Asymptotic stability of large energy harmonic maps under the wave map from 2D hyperbolic spaces to 2D hyperbolic spaces

Ze Li

Abstract In this paper, we prove that the large energy harmonic maps from $\mathbb{H}^2$ to $\mathbb{H}^2$ are asymptotically stable under the wave map. This result supports the soliton resolution conjecture for geometric wave equations in the large data case without equivariant assumptions.

1 Introduction

In this paper, we continue to study the asymptotic stability of harmonic maps under the wave equation from $\mathbb{R} \times \mathbb{H}^2$ to $\mathbb{H}^2$ along with the author’s previous collaborative work [49] where the small energy harmonic maps case was considered. Let $(M, h)$ and $(N, g)$ be two Riemannian manifolds without boundary. A wave map is a map from the Lorentz manifold $\mathbb{R} \times M$ into $N$,

$$u : \mathbb{R} \times M \to N,$$

which in the local coordinates $(x_1, ..., x_m)$ for $M$ and $(y_1, ..., y_n)$ for $N$ respectively satisfies

$$\Box u^k + \eta^{\alpha\beta} \Gamma^k_{ij}(u) \partial_\alpha u^i \partial_\beta u^j = 0,$$

(1.1)

Here $h = h_{ij} dx_i dx_j$, $g = g_{ij} dy_i dy_j$, $\eta = -dt dt + h_{ij} dx_i dx_j$ are the metric tensions for $M, N$ and $\mathbb{R} \times M$ respectively. Moreover, $\Box = -\partial_t^2 + \Delta_M$ is the D’Alembertian on $\mathbb{R} \times M$, $\Gamma^k_{ij}(u)$ are the Christoffel symbols at the point $u(t, x) \in N$. In this paper, we consider the case $M = \mathbb{H}^2$, $N = \mathbb{H}^2$.

The wave map equation on flat spacetimes known as the nonlinear $\sigma$-model, arises as a model problem in particle physics and is related to general relativity, see for instance Manton, Sutcliffe [51], Ionescu, Klainerman [24], Luk [50], Andersson, Gudapati, Szeftel [2]. The case where the background manifold is the hyperbolic space is of particular interest since the anti-de Sitter space (AdS$_n$) model is asymptotically hyperbolic.
The motivation of our paper is the so-called soliton resolution conjecture in dispersive PDEs which claims that every global bounded solution splits into the superposition of divergent solitons with a radiation part plus an asymptotically vanishing remainder term as $t \to \infty$. The version for geometric wave equations has been verified by Cote [10] and Jia, Kenig [27] for equivariant wave maps along a time sequence. Recently Duyckaerts, Jia, Kenig, Merle [13] obtained the universal blow up profile for type II blow up solutions to wave maps $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{S}^2$ with initial data of energy slightly above the ground state harmonic map. We also mention the works [31, 32] for outer-ball wave maps and [20] for wave maps on wormholes in the equivariant case.

For wave maps from $\mathbb{R} \times \mathbb{H}^2$ to $\mathbb{H}^2$, Lawrie, Oh, Shahshahani [46, 47] raised the following soliton resolution conjecture,

**Conjecture 1.1** (Soliton resolution for wave maps $\mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$). Consider the Cauchy problem for wave map $u : \mathbb{R} \times \mathbb{H}^2 \to \mathbb{H}^2$ with finite energy initial data $(u_0, u_1)$. Suppose that outside some compact subset $K$ of $\mathbb{H}^2$ for some harmonic map $Q : \mathbb{H}^2 \to \mathbb{H}^2$ we have

$$u_0(x) = Q_\lambda(x), \text{ for } x \in \mathbb{H}^2 \setminus K.$$  

Then the unique solution $(u(t), \partial_t u(t))$ to the wave map scatters to $(Q(x), 0)$ as $t \to \infty$.

In this paper, we consider the case when the initial data is a small perturbation of harmonic maps with large energy. Before stating our main result, we recall the notion of admissible harmonic maps used in our previous works [49].

**Definition 1.1.** Denote the Poincare disk by $\mathbb{D}$. We say the harmonic map $Q : \mathbb{D} \to \mathbb{D}$ is admissible if $\overline{Q(\mathbb{D})}$ is a compact subset of $\mathbb{D}$ covered by a geodesic ball centered at the origin of radius $R_0$, $\|\nabla^k dQ\|_{L^2} < \infty$ for $k = 0, 1, 2, 3$, and there exists some $\rho > 0$ such that $e^{\rho r}|dQ|^2 \in L^\infty$, where $r$ is the distance between $x \in \mathbb{D}$ and the origin.

**Remark 1.1** (Examples for the admissible harmonic maps) Any analytic function $f : \mathbb{C} \to \mathbb{C}$ with $\overline{f(\mathbb{D})} \subseteq \mathbb{D}$ is an admissible harmonic map.

For any given admissible harmonic map $Q$, we define the space $H^k \times H^{k-1}$ by (2.7). Our main theorem is as follows.

**Theorem 1.1.** Let $Q$ be an admissible harmonic map in Definition 1.1. Assume that the initial data $(u_0, u_1) \in H^3 \times H^2$ to (1.1) with $u_0 : \mathbb{H}^2 \to \mathbb{H}^2$, $u_1(x) \in T_{u_0(x)}N$ for each $x \in \mathbb{H}^2$ satisfy

$$\|(u_0, u_1) - (Q, 0)\|_{H^2 \times H^1} < \mu, \quad (1.2)$$

Then if $\mu > 0$ is sufficiently small, (1.1) has a global solution $(u(t), \partial_t u(t))$ which converges to the harmonic map $Q : \mathbb{H}^2 \to \mathbb{H}^2$ as $t \to \infty$, i.e.,

$$\lim_{t \to \infty} \sup_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(u(t, x), Q(x)) = 0.$$
Remark 1.2 We remark that the perturbation norm in Theorem 1.1 assume the initial data tends to $Q$ at infinity. By the uniqueness of harmonic maps with prescribed boundary map, one can expect the final asymptotic harmonic map of the solution $(u, \partial_t u)$ to (1.1) is exactly $Q$. This is the heuristic explanation why the asymptotic harmonic map coincide with the unperturbed one, which is different from other equations, for instance the nonlinear Schrödinger equation where we have moving and modulated solitons after the perturbation.

Remark 1.3 (Examples for the perturbations of admissible harmonic maps) It is easy to yield initial data satisfying (1.2). In fact, since we have global coordinates for $\mathbb{H}^2$ given by (2.1). The perturbation in the sense of (1.2) is nothing but perturbations of $\mathbb{R}^2$-valued functions.

Remark 1.4 The initial data considered in this paper are perturbations of harmonic maps in the $H^2$ norm. If one considers perturbations in the energy critical norm $H^1$, the $S_k$ v.s. $N_k$ norm constructed by Tataru [67] and Tao [63] should be built for the hyperbolic setting.

We first describe the outline of the proof. By constructing Tao’s caloric gauge in our setting, one obtains the nonlinear wave equation for the heat tension field. Separating the “effective” linear part from the nonlinear terms yields a linear wave equation with a magnetic potential. By establishing the Kato smoothing effect for the master linear equation, one obtains the corresponding non-endpoint Strichartz estimates. Applying a weighted Strichartz estimate for the free linear wave equation built in our previous work [49] gives us the endpoint Strichartz estimates. By bootstrap, the endpoint and non-endpoint Strichartz estimates, one can prove the heat tension filed enjoys a global space-time norm. Transforming the bounds of the heat tension field back to differential fields closes the bootstrap and thus finishing the whole proof.

The main contribution of this paper is using the freedom of the gauge fixed on the harmonic map and the geometric meaning of the master linear equation to rule out bottom resonance and the possibility of existence of any eigenvalue in the gap $[0, 1/4]$. In fact, the main difficulty for the large energy harmonic map case is to derive the Kato smoothing effect of a wave equation with large magnetic potentials. In order to derive the Kato smoothing effect estimate, one may divide the frequency into small frequency, mediate frequency and high frequency part. The enemy for the small frequency part is the possibility of existing bottom resonance. We use the Coulomb gauge on the harmonic map to obtain a nice spectrum distribution of the operator $V_1 + V_2 \nabla R_0(ie^2)$, where $V_1, V_2$ are the electric and magnetic potential respectively, $R_0$ is the free resolvent. In fact, by choosing the Coulomb gauge we have the spectrum of $V_1 + V_2 \nabla R_0(i e^2)$ lies on the right of the imaginary axis, then the resonance can be ruled out by a perturbation argument using Dunford-Schwartz projection operators. Moreover, We exclude the possible existence of eigenvalues in $(-\infty, 1/4)$ by calculating the numerical range of the magnetic Schrödinger operator in the language of covariant derivatives on $u^*(TN)$ instead of directly working with $C^2$-valued functions.

The high frequency part is always difficult in the large magnetic potential case, even in the
Euclidean case, see for instance [19]. In our case, we split the magnetic potential into a large long range part supported outside some geodesic ball and a remainder part supported near the original point. For the long range part we can put the magnetic Schrödinger operator uniformly bounded in a weighted $L^2$ space $w(x)L^2$ w.r.t. all high frequencies by a similar commutator method due to [6]. The important gain of this energy argument is the weight $w^{-1}$ can be chosen to vanish near the origin point. Thus we can view the Schrödinger operator with the whole magnetic potential as the perturbation of the long range Schrodinger operator due to the extra smallness gain from the vanishing of $w^{-1}$ and the closeness to the origin of the support to the remainder potential. Besides, we refine our previous estimates for high order covariant derivative terms along the heat flow where small energy assumption is essentially used and deal with some other technical issues.

There exist plenty of mathematical works on the Cauchy problem, the long dynamics and blow up for wave maps on $\mathbb{R}^{1+m}$. We just recall the following non-exhaustive lists of results, for more history remarks see for instance [46]. The sharp subcritical well-posedness theory was developed by Klainerman, Machedon [33, 34] and Klainerman, Selberg [36, 37]. The critical well-posedness theory in equivariant case was considered by Christodoulou, Tahvildar-Zadeh [9], Shatah, Tahvildar-Zadeh [57]. The critical small data global well-posedness theory was started by Tataru [67], Tao [62, 63] and generalized by [39, 38, 35, 56, 52, 66]. The below threshold critical global well-posedness theory was obtained by Krieger, Schlag [40], Sterbenz, Tataru [59, 58], Tao [64]. The bubbling theorem in the equivariant case was obtained by Struwe [60]. The explicit construction of blow up solutions was achieved by Krieger, Schlag, Tataru [41], Raphael, Rodnianski [54], and Rodnianski, Sterbenz [55]. And see D’Ancona, Georgiev [16] and Tao [61] for the ill-posedness theory.

The works on the wave map equations on curved spacetime were relatively less. D’Ancona, Zheng [17] studied critical small data global well-posedness of wave maps on rotationally symmetric manifolds in the equivariant case. The critical small data global well-posedness theory for wave maps on small asymptotically flat perturbations of $\mathbb{R}^4$ was studied by Lawrie [42]. The long time dynamics for wave maps from $\mathbb{R} \times \mathbb{H}^2$ to $S^2$ or $\mathbb{H}^2$ in the 1-equivariant case was studied by sequel works of Lawrie, Oh, Shahshahani [43, 44, 47, 45]. And Lawrie, Oh, Shahshahani [46] obtained the critical small data global well-posedness theory for wave maps from $\mathbb{R} \times \mathbb{H}^d$ to compact Riemann manifolds with $d \geq 4$.

This paper is organized as follows. In Section 2, we recall some results obtained in our previous works. Particularly, we recall the equivalence between the intrinsic and extrinsic Sobolev norm in some case and the existence of the caloric gauge. In addition, we prove the limit harmonic map for the heat flow is exactly the unperturbed one. In Section 3, we recall master equation and prove the corresponding Kato smoothing effect. In Section 4, we prove the non-endpoint and end-point Strichartz estimate for the linear magnetic wave equation. In Section 5, by bootstrap
we deduce the global spacetime bounds for the heat tension field and finish the proof of Theorem 1.1.

2 Preliminaries

In this section, we recall some background materials and recall known results obtained in our previous paper [49]. Lemma 2.4 is new.

2.1 The global coordinates and definitions of the function spaces

We collect some background materials on function theories on hyperbolic spaces. The covariant derivative in $TN$ is denoted by $\tilde{\nabla}$, the covariant derivative induced by $u$ in $u^*(TN)$ is denoted by $\nabla$. We denote the Riemann curvature tension of $N$ by $R$. The components of Riemann metric are denoted by $h_{ij}$ for $M$ and $g_{ij}$ for $N$ respectively. The Christoffel symbols on $M$ and $N$ are denoted by $\Gamma^k_{ij}$ and $\Gamma^l_{ij}$ respectively.

The hyperbolic space has several isometric models. One is the hyperboloid model defined by $H^2 = \{x \in \mathbb{R}^{2+1} : -|x_0|^2 + |x_1|^2 + |x_2|^2 = 1$ and $x^0 > 0\}$, with the metric being the pullback of the Minkowski metric $-(dx^0)^2 + (dx^1)^2 + (dx^2)^2$ in $\mathbb{R}^{1+2}$. For the hyperboloid model, Iwasawa decomposition induces a global system of coordinates, i.e., the diffeomorphism $\Psi : \mathbb{R} \times \mathbb{R} \rightarrow H^2$ given by

$$\Psi(x_1, x_2) = (\cosh x_2 + e^{-x_2}|x_1|^2/2, \sinh x_2 + e^{-x_2}|x_1|^2/2, e^{-x_2}x_1).$$

(2.1)

The Riemannian metric with respect to this coordinate system is given by

$$e^{-2x_2}(dx_1)^2 + (dx_2)^2.$$

The corresponding Christoffel symbols are

$$\Gamma^1_{2,2} = \Gamma^2_{2,1} = \Gamma^2_{2,2} = \Gamma^1_{1,1} = 0; \Gamma^1_{2,1} = -1, \Gamma^2_{1,1} = e^{-2x_2}.$$  

(2.2)

For any $(t, x)$ and $u : [0, T] \times H^2 \rightarrow H^2$, we define an orthonormal frame at $u(t, x)$ by

$$\Theta_1(u(t, x)) = e^{u^2(t,x)} \frac{\partial}{\partial y_1}; \Theta_2(u(t, x)) = \frac{\partial}{\partial y_2}.$$  

(2.3)

Recall also the identity for Riemannian curvature on $N = H^2$

$$\mathbf{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z = (X \wedge Y)Z,$$
where \( X, Y, Z \in TN \) and we adopt the simplicity notation

\[(X \wedge Y) Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X. \tag{2.4} \]

Let \( H^k(\mathbb{H}^2; \mathbb{R}) \) be the usual Sobolev space for scalar functions defined on manifolds. It is known that \( C_c^\infty(\mathbb{H}^2; \mathbb{R}) \) is dense in \( H^k(\mathbb{H}^2; \mathbb{R}) \). We also recall the norm of \( H^k \):

\[
\|f\|^2_{H^k} = \sum_{l=1}^{k} \|\nabla^l f\|^2_{L^2},
\]

where \( \nabla^l f \) is the covariant derivative. For maps \( u : \mathbb{H}^2 \to \mathbb{H}^2 \), we define the intrinsic Sobolev semi-norm \( \mathcal{H}^k \) by

\[
\|u\|^2_{\mathcal{H}^k} = \sum_{i=1}^{k} \int_{\mathbb{H}^2} |\nabla^{i-1} u|^2 \text{dvol}_h.
\]

We can associate the map \( u : \mathbb{H}^2 \to \mathbb{H}^2 \) with a vector-valued function \( u : \mathbb{H}^2 \to \mathbb{R}^2 \) by (2.1). Indeed, for \( P \in \mathbb{H}^2 \) the vector \( (u^1(P), u^2(P)) \) is defined by \( \Psi(u^1(P), u^2(P)) = u(P) \). Let \( Q : \mathbb{H}^2 \to \mathbb{H}^2 \) be an admissible harmonic map in Definition 1.1. Then the extrinsic Sobolev space is defined by

\[
H^k_Q = \{ u : u^1 - Q^1(x), u^2 - Q^2(x) \in H^k(\mathbb{H}^2; \mathbb{R}) \}, \tag{2.5}
\]

where \( \Psi(Q^1(x), Q^2(x)) = Q(x) \). Denote the set of smooth maps which coincide with \( Q \) outside of a compact set of \( M = \mathbb{H}^2 \) by \( \mathcal{D} \). Let \( H^k_Q \) be the completion of \( \mathcal{D} \) under the metric given by

\[
\text{dist}_{k,Q}(u, w) = \sum_{j=1}^{2} \|u^j - w^j\|_{H^k(\mathbb{H}^2; \mathbb{R})}, \tag{2.6}
\]

where \( u, w \in H^k_Q \). And for simplicity, we write \( H^k \) without confusions. If \( u \) is a map from \( \mathbb{R} \times \mathbb{H}^2 \) to \( \mathbb{H}^2 \), we define the space \( H^k \times H^{k-1} \) by

\[
H^k \times H^{k-1} = \left\{ u : \sum_{j=1}^{2} \|u^j - Q^j\|_{H^k(\mathbb{H}^2; \mathbb{R})} + \|\partial_t u^j\|_{H^{k-1}(\mathbb{H}^2; \mathbb{R})} < \infty \right\}. \tag{2.7}
\]

The distance in \( H^k \times H^{k-1} \) is given by

\[
\text{dist}_{H^k \times H^{k-1}}(u, w) = \sum_{j=1}^{2} \|u^j - w^j(x)\|_{H^k} + \|\partial_t u^j - \partial_t w^j\|_{H^{k-1}}. \tag{2.8}
\]
2.2 The Fourier transform on hyperbolic spaces and Sobolev embedding

The Fourier transform takes proper functions defined on $\mathbb{H}^2$ to functions defined on $\mathbb{R} \times S^1$, see Helgason [22]. For $b \in S^1$, and $\tau \in \mathbb{C}$, let $\mathbf{k}(b) = (1, b) \in \mathbb{R}^3$, we define

$$h_{\tau,b} : \mathbb{H}^2 \rightarrow \mathbb{C}, \quad h_{\tau,b} = [x, \mathbf{k}(b)]^i \tau^{-\frac{1}{2}},$$

where $[x, \mathbf{k}(b)] = -x_0 b_0 + x_1 b_1 + x_2 b_2$. The Fourier transform of $f \in C_0(\mathbb{H}^2)$ is defined by

$$\mathcal{F} f (\tau, b) = \int_{\mathbb{H}^2} f(x) h_{\tau,b}(x) dvol_h = \int_{\mathbb{H}^2} f(x)[x, \mathbf{k}(b)]^i \tau^{-\frac{1}{2}} dvol_h.$$

The corresponding Fourier inversion formula is given by

$$f(x) = \int_0^{\infty} \int_{S^1} \mathcal{F} f (\tau, b)[x, \mathbf{k}(b)]^{-i \tau - \frac{1}{2}} |c(\tau)|^{-2} d\tau db.$$

Here $c(\tau)$ is called the Harish-Chandra c-function on $\mathbb{H}^2$ and for some constant $C$ it is defined by

$$c(\tau) = C \frac{\Gamma(i\tau)}{\Gamma(\frac{3}{2} + i\tau)}.$$

The Plancherel theorem is known as

$$\int_{\mathbb{H}^2} f(x) \overline{g(x)} dvol_h = \frac{1}{2} \int_{\mathbb{R} \times S^1} \mathcal{F} f (\tau, b) \overline{\mathcal{F} g (\tau, b)} |c(\tau)|^{-2} d\tau db.$$

Thus any bounded multiplier $m : \mathbb{R} \rightarrow \mathbb{C}$ defines a bounded operator $T_m$ on $L^2(\mathbb{H}^2)$ by

$$\widehat{T_m(f)}(\tau, b) = m(\tau) \mathcal{F} f (\tau, b).$$

We define the operator $(-\Delta)^{\frac{p}{2}}$ by the Fourier multiplier $\lambda \rightarrow (\frac{1}{4} + \lambda^2)^{\frac{p}{2}}$. The Sobolev inequalities of functions in $H^k$ are recalled in Appendix A. Theorem 5.2 and Remark 5.5 of Anker, Pierfelice [3] obtained the Strichartz estimates for linear wave/Klein-Gordon equation. We state it in the following lemma. Let $\nu = 1/2$, and $\tilde{D} = (-\Delta - \nu^2 + \gamma^2)$ for some $\gamma > \nu$.

**Lemma 2.1.** Let $(p, q)$ and $(\tilde{p}, \tilde{q})$ be two admissible couples, i.e.,

$$\left\{(p^{-1}, q^{-1}) \in (0, 1/2] \times (0, 1/2) : \frac{1}{p} > \frac{1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) \right\} \cup \{(0, 1/2)\},$$

and similarly for $(\tilde{p}, \tilde{q})$. Meanwhile assume that

$$\sigma \geq \frac{3}{2} \left( \frac{1}{2} - \frac{1}{q} \right), \quad \tilde{\sigma} \geq \frac{3}{2} \left( \frac{1}{2} - \frac{1}{\tilde{q}} \right),$$

7
then the solution $u$ to the linear wave equation

$$\begin{cases}
\partial_t^2 u - \Delta u = F \\
\quad u(0, x) = f(x), \partial_t u|_{t=0} = g(x)
\end{cases}$$

satisfies the Strichartz estimate

$$\left\| \tilde{D}_x^{-\sigma+1/2} u \right\|_{L_t^p L_x^q} + \left\| \tilde{D}_x^{-\sigma-1/2} \partial_t u \right\|_{L_t^p L_x^q} \leq \left\| \tilde{D}_x^{1/2} f \right\|_{L^2} + \left\| \tilde{D}_x^{-1/2} g \right\|_{L^2} + \left\| \tilde{D}_x^{-1/2} F \right\|_{L_t^p L_x^q}.$$  

**Remark 2.1.** For all $\sigma \in \mathbb{R}$, $p \in (1, \infty)$, $\|\tilde{D}^\sigma \cdot\|_p$ is equivalent to $\|(-\Delta)^{\sigma/2} \cdot\|_p$.  

We recall the local well-posedness and conditional global well-posedness for (1.1) in $H^3 \times H^2$, see [49].

**Lemma 2.2.** For any initial data $(u_0, u_1) \in H^3 \times H^2$, there exists $T > 0$ depending only on $\|(u_0, u_1)\|_{H^3 \times H^2}$ such that (1.1) has a unique local solution $(u, \partial_t u) \in C([0, T]; H^3 \times H^2)$.  

The conditional global well-posedness is given by the following proposition.

**Proposition 2.1.** Let $(u_0, u_1) \in H^3 \times H^2$ be the initial data of (1.1), $T_*$ is the maximal lifespan determined by Lemma 2.2. If the solution $(u, \partial_t u)$ satisfies uniformly for all $t \in [0, T_*)$

$$\|\nabla u\|_{L^2_t L^2_x} + \|du\|_{L^2_t L^2_x} + \|\nabla \partial_t u\|_{L^2_t L^2_x} + \|\partial_t u\|_{L^2_t L^2_x} \leq C_1,$$

for some $C_1 > 0$ independent of $t \in [0, T_*)$ then $T_* = \infty$.  

### 2.3 Geometric identities related to Gauges

Let $\{e_1(t, x), e_2(t, x)\}$ be an orthonormal frame for $u^*(T\mathbb{H}^2)$. Let $\psi_\alpha = (\psi^1_\alpha, \psi^2_\alpha)$ for $\alpha = 0, 1, 2$ be the components of $\partial_t x u$ in the frame $\{e_1, e_2\}$

$$\psi^j_\alpha = \langle \partial_\alpha u, e_j \rangle.$$

For given $\mathbb{R}^2$-valued function $\phi$ defined on $[0, T] \times \mathbb{H}^2$, associate $\phi$ with a tangent filed $\phi e$ on $u^*(TN)$ by

$$\phi e = \sum_{j=1}^2 \phi^j e_j.$$

And the covariant derivative induced by $u$ is defined by

$$D_\alpha \phi = \partial_\alpha \phi + [A_\alpha] \phi,$$
which in the form of components reads as,

\[(D_{\alpha}\phi)^k = \partial_{\alpha}\phi^k + \sum_{j=1}^2 [A_{\alpha}]^k_j \phi^j,\]

with the induced connection coefficient matrix defined by \([A_{\alpha}]^k_j = \langle \nabla_{\alpha} e_j, e_k \rangle\). It is easy to check the torsion free identity,

\[D_{\alpha}\phi_{\beta} = D_{\beta}\phi_{\alpha}.\]  \hspace{1cm} (2.10)

and the commutator identity (in the two dimensional case)

\[((D_{\alpha}, D_{\beta})\phi)e = ((\partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha})\phi)e = R(u)(\partial_{\alpha} u, \partial_{\beta} u)(\phi e).\]  \hspace{1cm} (2.11)

**Remark 2.1** With a bit of abuse of notation, we define a matrix valued function \(a \wedge b\) by

\[(a \wedge b)c = \langle a, c \rangle b - \langle b, c \rangle a,\]  \hspace{1cm} (2.12)

where \(a, b, c\) are vectors on \(\mathbb{R}^2\). It is easy to see (2.12) coincide with (2.4) by letting \(X = a_1 e_1 + a_2 e_2, Y = b_1 e_1 + b_2 e_2, Z = c_1 e_1 + c_2 e_2\). Hence (2.11) can be written as

\[[D_{\alpha}, D_{\beta}]\phi = (\phi_{\alpha} \wedge \phi_{\beta})\phi.\]  \hspace{1cm} (2.13)

**Lemma 2.3.** With the notions and notations given above, (1.1) can be written as

\[D_t \phi_t - h^{ij} D_i \phi_j + h^{ij} \Gamma^k_{ij} \phi_k = 0\]  \hspace{1cm} (2.14)

**2.4 Caloric Gauge**

Denote the space \(C([0, T]; \mathbb{H}^3 \times \mathbb{H}^2)\) by \(\mathcal{X}_T\). The caloric gauge was first introduced by Tao [64] for wave maps from \(\mathbb{R}^{2+1}\) to \(\mathbb{H}^n\). When non-trivial harmonic maps occur, the caloric gauge can be defined as follows.

**Definition 2.1.** Let \(u(t, x) : [0, T] \times \mathbb{H}^2 \to \mathbb{H}^2\) be a solution of (1.1) in \(\mathcal{X}_T\). Suppose that the heat flow initiated from \(u_0\) converges to a harmonic map \(Q : \mathbb{H}^2 \to \mathbb{H}^2\). Then for a given orthonormal frame \(\Xi(x) \triangleq \{\Xi_j(Q(x))\}_{j=1}^2\) which spans the tangent space \(T_{Q(x)}\mathbb{H}^2\) for any \(x \in \mathbb{H}^2\), there exists a map \(\tilde{u} : \mathbb{R}^+ \times [0, T] \times \mathbb{H}^2 \to \mathbb{H}^2\) and an orthonormal frame \(\Omega \triangleq \{\Omega_j(\tilde{u}(s, t, x))\}_{j=1}^2\) such that

\[
\begin{cases}
\partial_s \tilde{u} = \tau(\tilde{u}) \\
\nabla_s \Omega_j = 0 \\
\lim_{s \to \infty} \Omega_j = \Xi_j
\end{cases}
\] \hspace{1cm} (2.15)
where the convergence of frames is defined by

\[
\begin{align*}
\lim_{s \to \infty} \tilde{u}(s, t, x) &= Q(x) \\
\lim_{s \to \infty} \langle \Omega_i(s, t, x), \Theta_j(\tilde{u}(s, t, x)) \rangle &= (\Xi_i(Q(x)), \Theta_j(Q(x)))
\end{align*}
\] (2.16)

2.5 Existence of Caloric Gauge

In this subsection, we recall the existence theorem of the caloric gauge in our previous work [49].

The equation of the heat flow is given by

\[
\begin{align*}
\partial_s u &= \tau(u) \\
u(s, x) \big|_{s=0} &= v(x)
\end{align*}
\] (2.17)

Consider the heat flow from \( H^2 \) to \( H^2 \) with a parameter

\[
\begin{align*}
\partial_s \tilde{u} &= \tau(\tilde{u}) \\
\tilde{u}(s, t, x) \big|_{s=0} &= u(t, x)
\end{align*}
\] (2.18)

The long time existence of the heat flow from \( H^2 \to H^2 \) in \( H^3 \) is given in Appendix A. We summarize the long time and short time behaviors as a proposition.

**Proposition 2.2.** Let \( u : [0, T] \times H^2 \to H^2 \) be a solution to (1.1) satisfying

\[
\| (\nabla du, \nabla \partial_t u) \|_{L^2 \times L^2} + \| (du, \partial_t u) \|_{L^2 \times L^2} \leq M_1.
\]

If \( \tilde{u} : \mathbb{R}^+ \times [0, T] \times H^2 \to H^2 \) is the solution to (2.18) with initial data \( u(t, x) \), then it holds uniformly for \( t \in [0, T] \) that

\[
\begin{align*}
\| s^{1/2} |\nabla \tilde{u}| \|_{L^\infty [0, 1]} &+ \| s^{1/2} e^{\delta s} |\nabla \partial_t \tilde{u}| \|_{L^\infty L^\infty} + \| s e^{\delta s} |\nabla \partial_s \tilde{u}| \|_{L^\infty L^\infty} + \| s^{1/2} e^{\delta s} |\partial_s \tilde{u}| \|_{L^\infty L^\infty} = M_1.
\end{align*}
\]

It has been prove in [49] that the heat flow initiated from all \( u(t, x) \) for different \( t \) converges to the same harmonic map say \( \tilde{Q} \). We want to prove \( \tilde{Q} \) is exactly \( Q \) in the definition of our working space \( H^k_Q \). But in [49], this was only verified for small energy harmonic maps. Thus we give a new proof to involve the large energy case. The key ingredient is the differential inequality concerning the distance between two harmonic maps proved by [28]. We remark that their main theorem 1.1 can not directly be applied to our case since it seems not easy to relate their boundary map setting to our function space in a reasonable and effective way.
Lemma 2.4. If \((u, \partial_t u)\) is a solution to (1.1) in \(X_T\), then as \(s \to \infty\),
\[
\lim_{s \to \infty} \sup_{(x,t) \in \mathbb{H}^2 \times [0,T]} \text{dist}_{\mathbb{H}^2}(\tilde{u}(s, x, t), Q(x)) = 0.
\]

Proof. As remarked above we only need to verify \(Q = \tilde{Q}\). First we note that due to Corollary 5.1 and \(Q, \tilde{Q} \in H^3\), one has \(\tilde{Q}(M)\) and \(Q(M)\) are contained in a geodesic ball of \(N = \mathbb{H}^2\) with radius \(R_0\). Hence the distance between \(Q(x)\) and \(\tilde{Q}\) is equivalent to \(|Q^1 - \tilde{Q}^1| + |Q^2 - \tilde{Q}^2|\) up to some large constant \(C(R_0)\) depending on \(R_0\). [28, Page 286] (the \(\kappa = 0\) case) has obtained the following inequality
\[
\Delta \left( \frac{1}{2} \text{dist}(Q(x), \tilde{Q}(x))^2 \right) \geq 0. \tag{2.19}
\]
Then the mean value inequality for nonnegative subharmonic functions yields
\[
\frac{1}{2\pi} \int_0^{2\pi} \left[ \text{dist}(Q(r, \theta), \tilde{Q}_1(r, \theta))^2 \right] d\theta \tag{2.20}
\]
is an nondecreasing function to \(r \in (0, \infty)\). If there exists some \(r_0\) such that (2.20) is strictly positive when \(r = r_0\), then integrating (2.20) with respect to \(r\) in \([r_0, \infty)\) gives
\[
\int_{r_0}^{\infty} \sinh r \left[ \text{dist}(Q(r, \theta), \tilde{Q}_1(r, \theta))^2 \right] d\theta dr = \infty. \tag{2.21}
\]
But the left hand side of (2.21) is bounded by
\[
C(R_0) \int_0^{\infty} \sinh r |(Q^1, Q^2) - (\tilde{Q}^1, \tilde{Q}^2)|^2 dr d\theta = C(R_0) \|\tilde{Q}\|_{H^3_0}^2 < \infty,
\]
which contradicts with (2.21). Hence \(\text{dist}(\tilde{Q}, Q) = 0\). \square

The existence of the caloric gauge defined in Definition 2.1 is given below.

Proposition 2.3. Given any solution \((u, \partial_t u)\) of (1.1) in \(X_T\), suppose that the limit harmonic map of the heat flow (2.18) with initial data \(u_0\) is \(Q(x)\). For any fixed frame \(\Xi \triangleq \{\Xi_1(Q(x)), \Xi_2(Q(x))\}\), there exists a unique corresponding caloric gauge defined in Definition 2.1.

The following lemma gives the expressions of the connection coefficients matrices \(A_{x,t}\) by differential fields. Since \(A_i\) are skew-symmetric, we can view \(A_i\) as a real-valued function \([A_i]_2\).

Lemma 2.5. Suppose that \(\Omega(s,t,x)\) is the caloric gauge constructed in Proposition 2.3, then
we have for \( i = 1, 2, s > 0, \)
\[
A_i(s, t, x) = \sqrt{h^{ii}(x)} \int_s^\infty \sqrt{h^{ii}(x)} R(\ddot{u}(s')) (\partial_s \ddot{u}(s'), \partial_t \ddot{u}(s')) ds' + \sqrt{h^{ii}(x)} (\nabla_i \Xi_1(x), \Xi_2(x)).
\]
(2.22)
\[
A_t(s, t, x) = \int_s^\infty R(\ddot{u}(s')) (\partial_s \ddot{u}(s'), \partial_t \ddot{u}(s')) ds'.
\]
(2.23)

**Remark 2.2.** For convenience, we rewrite (2.22) as
\[
A_i(s, t, x) = A_i^\infty(s, t, x) + A_i^{\text{con}}(s, t, x),
\]
where \( A_i^\infty \) denotes the limit part, i.e.,
\[
[A_i^\infty]^k_j = \langle \nabla_i \Xi_k(Q(x)), \Xi_j(Q(x)) \rangle,
\]
and \( A_i^{\text{con}} \) denotes the controllable part, i.e.,
\[
A_i^{\text{con}} = \int_s^\infty \phi_i \wedge \phi_i ds'.
\]

Similarly, we split \( \phi_i \) into \( \phi_i = \phi_i^\infty + \phi_i^{\text{con}} \), where
\[
\phi_i^{\text{con}} = \int_s^\infty \partial_s \phi_i ds',
\]
and
\[
\phi_i^\infty = ((\partial_i Q(x), \Xi_1(Q(x))), (\partial_i Q(x), \Xi_2(Q(x))))^t.
\]

### 2.6 The master equation for heat tension field

Recall that the heat tension filed \( \phi_s \) satisfies
\[
\phi_s = h^{ij} D_i \phi_j - h^{ij} \Gamma^k_{ij} \phi_k. \tag{2.24}
\]

And we define the wave tension filed as Tao by
\[
Z = D_t \phi_t - h^{ij} D_i \phi_j + h^{ij} \Gamma^k_{ij} \phi_k. \tag{2.25}
\]

In fact (2.24) is the gauged equation for the heat flow equation, and (2.25) is the gauged equation for the wave map (1.1), see Lemma 3.1.
Lemma 2.6. The heat tension field $\phi_s$ and the wave tension field $w$ satisfies

$$D_tD_t\phi_s - h^{ij}D_iD_j\phi_s + h^{ij}\Gamma^k_{ij}D_k\phi_s = \partial_s Z + h^{ij}(\phi_s \wedge \phi_i) + (\phi_t \wedge \phi_s)\phi_t.$$  

(2.26)

$$\partial_s Z = \Delta Z + 2h^{ij}A_i\partial_j Z + h^{ij}A_iZ + h^{ij}\partial_i A_iZ - h^{ij}\Gamma^k_{ij}A_kZ + h^{ij}(Z \wedge \phi_i)\phi_i + 3h^{ij}(\partial_t u \wedge \partial_i u)\nabla_t \partial_i u.$$  

(2.27)

Lemma 2.7. Let $Q$ be an admissible harmonic map. Recall the definitions of $A_i^\infty$ in Remark 2.2. Fix the frame $\Xi$ in Remark 2.2 by taking $\Xi(Q(x)) = \Theta(Q(x))$ given by (2.1). Then

$$|\sqrt{h^{ii}}A_i^\infty| \lesssim |dQ|, |\sqrt{h^{ii}}\phi_i^\infty| \lesssim |dQ|$$  

(2.28)

$$|h^{ii}(\partial_t A_i^\infty - \Gamma^k_{ii}A_k^\infty)| \lesssim |dQ|^2.$$  

(2.29)

Lemma 2.8. Given any fixed frame $\Xi$ in Proposition 2.3, we have the heat tension filed $\phi_s$ satisfies

$$(\partial_t^2 - \Delta)\phi_s + W\phi_s = -2A(t)\partial_t\phi_s - A(t)\partial_t A(t)\phi_s - \partial_t A(t)\phi_s + \partial_t s_w + R(\partial_t u, \partial_s w)(\partial_t u) + 2h^{ij}A_i^\text{con}\partial_i\phi_s + h^{ij}A_i^\infty A_i^\text{con}\phi_s + h^{ij}A_i^\text{con}A_i^\text{con}\phi_s + h^{ij}(\partial_t A_i^\text{con} - \Gamma^k_{ii}A_k^\text{con})\phi_s + h^{ij}(\partial_t \phi_i^\text{con} \wedge \phi_i^\infty) + h^{ij}(\phi_s \wedge \phi_i^\text{con})\phi_i^\infty + h^{ij}(\phi_s \wedge \phi_i^\text{con})\phi_i^\text{con},$$

where $A_i^\infty, A_i^\text{con}$ are defined in Remark 2.2, and $W$ is given by

$$W\varphi = -2h^{ij}A_i^\infty \partial_i \varphi - h^{ij}A_i^\infty A_i^\text{con} \varphi - h^{ij}(\varphi \wedge \phi_i^\infty)\phi_i^\infty - h^{ij}(\partial_t A_i^\infty - \Gamma^k_{ii}A_k^\infty)\varphi.$$  

(2.30)

Furthermore, $-\Delta + W$ is a self-adjoint operator in $L^2(\mathbb{H}; \mathbb{C})$.

3 Kato Smoothing Effect for Magnetic Wave Equation

In this section, we prove the Kato smoothing effect for $H = -\Delta + W$ given in Lemma 2.8. First of all, we point out the operator $H$ is independent of the orthogonal coordinates for $M$. In fact, viewing $\wedge A \equiv A_i^\infty dx^i$ to be the connection 1-form for $Q^*TN$, we have $h^{ij}A_i^\infty\partial_i \varphi = \langle \wedge A, d\varphi \rangle$. Moreover, it is easy to check $h^{ij}A_i^\infty A_i^\text{con}\varphi + h^{ij}(\varphi \wedge \phi_i^\infty)\phi_i^\text{con}$ is invariant if one changes $(x_1, x_2)$ to be $(z_1, z_2)$ with both of which being orthogonal. And for any orthogonal coordinate, we have $h^{ij}(\partial_t A_i^\infty - \Gamma^k_{ii}A_k^\infty) = d^*A$. Combining these three facts shows $H$ is independent of the orthogonal coordinates chosen for $M$. 

-
### 3.1 Free Resolvent Estimates

Denote $D = \sqrt{-\Delta_{\mathbb{R}^2}}$, its resolvent is denoted by $R_D(z) = (|D| - z)^{-1}$. The resolvent of $H$ is denoted by $R_H(z) = (H - z)^{-1}$. Define $|D| = \sqrt{-\Delta_{\mathbb{R}^2} + W}$ and its resolvent $(|D| - z)^{-1}$ is denoted as $R(z)$. Recall also $\nu = \frac{1}{2}$ is the spectrum gap. Let $\rho(x) = e^{-d(x,0)}$ for $x \in \mathbb{H}^2$.

The pointwise estimates for the free resolvent are given in Appendix A. We recall the kernel estimates of the shifted wave operator proved by [Theorem 4.1, Theorem 4.4 [3]] for reader’s convenience.

Let $\chi_\infty(\lambda)$ be a cutoff function which equals one when $\lambda \geq \nu + 1$ and vanishes near zero.

**Lemma 3.1** ([3]). Let $D_o = (-\Delta - \nu^2)^{1/2}$ be the shifted differential operator and let $\tilde{D} = (-\Delta - \nu^2 + \gamma^2)^{1/2}$ with any $\gamma > 0$. For $\sigma \in \mathbb{R}$, $\tau \in [0, \frac{3}{2})$, define the low frequency cutoff shifted wave operator

$$\tilde{W}_{t,0}^{\sigma,\tau} = (I - \chi_\infty(D_o))D_o^{-\tau} \tilde{D}^{\tau-\sigma} e^{itD_o},$$

and denote its kernel as $w_{t,0}^{\sigma,\tau}(r)$. The modified high frequency wave operator is defined by an analytic family of operators

$$\tilde{W}_{t,\infty}^{\sigma,\tau} = \frac{e^{\sigma^2}}{\Gamma(3/2 - \sigma)} \chi_\infty(D_o)D_o^{-\tau} \tilde{D}^{\tau-\sigma} e^{itD_o}$$

in the vertical strip $0 \leq \text{Re}\sigma \leq 3/2$. Denote its kernel by $\tilde{w}_{t,\infty}^{\sigma,\tau}(r)$. Then the two kernels satisfy

- Assume $|t| \leq 2$. Then for any $r \geq 0$

  $$|w_{t,0}^{\sigma,\tau}(r)| \lesssim \phi_0(r)$$

- Assume $|t| \geq 2$.
  - (a) If $0 \leq r \leq \frac{|t|}{2}$, then
    $$|w_{t,0}^{\sigma,\tau}(r)| \lesssim |t|^{-3}\phi_0(r)$$
  - (b) If $r \geq \frac{|t|}{2}$, then
    $$|w_{t,0}^{\sigma,\tau}(r)| \lesssim (1 + |t - r|)^{-2} e^{-\nu r}.$$

For any fixed $\tau \in \mathbb{R}$ and $\sigma \in C$ with $\text{Re}\sigma = \frac{3}{2}$, we have the following:

- Assume $0 < |t| \leq 2$.
  - (a) If $0 \leq r \leq 3$, then
    $$|\tilde{w}_{t,\infty}^{\sigma,\tau}(r)| \lesssim |t|^{-\frac{3}{2}} (1 - \log |t|)$$
  - (b) If $r \geq 3$, then $|w_{t,\infty}^{\sigma,\tau}(r)| = O(r^{-\infty} e^{-\nu r}).$
• Assume $|t| \geq 2$. Then for any $r \geq 0$

$$|w_{\sigma,\tau}^r(t)| \lesssim (1 + |r - |t||)^{-\infty} e^{-\nu r}.$$  

Now we use the kernel estimates for the shifted wave operator to deduce the resolvent estimates.

**Lemma 3.2.** Let $\alpha > 0$. Then for all $z \in \mathbb{C} \setminus [0, \infty)$:

- For $\frac{3}{5} < r < 2$ and $2 < p < 6$,

$$\|(-\Delta - \nu^2 - z)^{-1} f\|_{L^p_x} \lesssim \|f\|_{L^r_x} \tag{3.1}$$

- For $\frac{6}{5} \leq r \leq 2$ and $2 \leq p \leq 6$,

$$\|\rho^\alpha(-\Delta - \nu^2 - z)^{-1} \rho^\alpha f\|_{L^p_x} \lesssim \|f\|_{L^r_x} \tag{3.2}$$

**Proof.** We shall use the formula

$$(-\Delta - \nu^2 - (\lambda + i\mu)^2)^{-1} = \Lambda(\lambda, \mu) \int_0^\infty e^{i(\text{sgn}\mu)\lambda t} e^{-|\mu| t} D_o^{-1}(\sin t D_o) dt, \tag{3.3}$$

where $\Lambda(\lambda, \mu) = \text{sgn} \mu \frac{\lambda}{i(\lambda + \mu)}$. Consider the analytic family of operators

$$R_{0}^{\sigma,\tau} = \Lambda(\lambda, \mu) \int_0^\infty e^{i(\text{sgn}\mu)\lambda t} e^{-|\mu| t} \tilde{D}_\sigma D_\tau (\sin t D_o) \chi_0(D_o) dt. \tag{3.4}$$

$$R_\infty^{\sigma,\tau} = \Lambda(\lambda, \mu) \frac{e^{\sigma^2}}{\Gamma(3/2 - \sigma)} \int_0^\infty e^{i(\text{sgn}\mu)\lambda t} e^{-|\mu| t} \tilde{D}_\sigma D_\tau (\sin t D_o) \chi_\infty(D_o) dt. \tag{3.5}$$

Thus by Lemma 3.1 and Lemma 5.13 for $1 < r < 2 < p < \infty$, $\lambda, \mu \in \mathbb{R}$, $\tau < 2$, $\sigma \in \mathbb{R}$,

$$\|R_{0}^{\sigma,\tau} f\|_{L^p} \leq C \|f\|_{L^r}, \tag{3.6}$$

where $C$ is independent of $\lambda, \mu$. In particular, the low frequency part of (3.1) is done. When $\Re \sigma = \frac{3}{2}$, $\tau < 2$, for any $1 < r < 2 < p < \infty$, the kernel of $\tilde{D}_\sigma D_\tau (\sin t D_o) \chi_\infty(D_o)$ denoted by $w_{\sigma,\tau}^r$ satisfies

$$\|w_{\sigma,\tau}^r * f\|_{L^p_x} \leq t^{-\infty} \|f\|_{L^r_x}, \tag{3.7}$$

by applying Lemma 3.1 and Lemma 5.13 again. Meanwhile, when $\Re \sigma = 0$, we have the trivial $L^2 \to L^2$ bound. Hence one obtains by complex interpolation that $w_{\sigma,\tau}^r$ satisfies (3.7) as well.
when
\[ \frac{1}{2} - \frac{\sigma}{3} < \frac{1}{p} < \frac{1}{2}, \quad \frac{1}{2} < \frac{1}{r} < \frac{1}{2} + \frac{\sigma}{3}. \] (3.8)

Then let \( \tau = 1, \sigma = 1 \), we obtain that the high frequency part of (3.1). Thus, together with (3.6), (3.3) yields (3.1). (3.2) can be similarly proved by noticing the additional \( \rho^\alpha \) weight helps us to use the \( p = q = 2 \) case in Lemma 5.13.

**Remark 3.1.** The results as (3.1) are usually called uniform resolvent estimates. For high dimensional hyperbolic spaces, one needs a scaling balance condition for \( p, r \), see [23] for \( H_n, n \geq 3 \). We remark that the proof here is only available for \( n = 2 \) due to the \( t^{-1} \) singularity at \( t = 0 \) in (3.3) when one tries to apply dispersive estimates of wave operators.

The convolution kernel of the free resolvent \((−\Delta − s(1 − s))^{-1}\) in \( H^{n+1} \) is given by
\[ [n R]_0(s; x, y) = (2\pi)^{-\frac{n+1}{2}} e^{-i\eta \mu} (\sinh r)^{-\mu} Q_\mu(c \cosh r), \] (3.9)
where \( Q_\mu \) is the Legendre function, \( \mu = \frac{n-1}{2}, \eta = s - \frac{n+1}{2} \). We need to define \((−\Delta − \nu^2 − \lambda^2 \pm i0)^{-1}\) as what was done in the Euclidean case. The proof of the following lemma is totally analogous to the \( R^n \) case in [1] by using (3.22) to (3.24) below.

**Lemma 3.3.** For any \( \lambda > 0 \), the limit \( \lim_{\epsilon \to 0, \Im \epsilon > 0} (−\Delta − \nu^2 − \lambda^2 \pm \epsilon)^{-1} \) exists in the space \( L(\rho^{-\alpha}L^2, \rho^{-\alpha}L^2) \). And we denote
\[ \lim_{\epsilon \to 0, \Im \epsilon > 0} (−\Delta − \nu^2 − \lambda^2 \pm \epsilon)^{-1} = R_0(\lambda \pm i0). \] (3.10)

Moreover \( g = R_0(\lambda \pm i0) f \) satisfies
\[ (−\Delta − \nu^2 − \lambda^2)g = 0. \] (3.11)

and for all \( f \in \rho^\alpha L^2 \) it holds
\[ \Im \langle R_0(\lambda \pm i0) f, f \rangle(\tau) = \pm \frac{\pi}{2\lambda} \int_{|\xi| = \lambda} |\mathcal{F} f(\tau, b)|^2 |c(\tau)|^{-2} db, \] (3.12)

Assume \( f_n \rightharpoonup f_* \) weakly in \( \rho^\alpha L^2 \), \( z_n \to z_* \) with \( \Im z_n > 0 \).

- If \( \Im \epsilon_* = 0, \epsilon_* > 0 \), then it converges strongly in \( L^2 \) that
  \[ \rho^\alpha R_0(\nu^2 + z_n) \rho^\alpha f_n \to \rho^\alpha R_0(\sqrt{\epsilon_*} \pm i0) \rho^\alpha f_* \] (3.13)
  \[ \rho^\alpha \nabla R_0(\nu^2 + z_n) \rho^\alpha f_n \to \rho^\alpha \nabla R_0(\sqrt{\epsilon_*} \pm i0) \rho^\alpha f_. \] (3.14)
• if \( \Im \varepsilon > 0 \) or \( \Re \varepsilon < 0 \) then it converges strongly in \( L^2 \) that

\[
\rho^\alpha R_0(\nu^2 + z_n)\rho^\alpha f_n \to \rho^\alpha R_0(\nu^2 + z\ast)\rho^\alpha f_\ast \quad (3.15)
\]

\[
\rho^\alpha \nabla R_0(\nu^2 + z_n)\rho^\alpha f_n \to \rho^\alpha R_0(\nu^2 + z\ast)\rho^\alpha f_\ast. \quad (3.16)
\]

Denote the convolution operator with kernel \( [^1 R]_0(1/2, x, y) \) in (3.9) by \( G(0) \).

**Lemma 3.4.** For \( \varepsilon \in \{ \varepsilon : \Im \varepsilon^2 > 0, |\varepsilon| \ll 1 \} \) or \( \varepsilon \in \{ \varepsilon : \Im \varepsilon^2 < 0, |\varepsilon| \ll 1 \} \), there exists some universal constant \( C \) such that

\[
\left\| (-\Delta - \nu^2 + \varepsilon^2)^{-1} - G(0) \right\|_{L(\rho^{-\alpha}L^2, \rho^{-\alpha}L^2)} \leq C\varepsilon^{1/4}. \quad (3.17)
\]

\[
\left\| \nabla (-\Delta - \nu^2 + \varepsilon^2)^{-1} - \nabla G(0) \right\|_{L(\rho^{-\alpha}L^2, \rho^{-\alpha}L^2)} \leq C\varepsilon^{1/4}. \quad (3.18)
\]

**Proof.** We only prove the case when \( \Im \varepsilon^2 > 0 \). The proof of Lemma 3.4 is based on the corresponding estimates for \( \partial_t[^n R]_0(s, x, y) \). We will frequently use the identity

\[
(z^2 - 1)^{m/2} \frac{d^m}{dz^m} Q_\eta^0 = Q_\eta^m(z). \quad (3.19)
\]

Let \( s = 1/2 + e^{i\pi/2} \varepsilon \), then Lemma 5.10 implies

\[
|\partial_t[^1 R]_0(s, x, y)| \leq \begin{cases} 
\log |r|, |r| \leq 1 \\
|\varepsilon|^{-1/2}C_\delta \varepsilon^{-(1/2-\delta)r}, |r| \geq 1 
\end{cases} \quad (3.20)
\]

By Lemma 5.13 one easily obtains (3.17) from (3.20) and Newton-Leibniz formula. Since \( |\nabla_d d(x, y)| = 1 \) for any \( y \in \mathbb{H}^2 \), the key ingredient to prove (3.18) is the estimates for \( \partial_t \partial_r[^1 R]_0(s, x, y) \).

By using (3.19), (3.9) and Lemma 5.10, we have

\[
|\partial_t \partial_r[^1 R]_0(s, x, y)| \leq \begin{cases} 
(cosh^2 r - 1)^{-1/2}(\sinh^2 r)r^{-2}, |r| \leq 1 \\
C_\delta(cosh^2 r - 1)^{-1/2}(\sinh^2 r)|\varepsilon|^{1/2}e^{-(3/2-\delta)r}, |r| \geq 1 
\end{cases} \quad (3.21)
\]

Thus (3.18) follows by Lemma 5.13 and Newton-Leibniz formula.

The following lemma was proved in [49].

**Lemma 3.5.** Let \( \alpha > 0 \). Then for all \( \lambda \in (\delta, \infty) \), \( p \in [1, \infty] \), we have

\[
\|(-\Delta + \lambda)^{-1}\|_{L^p \to L^p} + \|\nabla (-\Delta + \lambda)^{-1}\|_{L^p \to L^p} \lesssim \min(\varepsilon(\delta), \lambda^{-\frac{\delta}{2}})
\]

\[
\|\rho^\alpha(-\Delta + \lambda)^{-1}\rho^\alpha\|_{L^p \to L^p} + \|\rho^\alpha \nabla(-\Delta + \lambda)^{-1}\rho^\alpha\|_{L^p \to L^p} \lesssim \min(1, \lambda^{-\frac{\delta}{4}}).
\]
For $z \in \mathbb{C}$ with $\Re z > 0$, we have for $\alpha > 0$ sufficiently small

$$\|\rho^\alpha R_0(\nu^2 - z^2)f\|_{L^2} \lesssim |z|^{-1}\|f \rho^{-\alpha}\|_{L^2} \quad (3.22)$$

$$\|\rho^\alpha R_0(\nu^2 - z^2)f\|_{L^2} \lesssim \|f \rho^{-\alpha}\|_{L^2} \quad (3.23)$$

$$\|\rho^\alpha \nabla R_0(\nu^2 - z^2)f\|_{L^2} \lesssim \|f \rho^{-\alpha}\|_{L^2}. \quad (3.24)$$

### 3.2 Kato Smoothing Effect for Magnetic Wave Equation

First, we recall the Kato smoothing theorem.

**Theorem 3.1.** ([30]) Let $X, Y$ be two Hilbert spaces and $H : X \to Y$ be a self-adjoint operator with resolvent denoted by $R(\lambda) = (H - \lambda)^{-1}$. Let $A : X \to Y$ be a closed densely defined operator. Suppose that for any $g \in D(A^*)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ it holds

$$\|AR(\lambda)A^*g\|_Y \leq C\|g\|_Y. \quad (3.25)$$

Then the operator $A$ is $H$-smooth, i.e., $e^{itH}f \in D(A)$ for all $f \in X$ and a.e. $t$, and

$$\int_{-\infty}^{\infty} \|Ae^{-itH}f\|_Y^2 dt \leq \frac{2}{\pi}C^2\|f\|_X^2. \quad (3.26)$$

**Proposition 3.1.** Let $\Xi(x)$ be any fixed frame on $Q(x)$ for $x \in \mathbb{H}^2$. Then the spectrum of the operator $-\Delta + W$ defined in Lemma 2.8 is contained in $[1/4, \infty)$.

**Proof.** Suppose that $\Xi(x) = (\Xi_1(Q(x)), \Xi_2(Q(x)))$ is the given frame on $Q(x)$ in Proposition 2.3. For $f, g \in L^2(\mathbb{H}^2, \mathbb{C}^2)$, we associate them with the vector fields $f\Xi \triangleq f_1\Xi_1 + f_2\Xi_2$ and $g\Xi \triangleq g_1\Xi_1 + g_2\Xi_2$ respectively. The corresponding induced covariant is $D_i = \partial_i + A_i^\infty$ with $[A_i^\infty]_p = \langle \nabla_i \Xi_p, \Xi_k \rangle$. Then we have

$$\nabla_i(f\Xi) = (\partial_i f_k + \sum_{p=1}^{2}[A_i^\infty]_{pj} f_p)\Xi_k = (D_i f)\Xi.$$ 

Furthermore, one has

$$h^{ii}\nabla_i \nabla_i(f\Xi) = h^{ii}(D_i D_i f)\Xi.$$ 

Meanwhile we see

$$h^{ii}(f \wedge \phi_i^\infty)\phi_i^\infty)\Xi = h^{ii}R(Q(x))(f\Xi, \phi_i\Xi)\phi_i\Xi. \quad (3.27)$$

Therefore, we conclude

$$h^{ii}\nabla_i \nabla_i(f\Xi) - h^{ii}\Gamma^k_{ii} \nabla_k(f\Xi) = h^{ii}(D_i D_i f)\Xi - h^{ii}\Gamma^k_{ii}(D_k f)\Xi.$$
\[
\Delta f - W f \Xi = h^{ii} R(Q(x))(f \Xi, \phi_i \Xi) \phi_i \Xi. \tag{3.28}
\]

Hence (3.27) and (3.28) give
\[
\Delta \mid f \Xi \mid^2 = h^{ii} \langle \nabla_i \nabla_i (f \Xi), f \Xi \rangle + h^{ii} \langle \nabla_i \nabla_i (f \Xi), \nabla_i (f \Xi) \rangle - h^{ii} \Gamma^k_{ii} \langle \nabla_i (f \Xi), f \Xi \rangle - h^{ii} \langle f \Xi, R(f \Xi, \phi_i \Xi) \phi_i \Xi \rangle.
\]

Then by integration by parts, the self-adjointness of \( \Delta - W \) and the non-positiveness of the sectional curvature, we obtain
\[
2 \langle (\Delta + W) f, f \rangle_{L^2} \geq \frac{1}{4} \int_{\mathbb{H}^2} |f|^2 \text{dvol}_h. \tag{3.29}
\]

By Kato's inequality \( |\nabla X| \leq |\nabla X| \) and the Sobolev inequality \( \| \nabla g \|_{L^2} \geq \frac{1}{4} \| g \|_{L^2}^2 \), one deduces
\[
\int_{\mathbb{H}^2} \langle (\Delta + W) f, f \rangle \text{dvol}_h \geq \frac{1}{4} \int_{\mathbb{H}^2} \langle (f \Xi), (f \Xi) \rangle \text{dvol}_h \tag{3.30}
\]
\[
= \frac{1}{4} \int_{\mathbb{H}^2} |f|^2 \text{dvol}_h. \tag{3.31}
\]

Since the spectrum is contained in the numerical range we obtain our lemma. \(\square\)

### 3.3 Weighted Elliptic Estimates for Coulomb Gauge

We will use the Coulomb gauge on the harmonic map to kill possibly existing resonances. Thus we first need to prove the point-wise estimates for the new connection matrices induced by the Coulomb gauge. The existence of Coulomb gauge in two dimensions is well-known, see for instance \[68\]. We give the detailed proof since it tells us the explicit form of the Coulomb gauge.

**Lemma 3.6.** There exists an orthonormal frame \( \{ \Xi_1, \Xi_2 \} \) which spans \( T_{Q(x)}N \) for any \( x \in M = \mathbb{H}^2 \) such that the corresponding connection 1-form \( A \in \Lambda^1(\text{Ad} Q^*TN) \) satisfies the Coulomb condition, i.e.,
\[
d^* A = 0. \tag{3.32}
\]

**Proof.** Since in the two dimensional case, the connection matrix \( A_i \) is of the form
\[
\begin{pmatrix}
0 & a_i \\
-a_i & 0
\end{pmatrix}
\]
for some real-valued function $a_i$. The mixture use of $A_i$ as a function or a matrix will not cause confusion in the following proof. Suppose $\Theta(Q(x))$ is the frame on $Q(x)$ given by (2.3). Then corresponding connection 1-form is

$$A = A_i^\infty dx^i, \quad [A_i^\infty]^k_i = \langle \nabla_i \Theta_l, \Theta_k \rangle.$$ (3.33)

Given any real valued function $u \in H^1(\mathbb{H}^2; \mathbb{R})$, we associate it with a matrix $U \in SO(2, \mathbb{R})$ defined by

$$U(x) = \begin{pmatrix} \sin u(x) & \cos u(x) \\ -\cos u(x) & \sin u(x) \end{pmatrix}.$$ 

Define the new frame $\Xi^*(x) = U(x)\Theta(Q(x))$. Then the new connection 1-from $A$ is given by

$$A = A + du.$$ (3.34)

Thus the Coulomb condition reduces to

$$d^*A = d^*A + \Delta u = 0.$$ (3.35)

Hence it suffices to set

$$u = (-\Delta)^{-1}d^*A.$$ (3.36)

**Proposition 3.2.** Assume that the given frame $\Xi$ in Proposition 2.3 is the Coulomb gauge constructed in Lemma 3.6. Then the associated Schrödinger operator $H = -\Delta + W$ reads as

$$H \varphi = -\Delta \varphi - 2h^{ii} A_i \partial_i \varphi - h^{ii} A_i A_i \varphi - h^{ii} (\varphi \wedge \phi_i^\infty,^*) \phi_i^\infty,^*.$$ (3.37)

where $A_i = A_i + \partial_i u$, and

$$\phi_i^\infty,^* = \langle \partial_i Q(x), \Xi^*_1(x) \rangle, \langle \partial_i Q(x), \Xi^*_2(x) \rangle.$$ 

Moreover, if we write $H$ as $H = -\Delta + V_1 \nabla + V_2$ with a bit of abuse of notations, then for any $0 < \beta < g$,

- (a) $V_2$ is nonnegative on $L^2(\mathbb{H}^2; \mathbb{C}^2)$
- (b) $\|\rho^{-\beta} V_1\|_{L^\infty} \leq C$
• (c) \( \| \rho^{-\beta} V_2 \|_{L^\infty_x} \leq C \)

Proof. Fix any orthogonal coordinates \( x_{1,2} \) for \( M = \mathbb{H}^2 \). Assume \( \mathcal{A} = \mathcal{A}_i dx^i \). Then the Coulomb condition is written as

\[
h^{ii}(\partial_i \mathcal{A}_i - \Gamma^{k}_{ii} \mathcal{A}_k) = 0, \tag{3.38}
\]

thus the (3.38) term in \( H = -\Delta + W \) vanishes and \( (3.37) \) follows. It remains to prove the three claims (a),(b),(c). Since the \( V_2 \) part reads as

\[
V_2 = -h^{ii} \mathcal{A}_i \varphi - h^{ii} (\varphi \wedge \phi_i^\infty)^* \phi_i^\infty, \tag{3.39}
\]

(a) is easy to verify by the negative sectional curvature of \( N \) and the skew-symmetry of \( \mathcal{A}_i \). With a bit of abuse of notation, \( V_1 \) can be written as

\[
V_1(x) = \sqrt{h^{ii}} \mathcal{A}_i(x). \tag{3.40}
\]

By (2.28) and \( \mathcal{A} = \mathcal{A} + du \), for (b) it suffices to prove

\[
|\sqrt{h^{ii}} \partial_i u| \leq \rho^\beta. \tag{3.41}
\]

With (3.36), one notices that (3.41) reduces to prove

\[
\| \rho^{-\beta} \nabla (-\Delta)^{-1} d^* \mathcal{A} \|_{L^\infty_2} \leq C. \tag{3.42}
\]

By the identity \( d^* \mathcal{A} = h^{ii} (\partial_i \mathcal{A}_i - \Gamma^{k}_{ii} \mathcal{A}_k) \) and (2.29), we see \( \rho^{-\beta} d^* \mathcal{A} \in L^2_2 \). Thus by (5.11) and Young’s convolution inequality, one has for any \( \beta \in (0, \rho) \)

\[
\| \rho^{-\beta} \nabla (-\Delta)^{-1} d^* \mathcal{A} \|_{L^\infty_2} \leq C. \tag{3.43}
\]

Until now we have obtained

\[
|\sqrt{h^{ii}} \mathcal{A}_i| \leq \rho^\beta. \tag{3.44}
\]

With (3.39), (c) follows by (2.28) and (3.44); (b) follows from (2.29) and (3.40).

Lemma 3.7. Fix the frame \( \Xi \) in Proposition 2.3 to be the Coulomb gauge built in Lemma 3.6. In the polar coordinates for \( M = \mathbb{H}^2 \), \( H \) can be written as

\[
H \varphi = -\Delta + 2 \mathcal{A}_r \partial_r \varphi + 2 \mathcal{A}_\theta \sinh^{-2} r \partial_\theta \varphi + \mathcal{U}_r \varphi + \mathcal{U}_\theta \varphi \tag{3.45}
\]
where we denote
\[ U_r \varphi = A_r A_r \varphi + (\phi_r^{\infty, *}) \wedge \varphi \phi_r^{\infty, *}, \]
\[ U_\theta \varphi = \sinh^{-2} r A_\theta A_\theta \varphi + \sinh^{-2} r (\phi_\theta^{\infty, *}) \wedge \phi_\theta^{\infty, *}. \]

Then one has when \( d(r, 0) \geq \delta \),

\[ |U_\theta| + |U_r| \leq C \rho^\beta \] (3.48)

\[ |\partial_r A_r| + |(\sinh^{-1} r) \partial_\theta A_\theta| + |(\sinh^{-2} r) \partial_\theta A_\theta| + |(\sinh^{-1} r) \partial_r A_\theta| \leq C(\delta) \] (3.49)

**Proof.** (3.48) is a trivial corollary of Proposition 3.2. It suffices to verify (3.49). By viewing \( A_{\theta, r} \) as a real-valued function we have

\[ \partial_i A_j = \langle \nabla_i \nabla_j \Theta_1, \Theta_2 \rangle + \langle \nabla_j \Theta_1, \nabla_i \Theta_2 \rangle, \]

(3.50)

where we associate \( r \) with \( i = 1 \) and \( \theta \) with \( i = 2 \) respectively. Then inserting the explicit formula for \( \Theta_{1, 2} \) given by (2.3), we get

\[
\begin{align*}
\partial_i A_j &= \left\langle \nabla_i \left( e^{Q^2} \partial_j Q^2 \frac{\partial}{\partial y_1} \right), \Theta_2 \right\rangle + \left\langle \nabla_j \left( e^{Q^2} \Gamma_{j1}^p \frac{\partial}{\partial y_p} \right), \Theta_2 \right\rangle + \left\langle e^{Q^2} \partial_j Q^2 \frac{\partial}{\partial y_1} + e^{Q^2} \Gamma_{j1}^p \frac{\partial}{\partial y_p}, \Gamma_{i2}^q \frac{\partial}{\partial y_q} \right\rangle \\
&= \left\langle \partial_i Q^2 \partial_j Q^2 e^{Q^2} \frac{\partial}{\partial y_1}, \Theta_2 \right\rangle + \left\langle e^{Q^2} \partial_j Q^2 \frac{\partial}{\partial y_1}, \Theta_2 \right\rangle + \left\langle e^{Q^2} \partial_j Q^2 \Gamma_{j1}^p \frac{\partial}{\partial y_p}, \Theta_2 \right\rangle \\
&\quad + \left\langle \partial_i Q^2 e^{Q^2} \Gamma_{j1}^p \frac{\partial}{\partial y_1}, \Theta_2 \right\rangle + \left\langle e^{Q^2} \Gamma_{j1}^p \frac{\partial}{\partial y_p}, \Theta_2 \right\rangle + \left\langle e^{Q^2} \Gamma_{j1}^p \Gamma_{k2}^q \frac{\partial}{\partial y_k}, \Theta_2 \right\rangle \\
&\quad + \left\langle e^{Q^2} \partial_j Q^2 \frac{\partial}{\partial y_1} + e^{Q^2} \Gamma_{j1}^p \frac{\partial}{\partial y_p}, \Gamma_{i2}^q \frac{\partial}{\partial y_q} \right\rangle.
\end{align*}
\]

Recall the explicit formula for \( |\nabla^k Q^i| \) for \( i = 1, 2, k = 1, 2 \), one obtains

\[
|\partial_r A_r| + |(\sinh^{-1} r) \partial_\theta A_\theta| + |(\sinh^{-2} r) \partial_\theta A_\theta| + |(\sinh^{-1} r) \partial_r A_\theta| \leq \sum_{k=1, i=1}^{2} |\nabla^k Q^i| + |\coth r \partial_r Q^i|
\]

(3.51)

When \( r \geq \delta \), by Sobolev embedding and Kato’s inequality, (3.51) further gives

\[
|\partial_r A_r| + |(\sinh^{-1} r) \partial_\theta A_\theta| + |(\sinh^{-2} r) \partial_\theta A_\theta| + |(\sinh^{-1} r) \partial_r A_\theta| \lesssim C(\delta) \sum_{k=1, i=1}^{2} \|\nabla^k Q^i\|_{L^\infty}
\]

\[
\lesssim C(\delta) \sum_{i=1}^{2} \|Q^i\|_{H^4}.
\]

(3.52)
Hence since $A_i = A_i + \partial_i u$, for (3.49) it suffices to bound $|h^{ij}\partial_i^2 u|$. Recall $u = (-\Delta)^{-1}d^* A$, we see

$$h^{ij}\partial_i^2 u| \leq |\nabla^2((\Delta)^{-1}d^* A)| + \coth r|\partial_r u|$$

$$\leq |\nabla^2((\Delta)^{-1}d^* A)| + \coth r|du|. \quad (3.53)$$

Applying Lemma 5.14 and Lemma 2.7 yields for $r \geq \delta$

$$h^{ij}\partial_i^2 u| \leq \|\nabla^2((\Delta)^{-1}d^* A)\|_{L^\infty} + C(\delta)\|du\|_{L^\infty} \quad (3.54)$$

Thus (3.49) follows.

### 3.4 Nonexistence of Resonance and Small Frequency Estimates

Denote the Schrödinger operator $-\Delta + W$ as $H = -\Delta + V_1 \nabla + V_2$.

**Lemma 3.8.** Let $\alpha > 0$, and $\Xi$ be the Coulomb gauge on $Q(x)$ in Proposition 2.3. And $H = -\Delta + W$ is the corresponding Schrödinger operator. Then we have

$$\langle (-\Delta - \nu^2 + W)g, g \rangle \geq \langle \nabla g, \nabla g \rangle. \quad (3.55)$$

**Proof.** From Proposition 3.2, $V_2$ is nonnegative and $H$ is self-adjoint. Thus one has

$$\langle -\Delta g + Wg, g \rangle = \Re\langle -\Delta g + Wg, g \rangle \geq \Re\langle \nabla g, \nabla g \rangle + \Re 2h^{ij}\langle A_i \partial_j g, g \rangle. \quad (3.56)$$

Meanwhile by viewing $A = A_i dx_i$ and the fact $d^*$ is the dual operator to $d$, we conclude

$$2\Re h^{ij}\langle A_i \partial_j g, g \rangle = \langle A, d(|g|^2) \rangle = \langle d^* A, |g|^2 \rangle = 0, \quad (3.57)$$

where the last equality is due to the Coulomb condition. Therefore, (3.57) and (3.56) yield

$$\langle -\Delta g + Wg, g \rangle \geq \langle \nabla g, \nabla g \rangle. \quad \square$$

**Proposition 3.3.** Let $\alpha > 0$, and $\Xi$ be the Coulomb gauge on $Q(x)$. Then $\rho^{-\alpha}(I + WG(0))^{-1}\rho^\alpha$ is invertible in $L^2$.

**Proof.** Let $T(\epsilon) = \rho^{-\alpha}W(\Delta - \nu^2 + i\epsilon)^{-1}\rho^\alpha$. We first calculate the spectrum range of $T(\epsilon)$. By Fredholm's alternative, $T(\epsilon)$ only has pure point spectrum. Assume $\lambda$ is an eigenvalue of
$T(\epsilon)$, then for some $u \in L^2$ it holds $T(\epsilon)u = \lambda u$. Let $g = (-\Delta - \nu^2 + i\epsilon^2)^{-1}\rho^a u$, then we have

$$
\lambda(-\Delta - \nu^2 + i\epsilon^2)g = Wg. 
$$

Thus we obtain

$$
(\lambda + 1)(-\Delta - \nu^2 + i\epsilon^2)g = (-\Delta - \nu^2 + i\epsilon^2 + W)g. 
$$

Since $\nu^2 - i\epsilon^2$ belongs to the resolvent set of $-\Delta$, $\rho^a u \in L^2$, we see $g \in H^2$. And thus integration by parts yields,

$$
(\lambda + 1)(\langle \nabla g, \nabla g \rangle - (\nu^2 - i\epsilon^2)\langle g, g \rangle) = \langle (-\Delta - \nu^2 + i\epsilon^2 + W)g, g \rangle. 
$$

Without loss of generality, we assume $\|g\|_{L^2} = 1$, and then it holds

$$
\lambda = \frac{(\Lambda_1 - \Lambda_2)(\Lambda_2 - i\epsilon^2)}{\Lambda_2^2 + \epsilon^4},
$$

where we denotes $\Lambda_1 = \langle (-\Delta - \nu^2 + W)g, g \rangle$ and $\Lambda_2 = \langle \nabla g, \nabla g \rangle - \nu^2\langle g, g \rangle$. In the Coulomb case, by Lemma [3.8] [5.61] gives

$$
\Re \lambda \geq 0. 
$$

Suppose that $1 + T(0)$ is not invertible in $L^2$, by Fredholm’s alternative, $-1$ is an eigenvalue of $T(0)$. Since the only possible accumulated point of $\sigma(T(0))$ is $0$ due to the compactness we see $-1$ is an isolated spectrum of $T(0)$. Then let $\partial B(-1, \delta)$ be a small circle centered at $-1$ with radius $\delta > 0$. Define the projection operator $P_0$ by

$$
P_0 = \int_{\partial B(-1, \delta)} (T(0) - z)^{-1}dz. 
$$

Since $-1$ is an isolated spectrum, we have a uniform bound for all $z \in \partial B(-1, \delta)$

$$
\|(T(0) - z)^{-1}\|_{L^2 \to L^2} \leq C(\delta). 
$$

Then by the resolvent identity, Lemma [3.31] and Neumann’s series argument, we have for all $z \in \partial B(-1, \delta)$ and $\epsilon \in (0, \epsilon(\delta))$ with $0 < \epsilon(\delta) \ll 1$,

$$
\|(T(\epsilon) - z)^{-1}\| \leq \|(T(0) - z)^{-1}(I + (T(\epsilon) - T(0))(T(0) - z)^{-1})^{-1}\|_{L^2 \to L^2} \leq C_1(\delta). 
$$
Similarly we define the projection operator $P_\epsilon$ by

$$P_\epsilon = \int_{\partial B(-1,\delta)} (T(\epsilon) - z)^{-1} dz.$$  \hfill (3.66)

Then by the resolvent identity, (3.64) and (3.65), we obtain for $\epsilon$ sufficiently small,

$$\|P_\epsilon - P_0\|_{L^2 \to L^2} \leq \int_{\partial B(-1,\delta)} \|(T(\epsilon) - z)^{-1}(T(\epsilon) - T(0))(T(0) - z)^{-1}\|_{L^2 \to L^2} dz \leq C_1(\delta)C(\delta)\delta \epsilon. \hfill (3.67)$$

Let $\epsilon \ll 1$, then one has $\|P_\epsilon - P_0\|_{L^2 \to L^2} \leq 1/2$. But we have shown $B(-1,\delta)$ is away from the spectrum of $T(\epsilon)$ for any $\epsilon > 0$, thus $P_\epsilon = 0$. Hence we arrive at $\|P_0\|_{L^2 \to L^2} \leq 1/2$, which contradicts with the assumption $-1 \in \sigma(T(0))$. Thus $I + T(0)$ is invertible and the lemma follows. \hfill \Box

Now we deal with the small frequency part.

**Lemma 3.9.** Let $\Xi$ be the Coulomb gauge on $Q(x)$, $\alpha > 0$. There exist $\delta > 0$ and $C > 0$ such that it holds uniformly for $0 < |\sigma| < \delta$, $\Im \sigma > 0$ that

$$\|\rho^\alpha(H - \nu^2 \pm \sigma)^{-1}\rho^\alpha\|_{L^2 \to L^2} \leq C.$$ \hfill (3.70)

$$\|WZ\sigma\rho^\alpha\|_{L^2 \to L^2} \leq C$$ \hfill (3.71)

Proof. Denote $(-\Delta - \nu^2 - \sigma)^{-1} = Z_\sigma$. By the formal identity

$$(H - \nu^2 - \sigma)^{-1} = Z_\sigma(I + WZ_\sigma)^{-1},$$

it suffices to prove for some $C$ independent of $0 < |\sigma| < \delta$, $\Im \sigma > 0$

$$\|\rho^\alpha Z_\sigma\rho^\alpha\|_{L^2 \to L^2} \leq C \hfill (3.72)$$

$$\|(I + \rho^{-\alpha}WZ_\sigma)\rho^\alpha\|_{L^2 \to L^2} \leq C \hfill (3.73)$$

(3.72) has been verified in Lemma 3.5. By resolvent identity, we have formally that

$$(I + \rho^{-\alpha}WZ_\sigma)\rho^\alpha)^{-1} = (I + \rho^{-\alpha}WG(0)\rho^\alpha)^{-1}(I + \tilde{Z}(I + \rho^{-\alpha}WG(0)\rho^\alpha)^{-1})^{-1},$$ \hfill (3.74)

where $\tilde{Z}$ denotes $\rho^{-\alpha}W(Z_\sigma - G(0))\rho^\alpha$. Then by Lemma 3.5, Proposition 3.3 and a Neumann
series argument we obtain for $0 < |\sigma| < \delta, \Im \sigma > 0$

$$\| (I + \tilde{Z})(I + \rho^{-\alpha}WG(0)\rho^\alpha)^{-1}\|_{L^2 \to L^2} \leq 2.$$ 

Hence (3.73) follows by Proposition 3.3, (3.74). The $(-\Delta - \nu^2 + \sigma)^{-1}$ case is the same and thus our lemma follows. \hfill \Box

3.5 Mediate Frequency Resolvent Estimates

The mediate frequency resolvent estimate is standard in our case by applying the original idea of Agmon \cite{1} and the Fourier restriction estimates obtained by Kaizuka \cite{29}.

Lemma 3.10. If $I + \rho^{-\alpha}W \mathfrak{R}_0(\lambda + i0)\rho^\alpha$ is not invertible in $L^2$ for some $\lambda > 0$, then $\nu^2 + \lambda^2$ is an eigenvalue of $-\Delta + W$ in $L^2$.

Proof. By Fredholm’s alternative, we can assume there exists $\tilde{f} \in L^2$ such that

$$\tilde{f} + \rho^{-\alpha}W \mathfrak{R}_0(\lambda + i0)\rho^\alpha \tilde{f} = 0. \quad (3.75)$$

Let $g = \mathfrak{R}_0(\lambda + i0)\rho^\alpha \tilde{f}$, then Lemma 3.5 with (3.1) shows $g \in W^{1,r}$ for $r \in (2,6)$ and

$$(-\Delta - \nu^2)g = \lambda^2 g + Wg \quad (3.76)$$

in the distribution sense. By density arguments and (3.75) one can verify

$$\langle \rho^\alpha \tilde{f}, \mathfrak{R}_0(\lambda + i0)\rho^\alpha \tilde{f} \rangle + \langle Wg, g \rangle = 0 \quad (3.77)$$

By the self-adjointness of $W$, we deduce

$$\Im \langle \rho^\alpha \tilde{f}, \mathfrak{R}_0(\lambda + i0)\rho^\alpha \tilde{f} \rangle = 0 \quad (3.78)$$

Let $f = \rho^\alpha \tilde{f}$. Hence (3.12) implies $|\mathcal{F}f(\tau, b)e^{-\nu(\tau)}|^2$ vanishes when $\tau = \lambda$. Then the Fourier restriction estimate in \cite[Equ. (4.4)]{29} gives for any $\theta \in (0,1)$

$$\left( \int_{\mathbb{S}^1} \left| c^{-1}(\tau)\mathcal{F}f(\tau, b) - c^{-1}(\lambda)\mathcal{F}f(\lambda, b) \right|^2 db \right)^{1/2} \leq C|\tau - \lambda|^\theta \|\langle x \rangle^{1/2 + \theta} f\|_{L^2}. \quad (3.79)$$

By the vanishing of $|\mathcal{F}f(\lambda, b)e^{-\lambda}|^2$, (3.79) further yields

$$\int_{\mathbb{S}^1} \left| c^{-1}(\tau)\mathcal{F}f(\tau, b) \right|^2 db \leq |\lambda - \tau|^{2\theta} \|\langle x \rangle^{1/2 + \theta} f\|_{L^2}^2,$$
Thus by Plancherel identity, one has for $0 < \theta_1 \ll 1$ and $\frac{1}{2} < \theta_2 < 1$,

$$\|g\|_{L^2}^2 \leq \int_0^{\infty} \int_{\mathbb{S}^1} (\tau^2 - \lambda^2)^{-2}\left| e^{-1}(\tau)\mathcal{F} f(\tau, b) \right|^2 db d\lambda$$

$$\leq C(\lambda) \|\langle x \rangle^{1/2 + \theta_1} f\|_{L^2}^2 + \|\langle x \rangle^{1/2 + \theta_2} f\|_{L^2}^2$$

$$\leq C(\lambda) \|\tilde{f}\|_{L^2}.$$

This implies $g \in L^2$, thus by (3.76), $\Delta g \in L^2$ and hence $g \in D(H)$ and $\lambda^2 + \nu^2$ is an eigenvalue of $H$.

The proof of the following lemma is quite standard, see [26], Lemma 4.6. For completeness, we give the detailed proof below.

**Lemma 3.11.** For all $\lambda > 0$ and $\epsilon \in [0, 1]$, we have

$$\sup_{\lambda \in [\delta, \delta^{-1}], \epsilon \in [0, 1]} \|\rho^\alpha (H - \nu^2 - \lambda^2 \pm i\epsilon)\rho^\alpha\|_{L^2 \to L^2} \leq C(\delta).$$  \hfill (3.80)

**Proof.** The non-existence of positive eigenvalue of $-\Delta + W$ in $(\nu^2, \infty)$ is standard by Mourre estimates, see [Prop. 5.2 [5]] for the electric potential case and [18] for the original idea. Thus by Lemma 3.10 for all $\lambda > 0$

$$I + \rho^{-\alpha} \mathcal{R}(\lambda \pm i\epsilon)\rho^\alpha$$

is invertible in $L^2$. \hfill (3.81)

By the identity $R_H(\nu^2 + \lambda^2 \pm i\epsilon) = R_0(\nu^2 + \lambda^2 \pm i\epsilon)(I + WR_0(\nu^2 + \lambda^2 \pm i\epsilon))^{-1}$ and Lemma 3.5 (3.80) follows by

$$\sup_{\lambda \in [\delta, \delta^{-1}], \epsilon \in [0, 1]} \|I + \rho^{-\alpha} WR_0(\nu^2 + \lambda^2 \pm i\epsilon)\rho^\alpha\|_{L^2 \to L^2} \leq C(\delta).$$  \hfill (3.82)

Denote $V(\lambda, \epsilon) = \rho^{-\alpha} W(H - \nu^2 - \lambda^2 \pm i\epsilon)\rho^\alpha$. Assume (3.82) fails, then there exists $f_n \in L^2$ with $\|f_n\|_{L^2} = 1$ and $(\lambda_n, \epsilon_n) \in [\delta, \delta^{-1}] \times [0, 1]$ such that

$$\|(I + V(\lambda_n, \epsilon_n))f_n\|_{L^2} \to 0.$$  \hfill (3.83)

Up to subsequence, we assume $\lambda_n \to \lambda_*$ and $\epsilon_n \to \epsilon_*$. And one may assume $f_n \to f_*$ weakly in $L^2$, then (3.13) to (3.16) give

$$V(\lambda_n, \epsilon_n)f_n \to V(\lambda_*, \epsilon_*)f_*$$

strongly in $L^2$. \hfill (3.84)
Thus (3.83) shows for $f_\ast \in L^2$ it holds in the distribution sense that

$$f_\ast + V(\lambda_\ast, \epsilon_\ast)f_\ast = 0. \quad (3.85)$$

If $\epsilon_\ast > 0$, it is obvious $f_\ast = 0$ by (3.83). If $\epsilon_\ast = 0$, we also have $f_\ast = 0$ by (3.81). Then (3.83) and (3.84) yield

$$\lim_{n \to \infty} \|f_n\|_{L^2} = 0, \quad (3.86)$$

which contradicts with $\|f_n\|_{L^2} = 1$.

### 3.6 High Frequency Estimates

Divide the magnetic potential into the long range and the remainder part by adding a cutoff function:

$$V_{i\text{far}} = \chi(x)V_i, \quad V_{i\text{near}} = (1 - \chi(x))V_i, \quad i = 1, 2, \quad (3.87)$$

where $1 - \chi(x) \in C_\infty_c(\mathbb{H}^2)$ is supported in $d(x, 0) < 2\delta$ and equals one in $\{x : d(x, 0) < \delta\}$.

Now we consider the operator $H_1 = -\Delta + V_{1\text{far}}\nabla + V_{2\text{far}}$. The high frequency resolvent estimates for $H_1$ in the weighted space is given by the following lemma. The proof of Lemma 3.12 is an energy argument based on [6] where high frequency resolvent estimates for Schrödinger operators with large long-range magnetic potentials in $\mathbb{R}^n$ are considered. Since polar coordinates will be used below, we first write $H_1$ in the following form

$$H_1 \varphi = -\Delta \varphi + V_r \partial_r \varphi + V_\theta \sinh^{-1} r \partial_\theta \varphi + U_r \varphi + U_\theta \varphi \quad (3.88)$$

where we denote

$$V_r = \chi(x)A_r, \quad V_\theta = \chi(x) \sinh^{-1} r A_\theta \quad (3.89)$$

$$U_r \varphi = \chi(x)A_r A_r \varphi + \chi(x) (\phi_r^{\infty,*} \wedge \varphi) \phi_r^{\infty,*} \quad (3.90)$$

$$U_\theta = \chi(x) \sinh^{-2} r A_\theta A_\theta \varphi + \chi(x) \sinh^{-2} r (\phi_\theta^{\infty,*} \wedge \varphi) \phi_\theta^{\infty,*} \quad (3.91)$$

Notice that due to the fact $H$ is independent of the orthogonal coordinates for $M$, the results obtained in the polar coordinates can be directly transformed to coordinates given by (2.1). We introduce a weight function $\psi_\alpha(r) = (\tanh r)^{\alpha-1} \rho^\alpha$.

Before stating the following lemma, we remark that although $H_1$ is not self-adjoint due to the cutoff function $\chi$, the numerical range of $H_1$ is still contained in the real line. This can be verified by applying the Coulomb condition. Thus $\nu^2 + \lambda^2 + i\epsilon$ lies in the resolvent set of $H_1$ for
The original idea is due to [6]. We will only prove (3.92) for 

\[ \| \rho^\alpha R_{H_1}(\nu^2 + \lambda^2 \pm i\epsilon)\rho^\alpha \|_{L^2 \rightarrow L^2} \leq C\lambda^{-1}, \]  

(3.92)

where \( C \) is independent of \( \delta, \epsilon. \)

**Proof.** The original idea is due to [6]. We will only prove (3.92) for \( R_{H_1}(\nu^2 + \lambda^2 + i\epsilon), \) the case is the same. Let \( (r, \theta) \in \mathbb{R}^+ \times [0, 2\pi] \) be the polar coordinates for \( \mathbb{H}^2, \) \( u = \sinh^{1/2} r f, \) \( X = (\mathbb{R}^+ \times [0, 2\pi], drd\theta). \) All the inner product \( \langle \cdot, \cdot \rangle \) in this proof denotes \( \langle \cdot, \cdot \rangle_{L^2(X)}. \) Denote \( D_r = i\lambda^{-1}\partial_r, \) \( D_\theta = i\lambda^{-1}\partial_\theta. \) Define \( P = \lambda^{-2} \sinh^{1/2} r (H_1 - \nu^2 - \lambda^2 + i\epsilon) \sinh^{-1/2} r. \) Since \( \rho^\alpha \leq \psi_\alpha, \) for (3.92) it suffices to prove

\[ \| \psi_\alpha u \|_{L^2(X)} \leq C\lambda \| \psi_\alpha^{-1} Pu \|_{L^2(X)}. \]

(3.93)

It is easy to verify,

\[ Pu = D_r^2 u + \sinh^{-2} r D_\theta^2 u - \lambda^{-2} \frac{1}{4} \frac{\cosh^2 r}{\sinh^2 r} u + \lambda^{-2} Lu + (i\epsilon\lambda^{-2} - 1)u + \lambda^{-2} V_r \partial_r u + \lambda^{-2} V_\theta \sinh^{-1} r \partial_\theta u, \]

(3.94)

where \( L = \frac{1}{2} + \frac{1}{2} V_r \frac{\cosh r}{\sinh r} + U. \) Divide \( L \) into the long range part \( L_0 \) and the short range part \( L_1 \) by

\[ L_0 = \frac{1}{2} + U, \quad L_1 = -\frac{1}{2} V_r \frac{\cosh r}{\sinh r}. \]

(3.95)

Define the energy functional \( E(r) \) by

\[ E(r) = \| D_r u \|^2_2 + \langle \nu^{-2} \sinh^{-2} r \Lambda_\theta u + u, u \rangle - \lambda^{-2} \langle L_0 u, u \rangle - \lambda^{-2} \Re \langle V_\theta \sinh^{-1} r \partial_\theta u, u \rangle, \]

(3.96)

where we denote \( \Lambda_\theta u = \partial_\theta^2 u - \frac{1}{4}(\cosh^2 r) u. \) Then we have by direct calculations

\[
\frac{dE}{dr} = \lambda^{-2} \langle \partial_r u, \partial_r^2 u \rangle + \lambda^{-2} \langle \partial_r^2 u, \partial_r u \rangle - 2\lambda^{-2} \langle (\nu^{-2}(\cosh r) \sinh^{-3} r \Lambda_\theta u + u, u \rangle \\
\quad - \lambda^{-2} \langle (\nu^{-2} L_0) u, u \rangle - \lambda^{-2} \Re \langle \nu^{-2} V_\theta \sinh^{-1} r \partial_\theta u, u \rangle \\
\quad + \langle (\sinh^{-2} \Lambda_\theta + 1) \partial_r u, u \rangle - \lambda^{-2} \langle L_0 \partial_r u, u \rangle - \lambda^{-2} \Re \langle V_\theta \sinh^{-1} r \partial_\theta \partial_r u, u \rangle \\
\quad + \langle \sinh^{-2} r \Lambda_\theta u + u, \partial_r u \rangle - \lambda^{-2} \langle L_0 u, \partial_r u \rangle - \lambda^{-2} \Re \langle V_\theta \sinh^{-1} r \partial_\theta u, \partial_r u \rangle \\
\quad - \frac{1}{2} \langle \sinh^{-1} r \lambda^{-2} \cosh r \rangle \langle u, u \rangle.
\]
Meanwhile, we have for $\tilde{P} = P - i\epsilon \lambda^{-2} - \lambda^{-2}L_1$,
\[
2\Re\langle \tilde{P} u, D_r u \rangle = K + \overline{K}, \tag{3.97}
\]
where $K$ denotes
\[
K = \langle \partial_r^2 u, u \rangle + \langle \sinh^{-2} r \partial_r^2 u + 1 \rangle u, \partial_r u \rangle - \lambda^{-2} \langle L_0 u, \partial_r u \rangle - \lambda^{-2} \langle V_\theta u, \partial_r u \rangle - \langle i\epsilon \lambda^2 u, \partial_r u \rangle - \lambda^{-2} \langle \partial_r L_0 u, \partial_r u \rangle - \lambda^{-2} \langle \partial_r V_\theta \sinh^{-1} r \partial_R u, \partial_r u \rangle - \frac{1}{2} \lambda^{-2} \coth r \langle u, u \rangle \tag{3.98}
\]
Integration by parts with respect to $\theta$ in $[0, 2\pi]$ yields
\[
\langle \partial_\theta^2 \partial_r u, u \rangle = \langle \partial_r u, \partial_\theta^2 u \rangle. \tag{3.100}
\]
By integration by parts and the skew-symmetry of $V_\theta$, one obtains
\[
\langle V_\theta \partial_\theta \partial_r u, u \rangle = \langle \partial_r u, V_\theta \partial_\theta u \rangle - \langle \partial_\theta (V_\theta) \partial_r u, u \rangle. \tag{3.101}
\]
Hence we obtain
\[
\Re\langle \sinh^{-1} r V_\theta \partial_\theta \partial_r u, u \rangle + \Re\langle \sinh^{-1} r V_\theta \partial_\theta u, \partial_r u \rangle
\]
\[
= -\Re\langle \sinh^{-1} r (\partial_\theta V_\theta) \partial_r u, u \rangle + 2\Re\langle \sinh^{-1} r V_\theta \partial_\theta u, \partial_r u \rangle. \tag{3.102}
\]
Since $L_0$ is self-adjoint, we have
\[
\langle L_0 \partial_r u, u \rangle + \langle L_0 u, \partial_r u \rangle - 2\Re\langle L_0 u, \partial_r u \rangle = 0. \tag{3.103}
\]
Thus by $\Re\langle V_r \partial_r u, \partial_r u \rangle = 0$ ($V_r$ is skew-symmetric), $3.103, 3.102$, we conclude
\[
\frac{dE}{dr} = 2\lambda\Re\langle \tilde{P} u, D_r u \rangle - 2\lambda^{-2} \langle \cosh r (\sinh^{-3} r) \Lambda_\theta u, u \rangle
\]
\[
- \lambda^{-2} \langle (\partial_r L_0) u, u \rangle - \lambda^{-2} \Re\langle \partial_r (V_\theta \sinh^{-1} r) \partial_R u, u \rangle
\]
\[
- \lambda^{-2} \Re\langle (\partial_\theta V_\theta) \sinh^{-1} r \partial_r u, u \rangle - \frac{1}{2} \lambda^{-2} \coth r \langle u, u \rangle \tag{3.104}
\]
Since $\Lambda_\theta$ is positive we define $\Lambda_\theta^{1/2}$ by $\langle \Lambda_\theta^{1/2} u, \Lambda_\theta^{1/2} u \rangle = \langle \partial_\theta u, \partial_\theta u \rangle + \cosh^2 r \langle u, u \rangle$. Then by Lemma $3.7$ the facts that the supports of $V_\theta, V_r, U$ are away from zero and the last term in $3.104$ is non-positive, one obtains
\[
\frac{dE}{dr} \geq \frac{1}{2} \lambda^{-2} \Re\langle \sinh^{-3} r \Lambda_\theta^{1/2} u, \Lambda_\theta^{1/2} u \rangle
\]
\[
- C(\delta) \lambda^{-1} \| \psi_\alpha u \|^2 - C(\delta) \lambda^{-1} \| \psi_\alpha D_r u \|^2 - 2\lambda N(r), \tag{3.105}
\]

30
where \(N(r) = \left| \langle \bar{P}u, \mathcal{D}u \rangle \right|\).

\[
E(r) = -\int_r^\infty E'(s)ds \leq C(\delta)\lambda^{-1}\|\psi_\alpha \mathcal{D}_r u\|^2_{L^2(X)} + C(\delta)\lambda^{-1}\|\psi_\alpha u\|^2_{L^2(X)} + 2\lambda \int_0^\infty N(r)dr,
\]

(3.106)

and for \(\lambda\) sufficiently large,

\[
-E(r) \leq \frac{1}{2} \lambda^{-2} (\sinh^{-1} r) \langle \Lambda^{1/2}_\delta u, \Lambda^{1/2}_\delta u \rangle.
\]

(3.107)

Multiplying (3.105) with \(\psi_\alpha \cosh^{-1} r \sinh r\), by integration by parts we deduce

\[
\int_0^\infty \psi_\alpha \cosh^{-1} r \sinh r E'(r)dr = -\int_0^\infty \frac{d}{dr} (\psi_\alpha \cosh^{-1} r \sinh r) E(r)dr.
\]

(3.108)

Since \(\frac{d}{dr}(\psi_\alpha \cosh^{-1} r \sinh r) < c\alpha \psi_\alpha\) with a universal constant \(c > 0\), then (3.108) shows

\[
\int_0^\infty \psi_\alpha \cosh^{-1} r \sinh r E'(r)dr \leq \alpha \int_0^\infty \psi_\alpha |E(r)| dr.
\]

(3.109)

Meanwhile (3.106) and (3.107) imply

\[
\int_0^\infty \psi_\alpha |E(r)| dr \leq C(\delta, \alpha)\lambda^{-1}\|\psi_\alpha \mathcal{D}_r u\|^2_{L^2(X)} + C(\delta, \alpha)\lambda^{-1}\|\psi_\alpha u\|^2_{L^2(X)}
\]

\[
+ 2C(\alpha)\lambda \int_0^\infty N(r)dr + \frac{1}{2} \lambda^{-2}\|\psi_\alpha^{1/2}(\sinh^{-1} r) \partial_\theta u\|^2_{L^2(X)}.
\]

(3.110)

Therefore by (3.110), (3.109) and (3.105), we obtain

\[
\frac{1}{2} \lambda^{-2}\|\psi_\alpha^{1/2}(\sinh^{-1} r) \partial_\theta u\|^2_{L^2(X)}
\]

\[
\leq c\alpha \lambda^{-2}\|\psi_\alpha^{1/2}(\sinh^{-1} r) \partial_\theta u\|^2_{L^2(X)} + C(\delta, \alpha)\lambda^{-1}\|\psi_\alpha u\|^2_{L^2(X)}
\]

\[
+ C(\delta, \alpha)\lambda^{-1}\|\psi_\alpha \mathcal{D}_r u\|^2_{L^2(X)} + 2C(\alpha)\lambda \int_0^\infty N(r)dr.
\]

(3.111)

Let \(0 < \alpha \ll 1\) be fixed say \(\alpha = 1/100\). Then we conclude

\[
\lambda^{-2}\|\psi_\alpha^{1/2}(\sinh^{-1} r) \partial_\theta u\|^2_{L^2(X)}
\]

\[
\leq C(\delta, \alpha)\lambda^{-1}\|\psi_\alpha u\|^2_{L^2(X)} + C(\delta, \alpha)\lambda^{-1}\|\psi_\alpha \mathcal{D}_r u\|^2_{L^2(X)} + 2C(\alpha)\lambda \int_0^\infty N(r)dr.
\]

(3.111)

Meanwhile, since \(\|L_0\|_{L^\infty}\) is bounded, one has by (3.96) that when \(\lambda \gg 1\)

\[
\int_0^\infty \psi_\alpha(r) E(r) dr \geq \|\psi_\alpha^{1/2} \mathcal{D}_r u\|^2_{L^2(X)} + \frac{1}{2} \|\psi_\alpha^{1/2} u\|^2_{L^2(X)}
\]
\[ -2\lambda^{-2}\|\psi_{\alpha}^{1/2}\sinh^{-1} r\Lambda_{\theta}^{1/2}u\|_{L^2(X)}^2. \quad (3.112) \]

Combining (3.111), (3.110) with (3.112), we arrive at

\[
\|\psi_{\alpha}^{1/2}u\|_{L^2(X)}^2 + \|\psi_{\alpha}^{1/2}\mathcal{D}_r u\|_{L^2(X)}^2 + \|\psi_{\alpha}^{1/2}\sinh^{-1} r\Lambda_{\theta}^{1/2}u\|_{L^2(X)}^2 \\
\leq C(\delta, \alpha)\lambda^{-1}\|\psi_{\alpha}^{1/2}\mathcal{D}_r u\|_{L^2(X)}^2 + C(\delta, \alpha)\lambda^{-1}\|\psi_{\alpha}^{1/2}u\|_{L^2(X)}^2 \\
+ C(\alpha)\lambda \int_0^\infty N(r)dr.
\]  

(3.113)

(3.114)

Define \( P^* = \tilde{P} + i\epsilon\lambda^{-2} = P - \lambda^{-2}L_1 \), \( M^* = |\langle Pu, \mathcal{D}_r u \rangle| \), and \( M(r) = |\langle Pu, \mathcal{D}_r u \rangle| \). By Lemma 3.7 and the support of \( V_{\text{far}} \), we see

\[
\lambda \int_0^\infty M^*(r)dr \leq C(\delta)\lambda^{-1}\|\psi_{\alpha}^{1/2}u\|_{L^2(X)}^2 + C(\delta)\lambda^{-1}\|\psi_{\alpha}^{1/2}\mathcal{D}_r u\|_{L^2(X)}^2 \\
+ \mu^{-1}\lambda^2\|\psi_{\alpha}^{1/2} Pu\|_{L^2(X)}^2 + \mu\|\psi_{\alpha}^{1/2}\mathcal{D}_r u\|_{L^2(X)}^2.
\]  

(3.115)

Notice that because of (3.114) and (3.115), (3.93) is a corollary the following claim: when \( \lambda \gg 1 \),

\[
\epsilon\lambda^{-2}\|u\|_{L^2(X)}^2 \leq \int_0^\infty |\langle Pu, u \rangle|dr + C(\delta)\lambda^{-2}\|\psi_{\alpha}^{1/2}u\|_{L^2(X)}^2
\]  

(3.116)

\[
\|\mathcal{D}_r u\|_{L^2(X)}^2 \leq 2\int_0^\infty |\langle Pu, u \rangle|dr + 4\|u\|_{L^2(X)}^2
\]  

(3.117)

In fact, inserting (3.116), (3.117) and (3.115) to the inequality

\[
N(r) \leq M^*(r) + \lambda^{-1}\epsilon\left(\|u\|_{L^2(X)}^2 + \|\mathcal{D}_r u\|_{L^2(X)}^2\right),
\]

one immediately obtains the right hand side of (3.114) is bounded by

\[
C(\delta, \alpha)\lambda^{-1}\|\psi_{\alpha}^{1/2}\mathcal{D}_r u\|_{L^2(X)}^2 + C(\delta, \alpha)\lambda^{-1}\|\psi_{\alpha}^{1/2}u\|_{L^2(X)}^2 \\
+ C(\alpha)\mu^{-1}\lambda^2\|\psi_{\alpha}^{1/2} Pu\|_{L^2(X)}^2 + \mu\|\psi_{\alpha}^{1/2} u\|_{L^2(X)}^2.
\]  

(3.118)

Let \( 0 < \alpha \ll 1 \) first be determined, then take \( 0 < \mu \ll 1 \), and finally let \( \lambda \gg 1 \) depending on the size of \( C(\delta, \alpha) \). Then (3.118) can be absorbed by the left of (3.114) and thus giving (3.93) with \( C \) independent of \( \delta \). Hence it remains to verify (3.116) and (3.117). Consider \( \int_0^\infty \Re(Pu, u)dr \), then (3.117) follows easily by integration by parts and the \( L^\infty \) bounds of \( \partial_r V_r \) and \( V_\theta \) implied by Lemma 3.7. We also have (3.116) by considering \( \int_0^\infty \Im(Pu, u)dr \), applying integration by parts and Lemma 3.7.

\[ \square \]

We also need the gradient resolvent estimates for \( H_1 \).

**Lemma 3.13.** Let \( 0 < \alpha \ll 1 \) be fixed and let \( \delta > 0 \). If \( \lambda_0 > 0 \) is sufficiently large depending
Thus by (3.123), (3.125) and (3.124), we conclude

\[ \| \psi_{\alpha}^{1/2} \nabla R_{H_1} (\nu^2 + \lambda^2 \pm i\epsilon) \psi_{\alpha}^{1/2} \|_{L^2 \to L^2} \leq C, \]  
(3.119)

where \( C \) is independent of \( \delta, \epsilon \).

**Proof.** As before we only prove \( R_{H_1} (\nu^2 + \lambda^2 \pm i\epsilon) \). (3.118) and (3.114) yield

\[
\| \psi_{\alpha}^{1/2} u \|_{L^2(X)} + \| \psi_{\alpha}^{1/2} \partial_r u \|_{L^2(X)} + \| \psi_{\alpha}^{1/2} (\sinh^{-1} r) \partial_\theta u \|_{L^2(X)} + \| \psi_{\alpha}^{1/2} (\coth r) u \|_{L^2(X)} \\
\leq \lambda \| \psi_{\alpha}^{1/2} Pu \|_{L^2(X)}. 
\]  
(3.120)

Recall \( u = \sinh^{1/2} r f, \ X = \mathbb{R}^+ \times [0, 2\pi] \). \( \mathcal{D}_r = i\lambda^{-1} \partial_r, \mathcal{D}_\theta = i\lambda^{-1} \partial_\theta, \ P = \lambda^{-2} \sinh^{1/2} r (H_1 - \nu^2 - \lambda^2 \pm i\epsilon) \sinh^{-1/2} r, \) then one has

\[
\lambda \| \psi_{\alpha}^{1/2} Pu \|_{L^2(X)} = \lambda^{-1} \| \psi_{\alpha}^{1/2} (H_1 - \nu^2 - \lambda^2 \pm i\epsilon) f \|_{L^2(\mathbb{H}^2)} 
\]  
(3.121)

\[
\| \psi_{\alpha}^{1/2} \sinh^{-1} r \partial_\theta u \|_{L^2(X)} = \lambda^{-1} \| \psi_{\alpha}^{1/2} (r) \nabla_\theta f \|_{L^2(\mathbb{H}^2)}, 
\]  
(3.122)

where we denotes \( \nabla_\theta f = \partial_\theta f d\theta, \ \nabla_r f = \partial_r f dr. \) Thus we have

\[
\| \psi_{\alpha}^{1/2} (r) \nabla_\theta f \|_{L^2(\mathbb{H}^2)} + \| \psi_{\alpha}^{1/2} f \|_{L^2} \leq \| \psi_{\alpha}^{1/2} (H_1 - \nu^2 - \lambda^2 \pm i\epsilon) f \|_{L^2(\mathbb{H}^2)}. 
\]  
(3.123)

By direct calculations, one has

\[
\| \lambda^{-1} \psi_{\alpha}^{1/2} \partial_r f \|_{L^2(\mathbb{H}^2)} \leq \| \psi_{\alpha}^{1/2} \partial_r u \|_{L^2(X)} + \lambda^{-1} \| \psi_{\alpha}^{1/2} (\sinh^{-1/2} r) \cosh r f \|_{L^2(X)}. 
\]  
(3.124)

Split \( X \) into \( X_I = [1, \infty) \times [0, 2\pi] \) and \( X_{II} = (0, 1] \times [0, 2\pi] \). Then it is easily seen that

\[
\| \psi_{\alpha}^{1/2} (\sinh^{-1/2} r) \cosh r \psi_{\alpha}^{1/2} f \|_{L^2(X_I)} \leq C \| \psi_{\alpha}^{1/2} f \|_{L^2(\mathbb{H}^2)}. 
\]

Meanwhile we have

\[
\| \psi_{\alpha}^{1/2} (\sinh^{-1/2} r) \cosh r \psi_{\alpha}^{1/2} f \|_{L^2(X_{II})} \leq \| \psi_{\alpha}^{1/2} (\sinh^{-1} r) \cosh r \psi_{\alpha}^{1/2} u \|_{L^2(X_{II})}. 
\]  
(3.125)

Thus by (3.123), (3.125) and (3.124), we conclude

\[
\| \psi_{\alpha}^{1/2} f \|_{L^2(\mathbb{H}^2)} + \| \psi_{\alpha}^{1/2} \nabla f \|_{L^2(\mathbb{H}^2)} \leq \| \psi_{\alpha}^{1/2} (-\Delta - \nu^2 - \lambda^2 \pm i\epsilon) f \|_{L^2(\mathbb{H}^2)}. 
\]  
(3.126)

Now we transform the resolvent estimates for the long range part \( H_1 \) to the full Schrödinger operator \( H \) by viewing the short range part as a \( \psi_{\alpha}^{1/2} L^\infty \) perturbation for \( H_1 \).
Lemma 3.14. Let $0 < \alpha \ll 1$ be fixed and let $\delta > 0$. If $\lambda_0 > 0$ is sufficiently large depending on $\alpha, \delta$, then for all $\lambda > \lambda_0$, $\epsilon \in [0,1]$, it holds that

$$\|\rho^\alpha R_H(\nu^2 + \lambda^2 \pm i\epsilon)\rho^\alpha\|_{L^2 \to L^2} \leq C\lambda^{-1}, \quad (3.127)$$

$$\|\rho^\alpha \nabla R_H(\nu^2 + \lambda^2 \pm i\epsilon)\rho^\alpha\|_{L^2 \to L^2} \leq C, \quad (3.128)$$

where $C$ is independent of $\epsilon, \lambda$.

Proof. We try to use the formal identity

$$R_H(z) = R_{H_1}(z)(I + (V_{\text{near}}^1 + V_{\text{near}}^2)\nabla R_{H_1})^{-1}. \quad (3.129)$$

Hence we first show for $\lambda > \lambda_0(\delta)$,

$$\|\rho^{-\alpha}(V_{\text{near}}^1 + V_{\text{near}}^2 \nabla) R_{H_1}\rho^\alpha\|_{L^2 \to L^2} \leq o(1). \quad (3.130)$$

where $o(1)$ denotes a quantity which tends to zero as $\delta \to 0$. By the identity,

$$R_{H_1}(z) = R_0 - R_0(V_{\text{far}}^1 + V_{\text{far}}^2 \nabla) R_{H_1}(z), \quad (3.131)$$

By Lemma 3.5,

$$\|(\rho^{-\alpha}(V_{\text{near}}^1 + V_{\text{near}}^2)R_0\rho^\alpha\|_{L^2 \to L^2} \leq \|\rho^{-\alpha\psi^{-1/2} V_{\text{near}}^1} R_0\rho^\alpha\|_{L^2 \to L^2} + \|\rho^{-\alpha\psi^{-1/2} V_{\text{near}}^1} R_0\rho^\alpha\|_{L^2 \to L^2} \leq C\delta^{1/2}, \quad (3.132)$$

where one uses $\psi^{-1}(r) \leq r^{1-\alpha}$ in the support of $V_{\text{near}}^1$ which lies in $\{x : d(x,0) \leq \delta\}$ and Lemma 3.7. The same arguments show

$$\|(\rho^{-\alpha}(V_{\text{near}}^1 + V_{\text{near}}^2)\nabla) R_{H_1}\rho^\alpha\|_{L^2 \to L^2} \leq C\delta^{1/2}. \quad (3.133)$$

Again due to Lemma 3.5, one has

$$\|\rho^{-\alpha} V_{\text{near}}^2 \nabla R_0(V_{\text{far}}^2 \nabla) R_{H_1}(z)\rho^\alpha\|_{L^2 \to L^2} \leq \|\rho^{-\alpha\psi^{-1/2} V_{\text{near}}^2} R_0\rho^\alpha\|_{L^2 \to L^2} + \|\rho^{-\alpha\psi^{-1/2} V_{\text{near}}^2} R_0\rho^\alpha\|_{L^2 \to L^2} \leq C\delta, \quad (3.134)$$

where we apply Lemma 3.13 in the last line. Hence (3.130) follows by (3.131) and similar
arguments as \((3.136)\). Thus by first choosing \(0 < \delta \ll 1\), then letting \(\lambda \gg 1\), we have
\[
\|\rho^{-\alpha}(V_1^{\text{near}} + V_2^{\text{near}} \nabla)R_{H, \rho}^{\alpha}\|_{L^2 \to L^2} \leq 1/2.
\]

We remark that this is possible because the constant \(C\) in Lemma \(3.12\) and Lemma \(3.13\) is independent of \(\delta\). Therefore \((3.129)\) makes sense and we have
\[
\|\rho^{-\alpha}(V_1^{\text{near}} + V_2^{\text{near}} \nabla)R_{H, \rho}^{\alpha}\|_{L^2 \to L^2} \leq 1/2.
\]

3.7 Assemble Resolvent Estimates in All Frequencies

Lemma \(3.9\), Lemma \(3.11\) and Lemma \(3.14\) give the desired resolvent estimates for \(H\).

**Lemma 3.15.** Let \(H\) be defined in Lemma \(2.8\) with \(\Xi\) being the Coulomb gauge. The limit
\[
\lim_{\xi \to 0, \Im \xi > 0} (H - \nu^2 - \lambda^2 \pm \xi)^{-1}
\]
exists strongly in \(L^2_x\). And if we denote
\[
\mathfrak{R}_H(\lambda \pm i0) = \lim_{\xi \to 0, \Im \xi > 0} (H - \nu^2 - \lambda^2 \pm \xi)^{-1},
\]
then we have
\[
\|\rho^{-\alpha}\mathfrak{R}_H(\lambda \pm i0)\rho^{-\alpha}\|_{L^2 \to L^2} \leq C \min(1, \lambda^{-1}).
\]

**Proof.** Lemma \(3.11\) shows \(\mathfrak{R}_0(\lambda \pm i0)(I + W\mathfrak{R}_0(\lambda \pm i0))^{-1}\) makes sense in \(L^2_x\) for any \(\lambda > 0\). Thus for \(\Im \xi > 0\) with \(\Im \xi < \delta_e\), we can pass the identity
\[
(H - \nu^2 - \lambda^2 \pm \xi)^{-1} = R_0(\lambda^2 + \nu^2 \pm \xi)(I + WR_0(\lambda^2 + \nu^2 \pm \xi))^{-1}
\]
to the following by Lemma \(3.3\)
\[
\mathfrak{R}_H(\lambda \pm i0) = \mathfrak{R}_0(\lambda \pm i0)(I + W\mathfrak{R}_0(\lambda \pm i0))^{-1}.
\]
Moreover the estimates of \((3.141)\) can be transformed to \(\mathfrak{R}_H(\lambda \pm i0)\). Thus Lemma \(3.9\), Lemma \(3.11\) and Lemma \(3.14\) yield \((3.140)\).

**Proposition 3.4.** The operator \(|D_H| = H^{\frac{1}{2}}\) satisfies the resolvent estimates,
\[
\|\rho^{-\alpha}(|D_H| - z)^{-1}\rho^{-\alpha}\|_{L^2 \to L^2} \leq C.
\]
where $C$ is independent of $z \in \mathbb{C}\setminus \mathbb{R}$.

**Proof.** The original idea is due to [14]. Since we have the identity for $z \geq 0$

$$\frac{1}{|D_H - z|} = \frac{1}{|D_H|} + 2z(H - z^2)^{-1}, \tag{3.144}$$

and $z \in (-\infty, 0)$ belongs to the resolvent of $|D_H|$, by Lemma 3.15, Lemma 3.3 and Phragmén-Lindelöf theorem, it suffices to prove (3.143) for $z \in \mathbb{R}$.

**Case 1.** Let $z \leq 0$. Proposition 3.1 shows

$$\|(|D_H| - z)f\|^2_{L^2_x} = \||D_H|f\|^2_{L^2_x} - 2\Re z \langle f, |D_H|f \rangle \geq \||D_H|f\|^2_{L^2_x} \geq \langle f, Hf \rangle \geq C\|f\|_{L^2_x}.$$  

Thus we have

$$\frac{1}{|D_H + z|} \|f\|^2_{L^2_x} \lesssim \|f\|_{L^2_x},$$

from which

$$\|\mu^\alpha(|D_H| - z)^{-1} \mu^\alpha f\|_{L^2_x} \leq C\|f\|_{L^2_x}. \tag{3.145}$$

**Case 2.** Let $z < 0$. Then (3.143) follows by (3.144), Lemma 3.15 and (3.145).

3.8 **Heat Semigroup Generated by the Magnetic Schrödinger Operator**

In this section, we shall consider the $L^p$-$L^p$ estimates for the heat semigroup $e^{-tH}$.

**Lemma 3.16.** Let $H$ be defined in Lemma 2.8. Then for any $p \in (1, \infty)$ and some $0 < \delta_1 < \nu^2$, we have

$$\|e^{-tH}f\|_{L^p_x} \leq e^{-\delta_1 t\|f\|_{L^p_x}. \tag{3.146}$$

**Proof.** Denote $e^{-tH}f(x) = u(t, x)$. Considering the tangent vector field $u\Xi$ defined by $u\Xi = u_1\Xi_1 + u_2\Xi_2$, one deduces by the proof in Proposition 3.1 that

$$\partial_t |u\Xi|^2 - \Delta |u\Xi|^2 + 2|\nabla (u\Xi)|^2 \leq 0, \tag{3.147}$$

which further yields

$$\partial_t |u\Xi| - \Delta |u\Xi| \leq 0. \tag{3.148}$$
Thus by maximum principle,

\[ |u(\Xi) (t, x)| \leq e^{t \Delta} f(x). \]  

(3.149)

Then (3.16) follows by (5.17). \( \square \)

**Lemma 3.17.** Let \( H \) be defined in Lemma 2.8. Then for \( \lambda = -\delta_2 + i\eta \) with \( 0 < \delta_2 < \delta_1 < \nu^2 \), \( \eta \in \mathbb{R}, p \in (1, \infty) \) we have

\[ \|(-H - \lambda)^{-1} f\|_{L^p_x} \leq C \|f\|_{L^p_x}, \]  

(3.150)

where \( C \) is independent of \( \eta \).

**Proof.** Assume \( f \in L^2 \cap L^p \), by the standard theories of semigroups of linear operators, for \( \lambda > 0 \)

\[ (-H - \lambda)^{-1} f = \int_0^\infty e^{-\lambda t} e^{-tH} f dt. \]  

(3.151)

Let \( E(\lambda) \) be the left hand side of (3.151) with \( \lambda \) slightly enlarged to \( \mathcal{O} = \{ \lambda \in \mathbb{C} : \Re \lambda > -\delta_2 \} \), i.e.,

\[ E(\lambda) = \int_0^\infty e^{-\lambda t} e^{-tH} dt. \]  

(3.152)

First we verify (3.152) converges in the uniform operator topology in \( \mathcal{L}(L^p; L^p) \) for \( \lambda \in \mathcal{O} \). In fact, Lemma 3.16 shows

\[ \int_0^\infty \left\| e^{-\lambda t} e^{-tH} \right\|_{L^p_x \to L^p_x} \, dt \leq \int_0^\infty e^{-\delta_1 t + \delta_2^t} dt. \]  

(3.153)

Since \( E(\lambda) \) is analytic with respect to \( \lambda \in \mathcal{O} \), (3.151) shows \((-H - \lambda)^{-1}\) coincides with \( E(\lambda) \) when \( \lambda \in \mathcal{O} \). Thus (3.153) implies (3.152). \( \square \)

**Lemma 3.18.** Let \( H \) be defined in Lemma 2.8. Then for any \( p \in (1, \infty), t > 0 \), we have

\[ \|He^{-tH} f\|_{L^p_x} \leq C_t t^{-1 - \epsilon} e^{-\delta_3 t} \|f\|_{L^p_x}, \]  

(3.154)

for some \( \delta_3 > 0 \) and any \( \epsilon \in (0, 1) \).

**Proof.** By the standard theories of semigroups of linear operators for instance [Theorem 7.7[53]], for any \( f \in L^p \cap L^2, t > 0 \),

\[ e^{-tH} f = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} (-H - \lambda)^{-1} f d\lambda, \]  

(3.155)
where $\Gamma$ is any curve lying in the resolvent set which connects $\infty e^{i\vartheta}$ and $-\infty e^{i\vartheta}$ with $\frac{\pi}{2} < \vartheta < \pi$. And the convergence of the integrand in (3.155) is in the $L^2$ norm. Let $\Gamma$ be made up of three components: one is $\Gamma_1$ defined by $\{a - iha : a \in (-\infty, -\delta]\}$, the other is $\Gamma_2$ defined by $\{-\delta + i\eta : \eta \in [-h\delta, h\delta]\}$, and the third one is $\Gamma_3$ defined by $\{a + iha : a \in [-\infty, -\delta]\}$. The constant $h > 0$ will be chosen to be sufficiently large later. Let $\lambda = \sigma^2 - \nu^2$, and if denote the angular of points in $\Gamma_1$ by $\vartheta$, it is easy to check for $\lambda \in \Gamma_1$ and $\Re \lambda > \nu^2$, we have

$$0 < \cot \vartheta < 4h\delta^2.$$  

Thus we get if $h > 10^6\delta^{-1}$, $\lambda \in \Gamma_1$ and $\Re \lambda > \nu^2$,

$$\Re \sigma \geq \frac{\sqrt{\delta^2}}{16} (1 + h^2)^{1/4} \geq 10.$$  

Similarly, if $h > 10^6\delta^{-1}$, $\lambda \in \Gamma_1$, $\Re \lambda \leq \nu^2$, then $0 < \cot(\pi - \vartheta) < \frac{2}{h\delta}$ and thus $|\cos \frac{\pi - \vartheta}{2} - \frac{\sqrt{2}}{2}| < \frac{8}{h\delta}$. Hence we deduce if $h > 10^6\delta^{-1}$, $\lambda \in \Gamma_1$ and $\Re \lambda \leq \nu^2$,

$$\Re \sigma \geq \frac{\sqrt{\delta^2}}{16} (1 + h^2)^{1/4} \geq 10.$$  

Therefore, by the resolvent estimates in Lemma 5.10 and Young’s convolution inequality, we have for $\lambda \in \Gamma_1$ and $r = 2, p$,

$$\|(-H - \lambda)^{-1}f\|_{L^r} \leq C\|f\|_{L^r}.$$  

(3.158)

The same arguments shows for $\lambda \in \Gamma_1$ and $r = 2, p$,

$$\|(-H - \lambda)^{-1}f\|_{L^r} \leq C\|f\|_{L^r}.$$  

(3.159)

Thus holds in $L^2 \cap L^p$ by using (3.158), (3.159), (3.150) and noticing

$$|e^{\lambda t}| < e^{-t\mu|\lambda|}$$  

(3.160)

for $\mu = \frac{1}{\sqrt{1+h^2}}$ and $\lambda \in \Gamma_1 \cup \Gamma_3$. Moreover, by (3.155) we have

$$He^{-tH}f = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}H(-H - \lambda)^{-1}f d\lambda$$

$$= -\frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t}f d\lambda - \int_{\Gamma} e^{\lambda t}\lambda(-H - \lambda)^{-1}f d\lambda.$$  

(3.161)

Notice that when $|\lambda| \to \infty$ along $\Gamma_{1,3}$, $\Re \sigma \sim |\lambda|^{1/2}$. Thus (3.158), (3.159) can be refined to be:

For $\lambda \in \Gamma_{1,3}$, $r = 2, p$,

$$\|(-H - \lambda)^{-1}f\|_{L^r} \leq C|\lambda|^{1-t}\|f\|_{L^r},$$  

(3.162)
for any \(\epsilon \in (0,1)\). Then combining (3.162), (3.160) with (3.150), we have by (3.161)
\[
\|H e^{-tH} f\|_{L_p^p} \leq C(\int_{\frac{1}{8}}^{\infty} e^{-\mu t^2} d\tau + \int_{\frac{1}{8}}^{\infty} e^{-\mu t^2 \epsilon^2} d\tau + e^{-t/8} h^2)\|f\|_{L_p^p}
\] (3.163)

Hence we arrive at
\[
\|H e^{-tH} f\|_{L_p^p} \leq t^{-1-\epsilon} e^{-\mu t} \|f\|_{L_p^p},
\] (3.164)

thus giving (3.154).

**Proposition 3.5.** For \(s \in (0, \frac{3}{2})\), \(p \in (1, \infty)\), we have
\[
\|(-\Delta)^{\frac{s}{2}} f\|_{L_p^p} \leq \|H^{\frac{s}{2}} f\|_{L_p^p} + C\|f\|_{L_p^p}
\] (3.165)
\[
\|H^{\frac{s}{2}} f\|_{L_p^p} \leq \|(-\Delta)^{\frac{s}{2}} f\|_{L_p^p} + C\|f\|_{L_p^p}.
\] (3.166)

**Proof.** By Lemma 3.16, \(e^{\delta_1 t} e^{-tH}\), whose infinitesimal generator is \(\delta_1 - H\), is a \(C_0\) semigroup of contractions in \(L^p\). Thus Lumer-Phillips theorem or Corollary 3.6 of [53] shows \(\{\lambda : \Re \lambda > 0\} \subset \rho(\delta_1 - H)\). And for \(\lambda > -\frac{1}{2}\delta_1\),
\[
\|(-H - \lambda)^{-1}\|_{L^p \to L^p} \leq (\Re \lambda + \delta_1)^{-1}.
\] (3.167)

Again due to Balakrishnan formula we deduce for \(s \in (0,2)\) (see Lemma 9.8, [49] for the proof)
\[
H^{\frac{s}{2}} = (-\Delta)^{\frac{s}{2}} + c(s) \int_{0}^{\infty} \lambda^{\frac{s}{2}}(\lambda - \Delta + W)^{-1} W(\lambda - \Delta)^{-1} d\lambda.
\] (3.168)

By (3.168), it is obviously that for (3.165) and (3.166), it suffices to prove
\[
\int_{0}^{\infty} \lambda^{\frac{s}{2}}(\lambda - \Delta + W)^{-1} W(\lambda - \Delta)^{-1}\|_{L^p \to L^p} d\lambda \leq C.
\]

Let \(\lambda = -\nu^2 + \sigma^2\), Lemma 3.5 and (3.167) give
\[
\int_{0}^{\infty} \lambda^{\frac{s}{2}}(\lambda + H)^{-1} W(\lambda - \Delta)^{-1}\|_{L^p \to L^p} d\lambda \leq C\delta_1 \int_{\frac{1}{2}}^{\infty} \sigma^{s-\frac{3}{2}} d\sigma \lesssim 1.
\]

**Proposition 3.6.** Let \(H\) be defined in Lemma 2.8. Then for any \(p \in (1, \infty)\), \(s \in [0, \frac{3}{2})\), and any \(\epsilon \in (0,1)\), \(t > 0\), we have
\[
\|(-\Delta)^{\frac{s}{2}} e^{-tH} f\|_{L_p^p} \leq C t^{s(1+\epsilon)} e^{-\delta t} \|f\|_{L_p^p}.
\] (3.169)
Proof. By Lemma 3.16, $e^{-tH}$ is a $C_0$ semigroup of contractions in $L^p$. Thus again by Lumer-Phillips theorem or Corollary 3.6 of [53], \{\lambda : \Re \lambda > 0\} \subseteq \rho(-H)$ and for such \( \lambda \), (3.167) holds. And thus by Balakrishnan formula (see for instance in the proof of [Theorem 6.10 [53]]), we have

\[ ||H^{\frac{s}{2}} f||_{L^p_x} \leq ||f||_1^{1-\frac{s}{2}} ||Hf||_p^{\frac{s}{2}}. \]  

(3.170)

Thus by Lemma 3.18 and Lemma 3.16,

\[ ||H^{\frac{s}{2}} e^{-tH} f||_{L^p_x} \leq e^{-\delta t} e^{-\frac{(1+\varepsilon)}{2}} ||f||_{L^p_x}. \]  

(3.171)

\[ ||e^{-tH} f||_{L^p_x} \leq e^{-\delta t} ||f||_{L^p_x}. \]  

(3.172)

Hence (3.165), (3.172), (3.171) give for \( s \in [0, \frac{3}{2}) \)

\[ ||(-\Delta)^{\frac{s}{2}} e^{-tH} f||_{L^p_x} \leq Ce^{-\delta t} \frac{t}{4} ||f||_{L^p_x}. \]  

(3.173)

When \( p = 2 \), Proposition 3.6 can be refined to be the following.

**Proposition 3.7.** Let $H$ be defined in Lemma 2.8 with $\Xi$ being the Coulomb gauge. Then for any \( s \in [0, 2) \), \( t > 0 \), we have

\[ ||H^{\frac{s}{2}} e^{-tH} f||_{L^2_x} \leq C t^{-s} e^{-\delta t} ||f||_{L^2_x} \]  

(3.174)

\[ ||(-\Delta)^{\frac{s}{2}} f||_{L^2_x} \leq ||H^{\frac{s}{2}} f||_{L^2_x} \]  

(3.175)

\[ ||H^{\frac{s}{2}} f||_{L^2_x} \leq ||(-\Delta)^{\frac{s}{2}} f||_{L^2_x}. \]  

(3.176)

**Proof.** We first prove (3.175) and (3.176). When \( s = 2 \), By (3.55) and Sobolev embedding,

\[ 4 ||Hu||_{L^2_x} ||\nabla u||_{L^2_x} \geq ||Hu||_{L^2_x} ||u||_{L^2_x} \geq \langle Hu, u \rangle \geq \langle \nabla u, \nabla u \rangle \geq \frac{1}{4} ||u||_{L^2_x}^2. \]  

(3.177)

Then we have

\[ ||u||_{L^2_x} + ||\nabla u||_{L^2_x} \lesssim ||Hu||_{L^2_x}. \]  

(3.178)

Hence one obtains by \( |A| \leq C \) and triangle inequality that

\[ ||\Delta u||_{L^2_x} \lesssim ||Hu||_{L^2_x} + ||\nabla u||_{L^2_x} \lesssim ||Hu||_{L^2_x} \]

\[ ||Hu||_{L^2_x} \lesssim ||\Delta u||_{L^2_x} + ||\nabla u||_{L^2_x} \lesssim ||\Delta u||_{L^2_x}, \]

from which (3.175) and (3.176) follow. (3.174) follows by the same arguments as (3.6) with the
following improved resolvent estimates in $L^2_x$:

$$\|(-H - z)^{-1}\|_{L^2 \to L^2} \leq \text{dist}(z, N\text{Line}),$$

(3.179)

where $N\text{Line} = \{x \in \mathbb{R} : x \leq -\nu^2\}$. Thus the additional $\epsilon$ in (3.162) can be removed and one gets

$$\|H e^{-\|H\|} f\|_{L^2_x} \leq t^{-1} e^{-\delta t} \|f\|_{L^2_x}.$$  

(3.180)

Then (3.174) follows by interpolation.

In addition, we need an almost equivalence lemma for $H^\frac{1}{2}$ and $(-\Delta)^\frac{1}{2}$ in $\rho^{-\alpha} L^2$.

**Lemma 3.19.** Let $H$ be defined in Lemma 2.8 with $\Xi$ being the Coulomb gauge. For $0 < \delta_4 \ll 1$, $\lambda > \nu^2 - \delta$, $(I + \rho^{-\alpha} W(-\Delta + \lambda)^{-1} \rho^\alpha)$ is invertible in $L^2_x$ and analytic with respect to $\lambda$ in any compact set of $\{\lambda : \lambda > \nu^2 - \delta_4\}$.

**Proof.** Assume $I + \rho^{-\alpha} W(-\Delta + \lambda)^{-1} \rho^\alpha$ is not invertible, then by Fredholm’s alternative, there exists $f \in L^2_x$ such that

$$f + \rho^{-\alpha} W(-\Delta + \lambda)^{-1} \rho^\alpha f = 0.$$

Let $(-\Delta + \lambda)^{-1} \rho^\alpha f = g$, then by Lemma 3.5, $g \in W^{1,r}$ for $2 < r < 6$. Moreover, we have

$$-\Delta g + \lambda g + W g = 0.$$  

(3.181)

By Hölder and $g \in W^{1,r}$, Proposition 3.2, it is easy to check $W g \in L^2_x$. Thus (3.181) shows $-\Delta g + (\sigma^2 - \nu^2) g \in L^2$ and consequently we have $g \in L^2$ due to $\sigma(-\Delta) \subset [\nu^2, \infty)$. Again by (3.181), $g$ is an eigenfunction of $-\Delta + W$ with eigenvalue $\nu^2 - \lambda$, which contradicts with Proposition 3.1. The analyticity of $(I + \rho^{-\alpha} W(-\Delta + \lambda)^{-1} \rho^\alpha)^{-1}$ claimed in our lemma follows by the fact $\nu^2 - \lambda$ lies in the resolvent set of $-\Delta$ when $\lambda > \nu^2 - \delta_4$. □

**Lemma 3.20.** Let $H$ be defined in Lemma 2.8 with $\Xi$ being the Coulomb gauge. We have

$$\|\nabla f\|_{\rho^{-\alpha} L^2} \leq \|H^\frac{1}{2} f\|_{\rho^{-\alpha} L^2} + \|f\|_{\rho^{-\alpha} L^2}$$  

(3.182)

$$\|\nabla f\|_{\rho^{-\alpha} L^2} \leq \|(-\Delta)^\frac{1}{2} f\|_{\rho^{-\alpha} L^2}$$  

(3.183)

$$\|(-\Delta)^\frac{1}{2} f\|_{\rho^{-\alpha} L^2} \leq \|H^\frac{1}{2} f\|_{\rho^{-\alpha} L^2} + \|f\|_{\rho^{-\alpha} L^2}.$$  

(3.184)

**Proof.** (3.183) has been proved in [49]. Since (3.182) is a corollary of (3.183) and (3.184), it suffices to prove (3.184). By (3.183), it suffices to verify

$$\int_0^\infty \lambda^\frac{3}{2} \|(-\Delta + W)^{-1} W(\lambda - \Delta)^{-1} f\|_{\rho^{-\alpha} L^2} d\lambda \leq C \|f\|_{\rho^{-\alpha} L^2}.$$  

(3.185)
For any fixed $K > 0$, by the formal identity
\[(\lambda + H)^{-1} = (-\Delta + \lambda)^{-1}(I + W(-\Delta + \lambda)^{-1})^{-1},\]  
(3.186)

Lemma \[3.19\] implies for $\lambda \in (0, K)$
\[
\|\rho^{\alpha}(\lambda + H)^{-1}W(\lambda - \Delta)^{-1}f\|_{L^2} \\
\leq \|\rho^{\alpha}(-\Delta + \lambda)^{-1}\rho^{\alpha}\|_{L^2 \to L^2}(I + \rho^{-\alpha}W(-\Delta + \lambda)^{-1}\rho^{\alpha})^{-1}\|_{L^2 \to L^2}\|\rho^{\alpha}W(\lambda - \Delta)^{-1}f\|_{L^2} \\
\leq C(K)\|\rho^{\alpha}(-\Delta + \lambda)^{-1}\rho^{\alpha}\|_{L^2 \to L^2}\|\rho^{\alpha}W(\lambda - \Delta)^{-1}f\|_{L^2}.
\]    
(3.187)

Then Lemma \[3.5\] gives
\[
\int_0^K \lambda^{\frac{4}{3}}\|(\lambda - \Delta + W)^{-1}W(\lambda - \Delta)^{-1}f\|_{\rho^{-\alpha}L^2}d\lambda \\
\leq C(K)\int_0^K \lambda^{\frac{4}{3}}\min(1, \lambda^{-\frac{3}{2}})d\lambda\|f\|_{\rho^{-\alpha}L^2}.
\]    
(3.188)

For $\lambda \in (K, \infty)$ with $K \gg 1$, Lemma \[3.5\] and Neumann series argument show \[3.186\] makes sense in $\rho^{-\alpha}L^2$ and
\[
\|\rho^{\alpha}(\lambda + H)^{-1}\rho^{\alpha}f\|_{L^2} \leq 2\|\rho^{\alpha}(-\Delta + \lambda)^{-1}\rho^{\alpha}\|_{L^2 \to L^2}\|f\|_{L^2} \\
\leq \lambda^{-\frac{1}{2}}\|f\|_{L^2}.
\]    
(3.189)

Thus Lemma \[3.5\] \[3.188\] and \[3.189\] imply that \[3.185\] is bounded by
\[
C\int_0^\infty \lambda^{\frac{4}{3}}\lambda^{-\frac{1}{2}}d\lambda\|f\|_{\rho^{-\alpha}L^2}.
\]    
(3.190)

### 3.9 Non-endpoint and endpoint Strichartz estimates

The non-endpoint Strichartz estimates for $H$ is a standard corollary of the Kato smoothing effect and the almost equivalent lemma in Section 6. The original idea of using Kato smoothing effect to obtain Strichartz estimates dates back to Rodnianski, Schlag [55]. We will state the Strichartz estimates in the following lemma without proof. But it is necessary to point out that although there exists a zero order term $\|f\|_{L^p}$ in Proposition \[3.5\] compared with the exact equivalence lemma in [49, Lemma 5.8, Lemma 5.9], it will cause no trouble in proving Strichartz estimates since we have $\|f\|_{L^p} \leq C\|(-\Delta)^{s}\|_{L^p}$ for any $p \in [2, \infty)$, $s \in (0, 1)$.

**Lemma 3.21.** Let $H = -\Delta + W$ be defined above. We have the Strichartz estimates for the
magnetic wave equation: If \( f \) solves the equation

\[
\begin{cases}
\partial_t^2 f - \Delta f + W f = g \\
f(0, x) = f_0, \partial_t f(0, x) = f_1
\end{cases}
\quad (3.191)
\]

then it holds for any admissible pair \((p, q)\) with \( p > 2 \)

\[
\left\| D^{1/2} f \right\|_{L^p_t L^q_x} + \left\| D^{-1/2} \partial_t f \right\|_{L^p_t L^2_x} + \left\| \partial_t f \right\|_{L^p_t L^2_x} + \left\| \nabla f \right\|_{L^p_t L^2_x} \lesssim \left\| \nabla f_0 \right\|_{L^2} + \left\| f_1 \right\|_{L^2} + \left\| g \right\|_{L^1_t L^2_x}.
\]

Proof. The proof relies on Lemma 3.20, Lemma 3.21, Proposition 3.4 and Theorem 3.1. See [Lemma 5.10, Proposition 5.1] for the details.

For the endpoint Strichartz estimates, we need a key lemma of our previous paper.

**Lemma 3.22.** Let \( u \) solves the linear wave equation

\[
\begin{cases}
\partial_t^2 u - \Delta u = g \\
0 < \alpha \ll 1
\end{cases}
\quad (3.192)
\]

Let \( 0 < \alpha \ll 1 \), then for \( q \in (2, 6) \)

\[
\left\| D^{1/2} u \right\|_{L^2_t L^q_x} \leq \left\| \rho^{-\alpha} g \right\|_{L^q_t L^2_x}.
\quad (3.193)
\]

Let \( \sigma > 0 \) be sufficiently small, and \( H = -\Delta + W \) be defined above. Assume that \( u \) solves

\[
\begin{cases}
\partial_t^2 u - \Delta u + W u = g \\
u(0, x) = f_0, \partial_t u(0, x) = f_1
\end{cases}
\quad (3.194)
\]

Then we have for \( 0 < \sigma \ll \rho \)

\[
\left\| \rho^\sigma \nabla u \right\|_{L^2_t L^2_x} \leq \left\| g(t) \right\|_{L^2_t L^2_x} + \left\| \nabla f_0 \right\|_{L^2_x} + \left\| f_1 \right\|_{L^2_x}.
\quad (3.195)
\]

Proof. (3.193) has been proved in [49]. (3.195) follows by applying Kato smoothing effect.

Lemma 3.21 and Lemma 3.22 yield the endpoint and weighted Strichartz estimates.

**Proposition 3.8.** Let \( H = -\Delta + W \) be defined above and \( 0 < \sigma \ll \alpha \ll 1 \). Then we have the weighted Strichartz estimates for the magnetic wave equation: If \( f \) solves the equation

\[
\begin{cases}
\partial_t^2 u - \Delta u + W u = g \\
u(0, x) = f_0, \partial_t u(0, x) = f_1
\end{cases}
\]
then it holds for any $p \in (2,6)$

$$\left\| D^{1/2}u \right\|_{L_t^2 L_x^p} + \|\phi \nabla f\|_{L_t^2 L_x^2} + \|\partial_t u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^\infty L_x^2} \lesssim \|\nabla f_0\|_{L^2} + \|f_1\|_{L^2} + \|g\|_{L_t^1 L_x^2},$$

(3.196)

4 The proof of Theorem 1.1.

4.1 Close the bootstrap of the heat tension field

We will prove Theorem 1.1 by bootstrap. In this section we always fix the frame $\Xi$ in Proposition 2.3 to be the Coulomb gauge. Suppose that the initial data $(u_0, u_1)$ satisfies

$$\| (du_0, u_1) \|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^2)} + \| (\nabla du_0, \nabla u_1) \|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^2)} \leq M_0.$$  (4.1)

Lemma 5.6 shows that

$$\|\phi_0(0,0,x)\|_{L_x^2} \leq \mu_1 C(R_0) C(M_0).$$  (4.2)

We fix the constants $\mu_1, \varepsilon_1, \rho, \sigma$ to be

$$0 < \mu_1 \ll \varepsilon_1 \ll e^{M_1 + 10}, 0 < \sigma \ll \rho \ll 1, \ M_0 \ll M_1.$$  (4.3)

Let $L > 0$ be sufficiently large say $L = 100$. Define $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ and $a : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\omega(s) = \begin{cases} s^{1/2} & \text{when } 0 \leq s \leq 1, \\ s^L & \text{when } s \geq 1 \end{cases}, \quad a(s) = \begin{cases} s^\frac{4}{3} + \varepsilon & \text{when } 0 \leq s \leq 1 \\ s^L & \text{when } s \geq 1 \end{cases}$$

Assume that $A$ is the set of $T \in (0, T_*)$ such that for any $2 < p < 6 + 2\gamma$ with $0 < \gamma \ll 1$,

$$\|du\|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^2)} + \|\nabla du\|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^2)} \leq M_1$$  (4.4)

$$\|\nabla \partial_t u\|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^2)} + \|\partial_t u\|_{L_t^\infty L_x^2([0,T] \times \mathbb{R}^2)} + \|\partial_t u\|_{L_t^2 L_x^p([0,T] \times \mathbb{R}^2)} + \left\| D^\frac{1}{2} \phi_t(0,t,x) \right\|_{L_t^2 L_x^p([0,T] \times \mathbb{R}^2)} \leq \varepsilon_1$$  (4.5)

$$\left\| \omega(s) D^\frac{1}{2} \partial_t \phi_s \right\|_{L_t^\infty L_x^2 L_x^2} + \left\| \omega(s) \partial_t \phi_s \right\|_{L_t^\infty L_x^2 L_x^2} \leq \varepsilon_1$$  (4.6)

$$\left\| \omega(s) \nabla \phi_s \right\|_{L_t^\infty L_x^\gamma L_x^2} + \left\| \phi_s \right\|_{L_t^\infty L_x^\gamma L_x^2} + \left\| \omega(s) D^\frac{1}{2} \phi_s \right\|_{L_t^\infty L_x^2 L_x^p} \leq \varepsilon_1.$$  (4.7)

Proposition 2.2 has given us the long time and short time behaviors of $|\nabla^k d\bar{u}|$ along the heat flow. The bounds of $|\nabla^k \partial_s \bar{u}|$ in Proposition 2.2 shall be improved by using (4.6) and (4.7).
Lemma 4.1. Assume (4.4) to (4.7) hold, then it holds uniformly for \((s, t) \in \mathbb{R}^+ \times [0, T]\) that

\[
e^{\delta_s} \|\phi_{t,s}\|_{L^2} + \frac{s}{2} e^{\delta_s} \|\phi_s\|_{L^\infty} + \frac{s^2}{4} e^{\delta_s} \|\phi_{t}\|_{L^\infty} \leq \varepsilon_1
\]  
\[
\|\sqrt{h_i} A_i\|_{L^\infty} \leq M_0 + M_1 \varepsilon_1
\]  
\[
\|A_i\|_{L^\infty} + \|A_t\|_{L^\infty} \leq \varepsilon_1
\]  
\[
s^2 e^{\delta_s} \|\nabla \partial_s \bar{u}\|_{L^2} + s e^{\delta_s} \|\nabla \partial_s \bar{u}\|_{L^\infty} \leq \varepsilon_1
\]  
\[
\|\sqrt{h_i} \phi_i^\text{con}\|_{L^\infty} \leq M_1 \varepsilon_1 e^{-\delta_s} \log s
\]  
\[
\omega(s) \|\nabla \partial_t \bar{u}\|_{L^2} + a_\frac{1}{4}(s) \|\nabla \partial_s \bar{u}\|_{L^\infty} \leq \varepsilon_1
\]  
\[
\beta_1(s) \|\sqrt{h_i} \partial_i A_i(s)\|_{L^\infty_{t,x}} + \beta_1(s) \|\sqrt{h_i} A_i^\text{con}(s)\|_{L^\infty_{t,x}} \leq M_1 \varepsilon_1
\]  
\[
1_{s \geq 1} e^{\delta_s} \|\sqrt{h_i} \partial_i A_i^\text{con}(s)\|_{L^\infty_{t,x}} + 1_{s \geq 1} e^{\delta_s} \|\sqrt{h_i} A_i^\text{con}(s)\|_{L^\infty_{t,x}} \leq M_1 \varepsilon_1
\]  
\[
\omega(s) \|\nabla^2 d\bar{u}\|_{L^\infty_{t,x}} + a_\frac{1}{4}(s) \|\nabla^2 d\bar{u}\|_{L^\infty_{t,x}} \leq M_1
\]

where \(\eta_1(s) = s^{\beta_1}\) when \(s \in [0, 1]\) with \(\beta_1\) being any constant in \((0, 1)\) and \(\beta_1(s) = e^{-\delta s}\) when \(s \in [1, \infty)\).

Proof. Since \(\partial_s \bar{u}\) satisfies \((\partial_s - \Delta)\partial_s \bar{u}\| \leq 0\), by maximum principle and (5.15), we deduce

\[
\|\partial_s \bar{u}\|_{L^2} \leq e^{-\frac{T}{2}} \|\partial_s \bar{u}(0, t, x)\|_{L^2},
\]

Meanwhile (5.14) and maximum principle show

\[
\|\partial_s \bar{u}\|_{L^\infty} \leq s^{-\frac{1}{2}} e^{-\frac{T}{2}} \|\partial_s \bar{u}(0, t, x)\|_{L^2}.
\]

Similarly the same results hold for \(\|\partial_t \bar{u}\|\) since we also have \((\partial_s - \Delta)\|\partial_t \bar{u}\| \leq 0\). Thus (4.8) follows by (4.7), (4.5), (4.9) follows by (4.8) and (5.34). And (4.10) follows by Lemma 2.5 and (4.8). Proposition 2.2 and (4.4) imply for \(s \geq 1\)

\[
\|\nabla d\bar{u}\|_{L^\infty}^2 + \|d\bar{u}\|_{L^\infty}^2 \leq C(M_1).
\]

Then (5.19) and (4.8) show

\[
\partial_s |\nabla \partial_s \bar{u}|^2 - \Delta |\nabla \partial_s \bar{u}|^2 \leq |\nabla \partial_s \bar{u}|^2(C(M_1) + 3) + C(M_1)^3 e^{-2\delta s} \epsilon_1^2 + C(M_1)^4 e^{-2\delta s} \epsilon_1^2.
\]

Fix any \(s_1 \geq 1\), let

\[
f(s) = |\nabla \partial_s \bar{u}|^2 e^{\int_{s_1}^s (C(M_1) + 3) ds'} + (C(M_1)^4 + C(M_1)^3) \delta^{-1} e^{-2\delta s} \epsilon_1^2.
\]
then \( f \) satisfies for \( s \in (s_1, \infty) \)

\[
(\partial_s - \Delta) f(s) \leq 0.
\] (4.22)

Hence Lemma 5.12 gives

\[
f(x, s) \lesssim \int_{s}^{s+1} \int_{B(x, 1)} f(\tau) d\nu d\tau \leq C(M_1)\|\nabla \partial_s \tilde{u}\|_{L^2_x}^2 + C(M_1)\epsilon_1^2.
\] (4.23)

Since \( |\nabla \partial_s \tilde{u}| \leq |\nabla \phi_s| + \sqrt{h^{ii}}|A_i||\phi_s| \), (4.11) follows by (4.6) and (4.10). By (5.18) and (5.20), the same arguments yield (4.13) and (4.14) respectively. (4.12) follows by \( |\sqrt{h^{ii}} \partial_i \phi_s| \leq |\sqrt{h^{ii}} A_i \phi_s| + |\nabla \partial_s \tilde{u}| \) and (4.10). Finally, by Lemma 2.5,

\[
|\sqrt{h^{ii}} \partial_t A_i(s)| \leq \int_{s}^{\infty} \sqrt{h^{ii}}|\nabla \partial_t \tilde{u} \wedge \partial_t \tilde{u}| ds' + \int_{s}^{\infty} \sqrt{h^{ii}}|\nabla \partial_s \tilde{u} \wedge \partial_s \tilde{u}| ds' + \sqrt{h^{ii}}|A_i A_i|,
\] (4.24)

\[
|h^{ii} \partial_t A_i^{\text{con}}(s)| \leq \int_{s}^{\infty} h^{ii}|\nabla \partial_t \tilde{u} \wedge \partial_t \tilde{u}| ds' + \int_{s}^{\infty} h^{ii}|\nabla \partial_s \tilde{u} \wedge \partial_s \tilde{u}| ds' + |h^{ii} A_i^{\text{con}}| |A_i|.
\] (4.25)

then (4.16) follows by (4.24), (4.25), (4.8)–(4.14). Since \( |d\tilde{u}| \) satisfies \( |d\tilde{u}| \leq e^{Cs} e^{s\Delta} |du| \), we obtain from \( ||d\tilde{u}||_{L^2_x} \leq M_1 \) shown by Proposition 2.2, Lemma 5.5 and Sobolev embedding that when \( s \in [0, 1] \)

\[
|d\tilde{u}| \leq s^{-\beta_1} M_1
\] (4.26)

for any \( \beta_1 \in (0, 1) \). Thus (4.15) follows by (4.24), (4.25), (4.8)–(4.14). By integration by parts, one has

\[
\|\nabla^2 d\tilde{u}\|_{L^2_x}^2 \leq \|\nabla \tau(\tilde{u})\|_{L^2_x}^2 + \|\nabla d\tilde{u}\|_{L^2_x}^2 \|d\tilde{u}\|_{L^2_x}^2 + \|d\tilde{u}\|_{L^2_x}^6.
\] (4.27)

Thus by \( \tau(\tilde{u}) = \partial_t \tilde{u} \), (4.11) and Proposition 2.2, the \( L^2_x \) part in (4.17) follows. The short time \( L^\infty_x \) part in (4.17) follows by applying maximum principle and smoothing effect of \( e^{t\Delta} \) to (5.21) with the help of the \( \omega(s)\|\nabla^2 d\tilde{u}\|_{L^\infty_x L^2_x} \) bound previously obtained. The large time \( L^\infty_x \) part in (4.17) follows by applying Lemma 5.12 and the bound \( \omega(s)\|\nabla^2 d\tilde{u}\|_{L^\infty_x L^2_x} \) to (5.21), see the proof of (4.11) above.

\[ \square \]

**Lemma 4.2.** Assume that (4.4) to (4.7) hold, then

\[
a_{\frac{1}{4}}(s)\|\nabla^2 \phi_s\|_{L^\infty_x L^2_x} \leq M_1 \epsilon_1
\] (4.28)

\[
a_{\frac{1}{4}}(s)\|\nabla^2 \partial_s \tilde{u}\|_{L^\infty_x L^2_x} \leq M_1 \epsilon_1
\] (4.29)

\[
a_{\frac{1}{4}}(s)\|\nabla^2 \partial_s \tilde{u}\|_{L^\infty_x L^2_x} \leq M_1 \epsilon_1
\] (4.30)
By Duhamel principle one deduces from the heat equation of φ

\[ a_1(s)\|\nabla^2 \phi_s\|_{L^\infty_x} \leq M_1 \varepsilon_1 \]  
\[ \|h^i h^{pp} \nabla \delta \phi_i^\infty\|_{L^\infty_x} \leq M_1 \varepsilon_1 \min (s^{-\frac{1}{2}}, s^{-L+1}) \]  

Denote the one order derivative term in G

Applying \(3.7\), we deduce

Then the homogeneous term in (4.36) is acceptable by Lemma 4.1. Again due to Proposition 3.7, we deduce

\[ \|H^2 \phi_s\|_{L_2^2} \leq s^{-1} e^{-\frac{4}{9}} \|\phi_s(\frac{s}{2})\|_{L_2^2} + \int_{\frac{s}{2}}^{s} (s-\tau)^{-\frac{5}{9}} \|H^2 G(\tau)\|_{L_2^2} d\tau \]  

Proof. Integration by parts gives

\[ \|\nabla^2 \phi_s\|_{L_2^2} \leq \|\Delta \phi_s\|_{L_2^2} + \|\nabla \phi_s\|_{L_2^2} \]  

For (4.28), due to (4.8)-(4.11) and \(|\nabla \phi_s| \leq |\nabla \partial_t \tilde{u}| + h^i |A_i \phi_s|\), it suffices to prove

\[ a_1(s)\|\Delta \phi_s\|_{L^\infty_x L_2^2} \leq M_1 \varepsilon_1. \]  

By Duhamel principle one deduces from the heat equation of φ that

\[ \phi_s = e^{-\frac{4}{9}H \phi_s(\frac{s}{2})} + \int_{\frac{s}{2}}^{s} e^{-H(s-\tau)} G(\tau) d\tau, \]  

where G denotes the inhomogeneous part, i.e.,

\[ G = 2h^i A_i^{con} \partial_t \phi_s + h^i (\partial_t A_i^{con}) \phi_s - h^i \Gamma^k_i A_k^{con} \phi_s + h^i A_i^{con} A_i^{con} \phi_s + h^i A_i^{con} A_i^{con} \phi_s + h^i (\phi_s \wedge \phi_i^{con}) \phi_i^{con} + h^i (\phi_s \wedge \phi_i^{con}) \phi_i^{con}. \]

Applying H to (4.35), we get by Proposition 3.7

\[ \|H \phi_s\|_{L_2^2} \leq s^{-1} e^{-\frac{4}{9}} \|\phi_s(\frac{s}{2})\|_{L_2^2} + \int_{\frac{s}{2}}^{s} (s-\tau)^{-\frac{5}{9}} \|H^2 G(\tau)\|_{L_2^2} d\tau, \]  

Then the homogeneous term in (4.30) is acceptable by Lemma 4.1. Again due to Proposition 3.7, we deduce

\[ \|H^2 G(\tau)\|_{L_2^2} \leq \|\nabla G(\tau)\|_{L_2^2}. \]  

Denote the one order derivative term in G by \(G_1 \triangleq 2h^i A_i^{con} \partial_t \phi_s \) and \(G_2 \triangleq h^i (\partial_t A_i^{con}) \phi_s - h^i \Gamma^k_i A_k^{con} \phi_s \). The remainder terms in G is denoted by \(G_3 \triangleq G - G_1 - G_2 \). By Lemma 5.15 and (4.16),

\[ \|\nabla G_1(\tau)\|_{L_2^x} \leq \|h^i A_i^{con}\|_{L^\infty_x} \|\nabla^2 \phi_s\|_{L_2^2} + \left( \int_{\tau}^{\infty} \|\nabla \phi_s\|_{L_2^x} \sqrt{h^i A_i^{con}} ds' \right) \|\nabla \phi_s\|_{L_2^2} \]  

\[ + \left( \int_{\tau}^{\infty} \|\phi_s\|_{L_2^x} \sqrt{h^i A_i^{con} \partial_t \phi_i^{con}} ds' \right) \|\nabla \phi_s\|_{L_2^2} \]  

\[ \|\nabla G_2(\tau)\|_{L_2^x} \leq M_1 \varepsilon_1 \tau^{-\frac{1}{2}} e^{-\frac{4}{9} \tau} \|\nabla \phi_s\|_{L_2^2} + \left( \int_{\tau}^{\infty} \|\nabla^2 \tilde{u}\|_{L_2^2} \|d\tilde{u}\|_{L_2^x} ds' \right) \|\phi_s\|_{L^\infty_x} \]  

47
We get from Lemma 5.15 and Lemma 4.1 that the $G_3$ term is bounded by

$$\| \nabla G_3(\tau) \|_{L^2} \leq \tau^{-\frac{1}{2}}e^{-\delta \tau} M_1^2 \varepsilon_1^2 \| \phi_s \|_{L^2} + e^{-\delta \tau} M_1^2 \varepsilon_1^2 \| \nabla \phi_s \|_{L^2} + \varepsilon_1^2 e^{-\delta \tau} (\log \tau)^2 \| \nabla \phi_s \|_{L^2}$$

Using the inequality

$$\int_\tau^\infty \| \nabla^2 \phi_s \|_{L^2} ds' \leq \| a_{\frac{1}{2}}(s) \nabla^2 \phi_s \|_{L^2 \rightarrow L^2} \min(\log \tau, \tau^{-L+1})$$

$$| \nabla^2 \partial_s u | \leq | \nabla^2 \phi_s | + \varepsilon^2 \partial_s \phi_s + \sqrt{h} \varepsilon^2 |\partial_s A_i| |\phi_s|$$

we conclude by Lemma 4.1 and (4.36)-(4.40) that

$$\| a_{\frac{1}{2}}(s) H \phi_s \|_{L^2 \rightarrow L^2} \leq \varepsilon_1 + M_1 \varepsilon_1 \| a_{\frac{1}{2}}(s) H \phi_s \|_{L^2 \rightarrow L^2},$$

thus finishing the proof of (4.29). (4.29) follows from (4.28), Lemma 4.1 and (4.42). Then the large time part of (4.30) follows from (4.29) and Lemma 4.1 by applying Lemma 5.12 to (5.22). The short time part of (4.30) follows by (4.29) and applying smoothing effect of $e^{t\Delta}$ to (5.22). And due to (4.42), (4.31) follows by (4.30) and Lemma 4.1. Finally (4.32) follows by (4.42), (5.33) and Lemma 4.1.

**Lemma 4.3.** Assume that (4.4) to (4.7) hold, then for $q \in (2, 6 + 2\gamma]$

$$\| \phi_t(s) \|_{L^q L^2 L^2} \leq M_1 \varepsilon_1$$

$$\| D^{\frac{1}{2}} \phi_t(s) \|_{L^q L^2 L^2} \leq M_1 \varepsilon_1$$

$$\| s^{L} D^{\frac{1}{2}} \phi_t(s) \|_{L^q[1, \infty) L^2 L^2} \leq M_1 \varepsilon_1.$$

**Proof.** (4.44) follows by the same arguments of [Lemma 7.2][49]. It suffices to prove (4.45). The differential field $\phi_t$ satisfies

$$(\partial_s - \Delta) \phi_t = 2h_i^i A_i \partial_s \phi_t + h_i^i A_i \phi_t + h_i^i \partial_i A_i \phi_t - h_i^i \Gamma_{ij}^k A_k \phi_t + h_i^i (\phi_t \wedge \phi_i) \phi_t.$$  

Separating the $H \phi_t$ part away from the nonlinearity terms yields

$$(\partial_s - H) \phi_t = 2h_i^i A_i^c \partial_i \phi_t + h_i^i A_i^c A_i \phi_t + h_i^i A_i^c A_i \phi_t + h_i^i A_i^\infty A_i \phi_t + h_i^i A_i^c A_i \phi_t$$

$$+ h_i^i (\partial_i A_i^c - \Gamma_{ik}^j A_j^c) \phi_t + h_i^i (\phi_t \wedge \phi_i^c) \phi_t + h_i^i (\phi_t \wedge \phi_i^c) \phi_t^c + h_i^i (\phi_t \wedge \phi_i^c) \phi_t^c.$$  

Denote the right hand side of (4.48) as $\mathcal{G}$. And denote the one order derivative term of $\phi_t$ by $\mathcal{G}_1,$
i.e., $G_1 = 2h^{ii}A_i^{con} \partial_i \phi_t$. The other zero order terms are denoted by $G_2$. Applying $H^{\frac{3}{2}}$ to (4.48), by Proposition 3.6 we have

$$
\|H^{\frac{3}{2}} \phi_t\|_{L_t^2 L_x^2} \leq \|H^{\frac{3}{2}} \phi_t(0, t, x)\|_{L^2_t L^2_x} + \int_0^t (s - \tau)^{-\frac{3}{2}} e^{-\delta'(s - \tau)} \|G_2(\tau)\|_{L_t^2 L_x^2} d\tau
$$

$$
+ \int_0^t e^{-(s - \tau)H} H^{\frac{3}{2}} G_1(\tau)\|_{L_t^2 L_x^2} d\tau
$$

(4.49)

Lemma 4.1 and (4.5) show

$$
\|G_2(\tau)\|_{L_t^2 L_x^2} \leq M_1 \varepsilon_1^2 \min(\tau^{-\frac{1}{2}}, \tau^{-L + 1})
$$

(4.50)

By Proposition 3.6 the $G_1$ term in (4.49) is bounded by

$$
\|e^{-(s - \tau)H} H^{\frac{3}{2}} G_1(\tau)\|_{L_t^2 L_x^2} \lesssim \|e^{-(s - \tau)H} H^{\frac{3}{2}} h^{ii} \partial_t (A_i^{con} \phi_t)\|_{L_t^2 L_x^2} + \|e^{-(s - \tau)H} H^{\frac{3}{2}} h^{ii} (\partial_t A_i^{con}) \phi_t\|_{L_t^2 L_x^2}.
$$

(4.51)

The second term in (4.51) has appeared in $G_2$. By (3.171), (3.172) and Proposition 3.6, one has for any $\varepsilon \in (0, 1)$

$$
\|e^{-(s - \tau)H} H^{\frac{3}{2}} h^{ii} \partial_t (A_i^{con} \phi_t)\|_{L_t^2}
\leq \|e^{-(s - \tau)H} H^{\frac{3}{2}} h^{ii} H^{\frac{3}{2}} (-\Delta)^{\frac{3}{2}} (\Delta)^{-\frac{3}{2}} h^{ii} \partial_t (A_i^{con} \phi_t)\|_{L_t^2}
$$

$$
\leq (s - \tau)^{-\frac{3}{2}} e^{-\delta'(s - \tau)} \|(-\Delta)^{-\frac{3}{2}} h^{ii} \partial_t (A_i^{con} \phi_t)\|_{L_t^2}
$$

$$
\leq (s - \tau)^{-\frac{3}{2}} e^{-\delta'(s - \tau)} \|\sqrt{h^{ii}} (\partial_t A_i^{con} \phi_t)\|_{L_t^2}
$$

(4.52)

where we use the boundedness of Riesz transform in the last line. Hence Lemma 4.1 (4.4)-(4.7), (4.49)-(4.52) give

$$
\|H^{\frac{3}{2}} \phi_t\|_{L_t^2 L_x^2} \leq M_1 \varepsilon_1.
$$

(4.53)

Then (4.45) follows from (4.53) and (4.6). Similar arguments yield (4.46).

Lemma 4.4. Assume that (4.4) to (4.7) hold, then for any $\varepsilon \in (0, 1)$, $\phi_t$ satisfies for $q \in (2, 6 + 2\gamma]$ with $0 < \gamma \ll 1$

$$
\left\|K_{\frac{q}{2}, \varepsilon}(s) \nabla \phi_t(s)\right\|_{L^\infty_t L^2_x L^2_x} \lesssim \varepsilon_1
$$

(4.54)

where $K_{\theta, \varepsilon}(s) = \theta + \varepsilon$ when $s \in [0, 1]$ and $K_{\theta, \varepsilon}(s) = s^L$ when $s \in (1, \infty)$.  

49
Proof. Applying $H^{\beta}$ to (4.48), Duhamel principle gives

$$
\|H^{\beta} \phi_t\|_{L^2} \leq \|H^{\beta} e^{-\frac{\beta}{2} H} \phi_t(S)\|_{L^2} + \int_2^S \sum_{j=1}^2 \|H^{\beta} e^{-(s-\tau)H} G_j(\tau)\|_{L^2} d\tau
$$

(4.55)

Proposition 3.6 gives the bound for the first term in (4.55)

$$
\kappa_{\frac{3}{8}, \epsilon}(s) \|H^{\beta} e^{-\frac{\beta}{2} H} \phi_t(S)\|_{L^2} \leq \|H^{\beta} \phi_t(S)\|_{L^2},
$$

(4.56)

which combined with (4.102), (3.171), (3.172) implies

$$
\kappa_{\frac{3}{8}, \epsilon}(s) \|H^{\beta} e^{-\frac{\beta}{2} H} \phi_t(S)\|_{L^2} \leq M_1 \epsilon_1.
$$

(4.57)

The second term in (4.55) is bounded by

$$
\int_2^S \sum_{j=1}^2 \|H^{\beta} e^{-(s-\tau)H} G_j(\tau)\|_{L^2} d\tau \leq M_1 \epsilon_1 \int_2^S e^{-\delta'(s-\tau)(s-\tau)^{-\frac{1}{2}} \epsilon} \|\nabla \phi_t(\tau)\|_{L^2} d\tau
$$

$$
+ M_1 \epsilon_1 \int_2^S e^{-\delta'(s-\tau)(s-\tau)^{-\frac{1}{2}} \epsilon} (s-\frac{1}{2}) \tau^{-\frac{1}{2}} d\tau.
$$

(4.58)

Thus (4.58), (4.57) and (4.55) yield

$$
\|\kappa_{\frac{3}{8}, \epsilon}(s) \|H^{\beta} \phi_t\|_{L^\infty L^2 L^2} \leq M_1 \epsilon_1 \|\kappa_{\frac{3}{8}, \epsilon}(s) \|H^{\beta} \phi_t\|_{L^\infty L^2 L^2} + M_1 \epsilon_1.
$$

(4.59)

Hence (4.54) follows by (4.59) and (3.171), (3.172).

\[\square\]

**Lemma 4.5.** Assume that (4.4) to (4.7) hold, then $\phi_t$ satisfies for $q \in (2, \frac{6}{2} + 2\gamma]$ with $0 < \gamma \ll 1$ and any $\epsilon \in (0, 1)$

$$
\|\kappa_{\frac{3}{8}, \epsilon}(s) \nabla^2 \phi_t(s)\|_{L^\infty L^2 L^2} \leq \epsilon_1.
$$

(4.60)

**Proof.** Applying $H$ to (4.48), Duhamel principle gives

$$
\|H \phi_t\|_{L^2} \leq \|He^{-\frac{\beta}{2} H} \phi_t(S)\|_{L^2} + \int_2^S \sum_{j=1}^2 \|He^{-(s-\tau)H} G_j(\tau)\|_{L^2} d\tau
$$

(4.61)

(3.171), (3.172) and Proposition 3.6 yield

$$
\|He^{-(s-\tau)H} G_j(\tau)\|_{L^2} \leq \|H^{\frac{1}{2}} e^{-(s-\tau)H} H^{\frac{1}{2}} G_j(\tau)\|_{L^2} \leq (s-\tau)^{-\frac{1}{2} - \epsilon} \|\nabla G_j(\tau)\|_{L^2} + \|G_j(\tau)\|_{L^2}.
$$

(4.62)
First we deal with the $G_1$ term. By the explicit formula of $\Gamma^i_{k j}$ and $h^{ij}$ in (2.2), we have

$$|\nabla(h^{ii}A^i_{con} \partial_i \phi_t)| \leq \sum_{p=1}^{2} \sqrt{h^{pp}} |\partial_p(h^{ii}A^i_{con} \partial_i \phi_t)|$$

$$\leq \sum_{p=1}^{2} \sqrt{h^{pp}} |h^{ii}A^i_{con} \partial_p \partial_i \phi_t| + \sqrt{h^{ii}} \sqrt{h^{pp}} |\partial_p A^i_{con} |\nabla \phi_t| + \sqrt{h^{ii}} |A^i_{con} |\nabla \phi_t|$$

$$\leq \sqrt{h^{ii}} A^i_{con} |\nabla^2 (\phi_t)| + \sqrt{h^{ii}} \sqrt{h^{pp}} |\partial_p A^i_{con} |\nabla \phi_t| + \sqrt{h^{ii}} |A^i_{con} |\nabla \phi_t|$$  \quad (4.63)

Thus Lemma 4.1 shows that the $G_1$ term in (4.62) is bounded by

$$\int_{\frac{s}{2}}^{s} \|H e^{-(s-\tau)} H G_1(\tau)\|_{L^2_x} d\tau$$

$$\leq \int_{\frac{s}{2}}^{s} (s - \tau)^{-\frac{1}{2} - \delta(s-\tau)} (\|\sqrt{h^{ii}} A^i_{con} \nabla^2 (\phi_t)\|_{L^2_x}) d\tau$$

$$+ \int_{\frac{s}{2}}^{s} \min(\tau^{-\beta_1}, \tau^{-L}) \|\nabla \phi_t\|_{L^2_x} d\tau,$$  \quad (4.64)

where $\beta_1$ is any sufficiently small constant in $(0, 1)$. Thus by Lemma 5.14 and Sobolev embedding, for $q < r$, $1 + \frac{1}{r} - \frac{1}{q} > \frac{3}{4}$ and $\frac{1}{m} + \frac{1}{r} = \frac{1}{q}$ we have

$$\|\sqrt{h^{ii}} A^i_{con} \nabla^2 (\phi_t)\|_{L^2_x} \leq \|\nabla^2 (\phi_t)\|_{L^2_x} \|\sqrt{h^{ii}} A^i_{con}\|_{L^m_x}$$

$$\leq M_1 \varepsilon_1 \|\Delta \phi_t\|_{L^2_x}.$$  \quad (4.66)

Then by the trivial inequality

$$\|\Delta f\|_{L^2_x} \leq \|H f\|_{L^2_x} + \|\nabla f\|_{L^2_x} + \|f\|_{L^2_x}$$  \quad (4.67)

and Lemma 4.1 (4.66) impies

$$\|\sqrt{h^{ii}} A^i_{con} \nabla^2 (\phi_t)\|_{L^2_x} \leq M_1 \varepsilon_1 (\|H \phi_t\|_{L^2_x} + \|\nabla \phi_t\|_{L^2_x} + \|\phi_t\|_{L^2_x}).$$  \quad (4.68)

Therefore, (4.54), (4.68) and (4.65) give the acceptable bound for $G_1$,

$$\int_{\frac{s}{2}}^{s} \|H e^{-(s-\tau)} H G_1(\tau)\|_{L^2_x} d\tau$$

$$\leq M_1 \varepsilon_1 \int_{\frac{s}{2}}^{s} (s - \tau)^{-\frac{1}{2} - \delta(s-\tau)} (\|H \phi_t\|_{L^2_x} + \|\nabla \phi_t\|_{L^2_x} + \|\phi_t\|_{L^2_x}) d\tau + M_1 \varepsilon_1.$$  \quad (4.69)

For the $G_2$ term, we first consider the tougher term $G_{21} \triangleq h^{ii} \partial_i A^i_{con} \phi_t - h^{ii} \Gamma^k_{ij} A^k_{con} \phi_t$. Lemma
\[ |\nabla G_{21}| \leq M_1 \varepsilon_1 \left( \min(\tau^{-\frac{1}{2}} - \beta_1, \tau^{-L})|\phi_t| + \min(\log \tau, \tau^{-L})|\nabla \phi_t| \right). \]  

We denote the remainder terms in \( G_2 \) by \( G_{22} \), then Lemma 4.1 and Lemma 4.2 show

\[ |\nabla G_{22}(\tau)| \leq M_1 \varepsilon_1 \left( \min(\tau^{-\frac{1}{2}} - \beta_1, \tau^{-L})|\phi_t| + \min(\log \tau, \tau^{-L})|\nabla \phi_t| \right). \]  

Hence (4.70), (4.71) give

\[ \kappa^\varepsilon(s) \int_0^s \| He^{-(s-\tau)H} G_2(\tau) \| L^2_\tau L^2_\tau L^2_\tau \, d\tau \leq M_1 \varepsilon_1 \| \kappa^\varepsilon(s) \nabla \phi_t \| L^\infty_\tau L^2_\tau L^2_\tau + \| \kappa^\varepsilon(s) \phi_t \| L^\infty_\tau L^2_\tau L^2_\tau \].  

(4.72)

Combining (4.62), (4.72), (4.62) with (4.54), we infer from (4.45) that

\[ \| \kappa^\varepsilon(s) H \phi_t \| L^\infty_\tau L^2_\tau L^2_\tau \leq M_1 \varepsilon_1 \| \kappa^\varepsilon(s) H \phi_t \| L^\infty_\tau L^2_\tau L^2_\tau + M_1 \varepsilon_1. \]

(4.73)

Therefore we get (4.60) from (4.54), (4.45) and (3.171), (3.172). \( \square \)

**Lemma 4.6.** Assume that (4.4) to (4.7) hold, then for \( p \in (2, 6) \) and any \( \varepsilon \in (0, 1) \),

\[ \left\| a_\varepsilon(s) \left\| \partial_\tau \phi_s \right\| L^2_\tau L^p_\tau \right\| L^\infty_\tau + \left\| a_\varepsilon(s) \left\| \nabla \phi_s \right\| L^2_\tau L^p_\tau \right\| L^\infty_\tau \leq M_1 \varepsilon_1. \]

(4.74)

Generally we have for \( \theta \in [0, 1/2) \), \( \theta_1 \in [0, 3/4] \)

\[ \left\| \beta_{\theta_1, \varepsilon}(s)(-\Delta)^{\theta} D^{1/2} \partial_\tau \phi_s \right\| L^\infty_\tau L^2_\tau L^p_\tau + \left\| \beta_{\theta_1, \varepsilon}(s)(-\Delta)^{\theta} D^{1/2} \phi_s \right\| L^\infty_\tau L^2_\tau L^p_\tau \leq M_1 \varepsilon_1. \]

(4.75)

where \( \beta_{\theta_1}(s) = s^{1/2 + \theta + \varepsilon} \), when \( s \in [0, 1] \) and \( \beta_{\theta_1}(s) = s^L \) when \( s \in [1, \infty) \).

**Proof. Step 1.** We prove the desired estimates for \( \phi_s \) in Step 1. First by (3.171), (3.172) and (4.7), we note that (4.75) follows by

\[ \left\| \beta_{\theta_1} \phi_s H^\theta \right\| L^\infty_\tau L^2_\tau L^p_\tau \leq M_1 \varepsilon_1. \]

(4.76)

**Step 1.1** It will be useful if we first obtain the following estimate

\[ \left\| \beta_{1/2} \phi_s \right\| L^p_\tau L^2_\tau L^2_\tau \leq M_1 \varepsilon_1. \]

(4.77)

Applying \( H^{1/2} \) to (4.35) by Proposition 3.6 we obtain for some \( \delta' > 0 \) and any \( \varepsilon \in (0, 1) \)

\[ \| H^{1/2} \phi_s \| L^p_\tau \lesssim s^{-1/2 - \varepsilon} e^{-s\delta'} \| H^{1/2} \phi_s \| L^p_\tau + \int_0^s \| e^{-H(t-\tau)} \nabla \phi_s \| L^p_\tau \, d\tau. \]

(4.78)
Split the $G$ into the one order term $G_1 \triangleq 2h^{ii}A_i^c \partial_i \phi_s$ and the zero order terms $G_2 \triangleq G - G_1$. Then for $G_2$, Proposition 3.6 gives

$$\int_{\frac{s}{2}}^{s} \| e^{-H(t-\tau)} H^{\frac{2}{3}} G(\tau) \phi_s \|_{L_x^2} d\tau \lesssim \int_{\frac{s}{2}}^{s} e^{-\delta'(s-\tau)} (s-\tau)^{-\frac{1}{2} - \epsilon} \| G_2(\tau) \|_{L_x^p} d\tau.$$  

Thus by (4.15), (4.12), we have

$$\beta_{\frac{1}{4}, \epsilon}(s) \int_{\frac{s}{2}}^{s} \| e^{-H(t-\tau)} H^{\frac{2}{3}} G(\tau) \|_{L_x^2 L_t^4} d\tau \leq M_1 \varepsilon_1 \| \omega(s) \phi_s \|_{L_x^{\infty} L_t^2 L_x^p}. \quad (4.79)$$

For the one order term $G_1$, Proposition 3.6, Lemma 4.1 yield

$$\beta_{\frac{1}{4}, \epsilon}(s) \int_{\frac{s}{2}}^{s} \| e^{-H(t-\tau)} H^{\frac{2}{3}} G_1(\tau) \|_{L_x^2 L_t^4} d\tau \leq M_1 \varepsilon_1 \| \beta_{\frac{1}{4}, \epsilon} \nabla \phi_s \|_{L_x^{\infty} L_t^2 L_x^p}. \quad (4.80)$$

Therefore (4.78) to (4.80) give

$$\| \beta_{\frac{1}{4}, \epsilon} H^{\frac{1}{4}} \phi_s \|_{L_x^{\infty} L_t^2 L_x^p} \leq M_1 \varepsilon_1 \| \beta_{\frac{1}{4}, \epsilon} \nabla \phi_s \|_{L_x^{\infty} L_t^2 L_x^p} + M_1 \varepsilon_1. \quad (4.81)$$

Thus by (3.170) and (4.7), (4.81) we get (4.77) and

$$\| \beta_{\frac{1}{4}, \epsilon} \nabla \phi_s \|_{L_x^{2} L_t^4 L_x^p} \leq M_1 \varepsilon_1. \quad (4.82)$$

**Step 1.2.** In this step, we prove (4.76). Applying $H$ to (4.35), by Proposition 3.6 we obtain

$$\| H \phi_s \|_{L_x^p} \lesssim s^{-\frac{1}{2} - \epsilon} e^{-s\delta'} H^{\frac{1}{4}} \phi_s \left(\frac{s}{2}\right) \|_{L_x^p} + \int_{\frac{s}{2}}^{s} e^{-\delta'(s-\tau)} (s-\tau)^{-\frac{1}{2} - \epsilon} \| H^{\frac{2}{3}} G(\tau) \|_{L_x^p} d\tau. \quad (4.83)$$

The same arguments in the proof of (4.60) give

$$\beta_{\frac{1}{4}, \epsilon}(s) \int_{\frac{s}{2}}^{s} e^{-\delta'(s-\tau)} (s-\tau)^{-\frac{1}{2} - \epsilon} \| \nabla G(\tau) \|_{L_t^2 L_x^p} \lesssim M_1 \varepsilon_1 \| \beta_{\frac{1}{4}, \epsilon}(s) \Delta \phi_s(\tau) \|_{L_x^{\infty} L_t^2 L_x^p} + M_1 \varepsilon_1 \| \beta_{\frac{1}{4}, \epsilon}(s) \nabla \phi_s(\tau) \|_{L_x^{\infty} L_t^2 L_x^p} + M_1 \varepsilon_1 \| \omega(s) \phi_s(\tau) \|_{L_x^{\infty} L_t^2 L_x^p} + M_1 \varepsilon_1. \quad (4.84)$$

Thus we arrive at (4.76) from (4.83), (4.84), (4.82) and (4.7).

**Step 2.** In this step we prove the desired estimates in (4.75) for $\partial_t \phi_s$. The proof is almost the same as Step 1 with the help of (4.15).

**Lemma 4.7.** Suppose that (4.4) to (4.7) hold, then the wave map tension field satisfies

$$\| s^{-\frac{1}{2}} Z(s) \|_{L_x^{\infty} L_t^1 L_x^2} \leq M_1 \varepsilon_1^2 \quad (4.85)$$
\[ \| \nabla Z(s) \|_{L^\infty_t L^1_x L^2_z} \leq M_1 \varepsilon_1 \]  
\[ \| s^{\frac{3}{2}} \Delta Z(s) \|_{L^\infty_t L^1_x L^2_z} \leq M_1 \varepsilon_1^2 \]  
\[ \| \omega(s) \partial_s Z(s) \|_{L^\infty_t L^1_x L^2_z} \leq M_1 \varepsilon_1^2. \]

**Proof.** (4.74) shows

\[
\partial_s Z - HZ = 2h^{ii} A^{con}_i \partial_i Z + h^{ii} A^{con}_i A^{\infty}_i Z + h^{ii} A^{\infty}_i A^{con}_i Z + h^{ii} A^{con}_i A^{\infty}_i Z + \Gamma^{\kappa k} (\partial_i A^{con}_i - \Gamma^{\kappa k} A^{con}_k) Z \\
+ h^{ii} (Z \wedge \phi^{\infty}_i) \phi^{\infty}_i + h^{ii} (Z \wedge \phi^{con}_i) \phi^{con}_i + 3h^{ii} (\partial_i u \wedge \partial_i \tilde{u}) \nabla x \partial_i \tilde{u}.
\]

Duhamel Principle gives

\[
\| Z(s) \|_{L^1_t L^2_x} \leq 3 \sum_{j=1}^{3} \int_0^s \| e^{-H(s-\tau)} \tilde{G}_j(\tau) \|_{L^1_t L^2_x} d\tau,
\]

where \( \tilde{G}_1 \) denotes the one derivative term of \( Z \), i.e., \( G_1 = 2h^{ii} A^{con}_i \partial_i Z \), \( \tilde{G}_2 \) denotes the zero order derivative terms of \( Z \), and \( \tilde{G}_3 \) denotes \( 3h^{ii} (\partial_i u \wedge \partial_i \tilde{u}) \nabla x \partial_i \tilde{u} \). When \( s \in [0, 1] \), by Lemma 4.1, the \( \tilde{G}_2 \) term in (4.90) is bounded by

\[
M_1 \varepsilon_1 \| w(s)s^{-1/2} \|_{L^\infty_{[0,1]} L^1_t L^2_x} (s^{\frac{3}{2}} + s + s^{\frac{3}{2}}).
\]

By (4.54) and Lemma 4.1, the \( \tilde{G}_3 \) term in (4.90) is bounded by

\[
\int_0^s \| d\tilde{u} \|_{L^\infty_t L^2_x} \| \nabla \phi_i \|_{L^2_t L^6_x} \| \partial_i \tilde{u} \|_{L^2_t L^6_x} ds' + \int_0^s \| d\tilde{u} \|_{L^\infty_t L^2_x} \| \nabla h^{ii} A_i \phi_i \|_{L^2_t L^6_x} \| \partial_i \tilde{u} \|_{L^2_t L^6_x} ds'.
\]

Thus (4.54) and Lemma 4.1 show that \( \tilde{G}_3 \) in (4.90) is bounded by

\[
\int_0^s \| \tilde{G}_3 \|_{L^1_t L^2_x} d\tau \leq s^{\frac{3}{2}} - \varepsilon M_1 \varepsilon_1.
\]

For the \( \tilde{G}_1 \) term in (4.90), direct calculations show

\[
\| e^{-(s-\tau)H} h^{ii} A^{con}_i \partial_i Z \|_{L^2_x} \leq \| e^{-(s-\tau)H} h^{ii} \partial_i (A^{con}_i Z) \|_{L^2_z} + \| e^{-(s-\tau)H} h^{ii} (\partial_i A^{con}_i Z) \|_{L^2_x}
\]

By Proposition 3.7 and the boundedness of Riesz transform, the first term in (4.94) is bounded by

\[
\| e^{-(s-\tau)H} h^{ii} \partial_i (A^{con}_i Z) \|_{L^2_z} \leq \| e^{-(s-\tau)H} h^{ii} H^{\frac{1}{2}} H^{-\frac{1}{2}} (\Delta)^{\frac{1}{2}} (\Delta)^{-\frac{1}{2}} \partial_i (A^{con}_i Z) \|_{L^2_z}
\]

\[
\leq (s - \tau)^{-\frac{3}{2}} e^{-\delta(s-\tau)} \| (\Delta)^{-\frac{1}{2}} h^{ii} \partial_i (A^{con}_i Z) \|_{L^2_z}
\]
\[
\leq (s - \tau)^{-\frac{T}{2}} e^{-\delta(s-\tau)} \| \sqrt{h^{ii}} A_i^{\text{con}} Z \|_{L_t^2} \tag{4.95}
\]

Thus by Lemma 3.1 (4.95) and the second term in (4.94) are bounded as

\[
\| e^{-(s-\tau)H} h^{ii} A_i^{\text{con}} \partial_i Z \|_{L_t^2} \leq M_1 \varepsilon_1 (s - \tau)^{-\frac{T}{2}} \| e^{-\delta(s-\tau)} \|_{L_t^2} + M_1 \varepsilon_1 \tau^{-\frac{T}{2}} e^{-\delta(s-\tau)} \| Z \|_{L_t^2}. \tag{4.96}
\]

Therefore, (4.90), (4.93), (4.92), (4.96) give (4.85) for \( s \in (0,1) \). Using the exponential decay of connection matrices and their one order derivatives in Lemma 4.1, one obtains (4.85) for \( s \in [1,\infty) \) by the same arguments above. For (4.86), applying \( H^\frac{1}{2} \) to (4.89) we have

\[
\| H^\frac{1}{2} Z(s) \|_{L_t^1 L_x^2} \leq \| H^\frac{1}{2} e^{-H(s-\tau)} Z(s) \|_{L_t^1 L_x^2} + \frac{3}{2} \int_0^s \| H^\frac{1}{2} e^{-H(s-\tau)} \tilde{G}_j(\tau) \|_{L_t^1 L_x^2} d\tau. \tag{4.97}
\]

And Proposition 3.7 and Lemma 4.1 give for \( s \in (0,1) \)

\[
\int_0^s \| H^\frac{1}{2} e^{-H(s-\tau)} \tilde{G}_1(\tau) \|_{L_t^1 L_x^2} d\tau \leq M_1 \varepsilon_1 \int_0^s (s - \tau)^{-\frac{T}{2}} \| \nabla Z(\tau) \|_{L_t^1 L_x^2} d\tau,
\]

\[
\int_0^s \| H^\frac{1}{2} e^{-H(s-\tau)} \tilde{G}_2(\tau) \|_{L_t^1 L_x^2} d\tau \leq \int_0^s (s - \tau)^{-\frac{T}{2}} \| \tilde{G}_1(\tau) \|_{L_t^1 L_x^2} d\tau,
\]

\[
\int_0^s \| H^\frac{1}{2} e^{-H(s-\tau)} \tilde{G}_3(\tau) \|_{L_t^1 L_x^2} d\tau \leq M_1 \varepsilon_1 \int_0^s (s - \tau)^{-\frac{T}{2}} \| \tilde{G}_1(\tau) \|_{L_t^1 L_x^2} d\tau.
\]

Thus (4.89) follows from (4.85), (3.171) and (3.172) when \( s \in (0,1) \). Similar arguments give (4.89) for \( s \in [0,\infty) \). Proposition 3.7, Lemma 5.15. The rest is to prove (4.87). Applying \( H \) to (4.89), we have by Proposition 3.7 that

\[
\| HZ \|_{L_t^1 L_x^2} \leq s^{-\frac{T}{2}} e^{-\delta(s-\tau)} \| \nabla Z(s) \|_{L_t^1 L_x^2} + \frac{3}{2} \int_0^s (s - \tau)^{-\frac{T}{2}} e^{-\delta(s-\tau)} \| \nabla \tilde{G}_3(\tau) \|_{L_t^1 L_x^2} d\tau. \tag{4.98}
\]

The same arguments as the proof of (4.60) with Proposition 3.7 give

\[
\sum_{j=1}^2 \omega(s) \int_0^s (s - \tau)^{-\frac{T}{2}} e^{-\delta(s-\tau)} \| \nabla \tilde{G}_3(\tau) \|_{L_t^1 L_x^2} d\tau \leq M_1 \varepsilon_1 \omega(s) \| \Delta Z \|_{L_t^\infty L_t^1 L_x^2} + M_1 \varepsilon_1. \tag{4.99}
\]

It remains to bound \( \tilde{G}_3 \). Since \( \nabla_i \partial_i \tilde{u} = \nabla_i \partial_i \tilde{u} \), we have \( \tilde{G}_3 = 3h^{ii}(\phi_i \wedge \phi_i)(\partial_i \phi_i + A_i \phi_i) \). Then the explicit formula for \( \Gamma^k_{ij} \) and \( h^{jk} \) yields

\[
| \nabla \tilde{G}_3(\tau) | \lesssim \sqrt{h^{pp}} \left| \left( (\partial_p h^{ii})(\phi_i \wedge \phi_i) + h^{ii}(\partial_p \phi_i) + h^{ii} \phi_i \wedge \partial_p \phi_i) (\partial_i \phi_i + A_i \phi_i) \right| 
+ \sqrt{h^{pp}} \left| h^{ii}(\phi_i \wedge \phi_i) \right| | \partial_p \phi_i | d\tau + (\partial_p A_i) \phi_i + A_i \partial_p \phi_i | 
\lesssim (|du| \nabla \phi_i + |\nabla du| \phi_i + |du| A |\phi_i|) (|\nabla \phi_i| + |A| \phi_i|)
\]

55
By (4.60), (4.54) and Lemma 4.1, the \( \tilde{G}_3 \) term in (4.98) is bounded by

\[
\int_{\frac{s}{2}}^{s} \| H e^{-H(s-\tau)} \tilde{G}_3(\tau) \|_{L_{1}^{1}L_{2}^{2}} \leq M_1 \epsilon_1 \int_{\frac{s}{2}}^{s} (s-\tau)^{-\frac{1}{2}} e^{-\delta(s-\tau)\tau^{-\frac{1}{2}-\epsilon}} d\tau, \quad \text{when } s \in [0, 1]
\]

\[
\int_{s}^{2s} \| H e^{-H(s-\tau)} \tilde{G}_3(\tau) \|_{L_{1}^{1}L_{2}^{2}} \leq M_1 \epsilon_1 \int_{s}^{2s} (s-\tau)^{-\frac{1}{2}} e^{-\delta(s-\tau)\tau^{-L}} d\tau, \quad \text{when } s \in [1, \infty)
\]

This combined with (4.99), Proposition 3.7 yields (4.87). (4.88) follows by (4.89), (4.87) and previously obtained bounds for \( \| \tilde{G}_i \|_{L_{1}^{1}L_{2}^{2}} \), \( i = 1, 2, 3 \).

\[\square\]

**Lemma 4.8.** Suppose that (4.4) to (4.7) hold, then for \( 0 < \gamma \ll 1 \)

\[
\left\| s^{-\frac{1}{2}+\epsilon} Z(s) \right\|_{L_{2}^{\infty}L_{1}^{1}L_{2}^{3+\gamma}} + \| \omega(\tau)Z(s) \|_{L_{2}^{\infty}L_{1}^{1}L_{2}^{3+\gamma}} \leq M_1 \epsilon_1^2
\]

(4.101)

\[
\| \omega(s)\partial_t \phi_i(s) \|_{L_{2}^{\infty}L_{1}^{1}L_{2}^{3+\gamma}} \leq M_1 \epsilon_1^2
\]

(4.102)

\[
\| \partial_t A_i(s) \|_{L_{2}^{\infty}L_{1}^{1}L_{2}^{3+\gamma}} \leq M_1 \epsilon_1^2
\]

(4.103)

\[
\| A_i(s) \|_{L_{2}^{\infty}L_{1}^{1}L_{2}^{\infty}} \leq M_1 \epsilon_1^2
\]

(4.104)

**Proof.** Applying Duhamel principle to (4.89), one obtains from Lemma 2.6 that

\[
\| Z(s) \|_{L_{1}^{1}L_{2}^{3+\gamma}} \leq \sum_{j=1}^{3} \int_{0}^{s} \| e^{-H(s-\tau)} \tilde{G}_i \|_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau \\
\leq \sum_{j=1}^{3} \int_{0}^{s} e^{-\delta(s-\tau)} \| \tilde{G}_j \|_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau.
\]

(4.105)

Lemma 4.1 and Proposition 3.6 give

\[
\int_{0}^{s} \| e^{-H(s-\tau)} \tilde{G}_1 \|_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau \\
\leq \int_{0}^{s} \| e^{-H(s-\tau)} H \tilde{H} H^{-\frac{1}{2}} (-\Delta)^{\frac{1}{2}} (-\Delta)^{-\frac{1}{2}} \tilde{G}_1 \|_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau \\
\leq \int_{0}^{s} (s-\tau)^{-\frac{1}{2}+\epsilon} e^{-\delta(s-\tau)} ||(-\Delta)^{-\frac{1}{2}} \tilde{G}_1 ||_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau.
\]

(4.106)

Due to the explicit expressions for \( h^{ij} \), we can write \( \tilde{G}_1 = 2h^{ii} A_i^i \partial_t Z \) in the form \( \tilde{G}_1 = 2\sqrt{h^{ii} \partial_t (\sqrt{h^{ii}} A_i^i \partial_t Z)} - 2(h^{ii} \partial_t A_i^i) Z \). Then (4.106), Lemma 4.1 and the boundedness of Riesz transform yield

\[
\int_{0}^{s} \| e^{-H(s-\tau)} \tilde{G}_1 \|_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau \leq \int_{0}^{s} M_1 \epsilon_1 (s-\tau)^{-\frac{1}{2}+\epsilon} \| Z \|_{L_{1}^{1}L_{2}^{3+\gamma}} d\tau.
\]

(4.107)
The $\tilde{G}_{2,3}$ are bounded by Lemma 4.1 (4.54) and (4.102):

$$\sum_{j=2,3} \int_0^S \|e^{-H(s-\tau)}\tilde{G}_j\|_{L_t^7L_x^{3+\gamma}} d\tau \leq M_1 \varepsilon_1 \int_0^S (s-\tau)^{-\frac{1}{2}-\epsilon} \|Z\|_{L_t^7L_x^{3+\gamma}} d\tau + M_1 \varepsilon_1 \int_0^S (s-\tau)^{-\frac{1}{2}-\epsilon} d\tau. \tag{4.108}$$

Thus we obtain (4.101) from (4.107), (4.108). By $Z = D_t \phi_t - \phi_s$, we get

$$\|\partial_t \phi_t(s)\|_{L_t^2L_x^{3+\gamma}} \leq \|\phi_s + A_t \phi_t + Z\|_{L_t^2L_x^{3+\gamma}}. \tag{4.109}$$

Hence (4.102) when $s \in [0,1]$ follows from (4.54), (4.102) and (4.7). (4.102) when $s \geq 1$ follows by the same arguments with (4.105) replaced by

$$\|Z(s)\|_{L_t^7L_x^{3+\gamma}} \leq e^{-s\frac{1}{2}} \left( \frac{S}{2} \right) \|Z\|_{L_t^7L_x^{3+\gamma}} + \sum_{j=1}^3 \int_{\frac{S}{2}}^S e^{-H(s-\tau)} \tilde{G}_j \|Z\|_{L_t^7L_x^{3+\gamma}} d\tau. \tag{4.103}$$

(4.103) follows from (4.102), (4.54), (4.7) and

$$\partial_t A_t = \int_s^\infty \partial_t \phi_t \wedge \phi_s ds' + \int_s^\infty \phi_t \wedge \partial_t \phi_s ds'. \tag{4.110}$$

(4.104) follows by Sobolev embedding, (4.54), (4.7).

**Proposition 4.1.** Suppose that (4.4) to (4.7) hold. Then we have for $p \in (2,6)$

$$\left\| \omega(s) D^{-1/2} \partial_t \phi_t \right\|_{L_x^p L_t^{\infty}} + \left\| \omega(s) D^{1/2} \phi_t \right\|_{L_x^p L_t^{\infty}} + \|\omega(s) \partial_t \phi_s\|_{L_x^p L_t^{\infty}} \leq \varepsilon_1^2 M_1. \tag{4.111}$$

**Proof.** By Lemma 2.8 and Proposition 3.1 we obtain for any $p \in (2,6)$

$$\omega(s) \|\partial_t \phi_s\|_{L_t^2L_x^6} + \omega(s) \|\nabla \phi_s\|_{L_t^\infty L_x^2} + \omega(s) \|\nabla^{1/2} \phi_s\|_{L_t^2L_x^6} \leq \omega(s) \|\partial_t \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|\nabla \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|\nabla \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|\nabla \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|\nabla \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|\nabla \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|\nabla \phi_s(0, s, x)\|_{L_x^2} + \omega(s) \|G\|_{L_x^1L_x^2}. \tag{4.112}$$

where we write the inhomogeneous term as $G$. First, the $\phi_s(0, s, x)$ term is bounded by Proposition 2.2 and (4.9)

$$\|\omega(s) \nabla_{x} \phi_s(0, s, x)\|_{L_x^2} \leq \|\omega(s) \nabla_{x} \partial_t U\|_{L_x^2} + \|\omega(s) \sqrt{H g} \partial_x U\|_{L_x^2} \leq \|(\nabla u_0, \nabla u_1)\|_{L_x^2} + M_1 \|\nabla u_0\|_{L_x^2} \leq M_0 + M_0^2 + M_1 \varepsilon_1 M_0. \tag{4.113}$$
where $U(s, x)$ is the heat flow initiated from $u_0$. Second, the three terms involved with $A_t$ are bounded by 

\[ \omega(s) \| A_t \partial_t \phi_s \|_{L^1_t L^2_x} \leq \| A_t \|_{L^1_t L^\infty_x} \omega(s) \| \partial_t \phi_s \|_{L^\infty_t L^2_x} \]

\[ \omega(s) \| A_t A_t \phi_s \|_{L^1_t L^2_x} \leq \| A_t \|_{L^1_t L^\infty_x} \| A_t \|_{L^\infty_t L^\infty_x} \omega(s) \| \phi_s \|_{L^1_t L^2_x} \]

\[ \omega(s) \| \partial_t A_t \phi_s \|_{L^1_t L^2_x} \leq \| \partial_t A_t \|_{L^2_t L^{3+\gamma}_x} \omega(s) \| \phi_s \|_{L^1_t L^2_x}, \]

where $\frac{1}{k} + \frac{1}{3+\gamma} = \frac{1}{2}$, and $k \in (2, 6)$. They are admissible by (4.4)-(4.7), (4.104) and (4.103). The $\partial_t u$ term is bounded by

\[ \omega(s) \| R(\partial_t u, \partial_s u)(\partial_t u) \|_{L^1_t L^2_x} \leq \| \partial_t u \|_{L^1_t L^{5+2\gamma}_x} \| \partial_t u \|_{L^\infty_t L^{5+2\gamma}_x} \omega(s) \| \phi_s \|_{L^1_t L^2_x}, \]

where $\frac{1}{k} + \frac{1}{3+\gamma} = \frac{1}{2}$, and $k \in (2, 6)$. This is acceptable due to (4.4)-(4.7). The $\partial_s Z$ term is bounded by (4.88). The $A_i^{con}$ terms should be dealt with separately. We present the estimates for these terms as a lemma.

**Lemma 4.9** (Continuation of Proof of Proposition 4.1). *Under the assumption of Proposition 4.1, we have*

\[ \omega(s) \| h^{ii} A_i^{con} \partial_i \phi_s \|_{L^1_t L^2_x} \leq \varepsilon_1 \omega(s) \| \rho^\sigma \nabla \phi_s \|_{L^1_t L^2_x} + M_1 \varepsilon_1^2 \]  

\[ \omega(s) \| h^{ii} A_i^{con} A_i^{\infty} \phi_s \|_{L^1_t L^2_x} \leq M_1 \varepsilon_1^2 \]  

\[ \omega(s) \| h^{ii} A_i^{con} A_i^{con} \phi_s \|_{L^1_t L^2_x} \leq M_1 \varepsilon_1^2 \]  

\[ \omega(s) \| h^{ii} \partial_i A_i^{con} \phi_s \|_{L^1_t L^2_x} \leq M_1 \varepsilon_1^2 \]  

\[ \omega(s) \| h^{ii} \Gamma_{ii}^{\kappa} A_k^{con} \phi_s \|_{L^1_t L^2_x} \leq M_1 \varepsilon_1^2. \]

**Proof.** **Step 1** Expanding $\phi_i$ as $\phi_i^\infty + \int_s^\infty \partial_s \phi_i ds'$ yields

\[ A_i^{con} = \int_s^\infty \phi_i \wedge \phi_s ds' = \int_s^\infty \left( \int_s^\infty \partial_s \phi_i(\tau) d\tau + \phi_i^\infty \right) \wedge \phi_s(s') ds'. \]

Thus we get

\[ \omega(s) \| h^{ii} A_i^{con} \partial_i \phi_s \|_{L^1_t L^2_x} \leq \omega(s) \left( \| h^{ii} \phi_i^\infty \wedge \int_s^\infty \phi_i(s') ds' \| \partial_i \phi_s \|_{L^1_t L^2_x} \right. \right. \]

\[ + \omega(s) \left( \| \int_s^\infty \phi_i(s') \wedge \int_s^\infty \partial_s \phi_i(\tau) d\tau \| ds' \right) h^{ii} \partial_i \phi_s \|_{L^1_t L^2_x} \]

\[ \Delta = II_1 + II_2 \]

58
The $II_1$ term is bounded by

\[
II_1 \lesssim \omega(s) \| \rho^\sigma \nabla \phi_s \|_{L_t^2 L_x^2} \left\| \int_s^\infty \rho^{-\sigma} \phi_{i\infty}^s \wedge \phi_s(s') ds' \right\|_{L_t^2 L_x^2} \\
\leq \omega(s) \| \rho^\sigma \nabla \phi_s \|_{L_t^2 L_x^2} \left\| \rho^{-\sigma} \phi_{i\infty}^s \int_s^\infty \| \phi_s(s') \|_{L_t^2 L_x^2} ds' \right\|
\lesssim \omega(s) \| \rho^\sigma \nabla \phi_s \|_{L_t^2 L_x^2} \left\| \rho^{-\sigma} \phi_{i\infty}^s \right\|_{L_t^2 L_x^2} \left\| a_\varepsilon(s) \| \nabla \phi_s(s) \|_{L_t^2 L_x^2} \right\|_{L_t^2 L_x^2},
\tag{4.120}
\]

where we have used the Sobolev embedding in the last line. Hence Lemma 4.6 gives an acceptable bound,

\[
II_1 \lesssim M_1 \varepsilon_1 \omega(s) \| \rho^\sigma \nabla \phi_s \|_{L_t^2 L_x^2}.
\]

The $II_2$ term is bounded by

\[
II_2 \lesssim \omega(s) \| \nabla \phi_s \|_{L_t^\infty L_x^2} \int_s^\infty \| \phi_s(s') \|_{L_t^2 L_x^2} \left( \int_s^{\infty} \| \nabla \phi_s(\tau) \|_{L_t^2 L_x^2} d\tau \right) ds'.
\]

Meanwhile, Sobolev embedding and Lemma 4.6 give for $\vartheta \in (\frac{7}{5}, \frac{3}{2})$

\[
\| \nabla \phi_s(\tau) \|_{L_t^2 L_x^\infty} = \left( \beta_{\frac{1}{\vartheta}, \varepsilon} \| \nabla \phi_s(\tau) \|_{L_t^2 L_x^2} \right)^{1-\varsigma} \left( \beta_{\frac{1}{\vartheta}, \varepsilon} \| D^\vartheta \phi_s(\tau) \|_{L_t^2 L_x^2} \right) \left( \beta_{\frac{1}{\vartheta}, \varepsilon} \right)^{-\varsigma} \left( \beta_{\frac{1}{\vartheta}, \varepsilon} \right)^{-\varsigma},
\tag{4.121}
\]

where $\varsigma = \frac{2}{5(\vartheta-1)}$. Similarly we deduce by Sobolev embedding $\| f \|_{L_t^\infty} \leq \| D^{\frac{3}{2}} f \|_{L_t^5}$ that

\[
\| \omega(s) \phi_s(\tau) \|_{L_t^2 L_x^\infty} \leq \varepsilon_1.
\tag{4.122}
\]

Therefore choosing $\vartheta$ slightly above $\frac{7}{5}$, we conclude from (4.121) and (4.122) that

\[
II_2 \leq \varepsilon_1^2 \omega(s) \| \nabla \phi_s \|_{L_t^\infty L_x^2}.
\tag{4.123}
\]

Lemma 4.6 together with (4.120), (4.123) yields (4.114).

**Step 2** Next we prove (4.113). Hölder yields

\[
\omega(s) \left\| h^{i} A^i A^\infty \phi_s \right\|_{L_t^1 L_x^2} \leq \left\| \sqrt{h^{i} A^i} \right\|_{L_t^2 L_x^{10/3}}^{\frac{10}{3}} \omega(s) \left\| \phi_s \right\|_{L_t^2 L_x^2}.
\]

Using the expression $A^i A^\infty = \int_s^\infty \phi_i \wedge \phi_s ds'$, we obtain

\[
\left\| \sqrt{h^{i} A^i} \right\|_{L_t^2 L_x^{10/3}} \lesssim \left\| \int_s^\infty \sqrt{h^{i} \phi_i \wedge \phi_s ds'} \right\|_{L_t^2 L_x^2} \lesssim \| d\bar{u} \|_{L_t^\infty L_x^{10}} \int_s^\infty \| \phi_s \|_{L_t^2 L_x^2} ds'\]

59
Finally we notice that (4.118) is a consequence of (4.124) and thus this is acceptable by Lemma 4.6 and interpolation between the constants in Proposition 2.2. Therefore Lemma 4.6 gives (4.115).

Step 3 Third, we verify (4.116). Hölder yields

\[
\omega(s)\| h^{ii} A_i^{con} A_i \phi_s \|_{L^1 L^2} \leq \| \sqrt{h^{ii}} A_i^{con} \|_{L^2 L^2} \| \sqrt{h^{ii}} A_i^{con} \|_{L^\infty L^\infty} \omega(s) \| \phi_s \|_{L^2 L^2}.
\]

The term \( \| \sqrt{h^{ii}} A_i^{con} \|_{L^2 L^2} \) has been estimated in (4.124). The \( \| \sqrt{h^{ii}} A_i^{con} \|_{L^\infty L^\infty} \) term is bounded by

\[
\| \sqrt{h^{ii}} A_i^{con} \|_{L^\infty L^\infty} \lesssim \| \int_s^\infty \| \delta \|_{L^\infty} \| \phi_s \|_{L^2 L^2} \mathrm{d}s' \|_{L^\infty}.
\]

This is acceptable by Proposition 2.2 and Lemma 5.4.

Step 4 Forth, we prove (4.117). Hölder yields

\[
\omega(s)\| h^{ii} (\partial_i A_i^{con}) \phi_s \|_{L^1 L^2} \leq \| h^{ii} \partial_i A_i^{con} \|_{L^2 L^2} \omega(s) \| \phi_s \|_{L^2 L^2}.
\]

The \( h^{ii} \partial_i A_i \) term is bounded by

\[
\| h^{ii} \partial_i A_i^{con} \|_{L^2 L^2} = \| \int_s^\infty h^{ii} \partial_i \phi_i \phi_s \mathrm{d}s' + \int_s^\infty h^{ii} \partial_i \phi_i \phi_s \mathrm{d}s' \|_{L^2 L^2}
\]

\[
\leq \| \int_s^\infty h^{ii} \partial_i \phi_i \|_{L^\infty L^\infty} \| \phi_s \|_{L^2 L^2} \mathrm{d}s' + \int_s^\infty \| \delta \|_{L^\infty} \| \nabla \phi_s \|_{L^2 L^2} \mathrm{d}s'
\]

\[
\leq \int_s^\infty \left( \| \nabla \delta \|_{L^\infty L^\infty} + \| h^{ii} A_i \phi_i \|_{L^\infty L^\infty} \right) \| \phi_s \|_{L^2 L^2} \mathrm{d}s' + \int_s^\infty \| \delta \|_{L^\infty L^\infty} \| \nabla \phi_s \|_{L^2 L^2} \mathrm{d}s'.
\]

Thus this is acceptable by Lemma 4.1 and interpolation between the \( \| \nabla \delta \|_{L^\infty} \) bound and the \( \| \nabla \delta \|_{L^2} \) bound in Proposition 2.2.

Step 5 Finally we notice that (4.118) is a consequence of (4.124) and

\[
\omega(s)\| h^{ii} A_i^{con} \|_{L^1 L^2} \leq \| A_i^{con} \|_{L^2 L^2} \| \phi_s \|_{L^2 L^2}.
\]

Lemma 4.6 and Proposition 4.1 yield

**Proposition 4.2.** Assume that the solution to (1.1) satisfies (4.2) to (4.3), then for any \( p \in (2,6), \theta \in [0, \frac{1}{2}), \theta_1 \in [0, \frac{3}{4}] \)

\[
\| \omega(s) \nabla \phi_s \|_{L^\infty L^2} + \| \omega(s) D^{1/2} \phi_s \|_{L^\infty L^2} \leq M_1 \varepsilon_1^2
\]
Lemma 4.10. Assume that the solution to (1.1) satisfies (4.3) and (4.5), then for any $p \in (2, 6 + 2\gamma)$

\[
\begin{align*}
\|D_t \phi_t(0, t, x)\|_{L_t^p L_x^q} &\leq \int_0^\infty \|D_t \phi_t\|_{L_t^p L_x^q} ds \\
&\leq \|D_t(\partial_t \phi_t)\|_{L_t^1 L_x^q} + \|D_t(A_t \phi_t)\|_{L_t^1 L_x^q}.
\end{align*}
\]  

(4.128)

Proof. Step 1. First we prove (4.127). Since $D_s \phi_t = D_t \phi_s$, $A_s = 0$, one has

\[
\|D_t \phi_t(0, t, x)\|_{L_t^p L_x^q} \leq \int_0^\infty \|D_t \partial_t \phi_t\|_{L_t^p L_x^q} ds \leq \|D_t \partial_t \phi_t\|_{L_t^1 L_x^q} + \|D_t(A_t \phi_t)\|_{L_t^1 L_x^q}.
\]

(4.129)

Step 2. We verify (4.125) in this step. (4.125) shows for $\vartheta \in (\frac{1}{12}, \frac{1}{2})$, $q \in (2, 6)$

\[
\|D^\vartheta \partial_t \phi_t\|_{L_t^1 L_x^q} \leq M_1 \varepsilon_1.
\]

(4.130)

Sobolev embedding yields

\[
\|D_t \partial_t \phi_t\|_{L_t^p L_x^q} \leq \|D_t \partial_t \phi_t\|_{L_t^{p-\eta}}
\]

where $\vartheta - \frac{1}{6} - \frac{1}{6+2\gamma} < 0 < \eta \ll 1, 0 < \gamma \ll 1$. Thus the first term in (4.128) is acceptable by (4.129) and (4.54). For the second term in (4.128), by Sobolev embedding

\[
\begin{align*}
\|D_t(A_t \phi_t)\|_{L_t^1 L_x^q} &\leq \|\nabla(A_t \phi_t)\|_{L_t^1 L_x^q} \\
&\leq \|\nabla \phi_t\|_{L_t^1 L_x^q} \|A_t\|_{L_t^p L_x^q} + \|\nabla A_t\|_{L_t^1 L_x^q} \|\phi_t\|_{L_t^\infty L_x^q}.
\end{align*}
\]

(4.131)

Meanwhile we have

\[
\begin{align*}
\|A_t\|_{L_t^p L_x^q} &\leq \int_0^\infty \|\phi_t\|_{L_t^p L_x^q} \|\phi_s\|_{L_t^\infty L_x^q} d\tau \\
&\leq M_1 \varepsilon_1^2 \min(1, s^{-L}) \\
\|\nabla A_t\|_{L_t^p L_x^q} &\leq \int_0^\infty \|\nabla \phi_t\|_{L_t^p L_x^q} \|\phi_s\|_{L_t^\infty L_x^q} d\tau \\
&\leq M_1 \varepsilon_1^2 \min(1, s^{-L})
\end{align*}
\]

Thus (4.75) and Lemma 3.1 imply that the second term in (4.128) is also acceptable, thus
proving (4.127).

**Step 3.1.** We prove (4.126) in this step. By Remark 3.2,

$$\phi_i(0, t, x) = \phi_i^\infty + \int_0^\infty \partial_s \phi_i ds'.$$

Since $|\tilde{u}| \leq \sqrt{h^{ii}}|\phi_i|$, $\|\sqrt{h^{ii}}\phi_i^\infty\|_{L^2} \leq \|dQ\|_{L^2} \leq M_0$, it suffices to prove for all $t, x \in [0, T] \times \mathbb{H}^2$

$$\int_0^\infty \|\sqrt{h^{ii}} \partial_s \phi_i\|_{L^2} ds' \leq M_1 \varepsilon_1.$$

This is acceptable by Proposition 4.2, Lemma 4.1 and $|\sqrt{h^{ii}} \partial_s \phi_i| \leq |\nabla \phi_s| + \sqrt{h^{ii}}|A_i||\phi_s|$. Hence we get

$$\|d\bar{u}\|_{L^2_2} \leq M_0. \tag{4.132}$$

**Step 3.2.** Recall the equation of $\phi_s$ evolving along the heat flow,

$$\partial_s \phi_s = h^{ij} D_i D_j \phi_s - h^{ij} \Gamma^{k}_{ij} D_k \phi_s + h^{ij} (\phi_s \wedge \phi_i) \phi_j, \tag{4.133}$$

$$\phi_s = h^{ij} D_i \phi_j - h^{ij} \Gamma^{k}_{ij} \phi_k. \tag{4.134}$$

Then we have by integration by parts,

$$\frac{d}{ds} \|\partial_s \bar{u}\|^2_{L^2_2} = \frac{d}{ds} \langle \phi_s, \phi_s \rangle = 2 \langle D_s \phi_s, \phi_s \rangle$$

$$= 2 h^{ii} \left( D_i D_i \phi_s - \Gamma^{k}_{ii} D_k \phi_s, \phi_s \right) + \langle h^{ij} (\phi_s \wedge \phi_i) \phi_j, \phi_s \rangle$$

$$= -2 h^{ii} \langle D_i \phi_s, D_i \phi_s \rangle + \langle h^{ij} (\phi_s \wedge \phi_i) \phi_j, \phi_s \rangle.$$

Hence we obtain by $\|\partial_s \bar{u}\|_{L^2} \leq e^{-\delta s}$,

$$\|\tau(\bar{u}(0, t, x))\|^2_{L^2_2} \leq 2 \int_0^\infty h^{ii} \langle D_i \phi_s, D_i \phi_s \rangle ds \leq 4 \int_0^\infty \langle \nabla \phi_s, \nabla \phi_s \rangle ds$$

$$+ 4 \int_0^\infty h^{ii} \langle A_i \phi_s, A_i \phi_s \rangle ds + 4 \int_0^\infty |\tilde{u}|^2 |\phi_s|^2 ds. \tag{4.135}$$

Notice that the nonnegative sectional curvature property of $N = \mathbb{H}^2$ with integration by parts yields

$$\|\nabla d\bar{u}\|^2_{L^2_2} \lesssim \|\tau(u)\|^2_{L^2_2} + \|d\bar{u}\|^2_{L^2_2}.$$

Thus (4.135) gives

$$\|\nabla d\bar{u}(0, t, x)\|^2_{L^2_2} \lesssim \int_0^\infty \langle \nabla \phi_s, \nabla \phi_s \rangle ds + \int_0^\infty |\tilde{u}|^2 |\phi_s|^2 ds.$$
Thus by Proposition 2.2, Proposition 4.2 and Lemma 4.1, we have

\[ + \int_0^\infty h^{ii} \langle A_t \phi_s, A_t \phi_s \rangle ds + \| d\vec{u}(0, t, x) \|_{L_x^2}^2. \] (4.136)

Since the \(|d\vec{u}|\) term has been estimated, by Proposition 4.2, Lemma 4.1 and (4.136),

\[ \| \nabla d\vec{u} \|_{L_t^\infty L_x^2([0, T] \times \mathbb{R}^2)} \leq M_0 + \varepsilon_1^2 M_1. \]

**Step 4.** We prove the desired estimates for \(|\nabla_t \vec{u}|\) of (4.126) in this step. Integration by parts yields,

\[ \frac{d}{ds} \| \nabla t \vec{u} \|_{L_x^2}^2 = \frac{d}{ds} h^{ii} \langle D_t \phi_t, D_t \phi_t \rangle = 2h^{ii} \langle D_s D_t \phi_t, D_t \phi_t \rangle \]

\[ = 2h^{ii} \langle D_t D_s \phi_s, D_t \phi_t \rangle + 2h^{ii} \langle (\phi_s \wedge \phi_t) \phi_t, D_t \phi_t \rangle \]

\[ = -2h^{ii} \langle D_t \phi_s, D_t D_s \phi_t \rangle + 2 \langle D_t \phi_s, D_t \phi_t \rangle + 2h^{ii} \langle (\phi_s \wedge \phi_t) \phi_t, D_t \phi_t \rangle \]

\[ = -2 \langle D_t \phi_s, h^{ii} D_t D_t \phi_t - h^{ii} \Gamma^k_{ii} D_k \phi_t \rangle + 2h^{ii} \langle (\phi_s \wedge \phi_t) \phi_t, D_t \phi_t \rangle. \]

Recall the parabolic equation satisfied by \(\phi_t\), then

\[ \frac{d}{ds} \| \nabla t \vec{u} \|_{L_x^2}^2 = -2 \langle D_t \phi_s, D_s \phi_t \rangle + 2h^{ii} \langle (\phi_s \wedge \phi_t) \phi_t, D_t \phi_t \rangle + 2h^{ii} \langle D_t \phi_s, (\phi_t \wedge \phi_t) \phi_t \rangle. \]

Consequently we obtain

\[ \| \nabla t \vec{u}(0, t, x) \|_{L_x^2}^2 \leq 4 \int_0^\infty \langle \partial_t \phi_s, \partial_t \phi_s \rangle ds' + 4 \int_0^\infty \langle A_t \phi_s, A_t \phi_s \rangle ds' \]

\[ + 2 \int_0^\infty \| \phi_s \|_{L_x^\infty}^2 \| d\vec{u} \|_{L_x^\infty} \| \nabla \partial_t \vec{u} \|_{L_x^2} ds' + 2 \int_0^\infty \| \partial_t \vec{u} \|_{L_x^2}^2 \| d\vec{u} \|_{L_x^\infty}^2 \| D_t \phi_s \|_{L_x^2} ds'. \]

Thus by Proposition 2.2, Proposition 4.2 and Lemma 4.1, we have

\[ \| \nabla t \vec{u}(0, t, x) \|_{L_x^2}^2 \leq \varepsilon_1^4 M_1. \]

Therefore, we have proved all estimates in (4.126) and (4.127).

**4.3 Proof of Theorem 1.1**

By Proposition 2.1 and Lemma 4.10, (4.2), (4.3), (1.1) has a global solution and \(\phi_s\) satisfies

\[ \| \phi_s \|_{L_t^p L_x^q} + \| \partial_t \phi_s \|_{L_t^p L_x^q} \leq C(s). \] (4.137)

Then Theorem 1.1 follows by the same arguments in [Section 8, 49].
5 Appendix A

Lemma 5.1. (23, 40) If \( f \in C_c^\infty(\mathbb{H}^2; \mathbb{R}) \), then for \( 1 < p < \infty, \ p \leq q \leq \infty, \ 0 < \theta < 1, \ 1 < r < 2, \ r \leq l < \infty, \ \alpha > 1 \) following inequalities hold

\[
\|f\|_{L^2} \lesssim \|\nabla f\|_{L^2} \quad (5.1)
\]
\[
\|f\|_{L^q} \lesssim \|\nabla f\|_{L^2}^\theta \|f\|_{L^p}^{1-\theta} \quad \text{when} \quad \frac{1}{p} - \frac{\theta}{2} = \frac{1}{q} \quad (5.2)
\]
\[
\|f\|_{L^l} \lesssim \|\nabla f\|_{L^r} \quad \text{when} \quad \frac{1}{r} - \frac{1}{2} = \frac{1}{l} \quad (5.3)
\]
\[
\|f\|_{L^\infty} \lesssim \left\|(-\Delta)^{\frac{\sigma_2}{2}}f\right\|_{L^2} \quad \text{when} \quad \alpha > 1 \quad (5.4)
\]
\[
\|\nabla f\|_{L^p} \sim \left\|(-\Delta)^{\frac{\sigma_1}{2}}f\right\|_{L^p}. \quad (5.5)
\]

We also recall a more generalized Sobolev inequality, see Proposition 2.2 of [3].

Lemma 5.2. Let \( 1 < p < q < \infty \) and \( \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \sigma_1 - \sigma_2 \geq n/p - n/q \geq 0 \). Then for all \( f \in C_c^\infty(\mathbb{H}^n; \mathbb{R}) \)

\[
\left\|(-\Delta)^{\frac{\sigma_2}{2}}f\right\|_{L^q} \lesssim \left\|(-\Delta)^{\frac{\sigma_1}{2}}f\right\|_{L^p}. \quad (5.6)
\]

The diamagnetic inequality which sometimes refers to Kato’s inequality as well was given in [46].

Lemma 5.3. If \( T \) is some \((r,s)\) type tension or tension matrix defined on \( \mathbb{H}^2 \), then in the distribution sense, one has the diamagnetic inequality

\[
|\nabla|T|| \leq |\nabla T|.
\]

Lemma 5.4. For \( f \in L^2 \) it holds that

\[
\int_0^\infty \|e^{-s\Delta}f\|_{L^2}^2 ds \lesssim \|f\|_{L^2}^2. \quad (5.6)
\]

Remark 5.1. Lemma 5.1 and Lemma 5.3 have several useful corollaries, for instance for \( f \in H^2 \),

\[
\|f\|_{L^\infty} \lesssim \|\nabla^2 f\|_{L^2} \quad (5.7)
\]
\[
\|f\|_{L^2} \lesssim \|\nabla^2 f\|_{L^2}. \quad (5.8)
\]

The intrinsic and extrinsic formulations are equivalent in the following sense.
Lemma 5.5. Suppose that $Q$ is an admissible harmonic map with $Q(H^2)$. If $u \in H^k_Q$ then for $k = 2, 3$

$$\|u\|_{H^k_Q} \sim \|u\|_{S^k},$$

(5.9) in the sense that there exist polynomials $P, Q$ such that

$$\|u\|_{H^k_Q} \leq P(\|u\|_{S^k}) C(R_0, \|u\|_{S^2})$$

(5.10)

$$\|u\|_{S^k} \leq Q(\|u\|_{H^k_Q}) C(R_0, \|u\|_{H^2_Q}).$$

(5.11)

Corollary 5.1. Suppose that $Q$ is an admissible harmonic map with $Q(H^2)$. If $u \in H^k_Q$ then for $k = 2, 3$, then $u(H^2)$ is compact in $N = H^2$.

Lemma 5.6. If $(u_0, u_1)$ with $u_0 : \mathbb{M} = H^2 \rightarrow N = H^2$, $u_1(x) \in T_{u_0(x)}N$ for any $x \in M$ is the initial data to (1.1) satisfying (1.2) and

$$\|\nabla du_0\|_{L^2_x} + \|du_0\|_{L^2_x} \leq M_0,$$

(5.12) then we have

$$\|\tau(u_0)\|_{L^2} \leq C(M_0) \mu_1,$$

(5.13) where $\tau(u_0)$ is the heat tension filed.

The estimate of the heat semigroup in $H^2$ is as follows.

Lemma 5.7. [11, 49, 46] The heat semigroup on $H^2$ denoted by $e^{t\Delta}$ satisfies the decay estimates

$$\|e^{s\Delta} f\|_{L^\infty_x} \lesssim e^{-\frac{4}{7}s^{-1}} \|f\|_{L^1_x},$$

(5.14)

$$\|e^{s\Delta} f\|_{L^2_x} \lesssim e^{-\frac{4}{7}} \|f\|_{L^2_x},$$

(5.15)

$$\|e^{s\Delta} f\|_{L^p_x} \lesssim \frac{1}{s^{p-1}} \|f\|_{L^p_x},$$

(5.16)

$$\|e^{s\Delta} (-\Delta)^{\frac{1}{2}} f\|_{L^2_x} \lesssim s^{-\frac{1}{2}} e^{-\frac{2}{7}} \|f\|_{L^2_x},$$

(5.17) where $1 \leq r \leq p \leq \infty$, $\alpha \in [0, 1]$, $1 < q < \infty$.

Lemma 5.8. If $(u, \partial_t u)$ solves (1.1) in $X_T$, then

$$\partial_s |\nabla \partial_s \bar{u}|^2 - \Delta |\nabla \partial_s \bar{u}|^2 + 2|\nabla \partial_s \bar{u}|^2 \lesssim |\nabla \partial_s \bar{u}|^2 |d\bar{u}|^2 + |\nabla \partial_t \bar{u}| |\partial_t \bar{u}| |\nabla \partial_s \bar{u}| |\partial_s \bar{u}| + |\nabla \partial_s \bar{u}| |\partial_t \bar{u}| |d\bar{u}| + |\nabla \partial_t \bar{u}| |\partial_s \bar{u}| |\nabla \partial_s \bar{u}| |d\bar{u}|,$$

(5.18)
and it holds
\[ \partial_s|\nabla \partial_s \tilde{u}|^2 - \Delta |\nabla \partial_s \tilde{u}|^2 + 2 |\nabla^2 \partial_s \tilde{u}|^2 \lesssim |\nabla \partial_s \tilde{u}|^2 |d\tilde{u}|^2 + |\nabla \partial_s \tilde{u}|^2 + |\nabla \partial_s \tilde{u}| |\partial_s \tilde{u}| |d\tilde{u}|^3 + |\nabla \partial_s \tilde{u}| |\partial_s \tilde{u}| |\nabla d\tilde{u}| |d\tilde{u}|. \]  
(5.19)

Moreover we have
\[ \partial_s|\nabla \partial_t \tilde{u}|^2 - \Delta |\nabla \partial_t \tilde{u}|^2 + 2 |\nabla^2 \partial_t \tilde{u}|^2 \lesssim |\nabla \partial_t \tilde{u}|^2 |d\tilde{u}|^2 + |\nabla \partial_t \tilde{u}|^2 + |\nabla \partial_t \tilde{u}| |\partial_t \tilde{u}| |d\tilde{u}|^5 + |\partial_t \tilde{u}| |d\tilde{u}|^2 |\nabla \partial_t \tilde{u}| + |\nabla \partial_t \tilde{u}| |\partial_t \tilde{u}| |\nabla d\tilde{u}| |d\tilde{u}|. \]  
(5.20)

**Lemma 5.9.** If \((u, \partial_t u)\) solves (1.1) in \(X_T\), then we have
\[ \partial_s|\nabla^2 d\tilde{u}|^2 - \Delta |\nabla^2 d\tilde{u}|^2 + 2 |\nabla^3 d\tilde{u}|^2 \lesssim |\nabla^2 d\tilde{u}|^2 |d\tilde{u}|^2 + |\nabla^2 d\tilde{u}|^2 |\nabla d\tilde{u}| \]  
\[ + |\nabla d\tilde{u}| |\nabla^2 d\tilde{u}| |\nabla d\tilde{u}| + |\nabla^2 d\tilde{u}|^2 |d\tilde{u}|^2 + |\nabla^2 d\tilde{u}|^2 |\nabla d\tilde{u}| |d\tilde{u}| \]  
\[ + |\nabla d\tilde{u}| |\nabla^2 d\tilde{u}|^2 \]  
(5.21)

and
\[ \partial_s|\nabla^2 \partial_s \tilde{u}|^2 - \Delta |\nabla^2 \partial_s \tilde{u}|^2 + 2 |\nabla^3 \partial_s \tilde{u}|^2 \lesssim |\nabla^2 \partial_s \tilde{u}|^2 |d\tilde{u}|^2 + |\partial_s \tilde{u}|^2 |\nabla^2 \partial_s \tilde{u}| |\nabla d\tilde{u}| \]  
\[ + |\partial_s \tilde{u}| |d\tilde{u}| |\nabla \partial_s \tilde{u}| |\nabla^2 \partial_s \tilde{u}| + |\nabla^2 \partial_s \tilde{u}|^2 |d\tilde{u}| |\partial_s \tilde{u}| + |\nabla \partial_s \tilde{u}| |\nabla^2 \partial_s \tilde{u}| |d\tilde{u}| |\nabla \tilde{u}| \]  
\[ + |\nabla d\tilde{u}| |\nabla^2 \partial_s \tilde{u}|^2 + |\nabla^2 d\tilde{u}| |d\tilde{u}| |\partial_s \tilde{u}| |\nabla^2 \partial_s \tilde{u}| \]  
(5.22)

**Lemma 5.10.** \((\mathbb{L})\) Let \(R_0(\tfrac{1}{2} + \sigma) = (-\Delta + \sigma^2 - \nu^2)^{-1}\) be the free resolvent in \(\mathbb{H}^{n+1}\) and denote its kernel by \(\left[n R\right]_0(\tfrac{n}{2} + \sigma, x, y)\). Then for \(\Re \sigma \geq 0, \ |\sigma| \geq 1, \ r \in (0, \infty), \) we have
\[ \left|\left[n R\right]_0(\tfrac{n}{2} + \sigma, x, y)\right| \leq \begin{cases}  C |\log r|, & |\sigma| \leq 1, n = 1 \\  C_n r^{1-n}, & |\sigma| \leq 1, n \geq 2 \\  C_n |\sigma|^{	frac{\nu}{2} - 1} e^{-(\tfrac{\nu}{2} + \Re \sigma) r}, & |\sigma| \geq 1 \end{cases} \]  
(5.23)

and for \(\Re \sigma \geq 0, \ |\sigma| \geq 1, \ r \in (0, \infty), \) any \(\epsilon \in (0,1)\), we have
\[ \left|\partial_s \left[n R\right]_0(\tfrac{n}{2} + \sigma, x, y)\right| \leq \begin{cases}  C |\log r|, & |\sigma| \leq 1, n = 1 \\  C_n r^{1-n}, & |\sigma| \leq 1, n \geq 2 \\  C_{n,\epsilon} |\sigma|^{	frac{\nu}{2} - 1} e^{-(\tfrac{\nu}{2} + \Re \sigma - \epsilon) r}, & |\sigma| \geq 1 \end{cases} \]  
(5.24)
Moreover, for $\Re \sigma \geq 0$, $|\sigma| \leq 1$, $r \in (0, \infty)$ we have

\[
\left| [^n R]_{0} \left( \frac{n}{2} + \sigma, x, y \right) \right| \leq \begin{cases} 
C |\log r|, & |r| \leq 1, \ n = 1 \\
C_n |r|^{1-n}, & |r| \leq 1, \ n \geq 2 \\
C_n |\sigma|^\frac{n}{2} e^{-\left(\frac{1}{2} + \Re \sigma\right)r}, & |r| \geq 1 
\end{cases} \tag{5.25}
\]

and for $\Re \sigma \geq 0$, $|\sigma| \leq 1$, $r \in (0, \infty)$

\[
\left| \partial_{\sigma} [^n R]_{0} \left( \frac{n}{2} + \sigma, x, y \right) \right| \leq \begin{cases} 
C |\log r|, & |r| \leq 1, \ n = 1 \\
C_n |r|^{1-n}, & |r| \leq 1, \ n \geq 2 \\
C_n, |\sigma|^\frac{n}{2} e^{-\left(\frac{1}{2} + \Re \sigma\right)r}, & |r| \geq 1 
\end{cases} \tag{5.26}
\]

**Lemma 5.11.** In the $\mathbb{H}^2$ case, for $\Re \sigma \geq 0$, $|\sigma| \geq 1$, $r \in (0, \infty)$, we have

\[
\left| \nabla_{x} R_{0} \left( \frac{1}{2} + \sigma, x, y \right) \right| \leq \begin{cases} 
C \Re^{-2} (\sinh r)^2 (\cosh^2 r - 1)^{-1/2}, & |r \sigma| \leq 1 \\
C |\sigma|^{1/2} e^{-\left(\frac{1}{2} + \Re \sigma\right)r} (\sinh r)^2 (\cosh^2 r - 1)^{-1/2}, & |r \sigma| \geq 1 
\end{cases} \tag{5.27}
\]

and for $\Re \sigma \geq 0$, $|\sigma| \leq 1$, $r \in (0, \infty)$ we have

\[
\left| \nabla_{x} R_{0} \left( \frac{1}{2} + \sigma, x, y \right) \right| \leq \begin{cases} 
C \Re^{-2} (\sinh r)^2 (\cosh^2 r - 1)^{-1/2}, & |r| \leq 1 \\
C |\sigma|^{1/2} e^{-\left(\frac{1}{2} + \Re \sigma\right)r} (\sinh r)^2 (\cosh^2 r - 1)^{-1/2}, & |r| \geq 1 
\end{cases} \tag{5.28}
\]

**Lemma 5.12.** If $v$ is a nonnegative function satisfying

\[
\partial_{t} v - \Delta v \leq 0,
\]

then for $t \geq 1$,

\[
v(x, t) \leq \int_{t-1}^{t} \int_{B(x, 1)} v(y, s) d\nu_{x} ds.
\]

This inequality is known in the heat flow literature and can be proved by Moser iteration.

The following is Lemma 5.1 of [4] whose proof bases on the Kunze-Stein phenomenon. We recall this for reader’s convenience.

**Lemma 5.13 ([4]).** There exists a constant $C > 0$ such that for any radial function $g$ on $\mathbb{H}^n$, for any $2 \leq q, p < \infty$ and $f \in L^{q'}(\mathbb{H}^n)$,

\[
\|f * g\|_{L^{q}} \leq C \|f\|_{L^{q'}} \left\{ \int_{0}^{\infty} (\sinh r)^{n-1} \varphi_0(r)^\mu |g(r)|^Q \right\}^{1/Q},
\]

where $\mu = \frac{2 \min (q, p)}{q + p}$ and $Q = \frac{qp}{q + p}$, and $|\varphi_0(r)| \leq C (1 + r)e^{-\nu r}$.
Lemma 5.14. For $1 \leq p < q \leq \infty$ and $1 + \frac{1}{q} - \frac{1}{p} > \frac{3}{4}$, we have

$$\|\nabla^2 f\|_{L^q_x} \leq \|\Delta f\|_{L^q_x}. \quad (5.29)$$

Proof. Fix $y \in \mathbb{H}^2$. Let $(r, \theta)$ be the polar coordinates with $y$ being the origin. Then (3.19) and (3.9) show

$$\partial_r^2 [1 R_0(\frac{1}{2} + \sigma, d(x,y))] = c_1 (\cosh^2 r - 1)^{-\frac{1}{2}} \sinh^3 r \cosh r [3 R_0(\frac{1}{2} + \sigma, r)$$

$$+ c_2 (\cosh^2 r - 1)^{-\frac{1}{2}} \sinh r \cosh r [3 R_0(\frac{1}{2} + \sigma, r)$$

$$+ c_3 (\cosh^2 r - 1)^{-1} \sinh^3 r \cosh r [5 R_0(\frac{1}{2} + \sigma, r)$$

$$+ c_4 (\cosh^2 r - 1)^{-1} \sinh^4 r [5 R_0(\frac{1}{2} + \sigma, r)$$

Thus when $\sigma = \frac{1}{2}$, Lemma 5.10 gives

$$|\partial_r^2 [1 R_0(1, d(x,y))]| \leq \begin{cases} C e^{-r}, & |r| \geq 2 \\ C r^{-\frac{3}{2}}, & |r| \leq 2 \end{cases} \quad (5.30)$$

Meanwhile, Lemma 5.11 yields

$$|\coth r \partial_r [1 R_0(1, d(x,y))]| \leq \begin{cases} C e^{-r}, & |r| \geq 1 \\ C r^{-\frac{3}{2}}, & |r| \leq 1 \end{cases} \quad (5.31)$$

Therefore, in the polar coordinates with $y$ being the origin, by (5.31) and (5.30), one has at $x \in \mathbb{H}^2$ that

$$|\nabla^2_x [1 R_0(1, d(x,y))]| \leq \begin{cases} C r^{-\frac{3}{2}}, & |r| \leq 1 \\ C e^{-r}, & |r| \geq 1 \end{cases} \quad (5.32)$$

Since the left and right hand side of (5.32) are free of coordinates charts, we obtain that (5.32) holds for all $(x, y) \in \mathbb{H}^2 \times \mathbb{H}^2$. Thus (5.29) follows by Young’s convolution inequality.

Lemma 5.15. Assume that (4.1) to (4.7) hold and the frame $\Xi$ in Proposition 2.3 is the Coulomb gauge. Let $A^\text{con}_i$ be defined in Remark 2.2. Then we have for the coordinates in (2.1) and any fixed

$$\sqrt{h^{\mu\nu}} h^\mu \partial_\mu \phi^\text{con}_i \leq \int_s^\infty |\nabla^2 \phi_s| + |\nabla \phi_s| ds' \quad (5.33)$$

$$\|A\|_{L^\infty_x} + \|\nabla A\|_{L^\infty_x} \leq C(M_0) \quad (5.34)$$
we obtain (5.34). By the definition of check for Proof.

\[
\int_\infty \sqrt{\int_\infty \partial_p \left( h^{ii} \partial_i A_i^{con} - h^{ii} \Gamma^k_{ii} A_k^{con} \right) \right)} \leq s^{-\frac{1}{2}} e^{-\delta s} M_1 \varepsilon_1 + \int_s^\infty |du| \nabla^2 \partial_s \bar{u} |ds'.
\]

(5.36)

**Proof.** (5.33) follows by direct calculations and (2.2). It suffices to prove (5.36). It is easy to check for \( p \in (2, \infty) \) \( \| A \|_{L^p} + \| \nabla A \|_{L^\infty} \leq C(M_0) \) as Lemma 3.7. Then by \( A = A + du \) and (5.29) we obtain (5.34). By the definition of \( A_i^{con} \), one obtains

\[
\sqrt{h^{pp}} \partial_p \left( h^{ii} \partial_i A_i^{con} - h^{ii} \Gamma^k_{ii} A_k^{con} \right)
= \sqrt{h^{pp}} h^{ii} \partial_p \partial_i A_i^{con} - \sqrt{h^{pp}} \partial_p (h^{ii} \Gamma^k_{ii}) A_k^{con} + \sqrt{h^{pp}} (\partial_p h^{ii}) \partial_i A_i^{con} - \sqrt{h^{pp}} h^{ii} \Gamma^k_{ii} \partial_p A_k^{con}
\]

(5.37)

\[
= \int_s^\infty \left( \sqrt{h^{pp}} (\partial_p h^{ii}) \partial_i \phi_i + \sqrt{h^{pp}} h^{ii} \partial_p \partial_i \phi_i - \sqrt{h^{pp}} h^{ii} \Gamma^k_{ii} \partial_p \phi_k \right) \wedge \phi_s |ds'.
\]

(5.38)

\[
+ \int_s^\infty \phi_i \wedge \left( \sqrt{h^{pp}} (\partial_p h^{ii}) \partial_i \phi_i + \sqrt{h^{pp}} h^{ii} \partial_p \partial_i \phi_i \right) ds'
\]

\[
+ \sqrt{h^{pp}} h^{ii} \int_s^\infty \partial_i \phi_i \wedge \partial_s \phi_s ds' + + \sqrt{h^{pp}} h^{ii} \int_s^\infty \partial_i \phi_i \wedge \partial_s \phi_s ds'.
\]

The second order derivative term in (5.38) can be expanded as

\[
\sqrt{\partial_p h^{ii} \partial_p \partial_i \phi_i}
= \sqrt{h^{pp}} h^{ii} \partial_p \partial_i \phi_i
\]

\[
= \sqrt{h^{pp}} h^{ii} \partial_p \partial_i \langle \partial_i \bar{u}, \Omega_j \rangle
\]

\[
= \sqrt{h^{pp}} h^{ii} \langle \nabla_p \nabla_i \partial_i \bar{u}, \Omega_j \rangle + \sqrt{h^{pp}} h^{ii} \langle \nabla_i \partial_i \bar{u}, \nabla_p \Omega_j \rangle
\]

(5.39)

\[
+ \sqrt{h^{pp}} h^{ii} \langle \nabla_p \partial_i \bar{u}, \Omega_k \rangle + \sqrt{h^{pp}} h^{ii} \langle \partial_i \bar{u}, \Omega_k \rangle
\]

Thus the third derivative term in (5.39) combined with the other two terms in (5.38) reduces to

\[
\sqrt{h^{pp}} h^{ii} \langle \nabla_p \nabla_i \partial_i \bar{u}, \Omega_j \rangle + \sqrt{h^{pp}} (\partial_p h^{ii}) \partial_i \langle \partial_i \bar{u}, \Omega_j \rangle - \sqrt{h^{pp}} \partial_p (h^{ii} \Gamma^k_{ii}) \langle \partial_k \bar{u}, \Omega_j \rangle
\]

\[
= \sqrt{h^{pp}} h^{ii} \langle \nabla_p \nabla_i \partial_i \bar{u}, \Omega_j \rangle + \sqrt{h^{pp}} (\partial_p h^{ii}) \langle \nabla_i \partial_i \bar{u}, \Omega_j \rangle + \sqrt{h^{pp}} (\partial_p h^{ii}) \langle \partial_i \bar{u}, \nabla_i \Omega_j \rangle
\]

(5.40)

\[
- \sqrt{h^{pp}} \partial_p (h^{ii} \Gamma^k_{ii}) \langle \partial_k \bar{u}, \Omega_j \rangle
\]

\[
= \sqrt{h^{pp}} \langle \nabla_p (h^{ii} \nabla_i \partial_i \bar{u} - h^{ii} \Gamma^k_{ii} \partial_k \bar{u}), \Omega_j \rangle + \sqrt{h^{pp}} (\partial_p h^{ii}) \langle \partial_i \bar{u}, \nabla_i \Omega_j \rangle
\]

\[
= \sqrt{h^{pp}} \langle \nabla_p \partial_s \bar{u}, \Omega_j \rangle + \sqrt{h^{pp}} (\partial_p h^{ii}) \langle \partial_i \bar{u}, \nabla_i \Omega_j \rangle.
\]
And we have
\[
\sqrt{h} \partial_p h^{ii} \partial_i \phi_s = \sqrt{h} \partial_p \langle \nabla_p \partial_i \bar{u}, \Omega_j \rangle + \sqrt{h} \partial_i \langle \nabla_i \partial_s \bar{u}, \nabla_p \Omega_j \rangle + \sqrt{h} \partial_i \langle \partial_s \bar{u}, \nabla_i \Omega_j \rangle + \sqrt{h} \partial_i \langle \partial_s \bar{u}, [A_i]_k^j \Omega_k \rangle + \sqrt{h} \partial_i \langle \partial_s \bar{u}, [A_i]_k^j \nabla_p \Omega_k \rangle.
\]
Hence we conclude by explicit formula in (2.2)
\[
\left| \sqrt{h} \partial_p \left( h^{ii} \partial_i A^\text{con}_i - h^{ii} \Gamma^k_{ii} A^\text{con}_k \right) \right| \leq \int_0^1 \left( |\nabla \partial_s u| + |\nabla du||\partial_s u| + |\nabla du| |A| + |du| |A|^2 + |du| (|\nabla A| + |A|) \right) ds' + \int_0^1 \left( |\nabla^2 \partial_s u| |du| + |\nabla \partial_s u| |du| |A| + |\partial_s u| |du| (|\nabla A| + |A|) + |\partial_s u| |du| |A|^2 \right) ds'.
\]
Then (5.36) follows by Lemma 4.1.

References

[1] S. Agmon. Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 2(4), 151-218, 1975.

[2] L. Andersson, N. Gudapati, J. Szeftel. Global regularity for the 2+1 dimensional equivariant Einstein-wave map system. ArXiv e-prints, 2015.

[3] J.P. Anker and V. Pierfelice. Wave and Klein-Gordon equations on hyperbolic spaces. Anal. PDE, 7(4), 953-995, 2014.

[4] J.P. Anker, V. Pierfelice and M. Vallarino. The wave equation on hyperbolic spaces. J. Differential Equations, 252(10), 5613-5661, 2012.

[5] D. Borthwick and J.L. Marzuola. Dispersive Estimates for Scalar and Matrix Schrödinger Operators on $H^{n+1}$. Math. Phys. Anal. Geom. 18(1), 22, 2015.

[6] F. Cardoso, C. Cuevas, G. Vodev. High frequency resolvent estimates for perturbations by large long-range magnetic potentials and applications to dispersive estimates. Ann. Henri Poincaré, 1(14), 95-117, 2013.

[7] E. Chiodaroli, J. Krieger and J. Luhrmann. Concentration Compactness for Critical Radial Wave Maps. arXiv preprint arXiv:1611.08557, 2016.

[8] D. Christodoulou and A.S. Tahvildar-Zadeh. On the asymptotic behavior of spherically symmetric wave maps. Duke Math. J., 71(1), 31-69, 1993.
[9] D. Christodoulou and A.S. Tahvildar-Zadeh. *On the regularity of spherically symmetric wave maps.* Comm. Pure Appl. Math., 46(7), 1041-1091, 1993.

[10] R. Cote. *On the soliton resolution for equivariant wave maps to the sphere.* Comm. Pure Appl. Math., 68(11), 1946-2004 (2015) Corrigendum: *On the Soliton Resolution for Equivariant Wave Maps to the Sphere.* Comm. Pure Appl. Math., 69(4), 609-612, 2016.

[11] E.B. Davies and N. Mandouvalos. *Heat kernel bounds on hyperbolic space and Kleinian groups.* Proceedings of the London Mathematical Society, 3(1), 182-208, 1988.

[12] T. Duyckaerts, C. Kenig, F. Merle. *Classification of radial solutions of the focusing, energy-critical wave equation.* Camb. J. Math., 1(1), 75-144, 2013.

[13] T. Duyckaerts, H. Jia, C. Kenig and F. Merle. *Universality of blow up profile for small blow up solutions to the energy critical wave map equation.* arXiv preprint [arXiv:1612.04927], 2016.

[14] P. D’Ancona and L. Fanelli. *Strichartz and smoothing estimates for dispersive equations with magnetic potentials.* Comm. Partial Differential Equations, 33(6), 1082-1112, 2008.

[15] P. D’Ancona, L. Fanelli, L. Vega and N. Visciglia. *Endpoint Strichartz estimates for the magnetic Schrödinger equation.* J. Funct. Anal., 258(10), 3227-3240, 2010.

[16] P. D’Ancona, V. Georgiev. *On the continuity of the solution operator to the wave map system.* Comm. Pure Appl. Math., 57(3), 357-383, 2004.

[17] P. D’Ancona, Q. Zhang. *Global existence of small equivariant wave maps on rotationally symmetric manifolds.* Int. Math. Res. Not. IMRN, 2016(4), 978-1025, 2015.

[18] H. Donnelly. *Eigenvalues embedded in the continuum for negatively curved manifolds.* Michigan Math. J., 28(1), 53-62, 1981.

[19] M.B. Erdogan, M. Goldberg, W. Schlag. *Strichartz and smoothing estimates for Schrödinger operators with large magnetic potentials in R³.* J. Eur. Math. Soc. (JEMS), 10(2), 507-531, 2008.

[20] C. Rodriguez. *Soliton resolution for equivariant wave maps on a wormhole: I-II.* arXiv preprint.

[21] M. Gell-Mann and M. L'evy. *The axial vector current in beta decay.* Nuovo Cimento (10), 16, 705-726, 1960.

[22] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces.* Academic press, 1979.
[23] S. Huang, C.D. Sogge. Concerning $L^p$ resolvent estimates for simply connected manifolds of constant curvature. J. Funct. Anal., 267(12), 4635-4666, 2014.

[24] A. D. Ionescu and S. Klainerman. On the global stability of the wave-map equation in kerr spaces with small angular momentum. ArXiv e-prints, 1412.5679, 12 2014.

[25] A. Ionescu, B. Pausader and G. Staffilani. On the global well-posedness of energy-critical Schrödinger equations in curved spaces. Anal. PDE, 5(4): 705-746, 2012.

[26] A.D. Ionescu, W. Schlag. Agmon-Kato-Kuroda theorems for a large class of perturbations. Duke Math. J., 131(3), 397-440, 2006.

[27] H. Jia and C. Kenig. Asymptotic decomposition for semilinear wave and equivariant wave map equations. Preprint, 03. 2015.

[28] W. Jager, H. Kaul, Uniqueness and stability of harmonic maps and their Jacobi fields. Manuscripta Mathematica, 28(1-3): 269-291, 1979.

[29] K. Kaizuka. Resolvent estimates on symmetric spaces of noncompact type. J. Math. Society of Japan, 66(3): 895-926, 2014.

[30] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Contributions to Functional Analysis. Springer Berlin Heidelberg, 1966: 258-279.

[31] C. Kenig, A. Lawrie, Schlag W., Relaxation of wave maps exterior to a ball to harmonic maps for all data. Geom. Funct. Anal., 24(2), 610-647, 2014.

[32] C. Kenig, A. Lawrie, B.P. Liu, W. Schlag. Stable soliton resolution for exterior wave maps in all equivariance classes. Adv. Math., 285, 235-300, 2015.

[33] S. Klainerman and M. Machedon. Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math., 46(9), 1221-1268, 1993.

[34] S. Klainerman and M. Machedon. Smoothing estimates for null forms and applications. Duke Math. J., 81(1), 99-133, 1995.

[35] S. Klainerman and I. Rodnianski. On the global regularity of wave maps in the critical Sobolev norm. Int. Math. Res. Not. IMRN, 2001(13), 655-677, 2001.

[36] S. Klainerman and S. Selberg. Remark on the optimal regularity for equations of wave maps type. Comm. Partial Differential Equations, 22(5-6):901-918, 1997.

[37] S. Klainerman and S. Selberg, Bilinear estimates and applications to nonlinear wave equations. Commun. Contemp. Math., 4(2):223-295, 2002.
[38] J. Krieger. Global regularity of wave maps from $R^{3+1}$ to surfaces. Comm. Math. Phys., 238(1-2), 333-366, 2003.

[39] J. Krieger. Global regularity of wave maps from $R^{2+1}$ to $H^2$. Small energy. Comm. Math. Phys., 250(3), 507-580, 2004.

[40] J. Krieger and W. Schlag. Concentration Compactness for critical wave maps. EMS Monographs. European Mathematical Society, Zürich, 2012.

[41] J. Krieger, W. Schlag and D. Tataru. Renormalization and blow up for charge one equivariant wave critical wave maps. Invent. Math., 171(3), 543-615, 2008.

[42] A. Lawrie. The Cauchy problem for wave maps on a curved background. Calculus of Variations and Partial Differential Equations, 45(3-4): 505-548, 2012.

[43] A. Lawrie, S.J. Oh and S. Shahshahani. Profile decompositions for wave equations on hyperbolic space with applications. Math. Ann., 365(1-2): 707-803, 2016.

[44] A. Lawrie, S.J. Oh and S. Shahshahani. Gap eigenvalues and asymptotic dynamics of geometric wave equations on hyperbolic space. J. Funct. Anal., 271 (11), 3111-3161, 2016.

[45] A. Lawrie, S.J. Oh, and S. Shahshahani. Equivariant wave maps on the hyperbolic plane with large energy. ArXiv e-prints, 2015.

[46] A. Lawrie, S.J. Oh and S. Shahshahani. The Cauchy problem for wave maps on hyperbolic space in dimensions $d \geq 4$. Int. Math. Res. Not. IMRN 2016rnw272.

[47] A. Lawrie, S.J. Oh, and S. Shahshahani. Stability of stationary equivariant wave maps from the hyperbolic plane. ArXiv e-prints, February 2014.

[48] P. Li and L. Tam. The heat equation and harmonic maps of complete manifolds. Invent. Math., 105(1), 1-46, 1991.

[49] Ze Li, Xiao Ma, Lifeng Zhao, Asymptotic stability of small energy harmonic maps under the wave map on 2D hyperbolic space, arXiv preprint.

[50] J. Luk. The null condition and global existence for nonlinear wave equations on slowly rotating kerr spacetimes. J. Eur. Math. Soc. (JEMS), 15(5):1629-1700, 2013.

[51] N.S. Manton and P. Sutcliffe, Topological Solitons, Cambridge University Press, 2004.

[52] A. Nahmod, A. Stefanov, and K. Uhlenbeck. On the well-posedness of the wave map problem in high dimensions. Comm. Anal. Geom., 11(1):49-83, 2003.
[53] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer Science and Business Media, 2012.

[54] P. Raphael and I. Rodnianski. *Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems*. Publ. Math. Inst. Hautes Etudes Sci., 1-122, 2012.

[55] I. Rodnianski and J. Sterbenz. *On the formation of singularities in the critical O(3) σ-model*. Ann. Math., 172:187-242, 2010.

[56] J. Shatah and M. Struwe. *On the Cauchy problem for wave maps*. Int. Math. Res. Not., 2002(11), 555-571, 2002.

[57] J. Shatah and A.S. Tahvildar-Zadeh. *On the Cauchy problem for equivariant wave maps*. Comm. Pure Appl. Math., 47(5), 719-754, 1994.

[58] J. Sterbenz and D. Tataru. *Energy dispersed large data wave maps in 2+1 dimensions*. Comm. Math. Phys., (1):139-230, 2010.

[59] J. Sterbenz and D. Tataru. *Regularity of wave maps in 2+1 dimensions*. Comm. Math. Phys., (1):231-264, 2010.

[60] M. Struwe. *Equivariant wave maps in two space dimensions*. Comm. Pure Appl. Math., 56(7):815-823, 2003.

[61] T. Tao. *Ill-posedness for one-dimensional wave maps at the critical regularity*. Amer. J. Math., 122(3): 451-463, 2000.

[62] T. Tao. *Global regularity of wave maps, I: small critical Sobolev norm in high dimension*. Int. Math. Res. Not. IMRN, 2001(6): 299-328, 2001.

[63] T. Tao. *Global Regularity of Wave Maps II. Small Energy in Two Dimensions*. Comm. Math. Phys., 224(2): 443-544, 2001.

[64] T. Tao. *Global regularity of wave maps. III-VII*. arXiv preprint [arXiv:0805.1666], 2008.

[65] T. Tao. *Geometric renormalization of large energy wave maps*. Journees equations aux derives partielles, 1-32, 2004.

[66] D. Tataru. *On global existence and scattering for the wave maps equation*. Amer. J. Math., 123(1):37-77, 2001.

[67] D. Tataru. *Rough solutions for the wave maps equation*. Amer. J. Math., 127(2), 293-377, 2005.
[68] K. Uhlenbeck. *Connections with $L^p$ bounds on curvature*. Comm. Math. Phys., **83**(1), 31-42, 1982.

Ze Li. rikudosennin@163.com
Academy of Mathematics and Systems Science (AMSS)
Chinese Academy of Sciences (CAS)
Beijing 100190, P. R. China