Faster Stochastic Algorithms via History-Gradient Aided Batch Size Adaptation

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Abstract

Various schemes for adapting batch size have been recently proposed to accelerate stochastic algorithms. However, existing schemes either apply prescribed batch size adaption or require additional backtracking and condition verification steps to exploit the information along optimization path. In this paper, we propose an easy-to-implement scheme for adapting batch size by exploiting history stochastic gradients, based on which we propose the Adaptive batch size SGD (AbaSGD), AbaSVRG, and AbaSPIDER algorithms. To handle the dependence of the batch size on history stochastic gradients, we develop a new convergence analysis technique, and show that these algorithms achieve improved overall complexity over their vanilla counterparts. Moreover, their convergence rates are adaptive to the optimization landscape that the iterate experiences. Extensive experiments demonstrate that our algorithms substantially outperform existing competitive algorithms.

1 Introduction

Many large-scale machine learning problems are modeled by the following finite-sum optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) := \frac{1}{n} \sum_{i=1}^{n} f_i(\mathbf{x}),$$

where each individual function $f_i(\mathbf{x})$ evaluates the loss on a particular $i$-th data sample, and is generally nonconvex due to the complex models. To solve these problems, gradient descent (GD) algorithms are popular choices due to their simplicity. Particularly, for large-scale problems, stochastic gradient descent (SGD) algorithms have been proposed to reduce the per-iteration complexity, but they suffer from a high optimization variance. To further reduce the variance, various variance reduction techniques were proposed, e.g., SAGA, SVRG, and SARAH/SPIDER. For all these stochastic algorithms, the judicious design of batch size significantly affects the overall computational complexity. Namely, large batch size provides better estimate of gradient and reduces the variance, but causes higher computational complexity.

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Recently, adaptively changing the batch size during the algorithm iterations emerges as a powerful approach for accelerating stochastic algorithms. Some studies adopted a prescribed way to adapt the batch size. For example, [8] showed that hybrid SGD (HSGD) with exponentially and polynomially increasing batch size converges respectively linearly for strongly convex optimization and sub-linearly for convex optimization. Furthermore, [28] studied the aforementioned two batch size adaptation schemes as well as the linearly increased batch size, and characterized the convergence rate of HSGD for sampling without replacement and for nonconvex optimization. Moreover, [10] proposed a SVRG scheme with exponentially increasing batch size at each outer-loop iteration for strongly convex optimization. [13] proposed an adaptive version of SCSG [14] with exponentially increasing batch size for nonsmooth convex optimization.

Though more efficient, the above prescribed batch size adaptation does not exploit the information along the optimization path, and hence cannot adapt to the actual optimization landscape that an algorithm experiences. To overcome such an issue, [4, 5] proposed Big Batch SGD, which requires the batch size to be adaptively chosen to ensure its resulting gradient and variance to satisfy certain conditions. Since the batch size is chosen at the beginning of each iteration before the gradient at the same iteration is calculated, the algorithm introduces the backtracking line search to iteratively check that the chosen batch size ensures the resulting gradient to satisfy a variance bound. Clearly, the backtracking step adds undesired complexity, but seems to be unavoidable under the given framework in [4, 5], because the convergence analysis exploits the instantaneous variance bound. Thus, the goal of this paper are two-fold.

• First, we wish to design an easy-to-implement scheme to adapt the batch size, which does not involve backtracking and condition verification, etc., but still incorporates the information along the optimization path, and at the same time, has the guaranteed convergence with improved complexity performance. This necessarily requires to develop new convergence analysis different from the existing studies.

• Second, we wish the resulting stochastic algorithms from our designed adaptive batch size scheme to have provable convergence under both the general nonconvex landscape and the local Polyak-Łojasiewicz (PL) geometry, and have improved complexity respectively compared to their counterparts with non-adaptive batch size. In this way, such an algorithm achieves a convergence rate adaptive to the underlying landscape even during the same optimization path.

In this paper, we provide affirmative answers to the above questions.

Adaptive Batch Size via History Gradients: We take SVRG as an example to briefly explain our idea of designing an adaptive batch size scheme. Our analysis of SVRG shows that the average function-value decrease over an epoch $s$ with length $m$ satisfies

$$\frac{1}{m} \mathbb{E}(f(\tilde{x}^{s+1}) - f(\tilde{x}^s)) \leq -\phi \eta \left( \frac{1}{m} \sum_{t=0}^{m-1} \mathbb{E} \|v_t\|^2 \right) + \frac{\psi \eta I_{|B_s|<n}}{|B_s|}$$

where $\phi, \psi$ are positive constants, $I_{\cdot}$ is the indicator function, $\eta$ is the stepsize, $\tilde{x}^s$ is the snapshot in epoch $s$, $v_t$ is a stochastic estimation of $\nabla f(x_t)$ within epoch $s$, and $|B_s|$ is the batch size used at the outer-loop iteration. The above bound naturally suggests that the batch size $|B_s|$ should be chosen such that the second term is at the same level as the first term of the average stochastic gradient. Thus, such optimization analysis suggests that the batch size should be made adaptive to the average gradient.
Though well motivated, such an adaptive scheme is not directly feasible in practice, because the batch size should be set at the beginning of the epoch at which time the average gradient in that epoch is not known yet. It is similar to the reason why \cite{4, 5} introduced the backtracking line search to guarantee the variance bound. Here, we solve such an issue differently. Instead of using the present gradient information, which is unavailable, we use the average stochastic gradients in the preceding epoch as an approximation. As a result, it becomes much less obvious that such an algorithm will have provable convergence guarantee. Necessarily, such a simple practice requires a new convergence analysis that differs from the existing analysis, which we develop in this paper.

1.1 Main Contributions

We highlight our following main contributions as below.

- We propose Adaptive batch size SGD (AbaSGD), AbaSVRG and AbaSPIDER algorithms, which adapt the batch size to the inverse of the average of stochastic gradients in the past iterations. Thus, these algorithms initially use small batch size (due to large gradient) and enjoy fast iteration, and then gradually increase batch size (due to the reduced gradient) and enjoy reduced variance and stable convergence. Such a scheme exploits the gradient information along the optimization path, and is landscape adaptive. It is also much simpler to implement than the previously designed gradient-aided schemes by avoiding the backtracking, and still retains the improved complexity in convergence.

- We develop a different and simpler convergence analysis for SVRG-type algorithms than previous studies \cite{15, 20} for nonconvex optimization, which allows very flexible choices of stepsize $\eta$, epoch length $m$ and mini-batch size $|I_b|$ (see Theorem 1 for details). More importantly, such an analysis enables to handle the adaptive batch size with dependence on the stochastic gradients in the previous epoch, whereas existing techniques do not seem to accommodate easily.

Based on this new analysis, we show that AbaSVRG achieves an improved complexity over vanilla SVRG, and such an improvement is significant especially at the initial optimization stage where the gradient has a large norm. The worst-case complexity of AbaSVRG matches the best known result $O\left(\epsilon^{-1}(n \wedge \epsilon^{-1})^{2/3}\right)$ (where $\wedge$ denotes the minimum) for SVRG-type algorithms for nonconvex optimization.

- We also show that AbaSPIDER achieves an improved complexity than vanilla SPIDER and SpiderBoost, and its worst-case complexity is $O\left(\epsilon^{-1}(n \wedge \epsilon^{-1})^{1/2}\right)$, which matches the near-optimal result \cite{7} for nonconvex optimization.

Due to the space limitations, we relegate our results on AbaSGD to the supplementary materials, which also achieves a better gradient complexity than mini-batch SGD.

- Our scheme of adaptive batch size enables to exploit the local Polyak-Łojasiewicz (PL) geometry for achieving a faster linear convergence rate. Specifically, under mild conditions, we show that AbaSVRG and AbaSPIDER automatically transfer to a linear convergence rate once the iteration enters a local PL region. Our proof does not require any assumption on the condition number, as opposed to existing analysis in \cite{15, 20} for SVRG-type algorithms and in \cite{24} for SpiderBoost.

- We provide extensive experiments for demonstrating that AbaSVRG and AbaSPIDER enjoy the fast initial descent as SGD and the high-accuracy estimation as variance-reduced algorithms.
at the later optimization stage, and achieve a significant overall improvement over exiting algorithms.

1.2 Related Work

Stochastic Algorithms for Nonconvex Optimization. Stochastic algorithms were proposed for large-scale nonconvex optimization, which include but not limited to SGD [22], SAG [23], SAGA [6], SVRG [11, 12, 20, 23], SARAH [18], SNVRG [27], SPIDER [7] and SpiderBoost [24]. Among them, SPIDER-type algorithms have been shown to achieve the existing best complexity performance both in the zeroth- and first-order optimization setting [7, 11]. This paper shows that SGD, SVRG and SPIDER can be equipped with the proposed adaptive batch size scheme and enjoy substantially improved performance.

Stochastic Algorithms with Adaptive Batch Size. [2, 8, 28] showed that HSGD converges linearly for strongly convex optimization with exponentially increasing batch size, sub-linearly for convex optimization with quadratically increasing batch size, and sub-linearly for nonconvex optimization with linearly increasing batch size. [4, 5] proposed Big Batch SGD with the batch size adaptive to the instantaneous gradient and variance information (which needs to be ensured, e.g., by backtracking line search) at each iteration. For variance-reduced algorithms, [10] proposed a practical SVRG with exponentially increasing batch size for strongly convex optimization, and [13] proposed an adaptive version of SCSG [14] with exponentially increasing batch size for nonsmooth convex optimization. As a comparison, our AbaSGD, AbaSVRG and AbaSPIDER adapt the batch size using history gradients, which in nature differs from the prescribed adaptive schemes such as linearly or exponentially increasing batch size, and is easier to implement than Big Batch SGD by eliminating backtraking line search and still retains the convergence guarantee.

Nonconvex Optimization under PL condition. Under the PL condition, [20, 21] proved the linear convergence for SVRG by incorporating a restart step, and [15] proposed ProxSVRG+ as an improved version of SVRG and proved its linear convergence without restart. More recently, [24] proved the linear convergence for a variant of SpiderBoost named Prox-SpiderBoost-PL. This paper shows that the proposed adaptive algorithms AbaSVRG and AbaSPIDER achieve the linear convergence rate under PL condition without restart.

Notations. Let ∧ and ∨ denote the minimum and the maximum, respectively. Let \( O(\cdot) \) hide constants that are independent of problem parameters. , and let \( \| \cdot \| \) denote the Euclidean norm of a vector. Let \([n] := \{1, ..., n\}\) and \(|S|\) denote the cardinality of a set \( S \). Given a set \( S \) whose elements are drawn from \([n]\), define \( f_S(\cdot) := \frac{1}{|S|} \sum_{i \in S} f_i(\cdot) \) and \( \nabla f_S(\cdot) := \frac{1}{|S|} \sum_{i \in S} \nabla f_i(\cdot) \).

2 Proposed Algorithms

In this section, we propose novel AbaSGD, AbaSVRG and AbaSPIDER algorithms. Due to the space limitations, we relegated the algorithmic description of AbaSGD and its main results to Appendix A. AbaSVRG and AbaSPIDER are described in Algorithms 1 and 2.

The general idea is to adapt the batch size based on the history gradient information as we describe in Section 1. Ultimately, it is desirable to adapt the batch size to the stochastic gradients calculated in the same loop in which case the convergence analysis follows easily. However, this is impossible, because the batch size should be chosen at the beginning of each outer loop when the gradients in the same loop have not been known yet. Such an issue was previously solved in [4, 5] via backtracking line search, which adds additional complexity. Our main idea here is to use the
Algorithm 1 AbaSVRG
1: **Input:** $x_0$, epoch length $m$, $S = mh, h \in \mathbb{N}$, mini-batch size $|B_s|$, stepsize $\eta, c_\beta, c_\epsilon, \beta_1 > 0$
2: $\bar{x}_0 = x_0$
3: for $s = 1, 2, \ldots, S$ do
4: $x_0^s = \bar{x}_0$
5: Sample $B_s$ from $[n]$ without replacement, where $|B_s| = \min\{c_\beta \beta_s^{-1}, c_\epsilon \epsilon_s^{-1}, n\}$
6: $g^s = \nabla f_{B_s}(\bar{x}^{s-1})$
7: Set $\beta_{s+1} = 0$
8: for $t = 1, 2, \ldots, m$ do
9: Sample $I_t$ from $[n]$ with replacement
10: $v_{t-1}^s = \nabla f_{I_t}(x_{t-1}^s) - \nabla f_{I_t}(\bar{x}^{s-1}) + g^s$
11: $x_t^s = x_{t-1}^s - \eta v_{t-1}^s$
12: $\beta_{s+1} = \beta_{s+1} + \frac{\|v_{t-1}^s\|^2}{m}$
13: end for
14: $\bar{x}^s = x_m^s$
15: end for
16: **Output:** choose $x_\xi$ from $\{x_{t}^s \}_{s \in [S]}$ uniformly at random

Algorithm 2 AbaSPIDER
1: **Input:** $x_0$, epoch length $m$, $S = mh, h \in \mathbb{N}$, mini-batch size $|B_s|$, stepsize $\eta, c_\beta, c_\epsilon, \beta_1 > 0$
2: $\bar{x}_0 = x_0$
3: for $s = 1, 2, \ldots, S$ do
4: $x_0^s = \bar{x}_0$
5: Sample $B_s$ from $[n]$ without replacement, where $|B_s| = \min\{c_\beta \beta_s^{-1}, c_\epsilon \epsilon_s^{-1}, n\}$
6: $v_0^s = \nabla f_{B_s}(\bar{x}^{s-1})$
7: $x_1^s = x_0^s - \eta v_0^s$
8: Set $\beta_{s+1} = \frac{\|v_0^s\|^2}{m}$
9: for $t = 1, 2, \ldots, m - 1$ do
10: Sample $I_t$ from $[n]$ with replacement
11: $v_t^s = \nabla f_{I_t}(x_t^s) - \nabla f_{I_t}(x_{t-1}^s) + v_{t-1}^s$
12: $x_{t+1}^s = x_t^s - \eta v_t^s$
13: $\beta_{s+1} = \beta_{s+1} + \frac{\|v_t^s\|^2}{m}$
14: end for
15: $\bar{x}^s = x_m^s$
16: end for
17: **Output:** choose $x_\xi$ from $\{x_{t}^s \}_{s \in [S]}$ uniformly at random

stochastic gradients calculated in the previous loop for adapting the batch size, and we show that such a scheme still retains the convergence guarantee and has improved computational complexity. More specifically, AbaSVRG/AbaSPIDER differs from SVRG/SPIDER in choosing the batch size $|B_s|$ at epoch $s$ adaptively to the average $\beta_s$ of stochastic gradients in the preceding epoch $s - 1$ as given below

$$(\text{Adaptive batch size:}) \quad |B_s| = \min\{c_\beta \beta_s^{-1}, c_\epsilon \epsilon_s^{-1}, n\}, \quad \text{where} \quad \beta_s = \frac{1}{m} \sum_{t=1}^{m} \|v_{t-1}^s\|^2,$$

where $c_\beta, c_\epsilon$ are positive constants. Note that the conventional SVRG/SPIDER algorithms pick a fixed batch size to be either $n$ or $\min\{c_\epsilon \epsilon_s^{-1}, n\}$. Note that parameters $c_\epsilon, c_\beta$ are correlated with gradient noise parameter $\sigma$, as will be seen in the convergence analysis.

3 Preliminaries

Throughout this paper, we adopt the following standard assumptions [7,14,15,20,24].

**Assumption 1.** The objective function in (1) satisfies:
(1) $\nabla f_i(x)$ is $L$-smooth for $i \in [n]$, i.e., for any $x, y \in \mathbb{R}^d$, $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\|$.
(2) The function $f(\cdot)$ is bounded below, i.e., $f^* = \inf_{x \in \mathbb{R}^d} f(x) > -\infty$.
(3) $\nabla f_i(\cdot)$ (with the index $i$ uniformly randomly chosen) has bounded variance, i.e., there exists a constant $\sigma > 0$ such that for any $x \in \mathbb{R}^d$, $\frac{1}{n} \sum_{i=1}^{n} \|\nabla f_i(x) - \nabla f(x)\|^2 \leq \sigma^2$.

The item (3) of the bounded variance assumption is commonly adopted for proving the convergence of SGD-type algorithms (e.g., SGD [9]) and stochastic variance reduced methods (e.g., SCSG [14])
that draw a sample batch of size less than $n$ for gradient estimation at each outer-loop iteration. In this paper, we use the gradient norm as the convergence criterion for nonconvex optimization.

**Definition 1.** We say that $x^\zeta$ is an $\epsilon$-accurate solution for the optimization problem if $E\|\nabla f(x^\zeta)\|^2 \leq \epsilon$, where $x^\zeta$ is an output returned by a stochastic algorithm.

To compare the efficiency of different stochastic algorithms, we adopt the following stochastic first-order oracle (SFO) for the analysis of the computational complexity.

**Definition 2.** Given an input $x \in \mathbb{R}^d$, SFO randomly takes an index $i \in [n]$ and returns a stochastic gradient $\nabla f_i(x)$ such that $E_i(\nabla f_i(x)) = \nabla f(x)$.

## 4 Analysis under General Nonconvex Landscape

In addition to the above intuitive explanation, we will provide a theoretical explanation on the choice of our adaptive batch size, as will be seen from the following subsections.

### 4.1 AbaSVRG: Convergence and Complexity Analysis

Since the batch size of AbaSVRG is adaptive to the history gradients due to the component $c_\beta \beta^{-1}$, the existing convergence analysis for SVRG type of algorithms in [15, 20] does not extend easily. Here, we develop a new analysis framework for SVRG algorithms which is simpler than that in [15, 20] (and can be of independent interest), and enables to handle the dependence of the batch size on the stochastic gradients in the past epoch in the convergence analysis for AbaSVRG. To compare more specifically, [20] introduced a Lyapunov function $R_\beta^s = E[f(x^\zeta_s)] + c_\beta \|x^\zeta_s - \bar{x}^s\|^2$ and proves that $R_\beta^s$ decreases by the accumulated gradient norms $\sum_{t=0}^{m-1} E\|\nabla f(x_t^s)\|^2$ within an epoch $s$, and [15] directly showed that $E f(x^\zeta)$ decreases by $\sum_{t=0}^{m-1} E\|\nabla f(x_t^s)\|^2$ using tighter bounds. Both studies adopted Young’s inequality, which involves more parameters required to be carefully tuned based on the relationship between $|I_b|$ and $m$. As a comparison, our analysis shows that $E f(x^\zeta)$ decreases by the accumulated stochastic gradient norms $\sum_{t=0}^{m-1} E\|\nabla f(x_t^s)\|^2$, and does not rely on Young’s inequality and extra tuning parameters. More details about our proof can be found in Appendix D. The following theorem provides a general convergence result for AbaSVRG.

**Theorem 1.** Suppose Assumption [1] holds. Let $\epsilon > 0$ and $c_\beta, c_\zeta \geq \alpha \sigma^2$ for certain constant $\alpha > 0$. Let $\psi = 2\eta^2 L^2 \frac{m^2}{|I_b|} + \frac{2}{\alpha} + 2$ and choose $\beta_1$, $\alpha$ and $\eta$ such that $\beta_1 \leq \epsilon S$ and $\phi = 1 - \frac{L_0}{2} - \frac{L_0^2 \frac{m^2}{|I_b|}}{2} > 0$. Then, the output $x^\zeta$ returned by AbaSVRG satisfies

$$E\|\nabla f(x^\zeta)\|^2 \leq \frac{\psi(f(x_0) - f^*)}{\phi \eta T} + \frac{\psi \epsilon}{\phi \alpha} + \frac{4 \epsilon}{\alpha},$$

where $f^* = \inf_{x \in \mathbb{R}^d} f(x)$ and $T = Sm$ is the total number of iterations.

Theorem 1 guarantees the convergence of AbaSVRG as long as $\phi$ is positive, i.e. $\frac{L_0}{2} + \frac{\eta^2 L^2 \frac{m^2}{|I_b|}}{2} \leq \frac{1}{2} - \frac{1}{2\alpha}$, and thus allows very flexible choices of stepsize $\eta$, epoch length $m$ and mini-batch size $|I_b|$. Such flexibility and generality are also due to the aforementioned simpler proof that we develop for SVRG-type algorithms.

In the following corollary, we provide the complexity performance of AbaSVRG under certain choices of parameters.
Corollary 1. Under the setting of Theorem 1, we choose the constant stepsize \( \eta = \frac{1}{4L\sqrt{m}} \), mini-batch size \( |I_b| = m^2 \) and \( c_\beta, c_\epsilon \geq 16\sigma^2 \). Then, to obtain an \( \epsilon \)-accurate solution \( x_\xi \), i.e., \( \mathbb{E}[\|\nabla f(x_\xi)\|^2] \leq \epsilon \), the total number of SFO calls required by AbaSVRG is given by

\[
\sum_{s=1}^{S} \min \left\{ c_\beta \beta_s^{-1}, c_\epsilon \epsilon^{-1}, n \right\} + T|I_b| < S \min \left\{ c_\epsilon \epsilon^{-1}, n \right\} + T|I_b| = O\left( \frac{n \wedge \epsilon^{-1}}{\sqrt{|I_b|\epsilon}} + \frac{|I_b|}{\epsilon} \right). \tag{2}
\]

If we specially choose \( |I_b| = n^{2/3} \wedge \epsilon^{-2/3} \), then the worst-case complexity is \( O\left( \frac{1}{\epsilon}(n \wedge \frac{1}{\epsilon})^{2/3} \right) \).

We make the following several remarks on Corollary 1.

First, the worst-case SFO complexity under the specific choice of \( |I_b| = n^{2/3} \wedge \epsilon^{-2/3} \) matches the best known result for SVRG-type algorithms. More importantly, since the adaptive component \( c_\beta \beta_s^{-1} \) can be much smaller than \( \min\{c_\epsilon \epsilon^{-1}, n\} \) during the optimization process particularly in the initial stage, the SFO complexity of AbaSVRG can be much less than that of SVRG with fixed batch size as well as the worst-case complexity of \( O\left( \frac{1}{\epsilon}(n \wedge \frac{1}{\epsilon})^{2/3} \right) \), as demonstrated in our experiments.

Second, our convergence and complexity results hold for any choice of mini-batch size \( |I_b| \), and thus we can safely choose a small mini-batch size rather than the large one \( n^{2/3} \wedge \epsilon^{-2/3} \) in the regime with large \( n \) and \( \epsilon^{-1} \). In addition, for a given \( |I_b| \), the resulting worst-case complexity \( O\left( \frac{n \wedge \epsilon^{-1}}{\sqrt{|I_b|\epsilon}} + \frac{|I_b|}{\epsilon} \right) \) still matches the best known order given by ProxSVRG+ [15] for SVRG-type algorithms.

Third, Corollary 1 sets \( |I_b| = m^2 \) to obtain the best complexity order. However, our experiments suggest that \( |I_b| = m \) has better performance. Hence, we provide analysis for this case in the following corollary.

Corollary 2. Let stepsize \( \eta = \frac{1}{4L\sqrt{m}} \), mini-batch size \( |I_b| = m \) and \( c_\beta, c_\epsilon \geq 16\sigma^2 \). Then, to obtain an \( \epsilon \)-accurate solution \( x_\xi \), the total number of SFO calls required by AbaSVRG is given by

\[
\sum_{s=1}^{S} \min \left\{ \frac{c_\beta}{\beta_s}, \frac{c_\epsilon}{\epsilon}, n \right\} + T|I_b| \leq O\left( \frac{n \wedge \epsilon^{-1}}{\sqrt{|I_b|\epsilon}} + \frac{|I_b|^{3/2}}{\epsilon} \right).
\]

If we specially choose \( |I_b| = n^{1/2} \wedge \epsilon^{-1/2} \), then the worst-case complexity is \( O\left( \frac{1}{\epsilon}(n \wedge \frac{1}{\epsilon})^{3/4} \right) \).

Note that under the choice of \( I_b = m \), the SFO complexity of AbaSVRG still improves that of SGD by an order of \( O(\epsilon^{-1/4}) \).

4.2 AbaSPIDER: Convergence and Complexity Analysis

In this subsection, we study AbaSPIDER, and compare our results with that for AbaSVRG. The following theorem provides a general convergence result for AbaSPIDER.

Theorem 2. Suppose Assumption 1 holds. Let \( \epsilon > 0 \) and \( c_\beta, c_\epsilon \geq \alpha\sigma^2 \) for certain constant \( \alpha > 0 \). Let \( \psi = \frac{2n^2L^2m}{|I_b|} + \frac{2}{\alpha} + 2 \) and choose \( \beta_1, \alpha \) and \( \eta \) such that \( \beta_1 \leq S\epsilon \) and \( \phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta}{2} - \frac{n^2L^2m}{2|I_b|} > 0 \). Then, the output \( x_\xi \) returned by AbaSPIDER satisfies

\[
\mathbb{E}[\|\nabla f(x_\xi)\|^2] \leq \frac{\psi(f(x_0) - f^*)}{\phi nT} + \frac{\psi\epsilon}{2\phi\alpha} + \frac{4\epsilon}{\alpha},
\]

where \( f^* = \inf_{x \in \mathbb{R}^d} f(x) \) and \( T = Sm \) is the total number of iterations.
To guarantee the convergence, AbaSPIDER needs a smaller mini-batch size $|I_b|$ than AbaSVRG, because AbaSPIDER requires $|I_b| \geq \Theta(mn^2)$ (see Theorem 2) to guarantee $\phi$ to be positive, whereas AbaSVRG requires $|I_b| \geq \Theta(mn^2)$ (see Theorem 1). Thus, to achieve the same-level of target accuracy, AbaSPIDER uses fewer mini-batch samples than AbaSVRG, and thus achieves a lower worst-case SFO complexity, as can be seen in the following corollary.

**Corollary 3.** Under the setting of Theorem 2, for any mini-batch size $|I_b| \leq n^{1/2} \wedge \epsilon^{-1/2}$, we choose the epoch length $m = (n \wedge \frac{1}{\epsilon})|I_b|^{1/2}$, the stepsize $\eta = \frac{1}{4T} \sqrt{|I_b| / m}$ and $c_\beta, c_\epsilon \geq 16\sigma^2$. Then, to obtain an $\epsilon$-accurate solution $x_\epsilon$, the total number of SFO calls required by AbaSPIDER is given by

$$\sum_{s=1}^S \min \left\{ c_\beta \beta_s^{-1}, c_\epsilon \epsilon^{-1}, n \right\} + T |I_b| = O \left( \frac{1}{\epsilon} \left( n \wedge \frac{1}{\epsilon} \right)^{1/2} \right).$$

Corollary 3 shows that for a wide range of mini-batch size $|I_b|$, AbaSPIDER achieves the near-optimal worst-case complexity $O \left( \frac{1}{\epsilon} \left( n \wedge \frac{1}{\epsilon} \right)^{1/2} \right)$ under a proper selection of $m$ and $\eta$. Thus, our choice of mini-batch size is less restrictive than $|I_b| = n^{1/2} \wedge \epsilon^{-1/2}$ used by SpiderBoost [21] to achieve the optimal complexity, which may be very large in practice. More importantly, our practical complexity can be much better than those of SPIDER [7] with a fixed batch size due to the batch size adaptivity.

## 5 Analysis under Local PL Geometry

Many nonconvex machine learning problems (e.g., phase retrieval [30]) and deep learning (e.g., neural networks [26, 29]) problems have been shown to satisfy the following Polyak-Łojasiewicz (PL) (i.e., gradient dominance) condition in local regions near local or global minimizers.

**Definition 3 ([17, 19]).** Let $x^* = \arg\min_{x \in \mathbb{R}^d} f(x)$. Then, the function $f$ is said to be $\tau$-gradient dominated if for any $x \in \mathbb{R}^d$, $f(x) - f(x^*) \leq \tau \|\nabla f(x)\|^2$.

Recently, [15] proposed a SVRG-type algorithm named ProxSVRG+, and showed that it achieves a linear converge rate under PL condition without restart. It means that ProxSVRG+ can be initialized in a general nonconvex landscape, and then transfers to a faster linear convergence rate once it enters the local PL regions. In this section, we explore whether under the adaptive batching scheme, our AbaSVRG and AbaSPIDER can transfer to a faster linear convergence rate once it enters the local PL regions.

### 5.1 AbaSVRG: Convergence under PL Geometry without Restart

The following theorem provides an analysis for AbaSVRG under the PL condition.

**Theorem 3.** Let $\eta = \frac{1}{c_\epsilon \epsilon^2}$, $|I_b| = m^2$ with $\frac{8L_T}{c_\epsilon - 2} \leq m < 4L_T$, $\beta_1 \leq \epsilon (\frac{1}{\epsilon})^{m(S-1)}$, and $c_\beta = c_\epsilon = \left( 2\tau \sigma^2 + \frac{2\tau^2 \sigma^2}{1 - \exp(\frac{1}{c_\eta (\epsilon^2 - 1)})} \right) \sqrt{\frac{16c_\eta L_T \sigma^2}{m}}$, where constants $c_\eta > 4$ and $\gamma = 1 - \frac{1}{8L_T} < 1$. Then under the PL condition, the final iterate $\tilde{x}^S$ of AbaSVRG satisfies $E(f(\tilde{x}^S) - f(x^*)) \leq \gamma^T (f(x_0) - f(x^*)) + \frac{\epsilon}{2}$.

To obtain an $\epsilon$-accurate solution $\tilde{x}^S$, the total number of SFO calls is given by

$$\sum_{s=1}^S \min \left\{ c_\beta \beta_s^{-1}, c_\epsilon \epsilon^{-1}, n \right\} + T |I_b| \leq O \left( \left( \frac{T}{\epsilon} \wedge n \right) \log \frac{1}{\epsilon} + \tau^3 \log \frac{1}{\epsilon} \right).$$  (3)
Our proof of Theorem 3 is different from and more challenging than previous techniques in [15, 20, 21] for SVRG-type algorithms, because we need to handle the adaptive batch size $|B_s|$ with the dependencies on the iterations at the previous epoch. In addition, we do not need extra assumptions for proving the convergence under PL condition, whereas [21] and [15] require $\tau \geq n^{1/3}$ and $\tau \geq n^{1/2}$, respectively. As a result, Theorem 3 can be applied to any condition number regime. For the small condition number regime where $1 \leq \tau \leq \Theta(n^{1/3})$, the worst-case complexity of AbaSVRG outperforms the result achieved by SVRG [21]. Furthermore, the actual complexity of our AbaSVRG can be much lower than the worst-case complexity due to the adaptive batch size.

Theorem 3 shows that the adaptive batch size $|B_s|$ still retains the desired local convergence property induced by the SVRG estimator, even though $|B_s|$ has an involved dependence on previous iterations.

5.2 AbaSPIDER: Convergence under PL Geometry without Restart

Although SPIDER-type algorithms (e.g., SPIDER [7], SpiderBoost and Prox-SpiderBoost [24]) have been shown to achieve the nearly optimal complexity performance for nonconvex optimization, it still remains unknown whether these algorithms have the similar local convergence property like ProxSVRG+ or our proposed AbaSVRG. The following theorem shows that AbaSPIDER achieves a linear convergence rate under the PL condition without restart. Our analysis can be of independent interest for other SPIDER-type methods.

**Theorem 4.** Let $\eta = \frac{1}{c_{\eta}L}$, $|I_b| = m$ with $\frac{8L\tau}{c_{\eta} - 2} \leq m < 4L\tau$, $\beta_1 \leq \epsilon(\frac{1}{\epsilon})^{m(S-1)}$, and $c_\beta = c_\epsilon = \left(2\tau\sigma^2 + \frac{2\tau\sigma^2}{1 - \exp(\frac{c_{\eta} - 2}{c_{\eta} - 2})}\right)^{\frac{1}{\epsilon}} \frac{\frac{2c_{\eta}L\tau\sigma^2}{m}}{m}$, where constants $c_{\eta} > 4$ and $\gamma = 1 - \frac{1}{8L\tau}$. Then under the PL condition, the final iterate $\tilde{x}^S$ of AbaSPIDER satisfies $E(f(\tilde{x}^S) - f(x^*)) \leq \gamma^T E(f(x_0) - f(x^*)) + \frac{\epsilon}{2}$. To obtain an $\epsilon$-accurate solution $\tilde{x}^S$, the total number of SFO calls is given by

$$\sum_{s=1}^{S} \min\{c_\beta \beta_1^{-1}, c_\epsilon \epsilon^{-1}, n\} + T|I_b| \leq O\left(\left(\frac{\tau}{\epsilon} \wedge n\right) \log \frac{1}{\epsilon} + \tau^2 \log \frac{1}{\epsilon}\right).$$

As shown in Theorem 4, AbaSPIDER achieves a lower worst-case SFO complexity than AbaSVRG by a factor of $\Theta(\tau)$, and matches the best result provided by Prox-SpiderBoost-gd [21]. However, Prox-SpiderBoost-gd is a variant of Prox-SpiderBoost with algorithmic modification, and has not been shown to achieve the near-optimal complexity for general nonconvex optimization. In addition, AbaSPIDER has a much lower complexity in practice due to the adaptive batch size.

6 Experiments

In this section, we compare our AbaSGD, AbaSVRG and AbaSPIDER with the state-of-the-art algorithms Mini-batch SGD [9], HSGD [28], SVRG+ [15] and SpiderBoost [21]. We conduct two nonconvex optimization problems, i.e., logistic regression and training multi-layer neural networks. To implement HSGD, we follow [28] and choose the linearly increasing mini-batch size at the $t^{th}$ iteration to be $c_b(t + 1)$, and tune $c_b$ to the best. We choose the epoch length $m = 10$ for all variance-reduced algorithms, and choose the batch size to be $\min\{n, c_\epsilon \epsilon^{-1}\}$ for SVRG+ and SpiderBoost, where $c_\epsilon$ is the same as our AbaSPIDER and AbaSVRG, and is tuned to be the best. For all algorithms, we

---

1We use SVRG+ to denote the non-regularized version of ProxSVRG+ proposed by [15].
tune the stepsize $\eta$ and mini-batch size $|I_b|$ to the best. The detailed hyper-parameter settings for all algorithms are specified in Appendix B.1 due to the space limitations.

6.1 Nonconvex Logistic Regression

In this subsection, we consider the following nonconvex logistic regression problem with two classes:

$$\min_{w \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \ell(w^T x_i, y_i) + \alpha \sum_{i=1}^{d} \frac{w_i^2}{1 + w_i^2},$$

where $x_i \in \mathbb{R}^d$ denote the features, $y_i \in \{\pm 1\}$ are the classification labels, $\ell$ is the cross-entropy loss, and we set $\alpha = 0.1$. For this problem, we use four datasets from LIBSVM [3]: the a8a dataset ($n = 22696, d = 123$), the a9a dataset ($n = 32561, d = 123$), the w8a dataset ($n = 43793, d = 300$) and the ijcnn1 dataset ($n = 49990, d = 22$).

As can be seen from Fig. 1 and Fig. 4 (in Appendix B.2), AbaSVRG and AbaSPIDER converges much faster than all other algorithms in terms of the total number of gradient evaluations (i.e., SFO complexity) on all four datasets. It can be seen that both of them take the advantages of the sample efficiency of SGD-type algorithm at the initial stage of optimization procedure and the high accuracy of variance-reduced methods at the final stage. This is consistent with the choice of our adaptive batching scheme. In addition, AbaSGD is faster than HSGD and mini-batch SGD at the initial optimization stage on all datasets, and converges more stably at the final stage.

6.2 Comparison to Variance Reduction Methods with Prescribed Adaptive Batch Size

In this subsection, we evaluate the performance of our history-gradient based adaptive batch size schemes with the other two prescribed adaptive batch size schemes, i.e., exponentially increasing batch size $|B_s| = \mu^s$ and linearly increasing batch size $|B_s| = \nu(s + 1)$. For all algorithms, we choose the same stepsize $\eta$, epoch length $m$ and mini-batch size $|I_b|$ for a fair comparison. Let SpiderBoost _exp_ $\mu$ and SpiderBoost _lin_ $\nu$ denote SpiderBoost algorithms with exponentially and linearly increasing batch sizes under parameters $\mu$ and $\nu$, respectively.

It can be observed from Fig. 2 that our adaptive batch size scheme achieves the best performance for a9a dataset, and performs better than all other algorithms for w8a dataset except SpiderBoost _lin_ 200, which, however, does not converge in the high-accuracy regime. Furthermore, the performance of prescribed batch size adaptation can be problem specific. For example, exponentially increased batch size (with $\mu = 2$ and $\mu = 2.1$) performs better than linearly increased batch size for a9a dataset, but worse for w8a dataset, whereas our gradient-based batch size can adapt to the optimization path, and hence performs the best in both cases.
6.3 Training Three-Layer Neural Networks

In this subsection, we compare our proposed algorithms with other competitive algorithms for training a three-layer ReLU neural network with a cross entropy loss on the dataset of MNIST ($n = 60000, d = 780$). The neural network has a size of $(d_{in}, 100, 100, d_{out})$, where $d_{in}$ and $d_{out}$ are the input and output dimensions and 100 is the number of neurons in the two hidden layers.

![Comparison of various algorithms for training a three-layer neural network on MNIST.](image)

Figure 3: Comparison of various algorithms for training a three-layer neural network on MNIST.

As shown in Fig. 3, our AbaSVRG achieves the best performance among all competing algorithms, and AbaSPIDER performs similarly to Mini-batch SGD for decreasing training loss, but converges faster in terms of gradient norm. Interestingly, the adaptive batch size used by AbaSVRG increases slower than both exponentially and linearly, and its scaling is close to the logarithmical increase. Such an observation further demonstrates that our gradient-based batch size scheme can also adapt to the neural network landscape with a differently (i.e., more slowly) increased batch size from that for regression problem over a9a and w8a datasets.

7 Conclusion

In this paper, we propose a novel history-gradient aided adaptive batching scheme, and develop three faster stochastic algorithms AbaSGD, AbaSVRG and AbaSPIDER. These algorithms achieve improved complexity than their vanilla counterparts, and their convergence rates are adaptive to the optimization landscape. Extensive experiments demonstrate the promising performances of proposed algorithms. We anticipate such a scheme can be applied to a wide range of other algorithms to accelerate their theoretical and practical performances.
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Supplementary Materials

A  AbaSGD: Faster SGD with Adaptive Mini-Batch Size

In this section, we propose a new SGD algorithm with mini-batch size depending on the stochastic gradients in the preceding $q$ steps. As shown in Algorithm 3. To simplify notations, we set norms of the stochastic gradients before the algorithm starts to be $\|v_{-1}\| = \|v_{-2}\| = \cdots = \|v_{-q}\| = \alpha_0$ and let $E_t(\cdot) = E(\cdot | x_0, ..., x_t)$.

Algorithm 3 AbaSGD

1: Input: $x_0$, stepsize $\eta$, $q > 0$, $\alpha_0 > 0$
2: for $t = 0, 1, ..., T$ do
3: Set $|B_t| = \min \{ \frac{2\sigma^2}{q} \sum_{i=1}^q \|v_{t-i}\|^2/q, \frac{24\sigma^2}{\epsilon}, n \}$.
4: if $|B_t| = n$ then
5: Compute $v_t = \nabla f(x_t)$
6: else
7: Sample $B_t$ from $[n]$ with replacement, and compute $v_t = \nabla f_{B_t}(x_t)$
8: end if
9: $x_{t+1} = x_t - \eta v_t$
10: end for
11: Output: choose $x_\zeta$ from $\{x_t\}_{t=0,...,T}$ uniformly at random

Theorem 5. Let Assumption 1 hold, $\epsilon > 0$ and choose a stepsize $\eta$ such that

$$\phi = \eta - \frac{L \epsilon^2}{2} > 0.$$ 

Then, the output $x_\zeta$ returned by AbaSGD satisfies

$$E \|\nabla f(x_\zeta)\|^2 \leq \frac{2(f(x_0) - f^*) + \eta q \alpha_0^2}{2T\phi} + \frac{\eta}{12\phi} \epsilon,$$

where $f^* = \inf_{x \in \mathbb{R}^d} f(x)$, and $T$ is the total number of iterations.

Theorem 5 shows that AbaSGD achieves a $O\left(\frac{1}{\epsilon} \wedge \frac{1}{\epsilon^2}\right)$ convergence rate for nonconvex optimization by using the adaptive mini-batch size. In the following corollary, we derive the SFO complexity of AbaSGD.

Corollary 4. Under the setting of Theorem 5, we choose the constant stepsize $\eta = \frac{1}{2L}$. Then, to obtain an $\epsilon$-accurate solution $x_\zeta$, the total number of iterations required by AbaSGD

$$T = \frac{16L (f(x_0) - f^*) + 4q \alpha_0^2}{\epsilon},$$

and the total number of SFO calls required by AbaSGD is given by

$$\sum_{t=0}^T |B_t| = \sum_{t=0}^T \min \left\{ \frac{2\sigma^2}{\sum_{i=1}^q \|v_{t-i}\|^2/q}, \frac{24\sigma^2}{\epsilon}, n \right\} \leq T \min \left\{ \frac{24\sigma^2}{\epsilon}, n \right\} = O\left(\frac{1}{\epsilon^2} \wedge \frac{n}{\epsilon}\right).$$
Corollary 4 shows that the worst-case complexity of AbaSGD is $O\left(\frac{1}{\epsilon^2} \wedge \frac{n}{\epsilon}\right)$, which is at least as good as those of SGD and GD. More importantly, the actual complexity of AbaSGD can be much lower than those of GD and SGD due to the adaptive batch size.

B Further Specification of Experiments and Additional Results

B.1 Hyper-parameter configuration for algorithms under comparison

For logistic regression, we select the stepsize $\eta$ from $\{0.1k, k = 1, 2, ..., 15\}$ and the mini-batch size $|I_b|$ from $\{10, 28, 64, 128, 256, 512, 1024\}$ for all algorithms, and we present the best performance among these parameters. For all variance-reduced algorithms, we select constants $c_\beta$ and $c_\epsilon$ from $\{1, 2, 3, ..., 10\}$, and present the best performance among these parameters. For HSGD algorithm, we select $c_\beta$ in its linearly increasing batch size $c_\beta(t + 1)$ from $\{1, 5, 10, 40, 100, 400, 1000\}$, and present the best performance among these parameters. For AbaSGD, we set its batch size as $\min\left\{\frac{c_\beta}{\sum_{i=1}^{n} \|v_{t-i}\|^2/5}, \frac{c_\epsilon}{\epsilon}, n\right\}$, and select the best $c_\beta$ and $c_\epsilon$ from $\{1, 2, 3, ..., 10\}$.

For training multi-layer neural networks, we select the stepsize $\eta$ from $\{10^{-4}k, k = 1, 2, ..., 15\}$ and the mini-batch size $|I_b|$ from $\{64, 96, 128, 256, 512\}$ for all algorithms, and we present the best performance among these parameters. For all variance-reduced algorithms, we set $c_\epsilon = 1$ and select the best $c_\beta$ from $\{10^3, 5 \times 10^3, 10^4\}$. For HSGD algorithm, we select $c_\beta$ from $\{1, 10, 50, 100, 500, 1000\}$, and present the best performance among these parameters.

B.2 Additional Results for Logistic Regression

Figure 4: Comparison of different algorithms for logistic regression problem on four datasets. All figures plot gradient norm v.s. # of gradient evaluations.
Technical Proofs

C Proofs for AbaSGD

C.1 Proof of Theorem \[5\]

Since the gradient $\nabla f$ is $L$-Lipschitz, we obtain that, for $t \geq 0$,

$$f(x_{t+1}) \leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2}\|x_{t+1} - x_t\|^2$$

\[= f(x_t) - \eta \langle \nabla f(x_t), v_t \rangle + \frac{L\eta^2}{2}\|v_t\|^2\]

\[\overset{(i)}{=} f(x_t) - \eta \langle \nabla f(x_t) - v_t, v_t + v_t, v_t \rangle + \frac{L\eta^2}{2}\|v_t\|^2\]

\[= f(x_t) - \eta \langle \nabla f(x_t) - v_t, v_t \rangle - \eta\|v_t\|^2 + \frac{L\eta^2}{2}\|v_t\|^2\]

\[= f(x_t) - \eta \langle \nabla f(x_t) - v_t, v_t - \nabla f(x_t) + \nabla f(x_t) \rangle - \eta \langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle - \eta \cdot \frac{L\eta^2}{2}\|v_t\|^2\]

\[= f(x_t) + \eta \|\nabla f(x_t) - v_t\|^2 - \eta \langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle - \eta \cdot \frac{L\eta^2}{2}\|v_t\|^2\]

where (i) follows from the fact that $x_{t+1} = x_t - \eta v_t$. Then, taking expectation $\mathbb{E}(\cdot)$ over the above inequality yields

$$\mathbb{E}f(x_{t+1}) \leq \mathbb{E}f(x_t) + \eta \mathbb{E}\|\nabla f(x_t) - v_t\|^2 - \eta \mathbb{E}\langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle - \left(\eta - \frac{L\eta^2}{2}\right) \mathbb{E}\|v_t\|^2$$

\[\overset{(i)}{=} \mathbb{E}f(x_t) + \eta \mathbb{E}\|\nabla f(x_t) - v_t\|^2 - \left(\eta - \frac{L\eta^2}{2}\right) \mathbb{E}\|v_t\|^2\]

\[= \mathbb{E}f(x_t) - \left(\eta - \frac{L\eta^2}{2}\right) \mathbb{E}\|v_t\|^2 + \eta \mathbb{E}\|\nabla f(x_t) - v_t\|^2,\]

(4)

where (i) follows from $\mathbb{E}\langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle = \mathbb{E}_{x_0, \ldots, x_t} (\mathbb{E}_t \langle \nabla f(x_t) - v_t, \nabla f(x_t) \rangle) = 0$.

Next, we upper-bound $\mathbb{E}\|\nabla f(x_t) - v_t\|^2$. For the case when $|B_t| < n$, we have

$$\mathbb{E}\|\nabla f(x_t) - v_t\|^2 = \mathbb{E}\left\| \nabla f(x_t) - \frac{1}{|B_t|} \sum_{i \in B_t} \nabla f_i(x_t) \right\|^2 = \mathbb{E}\left\| \frac{1}{|B_t|} \sum_{i \in B_t} (\nabla f(x_t) - \nabla f_i(x_t)) \right\|^2$$

\[= \mathbb{E}\left\| \frac{1}{|B_t|} \sum_{i \in B_t} (\nabla f(x_t) - \nabla f_i(x_t)) \right\|^2\]

\[= \mathbb{E}\left\| \frac{1}{|B_t|} \sum_{i \in B_t, j \in B_t} (\nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t)) \right\|^2\]

\[= \mathbb{E}_{x_0, \ldots, x_t} \left(\mathbb{E}_t \frac{1}{|B_t|^2} \sum_{i \in B_t, j \in B_t} (\nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t)) \right)\]

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\[
= \mathbb{E}_{x_0, \ldots, x_t} \frac{1}{|B_t|^2} \sum_{i \in B_t} \sum_{j \in B_t} \mathbb{E}_t (\nabla f(x_t) - \nabla f_i(x_t), \nabla f(x_t) - \nabla f_j(x_t))
\]

where (i) follows from the definition of \( i \neq j \), and \( \mathbb{E}_t (\nabla f_i(x_t), \nabla f_j(x_t)) = 0 \) for \( i \neq j \), and (ii) follows from item (3) in Assumption \ref{assumption}. For the case when \( |B_t| = n \), we have \( \mathbf{v}_t = \nabla f(x_t) \), and thus \( \mathbb{E}\|\nabla f(x_t) - \mathbf{v}_t\|^2 = 0 \). Combining the above two cases, we have

\[
\mathbb{E}\|\nabla f(x_t) - \mathbf{v}_t\|^2 \leq \mathbb{E}\left( \frac{I_{(|B_t|<n)}}{|B_t|} \sigma^2 \right).
\]  

Plugging (5) into (4), we obtain

\[
\left( \eta - \frac{L\eta^2}{2} \right) \mathbb{E}\|\mathbf{v}_t\|^2 \leq \mathbb{E}f(x_t) - \mathbb{E}f(x_{T+1}) + \mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|} \eta^2.
\]

Telescop ing the above inequality over \( t \) from 0 to \( T \) yields

\[
\sum_{t=0}^{T} \left( \eta - \frac{L\eta^2}{2} \right) \mathbb{E}\|\mathbf{v}_t\|^2 \leq \mathbb{E}f(x_0) - \mathbb{E}f(x_{T+1}) + \sum_{t=0}^{T} \mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|} \eta^2.
\]  

Next, we upper-bound \( \sum_{t=0}^{T} \mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|} \eta^2 \) in the above inequality through the following steps.

\[
\sum_{t=0}^{T} \mathbb{E}\frac{I_{(|B_t|<n)}}{|B_t|} \eta^2 \leq \sum_{t=0}^{T} \mathbb{E}\left( \sum_{i=1}^{q} \mathbb{E}\|\mathbf{v}_{t-i}\|^2 + \frac{\eta}{24} \right) \eta^2
\]

\[
= \frac{\eta}{2q} \sum_{t=0}^{T} \sum_{i=1}^{q} \mathbb{E}\|\mathbf{v}_{t-i}\|^2 + \sum_{t=0}^{T} \frac{\eta}{24}
\]

\[
= \frac{\eta}{2q} \sum_{t=q+1}^{T} \sum_{i=1}^{q} \mathbb{E}\|\mathbf{v}_{t-i}\|^2 + \frac{\eta}{2q} \sum_{t=0}^{q} \sum_{i=1}^{q} \mathbb{E}\|\mathbf{v}_{t-i}\|^2 + \sum_{t=0}^{T} \frac{\eta}{24}
\]

\[
\leq \frac{\eta}{2q} \sum_{t=q+1}^{T} \sum_{i=1}^{q} \mathbb{E}\|\mathbf{v}_{t-i}\|^2 + \sum_{t=0}^{T} \frac{\eta}{24}
\]

\[
= \frac{\eta}{2q} \sum_{t=0}^{T} \mathbb{E}\|\mathbf{v}_t\|^2 \sum_{i=1}^{\min(t+q,T)} 1 + \frac{\eta}{2} \sum_{t=0}^{T} \mathbb{E}\|\mathbf{v}_{t+1}\|^2 + \sum_{t=0}^{T} \frac{\eta}{24}
\]

\[
\leq \frac{\eta}{2} \sum_{t=0}^{T} \mathbb{E}\|\mathbf{v}_t\|^2 + \frac{\eta q}{2} \alpha_0^2 + \sum_{t=0}^{T} \frac{\eta}{24}
\]

where (i) follows from the definition of \( |B_t| \), (ii) follows from the fact that \( \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = \cdots = \|\mathbf{v}_q\| = \alpha_0 \).

Plugging (7) into (6), we obtain

\[
\sum_{t=0}^{T} \left( \eta - \frac{L\eta^2}{2} \right) \mathbb{E}\|\mathbf{v}_t\|^2 \leq \mathbb{E}f(x_0) - \mathbb{E}f(x_{T+1}) + \frac{\eta}{2} \sum_{t=0}^{T} \mathbb{E}\|\mathbf{v}_t\|^2 + \frac{\eta q}{2} \alpha_0^2 + \sum_{t=0}^{T} \frac{\eta}{24}.
\]
which further yields
\[
\sum_{t=0}^{T} \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E} \|v_t\|^2 \leq \mathbb{E} f(x_0) - \mathbb{E} f(x_{T+1}) + \frac{\eta q}{2} \alpha_0^2 + \sum_{t=0}^{T} \frac{\eta \epsilon}{24}.
\]
\[
\leq f(x_0) - f^* + \frac{\eta q}{2} \alpha_0^2 + \sum_{t=0}^{T} \frac{\eta \epsilon}{24}. \tag{8}
\]

Recall that \(\phi := \left( \eta - \frac{L\eta^2}{2} \right) > 0\). Then, we obtain from (8) that
\[
\sum_{t=0}^{T} \mathbb{E} \|v_t\|^2 \leq \frac{2(f(x_0) - f^*) + \eta q \alpha_0^2}{2\phi} + \frac{(T+1)\eta \epsilon}{24 \phi}. \tag{9}
\]

Recall that the output \(x_\zeta\) is chosen from \(\{x_t\}_{t=0,...,T}\) uniformly at random. Then, based on (9), we have
\[
\mathbb{E} \|\nabla f(x_\zeta)\|^2 = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \|\nabla f(x_t)\|^2 = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \|\nabla \mathbb{E}_t v_t\|^2 \leq \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(\mathbb{E}_t \|v_t\|^2)
\]
\[
= \frac{1}{T} \sum_{t=1}^{T} \mathbb{E} \|v_t\|^2 \leq \frac{2(f(x_0) - f^*) + \eta q \alpha_0^2}{2T\phi} + \frac{(T+1)\eta \epsilon}{24T\phi}
\]
\[
\leq \frac{2(f(x_0) - f^*) + \eta q \alpha_0^2}{2T\phi} + \frac{\eta \epsilon}{12\phi},
\]
where (i) follows from the Jensen’s inequality, and (ii) follows from (9).

C.2 Proof of Corollary 4

Since \(\eta = \frac{1}{2L}\), have
\[
\phi = \left( \eta - \frac{L\eta^2}{2} \right) = \frac{1}{8L} > 0.
\]

Then, plugging \(\eta = \frac{1}{2L\phi}\), \(\phi = \frac{1}{8L}\) and \(T = (16L(f(x_0) - f^*) + 4q\alpha_0^2) \epsilon^{-1}\) in Theorem 5 we have
\[
\mathbb{E} \|\nabla f(x_\zeta)\|^2 \leq \frac{8L(f(x_0) - f^*) + 2q\alpha_0^2}{T} + \frac{\epsilon}{3} \leq \frac{5}{6} \epsilon \leq \epsilon.
\]

Thus, the total SFO calls required by AbaSGD is given by
\[
\sum_{t=0}^{T} |B_t| = \sum_{t=0}^{T} \min\left\{ \frac{\sigma^2}{\sum_{i=1}^{q} \|v_{t-i}\|^2 / q}, \frac{24\sigma^2}{\epsilon} \right\}, n \leq (T + 1) \left( \frac{24\sigma^2}{\epsilon} \wedge n \right) = O\left( \frac{1}{\epsilon^2} \wedge \frac{n}{\epsilon} \right).
\]

D Proofs for Results in Section 4

D.1 Proof of Theorem 1

To prove Theorem 1 we first establish the following lemma to upper-bound the estimation variance \(\mathbb{E}_{0,s}\|\nabla f(x^*_t) - v^*_t\|^2\) for \(1 \leq t \leq m\), where \(\mathbb{E}_{t,s}(\cdot)\) denotes \(\mathbb{E}(\cdot|x_0^t, x_0^2, ..., x_2^t, ..., x_t^t)\)
Lemma 1. Let Assumption 1 hold. Then, for $1 \leq t \leq m$, we have

$$\mathbb{E}_{0,s} \| \nabla f(x_{t-1}^s) - v_{t-1}^s \|^2 \leq \frac{\eta^2 L^2(t-1)}{|I_b|} \mathbb{E}_{0,s} \sum_{i=0}^{t-2} \| v_i^s \|^2 + \frac{I(|B_s|<n)}{|B_s|} \sigma^2$$

(10)

where $I(A) = 1$ if the event $A$ occurs and 0 otherwise, and $\sum_{i=0}^{t-1} \| v_i^s \|^2 = 0$.

Proof. Based on line 10 in Algorithm 1 we have, for $1 \leq t \leq m$,

$$\| v_{t-1}^s - \nabla f(x_{t-1}^s) \|^2 = \| \nabla f_{t_b}(x_{t-1}^s) - \nabla f_{t_b}(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}) + g^s - \nabla f(\tilde{x}^{s-1}) \|^2.$$

Taking the expectation $\mathbb{E}_{0,s}(\cdot)$ over the above equality yields

$$\mathbb{E}_{0,s} \| v_{t-1}^s - \nabla f(x_{t-1}^s) \|^2 = \mathbb{E}_{0,s} \| \nabla f_{t_b}(x_{t-1}^s) - \nabla f_{t_b}(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}) \|^2 + 2 \mathbb{E}_{0,s} \langle \nabla f_{t_b}(x_{t-1}^s) - \nabla f_{t_b}(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}), g^s - \nabla f(\tilde{x}^{s-1}) \rangle$$

$$+ \mathbb{E}_{0,s} \| g^s - \nabla f(\tilde{x}^{s-1}) \|^2,$$

(11)

which, in conjunction with the fact that

\[ \text{(i)} = \mathbb{E}_{x_{t-1}^s, F_{t-1}, s} \langle \nabla f_{t_b}(x_{t-1}^s) - \nabla f_{t_b}(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}), g^s - \nabla f(\tilde{x}^{s-1}) \rangle = 0 \]

and letting $F_i := \nabla f_i(x_{t-1}^s) - \nabla f_i(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1})$, implies that

$$\mathbb{E}_{0,s} \| v_{t-1}^s - \nabla f(x_{t-1}^s) \|^2 = \mathbb{E}_{0,s} \| \nabla f_{t_b}(x_{t-1}^s) - \nabla f_{t_b}(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}) \|^2 + \mathbb{E}_{0,s} \| g^s - \nabla f(\tilde{x}^{s-1}) \|^2$$

$$+ \frac{1}{|I_b|^2} \mathbb{E}_{0,s} \sum_{i \in I_b} \| \nabla f_i(x_{t-1}^s) - \nabla f_i(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}) \|^2 + \mathbb{E}_{0,s} \| g^s - \nabla f(\tilde{x}^{s-1}) \|^2$$

$$+ \frac{2}{|I_b|^2} \sum_{i<j, i,j \in I_b} \mathbb{E}_{0,s} \langle F_i, F_j \rangle$$

$$\leq \frac{1}{|I_b|} \mathbb{E}_{0,s} \| \nabla f_{t_b}(x_{t-1}^s) - \nabla f_{t_b}(\tilde{x}^{s-1}) - \nabla f(x_{t-1}^s) + \nabla f(\tilde{x}^{s-1}) \|^2 + \mathbb{E}_{0,s} \| g^s - \nabla f(\tilde{x}^{s-1}) \|^2$$

$$\leq \frac{1}{|I_b|} \mathbb{E}_{0,s} \| \nabla f_i(x_{t-1}^s) - \nabla f_i(\tilde{x}^{s-1}) \|^2 + \mathbb{E}_{0,s} \| g^s - \nabla f(\tilde{x}^{s-1}) \|^2$$

$$\leq \frac{1}{|I_b|} \mathbb{E}_{0,s} \| \nabla f_i(x_{t-1}^s) - \nabla f_i(\tilde{x}^{s-1}) \|^2 + \frac{I(|B_s|<n)}{|B_s|} \sigma^2$$

(12)

where (i) follows from the fact that

$$\mathbb{E}_{0,s}(F_i, F_j) = \mathbb{E}_{x_{t-1}^s, x_{t-1}^j} (|F_{t-1,s}(F_i, F_j)|) = \mathbb{E}_{x_{t-1}^s, x_{t-1}^j} (|F_{t-1,s}(F_i), F_{t-1,s}(F_j)|) = 0,$$

(ii) follows from the fact that $\mathbb{E} \| y - \mathbb{E}(y) \|^2 \leq \mathbb{E} \| y \|^2$ for any $y \in \mathbb{R}^d$, and (iii) follows by combining Lemma B.2 in [12] and the fact that $|B_s|$ is fixed given $x_0^1, ..., x_0^s$. Then, we obtain from (12) that

$$\mathbb{E}_{0,s} \| v_{t-1}^s - \nabla f(x_{t-1}^s) \|^2 \leq \frac{L^2}{|I_b|} \mathbb{E}_{0,s} \| x_{t-1}^s - \tilde{x}^{s-1} \|^2 + \frac{I(|B_s|<n)}{|B_s|} \sigma^2.$$
where (i) follows from the Cauchy–Schwartz inequality that $\|\sum_{i=1}^{k} a_i\|^2 \leq k \sum_{i=1}^{k} \|a_i\|^2$. □

**Proof of Theorem** Based on Lemma we next prove Theorem

Since the objective function $f(\cdot)$ has a $L$-Lipschitz continuous gradient, we obtain that for $1 \leq t \leq m$,

$$f(x_t^s) \leq f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s), x_t^s - x_{t-1}^s \rangle + \frac{L\eta^2}{2} \|v_{t-1}^s\|^2$$

$$= f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s) - v_{t-1}^s, -\eta v_{t-1}^s \rangle - \eta \|v_{t-1}^s\|^2 + \frac{L\eta^2}{2} \|v_{t-1}^s\|^2$$

$$(i) \leq f(x_{t-1}^s) + \frac{\eta}{2} \|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 + \frac{\eta}{2} \|v_{t-1}^s\|^2 - \left(\eta - \frac{L\eta^2}{2}\right) \|v_{t-1}^s\|^2.$$  

where (i) follows from the inequality that $\langle a, b \rangle \leq \frac{1}{2}(\|a\|^2 + \|b\|^2)$. Then, taking expectation $\mathbb{E}_{0,s}(\cdot)$ over the above inequality yields

$$\mathbb{E}_{0,s}f(x_t^s) \leq \mathbb{E}_{0,s}f(x_{t-1}^s) + \frac{\eta}{2} \mathbb{E}_{0,s}\|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E}_{0,s}\|v_{t-1}^s\|^2.$$ (14)

Combining (14) and Lemma yields, for $1 \leq t \leq m$

$$\mathbb{E}_{0,s}f(x_t^s) \leq \mathbb{E}_{0,s}f(x_{t-1}^s) + \frac{\eta^2 L^2(t - 1)}{2|I_b|} \mathbb{E}_{0,s} \sum_{i=0}^{t-2} \|v_i^s\|^2 + \frac{\eta I(|B_s| < n)}{2|B_s|}\|v_{t-1}^s\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \mathbb{E}_{0,s}\|v_{t-1}^s\|^2.$$  

Telescoping the above inequality over $t$ from 1 to $m$ yields

$$\mathbb{E}_{0,s}f(x_m^s) \leq \mathbb{E}_{0,s}f(x_0^s) + \sum_{t=1}^{m} \frac{\eta^3 L^2(t - 1)}{2|I_b|} \mathbb{E}_{0,s} \sum_{i=0}^{t-2} \|v_i^s\|^2 + \frac{\eta \sigma^2 m I(|B_s| < n)}{2|B_s|}$$

$$(i) \leq \mathbb{E}_{0,s}f(x_0^s) + \frac{\eta^3 L^2 m^2}{2|I_b|} \mathbb{E}_{0,s} \sum_{i=0}^{m-1} \|v_i^s\|^2 + \frac{\eta \sigma^2 m I(|B_s| < n)}{2|B_s|} - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \sum_{t=0}^{m-1} \mathbb{E}_{0,s}\|v_t^s\|^2.$$ (15)
where (i) follows from the fact that $\frac{\eta^3 L^2(t-1)}{2|I_b|} \sum_{i=0}^{t-2} \|v_i^s\|^2 \leq \frac{\eta^3 L^2 m}{2|I_b|} \sum_{i=0}^{m-1} \|v_i^s\|^2$. Recall that $|B_s| = \min\{c_3\beta^{-1}s, c_4 \epsilon^{-1}, n\}$ and $c_3, c_4 \geq \alpha \sigma^2$. Then, we have

$$I_{(B_s|<n)} \leq \frac{1}{\min\{c_3\beta^{-1}s, c_4 \epsilon^{-1}\}} = \max \left\{ \beta_s \frac{\epsilon}{c_\beta c_\epsilon} \right\} \leq \max \left\{ \frac{\beta_s}{\alpha \sigma^2}, \frac{\epsilon}{2 \alpha} \right\},$$

which, in conjunction with (15), implies that

$$E_{0,s} f(x_m^s) \leq E_{0,s} f(x_0^s) + \frac{\eta m}{2\alpha} E \beta_s + \frac{\eta m \epsilon}{2\alpha} - \left( \frac{\eta}{2} - \frac{L \eta^2}{2} - \frac{\eta^3 L^2 m^2}{2|I_b|} \right) \sum_{t=0}^{m-1} E \|v_t^s\|^2.$$

Recall that $\beta_1 \leq \epsilon S$ and $\beta_s = \frac{1}{m} \sum_{t=1}^{m} \|v_{t-1}^s\|^2$ for $s = 2, \ldots, S$. Then, telescoping the above inequality over $s$ from 1 to $S$ and noting that $x_m^s = x_0^{s+1}$, we obtain

$$E f(x_m^s) \leq E f(x_0^s) - \left( \frac{\eta}{2} - \frac{L \eta^2}{2} - \frac{\eta^3 L^2 m^2}{2|I_b|} \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \|v_t^s\|^2 + \frac{\eta m |S| \epsilon}{2\alpha}.$$

Dividing the both sides of (18) by $\eta m$ and rearranging the terms, we obtain

$$\left( \frac{1}{2} - \frac{1}{2\alpha} - \frac{L \eta}{2} - \frac{\eta^2 L^2 m^2}{2|I_b|} \right) \frac{1}{Sm} \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \|v_t^s\|^2 \leq \frac{f(x_0) - f^*}{\eta Sm} + \frac{\epsilon}{\alpha},$$

where $f^* = \inf_{x \in \mathbb{R}^d} f(x) > -\infty$. Since the output $x_\xi$ is chosen from $\{x_t^s\}_{t=0,\ldots,m-1, s=1,\ldots,S}$ uniformly at random, we have

$$Sm E \|\nabla f(x_\xi)\|^2 = \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \|\nabla f(x_t^s)\|^2$$

$\leq 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \|\nabla f(x_t^s) - v_t^s\|^2 + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \|v_t^s\|^2$

$= 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} E x_0^s, \ldots, x_t^s (E_{0,s} \|\nabla f(x_t^s) - v_t^s\|^2) + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \|v_t^s\|^2$
Thus, to achieve

where (i) follows from Lemma 1 and (16), and (ii) follows from the definition of \( \beta_s \) for \( s = 1, \ldots, S \). Combining (19) and (20) and letting \( \phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L_0}{2} - \frac{\eta^2 L^2 m^2}{2|I_b|} \) and \( \psi = \frac{2\eta^2 L^2 m^2}{|I_b|} + \frac{2}{\alpha} + 2 \), we have

\[
\mathbb{E}[\|\nabla f(x_\ast)\|^2] \leq \frac{\psi(f(x_0) - f^*)}{\phi \eta S m} + \frac{\psi \epsilon}{\phi \alpha} + \frac{4\epsilon}{\alpha},
\]

which finishes the proof.

\[\square\]

### D.2 Proof of Corollary 1

Recall that \( \eta = \frac{1}{T}, |I_b| = m^2 \) and \( c_\beta, c_\epsilon \geq 16\sigma^2 \). Then, we obtain \( \alpha = 16, \phi \geq \frac{5}{16} > \frac{1}{4} \) and \( \phi \leq \frac{9}{4} \) in Theorem 1 and thus

\[
\mathbb{E}[\|\nabla f(x_\ast)\|^2] \leq \frac{36L(f(x_0) - f^*)}{T} + \frac{13}{16} \epsilon.
\]

Thus, to achieve \( \mathbb{E}[\|\nabla f(x_\ast)\|^2] < \epsilon \), AbaSVRG requires at most \( 192L(f(x_0) - f^*)\epsilon^{-1} = \Theta(\epsilon^{-1}) \) iterations. Then, the total number of SFO calls is given by

\[
\sum_{s=1}^{S} \min\{c_\beta \beta_s^{-1}, c_\epsilon \epsilon^{-1}, n\} + T|I_b| \leq S(c_\epsilon \epsilon^{-1} \land n) + T|I_b| \leq O \left( \frac{\epsilon^{-1} \land n}{\epsilon \sqrt{|I_b|}} + \frac{|I_b|}{\epsilon} \right).
\]

Furthermore, if we choose \( |I_b| = n^{2/3} \land \epsilon^{-2/3} \), then SFO complexity of AbaSVRG becomes

\[
O \left( \frac{\epsilon^{-2/3} \land n^{2/3}}{\epsilon} \right) \leq O \left( \frac{1}{\epsilon} \left( n \land \frac{1}{\epsilon} \right)^{2/3} \right).
\]

### D.3 Proof of Corollary 2

Since \( \eta = \frac{1}{4L\sqrt{m}}, |I_b| = m \) and \( c_\beta, c_\epsilon \geq 16\sigma^2 \), we obtain \( \alpha = 16, \phi = \frac{7}{16} - \frac{1}{8\sqrt{m}} \geq \frac{5}{16} > \frac{1}{4} \) and \( \psi \leq \frac{9}{4} \) in Theorem 1 and thus

\[
\mathbb{E}[\|\nabla f(x_\ast)\|^2] \leq \frac{36L\sqrt{m}(f(x_0) - f^*)}{T} + \frac{13}{16} \epsilon.
\]

To achieve \( \mathbb{E}[\|\nabla f(x_\ast)\|^2] < \epsilon \), AbaSVRG requires at most \( 192L\sqrt{m}(f(x_0) - f^*)\epsilon^{-1} = \Theta(\sqrt{m\epsilon^{-1}}) \) iterations. Then, the total number of SFO calls is given by

\[
\sum_{s=1}^{S} \min\{c_\beta \beta_s^{-1}, c_\epsilon \epsilon^{-1}, n\} + T|I_b| \leq S(c_\epsilon \epsilon^{-1} \land n) + T|I_b| \leq O \left( \frac{\epsilon^{-1} \land n}{\epsilon \sqrt{|I_b|}} + \frac{|I_b|^{3/2}}{\epsilon} \right).
\]

Furthermore, if we choose \( |I_b| = n^{1/2} \land \epsilon^{-1/2} \), then the SFO complexity is \( O \left( \frac{1}{\epsilon} (n \land \frac{1}{\epsilon})^{3/4} \right) \).
D.4 Proof of Theorem 2

In order to prove Theorem 2, we first use the following lemma to provide an upper bound on the estimation variance $\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2$ for $0 \leq t \leq m - 1$, where $\mathbb{E}_{t,s}(\cdot)$ denotes $\mathbb{E}(\cdot|x_0^s, x_1^s, ..., x_t^s)$.

**Lemma 2** (Adapted from [7]). Let Assumption 1 hold. Then, for $0 \leq t \leq m - 1$,

$$
\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2 \leq \frac{\eta^2 L^2}{|I_b|} \sum_{i=0}^{t-1} \mathbb{E}_{0,s}\|v_i^s\|^2 + \frac{I(|B_s|<n)}{|B_s|} \sigma^2.
$$  \hspace{1cm} (23)

where we define the stochastic gradients before the algorithm starts to satisfy $\sum_{i=0}^{t-1} \mathbb{E}_{0,s}\|v_i^s\|^2 = 0$ for easy presentation.

**Proof.** Combining A.3 and A.4 in [7] yields, for $1 \leq i \leq m - 1$,

$$
\mathbb{E}_{i,s}\|\nabla f(x_i^s) - v_i^s\|^2 \leq \frac{L^2}{|I_b|} \|x_i^s - x_{i-1}^s\|^2 + \|\nabla f(x_{i-1}^s) - v_{i-1}^s\|^2
$$

$$
= \frac{\eta^2 L^2}{|I_b|} \|v_{i-1}^s\|^2 + \|\nabla f(x_{i-1}^s) - v_{i-1}^s\|^2.
$$

Taking the expectation of the above inequality over $x_1^s, ..., x_i^s$, we have

$$
\mathbb{E}_{0,s}\|\nabla f(x_i^s) - v_i^s\|^2 \leq \frac{\eta^2 L^2}{|I_b|} \mathbb{E}_{0,s}\|v_i^s\|^2 + \mathbb{E}_{0,s}\|\nabla f(x_i^s) - v_i^s\|^2.
$$

Then, telescoping the above inequality over $i$ from 1 to $t$ yields

$$
\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2 \leq \frac{\eta^2 L^2}{|I_b|} \sum_{i=0}^{t-1} \mathbb{E}_{0,s}\|v_i^s\|^2 + \mathbb{E}_{0,s}\|\nabla f(x_0^s) - v_0^s\|^2.
$$  \hspace{1cm} (24)

Based on Lemma B.2 in [14], we have

$$
\mathbb{E}_{0,s}\|\nabla f(x_t^s) - v_t^s\|^2 \leq \frac{I(|B_s|<n)}{|B_s|} \sigma^2,
$$

which, combined with (24), finishes the proof.

**Proof of Theorem 2.** Based on Lemma 2, we now prove Theorem 2.

Since the objective function $f(\cdot)$ has a $L$-Lipschitz continuous gradient, we obtain that for $1 \leq t \leq m$,

$$
f(x_t^s) \leq f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s), x_t^s - x_{t-1}^s \rangle + \frac{L\eta^2}{2} \|v_{t-1}^s\|^2
$$

$$
= f(x_{t-1}^s) + \langle \nabla f(x_{t-1}^s) - v_{t-1}^s, -\eta v_{t-1}^s \rangle - \eta \|v_{t-1}^s\|^2 + \frac{L\eta^2}{2} \|v_{t-1}^s\|^2
$$

$$
\leq f(x_{t-1}^s) + \frac{\eta}{2} \|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 + \frac{\eta}{2} \|v_{t-1}^s\|^2 - \eta \|v_{t-1}^s\|^2 + \frac{L\eta^2}{2} \|v_{t-1}^s\|^2
$$

$$
\leq f(x_{t-1}^s) + \frac{\eta}{2} \|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 - \left(\frac{\eta}{2} - \frac{L\eta^2}{2}\right) \|v_{t-1}^s\|^2,
$$

24
where (i) follows from the inequality that $\langle a, b \rangle \leq \frac{1}{2} (\|a\|^2 + \|b\|^2)$. Then, taking expectation $\mathbb{E}_{0,s}$ over the above inequality and applying Lemma 2, we have, for $1 \leq t \leq m$,

$$
\mathbb{E}_{0,s} f(x_t^s) \leq \mathbb{E}_{0,s} f(x_{t-1}^s) + \frac{\eta}{2} \mathbb{E}_{0,s} \|\nabla f(x_{t-1}^s) - v_{t-1}^s\|^2 - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}_{0,s} \|v_{t-1}^s\|^2
$$

$$
\leq \mathbb{E}_{0,s} f(x_{t-1}^s) + \frac{\eta^3 L^2}{2|I_b|} \sum_{i=0}^{t-2} \mathbb{E}_{0,s} \|v_i^s\|^2 + \frac{I_{(|B_s| < n)} \eta \sigma^2}{|B_s|} - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}_{0,s} \|v_{t-1}^s\|^2
$$

$$
\leq \mathbb{E}_{0,s} f(x_{t-1}^s) + \frac{\eta^3 L^2}{2|I_b|} \sum_{i=0}^{m-1} \mathbb{E}_{0,s} \|v_i^s\|^2 + \max \left\{ \frac{\eta \beta_s}{2\alpha}, \frac{\eta \epsilon}{2\alpha} \right\} - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}_{0,s} \|v_{t-1}^s\|^2
$$

where (i) follows from $t - 2 < m - 1$ and (16). Telescoping the above inequality over $t$ from 1 to $m$ and using $\max(a, b) \leq a + b$ yield

$$
\mathbb{E}_{0,s} f(x_m^s) \leq \mathbb{E}_{0,s} f(x_0^s) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2 + \frac{\eta m \beta_s}{2\alpha} + \frac{\eta m \epsilon}{2\alpha}.
$$

Taking the expectation of the above inequality over $x_1^s, \ldots, x_0^s$, we obtain

$$
\mathbb{E} f(x_m^s) \leq \mathbb{E} f(x_0^s) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2 + \frac{\eta m \beta_s}{2\alpha} + \frac{\eta m \epsilon}{2\alpha}.
$$

Recall that $\beta_1 \leq \epsilon S$ and $\beta_s = \frac{1}{m} \sum_{s=2}^{m-1} \|v_{s-1}^s\|^2$ for $s = 2, \ldots, S$. Then, telescoping the above inequality over $s$ from 1 to $S$ and noting that $x_m^s = x_0^{s+1} = \bar{x}^s$, we have

$$
\mathbb{E} f(x_m^s) \leq \mathbb{E} f(x_0^s) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2 + \frac{\eta m S \epsilon}{2\alpha}
$$

$$
+ \frac{\eta}{2\alpha} \sum_{s=1}^{S-1} \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2
$$

$$
\leq \mathbb{E} f(x_0^s) - \left( \frac{\eta}{2} - \frac{\eta}{2\alpha} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m}{2|I_b|} \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2 + \frac{\eta m S \epsilon}{2\alpha}.
$$

Dividing the both sides of the above inequality by $\eta S m$ and rearranging the terms, we obtain

$$
\left( \frac{1}{2} - \frac{1}{2\alpha} - \frac{L\eta^2}{2} - \frac{\eta^3 L^2 m}{2|I_b|} \right) \frac{1}{S m} \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2 \leq \frac{f(x_0) - f^*}{\eta S m} + \epsilon.
$$

(25)

Since the output $x_t^s$ is chosen from $\{x_t^s\}_{t=0, \ldots, m-1, s=1, \ldots, S}$ uniformly at random, we have

$$
S m \mathbb{E} \|\nabla f(x_t^s)\|^2 = \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E} \|\nabla f(x_t^s)\|^2
$$

$$
\leq 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E} \|\nabla f(x_t^s) - v_t^s\|^2 + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2
$$

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\( \sum_{s=1}^{S} \sum_{t=0}^{m-1} E_{x_{s}^{3} \ldots x_{s}^{0}} \left( E_{0,s} \| \nabla f(x_{t}^{s}) - v_{t}^{s} \|^{2} \right) + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \| v_{t}^{s} \|^{2} \leq 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} \left( \frac{\eta^{2} L^{2}}{|I_{b}|} E_{0,s} \sum_{i=0}^{m-1} \| v_{i}^{s} \|^{2} + \frac{\beta_{s}}{\alpha} + \frac{\epsilon}{\alpha} \right) + 2 \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \| v_{t}^{s} \|^{2} \leq \left( \frac{2 \eta^{2} L^{2} m}{|I_{b}|} + \frac{2}{\alpha} + 2 \right) \sum_{s=1}^{S} \sum_{t=0}^{m-1} E \| v_{t}^{s} \|^{2} + \frac{4 Sm \epsilon}{\alpha} \right) \) (26)

where (i) follows from Lemma 2 and (16) and (ii) follows from the definition of \( \beta_{s} \) for \( s = 1, \ldots, S \).
Let \( \phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L \eta}{2} - \frac{\eta^{2} L^{2} m}{2 |I_{b}|} \), \( \psi = \frac{2 \eta L m}{|I_{b}|} + \frac{2}{\alpha} + 2 \) and \( T = Sm \). Then, combining (26) and (25), we finish the proof.

### D.5 Proof of Corollary 3

Recall that \( 1 \leq |I_{b}| \leq n^{1/2} \land \epsilon^{-1/2} \), \( m = (n \land \frac{1}{\epsilon}) |I_{b}|^{-1} \), \( \eta = \frac{1}{4L} \sqrt{|I_{b}|} \) and \( c_{3}, c_{\epsilon} \geq 16 \sigma^{2} \). Then, we have \( \alpha = 16, m \geq n^{1/2} \land \epsilon^{-1/2} \geq |I_{b}| \) and \( \eta \leq \frac{1}{4L} \). Thus, we obtain

\( \phi = \frac{1}{2} - \frac{1}{2\alpha} - \frac{L \eta}{2} - \frac{\eta^{2} L^{2} m}{2 |I_{b}|} \geq \frac{5}{16} > \frac{1}{4} \) and \( \psi \leq \frac{9}{4} \),

which, in conjunction with Theorem 2, implies that

\( E \| \nabla f(x_{\zeta}) \|^{2} \leq \frac{36L \sqrt{m}(f(x_{0}) - f^{*})}{\sqrt{|I_{b}|} T} + \frac{17}{32} \epsilon. \)

Thus, to achieve \( E \| \nabla f(x_{\zeta}) \|^{2} < \epsilon \), AbaSPIDER requires at most \( \frac{384L \sqrt{m}(f(x_{0}) - f^{*})}{5 \sqrt{|I_{b}|} \epsilon} = \Theta \left( \frac{\sqrt{m}}{|I_{b}| \epsilon} \right) \) iterations. Then, the total number of SFO calls is given by

\( \sum_{s=1}^{S} \min\{c_{3} \beta_{s}^{-1}, c_{\epsilon} \epsilon^{-1}, n\} + T |I_{b}| \leq S(c_{3} \epsilon^{-1} \land n) + T |I_{b}| \leq O \left( \frac{\epsilon^{-1} \land n}{\epsilon \sqrt{m |I_{b}|}} + \frac{\sqrt{m |I_{b}|}}{\epsilon} \right), \)

which, in conjunction with \( m |I_{b}| = n \land \frac{1}{\epsilon} \), finishes the proof.

### E Proofs for Results in Section 5

#### E.1 Proof for Theorem 3

To simplify notations, we let \( c_{3} = c_{\epsilon} = \alpha \sigma^{2} \) with \( \alpha = \left( 2 \tau + \frac{2 \tau}{1 - \exp(-\frac{1}{\epsilon^{2} (c_{\epsilon}^{2} - 2)})} \right) \sqrt{\frac{16 c_{3} \epsilon^{2} \sqrt{m}}{L \tau}}. \)

Since the objective function \( f(\cdot) \) has a \( L \)-Lipschitz continuous gradient, we obtain that for \( 1 \leq t \leq m \),

\( f(x_{t}^{s}) \leq f(x_{t-1}^{s}) + \langle \nabla f(x_{t-1}^{s}), x_{t}^{s} - x_{t-1}^{s} \rangle + \frac{L \eta^{2}}{2} \| v_{t-1}^{s} \|^{2} \)
\[ f(x_t^s) - f(x^*) \leq \left( 1 - \frac{\eta}{2\gamma} \right) (f(x_{t-1}^s) - f(x^*)) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \|v_{t-1}^s\|^2 + \frac{\eta^2 L^2 (t-1)}{2|I_b|} \|v_t^s\|^2 + \frac{\eta I(|B_s| < n)}{2|B_s|} \sigma^2. \]

Recall that \( \mathbb{E}_{t,s}(\cdot) \) denotes \( \mathbb{E}(|x_0^s, x_0^3, \ldots, x_2^1, \ldots, x_t^s) \). Then, taking expectation \( \mathbb{E}_{0,s}(\cdot) \) over the above inequality yields, for \( 1 \leq t \leq m \),

\[ \mathbb{E}_{0,s}(f(x_t^s) - f(x^*)) \leq \left( 1 - \frac{\eta}{2\gamma} \right) \mathbb{E}_{0,s}(f(x_{t-1}^s) - f(x^*)) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}_{0,s}\|v_{t-1}^s\|^2 + \frac{\eta^2 L^2 (t-1)}{2|I_b|} \mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta I(|B_s| < n)}{2|B_s|} \sigma^2. \]

Let \( \gamma := 1 - \frac{\eta}{2\gamma} \). Then, telescoping the above inequality over \( t \) from 1 to \( m \) and using the fact that \( t - 1 < m \), we have

\[ \mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left( \frac{\eta}{4} - \frac{L\eta^2}{2} \right) \sum_{t=0}^{m-1} \gamma^{m-1-t} \mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta^2 L^2 m}{2|I_b|} \sum_{t=0}^{m-1} \gamma^{m-2-t} \mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta I(|B_s| < n)}{2|B_s|} \sigma^2. \]

Note that \( \gamma^{m-1-t} \geq \gamma^m \) for \( 0 \leq t \leq m - 1 \) and \( \sum_{t=0}^{m-1} \gamma^t = \frac{1-\gamma^m}{1-\gamma} \leq \frac{1}{1-\gamma} = \frac{2\gamma}{\eta} \). Then, we obtain from \( (28) \) that

\[ \mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left( \frac{\eta}{4} - \frac{L\eta^2}{2} \right) \sum_{t=0}^{m-1} \gamma^{m-1-t} \mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta^2 L^2 m}{2|I_b|} \sum_{t=0}^{m-1} \gamma^{m-2-t} \mathbb{E}_{0,s}\|v_t^s\|^2 + \frac{\eta I(|B_s| < n)}{|B_s|} \sigma^2 \]
\[
\leq \gamma^m E_{0,s}(f(x_m^s) - f(x^*)) - \left( \left( \frac{\eta}{4} - \frac{L\eta^2}{2} \right) \gamma^m \cdot \frac{\eta^2 L^2 m_T}{|I_b|} \right) \sum_{t=0}^{m-1} E_{0,s} \|v_t^s\|^2 \\
+ \frac{\tau I(|B_s| < n)}{|B_s|} \sigma^2 - \frac{\eta}{4} \sum_{t=0}^{m-1} \gamma^{m-1-t} E_{0,s} \|v_t^s\|^2. 
\]

(29)

Recall \(\eta = \frac{1}{c_\eta L} (c_\eta > 4), \frac{8L^2}{c_\eta - 2} \leq m < 4L_T\) and \(|I_b| = m^2\). Then, we have

\[
\frac{\eta^2 L^2}{2} \geq \frac{\eta^2 L^2 m}{m} = \frac{\eta^2 L^2 m_T}{|I_b|}, 
\]

(30)

where (i) follows from the fact that \((1 - \frac{1}{2m})^m \geq \frac{1}{2}\) for any \(m \geq 1\). Recall that \(c_\beta = c_\varepsilon = \alpha \sigma^2\). Then, combining (16), (29) and (30) yields

\[
E_{0,s}(f(x_m^s) - f(x^*)) \leq \gamma^m E_{0,s}(f(x_0^s) - f(x^*)) + \frac{\tau I(|B_s| < n)}{|B_s|} \sigma^2 - \frac{\eta}{4} \sum_{t=0}^{m-1} \gamma^{m-1-t} E_{0,s} \|v_t^s\|^2.
\]

(31)

Further taking expectation of the above inequality over \(x_1^s, \ldots, x_0^s\), we obtain

\[
E(f(x_m^s) - f(x^*)) \leq \gamma^m E(f(x_0^s) - f(x^*)) + \frac{\tau}{\alpha} \beta_s + \frac{\tau \varepsilon}{\alpha} - \frac{\eta}{4} \sum_{t=0}^{m-1} \gamma^{m-1-t} E \|v_t^s\|^2 
\]

Recall that \(\beta_1 \leq \epsilon(\frac{1}{\gamma})^{m(S-1)}\) and \(\beta_s = \frac{1}{m} \sum_{t=1}^{m} \|v_{t-1}^s\|^2\) for \(s \geq 2\). Then, telescoping the above inequality over \(s\) from 1 to \(S\) yields

\[
E(f(x_m^S) - f(x^*)) \leq \gamma^S m E(f(x_0) - f(x^*)) + \sum_{s=1}^{S-1} \gamma^{m(S-1)-s} \frac{\tau \alpha m}{\alpha m} \sum_{t=0}^{m-1} E \|v_t^s\|^2 \\
+ \gamma^{m(S-1)-1} \frac{\beta_1}{\alpha} + \sum_{s=1}^{S} \gamma^{m(S-1)-s} \frac{\tau \varepsilon}{\alpha} - \frac{\eta}{4} \sum_{s=1}^{S} \gamma^{m(S-1)-s} \sum_{t=0}^{m-1} \gamma^{m-1-t} E \|v_t^s\|^2 \\
\leq \gamma^T E(f(x_0) - f(x^*)) - \left( \frac{\eta}{4} \gamma^{2m} - \frac{\tau}{\alpha m} \right) \sum_{s=1}^{S-1} \gamma^{m(S-1)-s} \sum_{t=0}^{m-1} E \|v_t^s\|^2 \\
+ \left( 1 + \frac{1}{1 - \exp(-\frac{4}{c_\eta (c_\eta - 2)m})} \right) \frac{\tau \varepsilon}{\alpha}, 
\]

(31)

where (i) follows from the fact that \(\gamma^{m-1-t} \geq \gamma^m\) for \(0 \leq t \leq m - 1\), \(\gamma^{m(S-1)} \leq 1\), \(\sum_{s=1}^{S} \gamma^{m(S-1)-s} \leq \frac{1}{1 - \gamma^m}\) and

\[
\gamma^m \leq \left( 1 - \frac{1}{2c_\eta T L} \right)^m \leq \left( 1 - \frac{4}{c_\eta (c_\eta - 2)m} \right)^m \leq \exp \left( -\frac{4}{c_\eta (c_\eta - 2)} \right).
\]

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which, in conjunction with Lemma 2, implies that

\[ (i) \text{ follows from } \gamma^m \geq (1 - \frac{1}{2^m})^m \geq \frac{1}{2}. \]

Note that \( x^S_m = \tilde{x}^S \). Then, combining (32) and (31) yields

\[ \mathbb{E}(f(\tilde{x}^S) - f(x^*)) \leq \gamma^T(f(x_0) - f(x^*)) + \frac{\epsilon}{2}. \]  

Let \( T = (2c_\eta T - 1) \log \left( \frac{2(f(x_0) - f(x^*))}{\epsilon} \right) \). Then, we have

\[ \gamma^T(f(x_0) - f(x^*)) = \exp \left[ (2c_\eta T - 1) \log \left( \frac{1}{\gamma} \log \frac{1}{2(f(x_0) - f(x^*))} \right) \right] (f(x_0) - f(x^*)) \leq \frac{\epsilon}{2}, \]

where (i) follows from the fact that \( \log \frac{1}{\gamma} = \log \left( 1 + \frac{1}{2c_\eta T - 1} \right) \leq \frac{1}{2c_\eta T - 1} \). Thus, the total number of SFO calls is

\[
\sum_{s=1}^{S} \min \left\{ \frac{c_\beta}{\beta_s}, \frac{c_\epsilon}{\epsilon}, n \right\} + T|I_b| \leq \mathcal{O} \left( \left( \frac{c_\epsilon}{\epsilon} \wedge n \right) \frac{\tau}{m} \log \frac{1}{\epsilon} + |I_b| \tau \log \frac{1}{\epsilon} \right) \leq \mathcal{O} \left( \left( \frac{\tau}{m} \wedge n \right) \log \frac{1}{\epsilon} + \tau^3 \log \frac{1}{\epsilon} \right),
\]

where (i) follows from the fact that \( m = \Theta(\tau) \) and \( c_\epsilon = \Theta(\tau) \).

### E.2 Proof of Theorem 4

To simplify notations, we let \( c_\beta = c_\epsilon = \alpha \sigma^2 \) with \( \alpha = \left( 2\tau + \frac{2\tau}{1 - \exp(-\frac{2\tau}{c_\eta(c_\eta - 2)})} \right) \sqrt{\frac{16c_\eta L \tau}{m}} \).

Using an approach similar to (27), we have, for \( 1 \leq t \leq m \)

\[ \mathbb{E}_{0,s}(f(x^s_t) - f(x^*)) \leq \left( 1 - \frac{\eta}{2\tau} \right) \mathbb{E}_{0,s}(f(x^s_{t-1}) - f(x^*)) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}_{0,s}\|v^s_{t-1}\|^2 + \frac{\eta}{2} \mathbb{E}_{0,s}(\nabla f(x^s_{t-1}) - v^s_{t-1})^2, \]

which, in conjunction with Lemma 2 implies that

\[ \mathbb{E}_{0,s}(f(x^s_t) - f(x^*)) \leq \left( 1 - \frac{\eta}{2\tau} \right) \mathbb{E}_{0,s}(f(x^s_{t-1}) - f(x^*)) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \mathbb{E}_{0,s}\|v^s_{t-1}\|^2 + \eta^3 L^2 \frac{t-2}{2|I_b|} \sum_{i=0}^{t-2} \mathbb{E}_{0,s}\|v^s_i\|^2 + \frac{\eta I(|B_s|\leq n)}{2|B_s|} \sigma^2. \]

Let \( \gamma := 1 - \frac{\eta}{2\tau} \). Then, telescoping the above inequality over \( t \) from 1 to \( m \) yields

\[ \mathbb{E}_{0,s}(f(x^s_m) - f(x^*)) \leq \gamma^m \mathbb{E}_{0,s}(f(x^s_0) - f(x^*)) - \left( \frac{\eta}{2} - \frac{L\eta^2}{2} \right) \sum_{t=0}^{m-1} \gamma^{m-1-t} \mathbb{E}_{0,s}\|v^s_t\|^2 \]
which, in conjunction with \( \sum_{t=0}^{m-1} \gamma^t = \frac{1-\gamma^m}{1-\gamma} \leq \frac{1}{1-\gamma} = \frac{2\tau}{\eta} \) and \( \gamma^{m-1} \geq \gamma^m \) for \( 0 \leq t \leq m-1 \), implies that

\[
\mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left( \frac{\eta}{4} - \frac{L \eta^2}{2} \right) \sum_{t=0}^{m-1} \gamma^m \mathbb{E}_{0,s} \|v_t^s\|^2 - \frac{\eta^2 L^2}{2|I_b|} \sum_{t=0}^{m-1} \gamma^m \mathbb{E}_{0,s} \|v_t^s\|^2 + \frac{\tau I(|B_s|<n)}{|B_s|} \sigma^2
\]

\[
\leq \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) - \left( \frac{\eta}{4} - \frac{L \eta^2}{2} \right) \sum_{t=0}^{m-1} \gamma^m \mathbb{E}_{0,s} \|v_t^s\|^2 + \frac{\eta^2 L^2 \tau}{m} \sum_{t=0}^{m-1} \gamma^m \mathbb{E}_{0,s} \|v_t^s\|^2 + \frac{\tau I(|B_s|<n)}{|B_s|} \sigma^2
\]

Recall that \( \eta = \frac{1}{c_\eta L} \) with \( c_\eta > 4 \) and \( |I_b| = m \) with \( \frac{8L^\tau}{c_\eta^2} \leq m \leq 4L^\tau \). Then, we have

\[
\left( \frac{\eta}{4} - \frac{L \eta^2}{2} \right) \gamma^m = \eta \left( \frac{1}{4} - \frac{1}{2c_\eta} \right) \left( 1 - \frac{1}{2c_\eta \eta L} \right)^m \geq \eta \left( \frac{1}{4} - \frac{1}{2c_\eta} \right) \left( 1 - \frac{1}{2m} \right)^m \geq \frac{\eta^2 L^2 \tau}{m} \]

which, combined with (34) and (16), implies that

\[
\mathbb{E}_{0,s}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}_{0,s}(f(x_0^s) - f(x^*)) + \tau \left( \frac{\beta_s}{\alpha} + \frac{\epsilon}{\alpha} \right) - \frac{\eta}{4} \sum_{t=0}^{m-1} \gamma^m \mathbb{E}_{0,s} \|v_t^s\|^2.
\]

Taking the expectation of the above inequality \( x_0^s, \ldots, x_m^s \), we obtain

\[
\mathbb{E}(f(x_m^s) - f(x^*)) \leq \gamma^m \mathbb{E}(f(x_0^s) - f(x^*)) + \tau \mathbb{E} \beta_s + \frac{\tau \epsilon}{\alpha} - \frac{\eta}{4} \sum_{t=0}^{m-1} \gamma^m \mathbb{E} \|v_t^s\|^2.
\]

Recall \( \beta_1 \leq \epsilon \left( \frac{1}{\gamma} \right) \gamma^{m(S-1)} \) and \( \beta_s = \frac{1}{m} \sum_{t=0}^{m-1} \|v_t^{s-1}\|^2 \) for \( s = 2, \ldots, S \). Then, telescoping the above inequality over \( s \) from 1 to \( S \) and using an approach similar to (31), we have

\[
\mathbb{E}(f(x_m^S) - f(x^*)) \leq \gamma^m \mathbb{E}(f(x_0^S) - f(x^*)) - \left( \frac{\eta}{4} \gamma^m \sum_{s=1}^{S-1} \gamma^m \mathbb{E}(f(x_0^s) - f(x^*)) \right) \sum_{t=0}^{m-1} \mathbb{E} \|v_t^s\|^2
\]

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\[
+ \left( 1 + \frac{1}{1 - \exp\left( -\frac{4}{c_\eta (c_\eta - 2)} \right)} \right) \frac{\tau \epsilon}{\alpha},
\]

which, in conjunction with (32), yields

\[
\mathbb{E}(f(x_m^S) - f(x^*)) \leq \gamma^T \mathbb{E}(f(x_0) - f(x^*)) + \frac{\epsilon}{2}, \tag{36}
\]

Let \( T = (2c_\eta \tau L - 1) \log \left( \frac{2(f(x_0) - f(x^*)))}{\epsilon} \right) \). Then, we have \( \gamma^T (f(x_0) - f(x^*)) \leq \frac{\epsilon}{2} \). Thus, the total number of SFO calls is

\[
\sum_{s=1}^{S} \min \left\{ \frac{c_\beta}{\beta_s}, \frac{c_\epsilon}{\epsilon}, n \right\} + T |I_b| \leq O\left( \left( \frac{c_\epsilon}{\epsilon} \wedge n \right) \frac{\tau}{m} \log \frac{1}{\epsilon} + |I_b| \tau \log \frac{1}{\epsilon} \right)
\]

\[
\leq O\left( \left( \frac{\tau}{\epsilon} \wedge n \right) \log \frac{1}{\epsilon} + \tau^2 \log \frac{1}{\epsilon} \right),
\]

where (i) follows from the fact that \(|I_b| = m = \Theta(\tau)\) and \( c_\epsilon = \Theta(\tau) \).