Relativistic quantum oscillators in the global monopole spacetime

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Abstract

We investigated the effects of the global monopole spacetime on the Dirac and Klein-Gordon relativistic quantum oscillators. In order to do this, we solve the Dirac and Klein-Gordon equations analytically and discuss the influence of this background which is characterized by the curvature of the spacetime on the energy profiles of these oscillators. In addition, we introduce a hard-wall potential and, for a particular case, determine the energy spectrum for relativistic quantum oscillators in this background.

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I. INTRODUCTION

Grand unified theories predict that, in the early universe, as consequence of vacuum symmetry breaking phase transitions, topological defects could be produced [1, 2] and the influence of these objects have been vastly investigated in various branches of physics [3–10]. In the context of condensed matter physics and gravitation, linear defects can be associated with the presence of torsion (dislocations) and curvature (disclinations) [5, 11]. Examples of well-known topological defects are the domain wall [12], cosmic string [13–15] and global monopole [16]. In particular, the global monopole has been studied in several cases: \(f(R)\) theory [17], in the scalar self-energy for a charged particle [18], in nonrelativistic scattering [19], in the vacuum polarization for a massless spin-\(\frac{1}{2}\) field [20] and a massless scalar field [21]. Also considering the presence of Wu-Yang magnetic monopole [22] and in the gravitating magnetic monopole [23].

The effects of the global monopole also have been studied in the context of quantum mechanics. Considering the nonrelativistic case, the Kratzer potential has been analyzed. In this case, the energy spectrum of the system is influenced by the defect [24], in a charged particle-magnetic monopole scattering [25] and on the harmonic oscillator [26]. For the relativistic case, there are studies on the hydrogen atom and pionic atom [27], on the exact solutions of scalar bosons in the presence of the Aharonov-Bohm and Coulomb potentials [28]. In addition, the exact solutions of the Klein Gordon equation in the presence of a dyon, magnetic flux and scalar potential [29] also have been analyzed in the nonrelativistic context. However, one point that has not been dealt with in the literature is the effects of the global monopole on the Dirac [30] and Klein-Gordon [31] relativistic quantum oscillators. Therefore, in order to obtain solutions of bound states, in this paper, we deal with a fermionic field and a scalar field subject to the Dirac oscillator (DO) and the Klein-Gordon oscillator (KGO), respectively, in the global monopole spacetime. We also consider the presence of a hard-wall potential, that, for a particular case, the relativistic energy spectrum can be obtained.

Proposed by Monshisky and Szczepaniak, the relativistic quantum oscillator model for the spin-\(\frac{1}{2}\) field, which it was known in the literature as DO [30], is derived from a nonminimum coupling in the Dirac equation, which preserves linearity in both the linear momentum and coordinates. In addition, at the non-relativistic limit, the modified Dirac equation is simplified in the Schrödinger equation for the simple harmonic oscillator, however, added with a strong spin-orbit coupling term. It is noteworthy that the DO has been extensively studied, for example, in (1 + 1)-dimensions [32–34], in (2 + 1)-dimensions [35, 36], in the cosmic string spacetime [37, 39], in an Aharonov-Casher...
system \[40\], in a rotating frame of reference \[41\], in thermodynamic properties \[42, 43\] and in the context of gravity’s rainbow \[44\].

Inspired by the DO \[39\], Bruce and Minning have proposed a relativistic quantum oscillator model for the scalar field which it was known in the literature as the KGO \[31\] that, in the non-relativistic limit, is reduced to the oscillator described by the Schrödinger equation \[45\]. The KGO has been studied by a \(\mathcal{PT}\)-symmetric Hamiltonian \[46\], in noncommutative space \[47, 48\], in spacetime with cosmic string \[49\], in a spacetime with torsion \[50, 51\], in a Kaluza-Klein theory \[52\], with noninertial effects \[53\], under effects of linear and Coulomb-type central potentials \[54–56\], in thermodynamic properties \[57, 58\] and in possible scenarios of Lorentz symmetry violation \[59, 60\].

The structure of this paper is organized as follows: in the Sec. \(\text{II}\), we obtain the relativistic energy levels of a spin-\(\frac{1}{2}\) fermionic field that interacts with the DO in the spacetime with a pointlike global monopole defect; in the Sec. \(\text{III}\), we analyze a scalar field subject to the KGO in the spacetime with a pointlike global monopole defect and obtain the energy spectrum; in the Sec. \(\text{IV}\), we extend our initial discussions to the presence of a hard-wall confining potential; and in the Sec. \(\text{V}\) we present our conclusions.

II. DO IN THE GLOBAL MONOPOLE SPACETIME

The global monopole spacetime is characterized by a line element that possesses a parameter related to the deficit angle \(\delta \Omega = 8\pi^2 G\eta_0^2\), with \(\eta_0\) being the dimensionless volumetric mass density of the pointlike global monopole and \(G\) is the gravitational Newton constant, and with the scalar curvature \(R = R^\mu_\mu = 2\frac{(1-\alpha^2)}{r^2}\). By working with the units \(c = \hbar = 1\) and the signature \((-+,+,+,+\)), the line element of the pointlike global monopole spacetime is written in the form \[27\]:

\[
ds^2 = -dt^2 + \frac{dr^2}{\alpha^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,
\]

where \(\alpha = 1 - 8\pi^2 G\eta_0^2 < 1\).

Henceforth, let us study the effects of the DO on the fermionic field in the global monopole spacetime. In curved spacetime, the Dirac equation is written in the form \[37\]

\[
[i\gamma^\mu(x)(\partial_\mu + \Gamma_\mu(x)) - m]\psi(x,t) = 0,
\]

where \(\Gamma_\mu(x)\) is the spinorial connection component given in terms of the tetrad components and of the Christoffel symbols

\[
\Gamma_\mu = \frac{1}{4} \gamma^{(a)} \gamma^{(b)} e_{(a)}^\nu (\partial_\mu e_{\nu(b)} + \Gamma^\lambda_{\mu\nu} e_{\lambda(b)}).
\]
The matrices $\gamma^\mu(x)$ are the generalized Dirac matrices for the background given by Eq. (1) in terms of the standard Dirac matrices, that is, of the Minkowski spacetime Dirac matrices $\gamma^{(a)}$

$$\gamma^\mu(x) = e^\mu_{(a)}(x)\gamma^{(a)},$$  \hspace{1cm} (4)

which obey the relation of anticomutation $\gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) = 2g^{\mu\nu}(x)$.

The tetrad components $e^\mu_{(a)}(x)$ obey the relation

$$g^{\mu\nu}(x) = e^\mu_{(a)}(x)e^\nu_{(b)}(x)\eta^{(a)(b)},$$  \hspace{1cm} (5)

where $\mu, \nu = 0, 1, 2, 3$ are the indices corresponding to the curved spacetime, while $(a), (b) = 0, 1, 2, 3$ are indices corresponding to the Minkowski spacetime. We consider the following choice for the tetrad base in the pointlike global monopole spacetime $[27]$

$$e^\mu_{(a)}(x) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha \sin \theta \cos \phi & \frac{\cos \theta \cos \phi}{r} & -\frac{\sin \phi}{r \sin \theta} \\
0 & \alpha \sin \theta \sin \phi & \frac{\cos \theta \sin \phi}{r} & \frac{\cos \phi}{r \sin \theta} \\
0 & \alpha \cos \theta & -\frac{\sin \theta}{r} & 0
\end{bmatrix}. \hspace{1cm} (6)$$

In this representation the matrices $\gamma^\mu(x)$ obey the relations

$$\gamma^0 = \gamma^t; \quad \gamma^1 = \alpha \gamma^r \hat{r} = \alpha \gamma^{(r)}; \quad \gamma^2 = \frac{1}{r} \gamma^\theta \hat{\theta} = \frac{1}{r} \gamma^{(\theta)}; \quad \gamma^3 = \frac{\gamma^\phi}{r \sin \theta} = \frac{\gamma^{(\phi)}}{r \sin \theta},$$  \hspace{1cm} (7)

where $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ are the spherical versors. In addition, the nonzero spinor connections are

$$\Gamma_2 = \frac{(\alpha - 1)}{2} [\gamma^1 \gamma^3 \cos \varphi + \gamma^2 \gamma^3 \sin \varphi];$$ \hspace{1cm} $\Gamma_3 = -\frac{(\alpha - 1)}{2} [\gamma^2 \sin \theta + \gamma^1 \gamma^3 \cos \theta \sin \varphi - \gamma^2 \gamma^3 \cot \theta \cos \varphi] \sin \theta,$$

where

$$\gamma^2 \Gamma_2 + \gamma^3 \Gamma_3 = \frac{(\alpha - 1)}{r} \gamma^r \hat{r}. \hspace{1cm} (8)$$

Given all this, we can now investigate the effects of the DO on a fermionic field in the pointlike global monopole spacetime. To introduce the DO in the Dirac equation (2) we must use the following coupling $[30, 37]$: $\partial_r \rightarrow \partial_r + m \omega \beta r$, where $\omega$ is the angular frequency of the DO. Then, the Dirac equation (2) for the spacetime under consideration given by the Eq. (1), with the Eqs. (6), (7), (8) and (9), becomes

$$i\gamma^{(t)} \partial_t + i\gamma^{(r)} \left( \alpha \partial_r + \frac{(\alpha - 1)}{r} + m \omega \alpha \beta r \right) + i\gamma^{(\theta)} \frac{\partial}{r \sin \theta} + i\gamma^{(\phi)} \partial r - m \right] \psi = 0, \hspace{1cm} (9)$$
with
\[
\gamma(t) = \begin{bmatrix}
I_2 & 0 \\
0 & -I_2
\end{bmatrix}; \quad \gamma(i) = \begin{bmatrix}
0 & \sigma^i \\
-\sigma^i & 0
\end{bmatrix},
\]
(11)
where \(I_2\) is order two identity matrix and \(\sigma^i\) are Pauli matrices.

The complete set of solutions for Eq. (10) is \([61, 62]\)
\[
\psi(\vec{r},t) = \frac{1}{r} \begin{bmatrix}
if(r)\Phi_{j,m}(\theta,\varphi) \\
g(r)(\vec{\sigma},\dot{r})\Phi_{j,m}(\theta,\varphi)
\end{bmatrix} e^{-i\mathcal{E}t},
\]
(12)
where \(\Phi_{j,m}\) are the spinor spherical harmonics, \(f(r)\) and \(g(r)\) are radial wave spinor functions and \(\mathcal{E}\) is the energy of the system. Note that the Eq. (11) presents a well defined parity under the transformation \(\vec{r} \rightarrow -\vec{r}\) \([61]\). Hence, by substituting the Eq. (12) into Eq. (10), we obtain radial differential equations
\[
(\mathcal{E} - m)f + \alpha \frac{dg}{dr} - \frac{\kappa}{r} g - m\omega r g = 0; \quad (13a)
\]
\[
(\mathcal{E} + m)g - \alpha \frac{df}{dr} - \frac{\kappa}{r} f - m\omega r f = 0, \quad (13b)
\]
where we use the definitions
\[
(\vec{\sigma},\vec{A})(\vec{\sigma},\vec{B}) = \vec{A}\vec{B} + i\vec{\sigma}.(\vec{A} \times \vec{B}); \quad \vec{\sigma}\vec{L}\Phi_{j,m} = -(1 + \kappa)\Phi_{j,m}; \quad \vec{L} = -i \left( \frac{\partial}{\partial \theta} - \frac{\dot{\theta}}{\sin \theta} \frac{\partial}{\partial \varphi} \right),
\]
(14)
with \(\kappa = \mp j + \frac{1}{2}, j = l \pm \frac{1}{2}\) and \(l = 0,1,2,\ldots\).

Multiplying (13a) by \((\mathcal{E} + m)\) and using (13b) we obtain
\[
\frac{d^2f}{dr^2} - \frac{\kappa(\kappa + \alpha)}{\alpha^2 r^2} f - m^2 \omega^2 r^2 f + \left[\frac{\mathcal{E}^2 - m^2 + m\omega(\alpha - 2\kappa)}{\alpha^2}\right] f = 0. \quad (15)
\]
With the purpose of solving this radial equation, let us write
\[
f(r) = \frac{F(r)}{\sqrt{r}},
\]
(16)
then, we obtain the following equation for the function \(F(r)\):\[
\frac{d^2F}{dr^2} - \frac{1}{r} \frac{dF}{dr} + \frac{3}{4r^2} F - \frac{\kappa(\kappa + \alpha)}{\alpha^2 r^2} F - m^2 \omega^2 r^2 F + \left[\frac{\mathcal{E}^2 - m^2 + m\omega(\alpha - 2\kappa)}{\alpha^2}\right] F = 0. \quad (17)
\]
We proceed with a change of variables given by \(s = m\omega r^2\), and thus we obtain:
\[
\frac{d^2F}{ds^2} + \frac{(1 - 4\mu^2)}{4s^2} F + \frac{\nu}{s} F - \frac{1}{4} F = 0,
\]
(18)
where we define new parameters given by
\[ \mu = \sqrt{\alpha^2 + 4\kappa(\kappa + \alpha)} / 4\alpha; \quad \nu = \frac{E^2 - m^2 + m\omega(\alpha - 2\kappa)}{4m\omega^2}. \] (19)

Hence, the Eq. (18) is the well-known Whittaker differential equation [63] and \( F(s) \) is the Whittaker function which can be written in terms of confluent hypergeometric function \( _1F_1(s) \) [63, 64]

\[ F(s) = s^{1+\mu}e^{-\frac{s}{2}} _1F_1 \left( \mu - \nu + \frac{1}{2}, 2\mu + 1; s \right). \] (20)

It is well known that the confluent hypergeometric series becomes a polynomial of degree \( n \) by imposing that \( \mu - \nu + \frac{1}{2} = -n \), where \( n = 0, 1, 2, 3, \ldots \). With this condition, we obtain

\[ \mathcal{E}_{l,n} = \pm \sqrt{m^2 + 4m\omega^2 \left( n + \frac{\sqrt{\alpha^2 + 4\kappa(\kappa + \alpha)} + 2\kappa}{4\alpha} + \frac{1}{4} \right)}. \] (21)

The Eq. (21) gives us the relativistic energy spectrum of a spin-\( \frac{1}{2} \) fermionic field subject to the DO in a pointlike global monopole spacetime. We can note that the spacetime topology influences the relativistic energy levels of the system through the presence of the parameter associated with the topological defect responsible by the curvature of the spacetime, that is, \( \alpha \). By making \( \alpha \to 1 \), we recover the energy spectrum of a spin-\( \frac{1}{2} \) fermionic field subject to the DO in \((3+1)\)-dimensions in the Minkowski spacetime.

In Fig. 1 we have plotted the profile of the energy (21) as function of \( \omega/m \). In the left plot, we consider different values of the parameter \( \alpha \) and \( n = 1 \). We note that the influence of the parameter \( \alpha \) is to reduce the energy of the oscillator. In the right plot we consider \( \alpha = 0.5 \) and different energy levels.

The set of autofunctions are given by
\[ f_{l,n}(s) = (m\omega)^{\frac{1}{2}} s^{\mu + \frac{1}{2}} e^{-\frac{s}{2}} _1F_1 \left( \mu - \nu + \frac{1}{2}, 2\mu + 1; s \right); \]
\[ g_{l,n}(s) = \frac{\sqrt{m\omega}}{(\mathcal{E}_{l,n} + m)} \left( 2\alpha\sqrt{s} \frac{d}{ds} + \kappa \sqrt{s} + \alpha\sqrt{s} \right) f_{l,n}(s), \] (22)
where we can see that these autofunctions are influenced by the spacetime topology through the presence of the parameter associated with the topological defect (pointlike global monopole).

III. KGO IN THE GLOBAL MONOPOLE SPACETIME

In this section, we investigate the scalar field under effects of the KGO in the pointlike global monopole spacetime. The KGO is described by introducing a coupling into the Klein-Gordon
Figure 1: Energy profile as function of $\omega/m$ for $n = 1$ (left plot) and $\alpha = 0.5$ (right plot). The full lines are the positive energy while the dashed lines are for the negative energy. In both plots we consider $\kappa = 1/2$.

equation as $\hat{p}_\mu \rightarrow \hat{p}_\mu + im\omega X_\mu$, where $m$ is the rest mass of the scalar field, $\omega$ is the angular frequency of the KGO and $X_\mu = (0, r, 0, 0)$ [50][52]. The Klein-Gordon equation in the spacetime with a pointlike global monopole (1) can be written as [52]

$$\frac{1}{\sqrt{-g}}(\partial_\mu + m\omega X_\mu)(\sqrt{-gg^{\mu\nu}})(\partial_\nu - m\omega X_\nu)\phi - \xi R\phi - m^2\phi = 0,$$

(23)

where $g = \det(g_{\mu\nu})$, $g^{\mu\nu}$ is inverse metric tensor, $\xi$ is an arbitrary coupling constant and $R$ the curvature scalar already reported in the previous section. In this way, from the Eq. (1), the Klein-Gordon equation (23) becomes

$$-\partial_t^2 \phi + \frac{\alpha^2}{r^2}(\partial_r + m\omega r)(r^2\partial_r \phi - m\omega r^3 \phi) + \frac{1}{r^2 \sin \theta} \partial_\theta (\sin \theta \partial_\phi \phi) + \frac{1}{r^2 \sin^2 \theta} \partial^2_\phi \phi$$

$$- \frac{2\xi(1 - \alpha^2)}{r^2} \phi - m^2\phi = 0.$$  

(24)

The solution to the Eq. (24) can be written in the form

$$\phi(r, \theta, \varphi, t) = e^{-i\xi t} R(r) Y_{l,m}(\theta, \varphi),$$

(25)

where $Y_{l,m}(\theta, \varphi)$ are the spherical harmonics and $f(r)$ is a radial wave function. Then, by substituting the solution (25) into the Eq. (24), we have

$$\alpha^2 \frac{a^2}{r^2 R} \left( \frac{d}{dr} + m\omega r \right) \left( r^2 \frac{dR}{dr} - m\omega r^3 R \right) - \frac{2\xi}{r^2} (1 - \alpha^2) + (\xi^2 - m^2)$$

$$- \frac{1}{r^2} \left[ - \frac{1}{\sin^2 \theta Y_{l,m}} \partial_\theta \left( \sin \theta \frac{\partial Y_{l,m}}{\partial \theta} \right) - \frac{1}{\sin \theta Y_{l,m}} \frac{\partial^2 Y_{l,m}}{\partial \varphi^2} \right] = 0,$$

(26)
\[
\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} - \frac{[2\xi(1 - \alpha^2) + l(l + 1)]}{\alpha^2 r^2} R - m^2 \omega^2 r^2 R + \left(\frac{\mathcal{E}^2 - m^2 - 3m\omega^2}{\alpha^2}\right) R = 0, \tag{27}
\]

where we use the definition \[62\]

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta}\right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] Y_{l,m}(\theta, \varphi) = -l(l+1) Y_{l,m}(\theta, \varphi). \tag{28}
\]

With the purpose of solving the radial differential equation (27), let us write

\[
R(r) = \frac{\bar{W}(r)}{r^\frac{3}{2}}, \tag{29}
\]

then, we obtain the following equation for the function \(W(r)\):

\[
\frac{d^2 \bar{W}}{dr^2} - \frac{1}{r} \frac{d\bar{W}}{dr} + \frac{3\bar{W}}{4r^2} - \frac{[2\xi(1 - \alpha^2) + l(l + 1)]}{\alpha^2 r^2} \bar{W} - m^2 \omega^2 r^2 \bar{W} + \left(\frac{\mathcal{E}^2 - m^2 - 3m\omega^2}{\alpha^2}\right) \bar{W} = 0. \tag{30}
\]

Let us define \(s = m\omega r^2\), then the Eq. (30) becomes

\[
\frac{d^2 \tilde{W}}{ds^2} + \frac{1 - 4\bar{\mu}^2}{4s^2} \tilde{W} + \frac{\bar{\nu}}{s} \tilde{W} - \frac{1}{4} \tilde{W} = 0, \tag{31}
\]

where we have defined the parameters

\[
\bar{\mu} = \sqrt{\frac{\alpha^2 + 8\xi(1 - \alpha^2) + 4l(l+1)}{4\alpha}}; \quad \bar{\nu} = \frac{\mathcal{E}^2 - m^2 - 3m\omega^2}{4m\omega^2}. \tag{32}
\]

Note that the Eq. (31) is also a Whittaker differential equation [63]. Then, by following the steps from Eqs. (20) to (21), we obtain

\[
\mathcal{E}_{l,n} = \pm \sqrt{m^2 + 4m\omega^2 \left(n + \frac{\alpha^2 + 8\xi(1 - \alpha^2) + 4l(l+1)}{4\alpha} + \frac{5}{4}\right)}. \tag{33}
\]

Then, the Eq. (33) is the relativistic energy spectrum that stems from the interaction of the scalar field with the KGO in the spacetime with a pointlike global monopole. As in the case of the DO, the presence of the parameter \(\alpha\) into the Eq. (33), modifies the relativistic energy levels of the system. By making \(\alpha \to 1\), we recover the relativistic energy levels of a scalar field subject to the KGO in \((3 + 1)\)-dimensions in the Minkowski spacetime. In the Fig. 2 we have plotted the energy profiles of the KGO as function of \(\omega/m\) for different values of the parameter \(\alpha\) (left plot) and different energy levels (right plot). As we can see, the presence of \(\alpha\) reduces the energy of the KGO.

The autofunctions, then, are given by

\[
R_{l,n}(s) = (m\omega)^{\frac{3}{2}} s^{\frac{\mu}{2} - \frac{1}{4}} e^{-\frac{1}{2}s} F_1 \left(\bar{\mu} - \bar{\nu} + \frac{1}{2}, 2\bar{\mu} + 1; s\right), \tag{34}
\]

that is, the system autofunctions are also influenced by the topology of the spacetime through the presence of the parameter associated with the topological defect, \(\alpha\), in the structure of the Eq. (34).
Figure 2: Energy profile of the KGO as function of $\omega/m$. In the left plot we have different values of $\alpha$ and $n = 1$, while in the right plot we consider $\alpha = 0.5$ and different energy levels. Also $l = 0$ and $\xi = 0$ in both plots. The full and dashed curves represent the positive and negative energies, respectively.

IV. EFFECTS OF A HARD-WALL POTENTIAL

Let us restrict the motion of the relativistic spin-$\frac{1}{2}$ fermionic and scalar fields to a region where a hard-wall confining potential is present. The hard-wall potential has been studied in rotating effects on the scalar field [10], in the KGO under effects of linear topological defects [50, 53], in noninertial effects on a nonrelativistic Dirac particle [65], in a Landau-Aharonov-Casher system [66], on a Dirac neutral particle in analogous way to a quantum dot [67], on the harmonic oscillator in an elastic medium with a spiral dislocation [68], on persistent currents for a moving neutral particle with no permanent electric dipole moment [69] and in a Landau-type quantization from a Lorentz symmetry violation background with crossed electric and magnetic fields [70].

A. DO

Here we confine the spin-$\frac{1}{2}$ fermionic field subject to the DO in the global monopole spacetime. In order to do that, we impose that the fermionic field obeys the MIT bag boundary condition at a finite radius $s_0$:

\[
(1 + in_\mu \gamma^\mu)\psi|_{s=s_0} = 0,
\]  

(35)

with $n_\mu$ is the outward oriented normal (with respect to the region under consideration) to the boundary. As we shall consider the region inside the bag, we have that $n_\mu = -\delta_\mu^1 / \alpha$. Taking into
consideration \cite{12}, the Eq. (35) becomes
\[ f_{n,l}(s_0) - g_{n,l}(s_0) = 0, \] (36)
and by using \cite{22} one finds
\[
\frac{2\alpha\sqrt{m\omega s_0}}{E + m} \left( \frac{\mu - \nu - 1/2}{2\mu + 1} \right) F_1 \left( \mu - \nu + \frac{3}{2}, 2\mu + 2, s_0 \right) + \left\{ \frac{\sqrt{m\omega}}{E + m} \left[ \frac{2\alpha}{s_0} (\mu + 1/4) + \frac{\kappa}{s_0^{1/2}} \right] - 1 \right\} F_1 \left( \mu - \nu + \frac{1}{2}, 2\mu + 1, s_0 \right) = 0 \] (37)

Let us consider \( \nu \gg 1 \). In this limit we can use the formula \cite{71}
\[
\lim_{a \to \infty} F_1(a, b; z) = \Gamma(b) \left( z \left( \frac{b}{2} - a \right) \right)^{1/4 - b/2} \pi^{-1/2} e^{z/2} \cos \left[ \sqrt{z} (2b - 4a) - \frac{b}{2} \pi + \frac{\pi}{4} \right], \] (38)
for the parameters \( b \) and \( z \) fixed. Taking into account the previous equation, we have for the confluent hypergeometric functions
\[
\lim_{\nu \to \infty} F_1 \left( \mu - \nu + \frac{3}{2}, 2\mu + 2, s_0 \right) \approx \frac{\Gamma(2\mu + 2)}{\sqrt{\pi}} \frac{\cos[2\sqrt{s_0(\nu - 1/2)} - \mu\pi - 3\pi/4]}{[s_0(\nu - 1/2)]^{\mu+3/4}} e^{s_0/2}, \] (39)
and
\[
\lim_{\nu \to \infty} F_1 \left( \mu - \nu + \frac{1}{2}, 2\mu + 1, s_0 \right) \approx \frac{\Gamma(2\mu + 1)}{\sqrt{\pi}} \frac{\cos[2\sqrt{s_0\nu} - \mu\pi - \pi/4]}{(s_0\nu)^{\mu+1/4}} e^{s_0/2}. \] (40)
By using these asymptotic forms for the confluent hypergeometric functions in Eq. (37), the main contribution reads
\[
\frac{2\alpha\sqrt{m\omega}}{E + m} \cos[2\sqrt{s_0\nu} - \mu\pi - 3\pi/4] \approx 0, \] (41)
which is satisfied if
\[
2\sqrt{s_0\nu} - \mu\pi - 3\pi/4 \approx (n + 1/2)\pi, \] (42)
with \( n \) being an integer number. Now, by using \cite{19} and noting that \( s_0 = m\omega r_0^2 \), the energy levels are written as
\[
E_{n,l} \approx \pm \sqrt{m^2 - m\omega\alpha(\alpha - 2\kappa) + \frac{\alpha^2\pi^2}{r_0^2} \left( n + \sqrt{\frac{\alpha^2 + 4\kappa(\kappa + \alpha)}{4\alpha} + \frac{5}{4}} \right)^2}. \] (43)
As expected, the energy spectrum of the fermionic field is modified by the confining potential. The Eq. (43) represents the energy spectrum of the DO subject to the hard-wall potential in the global monopole spacetime. We can see from the comparison between of Eqs. (21) and (43) that the presence of the hard-wall potential modifies the energy levels of the DO. We can also observe that
the energy spectrum is influenced by the spacetime topology through the presence of the parameter associated with the topological defect in the energy levels. We can note that by taking $\omega \to 0$ into Eq. (43) we obtain the energy spectrum of a Dirac field under effects of a hard-wall confining potential in the global monopole spacetime. In addition, by making $\alpha \to 1$, we recover the energy spectrum of the DO subject to the hard wall potential in Minkowski spacetime. We have plotted the behavior of the positive energy levels of this configuration in Fig. 3 as function $\omega/m$ (left plot) and the parameter $\alpha$ (right plot).

Figure 3: Positive energy profile of the DO subject to the MIT bag boundary condition as function of $\omega/m$ (left plot) and $\alpha$ (right plot). In the left plot we have different values of $\alpha$ and $n = 1$, while in the right plot we consider $\omega/m = 2.5$ and different energy levels. In both plots we have $m r_0 = 2.5$ and $\kappa = 5/2$.

B. KGO

In this subsection, we confine the scalar field subject to the KGO in the pointlike global monopole spacetime (1) to a wall-rigid potential. First, we will consider the Dirichlet boundary condition, where the scalar field vanishes at some fixed point $s_0$:

$$R(s_0) = 0.$$  \hspace{1cm} (44)

Considering the radial wave function (34), and the limit where $\nu \gg 1$, and proceeding in a similar way done in the previous section, we find

$$\mathcal{E}_{l,n} \approx \pm \left[ m^2 + 3 m \omega \alpha^2 + \frac{\alpha^2 \pi^2}{r_0^2} \left( n + \sqrt{\frac{\alpha^2 + 8 \xi (1 - \alpha^2)}{4 \alpha}} + \frac{4 l (l + 1)}{3} \right)^2 \right],$$  \hspace{1cm} (45)

which gives us the energy spectrum of a scalar field under effects of the KGO and to a hard-wall confining potential in the pointlike global monopole spacetime. We can note, by comparing Eqs.
and (45), that the presence of the hard-wall confining potential in the relativistic quantum system modifies the energy levels of the Klein-Gordon oscillator. We can also note that by taking \( \omega \to 0 \) in Eq. (45), we obtain the relativistic energy levels of a massive scalar field subject to the rigid-wall potential in the global monopole spacetime. In addition, for \( \alpha \to 1 \) into Eq. (45), we recover the energy spectrum of the KG oscillator in the Minkowski spacetime.

We also can analyze the behavior of the energy levels considering the Neumann boundary condition:

\[
\frac{d}{ds} R(s)|_{s=s_0} = 0. \tag{46}
\]

By making use of the Eq. (34), and following similar steps in comparison of the DO, the energy levels are given by

\[
\mathcal{E}_{l,n} \approx \pm \sqrt{m^2 + 3m\omega^2 + \frac{\alpha^2 - \pi^2}{r_0^2}} \left( n + \frac{\sqrt{\alpha^2 + 8\xi(1-\alpha^2) + 4l(l+1)}}{4\alpha} + \frac{5}{4} \right)^2. \tag{47}
\]

From the Eqs. (45) and (47) we note that the presence of a hard-wall confining potential modifies the relativistic energy levels of the system through the presence of the fixed radius \( r_0 \). In addition, the quantum numbers of the system have quadratic values, in contrast with Eq. (33).

The relativistic energy levels (45) and (47) are influenced by the spacetime topology through the presence of the parameter associated with the topological defect \( \alpha \). By making \( \alpha \to 1 \), we recover the relativistic energy levels of a scalar field subject to the effects of the KGO and to a hard-wall confining potential in the Minkowski spacetime. The positive energy profiles (45) and (47) are plotted in Fig. 4 as function of \( \omega/m \) (left plot) and the parameter \( \alpha \) (right plot) considering both Dirichlet (full curves) and Neumann (dashed curves) boundary conditions.

V. CONCLUSIONS

We have investigated the effects of the pointlike global monopole spacetime topology on the Dirac and Klein-Gordon relativistic quantum oscillators which have shown that in the two cases a discrete energy spectrum can be achieved. We have started our discussion with the interaction between spin-\( \frac{1}{2} \) fermionic field and the DO, where the relativistic energy spectrum depends of the parameter associated with the pointlike global monopole. In addition, the set of autofunctions also are influenced by the spacetime topology through the presence of the parameter associated with the topological defect. Then, we analyze the interaction between the scalar field and KGO, where we define the relativistic energy levels of the system which are influenced by the spacetime topology.
Figure 4: Positive energy profiles of the KGO subject to a hard wall potential. In the left plot we have the energy as function of $\omega/m$ considering different values of the parameter $\alpha$. The right plot represents the energy as function of the parameter $\alpha$ for different energy levels and $\omega/m$. In both plots we have $mr_0 = 2.5$, $l = 1$ and $\xi = 0$. The full and dashed curves represents the Dirichlet and Neumann boundary conditions, respectively.

through the presence of the parameter associated with the pointlike global monopole. In addition, the autofunctions are influenced by the spacetime topology, which characterizes the background, through the presence of the parameter associated with the pointlike global monopole.

We have extended our discussion to investigating the behavior of this system under the influence of a hard-wall confining potential. For the DO we have imposed that the fermionic field obeys the MIT-bag boundary condition at some fixed radius. We have considered the hard-wall confining potential for the KGO by imposing that the scalar field obeys the Dirichlet and Neumann boundary conditions. For both, fermionic and scalar fields, we have shown that the presence of the confining potential modifies the relativistic energy levels of the system under consideration and that there is the influence of the spacetime topology on the energy spectrum, as can be seen in the Figs. 3 and 4.

Recently, thermodynamic properties of quantum systems have been investigated, for example, in diatomic molecule systems, in a neutral particle system in the presence of topological defects in a magnetic cosmic string background, in exponential-type molecule potentials, on a 2D charged particle confined by a magnetic and Aharonov-Bohm flux fields under the radial scalar power potential and on a harmonic oscillator in an environment with a pontilike defect. In particular, the DO and KGO oscillators have been the subject of research on thermodynamic properties, as shown in Refs. Therefore, the systems analyzed in this work can be a
starting point for investigating the influence of the pointlike global monopole on the thermodynamic properties.

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