Anomaly-free chiral $U(1)_D$ and its scotogenic implication

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We consider an anomaly-free chiral hidden sector with additional gauge symmetry $U(1)_D$, such that after spontaneous symmetry breaking the residual symmetry stabilizes Dark Matter (DM) candidate and induces scotogenic neutrino masses. Charges and number of the new particles are nontrivially restricted by anomaly-free conditions and minimalitiy of scalar content. We search for available charge assignments within a given extent using computer program and identify the minimal solutions for model building. Two (one) charge assignments are found for Majorana (Dirac) neutrino scenario and corresponding models and phenomenologies have been briefly sketched. In all cases, observed DM abundance is contributed by three (two) components in Majorana (Dirac) neutrino case. Nature of neutrinos distinguishes nature of DM components and vice versa. Qualitative correlations between neutrinos and DM properties are established.

I. INTRODUCTION

Neutrino mass and Dark Matter (DM) are two well-established and pressing puzzles in particle physics and cosmology [1]. Both can be addressed by introducing new particle and/or symmetry in various extensions of the Standard Model (SM), convincing one the existence of a hidden sector [2] in nature. Without enough empirical information, the content of this sector can be highly model-dependent but some general theoretical guideline and analogue can be speculated. For example, if the new sector follows features found in the SM [3, 4], it should be an anomaly-free gauge theory, in which all fermions are chiral and are massless due to gauge invariant, until spontaneous symmetry breaking (SSB) at low energy induced by vacuum expectation value (VEV) of just one Higgs multiplet.

Anomaly cancellation is not only required for theoretical self-consistence, but also insightful to understand the charges of SM quarks and leptons [5, 6]. In the SM, the 15 chiral fermions per generation form the minimal set that satisfies all anomaly-free conditions for the SM gauge group $G_{SM} = SU(3)_C \times SU(2)_L \times U(1)_Y$. In some SM extensions, anomaly cancellation is taken as a tool to explore particle content when $G_{SM}$ is extended by a new $U(1)$ symmetry [7–27]. This approach is greatly predictive about the number, charges, and couplings of new fermions. In particular, an anomaly-free set of SM-singlet chiral fermions with exotic interactions is required if the SM degrees of freedom (DOFs) as a whole does not contribute to the anomalies [3, 4, 28–35].

The underlying mechanism of the interested phenomena is another important guideline to determine identities of new particles. In an economical idea called scotogenic mechanism, first proposed in the Ma model [36], origins of neutrino mass and DM are explained simultaneously by introducing a messenger scalar doublet and singlet fermions both are odd under a new global $Z_2$ parity. Majorana neutrino masses are generated at one-loop level by operators induced by these new particles. The $Z_2$ parity prevents tree level neutrino masses, and stabilizes the lightest new state that becomes a DM candidate. As nature of neutrinos has not yet been determined by current experiments, Dirac neutrino is also possible and corresponding scotogenic models are proposed [37, 38]. These models are similar to Majorana one, except the one-loop neutrino masses that are now generated by heavy Dirac sterile fermions and two messenger scalars (one doublet and one singlet), and the existence of right-handed neutrinos (RHNs).

In this work, we study the case where the SM is extended by a new gauge $U(1)_D$ symmetry and an anomaly-free set of SM-singlet chiral fermions that realizes the scotogenic mechanism after SSB. The charge assignment of chiral fermions is unusually restricted not only due to anomaly-free conditions, but also due to assuming the minimal number of Higgs singlet that breaks all above requirements, and study the minimal solutions to build scotogenic models for Majorana and Dirac neutrinos, respectively. We discuss their basic structure and phenomenologies, especially about neutrino masses and DM. We find some interesting qualitative correlations about the number and nature of light neutrinos to those of DM species. These models can also be regarded as examples of realizing gauged scotogenic models [39–49] (also Dirac counterpart [50–52]), and alter Renormalization Group (RG) running in the Ma model [53–56].

The paper is organized as followed: In Sec.II, constraints on charge assignments are discussed one by one and minimal solutions satisfying all of them are identified finally. In Sec.III minimal models are built and phenomenologies are sketched. Our results are discussed and concluded in Sec.IV and V respectively.

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II. CONSTRAINTS ON CHARGE ASSIGNMENT

Consider that the SM is extended by a gauge $U(1)_D$ symmetry under which all SM fields are neutral, and $N$ chiral fermions $\xi(z_1),\xi(z_2),\ldots,\xi(z_N)$ that are SM singlets but carrying their respective nonzero charges $z_1,z_2,\ldots,z_N$ under $U(1)_D$. These fermions are assumed right-handed without loss of generality. In this section, we search for candidates of $\{\vec{z}\} = \{z_1,\ldots,z_N\}$ which satisfies following properties.

A. Anomaly-free chiral fermions

The $\xi(z_i)$s contribute only to gauge anomaly [57–59] and mixed gauge-gravitational anomaly [60–62] of $U(1)_D$. The global Witten anomaly [63] is not relevant since there is no new $SU(2)_L$ doublet. Anomaly-free conditions impose following constrains on $\{\vec{z}\}$:

$$
\sum_{i} z_i^2 = 0, \quad \sum_{i} z_i = 0.
$$

Additional requirements are imposed on $\{\vec{z}\}$ for some physical considerations. We assume every $z_i \in \{\vec{z}\}$ is integer rather than real number, since $U(1)_D$ is believed embedded in a nonabelian group to avoid Landau pole [31], rendering rational charges [64, 65] that is rescalable to integers by redefining gauge coupling $g_D$. Eq.1 are therefore Diophantine equations. We also demand that $\{\vec{z}\}$ is chiral, i.e., no vector-like pair $\{z,-z\}$ contained, to avoid the arbitrariness from both number and charge values of such pairs that are not constrained by Eq.1; and arbitrariness of Dirac mass from $L \supset V_z \xi(z) \xi(-z)$ that is not protected by symmetry.

Solving these Diophantine equations can be highly nontrivial. Many efforts have been paid in literature. Group theoretical and algebraic methods have been proposed to construct analytic solutions in terms of some free parameters [3, 4, 23, 31, 34, 35]. However, for phenomenological uses and model-building, an explicit listing of these solutions could sometimes be more useful, especially when one searches for numerical patterns. For this sake, we build a computer program to look for solutions satisfying Eq.1 and aforementioned requirements. Given fermion number $N$, every $z_i$ is iterated over integers within a prescribed range $[-Z_{\text{max}},Z_{\text{max}}]$. For an acceptable running time, $Z_{\text{max}} = 12$ is chosen, and $N$ is taken from five to 12. The reason we ignore the cases with fermion number less than five is justified by following observations. If $N = 1$, the charge can only trivially zero; if $N = 2$, the charges are vector-like; if $N = 3$, there is never integer solution due to the Fermat last theorem applying on the cubic equation; if $N = 4$, the resulting charge assignment is just consisted of two vector-like pairs [25, 29].

In the computer survey defined above, we have found 1955 solutions. These solutions are coprime and non-composite, i.e., so-called “primitive” [34]. Charges in each solution are also arranged in increasing order according to absolute values, and the smallest absolute charge is made positive. These solutions are counted according to their respective length $N$ and the maximal absolute charge value Max($\vec{z}$), giving rise the birdeview shown in FIG. 1. Due to limiting space, also being sufficient for our discussion hereafter, we list only solutions corresponding to $N \leq 9$ and Max($\vec{z}$) $\leq 10$ explicitly in TABLE I. Some of the solutions have been explored in existing models [3, 28, 33] and similar tables [25, 30, 31].

![FIG. 1. A birdview of “density” distribution of anomaly-free chiral charge assignments, according to length $N$ and maximal absolute charge value Max($\vec{z}$). Solutions in un-shadowed region are listed in TABLE I.](image)

| Max($\vec{z}$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|---|---|----|
| N              | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 2              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 3              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 4              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 5              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 6              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 7              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 8              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 9              | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1  |
| 10             | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1   |

TABLE I. Some of the solutions have been explored in existing models [3, 28, 33] and similar tables [25, 30, 31].

Our list can be a double-check and complement to literature. Similar computer search on anomaly-free solutions has been done in [22], although in different physical context.

Anomaly-free conditions not only restrict the number and charges of new fermions, but also indirectly confine the value of gauge coupling $g_D$. The presence of chiral fermions contributes to RG running of $g_D$. At one-loop level, value of $g_D$ at energy $\mu$ is governed by $(4\pi)^2 g_D^2 / d\ln \mu = b g_D^3$, where $b$ is the beta function coefficient generally given by [33] (see also [66])

$$
b = \frac{2}{3} \sum_f z_f^2 + \frac{1}{3} \sum_s z_s^2.
$$

1 If $N = 5$, it can be proved that all charges are different [29].
TABLE I. Anomaly-free chiral fermion charges and associated Higgs charge. For those are 1HS (See Sec.II B), the Higgs charges are shown in column labelled by $z_1^S$. For those are composited by three identical elements and an 1HS set, the Higgs charges are shown in column labelled by $z_2^S$.

where $z_f$ and $z_s$ the $U(1)_D$ charges of Weyl fermionic and complex scalar DOFs respectively. Since always $b > 0$, $g_D$ reaches Landau pole at some energy $\Lambda_L$. This provides an upper limit of $g_D$ at lower energy $\Lambda$:

$$g_D(\Lambda) \leq \sqrt{\frac{8\pi^2}{b \ln(\Lambda_L/\Lambda)}}. \quad (3)$$

Assuming Landau pole energy at $10^{15}$GeV, upper limits of $g_D$ at electroweak scale $\Lambda \sim 100$GeV are evaluated for each solution and are shown in FIG. 2. As a comparison, perturbativity limits $g_D < \sqrt{4\pi/\text{Max}(\bar{z})}$ are also given.

We can see that Landau pole limit is always stronger than perturbativity limit. Perturbativity limit is controlled by merely the maximum charge value, while for Landau pole limit all charged DOFs contribute.

B. Minimal Higgs sector

We demand that all new fermions (except RHNs in Dirac scenario) are massive after SSB of $U(1)_D$ to avoid severe experimental constraints on massless particle, while we also demand that only one Higgs singlet is responsible for breaking $U(1)_D$ and generating all masses. This seemingly trivial analogue to the SM is highly non-trivial in our current setup, since the charges of chiral fermions are already constrained by anomaly-free conditions thus become exotic, rendering it opaque to see if such a Higgs singlet exists.

In general, any chiral fermion set $\{\vec{k}\} = \{k_1, \ldots, k_N\}$ can be fully-massive after SSB if the Higgs sector is constructed by charge assignment (see Appendix for details)

$$\{|k_1 + k_{P(1)}|, |k_2 + k_{P(2)}|, \ldots, |k_N + k_{P(N)}|\} \quad (4)$$

where $P$ is an arbitrary $N$-object permutation acting on index values $1, 2, \ldots, N$, equivalently on $\{\vec{k}\}$. Different permutation could lead to Higgs sector with fewer or more Higgs singlets. If there exists a special permuta-
tion $P_1$ by which we have
\[ |k_1 + k_{P(1)}| = |k_2 + k_{P(2)}| = \cdots = |k_N + k_{P(N)}|, \]
the resulting Higgs sector contains just one Higgs and fulfills our requirement. To determine whether this happens one needs to exhaust all possible permutations of $\{\tilde{k}\}$. We dub any chiral fermion set that can be made fully-massive by just one Higgs singlet an One-Higgs-Sufficient (1HS) set. For example, exhausting all possible permutations of fermion set $\{-4, -4, 5\}$ shows us that one needs Higgs sector $\{1, 8\}$ or $\{8, 10\}$ to generate all masses. So that is not an 1HS set. Instead, fermion set $\{2, 3, 3, -8\}$ is 1HS. The unique Higgs singlet is charged 5.

Based on this method, we build a computer program to search for anomaly-free chiral fermion sets in TABLE I satisfying following patterns for Majorana and Dirac neutrino models respectively:

1. For Majorana models: Since all new fermions are massive, the anomaly-free set must be itself 1HS. The Higgs singlet charge is shown in column labelled by “$z_S$” in the table if exists.

2. For Dirac models: Since all new fermions are massive except the three RHN candidates, the anomaly-free set must contain three identical elements and the rest form an 1HS set. The Higgs singlet charge is shown in column labelled by “$z_S$” in the table if exists.

Blank in these columns means that there is no such Higgs singlet capable to generate all will-be mass terms. Such requirement sieves out a significant portion of TABLE I.

C. Minimal messenger sector

1. Majorana case

Enough number of massive neutrinos required by neutrino oscillation data can be generated in scotogenic models by one messenger scalar doublet with multiple Majorana sterile fermions [36], or multiple messenger scalar doublets with one sterile fermion [67] (see also [68]). In this work we assume there is only one such doublet and one such singlet connecting the SM lepton doublets, and messenger scalar singlet connecting the will-be RHNs. In this work, we assume there are only one such doublet and one such singlet in Dirac models. This requires that the new fermionic DOFs must contain three identical elements massless at tree level and at least two Dirac mass eigenstates participating neutrino mass generation. There is only one candidate from TABLE I:

\[ \{1, 1, -4, -5, 9, 9, -10, -10\}. \]

Dirac model A:

Worth to mention that the anomaly-free set $\{1, -2, 3, 4, 6, -7, -7, -7, 9\}$ also contains three identical elements (i.e., charge –7) and the rest form an 1HS set (with the Higgs singlet charged +7) and there are three Dirac mass eigenstates (i.e., $\xi_{(1)} + \xi_{(6)}$, $\xi_{(-2)} + \xi_{(9)}$, and $\xi_{(3)} + \xi_{(-4)}$) presented after SSB of $U(1)_D$. However, only one of these Dirac state can participate the neutrino mass loop, if not introducing more messenger scalars.

III. MODELS

We apply the anomaly-free chiral charge assignments identified in last section (Eq. 6a, 6b and 7), one by one, to construct scotogenic models of Majorana and Dirac neutrinos. After working out the particle spectrum and interactions, phenomenologies will be briefly discussed, with emphasis on neutrino mass and DM physics where DM abundance is assumed thermal relics from the early universe.

A. Majorana model A

The SM is extended by a set of chiral fermions $\xi_{(1)1,2}$, $\xi_{(2)}$, $\xi_{(-4)1,2}$, $\xi_{(-5)}$, $\xi_{(6)}$, and an extended scalar sector

\[ \Phi \sim (2, 1, 0), \quad \eta \sim (2, 1, -1), \quad S \sim (1, 0, 2) \]

with their charges under $SU(2)_L, U(1)_Y$, and $U(1)_D$ indicated. Beta function coefficient $b = 74$, giving rise upper limit of $g_D$ at 100 GeV around 0.189 (0.165) for $\Lambda_L = 10^{15} (10^{19})$ GeV.

This particle content gives rise Yukawa couplings

\[ -\mathcal{L}_Y = Y_{a}^i T_{la} \bar{\eta} \xi_{i\alpha} + f_{ij} \bar{\xi}_{(1)}^i + \bar{\xi}_{(-4)}^i \xi_{(-4)}^j \]

\[ + h_i \bar{\xi}_{(-2)}^i \xi_{(-4)}^j S + h_i' \bar{\xi}_{(-4)}^i \xi_{(-4)}^j S^* + k \bar{\xi}_{(-3)}^i \xi_{(-5)}^j S + h.c. \]
where $i, j = 1, 2, 3$, and $\bar{\eta} = i\sigma_2\eta^*$ with $\sigma_2$ the second Pauli matrix. At low energy ($S \neq 0$), the model gives rise to two Majorana fermions $N_{1,2}$, and three Dirac fermions $\Psi_{1,2}$ and $\Sigma$. The charge 1 chiral states form Majorana fermions $N_i = \xi_{(i)}^1 + \xi_{(i)}$, with $i = 1, 2$, and they are responsible for generating neutrino masses. The charge 3 and $-5$ chiral states merge to be a Dirac fermion $\Sigma = \xi_{(3)} + \xi_{(-5)}$. The charge 2, $-4$, and 6 chiral states form the remaining two Dirac fermions through mass matrix

$$-\mathcal{L}_Y \supset \langle S \rangle \left( \xi^c_{(i)} \xi^c_{(j)} \right) \left( \begin{array}{cc} h_1 & h_2 \\ h_1^* & h_2^* \end{array} \right) \left( \begin{array}{c} \xi_{(-4)1} \\ \xi_{(-4)2} \end{array} \right) + \text{h.c}. \quad (10)$$

The mass matrix can be diagonalized by biunitary transformation consisted of $U_L$ and $U_R$, giving rise mass eigenstates $\Psi_{1,2} = U^T_L (\xi^c_{(2)}, \xi^c_{(6)}) T + U^T_R (\xi_{(4)1}, \xi_{(4)2}) T$. These mass eigenstates interact with new gauge bosons with $\mathcal{L} \supset g_D X_{\mu} j_{\mu}$, where $X_{\mu}$ is the $U(1)_D$ gauge field and

$$j_{\mu} = \frac{1}{2} N^a \gamma^5 N_i - \sum \gamma^\mu [3P_L + 5P_R] \Sigma$$

$$- (\bar{\Psi}_1 \bar{\Psi}_2) \gamma^\mu \left( \begin{array}{cc} 2 & 0 \\ 0 & 6 \end{array} \right) U_L P_L + 4P_R \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right), \quad (11)$$

with $P_L$ and $P_R$ are projection operators. Charge difference of $\xi_{(2)}$ and $\xi_{(6)}$ results in nontrivial off-diagonal coupling between $\Psi_1$ and $\Psi_2$ therefore a hidden Flavor Changing Neutral Current (see also [3]). As for the right-handed coupling, the identical charge of $\xi_{(-4)1}$ and $\xi_{(-4)2}$ makes $U_R$ vanishes and unphysical.

The most general renormalizable scalar potential contains only Hermitian operators and preserves any $U(1)$ number. Lepton number violation is achieved by including a dim-5 operator [69]:

$$V = - \mu_1^2 |\Phi|^2 + \mu_2^2 |\eta|^2 - \mu_3^2 |S|^2 + \lambda_1 |\Phi|^4 + \lambda_2 |\eta|^4 + \lambda_3 |S|^4 + \lambda_{12} |\Phi|^2 |\eta|^2 + \lambda_{13} |\Phi|^2 |S|^2 + \lambda_{23} |\eta|^2 |S|^2 + \frac{\Lambda}{(\Phi^4)^2} S^* + \text{h.c}. \quad (12)$$

with $\Lambda$ the cutoff scale. In unitary gauge, $\Phi = (0, (v + \phi)/\sqrt{2})^T$, $\eta = (\eta^+, \eta^0)^T$, and $S = (u + \phi^0)/\sqrt{2}$, where $v \approx 246$ GeV and $u \sim \mathcal{O}(\text{TeV})$. The VEV breaks electroweak symmetry and $U(1)_D$, leaving six physical bosons: scalar $H = \sqrt{2} Re(\eta^0)$, pseudoscalar $A = \sqrt{2} Im(\eta^0)$, charged scalars $\eta^\pm$, and the two Higgs bosons $h = \cos \theta_h \phi + \sin \theta_h \phi_S$ and $h' = \cos \theta_h \phi_S - \sin \theta_h \phi$ where $\theta_h$ is the rotation angle for diagonalization of mass matrix

$$2 \text{ Renormalizability can be recovered at tree level by introducing either scalar doublet (2, 1/2, -1) or singlet (1, 0, 1) [48], that opens the dim-5 operator without generating new nonzero VEV. We assume such additional scalar is much heavier (i.e., $S \ll \Lambda$), and the operator in Eq.12 is the only source of lepton number violation in our discussions.}$$

The light Higgs $h$ is identified as observed at LHC with mass 125 GeV [1].

Spontaneous breaking of $U(1)_D$ leaves three residual symmetries at low energy. Besides the Krauss-Wilczek $Z_2$ parity [70] carried by $H$, $A$, $\eta^\pm$, and $N_{1,2}$, there are two accidental global $U(1)$ symmetries, i.e., $U(1)_Y$ carried by $\Psi_{1,2}$ and $U(1)_\Sigma$ carried by $\Sigma$. Therefore, neutral particles carrying these new global charges form three distinct Dark Sectors (DSs):

DS-1: $\{N_1, N_2, H, A\} \quad (14a)$

DS-2: $\{\Psi_1, \Psi_2\} \quad (14b)$

DS-3: $\{\Sigma\} \quad (14c)$

The lightest state in each DS is stable and we consider that their thermal relics together explain the observed DM density [71].

Particles in DS-1 are also responsible for generating scotogenic neutrino masses through one-loop diagram in FIG. 3. That is [36]

$$m_{\nu} = \frac{2}{16\pi^2} \sum_{k=1}^{2} Y_{\nu k} Y_{\nu k} m_{\nu_k} \times$$

$$\left[ \frac{m_{H^0}^2}{m_{H^0}^2 - m_{N_k}^2} \ln \frac{m_{H^0}^2}{m_{N_k}^2} - \frac{m_{A^0}^2}{m_{A_0}^2 - m_{N_k}^2} \ln \frac{m_{A_0}^2}{m_{N_k}^2} \right] \quad (15)$$

and is proportional to difference between contributions from scalar $H$ and pseudoscalar $A$. If we assume $H$, $A$, and $N_{1,2}$ are close at mass, and splitting between $H$ and $A$, i.e., $\delta m^2 = m^2_H - m^2_A = 2\sqrt{2} v^2 \mu / \Lambda$, is tiny, neutrino masses can be simplifies to $(m_{\nu})_ab \approx (\delta m^2/32\pi^2) \sum_{k=1}^{2} (Y_{\nu k} Y_{\nu k})$. Therefore $m_{\nu} \sim 0.1 eV$ can be obtained by $Y \sim 0.01, m_{N_{1,2}} \sim 500$ GeV, and $v/\Lambda \sim 10^{-6}$, for example. Since there are only two Majorana fermions involved in the loop, the lightest neutrino is massless, still consistent to experiments [72]. With similar parameter values, $\mu \rightarrow e \gamma$ prediction is found satisfying current limit from MEG collaboration [73]. Detailed discussions including other Charged Lepton Flavor Violation (CLFV) processes may be referred to [74–76].

The first DM species (DM$_1$) is either $N_1$ or $H$ (assuming $m_A > m_H$). Various aspects of both candidates are well accounted for in the context of Inert Doublet Model and Ma model with a large body of literature which can be found in, e.g., reference of [77, 78]. For DM$_1 = N_1$, its annihilations into SM leptons in t-channel via $\tilde{L}_{\mu}\tilde{N}_1$ and coannihilation with $\eta$ are efficient enough if tension from CLFV experiments is alleviated (e.g., [72, 77, 79–82]). For DM$_1 = H$, observed relic abundance can be addressed by annihilations via $HHh$, $HHhh$, $HHWW$, $HHZZ$, and coannihilation via $HA$ and via Yukawa coupling (e.g., [78, 83–93]). The $h-h'$ mixing from Eq.13

$$2\lambda_{12}^2 \lambda_{13} vu \quad (13)$$
and $Z$-$Z'$ mixing between $X_\mu$ and $U(1)_Y$ gauge field $B_\mu$ induced by $\mathcal{L} \supset -c_{X_\mu} B^{\mu
u}/2$ [94–96] do not modify much on this picture, due to the small mixings constrained by LHC and Electroweak Precision Tests [95], and DM direct searches [96, 97]. General discussions on the extra gauge boson $Z'$ can be found in [98].

The second and the third DM species are $\text{DM}_2 = \Psi_1$ and $\text{DM}_3 = \Sigma$ respectively. Both communicate to SM species only through mediators $Z$, $Z'$, $h$, and $h'$, but necessary parameter space has almost ruled out by spin-independent DM-nuclei elastic scattering experiments [96, 97]. Therefore $\text{DM}_2$ and $\text{DM}_3$ reduce their densities during freeze-out primarily by DM conversion $\text{DM}_2 \leftrightarrow \text{DM}_3 \rightarrow \text{DM}_1 \Sigma$, mediated by $Z'$ and $h'$. For example, thermal averaged cross section of $\Psi_1 \Sigma_1 \rightarrow NN$ mediated by $Z'$ in s-channel is estimated by dimensional analysis

$$\langle \sigma v \rangle(\Psi_1 \Sigma_1 \rightarrow NN) \sim \frac{g_D^4 m_{\Psi_1}^2}{m_{Z'}^4} \approx 1 \text{pb} \times \left( \frac{m_{\Psi_1}}{400 \text{GeV}} \right)^2 \left( \frac{1.4 \text{ TeV}}{v} \right)^4 \quad (16)$$

where $m_{Z'} = 2g_D u$. The resulting abundance $\Omega_{\Psi_1} \approx 0.1 \text{pb}/\langle \sigma v \rangle \approx 0.1$ is at the level consistent with observations [71]. Mass hierarchy $m_{\text{DM}_2}, m_{\text{DM}_3} > m_{\text{DM}_1}$ is then necessary, otherwise the inverse conversion generates overdensity for $\text{DM}_2$ and $\text{DM}_3$. If $m_{\Psi_1}$ and $m_{\eta_1}$ are heavy enough, visible signals could be observed from cosmic ray resulted from present-day annihilations $\Psi_1 \Sigma_1 \rightarrow N_2 N_2, \eta_1^\pm \eta_1^\mp$ followed by decays $N_2 \rightarrow N_1 \tilde{\ell}, N_1 \nu \tilde{\sigma}$ and $\eta_1^\pm \rightarrow HW^\pm, AW^\pm$. DM conversion $\text{DM}_2 \leftrightarrow \text{DM}_3 \rightarrow \text{DM}_1 \Sigma$ happened in regions of high density, e.g., Galactic Center, may generate warm/hot DM particles and modify significantly small scale structure puzzle such as cusp-core problem [99]. In this model, $\text{DM}_2$ and $\text{DM}_3$ could interact indefinitely weak with quarks and leptons if the portal mixings are too small, leaving no signal in direct detections, even in colliders. Therefore pair productions of messenger scalar bosons $\eta_1^\pm, H$ and $A$ could become the most promising way to explore these new particles in colliders [77] (see also [100]).

### B. Majorana model B

The SM is extended by a set of chiral fermions $\xi^{(1)}, \xi^{(2)}_{1,2,3}, \xi^{(-3)}, \xi^{(-5)}, \xi^{(6)}, \xi^{(7)}$ and an extended scalar sector

$$\Phi \sim (2, \frac{1}{2}, 0), \quad \eta \sim (2, \frac{1}{2}, 2), \quad S \sim (1, 0, 4). \quad (17)$$

Beta function coefficient $b = 96$, giving rise upper limit of $g_D$ at 100 GeV around 0.160 (0.145) for $\Lambda_L = 10^{15}(10^{19})$ GeV.

![FIG. 3. Majorana neutrino masses generated at one loop level, where $\xi^{(2)}_{1,2,3}$ are the Majorana DOFs carrying $U(1)_D$ charge $z_s/2$.](image)

Yukawa sector can be determined as

$$-\mathcal{L}_Y = y_{ij} \bar{N}_a N_b \bar{\eta}_i \xi^{(2)}_{(j)} + f_{ij} \bar{\xi}^{(2)}_{(i)} \xi^{(2)}_{(j)} S^* + f'_{ij} \bar{\xi}^{(-6)} \xi^{(2)}_{(i)} S^* + f''_{ij} \bar{\xi}^{(-6)} S \xi^{(-3)}_j S^* + h.c.$$  \quad (18)

where $i, j = 1, 2, a = 1, 2, 3$. At low energy $\langle S \rangle \neq 0$, the model gives rise four Majorana fermions $N_{1,2,3,4}$, and two Dirac fermions $\Psi$ and $\Sigma$. The charge $1$ and $-6$ chiral states and charge $-3$ and $7$ chiral states, merge to be Dirac fermions $\Psi = \xi^{(1)} + \xi^{(-5)}$ and $\Sigma = \xi^{(-3)} + \xi^{(7)}$ respectively. The charge $2$ and $-6$ chiral states mix through mass matrix

$$\mathcal{M} = \begin{pmatrix} 0 & f_1' & f_2' & f_3' \\ f_1'' & f_{11} & f_{12} & f_{13} \\ f_2'' & f_{12} & f_{22} & f_{23} \\ f_3'' & f_{13} & f_{23} & f_{33} \end{pmatrix}. \quad (19)$$

The mass matrix is symmetric, and can be diagonalized by unitary transformation $U$ on basis $(\xi^{(-6)}, \xi^{(2)}_{1,2,3}, \xi^{(-3)}, \xi^{(-5)}$, resulting in four Majorana fermions $(N_1, N_2, N_3, N_4)^T = U(\xi^{(-6)}, \xi^{(2)}_{1,2,3}, \xi^{(-3)}, \xi^{(-5)})^T + U^T(\xi^{(-6)}, \xi^{(2)}_{1,2,3}, \xi^{(-3)}, \xi^{(-5)})^T$. Interestingly, the structure of $\mathcal{M}$ implies a possible Seesaw-like hierarchy among $N_i$. For example, a slight ratio $f''/f' \approx 0.1$ could result in two order of magnitude hierarchy between $N_1$ and the heavier ones, i.e., $m_{N_1} \sim (f''/f')^2 m_{N_{2,3,4}} \approx 10^{-2} m_{N_{2,3,4}}$.

The new gauge interaction current in $\mathcal{L} \supset g_D X_\mu j^\mu$
contributed by $N_i$, $\Psi$, and $\Sigma$ reads

\[
j^\mu = \overline{\Psi}[(3)P_L + (7)P_R]\Psi + \Sigma[(-1)P_L + (-5)P_R]\Sigma
\]
\[
+ \frac{1}{2}(N_1 N_2 N_3 N_4) \gamma^\mu \gamma^5 U^\dagger \left( \begin{array}{cccc}
-6 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{array} \right) U \left( \begin{array}{c}
N_1 \\
N_2 \\
N_3 \\
N_4
\end{array} \right).
\] (20)

The 4-by-4 unitary matrix $U$ contains 16 parameters, i.e., six rotation angles and 10 phases. Some of these parameters are unphysical. Since the charge matrix diag$(-6, 2, 2, 2)$ is diagonal and its 3-by-3 lower-right submatrix is proportional to identity, four phases and three rotation angles are always cancelled. Finally, only nine parameters (three angles, three Dirac-like phases, and three Majorana-like phases) are present in Lagrangian.

The scalar sector of this model gives rise to the same scalar potential (Eq.12), physical boson spectrum, and scalar couplings as in previous model. A distinction of gauge interaction since now $\eta$ and $S$ carry double charges under $U(1)_D$ in respect to those in Eq.8.

There are again three residual symmetries left in low energy theory. They are $Z_2$ parity carried by $H$, $A$, $\eta^\dagger$, and $N_{1,2,3,4}$: $U(1)_D$ carried by $\Psi$; and $U(1)_{\Sigma}$ carried by $\Sigma$. Their neutral members form three DSs:

\begin{align}
\text{DS-1:} & \quad \{N_1, N_2, N_3, N_4, H, A\} & (21a) \\
\text{DS-2:} & \quad \{\Psi\} & (21b) \\
\text{DS-3:} & \quad \{\Sigma\} & (21c)
\end{align}

Neutrino masses and CLFV $\mu \rightarrow e\gamma$ decay are induced by particles in DS-1, similar to Majorana model A (Sec.III A). However, number of loop fermions is now modified to four, thus all active neutrinos can be massive.

The three DM components are similar to those in Majorana model A: $\text{DM}_1$ being either $N_1$ or $H$; $\text{DM}_2 = \Psi$; and $\text{DM}_3 = \Sigma$. DM physics is then qualitatively similar to previous model, with different scattering strength due to different $U(1)_D$ charges carried, and apparent difference in number of particles in each DS.

### C. Dirac model A

The SM is extended by a set of chiral fermions $\xi(1)_{1,2}$, $\xi(-4)$, $\xi(-5)$, $\xi(-9)_{1,2,3}$, $\xi(-10)_{1,2}$, and an extended scalar sector

\[
\Phi \sim (2, \frac{1}{2}, 0), \quad \eta \sim (2, \frac{1}{2}, 1), \quad S \sim (1, 0, 9), \quad \sigma \sim (1, 0, 1).
\] (22)

Beta function coefficient $b = 352$, giving rise upper limit of $g_D$ at 100 GeV around 0.087 (0.076) for $\Lambda_L = 10^{15}(10^{19})$ GeV.

Yukawa sector reads:

\[
-\mathcal{L}_Y = Y_{\alpha}^{i} \overline{\psi}_{\alpha} \xi_{(i)j} + K_{\alpha j} \overline{\psi}_{(9)\alpha} \sigma^5 \xi_{(-10)j}^c \\
+ h_{ij} \xi_{(1)i} \xi_{(-10)j} S + k \xi_{(-4)i} \xi_{(-5)j} S + \text{h.c.}
\] (23)

where $i, j = 1, 2$ and $a = 1, 2, 3$. The 2-by-2 matrix $h_{ij}$ can be taken diagonal by redefining $\xi(1)_{i}$ and $\xi(-10)_{i}$ without loss of generality. At low energy $\langle S \rangle \neq 0$, three Dirac fermions and three Weyl fermions are generated. Charge 1 and $-10$ chiral states and charge $-4$ and $-5$ chiral states form Dirac fermions $\Psi_i = \xi(1)_{i} + \xi(-10)_{i}$ with $i = 1, 2$ and $\Sigma = \xi(-4) + \xi(-5)$ respectively. The three charge 9 chiral states do not receive mass from $\langle S \rangle$ and are identified as RHNs, i.e., $\nu_{9R} = \xi(9)_{a}^c$. Accordingly, gauge interactions of new fermions are described by the current

\[
j^\mu = \overline{\Psi} \gamma^\mu ((-1)P_L + (-10)P_R)\Psi_j \\
+ \Sigma \gamma^\mu ((4)P_L + (-5)P_R)\Sigma \\
+ \overline{\Psi} \gamma^\mu (9)P_R \nu_a
\] (24)

The most general renormalizable scalar potential

\[
V = -\mu_1^2 |\Phi|^2 + \mu_2^2 |\eta|^2 - \mu_3^2 |S|^2 + \mu_4^2 |\sigma|^2 \\
+ \lambda_1 |\Phi|^4 + \lambda_2 |\eta|^4 + \lambda_3 |S|^4 + \lambda_4 |\sigma|^4 \\
+ \lambda_{12} |\Phi|^2 |\eta|^2 + \lambda_{13} |\Phi|^2 |S|^2 + \lambda_{14} |\Phi|^2 |\sigma|^2 \\
+ \lambda_{23} |\eta|^2 |S|^2 + \lambda_{24} |\eta|^2 |\sigma|^2 + \lambda_{34} |S|^2 |\sigma|^2 \\
+ \kappa |\Phi|^4 |\sigma|^4 + \text{h.c.}
\] (25)

has the last term non-Hermitian. Its coefficient $\kappa$ is expected small, since the operator breaks a global symmetry $U(1)_{\Phi} \times U(1)_N \times U(1)_S \times U(1)_\sigma$ down to $U(1)_{\Phi+\eta+\sigma} \times U(1)_S$ in scalar potential [101]. In unitary gauge, $\Phi = (0, (v + \phi)/\sqrt{2})^T$, $\eta = (\eta^+, 0)^T$, $S = (u + \phi S)/\sqrt{2}$, $\sigma = \sigma$. After breaking of electroweak symmetry and $U(1)_D$, global symmetry in the scalar potential is further broken down to $U(1)_{\eta+\sigma}$. The presence of this symmetry implies that the real and imaginary parts in $\eta^0$ and $\sigma$ respectively are degenerate, giving rise complex scalar mass eigenstates. Finally there are eight physical bosons: charged bosons $\eta^\pm$, complex neutral scalar bosons $\phi_1 = \cos \theta_\eta \eta^0 + \sin \theta_\eta \sigma$ and $\phi_2 = \cos \theta_\phi \sigma - \sin \theta_\phi \eta^0$ with mixing angle $\theta_\phi$ diagonalizing mass matrix of $(\eta^0, \sigma)$:

\[
\begin{pmatrix}
2\mu_1^2 + (\lambda_{12} + \lambda_{13}) u^2 + \lambda_{23} u^2 + \sqrt{2\kappa} v \\
\sqrt{2\kappa} v \\
2\mu_2^2 + \lambda_{14} u^2 + \lambda_{34} u^2
\end{pmatrix}
\] (26)

\[\text{An alternative charge assignment can be taken, in which } \eta \sim (2, \frac{1}{2}, -10) \text{ and } \sigma \sim (1, 0, -10). \text{ Such assignment brings in 10-times larger gauge interactions for these fields.}\]

\[\text{The scalar potential has also studied in Ref. [50], up to a term } |\Phi|^2.\]
and the two Higgs bosons $h$ and $h'$ defined exactly as in Majorana model A thanks to the same mass matrix given in Eq.13.

Spontaneous breaking of $U(1)_D$ leaves two residual symmetries at low energy. Both of them are continuous symmetries. They are $U(1)_\phi$ carried by $\phi_{1,2}$, $\eta^\pm$, and $\Psi_{1,2}$; and $U(1)_\Sigma$ carried by $\Sigma$. Neutral members form two DSs:

$$\text{DS-1: } \{\Psi_1, \Psi_2, \phi_1, \phi_2\} \quad (27a)$$
$$\text{DS-2: } \{\Sigma\} \quad (27b)$$

Neutrino masses are generated by the one-loop diagram in FIG.4, calculated by

$$(m_\nu)_{ab} = \sum_{k=1}^{2} \frac{K_{ak}Y_{bk}m_{\Psi_k}}{16\pi^2} \sin 2\theta_\phi \times$$

$$\left[ \frac{m_{\phi_1}^2}{m_{\phi_1}^2 - m_{\Psi_k}^2} - \ln \frac{m_{\phi_1}^2}{m_{\Psi_k}^2} - \frac{m_{\phi_2}^2}{m_{\phi_2}^2 - m_{\Psi_k}^2} - \frac{m_{\phi_2}^2}{m_{\Psi_k}^2} \right].$$

(28)

Unlike Majorana models, neutrino masses are not proportional to difference between contributions from real and imaginary parts of the in-loop scalar, but between mass eigenstates of the complex scalars $\phi$'s. Since imaginary parts of both $\eta^0$ and $\sigma$ contribute, there is an implicit factor of two in above expression, distinct from Ref.[38] where $\sigma$ is a real scalar thus only $\text{Re}(\eta^0)$ is involved in the loop. If $\phi_{1,2}$ and $\Psi_{1,2}$ are close in mass and the splitting $\delta m^2 = m_{\phi_1}^2 - m_{\phi_2}^2 = 2\sqrt{2} v \sin 2\theta_\phi$ is small, we have $(m_\nu)_{ab} \approx \frac{\delta m^2/32\pi^2}{\sum_{i=1}^{2} (K_{ak}Y_{bk} \sin 2\theta_\phi/m_{\Psi_k})}$ and $m_\nu \sim 0.1eV$ can be obtained if $K \sim Y \sim 0.01$, $m_{\Psi_k} \sim 500GeV$, and $\kappa/v \sim 10^{-6}$. As in Majorana model A (Sec.III A), only two fermions are involved in the loop diagram (FIG.4), therefore the lightest neutrino is massless. Yukawa coupling $Y_{ai}$ induces CLFV $\mu \rightarrow e\gamma$ decay in the same way as Majorana models.

The first DM species (DM1) is either $\Psi_1$ or $\phi_1$. For $\Psi_1 = \Psi_1$, Yukawa coupling $K_{ai}$ could mediate $\Psi_1 \nu_i \rightarrow \nu_a h^0 h^0$ in $t$-channel, without constraint from CLFV experiments [38]. For $\Psi_1 = \phi_1$, the $\eta^0$ component is severely suppressed due to direct detection constraints on $\eta^0 h^0 Z$ coupling [83], thus $\phi_1$ is dominated by $\sigma$ and $\theta_\sigma \sim \pi/2$. Therefore $\phi_1$ can annihilate into RHNs in $t$-channel mediated by $\Psi_{1,2}$, with cross section [102]

$$\langle \sigma v_{\text{rel}} \rangle \simeq v_{\text{rel}}^2 \sum_{k=1}^{2} \frac{|K_{ak}^* K_{k}\phi|^2}{96\pi} \frac{m_{\phi_1}^2 \sin^4 \theta_\phi}{(m_{\phi_1}^2 + m_{\Psi_k}^2)^2}. \quad (29)$$

Take $k \equiv (|K_{ak}^* K_{k}\phi|^2)^{1/4} \sim 0.8$, $m_{\phi_1} \sim 500GeV$, $v_{\text{rel}} \sim 0.3$, $\theta_\phi \sim \pi/2$, and adding up all final state flavors, we obtain $\langle \sigma v_{\text{rel}} \rangle \sim 1pb$ as $m_{\Psi_k} \sim 540GeV$. The second DM species $\Sigma$ communicates with SM by Z, $Z'$, $h$, and $h'$, while all these portals are ineffective, as in Majorana models. Therefore DM conversion $\Sigma \Sigma \rightarrow \text{DM}_1 \text{DM}_1$ mediated by $Z'$ is relevant. Indeed, even if $m_{\Sigma} < m_{\text{DM}_1}$ and the above DM conversion doesn’t work, $\Sigma$ would not be overdensity in the early universe. Because for all DS particles, gauge boson $Z'$ could mediate annihilation into RHN pair, with cross section similar to that estimated in Eq.16.

RHNs are nearly massless and could play the role of dark radiation, hence the decoupling temperature of RHNs from SM plasma is constrained [103]. In this model, RHNs communicate with SM species via $\nu_R \sigma \leftrightarrow \nu_L \eta$ in $s$-channel and $\nu_L \eta^* \leftrightarrow \nu_L \sigma^*$ in $t$-channel, both mediated by $\Psi_i$. The lowest possible freeze-out temperature of these scatterings must be around mass scale of $\eta$ and $\sigma$, i.e., about $O(100GeV)$. Therefore we conclude that the model is consistent with the measured $\Delta N_{\text{eff}}$ [71].

IV. DISCUSSION

Given anomaly-free conditions and our minimality requirement, the briefly discussed phenomenologies in Sec.III reveals interesting correlations between neutrino and DM sector. Number of massive neutrinos could be used to distinguish models. If the absolute neutrino mass scale is found finite, Majorana model B (Sec.III B) is strongly flavored, implying neutrinos are Majorana and DM is three-component, i.e., two Dirac fermionic DMs and one Majorana fermionic or real scalar DM.

Moreover, nature of neutrinos can tell the number of DM components. For both Majorana neutrino models, DM is three-component, while for Dirac model, DM is two-component. Inversely, determining number of DM components can help finding out whether neutrinos are Majorana or Dirac. However, discovering number of DM species could be extremely difficult.

Nature of DM could also tell about nature of neutrinos. Record that Majorana model A and B provide DM components: two Dirac plus one Majorana or one real scalar; Dirac model A provides DM components: two Dirac, or
one Dirac plus one complex scalar. Therefore, only Majorana models provide Majorana fermion and real scalar as DM candidate. Similarly, only Dirac model provides complex scalar as DM candidate. In our approach, experimental confirmation on nature of DM species would become a “smoking-gun” to reveal nature of neutrinos, and it could be relatively easier in practice.

Some remarks about the relation between nature of neutrinos and that of heavy in-loop fermions is worth to point out (see FIG.3 and 4). It is possible to build scotogenic model for Majorana neutrinos with Dirac in-loop fermions [47], as well as that for Dirac neutrinos with Majorana in-loop fermions [104]. The former case can be realized in solutions found in our survey, for example, \{1, −2, −2, 4, 5, −7, −7, 8\} with Higgs singlet \(S \sim (9)\). By introducing scalar doublets \(\eta\) and \(\eta'\) carrying \(U(1)_D\) charges −2 and −7 respectively, Yukawa coupling \(\mathcal{L} \eta \xi_{(-2)}, \mathcal{L} \eta' \xi_{(-7)}\), and scalar coupling \((\Phi^+ \eta)(\Phi^+ \eta')S\) can mediate the neutrino mass loop but that is obviously not the minimal model since it needs two messenger scalars for Majorana neutrinos. For the latter case, within Table I, only solutions in Eq. 7 and \{1, −2, 3, 4, 6, −7, −7, −7, 9\} can provide right-handed neutrino candidate and heavy in-loop fermions, but one needs additional Higgs singlet to generate new Majorana masses and to connect the fermions into the loop. All in all, minimality of scalar particle number constrains the model space such that the nature of neutrinos and that of in-loop fermions are the same.

V. CONCLUSION

We have considered constraints on the particle number and charges of a new sector consisted of chiral fermions, Higgs singlet, messenger scalars, and gauge symmetry \(U(1)_D\), by which neutrino masses and DM are explained via scotogenic mechanism after SSB. We build a computer program to survey all anomaly-free chiral charges assignments up to some maximal absolute charge and particle number. Distribution of solutions is given in FIG. 1 and we have listed 65 of them in TABLE I. Considering the whole anomaly-free set of chiral fermions rather than some anomalous subsets in a standard model extension is not only meaningful to explore the particle spectrum, but also important to determine the upper limit of gauge coupling \(g_D\) if Landau pole is in mind, as shown in FIG. 2.

Since all new fermions are assumed massive (except right-handed neutrinos in Dirac scenario) and all such masses are from a common VEV, Higgs singlet charges fulfilling it are identified in Sec.II.B. Together with requirement from number of observed massive light neutrinos and minimality of messenger scalar sector, only two charge assignments from TABLE I are found appropriate for Majorana scotogenic model. It is a 2/65 probability of discovery in the table. For Dirac scotogenic model, that is even lower (i.e., 1/65). It may imply that either these models are predictive due to some degree of uniqueness, or the nature prefers some larger sets of anomaly-free chiral fermions containing greater absolute charge values although \(q_D\) will then be strongly suppressed (i.e., FIG. 2). These resulting minimal models provide interesting correlations between physics of neutrino and DM, rendering knowledge in one sector could reveal that in another (see Sec.IV).

Our approach can be regarded as follows. Unlike normal phenomenological models where parameter space is calculated and physical observables are correlated based on assuming a specific set of new particles and/or symmetry; we here employ a specific set of principles (i.e., analogue to the SM, anomaly freedom, minimality of new particle content) to restrict the model space, therefore to find out qualitative relations or linkages between new physics phenomena.

Many phenomenologies of the resulting models have not been discussed thoroughly, such as collider physics, theoretical constraints and finite-temperature effects of scalar potential, various scenarios of DM mass hierarchy, and baryon asymmetry in the universe. More precisely clarifying parameter space of each model could be done in future works.

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Appendix: Constructing Higgs sector

Consider a simpler case consisted of two chiral fermions \(\xi(a), \xi(b)\), and Higgs singlet \(S\) of charge \(q > 0\). Mass term \(\xi(a)\xi(b)\) presents in low energy Lagrangian only if \(a + b\) = \(q\), so the mass term can be written using Kronecker delta as \(\xi(a)\xi(b)\delta_{a+b, q}\). If there are \(K\) Higgs singlets carrying positive charges \(q_1, \ldots, q_K\), the above mass term exists only if \(a + b\) equals to anyone in \(\{q\} = \{q_1, \ldots, q_K\}\). Thus the mass term can be generalized to \(\xi(a)\xi(b)\Delta_{a+b}\), where

\[
\Delta_{a+b} = \delta_{a+b, \pm q_1} + \delta_{a+b, \pm q_2} + \cdots + \delta_{a+b, \pm q_K}.
\]  

(A.1)

We have \(\Delta_{a+b} = 1\) if and only if \(a + b\) \(\in\) \(\{q\}\).

For a general chiral fermion set \(z_1, \ldots, z_N\), all mass terms form a polynomial of fermion binaries, and a \(N \times N\) mass matrix

\[
M = \begin{pmatrix}
m_{11} \Delta_{z_{1}+z_{1}} & \cdots & m_{1N} \Delta_{z_{1}+z_{N}} \\
\vdots & \ddots & \vdots \\
m_{N1} \Delta_{z_{N}+z_{1}} & \cdots & m_{NN} \Delta_{z_{N}+z_{N}}
\end{pmatrix}
\]  

(A.2)
can be written down in basis \((\xi_{z_1}, \xi_{z_2}, \ldots, \xi_{z_N})\). Its determinant is

\[
\text{Det}(\mathbf{M}) = \sum_{P} m_{1P(1)}m_{1P(2)} \cdots m_{1P(N)} \times \Delta_{z\bar{z} P(1)} \Delta_{z\bar{z} P(2)} \cdots \Delta_{z\bar{z} P(N)} + \text{c.c.,} \tag{A.3}
\]

where \(P(i)\) denotes dummy permutation on index \(i\). If all fermions are massive, we have \(\text{Det}(\mathbf{M}) \neq 0\) that implies at least one term in Eq.A.3 is nonzero, in which must have \(\Delta_{z\bar{z} P(1)} \Delta_{z\bar{z} P(2)} \cdots \Delta_{z\bar{z} P(N)} = 1\). It is equivalent to \(|z_1 + z_{P(1)}|, |z_2 + z_{P(2)}|, \ldots, |z_N + z_{P(N)}| \in \{q_1, q_2, \ldots, q_K\}\). In other words, given a \(P\), a Higgs sector affordable for all masses of \(z_1, \ldots, z_N\) can be constructed by Higgs sector

\[
\{|z_1 + z_{P(1)}|, |z_2 + z_{P(2)}|, \ldots, |z_N + z_{P(N)}|\}. \tag{A.4}
\]

Different permutation may lead to different Higgs sector, with number of Higgs fewer or more. Iterating all possible permutations gives rise a complete list of Higgs sectors. One may like to employ some specific Higgs sector to generate desired Yukawa couplings, hence mass terms, nature (Majorana or Dirac), and accidental symmetry.
