Heat transport in overdamped quantum systems

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We obtain an analytical expression for the heat current between two overdamped quantum oscillators interacting with local thermal baths at different temperatures. The total heat current is split into classical and quantum contributions. We show how to evaluate both contributions by taking advantage of the timescale separation associated with the overdamped regime and without assuming the usual weak-coupling and Markovian approximations. We find that nontrivial quantum corrections survive even when the temperatures are high compared to the frequency scale relevant for the overdamped dynamics of the system.

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I. INTRODUCTION

In classical and statistical physics, the overdamped limit is an extremely useful approximation that allows us to simplify problems where the dynamics of a system is dominated by the friction due to its interaction with an environment. This can be understood based on the canonical example of a Brownian particle, where the limit of strong friction induces a timescale separation in which the momentum degree of freedom relaxes much faster than the position. In such a case, the Fokker-Planck equation describing the stochastic evolution of both degrees of freedom can be reduced to the Smoluchowski equation for the evolution of the probability density of the position alone [1].

For quantum systems, an analogous procedure proves to be more demanding. This is because, in general, tractable descriptions for the reduced dynamics of open quantum systems can be obtained only for weak coupling between the system and the environment. In contrast, by definition, the overdamped limit is a strong-coupling regime (however, this does not prevent weak-coupling master equations from providing approximate descriptions of overdamped dynamics under some conditions [2]). In spite of this, a quantum version of the Smoluchowski equation was first obtained by Ankerhold and collaborators [3,4] using path integral techniques. Those results, as well as later extensions to time-dependent systems [5], consider only equilibrium environments; that is, the system in question interacts only with a single thermal bath.

More recently, some efforts in stochastic and quantum thermodynamics have also focused on understanding the impact of strong-coupling effects in both in-equilibrium and out-of-equilibrium settings [6–18]. In this paper we explore the overdamped limit of a quantum system in contact with a nonequilibrium environment; that is, we consider a situation in which the system simultaneously interacts with two thermal baths at different temperatures. Specifically, we consider an electrical circuit composed of two parallel $RLC$ circuits coupled by a mutual inductance (see Fig. 1). Here, the resistors represent the thermal baths into which energy can be dissipated. If they are at different temperatures, then the system will reach a nonequilibrium stationary state in which heat flows from the hot to the cold resistor. We are interested in studying the properties of this heat current in the overdamped limit where dissipation dominates, which in this case is achieved for $C_i R_i^2 \ll L_i$. For this purpose, we will exploit the fact that for linear systems like this one an exact integral expression for the heat currents can be obtained, and in some cases it can be evaluated analytically [19,20]. In this way, we are able to split the heat current into classical and quantum contributions and to analyze their behavior in different regimes. We obtain analytical expressions for both contributions that fully take into account non-Markovian effects. Interestingly, we show that the quantum corrections to the heat current do not necessarily vanish in the limit where both temperatures are high with respect to the slow frequency scale (the only one relevant for the dynamics of the circuit in the overdamped regime). The surviving quantum corrections are nontrivial and depend logarithmically on the temperatures. We show that these results are, indeed, accurate by comparing them to exact numerical computations.

This paper is organized as follows. In Sec. II, we describe our model of quantum circuits and introduce the expression for the steady-state heat currents for general harmonic networks. Next, in Sec. III we give the result for the classical and quantum contributions to the steady-state heat current in terms of the circuit parameters. The evaluation of the heat currents in the overdamped regime is then done in Sec. IV together with the analysis of different regimes.

II. THE MODEL AND ITS SOLUTION

We begin by building a quantum model of the circuit in Fig. 1. For this, we will represent each resistor using the usual Caldeira-Leggett model. In this model, a resistor is considered an infinite array of independent $LC$ circuits or harmonic modes. In this way, using the usual procedure for canonical quantization [21,22], it is possible to obtain the following
Hamiltonian for the full system (see Appendix A for more details):

\[ H = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{L_1L_2}{L_1L_2 - M^2} \left( \frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2} - \frac{M}{L_1L_2} \phi_1 \phi_2 \right) + \sum_{m_1} \left( \frac{q_{m_1}^2}{2C_{m_1}} + \frac{(\phi_{m_1} - \phi_1)^2}{2L_{m_1}} \right) + \sum_{m_2} \left( \frac{q_{m_2}^2}{2C_{m_2}} + \frac{(\phi_{m_2} - \phi_2)^2}{2L_{m_2}} \right). \]  
(1)

Here, \( q_1 \) and \( q_2 \) are quantum-mechanical operators associated with the charge on the capacitors, while \( \phi_1 \) and \( \phi_2 \) are operators associated with the total magnetic flux through the inductors. They satisfy the usual commutation relations \([q_j, \phi_k] = i\hbar \delta_{jk}\). In a similar way, \([q_m, \phi_m]\) and \([q_m, \phi_m]\) are sets of conjugate operators associated with each of the individual modes in the Caldeira-Leggett model of each resistor. These individual modes are characterized by capacitances \(C_{m_1}\) and inductances \(L_{m_1}\), which are, in principle, arbitrary. They enter in the definition of the spectral density associated with the resistors, defined below.

In the following it will be convenient to write the different terms of the previous Hamiltonian in matrix form. In fact, we can write

\[ H = H_{\text{sys}} + \sum_{\alpha} (H_{\text{env},\alpha} + H_{\text{int},\alpha}), \]
(2)

with

\[ H_{\text{sys}} = q^T \frac{C^{-1}}{2} q + \phi^T \frac{L^{-1}}{2} \phi, \]
(3)

\[ H_{\text{env},\alpha} = q_{\alpha}^T \frac{C^{-1}}{2} q_{\alpha} + \phi_{\alpha}^T \frac{L^{-1}}{2} \phi_{\alpha}, \]
(4)

\[ H_{\text{int},\alpha} = -\phi^T L^{-1} \phi_{\alpha}. \]
(5)

Here, \( H_{\text{sys}} \), \( H_{\text{env},\alpha} \), and \( H_{\text{int},\alpha} \) are the Hamiltonians of the system, the thermal baths, and the interaction between the system and the baths, respectively. The index \( \alpha \in \{1, 2\} \) identifies a resistor or thermal bath in the environment. Moreover, \( q = (q_1, q_2)^T \) and \( \phi = (\phi_1, \phi_2)^T \) are column vectors of the charge and flux operators of the system, respectively, and the matrices appearing in \( H_{\text{sys}} \) are defined by

\[ C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \]
\[ L_0 = \begin{pmatrix} L_1 & -M \\ -M & L_2 \end{pmatrix}. \]
(6)

and

\[ L^{-1} = L_0^{-1} + \begin{pmatrix} \sum_{m_1} L_{m_1}^{-1} & 0 \\ 0 & \sum_{m_2} L_{m_2}^{-1} \end{pmatrix}. \]
(7)

In a similar way, \( q_{\alpha} \) and \( \phi_{\alpha} \) are column vectors formed by the charge and flux operators of the \( \alpha \)th bath, and \( C_{\alpha} \) and \( L_{\alpha} \) are diagonal matrices containing the capacitances and inductances of each bath. Finally, the matrices \( L_{\alpha}^{-1} \) are given by

\[ L_{\alpha}^{-1} = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1N} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_{2N} \end{pmatrix}. \]
(8)

The system described so far is a particular case of an open harmonic network. The nonequilibrium thermodynamics of these systems has been extensively studied before [7,8,13,19,20,23–26] since owing to their linearity exact analytical results can be obtained. The central quantities in this study will be the heat currents associated with each thermal bath, i.e., the average rates at which energy is interchanged between the system and each bath. They can be defined as (heat currents are considered to be positive when they enter the system)

\[ \dot{Q}_{\alpha} = -\frac{1}{\hbar} \{ [H_{\text{env},\alpha}, H_{\text{int},\alpha}] \}, \]
(10)

where the mean value is taken with respect to the instantaneous global state. Given an initial state, the heat currents \( \dot{Q}_{\alpha} \) will depend nontrivially on time during relaxation, after which they will reach stationary values. Typically, one assumes an uncorrelated initial state \( \rho_{0} = \rho_{\text{sys}} \otimes \rho_{\text{env}} \) in which each of the baths in the environment is in a thermal state \( \rho_{\text{th}} \). Under this assumption, it can be shown that in the long-time limit the average heat currents can be expressed as (see Appendix B)

\[ \dot{Q}_{\alpha} = \frac{\hbar}{2} \sum_{\alpha' \neq \alpha} \int_{0}^{\infty} d\omega \omega f_{\alpha'\alpha}(\omega) \times [\coth (\beta_{\alpha} \hbar \omega/2) - \coth (\beta_{\alpha'} \hbar \omega/2)], \]
(11)

where \( f_{\alpha'\alpha}(\omega) \) is the heat transfer matrix element and reads

\[ f_{\alpha'\alpha}(\omega) = \frac{\pi}{2} \text{Tr}[L_{\alpha}(\omega) g(\omega) L_{\alpha}(\omega) g(\omega)] \]
(12)

In the previous expression, \( L_{\alpha}(\omega) \) is the spectral density of the \( \alpha \)th bath. It is a \( 2 \times 2 \) matrix with elements

\[ [L_{\alpha}(\omega)]_{kj} = \sum_{n} (L_{\alpha}^{-1})_{kn} (L_{\alpha}^{-1})_{in} (\omega_{\alpha} C_{\alpha})_{mn} \delta [\omega - (\omega_{\alpha})_{mn}], \]
(13)

where \( \omega_{\alpha} = L_{\alpha}^{-1} C_{\alpha}^{-1} \) is a diagonal matrix with the squared natural frequencies of the modes in the \( \alpha \)th bath. Also, \( g(s) \) in Eq. (12) is the Laplace transform of the circuit Green’s function,

\[ g(s)^{-1} = C s^2 + \gamma(s) s + L_0^{-1}, \]
(14)
where $\gamma(s)$ is the Laplace transform of the dissipation kernel. It is given by
\begin{equation}
\gamma(s) = \int_0^\infty I(\omega) \frac{s}{\omega} e^{-s\omega} d\omega
\end{equation}
in terms of the total spectral density $I(\omega) = \sum_{\alpha} I_{\alpha}(\omega)$. Note that, according to these definitions, in the long-time limit we always have $\sum_{\alpha} Q_{\alpha} = 0$. This expresses nothing more than the conservation of energy, and it follows from Eq. (11) and the fact that the heat transfer matrix $f_{\alpha\alpha}(\omega)$ is symmetric upon $\alpha \leftrightarrow \alpha'$.

Even if the model is fully quantum, the fact that the system is linear makes its dynamics equivalent to the classical one [27]. Thus, the Green’s function of Eq. (14), which appears in the heat transfer matrix of Eq. (12), is just the one solving the classical equations of motion and has no dependence on $\hbar$, which appears only in the noise spectrum $2\hbar\omega\coth(\beta_{\alpha}\hbar\omega/2)$.

The frequency integral in Eq. (11) can be solved analytically in certain cases. As shown in Ref. [20], when the spectral densities of all baths are of the Lorentz-Drude form, the integral can be evaluated in terms of the eigenvalues and eigenvectors of a cubic eigenvalue problem. Thus, we will assume the following spectral density for the baths:
\begin{equation}
I_{\alpha}(\omega) = \frac{1}{\pi} \frac{\omega \omega_c^2}{\omega^2 + \omega_c^2} P_{\alpha},
\end{equation}
with
\begin{equation}
P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{equation}

Note that by Eq. (16), in a continuous limit of Eq. (13) the parameters $L_{\alpha}$ and $C_{\alpha}$ are determined by the values $R_{\alpha}$ of the resistances and also the frequency cutoff $\omega_c$. Interestingly, the previous results are valid for any value of $\omega_c$, which controls the autocorrelation time of the environment (the Markovian approximation corresponds to the limit $\omega_c \to \infty$). Thus, our results will automatically include non-Markovian effects. Finally, with the previous choice for the spectral densities, the function $\gamma(s)$ in Eq. (15) becomes
\begin{equation}
\gamma(s) = \left( \frac{P_1}{R_1} + \frac{P_2}{R_2} \right) \frac{\omega_c}{s + \omega_c}.
\end{equation}

III. CLASSICAL AND QUANTUM CONTRIBUTIONS TO THE HEAT CURRENT

The previous ingredients enable us to find the heat current in terms of the circuit parameters. By plugging the definitions of the spectral densities into Eq. (12) we find
\begin{equation}
f_{1,2}(\omega) = \frac{2}{\pi} \left( \frac{1}{\omega_{\alpha}^2 + \omega_{\alpha}^2} \right)^2 |g_{12}(i\omega)|^2,
\end{equation}
where $g_{1,2}(s)$ is the off-diagonal element of $g(s)$. For simplicity we will consider the case of a symmetric circuit, i.e., $R_1 = R_2 = R$, $C_1 = C_2 = C$, and $L_1 = L_2 = L$. Then, we obtain, from Eq. (14),
\begin{equation}
g_{12}(s) = \frac{M}{A} \left[ \left( C_{\alpha}^2 + \frac{1}{A \omega_{\alpha}} \right) - \left( \frac{M}{A} \right)^2 \right]^{-1},
\end{equation}
where $A = L^2 - M^2$. As a consequence, the transfer function can be finally written as
\begin{equation}
f_{1,2}(\omega) = \frac{2}{\pi} \omega^2 \omega_c^4 \left( \frac{RM}{A} \right)^2 \frac{1}{[u_+(i\omega) u_-(i\omega)]^2},
\end{equation}
with
\begin{equation}
u_\pm(s) = (s^3 + \omega_c s) RC + \left( \frac{R}{L \pm M} + \omega_c \right)s + \frac{R}{L \pm M} \omega_c.
\end{equation}

We can already see how an exact expression for the heat current can be obtained. Since the transfer function $f_{1,2}^\pm(\omega)$ was expressed as a rational function, the frequency integral in Eq. (11) can be evaluated via the residue theorem in terms of the poles of $f_{1,2}(\omega)$. In order to deal with the poles of the functions $\coth(\beta_\alpha\hbar\omega/2)$ at the Matsubara frequencies, it is convenient to write them in terms of digamma functions [30] (see Appendix C):
\begin{equation}
\pi \coth \left( \frac{\beta_\alpha\hbar\omega}{2} \right) = \frac{2\pi}{\beta_\alpha\hbar\omega} - i\psi \left( 1 - \frac{i\beta_\alpha\hbar\omega}{2\pi} \right) + i\psi \left( 1 + \frac{i\beta_\alpha\hbar\omega}{2\pi} \right).
\end{equation}

We note that the terms containing digamma functions vanish in the high-temperature limit. Thus, this decomposition induces a splitting of the heat current into a high-temperature contribution and a low-temperature correction, which we denote as classical and quantum contributions, respectively. Therefore, we have
\begin{equation}
\dot{Q}_1 = \dot{Q}_1^c + \dot{Q}_1^q,
\end{equation}
where
\begin{equation}
\dot{Q}_1^c = \frac{1}{\beta_1} - \frac{1}{\beta_2} \int_0^\infty d\omega f_{12}(\omega),
\end{equation}
\begin{equation}
\dot{Q}_1^q = \frac{i\hbar}{2\pi} \int_0^\infty d\omega \phi_{12}(\omega) \left[ \psi \left( 1 - \frac{i\beta_1\hbar\omega}{2\pi} \right) - \psi \left( 1 + \frac{i\beta_1\hbar\omega}{2\pi} \right) \right].
\end{equation}

Although the previous integrals could, in principle, be evaluated exactly [20], the procedure and the final result are greatly simplified in the overdamped limit in which we are interested. Thus, we now discuss this approximation and the frequency scales involved.
IV. Evaluation of the Heat Current in the Overdamped Limit

The classical equation of motion for a single parallel \( RLC \) circuit is

\[
\ddot{\phi} + \gamma \dot{\phi} + \frac{\omega_0^2}{\omega_0^2} \phi = 0,
\]

where \( \phi \) is the flux variable in the inductor and the relevant frequency scales are given by the damping rate \( \gamma = 1/RC \) and the natural frequency \( \omega_0 = 1/\sqrt{LC} \). The corresponding characteristic equation has roots \( \Gamma_{\pm} = (-\gamma/2) ± (\gamma^2/4 - \omega_0^2)^{1/2} \).

The overdamped limit corresponds to \( \gamma \gg \omega_0 \), and it can be reached, for instance, by reducing the value of the capacitance so that \( C \ll L/R^2 \). In that regime we have \( \Gamma_{+} \simeq -\omega_0^2/\gamma \) and \( \Gamma_{-} \simeq -\gamma + \omega_0^2/\gamma \), which in absolute value are the damping rates of the magnetic flux and charge \( q \), respectively, and therefore, \( |\Gamma_{+}| \ll |\Gamma_{-}| \). This is the timescale separation characteristic for overdamped systems, which means in this case that the charge relaxes much faster than the flux. Also, note that the flux damping rate \( \omega_{d} \simeq \omega_0^2/\gamma = R/L \) becomes independent of \( \omega_0 \). A similar analysis holds for each normal mode of the two coupled \( RLC \) circuits by just replacing \( L \) by \( L \pm M \). We can express the functions \( u_\pm(s) \) in Eq. (22) in terms of \( \gamma \) and \( \omega_\pm = \omega_{d}/(1 \pm M/L) \),

\[
u_\pm(s) = (s^2 + \omega_\pm^2)/\gamma + (\omega_\pm + \omega_c)s + \omega_\pm \omega_c.
\]

We see that the overdamped limit tends to reduce the weight of the cubic and quadratic terms, although they will always dominate for high frequencies. However, we also note that in the frequency integral of Eq. (11), the factor \( \coth(\beta_\alpha \hbar \omega/2) - \coth(\beta_\beta \hbar \omega/2) \) will cut off frequencies higher than \( \omega_{th} = k_b \max_\alpha \langle T_\alpha \rangle/\hbar \). From this it follows that the cubic and quadratic terms can be disregarded with respect to the other two whenever

\[
\omega_{th} \ll \gamma, (\gamma \omega_\pm)^{1/2}, (\gamma \omega_\pm \omega_c)^{1/3}.
\]

Thus, under those conditions, we can consider

\[
u_\pm(s) \simeq (\omega_\pm + \omega_c)s + \omega_\pm \omega_c,
\]

where the only remaining relevant frequency scales are \( \omega_c \) and \( \omega_\pm \). We note that the conditions in Eq. (29) can always be fulfilled by increasing \( \gamma \) and that they do not restrict in any way the ratios between \( \omega_{th}, \omega_\pm, \) and \( \omega_c \). However, they pose a restriction on the maximum value of the temperatures, and Table I specifies some temperature ranges relevant in the overdamped regime. This will become important later when we show that quantum effects survive even when the temperatures are high with respect to \( \hbar \omega_\pm/k_b \). Using the approximation of Eq. (30), the integrals in Eqs. (25) and (26) can be readily evaluated. For the classical contribution to the heat current, we obtain

\[
\tilde{Q}_c^q = \frac{k_b}{2} (T_1 - T_2) \frac{M \omega_c}{\omega_c + \omega_{d}} \frac{\lambda_+ \lambda_-}{\omega_0^2},
\]

where \( \lambda_{\pm} \) is the only root of \( \nu_\pm(s) \),

\[
\lambda_{\pm} = -\frac{\omega_\pm}{\omega_c + \omega_\pm}
\]

The evaluation of the quantum contribution is not as straightforward, and its details are explained in Appendix C. The final result is

\[
\tilde{Q}_q^q = \frac{\hbar}{\pi} \left( \frac{M}{L} \right)^2 \frac{\lambda_+ \lambda_-}{\omega_0^2} \ln \left( \frac{T_2}{T_1} \right) \frac{\omega_\pm}{\omega_c + \omega_{d}} \frac{1}{\lambda_+^2 + \frac{1}{2} \left[ \psi \left( 1 - \frac{\beta_\alpha \hbar \lambda_+}{\omega_c} \right) \right] - \frac{1}{2} \left( 1 - \frac{\beta_\beta \hbar \lambda_-}{\omega_c} \right)}.
\]

Equations (31) and (33) are the central results of this work. They make it possible to compute the heat current in the overdamped regime without assuming the weak-coupling or Markovian approximations and thus complement previous results in similar systems that are either numerical or limited by the mentioned approximations [23–26]. We observe that the classical contribution is proportional to the temperature difference \( \Delta T = T_1 - T_2 \), whereas the quantum contribution depends on \( T_1 \) and \( T_2 \) in a nonalgebraic way, as expected. In Fig. 2 we compare the exact heat current obtained by numerical integration of Eq. (11) with the one obtained by using Eqs. (31) and (33) for different values of \( T_1 \) and \( T_2 \) as a function of \( \gamma/\omega_d \) \( (M/L = 1/2) \) and \( \omega_c = 5\omega_d \). We see that the two results match as \( \gamma/\omega_d \) is increased.

We will now take some relevant limits in order to simplify the previous expressions. The Markovian limit \( (\omega_c \to \infty) \) can be easily obtained by replacing the factors \( \omega_\pm/\omega_0^2 \) by 1 in Eqs. (31) and (33) and noting that the roots \( \lambda_{\pm} \) satisfy

\[
\lim_{\omega_c \to \infty} \lambda_{\pm} = -\omega_{\pm}.
\]
digamma function for large $x$:

$$
\psi(x) \approx \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} + O(x^5).
$$

The contribution of the first logarithmic term cancels the first term in Eq. (33). Moreover, the contribution of the second term cancels the classical part of the heat current, while those coming from the third term vanish. Thus, the only remaining contributions originate from the term $\propto 1/x^4$, so the final result for the low-temperature heat current is

$$
\dot{Q}_{1}^{\text{low}} = \frac{2}{15} \left( \frac{\pi}{\hbar} \right)^3 \left( \frac{M}{L} \right)^2 \frac{\omega_d^4}{\omega_c^6} (T_1 - T_2) + O(T_{1/2}^6).
$$

Considering $T_{1/2} = T \pm \Delta T/2$ and to lower order in $\Delta T$, we can express this result as $\dot{Q}_{1}^{\text{low}} = \mathcal{T} G_0 \Delta T$, where $G_0 = \pi k_b T/3\hbar$ is the fundamental unit of thermal conductance [28,31,32] and $\mathcal{T} = (4\pi^2/5)(M/L)^2 (k_b T/\hbar\omega_d)^2$ is a temperature-dependent transmission coefficient. Also, we would like to point out that Eq. (36) is independent of the cutoff frequency. This is natural since for low temperatures only low-frequency modes contribute to the heat current, while the cutoff frequency controls the high-frequency region of the spectral densities. It is worth mentioning that one would also obtain the exact same expression for the heat currents at the low-temperature limit in the weak-coupling regime, i.e., $\gamma \ll \omega_c$ (see Appendix D). This again is due to the fact that at low temperatures only the low-frequency response of the system is relevant. In Fig. 3 we show the behavior of the total heat current when the bath temperatures are decreased. For low temperatures, we can see that the heat current is, indeed, well approximated by Eq. (36).

Turning to the regime of intermediate temperatures, where $|\lambda_{\pm}|/\omega_{bh} \ll 1$, we employ the following expansion of the digamma function for small values of $x$,

$$
\psi(1 + x) = -\eta + \frac{\pi^2 x}{6} + O(x^2),
$$

where $\eta$ is the Euler-Mascheroni constant. We then find the following high-temperature expansion of the quantum contribution:

$$
\dot{Q}_{1}^{\text{high}} = \frac{\hbar}{\pi} \left( \frac{M}{L} \right)^2 \left( \frac{\lambda_{+} + \lambda_{-}}{\omega_d} \right)^2 \ln \left( \frac{T_2}{T_1} \right) + \frac{\hbar^2}{48} \frac{\omega_r}{\omega_c + \omega_d}
\times \frac{M}{L} (\lambda_{+}^2 - \lambda_{-}^2) \left( \frac{1}{T_2} - \frac{1}{T_1} \right) + O(T_{1/2}^{-2}).
$$

Surprisingly, we see that the dominant term does not necessarily vanish for $|\lambda_{\pm}|/\omega_{bh} \to 0$. The reason for this is that under the constraints given in Eq. (29), one can assume that the temperature is high compared to the slow frequency scale $\omega_c$, but it must remain low compared to the fast frequency scale $\gamma$. In other words, the temperature sits in the middle of the timescale separation associated with the overdamped regime. Thus, to the first nontrivial order the total heat current for high temperatures is

$$
\dot{Q}_{1}^{\text{high}} = \dot{Q}_{1}^{\text{cl} \perp} + \frac{\hbar}{\pi} \left( \frac{M}{L} \right)^2 \left( \frac{\lambda_{+} - \lambda_{-}}{\omega_d} \right)^2 \ln \left( \frac{T_2}{T_1} \right).
$$

Figure 4 shows the behavior of the quantum contribution with respect to the growth of the temperature. When both bath...
FIG. 4. The quantum contribution to the heat currents for different values of \( T_1 \) is sketched with respect to \( T_2 \). We can see that, for high values of \( T_1 \) and \( T_2 \), the quantum correction is not vanishing and it will coincide with a nontrivial logarithmic expression (black dashed line). This plot is sketched for \( M = 1 \), \( L = 2 \), \( \omega_c = 5 \omega_d \), \( \omega_d = 1 \).

temperatures are increased, we can still observe a nonzero quantum correction to the heat currents.

V. CONCLUSIONS

We have investigated the heat current between two overdamped quantum harmonic oscillators interacting with local thermal baths without invoking the weak-coupling and Markovian approximations. Exploiting the timescale separation associated with the overdamped regime, we were able to obtain closed analytical expressions for the heat current, identifying quantum and classical contributions. These analytical results might offer a useful benchmark to test Markovian embedding schemes or other approximate methods, for example, the one developed in [33]. Although our results are valid for general harmonic systems, we have explicitly considered an electronic implementation. This is justified by the fact that low-temperature electronic circuits are a promising platform to study quantum energy transport [29,34–38]. We found that in the overdamped regime a range of intermediate temperatures opens up between the low-temperature and high-temperature regimes usually considered. Our results indicate that in this intermediate range there are significant quantum corrections to the classical heat current, which survive even if the temperatures are high compared to the only relevant frequency scale of the system dynamics.

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APPENDIX A: COUPLED RLC CIRCUIT HAMILTONIAN

The Hamiltonian of a LC circuit can be written such that

\[
H = \frac{\phi^2}{2L} + \frac{q^2}{2C}. \tag{A1}
\]

In this notation, \( q \) and \( \phi \) will play the role of momentum and position conjugate variables, respectively. To quantize the LC circuit, we need to replace the classical variables of the Hamiltonian (A1) with their quantum counterparts. In the other words, the Poisson bracket of the flux and charge in the circuit would be

\[
\{\phi, q\} = \frac{\partial \phi}{\partial \phi} \frac{\partial q}{\partial q} - \frac{\partial q}{\partial \phi} \frac{\partial \phi}{\partial q} = 1. \tag{A2}
\]

As shown by Dirac the value of a classical Poisson bracket imposes its corresponding quantum commutator

\[
\{\phi, q\} \rightarrow \frac{1}{\hbar} [\hat{\phi}, \hat{q}]. \tag{A3}
\]

Thus, we see that transforming the classical Hamiltonian into its quantum version will also be backed by the uncertainty relation between flux and charge variables as they play the role of position and momentum, respectively.

The dissipative part would be the resistor attached to the LC circuit. However, adding the Hamiltonian of this part is not trivial. To write the full Hamiltonian of an RLC circuit, we will employ the Caldeira-Leggett model for the Brownian motion. The resistor can be considered a circuit consisting of an infinite array of independent LC circuits, each playing the role of harmonic oscillators of the bath (Fig. 5). Considering the RLC circuit in Fig. 5, we can write the full Hamiltonian describing the circuit such that

\[
H = \frac{\Phi^2}{2L} + \frac{q^2}{2C} + \sum_m \frac{q_m^2}{2C_m} + \frac{(\phi_m - \Phi)^2}{2L_m}. \tag{A4}
\]

In the above expression, the flux variables \( \Phi \) and \( \phi_m \) correspond to the node fluxes. The node flux is defined as the time integral of the voltage along the path connecting the node and the ground. \( q \) and \( q_m \) are also the charges in capacitors.
The last term in the Hamiltonian can also be realized as the normalizing term to ensure that there will be no inconsistency in the minimum of the potential energy.

Next, we will magnetically couple two quantum RLC circuits by putting them in proximity to each other. Indeed, the coupling occurs due to the presence of a flux running in one circuit which is caused by the other inductor. This leads to the mutual inductance between the two inductors of the two circuits. Before considering two coupled RLC circuits, we first look at two simple coupled circuits in Fig. 6. We denote the mutual inductance between the two inductors of the two circuits by putting them in proximity to each other. Indeed, the mutual inductance between the two inductors of the two circuits by putting them in proximity to each other.

Now, we assume that \( i_1 = 0 \) and \( i_1 \) is increased up to an arbitrary value \( I_1 \). Then the power stored in the left circuit is

\[
p_1 = \int_0^t v_1(i_1 - I_1)dt = \int_0^t (v_1 - \dot{i}_1)dt.
\]

Then the total energy will be

\[
E_1 = \int p_1 dt = \int_0^t \dot{i}_1 dt = \frac{1}{2}L_1i_1^2.
\]

Now, assume that \( i_1 = I_1 \) is constant, and we change \( i_2 \) from zero to \( I_2 \). Since \( i_2 \) is changing, the mutual voltage induced in the left circuit is \( M\dot{i}_2/dt \), and therefore, the total power will become

\[
p_2 = v_2i_2 + i_2M\frac{\dot{i}_2}{dt} = i_2L_2\frac{\dot{i}_2}{dt} + i_1M\frac{\dot{i}_2}{dt};
\]

thus, the energy stored in the circuit can be written as

\[
E_2 = \int p_2 dt = L_2\int_0^t i_2\dot{i}_2 + I_1\int_0^t M\dot{i}_2
\]

\[
= \frac{1}{2}L_2i_2^2 + M1I_2.
\]

We can write the total energy of the circuits as the sum over \( E_1 \) and \( E_2 \) such that

\[
E_1 + E_2 = \frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 + M1I_2.
\]

Adding the energy with respect to the capacitors to this energy, we can write the Hamiltonian as

\[
H = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{\phi_{11}^2}{2L_1} + \frac{\phi_{22}^2}{2L_2} + \frac{M}{L_1L_2}\phi_{11}\phi_{22} + \sum_m \frac{q_m^2}{2C_m} + \frac{(\phi_m - \phi_1)^2}{2L_m}.
\]

Next, we will eliminate the flux terms \( \phi_{lk} \) to write it in terms of the total flux \( \phi_1 \). To do so we use the following relations between the fluxes:

\[
\phi_{12} = M_{12}\phi_2 = \frac{M}{L_2}\phi_{22},
\]

\[
\phi_{21} = M_{21}\phi_1 = \frac{M}{L_1}\phi_{11}.
\]

These relations together with Eq. (A5) give

\[
\phi_{11} = \frac{L_1L_2}{M^2 - L_1L_2}\left(\frac{M}{L_2}\phi_2 - \phi_1\right),
\]

\[
\phi_{22} = \frac{L_1L_2}{M^2 - L_1L_2}\left(\frac{M}{L_1}\phi_1 - \phi_2\right).
\]

We can now replace these transformations in Eq. (A15) to find the Hamiltonian of our model such that

\[
H = \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} + \frac{L_1L_2}{L_1L_2 - M^2}\left[\frac{\phi_1^2}{2L_1} + \frac{\phi_2^2}{2L_2} - \frac{M}{L_1L_2}\phi_1\phi_2\right]
\]

\[
+ \sum_m \left(\frac{q_m^2}{2C_m} + \frac{(\phi_m - \phi_1)^2}{2L_m}\right)
\]

\[
+ \sum_m \left(\frac{q_m^2}{2C_m} + \frac{(\phi_m - \phi_2)^2}{2L_m}\right).
\]
APPENDIX B: STEADY-STATE HEAT CURRENTS

Using the Heisenberg equations of motion, we can find the integro-differential equation for each variable of the system, and then they can be solved using the Green’s function matrix of the system $g(t, t')$, which satisfies the integro-differential equation,

$$
C \frac{\partial^2}{\partial t^2} g(t, t') + L_0^{-1} g(t, t') + \int_0^t \gamma(t - \tau) \frac{\partial}{\partial \tau} g(\tau, t') d\tau = \delta(t - t'),
$$

(B1)

with the initial condition $g(0, t') = 0$. Note that the term involving the dissipation kernel $\gamma(t)$ is nonlocal in time and takes into account the non-Markovian effects induced by the environment.

To calculate the local heat current, we first write $\sigma^{(\phi, \phi)}(t)$ by using

$$
\nu_a(t_1 - t_2) = \text{Re} \int_0^\infty \nu_a(\omega) e^{-i\omega(t_1 - t_2)} d\omega, \quad (B9)
$$

where $\nu_a(\omega) = L_a(\omega) \coth \frac{h\beta_a \omega}{2}$. Replacing this in Eq. (B4), we can see in the limit $t \to \infty$ we can write

$$
\int_0^\infty g(t, t_1) e^{-i\omega t_1} dt_1 = g(i\omega), \quad (B10)
$$

thus, we have

$$
\sigma^{\phi\phi}(t) = - \text{Re} \int_0^\infty \frac{h}{2} \nu_a(\omega) \nu_a(i\omega) \coth(-i\omega)^T C d\omega. \quad (B11)
$$

Inserting this equation into Eq. (B8), we have the local heat current expression for the steady-state limit,

$$
\dot{Q}_a = \frac{h}{2} \sum_{\alpha} \int_0^\infty \omega f_{a\alpha}(\omega) \coth \left( \frac{h\beta_a \omega}{2} \right) d\omega, \quad (B12)
$$

where we have used the fact that $\text{Re}(-iX) = \text{Im}(X)$. The heat transfer matrix element $f_{a\alpha}(\omega)$ is written such that

$$
f_{a\alpha}(\omega) = \text{Im} \text{Tr} \left[P_a L_a^{-1} g(i\omega) L_a(\omega) g(-i\omega)^T\right]. \quad (B13)
$$

Here, we have $P_a L_a^{-1} = P_a L_a^{-1} + L_a^{-1}$. Placing this relation into the above equation, we can see that $\text{Tr}[P_a L_a^{-1} g(i\omega) L_a(\omega) g(-i\omega)^T] = 0$ because $L_a^{-1}$ is a symmetric matrix and $g(i\omega) L_a(\omega) g(-i\omega)^T$ is antisymmetric, and the trace of their product will be vanishing. That said, we can write the heat transfer matrix as

$$
f_{a\alpha}(\omega) = \text{Im} \text{Tr} \left[P_a L_a^{-1} g(i\omega) L_a(\omega) g(i\omega)^T\right]. \quad (B14)
$$

To expand the above relation a bit further, we first take the Laplace transform of Eq. (B1) such that

$$
g(s)^{-1} = Cs^2 + \gamma(s)s + L_0^{-1}. \quad (B15)
$$

Writing $L_0^{-1}$ in terms of $g(s)^{-1}$, we have

$$
L_0^{-1} = g(s)^{-1} - Cs^2 + \gamma(s)s. \quad (B16)
$$

Placing the above equation into Eq. (B14) with $s = i\omega$, we have

$$
f_{a\alpha}(\omega) = \text{Im} \text{Tr} \left[P_a L_a^{-1} g(i\omega) L_a(\omega) g(i\omega)^T\right] + \text{Im}\omega^2 \text{Tr} \left[P_a g(i\omega) L_a(\omega) g(i\omega)^T\right] + \text{Im}\omega \text{Tr} \left[P_a g(i\omega) g(i\omega) L_a(\omega) g(i\omega)^T\right]. \quad (B17)
$$

The first term vanishes because $P_a L_a^{-1} = 0$ for $\alpha \neq \alpha'$. The second will also be vanishing because it is a product of two symmetric and antisymmetric matrices. In the third term, the matrix $g(i\omega) L_a(\omega) g(i\omega)^T$ is Hermitian, so we have to calculate only $\text{Im} [\text{Tr}(\omega g(i\omega))] = \text{Re} [\text{Tr}(\omega g(i\omega))] = \frac{\pi}{2} \text{Im}(\omega)$. Thus, we have

$$
f_{a\alpha}(\omega) = \frac{\pi}{2} \text{Tr} [L_a(\omega) g(i\omega) L_a(\omega) g(i\omega)^T]. \quad (B18)
$$

Inserting this matrix back into Eq. (B12), we have

$$
\dot{Q}_a = \frac{h}{2} \sum_{\alpha \neq \alpha} \int_0^\infty \omega d\omega f_{a\alpha}(\omega) \left[ \coth \left( \frac{h\beta_a \omega}{2} \right) - \coth \left( \frac{h\beta_a \omega}{2} \right) \right]. \quad (B19)
$$
APPENDIX C: QUANTUM CORRECTION TO THE HEAT CURRENT

To calculate the quantum contribution to the heat currents we will analytically solve the integral in Eq. (26). To do so, we will have to take into account the poles of the digamma function in addition to the poles of \( f_{12}(\omega) \). In fact the poles of the function \( \psi(1 - ix) \) are all located on the lower half of the imaginary axis, i.e., \( x = -i, -2i, -3i, \ldots \). The poles of the heat transfer matrix element \( \lambda_{\pm} \) and their conjugates \( \lambda_{\pm}^* \) are on the imaginary axis. However, since we would like to exclude the contribution from the digamma function poles, we choose the integration contour to run on the upper half plane, which covers only \( \lambda_{\pm} \). Thus, we can write the integral such that

\[
\dot{Q}_q^1 = \frac{i\hbar}{2\pi} \int_c d\omega \frac{\omega f_{12}(\omega)}{\omega} \left[ \psi \left( 1 - \frac{i\beta_2\hbar \omega}{2\pi} \right) - \psi \left( 1 - \frac{i\beta_1\hbar \omega}{2\pi} \right) \right] + \frac{i\hbar}{2\pi} \int_{\infty}^{\infty} d\omega \frac{\omega f_{12}(\omega)}{\omega} \left[ \psi \left( 1 - \frac{i\beta_2\hbar \omega}{2\pi} \right) - \psi \left( 1 - \frac{i\beta_1\hbar \omega}{2\pi} \right) \right].
\]

(C1)

The first integral is done over the contour \( c \) in Fig. 7 by using the residue theorem. The second integral is the contribution for \( \omega \to \infty \). In this limit we need to expand the digamma function using

\[
\psi(1 \pm ix) \simeq \ln(\pm ix) \pm \frac{i}{x}
\]

(C2)

for \( x \to \infty \). Since the integrand is vanishing as \( 1/\omega \), we need to keep only the logarithmic term in the asymptotic digamma functions. Placing this expansion into the second integral in Eq. (C1), we have

\[
\frac{i\hbar}{2\pi} \int_{\infty}^{\infty} d\omega \frac{\omega f_{12}(\omega)}{\omega} \left[ \psi \left( 1 - \frac{i\beta_2\hbar \omega}{2\pi} \right) - \psi \left( 1 - \frac{i\beta_1\hbar \omega}{2\pi} \right) \right] = -\frac{i\hbar}{\pi} \left( \frac{M}{\overline{L}} \right)^2 \left( \frac{\lambda_{\pm} \lambda_{\pm}}{\omega_d} \right)^2 \int d\omega \frac{1}{\omega} \ln \left( \frac{\beta_1}{\beta_2} \right).
\]

(C3)

We change the variable \( \omega = \Lambda e^{i\theta} \), and we integrate over the semicircle on the upper half plane for \( 0 \leq \theta \leq \pi \) and \( \Lambda \to \infty \); thus, we have

\[
\frac{i\hbar}{2\pi} \int_{\infty}^{\infty} d\omega \frac{\omega f_{12}(\omega)}{\omega} \left[ \psi \left( 1 - \frac{i\beta_2\hbar \omega}{2\pi} \right) - \psi \left( 1 - \frac{i\beta_1\hbar \omega}{2\pi} \right) \right] = \frac{i\hbar}{\pi} \left( \frac{M}{\overline{L}} \right)^2 \left( \frac{\lambda_{\pm} \lambda_{\pm}}{\omega_d} \right)^2 \ln \left( \frac{T_2}{T_1} \right).
\]

(C4)

Hence, by adding the above result and the integral over the contour we obtain Eq. (26).

APPENDIX D: WEAK-COUPLING LIMIT

To study the weak-coupling limit of the problem at low temperatures, we still use the Ohmic dissipation but in the Markovian limit \( (\omega_c \to \infty) \). In that case the spectral density and the dissipation kernel simplify into

\[
I_\alpha(\omega) = \frac{2}{\pi} \frac{\omega}{R_\alpha} P_\alpha,
\]

(D1)

\[
\gamma(s) = \frac{1}{R} (P_1 + P_2).
\]

(D2)
Placing the above equations into the heat transfer matrix element \( f_{\alpha\alpha}(\omega) \), we have

\[
f_{1,2}(\omega) = \frac{2}{\pi} \left( \frac{M}{L} \right)^2 \frac{2\omega^2}{\omega_d^2} (\omega_+ \omega_-)^2 \frac{1}{|u_+(i\omega) u_-(i\omega)|^2},
\]

where in the weak-coupling regime we have

\[
u_\pm(\omega) = -\frac{1}{\gamma} \left( \omega - \frac{i\gamma}{2} + \sqrt{\gamma\omega_\pm} \right) \left( \omega - \frac{i\gamma}{2} - \sqrt{\gamma\omega_\pm} \right).
\]

Employing the heat transfer matrix element, we may calculate the classical and quantum contributions to the heat currents in Eqs. (25) and (26) using the same method as in Appendix C. Nevertheless, we are interested in the low-temperature limit of the heat currents. Therefore, the classical contribution will be vanishing, and the heat current up to the first nonvanishing order of the quantum correction can be written as

\[
\dot{Q}_1^{\text{low}} = \frac{512}{15} \left( \frac{\pi}{\hbar} \right)^3 \left( \frac{M}{L} \right)^2 \frac{k_B^4}{\omega_d^4} \left( \frac{\omega_+ \omega_-}{(\gamma + 4\omega_+)(\gamma + 4\omega_-)} \right)^2 (T^4_1 - T^4_2) + O(T^{6/2}_1),
\]

where in the weak-coupling limit of \( \gamma \ll \omega_\pm \) the above expression is the same as (36).

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