Misère Hackenbush is NP-Hard

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Abstract

Hackenbush is a two player game, played on a graph with coloured edges where players take it in turns to remove edges of their own colour. It has been shown that under normal play rules Red-Blue Hackenbush (all edges are coloured either red or blue) is NP-hard. We will show that Red-Blue Hackenbush is in P, but that Red-Blue-Green Hackenbush is NP-Hard, when played under misère rules.

1 Introduction

Hackenbush is a game that is played on a graph, with coloured edges, that is connected to a ground defined arbitrarily before the game begins. The rules of Red-Blue Hackenbush are as follows:

1. Players take it in turn to remove edges.
2. Left may only remove blue edges and Right may only remove red edges.
3. Any edges not connected to the ground are also removed.
4. Under normal play the last player to move wins, under misère play the last player to move loses.

An example of Red-Blue Hackenbush positions is given in Figures 1. The vertices that are labelled with a “g” are the vertices that are connected to the ground.

In Winning Ways [2], the authors used different variants of Hackenbush to illustrate all parts of the theory for normal play games. For this reason it is worth studying when considering misère play games. However it has been
shown that determining the outcome of a general position of Red-Blue Hackenbush under normal play is NP-hard (for an explanation of NP-hardness see \cite{5}).

In this paper we will also be using the following definition for the four possible outcome classes of a normal or misère play game;

**Definition 1.** \cite{1} We define the following:

- \( \mathcal{L} = \{ G \mid \text{Left wins playing first or second in } G \} \).

- \( \mathcal{R} = \{ G \mid \text{Right wins playing first or second in } G \} \).

- \( \mathcal{P} = \{ G \mid \text{The second player to move wins in } G \} \).

- \( \mathcal{N} = \{ G \mid \text{The first player to move wins in } G \} \).

For further information about combinatorial game theory see \cite{1}, \cite{2} or \cite{3}.

\section{Red-Blue and Red-Blue-Green Misère Hackenbush}

You might expect that when we consider Red-Blue Hackenbush under misère rules that it is still hard to determine the winner. However it is actually very easy to determine the winner of a Red-Blue Hackenbush position under misère rules and it can be done in polynomial time as shown in Theorem \cite{3}.
Definition 2. A “grounded” edge, is an edge that is connected directly to the ground.

Theorem 3. Let $G$ be a game of Red-Blue misère Hackenbush, and let $B$ and $R$ be the number of grounded blue and red edges respectively, then the outcome of $G$ can be determined by the following formula:

$$G \in \begin{cases} \mathcal{L}, & \text{if } B > R \\ \mathcal{R}, & \text{if } R > B \\ \mathcal{N}, & \text{if } B = R \end{cases}$$

Proof. Let there be $R$ grounded Red edges and $B$ grounded Blue edges, and consider the case where $R \geq B$. Consider Left moving first. His winning move will be to remove one of his own grounded edges. Regardless of what Right does in response to this, Left can keep removing his grounded edges. Once Left has removed all of these edges, there will be at least one grounded Right edge, and Left wins regardless if he is to move first or second in this situation.

Left can only win moving second if $R > B$, since Right taking one of his grounded Red edges will be moving to the situation $R \geq B$, and it will be Left’s turn to move. If Right chooses not to take one of his grounded Red edges, then again Left takes one of his grounded Blue edges, and again wins.

The situation $B \leq R$ follows by symmetry. 

So this means that all we have to do to find the outcome class for a game of Red-Blue misère Hackenbush, we simply count the number of grounded red and blue edges and the difference will tell us the outcome class. This can clearly be done in polynomial time, which means that Red-Blue misère Hackenbush is neither NP-complete or NP-hard.

2.1 Red-Blue-Green Misère Hackenbush

The rules of Red-Blue-Green Hackenbush are identical to the rules of Red-Blue Hackenbush, but for one additional rule. That is green edges, which may be removed by both players, in the diagrams that follow green edges will be represented by thick edges. It turns out to be a far more complicated game.
PROBLEM: **RED-BLUE-GREEN MISÈRE HACKENBUSCH**

INSTANCE: A position of Red-Blue-Green Misère Hackenbush $G$.

QUESTION: What is the outcome of $G$?

**Theorem 4.** *Red-Blue-Green Misère Hackenbush is NP-hard.*

*Proof.* To prove this we will do a transformation from Red-Blue Hackenbush under normal play rules. First we note two things, as previously stated, it is known that determining the outcome of a general position of normal play Red-Blue Hackenbush is NP-hard. It is also known that we can think of the ground in Hackenbush as being a single vertex, which is drawn as a ground with separate vertices for clarity in diagrams, [4], page 40. With this in mind we will make our transformation.

The transformation will be as follows, start with a general Red-Blue Hackenbush position $G$. Next take the same position and replace the ground, and all the vertices that are on the ground with a single vertex and call this game $G'$. Lastly attach $G'$ to a single grounded green edge, and call this game $G_m$. This process is illustrated in Figure 2. The figure shows two red and blue edges, this is simply to illustrate the process, however the graph $G$ can be any graph only if the edges are all coloured red or blue.

![Figure 2: Transformation of $G$ to $G_m$.](image)

If we are playing $G_m$ under misère rules, then neither player will want to cut the single green edge, since doing so will remove every edge in the game, and thus the next player will be unable to move and therefore win under misère rules. So both players will want to move last on the graph $G'$ that is attached to the single green edge, thus forcing your opponent to remove the green edge, which will result in you winning the game. In other words, whoever wins $G'$ under normal play rules, will also win $G_m$ under misère play rules, and since determining the outcome of $G'$ is NP-hard, determining the outcome of $G_m$ is also NP-hard. So the theorem is proven.  

\[\square\]
Problem 5. If we restrict Red-Blue-Green misère Hackenbush to collections of strings with only grounded green edges is it still NP-hard to determine the outcome class? If not what is the solution?

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