The Angle Along a Curve and Range-Kernel Complementarity

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Abstract. We define the angle of a bounded linear operator $A$ along a curve emanating from the origin and use it to characterize range-kernel complementarity. In particular we show that if $\sigma(A)$ does not separate 0 from $\infty$, then $X = R(A) \oplus N(A)$ if and only if $R(A)$ is closed and some angle of $A$ is less than $\pi$. We first apply this result to invertible operators that have a spectral set that does not separate 0 from $\infty$. Next we extend the notion of angle along a curve to Banach algebras and use it to prove two characterizations of elements in a semisimple and in a $C^*$ commutative algebra respectively, whose spectrum does not separate 0 from $\infty$.

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1. Introduction

Let $X$ be a Banach space and $A : X \to X$ be a bounded linear operator. We will denote the range of $A$ by $R(A)$ and the kernel of $A$ by $N(A)$. The cosine of $A$ with respect to a semi-inner product $\langle \cdot, \cdot \rangle$ compatible with the norm of $X$, is defined by

$$\cos A = \inf \left\{ \frac{\Re [Ax, x]}{\|Ax\| \|x\|} : x \notin N(A) \right\}.$$  \hspace{1cm} (1)

Using this one can define the angle $\phi(A)$ of the linear operator $A$ by

$$\phi(A) = \arccos(\cos A).$$ \hspace{1cm} (2)

The angle $\phi(A)$ of $A$ has a geometric interpretation; it measures the maximum turning effect of $A$ along the positive real axis. This concept was introduced by Gustafson in [5] and independently by Krein in [8].
A moment’s thought reveals that $\phi(A)$ is just one of the many existing angles of the operator $A$. Indeed if $\theta \in [0, 2\pi]$, then $\phi(e^{i\theta}A)$ measures the maximum turning effect of $A$ along the ray emanating from the origin

$$\rho_\theta = 0 \cup \{z \in \mathbb{C} : \text{arg } z = \theta\}.$$ 

Since $\phi(e^{i\theta}A)$ measures the maximum turning effect of $A$ along $\rho_\theta$, it should be quite expected that it equals $\pi$ whenever $\rho_{\pi-\theta}$ contains an eigenvalue of $A$ or more generally a point in its approximate point spectrum. Note that $\phi(e^{i\theta}A)$ is related to points in the boundary of the normalized numerical range of $A$ and hence proofs of the aforementioned facts in the Hilbert space case may be found in [12, Proposition 1.2] and [4, Proposition 7].

Related to the above “abundance” of angles of $A$, is the so called “amplitude angle”, also introduced by Krein in [8], defined by

$$\text{am}(A) = \min \{ \phi(e^{i\theta}A) : \theta \in [0, 2\pi] \}.$$ 

The “amplitude angle” of $A$ compares the maximum turning effect of $A$, along every ray emanating from the origin, and provides us with the smallest one.

Note that if 0 lies in a hole of the spectrum of $A$, then every ray emanating from the origin meets the boundary of the spectrum which, as it is well-known, is contained in the approximate point spectrum. Hence every ray contains a point of the approximate point spectrum which, by what we said before, implies that $\text{am}(A) = \pi$ (for details and the general case see Proposition 4.2). This observation tells us that in cases where the spectrum separates 0 from $\infty$ all angles of $A$ are equal to $\pi$.

One could say that quasinilpotent operators, regarding their spectrum, are the complete opposite of operators whose spectrum separates 0 from $\infty$. Nevertheless the amplitude angle of a non-zero quasinilpotent operator is also equal to $\pi$. The reason here is not so apparent but still quite simple: if $A$ is a non-zero nilpotent operator, then $R(A) \cap N(A) \neq \{0\}$ and hence as we will see in Lemma 4.1 $\text{am}(A) = \pi$. For a quasinilpotent, but not nilpotent, operator the same conclusion follows by Theorem 4.1 and the fact that, as it was shown in [3], its range chain does not terminate. For the role quasinilpotent operators play in range-kernel complementarity see Remark 4.3.

Range-kernel complementarity i.e. the decomposition

$$X = R(A) \oplus N(A),$$

is the closest thing to the invertibility of $A$, since if (3) holds then $A$ is of the form “invertible $\oplus 0$”.

In finite dimensions (3) is equivalent to

$$R(A) \cap N(A) = \{0\},$$

which in turn is equivalent to the ascent (the length of the null-chain) of $A$ being lesser or equal to one. In infinite dimensions things are significantly different as (4) is no longer sufficient and one needs the additional assumption that $R(A) = R(A^2)$. Note that the latter is equivalent to the descent (the length of the range chain) of $A$ being lesser or equal to one. For other equivalent conditions to range-kernel complementarity, some of which we will use in the last section of this paper, we refer the interested reader to [9].
In [2, Theorem 3.4 and Remark 3.5 (vi)] it was shown that if \( \text{am}(A) < \pi \) and both \( R(A) \) and \( R(A) + N(A) \) are closed then (3) holds. The converse of this result is not true in general and the reason is simple: as is mentioned by M. Krein in [8] \( \text{am}(A) < \pi \) implies that, for some \( \theta \in [0, 2\pi] \), the spectrum of \( e^{i\theta}A \) is contained in the sector

\[
S = 0 \cup \{ z \in \mathbb{C} : |\arg z| \leq \phi(e^{i\theta}A) \}.
\]

Hence \( \text{am}(A) < \pi \) would imply that \( \sigma(A) \) is contained in the sector \( e^{-i\theta}S \) which does not cover the general case of an operator for which range-kernel complementarity holds in an infinite dimensional Banach space.

On the other hand having in mind that if 0 is in a hole of \( \sigma(A) \), then all angles are equal to \( \pi \) and invertible operators (a characteristic example being the bilateral shift) may have this property, one has to look for a compromise: does the converse hold provided the spectrum does not separate 0 from \( \infty \) (0 is not in a hole) and at the same time it is not confined in a sector of the complex plane? In other words is the converse true if instead of rays one allows curves “escaping” from \( \sigma(A) \)? As we will see the answer to this question turns out to be affirmative. To show that we define the angle of an operator \( A \) along a curve emanating from the origin and prove that if \( \sigma(A) \) does not separate 0 from \( \infty \), range-kernel complementarity holds if and only if \( R(A) \) is closed and some such angle of \( A \) is less than \( \pi \).

For the definition of such an angle the semi-inner product approach (through the cosine) is not suitable and so a quite natural “sine” along a curve for \( A \) had to be defined using the norm. Note that in a Hilbert space our sine with the cosine of (1) are compatible for rays emanating from the origin, as they satisfy the basic trigonometric identity.

The first application of our result deals with an invertible operator \( T \) that has a spectral set that does not separate 0 from \( \infty \). We show that if \( P \) is the spectral projection associated to this spectral set then there exists a curve emanating from the origin along which the angle of \( PT \) is less than \( \pi \). Next we define the angle along a curve of an element of a Banach algebra and use it to prove two characterizations: the first one is for elements in a semisimple commutative Banach algebra, whose spectrum does not separate 0 from \( \infty \), and 0 is an isolated point of their spectrum (or does not belong to it). The second is for elements in a commutative \( C^* \)-algebra, whose spectrum does not separate 0 from \( \infty \).

2. Preliminaries

The ascent \( \alpha(A) \) of \( A \) is the smallest positive integer \( k \) for which the null-chain of \( A \) stabilizes i.e. \( N(A^k) = N(A^{k+1}) \). If no such integer exists, then \( \alpha(A) = \infty \). The descent \( \delta(A) \) of \( A \) is the smallest positive integer \( k \) for which the range chain stabilizes i.e. \( R(A^k) = R(A^{k+1}) \). Again if no such integer exists, then \( \delta(A) = \infty \). It is well known, see for example [7, Proposition 38.4], that \( X = R(A) \oplus N(A) \) if and only if both \( \alpha(A) \) and \( \delta(A) \) are \( \leq 1 \).
Let \(d(x, N(A))\) be the distance of \(x\) from \(N(A)\). If \(A\) has closed range, then there exists \(M > 0\) such that
\[
\|Ax\| \geq Md(x, N(A)) , \text{ for all } x \in X .
\] (5)

We say that a compact subset \(K\) of the complex plane separates 0 from \(\infty\) if there is no continuous curve joining 0 to \(\infty\) that doesn’t intersect \(K\), except possibly at 0.

By \(\partial \sigma(A)\) and \(\sigma_{\text{app}}(A)\) we denote the boundary of the spectrum of the operator \(A\) and its approximate point spectrum respectively. It is well-known that
\[
\partial \sigma(A) \subseteq \sigma_{\text{app}}(A) .
\]

3. The Angle Along a Curve

3.1. Some Motivation

Let \(H\) be a complex Hilbert space. If \(x, y \in H\) are non-zero vectors then the number
\[
\theta(x, y) = \arccos \frac{\text{Re} \langle x, y \rangle}{\|x\| \|y\|}
\]

is usually called the angle between \(x\) and \(y\) (see for example [8]). Note that
\[
\inf_{\lambda \geq 0} \frac{\|x + \lambda y\|}{\|x\|} = 1
\]
is equivalent to \(\theta(x, y)\) being acute. Moreover it can be easily seen that if \(\frac{\pi}{2} \leq \theta(x, y) \leq \pi\), then
\[
\left(\inf_{\lambda \geq 0} \frac{\|x + \lambda y\|}{\|x\|}\right)^2 + \left(\frac{\text{Re} \langle x, y \rangle}{\|x\| \|y\|}\right)^2 = 1 .
\] (6)

Hence taking
\[
\inf_{\lambda \geq 0} \frac{\|x + \lambda y\|}{\|x\|}
\]
as our starting point we could define \(\theta(x, y)\) as
\[
\pi - \arcsin \inf_{\lambda \geq 0} \frac{\|x + \lambda y\|}{\|x\|}
\]
if the infimum is less than 1. Note that taking into account that \(H\) is a complex Hilbert space a more appropriate name for \(\theta(x, y)\) would be “the angle between \(x\) and \(y\) along the positive axis”.

3.2. The Angle Along a Curve

For the rest of this paper \(C\) will be an unbounded rectifiable curve in the complex plane emanating from the origin. If \(X\) is a complex Banach space and \(x, y \in X\), with \(x \neq 0\) then we define \(s_C(x, y)\) by
\[
s_C(x, y) = \inf_{\lambda \in C} \frac{\|x + \lambda y\|}{\|x\|} .
\]
Considering the above discussion we say that the angle \( \theta_C(x, y) \) of \( x, y \) along \( C \) is equal to \( \pi - \arcsin s_C(x, y) \) if \( s_C(x, y) < 1 \).

If \( s_C(Ax, x) < 1 \), for at least one \( x \in X \), then we define the sine of the operator \( A \) along \( C \) to be

\[
\sin_C A = \inf_{x \notin N(A)} s_C(Ax, x).
\]

If \( C \) is the positive axis instead of \( s_C(Ax, x) \) and \( \sin_C A \) we will just write \( s(Ax, x) \) and \( \sin A \).

It is worth noticing, although we are not going to need it in the sequel, that in a Hilbert space and if \( A \) is non-accretive, then \( \sin A \) together with \( \cos A \) defined by (1) satisfy the classical trigonometric identity. Recall that \( A \) is called accretive if

\[
\inf_{\lambda \geq 0} \frac{\|Ax + \lambda x\|}{\|Ax\|} = 1, \quad \text{for all } x \notin N(A)
\]

and in a Hilbert space \( X A \) is accretive if and only if \( \Re \langle Ax, x \rangle \geq 0, \) for all \( x \in X \). In particular we have the following.

**Proposition 3.1.** If \( X \) is a Hilbert space and \( A \in B(X) \) is not accretive, then

\[
\sin^2 A + \cos^2 A = 1,
\]

where \( \cos A \) is defined by (1).

**Proof.** If \( A \) is not accretive then the set

\[
M = \{x \notin N(A) : \Re \langle Ax, x \rangle < 0\} = \{x \notin N(A) : s(Ax, x) < 1\}
\]

is nonempty and obviously

\[
\sin A = \inf_{x \in M} s(Ax, x).
\]

If

\[
c(Ax, x) = \frac{\Re \langle Ax, x \rangle}{\|Ax\| \|x\|}, \quad \text{for } x \notin N(A).
\]

we have that

\[
\sin^2 A = \inf_{x \in M} s(Ax, x)
= \inf_{x \in M} \left[1 - c(Ax, x)^2\right]
= 1 - \sup_{x \in M} c(Ax, x)^2
= 1 - \left(\inf_{x \in M} c(Ax, x)\right)^2
\]

where second equality is due to (6) and the last because \( c(Ax, x) < 0, \) for all \( x \in M \). Hence

\[
\sin^2 A + \cos^2 A = 1.
\]

\( \Box \)
Remark 3.1. In [6, Theorem 3.2-1] it was shown by Gustafson that if $A$ is strongly accretive then another sine satisfying the basic trigonometric identity may be defined as $\min_{\varepsilon > 0} \| \varepsilon A - I \|$. We may now define the angle of $A$ along a curve.

Definition 3.1. Let $C$ be a curve emanating from the origin. The angle of $A$ along $C$ is

$$\phi_C(A) = \pi - \arcsin \sin C A,$$

if $s_C(Ax, x) < 1$, for at least one $x \in X$ and acute otherwise.

Remark 3.2. Note that although the above definition misses completely the case where all angles of $A$ along $C$ are acute, it is totally adequate for our purposes since what we will actually need is that the maximum turning effect of $A$ along $C$ is less than $\pi$ (and an acute angle obviously satisfies it).

For rays emanating from the origin and obtuse angles in a Hilbert space our definition coincides with the one given by (2).

Example. If $X$ is a Hilbert space, $\rho_\theta$ is a ray emanating from the origin and $\phi(e^{i(\pi - \theta)}A) > \pi$, then $\phi_{\rho_\theta}(A) = \phi(e^{i(\pi - \theta)}A)$,

where the angle $\phi(e^{i(\pi - \theta)}A)$ is the one defined by (2).

4. Results

Our starting point is the following Lemma.

Lemma 4.1. Let $A : X \to X$ be a bounded linear operator. If $\phi_C(A) < \pi$ for some curve $C$, then

$$\overline{R(A)} + N(A)$$

is closed and $R(A) \cap N(A) = \{0\}$.

Proof. If $\phi_C(A) < \pi$ then there exists $0 < c \leq 1$ such that

$$\|Ax + \lambda x\| \geq c\|Ax\|,$$

for all $x \in X$ and all $\lambda \in C$.

Let $x \in X$ and $y \in N(A)$. Since $C$ emanates from the origin there exists a sequence $(\lambda_n)$ of points in $C$ with $\lambda_n \to 0$, as $n \to \infty$ and the above inequality implies that

$$\left\| A \left( x + \frac{y}{\lambda_n} \right) + \lambda_n \left( x - \frac{y}{\lambda_n} \right) \right\| \geq c \left\| A \left( x + \frac{y}{\lambda_n} \right) \right\|,$$

hence

$$\|Ax + \lambda_n x + y\| \geq c\|Ax\|,$$

for all $n \in \mathbb{N}$.

Letting $n \to +\infty$ we get

$$\|Ax + y\| \geq c\|Ax\|,$$

(7)
for all $x \in X$, $y \in N(A)$ and thus $\overline{R(A)} + N(A)$ is closed. Moreover if $Ax \in N(A)$, then (7) implies that
\[ \|Ax - Ax\| \geq c\|Ax\| \]
and hence $Ax = 0$. \hfill \square

**Remark 4.1.** If we denote by $P$ the projection onto $\overline{R(A)}$ parallel to $N(A)$ under the assumption of Lemma 4.1, then $c \leq \|P\|^{-1}$. Hence if $\varphi = \arcsin \|P\|^{-1}$ is the angle between $R(A)$ and $N(A)$ we have that $\varphi = \pi/2$, if the angle of $A$ is acute and $\varphi \geq \pi - \phi C(A)$ otherwise. Note that $\varphi = \pi/2$ means that $R(A)$ is Birkhoff-James orthogonal to $N(A)$.

In what follows by $D_\infty$ we denote the unbounded component of the resolvent set of $A$. We will use the following result which is a corollary of the so called “filling the hole” theorem.

**Proposition 4.1.** [10, Theorem 0.8] If $M$ is a closed invariant subspace of $A$, then
\[ \sigma(A|_M) \cap D_\infty = \emptyset. \]

**Remark 4.2.** For an alternative proof of Proposition 4.1 see [1].

Note that the fact that $\sigma(A)$ does not separate 0 from $\infty$, i.e. the existence of the unbounded curve emanating from the origin, is equivalent to $0 \in \overline{D_\infty}$. Our main result is the following.

**Theorem 4.1.** Let $A : X \to X$ be a bounded linear operator and assume that $\sigma(A)$ does not separate 0 from $\infty$. Then
\[ X = R(A) \oplus N(A) \]
if and only if $R(A)$ is closed and $\phi C(A) < \pi$, for some $C$.

**Proof.** If $X = N(A) \oplus R(A)$ then there exists $\delta > 0$ such that
\[ \|Ax + y\| \geq \delta\|Ax\| \text{, for all } x \in X, y \in N(A). \]  
By Proposition 4.1 we have that
\[ \sigma(A|_{R(A)}) \cap D_\infty = \emptyset. \]
Hence $D_\infty$ lies in the resolvent of $A|_{R(A)}$ and thus there exists an unbounded curve $C_\infty$ emanating from the origin with
\[ C_\infty \subseteq \rho(A|_{R(A)}). \]
Since $A|_{R(A)}$ is invertible we claim that there exists $k > 0$ such that
\[ \|Ax - \lambda x\| \geq k\|x\|, \text{ for all } x \in R(A), \lambda \in C_\infty. \]  
Assume the contrary; i.e., that there exists a sequence $(\lambda_n)$ in $C_\infty$ and a sequence $(x_n)$ in $R(A)$, with $\|x_n\| = 1$, such that $\|Ax_n - \lambda_n x_n\| \to 0$, as $n \to \infty$. Then since $(\lambda_n)$ is bounded it has a subsequence, which for simplicity we denote again by $(\lambda_n)$, that converges to some $\lambda_0 \in C_\infty$. But then
\[ \|Ax_n - \lambda_0 x_n\| \leq \|Ax_n - \lambda_n x_n\| + |\lambda_n - \lambda_0| \]
and hence \( \|Ax_n - \lambda_0 x_n\| \to 0 \), as \( n \to \infty \), which is a contradiction since \( \lambda_0 \in \rho(A|_{R(A)}) \).

Let \( C = -C_\infty \) and we will show that \( \phi_C(A) < \pi \).

To this end let \( x \in X \). By hypothesis \( x = z + y \), with \( z \in R(A) \) and \( y \in N(A) \). Using (8) and (9) we have that for all \( \lambda \in C \)

\[
\|Ax + \lambda x\| = \|Az + \lambda z + \lambda y\| \geq \delta \|Az + \lambda z\| \geq \delta k \|z\| \geq \frac{\delta k}{\|A\|} \|Az\| = c \|Ax\|.
\]

Hence \( s_C(Ax,x) \geq c \), for all \( x \notin N(A) \) and so \( \phi_C(A) < \pi \).

Conversely, assume that \( \phi_C(A) < \pi \), for some \( C \). Then by Lemma 4.1 we have that \( R(A) \cap N(A) = \{0\} \) and hence the ascent \( \alpha(A) \) of \( A \) is lesser or equal to 1. To conclude the proof we have to show that the descent \( \delta(A) \) is finite. In particular we will show that \( R(A^2) = R(A) \).

By Lemma 4.1, using the fact that \( R(A) \) is closed, we have that \( R(A) + N(A) \) is a closed subspace of \( X \) which implies that \( R(A^2) \) is also closed. Hence since \( R(A) \cap N(A) = \{0\} \) by (5) we have that

\[
\|Ax\| \geq \|x\|, \text{ for all } x \in R(A).
\]

(10)

If \( 0 \in \sigma(A|_{R(A)}) \) and since by Proposition 4.1 we have that

\[
\sigma(A|_{R(A)}) \cap D_\infty = \emptyset
\]

we get that \( 0 \notin \partial\sigma(A|_{R(A)}) \subseteq \sigma_{app}(A|_{R(A)}) \) which contradicts (10).

So

\[
0 \notin \sigma(A|_{R(A)})
\]

and the proof is complete. \( \square \)

Remark 4.3. There is an important connection between range-kernel complementarity and quasinilpotency. In particular if \( 0 \) is an isolated point in the spectrum of \( A \), then by the Riesz decomposition \( X = M \oplus N \), where both \( M, N \) are invariant under \( A \) with \( 0 \notin \sigma(A|_M) \) and \( \sigma(A|_N) = \{0\} \).

Since \( M \subseteq R(A) \) and \( N(A) \subseteq N \) we have that a sufficient condition for range-kernel complementarity is that the quasinilpotent operator \( A|_N \) is 0 and range-kernel complementarity fails if and only if \( A = U \oplus S \), with \( S \neq 0 \) quasinilponent. Hence, as we have already mentioned in the Introduction, all angles of a quasinilpotent operator are equal to \( \pi \).

Recall that a hole in a compact subset of the complex plane is a bounded connected component of its complement. Using the above we can show that if we have range-kernel complementarity then, as we have already mentioned in the Introduction, \( \sigma(A) \) does not separate \( 0 \) from \( \infty \) if and only if some angle of \( A \) is less than \( \pi \).

Proposition 4.2. Let \( A \in B(X) \) and assume that \( X = N(A) \oplus R(A) \). Then \( \sigma(A) \) does not separate \( 0 \) from \( \infty \) if and only if \( \phi_C(A) < \pi \), for some \( C \).

Proof. If \( \sigma(A) \) does not separate \( 0 \) from \( \infty \), then \( 0 \in \partial \sigma(A) \) the result follows from Theorem 4.1. Conversely assume that \( 0 \) is inside a hole in \( \sigma(A) \). We will show that all angles of \( A \) are equal to \( \pi \). To see this note that in this case every unbounded curve \( C \) emanating from the origin intersects \( \partial \sigma(A) \).
and in particular $\sigma_{app}(A)$. But so does $-C$ and hence if $-\lambda_0$ is this point of intersection there exists a sequence $(x_n)$ in $X$ such that

$$\|Ax_n + \lambda_0x_n\| \to 0, \text{ as } n \to \infty.$$ 

Since $X = N(A) \oplus R(A)$ each $x_n$ may be written as $x_n = y_n + z_n$, with $y_n \in N(A)$ and $z_n \in R(A)$. But then and since the sum $N(A) \oplus R(A)$ is closed we have that

$$\|Ax_n + \lambda_0x_n\| = \|Az_n + \lambda_0y_n + \lambda_0z_n\| \geq c'\|Az_n + \lambda_0z_n\|$$

and hence

$$\|Az_n + \lambda_0z_n\| \to 0, \text{ as } n \to \infty. \tag{11}$$

On the other hand if $\phi_C(A) < \pi$, then by $X = N(A) \oplus R(A)$, the range of $A|_{R(A)}$ is closed and $N(A) \cap R(A) = \{0\}$. Hence

$$\|Ax + \lambda_0x\| \geq c\|Ax\| \geq k\|x\|,$$

for all $x \in R(A)$. But this contradicts (11) and thus $\phi_C(A) = \pi$, for all $C$. So $\phi_C(A) < \pi$ implies that $0 \in D_\infty$. \hfill \Box

**Remark 4.4.** The spectrum of the bilateral shift is the whole unit circle and hence by Proposition 4.2, all angles of it are equal to $\pi$.

## 5. Applications

Our first application of Theorem 4.1 is to show that if an invertible operator $T$ has a spectral set $\sigma$ that does not separate 0 from $\infty$ and $P$ is the spectral projection, in the Riesz decomposition corresponding to $\sigma$, then there exists a curve emanating from the origin for which

$$\phi_C(PT) < \pi.$$ 

To do so we will use the following factorization of an operator for which range-kernel complementarity holds (see [9, Theorem 3]).

**Theorem 5.1.** If $X$ is a Banach space and $A \in B(X)$, then

$$X = R(A) \oplus N(A)$$

if and only if $A = PT = TP$, where $T \in B(X)$ is invertible and $P \in B(X)$ is a projection.

Our result is the following.

**Theorem 5.2.** Assume that $T \in B(X)$ is invertible and has a spectral set $\sigma$ that does not separate 0 from $\infty$. If $P$ is the spectral projection corresponding to $\sigma$, then there exists a curve emanating from the origin for which

$$\phi_C(PT) < \pi.$$
Proof. Since the spectral projection $P$ corresponding to $\sigma$ commutes with $T$, we get by Theorem 5.1 that if $A = PT$, then

$$X = R(A) \oplus N(A).$$

Hence in order to use Theorem 4.1 and conclude the proof it is enough to show that $0 \in \overline{D}_\infty$ i.e $\sigma(A)$ does not separate $0$ from $\infty$. But this follows immediately by the fact that

$$A = T\big|_{R(P)} \oplus 0$$

with respect to the decomposition $X = R(P) \oplus N(P)$ and thus

$$\sigma(A) = \sigma(T|_{R(P)}) = \sigma.$$

□

Corollary 5.1. Assume that $T \in B(X)$ is invertible and $\sigma(T)$ that does not separate $0$ from $\infty$. Then there exists a curve emanating from the origin for which

$$\phi_C(T) < \pi.$$

Proof. If in Theorem 5.2 we have $\sigma = \sigma(T)$, then $P = I$ and hence in this case there exists a curve emanating from the origin for which $\phi_C(T) < \pi$. □

Corollary 5.2. If $P \in B(X)$ is a projection, then $\phi(P) < \pi$.

Proof. Taking $T = I$ in Theorem 5.2, we see that $\phi(P) < \pi$, since in this case we may choose $C$ to be the positive axis. □

Our second application concerns elements in a commutative semisimple Banach algebra, whose spectrum does not separate $0$ form $\infty$.

If $A$ is a Banach algebra then one may define the angle $\phi_C(a)$ of $a \in A$, along a curve $C$ emanating from the origin, as the corresponding angle $\phi_C(T_a)$ of the left regular representation $T_a$ defined as usual by

$$T_ab = ab,$$ for all $b \in A$,

Remark 5.1. The definition can be given directly on the algebra: let $C$ be an unbounded curve emanating from the origin. If there exists $0 \leq c < 1$ such that

$$\|ab + \lambda b\| \geq c\|ab\|,$$ for all $\lambda \in C$ and $b \in A$,

then the angle of $a$ along $C$ is

$$\phi_C(a) = \pi - \arcsin \inf_{\lambda \in C} \inf_{ab \neq 0} \frac{\|ab + \lambda b\|}{\|ab\|}.$$

If the above double infimum is equal to 1, then we say that the angle of $a$ is acute.
A Banach algebra \( \mathcal{A} \) is called semisimple if its radical (the intersection of all its maximal ideals) \( \text{rad} \mathcal{A} \) is trivial (see [11, Definition 11.8]). To proceed recall also that an operator \( T : \mathcal{A} \to \mathcal{A} \) is called a multiplier if
\[
b(Tc) = (Tb)c, \quad \text{for all } b, c \in \mathcal{A}.
\]

It turns out that there is a very simple characterization of range-kernel complementarity for multipliers in a semisimple Banach algebra (see [9, Theorem 10]).

**Theorem 5.3.** Let \( \mathcal{A} \) be a semisimple Banach algebra and \( T \) be a multiplier. Then
\[
\mathcal{A} = R(T) \oplus N(T)
\]
if and only if \( \text{dist}(0, \sigma(T) \setminus \{0\}) > 0 \).

**Remark 5.2.** As we have mentioned in Remark 4.3 if 0 is an isolated point in \( \sigma(T) \), then range-kernel complementarity fails if and only if \( T = U \oplus S \), with \( S \neq 0 \) quasinilpotent. The reason why the above result holds is that, as it was shown in [9, Lemma 9], in a semisimple Banach algebra there are no nontrivial quasinilpotent multipliers.

Our result is the following.

**Theorem 5.4.** Let \( \mathcal{A} \) be a semisimple commutative Banach algebra, \( a \in \mathcal{A} \) and assume that the spectrum \( \sigma(a) \) does not separate 0 from \( \infty \). Then
\[
\text{dist}(0, \sigma(a) \setminus \{0\}) > 0
\]
if and only if there exists a curve \( C \) emanating from the origin for which \( \phi_C(a) < \pi \).

**Proof.** In a commutative Banach algebra the left regular representation \( T_a \) of \( a \) is a closed range multiplier and hence the result follows by Theorems 4.1 and 5.3.

We conclude with a direct consequence of Proposition 4.2 to elements of a commutative \( C^* \)-algebra. This time we use the following theorem [9, Theorem 13].

**Theorem 5.5.** Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( T \) be a multiplier. Then
\[
\mathcal{A} = R(T) \oplus N(T)
\]
if and only if \( R(T) \) is closed.

Our result characterizes elements in a commutative \( C^* \)-algebra whose spectrum does not separate 0 from \( \infty \).

**Theorem 5.6.** Let \( \mathcal{A} \) be a commutative \( C^* \)-algebra. The spectrum of \( a \in \mathcal{A} \) does not separate 0 from \( \infty \) if and only if there exists a curve \( C \) emanating from the origin for which \( \phi_C(a) < \pi \).

**Proof.** As before in a commutative Banach algebra, the left regular representation \( T_a \) of \( a \) is a closed range multiplier and hence the result follows by Theorem 5.5 and Proposition 4.2.
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References

[1] Drivaliaris, D., Yannakakis, N.: The spectrum of the restriction to an invariant subspace. Oper. Matrices 14, 261–264 (2020)
[2] Drivaliaris, D., Yannakakis, N.: The angle of an operator and range-kernel complementarity. J. Optim. Theory 76, 205–218 (2016)
[3] Gabiner, S.: Ranges of quasinilpotent operators. Ill. J. Math. 15, 150–152 (1971)
[4] Gevorgyan, L.Z.: Some properties of the normalized numerical range. Izv. Nats. Akad. Nauk. Armenii Mat. 41, 41–48 (2006)
[5] Gustafson, K.: The angle of an operator and positive operator products. Bull. Amer. Math. Soc. 74, 488–492 (1968)
[6] Gustafson, K., Rao, D.: Numerical Range. The Field of Values of Linear Operators and Matrices. Springer, New York (1997)
[7] Heuser, H.: Functional Analysis. Wiley, New York (1982)
[8] Krein, M.: Angular localization of the spectrum of a multiplicative integral in a Hilbert space. Funct. Anal. Appl. 3, 73–74 (1969)
[9] Laursen, K.B., Mbekhta, M.: Closed range multipliers and generalized inverses. Stud. Math. 107, 127–135 (1993)
[10] Radjavi, H., Rosenthal, P.: Invariant Subspaces. Springer, New York (1973)
[11] Rudin, W.: Functional Analysis. McGraw-Hill, New Delhi (1992)
[12] Spitkovsky, I.M., Stoica, A.-F.: On the normalized numerical range. Oper. Matrices 11, 219–240 (2017)

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