CONFORMATIONALLY NATURAL EXTENSIONS REVISITED

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Abstract. In this note we revisit the notion of conformal barycenter of a measure on $S^n$ as defined by Douady and Earle [DE]. The aim is to extend rational maps from the Riemann sphere $\mathbb{C} \approx S^2$ to the (hyperbolic) three ball $B^3$ and thus to $S^3$ by reflection. The construction which was pioneered by Douady and Earle in the case of homeomorphisms actually gives extensions for more general maps such as entire transcendental maps on $\mathbb{C} \subset \mathbb{C}$. And it works in any dimension.

1. Introduction

Let $G = G_n$ denote the group of Möbius transformations of Möbius space $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ preserving the $n$-sphere $S^n$:

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \ldots + x_{n+1}^2 = 1\}.$$ 

as well as the enclosed ball $B^{n+1}$. Then each element $g \in G$ acts on $B^{n+1}$ as a hyperbolic isometry, that is it preserves the Riemannian metric $2|dx|/(1 - |x|^2)$. Moreover each $g$ is conformal and thus also acts as a conformal automorphism of both $S^n$ and $B^{n+1}$. Mostow [Mo] proved that any conformal isomorphism of $B^{n+1}$ and/or $S^n$ is an element of $G$, so that we may also define $G$ as the conformal automorphism group of $B^{n+1}$ and/or $S^n$.

We let $G_+ = G_{n,+}$ denote the index two subgroup consisting of orientation preserving conformal automorphisms. And we let $c$ denote the reflection in the coordinate plane $x_{n+1} = 0$, so that $G$ is generated by $G_+$ and $c$, i.e. $G = \langle G_+, c \rangle$.

We equip $S^n$ with the Spherical metric, which is the infinitesimal metric induced by the Euclidean metric on the ambient space $\mathbb{R}^{n+1}$. And we denote by $R = R_n$ the subgroup consisting of Euclidean isometries, and by $R_+ := G_+ \cap R$ the subgroup of orientation preserving rigid rotations. Then $R$ is also the stabilizer of the origin 0. For $w \in B^{n+1}$ define $g_w \in G_+$ by

$$g_w(x) = \frac{x(1 - |w|^2) + w(1 + |x|^2 + 2 \langle w, x \rangle)}{1 + |w|^2|w|^2 + 2 \langle w, x \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Then $g_w$ preserves the line segment $[-w/|w|, w/|w|]$, fixes $\pm w/|w|$ and $g_w^{-1} = g_{-w}$. Moreover for $0 < r < 1$ let $r = (r, 0, \ldots, 0) = re_1$ and write $g_r := g_r$, where in general $e_j$ denotes the $j$-th element of the standard orthonormal basis for $\mathbb{R}^{n+1}$. Any $g_w$ can be written $g_w = \rho \circ g_r$, where $r = |w|$ and $\rho = g_w \circ g_r^{-1} \in R_+$. Moreover any element $g \in G_+$ can be written in a unique way as

$$g = g_w \circ \rho.'$$

Date: January 12, 2013.
where \( w = g(0) \), and

\[
\rho' = g_w^{-1} \circ g = g_w \circ g \in R_+.
\]

So that in fact \( G_+ = \langle R_+, (g_r)_{0 < r < 1} \rangle \) and \( G = \langle R_+, (g_r)_{0 < r < 1}, c \rangle \).

We can identify \( \mathbb{R}^n \) with \( S^n \) via stereographic projection of the central plane \( x_{n+1} = 0 \) in \( \mathbb{R}^{n+1} \) or equivalently through reflection in the sphere \( S^n(e_{n+1}, \sqrt{2}) \). In the case \( n = 2 \) stereographic projection identifies \( \mathbb{C} = \mathbb{C}P^1 \) with \( S^2 \). And in the \( \mathbb{C} \) coordinate an orientation preserving Möbius-transformation \( g \) preserving the unit circle can be written \( g = g_w(\rho z) \), where \( |\rho| = 1 \) and

\[
g_w(z) = \frac{z + w}{1 + \overline{w}z}, \quad \text{where } w \in \mathbb{D}.
\]

Following Douady and Earle the group \( G \) operates on \( \mathbb{B}^{n+1}, \partial \mathbb{B}^{n+1} = S^n \), on the set of probability measures \( \mathcal{P}(S^n) \) and on the vector space \( \mathcal{F}(\mathbb{B}^{n+1}) \) of continuous vector fields on \( \mathbb{B}^{n+1} \). That is

\[
g \cdot z = g(z), \quad \text{for } z \in \mathbb{B}^{n+1},
\]

\[
(g \cdot \mu)(A) = g_\mu(A) = \mu(g^{-1}(A)), \quad \text{for } \mu \in \mathcal{P}(S^n) \text{ and } A \subset S^n \text{ a Borel subset},
\]

\[
(g \cdot \nu)(g(z)) = g_\nu(g(z)) = D_2g(\nu(z)), \quad \text{for } \nu \in \mathcal{F}(\mathbb{B}^{n+1}) \text{ and } z \in \mathbb{B}^{n+1}.
\]

Here \( D_2g \) denotes the differential of \( g \) at \( z \). The group \( G \times G \) operates on the spaces \( \text{End}(\mathbb{B}^{n+1}), \mathcal{C}(\mathbb{B}^{n+1}) \) and \( \text{End}(\mathbb{S}^n), \mathcal{C}(\mathbb{S}^n) \) of endomorphisms and continuous endomorphisms of \( \mathbb{B}^{n+1} \) and \( \mathbb{S}^n \) respectively by

\[
(g, h)\phi := g \circ \phi \circ h^{-1}.
\]

If \( G \) operates on the spaces \( X \) and \( Y \) then a map \( T : X \to Y \) is called \( G \)-equivariant or conformally natural if

\[
\forall g \in G, \quad \forall x \in X : T(g \cdot x) = g \cdot T(x).
\]

And if \( G \times G \) operates on both \( X \) and \( Y \) then conformal naturality of \( T \) is taken to mean \( G \times G \)-equivariance.

Douady and Earle introduced the idea of Conformal Barycenter for probability measures on \( S^1 \subset \mathbb{C} \) and more generally on \( S^n \subset \mathbb{R}^n \). They used the conformal barycenter to define conformally natural extensions of self-homeomorphisms of \( S^n \). We shall in this note study the application of the Douady-Earle extension operator to a much wider class of maps than just homeomorphisms. More precisely let \( \eta_0 \) denote the normalized standard Euclidean or Lebesgue probability measure on \( S^n \). And let \( f : S^n \to S^n \) be a measureable endomorphism, for which the push-forward \( f_*(\eta_0) \) of \( \eta_0 \) by \( f \) is absolutely continuous with respect to \( \eta_0 \). We shall show that the Douady-Earle extension operator also yields a conformally natural extensions of maps such as \( f \), to non-constant self maps also denoted by \( f, f : \overline{B}_{n+1} \to \overline{B}_{n+1} \), which are real-analytic in the interior and continuous, whenever the original map \( f \) is continuous. In particular we obtain extensions of rational and entire transcendental maps of \( \mathbb{S}^2 \cong \mathbb{C}P^1 \) to the hyperbolic three-space \( \mathbb{B}^3 \). And of course by reflection and renewed stereographic projection to an endomorphism of \( \mathbb{S}^3 \).

The motivation for this note comes from talks by Bill Thurston at a workshop in Roskilde 2010, where he asked: "What is the three-manifold of a rational map?"
"How can we in a natural way extend rational maps to $\mathbb{B}^{3n}$. The answer we propose to the second question is: Use the Douady-Earle extension.

In order to be self-contained we shall start by reviewing the Douady-Earle construction of the conformal barycenter and the Douady-Earle extension in any dimension.

2. Conformal Barycenters

2.1. Harmonic measure. Denote be $\eta_0$ the normalized Euclidean Lebesgue measure on $\mathbb{S}^n$,

$$\eta_0(A) = \frac{1}{\text{Vol}(\mathbb{S}^n)} \int_A L, \quad \text{Vol}(\mathbb{S}^n) = \int \ldots \int L,$$

where $L$ denotes Lebesgue measure. We shall henceforth also write $\eta_0(A) = \int_A d\eta_0$.

Then $\eta_0$ is invariant under $R$, i.e. $g^*(\eta_0) = \eta_0$ for every element $g \in R$.

For $w \in \mathbb{B}^{n+1}$ the harmonic measure with center $w$ is the measure $\eta_w = (g_w)_*(\eta_0)$. Note that by the above $\eta_w = g^*_w(\eta_0)$ for any $g \in G$ with $g(0) = w$.

Also note that since each $g \in G$ is conformal $|\text{Jac}_g(z)| = ||\text{Jac}_g(z)||^n$, where $|| \cdot ||$ denotes the operator norm and $| \cdot |$ denotes determinant. In the 2-dimensional and thus 1-complex dimensional case one computes for $g_w$ and $|z| = 1$:

$$|g'_w(z)| = \frac{1 - |w|^2}{|z + w|^2}.$$

Thus in real dimension $n$ we obtain for $z \in \mathbb{S}^n$ and $w \in \mathbb{B}_{n+1}$:

$$|\text{Jac}_{g_w}(z)| = \left(1 - \frac{|w|^2}{|z + w|^2}\right)^n$$

and hence by the change of variables formula

$$\eta_w(A) = \int_A 1 \, d\eta_w = \int_A 1(g_w)_* \eta_0 = \int_{g_w(A)} 1 \, d\eta_0 = \int_A \left(\frac{1 - |w|^2}{|z - w|^2}\right)^n \, d\eta_0.$$

The Conformal Barycenter of a measure. Let us define a probability measure to be admissible, if it has no atoms of mass greater than or equal to $1/2$. And let $\mathcal{P}'(\mathbb{S}^n)$ denote the space of admissible probability measures. To each admissible probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ we shall assign a point $B(\mu) \in \mathbb{B}_{n+1}$ so that the map $\mu \mapsto B(\mu) : \mathcal{P}'(\mathbb{S}^n) \to \mathbb{B}_{n+1}$ is conformally natural and normalized by

$$(2) \quad B(\mu) = 0 \quad \Leftrightarrow \quad \int_{\mathbb{S}^n} \zeta \, d\mu(\zeta) = 0.$$

**Proposition 1.** The mapping $V : \mathcal{P}(\mathbb{S}^n) \to \mathcal{F}(\mathbb{B}^{n+1})$, which to a probability measure $\mu \in \mathcal{P}'(\mathbb{S}^n)$ assigns the vector field

$$(3) \quad V_\mu(w) = \frac{1 - |w|^2}{2} \int_{\mathbb{S}^n} g_{-w}(\zeta) \, d\mu(\zeta), \quad w \in \mathbb{B}^{n+1}$$
is the unique conformally natural such map satisfying the normalizing condition

\[ V_\mu(0) = \frac{1}{2} \int_{S^n} \zeta \, d\mu(\zeta). \]

The normalizing factor \( \frac{1}{2} \) is inessential, but kept here in order to make \( V_\mu \) asymptotically a hyperbolic unit vector field at \( \infty \), when \( \mu \) has no atoms.

**Proof.** Equivariance or conformal invariance is equivalent to

\[ \forall g \in G, \forall w \in \mathbb{B}^{n+1} : \quad V_{g*\mu}(g(w)) = (g \cdot V_\mu)(g(w)) = D_w g(V_\mu(w)) \]

Thus the normalizing condition (4) is invariant under the subgroup \( R \) stabilizing the origin, because such maps are linear. And for \( g = g_{-w} = g_{w}^{-1} \) with \( g_{-w}(w) = 0 \), the above formula implies

\[ \forall w \in \mathbb{B}^{n+1} : \quad V_\mu(w) = D_{0g_w} V_{g_{-w} \mu}(0) \]

Thus the mapping \( \mu \mapsto V_\mu \) is conformally natural if and only if

\[
\begin{align*}
V_\mu(w) &= 1 - \frac{|w|^2}{2} \int_{S^n} \frac{\zeta}{g_{-w}*\mu}(\zeta) \\
&= 1 - \frac{|w|^2}{2} \int_{S^n} g_{-w}(\zeta) \, d\mu(\zeta) \\
&= 1 - |w|^2 \int_{S^n} \frac{\zeta(1 - |w|^2) - 2w(1- \zeta < \zeta, w >)}{1 + |w|^2 - 2 < \zeta, w >} \, d\mu(\zeta).
\end{align*}
\]

\[ (5) \]

Next we want to prove that

**Proposition 2.** For each admisible measure \( \mu \in \mathcal{P}'(S^n) \) the vector field \( V_\mu \) has a unique zero in \( \mathbb{B}^{n+1} \).

For the proof we shall use a few elementary lemmas, which are generalizations to dimension 3 and higher of the corresponding statements for the complex plane, as can be found in \[D-E, \text{Sections 2 and 11}].

**Lemma 3.** For any admisible probability measure \( \mu \in \mathcal{P}'(S^n) \) any zero \( v \in \mathbb{B}^{n+1} \) of the vector field \( V = V_\mu \) is an isolated stable equilibrium.

**Proof.** By conformal naturallity it suffices to consider the case \( v = 0 \). Expanding the above formula (5) for \( V_\mu(w) \) to first order in \( w \) we obtain:

\[
\begin{align*}
V_\mu(w) &= \frac{1}{2} \int_{S^n} \zeta - 2(w - \zeta < w, \zeta >) \, d\mu(\zeta) + o(|w|) \\
&= V_\mu(0) - \int_{S^n} (w - \zeta < w, \zeta >) \, d\mu(\zeta) + o(|w|) \\
&= - \int_{S^n} (w - \zeta < w, \zeta >) \, d\mu(\zeta) + o(|w|)
\end{align*}
\]
since $V_\mu(0) = 0$. Hence the Jacobian of $V$ at $v = 0$ is given by

$$\text{Jac}_V(0)(\zeta) = -\int_{S^n} (\zeta - \zeta < \zeta, \zeta >) \, d\mu(\zeta)$$

and thus $\text{Jac}_V(0)$ is non singular. In fact $v$ is a sink since

$$< \zeta, \text{Jac}_V(0)(\zeta) > = -\int_{S^n} (< \zeta, \zeta > - < \zeta, \zeta > < \zeta, \zeta >) \, d\mu(\zeta) < 0. \tag{7}$$

Douady and Earle showed that if $\mu(D_{S^n}(e_1, \pi/4)) \geq \frac{2}{3}$ then

$$< V_\mu(0), e_1 >> 0, \tag{8}$$

where $D_{S^n}(e_1, \delta)$ denotes the closed ball in $S^n$ of center $e_1$ and spherical radius $\delta$. This is sufficient to prove Proposition 2 if $\mu$ has no atoms of mass $\frac{1}{3}$ or higher. To prove the Proposition also, when no atom has mass $\frac{1}{2}$ or higher, we need the following slight refinement:

**Lemma 4.** Let $\delta \in ]0, \sqrt{2}[)$ and suppose $\mu(D_{S^n}(e_1, \delta)) \geq (1 + \frac{\sqrt{2}}{2})/2$. Then

$$< V_\mu(0), e_1 >> 0.$$

**Proof.**

$$< V_\mu(0), e_1 > = \int_{D_{S^n}(e_1, \delta)} < \zeta, e_1 > \, d\mu(\zeta) + \int_{S^n \setminus D_{S^n}(e_1, \delta)} < \zeta, e_1 > \, d\mu(\zeta) \geq (1 - \frac{\delta^2}{2})(1 + \frac{\delta^2}{2})/2 - 1 : (1 - \frac{\delta^2}{2})/2 = \frac{\delta^2}{4}(1 - \frac{\delta^2}{2}) > 0.$$

**Lemma 5.** Suppose that $\mu$ is admissible. Then there exists $r \in ]0, 1[$ such that $V_\mu(w)$ points inwards at any point $w \in B^{n+1}$ with $r \leq |w| < 1$, i.e. $< V_\mu(w), w >> 0$.

**Proof.** Choose $\delta \in ]0, \sqrt{2}[)$ such that for any $\zeta \in S^n : \mu(\{\zeta\}) < (1 - \frac{\delta^2}{2})/2$. Then there exists $\epsilon \in ]0, \pi[)$ such that for $\zeta \in S^n : \mu(D_{S^n}(\zeta, \epsilon)) \leq (1 - \frac{\epsilon^2}{2})/2$. Choose $r \in ]0, 1[$ such that $\forall w \in B^{n+1}$ with $r \leq |w| < 1$:

$$\eta_w(S^n \setminus D_{S^n}(\frac{w}{|w|}, \epsilon)) \leq \eta_0(D_{S^n}(e_1, \delta)).$$

Then it follows from Lemma 4 that $V_\mu(w)$ points into the sphere $S = |w|S^n$: Let $g \in G_+$ be any Möbius transformation mapping $w$ to $0$ and $-w/|w|$ to $e_1$. Then $g(S)$ is a sphere through $0$ and with $e_1$ as an inwards pointing normal vector at $0$. Moreover let $\nu = g_* \mu$ then by conformal naturality $g_*(V_\mu(w)) = V_\nu(0)$ and $\nu$ satisfies the hypotheses of Lemma 4.

**Proof.** of Proposition 2 Let $\mu \in P(S^n)$ be any admissible measure, i.e. with no atom of mass $1/2$ or higher. In Lemma 3 we have shown that any zero of the vector field $V_\mu$ is an isolated stable equilibrium, i.e. the vector field points inwards on small spheres around the zero. Moreover by Lemma 5 the vector field $V_\mu$ is pointing inwards near the boundary $S^n$ of $B^{n+1}$. Hence by the Poincaré-Hopf theorem [Mi] see also Lemma 3, p 36] $V_\mu$ has a unique zero $B(\mu) \in B^{n+1}$.\qed
Definition 6. Define a conformally natural mapping $B : \mathcal{P}(\mathbb{S}^n) \to \mathbb{B}^{n+1}$ by setting $B(\mu)$ equal to the unique zero $w \in \mathbb{B}^{n+1}$ of the vector field $V_{\mu}$. Then $B$ satisfies (2).

3. Extending continuous endomorphisms of $\mathbb{S}^n$.

Let $\mathcal{E}(\mathbb{S}^n)$ denote the space of endomorphisms $\phi : \mathbb{S}^n \to \mathbb{S}^n$ such that $\phi_*\eta_0$ has no atoms. For such mappings the measures $\phi_*\eta_z$ has no atoms neither for any $z \in \mathbb{B}^{n+1}$. And let $\text{End}(\mathbb{B}^{n+1})$ denote the space of endomorphisms of $\mathbb{B}^{n+1}$, whose restrictions to $\mathbb{B}^{n+1}$ are endomorphisms of $\mathbb{B}^{n+1}$.

The Douady-Earle extension operator $E$, in the following denoted the D-E extension, which Douady and Earle studied for homeomorphisms is the map $E : \mathcal{E}(\mathbb{S}^n) \to \text{End}(\mathbb{B}^{n+1})$ defined as follows: For $\phi \in \mathcal{E}(\mathbb{S}^n)$ the mapping $E(\phi) = \Phi : \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ is given by the formulas

\begin{equation}
\Phi(z) = \begin{cases} 
\phi(z), & z \in \mathbb{S}^n, \\
B((\phi \circ g_z)_*)(\eta_0)) = B(\phi_*(\eta_z)), & z \in \mathbb{B}^{n+1}
\end{cases}
\end{equation}

Clearly the mapping $\phi \mapsto E(\phi) = \Phi$ is conformally natural, i.e. for all $g, h \in G$:

$$E(g \circ \phi \circ h) = g \circ E(\phi) \circ h.$$ 

Moreover for any conformal automorphism $g \in G$ we have $E(g|_{\mathbb{S}^n}) = g$, by conformal naturality of $E$ and the fact that $B(\eta_0) = 0$. We can also formulate this as saying that the D-E extension operator extends the Poincaré extension operator. For $n = 1$ at least we have a much stronger property: For inner functions, [R, Def. 17.14] that is for holomorphic selfmaps $f : \mathbb{D} \to \mathbb{D}$ of the unit disc $\mathbb{D} \subset \mathbb{C}$ with boundary values in $\mathbb{S}^1$ a.e., the D-E extension simply recovers $f$ from its boundary values. More precisely it is well known that for bounded holomorphic functions (see [R, Th. 11.21]) the radial limit

$$f^#(\zeta) = \lim_{r \to 1} f(r\zeta)$$

exists for a.e. $\zeta \in \mathbb{S}^1$ and satisfies the Cauchy formula:

\begin{equation}
\forall z \in \mathbb{D} : \quad f(z) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \frac{f^#(\zeta)}{\zeta - z} \, d\zeta.
\end{equation}

Moreover the space of such functions $f^#$ is the space of bounded measureable functions, whose negative Fourier coefficients are all equal to zero. For inner functions where $|f^#(\zeta)| = 1$ a.e. the measure $f^#\eta_0$ is absolutely continuous with respect to $\eta_0$ (see [R, Th. 17.13]) and hence $f^#\eta_z$ is absolutely continuous with respect to $\eta_0$ for any $z \in \mathbb{D}$, so that $f^# \in \mathcal{E}(\mathbb{S}^1)$.

Proposition 7. If $f : \mathbb{D} \to \mathbb{D}$ is an inner function then

$$E(f^#)(z) = f(z), \quad \forall z \in \mathbb{D}.$$ 

Proof. Let $f$ be an arbitrary inner function and let $z \in \mathbb{D}$ be arbitrary. We need to show that $f(z) = E(f^#(z))$. By conformal naturality we can suppose $z = f(z) = 0$ as
we may precompose by $g_z$ and postcompose by $g_{-f(z)}$. That is it suffices to prove that $B(f_\ast \eta_0) = 0$ for any inner function $f$ with $f(0) = 0$. We compute

$$2V_{f_\ast \eta_0}(0) = \int_{S^1} \zeta(f_\ast \eta_0)(\zeta) = \int_{S^1} f_\ast(\zeta) d\eta_0(\zeta) = \frac{1}{2\pi i} \int_{S^1} \frac{f_\ast(\zeta)}{\zeta} d\zeta = f(0) = 0.$$ 

$\square$

**Lemma 8.** Let $\phi \in \mathcal{E}(\mathbb{S}^n)$ and let $\Phi = E(\phi)$. If $\phi$ is continuous at some point $\zeta_0 \in \mathbb{S}^n$ then so is $\Phi$. In particular if $\phi$ is continuous then $\Phi$ is continuous on $\mathbb{S}^n$.

We shall see in the next lemma that $\Phi$ is real-analytic in $\mathbb{B}^{n+1}$, so that in particular $\Phi$ is continuous whenever $\phi$ is continuous.

**Proof.** Recall that Euclidean balls $\mathbb{B}^{n+1}(\zeta, r)$ and spheres $\mathbb{S}^n(\zeta, r)$ are mapped to such balls and spheres (possibly half spaces and hyperplanes or complements of a closed ball union $\infty$) under any conformal automorphism $g \in G$.

Thus given a spherical ball $B_{\mathbb{S}^n}(\zeta, \delta) \subset \mathbb{S}^n$, $0 < \delta < \pi$ and $w \in \mathbb{B}^{n+1}$ there are two alternative ways of describing the size of $B_{\mathbb{S}^n}(\zeta, \delta)$ viewed from $w$. Either we can use the visual Poincaré radius from $w$, i.e. the spherical radius of the ball $g_{-w}(B_{\mathbb{S}^n}(\zeta, \delta))$ in $\mathbb{S}^n$ or we can use the $w$-harmonic measure $\eta_w(B_{\mathbb{S}^n}(\zeta, \delta)) = \eta_0(g_{-w}(B_{\mathbb{S}^n}(\zeta, \delta)))$.

Given $B_{\mathbb{S}^n}(\zeta, \delta)$ we denote by $W(B_{\mathbb{S}^n}(\zeta, \delta))$ the set $B_{\mathbb{S}^n}(\zeta, \delta)$ itself union the open subset of points $w \in \mathbb{B}^{n+1}$ for which the visual Poincaré radius from $w$ exceeds $\pi/4$. Similarly we denote by $U(B_{\mathbb{S}^n}(\zeta, \delta))$ the set $B_{\mathbb{S}^n}(\zeta, \delta)$ itself union the open subset of points $w \in \mathbb{B}^{n+1}$ for which $\eta_w(B_{\mathbb{S}^n}(\zeta, \delta)) > 2/3$. Then $U(B_{\mathbb{S}^n}(\zeta, \delta)) \subset W(B_{\mathbb{S}^n}(\zeta, \delta))$ and both sets are neighborhoods of $\zeta$ in $\mathbb{B}^{n+1} = \mathbb{B}^{n+1} \cup \mathbb{S}^n$. In fact for $\zeta = e_1$ and $\delta = \pi/4$ the set $W(B_{\mathbb{S}^n}(\zeta, \delta))$ equals the intersection of $\mathbb{B}^{n+1}$ with the open ball $\mathbb{B}^{n+1}(\sqrt{2}e_1, 1)$ and for $\delta = 2\pi/3$ the open set $U(B_{\mathbb{S}^n}(\zeta, \delta))$ is the complement $\mathbb{B}^{n+1} \setminus \mathbb{B}^{n+1}(-2e_1, \sqrt{3})$. Clearly any of the families of sets $U(B_{\mathbb{S}^n}(\zeta, \delta)), W(B_{\mathbb{S}^n}(\zeta, \delta)), 0 < \delta < \pi$ forms fundamental systems of neighbourhoods of $\zeta$ in $\mathbb{B}^{n+1}$. Suppose $\phi$ is continuous at $\zeta_0$ and let $0 < \epsilon < \pi$ be given. Choose $0 < \delta < \pi$ such that

$$\phi(B_{\mathbb{S}^n}(\zeta_0, \delta)) \subset B_{\mathbb{S}^n}(\phi(\zeta_0), \epsilon).$$

Then for any $w \in U(B_{\mathbb{S}^n}(\zeta_0, \delta))$ and any $z \in \partial W(B_{\mathbb{S}^n}(\phi(\zeta_0), \epsilon)) \cap \mathbb{B}^{n+1}$ the vector $V_{\phi, \eta_w}(z)$ points into $W(B_{\mathbb{S}^n}(\phi(\zeta_0), \epsilon))$. Hence $\Phi(w)$ the unique zero of $V_{\phi, \eta_w}$ belongs to $W(B_{\mathbb{S}^n}(\phi(\zeta_0), \epsilon)))$. This proves continuity at $\zeta_0$. $\square$

**Lemma 9.** Let $\phi \in \mathcal{E}(\mathbb{S}^n)$ and $E(\phi) = \Phi$ be as above. Then $\Phi$ is real-analytic in $\mathbb{B}^{n+1}$. 
Proof. Towards real-analyticity of $\Phi$ recall that $\Phi(z)$ is the unique zero of the vector field

$$V_{\phi_\ast(\eta_0)}(w) = \frac{1 - |w|^2}{2} \int_{S^n} \zeta(g_{-w} \circ \phi_\ast \eta_0)(\zeta)$$

$$= \frac{1 - |w|^2}{2} \int_{S^n} g_{-w}(\phi(\zeta))(g_{-z})_\ast \eta_0(\zeta)$$

$$= \frac{1 - |w|^2}{2} \int_{S^n} g_{-w}(\phi(\zeta)) \left(1 - \frac{|z|^2}{|z - \zeta|^2}\right)^n d\eta_0(\zeta).$$

Thus $\forall z \in \mathbb{B}^{n+1}$ the value $w = \Phi(z)$ is the unique point $w \in \mathbb{B}^{n+1}$ such that:

$$F(z, w) = \frac{2V_{\phi_\ast(\eta_0)}(w)}{1 - |w|^2} = \int_{S^n} g_{-w}(\phi(\zeta)) \left(1 - \frac{|z|^2}{|z - \zeta|^2}\right)^n d\eta_0(\zeta) = 0.$$

Clearly $F$ is a real-analytical function of $(z, w) \in \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$. Thus by the implicit function theorem we need only show that for any pair $(z, w) \in \mathbb{B}^{n+1} \times \mathbb{B}^{n+1}$ with $F(z, w) = 0$ the $w$ partial derivatives matrix $J_w F = \frac{\partial F}{\partial w}$ evaluated at $(z, w)$ is non-singular. By conformal naturality we can suppose $z = w = \Phi(z) = 0$. A straightforward computation analogous to the one leading to (6) yields that $J_w F$ evaluated at $(0, 0)$ and applied to the vector $\epsilon$ is given by the formula:

$$J_w F(\epsilon) = -2 \int_{S^n} (\zeta - <\zeta, \phi(\zeta) > \phi(\zeta)) d\eta_0(\zeta).$$

Similarly as for Jac$_V$ this shows that $J_w F$ is non singular at $(0, 0)$. So $\Phi$ is real-analytic by the implicit function theorem and

$$(11) \quad \text{Jac}_\Phi(0) = -(J_w F)^{-1} \circ J_z F$$

where both partial derivative matrices are evaluated at $(0, 0)$ and

$$J_z F = \int_{S^n} \phi(\zeta) \times \zeta d\eta_0(\zeta),$$

and where $\phi(\zeta) \times \zeta$ is the matrix valued mapping

$$A_{ij}(\zeta) = \phi_i(\zeta) \cdot \zeta_j.$$

4. Properties of the D-E extension of rational maps

The question that naturally arises is: For $f$ a rational map on the Riemann sphere. What are the geometric and dynamical properties of the D-E extension $E(f)$? How many of the properties of $f$ are inherited by $E(f)$. By elementary topology $E(f)$ is a proper map, that is the preimage of any compact set is compact. And moreover for any point $w \in \mathbb{B}^{n+1}$ the preimage $E(f)^{-1}(w)$ is a real analytic set.

Question 1: Is $E(f)$ a discrete map?
Question 2: Is $E(f)$ an open map?
Question 3: Is $E(f)$ a map of the same degree as $f$?
Question 4: Is the Julia set of $E(f)$ (the set of points $x$ for which the family of iterates does not form an equicontinuous family on any neighbourhood of $x$) equal to the convex hull of the Julia set for $\hat{f}$?

In certain elementary cases at least the immediate answer to the above questions are yes, but not completely satisfactory.

For the following discussion we shall identify $\mathbb{C}$ with the coordinate plane in $\mathbb{R}^3$, \{x = (x_1, x_2, x_3) | x_3 = 0\} and write $z = x + iy$ for the point $(x, y, 0)$. In particular we shall identify the complex unit disk $\mathbb{D}$ with the disc \{x $\in \mathbb{R}^3$ | $|z|^2 = x_1^2 + x_2^2 < 1, x_3 = 0$\} and the unit circle $S^1$ with the circle \{x $\in \mathbb{R}^3$ | $|z|^2 = 1, x_3 = 0$\}. Then stereographic projection $S$ of $\mathbb{C}$ on to $\mathbb{S}^2$ from the north pole $N = e_3 \in \mathbb{R}^3$ is the map

\[
z \mapsto S(z) = \left( \frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2} \right) = \left( 1 + \frac{2x}{1 + |z|^2}, \frac{1}{1 + |z|^2}(2x, 2y, 1 - |z|^2) \right).
\]

For $f : \mathbb{C} \to \mathbb{C}$ a holomorphic map we shall write $\hat{f}$ for its conjugate by $S$, i.e.:

\[
\hat{f}(S(z)) = S(f(z)).
\]

In the following we shall discuss finite Blaschke products

\[
f(z) = \sigma \prod_{j=1}^d \frac{z + a_j}{1 + \overline{a_j}z}, \quad |\sigma| = 1, \quad a_j \in \mathbb{D}
\]

**Proposition 10.** For $f$ a finite Blaschke product the D-E extension $E(\hat{f})$ maps $\mathbb{D}$ onto $\mathbb{D}$, preserves the upper and lower hemispheres, $S^2_+, S^2_-$ and further more on $\mathbb{D}$ we have $\partial E(\hat{f})/\partial x_3 = g(z)e_3$ for some positive real analytical function $g : \mathbb{D} \to \mathbb{R}_+$.

If moreover $f(z) = z^d$ (i.e. $a_j = 0$ for all $j$), then $E(\hat{f})(z) = z^d \cdot h(|z|^2)$ for some real analytical function $h$ with $h(r) \to 1$ as $r \to 1$.

**Proof.** The reflection $c(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ is the Poincaré extension of $\hat{\tau}$, where $\tau(z) = 1/\overline{z}$ denotes the reflection in $S^1$. Then $c \circ \hat{f} \circ c = \hat{f}$. Write $\Phi = E(\hat{f})$ for the D-E extension of $\hat{f}$. Then by conformal naturality of the D-E extension

\[
c \circ \Phi = \Phi \circ c.
\]

Hence $\Phi(\mathbb{D}) \subseteq \mathbb{D}$, since $c_1 : \mathbb{D} \to \mathbb{D}$ is the identity. Moreover if $\Phi(\mathbb{D}) \neq \mathbb{D}$, then a simple homotopy argument would imply that the restriction $\hat{f}_1 = \Phi_1$ to $S^1$ is homotopic to a constant map. Thus $\Phi(\mathbb{D}) = \mathbb{D}$.

To prove that $\Phi$ preserves the upper and hence the lower hemisphere it suffices to prove that for any $x \in S^2_+$ and any $w \in \mathbb{D}$: $e_3 \cdot V_{\hat{f}, \eta_0}(w) > 0$. Furthermore by conformal naturality it suffices to consider the case $x = te_3$ with $0 < t < 1$ and $w = 0$. Before we start computing let us note that since $c_\eta_0 = \eta_0$ we have for any measurable function $\phi : \mathbb{S}^n \to \mathbb{R}$ (C)

\[
\int_{\mathbb{S}^n} \phi(\zeta) \, d\eta_0 = \int_{\mathbb{S}^n_+} (\phi(\zeta) + \phi(c(\zeta))) \, d\eta_0(\zeta) = \int_{\mathbb{S}^n_-} (\phi(\zeta) + \phi(c(\zeta))) \, d\eta_0(\zeta).
\]
Applying this to $V_{f,\eta_z}(0)$ we obtain

$$e_3 \cdot V_{f,\eta_z}(0) = \int_{\mathbb{S}^2} e_3 \cdot \hat{f}(\zeta) \left( \frac{1 - |x|^2}{|x - \zeta|^2} \right)^2 \, d\eta_0(\zeta)$$

$$= \int_{\mathbb{S}^2} \hat{f}_3(\zeta) \left( \frac{1 - |x|^2}{|x - \zeta|^2} \right)^2 + \hat{f}_3(c(\zeta)) \left( \frac{1 - |x|^2}{|x - c(\zeta)|^2} \right)^2 \, d\eta_0(\zeta)$$

$$= \int_{\mathbb{S}^2} \hat{f}_3(\zeta) \left( \frac{1 - |x|^2}{|x - \zeta|^2} \right)^2 - \left( \frac{1 - |x|^2}{|x - c(\zeta)|^2} \right)^2 \, d\eta_0(\zeta) > 0,$$

since $|x - \zeta| < |x - c(\zeta)|$ and $\hat{f}_3$ is positive on $\mathbb{S}^2_+$. To compute the partial derivative vector $\partial \Phi/\partial x_3(z)$ for $z \in \mathbb{D}$ we equate the Jacobians of the two sides of (1.2) and obtain

$$\frac{\partial \Phi}{\partial x_3}(z) = \frac{\partial \Phi}{\partial x_2}(z) = \frac{\partial \Phi}{\partial x_1}(z) = \frac{\partial \Phi}{\partial x_2}(z) = 0.$$

Thus $\partial \Phi/\partial x_3(z) = \partial \Phi_3/\partial x_3(z)e_3 = g(z)e_3$. To complete the first set of statements we just need to show that $g$ is a positive function. By conformal naturality $z$ suffices to consider the case $\Phi(0) = 0$. Moreover in order to simply notation let us for $\phi : \mathbb{S}^n \to \mathbb{R} (\mathbb{C})$ a measureable function write

$$M(\phi) := \int_{\mathbb{S}^n} \phi(\zeta) \, d\eta_0(\zeta).$$

Then an elementary calculation, using (11) yields

$$g(0) = \frac{\partial \Phi_3}{\partial x_3}(0) = \frac{M(\hat{f}_3 \cdot \zeta_3)}{2M(1 - \hat{f}_3^2)} > 0,$$

since $\hat{f}_3 \cdot \zeta_3, (1 - \hat{f}_3^2) \geq 0$ with equality for $\zeta_3 = 0$ only in the first and for $f(z)$ equal to 0 or $\infty$ in the second.

In the special case $f(z) = z^d$ we have $f(e^{i\theta}z) = e^{id\theta}f(z)$. and thus by conformal naturality $\Phi(e^{i\theta}z) = e^{id\Phi}(z)$. Hence $\Phi(0) = 0$ and for $z \neq 0$:

$$\Phi(z) = \Phi(|z|) \frac{z^d}{|z|^d}.$$

Moreover $f$ commutes with complex conjugation, which translates to $\hat{f}$ commutes with the reflection $(x_1, x_2, x_3) \to (x_1, -x_2, x_3)$. As above this implies that $\Phi([-1, 1]) \subseteq [-1, 1]$, so that $\Phi$ is a real analytic real function on the reals. Expanding the real-analytic function in a power series in $z, \bar{z}$ on a neighbourhood of 0 and noting that $z\bar{z} = |z|^2$ we obtain:

$$\Phi(z) = \frac{z^d}{|z|^d} \sum_{m=0}^{\infty} b_m |z|^m = \sum_{n,k=0}^{\infty} a_{n,k} z^n \bar{z}^k.$$
By the uniqueness theorem for power series this implies that \( a_{n,k} = 0 \) for \( n - k \neq d \).

Thus we are left with

\[
\Phi(z) = \sum_{k=0}^{\infty} a_{k+d,k} z^{k+d}\varphi^k = z^d \sum_{k=0}^{\infty} a_{k+d,k}|z|^{2k}.
\]

\[ \square \]

Write \( M_t(z) = tz \) for \( 0 < t \) so that \( M_t \) is a homothety. Then the conformal automorphism \( h_t = E(\tilde{M}_t) = g_w \) with \( w = w(t) = \frac{t-1}{t+1}e_3 \) maps \( \mathbb{D} \) conformally onto the geodesic disk \( \mathbb{D}_t \) in \( \mathbb{B}^3 \) with boundary the circle \( \tilde{M}_t(\mathbb{S}^1) \).

**Corollary 11.** For \( f(z) = z^d \) the D-E extension \( E(\hat{f}) \) maps \( \mathbb{D}_t \) onto \( \mathbb{D}_{td} \) by a degree \( d \) ramified covering and the interval \([0, e_3]\) onto itself by an increasing diffeomorphism.

**Conjecture 1.** For all finite Blaschke products \( f \) we have \( f = E(\hat{f}) \) on \( \mathbb{D} \).

By conformal naturallity of the D-E extension this conjecture is equivalent to the seemingly simpler conjecture:

**Conjecture 2.** For all finite Blaschke products \( f \) with \( f(0) = 0 \) we have \( E(\hat{f})(0) = 0 \).

If true the following stronger conjecture would yield almost complete topological understanding of \( E(\hat{f}) \) for any finite Blaschke product \( f \):

**Conjecture 3.** For all finite Blaschke products \( f \) with \( f(0) = 0 \) the D-E extension \( E(\hat{f}) \) maps the interval \([0, e_3]\) diffeomorphically and increasingly onto itself.

Clearly the last conjecture implies the two previous ones. Moreover by conformal naturalty of the D-E extension, it would imply that for each \( z \in \mathbb{D} \) the unique hyperbolic geodesic through \( z \) and orthogonal to \( \mathbb{D} \) would be mapped diffeomorphically onto the unique such geodesic through \( f(z) \). And thus the dynamics of \( E(\hat{f}) \) would be conjugate to a skew product on \( \mathbb{D} \times [-1, 1] \). Which would be completely analogous to the case of Fuchsian groups.

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