Chekanov tori and pseudotoric structures

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Chekanov tori provide examples of exotic monotone Lagrangian tori which cannot be transformed to standard tori by symplectomorphisms, in \( \mathbb{R}^{2n}, \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 \), and certain projective spaces (see [1]). A sequence of such examples arises from the construction of the Lagrangian torus \( \Theta \subset \mathbb{R}^4 \), represented explicitly and implicitly in the now classical papers [2] and [3]. On the other hand, the papers [4], [5] proposed a generalization of the notion of toric structure on symplectic manifolds, called pseudotoric structure. Just like a toric structure, a pseudotoric structure generates a space of Lagrangian fibrations with smooth generic fibre, whose singular fibres are characterized by the existence of separatrix type solutions on them.

In the present brief note we construct the Chekanov torus \( \Theta_k \subset \mathbb{R}^{2k-2} \), originally defined in [1], as a smooth Lagrangian torus generated by a pseudotoric structure. This enables us to construct exotic monotone Lagrangian tori of Chekanov type on toric Fano varieties.

Consider \( \mathbb{R}^{2k+2} \) with the standard complex and symplectic structures, that is, \( \mathbb{C}^{k+1} \). Fix coordinates \((z_1, \ldots, z_{k+1})\); then one has a map \( \psi: \mathbb{C}^{k+1} \to \mathbb{C} \) defined by the formula

\[
(z_1, \ldots, z_{k+1}) \mapsto a = z_1 \times \cdots \times z_{k+1} \in \mathbb{C}_a.
\]

The fibres of \( \psi \), which are hypersurfaces in \( \mathbb{C}^{k+1} \), are smooth except for the fibre over zero. The Hamiltonian torus action of \( T^{k+1} \) can be chosen such that there is a subtorus \( T^0 \) whose Hamiltonian action preserves the fibres of \( \psi \). Just such a subtorus is defined by the condition

\[
\sum_{i=1}^{k+1} \alpha_i = 0,
\]

where \( \alpha_i \) are corresponding weights. The moment maps for the Hamiltonian action of this \( T^0 \) are given by the mean values \( F_i = \langle A_i \psi; \psi \rangle, \psi \in \mathbb{C}^{k+1} \), of self-adjoint operators \( A_i = \text{diag}(\lambda_1, \ldots, \lambda_{k+1}) \), where \( \lambda_i = 1, \lambda_{i+1} = -1, \) and \( \lambda_j = 0 \) for the remaining indices. Then it is not difficult to see that \( \{F_1, \ldots, F_k\} \) is a commuting set on \( \mathbb{C}^{k+1} \) whose restriction to any smooth fibre of \( \psi \) is a completely integrable system (see [6]). A structure of this type is called a pseudotoric structure (for a detailed definition see [5]).

If one fixes a set \((c_1, \ldots, c_k)\) of non-critical values for the restrictions \( F_1^a, \ldots, F_k^a \) on a smooth fibre \( N_a \) (so that \(-1 < c_i < 1\)), then the mutual level set \( S_{(c_1, \ldots, c_k)} = \{F_i^a = c_i\} \subset N_a \) is a smooth Lagrangian torus for any \( a \neq 0 \). Then choice of an arbitrary smooth loop \( \gamma \subset \mathbb{C}^* \) determines a smooth Lagrangian torus in \( \mathbb{C}^{k+1} \):

\[
T^{k+1} = S_{\gamma,(c_1, \ldots, c_k)} = \bigcup_{a \in \gamma} S_{(c_1, \ldots, c_n)}^a.
\]

There are two distinct cases: the standard type when \( \gamma \) is not contractible in \( \mathbb{C}_a^* \); the Chekanov type when \( \gamma \) is contractible in \( \mathbb{C}_a^* \). Let us explain how these types came about.
their names. Consider two smooth loops in $\mathbb{C}^*_a$: a non-contractible loop $\gamma_1$ is simply a circle centred at zero; for an arbitrary smooth loop $\gamma_0 \subset D \subset \mathbb{C}$ in the sector

$$D = \left\{ z \in \mathbb{C} : 0 < |z| < 2, \ 0 < \arg z < \frac{2\pi}{k+1} \right\}$$

(see the first formula of section 2 in [1]), a contractible loop $\gamma_2$ is $(\gamma_0)^{k+1}$.

**Theorem.** Lagrangian tori $T_1, T_2 \subset \mathbb{C}^{k+1}$ corresponding to the values $c_1 = \cdots = c_k = 0$ and the loops $\gamma_1, \gamma_2 \subset \mathbb{C}^*_a$, coincide with the standard torus and the torus $\Theta^k$, respectively.

The proof is by direct calculation (details can be found in [7]).

Furthermore, for the exotic monotone tori in $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ and projective spaces presented in [1], there also exist descriptions in terms of pseudotoric structures on these manifolds. It follows that every symplectic manifold with a pseudotoric structure admits tori of the Chekanov type. The paper [5] outlines a proof of the existence of a pseudotoric structure on an arbitrary toric Fano variety, which implies the existence of monotone Lagrangian tori of the Chekanov type on every toric Fano variety. Recall that in the original case the Chekanov tori are not equivalent to the standard Lagrangian tori induced by the toric structure; therefore, it is reasonable to expect that the Chekanov type tori in a toric Fano variety are not equivalent to the fibres of the action map modulo symplectomorphisms. Our plans for further study of pseudotoric structures include work on this conjecture.

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Presented by A. G. Sergeev  
Accepted 03/NOV/10  
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