Finite- and infinite-volume thermodynamics around the zero of the pressure in deconfining SU(2) Quantum Yang-Mills theory

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Abstract

We re-address the self-intersection region in a figure-eight shaped center-vortex loop containing a frequently perturbed BPS monopole subject to a core-oscillation frequency \( \omega_0 \), rectifying a numerical error in estimating the system’s radius \( r_0 \) in comparison to the spatial coarse-graining scale of infinite-volume thermodynamics. Implications are discussed. We also compute the lowest frequency \( \Omega_0 \) of a spherically symmetric plasma oscillation within a neutral and spatially homogeneous ball-like region of deconfining phase in dependence of its radius \( R_0 \). For \( r_0 = R_0 \) we compare \( \omega_0 \) with \( \Omega_0 \). We point out how the idealisations, which are assumed in this work, will have to be relaxed in order to address the emission of electromagnetic radiation and of non-intersecting as well as self-intersecting center-vortex loops away from the surface region of macroscopically sized plasma balls.
1 Introduction

The phase structure and the deconfining thermal ground state of (electric-magnetic dually interpreted) SU(2) Yang-Mills thermodynamics suggests the existence of a finite-extent particle with a magnetic moment provided by the electric flux along a figure-eight-like configuration of a self-intersecting center-vortex loop, immersed into the confining phase. This soliton is stable, possesses a mass $m_0$ mainly arising from the deconfining energy density of the ball-like, twice-Bohr-radius-$r_0$ sized self-intersection region whose radius $r_0$ is about fifty times larger than the Compton wave length $l_C$. This object exhibits one unit of electric charge, immersed into this deconfining region as a BPS monopole of mass $m_m$ whose core localises to within a radius matching $l_C$, which frequently is relocated and forced to oscillate at frequency $\omega_0$ by the action of a trivial-holonomy caloron/anticaloron in the deconfining thermal ground state. Such a model of the electron (and other charged leptons) concretely represents Louis de Broglie’s ideas on the (quantum) thermodynamics of the isolated particle, based on the important observation that its mass $m_0$ must be related to a time-periodic phenomenon of (circular) frequency $\omega_0$ as $m_0c^2 = \hbar\omega_0 - c$ the speed of light in vacuum, $\hbar$ Planck’s reduced unit of action – in the rest frame of this particle. Note that a Lorentz boost generates out of a spatially flat oscillation a propagating wave, whose (de Broglie) wavelength $\lambda$ relates to the spatial momentum $p$ of the particle as $\lambda = \frac{2\pi\hbar}{p}$, and the increase of rest energy $m_0^2c^2$ by virtue of the boost can be decomposed into a reduction of internal heat plus the mechanical work invested. The reason why this model of the electron complies the particle’s apparent structurelessness of charge distribution, as inferred from scattering experiments with energy-momentum transfers up to several TeV, is the thermal nature of the self-intersection region (maximum entropy), the locus of electric charge essentially being undetermined within the large volume $\sim \frac{4}{3}\pi r_0^3$.

There are implications of this model for high-temperature plasma physics (in considering transport properties in view of the turbulent behaviour induced by instable center-vortex loops with higher self-intersection numbers) and for the physics of strongly correlated electrons in a spatial plane (parts of the correlations intrinsically arising from the non-local magnetic interactions of vortex sectors and the thermal nature of electric charge distribution) underlying, e.g., the phenomenon of high-$T_c$ superconductivity. Both fields of research require long-term efforts, however.

One concern of the present paper is to discuss a few amendments to the above-sketched model of the electron was discussed. In particular, we re-state results on $r_0$ and $m_m$ and the ratio $m_m/m_0$ under the use of the proper (and not erroneous as in the monopole mass formula $m_m = \frac{4\pi}{e(T_0)}H_\infty(T_0)$ with $e(T_0) = 12.96$ the value of the effective gauge coupling at the temperature $T_0 = 1.32 T_c$ ($T_c$ the critical temperature for the deconfining-preconfining phase transition), where the deconfining pressure $P$ vanishes. Here $H_\infty(T_0) = \pi T_0$ denotes the asymptotic modulus of the adjoint Higgs field in the BPS monopole configuration linking it to (anti)caloron dissociation at maximum non-trivial holonomy.

There is no essential change in numerical values up to this point. However, in computing the ratio $\frac{r_0}{|\phi|^4(T_0)}$ a substantial error was committed in: Instead of $\frac{r_0}{|\phi|^4(T_0)} \sim 160$ the actual value turns out to be $\frac{r_0}{|\phi|^4(T_0)} \sim 0.104$ which corresponds to $\frac{r_0}{T_0} \sim 1.29$. (The
radius $r_0$ is denoted as $R_0$ in [4].) Therefore, it is incorrect to state, as was done in [4], that the self-intersection region of radius $r_0$ represents infinite-volume thermodynamics. Rather, this region is deeply contained within the center of the accommodating caloron or anticaloron, and, judged by the partial attainment of the asymptotic harmonic time dependence of $\phi$ when integrating the according field-strength correlator on a caloron or anticaloron in singular gauge over the instanton-scale parameter [14], the use of infinite-volume thermodynamics can thus be considered an approximation only.

The other objective of our present work is a discussion of an extended, neutral plasma ball at $T_0$, immersed into the confining phase, where infinite-volume thermodynamics is well saturated in the bulk (disregarding the non-thermal situation within a thin shell of preconfining/turbulent confining phase leading to surface-tension effects) for a sufficiently large ball radius $R_0$. In particular, we ask what the lowest frequency $\Omega_0$ of a spherically symmetric breathing mode for homogeneous plasma oscillations about $T_0$ is in dependence of $R_0$ and compare this with $\omega_0 = m_0 c^2/\hbar$. This requires the determination of the longitudinal sound speed $c_s$ at $T_0$ which turns out to be $c_s = 0.479 c$. Note that, in spite of the vanishing pressure at $T_0$, exhibiting a cancellation of the contributions from the thermal ground state and its (partially massive) excitations, this is not much below the ultrarelativistic-gas limit $c_s = \frac{1}{\sqrt{3}} c \sim 0.577 c$.

This paper is organised as follows. In Sec. 2 we re-address the computation of $r_0$ using the proper expression for the monopole mass $m_m$. Qualitatively, the value of $r_0$ is not affected in comparison to the one obtained in [4]. We also compute the ratio of $m_m$ to $m_0$. However, due to a calculational error the ratio of $r_0$ to the coarse-graining scale $|\phi|^{-1}$ turns out to be much smaller than what was found in [4] and what was taken as a proof that infinite-volume thermodynamics is valid at face value. We argue, however, that the ratio $r_0/T_0^{-1}$ permits an approximately thermodynamical treatment because the asymptotic harmonic time dependence of the integral over the field-strength correlator, rendering the Euclidean time dependence of field $\phi$ a mere gauge choice [14], is reasonably well approached when cutting off the instanton-scale-parameter integral at $r_0$. Sec. 3 we address a situation where the deconfining, homogeneous plasma around temperatures close to $T_0$ is void of an explicit monopole but undergoes spherically symmetric, oscillatory breathing by virtue of a finite longitudinal sound velocity $c_s$. We reliably compute the square of $c_s$ by appealing the derivatives w.r.t. temperature of the pressure and the energy density, employing the evolution equation for the mass of the off-Cartan modes. Both situations, monopole driven oscillation and spherically symmetric breathing are compared in terms of their frequencies, the former exhibiting a much more rapid oscillation than the latter. Finally, in Sec. 4 we summarise and discuss our results and provide an outlook on future work.

2 Self-intersection region of a figure-eight shaped center-vortex loop

In [4] we have proposed a model of the free electron, based on the phase structure of SU(2) Yang-Mills thermodynamics and the work in [1, 2], investigating the response of a BPS monopole to a spherically symmetric initial perturbation and the spectrum of normal modes. Here we would like to correct some numerical statements arising from an incorrect
monopole mass formula (Eq. (18) of \[4\]). Also, we point out an error in Eq. (21) of \[4\], entailing a conceptual re-interpretation of the physics associated with the self-intersection region. From now on we work in super-natural units: \(c = \hbar = k_B = 1\) where \(k_B\) denotes Boltzmann’s constant.

The mass \(m_m\) of a BPS monopole is given as \[15\]

\[
m_m = \frac{4\pi}{e} H_\infty ,
\]

(1)

where \(e\) denotes the defining gauge coupling of the adjoint Higgs model and \(H_\infty\) the spatially asymptotic modulus of the Higgs field. Assuming that the monopole was liberated by the dissociation of a maximum-holonomy caloron at \(T_0 = 1.32 T_c\), the temperature where the thermodynamical (one-loop) pressure vanishes, we have \[14, 13\]

\[
e(T_0) = 12.96, \quad H_\infty(T_0) = \pi T_0 .
\]

(2)

The electron’s rest mass \(m_0\) is, on one hand, given by the circular frequency \(\omega_0\) of monopole-core oscillation, found to be equal to the mass of the two off-Cartan modes in \[1, 2\]

\[
\omega_0 = eH_\infty .
\]

(3)

On the other hand, \(m_0\) decomposes into \(m_m\) and the energy contained within an approximately ball-like region of radius \(r_0\) (the contribution of the two vortex loops in negligible \[14\], \(r_0\) is denoted as \(R_0\) in \[4\]) due to the deconfining plasma of energy density \(\rho(T_0)\)

\[
m_0 = 12.96 H_\infty(T_0) = m_m + \frac{4\pi}{3} r_0^3 \rho(T_0)
\]

\[
= H_\infty(T_0) \left( \frac{4\pi}{12.96} + 8.31 \times \frac{128\pi}{3} \left( \frac{r_0}{18.31} \right)^3 H_\infty^3(T_0) \right) ,
\]

(4)

where \(\lambda_0 \equiv \frac{2\pi T_0}{\Lambda}\), and \(\Lambda\) denotes the Yang-Mills scale, related to \(m_0\) as \[4\]

\[
\Lambda = \frac{1}{118.6} m_0 .
\]

(5)

Solving Eq. (4) for \(r_0\) yields

\[
r_0 = 4.043 H_\infty^{-1}
\]

(6)

instead of \(r_0 = 4.10 H_\infty^{-1}\) as obtained in \[4\]. It is instructive to compute the relative contribution of \(m_m\) to \(m_0\) and the ratio of Compton wave length \(l_C = m_0^{-1} = \frac{1}{12.96 H_\infty(T_0)}\) \[1, 2\] to \(r_0\)

\[
\frac{m_m}{m_0} = \frac{4\pi}{(12.96)^2} = 0.0748, \quad \frac{l_C}{r_0} = \frac{1}{52.40} .
\]

(7)

Moreover, radius \(r_0\) compares to the spatial coarse-graining scale \(|\phi|^{-1}(T_0)\), turning the Euclidean time dependence of the field strength correlation within a caloron or anticaloron center into a mere choice of gauge \[14\], as follows

\[
\frac{r_0}{|\phi|^{-1}(T_0)} = \frac{4.043}{\sqrt{2 \left( \frac{118.6}{12.96} \right)^3}} = 0.1033 .
\]

(8)
Due to a simple calculational error in Eq. (22) of [4] this result is qualitatively different and implies that the “thermodynamics” we discussed so far actually occurs deep within the center of a caloron or anticaloron of a scale parameter $\rho \sim |\phi|^{-1}(T_0)$ being appropriate in the infinite-volume situation. Since

$$\frac{r_0}{\beta_0} = 1.29 \quad (\beta_0 = \frac{1}{T_0})$$

Fig. 1 suggests that the scale parameter integral, defining the phase of $\phi$ [14], does not quite saturate into a harmonic time dependence when cut off at $\rho \sim r_0$. Therefore, our above discussion, which assumes that such a saturation occurs within the volume $\frac{4\pi}{3}r_0^3$, is only an approximate account of the quantum physics within the self-intersection region: the monopole is always close to the locus of action at the inmost point of the caloron or anticaloron, rendering this region a highly jittery object, not only concerning the spherical shell of preconfining/confining phase that represents the boundary of the ball but also its deep bulk. Still, the thermodynamical approximation employed here should yield reasonable estimates of all spatial scales and the critical temperature $T_c = 13.87/(2\pi \times 118.6) \ m_0 = 9.49 \text{ keV}$ [4].

3 Lowest spherically symmetric breathing mode

Let us now address a situation, where thermodynamics is applicable without any restrictions. Namely, we would like to compute the frequency $\Omega_0$ of the lowest spherically symmetric breathing mode of a deconfining SU(2) Yang-Mills plasma ball of homogeneous energy density $\rho$ and pressure $P$, oscillating about temperature $T_0$. Surface effects, arising from the transition between the deconfining (bulk) and the confining (exterior of ball) phases can be neglected for a sufficiently large ball mass $M \equiv \frac{4}{3}\pi R_0^3 \rho(R_0)$, and the according expression for $\Omega_0$ reads [15, 17]

$$\Omega_0 = \frac{\pi c_s \lambda}{R_0},$$
where the dimensionless quantities $c_s$ (longitudinal sound velocity) and $\bar{R}_0$ ($R_0$ in units of the inverse Yang-Mills scale $\Lambda^{-1}$) are defined as

$$c_s^2 \equiv \left. \frac{d\bar{P}}{d\bar{\rho}} \right|_{\lambda = \lambda_0}$$

and $\bar{R}_0 \equiv R_0\Lambda$. In Eq. (11) the use of the one-loop pressure $P \equiv \Lambda^4 \hat{P}$ and of the one-loop energy density $\rho \equiv \Lambda^4 \hat{\rho}$ are excellent approximations [14], and we define the dimensionless temperature as $\lambda \equiv \frac{2\pi T}{\Lambda}$. Amusingly, an estimate of $\Omega_0$ by virtue of a linearisation of the force-balance equation

$$\frac{4}{3} \pi \bar{R}^3 \rho(R) \ddot{\bar{R}} = 4 \pi \bar{R}^2 P(R)$$

about $R_0$ and employing energy conservation,

$$\bar{R} \equiv R\Lambda = \left( \frac{3\bar{M}}{4\pi \bar{\rho}(R)} \right)^{1/3} \quad (\bar{M} \equiv \Lambda M),$$

replaces the factor of $\pi$ in Eq. (10) by a factor of three. For $\hat{P}$ and $\hat{\rho}$ we have [14]

$$\hat{P}(2a, \lambda) \equiv -\frac{2\lambda^4}{(2\pi)^6} \left[ 2\hat{P}(0) + 6\hat{P}(2a) \right] - 2\lambda,$$

$$\hat{\rho}(2a, \lambda) \equiv \frac{2\lambda^4}{(2\pi)^6} \left[ 2\hat{\rho}(0) + 6\hat{\rho}(2a) \right] + 2\lambda,$$

$$a = a(\lambda) \equiv 2\pi e(\lambda)\lambda^{-3/2} \quad (e(\lambda_0) = 12.96),$$

where

$$\hat{P}(y) \equiv \int_0^\infty dx x^2 \log \left[ 1 - \exp \left( -\sqrt{x^2 + y^2} \right) \right],$$

$$\hat{\rho}(y) \equiv \int_0^\infty dx x^2 \frac{\sqrt{x^2 + y^2}}{\exp \left( \sqrt{x^2 + y^2} \right) - 1},$$

Taking into account implicit (via $a(\lambda)$) and explicit dependences of $\hat{P}$ and $\hat{\rho}$ on $\lambda$ and employing the evolution equation [13]

$$1 = -\frac{24\lambda^3}{(2\pi)^6} \left( \lambda \frac{da}{d\lambda} + a \right) D(2a),$$

one derives

$$\frac{d\hat{P}}{d\lambda} = -\frac{\lambda^3}{(2\pi)^6} \left( 16\hat{P}(0) + 48(\hat{P}(2a) - a^2 D(2a)) \right),$$

$$\frac{d\hat{\rho}}{d\lambda} = \frac{\lambda^3}{(2\pi)^6} \left( 16\hat{\rho}(0) + 48(\hat{\rho}(2a) - a^2 (D(2a) - F(2a))) \right) + 2 \left( 1 - \frac{D(2a) - F(2a)}{D(2a)} \right),$$

where

$$D(y) \equiv \int_0^\infty dx \frac{x^2}{\sqrt{x^2 + y^2}} \frac{1}{\exp \left( \sqrt{x^2 + y^2} \right) - 1},$$

$$F(y) \equiv \int_0^\infty dx x^2 \frac{\exp \left( \sqrt{x^2 + y^2} \right)}{\left( \exp \left( \sqrt{x^2 + y^2} \right) - 1 \right)^2}.$$
Substituting Eqs. (16) into Eq. (11) at \(\lambda_0 = 18.31\), we numerically obtain

\[ c_s(\lambda_0) = 0.479. \]  

For Eq. (10) this yields

\[ \Omega_0 = 1.506 \frac{\Lambda}{R_0}. \]  

Let us now compare the monopole-core induced frequency \(\omega_0\) of the self-intersection region of the figure-eight shaped center-vortex loop (model of the electron) with \(\Omega_0\) at one and the same radius

\[ r_0 = R_0 = 4.043 H^{-1}(T_0), \]  

see Eq. (6). For \(\Omega_0\) this yields

\[ \Omega_0 = 0.372 H_\infty(T_0) \]  

such that \(\frac{\omega_0}{\Omega_0} = 34.84\). This large ratio is suggestive since the oscillation in the self-intersection region – quantum initiated by caloron or anticaloron action – is induced by the classical dynamics of a monopole core \([1, 2]\) whose size matches the Compton wave length \(l_C\) while the lowest symmetric breathing mode of the neutral deconfining ball is a consequence of sound propagation in bulk thermodynamics, spatially supported by a much larger system of Bohr-radius size.

Eq. (19) is the more reliable the larger \(\bar{R}_0\) is. Isotropy breaking effects, which associate with the neglected surface dynamics of the ball and/or the excitation of spherically non-symmetric oscillation states, cause this surface to (electromagnetically) radiate with a spectrum that is dominated by frequencies around \(\nu_0 \sim \frac{\Omega_0}{2\pi}\), corresponding to a wave length \(l_0 = \frac{1}{\nu_0} \sim 2\pi R_0 \frac{1}{1.506}\).

4 Summary and Discussion

This paper’s purpose was to compare two situations in which a ball-like region of deconfining phase in SU(2) Yang-Mills thermodynamics oscillates about the zero of the pressure at temperature \(T_0\): the self-intersection region of a figure-eight shaped, solitonic center-vortex loop (a model of the electron) containing a frequently perturbed BPS monopole, whose classical core dynamics drives this oscillation of (circular) frequency \(\omega_0\) (up to a factor \(\hbar\) coincident with the rest energy \(m_0 c^2\) of the soliton \([3, 6]\)). Furthermore, we have considered a homogeneous, electrically neutral region whose lowest spherically symmetric oscillatory excitation of (circular) frequency \(\Omega_0\) is supported by a finite speed of sound \(c_s\), in turn based on the thermal ground state’s quantum excitations which are microscopically mediated by the unit of action \(\hbar\) localised within the center of a caloron/anticaloron \([14]\). At the same radius, \(r_0 = R_0 = 4.043 H^{-1}(T_0)\), we obtain \(\frac{\omega_0}{\Omega_0} = 34.84\) which makes explicit the very different cause of oscillation in either case.

We have noticed a numerical error in \([4]\) concerning an estimate of the system size \(r_0\) in terms of the spatial coarse-graining scale \(|\phi|^{-1}\). The correct result states that finite-size effects cannot be excluded at face value since the region of self-intersection actually is contained deeply within the ball of spatial coarse-graining corresponding to the infinite-volume situation. However, the asymptotic harmonic time dependence of the integrated field strength correlation \([14]\), required for the introduction of the field \(\phi\), is approached
to some extent when cutting the instanton-scale-parameter integration off at $r_0$ such that, approximately, one can still rely on infinite-volume thermodynamics.

The present work only represents a first step in studying the plasma dynamics of a ball-like region of deconfining phase at $T_0$. More realistically, the physics of a certain boundary shell should be taken into account. The preconfining-phase part of this shell is superconducting (condensate of electrically charged, massless monopoles) [14] with implications for the stabilisation of the highly turbulent gas of self-intersecting vortex loops (electrically charged particles) within a region externally adjacent to this shell. Also, we did not address the evaporation physics (emission of non-intersecting and self-intersecting center-vortex loops from the surface of this shell) in case of macroscopically sized balls, see [19], and how this process affects the oscillation dynamics and electromagnetic emission. Our results on the spherically symmetric oscillations of the homogeneous and macroscopic plasma could be relevant for models of Cepheid variables beyond the usual kappa mechanism of star dynamics and the description of certain, quasi-stabilised, compact and radiating objects created within atmospheric discharges. It is not unlikely to provide for a conceptual impact on plasma stabilisation in terrestrial fusion experiments.

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