ON QUANTIZATION OF POLYNOMIAL MOMENTUM OBSERVABLES

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Abstract
The paper is devoted to quantization of polynomial momentum observables in the cotangent bundle of a smooth manifold. A quantization procedure is proposed allowing to quantize a wide class of functions which are polynomials of any order in momenta. In the last part of the paper the quantum mechanics of scalar particle in curved space-time is studied with the use of proposed approach.

INTRODUCTION
In the paper, the problem of quantization of polynomial classical observables is studied. It is well-known that it is impossible to fully quantize the algebra of polynomials on a Euclidean phase space \([1, 2, 3]\). Consequently, research has concentrated in two main directions. The first is weakening the notion of “full” quantization. In this direction various quantization schemes have been developed to the present time \([3, 4, 5, 6]\).

The second direction is quantizing of restricted classes of observables. Such classes are usually connected with symmetries of the phase space \([4, 5]\). If we are interested in phase spaces \(N\) which have a distinguished configuration space, i.e. \(N = T^*M\), then an important class of classical observables is one of functions polynomial in momenta. The results obtained previously in quantizing of such functions deals mainly with observables quadratic in momenta sniat1,gotay.

In present paper, a quantization procedure is proposed allowing to quantize a wide class of functions which are polynomials of any order in momenta. We use the geometric quantization procedure of Kirillov, Kostant and Souriau (see \([4, 4, 10]\) for review). As an application of proposed procedure, the quantization of the Hamiltonian of a scalar pointlike particle in curved space-time is considered.

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The plan of the paper is as follows. In Sec. 1 the main aspects of geometric quantization scheme are presented. Next section is devoted to the quantization of polynomial momentum observables in a phase space which is the cotangent bundle of a smooth manifold. In the last part of the paper the quantum mechanics of scalar particle in curved space-time is studied with the use of the proposed approach.

1. GEOMETRIC QUANTIZATION

In this section we shortly describe the geometric quantization method which will be used in following sections (see \[4, 7, 10\] for review).

Let \((N, \omega)\) be a symplectic manifold and \(C^\infty(N)\) be the set of \(C^\infty\)-functions on \(N\). For any \(f \in C^\infty(N)\) its Hamiltonian vector field \(\text{ad}(f)\) can be defined by the equation \(i(\text{ad}(f))\omega = -df\), where \(i\) is the internal product. The Poisson bracket

\[
\{f_1, f_2\} = \text{ad}(f_1)(f_2), \quad f_1, f_2 \in C^\infty(N)
\]

turns \(C^\infty(N)\) into a Lie algebra – the Poisson algebra of \(N\). If the Hamiltonian vector field \(\text{ad}(f)\) is complete then \(f\) generates via the Poisson brackets a one-parameter group \(\phi^t_f\) of canonical transformation of \(N\) \([11, 12]\).

A quantization is the linear map \(Q\) of a subalgebra \(\mathcal{F} \subset C^\infty(N)\) into the set of self-adjoint operators in a Hilbert space \(\mathcal{H}\) possessing the following quantization axioms

1. \(Q\) is the identity operator, \((Q1)\)
2. \(Q\) is a \(C^\infty\)-preserving map, \((Q2)\)
3. \(Q\) is unique, \((Q3)\)

\(Q\) is an explicit realization of the quantization map (see \[3, 8\] for review of another quantization approaches) providing a constructive way of quantizing subalgebras of the Poisson algebra.

The geometric quantization procedure includes the following main components \([4, 7, 11, 12]\).

a) Prequantization line bundle \(\mathcal{L}\) that is a Hermitian line bundle over \(N\) with connection \(D\), \(D\)-invariant\(\) Hermitian structure \(\langle,\rangle\). The connection form \(\alpha\) of \(D\) should obey the prequantization condition \(d\alpha = -\hbar^{-1}\omega\).

b) Polarization \(F\) that is an integrable Lagrange distribution in \(TN \otimes_R C\). The polarization plays an important role in the geometric quantization approach, because it determines the representation space.

c) Metaplectic structure which consists of the bundle of metalinear frames and the bundle \(\mathcal{L} \otimes \sqrt{\omega_1}F\) of \(\mathcal{L}\)-valued half-forms normal to the polarization \(F\).

If these structures are defined on a symplectic manifold \((N, \omega)\), then the Hilbert space \(\mathcal{H}\) is defined to be the space of sections \(\mu\) of the bundle \(\mathcal{L} \otimes \sqrt{\omega_1}F\) which are covariantly constant along the polarization \(F\). A function \(f \in C^\infty(N)\) is called polarization preserving if its Lie derivative \(L_X \text{ad}(f) \in F\) for any vector \(X \in F\). For such functions the quantization is defined by the Souriau-Kostant prequantization formula

\[
Q(f)\sigma = (f - i\hbar D_{\text{ad}(f)})\sigma, \quad D_X\sigma = 0, \quad \sigma \in \mathcal{H}, \quad X \in F. \quad (1.1)
\]

In order to quantize functions not preserving polarization one should construct a mapping connecting the Hilbert spaces \(\mathcal{H}_F\) corresponding to different polarizations. A way of doing it is provided by the Blattner-Kostant-Sternberg (BKS) kernel \([7, 10]\).

\(1\)Recall that \(D\)-invariance means that for each pair of sections \(\lambda\) and \(\mu\) of \(\mathcal{L}\) and each real vector field \(X\) on \(M\) holds \(X < \lambda, \mu > = < D_X\lambda, \mu > + < \lambda, D_X\mu >\).
Further we shall consider a particular case when the system under consideration has a physically distinguished \textit{configuration space} $M$, i.e. $N = T^*M$. Define the \textit{canonical} 1-form $\theta$ by $\theta(u) = \xi(\pi,(u))$ for each $\xi \in T^*M$, $u \in T\xi T^*M$ where $\pi : T^*M \to M$ is the cotangent bundle projection and $\pi_*$ is its differential. The 2-form $\omega = d\theta$ is closed and non-degenerate and turns $T^*M$ into a symplectic manifold.

For each smooth vector we define a function $P_X$ on $T^*M$ by $P_X(\xi) = \xi(X(\pi(\xi)))$, $\xi \in T^*M$. We shall refer to $P_X$ as the \textit{momentum associated to a vector field} $X$.

If $x^\alpha$, $\alpha = 1, \ldots, n$ are local coordinates in a domain $U \subset M$, then we can introduce the \textit{canonical coordinate system} $\left(\pi^{-1}(U), x^\alpha, p_\alpha\right)$ where $p_\alpha = P(\partial/\partial x^\alpha)$. In these coordinates on $T^*M$ the symplectic form $\omega$ takes the canonical form $\omega = \sum_\alpha dp^\alpha \wedge dx^\alpha$. Let $\pi : TM \to M$ be the tangent bundle projection, then the \textit{vertical polarization} $F = \text{Ker} \pi$ can be defined spanned by the vector fields $\partial/\partial x^\alpha$.

It is easy to demonstrate that the vertical polarization leads to the Schrödinger representation. For construction of the Hilbert space $\mathcal{H}_F$ corresponding to this representation we should define a global section of the bundle of metalin ear frames $\mathcal{L} \otimes \sqrt{n^*F}$ over $T^*M$. We demonstrate further (see [7] for details) how it can be done with the use of a non-degenerate symmetric tensor field $\gamma$ of second rank on the configuration space $M$. This tensor field can be considered as a (pseudo) Riemannian metric on $M$.

First, we note that there exists an isomorphism between the sections $\mu$ of the bundle $\mathcal{B} \equiv \sqrt{n^*F}$ and the set of complex-valued functions $\mu^\#$ on $\mathcal{B}^* \equiv \mathcal{B}\backslash \{0\}$ possessing the property $\mu^\#(cz) = c^{-1}\mu^\#(z)$, $c \in \mathbb{C}$. This isomorphism is given by $\mu(\pi z) = \mu^\#(z)z$ where $\pi : \mathcal{B} \to T^*M$ is the projection.

For each chart $(U, x^\alpha)$ on $M$, the metric $\gamma$ defines a matrix-valued function $\gamma_U = (\gamma_{\alpha\beta})$ on $U$. Let $\zeta^U = \{\zeta_1, \ldots, \zeta_n\}$ be an orthonormal frame for $TU$.

$$\zeta_\alpha = \sum_\beta C^\beta_\alpha \frac{\partial}{\partial x^\alpha}, \quad \det C > 0, \quad (C \gamma_U C^T)_{\alpha\beta} = \delta_{\alpha\beta}.$$  

From here $\det C = |\det \gamma_U|^{1/2}$. Let $\eta_U = \{\eta_1, \ldots, \eta_n\}$ be the frame field for $F$ defined by $\eta_U = \text{ad}(x)(C^T \circ \pi)^{-1}$ and let $\tilde{\eta}_U$ be a metalin ear frame field projecting onto $\eta_U$. Then we can define a local section $\mu_{\gamma_U}$ of $\mathcal{B}$ by $\mu_{\gamma_U}^\# \circ \eta = 1$ satisfying

$$\mu_{\gamma_U} = \pm |(\det \gamma_U) \circ \pi|^{1/4} \mu_U$$

where $\mu_U$ is the metalin ear frame field for $\zeta_U$. Using the transformation properties of $\gamma$ it can be shown that $\mu_{\gamma_U}$ defines a covariantly constant global section $\mu_\gamma$ of $\mathcal{B}$ such that, for each open domain $U \subset M$,

$$\mu_\gamma\big|_{\pi^{-1}(U)} = \pm |(\det \gamma_U) \circ \pi|^{1/4} \mu_U. \quad (1.2)$$

If $F$ is the vertical polarization, then the symplectic form is exact $\omega = d\theta$ and each section $\sigma$ covariantly constant along $F$ can be represented in the form

$$\sigma = \Psi \lambda_0 \otimes \mu_\gamma \quad (1.3)$$

where $\lambda_0$ is a non-vanishing section of $\mathcal{L}$ and $\Psi$ is a complex-valued function on $M$.

The most general quantization formula for a function $f \in C^\infty(N)$ by the geometric quantization procedure can be written in the form \cite{11, 7}

$$Q(f)\sigma = i\hbar \frac{d}{dt}(\partial_f \sigma)|_{t=0}, \quad \sigma \in \mathcal{H} \quad (1.4)$$
where $\tilde{\phi}_f^t$ is the one-parameter group of transformation in the Hilbert space $H$ generated by the function $f$. If the support of $\Psi$ is contained in some coordinate neighborhood $U$, then we can write
\[ \Psi \lambda_0 \otimes \mu_\gamma = \psi(x) \lambda_0 \otimes \mu_u \]
where $\psi$ is a function on $\mathbb{R}^n$ with support contained in the image of the chart $x : U \rightarrow \mathbb{R}^n$. Comparing (1.3) and (1.2) we find, for each $y \in U$,
\[ \psi(x(y)) = \pm \Psi(y) \left| \det \gamma_U(y) \right|^{1/4}. \]

For sufficiently small $t$
\[ \tilde{\phi}_f^t(\Psi \lambda_0 \otimes \mu_\gamma) = \psi_t(x) \lambda_0 \otimes \mu_u \]
where $\psi_t$ is given by the equation [7]
\[ \psi_t(x) = (i\hbar)^{-n/2} \int [\det \omega(\operatorname{ad}(x^\mu), \phi_f^t \operatorname{ad}(x^\nu))]^{1/2} \exp[i/\hbar \int_0^t (\theta(\operatorname{ad}(f)) - f) \circ \phi_f^{-1} ds] \psi(x \circ \phi_f^{-1}) d^n p. \] (1.8)

2. QUANTIZATION OF POLYNOMIAL MOMENTUM OBSERVABLES

In this section we consider the quantization of polynomial momentum observables in the phase space $N = T^*M$ using the geometric quantization procedure in the Schrödinger representation. Consider a classical system with an oriented configuration space $M$ and let $F$ be the vertical polarization on the phase space $T^*M$.

Let us denote $C_r(N)$ the subspace $C^\infty(N)$ consisting of the functions which are at most $r$-th degree polynomials along the fibers of the cotangent bundle projection. A function $f \in C_r(M)$ can be represented as a sum of homogeneous terms of the form
\[ f(k) = f^a_\ldots^k(x)p_{a_1} \ldots p_{a_k}, \quad k = 1, \ldots, r. \] (2.1)

Along with the functions $f(k)$ we shall consider further the associated tensor fields $\varphi^{(k)} = f^a_\ldots^k \partial_{a_1} \ldots \partial_{a_k}$.

In order to construct the quantization map for a function $f \in C_r(N)$ we should first consider the quantization for the homogeneous terms (2.1) of any order $k$.

At first, let us define a global non-vanishing section $\sigma_0$ of the bundle $\mathcal{L} \otimes \sqrt{\wedge^n F}$ of $\mathcal{L}$-valued half-forms normal to the (vertical) polarization. As it was shown in Sec. 2, such section can be constructed using a (pseudo)Riemannian metric $\gamma$ on the configuration space $M$.

In fact we can choose any such metric on $M$ in order to construct the section $\sigma_0$. However, in applications we are usually interesting in a natural choice of $\gamma$.

The problem simplifies if for given function $f$ the matrix $(f_{(2)}^{0\alpha \beta})$ in (2.1) is non-degenerate. In this case we can put $\gamma = f_{(2)}^{0\alpha \beta} dq^\alpha dq^\beta$, $\gamma_{(2)}^{0\alpha \beta} f_{(2)}^{\beta \gamma} = \delta_\gamma^\alpha$. However, if $(f_{(2)}^{0\alpha \beta})$ is degenerate, one should choose $\sigma_0$ using additional considerations. For example, a (pseudo)Riemannian structure on the configuration space $M$ could be used, motivated by some physical reasons.

When the global section of $\mathcal{L} \otimes \sqrt{\wedge^n F}$ is chosen, we have to develop an approach for quantization of the homogeneous terms (2.1) for any order $k$. In fact, we propose here a way of quantizing only a wide class of such terms.
For \( k = 0, 1 \) the functions are vertical polarization preserving and it is easy to write down the expressions for corresponding operators \cite{7} using the Kostant-Souriau prequantization formula (1.1)

\[
Q(f_0 + f_1^\alpha p_\alpha) \psi \lambda_0 \otimes \mu_\gamma = \]

\[
(-i\hbar \text{ad} (f_0 + f_1^\alpha p_\alpha) \psi + (f_0 - \frac{i\hbar}{2} \frac{\partial f_1^\alpha}{\partial q^\beta}) \psi) \lambda_0 \otimes \mu_\gamma.
\]

Consider now the case \( k \geq 2 \). Let us suppose that it is possible to introduce in a neighborhood of any point \( y \in M \) such coordinates \((x^\alpha)\) that the first partial derivatives of the functions \( f_1^{\alpha_1 \ldots \alpha_k} \) at \( y \) are equal to zero. We shall call such coordinates \textit{normal with respect to the tensor field} \( \varphi(k) = f_1^{\alpha_1 \ldots \alpha_k} \partial_{\alpha_1} \ldots \partial_{\alpha_k} \). The problem of finding the normal coordinates is equivalent to determining of a (local) linear connection in which the tensor field \( \varphi(k) \) is covariantly constant.

It is easy to see that in general case it is impossible to find such connection, because the corresponding system of algebraic equations on connection coefficients is overdetermined. However, the coordinates normal with respect to a tensor field can be introduced in many important cases. For example, if \( M \) is a (pseudo)Riemannian manifold with the metric \( g \) then it is possible to find the coordinates normal with respect to covariantly constant tensors \( \varphi(k) \) coincides with the Riemannian normal coordinates. In the next section we shall see how it allows us to quantize the Hamiltonian of the relativistic particle on curved space-time background.

In the following we suppose that the function \( f(k) \) is such that in a neighborhood of any point \( y \in M \) it is possible to introduce the coordinates normal with respect to a given tensor field \( \varphi(k) \).

Let us now quantize the function \( f = f(k) = f_1^{\alpha_1 \ldots \alpha_k} p_{\alpha_1} \ldots p_{\alpha_k} \) using (1.4) and (1.8). In this case the integral in (1.8) can be simplified if the coordinates \((x^\alpha)\) are normal at the point \( y \) with respect to the tensor field \( \varphi(k) \). In this case the functions \( x^\alpha \) depend linearly on \( t \) along the orbits of the group \( \phi_f^t \)

\[
x^\alpha \circ \phi_f^t = kt f_1^{\alpha_1 \ldots \alpha_k - 1 \alpha} p_{\alpha_1} \ldots p_{\alpha_k - 1}
\]

for each \( \xi \in T^*_y M \). At the same time \( p_\alpha \circ \phi_f^t = \text{const} \) in these coordinates.

Using the last equation and the formulae

\[
\frac{d}{dt} (x^\alpha \circ \phi_f^t) = k f_1^{\alpha_1 \ldots \alpha_k - 1 \alpha} p_{\alpha_1} \ldots p_{\alpha_k - 1},
\]

\[
\frac{d}{dt} (p_\alpha \circ \phi_f^t) = -\partial_\alpha f_1^{\alpha_1 \ldots \alpha_k} p_{\alpha_1} \ldots p_{\alpha_k}
\]

it is possible to approximate the integrand in (1.8) so that the integration will give results accurate to first order in \( t \). After that Eq. (1.8) takes the form

\[
\psi_t(0) = (i\hbar/k)^{-n/2} \int \left[ \det (f_1^{\alpha_1 \ldots \alpha_k - 2 \beta \tau(0)}) \right]^{1/2} \exp[i/\hbar t(k - 1)f(k)(0)] \psi(-kt f_1^{\alpha_1 \ldots \alpha_k - 1 \alpha}(0) p_{\alpha_1} \ldots p_{\alpha_k - 1}) d^n p.
\]

(2.2)

It is easy to see that together with (1.4) and (1.7) this formula defines a quantization coinciding with the quantization of the function

\[
f_1^{\alpha_1 \ldots \alpha_k}(0) p_{\alpha_1} \ldots p_{\alpha_k}.
\]
This function can be quantized using the von Neumann rule (see Appendix)

\[
Q(a^{\alpha_1\ldots\alpha_k} p_1 \ldots p_k) = a^{\alpha_1\ldots\alpha_k} Q(p_1) \ldots Q(p_k),
\]

\[a^{\alpha_1\ldots\alpha_k} = \text{const}.
\]  

From here using (1.3) we get

\[
Q(f(k)) = \psi_{\lambda_0} \otimes \mu_\gamma = (i\hbar)^k f^{\alpha_1\ldots\alpha_k}(0) \frac{\partial^k |g|^{1/4} \psi}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_k}} (0) \lambda_0 \otimes \mu_\gamma.
\]  

(2.4)

Because the point \(y \in M\) in these considerations was chosen to be appropriate, the formula (2.4) give the quantization operator corresponding to any homogeneous term of the form (2.1).

3. QUANTUM MECHANICS IN CURVED SPACE-TIME

As an application of the approach proposed in the previous section let us consider quantization of Hamiltonian of a particle in a curved space-time which is a general Riemannian manifold \((M, g)\).

We develop here the quantum theory of a pointlike scalar particle in curved space-time as quantization of its classical mechanics. This approach to construction of quantum theory is almost obvious, however it is badly developed in relativistic case because of well-known difficulties with quantizing of non-linear observables. Usually, the field theoretical approach to construction of quantum mechanics in curved space-time has been used instead \([13, 14]\) which deduces the (finite-dimensional) quantum mechanical description from the field equations describing corresponding infinite-dimensional field systems.

Consider first the formulation of classical mechanics of the system. It can be formulated by the use of an instant spacelike surface \(\Sigma \subset M\). If we choose the local coordinates \(v^\alpha, \alpha = 1, \ldots, 3\) on \(\Sigma\) then the Hamiltonian of the particle takes the form

\[
H = \sqrt{\rho^{\alpha_\beta} p_\alpha p_\beta + g_{00}} m^2
\]

where \(m\) is the particle’s mass and \(\rho\) is the induced metric on \(\Sigma\), \(\rho_{\alpha\beta} = g_{\alpha\beta}/g_{00}\).

Let us consider approximation of \(H\) for small values of momenta \(\rho^{\alpha_\beta} p_\alpha p_\beta << m^2\)

\[
H = H_1 + H_2 + H_3 + \text{higher order terms.}
\]  

(3.1)

where

\[
H_1 = m \sqrt{g_{00}}, \quad H_2 = \sqrt{g_{00}} \frac{g^{\alpha_\beta} p_\alpha p_\beta}{2m}, \quad H_3 = m \sqrt{g_{00}} \frac{(g^{\alpha_\beta} p_\alpha p_\beta)^2}{4m^3}.
\]

For simplicity further we consider only the case \(g_{00} = 1\) whence \(\rho_{\alpha\beta} = g_{\alpha\beta}\). Quantizing the first two terms in this expression we get \([3, 17]\)

\[
m - \frac{\hbar^2}{2m} g^{\alpha_\beta} \nabla_\alpha \nabla_\beta + \frac{\hbar^2}{12m} R
\]  

(3.2)

where \(\nabla_\alpha\) is the covariant derivative with respect to 3-metric \(\rho\) and \(R\) is the scalar curvature.
Let us consider the third term in (3.1). Introducing the Riemannian normal coordinates and applying the method developed in the previous section we see that quantization of the function \((g^{\alpha\beta}p_{\alpha}p_{\beta})^2\) leads in these coordinates to the quantization of the function \(\Sigma(p_{\alpha})^4\). From here

\[
Q(H_3)\psi(x) = \frac{\hbar^4}{4m^2} \sum_{\alpha} \frac{\partial^4 \psi(x)}{\partial v_{\alpha}^4}(0).
\]

(3.3)

Taking in account the equation (1.6) we can find the expression for \((QH_3)\psi\) in an appropriate coordinates. In order to do it we first list all possible independent invariant terms which could appear in this expression:

\[
I_1 = R^2\psi, \quad I_2 = \Delta R\psi, \quad I_3 = R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}\psi,
\]

\[
I_4 = R^{\alpha\beta}R_{\alpha\beta}, \quad I_5 = R^{\alpha}\psi,_{\alpha}, \quad I_6 = R^{\alpha\beta}\psi,_{\alpha\beta},
\]

\[
I_7 = R \triangle \psi, \quad I_8 = g^{\alpha\beta}g^{\mu\nu}\psi,_{\alpha\beta\mu\nu}.
\]

Here the comma denotes the covariant derivative \(\nabla_{\alpha}\), \(R_{\alpha\beta\mu\nu}\), \(R_{\alpha\beta}\) and \(R\) are Riemann, Ricci and scalar curvatures of \(\rho\) and \(\Delta = g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\). Using (1.6), in Riemannian normal coordinates we find from (3.3)

\[
g^{\alpha\beta}g^{\mu\nu}\partial_{\alpha\beta\mu\nu}(|g|^{1/4}\psi) = g^{\alpha\beta}g^{\mu\nu}(|g|^{-3/4}\partial_{\beta\mu}|g|\partial_{\alpha\lambda}\psi + |g|^{-3/4}\partial_{\beta\mu}|g|\partial_{\alpha}\psi) +
\]

\[
\frac{1}{2}|g|^{-3/4}\partial_{\lambda\mu}|g|\partial_{\alpha\beta}\psi + (\frac{3}{8}|g|^{-7/4}\partial_{\alpha\lambda}|g|\partial_{\beta\mu}\psi - \frac{3}{16}|g|^{-7/4}\partial_{\alpha\beta}|g|\partial_{\mu\lambda}\psi)
\]

(3.4)

where \(|g| = \det(g_{\alpha\beta})\).

Using the expansion for the components of the metric \(g\) in Riemannian normal coordinates

\[
g_{\mu\nu} = g_{\mu\nu}(0) + \frac{1}{3}R_{\mu\alpha\beta\gamma}(0)v^{\alpha}v^{\beta}v^{\gamma} +
\]

\[
\left(\frac{1}{20}R_{\mu\alpha\beta,\gamma\lambda}(0) + \frac{2}{45}R_{\mu\alpha\beta\sigma}R^{\sigma}_{\gamma\lambda}(0)\right)v^{\alpha}v^{\beta}v^{\gamma}v^{\lambda} + \text{higher order terms},
\]

Using this formula, after complicated but straightforward calculations, we get from (3.3) and (3.4)

\[
Q(H_3)\psi \lambda_0 \otimes \mu_\rho = |g|^{1/4}\left(-\frac{1}{12}I_1 + \frac{1}{5}I_2 + \frac{3}{10}I_3 + \frac{1}{30}I_4 - \frac{1}{3}I_5 + \frac{1}{3}I_7 + I_8\right)\lambda_0 \otimes \mu_\rho.
\]

By the use of the proposed approach it is in principle possible to calculate quantum operators corresponding to all terms in the expansion (3.1) of the particle Hamiltonian. In order to do it, for example, for the term \((g^{\alpha\beta}p_{\alpha}p_{\beta})^3\) one should calculate all possible invariants of 6th order in normal coordinates and, then rewrite the expression \(g^{\alpha\beta}g^{\mu\nu}\partial_{\alpha\beta\mu\nu\rho}(|g|^{1/4}\psi)\) in covariant form.

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APPENDIX

We consider here the von Neumann rule (2.3). Relations of such type appears in various quantization approaches (see e.g. [3, 19]). The simplest general form of von Neumann rule is

\[ Q(\varphi \circ f) = \varphi(Qf) \]  

(A.1)

for a distinguished class of observables \( C_D, f \in C_D \) and certain function \( \varphi \in C^\infty(\mathbb{R}) \). It is easy to see that (A.1) is not true for all observables \( C_D = C^\infty(\mathbb{N}) \). However, it is obviously true for restricted classes of functions on the phase space.

We shall prove here the relation

\[ Q(\Phi(p_{\alpha_1}, \ldots ,p_{\alpha_k})) = \Phi(Qp_{\alpha_1}, \ldots ,p_{\alpha_k}) \]  

(A.2)

where \( \Phi \) is a symmetric \( k \)-linear real-valued mapping of the momentum space.

Since \([Qp_{\alpha}, Qp_{\beta}] = 0\) we have

\[ [Q(p_{\alpha_1} \ldots p_{\alpha_k}), Qp_{\beta}] = i\hbar Q(\{p_{\alpha_1} \ldots p_{\alpha_k}, Qp_{\beta}\}) = 0 \]

for any \( \alpha_1, \ldots , \alpha_k = 1, \ldots , n \) because \([Qp_{\alpha}, Qp_{\beta}] = 0\). At the same time

\[ i\hbar Q(\delta^\beta_{\alpha_1} p_{\alpha_2} \ldots p_{\alpha_k} + \delta^\beta_{\alpha_k} p_{\alpha_1} \ldots p_{\alpha_{k-1}}) = [Q(p_{\alpha_1}) \ldots Q(p_{\alpha_k}), Qx^\beta]. \]

From here it follows that \( Q(p_{\alpha_1} \ldots p_{\alpha_k}) = Q(p_{\alpha_1}) \ldots Q(p_{\alpha_k}) + T \) where \( T \) is an operator commuting with both momenta and coordinates.

Supposing by induction that (A.2) holds for \( k - 1 \) and using the formula [4]

\[ Q(p_{\alpha}x^\beta) = \frac{1}{2}(Q(p_{\alpha})Q(x^\beta) + Q(x^\beta)Q(p_{\alpha})) \]

we get

\[ Q(p_{\alpha_1} \ldots p_{\alpha_k}) = Q(\{p_{\alpha_1} \ldots p_{\alpha_k}, p_{\beta}x^\beta\}) = -\frac{i}{\hbar} [Q(p_{\alpha_1} \ldots p_{\alpha_k}), Q(p_{\beta}x^\beta)] = \]

\[ -\frac{i}{2\hbar} [Q(p_{\alpha_1}) \ldots Q(p_{\alpha_k}) + T, Q(p_{\beta})Q(x^\beta) + Q(x^\beta)Q(p_{\beta})] = Q(p_{\alpha_1}) \ldots Q(p_{\alpha_k}) \]

By linearity of the quantization mapping this proves the generalised von Neumann rule (A.2) and (2.3).

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