INTRODUCTION.

This paper is a sequel to [12], where we laid the foundations to a program for studying mirror symmetry using logarithmic geometry. This program can be viewed as an algebro-geometric version of the Strominger-Yau-Zaslow (SYZ) program [35], and gives a way of passing via affine geometry to the two sides of the mirror symmetry picture.

We recall briefly the setup and goals of this program. We begin with $B$ an integral affine manifold with singularities: there is an open subset $B_0 \subseteq B$ with an atlas with transition maps in $\text{Aff}(\mathbb{Z}^n)$, and $\Delta := B \setminus B_0$ is codimension $\geq 2$ in $B$ (see [12], Def. 1.15). We also

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assume given a multi-valued convex potential function $K$ on $B$. As currently understood, one of the basic challenges in the SYZ picture of mirror symmetry is to associate a Kähler manifold $X$ to $(B, K)$. Mirror symmetry can then be accomplished by a Legendre transform: there is a Legendre transform $(\tilde{B}, \tilde{K})$ of the pair $(B, K)$, and one expects that the Kähler manifold $\tilde{X}$ associated to $(\tilde{B}, \tilde{K})$ is the mirror to $X$. One then wants to construct a dictionary between things one would like to compute on $X$, (such as Hodge numbers, periods, and Gromov-Witten invariants), and things one can compute on $B$. For example, one expects that in this picture, Gromov-Witten invariants on $X$ and periods on $\tilde{X}$ should both be expressible in terms of tropical curves on $B$.

Let us review very quickly the relationship between affine and complex manifolds. This has already been discussed in many places, see for example [27], [11], [26], [7]. There is a local system $\Lambda \subset T_{B_0}$ of integral constant vector fields with respect to the integral affine structure, and a torus bundle over $B_0$, $T_{B_0}/\Lambda$. This manifold carries a natural complex structure. One might furthermore consider a family of complex manifolds obtained by rescaling the lattice, $T_{B_0}/\epsilon \Lambda$, with $\epsilon \to 0$ corresponding to the so-called “large complex structure limit” of string theory. One would like to compactify $T_{B_0}/\epsilon \Lambda$ to a complex manifold fibering over $B$. While this can be done in many cases topologically, it is generally impossible to do so in the complex category, and one must deform the complex structure before a compactification can be performed. This problem was first considered directly by Fukaya [7]. An explicit description in two dimensions was found by Kontsevich and Soibelman by passing from the category of complex manifolds to the category of rigid analytic spaces. In [13], we used the program started in [12] to solve this problem in all dimensions, again in a slightly different category. It is worth noting that using the techniques in [12] and known deformation theory, one can in fact solve the two-dimensional problem quite easily, but without an explicit description.

We outline the program as begun in [12]. The basic idea is to discretize the problem. We consider toric polyhedral decompositions $\mathcal{P}$ of $B$ (see [12], Definition 1.22). Essentially this is a decomposition of $B$ into lattice polytopes, but there are some delicate conditions involving how these polytopes interact with the singular locus $\Delta$ of $B$. In particular, no vertex of $\mathcal{P}$ is contained in $\Delta$. By looking at $B$ and $\mathcal{P}$ in a neighbourhood of a vertex $v$ of $\mathcal{P}$, one obtains a rational polyhedral fan $\Sigma_v$, and hence a toric variety $X_v$. These toric varieties can then be glued together along toric strata using the combinatorics dictated by $\mathcal{P}$. This gluing can be modified by equivariant automorphisms of the strata, and so a gluing is specified by some additional data, which we called open gluing data in [12], Definition 2.25. A choice of open gluing data then specifies a scheme (or algebraic space) $X_0(B, \mathcal{P}, s)$, which is a union of toric varieties.

We then want to consider certain sorts of degenerations of Calabi-Yau varieties of the form $X \to S$ over a base scheme $S$, with some fibre isomorphic to $X_0(B, \mathcal{P}, s)$. The
basic idea then is to pass between \( B \) and genuine Calabi-Yau varieties by way of these degenerations.

There are two principal problems, in general, with such a degeneration approach. First, it is very important to note that this scheme \( X_0(B, \mathcal{P}, s) \) does not contain enough information by itself to carry out mirror symmetry. There may be many different smoothings of \( X_0(B, \mathcal{P}, s) \) to Calabi-Yau varieties, even with different Hodge numbers. Thus, at first glance, studying mirror symmetry via degenerations seems far-fetched.

The key innovation of [12] is the observation that one should consider \( X_0(B, \mathcal{P}, s) \) as a log scheme of Illusie-Fontaine and Kato (see [12], §3.1 and references therein for an introduction to log schemes as needed for this program). This extra structure is exactly what is needed to extract useful information from \( X_0(B, \mathcal{P}, s) \), such as moduli, Hodge numbers, and eventually, periods and Gromov-Witten invariants.

The second principal problem is that not all degenerations of Calabi-Yau varieties are degenerations to varieties of the type \( X_0(B, \mathcal{P}, s) \). There are many examples of degenerations which are, however. For example, all complete intersections in toric varieties have degenerations of this sort. More generally, one might conjecture that all large complex structure limit degenerations will be birational to a degeneration of the form discussed here. In any event, our belief is that the class of Calabi-Yau varieties for which our degeneration approach will be useful will turn out to be a far broader class than that of complete intersections in toric varieties. We leave this second problem for future work.

Returning to the program at hand, we recall it was shown in [12] that \( (B, \mathcal{P}) \) contains more information than just the scheme \( X_0(B, \mathcal{P}, s) \): it also says precisely how to put a log structure on \( X_0(B, \mathcal{P}, s) \), which we write as \( X_0(B, \mathcal{P}, s)^\dagger \), along with a suitable morphism \( X_0(B, \mathcal{P}, s)^\dagger \to \text{Spec} k^\dagger \log \) smooth away from a set \( Z \subset X_0(B, \mathcal{P}, s) \) which is codimension 2 and doesn’t contain any toric stratum of \( X_0(B, \mathcal{P}, s) \). Very roughly \( Z \) corresponds to \( \Delta \subset B \). (Here \( \text{Spec} k^\dagger \log \) denotes the standard log point over the field \( k \), which we always take to be an algebraically closed field of characteristic zero, and the daggers represent log schemes). Once we have done this, we obtain so-called log Calabi-Yau spaces which behave very much like non-singular Calabi-Yau varieties. Conversely, giving the log Calabi-Yau space structure on \( X_0(B, \mathcal{P}, s) \) gives enough information to reconstruct \( (B, \mathcal{P}) \); the scheme \( X_0(B, \mathcal{P}, s) \) by itself does not determine the affine structure on \( B \).

These log Calabi-Yau spaces can be viewed as playing an intermediate role between the affine manifold \( B \) and a complex manifold or non-singular variety. Without this extra log structure, there would be no way to extract useful information from \( X_0(B, \mathcal{P}, s) \) and much of the information contained in \( B \) would be lost.

A large part of [12] was devoted to classifying log Calabi-Yau spaces arising from \( (B, \mathcal{P}) \). With suitable hypotheses, namely that \( (B, \mathcal{P}) \) be simple (see [12], Definition 1.60), one finds the set of all log Calabi-Yau spaces arising from \( (B, \mathcal{P}) \) is \( H^1(B, i_*\Lambda \otimes k^\times) \), where
i : B_0 \hookrightarrow B$ is the inclusion. As we shall see in this paper, this moduli space has the expected dimension given by the first cohomology of the tangent bundle of a smoothing of $X_0(B, \mathcal{P}, s)^{\dagger}$ (if it exists).

More importantly, in [13], we show, with some more general hypotheses on $(B, \mathcal{P})$, that a log Calabi-Yau space $X_0(B, \mathcal{P}, s)^{\dagger} \rightarrow \text{Spec} k^{\dagger}$ can be put into a formal family $\mathcal{X}$ over $k[[t]]$. Assuming the family can be polarized, Grothendieck existence can be applied and one obtains a scheme $\mathcal{X}$ over $\text{Spec} k[[t]]$ which is a flat deformation of $X_0(B, \mathcal{P}, s)$. This accomplished one of the original aims of the program, allowing us to associate a scheme to $B$. Again, the log structure plays a vital role, providing the “initial conditions” to canonically determine the family $\mathcal{X}$.

Furthermore, in [12] we developed the discrete Legendre transform. Given a multi-valued strictly convex integral piecewise linear function $\varphi$ on $B$, one can construct from the triple $(B, \mathcal{P}, \varphi)$ the Legendre dual $(\tilde{B}, \tilde{\mathcal{P}}, \tilde{\varphi})$. This then should yield the mirror family. In [10], the first author checked that this agrees with Batyrev-Borisov duality.

Thus there remains the problem of building a dictionary between geometric objects on $B$ and geometric objects on $X_0(B, \mathcal{P}, s)$ and $\mathcal{X}$.

This paper starts this process. A great deal of the difficulty of this project is that the necessary theory of log structures has not been developed sufficiently in the literature for the types of log schemes that we have to deal with. As a result, both in [12] and this paper, some space has to be devoted to essentially foundational issues concerning log structures.

In this paper, the goal is two-fold: first, we wish to compute Dolbeault cohomology groups of log Calabi-Yau spaces and their smoothings. This will enable us to verify the usual exchange of Hodge numbers under mirror symmetry. Second, we wish to lay the groundwork for calculation of periods of the family of Calabi-Yaus $\mathcal{X} \rightarrow \text{Spec} k[[t]]$. This calculation will be carried out in a future paper, but many of the results in this paper will be needed. In fact, while these calculations are not visible, the results of this paper were essential for carrying out certain period calculations needed to make interesting new enumerative predictions of mirror symmetry which were stated and verified in the article [15].

To explain the results of this paper in further detail, we give an outline of the paper. In §1, we review the definition of logarithmic derivations and logarithmic differentials and give local calculations in a set-up suited for our needs. The only difficulty that arises here is the presence of the set $Z \subset X_0(B, \mathcal{P}, s)$ where the log structure breaks down. In logarithmic geometry parlance, the log structure fails to be coherent at these points. One consequence is that the sheaf of logarithmic differentials $\Omega_{X_0(B, \mathcal{P}, s)^{\dagger}}^{1}/k^{\dagger}$ fails to be coherent at $Z$. As a result, it turns out the proper sheaf to look at is $j_* \Omega_{X_0(B, \mathcal{P}, s)^{\dagger}}^{1}/k^{\dagger}$, where $j : X_0(B, \mathcal{P}, s) \setminus Z \hookrightarrow X_0(B, \mathcal{P}, s)$ is the inclusion. This is similar to the usual definition of Danilov differentials. §1 then is devoted to local calculations of these various sheaves.
§2 develops deformation theory for log Calabi-Yau spaces. Results currently available do not apply directly, see e.g. [20], again because of the lack of coherence. Unfortunately, we did not find a clean way of dealing with this problem. In general, there can be many unpleasant deformations of non-coherent log schemes. We restrict these deformations by only allowing deformations with a very specific sort of local model, and we call these divisorial deformations. We then prove that this deformation theory is controlled, as one would hope, by the sheaf of logarithmic derivations.

In a sense, §2 is a relic of our original hope, when we started this program, of proving the existence of smoothings of log Calabi-Yau spaces by proving a version of the Bogomolov-Tian-Todorov theorem. [36], [37], as had already been done for normal crossings Calabi-Yau varieties in [24]. However, there are technical problems in trying to prove such a result, and it resisted all our efforts. Instead, we constructed explicit smoothings in [13], and this is much more valuable anyway as it allows an explicit calculation of periods. It is important to know that those explicit smoothings are divisorial deformations, and to do so, we need the deformation theory results of §2. In this way we learn that the results of this paper, notably the basechange result of §4, apply to the smoothings constructed in [13]. Other than this point, the results of §2.2 are not needed elsewhere in the paper.

§3 is the heart of this paper. We investigate the Dolbeault cohomology groups

\[ H^q(X_0(B, \mathcal{P}, s), j_* \Omega^p_{X_0(B, \mathcal{P}, s)^\dagger / k^1}). \]

Our main results here follow from a lengthy explicit computation:

**Theorem 0.1.** Under a hypothesis slightly stronger than simplicity (see Theorem 3.21) we have

1. \[ H^q(X_0(B, \mathcal{P}, s), j_* \Omega^p_{X_0(B, \mathcal{P}, s)^\dagger / k^1}) \cong H^q(B, i_* \bigwedge^p \tilde{\Lambda} \otimes k), \]

   where \( \tilde{\Lambda} \) is the dual local system to \( \Lambda \).

2. (Hodge decomposition) The algebraic log de Rham cohomology groups satisfy the Hodge decomposition, i.e.

   \[ \mathbb{H}^r(X_0(B, \mathcal{P}, s), j_* \Omega^\bullet_{X_0(B, \mathcal{P}, s)^\dagger / k^1}) \cong \bigoplus_{p+q=r} H^q(B, i_* \bigwedge^p \tilde{\Lambda} \otimes k). \]

3. If in addition the holonomy of \( B_0 \subseteq B \) is contained in \( \mathbb{Z}^n \ltimes \text{SL}_n(\mathbb{Z}) \) rather than just \( \mathbb{Z}^n \ltimes \text{GL}_n(\mathbb{Z}) \), then

   \[ H^p(X_0(B, \mathcal{P}, s), \Theta_{X_0(B, \mathcal{P}, s)^\dagger / k^1}) \cong H^p(B, i_* \Lambda \otimes k). \]

By results of §2, under the milder assumption of simplicity, the tangent space of the log deformation functor is

\[ H^1(X_0(B, \mathcal{P}, s), \Theta_{X_0(B, \mathcal{P}, s)^\dagger / k^1}). \]
so this is \( H^1(B, i_* \Lambda \otimes \mathbb{k}) \). This fits with the description of the moduli space of log Calabi-Yau spaces with dual intersection complex \( B \), which is \( H^1(B, i_* \Lambda \otimes \mathbb{k}^\times) \).

The extra hypothesis, over and above simplicity, is important. Essentially, this hypothesis says that the mirror variety is non-singular, rather than an orbifold. As is well-known [1], to consider mirror pairs in dimension \( \geq 4 \), one needs to include orbifold singularities, as not all Gorenstein abelian quotient singularities in dimension \( \geq 4 \) have crepant resolutions. In the orbifold context, one needs to consider stringy Hodge numbers [3]. Calculation of stringy Hodge numbers would take us too far afield in this paper, and hence the extra hypothesis. However, stringy Hodge numbers are not necessary in dimension three, so the results of this paper are complete in this case. For exploration of the relationship between the calculations in this paper and stringy Hodge numbers in dimension 4, see the paper [32] of Ruddat.

This calculation of the log Dolbeault groups demonstrates that the mirror duality proposed in [12] in fact interchanges Hodge numbers. Indeed, consider a Legendre dual pair \((B, \mathcal{P}, \varphi)\) and \((\tilde{B}, \tilde{\mathcal{P}}, \tilde{\varphi})\). Then \( \Lambda^B \cong \tilde{\Lambda}^B \) (see [12], Proposition 1.50,(1)), where the superscripts denote which affine structure is being used to define the sheaf. Moreover, if \( \dim B = n \) and the holonomy of \( B \) is contained in \( \mathbb{Z}^n \rtimes \text{SL}_n(\mathbb{Z}) \), then \( \Lambda^q \Lambda^B \cong \Lambda^{n-q} \Lambda^B \) and hence \( \Lambda^q \Lambda^B \cong \Lambda^{n-q} \Lambda^B \). Thus the isomorphism \( H^p(B, i_* \Lambda^q \Lambda^B) \cong H^p(\tilde{B}, i_* \Lambda^{n-q} \Lambda^B) \) gives the usual exchange of Hodge numbers on the level of log Calabi-Yau spaces.

To relate these to the usual Hodge numbers of a smoothing, in §4 we prove a base-change theorem, which tells us that the log de Rham groups of \( X_0(B, \mathcal{P}, s) \) coincide with the ordinary algebraic de Rham groups of a smoothing. Again, with the extra hypothesis, the same holds for the Dolbeault groups. This demonstrates we have defined the log Dolbeault groups correctly, and also demonstrates that when one has non-singular Calabi-Yau varieties on both sides of the picture, the Hodge numbers are exchanged, as expected. This implies, for example, by the results of [10], the usual interchange of Hodge numbers in the context of Batyrev-Borisov duality [2]. In particular, this gives a new way of computing Hodge numbers in this case; it is not at all clear from a combinatorial viewpoint why these computations should give the same answer, but of course they must.

§5 calculates the Gauss-Manin connection in two different contexts. In §5.1, we consider the following situation. If \( \mathcal{X} \to \text{Spec} \mathbb{k}[\![t]\!] \) is a one-parameter deformation of \( X_0(B, \mathcal{P}, s)^\dagger \to \text{Spec} \mathbb{k}^\dagger \), then one would ideally like to calculate the Gauss-Manin connection on this family. Once this is done, one can write a family of holomorphic \( n \)-forms in terms of flat sections of the Gauss-Manin connection; this describes the periods of the holomorphic \( n \)-forms which
yields the $B$-model predictions of mirror symmetry. This must wait for further work, but one can easily calculate the monodromy of the system of flat sections of the Gauss-Manin connection as the exponential of the residue of the connection. If $\nabla$ is the Gauss-Manin connection, this residue is

$$\text{Res}_0(\nabla) = \nabla_{t\partial/\partial t}\big|_{t=0} : \mathbb{H}^r(X_0(B, \mathcal{P}, s), j_*\Omega^\bullet_{X_0(B, \mathcal{P}, s)^1/k^1}) \to \mathbb{H}^r(X_0(B, \mathcal{P}, s), j_*\Omega^\bullet_{X_0(B, \mathcal{P}, s)^1/k^1}).$$

In §5.1 this residue is calculated explicitly. It is given on the level of the summands of the Hodge decomposition by, for $r = p + q$, the map

$$H^q(B, i_*\bigwedge^p \tilde{\Lambda}) \to H^{q+1}(B, i_*\bigwedge^{p-1} \tilde{\Lambda})$$

$$\alpha \mapsto c_B \cup \alpha$$

where $c_B \in H^1(B, i_*\Lambda)$ is the radiance obstruction of $B$, an invariant of affine manifolds introduced in [8]. This description of monodromy coincides with that given in [9] for the SYZ picture.

In §5.2, we consider a universal family of log Calabi-Yau spaces over the moduli space $S = H^1(B, i_*\Lambda \otimes \mathbb{G}_m)$, and consider the Gauss-Manin connection of this family. In this way, we obtain a variation of mixed Hodge structures over $S$. This material is largely included for future applications to $B$-model calculations of enumerative predictions.

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**Notational summary**

By the very nature of the subject this paper involves a considerable amount of notation, which to a large part has been introduced in [12]. Although we must assume the reader has some familiarity with [12], we will give references to notation in [12] whenever it is first used in this paper. In addition, here we will survey the basic ideas and notation of [12]. While it would take up too much space to be completely self-contained, this section attempts to make this paper more accessible.

The setup for the bulk of this paper is as follows. We assume given an $n$-dimensional toric log Calabi-Yau space $X_0^\dagger := (X_0, \mathcal{M}_{X_0})$ over an algebraically closed field $k$ of characteristic 0, as defined in [12], Definition 4.3. One should think of $X_0$ as the central fibre of what we called a toric degeneration of Calabi-Yau varieties ([12], Definition 4.1). This is a Calabi-Yau variety $\mathcal{X}$ over a discrete valuation ring with closed fibre a union of toric varieties that mutually intersect in toric strata, and $\mathcal{M}_{X_0}$ is then the log structure defined by the embedding $X_0 \subseteq \mathcal{X}$. One of the central results of [12] (Theorem 5.4) shows that, under suitable hypotheses, toric log Calabi-Yau spaces are equivalent to discrete data $(B, \mathcal{P})$ and a cohomology class $s \in H^1(B, i_*\Lambda \otimes \mathbb{G}_m)$ called lifted gluing data. Here $B$ is an integral
affine manifold of real dimension \( n \), with singularities along a real codimension two subset \( \Delta \), together with a polyhedral decomposition \( \mathcal{P} \) consisting of integral lattice polytopes. Moreover, \( i : B \setminus \Delta \to B \) is the inclusion, \( \Lambda \) is the sheaf of integral tangent vectors on \( B \setminus \Delta \), and \( \check{\Lambda} \) is the dual local system.

As a rule, general elements of \( \mathcal{P} \) are denoted \( \tau \), while \( \omega \) usually denotes edges, \( \rho \) cells of codimension one and \( \sigma \) maximal cells. Vertices of \( \mathcal{P} \) are written \( v \) or \( w \). In \cite{12}, we allowed elements of \( \mathcal{P} \) to have self-intersections: For example, if \( B = \mathbb{R}/\mathbb{Z} \), we can take \( \mathcal{P} = \{ \{ 0 \}, B \} \), i.e., we view \( B \) as a line segment with opposite ends identified. Because of this decision, in order to be clear about how elements of \( \mathcal{P} \) attach, we viewed \( \mathcal{P} \) as a category, with morphisms \( \omega \to \tau \) indicating a choice of inclusion of a face \( \omega \) in the "normalization" of \( \tau \). To simplify notation, we shall assume here that cells do not self-intersect, the general case being straightforward. Most of the time, this will not make a difference, and we will write \( \omega \to \tau \) and \( \omega \subseteq \tau \) interchangeably, writing the latter when it makes notation cleaner.

The integral affine structure on the underlying topological space \( B \) is defined by the lattice polytopes together with the structure of a fan \( \Sigma_v \) at each vertex \( v \). In particular, \( \Delta \) is disjoint from vertices and the interiors of maximal cells of \( \mathcal{P} \). The toric variety \( X_v \) associated to \( \Sigma_v \) is one of the irreducible components of \( X_0 \). A higher dimensional cell \( \tau \in \mathcal{P} \) labels a lower dimensional toric stratum \( X_\tau = \bigcap_{v \in \tau} X_v \subseteq X_0 \), and \( q_\tau : X_\tau \to X_0 \) denotes the inclusion. Then \( X_0 \) is the categorical limit of the strata with attaching maps

\[
F_{\tau_1, \tau_2} : X_{\tau_2} \longrightarrow X_{\tau_1},
\]

for any two cells \( \tau_1 \subseteq \tau_2 \).

The meaning of the description of the cells of \( \mathcal{P} \) as lattice polytopes is that they define the discrete data of the log structure, given by \( \overline{\mathcal{M}}_{X_0} = \mathcal{M}_{X_0}/\mathcal{O}_{X_0}^\times \), along with a section \( \rho \) for the morphism to the standard log point. The cohomology class \( s \) takes care of the moduli of gluings of the irreducible components and the choice of log structure with given \( \overline{\mathcal{M}}_{X_0} \). In particular, the attaching maps \( F_{\tau_1, \tau_2} \) depend on \( s \), and hence were denoted \( F_{S,s}(\tau_1 \to \tau_2) \) in \cite{12}. In this paper we suppress \( s \) and the base scheme \( S \), which is \( \text{Spec} \, k \) except in §5, to simplify the notation.

Local monodromy of the affine structure around \( \Delta \) leads to singularities of the log structure on \( X_0 \), and this correspondence is quite important throughout. The local monodromy of \( \Lambda \) is completely determined by pairs \( \omega \subseteq \rho \) where \( \omega \) is an edge of the codimension one face \( \rho \). This data determines a closed loop passing from one vertex \( v^+ \) of \( \omega \) into one of the two maximal cells separated by \( \rho \) and back to \( v^+ \) via \( v^- \) and the other maximal cell bounding \( \rho \). Parallel transport along this loop leads to monodromy in \( \Lambda \) of the form

\[
m \longrightarrow m + \kappa_{\omega \rho}(d_{\rho}, m)d_{\omega}, \quad m \in \Lambda_{v^+},
\]
with \( \kappa_{\omega \rho} \in \mathbb{Z} \), \( d_\omega \) a primitive tangent vector of \( \omega \) and \( \tilde{\alpha}_\rho \in \Lambda_{\nu^+} \) a primitive linear form vanishing on tangent vectors of \( \rho \), see §1.5 in [12], where the symbol \( n_{\omega\rho} = \kappa_{\omega \rho} \tilde{\alpha}_\rho \) was used. Signs can be fixed in such a way that cases arising from actual degenerations fulfill \( \kappa_{\omega \rho} \geq 0 \). These are called positive ([12], Definition 1.54), and positivity of \((B, \mathcal{P})\) is an assumption throughout. More generally, if \( \omega \in \mathcal{P} \) is an edge and \( \sigma^\pm \) are two maximal cells containing \( \omega \), monodromy around a loop starting at a vertex \( v^+ \) of \( \omega \) into \( \sigma^+ \) to \( v^- \) and then via \( \sigma^- \) back to \( v^+ \) can be written as

\[
m \mapsto m + \langle n_{\omega \rho}^{\sigma^+ \sigma^-}, m \rangle d_\omega, \quad m \in \Lambda_{\nu^+},
\]

for some \( n_{\omega \rho}^{\sigma^+ \sigma^-} \in \Lambda_{\nu^+} \). Dually, if \( \rho \in \mathcal{P} \) is a codimension one cell contained in two maximal cells \( \sigma^\pm \), and \( v^\pm \) are two vertices of \( \rho \), then monodromy around a loop starting at \( v^+ \) into \( \sigma^+ \) to \( v^- \) and then via \( \sigma^- \) back to \( v^+ \) can be written as

\[
m \mapsto m + \langle \tilde{\alpha}_\rho, m \rangle m_{\nu^+,v^-}^\sigma, \quad m \in \Lambda_{\nu^+},
\]

for some \( m_{\nu^+,v^-}^\sigma \in \Lambda_{\nu^+} \).

We also assume a kind of indecomposability of the local monodromy, formalized in the concept of simplicity (see [12], Definition 1.60 for the full definition). A necessary condition is \( \kappa_{\omega \rho} \in \{0,1\} \) for all \( \omega, \rho \). Under this latter condition, local monodromy is determined completely combinatorially by the pairs \((\omega, \rho)\) with \( \kappa_{\omega \rho} = 1 \). In any case, for each \( \tau \in \mathcal{P} \) simplicity allows one to capture the local monodromy information around some \( \tau \in \mathcal{P} \) in terms of two collections of subsets

\[
\Omega_1, \ldots, \Omega_p \subseteq \{ \omega \subseteq \tau \mid \dim \omega = 1 \}, \quad R_1, \ldots, R_p \subseteq \{ \rho \supseteq \tau \mid \dim \rho = n - 1 \},
\]

of edges and facets with the same behaviour with respect to local monodromy satisfying several properties, including \( \kappa_{\omega \rho} = 1 \) if and only if there exists an \( i \) such that \( \omega \in \Omega_i \), \( \rho \in R_i \). There are then corresponding monodromy polytopes, well-defined after fixing a vertex \( v \in \tau \) and a maximal cell \( \sigma \) containing \( \tau \) and any \( \omega_i \in \Omega_i \), \( \rho_i \in R_i \):

\[
\Delta_i = \text{Conv} \left\{ m_{v'v_i}^{\rho_i} \mid v' \in \tau \right\} \subseteq \Lambda_{\tau,\mathbb{R}},
\]

\[
\tilde{\Delta}_i = \text{Conv} \left\{ n_{\omega_i \sigma'}^{\sigma} \mid \tau \subseteq \sigma' \in \mathcal{P}_{\max} \right\} \subseteq \tilde{\Lambda}_{\tau,\mathbb{R}}.
\]

In a sense, \( \Delta_i \) captures the \( i \)-th part of “inner monodromy” of \( \tau \), fixing \( \rho_i \) containing \( \tau \) and letting \( \omega \subseteq \tau \) vary, while \( \tilde{\Delta}_i \) captures the corresponding outer part. The partitioning into \( p \) parts is motivated by the study of the complete intersection case, see [10]. Simplicity is then defined by requiring the convex hulls of \( \bigcup_{i=1}^p \Delta_i \times \{e_i\} \) and of \( \bigcup_{i=1}^p \tilde{\Delta}_i \times \{e_i\} \) to be elementary simplices.

The monodromy polytopes are related to singularities of the log structure as follows. The charts for the log structure from [12] use the affine cover of \( X_0 \) given by the sets

\[
V(\sigma) = X_0 \setminus \bigcup_{\tau \cap \sigma = \emptyset} X_\tau,
\]
for $\sigma \in \mathcal{P}$ maximal cells. Now $V(\sigma)$ is canonically the boundary divisor of the affine toric variety $\text{Spec} \, k[\mathbf{P}]$, where $\mathbf{P}$ is the monoid of integral points of $C(\sigma)^\vee$, the dual of the cone generated by $\sigma \times \{1\}$ in $\Lambda_{\mathbb{R}} \oplus \mathbb{R}$, see Construction 2.15 in [12]. This embedding suggests a log structure on $V(\sigma)$ over the standard log point $\text{Spec} \, k^{\dagger}$, but these log structures are not locally isomorphic unless the local monodromy is trivial. The class $s \in H^1(B, \mathbb{i}_* \Lambda \otimes \mathbb{Z} \mathbb{G}_m)$ provides the necessary local changes to the standard log structures on $V(\sigma)$ to allow a consistent gluing. Explicitly, if $\omega \subseteq \sigma$ is an edge of integral length $e$ and endpoints $v^\pm$, then a chart for the standard log structure in a neighbourhood of the big cell of the codimension one stratum $X_\omega = X_{v^+} \cap X_{v^-}$ (an irreducible component of $(X_0)^{\text{sing}}$) is given by an equation $xy = t^e$. Here $t$ is the deformation parameter and $x, y$ monomials vanishing along $X_\omega \cap V(\sigma)$. The log structure modified by $s$ reads

$$xy = f(w)t^e$$

for some function $f$ depending only on monomials $w$ not vanishing identically along $X_\omega$. Along the zero locus of $f$ the induced log structure is singular (not finite). The global meaning of $f$ is as a section of a coherent sheaf $\mathcal{LS}_{\text{pre},X_0}^{+}$ on $(X_0)^{\text{sing}}$. The zero locus of this section is the $(n - 2)$-dimensional singular locus $Z \subseteq (X_0)^{\text{sing}}$ of the log structure of $X_0^{\dagger}$, see Theorems 3.22 and 3.27 in [12].

The meaning of the monodromy polytopes with respect to $Z$ is that $p$ is the number of irreducible components $Z_1, \ldots, Z_p$ of $Z \cap X_\tau$, the log structure locally along $Z_i$ is determined by $\Delta_i$, and $\hat{\Delta}_i$ is the Newton polytope of $Z_i$ on the big cell of $X_\tau$. Moreover, if the convex hull of $\bigcup_i \Delta_i \times \{e_i\}$ is an elementary simplex there is still a toric model defining a chart for the log structure locally along $Z$. This is shown in §2.1 of the present paper. We end this outline of the results of [12] with a list of relevant notation for easy reference.

**Relevant standard notation from [12]**

- $M, M_{\mathbb{R}}$: free abelian group of rank $n$, $M_{\mathbb{R}} = M \otimes \mathbb{Z} \mathbb{R}$
- $N, N_{\mathbb{R}}$: the dual groups $N = \text{Hom}(M, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes \mathbb{Z} \mathbb{R}$
- $B$: $n$-dimensional integral affine manifold with singularities
- $\Delta \subseteq B$, $i : B \setminus \Delta \to B$: discriminant locus and the inclusion of its complement
- $\Lambda_{\mathbb{R}}, \Lambda$: local system of flat (integral) tangent vector fields
- $\hat{\Lambda}_{\mathbb{R}}, \hat{\Lambda}$: duals to $\Lambda_{\mathbb{R}}$ and $\Lambda$
- $\mathcal{A}ff(B, \mathbb{R}), \mathcal{A}ff(B, \mathbb{Z})$: sheaf of continuous (integral) affine functions on $B \setminus \Delta$
- $\mathcal{P}$: polyhedral decomposition of $B$
- $v, w$: vertices of $\mathcal{P}$
- $\rho, \sigma, \tau, \omega$: cells of $\mathcal{P}$
- $\text{Int} \, \tau$: relative interior of $\tau \in \mathcal{P}$
- $\Lambda_{\tau, \mathbb{R}}, \Lambda_{\tau}$: space of (integral) tangent vector fields on $\tau \in \mathcal{P}$
- $\Sigma_v$: fan in $\Lambda_{\tau,v}$ induced from $\mathcal{P}$
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for $\tau \in \mathcal{P}$ generalizes to the fan $\Sigma_\tau$ in $\Lambda_{\tau, \mathbb{R}}$

$\Sigma_\tau$

normal fan of $\tau$ in $\Lambda_{\tau, \mathbb{R}}$

d$\omega$

for $\dim \omega = 1$, generator of $\Lambda_{\omega}$

d$\rho$

for $\dim \rho = n - 1$, generator of $\Lambda_{\rho}^\perp \subset \Lambda_{\rho}$

$W_{\tau_1 \rightarrow \tau_2}$

open star in the barycentric subdivision of $\mathcal{P}$ of the interior

of the edge connecting the barycenters of $\tau_1$ and $\tau_2$

$\kappa_{\omega, \rho}$

integer determining monodromy around loop given by $\omega \subseteq \rho$

$\Omega_i$, $\Delta_i$

dual data to $\Omega_i$, $\Delta_i$

$K\tau$  

for given $\tau \in \mathcal{P}$, set of edges and the associated monodromy polytope

$R_i$, $\check{\Delta}_i$

dual data to $\Omega_i$, $\Delta_i$

$\mathcal{C}(\tau)$

cone over $\tau \times \{1\}$ in $\Lambda_{\tau, \mathbb{R}} \oplus \mathbb{R}$ for $\tau \in \mathcal{P}$

$s$

$\in H^1(B, i^* \Lambda \otimes \mathbb{G}_m)$; represents moduli of gluing

of the strata and of the log structure

$X^\dagger_0 = (X_0, \mathcal{M}_{X_0})$

the toric log Calabi-Yau space associated to $(B, \mathcal{P})$

and $s \in H^1(B, i^* \Lambda \otimes \mathbb{G}_m)$.

$Z \subset (X_0)_{\text{sing}}$

singular locus of the log structure; $\dim Z = n - 2$

$\overline{\mathcal{M}}_{X_0}$

$\mathcal{M}_{X_0}/\mathcal{O}_{X_0}$, the discrete part of the log structure

$\text{Spec } \mathbb{k}^\dagger$

the standard log point $(\text{Spec } \mathbb{k}, \mathbb{k}^\times \times \mathbb{N})$

$q_\tau : X_\tau \rightarrow X_0$

inclusion of the toric stratum of $X_0$ isomorphic to $X(\Sigma_\tau)$

$E_{\tau_1, \tau_2}$

for $\tau_1 \subseteq \tau_2$, the attaching map $X_{\tau_2} \rightarrow X_{\tau_1}$

$V(\tau)$

canonical open affine neighbourhood in $X_0$ of the big cell of $X_\tau$

$V_{\tau_1 \rightarrow \tau_2}$

toric stratum of $V(\tau_2)$ corresponding to $\tau_1$, equal to $X_{\tau_1} \cap V(\tau_2)$

$\rho$

section of $\overline{\mathcal{M}}_{X_0}$ defining the morphism to $\text{Spec } \mathbb{k}^\dagger$ locally.

1. Derivations and differentials

Let $\pi : X^\dagger = (X, \mathcal{M}_X) \rightarrow S^\dagger = (S, \mathcal{M}_S)$ be a morphism of logarithmic spaces. Here $\mathcal{M}_X$ is a sheaf of monoids on $X$ and the dagger is always used to denote logarithmic spaces. See [12], §3.1 for an introduction to log schemes as needed here, and for further references.

Definition 1.1. A *log derivation* on $X^\dagger$ over $S^\dagger$ with values in an $\mathcal{O}_X$-modules $\mathcal{E}$ is a pair $(D, D\log)$, where $D : \mathcal{O}_X \rightarrow \mathcal{E}$ is an ordinary derivation of $X/S$ and $D\log : \mathcal{M}_X^{\mathbb{G}_P} \rightarrow \mathcal{E}$ is a homomorphism of abelian sheaves with $D\log \circ \pi^\# = 0$; these fulfill the following compatibility condition

$$D(\alpha_X(m)) = \alpha_X(m) \cdot D\log(m),$$

for all $m \in \mathcal{M}_X$, where $\alpha_X : \mathcal{M}_X \rightarrow \mathcal{O}_X$ is the log structure.

We denote by $\Theta_{X^\dagger/S^\dagger}$ the sheaf of log derivations of $X^\dagger$ over $S^\dagger$ with values in $\mathcal{O}_X$. □

Remark 1.2. Suppose $X$ has no embedded components, $\mathcal{M}_X$ has no sections with support contained in a codimension two subset, and the $\mathcal{O}_X$-module $\mathcal{E}$ is $S_2$, so that sections extend across codimension two subsets. Then if $Z \subseteq X$ is a codimension $\geq 2$ closed subset of
X, the module of log derivations on $X^\dagger$ with values in $\mathcal{E}$ is the same as the module of log derivations on $X^\dagger \setminus Z$ with values in $\mathcal{E}$. We will use this fact freely in what follows.

In many cases a log derivation $(D, D\text{log})$ is already determined by $D$.

**Proposition 1.3.** Assume that $\mathcal{M}_X = \pi^*(\mathcal{M}_S)$ holds on an open, dense subset $U \subset X$, and that $\mathcal{E}$ has no sections with support in $X \setminus U$. Then the forgetful map

$$(D, D\text{log}) \mapsto D$$

from the sheaf of log derivations on $X^\dagger / S^\dagger$ with values in $\mathcal{E}$ to the sheaf of usual derivations on $X / S$ with values in $\mathcal{E}$ is injective.

**Proof.** On $U$ each $m \in \mathcal{M}_X$ may be written as $h \cdot \pi^#(n)$ for $h \in \mathcal{O}_X^\times$ and $n \in \mathcal{M}_S$. Hence $D\text{log}(m)$ is determined by $D$ via Equation (1.1). Thus if $D = 0$ then $D\text{log}|_U = 0$, which under the assumption on $\mathcal{E}$ implies $D\text{log} = 0$. □

We may thus often think of log derivations as usual derivations with certain vanishing behaviour determined by the log structure:

**Example 1.4.** Let $Y$ be a normal integral scheme over $k$, and $X \subseteq Y$ a reduced Weil divisor. Endow $Y$ with the divisorial log structure $\mathcal{M}_{(Y,X)} := j_*\mathcal{O}_{Y \setminus X}^\times \cap \mathcal{O}_Y$, where $j : Y \setminus X \hookrightarrow Y$ is the inclusion. Then $\Theta_{Y/\kappa}$ consists of the usual derivations of $Y$ which preserve the ideal of $X$. Indeed, if $D$ is a log derivation and $f \in \mathcal{I}_{X/Y}$, then at the generic point $\eta$ of an irreducible component of $X$, we can write $f = f' \cdot t^p$ for $t$ a generator of $\mathcal{I}_{X/Y}$ at $\eta$, $p > 0$, and $f'$ a regular function. Then $t$ defines an element of $\mathcal{M}_{(Y,X)}$ in a neighbourhood of $\eta$, so $Df = t^pDf' + pf't^{p-1}Dt = t^p(Df' + pf'D\text{log} t)$ is in $\mathcal{I}_{X/Y}$ in a neighbourhood of $\eta$. Thus $Df$ vanishes along every component of $Y$, so is in $\mathcal{I}_{X/Y}$.

Conversely if $D$ is an ordinary derivation preserving $\mathcal{I}_{X/Y}$, then for $f \in \mathcal{M}_{(Y,X)}$, we define $D\text{log} f$ as $\frac{Df}{f}$; that this is a regular function is immediately checked again as above at the generic points of $\eta$. □

The sheaf of log derivations is well-understood for log-smooth morphisms of fine log schemes. However, the type of log schemes that arise in [12] are not in general fine: they fail to be coherent in the sense of Ogus [30], i.e. there may not be local charts of the form $P \rightarrow \mathcal{M}_X$ with $P$ a finitely generated monoid. However, the examples we wish to consider are relatively coherent [30]. We will not use the full formalism of [30] here as all the cases we need fall into a narrow class of examples, which we shall now introduce.

Let $P$ be a toric monoid, $F \subseteq P$ a face. Set $Y = \text{Spec}\ k[P]$, and let $j : U \hookrightarrow Y$ be the largest open subset where $z^p$ is invertible for all $p \in F$. Let $X := Y \setminus U$. There are two natural log structures on $Y$. The first is given by $\mathcal{M}_Y = \mathcal{M}_{(Y,X)} = j_*(\mathcal{O}_U^\times) \cap \mathcal{O}_Y$. The other one is induced by the chart $P \rightarrow \mathcal{K}[P]$, which is a fine log structure, which we
write as $\mathcal{M}_Y$. Note this can also be described as $\mathcal{M}_Y = j_*(\mathcal{O}_U^\times) \cap \mathcal{O}_Y$, where $\tilde{j} : \tilde{U} \to Y$ is the inclusion of the big torus orbit $\tilde{U}$ of $Y$. There is an obvious inclusion $\mathcal{M}_Y \subseteq \mathcal{M}_Y$, which is relatively coherent in the language of [30]. Such log structures still have good properties. We write $Y^\dagger$ and $\tilde{Y}^\dagger$ for the two log structures respectively. See Example 1.11 for a standard example.

Now let $M = \mathbb{Z}^n$, $M_\mathbb{R} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and suppose $P$ is given by $\sigma^\vee \cap N$ for a strictly convex rational polyhedral cone $\sigma$ in $M_\mathbb{R}$. Write $X = Y \setminus U = \bigcup_{i=1}^s X_i$, where the $X_i$’s are the toric divisors of $Y$ contained in $X$. Let

$$D = \bigcup_{j=1}^t D_j$$

be the union of toric divisors of $Y$ not contained in $X$. We take primitive generators of extremal rays of $\sigma$ to be $v_1, \ldots, v_{s+t}$, with $v_1, \ldots, v_s$ corresponding to $X_1, \ldots, X_s$, and $v_{s+1}, \ldots, v_{s+t}$ corresponding to $D_1, \ldots, D_t$. For ease of notation, we sometimes write

$$w_j = v_{s+j} \text{ for } 1 \leq j \leq t.$$

Let $P_1, \ldots, P_s$, $Q_1, \ldots, Q_t$ be the facets (maximal proper faces) of $P$ corresponding to $v_1, \ldots, v_s$, $w_1, \ldots, w_t$ respectively. Note that $X_i = \text{Spec}_k [P_i]$, the $P_i$’s are the facets of $P$ not containing $F$, and the face $F \subseteq P$ is given by

$$F = \langle w_1, \ldots, w_t \rangle^\perp \cap P.$$

**Proposition 1.5.** In the above situation, $\Gamma(Y, \Theta_{Y^\dagger/k})$ splits into $P_{\text{gp}}$-homogeneous pieces

$$\bigoplus_{p \in P_{\text{gp}}} z^p(\Theta_{Y^\dagger/k})_p,$$

where

$$(\Theta_{Y^\dagger/k})_p = \begin{cases} M \otimes_{\mathbb{Z}} k & \text{if } p \in P, \\ \mathbb{Z}v_i \otimes_{\mathbb{Z}} k & \text{if there exists an } i, s + 1 \leq i \leq s + t, \\ 0 & \text{with } \langle v_i, p \rangle = -1, \langle v_j, p \rangle \geq 0 \text{ for } j \neq i, \\ & \text{otherwise.} \end{cases}$$

We write an element $m \in (\Theta_{Y^\dagger/k})_p$ as $\partial_m$, and $z^p \partial_m$ acts on the monomial $z^q$ by

$$z^p \partial_m z^q = \langle m, q \rangle z^{p+q}.$$ 

**Remark 1.6.** Note that in the case that $F = P$, we have $\Theta_{Y^\dagger/k} = M \otimes_{\mathbb{Z}} \mathcal{O}_Y$. This is the standard case where $X$ is the toric boundary of $Y$, in which case $Y^\dagger$ is log smooth over $k$. Otherwise, $\Theta_{Y^\dagger/k}$ may not be locally free.
Proof of Proposition 1.5. We first identify ordinary derivations on $Y$ (which is the special case $F = \{0\}$). As any derivation on $Y$ restricts to a derivation on the big torus orbit of $Y$, 

$$
\Gamma(Y, \Theta_{Y/k}) \subseteq \bigoplus_{p \in P^P} z^p(M \otimes_\mathbb{k} k),
$$

where $m \in M \otimes_\mathbb{k} k$ corresponds to the derivation $\partial_m$. Furthermore, the torus $\text{Spec} \mathbb{k}[P^P]$ acts on $Y$, $\Theta_{Y/k}$, and $\Theta_{Y^\dagger/k}$, so $\Gamma(Y, \Theta_{Y/k})$ and $\Gamma(Y, \Theta_{Y^\dagger/k})$ both decompose into $P^P$-homogeneous pieces. Thus we need to determine for each $p \in P^P$ for which $m \in M \otimes \mathbb{k}$ is $z^p \partial_m$ a derivation on $Y$, i.e. when is $z^p \partial_m z^q \in \mathbb{k}[P]$ for all $q \in P$. But $\langle m, q \rangle z^{p+q}$ is regular if and only if $p + q \in P$ or $\langle m, q \rangle = 0$.

So first suppose $z^p \partial_m$ is an ordinary derivation on $Y$. There are two cases. Case 1. $m^\perp$ does not contain a facet of $P$. In this case, for each $i$ we can find a $q \in P$ such that $\langle v_i, q \rangle = 0$, $\langle m, q \rangle \neq 0$. Then $p + q \in P$, so $0 \leq \langle v_i, p + q \rangle = \langle v_i, p \rangle$. Thus $p \in P$. Case 2. $m$ is proportional to $v_i$ for some $i$. Then for each $j \neq i$, the same argument shows $\langle v_j, p \rangle \geq 0$, while if $q \in P$ with $\langle v_i, q \rangle = 1$, then $0 \leq \langle v_i, p + q \rangle = \langle v_i, p \rangle + 1$.

Reversing this argument, we see that 

$$
\Gamma(Y, \Theta_{Y/k}) = \bigoplus_{p \in P^P} z^p(\Theta_{Y/k})_p
$$

where 

$$(\Theta_{Y/k})_p = \begin{cases} 
M \otimes_\mathbb{k} k & \text{if } p \in P, \\
\mathbb{Z} v_i \otimes_\mathbb{k} k & \text{if there exists an } i \text{ with } \langle v_i, p \rangle = -1, \langle v_j, p \rangle \geq 0 \text{ for } j \neq i, \\
0 & \text{otherwise}.
\end{cases}
$$

By Proposition 1.3, $(\Theta_{Y^\dagger/k})_p \subseteq (\Theta_{Y/k})_p$. So now let’s consider log derivations. The ideal of $X$ (with the reduced induced scheme structure) is generated by $P \setminus (P_1 \cup \cdots \cup P_s)$. Let $z^p \partial_m \in \Gamma(Y, \Theta_{Y/k})$. Then $z^p \partial_m$ certainly preserves the ideal if $p \in P$. On the other hand, if $p \not\in P$, then we can take $m = v_i$ for some $i$ with $\langle v_i, p \rangle = -1$. For any given $i$, we can find a $q \in P \setminus (P_1 \cup \cdots \cup P_s)$ with $\langle v_i, q \rangle = 1$, $\langle v_j, q \rangle \geq 0$ for all $j \neq i$, and then $z^p \partial_m z^q = \langle v_i, q \rangle z^{p+q}$, but $\langle v_i, p + q \rangle = 0$. Thus if $1 \leq i \leq s$, we would have $p + q \in P \cap v_i^\perp = P_i$. So in this case $z^p \partial_m$ preserves the ideal of $X$ if and only if $i \not\in \{1, \ldots, s\}$, i.e. $s + 1 \leq i \leq s + t$. This gives the desired result. \hfill \square

Corollary 1.7. In the situation of Proposition 1.5 let $S = \text{Spec} \mathbb{k}[\mathbb{N}]$ with the log structure defined by the obvious chart $\mathbb{N} \to \mathbb{k}[\mathbb{N}]$, and let $\rho \in P$ lie in the interior of the face $F$, so that $\text{Spec} \mathbb{k}[P]/(z^p)$ yields a scheme with reduction $X$. Then $z^p$ induces a log morphism $Y^\dagger \to S^\dagger$, and 

$$
\Gamma(Y, \Theta_{Y^\dagger/S^\dagger}) = \bigoplus_{p \in P^P} z^p(\Theta_{Y^\dagger/S^\dagger})_p,
$$

where 

$$(\Theta_{Y^\dagger/S^\dagger})_p = \begin{cases} 
M \otimes_\mathbb{k} k & \text{if } p \in P, \\
\mathbb{Z} v_i \otimes_\mathbb{k} k & \text{if there exists an } i \text{ with } \langle v_i, p \rangle = -1, \langle v_j, p \rangle \geq 0 \text{ for } j \neq i, \\
0 & \text{otherwise}.
\end{cases}
$$
Proposition 1.8. Immediately from generalities.

\[
(\Theta_{Y^\dagger/S^\dagger})_p = \begin{cases} 
\rho^\perp \otimes \mathbb{Z} \otimes \mathbb{K} & \text{if } p \in P, \\
\mathbb{Z} v_i \otimes \mathbb{Z} \otimes \mathbb{K} & \text{if there exists an } i, s + 1 \leq i \leq s + t, \\
\text{with } \langle v_i, p \rangle = -1, \langle v_j, p \rangle \geq 0 \text{ for } j \neq i, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. This follows by imposing the condition that an element of \( \Theta_{Y^\dagger/S^\dagger} \) must annihilate \( z^\rho \).

The following result shows we obtain the same description for log derivations on suitable thickenings of \( X \); this is essentially a base-change result for derivations, but does not follow immediately from generalities.

**Proposition 1.8.** Under the same hypotheses as Corollary 1.7, assume in addition that \( Y \) is Gorenstein and that \( X = \text{Spec } \mathbb{K}[P]/(z^\rho) \) is reduced. Let \( \mathcal{X}_k = \text{Spec } \mathbb{K}[P]/(z^{k+1}) \), with the induced log structure from \( Y^\dagger \). Then \( \Gamma(\mathcal{X}_k, \Theta_{\mathcal{X}_k^\dagger/k}) \) splits into \( P^\text{gp} \)-homogeneous pieces

\[
\bigoplus_{p \in P^\text{gp}} z^p \left( \Theta_{\mathcal{X}_k^\dagger/k} \right)_p,
\]

where \( \left( \Theta_{\mathcal{X}_k^\dagger/k} \right)_p = 0 \) if there does not exist an \( i, 1 \leq i \leq s \), such that \( 0 \leq \langle v_i, p \rangle \leq k \); otherwise

\[
\left( \Theta_{\mathcal{X}_k^\dagger/k} \right)_p = (\Theta_{Y^\dagger/k})_p.
\]

In addition, let \( A_k = \mathbb{K}[t]/(t^{k+1}) \), with natural map \( \text{Spec } A_k \rightarrow S \). Pull back the log structure \( S^\dagger \) on \( S \) to \( \text{Spec } A_k \) to yield the log scheme \( \text{Spec } A_k^\dagger \). Then \( \Gamma(\mathcal{X}_k, \Theta_{\mathcal{X}_k^\dagger/A_k}) \) splits into \( P^\text{gp} \)-homogeneous pieces

\[
\bigoplus_{p \in P^\text{gp}} z^p \left( \Theta_{\mathcal{X}_k^\dagger/A_k} \right)_p,
\]

where \( \left( \Theta_{\mathcal{X}_k^\dagger/A_k} \right)_p = 0 \) if there does not exist an \( i, 1 \leq i \leq s \), such that \( 0 \leq \langle v_i, p \rangle \leq k \); otherwise

\[
\left( \Theta_{\mathcal{X}_k^\dagger/A_k} \right)_p = (\Theta_{Y^\dagger/S^\dagger})_p.
\]

Proof. We first observe there is a restriction map \( \Theta_{Y^\dagger/k} \rightarrow \Theta_{\mathcal{X}_k^\dagger/k} \). Indeed, given a log derivation \( (D, D\log) \) on \( Y \), for a function \( f \) on \( \mathcal{X}_k^\dagger \), we define \( D|_{\mathcal{X}_k} f = (D\tilde{f})|_{\mathcal{X}_k} \), where \( \tilde{f} \) is an extension of \( f \) to \( Y \). If \( \tilde{f}, \tilde{f}' \) are two such extensions, \( \tilde{f} - \tilde{f}' = h \cdot z^{k+1} \) for some function \( h \), and then

\[
D(\tilde{f} - \tilde{f}') = z^{(k+1)}Dh + h \cdot (k + 1)z^kD(z^\rho).
\]

Since \( D(z^\rho) \) is proportional to \( z^\rho \), this is in the ideal of \( \mathcal{X}_k \), hence vanishes on the restriction to \( \mathcal{X}_k \).
We also need to restrict $D\log$: this is important as the hypotheses of Proposition 1.3 don’t hold for $\mathcal{X}_k^\dagger$, so $\Theta_{\mathcal{X}_k^\dagger/k}$ is not contained in $\Theta_{\mathcal{X}_k/k}$. (For example, $\Theta_{\mathcal{X}_k/k}$ has a non-zero element defined by $D = 0$ and $D\log(n) = n$ for $n \in \mathbb{N}$.) But an element of $\mathcal{M}_{\mathcal{X}_k}$ is an equivalence class of pairs $(f, h)$ where $f \in \mathcal{M}_Y$, $h \in \mathcal{O}_{\mathcal{X}_k}$, and $(f, h) \sim (f \cdot g, h \cdot g|_{\mathcal{X}_k})$ for $g \in \mathcal{O}_Y$. Then we can define
\[D\log|_{\mathcal{X}_k}(f, h) = (D\log f)|_{\mathcal{X}_k} + h^{-1}D|_{\mathcal{X}_k}(h)\]
This is easily checked to be well-defined.

Now all monomials in $(k + 1)\rho + P$ restrict to zero on $\mathcal{X}_k$. Since $\langle v_i, \rho \rangle = 1$ for $1 \leq i \leq s$ and $\langle w_j, \rho \rangle = 0$, one sees that $p \in P$ is in $(k + 1)\rho + P$ if and only if $\langle v_i, p \rangle \geq k + 1$ for all $1 \leq i \leq s$. In case $z^p$ is one of the monomials with poles along some component of $D$ occurring in the description of $\Theta_{Y^\dagger/k}$ of Proposition 1.3, then the same condition tests to see if $z^p|_{\mathcal{X}_k} = 0$, as the poles allowed are not along any component of $X$. Thus we will obtain the desired description of $\Theta_{\mathcal{X}_k^\dagger/k}$ if we show the restriction map is surjective, and $z^p\partial_m$ restricts to zero on $\mathcal{X}_k$ if and only if $z^p$ restricts to zero.

To show this latter fact, note that $Y^\dagger \setminus D$ is log smooth over $k$ as the log structure on $Y^\dagger \setminus D$ is defined by the entire toric boundary. Thus $(\Theta_{Y^\dagger/k})|_{Y \setminus D}$ is locally free, and it is easy to see that
\[(\Theta_{\mathcal{X}_k^\dagger/k})|_{\mathcal{X}_k \setminus D} \cong (\Theta_{Y^\dagger/k})|_{Y \setminus D} \otimes \mathcal{O}_{\mathcal{X}_k \setminus D}.
\]
Now the restriction map $\Gamma(Y, \Theta_{Y^\dagger/k}) \to \Gamma(Y \setminus D, \Theta_{Y^\dagger/k})$ is injective, so a derivation of the form $z^p\partial_m$ restricts to zero on $\mathcal{X}_k \setminus D$ if and only if $z^p$ restricts to zero on $\mathcal{X}_k \setminus D$. This proves there are no derivations unexpectedly restricting to zero.

To show surjectivity of the restriction map $\Theta_{Y^\dagger/k} \to \Theta_{\mathcal{X}_k^\dagger/k}$, we make use of the Gorenstein condition. This is equivalent to the existence of a $\rho_K \in P$ such that $\langle v_i, \rho_K \rangle = 1$ for $1 \leq i \leq s + t$; this represents the canonical divisor on $Y$. Thus we can set
\[\tilde{\sigma} = \{m \in \sigma|\langle m, \rho_K \rangle = 1\}.
\]
This polytope is the convex hull of $v_1, \ldots, v_{s+t}$. Set $\tilde{\rho} = \rho_K - \rho$; because $X$ is reduced, this takes non-negative values on all the $v_i$’s and hence $\tilde{\rho} \in P$. In fact, $z^{\tilde{\rho}} = 0$ defines the divisor $D$. Then $\Gamma(Y \setminus D, \Theta_{Y^\dagger/k})$ is the localization of $\Gamma(Y, \Theta_{Y^\dagger/k})$ at $z^{\tilde{\rho}}$. Since $\mathcal{X}_k$ has no embedded components as $Y$ is Cohen-Macaulay, there can be no log derivations with support in $\mathcal{X}_k \cap D$, and hence $\Gamma(\mathcal{X}_k, \Theta_{\mathcal{X}_k^\dagger/k})$ injects into $\Gamma(\mathcal{X}_k \setminus D, \Theta_{\mathcal{X}_k^\dagger/k})$. In addition, $\Gamma(\mathcal{X}_k \setminus D, \Theta_{\mathcal{X}_k^\dagger/k})$ is generated by derivations of the form $z^{\tilde{n}\tilde{\rho} + n\rho + p}\partial_m$, where $\tilde{n}$ is any integer, $0 \leq n \leq k$, and $p \in P \setminus (\rho + P)$.

Now given such a derivation $z^{\tilde{n}\tilde{\rho} + n\rho + p}\partial_m$, set $I = \{i|p \in P_i\}$. Let us assume this derivation on $\mathcal{X}_k \setminus D$ extends to a derivation on $\mathcal{X}_k$, and see what restriction on $\tilde{n}\tilde{\rho} + n\rho + p$ we find. First assume $m$ is not proportional to $w_j$ for any $j$. Take an $i \in I$. Look at the vertices $v$ of $\tilde{\sigma}$ connected to $v_i$ by an edge. There are two cases. If $v = v_j$ for some
1 \leq j \leq s, then \langle v_j, \tilde{n}\rho + p \rangle = \langle v_j, p \rangle \geq 0. If v = w_j for some j, then X_i \cap D_j is a divisor on X_i. We can choose q \in Q_j such that \langle m, q \rangle \neq 0 and \langle v_i, q \rangle \neq 0 for all 1 \leq i \leq s. Let D_q = D \cap \{z^q = 0\}. Note D_j \not\subseteq D_q. Then z^q is a function on Y \setminus D_q which vanishes only along X \setminus D_q, and hence z^q is in \Gamma(Y \setminus D_q, \mathcal{M}_Y). Thus the pair (z^q, 1) represents an element of \Gamma(\mathcal{X}_k \setminus D_q, \mathcal{M}_{\mathcal{X}_k}). Computing, we see that
\[ z^{\tilde{n}\rho + n\rho + p} \partial_m \log(z^q, 1) = \langle m, q \rangle z^{\tilde{n}\rho + n\rho + p} \neq 0 \]
on \mathcal{X}_k \setminus D_q. If z^{\tilde{n}\rho + n\rho + p} \partial_m were then a log derivation on all of \mathcal{X}_k, the function z^{\tilde{n}\rho + n\rho + p} would not be allowed to have a pole on \mathcal{X}_k \setminus D_q. But the support of this scheme includes a dense open subset of X_i \cap D_j, so the absence of poles implies
\[ 0 \leq \langle w_j, \tilde{n}\rho + n\rho + p \rangle = \langle w_j, \tilde{n}\rho + p \rangle. \]
Thus we learn that \langle v, \tilde{n}\rho + p \rangle \geq 0 for all vertices v of \bar{\sigma} adjacent to v_i. As \langle v_i, \tilde{n}\rho + p \rangle = 0, it then follows that \langle v, \tilde{n}\rho + p \rangle \geq 0 for all vertices of \bar{\sigma}, hence \tilde{n}\rho + p \in P, hence \tilde{n}\rho + n\rho + p \in P.

If m is proportional to w_j for some j, we can assume m = w_j. The same argument shows that if i \in I, v adjacent to v_i in \bar{\sigma}, then \langle v, \tilde{n}\rho + p \rangle \geq 0 unless v = w_j. Suppose v = w_j is adjacent to v_i. Then P_i \cap Q_j is a maximal proper face of P_i, and we can find a q \in P_i such that \langle w_j, q \rangle = 1. Then
\[ z^{\tilde{n}\rho + n\rho + p} \partial_{w_j} z^q = z^{\tilde{n}\rho + n\rho + p + q}, \]
and again if this derivation extends to \mathcal{X}_k, the function on the right hand side cannot have a pole along X_i \cap D_j, i.e.
\[ 0 \leq \langle w_j, \tilde{n}\rho + n\rho + p + q \rangle = \langle w_j, \tilde{n}\rho + p \rangle + 1. \]
Thus either \langle w_j, \tilde{n}\rho + p \rangle \geq 0, and we finish the argument as before to show \tilde{n}\rho + n\rho + p \in P, or else \langle w_j, \tilde{n}\rho + p \rangle = -1, and then \langle \tilde{n} + 1, \rho \rangle + p \in P. In this latter case, \bar{\sigma} \cap ((\tilde{n} + 1)\rho + p)_\perp is a face of \bar{\sigma} spanned by S_1 = \{v_i|i \in I\} and S_2 = \{w_k|\langle w_k, \tilde{n}\rho + p \rangle = -1\}. Noting that the convex hulls of S_1 and S_2 are contained in the planes \langle \cdot, \rho \rangle = 1 and \langle \cdot, \tilde{\rho} \rangle = 1 respectively, it follows that if w_k \in S_2, then in the convex hull of S_1 \cup S_2, w_k is adjacent to some v_i \in S_1. But if w_k \neq w_j, w_k \in S_2, the above argument applied with initial choice of vertex v_i shows that \langle w_k, \tilde{n}\rho + p \rangle \geq 0, a contradiction. Thus \langle v_i, \tilde{n}\rho + n\rho + p \rangle \geq 0 for all 1 \leq i \leq s + t with v_i \neq w_j, and \langle w_j, \tilde{n}\rho + n\rho + p \rangle = -1. This shows that z^{\tilde{n}\rho + n\rho + p} \partial_{w_j} is in \langle (\Theta_{Y/k}) \tilde{n}\rho + n\rho + p \rangle, so the restriction map is surjective, as desired.

The second statement of the Proposition follows immediately from the first. \[ \square \]

There exists also a universal log-derivation (d, dlog) (see [22] for the coherent case):

**Lemma 1.9.** Given a morphism \( \pi: X^\dagger \rightarrow S^\dagger \) of log schemes, let
\[ \Omega^1_{X^\dagger/S^\dagger} = \left( \Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{M}_{X}^{\text{sp}}) \right)/\mathcal{K}, \]
with \( \mathcal{K} \) the \( \mathcal{O}_X \)-module generated by
\[
(\mathrm{d}\alpha_X(m), -\alpha_X(m) \otimes m), \quad \text{and} \quad (0, 1 \otimes \pi^*(n)),
\]
for \( m \in \mathcal{M}_X \), \( n \in \mathcal{M}_S \). Then the pair \((\mathrm{d}, \mathrm{dlog})\) of natural maps
\[
d : \mathcal{O}_X \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X^t/S^t}, \quad \mathrm{dlog} : \mathcal{M}^{\mathrm{gp}}_X \longrightarrow \mathcal{O}_X \otimes \mathcal{M}^{\mathrm{gp}}_X \longrightarrow \Omega^1_{X^t/S^t},
\]
is a universal log derivation.

**Proof.** We verify the universal property. Let \((D, \mathrm{Dlog})\) be a log derivation with values in the coherent \( \mathcal{O}_X \)-module \( \mathcal{E} \):
\[
D : \mathcal{O}_X \longrightarrow \mathcal{E}, \quad \mathrm{Dlog} : \mathcal{M}^{\mathrm{gp}}_X \longrightarrow \mathcal{E}.
\]
By the universal property of \( \Omega^1_{X/S} \) there is a unique morphism \( \varphi : \Omega^1_{X/S} \longrightarrow \mathcal{E} \) fulfilling
\[
D = \varphi \circ \mathrm{d}.
\]
Define
\[
\Phi : \Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{\mathrm{gp}}_X) \longrightarrow \mathcal{E}, \quad \Phi(\gamma, h \otimes m) = \varphi(\gamma) + h \cdot \mathrm{Dlog}(m).
\]
This descends to the quotient by \( \mathcal{K} \) because
\[
\varphi(\mathrm{d}\alpha_X(m)) - \alpha_X(m) \cdot \mathrm{Dlog}(m) = 0, \quad \mathrm{Dlog}(\pi^*(n)) = 0,
\]
for every \( m \in \mathcal{M}_X \), \( n \in \mathcal{M}_S \). Uniqueness follows since \( \Omega^1_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{\mathrm{gp}}_X) \) is generated as \( \mathcal{O}_X \)-modules by \( \Omega^1_{X/S} \) and by \( 1 \otimes \mathcal{M}^{\mathrm{gp}}_X \). On these subsets \( \Phi \) is defined by \( \varphi \) and by \( \mathrm{Dlog} \) respectively. \( \square \)

The \( \mathcal{O}_X \)-module \( \Omega^1_{X^t/S^t} \) is the module of log differentials. It is coherent for fine log structures if \( \pi \) is locally of finite type. If \( \pi \) is log smooth then \( \Omega^1_{X^t/S^t} \) is locally free \([22, \text{Proposition 3.10}]\). We define
\[
\Omega^r_{X^t/S^t} = \bigwedge^r \Omega^1_{X^t/S^t}.
\]

**Remark 1.10.** If \( \alpha : P \to \mathcal{O}_U \) is a chart for the log structure on \( X \), then in the formula for \( \Omega^1_{X^t/S^t} \) one may replace \( \mathcal{M}^{\mathrm{gp}}_X \) by \( P^{\mathrm{gp}} \) and \( \alpha_X \) by \( \alpha \). In fact, any \( h \in \mathcal{O}_X^\times \) gives a relation
\[
(\mathrm{d}h, -h \otimes \alpha^{-1}_X(h)) \in \mathcal{K}.
\]
Therefore, for any \( m \in \mathcal{M}^{\mathrm{gp}}_X \) the log differential \((0, 1 \otimes (\alpha^{-1}_X(h) \cdot m))\) may be written as
\[
h^{-1}(\mathrm{d}h, 0 \otimes 1) + (0, 1 \otimes m),
\]
which is the sum of an ordinary differential and a log differential involving only \( m \). \( \square \)

**Example 1.11.** For the relatively coherent examples we wish to consider, the sheaf of log differentials is poorly behaved at points where the log structure is not coherent. For example, take \( P \subseteq \mathbb{Z}^3 \) generated by \((1,0,0), (-1,0,1), (0,-1,1) \) and \( \rho = (0,1,0) \). If these generators correspond to variables \( x, y, w \) and \( t \) respectively, then \( k[P] \cong k[x, y, w, t]/(xy - \ldots) \).
wt) and $k[P]/(z^p) \cong k[x,y,w,t]/(xy,t)$. Let $Y = \Spec k[P]$, $X = \Spec k[P]/(z^p)$ as usual, with the log structure on $X$ induced by the inclusion $X \subseteq Y$. Note any function on a neighbourhood $0 \in Y$ with zero locus contained in $X$ is of the form $f \cdot t^l$, where $f$ is invertible. Then a section of $\mathcal{M}_X$ in a neighbourhood of 0 is necessarily induced by such a function: we write $(ft^l)|_X$ for the corresponding section of $\mathcal{M}_X$. But in $\Omega^{1}_{X'/k^1}$, $(0, 1 \otimes (ft^l)|_X) = (f|_X^{-1}df|_X, 0)$. Thus we see in fact that $\Gamma(X, \Omega^{1}_{X'/k^1}) = \Gamma(X, \Omega^{1}_{X/k})$. But if $\Omega^{1}_{X'/k^1}$ were quasi-coherent, we would then have $\Omega^{1}_{X'/k^1} = \Omega^{1}_{X/k}$, which is not the case. 

As a consequence, it is not natural to use this universal sheaf. Instead, we will use the push-forward of the sheaf of log differentials on the log smooth part of $X$, as in the following proposition.

**Proposition 1.12.** In the situation of Proposition 1.8, let $j : Z := D \cap X_{\text{Sing}} \hookrightarrow |\mathcal{X}_k| = |X|$ be the inclusion. (Here $|\mathcal{X}_k|$ denotes the underlying topological space.) Then $\Gamma(\mathcal{X}_k \setminus Z, \Omega^{r}_{\mathcal{X}_k})$ is naturally a $P$-module with decomposition into $P$-homogeneous pieces given as follows:

$$
\Gamma(\mathcal{X}_k \setminus Z, \Omega^{r}_{\mathcal{X}_k}) = \bigoplus_{p \in P \setminus (k+1)p+P} \bigwedge^r \left( \bigcap_{\{j \mid p \in Q_j\}} Q_j^{\text{gp}} \right) \otimes_Z k.
$$

Here $an_1 \wedge \cdots \wedge n_r$, for $a \in k$, $n_i \in P^{\text{gp}}$, in the summand of degree $p$ corresponds to the restriction of $az^p \log n_1 \wedge \cdots \wedge \log n_r \in \Gamma(Y \setminus Z, \Omega^{r}_{Y/k})$ to $\mathcal{X}_k$.

**Proof.** It is clear that $\Omega^{1}_{Y'/k} \cap Z \subseteq \Omega^{1}_{Y^+/k} \cap Z$. (See the discussion before Proposition 1.8 for the definition of $Y^+$. Note $Y^+$ is log smooth over Spec $k$. Denoting $\mathcal{X}_k^1$ the restriction of the log structure $Y^+$ to $\mathcal{X}_k$, we obtain

$$
\Gamma(\mathcal{X}_k \setminus Z, \Omega^{r}_{\mathcal{X}_k}) \subseteq \Gamma(\mathcal{X}_k \setminus Z, \Omega^{r}_{\mathcal{X}_k^1}) = \Gamma(\mathcal{X}_k, \Omega^{r}_{\mathcal{X}_k^1}).
$$

On the right-hand side we deal with a coherent sheaf on an affine space and hence there is a surjection

$$
\kappa : \Gamma(Y, \Omega^{r}_{Y^+/k}) \longrightarrow \Gamma(\mathcal{X}_k, \Omega^{r}_{\mathcal{X}_k^1}).
$$

Now $\Omega^{r}_{Y^+/k}$ is generated by $\log m$ for $m \in \bigwedge^r P^{\text{gp}}$ and hence

$$
\Gamma(Y, \Omega^{r}_{Y^+/k}) = \bigoplus_{p \in P} z^p \left( \bigwedge^r P^{\text{gp}} \right) \otimes_Z k.
$$

This description exhibits the $P$-grading. Letting $\mathcal{I} = (z^{(k+1)p})$ be the ideal sheaf of $\mathcal{X}_k \subseteq Y$ we see

$$
\ker(\kappa) = \Gamma(Y, \mathcal{I} \Omega^{r}_{Y^+/k}) = \Gamma(Y, \mathcal{I}) \cdot \Gamma(\Omega^{r}_{Y^+/k})
$$

$$
= \bigoplus_{p \in (k+1)p+P} z^p \left( \bigwedge^r P^{\text{gp}} \right) \otimes_Z k,
$$

where
and in turn
\[ \Gamma(\mathcal{X}_k, \Omega^r X^i_k/k) = \bigoplus_{p \in P \setminus ((k+1)\rho + P)} z^p \left( \bigwedge^r P^{sp} \right) \otimes_{\mathbb{Z}} \mathbb{k}. \]

Another way to view the $P$-grading is as weights with respect to the action of the algebraic torus $\text{Spec} \mathbb{k}[P^{sp}]$. As this action respects the inclusions $X \subseteq Y$ and $D \subseteq Y$ it induces an action on $\Gamma(Y, \Omega^r_{Y^1/k}) \subseteq \Gamma(Y, \Omega^r_{Y^{1/k}})$. From this it is clear that for each $p \in P$ there exists a $\mathbb{k}$-vector subspace $V^r_p \subseteq \bigwedge^r P^{sp} \otimes_{\mathbb{Z}} \mathbb{k}$ such that
\[ \Gamma(\mathcal{X}_k \setminus Z, \Omega^r X^i_k/k) = \bigoplus_{p \in P \setminus ((k+1)\rho + P)} z^p V^r_p. \]

To finish the proof it remains to describe $V^r_p$ for $p \in P \setminus ((k+1)\rho + P)$. An element of $z^p(\bigwedge^r P^{sp})$ is in $\Gamma(\mathcal{X}_k \setminus Z, \Omega^r X^i_k/k)$ if and only if the contraction of it with any element of $\Gamma(\mathcal{X}_k \setminus Z, \Theta_{X^i_k/k}) = \Gamma(\mathcal{X}_k, \Theta_{X^i_k/k})$ (described in Propositions 1.8 and 1.5) is in $\Gamma(\mathcal{X}_k \setminus Z, \Omega^{-1} X^i_k/k)$. Thus we can compute $V^r_p$ by induction. The result is obvious for $r = 0$. Assume now the formula for $V^{r-1}_p$ stated in the proposition is correct, and let $n \in \bigwedge^r P^{sp} \otimes \mathbb{k}$. Suppose $n \in V^r_p$, and in addition suppose $p \in Q_j$ for some $j$. Since $p \notin P \setminus ((k+1)\rho + P)$, there must be a $v_i$ connected to $w_j$ by an edge in $\mathcal{X} (\text{as defined in the proof of Proposition 1.8})$ such that $\langle v_i, p \rangle \leq k$. (Indeed, $\mathcal{X}$ is the convex hull of two polytopes, $\mathcal{X}_0$ and $\mathcal{X}_1$ on which $\rho$ takes the values 0 and 1 respectively. Then $\langle p, \cdot \rangle = 0$ defines a supporting hyperplane for $\mathcal{X}_0$. Let $l$ be the minimal value $p$ takes on $\mathcal{X}_0$. By assumption, $l \leq k$.) Then $p - lp$ defines a supporting hyperplane of $\mathcal{X}$ which contains at least one vertex of $\mathcal{X}_0$; one of these vertices will be connected by an edge of $\mathcal{X}$ to $w_j$, say $v_i$. Then $\langle v_i, p \rangle = l \leq k.$ We can then find a $q_j \in P^{sp}$ such that $\langle w_j, q_j \rangle = \langle v_i, q_j \rangle = -1$ and $\langle v_k, q_j \rangle \geq 0$ for $v_k \notin \{w_j, v_i\}$. Indeed, we just need $q_j + \rho$ to define a supporting hyperplane for the edge joining $w_j, v_i$. Replacing $q_j$ with $q_j + \rho$, we obtain a $q_j$ with $\langle w_j, q_j \rangle = -1$, $\langle v_i, q_j \rangle = 0$, $\langle v_k, q_j \rangle \geq 0$ for all $v_k \notin \{w_j, v_i\}$. Then $\langle v_i, q_j + \rho \rangle \leq k$, so $z^{q_j + \rho}$ does not vanish on $\mathcal{X}_k$. Now
\[
\iota(z^{q_j} \partial_{w_j})(z^p \text{dlog } n) = z^{p+q_j} \text{dlog } \iota(w_j)n \in \Gamma(\mathcal{X}_k \setminus Z, \Omega^{-1} X^i_k/k)
\]
if and only if $\iota(w_j)n \in V^{r-1}_{p+q_j}$. Since $p \in Q_j$, $\langle w_j, p + q_j \rangle = \langle w_j, q_j \rangle = -1$, so $p + q_j \notin P$ and $V^{r-1}_{p+q_j} = 0$. Hence if $p \in Q_j$, we must have $\iota(w_j)n = 0$, which is the case if and only if $n \in \bigwedge^r Q^{sp}_j \otimes \mathbb{k}$. Thus we see
\[
n \in \bigcap_{\{j \mid p \in Q_j\}} \bigwedge^r Q^{sp}_j \otimes \mathbb{k} = \bigwedge^r \bigcap_{\{j \mid p \in Q_j\}} Q^{sp}_j \otimes \mathbb{k}.
\]
(This equality is easily checked, say, for the intersection of two subspaces of any vector space.) Note this statement is true even if $p \notin Q_j$ for any $j$, as then it is vacuous.

Conversely, if $n \in \bigwedge^r \bigcap_{\{j \mid p \in Q_j\}} Q^{sp}_j \otimes \mathbb{k}$, then we need to check
\[ z^p \text{dlog } n \in \Gamma(\mathcal{X}_k \setminus Z, \Omega^{-1} X^i_k/k). \]
If \( q \in P, m \in P^{\text{gp}} \otimes \mathbb{k}, \) then
\[
\iota(m) n \in \bigwedge^{r-1} \left( \bigcap_{\{j|p \in Q_j\}} Q_j^{\text{gp}} \otimes \mathbb{k} \right) \subseteq \bigwedge^{r-1} \bigcap_{\{j|p+q \in Q_j\}} Q_j^{\text{gp}} \otimes \mathbb{k} = V_{p+q}^{r-1},
\]
so \( \iota(z^p \partial_m) z^p \dlog n \in \Gamma(\mathcal{X}_k \setminus Z, \Omega^{r-1}_{\mathcal{X}_k/k}) \). The other case to check is \( q \not\in P \), so there exists a \( j \) with \( \langle w_j, q \rangle = -1, \langle v_i, q \rangle \geq 0 \) for all \( v_i \neq w_j, 1 \leq i \leq s + t \); then
\[
\iota(z^p \partial_{w_j}) z^p \dlog n = \begin{cases} 0 & \text{if } p \in Q_j, \\ z^{p+q} \dlog \iota(w_j)n & \text{if } p \not\in Q_j. \end{cases}
\]
In the latter case, \( \langle w_j, p + q \rangle \geq 0 \) so \( p + q \in P \), and
\[
\iota(w_j) n \in \bigwedge^{r-1} \left( \bigcap_{\{l|p \in Q_l\}} Q_l^{\text{gp}} \otimes \mathbb{k} \right) \subseteq \bigwedge^{r-1} \bigcap_{\{l|p+q \in Q_l\}} Q_l^{\text{gp}} \otimes \mathbb{k} = V_{p+q}^{r-1}. \]

We then immediately obtain

**Corollary 1.13.** In the situation of Proposition \ref{prop:log-deformation}, \( \Gamma(\mathcal{X}_k \setminus Z, \Omega^r_{\mathcal{X}_k/k}) \) is naturally a \( P \)-module with decomposition into \( P^{\text{gp}} \)-homogeneous pieces
\[
\Gamma(\mathcal{X}_k \setminus Z, \Omega^r_{\mathcal{X}_k/k}) = \bigoplus_{p \in P \setminus ((k+1)P)} \bigwedge^r \left( \bigcap_{\{j|p \in Q_j\}} Q_j^{\text{gp}} / \mathbb{Z} \rho \right) \otimes_{\mathbb{Z}} \mathbb{k}.
\]

2. **Log Calabi-Yau spaces: local structure and deformation theory**

We now return to the type of log spaces considered in \cite{LlZ}. There, we defined the notion of a log Calabi-Yau space. In particular, we showed how to construct log Calabi-Yau spaces from integral affine manifolds with singularities with toric polyhedral decompositions \( (B, \mathcal{P}) \) which are positive and simple (see \cite{LlZ}, Definition 1.60). Our goal in this section are two-fold. First, we wish to show that the log Calabi-Yau spaces we constructed in \cite{LlZ} in fact are locally of the form considered in \S 1; we can then use the results of \S 1 to describe their sheaves of differentials, something we will do in \S 3. Second, we wish to understand the deformation theory of log schemes with local structure as described in \S 1. We will again need to make detailed use of the local descriptions of these log structures.

2.1. **Local structure.**

**Construction 2.1.** We will begin by refining the somewhat general examples considered in the previous section. We fix the following data. Let \( M' \) be a lattice, \( N' \) the dual lattice, and set \( M = M' \oplus \mathbb{Z}^{q+1}, N \) the dual lattice. We write \( e_0, \ldots, e_q \) for the standard basis of \( \mathbb{Z}^{q+1}, \) and we identify these with \( (0, e_0), \ldots, (0, e_q) \) in \( M. \) Thus we can write a general
element of $M$ as $m + \sum a_i e_i$ for $m \in M'$. Similarly, we write $e_0^*, \ldots, e_q^*$ for the dual basis, which we view as elements of $N$.

Fix a convex lattice polytope $\tau \subseteq M'_R$ with $\dim \tau = \dim M'_R$, with normal fan $\hat{\Sigma}_\tau$ living in $N'_R$ (see [12], Definition 1.38 for our convention concerning the normal fan). We obtain a cone $C'(\tau) \subseteq M'_R \oplus \mathbb{R}$, $C'(\tau) = \{(rm, r) | r \geq 0, m \in \tau\}$, and a monoid $P' = C'(\tau) \cap (N' \oplus \mathbb{Z})$. Define $\rho' \in P'$ to be given by the projection $M' \oplus \mathbb{Z} \to \mathbb{Z}$. We set $V'(\tau) = \text{Spec } k[P'] / (z^\rho') = \text{Spec } k[\partial P']$

(c.f. [12], Definition 2.13). Here $\partial P'$ is the monoid consisting of elements of the boundary of $P'$ and $\infty$, with $p + p'$ defined to be $p + p'$ if $p + p'$ lies in the boundary of $P'$ and $\infty$ otherwise. As in [12], we identify $\partial P'$ as a set with $N' \cup \{\infty\}$ via projection to $N'$. We always use the convention that $z^\infty = 0$.

Let $\hat{\psi}_1, \ldots, \hat{\psi}_q$ be integral piecewise linear functions on $\hat{\Sigma}_\tau$ whose Newton polytopes are $\Delta_1, \ldots, \Delta_q \subseteq M'_R$, i.e.

$$\hat{\psi}_i(n) = -\inf\{\langle n, m \rangle | m \in \Delta_i\}.$$

Similarly, let $\hat{\psi}_0$ have Newton polytope $\tau$, i.e.

$$\hat{\psi}_0(n) = -\inf\{\langle n, m \rangle | m \in \tau\}.$$

For convenience of notation, we set $\Delta_0 := \tau$.

Given this data, we can define a monoid $P \subseteq N$ given by

$$P = \left\{ n + \sum_{i=0}^q a_i e_i^* \middle| n \in N' \text{ and } a_i \geq \hat{\psi}_i(n) \text{ for } 0 \leq i \leq q \right\}.$$

Set $Y = \text{Spec } k[P]$ as in §1. Note that $P = K^\vee \cap N$ where $K$ is the cone in $M_R$ generated by

$$\bigcup_{i=0}^q (\Delta_i \times \{e_i\}).$$

In particular, we see $Y$ is Gorenstein because $\rho_K = \sum_{i=0}^q e_i^*$ takes the value 1 on each primitive integral generator of an extremal ray of $K$. Letting $X = \text{Spec } k[P] / (z^\rho)$ as usual with $\rho := e_0^*$, we describe $X$ explicitly by defining

$$Q = \left\{ n + \sum_{i=0}^q a_i e_i^* \in P \middle| a_0 = \hat{\psi}_0(n) \right\} \cup \{\infty\}$$

with addition on $Q$ defined by

$$q_1 + q_2 = \begin{cases} q_1 + q_2 & \text{if } q_1 + q_2 \in Q \\ \infty & \text{otherwise.} \end{cases}$$
Then $Q \setminus \{\infty\}$ is, as a set, $P \setminus (\rho + P)$, so it is clear that $X = \text{Spec}\, k[Q]$. Note that $Q \cong \partial P' \oplus \mathbb{N}^q$, via

$$(n, a_0, \ldots, a_p) \mapsto (n, 0, a_1 - \tilde{\psi}_1(n), \ldots, a_q - \tilde{\psi}_q(n)).$$

Thus $X = V'(\tau) \times \mathbb{A}^q$.

We define subschemes $Z_i$ of $X$ by their ideals, for $1 \leq i \leq q$, with $I_{Z_i/X}$ generated by the set of monomials

$$\{z^{e^*_i}\} \cup \left\{z^p \mid p = n + \sum a_j e^*_j \text{ such that there exists a unique vertex } w \text{ of } \Delta_i \text{ such that } \langle n, w \rangle = -\tilde{\psi}_i(n)\right\}.$$

The effect of the right-hand set is to select those irreducible components of the singular locus of $X$ corresponding to edges of $\Delta_i$, and $z^{e^*_i}$ defines a closed subscheme of this set of components. Set

$$Z = \bigcup_{i=1}^q Z_i.$$

This will be the locus where the log structure on $X$ fails to be coherent. Let

$$u_i := z^{e^*_i}$$

for $1 \leq i \leq q$. For any vertex $v$ of $\tau$, denote by Vert$_i(v)$ the vertex of $\Delta_i$ which represents the function $-\tilde{\psi}_i$ restricted to the maximal cone $\tilde{v}$ of $\Sigma_{\tau}$ corresponding to $v$. For every edge $\omega \subseteq \tau$, choose a primitive generator $d_\omega$ of the tangent space of $\omega$, and let $v_\omega^\pm$ be the two vertices of $\omega$, labelled so that $d_\omega$ points from $v_\omega^+$ to $v_\omega^-$ as in [12], §1.5. Set

$$\Omega_i = \{\omega \subseteq \tau \mid \text{dim } \omega = 1 \text{ and } \text{Vert}_i(v_\omega^+) \neq \text{Vert}_i(v_\omega^-)\}.$$ (This notation is compatible with that in the definition of simplicity, [12], Definition 1.60.)

It is then easy to see that

$$Z_i = \{u_i = 0\} \cap \bigcup_{\omega \in \Omega_i} V_\omega.$$

Here for $\omega \subseteq \tau$ any face, we define $V_\omega \subseteq X$ to be the closed toric stratum of $Y$ defined by the face of $K$ generated by $\omega \times \{e_0\}$.

Similarly we define $V'_\omega$, for any face $\omega \subseteq \tau$, to be the closed stratum of $V'(\tau)$ corresponding to $C'(\omega) \subseteq C'(\tau)$. □

For future use, we note

**Proposition 2.2.** Let $P$ be as in Construction 2.1, determined by data $\tau, \Delta_1, \ldots, \Delta_q$. Then the generic fibre of $f : \text{Spec}\, k[P] \rightarrow \text{Spec}\, k[\mathbb{N}]$ induced by $\rho = e^*_0$ is non-singular if and only if

$$\text{Conv} \left(\bigcup_{i=1}^q \Delta_i \times \{e_i\}\right)$$
is a standard simplex. If

\[ \text{Conv} \left( \bigcup_{i=1}^{q} \Delta_i \times \{e_i\} \right) \]

is an elementary simplex (i.e. its only integral points are its vertices) then the generic fibre of \( f \) has codimension at least four Gorenstein quotient singularities.

Proof. Since \( K = \text{Cone} \left( \bigcup_{i=0}^{q} \Delta_i \times \{e_i\} \right) \) is the cone defining the monoid \( P \) via \( P = K^+ \cap N \), it is a standard fact of toric geometry that the generic fibre of \( f \) is then defined similarly by the cone

\[ K \cap \rho^+ = \text{Cone} \left( \bigcup_{i=1}^{q} \Delta_i \times \{e_i\} \right) . \]

This cone is the standard simplicial cone corresponding to affine space if and only if \( \text{Conv} \left( \bigcup_{i=1}^{q} \Delta_i \times \{e_i\} \right) \) is the standard simplex.

The second part is then obvious, given that an elementary simplex is standard if it is dimension \( \leq 2 \). □

We now turn to the global situation considered in [12]. We will need to consider certain sorts of étale neighbourhoods of log schemes:

Definition 2.3. A morphism \( \phi : X^+ \to Y^+ \) is strict étale if it is étale as a morphism of schemes and is strict, i.e. the log structure on \( X^+ \) is the same as the pull-back of the log structure on \( Y^+ \).

Remark 2.4. This is consistent with standard notation: if a morphism of coherent log structures is log étale and strict, then it is also étale in the scheme theoretic sense. Thus in the coherent case the above definition fits with the standard one, and we are extending this to the non-coherent case.

Strict étale morphisms have the following standard property of étale morphisms: If \( Y^+ \) is a log scheme, and \( Y_0^+ \) is a closed subscheme of \( Y^+ \) defined by a nilpotent sheaf of ideals with the induced log structure on \( Y_0 \), then there is an equivalence between the categories of strict étale \( Y^+ \)-schemes and strict étale \( Y_0^+ \)-schemes. Indeed, \( X \mapsto X_0 = X \times_Y Y_0 \) gives an equivalence of categories between étale \( Y \)-schemes and étale \( Y_0 \)-schemes ([28], Chap. I, Theorem 3.23), and to obtain the log structures on \( X \) or \( X_0 \), one just pulls back the log structure on \( Y \) or \( Y_0 \).

In particular, if we have a strict étale morphism \( X_0^+ \to Y_0^+ \) and a thickening \( Y^+ \) of \( Y_0^+ \), we can talk about pulling back this thickening to \( X_0^+ \) giving \( X^+ \).

Note also that if \( f : X^+ \to Y^+ \) is a strict étale morphism over \( \text{Spec} k^+ \), then \( \Theta_{X^+/k} = f^* \Theta_{Y^+/k} \) and \( \Theta_{X_0^+/k} = f^* \Theta_{Y_0^+/k} \), as is easily checked. □
We now turn to the basic setup we will consider in the remainder of this paper, applying the above results to the log spaces constructed in [12]. In what follows, let \( B \) be an integral affine manifold with singularities with a toric polyhedral decomposition \( \mathcal{P} \) (Definitions 1.15, 1.22) and suppose \((B, \mathcal{P})\) is positive and simple (Definitions 1.54 and 1.60). Then a choice of lifted open gluing data \( s \in H^1(B, i^*_s \Lambda \otimes k^\times) \) (Definition 5.1) determines a toric log Calabi-Yau space (Definition 4.3) \( X_0(B, \mathcal{P}, s)\), with a morphism of log schemes \( X_0(B, \mathcal{P}, s) \rightarrow \text{Spec} k^\times \) which is log smooth off of a codimension two set \( Z \) as demonstrated in [12], Theorem 5.2. We will fix \((B, \mathcal{P}, s)\) now, and write \( X_{\mathcal{P}} \) instead of \( X_0(B, \mathcal{P}, s)\). We wish to describe local models for \( X_0 \) at points of \( Z \); because of the simplicity of \((B, \mathcal{P})\), the singularities of the log structure will be well-behaved, and essentially will be of the sort considered above.

We remark here that in [12], Remark 4.5, we mentioned we never really need to worry about the precise log structure of \( X_0 \) along \( Z \). In particular, the definition of an isomorphism of toric log Calabi-Yau spaces (in [12], Definition 4.3) ignores the log structure on \( Z \), and we will continue this policy below. We will never check isomorphisms of the log structure on the locus where the log structures fail to be coherent. This won’t affect any calculations below. Indeed, all of our log schemes \( X^\dagger \) are \( S_2 \), with \( \mathcal{M}_X \) having no sections with support in codimension two, and thus by Remark 1.2 modules of log derivations with values in \( \mathcal{O}_X \) are insensitive to the precise log structure on the singular locus \( Z \). Furthermore, sheaves of log differentials are only considered off of \( Z \).

Remark 2.5. We shall almost always assume \((B, \mathcal{P})\) is positive and simple in this paper (see [12], Definition 1.60). Thus for a cell \( \tau \in \mathcal{P} \) with \( 0 < \dim \tau < \dim B \), we always obtain associated to \( \tau \) the data \( \Omega_{1,\ldots,\Omega_p}, R_{1,\ldots,R_p}, \Delta_{1,\ldots,\Delta_p} \) and \( \Delta_{1,\ldots,\Delta_p} \), with \( \Delta_i \subseteq \Lambda_{\tau,R} \) and \( \Delta_i^\perp \subseteq \Lambda_{\tau,R}^\perp \) elementary simplices. (\( \Lambda_{\tau,R} \) is the tangent space to \( \tau \) in \( B \); see [12], Definition 1.31.) We call this data simplicity data for \( \tau \).

Theorem 2.6. Given \((B, \mathcal{P})\) positive and simple and \( s \) lifted open gluing data, let \( \bar{x} \rightarrow Z \subseteq X_0 \) be a closed geometric point. Then there exists data \( \tau, \psi_1, \ldots, \psi_q \) as in Construction 2.1 defining a monoid \( P \), and an element \( \rho \in P \), hence log spaces \( Y^\dagger, X^\dagger \rightarrow \text{Spec} k^\times \) as in §1, such that there is a diagram over \( \text{Spec} k^\times \)

\[
\begin{array}{ccc}
X^\dagger & \xrightarrow{\phi} & V^\dagger \\
\downarrow & & \downarrow \\
X_0^\dagger & \rightarrow & X^\dagger
\end{array}
\]

with both maps strict étale and \( V^\dagger \) an étale neighbourhood of \( \bar{x} \).

Proof. By [12], Lemma 2.29, for each \( \tau \in \mathcal{P} \) there is a map \( q_\tau : X_\tau \rightarrow X_0 \), where \( X_\tau \) is the toric variety defined in [12], Definition 2.7; this is the normalization of the stratum of
$X_0$ corresponding to $\tau$. There exists a unique $\tau \in \mathcal{P}$ with $\bar{x} \in q_\tau(X_\tau \setminus \partial X_\tau)$, and since $\bar{x} \in Z$, we have $0 < \dim \tau < \dim B$. We then obtain simplicity data associated to $\tau$ as in Remark 2.5. Set $M' = \Lambda_\tau$, $M'_\mathbb{R} = \Lambda_{\tau,\mathbb{R}}$, and identify $\tau$ as a polytope in $\Lambda_{\tau,\mathbb{R}}$, well-defined up to translation. (Without loss of generality we won’t worry about self-intersections of $\tau$.) By [12, Corollary 5.8], $q_\tau^{-1}(Z) \setminus \partial X_\tau = Z^r_1 \cup \cdots \cup Z^r_p$ where each $Z^r_i$ is an irreducible and smooth divisor on $\text{Int}(X_\tau) := X_\tau \setminus \partial X_\tau$, with Newton polytope given by $\tilde{\Delta}_i$. Furthermore, it is not difficult to see these divisors meet transversally, using the fact that simplicity implies the Newton polytopes $\tilde{\Delta}_i$ are elementary simplices and their tangent space $T_{\Delta_i}$ in $\Lambda_{\tau,\mathbb{R}}$ form an interior direct sum (see [12, Remark 1.61, (4)]). Assume that $\bar{x} \in q_\tau(Z^r_i)$, for $1 \leq i \leq r$ for some $r \leq p$ (reorder the indices if necessary) and redefine $\Delta_i = \{0\}$ for $i > r$. In addition, set $\Delta_i = \{0\}$ for $p < i \leq \dim B - \dim \tau$. This provides data $\tau, \Delta_1, \ldots, \Delta_q$ for $q = \dim B - \dim \tau$, and the $\Delta_i$’s define piecewise linear functions $\tilde{\psi}_i$ on the normal fan $\Sigma_\tau$ in $N^\vee_\mathbb{R}$ (see [12, Remark 1.59]: there is a typo in that Remark: $\varphi_\rho$ should be $\tilde{\psi}_\rho$.) This data produces monoids $P' \subseteq N'$ with $P' = C'(\tau)' \cap N'$, a monoid $P \subseteq N = N' \oplus \mathbb{Z}^{q+1}$ as constructed from this data in Construction 2.1, $\rho \in P$ given by $\rho = e^*_\sigma$, yielding $Y = \text{Spec} \mathbb{k}[P]$ and $X = \text{Spec} \mathbb{k}[P]/(z^\rho)$. We recall $X \cong \text{Spec} \mathbb{k}[\partial P' \oplus \mathbb{N}^q]$. We will now construct the diagram (2.1).

First, pick some $g : \tau \rightarrow \sigma \in \mathcal{P}_\text{max}$, so we obtain as in [12, Construction 2.15] an open set $V(\tau) \subseteq V(\sigma)$. As observed in [12, Construction 2.15], $V(\tau) \cong \text{Spec} \mathbb{k}[\partial P'] \times \mathbb{G}_m^q$. The factor $\mathbb{G}_m^q$ can be identified with $\text{Int}(X_\tau)$. In particular, there is a unique point $0 \in \text{Spec} \mathbb{k}[\partial P']$ whose ideal is generated by $\{z^p | p \in \partial P', p \neq 0\}$, and we identify the stratum $0 \times \mathbb{G}_m^q$ with $\text{Int}(X_\tau)$ also. Then there is an étale map $p_\sigma : V(\sigma) \rightarrow X_0$ by the construction of $X_0$: see [12, Definition 2.28]. Furthermore $p_\sigma$ is one-to-one on the interior of any toric stratum, so in particular is one-to-one on $\text{Int}(X_\tau)$. So there exists a unique point $\bar{v} \in \text{Int}(X_\tau)$ with $p_\sigma(\bar{v}) = \bar{x}$. In particular, $V(\tau)$ is an étale neighbourhood of $\bar{x}$.

We pull back the log structure on $X_0$ to $V(\sigma)$: along with the induced morphism to $\text{Spec} \mathbb{k}[\delta^r]$, this structure is then determined by a section $f_\sigma \in \Gamma(V(\sigma), \mathcal{L}S^+_{\text{pre}, V(\sigma)})$, (see [12, Definition 4.21]) or equivalently,

$$(f_{\sigma,e})_{e : \omega \rightarrow \sigma} \in \Gamma \left( V(\sigma), \bigoplus_{e : \omega \rightarrow \sigma, \dim \omega = 1} \mathcal{O}_{V_e} \right).$$

(Recall [12, Definition 2.22] that for $e : \omega \rightarrow \sigma$, $V_e$ is the open affine subset of $X_\omega$ determined by $\sigma$.) Furthermore, it is argued in the proof of [12, Theorem 5.2, (2)], that for any factorization $\omega \rightarrow \tau' \rightarrow \tau \rightarrow \sigma$ of $e$, $f_{\sigma,e}|_{V_{\sigma'}}$ is completely determined by $s$ and the Newton polytope of $f_{\sigma,e}|_{V_{\sigma'}}$.
We first claim the functions \( f_{\sigma,g,e} \) for \( e \in \Omega_i \) glue. Indeed, let \( e_1, e_2 \in \Omega_i \), with \( e_i : \omega_i \to \tau \), and suppose we have a diagram

![Diagram](https://via.placeholder.com/150)

Let \( \Omega'_1, \ldots, \Omega'_r, \Delta'_1, \ldots, \Delta'_\rho \) etc. be the simplicity data for \( \tau' \). Then as there exists \( l : \tau \to \rho \in \mathcal{P} \) with \( l \in R_i \) such that \( \kappa_{\omega_1,\rho}, \kappa_{\omega_2,\rho} \neq 0 \), (see \[12\], Remark 1.61, (3)) it follows that \( h_1, h_2 \) are both in the same \( \Omega'_j \) for some \( j \) by \[12\], Definition 1.60, (1). Thus \( f_{\sigma,g,e_1}|_{\nu,g,k} \) and \( f_{\sigma,g,e_2}|_{\nu,g,k} \) both have the same Newton polytope, namely \( \Delta'_j \), and hence must coincide, since as remarked in the previous paragraph, these functions are completely determined by \( s \) and their Newton polytope.

This gives rise, for each \( i \leq r \), to a function \( f_i \) on the closed subscheme \( \cup_{e \in \Omega_i} V_e \subseteq V(\tau) \). We can then extend each \( f_i \) to a function on \( V(\tau) \), which we also call \( f_i \).

Next, choose coordinates \( z_1, \ldots, z_q \) on \( \text{Int}(X_\tau) \cong \mathbb{G}_m^q \), and pull these back to functions on \( V(\tau) = \text{Spec} \mathbb{k}[\partial P'] \times \mathbb{G}_m^q \). Recalling from the first paragraph of this proof that \( Z_1', \ldots, Z_r' \) intersect transversally, one can find a subset \( \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, q\} \) such that \( \det(\partial f_i/\partial z_{i_j})_{1 \leq i, j \leq r} \neq 0 \) at \( \bar{v} \). By reordering the indices, we can assume \( \{i_1, \ldots, i_r\} = \{1, \ldots, r\} \), and set \( f_i := z_i - z_i(\bar{v}) \) for \( r + 1 \leq i \leq q \), where \( z_i(\bar{v}) \) denotes the value of the function \( z_i \) at the point \( \bar{v} \).

Now consider the section \( f'_\sigma \in \Gamma(V(\tau), \mathcal{L}S^+_{\text{pre}, V(\sigma)}) \) defined by

\[
f'_{\sigma,e} = \begin{cases} 
1 & \text{if } e \not\in \bigcup_{i=1}^r \Omega_i; \\
f|_{\nu,e} & \text{if } e \in \Omega_i, \ 1 \leq i \leq r.
\end{cases}
\]

Then essentially repeating the argument of the proof of \[12\], Theorem 5.2, (2), one sees that \( f'_\sigma \) satisfies the multiplicative condition of \[12\], Theorem 3.22. As \( f_\sigma \) itself satisfies this multiplicative condition, since \( f_\sigma \) defines a log smooth structure away from the singular locus \( p_{\sigma}^{-1}(Z) \), we see that the section \( a_\sigma := f_\sigma/f'_\sigma \in \Gamma(V(\tau), \mathcal{L}S^+_{\text{pre}, V(\sigma)}) \) also satisfies this multiplicative condition. Note

\[
a_{\sigma,e} = \begin{cases} 
f_{\sigma,e} & \text{if } e \not\in \bigcup_{i=1}^r \Omega_i; \\
1 & \text{if } e \in \Omega_i, \ 1 \leq i \leq r.
\end{cases}
\]

Also \( a_{\sigma,e} \) is always invertible at \( \bar{v} \), and so \( a_\sigma \) is a section of \( \mathcal{L}S_{V(\sigma)} \) (see \[12\], Definition 3.19) on some Zariski open neighbourhood \( V \subseteq V(\tau) \) of \( \bar{x} \). We can assume that \( \det(\partial f_i/\partial z_j)_{1 \leq i, j \leq r} \) is also invertible on \( V \). Then, possibly shrinking \( V \) more, the section
\( a_\sigma \) determines, via \([12]\), Theorem 3.22, a chart

\[ P' \to \Gamma(V, \mathcal{O}_V) \]

of the form

\[ p \mapsto h_p z^p \]

where \( h_p \) is some invertible function on \( V(p) \cap V \). (Here, \( V(p) \) is the closure of the open subset of \( V(\tau) \) where \( z^p \neq 0 \).) This is not a chart defining the log structure \( V^\dagger \) on \( V \).

We now define a morphism \( \phi : V \to X = \text{Spec} \, k[\partial P' \oplus \mathbb{N}^q] \) by

\[
\begin{align*}
\phi^*(z^p) &= h_p z^p \quad \text{for } p \in \partial P' \\
\phi^*(u_i) &= f_i \quad \text{for } 1 \leq i \leq q
\end{align*}
\]

where \( u_i \) is the monomial in \( k[\partial P' \oplus \mathbb{N}^q] \) corresponding to the \( i \)th standard basis vector of \( \mathbb{N}^q \).

**Claim:** Possibly after shrinking \( V \), \( \phi \) is \( \acute{e} \)tale.

**Proof.** We prove this via a slight modification of the argument in \([12]\), Proposition 3.20, which essentially dealt with the case when \( \bar{x} \not\in \mathbb{Z} \). First rewrite \( k[\partial P'] \) as \( k[X]/(G) \), where \( X \) denotes a collection of variables \( \{X_k\} \), with \( X_k = z^{p_k} \) for a set of generators \( \{p_k\} \) of \( \partial P' \). The relation ideal \( (G) \) is generated by a finite number of polynomials \( G_\lambda \) of the form \( \prod X_k^{a_k} - \prod X_k^{b_k} \) with \( \sum a_k p_k = \sum b_k p_k \), or \( \prod X_k^{a_k} \) with \( \sum a_k p_k = \infty \). For each \( k \), choose an extension \( h_k \) of \( h_{p_k} \) to \( V \); after shrinking \( V \) if necessary we can assume \( h_k \) is invertible on \( V \). Again, by shrinking \( V \) we can assume \( V \subseteq V(\tau) \) is given by \( G \neq 0 \) for some function \( G \) on \( V(\tau) \). Let \( A = k[\partial P' \oplus \mathbb{N}^q] = k[X, u_1, \ldots, u_q]/(G) \). Then the morphism \( \phi : V \to \text{Spec} \, A \) is given by the map of rings induced by the inclusion of \( A \) in the polynomial ring over \( A \),

\[
A \to A[S, z_1^{\pm 1}, \ldots, z_q^{\pm 1}, G^{-1}]/I
\]

where the variables \( S = \{S_k\} \) are in one-to-one correspondence with the variables \( \{X_k\} \), and the ideal \( I \) is generated by

\[
X - h' \cdot S \quad u_i - f_i(S, z) \quad 1 \leq i \leq q
\]

where \( h' \) denotes \( h = \{h_k\} \) with the \( X_k \)'s replaced by \( S_k \)'s. According to standard results concerning \( \acute{e} \)tale morphisms (see e.g. \([28]\), I. 3.16), a map of affine schemes induced by a map of rings \( C \to C[Z_1, \ldots, Z_n]/(F_1, \ldots, F_n) \) is \( \acute{e} \)tale if and only if \( \det(\partial F_i/\partial Z_j) \) is a unit in \( C[Z_1, \ldots, Z_n]/(F_1, \ldots, F_n) \). Furthermore, localizing a ring \( C \) at an element \( \alpha \) is the same thing as \( C[t]/(at - 1) \), so the same result holds if we further localize \( C[Z_1, \ldots, Z_n] \) before
dividing out by \((F_1, \ldots, F_n)\) with \(F_i\) in the localized ring. This is our situation, and in this case the Jacobian is

\[
\begin{pmatrix}
-D\text{diag}(h') - (\partial h'/\partial S) \cdot S & -(\partial h'/\partial z) \cdot S \\
-\partial f/\partial S & -\partial f/\partial z
\end{pmatrix}
\]

As \(S_k = 0\) for all \(k\) at \(v\), we see that at \(v\) this matrix is

\[
\begin{pmatrix}
-D\text{diag}(h') & 0 \\
-\partial f/\partial S & -\partial f/\partial z
\end{pmatrix}
\]

By our choice of the \(f_i\)'s, the determinant of this is non-zero at \(v\). Thus, possibly further shrinking \(V\), we get an étale morphism.

We now only need to show the log structure \(X^\dagger \to \text{Spec} k^\dagger\) induced by \(X \subseteq Y\) pulls back to the given structure \(V^\dagger \to \text{Spec} k^\dagger\). Since these structures are determined by sections of \(LS^+_{pre,X}\) and \(LS^+_{pre,V}\) respectively it is actually enough to show this on an open subset \(X^o \subseteq X\) which is dense on each codimension one stratum of \(X\). We define \(Y^o\) by localizing \(Y\) at \(\prod_{i=1}^q u_i\), and similarly define \(X^o\), so \(X^o = X \cap Y^o\). The diagram

\[
\begin{array}{c}
Y \leftarrow Y^o \\
\uparrow \\
X \leftarrow X^o
\end{array}
\]

corresponds to the diagram of rings

\[
\begin{array}{c}
k[P] \leftarrow k[P' \oplus \mathbb{Z}^q] \\
\downarrow \\
k[Q] \leftarrow k[\partial P' \oplus \mathbb{Z}^q]
\end{array}
\]

Here the horizontal inclusions are just

\[(n, a_0, \ldots, a_q) \mapsto (n, a_0, \ldots, a_q)\]

and the vertical maps are quotients by \(\rho\). The chart on \(X^o\) for the log structure given by the inclusion \(X^o \to Y^o\) is clearly just

\[P' \ni p \mapsto z^p.\]

However, the isomorphism of \(\text{Spec} k[\partial P' \oplus \mathbb{N}^q]\) with \(\text{Spec} k[Q]\) is given by the map of rings \(k[Q] \to k[\partial P' \oplus \mathbb{N}^q]\) given by

\[z^p \mapsto z^p \prod_{i=1}^q u_i^{-\psi_i(p)},\]

as described in Construction 2.1. Thus the chart for the log structure on \(X^o\) as a subset of \(\text{Spec} k[\partial P' \oplus \mathbb{N}^q]\) is

\[P' \ni p \mapsto z^p \prod_{i=1}^q u_i^{-\psi_i(p)}.\]
We then pull back this chart to $V$, getting
\[ P' \ni p \mapsto h_p z^p \prod_{i=1}^q f_i^{-\hat{\psi}_i(p)} = h_p z^p \prod_{i=1}^r f_i^{-\hat{\psi}_i(p)}, \]
the latter equality since $\hat{\psi}_i(p) = 0$ for $i > r$. Set $f_p = h_p \prod_{i=1}^r f_i^{-\hat{\psi}_i(p)}$, $f'_p = f_p / h_p$. We now have to compare this chart with the log structure on $V$. We do this by calculating the section of $\mathcal{L}S_V$ determined by this chart, using the construction of [12], Theorem 3.22, which associates to the data $(f_p)_{p \in \partial P'}$ a section of $\mathcal{L}S_V$. As this map is multiplicative, and by construction $f_p = f'_p h_p$ and $(h_p)_{p \in \partial P'}$ maps to the section $a_\sigma$ of $\mathcal{L}S_V$, it is enough to show that $(f'_p)_{p \in \partial P'}$ maps to the section $f'_\sigma$ of $\mathcal{L}S_V$.

Consider \( e : \omega \to \tau \) with $\omega$ one-dimensional, with endpoints \( e^\pm_\omega : v^\pm_\omega \to \omega \) as usual. On the maximal cones \( \check{v}^\pm_\omega \) of \( \check{\Sigma}_r \), suppose \( \psi_i \) is given by \( -m^\pm_i \). By definition of \( \psi_i \), \( m^\pm_i = \text{Vert}(v^\pm_\omega) \) are vertices of \( \Delta_i \). In order to show the construction of [12], Theorem 3.22 gives the section $f'_{\sigma}$, we have to show for $n \in N'$ we have on $V_e$
\[
(d_\omega \otimes f'_{\sigma,e})(n) = \prod_{i=1}^r f_i^{(m^- - m^+, n)} = \prod_{i=1}^r f_i^{(m^- - m^+, n)}. \]
But $m^- - m^+$ is always proportional to $d_\omega$, and by the fact that $T_{\Delta_1}, \ldots, T_{\Delta_r}$ are transversal (as follows from the simplicity assumption, [12], Remark 1.61, (4)) and the fact that $\Delta_i$ are elementary simplices, we have
\[
m^- - m^+ = \begin{cases} 0 & e \not\in \Omega_i, \\ d_\omega & e \in \Omega_i. \end{cases} \]
Thus on $V_e$
\[
\prod_{i=1}^r f_i^{(m^- - m^+, n)} = f_i^{(d_\omega, n)} = (f'_{\sigma,e})^{(d_\omega, n)} = (d_\omega \otimes f'_{\sigma,e})(n) \]
if \( e \in \Omega_i \), as desired; if there is no such \( i \), then this is 1, again as desired. \( \square \)

2.2. Deformation theory. We will now develop deformation theory for log Calabi-Yau spaces $\mathcal{f} : X^\dagger \to \text{Spec} k^\dagger$. This is foundational material, and it is possible that there is a much more general context in which to do deformation theory of relatively coherent log structures. Here, however, we restrict attention to log structures whose local models have been described in the previous section. We note, however, that the results of this section, while of interest by themselves, are only needed in order to be able to apply the results of Theorem 4.11 to the degenerations of Calabi-Yau manifolds constructed in [13]. In particular, Corollary 2.18 shows that those degenerations are of the correct sort to apply the results of §4. So this section is not necessary for the understanding of the remainder of the paper.
The difficulty is that as $f$ is not log smooth along $Z$, we can’t apply standard log smooth deformation theory as in [20],[22] immediately. As an abstract log space over $\text{Spec}\ k^\dagger$, $X^\dagger$ has many deformations which are fairly perverse and which we don’t wish to allow. So we consider the following restricted deformation problem.

Fix once and for all the $k$-algebra $R := k[[N]] = k[[t]]$. $\text{Spec}R$ carries the log structure induced by the natural chart $N \to R, n \to t^n$. We write this log scheme as $\text{Spec}\ R^\dagger$. Thus for any $R$-algebra $A$, $\text{Spec}\ A$ carries the pull-back log structure, which we write as $\text{Spec}\ A^\dagger$.

We denote by $C_R$ the category of Artin local $R$-algebras with residue field $k$. As usual, if $A, B \in \text{Ob}(C_R)$ with $B = A/I$, we say $A$ is a small extension of $B$ if $m_A I = 0$.

**Definition 2.7.** Let $X^\dagger_k$ be a toric log Calabi-Yau space over $\text{Spec}\ k^\dagger$, with positive and simple dual intersection complex $(B, \mathcal{P})$, and let $A \in \text{Ob}(C_R)$. Then a divisorial log deformation of $X^\dagger_k$ over $\text{Spec}\ A^\dagger$ is data

$f_A : X^\dagger_A \to \text{Spec}\ A^\dagger$ together with an isomorphism

$X^\dagger_A \times_{\text{Spec}\ A^\dagger} \text{Spec}\ k^\dagger \cong X^\dagger_k$ over $\text{Spec}\ k^\dagger$ such that

1. $f_A$ is flat as a morphism of schemes, and $f_A|_{X^\dagger_A \setminus Z}$ is log smooth.
2. For every closed geometric point $\bar{x} \in Z$, let $P, Y, X$ be the data of Theorem 2.6 giving a diagram (2.1) over $\text{Spec}\ k^\dagger$. Let $X^\dagger_A = Y^\dagger \times_{\text{Spec}\ k[[N]]^\dagger} \text{Spec}\ A^\dagger$. Then there exists a diagram over $\text{Spec}\ A^\dagger$

\[
\begin{array}{ccc}
X^\dagger_A & \xrightarrow{\phi} & \mathcal{V}^\dagger_A \\
\downarrow & & \downarrow \\
\mathcal{X}^\dagger_A & \xrightarrow{\sigma} & X^\dagger_A
\end{array}
\]

with both maps strict étale.

**Example 2.8.** The standard behaviour of a divisorial deformation in codimension two can be described as follows. Given $B, \mathcal{P}$ and $s$ as usual, let $\omega \in \mathcal{P}$ be an edge of affine length $l$. Let $Q_\omega = \Lambda y/\Lambda \omega$ for any $y \in \text{Int}(\omega) \setminus \Delta$, as in [12], Definition 1.38. Then $\omega$ defines an étale open neighbourhood $V(\omega)$ of $X_0(B, \mathcal{P}, s)$, which can be written as

$V(\omega) \cong \text{Spec}\ k[Q_\omega^\dagger][x, y]/(xy)$.

The log structure and log morphism to $\text{Spec}\ k^\dagger$ is determined by $f_\omega \in \Gamma(V(\omega), \mathcal{L}S^+_{\text{pre}, V(\omega)}) = k[Q_\omega^\dagger]$, and it is not difficult to describe the log structure: We have a morphism

$U = \text{Spec}\ k[Q_\omega^\dagger][x, y, t]/(xy - f_\omega t^l) \to \text{Spec}\ k[t] = \mathbb{A}^1$

given by $t \mapsto t$, and $V(\omega)$ is the fibre over the origin. The inclusion $V(\omega) \subseteq U$ induces a log structure on $U$, with $\mathcal{M}_U$ given as usual by the regular functions invertible away from $V(\omega)$. The restriction of this log structure to $V(\omega)$ is the desired log structure. If $\mathbb{A}^1$ is given the log structure induced by $0 \in \mathbb{A}^1$ (or equivalently by the chart $\mathbb{N} \mapsto k[t], n \mapsto t^n$),
and \( \text{Spec } k[t]/(t^{k+1}) \) is given a log structure via the same chart, then

\[
U^\dagger \times_{(A^\dagger)} \text{Spec } k[t]/(t^{k+1})^\dagger \to \text{Spec } k[t]/(t^{k+1})^\dagger
\]
gives a \( k \)th order divisorial deformation of \( V(\omega)^\dagger \to \text{Spec } k^\dagger \) (given by \( k = 0 \)). To check this, it is enough to check that \( U \to A^\dagger \) is étale locally equivalent to \( Y \to A^\dagger \) as given in Theorem 2.6, which is easily done as in the proof of that Theorem.

**Example 2.9.** Let us return to Example 2.11 and show there are many non-divisorial deformations of \( X^\dagger/k^\dagger \). Consider \( X[\epsilon] = \text{Spec } k[P][\epsilon]/(\epsilon^2) \) as a scheme over \( \text{Spec } k[\epsilon]/(\epsilon^2) \). Now the log structure on \( X \setminus \{0\} \) is fine and can be described using charts on open sets \( U_x, U_y \) and \( U_w \) of \( X \), the sets where \( x \neq 0, y \neq 0 \) and \( w \neq 0 \) respectively. Then the charts \( \mathbb{N} \to \mathcal{O}_{U_x} \) and \( \mathbb{N} \to \mathcal{O}_{U_y} \) given by

\[
n \mapsto \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}
\]

and \( \mathbb{N}^2 \to \mathcal{O}_{U_w} \) given by

\[
(n_1, n_2) \mapsto w^{-n_1}x^{n_1}y^{n_2}
\]

are easily seen to define isomorphic log structures on overlaps. Furthermore, these charts also define the log morphisms to \( \text{Spec } k^\dagger \), via \( \mathbb{N} \ni 1 \mapsto 1 \in \mathbb{N} \) for the first two charts and \( \mathbb{N} \ni 1 \mapsto (1, 1) \in \mathbb{N}^2 \) for the third chart. There are unique identifications of the three log structures on overlaps which identify these morphisms, and these identifications define the log structure \( X^\dagger \) on \( X \setminus \{0\} \). Furthermore \( \mathcal{M}_X = i^*\mathcal{M}_{X \setminus \{0\}} \), where \( i : X \setminus \{0\} \hookrightarrow X \) is the inclusion. We have now described the log morphism \( X^\dagger \to \text{Spec } k^\dagger \) in terms of charts.

We can then lift this log structure and morphism to \( X[\epsilon] \) by keeping the same charts on \( U_x \) and \( U_y \), but lifting \( \mathbb{N}^2 \to \mathcal{O}_{U_w} \) to \( \mathbb{N}^2 \to \mathcal{O}_{U_w}[\epsilon] \) given by

\[
(n_1, n_2) \mapsto (w + f(w, w^{-1})\epsilon)^{-n_1}x^{n_1}y^{n_2},
\]

for \( f(w, w^{-1}) \in k[w, w^{-1}] \). Furthermore, \( \text{Spec } k[\epsilon]/(\epsilon^2) \) has a log structure induced by the \( R \)-algebra structure \( t \mapsto 0 \). The same maps of monoids \( \mathbb{N} \to \mathbb{N} \) and \( \mathbb{N} \to \mathbb{N}^2 \) as above then induce log morphisms \( U_x[\epsilon]^\dagger, U_y[\epsilon]^\dagger, U_w[\epsilon]^\dagger \to \text{Spec } k[\epsilon]/(\epsilon^2)^\dagger \). These log structures and morphisms glue canonically. Note that applying an automorphism of \( X[\epsilon] \) fixing \( X \) can remove any positive powers of \( w \) from \( f \), but not negative powers. Thus this gives an infinite dimensional family of possible “exotic” deformations of the log structure. It follows from Lemma 2.13 below that none of these are divisorial. \( \square \)

**Lemma 2.10.** Let \( A' \in \text{Ob}(\mathcal{C}_R) \) and let \( X^\dagger_{A'} \to \text{Spec } A^\dagger \) be a morphism of log schemes which is flat as a morphism of schemes. If \( A \in \text{Ob}(\mathcal{C}_R) \) with a surjective homomorphism \( A' \to A \) and kernel \( I \) with \( I^2 = 0 \), then the set of log automorphisms of \( X^\dagger_{A'} \) fixing \( X^\dagger_A = \...
\(X^\dagger_A \times (A')^\dagger A^\dagger\) is in one-to-one correspondence with the log derivations of \(X^\dagger_A\) over \(\text{Spec}(A')^\dagger\) with values in \(O_{X_A'} \otimes_{A'} I\).

If \(A'\) is a small extension of \(A\), \(X^\dagger_A' \rightarrow \text{Spec} A^\dagger\) log smooth off of a codimension two closed subset \(Z\) of \(X^\dagger_A', X^\dagger_A\) has no embedded components and \(M_{X_A'}\) has no sections with support in \(Z\), then this space of log derivations is \(\Gamma(X_k, \Theta_{X_k^\dagger/k^\dagger} \otimes I)\), where \(X^\dagger_k = X^\dagger_A \times_{A'} k^\dagger\).

**Proof.** This is standard. An automorphism is induced by an automorphism of sheaves of rings \(\psi : O_{X_A'} \rightarrow O_{X_A'}\) and an automorphism \(\Psi : M_{X_A'} \rightarrow M_{X_A'}\) compatible with \(\psi\), which are the identity after restricting to \(X_A\). Then \(x \mapsto Dx = \psi(x) - x \in O_{X_A'} \otimes_{A'} I\) is an ordinary derivation, while if \(m \in M_{X_A'}\), then \(\Psi(m) = h \cdot m\) for \(h \in 1 + O_{X_A'} \otimes_{A'} I\) as \(\Psi\) induces the identity on \(\overline{M}_{X_A} = \overline{M}_{X_A'}\). Then we define \(\text{Dlog}(m) = h - 1\). It is easy to check \((D, \text{Dlog})\) yields a log derivation. Conversely, given a log derivation \((D, \text{Dlog})\), one can reverse this procedure to get an automorphism.

In the case of a small extension, we notice that \(I\) is a \(k = A'/m_A\)-vector space, and \(O_{X_A'} \otimes_{A'} I = O_{X_k} \otimes_k I\). Off of the codimension two locus \(Z\) where \(X^\dagger_A \rightarrow \text{Spec} A^\dagger\) fails to be log smooth, this module of derivations is \(\Theta_{X_A'/A'} \otimes_{O_{X_A}} (O_{X_k} \otimes_k I) = \Theta_{X_k^\dagger/k^\dagger} \otimes_k I\). By Remark \([12]\) this shows the module of derivations on \(X^\dagger_A\) with values in \(O_{X_A'} \otimes_{A'} I\) is in fact \(\Theta_{X_k^\dagger/k^\dagger} \otimes_k I\).

The main theorem for divisorial deformations is then the following.

**Theorem 2.11.** Let \((B, \mathcal{P})\) be an integral affine manifold with singularities and polyhedral decomposition \(\mathcal{P}\). Suppose \((B, \mathcal{P})\) is positive and simple, and let \(X^\dagger_k\) be a toric log Calabi-Yau space with dual intersection complex \((B, \mathcal{P})\) and log singular set \(Z \subseteq X_k\). Let \(A', A \in \text{Ob}(\mathcal{C}_R)\) with \(A'\) a small extension of \(A\), \(A = A'/I\). Let \(X^\dagger_A \rightarrow \text{Spec} A^\dagger\) be a divisorial deformation of \(X^\dagger_k \rightarrow \text{Spec} k^\dagger\). Then

1. Suppose this deformation lifts to a divisorial deformation \(X^\dagger_A' \rightarrow \text{Spec} A'^\dagger\). Then the set of log automorphisms of \(X^\dagger_A'\) over \(\text{Spec} A'^\dagger\) which is the identity on \(X^\dagger_A\) is \(H^0(X_k, \Theta_{X_k^\dagger/k^\dagger} \otimes_k I)\).

2. Under the same assumption as in (1), the set of equivalence classes of liftings \(X^\dagger_A' \rightarrow \text{Spec} A^\dagger\) of \(X^\dagger_A \rightarrow \text{Spec} A^\dagger\) is a torsor over \(H^1(X_k, \Theta_{X_k^\dagger/k^\dagger} \otimes_k I)\).

Here two liftings are equivalent if there is an isomorphism between them over \(\text{Spec}(A')^\dagger\) which is the identity on \(X^\dagger_A\).

3. The obstruction to the existence of a lifting as in (1) is in \(H^2(X_k, \Theta_{X_k^\dagger/k^\dagger} \otimes_k I)\).
For the proof of Theorem 2.11, we will need the following lemmas concerning derivations arising in Construction 2.7. The first will be needed to deal with general points of $Z$, and the last two will deal with higher codimensions.

**Lemma 2.12.** Let $M' = \mathbb{Z}$, $\Delta_0 = [0, l]$ for $l > 0$ an integer, and $\Delta_1 = [0, 1]$ determine data $P$, $\rho \in P$, as in Construction 2.7, so that $Y$ is the toric variety defined by the fan in $M_\mathbb{R}$ given by the cone over the convex hull of $(0, 0, 1), (1, 0, 1), (0, 1, 0)$ and $(l, 1, 0)$. Subdivide this cone as depicted:

![Diagram](image)

to define a blow-up $\bar{Y} \to Y$. Let $\pi : \bar{X} \to X$ be the restriction of this map to the inverse image of $X$, and let $E \subseteq \bar{X}$ be the exceptional locus. Let $\bar{X}^+$ be the log structure on $\bar{X}$ induced by the inclusion $\bar{X} \subseteq \bar{Y}$. Then

1. the composed morphism $\bar{X}^+ \to X^+ \to \text{Spec} k^+$ is log-smooth, and $\bar{X}^+ \setminus E \cong X^+ \setminus Z$.
2. $\pi_* \Theta_{\bar{X}^+/k^+} \cong \Theta_{X^+/k^+}$.
3. $R^p\pi_* \Theta_{\bar{X}^+/k^+} = 0$ for $p > 0$.

**Proof.** Let $\bar{Y}_1$ be the affine open subset of $\bar{Y}$ determined by the cone generated by

$$(0, 1, 0), (0, 0, 1), (1, 0, 1),$$

so that

$$\bar{Y}_1 = \text{Spec} k[z^{(1,0,0)}, z^{(0,1,0)}, z^{(-1,0,1)}] = \mathbb{A}^3_k,$$

and let $\bar{Y}_2$ be determined by the cone generated by $(0, 1, 0), (0, 0, 1), (l, 1, 0)$, so that

$$\bar{Y}_2 = \text{Spec} k[z^{(0,0,1)}, z^{(1,0,-1)}, z^{(-1,l,1)}, z^{(0,1,0)}] = \mathbb{A}^1_k \times \text{Spec} k[x, y, t]/(xy - t^l).$$

Recall $t = z^{(0,1,0)}$. If $\bar{X}_i = \bar{X} \cap \bar{Y}_i$, we have $\bar{X}_1 = \{z^{(1,0,0)} = 0\} \subseteq \mathbb{A}^3_k$ and $\bar{X}_2 = \mathbb{A}^1_k \times \{t = 0\} \subseteq \bar{Y}_2$. If we give $\bar{X}_i$ the log structure induced by the corresponding inclusion, we see $\bar{X}_i$ is clearly log smooth over $\text{Spec} k^+$. The second statement of (1) is obvious since $\bar{Y} \setminus E = Y \setminus Z$.

For (2) and (3), we use the open cover $\{\bar{X}_1, \bar{X}_2\}$ to compute Čech cohomology of $\Theta_{\bar{X}^+/k^+}$. Note $\Gamma(\bar{X}_1, \Theta_{\bar{X}^+/k^+})$ is the free $A_1 = k[z^{(1,0,0)}, z^{(-1,0,0)}]$-module generated by $z^{(1,0,-1)} \partial_{(0,0,1)}$ and $z^{(-1,0,0)} \partial_{(1,0,1)}$ from Proposition 1.8 while $\Gamma(\bar{X}_2, \Theta_{\bar{X}^+/k^+})$ is the free

$$A_2 = k[z^{(0,0,1)}, z^{(1,0,-1)}, z^{(-1,l,1)}, z^{(0,1,0)}]/(z^{(1,0,0)}).$$
module generated by $z^{(0,0,-1)} \partial_{(1,0,1)}$ and $\partial_{(0,0,1)}$. In addition $\Gamma(\tilde{X}_1 \cap \tilde{X}_2, \Theta_{\tilde{X}_1/k^t})$ is the free $A_{12} = \mathbb{k}[z^{(-1,0,1)}, z^{(1,0,-1)}, z^{(1,0,0)}]$-module generated by $\partial_{(0,0,1)}$ and $z^{(-1,0,0)} \partial_{(1,0,1)}$. From this one calculates easily that the kernel of the Čech coboundary map

\[ \Gamma(\tilde{X}_1, \Theta_{\tilde{X}_1/k^t}) \oplus \Gamma(\tilde{X}_2, \Theta_{\tilde{X}_1/k^t}) \to \Gamma(\tilde{X}_1 \cap \tilde{X}_2, \Theta_{\tilde{X}_1/k^t}) \]

is $\Gamma(X, \Theta_{X'/k^t})$ using the description of Proposition \[ \text{LNS} \] (which tells us $\Gamma(X, \Theta_{X'/k^t})$ is generated by $\partial_{(0,0,1)}, \partial_{(1,0,0)}, z^{(1,0,-1)} \partial_{(0,0,1)}$ and $z^{(-1,0,0)} \partial_{(1,0,1)}$), and the cokernel of this map is zero, proving (2) and (3). □

**Lemma 2.13.** In the situation of Construction \[ \text{LNS} \]

(1) the (ordinary) sheaf of derivations $\Theta_{V'(\tau)/k}$ is isomorphic to the sheaf whose sections over an open subset $V \subseteq V'(\tau)$ are

\[ \{ (h_p)_{p \in \partial P'} | h_p \in \Gamma(V, \mathcal{O}_V(p)) \text{ and } h_p + h_q = h_{p+q} \text{ on } V \cap V(p + q) \} . \]

Here $V(p) = \text{cl}(V'(\tau) \setminus \{ z^p = 0 \})$.

(2) Let $\mathfrak{l}_s_{V'(\tau)}$ denote the cokernel of the inclusion

\[ M' \otimes \mathcal{O}_{V'(\tau)} \to \Theta_{V'(\tau)/k} \]

given by $M' \ni m \mapsto (\langle m, p \rangle)_{p \in \partial P'}$. Then $\Gamma(V, \mathfrak{l}_s_{V'(\tau)})$ consists of

\[ (f_\omega)_\omega \in \bigoplus_{\dim \omega = 1, \omega \subseteq \tau} \Gamma(V, \mathcal{O}_{V'}) \]

such that for every two-dimensional face $\tau'$ of $\tau$,

\[ \sum_{\dim \omega = 1} \epsilon_{\tau'}(\omega) f_\omega|_{V'_{\omega'}} d_\omega = 0 \in \Gamma(V, M' \otimes \mathcal{O}_{V'}) . \]

Here $\epsilon_{\tau'}$ is a choice of sign vector for $\tau'$ (see [12], Definition 3.21), $d_\omega$ is a choice of primitive generator of the tangent space of an edge $\omega \subseteq \tau$, and $V'_{\omega'}$ is the toric stratum of $V'(\tau)$ corresponding to $\omega \subseteq \tau$. We think of $\mathfrak{l}_s_{V'(\tau)}$ as the Lie algebra of $\mathcal{L} \mathfrak{s}_{V'(\tau)}$.

**Proof.** (1) The data $(h_p)_{p \in \partial P'}$ on an open set $V$ defines a derivation $D : \mathcal{O}_V \to \mathcal{O}_V$ by $D(z^p) = h_p z^p$ for $p \in \partial P'$. This is immediately checked to be a derivation, and it gives an injective map from the sheaf of such data to $\Theta_{V(\tau)/k}$. Conversely, if $D$ is a derivation on an open set $V$, define for each $p \in \partial P'$ with $V \cap V(p) \neq \emptyset$,

\[ h_p = \frac{Dz^p}{z^p} . \]

This is clearly a well-defined function on $V'(\tau) \setminus \{ z^p = 0 \}$, and we need to show it extends to a function on $V(p)$. To show this, we first note we can restrict the derivation $D$ to irreducible components of $V$. Indeed, assuming $V$ is affine and $W$ is an irreducible component of $V$, let $f$ be a function on $W$. We can extend it to a function $f'$ on $V$, hence compute $(Df')|_W$. This
is independent of the lifting: if \( f', f'' \) are two different liftings of \( f \), then \( D(f' - f'')|_W = 0 \). Now if we denote by \( \partial W \) the toric boundary of \( W \) (which is \( \bigcup_{W'} (W' \cap W) \) where \( W' \) runs over all irreducible components of \( V \) not equal to \( W \)), then \( D \) must preserve the ideal of \( \partial W \) in \( W \) (extend a function in this ideal by zero off of \( W \)). Thus \( D \) is a log derivation for the pair \((W, \partial W)\) (see Example 1.4). Thus if \( W \) is in particular an irreducible component of \( V(p) \cap V \), it then follows that on \( W \), \( D \log p = \frac{Dp}{zp} \) is a regular function. Thus \( D \frac{Dp}{zp} \) is a regular function on \( V(p) \cap V \). This gives the data \((h_p)_{p \in \partial P'} \) as desired.

For (2), one repeats the proof of [12], Theorem 3.22, word for word, but everything is additive rather than multiplicative. \( \square \)

**Lemma 2.14.** In the situation of Construction 2.1, let \( \pi_1, \pi_2 \) be the projections of \( X \cong V'(\tau) \times \mathbb{A}^q \) to \( V'(\tau) \) and \( \mathbb{A}^q \) respectively.

1. The locus where \( \Theta_{X/\mathbb{A}^1} \) is not locally free is contained in \( Z = \bigcup_{i=1}^q Z_i \).
2. Suppose furthermore the convex hull of \( \bigcup_{i=1}^q (\Delta_i + e_i) \) in \( M' \oplus \mathbb{Z}^q \) is an elementary simplex. If \( \mathcal{B} \) is defined by the exact sequence

\[
0 \to \Theta_{X/\mathbb{A}^1} \to \Theta_{X/\mathbb{A}^1} \to \mathcal{B} \to 0,
\]

then \( \Gamma(V, \mathcal{B}) \) for any open subset \( V \subseteq X \) consists of those

\[
(f_\omega) \in \bigoplus_{\dim \omega = 1, \omega \subseteq \tau} \Gamma(V, \mathcal{L}_\omega),
\]

where

\[
\mathcal{L}_\omega = \begin{cases} 
\mathcal{O}_{V_\omega} \cdot u_i^{-1} & \text{if } \omega \in \Omega_i, \\
\mathcal{O}_{V_\omega} & \text{otherwise},
\end{cases}
\]

such that \((f_\omega)\) satisfies the condition that for every two-dimensional face \( \tau' \) of \( \tau \),

\[
\sum_{\dim \omega = 1} e_{\tau'}(\omega) f_\omega|_{V_\tau'} d_\omega = 0 \in M' \otimes K(V_{\tau'}).
\]

Here \( K(V_{\tau'}) \) denotes the function field of \( V_{\tau'} \).
Proof. We first show we have a commutative diagram of exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M' \otimes \mathcal{O}_X & \rightarrow & \Theta_{X^1/k'} & \rightarrow & \bigoplus_{i=1}^q \mathcal{I}_{Z_i/X} \partial / \partial u_i & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \pi_1^\ast \Theta_{V^1(r)/k} & \rightarrow & \Theta_{X/k} & \rightarrow & \pi_2^\ast \Theta_{A^q/k} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \pi_1^\ast \mathcal{L}_{V^1(r)} & \rightarrow & \mathcal{B} & \rightarrow & \bigoplus_{i=1}^q \mathcal{O}_{Z_i} \partial / \partial u_i & \rightarrow & 0 \\
\end{array}
\]

As \( \Theta_{X/k} = \pi_1^\ast \Theta_{V^1(r)/k} \oplus \pi_2^\ast \Theta_{A^q/k} \), and \( \pi_1^\ast \Theta_{V^1(r)/k} \) consists of those derivations of \( \Theta_{X/k} \) which are zero on all functions pulled back from \( \mathbb{A}^q \), one sees easily from the description of \( \Theta_{X^1/k'} \) of Proposition 1.8 that \( \Theta_{X^1/k'} \cap \pi_1^\ast \Theta_{V^1(r)/k} = M' \otimes \mathcal{O}_X \). Thus \( \text{coker}(M' \otimes \mathcal{O}_X \rightarrow \Theta_{X^1/k'}) \) injects into \( \pi_2^\ast \Theta_{A^q/k} \), and the only thing we need to do is to describe the image of this cokernel. Now \( \pi_2^\ast \Theta_{A^q/k} \) is generated by the coordinate vector fields \( \partial / \partial u_i, \, i = 1, \ldots, q \).

To see an element of \( \mathcal{I}_{Z_i/X} \partial / \partial u_i \) lifts to a logarithmic derivation, we consider the various generators of \( \mathcal{I}_{Z_i/X} \). Suppose \( z^p \in \mathcal{I}_{Z_i/X} \). First consider the case that \( p = e_i^\ast \). Recalling the notation of a log derivation from §1, we have \( \partial_{e_i} \in \Theta_{X^1/k'} \). Given a derivation \( D \in \Theta_{X/k} \), we can project it into \( \pi_2^\ast \Theta_{A^q/k} \) by evaluating it on functions pulled back from \( \mathbb{A}^q \), i.e. the image of \( D \) in \( \pi_2^\ast \Theta_{A^q/k} \) is

\[
\sum_{i=1}^q (D u_i)(\partial / \partial u_i).
\]

Thus in particular the image of \( \partial_{e_i} \) is \( u_i \partial / \partial u_i \), so \( \partial_{e_i} \) provides a lifting of the latter vector field. Next consider the case that \( p = n + \sum a_j e_j^\ast \in Q \setminus \{ \infty \} \) and there is a unique vertex \( w \) of \( \Delta \) with \( \langle n, w \rangle = -\tilde{\psi}_i(n) \). There are two subcases. If \( a_i > \tilde{\psi}_i(n) \), then \( z^p = z^{p'} u_i \) for some \( p' \in Q \), so we are done as before. If \( a_i = \tilde{\psi}_i(n) \), then we find that if \( v \) is a vertex of \( \Delta_j \) for \( j \neq i \), then

\[
\langle p - e_i^\ast, v + e_j \rangle = \langle p, v + e_j \rangle \geq 0,
\]

and if \( v \neq w \) is a vertex of \( \Delta_i \), then

\[
\langle p - e_i^\ast, v + e_i \rangle = \langle n, v \rangle + a_i - 1 > -\tilde{\psi}_i(n) + \tilde{\psi}_i(n) - 1 = -1,
\]

so \( \langle p - e_i^\ast, v + e_i \rangle \geq 0 \). Finally,

\[
\langle p - e_i^\ast, w + e_i \rangle = -1.
\]
As $w + e_i$ is a primitive generator of an extremal ray of the cone $K$, Proposition 1.8 tells us that $z^p u_i^{-1} \partial_{w + e_i} \in \Theta_{X'/k'}$. Now the image of this derivation in $\pi_{2*}^* \Theta_{X/k}$ is $z^p \partial / \partial u_i$, and hence provides a lifting of $z^p \partial / \partial u_i$.

Conversely, if $z^p \partial_m \in \Gamma(X, \Theta_{X'/k'})$ with $m = m' + \sum_{j=1}^q b_j e_j$, for $m' \in M'$, we consider its image $z^p \sum b_j u_j \partial / \partial u_j$ in $\pi_{2*}^* \Theta_{X/k}$. If $p \in P$, then clearly the latter is in $\bigoplus \mathcal{I}_{Z_i/X} \partial / \partial u_i$. Otherwise there is some unique $i \neq 0$ and unique vertex $w$ of $\Delta_i$ such that $\langle p, w + e_i \rangle = -1$, $\langle p, v + e_j \rangle \geq 0$ for all vertices $v \neq w$ of $\Delta_j$. Then $z^p u_i \in \Gamma(X, \mathcal{O}_X)$, and if $p = n + \sum a_j e_j$, we have $a_j \geq \psi_j(n)$ for all $j \neq i$ and $a_i = \psi_i(n) - 1$. Furthermore, $w$ is the only vertex of $\Delta_i$ for which $\langle w, n \rangle = -\psi_i(n)$. Thus $z^p u_i$ is in the ideal of $Z_i$, $m$ is proportional to $w + e_i$, and $z^p \partial_{w + e_i}$ projects to $z^p u_i \partial / \partial u_i$. This shows the image of $z^p \partial_m$ is in $\bigoplus \mathcal{I}_{Z_i/X} \partial / \partial u_i$. In particular, this shows (1).

To prove (2), first note the assumption that the $\Delta_i$’s yield an elementary simplex implies that the sets $\Omega_i$ are disjoint, so the description of the sheaf $\mathcal{L}_\omega$ in the statement makes sense. Call the sheaf claimed to be isomorphic to $\mathcal{B}$ in the statement $\mathcal{B}'$. To show $\mathcal{B}' \cong \mathcal{B}$, let $j : X \setminus Z \hookrightarrow X$ be the inclusion. Since $Z$ is codimension 2, $\Gamma(X \setminus Z, \Theta_{X'/k'}) = \Gamma(X, \Theta_{X'/k'})$, so $H^1_{\mathcal{L}}(X, \Theta_{X'/k'}) = 0$, while of course $H^0_{\mathcal{L}}(X, \Theta_{X/k}) = 0$. Thus $H^0_{\mathcal{L}}(X, \mathcal{B}) = 0$, and the adjunction map

$$\mathcal{B} \rightarrow j_* j^* \mathcal{B} = j_*(\pi_{2*}^* \mathcal{L}_{V'(\tau)})$$

is injective. Now $\mathcal{B}'$ is clearly a subsheaf of $j_*(\pi_{1*}^* \mathcal{L}_{V'((\tau))})$, and we want to show it is the image of $\mathcal{B}$. We first show the image of $\mathcal{B}$ is contained in $\mathcal{B}'$. Note that $\mathcal{B}$ is generated by $\pi_{1*}^* \mathcal{L}_{V'((\tau))}$ and the images of $\partial / \partial u_1, \ldots, \partial / \partial u_q$. We can determine the images of $\partial / \partial u_i$ in $j_*(\pi_{1*}^* \mathcal{L}_{V'((\tau))})$ as follows. After localizing at $u_i$, consider the log vector field $u_i^{-1} \partial_{e_i}$. This splits as a section of $\Theta_{X/k}$ as $D_i + \partial / \partial u_i$, where $D_i$ is a section of $\pi_{2*}^* \Theta_{V(\tau)/k}$. Thus $\partial / \partial u_i$ is congruent to $-D_i$ modulo $\Theta_{X'/k'}$ on $X \setminus \{u_i = 0\}$. To determine $D_i$, we evaluate $u_i^{-1} \partial_{e_i}$ on pull-backs of functions on $V'((\tau))$, i.e. for $p \in \partial P'$, identified with $N' \cup \{\infty\}$,

$$D_i(z^p \circ \pi_1) = u_i^{-1} \partial_{e_i}(z^p \circ \pi_1) = u_i^{-1} \partial_{e_i}(z^{p + \sum_{j=1}^q \psi_j(p) e_j}) = u_i^{-1} \psi_i(p)(z^p \circ \pi_1).$$

Thus $D_i$ corresponds, using the description of $\Theta_{V'(\tau)/k}$ of Lemma 2.13 to $(h_p)_{p \in \partial P'}$ with $h_p = u_i^{-1} \psi_i(p)$. The image of $D_i$ in $(\pi_{1*}^* \mathcal{L}_{V'((\tau))})_{X \setminus \{u_i = 0\}}$ is then given by $(f_\omega)_{\omega}$ with $d_\omega \otimes f_\omega = u_i^{-1}(\operatorname{Vert}_i(v^-) - \operatorname{Vert}_i(v^+_\omega))$. This follows from the additive version of the construction in the proof of [12], Theorem 3.22. By the assumption on the $\Delta_i$’s, if $\operatorname{Vert}_i(v^-) - \operatorname{Vert}_i(v^+_\omega) \neq 0$ then it is equal to $d_\omega$. Thus $f_\omega = u_i^{-1}$ if $\omega \in \Omega_i$ and $f_\omega = 0$ otherwise. So this section $(f_\omega)_{\omega}$ of $\pi_{1*}^* \mathcal{L}_{V'((\tau))}$ in fact extends to a section of $\mathcal{B}'$ on $X$. This section is the image of $-\partial / \partial u_i$ in $j_* j^* \mathcal{B}$. Thus the image of $\mathcal{B}$ in $j_* j^* \mathcal{B}$ is contained in $\mathcal{B}'$.

Conversely, let $(f_\omega)_{\omega}$ be a section of $\mathcal{B}'$. For a given $\omega$, if $\omega \notin \Omega_i$ for any $i$, then $f_\omega = f^0_\omega$ is a regular function on $V_\omega$. Otherwise, if $\omega \in \Omega_i$, then we can write $f_\omega = f^0_\omega + u_i^{-1} f^1_\omega$, where $f^0_\omega, f^1_\omega$ are regular functions on $V_\omega$ with $f^1_\omega$ containing no terms divisible by $u_i$. Then by testing the additive condition for each two-face $\tau'$, we find that $(f^0_\omega)_{\omega} \in \pi_{1*}^* \mathcal{L}_{V'((\tau))}$. On the
other hand, we claim there is a function \( f_i \) on \( X \) such that \( f_i^1 = f_i|_{V_i} \) for all \( \omega \in \Omega_i \). Indeed, let \( \tau' \subseteq \tau \) be a two-dimensional face. Then \( \tau' \) determines a face \( \tau'_1 \) of \( \Delta_i \), the convex hull of vertices of \( \Delta_i \), representing \( -\psi \) on the maximal cones of \( \Sigma_\tau \) corresponding to vertices of \( \tau' \). Since \( \Delta_i \) is an elementary simplex and \( \dim \tau'_1 \leq 2 \), \( \tau'_1 \) is a standard simplex. If \( \dim \tau'_1 = 0 \), then \( f_i^1 = 0 \) for all \( \omega \subseteq \tau' \); if \( \dim \tau'_1 = 1 \), then \( \tau' \) has exactly two edges in \( \Omega_i \), say \( \omega_1 \) and \( \omega_2 \), and then the additive condition tells us that \( f_{\omega_1}|_{V_{\omega_1}} = f_{\omega_2}|_{V_{\omega_2}} \). If \( \dim \tau'_1 = 2 \), then \( \tau' \) has exactly three edges \( \omega_1, \omega_2, \omega_3 \) contained in \( \Omega_i \), and the additive condition again says \( f_{\omega_i}|_{V_{\omega_i}} \) is independent of \( i = 1, 2, 3 \). Thus the \( f_i^1 \)'s glue for \( \omega \in \Omega_i \), yielding \( f_i \).

Lift \( (f_{\omega}^0)_{\omega} \) to a derivation \( D \in \pi_1^* \Theta_{V(\tau)}/k \). Then \( (f_\omega)|_{\omega} \) is the image of \( D - \sum_{i=1}^q f_i \partial/\partial u_i \), showing \( B = B' \).

**Proof of Theorem 2.11**. This result follows from standard methods of deformation theory (see for example [16], Exposée III) and Lemma 2.10 once we prove a local uniqueness statement for divisorial deformations:

**Lemma 2.15.** For every closed geometric point \( \bar{x} \in X_k \), there is a strict étale open neighbourhood \( V_k^f \) of \( \bar{x} \) such that for any divisorial deformation \( X^f_A \) of \( X_k^f \) over \( \text{Spec} A \), the induced deformation \( V_A^f \) of \( V_k^f \) is independent of the deformation of \( X_k^f \).

**Proof.** If \( \bar{x} \notin \mathcal{Z} \), then \( X_k \to \text{Spec} k^f \) is log smooth at \( \bar{x} \), and this local uniqueness is proved for log smooth morphisms in [20] Proposition 8.3 and [22], (3.14). If \( \bar{x} \in \mathcal{Z} \), let \( P \) be the monoid of Theorem 2.6 yielding \( Y^f \) and \( X^f \in \text{Spec} k^f \) and strict étale morphisms \( \chi_k^f \to \chi_k^f \to \text{Spec} k^f \). Then we will show that for any divisorial deformation \( X_A^f \to \text{Spec} A \) of \( X_k^f \to \text{Spec} k^f \), the induced deformation \( V_A^f \to \text{Spec} A \) is also induced by the deformation \( Y^f \times_{\text{Spec} k^f} \text{Spec} A \to \text{Spec} A^f \) of \( X^f \to X \).

Let \( \tau, \Delta_0, \ldots, \Delta_q \) be the data provided by Theorem 2.6. We first consider the case that \( \dim \tau = 1 \), so \( \bar{x} \in X_k \) is a double point of \( X_k \). Then we can assume \( \tau = [0, l] \) for some integer \( l > 0 \), and that either \( \Delta_1 = \cdots = \Delta_q = \{0\} \), in which case \( \bar{x} \notin \mathcal{Z} \), a contradiction, or else \( \Delta_1 = [0, 1], \Delta_2 = \cdots = \Delta_q = \{0\} \). This data determines \( Y, X \) as in Construction 2.1. By replacing \( V_k \) by an affine étale neighbourhood, we can use Condition (1) of Definition 2.7 to reduce the uniqueness statement to the following: given strict étale maps \( \phi, \phi' : V_k^f \to X^f \), let \( V_A^f \) and \( (V_\phi^f)^f \) be the pull-backs of the thickening \( X_A^f \) of \( X^f \) to \( V_k \) via \( \phi \) and \( \phi' \) respectively. We need to show there is an isomorphism over \( \text{Spec} A^f \)

\[
\psi : V_A^f \to (V_\phi^f)^f
\]

which is the identity on \( V_k^f \). Now \( Y \) can be written as \( Y' \times A_k^q-1 \), \( X = X' \times A_k^q-1 \), where \( Y', X' \) are the \( Y \) and \( X \) of Lemma 2.12. Let \( \tilde{Y}', \tilde{X}' \) be the blow-ups of Lemma 2.12 and set \( \tilde{Y} = \tilde{Y}' \times A_k^q-1 \), \( \tilde{X} = \tilde{X}' \times A_k^q-1 \), and let \( \tilde{V}_k = V_k \times \tilde{X} \tilde{X} \). We note here it is irrelevant whether we take this fibred product using the map \( \phi \) or the map \( \phi' \): either way, we are simply taking the normalization of \( V_k \), blowing up one of the two components along the
singular set \( p^{-1}(Z) \) of the log structure, and then regluing along the proper transform of the conductor locus.

Let \( \tilde{V}_A^+ \) be the pull-backs of the thickenings \( \tilde{Y}^+ \times_{\Spec k[\tau]} \Spec A^+ \) via \( \hat{\phi}, \hat{\phi}' : \tilde{V}_k \to \tilde{X} \). By Lemma 2.12, \( \tilde{V}_A^+ \) and \( \tilde{V}_{A'}^+ \) are log smooth over \( \Spec A^+ \), and these are both log smooth deformations of the same log smooth scheme \( \tilde{V}_k^+ \). We can apply F. Kato’s log smooth deformation theory here [20], and as \( H^1(\tilde{V}_k, \Theta_{\tilde{V}_k^+/k}) = 0 \) by Lemma 2.12 (3), it follows these deformations are unique, so there is an isomorphism \( \tilde{\psi} : \tilde{V}_A^+ \to (\tilde{V}_A^+)^{\dagger} \) which is the identity on \( \tilde{V}_k^+ \). This induces an isomorphism \( \psi : V_A^+ \to (V_A^+)^{\dagger} \) which is the identity on \( V_k^+ \); indeed, we get an isomorphism

\[
V_A^+ \setminus Z = \tilde{V}_A^+ \setminus E \to (\tilde{V}_A^+)^{\dagger} \setminus E = (V_A^+)^{\dagger} \setminus Z
\]

which is the identity on \( V_k^+ \setminus Z \), hence extends to an isomorphism \( \psi : V_A^+ \to (V_A^+)^{\dagger} \) because these spaces are \( S_2 \).

All that is needed to complete the proof of Theorem 2.11 is the general case, i.e. \( \dim \tau \geq 1 \). In this case, we will in fact show a stronger claim:

**Proposition 2.16.** Let \( Z_3 \) be the intersection of \( Z \) with the union of toric strata of \( X \) of codimension \( \geq 2 \), so that \( Z_3 \) is codimension 3 in \( X \). Let \( \tilde{x} \in Z_3 \), and let \( \tilde{Z}_3 \) be the pull-back of \( Z_3 \) to \( V_k \). Let \( V_A^+ \to \Spec A^+ \) be a flat deformation of \( V_k^+ \to \Spec k^+ \) such that \( V_A^+ \setminus \tilde{Z}_3 \to \Spec A^+ \) is a divisorial deformation of \( V_k^+ \setminus \tilde{Z}_3 \to \Spec k^+ \). Then \( V_A^+ \to \Spec A^+ \) is isomorphic to the deformation induced by \( X_A^+ \to \Spec A^+ \). This shows in particular that \( V_A^+ \to \Spec A^+ \) is divisorial, and also shows the local uniqueness in the dim \( \tau \geq 1 \) case.

**Proof.** We will proceed inductively. Assume \( A' \) is a small extension of \( A \) by an ideal \( I \), and assume the result is true for \( V_A^+ = V_{A'}^+ \times_{\Spec A'^+} \Spec A^+ \). Let \( (V_A^+)^{\dagger} \to \Spec A^+ \) be the pull-back of \( X_A^+ \to \Spec A^+ \) via \( \phi : V_k \to X \). We compare \( (V_A^+)^{\dagger} \) and \( V_{A'}^+ \). We see \( V_A^+ \setminus \tilde{Z}_3 \to \Spec A^+ \) and \( (V_A^+)^{\dagger} \setminus \tilde{Z}_3 \to \Spec A^+ \) are both liftings of the same deformation \( V_A^+ \setminus \tilde{Z}_3 \to \Spec A^+ \). As we have already shown the necessary local uniqueness result for points of \( Z \) not contained in \( Z_3 \), the set of all such deformations is a torsor over \( H^1(V_k \setminus \tilde{Z}_3, \Theta_{V_k^+/k} \otimes I) \), so we can write the difference between these two deformations as an element \( \theta \in H^1(V_k \setminus \tilde{Z}_3, \Theta_{V_k^+/k} \otimes I) \subset \Ext^1_{\O_{V_k}^{\dagger} / \tilde{Z}_3} (\Omega^1_{V_k^{\dagger} / k} |_{V_k^+} \otimes I, \O_{V_k} / \tilde{Z}_3) \). We wish to show \( \theta = 0 \). We first note that the natural map \( \Omega^1_{V_k / k} \to \Omega^1_{V_k^{\dagger} / k} \) induces a map

\[
\psi : \Ext^1_{\O_{V_k}^{\dagger} /	ilde{Z}_3} (\Omega^1_{V_k^{\dagger} / k} |_{V_k^+} \otimes I, \O_{V_k} / \tilde{Z}_3) \to \Ext^1_{\O_{V_k} /	ilde{Z}_3} (\Omega^1_{V_k / k} |_{V_k^+} \otimes I, \O_{V_k} / \tilde{Z}_3)
\]

The set of liftings of \( V_A \setminus \tilde{Z}_3 \to \Spec A \) to flat deformations of schemes over \( \Spec A' \) is a torsor over the latter group, by traditional deformation theory. In addition, \( \psi(\theta) \) is then the difference between the liftings \( V_{A'} \setminus \tilde{Z}_3 \) and \( V_{A'} \setminus \tilde{Z}_3 \) of \( V_A \setminus \tilde{Z}_3 \) as schemes. Now since \( V_{A'} \) and \( V_{A'} \) are both deformations of \( V_A \) as a scheme, we in fact have

\[
\psi(\theta) \in \im (\Ext^1_{\O_{V_k} / k} (\Omega^1_{V_k / k} \otimes I, \O_{V_k}) \to \Ext^1_{\O_{V_k} / \tilde{Z}_3} (\Omega^1_{V_k / k} |_{V_k^+} \otimes I, \O_{V_k} / \tilde{Z}_3)).
\]
We also have a commutative diagram with exact rows coming from local-global Ext spectral sequences

\begin{equation}
\begin{array}{c}
H^1(\mathcal{V}_k \setminus \tilde{Z}_3, \Theta_{\mathcal{V}_k/k} \otimes I) \to \text{Ext}^1_{\mathcal{O}_{\mathcal{V}_k}}(\Omega^1_{\mathcal{V}_k/k} \otimes I, \mathcal{O}_{\mathcal{V}_k}) \\
\downarrow \quad \downarrow \phi \\
H^1(\mathcal{V}_k \setminus \tilde{Z}_3, \Theta_{\mathcal{V}_k/k} \otimes I) \to \text{Ext}^1_{\mathcal{O}_{\mathcal{V}_k}}(\Omega^1_{\mathcal{V}_k/k} \otimes I, \mathcal{O}_{\mathcal{V}_k}) \to H^0(\mathcal{V}_k \setminus \tilde{Z}_3, \text{Ext}^1(\Omega^1_{\mathcal{V}_k/k} \otimes I, \mathcal{O}_{\mathcal{V}_k}))
\end{array}
\end{equation}

The lower right-hand vertical arrow is an injection as \( \mathcal{V}_k \) is étale locally isomorphic to a (reducible) toric variety, and hence the support of any section of \( \text{Ext}^1(\Omega^1_{\mathcal{V}_k/k} \otimes I, \mathcal{O}_{\mathcal{V}_k}) \) must be a toric stratum. By a diagram chase, it is then clear that \( \psi(\theta) \) is in fact in the image of \( H^1(\mathcal{V}_k, \Theta_{\mathcal{V}_k/k} \otimes I) \). Since we can assume \( \mathcal{V}_k \) is affine, the latter group is zero. Thus \( \mathcal{V}_A \) and \( \mathcal{V}_A' \) must be isomorphic as schemes over \( \text{Spec} A' \). So we only need to compare the log structures. In particular, this shows that \( \theta \in \ker(H^1(\mathcal{V}_k \setminus \tilde{Z}_3, \Theta_{\mathcal{V}_k/k} \otimes I) \to H^1(\mathcal{V}_k \setminus \tilde{Z}_3, \Theta_{\mathcal{V}_k/k} \otimes I)) \). We will show this kernel is zero, thus showing the two liftings are equivalent deformations. Indeed, this kernel is isomorphic to the cokernel of

\[ H^0(\mathcal{V}_k \setminus \tilde{Z}_3, \Theta_{\mathcal{V}_k/k} \otimes I) \to H^0(\mathcal{V}_k \setminus \tilde{Z}_3, \phi^* \mathcal{B} \otimes I), \]

where \( \mathcal{B} \) is defined in Lemma 2.14. By the description of \( \mathcal{B} \) there, the fact that \( \tilde{Z}_3 \) is codimension two in the codimension one strata of \( \mathcal{V}_k \), and the fact that these strata are \( S_2 \), it follows that any element of \( H^0(\mathcal{V}_k \setminus \tilde{Z}_3, \mathcal{B} \otimes I) \) lifts to an element of \( H^0(\mathcal{V}_k, \mathcal{B} \otimes I) \). However, \( H^0(\mathcal{V}_k, \Theta_{\mathcal{V}_k/k} \otimes I) \to H^0(\mathcal{V}_k, \mathcal{B} \otimes I) \) is surjective as \( \mathcal{V}_k \) is affine, so the cokernel of the map of sections over \( \mathcal{V}_k \setminus \tilde{Z}_3 \) is zero, as desired.

Having now proved Theorem 2.11 we note

**Corollary 2.17.** With the same hypotheses as in Theorem 2.11, if \( \mathcal{X}_k^\dagger \to \text{Spec} A^\dagger \) is a flat deformation of \( \mathcal{X}_k \to \text{Spec} k^\dagger \) which is divisorial off of \( Z_3 \), the intersection of \( Z \) with the union of codimension two strata of \( \mathcal{X}_k \), then \( \mathcal{X}_k^\dagger \to \text{Spec} A^\dagger \) is divisorial at all points of \( \mathcal{X}_A \).

**Proof.** This follows from Proposition 2.16. \( \square \)

Recall now that if there exists a strictly convex multi-valued integral piecewise linear function \( \varphi \) on \( (B, \mathcal{P}) \), then the main theorem of [13] yields a \( k \)-th order deformation \( \mathcal{X}_k^\dagger \to \text{Spec} k[t]/(t^{k+1})^\dagger \) of \( X_0(B, \mathcal{P}, s)^\dagger \to \text{Spec} k^\dagger \) (see [13], Theorem 1.29, Remark 1.28, and Remark 2.40). To know the results of this paper apply to this deformation, we need to know this deformation is divisorial.

**Corollary 2.18.** With the same hypotheses as in Theorem 2.11, the \( k \)-th order deformation \( \mathcal{X}_k^\dagger \to \text{Spec} k[t]/(t^{k+1})^\dagger \) of \( X_0(B, \mathcal{P}, s)^\dagger \to \text{Spec} k^\dagger \) constructed in [13] (given a strictly convex multi-valued integral piecewise linear function \( \varphi \) on \( (B, \mathcal{P}) \)) is divisorial.
Proof. By Corollary 2.17 it is enough to check divisoriality off of $\mathcal{Z}_3$. Proposition 2.32 shows log smoothness away from $\mathcal{Z}$, while divisoriality at points of $\mathcal{Z} \setminus \mathcal{Z}_3$ will follow from a local argument as in the proof of [13], Lemma 2.34. Indeed, given $\omega \in \mathcal{P}$ an edge, a thickening of the (étale) open set $V(\omega)$ of $X_0(B, \mathcal{P}, s)$ given by the construction of $[13]$ is

$$V^k(\omega) = \text{Spec } R_U,$$

where $R_U = R_- \times_{R_\gamma} R_+$ and

$$R_- = k[Q^\vee_\omega][x, y, t]/(xy - t^l, y^\beta t^\gamma | \beta l + \gamma \geq k + 1)$$

$$R_+ = k[Q^\vee_\omega][x, y, t]/(xy - t^l, x^\alpha t^\gamma | \alpha l + \gamma \geq k + 1)$$

$$R_\gamma = (k[Q^\vee_\omega][x, y, t]/(xy - t^l, x^\alpha y^\beta t^\gamma | \max \{\alpha, \beta\} l + \gamma \geq k + 1))_{f_\omega}.$$

Here $Q_\omega = \Lambda_y/\Lambda_\omega$ for any $y \in \text{Int}(\omega) \setminus \Delta$ (it is worth remembering in [13] we work with the Legendre dual $B$, hence we use the dimension one $\omega$ instead of the codimension one $\rho$ of [13], Lemma 2.34) and $l$ is the affine length of $\omega$. Furthermore, $f_\omega \in k[Q^\vee_\omega][t]/(t^{k+1})$ is a function with the property that $f_\omega \mod t$ is the section of $\mathcal{LS}^{+}_{\text{pre}, V(\omega)}$ defining the log structure on $V(\omega)$. (Compare with Example 2.8) The fibred product is defined using maps $R_+ \to R_\gamma$, the canonical quotient map followed by the localization map, while $R_- \to R_\gamma$ twists this by

$$x \mapsto f_\omega x, \quad y \mapsto f_\omega^{-1} y.$$

In particular, as shown in the proof of [13], Lemma 2.34,

$$R_U = k[Q^\vee_\omega][X, Y, t]/(XY - f_\omega t^l, t^{k+1}),$$

with $X = (x, f_\omega x), Y = (f_\omega y, y)$.

In addition, the log structure on $V^k(\omega)$ is defined by gluing together the two standard log structures on $\text{Spec } R_\pm$; these are given by the canonical charts $P_l \to R_\pm$ where $P_l$ is the monoid generated by $p_1, p_2$ and $\rho$ with $p_1 + p_2 = lp$ and $p_1 \mapsto x, p_2 \mapsto y, \rho \mapsto t$. Furthermore, these come with canonical log morphisms to $\text{Spec } k[t]/(t^{k+1})^\dagger$ (with log structure given by the chart $\mathbb{N} \ni n \mapsto t^n$) given by the monoid homomorphism $\mathbb{N} \to P_l, 1 \mapsto \rho$.

The glued log structure on $V^k(\omega)$ can be described using three charts, on the open sets $U_X, U_Y, U_{f_\omega} \subseteq V^k(\omega)$ defined by localizing at $X, Y$ and $f_\omega$ respectively. Since $U_X \cong \text{Spec}(R_-)_x$ and $U_Y \cong \text{Spec}(R_+)_y$, the charts for the log structure on these two open sets are just those given by the above standard charts $P_l \to R_\pm$. To see what the chart on $U_{f_\omega}$ is, we modify the chart on $R_-$ to be $p_1 \mapsto f_\omega^{-1} x, p_2 \mapsto f_\omega y, \rho \mapsto t$. This makes sense after localizing at $f_\omega$, and does not change the log structure or the log morphism to $\text{Spec } k[t]/(t^{k+1})^\dagger$. Now the charts $P_l \to R_\pm$ glue, giving $p_1 \mapsto f_\omega^{-1} X, p_2 \mapsto Y, \rho \mapsto t$. This describes the log structure on $V^k(\omega) \setminus \mathcal{Z}$, along with a log morphism to $\text{Spec } k[t]/(t^{k+1})^\dagger$. 


The log structure on $V_k^k(\omega)$ is then obtained by pushing forward the sheaf of monoids on $V_k^k(\omega) \setminus Z$ to $V_k^k(\omega)$.

It is then not difficult to check that this morphism $V_k^k(\omega) \to \text{Spec } k[t]/(t^{k+1})^\dagger$ coincides with that given by base-change of $\text{Spec } k[Q_\omega^\vee][X, Y, t]/(XY - f_\omega t^l) \to \text{Spec } k[t]$ as in Example 2.8. As in that Example, it is easy to check this is a divisorial deformation: the fact that here $f_\omega$ also depends on $t$ does not change the structure of the deformation étale locally. 

\[\square\]

Remark 2.19. When we began this project in 2001, it was our original hope that there would be an easy Bogomolov-Tian-Todorov type argument to show smoothability of log Calabi-Yau spaces. This approach failed, and instead, if one wants to show unobstructedness of divisorial deformation theory, one needs to appeal to the explicit smoothing results of [13]. We sketch an argument here, though we do not expect this will be important in the future for understanding mirror symmetry. Let $(B, \mathcal{P})$ be positive and simple. We assume that $X_0 = X_0(B, \mathcal{P}, s)$ comes along with a polarization, so that we can apply the results of [13]. We will also make a number of other assumptions we will include below.

First, there is a formal versal deformation space for divisorial deformations of $X_0^\dagger$. To make this precise we have to consider the corresponding functor of log Artin rings as in [21]. Let $S_0^\dagger := \text{Spec } k^\dagger$, (with the obvious $R$-algebra structure on $k$). We then look at the functor from $\mathcal{C}_R$ to sets associating to $A \in \mathcal{C}_R$ the set of isomorphism classes of divisorial deformations of $X_0^\dagger \to S_0^\dagger$ over $S^\dagger = \text{Spec } A^\dagger$. It is shown in [21] that if the analogues $(H^1_{\log}) - (H^3_{\log})$ of Schlessinger’s conditions [23] hold for a functor of log Artin rings then this functor has a hull. In view of Proposition 2.16 and the arguments for the log smooth deformation functor in [21] this is easy to verify for our functor. It is also proved in [21] that the hull can be chosen to pro-represent the functor if $H^0(X_0, \Theta_{X_0^\dagger/s_0^\dagger}) = 0$, that is, if $X_0^\dagger$ is infinitesimally rigid. Again the proof of this statement easily translates to our functor of divisorial deformations. This additional assumption is fulfilled in the most interesting cases of degenerate Calabi-Yau varieties. In particular, this holds when the holonomy of $B$ is contained in $\mathbb{Z}^n \times \text{SL}_n(\mathbb{Z})$ and $H^0(B, i_*\Lambda \otimes k) = 0$, by Theorem 3.23. We restrict the further discussion to the situation where this holds. We will also need one further restriction to guarantee that all deformations of $X_0^\dagger$ are projective. For this we need $H^2(X_0, \mathcal{O}_{X_0}) = 0$, or equivalently, $H^2(B, k) = 0$.

Thus we now have a formal model $\mathcal{X}^\dagger \to \mathcal{M}^\dagger$ for a moduli space of log spaces containing $X_0^\dagger$. One important feature of log deformation theory is that by construction, there is a morphism $\mathcal{M}^\dagger \to \text{Spf } k[t]^\dagger$, and $H^1(X_0, \Theta_{X_0^\dagger/s_0^\dagger})$ is actually the relative tangent space at the point corresponding to $X_0^\dagger$ in $\mathcal{M}^\dagger$; indeed, this relative tangent space is just the set of log morphisms from $\text{Spec } k[\epsilon]^\dagger$ to $\mathcal{M}$ over $T^\dagger$, with $\Lambda$-algebra structure given by $t \mapsto 0$. According to Theorem 2.11 this is $H^1(X_0, \Theta_{X_0^\dagger/s_0^\dagger})$. 

\[\square\]
The assumption that $H^2(X_0, \mathcal{O}_{X_0}) = 0$ implies that $\mathcal{X} \to \mathcal{M}$ carries a relative polarization, and hence can be algebraized. If $\mathcal{M} = \text{Spf } R$ for some complete ring $R$, then setting $\mathcal{M} = \text{Spec } R$, we obtain a morphism of schemes $\mathcal{X} \to \mathcal{M}$ whose completion along the closed fibre is $\mathcal{X} \to \mathcal{M}$. Second, by the main result of [13] and Corollary 2.18 there exists an irreducible component of $\mathcal{M}$ such that if $\eta \in \mathcal{M}$ is the generic point of this component, the fiber $X_\eta$ is a normal variety over $\kappa(\eta)$ with at worst codimension four Gorenstein quotient singularities, by Proposition 2.2. For such varieties the deformation theory is unobstructed by [31], thus showing that the fibers of $\text{cl}(\eta) \to \text{Spec } k[t]$ have dimension $h^1(X_\eta, \Theta_{X_\eta/\eta})$. (We note this requires a technical point of openness of versality for logarithmic deformation spaces, which we have not checked.)

Third, by Theorem 4.2 and the triviality of $\Omega^n_{\mathcal{X}/\mathcal{M}^\dagger}$, we know

$$h^1(X_0, \Theta_{X_0/S_0}) = h^1(X_0, \Omega^{n-1}_{X_0/S_0}) = h^1(X_\eta, \Omega^{n-1}_{X_\eta/\eta}) = h^1(X_\eta, \Theta_{X_\eta/\eta}).$$

Thus $\mathcal{M}$ contains a subspace of relative dimension $h^1(X_0, \Theta_{X_0/S_0})$ over $\text{Spec } k[t]$. Since this is the relative dimension of the Zariski tangent space of $\mathcal{M}$ at the closed point we conclude that $\mathcal{M} \to \text{Spec } k[t]$ is smooth of this relative dimension. In particular, every infinitesimal divisorial deformation can be extended to a divisorial deformation over $\text{Spec } k[t]$. This is the claimed unobstructedness result. □

### 3. Cohomology of log Calabi-Yau spaces

Our aim in this section is to calculate the logarithmic de Rham cohomology of a log Calabi-Yau space, along with its Hodge decomposition. We will succeed in doing so under certain hypotheses (see Theorem 3.21). The condition we require on the log Calabi-Yau spaces is slightly stronger than the requirement that the log Calabi-Yau space be positive and simple. We have partial results when the space is only positive and simple (Theorem 3.22).

As usual, we begin with calculations for our local models.

#### 3.1. Local calculations.
In what follows, we want to understand certain operations we will perform on the sheaves of differentials appearing in Proposition 1.12. Initially, we will think of these modules abstractly by considering modules over monoid rings graded by elements of the monoid, and consider the operations we will need to make use of in this more general context.

**Lemma 3.1.** Let $P$ be a toric monoid, $Q \subseteq P$ a face, $Y = \text{Spec } \mathbb{k}[P]$, $I \subseteq P$ a monoid ideal with radical $P \setminus Q$, $X = \text{Spec } \mathbb{k}[P]/I$. Suppose furthermore that $p \in P$, $q \in Q$, $p + q \in I$ implies $p \in I$. Consider a $\mathbb{k}[P]$-module $F$ of the form

$$F = \bigoplus_{p \in P} z^p F(p)$$

Then $H^1(X, \Theta_{X/\mathcal{M}}) = 0$ implies $H^1(Y, \Theta_{Y/\mathcal{M}}) = 0$. Finally, if $\mathcal{M} = \text{Spec } R$ is smooth over $\text{Spec } \mathbb{k}[t]$, then $H^1(X_\eta, \Theta_{X_\eta/\eta}) = 0$. □
where \((p)\) denotes the minimal face of \(P\) containing \(p\), \(F_{(p)}\) a \(k\)-vector space, and \(F_{P_1} \subseteq F_{P_2}\) whenever \(P_1 \subseteq P_2\). Then

\[ F \otimes_{k[P]} k[P]/I = \bigoplus_{p \in P} z^p \left( \frac{F_{(p)}}{\sum_{p' \in P, q \in I} F_{(p')}} \right). \]

(2) If \(F_Q = F_P\) then

\[ F_X := (F \otimes_{k[P]} k[P]/I) \text{Tors} = \bigoplus_{p \in P \setminus I} z^p F_{(p)}, \]

where \text{Tors} denotes the submodule of elements of \(F \otimes_{k[P]} k[P]/I\) with support on a proper closed subset of \(X\).

(3) Let \(J \subseteq P\) be a monoid ideal such that if \(Z \subseteq X\) is the closed subscheme defined by \((I + J)/I \subseteq k[P]/I\), then \(Z\) is codimension \(\geq 2\) in \(X\). Let \(\kappa : X \setminus Z \hookrightarrow X\) be the inclusion. If \(\mathcal{F}_X\) is the sheaf on \(X\) associated to \(F_X\), then

\[ \Gamma(X, \kappa_* \kappa^* \mathcal{F}_X) = \bigoplus_{p \in P^{ap}} z^p \bigcap_{q \in J \cap Q} \bigcup_{n \geq 0} n q F_{(p + n q)} \]

Proof. (1) is obvious. For (2), we first check that if \(p \in I\), then the degree \(p\) piece of \(F \otimes_{k[P]} k[P]/I\) is annihilated by an element \(z^q\) for some \(q \in Q\). Since \(\sqrt{I} = P \setminus Q\), \(X \not\subseteq V(z^q)\), and so this will imply the degree \(p\) piece is torsion. Choose an element \(q \in Q\) not contained in any proper face of \(Q\). Then \(F_P = F_Q = F_{(q)} \subseteq F_{(p + q)} \subseteq F_{P}\), so \(F_{(p + q)} = F_P\), and

\[ \sum_{p' \in P, q' \in I} \frac{F_{(p + q)}}{F_{(p')}} = F_P/F_Q = 0, \]

(take \(q' = p, p' = q\) in the sum). Thus \(z^q\) annihilates the degree \(p\) piece as desired.

On the other hand, if \(p \in P \setminus I\), then the degree \(p\) part of \(F \otimes_{k[P]} k[P]/I\) is \(F_{(p)}\) by (1), and by the assumption on \(I\), \(z^q\) does not annihilate any element of \(F_{(p)}\) for any \(q \in Q\). This gives (2).

(3) Note that \(X \setminus Z\) is covered by open subsets of the form

\[ D(z^q) = \{ x \in \text{Spec} k[P]/I | z^q \not\in x \} \]

with \(q \in J \cap Q\), as the reduced space \(X_{\text{red}}\) of \(X\) is \(\text{Spec} k[Q]\) and \(\text{Spec} k[Q]/J \cap Q\) has the same support in \(X_{\text{red}}\) as \(Z_{\text{red}}\). Thus we can write

\[ \Gamma(X, \kappa_* \kappa^* \mathcal{F}_X) = \bigcap_{q \in J \cap Q} \Gamma(D(z^q), \mathcal{F}_X) = \bigcup_{q \in J \cap Q} (F_X)_{z^q}. \]

From (2) and the assumption on \(I\), we can write

\[ (F_X)_{z^q} = \bigoplus_{p \in P^{ap}} \bigcup_{n \geq 0} \bigcap_{p + n q \in P \setminus I} z^p F_{(p + n q)}. \]
from which follows (3).

Suppose we are given data $\tau \subseteq M'_R$, $\Delta_1, \ldots, \Delta_q$ as in Construction 2.1 yielding a cone $K \subseteq M_R$, $P = K^\circ \cap N$, $\rho \in P$, $Y = \text{Spec} \, k[P]$, $X = \text{Spec} \, k[P]/(z^\rho)$, $\mathcal{X}_k = \text{Spec} \, k[P]/(z^{(k+1)\rho})$.

For every face $\omega$ of $\tau$, we have a stratum $V_\omega \subseteq X$, with $V_\omega = \text{Spec} \, k[P_\omega]$, where $P_\omega$ is the face of $P$ given by $P \cap (\omega + e_0)^\perp$. For every $k \geq 0$, consider the monoid ideal

$$I_\omega^k = \{ p \in P | \langle p, n \rangle > k \text{ for some } n \in \omega + e_0 \}.$$ 

This defines a thickening

$$V_\omega^k = \text{Spec} \, k[P]/I_\omega^k.$$ 

One sees easily that $V_\omega^k$ is a closed subscheme of $\mathcal{X}_k$, and that $I_\omega^k$ satisfies the hypotheses of Lemma 3.1 with $Q = P_\omega$. Set $q_\omega : V_\omega^k \to \mathcal{X}_k$ the embedding.

Let $Z = \bigcup_i Z_i$ be the subscheme of $X$ defined in Construction 2.1 with $j : X \setminus Z \to X$ the inclusion. Set $D_\omega = \bigcup_{\omega \subseteq \omega' \subseteq \tau} V_{\omega'}$. This is a subset of the toric boundary of $V_\omega$ consisting of proper intersections of the stratum $V_\omega$ with other strata of $X$. Let

$$\kappa_\omega : V_\omega^k \setminus (D_\omega \cap Z) \to V_\omega^k$$

be the inclusion. For $\Omega_\omega^k = j_* \Omega_\omega^\tau_{\mathcal{X}_k^1/k}$ or $\Omega_\omega^k = j_* \Omega_\omega^\tau_{\mathcal{X}_k^1/A^1_k}$, let $\Omega_\omega^k = \kappa_\omega^* \kappa_\omega^\tau(q_\omega^* \Omega_\omega^\tau / \text{Tors})$.

Our goal now is to construct a resolution of the sheaf $\Omega_\omega^k$. In the next subsection, we shall use this resolution for $k = 0$ to compute the log Hodge spaces of nice log Calabi-Yau spaces. The case of $k > 0$ will be needed for future period calculations.

**Lemma 3.2.** Given $\omega \subseteq \tau$, let $\omega_i \subseteq \Delta_i$ be the largest face of $\Delta_i$ such that $\langle n, m \rangle = -\psi_i(n)$ for all $n \in \omega$, $m \in \omega_i$. (Here $\check{\omega}$ is the cone in the normal fan $\hat{\Sigma}_\tau$ of $\tau$ corresponding to $\omega$). Then

$$\Gamma(V_\omega^k, \Omega_\omega^k) = \bigoplus_{p \in P_{\omega,k}} z^p \left( \bigwedge^r \bigcap_{\{ (v,j) | v \in \omega_j, p \in (v+e_j)^\perp \} } ((v + e_j)^\perp \cap N) \otimes_{\mathbb{Z}} k \right)$$

or

$$\bigoplus_{p \in P_{\omega,k}} z^p \left( \bigwedge^r \bigcap_{\{ (v,j) | v \in \omega_j, p \in (v+e_j)^\perp \} } \left( ((v + e_j)^\perp \cap N) / \mathbb{Z}\rho \right) \otimes_{\mathbb{Z}} k \right)$$

in the $/k$ or $/A^1_k$ cases respectively, where $v$ runs over vertices of $\omega_j$ for any $j$ and

$$P_{\omega,k} := \left\{ p \in P_{\mathbb{G}P} \left| \begin{array}{l} \langle p, v \rangle \geq 0 \text{ for all } v \in \omega_i + e_i, \ 1 \leq i \leq q \\ \langle p, v \rangle \leq k \text{ for all } v \in \omega + e_0 \\ \langle p, v \rangle \geq 0 \text{ for all } v \in \tau + e_0 \end{array} \right. \right\}.$$
Proof. We will do the /k case; the other case is identical.

Let $Q_1, \ldots, Q_t$ be the maximal proper faces of $P$ containing $p$. Set, for $p \in P$,
\begin{equation}
\Omega_p^r = \bigwedge_{j=1}^r \bigoplus_{q \in Q_j} Q_j^p \otimes \kappa,
\end{equation}
so that $\bigoplus_{p \in P} z^p \Omega_p^r$ defines a sheaf $\Omega_Y^r$ on $Y$. Note that $\Omega_p^r$ only depends on $\langle p \rangle$, so set $\Omega_{\langle p \rangle}^r := \Omega_p^r$. One checks easily that Proposition 1.12 implies $\Omega_{\langle \rho \rangle}|_{X_k} \cong j_* \Omega_\omega^{\omega,k}$. Then by Lemma 3.1 (2) applied with $I = I_k^\omega$ and $F_{\langle p \rangle} = \Omega_{\langle p \rangle}^r$, 
\begin{equation}
\Gamma(V_\omega^k, (q^* j_* \Omega_\omega^{\omega,k})/\text{Tors}) = \bigoplus_{p \in P \setminus I_k^\omega} z^p \Omega_p^r.
\end{equation}
Let $\Gamma(V_\omega^k, \Omega_{\omega,k})_p$, on the other hand, be the degree $p$ piece of $\Gamma(V_\omega^k, \Omega_{\omega,k})$.

First let us determine the structure of $D_\omega \cap Z$. The toric strata of $V_\omega$ are in one-to-one correspondence with faces $P'$ of $P_\omega$, which in turn are in one-to-one order reversing correspondence with cones $K'$ with $K_\omega \subseteq K' \subseteq K$, where $K_\omega = C(\omega + e_0)$. Now a stratum corresponding to $K'$ is in $D_\omega \cap Z$ if it is contained in $D_\omega$ and $Z_i$ for some $i$. The stratum is contained in $D_\omega$ provided $C(\omega' + e_0) \subseteq K'$ for some $\omega'$ with $\omega \subseteq \omega' \subseteq \tau$. On the other hand, it is contained in $Z_i$ if, firstly, $u_i = 0$ on the stratum, i.e. $K' \cap C(\Delta_i + e_i) \neq 0$ (otherwise $K' \subseteq (e_i^\perp)$); secondly, the stratum is contained in $V_\omega''$ for some $\omega'' \in \Omega_i$, this being equivalent to $\dim \omega'_i > 0$. Thus, let $P_{D_\omega \cap Z}$ be the union of faces of $P_\omega$ corresponding to cones $K'$ satisfying
\begin{enumerate}
\item $K' \cap C(\Delta_0 + e_0) = C(\omega' + e_0)$ for some $\omega' \supseteq \omega$;
\item $K' \cap C(\Delta_i + e_i) \neq 0$ and $\dim \omega'_i > 0$ for some $1 \leq i \leq q$.
\end{enumerate}
Then
\[ J := P \setminus P_{D_\omega \cap Z} \]
is the monoid ideal defining $D_\omega \cap Z$. In particular, by Lemma 3.1 (3),
\begin{equation}
\Gamma(V_\omega^k, \Omega_{\omega,k})_p = \bigcup_{q \in J \cap P_\omega} \bigcup_{n \geq 0} \Omega_{(p + nq)}^r.
\end{equation}
Let $q \in J \cap P_\omega$, and we consider the union in the above expression for this $q$. Then $Q := \langle q \rangle \subseteq P_\omega$ corresponds to some $K'$ with $K_\omega \subseteq K' \subseteq K$ such that $K'$ fails to satisfy either property (1) or property (2) above. We consider three cases.

Case 1. $K' \cap C(\Delta_0 + e_0) = C(\omega + e_0)$. Then $K'$ is contained in $K'_\omega := C((\omega + e_0) \cup \bigcup_{i=1}^q (\omega_i + e_i))$, as this is easily seen to be the largest face of $K$ satisfying $K'_\omega \cap C(\Delta_0 + e_0) = C(\omega + e_0)$. Now as $q \in (K')^\perp$ but $\langle q, v \rangle > 0$ for all $v \in K \setminus K'$, $p + nq \in P$ for large $n$ if and only if $\langle p, v \rangle \geq 0$ for all $v \in K'$. This condition becomes more restrictive the larger $K'$ is, so in order for $\Gamma(V_\omega^k, \Omega_{\omega,k})_p$ to be non-zero, we must certainly have $\langle p, v \rangle \geq 0$ for all $v \in K'_\omega$, i.e. $\langle p, v \rangle \geq 0$ for $v \in \omega_i + e_i$ for all $i$, which gives the first condition in the definition of $P_{\omega,k}$, and also $\langle p, v \rangle \geq 0$ for $v \in \omega + e_0$. Furthermore, assuming $p + nq \in P$ for large $n$, then
p + nq ∈ P \ I_k^b for large n if and only if \( \langle p, v \rangle \leq k \) for all \( v \in \omega + e_0 \). This is the second condition in the definition of \( P_{\omega,k} \).

Now suppose \( p + nq \in P \setminus I_k^b \) for large n, and look at (3.2). It is clear that for two choices of \( q, q' \) with \( \langle q \rangle \leq \langle q' \rangle \) and both \( q \) and \( q' \) falling into this first case, then \( \langle p + nq \rangle \leq \langle p + nq' \rangle \) for large \( n \), and hence \( \Omega^r_{(p+nq)} \subseteq \Omega^r_{(p+nq')} \) for large \( n \). Thus as far as the intersection is concerned, we can assume that \( \langle q \rangle \) is as small as possible, and hence \( K' = K'_w \).

In this case, \( q \in (v + e_i)^\perp \) for each vertex \( v \) of \( \omega_i \), but \( \langle q, v + e_i \rangle > 0 \) if \( v \in \Delta_i \setminus \omega_i \). Thus for large \( n \) and \( v \) a vertex of \( \omega_i \), \( p + nq \in (v + e_i)^\perp \) if and only if \( p \in (v + e_i)^\perp \). On the other hand, if \( v \in \Delta_i \setminus \omega_i \), \( p + nq \notin (v + e_i)^\perp \) for large \( n \). Since the \( Q_j^{gp} \)'s are the spaces \( (v + e_i)^\perp \) for \( v \) running over vertices of \( \Delta_i \), for all \( i \), we see that for large \( n \),

\[
\Omega^r_{(p+nq)} = \bigwedge^r \bigcap_{\{v,j\} | v \in \omega_i, p \in (v+e_j)^\perp} ((v + e_j)^\perp \cap N) \otimes_{\mathbb{Z}} k.
\]

**Case 2.** Here we consider a special case when property (2) doesn’t hold, namely \( K' = C(\omega' + e_0) \) for some \( \omega' \supseteq \omega \). In this case, \( p + nq \in P \) for large \( n \) if and only if \( \langle p, v \rangle \geq 0 \) for all \( v \in \omega' + e_0 \). In particular, if we take \( K' = C(\tau + e_0) \), we see the intersection in (3.2) is zero unless \( \langle p, v \rangle \geq 0 \) for all \( v \in \tau + e_0 \), hence the last condition in the definition of \( P_{\omega,k} \).

**Case 3.** We consider the general case when property (2) doesn’t hold, so \( K' \cap C(\Delta_0 + e_0) = C(\omega' + e_0) \) for some \( \omega' \supseteq \omega \), but \( \dim \omega' = 0 \) whenever \( K' \cap C(\Delta_i + e_i) \neq 0 \). Then \( K' \) must be contained in \( K'_w = C((\omega' + e_0) + \bigcup_{i=1}^N (\omega_i + e_i)) \), but the extra condition implies that in fact \( K' \) is contained in \( C((\omega' + e_0) + \bigcup_{i=1}^N (\omega_i + e_i)) \). Since \( \omega_i \subseteq \omega_i' \), we have \( \omega_i = \omega_i' \) whenever \( \dim \omega_i = 0 \), hence \( K' \subseteq C((\omega' + e_0) + \bigcup_{i=1}^N (\omega_i + e_i)) \). One sees that if \( p \in P_{\omega,k} \), then \( p + nq \in P \setminus I_k^b \) for large \( n \), and furthermore for large \( n \), \( \Omega^r_{(p+nq)} \) contains \( \Omega^r_{(p+nq')} \) for \( q' \) with \( \langle q' \rangle \) dual to \( K'_w \).

Putting together these three cases, one obtains the desired result. \( \square \)

**Proposition 3.3.** Given faces \( \omega \subseteq \omega' \subseteq \tau \), we have \( I_\omega^k \subseteq I_\omega'^k \), and hence a closed embedding \( V_{\omega'}^k \rightarrow V_\omega^k \). Then

\[
\Gamma(V_{\omega'}, \Omega^r_{\omega,k}|V_{\omega'}/\text{Tors}) = \bigoplus_{p \in P^r_{\omega',k}} z^p \left( \bigwedge^r \bigcap_{\{v,j\} | v \in \omega_j, p \in (v+e_j)^\perp} ((v + e_j)^\perp \cap N) \otimes_{\mathbb{Z}} k \right)
\]

or

\[
\bigoplus_{p \in P^r_{\omega',k}} z^p \left( \bigwedge^r \bigcap_{\{v,j\} | v \in \omega_j, p \in (v+e_j)^\perp} (((v + e_j)^\perp \cap N)/\mathbb{Z}p) \otimes_{\mathbb{Z}} k \right)
\]

in the \( /k \) or \( /A_k^1 \) cases respectively, where \( v \) runs over vertices of \( \omega_j \) and

\[
P_{\omega',k} := \left\{ p \in P^{gp} \mid \begin{array}{l}
\langle p, v \rangle \geq 0 \text{ for all } v \in \omega_i + e_i, 1 \leq i \leq q \\
\langle p, v \rangle \leq k \text{ for all } v \in \omega' + e_0 \\
\langle p, v \rangle \geq 0 \text{ for all } v \in \tau + e_0
\end{array} \right\}.
\]
Note the only difference between this set and $P_{\omega,k}$ defined in Lemma 3.3 is that in the latter, \( \langle p, v \rangle \leq k \) for all $v \in \omega + e_0$ instead of for all $v \in \omega' + e_0$.

Proof. Let $\tilde{P}$ be the monoid

\[
\tilde{P} := \left\{ p \in P_{\text{gp}} \; \middle| \; \langle p, v \rangle \geq 0 \text{ for all } v \in \omega_i + e_i, 1 \leq i \leq q \right\}
\]

with ideals

\[
\tilde{I}_\omega^k := \left\{ p \in \tilde{P} \; \middle| \; \langle p, v \rangle > k \text{ for some } v \in \omega + e_0 \right\}
\]

and

\[
\tilde{I}_{\omega'}^k := \left\{ p \in \tilde{P} \; \middle| \; \langle p, v \rangle > k \text{ for some } v \in \omega' + e_0 \right\}.
\]

Note $P \subseteq \tilde{P}$, $I_{\omega}^k = P \cap \tilde{I}_{\omega}^k$, $I_{\omega'}^k = P \cap \tilde{I}_{\omega'}^k$. If $F$ is the $\mathbb{k}[\tilde{P}]$-module defined by

\[
F = \bigoplus_{p \in \tilde{P}} \mathbb{Z}^p \left( \bigwedge^r \left( \bigcap_{\{(v,j) | v \in \omega_j, p \in (v+e_j) \geq 0\}} \right) \right) \otimes \mathbb{Z} \mathbb{k}
\]

(or a similar expression in the $/A_1^k$ case), then from Lemma 3.1 (2) and Lemma 3.2, we see that

\[
\Gamma(V_{\omega}^k, \Omega_{\omega,k}^r) \cong (F \otimes_{\mathbb{k}[\tilde{P}]} \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega}^k) / \text{Tors}.
\]

Now we claim that

\[
\Gamma(V_{\omega'}^k, \Omega_{\omega',k}^r|_{V_{\omega'}^k}) \cong (F \otimes_{\mathbb{k}[\tilde{P}]} \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega'}^k) / \text{Tors},
\]

from which will follow the result by Lemma 3.1 (2) again. To prove this claim, first note it is easy to check that Tors $\subseteq F \otimes_{\mathbb{k}[\tilde{P}]} \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega}^k$ contains no sections with support containing $V_{\omega'}^k$, from which it follows that

\[
(\Gamma(V_{\omega}^k, \Omega_{\omega,k}^r) \otimes_{\mathbb{k}[\tilde{P}]} \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega}^k) / \text{Tors} \cong (F \otimes_{\mathbb{k}[\tilde{P}]} \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega'}^k) / \text{Tors}.
\]

So we need to compare

\[
\tilde{M} = (\Gamma(V_{\omega}^k, \Omega_{\omega,k}^r) \otimes_{\mathbb{k}[\tilde{P}]} \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega}^k) / \text{Tors}
\]

with

\[
M = (\Gamma(V_{\omega}^k, \Omega_{\omega,k}^r) \otimes_{\mathbb{k}[P]} \mathbb{k}[P] / I_{\omega}^k) / \text{Tors}.
\]

Note $\mathbb{k}[P] / I_{\omega}^k \subseteq \mathbb{k}[\tilde{P}] / \tilde{I}_{\omega}^k$, so $\tilde{M}$ is also a $\mathbb{k}[P] / I_{\omega}^k$-module, and we need to compare $\tilde{M}$ and $M$ as $\mathbb{k}[P] / I_{\omega}^k$-modules. We then have a diagram of $\mathbb{k}[P] / I_{\omega}^k$-modules with $T$ and $\tilde{T}$.
the torsion submodules of the modules immediately below them, (also confusing the sheaf \( \Omega^r_{\omega,k} \) with its space of global sections),

\[
\begin{array}{ccccccc}
0 & 0 & \rightarrow & \tilde{T} & \rightarrow & T & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\rightarrow & & \rightarrow & \Omega^r_{\omega,k} & \rightarrow & \Omega^r_{\omega,k} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \rightarrow & \tilde{M} & \rightarrow & 0 \\
\end{array}
\]

We see that \( \tilde{I}^k_{\omega} \Omega^r_{\omega,k} / I^k_{\omega} \Omega^r_{\omega,k} \subseteq T \), as for any \( q \) in the interior of \( P_{\omega'} \) and \( p \in \tilde{P} \), \( p + nq \in P \) for \( n \gg 0 \), so \( z^{nq} \) annihilates \( \tilde{I}^k_{\omega} \Omega^r_{\omega,k} / I^k_{\omega} \Omega^r_{\omega,k} \) for \( n \gg 0 \). Thus by the snake lemma, \( M \cong \tilde{M} \) provided \( T \rightarrow \tilde{T} \) is surjective. But if an element of \( \tilde{T} \) is represented by \( \alpha \in \Omega^r_{\omega,k} \), we have \( z^{nq} \alpha \in \tilde{I}^k_{\omega} \Omega^r_{\omega,k} \) for \( q \) in the interior of \( P_{\omega'} \) and \( n \gg 0 \). But since \( z^{nq} \tilde{I}^k_{\omega} \subseteq I^k_{\omega} \), for \( n \gg 0 \), we see in fact \( \alpha \) represents an element of \( T \), proving surjectivity.

\[ \square \]

**Corollary 3.4.**

1. Given faces \( \omega_1 \subseteq \omega_2 \subseteq \omega_3 \) of \( \tau \), we have an inclusion

\[
(\Omega^r_{\omega_2,k} |_{V^k_{\omega_3}}) / \text{Tors} \subseteq (\Omega^r_{\omega_1,k} |_{V^k_{\omega_3}}) / \text{Tors}.
\]

2. Given \( \omega_1 \subseteq \omega_2 \) faces of \( \tau \),

\[
(\Omega^r_{\omega_1,k} |_{V^k_{\omega_2}}) / \text{Tors} = \bigcap_{v \in \omega_1} (\Omega^r_{v,k} |_{V^k_{\omega_2}}) / \text{Tors},
\]

where \( v \) runs over vertices of \( \omega_1 \), and the intersection can be viewed as taking place in \( j_*(\Omega^r_{v,k} |_{V^k_{\omega_2} \setminus Z}) \), which is independent of \( v \) since \( \Omega^r_k \) is locally free away from \( Z \).

**Proof.** These statements follow immediately from the explicit formula of the previous corollary.

\[ \square \]

We can now define a resolution of \( \Omega^r_k \). We define a barycentric complex (a slight variation of that of [12], Appendix A) by

\[
\mathcal{C}^p(\Omega^r_k) = \bigoplus_{\omega_0 \subseteq \cdots \subseteq \omega_p \subseteq \tau} (\Omega^r_{\omega_0,k} |_{V^k_{\omega_p}}) / \text{Tors}
\]

and a differential

\[
d_{\text{bct}} : \mathcal{C}^p(\Omega^r_k) \rightarrow \mathcal{C}^{p+1}(\Omega^r_k)
\]
Theorem 3.5. \( \mathcal{C} \) of \( \Omega \) Here we use the inclusions of Corollary 3.4, (1) to identify all these elements with elements of \( (\Omega^r_{\omega,k}|_{V^p_{p+1}})/\text{Tors} \).

Proof. The inclusion \( \Omega^r_k \to \mathcal{C}^0(\Omega^r_k) \) is defined in the obvious way: an element \( \alpha \) of \( \Omega^r_k \) yields for each face \( \omega \) of \( \tau \) an element of \( \Omega^r_k|_{V^p} \), and hence an element of \( \Omega^r_{\omega,k} \). This gives the map

\[
\epsilon : \Omega^r_k \to \mathcal{C}^0(\Omega^r_k) = \bigoplus_{\omega \subseteq \tau} \Omega^r_{\omega,k},
\]

which is clearly injective, as is easily seen by checking on \( X_k \setminus Z \), where everything is locally free. The fact that \( \epsilon(\Omega^r_k) \subseteq \ker(d_{\text{bct}} : \mathcal{C}^0(\Omega^r_k) \to \mathcal{C}^1(\Omega^r_k)) \) is also easily checked off of \( Z \), where all these sheaves are locally free. Conversely, if \( \alpha \in \Gamma(U, \mathcal{C}^0(\Omega^r_k)) \) with \( d_{\text{bct}}(\alpha) = 0 \), then \( d_{\text{bct}}(\alpha|_{U \setminus Z}) = 0 \), and again as everything is locally free on \( U \setminus Z \), it is obvious that by gluing we obtain an element \( \beta \in \Gamma(U \setminus Z, \Omega^r_k) \) mapping to \( \alpha|_{U \setminus Z} \). But \( \Gamma(U \setminus Z, \Omega^r_k) = \Gamma(U, \Omega^r_k) \) by definition, so we obtain an element \( \beta \in \Gamma(U, \Omega^r_k) \) such that \( \epsilon(\beta) = \alpha \) on \( U \setminus Z \). Thus \( \epsilon(\beta) = \alpha \) on \( U \) as these sheaves have no sections with support on \( Z \), so \( \ker(d_{\text{bct}}) = \epsilon(\Omega^r_k) \).

To check exactness of \( \mathcal{C}^* \), we can’t use [12], §A.1 directly because our complex takes a slightly different form: the module \( (\Omega^r_{\omega_0,k}|_{V^p_{p+1}})/\text{Tors} \) depends on both \( \omega_0 \) and \( \omega_p \), not just \( \omega_p \). However, the result follows from a slight modification of the argument given in [12], Proposition A.1. The question of exactness then reduces to a version of an easily checked compatibility condition, namely if we are given \( \omega_0 \subseteq \omega_{p-1} \) and elements of \( (\Omega^r_{\omega_0,k}|_{V^p_{p+1}})/\text{Tors} \) for all \( \omega_{p-1} \to \omega_p \) which agree under restriction, then these elements lift to an element of \( (\Omega^r_{\omega_0,k}|_{V^p_{p+1}})/\text{Tors} \). However, this is true again by the explicit description in Proposition 3.3. \( \square \)

Proposition 3.6. The differential \( d : j_*\Omega^r_{X^p_k/k} \to j_*\Omega^r_{X^p_k/k}+1 \) (or \( d : j_*\Omega^r_{X^p_k/k} \to j_*\Omega^r_{X^p_k/k}+1 \)) is given on the degree \( p \) piece of \( \Gamma(X_k, j_*\Omega^r_{X^p_k/k}) \) by \( z^p n \mapsto z^p p \wedge n. \) For any pair of faces \( \omega_1 \subseteq \omega_2 \subseteq \tau \), this induces a map \( d : (\Omega^r_{\omega_1,k}|_{V^p_{p+1}})/\text{Tors} \to (\Omega^r_{\omega_1,k}|_{V^p_{p+1}})/\text{Tors} \).

Proof. Of course \( d(z^p \text{dlog } n) = d(z^p) \wedge \text{dlog } n = z^p \text{ dlog } p \wedge \text{dlog } n \), giving the first formula. The second statement follows from the fact that the explicit description in Proposition 3.3 of \( (\Omega^r_{\omega_1,k}|_{V^p_{p+1}})/\text{Tors} \) is clearly closed under this operation. \( \square \)
3.2. Global calculations. Let \((B, \mathcal{P})\) be a positive and simple integral affine manifold with singularities with toric polyhedral decomposition. Let \(s\) be lifted open gluing data for \((B, \mathcal{P})\), yielding \(X_0 := X_0(B, \mathcal{P}, s)\). Then \(s\) also determines a well-defined log structure on \(X_0\) over \(\text{Spec} \, k\) with singular set \(Z \subseteq X_0\). In what follows, we will take \(\Omega^r\) to be the sheaf on \(X_0\) which is either \(j_* \Omega^r_{X_0^0/k} \) or \(j_* \Omega^r_{X_0^0/k^1}\), where \(j : X_0 \setminus Z \to X_0\) is the inclusion.

We refer to these as the \(/k\) and \(/k^1\) cases, respectively. Our goal is to calculate \(H^p(X_0, \Omega^r)\).

This section will be devoted to technical results which essentially lift the local descriptions of the previous subsection to the global situation.

Our first goal is to obtain a nice resolution for \(\Omega^r\). We have studied the local form of this resolution in \S3.1.

Let \(q_\tau : X_\tau \to X_0\) be the usual inclusion of strata maps, \([12]\), Lemma 2.29, \(D_\tau\) the toric boundary of \(X_\tau\), and let

\[
j_\tau : X_\tau \setminus q_\tau^{-1}(Z) \to X_\tau
\]

\[
k_\tau : X_\tau \setminus (D_\tau \cap q_\tau^{-1}(Z)) \to X_\tau
\]

be the inclusions. We then define

\[\Omega^r_\tau := k_\tau^* k_\tau^* (q_\tau^* \Omega^r / \text{Tors})\]

where Tors denotes the torsion subsheaf of \(q_\tau^* \Omega^r\).

Remark 3.7. While we do not need the result here, one can in fact show that \(\Omega^r_\tau\) is locally free, by showing the same result for \(\Omega^r_{\tau,0}\) in \S3.1. This is only true at 0th order, not for \(k > 0\). If one did know this fact, one could omit all the quotients by Tors appearing in this section, so the reader can safely ignore this torsion.

Recall that \(X_0\) can be viewed as the direct limit of a gluing functor \(F_{S,s}\) defined in \([12]\), Definition 2.11. Here we are taking \(S = \text{Spec} \, k\) as the base scheme, while in \S5 we shall use a different choice of \(S\). Since \(S\) and \(s\) are given, we shall write, for \(\tau_1 \subseteq \tau_2\),

\[F_{\tau_1, \tau_2} : X_{\tau_2} \to X_{\tau_1}\]

for

\[F_{S,s}(\tau_1 \to \tau_2) : X_{\tau_2} \to X_{\tau_1}\]

This introduces ambiguity in case there are several morphisms \(\tau_1 \to \tau_2\), but this only happens if the cell \(\tau_2\) has several faces identified inside of \(B\). If the reader wants to consider this case, he can return to the somewhat denser notation of \(F_{S,s}(\tau_1 \to \tau_2)\) instead of \(F_{\tau_1, \tau_2}\).

Recall that

\[q_{\tau_2} = q_{\tau_1} \circ F_{\tau_1, \tau_2}\]

Adapting the local results of \S3.1 to the global situation,
Proposition 3.8. If \( \tau_1 \subseteq \tau_2 \) with \( \tau_1, \tau_2 \in \mathcal{P} \), then the functorial isomorphism on \( X_{\tau_2} \setminus q_{\tau_2}^{-1}(Z) \)

\[
\Omega^r_{\tau_2} = q_{\tau_2}^* \Omega^r \cong F_{\tau_1, \tau_2}^* q_{\tau_1}^* \Omega^r = F_{\tau_1, \tau_2}^* \Omega^r_{\tau_1}
\]

extends to an inclusion

\[
F_{\tau_1, \tau_2}^* : \Omega^r_{\tau_2} \to (F_{\tau_1, \tau_2}^* \Omega^r_{\tau_1})/\text{Tors}.
\]

Proof. This can be checked in an étale neighbourhood of a point \( z \in Z \); by Theorem 2.6, this reduces to the case considered in Corollary 3.4, (1). \( \square \)

We are now able to define our explicit resolution of the sheaf \( \Omega^r \).

We define a barycentric complex with

\[
\mathcal{G}^k(\Omega^r) = \bigoplus_{\sigma_0 \subseteq \cdot \cdot \cdot \subseteq \sigma_k} q_{\sigma_k}^* ((F_{\sigma_0, \sigma_k}^* \Omega^r_{\sigma_0})/\text{Tors})
\]

and a differential \( d_{\text{bct}} : \mathcal{G}^k(\Omega^r) \to \mathcal{G}^{k+1}(\Omega^r) \) given by

\[
(d_{\text{bct}}(\alpha))_{\sigma_0, \cdot \cdot \cdot, \sigma_{k+1}} = \alpha_{\sigma_1, \cdot \cdot \cdot, \sigma_{k+1}} + \sum_{i=1}^{k} (-1)^i \alpha_{\sigma_0, \cdot \cdot \cdot, \hat{\sigma}_i, \cdot \cdot \cdot, \sigma_{k+1}}
\]

\[+ (-1)^{k+1} F_{\sigma_k, \sigma_{k+1}}^* \alpha_{\sigma_0, \cdot \cdot \cdot, \sigma_k}.
\]

Here \( \alpha_{\sigma_1, \cdot \cdot \cdot, \sigma_{k+1}} \in (F_{\sigma_1, \sigma_{k+1}}^* \Omega^r_{\sigma_1})/\text{Tors} \) can be viewed, by Proposition 3.8, as an element of \( (F_{\sigma_0, \sigma_{k+1}}^* \Omega^r_{\sigma_0})/\text{Tors} \).

Theorem 3.9. \( \mathcal{G}^*(\Omega^r) \) is a resolution of \( \Omega^r \).

Proof. This follows immediately from the local version, Theorem 3.5. \( \square \)

Corollary 3.10.

\[
H^p(X_0, \Omega^r) = \mathbb{H}^p(X_0, \mathcal{G}^*(\Omega^r)).
\]

Now the differential \( d : \Omega^r \to \Omega^{r+1} \) is defined on \( X_0 \setminus Z \), and hence on the pushforward, giving us a complex \( (\Omega^*, d) \), the log de Rham complex of \( X_0 \).

By Proposition 3.6, \( d \) induces maps, for \( e : \tau_1 \to \tau_2 \),

\[
d : (F_{\tau_1, \tau_2}^* \Omega^r_{\tau_1})/\text{Tors} \to (F_{\tau_1, \tau_2}^* \Omega^{r+1}_{\tau_1})/\text{Tors},
\]

and hence a map of complexes

\[
d : \mathcal{G}^*(\Omega^r) \to \mathcal{G}^*(\Omega^{r+1}).
\]

This gives us a double complex \( \mathcal{G}^*(\Omega^* \mathcal{G}) \), and the obvious

Corollary 3.11.

\[
\mathbb{H}^r(X_0, \Omega^*) = \mathbb{H}^r(X_0, \text{Tot}(\mathcal{G}^*(\Omega^*))),
\]

where \( \text{Tot} \) denotes the total complex of the double complex.
In order to compute these cohomology groups explicitly, we need a useful global description for the sheaves $\Omega^r_v$. We first describe $\Omega^r_v$ for $v$ a vertex of $\mathcal{P}$.

Pull back the log structure on $X_0$ via $q_v$ to obtain a log structure on $X_v \setminus q_v^{-1}(Z)$, with sheaf of monoids $\mathcal{M}_v$. By [12], Lemma 5.13, we have a split exact sequence

$$0 \to \mathcal{M}^{gp}_{(X_v,D_v)} \to \mathcal{M}^{gp}_v \to \mathbb{Z}\rho \to 0,$$

where $\mathcal{M}_{(X_v,D_v)}$ is the sheaf of monoids associated to the divisorial log structure given by $D_v \subseteq X_v$, and $\rho$ as usual is the image of $1 \in \mathbb{N}$ under the map of monoids induced by the log morphism $X_0^1 \to \text{Spec } \mathbb{N}$. Because $q_v^{-1}(Z) \subseteq X_v$ is codimension two, $j_! \mathcal{M}_v \to j_* \mathcal{O}_{X_v \setminus q_v^{-1}(Z)} = \mathcal{O}_{X_v}$ determines a log structure on $X_v$, which we write as $X_v^1$. Write $\mathcal{M}_v$ also for $j_! \mathcal{M}_v$. Then the exact sequence (3.3) still holds on $X_v$. From this exact sequence one sees that $\Omega^1_{X_v^1/k}$ coincides with the ordinary sheaf of log derivations for the pair $(X_v,D_v)$, which is canonically $\Lambda_v \otimes \mathcal{O}_{X_v}$ by [29], Proposition 3.1, while $\Omega^1_{X_v^1/k}$ is $(\Lambda_v \otimes \mathcal{O}_{X_v})/(\mathcal{O}_{X_v} \cap \mathcal{O}_{X_v}^1)$.

We will identify this with $\text{Aff}(B,Z) \otimes \mathcal{O}_{X_v}$ (see [12], Definition 2.12) for $P_v$. Then in the proof of [12], Lemma 5.13, the log structure on $V(\sigma) \setminus Z$ is given by charts $\varphi_i$ on an open cover $\{U_i\}$ of $V(\sigma) \setminus Z$, $\varphi_i : P_\sigma \to \mathcal{O}_{U_i}$ a monoid homomorphism. Restricting these charts to $U_i \cap X_v$ gives charts $\varphi_i : P_\sigma \to \mathcal{O}_{U_i \cap X_v}$, which were shown to be of the form

$$p \mapsto \begin{cases} 0 & p \notin P_v \\ h_p z^p & p \in P_v \end{cases}$$

where $P_v$ is the maximal proper face of $P_\sigma$ corresponding to $X_v \cap V(\sigma)$ and $P_v \ni p \mapsto h_p \in \mathcal{O}_{U_i \cap X_v}$ is a monoid homomorphism. Note $P_v^{gp} \cong \Lambda_v$ canonically. This chart lifts to a monoid homomorphism $\varphi_i : P_v \to \mathcal{M}_{U_i}$, so for $p \in P_v$, $d\log(\varphi_i(p)) \in \Gamma(U_i, \Omega^1)$ pulls back via $q_v^* \otimes \mathcal{O}_{X_v^1/k}$ to $\mathcal{O}_{U_i/k}$ via $d\log(h_p z^p) = \frac{d(h_p)}{h_p} + d\log(z^p)$ in $\Omega^1_{X_v^1/k}$. By extending $h_p$ to $U_i$, we see $dh_p$ is in the image of $q_v^* \Omega^1 \to \Omega^1_{X_v^1/k}$, so $d\log(z^p)$ is also for all $p \in \Lambda_v$. On the other hand $d\log(p)$ clearly pulls back to $d\log \rho \in \Omega^1_{X_v^1/k}$. Thus $q_v^*$ is surjective on each $U_i$, hence on $X_v \setminus q_v^{-1}(Z)$.

Now on $X_v \setminus q_v^{-1}(Z) = X_v \setminus (D_v \cap q_v^{-1}(Z))$, $\Omega^1_v = \kappa_v^* \kappa_v^* q_v^* \Omega^1$ is the ordinary sheaf of log derivations for the pair $(X_v,D_v)$. On the other hand $d\log \rho \in \Omega^1_{X_v^1/k}$. Thus $\kappa_v^* \kappa_v^* q_v^* \Omega^1$, so we get an isomorphism on $X_v$:

$$\Omega^1_v = \kappa_v^* \kappa_v^* q_v^* \Omega^1 \to \kappa_v^* (\Omega^1_{X_v^1/k}) = \Omega^1_{X_v^1/k}.$$
the latter equality as \( X_v \) is \( S_2 \) and \( q_v^{-1}(Z) \) is codimension at least two in \( X_v \). Similarly, we obtain \( \Omega^r_v \cong \Omega^r_{X_l/k} \).

One way to interpret this Lemma is that one can view \( \Omega^r \) as being obtained by gluing together trivial vector bundles on the irreducible components of \( X_0 \setminus Z \). Consider the situation where \( \omega \in \mathcal{P} \) is a cell of dimension one, with vertices \( e^\pm_\omega : v^\pm_\omega \to \omega \) arising from a choice of \( d_\omega \) a primitive generator of \( \Lambda_\omega \). On \( X_\omega \setminus q_\omega^{-1}(Z) \), there are of course canonical identifications

\[
F^*_\nu^\omega ,\omega \Omega^r_{\nu^\omega ,\omega } = F^*_\nu^\omega ,\omega q^*_\nu^\omega ,\omega \Omega^r = F^*_\nu^\omega ,\omega q^*_\nu^\omega ,\omega \Omega^r = F^*_\nu^\omega ,\omega \Omega^r_{\nu^\omega ,\omega }.
\]

On the other hand, using the isomorphism of Lemma \([3.12]\), on the left and right hand sides of the above identifications, we get on \( X_\omega \setminus q_\omega^{-1}(Z) \) a map

\[
(3.4) \quad \Gamma_\omega : F^*_\nu^\omega ,\omega \Omega^r_{\nu^\omega ,\omega }X^1_{\nu^\omega ,\omega }/k \to F^*_\nu^\omega ,\omega \Omega^r_{\nu^\omega ,\omega }X^1_{\nu^\omega ,\omega }/k
\]

(or \( /k^1 \)). Let’s describe \( \Gamma_\omega \) explicitly.

With \( \omega \in \mathcal{P} \) one-dimensional, pick \( \omega \to \sigma \) with \( \sigma \in \mathcal{P}_\text{max} \), so that we obtain \( V(\omega) \subseteq V(\sigma) \). Recall from \([12]\), Definition 4.21, that specifying a log smooth structure with singularities of the desired type on \( V(\sigma) \) means giving a section \( f_\sigma \in \Gamma(V(\sigma), \mathcal{L}S^+_\text{pre,V}(\sigma)) \) with certain properties. The sheaf \( \mathcal{L}S^+_\text{pre,V}(\sigma) \) restricted to \( V(\omega) \) is just the structure sheaf \( \mathcal{O}_{\text{Sing}(V(\omega))} \) of the singular locus of \( V(\omega) \), i.e. the intersection of the two irreducible components \( V_{\epsilon^\pm} \) of \( V(\omega) \). So on \( V(\omega) \), \( f_\sigma \) is just a function on \( \text{Sing}(V(\omega)) \).

**Lemma 3.13.** In the above situation, identify \( \tilde{\Lambda}_\nu^\omega \) and \( \tilde{\Lambda}_\nu^\omega \) via parallel transport through \( \sigma \), and identify these with a lattice \( N \). Then on \( \text{Sing}(V(\omega)) \), in the \( /k \) case, \( \Gamma_\omega \) is given by, for \( n \in \wedge^*(N \oplus \mathbb{Z}_p) \),

\[
\Gamma_\omega(d\log n) = -\left( \frac{df_\sigma}{f_\sigma} + l_\omega \log \rho \right) \wedge d\log(\nu(d_\omega)n) + d\log n
\]

where \( l_\omega \) is a positive integer such that there is an integral affine isomorphism \( [0, l_\omega] \to \omega \). The same formula holds modulo \( d\log \rho \) in the \( /k^1 \) case.

**Proof.** We will follow the notation used in \([12]\), Construction 2.15. We view \( \sigma \subseteq M_\mathbb{R} \) as a lattice polytope, with \( \omega \subseteq \sigma \) an edge, and set \( P := C(\sigma)^\vee \cap (N \oplus \mathbb{Z}) \), \( Q = C(\omega)^\vee \cap (N \oplus \mathbb{Z}) \).

We have the monoid \( \partial Q \) we can identify via projection to \( N \) with the set \( N \cup \{\infty\} \), with \( V(\omega) = \text{Spec} \mathbb{k}[\partial Q] \). If \( \Sigma \) is the normal fan to \( \sigma \), then \( \omega^{-1}\Sigma \) has two maximal cones, \( \tilde{v}_\omega^\pm \).

Addition on \( \partial Q = N \cup \{\infty\} \) is given by

\[
p + q = \begin{cases} 
p + q & \text{if } p, q \in \tilde{v}_\omega^\pm \text{ or } p, q \in \tilde{v}_\omega^-; \\
\infty & \text{otherwise.}
\end{cases}
\]

To describe the log structure on \( V(\omega) \), we write down a chart \( \varphi : Q \to \mathcal{O}_U \), for \( U \subseteq V(\omega) \) an open subset. Since any chart must take \( q \in Q \setminus \partial Q \) to \( 0 \in \mathcal{O}_U \), we just give values of
the chart on $N \subseteq \partial Q$:

$$\partial Q \ni N \ni q \mapsto \begin{cases} z^a f_{\sigma}(d_{\omega}, n) & q \in \tilde{v}_\omega^+; \\ z^a & q \in \tilde{v}_\omega^- . \end{cases}$$

Here, $f_{\sigma}$ is any extension of the section $f_{\sigma} \in \mathcal{L}S^+_{\text{pre}, V(\sigma)}|_{V(\omega)} = \mathcal{O}_{\text{Sing}(V(\omega))}$ to a function $k[\partial \Omega]$. It then follows from [12], Theorem 3.22 that the log structure induced by this chart is indeed given by $f_{\sigma} \in \Gamma(V(\omega), \mathcal{L}S^+_{\text{pre}, V(\sigma)})$. This chart can also be considered as giving a map $\varphi : Q \rightarrow \mathcal{M}_U$.

As $d\log \rho$ is a globally defined section of $\Omega^1$, $\Gamma_\omega(d\log \rho) = d\log \rho$. On the other hand, let $n \in \tilde{\Lambda}_{\tilde{v}_\omega} = N$, and let us calculate $\Gamma_\omega(d\log n)$. The monoids $\tilde{v}_\omega^+ \cap N$ are viewed as submonoids of $N$, and can be identified with the faces of $Q$ by the projection $Q \subseteq N \oplus \mathbb{Z} \rightarrow N$. By replacing $n$ by $-n$ if necessary, we can assume $n \in \tilde{v}_\omega^- \cap N$, so identifying $n$ with an element of $Q$ means lifting $n$ to $(v^-_\omega, 1) \subseteq N_{\mathbb{R}} \oplus \mathbb{R}$: this lifting is $n - \langle v^-_\omega, n \rangle \rho \in Q$, if we view $v^\pm_\omega$ as the endpoints of $\omega$ in $M$. Under the isomorphism of Lemma 3.12, $d\log n \in \Gamma(V_{e^\omega_\omega}, \Omega^1_{\mathcal{X}_{e^\omega_\omega}^1/k})$ is identified with $d\log \varphi(n - \langle v^-_\omega, n \rangle \rho) \in \Gamma(V_{e^\omega_\omega}, \Omega^1_{\mathcal{X}_{e^\omega_\omega}^1/k})$. On the other hand, $-n \in \tilde{v}_\omega^+$, and

$$n - \langle v^-_\omega, n \rangle \rho = -(n - \langle v^+_\omega, -n \rangle \rho) + \langle v^+_\omega - v^-_\omega, n \rangle \rho,$$

so on $V_{e^\omega_\omega}$,

$$d\log \varphi(n - \langle v^-_\omega, n \rangle \rho) = -d\log \varphi(-n - \langle v^+_\omega, -n \rangle \rho) + \langle v^+_\omega - v^-_\omega, n \rangle \rho d\log \rho$$

which is identified with

$$-d\log(z^{-n} f_{\sigma}(d_{\omega}, n)) + \langle v^+_\omega - v^-_\omega, n \rangle \rho d\log \rho = -\langle d_{\omega}, n \rangle \frac{df_{\sigma}}{f_{\sigma}} + d\log n + \langle v^+_\omega - v^-_\omega, n \rangle \rho d\log \rho$$

in $\Gamma(V_{e^\omega_\omega}, \Omega^1_{\mathcal{X}_{e^\omega_\omega}^1/k})$. Note that $v^+_\omega - v^-_\omega = -l_\omega d_{\omega}$. Thus for $n = n_1 \wedge \cdots \wedge n_r$,

$$\Gamma_\omega(d\log n) = \bigwedge_{i=1}^r \left(- \frac{df_{\sigma}}{f_{\sigma}} \langle d_{\omega}, n_i \rangle + d\log n_i - \langle d_{\omega}, n_i \rangle l_\omega d\log \rho \right)$$

$$= - \left( \frac{df_{\sigma}}{f_{\sigma}} + l_\omega d\log \rho \right) \wedge d\log(\nu(d_{\omega})n) + d\log n.$$

Next we describe $\Omega^r_\tau$ for $\tau \in \mathcal{P}$ arbitrary. We shall do this by picking a reference vertex $v \in \mathcal{P}$ with a morphism $g : v \rightarrow \tau$. We know by Proposition 3.3 that there is an inclusion of $\Omega^r_\tau$ in $F^*_{v, \tau} \Omega^r_v$, so we only need to describe this subsheaf. 

\[ \square \]
Recall that as we are assuming \((B, \mathcal{P})\) is simple, for every \(\tau \in \mathcal{P}\) with \(\dim \tau \neq 0, n\) we have as in \([12]\), Definition 1.60, the following data:

\[
\mathcal{P}_1(\tau) = \{ \omega \to \tau \mid \dim \omega = 1 \}
\]
\[
\mathcal{P}_{n-1}(\tau) = \{ \tau \to \rho \mid \dim \rho = n - 1 \}
\]

Simplicity allows us to find disjoint sets

\[
\Omega_1, \ldots, \Omega_q \subseteq \mathcal{P}_1(\tau),
\]
\[
R_1, \ldots, R_q \subseteq \mathcal{P}_{n-1}(\tau),
\]

and polytopes

\[
\Delta_1, \ldots, \Delta_q \subseteq \Lambda_{\tau, \mathbb{R}},
\]
\[
\tilde{\Delta}_1, \ldots, \tilde{\Delta}_q \subseteq \Lambda^\perp_{\tau, \mathbb{R}}.
\]

These have the property that if \(\omega \in \Omega_i, e : \omega \to \tau\), then the monodromy polytope \(\tilde{\Delta}_e(\tau) = \tilde{\Delta}_i\), and if \(\rho \in R_i, f : \tau \to \rho\), then \(\Delta_f(\tau) = \Delta_i\). (See \([12]\), Definition 1.58). Furthermore, the \(\Delta_i\)'s are the Newton polytopes of the functions \(\tilde{\psi}_i\) on \(\tilde{\Sigma}_\tau\), the normal fan to \(\tau\). (See \([12]\), Remark 1.59, where there is a typo: \(\varphi_\rho\) should be \(\tilde{\psi}_\rho\).) For any \(g' : v' \to \tau\), we obtain vertices \(\text{Vert}_i(g')\) of \(\Delta_i\) as in Construction 2.1. The reference vertex \(g : v \to \tau\) then gives reference vertices \(v_i := \text{Vert}_i(g) \in \Delta_i\). The sets \(\Omega_i\) are characterized by \(\omega \in \Omega_i\) if and only if \(\text{Vert}_i(\omega^+) \neq \text{Vert}_i(\omega^-)\).

In addition, simplicity includes the condition that the convex hulls of

\[
\bigcup_{i=1}^q \Delta_i \times \{ e_i \} \text{ and } \bigcup_{i=1}^q \tilde{\Delta}_i \times \{ e_i \}
\]

in \(\Lambda_{\tau, \mathbb{R}} \times \mathbb{R}^q\) and \(\Lambda^\perp_{\tau, \mathbb{R}} \times \mathbb{R}^q\) respectively are elementary simplices. In particular, \(\Delta_1, \ldots, \Delta_q\) and \(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_q\) are themselves elementary simplices, and their tangent spaces \(T_{\Delta_1}, \ldots, T_{\Delta_q}\) give a direct sum decomposition of \(\sum_{i=1}^q T_{\Delta_i} \subseteq \Lambda_{\tau, \mathbb{R}}\) and \(T_{\tilde{\Delta}_1}, \ldots, T_{\tilde{\Delta}_q}\) give a direct sum decomposition of \(\sum_{i=1}^q T_{\tilde{\Delta}_i} \subseteq \Lambda^\perp_{\tau, \mathbb{R}}\).

We will often use the obvious

\[\text{Lemma 3.14. If the convex hull of } \bigcup_{i=1}^q \Delta_i \times \{ e_i \} \text{ is an elementary simplex, then there is a one-to-one correspondence between faces } \sigma \text{ of } \Delta := \Delta_1 + \cdots + \Delta_q \text{ (Minkowski sum) and } q\text{-tuples } (\sigma_1, \ldots, \sigma_q) \text{ with } \sigma_i \text{ a face of } \Delta_i, \text{ with } \sigma = \sigma_1 + \cdots + \sigma_q. \text{ Furthermore,}
\]

\[\dim \sigma = \sum_{i=1}^q \dim \sigma_i.\]

\[\square\]

According to \([12]\), Corollary 5.8, \(q_\tau^{-1}(Z) = Z_1^\tau \cup \cdots \cup Z_q^\tau \cup Z'\) where \(Z' \subseteq D_\tau\) is of codimension at least two in \(X_\tau\) and \(Z_i^\tau\) is a hypersurface in \(X_\tau\), with Newton polytope \(\tilde{\Delta}_i\).
Furthermore, from the proof of [12], Corollary 5.8, $Z^\tau_i = F^\tau_{\omega_i}(Z_\omega)$, for any $\omega \in \Omega$, where $Z_\omega$ is the irreducible component of $Z$ contained in the codimension one stratum $X_\omega$ of $X$.

For an index set $I \subseteq \{1, \ldots, q\}$, set $Z^\tau_i := \bigcap_{i \in I} Z^\tau_i$. Pull back the log structure $X^\tau_i$ on $X_v$ to $X_\tau$ via $F_{v, \tau}$, and then restrict further to $Z^\tau_i$, for any $I$. We write these structures as $X^\tau_i$ and $(Z^\tau_i)^\dagger$, but keep in mind these are not intrinsic and depend on the choice of vertex $g : v \to \tau$. Note that these are all defined over $\text{Spec} k^\dagger$, by composing the inclusions into $X_v$ with $X^\tau_i \xrightarrow{q_v} X^\tau_i \rightarrow \text{Spec} k^\dagger$.

Viewing $Z^\tau_i \subseteq X_v$ via the inclusion $F_{v, \tau} : X_\tau \to X_v$, we have

**Lemma 3.15.**  

1. There are exact sequences

$$0 \to \bigoplus_{i \in I} \mathcal{O}_{Z^\tau_i}(-Z^\tau_i) \to \Omega^1_{X^\tau}/k \big|_{Z^\tau_i} \to \Omega^1_{(Z^\tau_i)^\dagger} \to 0$$

and

$$0 \to \bigoplus_{i \in I} \mathcal{O}_{Z^\tau_i}(-Z^\tau_i) \to \Omega^1_{X^\tau}/k \big|_{Z^\tau_i} \to \Omega^1_{(Z^\tau_i)^\dagger}/k \to 0.$$ 

Here $\mathcal{O}_{Z^\tau_i}(-Z^\tau_i)$ denotes the restriction of the line bundle $\mathcal{O}_{X_\tau}(-Z^\tau_i)$ to $Z^\tau_i$. In addition, $\Omega^1_{(Z^\tau_i)^\dagger}/k$ and $\Omega^1_{(Z^\tau_i)^\dagger}/k$ are locally free $\mathcal{O}_{Z^\tau_i}$-modules.

2. If $Y \subseteq X_\tau$ is a toric stratum, then $\Omega^r_{(Z^\tau_i)^\dagger}/k \mid_Y = \Omega^r_{(Z^\tau_i \cap Y)^\dagger}/k$ and

$$\text{Tor}^r_{X_\tau}(\Omega^r_{(Z^\tau_i)^\dagger}/k, \mathcal{O}_Y) = 0$$

for $j > 0$. Here the log structure on $Z^\tau_i \cap Y$ is the pull-back of the one on $Z^\tau_i$. The same holds for the $/k^\dagger$ case.

**Proof.** Again we do the $/k$ case. There is always a functorial map $\Omega^1_{X^\tau}/k \big|_{Z^\tau_i} \to \Omega^1_{(Z^\tau_i)^\dagger}/k$ as $(Z^\tau_i)^\dagger \to X^\tau_i$ is a morphism of log schemes. We also have a map $\mathcal{I}_{Z^\tau_i}/X_v \to \Omega^1_{X^\tau}/k \big|_{Z^\tau_i}$ by $f \mapsto df$. Then we first check exactness of

$$\mathcal{I}_{Z^\tau_i}/X_v \to \Omega^1_{X^\tau}/k \big|_{Z^\tau_i} \to \Omega^1_{(Z^\tau_i)^\dagger}/k \to 0$$

on affine pieces. Choosing $h : \tau \to \sigma \in \mathcal{P}_{\max}$, we obtain an affine open set $V_{\text{hog}} = \text{Spec} k[P_{\text{hog}}]$ of $X_v$, where $P_{\text{hog}}$ is the maximal proper face of $P_{\sigma}$ corresponding to $h \circ g : v \to \sigma$. The log structure $X^\tau_i$ is given by the chart $P_{\sigma} \to k[P_{\text{hog}}]$ that maps $p \mapsto z^p$ if $p \in P_{\text{hog}}$ and $p \mapsto 0$ otherwise. Then

$$\Omega^1_{X^\tau}/k = (\Omega^1_{X_v}/k \oplus (\mathcal{O}_{X_v} \otimes P_{\sigma}^{\text{gp}}))/K_{X_v}$$

where $K_{X_v}$ is the submodule generated by $(d(z^p), -z^p : p)$ for $p \in P_{\sigma}$, and

$$\Omega^1_{(Z^\tau_i)^\dagger}/k = (\Omega^1_{Z^\tau_i}/k \oplus (\mathcal{O}_{Z^\tau_i} \otimes P_{\sigma}^{\text{gp}}))/K_{Z^\tau_i}.$$
where \( \mathcal{K}_{Z_T^\tau} \) is defined analogously. Then we have a diagram

\[
\begin{array}{ccccccccc}
\mathcal{I}_{Z_T^\tau/X_v} & \to & \Omega^1_{X_v/k}|_{Z_T^\tau} & \oplus & (\mathcal{O}_{Z_T^\tau} \otimes P_{\sigma}) & \to & \Omega^1_{X_v/k}|_{Z_T^\tau} & \to & 0 \\
\nearrow & & \nearrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_{X_v}|_{Z_T^\tau} & \to & \Omega^1_{Z_T^\tau/k} & \oplus & (\mathcal{O}_{Z_T^\tau} \otimes P_{\sigma}) & \to & \Omega^1_{(Z_T^\tau)^\dagger/k} & \to & 0 \\
0 & \to & 0 & & 0 & & 0 \\
\end{array}
\]

where the map \( \mathcal{K}_{X_v}|_{Z_T^\tau} \to \mathcal{K}_{Z_T^\tau} \) is clearly surjective, and the middle column is exact from the standard result for Kähler differentials. So we see

\[
\mathcal{I}_{Z_T^\tau/X_v} \to \Omega^1_{X_v/k}|_{Z_T^\tau} \to \Omega^1_{(Z_T^\tau)^\dagger/k} \to 0
\]

is exact by the snake lemma.

Now view \( Z_T^\tau \subseteq X_\tau \subseteq X_v \). Looking at the composition \( \mathcal{I}_{X_\tau/X_v} \to \mathcal{I}_{Z_T^\tau/X_v} \to \Omega^1_{X_v/k}|_{Z_T^\tau} \), we see that as the former ideal is generated by monomials \( z^p \) for various \( p \), and \( d(z^p) = z^p \log(p) \), it follows that \( \mathcal{I}_{X_\tau/X_v} \) maps to zero in \( \Omega^1_{X_v/k}|_{Z_T^\tau} \), and thus the map

\[
\mathcal{I}_{Z_T^\tau/X_v} \to \Omega^1_{X_v/k}|_{Z_T^\tau}
\]

factors through \( \mathcal{I}_{Z_T^\tau/X_v} \to \mathcal{I}_{Z_T^\tau/X_\tau} \), so we have an exact sequence

\[
\mathcal{I}_{Z_T^\tau/X_\tau}/\mathcal{I}_{Z_T^\tau/X_v} \to \Omega^1_{X_v/k}|_{Z_T^\tau} \to \Omega^1_{(Z_T^\tau)^\dagger/k} \to 0.
\]

The conormal bundle is of course \( \bigoplus_{i \in I} \mathcal{O}_{Z_T^\tau}(-Z_T^\tau) \), and we just need to check injectivity. This can be done locally, where \( X_v = \text{Spec } \mathbb{k}[P] \), \( X_\tau = \text{Spec } \mathbb{k}[Q] \) for \( Q \) a face of \( P \), and \( Z_T^\tau \) is defined by equations \( \{ f_i = 0 \mid i \in I \} \) in \( X_\tau \), with \( f_i \) having, up to translation, Newton polytope \( \Delta_i \). We can write \( f_i = 1 + \sum_{j=1}^{l_i} a_{ij} z_i^{q_{ij}} \) where the set \( \{ q_{ij} \} \) is linearly independent in \( Q_{\sigma} \otimes \mathbb{Q} \) by simplicity. Now \( df_i = \sum_{j=1}^{l_i} a_{ij} z_i^{q_{ij}} \log(q_{ij}) \). We can’t have all \( z_i^{q_{ij}} \)’s vanishing at a point of \( Z_T^\tau \), so \( df_i \) is non-vanishing, and by the linear independence of the \( q_{ij} \)’s, \( \{ df_i \mid i \in I \} \) are an everywhere linearly independent set of sections of \( \Omega^1_{X_v/k}|_{Z_T^\tau} \). Thus the map \( \mathcal{I}_{Z_T^\tau/X_\tau}/\mathcal{I}_{Z_T^\tau/X_v} \to \Omega^1_{X_v/k}|_{Z_T^\tau} \) is injective, and the cokernel is locally free. (Recall that \( \Omega^1_{X_v/k} \) is locally free.) This shows (1).

For (2), just restrict the sequences of (1) of locally free sheaves to \( Z_T^\tau \cap Y \). This gives the sequence of (1) for \( \Omega^1_{(Z_T^\tau \cap Y)^\dagger/k} \), proving \( \Omega^1_{(Z_T^\tau \cap Y)^\dagger/k} = \Omega^1_{(Z_T^\tau)^\dagger/k} \) (or \( /k^\dagger \)). To show the vanishing of the Tor’s, by the local freeness statement of (1), it is enough to show that \( \text{Tor}^j_{X_v}(\mathcal{O}_{Z_T^\tau}, \mathcal{O}_Y) = 0 \) for \( j > 0 \). To do this, note using simplicity that \( Z_T^\tau \) is a complete
intersection, so we have the Koszul complex
\[(3.5) \quad 0 \to \bigwedge^i \bigoplus_{i \in I} \mathcal{O}_{X'}(-Z^r_j) \to \cdots \to \bigwedge^1 \bigoplus_{i \in I} \mathcal{O}_{X'}(-Z^r_1) \to \bigoplus_{i \in I} \mathcal{O}_{X'} \to \mathcal{O}_{Z^r_i} \to 0\]
giving a resolution for $\mathcal{O}_{Z^r_i}$. Tensoring with $\mathcal{O}_Y$ gives the Koszul resolution for $Y \cap Z^r_i$, which is still a complete intersection in $Y$. This gives the result. \[\square\]

Remark 3.16. We shall use on a number of occasions variations on the following simple observation: given a form $\alpha \in \Omega^r_{X/k}$ or $\Omega^r_{X'/k}$, if $f_j = 0$ locally defines $Z_j$ in an affine subset of $X_v$, then $\frac{df}{f_j} \wedge \alpha$ has no pole along $Z_j$ if and only if $\alpha|_{Z_j} = 0$. This follows from linear algebra: if $V$ is a vector space, $\beta \in V^*$, $\alpha \in \wedge^r V^*$, then $\beta \wedge \alpha = 0$ if and only if $\alpha|_{\ker \beta} = 0$.

**Proposition 3.17.** Given $v \to \tau_1 \to \tau_2$, the image of the inclusion $(F^*_v, Z_{\tau_2})/\text{Tors}$ in $F^*_v \Omega^r_v$ is
\[\ker \left( F^*_v, \Omega^r_v \xrightarrow{\delta_0} \bigoplus_{i=1, \ldots, q} \Omega^{r-1}_{(Z_i^{r_2})^\dagger/k} \right) \]
or \[\ker \left( F^*_v, \Omega^r_v \xrightarrow{\delta_0} \bigoplus_{i=1, \ldots, q} \Omega^{r-1}_{(Z_i^{r_2})^\dagger/k} \right) \]
in the $/k$ and $/k^\dagger$ cases respectively, where:

1. The direct sum is over all $i$ and all vertices $w_i$ of $\Delta_i$, $w_i \neq v$, and $\Delta_1, \ldots, \Delta_q$ are part of the simplicity data for $\tau_1$.
2. $Z^r_i = F^r_{\tau_1, \tau_2}(Z_i^{r_1})$ where $Z_1^{r_1}, \ldots, Z_q^{r_1}$ are as usual the codimension one irreducible components of $q_1^{-1}(Z)$ with Newton polytopes $\Delta_1, \ldots, \Delta_q$.
3. For $\alpha \in F^*_v, \Omega^r_v$, the component of $\delta_0(\alpha)$ in the direct summand $\Omega^{r-1}_{(Z_i^{r_2})^\dagger/k}$ or $\Omega^{r-1}_{(Z_i^{r_2})^\dagger/k^\dagger}$ corresponding to some $w_i$ is given by $\iota(\partial_{w_i - v_i})\alpha|_{(Z_i^{r_2})^\dagger}$.

**Proof.** Again we’ll do the $/k$ case. The result will follow by using the characterization of Corollary 3.14 (2) of $(F^*_v, Z_{\tau_2})/\text{Tors}$ as
\[\bigcap_{g': g' \to \tau_1} F^*_v, \Omega^r_v, \]
using Lemma 3.13 for the explicit identification of this intersection with a subsheaf of $F^*_v, \Omega^r_v$. Let $\alpha$ be a section of $F^*_v, \Omega^r_v$. Then for any $j$ and vertex $w_j \neq v_j$ of $\Delta_j$, we can find a sequence of edges $h_i : \omega_i \to \tau_1$, $i = 1, \ldots, m$ of $\tau_1$, with $d_{\omega_i}$ chosen appropriately, so that

- $v^+_{\omega_i} = v$;
- $v^-_{\omega_i} = v^-_{\omega_{i+1}}$ for $i < m$;
- $\text{Vert}_l(v^+_{\omega_i}) = v_l$ for $i < m$, for all $l$;
\begin{itemize}
  \item \(\text{Vert}_l(v^+_{\omega_m}) = \begin{cases} v_l & l \neq j, \\
w_j & l = j. \end{cases}\)
\end{itemize}

Choose a maximal cell \(\sigma\) containing \(\tau_2\) for reference, and let \(f_1, \ldots, f_q\) be the equations defining \(Z_1^{r_2}, \ldots, Z_q^{r_2}\) in the affine chart \(V(\sigma) \cap X_{\tau_2}\). Using Lemma 3.13 apply \(\Gamma_{\omega_1}, \ldots, \Gamma_{\omega_m}\) successively to \(\alpha\). Then in fact, for \(1 \leq i \leq m - 1\),
\[
\Gamma_{\omega_i}(\dlog n) = -l_{\omega_i} \dlog \rho \land \dlog(\iota(d_{\omega_i})n) + \dlog n.
\]
(Here and throughout this proof we write \(d_{\omega_i}\) rather than \(\partial_{d_{\omega_i}}\) etc. for typographical convenience.) Indeed, \(\text{Vert}_l(v^+_{\omega_i}) = \text{Vert}_l(v^-_{\omega_i})\) for each \(1 \leq i \leq m - 1\), so \(\omega_1, \ldots, \omega_{m-1}\) are not in any \(\Omega_i\). Thus \(F^{-1}_{\omega_i, \tau_2}(Z_{\omega_i}) = \emptyset\) (see discussion after Lemma 3.14 or [12], Cor. 5.8), so the function \(f_\sigma\) appearing in Lemma 3.13 is constant and hence does not play a role. On the other hand, \(\omega_m \in \Omega_j\), so
\[
\Gamma_{\omega_m}(\dlog n) = -\left(\frac{df_j}{f_j} + l_{\omega_m} \dlog \rho\right) \land \dlog(\iota(d_{\omega_m})n) + \dlog n.
\]
Putting this together, we see that
\[
\Gamma_{\omega_{m-1}} \circ \cdots \circ \Gamma_{\omega_1}(\alpha) = \dlog \rho \land \iota(v^+_{\omega_{m-1}} - v)\alpha + \alpha
\]
and
\[
\Gamma_{\omega_m} \circ \cdots \circ \Gamma_{\omega_1}(\alpha) = -\frac{df_j}{f_j} \land \iota(d_{\omega_m})[\dlog \rho \land \iota(v^+_{\omega_{m-1}} - v)\alpha + \alpha] + \dlog \rho \land \iota(v^+_{\omega_m} - v)\alpha + \alpha.
\]
Now \(\frac{df_j}{f_j} \land \iota(d_{\omega_m})\alpha\) has no pole along \(f_j = 0\) if and only if \(\iota(d_{\omega_m})\alpha\big|_{Z_j^{r_2}} = 0\) in \(\Omega^{-1}_{(Z_j^{r_2})^T/k}\) by Remark 3.16. If this latter condition holds, then of course \(\iota(d_{\omega_m} \land (v^+_{\omega_{m-1}} - v))\alpha\big|_{Z_j^{r_2}} = 0\) also, so \(\Gamma_{\omega_m} \circ \cdots \circ \Gamma_{\omega_1}(\alpha)\) has no pole if and only if \(\iota(v_j - w_j)\alpha\big|_{Z_j^{r_2}} = 0\). Hence we see that
\[
\bigcap_{v' \to \tau_1} F^*_{v', \tau_2} \Omega^r_{v'} \subseteq \ker \delta_0.
\]
Conversely, if \(\alpha \in \ker \delta_0\), let \(v'\) be any vertex of \(\tau_1\). Then we can find a sequence of edges \(\omega_i \to \tau_1, i = 1, \ldots, m\) of \(\tau_1\), with \(d_{\omega_i}\) chosen appropriately, so that
\begin{itemize}
  \item \(v^-_{\omega_1} = v;\)
  \item \(v^+_{\omega_i} = v^-_{\omega_{i+1}}\) for \(i < m;\)
  \item \(v^+_{\omega_m} = v';\)
  \item For each \(1 \leq l \leq q\), there is at most one \(i\) such that \(\text{Vert}_l(v^-_{\omega_i}) \neq \text{Vert}_l(v^+_{\omega_i})\), and for this \(i\), \(\text{Vert}_l(v^-_{\omega_i}) = v_l, \text{Vert}_l(v^+_{\omega_i}) = v'_l = \text{Vert}_l(v').\)
\end{itemize}

Then again using Lemma 3.13 repeatedly along each \(\omega_i\) to identify \(\alpha\) with a rational section of \(F^*_{v', \tau_2} \Omega^r_{v'}\), one sees that \(\alpha\) is identified with \(\alpha + \cdots\), where the terms in \(\cdots\) which may
not be regular sections of $F^*_{v',\tau_2} \Omega^r_{v'}$ are of the form

$$\left(\bigwedge_{i \in I} \frac{df_i}{f_i}\right) \wedge \iota \left(\bigwedge_{i \in I} (v_i - v'_i)\right) \alpha$$

or

$$\left(\bigwedge_{i \in I} \frac{df_i}{f_i}\right) \wedge \log \rho \wedge \iota \left(w \wedge \bigwedge_{i \in I} (v_i - v'_i)\right) \alpha$$

for $I \subseteq \{1, \ldots, q\}$ some index set and some $w \in \Lambda_{\tau_1}$. Here we use the fact that $v_i - v'_i \in \Lambda_{\tau_1}$, while the monomials appearing in the $f_i$’s are in $\Lambda^\perp_{\tau_2}$, so $\iota(v_i - v'_i)df_k = 0$. However, for $i \in I$, this has no pole along $Z^r_{i,\tau_2}$ again by Remark 3.16 because by assumption $\iota(v_i - v'_i)|_{Z^r_{i,\tau_2}}$ is zero. Thus we see that $\alpha$ is identified with a regular section of $F^*_{v',\tau_2} \Omega^r_{v'}$, and hence $\alpha$ is in $\bigcap_{g: v' \to \tau_1} F^*_{v',\tau_2} \Omega^r_{v'}$.

For $e: \tau_1 \to \tau_2$, we will now calculate the cohomology of $(F^*_{\tau_1,\tau_2} \Omega^r_{\tau_1})/\text{Tors}$ by building a convenient resolution of this sheaf. The first two terms of this resolution are given by Proposition 3.17; we need to extend this two-term complex.

For $V \subseteq \Lambda_{\tau_2}$, a subspace, we have a subsheaf $\Omega^r_{v}|_{X_r} \cap V^\perp$ of $\Omega^r_{v}|_{X_r}$, given by forms $\alpha$ with $\iota(\partial_m)\alpha = 0$ for all $m \in V$. We define $\Omega^r_{(Z^r_{\tau})^\perp}/k \cap V^\perp$ or $\Omega^r_{(Z^r_{\tau})^\perp/k^\perp}$ to be the image of $\Omega^r_{v}|_{X_r} \cap V^\perp$ in $\Omega^r_{(Z^r_{\tau})^\perp/k}$ (or $/k^\perp$). For $m \in \Lambda_r$, note that

$$\iota(\partial_m) \left(\text{im} \left(\bigoplus_{i \in I} \mathcal{O}_{Z^r_{\tau}} (-Z^r_{i}) \rightarrow \Omega^1_{v}|_{Z^r_{\tau}}\right)\right) = 0,$$

as all monomials occurring in the equations for the $Z^r_{i,\tau}$’s are in $\Lambda^\perp_r$. We thus in particular have from Lemma 3.15 an exact sequence

$$0 \to \bigoplus_{i \in I} \mathcal{O}_{Z^r_{\tau}} (-Z^r_{i}) \to \Omega^1_{v}|_{Z^r_{\tau}} \cap V^\perp \to \Omega^1_{(Z^r_{\tau})^\perp/k} \cap V^\perp \to 0$$

and a similar exact sequence for the $/k^\perp$ case.

Given $g: v \to \tau_1$ as usual, we can now define a complex $\mathcal{F}_v$ by

$$\mathcal{F}_v^{r,p} = \bigoplus_{\sigma \subseteq \Delta_{\tau_1}, \dim \sigma = p} \Omega^r_{(Z^r_{\tau})^\perp/k^\perp} \cap T^\perp_\sigma,$$

in the $/k$ case and

$$\mathcal{F}_v^{r,p} = \bigoplus_{\sigma \subseteq \Delta_{\tau_1}, \dim \sigma = p} \Omega^r_{(Z^r_{\tau})^\perp/k} \cap T^\perp_\sigma,$$
in the \( /k^\dagger \) case, where the sum is over all \( p \)-dimensional \( \sigma = \sigma_1 + \cdots + \sigma_q \) where \( \sigma_i \) is a face of \( \Delta_i \) containing \( v_i \), and

\[
\Delta_{\tau_1} = \Delta_1 + \cdots + \Delta_q;
\]

\[
\bar{v} = v_1 + \cdots + v_q;
\]

\( T_\sigma \) is the tangent space to \( \sigma \) in \( \Lambda_{\tau,R} \);

\[
I(\sigma) = \{ i | \sigma_i \neq \{ v_i \} \}.
\]

We use the convention that if \( I(\sigma) = \emptyset \) then \( \Omega_r^r(Z^r_{I(\sigma)})/k \) (or \( /k^\dagger \)) is \( \Omega_r^r|_\tau \).

We define differentials \( \delta_p : \mathcal{F}_{r,v}^p \rightarrow \mathcal{F}_{r,v}^p+1 \) by

\[
(\delta_p \alpha)_{\sigma'} = \sum_{\substack{\sigma \subseteq \sigma', v \in \sigma \atop \text{dim } \sigma = p}} \iota(\partial_{w_j-v}) \alpha_{\sigma} |_{\Omega^r_{\tau_1,\tau_2}}.
\]

Here \( \sigma' \) is a face of \( \Delta_{\tau_1} \) of dimension \( p+1 \), and we sum over all faces \( \sigma \) of \( \sigma' \) of dimension \( p \) containing \( v \). For each such \( \sigma' \), by Lemma 3.14 there is a unique \( j \) such that \( \sigma'_j \neq \sigma_j \), and \( w_j \) is the unique vertex of \( \sigma'_j \) not contained in \( \sigma_j \). By Proposition 3.17

\[
\Omega_r^r_{\tau_1} = \ker(\delta_0 : \mathcal{F}_{r,0}^0 \rightarrow \mathcal{F}_{r,1}^0).
\]

**Lemma 3.18.** For any \( \tau_1 \subseteq \tau_2 \),

\[
F^*_{\tau_1,\tau_2} \mathcal{F}_{v}^{r,\bullet}
\]

is a resolution of \( (F^*_{\tau_1,\tau_2} \Omega_r^r_{\tau_1})/\text{Tors} \).

**Proof.** We show this for \( \tau_1 = \tau_2 = \tau \), and then the complex remains a resolution under pull-back by Lemma 3.15 (2). We will do this for the \( /k \) case, the \( /k^\dagger \) case being essentially identical. We will proceed by induction. Consider faces \( v \in \omega \subseteq \omega' \subseteq \Delta_\tau \), and consider the complex \( \mathcal{F}^{r,\omega'} \) defined by

\[
\mathcal{F}^{r,\omega'} = \bigoplus_{\omega \subseteq \omega' \cap \dim \sigma = p} \Omega^r_{r,v} |_{\Omega^r_{Z^r}} \cap T^r_{\sigma},
\]

with differential \( \delta_p \) defined as before. Note that if \( \omega = v, \omega' = \Delta_\tau \), this is \( \mathcal{F}^{r,0} \). We will show \( H^i(\mathcal{F}^{r,\omega'}) = 0 \) for \( i > \dim \omega \) inductively on \( \dim \omega' - \dim \omega \). (Here \( H^i \) denotes cohomology of the complex.) If \( \omega = \omega' \), the statement is trivial. If \( \omega \neq \omega' \), let \( j \) be a vertex of \( \omega'_j \) not in \( w_j \), and write \( \omega \cup \{ w_j \} := \omega_1 + \cdots + \text{Conv} \{ w_j, w_j \} + \cdots + \omega_p \), and \( \omega' \setminus \{ w_j \} := \omega_1' + \cdots + \omega'_j \setminus \{ w_j \} + \cdots + \omega'_p \), where \( \omega'_j \setminus \{ w_j \} \) is the convex hull of those vertices of \( \omega'_j \) not equal to \( w_j \). Then we have an exact sequence of complexes

\[
0 \rightarrow \mathcal{F}^{r,\omega \cup \{ w_j \}} \rightarrow \mathcal{F}^{r,\omega} \rightarrow \mathcal{F}^{r,\omega' \setminus \{ w_j \}} \rightarrow 0,
\]

and hence an exact sequence

\[
0 \rightarrow H^{\dim \omega}(\mathcal{F}^{r,\omega}) \rightarrow H^{\dim \omega}(\mathcal{F}^{r,\omega' \setminus \{ w_j \}}) \rightarrow H^{\dim \omega+1}(\mathcal{F}^{r,\omega \cup \{ w_j \}}) \rightarrow H^{\dim \omega+1}(\mathcal{F}^{r,\omega'}) \rightarrow 0
\]
and $H^j(F_{\omega, \omega'}) = 0$ for $j > \dim \omega + 1$, by the induction hypothesis. To complete the result, we need to show the map $\delta$ is surjective.

There are two cases. First, suppose $\omega_j \neq \{v_j\}$. Then $j \in I(\omega) = I(\omega \cup \{w_j\})$, and we get a diagram with exact rows

$$0 \to H^{\dim \omega}(F_{\omega, \omega'} \setminus \{w_j\}) \to \Omega^{r=\dim \omega}(Z_{I(\omega)}')/k \cap T^\perp_\omega \to \bigoplus_{\omega \subseteq \sigma \subseteq \omega \setminus \{w_j\}, \dim \sigma = \dim \omega + 1} \Omega^{r=\dim \omega - 1}(\mathcal{Z}_{I(\sigma)}')/k \cap T^\perp_{\sigma} \to 0$$

The vertical maps are given by contraction by $\partial_{w_j - v_j}$. By simplicity, we can find an $n \in \Lambda_{v,R}$ such that $\langle n, w_j - v_j \rangle = 1$ and $\langle n, w'_i - v_i \rangle = 0$ for all $w'_i$ a vertex of $\Delta$, not equal to $w_j$. Let $\alpha \in H^{\dim \omega + 1}(F_{\omega, \omega'} \setminus \{w_j\})$. We claim that $d\log n \wedge \alpha \in H^{\dim \omega}(F_{\omega, \omega'} \setminus \{w_j\})$ and $\delta(d\log n \wedge \alpha) = \alpha$. First, for any $\omega \subseteq \sigma \subseteq \omega' \setminus \{w_j\}$ with $\dim \sigma = \dim \omega + 1$, $\sigma_i = w_i$ for $i \neq l$ and $\sigma_l = w_l \cup \{w_l\}$ for some unique $l$ and vertex $w_l$ of $\omega'_i \setminus \{w_j\}$. Computing the component of $\delta_{\dim \omega}(d\log n \wedge \alpha)$ labelled by $\sigma$ gives

$$\iota(\partial_{w_l - v_l})(d\log n \wedge \alpha) = -d\log n \wedge (\iota(\partial_{w_l - v_l})\alpha),$$

which restricts to zero on $(Z_{I(\sigma)}')^\perp$ because $\iota(\partial_{w_l - v_l})\alpha$ does. Thus

$$d\log n \wedge \alpha \in H^{\dim \omega}(F_{\omega, \omega'} \setminus \{w_j\}).$$

On the other hand $\delta(d\log n \wedge \alpha) = \alpha$, as $\partial_{w_j - v_j} \in T_{\omega \cup \{w_j\}}$, so $\iota(\partial_{w_j - v_j})\alpha = 0$. This shows surjectivity of $\delta$ in this case.

In the second case, suppose $\omega_j = \{v_j\}$. Then $j \notin I(\omega)$, and we get a diagram

$$0 \to H^{\dim \omega + 1}(F_{\omega, \omega'} \setminus \{v_j\}) \to \Omega^{r=\dim \omega + 1}(Z_{I(\omega)}')/k \cap T^\perp_\omega \to \bigoplus_{\omega \subseteq \sigma \subseteq \omega' \setminus \{v_j\}, \dim \sigma = \dim \omega + 1} \Omega^{r=\dim \omega - 1}(\mathcal{Z}_{I(\sigma)}')/k \cap T^\perp_{\sigma} \to 0$$

Note the differences between this and the previous diagram: the sheaves of log differentials in the lower line are now over subschemes of those on the upper line. We need a more explicit description of the kernels. In particular, we will describe a set of generators of $H^{\dim \omega + 1}(F_{\omega, \omega'} \setminus \{v_j\})$ as a sheaf of modules and give liftings of each of these generators. To do so, choose dual bases for $\Lambda_{v,R} \oplus (\mathcal{R})^*$ and $\Lambda_{v,R} \oplus \mathcal{R}$ as follows. Start with $e_0$ a generator of $(\mathcal{R})^*$, throw in a basis for $\sum_{i=1}^q T_{\Delta_i} \subseteq \Lambda_{v,R} \subseteq \Lambda_{v,R}$ with elements $\partial_{w - v_i}$, where $w$ is a vertex of $\Delta_i$, $w \neq v_i$, and $i$ runs from 1 to $q$. Extend to a basis for $\Lambda_{v,R} \oplus (\mathcal{R})^*$. Call this basis $e_0, \ldots, e_n$, with dual basis $e_0^*, \ldots, e_n^*$. Consider for $i = 1, \ldots, q$ the following subsets
of \{0, \ldots, n\}:
\[
I_i = \{l | e_l + v_i \text{ is a vertex of } \omega_i', \text{ but not } \omega_i, \text{ and } e_l + v_i \neq w_j \}
\]
\[
K_i = \{l | e_l + v_i \text{ is a vertex of } \omega_i, \text{ or } e_l + v_i = w_j \}.
\]
Now if \( \alpha \in H^{\dim \omega+1}(F_{\omega \cup \{w_j\}, \omega'}) \), let’s examine the property \( \delta_{\dim \omega+1}(\alpha) = 0 \) component by component. We have a component for each \( l \in I_i, i = 1, \ldots, q \). Then either \( i \in I(\omega \cup \{w_j\}) \) and \( \iota(e_l)\alpha = 0 \) because \( Z^*_I(\omega_{\iota(e_l+1)}) \cap Z^*_J = Z^*_I(\omega) \cap Z^*_J \), or else \( i \not\in I(\omega \cup \{w_j\}) \) and \( \iota(e_l)\alpha|_{Z^*_I(\omega) \cap Z^*_J} = 0 \). From this observation we will describe a set of generators for the sheaf \( H^{\dim \omega+1}(F_{\omega \cup \{w_j\}, \omega'}) \) as an \( \mathcal{O}_{Z^*_I(\omega) \cap Z^*_J} \)-module. For an index set \( J = \{j_1, \ldots, j_r\} \subseteq \{0, \ldots, n\} \), write \( e^*_j = e^*_j \cup \cdots \cup e^*_r \). Let \( I = \bigcup_{i \not\in I(\omega \cup \{w_j\})} I_i, K = \bigcup_{i=1}^q K_i \). Then a basis for \( T^\perp_{\omega \cup \{w_j\}} \) in \( \bigwedge^* (\mathbb{A}_{v, \mathbb{R}} \oplus \mathbb{R} \rho) \) is
\[
\{e^*_L | \text{L an index set with } L \cap K = \emptyset\}.
\]
Working on affine charts, let \( f_1, \ldots, f_q \) be the functions on an affine chart of \( X_\tau \) defining \( Z_1^*, \ldots, Z_q^* \). Then we will show locally a generating set for \( H^{\dim \omega+1}(F_{\omega \cup \{w_j\}, \omega'}) \) consists of forms of the shape
\[
\left( \prod_{i \in M} f_i \right) \text{dlog}(e^*_L) \wedge \text{dlog}(e^*_N) \wedge \bigwedge_{i \in \Omega \setminus M} df_i,
\]
where:
\[
L \subseteq I;
\]
\[
N \subseteq \{0, \ldots, n\} \setminus (K \cup \bigcup_{i=1}^q I_i);
\]
\[
O = \{i | L \cap I_i \neq \emptyset\};
\]
\[
M \subseteq O.
\]
Indeed, first note such an element is in \( \ker(\delta_{\dim \omega+1}) \): given \( l \in I_i \) for some \( i \), if \( i \not\in I(\omega \cup \{w_j\}) \) then \( e^*_i \) does not occur in \( e^*_L \) or \( e^*_N \) or any \( df_j \), so the contraction with \( e_l \) is zero. If \( i \not\in I(\omega \cup \{w_j\}) \), then either \( l \not\in L \) and the contraction is zero, or else \( l \in L \) and the contraction restricts to zero on \( Z^*_I(\omega) \cap Z^*_L \cap Z^*_N \), either because \( i \in M \) or \( i \not\in O \setminus M \).

Conversely, we need to show this set generates \( \ker(\delta_{\dim \omega+1}) \). If \( \alpha \in \ker(\delta_{\dim \omega+1}) \), then any term involving \( \text{dlog}(e^*_L) \) appearing in \( \alpha \) must satisfy \( L' \cap K = \emptyset \). Furthermore, as the contraction of \( \alpha \) with \( e_l \) for \( l \in I_i, i \not\in I(\omega \cup \{w_j\}) \) is zero, such an \( l \) cannot appear in \( L' \). Thus we can decompose \( L' \) as \( L \cup N \) as claimed. Furthermore, the condition that \( \iota(e_l)\alpha|_{Z^*_I(\omega) \cap Z^*_L \cap Z^*_N} = 0 \) for \( l \in I_i, i \not\in I(\omega \cup \{w_j\}) \) implies that \( \alpha \) must be a linear combination of the forms of the type given, with coefficients in \( \mathcal{O}_{Z^*_I(\omega) \cap Z^*_N} \).

We now have to lift these generators. So assume furthermore that \( e_1 = w_j - v_j \). Then the above generator can be lifted to \( \Omega^{\dim \omega}_{(Z^*_I(\omega))!/k} \cap T^\perp_\omega \) as \( \left( \prod_{i \in M} f_i \right) \text{dlog}(e^*_1) \wedge \text{dlog}(e^*_L) \wedge \text{dlog}(e^*_N) \wedge \bigwedge_{i \in \Omega \setminus M} df_i \), which as before is easily checked to be in \( \ker(\delta_{\dim \omega}) \). This shows surjectivity. \( \square \)
3.3. **The Hodge decomposition.** We continue with the notation of the previous section, and will use the technical results of that section to finish the calculation of the Hodge decomposition. Having now constructed the resolution of the sheaves \((F^n_{r_1,r_2} \Omega^r_{\tau_1})/\text{Tors}\), we wish to use these resolutions to compute the cohomology of these sheaves. As a first step, we calculate the cohomology of the individual sheaves appearing in the resolution. Under important additional hypotheses, these sheaves have no cohomology in degree \(\geq 1\), and their global sections are easily expressed in terms of data on \(B\).

**Lemma 3.19.** Suppose that for the cell \(\tau \in \mathcal{P}\), \(\text{Conv}(\bigcup_{i=1}^q \hat{\Delta}_i \times \{e_i\})\) is a standard simplex (as opposed to just an elementary simplex). Then

1. For \(\sigma \subseteq \Delta_\tau\) a face,
   \[
   \Gamma(X_\tau, \Omega^r_{(Z^\tau_i)^\dagger/k^1} \cap T_{\sigma}^\perp) = \frac{\Lambda^r T_{\sigma}^\perp}{\text{Top}(I)} \otimes k,
   \]
   for \(T_{\sigma}^\perp \subseteq \tilde{\Lambda}_{v,\mathbb{R}}\), \(\text{Top}(I)\), the degree \(r\) part of the ideal in the exterior algebra of \(T_{\sigma}^\perp\) generated by
   \[
   \bigcup_{i \in I} \bigwedge_{\text{top}} T_{\Delta_i}.
   \]

2. \(H^j(X_\tau, \Omega^r_{(Z^\tau_i)^\dagger/k^1} \cap T_{\sigma}^\perp) = 0\) for \(j > 0\).

**Proof.** Let \(W\) be a complementary subspace to \(\bigoplus_{i \in I} T_{\Delta_i} \subseteq T_{\sigma}^\perp\). Then we can split \(\Omega^1_{(Z^\tau_i)^\dagger/k^1} \cap T_{\sigma}^\perp\) as \((\mathcal{O}_X \otimes W) \oplus \bigoplus_{i \in I} (\mathcal{O}_X \otimes T_{\Delta_i})\), and in addition \(d(\mathcal{O}(-Z^\tau_i)) \subseteq \mathcal{O}_X \otimes T_{\Delta_i}\), as the polynomial defining \(Z^\tau_i\) only involves monomials in \(\hat{\Delta}_i\). Let \(d_i = \dim \hat{\Delta}_i\). Then we obtain a splitting of the exact sequence of Lemma 3.15

\[
0 \to \bigoplus_{i \in I} \mathcal{O}_{Z^\tau_i}(-Z^\tau_i) \to \Omega^1_{(Z^\tau_i)^\dagger/k} \cap T_{\sigma}^\perp \to \Omega^1_{(Z^\tau_i)^\dagger/k^1} \cap T_{\sigma}^\perp \to 0
\]

into exact sequences, for \(i \in I\),

\[
0 \to \mathcal{O}_{Z^\tau_i}(-Z^\tau_i) \to \mathcal{O}_{Z^\tau_i} \otimes T_{\Delta_i} \to \Omega^1_i \to 0,
\]

where this sequence defines locally free rank \(d_i - 1\) sheaves \(\Omega^1_i\). In addition, we have one remaining direct summand of the original exact sequence,

\[
0 \to 0 \to \mathcal{O}_{Z^\tau_i} \otimes W \to \mathcal{O}_{Z^\tau_i} \otimes W \to 0.
\]

If we show that \(H^j(Z^\tau_i, \bigotimes_{i \in I} \Omega^r_i) = 0\) for \(j > 0\) and any collection of \(r_i\)’s with \(0 \leq r_i \leq d_i - 1\), then that will show (2) of the Lemma. We will in fact show that

\[
H^j(Z^\tau_i, \bigotimes_{i \in I} \Omega^r_i) \left( - \sum_{i \in I} a_i Z^\tau_i \right) = 0 \text{ for } 0 \leq a_i \leq d_i - 1 - r_i,
\]

by induction on \(\sum_{i \in I} r_i\), starting with \(\sum_{i \in I} r_i = 0\).

**Base case:** \(\sum_{i \in I} r_i = 0\). We need to show \(H^j(Z^\tau_i, \mathcal{O}_{Z^\tau_i} \left( - \sum_{i \in I} a_i Z^\tau_i \right)) = 0\) for \(0 \leq a_i \leq d_i - 1\). For this we use toric methods. Let \(\psi_i\) be the piecewise linear function on the fan.
$\Sigma_\tau$ defining the line bundle $\mathcal{O}_{X_\tau}(Z_\tau^\tau)$. These are convex functions and $\tilde{\Delta}_i$ is the Newton polytope of $\psi_i$ ([12], Remark 1.59). Then $-\sum_{i \in I} a_i \psi_i$ corresponds to $\mathcal{O}_{X_\tau}(-\sum_{i \in I} a_i Z_i^\tau)$. With $Q_{\tau, \mathbb{R}} = (\Lambda^+_\tau)^*$ as in [12], Definition 1.33, by [29], §2.2,

$$H^j(\mathcal{O}_{X_\tau}(-\sum_{i \in I} a_i Z_i^\tau)) = \bigoplus_{n \in \Lambda^+_\tau} H^j_{Z(n)}(Q_{\tau, \mathbb{R}}, \mathbb{k}),$$

where

$$Z(n) := \{m \in Q_{\tau, \mathbb{R}} | \langle m, n \rangle \geq \sum_{i \in I} a_i \psi_i(m)\}.$$  

([29], p. 74. Note that our sign convention for the piecewise linear function corresponding to a divisor is the opposite of that used in [29].) Also $H^j_{Z(n)}(Q_{\tau, \mathbb{R}}, \mathbb{k}) \cong H^{j-1}(Q_{\tau, \mathbb{R}} \setminus Z(n), \mathbb{k})$ for $j \geq 2$, and we have an exact sequence

$$0 \to H^0_{Z(n)}(Q_{\tau, \mathbb{R}}, \mathbb{k}) \to \mathbb{k} \to H^0(Q_{\tau, \mathbb{R}} \setminus Z(n), \mathbb{k}) \to H^1_{Z(n)}(Q_{\tau, \mathbb{R}}, \mathbb{k}) \to 0.$$  

([29], p. 74.) Because $\sum_{i \in I} a_i \psi_i$ is convex for $a_i \geq 0$, it is easy to see $Q_{\tau, \mathbb{R}} \setminus Z(n)$ is contractible unless $Z(n)$ is a vector subspace of $Q_{\tau, \mathbb{R}}$, and this happens if and only if $\langle m, n \rangle \leq \sum_{i \in I} a_i \psi_i(m)$ for all $m \in Q_{\tau, \mathbb{R}}$ with equality if and only if $m$ is in the subspace $Z(n)$. This in turn is equivalent to $n$ being in the relative interior of $-\sum_{i \in I} a_i \tilde{\Delta}_i$, as

$$\tilde{\Delta}_i = \{n \in \Lambda^+_\tau | \langle m, n \rangle \geq -\psi_i(m) \forall m \in Q_{\tau, \mathbb{R}}\}.$$  

Now each $\tilde{\Delta}_i$ is assumed to be a standard simplex, and hence $\text{Int}(a_i \tilde{\Delta}_i) \cap \Lambda^+_\tau$ contains no integral points for $a_i \leq d_i$. By simplicity, $\sum_{i \in I} T_{\Delta_i}$ is an interior direct sum, and by the additional assumption that $\text{Conv}(\bigcup_{i \in I} \tilde{\Delta}_i \times \{e_i\})$ is a standard simplex, this direct sum decomposition also works over $\mathbb{Z}$. Thus if $n \in \text{Int}(\sum_{i \in I} a_i \tilde{\Delta}_i) \cap \Lambda^+_\tau$, we can write $n = \sum_{i \in I} n_i$ with $n_i \in \text{Int}(-a_i \tilde{\Delta}_i) \cap \Lambda^+_\tau$. Thus there is no such point $n$ if $0 \leq a_i \leq d_i$. In particular, $H^j(\mathcal{O}_{X_\tau}(-\sum_{i \in I} a_i Z_i^\tau)) = 0$ for $j > 0$, $0 \leq a_i \leq d_i$.

Now use the Koszul resolution of $\mathcal{O}_{Z_i^\tau}$. Tensoring (3.5) with $\mathcal{O}_{X_\tau}(-\sum_{i \in I} a_i Z_i^\tau)$ for $0 \leq a_i \leq d_i - 1$ we obtain the base case using the above vanishing.

**Induction case.** Suppose $0 \leq a_i \leq d_i - 1 - r_i$ for all $i \in I$. Pick $i_1 \in I$ with $r_{i_1} > 0$. The exact sequence (3.6) yields an exact sequence (see [18], Exercise II 5.16)

(3.7) \hspace{1cm} 0 \to \Omega^{r_{i_1}-1}_{i_1}(-Z_i^\tau) \to \Omega^{r_{i_1}}_{i_1} \bigotimes_{i \in I \setminus \{i_1\}} \Omega^{r_{i_1}}_{i_1} \to \Omega^{r_{i_1}}_{i_1} \mathcal{T}_{\tilde{\Delta}_{i_1}} \to 0,

which we tensor with $\left(\bigotimes_{i \in I \setminus \{i_1\}} \Omega^{r_{i_1}}_{i_1}\right) (-\sum_{i \in I} a_i Z_i^\tau)$. Then

$$H^j\left(Z_i^\tau, \left(\bigotimes_{i \in I \setminus \{i_1\}} \Omega^{r_{i_1}}_{i_1}\right) (-\sum_{i \in I} a_i Z_i^\tau)\right)$$

and

$$H^{j+1}\left(Z_i^\tau, \left(\bigotimes_{i \in I \setminus \{i_1\}} \Omega^{r_{i_1}}_{i_1}\right) \otimes \Omega^{r_{i_1}-1}_{i_1}(-Z_i^\tau - \sum_{i \in I} a_i Z_i^\tau)\right)$$
both vanish for \( j > 0 \) by the induction hypothesis, hence
\[
H^j \left( Z^\tau_I, \left( \bigotimes_{i \in I} \Omega_i^{r_i} \right) \left( - \sum_{i \in I} a_i Z^\tau_i \right) \right)
\]
vanishes, completing the induction step.

This proves (2).

To prove (1), we use
\[
\Omega^r_{(Z^\tau_I)_{/k^1}} \cap T^\perp_{\sigma} = \bigoplus_{t, r_i, \sum_{i \in I} r_i = r} \left( \bigwedge^t W \otimes \bigotimes_{i \in I} \Omega_i^{r_i} \right).
\]

Since \( \text{rank} \, \Omega_i = d_i - 1 \), the direct summand is zero unless \( r_i \leq d_i - 1 \) for each \( i \). Thus, if we show
\[
\Gamma(Z^\tau_I, \bigwedge^t W \otimes \bigotimes_{i \in I} \Omega_i^{r_i}) = \bigwedge^t W \otimes \bigotimes_{i \in I} \bigwedge^{r_i} T_{\Delta_i}
\]
whenever \( r_i \leq d_i - 1 \) for all \( i \in I \), the result will follow. We will show inductively that
\[
H^0(Z^\tau_I, \left( \bigotimes_{i \in I} \Omega_i^{r_i} \right)(- \sum_{i \in I} a_i Z^\tau_i)) = 0
\]
for \( 0 \leq a_i \leq d_i - 1 - r_i \) if at least one \( a_i > 0 \), and
\[
H^0(Z^\tau_I, \bigotimes_{i \in I} \Omega_i^{r_i}) = \bigotimes_{i \in I} \bigwedge^{r_i} T_{\Delta_i}.
\]

Again, induction is on \( \sum_{i \in I} r_i \). The base case follows immediately from
\[
H^0(X, \mathcal{O}_X, (- \sum_{i \in I} a_i Z^\tau_i)) = 0
\]
for \( a_i \geq 0 \) and at least one \( a_i > 0 \), and the Koszul resolution \([3.5]\). For the induction step, we tensor \([3.7]\) with \( (\bigotimes_{i \in \Gamma \setminus \{i_1\}} \Omega_i^{r_i})(- \sum_{i \in I} a_i Z^\tau_i) \) as before; if at least one \( a_i > 0 \), then the vanishing of \( H^0 \) follows from the induction hypothesis, while if \( a_i = 0 \) for all \( i \), we get
\[
0 \to H^0(Z^\tau_I, \Omega_i^{r_i-1}(-Z^\tau_i) \otimes \bigotimes_{i \in \Gamma \setminus \{i_1\}} \Omega_i^{r_i}) \to H^0(Z^\tau_I, \bigwedge^{r_{i_1}} T_{\Delta_{i_1}} \otimes \bigotimes_{i \in \Gamma \setminus \{i_1\}} \Omega_i^{r_i}) \to 0.
\]

From the induction hypothesis and part (2) already proved, we get \( H^0(Z^\tau_I, \bigotimes_{i \in I} \Omega_i^{r_i}) \cong \bigotimes_{i \in I} \bigwedge^{r_i} T_{\Delta_i} \otimes k \), as desired. \( \square \)

**Lemma 3.20.** With the same hypotheses as in Lemma \([3.19]\) in the \( /k^1 \) case, we have for any morphism \( e: \tau_1 \to \tau_2 \), \( W_e \subseteq B \) the open subset defined in \([12]\), Lemma 2.9,
\[
\Gamma(W_e, i_* \bigwedge^r \Lambda \otimes k) \cong H^0(X_{\tau_2}, (F_{\tau_1, \tau_2}^* \Omega_{\tau_1}^r)/\text{Tors})
\]
and
\[
H^j(X_{\tau_2}, (F_{\tau_1, \tau_2}^* \Omega_{\tau_1}^r)/\text{Tors}) = 0
\]
for \( j > 0 \). Without the additional hypotheses on \( \bigcup_{i=1}^{n} \Delta_i \times \{e_i\} \), there is still an injective map

\[
\Gamma(W_{e_i}, i_s \bigwedge^r \Lambda \otimes k) \to H^0(X_{\tau_2}, (F^s_{\tau_1, \tau_2} \Omega_{\tau_1}^r) / \text{Tors}).
\]

**Proof.** Pick a vertex \( g : \nu \to \tau_1 \). Then

\[
H^j(X_{\tau_2}, (F^s_{\tau_1, \tau_2} \Omega_{\tau_1}^r) / \text{Tors}) \cong \bigoplus_{\substack{\sigma \subseteq \Delta_1, \tau \in \sigma \atop \dim \tau = s}} \left( \bigwedge^{r-s} T_{\sigma} \right) / \text{Top}(e, I(\sigma))_{r-s},
\]

where \( \sigma = \sigma_1 + \cdots + \sigma_q, I(\sigma) = \{ i | \sigma_i \neq \{ v_i \} \} \) as before, and \( \text{Top}(e, I(\sigma))_{r-s} \) is the degree \( r - s \) part of the ideal of the exterior algebra of \( T_{\sigma} \) generated by

\[
\bigcup_{i \in I(\sigma)} \bigwedge^{\text{top}} (T_{\Delta_i} \cap \Lambda_{\tau_2}^t).
\]

The term \( \Lambda_{\tau_2}^t \) appears because \( \Delta_i \) is the Newton polytope for \( Z_{\tau_1} \subseteq X_{\tau_1} \), while in Lemma 3.19 (1), one uses the tangent space to the Newton polytope for the corresponding divisor in \( X_{\tau_2} \), i.e. \( F_{\tau_1, \tau_2}^{-1}(Z_{\tau_1}) \). This Newton polytope is a face of \( \Delta_i \), and its tangent space is precisely \( T_{\Delta_i} \cap \Lambda_{\tau_2}^t \). We interpret the ideal to be the unit ideal if \( T_{\Delta_i} \cap \Lambda_{\tau_2}^t = \{ 0 \} \) for some \( i \in I(\sigma) \), i.e. if \( F_{\tau_1, \tau_2}^{-1}(Z_{\tau_1}) = \emptyset \).

Furthermore, the differential \( \delta : F^s \to F^{s+1} \) is defined by

\[
(\delta \alpha)_\sigma = \sum_{\substack{\sigma \subseteq \sigma \atop \dim \tau = s, \dim \omega = \dim + 1}} t(\partial w_j \omega) \alpha_{\sigma}
\]

as before. We can then show \( H^j(F^* \simeq 0 \) for \( j > 0 \) by repeating the argument of Lemma 3.18 defining analogous complexes \( F_{\omega, \omega'} \) and proceeding by induction. The key stage looks at the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & H^{\dim \omega}(F_{\omega, \omega'} \setminus \{ w_j \}) \\
\delta & \downarrow & \bigoplus_{\substack{\omega \subseteq \omega \setminus \{ w_j \} \atop \dim \omega = \dim + 1}} \bigwedge^{r-\dim \omega - 1} T_{\sigma} \\
0 & \longrightarrow & H^{\dim \omega + 1}(F_{\omega, \omega'} \setminus \{ w_j \}, \omega')
\end{array}
\]

and we need to show surjectivity of \( \bar{\delta} \). We explicitly write down a basis for

\[
H^{\dim \omega + 1}(F_{\omega, \omega'} \setminus \{ w_j \}, \omega').
\]
We can choose a basis $e_1, \ldots, e_n$ of $\Lambda_\ast \mathbb{A}$, $e'_1, \ldots, e'_n$ the dual basis for $\Lambda_\ast \mathbb{A}$, with the properties listed in the proof of Lemma 3.18, but in addition make a choice so that there are index sets $J_i$, for $i = 1, \ldots, q$ with $\{e'_i k : k \in J_i\}$ a basis for $T_{\Delta, \cdot} \cap \Lambda^\perp_{r_2}$. Then a basis for $\bigwedge^{r-\dim \omega-1} T_{\omega \cup \{w_j\}}^1 / \text{Top}(e, I(\omega \cup \{w_j\}))_{r-\dim \omega-1}$ is
\[
\{e'_i | L \text{ satisfies } L \cap K = \emptyset \text{ and } J_i \not\subseteq L \text{ for any } i \in I(\omega \cup \{w_j\})\}.
\]

A basis for $H^{\dim \omega + 1}(F_{\omega \cup \{w_j\}, \omega'})$ then is
\[
\{e'_i | L \cap K = \emptyset, J_i \not\subseteq L \text{ for any } i \in I(\omega \cup \{w_j\}) \text{, and for each } i, L \cap I_i \neq \emptyset \text{ implies } J_i \subseteq L\}.
\]

One then checks as in the proof of Lemma 3.18 that each basis vector lifts.

Finally, we calculate $H^0(F^\ast)$, and compare this with $\Gamma(W^e, i^\ast \bigwedge^r \Lambda \otimes k)$. We identify this with monodromy invariant elements of $i^\ast \bigwedge^r \Lambda \otimes k$ for loops based at $v$ whose interior is in $W^e$. The monodromy action is then generated by transformations of the form $T^\rho_{feg, feg'} : \Lambda \to \Lambda$, (see [12], §1.5) where we have $f : \tau \to \rho$ with $\rho$ codimension one, and $g' : v' \to \tau_1$ a vertex. Then as in [12], §1.5,
\[
T^\rho_{feg, feg'}(m) = m + (\hat{d}_\rho, m)m^\rho_{feg, feg'},
\]
and hence the action on $\bigwedge^r \Lambda \otimes k$ is
\[
T^\rho_{feg, feg'}(n) = n + \hat{d}_\rho \wedge (m^\rho_{feg, feg'})n.
\]

Thus $n \in \bigwedge^r \Lambda \otimes k$ is invariant under all such monodromy operations if and only if $\hat{d}_\rho \wedge (m^\rho_{feg, feg'})n = 0$ for all choices of $f$ and $g'$. Note that as $f \circ e$ runs through elements of $R_i$ which factor through $e$, $\hat{d}_\rho$ runs through a generating set for $T_{\Delta, \cdot} \cap \Lambda^\perp_{r_2}$, and for any given $f$ with $f \circ e \in R_i$, $\rho$ varies over all vertices of $\tau_1$. $m^\rho_{feg, feg'}$ runs over $\{v'_i - v_i | v'_i := \text{Vert}_i(g') \text{ a vertex of } \Delta_i\}$. From this description, it is then clear that $\hat{d}_\rho \wedge (m^\rho_{feg, feg'})n = 0$ for all $f, g'$ if and only if $n \in H^0(F^\ast)$.

Note that in any event, without the extra hypotheses, if $n \in \Gamma(W^e, i^\ast \bigwedge^r \Lambda \otimes k)$ is viewed as an element $n \in \bigwedge^r \Lambda \otimes k$, hence defining $\text{dlog} n$ in $F^\ast \otimes_{v, \tau_2} \Omega^r_{v}$, it is easy to check from the above formulae that monodromy invariance of $n$ implies $\text{dlog} n \in \ker \delta_0$, i.e., $\text{dlog} n \in \Gamma(X_{\tau_2}, F^\ast \otimes_{\tau_2} \Omega^r_{\tau_1})/\text{Tors}$, giving the map in general, which is clearly injective, as $\text{dlog} n = 0$ if and only if $\text{dlog} n = 0$ in $F^\ast \otimes_{v, \tau_2} \Omega^r_{v}$.

We can now prove the main theorem of this section: the identification of the logarithmic Dolbeault groups with the expected cohomology groups on $B$.

**Theorem 3.21.** Let $B$ be an integral affine manifold with singularities, with polyhedral decomposition $\mathcal{P}$, and suppose $(B, \mathcal{P})$ is positive and simple. Assume furthermore that for all $r \in \mathcal{P}$, $\text{Conv}(\bigcup_{i=1}^q \Delta_i \times \{e_i\})$ is a standard simplex. Let $s$ be lifted gluing data, with $X_0 = X_0(B, \mathcal{P}, s)$. Then there is a canonical isomorphism
\[
H^p(X_0, j^\ast \Omega^r_{X_0/k}) \cong H^p(B, i^\ast \bigwedge^r \Lambda \otimes k).
\]
Proof. By Corollary 3.10 and Lemma 3.20,
\[ H^p(X_0, j_* \Omega^r_{X_0/k^{1}}) = H^p(\Gamma(X_0, \mathcal{E}^\bullet(\Omega^r))) , \]
and
\[ \Gamma(X_0, \mathcal{E}^p(\Omega^r)) = \bigoplus_{\sigma_0 \subseteq \cdots \subseteq \sigma_p} \Gamma(W_{\sigma_0 \rightarrow \sigma_p}, i_* \wedge^r \Lambda \otimes k). \]
However, \( \Gamma(W_{\sigma_0 \rightarrow \sigma_p}, i_* \wedge^r \Lambda \otimes k) = \Gamma(W_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_p}, i_* \wedge^r \Lambda \otimes k) \) (where \( W_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_p} \) is the connected component of \( W_{\sigma_0} \cap \cdots \cap W_{\sigma_p} \) indexed by \( \sigma_0 \rightarrow \cdots \rightarrow \sigma_p \); if \( \mathcal{P} \) has no self-intersecting cells, then \( W_{\sigma_0} \cap \cdots \cap W_{\sigma_p} \) only has one connected component anyway) because the relevant monodromy operators, as considered in the proof of Lemma 3.20 only depend on \( \sigma_0 \rightarrow \sigma_p \). Under this identification, the differential then agrees with the Čech differential for \( i_* \wedge^r \Lambda \otimes k \) with respect to the standard open covering \( \{W_{\sigma}\} \). This proves the theorem.

We can still prove more limited results without the additional hypotheses.

Theorem 3.22. Let \( B \) be an integral affine manifold with singularities with polyhedral decomposition \( \mathcal{P} \) and \( \dim B = n \), with \((B, \mathcal{P})\) positive and simple. Let \( s \) be lifted gluing data, with \( X_0 = X_0(B, \mathcal{P}, s) \). Then there is a canonical isomorphism
\[ H^p(X_0, j_* \Omega^r_{X_0/k^{1}}) \cong H^p(B, i_* \wedge^r \Lambda \otimes k) \]
for \( 0 \leq p \leq n \), \( r = 0, 1, n - 1 \) or \( n \).
In addition there is always a canonical homomorphism
\[ H^p(B, i_* \wedge^r \Lambda \otimes k) \rightarrow H^p(X_0, \Omega^r_{X_0/k^{1}}). \]

Proof. The case when \( r = 0 \) is already [12], Proposition 2.37.

We need to show the proof of Theorem 3.21 goes through for forms of degree one, \( n - 1 \), and \( n \). For \( \Omega^1 \), we just prove Lemma 3.20 in degree one. The resolution \( F_v \cdot \) gives a resolution
\[ 0 \rightarrow \Omega^1_{\tau} \rightarrow F^*_{v,v} \Omega^1_v \rightarrow \bigoplus_{v \in \sigma \subseteq \Delta_v \dim \sigma = 1} \mathcal{O}_{Z_{I(\sigma)}} \rightarrow 0. \]
Note \( \#I(\sigma) = 1 \) if \( \dim \sigma = 1 \). Also \( H^p(X_{\tau}, \mathcal{O}_{Z_{\sigma}}) = 0 \) for \( p > 0 \), as follows from \( H^p(X_{\tau}, \mathcal{O}_{X_{\tau}}(-Z_i)) = 0 \) for all \( p \), which comes from the same toric argument given in the base case of the proof of Lemma 3.19 (2), as \( -\Delta_i \) contains no interior points. From this it immediately follows that Lemma 3.20 holds in general in degree one. Thus Theorem 3.21 holds for \( r = 1 \), as desired.

Now consider the resolution for \( \Omega^{n-1}_{\tau} \). The \( p \)-th term of this resolution is a direct sum of terms of the form \( \Omega^{n-1-p}_{Z_{I(\sigma)}(\tau)} \cap T^\perp_{\sigma} \), for \( v \in \sigma \subseteq \Delta_v \), with \( \dim \sigma = p \). Now \( \Omega^{1}_{Z_{I(\sigma)}(\tau)} \cap T^\perp_{\sigma} \) is a locally free sheaf of rank \( (n - p) - \#I(\sigma) \), so \( \Omega^{n-1-p}_{Z_{I(\sigma)}(\tau)} \cap T^\perp_{\sigma} = \wedge^{n-1-p}(\Omega^{1}_{Z_{I(\sigma)}(\tau)} \cap T^\perp_{\sigma}) = 0 \) unless \( \#I(\sigma) = 1 \). In this case, if \( I(\sigma) = \{i\} \), split \( T^\perp_{\sigma} = T_{\Delta_i} \oplus W \). Then as in the
proof of Lemma 3.19 with $d_i = \dim T_{\Delta_i}, \Omega^{i}_{Z_i} \cap T_\sigma \cong \Omega^{i}_{Z_i} \oplus (\mathcal{O}_{X_r} \otimes W)$ and $\Omega^{n-1-p}_{Z_i} \cong \Omega^{d_i-1}_i \otimes \wedge^{\text{top}} W$. On the other hand, from the definition of $\Omega_i$, (3.6), we see $\Omega^{d_i-1}_i \cong \mathcal{O}_{Z_i}(Z_i)$. As $H^j(X_r, \mathcal{O}_{X_r}) = H^j(X_r, \mathcal{O}_{X_r}(Z_i)) = 0$ for $j > 0$, $H^j(X_r, \Omega^{n-1-p}_{Z_i} \cap T_\sigma) = 0$ for $j > 0$. In addition, $H^0(X_r, \mathcal{O}_{Z_i}(Z_i)) = \text{coker}(H^0(X_r, \mathcal{O}_{X_r}) \to H^0(X_r, \mathcal{O}_{X_r}(Z_i)))$, and $\dim H^0(X_r, \mathcal{O}_{X_r}(Z_i)) = d_i + 1$: there is a basis of sections corresponding to the vertices of $\Delta_i$. Thus $\dim H^0(X_r, \mathcal{O}_{Z_i}(Z_i)) = d_i$. On the other hand, the map $\wedge^{d_i-1} T_{\Delta_i} \otimes \mathbb{k} \to H^0(X_r, \Omega^{d_i-1}_i)$ is injective. To see this, take a basis $e_{1, \ldots, e_{d_i}}$ of $T_{\Delta_i}$, chosen so that the vertices of $\Delta_i$ are $v_i, v_i + e_1, \ldots, v_i + e_{d_i}$. Then a basis for $\wedge^{d_i-1} T_{\Delta_i}$ is $\{ \hat{e}_j := (-1)^{j-1} e_1^* \land \cdots \land \hat{e}_j^* \land \cdots \land e_{d_i}^* \}$. If $f_i = 0$ locally defines $Z_i$, $f_i = 1 + \sum_{j=1}^{d_i} a_j e_j^*$, then for $n \in \wedge^{d_i-1} T_{\Delta_i}$, $\deg n$ maps to zero in $H^0(X_r, \Omega^{d_i-1}_i)$ if and only if $df_i \land \deg n = 0$ along $Z_i$. But writing $n = \sum b_j \hat{e}_j^* \land df_i \land \deg n = \sum_{j=1}^{d_i} b_j a_j e_j^* \land \deg (e_1^* \land \cdots \land e_{d_i}^*)$, which is never zero everywhere on $Z_i$ unless $n = 0$, since all the $a_j$’s are non-zero. Thus by comparing dimensions, we see $H^0(X_r, \Omega^{d_i-1}_i) \cong \wedge^{d_i-1} T_{\Delta_i} \otimes \mathbb{k}$, and thus $H^0(X_r, \Omega^{n-1-p}_{Z_i} \cap T_\sigma) = \wedge^{n-1-p} T_{\text{top}(\{i\})} \otimes \mathbb{k}$. So Lemma 3.19 holds for the appropriate degrees and Lemma 3.20 follows for degree $n - 1$ forms. This proves the result for $r = n - 1$.

The case $q = n$ is easier: in this case $\Omega^n_r \cong \mathcal{O}_r$ for each $r$, and the proof goes through immediately.

In any event, by Lemma 3.20, we always have maps, for $e : \tau_1 \to \tau_2$,  
$$
\Gamma(W_e, i_* \wedge^r \Lambda \otimes \mathbb{k}) \to H^0(X_{\tau_2}, (F_{\tau_1, \tau_2}^* \Omega^r_{\tau_1})/\text{Tors})
$$
giving a morphism of complexes from the Čech complex of $i_* \wedge^r \Lambda \otimes \mathbb{k}$ to the complex $\Gamma(X_0, \mathcal{C}^*(\Omega^r))$, which in turn maps to the complex $\mathbb{R}\Gamma(X_0, \mathcal{C}^*(\Omega^r))$. After taking the cohomology of this composition, we get a morphism $H^p(B, i_* \wedge^r \Lambda \otimes \mathbb{Z} \mathbb{k}) \to H^p(X_0, j_* \Omega^n_{X_0/k})$, as desired.

Relating this to the cohomology of the tangent bundle, we get

**Theorem 3.23.** Let $(B, \mathcal{P})$ be positive and simple, and suppose the holonomy of $B$ is contained in $\mathbb{Z}^n \rtimes \text{SL}_n(\mathbb{Z})$ (rather than $\text{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \text{GL}_n(\mathbb{Z})$). Then with $X_0 = X_0(B, \mathcal{P}, s)$,

1. $j_* \Omega^n_{X_0/k} \cong \mathcal{O}_X$, so that $j_* \Omega^{n-r}_{X_0/k} \cong j_* \wedge^r \Theta_{X_0/k}$.
2. $i_* \wedge^n \Lambda \cong \mathbb{Z}$ so that $i_* \wedge^{n-r} \Lambda \cong i_* \wedge^r \Lambda$.
3. There is a canonical isomorphism $H^p(X, \Theta_{X_0/k}) \cong H^p(B, i_* \Lambda \otimes \mathbb{k})$.

**Proof.** The statement about $i_* \wedge^n \Lambda$ is obvious, since the monodromy of $\Lambda$ is the linear part of the holonomy. Thus by Theorem 3.22, $H^0(X_0, \Omega^n_{X_0/k}) = \Gamma(B, i_* \wedge^n \Lambda \otimes \mathbb{k}) = \mathbb{k}$, so $\Omega^n_{X_0/k}$ has a section which is nowhere zero, as $\Omega^n_v = \mathcal{O}_{X_v}$ for each $v$. Thus on $X_0 \setminus Z$, $j_* \Omega^n_{X_0/k}$ is trivial. The remaining statements are then obvious from previous results. \( \square \)

We also obtain the proverbial interchange of Hodge numbers of mirror symmetry:
Theorem 3.26. With the hypotheses of Theorem 3.21, there is a canonical isomorphism

\[ H^p(X_0(B, \mathcal{P}, s), j_*\Omega^q_{X_0(B, \mathcal{P}, s)^\dagger/\mathbb{k}}) \cong H^p(X_0(\tilde{B}, \tilde{\mathcal{P}}, \tilde{s}), j_*\Omega^{q-n} \cdot \Lambda^B \otimes \mathbb{k}) \]

Proof. Using successively Theorem 3.21 [12], Proposition 1.50, (1), and Theorem 3.23, (2),

\[ H^p(X_0(B, \mathcal{P}, s), j_*\Omega^q_{X_0(B, \mathcal{P}, s)^\dagger/\mathbb{k}}) \cong H^p(\tilde{B}, i_*\Lambda^B \otimes \mathbb{k}) \]

\[ \cong H^p(\tilde{B}, i_*\Lambda^\tilde{B} \otimes \mathbb{k}) \]

\[ \cong H^p(X_0(\tilde{B}, \tilde{\mathcal{P}}, \tilde{s}), j_*\Omega^{n-q} \cdot \Lambda^B \otimes \mathbb{k}) \]

\[ \cong H^p(X_0(\tilde{B}, \tilde{\mathcal{P}}, \tilde{s}), j_*\Omega^{n-q} \cdot \Lambda^\tilde{B} \otimes \mathbb{k}). \]

Here the superscript \( B \) or \( \tilde{B} \) on \( \Lambda \) indicates which affine structure we are using to define \( \Lambda \). \( \square \)

Remark 3.25. The discrete Legendre transform interchanges the role of the \( \Delta_i \)'s and \( \tilde{\Delta}_i \)'s in simplicity data, as follows easily from the definition in [12], §1.5. Thus by Proposition 2.2 and Corollary 2.18 the deformation of \( X_0(\tilde{B}, \tilde{\mathcal{P}}, \tilde{s}) \) constructed in [13] over \( \operatorname{Spec} \mathbb{k}[[t]] \) has non-singular generic fibre if \((B, \mathcal{P})\) satisfies the hypotheses of Theorem 3.21. Thus if both \((B, \mathcal{P})\) and \((\tilde{B}, \tilde{\mathcal{P}})\) satisfies the hypotheses of Theorem 3.21, one obtains a non-singular mirror pair. If \((B, \mathcal{P})\) is only simple, then this generic fibre may have orbifold singularities. As is well-known [1], in dimensions \( \geq 4 \), orbifold singularities are often necessary, in which case stringy Hodge numbers are needed [3]. It is not yet clear what the relationship between stringy Hodge numbers and the affine geometry of \( B \) is, but see [32].

We now obtain the Hodge decomposition:

Theorem 3.26. With the hypotheses of Theorem 3.21, there is a canonical isomorphism

\[ H^\tau(X_0, j_*\Omega^\bullet_{X_0^\dagger/\mathbb{k}}) \cong \bigoplus_{p+q=\tau} H^p(X_0, j_*\Omega^q_{X_0^\dagger/\mathbb{k}}). \]

Proof. By Corollary 3.11 and Lemma 3.20,

\[ H^\tau(X_0, \Omega^\bullet) = H^\tau(\Gamma(X_0, \operatorname{Tot}(\mathcal{C}^\bullet(\Omega^\bullet)))) \]

But as \( \Gamma(X_0, (F_{r,s}^\bullet, \Omega^\bullet)/\text{Tors}) \) consists entirely of differentials of the form \( \log n, \) \( d \) is in fact zero in \( \Gamma(X_0, \mathcal{C}^\bullet(\Omega^\bullet)) \), and thus the global sections of the total complex split as a direct sum \( \bigoplus_q \Gamma(X_0, \mathcal{C}^\bullet(\Omega^q)[-q]) \), hence the result. \( \square \)
4. Basechange and the cohomology of smoothings

**Theorem 4.1.** Let $A$ be a local Artinian $k[t]$-algebra with residue class field $k$ and $\text{Spec } A^\dagger$ the scheme $\text{Spec } A$ with log structure induced by $\mathbb{N} \to A$, $1 \mapsto t$. Assume that $\pi : X^\dagger = (X, \mathcal{M}_X) \to \text{Spec } A^\dagger$ is a divisorial deformation of a positive and simple toric log Calabi-Yau space $X^\dagger_0 \to \text{Spec } k^\dagger$. As before denote by $Z \subseteq X$ the singular set of the log structure of relative codimension two, $j : X \setminus Z \to X$ the inclusion of the complement and write $\Omega^\bullet_X := j_* \Omega^\bullet_{X^\dagger/A^\dagger}$. Then $\mathbb{H}^p(X, \Omega^\bullet_X)$ is a free $A$-module, and it commutes with base change.

**Proof.** We follow [34], [24]. By the cohomology and base change theorem it suffices to prove the surjectivity of the restriction map $H^p(X, \Omega^\bullet_X) \to H^p(X_0, \Omega^\bullet_{X_0})$. Here $\Omega^\bullet_{X_0} = j_* \Omega^\bullet_{X^\dagger_0/k}$. As in [24], p.404 it suffices to prove this for $A = k[t]/(t^{k+1})$ with the obvious $k[t]$-algebra structure. For structural clarity we nevertheless keep the notation $A$ for the base ring.

Consider the complex of $\mathcal{O}_X$-modules

$$L^\bullet = j_* \Omega^\bullet_{X^\dagger/k} = \bigoplus_{s=0}^\infty j_* \Omega^\bullet_{X^\dagger/k} \cdot u^s$$

with differential

$$d\left(\sum_{s=0}^N \alpha_s u^s\right) = \sum_{s=0}^N d\alpha_s \cdot u^s + s \cdot d\log \rho \wedge \alpha_s \cdot u^{s-1}$$

$$= d\alpha_N \cdot u^N + \sum_{s=0}^{N-1} (d\alpha_s + (s+1) \cdot d\log \rho \wedge \alpha_{s+1}) \cdot u^s,$$

where $\rho \in \Gamma(\mathcal{M}_X)$ is the pull-back of the section of $\mathcal{M}_A$ induced by $t$. Note that these are differentials relative $\text{Spec } k$ rather than relative $\text{Spec } A^\dagger$, so $d\log \rho \neq 0$ unlike in $\Omega_X$. In this complex the dummy variable $u$ formally behaves like $\log t$, and the use of considering this complex is to trade powers of $d\log \rho$ with powers of $u$.

Now projection $\sum \alpha_s u^s \mapsto \alpha_0$ defines a map

$$L^\bullet \to \Omega^\bullet_X.$$

To finish the proof it suffices to show that the composition

$$\varphi^\bullet : L^\bullet \to \Omega^\bullet_X \to \Omega^\bullet_{X_0}$$
is a quasi-isomorphism, that is, induces isomorphisms of cohomology sheaves \( H^p(\mathcal{L}^\bullet) \to H^p(\Omega_{X_0}^\bullet) \). In fact, then the induced composed map of hypercohomology

\[
\mathbb{H}^p(\mathcal{L}^\bullet) \longrightarrow \mathbb{H}^p(\Omega_X^\bullet) \longrightarrow \mathbb{H}^p(\Omega_{X_0}^\bullet)
\]

is an isomorphism, and hence the second map is surjective as needed.

By this argument and since \( X^\dagger \to \text{Spec} A^\dagger \) is a divisorial deformation of \( X_0^\dagger \to \text{Spec} \mathcal{k}^\dagger \), for the rest of the proof we consider the following local situation. There is a toric variety \( Y = \text{Spec} \mathcal{k}[P] \) containing \( X_0 \) as a toric Cartier divisor \( V(\omega^\rho) \) such that the deformation \( X^\dagger \to \text{Spec} A^\dagger \) is given by

\[
\pi : \text{Spec} \mathcal{k}[P]/(z^{(k+1)}\rho) \longrightarrow \text{Spec} \mathcal{k}[t]/(t^{k+1}), \quad \pi^*(t) = \omega^\rho.
\]

Since \( \varphi^r : \mathcal{L}^r \to \Omega_{X_0}^r \) is surjective for any \( r \) we obtain a short exact sequence

\[
0 \longrightarrow \mathcal{K}^\bullet \longrightarrow \mathcal{L}^\bullet \overset{\varphi^\bullet}{\longrightarrow} \Omega_{X_0}^\bullet \longrightarrow 0
\]

of complexes by defining \( \mathcal{K}^\bullet = \ker \varphi^\bullet \). Now \( \varphi^\bullet \) is a quasi-isomorphism if and only if \( \mathcal{K}^\bullet \) is acyclic, and this is what we are going to show.

For an explicit description of \( \mathcal{K}^r \) let \( \sum_{s=0}^N \alpha_s u^s \in \mathcal{L}^r \), that is, \( \alpha_s \in j_s \Omega_{X^\dagger/k}^r \) for all \( s \). Then \( \sum_{s=0}^N \alpha_s u^s \in \ker \varphi^r \) iff \( \alpha_0|_{X_0} = 0 \). On the other hand, the closedness equation \( d(\sum \alpha_s u^s) = 0 \) is equivalent to the system of equations

\[
\begin{align*}
d\alpha_N &= 0 \\
d\alpha_s + (s + 1) d\log \rho \wedge \alpha_{s+1} &= 0, \quad s < N.
\end{align*}
\]

(4.1)

It is easy to solve these equations after decomposing the coefficients \( \alpha_s \) according to weights, that is, according to the \( P \)-grading. First, Proposition 1.12 gives a decomposition of \( \Gamma(X, j_* \Omega_{X^\dagger/k}^r) \) into homogeneous pieces as follows:

\[
\Gamma(X, j_* \Omega_{X^\dagger/k}^r) = \bigoplus_{p \in P \setminus ((k+1)\rho + P)} z^p \cdot \bigwedge^r \left( \bigcap_{\{j \mid j \in Q_j\}} Q_j^{\text{gp}} \right) \otimes_{\mathcal{Z}} \mathcal{k}.
\]

The \( Q_j \) in this formula are the submonoids of \( P \) defined before Proposition 1.13. Thus we obtain \( P \)-gradings on \( \mathcal{L}^\bullet \) by imposing the \( P \)-grading on each direct summand \( j_s \Omega_{X^\dagger/k}^r \cdot u^s \subset \mathcal{L}^\bullet \), and on each \( \Omega_{X_0}^\bullet \) by plugging \( k = 0 \) into the formula above and dividing by \( \mathcal{Z}_{\rho} \). Second, the differentials on \( \mathcal{L}^\bullet \) and on \( \Omega_{X_0}^\bullet \) commute with the respective \( P \)-gradings, and so does \( \varphi^\bullet \). In fact, this follows from

\[
d(z^p d\log \omega \cdot u^s) = z^p (d\log (\rho \wedge \omega) \cdot u^s + s d\log (\rho \wedge \omega) \cdot u^{s-1}).
\]

(4.2)

Third, all sheaves involved are pull-backs under the morphism \( \text{Et} : X^{\text{et}} \to X^{\text{Zar}} \) relating the Zariski site on \( X \) to the étale site.

We may thus assume that the \( \alpha_s \) in (4.1) are of the form

\[
\alpha_s = z^p d\log \omega_s, \quad s = 0, \ldots, N,
\]
with \( \omega_s \in \bigwedge^r V_p \otimes_{\mathbb{Z}} k \), \( V_p := \bigcap_{j | p \in Q_j} Q_j^{p} \). Taking into account (4.2) the closedness condition (4.1) now reads

\[
\begin{align*}
  p \wedge \omega_N &= 0 \\
  p \wedge \omega_s + (s + 1)p \wedge \omega_{s+1} &= 0, \quad s = 0, \ldots, N - 1.
\end{align*}
\]

Let us first assume \( p \neq 0 \). We claim that \( \sum \alpha_s u^s = d(\sum z^p \log_\tau_s \cdot u^s) \) for some \( \sum z^p \log_\tau_s \cdot u^s \in \mathcal{L}^{r-1} \) if and only if \( d(\sum \alpha_s u^s) = 0 \). To show this, given that \( \sum \alpha_s u^s \) is closed, we need to find \( \tau_0, \ldots, \tau_N \) such that

\[
\begin{align*}
  p \wedge \tau_N &= \omega_N \\
  p \wedge \tau_s + (s + 1)p \wedge \tau_{s+1} &= \omega_s, \quad s = 0, \ldots, N - 1.
\end{align*}
\]

Recall for a vector space \( V \) and \( \omega \in \bigwedge^r V \), \( p \in V \setminus \{0\} \), a necessary and sufficient condition for solvability of the equation \( p \wedge \tau = \omega \) is \( p \wedge \omega = 0 \) (integrability condition). By the first line of (4.3) we can therefore find \( \tau_N \in \bigwedge^{r-1} V_p \) with \( p \wedge \tau_N = \omega_N \). Assuming inductively that \( \tau_N, \ldots, \tau_{s+1} \) fulfilling (4.4) have already been found the integrability condition for \( \tau_s \) reads

\[
p \wedge (\omega_s - (s + 1)p \wedge \tau_{s+1}) = 0.
\]

This follows from (4.3) for \( s \) and from (4.4) for \( s + 1 \):

\[
p \wedge \omega_s = -(s + 1)p \wedge \omega_{s+1} = -(s + 1)p \wedge p \wedge \tau_{s+1} = (s + 1)p \wedge p \wedge \rho \wedge \tau_{s+1}.
\]

Thus there exists a \( \tau_s \) satisfying the second line of (4.4). Note that \( \tau_s \in \bigwedge^{r-1} V_p \) and hence \( \sum z^p \log_\tau_s \cdot u^s \in \mathcal{L}^{r-1} \), proving the claim.

Now suppose furthermore that

\[
\sum z^p \log \omega_s \cdot u^s \in \ker \varphi^r.
\]

This is the case if and only if the image of \( z^p \log \omega_0 \) in \( \Omega^r_{X_0} \) is zero, which by Proposition 1.12 and Corollary 1.13 holds if and only if \( p \in \rho + P \) or \( \rho \wedge \omega_0 = 0 \). If \( p \in \rho + P \), then also \( \sum z^p \log_\tau_s \cdot u^s \in \ker \varphi^{r-1} \). On the other hand, if \( \rho \wedge \omega_0 = 0 \), then by the second line of (4.4),

\[
\rho \wedge p \wedge \tau_0 = \rho \wedge \omega_0 = 0,
\]

so \( \tau_0 = \rho \wedge \tau_0' + p \wedge \tau_0'' \). We can replace \( \tau_0 \) by \( \rho \wedge \tau_0' \) without affecting (4.4), and hence \( \sum z^p \log_\tau_s \cdot u^s \in \ker \varphi^{r-1} \) also. This shows acyclicity of the \( p \)-graded component of \( \mathcal{K}^r \) for \( p \neq 0 \).

For \( p = 0 \) Equation (4.3) simply says \( \rho \wedge \omega_s = 0 \) for \( s = 1, \ldots, N \). Now \( V_0 = \bigcap_j Q_j \otimes_{\mathbb{Z}} k \), but as \( \rho \in Q_j \) for all \( j \), for \( s > 0 \) there exists \( \tau_{s+1} \in \bigwedge^{r-1} V_0 \) with \( \omega_s = (s + 1)\rho \wedge \tau_{s+1} \). Finally, \( \varphi^r(\sum z^0 \log_\tau \omega_s \cdot u^s) = 0 \) implies \( \rho \wedge \omega_0 = 0 \). Hence there exists a \( \tau_1 \) such that \( \rho \wedge \tau_1 = \omega_0 \), and \( d(\sum_{s=1}^N \log_\tau_s \cdot u^s) = \sum \log_\omega_s \cdot u^s \). \( \square \)
Theorem 4.2. With the same hypotheses as Theorem 4.1, with  
\( X_0 = X_0(B, \mathcal{P}, s) \), suppose  
\( (B, \mathcal{P}) \) satisfies the hypotheses of Theorem 3.21. Then  
\( H^p(\mathcal{X}, j_* \Omega^q_{\mathcal{X}/\mathcal{A}}) \) is a locally free  
\( \mathcal{A} \)-module, and it commutes with base change.

Proof. This follows from Theorems 3.26 and 4.1 in a standard way, see [5], §5. \( \square \)

Remark 4.3. Combining this base-change result with Corollary 3.24, we obtain the interchange of Hodge numbers for non-singular Calabi-Yau varieties obtained as smoothings of mirror pairs of log Calabi-Yau spaces, of course subject to the hypotheses of Corollary 3.24: see the discussion of Remark 3.25. For example, as it was shown that the Batyrev construction of [1] is a special case of our mirror symmetry construction using the discrete Legendre transform on affine manifolds, the exchange of Hodge numbers in the Batyrev construction follows from the results in this paper. Of course, our proof is much more involved, but it covers a much more general construction and will ultimately give deeper insight into the role that periods play in calculating Gromov-Witten invariants of the mirror.

5. Monodromy and the logarithmic Gauss-Manin connection

5.1. Monodromy. We begin with a few observations about  
\( B \), if  \( B \) is an integral affine manifold with singularities and toric polyhedral decomposition  \( \mathcal{P} \). Recall on  \( B_0 \) the extension class of the exact sequence

\[
0 \to \mathbb{Z} \to \text{Aff}(B_0, \mathbb{Z}) \to \Lambda \to 0
\]

is the radiance obstruction ([12], Proposition 1.12)  \( c_{B_0} \in H^1(B_0, \Lambda) = \text{Ext}^1(\Lambda, \mathbb{Z}) \). Here  \( \text{Aff}(B_0, \mathbb{Z}) \) is the sheaf of affine linear functions on  \( B_0 \) with integral slope and integral constant term, see [12], Definitions 1.13 and 1.39. As a consequence, the extension class of the natural exact sequence

\[
0 \to \mathbb{Z} \otimes \bigwedge^{r-1} \Lambda \to \bigwedge^r \text{Aff}(B_0, \mathbb{Z}) \to \bigwedge^r \Lambda \to 0
\]

in  \( \text{Ext}^1(\bigwedge^r \Lambda, \mathbb{Z} \otimes \bigwedge^{r-1} \Lambda) \) is by direct check the image of  \( c_{B_0} \) under the natural map  \( \Lambda \to \text{Hom}(\bigwedge^r \Lambda, \mathbb{Z} \otimes \bigwedge^{r-1} \Lambda) \) given by

\[
m \mapsto \left( \bigwedge^r \Lambda \ni n \mapsto 1 \otimes \iota(m)n \right).
\]

Pushing forward by  \( i \), we obtain the exact sequence

\[
0 \to \mathbb{Z} \otimes i_* \bigwedge^{r-1} \Lambda \to i_* \bigwedge^r \text{Aff}(B_0, \mathbb{Z}) \to i_* \bigwedge^r \Lambda \to 0. \tag{5.1}
\]

We have surjectivity on the right because by local triviality of the radiance obstruction on  \( B \) ([12], Prop. 1.29), one can find an open cover \( \{U_j\} \) of  \( B \) such that  \( \text{Aff}(B_0, \mathbb{Z}) \cong \mathbb{Z} \oplus \Lambda \) on  \( U_j \cap B_0 \). This also implies the extension class of (5.1) lies in  \( H^1(B, \text{Hom}(i_* \bigwedge^r \Lambda, \mathbb{Z} \otimes
\[ i_* (\Lambda^r \Lambda^{-1}) \subseteq \text{Ext}^1(i_* \Lambda^r \Lambda, \mathbb{Z} \otimes i_* \Lambda^{-1} \Lambda). \] Since \( c_{B_0} \in H^1(B_0, \Lambda) \) is in \( H^1(B, i_* \Lambda) \subseteq H^1(B_0, \Lambda) \) (by local triviality of \( c_{B_0} \)) we see the image of \( c_{B_0} \) in
\[ H^1(B, \mathcal{H}om(i_* \Lambda^r \Lambda, \mathbb{Z} \otimes i_* \Lambda^{-1} \Lambda)) \]
induced by the natural map \( i_* \Lambda \rightarrow \mathcal{H}om(i_* \Lambda^r \Lambda, \mathbb{Z} \otimes i_* \Lambda^{-1} \Lambda) \) as above is the extension class of \( (5.1) \). The connecting homomorphisms for \( (5.1) \)
\[ H^p(B, i_* \Lambda^r \Lambda) \rightarrow H^{p+1}(B, \mathbb{Z} \otimes i_* \Lambda^{-1} \Lambda) \]
is then given by \( c_{B_0} \cup \) using the natural cup product
\[ H^1(B, i_* \Lambda) \otimes H^p(B, i_* \Lambda^r \Lambda) \rightarrow H^{p+1}(B, \mathbb{Z} \otimes i_* \Lambda^{-1} \Lambda). \]

**Theorem 5.1.** Let \( (B, \mathcal{P}) \) be a positive and simple integral affine manifold with singularities and polyhedral decomposition \( \mathcal{P} \). Let \( s \) be lifted gluing data. Let \( X_0 \) := \( X_0(B, \mathcal{P}, s)^{\dagger} \). Then there is an exact sequence of complexes
\[ 0 \rightarrow f^* \Omega^1_{k/\mathbb{k}} \otimes j_* \Omega^0_{X_0/k}|[-1] \rightarrow j_* \Omega^0_{X_0/k} \rightarrow j_* \Omega^0_{X_0/k} \rightarrow 0. \]
If \( (B, \mathcal{P}) \) satisfies the hypotheses of Theorem 3.21, then there are canonical isomorphisms and equalities
1. \[ H^p(X_0, j_* \Omega^r_{X_0/k}) \cong H^p(B, i_* \Lambda^r \Lambda) \otimes \Omega^0_{k/\mathbb{k}}. \]
2. \[ \mathbb{H}^r(X_0, j_* \Omega^0_{X_0/k}) = \bigoplus_{p+q=r} H^p(X_0, j_* \Omega^q_{X_0/k}). \]
3. \[ H^p(X_0, f^* \Omega^1_{k/\mathbb{k}} \otimes \Omega^r_{X_0/k}) \cong \Omega^1_{k/\mathbb{k}} \otimes H^p(X_0, \Omega^r_{X_0/k}). \]
4. Let \( \text{Res}_0 \nabla \) denote the coboundary map
\[ \mathbb{H}^r(X_0, j_* \Omega^0_{X_0/k}) \rightarrow \Omega^1_{k/\mathbb{k}} \otimes \mathbb{H}^r+1(X_0, j_* \Omega^0_{X_0/k}|[-1]) = \Omega^1_{k/\mathbb{k}} \otimes \mathbb{H}^r(X_0, j_* \Omega^0_{X_0/k}) \]
followed by evaluation on \( \partial_t \), where \( \partial_t \) is the log derivation on \( \text{Spec} k^\dagger \) with \( \partial_t \log : \mathcal{M}_k = k^\times \times \mathbb{N} \rightarrow k \) given by \( (\alpha, n) \mapsto n \). Then under the identifications of Theorem 3.21 and Theorem 3.26,
\[ \text{Res}_0 \nabla (\alpha) = c_B \cup \alpha. \]

**Remark 5.2.** Before proving this theorem, let us explain its importance. Suppose that we have a flat deformation \( \mathcal{X} \rightarrow D \) over a disk \( D \) of \( \mathcal{X}_0 = X_0(B, \mathcal{P}, s) \) such that the log structure on \( \mathcal{X}_0 \) induced by the divisorial log structure given by \( \mathcal{X}_0 \subseteq \mathcal{X} \) coincides with \( X_0(B, \mathcal{P}, s)^\dagger \). Then there is a monodromy operator acting on the de Rham cohomology
of a fibre $X_t$, $t \neq 0$, and this monodromy operator is determined by the behaviour of the Gauss-Manin connection of this family at 0. In particular, the residue of the Gauss-Manin connection at 0, $\text{Res}_0(\nabla)$, coincides with the map defined above (see [34], (2.19)–(2.21) for the normal crossings case) and the monodromy operator satisfies

$$T = e^{-2\pi i \text{Res}_0(\nabla)}.$$  

This description of the monodromy operator should be compared to that of [9], Theorem 4.1, which calculated the effect of monodromy assuming the topological monodromy is described by translation by a section of an SYZ fibration. In fact, in the current context, there is a canonical section to consider: one expects the topological SYZ fibration on $X$ to be a compactification of $X(B_0) := T_{\mu_0}/\Lambda$. The torus fibration $f_0 : X(B_0) \to B_0$ has a canonical section which can be described locally as the graph of the developing map $\tilde{B}_0 \to \mathbb{R}^n$ of the affine structure (see [12], §1.1). As in [9], this section defines a class in $H^1(B_0, R^{n-1} f_* \mathbb{Z}) \cong H^1(B_0, \Lambda)$. One can show this class coincides with $c_{B_0}$. Hence, up to insignificant differences of sign and factors of $2\pi i$, the description of monodromy given above and in [9], Theorem 4.1, agree.

**Proof.** Off of $Z$, we always have an exact sequence

$$0 \to f^* \Omega^1_{k^1/k} \to \Omega^1_{X^1_0/k} \to \Omega^1_{X^1_0/k^1} \to 0;$$

hence, as $\Omega^1_{k^1/k}$ is rank 1,

$$0 \to f^* \Omega^1_{k^1/k} \otimes j^* \Omega^r_{X^1_0/k^1} \to j^* \Omega^r_{X^1_0/k} \to j^* \Omega^r_{X^1_0/k^1}$$

is exact. Surjectivity on the right can be checked étale locally, which can be done immediately from Proposition 1.12 and Corollary 1.13. This gives the exact sequence of the theorem. This sequence induces, for $\tau \in \mathcal{P}$, homomorphisms $\Omega^1_{k^1/k} \otimes \Omega^r_{\tau/k^1} \to \Omega^r_{\tau/k}$ and $\Omega^r_{\tau/k} \to \Omega^r_{\tau/k^1}$, where we now distinguish between the two different types of $\Omega^r_{\tau}$’s by adding the /$k$ or /$k^1$. One sees immediately from Lemma 3.2 that these give an exact sequence

$$0 \to \Omega^1_{k^1/k} \otimes \Omega^r_{\tau/k^1} \to \Omega^r_{\tau/k} \to \Omega^r_{\tau/k^1} \to 0$$

for each $\tau$, and hence an exact sequence of complexes

$$0 \to \Omega^1_{k^1/k} \otimes \mathcal{O}^* (j^* \Omega^r_{\tau/k^1}) \to \mathcal{O}^* (j^* \Omega^r_{\tau/k}) \to \mathcal{O}^* (j^* \Omega^r_{\tau/k^1}) \to 0.$$  

We wish to understand the long exact cohomology sequence of this sequence. Consider $g : v \to \tau$. We can write $\Omega^r_{\tau/k} = \Omega^r_{v/k^1} \oplus \text{dlog } \rho \wedge \Omega^r_{v/k^1}$. Under this splitting, it follows from Proposition 3.14 that $\Omega^r_{\tau/k}$ as a subsheaf of $F^*_{v,\tau} \Omega^r_{v/k}$ splits as $\Omega^r_{\tau/k^1} \oplus \text{dlog } \rho \wedge \Omega^r_{\tau/k^1}$. This splitting is not canonical, but depends on the choice of $g$. 
Now assume \((B, \mathcal{P})\) satisfies the hypotheses of Theorem 3.21. To show (1), we need, for \(e : \tau_1 \to \tau_2\), a canonical isomorphism
\[
H^0(X_{\tau_2}, (F^*_e \Omega^r_{\tau_1/k})/\text{Tors}) \cong \Gamma(W_e, i_* \bigwedge^r \text{Aff}(B_0, \mathbb{Z}) \otimes k).
\]
Now the sequence (5.1) splits on \(W_e\) by local vanishing of the radiance obstruction (12, Proposition 1.29), and given any splitting \(\varphi : i_* \Lambda \to \text{Aff}(B_0, \mathbb{Z})\) on \(W_e\), a choice of vertex \(v \to \tau_1\) gives a well-defined splitting \(\varphi_v\) of \(\Gamma(W_e, \text{Aff}(B_0, \mathbb{Z}))\) by \(\varphi_v(n) = \varphi(n) - \varphi(n)(v)\) for \(n \in \Gamma(W_e, i_* \Lambda)\). Thus we lift sections of \(i_* \Lambda\) to affine linear functions vanishing on \(v\); the resulting splitting is independent of the original choice \(\varphi\). Thus we get an isomorphism, depending only on \(v\),
\[
\Gamma(W_e, i_* \bigwedge^r \text{Aff}(B_0, \mathbb{Z})) \cong \Gamma(W_e, i_* \bigwedge^r \Lambda) \oplus (\mathbb{Z} \otimes \Gamma(W_e, i_* \bigwedge^{r-1} \Lambda)).
\]
Now let \(h : \omega \to \tau_1\) be an edge. Then for \(n \in \Gamma(W_e, i_* \Lambda)\),
\[
\varphi_{\omega_+}(n) = \varphi_{\omega_-}(n) - \varphi_{\omega_-}(n)(v^+_{\omega_-}) = \varphi_{\omega_-}(n) - (n, v^+_{\omega_-} - v^-_{\omega_-}) = \varphi_{\omega_-}(n) + (n, l_{\omega} d_{\omega}),
\]
where the notation is as in Lemma 3.13. Thus we have two isomorphisms
\[
\Gamma(W_e, i_* \bigwedge^r \Lambda) \oplus (\mathbb{Z} \otimes \Gamma(W_e, i_* \bigwedge^{r-1} \Lambda)) \xrightarrow{\cong} \Gamma(W_e, i_* \bigwedge^r \text{Aff}(B_0, \mathbb{Z}))
\]
given by the splittings \(\varphi_{\omega_-}\) and \(\varphi_{\omega_+}\) respectively. The composition of the inverse of the second isomorphism with the first isomorphism is
\[
\Gamma_h : (\alpha_1, \alpha_2) \mapsto (\alpha_1, -l_{\omega} d_{\omega} \alpha_1 + \alpha_2).
\]
We now have by Lemma 3.20 and (5.2) a diagram
\[
\begin{array}{ccc}
\Gamma(X_{\tau_2}, (F^*_e \Omega^r_{\tau_1/k})) / \text{Tors} & \xrightarrow{\cong} & \Gamma(W_e, i_* \bigwedge^r \Lambda \otimes k) \oplus \text{dlog}\rho \wedge \Gamma(W_e, i_* \bigwedge^{r-1} \Lambda \otimes k) \\
& & \Gamma_h \\
\Gamma(X_{\tau_2}, (F^*_e \Omega^r_{\tau_1/k})) / \text{Tors} & \xrightarrow{\cong} & \Gamma(W_e, i_* \bigwedge^r \Lambda \otimes k) \oplus \text{dlog}\rho \wedge \Gamma(W_e, i_* \bigwedge^{r-1} \Lambda \otimes k)
\end{array}
\]
where we think of the two occurrences of \(\Gamma(X_{\tau_2}, (F^*_e \Omega^r_{\tau_1/k})) / \text{Tors}\) as being contained in
\[
\Gamma(X_{\tau_2}, F^*_{i\omega_-^{\tau_2}} \Omega^r_{\omega_-^{\tau_2/k}}) \quad \text{and} \quad \Gamma(X_{\tau_2}, F^*_{i\omega_+^{\tau_2},\tau_2} \Omega^r_{\omega_+^{\tau_2/k}})
\]
respectively. This diagram is then commutative by Lemma 3.13. Thus we can canonically identify \(\Gamma(X_{\tau_2}, (F^*_e \Omega^r_{\tau_1/k})) / \text{Tors}\) with \(\Gamma(W_e, i_* \bigwedge^r \text{Aff}(B_0, \mathbb{Z}) \otimes k)\). Item (1) then follows because \(\Gamma(X_0, G^* (j_* \Omega^r_{X_0/k}))\) now coincides with the Čech complex for \(i_* \bigwedge^r \text{Aff}(B_0, \mathbb{Z}) \otimes k\) with respect to the standard open cover \(\{W_r\}\). Item (2) follows as in Theorem 3.26, item (3) is obvious, and item (4) follows from the discussion of \(c_B\) and (1)–(3).
5.2. The connection on moduli. Let \((B, \mathcal{P})\) be positive and simple. Recall from \cite{12}, Theorem 5.4, that the set of positive log Calabi-Yau spaces with dual intersection complex \((B, \mathcal{P})\), modulo isomorphisms preserving \(B\), is canonically \(H^1(B, i_*\Lambda \otimes \mathbb{k}^\times)\). By the universal coefficient theorem, there is an exact sequence

\[0 \to H^1(B, i_*\Lambda) \otimes \mathbb{k}^\times \to H^1(B, i_*\Lambda) \otimes \mathbb{k}^\times \to H^2(B, i_*\Lambda)_{\text{tors}} \to 0,\]

so \(H^1(B, i_*\Lambda \otimes \mathbb{k}^\times)\) can be viewed as a disjoint union of torsors over \(H^1(B, i_*\Lambda) \otimes \mathbb{k}^\times\). This latter algebraic torus can be viewed as \(S := \text{Spec} \mathbb{k}[H^1(B, i_*\Lambda)]_f\), where \(G_f := G/G_{\text{tors}}\) denotes the torsion free part of the finitely generated abelian group \(G\). Observe \(\mathbb{G}_m(S) = \mathbb{k}^\times \times H^1(B, i_*\Lambda)_f^*\), and

\[H^1(B, i_*\Lambda \otimes \mathbb{G}_m(S)) = H^1(B, i_*\Lambda \otimes \mathbb{k}^\times) \times (H^1(B, i_*\Lambda) \otimes H^1(B, i_*\Lambda)^*_f).

For the remainder of this section fix an element

\[s \in H^1(B, i_*\Lambda \otimes \mathbb{G}_m(S))\]

of the form

\[s = (s_0, s_{id}),\]

where \(s_0 \in H^1(B, i_*\Lambda \otimes \mathbb{k}^\times)\) is arbitrary and \(s_{id} \in H^1(B, i_*\Lambda) \otimes H^1(B, i_*\Lambda)^*_f\) corresponds to a choice of splitting \(H^1(B, i_*\Lambda)_f \to H^1(B, i_*\Lambda)\). One can view \(s_0\) as selecting the component of \(H^1(B, i_*\Lambda \otimes \mathbb{k}^\times)\) containing it, and \((s_0, s_{id})\) can then be viewed as a “universal element” on this component, in the following sense. We can view \(s\) as an \(H^1(B, i_*\Lambda \otimes \mathbb{k}^\times)\)-valued function on \(S\). At a closed point \(s\) of \(S\) corresponding to a group homomorphism \(\chi_s : H^1(B, i_*\Lambda)_f^* \to \mathbb{k}^\times\), the value of \(s\) is \(s_0 \cdot (1 \otimes \chi_s)(s_{id}) = s_0 \cdot s\).

Choose a Čech representative \((s_e) \in \bigoplus_e \Gamma(W_e, i_*\Lambda \otimes \mathbb{G}_m(S))\) for \(s\) by choosing Čech representatives \((s_{0, e})\) and \((s_{id, e})\) for \(s_0\) and \(s_{id}\) respectively. As in \cite{12}, Definition 5.1, \((s_e)\) can be viewed as open gluing data for \((B, \mathcal{P})\), and hence we obtain a family of algebraic spaces \(\pi : X_0(B, \mathcal{P}, s) \to S\) by \cite{12}, Definition 2.28.

**Remark 5.3.** The choice of splitting \(s_{id}\) does not affect the isomorphism class of each fibre of \(\pi : X_0(B, \mathcal{P}, s)^\dagger \to S^\dagger\). However, it does affect the total space \(X_0(B, \mathcal{P}, s)^\dagger\). The results we give here apply to any choice.

We need to put a log structure on this space. Unfortunately, in \cite{12}, we did not deal with log structures for families; however, since the spaces are reduced, the methods of \cite{12}, §3.2, 3.3 still apply. Let \(X_0 = X_0(B, \mathcal{P}, s)\). Then \(X_0\) is covered by open sets of the form \(V(\sigma) \times S\) for \(\sigma \in \mathcal{P}_{\text{max}}\), and the sheaf \(\mathcal{L}S_{V(\sigma) \times S}\) on \(V(\sigma) \times S\) is determined by \cite{12}, Theorem 3.24; in particular, \(\mathcal{L}S_{V(\sigma) \times S}^+\) is just the pull-back of \(\mathcal{L}S_{V(\sigma)}^+\) under the projection to \(V(\sigma)\). Sections of \(\mathcal{L}S_{V(\sigma) \times S}^+\) transform under the gluing map exactly as in \cite{12}, Theorem 3.27, keeping in mind quantities involving \(s\) are now functions on \(S\). This defines the sheaf \(\mathcal{L}S_{X_0, \mathcal{P}_0}^+\), and as in \cite{12}, Theorem 5.2, (2), \(s\) determines a unique
normalized section \( f \in \Gamma(X_0, LS^{+}_{\text{pre}, X_0}) \) which in fact is a section of \( LS_{X_0} \) off of some singular set of relative codimension \( \geq 2 \) over \( S \) not containing any toric stratum of \( X_0 \). This defines a log structure \( X_0^\dagger \) on \( X_0 \). Furthermore, if \( S^\dagger \) denotes the log structure on \( S \) given by the chart
\[
\mathbb{N} \to \mathcal{O}_S, \quad n \mapsto \begin{cases} 
1 & n = 0 \\
0 & n > 0,
\end{cases}
\]
then \( \pi \) lifts to a log morphism \( \pi : X_0^\dagger \to S^\dagger \), smooth away from \( Z \). Note in addition there is an obvious canonical map \( S^\dagger \to \text{Spec} k^\dagger \).

We have the following generalisation of Theorem 2.6, whose proof is identical.

**Theorem 5.4.** In the above situation, let \( \bar{x} \to Z \subseteq X_0 \) be a closed geometric point. Then there exists data \( \tau, \tilde{\psi}_1, \ldots, \tilde{\psi}_q \) as in Construction 2.1 defining a monoid \( P \), hence log spaces \( Y^\dagger, X^\dagger \to \text{Spec} k^\dagger \) as in \( \S 1 \), such that there is a diagram over \( S^\dagger \)
\[
\begin{array}{ccc}
V^\dagger & \xrightarrow{\phi} & X^\dagger \times_{\text{Spec} k^\dagger} S^\dagger \\
\end{array}
\]
with both maps strict étale and \( V^\dagger \) an étale neighbourhood of \( \bar{x} \).

Our goal now is to compute the Gauss-Manin connection for the family \( \pi \),
\[
\nabla_{GM} : \mathbb{R}^r \pi_* (j_* \Omega^\bullet_{X_0^\dagger/S^\dagger}) \to \Omega^1_{S^\dagger/k^\dagger} \otimes \mathbb{R}^r \pi_* (j_* \Omega^\bullet_{X_0^\dagger/\text{Spec} k^\dagger}),
\]
where \( j : X_0 \setminus Z \to X_0 \) is the inclusion as usual. Just as in \( [23] \), this connection can be defined as follows. We have on \( X_0 \setminus Z \) an exact sequence of locally free sheaves
\[
0 \to \pi^* \Omega^1_{S^\dagger/k^\dagger} \to \Omega^1_{X_0^\dagger/k^\dagger} \to \Omega^1_{X_0^\dagger/S^\dagger} \to 0,
\]
where a filtration of complexes
\[
\Omega^\bullet_{X_0^\dagger/k^\dagger} = F^0(\Omega^\bullet_{X_0^\dagger/k^\dagger}) \supseteq F^1(\Omega^\bullet_{X_0^\dagger/k^\dagger}) \supseteq \cdots
\]
where
\[
F^i := F^i(\Omega^\bullet_{X_0^\dagger/k^\dagger}) \supseteq \text{im}(\pi^* (\Omega^i_{S^\dagger/k^\dagger}) \otimes \Omega^\bullet_{X_0^\dagger/k^\dagger}[-i] \to \Omega^\bullet_{X_0^\dagger/k^\dagger})
\]
with
\[
F^i/F^{i+1} = \pi^* (\Omega^i_{S^\dagger/k^\dagger}) \otimes \Omega^\bullet_{X_0^\dagger/S^\dagger}[-i].
\]
Noting that by Theorem 5.4 (5.3) is locally split after applying \( j_* \), we get an exact sequence on \( X_0 \)
\[
0 \to \pi^* \Omega^1_{S^\dagger/k^\dagger} \to j_* \Omega^1_{X_0^\dagger/k^\dagger} \to j_* \Omega^1_{X_0^\dagger/S^\dagger} \to 0,
\]
as well as
\[ j_*F^i/j_*F^{i+1} = \pi^*(\Omega^i_{S^1/k'}) \otimes j_*\Omega^{\bullet}_{X_0^1/S^1}[-i]. \]

Then the Gauss-Manin connection is the boundary map coming from the exact sequence
(5.4) \[ 0 \to j_*F^1/j_*F^2 \to j_*F^0/j_*F^2 \to j_*F^0/j_*F^1 \to 0, \]
i.e.
\[ \nabla_{GM} : \mathbb{R}^r \pi_*(j_*\Omega^p_{X_0^1/S^1}) \to \mathbb{R}^{q+1} \pi_*(\pi^*(\Omega^1_{S^1/k'}) \otimes j_*\Omega^{\bullet}_{X_0^1/S^1}[-1]) = \Omega^1_{S^1/k'} \otimes \mathbb{R}^r \pi_*(j_*\Omega^{\bullet}_{X_0^1/S^1}). \]

**Theorem 5.5.** Let \((B, \mathcal{P})\) be positive and simple, satisfying the hypotheses of Theorem \[3.27.\] Let \(s\) be as above. Let \(X_0 = X_0(B, \mathcal{P}, s)\). Then
(1) there exists a canonical isomorphism
\[ \mathbb{R}^q \pi_*(j_*\Omega^p_{X_0^1/S^1}) \cong H^q(B, i_*\bigwedge^p \Lambda) \otimes \mathcal{O}_S. \]
(2) \[ \mathbb{R}^p \pi_*(j_*\Omega^p_{X_0^1/S^1}) \cong \bigoplus_{p+q=r} \mathbb{R}^q \pi_*(j_*\Omega^p_{X_0^1/S^1}). \]
(3) (Griffiths Transversality)
\[ \nabla_{GM}(\mathbb{R}^q \pi_*(j_*\Omega^p_{X_0^1/S^1})) \subseteq \Omega^1_{S^1/k'} \otimes \mathbb{R}^{q+1} \pi_*(j_*\Omega^{1}_{X_0^1/S^1}). \]
(4) Identifying \(\Omega^1_{S^1/k'} = \Omega^1_{S/k} \otimes H^1(B, i_*\Lambda) \otimes \mathcal{O}_S\), via \(\alpha \in H^1(B, i_*\Lambda) \otimes \mathcal{O}_S\) yielding the differential \(d\log \alpha\), then \(-\nabla_{GM}\) restricted to the constant sections (under the isomorphism of (1)) of \(\mathbb{R}^q \pi_*(j_*\Omega^p_{X_0^1/S^1})\) is the map
\[-\nabla_{GM} : H^q(B, i_*\bigwedge^p \Lambda \otimes k) \to H^1(B, i_*\Lambda \otimes k)^* \otimes H^{q+1}(B, i_*\bigwedge^{p-1} \Lambda \otimes k) \]
induced by the cup product composed with contraction:
\[ H^1(B, i_*\Lambda \otimes k) \otimes H^q(B, i_*\bigwedge^p \Lambda \otimes k) \to H^{q+1}(B, i_*\bigwedge^{p-1} \Lambda \otimes k). \]
We denote this latter map by \(\eta^* \otimes n \mapsto \iota(\eta^*)n\), for \(\eta^* \in H^1(B, i_*\Lambda \otimes k)\) and \(n \in H^q(B, i_*\bigwedge^p \Lambda \otimes k)\).

**Proof.** Step 1. Review of how the lifted gluing data \((s_\sigma)\) determines the section \(f \in \Gamma(X_0, \mathcal{L}^{\sigma}_{\text{pre}, X_0})\).

For \(\sigma \in \mathcal{P}_{\text{max}}\), denote by \(f_\sigma\) the pull-back of \(f\) to \(V(\sigma) \times S\). We follow the proof of \[12.\]
Prop. 4.25. We can write
\[ \mathcal{L}^{\sigma}_{\text{pre}, V(\sigma) \times S} = \bigoplus_{\omega \in \mathcal{H}(\sigma) \times S} \mathcal{O}_{V(\sigma) \times S}. \]
which allows us to decompose $f_\sigma$ into components $f_{\sigma,e} \in \Gamma(O_{V_e \times S})$. We can then write, for $e : \omega \to \sigma$,

$$f_{\sigma,e} = \sum_{p \in \Delta(\omega) \cap \Lambda_y} f_{\sigma,e,p} z^p,$$

where $y \in \omega$ is a point near $v^+_\omega$ and $\Delta(\omega)$ is the convex hull in $\Lambda_y^+ \subseteq \Lambda_R$ of

$$\{n^\omega_{\sigma'}|e' : \omega \to \sigma' \in \mathcal{P}_{\text{max}}\}.$$

(See [12], §1.5). By simplicity this is an elementary simplex. Given $e' : \omega \to \sigma'$, we have a diagram

$$\begin{array}{ccc}
\omega & \overset{h}{\rightarrow} & \tau \\
\downarrow{g} & & \downarrow{g'} \\
\sigma & \overset{e}{\rightarrow} & \sigma'
\end{array}$$

with $(g,g')$ maximal ([12], Definition 2.21). Then by [12], (4.4),

$$f_{\sigma,e,n^\omega_{\sigma'}} = \frac{D(s_g,\omega,\sigma)}{D(s_{g'},\omega,\sigma')} s_g(n^\omega_{\sigma'}).$$

However, since $s_g$, $s_{g'}$ are multiplicative maps on $\Lambda_y$, rather than piecewise multiplicative (see [12], Definition 5.1), $D(s_g,\omega,\sigma) = D(s_{g'},\omega,\sigma') = 1$. On the other hand, by the Čech cocycle condition, $s_h \cdot s_g = s_e$. Since $s_h \in \Gamma(W_h, i_*\Lambda \otimes \mathbb{G}_m(S))$, $s_h$ is invariant under the monodromy operator $T^\omega_{\sigma'}$ (see [12], §1.5, where we write $T^{\text{exc}}_\omega$), which takes the form (expressed multiplicatively on $\Lambda_y \otimes \mathbb{G}_m(S)$)

$$T^\omega_{\sigma'}(s_h) = s_h \cdot (d_\omega \otimes s_h(n^\omega_{\sigma'})).$$

Thus

$$s_h(n^\omega_{\sigma'}) = 1,$$

and we see

$$f_{\sigma,e,n^\omega_{\sigma'}} = s_e(n^\omega_{\sigma'}).$$

Since as $e'$ varies, $n^\omega_{\sigma'}$ runs over all integral points of $\tilde{\Delta}(\omega)$, we in fact see

$$f_{\sigma,e} = \sum_{p \in \Delta(\omega) \cap \Lambda_y} s_e(p) z^p.$$

**Step 2.** The resolutions of $j_*\Omega^r_{X_0/k^t}$ and $j_*\Omega^r_{X_0/S^t}$ and the proof of (1) and (2).

We have maps $q_r : X_r \times S \to X_0$ from the construction of $X_0$ in [12], §2.2, as usual, and using this, we can define the sheaves $\Omega^r_{r/k^t}$ and $\Omega^r_{r/S^t}$ on $X_r$ using $\Omega^r_{X_0/k^t}$ and $\Omega^r_{X_0/S^t}$ just as in [3,2]. As usual these yield resolutions $\mathcal{C}^\bullet(j_*\Omega^r_{X_0/k^t})$ and $\mathcal{C}^\bullet(j_*\Omega^r_{X_0/S^t})$ of $j_*\Omega^r_{X_0/k^t}$ and
respectively. This follows from Theorem 5.4 and the local form of these results (Theorem 3.5).

Now if \( s \in S \), then \( \Omega^r_{X_0(B, \mathcal{P}, s)^! / S^!} \left|_{\pi^{-1}(s)} \right. = \Omega^r_{X_0(B, \mathcal{P}, s_0 \cdot s)^! / k^!} \) on \( X_0(B, \mathcal{P}, s_0 \cdot s) \setminus Z \). One checks locally that the same continues to hold after pushing forward by \( j \), i.e.

\[
(j_* \Omega^r_{X_0(B, \mathcal{P}, s)^! / S^!}) \mid_{\pi^{-1}(s)} = j_* \Omega^r_{X_0(B, \mathcal{P}, s_0 \cdot s)^! / k^!}.
\]

Thus \( R^q \pi_* (j_* \Omega^P_{X_0^! / S^!}) \) is locally free. Given a Čech cocycle

\[
(W_{\sigma_0, \ldots, \sigma_q}, \eta_{\sigma_0, \ldots, \sigma_q})
\]

representing an element \( n \in H^q(B, i_* \check{\Lambda} \otimes k) \), \( \eta_{\sigma_0, \ldots, \sigma_q} \) determines a section of

\[
\Gamma(X_{\sigma_q} \times S, F^*_{\sigma_0, \ldots, \sigma_q} \Omega^P_{\sigma_0 \cdot \sigma_q / S^!}),
\]

namely \( \text{dlog} \, n_{\sigma_0, \ldots, \sigma_q} \), exactly as in Lemma 3.20 and hence \( n \) determines an element of

\[
H^q(\Gamma(X_0, \mathcal{F}^* (j_* \Omega^P_{X_0^! / S^!})))
\]

which restricts to the class \( n \) on each fibre under the isomorphism of Theorem 3.21. This shows item (1), while item (2) follows exactly as in the proof of Theorem 3.26.

**Step 3. Describing \( \Omega^r_{\tau/k^!} \).**

The methods of §3.2 apply to calculate \( \Omega^r_{\tau/k^!} \), as nowhere did we assume properness while calculating these sheaves. However, there is an important subtlety. Consider first an irreducible component \( X_v \times S \) of \( X_0 \). Lemma 3.12 still applies, so \( \Omega^r_{\tau/k^!} \) is a trivial vector bundle, but we need to be careful about how we identify this bundle. The reason is that \( X_v \times S \) is obtained by gluing together affine schemes of the form \( V_e \times S \), with \( e : v \rightarrow \sigma \in \mathcal{P}_{\text{max}} \). However, they are not glued in the canonical way, but rather with a twist induced by \( s \), and this needs to be taken into account. As a result, this bundle is abstractly

\[
\Omega^r_{X_v \times S^! / k^!} = \bigwedge^r (\check{\Lambda}_v \oplus H^1(B, i_* \Lambda)^*_f) \otimes \mathcal{O}_{X_v \times S};
\]

however there is only a canonical identification after choosing a maximal cell \( \sigma \in \mathcal{P}_{\text{max}} \) containing \( v \). We shall now see how this representation depends on \( e : v \rightarrow \sigma \in \mathcal{P}_{\text{max}} \). We have \( V_v \times S \) is an open affine subset of \( X_v \times S \), and given a diagram

\[
\begin{array}{ccc}
v & \xrightarrow{h} & \sigma \\
| & \nearrow{g} & \searrow{g'} \\
\tau & & \sigma'
\end{array}
\]

with \( (g, g') \) a maximal pair, \( V_v \times S \) and \( V_{v'} \times S \) are glued along the open subset \( V_h \times S \) using \( \Phi_{gg'}(s) \). This is given on the level of rings as follows, as explained in [12], §2.2. The
vertex \( v \) determines (different) cones \( \bar{v} \) in the normal fans \( \Sigma_\sigma \) and \( \Sigma_{\sigma'} \) of \( \sigma \) and \( \sigma' \) in \( \Lambda_{\sigma,R}^* \) and \( \Lambda_{\sigma',R}^* \) respectively. Furthermore, \( V_\epsilon = \text{Spec} \ k[\bar{v} \cap \Lambda_{\sigma}^*] \) and \( V_{\epsilon'} = \text{Spec} \ k[\bar{v} \cap \Lambda_{\sigma'}^*] \). Then \( V_\eta \subseteq V_\epsilon, V_{\epsilon'} \) is described via the localizations \( \text{Spec} \ k[(\bar{v}+\mathbb{R}\bar{\tau}) \cap \Lambda_{\sigma}^*] \) and \( \text{Spec} \ k[(\bar{v}+\mathbb{R}\bar{\tau}) \cap \Lambda_{\sigma'}^*] \) respectively. Identifying \( \Lambda_\sigma \) and \( \Lambda_{\sigma'} \) via parallel transport through \( \nu \) yields an isomorphism

\[
\Phi_{gg'} : \text{Spec} \ k[(\bar{v}+\mathbb{R}\bar{\tau}) \cap \Lambda_{\sigma}^*] \rightarrow \text{Spec} \ k[(\bar{v}+\mathbb{R}\bar{\tau}) \cap \Lambda_{\sigma'}^*],
\]

hence defining an identification of \( V_\eta \subseteq V_\epsilon \) with \( V_\eta \subseteq V_{\epsilon'} \), which we also denote by \( \Phi_{gg'} \).

Then to construct \( X_0 \), we glue \( V_\eta \times S \subseteq V_\epsilon \times S \) and \( V_\eta \times S \subseteq V_{\epsilon'} \times S \) via

\[
\Phi_{gg'}(s) := s_g^{-1} \circ (\Phi_{gg'} \times \text{id}) \circ s_g',
\]

Here \( s_g \) acts on the coordinate ring of \( V_\eta \times S \subseteq V_\epsilon \times S \) via \( z^n \mapsto s_g(n)z^n \), where as usual we view \( s_g \) as a multiplicative function on \( \Lambda_\sigma^* \) with values in \( \mathbb{G}_m(S) \).

Now if \( n \in \Lambda_\sigma^* \) represents a logarithmic 1-form \( \log \) on \( V_\epsilon \), let us consider the pulled-back 1-form \( \Phi_{gg'}(s)(\log n) \). Clearly \( (\Phi_{gg'} \times \text{id})(\log n) \) just has the effect of parallel transporting \( n \) from \( \Lambda_\sigma^* \) to \( \Lambda_{\sigma'}^* \) through \( v \), so identify both those spaces with \( \hat{\Lambda}_v \). We note

\[
\Phi_{gg'}(s)^*(\log n) = s_g'(s_g^{-1})^*(\log n).
\]

The upshot of this is that whenever we specify a logarithmic form on \( X_v \times S \) defined over \( k^\dagger \) as \( \log n \) for some \( n \in \Lambda^r \mathcal{A}_v \), we need to specify \( \sigma \in \mathcal{P}_{\text{max}} \) containing \( v \) to indicate in which affine piece of \( X_v \times S \) we are representing this form. Furthermore, if we wish to represent an element of \( \Omega_{\tau/k_1}^r \) we should pick \( v \rightarrow \tau \rightarrow \sigma \in \mathcal{P}_{\text{max}} \), describe \( \Omega_{\tau/k_1}^r \) as a subsheaf of \( F_{v,\tau}^*\Omega_v^r \), but describe the latter sheaf in the affine open subset \( V_\eta \times S \) of \( X_\tau \times S \).

**Step 4. The Gauss-Manin connection in terms of our resolutions.**

We note that \( \Omega_{\tau/k_1}^r \) has a filtration

\[
F^i(\Omega_{\tau/k_1}^r) = \text{im}(\pi^*\Omega_{S^1/k_1}^i \otimes \Omega_{\tau/k_1}^{r-i} \rightarrow \Omega_{\tau/k_1}^r)
\]

and similarly for \( e : \tau_1 \rightarrow \tau_2 \),

\[
F^i((F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^r)/\text{Tors}) = \text{im}(\pi^*\Omega_{S^1/k_1}^i \otimes (F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^{r-i})/\text{Tors} \rightarrow (F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^r)/\text{Tors})
\]

yields a filtration of \( (F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^r)/\text{Tors} \). We have

\[
F^i((F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^r)/\text{Tors})/F^{i+1}((F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^r)/\text{Tors}) \cong \pi^*\Omega_{S^1/k_1}^i \otimes (F_{\tau_1,\tau_2}^*\Omega_{\tau_1/k_1}^{r-i})/\text{Tors}.
\]
This follows from Theorem 5.4, which tells us étale locally $\Omega^1_{\tau/k^1}$ splits as $\Omega^1_{S^1/k^1} \oplus \Omega^1_{\tau/S^1}$. Similarly, this gives a resolution of the exact sequence 5.4:

\[
\begin{array}{ccccccc}
0 & \rightarrow & j_\ast F^1 / j_\ast F^2 & \rightarrow & \pi^\ast \Omega^1_{S^1/k^1} \otimes \mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/k^1}[-1]) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & j_\ast F^0 / j_\ast F^2 & \rightarrow & \mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/k^1}) & \rightarrow & F^2(\mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/k^1})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & j_\ast F^0 / j_\ast F^1 & \rightarrow & \mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/S^1}) & \rightarrow & 0 \\
\end{array}
\]

where $F^2(\mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/k^1}))$ is obtained by replacing each $\Omega^r_{\tau/k^1}$ with $F^2(\Omega^r_{\tau/k^1})$.

**Step 5. Calculating $\nabla_{GM}$ and the proof of (3) and (4).**

Consider $n \in H^q(B, i_\ast \wedge^{p} \tilde{\Lambda} \otimes k)$, represented as before by a Čech cocycle $(W_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_q}, n_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_q})$.

Then as before, $d\log n_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_q} \in \Gamma(X_{\sigma_q} \times S, (F_{\sigma_0, \sigma_q}^\ast \Omega^p_{\sigma_0/S^1})/\text{Tors})$, and these elements represent a section of $\mathbb{R}^{p+q} \pi_\ast(j_\ast \Omega^\ast_{X_0/S^1})$ over $S$ precisely because they yield a $(p+q)$-cocycle in $\Gamma(X_0, \text{Tot}(\mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/S^1})))$, keeping in mind

\[
d\log n_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_q} = 0.
\]

To compute the value of the boundary homomorphism on this class, we first lift this to a $(p+q)$-cochain in

\[
\Gamma(X_0, \text{Tot} \left( \frac{\mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/S^1})}{F^2(\mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/S^1}))} \right)
\]

and then apply the total differential to this cochain to get a $(p+q+1)$-cochain for the complex $\pi^\ast \Omega^1_{S^1/k^1} \otimes \text{Tot}(\mathcal{E}^\ast(j_\ast \Omega^\ast_{X_0^1/S^1}[-1]))$.

We now carry out this procedure. We need to lift $d\log n_{\sigma_0 \rightarrow \cdots \rightarrow \sigma_q}$. For convenience, we drop the subscript for the moment. Pick $e : v \rightarrow \sigma_0$, $g : \sigma_q \rightarrow \sigma \in \mathcal{P}_{\text{max}}$, let $g_0$ be the composition $\sigma_0 \rightarrow \sigma_q \rightarrow \sigma$, and set

\[
\tilde{n} = s^e_{g_0}(d\log n)
\]
viewed as an element of $F^*_{\sigma,q} \Omega^p_{\sigma/q/k^1}$. From (5.8), $\tilde{n}$ is a lifting of $d\log n$ in $F^*_{\sigma,q} \Omega^p_{\sigma/q/k^1}$.

We claim that in fact $\tilde{n}$ is a section of $(F^*_{\sigma,0} \Omega^p_{\sigma/q/k^1})/\text{Tors}$. We use Proposition 3.17 to do this. Let $\Delta_i, \tilde{\Delta}_i$ be the simplicity data for $\sigma_0$, yielding $Z_{\tilde{\sigma}} \subseteq X_{\tilde{\sigma}} \times S$. Let $f_i$ be the equation determining $Z_{\tilde{\sigma}}$ in the affine subset $V_g \times S$ of $X_{\tilde{\sigma}} \times S$ determined by $g$. Each $f_i$ is determined as follows: there is some $\omega_i \to \sigma_0$, dim $\omega_i = 1$, $e_i : \omega_i \to \sigma$ such that $f_{\sigma,e_i} | V_g \times S = f_i$, and by (5.6),

$$f_i = \sum_{p \in \Delta(\omega_i) \cap \Lambda_{\tilde{\sigma}}} s_{e_i}(p) z^p = \sum_{p \in \Delta_i \cap \Lambda_{\tilde{\sigma}}} s_{e_i}(p) z^p.$$

But $e_i$ factors as $\omega_i \to \sigma_0 \to g_0 \to \sigma$, and $s_h(p) = 1$ for $p \in \tilde{\Delta}_i \cap \Lambda_{\tilde{\sigma}}$ by (5.5), so we can write

$$f_i = \sum_{p \in \Delta_i \cap \Lambda_{\tilde{\sigma}}} s_{g_0}(p) z^p = s_{g_0}^* (f_i^0),$$

where

$$f_i^0 = \sum_{p \in \Delta_i \cap \Lambda_{\tilde{\sigma}}} z^p.$$

Since $d\log n \in \Gamma(X_{\sigma} \times S, F^*_{\sigma,\sigma} \Omega^p_{\sigma/q/k^1}/\text{Tors})$, it follows from Proposition 3.17, Remark 3.16 and Lemma 3.15 that for each $v'_i \neq v_i$ a vertex of $\Delta_i$, $\frac{df_i}{f_i} \wedge d\log \iota(v'_i - v_i)n$ has no pole along the locus $f_i = 0$, as a form defined over $S^\dagger$. However, over $S^\dagger$, this is equivalent to the statement that $\frac{df_i}{f_i} \wedge d\log \iota(v'_i - v_i)n$ has no pole along the locus $f_i^0 = 0$ (i.e. the coefficients don’t affect whether or not this holds). As forms over $k^\dagger$, there is some invertible function $g$ pulled back from $S$ such that

$$s_{g_0}^* \left( \frac{df_i^0}{f_i^0} \wedge \log \iota(v'_i - v_i)n \right) = g \frac{df_i}{f_i} \wedge \log \iota(v'_i - v_i) s_{g_0}^* (d\log n).$$

Thus this latter form has no poles along $Z_{\tilde{\sigma}}$. So

$$d\log \tilde{n} \in \Gamma(X_{\sigma} \times S, (F^*_{\sigma,\sigma} \Omega^p_{\sigma/q/k^1})/\text{Tors}),$$

by Proposition 3.17. Note in fact this lifting is independent of the choice of $\sigma$: given a diagram

\[
\begin{array}{c}
\sigma \\
\downarrow \sigma_0 \\
\sigma_q \\
\downarrow g \\
\sigma_q' \\
\downarrow g' \\
\sigma_0'
\end{array}
\]

\[
\begin{array}{c}
\sigma \\
\downarrow g_0 \\
\sigma_q \\
\downarrow g \\
\sigma_q' \\
\downarrow g_0' \\
\sigma_q'
\end{array}
\]
we note that we get liftings \( s_{g_0}^* (\log n) \), \( s_{g_1}^* (\log n) \) on \( V_q \times S \) and \( V_{q'} \times S \) respectively. These are in fact identified under the transformation (5.9).

Thus we have a well-defined lifting of our \((p + q)\)-cocycle, to get the \((p + q)\)-cochain \((\tilde{n}_{\sigma_0} \cdots \sigma_q)\). We now apply the total differential to this. Since the liftings \( \tilde{n} \) are themselves \(d\)-closed, this just means applying \(d_{\text{bct}}\). In particular, (3) is now clear. For (4) to see what the contribution to \( \sigma_0 \to \cdots \to \sigma_{q+1} \) is, choose \( g : \sigma_{q+1} \to \sigma \), and let \( g_i : \sigma_i \to \sigma_{q+1}^g \to \sigma \) be the composition. Then using the representation determined by \( \sigma \), the contribution is

\[
\sum_{i=1}^{q+1} (-1)^i s_{g_0}^* (\log n_{\sigma_0} \cdots \sigma_i \cdots \sigma_{q+1}).
\]

We view this modulo \( \Omega^2_{S/k^1} \), noting from (3.8) that

\[
s_g^* (\log n) = \log (s_{id,g}) n + \log n \mod \pi^n \Omega^2_{S/k^1},
\]

so we obtain

\[
- \log (s_{id,\sigma_0} \to \sigma_1) n_{\sigma_1} \cdots \sigma_{q+1} + \sum_{i=0}^{q+1} (-1)^i \log n_{\sigma_0} \cdots \sigma_i \cdots \sigma_{q+1}
\]

\[
+ \sum_{i=0}^{q+1} (-1)^i \log (s_{id,g_0}) n_{\sigma_0} \cdots \sigma_i \cdots \sigma_{q+1}.
\]

Now as \((\sigma_0 \to \cdots \to \sigma_{q}, \log n_{\sigma_0} \cdots \sigma_{q})\) is already \(d_{\text{bct}}\)-closed over \(S^1\), the last two terms in fact vanish, and we are left with the \((p + q + 1)\)-cocycle

\[
(\sigma_0 \to \cdots \to \sigma_{q+1}, - \log (s_{id,\sigma_0} \to \sigma_1) n_{\sigma_1} \cdots \sigma_{q+1}).
\]

By [4], III, 4.15, this represents in Čech cohomology the negative of the image of \( s_{id} \otimes n \) under the composition

\[
H^1(B, i_* \Lambda \otimes H^1(B, i_* \Lambda)^*_f) \otimes H^q(B, i_* \bigwedge^p \tilde{\Lambda} \otimes k) \to H^{q+1}(B, (i_* \Lambda \otimes H^1(B, i_* \Lambda)^*_f) \otimes i_* \bigwedge^p \tilde{\Lambda} \otimes k)
\]

\[
\to H^{q+1}(B, H^1(B, i_* \Lambda)^*_f \otimes i_* \bigwedge^p \tilde{\Lambda} \otimes k)
\]

\[
= H^1(B, i_* \Lambda \otimes k)^* \otimes H^{q+1}(B, i_* \bigwedge^p \tilde{\Lambda} \otimes k),
\]

where the first map is cup product and the second is contraction. However, this map is the same as the map

\[
H^1(B, i_* \Lambda \otimes k) \otimes H^q(B, i_* \bigwedge^p \tilde{\Lambda} \otimes k) \to H^{q+1}(B, i_* \bigwedge^p \tilde{\Lambda} \otimes k),
\]

tensored with \(H^1(B, i_* \Lambda \otimes k)^*\). Now in general, if \(U, V\) and \(W\) are vector spaces, the canonical identification \(\text{Hom}(U \otimes V, W) = \text{Hom}(V, U^* \otimes W)\) can be realised by, given a map \(\varphi : U \otimes V \to W\), tensoring with \(U^*\) to get \(\text{id}_U \otimes \varphi : U^* \otimes U \otimes V \to U^* \otimes W\), which induces a map \(V \to U^* \otimes W\) via \(v \mapsto (\text{id}_U \otimes \varphi)(\text{id}_U \otimes v)\), thinking of \(\text{id}_U \in U^* \otimes U\). Since \(s_{id}\) is just the identity in \(H^1(B, i_* \Lambda \otimes k) \otimes H^1(B, i_* \Lambda \otimes k)^*\), the result stated in (4) follows. \(\square\)
Corollary 5.6. Choose a basis $\eta_1, \ldots, \eta_h$ of $H^1(B, i_*\Lambda)^*_f$, and let $\eta_1^*, \ldots, \eta_h^*$ be the dual basis. Let $q_1, \ldots, q_h$ be the corresponding monomials in $k[H^1(B, i_*\Lambda)^*_f]$. If

$$n \in H^q(B, i_* \bigwedge^p \Lambda \otimes k),$$

then

$$n + \sum_{i_1=1}^h (\iota(\eta_{i_1}^*)n) \log q_{i_1} + \sum_{i_1, i_2=1}^h (\iota(\eta_{i_1}^*) \iota(\eta_{i_2}^*)n) \log q_{i_1} \log q_{i_2} \frac{1}{2}$$

$$\cdots + \sum_{i_1, \ldots, i_p=1}^h (\iota(\eta_{i_1}^*) \cdots \iota(\eta_{i_p}^*)n) \frac{\log q_{i_1} \cdots \log q_{i_p}}{p!}$$

is flat with respect to $\nabla_{GM}$. Here $\log q_i$ can be viewed formally via the property $d(\log q_i) = \log q_i$. 

Proof. Simply apply $\nabla_{GM}$ to this section. \qed

Remark 5.7. We have now in fact described a variation of mixed Hodge structures over $S$: indeed, $\mathbb{R}^r \pi_* (j_* \Omega^p_{X^1_{u/S^1}})$ is a vector bundle on $S$ with $\nabla_{GM}$ a flat connection defining a local system of flat sections of this vector bundle. We have the Hodge filtration

$$F^i = \bigoplus_{p \geq i} R^r \pi_* (j_* \Omega^p_{X^1_{u/S^1}})$$

and weight filtration

$$W_{2i} = W_{2i+1} = \bigoplus_{p \leq i} R^r \pi_* (j_* \Omega^p_{X^1_{u/S^1}}).$$

What is missing from this description is the integral structure on the local system. This will be addressed in [14].
References

[1] Batyrev, V., “Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties,” J. Algebraic Geom. 3 (1994), 493–535.

[2] Batyrev, V., and Borisov, L., “On Calabi-Yau complete intersections in toric varieties,” in Higher-dimensional complex varieties (Trento, 1994), 39–65, de Gruyter, Berlin, 1996.

[3] Batyrev, V., and Dais, D., “Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry,” Topology, 35 (1996), 901–929.

[4] Bredon, G., Sheaf Theory, 2nd Edition, Springer-Verlag, 1997.

[5] Deligne, P., “Théorème de Lefschetz et critères de dégénérescence de suites spectrales,” Inst. Hautes études Sci. Publ. Math. 35, (1968) 259–278.

[6] Friedman, R., “Global smoothings of varieties with normal crossings,” Ann. Math. 118, 75–114 (1983).

[7] Fukaya, K., “Multivalued Morse theory, asymptotic analysis and mirror symmetry,” in Graphs and patterns in mathematics and theoretical physics, 205–278, Proc. Sympos. Pure Math., 73, Amer. Math. Soc., Providence, RI, 2005.

[8] Goldman, W., and Hirsch, M., “The radiance obstruction and parallel forms on affine manifolds,” Trans. Amer. Math. Soc., 286 (1984), 629–649.

[9] Gross, M., “Special Lagrangian Fibrations I: Topology,” in Integrable Systems and Algebraic Geometry, eds. M.-H. Saito, Y. Shimizu and K. Ueno, World Scientific, 1998, 156–193.

[10] Gross, M., “Toric degenerations and Batyrev-Borisov duality,” Math. Ann. 333 (2005), 645–688.

[11] Gross, M., and Siebert, B., “Affine manifolds, log structures, and mirror symmetry,” Turkish J. Math., 27, (2003), 33-60.

[12] Gross, M., and Siebert, B., “Mirror symmetry via logarithmic degeneration data. I,” J. Differential Geom. 72 (2006), 169–338.

[13] Gross, M., and Siebert, B., “From affine geometry to complex geometry,” preprint (2007).

[14] Gross, M., and Siebert, B., “Toric fibrations and toric degenerations,” in preparation.

[15] Gross, M., Pandharipande, R., and Siebert, B., “The tropical vertex,” preprint (2009), to appear in Duke Math. J.

[16] Grothendieck, A., Revêtements étales et groupe fondamental (SGA I), Lecture notes in Mathematics, 224, Springer-Verlag, 1963.

[17] Haase, C., and Zharkov, I., “Integral affine structures on spheres and torus fibrations of Calabi-Yau toric hypersurfaces I,” preprint, 2002, math.AG/0205321.

[18] Hartshorne, R., Algebraic geometry, Graduate Texts in Mathematics, No. 52., Springer-Verlag, New York-Heidelberg, 1977.

[19] Hitchin, N., “The Moduli Space of Special Lagrangian Submanifolds,” Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 25 (1997), 503–515.

[20] Kato, F., “Log smooth deformation theory,” Tohoku Math. J. 48, 317–354 (1996).

[21] Kato, F., “Functors of log Artin rings,” Manuscripta Math. 96 (1998), 97–112.

[22] Kato, K., “Logarithmic structures of Fontaine–Illusie,” in: Algebraic analysis, geometry, and number theory (J.-I. Igusa et. al. eds.), 191–224, Johns Hopkins Univ. Pr., Baltimore, 1989.

[23] Katz, K., and Oda, T., “On the differentiation of de Rham cohomology classes with respect to a parameter,” J. Math. Kyoto Univ. 8, (1968), 199-213.

[24] Kawamata, K., and Namikawa, Y., “Logarithmic deformations of normal crossing varieties and smoothering of degenerate Calabi-Yau varieties,” Invent. Math. 118, 395–409 (1994).
[25] Kontsevich, M., and Soibelman, Y., “Homological mirror symmetry and torus fibrations,” in: Symplectic geometry and mirror symmetry (Seoul, 2000), 203–263, World Sci. Publishing, River Edge, NJ, 2001.

[26] Kontsevich, M., and Soibelman, Y., “Affine structures and non-Archimedean analytic spaces,” in The unity of mathematics, 321–385, Progr. Math., 244, Birkhäuser Boston, Boston, MA, 2006.

[27] Leung, N.C., “Mirror symmetry without corrections,” Comm. Anal. Geom. 13, (2005), 287–331.

[28] Milne, J.S., Étale Cohomology, Princeton University Press 1980.

[29] Oda, T., Convex bodies and algebraic geometry. An introduction to the theory of toric varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 15. Springer-Verlag, Berlin, 1988.

[30] Ogus, A., Lectures on logarithmic algebraic geometry, in progress.

[31] Ran, Z., “Unobstructedness of Calabi-Yau orbi-Kleinfolds,” J. Math. Phys. 39 (1998), 625–629.

[32] Ruddat, H., “Log Hodge groups on a toric Calabi-Yau degeneration,” preprint, 2009.

[33] Schlessinger, M., “Functors of Artin rings,” Trans. Amer. Math. Soc. 130 (1968), 208–222.

[34] Steenbrink, J., “Limits of Hodge structures”, Inv. Math. 31, 229–257 (1976).

[35] Strominger, A., Yau, S.-T., and Zaslow, E., “Mirror symmetry is $T$-duality,” Nucl. Phys. B, 479, (1996), 243–259.

[36] Tian, G., “Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Petersson-Weil Metric,” in Mathematical Aspects of String Theory, 629-646, ed. S.-T. Yau, World Scientific, Singapore, 1987.

[37] Todorov, A., “The Weil-Petersson Geometry of the Moduli Space of $SU(n \geq 3)$ (Calabi-Yau) Manifolds I,” Commun. Math. Phys 126, (1989) 325–346.

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