On a Pólya functional for rhombi, isosceles triangles, and thinning convex sets.

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Abstract

Let $\Omega$ be an open convex set in $\mathbb{R}^m$ with finite width, and let $v_\Omega$ be the torsion function for $\Omega$, i.e. the solution of $-\Delta v = 1$, $v \in H^1_0(\Omega)$. An upper bound is obtained for the product of $\|v_\Omega\|_{L^\infty(\Omega)} \lambda(\Omega)$, where $\lambda(\Omega)$ is the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$. The upper bound is sharp in the limit of a thinning sequence of convex sets. For planar rhombi and isosceles triangles with area 1, it is shown that $\|v_\Omega\|_{L^1(\Omega)} \lambda(\Omega) \geq \frac{\pi^2}{24}$, and that this bound is sharp.

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1 Introduction

Let $\Omega$ be an open set in $\mathbb{R}^m$, and denote the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ by

$$\lambda(\Omega) = \inf_{\varphi \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_\Omega |D\varphi|^2}{\int_\Omega \varphi^2}.$$ 

It was shown in [3] and [1] that if $\lambda(\Omega) > 0$, then the torsion function, defined by $v_\Omega : \Omega \mapsto \mathbb{R}^+$

$$-\Delta v = 1, \quad v \in H^1_0(\Omega),$$

satisfies

$$1 \leq \lambda(\Omega) M(\Omega) \leq c_m,$$ (1.1)
where \( M(\Omega) = \|v_\Omega\|_{L^\infty(\Omega)}. \)

In [12] it was shown that
\[
c_m \leq \frac{1}{8} (m + (5(4 + \log 2))^{1/2} m^{1/2} + 8)
\]
The sharp constant in the right-hand side of (1.1) is not known. However, for an open ball \( B \subset \mathbb{R}^2 \), an open square \( S \), and an equilateral triangle \( E \),
\[
\lambda(B)M(B) < \lambda(S)M(S) < \lambda(E)M(E).
\] (1.2)
The fact that \( \lambda(B)M(B) < \lambda(E)M(E) \) was shown in [6]. The full inequality (1.2) follows from numerical evaluation of the series for the square, pp. 275-277 in [13].

In [2] it was shown that the left-hand side of (1.1) is sharp: for \( \epsilon > 0, m \geq 2 \), there exists an open bounded and connected set \( \Omega_\epsilon \subset \mathbb{R}^m \) such that
\[
\lambda(\Omega_\epsilon)M(\Omega_\epsilon) < 1 + \epsilon.
\]

For open, bounded convex sets in \( \mathbb{R}^m \) it was shown in (3.12) of [7] that
\[
\lambda(\Omega)M(\Omega) \geq \frac{\pi^2}{8},
\] (1.3)
with equality in the limit of an infinite slab (the open set with finite width bounded by two parallel \( (m - 1) \)-dimensional planes). The latter assertion has been made precise in [6] where it was shown that if
\[
S_n = (-n, n)^{m-1} \times (0, 1), \ n \geq 1,
\] (1.4)
then
\[
\lambda(S_n)M(S_n) \leq \frac{\pi^2}{8} + \frac{m - 1}{8(n - \frac{3}{2})}.
\] (1.5)

For bounded planar convex sets with width \( w(\Omega) \), and diameter \( \text{diam}(\Omega) \), it was shown in [2] that
\[
\lambda(\Omega)M(\Omega) \leq \frac{\pi^2}{8} \left( 1 + 3^{2/3} \frac{m - 1}{\text{diam}(\Omega)} \right)^{2/3}.
\] (1.6)

In Theorem 1.1 below we put (1.4)-(1.5), and (1.6) in a more general setting. We introduce the following notation. For an open, bounded, convex set with finite width \( w(\Omega) \), and boundary \( \partial \Omega \) we let \( H_\mu = \{ x \in \mathbb{R}^m : x_m = \mu w(\Omega) \} \), \( 0 \leq \mu \leq 1 \), be a family of parallel hyper-planes such that \( H_0 \) and \( H_1 \) are tangent to \( \partial \Omega \) at two points \( z_0 \in H_0 \) and \( z_1 \in H_1 \) respectively, where \( |z_0 - z_1| = w(\Omega) \), and \( z_0 - z_1 \) is orthogonal to \( H_0 \). That this is always possible was shown in Theorem 1.5 in [4]. We identify sets in \( H_0 \) with sets in \( \mathbb{R}^{m-1} \). Let
\[
\Omega_\mu = \{ (x', x_m - \mu w(\Omega)) : (x', x_m) \in \Omega \cap H_\mu \}.
\]
The projection of \( \Omega \) onto \( H_0 \) is denoted by
\[
\Pi(\Omega) := \bigcup_{0 \leq \mu \leq 1} \Omega_\mu.
\]

We denote the inradius of this \( (m - 1) \)-dimensional set by \( \rho(\Omega) \). The measure of \( \Omega \) is denoted by \( |\Omega| \).

**Theorem 1.1** If \( \Omega \) is an open, bounded, convex set in \( \mathbb{R}^m, m \geq 2 \), then
\[
\lambda(\Omega)M(\Omega) \leq \frac{\pi^2}{8} \left( 1 + d_m \left( \frac{w(\Omega)}{\rho(\Omega)} \right)^{2/3} \right),
\] (1.7)
where
\[
d_m = 3^{2/3} 28 \pi^{2/3} j_{(m-3)/2}^2.
\] (1.8)
The torsional rigidity (or torsion) of an open set $\Omega$ is defined by

$$T(\Omega) = \|v_\Omega\|_{L^1(\Omega)} = \int_\Omega v_\Omega. \quad (1.9)$$

In Pólya and Szegö [8], it was shown that for sets $\Omega$ with finite measure $|\Omega|$, 

$$\frac{T(\Omega)\lambda(\Omega)}{|\Omega|} \leq 1. \quad (1.10)$$

It was subsequently shown in [4] that the constant 1 in the right-hand side above is sharp: for $\epsilon > 0$, there exists an open, bounded, and connected set $\Omega_\epsilon \subset \mathbb{R}^m$ such that

$$T(\Omega_\epsilon)\lambda(\Omega_\epsilon)|\Omega_\epsilon| - 1 \geq 1 - \epsilon. \quad (1.11)$$

The quantity in the left-hand side of (2.4) is invariant under the homothety transformation $t \mapsto t\Omega$. This implies for example that in Theorems 1.2-1.5 below we do not have to specify the actual lengths of the edges of the rhombi and triangles. In the proofs of these theorems we fix the various lengths as a matter of convenience.

It was shown in Theorem 1.5 of [4] that for a thinning (collapsing) sequence $(\Omega_n)$ of bounded convex sets

$$\limsup_{n \to \infty} \frac{T(\Omega_n)\lambda(\Omega_n)}{|\Omega_n|} \leq \frac{\pi^2}{24}. \quad (1.11)$$

This supports the conjecture that for bounded, convex sets the sharp constant in the right-hand side of (2.4) is $\frac{\pi^2}{24}$.

It was shown in Theorem 1.4 in [4] that for bounded convex sets in $\mathbb{R}^m, m \geq 3$

$$\frac{T(\Omega)\lambda(\Omega)}{|\Omega|} \geq \frac{\pi^2}{4m^{m+2}(m+2)}, \quad (1.12)$$

and that for planar, bounded convex sets the inequality holds with constant $\frac{\pi^2}{48}$. In [6] it was conjectured that for planar, bounded, convex sets

$$\frac{T(\Omega)}{M(\Omega)|\Omega|} \geq \frac{1}{3}, \quad (1.13)$$

and that this constant is sharp for a thinning (collapsing) sequence of isosceles triangles. See also [5].

By (1.3) and (1.6) above, we have that for a thin isosceles triangle $M(\Omega)\lambda(\Omega) \approx \frac{\pi^2}{8}$. This suggests that the sharp constant for planar convex sets in the right-hand side of (1.12) is $\frac{\pi^2}{24}$. We have the following.

**Theorem 1.2** If $\triangle_\beta$ is an isosceles triangle with angles $\beta, \beta, \pi - 2\beta$, and if $0 < \beta \leq \frac{\pi}{3}$ then

$$\frac{T(\triangle_\beta)\lambda(\triangle_\beta)}{|\triangle_\beta|} \leq \frac{\pi^2}{24} \left(1 + 81 \left(\tan \beta \right)^{2/3}\right). \quad (1.14)$$

**Theorem 1.3** If $\diamond_\beta$ is a rhombus with angles $\beta, \pi - \beta, \beta, \pi - \beta$, and if $\beta \leq \frac{\pi}{3}$ then

$$\frac{T(\diamond_\beta)\lambda(\diamond_\beta)}{|\diamond_\beta|} \leq \frac{\pi^2}{24} \left(1 + 15 \left(\tan \beta \right)^{2/3}\right). \quad (1.15)$$

**Theorem 1.4** If $\diamond_\beta$ is as in Theorem 1.3 then

$$\frac{T(\diamond_\beta)\lambda(\diamond_\beta)}{|\diamond_\beta|} \geq \frac{\pi^2}{24}. \quad (1.16)$$

**Theorem 1.5** If $\triangle_\beta$ is an isosceles triangle with angles $\beta, \beta, 2\pi - \beta$, then

$$\frac{T(\triangle_\beta)\lambda(\triangle_\beta)}{|\triangle_\beta|} \geq \frac{\pi^2}{24}. \quad (1.17)$$
This paper is organised as follows. In Section 2, we prove Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are deferred to Section 3. The proof of Theorem 1.4 is deferred to Section 4. The proof of Theorem 1.5 consists of two parts. In Section 5, part 1, we show that inequality (1.17) holds for all \( \beta \in (0, \pi/3] \cup [\beta_0, \pi/2) \), where

\[
\beta_0 = \frac{\pi}{2} - \frac{33}{200} \tag{1.18}
\]

In Section 5, part 2, we use interval arithmetics to verify that (1.17) also holds for \( \beta \in (\pi/3, \beta_0) \).

## 2 Proof of Theorem 1.1

**Proof.** We first observe that by domain monotonicity of the torsion function, \( v_{\Omega} \) is bounded by the torsion function for the (connected) set bounded by \( H_0 \) and \( H_1 \). Hence

\[
v_{\Omega}(x) \leq \frac{1}{2} x_m(w(\Omega) - x_m) \leq \frac{w(\Omega)^2}{8}, \quad (x', x_m) \in \Omega.
\]

It suffices to obtain an upper bound for \( \lambda(\Omega) \). We choose the \( x' \)-coordinates such that \( x'(z_1) = x'(z_2) = 0 \). By convexity we have that the convex hull of \( z_1, z_2, \Omega \cap H_{1/2} \) is contained in \( \Omega \). This convex hull in turn contains a cylinder with height \( z \in [0, w(\Omega)] \), and base \( \left( 1 - \frac{z}{w(\Omega)} \Omega \cap H_{1/2} \right) \). Denote the first \((m - 1)-\)dimensional Dirichlet eigenvalue of \( \Omega_{1/2} \) by \( \lambda_{1/2} \). Then, by separation of variables, we have that

\[
\lambda(\Omega) \leq \frac{\pi^2}{z^2} + \left( 1 - \frac{z}{w(\Omega)} \right)^{-2} \lambda_{1/2}. \tag{2.1}
\]

The right-hand side of (2.1) is minimised for

\[
\frac{1}{z} = \frac{1}{w(\Omega)} + \left( \frac{\lambda_{1/2}}{\pi^2 w(\Omega)} \right)^{1/3}.
\]

This gives that

\[
\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left( 1 + 3 \left( \frac{\lambda_{1/2} w(\Omega)^2}{\pi^2} \right)^{1/3} + 3 \left( \frac{\lambda_{1/2} w(\Omega)^2}{\pi^2} \right)^{2/3} + \frac{\lambda_{1/2} w(\Omega)^2}{\pi^2} \right). \tag{2.2}
\]

Denote the inradius of \( \Omega \), and the centre of the inball by \( r(\Omega) \) and \( c(\Omega) \), respectively:

\[
r(\Omega) = \sup_{x \in \Omega} \text{dist}(x, \partial \Omega) = \text{dist}(c(\Omega), \partial \Omega).
\]

Then

\[
r(\Omega) \leq x_m(c(\Omega)) \leq w(\Omega) - r(\Omega),
\]

and

\[
\text{dist}(c(\Omega), H_{1/2}) \leq \left| \frac{w(\Omega)}{2} - r(\Omega) \right|.
\]

The inball intersects \( \Omega \cap H_{1/2} \) in a \((m - 1)\)-dimensional disc with radius bounded from below by

\[
\left( |r(\Omega)^2 - \frac{w(\Omega)^2}{2} - r(\Omega)|^2 \right)^{1/2} = \left( w(\Omega) r(\Omega) - \frac{w(\Omega)^2}{4} \right)^{1/2} \geq \frac{w(\Omega)}{2\sqrt{3}},
\]

where we have used that \( r(\Omega) \geq w(\Omega)/3 \), see Blaschke’s theorem, p. 215 in [14]. Hence

\[
\lambda_{1/2} \leq 12 \frac{j_{(m-3)/2}}{w(\Omega)^2} \tag{2.3}
\]
where \( j_\nu \) is the first positive zero of the Bessel function \( J_\nu \). By (2.2) and (2.3) we obtain that
\[
\lambda(\Omega) \leq \frac{\pi^2}{w(\Omega)^2} \left( 1 + \left( \frac{\lambda_{1/2} w(\Omega)^{1/2}}{\pi^2} \right)^{1/3} \left( 3 + 3 \left( \frac{12 j_{(m-3)/2}^2}{\pi^2} \right)^{1/3} + \left( \frac{12 j_{(m-3)/2}^2}{\pi^2} \right)^{2/3} \right) \right)
\]
\[
\leq \frac{\pi^2}{w(\Omega)^2} \left( 1 + 7 \left( \frac{12 j_{(m-3)/2}^2}{\pi^2} \right)^{2/3} \left( \frac{\lambda_{1/2} w(\Omega)^{1/2}}{\pi^2} \right)^{1/3} \right).
\]  \quad (2.4)

By convexity we have that \( \Pi(\Omega) \subseteq 2\Omega_{1/2} \). Hence \( \lambda_{1/2} \) is bounded from above by 4 times the bottom of the spectrum of \( \Pi(\Omega) \). The latter contains a \((m-1)\)-dimensional ball with radius \( \rho(\Omega) \). So
\[
\lambda_{1/2} \leq 4 j_{(m-3)/2}^2 \rho(\Omega)^{-2},
\]  \quad (2.5)
and \((1.7), (1.8)\) follows by (2.4) and (2.5).

3 Proofs of Theorem 1.2 and Theorem 1.3

**Proof of Theorem 1.2** Let \( \triangle_\beta \) be an isosceles triangle with a base of length 2 and width (height) of length \( d \), and angles \( \beta, \beta, \) and \( \pi - 2\beta \) respectively. By hypothesis, \( \beta = \arctan \frac{d}{2} \leq \frac{\pi}{3} \) so that \( d \leq \sqrt{3} \).

We denote the infinite sector with opening angle \( \beta \) by \( \Omega_\beta = \{(r, \phi) : r > 0, -\beta/2 < \phi < \beta/2 \} \).

It is straightforward to verify that the torsion function for \( \Omega_\beta \) is given by,
\[
v_{\Omega_\beta}(r, \phi) = \frac{r^2}{4} \left( \frac{\cos(2\phi)}{\cos \beta} - 1 \right), \quad r > 0, -\beta/2 < \phi < \beta/2.
\]

Let
\[
R = (1 + d^2)^{1/2}.
\]

We can cover \( \triangle_\beta \) with two sectors of opening angles \( \beta \) and radii \( R \) each. By monotonicity and positivity of the torsion function we have that
\[
T(\triangle_\beta) = \int_{\triangle_\beta} v_{\triangle_\beta} \leq 2 \int_0^R dr \int_{-\beta/2}^{\beta/2} d\phi v_{\Omega_\beta}(r, \phi)
\]
\[
= \frac{1}{8} (1 + d^2)^2 \left( \tan \beta - \beta \right)
\]
\[
= \frac{1}{8} (1 + d^2)^2 (d - \arctan d)
\]
\[
\leq \frac{d^3}{24} (1 + d^2)^2,
\]  \quad (3.1)
where we have used that \( d - \arctan d \leq d^3/3 \). By adapting formula (31) in the proof of Theorem 2 in [2] to the geometry of \( \triangle_\beta \) we find that
\[
\lambda(\triangle_\beta) \leq \frac{\pi^2}{d^2} \left( 1 + 7 \left( \frac{d}{2} \right)^{2/3} \right).
\]  \quad (3.2)

By (3.1), (3.2), and \(|\triangle_\beta| = d\), we obtain that
\[
\frac{T(\triangle_\beta) \lambda(\triangle_\beta)}{|\triangle_\beta|} \leq \frac{\pi^2}{24} (1 + d^2)^2 \left( 1 + 7 \left( \frac{d}{2} \right)^{2/3} \right)
\]
\[
\leq \frac{\pi^2}{24} (1 + 81d^2/3)
\]
\[
= \frac{\pi^2}{24} (1 + 81(\tan \beta)^{2/3}), \quad 0 < \beta \leq \frac{\pi}{3}.
\]
Figure 1: The rhombus of diagonals 2 and $d$. To estimate the torsion we construct a test function $v$ which is symmetric with respect to the minor diagonal.

which completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let $\hat{\beta}$ be a rhombus with angles $\beta, \pi - \beta, \beta, \pi - \beta$, and diagonals of length 2 and $d$ respectively. By hypothesis we have that $\beta \leq \pi/3$, and $d \leq 2/\sqrt{3}$. This rhombus is covered by two sectors of opening angle $\beta = 2 \arctan(d/2)$, and radius $R = (1 + (d/2)^2)^{1/2}$. By the calculations in the proof of Theorem 1.2 we find that

$$T(\hat{\beta}) \leq \frac{1}{8} R^4 (\tan \beta - \beta)$$

$$= \frac{1}{8} \left(1 + \frac{d^2}{4}\right)^2 \left(\frac{d}{1 - \frac{d^2}{4}} - 2 \arctan \left(\frac{d}{2}\right)\right)$$

$$\leq \frac{1}{8} \left(1 + \frac{d^2}{4}\right)^2 \left(\frac{d}{1 - \frac{d^2}{4}} - d + \frac{d^3}{12}\right)$$

$$\leq \frac{d^3}{24} \left(1 + \frac{d^2}{4}\right)^2 \left(1 + \frac{9d^2}{32}\right), 0 < d \leq \frac{2}{\sqrt{3}}.$$

By adapting formula (31) in the proof of Theorem 2 in [2] to the geometry of $\hat{\beta}$ we find that

$$\lambda(\hat{\beta}) \leq \frac{\pi^2}{d^2} \left(1 + 7 \left(\frac{d}{2}\right)^{2/3}\right).$$

This, together with $|\hat{\beta}| = d$ gives that,

$$\frac{T(\hat{\beta}) \lambda(\hat{\beta})}{|\hat{\beta}|} \leq \frac{\pi^2}{24} \left(1 + \frac{d^2}{4}\right)^2 \left(1 + \frac{9d^2}{32}\right) \left(1 + 7 \left(\frac{d}{2}\right)^{2/3}\right)$$

$$\leq \frac{\pi^2}{24} \left(1 + 15 \left(\frac{d}{2}\right)^{2/3}\right)$$

$$= \frac{\pi^2}{24} \left(1 + 15 \left(\tan \beta\right)^{2/3}\right), 0 < \beta \leq \frac{\pi}{3}.$$

This concludes the proof of Theorem 1.3.

4 Proof of Theorem 1.4

Let $\hat{\beta}$ be a rhombus such that major and minor diagonals have lengths 2 and $d$, respectively (see Figure 1). We want to estimate the torsion and to this aim we use a test function

$$v(x, y) = \begin{cases} 
\frac{d^2 x^2}{4} - y^2, & 0 \leq x \leq 1, \\
\frac{d^2 (2 - x)^2}{4} - y^2, & 1 \leq x \leq 2.
\end{cases}$$
Figure 2: The rectangle shaded in grey is obtained by Steiner symmetrization. The Dirichlet Laplacian eigenvalue of the rectangle provides an estimate from below for the one on the rhombus.

In view of the variational definition of the torsion we have
\[
\frac{1}{T(\hat{\beta})} \leq \frac{\int_{\hat{\beta}} |Dv|^2}{\left( \int_{\hat{\beta}} v \right)^2} = \frac{24 + 18d^2}{d^3}.
\]

On the other hand we can estimate from below the first Dirichlet Laplacian eigenvalue of any rhombus by means of the Dirichlet Laplacian eigenvalue of a rectangle obtained by Steiner symmetrising the rhombus along a direction parallel to one of the sides (see Figure 2). We denote by \( b \) and \( h \) the base and the height of the rectangle, respectively. Since the base \( b \) coincides with the side of the rhombus we know that \( b^2 = 1 + \frac{d^2}{4} \), the height \( h = \frac{d}{\sqrt{1 + \frac{d^2}{4}}} \).

We have that
\[
\lambda(\hat{\beta}) \geq \frac{\pi^2}{b^2 + h^2} = \frac{\pi^2}{d^2(16 + 4d^2)}.
\]

Observing that the area of the rhombus is equal to \( d \), we have that
\[
\frac{\lambda(\hat{\beta})}{|\hat{\beta}|} \geq \frac{\pi^2}{24} \frac{16 + 24d^2 + d^4}{(1 + \frac{3}{4}d^2)(16 + 4d^2)} \geq \frac{\pi^2}{24}, \quad 0 \leq d \leq 2. \tag{4.1}
\]

5 Proof of Theorem 1.5

5.1 Proof for the case \( \beta \in (0, \pi/3] \cup [\beta_0, \pi/2) \)

Let \( \Delta_\beta \) be an isosceles triangle with angles \( \beta, \beta, \alpha = \pi - 2\beta \). We first consider the case \( \frac{\pi}{3} \leq \alpha < \pi \). We denote the height by \( d \), and we fix the length of the basis equal to 2. See Figure 3.

We use the function
\[
u(x, y) = \begin{cases} \frac{d^2x^2}{4} - \left( y - \frac{dx}{2} \right)^2, & 0 \leq x \leq 1, \\ \frac{d^2(2-x)^2}{4} - \left( y - \frac{d(2-x)}{2} \right)^2, & 1 \leq x \leq 2, \end{cases}
\]
as a test function for the torsion of $\triangle_\beta$. We find that
\[
\frac{2}{T(\triangle_\beta)} \leq \frac{48(1 + d^2)}{d^3}.
\] (5.1)

Hence
\[
\frac{T(\triangle_\beta)}{|\triangle_\beta|} \geq \frac{1}{24} \left(1 + \frac{1}{d^2}\right)^{-1}.
\] (5.2)

We wish to estimate $\lambda(\triangle_\beta)$ from below. To this aim we consider the first Dirichlet eigenfunction of $\triangle_\beta$ restricted to $x \in [0, 1]$ and we reflect it, anti-symmetrically, with respect to the line $y = dx$ (see Figure 4). This new function is a test function defined on the rectangle of sides 1, $d$ (shaded in grey in Figure 4) orthogonal to the first eigenfunction of the Laplacian with the mixed boundary conditions described in Figure 4.

For $\frac{\pi}{3} \leq \alpha \leq \pi$ we find that
\[
\lambda(\triangle_\beta) \geq \min \left\{ \pi^2 \left(1 + \frac{1}{d^2}\right), \frac{4\pi^2}{d^2} \right\} = \pi^2 \left(1 + \frac{1}{d^2}\right).
\] (5.3)

Combining [5.2] and [5.3] we obtain that
\[
\frac{T(\triangle_\beta)\lambda(\triangle_\beta)}{|\triangle_\beta|} \geq \frac{\pi^2}{24}, \quad 0 < \beta \leq \frac{\pi}{3}.
\]

Next we consider the case $0 < \alpha \leq \frac{\pi}{3}$ or $\pi/3 \leq \beta < \pi/2$. We have that
\[
|\triangle_\beta| = 1/ \tan(\alpha/2).
\] (5.4)

Let
\[
S(\rho, \alpha) = \{(r, \phi) : 0 < r < \rho, -\alpha/2 < \phi < \alpha/2\}
\]
be the circular sector with radius $\rho$ and opening angle $\alpha$. Siudeja’s Theorem 1.3 in [9] asserts that for $0 < \beta \leq \pi/3$, $\lambda(T(\triangle_{\pi/2-\alpha/2}) \geq \lambda(S(\rho, \alpha))$ where $d$ is such that $|T(\triangle_\beta)| = |S(\rho, \alpha)|$. It follows that

$$\rho^2 = 2/(\alpha \tan(\alpha/2)). \quad (5.5)$$

Hence

$$\lambda(\triangle_\beta) \geq 2^{-1} \alpha \tan(\alpha/2) j_{\pi/\alpha}^2,$$

where we have used that the first Dirichlet eigenvalue of a circular sector of opening angle $\beta$ and radius $\rho$ equals $j_{\pi/\beta}^2 \rho^{-2}$. See [8]. Moreover by (1.2) and (4.3) for $k = 1$ and $\nu = \pi/\alpha$ in [11] we have that,

$$j_{\pi/\alpha}^2 > \left( \frac{\pi}{\alpha} - \frac{a_1}{21/3} \left( \frac{\pi}{\alpha} \right)^{1/3} \right)^2, \quad -a_1 \geq \left( \frac{9\pi}{8} \right)^{2/3},$$

where $a_1$ is the first negative zero of the Airy function. It follows that

$$j_{\pi/\alpha}^2 \geq \frac{\pi^2}{\alpha^2} \left( 1 + C \left( \frac{\alpha}{\pi} \right)^{2/3} \right)^2 \geq \frac{\pi^2}{\alpha^2} \left( 1 + C_1 \alpha^{2/3} \right), \quad (5.6)$$

where

$$C = (9\pi/8)^{2/3} 2^{-1/3}, \quad C_1 = (9/4)^{2/3}. \quad (5.7)$$

The torsion function for $S(\rho, \alpha)$, $\alpha < \pi/2$, is given by (p.279 in [13]),

$$v_{S(\rho, \alpha)}(r, \phi) = \frac{r^2}{4} \left( \cos(2\phi) - 1 \right)$$

$$+ \frac{4\rho^2 \alpha^2}{\pi^3} \sum_{n=1,3,5,...} (-1)^{(n+1)/2} \left( \frac{r}{\rho} \right)^{n\pi/\alpha} \cos \left( \frac{n\pi\phi}{\alpha} \right) n^{-1} \left( n + \frac{2\alpha}{\pi} \right)^{-1} \left( n - \frac{2\alpha}{\pi} \right)^{-1}. \quad (5.8)$$

By monotonicity of the torsion we obtain that

$$T(\triangle_\beta) \geq T(S(\rho, \alpha))$$

$$= \int_{(0,d)} r \ dr \int_{(-\alpha/2,\alpha/2)} d\phi v_{S(d,\alpha)}(r, \phi)$$

$$= \frac{d^4}{16} \left( \tan \alpha - \alpha - \frac{128\alpha^4}{\pi^5} \sum_{n=1,3,...} n^{-2} \left( n + \frac{2\alpha}{\pi} \right)^{-2} \left( n - \frac{2\alpha}{\pi} \right)^{-1} \right), \quad (5.8)$$

We have that for $0 < \alpha \leq \pi/3$, $(n + 2\alpha/\pi)^2 (n - 2\alpha/\pi) \geq n^2$, $n \in \mathbb{N}$. This gives that

$$T(\triangle_\beta) \geq \frac{\rho^4}{16} \left( \tan \alpha - \alpha - \frac{2^{23} 31 \zeta(5) \alpha^4 d^4}{25 \pi^5} \right)$$

$$\geq \frac{\rho^4}{48} \left( 1 - C_2 \alpha \right), \quad (5.9)$$

where

$$C_2 = \frac{2^{23} 31 \zeta(5)}{5^2 \pi^5}.$$ 

By (5.6), (5.8), (5.9), and (5.4) we obtain that

$$\frac{T(\triangle_\beta) \lambda(\triangle_\beta)}{|\triangle_\beta|} \geq \frac{\rho^2}{24} \left( 1 - C_2 \alpha \right) \left( 1 + C_1 \alpha^{2/3} \right). \quad (5.10)$$

The right-hand side of (5.5) is greater or equal than $\frac{\rho^2}{24}$ for

$$C_1 \geq C_1 C_2 \alpha + C_2 \alpha^{1/3}. \quad (5.11)$$

Inequality (5.11) holds for all $\alpha \leq 33/100$. 


5.2 Computer validation for the case $\beta \in (\pi/3, \beta_0)$ via interval arithmetic.

We consider a triangle $\triangle^\alpha$ of height 1 and opening angle $\alpha$, where $\alpha = \pi - 2\beta$. Let

$$F(\alpha) = \frac{24}{\pi^2} \frac{\lambda(\triangle^\alpha) T(\triangle^\alpha)}{|\triangle^\alpha|}.$$  

We wish to show that $F(\alpha) > 1.01$ in the range $0.33 \leq \alpha \leq \pi/3$.

We present here a computer assisted proof of the result using Interval Arithmetic.

We once more use Siudeja’s lower bound, comparing with the sector having the same opening angle $\alpha$ of height 1 and opening angle $\beta$. Therefore

$$\lambda(\triangle^\alpha) \geq \tilde{\lambda}(\alpha) = \cos^2 \left( \frac{\alpha}{2} \right) \left( \frac{\alpha}{\sin \alpha} \right) \left( \frac{\pi}{\alpha} + C \left( \frac{\pi}{\alpha} \right)^{\frac{3}{2}} \right)^2,$$

where $C$ is given by (5.7).

The area is given by

$$|\triangle^\alpha| = \tan \left( \frac{\alpha}{2} \right),$$

(5.13)

The monotonicity of $T$ with respect to inclusion allows us to estimate from below using the torsion of a tangent sector with same opening angle $\alpha$. We use (5.8) and find that

$$T(\triangle^\alpha) \geq \frac{1}{16} (\tan \alpha - \alpha) - \frac{8}{\pi^5} \alpha^4 \sum_{n=1,3,5,...} n^{-2} \left( n + \frac{2\alpha}{\pi} \right)^{-2} \left( n - \frac{2\alpha}{\pi} \right)^{-1}.$$

In order to perform a numerical evaluation we truncate the series in the following way

$$\sum_{n=1,3,5,...} n^{-2} \left( n + \frac{2\alpha}{\pi} \right)^{-2} \left( n - \frac{2\alpha}{\pi} \right)^{-1} = \sum_{n=0}^\infty (2n+1)^{-2} \left( 2n + 1 + \frac{2\alpha}{\pi} \right)^{-2} \left( 2n + 1 - \frac{2\alpha}{\pi} \right)^{-1}$$

$$\leq \sum_{n=0}^N (2n+1)^{-2} \left( 2n + 1 + \frac{2\alpha}{\pi} \right)^{-2} \left( 2n + 1 - \frac{2\alpha}{\pi} \right)^{-1} + \frac{1}{2^5} \sum_{n=N+1}^\infty \frac{1}{n^4}$$

$$\leq \sum_{n=0}^N (2n+1)^{-2} \left( 2n + 1 + \frac{2\alpha}{\pi} \right)^{-2} \left( 2n + 1 - \frac{2\alpha}{\pi} \right)^{-1} + \frac{1}{2^7} N^4.$$ 

It follows that

$$T(\triangle^\alpha) \geq \tilde{T}(\triangle^\alpha) = \frac{1}{16} (\tan \alpha - \alpha) - \frac{8}{\pi^5} \alpha^4 \left( \sum_{n=0}^{10} (2n+1)^{-2} \left( 2n + 1 + \frac{2\alpha}{\pi} \right)^{-2} \left( 2n + 1 - \frac{2\alpha}{\pi} \right)^{-1} + \frac{1}{2^7} \cdot 10^4 \right).$$

Therefore

$$F(\alpha) \geq G(\alpha) = \frac{24}{\pi^2} \frac{\tilde{\lambda}(\triangle^\alpha) \tilde{T}(\triangle^\alpha)}{|\triangle^\alpha|}.$$

At this point we can prove that $G(\alpha) > 1.01$ for all values $0.33 \leq \alpha \leq \pi/3$ by using Interval Arithmetic. There are many softwares and libraries which can be employed for this purpose. We selected Octave\footnote{John W. Eaton, David Bateman, Sren Hauberg, Rik Wehbring (2018). GNU Octave version 4.4.1 manual: A high-level interactive language for numerical computations. URL https://www.gnu.org/software/octave/doc/v4.4.1/} (A free software that runs on GNU/Linux, macOS, BSD, and Windows) which provides a specific package called Interval\footnote{Oliver Heimlich, GNU Octave Interval Package, https://octave.sourceforge.io/interval/, version 3.2.0, 2018-07-01. The interval package is a collection of functions for interval arithmetic. It is developed at Octave Forge, a sibling of the GNU Octave project.}.
We covered the interval $[\frac{33}{100}, \pi]$ by a collection of 1001 intervals $I_n$ with $n = 0, \ldots, 1000$, so that

$$I_n = \left[\frac{33}{100} + \frac{(n-1)(\pi/3 - \frac{33}{100})}{1000}, \frac{33}{100} + \frac{(n+1)(\pi/3 - \frac{33}{100})}{1000}\right].$$

We observe that the intersection of consecutive intervals is intentionally non-empty. Using the Interval package, we designed a code that for $n$ going from 0 to $10^3$ provides upper and lower bounds for $G(I_n)$ in terms of floating point numbers. This is performed in an automated way by standard and reliable algorithms. We established that the inequality $F(\alpha) > 1.01$ holds true on the whole interval $[\frac{33}{100}, \pi]$ by verifying it on $I_n$ for all $n = 0, \ldots, 10^3$.

For completeness we include the code:

```plaintext
pkg load interval  # load the package Interval
output_precision (6)  # number of digits displayed
C=(9*pi/8)^(2./3) * 2^(-1./3);

function K=G(x)  # this is the definition of the function G(alpha)
    Sum=0;
    for n = 0:10
        Sum = Sum + (2*n+1)^(-2)*(2*n+1+2*x/pi)^(-2)*(2*n+1-2*x/pi)^(-1);
    endfor
    K=(24./pi^2)*(1./16*(tan(x)-x)-8*x^4/pi^5*(Sum+1./(2.7*10.4))*(cos(x/2))^2*x/sin(x))*(pi/x+((9*pi/8)^(2./3)*2^(-1./3))*(pi/x)^(1/3))^2)/tan(x/2);
endfunction

control="OK";  # the variable control is set to "OK"
N=1000;  # Number of intervals
Delta=(pi/3-0.33)/N;  # 2Delta is the size of each interval
for n = 0:N
    n  #print the value of n
    a=0.33+n*Delta;
    I=midrad(a,Delta)  # I = interval with center in a and radius Delta
    J=G(I)  # J is an interval which includes the image of I
    if(J>1.01)  # check that (min J) > 1.01
        "so_far_inequality_holds"  # tell that everything is working fine
    elseif
        control="failure"  # the variable control is set to "failure"
        break  # in case of failure the cycle breaks
    endif
endfor
```

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