Testing Quantum Origin of Primordial Gravitational Waves and Magnetic Field

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Abstract

The universe is believed to be originated from a quantum state. However, defining measurable quantities for the quantum properties in the present universe has gained interest recently. In this submission, we propose a quantum Poincare sphere as an observable quantity that can hint at the quantumness of primordial gravitational waves and large-scale magnetic fields. The Poincare sphere is defined in terms of quantum stokes operators associated with the polarization of those fields, which can be measured directly. To support our results, we further explored the possible Bell violation test for a set of generalized pseudo spin operators defined in the polarization space of those fields.

1. INTRODUCTION

The origin of our universe is believed to be a quantum mechanical phenomenon. Any experimental verification of this quantumness would be extremely important to understand the underlying principle of nature. Therefore, how one can confirm our universe to be of quantum mechanical origin is an important question that has gained significant interest in the recent past. One of the recent endeavors towards understanding this question has been to re-examine the underlying mechanism of the large-scale structure of our universe and look for the observable quantum signatures into it. The most successful inflation [1–6] paradigm, which explains the observed large scale structure of our universe to a great precession, is intrinsically assumed to be quantum mechanical phenomena. Though the observed fluctuations of the matter distribution is successfully explained by quantum fluctuation during inflation, any kind of classical statistical origin of such fluctuation can not be ruled out. Therefore, looking for any unambiguous quantum mechanical signature encoded in the distribution of matter would be a logical step towards establishing the inflationary paradigm itself. Cosmic Microwave Background (CMB) anisotropy, which directly corresponds to the matter distribution of the universe, is shown to be related to the scalar quantum fluctuations during inflation. Therefore, it has been widely considered as a potential test-bed to look for quantum signature in the early universe [7–11]. Inflation is believed to be a fundamental mechanism that produces primordial gravitational field as well. Hence, further studies have been performed in the context of gravitational waves [12]. The primordial magnetic field is also believed to be originated from inflation [13–19]. Therefore, besides scalar and gravitational waves, the primordial magnetic field can also be an interesting observable which may encode the quantum signature of the early universe. In this paper, we aim to understand this through defining an appropriate observable called quantum Poincare sphere. We will consider both gravitational and electromagnetic field in a unified framework for our present discussion.

Measuring the gravitational waves and cosmic magnetic field is long standing endeavor with resounding successes. Detection of gravitational waves over the last few years by LIGO [20] has opened up a new era that has the potential to answer all these fundamental questions about the origin of our universe in the near future. Magnetic fields of order a few micro-Gauss (µG) with coherence length of hundred kilo-parsecs (Kpc) scale [21–23] has been observed in the galaxies and galaxy clusters. The intergalactic void may also host a weak ~ 10^{-16} Gauss magnetic field, with the coherence length as large as Mpc scales [24]. The magnetic field of Mpc coherence scale is difficult to explain by any known classical processes. Further, at the astrophysical scale, to generate a magnetic field of µG order, one needs a very weak seed magnetic field. To explain the fundamental origin of such magnetic field at different scales, inflationary magnetogenesis [25–28, 31] is believed to be a natural mechanism. Similar to scalar and gravitational perturbation, the quantum electromagnetic fluctuations during inflation can be amplified to produce large-scale magnetic field. Apart from being present in the void, such magnetic field can then act as a seed at small scales and get enhanced to the galactic scale of µG order by the well known Galactic dynamo mechanism [22, 29, 30]. Motivated by these facts, our primary goal of this letter would be to construct unique observables for both gravitational waves and magnetic field, which can in principle be measured in the present universe and provide important quantum signatures. Our analysis will be along the line of the squeezed state formalism [7–9]. Furthermore, by constructing appropriate observables measured in this state, we will solidify our proposal for quantum signatures.

2. MAGNETOGENESIS IN SQUEEZED STATE FORMALISM

We begin with detailed account of the primordial magnetogenesis in the squeezed state formalism. The conformal symmetry broken electromagnetic (EM) Lagrangian is taken as [31],

\[ S = - \frac{1}{4} \int d^4x \sqrt{-g} I(\tau) F_{\mu\nu} F^{\mu\nu}, \]

where, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is EM field strength tensor with \( A_\mu \) being the vector potential. \( I(\tau) \) is arbitrary EM coupling which breaks the conformal invariance of the
EM field. Generally in case of inflationary magnetogenesis this coupling is taken as a function of the inflaton field, which goes to 1 at the end of inflation. Following the literature it can be written as

\[ I(\tau) = \begin{cases} 
\left( \frac{a_{\text{end}}}{a} \right)^{2n}, & \text{if } \tau < \tau_{\text{end}} \\
1, & \text{if } \tau \geq \tau_{\text{end}}
\end{cases} \] (2)

Where \( a \) is the scale factor at any arbitrary time \( \tau \), \( a_{\text{end}} \) is the scale factor at the end of inflation, and \( n \) is the coupling parameter. The background is taken to be spatially flat FLRW metric with scale factor \( a(\tau) \),

\[ ds^2 = a(\tau)^2(-d\tau^2 + dx^2 + dy^2 + dz^2), \] (3)

Because of spatial flatness, components of the vector potential can be defined in terms of irreducible components as \( A_\mu = (A_0, \partial_i S + V_i) \) with the traceless condition \( \delta^{ij} \partial_i V_j = 0 \). In terms of these components the action Eq.(1) becomes,

\[ S = \frac{1}{2} \int d\tau d^3x I(\tau)(V_i'V_i' - \partial_i V_j \partial^j V_i). \] (4)

Important to note that the action does not depend on the scale factor explicitly because of the inherent conformal invariance of free electrodynamics. The spatial index will be raised or lowered by the usual Kronecker delta function. The Fourier expansion of \( V_i \) is taken as

\[ V_i(\tau, x) = \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} c_i^{(p)}(k) e^{ik \cdot x} u_k^{(p)}(\tau), \] (5)

where the reality condition of the field implies \( u_{-k}^p = u_{k}^{p*} \).

Here, \( c_i^{(p)}(k) \) is the polarization vectors corresponding to two modes \( p = 1, 2 \), which satisfy \( c_i^{(p)}(k)k_i = 0 \) and \( \epsilon_i^{(p)}(k)\epsilon_{i'}^{(q)}(k) = \delta_{pq} \). Dynamics of the field will be governed by following mode function equation,

\[ u_k^{(p)''} + \frac{1}{T} u_k^{(p)'} + k^2 u_k^p = 0 \] (6)

Where the prime denotes derivative with respect to the proper time \( \tau \). In terms of \( u_k^p \) and its associated conjugate momentum \( \pi_k^p = I u_k^{p*} \), the Hamiltonian becomes,

\[ \mathcal{H} = \frac{1}{2} \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} \left( \frac{\pi_k^p \pi_k^{p*}}{I} + I k^2 u_k^p u_k^{p*} \right). \] (7)

Treating the above canonically conjugate variable as operators, we express those in terms of creation and annihilation operators,

\[ \pi_k = -i \sqrt{\frac{k}{2}} (a_k^{(p)} - a_k^{(p)*}), \quad u_k = \sqrt{\frac{1}{2k}} (a_k^p + a_k^{(p)*}), \] (8)

with \( \begin{bmatrix} a_k^{(p)} & a_k^{(p)*} \end{bmatrix} = (2\pi)^3 \delta^{(3)}(k-h) \) being the fundamental commutation relation. In terms of the operators, Hamiltonian of each mode \( k \) becomes,

\[ H_k = \frac{k}{2} \sum_p \left( \frac{I_1}{2} (a_k^{(p)} u_k^p + a_k^{(p)*} u_k^{p*}) + \frac{I_2}{2} (a_k^p a_{-k}^{(p)} + a_{-k}^{(p)*} a_k^{(p)*}) \right), \]

where, \( I_1 = I + 1/2, \quad I_2 = I - 1/2 \). To solve the problem we consider following Bogoliubov transformations, \( a_k^p(\tau) = \alpha_k(\tau) a_k^0(0) + \beta_k(\tau) a_k^{(p)*}(0) \), where, Bogoliubov coefficients satisfy the relation \( \alpha_k^2 - |\beta_k|^2 = 1 \), which is parametrized by,

\[ \alpha_k = e^{i\phi_k} \cos(\phi_k) ; \quad \beta_k = -e^{-i\phi_k + 2i\phi_k} \sin(\phi_k) \] (9)

\( (\phi_k, \phi_k) \) symbolize the squeezing parameter and squeezing angle of the quantum state respectively. They satisfy

\[ \phi_k' = -k \frac{I_1}{2} + k \frac{I_2}{2} \cos 2\phi_k \cosh 2r_k, \quad r_k' = -k \frac{I_1}{2} + k \frac{I_2}{4} \cos 2\phi_k \tanh r_k. \] (10)

If one chooses the Bunch-Davies vacuum with the coupling function mentioned in Eq.(2) then, the solution of those parameters can be expressed as

\[ r_k = \sinh^{-1} |\beta_k|; \quad \phi_k = \frac{1}{2} \text{Arg}(\alpha_k^2) \]

\[ \beta_k^2 = \left( \frac{\pi z}{8T} \right)^{1/2} \left( H_{n+1/2}^1(z) + i H_{n-1/2}^1(z) \right) \] (11)

\[ \alpha_k = \left( \frac{\pi z}{8T} \right)^{1/2} \left( H_{n+1/2}^1(z) - i H_{n-1/2}^1(z) \right) \]

Where \( z = k \tau \). We can find out the time evolution of the squeezing parameter \( r_k \) from Eqs.(11). The evolution of \( r_k \) with \( z = k \tau \) is shown in Fig.1 for different values of coupling parameter \( n \).

To this end, let us give a connection formula with the observable quantities such as electric and magnetic correlation function with the aforementioned squeezing parameters as

\[ <E_\mu(x)E^\mu(y)> = \int \frac{d^3k}{4\pi k^3} e^{ik \cdot (x-y)} \mathcal{P}_E(\tau, k) \]

\[ <B_\mu(x)B^\mu(y)> = \int \frac{d^3k}{4\pi k^3} e^{ik \cdot (x-y)} \mathcal{P}_B(\tau, k) \] (12)

Where, electric and magnetic power spectrum defined in momentum space turned out to be

\[ \mathcal{P}_E(\tau, k) = \frac{k^4}{4\pi^2 a^4 T^2} \sum_{p=1,2} (\cos 2\tau_k^p - \sin 2\tau_k^p \cos 2\phi_k^p) \] (13)

\[ \mathcal{P}_B(\tau, k) = \frac{k^4}{4\pi^2 a^4 T^2} \sum_{p=1,2} (\cos 2\tau_k^p + \sin 2\tau_k^p \cos 2\phi_k^p). \]
After the end of inflation, the conformal invariance is restored, and thus there is no further production of the EM field. If we consider the instantaneous reheating scenario, then the universe readily transforms into a plasma state. Which makes the electric field vanish due to the high conductivity of the medium. This also freezes the magnetic field produced in the inflationary era. As there is no production of magnetic field in the post-inflationary era, the magnetic field power spectrum decays as $a^{-4}$.

At the present time, the magnetic power spectrum turns out to take the following form,

$$ P_{BD} \propto \left( \frac{k}{a_{eq}} \right)^{6-2n} e^{2(n-1)N} \quad (14) $$

Where $(N, a_0)$ are the inflationary e-folding number and the present value of the scale factor, respectively. By this mechanism, therefore, a large scale magnetic field of order $10^{-15} - 10^{-22}$ Gauss can be generated on Mpc scales [28, 32], once an inflation model is considered. The magnetic field generated during the inflationary era is produced from quantum fluctuations. Subsequently, those modes will evolve throughout the entire cosmological evolution. As mentioned earlier, due to large conductivity electric field vanishes, but magnetic field freezes after the end of inflation. However, subsequent dynamics of this frozen magnetic field will depend on the individual modes. The magnetic modes which re-enters the Hubble horizon during radiation era, interact with the plasma and further evolve through magnetohydrodynamic (MHD) evolution. Due to this MHD effect the information of inflationary origin of magnetic field from Bunch-Davis vacuum will be erased for those magnetic modes. On the other hand the modes with larger wavelength will re-enter the horizon during matter dominated era and will not interact with the plasma. Hence, no MHD evolution will take place. Therefore, those large scale magnetic fields are expected to preserve the information of their quantum origin during inflation. The above mentioned fact is schematically illustrated in the Fig. 2. In the following discussion we estimate the approximate scale above which information of quantum origin will be preserved. We calculate the wavenumber of the mode which re-enters the horizon during matter-radiation equality, where, following relation is satisfied,

$$ \frac{T_{eq}}{T_{re}} = \frac{a_{re}}{a_{eq}}, \quad (15) $$

where $a_{eq}$ is the scale factor at the matter-radiation equality, and $H_{eq}$ is the Hubble parameter at that point. The temperature at that point is $T_{eq} \approx 10^{-9}$ GeV, $T_{re}$ is the reheating temperature. Considering instantaneous reheating scenario it is generically observed that $T_{re} \approx 10^{15}$ GeV. Furthermore, the entropy conservation leads to the following relation,

$$ a_{eq}H_{eq} = a_{re}H_{re} \frac{a_{re}}{a_{eq}} \implies k_{eq} = 10^{-24} k_{re}, \quad (16) $$

where, $k_{eq}$ and $k_{re}$ are the modes associated with the radiation-matter equality and the end of reheating.

For instantaneous reheating, $k_{eq} \sim 10^{-15}$ GeV, which immediately gives $k_{eq} \sim 10^{-39}$ GeV $\sim 1$ Mpc$^{-1}$. From this estimation one observes that any mode $k < k_{eq} \sim 1$ Mpc$^{-1}$ re-enters the horizon during the matter dominated era, is expected to carry information about its quantum origin, as those are not affected by causal MHD processes. A schematic of the modes that re-enters the horizon during the different era is given in Fig. 2.

Therefore, from the observation of magnetic field strength and structure for large scale modes we may be able to decode the quantum nature early universe trough observables such as squeezing parameters, which encodes the evolution of the field.

3. PRIMORDIAL GRAVITATIONAL WAVES IN SQUEEZED STATE FORMALISM

Primordial gravitational wave (GW) has been studied already in the squeezed state formalism [12, 33, 34]. Hence we will quote the main results here. The formalism would be the same as discussed for the electromagnetic case. The physical component of the gravitational waves are defined by the transverse traceless part of the metric perturbation,

$$ ds^2 = a^2(\tau) \left[ -dz^2 + (\delta_{ij} + h_{ij})dx^i dx^j \right] \quad (17) $$

The tensor perturbation here $h_{ij}$ satisfies the relation $\partial_t h^{ij} = 0$, $h_i^j = 0$ and $|h_{ij}| \ll \delta_{ij}$.

$$ h_{ij}(x, \tau) = \frac{1}{(2\pi)^3} \int d^3k \epsilon_{ij}^k p^k e^{ik \cdot x} \quad (18) $$

Where $\epsilon_{ij}$ is the polarization tensor for the GW and it satisfies the relation $e_i^m e_j^n = 2\delta^{mn}$. Here $p$ is the polarization index corresponding to the GW. Like the electromagnetic fields, $p$ assumes two values corresponding to
the two polarization modes. The conjugate momentum corresponding to $h_k$ is

$$\pi_k^{(p)} = h_k^{(i(p)} - \frac{\alpha}{a} h_k^{(p)}$$

(19)

With this in hand the Hamiltonian turns out as

$$\mathcal{H} = \frac{1}{2} \sum_{p=1,2} \int \frac{d^3k}{(2\pi)^3} k \left( \alpha_k^{(p)} a_k^{(i(p)} + a_k^{(i(p)} \alpha_k^{(p)} \right)$$

$$+ i \frac{\alpha}{a} \left( \alpha_k^{(p)} a_k^{(i(p)} - a_k^{(i(p)} \alpha_k^{(p)} \right)$$

(20)

Now as in the case of electromagnetic case we can calculate the Bogoliubov coefficients as

$$\alpha_k = \left( 1 + \frac{i}{k \tau} \right). \beta_k = \frac{1}{2k^2 \tau^2}$$

(21)

Here also we the Bogoliubov coefficients are represented in terms of the squeezing parameter and the squeezing angle as in Eq. (9). Moreover, the evolution of the squeezing parameter with the conformal time turns out as shown in Fig. 3.

And the GW power spectra in terms of the squeezing parameter ($r_k$) and squeezing angle ($\phi_k$) turns out as

$$\mathcal{P}_k^{GW}(\tau, k) = \frac{k^3}{2\pi^2 M_T a_0} \sum_{p=1,2} (\cosh(2r_k^{(p)}))$$

$$- \sinh(2r_k^{(p)}) \sin(2\phi_k^{(p)}))$$

(22)

From the above expression, it is evident that the power spectra for GW also depend on the squeezing parameter $r_k$, which can be measured from the strengths.

4. SQUEEZED QUANTUM STATES

In order to perform experiments for quantum signatures, following [10], we construct the quantum state generalizing for two polarization states of the photon and graviton. For each polarization mode we already defined associated squeezing parameters, $(r_k^{1,2}, \phi_k^{1,2})$. A generic quadratic Hamiltonian can always be expressed in terms of those parameters and associated so called squeezing operator $S(r_k, \phi_k) = e^{B_k}$ and rotation operator $R_k = e^{\frac{i}{\hbar}A}$ for each individual polarization mode, with [7, 8]

$$B_k = r_k e^{-2i\phi_k} a_{-k} a_k - r_k e^{2i\phi_k} a_k^{\dagger} a_{-k}$$

$$D_k = -i \theta_k a_k^{\dagger} a_k - i \theta_k a_k^{\dagger} a_{-k}$$

(23)

These are unitary operators which can be shown to form the unitary time evolution operator for the quantum state [10]. During inflation, it is generically assumed that all the quantum state start to evolve from the Bunch-Davies vacuum. Because of the coupling of the fields under study with the classical background, the vacuum state will evolve to squeezed state, which can be expressed as follows, $|\psi\rangle = S_2(r_k^2, \phi_k^2) D_2(\theta_k^2)|0\rangle \otimes S_1(r_k^1, \phi_k^1) D_1(\theta_k^1)|0\rangle$

(24)

$$= \frac{1}{\mathcal{A}} \sum_{n, m=0}^{\infty} e^{-2i(n\phi_k^1 + m\phi_k^2)} \tanh^n r_k^1 \tanh^m r_k^2 \eta_{n, m}(-k, m^2)$$

Where, $\mathcal{A} = \cosh(r_k^1) \cosh(r_k^2)$. The position space representation of Eq. (24) assume the following form,

$$\langle q_A^{1,2}, q_B^{1,2} | \Psi \rangle = \frac{1}{\pi \cosh r_k^1 \sqrt{1 - e^{4i\phi_k^1} \tanh^2 r_k^p}} \frac{1}{\pi \cosh r_k^1 \sqrt{1 - e^{4i\phi_k^1} \tanh^2 r_k^p}}$$

(25)

Here $X^{1,2}$ and $Y^{1,2}$ are defined as

$$X_p = e^{-4i\phi_k^1} \tanh^2 r_k^p + 1 \over 2 (e^{-4i\phi_k^1} \tanh^2 r_k^p - 1); Y_p = e^{-4i\phi_k^1} \tanh^2 r_k^p - 1$$

Where the subscript 'A' stands for momentum mode $k$ and 'B' stands for the momentum mode $-k$. Superscripts 1, 2 are the polarization index. Therefore, it is obvious from the present discussion that the large-scale magnetic field and gravitational waves should be in a highly non-classical state at present. Quantum discord, an important measure of quantumness, is indeed shown to increase [8] for such state with increasing $r_k$. However, its measurement in the cosmological setting is not obvious. Therefore, in order to quantify this quantumness, we will perform fuzziness of the quantum Poincare sphere on this state Eq. (24) and to show the concreteness of this proposal, we also perform the well established Bell test on this state. For this purpose, we consider two sets of observables with their unique algebraic properties.

5. QUANTUM STOKES OPERATORS

The Stokes parameters [35] are known to be important description of the polarization properties of electromagnetic field. Same can be applied to gravitational waves also. The Quantum Stokes parameters are the operator representations of the polarization that can be applied to non-classical waves. Most interestingly these operators satisfy angular momentum like commutation alge-
bra. Stokes operator are defined as follows,

\[
\begin{align*}
S_k^{(0)} &= a_k^\dagger a_k + a_k a_k^\dagger \quad ; \\
S_k^{(1)} &= a_k^\dagger - a_k \quad ; \\
S_k^{(2)} &= t_k a_k^\dagger a_k + t_k^* a_k a_k^\dagger \quad ; \\
S_k^{(3)} &= i(t_k^* a_k^\dagger a_k - t_k a_k a_k^\dagger)
\end{align*}
\]  

(26)

Where \( t_k = e^{i\psi_k} \) measures the relative phase between the two modes. According to the above definition \( p = (1,2) \) correspond to two different polarization modes associated with electromagnetic and gravitational field. \( S_k^0 \) encodes the intensity of a particular mode \( k \) or the total number of photons or graviton. \( S_k^1 \) and \( S_k^2 \) measures the linear polarization and \( S_k^3 \) measures the circular polarization. Interestingly last three Stokes parameters parametrizing the polarization states satisfy \( SU(2) \) spin algebra \([S_k^{(p)}, S_k^{(q)}] = 2i\epsilon^{pqr} S_k^{(r)}\) , where \( \epsilon^{pqr} \) is the three dimensional levicivita tensor density. \((p,q,r) = 1,2,3\).

In order to prove the quantumness of a system through the Stokes parameters, we examine it via the quantum Poincare sphere\[36, 37\]. For the classical case we have the usual relation \( (|S_k^1|^2 + |S_k^2|^2 + |S_k^3|^2) = (|S_k^0|^2)\)\[38, 39\]. But in case of quantum Poincare sphere we can’t precisely define the radius of the Poincare sphere due to the non commutativity of \( S_k^0 \). And we have \(|S_k^1|^2 + |S_k^2|^2 + |S_k^3|^2 - (|S_k^0|^2) > 0\). We define fuzziness of quantum Poincare sphere (can be also called fuzzy sphere) as a measure of quantumness

\[
(|S_k^1|^2 + |S_k^2|^2 + |S_k^3|^2 - (|S_k^0|^2))
\]

(27)

In order to calculate the fuzziness of the Poincare sphere constructed by the stokes operators we need to explicitly calculate the expectation values of the operators \( S_k^j \). We have

\[
\begin{align*}
(S_k^0)^2 &= (n_k^1)^2 + n_k^2 n_k^1 + n_k^1 n_k^2 + (n_k^2)^2 \\
(S_k^1)^2 &= (n_k^1)^2 - n_k^1 n_k^2 - n_k^2 n_k^1 + (n_k^2)^2 \\
(S_k^2)^2 &= (t_k^1)^2 (a_k^1)^2 (a_k^2)^2 + t_k^* t_k^1 n_k^1 a_k^2 a_k^1 + t_k^1 t_k^* n_k^2 a_k^1 a_k^1 + (t_k^1)^2 (a_k^2)^2 (a_k^1)^2 \\
(S_k^3)^2 &= -(t_k^1)^2 (a_k^1)^2 (a_k^2)^2 + t_k^* t_k^1 n_k^1 a_k^2 a_k^1 + t_k^1 t_k^* n_k^2 a_k^1 a_k^1 - (t_k^1)^2 (a_k^2)^2 (a_k^1)^2
\end{align*}
\]

(28)

Where \( n_k^1 = a_k^1 a_k^1 \) is the number operator corresponding to polarization 1 and \( n_k^2 = a_k^2 a_k^2 \) is the number operator corresponding to polarization 2. Calculating the expectation values of the operators in Eq.(28) on the state defined in Eq.(24) we get

\[
\langle (S_k^0)^2 \rangle = \frac{1}{\mathcal{A}} \sum_{m,n=0}^{\infty} \tanh^{2n}(r_k^1) \tanh^{2m}(r_k^2)(n^2 + 2mn + m^2)
\]

\[
= \frac{1}{4} \left( \cosh(4r_k^1) + 2 \cosh(2r_k^1) \cosh(2r_k^2) + \cosh(4r_k^2) \right)
\]

\[
\langle (S_k^0)^2 \rangle = \frac{1}{\mathcal{A}} \sum_{m,n=0}^{\infty} \tanh^{2n}(r_k^1) \tanh^{2m}(r_k^2)(n^2 - 2mn + m^2)
\]

\[
= \frac{1}{4} \left( \cosh(4r_k^1) - 2 \cosh(2r_k^1) \cosh(2r_k^2) + \cosh(4r_k^2) \right)
\]

\[
\langle (S_k^0)^2 \rangle = \frac{1}{\mathcal{A}} \sum_{m,n=0}^{\infty} \tanh^{2n}(r_k^1) \tanh^{2m}(r_k^2)m(n+1)
\]

\[
= \cosh^2(r_k^1) \sinh^2(r_k^2)
\]

(29)

Where \( \mathcal{A} = \cosh(r_k^1) \cosh(r_k^2) \) and \( n,m \) are the number operator expectation values on the state for polarization 1 and polarization 2 respectively. Calculating the fuzziness on the state from the expectation values of the operators we get

\[
\langle (S_k^1)^2 + (S_k^2)^2 + (S_k^3)^2 - (S_k^0)^2 \rangle = 4 \sinh^2(r_k^1)
\]

(30)

Where for simplicity we have taken \( r_k^1 = r_k^2 = r_k \). We can see that the quantity is dependent on the squeezing parameter \( r_k \). When \( r_k \) goes to zero, we can consider the state to be classical in nature. Therefore, the higher the squeezing parameter, the higher would be the fuzziness of the Poincare sphere, and thus the quantumness increases.

6. MEASUREMENT AND POINCARÉ SPHERE

Performing the test for quantumness is not an easy task as for cosmological scenarios. However, the primordial gravitational Eq.(22) and magnetic power spectrum Eq.(13) provide us a good observational measure of the squeezing parameters \((r_k, \phi_k)\), in terms of their strength and index. Using those measured values of \((r_k, \phi_k)\) we can theoretically obtain the fuzziness of the Poincare sphere from (27). Therefore, for a particular value of squeezing parameter, if that quantity is greater than zero, we can infer about the quantumness of the system. Further, Stokes parameters are also measurable quantities \([40, 41]\), which are directly related to the fuzziness by Eq.(27). Therefore, experimentally measured value of the fuzziness \(\langle (S_k^1)^2 + (S_k^2)^2 + (S_k^3)^2 - (S_k^0)^2 \rangle\) in terms of stokes parameters \((S_k^1, S_k^2, S_k^3)\), can now be compared with its classical counterpart (which is zero)\[38, 39\]. Upon observing \(\langle (S_k^1)^2 + (S_k^2)^2 + (S_k^3)^2 - (S_k^0)^2 \rangle > 0\), we can infer the quantum origin of those primordial fluctuations. In principle when we make the measurement of either gravitational wave or primordial magnetic field, we cannot measure a particular wave number \( k \). We measure a superposition of several wave numbers. Therefore the
actual measurable quantity is defined as
\[
\sum_{k=k_{min}}^{k_{max}} (s^k_1)^2 + (s^k_2)^2 + (s^k_3)^2 - (s^k_0)^2 \geq 0
\]  
(32)
We know for classical case the quantity defined in Eq.(27) is zero, so it is never negative for any scenario. Thus the measurement will give us
\[
\sum_{k=k_{min}}^{k_{max}} (s^k_1)^2 + (s^k_2)^2 + (s^k_3)^2 - (s^k_0)^2 \geq 0
\]  
(32)
After the measurement, if the measured quantity is greater than zero, we can conclude that the system under consideration has a quantum origin.

7. Pseudospin Operators and Bell Violation

Now we embark on a different set of operators called pseudo spin operators and their Bell violation test. When dealing with the system having continuous variables, there exists an elegant way [11, 42, 43] of constructing a set of position dependent pseudo spin operators which satisfy the same algebra as the spin angular momentum. To the best of our knowledge, such operators were constructed only in one dimension. Our present study demands the generalization of these operators in two dimensions corresponding to two independent polarization modes. As a prerequisites of the spin like operators following operators are defined in position \((q,q')\) space,
\[
P_{n,m}(l) = \int_{n-l}^{n+l} dq \int_{m-l}^{m+l} dq' |q,q'\rangle \langle q,q'|,
\]
\[
T_{n,m}(l) = \int_{n-l}^{n+l} dq \int_{m-l}^{m+l} dq' |q,q'\rangle \langle q+l,q'+l|,
\]
\[
T^\dagger_{n,m}(l) = \int_{n-l}^{n+l} dq \int_{m-l}^{m+l} dq' |q,q'\rangle \langle q-l,q'-l|.
\]
Those are called on-site projection, forward and backward hopping operators respectively. When the projection and the hopping operators operates on a state they behave as
\[
P_{n,m}(l) \psi(q) = \begin{cases} 
\psi(q) & ml, nl \leq q < ml + l, nl + l \\
0 & \text{otherwise}
\end{cases}
\]
\[
T_{n,m}(l) \psi(q) = \begin{cases} 
\psi(q+l) & ml, nl \leq q < ml + l, nl + l \\
0 & \text{otherwise}
\end{cases}
\]
\[
T^\dagger_{n,m}(l) \psi(q) = \begin{cases} 
\psi(q-l) & ml + l, nl + l \leq q < ml + 2l, nl + 2l \\
0 & \text{otherwise}
\end{cases}
\]
The generalized pseudo spin operators are constructed from the aforementioned position space operators and generalizing from [42] we define them as
\[
s_z = \sum_{c=-\infty}^{\infty} (-1)^{n+m} P_{n,m}
\]
\[
s_+ = \sum_{c=-\infty}^{\infty} T_{2n,2m}
\]
\[
s_- = \sum_{c=-\infty}^{\infty} T^\dagger_{2n,2m}
\]
where, \(\mathcal{C} \equiv (m,n)\), and \(l\) is the length of the interval. We define \(s_+ = s_z + i s_y\) and \(s_- = s_z - i s_y\). It can be easily shown that these three operators \(s_z, s_y\) and \(s_x\) are dichotomous i.e \((s_z^2 = s_y^2 = s_z^2 = 1)\). And they follow the usual \(SU(2)\) spin algebra. Exploiting the nature of the operators we can define the Bell-CHSH operator in terms of the pseudospin operators. For two observer ‘A’ having momentum \(k\) and observer ‘B’ having momentum \(-k\), the maximal Bell-CHSH operator can be constructed as
\[
B^\chi_{CHSH} = 2 \sqrt{(s_z^A \otimes s_z^B)^2 + (s_z^A \otimes s_z^B)^2 + (s_z^A \otimes s_z^B)^2}
\]
(38)
On the quantum state Eq.(25) we now calculate the expectation value of the Bell operator \(B^\chi_{CHSH}\) with \(r_k = r\) presented in the Fig.4. We know that the Bell inequality suggests that for a system having the Bell-CHSH operator expectation value grater than 2 can only be described by means of quantum mechanics. Thus it is evident from the figure that the system behaves quantum mechanically, i.e \((B^\chi_{CHSH}) > 2\), above a certain value of the squeezing parameter \(r_k\), which is in agreement with conclusion from our previous Poincare sphere measurement.

8. Conclusion

In this letter, we have proposed a Poincare sphere fuzziness or Fuzzy sphere in terms of observable quantities
to verify the quantum origin of the large scale magnetic field and gravitational waves. For this purpose, we first constructed the squeezed quantum states for the electromagnetic and gravitational wave fluctuation during inflation and the associated power spectrum. We have then considered the Poincare sphere fuzziness or the uncertainty of the radius in terms of quantum stokes operators, which assume a positive value for the cosmologically evolved squeezed quantum states. Our theoretical calculation reveals that the larger the squeezing, the stronger the quantumness. Importantly this can be confirmed by the direct measurement of Stokes parameters and primordial power spectrums. Finally, we confirmed our conclusion by further performing the Bell test considering dichotomic pseudospin operators. Other quantum diagnostics such as quantum discord, Fano factors could be interesting to study in this present context.

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