Commuting integral and differential operators and the master symmetries of the Korteweg–de Vries equation

F Alberto Grünbaum*

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720, United States of America

E-mail: grunbaum@math.berkeley.edu

Received 16 March 2021, revised 17 June 2021
Accepted for publication 6 July 2021
Published 27 July 2021

Abstract

The singular value decomposition going with many problems in medical imaging, non-destructive testing, geophysics, etc is of central importance. Unfortunately the effective numerical determination of the singular functions in question, is a very ill-posed problem. The best known remedy to this problem goes back to the work of Slepian et al (1960–1965 Bell Labs). We show that the master symmetries of the Korteweg–de Vries equation give away to extend the remarkable result of Slepian in connection with the Bessel integral kernel and the existence of a differential operator that commutes with the corresponding integral operator. The original results of Bell Labs group has already played an important role in the study of the limited angle problem in x-ray tomography as well as in random matrix theory.

Keywords: commuting, integral, differential, operators, master, symmetries

1. Introduction

This section starts with a reader friendly introduction to the two themes mentioned in the title and the relevance of the first one to an important image reconstruction problem. This is followed by a more detailed introduction and a description of the new results in the paper.

1.1. The work of the Bell Labs group, commuting integral and differential operators

Starting in 1961, the Bell Labs group of Landau, Pollak and Slepian produced a series of remarkable papers under the general title ‘Prolate spheroidal wave functions, Fourier analysis and uncertainty’ [22, 23, 34–36]. The first few of these papers, papers I, II and III, deal with the analysis of signals in continuous time. Paper IV in the series is a paper by Slepian and it
considers the multidimensional case. His results will be extended in the present paper, as will be explained later in this section.

The last paper in the series, paper V, [36], is once again by Slepian and it considers the case when the time series is given by discrete samples of a signal. This naturally leads him to consider the amplitude spectrum given by a function defined on the unit circle, i.e. one replaces the Fourier transform in one or several variables by the Fourier series.

When one looks at these different scenarios for a unified signal processing point of view, the different situations deal with various incarnations of the same problem which originated with Shannon: what is the best use one can make of a limited duration portion of a signal if one knows that the signal is bandlimited, i.e. its Fourier content vanishes outside of an interval on the real line, a disc in $\mathbb{R}^N$ or, in the last case, an interval on the unit circle?

The only setup of the Fourier picture not considered by the Bell Labs group is the purely finite one, namely the case when both physical and frequency space is the set of $N$-roots of unity, i.e. the discrete Fourier transform, which was later considered in [10].

Returning to paper V in the series, see [36], involving discrete time and the Fourier series, Slepian looks at the appropriate Toeplitz matrix in section 2 of the paper and establishes a series of asymptotic results for its eigenvectors and eigenvalues, see in particular section 2.5. His Toeplitz matrix has entries given by

$$\frac{2\phi}{\pi}$$

on the main diagonal and by

$$\frac{2\sin(l\phi)}{\pi l}$$

for the $l$th diagonal.

After section 2 in [36] one finds a long section on applications of the analytical results to problems in signal processing. I reproduce the list of applications discussed by Slepian to indicate its wide scope. In section 3.1 one finds: discussions of extremal properties, most concentrated bandlimited sequence, index limited sequence with most concentrated spectrum, simultaneously achievable concentration, minimum energy bandlimited extension of a finite sequence, trigonometric polynomial with greatest fractional energy in an interval-optimal window. Section 3.2 deals with a prediction problem, and finally section 3.2 deals with the approximatedimension of signalspace.

It is remarkable that the problem considered by Slepian has (at least) one more application, maybe closer to the interests of workers in inverse problems.

In the book the mathematics of computerized tomography, [29] section 6.2, entitled ‘the limited angle problem’, Natterer gives a full discussion of the relevance of Slepian’s work to analyze the degree of ill-conditioning of the reconstruction problem when only projections limited to the angle $\phi < \pi/2$ are considered. The matrix that is analyzed in [29] is exactly the same as in [36] and the main result quoted from this reference in page 161 of [29], shows in a sharp quantitative fashion that when the size of the matrix is large a small fraction of its eigenvalues are close to 1 and most of the eigenvalues are very close to 0 with a very small transition region in between. The fraction in question depends on the angle $\phi$ in ‘the limited angle problem’. For a fuller discussion of the limited angle problem the reader should consult [29] as well as [11].

Needless to say the fact that most of the eigenvalues are very close to 0 makes the computation of the corresponding eigenvectors a difficult problem, which is handled, as in all the other papers of the Bell Labs group by the existence of a second order differential (or a tridiagonal
matrix in paper V) that commutes with the original convolution integral operator (or a Toeplitz matrix in the case of paper V).

The Bell Labs group found a mathematical miracle: one can produce a differential operator with simple spectrum that shares its eigenfunctions with the integral operator in question and has a very spread out spectrum, resulting in a numerically stable problem when it comes to computing these eigenfunctions.

The discussion above illustrates very well the fact that long series of papers mentioned earlier, see [22, 23, 34–36] have found important applications in areas far removed from the original problem considered in them. A few more instances of this are seen in [32, 33, 41].

In 1982 Slepian delivered the John von Neumann lecture at the SIAM meeting, see [37]. After giving a short view of the work done by the Bell Labs group he closes the introduction with ‘the mystery of this serendipity grows. Most of us feel that there is something deeper here that we currently understand—that there is a way of viewing these problems more abstractly that will explain their elegant solution in a more natural and profound way, so that these nice results will not appear so much as a lucky accident’.

The new results given in this paper, to be described in section 5, are an extension of the results in paper IV of the series of papers discussed above and could be considered in the spirit of the comments above, even if no one has found a concrete application of them yet. Presenting them to an audience of people interested in applications should increase the chance that an application may be found.

1.2. The Korteweg–de Vries equation

Starting around 1967 people started finding ways to produce explicit solutions of a non linear partial differential equation in the theory of water waves, namely

\[ u_t = u_{xxx} - 6uu_x. \]

Solitary wave solutions had been found much earlier but the discovery of ‘soliton type solutions’ opened up a remarkable period of interesting developments. This is not the place to take even a brief look at this area, but the interested reader may want to look (for instance) at [8].

A look at history shows that the development of methods to solve the heat and the wave equations produced tools that went way beyond the original problem. One can only hope that some of the developments centered around the Korteweg–de Vries equation (KdV equation from now on) could have payoffs in areas very far from their birthplace. The execution of this program is a challenge.

One way to make the statement above less controversial is to notice that the KdV equation can be seen as the second in a hierarchy of partial differential equations of which the first one is given by

\[ u_t = u_x. \]

This linear partial differential equation is trivial to solve and its solution is given by the time translation of the initial data. The point is that translation is the underlying structure behind Fourier analysis.
1.3. Statement of the new results

The operator

\[ L_\nu = -D_x^2 + \frac{\nu^2 - 1/4}{x^2} \quad x > 0 \]  

(1.1)

plays a crucial role in mathematical physics, geometry and many other areas. The reason behind this is very simple: after conjugation it gives the radial part of the (negative) Laplacian in \( \mathbb{R}^N \) when \( \nu = \frac{N-2}{2} \).

The space of its eigenfunctions, i.e. solutions of

\[ L_\nu \varphi(x, z) = z^2 \varphi(x, z) \]

is given in terms of the well known Bessel functions and the bounded solution at \( x = 0 \) is

\[ f_\nu(x, z) = \sqrt{z} J_\nu(xz). \]  

(1.2)

In this paper we will revisit a remarkable property of \( L_\nu \) discovered by Slepian a long time ago [34]. This has important ‘signal processing’ applications. His result is an extension to the Bessel case of a property of the Fourier transform in \( \mathbb{R} \) which had been studied by the Bell Labs group [22, 35]. In the Fourier case (obtained by setting \( \nu = -\frac{1}{2} \)) they showed that the integral operator with kernel

\[ K_T(z_1, z_2) = \int_{-T}^{T} e^{i z_1 x} e^{-i z_2 x} \, dx = \frac{\sin T(z_1 - z_2)}{z_1 - z_2} \]

acting on \( L^2([-G, G], dz) \) commutes with the differential operator

\[ -D_z(G^2 - z^2)D_z + z^2 T^2. \]

Slepian found that if one considers the eigenfunction of \( L_\nu \) given by \( f_\nu(x, z) = \sqrt{z} J_\nu(xz) \) then the integral operator with kernel

\[ K(z_1, z_2) \equiv \int_0^T f_\nu(x, z_1) f_\nu(x, z_2) \, dx \]  

(1.3)

acting in \( L^2((-G, G), dz) \) admits a commuting differential operator, namely

\[ A_\nu = -D_z(G^2 - z^2)D_z + z^2 T^2 + G^2 \nu^2 - 1/4. \]  

(1.4)

The eigenfunctions of the integral operators with kernel (1.3) are important in signal processing since they are the singular functions of important time-and-band limiting problems.

It was mentioned earlier in this introduction that in the context of inverse problems such as x-ray tomography these ideas have played an important role, specifically in the ‘limited-angle problem’, see chapter 6 of Natterer’s book, [29] section VI.2, where one can see the connections with the work of Slepian, [36]. For a very good discussion of the general problem in signal processing, see [37, 38].

In the setup of limited angle tomography ‘time limiting’ is replaced by the fact that the function that one wants to reconstruct (a slice of the patient’s body) has compact support while the knowledge of the line integrals orthogonal to directions in a limited angle amount the knowledge of the Fourier transform of the object in this limited angle because of the ‘central section
theorem’. This plays the role of ‘band limiting’. For a more complete discussion connecting time and band limiting to the integral operators above in the context of x-ray tomography see [12].

For a discussion of several important cases in medical imaging where the singular value decomposition has been determined the reader can see [6, 18, 19, 25–27, 31] and their references. It remains as a challenge to find commuting differential operators in most of these cases. For a case where this has been found in geophysical applications see [15, 32, 33].

The need to compute the singular value decomposition is, at times, of paramount importance. A good reconstruction algorithm should ONLY try to find the projection of the object onto the span of the singular functions going with eigenvalues that are sensibly away from zero.

This applies in particular to iterative algorithms such as those used in phase determination, see [9]. The same is true in the limited angle tomography problem. If this linear span is not satisfactory for the spatial resolution that one wants to achieve, one needs to measure over a larger range of angles, i.e. increase $\phi$, and compute the new singular value decomposition from scratch.

In numerical simulations, where the phantom is known, the error (in well chosen examples) is seen to decrease monotonically as the number of iterations increases, but eventually the error starts increasing erratically with the number of iterations. A good stopping criterion for deciding when to stop iterating is given by a detailed knowledge of the spectral properties of the integral operator in question.

These integral operators with kernel given as in (1.3) have simple spectrum that rapidly accumulate at a point, making the effective computation of their eigenfunctions a very tricky problem. This problem is handled in [22, 23, 34–36] very effectively by the miraculous existence of a commuting differential operator: not only do these operators have the same eigenfunctions but the differential one has a very spread out spectrum yielding a numerically stable way to obtain the desired singular functions. For a fuller discussion of many important numerical aspects see [30].

It is appropriate to point out that the existence of these commuting differential operators for the corresponding integral operators is a very exceptional situation. It plays an important role in discussions in random matrix theory, see [28] as well as for the Bessel and Airy cases, see [39–41].

The search for other situations where this (albeit exceptional) miracle holds is the motivation of this paper.

We are now in a position to state the new results in this paper.

We describe a method to extend the results of Slepian, [34], to families of differential operators that are deformations of the Bessel cases in (1.1). In the very short note [14] the first two examples to be discussed in section 5 were displayed without any indication of the method used to produce them. These examples depend on one deformation parameter, whereas the new examples presented in this paper depend on two parameters. All examples, new and old ones, are derived ab initio in section 5.

We close this introduction by noticing that the presence of a deformation parameter in the operators (5.2), (5.16), (5.21) and (5.25) for which we extend the results in [34] is the reason for our interest in the KdV equation. The potentials $V(x)$ will be seen to evolve in time by an evolution equation very much related to the Korteweg–de Vries equation, as discussed in section 4. Before dealing with this point we will see in section 2 how ‘free parameters’ enter naturally in very basic constructions related to a general Schrödinger operator.
2. The Darboux process

We recall the ‘Darboux process’, namely one that produces out of an initial second order (Schrödinger) operator

$$L = -D_x^2 + V(x)$$

(2.1)

for which the eigenfunctions $\psi(x, z)$ are known, a new one parameter family of operators $\tilde{L}(t)$ whose eigenfunctions can be written in terms of those of $L$.

We can express $L$ in factorized form

$$L = \left(-D_x - \frac{\phi'(x)}{\phi(x)}\right) \left(D_x - \frac{\phi'(x)}{\phi(x)}\right),$$

where $\phi(x)$ is any eigenfunction of $L$ with zero eigenvalue and $\frac{\phi'(x)}{\phi(x)} = \partial_x (\log \phi)$. Since only the ratio $\frac{\phi'}{\phi}$ enters here one has

$$\phi(x) = \phi^{(1)}(x) + t\phi^{(2)}(x),$$

where $\phi^{(1)}(x), \phi^{(2)}(x)$ form a basis of the two-dimensional space of eigenfunctions of $L$ with 0 eigenvalue. This generic eigenfunction $\phi(x)$ depends now on the free deformation parameter $t$ and we will write it from now on as $\phi(x, t)$.

The operator $\tilde{L}$, i.e. the family of operators produced by the Darboux process, denoted by $\tilde{L}(t)$, is now given by

$$\tilde{L}(t) \equiv \left(D_x - \frac{\phi'(x, t)}{\phi(x, t)}\right) \left(-D_x - \frac{\phi'(x, t)}{\phi(x, t)}\right)$$

(2.2)

and it is easy to see that

$$\tilde{L}(t) = L - 2\partial_x^2 \log \phi(x, t).$$

(2.3)

If $\psi(x, z)$ is any solution of

$$L\psi(x, z) = z^2 \psi(x, z)$$

then

$$\tilde{L}(t) \left(D_x - \frac{\phi'(x, t)}{\phi(x, t)}\right) \psi(x, z) = z^2 \left(D_x - \frac{\phi'(x, t)}{\phi(x, t)}\right) \psi(x, z),$$

i.e. the eigenfunctions of $\tilde{L}(t)$ are given by

$$\left(D_x - \frac{\phi'(x, t)}{\phi(x, t)}\right) \psi(x, z).$$

(2.4)

We will make repeated use of the method in this section later in the paper.

For later use, see section 5, we note that if we define a function $\theta(x)$ by

$$V(x) = -\frac{1}{4x^2} - 2\partial_x^2 \log \theta(x)$$
and we put
\[ \phi(x, t) = \tilde{\theta}(x, t) \frac{\theta(x)}{\theta(x)} \]
we can express the operator in (2.3) above as
\[ \tilde{L}(t) = -D^2_x - \frac{1}{4x^2} - 2\partial^2_x \log \tilde{\theta}(x, t) \]
and the eigenfunctions of this operator are given in terms of those of \( L \), namely \( \psi(x, z) \), by the expression
\[ \left( \frac{1}{z} \right) \left( D_x - \partial_x \log \frac{\tilde{\theta}(x, t)}{\theta(x)} \right) \psi(x, z). \]
This expression, with the factor \( \frac{1}{z} \) included here for convenience will be useful in section 5.

3. The bispectral problem

My interest in extending the results of the Bell Labs group mentioned earlier was the main motivation behind the problem posed and solved in [7], see the comments in page 178 of that paper. That problem took a life of its own and the present paper is an attempt to return to this open question motivated (as mentioned earlier) by limited angle x-ray tomography.

For the benefit of the reader, we reproduce a few results from [7] that will play an important role here.

The sequence of functions of \( x \)

\[ \begin{align*}
\theta_0 &= 1, \\
\theta_1 &= x^{1/2}, \\
\theta_2 &= x^2 + t_1, \\
\theta_3 &= \frac{3}{4} x^{3/2} + t_2 x^{1/2}, \\
\theta_4 &= \frac{15}{32} x^9 + \frac{15t_2}{4} x^4 + t_3 x^2 - \frac{5}{2} t_2^2, \\
\theta_5 &= \frac{525}{2048} x^{25/2} + \frac{35t_3 x^{13/2}}{8} + \frac{3t_4 x^{9/2}}{4} - \frac{7t_2^3}{3} x^{1/2}, \\
\theta_6 &= \frac{33075}{262144} x^{18} + \frac{19845t_4}{2048} x^{12} + \frac{945t_5}{256} x^{10} + \frac{15t_6}{32} x^8 - \frac{2205t_2^3}{32} x^6 - \frac{63t_3 t_4}{4} x^4 + \left( t_3 t_5 - \frac{9t_2^2}{5} \right) x^2 - \frac{49t_2^3}{2},
\end{align*} \]

were obtained in [7] by using the recursion
\[ \theta_{k+1} \theta_{k-1} - \theta_{k+1} \theta_{k-1} = (2k-1) \theta_k^2. \]

It is clear that at each step we can choose a new integration constant \( t_i \). These give the deformation parameters mentioned earlier. The examples discussed in section 5 will only require
the functions $\theta_2, \theta_3, \theta_4, \theta_5$ but we include $\theta_6$ here to make the dependence of these functions on the free parameters more evident, see (3.4) and (3.5), below.

The functions above are shown in [7] to give rise to one half of the solutions of the ‘bispectral problem’, i.e. they allow one to define

$$V_k = -\frac{1}{4x^2} - 2\partial^2_k \log \theta_k$$

such that the eigenfunctions $\phi(x, z)$ of

$$L = -D^2_k + V_k$$
satisfy not only

$$L\phi(x, z) = z\phi(x, z)$$

but also

$$B(z, \partial_z)\phi(x, z) = \Theta(x)\phi(x, z)$$

for some differential operator $B$.

In [7] one proves that the potentials $V(x)$ in $L = -D^2_k + V$ need to be rational functions for this very exceptional bispectral property to hold. The functions $\theta_k$ above meet this requirement if, as explained in [7], at every other step of the recursion we set some of the earlier $t_i$ equal to zero. This has the effect that the dependence of the functions $\theta_k$ on the free parameters $t_i$ is given as follows

$$\theta_{2n} = \theta_{2n}(x; t_n, t_{n+1}, \ldots, t_{2n-1})$$

and

$$\theta_{2n-1} = \theta_{2n-1}(x; t_n, t_{n+1}, \ldots, t_{2n-2}),$$

as can be seen in the explicit examples displayed above.

It may be appropriate to note that the other half of the solutions of the bispectral problem considered in [7]-and related to the Korteweg–de Vries hierarchy-will play no role in this paper. Their corresponding theta functions are obtained from the same recursion relation as above with $\theta_1 = x$. In that case there is no need to set some of the $t_i$ equal to zero to get a rational potential $V(x)$. For different but related results dealing with this KdV hierarchy, or more generally the Kadomtsev–Petviashvili hierarchy of equations, usually referred to as the KP family of evolution equations, one can see [2–4]. For deep results on rational solutions of the KdV and the KP hierarchies, see [1, 21]. For other work on commuting integral-differential operators without the presence of deformation parameters but in a matrix valued context see [5, 16, 17].

For the potentials $V_k(x)$ that we will be concerned with here, it was observed in [43] that the relevant evolution equations are the so called ‘master symmetries of Korteweg–de Vries’. The potentials $V_k$ were given in [7] but the role of the master symmetries was only uncovered in [43]. The theta functions going with the KdV hierarchy are given by the characters of certain representations of $GL(n, R)$. Finding a similar group theoretical interpretation for the theta functions given above remains a challenge.

We close this section with the remark that the desire to extend the results of [22, 34–36] motivated by the consideration of the limited-angle x-ray tomography problem has taken us
and other authors pretty far into what some people may call 'pure' mathematics. The sharp distinction between pure and applied mathematics is, at times, not a productive one. The notion of bispectrality—explained above—surfaces from time to time in unexpected areas. For a recent instance see [24], and references [47–49] in that paper.

4. The KdV hierarchy and its master symmetries

The celebrated KdV equation from water wave theory is given, as was mentioned in the introduction, by

\[ u_t = u_{xxx} - 6uu_x. \]

This is not the place to give any details about this subject, and we just mention a few relevant facts. There are many references that could be cited, but for our purposes it is enough to refer the reader to [8, 43] and the references there.

It is known that the KdV partial differential equation is the second one in a hierarchy of which the first equation is just translation, given by solving

\[ u_t = u_x \]

with initial data

\[ u(x, 0) = f(x) \]

and solution

\[ u(x, t) = f(x + t). \]

It is useful to denote the partial derivative operator acting on the function \( u \) above as follows

\[ u_x = \mathcal{X}_0(u) \]

so that (4.1) becomes

\[ u_t = \mathcal{X}_0(u). \]

The remaining non-linear equations in the hierarchy, namely

\[ u_t = \mathcal{X}_j(u), \]

can be obtained by applying higher order powers of the tensor \( \mathbb{N}_u \) to the generator of translations \( u_x = \mathcal{X}_0(u) \), more explicitly

\[ \mathcal{X}_j(u) = \mathbb{N}_u^j \mathcal{X}_0(u) = \mathbb{N}_u^j u_x, \quad j = 2, 3, 4, \ldots \]

where the tensor \( \mathbb{N}_u \) is given by

\[ \mathbb{N}_u = -\partial_x^2 + 4u + 2u_x\partial_x^{-1}. \tag{4.2} \]

One word about the operator of integration, denoted here by \( \partial_x^{-1} \). The arbitrary constant is chosen so that the resulting function vanishes at plus infinity.

An important property of the KdV evolution is that it is given by a Hamiltonian in the appropriate phase space of functions, an observation of Faddeev and Zakharov. Moreover it is
what is called a bi-Hamiltonian flow, a notion going back to Magri and independently Dorfman, Gelfand and Dikii. For a very good guide to all of these developments and the work of many other authors who have made very important contributions in this area the reader may consult [8].

The consequence of this is that there exists another hierarchy of evolution equations obtained in this case by applying powers of the tensor (4.2) to the infinitesimal generator of dilations, namely

$$\tau_0(u) = \frac{1}{2} xu_x + u.$$  

This other hierarchy has generators given by

$$\tau_j(u) = N_j^i \tau_0(u),$$

where the first few explicit examples are given by

$$\tau_0(u) = \frac{1}{2} xu_x + u,$$

$$\tau_1(u) = -\frac{x}{2} (u_{xxx} - 6u u_x) - 2u_x u_x + u_x \partial_x^{-1} u + 4u^2,$$

$$\tau_2(u) = \frac{x}{2} (u_{xxxx} - 10u u_{xxx} - 18u_x u_{xx} + 24u^2 u_x) + 5u_{xxx} - u_{xx} \partial_x^{-1} u$$

$$- 24u u_{xx} - 15u_x^2 + u_x (4u \partial_x^{-1} u + 2 \partial_x^{-1} \tau_1(u)) + 16u^3,$$

and the new family of evolution equations takes the form

$$u_t = \tau_j(u).$$

As pointed out in [42], see page 166, very few explicit solutions are known for these evolution equations which are transversal to the more familiar KdV hierarchy. Here we record a few such solutions in terms of the potentials $V_k$ in (3.1). This establishes a remarkable connection between operators for which we extend in section 5 the result of Slepian, [34], and the master symmetries of the KdV equation.

We have

$$\tau_1(V_3) = 0$$

and

$$\tau_0(V_3) = -2t_2 \frac{\partial V_3}{\partial t_2}.$$  

We also have

$$\tau_1(V_4) = -120t_2 \frac{\partial V_4}{\partial t_3}$$

as well as

$$\tau_1(V_5) = -420t_3 \frac{\partial V_5}{\partial t_4}.$$
and

\[ \tau_2(V_6) = -105,840 t_3 \frac{\partial V_6}{\partial t_5}. \]

While the vector fields

\[ X_j(u) \]

going with the KdV hierarchy all commute with each other, the new vector fields

\[ \tau_j(u) \]
satisfy the commutation relations

\[ [X_j, \tau_l] = -\left( j + \frac{1}{2} \right) \tau_{l+j} \]

and

\[ [\tau_j, \tau_l] = (l - j) \tau_{l+j}. \]

5. Commuting differential operators

We have recalled in section 3 the results from [7] to obtain a sequence of families of operators of the form

\[ L(t) = -D_x^2 + V_k(x), \]

where \( V_k(x) \) is given by (3.1) and the dependence on

\[ t = (t_1, t_2, t_3, \ldots) \]

results from the dependence of the functions \( \theta_k \) on these parameters, and such that we have solutions of the bispectral problem.

In the first two families the operators \( L_2(t_1) \) and \( L_3(t_2) \) depend on one free parameter, in the next two families we have two free parameters, in the following pair we have three free parameters, etc.

The purpose of this section is to discuss in detail a method to study the first four examples, starting with \( \theta_2 \), of the list of the \( \theta_j \) given in section 3 and in each case address the ‘time-and-band limiting’ problem considered in [22, 23, 34–36]. More explicitly, we demonstrate the existence of a commuting differential operator that provides an effective and numerically stable way to compute the eigenfunctions of the naturally appearing integral operator in each case. For a careful look at the numerical issues involved here, see [30]. In the first two cases these results, without any indication of the method used to obtain them were given in [14].

5.1. First example

We start with the operator

\[ L = -D_x^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \theta_1, \]

(5.1)
which can be expressed as

\[ L = -D_x^2 - \frac{1}{4x^2} + \frac{1}{x^2} = -D_x^2 + \frac{3}{4x^2}. \]

This is one of the situations (1.1) (\( \nu = 1 \)) covered by Slepian’s result, see [34], with (regular) eigenfunction given by (1.2).

One application of the Darboux method of section 2 produces the family of operators

\[ L_2(t_1) = -D_x^2 - \frac{1}{4x^2} - 2\partial^2 x \log \theta_2. \] (5.2)

In fact, as one can see in [7], this is the reason for introducing the sequence of functions \( \theta_k \) that were reproduced in section 3.

As we observed at the end of section 2, the eigenfunctions of this operator can be written in terms of those of \( L \), namely \( f_1(x, z) \).

The new eigenfunction is

\[ \tilde{f}_1(x, z) = \left( \frac{1}{z} \right) \left( D_x - \partial_x \log \frac{\theta_1}{\theta_2} \right) f_1(x, z), \] (5.3)

where, as we recall, \( f_1 \) is given in terms of the Bessel function \( J_1 \) and thus \( \tilde{f}_1 \) is written in terms of \( J_1 \) and \( J_2 \). The reason for this is the relation

\[ \frac{\partial J_1}{\partial x} = \frac{1}{x} J_1 - J_2. \] (5.4)

These eigenfunctions depend on the free parameter \( t_1 \) brought in by the dependence of \( \theta_1 \) on it.

The integral kernel in question

\[ K(z_1, z_2) \equiv \int_0^T \tilde{f}_1(x, z_1)\tilde{f}_1(x, z_2)dx \] (5.5)

can be written out explicitly in terms of \( f_1(T, z_1), f_1(T, z_2), f_2(T, z_1) \) and \( f_2(T, z_2) \), namely we have

\[ K(z_1, z_2) = \frac{z_1 f_1(T, z_1)f_2(T, z_2) - z_2 f_2(T, z_1)f_1(T, z_2)}{z_1^2 - z_2^2} + \frac{2t_1 f_1(T, z_1)f_1(T, z_2)}{(t_1 + T^2)Tz_1z_2}. \]

A word about the derivation of this expression: we note the fact that if a function \( f(x, z) \) satisfies

\[ Lf(x, z) = (-D_x^2 + V(x))f(x, z) = \partial^2 f(x, z) \]

it is straightforward to see that the expression

\[ f(x, z_1)\frac{\partial f(x, z_2)}{\partial z_2} - \frac{\partial f(x, z_1)}{\partial z_1} f(x, z_2) \]
(5.6)

is an antiderivative for the product \( f(x, z_1)f(x, z_2) \).
Using this observation, the kernel (5.5) has the traditional Christoffel–Darboux expression that follows from (5.6). From it one can obtain the version of the kernel given above, free of any derivatives. This expression is more useful to us.

The same procedure will be used for the kernels in the next few examples. In every case we are making repeated use of the relations (5.7) and (5.8) given below and involving (1.2)

\[
\frac{\partial f_\nu}{\partial x} = \frac{1 + 2\nu}{2x} f_\nu - z f_{\nu+1}
\]

and

\[
\frac{\partial f_\nu}{\partial x} = \frac{1 - 2\nu}{2x} f_\nu + z f_{\nu-1}.
\]

Using the fact that \( f_\nu \) depends only on the product \( xz \) we also have

\[
\frac{\partial f_\nu}{\partial z} = \frac{1 + 2\nu}{2z} f_\nu - x f_{\nu+1}
\]

and

\[
\frac{\partial f_\nu}{\partial z} = \frac{1 - 2\nu}{2z} f_\nu + x f_{\nu-1}.
\]

The second set of equations (5.9) and (5.10) will be used later in this section to get rid of \( z \) derivatives.

The operator with kernel (5.5) acts on \( L^2([-G, G], dz) \).

We set out to look for an operator of the form

\[
op(z, \partial_z) = \sum_{k=0}^{2} \partial^k(z^2 - G^2)^k a_k(z) \partial_z^k,
\]

with \( a_k(z) \) an even Laurent polynomial of degree \( 4 - 2k \), to be determined, such that

\[
op(z_1, \partial_{z_1})K(z_1, z_2) = \nop(z_2, \partial_{z_2})K(z_1, z_2).
\]

More explicitly, we look for \( a_k(z) \) of the form \( a_2(z) = 1, a_1(z) = az^2 + b + \frac{h}{z} \) and \( a_0(z) = d\frac{dz}{z} + e\frac{z^2}{x} + f \frac{x}{z} + g \) so that (5.12) will be satisfied.

The inclusion of the factors \( (z^2 - G^2)^k \) in the coefficients of the differential operator \( \nop(z, \partial_z) \) guarantees that the domain where commutativity holds can be taken as the set of smooth functions that are bounded at the endpoints \( z = G \) and \( z = -G \). This consideration is important starting with the classical examples in \([22, 34, 35]\). A careful discussion of the domain of the differential operator even in the classical cases is—to the best of my knowledge—only found in \([20]\).

At this point we have another chance to use the well known relations among the functions \( f_i(x, z) \) introduced earlier, see (5.9) and (5.10) (and very related to properties of the Bessel functions). Using these relations repeatedly we can express any partial derivative (of any order) of \( f_1(x, z) \) and \( f_2(x, z) \) with respect to \( z \) as linear combinations of \( f_1(x, z) \) and \( f_2(x, z) \) with coefficients that are rational functions of \( x \) and \( z \).

At the end of the day the commutativity relation (5.12) we are trying to satisfy (by a proper choice of the \( a_k \)) amounts to the vanishing of a linear combination of the four products

\[
f_1(T, z_1) f_1(T, z_2).
\]
\[ f_2(T, z_1) f_1(T, z_2), \]
\[ f_1(T, z_1) f_2(T, z_2), \]
and
\[ f_2(T, z_1) f_2(T, z_2). \]

with coefficients which are polynomials in \( z_1, z_2, t_1, T \). The simplest such coefficient is the one going with the product \( f_2(T, z_1) f_2(T, z_2) \), and it is given by
\[ 32z_1^2 z_2^2(z_2^2 - z_1^2)T^2(T^2 + t_1)(2T^2 - a), \]
where as we recall \( a_1(z) = a_2 z + b + c. \)

From here we see that \( a = 2T^2 \), and with the assistance of symbolic computation we can get that the vanishing of each of the four coefficients alluded to above gives an essentially unique solution for \( a_0(z) \) and \( a_1(z) \) once we take \( a_2(z) = 1. \)

One gets
\[ a_1(z) = 2T^2 z^2 + \frac{4G^2 t_1 + 1}{2} + \frac{15G^2}{2z^4} \] (5.13)
and
\[ a_0(z) = T^4 z^4 + (4G^2 t_1 + 9)T^2 z^2 - \frac{4G^4 t_1 - 15G^2}{8z^2} - \frac{135G^4}{16z^6} \] (5.14)

It is probably more interesting to notice that \( \mathfrak{o} \) can be written in terms of the operators \( \mathbb{A}_\nu \), see (1.4), that Slepian found in the Bessel case, namely
\[ \mathfrak{o} = \mathbb{A}_2^2 - \frac{3}{2} \mathbb{A}_2 - \frac{11G^2 T^2}{2} + 2t_1 G^2 \mathbb{A}_0. \] (5.15)

Notice that only \( \mathbb{A}_\nu \) with even \( \nu \) enter in (5.15).

These results are summarized as follows

**Theorem 5.1.** The integral operator with the kernel given by (5.5) acting in \( L^2([-G, G], dz) \) and the differential operator \( \mathfrak{o}(z, \partial_z) \), see (5.11), with \( a_2(z) = 1, a_1(z) \) and \( a_0(z) \) given as in (5.13), (5.14) commute with each other. Moreover the operator (5.11) is given by a polynomial (5.15) in the operators \( \mathbb{A}_\nu \), see (1.4), with coefficients that are polynomials in all the parameters involved, namely \( G, T \) and \( t_1 \).

Notice that the family of operators (5.2) for which we have extended Slepian’s result, [34], is explicitly given by
\[ L_2(t_1) = -D_x^2 + \frac{15x^4 - 18t_1 x^2 - t_1^2}{4x^6 + 8t_1 x^4 + 4t_1^2 x^2}, \]
and it interpolates by letting \( t_1 \) range from 0 to \( \infty \) between the operators \( L_0 \) and \( L_2 \) considered by Slepian. Finally, notice that this family, which is obtained by applying the Darboux process of section 2 to \( L_1 \) does not include this operator.

A natural question here and in the examples that follow is: how do you come up with a proposal for the order of the commuting differential operator \( \mathfrak{o}(z, \partial_z) \), in (5.11)? The answer is that in each one of these ‘bispectral’ situations, see section 3 we adopt the order of the operator \( B(z, \partial_z) \) (3.3), worked out in [7].
5.2. Second example

Here we start with the operator $L_2$, see (1.1), and apply the Darboux process to it. This happens to be the operator obtained in the example above when $t_1 = 0$.

We obtain the family of operators

$$L_3(t_2) = -D_x^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \theta_3.$$  \hfill (5.16)

This family of operators, has eigenfunctions given by

$$\tilde{f}_2(x, z) = \left(\frac{1}{z}\right) \left(D_x - \partial_x \log \frac{\theta_3}{\theta_2}\right) f_2(x, z),$$

where, as we recall, $f_2$ is given in terms of the Bessel function $J_2$ and thus $\tilde{f}_2$ is given in terms of $J_2$ and $J_3$. In the previous example the important functions were $J_1$ and $J_2$. In this and the examples in the next subsections this role is taken up by $J_2$ and $J_3$.

In the expression above for $\tilde{f}_2$ we have set $t_1 = 0$ and the eigenfunctions depend on the free parameter $t_2$ brought in by the function $\theta_3$ in one application of the Darboux process.

The integral kernel in question

$$K(z_1, z_2) \equiv \int_0^T \tilde{f}_2(x, z_1) \tilde{f}_2(x, z_2) dx$$  \hfill (5.17)

can be written out explicitly in terms of $f_2(T, z_1)$, $f_3(T, z_2)$, $f_2(T, z_1)$ and $f_3(T, z_2)$, namely we have

$$K(z_1, z_2) = \frac{z_1 f_2(T, z_1) f_3(T, z_2) - z_2 f_3(T, z_1) f_2(T, z_2)}{z_1^2 - z_2^2} - \frac{4t_2 f_3(T, z_1) f_3(T, z_2)}{(t_2 + T^2)Tz_1z_2}.$$  \hfill (5.18)

We set out to look for an operator of the form

$$\text{op}(z, \partial_z) = \sum_{k=0}^3 \partial_z^k (z^2 - G^2)^k a_k(z) \partial_z^k,$$  \hfill (5.19)

with $a_k(z)$ an even Laurent polynomial of degree $6 - 2k$ such that

$$\text{op}(z_1, \partial_{z_1}) K(z_1, z_2) = \text{op}(z_2, \partial_{z_2}) K(z_1, z_2).$$

By using the same strategy as in the first example we can express all partial derivatives of $f_2(x, z)$ and $f_3(x, z)$ with respect to $z$ as linear combinations of $f_2(x, z)$ and $f_3(x, z)$ with coefficients that are rational functions of $x$ and $z$.

Finally we need to assure that the coefficients of the four products

$$f_2(T, z_1) f_3(T, z_2),$$
$$f_3(T, z_1) f_2(T, z_2),$$
$$f_2(T, z_1) f_3(T, z_2),$$
$$f_3(T, z_1) f_3(T, z_2),$$

are rational functions of $x$ and $z$. We conclude with the result:
and

\[ f_3(T, z_1) f_3(T, z_2), \]

will vanish. The resulting set of equations for \( a_k \) can be solved uniquely once we take \( a_3(z) = 1 \).

Once again we can write \( \text{op} \), see (5.19), in terms of the operators \( \mathcal{A}_\nu \) that Slepian found in the Bessel case, namely

\[
\text{op} = \mathcal{A}_3^3 - \frac{29}{4} \mathcal{A}_3^2 + \frac{195 - 256G^2T^2}{16} \mathcal{A}_3 + \frac{435G^2T^2}{8} + 3G^2t_2 \mathcal{A}_1. \tag{5.20}
\]

Notice that only \( \mathcal{A}_\nu \) with odd \( \nu \) enter in the expression (5.20).

These results are summarized as follows

**Theorem 5.2.** The integral operator with kernel given by (5.17) acting in \( L^2([-G, G], dz) \) and the differential operator \( \text{op}(z, \partial_z) \), see (5.19), with \( a_3(z) = 1, a_2(z), a_1(z) \) and \( a_0(z) \) properly chosen commute with each other. Moreover the operator (5.19) is given by a polynomial (5.20) in the operators \( \mathcal{A}_\nu \), see (1.4), with coefficients that are polynomials in all the parameters involved, namely \( G, T \) and \( t_2 \).

Notice that the family of operators (5.16) for which we have now extended Slepian’s result, [34], is explicitly given by

\[
L_3(t_2) = -D^2 - \frac{35x^8 - 90t_2x^4 + 3t_2^2}{4x^{10} + 8t_2x^6 + 4t_2^2x^2},
\]

and it interpolates by letting \( t_2 \) range from 0 to \( \infty \) between the operators \( L_1 \) and \( L_3 \) considered by Slepian. Finally, notice that this family, which obtained by applying the Darboux process of section 2 to \( L_2 \) does not include this operator.

**5.3. Third example**

In this subsection we start with any of the operators in (5.16) considered in the previous example and perform a step of the Darboux process to it, bringing in a new parameter \( t_3 \) besides \( t_2 \).

This results in the two parameter family of operators

\[
L_4(t_2, t_3) = -D^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \theta_4. \tag{5.21}
\]

This family of operators, has eigenfunctions given by

\[
\tilde{f}_3(x, z) = (1/z) \left( D_x - \partial_x \log \frac{\theta_4}{\theta_3} \right) \tilde{f}_2(x, z),
\]

where \( \tilde{f}_2 \) are the eigenfunctions of the previous example. Eventually they can be expressed in terms of the Bessel functions \( J_2 \) and \( J_3 \).

The integral kernel in question

\[
K(z_1, z_2) \equiv \int_0^T \tilde{f}_3(x, z_1) \tilde{f}_3(x, z_2) dx \tag{5.22}
\]

can be written out explicitly in terms of \( \tilde{f}_3(T, z_1), \tilde{f}_3(T, z_2), \tilde{f}_3(T, z_1) \) and \( \tilde{f}_3(T, z_2) \).
We set out to look for an operator of the form

\[ \text{op}(z, \partial_z) = \sum_{k=0}^{5} \partial_z^k (z^2 - G^2)^k a_k(z) \partial_z^k, \quad (5.23) \]

with \(a_k(z)\) an even Laurent polynomial of degree \(10-2k\) such that

\[ \text{op}(z_1, \partial_{z_1}) K(z_1, z_2) = \text{op}(z_2, \partial_{z_2}) K(z_1, z_2). \]

By now the strategy should be clear. We can express all partial derivatives of \(f_2(x, z)\) and \(f_3(x, z)\) with respect to \(z\) as linear combinations of \(f_2(x, z)\) and \(f_3(x, z)\) with coefficients that are rational functions of \(x\) and \(z\) and then try to insure the vanishing of each coefficient going with the four products

\[ f_3(T, z_1) f_3(T, z_2), \]
\[ f_3(T, z_1) f_2(T, z_2), \]
\[ f_2(T, z_1) f_3(T, z_2), \]

and

\[ f_2(T, z_1) f_3(T, z_2). \]

The vanishing of the individual coefficients determines the differential operator (5.23) once we set \(a_5(z) = 1\).

Once again we can write \(\text{op}\) in terms of the operators \(A_{\nu}\) that Slepian found in the Bessel case, namely

\[ \text{op} = \sum_{j=0}^{5} w_j A^j + t_2 \left( -\frac{320}{9} G^8 A_2 + \frac{80}{9} G^8 A_4 \right) + t_3 \left( \frac{16G^9}{3} A_2^2 - 8G^6 A_2 - \frac{88G^8 T^2}{3} \right) + t_4 \left( v_1 A_2 + v_2 A_0^3 + v_3 A_4 + v_4 A_0^2 + v_5 A_2^2 + v_6 A_3^2 + v_7 I \right), \quad (5.24) \]

with the \(w_j\) given by

\[ w_5 = 1, \quad w_4 = -\frac{95}{4}, \quad w_3 = -\frac{320 G^2 T^2 - 1549}{8}, \]
\[ w_2 = \frac{22336 G^2 T^2 - 19703}{32}, \quad w_1 = \frac{36864 G^4 T^4 - 971648 G^2 T^2 + 162645}{256}, \]
\[ w_0 = -\frac{84726 G^4 T^4 - 399805 G^2 T^2}{64}, \]

and the \(v_i\) given by

\[ v_1 = -\frac{5 G^4 (64 G^2 T^2 - 363)}{9}, \quad v_2 = -\frac{40 G^4}{3}, \quad v_3 = \frac{5 G^4 (64 G^2 T^2 + 69)}{18}, \]
\[ v_4 = -\frac{70 G^4}{3}, \quad v_5 = -\frac{340 G^4}{3}, \quad v_6 = \frac{80 G^4}{3}, \quad v_7 = -\frac{305 G^6 T^2}{3}. \]
These results are summarized as follows

**Theorem 5.3.** The integral operator with kernel given by (5.22) acting in $L^2([-G, G], dz)$ and the differential operator $\mathfrak{op}(z, \partial_z)$, see (5.23), with $a_5(z) = 1$, $a_4(z)$, $a_3(z)$, $a_2(z)$, $a_1(z)$, $a_0(z)$ properly chosen commute with each other. Moreover the operator (5.23) is given by a polynomial (5.24) in the operators $A_\nu$, see (1.4), with coefficients that are polynomials in all the parameters involved, namely $G, T$ and $t_2, t_3$.

Notice that only $A_\nu$ with even $\nu$ enter in (5.24).

### 5.4. Fourth example

Here we start with the operator (5.21) but we set the parameter $t_2$ equal to 0 and apply one step of the Darboux process to get a family of operators depending on the old free parameter $t_3$ and a new parameter $t_4$.

This results in the family of operators

$$L_4(t_3, t_4) = -D^2 - \frac{1}{4x^2} - 2\partial_x^2 \log \theta_5,$$

(5.25)

This operator, has eigenfunctions given by

$$\tilde{f}_4(x, z) = \frac{1}{z} \left(D - \partial_x \log \theta_5\right) \tilde{f}_3(x, z),$$

where $\tilde{f}_3$ are the eigenfunctions of the previous example, with the understanding that $t_2 = 0$.

Eventually they can be expressed in terms of the Bessel functions $J_2$ and $J_3$.

The integral kernel in question

$$K(z_1, z_2) \equiv \int_0^T \tilde{f}_4(x, z_1) \tilde{f}_4(x, z_2) dx$$

(5.26)

can be written out explicitly in terms of $\bar{f}(T, z_1), \bar{f}(T, z_2), \bar{f}(T, z_1)$ and $\bar{f}(T, z_2)$.

We set out to look for an operator of the form

$$\mathfrak{op}(z, \partial_z)^{\mathfrak{i}} = \sum_{k=0}^{7} \partial_x^k \left(\partial_x^2 - G^2 a_5(z)\partial_x^2\right)^k,$$

(5.27)

with $a_5(z)$ an even Laurent polynomial of degree $14 - 2k$ such that

$$\mathfrak{op}(z_1, \partial_{z_1}) K(z_1, z_2) = \mathfrak{op}(z_2, \partial_{z_2}) K(z_1, z_2).$$

We follow the same strategy as above and determine the differential operator.

As in the previous examples we can write $\mathfrak{op}$ in terms of the operators $\mathfrak{h}_\nu$ that Slepian found in the Bessel case, namely

$$\mathfrak{op} = \sum_{j=0}^{7} w_j \mathfrak{h}_5^j + t_3^2 \left(-\frac{16}{225} G^{12} (3 \mathfrak{h}_3 - \mathfrak{h}_5)\right) + t_4 \sum_{j=0}^{3} v_j \mathfrak{h}_3^j + t_3$$

$$\times (u_1 \mathfrak{h}_3^4 + u_2 \mathfrak{h}_3^3 + u_3 \mathfrak{h}_3^2 + u_4 \mathfrak{h}_3 + u_5 \mathfrak{h}_3 + u_6 \mathfrak{h}_3^2 + u_7 \mathfrak{h}_3 + u_8 \mathfrak{h}_3 + u_9 \mathfrak{h}_3),$$

(5.28)

where the explicit expression for the coefficients $w_j, v_j, u_j$ is rather unilluminating and will not be displayed.
These results are summarized as follows

**Theorem 5.4.** The integral operator with kernel given by (5.26) acting in $L^2([-G, G], dz)$ and the differential operator $\partial_t^2 \partial_x + \partial_x^2$, see (5.27), with $a_7(z) = 1, a_6(z), a_5(z), a_4(z), a_3(z), a_2(z), a_1(z), a_0(z)$ properly chosen commute with each other. Moreover the operator (5.27) is given by a polynomial (5.28) in the operators $\partial_x^\nu$, see (1.4), with coefficients that are polynomials in all the parameters involved, namely $G, T$ and $t_3, t_4$.

Notice that only $\partial_x^\nu$ with odd $\nu$ enter in (5.24).

6. Final comments

The results in [2–4] deal with the large class of situations related to the KP hierarchy, we have only dealt here with the master symmetries of the KdV evolution equations. These are related to the Schroedinger (second order) differential operator. The paper [13] considers solutions of the bispectral problem when this second order differential operator is replaced by a third order one. The possible existence of commuting pairs of integral and differential operators in this case is—to the best of knowledge—largely unexplored territory. For interesting examples and theoretical tools that could be applied in this case, see [2].

The problem considered here is an extension in the case of higher dimensional Euclidean spaces of the work in [34]. In the paper [15] one finds an excursion into the non-commutative situation that arises when Euclidean space is replaced by spheres. That study was motivated purely by mathematical reasons, and yet several years later, people working in geophysics found these results to have practical use, see [32, 33].

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

F Alberto Grünbaum [https://orcid.org/0000-0001-9663-4283](https://orcid.org/0000-0001-9663-4283)

References

[1] Airault H, McKean H P and Moser J 1977 Rational and elliptic solutions of the Korteweg–de Vries equation and a related many-body problem Commun. Pure Appl. Math. 30 95–148
[2] Casper W R and Yakimov M T 2020 Integral operators, bispectrality and growth of Fourier algebras I. Reine Angew. Math. 2020 151–94
[3] Casper W R, Grünbaum F A, Yakimov M and Zurrián I 2019 Reflective prolate-spheroidal operators and the KP/KdV equations Proc. Natl Acad. Sci. USA 116 18310–5
[4] Casper W R, Grünbaum F A, Yakimov M and Zurrián I 2020 Reflective prolate-spheroidal operators and the adelic Grassmanian (arXiv:2003.11616)
[5] Castro M and Grünbaum F A 2017 Time-and-band limiting for matrix orthogonal polynomials of Jacobi type Random Matrices: Theor. Appl. 06 1740001
[6] Davison M E 1981 A singular value decomposition for the radon transform in n-dimensional Euclidean space Numer. Funct. Anal. Optim. 3 321–40
[7] Duistermaat J J and Grünbaum F A 1986 Differential equations in the spectral parameter Commun. Math. Phys. 103 177–240
[8] Fokas A and Zakharov V E (ed) 1993 *Important Developments in Soliton Theory* (Springer Series in Nonlinear Dynamics) (Berlin: Springer).

[9] Gerchberg R and Saxton W 1972 A practical algorithm for the determination of phases from image and diffraction plane pictures *Optik* **35** 237–46.

[10] Grünbaum F A 1981 Eigenvectors of a Toeplitz matrix: discrete version of the prolate spheroidal wave functions *SIAM. J. Algebra: Discrete Methods* **2** 136–41.

[11] Grünbaum F A 1982 The limited angle reconstruction problem *Proc. Symposia Applied Mathematics* vol 27 pp 43–61.

[12] Grünbaum F A 1980 A study of Fourier space methods for ‘limited angle’ image reconstruction *Numer. Funct. Anal. Optim.* **2** 31–42.

[13] Grünbaum F A 1987 Differential equations in the spectral parameter: the higher order case. Inverse problems: an interdisciplinary study (Montpellier, 1986), 307–322 *Advanced in Electronics and Electron Physics* vol 19 (London: Academic).

[14] Grünbaum F A 1996 Band-time-band limiting integral operators and commuting differential operators *Algebr. Anal.* **8** 122–6.

[15] Grünbaum F A, Longhi L and Perlstäd M 1982 Differential operators commuting with finite convolution integral operators: some nonabelian examples *SIAM J. Appl. Math.* **42** 941–55.

[16] Grünbaum F A, Pacharoni I and Zurrián I 2017 Time and band limiting for matrix valued functions: an integral and a commuting differential operator *Inverse Problems* **33** 025005.

[17] Karantsev S G 2015 Singular value decomposition for the cone-beam transform in the ball *J. Inverse Ill-Posed Problems* **23** 173–85.

[18] Katsnelson V 2018 Self-adjoint boundary conditions for the prolate spheroid differential operator *Indefinite Inner Product Spaces, Schur Analysis, and Differential equations* (Operator Theory Advance and Applications vol 263) (Basel: Birkhäuser) pp 357–86.

[19] Krichever I M 1978 Rational solutions of the Kadomcev–Petviashvili equation and the integrable systems of N particles on a line *Funk. Anal. Prilozhen.* **12** 76–8.

[20] Landau H J and Pollak H O 1961 Prolate spheroidal wave functions, Fourier analysis and uncertainty–II *Bell Syst. Tech. J.* **40** 65–84.

[21] Landau H J and Pollak H O 1962 Prolate spheroidal wave functions, Fourier analysis and uncertainty-III: the dimension of the space of essentially time- and band-limited signals *Bell Syst. Tech. J.* **41** 1295–336.

[22] Lee N and Nekrasov N 2021 Quantum spin systems and supersymmetric gauge theories. Part I *J. High Energy Phys.* JHEP03(2021)093.

[23] Louis A K 1985 Orthogonal function series expansion and the null space of the Radon transform *SIAM J. Math. Anal.* **15** 621–33.

[24] Maass P 1987 The x-ray transform: singular value decomposition and resolution *Inverse Problems* **3** 729–41.

[25] Mehta M L 2004 *Random Matrices* 3rd edn (Amsterdam: Elsevier).

[26] Natterer F 2001 The mathematics of computerized tomography *Classics in Applied Mathematics* vol 32 (Philadelphia, PA: SIAM).

[27] Osipov A, Rokhlin V and Xiao H 2013 *Prolate Spheroidal Wave Functions of Order Zero* (Applied Mathematical Science vol 187) (New York: Springer).

[28] Quellmalz M, Hielsmacher M and Louis A K 2018 The cone-beam transform and spherical convolution operators *Inverse Problems* **34** 10.

[29] Simons F J and Dahlen F A 2006 Spherical Slepian functions and the polar gap in geodesy *Geophys. J. Int.* **166** 1039–61.

[30] Simons F J, Dahlen F A and Wieczorek M A 2006 Spatiospectral concentration on a sphere *SIAM Rev.* **48** 504–36.

[31] Slepian D 1964 Prolate spheroidal wave functions, fourier analysis and uncertainty-IV: extensions to many dimensions; generalized prolate spheroidal functions *Bell Syst. Tech. J.* **43** 3009–57.

[32] Slepian D and Pollak H O 1961 Prolate spheroidal wave functions, fourier analysis and uncertainty-I *Bell Syst. Tech. J.* **40** 43–63.

[33] Slepian D 1978 Prolate spheroidal wave functions, fourier analysis, and uncertainty-V: the discrete case *Bell Syst. Tech. J.* **57** 1371–430.
[37] Slepian D 1983 Some comments on fourier analysis, uncertainty and modeling SIAM Rev. 25 379–93
[38] Slepian D 1976 On bandwidth Proc. IEEE 64
[39] Tracy C A and Widom H 1994 Fredholm determinants, differential equations and matrix models Commun. Math. Phys. 163 33–72
[40] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel Commun. Math. Phys. 159 151–74
[41] Tracy C A and Widom H 1994 Level spacing distributions and the Bessel kernel Commun. Math. Phys. 161 289–309
[42] van Moerbeke P 1991 Integrable foundations of string theory Lectures on Integrable Systems (Sophia-Antipolis) ed O Babelon, P Cartier and Y Kosmann-Schwarzbach (Singapore: World Scientific) pp 163–267
[43] Zubelli J P and Magri F 1991 Differential equations in the spectral parameter, Darboux transformations and a hierarchy of master symmetries for KdV Commun. Math. Phys. 141 329–51