mathcal{M})$ is generically injective, then $\Phi_M : \mathcal{M} \to \Phi_M(\mathcal{M})$ is a degree one covering map, which must be globally injective. \hfill \Box

Note that in many cases it is possible to find a subgroup $\Gamma_0$ of $\Gamma$, which is of finite index in $\Gamma$, such that its action on $D$ is free and $D/\Gamma_0$ is smooth. In such cases we can consider the lift $\Phi_{\mathcal{M}_0} : \mathcal{M}_0 \to D/\Gamma_0$ of the period map $\Phi_M$, with $\mathcal{M}_0$ a finite cover of $\mathcal{M}$. Then our argument can be applied to prove that $\Phi_{\mathcal{M}_0}$ is actually a covering map onto its image for polarized Calabi–Yau type manifolds with smooth moduli spaces.

In this subsection, we require that the moduli space $\mathcal{M}$ of Calabi–Yau type manifolds are smooth, and there are many nontrivial examples satisfying this requirement. As in the special cases of Calabi–Yau manifolds, one may see from Popp [17], Viehweg [24] and Szendroi [23] that the moduli spaces $\mathcal{Z}_m$ of Calabi–Yau manifolds are smooth.
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