COUNTING ALGEBRAIC CURVES
WITH TROPICAL GEOMETRY

FLORIAN BLOCK

ABSTRACT. Tropical geometry is a piecewise linear “shadow” of algebraic geometry. It allows for the computation of several cohomological invariants of an algebraic variety. In particular, its application to enumerative algebraic geometry led to significant progress.

In this survey, we give an introduction to tropical geometry techniques for algebraic curve counting problems. We also survey some recent developments, with emphasis on the computation of the degree of Severi varieties of the complex projective plane and other toric surfaces as well as Hurwitz numbers and applications to real enumerative geometry. This paper is based on the author’s lecture at the Workshop on Tropical Geometry and Integrable Systems in Glasgow, July 2011.

1. Enumerative Algebraic Geometry

1.1. Overview. Enumerative algebraic geometry is the study of enumerations of algebro-geometric objects with certain properties. In this article, we mostly consider the enumeration of complex algebraic curves. A typical question is: “What is the number \( N_{d,0} \) of irreducible rational curves in the complex plane \( \mathbb{C}P^2 \) of degree \( d \) passing through \( 3d - 1 \) points in general position?”

Enumeration of algebraic curves in a given algebraic variety \( X \) is closely related to its Gromov-Witten theory. If \( X \) is a del Pezzo surface (i.e., a projective algebraic surface with ample anticanonical bundle) its Gromov-Witten invariants are enumerative, which means that they can be computed by a curve enumeration [37]. For example, the numbers \( N_{d,0} \) are the rational Gromov-Witten invariants of \( X = \mathbb{C}P^2 \). Classically, we have \( N_{1,0} = N_{2,0} = 1 \) and \( N_{3,0} = 12 \). In the late 19th century, Zeuthen computed \( N_{4,0} = 620 \). The number \( N_{5,0} = 87304 \) was computed in the mid 20th century. For larger \( d \), no progress was made until Kontsevich [23], in 1995, computed \( N_{d,0} \), for all \( d \), by his famous recursion

\[
N_{d,0} = \sum_{d_1 + d_2 = d \atop d_1, d_2 > 0} \left( d_1^2 d_2 \left( \frac{3d - 4}{3d_1 - 2} \right) - d_1^3 d_2 \left( \frac{3d - 4}{3d_1 - 1} \right) \right) N_{d_1,0} N_{d_2,0}.
\]

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Using tropical geometry, Gathmann and Markwig [17] reproved Kontsevich’s formula, based on Mikhalkin’s Correspondence Theorem between algebraic and tropical plane curves (see Theorem 3.1). An outline of their proof is given in [15, Section 3.2].

More generally, one can allow curves of arbitrary genus and ask for the number $N_{d,g}$ of irreducible degree-$d$ genus-$g$ plane curves passing through $3d + g - 1$ points in general position. The numbers $N_{d,g}$ are the Gromov-Witten invariants of $\mathbb{CP}^2$. The Gromov-Witten invariants $N_{d,g}$ were computed by Caporaso and Harris [6] for all $d$ and $g$ in 1998. Even more generally, one can consider appropriate counts of genus-$g$ curves on other surfaces, or in algebraic varieties of higher dimension. The enumerative meaning, however, can in general be quite subtle.

Closely related to the Gromov-Witten invariant $N_{d,g}$ is the Severi degree $N^{d,\delta}$, counting plane curves of degree $d$ with exactly $\delta$ nodes as singularities (we call such curves $\delta$-nodal) passing through $\frac{(d+3)d}{2} - \delta$ points in $\mathbb{CP}^2$ in general position. Equivalently, $N^{d,\delta}$ is the degree of the Severi variety parametrizing such curves. Enrique [12] and Severi [30] introduced these varieties around 100 years ago.

Tropical geometry techniques have been applied successfully also to problems in real enumerative geometry. Later in this article, in Section 4.4, we briefly mention Welschinger invariants, a real analog of rational Gromov-Witten invariants, and how they can be computed by tropical means.

Enumerative algebraic geometry includes many further subjects, such as Schubert Calculus. There, one considers questions of the form “How many lines in $\mathbb{CP}^3$ simultaneously intersect four given generic lines?” (The answer is, maybe surprisingly, two.) More generally, one counts linear subspaces, or flags of subspaces, that meet given linear subspaces in a prescribed way. One may expect that some of these question can also be answered tropically in the future [33].

1.2. Enumerative Geometry on Toric Surfaces. We now generalize the definitions of $N_{d,g}$ and $N^{d,\delta}$ to toric surfaces. Such invariants can still be computed solely in terms of tropical geometry (see Theorem 3.4). In Section 1.2, we discuss an application of the resulting combinatorics implying polynomiality of the curve counts, in some parameters of the surface. This may suggest a generalization of the Göttsche conjecture [18] Conjecture 2.1] to a family of possibly non-smooth surfaces.

Fix a lattice polygon $\Delta$ in $\mathbb{R}^2$, i.e., $\Delta$ is the convex hull of a finite subset of $\mathbb{Z}^2$. As is well-known in toric geometry, $\Delta$ determines, via its normal fan, a projective toric variety $X = X(\Delta)$, together with an ample line bundle $L = L(\Delta)$ on $X(\Delta)$. Conversely, any such data $(X, L)$ determines a lattice polygon. A common theme in toric geometry is that many geometric invariants of $X(\Delta)$, such as smoothness or its Chow groups, can be directly read off from the combinatorics of $\Delta$. For a detailed introduction to toric varieties see Cox, Little, and Schenk’s recent book [9] or Fulton’s classical introduction [14].

Counting curves on $X(\Delta)$ of a given “degree” means counting curves in the complete linear system $|L(\Delta)|$ of $L(\Delta)$ (or a subsystem thereof). A concrete way to think about a curve $C$ in $|L(\Delta)|$ is as follows. Let $f$ be a polynomial (or Laurent polynomial) with Newton polygon $\Delta$. Then the closure, in $X(\Delta)$, of the vanishing set of $f$ in the complex torus $(\mathbb{C}^*)^2$ is an element of $|L(\Delta)|$. 
Given a lattice polygon $\Delta$, the Severi degree $N^{\Delta,\delta}$ of the toric surface $X(\Delta)$, together with the line bundle $L(\Delta)$, is the number of (not necessarily irreducible) $\delta$-nodal curves in $|L(\Delta)|$ passing through $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ points in general position.

**Example 1.1.** Let $\Delta = \text{conv}\{(0,0), (3,0), (0,2), (3,2)\}$ be the lattice polygon shown on the left of Figure 4. Then $\Delta$ defines the toric surfaces $X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1$ and the line bundle $L(\Delta)$ equals $O(3,2)$. The elements of the linear system $|O(3,2)|$ are the divisors in $\mathbb{P}^1 \times \mathbb{P}^1$ of polynomials in $x_0, x_1, y_0, y_1$ of bi-degree $(3,2)$, where $x_i$ and $y_i$ have degree $(1,0)$ and $(0,1)$, respectively. Thus, $N^{\Delta,\delta}$ counts $\delta$-nodal curves in $\mathbb{P}^1 \times \mathbb{P}^1$ of bi-degree $(3,2)$ through $(3 + 1)(2 + 1) - 1 - \delta = 11 - \delta$ points in general position.

Notice that, unlike in the case of $\mathbb{P}^2$, the number $N_{\Delta,g}$ of irreducible genus-$g$ curves in $|L(\Delta)|$ through sufficiently many points, in general, does not equal a Gromov-Witten invariant of $X(\Delta)$.

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## 2. Tropical Geometry

Tropical geometry is a piecewise linear analog (or “shadow”) of algebraic geometry. The main objects of study are tropical varieties, i.e., weighted, balanced, polyhedral complexes in a real vector space $\mathbb{R}^n$, equipped with a lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. We won’t give the general definition here, and only discuss the case of tropical curves in toric surfaces; see Definition 2.1 for the $\mathbb{P}^2$ case and Definition 2.4 for any toric surface.

Introductory texts on tropical geometry include two book drafts, one by Maclagan and Sturmfels [25], the second by Mikhalkin and Rau [28]. The former text is more extrinsic, with tropical varieties often given as “tropicalizations” of algebraic varieties given by polynomial equations, and is more computationally oriented. The latter takes a more intrinsic approach, with focus on developing a theory of tropical geometry in analogy with (non-tropical) algebraic geometry. For a shorter introduction, with an emphasis on tropical curves, see Gathmann’s excellent survey [15].

### 2.1. Tropical Curves for $\mathbb{P}^2$.

**Definition 2.1.** A tropical plane curve of degree $d$ is a piecewise linear, weighted graph $\Gamma$ in $\mathbb{R}^2$ satisfying:

1. all edges $e$ of $\Gamma$ have weights $\text{wt}(e) \in \mathbb{Z}_{\geq 1}$,
2. all edges have rational slopes,
3. the total weight of the edges of $\Gamma$ in each of the directions $(-1,0)$, $(0,-1)$, and $(1,1)$ equals $d$, and $\Gamma$ has no other unbounded edges,
4. all vertices $v$ of $\Gamma$ are balanced, i.e.,

$$\sum_{\text{edges } e \atop v \in e} \text{wt}(e) \cdot \text{primitive}(e, v) = 0,$$

where $\text{primitive}(e, v)$ is the primitive vector of the edge $e$ at the vertex $v$, i.e., the shortest non-zero integral vector in the ray spanned by $e$. 
Figure 1. A smooth tropical plane cubic. All edge weights are equal to 1. A close-up of the highlighted vertex is shown in Figure 2.

See Figure 1 for an illustration of a tropical plane curve of degree 3 and Figure 2 for a balanced vertex. Condition (4) in Definition 2.1, also known as the zero-tension condition, says that at each vertex a “tug of war”, with directions given by the edges and forces given by the weights, results in no net movement.

Figure 2. A balanced vertex. The weighted sum of the adjacent primitive vectors vanishes: \(1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

Every tropical plane curve \(\Gamma\) has a number of numerical invariants associated to it. We say that \(\Gamma\) is irreducible if \(\Gamma\) is not a union of two (non-empty) tropical plane curves. If \(\Gamma\) is irreducible, its genus \(g(\Gamma)\) is the minimal first Betti number (or rank of the fundamental group) of any topological graph \(\Gamma'\) such that there exists a surjective continuous map \(\Gamma' \to \Gamma\). If \(\Gamma\) has irreducible components \(\Gamma_1, \ldots, \Gamma_r\), then its genus \(g(\Gamma)\) is \(\sum_{i=1}^r g(\Gamma_i) + 1 - r\).

The number of nodes \(\delta(\Gamma)\) of a tropical plane curve of degree \(d\) is \(\frac{(d-1)(d-2)}{2} - g(\Gamma)\). This formula is motivated by the corresponding genus-degree formula for algebraic plane curves. Equivalently, if \(\Gamma\) has irreducible components of degree \(d_1, d_2, \ldots\) and number of nodes \(\delta_1, \delta_2, \ldots\), respectively, then the number of nodes of \(\Gamma\) equals \(\sum_i \delta_i + \sum_{i<j} d_id_j\). This formula parallels the classical Bézout theorem for plane curves. The interested reader wanting to “find” the locations of the tropical nodes can have a look at [11, Theorem 2.9].

Example 2.2. The tropical plane degree-3 curve \(\Gamma\) in Figure 1 has genus \(g = 1\), the cycle is realized by the hexagon. The number of nodes of \(\Gamma\) is \(\delta = \frac{(3-1)(2-1)}{2} - 1 = 0\). Thus \(\Gamma\) is a smooth tropical plane cubic.
Example 2.3. Consider now the curve $\Gamma$ in Figure 3. It differs from the curve in Figure 1 only around the transverse crossing of two edges. Although it looks like $\Gamma$ could have genus 1, there is in fact a surjective continuous map from a tree (considered as a topological space) with nine leaves onto $\Gamma$. Thus $\Gamma$ has genus 0, and we have $\delta(\Gamma) = \frac{(3-1)(3-2)}{2} - 0 = 1$. (Here, the “tropical node” is at the transverse intersection of the two edges.) Thus $\Gamma$ is a rational tropical plane cubic.

Figure 3. A 1-nodal tropical plane cubic.

The point is that each tropical plane degree-$d$ curve has a notion of genus and a number of nodes, paralleling those for algebraic plane curves. These notions let us compute the Gromov-Witten invariant $N_{d,g}$ and the Severi degree $N^{d,\delta}$ tropically, by Mikhalkin’s Correspondence Theorem (Theorem 3.1) in Section 3.

2.2. Tropical Curves for Toric Surfaces. There is also a notion of tropical curves for any projective toric surface (in this article all toric surfaces are assumed to be projective).

Definition 2.4. A tropical curve of degree $\Delta$ is a piecewise linear, weighted graph $\Gamma$ in $\mathbb{R}^2$ satisfying conditions (1), (2), and (4) of Definition 2.1, and additionally (3') the directions of the unbounded edges of $\Gamma$ are “dual to $\partial \Lambda$”, i.e., the total weight of the unbounded edges of $\Gamma$ in direction $v$ equals the lattice length of the edge of $\Lambda$ with outer normal vector $v$.

Figure 4. Left: the Newton polygon $\Delta$ of curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bi-degree $(3,2)$. Right: a corresponding tropical curve of degree $\Delta$. All but one of its edges have weight 1.
Definition 2.4 agrees with Definition 2.1 for \( \mathbb{CP}^2 \) and degree \( d \) curves: in that case, \( \Delta = \text{conv}\{(0, 0), (d, 0), (0, d)\} \) defines the toric surfaces \( X(\Delta) = \mathbb{CP}^2 \) with the ample line bundle \( L(\Delta) = \mathcal{O}_{\mathbb{CP}^2}(d) \). The degree-\( \Delta \) tropical curve has \( d \) rays in each of the directions \((-1, 0), (0, 1), (1, 1)\).

Example 2.5. A tropical curve of degree \( \Delta \), with \( \Delta = \text{conv}\{(0, 0), (3, 0), (0, 2), (3, 2)\} \), is shown in Figure 4. Such tropical curves count algebraic curves in \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) of bi-degree \( (3, 2) \), see Theorem 3.4.

Just as in the \( \mathbb{CP}^2 \) case, we associate numerical invariants to a tropical curve \( \Gamma \) of degree \( \Delta \). We say that \( \Gamma \) is irreducible, if \( \Gamma \) is not a union of two (non-empty) tropical curves. The algebraic analog here is that the polynomial does not factor. If \( \Gamma \) is irreducible, its genus \( g(\Gamma) \) is again defined as the minimal first Betti number of any topological graph mapping continuously and surjectively onto \( \Gamma \). As before, if \( \Gamma \) has irreducible components \( \Gamma_1, \ldots, \Gamma_r \), then its genus is \( \sum_{i=1}^{r} g(\Gamma_i) + 1 - r \). The number of nodes of a tropical curve \( \Gamma \) is the number of interior lattice points \( \text{int}(\Delta) \cap \mathbb{Z}^2 \) in \( \Delta \) minus the genus \( g(\Gamma) \) of \( \Gamma \). This formula mimics the algebraic analog of the relation between genus and the number of nodes of a nodal algebraic curve in a toric surface.

Remark 2.6. Despite allowing unbounded edges in Definitions 2.1 and 2.4 of weight bigger than 1, all weights of such edges are forced to be 1 after we fix sufficiently many point conditions (see [27, Lemma 4.20]). Here, sufficiently many means that we require a finite curve count. This will always be the case in the sequel. Furthermore, by the same lemma, all vertices are 3-valent in such situations (where we view transversely crossing edges as such).

3. Counting Algebraic Curves tropically

3.1. Tropical Curve Enumeration for \( \mathbb{CP}^2 \). Now we discuss how to enumerate algebraic curves tropically, at least in some special cases. That this is indeed possible can be seen as an instance of the “shadow” tropical geometry containing enough information about the algebraic counterpart. We begin by considering the complex projective plane \( \mathbb{CP}^2 \). Recall that the Gromov-Witten invariant \( N_{d,g} \) counts irreducible plane curves of degree \( d \) and genus \( g \) passing through \( 3d + g - 1 \) points in general position. The Severi degree \( N^{d,\delta} \) enumerates (not necessarily irreducible) plane curves of degree \( d \) with exactly \( \delta \) nodes as singularities through \( \frac{(d+3)d}{2} - \delta \) points in general position.

It will turn out that, to enumerate algebraic curves with tropical curves, we need to count them with a multiplicity. We define this multiplicity for tropical curves with only 3-valent vertices (and possibly some transversely crossing edges). This will suffice to compute \( N_{d,g} \) and \( N^{d,\delta} \) by Remark 2.6 as the tropical curves passing through sufficiently many points in general position are automatically of this form.

Let \( \Gamma \) be a tropical curve with 3-valent vertex \( v \). Let \( e_1 \) and \( e_2 \) be two of the \( v \)-adjacent edges of \( \Gamma \). The multiplicity \( \text{mult}(v) \) of \( v \) is
\[
\text{mult}(v) = \text{wt}(e_1) \cdot \text{wt}(e_2) \cdot |\text{primitive}(e_1, v) \land \text{primitive}(e_2, v)|,
\]
thus \( \text{mult}(v) \) is \( \text{wt}(e_1) \cdot \text{wt}(e_2) \) times the Euclidean area of the parallelogram spanned by the two primitive vectors of \( e_1 \) and \( e_2 \). The balancing condition implies the independence of which two edges we choose. The \textit{multiplicity} \( \text{mult}(\Gamma) \) of \( \Gamma \) is the product over the multiplicities of the 3-valent vertices of \( \Gamma \):

\[
\text{mult}(\Gamma) = \prod_{v \text{ 3-valent}} \text{mult}(v).
\]

In close analogy with the algebraic curve count, the \textit{tropical Gromov-Witten invariant} \( N_{d,g}^{\text{trop}} \) is the number of irreducible tropical plane curves \( \Gamma \) of degree \( d \) and genus \( g \) passing through \( 3d + g - 1 \) points in \( \mathbb{R}^2 \) in general position, counted with multiplicity \( \text{mult}(\Gamma) \). Similarly, the \textit{tropical Severi degree} \( N_{d,\delta}^{\text{trop}} \) is the number of (possibly reducible) tropical plane curves \( \Gamma \) of degree \( d \) with \( \delta \) nodes through \( \frac{(d+3)d}{2} - \delta \) points in \( \mathbb{R}^2 \) in general position, counted with multiplicity \( \text{mult}(\Gamma) \). The following influential theorem, suggested by Kontsevich and proved by Mikhalkin, says that the tropical “shadow” suffices to compute the classical numbers \( N_{d,g} \) and \( N_{d,\delta} \), and contributed a great deal to the success of tropical geometry [1, 2, 7, 8, 13, 16, 17, 26].

**Theorem 3.1** (Mikhalkin’s Correspondence Theorem for \( \mathbb{C}\mathbb{P}^2 \) [27, Theorem 1]).

1. We have \( N_{d,g} = N_{d,g}^{\text{trop}} \).
2. We have \( N_{d,\delta} = N_{d,\delta}^{\text{trop}} \).

The Correspondence Theorem reduces the computation of \( N_{d,g} \) and \( N_{d,\delta} \) to a piecewise-linear, combinatorial problem, which we can think of as the “cartoon” in Figure 5.

![Figure 5](image_url)

**Figure 5.** Computing the Gromov-Witten invariant \( N_{d,g} \) with piecewise-linear geometry: \( N_{d,g} \) equals the number of piecewise-linear tropical “interpolation” solutions, counted with multiplicity \( (1) \).

To compute \( N_{3,0}^{\text{trop}} \), for example, we need to fill in the picture in Figure 5 according to the rules of Definition 2.1 with 3 rays each going “west”, “south”, and “north-east”, respectively, together with 8 point conditions.

For larger \( d \), this combinatorial problem is, however, still subtle and difficult to carry out. A choice of a suitable point configuration further simplifies the problem as the tropical curves can be organized in a more manageable way. Specifically, if the points are \textit{vertically stretched}, i.e., lie on a (classical) line of very large, irrational slope (such a configuration is tropically generic), then the arising tropical curves can be encoded by \textit{floor diagrams}, a family of decorated graphs. For more details see [2, 5, 13].
Remark 3.2. Mikhalkin shows a stronger statement than Theorem 3.1, namely an actual “correspondence” between algebraic curves through a particular configuration and their tropicalizations. On the complex 2-torus \((\mathbb{C}^*)^2\), we define, for any \(t > 0\),

\[
\text{Log}_t : (\mathbb{C}^*)^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (\log_t |x|, \log_t |y|).
\]

Let \(\mathcal{P}_R\) be a configuration of \(3d + g - 1\) tropically generic points in \(\mathbb{R}^2\) (see [27, Definition 4.7]). For \(t > 0\), let \(\mathcal{P}_C^t\) be a configuration of \(3d + g - 1\) points in \(\mathbb{CP}^2\) in general position such that \(\text{Log}_t(\mathcal{P}_C^t) = \mathcal{P}_R\). Let \(\mathcal{C}_{\text{trop}}\) be the set of irreducible tropical plane curves of degree \(d\) and genus \(g\) through the points \(\mathcal{P}_R\). Let \(\mathcal{C}^t\) be the set of irreducible complex plane curves of degree \(d\) and genus \(g\) through the points \(\mathcal{P}_C^t\). If we weight the tropical curves in \(\mathcal{C}_{\text{trop}}\) by their multiplicity \(1\), then, by Theorem 3.1, both sets \(\mathcal{C}^t\) and \(\mathcal{C}_{\text{trop}}\) have cardinality \(N_{d,g}\).

We can identify the curves in \(\mathcal{C}^t\) and \(\mathcal{C}_{\text{trop}}\), for \(t\) large enough, as follows [27, Lemmas 8.3 and 8.4]: For each \(\varepsilon > 0\), there is a \(T > 0\) such that, for \(t \geq T\) and each tropical curve \(\Gamma\) in \(\mathcal{C}_{\text{trop}}\), there are precisely \(\text{mult}(\Gamma)\) complex curves \(C\) in \(\mathcal{C}^t\) with \(\text{Log}_t(C) \subset \mathcal{N}_\varepsilon(\Gamma) \subset \mathbb{R}^2\), where \(\mathcal{N}_\varepsilon(\Gamma)\) is an \(\varepsilon\)-neighborhood of \(\Gamma\). Furthermore, each curve \(C\) in \(\mathcal{C}^t\) maps into \(\mathcal{N}_\varepsilon(\Gamma)\), for some \(\Gamma\) in \(\mathcal{C}_{\text{trop}}\).

The content here is that one can read off the cardinality of a fiber over \(\Gamma\) of the tropicalization map \(\mathcal{C}^t \to \mathcal{C}_{\text{trop}}\), for large \(t\), from the tropical curve \(\Gamma\): it equals the tropical multiplicity \(\text{mult}(\Gamma)\).

Remark 3.3. There is also a notion of “parametrized” tropical curves, which are maps \(\pi : \Gamma^{\text{abs}} \to \mathbb{R}^2\) satisfying a balancing conditions, from an “abstract” tropical curve \(\Gamma^{\text{abs}}\); see [16, Section 2.2] for the precise definition. In this language, the tropical curves in Definitions 2.1 and 2.4 are the images \(\pi(\Gamma^{\text{abs}})\). This notion is the tropical analog of stable maps in Gromov-Witten theory and is, thus, more natural and flexible than embedded curves in this setting. In this paper, we chose to restrict to the simpler notion of (embedded) tropical curves as in Definitions 2.1 and 2.4 as those are sufficient for our purposes.

3.2. Tropical Curve Enumeration for Toric Surfaces. A very similar approach works for arbitrary toric surfaces as well. Recall that a lattice polygon \(\Delta\) determines a projective toric surface \(X = X(\Delta)\), together with an ample line bundle \(\mathcal{L} = \mathcal{L}(\Delta)\). The number of irreducible genus-\(g\) curves in the complete linear system \(|\mathcal{L}|\) passing through sufficiently many points in general position is denote \(N_{\Delta,g}\); the number of (possibly reducible) curves in \(|\mathcal{L}|\) through \(|\Delta \cap \mathbb{Z}^2| - 1 - \delta\) points in general position is denoted by \(N_{\Delta,\delta}\). The latter number \(N_{\Delta,\delta}\) is known as the Severi degree of the surface \(X(\Delta)\).

As before, we define a tropical analog of the numbers \(N_{\Delta,g}\) and \(N_{\Delta,\delta}\). Let \(N_{\Delta,g}^{\text{trop}}\) be the number of irreducible tropical degree-\(\Delta\) curves \(\Gamma\) with genus \(g\) through sufficiently many points in \(\mathbb{R}^2\) in general position, counted with multiplicity \(\text{mult}(\Gamma)\) (see [11]). Let \(N_{\Delta,\delta}^{\text{trop}}\) be the number of (possibly reducible) tropical degree-\(\Delta\) curves \(\Gamma\) with \(\delta\) nodes through \(|\Delta \cap \mathbb{Z}^2| - 1 - \delta\) points in \(\mathbb{R}^2\) in general position, counted with multiplicity \(\text{mult}(\Gamma)\).
Theorem 3.4 (Correspondence Theorem for Toric Surfaces [27, Theorem 1]).

1. We have $N_{\Delta,g} = N_{\Delta,g}^{\text{trop}}$.
2. We have $N_{\Delta,\delta} = N_{\Delta,\delta}^{\text{trop}}$.

4. Applications

While it’s certainly nice to have combinatorial descriptions of classical curve enumeration problems (as in Section 3), the power of tropical techniques comes with their ability to prove deep and new theorems in enumerative algebraic geometry. In this section, we collect a few of these applications. For more, see for example [3, 16, 17].

4.1. Node Polynomials for Plane Curves. Steiner [34], in 1848, computed the degree $N^{d,1} = 3(d - 1)^2$ of the discriminant of $\mathbb{C}P^2$. A few decades later, in 1863 resp. 1867, Cayley resp. Roberts, gave polynomial expressions for $N^{d,2}$ resp. $N^{d,3}$ (in the latter case for $d \geq 3$).

Much later, in 1994, Di Francesco and Itzykson [10], conjectured the numbers $N^{d,\delta}$ to be polynomial in $d$, for fixed $\delta$ and $d$ large enough. For $\delta = 4, 5, 6$, this was affirmed by Vainsencher [36] in 1995 using deformation theory. In 2001, Kleiman and Piene [22] settled the cases $\delta = 7, 8$ utilizing similar techniques. Fomin and Mikhalkin [13], in 2009, proved Di Francesco and Itzykson’s conjecture, using tropical geometry techniques.

Theorem 4.1 ([13, Theorem 5.1]). For fixed $\delta \geq 1$, there is combinatorially defined polynomial $N_{\delta}(d)$ in $d$, such that

$$N_{\delta}(d) = N^{d,\delta},$$

provided that $d \geq 2\delta$.

Here “combinatorially defined” means that Fomin and Mikhalkin’s description of the polynomials $N_{\delta}(d)$ gives rise to a combinatorial algorithm computing $N_{\delta}(d)$. Their method was improved and implemented by the author [2], who computed $N_{\delta}(d)$ for $\delta \leq 14$. Following Kleiman and Piene [22], the $N_{\delta}(d)$ are called node polynomials.

Fomin and Mikhalkin’s proof is mostly combinatorial and uses a description of $N^{d,\delta}$ in terms of floor diagrams. These purely combinatorial gadgets, introduced by Brugallé and Mikhalkin [4, 5], are a family of enriched graphs, arising from tropical plane curves by topological contractions (see [13, Section 3] or [5, Section 4]).

There are now also alternate proofs of Theorem 4.1. Tzeng, in her celebrated work [35], proved the Göttsche conjecture [18] using algebraic cobordism. (For the precise statement of the conjecture, see Section 4.2 below.) Tzeng thus established universal polynomiality (in certain Chern numbers) of the Severi degree of any smooth projective surface. A second proof of the Göttsche conjecture was given soon after by Kool, Shende, and Thomas [24] using BPS calculus [29]. Although the algebro-geometric techniques give rise to an algorithm to compute node polynomials, the tropical approach seems to be the most efficient, at least in the case of $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. By the universality of the node polynomials, the two cases suffice to determine the Severi degree of any smooth projective surface (and the two cases can indeed be computed by tropical geometry).
4.2. Node Polynomials for Toric Surfaces. Tropical geometry techniques can also be used to compute node polynomials for a large family of toric surfaces that are, in general, non-smooth. Recall that a lattice polygon $\Delta$ determines a toric surface $X(\Delta)$, together with a line bundle $L(\Delta)$. Given $\Delta$, we are interested in the number of $\delta$-nodal curves in the complete linear system $|L| \cap \Delta \cap \mathbb{Z}^2$ through $|\Delta \cap \mathbb{Z}^2| - 1 - \delta$ points in general position. As mentioned before, this number is the Severi degree $N^{\Delta,\delta}$. This section is based on [1].

In the following, we restrict the presentation to the special case of Hirzebruch surfaces in favor of simpler notation. Such surfaces are smooth, but exhibit (see Theorem 4.2) already the main features of the more general case: the Severi degrees are polynomial in the “multi-degree” $L(\Delta)$ and parameters of the surface $X(\Delta)$.

Let $N_m^{(a,b),\delta}$ be the number of $\delta$-nodal curves in the linear system determined by a divisor of bi-degree $(a, b)$ on the Hirzebruch surface $F_m$, i.e., with Newton polygon conv($(0,0), (0,b), (a,b), (a+bm,0)$), up to translation. For $m = 0$, this means enumerating the $\delta$-nodal curves in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of bi-degree $(a,b)$ through $(a+1)(b+1) - 1 - \delta$ points in general position. The polygon determining $F_m$ and the corresponding line bundle is shown in Figure 6.

![Figure 6. The polygon $\Delta$ of the Hirzebruch surfaces $F_2$ and a divisor of bi-degree $(2,3)$.](image)

**Theorem 4.2.** For every $\delta \geq 1$, there is a combinatorially defined polynomial $p_{\delta}(m,a,b)$ such that $N_m^{(a,b),\delta} = p_{\delta}(m,a,b)$ provided $a + m \geq 2\delta$ and $b \geq 2\delta$.

Göttsche [18, Conjecture 2.1] famously conjectured the existence of universal polynomials $T_\delta(x,y,z,t)$ that compute the Severi degree for any smooth projective surface $S$ and any sufficiently ample line bundle $L$ on $S$. According to the conjecture, the number of $\delta$-nodal curves in the linear system $|L|$ through an appropriate number of points is given by evaluating $T_\delta$ at the four topological numbers $L^2, L K_S, K_S^2$ and $c_2(S)$. Here, $K_S$ denotes the canonical bundle, $c_1$ and $c_2$ represent Chern classes, and $LM$ denotes the degree of $c_1(L) \cdot c_1(M)$ for line bundles $L$ and $M$. In the setting of Theorem 4.2, the four topological numbers are polynomial in $m$, $a$, and $b$. The theorem thus also follows from the Göttsche conjecture.

One can prove a similar result as in Theorem 4.2 for “$h$-transverse” polygons $\Delta$ [1, Theorem 1.3]. Such polygons are allowed to have only edges of slope $1/n$, for $n \in \mathbb{Z} \cup \{\infty\}$. The resulting toric surfaces are not smooth in general and are, thus, outside the realm of the Göttsche conjecture. Still, the Severi degree is polynomial in parameters of $\Delta$, and we can use tropical geometry to prove it.
4.3. **Double Hurwitz Numbers.** Fix two partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m > 0) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_n > 0) \) of a positive integer \( d \). The double Hurwitz number \( H_g(\lambda; \mu) \) counts degree-\( d \) maps \( \pi : C \to \mathbb{C}P^1 \), where \( C \) is a connected, genus \( g \) curve and \( \pi \) has ramification profiles \( \lambda \) resp. \( \mu \) over 0 resp. \( \infty \), and simple ramification over \( r = 2g - 2 + m + n \) fixed other points. Each cover is counted with weight \( 1/|\text{Aut}(\pi)| \).

The main reference for this section is Cavalieri, Johnson, and Markwig’s paper [8].

We can think of double Hurwitz numbers as the \( \mathbb{C}P^1 \)-analog of the Gromov-Witten invariants \( N_{d,g} \) of \( \mathbb{C}P^2 \): instead of counting maps to \( \mathbb{C}P^2 \) (each degree-\( d \) plane curve of genus \( g \) is the image of a degree-\( d \) map from an abstract genus-\( g \) curve to \( \mathbb{C}P^2 \)), we now count such maps to \( \mathbb{C}P^1 \).

Alternatively, \( H_g(\lambda; \mu) \) counts tuples of permutations \( \sigma_0, \sigma_1, \ldots, \sigma_r, \sigma_\infty \in S_d \) with

- \( \sigma_0 \) and \( \sigma_\infty \) have cycle type \( \lambda \) and \( \mu \), respectively,
- \( \sigma_1, \ldots, \sigma_r \) are transpositions,
- \( \sigma_0 \sigma_1 \cdots \sigma_r \sigma_\infty = \text{id} \in S_d \),
- the subgroup generated by \( \sigma_0, \ldots, \sigma_\infty \) acts transitively on \( \{1, \ldots, d\} \).

We weight the count by \( d! \cdot |\text{Aut}(\sigma_0)| \cdot |\text{Aut}(\sigma_\infty)| \). Here, the number \( r \) of points with simple ramification is determined by the Riemann-Hurwitz formula: \( r = 2g - 2 + |\lambda| + |\mu| \).

Goulden, Jackson, and Vakil [19] showed that, for \( \lambda \) and \( \mu \) of fixed length, double Hurwitz numbers \( H_g(\lambda; \mu) \) are piecewise polynomial in the entries of \( \mu \) and \( \nu \). This means that there is a hyperplane arrangement in \( \mathbb{R}^{l(\lambda)+l(\mu)} \) such that \( H_g(\lambda; \mu) \) is polynomial on each connected component of the complement of the arrangement. Here, \( l(\lambda) \) and \( l(\mu) \) are the number of parts of \( \lambda \) and \( \mu \). Using a tropical analog of Hurwitz numbers, Cavalieri, Johnson, and Markwig [7] confirmed Goulden, Jackson, and Vakil’s result.

**Theorem 4.3** ([19, Theorem 2.1], [8, Theorem 1.1]). For fixed \( l(\lambda) \) and \( l(\mu) \), the function \( H_g(\lambda; \mu) : (\mathbb{Z} - \{0\})^{l(\lambda)+l(\mu)} \to \mathbb{Q} \) is piecewise polynomial.

Shadrin, Shapiro, and Vainshtein [31] computed, in genus 0, the chamber structure of \( H_0(\lambda; \mu) \) (i.e., the domains of polynomiality) as well as an explicit “wall crossing formula.” The latter describes how \( H_0(\lambda; \mu) \) changes when one moves from a chamber to an adjacent chamber.

Cavalieri, Johnson, and Markwig [8] generalized these results to double Hurwitz numbers for all genera, using tropical geometry techniques, and thus giving a common approach to both [19] and [31].

**Theorem 4.4** ([8, Theorems 1.3 and 1.5]).

1. The chambers of polynomiality of \( H_g(\lambda; \mu) \) are the complements of hyperplanes given by explicit formulas.
2. The wall crossing formulas are computed explicitly in terms of Hurwitz numbers with fewer simple ramification points.

For the explicit formulas, see [8].
4.4. **Real Enumerative Geometry via Tropical Geometry.** Tropical geometry was also successfully applied to problems in real algebraic geometry. There, one studies, for example, those complex solutions to polynomial equation that are invariant (as a set) under complex conjugation. In general, the real analogs of complex enumerative problems are even more difficult. Nevertheless, some of them can be addressed nicely by tropical means. For a general reference on real algebraic geometry, see [32], for some further tropical application see for example [20, 21, 27].

In this section, we focus on enumeration of **real plane curves**, i.e., complex algebraic curves in $\mathbb{C}P^2$ invariant under complex conjugation. A natural question is about a real analog of Gromov-Witten invariants of $\mathbb{C}P^2$: real plane curves of fixed degree and genus passing through a real point configuration in general position. We quickly run into difficulty however: if we try to naively count such curves, we find that their number does depend on the point configuration! For example, the number of real rational cubics through 8 real points in general position can be 8, 10, or 12.

Welschinger [38] resolved this problem by proposing to count real plane curves with a sign. To a real plane curve $C$, Welschinger associated a multiplicity $(-1)^s$, where $s$ is the number of real double points of $C$ (a real double point is locally given by $\{x^2 + y^2 = 0\}$, with $x$ and $y$ some local coordinates). For $d \geq 1$, let the **Welschinger invariant** $W_d$ be the number of rational degree-$d$ real plane curves, counted with Welschinger’s multiplicity, passing through $3d - 1$ points in general position. Welschinger showed that $W_d$ is indeed independent of the point configuration [38, Theorem 2.1]. In particular, there are always $W_3 = 8$ real rational plane cubics through 8 points in general position, if one counts them with Welschinger’s multiplicity.

From the definition, it is not at all obvious whether $W_d$ is positive or negative or zero. With tropical geometry, Itenberg, Kharlamov, and Shustin resolved this. To the author’s knowledge, no non-tropical proof of the following has been discovered yet.

**Theorem 4.5** ([20, Theorem 1.1]). For all $d \geq 1$, the Welschinger invariant $W_d$ is positive.

The proof is based on a real analog of Mikhalkin’s Correspondence Theorem: Mikhalkin associates in [27] to a tropical curve $\Gamma$ not only a (complex) multiplicity $\text{mult}(\Lambda)$ (as we do in Section 3.1), but also a **real multiplicity** $\text{mult}_R(\Lambda)$. For the precise definition, see [27, Definition 7.19]. Similarly to the complex case, we define the **tropical Welschinger invariant** $W_d^{\text{trop}}$ as the number of tropical degree-$d$ genus-0 plane curves $\Gamma$ passing though $3d - 1$ points in general position, but now counted with multiplicity $\text{mult}_R(\Gamma)$.

**Theorem 4.6** (Mikhalkin’s Real Correspondence Theorem [27, Theorem 6]). For any $d \geq 1$, we have

$$W_d = W_d^{\text{trop}}.$$
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Florian Block, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom.
E-mail address: f.s.block@warwick.ac.uk