1. INTRODUCTION

There has been intense recent speculation on the form of the QCD phase diagram at small temperature $T$ but large baryon chemical potential $\mu$. The conventional view is that for sufficiently large $\mu$, chiral symmetry is spontaneously restored, resulting in a degenerate system of relativistic quarks known as quark matter, with Fermi energy $E_F = \mu/3$. Since the $qq$ interaction can be attractive due to gluon exchange [1] and/or instanton effects [2], however, this simple picture may have a BCS instability with respect...
to condensation of diquark pairs at the Fermi surface, resulting in a color superconducting ground state \[3\], and the formation of an energy gap \(\Delta \neq 0\) to the first excited state of the system (see Fig. \[1\]). Model calculations \[4\] suggest that \(\Delta\) could be as large as 100 MeV, comparable with the constituent quark scale.

It is clearly desirable to examine the various scenarios in non-perturbative lattice QCD simulations, where systematic control is at least in principle possible. Unfortunately, this has so far proved impracticable because the Euclidean functional measure is no longer positive definite for \(\mu \neq 0\), rendering the usual importance sampling methods ineffective. There are, however, simpler four-fermion models where Monte Carlo methods can be applied and yield a plausible description of degenerate matter \[5\]. In this talk I describe numerical studies of the Nambu – Jona-Lasinio (NJL) model in 2+1 spacetime dimensions; the Lagrangian in continuum notation reads

\[
\mathcal{L} = \bar{\psi}(\partial \psi + \mu \gamma_0 + m) - \frac{g^2}{2} \left[ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \tau \psi)^2 \right].
\]

(1)

The model in 3+1d has a long history as an effective description of strong interaction physics \[9\]. For coupling \(g^2\) stronger than some critical \(g^2_c\) the SU(2)_L \(\otimes\) SU(2)_R \(\otimes\) U(1)_B global symmetry present for quark mass \(m \to 0\) spontaneously breaks to SU(2)_I \(\otimes\) U(1)_B, accompanied by the generation of a constituent quark mass \(\Sigma = g^2 \langle \bar{\psi} \psi \rangle\). The behaviour in 2+1d has the same pattern, except that this time there is an interacting continuum limit at \(g^2 \to g^2_c, \Sigma/\Lambda \to 0\) \[5\]. Once a chemical potential is introduced, for low \(T\) an expansion in the inverse number of flavors \(1/N_f\) predicts \(\Sigma(\mu)\) to remain roughly unchanged out to a critical \(\mu_c \simeq \Sigma(0)\), whereupon chiral symmetry is abruptly restored in a strong first-order transition \[7\]. At the same point the baryon density \(n_B = \langle \bar{\psi} \gamma_0 \psi \rangle\) rises from zero to a non-zero \(n_B \propto \mu^2 \theta(\mu - \mu_c)\), consistent with filling a two-dimensional Fermi ball of radius \(\mu\). This is confirmed by lattice simulations \[5\]; in particular, unlike other simulations with a real measure, there is a clear separation of scales between \(\mu_c\) and \(m_\pi\) \[8\].

2. THE DIQUARK SECTOR

The question to consider is whether for \(\mu > \mu_c\) the baryon number U(1)_B symmetry is spontaneously broken by a diquark condensate \(\langle qq \rangle \neq 0\). Since the NJL model is not a gauge theory, this leads not to superconductivity but to the associated phenomenon of superfluidity. In fact, numerical studies \[9\] reveal that the preferred channel for pairing in this regime is (in the notation of staggered fermions) via the scalar SU(2)_L \(\otimes\) SU(2)_R singlet \(\chi^{tr} \tau_2 \chi\). To find unambiguous evidence for BCS condensation on a finite system, however, the correct procedure \[10\] is to add to \(\mathcal{L}\) a diquark source term

\[
j_{\pm}qq_{\pm} \equiv j_{\pm}(\chi^{tr} \tau_2 \chi \pm \bar{\chi} \tau_2 \bar{\chi}^{tr}).
\]

(2)

It is then possible to measure the diquark condensate using methods similar to those used in lattice studies of chiral symmetry breaking:

\[
\langle qq_+ \rangle = \frac{1}{V} \frac{\partial \ln Z}{\partial j_+}
\]

(3)

together with the associated susceptibilities

\[
\chi_{\pm} = \sum_x \langle qq_+(0)qq_+(x)\rangle.
\]

(4)
In a conventional BCS condensation the + describes a “Higgs” mode while the – is a “Goldstone”, which is constrained by a Ward identity:

\[ \chi_-(j=0) = \frac{\langle qq_+ \rangle}{j_+}. \]  

(5)

In collaboration with Biagio Lucini and Susan Morrison I have simulated a lattice NJL model including the source term (2) \[11\]. Our parameter choice yields a \( \mu_c \approx 0.65 \). Fig. 2 plots \( \langle qq_+ \rangle \) extrapolated to the zero temperature \( L_t \to \infty \) limit; for \( \mu < \mu_c \) the data support a linear relation between \( \langle qq_+ \rangle \) and \( j \), consistent with a U(1)\(_B\)-symmetric ground state. For \( \mu > \mu_c \) by contrast, the data suggest

\[ \langle qq_+ \rangle \propto j^\alpha \]  

with \( \alpha = \alpha(\mu) \) falling in the range 0.2 - 0.3 for the \( \mu \) values studied. Evidence for power law behaviour is reinforced by considering the susceptibility ratio \( |\chi_+ / \chi_-| \): eqns. (5) and (6) together imply that this ratio should take the constant value \( \alpha \) in the high density phase, and the plateaux of Fig. 3 for \( \mu = 0.8, 0.9 \) support this.

The simulation data suggest that the high density phase \( \mu > \mu_c \) is critical, characterised by continuously varying exponents \( \delta(\mu) = \alpha^{-1} \) and \( \eta(\mu) \) defined via

\[ \langle qq(0)qq(x) \rangle \propto \frac{1}{|x|^\eta}, \quad x \in \mathbb{R}^2, \]  

(7)

and is thus qualitatively similar to the low-\( T \) phase of the 2d XY model. If we write \( qq(x) = \phi_0 e^{i \theta(x)} \), then long range order is washed out by IR fluctuations of \( \theta \), but long-range phase coherence persists via (7). This is precisely what gives rise to persistent currents and hence superfluidity in 2d systems. The supercurrent \( \vec{J}_s \) is related to \( qq \) via

\[ \vec{J}_s = K_s \vec{\nabla} \theta. \]  

(8)
The only way to change the circulation $\kappa = \oint \vec{J}_s \cdot d\vec{l}$ around a periodic volume is to create a vortex – anti-vortex in the $\{\theta\}$ configuration and translate one of the pair around the universe in the perpendicular direction until they reannihilate, in so doing incrementing the quantised $\kappa$ by $2\pi K_s/L$. The energy needed increases logarithmically with $L$, implying that the circulation is metastable [12].

3. THE SPIN-$\frac{1}{2}$ SECTOR

![Figure 4](image1.png)  ![Figure 5](image2.png)

Figure 4. Amplitudes for hole (A), particle (B) and anomalous (C) propagation. Figure 5. Dispersion relation $E(k)$ for both free and interacting fermions.

The conclusions of the previous section beg the question of why the observed $\delta \simeq 3 - 5$ is not consistent with the $2d$ XY value $\delta(T) \geq 15$, in contrast to a recent study of superfluidity in the attractive Hubbard model [13]. Further insight is gained from studying the spin-$\frac{1}{2}$ sector via the fermion propagator, which for $j \neq 0$ contains both normal $\langle q(0)q(x) \rangle$ and anomalous $\langle q(0)\bar{q}(x) \rangle$ components. To probe the Fermi surface, data at non-zero momentum $\vec{k}$ are needed. In the normal sector a sharp transition between forwards $Ae^{-Et}$ and backwards $Be^{-E(Lt-t)}$ propagation at a rather well-defined momentum $k/\pi \approx 0.28$ on a $32^3$ system at $\mu = 0.8$, as shown in Fig. 4. We interpret this as a crossover between hole and particle excitations at the Fermi surface. The anomalous propagators signifying particle-hole mixing peak at the same momentum, but appear to vanish as $j \to 0$.

The results for the mass gap are shown in Fig. 4, with hole energies plotted as negative. The resulting curve is the quasiparticle dispersion relation; its detailed form indicates a Fermi momentum $k_F$ slightly less than $\mu$ and a Fermi velocity $\beta_F = \partial E/\partial k|_{k=k_F} \approx 0.7$, somewhat less than the speed of light. Both results differ from the corresponding free-field values, but are consistent with a relativistic normal Fermi liquid with repulsive interaction between quasiparticles with parallel momenta [14]. Most importantly, there
is no signal for a gap $\Delta \neq 0$. The origin of the non-standard critical behaviour may therefore be attributed to massless fermions; the presence of a Fermi surface leads to a new 2$d$ universality class. This situation should be contrasted with dimensional reduction in systems with $T > 0$ \cite{15}; because the fermi degrees of freedom in this case acquire masses $\pi T$ and decouple, the universality class is that of the spin model with the corresponding global symmetry.

4. CONCLUSION

Both $\langle qq \rangle$ and $\Delta$ vanish in the limit $j \to 0$ implying that the conventional BCS description appropriate for, say, superfluid $^3$He, does not apply in this case. The most economic hypothesis explaining the results of Figs. 2 – 5 is that the NJL$_{2+1}$ model in its high density phase describes a gapless relativistic system with superfluidity realised in the way first suggested by Kosterlitz and Thouless for thin films of $^4$He. Further numerical confirmation for this picture would come from observation of current quantisation in the presence of a source $j(x)$ with built-in winding number. The more urgent problem, however, is to extend the calculations to NJL$_{3+1}$ and test the exciting scenarios predicted in \cite{2,4} beyond the self-consistent approximation.

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