On domination perfect graphs

JERZY TOPP AND PAWEL ŻYLIŃSKI
University of Gdańsk, 80-308 Gdańsk, Poland
{j.topp,zylinski}@inf.ug.edu.pl

Abstract

Let \( \gamma(G) \) and \( \beta(G) \) denote the domination number and the covering number of a graph \( G \), respectively. A connected non-trivial graph \( G \) is said to be \( \gamma\beta \)-perfect if \( \gamma(H) = \beta(H) \) for every non-trivial induced connected subgraph \( H \) of \( G \). In this note we present an elementary proof of a characterization of the \( \gamma\beta \)-perfect graphs.

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In this note, we follow the notation of [2]. In particular, a subset \( D \subseteq V_G \) is a dominating set of a graph \( G = (V_G, E_G) \) if each vertex belonging to the set \( V_G - D \) has a neighbor in \( D \). The cardinality of a minimum dominating set of \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \). A subset \( C \subseteq V_G \) is a vertex cover of \( G \) if each edge of \( G \) has an end-vertex in \( C \). (Note that in [1] a vertex cover is called a transversal of \( G \).) The cardinality of a minimum vertex cover of \( G \) is called the covering number of \( G \) and is denoted by \( \beta(G) \). A connected non-trivial graph \( G \) is said to be \( \gamma\beta \)-perfect if \( \gamma(H) = \beta(H) \) for every non-trivial induced connected subgraph \( H \) of \( G \). Such graphs have been studied in [1] and [3]. In this note we compose Theorem 3.9 in [1] with Theorem 9 in [3] and present an elementary proof of the unified result.

We start with two assertions, then give a characterization of the \( \gamma\beta \)-perfect graphs.

Proposition 1. Every non-trivial tree of diameter at most four and every non-trivial connected subgraph of \( K_{2,n} \) is a \( \gamma\beta \)-perfect graph, while no one of the graphs \( C_3, C_5 \) and \( P_6 \) is a \( \gamma\beta \)-perfect graph.

Proposition 2. If \( F \) is a connected spanning subgraph of a graph \( H \) of order at least three and \( \gamma(F) < \beta(F) \), then \( \gamma(H) < \beta(H) \) and, therefore, \( H \) is not a \( \gamma\beta \)-perfect graph.

Proof. Since a dominating set of \( F \) is a dominating set of \( H \), we have \( \gamma(H) \leq \gamma(F) \). Similarly, \( \beta(F) \leq \beta(H) \), since a vertex cover of \( H \) is a vertex cover of \( F \). Consequently, \( \gamma(H) \leq \gamma(F) < \beta(F) \leq \beta(H) \) and \( H \) is not a \( \gamma\beta \)-perfect graph. \( \square \)
**Theorem.** The following statements are equivalent for a non-trivial connected graph $G$:

1. $G$ is a tree of diameter at most four or $G$ is a connected subgraph of $K_{2,n}$.
2. $G$ is a $\gamma\beta$-perfect graph.
3. $G \neq C_5$ and neither $C_3$ nor $P_6$ is a subgraph of $G$.

**Proof.** The implication (1) $\Rightarrow$ (2) is obvious from Proposition 1. Assume that $G$ is a $\gamma\beta$-perfect graph. Then, by Proposition 1 no one of the graphs $C_3$, $C_5$ and $P_6$ is an induced subgraph of $G$. Consequently, $G \neq C_5$ and $C_3$ is not a subgraph of $G$. We claim that also $P_6$ is not a subgraph of $G$. Otherwise $P_6$ is a spanning subgraph of some 6-vertex induced subgraph $H$ of in $G$. Then, since $\gamma(P_6) < \beta(P_6)$, we have $\gamma(H) < \beta(H)$ (by Proposition 2), which contradicts the premise that $G$ is $\gamma\beta$-perfect. This proves the implication (2) $\Rightarrow$ (3). To prove (3) $\Rightarrow$ (1), assume that $G \neq C_5$ and neither $C_3$ nor $P_6$ is a subgraph of $G$. If $G$ is a tree, then, since $P_6$ is not a subgraph of $G$, $G$ is of diameter at most 4. Thus assume that $G$ has a cycle, say $C$. Since $G \neq C_5$, the absence of $C_3$ and $P_6$ in $G$ guarantees that $C$ is a chordless 4-cycle. If $G = C$, then $G = K_{2,2}$. Thus assume that the cycle $C$ is a proper subgraph of $G$. Let $v_1, v_2, v_3, v_4$ be the consecutive vertices of $C$. We may assume without loss of generality that $d_G(v_1) > 2$. This time from the absence of $C_3$ and $P_6$ in $G$ it follows that $d_G(v_2) = d_G(v_4) = 2$. Now, since $G$ is connected and $P_6$ is not a subgraph of $G$, $N_G(v) \subseteq \{v_1, v_3\}$ for every vertex $v$ belonging to $V_G - \{v_1, v_2, v_3, v_4\}$. Consequently, $V_G - \{v_1, v_3\}$ is independent and $G$ is a subgraph of the complete bipartite graph $K_{2,n}$, where $n = |V_G - \{v_1, v_3\}|$. 

**References**

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