Abstract

The Hylland-Zeckhauser (HZ) rule is a well-known rule for probabilistic assignment of items. The complexity of the rule has received renewed interest recently with Vazirani and Yannakakis (2020) proposing a strongly polynomial-time algorithm for the rule under bi-valued utilities, and making several general insights. We study the rule under the case of agents having bi-valued utilities. We point out several characterizations of the HZ rule, drawing clearer relations with several well-known rules in the literature. As a consequence, we point out alternative strongly polynomial-time algorithms for the HZ solution. We also give reductions from computing the HZ solution to computing well-known solutions based on leximin or Nash social welfare. An interesting contrast is that the HZ rule is group-strategyproof whereas the unconstrained competitive equilibrium with equal incomes rule is not even strategyproof. We also clarify which results change when moving from 1-0 binary utilities to the more general bi-valued utilities.

Keywords: Fair Division, Computational Complexity, Competitive Equilibrium with Equal Incomes

1. Introduction

Competitive Equilibrium with Equal Incomes (CEEI) is one of the most fundamental solution concepts in resource allocation [7, 14, 16]. The concept is based on the idea of a market-based equilibrium which underpins classical economics and has been referred to as the crown jewel of mathematical economics. In CEEI, each agent is considered to have equal budget of unit one to spend. An assignment of items satisfies CEEI if, for some price vector for the items, the supply meets demand. In other words, the agents get allocations that give them the maximum possible utility. CEEI is a well-established in economics because
it is based on the idea of a market equilibrium. It is an attractive solution concept because it implies envy-freeness and Pareto optimality.

We consider the problem of allocating $n$ items among $n$ agents. Agents have linear cardinal utilities over the items. If we view the items as divisible and do not impose any limits on the amount of items given to agents, then CEEI is well-understood. Under additive utilities, CEEI is equivalent to maximizing Nash social welfare. The equivalence follows from an analysis of the Eisenberg-Gale convex program that maximizes the Nash welfare within the set of divisible items. However, such a characterization disappears when each agent has a demand for exactly one unit of items. The constraint of unit capacities is especially critical when the fractions of items given to agents are interpreted as probabilities and the goal is to probabilistically find an assignment in which each agent gets one item. If each agent gets one unit of items, then the given probabilistic assignment can be instantiated into a lottery over perfect matchings using Birkhoff’s theorem.

In this paper, we focus on the pseudo-market rule proposed by Hylland and Zeckhauser [11] that is inspired by CEEI. We will refer to the rule as the HZ rule. HZ can be viewed as the suitable CEEI solution for probabilistic assignment of indivisible items. The HZ rule has been referred to as CEEI in the literature. We will not use the term CEEI for HZ so that it is clear that we assume unit-demand requirement when referring to the HZ solution. The complexity of computing the HZ solution has been open for 40 years. For example, it was mentioned as open by Sethuraman [15]. Recently, Vazirani and Yannakakis [18] explored the computational complexity of the HZ rule. They make several general insights including the fact that the HZ solution can be irrational. They present a strongly polynomial-time algorithm to compute the HZ solution under bi-valued utilities.

Contributions. We also focus on the case of bi-valued utilities. We prove several characterizations of the HZ rule, drawing clearer relations with several well-known rules in the literature. As a consequence, we point out alternative strongly polynomial-time algorithms for the HZ solution. In particular, we show that the Extended Probabilistic Serial (EPS) by Katta and Sethuraman [12] (which is designed for egalitarian objectives) also returns the HZ solution. Some of our key observations are based on the brilliant paper by Bogomolnaia and Moulin [6]. For bi-valued utilities, we also provide a reduction from computing the HZ rule to computing the leximin or maximum Nash welfare solution. Another structural insight we have is the following one: for all HZ solutions under bi-valued utilities, each item gets the same price in all the solutions as long as the underlying dichotomous preferences do not change. We also show the following interesting contrast. Under bi-valued utilities, the HZ rule is group-strategyproof whereas the (unconstrained) CEEI rule is not even strategyproof. Therefore, an innocuous-looking relaxation of the unit-demand

\[\text{1There are several works on probabilistic assignment of items to agents:}\ 5, 6, 7, 13, 10, 12.\]
requirement leads to completely different strategic properties. Our result regarding the manipulability of the CEEI rule also implies that for bi-valued utilities, the MNW rule of Caragiannis et al. [8] for indivisible goods is not strategyproof.

2. Preliminaries

An assignment problem is a triple \((N, O, u)\) such that \(N = \{1, \ldots, n\}\) is the set of agents, \(O = \{o_1, \ldots, o_n\}\) is the set of items, and \(u = (u_1, \ldots, u_n)\) is the utility profile which specifies for each agent \(i \in N\) utility function \(u_i\) where \(u_{ij}\) denotes the utility of agent \(i\) for item \(o_j\). We assume that for each \(j \in \{1, \ldots, m\}\), \(u_{ij} > 0\) for some \(i \in N\) and for each \(i \in N\), \(u_{ij} > 0\) for some \(j \in \{1, \ldots, m\}\).

A fractional assignment \(x\) is a \((n \times n)\) matrix \([x_{ij}]\) such that \(x_{ij} \in [0, 1]\) for all \(i \in N\), and \(o_j \in O\), and \(\sum_{i \in N} x_{ij} = 1\) for all \(o_j \in O\). The value \(x_{ij}\) represents the fraction of item \(o_j\) being allocated to agent \(i\). Each row \(x_i = (x_{i1}, \ldots, x_{im})\) represents the allocation of agent \(i\). We will denote the set of all allocations by \(\mathcal{A}\). An allocation \(x_i\) is balanced if \(\sum_{o_j \in O} x_{ij} = 1\). We will denote the set of all balanced allocations by \(\mathcal{A}_b\). For any allocation \(x_i\), we will refer to \(\sum_{o_j \in O} x_{ij}\) as the size of the allocation. The set of columns correspond to the items \(o_1, \ldots, o_n\).

A fractional assignment is discrete if \(x_{ij} \in \{0, 1\}\) for all \(i \in N\) and \(o_j \in O\). Let the set of all fractional assignments be \(\mathcal{F}\). A fractional assignment is balanced if \(\sum_{o_j \in O} x_{ij} = 1\) for all \(i \in N\). Let the set of all balanced fractional assignments be \(\mathcal{F}_b\).

The expected utility received by agent \(i\) from assignment \(x\) is \(u_i(x_i) = \sum_{o_j \in O} x_{ij} u_{ij}\).

We say that the utility functions are binary or 1-0 if \(u_{ij} \in \{0, 1\}\) for all \(i, j \in N\). We say that the utility functions are bi-valued if for all \(i\), \(u_{ij} \in \{\alpha_i, \beta_i\}\) where \(\alpha_i > \beta_i \geq 0\). Both binary and bi-valued preferences are forms of dichotomous preferences. We will denote by \(D_i\) the set of items most preferred by agent \(i\). For any bi-valued utility function involving values \(\alpha_i, \beta_i\), we call by binary-reduced those utility functions in which \(\alpha_i\) is turned into 1 and \(\beta_i\) is turned into zero.

An assignment \(x\) is Pareto optimal (PO) if there exists no other assignment \(y\) such that \(u_i(y_i) \geq u_i(x_i)\) for all \(i \in N\) and \(u_i(y_i) > u_i(x_i)\) for some \(i \in N\). An assignment \(x\) is Pareto optimal among balanced assignments if there exists no balanced assignment \(y\) such that \(u_i(y_i) \geq u_i(x_i)\) for all \(i \in N\) and \(u_i(y_i) > u_i(x_i)\) for some \(i \in N\).

We present a simple routine to turn an unbalanced assignment into a balanced one. We will refer to it as the balancing operation.

Balancing operation. If an assignment is not balanced, we consider sets \(N^+ = \{i \in N : \sum_{o_j \in O} x_{ij} > 1\}\) and \(N^- = \{i \in N : \sum_{o_j \in O} x_{ij} < 1\}\). Each agent \(i \in N^+\) gives away the least preferred items from her allocation so as to ensure that her allocation \(x_i\) has size 1. The donated items are then given to \(N^-\) arbitrarily to ensure that \(\sum_{o_j \in O} x_{ij} = 1\) for all \(i \in N\).
3. Solution Concepts

We present a few prominent solution concepts starting with the HZ solution.

An assignment $x$ is an HZ solution if $x \in \mathcal{F}_b$ and there exists a price vector $p = (p_1, \ldots, p_n)$ that specifies the price $p_j$ of item $o_j$ such that the maximal share that each $i \in N$ can get with budget 1 is $x_i \in \{x'_i \in \mathcal{A}_b : x'_i \in \text{argmax}\{u_i(x'_i) : \sum_{o_j \in O} x'_{ij} \cdot (p_j) \leq 1\}\}$. We will refer to the rule that returns the HZ solution as the HZ rule.

A closely related concept is CEEI. An assignment $x$ satisfies competitive equilibrium with equal incomes (CEEI) if there exists a price vector $p = (p_1, \ldots, p_n)$ that specifies the price $p_j$ of item $o_j$ such that the maximal share that each $i \in N$ can get with budget 1 is $x_i \in \{x'_i \in \mathcal{A}_b : x'_i \in \text{argmax}\{u_i(x'_i) : \sum_{o_j \in O} x'_{ij} \cdot (p_j) \leq 1\}\}$. The CEEI rule returns a CEEI assignment. CEEI [16] coincides with the market equilibrium notion studied in [17]. Note that the HZ solution can be viewed as CEEI with the additional constraint that each agent gets one unit of items.

The MNW (Maximum Nash Welfare) rule returns an assignment $x$ that maximizes the Nash social welfare: $x \in \text{arg max}_{x' \in \mathcal{F}_b} \prod_{i \in N} u_i(x'_i)$.

For two vectors $\vec{u}, \vec{v} \in \mathbb{R}^k$, we say that $\vec{u}$ leximin-dominates $\vec{v}$, written $\vec{u} \succeq_{\text{lex}} \vec{v}$, if there exists an $i \leq k$ such that $\vec{u}_j = \vec{v}_j$, for all $j < i$, and $\vec{u}_i > \vec{v}_i$. Finally, $\pi$ is leximin if there is no $\pi'$ such that $\vec{u}(\pi') \succeq_{\text{lex}} \vec{u}(\pi)$. The leximin rule is the rule that returns a leximin optimal assignment.

CEEI, MNW, and leximin may not return a balanced assignment.

We will say that two rules are equivalent if they result in the same utilities for the agents.

Example 1. Consider the following instance with two agents and items.

|   | a | b |
|---|---|---|
| 1 | 3 | 2 |
| 2 | 1 | 0 |

For this instance, the HZ, MNW constrained to balanced assignments, and the MNW solution are as follows.

$\text{HZ solution} = \frac{1}{2} \left( \begin{array}{cc} a & b \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right)$

$\text{MNW solution constrained to balanced assignments} = \frac{1}{2} \left( \begin{array}{cc} a & b \\ 0 & 1 \\ 1 & 0 \end{array} \right)$

$\text{CEEI} = \text{MNW solution} = \frac{1}{2} \left( \begin{array}{cc} a & b \\ \frac{1}{6} & \frac{5}{6} \\ 1 & 0 \end{array} \right)$
Leximin solution = Balanced Leximin = \[ \begin{pmatrix} \frac{a}{1} & \frac{b}{2} \\ \frac{0}{1} & \frac{1}{0} \end{pmatrix} \]

Note that if all the outcomes are balanced (as is the case under the HZ rule) and an agent’s preferences are dichotomous, then an agent’s preferences over the allocations only depends on the stochastic dominance relation over outcomes.

4. Bi-valued Utilities: Relations of HZ with other Rules and Algorithms

Binary (1-0) utilities are a special class of utilities under which many rules and algorithms coincide.

Fact 1. Under 1-0 utilities, the following rules are equivalent even if the number of items is different than the number of agents:

(i) Leximin rule

(ii) Maximum Nash Welfare (MNW) rule

(iii) Competitive Equilibrium with Equal Incomes (CEEI) \[16\]

(iv) Controlled Cake Eating Algorithm (CCEA) \[4\]

(v) Mechanism 1 of Chen et al. \[9\].

For example, CEEI and MNW are well-known to be equivalent even for general additive utilities. Under binary utilities, all the rules were shown to be equivalent \[4\]. Since the leximin rule gives rise to a unique utility profile (agents’ utilities do not change under different leximin outcomes), it follows that all the rules above give rise to a unique utility profile. The rules above may not return a balanced assignment even for 1-0 utilities. Therefore, they are most suitable when the items are viewed as divisible.

Next we highlight the intimate connection between the HZ rule under bi-valued utilities and the elegant Extended Probabilistic Serial (EPS) algorithm of Katta and Sethuraman \[12\]. EPS is well-defined for any weak orders but we will stick to its presentation for the case of dichotomous preferences. The running time is \(O(n^3 \log n)\). Katta and Sethuraman \[12\] note that EPS for dichotomous preferences is equivalent to the egalitarian rule studied by Bogomolnaia and Moulin \[6\]. The egalitarian rule studied by Bogomolnaia and Moulin \[6\] is the leximin rule applied to the set of balanced (unit-demand) assignments. Bogomolnaia and Moulin \[6\] studied the rule in the context of two-sided matching.

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\[2\] The Controlled Cake Eating Algorithm (CCEA) algorithm \[4\] and Mechanism 1 of Chen et al. \[9\] are described in the context of cake cutting. They also apply to allocation of items: each cake segment can be treated as a separate item.
with men on one side and women on the other side. We adapt the presentation of Katta and Sethuraman \[12\] which is more algorithmic in nature and is directly focussed on the assignment problem with one-sided preferences.

**Extended Probabilistic Serial (EPS) for dichotomous preferences.** For each agent \(i\), let \(D_i\) denote the set of items most preferred by \(i\). Agents gradually guarantee more and more fractional amount of liked items until agents cannot guarantee more. At this point there is a bottleneck set of agents who cannot fractionally get more amount of liked items. Let \(v = \min_{C \subseteq N} \frac{|\cup_{i \in C} D_i|}{|C|}\) with \(X_1\) denoting the largest cardinality set \(X_1 \subseteq N\) for which \(\frac{|\cup_{i \in X_1} D_i|}{|X_1|} = v\). Such a bottleneck set can be computed via network flows as explained by Katta and Sethuraman \[12\]. When we are allowed fractional allocations, then agents in \(X_1\) can each get utility \(v\). These agents \(X_1\) and the items that they like \(O_1\) are removed from the market. The same process is recursively applied to the remaining market until all items are allocated. Once all agents exit from the market, then the remaining items are allocated among those agents who got less than one unit of items to ensure that the allocation is balanced. Note that each successive bottleneck set has a strictly higher utility guarantee \(v\): \(v_1 < v_2 < \cdots < v_k\).

Next, we present a theorem which clarifies the relations between many rules.

**Theorem 1.** For 1-0 utilities, the following rules are equivalent.

(i) HZ rule

(ii) EPS rule

(iii) Egalitarian rule of Bogomolnaia and Moulin \[6\]

(iv) Leximin applied to the set of balanced assignments

(v) MNW applied to the set of balanced assignments

(vi) The rule that applies the balancing operation to a leximin solution

(vii) The rule that applies the balancing operation to a MNW solution

(viii) The rule that applies the balancing operation to a CEEI solution.

**Proof.**

\((i) \implies (ii)\): Bogomolnaia and Moulin \[6\] proved that their egalitarian rule (leximin solution) has a competitive rule interpretation for the two-sided matching problem. From their Theorem 1, it follows that the egalitarian rule of Bogomolnaia and Moulin \[6\] gives the HZ solution when one side can be treated as items who are indifferent over the members of the other side. We provide a direct proof for why the HZ solution under 1-0 utilities is implied by the EPS rule. The direct proof is also useful in proving Theorem 2. Let
Algorithm 1: EPS rule for dichotomous preferences

**Input:** \((N, O, u)\) under 1-0 utilities

**Output:** A balanced assignment

1. \(N' \leftarrow N; O' \leftarrow O\)
2. \(k \leftarrow 1\)
3. while \(N' \neq \emptyset\) do
4.   if each agent \(i \in N'\) can get one unit of items in \(D_i \cap O'\) then
5.     Give each agent \(i \in N'\) one unit of items from \(D_i \cap O'\) [can be done via an algorithm to compute a maximum size matching]
6.   else
7.     Let 
8.     \[v_k = \min_{C \subseteq N'} \frac{|\bigcup_{i \in C} (D_i \cap O')|}{|C|}\]
9.     with \(X_k\) denoting the largest cardinality set \(X_k \subseteq N'\) for which 
10. \[\frac{|\bigcup_{i \in X_k} (D_i \cap O')|}{|X_k|} = v_k\]. Such a bottleneck set can be computed via network flows as explained by Katta and Sethuraman [12]. Let \(\bigcup_{i \in X_k} (D_i \cap O')\) be \(O_k\).
11.     Agents in \(X_k\) can each get utility \(v_k\) by getting items from \(O_k\). Assignment \(x\) for agents in \(X_k\) is finalized. These agents \(X_k\) and the items that they like \(O_k\) are removed from the market: \(N' \leftarrow N' \setminus X_k'; O' \leftarrow O' \setminus O_k\);
12.     end if
13.     \(k \leftarrow k + 1\)
14. end while
15. For agents in \(\{i \in N : \sum_{o_j \in O} x_{ij} < 1\}\), give them any remaining unallocated items in \(O'\) to ensure that the assignment \(x\) is balanced.
16. return \(x\).

The EPS outcome be \(x\). We compute the prices \(p\) of the items such that 
\[x_i \in \{x_i' \in \mathcal{A}_i : x_i' \in \text{argmax}\{u_i(x_i') : \sum_{o_j \in O} x_{ij}' \cdot (p_j) \leq 1\}\}.\]
Consider the run of EPS on dichotomous preferences. When a set of agents \(X_k \subseteq N\) in EPS becomes a bottleneck set and each agent gets utility \(v\), we can set the the individual prices of the goods allocated to agents in \(X_k\) to \(p_j = 1/v_k\) for all \(o_j\) allocated to the agents in \(X_k\). Each agent in \(X_k\) at this point gets an allocation that does not exceed size constraints. Moreover, it gets total utility \(v_k\) for items each of which cost \(1/v_k\). An agent \(i \in X_k\) does not like any items after items \(O_k\) are removed from \(O'\). It may most prefer items that were removed before \(O_k\) were removed. However, those items have even higher prices because the \(v\) value progressively becomes more with the next bottleneck set and hence the prices keep going lower. Therefore for each agent in \(X_k\), the utility \(v_k\) is the maximum utility that can be achieved if \(i \in N\) was to buy liked items at their prices according to \(p\). For any extraneous items that are allocated in Step 12, they can get price 0. We have proved that \((ii)\) implies \((i)\).
Consider an HZ solution. Note that no agent pays anything for a zero utility item because if it did, it can get a bit more of a one utility item. Therefore, if an agent gets a zero utility item, its price is zero. We show that there is a CEEI solution under the same prices. Consider all the agents who get utility less than 1 in an HZ solution. For such agents, the size constraint does not impact in specifying their demand set (best possible feasible allocations within the budget). Now consider the agents who get utility 1 in the HZ solution. Their demand set changes when the size constraints are removed because they can avail additional items. In the HZ solution, all such additional items are given to agents with utility less than one to ensure than every agent has total amount one. Hence, these additional items have zero price. We claim that there is a CEEI outcome under the same prices. Each agent who gets utility less than one in the HZ outcome clearly maximizes her utility even if the size constraints are removed. The only agents who can get more utility under the CEEI outcomes are the ones who got utility one in the HZ solution but can benefit from zero price items. Therefore the market clears under the same prices even if there are no size constraints. From Fact 1, it follows all agents that get utility less than 1 in an HZ solution, will get the same utility in a CEEI/leximin/MNW solution. As for agents who get utility 1 in an HZ solution, they cannot get any more utility in a balanced assignment.

The EPS rule of Katta and Sethuraman [12] returns a balanced random assignment. Under dichotomous preferences, the EPS rule has a direct connection with the egalitarian rule proposed by Bogomolnaia and Moulin [6] for two-sided matching problem with equal number of men and women with dichotomous preferences. If men are treated as items who are completely indifferent among women, then the setting studied by Bogomolnaia and Moulin [6] reduces to the random assignment problem with one side having dichotomous preferences and the EPS rule coincides with the egalitarian rule of Bogomolnaia and Moulin [6]. The egalitarian rule of Bogomolnaia and Moulin [6] is equivalent to the leximin rule on the set of balanced assignments. Bogomolnaia and Moulin [6] also note that their egalitarian rule is equivalent to the MNW rule applied to the set of balanced assignments (page 259). Thus the equivalence between (ii), (iii), (iv), and (v) follows from the papers of Bogomolnaia and Moulin [6] and Katta and Sethuraman [12].

We also know from Fact 1 that under 1-0 utilities, MNW, CEEI, and leximin coincide. Therefore, (vi), (vii), and (viii) are equivalent.

(i) \(\implies\) (viii)

(ii) \(\iff\) (iii) \(\iff\) (iv) \(\iff\) (v)

We also know from Fact 1 that under 1-0 utilities, MNW, CEEI, and leximin coincide. Therefore, (vi), (vii), and (viii) are equivalent.

(i) \(\iff\) (vi)
Consider an assignment \( x \) that is balanced and leximin among balanced assignments. Suppose it is not globally leximin. Then observe how a leximin assignment among balanced assignments is achieved during the run of the EPS algorithm. For all agents in bottleneck sets who get utility less than 1, their utilities are exactly the same as they would get in a globally leximin random assignment. The reason \( x \) is not a globally leximin assignment is that the last set of agents who exit the market get utility 1 but some of them could have got utility strictly more than 1 (without decreasing the utility of other agents) if the balancedness condition is not imposed. These items are not additionally allocated to the agents who already have utility 1 and these items are only distributed in Step 12 of Algorithm 1. It follows that assignment \( x \) can be achieved by first computing a balanced assignment that is leximin and then implementing the balancing operation on it.

This completes the proof.

A corollary of the theorem above is the following one.

**Corollary 1.** For all HZ solutions under 1-0 utilities, each agent gets the same utility in all the solutions and each item gets the same price in all the solutions.

*Proof.* We proved that under 1-0 utilities, the HZ solution is equivalent to applying the balancing operation to a CEEI solution. The utilities of each agent in invariant under all CEEI solutions. Hence, it follows that the utilities of each agent in invariant under all HZ solutions \( [17] \). It also follows that the prices of items are invariant under all HZ solutions.

Next, we present the following theorem for the case of bi-valued utilities.

**Theorem 2.** Under bi-valued utilities, the HZ rule is equivalent to

(i) EPS rule

(ii) Egalitarian rule of Bogomolnaia and Moulin \( [6] \) applied with respect to the binary-reduced utilities

(iii) Leximin applied with respect to the binary-reduced utilities to the set of balanced assignments

(iv) MNW applied with respect to the binary-reduced utilities to the set of balanced assignments

(v) The rule that applies the balancing operation to a leximin solution (with respect to the binary-reduced utilities)

(vi) The rule that applies the balancing operation to a MNW solution (with respect to the binary-reduced utilities)

(vii) The rule that applies the balancing operation to a CEEI solution (with respect to the binary-reduced utilities).
Proof. We know from Theorem 1 that HZ and EPS are equivalent under 1-0 utilities. Note that EPS is an ordinal algorithm so it gives the same outcome under dichotomous preferences. The same argument that is used to prove that EPS gives an HZ solution under 1-0 utilities can be used verbatim to prove that EPS gives an HZ solution under bi-valued utilities. At any point at which a bottleneck set $X_k$ is removed, the agents in the set are able to get their most preferred items at the lowest price. Any most preferred items that are not available were sold at a higher price. As for lesser preferred items, they are given to the agents for free because they are the under-demanded items that are given price zero.

We have shown that HZ and EPS are equivalent under 1-0 utilities; EPS solutions are equivalent under 1-0 utilities and bi-valued utilities; and EPS under bi-valued utilities gives a HZ solution under bi-valued utilities. It follows that HZ under 1-0 utilities implies HZ under bi-valued utilities.$^3$

Next, we prove that any HZ solution under bi-valued utilities is the HZ solution under binary-reduced utilities for the same item prices. Suppose there is an HZ solution under bi-valued utilities that is not an outcome of HZ under binary-reduced utilities. This means that the market does not clear under binary-reduced utilities for the same prices. For the base case, consider the items in $O_1$ in the corresponding EPS outcome. The agents $X_1$ pay nothing for the lesser preferred items. If some item in $O_1$ has a different price, then for the market to clear, at least some item in $O_1$ has lesser price. Consider the item $o \in O_1$ whose price dropped the most. But then all agents in $X_1$ who most prefer $o$ want to get more of $o$ which implies that the market does not clear. The same argument works inductively for the items in $O_2, \ldots, O_k$. Hence, the market clears for binary-reduced utilities for the original prices which contradicts that the HZ solution under bi-valued utilities that is not an outcome of HZ under binary-reduced utilities.

The remaining equivalences follow from Theorem 1 that EPS is equivalent to the rules under 1-0 utilities.

Corollary 2. For all HZ solutions under bi-valued utilities, the solution is rational, and each item gets the same price in all the solutions for problem instances in which the underlying ordinal preferences do not change. For a given problem instance with bi-valued utilities, each agent gets the same utility in all the solutions of the problem instance. Under bi-valued utilities, the set of HZ solutions does not change even if the agents’ utilities are shifted or scaled.

Proof. We have proved that HZ solutions under bi-valued utilities is equivalent to applying the balancing operation to a CEEI solution (with respect to the binary-reduced utilities). It is well-known that the CEEI solution is always

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$^3$The proof is also an alternative argument that HZ under bi-valued utilities is invariant under scaling and shifting of the utility functions as their outcomes are equivalent to HZ under 1-0 utilities.
rational and for the problem instance, each agent gets the same utility in all the solutions of the instance and each item gets the same price in all the solutions of the instance.

The theorem above also has the several algorithmic consequences. Firstly, in order to compute the HZ solution for bi-valued utilities, one only needs to consider the underlying dichotomous preferences and run the EPS algorithm! The characterizations combined with the EPS algorithm provide an alternative route to proving that the HZ solution can be computed in strongly polynomial time.

**Corollary 3.** Under bi-valued utilities, the HZ solution can be computed in time $O(n^3 \log n)$.

*Proof.* Computing the HZ solution reduces to computing the outcome of the EPS algorithm. The running time of EPS is $O(n^3 \log n)$.

Another algorithmic consequence is a reduction from HZ to CEEI under the case of bi-valued utilities.

**Corollary 4.** There is a linear-time reduction from computing the HZ solution under bi-valued utilities to computing the MNW/CEEI/Leximin solution under binary utilities.

*Proof.* The reduction is specified as Algorithm 2.

**Algorithm 2** Reduction from HZ for bi-valued utilities to MNW/CEEI/Leximin

*Input:* $(N, O, u)$ where $u$ is bi-valued

*Output:* A balanced assignment

1. Turn the bi-valued utilities $u$ to binary reduced utilities $u'$.
2. Apply an algorithm for the (unconstrained) MNW/CEEI/Leximin for utilities $u'$ to compute a solution $x$
3. Apply the balancing operation on $x$ to derive the HZ outcome.
4. return $x$.

5. **Efficiency and Strategyproofness**

    We have already pointed out that in EPS, an agent who gets one unit of most-preferred items, cannot get more even though he may be the only agent liking the additional item. Due to the adherence to the size constraints, the HZ solution may not be ex-ante Pareto optimal (PO) among the set of all assignments if we assume that agents have additive utilities. However, it is PO among all balanced assignments in which each agent gets exactly one unit of items. Ex-ante Pareto optimality among balanced assignments follows from the fact that each agent maximizes his utility among all allocations that are balanced.
Remark 1. The HZ solution is PO among the set of balanced assignments. It may not be PO among the set of all assignments if we assume that agents have additive utilities.

We recall that CEEI is equivalent to the MNW rule. However, the HZ rule is not equivalent to maximizing Nash welfare while imposing equal size constraints. This is evident from Example 1 that showed that applying MNW to balanced assignments may give highly unfair assignments.

Next, we look at issues around strategyproofness and group-strategyproofness. A rule is strategyproof if no agent can misreport her preferences to get higher utility. A rule is ex-ante group-strategyproof if no group of agents can misreport their preferences so that all agents get at least as much utility and at least one agent gets strictly more utility.

For dichotomous preferences, EPS is ex-ante group-strategyproof (no group of agents can misreport their preferences so that all agents get at least as much utility and at least one agent gets strictly more utility). This fact has been shown before as well (see, e.g. Theorem 1 of Katta and Sethuraman [12] who refer to the argument by Bogomolnaia and Moulin [6]). Group-strategyproofness for EPS is established by induction on the bottleneck sets created: it can be proved that no agent in a bottleneck set will be a member of a manipulating coalition. Hence, it follows that the HZ solution is also group-strategyproof under bi-valued utilities.

Corollary 5. Under bi-valued utilities, the HZ rule is group-strategyproof.

For 1-0 utilities, all the rules leximin/MNW/CEEI that may return unbalanced assignments are group-strategyproof as well. See for example Theorem 3 of Aziz and Ye [4] that shows that leximin/MNW/CEEI are group-strategyproof. Within the class of bi-valued utilities, it is clear that leximin is not strategyproof or envy-free: an agent with scaled-down utilities get predominant importance under the leximin rule. On the other hand, scaling down of utilities has no effect in the case of MNW (equivalently CEEI). Despite resistance to manipulation to scaling, we show that MNW is not strategyproof under bi-valued utilities even if the underlying ordinal preferences remain unchanged.

Theorem 3. Under bi-valued utilities, MNW/CEEI is not strategyproof even if the underlying ordinal preferences remain unchanged.

Proof. We provide an example with 5 agents and 5 items.

---

4The HZ rule is not strategyproof for general utilities. See, for example, further discussion by Abebe et al. [3].
Under the valuations, the MNW/CEEI outcome $x$ is as follows.

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
2 & 0 & \frac{1}{12} & 1 & \frac{3}{12} & 0 \\
3 & 0 & \frac{11}{12} & 0 & \frac{11}{12} & 0 \\
4 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
5 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\]

Agent 1 gets utility 10.

Suppose agent 1 misreports as follows by raising her value for the lower preferred items.

\[
\begin{bmatrix}
1 & 10 & 8 & 8 & 8 & 8 \\
2 & 6 & 6 & 10 & 6 & 6 \\
3 & 4 & 10 & 4 & 10 & 4 \\
4 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Under the misreport, the MNW/CEEI outcome $y$ is as follows.

\[
\begin{bmatrix}
1 & 3/16 & 0 & 3/16 & 0 \\
2 & 0 & 0 & 1 & 0 & 0 \\
3 & 0 & 13/16 & 0 & 13/16 & 0 \\
4 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
5 & 0 & 0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}
\]

The assignment $y$ gives agent 1 utility more than 10 with respect to her original utilities. Hence, MNW/CEEI is not strategyproof under bi-valued utilities.

The theorem above can be recast in the context of indivisible goods to state that for bi-valued utilities, the MNW rule of Caragiannis et al. \cite{8} is not strategyproof for any tie-breaking over the set of possible outcomes.

**Corollary 6.** Under bi-valued utilities, and for indivisible goods, the MNW rule of Caragiannis et al. \cite{8} is not strategyproof for any tie-breaking over the set of possible outcomes.
The argument follows from the observation that each divisible good can be approximately modelled as small enough multiple indivisible goods.

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