ON UNIQUENESS OF WEAK SOLUTIONS OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. In this article the question on uniqueness of weak solution of the incompressible Navier-Stokes Equations in the 3-dimensional case is studied. Here the investigation is carried out with use of another approach. The uniqueness of velocity for the considered problem is proved for given functions from spaces that possess some smoothness. Moreover, these spaces are dense in respective spaces of functions, under which were proved existence of the weak solutions. In addition here the solvability and uniqueness of the weak solutions of auxiliary problems associated with the main problem is investigated, and also one conditional result on uniqueness is proved.

1. INTRODUCTION

In this article we investigate the question on uniqueness of the weak solutions of the incompressible Navier-Stokes equations, namely is investigated question: when the weak solution of the following problem is unique?

\begin{equation}
\frac{\partial u_i}{\partial t} - \nu \Delta u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i, \quad i = 1, n,
\end{equation}

\begin{equation}
\text{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0, \quad x \in \Omega \subset R^n, t > 0 ,
\end{equation}

\begin{equation}
u(0, x) = u_0(x), \quad x \in \Omega; \quad u \mid_{(0,T) \times \partial \Omega} = 0
\end{equation}

where \( \Omega \subset R^n \) is a bounded domain with sufficiently smooth boundary \( \partial \Omega \), \( T > 0 \) is a positive number. In this work for study of the posed question two distinct way are used, therefore it consist of two parts.

As is well-known Navier-Stokes equations describe the motion of a fluid in \( R^n \) \((n = 2 \text{ or } 3)\). Consequently, in this problem \( u(x, t) = \{u_i(x, t)\}_1^n \in R^n \) is an unknown velocity vector and \( p(x, t) \in R \) is an unknown pressure, at the position \( x \in R^n \) and time \( t \geq 0 \); \( f_i(x, t) \) are the components of a given, externally applied force (e.g. gravity), \( \nu \) is a positive coefficient (the viscosity), \( u_0(x) \in R^n \) is a sufficiently smooth vector function (vector field).

As is well-known in [1] is shown that the Navier-Stokes equations (1.1), (1.2), (1.3) in three dimensions case has a weak solution \((u, p)\) with suitable properties (see, also, [2], [3], [4], [5], [7]). It is known that uniqueness of weak solution of the Navier-Stokes equation in two space dimensions case were proved (8), (7), see

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also [9], but the result of such type for the uniqueness of weak solutions in three space dimensions case as yet isn’t known. It should be noted that in three dimensional case the uniqueness was studied also, but under complementary conditions on the smoothness of the solution (see, e.g. [7], [28], [10], etc.). It is known for the Euler equations were shown that uniqueness of weak solutions isn’t (see, [10], [11]).

We need to note the regularity of solutions in three dimensional case were investigated and partial regularity of the suitable weak solutions of the Navier–Stokes equation were obtained (see, e.g. [12], [14], [15], [7], [2], [28], etc.). Here we would like to note the result of article [13] that possesses

where \( B \), \( \gamma \), \( \nu \) are constants. In [13] the problem in the following form was studied

where \( B(u) \equiv \sum_{j=1}^{3} u_j \frac{\partial u}{\partial x_j} \) and \( \gamma_0 u \equiv u(0) \). In which the author shows that \( (N, \gamma_0) : Z \rightarrow L^2(0, T : H^{-1/2}(\Omega)) \times H^{1/2}(\Omega) \) is the continuous operator under the condition that \( \Omega \subset R^d \) is a bounded region whose boundary \( \partial \Omega \) is a closed manifold of class \( C^\infty \), where

Moreover, here is proved that if to denote by \( F_{\gamma_0} \) the image: \( N(Z_{u_0}) = F_{\gamma_0} \), for \( u_0 \in H^{1/2}(\Omega) \) then for each \( f \in F_{\gamma_0} \) there exists only one solution \( u \in Z \) such that \( Nu = f \) and \( \gamma_0 u = u_0 \), here \( Z_{u_0} = \{ u \in Z \mid \gamma_0 u = u_0 \} \), and also the density of set \( F_{\gamma_0} \) in \( L^2(0, T : H^{-1/2}(\Omega)) \) in the topology of \( L^p(0, T : H^{-1}(\Omega)) \) under certain conditions on \( p, l \). The proof given in [13] is similar to the proof of [7] and [28], but the result not follows from their results.

In the beginning in this paper certain explanation why for study of the posed question is enough to investigate the problem (1.1) - (1.3) is provided. Here the approach Hopf-Leray (with taking into account of the result of de Rham) for study the existence of the weak solution of the considered problem is used, as usually.

Unlike above investigations here we study the question on the uniqueness in the case when the weak solution \( u \) of the problem (1.1) - (1.3) is contained in \( \mathcal{V}(Q^2) \) (in 3D case), consequently, as is known, for this the following condition is sufficiently: functions \( u_0 \) and \( f \) satisfy conditions

\[
u_0 \in H(\Omega), \quad f \in L^2(0, T; V^*(\Omega)).\]

**Notation 1.** The result obtained for the problem (1.1) - (1.3) allows us to respond to the posed question, namely this shows the uniqueness of the velocity vector \( u \).
So, in this article the uniqueness of the weak solutions $u$ obtaining by the Hopf-Leray’s approach of the mixed problem with Dirichlet boundary condition for the incompressible Navier-Stokes system in the $3D$ case is investigated. For investigation we use an approach that is different from usual methods used for study of the questions of such type. The approach used here allows us to receive more general result on the uniqueness of the weak solution (of the velocity vector $u$) of the mixed problem for the incompressible Navier–Stokes equation under more general conditions. In addition, here in order to carry out of the proof of the main result, in the beginning the existence and uniqueness of the weak solutions of auxiliary problems are studied.

For study of the uniqueness of the solution of the problem we also use of the formulation of the problem in the weak sense according to J. Leray [1]. As well-known, problem (1.1) - (1.3) and (1.1') - (1.3) was investigated in many works (see, [7], [28] and [6]). Here we will bring the result on weak solvability from the book [28].

**Theorem 1.** ([28]) Let $\Omega$ be a Lipschitz open bounded set in $\mathbb{R}^n$, $n \leq 4$. Let there be given $f$ and $u_0$ which satisfy $f \in L^2 \left( 0, T; V^* (\Omega) \right)$ and $u_0 \in H (\Omega)$. Then there exists at least one function $u$ which satisfies $u \in L^2 \left( 0, T; V (\Omega) \right)$, $\frac{du}{dt} \in L^1 \left( 0, T; V^* (\Omega) \right)$, $u \left( 0 \right) = u_0$ and the equation

$$
\frac{d}{dt} \langle u, v \rangle - \langle \nu \Delta u, v \rangle + \left( \sum_{j=1}^{n} u_j \frac{\partial u}{\partial x_j}, v \right) = \langle f, v \rangle
$$

for any $v \in V (\Omega)$. Moreover, $u \in L^\infty \left( 0, T; H (\Omega) \right)$ and $u (t)$ is weakly continuous from $[0, T]$ into $H (\Omega)$ (i. e. $\forall v \in H (\Omega)$, $t \rightarrow \langle u (t), v \rangle$ is a continuous scalar function, and consequently, $\langle u (0), v \rangle = \langle u_0, v \rangle$).

"Moreover, in the case when $n = 3$ a weak solution $u$ satisfy

$$
u L^4 \left( 0, T; V^* (\Omega) \right),
$$

and also is almost everywhere (a.e.) equal to some continuous function from $[0, T]$ into $H$, so that (1.3) is meaningful, with use of the obtained properties that any weak solution belong to the bounded subset of

$$
\mathcal{V} \left( Q^T \right) \equiv V \left( Q^T \right) \cap W^{1,4/3} \left( 0, T; V^* (\Omega) \right)
$$


and satisfies the equation (1.4)."

In what follows we will base on the mentioned theorem about the existence of the weak solution of problem (1.1') - (1.3) and the added remarks as principal results, since here is investigated the question related to the weak solution of the problem that is studied in Theorem 1.

The main result of this paper is the following uniqueness theorem.

**Theorem 2.** Let $\Omega \subset \mathbb{R}^3$ be a domain of $\text{Lip}_{\text{loc}}$ (will be defined below; see, Section 4), $T > 0$ be a number. If given functions $u_0$, $f$ satisfy of conditions $u_0 \in H^{1/2} \left( \Omega \right)$, $f \in L^2 \left( 0, T; V^* (\Omega) \right)$, $u \in L^2 \left( 0, T; V (\Omega) \right)$, $\nu \Delta u \in L^1 \left( 0, T; V^* (\Omega) \right)$, and further denote $\langle g, h \rangle = \sum_{i=1}^{3} \int_{\Omega} g_i h_i \, dx$ for any $g, h \in \left( H (\Omega) \right)$, or $g \in V (\Omega)$ and $h \in V^* (\Omega)$, respectively.
$f \in L^2(0, T; H^{1/2}(\Omega))$ then weak solution $u \in \mathcal{V}(Q^T)$ of the problem (1.1) - (1.3) given by the above mentioned theorem is unique.

This article is organized as follow. In Part I the question is studied under certain smoothness conditions onto given functions. In Section I.2 some known results and the explanation of the relation between problems (1.1) - (1.3) and (1.1') - (1.3') is adduced, and also the necessary auxiliary results, namely lemmas are proved. These lemmas are need us for the study of the main problem. In Section I.3 the auxiliary problems determined that posed on the cross-sections of $\Omega$, which are obtained from problem (1.1') - (1.3'). Here is explained how these problems are obtained from problem (1.1') - (1.3'), and also is suggested to study the main question for the auxiliary problems on the cross-sections instead of the investigation of this question on whole of $\Omega$. In Section I.4 the existence of the solution and, in Section I.5 the uniqueness of solution of the auxiliary problem are studied. In Section I.6 the main result Theorem 2 is proved. In Section II.7 of Part II one conditional result on uniqueness of weak solution of problem (1.1') - (1.3) by use of certain modification of the well-known approach is proved.

**Part 1. One new approach for study of the uniqueness**

2. **Preliminary results**

In this section the background material, definitions of the appropriate spaces, that will be used in the next sections are briefly recalled. In addition, here some notations are introduced, and also the necessary auxiliary results are proved that in the follow will be employed. Moreover, we recall the basic setup and results regarding of the weak solutions of the incompressible Navier–Stokes equations used throughout this paper.

As is well-known, problem (1.1) - (1.3) possesses weak solution in the space $\mathcal{V}(Q^T) \times L^2(Q^T)$ for each $u_{0i}(x), f_i(x,t) \ (i = 1, 3)$, which are contained in the suitable spaces (see, e.g. [7], [28] and references therein), (the space $\mathcal{V}(Q^T)$ will be defined later on). Here our main problem is the investigation of the posed question in the case $n = 3$, consequently, here problems will be studied mostly in the case $n = 3$.

**Definition 1.** Let $\Omega \subset \mathbb{R}^3$ be an open bounded Lipschitz domain and $Q^T \equiv (0, T) \times \Omega, T > 0$ be a number. Let $\mathcal{V}(Q^T)$ be the space determined as

$$
\mathcal{V}(Q^T) \equiv L^2(0, T; \mathcal{V}(\Omega)) \cap L^\infty(0, T; \mathcal{H}(\Omega)),
$$

where $\mathcal{V}(\Omega)$ and $\mathcal{H}(\Omega)$ are the closure of

$$
\left\{ \varphi \mid \varphi \in (C^\infty_0(\Omega))^3, \, \text{div} \, \varphi = 0 \right\}
$$

in the topology of $\left(W^{1,2}_{0}(\Omega)\right)^3$ and in the topology of $\left(L^2(\Omega)\right)^3$, respectively;

the dual $\mathcal{V}(\Omega)$ determined as $\mathcal{V}^\ast(\Omega)$ and is the closure of the linear continuous functionals defined on $\mathcal{V}(\Omega)$ in the sense of the Lax dual relative to $\mathcal{H}(\Omega)$.

Moreover we set also the space $\mathcal{V}(Q^T) \equiv \mathcal{V}(Q^T) \cap W^{1,4/3}(0, T; \mathcal{V}^\ast(\Omega))$ for $n = 3$ ([28]).
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Here as is well-known $L^2(\Omega)$ is the Lebesgue space and $W^{1,2}(\Omega)$ is the Sobolev space, that are the Hilbert spaces and

$$W^{1,2}_0(\Omega) \equiv \{ v \mid v \in W^{1,2}(\Omega), \; v |_{\partial \Omega} = 0 \}.$$  

As is well-known in this case $H(\Omega)$ and $V(\Omega)$ also are the Hilbert spaces, therefore

$$V(\Omega) \subset H(\Omega) \equiv H^*(\Omega) \subset V^*(\Omega).$$

So, assume the given functions $u_0$ and $f$ satisfy

$$u_0 \in H(\Omega), \; f \in L^2(0,T; V^*(\Omega))$$

where $V^*(\Omega)$ is the dual space of $V(\Omega)$.

Consider the problem for which the existence of the weak solution directly connected with the existence of the weak solution of problem (1.1) - (1.3) as will be shown below

\[(1.1^1) \quad \frac{\partial u_i}{\partial t} - \nu \Delta u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = f_i(t,x), \quad i = 1, n, \; \nu > 0 \]

\[(1.2) \quad \text{div} \; u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{n} D_i u_i = 0, \quad x \in \Omega \subset R^n, \; t > 0, \]

\[(1.3) \quad u(0,x) = u_0(x), \quad x \in \Omega; \quad u \mid_{(0,T) \times \partial \Omega} = 0.\]

\[\text{Proposition 1. (28)} \] Let $\Omega$ be a bounded Lipschitz open set in $R^n$ and $g = (g_1, \ldots, g_n)$, $g_i \in \mathcal{D}'(\Omega), \; 1 \leq i \leq n$. A necessary and sufficient condition that $g = \text{grad} \; p$ for some $p$ in $\mathcal{D}'(\Omega)$, is that $\langle g, v \rangle = 0 \; \forall v \in V(\Omega)$.

\[\text{Note: All equations are needed to understand in the sense of the corresponding spaces, e.g. the equation (1.1) is understood in the sense of the dual space of V(Q^T).}\]
Proposition 2. \((\ref{28})\) Let \(\Omega\) be a bounded Lipschitz open set in \(\mathbb{R}^n\).

(i) If a distribution \(p\) has all its first-order derivatives \(D_i p, 1 \leq i \leq n\) in \(L^2(\Omega)\), then \(p \in L^2(\Omega)\) and

\[
\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c(\Omega) \|\text{grad} p\|_{(L^2(\Omega))^n};
\]

(ii) If a distribution \(p\) has all its first derivatives \(D_i p, 1 \leq i \leq d\) in \(H^{-1}(\Omega)\), then \(p \in L^2(\Omega)\) and

\[
\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c(\Omega) \|\text{grad} p\|_{H^{-1}(\Omega)}.
\]

In both cases, if \(\Omega\) is any open set in \(\mathbb{R}^n\), then \(p \in L^2_{\text{loc}}(\Omega)\).

Combining these results, one can note that if \(g \in H^{-1}(\Omega)\) (or \(g \in L^2(\Omega)\)) and \((g, v) = 0\), then \(g = \text{grad} p\) with \(p \in L^2(\Omega)\) (or \(p \in H^1(\Omega)\)) if \(\Omega\) is a Lipschitz open bounded set.

Theorem 3. \((\ref{28})\) Let \(\Omega\) be a Lipschitz open bounded set in \(\mathbb{R}^n\). Then

\[
H^1(\Omega) = \left\{ w \in (L^2(\Omega))^n \mid w = \text{grad} p, \; p \in H^1(\Omega) \right\};
\]

\[
H^j(\Omega) = \left\{ u \in (L^2(\Omega))^n \mid \|u\|_{L^2(\Omega)} \leq c(\Omega), \; u_{\partial\Omega} = 0 \right\}.
\]

Lemma 1. \((\text{see, e.g.} \; \ref{7}, \; \ref{28} \; \text{and also} \; \ref{27}, \; \ref{33})\) Let \(B_0, B, B_1\) be three Banach spaces, each space continuously included in the following one \(B_0 \subset B \subset B_1\) and \(B_0, B_1\) are reflexive, moreover, the inclusion \(B_0 \subset B\) is compacts.

Let \(X\) be

\[
X \equiv \{ u \mid u \in L^{p_0}(0, T; B_0), \; u \in L^{p_1}(0, T; B_1) \},
\]

where \(1 < p_j < \infty, \; j = 0, 1\) and \(0 < T < \infty\). Hence \(X\) is Banach space with the norm

\[
\|u\|_X = \|u\|_{L^{p_0}(0, T; B_0)} + \|u\|_{L^{p_1}(0, T; B_1)}.
\]

Then under these conditions the inclusion \(X \subset L^{p_0}(0, T; B)\) is compact.

Moreover, the inclusion \(X \subset C(0, T; B_1)\) holds, due of Lebesgue theorem.

Consequently, if one will seek of weak solution of the problem \((\ref{11}) - (\ref{13})\) by according Hopf-Leray then one can get the following equation

\[
\frac{d}{dt} \langle u, v \rangle - \langle \nu \Delta u, v \rangle + \sum_{j=1}^{n} \langle u_j \frac{\partial u}{\partial x_j}, v \rangle = \langle f, v \rangle - \langle \nabla p, v \rangle,
\]

where \(v \in V(\Omega)\) is arbitrary.

So, if to consider of the last adding in the right side then at illumination of above results (Propositions \((\ref{11}) - (\ref{22})\) and Theorem \((\ref{23})\) using the integration by parts and taking into account that \(v \in V(\Omega)\), i.e. \(\text{div} \; v = 0\) and \(v \big|_ {\partial(0, T) \times \partial\Omega} = 0\) we will get the equation

\[
\langle \nabla p, v \rangle \equiv \int_{\Omega} \nabla p \cdot v \; dx = \int_{\Omega} p \text{div} \; v \; dx = 0, \; \forall v \in V(\Omega)
\]

by virtue of de Rham result. Consequently, taking into account \((\ref{23})\) in \((\ref{22})\) we obtain equation \((\ref{21})\) that shows why for study of the posed question is enough to study this question for problem \((\ref{11}) - (\ref{13})\).

So, we can continue the investigation of the posed question for problem \((\ref{11}) - (\ref{13})\) in the case \(n = 3\).
Let $\Omega \subset R^3$ be a bounded open domain with the boundary $\partial \Omega$ of the Lipschitz class. We will denote by $H^{1/2} (\Omega)$ the vector space defined as in Definition 1 by
\[
\left( W^{1/2} (\Omega) \right)^3 \equiv \left\{ w \mid w_i \in W^{1/2} (\Omega), \ i = 1, 2, 3 \right\}, \ w = (w_1, w_2, w_3)
\]
where $W^{1/2,2} (\Omega)$ is the Sobolev-Slobodeskij space (see, [29], etc.). As well-known the trace for the function of the space $H^{1/2} (\Omega)$ is definite for each smooth surface from $\Omega$ (see, e.g. [29], [30] and references therein), which is necessary for application of our approach to the considered problem. The main theorem will be proved under this additional condition that is the sufficient condition for present investigation.

**Definition 2.** A $u \in \mathcal{V} (Q^T)$ is called a solution of problem (1.1) - (1.3) if $u(t)$ almost everywhere in $(0,T)$ satisfies the following equation
\[
\frac{d}{dt} \langle u, v \rangle - \langle \nu \Delta u, v \rangle + \sum_{j=1}^{n} u_j D_j u, v \rangle = \langle f, v \rangle
\]
for any $v \in V (\Omega)$ and $u(t)$ is weakly continuous from $[0,T]$ into $H (\Omega)$ (i.e. $\forall v \in H (\Omega), t \rightarrow \langle u(t), v \rangle$ is a continuous scalar function, and consequently, $\langle u(0), v \rangle = \langle u_0, v \rangle$).

In what follows we will understand of an existing solutions be functions that satisfy this definition together with the standard notations that are used usually. Moreover, as above were noted if $\Omega$ be a Lipschitz open bounded set in $R^3$, functions $f$ and $u_0$ satisfy $f \in L^2 (0,T;V^* (\Omega))$ and $u_0 \in H (\Omega)$, respectively, then the vector function $u$ is the solution of problem (1.1) - (1.3) if it satisfies of conditions of Definition 2 in addition, $u \in L^\infty (0,T;H (\Omega))$ and the term $\sum_{j=1}^{3} u_j D_j u \equiv B (u)$ belong to $L^{4/3} (0,T;V^* (\Omega))$.

Now we go over into main question: let problem (1.1) - (1.3) have two different solutions $u, v \in \mathcal{V} (Q^T)$ then within the known approach we derive that the function $w(t,x) = u(t,x) - v(t,x)$ must satisfies the following problem
\[
\frac{1}{\nu} \frac{\partial}{\partial t} \| w \|^2 + \nu \| \nabla w \|^2 + \sum_{j,k=1}^{3} \left( \frac{\partial v_j}{\partial x_j} w_k, w_j \right) = 0,
\]
\[
w(0,x) \equiv w_0 (x) = 0, \quad x \in \Omega; \quad w \big|_{(0,T) \times \partial \Omega} = 0,
\]
where $\langle g, h \rangle = \sum_{i=1}^{3} \int f g_i h_i dx$ for any $g, h \in (H (\Omega))^3$, or $g \in V (\Omega)$ and $h \in V^* (\Omega)$, respectively. So, for the proof of the uniqueness of solution it is follows to show that $w \equiv 0$ in the sense of needed space.

Later in this section one will studied questions and provided certain results that are necessary for employing of the basic approach to study of the requered question. More exactly, these reasonings and results will be used in sections 3-6 for study of the posed question.
Therefore we will conduct our study under the condition that problem \((1.1^1) - (1.3)\) have weak solutions and they belong to \(\mathcal{V} (Q^T)\). For the study of the uniqueness of solution of problem \((1.1^1) - (1.3)\) as above assume that problem \((1.1^1) - (1.3)\) has, at least, two different solutions \(u, v \in \mathcal{V} (Q^T)\). But for demonstrate that this isn’t possible we will employ a different procedure.

Consequently, if we assume that problem \((1.1^1) - (1.3)\) have two different solutions then they need to be different at least on some subdomain \(Q^T \subseteq (t_1, t_2) \times \Omega \subseteq Q^T\) with \(\text{mes}_4 (Q^T) > 0\) for which

\[
\text{mes}_4 \left( \{ (t, x) \in Q^T \mid |u(t, x) - v(t, x)| > 0 \} \right) = \text{mes}_4 (Q^T) > 0
\]

holds, where \(\text{mes}_4 (Q^T)\) denote the measure of \(Q^T\) in \(R^4\) (i.e. \(\text{mes}_k\) denote the Lebesgue measure on \(k\) dimensional space \(R^n\)). Whence follows, that subdomain \(\Omega_1\) must have of the positive Lebesgue measure, i.e. \(\text{mes}_3 (\Omega_1) > 0\).

The following lemmas will proved even though for \(n > 1\), but mostly these will use for the case \(n = 4\).

So, it is need to prove the following lemmas, which will use later on.

**Lemma 2.** Let \(G \subset R^n\) be Lebesgue measurable subset then the following statements are equivalent:

1) \(\infty > \text{mes}_n (G) > 0\);

2) there exist a subsets \(I \subset R^3\), \(\infty > \text{mes}_1 (I) > 0\) and \(G_\beta \subset L_{\beta, n-1}\), \(\infty > \text{mes}_{n-1} (G_\beta) > 0\) such that \(G = \bigcup_{\beta \in I} G_\beta \cup N\), where \(N\) is a set with \(\text{mes}_{n-1} (N) = 0\), and \(L_{\beta, n-1}\) is the hyperplane of \(R^n\), with \(\text{codim} n L_{\beta, n-1} = 1\), for any \(\beta \in I\), which is generated by the arbitrary fixed vector \(y_0 \in R^n\) and defined as follow

\[
L_{\beta, n-1} \equiv \{ y \in R^n \mid \langle y_0, y \rangle = \beta \}, \quad \forall \beta \in I.
\]

**Proof.** Let \(\text{mes}_n (G) > 0\) and consider the class of hyperplanes \(L_{\gamma, n-1}\) for which \(G \cap L_{\gamma, n-1} \neq \emptyset\) and \(\gamma \in I_1\), where \(I_1 \subset R^1\) be some subset. It is clear that

\[
G = \bigcup_{\gamma \in I_1} \{ x \in G \cap L_{\gamma, n-1} \mid \gamma \in I_1 \}.
\]

Then there exists a subclass of hyperplanes \(\{ L_{\gamma, n-1} \mid \gamma \in I_1 \}\) for which \(\text{mes}_{n-1} (G \cap L_{\gamma, n-1} > 0\) is fulfilled. The number of such type hyperplanes cannot be less than countable or equal it because \(\text{mes}_n (G) > 0\), moreover this subclass of \(I_1\) must possess the \(R^1\) measure greater than 0 since \(\text{mes}_n (G) > 0\).

Indeed, let \(I_{1,0}\) be this subclass and \(\text{mes}_1 (I_{1,0}) = 0\). In this case we get subset

\[
\{ (\gamma, y) \in I_{1,0} \times G \cap L_{\gamma, n-1} \mid \gamma \in I_{1,0}, y \in G \cap L_{\gamma, n-1} \} \subset R^n
\]

where \(\text{mes}_{n-1} (G \cap L_{\gamma, n-1} > 0\) for all \(\gamma \in I_{1,0}\), but \(\text{mes}_1 (I_{1,0}) = 0\), then

\[
\text{mes}_n \left( \{ (\gamma, y) \in I_{1,0} \times G \cap L_{\gamma, n-1} \mid \gamma \in I_{1,0} \} \right) = 0.
\]

On the other hand if \(\text{mes}_{n-1} (G \cap L_{\gamma, n-1} = 0\) for all \(\gamma \in I_1 - I_{1,0}\) then

\[
\text{mes}_n \left( \{ (\gamma, y) \in I_1 \times G \cap L_{\gamma, n-1} \mid \gamma \in I_1 \} \right) = 0,
\]

whence follows

\[
\text{mes}_n (G) = \text{mes}_n \left( \{ (\gamma, y) \in I_1 \times G \cap L_{\gamma, n-1} \mid \gamma \in I_1 \} \right) = 0.
\]
But this contradicts the condition $\text{mes}_n(G) > 0$. Consequently, the statement 2 holds.

Let the statement 2 holds. It is clear that the class of hyperplanes $L_{\beta,n-1}$ defined by such way are parallel and also we can define the class of subsets of $G$ as its cross-section with hyperplanes, i.e. in the form: $G_{\beta} \equiv G \cap L_{\beta,n-1}$, $\beta \in I$. Then $G_{\beta} \neq \emptyset$ and we can write $G_{\beta} \equiv G \cap L_{\beta,n-1}$, $\beta \in I$, moreover $G \equiv \bigcup_{\beta \in I} \{x \in G \cap L_{\beta,n-1} \mid \beta \in I\} \cup N$. Whence we get

$$G \equiv \{(\beta,x) \in I \times G \cap L_{\beta,n-1} \mid \beta \in I, x \in G \cap L_{\beta,n-1}\} \cup N.$$ 

Consequently, $\text{mes}_n(G) > 0$ by virtue of conditions: $\text{mes}_1(I) > 0$ and $\text{mes}_{n-1}(G_{\beta}) > 0$ for any $\beta \in I$. $\square$

Lemma 2 shows that for the study of the measure of some subset $\Omega \subseteq R^n$ it is enough to study its stratifications by a class of corresponding hyperplanes.

**Lemma 3.** Let problem (1.1)$^1$ - (1.3) has, at least, two different solutions $u, v$ that are contained in $V(Q^T)$ and assume that $Q^T_1 \subseteq Q^T$ is one of a subdomain of $Q^T$ where $u$ and $v$ are different. Then there exists, at least, one class of parallel hyperplanes $L_{\alpha}$, $\alpha \in I \subseteq (\alpha_1, \alpha_2) \subset R^1$ $(\alpha_2 > \alpha_1)$ with $\text{co dim}_{R^3} L_{\alpha} = 1$ such that $u \neq v$ on $Q^T_{L_{\alpha}} \equiv [(0,T) \times (\Omega \cap L_{\alpha})] \cap Q^T_1$, and vice versa, here $\text{mes}_1(I) > 0$, $\text{mes}_2(\Omega \cap L_{\alpha}) > 0$ and $L_{\alpha}$ are hyperplanes which are defined as follows: there is vector $x_0 \in S^1(0)$ such that

$$L_{\alpha} \equiv \{x \in R^3 \mid \langle x_0, x \rangle = \alpha, \forall \alpha \in I\}.$$ 

**Proof.** Let problem (1.1)$^1$ - (1.3) have two different solutions $u, v \in V(Q^T)$ then there exist a subdomain of $Q^T$ on which these solutions are different. Then there are $t_1, t_2 > 0$ such that

$$(2.8) \quad \text{mes}_3 \{x \in \Omega \mid |u(t,x) - v(t,x)| > 0\} > 0$$

holds for any $t \in I \subseteq [t_1, t_2] \subseteq [0,T)$, where $\text{mes}_1(J) > 0$ by the virtue of the condition

$$(2.9) \quad \text{mes}_4 \{(t,x) \in Q^T \mid |u(t,x) - v(t,x)| > 0\} > 0$$

and of Lemma 2

Whence follows, that there exist, at least, one class of the parallel hyperplanes $L_{\alpha}$, $\alpha \in I \subseteq (\alpha_1, \alpha_2) \subset R^1$ such that $\text{co dim}_{R^3} L_{\alpha} = 1$ and

$$(2.9) \quad \text{mes}_2 \{x \in \Omega \cap L_{\alpha} \mid |u(t,x) - v(t,x)| > 0\} > 0, \forall \alpha \in I$$

hold for $\forall t \in J$, where subsets $I$ and $J$ are satisfy inequations: $\text{mes}_1(I) > 0$, $\text{mes}_1(J) > 0$, and also (2.9) holds, by virtue of (2.8). This proves the "if" part of Lemma.

Now consider the converse assertion. Let there exist a class of hyperplanes $L_{\alpha}$, $\alpha \in I_1 \subseteq (\alpha_1, \alpha_2) \subset R^1$ with $\text{co dim}_{R^3} L_{\alpha} = 1$ that fulfills the condition of Lemma and the subset $I_1$ satisfies of same condition as $I$. Then there exist, at least, one subset $J_1$ of $[0,T)$ such that $\text{mes}_1(J_1) > 0$ and the inequation $u(t,x) \neq v(t,x)$ holds onto $Q^T_{L_{\alpha}}$ with $\text{mes}_4(Q^T_{L_{\alpha}}) > 0$, which is defined as $Q^T_{L_{\alpha}} \equiv J_1 \times U_{L_{\alpha}}$, where

$$(2.10) \quad U_{L_{\alpha}} \equiv \bigcup_{\alpha \in I_1} \{x \in \Omega \cap L_{\alpha} \mid u(t,x) \neq v(t,x)\} \subset \Omega, \quad t \in J_1$$

for which the inequation $\text{mes}_{R^3}(U_{L_{\alpha}}) > 0$ is fulfilled by virtue of the condition and of Lemma 2.
So we get

\[ u(t,x) \neq v(t,x) \text{ onto } Q_T^\ast \equiv J_1 \times U_L, \text{ with } \text{mes}_4(Q_T^\ast) > 0. \]

Thus, we obtain the fact that \( u(t,x) \) and \( v(t,x) \) are different functions in \( \mathcal{V}(Q_T^\ast) \).

\[ \square \]

It is not difficult to see that result of Lemma 3 is independent of assumption: \( Q_T^\ast \subseteq Q_T \) or \( Q_T = Q_T^\ast \).

Likely one could be to prove more general results of such type using of the regularity properties of weak solutions of this problem (see, [10], [14], [15], [39], etc.).

3. Investigation of the auxiliary problem

In this section we will transform problem (1.1) - (1.3) to the auxiliary problems in order to use of the result of Lemma 3. In other words, here our concept of the investigation of the posed question will presented. This concept is based to result of Lemma 3 which shows, that for study of posed problem it is enough to investigate this problem on the cross-sections of the domain \( Q_T = (0,T) \times \Omega \).

So, we will begin with the definition of the domain \( \Omega \subset \mathbb{R}^3 \) on which will be study of the problem.

**Definition 3.** A bounded open domain \( \Omega \subset \mathbb{R}^3 \) with the boundary \( \partial \Omega \) is spoken from the class Liploc iff \( \partial \Omega \) is a locally Lipschitz hypersurface. (This means: any point \( x \in \partial \Omega \) possesses a neighbourhood in \( \partial \Omega \) that admits a representation as a hypersurface \( y_3 = \psi(y_1,y_2) \), where \( \psi \) is a Lipschitz function, and \( (y_1,y_2,y_3) \) are rectangular coordinates in \( \mathbb{R}^3 \). In a coordinate basis that may be different from the canonical basis \( (e_1,e_1,e_3) \).)

According to \( \Omega \) is a locally Lipschitz and bounded one can draw the conclusion: each point \( x_j \in \partial \Omega \), has an open neighbourhood \( U_j \) such that \( U_j' = \Omega \cap U_j \), moreover, \( \partial \Omega \) can be covered by a finite family of such sets \( U_j' \), \( j \in J \), that boundary \( U_j' \), \( j \in J \) is Lipschitz, or \( \partial \Omega \subset \text{Liploc} \). Consequently for every "cross-section" \( \Omega_L \equiv \Omega \cap L \neq \emptyset \) of \( \Omega \) with arbitrary hyperplain \( L \) exists, at least, one coordinate subspace \( \{(x_j,x_k)\} \) which possesses a domain \( P_{x_j} \Omega_L \) (or union of domains) whith the Lipschitz class boundary since \( \partial \Omega_L \equiv \partial \Omega \cap L \neq \emptyset \) and isomorphically defined of \( \Omega_L \) with the affine representation, in addition \( \partial \Omega_L \leftrightarrow \partial P_{x_j} \Omega_L \).

Thus, with use of the representation \( P_{x_j} \) of the hyperplane \( L \) we get that \( \Omega_L \) can be written in the form \( P_{x_j} \Omega_L \), therefore an integral on \( \Omega_L \) also will defined by the respective representation, i. e. as the integral on \( P_{x_j} \Omega_L \).

It should be noted that \( \Omega_L \) can consist of many parts then \( P_{x_j} \Omega_L \) will be such as \( \Omega_L \). Consequently in this case \( \Omega_L \) will be as the union of domains and the following relation will be holds

\[ \Omega_L = \bigcup_{r=1}^m \Omega_L^r \quad \leftrightarrow \quad P_{x_j} \Omega_L = \bigcup_{r=1}^m P_{x_j} \Omega_L^r, \quad \infty > m \geq 1, \]

by virtue of the definition 3. Therefore, each of \( P_{x_j} \Omega_L^r \) will be the domain and one can investigate these separately, as the following inclusions take place: \( \Omega_L^r \subset \Omega \) and \( \partial \Omega_L^r \subset \partial \Omega \).

So, we will define subdomains of \( Q_T^\ast = (0,T) \times \Omega \) as follows \( Q_T^\ast = (0,T) \times (\Omega \cap L) \), where \( L \) is arbitrary fixed hyperplane of the dimension two and \( \Omega \cap L \neq \emptyset \). Therefore, we will study the problem onto the subdomain defined by use of the
"cross-section" of $\Omega$ with arbitrary fixed hyperplane of the dimension two $L$, i.e. the $\text{codim}_{R^3} L = 1$ ($\Omega \cap L$, namely on $Q^T_L \equiv (0, T) \times (\Omega \cap L)$).

Consequently, we will investigate uniqueness of the problem (1.1) - (1.3) on the "cross-section" $Q^T_L$ defined by the "cross-section" of $\Omega$, where $\Omega \subset R^3$. This "cross-section" is understood in the following sense: Let $L$ be a hyperplane in $R^3$ with $\text{codim}_{R^3} L = 1$, clearly that $L$ is certain shift of $R^2$ or $R^2$. Denote by $\Omega_L$ of the "cross-section" $\Omega_L \equiv \Omega \cap L \neq \varnothing$, $\text{mes}_{R^2} (\Omega_L) > 0$, e.g. $L$ can be $L \equiv \{(x_1, x_2, 0) \mid x_1, x_2 \in R^3\}$. In other words, if $L$ is the hyperplane in $R^3$ then we can determine it as

$$L \equiv \{x \in R^3 \mid \langle a, x \rangle = a_1x_1 + a_2x_2 + a_3x_3 = b\},$$

where $a \in S^R(0)$ is arbitrary fixed unit vector of $R^3$ and $b \in R$ is arbitrary fixed constant, furthermore each $a \in S^R(0)$ and $b \in R$ define of single $L_b(a)$ and vice versa. Whence follows $a_3x_3 = b - a_1x_1 - a_2x_2$, if we assume $a_3 \neq 0$ then $x_3 = \frac{b}{a_3} - \frac{a_1}{a_3}x_1 - \frac{a_2}{a_3}x_2$, moreover, if we takes of substitutions: $\frac{a}{a_3} \Rightarrow b, \frac{a_1}{a_3} \Rightarrow a_1$ and $\frac{a_2}{a_3} \Rightarrow a_2$ then we derive $x_3 \equiv \psi_3(x_1, x_2) = b - a_1x_1 - a_2x_2$ in the new coefficients.

Since we will investigate the problem (1.1) - (1.3) on $Q^T_L$, in the beginning we need define the problem that we will derive after using this projection to the problem (1.1) - (1.3). In other words, if we denote by $F : D(F) \subset V(Q^T_L) \rightarrow L^2(0, T; V^*(\Omega)) \times L^2(\Omega)$ the operator generated by problem (1.1) - (1.3), then we must determine of the derived problem after projection of the operator $F$ on $Q^T_L$. Clearly under this projection some of the expressions in the problem (1.1) - (1.3) will change according of above relation, and we will derive the problem that we need to study. Consequently, now we will derive these expressions.

Thus, we get

\[
D_3 = \frac{\partial x_1}{\partial x_3}D_1 + \frac{\partial x_2}{\partial x_3}D_2 = -a_1^{-1}D_1 - a_2^{-1}D_2 \quad \& \\
D_3^2 = a_1^{-2}D_2^2 + a_2^{-2}D_2^2 + 2a_1^{-1}a_2^{-1}D_1D_2, \quad D_i = \frac{\partial}{\partial x_i}, i = 1, 2, 3,
\]

according to above mentioned reasoning.

We will assume that functions $u_0$ and $f$ satisfy of conditions of Theorem 2, namely $u_0 \in H^{1/2}(\Omega)$, $f \in L^2(0, T; H^{1/2}(\Omega))$ that are needed for the application of our approach. Consequently, functions $u_0$ and $f$ are correctly defined on $(0, T) \times \Omega_L$.

Let $L$ be arbitrary hyperplane intersecting with $\Omega$, i.e. $\Omega_L \neq \varnothing$ and $u \in V(Q^T_L)$ is the solution of the problem (1.1) - (1.3). We will be investigate of the posed question according of Lemma 3. More precisely, we will study of the posed question for the problem generated by the "projection" (or "trace") of problem (1.1) - (1.3) onto $(0, T) \times \Omega_L$.

So, we would like to apply of Lemma 3 to solutions of the problem (1.1) - (1.3), for that it is necessary to study of properties of solutions of the problem (1.1) - (1.3) in "cross-section" $(0, T) \times \Omega_L$. Consequently, one need to study the problem which is received from the problem (1.1) - (1.3) by "projection" (or "trace") it to $(0, T) \times \Omega_L$ in order to investigate of the needed properties of solutions of the problem (1.1) - (1.3) on $(0, T) \times \Omega_L$. 
As function $u$ belong to $\mathcal{V}(Q^T)$, therefore the function $u$ on $(0, T] \times \Omega_L$ is well defined. Thus, we obtain the following problem on $(0, T] \times \Omega_L$

$$\frac{\partial u}{\partial t} - \nu \Delta u + \sum_{j=1}^{3} u_j D_j u = \frac{\partial u_L}{\partial t} - \nu \left( D_1^2 + D_2^2 + D_3^2 \right) u_L +$$
$$u_{L1} D_1 u_L + u_{L2} D_2 u_L + u_{L3} D_3 u_L = \frac{\partial u_L}{\partial t} - \nu \left[ D_1^2 + D_2^2 + a_1^{-2} D_1^2 \right] +$$
$$a_2^{-2} D_2^2 + 2a_1^{-1} a_2^{-1} D_1 D_2 \left[ u_L + u_{L1} D_1 u_L + u_{L2} D_2 u_L - u_{L3} a_1^{-1} D_1 u_L \right] -$$
$$u_{L3} a_2^{-1} D_2 u_L = \frac{\partial u_L}{\partial t} - \nu \left[ (1 + a_1^{-2}) D_1^2 + (1 + a_2^{-2}) D_2^2 \right] u_L -$$

(3.3) \hspace{1cm} 2\nu a_1^{-1} a_2^{-1} D_1 D_2 u_L + (u_{L1} - a_1^{-1} u_{L3}) D_1 u_L + (u_{L2} - a_2^{-1} u_{L3}) D_2 u_L = f_L$$

on $(0, T) \times \Omega_L$, by virtue of the above reasons, of the conditions of the main theorem, and also of the presentations (3.1) and (3.2). We get

(3.4) \hspace{1cm} \text{div } u_L = D_1 (u_L - a_1^{-1} u_{L3}) + D_2 (u_L - a_2^{-1} u_{L3}) = 0, \quad x \in \Omega_L, \; t > 0$$

(3.5) \hspace{1cm} u_L(0, x) = u_{L0}(x), \quad (t, x) \in [0, T] \times \Omega_L; \quad u_L \big|_{(0, T) \times \partial \Omega_L} = 0.$$

by using of same way.

Thus, we derived the problem (3.3) - (3.5) the study of which will give we possibility to define properties of solutions \(u\) of problem \((1.1^1) - (1.3)\) on each ”cross-section” \([0, T] \times \Omega_L \equiv Q^T_L\).

In the beginning it is necessary to investigate the existence of the solution of problem (3.3) - (3.5) and determine the space where the existing solutions are contained. Consequently, for study of the uniqueness of the solution of problem \((1.1^1) - (1.3)\) at first it is necessary to investigate the existence and uniqueness of the solution for the derived problem (3.3) - (3.5). Therefore we will to investigate of problem (3.3) - (3.5).

We must to note: For each hyperplane $L \subset R^3$ there exists, at least, one 2-dimensional subspace of $R^3$ that in the given coordinat system one can determine as \((x_i, x_j)\) and $P_{x_k} L = R^2$ (e.g. \(i, j, k = 1, 2, 3\)), i.e.

$$L \equiv \{ x \in R^3 \mid x = (x_i, x_j, \psi_L(x_i, x_j)), \; (x_i, x_j) \in R^2 \}$$

and

$$\Omega \cap L \equiv \{ x \in \Omega \mid x = (x_i, x_j, \psi_L(x_i, x_j)), \; (x_i, x_j) \in P_{x_k} (\Omega \cap L) \}$$

hold, where $\psi_L$ is the affine function that is the bijection.

Thereby, in this case functions $u(t, x)$, $f(t, x)$ and $u_0(x)$ can be represented as

$$u(t, x_i, x_j, \psi_L(x_i, x_j)) \equiv v(t, x_i, x_j), \; f(t, x_i, x_j, \psi_L(x_i, x_j)) \equiv \phi(x_i, x_j)$$

and

$$u_0(x_i, x_j, \psi_L(x_i, x_j)) \equiv v_0(x_i, x_j) \quad \text{on } (0, T) \times P_{x_1} \Omega_L,$$

respectively.

So, each of these functions can be represented as functions from the independent variables: $t$, $x_i$ and $x_j$. 
3.1. On Dirichlet to Neumann map. As is known ([34], [35], [38], [36], [37] etc.) the Dirichlet to Neumann map is single-value mapping if the homogeneous Dirichlet problem for elliptic equation has only trivial solution, i.e. zero not is eigenvalue of this problem. Consequently, it is enough to show that the homogeneous Dirichlet problem for elliptic equation associated to considered problem satisfies of the corresponding conditions of the results of the mentioned articles. So, we will prove the following

**Proposition 3.** The homogeneous Dirichlet problem for elliptic part of problem (3.3) - (3.5) has only trivial solution.

*Proof.* If consider the elliptic part of problem (3.3) - (3.5) then we get the problem

\[-\Delta u_L + Bu_L = -\nu \left[(1 + a_1^{-2}) D_1^2 + (1 + a_2^{-2}) D_2^2 + 2a_1^{-1} a_2^{-1} D_1 D_2\right] u_L +
\]

\n
\[\{u_{L1} - a_1^{-1} u_{L3}\} D_1 u_L + \{u_{L2} - a_2^{-1} u_{L3}\} D_2 u_L = 0, \ x \in \Omega_L, \ u_L |_{\partial \Omega_L} = 0,\]

where \(\Omega_L = \Omega \cap L\).

Let’s show that this problem cannot have nontrivial solutions. This will be to prove by method of contradiction. Let \(u_L \in V(\Omega_L)\) be nontrivial solution of this problem then we get the following equation

\[0 = (-\Delta u_L + Bu_L, u_L)_{P_{x_3}{\Omega}_L}\]

hence

\[-\nu \sum_{i=1}^{3} \left\langle \left[D_1^2 + D_2^2\right] + \left(a_1^{-1} D_1 + a_2^{-1} D_2\right)^2\right\rangle u_{L1}, u_{L1}\right\rangle_{P_{x_3}{\Omega}_L} +
\]

\[\sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} \left[u_{L1} D_1 u_{L1}, u_{L1} + u_{L2} D_2 u_{L1}, u_{L1}\right] dx_1 dx_2 +
\]

\[\nu \sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} \left[(D_1 u_{L1})^2 + (D_2 u_{L1})^2 + \left((a_1^{-1} D_1 + a_2^{-1} D_2) u_{L1}\right)^2\right] dx_1 dx_2 +
\]

\[\frac{1}{2} \sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} \left[u_{L1} D_1 (u_{L1})^2 + u_{L2} D_2 (u_{L1})^2 +
\]

\[u_{L3} \left(-a_1^{-1} D_1 - a_2^{-1} D_2\right) (u_{L1})^2\right] dx_1 dx_2 \geq
\]

\[\nu \sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} \left[D_1 u_{L1}^2 + D_2 u_{L1}^2\right] dx_1 dx_2 +
\]

\[-\frac{1}{2} \sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} \left[D_1 u_{L1} + D_2 u_{L2} + (-a_1^{-1} D_1 - a_2^{-1} D_2) u_{L3}\right] |u_{L1}|^2 dx_1 dx_2 =
\]

by ([5.3])

\[\nu \sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} \left[D_1 u_{L1}^2 + D_2 u_{L1}^2\right] dx_1 dx_2 - \frac{1}{2} \sum_{i=1}^{3} \int_{P_{x_3}{\Omega}_L} |u_{L1}|^2 \div u_L dx_1 dx_2 =
\]
Thus, the obtained contradiction shows that function $u_L$ need be zero, i.e. $u_L = 0$ holds.

Consequently, the Dirichlet to Neumann map is single-value operator. \( \square \)

It is well-known that operator $-\Delta : H_0^1(\Omega_L) \rightarrow H^{-1}(\Omega_L)$ generates of the $C_0$
semigroup on $H(\Omega_L)$, and since inclusion $H_0^1(\Omega_L) \subset H^{-1}(\Omega_L)$ is compact, therefore $(-\Delta)^{-1}$ is the compact operator in $H^{-1}(\Omega_L)$. Moreover, $-\Delta : H^{1/2}(\partial\Omega_L) \rightarrow H^{-1/2}(\partial\Omega_L)$ and the operator $B : H^{1/2}(\partial\Omega_L) \rightarrow H^{-1/2}(\partial\Omega_L)$ also possess appropriate properties of such types.

4. Existence of Solution of Problem (3.3) - (3.5)

So, assume conditions of Theorem\(^2\) fulfilled, i.e.

$$u_0 \in H^{1/2}(\Omega), \ f \in L^2\left(0, T; H^{1/2}(\Omega)\right),$$

then these functions on $\Omega_L, Q_T^L$ are correctly defined and belong to $H(\Omega_L), L^2(0, T; H(\Omega_L))$, respectively. Consequently, we can study problem (3.3) - (3.5) under conditions $u_{0L} \in H(\Omega_L)$ and $f_L \in L^2(0, T; H(\Omega_L))$, as independent problem.

By executing according the known argument started by Leray (\cite{1}, see, also \cite{7}, \cite{17}, \cite{5}), the space $V(\Omega_L)$ of the vector functions $u$ one can determine by same way as in Definition 4 the space $V(\Omega_L)$ is the closure in $(H_0^1(\Omega_L))^3$ of

$$\left\{ \varphi \mid \varphi \in (C_0^\infty(\Omega_L))^3, \ \text{div} \varphi = 0 \right\}$$

$$\left( W_0^{1,2}(\Omega_L) \right)^3,$$

where div is regarded in the sense (3.4), in this case the dual space $V^*(\Omega_L)$ is determined as $V^*(\Omega_L)$, the space $H(\Omega_L)$ also is determined as the closure in $(L^2(\Omega_L))^3$ of

$$\left\{ \varphi \mid \varphi \in (C_0^\infty(\Omega_L))^3, \ \text{div} \varphi = 0 \right\}$$

in the topology of $(L^2(\Omega_L))^3$.

Consequently, one can determine of space $V(Q_T^L)$ as

$$V(Q_T^L) = L^2(0, T; V(\Omega_L)) \cap L^\infty(0, T; H(\Omega_L)).$$

Here $\Omega \subset R^3$ is bounded domain of $Lip_{loc}$ and $\Omega_L \subset R^2$ is subdomain defined in the beginning of Section 3 therefore, $\Omega_L$ is Lipschitz, $Q_T^L = (0, T) \times \Omega_L$.

Let $f_L \in L^2(0, T; V^*(\Omega_L))$ and $u_{0L} \in H(\Omega_L)$. Consequently, a solution of problem (3.3) - (3.5) will be understood as follows.

So, we can call the solution of this problem: A function $u_L \in V(Q_T^L)$ is called a solution of the problem (3.3) - (3.5) if $u_L(t, x')$ satisfy the equality

$$\frac{d}{dt} \langle u_L, v \rangle_{\Omega_L} - \langle \nu \Delta u_L, v \rangle_{\Omega_L} + \sum_{j=1}^3 u_{Lj} D_j u_L, v \rangle_{\Omega_L} = \langle f_L, v \rangle_{\Omega_L},$$

for any $v \in V(\Omega_L)$ and almost everywhere in $(0, T)$ and initial condition

$$\langle u_L(t), v \rangle_{\Omega_L} = \langle u_{0L}, v \rangle,$$
in the sense of $H$, where $\langle \cdot , \cdot \rangle_{\Omega_L}$ is the dual form for the pair of spaces $(V (\Omega_L), V^* (\Omega_L))$ and $\Omega_L$ is Lipschitz. Where $x^i \in \Omega_L$ is $x^i = (x_1, x_2)$ (according to our selection of the $L$) and $V (Q_T^I)$ is

$$V (Q_T^I) \equiv \{ w \mid w \in V (Q_T^I) , \ w' \in L^+ (0 ; V^* (\Omega_L)) \} .$$

We will lead of the proof of this problem in five-steps as independent problem.

4.1. A priori estimates. In order to derive of the a priori estimates for the possible solutions of the problem we will apply of the usual approach. By substituting (4.1) A priori estamates.

in order to derive of the a priori estimates for the possible solutions of the problem we will apply of the usual approach. By substituting (4.1) of the function $u_L$ instead of the function $v$, we get

$$\frac{d}{dt} \langle u_L, u_L \rangle_{\Omega_L} - \langle \nu \Delta u_L, u_L \rangle_{\Omega_L} + \left( \sum_{j=1}^{3} u_{Lj} D_j u_L, u_L \right)_{\Omega_L} = \langle f_L, u_L \rangle_{\Omega_L} .$$

Thence, by making the known calculations, taking into account of the condition on $\nu$ and $\Omega_L$ and (3.4), and also of calculations (3.1) that carried out in the previous Section, we derive

$$\frac{1}{2} \frac{d}{dt} \| u_L \|_{H^1 (\Omega_L)}^2 (t) + \nu \left( 1 + a_1^{-2} \right) \| D_1 u_L \|_{H^1 (\Omega_L)}^2 (t) +$$

(4.3) $\nu \left( 1 + a_2^{-2} \right) \| D_2 u_L \|_{H^1 (\Omega_L)}^2 (t) + 2 \nu a_1^{-1} a_2^{-1} \langle D_1 u_L, D_2 u_L \rangle_{\Omega_L} (t) = \langle f_L, u_L \rangle_{\Omega_L}$,

where $\langle g, h \rangle_{\Omega_L} = \sum_{i=1}^{3} \int_{P_{3}, \Omega_L} g_i h_i dx_1 dx_2$ for any $g, h \in H (\Omega_L)$, or $g \in (W^{1,2} (\Omega_L))^3$

and $h \in (W^{-1,2} (\Omega_L))^3$, respectively. We will show the correctness of (4.3), and to this end we shall prove the correctness of each term of this sum, separately.

So, using (4.2) we get

$$- \nu \langle \Delta u_L (t), u_L (t) \rangle_{\Omega_L} =$$

$$- \nu \sum_{i=1}^{3} \langle \left( (1 + a_1^{-2}) D_i^2 + (1 + a_2^{-2}) D_i^2 + 2 a_1^{-1} a_2^{-1} D_1 D_2 \right) u_{Li}, u_{Li} \rangle_{P_{3}, \Omega_L} =$$

$$\nu \sum_{i=1}^{3} \int_{P_{3}, \Omega_L} \left[ (1 + a_1^{-2}) \left( D_1 u_{Li} \right)^2 + (1 + a_2^{-2}) \left( D_2 u_{Li} \right)^2 + 2 a_1^{-1} a_2^{-1} D_1 u_{Li} D_2 u_{Li} \right] dx_1 dx_2$$

thus is obtained the sum reducible in (4.3).

Whence isn’t difficult to seen, that if to estimate of the last adding in the above mentioned sum then one will received

$$- \nu \langle \Delta u_L (t), u_L (t) \rangle_{\Omega_L} \geq$$

(4.4) $\nu \left[ \| D_1 u_L \|_{H^1 (\Omega_L)}^2 (t) + \| D_2 u_L \|_{H^1 (\Omega_L)}^2 (t) \right]$.

Now consider the trilinear form from (4.2)

$$\left( \sum_{j=1}^{3} u_{Lj} D_j u_L, u_L \right)_{\Omega_L} =$$
due to (3.3) we get
\[
\sum_{i=1}^{3} \int_{\Omega_L} \left[ u_{L1} D_1 u_{L1} u_{L1} + u_{L2} D_2 u_{L2} u_{L1} + u_{L3} \left( -a_1^{-1} D_1 - a_2^{-1} D_2 \right) u_{L1} \right] \, dx_1 \, dx_2 =
\]
\[
\frac{1}{2} \sum_{i=1}^{3} \int_{\Omega_L} \left[ u_{L1} D_1 (u_{L1})^2 + u_{L2} D_2 (u_{L1})^2 + u_{L3} \left( -a_1^{-1} D_1 - a_2^{-1} D_2 \right) (u_{L1})^2 \right] \, dx_1 \, dx_2 =
\]
\[
\frac{1}{2} \sum_{i=1}^{3} \int_{\Omega_L} \left[ (D_1 u_{L1})^2 + (D_2 u_{L2})^2 \right] \, dx_1 \, dx_2 \leq \int_{\Omega_L} |(f_L \cdot u_L)| \, dx_1 \, dx_2
\]

hence by (3.4)
\[
(4.5) \quad - \frac{1}{2} \sum_{i=1}^{3} \int_{\Omega_L} (u_{L1})^2 \, \text{div} \, u_{L1} \, dx_1 \, dx_2 = 0.
\]

Consequently, the correctness of equation (4.3) is proved.

From (4.3) in view of (4.4)-(4.5) is derived the following inequality
\[
\frac{1}{2} \frac{d}{dt} \|u_L\|_{H(\Omega_L)}^2 (t) + \nu \sum_{i=1}^{3} \int_{\Omega_L} \left[ (D_1 u_{L1})^2 + (D_2 u_{L2})^2 \right] \, dx_1 \, dx_2 \leq \int_{\Omega_L} |(f_L \cdot u_L)| \, dx_1 \, dx_2
\]

which gives the following a priori estimates
\[
(4.7) \quad \|u_L\|_{H(\Omega_L)}^2 (t) \leq C (f_L, u_{L0}, \text{mes} \Omega),
\]
\[
(4.8) \quad \|D_1 u_L\|_{H(\Omega_L)} + \|D_2 u_L\|_{H(\Omega_L)} \leq C (f_L, u_{L0}, \text{mes} \Omega),
\]

where \( C (f_L, u_{L0}, \text{mes} \Omega) > 0 \) is the constant that is independent of \( u_L \). Consequently, any possible solution of this problem belong to a bounded subset of the space \( V (\Omega_L^T) \).

Thus, it is remain to receive of the necessary a priori estimate for \( \frac{\partial u_L}{\partial t} \) and to study of properties of the thrilinear term in order to prove of the existence theorem.

---

3It should be noted that if the representation of \( \Omega_L \) by coordinates \((x_1, x_2)\) not is best for the definition of the appropriate integral, then we will select other coordinates: either \((x_1, x_3)\) or \((x_2, x_3)\) instead of \((x_1, x_2)\), which is best for our goal (that must exist by virtue of the definition of \( \Omega \)).
4.2. Boundedness of the trilinear form. Boundedness of the trilinear form $b_L(u_L, u_L, v)$ from (4.1) follows from the next result.

**Proposition 4.** Let $u_L \in V (Q_L^T) \cap L^\infty (0, T; H)$, $v \in V (\Omega_L)$ and $B$ is the operator defined by

$$
\langle B(u_L), v \rangle_{\Omega_L} = b_L(u_L, u_L, v) = \left\langle \sum_{j=1}^3 u_{Lj} D_j u_L, v \right \rangle_{\Omega_L}
$$

then $B(u_L)$ belongs to bounded subset of $L^2 (0, T; V^* (\Omega_L))$.

**Proof.** At first we will show boundedness of the operator $B$ acting from $V (\Omega_L) \times V (\Omega_L)$ to $V^* (\Omega_L)$ for a. e. $t \in (0, T)$. We have

$$
\langle B(u_L), v \rangle_{\Omega_L} = \left\langle \sum_{j=1}^3 u_{Lj} D_j u_L, v \right \rangle_{\Omega_L} =
$$

$$
\sum_{i=1}^3 \int_{P_{x_j} \Omega_L} \left[ u_{L1} D_1 u_{L1} v_i + u_{L2} D_2 u_{L2} v_i + u_{L3} (-a_1^{-1} D_1 - a_2^{-1} D_2) u_{L3} v_i \right] dx_1 dx_2 =
$$

$$
\sum_{i=1}^3 \int_{P_{x_j} \Omega_L} \left[ (u_{L1} - a_1^{-1} u_{L3}) D_1 u_{L1} v_i + (u_{L2} - a_2^{-1} u_{L3}) D_2 u_{L2} v_i \right] dx_1 dx_2 =
$$

$$
\sum_{i=1}^3 \int_{P_{x_j} \Omega_L} \left[ (u_{L1} - a_1^{-1} u_{L3}) D_1 + (u_{L2} - a_2^{-1} u_{L3}) D_2 \right] u_{L3} v_i dx_1 dx_2
$$

(4.9)

due of (3.4) and of the definition of $B$

Hence follows

$$
|\langle B(u_L), v \rangle| \leq \sum_{i=1}^3 \int_{P_{x_j} \Omega_L} c (|u_{L1}| + |u_{L2}| + |u_{L3}|) (|D_1 u_{L1}| + |D_2 u_{L2}|) v_i dx_1 dx_2 \leq
$$

$$
c \|u_L\|_{L^4(\Omega_L)} \|u_L\|_V \|v\|_{L^4(\Omega_L)} \leq \|B(u_L)\|_{V^*} \leq c \|u_L\|_{L^4(\Omega_L)} \|u_L\|_V
$$

(4.10)

due of $V (\Omega_L) \subset L^4 (\Omega_L)$. This also shows that operator $B : V (\Omega_L) \rightarrow V^* (\Omega_L)$ is bounded, and continuous for a. e. $t > 0$.

Finally, we obtain needed result using above inequality and the well-known inequality, which is valid in two-dimension space (see, [2], [7], [28])

$$
\int_0^T \|B(u_L(t))\|_{V^*}^\frac{4}{3} dt \leq c \int_0^T (\|u_L(t)\|_{L^4} \|u_L\|_V^\frac{2}{3})^\frac{4}{3} dt \leq
$$

according to Gagliardo-Nirenberg inequality we get

$$
c_1 \int_0^T \|u_L(t)\|_{L^2}^\frac{2}{3} \|u_L\|_V^\frac{2}{3} dt \leq c_1 \|u_L\|_{L^\infty(0,T;H)}^\frac{2}{3} \int_0^T \|u_L\|_V^\frac{2}{3} dt \leq
$$

$$
|B(u_L)| \|u_L\|_{L^\frac{4}{3}(0,T;V^*)} \leq c_1 \|u_L\|_{L^\infty(0,T;H)} \|u_L\|_{L^2(0,T;V^*)}^\frac{4}{3}
$$

(4.11)

□
Moreover, is proved that
\[ B : L^2 (0, T; V (\Omega_L)) \cap L^\infty (0, T; H) = V (Q^T_L) \rightarrow L^\frac{4}{3} (0, T; V^*) \]
is bounded operator.

4.3. **Boundedness of** \( u' \). Sketch of the proof that \( u' \) belongs to bounded subset of the space \( L^4 (0, T; V^* (\Omega_L)) \). It is possible to draw the following conclusion based due received of a priori estimates, proposition 4 and reflexivity of all used spaces: If we will use of the Faedo-Galerkin’s method for investigation then for the approximate solutions we obtain estimates of such type as (4.7), (4.8) and (4.11). Indeed since \( V (\Omega_L) \) is a separable there exists a countable subset of linearly independent elements \( \{w_i\}_{i=1}^\infty \subset V (\Omega_L) \), which is total in \( V (\Omega_L) \). For each \( m \) we can define an approximate solution of \( u_{Lm} \) as follows
\[
(4.12) \quad u_{Lm} = \sum_{i=1}^{m} u_{Lm}^i (t) w_i, \quad m = 1, 2, \ldots ,
\]
where \( u_{Lm}^i (t), i = 1, m \) be unknown functions that will be determined as solutions of the following system of the differential equations that is received according to equation (4.11)
\[
\left\langle \frac{d}{dt} u_{Lm}, w_j \right\rangle_{\Omega_L} = \langle \nu \Delta u_{Lm}, w_j \rangle_{\Omega_L} + b_L (u_{Lm}, u_{Lm}, w_j) + \langle f_L, w_j \rangle_{\Omega_L}, \quad t \in (0, T], \quad j = 1, m,
\]
\[
(4.13) \quad u_{Lm} (0) = u_{0Lm}.
\]
Here we assume \( \{u_{0Lm}\}_{m=1}^\infty \subset H (\Omega_L) \) be such sequence that \( u_{0Lm} \rightarrow u_{0L} \) in \( H (\Omega_L) \) as \( m \rightarrow \infty \). (Since \( V (\Omega_L) \) is everywhere dense in \( H (\Omega_L) \) one can determine \( u_{0Lm} \) by using the total system \( \{w_i\}_{i=1}^\infty \)).

So, with use (4.12) in (4.13) we obtain
\[
\sum_{j=1}^{m} \langle w_j, w_i \rangle_{\Omega_L} \frac{d}{dt} u_{Lm}^j (t) - \nu \sum_{j=1}^{m} \langle \Delta w_j, w_i \rangle_{\Omega_L} u_{Lm}^j (t) + \sum_{j,k=1}^{m} b_L (w_j, w_k, w_i) u_{Lm}^j (t) u_{Lm}^k (t) = \langle f_L (t), w_i \rangle_{\Omega_L}, \quad i = 1, m.
\]
As the matrix generated by \( \langle w_i, w_j \rangle_{\Omega_L}, i, j = 1, m \) is nonsingular then its inverse exists. Thanks this from the previous equations we will derive the following Cauchy problem for the system of the nonlinear ordinary differential equations for unknown functions \( u_{Lm}^i (t), i = 1, \ldots, m \).
\[
\frac{du_{Lm}^i (t)}{dt} = \sum_{j=1}^{m} c_{i,j} \langle f_L (t), w_j \rangle_{\Omega_L} - \nu \sum_{j=1}^{m} d_{i,j} u_{Lm}^j (t) + \sum_{j,k=1}^{m} h_{ijk} u_{Lm}^j (t) u_{Lm}^k (t),
\]
\[
(4.14) \quad u_{Lm}^i (0) = u_{i0Lm}^i, \quad i = 1, \ldots, m, \quad m = 1, 2, \ldots
\]
where $u_{0Lm}$ is $i^{th}$ component of $u_{0L}$ in representation $u_{0L} = \sum_{k=1}^{\infty} u_{0Lm}^k w_k$.

The Cauchy problem for the system of the nonlinear ordinary differential equations (4.14) has solution, which defined on whole of interval $(0, T]$ due of uniformity of estimations received in subsections 4.1 and 4.2. Consequently, the approximate solutions $u_{Lm}$ exist and belong to a bounded subset of $W^{1, \frac{4}{3}} (0, T; V^* (\Omega_L))$ for every $m = 1, 2, \ldots$ since the right side of (4.14) belong to a bounded subset of $L^2 (0, T; V^* (\Omega_L))$ as were proved in subsections 4.1 and 4.2, and also by virtue of the next lemma.

Lemma 4. (28) Let $X$ be a given Banach space with dual $X^*$ and let $u$ and $g$ be two functions belonging to $L^1 (a, b; X)$. Then, the following three conditions are equivalent

(i) $u$ is a.e. equal to a primitive function of $g$,

$$u(t) = \xi + \int_a^t g(s) \, ds, \quad \xi \in X, \quad \text{a.e.} \ t \in [a, b]$$

(ii) For each test function $\varphi \in D ((a, b))$,

$$\int_a^b u(t) \varphi'(t) \, dt = - \int_a^b g(t) \varphi(t) \, dt, \quad \varphi' = \frac{d \varphi}{dt}$$

(iii) For each $\eta \in X^*$,

$$\frac{d}{dt} \langle u, \eta \rangle = \langle g, \eta \rangle$$

in the scalar distribution sense, on $(a, b)$. If (i) - (iii) are satisfied $u$, in particular, is a.e. equal to a continuous function from $[a, b]$ into $X$.

It isn’t difficult to see that if take $\forall v \in V (\Omega_L)$ instead of $w_k$ and pass to limit according to $m \to \infty$ in equation (4.13) (may be by subsequence $\{u_{Lm_l}\}_{l=1}^{\infty}$ of this sequence, is known that such subsequence exists) then we get

$$\left( \frac{d}{dt} u_L, v \right)_{\Omega_L} = \left( f_L + \nu \Delta u_L - \chi, v \right)_{\Omega_L},$$

due of fullness of the class $\{w_i\}_{i=1}^{\infty}$ in $V (\Omega_L)$. Where function $\chi$ belongs to $L^4 (0, T; V^* (\Omega_L))$ and is determined by equality

$$\lim_{l \to \infty} \left( B(u_{Lm_l}), v \right)_{\Omega_L} = \left( \chi, v \right)_{\Omega_L}$$

that shown in the above section. So, we obtain that in (4.16) the right side belong to $L^4 (0, T)$ then the left side also belongs to $L^2 (0, T)$ according to above a priori estimates and Proposition 4 i.e.

$$\frac{du_L}{dt} \in L^4 (0, T; V^* (\Omega_L)).$$

Consequently, the following result is proven.

Proposition 5. Under above mentioned conditions $u'_L$ belongs to a bounded subset of the space $L^\frac{4}{3} (0, T; V^* (\Omega_L))$.

From above results of this section by virtue of the abstract form of Riesz-Fischer theorem follows
Corollary 1. Under above mentioned conditions function $u_L$ belongs to a bounded subset of the space $V(Q^*_L)$, where

$$V(Q^*_L) \equiv V(Q^*_L) \cap W^{1,\frac{1}{2}}(0, T; V^*(\Omega_L)).$$

Thus for the proof that $u_L$ is the solution of problem (3.3) - (3.5) or (4.4) remains to show that $\chi = B(u_L)$ or $\langle \chi, v \rangle_{\Omega_L} = b_L(u_L, u_L, v)$ for $v \in V(\Omega_L)$.

4.4. Weakly compactness of operator $B$.

Proposition 6. Operator $B : V(Q^*_L) \rightarrow L^\frac{1}{2}(0, T; V^*(\Omega_L))$ is weakly compact operator, i.e. any weakly convergent sequence $\{u^m_L\} \subset V(Q^*_L)$ possess such subsequence $\{u^m_L\} \subset \{u^m_L\} \subset \{u^m_L\}$ weakly converged in $L^\frac{1}{2}(0, T; V^*(\Omega_L))$.

Proof. Let sequence $\{u^m_L\} \subset V(Q^*_L)$ be weakly converge to $u^0_L$ in $V(Q^*_L)$. Then there exists such subsequence $\{u^m_L\} \subset \{u^m_L\}$ that $u^m_L \rightarrow u^0_L$ in $L^2(0, T; H)$, due to the known theorems on the compactness of the embedding, particularly, as known the following embedding

$$V(Q^*_L) \equiv L^2(0, T; V(\Omega_L)) \cap W^{1,\frac{1}{2}}(0, T; V^*(\Omega_L)) \subset L^2(0, T; H)$$

is compact (see, e. g. [7], [28], [31]).

Actually it is enough to show that the operator defined by expression $\sum_{j=1}^{3} u_{Lj} D_j u_L$

is weakly compact from $V(Q^*_L)$ to $L^\frac{1}{2}(0, T; V^*(\Omega_L))$. From a priori estimations and Proposition 4 follow that operator $B : V(Q^*_L) \rightarrow L^\frac{1}{2}(0, T; V^*(\Omega_L))$ is bounded, i.e. the image of operator $B$ of each bounded subset of space $V(Q^*_L)$ is the bounded subset of space $L^\frac{1}{2}(0, T; V^*(\Omega_L))$.

From above compactness theorem follows the sequence $\{u^m_L\}$ posses some subsequence $\{u^m_L\} \subset \{u^m_L\}$ strongly convergent to some element $u_L$ of $L^2(0, T; H)$ in the space $L^2(0, T; H)$. Consequently, $B(\{u^m_L\})$ belongs of bounded subset of space $L^\frac{1}{2}(0, T; V^*(\Omega_L))$. Thence lead that there is such element $\chi \in L^\frac{1}{2}(0, T; V^*(\Omega_L))$ that sequence $B(\{u^m_L\})$ weakly converges to $\chi$ when $m \rightarrow \infty$, i.e.

$$B(u^m_L) \rightharpoonup \chi \quad \text{in} \quad L^\frac{1}{2}(0, T; V^*(\Omega_L))$$

due to the reflexivity of this space (there exists, at least, such subsequence that this occurs).

If we set the vector space

$$C^1(\Omega_L) \equiv \{ v \mid v_i \in C^1([0, T]; C_0^1(\Omega_L)), \quad i = 1, 2, 3 \}$$

and consider the trilinear form

$$\int_0^T \langle B(u^m_L), v \rangle_{\Omega_L} dt = \int_0^T b(u^m_L, u^m_L, v) dt = \int_0^T \left( \sum_{j=1}^{3} u_{Lj} D_j u^m_L, v \right)_{\Omega_L} dt = \int_0^T \left\langle \sum_{j=1}^{3} u_{Lj} D_j u^m_L, v \right\rangle_{\Omega_L} dt$$

for $v \in C^1(\Omega_L)$, then we get

$$-\sum_{i=1}^{3} \int_0^T \int_{\Omega_{L}} \left( (u^m_L, u^m_L) - a_1^{-1} u^m_L, u^m_L ) D_1 v_i + (u^m_L, u^m_L) - a_2^{-1} u^m_L, u^m_L ) D_2 v_i \right) dx_1 dx_2 dt.$$
according to (4.19). Now if we take arbitrary term in this sum separately then it isn’t difficult to see that the following convergences are true, because \( u_{L_i}^{m_k} \rightarrow u_{L_i} \) in \( L^2(0,T;H) \) and \( u_{L_i}^{m} \rightarrow u_{L_i} \) in \( L^\infty(0,T;H) \) strongly since \( u_{L}^{m} \) belong to a bounded subset of \( V(Q_T^1) \) and (4.18) is fulfill for each term.

Thus passing to the limit when \( m_k \not\rightarrow \infty \) we obtain
\[
\chi = B(u_L) \quad \text{implies} \quad B(u_{L_k}^{m_k}) \rightarrow B(u_L) \quad \text{in the distribution sense.}
\]
Whence using the density of \( C^1(\bar{Q}_L) \) in \( V(Q_T^1) \), and as \( B(u_{L_k}^{m}) \rightarrow \chi \) takes place in the space \( L^2(0,T;V^*(\Omega_L)) \) we get that \( \chi = B(u_L) \) also takes place in this space.

Consequently, we proved the existence of the function \( u_L \in V(Q_T^1) \) that satisfies equation (4.13) by applying to this problem of the Faedo-Galerkin method and using the above mentioned results.

4.5. Realisation of the initial condition. We will lead the proof of the realisation of initial condition according to same way as in [28] (see, also [2], [7]).

Let \( \phi \) be a continuously differentiable function on \([0,T]\) with \( \phi(T) = 0 \). With multiplying (4.13) by \( \phi(t) \), and then the first term integrating by parts we leads to equation
\[
-\int_0^T \left\langle u_{L,m}, \frac{d}{dt} \phi(t) w_j \right\rangle_{\Omega_L} dt = \int_0^T \left\langle \nu \Delta u_{L,m}, \phi(t) w_j \right\rangle_{\Omega_L} dt + \int_0^T b(u_{L,m}, u_{L,m}, \phi(t) w_j) dt + \int_0^T f_L, \phi(t) w_j\rangle_{\Omega_L} dt + \langle u_{0L,m}, \phi(0) w_j\rangle_{\Omega_L}.
\]

One can pass to the limit with respect to subsequence \( \{u_{L,m}\}_{m=1}^\infty \) of the sequence \( \{u_{L,m}\}_{m=1}^\infty \) in the equality mentioned above owing to the results proved in the previous subsections. Then we find the equation
\[
-\int_0^T \left\langle u_L, \frac{d}{dt} \phi(t) w_j \right\rangle_{\Omega_L} dt = \int_0^T \left\langle \nu \Delta u_L, \phi(t) w_j \right\rangle_{\Omega_L} dt + \int_0^T b(u_L, u_L, \phi(t) w_j) dt + \int_0^T f_L, \phi(t) w_j\rangle_{\Omega_L} dt + \langle u_{0L}, \phi(0) w_j\rangle_{\Omega_L}.
\]
that holds for each \( w_j, j = 1, 2, \ldots \). Consequently, this equality holds for any finite linear combination of the \( w_j \) and moreover due of continuity (4.19) remains true and for any \( v \in V(\Omega_L) \).

Whence, one can draw conclusion that function \( u_L \) satisfies equation (4.11) in the distribution sense.

Now if multiply (4.11) by \( \phi(t) \), and integrate with respect to \( t \) after integrating the first term by parts, then we get
\[
-\int_0^T \left\langle u_L, \frac{d}{dt} \phi(t) \right\rangle_{\Omega_L} dt - \int_0^T \left\langle \nu \Delta u_L, \phi(t) v \right\rangle_{\Omega_L} dt +
\]
\[
\int_0^T \left( \sum_{j=1}^3 u_{Lj} D_j u_L, \phi(t) v \right) dt = \int_0^T \left( f_L, \phi(t) v \right)_{\Omega_L} dt + \langle u_L(0), \phi(0) v \rangle_{\Omega_L}.
\]

If we compare this with (3.11) after replacing \( w_j \) with any \( v \in V(\Omega_L) \) then we obtain
\[
\phi(0) \langle u_L(0) - u_0L, v \rangle_{\Omega_L} = 0.
\]

Whence, we get the realisation of the initial condition by virtue of arbitrariness of \( v \in V(\Omega_L) \) and \( \phi \), since function \( \phi \) one can choose as \( \phi(0) \neq 0 \).

Consequently, the following result is proven.

**Theorem 4.** Under above mentioned conditions for any
\[
u_0L \in (H(\Omega_L))^3, \quad f_L \in L^2(0, T; V^*(\Omega_L))
\]
problem (5.1) - (5.2) has weak solution \( u_L(t, x) \) that belongs to \( V(Q_T^L) \).

**Remark 1.** From the obtained a priori estimates and Propositions 4 and 6 follows of the fulfillment of all conditions of the general theorem of the compactness method (see, e.g., [31], [32], and for complementary informations see, [27], [33]). Consequently, one could be to study the solvability of problem (5.1) - (5.2) with use of this general theorem.

### 5. Uniqueness of Solution of Problem (3.3) - (3.5)

For the study of the uniqueness of the solution as usually: we will assume that posed problem have, at least, two different solutions \( u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3) \). Below will show that this isn’t possible, and for which one need to investigate their difference, i.e. \( w = u - v \). (Here for brevity we won’t specify indexes for functions, which showing that here is investigated the system of equations (3.3) - (3.5) on \( Q_T^L \).)

So, we obtain the following problem for \( w = u - v \)

\[
\frac{\partial w}{\partial t} - \nu \left[ (1 + a_1^{-2}) D_1^2 + (1 + a_2^{-2}) D_2^2 \right] w - 2\nu a_1^{-1} a_2^{-1} D_1 D_2 w + (u_1 - a_1^{-1} u_3) D_1 u - (v_1 - a_1^{-1} v_3) D_1 v + (u_2 - a_2^{-1} u_3) D_2 u - (v_2 - a_2^{-1} v_3) D_2 v = 0,
\]
\[
\text{div } w = D_1 \left[ (u - a_1^{-1} u_3) - (v - a_1^{-1} v_3) \right] + D_2 \left[ (u - a_2^{-1} u_3) - (v - a_2^{-1} v_3) \right] = D_1 w + D_2 w - (a_1^{-1} D_1 + a_2^{-1} D_2) w = 0,
\]
\[
w(0, x) = 0, \quad x \in \Omega \cap L; \quad w|_{(0, T) \times \partial \Omega_L} = 0.
\]

Hence we derive
\[
\frac{1}{2} \frac{d}{dt} \| w \|_2^2 + \nu \left( \| D_1 w \|_2^2 + \| D_2 w \|_2^2 \right) + 2\nu a_1^{-1} a_2^{-1} \langle D_1 w, D_2 w \rangle_{\Omega_L} + \langle (u_1 - a_1^{-1} u_3) D_1 u - (v_1 - a_1^{-1} v_3) D_1 v, w \rangle_{\Omega_L} + \langle (u_2 - a_2^{-1} u_3) D_2 u - (v_2 - a_2^{-1} v_3) D_2 v, w \rangle_{\Omega_L} = 0
\]
or
\[
\frac{1}{2} \frac{d}{dt} \| w \|_2^2 + \nu \left( \| D_1 w \|_2^2 + \| D_2 w \|_2^2 \right) + \nu \left[ a_1^{-2} \| D_1 w \|_2^2 + a_2^{-2} \| D_2 w \|_2^2 \right] + 2a_1^{-1} a_2^{-1} \langle D_1 w, D_2 w \rangle_{\Omega_L} + \langle u_1 D_1 u - v_1 D_1 v, w \rangle_{\Omega_L} + \langle u_2 D_2 u - v_2 D_2 v, w \rangle_{\Omega_L} = 0.
\]
\[ v \] by virtue of (4.16).

Thus we obtain the Cauchy problem for equation (5.4) with the initial condition (5.5).

If takes into account this equality in equation (5.3) then we get

\[
\begin{aligned}
&\frac{1}{2} a_2^{-1} \langle v_3, D_2 w^2 \rangle_{\Omega_L} = \frac{1}{2} \langle v_1 - a_1^{-1} v_3, D_1 w^2 \rangle_{\Omega_L} + \frac{1}{2} \langle v_2 - a_2^{-1} v_3, D_2 w^2 \rangle_{\Omega_L} + \\
&\langle (w_1 - a_1^{-1} w_3) w, D_1 u \rangle_{\Omega_L} + \langle (w_2 - a_2^{-1} w_3) w, D_2 u \rangle_{\Omega_L} = 0, \quad (t, x) \in (0, T) \times \Omega_L.
\end{aligned}
\]

In the last equality were used the equation \( \text{div} v = 0 \) (see, (3.14)) and the condition (5.2).

If takes into account this equality in equation (5.3) then we get

\[
\begin{aligned}
&\frac{1}{2} a_2^{-1} \langle v_3, D_2 w^2 \rangle_{\Omega_L} = \frac{1}{2} \langle v_1 - a_1^{-1} v_3, D_1 w^2 \rangle_{\Omega_L} + \frac{1}{2} \langle v_2 - a_2^{-1} v_3, D_2 w^2 \rangle_{\Omega_L} + \\
&\langle (w_1 - a_1^{-1} w_3) w, D_1 u \rangle_{\Omega_L} + \langle (w_2 - a_2^{-1} w_3) w, D_2 u \rangle_{\Omega_L} = 0, \quad (t, x) \in (0, T) \times \Omega_L.
\end{aligned}
\]

Thus we obtain the Cauchy problem for equation (5.4) with the initial condition (5.5)

\[ \| w \|_2 (0) = 0. \]

We get the following Cauchy problem for the differential inequation using the appropriate estimates

\[ \frac{1}{2} \frac{d}{dt} || w ||^2 + \nu \left( || D_1 w ||^2 + || D_2 w ||^2 \right) \leq \]

\[ \left( || w_1 - a_1^{-1} w_3 || + || w_2 - a_2^{-1} w_3 || \right) || w ||_4 || \nabla u ||_2 \leq \]

with the initial condition (5.5).

Then for the right side of (5.6) we get the following estimate

\[ \left( || w_1 - a_1^{-1} w_3 || + || w_2 - a_2^{-1} w_3 || \right) || w ||_4 || \nabla u ||_2 \leq \]

whence we derive

\[ (1 + \max \{ |a_1^{-1}|, |a_2^{-1}| \}) || w ||_4^2 || \nabla u ||_2 \leq c || w ||_2 || \nabla w ||_2 || \nabla u ||_2 \]

thanks of Gagliardo-Nirenberg inequality (30).

It need to note that

\[ (w_1 - a_1^{-1} w_3), (w_2 - a_2^{-1} w_3) w \in L^2 (0, T; V^* (\Omega_L)), \]

by virtue of (4.16).
Now taking into account this in (5.6) one can arrive the following Cauchy problem for differential inequality

\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_2(t) + \nu \|\nabla w\|^2_2(t) \leq C(c,\nu) \|\nabla u\|^2_2(t), \quad \|w\|^2_2(0) = 0,
\]

since \(w \in L^\infty(0,T; H(\Omega_L))\). Consequently, \(\|w\|^2_2 \|\nabla w\|^2_2 \in L^2(0,T)\) by virtue of the proved above existence theorem \(w \in \mathcal{V}(Q^T_L)\), where \(C(c,\nu) > 0\) is constant.

Thus we obtain the problem

\[
\frac{d}{dt} \|w\|^2_2(t) \leq 2C(c,\nu) \|\nabla u\|^2_2(t) y(t), \quad y(0) = 0.
\]

If to denote \(\|w\|^2_2(t) \equiv y(t)\) then

\[
\frac{d}{dt} y(t) \leq 2C(c,\nu) \|\nabla u\|^2_2(t) y(t), \quad y(0) = 0.
\]

Whence follows \(\|w\|^2_2(t) \equiv y(t) = 0\), and consequently the following result is proven:

**Theorem 5.** Under above mentioned conditions for any

\[(f,u_0) \in L^2(0,T; V^*(\Omega_L)) \times H(\Omega_L)\]

problem (3.3) - (3.5) has a unique weak solution \(u(t,x)\) that is contained in \(\mathcal{V}(Q^T_L)\).

6. **Proof of Theorem 2**

**Proof.** (of Theorem 2). As were noted in introduction, under the above mentioned conditions problem (1.1) - (1.3) is weakly solvable and any solution belongs to the space \(\mathcal{V}(Q^T)\). Consequently, under the conditions of Theorem 2 this problem also has weak solution that belongs, at least, to the space \(\mathcal{V}(Q^T)\). But as shown in Sections 4 under conditions of Theorem 2 the auxiliary problems of problem (1.1) - (1.3) are weakly solvable and any solution belongs to the space \(\mathcal{V}(Q^T_L)\). Moreover, as shown in Section 5 weak solution of each of these problems is unique. Hence follows, that we can employ of Lemma 3 to solutions of problem (1.1) - (1.3) on \(Q^T_L\) due of the smoothness of solutions of this problem.

So, assume problem (1.1) - (1.3) has, at least, two different weak solutions under conditions of Theorem 2. It is clear that if the problem have more than one solution then there is, at least, some subdomain of \(Q^T \equiv (0,T) \times \Omega\), on which this problem have, at least, two solutions that different. Consequently, starting from the above Lemma 3 is sufficiently to investigate the existence and uniqueness of the posed problem on arbitrary fixed subdomain in order to shows that exist or unexist such subdomains, on which the studied problem can possess more than one solutions. More exactly it is sufficiently to study of this question in the case when subdomains are generated by arbitrary fixed hyperplanes by virtue of Lemma 3. For this aim it is enough to prove, that isn’t exist such subdomains, on which the problem (1.1) - (1.3) could has of more than one solution by virtue of Lemma 3. Thus, in order to end of the proof is remains to use the above results (i.e. Theorems 4 and 5).

Indeed, as follows from theorems that were proved in the previous sections there not are exist subdomains, on which the problem (1.1) - (1.3) could be possesses more than one weak solution.
Consequently, according of Lemma we obtain, that the problem (1.11) - (1.3) under conditions of Theorem possesses only one weak solution.

Whence can make the following conclusion.

6.1. Conclusion. Let’s
\[ f \in L^2(0,T;V^*(\Omega)), \ u_0 \in H(\Omega). \]

It well-known that following inclusions are dense
\[ L^2 \left(0, T; H^{1/2}(\Omega)\right) \subset L^2 \left(Q_T^\prime\right) ; \ H^{1/2}(\Omega) \subset H(\Omega) & \]
\[ L^2 \left(0, T; H^{1/2}(\Omega)\right) \subset L^2 \left(0, T; H^{-1}(\Omega)\right) . \]

Hence, there exist such sequences
\[ \{u_{0m}\}_{m=1}^\infty \subset H^{1/2}(\Omega) ; \ \{f_m\}_{m=1}^\infty \subset L^2 \left(0, T; H^{1/2}(\Omega)\right) \]
that \( u_{0m} \rightarrow u_0 \) in \( H(\Omega) \) , \( f_m \rightarrow f \) in \( L^2 \left(0, T; H^{-1}(\Omega)\right) . \)

Thus, we establish following result.

**Theorem 6.** Let \( \Omega \) be a Lipschitz open bounded domain in \( \mathbb{R}^3 \) and the given functions \( f \) and \( u_0 \) satisfy of conditions \( f \in L^2(0,T;H^{1/2}(\Omega)) \) and \( u_0 \in H^{1/2}(\Omega) \), respectively. Then there exists unique function \( u \in V(Q_T^\prime) \) that is the weak solution of the considered problem, in the sense of Definition.

Roughly speaking, since \( L^2 \left(0, T; H^{1/2}(\Omega)\right) \) and \( H^{1/2}(\Omega) \) are everywhere dense in spaces \( L^2 \left(0, T; H^{1/2}(\Omega)\right) \) and \( H^{1/2}(\Omega) \), respectively, then if functions \( f \) and \( u_0 \) are any given functions from \( L^2 \left(0, T; V^*(\Omega)\right) \) and \( H(\Omega) \), respectively then in their any neighborhoods there are functions \( f \) and \( u_0 \) from \( L^2 \left(0, T; H^{1/2}(\Omega)\right) \) and \( H^{1/2}(\Omega) \), respectively that the problem (1.11) - (1.3) has unique weak solution \( u \), that belongs to a bounded subset of \( V(Q_T^\prime) \), where a weak solution be understood in the sense of Definition 6.

So, under conditions of Theorem 2 the uniqueness of weak solution \( u(x,t) \) (of velocity vector) of the problem (1.11) - (1.3) obtained from the mixed problem for the incompressible Navier-Stokes 3D-equation proved (explanations of the last proposition see Notation 1 and next paragraph of this Notation 1).

**Part 2. Employment of modified approach to study of uniqueness**

7. ONE CONDITIONAL UNIQUENESS THEOREM FOR PROBLEM (1.11) - (1.3)

We believed there have the sense to provide here yet one result connected with same question for problem (1.11) - (1.3), but with conditions onto the given functions under which the existence theorem of the weak solution of this problem is proven. Here the known approach for the investigation of the uniqueness of solution of problem (1.11) - (1.3) is applied, but with use also other properties of this problem.

Let posed problem have two different solutions: \( u, v \in V(Q_T^\prime) \), then within known approach we get the following problem for vector function \( w(t,x) = u(t,x) - v(t,x) \)

\[
\frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 + \nu \|\nabla w\|_2^2 + \sum_{j,k=1}^3 \left< \frac{\partial v_k}{\partial x_j}, w_k, w_j \right> = 0,
\]

\[
\text{(7.1)}
\]
\[ w(0, x) = w_0(x) = 0, \quad x \in \Omega; \quad w \mid [0, T] \times \partial \Omega = 0, \]

where \( \Omega \subset \mathbb{R}^3 \) is above-mentioned domain.

So, for the proof of triviality of solution of problem (7.1)-(7.2), as usually will used method of contradiction. Consequently, one will start with assume that problem have nontrivial solution.

In addition, it is need to noted here will used of the peculiarity of having non-linearity of this problem.

In the beginning we will study the following quadratic form (21) for examination of problem (7.1)-(7.2)

\[ B(w, w) = \sum_{j, k=1}^{3} \left( \frac{\partial v_k}{\partial x_j} w_j w_k \right) (t, x), \]

denote it as

\[ B(w, w) \equiv \sum_{j, k=1}^{3} (a_{jk} w_k w_j) (t, x). \]

It is clear that behavior of the surface generated by function \( B(w, w) \) respect to the variables \( w_k, k = 1, 2, 3 \) depende of the accelerations of the flow on the different directions.

Consider the question: it would possible to transform the quadratic form \( B(w, w) \) to the canonical form, namely to the following form

\[ B(w, w) = \sum_{i=1}^{3} (b_i w_i^2) (t, x), \quad b_i (t, x) \equiv b_i \left( \frac{\partial v_k}{\partial x_i} \right), \]

where \( D_i v_k = \frac{\partial v_k}{\partial x_i}, \quad i, k = 1, 2, 3, \quad b_i : \mathbb{R}^3 \rightarrow \mathbb{R} \) be functions?

The matrix \( \|a_{jk}\| \) of coefficients of the quadratic form \( B(w, w) \) can be represented in the following form

\[ \|a_{jk}\|_{j, k=1}^{3} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \quad \text{where } a_{jk} = a_{kj} = \frac{1}{2} \left( D_j v_k + D_k v_j \right), \]

hence it is symmetric matrix. As known in this case the above transformation exists according of symmetricness of matrix \( \|a_{jk}\|_{j, k=1}^{3} \) (see, [21]). Consequently, coefficients \( b_i \) are defined and have the following presentations

\[ b_1 = D_1 v_1; \quad b_2 = D_2 v_2 - \frac{(D_1 v_2 + D_2 v_1)^2}{4b_1}; \quad b_3 = \frac{\det \|D_i v_k\|_{i, k=1}^{3}}{\det \|D_i v_k\|_{i, k=1}^{2}}, \]

where \( \|D_i v_k\|_{i, k=1}^{3} \) and \( \|D_i v_k\|_{i, k=1}^{2} \) define by equalities

\[ \|D_i v_k\|_{i, k=1}^{3} \equiv \begin{vmatrix} D_1 v_1 \\ \frac{1}{2} (D_1 v_2 + D_2 v_1) \\ \frac{1}{2} (D_1 v_3 + D_3 v_1) \end{vmatrix}, \quad \|D_i v_k\|_{i, k=1}^{2} \equiv \begin{vmatrix} D_1 v_1 \\ \frac{1}{2} (D_1 v_2 + D_2 v_1) \\ \frac{1}{2} (D_1 v_3 + D_3 v_1) \end{vmatrix} \]

and

\[ \|D_i v_k\|_{i, k=1}^{3} \equiv \begin{vmatrix} \frac{1}{2} (D_1 v_2 + D_2 v_1) \\ \frac{1}{2} (D_1 v_3 + D_3 v_1) \\ \frac{1}{2} (D_1 v_3 + D_3 v_2) \end{vmatrix}, \quad \|D_i v_k\|_{i, k=1}^{2} \equiv \begin{vmatrix} \frac{1}{2} (D_1 v_2 + D_2 v_1) \\ \frac{1}{2} (D_1 v_3 + D_3 v_1) \\ \frac{1}{2} (D_1 v_3 + D_3 v_2) \end{vmatrix}. \]
for any \((t, x) \in Q^T \equiv (0, T) \times \Omega\).
Therefore we have

\[
(7.3) \quad (B (w, w)) (t, x) \equiv \sum_{j, k=1}^{3} (a_{jk} w_k w_j) (t, x) \equiv \sum_{j=1}^{3} b_j (t, x) \cdot w_j^2 (t, x)
\]

that one can rewrite in the following open form

\[
B (w, w) \equiv \frac{1}{D_1 v_1} \left[ 2 D_1 v_1 w_1 + (D_1 v_2 + D_2 v_1) w_2 + (D_1 v_3 + D_3 v_1) w_3 \right]^2 + \\
\frac{1}{4D_1 v_1^2} \left[ 4 D_1 v_1 D_2 v_2 - (D_1 v_2 + D_2 v_1)^2 \right] \times \\
\left[ (4 D_1 v_1 D_2 v_2 - (D_1 v_2 + D_2 v_1)^2) w_2 + \\
(2 D_1 v_1 (D_2 v_3 + D_3 v_2) - (D_1 v_2 + D_2 v_1) (D_1 v_3 + D_3 v_1)) w_3 \right] + \\
\frac{1}{4} \left[ 4 D_1 v_1 D_2 v_2 D_3 v_3 + (D_1 v_2 + D_2 v_1) (D_1 v_3 + D_3 v_1) (D_2 v_3 + D_3 v_2) - \\
D_1 v_1 (D_2 v_3 + D_3 v_2)^2 - D_2 v_2 (D_1 v_3 + D_3 v_1)^2 - D_3 v_3 (D_1 v_2 + D_2 v_1)^2 \right] w_3^2.
\]

If take account (7.3) in the equation (7.1) then we get

\[
\frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 + \nu \|\nabla w\|_2^2 + \sum_{j=1}^{3} \langle b_j w_j, w_j \rangle = 0, \quad \|w_0\|_2 = 0,
\]
or

\[
(7.4) \quad \frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 = -\nu \|\nabla w\|_2^2 - \sum_{j=1}^{3} \langle b_j w_j, w_j \rangle, \quad \|w_0\|_2 = 0.
\]

This shows that if \(b_j (t, x) \geq 0\) for a.e. \((t, x) \in Q^T\) then the posed problem have unique solution. It is need noted that images of functions \(b_j (t, x)\) and \(D_i v_k\) belong to the bounded subset of the same space.

So, is remains to investigate the cases when the mentioned isn’t fulfill.

Here the following variants are possible:

1. Integral of \(B (w, w)\) is determined and non-negative

\[
\int_{\Omega} B (w, w) \, dx = \sum_{j=1}^{3} \langle b_j w_j, w_j \rangle \equiv \sum_{j=1}^{3} \int_{\Omega} b_j w_j^2 \, dx \geq 0;
\]

In this case one can conclude the main problem have unique solution (and this solution is stable).

2. Integral of is undetermined and \(\sum_{j=1}^{3} \int_{\Omega} b_j w_j^2 \, dx \neq 0\).

In this case for investigation of problem (7.4) it is necessary to derive suitable estimates for \(B (w, w) \equiv \sum_{j, k=1}^{3} (D_i v_k w_k w_j)\).

So, let \(\int_{\Omega} B (w, w) \, dx\) is undetermined. Therefore we need estimate the right part of the equation from (7.4).
\[
\frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 = -\nu \|\nabla w\|_2^2 + \sum_{j,k=1}^{3} (D_i v_k w_k, w_j) \leq \\
(7.5) \quad - \sum_{j=1}^{3} \int_{\Omega} \nu |\nabla w_j (t, x)|^2 \, dx + \sum_{j,k=1}^{3} \int_{\Omega} |(D_i v_k w_k w_j) (t, x)| \, dx,
\]

more precisely, we need estimate the second adding in the right part of (7.5). So, for one of the trilinear terms we obtain\[4\]
\[\|(D_i v_j w_i, w_j)\| \leq \|D_i v_j\|_2 \|w_i\|_{p_1} \|w_j\|_{p_2},\]

with use of the Hölder inequality, where is sufficient to choose, \(p_1 = p_2 = 4\). Consequently, one can estimate \(\int_{\Omega} B (w, w) \, dx\) as follows

\[
\int_{\Omega} |B (w, w)| \, dx \leq \sum_{i,j=1}^{3} \|D_j v_i\|_2 \|w_i\|_4 \|w_j\|_4.
\]

Hence, use Gagliardo-Nirenberg inequality (see, e.g., [30]) we get

\[
\|w_j\|_4 \leq c \|w_j\|_2^{1-\sigma} \|\nabla w_j\|_2^\sigma, \quad \sigma = \frac{3}{4},
\]

where \(c \equiv C (4, 2, 2, 0, 1)\), and for this case

\[
\|w_j\|_4 \leq c \|w_j\|_2^{\frac{3}{2}} \|\nabla w_j\|_2^{\frac{3}{2}} \implies \|w_j\|_4^2 \leq c^2 \|w_j\|_2^{3} \|\nabla w_j\|_2^{\frac{9}{2}}.
\]

Therefore

\[
\int_{\Omega} |B (w, w)| \, dx \leq c^2 \sum_{i,j=1}^{3} \|D_j v_i\|_2 \|w_i\|_2^{\frac{3}{2}} \|\nabla w_i\|_2^{\frac{3}{2}} \|w_j\|_2^{\frac{9}{2}} \|\nabla w_j\|_2^{\frac{3}{2}}
\]

holds. Now taking into account the above estimate in (??) we derive

\[
\frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 \leq - \sum_{j=1}^{3} \nu \|\nabla w_j (t)\|_2^2 + c^2 \sum_{i,j=1}^{3} \|D_j v_i (t)\|_2 \|w_i (t)\|_2^{\frac{3}{2}} \|\nabla w_i (t)\|_2^{\frac{3}{2}} \\
\leq - \sum_{j=1}^{3} \|\nabla w_j (t)\|_2^{\frac{3}{2}} \left[ \nu \|\nabla w_j (t)\|_2^{\frac{1}{2}} - c^2 \sum_{i=1}^{3} \|D_i v_j (t)\|_2 \|w_j (t)\|_2^{\frac{1}{2}} \right] \\
\leq - \sum_{j=1}^{n} \|\nabla w_j (t)\|_2^{\frac{3}{2}} \left[ \nu \lambda_1^{\frac{1}{2}} - c^2 \sum_{i=1}^{n} \|D_i v_j (t)\|_2 \right] \|w_j (t)\|_2^{\frac{1}{2}}.
\]

Whence follows, that if \(\nu \lambda_1^{\frac{1}{2}} \geq c^2 \sum_{i=1}^{n} \|D_i v_j (t)\|_2\) then problem (1.1)-(1.3) has only unique solution (and solution is stable), where \(\lambda_1\) is minimum of the spectrum of the operator Laplace. Thus is proved

\[4\text{It is known that} (2, 7) \quad |(u_k D_iv_j, w_i)| \leq \|u_k\|_q \|D_i v_j\|_2 \|w_i\|_n, \quad n \geq 3; \quad \|v_j\|_4 \leq C (\text{mes} \, \Omega) \|Dv_j\|_2^{\frac{1}{2}} \|v_j\|_2^{\frac{3}{2}}, \quad n = 2\]
Theorem 7. Let $\Omega \in \mathbb{R}^3$ be an open bounded domain of Lipschitz class, $(u_0, f) \in H(\Omega) \times L^2(\Omega)$ and weak solution $u(t, x)$ of problem (1.1) exists and $u \in V(Q^T)$. Then if either $\int_{\Omega} |B(w, w)| \, dx \geq 0$ or $\int_{\Omega} |B(w, w)| \, dx \neq 0$ (is undetermined) and $\nu \lambda_1^\frac{1}{4} \geq c^2 \sum_{i=1}^{3} \|D_i u_j(t)\|^2$ fulfilled then weak solution $u(t, x)$ is unique.

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