Quantum Euler class and virtual Tevelev degrees of Fano complete intersections

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Abstract

We compute the quantum Euler class of Fano complete intersections $X$ in a projective space. In particular, we prove a recent conjecture of A. Buch and R. Pandharipande, namely [5, Conjecture 5.14]. Finally we apply our result to obtain formulas for the virtual Tevelev degrees of $X$. An algorithm computing all genus 0 two-point Hyperplane Gromov Witten invariants of $X$ is illustrated along the way.

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1 Introduction

1.1 Quantum Euler class of a variety

Let $X$ be a nonsingular, projective, algebraic variety over $\mathbb{C}$ of dimension $r$ and let $\{\gamma_j\}_{j=0}^N \subset H^*(X)$ be a homogeneous basis with $\gamma_0 = 1$ and $\gamma_N = P$ the point class. The small quantum cohomology ring $QH^*(X)$ of $X$ is defined via the 3-point genus 0 Gromov-Witten invariants:

$$\gamma_i \ast \gamma_j = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_k \langle \gamma_i, \gamma_j, \gamma_k^\vee \rangle_{0,\beta} q^\beta \gamma_k$$

where $\gamma_k^\vee \in H^*(X)$ is the dual of $\gamma_k$ with respect to the intersection form on $X$, defined by the conditions

$$\int_X \gamma_j \cup \gamma_k^\vee = \delta_{j,k} \text{ for } j = 0, \ldots, N.$$

Here we are following the notation of [10].

Let also

$$\Delta = \sum_j \gamma_j^\vee \otimes \gamma_j \in H^*(X) \otimes H^*(X)$$

be the K"unneth decomposition of the diagonal class of $X \times X$.

The quantum Euler class of $X$ is the image of $\Delta$ under the product map

$$H^*(X) \otimes H^*(X) \rightarrow QH^*(X).$$

This is a canonically defined element of $QH^*(X)$, first introduced by Abrams in [2]. In terms of the basis $\{\gamma_j\}$, we have

$$E = \sum_j \gamma_j^\vee \ast \gamma_j.$$

Note that in particular

$$E \equiv \chi(X)P \mod q$$

where $\chi(X)$ is the Euler characteristic of $X$.

In this paper we compute the quantum Euler class of all Fano nonsingular complete intersections of dimension at least 3 in a projective space (see Theorem 5 below). In particular, we prove a conjecture of Buch-Pandharipande, namely [5, Conjecture 5.14].

It is worth noting that although a priori the definition of $E$ involves also the primitive cohomology of $X$, in our case of interest, this class actually lies in the restricted quantum cohomology ring $QH^r(X)^{res}$ of $X$, that is the quantum cohomology ring coming from the projective space (see Proposition 1 below for the exact definition of $QH^r(X)^{res}$). This is a key reason we were able to obtain so explicit a formula for $E$.

Finally, in [5] the quantum Euler class $E$ of any variety $X$ is related via a very simple formula (see [5, Theorem 1.4]) to the virtual Tevelev degrees of $X$, that is the virtual count of genus $g$ maps of fixed complex structure in a given curve class $\beta$ through $n$ general points of $X$. Exploiting their formula and our explicit expression of $E$ for $X$ a Fano nonsingular complete intersections of dimension at least 3, we are able to compute all the virtual Tevelev degrees of such varieties $X$ (see Theorem 10 below).

\footnote{Unless otherwise specified, (co)homology and quantum cohomology will always be taken with $\mathbb{Q}$-coefficients in this paper.}
1.2 Preliminary results on complete intersections

We now specialize to smooth complete intersections of dimension at least 3. Let \( X = V(f_1, \ldots, f_L) \subset \mathbb{P}^r + L \) be a nonsingular complete intersection of dimension \( r \). Assume for the rest of the paper that \( r \geq 3 \) and that for \( i = 1, \ldots, L \),

\[ f_i \in H^0(\mathbb{P}^r + L, \mathcal{O}(m_i)) \]

where \( m_i \geq 2 \).

Let \( m = (m_1, \ldots, m_L) \) be the vector of degrees and, for \( a, b \in \mathbb{Z} \) adopt the following notation:

\[ |m| = \sum_{i=1}^L m_i, \quad m^{am+b} = \prod_{i=1}^L m_i^{am_i+b}. \]

1.2.1 Cohomology of complete intersections

Consider the map

\[ H^i(\mathbb{P}^r + L) \to H^i(X) \]  

induced by the inclusion \( X \subset \mathbb{P}^r + L \). By the Lefschetz Hyperplane Theorem, this map is an isomorphism for all \( i \leq 2r, i \neq r \) and is injective for \( i = r \). Also, for \( i = r \), we have a canonical decomposition

\[ H^r(X) = H^r(X)\text{prim} \oplus H^r(X)\text{res} \]

as a direct sum of the primitive cohomology and the restricted cohomology of degree \( r \).

Explicitly

\[ H^r(X)\text{res} = \text{Im}(H^r(\mathbb{P}^r + L) \hookrightarrow H^r(X)) \]

and

\[ H^r(X)\text{prim} = \text{Ker}(H \cup - : H^r(X) \to H^{r+2}(X)) \]

where \( H \in H^*(X) \) is the hyperplane class.

Note that

\[ \dim H^r(X)\text{prim} = (-1)^r(\chi(X) - (r + 1)) \]

where \( \chi(X) \) is the Euler characteristic of \( X \).

1.2.2 Quantum cohomology of Fano complete intersections

From now on, we will further restrict our attention to the Fano case

\[ |m| \leq r + L. \]

Since \( r \geq 3 \), the map in Equation 1 is an isomorphism when \( i = 2 \) and thus

\[ H_2(X) = \mathbb{Q}L. \]

where \( L \in H^*(X) \) is the class of a line in \( X \). It follows that \( \mathbb{Q}H^*(X) \) is a graded algebra over the polynomial ring \( \mathbb{Q}[q] \) in one variable \( q \) and as a \( \mathbb{Q}[q] \)-modules we have

\[ \mathbb{Q}H^*(X) = H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q]. \]
The degree of $q$ is equal to $2d$ where
\[ d = r + L + 1 - |m|. \]
which is greater than 0 by Equation 2.

Depending on the degree $|m|$ of $X$, the ring $QH^*(X)$ satisfies the following magic relation (due to A. Givental):

- if $|m| \leq r + L - 1$ we have:
  \[ H^{*(r+1)} = m^{|m|}qH^{*(r - |m| - L)} \]  
  \[ (3) \]
- if $|m| = r + L$ we have:
  \[ (H + m!q)^{*(r+1)} = m^{|m|}q(H + m!q)^r \]  
  \[ (4) \]
Some cases of Equation 3 are proved in [3]. A complete proof of both relations can be found in Givental’s paper [11]. There also are two very nice expositions of Givental’s work, see [15, Section 3.2] and [4, Corollary 4.4 and Corollary 4.19]. Relations 3 and 4 will be essential for the object of this paper.

1.3 Statement of the main Theorem

In Theorem 5 below we will give explicit formulas for the quantum Euler class of any smooth Fano complete intersection $X \subseteq \mathbb{P}^{r+L}$ as above. Much of our work starts with the results in [5], some of which we now recall for the reader’s convenience.

**Proposition 1** (due to T. Graber). Let $R = \text{Span}\{1, H, \ldots, H^r\} \subset H^*(X)$. Then, $(R \otimes_{\mathbb{Q}} \mathbb{Q}[q], *)$ is a subring of $QH^*(X)$.

**Proof.** This is [5, Proposition 5.1].\[ \]

This ring is denoted by $QH^*(X)^{\text{res}}$.

**Remark 2.** The elements $1, H, \ldots, H^r$ form a basis of $QH^*(X)^{\text{res}}$ as $\mathbb{Q}[q]$-module. This follows from the fact that $H^i_{*} = H^{*+i} \text{ mod } q$ and $H^i \in QH^*(X)^{\text{res}}$ for $i = 0, \ldots, r$.

Let $E$ be the quantum Euler class of $X$.

**Lemma 3.** We have $E \in QH^*(X)^{\text{res}}$.

**Proof.** See [5, Proof of Proposition 5.5].\[ \]

By Remark 2 and Lemma 3 we can uniquely write
\[ E = \sum_{i=0}^{|r|} \text{Coeff}(E, q^iH^{*(r-i)d})q^iH^{*(r-id)}. \]
where $\text{Coeff}(E, q^iH^{*(r-id)}) \in \mathbb{Q}$. Our goal is to make this coefficients explicit.

**Remark 4.** We have
\[ \text{Coeff}(E, H^r) = m^{-1} \sum_j \int_X \gamma_j \cup \gamma_j = m^{-1} \sum_j (-1)^{\text{deg}(\gamma_j)} = m^{-1} \chi(X). \]
The main result of the paper is the following:

**Theorem 5 (Main Theorem).** The following equalities hold:

- if $|m| \leq r + L - 1$ then
  $$E = m^{-1}\chi(X)H^*r + (r + L + 1 - |m| - \chi(X))m^{m-1}qH^*|m|-L-1,$$

- if $|m| = r + L$ then
  $$E = m^{-1}\chi(X)H^*r + \sum_{j=1}^{r} m^{-1}(j - \chi(X)) \binom{r}{j-1} (m!)^{j-1} \left[ m^m - \frac{m!}{j} (r + 1) \right] q^j H^* - j.$$

The case $|m| \leq r + L - 1$ in the theorem is exactly [5, Conjecture 5.14] and is already shown to be true mod $q^2$ in [5, Corollary 5.13]. The proof of this theorem is given in Section 3.

### 1.4 Application: virtual Tevelev degrees of Fano complete intersections

#### 1.4.1 Virtual Tevelev degrees

Let $X$ be a nonsingular, projective, algebraic variety over $\mathbb{C}$ of dimension $r$. Fix integers $g, n \geq 0$ satisfying the stability condition $2g - 2 + n > 0$ and fix $\beta \in H_2(X, \mathbb{Z})$ an effective curve class satisfying the condition

$$\int_\beta c_1(T_X) > 0.$$

Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of genus $g$, $n$-pointed stable maps to $X$ in class $\beta$ and assume that the dimensional constraint

$$\text{vdim}(\overline{M}_{g,n}(X, \beta)) = \dim(\overline{M}_{g,n} \times X^n)$$

holds. This is equivalent to

$$\int_\beta c_1(T_X) = r(n + g - 1). \tag{5}$$

Let

$$\tau : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n} \times X^n$$

be the canonical morphism obtained from the domain curve and the evaluation maps:

$$\pi : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}, \quad \text{ev} : \overline{M}_{g,n}(X, \beta) \to X^n.$$

Then the virtual Tevelev degree $\text{vTevelev}_{g,n, \beta}^X \in \mathbb{Q}$ of $X$ is defined by the equality

$$\tau_*(\overline{M}_{g,n}(X, \beta))^\text{vir} = \text{vTevelev}_{g,n, \beta}^X(\overline{M}_{g,n} \times X^n) \in A^0(\overline{M}_{g,n} \times X^n).$$

Alternatively, denoting by $\Omega_{g,n, \beta}^X : H^*(X)^\otimes n \to H^*(\overline{M}_{g,n})$ the Gromov Witten class

$$\Omega_{g,n, \beta}^X : (\alpha) := \pi_*(\text{ev}^*(\alpha) \cap [\overline{M}_{g,n}(X, \beta)])^\text{vir},$$
we have
\[ \nuTev_{g,n,\beta}(\overline{M}_{g,n}) = \Omega_{g,n,\beta}(\mathbb{P}^n). \]

Tevelev degrees were essentially introduced in [16], starting from which a lot of work has been done to study this degrees. In [7] these degrees were formally defined and computed via Hurwitz theory for the case of \( \mathbb{P}^1 \); then, in [9], using Schubert calculus the problem was posed and solved for the case of \( \mathbb{P}^n \); in [5] a generalization of these degrees is presented for \( \mathbb{P}^1 \); in [6] a virtual perspective is adopted via Gromov-Witten theory; in [14] an equality between virtual and geometric Tevelev degrees is proven for certain Fano varieties and large degree curve classes and finally in [13] geometric Tevelev degrees are computed for low degree hypersurfaces and large degree curve class via projective geometry.

### 1.4.2 Virtual Tevelev degrees of Fano complete intersections

In this paper, we concern about exact computations of virtual Tevelev degrees of Fano complete intersections following the perspective presented in [5] and described above in Section 1.4.1.

Let \( X \) be a smooth Fano complete intersection in \( \mathbb{P}^{r+L} \) of dimension \( r \geq 3 \) and vector of degrees \( m \). Writing \( \beta = kL \) with \( k > 0 \), condition 5 becomes
\[ k = k[g,n] := \frac{n + g - 1}{d} - r. \]

For us, the main ingredient to compute \( \nuTev_{g,n,k} \) will be the following result:

**Theorem 6.** Suppose \( k = k[g,n] \). Then
\[ \nuTev_{g,n,k} = m^1 \text{Coeff}(P^{*n} \ast E^{*g}, q^kH^r). \]

**Proof.** This is [5, Theorem 1.4]. \( \square \)

Before stating our theorem, we require a remark and some additional notation.

**Remark 7.** Given the form of Equation 4, when \( |m| = r + L \) it will be more convenient to use \( 1, (H + mLq), ..., (H + mLq)^{r} \) instead of \( 1, H, ..., H^r \) as a basis of \( QH^r(X) \) as \( \mathbb{Q}[q] \)-module.

**Definition 8.** Define
\[ \mathbb{Q} \ni P_i = \begin{cases} \text{Coeff}(P, q^iH^{r-id}) & \text{when } |m| \leq r + L - 1, \\ \text{Coeff}(P, q^i(H + mLq)^{r-i}) & \text{when } |m| = r + L, \end{cases} \]
for \( i = 0, ..., \lfloor \frac{r}{d} \rfloor \) and
\[ \mathbb{Q} \ni b_i = \begin{cases} \text{Coeff}(P^{*n} \ast E^{*g}, q^{i+k}H^{r-id}) & \text{when } |m| \leq r + L - 1, \\ \text{Coeff}(P^{*n} \ast E^{*g}, q^{i+k}(H + mLq)^{r-i}) & \text{when } |m| = r + L, \end{cases} \]
for \( i = 0, ..., \lfloor \frac{r}{d} \rfloor \).

Note that, by Theorem 5
\[ \nuTev_{g,n,k} = m^1 b_0 \]
and that by Theorem 6 the \( b_i \)'s are determined by the \( P_i \)'s.
Definition 9. Following [5, Definition 5.15], we define the discrepancy of $P^n \ast E^g$ to be

$$\text{Disc}(P^n \ast E^g) = \sum_{i=1}^{\lfloor \frac{|m|}{r} \rfloor} b_i m^{-im+1}.$$  

Putting all together we obtain explicit formulas for all virtual Tevelev degrees of $X$ (once all the coefficients $P_i$ are known):

Theorem 10. Suppose $k = k[g,n]$. Then, the virtual Tevelev degrees of $X$ are as follows:

- if $|m| \leq r + L - 1$ then
  $$v_{\text{Tev}}^{X,g,n,k} = \left( \sum_{i=0}^{\lfloor \frac{|m|}{r} \rfloor} P_i m^{-im} \right)^n (r + L + 1 - |m|)^g m^{km-g+1} - \text{Disc}(P^n \ast E^g),$$

- if $|m| = r + L$ then
  $$v_{\text{Tev}}^{X,g,n,k} = \left( \sum_{i=0}^{r} P_i m^{-im} \right)^n (1 - m^{-rm(m!)}(r+1-\chi(X)))^g m^{km-g+1} - \text{Disc}(P^n \ast E^g).$$

The case $|m| \leq r + L - 1$ already appears in [5, Proposition 5.16] (where they assume that [5, conjecture 5.14] holds for $X$), the case $|m| = r + L$ is instead completely new.

The last question would be if we can actually express the coefficients $P_i$ appearing in Theorem 10 in a closed formula obtaining in this way a closed formula for the virtual Tevelev degrees.

Partial results have been obtained in [5], where they gave a complete answer in the following cases:

- for quadric hypersurfaces (see [5, Theorem 1.5 and Example 2.4]);
- for low degree complete complete intersections $r > 2|m| - 2L - 2$ which are not quadrics (see [5, Corollary 5.11 and Theorem 5.19]);
- for the border case $r = 2|m| - 2L - 2$ (see [5, Lemma 5.21 and Corollary 5.23]).

Here we will content ourselves with illustrating in Section 5 an algorithm that calculates all the coefficients $P_i$’s. It should be noted here that the method we will describe is more effective than the general result in [1], where they deal with Gromov-Witten invariants in all genera with arbitrary insertions.

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2 A preliminary computation

We start with expressing $H^i$ for $i = 1, ..., r$ as a linear combination of $1, ..., H^r$ with coefficients in $\mathbb{Q}[q]$.

The following notation will be convenient. For $k \geq 0$ and $0 \leq j \leq r$ let

$$\alpha_{r-j}^k := m^{-1}(H^{kd+j-1}, H^{r-j})_{0,k} = m^{-1} \int_{\overline{\mathcal{M}}_{0,2}(X,kL)}^{\text{vir}} ev_i^*H^{kd+j-1} \cup ev_r^*H^{r-j}$$

where $L$ is the class of a line in $X$ and $ev_i : \overline{\mathcal{M}}_{0,2}(X,kL) \to X$ are the evaluation maps for $i = 1, 2$. Note the following symmetry:

$$\alpha_{r-j}^k = \alpha_{kd+j-1}^k.$$

**Proposition 11.** Let $0 \leq i \leq r$. Then, for $1 \leq j \leq \lfloor \frac{i}{d} \rfloor$, we have

$$\text{Coeff}(H^i, q^j H^{(i-jd)}) =$$

$$= \sum_{\ell : 1 \leq \ell \leq j} (-1)^{\ell} \sum_{(i_1, ..., i_\ell) \in \mathbb{Z}_{\geq 1}^\ell : (u_1, ..., u_\ell) \in (\mathbb{Z}_{\geq 0})^{\ell	imes: a=1}} \prod_{a=1}^\ell i_a \alpha_{r-(j-i_1-...-i_\ell)d-u_a}.$$ 

**Proof.** We proceed by induction on $i$.

When $i < d$ there is nothing to prove. When $i = d$, it must be $j = 1$ and the right-hand side of the equation is just $\alpha_1^1$. To compute the left-hand side note that $H^d = H^\star$ for $i < d$ and thus

$$H^\star d = H^\star H^{d-1} = H^d + q(H, H^{d-1}, H^r)_{0,1} m^{-1}.$$ 

Note that, since $H^\star H^{d-1} \in QH^\star(X)^{\text{res}}$, the primitive cohomology contributions in $H^\star H^{d-1}$ is 0.

Use now the divisor equation in Gromov-Witten theory to obtain

$$(H, H^{d-1}, H^r)_{0,1} m^{-1} = \alpha_1^1.$$ 

Assume now that the Theorem is true for $i = d, ..., t - 1 < r$.

Write

$$H^\star H^{t-1} = H^t + \sum_{k=1}^{\lfloor \frac{t}{d} \rfloor} (H, H^{t-1}, H^{kd+t-1})_{0,k} m^{-1} q^k H^{t-kd}.$$ 

where again the primitive cohomology contributions in $H^\star H^{t-1}$ is 0 and by the divisor equation

$$m^{-1}(H, H^{t-1}, H^{kd+t-1})_{0,k} = k \alpha_{r-(t-kd)}^k.$$ 

Note now that, by induction, we know how to write the $H^{t-kd}$ and $H^{t-1}$ in terms of
1, H, ..., H^r. Putting all together we obtain for 1 ≤ j ≤ \lfloor \frac{r}{d} \rfloor$

\[
\text{Coeff}(H^t, q^j H^{(t-j)d}) = \text{Coeff}(H^{t-1}, q^j H^{(t-1-j)d}) - \sum_{k=1}^{j} k\alpha_{r-(t-kd)} \text{Coeff}(H^{t-kd}, q^{j-k} H^{(t-jd)})
\]

\[
= \sum_{\ell=1}^{j} (-1)^\ell \sum_{i_1+...+i_\ell = j, 0 \leq u_1 \leq ... \leq u_\ell \leq t-1-jd} \prod_{a=1}^{\ell} i_a \alpha_{r-(j-i_1-...-i_a)d-u_a}
\]

and, since

r - (t - kd) = r - (j - k)d - (t - jd),

this is exactly the right-hand side appearing in the statement above with i = t.

**Corollary 12.** For 1 ≤ j ≤ \lfloor \frac{r}{d} \rfloor we have

\[
\text{Coeff}(H^{*r+1}, q^j H^{*r+1-jd}) = \sum_{\ell: 1 \leq \ell \leq j} (-1)^{\ell+1} \sum_{(i_1, ..., i_\ell) \in \mathbb{Z}_{\geq 1}^\ell: (u_1, ..., u_{\ell}) \in (\mathbb{Z}_{\geq 0}^\ast)^\ell} \prod_{a=1}^{\ell} i_a \alpha_{r-(j-i_1-...-i_a)d-u_a}.
\]

**Proof.** Proceeding as in the proof of Proposition [11] we write $H^{*r+1} = H \ast H^r$ and

\[
H^r = H^r - \sum_{j=1}^{\lfloor \frac{r}{d} \rfloor} \text{Coeff}(H^r, q^j H^{r-jd}) q^j H^{r-jd}.
\]

Since

\[
H \ast H^r = \sum_{k=1}^{\lfloor \frac{r+1}{d} \rfloor} k\alpha^k_{kd-1} q^k H^{r-kd+1},
\]

we have

\[
\text{Coeff}(H^{*r+1}, q^j H^{*r+1-jd}) = -\text{Coeff}(H^r, q^j H^{*r-jd}) + \sum_{k=1}^{j} k\alpha_{r-(r+1-kd)} \text{Coeff}(H^{r+1-kd}, q^{j-k} H^{r+1-jd})
\]

\[
= \sum_{1 \leq \ell \leq j} (-1)^{\ell+1} \sum_{i_1+...+i_\ell = j, 0 \leq u_1 \leq ... \leq u_\ell \leq r-jd} \prod_{a=1}^{\ell} i_a \alpha_{r-(j-i_1-...-i_a)d-u_a}
\]

and

\[
= \sum_{k=1}^{j} k\alpha_{kd-1} \sum_{1 \leq \ell \leq k} (-1)^{\ell+1} \sum_{i_1+...+i_\ell = j-k, 0 \leq u_1 \leq ... \leq u_\ell \leq r+1-jd} \prod_{a=1}^{\ell} i_a \alpha_{r-(j-k-i_1-...-i_a)d-u_a}
\]

for j = 1, ..., \lfloor \frac{r}{d} \rfloor. Finally, since kd - 1 = r - (j - k)d - (r + 1 - jd), we are done.

Using the previous Corollary and Equations [3] and [4] we get interesting equalities.
3 Proof of the main theorem

3.1 Plan of the Proof

In this subsection we explain how the proof of Theorem 5 goes.

Define

\[ \Gamma := \sum_{j: \gamma_j \in H^r(X)_{prim}} \gamma_j^\vee \star \gamma_j \]

and

\[ E' := m^{-1} \sum_{i=0}^r H^i \star H^{r-i} \]

Then

\[ E = \Gamma + E' \]

and so by Lemma 3 we see that \( \Gamma \in QH^*(X)_{res} \).

Using relations 3 and 4 the proof of Theorem 5 becomes an easy algebraic count (done in Section 3.4) once we know the following two propositions.

**Proposition 13.** For \( j = 1, \ldots, \lfloor \frac{r}{d} \rfloor \) we have

\[ \text{Coeff}(\Gamma, q^j H^{r-jd}) = m^{-1}(r + 1 - \chi(X)) \text{Coeff}(H^{r+1}, q^j H^{r+1-jd}). \]

The proof is presented in Section 3.2.

**Proposition 14.** For \( j = 1, \ldots, \lfloor \frac{r}{d} \rfloor \) we have

\[ \text{Coeff}(E', q^j H^{r-jd}) = -m^{-1}(r - jd + 1) \text{Coeff}(H^{r+1}, q^j H^{r+1-jd}). \]

The proof is presented in Section 3.3.

We remark here that the way we prove Proposition 14 is purely algebraic. It would be interesting to find a more conceptual explanation for this equality.

3.2 Computation of \( \Gamma \)

The proof of Proposition 13 relies on the following preliminary lemma which is very similar to [5, Lemma 5.2].

**Lemma 15.** Let \( \Lambda \in QH^*(X)_{res} \) be a degree 2r class such that

\[ \Lambda = aH^r \mod q \text{ and } H \star \Lambda = 0 \]

where \( a \in \mathbb{Q} \). Then

\[ \text{Coeff}(\Lambda, q^i H^{r-id}) = -a \text{Coeff}(H^{r+1}, q^i H^{r+1-id}) \]

for \( i = 1, \ldots, \lfloor \frac{r}{d} \rfloor \).

**Proof.** Write

\[ \Lambda = aH^r + \sum_{i=1}^{\lfloor \frac{r}{d} \rfloor} \text{Coeff}(\Lambda, q^i H^{r-id})q^i H^{r-id}. \]
Then we have
\[ 0 = H \ast \Lambda = aH^{r+1} + \sum_{i=1}^{\left\lfloor \frac{r}{d} \right\rfloor} \text{Coeff}(\Lambda, q^iH^{r-id})q^iH^{r+1-id} \]
from which we obtain the lemma. \qed

**Proof of Proposition 14** By [5, Corollary 5.3] (the same proof of that corollary also applies when \( |m| = r + L \)), we have \( H \ast \Gamma = 0 \). Moreover

\[ \Gamma = \sum_{j: \gamma_j \in H'(X)_{\text{prim}}} \gamma_j \cup \gamma_j = \dim(H'(X)_{\text{prim}})(-1)^r m^{-1} H' \mod q. \]

Note that we \( \dim(H'(X)_{\text{prim}}) = (-1)^r (\chi(X) - r - 1) \). The proposition now follows from an application of Lemma 15. \qed

### 3.3 Computation of \( E' \)

In this subsection we prove Proposition 14 by showing the following equivalent result:

**Lemma 16.** For \( j = 1, \ldots, \left\lfloor \frac{r}{d} \right\rfloor \) we have

\[ \sum_{i=0}^{r} \text{Coeff}(H^i \ast H'^{-i}, q^iH^{(r-jd)}) = -(r - jd + 1) \text{ times the RHS of Equation 6} \]

**Proof.** For \( 0 \leq i \leq r \) and \( 1 \leq j \leq \left\lfloor \frac{r}{d} \right\rfloor \) we have

\[ \text{Coeff}(H^i \ast H'^{-i}, q^iH^{(r-jd)}) = \sum_{(h,s) \in \mathbb{Z}_{\geq 0}^2: h+s=j} \text{Coeff}(H^i, q^hH^{(i-hd)})\text{Coeff}(H'^{-i}, q^sH^{(r-i-sd)}) \]

and for each \((h,s)\) as above such that \( hd \leq i \) and \( r - i \geq sd \), the product

\[ \text{Coeff}(H^i, q^hH^{(i-hd)})\text{Coeff}(H'^{-i}, q^sH^{(r-i-sd)}) \]

is equal to

\[ \sum_{1 \leq w \leq h} (-1)^{w+z} \sum_{1 \leq z \leq s} \prod_{a=1}^{w} i_a \alpha_{r-(h-y_1-\ldots-y_a)d-p_a} \prod_{b=1}^{z} x_b \alpha_{r-(s-x_1-\ldots-x_b)d-v_b} \] (7)

Note that it could be \( h = 0 \) or \( s = 0 \) (but not \( h = s = 0 \) being \( h + s = j > 0 \)). Observe the symmetry

\[ \alpha_{r-(s-x_1-\ldots-x_b)d-v_b} = \alpha_{r-(s-x_1-\ldots-x_{b-1})d+v_b-1}; \]

and that

\[ (s - x_1 - \ldots - x_{b-1})d + v_b - 1 = r - [(j - x_b - \ldots - x_2)d + r - v_b + 1 - jd] \]
where \( r - v_b + 1 - jd \) varies in \([i - hd + 1, r + 1 - jd]\) for \( 0 \leq v_b \leq r - i - sd \). Therefore we can rewrite the quantity in Equation \( \text{1} \) as

\[
\sum_{1 \leq w \leq s, 1 \leq z \leq s} (-1)^{w+z} \sum_{y_1 + \ldots + y_w = h} \prod_{a=1}^{w} i_a \alpha_{r-(h-y_1-\ldots-y_w)d-p_{a-z}} \prod_{b=1}^{z} x_b \alpha_{r-(j-x_1-\ldots-x_z)d-v_b}.
\] (8)

Note that the quantity \( i - hd \) appearing in Equation \( \text{8} \) under the third summation symbol varies in \([0, r - jd]\) and not in \([0, r]\) (if \( i - hd > r - jd \), then \( r - i < (j - h)d = sd \) and so \( \text{Coeff}(H^{r-i}, q^rH^{s(i-hd)}) = 0 \)).

Fix \( j \in \{1, \ldots, \lfloor \frac{r}{d} \rfloor \} \) and let \( (i_1, \ldots, i_\ell) \) and \( (u_1, \ldots, u_\ell) \) be such that

\[
0 \leq \ell \leq j, \quad i_1 + \ldots + i_\ell = j \quad \text{and} \quad 0 \leq u_\ell \leq \ldots \leq u_1 \leq r + 1 - jd.
\]

We want to count how many times the term

\[
(-1)^\ell \prod_{a=1}^{\ell} i_a \alpha_{r-(j-i_1-\ldots-i_\ell)d-u_\ell}
\] (9)

appears in

\[
b_j := \sum_{i=0}^{r} \text{Coeff}(H^i, q^iH^{r-i}) = \sum_{i=0}^{r} \text{Coeff}(H^i, q^iH^{r-i}) \text{Coeff}(H^{r-i}, q^{r-i-sd}).
\]

First of all we observe that it must be \( w + z = \ell \) and \( (x_1, \ldots, x_w, y_1, \ldots, y_w) = (i_1, \ldots, i_\ell) \).

Moreover, fixed any integer \( g \in [0, r - jd] \), if \( i - hd = g \) then in Equation \( \text{8} \) it must be

\[
z = \min\{f : u_f \leq g\} - 1 \quad \text{and} \quad w = \ell - z
\]

where if \( \{f : u_f \leq g\} = \emptyset \), we set \( z = \ell \) and \( w = \ell \).

Therefore

\[
u_z = u_1, \ldots, v_1 = u_z, p_1 = u_{z+1}, \ldots, p_w = u_\ell
\]

and

\[
x_z = i_1, \ldots, x_1 = i_z, y_1 = i_{z+1}, \ldots, y_w = i_\ell
\]

and finally

\[
h = y_1 + \ldots + y_w \quad \text{and} \quad s = x_1 + \ldots + x_z.
\]

This means that the term in Equation \( \text{8} \) appears in \( b_j \) once for every \( g \in [0, r - jd] \), and thus a total of \( r - jd + 1 \) times. This concludes the proof of the lemma. \( \square \)

### 3.4 Computation of \( E \)

We finally prove Theorem \( \text{3} \). We will distinguish two cases.

- **Case** \( |m| \leq r + L - 1 \).

  By Relation \( \text{3} \) in this case we have

  \[
  \text{Coeff}(H^{r+1}, q^rH^{r+1-jd}) = 0 \text{ for } j > 1 \text{ and } \text{Coeff}(H^{r+1}, q^{r+1-d}) = m^m.
  \]
Therefore, by Propositions 13 and 14 we have
\[ \text{Coeff}(E, q^j H^{r-j-d}) = 0 \] for \( j > 1 \) and
\[ \text{Coeff}(E, q H^{r-d}) = m^{m-1}(r + L + 1 - |m| - \chi(X)). \]
This is what we wanted to prove.

- **Case** \( |m| = r + L. \)

  Note that in this case \( d = 1. \) Relation 4 can be rewritten as
  \[ \text{Coeff}(H^{r+1}, q^j H^{r+1-j}) = \left( \begin{array}{c} r \\ j \\ -1 \end{array} \right) (m!)^{j-1} \left( m^m - \frac{m!}{j} (r + 1) \right) \]
  for \( j = 1, ..., r + 1. \) Therefore for \( j = 1, ..., r \) we have
  \[ \text{Coeff}(E, q^j H^{r-j}) = m^{-1}(j - \chi(X)) \text{Coeff}(H^{r+1}, q^j H^{r+1-j}) \]
  \[ = m^{-1}(j - \chi(X)) \left( \begin{array}{c} r \\ j \\ -1 \end{array} \right) (m!)^{j-1} \left( m^m - \frac{m!}{j} (r + 1) \right). \]
This concludes the proof.

### 4 Virtual Tevelev degrees

We now apply our computations to prove Theorem 10. We distinguish two cases again.

- **Case** \( |m| \leq r + L - 1. \)

  This case follows from [5, Proposition 5.16] and Theorem 5 above.

- **Case** \( |m| = r + L. \)

  The first step is to express \( E \) in terms of the basis \( 1, H + m!q, ..., (H + m!q)^{r}. \) This will use the following simple combinatorial lemma.

**Lemma 17.** For \( j = 2, ..., r \) the following two equalities hold:
\[ \sum_{i=1}^{j} \binom{r}{i-1} \binom{r-i}{j-i} (-1)^{j-i} = 1 \quad (10) \]
and
\[ \sum_{i=1}^{j} i \binom{r}{i-1} \binom{r-i}{j-i} (-1)^{j-i} = r + 1. \quad (11) \]

**Proof.** The proof is left to the reader. \( \square \)

**Lemma 18.** We have
\[ E = m^{-1} \chi(X)(H + m!q)^r + [m^{-1}(r + 1 - \chi(X))(m^m - m!) - m^{m-1}r]q(H + m!q)^{r-1} \]
\[ + \sum_{j=2}^{r} [m^{-1}(m!)^{j-1}(r + 1 - \chi(X))(m^m - m!)]q^j(H + m!q)^{r-j}. \]
Proof. This is an algebraic check substituting

\[ H = (H + m!q) - m!q \]

in the expression of \( E \) found in Theorem 5. Here we will deal with \( \text{Coeff}(E, q^j(H + m!q)^{r-j}) \) for \( j = 2, \ldots, r \). The cases \( j = 0, 1 \) are instead left to the reader.

Using Theorem 5 we see that for \( j = 2, \ldots, r \) the coefficient \( \text{Coeff}(E, q^j(H + m!q)^{r-j}) \) is equal to

\[
\frac{m - 1}{r!} (r - i)(m!)^{j-1} \frac{(r - i)(r + 1)}{j - i} \left( -1 \right)^{j-i} (m!)^{j-i}
\]

which we now rewrite as a sum of four terms. The first one is

\[
\frac{m - 1}{r!} (r - i)(m!)^{j-1} m^m \left( -1 \right)^{j-i} (r - i)(r + 1) \left( -1 \right)^{j-i} (m!)^{j-i}
\]

where we used Equation 10. The second one is

\[
- \frac{m - 1}{r!} (r - i)(m!)^{j-1} m^m \sum_{i=1}^{j} \left( -1 \right)^{j-i} \left( r - i \right) \left( j - i \right) \left( -1 \right)^{j-i} (m!)^{j-i}
\]

where we used Equation 10. The third term is

\[
- \frac{m - 1}{r!} (r - i)(m!)^{j-1} m^m \sum_{i=1}^{j} \left( r - i \right) \left( j - i \right) \left( -1 \right)^{j-i} (r + 1)
\]

where we used again Equation 10. Finally the last term is

\[
\frac{m - 1}{r!} (r - i)(m!)^{j-1} m^m \sum_{i=1}^{j} \left( r - i \right) \left( j - i \right) \left( -1 \right)^{j-i} (r + 1)
\]

where instead we used Equation 11. Summing everything up we obtain the desired conclusion.

Although the full expression of \( E \) might be a bit complicated, the product \((H + m!q)^r \ast E\) is quite simple.

**Corollary 19.** We have

\[
(H + m!q)^r \ast E = [m^{-1} - m^{-r-1}(m!)^r(r + 1 - \chi(X))] (H + m!q)^{2r}.
\]

Proof. Use \( r \) times Equality 4.

We can now finish the proof of Theorem 10.
Proof of Theorem 10 when \( |m| = r + L \). From Definition 8 and Equation 4, we see that
\[
P^* \star n \star E \star g \star (H + mL^r + q) \star r = \left( \sum_{i=0}^{r} b_i m^{-(k+i)m} \right) (H + qm^r + nr)^{r + r + n + r}.
\] (12)

Using Definition 8 and Equation 4, we also have
\[
P^* \star n \star E \star g \star (H + mL^r + q) \star r = \left( \sum_{i=0}^{r} P_i m^{-im} \right) n (H + qm^r + nr)^{n + r}.
\]

and so by Corollary 19
\[
P^* \star E \star (H + mL^r + q) \star r = \left( \sum_{i=0}^{r} P_i m^{-im} \right) n (m^{-1} - m^{-rm-1}(m!)^r (r + 1 - \chi(X)))^g (H + mL^r + nr)^{r + gr + nr}.
\] (13)

The theorem follows by comparing Equation 12 and Equation 13. \( \square \)

5 An algorithm for the calculation of the coefficients \( P_i \)

In this final section we propose a method to compute the coefficients \( P_i \) appearing in Definition 8. In this way, up to implementing the algorithm with a computer, all the virtual Tevelev degrees of \( X \) can be explicitly calculated.

It is possible that our is known to the experts, but we preferred to include it anyway for completeness.

5.1 Recursion for genus 0 two-pointed Hyperplane Gromov-Witten invariants

Proposition 11 reduces the computation of the \( P_i \)s to the computation of genus 0 two-pointed Hyperplane Gromov Witten invariants of \( X \). These invariants satisfies a recursion involving more general integrals which we now recall.

5.1.1 The recursion

For \( g \geq 0 \), \( k > 0 \) and \( n > 0 \) the gravitational descendant invariants of \( X \) are defined by:
\[
\langle \tau_{a_1}(\gamma_1), ..., \tau_{a_n}(\gamma_n) \rangle_{g,k} := \int_{\overline{M}_{g,n}(X,kl)} \text{ev}_1^{*}(\gamma_1) \cup \psi_1^{a_1} \cup ... \cup \text{ev}_n^{*}(\gamma_n) \cup \psi_1^{a_n}
\]
where \( \gamma_1, ..., \gamma_n \in H^*(X) \) and \( \psi_i = c_1(L_i) \in H^2(\overline{M}_{g,n}(X,kl)) \) is the first Chern class of the cotangent line
\[
L_i \big|_{f:C(p_1,...,p_n) \rightarrow X} = (T_{p_i}C)^\vee
\]
for \( i = 1, ..., n \).

We start with a monodromy result.
Lemma 20. For any $\gamma \in H^*(X)^{\text{prim}}, \gamma_1, ..., \gamma_n \in H^*(X)^{\text{res}}$ (with $n \geq 0$), $a_1, ..., a_n \in \mathbb{Z}_{\geq 0}$ and $k > 0$ we have

$$\langle \tau_{a_1}(\gamma_1), ..., \tau_{a_n}(\gamma_n), \gamma \rangle^{X}_{0,k} = 0.$$ 

Proof. The proof is a monodromy argument. Let

$$U = \prod_{i=1}^{L} \mathbb{P}(H^0(\mathbb{P}^{r+L}, \mathcal{O}(m_i)))$$

be the open subscheme parametrizing smooth complete intersection in $\mathbb{P}^{r+L}$ of dimension $r$ and degree $m$. Call $V = V^{\text{prim}} \oplus V^{\text{res}}$ where $V^{\text{prim}} = H^*(X)^{\text{prim}} \otimes_{\mathbb{Q}} \mathbb{R}$ and $V^{\text{res}} = H^*(X)^{\text{res}} \otimes_{\mathbb{Q}} \mathbb{R}$, and

$$\rho : \pi_1(U, u) \to \text{Aut}(V).$$

the monodromy homomorphism (here $u \in U$ is the point corresponding to $X$). The homomorphism $\rho$ preserves the decomposition $V = V^{\text{prim}} \oplus V^{\text{res}}$ and actually its invariant subspace is exactly $V^{\text{res}}$. Let $G \subseteq \text{GL}(V^{\text{prim}})$ be the algebraic monodromy group defined as the Zariski closure of the image of $\pi_1(U, u) \to \text{Aut}(V^{\text{prim}})$. The lemma will follow from the following two standard facts:

- the invariance under deformations of $X$ in Gromov Witten theory tells us that for any $\alpha \in \pi_1(U, u)$ we have:

$$\langle \tau_{a_1}(\alpha, \gamma_1), ..., \tau_{a_n}(\alpha, \gamma_n), \alpha, \gamma \rangle^{X}_{0,k} = \langle \tau_{a_1}(\gamma_1), ..., \tau_{a_n}(\gamma_n), \gamma \rangle^{X}_{0,k};$$

- the intersection form $Q$ on $V^{\text{prim}}$ is preserved by the monodromy action. When $r$ is odd, $Q$ is a non-degenerate skew-symmetric bilinear form, it follows that in this case $\dim(V^{\text{prim}})$ is even and that $G \subseteq \text{Sp}(V^{\text{prim}})$. When instead $r$ is even, $Q$ is a non-degenerate symmetric bilinear form and we have $G \subseteq \text{O}(V^{\text{prim}})$. Since for us $r \geq 3$, by [8, Theorem 4.4.1] (see also [1, Proposition 4.2]), the previous inclusions are actually equalities except for the case when $r$ is even and $m = (2, 2)$. In this latter case, $\dim(V^{\text{prim}}) = r + 3$ and $G$ is the Weil group $W$ of $D_{r+3}$.

Since $-\text{Id} \in \text{Sp}(V^{\text{prim}})$ and $-\text{Id} \in \text{O}(V^{\text{prim}})$ the proof is complete in all cases except for the case $r$ even and $m = (2, 2)$. In this case, note that if $L : V^{\text{prim}} \to V^{\text{prim}}$ is any $\mathbb{R}$-linear map invariant under $\mathcal{W}$ then it must be $L = 0$ (reason: if $\Phi \subseteq V^{\text{prim}}$ is the root system corresponding to $D_{r+3}$ then for all $v \in \Phi$ the reflection $r_v$ along the hyperplane $v^+ \subseteq \mathcal{W}$ and sends $v$ to $-v$, thus $L(v) = L(-v) = -L(v)$, from which $L(v) = 0$. Since $\text{Span}_{\mathbb{R}}(\Phi) = V^{\text{prim}}$ we are done). To conclude the proof of the lemma, apply this observation with $L = \langle \tau_{a_1}(\gamma_1), ..., \tau_{a_n}(\gamma_n), -\rangle^{X}_{0,k}$. 

$$\square$$

Proposition 21. Let $i, a \geq 0$ and $j, k > 0$ be integers satisfying

$$i + j + a = \text{vdim}(\overline{M}_{0,2}(X, kL)).$$

Then we have

$$\langle \tau_a(H^i), H^j \rangle^{X}_{0,k} = \langle \tau_a(H^{i+1}), H^{j-1} \rangle^{X}_{0,k} + k \langle \tau_{a+1}(H^i), H^{j-1} \rangle^{X}_{0,k} - \sum_{\ell=1}^{k-1} m^{-1} \ell \langle \tau_a(H^i), H^{d+r-1-i-a}_{\ell} \rangle^{X}_{0,k} \langle \tau_{a-1}(H^{j-1}), H^{(k-\ell)d+r-j} \rangle^{X}_{0,k}. $$

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Proof. An application of \cite{12} Corollary 1 and the splitting principle in Gromov Witten theory yields

\begin{equation}
\langle \tau_a(H^i), H^j \rangle_0^X = \langle \tau_a(H^{i+1}), H^{j-1} \rangle_0^X + k \langle \tau_{a+1}(H^i), H^{j-1} \rangle_0^X
- \sum_{\ell=1}^{k-1} \sum_{j=0}^{N} \ell \langle \tau_a(H^i), \gamma_j \rangle_0^X, \gamma_j(\gamma^j_0, k-\ell)
\end{equation}

where as always \{\gamma_j\}_{j=0}^1 \text{ is any homogeneous basis of } H^*(X) \text{ with } \gamma_0 = 1 \text{ and } \gamma_N = P.

Finally, apply Lemma \ref{201} to conclude the proof.

5.1.2 The base case
Consider the recursion of Proposition \ref{21}. In each two pointed Gromov Witten integral on the right-hand side, either the quantity \(j\) decreased or the quantity \(k\) decreased (when compared to those appearing in the left-hand side). Note also that when \(k = 1\), the recursion becomes simply

\begin{equation}
\langle \tau_a(H^i), H^j \rangle_0^X = \langle \tau_a(H^{i+1}), H^{j-1} \rangle_0^X + \langle \tau_{a+1}(H^i), H^{j-1} \rangle_0^X.
\end{equation}

So, when \(k = 1\), \(k\) stabilizes while \(j\) keeps going down. It follows that the recursion completely determines all the integrals

\begin{equation}
\langle \tau_a(H^i), H^j \rangle_0^X \text{ for } a, i, j \geq 0 \text{ such that } a + i + j = vdim(M_{0,2}(X, kL))
\end{equation}

once the integrals \(\langle \tau_a(H^i), 1 \rangle_0^X \) are given for all \(a, i \geq 0\) and \(k > 0\). These last invariants are indeed known as the next proposition shows.

Proposition 22. Let \(a, i \geq 0\) and \(k > 0\) be integers such that

\begin{equation}
a + i = vdim(M_{0,2}(X, kL)).
\end{equation}

Then

- for \(|m| \leq r + L - 1\) and \(i = 0, ..., r\) we have

\begin{equation}
\langle \tau_{r+kd-1-i}(H^i), 1 \rangle_0^X = \text{Coeff} \left( \frac{\prod_{j=1}^{L} \prod_{h=0}^{km_j} (m_j x + \ell)}{\prod_{\ell=1}^{k} (x + \ell)^{r+L-1}} \right)
\end{equation}

- for \(|m| = r + L\) and \(i = 0, ..., r\) we have

\begin{equation}
\langle \tau_{r+k-1-i}(H^i), 1 \rangle_0^X = \sum_{h=0}^{k} \frac{(-m!)^{k-h}}{(k-h)!} \text{Coeff} \left( \frac{\prod_{j=1}^{L} \prod_{h=0}^{hm_j} (m_j x + \ell)}{\prod_{\ell=1}^{h} (x + \ell)^{r+L-1}} \right)
\end{equation}

where in both cases the coefficient of \(x^{r+L-i}\) is meant to be the coefficient of the Taylor expansion in \(x\) at 0.

Proof. This is just a way of rephrasing \cite{4} Theorem 4.2 and Theorem 4.17. Note that in \cite{4} Theorem 4.17 there is typo: in their notation, their index \(m\) in the product appearing in the numerator should range from 0 to \(d_{l_j}\), instead of from 1 to \(d\). \hfill \(\square\)
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