FAST COMPUTATION OF SECONDARY INVARIANTS

SIMON A. KING

ABSTRACT. A very classical subject in Commutative Algebra is the Invariant Theory of finite groups. In our work on 3–dimensional topology [12], we found certain examples of group actions on polynomial rings. When we tried to compute the invariant ring using SINGULAR [9] or MAGMA [1], it turned out that the existing algorithms did not suffice.

We present here a new algorithm for the computation of secondary invariants, if primary invariants are given. Our benchmarks show that the implementation of our algorithm in the library finvar of SINGULAR [9] marks a dramatic improvement in the manageable problem size. A particular benefit of our algorithm is that the computation of irreducible secondary invariants does not involve the explicit computation of reducible secondary invariants, which may save resources.

The implementation of our algorithm in SINGULAR is for the non-modular case; however, the key theorem of our algorithm holds in the modular case as well and might be useful also there.

KEYWORDS: Invariant Ring, Secondary Invariant, irreducible Secondary Invariant, Gröbner basis.
MSC: 13A50 (primary), 13P10 (secondary)

1. Introduction

Let $G$ be a finite group, linearly acting on a polynomial ring $R$ with $n$ variables over some field $K$. We denote the action of $g \in G$ on $r \in R$ by $g.r \in R$.

Let $R^G = \{ r \in R : g.r = r, \forall g \in G \}$ be the invariant ring. Obviously, it is a sub-algebra of $R$, and one would like to compute generators for $R^G$. We study here the non-modular case, i.e., the characteristic of $K$ does not divide the order of $G$. Note that according to [8], algorithms for the non-modular case are useful also in the modular case.

For any subset $S \subset R$, we denote by $\langle \langle S \rangle \rangle \subset R$ the sub-algebra generated by $S$, and by $\langle S \rangle \subset R$ the ideal generated by $S$. It is well known [4] that there are $n$ (the number of variables) algebraically independent homogeneous invariant polynomials $P = \{p_1, ..., p_n \} \subset R^G$ such that $R^G$ is a finitely generated $\langle \langle P \rangle \rangle$–module. The elements of $P$ are called primary invariants. Of course, they are not uniquely determined. There are various algorithms to compute primary invariants [8]. Since the primary invariants are algebraically independent, the sub-algebra $\langle \langle P \rangle \rangle$ is isomorphic to a polynomial ring with $n$ variables. It is called (homogeneous) Noetherian normalization of $R^G$.

Let $S \subset R^G$ be a minimal set of homogeneous $\langle \langle P \rangle \rangle$–module generators of $R^G$. The elements of $S$ are called secondary invariants. Note that the number of secondary invariants depends on the degrees of the primary invariants. Hence, it is advisable to minimize the degrees of the primary invariants. Irreducible secondary invariants are those non-constant secondary invariants that can not be written as
a polynomial expression in the primary invariants and the other secondary invariants. The set of secondary invariants is not unique, even if one fixes the primary invariants. It is easy to see that one can choose secondary invariants so that all of them are power products of irreducible secondary invariants.

The aim of this paper is to present a new algorithm for the computation of (irreducible) homogeneous secondary invariants, if homogeneous primary invariants $P$ are given. The key theorem for our algorithm concerns Gröbner bases and holds in arbitrary characteristic; however, the algorithm assumes that we are in the non-modular case. For simplicity, we even assume that $K$ is of characteristic 0, but this is not crucial.

The rest of this paper is organised as follows. In Section 2 we briefly expose our motivating examples arising in low-dimensional topology. In the Section 3 we recall the basic scheme for computing secondary invariants. In Section 4 we state our key result and formulate our new algorithm for the computation of (irreducible) secondary invariants. In Section 5 we provide some examples (partially inspired by our study of problems in low-dimensional topology) and compare the implementation of our algorithm in SINGULAR [6] with previously implemented algorithms in SINGULAR by A. Heydtmann [7] respectively in MAGMA [1] by A. Steel [10].

2. Motivating examples

The starting point of our work was the study of generalisations of Turaev–Viro invariants [11], [12]. These are homeomorphism invariants of compact 3-dimensional manifolds. Their construction is (with some simplifications) as follows. Let $\mathcal{F}$ be some finite set, and let $\mathcal{T}$ be a triangulation of a compact 3-manifold $M$. An $\mathcal{F}$-colouring of $\mathcal{T}$ assigns to any edge of $\mathcal{T}$ an element of $\mathcal{F}$. Tetrahedra have six edges. So, for any tetrahedron of $\mathcal{T}$, an $\mathcal{F}$-colouring of $\mathcal{T}$ gives rise to a six-tuple of colours, that is called 6j-symbols and denoted by $|a b c d e f\rangle$, for $a, b, c, d, e, f \in \mathcal{F}$. The equivalence classes of 6j-symbols with respect to tetrahedral symmetry are variables of some polynomial ring, $R$. The ring also contains one variable $w_f$ for any $f \in \mathcal{F}$, called the weight of $f$. For any $\mathcal{F}$-colouring, we form the product over the weights of the coloured edges and over the 6j-symbols of the coloured tetrahedra of $\mathcal{T}$. By summation over all possible $\mathcal{F}$-colourings of $\mathcal{T}$, we obtain a polynomial $TV(\mathcal{T})$ called the state sum of $\mathcal{T}$. Due to the tetrahedral symmetry of the 6j-symbols, the state sum is well-defined. However, it depends on the choice of $\mathcal{T}$ rather than on the homeomorphism type of $M$. It was shown by V. Turaev and O. Viro [10] that an appropriate evaluation of the state sum (yield by the representation theory of Quantum Groups) is independent of the choice of $\mathcal{T}$. This is called a Turaev–Viro invariant.

In [11] and [12], we define an ideal $I \subset R$, the Turaev–Viro ideal. We show that the coset $te(M) = TV(\mathcal{T}) + I$ is independent of $\mathcal{T}$, hence, a homeomorphism invariant of $M$. This generalises the classical Turaev–Viro invariants. By extensive computations, we show in [12] that these so-called ideal Turaev–Viro invariants are much stronger than the classical Turaev–Viro invariants. For this, it was necessary to compute Gröbner bases of Turaev–Viro ideals. It turns out that different algorithms for the computation of Gröbner bases differ widely in their performance. The algorithm slimgb in SINGULAR [6] of M. Brickenstein [2] performs particularly well.
We obtain a lower bound for the number of tetrahedra of any triangulation of $M$, in terms of the minimal degree of polynomials in the coset $tv(M)$. However, in our computations, the bound appears to be trivial \[12\]. There was some hope to improve the lower bound as follows, using computations of invariant rings. Let $G$ be the symmetric group of $F$. In the obvious way, $G$ acts on the tetrahedral symmetry classes of $6j$–symbols and on the weights, and hence, on $R$. The $G$–action permutes the summands of the state sum. So, the state sum belongs to $\mathcal{R}$. This is how we became interested in the computation of invariant rings. The existing implementations in MAGMA and SINGULAR could not compute the secondary invariants in several of our examples. This motivated us to develop a new algorithm for the computation of secondary invariants. It has been part of the finvar library of SINGULAR \[6\] since release 3-0-2 (July 2006). Unfortunately, in our topological applications, we did not find an improvement of the lower bound for the number of tetrahedra. However, our new algorithm for the computation of secondary invariants certainly is of independent interest.

3. Generalities on the computation of secondary invariants

In the non-modular case, we can use the Reynolds operator $\text{Rey}: R \rightarrow R^G$, which is defined by

$$\text{Rey}(r) = \frac{1}{|G|} \sum_{g \in G} g.r$$

for $r \in R$. By construction, the restriction of the Reynolds operator to $R^G$ is the identity. Let $B_d \subset R^G$ be the images under the Reynolds operator of all monomials of $R$ of degree $d$. It is well known that one can find a system of homogeneous secondary invariants of degree $d$ in $B_d \[13\]$. But how can one determine what elements of $B_d$ are eligible as secondary invariants?

Let $S_0, S_1, S_2, ..., S_{d-1} \subset R^G$ be the homogeneous secondary invariants of degree $0, 1, 2, ..., d-1$, respectively (we can take $S_0 = \{1\}$), and let $IS_i \subset S_i$ be the irreducible ones, for $i = 1, ..., d-1$. Let $s_1, ..., s_m \in R^G$ be some homogeneous secondary invariants of degree $d$. Let $b \in B_d$. We can choose $b$ as a new homogeneous secondary invariant of $R^G$, if $b$ is not contained in the $\langle \langle P \rangle \rangle$–module generated by $S_0 \cup S_1 \cup \cdots \cup S_{d-1} \cup \{s_1, ..., s_m\}$. It is not difficult to show that this is the case if and only if $b$ is not contained in the ideal $\langle P \cup \{s_1, ..., s_m\} \rangle \subset R$; see \[15\].

Ideal membership can be tested using Gröbner bases. For $p \in R$ and a finite subset $G \subset R$, we denote the remainder of $p$ by reduction modulo $G$ by $\text{rem}(b; G)$. The remainder is iteratively defined, depends on the choice of a monomial order, and in general depends on the order of the elements of $G$. For a definition of remainder, of Gröbner bases, and for a proof of the following classical result, we refer to \[5\] or \[14\].

**Theorem 1.** Let $G$ be a Gröbner basis of $\langle G \rangle \subset R$, and let $p \in R$. Then, $\text{rem}(p; G)$ does not depend on the order of polynomials in $G$, and we have $\text{rem}(p; G) = 0$ if and only if $p \in \langle G \rangle$. \[\square\]
We thus obtain the following very basic algorithm for finding homogeneous secondary invariants $S_d$ of degree $d$, provided those of smaller degrees have been computed before.

**Basic Algorithm**

1. Let $S_d = \emptyset$. Let $\mathcal{G}$ be a Gröbner basis of $\langle P \rangle$.
2. For all $b \in B_d$:
   - If $b \not\in \langle P \cup S_d \rangle$ (which is tested by reduction modulo $\mathcal{G}$) then replace $S_d$ by $S_d \cup \{b\}$; compute a Gröbner basis of $\langle P \cup S_d \rangle$ and replace $\mathcal{G}$ with it.
3. Return $S_d$.

There are several ways to improve this algorithm. One way is an application of Molien’s Theorem [15], [8], [7]. We will not go into details here. Molien’s Theorem allows to compute the number $m_d$ of secondary invariants of degree $d$. In other words, if in the above algorithm we got $m_d$ secondary invariants, we can immediately break the loop in Step (2).

We also would like to see which of the secondary invariants in $S_d$ are irreducible, since these, together with $P$, generate $R^G$ as a sub-algebra of $R$. For that purpose, one forms all power products of degree $d$ of elements of $IS_1 \cup IS_2 \cup \cdots \cup IS_{d-1}$ and chooses from them as many secondary invariants as possible (compare [7] or [8]). If there are further secondary invariants (which we know from computation of $m_d$), then one proceeds as above with $B_d$, and obtains all irreducible secondary invariants $IS_d$ of degree $d$. So, the algorithm is as follows.

**Refined Algorithm**

1. Compute $m_d$. Let $S_d = IS_d = \emptyset$ and let $\mathcal{G}$ be a Gröbner basis of $\langle P \rangle$.
2. For all power products $b$ of degree $d$ of elements of $IS_1 \cup IS_2 \cup \cdots \cup IS_{d-1}$:
   - (a) If $b \not\in \langle P \cup S_d \rangle$ (which is tested using $\mathcal{G}$) then replace $S_d$ by $S_d \cup \{b\}$; compute a Gröbner basis of $\langle P \cup S_d \rangle$ and replace $\mathcal{G}$ with it.
   - (b) If $|S_d| = m_d$ then break and return $(S_d, IS_d)$.
3. For all $b \in B_d$:
   - (a) If $b \not\in \langle P \cup S_d \rangle$ (which is tested using $\mathcal{G}$) then replace $S_d$ by $S_d \cup \{b\}$, and $IS_d$ by $IS_d \cup \{b\}$; compute a Gröbner basis of $\langle P \cup S_d \rangle$ and replace $\mathcal{G}$ with it.
   - (b) If $|S_d| = m_d$ then break and return $(S_d, IS_d)$.

Eventually, $S_d$ contains homogeneous secondary invariants of degree $d$, and $IS_d$ contains the irreducible ones. In this form, the algorithm has been implemented in 1998 by A. Heydtmann [7] as the procedure `secondary_char0` of the library `finvar` of SINGULAR. In Step (2), the ideal membership is tested by computing the remainder modulo some Gröbner basis of the ideal. This ideal changes once a new secondary invariant has been found. So, the algorithm involves many Gröbner basis computations. This is its main disadvantage and limits the applicability of the Basic and the Refined Algorithm.

An alternative algorithm was proposed by Kemper and Steel (see [8], [10] or [3]) and implemented in MAGMA [1]. Here, new secondary invariants are detected not by a general solution of the ideal membership problem but by Linear Algebra. This algorithm only involves one Gröbner basis computation, namely for the ideal $\langle P \rangle$. But for computing some of the invariant rings that arise in our study of homeomorphism invariants of 3–dimensional manifolds [12], this does not suffice either.
4. The New Algorithm

The main feature of our new algorithm is that, after computing some (homogeneous) Gröbner basis of \( \langle P \rangle \), we can directly write down a "homogeneous Gröbner basis up to degree \( d \) of \( \langle P \cup S_d \rangle \), once a new secondary invariant of degree \( d \) has been found. We can do so without any lengthy computations (in contrast to \cite{10}, \cite{8}, \cite{3}). This allows to solve the ideal membership problem in a very quick way. We recall the notion of "homogeneous Gröbner bases up to degree \( d \)" in the following paragraphs. At the end of the section, we provide our key theorem and formulate our new algorithm.

For \( p \in R \), let \( \text{lm}(p) \) the leading monomial of \( p \), let \( \text{lc}(p) \) be the coefficient of \( \text{lm}(p) \) in \( p \), and let \( \text{lt}(p) = \text{lc}(p)\text{lm}(p) \) be the leading term of \( p \). The least common multiple is denoted by \( \text{LCM}(\cdot, \cdot) \). Now we can recall the definition of the \( S \)-polynomial of \( p, q \in R \):

\[
S(p, q) = \frac{\text{LCM}(\text{lm}(p), \text{lm}(q))}{\text{lt}(p)} \cdot p - \frac{\text{LCM}(\text{lm}(p), \text{lm}(q))}{\text{lt}(q)} \cdot q
\]

Obviously, the \( S \)-polynomial of \( p \) and \( q \) belongs to the ideal \( \langle p, q \rangle \subset R \). The leading terms of \( p \) and \( q \) are canceling one another, so, the leading monomial of \( S(p, q) \) corresponds to monomials of \( p \) or \( q \) that are not leading. The following result can be found, e.g., in \cite{5} or \cite{14}.

**Theorem 2** (Buchberger’s Criterion). A set \( g_1, \ldots, g_k \in R \) of polynomials is a Gröbner basis of the ideal \( \langle g_1, \ldots, g_k \rangle \subset R \) if and only if \( \text{rem}(S(g_i, g_j); g_1, \ldots, g_k) = 0 \) for all \( i, j = 1, \ldots, k \).

Buchberger’s Criterion directly leads to Buchberger’s algorithm for the construction of a Gröbner basis of an ideal: One starts with any generating set of the ideal. If the remainder modulo the generators of the \( S \)-polynomial of some pair of generators does not vanish, then the remainder is added as a new generator. This will be repeated until all \( S \)-polynomials reduce to 0; it can be shown that this will eventually be the case, after finitely many steps.

Here, we are in a special situation: We work with homogeneous polynomials. It is easy to see that if \( p \) and \( q \) are homogeneous then so is \( S(p, q) \), and its degree is higher than the maximum of the degrees of \( p \) and \( q \), unless \( \text{lm}(p) = \text{lm}(q) \). If \( p, g_1, g_2, \ldots, g_k \in R \) are homogeneous then so is \( \text{rem}(p; g_1, \ldots, g_k) \). Moreover, either \( \text{rem}(p; g_1, \ldots, g_k) = 0 \) or \( \deg(\text{rem}(p; g_1, \ldots, g_k)) = \deg(p) \). For computing \( \text{rem}(p; g_1, \ldots, g_k) \), only those \( g_i \) play a role with \( \deg(g_i) \leq \deg(p) \), for \( i = 1, \ldots, k \).

It follows: If an ideal \( I \subset R \) is homogeneous (i.e., it can be generated by homogeneous polynomials) then it has a Gröbner basis of homogeneous polynomials. Such a Gröbner basis can be constructed degree-wise.

**Definition 1.** A finite set \( \{g_1, \ldots, g_k\} \subset R \) of homogeneous polynomials is a **homogeneous Gröbner basis up to degree** \( d \) of the ideal \( \langle g_1, \ldots, g_k \rangle \), if

\[
\text{rem}(S(g_i, g_j); g_1, \ldots, g_k) = 0
\]

or \( \deg(S(g_i, g_j)) > d \), for all \( i, j = 1, \ldots, k \).

**Lemma 1.** Let \( \{g_1, \ldots, g_k\} \subset R \) be a homogeneous Gröbner basis up to degree \( d \), and let \( p \in R \) be a homogeneous polynomial of degree at most \( d \). Then, \( p \in \langle g_1, \ldots, g_k \rangle \) if and only if \( \text{rem}(p; g_1, \ldots, g_k) = 0 \).
Proof. The paragraph preceding the definition implies that \( \{g_1, ..., g_k\} \) can be extended to a Gröbner basis \( G \) of \( \langle g_1, ..., g_k \rangle \) by adding homogeneous polynomials whose degrees exceed \( d \). Since \( \deg(p) \leq d \), we have \( \text{rem}(p; G) = \text{rem}(p; g_1, ..., g_k) \). Since \( p \in \langle G \rangle \) if and only if \( \text{rem}(p; G) = 0 \) by Theorem 1, the result follows. \( \square \)

We see that in order to do Step (2) in the \textsc{Basic Algorithm} (or the corresponding steps in the \textsc{Refined Algorithm}) it suffices to know a homogeneous Gröbner basis up to degree \( d \) of \( \langle P \cup S_d \rangle \). Our key theorem states that this Gröbner basis can be constructed iteratively, as follows.

**Theorem 3.** Let \( G \subset R \) be a homogeneous Gröbner basis up to degree \( d \) of \( \langle G \rangle \). Let \( p \in R \) be a homogeneous polynomial of degree \( d \), and \( p \notin \langle G \rangle \). Then \( G \cup \{\text{rem}(p; G)\} \) is a homogeneous Gröbner basis up to degree \( d \) of \( \langle G \cup \{p\} \rangle \).

Proof. Let \( r = \text{rem}(p; G) \). Since \( p \notin \langle G \rangle \) and all polynomials are homogeneous, we have \( r \neq 0 \), \( \deg(r) = d \), and \( \langle G \cup \{p\} \rangle = \langle G \cup \{r\} \rangle \).

By hypothesis, the \( S \)–polynomials of pairs of elements of \( G \) are of degree \( > d \) or reduce to 0 modulo \( G \). We now consider the \( S \)–polynomials of \( r \) and elements of \( G \). Let \( g \in G \). By definition of the remainder, we have \( \text{lm}(r) \neq \text{lm}(g) \). Therefore the \( S \)–polynomial of \( r \) and \( g \) is of degree \( > d = \deg(r) \). Thus the claim follows. \( \square \)

We obtain the \textsc{New Algorithm}

1. Compute \( m_d \) and a homogeneous Gröbner basis \( G \) of \( \langle P \rangle \). Let \( S_d = IS_d = \emptyset \).
2. For all power products \( b \) of degree \( d \) of elements of \( IS_1 \cup IS_2 \cup \cdots \cup IS_{d-1} \):
   a. If \( \text{rem}(b; G) \subsetneq 0 \) then replace \( S_d \) by \( S_d \cup \{b\} \) and \( G \) by \( G \cup \{\text{rem}(b; G)\} \).
   b. If \( |S_d| = m_d \) then break and return \( (S_d, IS_d) \).
3. For all \( b \in B_d \):
   a. If \( \text{rem}(b; G) < 0 \) then replace \( S_d \) by \( S_d \cup \{b\} \), \( IS_d \) by \( IS_d \cup \{b\} \) and \( G \) by \( G \cup \{\text{rem}(b; G)\} \).
   b. If \( |S_d| = m_d \) then break and return \( (S_d, IS_d) \).

By Theorem 3 and induction, \( G \) is a homogeneous Gröbner basis up to degree \( d \) of \( \langle P \cup S_d \rangle \). Hence, in Step (2)(a) and (3)(a) one has \( \text{rem}(b; G) \subsetneq 0 \) if and only if \( b \notin \langle P \cup S_d \rangle \). The \textsc{New Algorithm} is a dramatic improvement of the \textsc{Refined Algorithm}. However, in our examples this was still not enough.

One should take more care in Step (2) of the \textsc{New Algorithm}. It simply says “For all power products \( b \) of degree \( d \) of elements of \( IS_1 \cup IS_2 \cup \cdots \cup IS_{d-1} \)”.

Two questions arise:

1. How shall one generate the power products?
2. Is it necessary to generate all possible power products, or can one restrict the search?

In very complex computations, the number of power products is gigantic. But usually only a small proportion of them will be eligible as secondary invariant. So, for saving computer’s memory, it is advisable to generate the power products one after the other (or in small packages), rather than generating all power products at once; this answers Question (1).

Apparently Question (2) was never addressed in the literature. However, it turns out that a careful choice of power products provides another dramatic improvement of the performance of the algorithm. Our choice is based on the following lemma.
This lemma seems to be well known, but to the best of the author’s knowledge it did not appear in the literature and it was not used in implementations.

**Lemma 2.** Assume that secondary invariants of degree \(< d\) are computed such that all of them are power products of irreducible secondary invariants. In the quest for reducible homogeneous secondary invariants of degree \(d\), it suffices to consider power products of the form \(i \cdot s\), where \(i\) is a homogeneous irreducible secondary invariant of degree \(< d\), and \(s\) is some secondary invariant of degree \(d - \deg(i)\).

**Proof.** Let \(p \in R\) be a power product of degree \(d\) of irreducible secondary invariants. Hence, it can be written as \(p = iq\), with an irreducible homogeneous secondary invariant \(i\) of degree \(< d\) and some homogeneous \(G\)–invariant polynomial \(q\) of degree \(d - \deg(i)\) (we do not use that \(q\) is a power product of irreducible secondary invariants).

Recall that the secondary invariants generate the invariant ring as a \(\langle\langle P\rangle\rangle\)–module. Hence one can rewrite \(q = q_0 + k_1s_1 + \cdots + k_ts_t\), where \(q_0 \in \langle\langle P\rangle\rangle\) and \(k_1, \ldots, k_t \in K\), and \(s_1, \ldots, s_t\) are homogeneous secondary invariants of degree \(\deg(q)\). We obtain \(p = iq_0 + k_1(is_1) + \cdots + k_t(is_t)\). Hence, rather than choosing \(p\) as a \(\langle\langle P\rangle\rangle\)–module generator of \(R^S\), we may choose \(is_1, \ldots, is_t\), which, by induction, are all power products of irreducible secondary invariants. \(\square\)

**Improved New Algorithm**

1. Compute \(m_d\). Let \(G\) be a Gröbner basis of \(\langle P\rangle\). Let \(S_d = IS_d = \emptyset\).
2. For all products \(b = i \cdot s\) with \(i \in IS_1 \cup \cdots IS_{d-1}\) and \(s \in S_{d-\deg(i)}\):
   - (a) If \(\text{rem}(b; G) \neq 0\) then replace \(S_d\) by \(S_d \cup \{b\}\) and \(G\) by \(G \cup \{\text{rem}(b; G)\}\).
   - (b) If \(|S_d| = m_d\) then break and return \((S_d, IS_d)\).
3. For all \(b \in B_d\):
   - (a) If \(\text{rem}(b; G) \neq 0\) then replace \(S_d\) by \(S_d \cup \{b\}\), \(IS_d\) by \(IS_d \cup \{b\}\) and \(G\) by \(G \cup \{\text{rem}(b; G)\}\).
   - (b) If \(|S_d| = m_d\) then break and return \((S_d, IS_d)\).

This is the algorithm that is implemented as \texttt{secondary_char0}\footnote{This algorithm can be found in the library \texttt{finvar} of \textsc{Singular} 3-0-2 [6], released in July 2006.} in the library \texttt{finvar} of \textsc{Singular} 3-0-2 [6], released in July 2006. In Step (2), the secondary invariant \(s\) may be a non-trivial powerproduct itself, hence, can be expressed as \(s = is's'\), where \(is\) is an irreducible secondary invariant and \(s'\) is (by induction) some other secondary invariant. Of course one should consider only one of the two products \(is(is')\) and \(i(is's')\) in the enumeration.

Often one is only interested in the irreducible secondary invariants, which, together with the primary invariants, generate the invariant ring as a sub-algebra. Therefore we implemented yet another version of the Improved New Algorithm in \textsc{Singular} 3-0-2, namely \texttt{irred.secondary_char0}. This algorithm computes irreducible secondary invariants, but does not explicitly compute the reducible secondary invariants. That works as follows.

Let \(G_P\) be a Gröbner basis of \(\langle P\rangle\). In Step (2)(a) of the Improved New Algorithm, one replaces \(S_d\) by \(S_d \cup \{\text{rem}(b; G_P)\}\), rather than by \(S_d \cup \{b\}\). In Step (3)(a) one replaces \(S_d\) by \(S_d \cup \{\text{rem}(b; G_P)\}\) and \(IS_d\) by \(IS_d \cup \{b\}\). In the end, \(S_d\) does not contain secondary invariants, but normal forms of secondary invariants with respect to \(G_P\). Since \(\text{rem}(\text{rem}(p_1; G_P); \text{rem}(p_2; G_P); G_P) = \text{rem}(p_1 \cdot p_2; G_P)\) and since a reduction modulo \(G\) in Steps (2)(a) and (3)(a) also comprises a reduction modulo \(G_P\), this maintains all informations that one needs for determining how many secondary invariants are reducible in Step (2) and for finding the irreducible
secondary invariants in Step (3). So in the end, \( IS_d \) contains the irreducible secondary invariants in degree \( d \). This detail of our implementation very often saves much memory and computation time, as can be seen in Table 1 in Examples (1) and (6)–(9). In Example (8), we can compute the irreducible secondary invariants although the computation of all 31104 secondary invariants exceeds the resources.

An example of Kemper (example (9) in the next Section) motivated us to further refine the implementation of the Improved New Algorithm. It concerns the generation of \( B_d \): If there are irreducible secondary invariants in rather high degrees \( d \) (in Kemper’s example, there are two irreducible secondary invariants of degree 9), it is advisable to generate not all of \( B_d \) at once, but in small portions. This will be part of release 3-0-3 of Singular.

5. Benchmark Tests for the Computation of Invariant Rings

5.1. The Test Examples. We already mentioned that some of our test examples arise in low-dimensional topology. This yields Examples (1), (7) and (8). For background information, see [12]. We will not go into details here, but just provide the matrices and primary invariants of our nine test examples. They are roughly ordered by increasing computation time. The ring variables are called \( x_1, x_2, \ldots \).

Let \( e_i \) be the column vector with 1 in position \( i \) and 0 otherwise. Our focus was not on the computation of primary invariants; note that in various examples the primary invariants are not optimal.

(1) A 13–dimensional representation of the symmetric group \( S_2 \) is given by the matrix

\[
M = (e_2 e_1 e_{13} e_{12} e_{11} e_8 e_{10} e_9 e_7 e_5 e_4 e_3)
\]

Our primary invariants are

\[
x_9, x_7 + x_{10}, x_6 + x_8, x_5 + x_{11}, x_4 + x_{12}, x_3 + x_{13}, \]
\[
x_1 + x_2, x_3 x_{13}, x_4 x_{12}, x_5 x_{11}, x_7 x_{10}, x_6 x_8, x_1 x_2
\]

There are 32 secondary invariants of maximal degree 6, among which are 15 irreducible secondary invariants up to degree 2.

(2) A 6–dimensional representation of \( S_4 \) is given by the matrices

\[
M_1 = (e_1 e_5 e_2 e_3 e_6)
\]
\[
M_2 = (e_1 e_5 e_2 e_6 e_3)
\]

Our primary invariants are

\[
x_3 + x_5 + x_6, x_1 + x_2 + x_4, x_3 x_5 + x_3 x_6 + x_5 x_6, \]
\[
x_3 x_4 + x_2 x_5 + x_4 x_6, x_1 x_2 x_4, x_1^3 x_2^3 + x_1^3 x_4^3 + x_2^3 x_4^3 + x_3^2 x_5 x_6^2
\]

There are 12 secondary invariants of maximal degree 9, among which are 4 irreducible secondary invariants of maximal degree 3.

(3) A 6–dimensional representation of the alternating group \( A_4 \) is given by the matrices

\[
M_1 = (e_4 e_5 e_2 e_6 e_3)
\]
\[
M_2 = (e_2 e_3 e_1 e_6 e_4 e_5)
\]
Our primary invariants are
\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6, \quad x_3 x_4 + x_2 x_5 + x_1 x_6, \]
\[ x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + x_1 x_5 + x_2 x_5 + x_3 x_5 \]
\[ + x_4 x_5 + x_2 x_6 + x_3 x_6 + x_4 x_6 + x_5 x_6, \]
\[ x_3^2 x_4 + x_3 x_4^2 + x_2^2 x_5 + x_2 x_5^2 + x_1^2 x_6 + x_1 x_6^2, \]
\[ x_1 x_2 x_4 + x_1 x_3 x_5 + x_2 x_3 x_6 + x_4 x_5 x_6, \]
\[ x_2^2 + x_1^3 + x_2^3 + x_3^2 + x_2^4 \]
\[ + x_3 x_5^2 + x_4 x_5^2 + x_2^4 + x_2^2 x_6 + x_3^2 x_6 + x_2^4 x_6 \]

There are 18 secondary invariants of maximal degree 11, among which are 8 irreducible secondary invariants of maximal degree 5.

(4) A 6–dimensional representation of the dihedral group \( D_6 \) is given by the matrices
\[
M_1 = (e_6 e_5 e_4 e_3 e_2 e_1) \\
M_2 = (e_3 e_1 e_2 e_6 e_4 e_5)
\]

Our primary invariants are the elementary symmetric polynomials. There are 120 secondary invariants of maximal degree 14, among which are 10 irreducible secondary invariants of maximal degree 4.

(5) A 8–dimensional representation of \( D_8 \) is given by the matrices
\[
M_1 = (e_8 e_7 e_6 e_5 e_4 e_3 e_2 e_1) \\
M_2 = (e_4 e_1 e_2 e_3 e_7 e_6 e_5 e_7)
\]

Our primary invariants are
\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8, \]
\[ x_1 x_5 + x_1 x_6 + x_2 x_7 + x_3 x_8, \quad x_3 x_5 + x_4 x_6 + x_1 x_7 + x_2 x_8, \]
\[ x_2 x_5 + x_3 x_6 + x_4 x_7 + x_1 x_8, \quad x_1 x_5 + x_2 x_6 + x_3 x_7 + x_4 x_8, \]
\[ x_1 x_3 + x_2 x_4 + x_5 x_7 + x_6 x_8, \quad x_1 x_2 x_3 x_4 + x_5 x_6 x_7 x_8, \]
\[ x_1 x_2^2 + x_2 x_3^2 + x_1^2 x_4 + x_3 x_4^2 + x_2^2 x_6 + x_3^2 x_6 + x_2^2 x_8 + x_3 x_8^2 \]

There are 64 secondary invariants of maximal degree 11, among which are 24 irreducible secondary invariants of maximal degree 5.

(6) A 7–dimensional representation of \( D_{14} \) is given by the matrices
\[
M_1 = (e_2 e_3 e_4 e_5 e_6 e_7 e_1) \\
M_2 = (e_1 e_7 e_6 e_5 e_4 e_3 e_2)
\]

Our primary invariants are the elementary symmetric polynomials. There are 360 secondary invariants of maximal degree 18, among which are 19 irreducible secondary invariants of maximal degree 7.

(7) A 15–dimensional representation of \( S_3 \) is given by the matrices
\[
M_1 = (e_2 e_3 e_4 e_7 e_1 e_4 e_5 e_8 e_{11} e_{13} e_6 e_{15} e_{10} e_6 e_{12}) \\
M_2 = (e_1 e_3 e_2 e_4 e_5 e_9 e_8 e_7 e_6 e_{13} e_{12} e_{11} e_{10} e_{15} e_{14})
\]
Our primary invariants are
\[ x_1 + x_2 + x_3, \ x_1x_2 + x_1x_3 + x_2x_3, \ x_1x_2x_3, \]
\[ x_{10} + x_{13}, \ x_{10}x_{13}, \ x_6 + x_9 + x_{11} + x_{12} + x_{14} + x_{15}, \]
\[ x_{11}x_{12} + x_6x_{14} + x_9x_{15}, \ x_9x_{11} + x_6x_{12} + x_{14}x_{15}, \]
\[ x_6x_{11} + x_9x_{12} + x_9x_{14} + x_{12}x_{14} + x_6x_{15} + x_{11}x_{15}, \]
\[ x_6x_9x_{14} + x_6x_{11}x_{14} + x_{11}x_{12}x_{14} + x_6x_9x_{15} + x_9x_{12}x_{15} + x_{11}x_{12}x_{15}, \]
\[ x_6^6 + x_9^6 + x_{11}^6 + x_{12}^6 + x_{14}^6 + x_{15}^6, \ x_4, \ x_5 + x_7 + x_8, \]
\[ x_5x_7 + x_5x_8 + x_7x_8, \ x_5x_7x_8 \]
There are 1728 secondary invariants of maximal degree 17, among which are 76 irreducible secondary invariants of maximal degree 4.

(8) A 18-dimensional representation of \(S_3\) is given by the matrices
\[ M_1 = \left( e_2 e_3 e_4 e_5 e_6 e_7 e_8 e_9 e_{15} e_{16} e_{17} e_{18} \right) \]
\[ M_2 = \left( e_1 e_3 e_4 e_5 e_6 e_7 e_8 e_{10} e_{15} e_{16} e_{17} e_{18} \right) \]

Our primary invariants are
\[ x_1 + x_2 + x_3, \ x_1x_2 + x_1x_3 + x_2x_3, \ x_1x_2x_3, \]
\[ x_4 + x_9 + x_{14}, \ x_4x_9 + x_4x_{14} + x_9x_{14}, \ x_4x_9x_{14}, \]
\[ x_{16} + x_{17} + x_{18}, \ x_{16}x_{17} + x_{16}x_{18} + x_{17}x_{18}, \ x_{16}x_{17}x_{18}, \]
\[ x_6 + x_7 + x_{10}, \ x_6x_7 + x_6x_{10} + x_7x_{10}, \]
\[ x_6x_7x_{10}, \ x_5 + x_8 + x_{11} + x_{12} + x_{13} + x_{15}, \]
\[ x_5x_{12} + x_8x_{13} + x_{11}x_{15}, \ x_8x_{11} + x_{12}x_{13} + x_5x_{15}, \]
\[ x_5x_{11} + x_8x_{12} + x_5x_{13} + x_{11}x_{13} + x_8x_{15} + x_{12}x_{15}, \]
\[ x_5x_8x_{12} + x_5x_{11}x_{12} + x_5x_8x_{13} + x_{11}x_{12}x_{15} + x_8x_{13}x_{15} + x_{11}x_{13}x_{15}, \]
\[ x_5^6 + x_8^6 + x_{11}^6 + x_{12}^6 + x_{13}^6 + x_{15}^6 \]
There are 31104 secondary invariants of maximal degree 22, among which are 137 irreducible secondary invariants of maximal degree 4.

(9) A 10-dimensional representation of \(S_5\) is given by the matrices
\[ M_1 = \left( \begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) \]
\[ M_2 = \left( \begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right) \]

We are not listing the primary invariants here, as they are too big polynomials. There are 720 secondary invariants of maximal degree 22, among which are 46 irreducible secondary invariants of maximal degree 9.
Examples (2), (3) and (9) belong to a very interesting class of examples that was shown to us by G. Kemper [9]. For \( n \in \mathbb{N} \), let \( M_n \) be the set of two-element subsets of \( \{1, \ldots, n\} \). Then, one studies the obvious \( S_n \) action on \( M_n \) (or similarly, the obvious \( A_n \) action), and one can try to compute the invariant ring \( \mathbb{Q}[M_n]^{S_n} \) (resp. \( \mathbb{Q}[M_n]^{A_n} \)).

The 10–dimensional representation of \( S_5 \) in Example (9) is a surprisingly challenging problem. To simplify the computations, Kemper provided a decomposition of the representation into a direct sum of a 1–, a 4– and a 5–dimensional representation. Without ad-hoc methods, the computation of secondary invariants for that problem has been beyond reach. The procedure (Irreducible)SecondaryInvariants of Magma V2.13-8 breaks immediately, since it requests 55.62 GB memory, while the memory limit of our computer is 16 GB. Our algorithm irred_secondary_char0 in Singular version 3-0-2 exceeds the limit of 16 GB while computing secondary invariants in degree 8.

The total number of secondary invariants in Example (9) is not particularly large. The difficulties in Example (9) come from the fact that there are irreducible secondary invariants of rather high degrees.

5.2. Comparison. We describe here how different algorithms perform on Examples (1) up to (9). All computations had been done on a Linux x86_64 platform with two AMD Opteron 248 processors (2.2 GHz) and a memory limit of 16 GB. The computation of primary invariants is not part of our tests. Hence, in each example we use the same primary invariants for all considered implementations. We compare the following implementations:

(1) \texttt{secondary_char0} as in Singular release 2-0-6. In Table 1 we refer to it as “Singular (1998)”.

(2) \texttt{secondary_char0} as in Singular release 3-0-2, with a small refinement. In Table 1 we refer to it as “Singular (all sec.)”.

(3) \texttt{irred_secondary_char0}, as in Singular release 3-0-2, with a small refinement. In Table 1 we refer to it as “Singular (irr. sec.)”.

(4) \texttt{SecondaryInvariants} in Magma V2.13-8.

Implementation (1) is due to A. Heydtmann [7] (1998) and has been part of Singular up to release 3-0-1.

Implementations (2) and (3) are our implementations of the Improved New Algorithm explained in Section 4. They are part of Singular 3-0-2, released in July, 2006. Here, we test a slightly improved version, that saves memory when generating irreducible secondary invariants in high degrees. However, this only affects example (9): the performance in the other eight examples remains essentially the same, as the degrees of their irreducible secondary invariants are not high enough.

Implementation (4) is due to A. Steel, based on [10] or [8] or [3]. We consider here the Magma-version V2.13-8, released in October, 2006. There is also a function IrreducibleSecondaryInvariants in Magma, but computation time and memory consumption are essentially the same, in our examples. So, for the sake of simplicity, we do not provide separate timings for that function.

Note that, after posting the first version of this manuscript, there was a new release of Magma containing an algorithm that G. Kemper developed in 2006. However, it seems that Kemper did not describe his algorithm in a paper yet.
Meanwhile we implemented another, completely different algorithm in SINGULAR. It will be part of SINGULAR release 3-0-3 and often works much faster. E.g., it can compute Example (8) in 1.06 seconds. We describe this algorithm in [13] and also provide there comparative benchmarks using the new versions of SINGULAR and MAGMA.

Interestingly, in contrast to the corresponding MAGMA functions, `irred_secondary_char0` often works much faster and needs much less memory than `secondary_char0`; see Examples (1) and (6)–(9). However, this is not always the case, as can be seen in Examples (4) and (5).

In Table 1, “—” means that the computation fails since the process exceeds the memory limit; in examples (8) and (9), MAGMA requests the amount of memory that we indicate in round brackets. In some cases, we stopped the computation when it was clear that it takes too much time; this is indicated in the table by “> ...”.

In conclusion, our benchmarks provide some evidence that the IMPROVED NEW ALGORITHM has great advantages in the computation of invariant rings with many secondary invariants. Here, it marks a dramatic improvement compared with previous algorithms in SINGULAR or algorithms in MAGMA. In 3 of our 9 examples, it is the only algorithm that terminates in reasonable time with a memory limit of 16 GB. A particular benefit or our algorithm is that the computation of irreducible secondary invariants does not involve the explicit computation of reducible secondary invariants, which may save resources.

| Algorithm: | (1) SINGULAR (1998) | (2) SINGULAR (all sec.) | (3) SINGULAR (irr. sec.) | (4) MAGMA |
|------------|------------------|------------------------|------------------------|------------|
| Expl. (1)  | 0.55 s 8.62 MB   | 0.05 s 1.49 MB         | 0.03 s 1.0 MB          | 0.05 s 10.3 MB |
| Expl. (2)  | 0.05 s 0.99 MB   | 0.04 s 0.96 MB         | 0.04 s 0.97 MB         | 0.01 s 7.05 MB |
| Expl. (3)  | 0.48 s 2.97 MB   | 0.33 s 1.95 MB         | 0.3 s 1.96 MB          | 0.19 s 8.96 MB |
| Expl. (4)  | 6.55 s 12.29 MB  | 0.63 s 2.47 MB         | 0.32 s 2.97 MB         | 0.48 s 9.09 MB |
| Expl. (5)  | 18.15 s 45.79 MB | 10.53 s 10.61 MB       | 9.69 s 17.0 MB         | 6.66 s 31.82 MB |
| Expl. (6)  | > 984 m 167 MB   | 100.4 s 110.0 MB       | 16.55 s 39.0 MB        | 118.51 s 54.0 MB |
| Expl. (7)  | — 268.9s  > 167 MB | 20.94 s 872.7 MB       | > 7 h 35.1 MB          | > 15 GB |
| Expl. (8)  | — 872.7 MB  > 10 GB | 50.7 m  > 10 GB        | —                     | (259.5 GB) |
| Expl. (9)  | — 6.42 h 10.74 GB | 99.2 m 7.35 GB         | —                     | (55.62 GB) |

Table 1. Comparison of different implementations
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Simon A. King, Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstr. 9–11, D-77709 Oberwolfach, Germany
E-mail address: king@mfo.de