Numerical analysis of a time discretized method for nonlinear filtering problem with Lévy process observations

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Abstract

In this paper, we consider a nonlinear filtering model with observations driven by correlated Wiener processes and point processes. We first derive a Zakai equation whose solution is an unnormalized probability density function of the filter solution. Then we apply a splitting-up technique to decompose the Zakai equation into three stochastic differential equations, based on which we construct a splitting-up approximate solution and prove its half-order convergence. Furthermore, we apply a finite difference method to construct a time semi-discrete approximate solution to the splitting-up system and prove its half-order convergence to the exact solution of the Zakai equation. Finally, we present some numerical experiments to demonstrate the theoretical analysis.

Keywords: nonlinear filter, splitting-up technique, difference method, convergence order
1 Introduction

The aim of a nonlinear filtering problem is to seek the conditional expectation, which is the best estimate of the unobserved state of a stochastic dynamical system given its partial observation. The observation is usually described as a nonlinear stochastic differential equation driven by a noise process. In many applications, such as biology [1], physics [2], target tracking [3] and weather forecast [4], the noise can be characterized by a standard Wiener process. However, in some applications, such as the number of customers arriving at a supermarket [5] and the number of births in a given period of time [6], the noise is governed by a point process. In some other applications such as the credit risk models [7, 8], mathematical finance [9, 10] and insurance [11, 12], the noise can be described by a mixture of a Wiener process and a point process, which is usually called Lévy process.

There have been a few theoretical and numerical studies on nonlinear filtering problems driven by Lévy processes. Qiao and Duan [13] studied a nonlinear filtering model where both the state and observation involve point processes. They simultaneously derived the Zakai equations and Kushner-Stratonovich equations and proved their well-posedness. Fernando and Hausenblas [14] investigated a nonlinear filter model with correlated point processes for the state and observation. They provided sufficient conditions for the well-posedness of the corresponding Zakai equation. Frey etc. [8] used the PDF filter method to approximate a nonlinear filter model driven by point processes and independent Wiener processes. The PDF filter method is designed to directly approach the conditional density function, which satisfies a stochastic partial differential equation, namely Zakai equation [15]. In [8] the authors applied a spectral Galerkin method to set up a spatial semi-discrete equation and proved that its solution converges to the exact solution of the Zakai equation. However, they did not provide the convergence order. They also used the Euler-Maruyama scheme and a splitting-up method, to discretize temporal variables.

In this paper, we use the splitting-up method to investigate the numerical approximation of a nonlinear filtering model where the observation is driven by the mixture of point processes and correlated Wiener processes. The splitting-up method [16–18] is a well-known strategy for solving Zakai equations. It decomposes the Zakai equation into a system consisting of deterministic PDEs and stochastic differential equations (SDEs) [4, 16, 17, 19, 20]. Our contribution in this paper is twofold. First, we decompose the Zakai equation into three equations: an SDE driven by the Wiener process, a second order parabolic equation satisfying the uniform elliptic condition, and an SDE driven by a point process. Through the solution operators of the three equations and their a priori estimates, we construct a splitting-up approximation and prove that it converges to the Zakai solution with first order accuracy. We note that in some references [18, 21, 22] concerning the nonlinear filtering models with correlated noises, the decomposed second-order parabolic equation is possibly degenerate, which may cause difficulty for numerical implementations. Our second
contribution is the derivation of the half-order convergence of the time semi-
discrete approximation. To the best of our knowledge, this is the first time a
convergence order of a numerical method for nonlinear filtering problems with
jump processes has been provided.

This paper is organized as follows. In section 2, we introduce a nonlinear
filtering model with the mixed noise of point process and correlated Wiener
process and then derive the corresponding Zakai equation. In section 3, we
apply a splitting-up method to construct a splitting-up approximate solution
to the Zakai equation and establish a priori estimates for the splitting-up
solution and show that the convergence is of half order. In section 4, we use
finite difference methods to construct a time semi-discrete approximation and
prove that the semi-discrete solution converges to the exact solution with half
order. Finally in section 5, we present some numerical experiments to illustrate
our theoretical analysis.

2 A nonlinear filtering model with jump
observations and its Zakai equation

In this section, we first introduce a nonlinear filtering model whose obser-
vations are driven by Lévy processes. Then we derive the corresponding Zakai equation
which characterizes the development of the density function of the filtering
solution process. Finally, we investigate the regularity of the solution of the
Zakai equation.

2.1 A nonlinear filtering model

In this subsection, we introduce a nonlinear filtering model with noises simul-
taneously driven by a point and correlated Wiener processes and then discuss
some basic assumptions.

Let \((\Omega, \mathcal{F}, P)\) be a given probability space. Consider a nonlinear filtering
model whose state (or signal) process \(X_t\) and two observation processes \(Y_t\) and
\(Z_t\) are given by

\[
X_t = X_0 + \int_0^t g(X_s, Y_s)ds + \int_0^t \sigma(X_s)dw_s, \quad 0 \leq t \leq T, \tag{1}
\]

\[
Y_t = Y_0 + \int_0^t h(X_s)ds + \int_0^t b(Y_s)dw_s + \int_0^t \tilde{b}(Y_s)dv_s, \quad 0 \leq t \leq T, \tag{2}
\]

\(Z_t\) is a doubly stochastic Poisson process with density function \(\lambda(X_t)\), \(\tag{3}
\)

where \(w_t \in \mathbb{R}^{m_1}\) and \(v_t \in \mathbb{R}^{m_2}\) are two standard independent Winner processes,
\(Z_t\) is a doubly stochastic Poisson process with a continuous density function
\(\lambda : \mathbb{R}^d \to [\omega_1, \omega_2] \subset \mathbb{R}_+\) such that \(Z_t - \int_0^t \lambda(X_s)ds\) is a martingale. The
corresponding jump times for \(Z_t\) are random variables denoted by \(\tau_1 < \tau_2 < \ldots < \tau_{n_0}\), where \(n_0\) is an integer-valued random variable.
4 Numerical analysis of a time discretized method for nonlinear filtering problem

The objective of the nonlinear filtering problem is to seek an optimal estimation of $X_t$ based on observations $Y_t$ and $Z_t$, which is characterized by the conditional expectation $E[X_t|Y_t, Z_t]$. Now, we describe in detail the assumptions used in this work.

**H1** $E|X_0|^2 + E|Y_0|^2 < \infty$.

**H2** $g : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ and $h : \mathbb{R}^d \to \mathbb{R}^q$ are bounded, continuous and square integrable, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m_1}$ is in $C^2$ with bounded first and second order derivatives, and $b, \tilde{b}$ are in $C^1$ with bounded first order derivatives.

Define two families of symmetric non-negative matrix

$$A(x) = (a_{i,j}(x))_{d \times d} = \frac{1}{2} \sigma(x)\sigma^T(x), \quad D(y) = b(y)b(y)^T + \tilde{b}(y)\tilde{b}(y)^T.$$  

**H3** There exist two constants $0 < \alpha_1 < \alpha_2$ such that for any $x \in \mathbb{R}^d$, $y \in \mathbb{R}^q$ and $u \in \mathbb{R}^q$, there hold

$$\alpha_1 \|u\|^2 \leq u^T A(x) u \leq \alpha_2 \|u\|^2,$$

$$\alpha_1 \|u\|^2 \leq u^T D(y) u \leq \alpha_2 \|u\|^2.$$  

Define two Sobolev spaces $H = L^2(\mathbb{R}^d)$ with norm $\| \cdot \|$ and $V = H^1_0(\mathbb{R}^d)$ with norm $\| \cdot \|_1$. By $V'$, we denote the dual space of $V$. Obviously, for any $\forall \phi \in V \subset H \subset V'$ there holds

$$\|\phi\|_{V'} \leq \|\phi\| \leq \|\phi\|_1. \tag{4}$$

Define a filtration associated with the observations by

$$\mathcal{F}_t = \sigma(Y_s, Z_s, \ 0 \leq s \leq t), \quad t \in [0, T],$$

which is right continuous and complete. By $L^2(0, T; V)$ we denote a Hilbert space consisting of $\mathcal{F}_t$ progressively measurable $V$-valued stochastic process $z(t)$ with $E \int_0^T \|z(t)\|^2 dt < \infty$.

2.2 Zakai equation and its regularity

The main task of this subsection is to derive the Zakai equation of the nonlinear filtering model (1)-(3) and study the regularity of its solution.

Assume that $X_t$ is the solution process of (1). For any $\phi \in C^\infty_0(\mathbb{R}^d)$, define $\pi_t(\phi)$ as the conditional expectation of $\phi(X_t)$ given $\mathcal{F}_t$, i.e.,

$$\pi_t(\phi) := E(\phi(X_t)|\mathcal{F}_t). \tag{5}$$

Numerical analysis of a time discretized method for nonlinear filtering problem

For any \( t \in [0, T] \), define

\[
\eta_t = \prod_{\tau_m \leq t} \lambda(X_{\tau_m}) \cdot \exp \left( \int_0^t h^T(X_s) D^{-1}(b(Y_s) dw_s + \tilde{b}(Y_s) dv_s) \right) + \frac{1}{2} \int_0^t h^T(X_s) D^{-1} h(X_s) ds - \int_0^t (\lambda(X_s) - 1) ds.
\]

According to Novikov Criterion [23, Theorem 41], \( \eta_t \) is a nonnegative martingale if H1-H3 hold. Define a new probability measure \( \tilde{P} \) by virtue of the Radon-Nikodym derivative \( \frac{d\tilde{P}}{dP} = \eta_t^{-1} \). The Girsanov theorem [24] implies that \( \tilde{Y}_t = \int_0^t D(Y_s)^{-1/2} dY_s \) is a standard Wiener process and \( Z_t \) is a Poisson process with intensity equal to 1 under the probability measure \( \tilde{P} \). Furthermore, in the probability space \((\Omega, \mathcal{F}, \tilde{P})\) the three stochastic processes \( X_t, Y_t \) and \( Z_t \) are independent of each other, and the compensated Poisson process \( N_t := Z_t - t \) is a martingale.

Define a stochastic process

\[
\tilde{Y}_t = \int_0^t [I - b^T D^{-1} b]^{-1/2} (d\tilde{w}_t - b^T D^{-1} dY_t),
\]

where \( \tilde{w}_t = w_t + \int_0^t b^T(Y_s) D^{-1}(Y_s) h(X_s) ds \) is a standard Wiener process under \( \tilde{P} \). From [25], we have

**Lemma 1** Assume H1-H3. Then \( \tilde{Y}_t \) is a standard Wiener process independent of \( Y_t \) under \( \tilde{P} \) and equation (1) is equivalent to

\[
X_t = X_0 + \int_0^t [g(X_s) - B_1(X_s, Y_s) h(X_s)] ds + \int_0^t B_1(X_s, Y_s) dY_s + \int_0^t B_2(X_s, Y_s) d\tilde{Y}_s,
\]

where \( B_1(x, y) = \sigma(x) b^T(y) D^{-1}(y) \) and \( B_2(x, y) = \sigma(x)(I - b^T(y) D^{-1}(y) b(y))^{1/2} \).

Denote by \( \tilde{E} \) the expectation under the probability measure \( \tilde{P} \). The next proposition plays an important role in the forthcoming analysis.

**Proposition 2** [24, Proposition 3.15] Assume that H1-H3 hold and let \( U \) be an \( \mathcal{F}_t \)-measurable and integrable random variable. Then we have

\[
\tilde{E}[U|\mathcal{F}_t] = \tilde{E}[U|\mathcal{F}_T].
\]

By the Kallianpur-Striebel formula [24, Proposition 3.16], we have

\[
\pi_t(\phi) = \frac{\tilde{E}(\phi(X_t)|\mathcal{F}_t)}{\tilde{E}(\eta_t|\mathcal{F}_t)} = \frac{(p(t), \phi)}{(p(t), 1)},
\]

where \( p(t) \) is the unnormalized conditional density function of \( \tilde{E}(X_t|\mathcal{F}_t) \).
Using Itô’s formula for the jump process, we have
\begin{equation}
\rho_t(\phi) = \rho_0(\phi) + \int_0^t \rho_s(\mathcal{L}\phi)ds + \int_0^t \rho_s(\mathcal{B}\phi)dY_s + \int_0^t \rho_s(\phi(\lambda-1))d(Z_s - s), \quad \tilde{P} - a.s.,
\end{equation}
where
\begin{align*}
\mathcal{L}\phi &= \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^d g_i(x) \frac{\partial \phi}{\partial x_i}, \\
\mathcal{B}\phi &= \phi^T \Gamma^{-1} + \sum_{i=1}^d \frac{\partial \phi}{\partial x_i} b_{1,i},
\end{align*}
and $b_{1,i}$ denotes the $i$-th row of matrix $B_1(x,y)$.

**Proof** We approximate \( \eta_t \) with \( \tilde{\eta}_t = \frac{\eta_t}{1+\varepsilon \eta_t} \). By Itô formula
\begin{equation}
d\tilde{\eta}_t = \eta_t - [h(X_t)^T \Gamma^{-1} dY_t + (\lambda(X_t)-1)d(Z_t - t)].
\end{equation}
Using Itô’s formula for the jump process, we have
\begin{equation}
\tilde{\eta}_t = \tilde{\eta}_0 + \int_0^t \frac{\eta_s}{(1 + \varepsilon \eta_s)^2} h(X_s)^T \Gamma^{-1} dY_s - \int_0^t \frac{\varepsilon \eta^2_t}{(1 + \varepsilon \eta_t)^3} h(X_s)^T \Gamma^{-1} h(X_s)ds
- \int_0^t \frac{\eta_s}{(1 + \varepsilon \eta_s)^2} (\lambda(X_s) - 1)ds + \sum_{\tau_m \leq t} \Delta \frac{\eta_{\tau_m}}{1 + \varepsilon \eta_{\tau_m}}.
\end{equation}
Let $X_t$ satisfy (1) and for any $\phi \in C_0^\infty(\mathbb{R}^d)$
\begin{equation}
d\phi(X_t) = \mathcal{L}\phi dt - \sum_{i=1}^d (B_1 h)^i \frac{\partial \phi}{\partial x_i} dt + \nabla \phi^T B_1 dY_t + \nabla \phi^T B_2 d\tilde{Y}_t.
\end{equation}
Applying the product rule for semi-martingales to (10) and (12), we obtain
\begin{align}
\tilde{\eta}_t \phi(X_t) &= \tilde{\eta}_0 \phi(X_0) + \int_0^t \tilde{\eta}_s \mathcal{L}\phi(X_s)ds - \int_0^t \frac{\varepsilon \eta^2_t}{(1 + \varepsilon \eta_s)^2} \nabla \phi^T B_1 h ds + \int_0^t \tilde{\eta}_s \nabla \phi^T B_1 dY_s \\
&\quad + \int_0^t \tilde{\eta}_s \nabla \phi^T B_2 d\tilde{Y}_s + \int_0^t \frac{\eta_s \phi(X_s)}{(1 + \varepsilon \eta_s)^2} h^T \Gamma^{-1} dY_s - \int_0^t \frac{\varepsilon \eta^2_t \phi(X_s)}{(1 + \varepsilon \eta_s)^2} h^T \Gamma^{-1} h ds \\
&\quad - \int_0^t \frac{\eta_s \phi(X_s)}{(1 + \varepsilon \eta_s)^2} (\lambda(X_s) - 1)ds + \int_0^t \frac{\eta_s - (\lambda(X_s) - 1)\phi(X_s)}{(1 + \varepsilon \eta_s - \lambda(X_s))(1 + \varepsilon \eta_s)} dZ_s.
\end{align}

According to Proposition 2, we only need to compute the conditional expectation based on the filtration $\mathcal{F}_T$. Take conditional expectation about $\tilde{\eta}_t \phi(X_t)$ based on
the observation $F_T$, then we have
\[
\hat{E}(\eta_t \phi(X_t)|F_T) \\
= \hat{E}(\eta_0 \phi(X_0)|F_T) + \hat{E}\left(\int_0^t \eta_s \mathcal{L}\phi(X_s)ds|F_T\right) \\
- \hat{E}\left(\int_0^t \frac{\varepsilon \eta_s^2}{(1 + \varepsilon \eta_s)^2} \nabla \phi^T B_1 hds|F_T\right) + \hat{E}\left(\int_0^t \eta_s \nabla \phi^T B_1 dY_s|F_T\right) \\
+ \hat{E}\left(\int_0^t \frac{\eta_s \phi}{(1 + \varepsilon \eta_s)^2} h^T D^{-1} dY_s|F_T\right) \quad \text{and} \\
- \hat{E}\left(\int_0^t \frac{\eta_s \phi}{(1 + \varepsilon \eta_s)^3} h^T D^{-1} hds|F_T\right) - \hat{E}\left(\int_0^t \frac{\eta_s \phi}{(1 + \varepsilon \eta_s)^2} (\lambda(X_s) - 1) ds|F_T\right) \\
+ \hat{E}\left(\int_0^t \frac{\eta_s - (\lambda(X_s) - 1) \phi(X_s)}{(1 + \varepsilon \eta_s - \lambda(X_s))(1 + \varepsilon \eta_s)} dZ_s|F_T\right) \\
:= E_1 + E_2 - E_3 + E_4 + E_5 + E_6 - E_7 - E_8 + E_9. 
\]
(14)

Now, we show that as $\varepsilon \to 0$, the following limits hold in the sense $\tilde{P}$-a.s.,
\[
\hat{E}(\eta_0 \phi)|F_T) \to \rho_t(\phi), \quad E_1 \to \rho_0(\phi), \quad E_2 \to \int_0^t \rho_s(\mathcal{L}\phi)ds, \quad E_3 \to 0, \\
E_4 \to \int_0^t \rho_s(\nabla \phi^T c) dY_s, \quad E_5 \to \int_0^t \rho_s(h^T D^{-1} \phi) dY_s, \quad E_6 \to 0, \quad E_7 \to 0, \\
E_8 \to \int_0^t \rho_s(\phi(\lambda - 1)) ds, \quad E_9 \to \int_0^t \rho_s(\rho(\lambda - 1)) dZ_s. 
\]

From the pointwise convergence of $\eta_t$ to $\eta$ as $\varepsilon \to 0$, it follows that $\lim_{\varepsilon \to 0} \hat{E}(\eta_t \phi) = \eta \phi$. In addition,
\[
\hat{E}\|\eta_0 \phi\| \leq \|\phi\| \hat{E}(\eta_0) = \|\phi\| \hat{E}(\eta_t^{-1}) = \|\phi\| < \infty.
\]

Due to the dominated convergence theorem [26, page 152], we obtain
\[
\lim_{\varepsilon \to 0} \hat{E}(\eta_t \phi|F_T) = \hat{E}(\eta_0 \phi|F_T) = \rho_0(\phi), \quad \tilde{P} - a.s.
\]

Similarly, there holds
\[
\lim_{\varepsilon \to 0} E_1 = \hat{E}(\eta_0 \phi(X_0)|F_T) = \rho_0(\phi), \quad \tilde{P} - a.s.
\]

Now, we consider item $E_2$. Notice that for any $\varepsilon > 0$, there holds
\[
\hat{E}\left(\int_0^t \eta_s \mathcal{L}\phi ds\right) = \hat{E}\left(\int_0^t \frac{\varepsilon \eta_s}{1 + \varepsilon \eta_s} \frac{1}{\varepsilon} \mathcal{L}\phi ds\right) \leq \frac{1}{\varepsilon} \|\mathcal{L}\phi\| \mathcal{T} < \infty.
\]

By Fubini’s theorem, we exchange the integral order in $E_2$ to obtain
\[
\|E_2\| = \|\hat{E}\left(\int_0^t \eta_s \mathcal{L}\phi ds\right)\| = \|\int_0^t \hat{E}(\eta_s \mathcal{L}\phi|F_T) ds\| \\
\leq \hat{E}\left(\int_0^t \hat{E}(\|\eta_s \mathcal{L}\phi\|) ds\right) \leq \hat{E}\left(\int_0^t \hat{E}(\|\mathcal{L}\phi\| \|F_T\|) ds\right) \\
= \|\mathcal{L}\phi\| \int_0^t \hat{E}(\eta_s) ds \leq \|\mathcal{L}\phi\| \mathcal{T} < \infty.
\]

By the dominated convergence theorem, we get
\[
\lim_{\varepsilon \to 0} E_2 = \int_0^t \hat{E}(\eta_s \mathcal{L}\phi|F_T) ds = \int_0^t \rho_s(\mathcal{L}\phi)ds, \quad \tilde{P} - a.s.
\]
Next, we study item $E_3$. Notice that
\[
\hat{E}\left(\int_0^t \frac{\varepsilon \eta_s^2}{(1 + \varepsilon \eta_s)^2} \nabla \phi^T B_1 h ds\right) \leq \|\nabla \phi\| \hat{E}\left(\int_0^t \eta_s \|B_1 h\| ds\right) = \|\nabla \phi\| \int_0^t \hat{E}(\|B_1 h\|) ds < \infty.
\]
This estimate, together with the pointwise convergence $\lim_{\varepsilon \to 0} \frac{\varepsilon \eta_s^2}{(1 + \varepsilon \eta_s)^2} \nabla \phi^T B_1 h = 0$ and the dominated convergence theorem implies
\[
\lim_{\varepsilon \to 0} \hat{E}(E_3) = 0, \quad \hat{P} - a.s.
\]
In a similar way, we obtain $\lim_{\varepsilon \to 0} E_i = 0, \quad \hat{P} - a.s.$ for $i = 6, 7$.

By isometry formula, we have that for any $\varepsilon > 0$
\[
\hat{E}\left(\left(\int_0^t \eta_s \nabla \phi^T B_1 dY_s\right)^2\right) = \hat{E}\left(\int_0^t \left(\frac{\eta_s}{1 + \varepsilon \eta_s}\right)^2 \nabla \phi^T B_1 DB_1^T \nabla \phi ds\right) = \frac{1}{\varepsilon} \hat{E}\left(\int_0^t \varepsilon \eta_s \nabla \phi^T B_1 DB_1^T \nabla \phi ds\right) \leq \frac{\|\nabla \phi\|^2}{\varepsilon} \hat{E}\left(\int_0^t \eta_s \|B_1 DB_1^T\| ds\right) = \frac{\|\nabla \phi\|^2}{\varepsilon} \int_0^t \hat{E}(\|B_1 DB_1^T\|) ds < \infty.
\]
According to [24, Lemma 3.21], we can change the order between conditional expectation and stochastic integral to obtain
\[
E_4 = \int_0^t \hat{E}\left(\frac{\eta_s}{1 + \varepsilon \eta_s} \nabla \phi^T B_1 |\mathcal{F}_T\right) dY_s. \quad (15)
\]
Using Jensen’s inequality and Fubini’s Theorem, for any $\varepsilon > 0$, we have
\[
\hat{E}(E_4^2) = \hat{E}\left\{\int_0^t \left[\hat{E}(\frac{\eta_s}{1 + \varepsilon \eta_s} \nabla \phi^T B_1 |\mathcal{F}_T\right)] D\left[\hat{E}(\frac{\eta_s}{1 + \varepsilon \eta_s} \nabla \phi^T B_1 |\mathcal{F}_T\right)]^T ds\right\} \\
\leq \hat{E}\left\{\int_0^t \hat{E}\left[\frac{\eta_s}{1 + \varepsilon \eta_s} \nabla \phi^T B_1\right] D\left(\frac{\eta_s}{1 + \varepsilon \eta_s} \nabla \phi^T B_1\right)^T |\mathcal{F}_T\right] ds\right\} \\
\leq \hat{E}\left\{\int_0^t \hat{E}[\|\nabla \phi\|^2 \|B_1 DB_1^T\| \frac{1}{\varepsilon} \frac{\eta_s}{1 + \varepsilon \eta_s} |\mathcal{F}_T\right] ds\right\} \\
\leq \frac{\|\nabla \phi\|^2}{\varepsilon} \hat{E}\left(\int_0^t \hat{E}(\eta_s \|B_1 DB_1^T\|) ds\right) = \frac{\|\nabla \phi\|^2}{\varepsilon} \int_0^t \hat{E}(\eta_s \|B_1 DB_1^T\|) ds \leq \frac{\|\nabla \phi\|^2}{\varepsilon} \int_0^t \hat{E}(\|B_1 DB_1^T\|) ds < \infty.
\]
This implies that $E_4$ is a martingale, c.f. [27, Theorem 4.3.1] and then the process $\int_0^t \rho_s(\nabla \phi^T B_1) dY_s$ is a local martingale. Thus, the following difference is a local martingale
\[
\int_0^t \rho_s(\nabla \phi^T B_1) dY_s - E_4 = \int_0^t \hat{E}\left(\frac{\varepsilon \eta_s^2}{1 + \varepsilon \eta_s} \nabla \phi^T B_1 |\mathcal{F}_T\right) dY_s. \quad (16)
\]
Set $\xi^\varepsilon = \frac{\varepsilon^2}{1 + \varepsilon \eta_s} \nabla \phi^T B_1$, then $\lim_{\varepsilon \to 0} \xi^\varepsilon = 0$, $\tilde{P} - a.s.$ Obviously, for any $\phi \in V$ we have

$$\|\xi^\varepsilon\| \leq \|\nabla \phi\| \|B_1\| \eta_s.$$  

Due to the dominated convergence theorem, we obtain

$$\lim_{\varepsilon \to 0} \tilde{E}(\xi^\varepsilon | \mathcal{F}_T) = \tilde{E}(\lim_{\varepsilon \to 0} \xi^\varepsilon | \mathcal{F}_T) = 0, \tilde{P} - a.s.$$  

Furthermore,

$$\|\tilde{E}(\xi^\varepsilon | \mathcal{F}_T)\| \leq \tilde{E}(\|\xi^\varepsilon\| | \mathcal{F}_T) \leq \|\nabla \phi\| \|B_1\| \tilde{E}(\eta_s | \mathcal{F}_T) \leq \|\nabla \phi\| \|B_1\|.$$  

(17)

Applying the stochastic dominated convergence theorem [23, Theorem 32], we have

$$\lim_{\varepsilon \to 0} \int_0^t \tilde{E}(\xi^\varepsilon | \mathcal{F}_T) dY_s = \int_0^t \lim_{\varepsilon \to 0} \tilde{E}(\xi^\varepsilon | \mathcal{F}_T) dY_s = 0, \tilde{P} - a.s.$$  

Hence, we obtain

$$\lim_{\varepsilon \to 0} E_4 = \int_0^t \rho_s(\nabla \phi B_1) dY_s, \tilde{P} - a.s.$$  

Similarly, we can prove that

$$\lim_{\varepsilon \to 0} E_5 = \int_0^t \rho_s(\phi^T D^{-1}) dY_s, \quad \lim_{\varepsilon \to 0} E_8 = \int_0^t \rho_s(\phi(\lambda - 1)) ds, \tilde{P} - a.s.$$  

Finally, we investigate the term $E_9$. Let

$$G^\varepsilon_s := \frac{\eta_s - (\lambda(X_{s-} - 1))}{(1 + \varepsilon \eta_s)X_{s-})/(1 + \varepsilon \eta_s)}.$$  

Then $\lim_{\varepsilon \to 0} G^\varepsilon_s = \eta_s - (\lambda(X_{s-} - 1))\phi(X_{s-})$, $\tilde{P} - a.s.$

It is easy to see that for any $\varepsilon > 0$

$$|G^\varepsilon_s| \leq \|\phi\| \|\lambda - 1\| \|\varepsilon\| < \infty. \quad (18)$$

This estimate, together with the stochastic Fubini’s theorem [23, Theorem 64] implies that we can change the order between the stochastic integral and conditional expectation in $E_9$ to obtain $E_9 = \int_0^t \tilde{E}(G^\varepsilon_s | \mathcal{F}_T) dZ_s$.

Using the same argument as above, we obtain

$$\lim_{\varepsilon \to 0} E_9 = \int_0^t \tilde{E}(\phi(X_{s-} - \eta_s - (\lambda(X_{s-} - 1)| \mathcal{F}_T) dZ_s.$$  

According to [28, Theorem 1.6], we have

$$\tilde{E}(\phi(X_{s-} - \eta_s - (\lambda(X_{s-} - 1)| \mathcal{F}_T) = \lim_{\mathcal{T} \uparrow s} \tilde{E}(\phi(X_r) \eta_r (\lambda(X_r) - 1)| \mathcal{F}_T)$$  

$$= \lim_{\mathcal{T} \uparrow s} \rho_r(\phi(\lambda - 1)) = \rho_s - (\phi(\lambda - 1))$$  

(19)

Hence, $\lim_{\varepsilon \to 0} E_9 = \int_0^t \rho_s(\phi(\lambda - 1)) dZ_s$, $\tilde{P} - a.s.. \quad \square$

The next theorem follows from Theorem 3.

**Theorem 4** Assume H1-H3. Then $p(t)$ satisfies Zakai equation:

$$dp(t) = L^* p(t) dt + B^* p(t) dY_t + C(p(t^-))(dZ_t - dt), \quad p(0) = p_0 \in H, \quad (20)$$
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where for any \( \phi \in V \)

\[
\mathcal{L}^* \phi = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{i,j} \phi) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (g_i \phi),
\]

\[
\mathcal{B}^* \phi = \phi h^T D^{-1} - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_{1,i} \phi),
\]

\[
C \phi = (\lambda - 1) \phi.
\]

The differential operator \( \mathcal{B}^* \) is not bounded in the usual sense. We now study its boundedness in \( L(V, V') \). Due to \( H_2 \), for any \( \phi, \psi \in V \), \( \mathcal{B}^* \phi \in H \subset V' \) and satisfies

\[
\| \mathcal{B}^* \phi \|_{V'} = \sup_{\| \psi \| = 1} (\phi, \mathcal{B} \psi) \leq \sup_{\| \psi \| = 1} \| \phi \| \| \mathcal{B} \psi \| \leq C \| \phi \|, \tag{21}
\]

where \( C > 0 \) is a constant.

The following lemma is concerned with the regularity of the second-order differential operator \( -\mathcal{L}^* + C \).

**Lemma 5** Assume \( H_1-H_3 \). Then there exist constants \( \beta_1, \beta_2 > 0 \) and \( \alpha > 0 \) such that \( \forall \phi, \psi \in V \)

\[
|\langle (\mathcal{L}^* - C) \phi, \psi \rangle| \leq \beta_1 \| \phi \|_1 \| \psi \|_1,
\]

\[
\beta_2 \| \phi \|_1^2 \leq -\langle (\mathcal{L}^* - C) \phi, \phi \rangle + \alpha \| \phi \|^2.
\]

**Proof** The first inequality directly follows from assumption \( H_2 \). Thus we only need to prove the second one.

Direct computation gives, for any \( \phi \in V \),

\[
\mathcal{L}^* \phi = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{i,j} \phi) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (\delta_i \phi)
\]

where \( \delta \) is a \( \mathbb{R}^d \)-valued function with \( \delta_i = g_i - \sum_{j=1}^{d} \frac{\partial a_{i,j}}{\partial x_j} \). Therefore,

\[
-\langle \mathcal{L}^* \phi, \phi \rangle = (A \nabla \phi, \nabla \phi) - (\delta \phi, \nabla \phi).
\]

From assumption \( H_3 \), it follows that \( \| \delta \| < \infty \). Take a positive number \( \alpha > -(\varpi_2 - 1 - \frac{1}{2 \alpha_1} \| \delta \|^2) \) and let \( \beta_2 = \min(\frac{1}{2} \alpha_1, \alpha + \varpi_2 - 1 - \frac{1}{2 \alpha_1} \| \delta \|^2) \), then we obtain

\[
-\langle (\mathcal{L}^* - C) \phi, \phi \rangle + \alpha \| \phi \|^2 \geq \alpha_1 \| \nabla \phi \|^2 - (\delta \phi, \nabla \phi) + (\alpha + \varpi_2 - 1) \| \phi \|^2 \geq \frac{1}{2} \alpha_1 \| \nabla \phi \|^2 + (\alpha + \varpi_2 - 1 - \frac{1}{2 \alpha_1} \| \delta \|^2) \| \phi \|^2 \geq \beta_2 \| \phi \|^2.
\]

\[\square\]
The existence, uniqueness, and regularity of the solution to the Zakai equation (20), which we summarize in the next lemma., can be obtained following the approaches in [29, 30],

**Lemma 6** Assume $\textbf{H1-H3}$. For each $p_0 \in H$, there exists a unique solution $p$ of (20) satisfying

$$p \in L^2(0, T; V) \cap L^2(\Omega; C(0, T; H)).$$

Furthermore, if $p_0 \in L^4(\Omega; H)$, there holds $p \in L^\infty(0, T; L^4(\Omega; H))$.

In order to construct a stable and efficient numerical method, we take a transformation $p(t) \to p(t)e^{-\mu t}$ in (20) and obtain a well-posed Zakai equation

$$dp(t) + (\mathcal{L}^* p(t) + \mu p(t))dt = \mathcal{B}^* p(t) dY_t + \mathcal{C} p(t)(dZ_t - dt), \quad p(0) = p_0 \in H.$$ (22)

### 3 A splitting-up scheme and error estimation

In this section, we apply a splitting-up method to construct a temporal semi-discretized approximation of equation (22) and derive corresponding error estimations.

#### 3.1 A splitting-up approximate solution

Consider three equations

$$dp_1(t) + \frac{\mu}{3} p_1(t)dt = \mathcal{B}^* p_1(t) dY_t, \quad (23)$$

$$dp_2(t) + \frac{\mu}{3} p_2(t)dt = (\mathcal{L}^* - \mathcal{C}) p_2(t)dt, \quad (24)$$

$$dp_3(t) + \frac{\mu}{3} p_3(t)dt = \mathcal{C} p_3(t)(dZ_t - dt). \quad (25)$$

Equation (23) is a first-order SPDE. We denote by $\{Q_t^s, 0 \leq s \leq t\}$ with $Q_t^s = I$ its solution operator which is a Markov semigroup, cf [31]. (24) is a determined second-order PDE satisfying uniform elliptic condition if $\mu$ is large. Denote by $\{R_t^s, t \geq 0\}$ with $R_t^s = I$ its strongly continuous semigroup. (25) is a stochastic differential equation driven by a point process. The existence and uniqueness of the solution to (25) are obtained in [23]. We denote its solution operator by $\{\Gamma_t^s, t \geq 0\}$ with $\Gamma_t^s = I$. According to [23, 32], there exists a constant $C = C(T)$ such that for any $\phi \in L^2(\Omega; H)$ and $0 \leq s < t \leq T$, there holds

$$\tilde{E}\|Q_t^s \phi\| \leq C\tilde{E}\|\phi\|, \quad \tilde{E}\|R_t^s \phi\| \leq C\tilde{E}\|\phi\|, \quad \tilde{E}\|\Gamma_t^s \phi\| \leq C\tilde{E}\|\phi\|. \quad (26)$$

For any given integer $N > 0$, let $t_r = r\kappa$ ($r = 0, 1, \cdots, N$) be the uniformly partition of interval $[0, T]$ with stepsize $\kappa = \frac{T}{N}$. By virtue of the solutions of
On each interval \([t_r, t_{r+1}]\) we define a splitting-up solution \(p^r_{\kappa} + 1\) to (22) at each node point \(t_{r+1}\) by

\[
p_{\kappa}^r + 1 = \Gamma_{t_{r+1}}^{t_r} R_{t_{r+1}}^{t_r} Q_{t_{r+1}}^{t_r} p_{\kappa}^r, \quad r = 0, 1, \ldots, N - 1, \quad p_{\kappa}^0 = p_0.
\]

(27)

Meanwhile, we also define three solution process to equations (23)-(25) in each interval \([t_r, t_{r+1}]\),

\[
p_{1\kappa}(t) = Q_{t}^{t_r} p_{\kappa}^r, \quad p_{2\kappa}(t) = R_{t}^{t_r} Q_{t_{r+1}}^{t_r} p_{\kappa}^r, \quad p_{3\kappa}(t) = \Gamma_{t}^{t_r} R_{t_{r+1}}^{t_r} Q_{t_{r+1}}^{t_r} p_{\kappa}^r.
\]

Obviously, \(p_{\kappa}^{r+1} = p_{3\kappa}(t_{r+1})\). Then we have that \(p_{1\kappa}, p_{3\kappa} \in \mathbb{L}^2(0, T; H)\) and \(p_{2\kappa} \in \mathbb{L}^2(0, T; V)\) are right continuous and their discontinuity only occurs at node points. Furthermore, there hold

\[
p_{1\kappa}(t) \in \mathcal{F}_t \text{ measurable for } t \in [t_r, t_{r+1}], \quad p_{2\kappa}(t) \text{ and } p_{3\kappa}(t) \text{ are } \mathcal{F}_{t_{r+1}} \text{ measurable for } t \in [t_r, t_{r+1}]. \tag{28}
\]

### 3.2 A priori estimate

**Theorem 7** Assume H1-H3. Then there exist constants \(\mu > 0\) and \(C = C(T) > 0\) such that the three processes \(p_{i\kappa}(t)\) \((i = 1, 2, 3)\) satisfy

\[
\tilde{E} \int_{t_{r+1}}^{T} ||p_{1\kappa}||^2 dt \leq C, \quad \tilde{E} \int_{t_{r+1}}^{T} ||p_{2\kappa}||^2 dt \leq C, \quad \tilde{E} \int_{t_{r+1}}^{T} ||p_{3\kappa}||^2 dt \leq C. \tag{29}
\]

Furthermore, if \(p_0 \in L^4(\Omega; H)\), then there hold

\[
\tilde{E}||p_{1\kappa}(t)||^4 \leq C, \quad \tilde{E}||p_{2\kappa}(t)||^4 \leq C, \quad \tilde{E}||p_{3\kappa}(t)||^4 \leq C, \quad \forall t \in [0, T]. \tag{30}
\]

**Proof** On each interval \((t_r, t_{r+1})\), by Itô formula there holds

\[
d(||p_{1\kappa}||^2 + (\frac{2}{3}\mu ||p_{1\kappa}||^2) + |\mathcal{B}^* p_{1\kappa}(t)|^2) dt = 2(p_{1\kappa}(t), \mathcal{B} p_{1\kappa}(t)) dY_t, \tag{31}
\]

\[
d(||p_{2\kappa}||^2 + (\frac{2}{3}\mu ||p_{2\kappa}||^2) - 2(\langle \mathcal{L}^* - \mathcal{C} \rangle p_{2\kappa}(t), p_{2\kappa}(t)) dt = 0, \tag{32}
\]

\[
d(||p_{3\kappa}||^2 + (\frac{2}{3}\mu ||p_{3\kappa}||^2) dt = (\lambda^2 - 1) ||p_{3\kappa}(t^-)||^2 dZ_t, \tag{33}
\]

where \(\mathcal{B}^* p(t)^2 \frac{d}{dt} = \int_{\mathbb{R}^d} \mathbb{E}[\mathcal{B}^* p(t)] D[\mathcal{B}^* p(t)]^T dx \leq M ||p||^2\) due to (21) and assumption H3.

Choose \(\mu = \max\{3\alpha, \frac{3}{2}M\} + 1\) and \(\gamma = \min\{\frac{2}{3}\mu - M, 2\beta_2, \frac{2}{3}\mu\}\). By Lemma 5, we obtain

\[
d(||p_{1\kappa}||^2 + ||p_{2\kappa}||^2 + ||p_{3\kappa}||^2) + \gamma(||p_{1\kappa}||^2 + ||p_{2\kappa}||^2 + ||p_{3\kappa}||^2) dt \\
\leq 2(2p_{2\kappa}, \mathcal{B} p_{2\kappa}) dY_t + (\lambda^2 - 1)||p_{3\kappa}(t^-)||^2 dZ_t.
\]

Integrating this equation over \((t_i, t_{i+1})\) and taking expectation yields

\[
\tilde{E}(||p_{1\kappa}(t_{i+1})||^2 + ||p_{2\kappa}(t_{i+1})||^2 + ||p_{3\kappa}(t_{i+1})||^2) - \tilde{E}(||p_{1\kappa}(t_{i})||^2 + ||p_{2\kappa}(t_{i})||^2 + ||p_{3\kappa}(t_{i})||^2) \\
+ \gamma \tilde{E} \int_{t_{i}}^{t_{i+1}} (||p_{1\kappa}||^2 + ||p_{2\kappa}||^2 + ||p_{3\kappa}||^2) dt \leq C \tilde{E} \int_{t_{i}}^{t_{i+1}} ||p_{3\kappa}(t^-)||^2 dZ_t.
\]
Together with (23)-(25), we get
\[ \tilde{E}\|p_{k}^{i+1}\|^2 - \tilde{E}\|p_{k}^{i}\|^2 + \gamma \tilde{E} \int_{t_{i}}^{t_{i+1}} (\|p_{1\kappa}\|^2 + \|p_{2\kappa}\|^2 + \|p_{3\kappa}\|^2) dt \leq C \tilde{E} \int_{t_{i}}^{t_{i+1}} \|p_{3\kappa}(t)\|^2 dZ_t. \]
Summing up this equation from \( i = 0 \) to \( r - 1 \) gives
\[ \tilde{E}\|p_{k}^{r}\|^2 + \gamma \tilde{E} \int_{0}^{t_{r}} (\|p_{1\kappa}\|^2 + \|p_{2\kappa}\|^2 + \|p_{3\kappa}\|^2) dt \leq \tilde{E}\|p_{0}\|^2 + C \tilde{E} \int_{0}^{t_{r}} \|p_{3\kappa}(t)\|^2 dZ_t. \quad (34) \]
Since the Poisson process \( Z_t \) has only finite jump times, we have
\[ \tilde{E} \int_{0}^{T} \|p_{3\kappa}(t)\|^2 dZ_t \leq C. \]
Thus, we obtain \( \tilde{E}\|p_{k}^{r}\|^2 \leq C, \ r = 1, \cdots, N, \) and estimates (29) follow from (34).

Next, we integrate (31) and (32) over \( (t_{r}, t_{r+1}) \) and then take expectation to obtain
\[ \tilde{E}\|p_{1\kappa}(t_{r+1} - 0)\|^2 \leq \tilde{E}\|p_{1\kappa}\|^2 \leq C, \quad \tilde{E}\|p_{2\kappa}(t_{r+1} - 0)\|^2 \leq \tilde{E}\|p_{1\kappa}(t_{r+1} - 0)\|^2 \leq C. \]
For any \( t \in (t_{r}, t_{r+1}) \), integrating (31), (32) and (33) over \([t_{r}, t]\), we obtain
\[ \tilde{E}\|p_{1\kappa}(t)\|^2 \leq \tilde{E}\|p_{k}\|^2 \leq C, \quad \tilde{E}\|p_{2\kappa}(t)\|^2 \leq \tilde{E}\|p_{1\kappa}(t_{r+1} - 0)\|^2 \leq C, \]
\[ \tilde{E}\|p_{3\kappa}(t)\|^2 \leq \tilde{E}\|p_{2\kappa}(t_{r+1} - 0)\|^2 + \tilde{E} \int_{t_{r}}^{t} (\lambda^2 - 1) \|p_{3\kappa}(t)\|^2 dZ_t \leq C, \]
where the constant \( C \) is independent of \( r \). Therefore,
\[ \tilde{E}\|p_{ik}(t)\|^2 \leq C, \quad \forall t \in [0, T], \quad i = 1, 2, 3. \quad (35) \]
By Ito’s formula, (31), (32) and (33), we have for any \( t \in [t_{r}, t_{r+1}], \ r = 0, 1, 2, \cdots, N - 1 \)
\[ d\|p_{1\kappa}\|^2 + 2\|p_{1\kappa}\|^2 (2 \lambda \mu \|p_{1\kappa}\|^2 - |B^* p_{1\kappa} (\mathcal{D} B_{1\kappa})|) - 4 |B^* p_{1\kappa} (\mathcal{D} B_{1\kappa})| dY_t = 4 \|p_{1\kappa}\|^2 (p_{1\kappa}, B p_{1\kappa}) dY_t, \]
\[ d\|p_{2\kappa}\|^2 + 2\|p_{2\kappa}\|^2 (2 \lambda \mu \|p_{2\kappa}\|^2 - 2 (|\mathcal{L}^* - C| p_{2\kappa}, p_{2\kappa})) dt = 0, \]
\[ d\|p_{3\kappa}\|^4 + 4 \|p_{3\kappa}\|^4 dt = (\lambda^4 - 1) \|p_{3\kappa}(t)\|^4 dZ_t. \]
Following the same argument above, we can also derive the estimates (30). \( \square \)

3.3 Convergence of splitting-up solution

In this section, we shall investigate the convergence and convergence order of the splitting-up solution.

**Theorem 8** Assume that H1-H3 hold and \( p_0 \in L^4(\Omega; H) \). Then as \( \kappa \to 0 \), there holds
\[ p_{1\kappa}(t), p_{2\kappa}(t), p_{3\kappa}(t) \to p(t) \text{ in } L^2(\Omega; H), \text{ uniformly for } t \in [0, T], \]
\[ p_{1\kappa}, p_{3\kappa} \to p \text{ in } L^2(0, T; H), \quad p_{2\kappa} \to p \text{ in } L^2(0, T; V). \]

Before proving the theorem, we notice that, according to Theorem 7, the three sequences \( p_{1\kappa}(t), p_{2\kappa}(t) \) and \( p_{3\kappa}(t) \) are bounded in spaces \( L^2(0, T; H), \ L^2(0, T; V) \) and \( L^2(0, T; H) \), respectively. By the weakly compactness of these
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spaces, we can extract three subsequences, still denoted by \( p_{1\kappa}(t), p_{2\kappa}(t) \) and \( p_{3\kappa}(t) \), such that as \( \kappa \to 0 \)

\[
(p_{1\kappa}, p_{3\kappa}) \to (p_1, p_3) \text{ in } L^2(0, T; H) \text{ weakly,}
\]

\[
p_{2\kappa} \to p_2 \text{ in } L^2(0, T; V) \text{ weakly,}
\]

\[
(p_{i\kappa} - p_{1\kappa}) \to (p_i - p_1) \text{ in } L^2(0, T; V') \text{ weak star for } i = 2, 3.
\]  \( (36) \)

Furthermore, if \( p_0 \in L^4(\Omega; H) \)

\[
p_{i\kappa} \to p_i \text{ in } L^\infty(0, T; L^4(\Omega; H)) \text{ for } i = 1, 2, 3 \text{ weak star as } \kappa \to 0. \]  \( (37) \)

To prove the theorem, we need a series of lemmas.

**Lemma 9** Assume that \( \textbf{H1-H3} \) hold and \( p_0 \in L^4(\Omega; H) \). Then the functions \( p_1, p_2 \) and \( p_3 \) are equal to a common function \( \xi \in L^2(0, T; V) \cap L^\infty(0, T; L^4(\Omega; H)) \).

**Proof** Integrating \((23)\) over \((t, t_{r+1})\), \((24)\) over \((t_r, t_{r+1})\) and \((25)\) over \((t_r, t)\), yields

\[
p_{1\kappa}(t_{r+1} - 1) - p_{1\kappa}(t) + \int_t^{t_{r+1}} \frac{\mu}{3} p_{1\kappa}(s)ds = \int_t^{t_{r+1}} B^* p_{1\kappa} dY_s,
\]

\[
p_{2\kappa}(t_{r+1} - 1) - p_{1\kappa}(t_{r+1} - 1) + \int_{t_r}^{t_{r+1}} \frac{\mu}{3} p_{2\kappa}(s)ds = \int_{t_r}^{t_{r+1}} (L^* - C)p_{2\kappa}(s)ds,
\]  \( (38) \)

\[
p_{3\kappa}(t) - p_{2\kappa}(t_{r+1} - 1) + \int_{t_r}^{t} \frac{\mu}{3} p_{3\kappa}(s)ds = \int_{t_r}^{t} C p_{3\kappa}(s-)dZ_s.
\]

Adding them up, we get

\[
p_{3\kappa}(t) - p_{1\kappa}(t) + \int_{t_r}^{t_{r+1}} \frac{\mu}{3} p_{1\kappa}(s)ds + \int_{t_r}^{t_{r+1}} \frac{\mu}{3} p_{2\kappa}(s) - (L^* - C)p_{2\kappa}(s)ds
\]

\[
+ \int_{t_r}^{t} \frac{\mu}{3} p_{3\kappa}(s)ds = \int_{t_r}^{t} B^* p_{1\kappa} dY_s + \int_{t_r}^{t} C p_{3\kappa}(s-)dZ_s.
\]

Then we have, for \( t \in [t_r, t_{r+1}] \),

\[
\tilde{E}\|p_{3\kappa}(t) - p_{1\kappa}(t)\|_{V'}^2
\]

\[
\leq 5\tilde{E}\left( \int_t^{t_{r+1}} \frac{\mu}{3} \|p_{1\kappa}(s)\|_{V'} ds \right)^2 + 5\tilde{E}\left( \int_{t_r}^{t_{r+1}} \|p_{2\kappa}(s) - (L^* - C)p_{2\kappa}(s)\|_{V'} ds \right)^2
\]

\[
+ 5\tilde{E}\left( \int_{t_r}^{t} \frac{\mu}{3} \|p_{3\kappa}(s)\|_{V'} ds \right)^2 + 5\tilde{E}\left( \int_{t_r}^{t_{r+1}} (B^* p_{1\kappa} dY_s \right)_{V'}^2 + 5\tilde{E}\left( \int_{t_r}^{t} C p_{3\kappa}(s-)dZ_s \right)_{V'}^2
\]

\[
:= 5I_1 + 5I_2 + 5I_3 + 5I_4 + 5I_5.
\]

Due to Theorem 7 and \((4)\), we have

\[
I_1 \leq \tilde{E}\left( \int_{t_r}^{t_{r+1}} \frac{\mu}{3} \|p_{1\kappa}(s)\| ds \right)^2 \leq C\kappa \tilde{E}\left( \int_{t}^{t_{r+1}} \|p_{1\kappa}(s)\|^2 ds \leq C\kappa.
\]

Similarly, we get

\[
I_2 \leq C\kappa \tilde{E}\int_{t_r}^{t_{r+1}} \|p_{2\kappa}(s)\|^2 ds \leq C\kappa, \quad I_3 \leq C\kappa \tilde{E}\int_{t_r}^{t} \|p_{3\kappa}(s)\|^2 ds \leq C\kappa.
\]
Applying Itô isometry formula to $I_4$, we have
\begin{equation}
I_4 = \int_{t_r}^{t_{r+1}} \tilde{E}\|B^*p_{1\kappa}(s)\|_{V'}^2 \, ds \leq C \int_{t_r}^{t_{r+1}} \tilde{E}\|p_{1\kappa}(s)\|^2 \, ds \\
\leq C\kappa^{1/2} \left( \int_{t_r}^{t_{r+1}} \tilde{E}\|p_{1\kappa}(s)\|^4 \, ds \right)^{1/2} \leq C\kappa^{1/2}.
\end{equation}

Since $Z_t - t$ is a martingale under measure $\tilde{P}$, we have
\begin{align*}
I_5 &\leq \tilde{E}\left\| \int_{t_r}^t C_{3\kappa}(s-)d(Z_s - s) \right\|_{V'}^2, \\
&= \tilde{E}\left( \int_{t_r}^t \|C_{3\kappa}(s-)\|_2^2 \, ds + \tilde{E}\left( \int_{t_r}^t C_{3\kappa}(s-) \, ds \right)^2 \right) \\
&\leq C\kappa^{1/2} \left( \tilde{E}\left( \int_{t_r}^t \|C_{3\kappa}(s-)\|^4 \, ds \right)^{1/2} + C\kappa \tilde{E}\left( \int_{t_r}^t \|C_{3\kappa}(s-)\|^2 \, ds \right) \right) \\
&\leq C\kappa^{1/2}.
\end{align*}

Therefore, we have proved
\begin{equation}
\tilde{E}\|p_{3\kappa}(t) - p_{1\kappa}(t)\|_{V'}^2 \leq C\kappa^{1/2}, \quad t \in [t_r, t_{r+1}]. \tag{39}
\end{equation}

This estimate leads to
\begin{equation}
\lim_{\kappa \to 0} \tilde{E}\|p_{3\kappa} - p_{1\kappa}\|_{V'} = 0, \text{ uniformly for } t \in [0, T].
\end{equation}

Thus we have proved $p_3 = p_1$. Similarly we prove $p_2 = p_1$. Thus $\xi = p_1 = p_2 = p_3$, which completes the proof. \hfill \square

**Lemma 10** Assume that H1-H3 hold and $p_0 \in L^4(\Omega; H)$. Then $p = \xi$ is the unique solution of (22).

**Proof** Integrating equations (23)-(25) over $(t_{i-1}, t_i)$ and adding up, we get
\begin{equation}
p^i_{\kappa} - p^{i-1}_{\kappa} + \int_{t_{i-1}}^{t_i} \left( \frac{\mu}{3} (p_{1\kappa}(s) + p_{2\kappa}(s) + p_{3\kappa}(s)) - (L^* - C)p_{1\kappa}(s) \right) \, ds \\
= \int_{t_{i-1}}^{t_i} B^* p_{1\kappa}(s) \, dY_s + \int_{t_{i-1}}^{t_i} C_{3\kappa}(s-) \, dZ_s. \tag{40}
\end{equation}

Sum up this equation from $i = 0$ to $r$, we get
\begin{align*}
p^r_{\kappa} - p^0_{\kappa} + \int_0^{t_r} \left( \frac{\mu}{3} (p_{1\kappa}(s) + p_{2\kappa}(s) + p_{3\kappa}(s)) - (L^* - C)p_{1\kappa}(s) \right) \, ds \\
= \int_0^{t_r} B^* p_{1\kappa} \, dY_s + \int_0^{t_r} C_{3\kappa}(s-) \, dZ_s. \tag{41}
\end{align*}

For any $t \in (t_r, t_{r+1})$, integrating (23) on $[t_r, t]$ leads to
\begin{equation}
p_{1\kappa}(t) - p^r_{\kappa} + \int_{t_r}^{t} \frac{\mu}{3} p_{1\kappa}(s) \, ds = \int_{t_r}^{t} B^* p_{1\kappa}(s) \, dY_s. \tag{42}
\end{equation}
Adding up (41) and (42), we have
\[
\begin{align*}
&\quad p_{1\kappa}(t) - p_0 + \int_0^t \frac{\mu}{3} p_{1\kappa}(s) ds - \int_0^{[t/\kappa]} (L^* - C) p_{2\kappa}(s) ds \\
&+ \int_0^{[t/\kappa]} \frac{\mu}{3} (p_{2\kappa}(s) + p_{3\kappa}(s)) ds = \int_0^t B^* p_{1\kappa}(s) dY_s + \int_0^{[t/\kappa]} C p_{3\kappa}(s) dZ_s.
\end{align*}
\] (43)

Noticing that as \( \kappa \to 0 \), for \( i = 2, 3 \)
\[
\tilde{E} \left\| \int_{[t/\kappa]}^{t} p_{i\kappa}(s) ds \right\|^2 \leq (t - [t/\kappa]) \left( \int_{[t/\kappa]}^{t} \tilde{E} \| p_{i\kappa}(s) \|^2 ds \right) \leq C (t - [t/\kappa]) \to 0.
\]
According to Itô isometry formula, we have, as \( \kappa \to 0 \),
\[
\tilde{E} \left\| \int_{[t/\kappa]}^{t} B^* p_{1\kappa}(s) dY_s \right\|^2_{\mathcal{V'}} = \tilde{E} \left\| B^* p_{1\kappa}(s) \right\|^2_{\mathcal{V'}} ds \leq (t - [t/\kappa])^{1/2} \tilde{E} \left( \int_{[t/\kappa]}^{t} \| p_{1\kappa}(s) \|^4 ds \right)^{1/2} \to 0,
\]
and
\[
\begin{align*}
&\quad \tilde{E} \left\| \int_{[t/\kappa]}^{t} C p_{3\kappa}(s) dZ_s \right\|^2 \leq 2 \tilde{E} \left\| \int_{[t/\kappa]}^{t} C p_{3\kappa}(s) (dZ_s - ds) \right\|^2 + 2 \tilde{E} \left\| \int_{[t/\kappa]}^{t} C p_{3\kappa}(s) ds \right\|^2 \\
&= 2 \tilde{E} \int_{[t/\kappa]}^{t} \| C p_{3\kappa}(s) \|^2 ds + 2 \tilde{E} \left\| \int_{[t/\kappa]}^{t} C p_{3\kappa}(s) ds \right\|^2 \\
&\leq C (t - [t/\kappa])^{1/2} \tilde{E} \left( \int_{[t/\kappa]}^{t} \| p_{3\kappa}(s) \|^4 ds \right)^{1/2} + C (t - [t/\kappa]) \tilde{E} \int_{[t/\kappa]}^{t} \| p_{3\kappa}(s) \|^2 ds \\
&\quad \to 0.
\end{align*}
\]

Taking limit in (43) in weak star sense as \( \kappa \to 0 \), we obtain
\[
\xi(t) - p(0) + \int_0^t \mu \xi(s) ds - \int_0^t (L^* - C) \xi(s) ds = \int_0^t B^* \xi(s) dY_s + \int_0^t C \xi(s) dZ_s.
\]
This is precisely equation (22). Then the proof of this lemma follows from Lemma 6. \( \square \)

**Proof of Theorem 8.** We integrate (31), (32) and (33) over interval \([t_i, t_{i+1}]\), then take expectation and sum them up to obtain
\[
\tilde{E} \left\| p_{1\kappa}^{t_{i+1}} \right\|^2 - \tilde{E} \left\| p_{1\kappa}^{t_i} \right\|^2 + \tilde{E} \int_{t_i}^{t_{i+1}} \left( \frac{2}{3} \mu (\| p_{1\kappa}(s) \|^2 + \| p_{2\kappa}(s) \|^2 + \| p_{3\kappa}(s) \|^2) \\
- \| B^* p_{1\kappa}(s) \|^2_D - 2 \langle (L^* - C) p_{2\kappa}(s), p_{2\kappa}(s) \rangle \right) ds = \tilde{E} \int_{t_i}^{t_{i+1}} (\lambda^2 - 1) \| p_{3\kappa}(s) \|^2 dZ_s.
\]
Summing up this equation in \( i \) from 0 up to \( r - 1 \), we get
\[
\tilde{E} \left\| p_{1\kappa}^{t_r} \right\|^2 - \tilde{E} \left\| p_{1\kappa}^{0} \right\|^2 + \tilde{E} \int_0^{t_r} \left( \frac{2}{3} \mu (\| p_{1\kappa}(s) \|^2 + \| p_{2\kappa}(s) \|^2 + \| p_{3\kappa}(s) \|^2) \\
- \| B^* p_{1\kappa}(s) \|^2_D - 2 \langle (L^* - C) p_{2\kappa}(s), p_{2\kappa}(s) \rangle \right) ds = \tilde{E} \int_0^{t_r} (\lambda^2 - 1) \| p_{3\kappa}(s) \|^2 dZ_s
\] (44)
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For any \( t \in [t_r, t_{r+1}] \), integrating (32) on \([t_r, t] \), we have

\[
\hat{E}\|p_{1\kappa}(t)\|^2 - \hat{E}\|p_0\|^2 + \hat{E} \int_{t_r}^t \frac{2}{3} \mu\|p_{1\kappa}(s)\|^2 ds = \hat{E} \int_{t_r}^t |\mathcal{B}^* p_{1\kappa}(s)|_D^2 ds. \tag{45}
\]

Adding (44) to (45), we obtain

\[
\hat{E}\|p_{1\kappa}(t)\|^2 - \hat{E}\|p_0\|^2 + \hat{E} \int_{t_r}^t \frac{2}{3} \mu\|p_{1\kappa}(s)\|^2 ds - 2\hat{E} \int_{t_r}^t (\langle \mathcal{L}^* - \mathcal{C} \rangle p_{2\kappa}(s), p_{2\kappa}(s)) ds + \hat{E} \int_{t_r}^t \|p_{2\kappa}(s)\|^2 ds + \hat{E} \int_{t_r}^t \|p_{3\kappa}(s)\|^2 ds
\]

\[= \hat{E} \int_{t_r}^t |\mathcal{B}^* p_{1\kappa}(s)|_D^2 ds + \hat{E} \int_{t_r}^t (\lambda^2 - 1)\|p_{3\kappa}(s-\cdot)\|^2 dZ_s. \tag{46}
\]

Define

\[
S_1^\kappa := \hat{E}\|p(t)\|^2 - 2\hat{E} \int_{t_r}^t (\langle \mathcal{L}^* - \mathcal{C} \rangle p(s), p(s)) ds + \hat{E} \int_{t_r}^t \frac{2}{3} \mu\|p(s)\|^2 ds
\]

\[+ \hat{E} \int_{t_r}^t \frac{4}{3} \mu\|p(s)\|^2 ds - \hat{E} \int_{t_r}^t |\mathcal{B}^* p(s)|_D^2 ds - \hat{E} \int_{t_r}^t (\lambda^2 - 1)\|p(s-\cdot)\|^2 dZ_s,
\]

\[
S_2^\kappa := -2\hat{E}(p(t), p_{1\kappa}(t)) + 4\hat{E} \int_{t_r}^t \langle (\mathcal{L}^* - \mathcal{C}) p_{2\kappa}(s), p(s) \rangle ds
\]

\[- \frac{4}{3} \mu\hat{E} \int_{t_r}^t (p(s), p_{1\kappa}(s)) ds - \frac{4}{3} \mu\hat{E} \int_{t_r}^t (p(s), p_{2\kappa}(s) + p_{3\kappa}(s)) ds + 2\hat{E} \int_{t_r}^t \mathcal{B}^* p(s) D(\mathcal{B}^* p_{2\kappa}(s))^T dx ds + 2\hat{E} \int_{t_r}^t (\lambda^2 - 1)(p(s-\cdot), p_{3\kappa}(s-\cdot)) dZ_s,
\]

\[
S_3^\kappa := \hat{E}\|p_{1\kappa}(t)\|^2 + \hat{E} \int_{t_r}^t \frac{2}{3} \mu\|p_{1\kappa}(s)\|^2 ds - 2\hat{E} \int_{t_r}^t (\langle \mathcal{L}^* - \mathcal{C} \rangle p_{2\kappa}(s), p_{2\kappa}(s)) ds
\]

\[+ \hat{E} \int_{t_r}^t \frac{2}{3} \mu\|p_{2\kappa}(s)\|^2 + \|p_{3\kappa}(s)\|^2 ds - \hat{E} \int_{t_r}^t |\mathcal{B}^* p_{2\kappa}(s)|_D^2 ds
\]

\[+ \hat{E} \int_{t_r}^t (\lambda^2 - 1)\|p_{3\kappa}(s-\cdot)\|^2 dZ_s.
\]
We now consider the convergence of these items in $L^2(\Omega; H)$ as $\kappa \to 0$

\[
S^1_\kappa \to \hat{E}\|p(t)\|^2 - 2\hat{E}\int_0^t \langle (\mathcal{L}^* - \mathcal{C})p(s), p(s) \rangle ds + 2\hat{E}\int_0^t \mu\|p(s)\|^2 ds
\]

\[
- \hat{E}\int_0^t |B^*p(s)|^2 ds - \hat{E}\int_0^t (\lambda^2 - 1)\|p(s)\|^2 dZ_s
\]

\[
= \|p_0\|^2,
\]

\[
S^2_\kappa \to -2\hat{E}\|p(t)\|^2 + 4\hat{E}\int_0^t \langle (\mathcal{L}^* - \mathcal{C})p(s), p(s) \rangle ds - 4\mu\hat{E}\int_0^t \|p(s)\|^2 ds
\]

\[
+ 2\hat{E}\int_0^t |Bp(s)|^2 ds + 2\hat{E}\int_0^t (\lambda^2 - 1)\|p(s)\|^2 dZ_s
\]

\[
= -2\|p_0\|^2.
\]

Notice that $\lim_{\kappa \to 0} S^3_\kappa = \|p_0\|^2$ also follows from (46). Therefore we have $S_\kappa := S^1_\kappa + S^2_\kappa + S^3_\kappa \to 0$ as $\kappa \to 0$.

Choosing $\mu = \max\{\frac{3}{2}M, 3\alpha, \frac{3}{2}(\omega^2 - 1)\} + 1$ and by the uniform elliptic condition, we have

\[
-\langle (\mathcal{L}^* - \mathcal{C})(p(t) - p_{2\kappa}(t)), p(t) - p_{2\kappa}(t) \rangle + \alpha\|p(t) - p_{2\kappa}(t)\|^2 \geq \beta_2\|p(t) - p_{2\kappa}(t)\|^2.
\]

Thus

\[
S_\kappa \geq \hat{E}\|p(t) - p_{1\kappa}(t)\|^2 + \left(\frac{2}{3}\mu - M^2\right)\hat{E}\int_0^t \|p(s) - p_{1\kappa}(s)\|^2 ds
\]

\[
+ \left(\frac{2}{3}\mu - 2\alpha\right)\hat{E}\int_0^{[t/\kappa]} \|p(s) - p_{2\kappa}(s)\|^2 ds
\]

\[
+ \left(\frac{2}{3}\mu - \lambda^2 + 1\right)\hat{E}\int_0^{[t/\kappa]} \|p(s) - p_{3\kappa}(s)\|^2 ds \geq 0,
\]

which implies that for any $\forall t \in [0, T]$, as $\kappa \to 0$

\[
\hat{E}\|p(t) - p_{1\kappa}(t)\|^2 \to 0, \quad \hat{E}\int_0^t \|p(s) - p_{1\kappa}(s)\|^2 ds \to 0, \quad S_\kappa \to 0.
\]

Hence as $\kappa \to 0$

\[
p_{1\kappa}(t) \to p(t) \text{ in } L^2(\Omega; H) \text{ uniformly for } t \in [0, T],
\]

\[
p_{1\kappa} \to p \text{ in } L^2(0, T; H).
\]

Similarly, we obtain the convergence of $p_{2\kappa}$ and $p_{3\kappa}$ as $\kappa \to 0$. 

As an application of Theorem 8, we immediately obtain a convergence property for splitting-up solution $p_{\kappa+1}^r$. \qed
Theorem 11 Assume that H1-H3 hold and \( p_0 \in L^4(\Omega; H) \). Then the splitting-up solution \( p_{\kappa} \) converges to the exact solution \( p(t_{\kappa+1}) \) in \( L^2(\Omega; H) \) as \( \kappa \to 0 \).

For \( \phi \in V \) and \( \tau \in [0, T] \), define two processes \( \psi \) and \( \zeta \) by

\[
\psi(\tau) = (\Gamma^s_c(\mathcal{L}^* - C) - (\mathcal{L}^* - C)\Gamma^s_r)R^s_rQ^s_r\phi,
\]

\[
\zeta(\tau) = (\Gamma^s_cR^s_rB^* - B^*\Gamma^s_cR^s_r)Q^s_r\phi.
\]

We now estimate the two processes, which will play an important role in the subsequent analysis.

Lemma 12 Assume that H1-H3 hold and \( \phi \in V \cap H^3 \). Then for \( 0 \leq s < \tau \leq T \)

\[
\tilde{E}\|\psi(\tau)\|^2 \leq C(\tau - s)\tilde{E}\|\phi\|^2_3, \quad \|\tilde{E}(\psi(\tau))\| \leq C(\tau - s)\tilde{E}\|\phi\|_3,
\]

\[
\tilde{E}\|\zeta(\tau)\|^2 \leq C(\tau - s)\tilde{E}\|\phi\|^2_3, \quad \|\tilde{E}(\zeta(\tau))\| \leq C(\tau - s)\tilde{E}\|\phi\|_3.
\]

Proof By Lemma 5, we have

\[
\|\mathcal{L}^*\phi\| \leq C\|\phi\|_1, \quad \|B^*\phi\| \leq C\|\phi\|_1.
\]

From (25) it follows that

\[
\Gamma^s_c(\mathcal{L}^* - C)R^s_rQ^s_r\phi = (\mathcal{L}^* - C)R^s_rQ^s_r\phi - \frac{\mu}{3} \int_s^\tau \Gamma^s_c(\mathcal{L}^* - C)R^s_rQ^s_r\phi d\tau' + \int_s^\tau C\Gamma^s_c(\mathcal{L}^* - C)R^s_rQ^s_r\phi d\tau',
\]

\[
(\mathcal{L}^* - C)\Gamma^s_cR^s_rQ^s_r\phi = (\mathcal{L}^* - C)R^s_rQ^s_r\phi - \frac{\mu}{3} \int_s^\tau (\mathcal{L}^* - C)\Gamma^s_cR^s_rQ^s_r\phi d\tau' + \int_s^\tau (\mathcal{L}^* - C)\Gamma^s_cR^s_rQ^s_r\phi d\tau'.
\]

Let \( U_1 = \Gamma^s_c(\mathcal{L}^* - C) - (\mathcal{L}^* - C)\Gamma^s_r, U_2 = C\Gamma^s_c(\mathcal{L}^* - C) - (\mathcal{L}^* - C)\Gamma^s_r \), then

\[
\psi(\tau) = -\frac{\mu}{3} \int_s^\tau U_1R^s_rQ^s_r\phi d\tau' + \int_s^\tau U_2R^s_rQ^s_r\phi d\tau'. \tag{47}
\]

Since \( Z_{\tau'} - \tau' \) is a martingale, we have

\[
\tilde{E}\|\psi(\tau)\|^2 \leq 2\tilde{E} \left( \int_s^\tau \| - \frac{\mu}{3} U_1R^s_rQ^s_r\phi\| d\tau' \right)^2 + 4\tilde{E} \int_s^\tau \| U_2R^s_rQ^s_r\phi\|^2 d\tau' + 4\tilde{E} \int_s^\tau \| U_2R^s_rQ^s_r\phi\| d\tau' \tag{48}
\]

\[
\leq C(\tau - s)\tilde{E}\|\phi\|^2_3
\]

From (47), we have

\[
\|\tilde{E}(\psi(\tau))\| \leq \frac{\mu}{3} \int_s^\tau \| \tilde{E}(U_1R^s_rQ^s_r\phi)\| d\tau' + \int_s^\tau \| \tilde{E}(U_2R^s_rQ^s_r\phi)\| d\tau' \leq C(\tau - s)\tilde{E}\|\phi\|_3.
\]

Similarly, we obtain the estimates for \( \zeta \).
Theorem 13 Assume that H1-H3 hold and $p(t) \in V \cap H^3$ is the solution of (22). Then $p_{k+1}^t$ converges to $p(t_{r+1})$ as $\kappa \to 0$, and satisfies

$$
(\mathcal{E}(p(t_{r+1}) - p_{k+1}^t)^2)^{1/2} \leq C\kappa^{1/2},
$$
$$
\|\mathcal{E}(p(t_{r+1}) - p_{k+1}^t)\| \leq C\kappa.
$$

Proof Let $\varepsilon(t) = \Gamma^s_0 R^s_0 Q^s_0 \phi$ for $\phi \in C_0^\infty$ and $0 \leq s < t \leq T$. Then

$$
d\varepsilon(t) = \Gamma^s_0 (-\frac{\mu}{3} R^s_0 Q^s_0 \phi dt + (\mathcal{L}^* - C) R^s_0 Q^s_0 \phi dt + R^s_0 (-\frac{\mu}{3} Q^s_0 \phi dt + B^* Q^s_0 \phi dY_t))
$$
$$
- \frac{\mu}{3} \varepsilon(t) dt + C\varepsilon(t-1) dZ_t
$$
$$
= - \mu \varepsilon(t) dt + \Gamma^s_0 (\mathcal{L}^* - C) R^s_0 Q^s_0 \phi dt + \Gamma^s_0 R^s_0 B^* Q^s_0 \phi dY_t + \varepsilon(t-) dZ_t.
$$
$$
= - \mu \varepsilon(t) dt + (\mathcal{L}^* - C) \varepsilon(t) dt + B^* \varepsilon(t) dY_t + C\varepsilon(t-) dZ_t
$$
$$
+ \psi(t) dt + \zeta(t) dY_t.
$$

Let $\gamma(t) = \varepsilon(t) - p(t)$. By Itô's formula, we have

$$
\mathcal{E}(\gamma(t))^2 = \mathcal{E}(\gamma(s))^2 - 2\mu\mathcal{E} \int_s^t \|\gamma(t)\|^2 d\tau + 2\mathcal{E} \int_s^t \langle(\mathcal{L}^* - C)\gamma(t), \gamma(t)\rangle d\tau
$$
$$
+ 2\mathcal{E} \int_s^t (\gamma(t), \psi(t)) d\tau + \int_s^t \|B^* \gamma(t) + \zeta(t)\|^2 d\tau + \int_s^t (\lambda^2 - 1) \|\gamma(t)\|^2 d\tau.
$$

From Lemma 5 and inequality (21), we have

$$
\mathcal{E}(\gamma(t))^2 \leq \mathcal{E}(\gamma(s))^2 + C \int_s^t \mathcal{E}(\gamma(t))^2 d\tau + \int_s^t \mathcal{E}(\psi(t))^2 d\tau + 2 \int_s^t \mathcal{E}(\zeta(t))^2 d\tau.
$$

Applying the Gronwall lemma and Lemma 12 gives

$$
\mathcal{E}(\gamma(t))^2 \leq \left(\mathcal{E}(\gamma(s))^2 + \int_s^t \mathcal{E}(\psi(t))^2 d\tau + \int_s^t \mathcal{E}(\zeta(t))^2 d\tau\right) e^{C(t-s)}
$$
$$
\leq (\mathcal{E}(\gamma(s))^2 + C(t-s)^2 \mathcal{E}(\phi^2_3) e^{C(t-s)}).
$$

Taking $s = t_i$, $t = t_{i+1}$ and $\phi = p^i_k$, we obtain

$$
\mathcal{E}(p_{k+1}^{r+1} - p(t_{r+1}))^2 \leq \left(\mathcal{E}(p_{k+1}^i - p(t_i))^2 + C\kappa^2 \mathcal{E}(p_{k+1}^i)^2\right) e^{C\kappa}.
$$

(49)

Iterating the above equation in $i$ from $i = 0$ to $r$ and applying Theorem 7, we have

$$
\mathcal{E}(p_{k+1}^{r+1} - p(t_{r+1}))^2
$$
$$
\leq \left[\left(\mathcal{E}(p_{k+1}^{r-1} - p(t_{r-1}))^2 + C\kappa^2 \mathcal{E}(p_{k+1}^{r-1})^2\right) e^{C\kappa} + C\kappa^2 \mathcal{E}(p_{k+1}^r)^2\right] e^{C\kappa}
$$
$$
\leq \cdots \leq C\kappa^2 \sum_{i=0}^r \mathcal{E}(p_{k+1}^i)^2 e^{CT} \leq C\kappa.
$$

Noticing

$$
\mathcal{E}(\gamma(t)) = \mathcal{E}(\gamma(s)) - \mu \int_s^t \mathcal{E}(\gamma(t)) d\tau + \int_s^t \mathcal{E}(\mathcal{L}^* \gamma(t)) d\tau + \int_s^t \mathcal{E}(\psi(t)) d\tau,
$$

we have

$$
\|\mathcal{E}(\gamma(t))\| \leq \|\mathcal{E}(\gamma(s))\| + C \int_s^t \|\mathcal{E}(\gamma(t))\| d\tau + \int_s^t \|\mathcal{E}(\psi(t))\| d\tau.
$$
Then by the Gronwall lemma, from Lemma 12, we have
\[
\|\bar{E}(\gamma(t))\| \leq (\|\bar{E}(\gamma(s))\| + \int_s^t \|\bar{E}(\psi(\tau))\|d\tau)e^{C(t-s)}
\]
\[
\leq (\|\bar{E}(\gamma(s))\| + C(t-s)^2 \bar{E}\|\phi\|_3)e^{C(t-s)}.
\]
Taking \(s = t_i, t = t_{i+1}\) and \(\phi = \check{p}_i\), we obtain
\[
\|\bar{E}(p_{i+1}^r - p(t_{i+1}))\| \leq (\|\bar{E}(p_i^r - p(t_i))\| + C\kappa^2 \bar{E}\|p_i^r\|_3)e^{C\kappa}.
\]
(50)
Integrating the above equation in \(i\) from \(i = 0\) to \(r\) and applying Theorem 7, we have
\[
\|\bar{E}(p_{i+1}^r - p(t_{r+1}))\|
\leq \left[\|\bar{E}(p_i^r - p(t_{r-1}))\| + C\kappa^2 \bar{E}\|p_i^r\|_3e^{C\kappa} + C\kappa^2 \bar{E}\|p_i^r\|_3\right]e^{C\kappa}
\leq \cdots \leq C\kappa^2 \sum_{i=0}^r \bar{E}\|p_i^r\|_3e^{C\kappa t} \leq C\kappa.
\]
The proof is complete. \(\square\)

**Remark** We note that \(p_{1\kappa}, p_{2\kappa}\) and \(p_{3\kappa}\) are defined by the continuous solution operators \(Q_1^r, R_2^r\) and \(\Gamma_3^r\) for (23), (24) and (25), respectively. They are splitting up solutions for continuous problems, not numerical ones. In the next section, we will consider temporal discretizations of (23), (24), (25) and construct semi-discretized splitting-up approximations for the exact solution \(p\).

## 4 Semi-discretization and error analysis

In this section, we construct a semi-discretized splitting-up scheme by discretizing (23)-(25) with the finite difference method and investigate its error estimate.

On each interval \([t_r, t_{r+1}]\) \((r = 0, 1, 2, \cdots , N - 1)\), we apply the Euler-Maruyama scheme to (23), backward implicit Euler method to (24) and forward explicit Euler method to (25) to obtain a semi-discrete scheme:

\[
p_{1\kappa,r+1} - p_{1\kappa,r} = -\frac{\mu}{3} p_{1\kappa,r} + B^* p_{1\kappa,r} (Y_{t_{r+1}} - Y_{t_r}),
\]
(51)
\[
p_{2\kappa,r+1} - p_{2\kappa,r} = ((L^* - C) p_{2\kappa,r+1} - \frac{\mu}{3} p_{2\kappa,r+1})\kappa,
\]
(52)
\[
p_{3\kappa,r+1} - p_{3\kappa,r} = -\frac{\mu}{3} p_{3\kappa,r} + C p_{3\kappa,r} (Z_{t_{r+1}} - Z_{t_r}),
\]
(53)
where \(Z_{t_{r+1}} - Z_{t_r}\) is the number of jumps of Poisson process \(Z_t\) within time interval \([t_r, t_{r+1}]\). The iterative solutions \(p_{i\kappa,r}\) of equations (51)-(53) are numerical approximations to \(p_{i\kappa}(t_r)\) for \(i = 1, 2, 3\). And each jump quantity is approximated by \(C p_{3\kappa,r} = (\lambda - 1)p_{3\kappa,r}\).

Let \(\bar{Q}_{t_{r+1}}^r, \bar{R}_{t_{r+1}}^r\) and \(\bar{\Gamma}_{t_{r+1}}^r\) successively denote the solution operators of above equations. In terms of these settings, we define a discrete splitting-up approximate solution \(p_{\kappa,r}\) \((r = 0, 1, 2, \cdots , N)\) of Zakai equation (20) as

\[
p_{\kappa,r+1} = \bar{\Gamma}_{t_{r+1}}^r \bar{R}_{t_{r+1}}^r \bar{Q}_{t_{r+1}}^r p_{\kappa,r}, \quad p_{\kappa,0} = p_0.
\]
(54)
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Now, we are ready to state and prove the main result of this work.

**Theorem 14** Assume H1-H3. Then the discrete splitting-up solution $p_{k,r+1}$ converges to the exact solution $p(t_{r+1})$ as $\kappa \to 0$ and satisfies

$$\{\mathbf{E}\|p(t_{r+1}) - p_{k,r+1}\|^2\}^{1/2} \leq C\sqrt{\kappa}. \tag{55}$$

**Proof** By the Milstein Theorem in [33, Theorem 1.1, page 12], we only need to show that

$$\{\mathbf{E}\|p(t_{r+1}) - p_{k,r+1}\|^2\}^{1/2} \leq C\|p(t_r)\|\kappa,$$

$$\mathbf{E}\|p(t_{r+1}) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\| \leq C\|p(t_r)\|^2\kappa.$$

First, we estimate the truncation error in the mean square sense. Assume $p(t_r) = p_{k,r}$, then

$$\mathbf{E}\|p(t_{r+1}) - p_{k,r+1}\|^2 = \mathbf{E}\|p(t_{r+1}) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\|^2$$

$$\leq 3\mathbf{E}\|p(t_{r+1}) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\|^2$$

$$+ 3\mathbf{E}\|\Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\|^2$$

$$+ 3\mathbf{E}\|\Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\|^2$$

$$+ 3\mathbf{E}\|\Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\|^2$$

$$:= S_0 + S_1 + S_2 + S_3.$$

Inequality (49) directly implies that $S_0 \leq C\kappa^2\|p(t_r)\|^2$. From [32, Theorem 4.6] and [23, Theorem 67] it follows that

$$\mathbf{E}\|\Gamma^r_{t_{r+1}} \phi\|^2 \leq C\mathbf{E}\|\phi\|^2, \quad \mathbf{E}\|R^r_{t_{r+1}} \phi\|^2 \leq C\mathbf{E}\|\phi\|^2, \quad \forall \phi \in L^2(\Omega; H), \tag{56}$$

which leads to

$$S_1 \leq C\mathbf{E}\|Q^r_{t_{r+1}} p(t_r) - Q^r_{t_{r+1}} p(t_r)\|^2 \leq C\|p(t_r)\|^2 \kappa^2.$$

Here we have used the convergence theorem of the Euler-Maruyama method [34]. Similarly, we obtain $S_3 \leq C\|p(t_r)\|^2\kappa^2$.

According to the convergence theorem of the backward Euler scheme [35, page 343], we have

$$S_2 \leq C\mathbf{E}\|R^r_{t_{r+1}} - R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r)\|^2 \leq C\|Q^r_{t_{r+1}} p(t_r)\|^2 \kappa^4 \leq C\|p(t_r)\|^2 \kappa^4.$$

Therefore, we have proved that

$$\{\mathbf{E}\|p(t_{r+1}) - p_{k,r+1}\|^2\}^{1/2} \leq C\|p(t_r)\|\kappa. \tag{57}$$

Next, we estimate the truncation error in expectation. Notice that

$$\mathbf{E}(p(t_{r+1}) - p_{k,r+1}) = \mathbf{E}(p(t_{r+1}) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r))$$

$$\leq \mathbf{E}(p(t_{r+1}) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r))$$

$$+ \mathbf{E}(\Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r))$$

$$+ \mathbf{E}(\Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r))$$

$$+ \mathbf{E}(\Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r))$$
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From \((50)\) it follows that \(S'_0 \leq C\|p(t_r)\|\kappa^2\). By \((56)\) and the convergence theory of Euler-Maruyama method in \([34]\), we have

\[
S'_1 \leq C\|\bar{E}[Q^r_{t_{r+1}} - \bar{Q}^r_{t_{r+1}}]p(t_r)\| \leq C\|p(t_r)\|\kappa^2.
\]

Similarly, we obtain \(S'_2 \leq C\|p(t_r)\|\kappa^2\).

From \([35]\), page 343 and the convergence theory of implicit Euler scheme, we get

\[
S'_2 \leq C\|\bar{E}[R^r_{t_{r+1}} - \bar{R}^r_{t_{r+1}}]Q^r_{t_{r+1}} p(t_r)\| \leq C\bar{E}\|Q^r_{t_{r+1}} p(t_r)\|\kappa^2 \leq C\|p(t_r)\|\kappa^2.
\]

Summing the above estimates, we obtain

\[
\|\bar{E}(p(t_{r+1}) - \Gamma^r_{t_{r+1}} R^r_{t_{r+1}} Q^r_{t_{r+1}} p(t_r))\| \leq C\|p(t_r)\|\kappa^2. \tag{58}
\]

By the estimates in \((57)\) and \((58)\), and Milstein’s theorem \([33]\), we obtain \((55)\). Therefore, the proof is completed. \(\Box\)

5 Numerical experiments

In this section, we apply our algorithm to a linear filtering model and a nonlinear filtering model to illustrate our theoretical results on error estimates.

We use a spectral Galerkin method to discretize the spatial variable with an \(n\)-dimensional subspace whose basis functions are given by

\[
e_i(x) = \sqrt{\frac{\phi(x)}{(i-1)!}} f_{i-1}(x), \quad x \in \mathbb{R}, \quad i = 1, 2, \ldots n,
\]

where \(f_{i-1}(x)\) is a Hermite polynomial of order \(i - 1\) and \(\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}\). Due to \([36]\), the spatial Galerkin discretization errors are expected to decrease exponentially with respect to the dimension \(n\). To proceed with numerical experiments, we need to simulate the sample trajectories of the Poisson process with density function \(\lambda(X_t)\). For convenience, we assume the jump time \(\tau_i\) may occur only at \(t_r\). To determine the jump time, we first calculate a discrete sample trajectory \(X_{t_r}\) \((r = 0, 1, \ldots, N)\) in terms of equation \((1)\) and compute the density function \(\lambda(X_{t_r})\). For any \(i \leq r\), define \(T_i(t_r) := \kappa \sum_{j=1}^{r-1} \lambda(X_{t_j})\). Assuming \(\tau_0 = 0\) and \(\tau_i\) is the present jump time; we will describe a criterion for finding the next jump time \(\tau_{i+1}\). Let \(E\) be a random number generated by a random variable with unit exponential distribution and independent of \(X_{t_r}\). Then the next jump time \(\tau_{i+1}\) is defined as the first time that \(T_i(t_r)\) exceeds \(E\).

**Example 1.** Consider a linear filtering model

\[
\begin{align*}
dX_t &= 0.5X_t dt + 2dw_t, \quad X_0 \sim \mathcal{N}(5, 0.01), \\
dY_t &= X_t dt + 0.5dw_t + dv_t, \quad Y_0 = 0,
\end{align*}
\]

\(Z_t\) is a doubly stochastic Poisson process with the intensity of \(\lambda X_t^2\).
The corresponding Zakai equation reads

\[ dp(t) = \left( 2 \frac{\partial^2 p(t)}{\partial x^2} - \frac{1}{2} \frac{\partial p(t)}{\partial x} - \lambda x^2 p(t) + p(t) \right) dt \]

\[ + \left( - \frac{2\sqrt{5}}{5} \frac{\partial p(t)}{\partial x} + \frac{11}{5} \sqrt{5} x p(t) \right) dY_t + (\lambda x^2 - 1)p(t-)dZ_t. \]

Taking a time stepsize \( \kappa = 0.5 \times 10^{-5} \) and choosing \( \lambda = 3 \), we trace a sample trajectory using the Zakai filter and depict it in Fig. 1.

\[ \text{Fig. 1} \quad \text{The Zakai filter and signal for } \lambda = 3. \]

In addition, we simulate a sample path for only continuous observation \( Y_t \) with \( \lambda = 0 \) and mixed observations of \( Y_t \) and \( Z_t \) with \( \lambda = 3 \). We trace their corresponding conditional standard deviation versus time \( t \), see Fig. 2. It shows that including information on point process observation reduces the conditional standard deviation.
We fix $T = 0.25$, $\lambda = 2$ and a stepsize $\kappa = N^{-1} = 2^{-20}$ and then compute $m = 500$ reference 'exact' sample paths. Simultaneously we compute $m$ Zakai filter approximate solutions for each stepsize $\kappa_i = N_i^{-1} = 2^{-i}$ with $i = 16, 17, 18, 19$. Define an error function

$$d(\kappa_i) = \left\{ \frac{1}{N_i m} \sum_{r=1}^{N_i} \sum_{j=1}^{m} |\hat{X}^j(t_r) - \tilde{X}^j_r|^2 \right\}^{1/2}.$$ 

The dynamic behavior of the errors as varying stepsize $\kappa_i$ is exhibited in Fig. 3, demonstrating the half order convergence rate.
Example 2. Consider a nonlinear filtering model

\[ dX_t = \sin(X_t)dt + 2dw_t, \quad X_0 \sim \mathcal{N}(5, 0.01), \]
\[ dY_t = 5.5X_t dt + 0.5dw_t + dv_t, \quad Y_0 = 0, \]

\( Z_t \) is a doubly stochastic Poisson process with the intensity of \( \lambda X_t^2 \).

The corresponding Zakai equation is

\[
dp(t) = \left(2 \frac{\partial^2 p(t)}{\partial x^2} - \sin(x) \frac{\partial p(t)}{\partial x} - \cos(x)p(t) - (\lambda x^2 - 1)p(t)\right)dt
+ \left(-\frac{2}{5} \sqrt{5} \frac{\partial p(t)}{\partial x} + \frac{11}{5} \sqrt{5} xp(t)\right)dY_t + (\lambda x^2 - 1)p(t-)dZ_t.
\]

Set \( \lambda = 3, \ T = 0.5, \ \kappa = 0.5 \times 10^{-5} \). We trace a sample signal trajectory using our Zakai filter and depict the approximations in Fig. 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{The Zakai filter and signal process.}
\end{figure}

Next, we verify the convergence order in a temporal variable. We compute \( m = 500 \) reference “exact” Zakai filter solutions by fixing the stepsize \( \kappa = 2^{-20} \).

Then we calculate \( m \) numerical Zakai filter solutions for each stepsize \( \kappa_i = 2^{-i} \), \( i = 14, 15, 16, 17 \). We plot the errors in log-log scale, cf. Fig. 5, which shows that the convergence order is of \( \frac{1}{2} \).
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6 Conclusion and discussion

In this paper, we considered a nonlinear filtering model with observations involving a mixture of a Wiener process and a point process. After deriving the corresponding Zakai equation, we constructed the splitting-up scheme where the Zakai equation is decomposed into three equations: A deterministic PDE, an SDE driven by a Wiener process, and an SDE driven by a point process. Then we discretized these equations in the temporal direction by finite difference methods. By estimating the errors of these splitting up equations and the errors of the temporal discretization, we derived the half-order convergence result for the proposed numerical scheme using Milstein’s fundamental theorem on numerical methods for SDEs. Our current work focuses on semi-discretizations in time. Future research on this topic includes the construction and error estimates for fully discretized numerical schemes.

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