A Survey of Application and Development of Small Disturbance Equation Method in the Aviation Area

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Abstract. All the small disturbance equation (SDE) methods, which are a useful technique to enhance the stability and security of the aircraft, play a significant role in the development of the aviation area. This review paper briefly introduces the development of the SDE and summarizes the two calculation methods combining with the SDE and application of the SDE method in the aviation area. The calculation methods mainly focus on the approximate factorization (AF) algorithm with the SDE and Newton-Orthomin method with the SDE. AF algorithm with the SDE is robust enough, and it can choose the calculation methods according to accuracy rather than stability that significantly saves the calculation cost. The newton-Orthomin method with the SDE can improve computing efficiency by 2.1 to 4.5 times compared with the monotone AF method. The direction of application focus on the flutter analysis, gust analysis and resizing of some component in the aircraft and the disadvantages are explored for each type. In conclusion, the combination of the SDE with the other algorithms can increase calculation efficiency, and the SDE has a wide range of applications in the aviation area. This paper can provide relevant references for the subsequent combination of the SDE and other algorithms development or practical application of SDE.

1. Introduction

These guidelines, powered engines, influence the aircraft's stability through mainly two ways. The unstable force generated by the engine's oscillation will significantly affect the aeroelastic stability, and the irregular shape of the engine will change the trajectory of the air and thereby influence unstable aerodynamics [1]. In order to solve this problem, the small disturbance equation (SDE) method was introduced, which incorporates wing, engine, and other disturbances into the SDE model framework to improve the accuracy and security of aircraft manufacturing [1]. Moreover, it provides higher fidelity for the simulation of geometrically complex configurations or transonic flows [2].

The SDE Computational Fluid Dynamics (CFD) was first developed for turbomachinery applications and subsequently applied to external aerodynamics [3-6]. Then, Reynolds-averaged Navier-Stokes (RANS) equations also use the SDE CFD to solve relevant problems [7]. In early
aircraft development, the SDE method was used to provide more accurate datasets for aeroelastic analyses [1]. To be specific, taking advantage of the combination of classical potential theory, the SDE method can be applied to dynamical aeroelastic analyses. SDE method plays a significant role in improving the precision and speed of the Doublet-Lattice Method, especially for more complex devices, making use of linearization technique to save the simulation cost [8]. For example, compared with Navier-Stokes (NS) method only, the calculation time of the small disturbance NS method is reduced by as much as half of the order of magnitude [9]. The gradual maturity of the SDE method can be used for many practical engineering problems, from calculating dynamic stability derivatives, aeroelastic, and aeroservoelastic research to generating reduced-order models [1]. However, the early SDE method has an obvious disadvantage. Despite its high efficiency compared to other CFD methods, the time of SDE computation is still too long. Using the SDE method to generate a complete dataset may take from a few days to some weeks [10]. Over the years, lots of methods are proposed to decrease the number of SDE computations. For instance, the Akaike Information Criterion-correction method can make the computation more efficient by using different matrices than the matrices of the traditional SDE method [10].

This article aims to review the advances of the SDE method and briefly summarize some combinations of SDE and other algorithms for the steady two-dimensional transonic flow. For this paper's organization, we will first introduce the SDE method combining with the Approximate Factorization (AF) algorithm and Newton-Orthomin method. After that, the application of the SDE method will be explored in the aviation area.

2. Background
The general form of SDE can be written as
\[
\frac{\partial w_0}{\partial t} + \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z} = 0
\]  
where \( t \) is the time, \( x, y, z \) are the nondimensional Cartesian coordinates in streamwise, spanwise, and vertical directions, respectively; \( w_0 \) is the function defining the instantaneous position of the wing; \( w_1, w_2, w_3 \) are the instantaneous position functions of the wing in the \( x, y \) and \( z \) directions, respectively.

When we consider the flow in the case of the thin obstacle, such as the thin airfoils, the flow can be simplified in a two-dimensional flow because the flow variation in the \( z \) direction is negligible. Moreover, the steady flow means that physical quantities such as speed, density will not change over time. Therefore, Eq. (1) becomes:
\[
[1 - M_\infty^2 - (y + 1)M_\infty^2 \Phi_x] \Phi_{xx} + \Phi_{yy} = 0
\]  
where \( M_\infty \) is the free stream Mach number, \( y \) is the ratio of specific heats and \( \Phi_x \) represents the derivative of the disturbance velocity potential in the \( x \) direction; \( \Phi_{xx} \) and \( \Phi_{yy} \) represents the second derivative of the disturbance velocity potential in the \( x, y \) directions. For purpose of studying the equation easily, we can define a parameter in Eq. (2).
\[
K = 1 - M_\infty^2 - (y + 1)M_\infty^2 \Phi_x
\]  
The value of \( K \) determines the category of the flow. When the \( K \) is negative, \( M_\infty > 1 \), as a result, the flow will be the supersonic flow. When the \( K \) is positive, the flow will switch to the subsonic flow (\( M_\infty < 1 \)).

Partial differential equations do not have analytical solutions, so it is necessary to combine numerical methods to compute the approximate solutions. The finite difference method is the commonly used approximate solution method. Eq. (2) is discretized with finite differences over a solution mesh. We used the chord length of the airfoil to normalize the grid and expanded the computational domain by five chord lengths in both the \( x \) and \( y \) directions as suggested in [11]. The subsonic flow is supposed to use the central differences method, and the supersonic flow should use the backward differences method [11].

When it comes to the far-field boundary condition of Eq. (2), velocity potential is equal to zero along the inlet, outlet, or other boundaries of the mesh. The boundary condition along the surface of the airfoil at the bottom boundary can be written as
\[ \Phi_y = \frac{\partial f}{\partial x} \]  

(4)

where \( f \) is the function defining shape of the airfoil; \( \Phi_y \) represents the derivative of the disturbance velocity potential in the \( y \) direction. The velocity potential under the airfoil is equal to zero. Furthermore, the velocity potential of elsewhere at the bottom edge is also equal to zero.

3. Main body

3.1. Approximate Factorization algorithm with the SDE

The approximate factorization (AF) algorithm consists of Newton linearization and the internal iteration technique. In 1977, Ballhaus et al. [12] proposed an implicit AF scheme for solving the steady transonic flow problems and compared it with the standard solution for transonic flow problems. Results indicated that the computational cost required for the AF method to solve the nonlinear finite difference matrix equations of the transonic flow field is reduced significantly. The improvement of computing efficiency is realized without increasing the complexity of computer storage or coding. In 1987, T. Batina [13] proposed a time accurate AF algorithm for solving three-dimensional unsteady transonic SDE. It solves the steady and oscillatory flow problems of the F-5 fighter wing in subsonic and supersonic free flow conditions. It is found that the time step size is cyclic for steady flow calculation so as to achieve fast convergence. For unsteady flow calculation, the AF algorithm is robust enough to choose the step size according to accuracy rather than stability. Only a few hundred times steps are needed to obtain the exact solution, which significantly saves the calculation cost [13].

The AF algorithm includes four steps: mathematical formula, finite difference discretization, boundary conditions, and solution. The computation of unsteady flow includes two steps: the first step is to perform time linearization to determine the estimation of potential field, and the second step is to perform internal iteration to provide time accuracy [13].

Firstly, the form of SDE is considered as follow:

\[ R(\Phi^{n+1}) = 0 \]  

(5)

where \( \Phi^{n+1} \) represents the unknown potentials at time level \((n+1)\). Given the Newton linearization about \( \Phi^* \), we can obtain:

\[ R(\Phi^*) + \left( \frac{\partial R}{\partial \Phi} \right)_{\Phi=\Phi^*} \Delta \Phi = 0 \]  

(6)

where \( \Phi^* \) is the currently available value of \( \Phi^{n+1} \) and \( \Delta \Phi = \Phi^{n+1} - \Phi^* \). During the convergence of the iteration procedure, \( \Delta \Phi \) will approach zero so that the solution will be given by \( \Phi^{n+1}=\Phi^* \). The AF algorithm is formulated by firstly approximating the time derivative terms (\( \Phi_{tt} \) and \( \Phi_{xt} \)) via second-order-accurate finite-difference formulas [13]. By substituting \( \Phi = \Phi^* + \Delta \Phi \) into the SDE and neglecting squares of derivatives of \( \Delta \Phi \), terms in Eq. (1) can be rewritten as

\[ \left( \frac{\partial w_0}{\partial t} \right) = -A \frac{2\Phi^* - 5\Phi^n + 4\Phi^{n-1} - \Phi^{n-2}}{\Delta t^2} - \frac{2A}{\Delta t^2} \Delta \Phi_x \]

\[ -B \frac{3\Phi_x^* - 4\Phi_x^n + \Phi_x^{n-1}}{2\Delta t} - \frac{3B}{2\Delta t} \Delta \Phi_x \]

\[ \left( \frac{\partial w_2}{\partial x} \right) = \frac{\partial}{\partial x} \left( E\Phi_x^* + F\Phi_x^n + G\Phi_x^{n-1} \right) + \frac{\partial}{\partial x} \left( E\Delta \Phi_x + 2F\Phi_x^* \Delta \Phi_x + 2G\Phi_x^n \Delta \Phi_y \right) \]

(7)

\[ \left( \frac{\partial w_2}{\partial y} \right) = \frac{\partial}{\partial y} \left( \Phi_y^* + H\Phi_y^* \Phi_y^* + \Phi_y^n + \Phi_y^{n-1} \Delta \Phi_y + H\Phi_y^n \Delta \Phi_y + H\Phi_y^* \Delta \Phi_y \right) \]

(8)

\[ \left( \frac{\partial w_3}{\partial z} \right) = \frac{\partial}{\partial z} (\Phi_z^* + \Phi_z^n) \]

(9)

The coefficients \( A, B, E, F, G, H \) are respectively defined as \( A=M^2, \ B=2M^2, \ E=1-M^2, \ F=-1/(y+1)M^2, \ G=1/(y-3)M^2, \ H=-(y-1)M^2 \), where \( M \) denotes Mach number.

Summing these four terms and rearranging them, the following equation can be obtained:
\[
\frac{2A}{\Delta t^2} \Delta \Phi + \frac{3B}{2\Delta t} \Delta \Phi_x - \frac{\partial}{\partial x} \left( E \Delta \Phi_x + 2F \Phi_x^* \Delta \Phi_x + 2G \Phi_x^* \Delta \Phi_y \right) \\
- \frac{\partial}{\partial y} \left( H \Phi_y + H \Phi_y^* \Delta \Phi_y \right) - \frac{\partial}{\partial z} \left( \Delta \Phi_z \right) \\
= -A \frac{2\Phi^* - 5\Phi^n + 4\Phi^{n-1} - \Phi^{n-2}}{\Delta t^2} - B \frac{3\Phi_x^* - 4\Phi_x^{n-1} + \Phi_x^{n-2}}{2\Delta t} \\
+ \frac{\partial}{\partial x} \left( E \Phi_x^* + F \Phi_x^{*2} + G \Phi_y^{*2} \right) + \frac{\partial}{\partial y} \left( F \Phi_x^* + H \Phi_y^* \right) + \frac{\partial}{\partial z} \left( \Phi_z^* \right)
\]

(11)

Eq. (11) is transformed into computational coordinates and is rewritten in conservation form for solution by approximate factorization.

\[ L_\xi L_\eta L_\zeta \Delta \Phi = -R(\Phi^*, \Phi^n, \Phi^{n-1}, \Phi^{n-2}) \]

(12)

Eq. (12) is solved using three sweeps through the grid by sequentially applying the operators \( L_\xi \), \( L_\eta \) and \( L_\zeta \) as

\[ \xi \text{ sweep: } L_\xi \Delta \Phi = -R \]

(13)

\[ \eta \text{ sweep: } L_\eta \Delta \Phi = \Delta \Phi \]

(14)

\[ \zeta \text{ sweep: } L_\zeta \Delta \Phi = \Delta \Phi \]

(15)

In AF algorithm, the time derivatives are implemented for variable time stepping to allow for step-size cycling to accelerate convergence to steady-state. In these calculations, the step size is cycled using a standard geometric sequence. Also, since the \( L_\xi \), \( L_\eta \) and \( L_\zeta \) operators only contain derivatives in their respective coordinate directions, all three sweeps may be vectorized.

An initial estimate of the potentials at time level \((n+1)\) is required to start the iteration process. This estimate is provided by performing a time-linearization calculation. The equations governing the time-linearization step are derived in a similar fashion as the equations for iteration. Time-linearization step may be written as:

\[ L_\xi L_\eta L_\zeta \Delta \Phi = -R(\Phi^n, \Phi^{n-1}, \Phi^{n-2}) \]

(16)

The boundary conditions are numerically imposed by redefining the \( L_\xi \), \( L_\eta \) and \( L_\zeta \) operators as well as the right-hand side \( R \) at the appropriate grid points. The equation to be solved at boundary grid points may then be written symbolically as

\[ L_\xi L_\eta L_\zeta \Delta \Phi = -R \]

(17)

3.2. Newton-Orthomin method with the SDE

The Newton-Orthomin method is based on Newton’s method with the incomplete LU decomposition. It is a preconditioner technique but much more efficient than Monotone Approximate Factorization (MAF code) method [14]. The Newton-Orthomin uses Newton’s method to solve the nonlinear system of equations resulting from the discretization via finite differences. An efficient iterative linear solver (i.e., Orthomin) is used to solve the sparse linear system of equations in each Newton step. The Newton-Orthomin have 2.1 to 4.5 speedups (the CPU calculated SDE speed) than the MAF method for various cases and mesh sizes [11]. The Orthomin method in subsonic and transonic flow fields for a parabolic and a NACA 64A006 airfoil for different mesh sizes, yields better efficiency.

In the Newton-Orthomin algorithm, the resulting nonlinear system of equations is solved directly using Newton’s method. The method can be written as

\[ F(\Phi^n)\Delta \Phi^{n+1} = -F(\Phi^n) \]

(18)

where

\[ F(\Phi) = \left( K_{i,j}D_x \delta_{x,i,j} + \delta_{yy} \right) \Phi^n_{i,j} \]

(19)

where \( D_x \) represents the operator in the \( x \) direction and \( \delta_{x,i,j} \) represents the difference operator in the \( x \) direction.
The Newton-Orthomin algorithm can be summarized as follows [11]: 1) Evaluate $F$ and $F'$ for a given $\Phi^n$; 2) Substitute in SDE and solve the linear system using Orthomin to obtain $\Delta\Phi^{n+1}$; 3) Repeat steps 1 and 2 until convergence.

The Newton-Orthomin algorithm is 2.7 times as fast as MAF in subsonic flows and 3.8 times as fast in transonic flows. And using Newton-Orthomin, as the size of the system is increased, the ratio of CPU times required for the larger mesh is much less for the Newton-Orthomin algorithm. Speedups increase from 2.1 for the regular mesh to 3.9 for the fine mesh [11]. The algorithm idea can be extended for different switches, more complex flow models (i.e., Euler and Navier-Stokes equations), and configurations (i.e., three-dimensional flow).

3.3. Applications of the SDE

3.3.1. Flutter analysis. Flutter refers to the big vibration of aircraft produced by the coupling effect of aerodynamic forces and other forces during the flight, which is undoubtedly threatening security [15]. Vidy et al. proved that the SDE CFD tool (AER-SDNS) could be used to perform the flutter analysis [10]. Specifically, the SDE method manages to calculate the vibration equation of the wing near the equilibrium position, which is profitable for flutter analysis [16]. In order to testify this opinion, the researcher uses the AGARD 445.6 (weak. 3) as the example to compare its flutter speed using the SDE method with the actual value [10]. Table 1 is the flutter analysis of this wing (AGARD 445.6 (weak. 3)).

| Mach number | 0.499 | 0.678 | 0.901 | 0.960 |
|-------------|-------|-------|-------|-------|
| Experimental data of flutter speed (m/s) | 172.45 | 231.36 | 296.68 | 307.35 |
| AER-SDNS data of flutter speed (m/s) | 171.33 | 233.43 | 285.83 | 294.80 |

The results are given in Table 1. The data of AER-SDNS is very similar to that of the experiment, indicating that the SDE CFD tool matches very well with the actual flutter speed. Therefore, the SDE method can perform the flutter analysis. When it comes to the disadvantage, it will take a huge time to perform the analysis [10].

3.3.2. Gust analysis. Gust response is a nonnegligible dynamic loading condition in the aircraft design [18]. Generally, the researchers always employ the time and frequency domain method to analyze the gust [19]. The SDE method demonstrated that it can perform the gust analysis by Vidy, Katzenmeier, Winter, and Breitsamter [8]. Because the "gust-shape" is the standard cosine plot, the gust's characteristic can be determined by the gust time and maximum modal amplitudes. Specifically, they also used the AGARD 445.6 (weak. 3) as the experimental object. Then, they assumed the gust times and adopted the SDE method to calculate the maximum modal amplitudes and then compared it with the actual maximum modal amplitudes in the same gust times [8]. Table 2 and Table 3 summarize the gust results of this wing (AGARD 445.6 (weak. 3)).

| SDE method | Gust time (s) | Maximum modal amplitudes (-) |
|------------|---------------|------------------------------|
| M=0.499    | 0.5           | -0.3                         |
| M=0.499    | 0.1           | -0.42                        |
| M=0.960    | 0.5           | -0.065                       |
| M=0.960    | 0.1           | -0.1                         |
Table 3. Experimental gust results of the wing (AGARD 445.6 (weak. 3)) [8]

| Experimental data | Gust time (s) | Maximum modal amplitudes (-) |
|-------------------|--------------|-----------------------------|
| M=0.499           | 0.5          | -0.25                       |
| M=0.499           | 0.1          | -0.38                       |
| M=0.960           | 0.5          | -0.05                       |
| M=0.960           | 0.1          | -0.82                       |

Table 2, Table 3 summarize the gust results of the AGARD. From the maximum modal amplitudes, we can get that the gust plot of the SDE method is similar to that of the experiment, which strongly proves that the SDE method can basically simulate the gust situation and perform the gust analysis [8]. When it comes to the disadvantages of this method, the cost of SDE based simulation is expensive. Moreover, it is not suitable for the analysis of gust with different frequencies and shapes in aircraft development [19].

3.3.3. Resizing of some components in the aircraft. The size of components can influence the balance and stability of the aircraft because of the data imbalance distribution issue [20]. Katzenmeier et al. proved that SDE contributes to the resizing of aircraft's components without altering its aerodynamic features [10]. They use the SDE CFD tool to construct the related dataset. For instance, resizing the T-tail, which is widely used in large military transport aircraft, would require the substitution of the 18 component modes through the SDE analyses [10]. However, it also exists some disadvantages. For example, the cost of the SDE analysis on resizing components is very expensive [8]. Fortunately, some strategies have been developed to solve that issue. When we use the SDE CFD tool with the component-wise modal basis to construct the dataset, the cost will decrease a lot. As an example, the replacement of resizing the T-tail (referred above) would decrease from 18 to 4 component modes [10].

4. Conclusions and prospects
In this paper, the development of the SDE and the two calculation methods combining with the SDE and application of the SDE method in the aviation area are summarized. The main conclusion that can be drawn is that both AF and Newton-Orthomin combine with the SDE to increase the computation efficiency and decrease the cost. AF algorithm, which is made up of the Newton linearization and internal iteration method, can choose the step size according to accuracy instead of stability. It only needs a few hundred times steps to obtain an accurate solution, which greatly reduces the computation cost. The Newton-Orthomin method contains the Newton's method and the incomplete LU decomposition preconditioner technique. Compared with the original Monotone AF method, it can enhance computing efficiency by 2.1 to 4.5 times. Moreover, the SDE has a wide range of applications. Taking advantage of the SDE, we can perform the flutter analysis, gust analysis, and the resizing of some components in the aircraft. However, they have a common shortcoming that the calculation cost is too high. Fortunately, we can use a reasonable modal basis to reduce the calculation scale of SDE. SDE still has considerable development potential. The combination of advanced calculation methods with SDE will further improve the calculation efficiency of SDE and contribute to the stability and security of the aircraft.

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