LIMIT VARIETIES OF MONOIDS SATISFYING A CERTAIN IDENTITY

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Abstract. A limit variety is a variety that is minimal with respect to being non-finitely based. Since the turn of the millennium, much attention has been given to the classification of limit varieties of aperiodic monoids. Seven explicit examples have so far been found, and the task of locating other examples has recently been reduced to two subproblems, one of which is concerned with monoids that satisfy the identity $xxt \approx xxtx$. In the present article, we provide a complete solution to this subproblem by showing that there are precisely two limit varieties that satisfy this identity. One of them turns out to be the first example having infinitely many subvarieties.

It is also deduced that the variety generated by any monoid of order five or less contains at most countably many subvarieties.

1. Introduction

A variety of algebras is finitely based (FB) if it has a finite basis of identities; otherwise, it is non-finitely based (NFB). The finite basis problem—to determine which varieties are finitely based—has been the focus of investigation for many years. In the 1930s, Neumann [26] questioned if every variety of groups is FB, and it was not until 1970 when the existence of NFB examples was independently confirmed by Adyan [1], Ol’shanski˘ı [28], and Vaughan-Lee [35]. The finite basis problem for varieties of semigroups and of monoids gained much interest in the 1960s after Perkins [29] exhibited the first examples of NFB varieties generated by a finite semigroup. For more information on the finite basis problem for these varieties, refer to Gupta and Krasilnikov [5] for varieties of groups and to Volkov [36] for varieties of semigroups and of monoids.

A FB variety satisfies the stronger property of being hereditarily finitely based (HFB) if all its subvarieties are FB. A limit variety is a NFB variety whose proper subvarieties are all FB. By Zorn’s lemma, every NFB variety contains some limit subvariety; in other words, a variety is HFB if and only if it excludes all limit varieties. Therefore, classifying HFB varieties in a certain sense reduces to classifying limit varieties. But finding explicit examples of limit varieties turns out to be highly nontrivial. For instance, no explicit example of limit variety of groups is known so far, even though uncountably many of them exist [15]. Locating such an example
remains one of the foremost unsolved problems in the theory of varieties of groups; see Gupta and Krasilnikov [5].

One of the main goals of the present paper is to study limit varieties of monoids, considered as algebras of type \(\langle 2, 0 \rangle\). A complete classification of all limit varieties of monoids is highly infeasible since that would include a description of all limit varieties of groups. Therefore it is natural to focus on the class \(A\) of aperiodic monoids, that is, monoids whose subgroups are all trivial. The first explicit examples of limit subvarieties of \(A\) were exhibited by Jackson [11] in the early 2000s. Since then, limit subvarieties of \(A\) have received much attention and a few more examples have been found; see, for example, Gusev [6, 7], Gusev and Sapir [9], Lee [18, 19], Sapir [32, 33], Zhang [37], and Zhang and Luo [38].

Presently, seven explicit examples of limit subvarieties of \(A\) are known, and it follows from Gusev and Sapir [9, Sorting Lemma 2] that any other limit subvariety of \(A\) is contained in one of the following varieties or their duals:

\[ P = \text{var}\{xsxt \approx xsxtx\} \quad \text{and} \quad Q = \text{var}\{xtx \approx xtx^2, xytx \approx (xy)^2tx\}. \]

Therefore, to completely classify all limit subvarieties of \(A\), it suffices to consider only subvarieties of \(P\) and of \(Q\). Sapir [33] made some progress by exhibiting three new examples of limit subvarieties of \(Q\).

The present paper is concerned with the variety \(P\), where the main objective is to provide a complete description of all its limit subvarieties (Theorem 5.11). Specifically, we present two new limit varieties of monoids and show that they exhaust all limit subvarieties of \(P\). One of these limit varieties contains countably infinitely many subvarieties (Corollary 5.10)—a result that is quite surprising given that every limit variety of monoids previously found is small in the sense that it contains only finitely many subvarieties.

Our main result also implies an important fundamental result concerning varieties generated by a monoid of small order. Finite members from several classical classes of algebras, such as groups [27] and associative rings [16, 25], generate small HFB varieties. But this result does not hold for finite semigroups or monoids. The smallest monoids currently known to generate a variety with continuum many subvarieties are of order six; the well-known Brandt monoid

\[ B_1^6 = \langle a, b, 1 \mid aba = a, bab = b, a^2 = b^2 = 0 \rangle = \{a, b, ba, b^2, 0, 1\} \]

is one such example [12, 14]. As for monoids of order five or less, up to isomorphism and anti-isomorphism, every one of them, with the exception of

\[ P_2^5 = \langle a, b, 1 \mid a^2 = ab = a, b^2a = b^2 \rangle = \{a, b, ba, b^2, 1\}, \]

generates a small HFB variety [24]. The variety \(P_2^5\) generated by \(P_2^5\) is not small, but not much else is known about it. An answer to the following question is thus desirable.

**Question 1** (Jackson and Lee [12, Question 6.1]). Does the variety \(P_2^5\) contain uncountable many subvarieties?

The monoid \(P_2^5\) satisfies the identity \(xsxt \approx xsxtx\) and so \(P_2^5\) is a subvariety of \(P\). It follows from our classification of limit subvarieties of \(P\) that \(P_2^5\) is HFB (Proposition 6.1). Consequently, the variety generated by any monoid of order five or less is HFB and so contains at most countably many subvarieties.
Remark 1.1. It is relevant to note that $P_2^1$ was originally investigated as a semigroup by Lee, who first questioned if it generates a HFB variety of semigroups; see Edmunds et al. [4, Question 4.2] and Lee [21, Problem 1.4]. The answer to this question, however, remains elusive.

This article consists of six sections. Background information and some basic results are first given in Section 2. Then several important results on the variety $P$ and its subvarieties are established in Section 3. Results that are crucial to the proof of the main results are established in Section 4. Limit subvarieties of $P$ are then classified in Section 5, while results concerning subvarieties of $P_2^1$ are established in Section 6.

Many identities will be introduced and used throughout this article. For the reader’s convenience, these identities are collected in the appendix for quick referencing.

2. Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to Burris and Sankappanavar [3] for more information.

2.1. Words, identities, and deduction. Let $A^*$ denote the free monoid over a countably infinite alphabet $A$. Elements of $A$ are called letters and elements of $A^*$ are called words. The empty word, denoted by $\lambda$, is the identity element of $A^*$.

An identity is written as $u \approx v$, where $u, v \in A^*$; it is non-trivial if $u \neq v$. An identity $u \approx v$ is directly deducible from an identity $s \approx t$ if there exist some words $a, b \in A^*$ and substitution $\phi: A \to A^*$ such that $\{u, v\} = \{a\phi(s)b, a\phi(t)b\}$. A non-trivial identity $u \approx v$ is deducible from a set $\Sigma$ of identities if there exists some finite sequence $u = w_0, w_1, \ldots, w_m = v$ of distinct words such that each identity $w_i \approx w_{i+1}$ is directly deducible from some identity in $\Sigma$.

The following assertion is a specialization for monoids of a well-known universal-algebraic fact (see Burris and Sankappanavar [3, Theorem II.14.19]).

Proposition 2.1. Let $V$ be the variety defined by some set $\Sigma$ of identities. Then $V$ satisfies an identity $u \approx v$ if and only if $u \approx v$ is deducible from $\Sigma$. \hfill $\square$

2.2. $k$-decomposition of a word. Here we introduce a series of notions such as $k$-divider, $k$-block, $k$-decomposition, and some others, which appeared in Gusev and Vernikov [10, Chapter 3].

The content of a word $w$, denoted by $\text{con}(w)$, is the set of all letters occurring in $w$. A letter is simple [respectively, multiple] in a word $w$ if it occurs in $w$ once [respectively, at least twice]. The set of all simple [respectively, multiple] letters in a word $w$ is denoted by $\text{sim}(w)$ [respectively, $\text{mul}(w)$]. If $w$ is a word and $X$ is a set of letters then $w_X$ denotes the word obtained from $w$ by deleting all letters from $X$. If $X = \{x\}$, then we write $w_x$ rather than $w_{\{x\}}$. For a word $w$ and letters $x_1, x_2, \ldots, x_k \in \text{con}(w)$, let $w(x_1, x_2, \ldots, x_k)$ denote the word obtained from $w$ by retaining the letters $x_1, x_2, \ldots, x_k$. Equivalently,

$$w(x_1, x_2, \ldots, x_k) = w_{\text{con}(w) \setminus \{x_1, x_2, \ldots, x_k\}}.$$  

Let $w$ be a word such that $\text{sim}(w) = \{t_1, t_2, \ldots, t_m\}$. We may assume without loss of generality that $w(t_1, t_2, \ldots, t_m) = t_1 t_2 \cdots t_m$. Then

$$w = t_0 w_0 t_1 w_1 \cdots t_m w_m,$$

(2.1)
where \( w_0, w_1, \ldots, w_m \in \mathbb{A}^* \) and \( t_0 = \lambda \). The words \( w_0, w_1, \ldots, w_m \) are 0-blocks of \( w \), while \( t_0, t_1, \ldots, t_m \) are 0-dividers of \( w \). The representation of the word \( w \) as a product of alternating 0-dividers and 0-blocks, starting with the 0-divider \( t_0 \) and ending with the 0-block \( w_m \), is called the 0-decomposition of \( w \).

For \( k \geq 1 \), we define the \( k \)-decomposition of a word \( w \) recursively as follows. Suppose that (2.1) is the \((k-1)\)-decomposition of \( w \) with \((k-1)\)-dividers \( t_0, t_1, \ldots, t_m \) and \((k-1)\)-blocks \( w_0, w_1, \ldots, w_m \). For any \( i = 0, 1, \ldots, m \), let \( s_{i1}, s_{i2}, \ldots, s_{ir_i} \) be all the letters in \( \text{sim}(w_i) \) that do not occur to the left of \( w_i \). We may assume that \( \mu(w_i(s_{i1}, s_{i2}, \ldots, s_{ir_i})) = s_{i1}s_{i2}\cdots s_{ir_i} \). Then

\[
(2.2) \quad w_i = v_{i0}s_{i1}v_{i1}s_{i2}v_{i2}\cdots s_{ir_i}v_{ir_i}
\]

for some \( v_{i0}, v_{i1}, \ldots, v_{ir_i} \in \mathbb{A}^* \). Put \( s_{i0} = t_i \). Then the words \( v_{i0}, v_{i1}, \ldots, v_{ir_i} \) are \( k \)-blocks of \( w \), while the letters \( s_{i0}, s_{i1}, \ldots, s_{ir_i} \) are \( k \)-dividers of \( w \).

As noted in Gusev and Vernikov [10, Remark 3.1], only the first occurrence of a letter in a given word might be a \( k \)-divider of the word for some \( k \). In view of this observation, we use below an expression like “a letter \( x \) is (or is not) a \( k \)-divider of a word \( w \)” to mean that the first occurrence of \( x \) in \( w \) has the specified property.

Now decompose every \((k-1)\)-block \( w_i \) from (2.1) in the form (2.2). Then the word \( w \) in (2.1) is decomposed as a product of alternating \( k \)-dividers and \( k \)-blocks, starting with the \( k \)-divider \( s_{00} = t_0 \) and ending with the \( k \)-block \( v_{mr_m} \). This is called the \( k \)-decomposition of \( w \).

**Remark 2.2.** Since the length of the word \( w \) is finite, there is a number \( k \) such that the \( k \)-decomposition of \( w \) coincides with its \( n \)-decomposition for each \( n > k \).

We say that the \( k \)-decomposition of \( w \) is maximal if it coincides with the \( n \)-decomposition of \( w \) for all \( n > k \).

Let \( \text{occ}_x(w) \) denote the number of occurrences of a letter \( x \) in a word \( w \). For a given word \( w \), a letter \( x \in \text{con}(w) \), a natural number \( i \leq \text{occ}_x(w) \), and an integer \( k \geq 0 \), we denote by \( h^k_x(w, x) \) the right-most \( k \)-divider of \( w \) that precedes to the \( i \)th occurrence of \( x \) in \( w \).

For a given word \( w \) and a letter \( x \in \text{con}(w) \), the depth of \( x \) in \( w \) is the number \( D(w, x) \) defined as follows. If \( x \in \text{sim}(w) \), then we put \( D(w, x) = 0 \). Suppose now that \( x \in \text{mul}(w) \). If there is a natural \( k \) such that the first and the second occurrences of \( x \) in \( w \) lie in different \((k-1)\)-blocks of \( w \), then the depth of \( x \) in \( w \) is equal to the minimal number \( k \) with such a property. If, for any natural \( k \), the first and the second occurrences of \( x \) in \( w \) lie in the same \( k \)-block of \( w \), then we put \( D(w, x) = \infty \). In other words, \( D(w, x) = k \) if and only if \( h^{-1}_1(w, x) \neq h^{-1}_2(w, x) \) and \( k \) is the least number with such a property, while \( D(w, x) = \infty \) if and only if \( h^{-1}_1(w, x) = h^{-1}_2(w, x) \) for any \( k \).

**Lemma 2.3** (Gusev and Vernikov [10, Lemma 3.7]). A letter \( t \) is a \( k \)-divider of a word \( w \) if and only if \( D(w, t) \leq k \). \( \square \)

The \( i \)th occurrence of a letter \( x \) in a word \( w \) is denoted by \( i_w x \). We write \((i_w x) < (j_w y)\) to indicate that within \( w \), the \( i \)th occurrence of \( x \) precedes the \( j \)th occurrence of \( y \).

**Lemma 2.4** (Gusev and Vernikov [10, Lemma 3.9]). Let \( w \) be a word, \( x \) be a multiple letter in \( w \) with \( D(w, x) = k \) and \( t \) be a \((k-1)\)-divider of \( w \).

(i) If \( t = h^{-1}_2(w, x) \), then \((1_w x) < (1_w t)\).
(ii) If \((1_1w) < (1_1w^t) < (2_1w)\), then \(D(w, t) = k - 1\). Further, if \(k > 1\), then \((2_1w) < (2_1w^t)\). □

Lemma 2.5 (Gusev and Vernikov [10, Lemma 3.13]). Let \(w\) be a word, \(r > 1\) be a number and \(y\) be a letter such that \(D(w, y) = r - 2\). If \((1_1w^z) < (1_1w^y)\) for some letter \(z\) with \(D(w, z) \geq r\), then \((2_1w^z) < (1_1w^y)\). □

In the following, to facilitate understanding of the form of words, we will sometimes explicitly display the occurrence of a letter, for example,

\[
W = x_1 x_2 x_1 x_3 x_2 x_1 .
\]

Lemma 2.6 (Gusev and Vernikov [10, Lemma 3.14]). Let \(u \approx v\) be an identity and \(\ell\) be a natural number. Suppose that

\[
\begin{align*}
(2.3) & \quad \text{sim}(u) = \text{sim}(v) \text{ and } \text{mul}(u) = \text{mul}(v), \\
(2.4) & \quad h_i^r(u, x) = h_i^r(v, x) \text{ for } i = 1, 2 \text{ and all } x \in \text{con}(u),
\end{align*}
\]

and there is a letter \(x_0\) such that \(D(u, x_0) = \ell\). Then there are letters \(x_0, x_1, \ldots, x_{\ell - 1}\) such that \(D(u, x_s) = D(v, x_s) = s\) for any \(0 \leq s < \ell\) and the identity \(u \approx v\) has the form

\[
\begin{align*}
& u_{2_1\ell + 1} x_\ell u_{2_1\ell} x_{\ell - 1} u_{2_1\ell - 1} x_{\ell} u_{2_1\ell - 2} x_{\ell - 2} u_{2_1\ell - 3} x_{\ell - 3} u_{2_1\ell - 4} x_{\ell - 3} \ldots \\
& \hspace{1cm} \cdot u_{2_1\ell - 5} x_{\ell - 2} \ldots u_4 x_1 u_3 x_2 u_2 x_0 u_1 x_1 u_0 \ldots \\
& \approx v_{2_1\ell + 1} x_\ell v_{2_1\ell} x_{\ell - 1} v_{2_1\ell - 1} x_{\ell} v_{2_1\ell - 2} x_{\ell - 2} v_{2_1\ell - 3} x_{\ell - 3} v_{2_1\ell - 4} x_{\ell - 3} \ldots \\
& \hspace{1cm} \cdot v_{2_1\ell - 5} x_{\ell - 2} \ldots v_4 x_1 v_3 x_2 v_2 x_0 v_1 x_1 v_0
\end{align*}
\]

for some \(u_0, u_1, \ldots, u_{2_1\ell + 1}, v_0, v_1, \ldots, v_{2_1\ell + 1} \in A^*\). □

Remark 2.7. Analyzing the proof of Lemma 3.14 in Gusev and Vernikov [10], one can notice that, in Lemma 2.6, the letters \(x_0, x_1, \ldots, x_{\ell - 1}\) can be chosen so that \(u_{2_1s + 2}\) and \(v_{2_1s + 2}\) do not contain any \(s\)-dividers of \(u\) and \(v\), respectively, for any \(s = 0, 1, \ldots, \ell - 1\). □

Evidently, if \(u = v\), then (2.3) and (2.4) are true for all \(\ell\). So, one can apply Lemma 2.6 and Remark 2.7 for the trivial identity and obtain the following corollary.

Corollary 2.8. Let \(u\) be a word and \(k\) be a natural number. Suppose that there is a letter \(x_0\) such that \(D(u, x_0) = k\). Then there exist letters \(x_0, x_1, \ldots, x_{k - 1}\) such that the word \(u\) has the form

\[
\begin{align*}
& u_{2k + 1} x_k u_{2k} x_{k - 1} u_{2k - 1} x_k u_{2k - 2} x_{k - 2} u_{2k - 3} x_{k - 3} u_{2k - 4} x_{k - 3} \ldots \\
& \hspace{1cm} \cdot u_{2k - 5} x_{k - 2} \ldots u_4 x_1 u_3 x_2 u_2 x_0 u_1 x_1 u_0
\end{align*}
\]

for some \(u_0, u_1, \ldots, u_{2k + 1} \in A^*\) such that \(D(u, x_s) = s\) and \(u_{2_1s + 2}\) does not contain any \(s\)-dividers of \(u\) for any \(s = 0, 1, \ldots, k - 1\). □

Lemma 2.9. Let \(k, m\) be natural numbers and \(u\) be a word of the form

\[
\begin{align*}
& u_{2k + 2} y_m u_{2k + 1} x_k u_{2k} x_{k - 1} u_{2k - 1} x_k u_{2k - 2} x_{k - 2} u_{2k - 3} x_{k - 3} u_{2k - 4} x_{k - 3} \ldots \\
& \hspace{1cm} \cdot u_{2k - 5} x_{k - 2} \ldots u_4 x_1 u_3 x_2 u_2 x_0 u_1 x_1 u_0,
\end{align*}
\]
Proof. Since $D(u; y_m) = m > 0$, we have $y_m \in \text{mul}(u)$. Let $y_{m-1} = h_1^{m-1}(u; y_m)$. Since $(1u; y_m) < (1u; y_{m-1}) < (2u; y_m)$ by Lemma 2.4(i), the letter $1u; y_{m-1}$ does not occur in $u_{2k+2}$. By the hypothesis, $1u; y_{m-1}$ does not occur in $u_{2k+1}$. Since $(1u; y_{m-1}) < (2u; y_m)$, the letter $2u; y_m$ occurs in neither $u_{2k+2}$ nor $u_{2k+1}$. Then $(1u; x_k) < (2u; y_m)$. It follows that $m \leq k + 1$. If $m = k + 1$, then $2u; y_m$ occurs in $u_{2m-2} = u_{2k}$ because $h_1^k(u; y_m) \neq h_2^k(u; y_m)$ otherwise. If $m = k$, then $2u; y_m$ occurs in either $u_k$ or $u_{2k-1}$ or $u_{2k-2}$ because $h_1^{k-1}(u; y_m) \neq h_2^{k-1}(u; y_m)$ otherwise. If $k > m$, then $1u; y_{m-1}$ and so $2u; y_m$ do not occur in both $u_{2k}$ and $u_{2k-1}$ because $h_1^{k-1}(u; x_k) \neq h_2^{k-1}(u; x_k)$ otherwise. Further, if $k > m + 1$, then $1u; y_{m-1}$ and so $2u; y_m$ do not occur in the subwords $u_{2r+1}$, $u_{2r}$, and $u_{2r-1}$ of $u$ for any $r = m + 1, m + 2, \ldots, k - 1$ because $h_1^{k-2}(u; x_r) \neq h_2^{k-2}(u; x_r)$ otherwise. Finally, if $m > 1$, then $(2u; y_m) < (1u; x_{m-2})$ by Lemma 2.5. This means that if $m > 1$, then $2u; y_m$ does not occur in $u_{2m-3}, u_{2m-4}, \ldots, u_0$. Therefore, $2u; y_m$ occurs in either $u_{2m}$ or $u_{2m-1}$ or $u_{2m-2}$. \hfill \qed

2.3. Some known facts. For an identity system $\Sigma$, we denote by $\text{var} \Sigma$ the variety of monoids defined by $\Sigma$. It is well known that the variety

$$LRB = \text{var}\{xy \approx x y x\}$$

of left regular bands is generated by the monoid obtained by adjoining an identity element to the left zero semigroup of order two. The initial part of a word $w$, denoted by $\text{ini}(w)$, is the word obtained from $w$ by retaining the first occurrence of each letter. The following statement is well known and can be easily verified.

Lemma 2.10. A non-trivial identity $u \approx v$ holds in the variety $LRB$ if and only if $\text{ini}(u) = \text{ini}(v)$. \hfill \qed

We use the standard symbol $\mathbb{N}$ to denote the set of all natural numbers. For any $s \in \mathbb{N}$ and $1 \leq q \leq s$, put

$$b_{s,q} = x_{s-1}x_s x_{s-2}x_{s-1} \cdots x_{q-1}x_q.$$ 

For brevity, we will write $b_s$ rather than $b_{s,1}$. We put also $b_0 = \lambda$ for convenience. We introduce the following four countably infinite series of identities:

$$\alpha_k : x_{k} x_{k-1} x_{k} y_{k} y_{k} b_{k-1} = y_{k} x_{k} x_{k-1} x_{k} y_{k} b_{k-1},$$

$$\beta_k : x_{k} x_{k} b_{k} \approx x_{k} x^{2} b_{k},$$

$$\gamma_k : y_{1} y_{0} x_{k} b_{k} \approx y_{1} y_{0} y_{1} x_{k} b_{k},$$

$$\epsilon^{m}_{k} : y_{m+1} y_{m} x_{k} y_{m+1} b_{k} = y_{m+1} y_{m} x_{k} b_{k} = y_{m+1} y_{m} y_{m+1} x_{k} b_{k} = y_{m+1} y_{m} y_{m} b_{m-1},$$

where $k \in \mathbb{N}$ and $1 \leq m \leq k$. The subvariety of a variety $V$ defined by a set $\Sigma$ of identities is denoted by $V\Sigma$. Let

$$F = F\{x^2 y^2 \approx y^2 x^2\},$$

$$F_k = F\{\alpha_k\},$$

$$H_k = F\{\beta_k\},$$

$$I_k = F\{\gamma_k\},$$

and $J^{m}_{k} = F\{\epsilon^{m}_{k}\},$
where \( k \in \mathbb{N} \) and \( 1 \leq m \leq k \). The trivial variety of monoids is denoted by \( T \). Let \( SL \) denote the variety of all semilattice monoids. Put

\[
C = \text{var}\{x^2 \approx x^3, xy \approx yx\},
\]
\[
D = \text{var}\{x^2 \approx x^3, x^2y \approx yx \approx y^2x^2\},
\]
\[
E = \text{var}\{x^2 \approx x^3, x^2y \approx yx, x^2y^2 \approx y^2x^2\}.
\]

The subvariety lattice of a monoid variety \( V \) is denoted by \( \mathcal{L}(V) \).

**Lemma 2.11** (Gusev and Vernikov [10, Proposition 6.1]). The chain

\[
T \subset SL \subset C \subset D \subset E \subset F_1 \subset H_1 \subset I_1 \subset J_1^1
\]
\[
\subset F_2 \subset H_2 \subset I_2 \subset J_2^1 \subset J_2^2
\]
\[
\vdots
\]
\[
\subset F_k \subset H_k \subset I_k \subset J_k^1 \subset J_k^2 \subset \cdots \subset J_k^k
\]
\[
\vdots
\]
\[
\subset F
\]

is the lattice \( \mathcal{L}(F) \). \( \Box \)

**Lemma 2.12** (Gusev and Vernikov [10, Proposition 6.9(i)]). A non-trivial identity \( u \approx v \) holds in the variety \( F_k \) if and only if (2.3) and (2.4) hold with \( \ell = k \).

An identity \( u \approx v \) is \( k \)-well-balanced if the words \( u \) and \( v \) have the same set of \( k \)-dividers, these \( k \)-dividers appear in \( u \) and \( v \) in the same order, and \( iu \) occurs in some \( k \)-block of \( u \) if and only if \( \nu x \) occurs in the corresponding \( k \)-block of \( v \) for any letter \( x \) and \( i \in \{1, 2\} \).

**Corollary 2.13.** Let \( u \approx v \) be an identity of \( F_k \). Then the identity \( u \approx v \) is \((k - 1)\)-well-balanced.

**Proof.** Let \( t_0 u_0 t_1 u_1 \cdots t_m u_m \) be the \((k - 1)\)-decomposition of \( u \). According to Lemma 2.12, (2.3) and (2.4) hold with \( \ell = k \). Then the \((k - 1)\)-decomposition of \( v \) has the form \( t_0 v_0 t_1 v_1 \cdots t_m v_m \) by Gusev and Vernikov [10, Lemma 3.8]. Suppose that \( iu \) occurs in the \((k - 1)\)-block \( u_i \) of \( u \), where \( i \in \{1, 2\} \) and \( j \in \{0, 1, \ldots, m\} \). Then \( h_{i-1}^k(u, x) = t_j \). Since (2.4) holds with \( \ell = k \), we have \( h_{i-1}^k(v, x) = t_j \) and so \( \nu x \) occurs in the \((k - 1)\)-block \( v_j \) of \( u \). Therefore, the identity \( u \approx v \) is \((k - 1)\)-well-balanced. \( \Box \)

3. Some properties of the variety \( P \)

Recall that the variety \( P \) is defined by the identity

\[
xsxt \approx xsxtx.
\]

For any word \( w \), let \( ini_2(w) \) denote the word obtained from \( w \) by retaining the first and second occurrences of each letter.

The following result is evident.

**Lemma 3.1.** For any \( w \in A^* \), the identity \( w \approx ini_2(w) \) holds in the variety \( P \). \( \Box \)
Let \( \phi: \mathfrak{A} \to \mathfrak{A}^* \) be a substitution, \( x_1, x_2, \ldots, x_n \in \mathfrak{A} \) and \( w_1, w_2, \ldots, w_n \in \mathfrak{A}^* \). For brevity, we will say that \( \phi \) is the substitution

\[
(x_1, x_2, \ldots, x_n) \mapsto (w_1, w_2, \ldots, w_n)
\]

if \( \phi(x_i) = w_i \) for any \( i = 1, 2, \ldots, n \) and \( \phi(a) = a \) otherwise.

**Lemma 3.2.** Let \( \mathbf{V} \) be a subvariety of \( \mathbf{P} \).

(i) If \( \mathbf{LRB} \not\subseteq \mathbf{V} \), then \( \mathbf{V} \subseteq \mathbf{F} \).

(ii) If \( \mathbf{F}_1 \not\subseteq \mathbf{V} \), then \( \mathbf{V} \subseteq \mathbf{LRB} \lor \mathbf{C} = \text{var}\{x^2 \approx x^3, x^2y \approx xyx\} \).

**Proof.** (i) It follows from Lee et al. [23, Theorem 5.17] that \( \mathbf{V} \) satisfies the identity \( x^2(y^2x^2)^2 \approx (y^2x^2)^2 \). It is then easily shown that \( \mathbf{V} \) satisfies \( x^2y^2 \approx y^2x^2 \). Thus, \( \mathbf{V} \subseteq \mathbf{F} \).

(ii) If \( \mathbf{C} \not\subseteq \mathbf{V} \), then by Gusev and Vernikov [10, Corollary 2.6], the variety \( \mathbf{V} \) is completely regular, that is, it consists of unions of groups. Then \( \mathbf{V} \) satisfies the identity \( x \approx x^n \) for some \( n \geq 2 \). Clearly, this identity together with (3.1) imply the identity \( xy \approx xyx \), whence \( \mathbf{V} \subseteq \mathbf{LRB} \). Therefore, we may assume that \( \mathbf{C} \subseteq \mathbf{V} \).

There is an identity \( u \approx v \) of \( \mathbf{V} \) that is not satisfied by \( \mathbf{F}_1 \). According to Gusev and Vernikov [10, Proposition 2.2] and Lemma 2.12, (2.3) holds but (2.4) with \( \ell = 1 \) does not. Then \( h_3^2(u, x) \neq h_3^2(v, x) \) for some \( x \in \text{con}(u) \) and \( i \in \{1, 2\} \). If \( i = 1 \), then we can multiply both sides of the identity \( u \approx v \) on the left by \( xt \), where \( t \notin \text{con}(u) \).

So, we may assume that \( i = 2 \). Put \( y = h_3^2(u, x) \) and \( z = h_3^2(v, x) \). If \( y \) and \( z \) occur in \( u \) and \( v \) in different order, then \( \mathbf{V} \) is commutative and so \( \mathbf{V} \subseteq \mathbf{C} \). Therefore, we may assume without loss of generality that \( \langle u \rangle \not< \langle v \rangle \) and \( \langle v \rangle \neq \langle u \rangle \).

Then the identity \( u(x, y) \approx v(x, y) \) is equivalent modulo (3.1) to \( xyx \approx x^2y \). It follows from Lee [20, Lemma 3.3] that \( \mathbf{LRB} \lor \mathbf{C} = \text{var}\{x^2 \approx x^3, x^2y \approx xyx\} \). So, \( \mathbf{V} \subseteq \mathbf{LRB} \lor \mathbf{C} \). \( \square \)

**Lemma 3.3.** Let \( \mathbf{V} \) be a subvariety of \( \mathbf{P} \) that does not contain \( \mathbf{F}_{k+1} \). Then \( \mathbf{V} \) satisfies the identity

\[
\kappa_k: \quad xx_kxb_k \approx x^2x_kb_k.
\]

**Proof.** If \( \mathbf{LRB} \not\subseteq \mathbf{V} \), then \( \mathbf{V} \subseteq \mathbf{F} \) by Lemma 3.2(i). Since \( \mathbf{F}_{k+1} \not\subseteq \mathbf{V} \), it follows from Lemma 2.11 that \( \mathbf{V} \subseteq \mathbf{J}_k^k \). It is verified in Gusev and Vernikov [10, Lemma 6.3] that \( \mathbf{J}_k^k \) satisfies \( \kappa_k \). Then \( \mathbf{V} \) satisfies \( \kappa_k \) as well. Therefore, we may assume that \( \mathbf{LRB} \subseteq \mathbf{V} \). Further, if \( \mathbf{F}_1 \not\subseteq \mathbf{V} \), then \( \mathbf{V} \) satisfies \( x^2y \approx xyx \) by Lemma 3.2(ii).

Clearly, \( x^2y \approx xyx \) implies \( \kappa_k \). Therefore, we may assume that \( \mathbf{F}_1 \subseteq \mathbf{V} \).

Let \( r \) be the least number such that \( \mathbf{F}_r \subseteq \mathbf{V} \). Then \( \mathbf{F}_{r+1} \not\subseteq \mathbf{V} \). Evidently, \( r \leq k \). There is an identity \( u \approx v \) of \( \mathbf{V} \) that is not satisfied by \( \mathbf{F}_{r+1} \). According to Lemma 2.12, (2.3) and (2.4) hold with \( \ell = 1, 2, \ldots, r \), while (2.4) with \( \ell = r+1 \) does not. Then \( h_1^2(u, x) \neq h_1^2(v, x) \) for some \( x \in \text{con}(u) \) and \( i \in \{1, 2\} \). If \( i = 1 \), then we multiply both sides of the identity \( u \approx v \) on the left by \( xt \), where \( t \notin \text{con}(u) = \text{con}(v) \).

So, we may assume that \( i = 2 \). Put \( a = h_1^2(u, x) \) and \( b = h_1^2(v, x) \). We may assume without loss of generality that \( \langle u \rangle \not< \langle v \rangle \). Then \( \langle v \rangle < \langle u \rangle \) by Lemma 2.10.

In view of Lemma 2.3, \( D(u, a) = s \) for some \( s \leq r \). Put \( x_s = a \). According to Lemma 2.6, there exist letters \( x_0, x_1, \ldots, x_{s-1} \) such that \( D(u, x_j) = D(v, x_j) = j \).
for any \(0 \leq j < s\) and the identity \(u \approx v\) is of the form

\[
\begin{align*}
&u_{2s+1} \cdot u_{2s} \cdot u_{2s-1} \cdot u_{2s-2} \cdot u_{2s-3} \cdot u_{2s-4} \cdot x_{s-3} \\
&\quad \cdot u_{2s-5} \cdot x_{s-2} \cdots u_1 \cdot x_1 \cdot x_0 \cdot u_1 \cdot x_1 \cdot u_0 \\
&\approx v_{2s+1} \cdot v_{2s} \cdot v_{2s-1} \cdot v_{2s-2} \cdot v_{2s-3} \cdot v_{2s-4} \cdot x_{s-3} \\
&\quad \cdot v_{2s-5} \cdot x_{s-2} \cdots v_1 \cdot x_1 \cdot v_1 \cdot x_1 \cdot v_0
\end{align*}
\]

for some \(u_0, u_1, \ldots, u_{2s+1}, v_0, v_1, \ldots, v_{2s+1} \in \mathbb{A}^*\). Clearly, \(1u x\) occurs in the subword \(u_{2s+1}\) of \(u\), while \(1v x\) and \(2v x\) occur in the subword \(v_{2s+1}\) of \(v\). Since \(x_s = h_2^*(u, x)\) and \(x_{s-1}\) is an \(r\)-divider by Lemma 2.3, the letter \(2u x\) occurs in \(u_{2s}\). Therefore, the identity

\[u(x, x_1, x_2, \ldots, x_s) \approx v(x, x_1, x_2, \ldots, x_s)\]

is equivalent modulo (3.1) to \(\kappa_s\). Clearly, \(\kappa_s\) implies \(\kappa_k\). Therefore, \(\kappa_k\) is satisfied by \(V\). \(\square\)

For any word \(w\), let \(\text{ini}^2(w)\) denote the word obtained from \(\text{ini}(w)\) by replacing each letter \(x\) with \(x^2\). For example, if \(w = x^3 y z y z x\), then \(\text{ini}(w) = xyz\) and so \(\text{ini}^2(w) = x^2 y^2 z^2\). A word \(w\) is said to be 2-limited if \(\text{occ}_x(w) \leq 2\) for any letter \(x\).

**Lemma 3.4.** Let \(w\) be a word that does not contain simple letters. Then the identities (3.1) and

\[ (xy)^2 \approx x^2 y^2. \tag{3.2} \]

imply the identity \(w \approx \text{ini}^2(w)\).

**Proof.** First, we note that \(P\{3.2\}\) satisfies the identity

\[ x_1^2 x_2^2 \cdots x_n^2 \approx (x_1 x_2 \cdots x_n)^2 \tag{3.3} \]

for any \(n \geq 2\) because the identities

\[ (x_1 x_2 \cdots x_n)^2 \approx x_1^2 (x_2 x_3 \cdots x_n)^2 \approx x_1^2 x_2^2 (x_3 x_4 \cdots x_n)^2 \approx \cdots \approx x_1^2 x_2^2 \cdots x_n^2 \]

follow from (3.2).

In view of Lemma 3.1, we may assume that the word \(w\) is 2-limited. We say that a letter \(x\) forms an island in \(w\) if \(w = w_1 x^2 w_2\) for some words \(w_1, w_2 \ {\mathbb{A}^*}\) such that \(x \notin \text{con}(w_1w_2)\). Let \(k\) be the number of distinct letters that do not form an island in \(w\). We will use induction on \(k\).

**Induction base:** \(k = 0\). Then \(w = \text{ini}^2(w)\), and we are done.

**Induction step:** \(k > 0\) and \(P\{3.2\}\) satisfies the identity \(v' \approx \text{ini}^2(v')\) for any 2-limited word \(v'\) with at most \(k-1\) letters that do not form an island in \(v'\). Then, since \(w\) is 2-limited, there is a letter \(x\) such that \(w = w_1 x y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n} x w_2\), where
Let $w = w_1x_{k_1}y_1^2y_2^2 \cdots y_n^2xw_2$ be a word such that the $k$-decomposition of $w$ is maximal and $w$ satisfies the identity $w \approx w_1x(y_1y_2 \cdots y_n)^2x(y_1y_2 \cdots y_n)^2w_2$.

Corollary 3.5. Let $w$ be a word, $k$ be a number such that the $k$-decomposition of $w$ is maximal and $a$ be a $k$-block of $u$. If $w = w'aw''$, then the variety $P\{\text{(3.2)}\}$ satisfies the identity $w \approx w' \text{ini}^2(a)w''$.

Proof. Let $x$ be a letter such that $1_xw$ occurs in the $k$-block of $a$ of $w$. If $2_xw$ does not occur in $a$, then $h_1^k(u, x) \neq h_2^k(w, x)$ and so $x$ is a $(k+1)$-divider of $w$. This contradicts the fact that the $k$-decomposition of $w$ is maximal. Therefore, $2_xw$ occurs in $a$ as well. We see that $\text{con}(a) \subseteq \text{mul}(w'a)$. Then we can apply the identity (3.1) and add some occurrences of letters in the $k$-block $a$ so that the resulting $k$-block $b$ contains multiple letters only and $\text{ini}(a) = \text{ini}(b)$. Then identity $b \approx \text{ini}^2(b)$ follows from (3.1) and (3.2) by Lemma 3.4. Evidently, $\text{ini}^2(a) = \text{ini}^2(b)$. We see that the identities $w = w'aw'' \approx w'bw'' \approx w' \text{ini}^2(b)w'' = w' \text{ini}^2(a)w''$ follow from the identities (3.1) and (3.2), and we are done. 

4. Critical pairs

An identity $u \approx v$ is said to be balanced if $\text{occ}_u(u) = \text{occ}_v(v)$ for any $x \in A$. We say that a pair $\{i_x, j_y\}$ of occurrences of letters $x$ and $y$ in a balanced identity $u \approx v$ is critical if $u$ contains $i_u x_{j_u} y$ as a subword and $j_v y$ precedes $i_v x$ in $v$. Let $w$ denote the word obtained from $u$ by replacing $i_u x_{j_u} y$ with $j_u y_{i_u} x$. Given a set $\Delta$ of identities and a balanced identity $u \approx v$, we say that the critical pair $\{i_x, j_y\}$ is $\Delta$-removable in $u \approx v$ if $\Delta$ implies $u \approx w$.

The following special case of Sapir [31, Lemma 3.4], which can be easily verified by induction, describes the standard method of deriving identities by removing critical pairs. This method traces back to Jackson and Sapir [13] and Sapir [30], for instance.

Lemma 4.1. Let $V$ be a monoid variety and $\Delta$ be a set of identities. Suppose that each critical pair in every balanced identity of $V$ is $\Delta$-removable. Then every balanced identity of $V$ can be derived from $\Delta$. 

$\square$
4.1. **Removing critical pairs of the form** \( \{1x, 2y\} \). If \( \phi: \mathbb{A} \to \mathbb{A}^* \) is a substitution and \( \varepsilon \) denote an identity \( u \approx v \), then, for convenience, we will write \( \phi(\varepsilon) \) rather than \( \phi(u) \approx \phi(v) \).

The following lemma generalizes Gusev and Vernikov [10, Lemma 6.3].

**Lemma 4.2.** For any \( k \in \mathbb{N} \), the identities \( \kappa_k \) and \( \delta_k^k \) are equivalent within \( P \).

**Proof.** Let \( \phi \) be the substitution \((y_k, y_{k+1}) \mapsto (1, x)\). Then the identity \( \phi(\delta_k^k) \) is equal to \( \kappa_k \). Therefore, \( \delta_k^k \) implies \( \kappa_k \). Further, \( \delta_k^k \) follows from (3.1) and \( \kappa_k \) because

\[
y_{k+1}y_kx_{k}x_{k+1}b_{k,k}y_kb_{k-1} = y_{k+1}y_kx_{k}y_{k+1}x_{k-1}x_ky_kb_{k-1}
\]

(3.1)

\[
y_{k+1}y_kx_{k}y_{k+1}x_{k-1}x_ky_kc \\ 
\approx y_{k+1}y_kx_{k}x_{k-1}x_ky_kc
\]

\( \approx y_{k+1}y_kx_{k}x_{k-1}x_ky_kd \\ 
\frac{\varepsilon_k^k}{(3.1)} \\
\left( y_{k+1}y_kx_{k}x_{k-1}x_ky_kb_{k-1}
\right)
\]

where

\[
c = \begin{cases}
\lambda & \text{if } k = 1, \\
x_k-2x_{k-1}x_kb_{k-2} & \text{if } k > 1
\end{cases}
\]

and

\[
d = \begin{cases}
\lambda & \text{if } k = 1, \\
x_k-2x_{k-1}x_kb_{k-2} & \text{if } k > 1
\end{cases}
\]

The lemma is thus proved. \( \square \)

We introduce the following countably infinite series of identities:

\[
\varepsilon_{k-1} : y_kx_k-1x_ky_kb_{k-1} \approx y_kx_k-1y_kx^2b_{k-1},
\]

where \( k \in \mathbb{N} \).

**Lemma 4.3.** The following proper inclusions hold:

(i) \( P\{\gamma_1\} \subset P\{\gamma_2\} \subset \cdots \subset P\{\gamma_k\} \subset \cdots \subset P\{\varepsilon_0\} \);

(ii) \( P\{\gamma_k\} \subset P\{\delta_k^k\} \subset P\{\varepsilon_k^k\} \subset \cdots \subset P\{\delta_k^{k+1}\} \subset \cdots \subset P\{\varepsilon_k\} \);

(iii) \( P\{\varepsilon_0\} \subset P\{\varepsilon_1\} \subset \cdots \subset P\{\varepsilon_k\} \subset \cdots \subset P\{(3.2)\} \).

**Proof.** (i) Evidently, \( P\{\gamma_1\} \subset P\{\gamma_2\} \subset \cdots \subset P\{\gamma_k\} \subset \cdots \subset P\{\varepsilon_0\} \). These inclusions are proper because \( I_{k+1} \subset P\{\gamma_{k+1}\} \subset P\{\varepsilon_0\} \) but \( I_{k+1} \not\subset P\{\gamma_k\} \) for any \( k \in \mathbb{N} \).

(ii) The inclusion \( P\{\gamma_k\} \subset P\{\delta_k^k\} \) is evident. Let \( \phi_1 \) and \( \phi_2 \) be the substitutions

\[
(x_{m-1}, x_m, x_{m+1}, \ldots, x_k) \mapsto (x_m, x_{m+1}, x_{m-1}, x_m, x_{m+2}, \ldots, x_{k+1})
\]

and

\[
(x_{m-1}, x_m, x_{m+1}, \ldots, x_{k-1}, x_k, y_m) \mapsto (x_{m-1}, 1, 1, \ldots, 1, 1, x_m),
\]

respectively. Then the identities \( \phi_1(\delta_k^m) \) and \( \phi_2(\delta_k^m) \) are equivalent modulo (3.1) to \( \delta_{k+1}^m \) and \( \varepsilon_m \), respectively. Therefore, \( P\{\delta_k^m\} \subset P\{\delta_k^{k+1}\} \) and \( P\{\delta_k^m\} \subset P\{\varepsilon_m\} \). If \( 1 \leq m < k \) and \( \phi_2 \) is the substitution

\[
(x_{m-1}, y_m, y_{m+1}) \mapsto (y_m, x_{m-1}, y_{m+1}, y_{m+2}),
\]

respectively.
Lemma 2.3 implies that subwords for some . Then let the first occurrence of . Then the identity (4.1) □

Proof. We also omit the proof of Part (i) because it is very similar to (and in fact simpler) than that of Part (ii).

(ii) Put and . It follows from Corollary 2.8 that there exist letters such that for any and and

where

for some . Clearly, for any , there exist such that , where does not contain any -divider of , while either or is an -divider of . Put and . Let

for any . Let be the substitution

Then the identity coincides with the identity (4.1)

where

We note that . Since , Lemma 2.3 implies that are not -dividers of . If one of the subwords or of contains some -divider of , then (1u) < (2u) and so contradicting . Therefore, and do not contain any -divider of . Finally, the subword
\(v'_{2s}\) of \(u\) does not contain any \((s - 1)\)-dividers of \(u\) by its definition. We see that the subword \(q_s\) of \(u\) does not contain \((s - 1)\)-dividers of \(u\) for any \(s = 1, 2, \ldots, k - 1\).

Let \(z\) be a letter such that \(u_1z\) occurs in the subword \(v_{2k+1}\) of \(u\). Since \(D(u, y_{m+1}) = m + 1 > 1\), we have \(z \in \text{mul}(u)\). We are going to verify that \(z \in \text{mul}(v_{2k+1}q_{k-1} \cdots q_m q_{m-1})\). If \(m = 1\), then this claim is evident because \(v_{2k+1}q_{k-1} \cdots q_m q_{m-1}\) is a suffix of \(u\) in this case. So, we may assume that \(m > 1\). Let \(z' = h(q_{m-2})\). Since \(z' = h(v_{2m-2}')\) or \(z' = x_{m-2}\), the letter \(z'\) is an \((m - 2)\)-divisor of \(u\). If \((1u z') < (2u z)\), then \(h_1^{m-2}(u, z) \neq h_2^{m-2}(u, z)\) and so \(D(u, z) \leq m - 1\), whence \(z\) is an \((m - 1)\)-divisor of \(u\) by Lemma 2.3. But this contradicts the fact that \(D(u, y_{m+1}) = m + 1\) because \((1u y_{m+1}) < (2u z)\). Therefore, \((2u z) < (1u z')\). We see that if a letter occurs in \(v_{2k+1}\), then the subword \(v_{2k+1}x_k y_{m+1}q_{k-1} \cdots q_m q_{m-1}\) of \(u\) contains some non-first occurrence of this letter in \(u\).

Let \(z\) be a letter such that \(u_1z\) occurs in the subword \(q_s\) of \(u\) for some \(s \in \{1, 2, \ldots, k - 1\}\). We are going to verify that \(z \in \text{mul}(q_s q_{s-1})\). If \(s = 1\), then this claim is evident because \(q_1\) does not contain \(0\)-dividers of \(u\) and \(q_1 q_0\) is a suffix of \(u\). So, we may assume that \(s > 1\). Clearly, either \(z = x_s\) or \(z \in \text{con}(v_{2s+2}^\prime v_{2s+1} v_{2s+2})\). Evidently, if \(z = x_s\), then \(z \in \text{con}(q_s q_{s-1})\). Suppose now that \(z \in \text{con}(V_{2s+2}^\prime v_{2s+1} v_{2s})\). Since the subword \(q_s\) of \(u\) does not contain \((s - 1)\)-dividers of \(u\), this subword does not contain \(0\)-dividers of \(u\) as well by Lemma 2.3. Therefore, the letter \(z\) is multiple in \(u\). Let \(z' = h(q_{s-2})\). Since \(z' = h(v_{2s-2}')\) or \(z' = x_{s-2}\), the letter \(z'\) is an \((s - 2)\)-divisor of \(u\). If \((1u z') < (2u z)\), then \(h_1^{s-2}(u, z) \neq h_2^{s-2}(u, z)\) and so \(D(u, z) \leq s - 1\), whence \(z\) is an \((s - 1)\)-divisor of \(u\) by Lemma 2.3. But \(q_s\) does not contain \((s - 1)\)-dividers of \(u\). Therefore, \((2u z) < (1u z')\). This means that \(2u z\) occurs in the subword \(q_s q_{s-1}\) of \(u\). We see that if a letter occurs in \(q_s\), then the subword \(q_s q_{s-1}\) of \(u\) contains some non-first occurrence of this letter in \(u\).

Finally, we recall that \(q_2 x_k\) occurs in the subword \(q_{k-1}\) of \(u\).

In view of the previous three paragraphs, the identity (3.1) implies

\[u = v_{2k+2} y_{m+1} v_{2k+1} x_k y_{m+1} p \approx v_{2k+2} y_{m+1} v_{2k+1} x_k y_{m+1} q_u\]

By a similar argument we can show that the identity (3.1) implies

\[v_{2k+2} y_{m+1} v_{2k+1} x_k y_{m+1} x_k p \approx v_{2k+2} y_{m+1} v_{2k+1} x_{k+1} x_k q_u\]

These identities together with (4.1) imply the identity \(u \approx u' y u''\). Thus, it is satisfied by \(V\).

(iii) There are two cases.

Case 1: \(h(u') = x\). If \(k = 0\), then \(\varepsilon_k\) is nothing but the identity \(y_1 x_0 y_1 x \approx y_1 x_0 y_1 x\). Since \(y \in \text{con}(u')\), this identity implies \(u \approx u' y u''\). So, we may assume that \(k > 0\). Put \(y_{k+1} = y\). In view of Corollary 2.8, there are letters \(x_0, x_1, \ldots, x_k\) such that \(D(u, x_s) = s\) for any \(0 \leq s \leq k\) and

\[u = v_{2k+3} y_{k+1} v_{2k+2} x_k v_{2k+1} x y_{k+1} x p,\]

where

\[p = v_{2k} x_{k-1} v_{2k-1} x_k \cdots v_4 x_1 v_3 x_2 v_2 x_0 v_1 x_1 v_0\]

for some \(v_0, v_1, \ldots, v_{2k+3} \in \mathcal{A}^*\). Clearly, for any \(s = 1, 2, \ldots, k - 1\), there exist \(v'_s, v''_s \in \mathcal{A}^*\) such that \(v_{2s} = v'_s v''_s\) where \(v'_s\) does not contain any \((s - 1)\)-divider of \(u\), while either \(v''_s = \lambda\) or \(h(v''_s)\) is an \((s - 1)\)-divider of \(u\). Put \(v''_{2k} = v_{2k}\)
and \( v'_0 = v_0 \). Let
\[
q_s = v'_{2s+2}x_sv_{2s+1}x_{s+1}v'_{2s}
\]
for any \( s = 0, 1, \ldots, k - 1 \). Let \( \phi_2 \) be the substitution
\[
(x_0, x_1, \ldots, x_{k-1}, x_k) \mapsto (q_0, q_1, \ldots, q_{k-1}, V_2, v_{2k+2}xkV_{2k+1}).
\]
Then the identity \( \phi_2(\varepsilon_k) \) coincides with the identity
\[
y_{k+1}V_{2k+2}xkV_{2k+1}y_{k+1}x^2q \equiv y_{k+1}V_{2k+2}xkV_{2k+1}y_{k+1}x^2q
\]
where
\[
q = q_{k-1}v_{2k+2}xkV_{2k+1}q_{k-2}q_{k-1} \cdots q_1q_2q_0q_1.
\]
We note that \( p = q_{k-1}q_{k-2} \cdots q_0 \). By the same arguments as in Part (ii) one can show that
\begin{itemize}
  \item if a letter occurs in the word \( v_{2k+2}xkV_{2k+1} \), then the subword \( v_{2k+2}xkV_{2k+1}y_{k+1}x^2q_{k-1} \) of \( u \) contains some non-first occurrence of this letter in \( u \);
  \item if a letter occurs in \( q_s \), then the subword \( q_sq_{s-1} \) of \( u \) contains some non-first occurrence of this letter in \( u \).
\end{itemize}
Then the identity (3.1) implies
\[
u = v_{2k+3}y_{k+1}v_{2k+2}xkV_{2k+1}y_{k+1}x^2p \equiv v_{2k+3}y_{k+1}v_{2k+2}xkV_{2k+1}y_{k+1}x^2q.
\]
By a similar argument one can show that the identity (3.1) implies
\[
v_{2k+3}y_{k+1}v_{2k+2}xkV_{2k+1}y_{k+1}x^2p \equiv v_{2k+3}y_{k+1}v_{2k+2}xkV_{2k+1}y_{k+1}x^2q.
\]
These identities together with (4.2) imply the identity \( u \approx u'y_xu'' \). Thus, it is satisfied by \( V \).

**Case 2:** \( h(u'') \neq x \). In view of Remark 2.2, there is a number \( r \) such that the \( r \)-decomposition of \( u \) is maximal. Since \( D(u, x) = \infty \), the subword \( 1u_2x_2y \) is contained in some \( r \)-block \( a \) of \( u \). Then \( a = a_1x_2a_2 \) and \( u = u_1a_1a_2u_2 \) for some \( a_1, a_2, u_1, u_2 \in \mathfrak{A}^* \).

Let \( u^* = u_1a_1x_1y_xa_2u_2 \). The word \( u^* \) differs from \( u \) by the occurrences of \( x \) only. Then, since \( D(u, x) = D(u', x) = \infty \), the identity \( u \approx u^* \) is \( s \)-well-balanced for any \( s \). Since (3.2) is a consequence of \( \varepsilon_k \) by Lemma 4.3(iii), it follows from Corollary 3.5 that the identities
\[
u \approx u_1a_1x_1y_xa_2u_2 \quad \text{and} \quad u^* \approx u_1a_1x_1y_xa_2u_2
\]
hold in \( V \). Clearly, \( \text{ini}^2(a_1x_1y_xa_2) = \text{ini}^2(a_1x_1y_xa_2) \) because \( x \in \text{con}(a_2) \setminus \text{con}(a_1) \).
This implies that \( V \) satisfies \( u \approx u^* \). By a similar argument we can show that \( u_1a_1y_xa_2u_2 \approx u_1a_1y_xa_2u_2 \) is satisfied by \( V \). Then \( D(u^*, y) = k + 1 \) because the identity \( u \approx u^* \) is \( s \)-well-balanced for any \( s \). In view of Case 1, the identity \( u^* \approx u_1a_1x_1y_xa_2u_2 \) is satisfied by \( V \). Therefore, \( u \approx u'y_xu'' \) holds in \( V \), and we are done.

(iv) In view of Remark 2.2, there is a number \( r \) such that the \( r \)-decomposition of \( u \) is maximal. Since \( D(u, x) = D(u, y) = \infty \), the subword \( 1u_2x_2y \) is contained in some \( r \)-block \( a \) of \( u \) and \( a = a_1y_xa_2x_3a_4 \) for some \( a_1, a_2, a_3, a_4 \). Evidently,
\[
\text{ini}^2(a_1y_xa_2x_3a_4) = \text{ini}^2(a_1y_xa_2x_3a_4).
\]
This fact and Corollary 3.5 imply that \( u \approx u'y_xu'' \) is satisfied by \( V \). \( \square \)
Corollary 4.5. Let $V$ be a subvariety of $P\{\kappa_k\}$ and $u$ be a word. Further, let $u = u'xyu''$ for some $u', u'' \in \mathfrak{A}^*$ such that $x, y \in \text{sim}(u')$. Suppose that either $D(u, x) > k$ or $D(u, y) > k$. Then $V$ satisfies the identity $u \approx u'yxu''$.

Proof. We may assume without loss of generality that $(1u, x) < (1u, y)$. Then $u' = u_1xu_2y_3u_4$ for some $u_1, u_2, u_3 \in \mathfrak{A}^*$. Evidently, $D(u, x) \leq D(u, y)$. It follows that $k < D(u, y)$. Let $\phi$ be a substitution defined as follows: $\phi(c) = c^2$ for any $c \in \text{con}(u_1)$ and $\phi(c) = c$ otherwise. Put $w = \phi(u_2)y\phi(u_3)x\phi(u')$. We are going to verify that $D(u, a) \leq D(w, a)$ for any $a \in \text{con}(w)$. This fact is evident whenever $D(w, a) = \infty$. So, we may assume that $D(w, a) = k < \infty$. We will use induction on $k$.

Induction base: $k = 0$. Then $a \in \text{sim}(w)$. The definition of $w$ implies that $a \notin \text{sim}(u_1)$ and so $a \in \text{sim}(u)$. Therefore, $D(u, a) = 0$, and we are done.

Induction step: $k > 0$. Then $a \in \text{mul}(w)$. Let $b = h_k^{-1}(w, a)$. In view of Lemma 2.4, $D(w, b) = k - 1$ and $(1w, a) < (1w, b) < (2w, a)$. By the induction assumption $D(u, b) \leq k - 1$. The definition of $w$ implies that $b \notin \text{sim}(u_1)$ and $w \in \text{mul}(w)$. Therefore, $D(u, a) \leq k$, and we are done.

We have proved that $D(u, a) \leq D(w, a)$ for any $a \in \text{con}(w)$. In particular, $k < D(w, y)$. According to Lemmas 4.2 and 4.3(i),(ii), the variety $P\{\kappa_k\}$ satisfies the identities (3.2) and $\varepsilon_r$ for any $r \geq k$. Hence if $k < D(w, y) < \infty$, then

(4.3) $w = \phi(u_2)y\phi(u_1)x\phi(u') = \phi(u_2)y\phi(u_3)x^2\phi(u''\right)$

is satisfied by $P\{\kappa_k\}$ by Lemma 4.4(iii), and if $D(w, y) = \infty$, then (4.3) holds in $P\{\kappa_k\}$ by Lemma 4.4(iv). It remains to note that $P\{\kappa_k\}$ satisfies

(3.1) $u \approx u_1xw \approx u_1x\phi(u_2)y\phi(u_3)x^2\phi(u'') \approx u'yxu''$

and so $u \approx u'yxu''$. □

Lemma 4.6. Suppose that $k \in \mathbb{N}$ and $a \in \mathfrak{A}^*$ are such that $x_0, x_1, \ldots, x_{k-1} \notin \text{con}(a)$ and $\text{sim}(a) = \{x_k\}$. Then the variety $P\{\kappa_k\}$ satisfies the identities

$$ab_k \approx \text{ini}(a)(x_{s_k})b_k,$$

$$ymab_{k, m + 1}x_{m - 1}cb_{m - 1} \approx ym\text{ini}(a)(x_{s_k})b_{k, m + 1}x_{m - 1}cb_{m - 1},$$

where $c \in \{x_ym, yxm, yxm\}$.

Proof. It is routine to verify that $D(ab_k, x_s) = s$ for any $0 \leq s \leq k - 1$. Let $z$ be a letter from $\text{con}(a)$ with minimal depth in $ab_k$, which we denote by $d$. Clearly, $d < \infty$ because $h_k^{-1}(ab_k, x_k) \neq f_k^{-1}(ab_k, x_k)$ and so $D(ab_k, x_k) \leq k < \infty$. Then, since $\text{con}(a) \subseteq \text{mul}(ab_k)$, we have $d > 0$ and there is a $(d-1)$-divisor $t$ between the first and the second occurrences of $z$ in $ab_k$. In view of the choice of $z$, we have $t \notin \text{con}(a)$. It follows that $z \in \text{sim}(a)$. Hence $z = x_k$. According to Lemma 2.3, $x_{k-1}$ is a $(k-1)$-divisor of the word $ab_k$. Now Lemma 2.4(iii) applies and we conclude that $D(ab_k, z) = k$. In view of the choice of $z$, we have $D(ab_k, a) > k$ for any $a \in \text{con}(a) \setminus \{x_k\}$ and $D(ab_k, x_k) = k$. Let $\text{con}(a) = \{a_1, a_2, \ldots, a_r\}$. We may assume without loss of generality that

$$a = (1) \cdot a_1 \cdot (1) \cdot a_2 \cdot (1) \cdot a_r \cdot a_r$$

for some $a_1, a_2, \ldots, a_r \in \mathfrak{A}^*$. The identity (3.1) allows us to remove any third or subsequent occurrence of letters from the subwords $a_1, a_2, \ldots, a_r$ of $a$. So, we may
assume that these subwords contain second occurrences of letters only. According to Lemmas 4.2 and 4.3(ii), (iii), the identities (3.2), $\delta^p_k$ and $\varepsilon_p$ are satisfied by $P\{k\}$ for all $p \geq q \geq k$. Then, since $D(ab_k, x_k) = k$ and $D(ab_k, a) > k$ for any $a \in \text{con}(a) \setminus \{x_k\}$, the identities

$$ab_k = a_1 a_2 a_2 \cdots a_r a_r b_k \approx a_1 a_2 \cdots a_r a_1 a_2 \cdots a_r b = \text{ini}(a_1 a_2 \cdots a_r b_k)$$

hold in $P\{k\}$ by Parts (ii)–(iv) of Lemma 4.4. According to Corollary 4.5, $P\{k\}$ satisfies $\text{ini}(a_1 a_2 \cdots a_{m+1} b_k) \approx \text{ini}(a) \text{ini}(a_{x_k}) b_k$ and so $ab_k \approx \text{ini}(a) \text{ini}(a_{x_k}) b_k$. Substituting $x_{m-1} c$ for $x_{m-1}$ in the last identity, we obtain the identity

$$(4.4) \quad ab_{k,m+1} x_{m-1} c x_m d \approx \text{ini}(a) \text{ini}(a_{x_k}) b_{k,m+1} x_{m-1} c x_m d$$

where

$$d = \begin{cases} b_0 & \text{if } m = 1, \\ b_{m-1,m-1} c b_{m-2} & \text{if } m > 1. \end{cases}$$

Then $P\{k\}$ satisfies

$$y_m ab_{k,m+1} x_{m-1} c b_{m-1} \approx y_m ab_{k,m+1} x_{m-1} c x_m d \quad \text{(3.1)}$$

$$\approx y_m \text{ini}(a) \text{ini}(a_{x_k}) b_{k,m+1} x_{m-1} c x_m d \quad \text{(4.4)}$$

$$\approx y_m \text{ini}(a) \text{ini}(a_{x_k}) b_{k,m+1} x_{m-1} c b_{m-1} \quad \text{(3.1)}$$

and we are done. \hfill \Box

Let

$$\Phi = \{(3.1), (3.2), \gamma_k, \delta^m_k, \varepsilon_k \mid k \in \mathbb{N}, \ 1 \leq m \leq k\}.$$ 

Given a monoid variety $V$ and a system of identities $\Sigma$ we use $\Sigma(V)$ to denote the set of those identities from $\Sigma$, which hold in $V$.

**Proposition 4.7.** Let $V$ be a monoid variety from the interval $[\text{LRB} \lor F_1, P]$, $u \approx v$ be an identity of $V$ and $\{i, j \mid y\}$ be a critical pair in $u \approx v$. If $\{i, j\} = \{1, 2\}$, then the critical pair $\{i, j \mid y\}$ is $\Delta$-removable for some $\Delta \subseteq \Phi(V)$.

**Proof.** It suffices to assume that $(i, j) = (1, 2)$ since the case when $(i, j) = (2, 1)$ is similar. Let $w$ denote the word obtained from $u$ by replacing $1u^r x 2a^q y$ with $2a^q y 1u^r x$. Since $\{i, j \mid y\}$ is a critical pair in $u \approx v$, the identity $u(x, y) \approx v(x, y)$ is equivalent modulo (3.1) to (3.2). Therefore, (3.2) is satisfied by $V$.

Suppose that $D(u, y) = \infty$. Then $x$ is not an $r$-divider of $u$ for any $r \geq 0$. Therefore, $D(u, x) = \infty$ by Lemma 2.3. Then $u \approx w$ follows from $\{(3.1), (3.2)\} \subseteq \Phi(V)$ by Lemma 4.4(iv). So, we may assume that $D(u, y) < \infty$, say $D(u, y) = s$ for some $s \in \mathbb{N}$. Further, Lemma 3.1 allows us to assume that the identity $u \approx v$ is 2-limited.

There are two cases.

**Case 1: $F \subseteq V$.** In view of Remark 2.2, there is a number $k$ such that the $k$-decompositions of $u$ and $v$ are maximal. According to Corollary 2.13 and the inclusion $F_{k+1} \subseteq F$, the identity $u \approx v$ is $k$-well-balanced. It follows that $x$ is not an $r$-divider of $u$ for any $r \geq 0$. Therefore, $D(u, x) = \infty$ by Lemma 2.3.

Now we apply Lemma 2.6 and conclude that there are letters $x_0, x_1, \ldots, x_{s-1}$ such that $D(u, x_r) = D(v, x_r) = r$ for any $0 \leq r < s$ and the identity $u \approx v$ has
the form
\[
\begin{align*}
&\mathbf{u}_{2s+1} = \mathbf{u}_{2s} \cdot \mathbf{x}_{s-1} \quad \mathbf{u}_{2s-1} = \mathbf{u}_{2s-2} \cdot \mathbf{x}_{s-2} \quad \mathbf{u}_{2s-3} = \mathbf{x}_{s-1} \quad \mathbf{u}_{2s-4} = \mathbf{x}_{s-3} \\
&\quad \cdots \quad \mathbf{u}_{2s-2} = \mathbf{x}_{s-2} \quad \mathbf{u}_{2s-3} = \mathbf{x}_{s-1} \quad \mathbf{u}_{2s-4} = \mathbf{x}_{s-3} \\
\end{align*}
\]

for some words \( \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2s+1} \) and \( \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_{2s+1} \). Clearly, \( \mathbf{1u}x \) is the last letter of \( \mathbf{u}_{2s-1} \). Since \( D(\mathbf{u}, x) = \infty \), we have \( (2u) < (1u) \mathbf{u}_{2s} \). It follows that \( 2u \) occurs in \( \mathbf{u}_{2s-2} \). Further, \( D(\mathbf{v}, x) = \infty \) by Corollary 2.13. Since \( 2v \) \( \mathbf{1y} \) \( \mathbf{u} \approx \mathbf{v} \) is \( k \)-well-balanced, both \( 1v \) and \( 2v \) occur in \( \mathbf{v}_{2s-2} \). Then we substitute \( y \) \( \mathbf{u} \) \( \mathbf{v} \) in the identity
\[
\mathbf{u}(x_0, x_1, \ldots, x_{s-1}, x, y) \approx \mathbf{v}(x_0, x_1, \ldots, x_{s-1}, x, y)
\]
and obtain the identity \( \varepsilon_{s-1} \). Therefore, \( \varepsilon_{s-1} \) is satisfied by \( \mathbf{V} \). Now we apply Lemma 4.4(iii) and conclude that the identity \( \mathbf{u} \approx \mathbf{w} \) follows from \( \{ (3.1), \varepsilon_{s-1} \} \subseteq \Phi(\mathbf{V}) \).

**Case 2:** \( \mathbf{F} \not\subseteq \mathbf{V} \). In view of Lemma 2.11, there is \( k \in \mathbb{N} \) such that \( \mathbf{F}_{k+1} \not\subseteq \mathbf{V} \). Let \( k \) be the least number with such a property. Then \( \mathbf{F}_k \subseteq \mathbf{V} \). According to Lemma 3.3, \( \mathbf{V} \) satisfies \( \varepsilon_k \). Then \( \varepsilon_k \) holds in \( \mathbf{V} \) by Lemma 4.2.

Suppose that \( k < s \). Then, by Parts (ii) and (iii) of Lemma 4.3, \( \mathbf{V} \) satisfies \( \varepsilon_{s-1} \) and \( \varepsilon'_s \) for any \( r \geq s \). Clearly, \( D(\mathbf{u}, x) \leq k \) because \( D(\mathbf{u}, y) \leq k < s \) otherwise. If \( D(\mathbf{u}, x) = \infty \), then Lemma 4.4(iii) applies and we conclude that the identity \( \mathbf{u} \approx \mathbf{w} \) follows from \( \{ (3.1), \varepsilon_{s-1} \} \subseteq \Phi(\mathbf{V}) \). If \( D(\mathbf{u}, x) = r \) for some \( k \leq r < \infty \), then, by Lemma 4.4(ii), the identity \( \mathbf{u} \approx \mathbf{w} \) follows from \( \{ (3.1), \varepsilon'_s \} \subseteq \Phi(\mathbf{V}) \). So, we may assume without loss of generality that \( s \leq k \).

If \( D(\mathbf{u}, x) = \infty \), then since (2.3) and (2.4) hold for all \( \ell = 1, 2, \ldots, k \) by Lemma 2.12, one can repeat literally arguments from the second paragraph of Case 1 and obtain that the identity \( \mathbf{u} \approx \mathbf{w} \) follows from \( \{ (3.1), \varepsilon_{s-1} \} \subseteq \Phi(\mathbf{V}) \). So, we may assume without loss of generality that \( D(\mathbf{u}, x) = p \) for some \( p \in \mathbb{N} \).

Then by Corollary 2.8, there exist letters \( x_0, x_1, \ldots, x_{p-1} \) such that
\[
\mathbf{u} = \mathbf{u}_{2p+1} = \mathbf{x}_p(1) \quad \mathbf{u}_{2p} = \mathbf{x}_{p-1}(2) \quad \mathbf{u}_{2p-1} = \mathbf{x}_{p-2}(1) \quad \mathbf{u}_{2p-2} = \mathbf{x}_{p-3}(2) \quad \mathbf{u}_{2p-3} = \mathbf{x}_{p-4}(1) \\
\quad \cdots \quad \mathbf{u}_{2p-5} = \mathbf{x}_3(1) \quad \mathbf{u}_3 = \mathbf{x}_2(2) \quad \mathbf{u}_2 = \mathbf{x}_1(1) \quad \mathbf{u}_1 = \mathbf{x}_0(2)
\]
for some \( \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{2p+1} \in \mathcal{A}^* \), with \( D(\mathbf{u}, x_r) = r \) and \( \mathbf{u}_{2r+2} \) not containing any \( r \)-dividers of \( \mathbf{u} \) for any \( r = 0, 1, \ldots, k-1 \).

Clearly, \( \mathbf{1y} \) occurs in \( \mathbf{u}_{2p+1} \), while \( 2\mathbf{u} \) is the first letter of \( \mathbf{u}_{2p} \). In view of Lemma 2.4, there is a letter \( y_{s-1} \) such that \( D(\mathbf{u}, y_{s-1}) = s - 1 \) and \( (1\mathbf{u}y_{s-1}) < (2\mathbf{u}y) \). Let \( X = \{ x_0, x_1, \ldots, x_{p-1}, x, y_{s-1}, y \} \).

**Subcase 2.1:** \( s = 1 \). Then \( y_{s-1} = y_0 \in \text{sim}(\mathbf{u}) = \text{sim}(\mathbf{v}) \) and
\[
\mathbf{u}(X) = \text{yy}_0 \mathbf{x} y \mathbf{x}_{p-1} \mathbf{x} \mathbf{b}_{p-1}.
\]

Let \( p \leq k \). Since \( (2\mathbf{u}y) < (1\mathbf{u}x) \) and the identity \( \mathbf{u} \approx \mathbf{v} \) is \( (k-1) \)-well-balanced by Corollary 2.13, we have \( \mathbf{v}(X) = \text{yy}_0 \mathbf{x} y \mathbf{x}_{p-1} \mathbf{x} \mathbf{b}_{p-1} \). Let \( \phi \) be the substitution \( (x, y) \mapsto (x_p, y_s) \). Then the identity \( \phi(\mathbf{u}(X)) \approx \phi(\mathbf{v}(X)) \) coincides with \( \gamma_p \).
Now let \( p > k \). Since the identity \( u \approx v \) is \((k-1)\)-well-balanced by Corollary 2.13, we have \( v(X) = y y_0 y a' b_k \) for some word \( a' \) with \( \text{sim}(a') = \{x_k, y\} \) and \( \text{mul}(a') = \{x_{k+1}, x_{k+2}, \ldots, x_{p-1}, x\} \). Now Lemma 2.10 and the fact that \((2v y) < (1v x)\) imply that \( \text{ini}(a') = y x x_{p-1} x_{p-2} \cdots x_k \). It follows that \( a' = y a \) for some \( a \). Since \( c_k \) holds in \( V \), Lemma 4.6 implies that \( V \) satisfies the identities

\[
y y_0 y a b_k \approx y y_0 y a a(x_k) b_k = y y_0 y x x_{p-1} x_{p-2} \cdots x_k x_{p-1} x_{p-2} \cdots x_{k+1} b_k
\]

and so the identity \( y p \).

We see that \( y p \) is satisfied by either case. Now we apply Lemma 4.4(i) and conclude that the identity \( u \approx w \) follows from \{3.1, \( y p \} \subseteq \Phi(V)\).

**Subcase 2.2:** \( s > 1 \). Clearly, \( p \geq s - 1 \) because \( h^p_k(u, y) \neq h^p_k(u, y) \) and so \( D(u, y) < s \) otherwise. Suppose that \( p = s - 1 \). Then \( p \leq k \). In view of Corollary 2.13, the identity \( u \approx v \) is \((k-1)\)-well-balanced. Hence

\[
v = v_{2p+1} x v_{2p} x_{p-1} v_{2p-1} x v_{2p-2} x_{p-2} v_{2p-3} x_{p-3} v_{2s-4} x_{p-3} \cdot v_{2p-5} x_{p-2} \cdots v_1 x_1 v_0 x_0 v_1 x_1 v_0
\]

for some words \( v_0, v_1, \ldots, v_{2p+1} \). Let \( \phi \) be the substitution \((x, y) \mapsto (x, p, x)\). Then the identity

\[
\phi(u(x_0, x_1, \ldots, x_{p-1}, x, y)) \approx \phi(v(x_0, x_1, \ldots, x_{p-1}, x, y))
\]

is equal to \( \kappa_{s-1} \). Then \( V \) satisfies \( \delta_{s-1}^{\gamma} \) by Lemma 4.2. Now we apply Lemma 4.4(ii) and conclude that the identity \( u \approx w \) follows from \{3.1, \( \delta_{s-1}^{\gamma} \} \subseteq \Phi(V)\). So, we may assume that \( p > s - 1 \). In particular, \( x \neq y_{s-1} \).

Since \( s > 1 \), the letter \( y_{s-1} \) is in \( \text{mul}(u) = \text{mul}(v) \). Since \( D(u, y) = s \), there are no \((s - 2)\)-dividers of \( u \) between \( u_{y} \) and \( 2 u_{y} \). This fact and Lemma 2.9 imply that \( 2 u_{y_{s-1}} \) occurs in either \( u_{2s-2} \) or \( u_{2s-3} \) or \( u_{2s-4} \). Let \( y_{s-2} = h^p_k(u, y_{s-1}) \). Using Lemmas 2.4 and 2.5, one can show that \( u_{1u_{y_{s-2}}} \) follows \( 2 u_{x_2} \). Since \( u_{2s-2} \) does not contain any \((s - 2)\)-dividers of \( u \), \( 1 u_{y_{s-2}} \) does not occur in \( u_{2s-2} \). Therefore, \( 2 u_{y_{s-1}} \) occurs in either \( u_{2s-3} \) or \( u_{2s-4} \). Since \( s \leq k \) and \( u \approx v \) is \((k-1)\)-well-balanced by Corollary 2.13, the letter \( 2 v y_{s-1} \) lies between \( 2 v x_{2s-2} \) and \( 1 v x_{s-3} \). This means that \( 2 v y_{s-1} \) occurs in either \( v_{2s-3} \) or \( v_{2s-4} \).

Clearly,

\[
u(X) = y y_{s-1} x y x_{p-1} x b_{p-1, s} x_{s-2} c_{s-2},
\]

where \( c \in \{x_{s-1} y_{s-1}, y_{s-1} x_{s-1}\} \).

Let \( p \leq k \). Since the identity \( u \approx v \) is \((k-1)\)-well-balanced and \((2v y) < (1v x)\), we have

\[
v(X) = y y_{s-1} y x x_{p-1} x b_{p-1, s} x_{s-2} d_{s-2},
\]

where \( d \in \{x_{s-1} y_{s-1}, y_{s-1} x_{s-1}\} \). Then the identity \( u(X) \approx v(X) \) coincides with

\[
y y_{s-1} y x x_{p-1} x b_{p-1, s} x_{s-2} c_{s-2} \approx y y_{s-1} y x x_{p-1} x b_{s-1, s} x_{s-2} d_{s-2}.
\]

Now let \( p > k \). Since the identity \( u \approx v \) is \((k-1)\)-well-balanced and \((2v y) < (1v x)\), we have

\[
v(X) = y y_{s-1} a' b_{k, s} x_{s-2} d_{s-2},
\]

where \( d \in \{x_{s-1} y_{s-1}, y_{s-1} x_{s-1}\} \) and \( a' \) is a word with \( \text{sim}(a') = \{x_k, y\} \) and \( \text{mul}(a') = \{x_{k+1}, x_{k+2}, \ldots, x_{p-1}, x\} \). Now Lemma 2.10 and the fact that \((2v y) <
Lemma 2.3). If, for any such a property, Equivalently, the depth of the subword \( w \) is

\[
y y_{s-1} y a b_{k,s} x_{s-2} d b_{s-2} \approx y y_{s-1} y \text{ini}(a) \text{ini}(a_{x_k}) b_{k,s} x_{s-2} d b_{s-2}
\]

and so (4.5).

We see that (4.5) is satisfied by \( V \) in either case. Then \( V \) satisfies the identity \( \delta_p^{-1} \) because

\[
y y_{s-1} x_p y_s \cdots x_{s-2} x_{s-1} y_{s-1} x_{s-3} x_{s-2} \cdots x_1 x_2 x_0 x_1
\]

\[
(3.1) \approx y y_{s-1} x_p y_s \cdots x_{s-2} x_{s-1} y_{s-1} x_{s-3} x_{s-2} x_{s-1} y_{s-1} \cdots x_1 x_2 x_0 x_1
\]

\[
(4.5) \approx y y_{s-1} x_p y_s \cdots x_{s-2} x_{s-1} y_{s-1} x_{s-3} x_{s-2} x_{s-1} y_{s-1} \cdots x_1 x_2 x_0 x_1
\]

Now we apply Lemma 4.4(ii) and conclude that the identity \( u \approx w \) follows from

\[
\{ (3.1), \delta_p^{-1} \} \subseteq \Phi(V).
\]

4.2. Removing critical pairs of the form \( \{2x, 2y\} \). We introduce the following countably infinite series of identities:

\[
\zeta_{k-1} : x y_0 y x_{k-1} x y b_{k-1} \approx x y_0 y x_{k-1} y x b_{k-1},
\]

\[
\lambda^m_k : x y_m y x_k y x b_{k,m} y m b_{m-1} \approx x y_m y x_k y x b_{k,m} y m b_{m-1},
\]

\[
\eta_{k-1} : x y_1 y y_0 y x_{k-1} y_1 b_{k-1} \approx x y_1 y y_0 y x_{k-1} y_1 b_{k-1},
\]

\[
\mu^m_k : x y_{m+1} y y_0 y x_k y m y m b_{m-1} \approx x y_{m+1} y y_0 y x_k y m y m b_{m-1},
\]

\[
\nu_{k-1} : x x_{k-1} y 2 x y z_\infty \approx x x_{k-1} y 2 x y z_\infty b_{k-1},
\]

where \( k \in \mathbb{N} \) and \( 1 \leq m \leq k \).

We omit the proof of the following statement because it is very similar to the proof of Lemma 4.3.

**Lemma 4.8.** The following proper inclusions hold:

(i) \( P \{ g_0 \} \subset P \{ g_1 \} \subset \cdots \subset P \{ g_k \} \subset \cdots \subset P \{ \nu_0 \} \subset P \{ \nu_1 \} \subset \cdots \subset P \{ \nu_k \} \subset \cdots \);

(ii) \( P \{ \zeta_k \} \subset P \{ \zeta_k \} \subset P \{ \lambda_1^k \} \subset \cdots \subset P \{ \lambda_k^k \} \subset P \{ \lambda_{k+1}^k \} \subset \cdots \subset P \{ \lambda_{k+m}^k \} \subset \cdots \);

(iii) \( P \{ \eta_0 \} \subset P \{ \eta_1 \} \subset \cdots \subset P \{ \eta_k \} \subset \cdots \);

(iv) \( P \{ \mu_1^k \} \subset P \{ \mu_1^k \} \subset \cdots \subset P \{ \mu^k_k \} \subset \cdots \subset P \{ \mu^k_{k+1} \} \subset \cdots \).

Let \( w = w_1 w_2 w_3 \). We say that the depth of the subword \( w_2 \) in \( w \) is equal to \( k \) if \( D(w, a) = k \) for some \( a \in \text{con}(w_2) \setminus \text{con}(w_1) \) and \( k \) is the least number with such a property. Equivalently, the depth of the subword \( w_2 \) in \( w \) is equal to \( k \) if \( w_2 \) contains a \( k \)-divider of \( w \) but does not contain any \((k - 1)\)-divider of \( w \) (see Lemma 2.3). If, for any \( k \geq 0 \), \( w_2 \) does not contain any \( k \)-divider of \( w \), then we say that the depth of the subword \( w_2 \) in \( w \) is infinite. Equivalently, the depth of the subword \( w_2 \) in \( w \) is infinite if \( D(w, a) = \infty \) for any \( a \in \text{con}(w_2) \setminus \text{con}(w_1) \) (see Lemma 2.3).
Lemma 4.9. Let $V$ be a subvariety of $P$ and $u = u'^{(2)(2)} x' y$ for some $u', u'' \in \mathfrak{X}^*$. Further, let $u' = u_1^{(1)} \ x \ u_2^{(1)} y \ u_3$, where the depth of the subwords $u_2$ and $u_3$ in $u$ is equal to $m$ and $k$, respectively. Suppose also that one of the following holds:

(i) $k \geq 0$, $m = 0$ and $V$ satisfies $\zeta_k$;
(ii) $1 \leq m \leq k$ and $V$ satisfies $\lambda^m_k$.

Then $V$ satisfies the identity $u \approx u'y xu''$.

Proof. We omit the proof of Part (i) because it is very similar to (and in fact simpler than) that of Part (ii).

(ii) The fact that the depth of $u_3$ is equal to $k$ in $u$ and Lemma 2.3 imply that $u_3$ contains a $k$-divider $x_k$ of $u$ such that $D(u, x_k) = k$. It follows from Corollary 2.8 that there exist letters $x_0, x_1, \ldots, x_{k-1}$ such that $D(u, x_s) = s$ for any $0 \leq s < k$ and

$$u'' = v_{2k} x_{k-1} v_{2k-1} x_k v_{2k-2} x_{k-2} v_{2k-3} x_{k-3} \cdots v_4 x_1 v_3 x_2 v_2 x_0 v_1 x_1 v_0,$$

for some $v_0, v_1, \ldots, v_{2k} \in \mathfrak{X}^*$. Clearly, for any $s = 1, 2, \ldots, k - 1$, there exist $v'_{2s}, v''_{2s} \in \mathfrak{X}^*$ such that $v_{2s} = v'_{2s} v''_{2s}$, where $v'_{2s}$ does not contain any $(s - 1)$-divider of $u$, while either $v''_{2s} = \lambda$ or $h(v''_{2s})$ is an $(s - 1)$-divider of $u$. Put $v''_{2k} = v_{2k}$ and $v_0 = v_0$. Let

$$q_s = v''_{2s+2} x_s v_{2s+1} x_{s+1} v'_{2s}$$

for any $s = 0, 1, \ldots, k - 1$.

Let $\phi$ be the substitution

$$(x_0, \ldots, x_{k-1}, x_k, y_m) \mapsto (q_0, \ldots, q_{k-1}, u_3, u_2).$$

Then the identity $\phi(\lambda^m_k)$ coincides with

$$(4.6) \quad xu_2 y u_3 xy q \approx xu_2 y u_3 y x q$$

where

$$q = q_{k-1} u_3 q_{k-2} q_{k-1} \cdots q_m u_2 q_{m-2} q_{m-1} \cdots q_1 q_0 q_1.$$

We note that $u'' = q_{k-1} q_{k-2} \cdots q_0$. By the same arguments as in the proof of Lemma 4.4(ii) we can show that

- if a letter occurs in $u_2$, then the subword $u_2 y u_3 x y q_{k-1} \cdots q_m q_{m-1}$ of $u$ contains some non-first occurrence of this letter in $u$;
- if a letter occurs in $u_3$, then the subword $u_3 x y q_{k-1}$ of $u$ contains some non-first occurrence of this letter in $u$;
- if a letter occurs in $q_s$, then the subword $q_s q_{s-1}$ of $u$ contains some non-first occurrence of this letter in $u$ for any $s = 1, 2, \ldots, k - 1$.

Then the identity (3.1) implies $u = u' xy u'' \approx u' xy q$. By a similar argument we can show that $u' y xu'' \approx u' y u q$ follows from (3.1). These identities together with (4.6) imply the identity $u \approx u' y xu''$. Thus, it is satisfied by $V$. □

The proof of the following lemma is very similar to the proof of Lemma 4.9. We provide it for the sake of completeness.

Lemma 4.10. Let $V$ be a subvariety of $P$ and $u = u'^{(2)(2)} x' y$ for some $u', u'' \in \mathfrak{X}^*$. Further, let

$$u' = u_1^{(1)} x \ u_2^{(1)} y \ u_{m+1}^{(1)} u_3^{(1)} y \ u_4^{(1)}$$

and $u'' = u_5^{(1)} x_k u_6^{(1)} y_{m+1} u_7^{(1)}$. 

where \( D(u, y_{m+1}) = m + 1 \), \( D(u, x_k) = k \) and \( y_{m+1} \) is a letter such that \((2u) \leq (2u_{m+1})\) for any \( a \in \{c | \text{ first occurrence of } c \text{ lies in } u_{2y_{m+1}u_3} \}.\) Suppose that the depth of the subwords \( u_{2y_{m+1}u_3}, u_4 \) and \( u_5x_ku_6 \) is equal to \( m + 1 \), \( m \) and \( k \), respectively. Suppose also that one of the following holds:

(i) \( k \geq 0, m = 0 \) and \( V \) satisfies \( \eta_k \);

(ii) \( 1 \leq m \leq k \) and \( V \) satisfies \( \mu^m_k \).

Then \( V \) satisfies the identity \( u \approx u'yxu'' \).

Proof. We omit the proof of Part (i) because it is very similar to (and in fact simpler than) that of Part (ii).

(ii) It follows from Corollary 2.8 and the fact that the depth of \( u_{5x_ku_6} \) is equal to \( k \) that there exist letters \( x_0, x_1, \ldots, x_{k-1} \) such that \( D(u, x_s) = s \) for any \( 0 \leq s < k \) and

\[
u = v_{2k-1}v_{2k-2}v_{2k-3}v_{2k-4} \cdots v_1 v_0 v_1 v_2 \cdots v_{2k} \]

for some \( v_0, v_1, \ldots, v_{2k} \in A^* \). Clearly, for any \( s = 1, 2, \ldots, k - 1 \), there exist \( v_{2s}, v'_{2s} \in A^* \) such that \( v_{2s} = v_{2s}v'_{2s} \), where \( v'_{2s} \) does not contain any \((s - 1)\)-divider of \( u \), while either \( v'_{2s} = \lambda \) or \( h(v''_{2s}) \) is an \((s - 1)\)-divider of \( u \). Put \( v''_{2s} = v_{2k} \) and \( v_0 = v_0 \). Let

\[
q_s = v_{2s+1}v_{2s+2}v_{2s+3}v_{2s+4}v_{2s+5}v_{2s+6} \]

for any \( s = 0, 1, \ldots, k - 1 \).

Let \( \phi \) be the substitution

\[
(x_0, \ldots, x_{k-1}, x_k, y_m, y_{m+1}) \mapsto (q_0, \ldots, q_{k-1}, u_{5x_ku_6}, u_4, u_{2y_{m+1}u_3}).
\]

Then the identity \( \phi(\mu^m_k) \) coincides with

\[
xu_2y_{m+1}u_3y_4xy_5x_ku_6q \approx xu_2y_{m+1}u_3y_4yu_5x_ku_6q
\]

where

\[
q = u_{2y_{m+1}u_3}q_{k-1}u_{5x_ku_6}q_{k-2}q_{k-1} \cdots q_{m}q_{m-1}q_{m-2}q_{m-1} \cdots q_1 q_0 q_1.
\]

We note that \( u_7 = q_k \cdots q_1 \cdots q_0 \). The choice of \( y_{m+1} \) implies that if \( z \) is a letter such that \( u_z \) occurs in the subword \( u_{2y_{m+1}u_3} \) of \( u \), then \((2u_z) \leq (2u_{y_{m+1}})\). We see that if a letter lies in \( u_{2y_{m+1}u_3} \), then the subword \( u_{2y_{m+1}u_3}u_{3}y_4x_5x_ku_6y_{m+1} \) of \( u \) contains some non-first occurrence of this letter in \( u \). By the same arguments as in the proof of Lemma 4.4(ii) we can show that

- if a letter occurs in \( u_4 \), then the subword \( u_{4}xyu_{5}x_ku_6y_{m+1}q_{k-1} \cdots q_{m}q_{m-1} \) of \( u \) contains some non-first occurrence of this letter in \( u \);
- if a letter occurs in \( u_{5x_ku_6} \), then the subword \( u_{5x_ku_6y_{m+1}q_{k-1}} \) of \( u \) contains some non-first occurrence of this letter in \( u \);
- if a letter occurs in \( q_s \), then the subword \( q_s q_{s-1} \) of \( u \) contains some non-first occurrence of this letter in \( u \) for any \( s = 1, 2, \ldots, k - 1 \).

Then the identity (3.1) implies

\[
u = u'xyu_5x_ku_6y_{m+1}u_7 \approx u'xyu_5x_ku_6y_{m+1}q.
\]

By a similar argument we can show that

\[
u'yxu_5x_ku_6y_{m+1}u_7 \approx u'yxu_5x_ku_6y_{m+1}q
\]

follows from (3.1). These identities together with (4.7) imply the identity \( u \approx u'yxu'' \). Thus, it is satisfied by \( V \).
Lemma 4.11. Let $\mathbf{V}$ be a subvariety of $\mathbf{P}$ and $\mathbf{u} = \mathbf{u}_{(2)2}^{(2)} \mathbf{u}''$ for some $\mathbf{u}', \mathbf{u}'' \in \mathfrak{X}^*$. Further, let $\mathbf{u}' = \mathbf{u}_1 \mathbf{x}_2 \mathbf{u}_3$, where the depth of the subword $\mathbf{u}_2$ in $\mathbf{u}$ is equal to $k$, while the depth of the subword $\mathbf{u}_3$ in $\mathbf{u}$ is infinite. Suppose that $\mathbf{V}$ satisfies $\nu_k$. Then $\mathbf{V}$ satisfies the identity $\mathbf{u} \approx \mathbf{u}' \mathbf{y} \mathbf{x} \mathbf{u}''$.

Proof. The fact that the depth of $\mathbf{u}_2$ is equal to $k$ in $\mathbf{u}$ and Lemma 2.3 imply that $\mathbf{u}_2$ contains a $k$-divider $\mathbf{x}_k$ of $\mathbf{u}$ such that $D(\mathbf{u}, \mathbf{x}_k) = k$. It follows from Corollary 2.8 and the fact that the depth of $\mathbf{u}_3$ is infinite that there exist letters $x_0, x_1, \ldots, x_{k-1}$ such that $D(\mathbf{u}, x_s) = s$ for any $0 \leq s < k$ and

$$\mathbf{u}'' = \mathbf{v}_{2s+2} x_s \mathbf{v}_{2s+1} x_{s+1} \mathbf{v}'_{2s}$$

for any $s = 0, 1, \ldots, k - 1$.

Let $\phi$ be the substitution

$$(x_0, x_1, \ldots, x_{k-1}, x_k, z, \zeta) \mapsto (q_0, q_1, \ldots, q_{k-1}, \mathbf{u}_2, \mathbf{v}'_{2k}, \mathbf{u}_3).$$

Then the identity $\phi(\nu_k)$ coincides with

$$(4.8) \quad \mathbf{x} \mathbf{u}_2 \mathbf{y} \mathbf{u}_3 \mathbf{y} \mathbf{x} \mathbf{q} \approx \mathbf{x} \mathbf{u}_2 \mathbf{y} \mathbf{u}_3 \mathbf{y} \mathbf{x} \mathbf{q}$$

where

$$\mathbf{q} = \mathbf{v}_{2k} q_{k-2} q_{k-1} q_k.$$ 

We note that $\mathbf{u}'' = \mathbf{v}_{2k} q_{k-2} q_k q_0$. Since the subword $\mathbf{u}_3 \mathbf{xy} \mathbf{v}_{2k}'$ of $\mathbf{u}$ does not contain $r$-dividers of $\mathbf{u}$ for any $r \in \mathbb{N} \cup \{0\}$ and the first letter of $q_{k-1}$ is an $r'$-divider of $\mathbf{u}$ for some $r' \in \mathbb{N} \cup \{0\}$, if a letter occurs in the subword $\mathbf{u}_3 \mathbf{xy} \mathbf{v}_{2k}'$ of $\mathbf{u}$, then this subword contains some non-first occurrence of this letter. By the same arguments as in the proof of Lemma 4.4(ii) we can show that

- if a letter occurs in $\mathbf{u}_2$, then the subword $\mathbf{u}_2 \mathbf{y} \mathbf{u}_3 \mathbf{y} \mathbf{v}_{2k}' q_{k-1}$ of $\mathbf{u}$ contains some non-first occurrence of this letter in $\mathbf{u}$;
- if a letter occurs in $\mathbf{q}_s$, then the subword $\mathbf{q}_s \mathbf{q}_{s-1} \mathbf{u}_2 \mathbf{q}_{k-2} \mathbf{q}_{k-1}$ of $\mathbf{u}$ contains some non-first occurrence of this letter in $\mathbf{u}$ for any $s = 1, 2, \ldots, k - 1$.

Then the identity $(3.1)$ implies $\mathbf{u} = \mathbf{u}' \mathbf{x} \mathbf{y} \mathbf{u}'' \approx \mathbf{u}' \mathbf{x} \mathbf{y} \mathbf{q}$. By a similar argument we can show that $\mathbf{u}' \mathbf{x} \mathbf{y} \mathbf{x} \mathbf{u}'' \approx \mathbf{u}' \mathbf{x} \mathbf{y} \mathbf{x} \mathbf{q}$ follows from $(3.1)$. These identities together with $(4.8)$ imply the identity $\mathbf{u} \approx \mathbf{u}' \mathbf{y} \mathbf{x} \mathbf{u}''$. Thus, it is satisfied by $\mathbf{V}$. $\square$

Let $\mathbf{w} = \mathbf{w}_1 \mathbf{w}_2 \mathbf{w}_3$. The depth of $\mathbf{w}_2$ in $\mathbf{w}$ is strictly infinite if

- the depth of $\mathbf{w}_2$ in $\mathbf{w}$ is infinite, in particular, $\text{con}(\mathbf{w}_2) \subseteq \text{mul}(\mathbf{w})$;
- for any $a \in \text{sim}(\mathbf{w}_2) \setminus \text{con}(\mathbf{w}_1)$, there are no first occurrences of letters between the last letter of $\mathbf{w}_2$ and the second occurrence of $a$ in $\mathbf{w}$.

For an arbitrary $n \in \mathbb{N}$, we denote by $S_n$ the full symmetric group on the set $\{1, 2, \ldots, n\}$. 
Lemma 4.12. Let \( u = u_1 x u_2 y u_3 y u_4 \). If the depth of \( u_3 \) in \( u \) is strictly infinite, then the identity \( u \cong u_1 x u_2 y u_3 y u_4 \) follows from the identities (3.1) and (4.9) \( x t y x y \cong x t y^2 x \).

Proof. First, we note that if \( w_1, w_2, w_3 \) are words with \( \text{sim}(w_3) \subseteq \text{con}(w_1 w_2) \), then the identities (3.1) and (4.9) imply the identity \( w_1 x w_2 y w_3 x y \cong w_1 x w_2 y w_3 y x \) because

\[
w_1 x w_2 y w_3 x y \cong w_1 x w_2 y w_3 x y w_3 \cong w_1 x w_2 (y w_3)^2 x \cong w_1 x w_2 y w_3 y x.
\]

By the hypothesis, \( u_3 = a_0 z_1 a_1 z_2 a_2 \cdots z_n a_n \) and \( u_4 = b_1 z_1 b_2 z_2 \cdots b_n z_n b' \), where \( \pi \in S_n \) and \( a_0, a_1, b_1, b_2, \ldots, a_n, b_n \) do not contain first occurrences of letters in \( u \). In view of the saying in previous paragraph, the identities (3.1) and (4.9) imply the identities

\[
u \cong u_1 x u_2 y u_3 z_n x y (b_1 z_1 b_2 z_2 \cdots b_n z_n \pi \{z_n\}) b'
\]

\[
\cong u_1 x u_2 y u_3 z_n z_{n-1} x y (b_1 z_1 b_2 z_2 \cdots b_n z_n \pi \{z_n, z_{n-1}\}) b'
\]

\[
\vdots
\]

\[
\cong u_1 x u_2 y u_3 z_n z_{n-1} \cdots z_1 x y b_1 b_2 \cdots b_n b'.
\]

By a similar argument we can show that (3.1) and (4.9) imply

\[
u_1 x u_2 y u_3 y u_4 \cong u_1 x u_2 y u_3 z_n z_{n-1} \cdots z_1 x y b_1 b_2 \cdots b_n b'.
\]

We see that \( u \cong u_1 x u_2 y u_3 y u_4 \) follows from (3.1) and (4.9). \( \square \)

Lemma 4.13. Let \( V \) be a monoid variety from the interval \([F_1, P]\) and \( u \cong v \) be an identity of \( V \) such that

\[
u = w_1 x w_2 y w_3 x y w_4
\]

for some words \( w_1, w_2, w_3, w_4 \) and \( (2x y) < (2x y) \). Suppose that the depth of the subwords \( w_2 \) and \( w_3 \) of \( u \) is equal to \( m \) and \( k \), respectively.

(i) If \( k \geq 0 \) and \( m = 0 \), then \( V \) satisfies \( \zeta_k \).

(ii) If \( 1 \leq m \leq k \), then \( V \) satisfies \( \lambda_k^m \).

Proof. In view of Lemma 3.1, we may assume without loss of generality that the identity \( u \cong v \) is 2-limited.

Let \( y_m \) be an \( m \)-divisor of \( u \) that occurs in \( w_2 \) and let \( x_k \) be a \( k \)-divisor of \( u \) that occurs in \( w_3 \). In view of Lemma 2.3, \( D(u, x_k) = k \) and \( D(u, y_m) = m \).

If \( k = 0 \), then \( m = 0 \) as well. Then Corollary 2.13 and the inclusion \( F_1 \subseteq V \) imply that the identity \( u(x, y, x_0, y_0) \approx v(x, y, x_0, y_0) \) coincides with the identity \( \zeta_0 \), and we are done. So, we may assume that \( k > 0 \).

Suppose that the identity \( u \cong v \) is not \( k \)-well-balanced. In view of Corollary 2.13 and the inclusion \( F_1 \subseteq V \), the identity \( u \cong v \) is 0-well-balanced. Then there is the smallest \( p \) such that \( 0 < p \leq k \) and the identity \( u \cong v \) is not \( p \)-well-balanced. In view of Corollary 2.13, \( F_{p+1} \not\subseteq V \). According to Lemma 3.3, \( V \) satisfies \( \kappa_p \). Then Lemmas 4.2, 4.3(ii) and 4.8(ii) imply that \( V \) satisfies \( \zeta_k \) and \( \lambda_k^m \) for any
$r = 1, 2, \ldots, m$, and we are done. So, we may assume that the identity $u \approx v$ is $k$-well-balanced.

Now we apply Corollary 2.8 to deduce that there exist letters $x_0, x_1, \ldots, x_{k-1}$ such that

\begin{align*}
    u &= u_{2k+1} x_k u_{2k} x_{k-1} u_{2k-1} x_k u_{2k-2} x_{k-2} u_{2k-3} x_{k-3} u_{2k-4} x_{k-4} u_{2k-5} x_{k-5} \cdots u_1 x_0 u_1 u_0 \\
    &= u_{2k+1} x_k u_{2k} x_{k-1} u_{2k-1} x_k u_{2k-2} x_{k-2} u_{2k-3} x_{k-3} u_{2k-4} x_{k-4} u_{2k-5} x_{k-5} \cdots u_1 x_0 u_1 u_0 \bmod \mathcal{F}
\end{align*}

for some $u_0, u_1, \ldots, u_{2k+1} \in \mathcal{F}^*$, $D(u, x_r) = r$ and $u_{2r+2}$ does not contain any $r$-dividers of $u$. Then the letter $u^r$ lies in $u_{2k}$. Put

\[ X = \{x_0, x_1, \ldots, x_k, x, y, m, y\}. \]

Suppose that $m = 0$. Then $y_0 \in \sim(u) = \sim(v)$ and $u(X) = xy_0xykxy b_k$. Since the identity $u \approx v$ is $k$-well-balanced, $v(X) = xy_0xykxy b_k$ and, therefore, the identity $u(X) \approx v(X)$ coincides with $\zeta_k$, we are done.

Suppose now that $1 \leq m \leq k$. Then $y_m \in \mul(u) = \mul(v)$. According to the choice of $y_m$ and $x_k$, the subwords $w_2$ and $w_3$ do not contain any $(m-1)$-divider of $u$. It follows from Lemma 2.9 that $2uy_m$ occurs in either $u_{2m}$ or $u_{2m-1}$ or $u_{2m-2}$. Let $y_{m-1} = h^{-1}_m (u, y_m)$. Since $u_{2m}$ does not contain any $(m-1)$-dividers of $u$, the letter $u_m$ lies between $u_{2m}$ and $u_{2m-1}$ or $u_{2m-2}$. Then

\[ u(X) = xy_m y_k x y b_{k, m+1} x_{m-1} cb_{m-1} \]

where $c \in \{x_m y_m, y_m x_m\}$. Since $u \approx v$ is $k$-well-balanced, $2v y_m$ lies between $1v x_{m-1}$ and $1v x_m$ and

\[ v(X) = xy_m y_k x y b_{k, m+1} x_{m-1} db_{m-1} \]

where $d \in \{x_m y_m, y_m x_m\}$. We see that the identity $u(X) \approx v(X)$ coincides with

\[ xy_m y_k x y b_{k, m+1} x_{m-1} cb_{m-1} \approx xy_m y_k x y b_{k, m+1} x_{m-1} cb_{m-1} \]

Then $V$ satisfies $\lambda_k^m$ because

\begin{align*}
    (3.1) & \implies xy_m y_k x y \cdots x_{m-1} x_{m-2} x_{m-1} \cdots x_1 x_2 x_0 x_1 \\
    (3.2) & \implies xy_m y_k x y \cdots x_{m-1} x_{m-2} x_{m-1} \cdots x_1 x_2 x_0 x_1 \\
    (4.10) & \implies xy_m y_k x y \cdots x_{m-1} x_{m-2} x_{m-1} x_{m-1} x_1 x_2 x_0 x_1 \\
    (4.10) & \implies xy_m y_k x y \cdots x_{m-1} x_{m-2} x_{m-1} x_{m-1} x_1 x_2 x_0 x_1 \\
    (3.1) & \implies xy_m y_k x y \cdots x_{m-1} x_{m-2} x_{m-1} \cdots x_1 x_2 x_0 x_1,
\end{align*}

and we are done. \hfill £

\begin{lemma}
Let $V$ be a monoid variety from the interval $[F_1, P]$ and $u \approx v$ be an identity of $V$ such that

\[ u = w_1 x w_2 y_{m+1} w_3 y w_4 x y w_5 y_{m+1} w_6 \]

for some words $w_1, w_2, w_3, w_4, w_5, w_6$ and $(2v x) < (2v y)$. Suppose that $(2u a) < (2v y_{m+1})$ for any $a \in \con(w_2 w_3) \setminus \con(w_1)$. Suppose also that the depth of the subwords $w_{2m-1} w_3$, $w_4$ and $w_5$ of $u$ is equal to $m + 1$, $m$ and $k$, respectively.

(i) If $k \geq 0$ and $m = 0$, then $V$ satisfies $\eta_k$.

(ii) If $1 \leq m \leq k$, then $V$ satisfies $\mu_k^m$.
\end{lemma}
Proof. If \( \text{LRB} \not\subseteq \text{V} \), then \( \text{V} \subseteq \text{F} \) by Lemma 3.2. Evidently, \( \text{F} \) satisfies \( \eta_k \) and \( \mu_k^m \) for any \( k \geq 0 \) and \( m \leq k \). So, we may further assume that \( \text{LRB} \subseteq \text{V} \). Further, Lemma 3.1 allows us to assume that the identity \( u \approx v \) is 2-limited.

Let \( x_k \) be a \( k \)-divisor of \( u \) that occurs in \( w_5 \). In view of Lemma 2.3, \( D(u, x_k) = k \) and \( D(u, y_{m+1}) = m + 1 \). Now by Corollary 2.8, there exist letters \( x_0, x_1, \ldots, x_{k-1} \) such that \( D(u, x_r) = r \) and

\[
\begin{align*}
  u &= u_{2k+1} x_k u_{2k} x_{k-1} u_{2k-1} x_k u_{2k-2} x_{k-2} u_{2k-3} x_{k-3} u_{2k-4} x_{k-4} x_{k-3} u_{2k-5} x_{k-2} \cdots u_1 x_1 u_0,
\end{align*}
\]

for some \( u_0, u_1, \ldots, u_{2k+1} \in \mathfrak{F} \) such that \( u_{2r+2} \) does not contain any \( r \)-dividers of \( u \) for any \( r = 0, 1, \ldots, k - 1 \). Since the depth of \( w_5 \) in \( u \) is equal to \( k \), the letter \( 2u_{ym+1} \) lies in \( u_{2k} \). Let \( y_m \) be an \( m \)-divisor of \( u \) that occurs in \( w_4 \). Clearly,

\[
(1u^x) < (1uym+1) < (1uy) < (1uy_m) < (2ux) < (2uy) < (1uxk).
\]

Put

\[
X = \{x_0, x_1, \ldots, x_k, x, y_m, y_{m+1}, y\}.
\]

There are two cases.

Case 1: the identity \( u \approx v \) is \( k \)-well-balanced.

Suppose that \( k \geq 0 \) and \( m = 0 \). Then \( y_m = y_0 \in \text{sim}(u) = \text{sim}(v) \) and \( u(X) = xy_1y_0xyk_1y_1b_k \). Since the identity \( u \approx v \) is \( k \)-well-balanced, we have \( v(X) = xy_1y_0yxxk_1y_1b_k \). Therefore, \( u(X) \approx v(X) \) coincides with \( \eta_k \), and we are done.

Suppose now that \( 1 \leq m \leq k \). Then \( y_m \in \text{mul}(u) = \text{mul}(v) \). Clearly, the subwords \( w_4 \) and \( w_5 \) do not contain any \( (m - 1) \)-dividers of \( u \). It follows from Lemma 2.9 that \( 2u_{ym} \) occurs in either \( u_{2m} \) or \( u_{2m-1} \) or \( u_{2m-2} \). Let \( y_m = h_2^{-1}(u, y_m) \). Since \( u_{ym} \) does not contain any \( (m - 1) \)-dividers of \( u \), the letter \( 1u_{ym} \) does not occur in \( u_{2m} \). Therefore, \( 2u_{ym} \) occurs in either \( u_{2m-1} \) or \( u_{2m-2} \). Then

\[
u(X) = xy_{ym+1}y_{ym}xyk_{ym+1}y_{ym}b_{k, m+1} x_{m-1} \ c_{m-1} \ b_{m-1},
\]

where \( c \in \{x_{ym}, y_{ym}x_{ym}\} \). Since \( u \approx v \) is \( k \)-well-balanced, \( 2v_{ym} \) lies between \( x_{ym}x_{m-1} \) and \( x_{ym}x_{m-2} \) and

\[
v(X) = xy_{ym+1}y_{ym}xyk_{ym+1}y_{ym}b_{k, m+1} x_{m-1} \ d_{m-1},
\]

where \( d \in \{x_{ym}, y_{ym}x_{ym}\} \). We see that the identity \( u(X) \approx v(X) \) coincides with

\[
\begin{align*}
  &xy_{ym+1}y_{ym}xyk_{ym+1}y_{ym}b_{k, m+1} x_{m-1} \ c_{m-1} \ b_{m-1}^m \\
  &\approx xy_{ym+1}y_{ym}xyk_{ym+1}y_{ym}b_{k, m+1} x_{m-1} \ c_{m-1} \ b_{m-1}^m,
\end{align*}
\]

Then \( V \) satisfies \( \mu_k^m \) because

\[
\begin{align*}
  (4.1) &\approx x_{ym+1}y_{ym}xyk_{ym+1}\cdots x_{m-1}y_{ym}x_{m-2}x_{m-1}\cdots x_1x_2x_0x_1
  \quad (4.11) \approx x_{ym+1}y_{ym}xyk_{ym+1}\cdots x_{m-1}y_{ym}x_{m-2}x_{m-1}\cdots x_1x_2x_0x_1,
\end{align*}
\]

and we are done.
Case 2: the identity \( u \approx v \) is not \( k \)-well-balanced. In view of Corollary 2.13 and the inclusion \( F_1 \subset V \), the identity \( u \approx v \) is \( 0 \)-well-balanced. Then there is the smallest \( n \) such that \( 0 < n \leq k \) and the identity \( u \approx v \) is not \( n \)-well-balanced. In view of Corollary 2.13, \( F_{n+1} \not\subset V \). Then \( F_n \subset V \). According to Lemma 3.3, \( V \) satisfies \( \kappa_n \).

Suppose that \( k \geq 0 \) and \( m = 0 \). Then \( y_n = y_0 \in \text{sim}(u) = \text{sim}(v) \) and \( u(X) = xy_1yy_0xy_ky_1b_k \). Lemma 2.10 and the inclusion \( \text{LRB} \subset V \) together with the fact that the identity \( u \approx v \) is \((n - 1)\)-well-balanced imply \( v(X) = xy_1yy_0ab_n \) for some word \( a \) with \( \text{sim}(a) = \{x_n, x, y_1, y\} \) and \( \text{mul}(a) = \{x_{n+1}, x_{n+2}, \ldots, x_k\} \). According to Lemma 2.10, \( \text{ini}(a(x,y)) = x_kx_{k-1} \cdots x_n \). Further, \( a(x,y) = yx \) because \((2v y) < (2v x)\). Then \( V \) satisfies the identities

\[
xy_1yy_0ab_n \approx xy_1yy_0axyy_1b_n \quad \text{by (3.1)}
\]

\[
\approx xy_1yy_0 \text{ini}(axyy_1) \text{ini}((axyy_1)_x_n) b_n \quad \text{by Lemma 4.6}
\]

\[
= xy_1yy_0 \text{ini}(a) \text{ini}(a_n) b_n.
\]

Since \((1a y) < (1a x)\), it follows from Proposition 4.7 that \( \Phi(V) \) together with (4.12)

\[
xytxy \approx xytyx
\]

imply the identity

\[
\text{ini}(a) \text{ini}(a_n) b_n \approx a^0b_n,
\]

where \( \text{sim}(a^0) = \{x_n\} \) and \( \text{ini}(a^0) = yxx_ky_1x_{k-1} \cdots x_n \). Now we apply Lemma 4.6 again and obtain that \( V \) satisfies the identities

\[
a^0b_n \approx \text{ini}(a^0) \text{ini}(a_n) b_n = yxx_ky_1x_{k-1} \cdots x_nyxx_ky_1x_{k-1} \cdots x_{n+1}b_n
\]

\[
= \text{ini}(y^2x^2 x_ky_1^2b_{k,n+1}) \text{ini}((y^2x^2 x_ky_1^2b_{k,n+1})_{x_n}) b_n \approx y^2x^2 x_ky_1^2b_{k,n+1}b_k
\]

\[
= y^2x^2 x_ky_1^2b_k.
\]

Then the identities

\[
xy_1yy_0a^0b_n \approx xy_1yy_0 y^2x^2 x_ky_1^2b_k \quad (3.1) \approx xy_1yy_0 yxx_ky_1b_k
\]

hold in \( V \). Therefore, \( V \) satisfies

\[
xy_1yy_0ab_n \approx xy_1yy_0 yxx_ky_1b_k
\]

and so \( \eta_k \).

Suppose now that \( 1 \leq m \leq k \). Then \( y_m \in \text{mul}(u) = \text{mul}(v) \). It is routine to verify that the depth of the letter \( y \) in both sides of the identity \( \mu^m_k \) is equal to \( m + 1 \). Thus, if \( n \leq m \), then \( V \) satisfies \( \mu^m_k \) by Corollary 4.5. So, we may assume that \( m < n \). Clearly, the subwords \( w_4 \) and \( w_5 \) do not contain any \((m - 1)\)-divider of \( u \). It follows from Lemma 2.9 that \( 2u y_m \) occurs in either \( u_{2m} \) or \( u_{2m-1} \) or \( u_{2m-2} \). Let \( y_{m-1} = h_{m-1}^2(u, y_m) \). Since \( u_{2m} \) does not contain any \((m - 1)\)-divider of \( u \), the letter \( 1u y_{m-1} \) does not occur in \( u_{2m} \). Therefore, \( 2u y_m \) occurs in either \( u_{2m-1} \) or \( u_{2m-2} \). Then

\[
u(X) = xy_{m+1}yy_mxy_ky_{m+1}b_{k,m+1} x_{m-1} c b_{m-1},
\]

where \( c \in \{x_m y_m, y_m x_m\} \). Lemma 2.10 and the inclusion \( \text{LRB} \subset V \) together with the facts that \( m < n \) and \( u \approx v \) is \((n - 1)\)-well-balanced imply that \( 2v y_m \) lies between \( 1v x_{m-1} \) and \( 1v x_{m-2} \) and

\[
v(X) = xy_{m+1}yy_mab_{n,m+1} x_{m-1} db_{m-1},
\]
sim(a) = \{x_n, x, y, y_{m+1}\}, mul(a) = \{x_{n+1}, x_{n+2}, \ldots, x_k\} and \(d \in \{x_my, y_mx\}\). As in the previous paragraph, one can show that \(V\) satisfies the identity

\[
xy_{m+1}yy_m a b_{n, m+1} x_{m-1} dB_{m-1} \approx xy_{m+1}yy_m x y_k y_{m+1} b_{k, m+1} x_{m-1} dB_{m-1}
\]

and so the identity (4.11). As we have verified in Case 1, the last identity together with (3.1) imply \(\mu_k^m\). We see that the identity \(\mu_k^m\) holds in \(V\). □

**Lemma 4.15.** Let \(V\) be a variety from the interval \([LRB, P]\). Suppose that \(V\) satisfies the identity

\[
(4.13) \quad xx_p yzyx^2 z^2 b_{p} \approx xx_p cb_{p},
\]

where

\[
(4.14) \quad c = c_1 y c_2 y x c_4 z^2 c_5 \quad \text{and} \quad c_1 c_2 c_3 c_4 c_5 = z^2.
\]

Then the identity \(\nu_p\) holds in \(V\).

**Proof.** According to Lemma 2.10 and the inclusion \(LRB \subset V\), \(ini(c) = yz z_\infty\). Therefore, \(c_1 = \lambda\) and \(c_2 c_3 c_4 \neq \lambda\). If \(c_2 = \lambda\), then it follows from Proposition 4.7 that \(\Phi(V)\) implies the identity \(xx_p cb_p \approx xx_py z b_p\), where \(d\) is a word obtained from \(y c_4 x c_4 z c_5\) by removing the first occurrence of \(z\). So, we may assume that \(z \in \text{con}(c_2)\). If \(z \in \text{con}(c_5)\), then (4.13) is nothing but \(\nu_p\). So, it remains to consider the case when \(c_5 = \lambda\). In this case, we substitute \(z_\infty = z\in\infty\) in the identity \(\mu(z_\infty) = v(z, z_\infty)\). As a result, we obtain an identity that is equivalent modulo (3.1) to (3.2). Since \(ini^2(c) = y^2 z \approx x^2 z_\infty^2 = ini^2(yz x z_\infty^2 z)\), Corollary 3.5 implies that \(V\) satisfies the identity \(xx_p cb_p \approx xx_py_z z_\infty^2 b_p\). Therefore, the identity \(\nu_p\).

□

Let \(O_1\) denote the variety defined by the identities (3.1), (4.9) and

\[
(4.15) \quad xzytxy z_\infty^2 z \approx xzytxy z_\infty^2 z.
\]

Put

\[
\Psi_1 = \{(3.1), (4.9), (4.15), \zeta_{k-1}, \lambda_k^m, \eta_{k-1}, \mu_k^m, \nu_{k-1} | k \in \mathbb{N}, 1 \leq m \leq k\}.
\]

**Proposition 4.16.** Let \(V\) be a monoid variety from the interval \([LRB \land F_1, O_1]\), \(u \approx v\) be an identity of \(V\) and \(\{x_2, x_2\}\) be a critical pair in \(u \approx v\). Then the critical pair \(\{x_2, x_2\}\) is \(\Delta\)-removable for some \(\Delta \subseteq \Psi_1(V)\).

**Proof.** We consider only the case when \((1u) < (1y)\) since the case when \((1y) < (1u)\) is similar. Then \(u = w_1 x w_2 y w_3 x y w_4\) for some words \(w_1, w_2, w_3, w_4\). According to the inclusion \(LRB \subset V\) and Lemma 2.10, \((1v) < (1y)\). Let \(w\) be the word obtained from \(u\) by replacing \(2u x 2u y\) with \(2u y 1u x\). Lemma 3.1 allows us to assume that the identity \(u \approx v\) is 2-limited.

In view of Remark 2.2, there is a number \(d\) such that the \(d\)-decomposition of \(u\) is maximal. Let \(a\) denote the \(d\)-block of \(u\), which contains the critical pair \(\{x_2, x_2\}\). Then \(a = a' x y a''\) and \(u = u' a u''\) for some \(a', a', u', u'' \in \mathbb{N}\). Clearly, \(\text{con}(a) \subseteq \text{mul}(u'a)\).

If \(\text{con}(w_2) \subseteq \text{mul}(u'a)\), then the identity \(u \approx w\) follows from \(\{(3.1), (4.15)\} \subseteq \Psi_1(V)\) because

\[
(4.14) \quad u \approx w_1 x w_2 y w_3 x y (a'')^2 w_2 u'' \approx w_1 x w_2 y w_3 x (a'')^2 w_2 u'' \approx w.
\]
So, we may assume that con$(w_2) \not\subseteq \text{mul}(u'a)$. Then there exist a letter $x_p$ and a number $p \in \mathbb{N} \cup \{0\}$ such that the following claims hold:

- $1u^p$ occurs in the subword $w_2$ of $u$;
- $D(u, x_p) = p$;
- the depth of $w_2$ in $u$ is equal to $p$;
- if $p > 0$, then $2u^p$ occurs in the subword $u''$ of $u$.

There are two subcases.

Case 1: $D(u, y) = \infty$. Then the first occurrence of $y$ in $u$ lies in the $d$-block $a$. This means that $a' = a_1 \cdot y \cdot w_3$ for some $a_1 \in \mathbb{A}^*$ such that $y \notin \text{con}(a_1)$.

Clearly, the depth of $w_3$ in $u$ is infinite. If the depth of $w_3$ in $u$ is strictly infinite, then the identity $u \approx w$ follows from $\{(3.1), (4.9)\} \subseteq \Psi_1(V)$ by Lemma 4.12. So, it remains to consider the case when the depth of $w_3$ in $u$ is not strictly infinite. Then there are letters $z$ and $z_{\infty}$ such that $(1u^z) < (1u^z_{\infty}) < (2u^z)$ and $1u^z$ lies in the subword $w_3$ of $u$, while $1u^z_{\infty}$ and $2u^z$ lie in the subword $a''$ of $u$.

Now by Corollary 2.8, there exist letters $x_0, x_1, \ldots, x_{p-1}$ such that

$$u = u_{2p+1} x_p u_{2p} x_{p-1} u_{2p-1} x_p u_{2p-2} x_{p-2} u_{2p-3} x_{p-3} u_{2p-4} x_{p-4} + \cdots + u_{2p-5} x_{p-5} \cdots u_1 x_1 u_0$$

for some $u_0, u_1, \ldots, u_{2p+1} \in \mathbb{A}^*$ with $D(u, x_r) = r$. Clearly, $(1u^{x}) < (1u^{y}) < (2u^{z}) < (1u^{x_{p-1}})$.

Corollary 2.13 and the inclusion $F_1 \subseteq V$ imply that the identity $u \approx v$ is 0-well-balanced. Then if the identity $u \approx v$ is not $p$-well-balanced, then there is the smallest $k$ such that $0 < k \leq p$ and the identity $u \approx v$ is not $k$-well-balanced. In view of Corollary 2.13, $F_{k+1} \not\subseteq V$. According to Lemma 3.3, $V$ satisfies $\kappa_k$. Then Lemmas 4.2, 4.3(ii) and 4.8(i) imply that $V$ satisfies $\nu_p$. Now we apply Lemma 4.11 and conclude that the identity $u \approx w$ follows from $\{(3.1), \nu_p\} \subseteq \Psi_1(V)$ by Lemma 4.11.

So, we may assume that the identity $u \approx v$ is $p$-well-balanced. Now we substitute $yxz_{\infty}$ for $z_{\infty}$ in the identity

$$u(x_0, x_1, \ldots, x_p, x, y, z_{\infty}) \approx v(x_0, x_1, \ldots, x_p, x, y, z, z_{\infty})$$

Since $(2u^z) < (2u^x)$, we obtain an identity that is equivalent modulo (3.1) to the identity (4.13) such that (4.14) is true. Now we apply Lemma 4.15 and conclude that $\nu_p$ is satisfied by $V$. Then the identity $u \approx w$ follows from $\{(3.1), \nu_p\} \subseteq \Psi_1(V)$ by Lemma 4.11.

Case 2: $D(u, y) < \infty$, say $D(u, y) = s + 1$ for some $s \in \mathbb{N} \cup \{0\}$. This means that the depth of the subword $w_3$ in $u$ is equal to $s$. The choice of $x_p$ implies that $p \leq s + 1$.

Suppose that $p \leq s$. If $p = 0$, then $V$ satisfies $\zeta_s$ by Lemma 4.13(i). Then we apply Lemma 4.9(ii) and conclude that the identity $u \approx w$ follows from $\{(3.1), \zeta_s\} \subseteq \Psi_1(V)$. If $p > 0$, then $V$ satisfies $\lambda_p^s$ by Lemma 4.13(ii). Then the identity $u \approx w$ follows from $\{(3.1), \lambda_p^s\} \subseteq \Psi_1(V)$ by Lemma 4.9(ii).

Suppose now that $p = s + 1$. Evidently, we may assume without loss of generality that $(2a^u) \leq (2a^u x_p)$ for any $a \in \text{con}(w_2) \setminus \text{con}(w_3)$. Let $uu'' = u_1^{u''} u_2 u_p u_3^u$. Since the first letter of $u''$ is a $d$-divider of $u$, the depth of the subword $a'u''$ in $u$ is equal to some number, say $k$. Clearly, $p - 1 = s \leq k$. If $p = 1$, then $V$ satisfies $\eta_k$ by Lemma 4.14(i). Then we apply Lemma 4.10(i) and conclude that the identity $u \approx w$
follows from \{(3.1), \eta_k\} \subseteq \Psi_1(V). If p > 1, then \(V\) satisfies \(\mu_k^i\) by Lemma 4.14(ii).
Then the identity \(u \approx w\) follows from \{(3.1), \mu_k^i\} \subseteq \Psi_1(V) by Lemma 4.10(ii). □

Let \(O_2 = P\{(4.15), \nu_1\}\). Put
\[
\Psi_2 = \{\{(3.1), \eta_k\}, \zeta_{k-1}, \lambda_k^m, \eta_k^{-1}, \mu_k^m, \nu_0, \nu_1 | k \in \mathbb{N}, 1 \leq m \leq k\}.
\]

**Proposition 4.17.** Let \(V\) be a monoid variety from the interval \([\text{LRB} \lor F_1, O_2]\), \(u \approx v\) be an identity of \(V\) and \(\{2x, 2y\}\) be a critical pair in \(u \approx v\). Then the critical pair \(\{2x, 2y\}\) is \(\Delta\)-removable for some \(\Delta \subseteq \Psi_2(V)\).

**Proof.** We consider only the case when \((1u_x) \prec (1u_y)\) since that case when \((1u_y) \prec (1u_x)\) is similar. Then \(u = w_1 x y w_3 x y w_4\) for some words \(w_1, w_2, w_3, w_4\).

The inclusion \(\text{LRB} \subseteq V\) and Lemma 2.10 imply that \((1v_x) \prec (1v_y)\).
Let \(w\) be the word obtained from \(u\) by replacing \(2ux 2uy\) with \(2ux 1u_x 1u_y\). Lemma 3.1 allows us to assume that the identity \(u \approx v\) is 2-limited.

In view of Remark 2.2, there is a number \(d\) such that the \(d\)-decomposition of \(u\) is maximal. Let \(a\) denote the \(d\)-block of \(u\), which contains the critical pair \(\{2x, 2y\}\).
Then \(a = a' 2ux 2uy a''\) and \(u = u'au''\) for some \(a', a'', u', u'' \in \mathbb{A}^*\).
Clearly, \(\text{con}(a) \subseteq \text{mul}(u'a)\).

Case 1: \(D(u, y) = \infty\). Then the first occurrence of \(y\) in \(u\) lies in the \(d\)-block \(a\).
This means that \(a' = a_1 y w_3\) for some \(a_1 \in \mathbb{A}^*\) such that \(y \not\in \text{con}(a_1)\).

Suppose that \(w_2\) contains some simple letter \(t\) of \(u\). In view of the inclusion \(F_1 \subseteq V\) and Corollary 2.13, the identity \(u \approx v\) is \(0\)-well-balanced. Hence \((1v_x) \prec (1v_y)\). Then the \(V\) satisfies (4.9) because it coincides with \(u(x, y, t) \approx v(x, y, t)\).
Hence \(V\) is a subvariety of \(O_1\). Now Proposition 4.16 applies and we conclude that \(u \approx w\) follows from \(\Psi_1(V)\). In view of Lemma 4.8(i), \(P\{\nu_1\} \subseteq P\{\nu_r\}\) for any \(r \in \mathbb{N}\).
Then, since \(\nu_1\) holds in \(V\), the identities in \(\Psi_1(V)\) follow from \(\Psi_2(V)\). Therefore, \(\Psi_2(V)\) implies \(u \approx w\).

So, it remains to consider the case when \(w_2\) does not contain simple letters of \(u\). Let \(\phi\) be the substitution \((x_0, x_1, z, z, z) \mapsto (u'', w_2, a'', w_3)\). Then the identity \(\phi(\nu_1)\) coincides with
\[
(4.16) \quad xw_2yw_3xy(a'')^2w_3u''w_2 \approx xw_2yw_3y(x(a'')^2w_3u''w_2.
\]
Then the identity \(u \approx w\) follows from \{(3.1), \nu_1\} \subseteq \Psi_2(V) because
\[
(3.1) \quad u \approx w_1xw_2yw_3xy(a'')^2w_3u''w_2 \approx xw_1xw_2yw_3yw_3y(x(a'')^2w_3u''w_2 \approx w,
\]
and we are done.

Case 2: \(D(u, y) < \infty\), say \(D(u, y) = s + 1\) for some \(s \in \mathbb{N} \cup \{0\}\). This means that the depth of the subword \(w_2\) in \(u\) is equal to \(s\). If \(\text{con}(w_2) \subseteq \text{mul}(u'a)\), then the identity \(u \approx w\) follows from \{(3.1), (4.15)\} \subseteq \Psi_2(V) because
\[
(3.1) \quad u \approx w_1xw_2yw_3xy(a'')^2w_3u''w_2 \approx xw_1xw_2yw_3yw_3y(x(a'')^2w_3u''w_2 \approx w,
\]
and we are done.

So, we may assume that \(\text{con}(w_2) \not\subseteq \text{mul}(u'a)\). Then there exist a letter \(x_p\) and a number \(p \in \mathbb{N} \cup \{0\}\) such that \(D(u, x_p) = p\); the letters \(1ux_p\) and \(2ux_p\) occur in the subwords \(w_2\) and \(u''\) of \(u\), respectively; and the depth of \(w_2\) in \(u\) is equal to \(p\). The choice of \(x_p\) implies that \(p \leq s + 1\). Further, by similar arguments as in the last two paragraphs of the proof of Proposition 4.16, one can show that \(u \approx v\) follows from \(\Phi_2(V)\). □
Corollary 4.18. For any \( i = 1, 2 \), each variety from the interval \([\text{LRB} \lor F_1, O_i]\) can be defined within \( O_i \) by identities from \( \Phi \cup \Psi_i \). Consequently, the variety \( O_i \) is HFB.

Proof. Let \( V \) be a variety from \([\text{LRB} \lor F_1, O_i]\). The inclusion \( O_i \subseteq P \) and Lemmas 2.12 and 3.1 imply that \( V \) can be defined within \( P \) by a set of 2-limited balanced identities. In view of Lemma 2.10, \( \text{ini}(u) = \text{ini}(v) \) for any identity \( u \approx v \) of \( V \). Then Lemma 4.1 and Propositions 4.7, 4.16 and 4.17 imply that the identity system \((\Phi \cup \Psi_i)(V)\) forms an identity basis for \( V \). It follows from Lemmas 4.3 and 4.8 that every subset of \( \Phi \cup \Psi_i \) defines a FB subvariety of \( P \). Therefore, \( V \) is FB.

So, it remains to show that if \( U \) is a subvariety of \( O_i \) that does not contain \([\text{LRB} \lor F_1] \), then \( U \) is FB. Clearly, either \( \text{LRB} \not\subset U \) or \( F_1 \not\subset U \). Then by Lemma 3.2, either \( U \subseteq F \) or \( U \subseteq \text{LRB} \lor C \). Since \( F \) is HFB by Lemma 2.11 and \( \text{LRB} \lor C \) is also HFB [20, Proposition 4.1], \( U \) is HFB in any case. \( \square \)

5. Limit subvarieties of \( P \)

5.1. The limit variety \( J_1 \).

Lemma 5.1. The variety \( P\{\kappa_1\} \) violates the identity (4.12).

Proof. Let \( u \) be a word such that \( P\{\kappa_1\} \) satisfies \( xytxy \approx u \) and \( \text{ini}_2(u) = xytxy \). In view of Proposition 2.1, it suffices to show that if the identity \( u \approx v \) is directly deducible from some identity \( s \approx t \in \{(3.1), \kappa_1\} \), that is, \( \{u, v\} = \{a\zeta(s)b, a\zeta(t)b\} \) for some words \( a, b \in A^* \) and some endomorphism \( \zeta \) of \( A^* \), then \( \text{ini}_2(v) = xytxy \). Arguing by contradiction suppose that \( \text{ini}_2(v) \neq xytxy \). According to Lemmas 2.10 and 2.12 and the inclusion \( \text{LRB} \lor F_1 \subset P\{\kappa_1\} \), \( \text{ini}_2(v) = xytxy \). In particular, \( (2xy) < (2xy) \).

Clearly, \( s \approx t \) does not coincide with \( (3.1) \) because it does not change the first and the second occurrences of letters in a word. Suppose that \( s \approx t \) coincides with \( \kappa_1 \). We consider only the case when \( (s, t) = (xx_1x_2x_1x_2, x^2x_1x_2x_1) \) because the other case is similar. Since the left side of \( \kappa_1 \) differs from the right side of \( \kappa_1 \) only in swapping of the first occurrence of \( x_1 \) and the second occurrence of \( x \) and \( (2xy) < (2xy) \), there are three possibilities:

- \( \xi(1x_1) \) contains \( 2ax \) and \( \xi(2ax) \) contains \( 2ay \);
- \( 2ux_2uy \) is a subword of \( \xi(1ax) \) and \( \xi(2ax) \) contains some non-first and non-second occurrence of \( y \);
- \( \xi(1ux) \) contains \( 1ax \) and \( 2ux_2uy \) is a subword of \( \xi(2ax) \).

In any case, \( \xi(1ux) \) contains \( 1ay \), while \( \xi(2ax) \) contains some non-first occurrence of \( y \). Then \( t \in \text{sim}(\xi(x_1x)) \) because \( (1ux) < (1ut) < (2uy) \). But this is impossible because \( \xi(x_1) \neq \text{ini}(u) \), while \( x, x_1 \in \text{mul}(s) \). Therefore, \( s \approx t \) cannot coincide with \( \kappa_1 \) as well. \( \square \)

For arbitrary \( n \in \mathbb{N} \) and \( \pi, \tau \in S_{2n} \), we put

\[
\begin{align*}
\mathbf{w}_u[\pi, \tau] &= \left( \prod_{i=1}^{n} x_i t_i \right) x \left( \prod_{i=1}^{2n} z_i \right) y \left( \prod_{i=n+1}^{2n} t_i x_i \right) t x y \left( \prod_{i=1}^{2n} x_i z_i t_i \right), \\
\mathbf{w}_e[\pi, \tau] &= \left( \prod_{i=1}^{n} x_i t_i \right) x \left( \prod_{i=1}^{2n} z_i \right) y \left( \prod_{i=n+1}^{2n} t_i x_i \right) t y x \left( \prod_{i=1}^{2n} x_i z_i t_i \right).
\end{align*}
\]
Let $J_1 = \text{var} \Omega_1$, where
\[
\Omega_1 = \{(3.1), \kappa_1, w_n[n, \tau] \approx w'_n[n, \tau] \mid n \in \mathbb{N}, \; \tau \in S_{2n}\}.
\]

**Lemma 5.2.** The variety $J_1$ violates the identity
\[
xsytxy \approx xsytxy.
\]

**Proof.** Let $u$ be a word such that $J_1$ satisfies $xsytxy \approx u$ and $\text{ini}_2(u) = xsytxy$.

In view of Proposition 2.1, to verify that $J_1$ violates (5.1), it suffices to show that if the identity $u \approx v$ is directly deducible from some identity $s \approx t \in \Omega_1$, that is, \{s, t\} = \{a\zeta(s)b, a\zeta(t)b\} for some words $a, b \in \mathbb{A}^*$ and some endomorphism $\xi$ of $\mathbb{A}^*$, then $\text{ini}_2(v) = xsytxy$. Arguing by contradiction suppose that $\text{ini}_2(v) \neq xsytxy$. According to Lemma 2.12 and the inclusion $F_1 \subset J_1$, $\text{ini}_2(v) = xsytxy$.

In particular, $(2v, y) < (2v, x)$. However, every word $w$ of length $\leq 2v$ belongs to $\mathbb{A}^*$ and if $w$ contains a subword of the form $b^m$, then $b$ contains a subword of the form $b^m$, contradicting (3.1). Therefore, $\text{ini}_2(v) = xsytxy$.

In view of Lemma 5.1, $s \approx t$ cannot coincide with (3.1) or $\kappa_1$. Therefore, $s \approx t$ coincides with $w_n[n, \tau] \approx w'_n[n, \tau]$ for some $n \in \mathbb{N}$ and $\tau \in S_{2n}$. We consider only the case when \(s, t = (w_n[n, \tau], w'_n[n, \tau])\) because the other case is similar. Since $w_n[n, \tau]$ differs from $w'_n[n, \tau]$ only in swapping of the second occurrences of letters $x$ and $y$ and $(2v, y) < (2v, x)$, there are three possibilities:

(a) $\xi(2x)$ contains $2ux$ and $\xi(2y)$ contains $2uy$;

(b) $2ux2uy$ is a subword of $\xi(2x)$ and $\xi(2y)$ contains some occurrence of $y$;

(c) $\xi(2x)$ contains $1ux$ and $2ux2uy$ is a subword of $\xi(2y)$.

Suppose that (a) holds. Then $\xi(1ux)$ contains $1ux$ and $\xi(1uy)$ contains $1uy$. Therefore, $s \in \text{con}(\xi(xz_1z_2 \cdots z_{2k}))$, contradicting $s \in \text{sim}(u)$ and $\{x, y, z_1, z_2, \ldots, z_{2k}\} \subset \text{mul}(s)$. Suppose that (b) holds. Then $\xi(1ux)$ contains both $1ux$ and $1uy$. Hence $s \in \text{con}(\xi(u))$, which contradicts $s \in \text{sim}(u)$ and $x \in \text{mul}(s)$. Finally, (c) is impossible because $\xi(2x)$ cannot contain any first occurrence of a letter in $u$. We see that $s \approx t$ cannot coincide with $w_n[n, \tau] \approx w'_n[n, \tau]$ as well. Therefore, $J_1$ violates (5.1). $\Box$

**Lemma 5.3.** Let $V$ be a subvariety of $P$. If $V$ does not contain $J_1$, then $V$ satisfies the identity (4.15).

**Proof.** If $V$ does not contain $LRB$ or $F_1$, then $V$ is contained in either $F$ or $LRB \lor C$ by Lemma 3.2. Evidently, both $F$ and $LRB \lor C$ satisfy (4.15). So, we may assume that $V$ belongs to the interval $[LRB \lor F_1, P]$.

In view of Lemma 2.12 and the inclusion $F_1 \subset V$, we have $\text{sim}(a) = \text{sim}(b)$ and $\text{mul}(a) = \text{mul}(b)$ for any identity $a \approx b$ of $V$. This fact and Lemma 3.1 imply that any identity of $V$ is equivalent to $\text{ini}_2(v) < (2v, x)$ containing $2ux2uy$.

If there exists a 2-limited balanced identity $u \approx v$ of $V$ with a critical pair $\{i, j\}$ that is not $\Omega_1$-removable. It follows from the inclusion $LRB \subset V$ and Lemma 2.10 that $(i, j) \neq (1, 1)$. Then $(i, j) \in \{(1, 2), (2, 1), (2, 2)\}$ because the identity $u \approx v$ is 2-limited.

Suppose that $(i, j) = (1, 2)$. According to Proposition 4.7, the identity $u \approx v$ may be chosen from the set
\[
\Phi = \{(3.1), (3.2), \gamma_k, \delta^m_k, \varepsilon_{k-1} \mid k \in \mathbb{N}, \; 1 \leq m \leq k\}.
\]

Evidently, $u \approx v$ does not coincide with (3.1) because it holds in $J_1$. Since $J_1$ satisfies $\kappa_1$, it also satisfies $\delta^m_k$ and $\varepsilon_k$ for any $k \in \mathbb{N}$ and $1 \leq m \leq k$ by Lemmas 4.2 and 4.3(ii),(iii). Besides that, (3.2) holds in $J_1$ because it is a consequence of $\kappa_1$. "$\Box"
It follows that $u \approx v$ coincides with either $e_0$ or $\gamma_k$ for some $k \in \mathbb{N}$. It is easy to see that (4.15) is a consequence of $e_0$ and so $\gamma_k$ for any $k \in \mathbb{N}$.

So, it remains to consider the case when $(i, j) = (2, 2)$. By symmetry, we may assume that $(1u^ix) < (1u^iy)$. Then $(1v^i x) < (1v^i y)$ by Lemma 2.10 and the inclusion $LRB \subset V$. If there are no simple letters between $1u^i y$ and $2u^i y$, then $D(u, y) > 1$ and so the critical pair $(x, 2y)$ is $(3.1, \kappa_1)$-removable in $u \approx v$ by Corollary 4.5. So, we may assume that there is a simple letter $t$ between $1u^i y$ and $2u^i y$.

If, for any letter $a$ with $(1u^i x) < (1u^i a) < (1u^i y) < (2u^i y) < (2u^i a)$,

there are no first occurrences of letters between $2u^i y$ and $2u^i a$, then it is easy to see that one can choose $n \in \mathbb{N}$ and $\pi, \tau \in S_{2n}$ so that the critical pair $(x, 2y)$ is $(3.1, \kappa_1)$-removable in $u \approx v$. So, we may assume that there are letters $z$ and $z^\infty$ such that

$(1u^i x) < (1u^i z) < (1u^i y) < (2u^i y) < (1u^i z^\infty) < (2u^i z)$.

Now we substitute $y^2$ for $z^\infty$ in the identity $u(x, y, t, z, z^\infty) \approx v(x, y, t, z, z^\infty)$. Since $(2y^i x) < (2y^i x)$, we obtain an identity that is equivalent modulo (3.1) to

$xztzy^2 \approx xztwy$,

where $w \in \{y^2, x^2, yxz^2, x^2, x^2, yx^2, yxz^2\}$. If $w = y^2z^2$, then $V$ satisfies (4.15).

So, it remains to consider the case when

$w \in \{y^2, yxz^2, x^2, yxz^2\}$.

In this case, the variety $V$ satisfies the identity

$xztwy \approx xztwy^2$.

because this identity is a consequence of the identity (4.12), which is evidently satisfied by $V$. Now Proposition 4.7 applies and we conclude that the identity $xztzy^2 \approx xztwy^2$ follows from $\Phi(V)$. Hence $V$ satisfies (5.2)

$xztzy^{2}z \approx xztwy^{2}z^\infty$.

Finally, we substitute $z^2$ for $z^\infty$ in the identity (5.2). We obtain an identity that is equivalent modulo (3.1) to (4.15). We see that (4.15) is satisfied by $V$ in either case. $\Box$

Proposition 5.4. The variety $J_1$ is a limit variety.

Proof. According to Lemmas 4.2, 4.3(ii) and 4.8(i),(ii), $\nu_1$ is satisfied by $J_1$. Then Lemma 5.3 implies that any proper subvariety of $J_1$ is contained in $O_2$ and so is FB by Corollary 4.18.

So, it remains to establish that $J_1$ is NFB. It suffices to verify that, for any $n \in \mathbb{N}$, the set of identities

$\Delta_n = \{(3.1, \kappa_1, w_k[\pi, \tau] \approx w_k'[\pi, \tau] \mid k < n, \pi, \tau \in S_{2k}\}$

does not imply the identity $w_n[e, e] \approx w_n'[e, e]$, where $e$ is the trivial permutation on $\{1, 2, \ldots, 2n\}$. Let $w$ be a word such that $J_1$ satisfies $w_n[e, e] \approx w$. According to Lemmas 2.10 and 2.12 and the inclusion $LRB \lor F \subseteq J_1$,

$w = (\prod_{i=1}^{n} x_i t_i) x (\prod_{i=1}^{2n} z_i) y (\prod_{i=n+1}^{2n} t_i x_i) t w'$.
for some word $w'$ with $\mathrm{con}(w') = \{x, y, x_1, z_1, x_2, z_2, \ldots, x_{2n}, z_{2n}\}$. If $1w'x_1$ in $w'$, then the identity $w_n[\varepsilon, \varepsilon](x, t) = 1$ is equivalent modulo $(3.1)$ to $(5.1)$. But this is impossible by Lemma 5.2. Therefore, $1w'x$ precedes $1w'x_1$ in $w'$. By a similar argument we can show that $(1w'y) < (1w'x_1)$ and

$$(1w'x_1) < (1w'z_1) < (1w'x_2) < (1w'z_2) < \cdots < (1w'x_{2n}) < (1w'z_{2n}).$$

This means $\mathrm{ini}(w') = ax_1x_2z_1\cdots x_{2n}z_{2n}$, where $a \in \{xy, yx\}$. So, we have proved that $\mathrm{ini}(w) = \{w_n[\varepsilon, \varepsilon], w_n[\varepsilon, \varepsilon]\}$.

Let $u$ be a word such that $J_1$ satisfies $w_n[\varepsilon, \varepsilon] \approx u$ and $\mathrm{ini}(u) = w_n[\varepsilon, \varepsilon]$. According to Proposition 2.1, to verify that $w_n[\varepsilon, \varepsilon] \approx w_n[\varepsilon, \varepsilon]$ does not follow from $\Delta_n$, it suffices to establish that if the identity $u \approx v$ is directly deducible from some identity $s \approx t$ in $\Delta_n$, that is, $\{u, v\} = \{\alpha(s), \alpha(t)\}$ for some words $a, b \in \mathfrak{A}^*$ and some endomorphism $\xi$ of $\mathfrak{A}$, then $\mathrm{ini}(v) = w_n[\varepsilon, \varepsilon]$. Arguing by contradiction suppose that $\mathrm{ini}(v) \neq w_n[\varepsilon, \varepsilon]$. In view of the previous paragraph, $\mathrm{ini}(v) = w'_n[\varepsilon, \varepsilon]$. In particular, $(2v_y) < (2v_x)$.

Since $\mathrm{ini}(u) = \{x, y, t\} = xyt(xy)$, Lemma 5.1 implies that $s \approx t$ cannot coincide with $(3.1)$ and $\kappa_1$. Suppose that $s \approx t$ coincides with $w_k[\pi, \tau] \approx w_k'[\pi, \tau]$ for some $k < n$ and $\pi, \tau \in S_{2k}$. We consider only the case when $(s, t) = (w_k[\pi, \tau], w_k'[\pi, \tau])$ because the other case is similar. Since $w_k[\pi, \tau]$ differs from $w_k'[\pi, \tau]$ only in swapping of the second occurrences of letters $x$ and $y$ and $(2v_x) < (2v_y)$, there are three possibilities:

(a) $\xi(2x) = 2u$ and $\xi(2y) = 2u$;

(b) $2u x 2u y$ is a subword of $\xi(2x)$ and $\xi(2y)$ contains some occurrence of $y$;

(c) $\xi(2x)$ contains $1u x$ and $2u x 2u y$ is a subword of $\xi(2y)$.

Suppose that (a) holds. Then $\xi(1a) = 1u x$ and $\xi(1a) = 1u y$. Hence $t_n, z_1 \notin \mathrm{con}(\xi(x))$ and $z_{2n}, t_{n+1} \notin \mathrm{con}(\xi(y))$ because $\mathrm{ini}(u) \neq w_n[\varepsilon, \varepsilon]$ otherwise. Therefore, $\xi(x) = x$ and $\xi(y) = y$. It follows that $\xi\left(\prod_{i=1}^{2k} z_i\right) = \prod_{i=1}^{2k} z_i$. Then $\xi(1a) = 1u z_q u z_q + 1$ for some $p \leq 2k$ and $q < 2n$. Since $\mathrm{ini}(u) = w_n[\varepsilon, \varepsilon]$, the word $\xi(2a)$ does not contain $2u z_q 2u z_q + 1$. Hence $2u z_q$ is contained in the subword $\xi\left(\prod_{i=1}^{2k} (x_i, y_i)\right)$ of $u$, which is $p$. Then $1u z_q$ must be contained in $\xi(1a)$ for some $a \in \{z_1, z_2, z_2, z_2, \ldots, z_{r-1}, x_{r-1}, x_{r-1}, x_{r-1}\}$. This contradicts the fact that $1u z_q$ is contained in $\xi(1a)$. Therefore, (a) is impossible. Suppose that (b) holds. Then $\xi(1a)$ contains both $1u x$ and $1u y$. This is only possible when $x_{21} z_2 \cdots z_{2n} y$ is a subword of $\xi(x)$. But this contradicts the fact that $\mathrm{ini}(u) = w_n[\varepsilon, \varepsilon]$ and, therefore, (b) is impossible as well. Finally, (c) is impossible because $\xi(2u)$ cannot contain any first occurrence of a letter in $u$. We see that $s \approx t$ cannot coincide with $w_k[\pi, \tau] \approx w_k'[\pi, \tau]$. So, we have proved that $\Delta_n$ does not imply $w_n[\varepsilon, \varepsilon] \approx w_n'[\varepsilon, \varepsilon]$ and so $J_1$ is NFU.

5.2. The limit variety $J_2$. For arbitrary $n \in \mathbb{N}$ and $\pi, \tau \in S_{2n}$, define

$$z_n[\pi, \tau] = \left(\prod_{i=1}^{2n} x_i t_i x \prod_{i=1}^{n} z_i s_i y \prod_{i=1}^{2n} x_i t_i \right) x t \left(\prod_{i=1}^{n} s_i\right)$$

$$z'_n[\pi, \tau] = \left(\prod_{i=1}^{2n} x_i t_i x \prod_{i=1}^{n} z_i s_i y \prod_{i=1}^{2n} x_i t_i \right) y t \left(\prod_{i=1}^{n} s_i\right).$$
Let $J_2 = \text{var} \omega_2$, where

$$\omega_2 = \{(3.1), \eta_1, z_n[\pi, \tau] \approx z'_n[\pi, \tau] \mid n \in \mathbb{N}, \pi, \tau \in S_{2n}\}.$$

Lemma 5.5. Let $u \approx v$ be an identity of $J_2$. Then $(1u) < (2u)v$ if and only if $(1v) < (2v)w$ for any letters $x$ and $y$.

Proof. In view of Proposition 2.1, it suffices to show that the required conclusion is true for an identity $u \approx v$ of $J_2$ that is directly deducible from some identity $s \approx t \in \omega_2$, that is, \{u, v\} = \{a\xi(s)b, a\xi(t)b\} for some words $a, b \in \mathcal{A}^*$ and some endomorphism $\xi$ of $\mathcal{A}^*$.

If $s \approx t$ coincides with (3.1), then the required conclusion is true because (3.1) does not change the first and the second occurrences of letters in a word. Suppose that $s \approx t$ coincides with some identity from $\omega_2 \setminus \{(3.1)\}$. Then $s$ differs from $t$ only in swapping of the second occurrences of $x$ and $y$. Clearly, both $\xi(2x)$ and $\xi(2y)$ cannot contain first occurrences of letters. It follows that if $(1u) < (2u)v$ for some letters $x$ and $y$, then $(1v) < (2v)w$. □

Lemma 5.6. The variety $P\{\eta_1\}$ violates the identity

$$xx_1yy_0xx_0x_1 \approx xx_1y^2xx_0x_1.$$  

Proof. Let $u$ be a word such that $P\{\eta_1\}$ satisfies $xx_1yy_0xx_0x_1 \approx u$ and $\text{ini}_2(u) = xx_1yy_0xx_0x_1$. In view of Proposition 2.1, it suffices to establish that if the identity $u \approx v$ is directly deducible from some identity $s \approx t \in \{(3.1), \eta_1\}$, that is, \{u, v\} = \{a\xi(s)b, a\xi(t)b\} for some words $a, b \in \mathcal{A}^*$ and some endomorphism $\xi$ of $\mathcal{A}^*$, then $\text{ini}_2(v) = xx_1yy_0xx_0x_1$. Arguing by contradiction suppose that $\text{ini}_2(v) \neq xx_1yy_0xx_0x_1$. According to Lemmas 2.10, 2.12 and 5.5 and the inclusions $\text{LRB} \cap F_1 \subseteq J_2 \subseteq P\{\eta_1\}$, we have $\text{ini}_2(v) = xx_1y^2xx_0x_1$. In particular, $(2v) < (2v)w$.

Clearly, $s \approx t$ does not coincide with (3.1) because it does not change the first and the second occurrences of letters in a word. Suppose that $s \approx t$ coincides with $\eta_1$. We consider only the case when

$$(s, t) = (xy_1y_0xy_1x_1x_0x_1, xy_1y_0yxx_1y_1x_0x_1)$$

because the other case is similar. Since the left side of $\eta_1$ differs from the right side of $\eta_1$ only in swapping of the second occurrences of $x$ and $y$ and $(2v) < (2v)w$, there are three possibilities:

- (a) $\xi(2x)$ contains $2u(x)$ and $\xi(2y)$ contains $2u(y)$;
- (b) $2u(x)2u(y)$ is a subword of $\xi(2x)$ and $\xi(2y)$ contains some non-first and non-second occurrence of $y$;
- (c) $\xi(2x)$ contains $1u(x)$ and $2u(x)2u(y)$ is a subword of $\xi(2y)$.

Suppose that (a) holds. Then $\xi(1x)$ contains $1u(x)$ and $\xi(1y)$ contains $1u(y)$. It follows that $x_1 \in \text{con}(\xi(xy_1))$. Clearly, $x_1 \notin \text{con}(xy)$ because otherwise, some occurrence of $x_1$ lies between $2u(x)$ and $2u(y)$, contradicting $\text{ini}_2(u) = xx_1yy_0xx_0x_1$. Therefore, $1u(x)$ coincides with $\xi(1x)$ and so $\xi(x) = x$, $\xi(y) = y$ and $\xi(y_1) = x_1$. It follows that $x_0 \in \text{con}(\xi(x))$, contradicting $x_0 \in \text{ini}(u)$ and $x_1 \in \text{ini}(s)$. Suppose now that (b) holds, then $\xi(1x)$ contains both $1u(x)$ and $1u(y)$. It follows that $x_1 \in \text{con}(\xi(x))$. But this is impossible because there are no occurrences of $x_1$ between $2u(x)$ and $2u(y)$.

Finally, (c) is impossible because $\xi(2x)$ cannot contain any first occurrence of a letter in $u$. We see that $s \approx t$ cannot coincide with $\eta_1$ as well. Therefore, $P\{\eta_1\}$ violates (5.3). □
Lemma 5.7. The variety $J_2$ violates the identity (4.9).

Proof. Let $u$ be a word such that $J_2$ satisfies $xty(xyst) ≈ u$ and $ini_2(u) = xty(xyst)$. In view of Proposition 2.1, to verify that $J_2$ violates (4.9), it suffices to show that if the identity $u ≈ v$ is directly deducible from some identity $s ≈ t$ in $J_2$, that is \( \{u, v\} = \{a, c(s), a(c(t))\} \) for some words $a, b, c \in A^*$ and some endomorphism $\xi$ of $A^*$, then $ini_2(v) = xty(xyst)$. Arguing by contradiction suppose that $ini_2(v) \neq xty(xyst)$. According to Lemmas 2.10, 2.12 and 5.5 and the inclusion $LRB \lor (J_2 \lor F_1 \lor J_2)$, $ini_2(v) = xty(xyst)$. In particular, $(2uv) < (2uv)$. Since $ini_2(u) ≈ ini_2(v)$ coincides with (4.9), in view of Lemma 5.6 and the fact that (4.9) implies (5.3), the identity $s ≈ t$ cannot coincide with (3.1) or $\eta_2$. Therefore, $s ≈ t$ coincides with $\zeta_k[\pi, \tau] \approx \zeta_k[\pi, \tau]$ for some $k \in \mathbb{N}$ and $\pi, \tau \in S_{2k}$. We consider only the case when $(s, t) = (\zeta_k[\pi, \tau], \zeta_k[\pi, \tau])$ because the other case is similar. Since $\zeta_k[\pi, \tau]$ differs from $\zeta_k[\pi, \tau]$ only in swapping of the second occurrences of letters $x$ and $y$ and $(2uv) < (2uv)$, there are three possibilities:

(a) $\zeta_k[\pi, \tau]$ contains $2ux$ and $\zeta_k[\pi, \tau]$ contains $2uy$;

(b) $2ux2uy$ is a subword of $\zeta_k[\pi, \tau]$ and $\zeta_k[\pi, \tau]$ contains some occurrence of $y$;

(c) $\zeta_k[\pi, \tau]$ contains $1ux$ and $\zeta_k[\pi, \tau]$ contains $2ux2uy$.

Suppose that (a) holds. Then $\zeta_k[\pi, \tau]$ contains $1ux$ and $\zeta_k[\pi, \tau]$ contains $1uy$, so that $t \in \text{con}(\zeta_k[\pi, \tau])$, contradicting \( \{x, y, z_1, s_1, \ldots, z_k, s_k\} \subset \text{mul}(s) \) and $t \in \text{sim}(u)$. Suppose that (b) holds. Then $\zeta_k[\pi, \tau]$ contains both $1ux$ and $1uy$, so that $t \in \text{con}(\zeta_k[\pi, \tau])$, contradicting $t \in \text{sim}(u)$ and $x \in \text{mul}(s)$. Finally, (c) is impossible because $\zeta_k[\pi, \tau]$ cannot contain any first occurrence of a letter in $u$. We see that $s ≈ t$ cannot coincide with $\zeta_k[\pi, \tau]$ as well. Therefore, $J_2$ violates (4.9).

Lemma 5.8. Let $V$ be a subvariety of $P$. If $V$ does not contain $J_2$, then $V$ satisfies one of the identities (4.9) or $\eta_1$.

Proof. If $V$ does not contain $LRB$ or $F$, then $V$ is contained in either $F$ or $P \{\kappa_r\}$ for some $r \in \mathbb{N}$ by Lemma 3.2(i), 2.11 and 3.3. Evidently, $F$ satisfies the identity (4.9), while $P \{\kappa_r\}$ satisfies this identity by Corollary 4.5. So, we may assume that $V$ belongs to the interval $[LRB \lor F, P]$. In particular, Corollary 2.13 implies that every identity of $V$ is $r$-well-balanced for any $r \geq 0$. In view of this fact and Lemma 3.1, any identity of $V$ is equivalent to a 2-limited balanced identity. Then Lemma 4.1 implies that there exist a 2-limited balanced identity $u \approx v$ of $V$ and a critical pair $\{i, j\}$ in $u \approx v$ that is not $\Omega$-removable in $u \approx v$. It follows from the inclusion $LRB \subset V$ and Lemma 2.10 that $(i, j) \neq (1, 1)$. Then $(i, j) \in \{(1, 2), (2, 1), (2, 2)\}$ because the identity $u \approx v$ is 2-limited.

Suppose that $(i, j) = (1, 2)$. According to Proposition 4.7, the identity $u \approx v$ may be chosen from the set

$$\Phi = \{(3.1), (3.2), \gamma_k, \delta_k^m, \varepsilon_{k-1} \mid k \in \mathbb{N}, 1 \leq m \leq k\}.$$

Evidently, $u \approx v$ does not coincide with (3.1) because it holds in $J_2$. It follows from Lemma 4.3 that every identity from $\Phi \setminus \{(3.1)\}$ together with (3.1) imply the identity (3.2). It is routine to verify that (4.9) follows from $\{(3.1), (3.2)\}$. Therefore, (4.9) is satisfied by $V$, and we are done.

So, it remains to consider the case when $(i, j) = (2, 2)$. By symmetry, we may assume that $(1uv) < (1uv)$. Then $(1vx) < (1vx)$ by Lemma 2.10 and the inclusion $LRB \subset V$. If there is a simple letter $t$ between $1ux$ and $1uy$, then the identity $u(x, y, t) \approx v(x, y, t)$ coincides with (4.9) because $u \approx v$ is 0-well-balanced. So, we
may assume that there are no simple letters between \(1_u x\) and \(1_u y\). If, for any letter \(a\) with
\[
(1_u x) < (1_u a) < (1_u y) < (2_u a),
\]
there are no simple letters between \(2_u y\) and \(2_u a\), then it is routine to show that the critical pair \(\{x_2, y_2\}\) is \((3.1), \eta_1\)-removable. So, we may assume that there are letters \(x_0 \in \text{sim}(u)\) and \(x_1 \in \text{mul}(u)\) such that
\[
(1_u x) < (1_u x_1) < (1_u a) < (2_u a) < (1_u y) < (2_u x_0) < (2_u x_1).
\]
If there is a simple letter \(t\) between \(1_u y\) and \(2_u x\), then the identity \(u(x_0, x_1, x, y, t) \approx v(x_0, x_1, x, y, t)\) coincides with the identity \(\eta_0\) (up to renaming of letters) because \(u \approx v\) is \(1\)-well-balanced. It is easy to see that \(\eta_0\) implies \(\nu_1\), and we are done. So, it remains to consider the case when there are no simple letters between \(1_u y\) and \(2_u x\). If, for any letter \(a\) with
\[
(1_u y) < (1_u a) < (2_u a) < (2_u x),
\]
there are no first occurrences of letters between \(2_u y\) and \(2_u a\), then it is easy to see that one can choose \(n \in \mathbb{N}\) and \(\pi, \tau \in S_{2n}\) so that the critical pair \(\{x_2, y_2\}\) is \((3.1), z_n[\pi, \tau] \approx z_n[\pi, \tau]'\)-removable in \(u \approx v\). So, we may assume that there are letters \(z\) and \(z_\infty\) such that
\[
(1_u y) < (1_u z) < (2_u a) < (2_u x) < (1_u z_\infty) < (2_u z) < (2_u x).
\]
We may assume without loss of generality that \((1_u z_\infty) < (2_u z) < (1_u x_0)\) because we can substitute \(z_\infty x_0\) for \(x_0\). Then we substitute \(y x z_\infty^2\) for \(z_\infty\) in the identity
\[
u(x_0, x_1, x, y, z_\infty, z) \approx v(x_0, x_1, x, y, z_\infty, z).
\]
Since the identity \(u \approx v\) is \(1\)-well-balanced and \((2_u y) < (2_u x)\), we obtain an identity, which is equivalent modulo \((3.1)\) to
\[
x x_1 y z x y z_\infty^2 z x_0 x_1 \approx x x_1 c x_0 x_1
\]
such that \((4.14)\) is true. Now Lemma 4.15 applies and we conclude that \(\nu_1\) is satisfied by \(V\). We see that \(V\) satisfies either \(\nu_1\) or \((4.9)\) in either case.

\textbf{Proposition 5.9.} The variety \(J_2\) is a limit variety.

\textit{Proof.} It is routine to verify that \(J_2\) satisfies \((4.15)\). Then Lemma 5.8 implies that any proper subvariety of \(J_2\) is contained in either \(O_1\) or \(O_2\) and so is FB by Corollary 4.18.

So, it remains to establish that \(J_2\) is NFB. It suffices to verify that, for any \(n \in \mathbb{N}\), the set of identities
\[
\Delta_n = \{(3.1), \eta_1, z_k[\pi, \tau] \approx z_k'[\pi, \tau] \mid k < n, \pi, \tau \in S_{2k}\}
\]
does not imply the identity \(z_n[\varepsilon, \varepsilon] \approx z_n'[\varepsilon, \varepsilon]\), where \(\varepsilon\) is the trivial permutation on \(\{1, 2, \ldots, 2n\}\). Let \(w\) be a word such that \(J_2\) satisfies \(z_n[\varepsilon, \varepsilon] \approx w\). Accordingly to the inclusion \(LRB \subset J_2\) and Lemmas 2.10 and 5.5,
\[
w = \left( \prod_{i=1}^{2n} x_i t_i \right) x \left( \prod_{i=1}^{n} z_i s_i \right) y \left( \prod_{i=1}^{n} z_{n+i} \right) w' t w''
\]
for some words \(w'\) and \(w''\) such that \(\text{con}(w') = \{x, y, x_1, z_1, x_2, z_2, \ldots, x_{2n}, z_{2n}\}\) and \(\text{con}(w'') = \{s_1, s_2, \ldots, s_n\}\). If \(1_{w'} x_1\) follows \(1_w x_1\) in \(w'\), then the identity \(w_n[\varepsilon, \varepsilon](x, x_1, t_1) \approx w(x, x_1, t_1)\) is equivalent modulo \((3.1)\) to \((4.9)\). But this is
impossible by Lemma 5.7. Therefore, $1w'x$ precedes $1w'x_1$ in $w'$. By a similar argument we can show that $(1w'y < (1w'x_1)$ and

$$(1w'x_1) < (1w'z_1) < (1w'z_2) < (1w'z_2) < \cdots < (1w'z_{2n}) < (1w'z_{2n}).$$

This means that $\text{ini}(w') = a_1x_1z_1x_2z_2 \cdots z_{2n}z_{2n}$, where $a \in \{xy, yx\}$. So, we have proved that $\text{ini}_2(w) = z_n[\theta]$ for some $\theta \in S_n$, where

$$z_n[\theta] = \left( \prod_{i=1}^{2n} x_i, t_i \right) x \left( \prod_{i=1}^{n} z_i, s_i \right) y \left( \prod_{i=1}^{n} z_{n+1} \right) a \left( \prod_{i=1}^{2n} x_i, z_1 \right) t \left( \prod_{i=1}^{n} s_i, \theta \right).$$

Let $u$ be a word such that $J_2$ satisfies $z_n[\theta] = u$ and $(2n)x < (2n)y$. In view of the previous paragraph, $\text{ini}_2(u) = z_n[\theta_1]$ for some $\theta_1 \in S_n$. According to Proposition 2.1, to verify that $z_n[\theta, \varepsilon, \xi] = z_n[\theta, \xi, \varepsilon]$ does not follow from $\Delta_n$, it suffices to show that if an identity $u \approx v$ is directly deducible from some identity $s \approx t$ in $\Delta_n$, that is, $(u, v) = (\xi_1 s_1 b_1, \xi_1 t_1 b_1)$ for some words $a, b, \varepsilon, \xi \in \mathfrak{A}$ and some endomorphism $\xi$ of $\mathfrak{A}$, then $(2n) < (2n)$. Arguing by contradiction suppose that $(2n) < (2n)$. Then, in view of the observation in the previous paragraph, $\text{ini}_2(v) = z_n[\theta_2]$ for some $\theta_2 \in S_n$. Further, since $\text{ini}_2(u(x, \varepsilon, y, s_1, t)) = x_1s_1xyts_1$ and $\text{ini}_2(v(x, \varepsilon, y, s_1, t)) = x_1s_1y^2xts_1$, Lemma 5.6 implies that $s \approx t$ cannot coincide with $(3.1)$ and $n_1$. Suppose that $s \approx t$ coincides with $z_k[\xi, \tau] \approx z_k[\xi, \tau]$ for some $k < n$ and $\pi, \tau \in S_{2k}$. We consider only the case when $(s, t) = (z_k[\xi, \tau], z_k[\xi, \tau])$ because the other case is similar. Since $z_k[\xi, \tau]$ differs from $z_k[\xi, \tau]$ only in swapping the second occurrences of letters $x$ and $y$, there are three possibilities:

(a) $\xi((2n)x)$ contains $2u$ and $\xi((2n)y)$ contains $2u$;

(b) $2u \times 2u$ is a subword of $\xi((2n)x)$ and $\xi((2n)y)$ contains some occurrence of $x$;

(c) $\xi((2n)x)$ contains $1u$ and $2u \times 2u$ is a subword of $\xi((2n)y)$.

Suppose that (a) holds. Then $\xi((1u)x)$ contains $1u$ and $\xi((1u)y)$ contains $1u$. Hence $t_{2n}, z_1 \notin \text{cons}(\xi(x))$ and $s_n, z_{n+1} \notin \text{cons}(\xi(y))$ because the equality $\text{ini}_2(u) = z_n[\theta]$ is false otherwise. Therefore, $\xi(x) = x$ and $\xi(y) = y$, whence $\xi((\prod_{i=1}^{n} z_{n+1}) = (\prod_{i=1}^{n} z_{n+1})$. Then $\xi((1u)z_p)$ contains $1u z_q 1u z_{n+1}$ for some $k < n$ and $1 < q < n + 1 < q < n$. Since $\text{ini}_2(u) = z_n[\theta]$, the word $\xi((2n)z_p)$ cannot contain $2u$. Then $2u$ is contained in the subword $\xi((\prod_{i=1}^{n} z_{n+1}) z_{n+1})$ of $u$, where $r = p$. Hence $1u z_q$ must be contained in $\xi((1u) \theta)$ for some $a \in \{z_{1r}, x_1z_1, z_2, x_2z_2, \ldots, z_{n-1}x_{n-1}, x_{n-1}z_{n-1}, x_{n-1} \}$.

This contradicts the fact that $1u z_q$ is contained in $\xi((1u)z_p)$. Therefore, (a) is impossible. Suppose that (b) holds. Then $\xi((1u)x)$ contains both $1u$ and $1u y$. This is only possible when $x z_1 z_2 z_3 \cdots z_{2n}$ is a subword of $\xi(x)$. But this contradicts the fact that $\text{ini}_2(u) = z_n[\theta_1]$ and, therefore, (b) is impossible as well. Finally, (c) is impossible because $\xi((2n)x)$ cannot contain any first occurrence of a letter in $u$. We see that $s \approx t$ cannot coincide with $z_k[\xi, \tau] \approx z_k[\xi, \tau]$. So, we have proved that $\Delta_n$ does not imply $z_n[\varepsilon, \xi] \approx z_n[\varepsilon, \xi]$ and so $J_2$ is NFB.

The following fact provides the first example of a limit variety of monoids with infinitely many subvarieties.

**Corollary 5.10.** The limit variety $J_2$ have countably infinitely many subvarieties.
Proof. Since $J_2$ is a limit variety by Proposition 5.9, it contains at most countably infinitely many subvarieties. It remains to note that $J_2$ has infinitely many subvarieties because $F_1 \subset F_2 \subset \cdots \subset F_k \subset \cdots \subset J_2$ by Lemma 2.12. □

5.3. Classification of limit subvarieties of $P$. The following is the main result of the paper.

Theorem 5.11. The following statements on any subvariety $V$ of $P$ are equivalent:

(i) $V$ is HFB;
(ii) $V \subseteq O_1$ or $V \subseteq O_2$;
(iii) $J_1, J_2 \not\subseteq V$.

Consequently, $J_1$ and $J_2$ are the only limit subvarieties of $P$.

Proof. The implication (ii) $\rightarrow$ (i) holds by Corollary 4.18, while the implication (i) $\rightarrow$ (iii) holds because the varieties $J_1$ and $J_2$ are NFB by Propositions 5.4 and 5.9, respectively. By Lemmas 5.3 and 5.8, the statement $J_1, J_2 \not\subseteq V$ implies that $V$ satisfies either $\{4.15, 4.9\}$ or $\{4.15, \nu_1\}$, whence either $V \subseteq O_1$ or $V \subseteq O_2$. Therefore the implication (iii) $\rightarrow$ (ii) holds. □

6. The monoid $P_2^1$

6.1. The variety $P_2^1$ is HFB.

Proposition 6.1. The monoid variety $P_2^1$ is HFB.

Proof. Since the identities (3.1) and (5.1) constitute an identity basis for the variety $P_2^1$ [22, Corollary 6.6], the inclusion $P_2^1 \subseteq O_1$ is easily deduced. Then $P_2^1$ is HFB because $O_1$ is HFB by Corollary 4.18. □

Lemma 6.2 (Lee and Zhang [24]). Up to isomorphism and anti-isomorphism, every monoid of order five or less, with the possible exception of $P_2^1$, generates a small HFB variety.

Corollary 6.3. The variety generated by any monoid of order five is HFB and so contains at most countably many subvarieties. □

In contrast, as we have mentioned in the introduction, there exist monoids of order six that generate varieties with continuum many subvarieties [12, 14].

Remark 6.4. Up to isomorphism and anti-isomorphism, every semigroup of order five or less that is different from $P_2^1$ generates a HFB variety of semigroups [21]. As observed in Remark 1.1, whether or not $P_2^1$ generates a HFB variety of semigroups is presently an open question.

6.2. Subvarieties of $P_2^1$. In this subsection, we describe the subvariety lattice of $P_2^1$. Recall from Subsection 4.1 that

$$\Phi = \{(3.1), (3.2), \gamma_k, \delta_k^m, \varepsilon_{k-1} \mid k \in \mathbb{N}, 1 \leq m \leq k\}.$$

Proposition 6.5. Each variety $V$ from the interval $[LRB \lor F_1, P_2^1]$ can be defined within $P_2^1$ by identities from $\Phi$.

Proof. The inclusion $F_1 \subset V$ and Lemma 2.12 imply that any 2-limited identity of $V$ is balanced. In view of this fact and Lemma 3.1, $V$ can be defined within $P_2^1$ by a set of 2-limited balanced identities.
Let $u \approx v$ be a non-trivial 2-limited balanced identity of $V$. Then there exists a critical pair $\{i, j\}$ in $u \approx v$. Let $w$ denote the word obtained from $u$ by replacing $iuju$ with $juuia$. It follows from the inclusion $\text{LRB} \subset V$ and Lemma 2.10 that $(i, j) \neq (1, 1)$. Then $(i, j) \in \{(1, 2), (2, 1), (2, 2)\}$ because the identity $u \approx v$ is 2-limited. If $(i, j) = (2, 2)$, then the critical pair $\{i, j\}$ is $(5.1)$-removable. Now let $(i, j) = \{1, 2\}$. Then the critical pair $\{i, j\}$ is $\Phi(V)$-removable by Proposition 4.7. We see that $\Phi(V) \cup \{(5.1)\}$ implies $u \approx w$ in either case. According to Lemma 4.1, $u \approx v$ is a consequence of $\Phi(V) \cup \{(5.1)\}$ and, therefore, $V$ is defined within $P^2_1$ by $\Phi(V)$. □

Lemma 6.6. For any $1 \leq m \leq k$, we have $P\{\delta^m_k, \varepsilon_0\} \subseteq P\{\gamma_{k+1}\}$.

Proof. By Lemmas 4.2 and 4.3(ii), the identity $\kappa_k$ is satisfied by $P\{\delta^m_k, \varepsilon_0\}$. Then the identity $\gamma_{k+1}$ holds in $P\{\delta^m_k, \varepsilon_0\}$ because

$$
\begin{align*}
\kappa_k & \approx y_1y_0x_{k+1}+1x_{k}x_{k+1}b_k \approx y_1y_0x_{k+1}+1x_{k}x_{k+1}b_k \\
(3.1) & \approx y_1y_0x_{k+1}+1x_{k}x_{k+1}b_k,y_{b_{m-1}} \approx y_1y_0x_{k+1}+1x_{k}x_{k+1}b_{k,b_{m-1}} \\
(1.1) & \approx y_1y_0x_{k+1}+1x_{k}x_{k+1}b_k.
\end{align*}
$$

The lemma is thus proved. □

Lemma 6.7. For any $1 \leq m \leq \ell \leq k$, we have $P\{\delta^\ell_k, \varepsilon_m\} \subseteq P\{\delta^m_{k+1}\}$.

Proof. For convenience, let $b_{p-1,p} = \lambda$ for any $p \in \mathbb{N}$. We notice that the identity $\kappa_k$ is satisfied by $P\{\delta^\ell_k, \varepsilon_m\}$ by Lemmas 4.2 and 4.3(ii). Then the identity $\delta_{k+1}$ holds in $P\{\delta^\ell_k, \varepsilon_m\}$ because

$$
\begin{align*}
\delta_{k+1} & \approx y_{m+1}y_{m+1}x_{k+1}+1x_{k}x_{k+1}b_{k,m}y_{b_{m-1}} \\
(3.1) & \approx y_{m+1}y_{m+1}x_{k+1}+1x_{k}x_{k+1}b_{k,m}d_1 \\
(3.1) & \approx y_{m+1}y_{m+1}x_{k+1}+1x_{k}x_{k+1}b_{k,m}d_2 \\
(3.1) & \approx y_{m+1}y_{m+1}x_{k+1}+1x_{k}x_{k+1}b_{k,m}d_3 \\
(3.1) & \approx y_{m+1}y_{m+1}x_{k+1}+1x_{k}x_{k+1}b_{k,m}d_4 \\
(3.1) & \approx y_{m+1}y_{m+1}x_{k+1}+1x_{k}x_{k+1}b_{k,m}d_5
\end{align*}
$$

where

$$
\begin{align*}
d_1 & = \begin{cases} 
y_m & \text{if } m = 1, \\
y_{m-2}x_{m-2}x_{m-1}y_{b_{m-3}} & \text{if } m = 2, \\
y_{m-2}x_{m-2}x_{m-3}y_{m-2}y_{m-3}b_{m-3} & \text{if } m \geq 3, 
\end{cases} \\
d_2 & = \begin{cases} 
y_m & \text{if } m = 1, \\
x_{m-2}x_{m-2}x_{m-2}y_{b_{m-2}} & \text{if } m \geq 2.
\end{cases}
\end{align*}
$$

To complete the proof, it remains to notice that $P\{\delta^\ell_k, \varepsilon_m\} \subseteq P\{\delta^\ell_k, \varepsilon_m\}$ by Lemma 4.3(ii). □
Figure 1. The lattice $\mathcal{L}(P_2^1)$
Proposition 6.8. The lattice $\mathcal{L}(P_2^1)$ is as shown in Fig. 1.\(^1\)

Proof. Lemma 3.2(ii) implies that the lattice $\mathcal{L}(P_2^1)$ is the disjoint union of the lattice $\mathcal{L}(LRB \lor C)$ and the interval $[F_1, P_2^1]$. As verified in Lee [19, Fig. 2], the lattice $\mathcal{L}(LRB \lor C)$ is as shown in Fig. 1. In view of Lemma 3.2(i), the interval $[F_1, P_2^1]$ is a disjoint union of the intervals $[F_1, F]$ and $[LRB \lor F_1, P_2^1]$. According to Lemma 2.11, the interval $[F_1, F]$ has the form shown in Fig. 1. Since any identity from $\Phi \setminus \{ (3.1) \}$ together with (3.1) imply (3.2) by Lemma 4.3, the variety $P_2^1(3.2)$ is the greatest proper subvariety of $P_2^1$. (This result is also deducible from Lee and Li [22, Lemma 6.7].) Let $V$ be a variety from $[LRB \lor F_1, P_2^1(3.2)]$.

Suppose that $P_2^1(\delta^k_1) \not\subseteq V$. Since $P_2^1(\delta^k_1)$ satisfies the identities (3.2), $\delta^m_k$ and $\varepsilon_k$ for any $k \in \mathbb{N}$ and $1 \leq m \leq k$ by Parts (ii) and (iii) of Lemma 4.3, Proposition 6.5 implies that $V$ satisfies $\varepsilon_0$ or $\gamma_1$ for some $\ell \in \mathbb{N}$. Now we apply Lemma 4.3(i) and obtain that $P_2^1(\gamma_1) \subseteq P_1^1(\varepsilon_0)$ for any $\ell \in \mathbb{N}$. Hence $V$ satisfies $\varepsilon_0$. Thus, $V$ belongs to the interval $[LRB \lor F_1, P_2^1(\varepsilon_0)]$.

Suppose now that $P_2^1(\delta^k_1) \subseteq V$ for all $k \in \mathbb{N}$. Then, since $P_2^1(\delta^k_1)$ violates $\varepsilon_{k-1}$, Lemma 4.3 and Proposition 6.5 imply that $V = P_2^1(3.2)$.

Finally, suppose that $P_2^1(\delta^k_1) \subseteq V$ but $P_2^1(\delta^{k+1}_1) \not\subseteq V$ for some $k \in \mathbb{N}$. Notice that

(a) $J^1_s \not\subseteq P_2^1(\gamma_s)$ but $J^1_s \subseteq P_2^1(\delta^1_1)$ for any $s \in \mathbb{N}$;
(b) $J^1_{s+1} \not\subseteq P_2^1(\delta^s_s)$ but $J^1_{s+1} \subseteq P_2^1(\gamma_{s+1})$ for any $s \in \mathbb{N}$;
(c) $J^{s+1}_s \not\subseteq P_2^1(\delta^s_s)$ but $J^{s+1}_s \subseteq P_2^1(\delta^{s+1}_{s+1})$ for any $1 \leq t < s$;
(d) $J^{s+1}_{s+1} \not\subseteq P_2^1(\delta^s_s)$ but $J^{s+1}_{s+1} \subseteq P_2^1(\delta^{s+1}_{s+1})$ for any $1 \leq t \leq s$.

These facts together with Lemma 4.3(ii),(iii) imply that $P_2^1(\delta^k_1)$ violates the identities $\gamma_p$ and $\delta^k_s$, where $p \in \mathbb{N}$ and $1 \leq q < k$. Now we apply Lemma 4.3(ii),(iii) again and conclude that $P_2^1(\delta^{k+1}_1)$ satisfies $\delta^m_k$ and $\varepsilon_k$ for all $r$ and $m$ such that $k + 1 \leq m \leq r$. Then, since $P_2^1(\delta^k_1) \subseteq V$ and $P_2^1(\delta^{k+1}_1) \not\subseteq V$, Proposition 6.5 implies that $V$ satisfies one of the identities $\varepsilon_k$ or $\delta^k_p$ for some $p \geq k$. Now we apply Lemma 4.3(ii) again and obtain that $P_2^1(\delta^k_1) \subseteq P_1^1(\varepsilon_k)$. Hence $V$ satisfies $\varepsilon_k$. Thus, $V$ belongs to the interval $[P_2^1(\delta^k_1), P_1^1(\varepsilon_k)]$.

We see that the interval $[LRB \lor F_1, P_2^1(\varepsilon_0)]$ is a disjoint union of the intervals $[LRB \lor F_1, P_1^1(\varepsilon_0)]$, $[P_2^1(3.2)]$ and $[P_2^1(\delta^k_1), P_1^1(\varepsilon_k)]$ for all $k \in \mathbb{N}$. So, it remains to verify that these intervals are as shown in Fig. 1.

First, we will show that the interval $[LRB \lor F_1, P_1^1(\varepsilon_0)]$ has the form shown in Fig. 1. In view of Lemma 4.3, $P_1^1(\varepsilon_0)$ satisfies $\{ (3.2), \varepsilon_r \mid r \in \mathbb{N} \}$. Then Proposition 6.5 implies that every variety from the interval $[LRB \lor F_1, P_1^1(\varepsilon_0)]$ can be defined within $P_1^1(\varepsilon_0)$ by some possibly empty subset of $\{ \gamma_p, \delta^q_r \mid 1 \leq q \leq p \}$: in particular $LRB \lor F_1 = P_1^1(\gamma_1)$. It follows from Lemma 6.6 that $P_1^1(\gamma_p, \varepsilon_0) \subseteq P_1^1(\gamma_{p+1})$ for any $1 \leq q \leq p$. Besides that, Lemma 4.3(ii) implies that $P_2^1(\gamma_p) \subseteq P_2^1(\delta^q_r, \varepsilon_0)$. Therefore, the interval $[LRB \lor F_1, P_1^1(\varepsilon_0)]$ is a union of the singleton interval $P_1^1(\varepsilon_0)$ and the intervals of the form $P_2^1(\gamma_p)$, $P_2^1(\gamma_{p+1})$, where $p \in \mathbb{N}$. In view of Lemma 4.3, $P_2^1(\gamma_{p+1})$ satisfies $\gamma_s$, $\delta^s_t$ and $\varepsilon_r$ for all $s$, $t$ and $r$ such that $p + 1 \leq s$, $1 \leq t \leq s$ and $r \geq 0$. It follows from (a)–(d) that $P_1^1(\gamma_p)$ violates $\gamma_s$ and $\delta^s_t$ for all $s$ and $t$ such that $1 \leq t \leq s < p$. In view of these facts and Proposition 6.5, every variety from the interval $[P_2^1(\gamma_p), P_2^1(\gamma_{p+1})]$ can

\(^1\)For convenience, some varieties in Fig. 1 are marked by the identities, which define them within the variety $P_2^1$. For instance, the identity $\gamma_1$ marks the variety $P_2^1(\gamma_1)$.\)
be defined within $P^1_2(\gamma_{p+1})$ by some possibly empty subset of $\{\gamma_p, \delta^1_p, \delta^2_p, \ldots, \delta^p_p\}$. Now we apply Lemma 4.3 again and obtain that the interval $[P^1_2(\gamma_p), P^1_2(\gamma_{p+1})]$ forms the chain

$$P^1_2(\gamma_p) \subseteq P^1_2(\delta^1_p, \gamma_{p+1}) \subseteq P^1_2(\delta^2_p, \gamma_{p+1}) \subseteq \cdots \subseteq P^1_2(\delta^p_p, \gamma_{p+1}) \subseteq P^1_2(\gamma_{p+1}).$$

It remains to notice that all these inclusions are strict by (a)–(d). We see that the finitely universal bras is 6.3. □

Proof. Proposition 6.9.

Besides that, Lemma 4.3 implies that the chain $P^1_2(\gamma_p)$ we apply Lemma 4.3 again and obtain that the interval $[P^1_2(\gamma_p), P^1_2(\gamma_{p+1})]$ and, therefore, the interval $[\text{LRB} \lor F_1, P^1_2(\varepsilon_0)]$ are as shown in Fig. 1.

It remains to show that, for any $k \in \mathbb{N}$, the interval $[P^1_2(\delta^k_p), P^1_2(\varepsilon_k)]$ has the form shown in Fig. 1. The proof of this fact is very similar to the arguments from the previous paragraph but we provide it for the sake of completeness. It follows from (a)–(d) that $P^1_2(\delta^k_p)$ violates $\gamma_r, \delta^t_s$ and $\varepsilon_t$ for all $r, s$ and $t$ such that $r \geq 0, s \geq 1, t \leq s$ and $1 \leq t < k$. In view of Lemma 4.3, $P^1_2(\varepsilon_k)$ satisfies $\{(3,2), \varepsilon_r | r \geq k\}$. Then Proposition 6.5 implies that every variety from the interval $[P^1_2(\delta^k_p), P^1_2(\varepsilon_k)]$ can be defined within $P^1_2(\delta^k_p)$ by some possibly empty subset of $\{\delta^q_p | k \leq q \leq p\}$. It follows from Lemma 6.7 that $P^1_2(\delta^k_p, \varepsilon_k) \subseteq P^1_2(\delta^k_{p+1})$ for any $k \leq q \leq p$. Besides that, Lemma 4.3 implies that $P^1_2(\delta^k_p) \subseteq P^1_2(\delta^k_{p+1})$. Therefore, the interval $[P^1_2(\delta^k_p), P^1_2(\varepsilon_k)]$ is the union of the singleton interval $\{P^1_2(\varepsilon_k)\}$ and the intervals of the form $[P^1_2(\delta^k_p), P^1_2(\delta^k_{p+1})]$, where $p \geq k$. Finally, Lemma 4.3, Proposition 6.5 and (a)–(d) imply that every variety from the interval $[P^1_2(\delta^k_p), P^1_2(\delta^k_{p+1})]$ can be defined within $P^1_2(\delta^k_{p+1})$ by some possibly empty subset of $\{\delta^q_p, \delta^q_{p+1}, \ldots, \delta^p_p\}$. Now we apply Lemma 4.3 again and obtain that the interval $[P^1_2(\delta^k_p), P^1_2(\delta^k_{p+1})]$ forms the chain

$$P^1_2(\delta^k_p) \subseteq P^1_2(\delta^k_{p+1}, \delta^k_{p+1}) \subseteq P^1_2(\delta^k_{p+1}, \delta^k_{p+1}) \subseteq \cdots \subseteq P^1_2(\delta^p_p, \delta^p_{p+1}) \subseteq P^1_2(\delta^p_{p+1}).$$

It remains to notice that all these inclusions are strict by (a)–(d). We see that the interval $[P^1_2(\delta^k_p), P^1_2(\delta^k_{p+1})]$ and so the interval $[P^1_2(\delta^k_p), P^1_2(\varepsilon_k)]$ are as shown in Fig. 1. □

6.3. Finitely universal varieties. Following Shevrin et al. [34], a variety of algebras is finitely universal if every finite lattice is order-embeddable into its subvariety lattice. The first example of a finitely universal variety of semigroups was presented by Burris and Nelson [2] half a century ago. In contrast, the first example of a finitely universal variety of monoids was just recently found, in particular, there exist finitely universal varieties of monoids that are finitely generated [8, Section 5]. In view of this result, it is natural to find the least order of a monoid that generates a finitely universal variety. The following statement shows that this order is more than five.

Proposition 6.9. Every monoid of order five or less generates a variety that is not finitely universal.

Proof. In view of Lemma 6.2, it suffices to show that $P^1_2$ is not finitely universal. It is clear from Fig. 1 that every proper subvariety of $P^1_2$ has at most two coverings, so that $P^1_2$ is indeed not finitely universal. For instance, the modular non-distributive lattice in Fig. 2 is not embeddable in the lattice $\Sigma(P^1_2)$. □

Remark 6.10. In contrast, up to isomorphism and anti-isomorphism, there exist precisely four distinct semigroups of order four that generate a finitely universal
variety, while the variety generated by any other semigroup of order four or less is not finitely universal [17].

APPENDIX. TABLES OF IDENTITIES

Table 1. Identities labeled by Greek letters

| Identity | Page number |
|----------|-------------|
| $\alpha_k$ | 6 |
| $\beta_k$ | 6 |
| $\gamma_k$ | 6 |
| $\delta_k^m$ | 6 |
| $\varepsilon_k$ | 11 |
| $\zeta_k$ | 19 |
| $\eta_k$ | 19 |
| $\kappa_k$ | 8 |
| $\lambda_k^m$ | 19 |
| $\mu_k^m$ | 19 |
| $\nu_k$ | 19 |

Table 2. “Non-local” identities labeled by numbers

| Identity | Page number |
|----------|-------------|
| (3.1) | 7 |
| (3.2) | 9 |
| (4.9) | 23 |
| (4.12) | 26 |
| (4.15) | 27 |
| (5.1) | 31 |

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