Application of Lie point symmetry and Adomain decomposition techniques to thermal-storage nonlinear diffusion models

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Abstract. Classical Lie point symmetry techniques are employed to time dependent nonlinear heat diffusion equations describing thermal energy storage in a medium subjected to a convective heat transfer to the surrounding environment at the boundary through a variable heat transfer coefficient. Exponential temperature-dependent thermal conductivity and heat capacity are assumed. Group classification for the source term is performed and some exciting large symmetry algebras are admitted. It turns out that the principal Lie algebra extends when the source term vanishes and when it is given as the exponential function of temperature. Reduction by one of the independent variables is performed for some realistic choices of the source term. In some case the resulting nonlinear ordinary differential equation with appropriate corresponding conditions are solved using Adomian decomposition method.

1. Introduction
The operation of hot water storage tanks for thermal energy storage is classified as static mode and dynamic mode \cite{7, 12, 8}. The static mode is further classified into fully charged and partially charged mode. Dynamic mode of operation includes charging and discharging cycle. In static fully charged mode, the storage tank is initially completely filled with hot water at a constant temperature and is subjected to convective heat loss from the tank walls to the ambient \cite{18}. In static partially charged mode, the storage tank is initially charged with different levels of hot and cold water separated by thermocline \cite{12}. The heat loss from the stored fluid to the ambient decreases the temperature of the fluid near to the tank wall, thereby increasing its density. Therefore, accurate and efficient design of solar heating or cooling thermal energy storage tanks generally requires adequate theoretical framework from which insight might be gleaned into an inherently complex physical process \cite{18}. The solution of the transient nonlinear heat conduction for thermal energy storage applicable in rectangular, cylindrical and spherical coordinates remains an extremely important problem of practical relevance in the engineering sciences \cite{6, 3}. The mathematical techniques employed in the present investigation include the parameter-group transformation \cite{16} and Adomain decomposition method \cite{2}. The foundation of the group-theoretical method is contained in the general theory of continuous transformation
groups that were introduced and treated extensively by Lie (see e.g. [10]). This paper is arranged as follows, Section 2 provides a mathematical formulation of the problem, In Section 3 a brief theory of symmetry analysis of differential equations is discussed. Solutions to the problem is discussed in section 4, and Section 5 and 6 deals with results and conclusions respectively.

2. Mathematical models

The transient heat conduction equation with heat source term modelling the thermal storage problem in a rectangular, cylindrical or spherical coordinate system is given as [12, 3];

\[
\rho C(T) \frac{\partial T}{\partial t} = 1 \frac{1}{r^m} \frac{\partial}{\partial r} \left( K(T) r^m \frac{\partial T}{\partial r} \right) + S(T),
\]

(1)

with the initial condition

\[ T(r, t) = T_0 \quad \text{at} \quad t = 0, \]

(2)

and the boundary condition

\[
\begin{align*}
\frac{\partial T}{\partial r} &= 0; \quad \text{at} \quad r = 0 \\
K(T) \frac{\partial T}{\partial r} &= -h(t) (T - T_\infty); \quad \text{at} \quad r = a
\end{align*}
\]

(3)

where \( T \) is the temperature, \( \rho \) is the density, \( r \) is the space variable, \( t \) is time, \( T_0 \) is the initial temperature of the body and \( S(T) \) is the temperature-dependent heat source term. \( \rho \) is the density and \( h(t) = h_0 f(t) \) is the time-dependent heat transfer coefficient. Following [3], the exponential temperature-dependent thermal conductivity and heat capacity are taken as

\[
K(T) = K_0 \exp \left\{ p \left( \frac{T - T_\infty}{T_0 - T_\infty} \right) \right\} \quad \text{and} \quad C(T) = C_0 \exp \left\{ q \left( \frac{T - T_\infty}{T_0 - T_\infty} \right) \right\},
\]

where \( K_0, C_0, p \) and \( q \) are constants, \( T_0 \) is the initial temperature of the body, \( T_\infty \) is the ambient temperature of the surrounding environment. The geometry of the body is specified by \( m = 0, 1, 2 \) representing rectangular, cylindrical and spherical coordinates respectively. Equations (1-2) are made dimensionless by introducing the following quantities:

\[
\bar{r} = \frac{r}{a}, \quad \bar{t} = \frac{K_0 t}{\rho C_0 a^2}, \quad \bar{T} = \frac{T - T_\infty}{T_0 - T_\infty}, \quad \bar{S} = \frac{a^2 S}{K_0 (T_0 - T_\infty)}, \quad Bi = \frac{ah_0}{K_0}
\]

Neglecting the bar symbol for clarity, the dimensionless governing equations become

\[
\epsilon^{\rho \gamma} \frac{\partial \bar{T}}{\partial \bar{t}} = 1 \frac{1}{r^m} \frac{\partial}{\partial r} \left( \epsilon^{\gamma} r^m \frac{\partial \bar{T}}{\partial r} \right) + S(T),
\]

(4)

\[
\begin{align*}
\frac{\partial \bar{T}}{\partial \bar{r}} &= 0; \quad \text{at} \quad \bar{r} = 0 \\
\epsilon^{\gamma} \frac{\partial \bar{T}}{\partial \bar{r}} &= -Bi f(t) \bar{T}; \quad \text{at} \quad \bar{r} = 1
\end{align*}
\]

(5)
3. Symmetry techniques for differential equations

The theory and applications of continuous symmetry groups were founded by Lie in the 17th century [13]. The modern accounts of this theory may be found in excellent text such as those of [4, 14]. We restrict our discussion to classical point symmetries, since we will only use such symmetries. The reader is referred to [4, 10, 14, 16] for more details on this theory.

Given a continuous one parameter symmetry group, it is possible to reduce the number of independent variables by one. Lie’s fundamental result is that the whole of one parameter group can be determined from the transformation laws up to the first degree of the parameter $\epsilon$, i.e. determination of symmetry groups involve finding transformation of the form

$$
\begin{align*}
\tau_j &= x_j + \epsilon X_j(x, u) + O(\epsilon^2) \\
\tau &= u + \epsilon U(x, u) + O(\epsilon^2)
\end{align*}
$$

that leave the $m$th order governing partial differential equation,

$$
\Delta(x, u, u^{(1)}, ..., u^{(m)}) = 0,
$$

invariant. Here, $x$ is set of $n$ independent variables $(x_1, x_2, ..., x_n)$ and $u^{(k)}$ denotes a set of coordinates corresponding to all the $k$th order partial derivatives of $u$ with respect to $x_1, x_2, ..., x_n$. The coefficients $X_j$ and $U$ are the components of the infinitesimal symmetry generator, which is one of the vector fields

$$
\Gamma = X_j(x, u) \frac{\partial}{\partial x_j} + U(x, u) \frac{\partial}{\partial u},
$$

which span the associated Lie algebra. Here we sum over repeated index (see e.g. [4]). Calculations of these generators or symmetry groups are very long and tedious. However, we use the freely available program Dimsym [17], which is written as a subprogram for the computer algebra package Reduce [9] to construct the admitted symmetries.

If a differential equation is invariant under some point symmetry, one can often construct similarity solutions which are invariant under some subgroup of the full group admitted by the equation in question. These solutions result form solving a reduced equation in fewer variables.

More often, differential equations arising in real world problem involve one or more functions depending on either the independent variable or on the dependent variables. It is possible by symmetry techniques to determine the cases which allow the equation in question to admit extra symmetries. The exercise of searching for the forms of arbitrary functions that extend the principal Lie algebra is called group classification. The problem of group classification was introduced by Ovsiannikov [15] and recent accounts on this topic may be found for example in [5, 11, 19, 21]. We adopt methods in [5] to perform group classification.

4. Solution to the problem

Firstly, we reduce the boundary value problem (BVP) (4)-(5) to a system of ordinary differential equations using the Lie group methods. This will be followed by the application of Adomian decomposition method [1, 2] in order to obtain a semi-analytical non-perturbative approximate solution to the problem. Instead of determining symmetries for the entire boundary value problem, we consider the governing equation (4). It is well known (e.g. [5]) that symmetry analysis of the entire system such as (4)-(5) may result in reduced admitted group of transformations.
Classical Lie symmetry reductions

In the initial symmetry analysis of equation (4) where all the constants are arbitrary, we obtain nothing beyond translation in time $t$. It turns out using Dimsym [17] that $p = q$. Adopting techniques in [5] we observe that extra Lie point symmetries are admitted for the cases of the source term listed below;

**Case (a) $S=0$.** In this case Equation (4) admits a finite four dimensional Lie algebra

\[
\begin{align*}
\Gamma_1 &= -\frac{1}{4q} \left( r^2 - 2(m+1)t \right) \frac{\partial}{\partial T} + r t \frac{\partial}{\partial r} + t^2 \frac{\partial}{\partial t}; \\
\Gamma_2 &= r \frac{\partial}{\partial r} + 2t \frac{\partial}{\partial t}; \\
\Gamma_3 &= \frac{\partial}{\partial t}; \\
\Gamma_4 &= \frac{\partial}{\partial T}; \\
\end{align*}
\]

and the infinite symmetry generator

\[
\Gamma_\infty = \frac{H(r,t)}{q e^{qT}} \frac{\partial}{\partial T};
\]

where $H$ is a function of $r$ and $t$ satisfies the equation

\[
\frac{\partial^2 H}{\partial r^2} + \frac{m}{r} \frac{\partial H}{\partial r} = \frac{\partial H}{\partial t}.
\]

**Case (b) $S=0$, $m=2$.** Other than the infinite symmetry generator, Equation (4) admits a six dimensional finite Lie algebra for this case, namely;

\[
\begin{align*}
\Gamma_1 &= -\frac{1}{4q} \left( r^2 - 2(m+1)t \right) \frac{\partial}{\partial T} + r t \frac{\partial}{\partial r} + t^2 \frac{\partial}{\partial t}; \\
\Gamma_2 &= r \frac{\partial}{\partial r} + 2t \frac{\partial}{\partial t}; \\
\Gamma_3 &= \frac{\partial}{\partial t}; \\
\Gamma_4 &= \frac{\partial}{\partial T}; \\
\Gamma_5 &= -\frac{1}{q r} \frac{\partial}{\partial T} + \frac{\partial}{\partial r}; \\
\Gamma_6 &= -\frac{1}{2qr} \left( r^2 + 2t \right) \frac{\partial}{\partial T} + t \frac{\partial}{\partial r} \\
\end{align*}
\]

and

\[
\Gamma_\infty = \frac{H(r,t)}{q e^{qT}} \frac{\partial}{\partial T};
\]

where $H$ satisfies

\[
\frac{\partial^2 H}{\partial r^2} + \frac{2}{r} \frac{\partial H}{\partial r} = \frac{\partial H}{\partial t}.
\]

**Case (c) $S=0$, $m=0$.** Similar to case (b), the governing equation admits a six dimensional Lie algebra,

\[
\begin{align*}
\Gamma_1 &= -\frac{1}{4q} \left( r^2 - 2t \right) \frac{\partial}{\partial T} + r t \frac{\partial}{\partial r} + t^2 \frac{\partial}{\partial t}; \\
\Gamma_2 &= r \frac{\partial}{\partial r} + 2t \frac{\partial}{\partial t}; \\
\Gamma_3 &= \frac{\partial}{\partial t}; \\
\Gamma_4 &= \frac{\partial}{\partial T}; \\
\Gamma_5 &= \frac{\partial}{\partial r}; \\
\Gamma_6 &= -\frac{r}{2qr} \frac{\partial}{\partial T} + t \frac{\partial}{\partial r} \\
\end{align*}
\]


\[ \Gamma_\infty = \frac{H(r,t)}{q e^t} \frac{\partial}{\partial T}; \]

where \( H \) satisfies

\[ \frac{\partial^2 H}{\partial r^2} = \frac{\partial H}{\partial t}. \]

**Case (d) \( S = A e^{kT} \).**

\[
\begin{align*}
\Gamma_1 &= -\frac{1}{(k-q)} \frac{\partial}{\partial T} + \frac{r}{2} \frac{\partial}{\partial r} + t \frac{\partial}{\partial t}; \\
\Gamma_2 &= \frac{\partial}{\partial t} \\
\end{align*}
\]

(9)

**Case (e) \( S = A e^{kT}, \ k = q \).** In this case the admitted finite algebra is four dimensional;

\[
\begin{align*}
\Gamma_1 &= \frac{1}{4q} \left( 4q A t^2 - r^2 - 2(m+1)t \right) \frac{\partial}{\partial T} + rt \frac{\partial}{\partial r} + t^2 \frac{\partial}{\partial t}; \\
\Gamma_2 &= 2A t \frac{\partial}{\partial T} + r \frac{\partial}{\partial r} + 2t \frac{\partial}{\partial t}; \\
\Gamma_3 &= \frac{\partial}{\partial t}; \\
\Gamma_4 &= \frac{\partial}{\partial T}; \\
\end{align*}
\]

(10)

and

\[ \Gamma_\infty = \frac{H(r,t)}{q e^t} \frac{\partial}{\partial T}; \]

where \( H \) satisfies

\[ A q H + \frac{\partial^2 H}{\partial r^2} + \frac{m}{r} \frac{\partial H}{\partial r} = \frac{\partial H}{\partial t}. \]

**Case (f) \( S = A e^{kT}, \ k = q, \ m = 2 \).**

\[
\begin{align*}
\Gamma_1 &= \frac{1}{4q} \left( 4q A t^2 - r^2 - 6t \right) \frac{\partial}{\partial T} + rt \frac{\partial}{\partial r} + t^2 \frac{\partial}{\partial t}; \\
\Gamma_2 &= 2A t \frac{\partial}{\partial T} + r \frac{\partial}{\partial r} + 2t \frac{\partial}{\partial t}; \\
\Gamma_3 &= \frac{\partial}{\partial t}; \\
\Gamma_4 &= \frac{\partial}{\partial T}; \\
\Gamma_5 &= -\frac{1}{q r} \frac{\partial}{\partial T} + \frac{\partial}{\partial r}; \\
\Gamma_6 &= -\frac{1}{2 qr} \left( r^2 + 2t \right) \frac{\partial}{\partial T} + t \frac{\partial}{\partial r}; \\
\end{align*}
\]

(11)

and

\[ \Gamma_\infty = \frac{H(r,t)}{q e^t} \frac{\partial}{\partial T}; \]

where \( H \) satisfies

\[ A q H + \frac{\partial^2 H}{\partial r^2} + \frac{2}{r} \frac{\partial H}{\partial r} = \frac{\partial H}{\partial t}. \]
Case (f) \( S = Ae^{kT}, \quad k = q, \quad m = 0. \)

\[
\begin{align*}
\Gamma_1 &= \frac{1}{4q} \left( 4qA t^2 - r^2 - 2t \right) \frac{\partial}{\partial T} + rt \frac{\partial}{\partial r} + t^2 \frac{\partial}{\partial t}; \\
\Gamma_2 &= 2At \frac{\partial}{\partial T} + r \frac{\partial}{\partial r} + 2t \frac{\partial}{\partial t}; \quad \Gamma_3 = \frac{\partial}{\partial t}; \quad \Gamma_4 = \frac{\partial}{\partial T}; \\
\Gamma_5 &= \frac{\partial}{\partial r}; \quad \Gamma_6 = -\frac{r}{2q} \frac{\partial}{\partial T} + t \frac{\partial}{\partial r};
\end{align*}
\]

and

\[
\Gamma_\infty = \frac{H(r,t)}{q e^{kT}} \frac{\partial}{\partial T};
\]

where \( H \) satisfies

\[
AqH + \frac{\partial^2 H}{\partial r^2} = \frac{\partial H}{\partial t},
\]

Variable reductions

In most cases, one may reduce the number of independent variables of the partial differential equation by one, using the symmetries admitted by such a PDE. In some cases it is also possible to reduce or transform the imposed boundary conditions. We concentrate on the following examples;

Example 1(a)

In case (d) where \( S \) is also given by an exponential, we observe that the \( \Gamma_1 \) invariant solution is of the functional form

\[
T = -\frac{1}{k - q} \ln t + G(\gamma)
\]

where \( \gamma = \frac{r}{vt} \) is the invariant or the Boltzmann similarity variable, and \( G \) satisfies the O.D.E.

\[
G'' + \frac{m}{\gamma} G' + q(G')^2 + \frac{\gamma}{2} G' + \left( A + \frac{1}{k - q} \right) = 0
\]

It appears the boundary conditions are impossible to transform except Equation (5(a)) becomes \( G' = 0, \quad \gamma = 0. \)

Example 1(b)

In case (d), the operator \( \Gamma_2 \) suggests that both \( T \) and \( r \) are invariants. In fact the functional form of the invariant solution is simply \( T = G(r) \) (see e.g. [20]), where \( G \) satisfies the O.D.E.

\[
\frac{1}{v^m} \frac{d}{dr} \left( e^{G} r^m \frac{dG}{dr} \right) + Ae^{kG} = 0,
\]

and the boundary conditions transforms to

\[
dG \frac{dr}{dr} = 0, \quad r = 0 \quad \text{and} \quad G = 0, \quad r = 1 \quad (\text{i.e. as} \quad Bi \to \infty). \]

Example 2

In this example we consider the \( \Gamma_2 \) reduction in case (a) \( S = 0. \) \( \Gamma_2 \) leads to the solution \( T = G(\gamma) \) where \( \gamma = \frac{r}{vt} \) and \( G \) satisfies the O.D.E.

\[
-\frac{1}{2} \gamma G' = q(G')^2 + \frac{m}{\gamma} G' + G''.
\]
The initial condition (2) reduces to $G \rightarrow G_a$ as $\gamma \rightarrow \infty$, with $G_a$ being a constant and the boundary conditions become
\[ \frac{dG}{d\gamma} = 0, \quad \gamma = 0. \]

The boundary condition (5b) has proven to be difficult to transform in terms of the obtained invariant $\gamma$. One may note that the invariant (similarity) solution to the above reduced O.D.E. subject to the transformed conditions is merely trivial i.e. $G = \text{constant}$.

**Numerical methods**

In this section, Adomian decomposition technique is employed in order to perform a numerical experiment on the above transformed nonlinear ordinary differential equations. The advantage of this method is that it provides a direct scheme for solving the problem, i.e., without the need for linearization, perturbation or massive computation. For illustration purpose, we consider the transformed nonlinear ordinary differential equation in example 1(b) above depicting the energy storage problem subject to exponentially increasing heat source with very large Biot number:

\[ \frac{1}{r^{m}} \frac{d}{dr} \left( r^{m} \frac{dG}{dr} \right) + q \left( \frac{dG}{dr} \right)^{2} + Ae^{(k-q)G} = 0, \quad (13) \]

with
\[ \frac{dG}{dr} = 0, \quad r = 0 \quad \text{and} \quad G = 0, \quad r = 1, \quad (\text{i.e. as } Bi \rightarrow \infty) \quad (14) \]

Following [7], we rewrite Equation (13) with respect to Equation (14) in the form
\[ L_r G = -q(G_r)^2 - Ae^{(k-q)G}, \quad (15) \]

where the subscript $r$ represents derivatives with respect to $r$ and the differential operator employs the first two derivatives in the form
\[ L_r = \frac{1}{r^{m}} \frac{d}{dr} \left( r^{m} \frac{d}{dr} \right), \quad (16) \]

in order to overcome the singularity behaviour at the point $r = 0$. In view of Equation (16), the inverse operator $L_r^{-1}$ is considered a twofold integral operator defined by
\[ L_r^{-1} = \int_{0}^{r} r^{-m} \int_{0}^{r} r^{m}(\cdot) dr dr, \quad (17) \]

Applying $L_r^{-1}$ to both sides of Equation (15), using the boundary conditions in Equation (14), we obtain
\[ G(r) = G(0) - qL_r^{-1} \left( G_r^2 \right) - AL_r^{-1} \left( e^{(k-q)G} \right). \quad (18) \]

As usual in Adomian decomposition method the solution of Equation (18) is approximated as an infinite series
\[ G(r) = \sum_{j=0}^{\infty} G_j, \quad (19) \]

and the nonlinear terms are decomposed as follows:
\[ G_r^2 = \sum_{j=0}^{\infty} H_j, \quad e^{(k-q)G} = \sum_{j=0}^{\infty} M_j, \quad (20) \]
where $H_j$ and $M_j$ are polynomials (called Adomian polynomials) given by

$$H_j = \frac{1}{j!} \frac{d^j}{d\lambda^j} \left[ \left( \sum_{i=0}^{\infty} G_{ri} \lambda^i \right)^2 \right]_{\lambda=0}, \quad M_j = \frac{1}{j!} \frac{d^j}{d\lambda^j} \left[ (e^{(k-q)}(\sum_{i=0}^{\infty} G_{ri} \lambda^i)) \right]_{\lambda=0}. \quad (21)$$

Thus, we can identify

$$G_0 = G(0),
G_{j+1} = -q L_{r-1}(H_j) - AL_{r-1}(M_j), \quad j \geq 0. \quad (22)$$

Substituting Equations (20)-(22) into Equation (19), and using MAPLE we obtained a few terms approximation to the solution as

$$\Psi_N = \sum_{n=0}^{N} G_n, \quad (23)$$

where $G(r) = \lim_{N \to \infty} (\Psi_N)$. Using the above procedure, we obtain

$$G(r) := G_0 - \frac{1}{2} \frac{A e^{kG_0} r^2}{(m+1) e^{G_0}} + \frac{1}{8} \frac{A^2 (k-q) r^4 e^{2(k-q)G_0}}{(m+1)(m+3)}
- \frac{1}{4} \frac{q A^2 \left( e^{kG_0} \right)^2 r^4}{(e^{G_0})^2 (m+1)(m+4m+3)}
- \frac{1}{24} \frac{A^3 (mk^2 - 2kmq + mq^2 + 2k^2 - 4kq + 2q^2) \left( e^{kG_0} \right)^3 r^6}{(m+1)^2(m+8m+15) (e^{G_0})^3} + ... \quad (24)$$

By applying the boundary conditions in Equation (14) to the expression for in Equation (24), we obtain approximately the values for $G_0$ as shown graphically in the following section. Usually, the decomposition method yields rapidly convergent series solutions by using a few iterations for the nonlinear deterministic equations [2].

5. Results and discussion
The approximate solution given in Equation (24) is valid for energy storage systems in an interval $0 \leq r \leq 1$ under exponentially increasing heat source. For the numerical validation of our results we have chosen physically meaningful values of the parameters for the problem. Unless otherwise stated we have taken: $k = q = 1; Bi \to \infty$. In Figures 1 and 2, we depict the effect of material parameters on the medium temperature profiles. For all different geometrical regions ($m = 0, 1, 2$), we observe that the temperature profiles decrease transversely with maximum value along the central region and minimum value at the surface as shown in Figure 2. The minimum value at the surface temperature can be attributed to the convective cooling at the region. However, it is noteworthy that the medium temperature is highest with rectangular configuration, followed by that of cylindrical geometry and the lowest is observed with spherical geometry. Figure 2 illustrates the effects of increasing values of heat source parameter (A) on the temperature profile. It is interesting to note that the medium temperature increases as the heat source parameter increases due to increasing heat generation as expected.
Figure 1. Medium temperature profile as Bi $\rightarrow \infty$, $k = 0$; $q = 1$; $m = 0$; $A = 0.1$; $m = 1$; $m = 2$

Figure 2. Medium temperature profile as Bi $\rightarrow \infty$, $k = 0$; $q = 1$; $m = 0$; $A = 0.2$; $A = 0.3$

6. Conclusion
Classical Lie point symmetry analysis resulted in some large symmetries being admitted for special cases of the source term. It is possible to reduce the governing P.D.E. to O.D.E.s using any linear combination of the admitted symmetries. In fact one may determine the optimal systems of sub algebras (symmetries) obtained in each case, to find reductions that are nonequivalent, and so one may classify group invariant solutions (see e.g. [14]). In Section 4, reductions were performed for some examples. Not surprisingly, the boundary condition seemed difficult to be transformed in terms of the new variables for many examples. However, in Example 1(b) the problem resulted in an exiting transformed system of O.D.E.s which was completely solved using Adomain decomposition techniques.

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7. References
[1] Abboui K and Cherruault Y 1995 New ideas for proving convergence of decomposition methods, *Comput. Appl. Math.* **29** (7), 103-105.
[2] Adomian G 1994 Solving frontier problems of physics: The Decomposition method, Kluwer Academic Publishers, Dordrecht.
[3] Badran N A and Abd-el-Malek M B 1995 Group analysis of nonlinear heat-conduction problem for a semi-infinite body, *Nonlinear Math. Phys.* **2**(3-4), 319-328.
[4] Bluman G W and Anco S C 2002 Symmetry and integration methods for differential equations, New York: Springer-Verlag.
[5] Bluman G W and Kumei S 1989 Symmetry and differential equations, Springer-Verlag, New York.
[6] Davies T W 1985 Transient conduction in a plate with counteracting convection and thermal radiation at the boundary, *Appl. Math. Model.* **9**, 337-340.
[7] Duffie J A and Becham W A 1980 Solar Engineering of thermal processes, 485, Wiley.
[8] Hawlader M N A and Brinkworth B J 1981 An analysis of the non-convective solar pond, Solar Energy **27** (3), 195.
[9] Hearn A C 1985 Reduce user’s manual version 3.4, Santa Monica, California: Rand Publication CP78, The Rand Corporation.
[10] Ibragimov N H 1999 Elementary Lie group analysis and ordinary differential equations, John Wiley & sons, New York.
[11] Ivanova N M and Sophocleous C 2006 On the group classification of variable nonlinear diffusion-convection equations, *J. Comput. Appl. Math.*, **197**(2), 322-344.
[12] Jahiria Y and Gupta S K 1982 Decay of thermal stratification in a water body for solar energy storage, Solar Energy **28** (2) 137.
[13] Lie S 1880 Theorie der Transformationgruppen, *Math. Ann.*, **16**, 441-528.
[14] Olver P J 1972 Applications of Lie groups to differential equations, New York: Springer-Verlag.
[15] Ovsiannikov L V 1959 Group relations of the equation of non-linear heat conductivity, *Dokl. Akad. Nauk SSSR.* **125**, 492-495.
[16] Ovsiannikov L V 1982 Group analysis of differential equations, New York: Academic Press.
[17] Sherring J 1993 Dimsym: symmetry determination and linear differential equation package, Latrobe University Mathematics Dept., Melbourne.
[18] Shin M S, Kim H S, Jang D S, Lee S N and Yoon H G 2004 Numerical and experimental study on the design of a stratified thermal storage system, Applied Thermal Engineering, **24**, 17-27.
[19] Sophocleous C 2003 Symmetries and form-preserving transformation of generalised inhomogeneous nonlinear diffusion equations *Physica A.*, **324**, 509-529.
[20] Stephani H 1989 Differential equations, their solution using symmetries, Cammbridge University Press, New York.
[21] Vaneeva O O, Johnpillai A G, Popovych R O and Sophocleous C 2007 Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities, *J. Math. Anal. Appl.* **330**(2): 1363-1386.