Higgs branches of Argyres-Douglas theories as quiver varieties

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ABSTRACT: We present a general prescription for constructing 3d $\mathcal{N} = 4$ Lagrangians for the IR SCFTs that arise from the circle reduction of a large class of Argyres-Douglas theories. The resultant Lagrangian gives a realization of the Higgs branch of the 4d SCFT as a quiver variety, up to a set of decoupled interacting SCFTs with empty Higgs branches. As representative examples, we focus on the families $(A_{p-N-1}, A_{N-1})$ and $D_p(SU(N))$. The Lagrangian in question is generically a non-ADE-type quiver gauge theory involving only unitary gauge nodes with fundamental and bifundamental hypermultiplets, as well as hypermultiplets which are only charged under the $U(1)$ subgroups of certain gauge nodes. Our starting point is the Lagrangian 3d mirror of the circle-reduced Argyres-Douglas theory, which can be read off from the class $S$ construction. Using the toolkit of the $S$-type operations, developed in [1], we show that the mirror of the 3d mirror for any Argyres-Douglas theory in the aforementioned families is guaranteed to be a Lagrangian theory of the above type, up to some decoupled free sectors. We comment on the extension of this procedure to other families of Argyres-Douglas theories. In addition, for the case of $D_p(SU(N))$ theories, we compare these 3d Lagrangians to the ones found in [2] and propose that the two are related by an IR duality. We check the proposed IR duality at the level of the three-sphere partition function for specific examples. In contrast to the 3d Lagrangians in [2], which are linear chains involving unitary-special unitary nodes, we observe that the Coulomb branch global symmetries are manifest in the 3d Lagrangians that we find.

KEYWORDS: Duality in Gauge Field Theories, Field Theories in Lower Dimensions, Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction and summary

1.1 Overview and the basic idea of the paper

1.1.1 AD theories and 3d mirrors from the class $S$ construction

Argyres-Douglas (AD) theories constitute a large class of 4d $\mathcal{N} = 2$ SCFTs for which the Coulomb branch operators have fractional scaling dimensions. The first examples
of such theories were realized as special points on the Coulomb branch of Lagrangian theories in 4d [3, 4]. Over the years, various important subclasses of such theories have been constructed via the geometric engineering technique [5, 6] of compactifying Type IIB superstring theory on certain singular Calabi-Yau hypersurfaces, as well as by the class $S$ construction [7, 8]. As is well known, the latter involves compactifying a 6d (2,0) theory (which has an ADE classification) on a Riemann sphere with either a single irregular puncture, or a single irregular puncture with a single regular puncture [8].

One can compactify a given 4d $\mathcal{N} = 2$ SCFT to 3d by putting the theory on $\mathbb{R}^3 \times S^1$, and flowing to the deep IR. The resultant 3d $\mathcal{N} = 4$ SCFT may or may not have a Lagrangian description, but is known to have a Lagrangian 3d mirror for a large subset of class $S$ theories. Given the class $S$ construction of a 4d $\mathcal{N} = 2$ SCFT, the basic idea is to show that the associated Hitchin moduli space (which is the Coulomb branch of the theory on $\mathbb{R}^3 \times S^1$) in the IR limit is isomorphic to the Higgs branch of a 3d $\mathcal{N} = 4$ quiver gauge theory. The aforementioned 3d quiver gauge theory can be explicitly built from the class $S$ data in the following generic steps:

- To every puncture on the Riemann surface, one associates a quiver gauge theory, which can be read off from the singularity data of the puncture.
- Given the set of quivers associated with the Riemann surface with punctures, one glues them together, following certain rules which depend on the specific punctures involved.

The 3d mirrors for 4d $\mathcal{N} = 2$ SCFTs, which arise from 6d $A_{N-1}$ (2,0) theories compactified on a Riemann surface with regular punctures, was found in [10, 11]. For an important subclass of irregular punctures, which appear in the realization of AD theories, the analogous result was first obtained in [12] and was extended in subsequent papers [8, 13, 14]. For a recent review of the subject, we would like to refer the reader to [15].

In this work, we would be primarily interested in the 3d mirrors of the AD theories of the following families: $(A_{p-N-1}, D_{N-1})$ and $D_{p}(SU(N))$. The first family is realized by a single irregular puncture on a Riemann sphere, while the second is realized by a single irregular puncture and a single regular puncture. The 3d mirror for each family is discussed in detail in section 3.1 and section 3.2.

1.1.2 $S$-type operations and 3d Lagrangians for AD theories

In [1], a systematic field theory construction for engineering new 3d mirror dualities was presented, starting from a well-defined set of basic dualities. Let us briefly summarize the construction. We consider a class of 3d SCFTs (which we refer to as class $U$) with UV Lagrangians for which the Higgs branch global symmetry has a subgroup $G_{\text{global}}^{\text{sub}} = \prod_\gamma U(M_\gamma) \subset G_H$.  

\footnote{Certain AD theories can also be realized by the compactification of twisted ADE theories on a Riemann surface with only regular punctures, as pointed out in [9].}
Figure 1. Generating new dual pairs using an elementary $S$-type operation.

Given a UV Lagrangian $X$ in class $\mathcal{U}$, one can define a map which acts on $X$ to give a generically different theory $X'$, i.e.

$$O_{\mathcal{P}}^\alpha : X[\hat{A}] \mapsto X'[\hat{B}], \quad (1.1)$$

where the phrase “different theory” implies that $X$ and $X'$ flow to different SCFTs in the IR. The precise definition of the map $O_{\mathcal{P}}^\alpha$ is reviewed in section 2. In (1.1), $\hat{A}$ and $\hat{B}$ collectively represent the background fields associated with the global symmetries of the theories $X$ and $X'$ respectively. We refer to the above map as an “elementary $S$-type operation”. The name arises from the fact that in the special case, where $G_{\text{sub}} = U(1)$ and $O_{\mathcal{P}}^\alpha$ is a gauging operation of the said $U(1)$, the above map coincides with the $S$ generator of Witten’s $\text{SL}(2,\mathbb{Z})$ action [16] on a 3d CFT.

Suppose the theory $X[\hat{A}]$ has a 3d mirror $Y[\hat{A}]$, such that both theories have Lagrangian realizations. For a given elementary $S$-type operation $O_{\mathcal{P}}^\alpha$ on $X[\hat{A}]$, one can define a dual operation $\tilde{O}_{\mathcal{P}}^\alpha$ on $Y[\hat{A}]$:

$$\tilde{O}_{\mathcal{P}}^\alpha : Y[\hat{A}] \mapsto Y'[\hat{B}], \quad (1.2)$$

which maps it to a different theory $Y'[\hat{B}]$, such that the pair of theories $(X'[\hat{B}], Y'[\hat{B}])$ are 3d mirrors. The four theories $X[\hat{A}], Y[\hat{A}], X'[\hat{B}]$, and $Y'[\hat{B}]$, are therefore related as shown in figure 1.

The dual operation $\tilde{O}_{\mathcal{P}}^\alpha$ can be determined explicitly, with the help of RG-invariant supersymmetric observables computed using localization, as discussed in [1]. The procedure is reviewed in section 2. Given the pair $(X[\hat{A}], Y[\hat{A}])$ and the map $O_{\mathcal{P}}^\alpha$, one can then read off the Lagrangian for the theory $Y'[\hat{B}]$, assuming a weakly coupled description of the theory exists. Given the recipe for finding the dual of the $S$-type operation, the strategy for engineering new mirror pairs can be stated as follows. We first define a conveniently small subset of mirror pairs, which are well understood from the String Theory and/or the QFT perspective, and refer to it as the set of “basic dualities”. Starting with a dual pair $(X, Y)$ in this set of basic dualities, one can then implement the prescription of figure 1 sequentially to generate new mirror pairs. We will pick the set of basic dualities to be the set of mirror pairs involving linear (A-type) quiver gauge theories with unitary gauge groups. Additionally, we will demand that the quivers are good theories in the Gaiotto-Witten sense [17]. Mirror symmetry for this class of theories has been studied in a lot of detail using the Type IIB Hanany-Witten brane construction [18] on the one hand and
QFT approaches on the other. In particular, it was shown in [17] that the mirror of a good linear quiver is also a good linear quiver. Using the construction outlined above, one can then construct mirror pairs involving quivers of arbitrary shapes.

In this paper, we will use this construction to determine 3d $\mathcal{N} = 4$ Lagrangians for families of AD theories which have Lagrangian 3d mirrors. Let this Lagrangian 3d mirror be $X'$ (see figure 1 above). The strategy is to show that $X'$ can be engineered from a good linear quiver $X$ by a sequence of $S$-type operations. Since $X$ has a 3d mirror $Y$, which is also a good linear quiver, the 3d mirror of $X'$ can be obtained by a sequence of dual $S$-type operations on the theory $Y$. If the resultant theory $Y'$ is Lagrangian, then it gives a Lagrangian realization for the 3d SCFT obtained from the circle-reduction of the AD theory. As mentioned above, for a completely generic quiver gauge theory $X'$, the theory $Y'$ is not Lagrangian. However, for the AD theories under consideration, $X'$ can be engineered from a linear quiver $X$ by a very special class of $S$-type operations, for which the theory $Y'$ is guaranteed to have a Lagrangian description.

1.1.3 Another 3d Lagrangian for a $D_p(SU(N))$ theory

In [2, 19], the authors derived 3d Lagrangians for $D_p(SU(N))$ theories, by making use of a partially weakly-coupled description of the 4d $\mathcal{N} = 2$ SCFTs. We briefly summarize the results relevant for our paper below.

The $D_p(SU(N))$ theories are specified by the SW curve:

$$x^N + z^p + \sum_{i,j \geq 0} u_{ij} z^i x^j = 0, \quad \lambda_{SW} = \frac{x dz}{z}, \quad (1.3)$$

where the scaling dimension of the SW differential is $[\lambda_{SW}] = 1$, and this implies that the other scaling dimensions are given as

$$[x] = 1, \quad [z] = \frac{N}{p}, \quad [u_{ij}] = N - j - \frac{N}{p} i. \quad (1.4)$$

If $p, N$ are coprime, then $D_p(SU(N))$ is an isolated SCFT, i.e. it has no conformal manifold. There is another isolated SCFT that will be important in this story: $D_q(N, N - n)$, which has the following SW curve

$$x^N + x^{N-n} z^q + \sum_{i,j \geq 0} u_{ij} z^i x^j = 0, \quad \lambda_{SW} = \frac{x dz}{z}, \quad (1.5)$$

If $n, q$ are chosen such that $n = N/\text{GCD}(p, N)$ and $q = p/\text{GCD}(p, N)$, then $D_q(N, N - n)$ is an isolated SCFT, as can be seen by counting the number of deformations of scaling dimension zero. These theories have a global symmetry $\text{SU}(N) \times U(1) \times \text{SU}(N - n)$.

Given an SCFT $D_p(SU(N))$, the monomial $x^{N-n} z^q$ in the SW curve corresponds to a marginal deformation. Also, the monomials $x^j z^q$ with $j = 0, 1, \ldots, N - n - 1$ correspond to integer scaling dimensions $\Delta_{qj} = N-n, N-n-1, \ldots, 1$. Leaving out the mass deformation,
the remaining deformations correspond to CB operators of a conformally gauged SU(N−n) factor of the 4d SCFT. Together with the marginal deformation term, these terms in the SW curve can be organized as follows (after appropriate rescaling):

\[ x^{N−n}z^q + \sum_{j=0}^{N−n−2} u_{jq}x^j z^q =: (x^{N−n} + \sum_{j=2}^{N−n} u_j x^{N−n−j}) z^q =: P_{N−n}(x) z^q. \]  

(1.6)

The SW curve can then be written as

\[ x^N + \ldots + P_{N−n}(x) z^q + \ldots + z^p = 0, \quad \Rightarrow \quad \frac{x^N}{z^q} + \ldots + P_{N−n}(x) + \ldots + z^{p−q} = 0. \]  

(1.7)

In this form, the above SW curve represents a \( D_q(N, N−n) \) system and a \( D_{p−q}(SU(N−n)) \) system with the diagonal SU(N−n) flavor symmetry being conformally gauged. This allows us to write a partially weakly-coupled description of the SCFT as follows:

\[ D_p(SU(N)) = D_q(N, N−n) \leftarrow SU(N−n) \rightarrow D_{p−q}(SU(N−n)). \]  

(1.8)

Applying this argument iteratively, one can write the following partially weakly coupled description:

\[
D_p(SU(N)) = D_q(N, N−n) \leftarrow SU(N−n) \rightarrow D_q(N−n, N−2n) \leftarrow SU(N−2n) \rightarrow D_q(N−2n, N−3n) \leftarrow SU(N−3n) \rightarrow D_q(N−3n, N−4n) \leftarrow SU(N−4n) \rightarrow D_q(N−4n, N−5n) \leftarrow \ldots \leftarrow SU(n) \rightarrow D_q(n,0),
\]

(1.9)

where \( D_q(n,0) = D_q(SU(n)) \). This allows one to write the SCFT \( D_p(SU(N)) \) in terms of a set of isolated SCFTs with conformally-gauged special unitary flavor subgroups.

The isolated SCFT \( D_q(N,K) \) can be realized as the low energy effective field theory at the origin of the Coulomb branch of a 4d Lagrangian theory, with appropriately tuned flavor masses. The 4d Lagrangian theory in question is a linear chain of special unitary gauge groups with \( N \) fundamental flavors at one end, and \( K \) fundamental flavors at the other. For example, with \( K = N−1 \), i.e. \( n = N/GCD(p,N) = 1 \), the 4d Lagrangian is given as

\[ D_q(N, N−1) = [N] − SU(N)_1 − SU(N)_2 − \ldots − SU(N)_{q−1} − [N−1]. \]  

(1.10)

The case for general \( (N,K) \) can be found in [2]. Given the description in (1.9) and the above realization of the SCFTs \( D_q(N,K) \), the 3d Lagrangian can be obtained as follows:

1. The 3d limit of a given isolated SCFT \( D_q(N,K) \) is obtained by studying the low energy effective theory of the dimensionally reduced linear quiver at the origin of the Coulomb branch, with the flavor masses turned on. Let us call these 3d theories \( T_i \), where \( i = 1, \ldots, m−1 \). The theory \( T_i \) associated with an isolated SCFT \( D_q(N,K) \) turns out to be a linear quiver with unitary gauge nodes and Higgs branch global symmetry \( SU(N) \times U(1) \times U(K) \).

2. The 4d vector multiplets of the conformally-gauged \( SU \) factors become 3d vector multiplets under dimensional reduction.
3. Given the set of 3d theories $\{T_i\}$, the 3d limit of $D_p(SU(N))$ is then given by a linear chain of unitary and special unitary nodes with bifundamental and fundamental hypers:

$$[D_p(SU(N))]_{3d} : T_1 - SU(N - n) - T_2 - SU(N - 2n) - \ldots - SU(n) - T_{m-1}. \quad (1.11)$$

In certain cases, the 3d theory can have ugly nodes, but one can always reduce the theory to a quiver of the form (1.11) and a set of free hypermultiplets.

In section 3.2, we note that the 3d Lagrangian in (1.11) and the 3d Lagrangian constructed using $S$-type operations are different quiver gauge theories, i.e. they have different gauge groups and matter content. We propose that the two 3d Lagrangians are related by an IR duality, and perform certain checks of the proposed duality for specific examples in section 4.

1.2 Summary of the main results

The main results of the paper can be summarized as follows:

- **Existence of 3d Lagrangians for AD theories on circle reduction:** we present a general proof of the fact that the 3d SCFTs which arise in the IR limit of the AD theories $(A_{p-N-1}, A_{N-1})$ $(p > N)$ and $D_p(SU(N))$ $(p \geq N)$, compactified on a circle, always have an $\mathcal{N} = 4$ UV Lagrangian description (up to a free sector). Using the strategy outlined in section 1.1.2, the basic idea is to show that the 3d mirrors of the circle-reduced AD theories, which can be constructed from the class $S$ construction (which we will refer to as class $S$ mirrors), themselves have a Lagrangian mirror. The latter then gives a 3d Lagrangian description of the IR SCFT that arises from the circle reduction of the AD theory in question.

For the two families of AD theories mentioned above, we show that the class $S$ mirror can be constructed from an $A$-type quiver gauge theory by a sequence of Abelian $S$-type operations. Using the properties of Abelian $S$-type operations, reviewed in section 2, we then show that the 3d mirror of the class $S$ mirror is manifestly Lagrangian.

- **General features of the 3d Lagrangian and examples:** the construction outlined above gives a systematic procedure for constructing the 3d Lagrangians. In section 3.1 and section 3.2, we outline some of the general features of the 3d Lagrangian for different families of AD theories under consideration. We present concrete examples in section 3.1, section 3.3–section 3.4 where we explicitly write down the 3d Lagrangian. The results can be summarized as follows:

1. $(A_{p-N-1}, A_{N-1})$ theories: in this case, the 3d Lagrangian is a generically non-linear quiver built out of $U(1)$ gauge nodes with fundamental and bifundamental matter, along with a set of free twisted hypermultiplets. After presenting the general procedure for obtaining the 3d quiver in section 3.1, we work out an
example for which \( m = GCD(p, N) = 3 \) and \( m_G = \frac{N(p-N)}{9} \). The 3d Lagrangian for this example is given as:

and the number of free twisted hypermultiplets is \( H = \frac{(N-3)(p-N-3)}{6} \).

2. \( D_p(\text{SU}(N)) \) theories: in this case, the 3d Lagrangian is generically a non-linear quiver built out of unitary gauge nodes with fundamental and bifundamental hypermultiplets, as well as certain hypermultiplets charged only under certain \( U(1) \) subgroups of the unitary groups. We will refer the latter as “Abelian hypermultiplets”. After presenting the general procedure for obtaining the 3d quiver in section 3.2, we present an example each for the cases \( p < N \) and \( p > N \) in section 3.3 and section 3.4 respectively. The 3d Lagrangians in these cases are given as:

The blue lines in the quiver diagrams denote the Abelian hypermultiplets. The quiver notation is explained in detail in section 2.1.

- Relation to 3d Lagrangians of Closset et al. for \( D_p(\text{SU}(N)) \) theories and IR duality: the authors of [2] showed that the circle reduction of the \( D_p(\text{SU}(N)) \) theories admit a Lagrangian realization in terms of a linear chain of unitary and special unitary gauge nodes with fundamental and bifundamental hypermultiplets (reviewed in section 1.1.3). Therefore, comparing the 3d Lagrangian of [2] with that obtained by the construction above, one expects that for a given \( D_p(\text{SU}(N)) \) AD theory, there exists an IR duality (not mirror symmetry) between two theories belonging to the following distinct classes:
– Theory $\mathcal{T}$: a linear chain of unitary and special unitary gauge groups with fundamentals and bifundamentals.

– Theory $\mathcal{T}^\vee$: a generically non-linear quiver involving only unitary gauge groups, decorated with Abelian hypermultiplets in addition to fundamental/bifundamental hypers.

In section 4.2.1, we present a general prescription for showing that the three-sphere partition function of the theory $\mathcal{T}^\vee$ agrees with that of the theory $\mathcal{T}$, associated with the circle reduction of a given $D_p(SU(N))$. Generically, the partition functions of the proposed duals agree for generic mass deformations, but only for a subspace of FI deformations for the theory $\mathcal{T}^\vee$. The following IR duality between an SU($N$) gauge theory and a U($N - 1$) gauge theory plays an important role in the prescription (see the quiver notation in section 2.1):

$$\eta = 0$$

As explained in section 4.2.1, one can arrive at the theory $\mathcal{T}^\vee$ by using this duality multiple times in the theory $\mathcal{T}$, at the level of the partition function. In section 4.2.2–section 4.2.3, we present explicit examples of this IR duality associated with the $D_4(SU(6))$ and the $D_9(SU(3))$ respectively, and check the duality by showing that the three-sphere partition functions of the theories agree. Our computations show that the partition functions agree only if certain linear combinations of FI parameters in the $\mathcal{T}^\vee$ theory are set to zero. The proposed IR dualities, for the examples studied in section 4.2, can be summarized as follows:

$D_4(SU(6))$: \vspace{1em}

$D_9(SU(3))$: \vspace{1em}

\footnote{These correspond to certain emergent U(1) Coulomb branch global symmetries that appear in the IR for unitary-special unitary quivers. For the SU($N$) theory with $N_f = 2N - 1$, this was verified in [19].}
For the above dual pairs, we have indicated how the FI parameters map across the duality.

For the class of Lagrangians $\mathcal{T}'$, the Coulomb branch symmetry can be read off from the quiver gauge theory. This is, however, not the case for the class of Lagrangians $\mathcal{T}$, where the Coulomb branch symmetry (or some subgroup of it) appears as an emergent symmetry in the IR [19].

1.3 Concluding remarks

Before moving on to the main text of the paper, we would like to make a few comments about future directions broadly related to the content of this work:

1. **3d Lagrangians for other families of Argyres-Douglas theories:** the construction using the $S$-type operation, as summarized in figure 1, can be deployed whenever the 3d mirror of the AD theory reduced on a circle has a Lagrangian description. In the generic case, where the 3d mirror can only be engineered from a linear quiver with unitary gauge groups by a non-Abelian $S$-type operation, there is no guarantee that the dual operation will produce a Lagrangian theory. However, for families of AD theories for which the aforementioned 3d mirror can be engineered from a linear quiver by a sequence of elementary Abelian $S$-type operations, the mirror of the 3d mirror is guaranteed to be Lagrangian. This in turn implies that the 3d SCFT obtained by the circle-reduction of the associated AD theory is guaranteed to have a Lagrangian description. Aside from the two infinite families — $(A_{p-N-1}, A_{N-1})$ and $D_p(SU(N))$ — that we study in the present work, there are other infinite families for which the 3d mirrors have the same property and can therefore be treated in an analogous fashion. These include the families $(A_s, D_{(s+1)p+2})$ and $D_p^{N-1}(SU(N))$ (for $p \geq N - 1$). In the first case, the 3d Lagrangians can be worked out in a fashion very similar to the $(A_{p-N-1}, A_{N-1})$ theories, while in the second case, the construction closely mimics the $D_p(SU(N))$ theories. It is unclear if our analysis has an extension for theories whose 3d mirrors involve orthosymplectic gauge groups.

2. **IR Dualities:** the IR duality in section 4.1, which plays a crucial role in our analysis, is a representative of a rich class of 3d IR dualities. These have been studied in a series of later (and upcoming) papers [20–22].

3. **Maruyoshi-Song Flow:** it is well known that the $D_p(SU(N))$ theories flow to the $(A_{N-1}, A_{p-1})$ theories [23] in the IR, when an appropriate relevant deformation [24] is turned on. In [19], the authors studied the effect of the RG flow at the level of the 3d mirrors of the respective theories. It would be interesting to investigate if the RG flow can be studied in terms of the new proposed 3d Lagrangians of $D_p(SU(N))$ presented in this paper.\(^3\)

\(^3\)We thank the anonymous referee for pointing this out to us.
2 Review of $S$-type operations and a result for Abelian operations

In this section, we review the $S$-type operations introduced in [1] and a field-theory algorithm to construct new pairs of mirrors starting from a given pair $(X,Y)$.

In particular, we discuss a result on Abelian $S$-type operations which will be central to our construction of the 3d Lagrangians for the AD theories.

2.1 Quiver notation

To begin with, we set up the 3d $\mathcal{N} = 4$ quiver gauge theory notation that we will use for the rest of the paper. The quiver notation, relevant for this paper, is summarized in figure 2.

In this work, we will also encounter hypermultiplets charged only under the $U(1)$ subgroup of certain unitary factors in the quiver gauge group. Given their importance in the current discussion, we will distinguish these hypermultiplets in the quiver diagram with the following notation:

![Quiver diagram](image)

The blue line in the above quiver diagram, connecting the gauge nodes $U(N_1)$, $U(N_2)$ and $U(N_\alpha)$, denotes a single hypermultiplet charged under the gauge subgroup $U(1)_1 \times U(1)_2 \times U(1)_\alpha$, where $U(1)_i \subset U(N_i)$, with the charges $(Q^1, Q^2, Q^\alpha)$. The blue line connected to the flavor node denotes $P$ copies of a hypermultiplet with charge $Q^{\alpha+1}$ under $U(1)_{\alpha+1}$.

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4In a parallel set of work, the authors of [25–27] proposed a field-theoretic algorithm for constructing new 3d mirror pairs starting from known ones. In contrast to the exclusively 3d approach of [1], this algorithm works with an appropriate set of 4d $\mathcal{N} = 1$ theories, which on compactification to 3d and RG flow gives 3d $\mathcal{N} = 4$ theories.

5For a short recent review on 3d $\mathcal{N} = 4$ Lagrangian theories, we refer the reader to the paper [28].

6For the examples treated in this paper, it is more appropriate to think of these hypermultiplets as transforming in determinant/anti-determinant representation of the unitary gauge group.
We will occasionally use another notation for quiver gauge theories, which is particularly useful for representing linear chains of unitary and special unitary gauge groups with fundamental and bifundamental matter:

\[ T = [N] - SU(N_1) - SU(N_2) - \ldots - N_k - N_{k+1} - [M], \]

where the unitary gauge group factor \( U(N) \) is denoted by the integer \( N \), the special unitary factors are explicitly written, \([M], [N]\) denote fundamental hypers for the associated gauge group factor, while consecutive gauge group factors are connected by a single bifundamental hyper.

### 2.2 Review of \( S \)-type operations

Let us now review the \( S \)-type operations \([1]\). Consider a class of 3d quiver gauge theories for which the Higgs branch global symmetry has a subgroup \( G_{\text{sub}} = \prod \gamma U(M_{\gamma}) \subset G_{H} \), and are also good theories in the Gaiotto-Witten sense \([17]\). We will refer to them as class \( U \). Let \( X[\hat{A}] \) be a generic theory in this class, where \( \hat{A} \) collectively denotes the background vector multiplets (including twisted ones) for the global symmetries. Given such a quiver \( X[\hat{A}] \), one can define a set of three basic quiver operations\(^7\) (see \([1]\) for details), which are shown in figure 3:

1. **Gauging** \((G_\alpha^\alpha)\)
2. **Flavoring** \((F_\alpha^\alpha)\)
3. **Identification** \((I_\alpha^\alpha)\)

The superscript \( \alpha \) denotes the flavor node at which the operation is being performed. Note that each operation involves splitting the flavor node \( U(M_\alpha) \) into two, corresponding to \( U(r_\alpha) \times U(M_\alpha - r_\alpha) \) flavor nodes. The U(1)\(M_\alpha\) masses are related to the U(1)\(r_\alpha \times U(1)^{M_\alpha - r_\alpha}\) masses by the following map:

\[
\overline{m}_{i_\alpha} = P_{i_\alpha i} \overline{r}_{i_\alpha} + P_{i_\alpha r_\alpha + j} \overline{r}_{j}, \quad (i_\alpha = 1, \ldots, M_\alpha, \quad i = 1, \ldots, r_\alpha, \quad j = 1, \ldots, M_\alpha - r_\alpha),
\]

where \( P \) is a permutation matrix of order \( M_\alpha \). The resultant theory deformed by the \( U(r_\alpha) \times U(M_\alpha - r_\alpha) \) mass parameters is denoted as \((X, P)\). For the identification operation, this procedure is carried out at multiple flavor nodes, involving a permutation matrix for each. Therefore, in the most general case, we will refer the above theory as \((X, \{P_\beta\})\) or \((X, P)\).

\(^7\)One can also include a defect operation in this list, but we will not need it for this work. See \([29]\) for details on the defect operation.
Figure 3. The gauging, flavoring and identification quiver operations on a generic quiver $X$ of class $U$. Each operation involves two steps. In the first step, one splits the flavor node $U(M_{\alpha})$ into two, corresponding to $U(r_{\alpha}) \times U(M_{\alpha} - r_{\alpha})$ flavor nodes. In the second step, one gauges, adds hypermultiplets to, or identifies the $U(r_{\alpha})$ flavor node respectively. The identification operation may involve two or more $U(r_{\alpha})$ flavor nodes.

**Definition.** An elementary $S$-type operation $O_{\alpha}^{\alpha}$ on $X$ at a flavor node $\alpha$, is defined as any possible combination of the identification ($I_{\alpha}^{\alpha}$) and the flavoring ($F_{\alpha}^{\alpha}$) quiver operations followed by a single gauging operation $G_{\alpha}^{\alpha}$, i.e.

$$ O_{\alpha}^{\alpha}(X) := (G_{\alpha}^{\alpha}) \circ (F_{\alpha}^{\alpha})^{n_2} \circ (I_{\alpha}^{\alpha})^{n_1}(X), \quad (n_i = 0, 1, \forall i). \quad (2.3) $$

The operation $O_{\alpha}^{\alpha}$ generically maps a quiver gauge theory $X[\hat{A}]$ to a new quiver gauge theory $X'[\hat{B}]$:

$$ O_{\alpha}^{\alpha} : X[\hat{A}] \mapsto X'[\hat{B}]. \quad (2.4) $$

The order of the different constituent operations in the above definition is important to emphasize. The composition of the identification and the flavoring operations is commutative, while none of the operations commute with the gauging operation. From the above definition, one finds that there are four distinct types of elementary $S$-type operations depending on the constituent quiver operations:
Figure 4. Generating new dual pairs using an elementary $S$-type operation.

- Gauging.
- Flavoring-gauging.
- Identification-gauging.
- Identification-flavoring-gauging.

**Definition.** An elementary Abelian $S$-type operation is defined as an elementary $S$-type operation where the gauging operation $G_\mathcal{P}$ involves a $U(1)$ global symmetry.

Note that the elementary Abelian gauging operation gives the action of the $SL(2, \mathbb{Z})$ $S$-generator [16] on a 3d SCFT with a $U(1)$ global symmetry.

**Definition.** A generic $S$-type operation is a combination of any number of elementary $S$-type operations:

$$O_{(\alpha_1, \ldots, \alpha_l)}(\mathcal{P}_1, \ldots, \mathcal{P}_l)(X) := O_{\alpha_1}^{\alpha_l} \circ O_{\mathcal{P}_1}^{\alpha_{l-1}} \circ \ldots \circ O_{\mathcal{P}_2}^{\alpha_2} \circ O_{\mathcal{P}_1}^{\alpha_1}(X).$$  

(2.5)

The superscript $(\alpha_1, \ldots, \alpha_l)$ labels the flavor nodes at which the elementary $S$-type operations are performed, with $(\mathcal{P}_1, \ldots, \mathcal{P}_l)$ denoting the associated permutation matrices.

One of the main applications of the $S$-type operations, as discussed in [1], is to construct new pairs of 3d mirror theories starting from a given pair of Lagrangian mirrors $(X[\hat{A}], Y[\hat{A}])$, with $X$ in class $U$. We now briefly review the construction. Given the operation

$$O_{\hat{B}}^x : X[\hat{A}] \rightarrow X'[\hat{B}],$$

(2.6)

one can then define a dual operation $\tilde{O}_{\hat{B}}^x$ which acts on the quiver gauge theory $Y$:

$$\tilde{O}_{\hat{B}}^x : Y[\hat{A}] \rightarrow Y'[\hat{B}],$$

(2.7)

such that the pair $(X'[\hat{B}], Y'[\hat{B}])$ is IR dual. This can be summarized in figure 4.

The goal now is to precisely determine the dual operation $\tilde{O}_{\hat{B}}^x$ on the theory $Y$ and therefore determine the theory $Y'$. This is achieved using supersymmetric observables computed using localization. Below, we illustrate the construction using the three-sphere partition function, as discussed in section 4.1 of [1]. For a similar construction using superconformal index, we refer the reader to appendix C of [1].
The three-sphere partition function of a 3d $\mathcal{N} = 4$ Lagrangian theory with gauge group $G$ is given by a matrix integral of the following form [30]:

\[ Z^{(T)}(m; \eta) = \int [ds] Z_{\text{int}}(s, m, \eta) =: \int [ds] Z_{\text{FI}}(s, \eta) Z_{1\text{-loop}}^{\text{vector}}(s) Z_{1\text{-loop}}^{\text{hyper}}(s, m), \quad (2.8) \]

where the measure $[ds] = \frac{ds}{|W(G)|}$, and $|W(G)|$ is the order of the Weyl group of $G$. The partition function is a function of the real masses $m$ and real FI parameters $\eta$, which are the real scalar components inside background 3d $\mathcal{N} = 4$ vector multiplets and twisted vector multiplets respectively. The individual terms in the integrand on the r.h.s. consist of classical and one-loop contributions, and are given as follows.

- The classical contribution to the matrix integral comes from the $l$ U(1) factors in the gauge group $G$:

\[ Z_{\text{FI}}(s, \eta) = \prod_{\gamma=1}^{l} e^{2\pi i \eta_{\gamma} \text{Tr}(s^{\gamma})}, \quad (2.9) \]

where $\gamma$ runs over the $l$ unitary gauge nodes of the theory and $\eta_{\gamma}$ is the FI parameter for a given $\gamma$.

- The one-loop contribution of the $\mathcal{N} = 4$ vector multiplet is given as:

\[ Z_{1\text{-loop}}^{\text{vector}}(s) = \prod_{\alpha} \sinh \pi \alpha(s), \quad (2.10) \]

where the product extends over the roots of the Lie algebra of $G$.

- The one-loop contribution of an $\mathcal{N} = 4$ hypermultiplet transforming in a representation $\mathcal{R}$ of $G \times G_H$ is given as:

\[ Z_{1\text{-loop}}^{\text{hyper}}(s, m) = \prod_{\rho(\mathcal{R})} \frac{1}{\cosh \pi \rho(s, m)}, \quad (2.11) \]

where the product extends over the weights of the representation $\mathcal{R}$.

An elementary $S$-type operation can be implemented in terms of the three-sphere partition function as follows:

\[ Z_{\mathcal{O}_{\mathcal{P}}(X)}^{\mathcal{O}_{\mathcal{P}}(X)}(m^{\mathcal{O}_{\mathcal{P}}}, \ldots; \eta, \eta_{\alpha}) = \int [du^{\alpha}] Z_{\mathcal{O}_{\mathcal{P}}(X)}(u^{\alpha}, \{u^{\beta}\}_{\beta \neq \alpha}, \eta_{\alpha}, m^{\mathcal{O}_{\mathcal{P}}}) \cdot Z^{(X, \{P_{\beta}\})}(\{u^{\beta}\}, \ldots; \eta), \quad (2.12) \]

where $Z_{\mathcal{O}_{\mathcal{P}}(X)}^{\mathcal{O}_{\mathcal{P}}(X)}$ denotes the partition function of the theory $X' = \mathcal{O}_{\mathcal{P}}(X)$, and $Z_{\mathcal{O}_{\mathcal{P}}(X)}$ is an operator that we will momentarily specify. The parameter $\eta_{\alpha}$ is an FI parameter associated with the gauging operation, and the parameters $m^{\mathcal{O}_{\mathcal{P}}}$ collectively denote the hypermultiplet masses associated with the flavoring and/or the identification operations. The operator $Z_{\mathcal{O}_{\mathcal{P}}(X)}$ can be built out of the partition function contributions of the constituent gauging $(G_{\mathcal{P}})$, flavoring $(F_{\mathcal{P}})$, and identification $(I_{\mathcal{P}})$ operations following the definition (2.3):

\[ Z_{\mathcal{O}_{\mathcal{P}}(X)}(u^{\alpha}, \{u^{\beta}\}_{\beta \neq \alpha}, \eta_{\alpha}, m^{\mathcal{O}_{\mathcal{P}}}) = Z_{G_{\mathcal{P}}(X)} \cdot \left(Z_{F_{\mathcal{P}}(X)}\right)^{n_{2}} \cdot \left(Z_{I_{\mathcal{P}}(X)}\right)^{n_{1}}, \quad (n_{1}, n_{2} = 0, 1), \quad (2.13) \]
where the individual constituents are:

\[
Z_{G^p, (X)}(u^\alpha, \eta_\alpha) = Z_{\text{FI}}(u^\alpha, \eta_\alpha) Z_{1\text{-loop}}^{\text{vector}}(u^\alpha) = e^{2\pi i \eta_\alpha \sum_i u^\alpha_i \prod_{i<j} \sinh^2 \pi (u^\alpha_i - u^\alpha_j)},
\]

(2.14)

\[
Z_{F^p, (X)}(u^\alpha, m^\alpha_F) = Z_{1\text{-loop}}^{\text{hyper}}(u^\alpha, m^\alpha_F) = \prod_{\rho(R^\alpha)} \frac{1}{cosh \pi \rho(u^\alpha, m^\alpha_F)},
\]

(2.15)

\[
Z_{I^p, (X)}(u^\alpha, \{u^\beta\}, \mu) = \int \prod_{j=1}^{r_\alpha} du^\gamma \prod_{j=1}^{p} \delta(r_\alpha) \left( u^\alpha - u^\gamma + \mu^\gamma \right).
\]

(2.16)

In (2.15), the parameters \( m^\alpha_F \) denote the masses for the added flavors. In (2.16), the identified nodes (we assume that there are \( p \) of them) are labelled as \( \{\gamma_j\}_{j=1,\ldots,p} \), while \( \{\mu^\gamma\}_{j=1,\ldots,p} \) denote the masses introduced by the identification operation. The set of masses \( \{m^\alpha_F, \{\mu^\gamma\}_{j=1,\ldots,p} \} \) are collectively denoted as \( m^{\Omega_F} \) in the subsequent formulas.

The index \( i \) runs from \( 1 \) to \( r_\alpha \), where \( r_\alpha \) denotes the rank of the group being gauged.

We can now proceed to work out the dual operation in terms of the partition function. Mirror symmetry implies that the partition functions of \( X \) and \( Y \) are related as

\[
Z^{(X,\{P_\beta\})}(\{u^\beta\}, \{v^\beta\}, \ldots; \eta) = e^{2\pi i \sum_{i,l,\beta} b^\beta_{il} u^\gamma_i \eta_i \cdot C_{XY}(\{u^\beta = 0\}, \ldots, \eta \cdot Z^{(Y,\{P_\beta\})}(m^\gamma, \eta^\gamma)},
\]

(2.17)

where \( C_{XY}(\{u^\beta\}, \ldots, \eta) \) collectively denotes the contact terms [31]. In the above equation, we have separated the \( u^\beta \)-dependent contact terms from the \( u^\beta \)-independent ones, for reasons that will become apparent momentarily. The coefficients in the exponent \( \{b^\beta_{il}\} \) are integer entries that can be explicitly computed, given the matrix integrals for \( X \) and \( Y \).

The partition function of the theory \( Y \) has the following form

\[
Z^{(Y,\{P_\beta\})} = \prod_{\gamma'} \left[ d\sigma^{\gamma'} \right] Z^{(Y,\{P_\beta\})}_{\text{int}}(\{\sigma^{\gamma'}\}, m^Y(\eta), \eta^Y(\{u^\beta\}, \ldots))
\]

\[
= \prod_{\gamma'} \left[ d\sigma^{\gamma'} \right] e^{2\pi i \sum_{i,\beta} g^\beta_{i\beta}(\sigma^{\gamma'}, P_\beta) u^\gamma_i Z^{(Y,\{P_\beta\})}_{\text{int}}(\{\sigma^{\gamma'}, \sigma^Y(\{P_\beta\}), m^Y(\eta), \eta^Y(\{u^\beta = 0\}, \ldots))},
\]

(2.18)

where \( \gamma' \) labels the gauge nodes of the quiver \( Y \), and in the second equality, we have isolated the \( \{u^\beta\} \)-dependent terms of the integrand of the matrix model. Recall that the parameters \( \{u^\beta\} \), which are mass deformations of theory \( X \), will map to FI parameters of the theory \( Y \), and contribute a term of the form (2.9) in the matrix integral. The term \( e^{2\pi i \sum_{i,\beta} g^\beta_{i\beta}(\sigma^{\gamma'}, P_\beta) u^\gamma_i \} \) in the second line denotes the \( \{u^\beta\} \)-dependent part of the FI contribution in the matrix integral of \( Y \). For a generic mirror pair \((X,Y)\), determining the functions \( g^\beta_{i\beta} \) will require knowing the mirror map between the masses of \( X \) and the FI parameters of \( Y \), which may not be readily known. However, our starting point will always

---

*Generically, such a computation can be cumbersome. However, our starting point will always be a pair of good linear quivers with unitary gauge groups, for which the computation is relatively straightforward. See section A.1 of [29] for a concrete example of such a computation.*
be a pair of good linear quivers with unitary gauge groups, for which the mirror map can be read off easily from the Type IIB construction (or from a matrix model computation).

Following the logic of figure 1, the partition function of the dual theory \( Y' = \hat{\mathcal{O}}_{\mathcal{P}}(Y) \) can be written in the following form:

\[
Z^{\hat{\mathcal{O}}_{\mathcal{P}}(Y)}(m', \eta') = \prod_{\gamma'} \left[ d\sigma' \right] Z_{\hat{\mathcal{O}}_{\mathcal{P}}(Y)}(\{\sigma'\}, m' \mathcal{O}^a_{\mathcal{P}}, \eta_a, \eta) \cdot C_{XY}(\{u^\beta = 0\}, \ldots, \eta) \times Z^{(Y,\{P_s\})}_{\text{int}}(\{\sigma'\}, m' Y(\eta), \eta'(u^\beta = 0), \ldots), \tag{2.19}
\]

where \( m' = m'(\eta, \eta_a), \eta' = \eta'(m' \mathcal{O}^a_{\mathcal{P}}) \) are linear functions of their arguments. The function \( Z_{\hat{\mathcal{O}}_{\mathcal{P}}(Y)} \) can be formally written as a Fourier-transform of the operator \( Z_{\mathcal{O}_{\mathcal{P}}(X)} \) defined in (2.13):

\[
Z_{\hat{\mathcal{O}}_{\mathcal{P}}(Y)} = \int d\gamma \left[ du^\gamma \right] Z_{\mathcal{O}_{\mathcal{P}}(X)}(u^\gamma, \{u^\beta\}_{\beta \neq \alpha}, \eta_a, m' \mathcal{O}^a_{\mathcal{P}}) \cdot e^{2\pi i \sum_{i,\beta}(g_{i,\beta}^{(a)}(\sigma'')); P_{\beta} + \sum_{i,\beta} \eta_{i,\beta} u_{\beta}}, \tag{2.20}
\]

where \( g_{ij}^{(a)} \) is the set of functions defined in (2.18), and \( b_{ij}^{(a)} \) are a set of integers defined in (2.17). The equations (2.19)–(2.20) gives a working definition of the dual S-type operation on the theory \( Y \). Although the r.h.s. of (2.19) is written as an operation on a Lagrangian theory \( Y \), the theory \( Y' \) may not turn out to be Lagrangian. However, if the theory \( Y' \) is indeed Lagrangian, one can rewrite the r.h.s. of (2.19) in the standard form of (2.8), after some manipulations of the matrix integral. The gauge group and matter content of the theory \( Y' \) can then be read off from (2.19), as well as the contact term \( C_{XY'}. \)

### 2.3 Dual operations for the Abelian case

The dual operation greatly simplifies for the special case where \( \mathcal{O}_{\mathcal{P}} \) is an Abelian S-type operation. Since the gauging operation is Abelian (i.e. one is integrating over a single variable \( u \)), the integers \( \{b_{ij}^{(a)}\} \) and the functions \( \{g_{ij}^{(a)}\} \) simplify as:

\[
b_{ij}^{(a)} \to b_i^{(a)}; \quad g_{ij}^{(a)} \to g_i^{(a)}. \]

The dual theory for the four types of elementary operations can be explicitly determined from the general expression in (2.19) and turn out to be Lagrangian. The details of the computation in terms of the sphere partition function can be found in section 4.1 of [1], and we summarize the results below.\(^9\) The general features of the dual theory in each case are as follows:

- **Gauging and Identification-gauging:** the dual partition function for the gauging operation is given as

\[
Z^{\hat{\mathcal{O}}_{\mathcal{P}}(Y)} = \prod_{\gamma'} \left[ d\sigma' \right] \delta(\eta_a + \sum_{i} b_i^{(a)} \eta_i + g_a(\{\sigma'\}, P)) \times Z^{(Y,P)}_{\text{int}}(\{\sigma'\}, m Y(\eta), \eta Y(u^\alpha = 0, v^\alpha, \ldots)), \tag{2.21}
\]

\(^9\)The analogous computation in terms of thesuperconformal index can be found in appendix C of [1].
while that for the identification-gauging operation is given as
\begin{equation}
Z^{(G^\alpha_p \circ \mathcal{F}_p^\beta)(Y)} = \int \prod_{\gamma'} [d\sigma^{\gamma'}] \delta(\eta_{\alpha} + \sum_{\beta} \sum_{l} b_{\beta l} \eta_{l} + \sum_{\beta} \frac{g_{\beta}}{2} (\sigma^{\gamma'}, P_{\beta})) \\
\times Z^{[Z^{(Y(P_{\beta})})]}_{\text{int}}(\sigma^{\gamma'}, m^{Y}(\eta), \eta^{Y}(\{u^{3} = \mu^{\beta}\}, \{v^{\beta}\}, \ldots)).
\end{equation}

In both cases, the dual operation amounts to gauging a $U(1)$ topological symmetry of the theory $Y$, which is equivalent to ungauging a $U(1)$ subgroup of the group of $Y$. The precise $U(1)$ subgroup being ungauged can be read off from the argument of the delta function in the matrix model on the r.h.s. of (2.21) and (2.22) respectively.

- **Flavoring-gauging and Identification-flavoring-gauging**: consider the flavoring-gauging operation first. For the sake of concreteness (and relevance to theories studied in this paper), we will assume that the flavoring involves adding $N_{P}^{\alpha}$ hypermultiplets with charge 1 under the $U(1)$ flavor symmetry to be gauged. In this case, the dual partition function can be written in the following form:
\begin{equation}
Z^{(G^\alpha_p \circ \mathcal{F}_p^\beta)(Y)} = \\
\int \prod_{\gamma'} [d\sigma^{\gamma'}] \prod_{k=1}^{N_{P}^{\alpha} - 1} d\tau_{k}^{\alpha} Z^{[\mathcal{T}[N_{P}^{\alpha} - 1]]}_{\text{int}}(\tau^{\alpha}, 0, -m_{\alpha}^{F}) \cdot Z_{\text{hyper}}^{(G^\alpha_p \circ \mathcal{F}_p^\beta)(Y)}(\sigma^{\gamma'}, \tau_{1}^{\alpha}, \eta_{\alpha}, \eta) \\
\times Z^{[\mathcal{T}[N_{P}^{\alpha} - 1]]}_{\text{int}}(\sigma^{\gamma'}, m^{Y}(\eta), \eta^{Y}(u^{\alpha} = m_{1}^{\alpha} F, v^{\alpha}, \ldots)),
\end{equation}

with $Z^{[\mathcal{T}[N_{P}^{\alpha} - 1]]}_{\text{int}}(\tau^{\alpha}, 0, m_{\alpha}^{F})$ being the partition function of the theory $\mathcal{T}[N_{P}^{\alpha} - 1]$: \\
\begin{center}
\begin{array}{cccccccc}
& 1 & 2 & N_{P}^{\alpha} - 2 & N_{P}^{\alpha} - 1 \\
\mathcal{T}[N_{P}^{\alpha} - 1]: & 1 & \cdot & \cdots & 1 & 1 & 1
\end{array}
\end{center}

where the mass parameter for the single fundamental hyper is set to zero, the FI parameters (from left to right) are given by the parameters $\{m_{\alpha}^{F}\}_{1, \ldots, N_{P}^{\alpha}}$, and the matrix model integration variables (from left to right) are labelled as $\{\tau_{k}^{\alpha}\}_{1, \ldots, N_{P}^{\alpha} - 1}$. Finally, $Z_{\text{hyper}}^{(G^\alpha_p \circ \mathcal{F}_p^\beta)(Y)}$ is the contribution of a single hypermultiplet which is charged under the $U(1)$ node of $\mathcal{T}[N_{P}^{\alpha} - 1]$ and under $U(1)$ subgroups of certain unitary gauge nodes of the quiver $Y$. Explicitly, the function $Z_{\text{hyper}}^{(G^\alpha_p \circ \mathcal{F}_p^\beta)(Y)}$ is given as:
\begin{equation}
Z_{\text{hyper}}^{(G^\alpha_p \circ \mathcal{F}_p^\beta)(Y)}(\sigma^{\gamma'}, \tau_{1}^{\alpha}, \eta_{\alpha}, \eta) = \frac{1}{\cosh \pi \{g_{\alpha}(\sigma^{\gamma'}), P\} + \tau_{1}^{\alpha} + \eta_{\alpha} + \sum_{l} b_{\alpha l} \eta_{l}},
\end{equation}

where the various $U(1)$ charges can be read off from the argument of the cosh-function.
In a similar fashion, the dual partition function for the identification-flavoring-gauging operation is given as:

\[
Z^{(G^αP ◦ F^αP ◦ I^αP)}(Y) = \int \prod_{γ'} [dσ^{γ'}] \prod_{k=1}^{N^α_F-1} dτ^{α}_k \ Z^{(T[N^α_F-1])}(τ^{α}, 0, -m^α_{F}) \cdot Z^{(G^αP ◦ F^αP ◦ I^αP)}_{\text{hyperm}}\left(\left\{σ^{γ'}\right\}, \tau^{α}_1, η_{α}, η\right) \\
\times Z^{(Y, \left\{P^{β}\right\})}(\left\{σ^{γ'}\right\}, τ^{α}_1, η_{α}, η) \cdot Z^{(Y', \left\{P^{β}\right\})}(\left\{σ^{γ'}\right\}, τ^{α}_1, η_{α}, η) \cdot Z^{\left(\sum_{β} g^{β}(\left\{σ^{γ'}\right\}, P^{β}) \right)}^{(G^αP ◦ F^αP ◦ I^αP)}(Y), \tag{2.25}
\]

where \(Z^{(T[N^α_F-1])}\) is defined as above, and \(Z^{(G^αP ◦ F^αP ◦ I^αP)}_{\text{hyperm}}\) is the contribution of a single hypermultiplet:

\[
Z^{(G^αP ◦ F^αP ◦ I^αP)}_{\text{hyperm}}\left(\left\{σ^{γ'}\right\}, τ^{α}_1, η_{α}, η\right) = \frac{1}{\cosh \left(\sum_{β} g^{β}(\left\{σ^{γ'}\right\}, P^{β}) \right)} \cdot \frac{1}{\cosh \left(\sum_{β} g^{β}(\left\{σ^{γ'}\right\}, P^{β}) + τ^{α}_1 + η_{α} + \sum_{β,l} b^{β}_l η_l\right)}. \tag{2.26}
\]

Therefore, for both types of operations, one can write down the general form of the quiver \(Y'\). Assuming that the flavoring operation involves adding \(N^{α}_{F}\) hypermultiplets of gauge charge 1, \(Y'\) has the following form:

The quiver \(Y'\) can be built from the two subquivers - \(Y\) and the linear quiver \(T[N^α_F-1]\), which are connected by a single hypermultiplet which has charge 1 under \(U(1)_1\) gauge node of \(T[N^α_F-1]\), and is also charged under the \(U(1)\) subgroups of certain unitary factors in the gauge group of \(Y\). The charge vector corresponding to the latter is denoted by \(Q^{P}\), so that the full charge vector is given by \(Q^{P}, 1\). The precise charges can be read off from the formula \(2.24\) and \(2.26\) respectively.\(^{10}\)

This leads directly to the following result, which will be the main tool in this paper for constructing 3d Lagrangians for the AD theories:

**Result.** If a quiver gauge theory \(X'\) is related to another quiver \(X\) by a sequence of elementary Abelian \(S\)-type operations, and \(X\) has a Lagrangian mirror \(Y\), then \(X'\) is guaranteed to have a Lagrangian mirror \(Y'\).

The proof of this statement is straightforward. Starting from a dual pair of Lagrangian theories \((X, Y)\), the action of a single elementary Abelian \(S\)-type operation always gives a new pair of Lagrangian dual theories, as we discussed above. Therefore, implementing any

\(^{10}\)For a more intuitive description of the Abelian dual operations, see section 4.1.5 of [1].
number of elementary Abelian $S$-type operations on the dual pair $(X,Y)$ will obviously lead to a Lagrangian dual pair $(X',Y')$.

An important corollary of the above construction is as follows. Assume that the pair of dual theories $(X,Y)$ are quivers with unitary gauge groups. If a generic Abelian $S$-type operation $O^{(\alpha_1,\ldots,\alpha_l)}(P_1,\ldots,P_l)$ acting on $(X,Y)$ consists only of flavoring-gauging and identification-flavoring-gauging operations, then the new mirror pair $(X',Y')$ are guaranteed to have only unitary gauge groups. The generic structure of the quiver $Y'$ is given as above. We will use this corollary to infer certain general features of 3d Lagrangians for AD theories in section 3.

3 Unitary 3d Lagrangians from $S$-type operations

In this section, we discuss the general strategy for writing down the 3d Lagrangians for the $(A_{p-N-1},A_{N-1})$ and the $D_p(SU(N))$ theories using $S$-type operations, and work out a few examples explicitly.

3.1 Constructing the 3d Lagrangians for the $(A_{p-N-1},A_{N-1})$ theories

In this subsection, we discuss the procedure for constructing the 3d Lagrangians for $(A_{p-N-1},A_{N-1})$ on circle reduction. We give a systematic prescription for writing down the 3d Lagrangian for the generic case, and then present a specific example.

The basic idea is to study the 3d mirror of $(A_{p-N-1},A_{N-1})$, which can be explicitly written down from the class $S$ construction of the 4d theory [8]. Given these 3d mirrors, one observes that they can be engineered from a $U(1)$ gauge theory with fundamental hypermultiplets, by a sequence of elementary Abelian $S$-type operations. Since the dual operation, following the discussion in section 2, is guaranteed to produce a Lagrangian theory, we can explicitly write down a Lagrangian for the mirror of the 3d mirror i.e. a Lagrangian for the AD theory reduced on a circle.

Given the $(A_{p-N-1},A_{N-1})$ theory, one can define the following parameters:

\[
\begin{align*}
m = \text{GCD}(p,N), & \quad n = \frac{N}{m}, \quad q = \frac{p}{m}, \quad m_G = n(q - n).
\end{align*}
\] (3.1)

The 3d mirror of the $(A_{p-N-1},A_{N-1})$ theory is given by a quiver gauge theory with a set of free hypermultiplets. The former is given by a complete graph of $m$ vertices and edge multiplicity $m_G$, where each vertex denotes a $U(1)$ gauge node and an edge of multiplicity $m_G$ denotes $m_G$ bifundamental hypermultiplets, as shown in figure 5. The number of free multiplets is $H_{\text{free}} = \frac{(N-m)(p-N-m)}{2m}$.

We now observe that a generic quiver gauge theory of the form given in figure 5 can be constructed from a $U(1)$ theory with fundamental hypermultiplets by a sequence of elementary $S$-type operations. This can be done using the following steps:
Figure 5. The quiver gauge theory corresponding to the case with $m = 3$ and generic $m_G$. The quiver on the right is obtained by decoupling a U(1) vector multiplet associated with one of the vertices of the complete graph on the left.

1. The starting point is the quiver gauge theory $X$:

\[ X : \]

\[ \begin{array}{c}
1 \\
(m-1)m_G
\end{array} \]

2. Consider splitting the flavor node $U((m-1)m_G) \to U((m-2)m_G) \times U(m_G)$, and the resultant $U(m_G)$ flavor node into $m_G \ U(1)$ flavor nodes, as shown below. Now, we implement the following elementary $S$-type operation — the $m_G \ U(1)$ flavor nodes are identified to a single $U(1)$ flavor node, which is then flavored by $(m-2)m_G$ fundamental hypermultiplets, followed by gauging the identified $U(1)$. This identification-flavoring-gauging operation $O_P^{(1)}$ leads to the following quiver gauge theory:

3. In the next step, we split the two flavor nodes as $U((m-2)m_G)_1 \to U((m-3)m_G)_1 \times U(m_G)_1$ and $U((m-2)m_G)_2 \to U((m-3)m_G)_2 \times U(m_G)_2$. In addition, we split each of the two $U(m_G)$ flavor nodes into $m_G \ U(1)$ nodes. Now, we implement the following elementary $S$-type operation — the $2m_G \ U(1)$ flavor nodes are identified to a single $U(1)$ flavor node, which is then flavored by $(m-3)m_G$ fundamental hypermultiplets, followed by gauging the identified $U(1)$. This identification-flavoring-gauging
Moduli space data & Theory $X'$ & Theory $Y'$ \\
$\dim M_H$ & $3m_G-2$ & $2$ \\
$\dim M_C$ & $2$ & $3m_G-2$ \\
$G_H$ & $SU(m_G) \times SU(m_G) \times SU(m_G) \times U(1)$ & $U(1) \times U(1)$ \\
$G_C$ & $U(1) \times U(1)$ & $SU(m_G) \times SU(m_G) \times SU(m_G) \times U(1)$ \\

Table 1. Summary table for the moduli space dimensions and global symmetries for the mirror pair in figure 6.

operation leads to the following quiver gauge theory:

![Quiver Diagram]

4. Proceeding in the same fashion, we implement $(m-2)$ identification-flavoring-gauging operation to engineer the quiver gauge theory $X'$ from the quiver $X$, i.e.

$$X' = \mathcal{O}_{P_{m-2}}^{(m-2)} \circ \mathcal{O}_{P_{m-3}}^{(m-3)} \circ \ldots \circ \mathcal{O}_{P_2}^{(2)} \circ \mathcal{O}_{P_1}^{(1)}(X),$$  

where $X'$ is the quiver gauge theory given in figure 5, for generic $m$ and $m_G$.

Using the general prescription in section 2.3 for finding out the dual of a given elementary Abelian $S$-type operation, one can explicitly determine the 3d mirror of $X'$. Since $X'$ can be engineered from the linear quiver $X$ by a sequence of elementary Abelian $S$-type operations, the dual quiver $Y'$ is guaranteed to be a Lagrangian theory. The quiver $Y'$, for generic $p$ and $N$, will be a non-linear quiver involving $U(1)$ gauge groups. As discussed earlier, the quiver $Y'$, along with $H$ free twisted hypermultiplets, then gives a Lagrangian description for the $(A_p-N-1,A_N-1)$ AD theory reduced on a circle. We will now explicitly find the quiver $Y'$ in a specific example.

Consider the case where $m = 3$ and $m_G$ is arbitrary. The 3d mirror of the associated AD theory is then given by the theory $(X')$ in figure 5. One can now construct the quiver $X'$ starting from a simple linear quiver using the $S$-type operation technique, and determine its mirror dual $Y'$. The result is shown in figure 6, where the dual theory $Y'$ has the shape of a star-shaped quiver consisting of three linear quivers glued at a common $U(1)$ gauge node. The dimensions of the respective Higgs and Coulomb branches, and the associated global symmetries are shown in table 1.

Let us now construct the quiver $Y'$ explicitly, using the $S$-type operations. The starting point is the mirror pair $(X,Y)$, as shown in figure 7. The partition functions of the theories
Figure 6. The quiver $X'$ is the 3d mirror of $(A_{p-N-1}, A_{N-1})$ (with $m = 3$ and $m_G$ arbitrary) reduced on a circle, constructed from the class $S$ prescription. The quiver $Y'$ is the 3d mirror of $X'$, i.e. a Lagrangian realization of the SCFT obtained by the reduction of $(A_{p-N-1}, A_{N-1})$ (with $m = 2$ and $m_G$ arbitrary) on a circle.

Figure 7. A pair of mirror dual Abelian linear quivers.

$(X,Y)$ are given as

\begin{align}
Z^{(X)}(m,t) &= \int ds \frac{e^{2\pi i (t_1-t_2)s}}{\prod_{k=1}^{2m_G} \cosh \pi (s-m_k)} =: \int ds Z^{(X)}_{\text{int}}(s,m,t), \quad (3.3) \\
Z^{(Y)}(t,m) &= \int \prod_{k=1}^{2m_G-1} d\sigma_k \frac{e^{2\pi i (m_k-m_{k+1})\sigma_k}}{\prod_{k=1}^{2m_G-1} \cosh \pi (\sigma_k - \sigma_{k+1}) \cosh \pi (\sigma_{2m_G-1} - t_2)} \\
&=: \int \prod_{k=1}^{2m_G-1} d\sigma_k Z^{(Y)}_{\text{int}}(\{\sigma_k\},m,t), \quad (3.4)
\end{align}

where $Z^{(X)}_{\text{int}}$ and $Z^{(Y)}_{\text{int}}$ are the respective matrix model integrands for $X$ and $Y$, and the parameters $m,t$ are unrestricted. Mirror symmetry of $X$ and $Y$ implies that

\begin{align}
Z^{(X)}(m,t) &= C_{XY}(m,t) Z^{(Y)}(t,-m), \quad (3.6) \\
C_{XY}(m,t) &= e^{2\pi i (m_1 t_1 - m_2 m_G t_2)}, \quad (3.7)
\end{align}

where $C_{XY}$ is a contact term, which reminds us that the duality holds for unrestricted $m$ and $t$, only when we turn on a certain background Chern-Simons sector.

To obtain the quiver $X'$ from the linear quiver $X$, one implements an elementary Abelian identification-flavoring-gauging operation $O^\alpha_P$ on $X$ (as stated in Step 2 of the general procedure above):

\begin{align}
O^\alpha_P(X) &= G^\alpha_P \circ F^\beta_P \circ I^\beta_P(X), \quad (3.8)
\end{align}
where $O_\alpha^\alpha$ is shown explicitly in figure 8. The $m_G$ $U(1)$ nodes being identified are shown in red. The mass parameters associated with the $U(1)^m_G$ global symmetry being identified are chosen as:

$$u_j = m_{m_G + j}, \quad j = 1, \ldots, m_G,$$

which corresponds to a specific choice of the permutation matrix $P$. The partition function of the theory $X' = O_\alpha^\alpha(X)$ is then given as

$$Z^{O_\alpha^\alpha}_P(X) = \int du^\alpha Z_{O_\alpha^\alpha}(X(u^\alpha, \eta, m_f, \mu) \cdot Z^{(X, \alpha)}(u, \ldots, t),$$

where the operator $Z_{O_\alpha^\alpha}(X)$ can be constructed from the general prescription of (2.13) and has the following form:

$$Z_{O_\alpha^\alpha}(X) = Z_{FI}(u^\alpha, \eta, \mu) Z_{\text{hyper}_1}^{(X)}(u^\alpha, m_f) \prod_{j=1}^{m_G} du_j \delta(u^\alpha - u_j + \mu_j)$$

$$= \frac{e^{2\pi i \eta - u^\alpha}}{\prod_{a=1}^{m_G} \cosh \pi(u^\alpha - m_f) \prod_{j=1}^{m_G} du_j \delta(u^\alpha - u_j + \mu_j).$$

Using the above expression, the partition function of the theory $X'$ can be explicitly written as

$$Z^{(X')} = \int ds du^\alpha e^{2\pi i \eta - u^\alpha} e^{2\pi i (t_1 - t_2) s} \prod_{i=1}^{m_G} \cosh \pi(s - m_i^{(1)}) \prod_{a=1}^{m_G} \cosh \pi(u^\alpha - m_a^{(2)}) \prod_{j=1}^{m_G} \cosh \pi(s - u^\alpha - m_j^{\text{bif}}),$$

where the fundamental masses $m_i^{(1)}$, $m_i^{(2)}$ and the bifundamental masses $m_j^{\text{bif}}$ are given in terms of mass parameters introduced above in the following fashion:

$$m_i^{(1)} = m_i, \quad i = 1, \ldots, m_G, \quad (3.14)$$

$$m_a^{(2)} = m_a^{f}, \quad a = 1, \ldots, m_G, \quad (3.15)$$

$$m_j^{\text{bif}} = \mu_j, \quad j = 1, \ldots, m_G. \quad (3.16)$$

The matrix integral on the r.h.s. of (3.13) manifestly corresponds to the quiver $X'$ in figure 6. By shifting the integration variables, one can check that the matrix integral can be written in terms of $3m_G - 2$ independent real masses, which live in the Cartan subalgebra of the Higgs branch global symmetry group of $X'$, i.e. $G^X_H = SU(m_G)^3 \times U(1)$. The Coulomb branch global symmetry can be read off from the quiver itself — it has two unbalanced nodes (unless $m_G = 1$, in which case $X'$ is a linear quiver) leading to a $G^X_C = U(1)^2$. 

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Figure 8. Construction of the pair of mirror dual theories \((X', Y')\), by a single elementary Abelian \(S\)-type operation of the identification-flavoring-gauging type. The dual operation involves attaching a balanced linear chain of \(U(1)\) gauge nodes to the \(U(1)_{mG}\) gauge node of \(Y\).

The dual theory can be read off from the general construction in (2.19)–(2.20), with the function \(\tilde{Z}_{\tilde{O}_\alpha P}(Y)\) being given as\(^{11}\)

\[
\tilde{Z}_{\tilde{O}_\alpha P}(Y) = \int du^\alpha Z_{\tilde{O}_\alpha P}(X) \cdot \prod_{j=1}^{mG} e^{2\pi i g_j(\{\sigma_k\}, P) + \sum_l b_l(t_l) u_j},
\]

\[
\prod_{j=1}^{mG} e^{2\pi i g_j(\{\sigma_k\}, P) u_j} = \left( \prod_{j=1}^{mG-1} e^{2\pi i u_j (\sigma_{mG+j-1} - \sigma_{mG+j})} \right) e^{2\pi i u_{mG} \sigma_{2mG-1}},
\]

\[
\prod_{j=1}^{mG} e^{2\pi i t_j} \sum_l b_l(t_l) = e^{-2\pi i u_{mG} t_2}.
\]

This leads to the following dual partition function

\[
Z_{\tilde{O}_\alpha P}(Y)(m', \eta') = C_{X'Y'} \cdot \int \prod_{k=1}^{2mG-1} d\sigma_k \prod_{b=1}^{mG-1} d\tau_b Z_{1-\text{loop}}^{\text{bif}}(\tau_1, \sigma_{mG}, 0) Z_{\text{int}}^{(mG-1)}(\{\tau_b\}, \eta_\alpha - t_2, -m')
\]

\[
\times Z_{\text{int}}^{(Y)}(\{\sigma_k\}, t, \{-m_1, \ldots, -m_{mG}, -\mu_1', \ldots, -\mu_{mG}'\}),
\]

where the mass parameters \(\mu'_j = \mu_j + m'_1\) for \(j = 1, \ldots, m_G\). The function \(C_{X'Y'}\) and the

\(^{11}\)The functions \(g_j(\{\sigma_k\}, P)\) can be read off from the numerator of the r.h.s. of (3.4), while the integers \(b_l\) can be read off from the contact term in (3.6)–(3.7).
function $Z_{\text{int}}^{(T[m_G-1])}$ (defined in section 2.3) are given as
\[ C_{X'Y'}(m^f, \mu, t) = e^{\frac{2\pi i m}{m_G - 1}(m_{\eta} - t_2)} e^{\frac{2\pi i (m_1 t_1 - \mu m_G t_2)}{m_G - 1}}, \]
(3.21)
\[ Z_{\text{int}}^{(T[m_G-1])}([\eta], \eta - t_2, m^f) = \prod_{b=1}^{m_G-1} e^{2\pi i m (m'_b - m'_{b+1})} \prod_{b=1}^{m_G-2} \cosh \pi (\tau_b - \tau_{b+1}) \cosh \pi (\tau_{m_G-1} - \eta + t_2). \]
(3.22)

From the r.h.s. of (3.20), the gauge group and the matter content of the dual theory can be read off:
\[ Z_{\tilde{O}^p}^{(Y)}(m', \eta') = C_{X'Y'}(m^f, \mu, t) \cdot Z^{(Y')}(m'(t, \eta); \eta'(m, m^f, \mu)), \]
(3.23)
where $Y'$ is the quiver gauge theory shown in figure 8, and $C_{X'Y'}$ is a contact term associated with the new duality. The mirror map for the masses and FI parameters can be read off from the r.h.s. of (3.20). After a shift in the integration variables, the matrix integral can be written in terms of two independent real masses, associated with the Higgs branch global symmetry $G_H^{(Y')} = U(1)^2$. The Coulomb branch global symmetry can be read off from $Y'$ as follows. The quiver has three subquivers each consisting of a linear chain of $m_G - 1$ gauge nodes and a single unbalanced gauge node at the center. This leads to the Coulomb branch global symmetry $G_{H}^{(Y')} = SU(m_G)^3 \times U(1)$.

### 3.2 Constructing the 3d Lagrangians for the $D_p(SU(N))$ theories

In this subsection, we discuss the procedure for constructing the 3d Lagrangians for $D_p(SU(N))$ on circle reduction, using S-type operations. We make a general argument that the resultant quiver gauge theory always has unitary gauge group factors with fundamentals and bifundamentals, as well as the Abelian hypermultiplets.

The starting point is the 3d mirror of $D_p(SU(N))$, which can be explicitly written down from the class S construction of the 4d theory [19, 32]. Below, we treat the two cases $p < N$ and $p > N$ separately.

#### 3.2.1 The case of $p < N$

Let us first consider the case of $p < N$, for which we define the parameters:
\[ x = \left\lfloor \frac{N}{p} \right\rfloor, \quad M = N - (x + 1), \quad m = \text{GCD}(p, N), \quad n = \frac{N}{m}, \quad q = \frac{p}{m}, \]
\[ m_1 = q(1 + x) - n, \quad m_2 = n - qx. \]
(3.24)

The 3d mirrors for $D_p(SU(N))$ with $p < N$ can be constructed using the following steps.

1. Consider three subquivers from which the 3d quiver will be built out:
   - A linear chain of unitary gauge groups with bifundamental matter:
   \[ T_1 = (p - 1) - 2(p - 1) - \ldots - (x - 1)(p - 1) - x(p - 1) \]
• Another linear chain of unitary gauge groups with bifundamental matter:
  \[ T_2 = M - (M - 1) - \ldots - 2 - 1 \]

• A complete graph \( T_3 \) of \( m \) vertices, where each vertex is a U(1) gauge node, and edge multiplicity \( m_G = m_1 m_2 \).

2. Connect the U\((x(p - 1))\) gauge node of \( T_1 \) and the U\((M)\) gauge node of \( T_2 \) with an edge of multiplicity 1, i.e. single bifundamental hyper.

3. Connect the U\((x(p - 1))\) gauge node of \( T_1 \) to each of the U(1) gauge nodes of \( T_3 \) with an edge of multiplicity \( m_1 \).

4. Connect the U\((M)\) gauge node of \( T_2 \) to each of the U(1) gauge nodes of \( T_3 \) with an edge of multiplicity \( m_2 \).

5. Finally, one needs to decouple an overall U(1) factor from the quiver, and we will choose to implement this by ungauging one of the U(1) gauge nodes of \( T_3 \).

6. In addition, one can have a free sector consisting of \( H \) free hypers where

\[
H = \frac{(N - px - m)(p(1 + x) - N - m)}{2m}.
\]

In the special cases, where \( N - px - m = 0 \), or \( p(1 + x) - N - m = 0 \), the 3d mirror is given by the quiver gauge theory constructed above.

It is convenient to rewrite the linear chains \( T_1 \) and \( T_2 \) in the following fashion:

\[ T_1 : \quad A \quad x(p - 1) \]

\[ T_2 : \quad M \quad B \]

The grey nodes \( A \) and \( B \) are linear quivers that can be read off from the linear quivers \( T_1 \) and \( T_2 \) respectively, given above. The 3d mirror for \( D_p(SU(N)) \) for the case \( p < N \) can then be written down explicitly, as shown figure 9.

We now observe that the quiver gauge theory in figure 9 can be constructed from a linear quiver with unitary gauge groups by a sequence of elementary \( S \)-type operations. This can be done using the following steps:

1. The starting point is the linear quiver \( X \):

\[ X : \quad A \quad x(p - 1) \quad M \quad B \]

\[ m m_1 \quad m m_2 \]
Figure 9. The 3d mirror for the $D_p(SU(N))$ ($p < N$) AD theory reduced on a circle, for the case $m = \text{GCD}(p, N) = 3$. The quiver on the right is obtained by decoupling a $U(1)$ vector multiplet associated with one of the vertices of the complete graph from the quiver on the left. The multiplicity of the edges connecting $U(x(p-1))$ with the $U(1)$ nodes is $m_1$, and that of the edges connecting $U(M)$ with the $U(1)$ nodes is $m_2$. The multiplicity of edges connecting two $U(1)$ nodes is $m_G = m_1 m_2$.

2. Consider splitting the flavor nodes $U(mm_1) \rightarrow U((m-1)m_1) \times U(m_1)$ and $U(mm_2) \rightarrow U((m-1)m_2) \times U(m_2)$. In addition, let us split the new $U(m_1)$ and $U(m_2)$ flavor nodes into $m_1 U(1)$ nodes and $m_2 U(1)$ nodes respectively. Now, we implement the following elementary $S$-type operation — the $m_1 U(1)$ nodes and $m_2 U(1)$ nodes are identified to a single $U(1)$ flavor node, which is then flavored by $(m-1)m_G$ fundamental hypermultiplets, followed by gauging the identified $U(1)$. This identification-flavoring-gauging operation $O_{P_1}^1$ leads to the following quiver gauge theory:

3. In the next step, we split the flavor nodes $U((m-1)m_1) \rightarrow U((m-2)m_1) \times U(m_1)$, $U((m-1)m_2) \rightarrow U((m-2)m_2) \times U(m_2)$, and $U(m_G(m-1)) \rightarrow U(m_G(m-2)) \times U(m_G)$. In addition, we split the new $U(m_1)$, $U(m_2)$, and $U(m_G)$ flavor nodes into $m_1 U(1)$ nodes, $m_2 U(1)$ nodes, and $m_G U(1)$ nodes respectively. Now, we implement the second elementary $S$-type operation — the $m_1 U(1)$ nodes, $m_2 U(1)$ nodes, and $m_G U(1)$ nodes are identified to a single $U(1)$ flavor node, which is then flavored by $(m-2)m_G$ fundamental hypermultiplets, followed by gauging the identified $U(1)$. This identification-flavoring-gauging operation $O_{P_2}^2$ leads to the following quiver gauge
theory:

Note that the quiver on the right is the 3d mirror for $D_p(SU(N))$ ($p < N$) for the special case $m = 3$.

4. Proceeding in the same fashion, we implement a total of $(m - 1)$ identification-flavoring-gauging operations on the linear quiver $X$ leading to the quiver gauge theory $X'$, i.e.

$$X' = \mathcal{O}_{\mathcal{P}_{m-1}}^{(m-1)} \circ \mathcal{O}_{\mathcal{P}_{m-2}}^{(m-2)} \circ \ldots \circ \mathcal{O}_{\mathcal{P}_2}^{(2)} \circ \mathcal{O}_{\mathcal{P}_1}^{(1)}(X),$$

where $X'$ is the quiver gauge theory given in figure 9.

Using the general prescription in section 2 for finding out the dual of a given elementary $S$-type operation, one can explicitly determine the 3d mirror of $X'$. One can make some general statements about the mirror $Y'$:

- Since $X$ is a good linear quiver (a balanced quiver) with unitary gauge groups, its mirror dual $Y$ is also a Lagrangian theory. Since the quiver $X'$ is obtained from $X$ by a sequence of Abelian elementary $S$-type operations, we conclude that the 3d mirror $Y'$ must be a Lagrangian theory, using the main result of section 2.

- In addition, we note that the elementary $S$-type operations in (3.25) are of the identification-flavoring-gauging type. Using the corollary to the main result, this implies that the quiver $Y'$ will only have unitary gauge groups.

- In addition to fundamental and bifundamental matter, the quiver will involve several hypermultiplets which are only charged under the $U(1)$ subgroups of certain unitary factors, since this is a general feature of the dual of an identification-flavoring-gauging operation on a theory with unitary gauge group.

As discussed earlier, the quiver $Y'$, along with $H$ free twisted hypermultiplets, gives a Lagrangian description for the $D_p(SU(N))$ AD theory reduced on a circle. We will write down the quiver $Y'$ for an explicit example in section 3.3.

3.2.2 The case of $p > N$

Now let us consider the case $p > N$, for which we define the following parameters:

$$m = \text{GCD}(p, N), \quad n = \frac{N}{m}, \quad q = \frac{p}{m}$$

(3.26)

The 3d mirror for $D_p(SU(N))$ with $p > N$ can now be constructed in the following fashion:
Figure 10. The 3d mirror for the $D_p(SU(N))$ ($p > N$) AD theory reduced on a circle, for the case $m = \text{GCD}(p,N) = 4$. The quiver on the right is obtained by decoupling a $U(1)$ vector multiplet associated with one of the vertices of the complete graph from the quiver on the left. The multiplicity of the edges connecting $U(N-1)$ with the $U(1)$ nodes is $n$, and that of the edges connecting two $U(1)$ nodes is $m_G = n(q-n)$.

1. Consider the following subquivers from which the 3d quiver can be built:
   - A linear chain of unitary gauge groups with bifundamental matter:
     \[ \mathcal{T}_1 = (N-1) - (N-2) - \ldots - 2 - 1 \]
   - A complete graph $\mathcal{T}_2$ of $m$ vertices, where each vertex is a $U(1)$ gauge node, and edge multiplicity $m_G = n(q-n)$.

2. Connect the $U(N-1)$ gauge node of $\mathcal{T}_1$ to each of the $U(1)$ gauge nodes of $\mathcal{T}_2$ with an edge of multiplicity $n$.

3. Finally, one needs to decouple an overall $U(1)$ factor from the quiver, and we will choose to implement this by ungauging one of the $U(1)$ gauge nodes of $\mathcal{T}_2$.

4. In addition, one can have a free sector consisting of $H$ free hypers where
   \[ H = \frac{(N-m)(p-N-m)}{2m}. \]

In the special cases, where $N - m = 0$, or $p - N - m = 0$, the 3d mirror is given by the quiver gauge theory constructed above.

It is convenient to rewrite the linear chain $\mathcal{T}_1$ in the following fashion:

\[ \mathcal{T}_1 : \quad \begin{array}{c}
1 \\
\vdots \\
N-1
\end{array} \]

The quiver $C$ is a linear chain $C : 1 - 2 - \ldots - (N-2)$. The 3d mirror for $D_p(SU(N))$ for the case $p < N$ can then be written down explicitly, as shown figure 10.

We observe that the quiver gauge theory in figure 10 can be constructed from a linear quiver with unitary gauge groups by a sequence of elementary Abelian $S$-type operations. This can be performed using the following steps:
1. The starting point is the following linear quiver $X$ (the $T(U(N))$ quiver with $N = mn$):

$$X : \begin{array}{c}
\text{C} \\
\text{N-1} \\
\text{N}
\end{array}$$

2. Consider splitting the $U(mn)$ flavor node as $U(mn) \rightarrow U((m-1)n) \times U(n)$, and further splitting the new $U(n)$ node as $U(n) \rightarrow U(1)^n$. Now, we implement the following elementary $S$-type operation — the $n$ $U(1)$ nodes are identified to a single $U(1)$, which is then flavored by $(m-1)m_G$ fundamental hypers, followed by gauging the identified $U(1)$. The identification-flavoring-gauging operation $\mathcal{O}^{(1)}_{P_1}$ leads to the following quiver gauge theory:

$$\begin{array}{c}
\text{C} \\
\text{N-1} \\
\text{n}
\end{array} \quad \mathcal{O}^{(1)}_{P_1} \quad \begin{array}{c}
\text{C} \\
\text{N-1} \\
\text{n}
\end{array} \quad \begin{array}{c}
\text{(m-1)m_G} \\
\text{(m-1)n}
\end{array}$$

3. In the next step, we split the $U((m-1)m_G)$ flavor node as $U((m-1)m_G) \rightarrow U((m-2)m_G) \times U(m_G)$, and the $U((m-1)n)$ flavor node as $U((m-1)n) \rightarrow U((m-2)n) \times U(n)$. In addition, we split the new $U(m_G)$ and $U(n)$ flavor nodes into $m_G$ $U(1)$ nodes and $n$ $U(1)$ nodes respectively. Then, we implement the following elementary $S$-type operation — the $m_G$ $U(1)$ nodes and $n$ $U(1)$ nodes are identified to a single $U(1)$, which is then flavored by $(m-2)m_G$ fundamental hypers, followed by gauging the identified $U(1)$. The identification-flavoring-gauging operation $\mathcal{O}^{(2)}_{P_2}$ leads to the following quiver gauge theory:

$$\begin{array}{c}
\text{C} \\
\text{N-1} \\
\text{n} \quad \text{m_G}
\end{array} \quad \mathcal{O}^{(2)}_{P_2} \quad \begin{array}{c}
\text{C} \\
\text{N-1} \\
\text{n} \quad \text{m_G}
\end{array} \quad \begin{array}{c}
\text{(m-2)m_G} \\
\text{(m-2)n}
\end{array}$$

4. Proceeding in the same fashion, we implement a total of $(m-1)$ identification-flavoring-gauging operations on the linear quiver $X$ leading to the quiver gauge theory $X'$, i.e.

$$X' = \mathcal{O}^{(m-1)}_{P_{m-1}} \circ \mathcal{O}^{(m-2)}_{P_{m-2}} \circ \ldots \circ \mathcal{O}^{(2)}_{P_2} \circ \mathcal{O}^{(1)}_{P_1}(X),$$

(3.27)

where $X'$ is the quiver gauge theory given in figure 10.

Similar to the case of $p < N$, one can make some general statements about the 3d mirror of $X'$, i.e. $Y'$ for the $p > N$ as follows:
• The 3d mirror $Y'$ of the quiver gauge theory $X'$ must be a Lagrangian theory.

• The gauge group of $Y'$ will be a product of unitary factors.

• In addition to fundamental and bifundamental matter, the quiver will involve several hypermultiplets which are only charged under U(1) subgroups of certain unitary factors of the gauge group.

The resultant quiver gauge theory $Y'$, along with $H$ free twisted hypermultiplets, gives a Lagrangian realization of the $D_p$(SU($N$)) ($p > N$) AD theory reduced on a circle. In section 3.4, we work out the explicit form for the quiver $Y'$ in a specific example.

3.3 An example with $p < N$: $D_4$(SU($6$))

In this section, we consider a specific example of the class of quiver gauge theories discussed in section 3.2.1 and explicitly determine its 3d mirror following the general prescription given in section 2.3.

3.3.1 The unitary Lagrangian

Consider the quiver gauge theory $X'$ in figure 9 for $p = 4$, $N = 6$. The various parameters that encode the quiver data are given as:

$$x = 1, \quad M = 4, \quad m = 2, \quad n = 3, \quad q = 2,$$
$$m_1 = 1, \quad m_2 = 1.$$  \hspace{1cm} (3.28)

This corresponds to the quiver gauge theory:

\begin{center}
\begin{tikzpicture}[node distance=1.5cm, thick, main/.style = {draw, circle}]

\node[main] (1) at (0,0) {$1$};
\node[main] (2) at (1,0) {$2$};
\node[main] (3) at (2,0) {$3$};
\node[main] (4) at (3,0) {$4$};
\node[main] (3') at (2,-1) {$3'$};
\node[main] (1') at (0,-2) {$1'$};

\draw[->] (1) -- (2);
\draw[->] (2) -- (3);
\draw[->] (3) -- (4);
\draw[->] (4) -- (3');
\draw[->] (3') -- (1');
\end{tikzpicture}
\end{center}

Note that we have decoupled a U(1) vector multiplet to write the quiver gauge theory on the r.h.s. Also, note that we have $N - px - m = 0 = p(1 + x) - N - m = 0$, which implies that there are no free hypermultiplets. One can now deploy the S-type operation technique, the details of which are described below, to construct the 3d mirror of $X'$. The result is shown in figure 11. Note that the quiver $Y'$ has as gauge group U(4) x U(2) x U(1) with fundamental and bifundamental hypers. In addition, it involves a single hypermultiplet with charges (4,1) under the U(1) subgroup of the U(4) gauge node and the U(1) gauge node respectively. This is consistent with our analysis of the general features of the theory $Y'$, as discussed in section 3.2.1. The moduli space dimensions and global symmetries of the dual theories are summarized in table 2.
Figure 11. The quiver $X'$ is the 3d mirror of $D_4(\text{SU}(6))$ reduced on a circle, constructed from a class $S$ prescription. The quiver $Y'$ is the 3d mirror of $X'$, i.e. a Lagrangian realization of the SCFT obtained by the reduction of $D_4(\text{SU}(6))$ on a circle.

| Moduli space data | Theory $X'$ | Theory $Y'$ |
|-------------------|-------------|-------------|
| $\dim \mathcal{M}_H$ | 7           | 14          |
| $\dim \mathcal{M}_C$ | 14          | 7           |
| $G_H$ | $\text{U}(1)^3$ | $\text{SU}(6) \times \text{U}(1)$ |
| $G_C$ | $\text{SU}(6) \times \text{U}(1)$ | $\text{U}(1)^3$ |

Table 2. Summary table for the moduli space dimensions and global symmetries for the mirror pair in figure 11.

3.3.2 Derivation via $S$-type operations

The starting point for obtaining the mirror pair $(X', Y')$ by an $S$-type operation is the linear mirror pair $(X, Y)$, as shown in figure 12. The $S^3$ partition functions of the theories $(X, Y)$ can be written down from the rules given in section 2 as follows:

\[
Z^{(X)}(m; t) = \int \prod_{\gamma=1}^5 [d s^\gamma] Z^{(X)}_{\text{int}}(\{ s^\gamma \}, m, t) = \int \prod_{\gamma=1}^5 [d s^\gamma] Z^{(X)}_{\text{FI}}(\{ s^\gamma \}, t) Z^{(X)}_{1-\text{loop}}(\{ s^\gamma \}, m),
\]

\[
Z^{(Y)}(t; m) = \int \prod_{\gamma'=1}^3 [d \sigma^\gamma'] Z^{(Y)}_{\text{int}}(\{ \sigma^\gamma' \}, t, m)
= \int \prod_{\gamma'=1}^3 [d \sigma^\gamma'] Z^{(Y)}_{\text{FI}}(\{ \sigma^\gamma' \}, m) Z^{(Y)}_{1-\text{loop}}(\{ \sigma^\gamma' \}, t),
\]

(3.30)

where $\gamma, \gamma'$ label the gauge nodes of the linear quivers $X, Y$ respectively, with the count increasing from left to right in both cases. The functions $Z^{(X)}_{\text{int}}$ and $Z^{(Y)}_{\text{int}}$ are the respective matrix model integrands for $X$ and $Y$, each consisting of an FI term and a one-loop term with contributions from vector multiplets and hypermultiplets. We will, in particular, need the FI term of $Y$, which is given as

\[
Z^{(Y)}_{\text{FI}}(\{ \sigma^\gamma' \}, t) = e^{2\pi i(m_1-m_2) \text{Tr} \sigma^1} e^{2\pi i(m_2-m_3) \text{Tr} \sigma^2} e^{2\pi i(m_3-m_4) \sigma^3}.
\]

(3.32)
Figure 12. A pair of mirror dual linear quivers with unitary gauge groups.

Figure 13. The construction of the quiver gauge theory \((X')\) from the theory \((X)\) by the action of the \(S\)-type operation \(O_P\). The flavor nodes to be identified in the quiver \((X)\) are shown in red. The quiver gauge theory \((Y')\) is constructed from the theory \((Y)\) by the action of the dual operation \(\bar{O}_P\).

Mirror symmetry of \(X\) and \(Y\) implies that

\[
Z^{(X)}(m; t) = C_{XY}(m, t) Z^{(Y)}(t; -m),
\]

(3.33)

where \(C_{XY}\) is a contact term, which reminds us that the duality holds for unrestricted \(m\) and \(t\), only when we turn on a certain background Chern-Simons sector. We will not need the precise form of the contact term to read off the gauge group and matter content of theories we will construct using \(S\)-type operations.

The quiver \(X'\) can be obtained from the linear quiver \(X\) by implementing an \(S\)-type operation \(O_P\), which constitutes a single elementary Abelian operation of the identification-flavoring-gauging type, as shown in figure 13. The mass parameters \(\{u_i\}\) associated the \(S\)-type operation are given as

\[
u_1 = m_2, \quad u_2 = m_3, \quad v_1 = m_1 v_2 = m_4.
\]

(3.34)
The partition function of the theory $X' = \mathcal{O}_p(X)$ is then given as
\[
Z^{\mathcal{O}_p(X)} = \int du \, \mathcal{Z}_{\mathcal{O}_p(X)}(u, u_1, u_2, \eta, m^f, \bm{\mu}) \cdot Z^{X_p}(u_1, u_2, v_1, v_2; \bm{t}),
\] (3.35)
where the operator $\mathcal{Z}_{\mathcal{O}_p(X)}$ can be constructed from the general prescription in (2.13) and has the following form:
\[
\mathcal{Z}_{\mathcal{O}_p(X)} = \int \prod_{i=1}^2 du_i \frac{e^{2\pi i u \mu}}{\cosh \pi (u - m^f)}.
\] (3.36)

Using the above expression, the partition function of the theory $X'$ can be explicitly written as
\[
Z^{(X')} = \int du \, \prod_{\gamma=1}^5 (ds''^\gamma) \, Z_{FI}^{(X')}(\{s''^\gamma, t, \eta\}) \prod_{\gamma=1}^5 Z_{1-loop}^{\text{vector}}(s''^\gamma) \prod_{\gamma=1}^4 Z_{1-loop}^{\text{bifund}}(s''^\gamma, s''^{\gamma+1}, 0) 
\times Z_{1-loop}^{\text{fund}}(s''^1, v_1) \, Z_{1-loop}^{\text{fund}}(s''^5, s''^1, u, \mu_1) \, Z_{1-loop}^{\text{fund}}(s''^5, u, \mu_2) \, Z_{1-loop}^{\text{fund}}(u, m^f),
\] (3.37)
where $Z_{FI}^{(X')} = e^{2\pi i u} \mathcal{Z}_{FI}^{(X')}$. The matrix integral on the r.h.s. manifestly corresponds to the quiver $X'$ in figure 11. By shifting the integration variables, one can check that the matrix integral can be written in terms of three independent real masses, which live in the Cartan subalgebra of the Higgs branch global symmetry group of $X'$, i.e. $G^{X'}_H = U(1)^3$. The Coulomb branch global symmetry can be read off from the quiver itself. It has a linear subquiver of 5 balanced nodes giving an SU(6) factor and a single unbalanced gauge node which gives an additional U(1) factor, giving $G^{X'}_C = SU(6) \times U(1)$.

The dual theory can be read off from the general equations in (2.19)–(2.20), with the function $\widehat{Z}_{\mathcal{O}_p(Y)}$ being given as
\[
\widehat{Z}_{\mathcal{O}_p(Y)} = \int du \, \mathcal{Z}_{\mathcal{O}_p(X)} \cdot \prod_{i=1}^2 e^{2\pi i (g'(\{s''^i\}, \mathcal{P}) + b^i t_1)} u_i,
\] (3.38)
\[
\prod_{i=1}^2 e^{2\pi i (g'(\{s''^i\}, \mathcal{P}) + b^i t_1) u_i} = e^{2\pi i u_1 (\text{Tr} \sigma^1 - \text{Tr} \sigma^3 + b^3 t_1)} e^{2\pi i u_2 (\text{Tr} \sigma^2 - \sigma^3 + b^2 t_1)}.
\] (3.39)

Integrating over $u$, the function $\widehat{Z}_{\mathcal{O}_p(Y)}$ can be written in the following form:
\[
\widehat{Z}_{\mathcal{O}_p(Y)} = e^{2\pi i \mu_b t_1} \frac{e^{2\pi i m^f (\eta + \text{Tr} \sigma^1 - \sigma^3)}}{\cosh \pi (\eta + \text{Tr} \sigma^1 - \sigma^3)} e^{2\pi i \mu_1 \text{Tr} \sigma^1} e^{-2\pi i (\mu_1 - \mu_2) \text{Tr} \sigma^2} e^{-2\pi i \mu_2 \sigma^3}.
\] (3.40)

After some minor rearrangement of terms, the dual partition function can be written as (up to contact terms)
\[
\widehat{Z}_{\mathcal{O}_p(Y)}(\bm{m'}; \eta') = \int \prod_{\gamma'=1}^3 d\sigma''^{\gamma'} \, Z_{1-loop}^{\text{hyper}}(\sigma''^1, \sigma''^3, \eta)
\]
\[
Z_{\text{int}}^{(Y)}(\{\sigma''^{\gamma'}\}, t, \{-v_1, -(\mu_1 + m^f), -(\mu_2 + m^f), -v_2\}),
\] (3.41)
\[
Z_{1-loop}^{\text{hyper}}(\sigma''^1, \sigma''^3, \eta) = \frac{1}{\cosh \pi (\text{Tr} \sigma''^1 - \sigma''^3 + \eta').
\] (3.42)
The resultant dual quiver gauge theory can be read off from the matrix model integrand in (3.41) — it is given by the quiver $Y$ with an additional hypermultiplet which is charged under the U(1) subgroup of the gauge group U(4) as well under the gauge group U(1).

The respective charges (4 and 1) can be read off from the r.h.s. of (3.42). Therefore, one can identify the 3d mirror of $X'$ as the quiver gauge theory $Y'$ in figure 13. In addition, by shifting the integration variables, one can check that the dual partition function is a function of 6 independent real masses, which live in the Cartan subalgebra of the Higgs branch global symmetry of $Y'$. i.e. $G^Y_H = SU(6) \times U(1)$. The Coulomb branch symmetry can be read off from the quiver $Y'$. Since none of the gauge nodes are balanced, it is given as $G^Y_C = U(1)^3$.

3.4 An example with $p > N$: $D_9(SU(3))$

In this section, we consider a specific example of the class of quiver gauge theories discussed in section 3.2.2 and explicitly determine its 3d mirror following the general prescription given in section 2.

3.4.1 The unitary Lagrangian

Consider the quiver gauge theory $X'$ in figure 10 for $p = 9, N = 3$. The various parameters that encode the quiver data are given as:

$$m = 3, \quad n = 1, \quad q = 3. \quad (3.43)$$

This corresponds to the quiver gauge theory:

Note that we have decoupled a U(1) vector multiplet to write the quiver gauge theory on the r.h.s. Also, note that we have $N - m = 0$, which implies that there are no free hypermultiplets. We can again use the S-type operation technique, the details of which are described below, to construct the 3d mirror of $X'$. The result is shown in figure 14. Note that the quiver $Y'$ has as gauge group $U(2) \times U(1)^5$ with fundamental and bifundamental hypers. In addition, it involves a single hypermultiplet with charges (2,1) under the subgroup $U(1) \subset U(2)$ and one of the U(1) gauge groups respectively. This is consistent with our analysis of the general features of the theory $Y'$, as discussed in section 3.2.2. The moduli space dimensions and global symmetries of the dual theories are summarized in table 3.

3.4.2 Derivation via S-type operations

The starting point for obtaining the mirror pair $(X', Y')$ by an S-type operation will again be a linear mirror pair $(X, Y)$. As discussed in section 3.2.2, we choose $X$ to be the self-dual
Figure 14. The quiver $X'$ is the 3d mirror of $D_9(SU(3))$ reduced on a circle, constructed from a class $S$ prescription. The quiver $Y'$ is the 3d mirror of $X'$, i.e. a Lagrangian realization of the SCFT obtained by the reduction of $D_9(SU(3))$ on a circle.

| Moduli space data | Theory $X'$ | Theory $Y'$ |
|-------------------|-------------|-------------|
| dim $M_H$         | 7           | 5           |
| dim $M_C$         | 5           | 7           |
| $G_H$             | $SU(2)^3 \times U(1)^3$ | $SU(3) \times U(1)^2$ |
| $G_C$             | $SU(3) \times U(1)^2$ | $SU(2)^3 \times U(1)^3$ |

Table 3. Summary table for the moduli space dimensions and global symmetries for the mirror pair in figure 14.

theory $T(U(3))$ as shown in figure 15. The partition functions of the theories $(X,Y)$ are given as

$$Z^{(X)}(m; t) = \int \prod_{\gamma=1}^{2} \left[ ds^\gamma \right] Z^{(X)}_{\text{int}}(\{s^\gamma\}, m, t) = \int \prod_{\gamma=1}^{2} \left[ ds^\gamma \right] Z^{(X)}_{\text{FI}}(\{s^\gamma\}, t) Z^{(X)}_{1\text{-loop}}(\{s^\gamma\}, m),$$

(3.44)

$$Z^{(Y)}(t; m) = \int \prod_{\gamma'=1}^{2} \left[ d\sigma^{\gamma'} \right] Z^{(Y)}_{\text{int}}(\{\sigma^{\gamma'}\}, t, m)$$

$$= \int \prod_{\gamma'=1}^{2} \left[ d\sigma^{\gamma'} \right] Z^{(Y)}_{\text{FI}}(\{\sigma^{\gamma'}\}, m) Z^{(Y)}_{1\text{-loop}}(\{\sigma^{\gamma'}\}, t),$$

(3.45)

where $\gamma, \gamma'$ label the gauge nodes of the linear quivers $X, Y$ respectively, with the count increasing from left to right in both cases. The FI contribution for $Y$ in the matrix integral is given as

$$Z^{(Y)}_{\text{FI}}(\{\sigma^{\gamma'}\}, t) = e^{2\pi i (m_1-m_2)} \text{Tr} \sigma^{1} e^{2\pi i (m_2-m_3)} \sigma^{2}. \quad (3.46)$$

The fact that $T(U(3))$ is self-mirror implies that

$$Z^{(X)}(m; t) = C_{XY}(m, t) Z^{(Y)}(t; -m),$$

(3.47)

where $C_{XY}$ is a contact term, and as before, we will not need the precise form of the contact term to read off the gauge group and matter content of theories that we will construct using $S$-type operations.
Figure 15. The starting point for the $S$-type operation construction is the self-dual theory $T(U(3))$.

Figure 16. The construction of the quiver gauge theory $X'$ of figure 14 by a sequence of two elementary Abelian $S$-type operation. The labels $i = 1, 2$ on the flavor nodes correspond to the mass parameters $u_i$. At each step, the flavor node on which the elementary operation acts is shown in blue.
The quiver $X'$ can be obtained from the linear quiver $X$ by implementing an $S$-type operation $\mathcal{O}_P$, which includes two elementary Abelian $S$-type operations, i.e.

$$\mathcal{O}_P(X) = \mathcal{O}_{P_2}^2 \circ \mathcal{O}_{P_1}^1(X), \quad (3.48)$$

where $\mathcal{O}_{P_i}^j$ ($i = 1, 2$) are elementary Abelian $S$-type operations shown in figure 16. Note that $\mathcal{O}_{P_1}^1$ is a flavoring-gauging operation, while $\mathcal{O}_{P_2}^2$ is an identification-flavoring-gauging operation. The mass parameters $\{u_i\}$ associated the $S$-type operation $\mathcal{O}_{P_i}^j$, are given as

$$u_1 = m_3, \ u_2 = m_2, \ v = m_1. \quad (3.49)$$

The partition function of the theory $X' = \mathcal{O}_P(X)$ is then given as

$$Z^{\mathcal{O}_P(X)} = \int \prod_{i=1}^2 du_i \ Z_{\mathcal{O}_P(X)}(u, \eta, m^f, x^f, \mu) \cdot Z^{(X, P)}(u, v, t), \quad (3.50)$$

where the operator $Z_{\mathcal{O}_P(X)}$ can be constructed from the general prescription in (2.13) and has the following form:

$$Z_{\mathcal{O}_P(X)} = Z_{\mathcal{O}_{P_2}^2(\mathcal{O}_{P_1}^1(X))} \circ Z_{\mathcal{O}_{P_1}^1(X)} = \left( \int \prod_{b=1}^2 dm^f_b \ \frac{e^{2\pi i u_2} \prod_{b=1}^2 \delta(m^f_b - u_2 - \mu_b)}{\prod_{c=1}^2 \cosh(\pi(u_2 - x^f_c))} \right) \frac{e^{2\pi i u_1}}{\prod_{a=1}^4 \cosh(\pi(u_1 - m^f_a))}, \quad (3.51)$$

where $m^f_a$ ($a = 1, \ldots, 4$) and $x^f_c$ ($c = 1, 2$) are the masses of hypers added in the two flavoring operations. Note that the second equality involves specific choices of the permutation matrices $P_i$. Using the above expression, the partition function of the theory $X'$ can be explicitly written as

$$Z^{(X')} = \int \prod_{i=1}^2 du_i \ \prod_{\gamma=1}^2 (ds^{\gamma}) \ Z_{\mathcal{P}_1}^{(X')}(s^{\gamma}, u, \{t, \eta\}) \ Z_{1-loop}^{vector}(s^1, s^2) \ Z_{1-loop}^{bif}(s^1, s^2, 0) \ Z_{1-loop}^{fund}(s^2, v)$$

$$\times \prod_{i=1}^2 Z_{1-loop}(s^2, u_i, 0) \ Z_{1-loop}^{fund}(u_1, m^f_{2+a}) \ Z_{1-loop}^{fund}(u_2, x^f_c) \ \prod_{b=1}^2 Z_{1-loop}^{bif}(u_1, u_2, \mu_b), \quad (3.52)$$

where $Z_{\mathcal{P}_1}^{(X')} = e^{2\pi i u_2} e^{2\pi i u_1} Z_{\mathcal{P}_1}^{(X)}$. The matrix integral on the r.h.s. manifestly corresponds to the quiver $X'$ in figure 16. By shifting the integration variables, one can check that the matrix integral can be written in terms of 6 independent real masses, which live in the Cartan subalgebra of the Higgs branch global symmetry group of $X'$, i.e. $G^{X'}_H = SU(2)^3 \times U(1)^3$. The Coulomb branch global symmetry can be read off from the quiver itself. It has a linear subquiver of 2 balanced nodes giving an SU(3) factor and two unbalanced gauge node which gives an additional U(1)$^2$ factor, giving $G^{X'}_C = SU(3) \times U(1)^2$. 


The dual theory can be read off from the general equations in (2.19)–(2.20), with the function \( Z_{\tilde{\mathcal{C}}(Y)} \) being given as

\[
Z_{\tilde{\mathcal{C}}(Y)} = Z_{\tilde{\mathcal{C}}_{p_2}(\tilde{\mathcal{C}}_{p_1}(Y))} \cdot Z_{\tilde{\mathcal{C}}_{p_1}(Y)} = \int \prod_{i=1}^{2} du_i \, Z_{\mathcal{O}(Y)} \cdot \prod_{i=1}^{2} e^{2\pi i (g'(\sigma^k), \mathcal{P}) + b^i t_i) u_i}
\]

\[
= \int du_2 \left( \int \prod_{b=1}^{2} \frac{du_b}{m_b} e^{2\pi i u_b (Tr \sigma^1 - \sigma^2 + \eta_2 + b_2 t_2)} \prod_{b=1}^{2} \delta (m_b u - u_2 - \mu_b) \right)
\]

\[
\cdot \int du_1 \frac{e^{2\pi i u_1 (\sigma^2 + \eta_1 + b^1 t_1)}}{\prod_{a=1}^{4} \cosh \pi (u_1 - m_a)},
\]

where the functions \( g'(\{\sigma^k\}, \mathcal{P}) \) are given as:

\[
\prod_{i=1}^{2} e^{2\pi i (g'(\{\sigma^k\}, \mathcal{P}) + b^i t_i) u_i} = e^{2\pi i u_2 (Tr \sigma^1 - \sigma^2 + b_2 t_2)} e^{2\pi i u_1 (\sigma^2 + b_1 t_1)}.
\]

In the first step, integrating over \( u_1 \), the function \( Z_{\tilde{\mathcal{C}}_{p_2}(\tilde{\mathcal{C}}_{p_1}(Y))} \) can be written in the following form:

\[
Z_{\tilde{\mathcal{C}}_{p_1}(Y)} = \int \prod_{j=1}^{3} d\tau_j \frac{e^{2\pi i m^i (\sigma^2 + \eta_1 + b^1 t_1)}}{\cosh \pi (\sigma^2 - \tau_1 + \eta_1 + b^1 t_1)} \cdot \prod_{j=1}^{3} e^{-2\pi i \tau_j (m^j - m^j + 1)} \prod_{j=1}^{3} \cosh \pi (\tau_1 - \tau_1 + 1) \cosh \pi \tau_3
\]

\[
= \int \prod_{j=1}^{3} d\tau_j e^{2\pi i m^i (\sigma^2 + b^1 t_1)} Z_{\text{1-loop}}^{\text{bif}}(\tau_1, \sigma^2, \eta_1 + b^1 t_1) Z_{\text{int}}^{\text{3}}(\tau, -m^f, 0),
\]

where \( \mathcal{T}[N] \) is a linear chain of \( N \) U(1) gauge nodes with bifundamental hypers and a single fundamental hyper at one end. After some minor rearrangement of terms, the dual partition function can be written as (up to contact terms)

\[
Z_{\tilde{\mathcal{C}}_{p_1}(Y)}(m^i, \eta^f) = \int \prod_{\gamma'=1}^{2} \left[ d\sigma^{\gamma'} \right] \prod_{j=1}^{3} d\tau_j \, Z_{\text{int}}^{\text{3}}(\{\sigma^{\gamma'}\}, t, \{-v_1, -u_2, -m^f\})
\]

\[
\times Z_{\text{1-loop}}^{\text{bif}}(\tau_1, \sigma^2, \eta_1 + b^1 t_1) Z_{\text{int}}^{\text{3}}(\tau, -m^f, 0),
\]

The resultant dual quiver gauge theory can be read off from the matrix model integrand in (3.57). It is given by the quiver \( Y \) and the quiver \( \mathcal{T}[3] \) connected by a single bifundamental hyper, where the latter is charged under the U(1) gauge factor of \( Y \) and one of the U(1) gauge nodes of \( \mathcal{T}[3] \), as shown in figure 16.

In the final step, integrating over \( u_2 \), the function \( Z_{\tilde{\mathcal{C}}_{p_2}(\tilde{\mathcal{C}}_{p_1}(Y))} \) can be written in the following form, after a change of variables \( \tilde{\tau} \rightarrow -\tilde{\tau} + \tau_2 \):

\[
Z_{\tilde{\mathcal{C}_{p_2}(\tilde{\mathcal{C}}_{p_1}(Y))}} = \int d\tilde{\tau} \prod_{j=1}^{3} d\tau_j e^{2\pi i x^i_1 (Tr \sigma^1 - s^2 + b_2 t_2 + \eta_2)} e^{-2\pi i \tilde{\tau}(x^f_1 - x^f_2)} e^{2\pi i (\mu_1 + x^f_1) (\sigma^2 + \eta_1 + b_1 t_1)}
\]

\[
\times Z_{\text{1-loop}}^{\text{bif}}(\tilde{\tau}, \tau_2, 0) Z_{\text{hyper}}^{\text{bif}}(\sigma^1, \tilde{\tau}, \sum_i (\eta_1 + b_1 t_1)) Z_{\text{1-loop}}^{\text{bif}}(\tilde{\tau}, \sigma^2, \eta_1 + b_1 t_1)
\]

\[
\times Z_{\text{int}}^{\text{3}}(\tau, -\{\mu_1 + x^f_1, \mu_2 + x^f_2, m^f_3, m^f_4\}, 0),
\]

(3.58)
where the function $Z_{1\text{-loop}}^{\text{hyper}}$ is given as:

$$Z_{1\text{-loop}}^{\text{hyper}} = \frac{1}{\cosh \pi (\text{Tr} \sigma^1 - \tilde{\tau} - \sum_i (\eta_i + b^it_i))}.$$  \hfill (3.59)

After some minor rearrangement of terms, the dual partition function can be written as (up to contact terms)

$$Z_{\tilde{\mathcal{O}}_P} (Y) (m'; \eta') = \int \prod_{\gamma' = 1}^{2} d\sigma' \prod_{j = 1}^{3} d\tau_j e^{-2\pi i \tilde{\tau} (x_1' - x_2')} Z_{\text{int}}' \left( \{ \sigma' \}, t, -\{ v_1, x_1', \mu_1 + x_1' \} \right)$$

$$\times Z_{1\text{-loop}}^{\text{hyper}} (\sigma^1, \tilde{\tau}, \sum_i (\eta_i + b^it_i)) Z_{1\text{-loop}}^{\text{bif}} (\tau_1, \tau_2, 0) Z_{1\text{-loop}}^{\text{bif}} (\sigma^2, \eta_1 + b^it_1)$$

$$\times Z_{\text{int}}^{(3)} (\tau, -\{ \mu_1 + x_1', \mu_2 + x_2', m_3', m_4' \}, 0).$$ \hfill (3.60)

The resultant dual quiver gauge theory can be read off from the matrix model integrand in (3.57), and manifestly reproduces the quiver $Y'$ in figure 16. By shifting the integration variables, one can check that the dual partition function is a function of 4 independent real masses, which live in the Cartan subalgebra of the Higgs branch global symmetry of $Y'$. i.e. $G_{Y'}^H = SU(3) \times U(1)^2$. The Coulomb branch symmetry can be read off from the quiver $Y'$. It includes three isolated balanced gauge nodes giving a factor $SU(2)^3$, and three unbalanced nodes giving a factor $U(1)^3$, so that $G_{Y'}^C = SU(2)^3 \times U(1)^3$.

4 An infinite class of IR dualities

From section 3.2.1 and section 3.2.2, we showed that the 3d SCFT obtained by compactifying $D_p(SU(N))$ on a circle has a Lagrangian description in terms of a quiver gauge theory. The gauge group of the theory in question has only unitary factors, and involves a set of Abelian hypermultiplets in addition to fundamental and bifundamental hypers. Comparing our results with [2] suggests that we have an IR duality for every $D_p(SU(N))$, where the dual theories belong to the following classes of quiver gauge theory respectively:

- Theory $\mathcal{T}$: a linear chain of unitary and special unitary gauge groups with fundamentals and bifundamentals.

- Theory $\mathcal{T}^\vee$: a generically non-linear quiver involving only unitary gauge groups, decorated with Abelian matter in addition to fundamental/bifundamental hypers.

In this section, we present a couple of simple examples of this conjectured duality, and check that the three-sphere partition function of the dual theories do agree.

4.1 An IR duality for $SU(N)$ with $2N - 1$ flavors

Consider the simplest possible duality of the type mentioned above — an $SU(N)$ gauge theory with $2N - 1$ fundamental hypers, and a $U(N - 1)$ gauge theory with $2N - 1$ fundamental hypers and a single hypermultiplet charged under $U(1) \subset U(N - 1)$ with charge $N - 1$, as shown in figure 17. This specific duality is not associated with any $D_p(SU(N))$
theory, but will play a central role in checking the dualities of the class described above. Firstly, note that the quaternionic dimensions of the Higgs and the Coulomb branches do agree:

\[ \dim M_H^{(T)} = \dim M_H^{(T^\vee)} = N^2 - N + 1, \]  
\[ \dim M_C^{(T)} = \dim M_C^{(T^\vee)} = N - 1, \] 

and that the Higgs branch global symmetries \( G_H^{(T)} = G_H^{(T^\vee)} = U(2N - 1) \) manifestly agree. The theory \( T^\vee \) has a \( U(1) \) Coulomb branch global symmetry manifest in the UV, while the theory \( T \) has trivial Coulomb branch global symmetry in the UV. However, it can be shown [19] that \( T \) has an emergent \( U(1) \) Coulomb branch global symmetry in the IR.

Note that the first non-trivial duality in this family, for \( N = 2 \), is the familiar duality between an \( SU(2) \) theory with three fundamental hypers and a \( U(1) \) theory with four fundamental hypers [33].

The equality of the partition functions of the two theories can be demonstrated as follows. Recall that a \( U(N) \) gauge theory with \( N_f = 2N - 1 \) fundamental hypers is an ugly theory in the Gaiotto-Witten sense. This implies that the IR theory factorizes into an SCFT, which in turn has a good UV description, and a single twisted hypermultiplet. The good SCFT in question is the IR SCFT of a \( U(N - 1) \) gauge theory with \( N_f = 2N - 1 \) fundamental hypers. In terms of the partition function, this translates to the following identity (derived in [34] with a sign mistake that we have corrected below):

\[ Z_{2N-1}^{U(N)}(m, \eta) = \left( \frac{e^{2\pi i \eta \sum_{i=1}^{2N-1} m_i}}{\cosh \pi \eta} \right) \cdot Z_{2N-1}^{U(N-1)}(m, -\eta). \]  

\( \eta = 0 \)

\( 2N-1 \)

\( N-1 \)

\( 1 \)

\( 1 \)

\( 2 \)

\( 3 \)

\( N-1 \)

\( 1 \)

\( 3 \)

\( 2 \)

\( 1 \)

\( (T) \)

\( (T^\vee) \)

\( 3d \) mirror

\( \eta = 0 \)
The partition function of an SU(N) theory can be obtained from the partition function of the unitary theory on the l.h.s. by integrating over \( \eta \), i.e.

\[
Z_{2N-1}^{SU(N)}(m) = \int d\eta Z_{2N-1}^{U(N)}(m, \eta) = \int [ds] \delta(Trs) \frac{\prod_{j<k} \sinh^2 \pi(s_j - s_k)}{\prod_{i=1}^{N} \prod_{i=1}^{2N-1} \cosh \pi(s_j - m_i)}. \tag{4.4}
\]

Integrating both sides of the identity over \( \eta \), we then have

\[
Z_{2N-1}^{SU(N)}(m) = \int d\eta \left( \frac{e^{2\pi i \eta \sum_{i=1}^{2N-1} m_i}}{\cosh \pi \eta} \right) \int [d\sigma] \frac{\prod_{j<k} \sinh^2 \pi(\sigma_j - \sigma_k)}{\prod_{i=1}^{N} \prod_{i=1}^{2N-1} \cosh \pi(\sigma_j - m_i)}
\]

\[
= \int [d\sigma] \frac{1}{\cosh \pi(Tr\sigma - \sum_{i=1}^{2N-1} m_i)} \frac{\prod_{j<k} \sinh^2 \pi(\sigma_j - \sigma_k)}{\prod_{i=1}^{N} \prod_{i=1}^{2N-1} \cosh \pi(\sigma_j - m_i)}. \tag{4.5}
\]

The final form of the matrix integral can be identified as the partition function of a U(N−1) theory with 2N−1 fundamental hypers and a single hyper charged under U(1) ⊂ U(N−1) with charge N−1. We therefore have the following relation between the two partition functions:

\[
Z_{2N-1}^{SU(N)}(m) = Z_{2N-1,1,1,1}^{U(N)}(m, \tilde{m} = \sum_{i} m_i, \eta = 0), \tag{4.6}
\]

where \( \tilde{m} \) is the real mass for the Abelian hypermultiplet. Note that the equality of the partition function holds only when the FI parameter of the U(N−1) vector multiplet is tuned to zero. Also, note that all the masses can be shifted by a real parameter without changing the partition function relation (this does not even induce a contact term since the FI parameter is set to zero). The 2N−1 masses for the SU(N) gauge theory live in the Cartan subalgebra of the Higgs branch global symmetry U(2N−1). The 2N−1 independent masses for the U(N−1) gauge theory also live in the Cartan subalgebra of the Higgs branch global symmetry U(2N−1).

The 3d mirror for the theories \( T \) and \( T^\vee \) is shown in figure 17. This fact can be understood from an S-type operation construction starting from the mirror linear quiver pairs in the first row of figure 18. An Abelian flavoring-gauging operation on the U(1) flavor node (marked in red) and its dual leads to the quiver gauge theory \( T^\vee \) and its 3d mirror. The latter is then also the 3d mirror of \( T \), since we have already shown (at least at the level of the three-sphere partition function) that \( T \) and \( T^\vee \) are IR dual.

### 4.2 IR dual pairs associated with \( D_p(SU(N)) \)

#### 4.2.1 General strategy for obtaining the duality

We can now write down the general prescription for relating the quiver \( T \) and the quiver \( T^\vee \), associated with the 3d SCFT that arises from the circle reduction of \( D_p(SU(N)) \), for a given \((p, N)\). Firstly, let us recall the general structure of 3d quiver \( T \) that arise from
the analysis of [2, 19]. For \( p \geq N \), we have

\[
\left[ D_p(SU(N)) \right]_{3d} = \left[ N \right] - (N - 1) - \ldots - (N - n + 1) - (N - n)^{q-n} - SU(N - n)
- (N - n - 1) - \ldots - (N - 2n)^{q-n} - SU(N - 2n) - \ldots - SU(2n)
- (2n - 1) - \ldots - (n + 1) - (n)^{q-n} - SU(n) - (n - 1) - \ldots - 2 - 1,
\]

where the integers \( m, n \) and \( q \) are given in (3.26). For \( p \leq N \), the 3d quiver \( \mathcal{T} \) has the form

\[
\left[ D_p(SU(N)) \right]_{3d} = \left[ N \right] - (N - (x+1)) - \ldots - (N - m_B(x+1)) - C[N - n] - SU(N - n)
- (N - n - (x+1)) - \ldots - (N - n - m_B(x+1)) - C[N - 2n] - SU(N - 2n)
- \ldots - C[n] - SU(n) - (n - (x+1)) - \ldots - (n - m_B(x+1)) - C[0],
\]

where the various parameters are given in (3.24), and \( C[Y] \) (for a positive integer \( Y \)) is a linear chain of unitary gauge groups:

\[
C[Y] = (Y + (m_A - 1)x) - (Y + (m_A - 2)x) - \ldots - (Y + 2x) - (Y + x).
\]

The quiver \( \mathcal{T}^\vee \) consisting of unitary gauge groups can then be obtained from the quiver \( \mathcal{T} \) using the following steps:

1. Note that all the SU\((N_c)\) factors in the 3d quiver \( \mathcal{T} \) have \( 2N_c - 1 \) flavors. One can therefore use the identity (4.6) to replace each SU\((N_c)\) factor with a U\((N_c - 1)\) factor with an additional Abelian hypermultiplet. The resultant quiver has only unitary gauge groups with a set of Abelian hypermultiplets.
2. To the left of the replaced SU($N_c$) node, there will generically be a unitary node U($N'_c$) with $2N'_c - 1$ fundamental hypers, in addition to an Abelian hyper. Note that the presence of the Abelian hypermultiplet ensures that the unitary node is not ugly. At the level of the matrix integral, one can treat the U($N'_c$) node as a $U(1) \times SU(N'_c)$ node, where the SU($N'_c$) factor has $2N'_c - 1$ fundamental hypers. Therefore, one can apply the identity (4.6) again to replace the SU($N'_c$) factor with a $U(N'_c - 1)$ factor.\footnote{This is in fact the partition function realization of another duality that we will discuss in \cite{20}.}

3. This procedure is repeated from right to left along the quiver tail until we encounter a unitary gauge node U($N''_c$) with $N''_c \geq 2N'_c$ fundamental hypers. This is the quiver $\mathcal{T}^\vee$ that we derived from the class $S$ mirror of $D_p(SU(N))$ by the $S$-type operations.

4.2.2 Example I: dual pairs from $D_4(SU(6))$

Now, let us consider the IR duality between the two Lagrangian theories associated with the 3d SCFT that arises from the circle reduction of $D_4(SU(6))$. The quiver gauge theories are shown in figure 19 — the theory $\mathcal{T}$ arises from the dimensional reduction in \cite{2,19} while the theory $\mathcal{T}^\vee$ was derived from the $S$-type operation construction in section 3.3. The 3d mirror of these theories is given in figure 20. The quaternionic dimensions of the Higgs and the Coulomb branches of the theories are given as

\begin{align*}
\dim \mathcal{M}_H^{(T)} &= \dim \mathcal{M}_H^{(T^\vee)} = 14, \\
\dim \mathcal{M}_C^{(T)} &= \dim \mathcal{M}_C^{(T^\vee)} = 7,
\end{align*}

and the Higgs branch global symmetries $G_H^{(T)} = G_H^{(T^\vee)} = SU(6) \times U(1)$ of the theories manifestly agree. The Coulomb branch symmetry for $\mathcal{T}^\vee$ is manifestly $G_C^{(T^\vee)} = U(1)^3$, which agrees with the Higgs branch global symmetry of the 3d mirror in figure 20. However, for the theory $\mathcal{T}$, one only has a $U(1)^2$ Coulomb branch symmetry manifest in the UV Lagrangian. Similar to the example of SU($N$) gauge theory above, the theory has an additional $U(1)$ global symmetry which arises as an emergent symmetry in the IR. One can confirm this for example from a Hilbert Series computation in \cite{19}.

Let us now demonstrate how the partition functions of the theory $\mathcal{T}$ and $\mathcal{T}^\vee$ are related.
The partition function for $\mathcal{T}$ is given as

$$Z^{(\mathcal{T})}(m, \eta) = \int \left[ d\sigma^1 \right] \left[ ds^2 \right] d\sigma^3 Z_{\text{FI}}(\sigma^1, \eta_1) Z_{\text{FI}}(\sigma^3, \eta_3) \delta(\text{Tr} s^2) \prod_{\gamma=1,3} Z_{\text{1-loop}}^{\text{vec}}(\sigma^\gamma)$$

$$\times Z_{\text{1-loop}}^{\text{vec}}(s^2) Z_{\text{1-loop}}^{\text{fund}}(\sigma^1, m) Z_{\text{1-loop}}^{bif}(\sigma^1, s^2, 0) Z_{\text{1-loop}}^{bif}(s^2, \sigma^3, 0),$$

(4.12)

where $\sigma^1, \sigma^3$ denote the integration variables associated with the U(4) and U(1) gauge groups respectively, while $s^2$ denotes the integration variable associated with the SU(3) gauge group. Isolating the $s^2$-dependent part of the matrix integral, and using the relation (4.5) (for the special case of $N = 3$), we have the following identity:

$$\int [ds^2] \delta(\text{Tr} s^2) Z_{\text{1-loop}}^{\text{vec}}(s^2) Z_{\text{1-loop}}^{\text{bif}}(\sigma^1, s^2, 0) Z_{\text{1-loop}}^{\text{bif}}(s^2, \sigma^3, 0)$$

$$= \int [d\sigma^2] \frac{1}{\cosh \pi (\text{Tr} \sigma^2 - \text{Tr} \sigma^1 - \sigma^3)} Z_{\text{1-loop}}^{\text{vec}}(\sigma^2) Z_{\text{1-loop}}^{\text{bif}}(\sigma^2, \sigma^1, 0) Z_{\text{1-loop}}^{\text{bif}}(\sigma^2, \sigma^3, 0).$$

(4.13)

Substituting the above identity in (4.12), and performing the change of variable $\sigma^3 \to -\sigma^3 + \text{Tr} \sigma^2$, the partition function of the theory $\mathcal{T}$ is given as:

$$Z^{(\mathcal{T})}(m, \eta) = \int \prod_{\gamma=1}^3 [d\sigma^\gamma] Z_{\text{FI}}(\sigma^1, \eta_1) Z_{\text{FI}}(\sigma^3, \eta_3) Z_{\text{FI}}(\sigma^3, -\eta_3) \prod_{\gamma=1}^3 Z_{\text{1-loop}}^{\text{vec}}(\sigma^\gamma)$$

$$\times Z_{\text{1-loop}}^{\text{bif}}(\sigma^2, \sigma^1, 0) Z_{\text{1-loop}}^{\text{bif}}(\sigma^2, \sigma^3, 0) Z_{\text{1-loop}}^{\text{fund}}(\sigma^1, m) Z_{\text{1-loop}}^{\text{hyper}}(\sigma^1, \sigma^3, 0),$$

(4.14)

where $Z_{\text{1-loop}}^{\text{hyper}}$ is the 1-loop contribution of a single hyper to the matrix integral integrand given as

$$Z_{\text{1-loop}}^{\text{hyper}}(\sigma^1, \sigma^3, 0) = \frac{1}{\cosh \pi (\text{Tr} \sigma^1 - \sigma^3)}.$$  

(4.15)

The r.h.s. of (4.14) can be identified with the partition function of theory $\mathcal{T}^\vee$, which leads to the following relation between the partition functions of the theories $\mathcal{T}$ and $\mathcal{T}^\vee$:

$$Z^{(\mathcal{T})}(m; \eta_1, \eta_3) = Z^{(\mathcal{T}^\vee)}(m; \eta_1, \eta_3, -\eta_3),$$

(4.16)

where $m$ are the masses of the fundamental hypermultiplets on both sides, $\eta_1, \eta_3$ are the FI parameters of the U(4) and the U(1) gauge group factors of the theory $\mathcal{T}$ respectively,
while $\eta_1, \eta_3, -\eta_3$ are the FI parameters of the U(4), U(2) and the U(1) gauge group factors of the theory $T'$ respectively. Note that the partition functions do not agree for completely generic FI parameters of $T'$ — they only do so if the FI parameters of the U(2) and the U(1) gauge group factors of $T'$ are related as above.

### 4.2.3 Example II: dual pairs from $D_9(SU(3))$

Now, let us consider the IR duality between the two Lagrangian theories associated to the 3d SCFT that arises from the circle reduction of $D_9(SU(3))$. The quiver gauge theories are shown in figure 21 — the theory $T$ arises from the dimensional reduction in [2, 19] while the theory $T'$ was derived from the S-type operation construction in section 3.4. The 3d mirror of these theories is given in figure 22. The quaternionic dimensions of the Higgs and the Coulomb branches of the theories are given as

$$\dim M^{(T)}_H = \dim M^{(T')}_H = 5, \quad (4.17)$$
$$\dim M^{(T)}_C = \dim M^{(T')}_C = 7, \quad (4.18)$$

and the Higgs branch global symmetries $G^{(T)}_H = G^{(T')}_H = SU(3) \times U(1)^2$ of the theories manifestly agree. The Coulomb branch symmetry for $T'$ is manifestly $G^{(T')}_C = SU(2)^3 \times U(1)^3$, which agrees with the Higgs branch global symmetry of the 3d mirror in figure 22. As before, for the theory $T$, the full Coulomb branch symmetry arises in the IR as an emergent symmetry. This can be confirmed from a Hilbert Series computation, for example, as discussed in [19].
Let us now demonstrate how the partition functions of the theory $\mathcal{T}$ and $\mathcal{T}^\vee$ are related. The partition function for $\mathcal{T}$ is given as

\[
Z^{(\mathcal{T})}(m, \eta) = \int \prod_{\gamma=1}^{5} \left[ d\sigma^\gamma \right] \prod_{\gamma \neq 3} \frac{\int F_{\gamma}(\sigma^\gamma, \eta_\gamma)}{\prod_{\gamma=1}^{2} Z^\text{vec}_{1-\text{loop}}(\sigma^\gamma) Z^\text{vec}_{1-\text{loop}}(s^3) Z^\text{bif}_{1-\text{loop}}(\sigma^1, m^1) Z^\text{bif}_{1-\text{loop}}(\sigma^1, \sigma^2, 0) Z^\text{bif}_{1-\text{loop}}(\sigma^2, s^3, 0) Z^\text{bif}_{1-\text{loop}}(\sigma^3, \sigma^4, 0) Z^\text{bif}_{1-\text{loop}}(\sigma^3, \sigma^4, 0)}.
\]

where $\{\sigma^\gamma\}_{\gamma \neq 3}$ denote the integration variables associated with the unitary gauge groups, while $s^3$ denotes the integration variable associated with the SU(2) gauge group. Isolating the $s^3$-dependent part of the matrix integral, and using the relation (4.5) (for the special case of $N = 2$), we have the following identity:

\[
\begin{align*}
\int [d\sigma^3] \delta(Tr s^3) Z^\text{vec}_{1-\text{loop}}(s^3) Z^\text{bif}_{1-\text{loop}}(s^3, \sigma^2, 0) Z^\text{bif}_{1-\text{loop}}(s^3, \sigma^4, 0) \\
= \int d\sigma^3 \frac{1}{\cosh \pi(\sigma^3 - Tr \sigma^2 - \sigma^4)} Z^\text{vec}_{1-\text{loop}}(s^3) Z^\text{bif}_{1-\text{loop}}(s^3, \sigma^2, 0) Z^\text{bif}_{1-\text{loop}}(s^3, \sigma^4, 0).
\end{align*}
\]

Substituting the above identity in (4.19), the partition function for $\mathcal{T}$ can be written as

\[
Z^{(\mathcal{T})}(m, \eta) = \int \prod_{\gamma=1}^{5} \left[ d\sigma^\gamma \right] \prod_{\gamma=1}^{5} F_{\gamma}(\sigma^\gamma, \eta_\gamma) \prod_{\gamma=1}^{2} Z^\text{vec}_{1-\text{loop}}(\sigma^\gamma) Z^\text{fund}_{1-\text{loop}}(\sigma^1, m^1) \times \prod_{\gamma=1}^{4} Z^\text{bif}_{1-\text{loop}}(\sigma^\gamma, \sigma^\gamma+1, 0) Z^\text{fund}_{1-\text{loop}}(\sigma^5, m^5) Z^\text{hyper}_{1-\text{loop}}(\sigma^3, \sigma^2, \sigma^4),
\]

where $Z^\text{hyper}_{1-\text{loop}}$ is the contribution of a single hypermultiplet charged under U(1)$_2 \subset$ U(2)$_2$, U(1)$_3$ and U(1)$_4$ to the matrix model integrand:

\[
Z^\text{hyper}_{1-\text{loop}}(\sigma^3, \sigma^2, \sigma^4) = \frac{1}{\cosh \pi(Tr \sigma^2 - \sigma^3 + \sigma^4)}.
\]

After a simple change of variables: $\sigma^4 \rightarrow -\sigma^4 + \sigma^3$ and $\sigma^5 \rightarrow -\sigma^5 + \sigma^3$, one can check that the matrix integral on the r.h.s. of (4.21) is the partition function of the following quiver:

![Quiver Diagram]

Note that after the change of variables, $\sigma^4$ labels the rightmost U(1) gauge node, while $\sigma^5$ labels the U(1) gauge node immediately to its left. The contribution of the Abelian hypermultiplet to the matrix integral also transforms as:

\[
Z^\text{hyper}_{1-\text{loop}}(\sigma^3, \sigma^2, \sigma^4) \rightarrow Z^\text{hyper}_{1-\text{loop}}(\sigma^2, \sigma^4) = \frac{1}{\cosh \pi(Tr \sigma^2 - \sigma^4)}.
\]
The second U(2) gauge node from the left has three fundamental hypers and a single hyper charged only under the U(1) subgroup with charge 2. One can again use the identity (4.5) to rewrite the matrix integral in a way that the desired duality becomes manifest.

To do this, let us focus on the $\sigma^2$-dependent part of the matrix integral in (4.21). We can rewrite the U(2) matrix integral explicitly as a $U(1) \times SU(2)$ integral by the following change of variables:

$$u = \frac{1}{2} \text{Tr} \sigma^2, \quad s = \frac{1}{2} (\sigma_1^2 - \sigma_2^2).$$

In terms of the new integration variables, the $\sigma^2$-dependent part of the matrix integral can be written as

$$\int [d\sigma^2] Z_{\text{FT}}(\sigma^2, \eta_2) Z_{1-\text{loop}}^{\text{vec}}(\sigma^2) Z_{1-\text{loop}}^{\text{bf}}(\sigma^1, \sigma^2, 0) Z_{1-\text{loop}}^{\text{bf}}(\sigma^2, \sigma^3, 0) Z_{1-\text{loop}}^{\text{hyper}}(\sigma^2, \sigma^4)$$

$$= 2 \int \frac{ds}{2!} \prod_j \cosh \pi(s \pm (\sigma_j - u)) \cosh \pi(s \pm (\sigma_j - u)) \int du \frac{e^{4\pi i \eta_2 u}}{\cosh \pi(2u - \sigma^2)}$$

$$= 2 \int dv \frac{\cosh \pi(v - \text{Tr}\sigma^1 + \sigma^3 + u)}{\prod_j \cosh \pi(v + u - \sigma_j^3)} \cosh \pi(v - u + \sigma^3)$$

$$\times \int du \frac{e^{4\pi i \eta_2 u}}{\cosh \pi(2u - \sigma^4)},$$

where for the final equality we have used (4.5) (for $N = 2$) with the masses of the three hypers in the fundamental representation of SU(2) as $m_1 =: \sigma_1 - u$, $m_2 =: \sigma_2^3 - u$, and $m_3 =: -\sigma^3 + u$. Substituting the identity (4.25) in (4.21), and implementing the following change of integration variables:

$$\bar{\sigma}^0 = u + v + \sigma^3,$$  

$$\bar{\sigma}^1 = \sigma_1^1,$$  

$$\bar{\sigma}^2 = u + v,$$  

$$\bar{\sigma}^3 = -\sigma^5 + \sigma^3 + \sigma^2 = -\sigma^5 + \sigma^3 + u + v,$$  

$$\bar{\sigma}^4 = -\sigma^4 + \sigma^3 + \sigma^2 = -\sigma^4 + \sigma^3 + u + v,$$  

$$\bar{\sigma}^5 = v - u + \sigma^3,$$

the partition function of $\mathcal{T}$ can be written in the following form:

$$Z^{(T)}(m, \eta) = \prod_{\gamma = 0}^5 \prod_{\gamma = 0}^\gamma \prod_{\gamma = 0}^{\delta} Z_{\text{FT}}(\bar{\sigma}^\gamma, \eta_\gamma) Z_{1-\text{loop}}^{\text{vec}}(\bar{\sigma}^1) Z_{1-\text{loop}}^{\text{fund}}(\bar{\sigma}^1, m^1) Z_{1-\text{loop}}^{\text{hyper}}(\bar{\sigma}^0, \bar{\sigma}^1, m^5)$$

$$\times \prod_{\gamma = 1}^4 Z_{1-\text{loop}}^{\text{bf}}(\sigma^\gamma, \sigma^\gamma + 1, 0) Z_{1-\text{loop}}^{\text{bf}}(\sigma^0, \sigma^1, 0) Z_{1-\text{loop}}^{\text{fund}}(\sigma^5, m^5),$$

where the contribution of the Abelian hypermultiplet to the integrand is given as

$$Z_{1-\text{loop}}^{\text{hyper}}(\sigma^0, \sigma^1, m^5) = \frac{1}{\cosh \pi(\sigma^0 - \text{Tr}\sigma^1 + m^5)}.$$  

\footnote{Note that the Jacobian factor of 1/2 precisely cancels the factor of 2 sitting in front of the integral in (4.25).}
The r.h.s. of (4.32) can now be identified as the partition function of the theory \( \mathcal{T}^\vee \), with the gauge nodes labelled as:

\[
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\text{1} \\
\text{4} \\
\text{1} \\
\text{1} \\
\text{2} \\
\text{1} \\
\end{array}
\]

The FI parameters of \( \mathcal{T}^\vee \) are given in terms of the FI parameters of \( \mathcal{T} \) as follows:

\[ \eta_0 = \eta_2, \quad (4.34) \]
\[ \eta_1 = \eta_1, \quad (4.35) \]
\[ \eta_2 = - (\eta_4 + \eta_5), \quad (4.36) \]
\[ \eta_3 = \eta_5, \quad (4.37) \]
\[ \eta_4 = \eta_4, \quad (4.38) \]
\[ \eta_5 = - \eta_2. \quad (4.39) \]

The fundamental hypermultiplet masses \( m^1 \) of the U(2) gauge group in \( \mathcal{T} \) are mapped to the masses \( \overline{m}^1 \) of fundamental hypers for the U(2) gauge group, i.e. \( \overline{m}^1 = m^1 \). The mass of the fundamental hyper for U(1) \(_5\) (i.e. the gauge group labelled by \( \sigma^5 \)) and the mass of the Abelian hyper are given as:

\[ m^5 = - \overline{m}^5, \quad m_{\text{hyper}} = m^5. \quad (4.40) \]

Therefore, the partition functions of the theories \( \mathcal{T} \) and \( \mathcal{T}^\vee \) are related as follows:

\[ Z^{(\mathcal{T})}(m^1, m^5; \eta_1, \eta_2, \eta_4, \eta_5) = Z^{(\mathcal{T}^\vee)}(\overline{m}^1, \overline{m}^5, m_{\text{hyper}}; \overline{\eta}), \quad (4.41) \]

where the relation between themasses and the FI parameters of \( \mathcal{T} \) and \( \mathcal{T}^\vee \) are given above. As we observed in the previous example, the partition functions do not agree for completely generic FI parameters of \( \mathcal{T}^\vee \) — they only do so if the FI parameters of \( \mathcal{T}^\vee \) obey two non-trivial relations so that the number of parameters match on both sides.

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