Torelli’s theorem from the topological point of view

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Torelli’s theorem states, that the isomorphism class of a smooth projective curve of genus \( g \geq 2 \) over an algebraically closed field \( k \) is uniquely determined by the isomorphism class of the associated pair \((X, \Theta)\), where \( X \) is the Jacobian variety of \( C \) and \( \Theta \) is the canonical theta divisor. The aim of this note is to give a ‘topological’ proof of this theorem. Although Torelli’s theorem is not a topological statement the proof to be presented gives a characterization of \( C \) in terms of perverse sheaves on the Jacobian variety \( X \), which are attached to the theta divisor by a ‘topological’ construction.

For complexes \( K, L \in D^b_c(X, \mathbb{Q}_l) \) define \( K * L \in D^b_c(X, \mathbb{Q}_l) \) by the direct image complex \( Ra^*(K \boxtimes L) \), where \( a : X \times X \to X \) is the addition law of \( X \). Let \( K^0_*(X) \) be the tensor product of the Grothendieck group of perverse sheaves on \( X \) with the polynomial ring \( \mathbb{Z}[t^{1/2}, t^{-1/2}] \). \( K^0_*(X) \) is a commutative ring with ring structure defined by the convolution product, hence also the quotient ring \( K_*^*(X) \) obtained by dividing the principal ideal generated by the constant perverse sheaf \( \delta_X \) on \( X \). Both rings \( K^0_*(X) \) and \( K_*^*(X) \) resemble properties of the homology ring of \( X \) endowed with the \(*\)-product, but have a much richer structure. A sheaf complex \( L \in D^b_c(X, \mathbb{Q}_l) \) defines a class in \( K_*^*(X) \) by the perverse Euler characteristic \( \sum_{\nu} (-1)^\nu \cdot p\nu(L) \cdot t^{\nu/2} \). Similar to the homology ring every irreducible closed subvariety \( Y \) has a class in \( K_*^*(X) \) defined as the class of the perverse intersection cohomology sheaf \( \delta_Y \) of \( Y \). For details see [W]. This allows to consider the product

\[
\delta_\Theta * \delta_\Theta \in K_*^*(X).
\]

Whereas the corresponding product in the homology ring of \( X \) is zero, this product turns out to be nonzero. \( \delta_\Theta * \delta_\Theta \) is of the form \( \sum_{\nu, \mu} A_{\nu, \mu} t^{\nu/2} \). Recall, the coefficients are irreducible perverse sheaves \( A_{\nu, \mu} \) on \( X \). For a perverse sheaf \( A \) on \( X \), which is a sheaf complex on \( X \), let \( \mathcal{H}^i(A) \) denote the associated cohomology sheaves for \( i \in \mathbb{Z} \). Let \( \kappa \in X(k) \) be the Riemann constant defined by \( \Theta = \kappa - \Theta \). It depends on the choice of the Abel-Jacobi map \( C \to X \).

**Theorem:** Let \( C \) be a curve of genus \( g \geq 3 \). There exists a unique irreducible perverse sheaf \( A = A_{\nu,0} \), among the coefficients of \( \delta_\Theta * \delta_\Theta \), characterized by one of the following equivalent properties

1. \( \mathcal{H}^{-1}(A) \) is nonzero, but not a constant sheaf on \( X \).

2. \( \mathcal{H}^{-1}(A) \) is the skyscraper sheaf \( H^1(C) \otimes \delta_{\{\kappa\}} \) with support in the point \( \kappa \in X \).

Furthermore the support of the perverse sheaf \( A \) is \( \kappa + C - C \subseteq X \).
Another case is particular, convolution where $a$ since we apply subvariety where $p$ to the addition law of $X$ is the Brill-Noether subvariety $W_r = C + \cdots + C \; (r \; \text{copies})$ of $X$. If $C$ is not hyperelliptic, then

$$δ_{W_r} = δ_r ,$$

since $p_r$ is a small morphism by the theorem of Martens [M] for $r \leq g - 1$. In particular $δ_Θ = δ_{g - 1}$, which will be used in the proof. (In the hyperelliptic case $δ_Θ = δ_{g - 1} - δ_{g - 3}$. For this and further details we refer to [W]).

**Proof of the theorem:** Suppose $C$ is not hyperelliptic.

1) Since the canonical morphism

$$τ : C(\iota) \times C(\j) \to C(\i+j)$$

is a finite ramified covering map, the direct image $Rτ_∗δ_{C(\i+j)}$ decomposes into a direct sum of etale sheaves $⊕μ \cdot m(i, j, ν) \cdot F_{i+j-ν,i}$ by keeping track of the underlying action of the symmetric group $Σ_{i+j}$ for the map $C(\i+j) \to C(\i+j)$ (see [W],4.1). If we apply $Rp_{i+j,*}$, this gives a formula for $δ_i * δ_j$. From $p_{i+j} ◦ τ = a ◦ (p_i × p_j)$, where $a : X \times X \to X$ is the addition law of $X$, one obtains for $i \geq j$ that the convolution $δ_i * δ_j$ is $δ_{i+j} \oplus δ_{i+j-1,1} \oplus \cdots \oplus δ_{i-j,j}$, where $δ_{r,s} = Rp_{i+j,*}(F_{r,s})$. A special case is

$$δ_Θ * δ_Θ = δ_{g - 1} * δ_{g - 1} = δ_{2g - 2} \oplus δ_{2g - 3,1} \oplus \cdots \oplus δ_{g - 1,g - 1} .$$

Another case is $δ_1 * δ_{2g - 3} = δ_{2g - 2} \oplus δ_{2g - 3,1}$, and together this implies

$$δ_{2g - 3} * δ_1 \leftrightarrow δ_Θ * δ_Θ .$$
2) The morphism \( f : C \times C \to \kappa + C - C \subseteq X \), defined by \((x, y) \mapsto \kappa + x - y\), is semi-small. If \( C \) is not hyperelliptic, then \( f \) is a birational map, which blows down the diagonal to the point \( \kappa \), and is an isomorphism otherwise. Hence the direct image \( Rf_*(\delta_C \boxtimes \delta_C) \) is perverse on \( X \), and necessarily decomposes \( Rf_*(\delta_C \boxtimes \delta_C) = \delta_C \ast \delta_{\kappa - C} = \delta_{\{\kappa\}} \oplus \delta_{\kappa + C - C} \) such that

\[
\mathcal{H}^{-1}(\delta_{\kappa + C - C}) \cong H^1(C) \otimes \delta_{\{\kappa\}}.
\]

3) We claim \( \delta_{2g-3} \equiv \delta_{\kappa - C} \) and \( \delta_{2g-2} \equiv \delta_{\{\kappa\}} \) in \( K_*(X) \) (ignoring Tate twists). These are the simplest cases of the duality theorem \([W]\) 5.3. This implies

\[
\delta_{2g-3} \ast \delta_1 \equiv \delta_{\{\kappa\}} + \delta_{\kappa + C - C},
\]

in \( K_*(X) \) using step 2.

Proof of the claim: By the theorem of Riemann-Roch \( C^{(2g-3)} \) \( \to X \) is a \( \mathbb{P}^{g-2} \)-bundle over \( \kappa - C \) and a \( \mathbb{P}^{g-3} \)-bundle over the open complement \( X \setminus (\kappa - C) \). Hence \( Rp_*(\delta_{C^{(2g-3)}}) \) is a direct sum of \( \delta_{\kappa - C} \) and a sum of translates of constant sheaves on \( X \). Similarly \( pr_2^{-1}2g-2(\{\kappa\}) = \mathbb{P}^{g-1} \), and \( pr_{2g-2} \) is a \( \mathbb{P}^{g-2} \)-bundle over the open complement \( X \setminus \{\kappa\} \). Hence \( \delta_{2g-2} \equiv \delta_{\{\kappa\}} \) in \( K_*(X) \).

4) \( \Theta = \kappa - \Theta \) and the definition of the convolution product implies

\[
\mathcal{H}^{-1}(\delta_\Theta \ast \delta_\Theta) \cong IH^{2g-1}(\Theta) \otimes \delta_{\{\kappa\}}
\]

for an arbitrary principally polarized abelian varieties \((X, \Theta)\), where \( IH^{2g-1}(\Theta) \) denotes the intersection cohomology group of \( \Theta \). If the singularities of \( \Theta \) have codimension \( \geq 3 \) in \( \Theta \), then ignoring Tate twists this implies (see \([W]\) 2.9 and \([W2]\))

\[
\mathcal{H}^{-1}(\delta_\Theta \ast \delta_\Theta) \cong H^1(X) \otimes \delta_{\{\kappa\}}.
\]

In fact by the Hard Lefschetz theorem \( IH^{2g-3}(\Theta) \) and \( H^1(X) \) have the same dimensions. A more elementary argument proves this for Jacobians including the case of hyperelliptic curves: For example for non-hyperelliptic curves we have \( IH^\bullet(W_d) = H^\bullet(X, Rp_{d,*}\delta_{C(\omega)}[-d]) = H^\bullet(C(d)) = (\bigotimes^d H^\bullet(C))^\Sigma_d \), since \( p_d \) is a small morphism. Thus

\[
IH^{d+\bullet}(W_d) \cong \bigoplus_{a+b=d} Sym^a \left( H^0(C)[1] \oplus H^2(C)[-1] \right) \otimes \Lambda^b(H^1(C)).
\]
For $IH^{2d-1}(W_d)$ only $a = d - 1$ contributes, hence $IH^{2d-1}(W_d) \cong H^1(C) \cong H^1(X)$. (For the hyperelliptic case see [W] 4.2).

**Conclusion:** For curves $C$, which are not hyperelliptic, the perverse sheaf $A$ defined by $\delta_{\kappa+C-C}$ satisfies all the assertions of the theorem. $\delta_{\kappa+C-C}$ is a direct summand of $\delta_\Theta \ast \delta_\Theta$ by step 1 and 3. By step 2 and 4 we obtain modulo constant sheaves on $X$

$$\mathcal{H}^{-1}(\delta_\Theta \ast \delta_\Theta) \equiv H^1(X) \otimes \delta_{\{\kappa\}} \equiv H^1(C) \otimes \delta_{\{\kappa\}} \equiv \mathcal{H}^{-1}(\delta_{\kappa+C-C}) .$$

Since $K_*(X)$ is a quotient of $K^0_*(X)$, the last identity only holds modulo constant sheaves on $X$. But this suffices to imply the theorem.

**Remark:** In [W] we constructed a $\overline{Q}_l$-linear Tannakian category $BN$ attached to $C$ equivalent to the category of finite dimensional $\overline{Q}_l$-representations $Rep(G)$, where $G$ is $Sp(2g-2, \overline{Q}_l)$ or $Sl(2g-2, \overline{Q}_l)$ depending on whether $C$ is hyperelliptic or not. In this category $\delta_\Theta$ corresponds to the alternating power $\Lambda^{g-1}(st)$ of the standard representation, and $A$ corresponds to the adjoint representation.

**Bibliography**

[M] Martens H.H, On the variety of special divisors on a curve, Crelle 227 (1967), p.111 – 120.

[W] Weissauer R., Brill-Noether sheaves (preprint)

[W2] Weissauer R., Inner cohomology (preprint)