On vibrational modes of Q-balls

A. Kovtun, E. Nugaev and A. Shkerin

INR RAS

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Overview

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Overview of Q-balls

- Q-balls are a class of non-topological solitons existing in theories of complex scalar field possessing (global) $U(1)$-symmetry (Rosen, G.’68; Coleman, S.’85)

- They are well-studied objects with numerous applications in astrophysics and cosmology, condensed matter physics, even non-linear optics.

- They allow for simple (sometimes analytical) treatment (Theodorakis, S’00)

- They can serve as prototypes for more complicated objects arising in realistic setting (e.g. boson stars).
We will be interested in the models of one complex scalar field in 3+1 dimensions with a potential of a special form,

\[ \mathcal{L} = |\partial_\mu \phi|^2 - V(|\phi|) \]

The ansatz, energy and charge of a Q-ball

\[ \phi = f(r)e^{i\omega t}, \quad E = \int d^3x \left( (\vec{\nabla} f)^2 - \omega^2 f^2 + V(f) \right), \quad Q = 2\omega \int d^3x f^2 \]

We will study small perturbations on top of these configurations.

- The potential will be chosen so that to allow for analytical treatment of both the solitons and their perturbations.
- Note that the problem of finding a spectrum of bound states of a Q-ball is not an eigenvalue problem for an Hermitian operator.
Q-balls in the flat potential

Consider a potential consisting of two parabolic branches joined at some point $|\varphi| = \nu$. Require the presence of a flat direction,

**Parabolic potential with the flat direction**

$$V(|\phi|) = m^2 |\phi|^2 \theta \left( 1 - \frac{|\phi|^2}{\nu^2} \right) + m^2 \nu^2 \theta \left( \frac{|\phi|^2}{\nu^2} - 1 \right)$$

The set of Q-balls split on two branches. One of them (with $\omega < \omega_c$) contains *classically stable* solutions. Another (with $\omega > \omega_c$) corresponds to unstable “Q-clouds” (Alford, M. G.’88)

The critical frequency $\omega_c \approx 0.960m$ corresponds to the soliton with the minimal possible energy and charge.

*Figure: $E(Q)$ for the Q-balls in the flat parabolic potential (Gulamov, I. E., Nugaev, E. Ya. and Smolyakov, M. N.’13)*
Perturbations of Q-balls in the flat potential

An appropriate ansatz governing the dynamics of small oscillations on top of the classically stable Q-balls reads as follows (M. N. Smolyakov’18)

**Perturbation ansatz**

\[ \phi = \phi_0 + \psi e^{i\omega t}, \quad \psi(\vec{x}, t) = (\psi_1^{(l)}(r)e^{i\gamma t} + \psi_2^{(l)}(r)e^{-i\gamma t})Y_{l,m}(\theta, \varphi) \]

where the parameter \( \gamma \) is taken to be real and positive, \( \psi_1^{(l)}, \psi_2^{(l)} \) are real functions of the radial coordinate and \( Y_{l,m} \) are spherical harmonics.

Substituting this into the linearized equations of motion, one gets

**Equations for perturbations**

\[
\begin{align*}
\left( \Delta_r - \frac{l(l + 1)}{r^2} + (\omega + \gamma)^2 - g(r) \right) \psi_1^{(l)}(r) - h(r) \psi_2^{(l)*}(r) &= 0 \\
\left( \Delta_r - \frac{l(l + 1)}{r^2} + (\omega - \gamma)^2 - g(r) \right) \psi_2^{(l)}(r) - h(r) \psi_1^{(l)*}(r) &= 0
\end{align*}
\]

The functions \( g \) and \( h \) are determined by the potential. In our case

\[ h(r) = -\frac{m^2}{2} \delta \left( \frac{f(r)}{v} - 1 \right), \quad g(r) = m^2 \theta \left( 1 - \frac{f^2(r)}{v^2} \right) + h(r) \]

Hence, equations are disentangled everywhere except the single point \( R \) such that \( f(R) = v \).
Perturbations of Q-balls in the flat potential

To study bound states, one imposes

**Boundary conditions**

\[ \psi_{1,2}^{(l)}(\infty) = 0, \quad \partial_r \psi_{1,2}^{(l)} \bigg|_{r=0} = 0 \]

and also \( \gamma + \omega < m \).

**Figure:** The discrete spectrum of linear perturbations of classically stable Q-balls in the flat potential, at \( l = 0 \). All quantities are normalized to the parameter \( m \).
Perturbations of Q-balls in the flat potential

Features of the spectrum:

- At $\omega \to 0$, one has $Q \to \infty$. Hence, large Q-balls possess soft modes. In this limit, the spectrum linearizes,

  $$\gamma_n = k_n \omega , \quad k_n \approx \frac{n\omega}{2} , \quad n = 1, 3, 4, 5, \ldots$$

- The number of bound states of large Q-balls is proportional to its size$^3$.

- At intermediate frequencies the Q-balls do not support bound states.

- Close to the stability bound $\omega = \omega_c$ one vibrational spherically-symmetric mode reappears. For it

  $$\gamma \sim \sqrt{\omega_c - \omega}$$

  This mode continues analytically into the instability region where it becomes the decay mode.
The structure of the spectrum with a non-zero orbital momentum is similar to that with $l = 0$:

![Graph showing the discrete spectrum of linear perturbations of classically stable Q-balls in the flat potential, at $l = 1$ (the left panel) and $l = 2$ (the right panel).](image)

**Figure:** The discrete spectrum of linear perturbations of classically stable Q-balls in the flat potential, at $l = 1$ (the left panel) and $l = 2$ (the right panel).

Note the absence of vibrational modes near the cusp point $\omega = \omega_c$. 


In order to allow Q-balls in a theory of one scalar field with a polynomial potential, it is necessary to include non-renormalizable self-interactions in the latter. Here we consider the simplest bounded below potential of the sixth degree,

\[ V(|\phi|) = \left( \delta (|\phi|^2 - v^2)^2 + \omega_{\text{min}}^2 \right) |\phi|^2, \quad \delta > 0 \]

The frequencies of Q-balls are confined in the region

\[ \omega_{\text{min}} < \omega < m = \sqrt{\omega_{\text{min}}^2 + \delta v^4} \]

The thin-wall approximation is applicable near the lower limit. It is controlled by the small parameter

\[ \epsilon = \omega - \omega_{\text{min}} \]
Q-balls in the thin-wall regime

In the thin-wall regime, the properties of a Q-ball are well captured by few quantities — the distance $R$ to the wall and the magnitude $f_0$ of the field in the interior region.

In order to justify the description of a soliton in terms of a finite set of variables, a suitable thin-wall ansatz must be adopted.

To study perturbations on top of a Q-ball, it suffices to choose the simplest ansatz:

**Thin-wall ansatz**

$$f(r) = f_0 \theta \left( 1 - \frac{r}{R} \right)$$

With this ansatz the energy and the charge of the Q-ball are

$$Q = \frac{8}{3} \pi R^3 \omega f_0^2 , \quad E = 8 \pi R^2 \sqrt{\delta} v^4 + \frac{4}{3} \pi R^3 \left( \omega^2 + \omega_{\text{min}}^2 \right) f_0^2$$

Minimizing $E$ while keeping $Q$ fixed, one gets

$$f_0 = v + O(\epsilon) , \quad R = \frac{\sqrt{\delta} v^2}{2 \omega_{\text{min}}} \frac{1}{\epsilon} + O(1)$$
Perturbations in the thin-wall regime

The equations for perturbations are the same as before. The functions $f$ and $g$ are now given by

$$h(r) = -4\delta v^2 f^2(r) + 6\delta f^4(r), \quad g(r) = m^2 - 8\delta v^2 f^2(r) + 9\delta f^4(r)$$

The equations are disentangled in the exterior of the Q-ball, $r > R$. In the interior, $r < R$, one obtains separate equations for the rotated vector $\Xi = (\xi_1, \xi_2)^T$ such that

$$\Psi = U\Xi,$$

where $U$ diagonalizes the non-diagonal part of the linearized equations.

The resulting solutions are joined at $r = R$. This gives the spectrum of allowed values of $\gamma$. 
Perturbations in the thin-wall regime

Figure: The spectrum of vibrational modes of stable Q-balls in the thin-wall approximation. The parameters of the potential are $\delta = 1.5$, $\nu = 0.9$, $\omega_{\text{min}} = 0.126$. All quantities are normalized to $m$. The left panel shows the full spectrum of the spherically-symmetric modes, $l = 0$. The right panel compares the modes of the 1st “energy level” and with different orbital momenta.

The features of the spectrum near the bound $\omega = \omega_{\text{min}}$ are the same as for the flat parabolic potential.
The spectra of vibrations of the Q-balls in our examples have some properties in common. In fact, those properties are model-independent.

Large Q-balls in the model with the flat potential possess soft modes with $\gamma \sim \omega \to 0$, well below the mass $m$ of the free boson in vacuum.

Q-balls with the near-critical charge have the vibrational mode related to the decay mode of Q-clouds.

It is important to note that the near-critical regime of these (in general, relativistic) solitons can be analyzed by the means of the perturbation theory with respect to the relative frequency $\gamma$ of an excitation.
Thank you!
The decay mode is captured by the following spherically-symmetric ansatz,

\[ \psi(\vec{x}, t) = \zeta(r)e^{\gamma t}, \quad \gamma > 0 \]

Define \( \tilde{\gamma} \) as

\[ \tilde{\gamma}^2 \equiv \gamma^2 \quad \text{for} \quad \omega < \omega_c, \quad \tilde{\gamma}^2 \equiv -\gamma^2 \quad \text{for} \quad \omega \geq \omega_c \]

**Figure:** *Left panel:* the decay rate of unstable Q-balls in the flat parabolic potential. *Right panel:* the transition between the decay and the vibrational modes.
Perturbation theory near the cusp point

Whenever $\gamma$ is small, one can make use of the perturbation theory with respect to $\gamma$. Then, the linear perturbations of a Q-ball take a simple form

$$\psi_1 \sim f + \gamma \frac{\partial f}{\partial \omega} + O(\gamma^2), \quad \psi_2 \sim -f + \gamma \frac{\partial f}{\partial \omega} + O(\gamma^2)$$

Similarly, for the decay mode we have

$$\psi e^{-\gamma t} \sim if + \gamma \frac{\partial f}{\partial \omega} + O(\gamma^2)$$

In this expression, the first term represents the Goldstone mode corresponding to the global $U(1)$-symmetry of the theory.