ADJOINTS OF ELLIPTIC CONE OPERATORS

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ABSTRACT. We study the adjointness problem for the closed extensions of a general $b$-elliptic operator $A \in x^{-\nu}\text{Diff}_b^\infty(M;E)$, $\nu > 0$, initially defined as an unbounded operator $A : C_c^\infty(M;E) \subseteq x^\mu L_2(M;E) \to x^{\mu-2}L_2^2(M;E)$, $\mu \in \mathbb{R}$. The case where $A$ is a symmetric semibounded operator is of particular interest, and we give a complete description of the domain of the Friedrichs extension of such an operator.

1. INTRODUCTION

Let $M_0$ be a smooth paracompact manifold, $m$ a smooth positive measure on $M_0$. Suppose $A : C_c^\infty(M_0) \to C_c^\infty(M_0)$ is a scalar linear partial differential operator with smooth coefficients. Among all possible domains $D \subset L^2(M_0,m)$ for $A$ as an unbounded operator on $L^2(M_0)$ there are two that stand out:

$$D_{\text{max}}(A) = \{ u \in L^2(M_0,m) \mid Au \in L^2(M_0,m) \}$$

where $Au$ is computed in the distributional sense, and

$$D_{\text{min}}(A) = \text{completion of } C_c^\infty(M_0) \text{ with respect to the norm } \|u\| + \|Au\|,$$

which can be regarded as a subspace of $L^2(M_0,m)$. Both domains are dense in $L^2(M_0,m)$, since $C_c^\infty(M_0) \subset D_{\text{min}}(A) \subset D_{\text{max}}(A)$, and with each domain $A$ is a closed operator. Clearly $D_{\text{min}}(A)$ is the smallest domain containing $C_c^\infty(M_0)$ with respect to which $A$ is closed, and $D_{\text{max}}(A)$ contains any domain on which the action of the operator coincides with the action of $A$ in the distributional sense. Also clearly, $A$ with domain $D_{\text{max}}(A)$ is an extension of $A$ with domain $D_{\text{min}}(A)$.

If $M_0$ is compact without boundary and $A$ is elliptic then $D_{\text{max}}(A) = D_{\text{min}}(A)$; the interesting situations occur when $A$ is non-elliptic or $M_0$ is noncompact. In this paper we shall analyze the latter problem, assuming that $M_0$ is the interior of a smooth compact manifold $M$ with boundary and that $A \in x^{-\nu}\text{Diff}_b^\infty(M;E)$, $\nu > 0$, is a $b$-elliptic ‘cone’ operator acting on sections of a smooth vector bundle $E \to M$; here $x : M \to \mathbb{R}$ is a smooth defining function for $\partial M$, positive in $M_0$.

The elements of $\text{Diff}_b^\infty(M;E)$ are the totally characteristic differential operators introduced and analyzed systematically by Melrose [8]. These are linear operators with smooth coefficients which near the boundary can be written in local coordinates $(x,y_1,\ldots,y_n)$ as $P = \sum_{k+|\alpha| \leq m} a_{k\alpha}(x) D_x^k D_y^\alpha$. Such an operator is $b$-elliptic if it is elliptic in the interior in the usual sense and in addition $\sum_{k+|\alpha|=m} a_{k\alpha}(0,y) \xi^k \eta^\alpha$ is invertible for $(\xi,\eta) \neq 0$; this is expressed more concisely by saying that the principal symbol of $P$, as an object on the compressed cotangent bundle (see Melrose, op.cit.), is an isomorphism. It follows from the definition of $b$-ellipticity that the family of differential operators on $\partial M$ given locally by $P_0(\sigma) = \sum_{j+|\alpha| \leq m} a_{\alpha,j}(0,y) \sigma^j D_y^\alpha$, called the indicial operator or conormal symbol



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of $P$, is elliptic of order $m$ for any $\sigma \in \mathbb{C}$. For the general theory of these operators and the associated pseudodifferential calculus the reader is referred to the paper of Melrose cited above, his book \[9\], as well as Schulze \[1\] \[12\]. An operator $A \in x^{-\nu} \text{Diff}^m_0(M; E)$ is $b$-elliptic if $P = x^\nu A$ is $b$-elliptic. By definition the conormal symbol of $A$ is that of $P$. For more details see Section \[2\].

The measure used to define the $L^2$ spaces will be of the form $x^\mu \mathbf{m}$ for some real $\mu$, where $\mathbf{m}$ is a $b$-density, that is, $x\mathbf{m}$ is a smooth positive density. The spaces $L^2(M; E; x^\mu \mathbf{m})$ are defined in the usual way with the aid of a smooth but otherwise arbitrary hermitian metric on $E$. These spaces are related among themselves by canonical isometries with which $L^2(M; E; x^\mu \mathbf{m}) = x^{-\nu/2} L^2_0(M, E)$, where $L^2_0(M, E)$ is the space defined by the measure $\mathbf{m}$ itself.

This said, the general problem we are concerned with is the description of the adjoints of the closed extensions of a general $b$-elliptic operator $A \in x^{-\nu} \text{Diff}^m_0(M; E)$ initially defined as an unbounded operator

\begin{equation}
A : C^\infty_c(M; E) \subset x^\mu L^2_0(M; E) \rightarrow x^\mu L^2_0(M; E).
\end{equation}

The case where $A$ is a symmetric semibounded operator is of particular interest, and we give, in Theorem \[8.12\], a complete description of the domain of the Friedrichs extension of such an operator.

Differential operators in $x^{-\nu} \text{Diff}^m_0(M; E)$ arise in the study of manifolds with conical singularities. The study of such manifolds from the geometric point of view began with Cheeger \[3\], and by now there is an extensive literature on the subject. In the specific context of our work, probably the most relevant references, aside from those cited above, are the book by Lesch \[7\], and the papers by Brüning and Seeley \[1\] \[8\] \[9\], and Mooers \[11\]. See also Coriasco, Schrohe, and Seiler \[13\].

As already mentioned, the domains $\mathcal{D}_{\text{min}}(A)$ and $\mathcal{D}_{\text{max}}(A)$ need not be the same. The object determining the closed extensions of $A$ is its conormal symbol $\hat{P}_0(\sigma) : C^\infty(\partial M) \rightarrow C^\infty(\partial M)$. Because of the $b$-ellipticity, this operator is invertible for all $\sigma \in \mathbb{C}$ except a discrete set $\text{spec}_b(A)$, the boundary spectrum of $A$ (or $P$), a set which (again due to the $b$-ellipticity) intersects any strip $a < \Im \sigma < b$ in a finite set.

It was noted by Lesch \[3\] that $\mathcal{D}_{\text{min}}(A) = \mathcal{D}_{\text{max}}(A)$ if and only if $\text{spec}_b(A) \cap \{-\mu - \nu < \Im \sigma < -\mu\} = \emptyset$. Also proved by Lesch [op.cit., Proposition 1.3.16] was the fact that $\mathcal{D}_{\text{max}}(A)/\mathcal{D}_{\text{min}}(A)$ is finite dimensional. This provides a simple description of the closed extensions of $A$: they are given by the operator $A$ acting in the distributional sense on subspaces $\mathcal{D} \subset x^\nu L^2_0(M; E)$ with $\mathcal{D}_{\text{min}}(A) \subset \mathcal{D} \subset \mathcal{D}_{\text{max}}(A)$. These results and others, also due to Lesch (op.cit.) are reproved in Section \[8\] for the sake of completeness, with a slightly different approach emphasizing the use of the pseudodifferential calculus, for totally characteristic operators \[1\] \[14\], or for the cone algebra \[1\] \[14\]. In that section we also prove a relative index theorem for the closed extensions of $A$.

From the point of view of closed extensions there is nothing more to understand than that they are in one to one correspondence with the subspaces of $\mathcal{E}(A) = \mathcal{D}_{\text{max}}(A)/\mathcal{D}_{\text{min}}(A)$. The problem of finding the domain of the adjoint is more delicate, and forces us to pass to the Mellin transforms of representatives of elements of $\mathcal{E}$. Our approach entails rewriting the pairing $[u, v]_A = (Au, v) - (u, A^* v)$, defined for $u \in \mathcal{D}_{\text{max}}(A)$ and $v \in \mathcal{D}_{\text{max}}(A^*)$, where $A^*$ is the formal adjoint of $A$, in terms of the Mellin transforms of $u$ and $v$, and proving certain specific nondegeneracy properties of the pairing. It is generally true (and well known) that in a general abstract setting, $[\cdot, \cdot]_A$ induces a nonsingular pairing of $\mathcal{E}(A)$ and $\mathcal{E}(A^*)$. In the
case at hand, there is an essentially well defined notion of an element $u \in D_{\text{max}}(A)$ with pole ‘only’ at $\sigma_0 \in \text{spec}_b(A) \cap \{-\mu - \nu < 3\sigma < -\mu\}$. If $\sigma_0 \in \text{spec}_b(A)$ then $\sigma_0 - i(\nu + 2\mu) \in \text{spec}_b(A^*)$, and we show, for example, that the restriction of the pairing to elements $u \in D_{\text{max}}(A)$ with pole ‘only’ at $\sigma_0$ and elements $v$ in $D_{\text{max}}(A^*)$ with pole ‘only’ at $\sigma_0 - i(\nu + 2\mu)$ is nonsingular (modulo the respective minimal domains).

Our analysis of the pairing begins in Section 6 with a careful description of the Mellin transforms of elements in $D_{\text{max}}(A)$. The main result in that section is Proposition 5.9, a result along the lines of part of the work of Golberg and Sigal [4]. In Section 6 we prove in a somewhat abstract setting that the pairing alluded to above for solutions with poles at conjugate points is nonsingular (Theorem 6.4). In Section 7 we link the pairing $\llbracket \cdot, \cdot \rrbracket_A$ with the pairing of Section 6. The main results are Theorems 7.11 and 7.17. The first of these gives a formula for the pairing which in particular shows that the pairing is null when the poles in question are not conjugate, and the second, which is based on the formula in Theorem 7.11 and Theorem 6.4, shows that the pairing of elements associated to conjugate poles is nonsingular. The formulas are explicit enough that in simple cases it is easy to determine the domain of the adjoint of a given extension of $A$.

We undertake the study of the domain of the Friedrichs extension of $b$-elliptic semibounded operators in Section 8. The main result there is Theorem 8.12 which is a complete description of the domain of the Friedrichs extension. Loosely speaking, the domain consists of the sum of those elements $u \in D_{\text{max}}(A)$ with poles ‘only’ in $-\mu - \nu < 3\sigma < -\mu - \nu/2$ and those with pole ‘only’ at $3\sigma = -\mu - \nu/2$ and ‘half’ of the order of the pole. Finally, in Section 8 we collect a number of examples that illustrate the use of Theorems 7.11, 7.17, and 8.12.

Any closed extension of a $b$-elliptic operator $A \in x^{-\nu} \text{Diff}_b^m(M; E)$ as an operator $\mathcal{D} \subset x^{m} L^2_b(M; E) \to x^{m} L^2_b(M; E)$ has the space $x^{m+\nu} H^m_b(M; E)$ in its domain. The space $H^m_b(M; E)$ is the subspace of $L^2_b(M; E)$ whose elements $u$ are such that for any smooth vector fields $V_1, \ldots, V_k$, $k \leq m$, on $M$ tangent to the boundary, $V_1 \cdots V_k u \in L^2_b(M; E)$. Other than this, the set of domains of the closed extensions of different $b$-elliptic operators in $x^{-\nu} \text{Diff}_b^m(M; E)$ are generally not equal. In Section 8 we provide simple sufficient conditions for the domains of two such operators to be the same. Not unexpectedly, these conditions are on the equality of Taylor expansion at the boundary of the operators involved, up to an order depending on $\nu$. We prove in particular that under the appropriate condition the domains of the Friedrichs extensions of two different symmetric semibounded operators are the same. This is used in Section 8 as an intermediate step to determine the domain of the Friedrichs extension of such an operator, and a refinement of the condition is obtained as a consequence.

Operators of the kind we investigate arise naturally as geometric operators. In such applications $\mu$ is determined by the actual situation. It is convenient for us, however, to work with the normalization

$$x^{-\mu-\nu/2} Ax^{\mu+\nu/2} : C_c^\infty(M; E) \subset x^{-\nu/2} L^2_b(M; E) \to x^{-\nu/2} L^2_b(M; E).$$

rather than (6.11). Since the mappings $x^s : x^{m} L^2_b(M; E) \to x^{m+s} L^2_b(M; E)$ are surjective isometries, this represents no loss. In particular, adjoints and symmetry properties of operators are preserved. Also to be noted is that these transformations represent translations on the Mellin transform side, so it is a simple exercise to
recast information presented in terms of Mellin transforms of the modified operator as information on the original operator.

2. GEOMETRIC PRELIMINARIES

Throughout the paper $M$ is a compact manifold with (nonempty) boundary with a fixed positive $b$-density $m$, that is, a smooth density $m$ such that for some (hence any) defining function $x$, $x m$ is a smooth positive density. We will also fix a hermitian vector bundle $E \to M$.

Fix a collar neighborhood $U_Y$ for each boundary component $Y$ of $M$, so we have a trivial fiber bundle $\pi_Y : U_Y \to Y$ with fiber $[0,1)$. We can then canonically identify the bundle of 1-densities over $U_Y$ with $\left| \bigwedge \right| [0,1) \otimes \left| \bigwedge \right| Y$ (the tensor product of the pullback to $[0,1) \times Y$ of the respective density bundles).

Let $m$ be a $b$-density on $\left| \bigwedge \right| [0,1) \otimes \left| \bigwedge \right| Y$. Then there is a smooth defining function $x : Y \times [0,1) \to \mathbb{R}$ vanishing at $Y \times 0$, and a smooth density $m_Y$ on $Y$, such that $m = dx \otimes m_Y$.

Indeed, let $\xi$ be the variable in $[0,1)$. Over the boundary of $[0,1) \times Y$ we then get, canonically, $\xi m = d\xi \otimes m_Y$. Then, on $[0,1) \times Y$, $m = h \frac{d\xi}{\xi} \otimes m_Y$ with $h$ smooth, positive, $h(0,y) = 1$. Let $x = g \xi$ where $g$ is determined modulo constant factor by the requirement that $\frac{dx}{x} = h \frac{d\xi}{\xi}$. Then, $g(\xi,y)$ should satisfy the equation

$$\frac{\partial g}{\partial \xi} + \frac{1}{\xi} h g = 0$$

Since $h = 1$ when $\xi = 0$, the solutions $g$ are smooth across $\xi = 0$. Pick the one which is 1 when $\xi = 0$.

We fix the choice of $x$ for each boundary component. When working near the boundary we will always assume that the defining function was chosen above, and that the coordinates, if at all necessary, are consistent with a choice of product structure as above. By $\partial_x$ we mean the vector field tangent to the fibers of $U_Y \to Y$ such that $\partial_x x = 1$.

If $E, F \to M$ are (smooth) vector bundles and $P \in \text{Diff}^m_b(M; E, F)$ is a $b$-elliptic differential operator, then $E$ and $F$ are isomorphic. This follows from the fact that the principal symbol of $P$ is an isomorphism $\pi^* E \to \pi^* F$ where $\pi : \mathcal{b}T^* M \setminus 0 \to M$ is the projection, and the fact that the compressed cotangent bundle $\mathcal{b}T^* M$ admits a global nonvanishing section (since it is isomorphic to $T^* M$ and $M$ is a manifold with boundary). Thus when analyzing $b$-elliptic operators in $\text{Diff}^m_b(M; E, F)$ we may assume $F = E$ (for more on this see [3]).

The Hilbert space structure of the space of sections of $E \to M$ is the usual one, namely integration with respect to $m$ of the pointwise inner product in $E$:

$$(u, v)_{L^2_b(M; E)} = \int (u, v)_E \, m \quad \text{if } u, v \in L^2_b(M; E).$$
Fix a hermitian connection $\nabla$ on $E$. If $P \in \text{Diff}^m_0(M; E)$, then near a boundary component one can write

$$P = \sum_{\ell=0}^{m} P_\ell \circ (\nabla_{xD_x})^\ell$$

where the $P_\ell$ are differential operators of order $m - \ell$ (defined on $U_Y$) such that for any smooth function $\phi(x)$ and section $u$ of $E$ over $U_Y$, $P_\ell(\phi(x)u) = \phi(x)P_\ell(u)$, in other words, of order zero in $\nabla_{xD_x}$. $P$ is said to have coefficients independent of $x$ near $Y$ if $\nabla_{\partial_x} P_k(u) = P_k(\nabla_{\partial_x} u)$ for any smooth section $u$ of $E$ supported in $U_Y$. By means of parallel transport along the fibers of $U_Y \to Y$ one can show that if $P \in \text{Diff}^m_0(M; E)$, then for any $N$ there are operators $P_k, \tilde{P}_N \in \text{Diff}^m_0(M; E)$ such that

$$P = \sum_{k=0}^{N} P_k x^k + \tilde{P}_N x^N$$

where $P_k$ has coefficients independent of $x$ near $Y$. If $P_k$ has coefficients independent of $x$ near $Y$ then so does its formal adjoint $P_k^*$. To see this recall that since the connection is hermitian, $\partial_x (u, v)_E = (\nabla_{\partial_x} u, v) + (u, \nabla_{\partial_x} v)$ if $u$ and $v$ are supported near $Y$, so if they vanish on $Y$ then

$$(\nabla_{\partial_x} u, v)_{L^2(M; E)} = -(u, \nabla_{\partial_x} v)_{L^2(M; E)}.$$  

One derives the assertion easily from this.

Fix $\omega \in C^\infty_c(-1, 1)$ real valued, nonnegative and such that $\omega = 1$ in a neighborhood of 0. The Mellin transform of a section of $C^\infty_c(\hat{M}; E)$ is defined to be the entire function $u : \mathbb{C} \to C^\infty(Y; E)$ such that for any $v \in C^\infty(Y; E|_Y)$

$$\omega \phi(u) = \frac{1}{2\pi} \int_{3\sigma=0} (\hat{u}(\sigma, y), v(y))_{L^2(Y; E|_Y)} d\sigma$$

By $\pi_Y^* v$ we mean the section of $E$ over $U_Y$ obtained by parallel transport of $v$ along the fibers of $\pi_Y$. Thus if $u \in C^\infty(U_Y; E)$ is such that $\nabla_{\partial_x} u = 0$ and $\phi \in C^\infty(0, 1)$, then

$$\hat{\phi} u(\sigma) = \hat{\omega} \hat{\phi}(\sigma) u$$

where $\hat{\phi} \omega(\sigma)$ is the “usual” Mellin transform of $\omega \phi$. The only point here is that we incorporate the cut-off function into the definition. As is well known, the Mellin transform extends to the spaces $x^\mu L^2(M, E)$ in such a way that if $u \in x^\mu L^2(M, E)$ then $\hat{u}(\sigma)$ is holomorphic in $\{3\sigma > -\mu\}$ and in $L^2(\{3\sigma = -\mu\} \times Y)$ with respect to $d\sigma \otimes \frac{m_Y}{\sigma}$.

The conormal symbol $\tilde{P}_0$ of $P \in \text{Diff}^m_0(M)$ is the operator valued polynomial defined by

$$\tilde{P}_0(\sigma)(u) = x^{-i\sigma} P(x^{i\sigma} \pi_Y^* u)|_Y, \quad u \in C^\infty(Y; E_Y), \sigma \in \mathbb{C}.$$  

It is easy to prove that $\tilde{P}_0^*(\sigma) = (\tilde{P}_0(\sigma))^*$.

3. Closed Extensions

Recall first the abstract situation (cf. [2]), where $A : D_{\text{max}} \subset H \to H$ is a densely defined closed operator in a Hilbert space $H$. Thus $D_{\text{max}}$ is complete with the graph norm $\| \cdot \|_A$ induced by the inner product $(u, v) + (Au, Av)$, and if $D \subset D_{\text{max}}$ is a subspace, then $A : D \subset H \to H$ is a closed operator if and only if $D$
is closed with respect to $\| \cdot \|_A$. Fix $D_{\text{min}} \subset D_{\text{max}}$, suppose $D_{\text{min}}$ is dense in $H$ and closed with respect to $\| \cdot \|_A$. Let

$$\mathcal{D} = \{D \subset D_{\text{max}} \mid D_{\text{min}} \subset D \text{ and } D \text{ is closed w.r.t. } \| \cdot \|_A\}$$

Thus $\mathcal{D}$ is in one to one correspondence with the set of closed operators $A : D \subset H \rightarrow H$ such that $D_{\text{min}} \subset D \subset D_{\text{max}}$. For our purposes the following restatement is more appropriate.

**Proposition 3.2.** The set $\mathcal{D}$ is in one to one correspondence with the set of closed subspaces of the quotient

$$\mathcal{E} = D_{\text{max}}/D_{\text{min}}$$

In our concrete case $A \in x^{-\nu} \text{Diff}^m_b(M; E)$, $\nu > 0$, is a $b$-elliptic cone operator, considered initially as a densely defined unbounded operator

$$A : C^\infty_c(M; E) \subset x^{-\nu/2}L^2_b(M; E) \rightarrow x^{-\nu/2}L^2_b(M; E).$$

We take $D_{\text{min}}(A)$ as the closure of $\mathcal{B}$ with respect to the graph norm, and

$$D_{\text{max}}(A) = \{u \in x^{-\nu/2}L^2_b(M; E) \mid Au \in x^{-\nu/2}L^2_b(M; E)\},$$

which is also the domain of the Hilbert space adjoint of $A^* : D_{\text{min}}(A^*) \subset x^{-\nu/2}L^2_b(M; E) \rightarrow x^{-\nu/2}L^2_b(M; E)$.

Thus $D_{\text{max}}$ is the largest subspace of $x^{-\nu/2}L^2_b(M; E)$ on which $A$ acts in the distributional sense and produces an element of $x^{-\nu/2}L^2_b(M; E)$; one can define $A$ on any subspace of $D_{\text{max}}$ by restriction. These definitions have nothing to do with ellipticity. The following almost tautological lemma is based on the continuity of $A : x^\nu H^m_b(M; E) \rightarrow x^{-\nu/2}L^2_b(M; E)$ and the fact that $C^\infty_c(M; E)$ is dense in $x^\nu H^m_b(M; E)$.

**Lemma 3.5.** Suppose $A \in x^{-\nu/2}L^2_b(M; E)$, let $D \subset D_{\text{max}}(A)$ be such that $A : D \subset x^{-\nu/2}L^2_b(M; E) \rightarrow x^{-\nu/2}L^2_b(M; E)$ is closed. If $D$ contains $x^\nu H^m_b(M; E)$ then $D$ contains $D_{\text{min}}(A)$. In particular, $x^\nu H^m_b(M; E) \subset D_{\text{min}}(A)$.

Adding the $b$-ellipticity of $A$ as a hypothesis provides the following precise characterization of $D_{\text{min}}(A)$:

**Proposition 3.6.** If $A \in x^{-\nu/2}L^2_b(M; E)$ is $b$-elliptic, then

1. $D_{\text{min}}(A) = D_{\text{max}}(A) \cap \left(\bigcap_{\varepsilon > 0} x^{\nu/2 - \varepsilon} H^m_b(M; E)\right)$
2. $D_{\text{min}}(A) = x^{\nu/2}H^m_b(M; E)$ if and only if $\text{spec}_b(A) \cap \{3\sigma = -\nu/2\} = \emptyset$.

The proof requires a number of ingredients, beginning with the following fundamental results [10, 11].

**Theorem 3.7.** Let $A \in x^{-\nu} \text{Diff}^m_b(M; E)$ be $b$-elliptic. For every real $s$ and $\gamma$,

$$A : x^\gamma H^s_b(M; E) \rightarrow x^{\gamma - \nu}H^{s - m}_b(M; E)$$

is Fredholm if and only if $\text{spec}_b(A) \cap \{3\sigma = -\gamma\} = \emptyset$. In this case, one can find a bounded pseudodifferential parametrix

$$Q : x^{\gamma - \nu}H^{s - m}_b(M; E) \rightarrow x^\gamma H^s_b(M; E)$$

such that $R = QA - 1$ and $\bar{R} = AQ - 1$ are smoothing cone operators.
Note that if $A$ is $b$-elliptic we always can find an operator $Q$ such that
\[ QA - 1 : x^\gamma H^s_b(M; E) \to x^\gamma H^{\infty}_b(M; E) \quad \text{and} \quad AQ - 1 : x^{\gamma - \nu} H^s_b(M; E) \to x^{\gamma - \nu} H^{\infty}_b(M; E) \]
are bounded for every $s \in \mathbb{R}$, even if the boundary spectrum intersects the line \( \{ \Im \sigma = -\gamma \} \). Also in this case \( \ker A \) and \( \ker A^* \) are finite dimensional spaces. Moreover, for every $u \in x^\gamma H^s_b(M; E)$,
\[ \| u \|_{x^\gamma H^s_b} \leq \|(QA - 1)u\|_{x^\gamma H^s_b} + \|Au\|_{x^\gamma H^s_b}, \]
which implies
\[ (3.8) \quad \| u \|_{x^\gamma H^s_b} \leq C_{s, \gamma} \left( \| u \|_{x^\gamma H^s_b} + \|Au\|_{x^{\gamma - \nu} H^{s - m}_b} \right) \]
for some constant $C_{s, \gamma} > 0$.

As noted above, there is a bounded operator $Q : x^{\gamma - \nu} H^{s - m}_b \to x^{\gamma} H^s_b$ such that $R = QA - 1 : x^{\gamma} H^{s - m}_b \to x^{\gamma} H^{\infty}_b$ is bounded. Hence for $u \in x^{\gamma} H^s_b$
\[ \| u \|_{x^{\gamma} H^s_b} \leq \| Ru\|_{x^{\gamma} H^s_b} + \|QAu\|_{x^{\gamma} H^s_b}, \]
which implies the estimate (3.8).

**Lemma 3.9.** There exists $\varepsilon > 0$ such that
\[ \mathcal{D}_{\max}(A) \hookrightarrow x^{-\nu / 2 + \varepsilon} H^m_b(M; E). \]

**Proof.** The inclusion follows from (3.8) and the fact that, if $u \in \mathcal{D}_{\max}(A)$ then $\hat{u}$ has no poles on $\{ \Im \sigma = \nu / 2 \}$. Choose $\varepsilon > 0$ smaller than the distance between \( \text{spec}_b(A) \cap \{ \Im \sigma < \nu / 2 \} \) and the line $\{ \Im \sigma = \nu / 2 \}$. The continuity of the embedding is a consequence of the closed graph theorem since $\mathcal{D}_{\max}(A)$ and $x^{-\nu / 2 + \varepsilon} H^m_b$ are both continuously embedded in $x^{-\nu / 2} L^2_b$. \( \square \)

Recall that $x^{-\nu / 2 + \varepsilon} H^m_b(M; E)$ is compactly embedded in $x^{-\nu / 2} L^2_b(M; E)$. Hence if $A$ with domain $\mathcal{D}$ is closed, then
\[ (\mathcal{D}, \| \cdot \|_A) \hookrightarrow x^{-\nu / 2} L^2_b(M; E) \quad \text{compactly}. \]
That this embedding is compact is a fundamental difference between the situation at hand and b-elliptic totally characteristic operators and is due to the presence of the factor $x^{-\nu}$ in $A$.

**Lemma 3.11.** Let $\gamma \in [-\nu / 2, \nu / 2]$ be such that $\text{spec}_b(A) \cap \{ \Im \sigma = -\gamma \} = \emptyset$. Then $A$ with domain $\mathcal{D}_{\max}(A) \cap x^\gamma H^m_b(M; E)$ is a closed operator on $x^{-\nu / 2} L^2_b(M; E)$.

**Proof.** Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}_{\max}(A) \cap x^\gamma H^m_b$ with $u_n \to u$ and $Au_n \to f$ in $x^{-\nu / 2} L^2_b$. Further, let $Q$ be a parametrix of $A$ as in Theorem 3.7. In particular,
\[ QA = 1 + R : x^\gamma H^m_b(M; E) \to x^\gamma H^m_b(M; E) \]
is Fredholm.

So, $Au_n \to f$ implies $(1 + R)u_n \to Qf$ in $x^\gamma H^m_b$, and $Qf = (1 + R)\tilde{u}$ for some $\tilde{u} \in x^\gamma H^m_b$. Now, since $\dim \ker(1 + R) < \infty$, there is a closed subspace $H \subset x^{-\nu / 2} L^2_b$ such that $x^{-\nu / 2} L^2_b(M; E) = H \oplus \ker(1 + R)$. If $\pi_H$ denotes the orthogonal projection onto $H$, then $\pi_H u_n \to \pi_H u$ in $x^{-\nu / 2} L^2_b$ and, as above,
\[ (1 + R)\pi_H u_n \to (1 + R)\pi_H \tilde{u} \quad \text{in } x^\gamma H^m_b. \]
Thus $\pi_H u_n \to \pi_H \tilde{u}$ in $x^\gamma H^m_b \hookrightarrow x^{-\nu / 2} L^2_b$ which implies that $\pi_H u = \pi_H \tilde{u} \in x^\gamma H^m_b$. 

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Then pair \( \epsilon > \) Indeed, suppose \( u \). Likewise, if \( v \). Let \( n \) for \( A \). Thus, we have \( D \subset D^* \). Let \( D^* \) and \( E^* \) be the analogues of (3.1) and (3.3) for \( A^* \). Define
\[
[s,v]_A : D^* \cap D^* \to \mathbb{C}
\]
by
\[
[s,v]_A = (Au,v) - (u,A^*v).
\]
Then \( [s,v]_A = 0 \) if either \( u \in D \) or \( v \in D^* \), and \([s,v]_A \) induces a nondegenerate pairing
\[
[s,v]_A : E \times E^* \to \mathbb{C}.
\]
Indeed, suppose \( u \in D^* \) is such that \( [s,v]_A = 0 \) for all \( v \in D^* \). Then \( (Au,v) = (u,A^*v) \) for all \( u,v \in D^* \), which implies that \( u \) belongs to the domain of the adjoint of \( A^* : D^* \subset H \to H \), that is, \( u \in D \). Thus the class of \( u \) in \( E \) is zero. Likewise, if \( v \in D^* \) and \( u,v = 0 \) for all \( u \in D^* \) then \( v \in D^* \).

Given \( D \in \mathcal{D} \), let \( D^\perp \subset D^* \) be the orthogonal of \( D \) with respect to \([s,v]_A \).

**Proposition 3.13.** Let \( D \in \mathcal{D} \). The adjoint \( A^* : D \subset H \to H \) is precisely the operator \( A^\perp \) restricted to \( D^\perp \). Consequently, \( A^\perp \) is selfadjoint if and only if \( D = D^\perp \).

**Proof.** Let \( A^* : D^* \subset H \to H \) be the adjoint of \( A|_D \). We have \( (Au,v) = (u,A^*v) \) for all \( u \in D \). Thus \( D^\perp \subset D^* \) and \( A^*v = A^*v \) if \( v \in D^\perp \). On the other hand, since \( D^* \subset D^* \) (because \( D \subset D^* \)), it makes sense to compute \([s,v]_A \) for \( u \in D \) and \( v \in D^* \). For such \( u, v \) we then have \([s,v]_A = 0 \) since \( A^*v = A^*v \) for every \( v \in D^* \). Thus \( D^* \subset D^\perp \) which completes the proof that \( D^* = D^\perp \).

**Proof of Proposition 3.6.** To prove part \([i] \) we will show inclusion in both directions. If \( \epsilon > 0 \) is such that \( \text{spec}_\epsilon(A) \cap \{ \delta \sigma = -\nu/2 + \epsilon \} = \emptyset \), then
\[
x^{\nu/2}H^m_b(M,E) \subset D^max \cap x^{\nu/2-\epsilon}H^m_b(M,E),
\]
so from Lemmas 3.11 and 3.3 we deduce \( D_{min}(A) \subset D_{max}(A) \cap x^{\nu/2-\epsilon}H^m_b(M,E) \). Thus,
\[
D_{min}(A) \subset D_{max}(A) \cap \bigcap_{\epsilon > 0} x^{\nu/2-\epsilon}H^m_b(M,E).
\]
To prove the reverse inclusion let \( u \in D_{max}(A) \) and set \( u_n = x^{1/n}u \) for \( n \in \mathbb{N} \). Then \( \{u_n\}_{n\in\mathbb{N}} \) is a sequence in \( x^{\nu/2}H^m_b \) and as \( n \to \infty \)
\[
u_n \to u \text{ in } x^{\nu/2}H^m_b, \text{ and } Au_n \to Au \text{ in } x^{-\nu/2-\epsilon}H^m_b
\]
for every \( \epsilon > 0 \). In particular, \( x^\nu Au_n \to x^\nu Au \) in \( x^{-\nu/2}\mathcal{L}^2_b \). Choose \( \epsilon \) sufficiently small such that \( D_{max}(A^*) \subset x^{-\nu/2+\epsilon}H^m_b \) (Lemma 3.9). Then for \( v \in D_{max}(A^*) \)
\[
(Au_n,v) = (x^\nu Au_n,x^{-\epsilon}v) \to (x^\nu Au,x^{-\epsilon}v) = (Au,v) \text{ as } n \to \infty.
\]
On the other hand, \( (u_n,A^*v) \to (u,A^*v) \) since \( u_n \in D_{min}(A) \). Hence \( (Au,v) = (u,A^*v) \) for all \( v \in D_{max}(A^*) \), that is, \([s,v]_A = 0 \) for all \( v \in D_{max}(A^*) \) which implies \( u \in D_{min}(A) \) since \( D_{min}(A) = D_{max}(A^*)^\perp \).

On the other hand, \( (u_n,A^*v) \to (u,A^*v) \) and \( (Au_n,v) = (u_n,A^*v) \) since \( u_n \in D_{min}(A) \). Hence \( (Au,v) = (u,A^*v) \) for all \( v \in D_{max}(A^*) \), that is, \([s,v]_A = 0 \) for all \( v \in D_{max}(A^*) \) which implies \( u \in D_{min}(A) \) since \( D_{min}(A) = D_{max}(A^*)^\perp \).
To prove part 3, suppose first that $\text{spec}_b(A) \cap \{\Im \sigma = -\nu/2\} = \emptyset$. Then Lemma 3.11 gives that $A$ with domain $\mathcal{D}_{\max}(A) \cap x^{\nu/2}H^m_b(M;E)$ is closed. Since $x^{\nu/2}H^m_b(M;E) \subset \mathcal{D}_{\max}(A)$, Lemma 3.3 implies $\mathcal{D}_{\min}(A) = x^{\nu/2}H^m_b(M;E)$. On the other hand, if $\mathcal{D}_{\min}(A) = x^{\nu/2}H^m_b(M;E)$, then $A$ with domain $x^{\nu/2}H^m_b(M;E)$ is Fredholm, so by Theorem 3.7, $\text{spec}_b(A) \cap \{\Im \sigma = -\nu/2\} = \emptyset$.

We will now prove that $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ is finite dimensional. This is a consequence of the following proposition, which is interesting on its own, see Lesch [7, Lemma 1.3.15, Prop. 1.3.16].

**Proposition 3.14.** If $A \in x^{-\nu}\text{Diff}^m_b(M;E)$ is $b$-elliptic, every closed extension

$$A : \mathcal{D} \subset x^{-\nu/2}L^2_b(M;E) \to x^{-\nu/2}L^2_b(M;E)$$

is a Fredholm operator. Moreover, $\dim \mathcal{D}(A)/\mathcal{D}_{\min}(A)$ is finite, and

$$\text{ind } A|_{\mathcal{D}} = \text{ind } A|_{\mathcal{D}_{\min}} + \dim \mathcal{D}(A)/\mathcal{D}_{\min}(A).$$

In particular,

$$\mathcal{E}(A) = \mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$$

is finite dimensional.

Below we will give a proof different from that of Lesch. That the dimension of $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$ is finite can also be proved by observing that if $A = x^{-\nu}P$ with $P \in \text{Diff}^m_b(M;E)$ and $Au \in x^{-\nu/2}L^2_b(M;E)$, then $Pu \in x^{\nu/2}L^2_b(M;E)$, so the Mellin transform of $Pu$ is holomorphic in $\Im \sigma > -\nu/2$, from which it follows that $\hat{u}(\sigma)$ is meromorphic in $\Im \sigma > -\nu/2$, and holomorphic in $\Im \sigma > \nu/2$ since $u \in x^{-\nu/2}L^2_b(M;E)$ (see [11], also [14]). On the other hand, if $u \in \mathcal{D}_{\min}$, then $\hat{u}(\sigma)$ is holomorphic in $\Im \sigma > -\nu/2$. This type of argument leads to

**Corollary 3.15.** If $A \in x^{-\nu}\text{Diff}^m_b(M;E)$ is $b$-elliptic then

$$\mathcal{D}_{\min}(A) = \mathcal{D}_{\max}(A)$$

if and only if $\text{spec}_b(A) \cap \{\Im \sigma \in (-\nu/2,\nu/2)\} = \emptyset$.

We will analyze $\mathcal{E}(A)$ more carefully in the next sections. This will entail some repetition of work done by Golberg and Sigal [8].

Our proof of Proposition 3.14 requires the following classical result [10]:

**Lemma 3.16.** Let $X$, $Y$ and $Z$ be Banach spaces such that $X \hookrightarrow Y$ is compact. Further let $T \in \mathcal{L}(X,Z)$. Then the following conditions are equivalent:

1. $\dim \ker T < \infty$ and $\text{rg } T$ is closed,
2. there exists $C > 0$ such that for every $u \in X$

$$\|u\|_X \leq C(\|u\|_Y + \|Tu\|_Z)$$

(a-priori estimate).

**Proof of Proposition 3.14.** Using (3.11) and Lemma 3.16 with $X = (\mathcal{D}, \| \cdot \|_A)$ and $Y = Z = x^{-\nu/2}L^2_b(M;E)$, we obtain $\dim \ker A|_{\mathcal{D}} < \infty$ and $\text{rg } A|_{\mathcal{D}}$ closed. On the other hand, the adjoint $A^*$ of a cone operator $A$ is just a closed extension of its $b$-elliptic formal adjoint $A^*$, cf. Proposition 3.13. Applying again Lemma 3.16 we get $\dim \ker (A|_{\mathcal{D}})^* < \infty$.

To verify the index formula consider the inclusion $i : \mathcal{D}_{\min} \to \mathcal{D}$ which is clearly Fredholm (use the same argument but now with $X = (\mathcal{D}_{\min}, \| \cdot \|_A), Y = x^{-\nu/2}L^2_b(M;E)$ and $Z = (\mathcal{D}, \| \cdot \|_A)$). Then

$$\dim \mathcal{D}/\mathcal{D}_{\min}$$

and

$$\dim \mathcal{D}/\mathcal{D}$$

hence

$$\text{ind } A|_{\mathcal{D}_{\min}} = \text{ind } (A|_{\mathcal{D}} \circ i) = \text{ind } A|_{\mathcal{D}} + \text{ind } i = \text{ind } A|_{\mathcal{D}} - \dim \mathcal{D}/\mathcal{D}_{\min}.$$
As a consequence of Propositions 3.2 and 3.14, for any closed extension of
\[ A : C_c^\infty(\mathcal{M}; E) \subset x^{-\nu/2}L_b^2(M; E) \to x^{-\nu/2}L_b^2(M; E) \]
with domain \( \mathcal{D}(A) \) there is a finite dimensional space \( \mathcal{E}' \subset \mathcal{D}_{\text{max}}(A) \) such that
\[ \mathcal{D}(A) = \mathcal{D}_{\text{min}}(A) \oplus \mathcal{E}' \] (algebraic direct sum).

From Proposition 3.14 we also get the following two corollaries

**Corollary 3.17.** Let \( A : \mathcal{D}_1 \to x^{-\nu/2}L_b^2(M; E) \) and \( A : \mathcal{D}_2 \to x^{-\nu/2}L_b^2(M; E) \) be
closed extensions of \( A \) such that \( \mathcal{D}_1 \subset \mathcal{D}_2 \), \( \text{ind} A|_{\mathcal{D}_1} < 0 \) and \( \text{ind} A|_{\mathcal{D}_2} > 0 \). Then
there exists a domain \( \mathcal{D} \) with \( \mathcal{D}_1 \subset \mathcal{D} \subset \mathcal{D}_2 \) such that \( \text{ind} A|_{\mathcal{D}} = 0 \).

The interest of this corollary lies in the fact that the vanishing of the index is
necessary for the existence of the resolvent and of selfadjoint extensions.

**Corollary 3.18.** Let \( A : \mathcal{D}_1 \to x^{-\nu/2}L_b^2(M; E) \) and \( A : \mathcal{D}_2 \to x^{-\nu/2}L_b^2(M; E) \) be
closed extensions of \( A \). Then
\[ \text{ind} A|_{\mathcal{D}_2} - \text{ind} A|_{\mathcal{D}_1} = \dim \mathcal{D}_2/\mathcal{D}_{\text{min}} - \dim \mathcal{D}_1/\mathcal{D}_{\text{min}}. \]

In particular, if \( \mathcal{D}_1 \subset \mathcal{D}_2 \) then
\[ \text{ind} A|_{\mathcal{D}_2} - \text{ind} A|_{\mathcal{D}_1} = \dim \mathcal{D}_2/\mathcal{D}_1. \]

This corollary implies the well-known relative index theorems for operators acting
on the weighted Sobolev spaces \( x^\nu H_b^m(M; E) \), cf. [10], [14].

4. **Equality of Domains**

In this section we give sufficient conditions for the domains of different operators
to be equal. Of these results, only the one concerning Friedrichs extensions will be
used later on.

An operator \( A \in x^{-\nu} \text{Diff}_b^m(M; E) \) is said to vanish on \( \partial M \) to order \( k \) \((k \in \mathbb{N})\) if
for any \( u \in C_c^\infty(M; E) \), \( x^\nu Au \) vanishes to order \( k \) on \( \partial M \). Let
\[ [s] = \min \{ k \in \mathbb{N} \mid s \leq k \}. \]

**Proposition 4.1.** Let \( A_0, A_1 \in x^{-\nu} \text{Diff}_b^m(M; E) \) be \( b \)-elliptic.

1. If \( A_0 - A_1 \) vanishes on \( \partial M \), then \( \mathcal{D}_{\text{min}}(A_0) = \mathcal{D}_{\text{min}}(A_1) \).
2. If \( A_0 - A_1 \) vanishes to order \( \ell \leq \nu - 1 \) on \( \partial M \), \( \ell \in \mathbb{N} \), then
\[ \mathcal{D}_{\text{max}}(A_0) \cap x^{\nu - 1 - \ell} H_b^m(M; E) = \mathcal{D}_{\text{max}}(A_1) \cap x^{\nu - 1 - \ell} H_b^m(M; E). \]
3. If \( A_0 - A_1 \) vanishes to order \([\nu - 1]\) on \( \partial M \), then \( \mathcal{D}_{\text{max}}(A_0) = \mathcal{D}_{\text{max}}(A_1) \).
4. If \( A_0 \) and \( A_1 \) are symmetric and bounded from below and \( A_0 - A_1 \) vanishes to
order \([\nu - 1]\) on \( \partial M \), then the domains of their Friedrichs extensions coincide,
that is, \( \mathcal{D}_F(A_0) = \mathcal{D}_F(A_1) \).

**Proof.** First of all, observe that in all cases it is enough to prove only one inclusion;
the equality of the sets follows then by exchanging the roles of \( A_0 \) and \( A_1 \).

To prove part 1, write
\[ A_1 = A_0 + (A_1 - A_0) = A_0 + x^{-\nu} Px \]
with \( P \in \text{Diff}_b^m(M; E) \) and suppose \( u \in \mathcal{D}_{\text{min}}(A_0) \). There is then a sequence
\( \{ u_n \}_{n \in \mathbb{N}} \subset C_c^\infty(M; E) \) such that \( u_n \to u \) and \( A_0 u_n \to A_0 u \), in \( x^{-\nu/2}L_b^2 \). Consequently,
\( xu_n \to xu \) in \( x^{\nu/2}H_b^m \) and \( x^{-\nu} Px u_n \to x^{-\nu} Px u \) in \( x^{-\nu/2}L_b^2 \). Thus
\( A_1 u_n \to A_0 u + x^{-\nu} Px u \) which implies \( \mathcal{D}_{\text{min}}(A_0) \subset \mathcal{D}_{\text{min}}(A_1) \).
Now, let $\ell \in \mathbb{N}$, $\ell \leq \nu - 1$, and let $u \in \mathcal{D}_{\max}(A_0) \cap x^{\nu/2-\ell-1}H^m_{2\nu}$. This means $u \in x^{\nu/2-\ell-1}H^m_{2\nu}$ and $A_0 u \in x^{-\nu/2}L^2_b$. To prove that $u \in \mathcal{D}_{\max}(A_1) \cap x^{\nu/2-\ell-1}H^m_{2\nu}$ we only need to show that $A_1 u$ belongs to $x^{-\nu/2}L^2_b$. Let $P \in \text{Diff}^m(M; E)$ be such that $A_1 = A_0 + x^{-\nu} P x^{\ell+1}$. Since $u \in x^{\nu/2-\ell-1}H^m_{2\nu}$ then $x^{-\nu} P x^{\ell+1} u \in x^{-\nu/2}L^2_b$. Hence $A_1 u \in x^{-\nu/2}L^2_b$ which proves the second statement.

To prove the third statement, let $u \in \mathcal{D}_{\max}(A_0)$, $\ell = [\nu - 1]$, and $P$ as above. Then $u \in x^{-\nu/2}H^m_{2\nu}$ and $x^{-\nu} P x^{\ell+1} u \in x^{-\nu/2}L^2_b$. Thus $A_1 u \in x^{-\nu/2}L^2_b$ and $\mathcal{D}_{\max}(A_0) \subset \mathcal{D}_{\max}(A_1)$.

To prove part 4 we first prove the rather useful and well known abstract characterization of the domain of the Friedrichs extension of a symmetric semibounded operator given in Lemma 4.3 below. Suppose $A : \mathcal{D}_{\min} \subset H \to H$ is a densely defined closed operator which is symmetric and bounded from below. Let

$$
D_A = \{ u \in H : (u, u) = (u, Au) \text{ for } u \in \mathcal{D}_{\min} \}.
$$

Then $D_A$ is a closed subspace of $H$, and let $A^* : D_{A^*} \subset H \to H$ be its adjoint and let $A : \mathcal{D}_{A^*} \subset H \to H$ be the Friedrichs extension of $A$. Define

$$(u, v)_{A^*} = (u, v) + (A^* u, v) \quad \text{for } u, v \in \mathcal{D}_{\max},$$

where $c = 1 - c_0$ and $c_0 \leq 0$ is a lower bound of $A$.

**Lemma 4.3.** $u \in \mathcal{D}_{\max}$ belongs to $\mathcal{D}_{\max}$ if and only if there exists a sequence $\{ u_n \}_{n \in \mathbb{N}}$ in $\mathcal{D}_{\min}$ such that

$$(u - u_n, u - u_n)_{A^*} \to 0 \quad \text{as } n \to \infty.$$

**Proof.** Because of the fact that $\mathcal{D}_{\min}$ is dense in $\mathcal{D}_{A^*}$ with respect to the norm $\| \cdot \|_{A^*}$ induced by (4.2), every $u \in \mathcal{D}_{\max}$ can be approximated as claimed.

Let now $u \in \mathcal{D}_{\max}$ and let $\{ u_n \}_{n \in \mathbb{N}} \subset \mathcal{D}_{\min}$ be such that $(u - u_n, u - u_n)_{A^*} \to 0$, so $u_n \to u$ in $H$. Let $K \subset H$ be the domain of the positive square root $R$ of $A_F + c I$. Recall that $\mathcal{D}_{A^*} = \mathcal{D}_{\max} \cap K$. Hence $u \in \mathcal{D}_{\max}$ belongs to $\mathcal{D}_{A^*}$ if $u \in K$, and the identity

$$
\| u_n - u \|_{A^*} = \| R(u_n - u) \|
$$

implies that $\{ R u_n \}_{n \in \mathbb{N}}$ also converges in $H$. Thus $u \in K$ since $R$ is closed. \hfill $\Box$

We now prove part 4 of Proposition 4.3. Suppose that $A_0$ and $A_1$ satisfy the hypotheses there. Then by parts 1 and 3, $\mathcal{D}_{\min} = \mathcal{D}_{\min}(A_0) = \mathcal{D}_{\min}(A_1)$ and $\mathcal{D}_{\max} = \mathcal{D}_{\max}(A_0) = \mathcal{D}_{\max}(A_1)$, and from the fact that $A_0 + A_1 - 2 A_0$ vanishes to order $[\nu - 1]$ we also get that $\mathcal{D}_{\min}(A_0 + A_1) = \mathcal{D}_{\min}$ and $\mathcal{D}_{\max}(A_0 + A_1) = \mathcal{D}_{\max}$. Since $A_0$ and $A_1$ are symmetric and bounded from below, so is $A_0 + A_1$. We will show that these three operators share the same Friedrichs domain by showing that

$$(A_0 + A_1) : \mathcal{D}_{\min}(A_0 + A_1) \subset \mathcal{D}_{\max}(A_0) \cap \mathcal{D}_{\max}(A_1).$$

Suppose this has been shown. Since $[u, v]_{A_0} = [u, v]_{A_1} = \frac{1}{2} [u, v]_{A_0 + A_1}$, Proposition 3.13 implies that $A_0$ is selfadjoint with either of the domains $\mathcal{D}_{\max}(A_0)$ or $\mathcal{D}_{\max}(A_0 + A_1)$, and from the inclusion of the latter in the former one deduces the equality of these spaces, hence, that $\mathcal{D}_{\max}(A_0) = \mathcal{D}_{\max}(A_1)$. To prove (4.4), suppose $(A_i u, v)_{x^{-\nu/2}L^2_b} \geq c_i (u, v)_{x^{-\nu/2}L^2_b}, \ i = 0, 1$, on $\mathcal{D}_{\min}$, let $c = 1 - c_0 - c_1$. If $u \in \mathcal{D}_{\max}(A_0 + A_1)$, then by Lemma 4.3 there is a sequence $\{ u_n \}_{n \in \mathbb{N}} \subset \mathcal{D}_{\min}$ such that

$$(A_0(u - u_n), u - u_n)_{x^{-\nu/2}L^2_b} + (A_1(u - u_n), u - u_n)_{x^{-\nu/2}L^2_b} + c (u - u_n, u - u_n)_{x^{-\nu/2}L^2_b} \to 0 \quad \text{as } n \to \infty.$$
But then also
\[(1 - e_i)(u - u_n, u - u_n)_{x - x/2L_i^2} + (A_i(u - u_n), u - u_n)_{x - x/2L_i^2}\]
as \(n \to \infty, i = 0, 1\), so \(u \in \mathcal{D}_F(A_0) \cap \mathcal{D}_F(A_1)\).

5. Spaces of Meromorphic Solutions

If \(K\) is a finite dimensional complex vector space, we let \(\mathcal{M}_{\sigma_0}(K)\) be the space of germs of \(K\)-valued meromorphic functions with pole at \(\sigma_0\) and \(\mathfrak{H}l_{\sigma_0}(K)\) be the subspace of holomorphic germs. These are naturally modules over the ring \(\mathfrak{H}l_{\sigma_0}((\mathbb{C})\).

Let \(R^\perp\) be another finite dimensional complex vector space. If \(\mathcal{P}(\sigma) : K \to R^\perp\) is a linear map depending holomorphically on \(\sigma\) in a neighborhood of \(\sigma_0\), then \(\mathcal{P}\) defines a map \(\mathcal{M}_{\sigma_0}(K) \to \mathcal{M}_{\sigma_0}(R^\perp)\), which we also denote by \(\mathcal{P}\).

**Lemma 5.1.** Suppose that \(\mathcal{P}(\sigma)\) is defined near \(\sigma = \sigma_0\), is invertible for \(\sigma \neq \sigma_0\) but \(\mathcal{P}(\sigma_0) = 0\). Then there are \(\psi_1, \ldots, \psi_d \in \mathcal{M}_{\sigma_0}(K)\) such that \(\beta_j(\sigma) = \mathcal{P}(\sigma)(\psi_j(\sigma))\) is holomorphic and \(\beta_1(\sigma_0), \ldots, \beta_d(\sigma_0)\) is a basis of \(R^\perp\). For any such \(\psi_j\), if \(u \in \mathcal{M}_{\sigma_0}(K)\) and \(\mathcal{P}u \in \mathfrak{H}l_{\sigma_0}(R^\perp)\), then there are \(f_j \in \mathfrak{H}l_{\sigma_0}(\mathbb{C})\) such that \(u = \sum_{j=1}^d f_j \psi_j\).

**Proof.** Let \(\{b_j\}_{j=1}^d\) be a basis of \(R^\perp\) and define \(\psi_j = \mathcal{P}^{-1}(b_j)\). Then the \(\psi_j\) are meromorphic with pole at \(\sigma_0\), \(\mathcal{P}\psi_j = \beta_j = b_j\) is holomorphic, and the \(\beta_j(\sigma_0)\) form a basis of \(R^\perp\).

Let now \(\psi_1, \ldots, \psi_d \in \mathcal{M}_{\sigma_0}(K)\) be such that \(\beta_j(\sigma) = \mathcal{P}(\sigma)(\psi_j(\sigma))\) is holomorphic and \(\beta_1(\sigma_0), \ldots, \beta_d(\sigma_0)\) is a basis of \(R^\perp\). If \(f \in \mathfrak{H}l_{\sigma_0}(R^\perp)\) then \(f = \sum_{j=1}^d f_j \beta_j\) for some \(f_j \in \mathfrak{H}l_{\sigma_0}(\mathbb{C})\), because the \(\beta_j(\sigma_0)\) form a basis, and each \(\beta_j(\sigma_0)\) can be written as a linear combination (over \(\mathfrak{H}l_{\sigma_0}(\mathbb{C})\)) of \(\beta_1, \ldots, \beta_d\). If \(\mathcal{P}u = f\) then \(\mathcal{P}(u - \sum_{j=1}^d f_j \psi_j) = 0\), so \(u = \sum_{j=1}^d f_j \psi_j\) for \(\sigma \neq \sigma_0\), which is the equality of meromorphic functions.

The lemma asserts that \(\mathcal{P}^{-1}(\mathfrak{H}l_{\sigma_0}(R^\perp))\) is finitely generated as a submodule of \(\mathcal{M}_{\sigma_0}(K)\) over \(\mathfrak{H}l_{\sigma_0}(\mathbb{C})\). We will be interested in \(\mathcal{E}_{\sigma_0} = \mathcal{P}^{-1}(\mathfrak{H}l_{\sigma_0}(R^\perp))/\mathfrak{H}l_{\sigma_0}(K)\) as a vector space over \(\mathbb{C}\). The following fundamental lemma paves the way to describing a basis of \(\mathcal{E}_{\sigma_0}\).

**Lemma 5.2.** Suppose that \(\mathcal{P}(\sigma)\) is defined near \(\sigma = \sigma_0\), is invertible for \(\sigma \neq \sigma_0\) but \(\mathcal{P}(\sigma_0) = 0\). There are \(\psi_1, \ldots, \psi_d \in \mathcal{M}_{\sigma_0}(K)\) such that each \(\beta_j(\sigma) = \mathcal{P}(\sigma)(\psi_j(\sigma))\) is holomorphic, \(\beta_1(\sigma_0), \ldots, \beta_d(\sigma_0)\) form a basis of \(R^\perp\), and if

\[
\psi_j = \sum_{\ell=0}^{\mu_j-1} \frac{1}{(\sigma - \sigma_0)^{\mu_j-\ell}} \psi_{j\ell} + h_j
\]

with holomorphic \(h_j\) then the \(\psi_{j0}\) are linearly independent.

**Proof.** Without loss of generality assume \(\sigma_0 = 0\). Pick a basis \(\{b_j\}_{j=1}^d\) of \(R^\perp\) and define \(\psi_j = \mathcal{P}^{-1}(b_j)\). Then the \(\psi_j\) are meromorphic with pole at 0, \(\mathcal{P}\psi_j = \beta_j = b_j\) is holomorphic, and the \(\beta_j(0)\) are independent. Each \(\psi_j\) can be written as (5.3).
Order them so that \( \{\mu_j\}_{j=1}^d \) is nonincreasing and let
\[
\tilde{\mu}_1 = \max\{\mu_j \mid j = 1, \ldots, d\},
\]
\[\tilde{\mu}_i = \max\{\mu_j \mid \mu_j < \tilde{\mu}_{i-1}\} \quad i = 2, \ldots, L\]
\[s_i = \max\{j \mid \mu_j = \tilde{\mu}_i\},\]
that is,
\[\tilde{\mu}_1 = \mu_1 = \cdots = \mu_{s_1} > \tilde{\mu}_2 = \mu_{s_1+1} = \cdots = \mu_{s_2} > \cdots > \tilde{\mu}_L = \mu_{s_L+1} = \cdots = \mu_d\]

If the vectors \( \psi_{j0}, j = 1, \ldots, s_1 \), are not linearly independent, then order the \( \psi_j \) with \( j \leq s_1 \) so that \( \psi_{10}, \ldots, \psi_{s_1,0} \) is a maximal set of linearly independent vectors among \( \{\psi_{j0} \mid 1 \leq j \leq s_1\} \), write
\[
\psi_{k0} = \sum_{j=1}^{s_1'} a_{kj} \psi_{j0} \quad \text{for} \quad k = s_1' + 1, \ldots, s_1,
\]
and replace \( \psi_k \) by \( \psi_k - \sum_{j=1}^{s_1'} a_{kj} \psi_{j0} \) for \( k = s_1' + 1, \ldots, s_1 \). Now \( \mathcal{P}(\psi_k) = \beta_k - \sum_{j=1}^{s_1'} a_{kj} \beta_j \) for these indices, so it is still true that the \( \mathcal{P}(\psi_j)(0) \) form a basis. With \( \mu_j \) denoting the order of the pole of the new \( \psi_j \), and again assuming the orders form a nonincreasing sequence, let \( \tilde{\mu}_i \) and \( s_i \) be defined as above. Suppose that already \( \psi_{j0}, j = 1, \ldots, s_1 \) is an independent set. If \( \psi_{s_1+1,0} \) depends linearly on \( \psi_{1,0}, \ldots, \psi_{s_1,0} \) then put \( s_{i+1}' = s_1 \). Otherwise, reorder \( \psi_{s_1+1,0}, \ldots, \psi_{s_{i+1},0} \) so that \( \psi_{s_1+1,0}, \ldots, \psi_{s_{i+1},0} \) together with \( \psi_{j0}, j = 1, \ldots, s_i \) are a maximally independent set in \( \{\psi_{j0} \mid 1 \leq j \leq s_{i+1}\} \). If \( s_{i+1}' < s_{i+1} \) write
\[
\psi_{k0} = \sum_{j=1}^{s_{i+1}'} a_{kj} \psi_{j0}, \quad k = s_{i+1}' + 1, \ldots, s_{i+1}.
\]
Replacing \( \psi_k \) by \( \psi_k - \sum_{j=1}^{s_{i+1}'} a_{kj} \sigma^{\mu_j - \tilde{\mu}_{i+1}} \psi_{j} \) (\( 1 \leq k \leq s_{i+1} \)), reordering by decreasing order of the pole (which reorder only \( \psi_j, j > s_{i+1}' \)), now have that the leading coefficients of the \( \psi_j, j \leq s_{i+1} \), are independent.

\begin{lemma}
With the setup of Lemma 5.2, let \( \psi_1, \ldots, \psi_d \in \mathcal{M}_{\sigma_0}(K) \) be as stated there, and let \( \mu_j \) be the order of the pole of \( \psi_j \). Let
\[
\hat{\mathcal{E}}_{\sigma_0} = \mathcal{P}^{-1}(\mathfrak{Do}t_{\sigma_0}(R^\perp)) / \mathfrak{Do}t_{\sigma_0}(K),
\]
regarded as a vector space over \( \mathbb{C} \). Then the images in \( \hat{\mathcal{E}}_{\sigma_0} \) of the elements
\[
(\sigma - \sigma_0)^\ell \psi_j, \quad j = 1, \ldots, d, \ell = 0, \ldots, \mu_j - 1
\]
form a basis of this space.
\end{lemma}

\begin{proof}
As before assume \( \sigma_0 = 0 \). Because of Lemma 5.1, the images of the \( \sigma^j \psi_j \) span, and we only need to prove linear independence. Suppose \( \sum_{j=1}^d \sum_{k=0}^{\mu_j-1} u_{jk} \sigma^k \psi_j \) is holomorphic. Modulo holomorphic functions,
\[
\psi_j = \sum_{\ell=0}^{\mu_j-1} \frac{1}{\sigma^{\mu_j-\ell}} \psi_j \ell
\]
(where the $\psi_{j0}$ are independent) so

$$u(\sigma) = \sum_{j=1}^{d} \sum_{k=0}^{\mu_j-1} \sum_{\ell=0}^{\mu_j-k} u_{jk} \sigma^{\mu_j-\ell-k} \psi_{j\ell}$$

is holomorphic. Thus $\sigma^{\nu} u(\sigma)$ vanishes at $0$ for $\nu > 0$. Let the $\tilde{\mu}_1$ be as in (5.4). We have that $\sigma^{\tilde{\mu}_1} \psi_j(\sigma)$ vanishes at $0$ for $j > s_1$, and so does $\sigma^{\tilde{\mu}_1} \sigma^k \psi_j(\sigma)$ for $k > 0$. Hence

$$0 = \left( \sigma^{\tilde{\mu}_1} u(\sigma) \right)_{\sigma=0} = \sum_{j=1}^{s_1} u_{j0} \left( \sigma^{\tilde{\mu}_1} \psi_j(\sigma) \right)_{\sigma=0} = \sum_{j=1}^{s_1} u_{j0} \psi_{j0},$$

and so $a_{j0} = 0$ for $j = 1, \ldots, s_1$, since the $\psi_{j0}$ are independent. If $\tilde{\mu}_2 < \tilde{\mu}_1 - 1$ then by the same argument one concludes that $a_{1j} = 0$ for $j = 1, \ldots, s_1$, and if $\tilde{\mu}_2 = \tilde{\mu}_1 - n, \ (n \geq 1)$ then the conclusion is that $a_{kj} = 0$ for $j = 1, \ldots, s_1$ and $k = 0, \ldots, n - 1$. Having proved this, we conclude

$$u(\sigma) = \sum_{j=1}^{s_1} \sum_{k=n}^{1} \sum_{\ell=0}^{\mu_j-k} \frac{u_{jk}}{\sigma^{\mu_j-\ell-k}} \psi_{j\ell} + \sum_{j=1}^{s_1} \sum_{k=n}^{1} \sum_{\ell=0}^{\mu_j-k} \frac{u_{jk}}{\sigma^{\mu_j-\ell-k}} \psi_{j\ell}.$$ 

Now, since $\sigma^{\tilde{\mu}_2} u(\sigma)$ also vanishes at $0$ then

$$\sum_{j=1}^{s_1} u_{jn} \psi_{j0} + \sum_{j=s_1+1}^{s_2} u_{j0} \psi_{j0} = 0,$$

therefore $u_{jn} = 0$ for $j = 1, \ldots, s_1$, and $u_{j0} = 0$ for $j = s_1 + 1, \ldots, s_2$. Continuing in this manner, one obtains $u_{jk} = 0$ for all $j, k$. \hfill \Box

**Lemma 5.6.** With the setup of Lemma 5.4, let $\psi_1, \ldots, \psi_d$ be as stated there, let $\mu_j$ be the order of the pole of $\psi_j$. Suppose the $\psi_j$ ordered so that $\{\mu_j\}_{j=1}^{d}$ is nonincreasing. With the notation in formulas (5.3) and (5.4) let

$$K_{\tilde{\mu}_\ell} = \text{span}_{\mathbb{C}} \{ \psi_{j0} | \mu_j \geq \tilde{\mu}_\ell \}.$$ 

The spaces $K_{\tilde{\mu}_\ell}$ are independent of the choice of $\psi_j$.

**Proof.** Let $\tilde{R}_\mu = \{ \psi \in \mathcal{P}^{-1}(\tilde{R}_\sigma) \cap \mathcal{O}(R_\mu) | \text{ord}(\psi) \leq \mu \}$. Thus if $\psi \in \tilde{R}_\mu$ then $(\sigma - \sigma_0)^\mu \psi$ is regular; let $m_\mu : \tilde{R}_\mu \rightarrow K$ be defined by setting

$$m_\mu(\psi) = (\sigma - \sigma_0)^\mu \psi(\sigma) \big|_{\sigma = \sigma_0}.$$ 

We will show that $K_{\tilde{\mu}_\ell} = m_{\tilde{\mu}_\ell}(\tilde{R}_{\tilde{\mu}_\ell})$. To see this, set

$$\tilde{R}_\sigma^\psi = \text{span}_{\mathbb{C}} \{ \psi_{j0} \ | \ \text{ord}(\psi_{j0}) = \tilde{\mu}_\ell \},$$

and note that if $\mu \geq \tilde{\mu}_1$ then

$$\tilde{R}_\mu = \tilde{R}_{\tilde{\mu}_1}^\psi + \tilde{R}_{\tilde{\mu}_2}^\psi + \cdots + \tilde{R}_{\tilde{\mu}_L}^\psi$$

and if $\tilde{\mu}_{\ell-1} \geq \mu \geq \tilde{\mu}_\ell$ then

$$\tilde{R}_\mu = (\sigma - \sigma_0)^{\tilde{\mu}_{\ell-1} - \mu} \tilde{R}_{\tilde{\mu}_1}^\psi + \cdots + (\sigma - \sigma_0)^{\tilde{\mu}_{\ell-1} - \mu} \tilde{R}_{\tilde{\mu}_{\ell-1}}^\psi + \tilde{R}_{\tilde{\mu}_\ell}^\psi + \cdots + \tilde{R}_{\tilde{\mu}_L}^\psi.$$ 

This is proved using Lemma 5.3. Thus if $\tilde{\mu}_{\ell-1} \geq \mu > \tilde{\mu}_\ell$ then

$$m_\mu(\tilde{R}_\mu) = \text{span}_{\mathbb{C}} \{ \psi_{j0} | \mu_j > \tilde{\mu}_\ell \}$$

and if $\mu = \tilde{\mu}_\ell$ then

$$m_\mu(\tilde{R}_\mu) = \text{span}_{\mathbb{C}} \{ \psi_{j0} | \mu_j \geq \tilde{\mu}_\ell \}.$$
Note that $K_{\bar{\mu}_i} \subset \cdots \subset K_{\bar{\mu}_L} = K$, $\dim K_{\bar{\mu}_L} = s_\ell$ and $m_{\bar{\mu}_L} : \mathcal{R}_{\bar{\mu}_L} \to K_{\bar{\mu}_L}$ is surjective. As in see Gohberg and Sigal [4], the numbers $\mu_j$ will be called the partial multiplicities of $P$ (at $\sigma_0$).

**Lemma 5.7.** Let $\{\psi_i^*\}_{i=1}^d$, $\{\psi_j\}_{j=1}^d \subset \mathcal{M}_{\sigma_0}(K)$ be as in Lemma 5.2, both sequences ordered so that the sequences $\{\mu_i^*\}$, $\{\mu_j\}$ of the orders of the poles is nonincreasing. Then $\mu_i^* = \mu_i$ for all $i$ and

$$\psi_i^* = \sum_{j=1}^d f_{ij} \psi_j$$

where the $f_{ij}$ are holomorphic, form a nonsingular matrix and $f_{ij} = (\sigma - \sigma_0)^{\mu_i - \mu_j} \tilde{f}_{ij}$ for some holomorphic $\tilde{f}_{ij}$ if $\mu_i > \mu_j$. Conversely, given holomorphic functions $\tilde{f}_{ij}$ forming a nonsingular matrix and with $f_{ij}/(\sigma - \sigma_0)^{\mu_i - \mu_j}$ holomorphic when $\mu_i > \mu_j$, then the $\psi_j^*$ defined by the formula above satisfy the conclusion of Lemma 5.2.

We leave the proof of this to the reader. It uses the previous lemma and its proof.

If $K$ is a hermitian vector space, let $\phi_1, \ldots, \phi_d$ be an orthonormal basis of $K$ such that for each $\ell = 1, \ldots, L$,

$$\phi_1, \ldots, \phi_{s_\ell} \in K_{\bar{\mu}_\ell}$$

Then for $j = s_{\ell - 1} + 1, \ldots, s_\ell$ we can pick $\psi_j \in \mathcal{R}_{\bar{\mu}_\ell}$ such that $m_{\bar{\mu}_\ell}(\psi_j) = \phi_j$, that is, if $K$ is hermitian then the $\psi_j$ can be chosen to have orthogonal leading coefficients.

**Proposition 5.8.** Let $P(\sigma) : K \to R^+$ be defined and holomorphic near $\sigma = \sigma_0$, invertible for $\sigma \neq \sigma_0$ but $P(\sigma_0) = 0$. Then

1. there are $\psi_1, \ldots, \psi_d \in \mathcal{M}_{\sigma_0}(K)$ such that each $\beta_j = P \psi_j \in \mathfrak{Hol}_{\sigma_0}(R^\perp)$, $\beta_1(\sigma_0), \ldots, \beta_d(\sigma_0)$ form a basis of $\mathbb{C}^d$, and if

$$\psi_j = \sum_{\ell=0}^{\mu_j - 1} \frac{1}{(\sigma - \sigma_0)^{\mu_j - \ell}} \psi_{j\ell} + h_j$$

with holomorphic $h_j$ then the $\psi_{j0}$ are linearly independent,

2. if $K$ is a hermitian vector space, then the $\psi_j$ can be chosen such that the $\psi_{j\ell}$ form an orthonormal basis of $K$ and for $\ell > 0$, $\psi_{j\ell}$ is orthogonal to $\psi_{k0}$ whenever $\mu_k \geq \mu_j - \ell$.

**Proof.** Because of Lemma 5.2 there are $\psi_1, \ldots, \psi_d$ satisfying the first statement.

Let now $K$ be a hermitian vector space. We may assume that already the leading coefficients form an orthonormal basis of $K$. If a coefficient $\psi_{j\ell}$ with $\ell > 0$ is not already orthogonal to those $\psi_{k0}$ such that $\mu_k \geq \mu_j - \ell$, then write

$$\psi_{j\ell} = \psi_{j\ell}^0 + \sum_{\{k : \mu_k \geq \mu_j - \ell\}} a_k \psi_{k0}$$

where $\psi_{j\ell}^0$ is orthogonal to the $\psi_{k0}$ such that $\mu_k \geq \mu_j - \ell$. Then

$$\chi(\sigma) = \sum_{\{k : \mu_k \geq \mu_j - \ell\}} a_k (\sigma - \sigma_0)^{\mu_k - \mu_j + \ell} \psi_{k}(\sigma) \in \mathcal{R}_{\mu_j - \ell} \subset \mathcal{R}_{\mu_j}$$
(d\mathcal{R}_\mu being defined using \sigma_0) and \((\sigma - \sigma_0)^\mu \chi(\sigma)|_{\sigma=\sigma_0} = 0\). So \psi_j - \chi has the same leading term as \psi_j but now the coefficient of \((\sigma - \sigma_0)^{-\mu_j + \ell}\psi_j^0\) is orthogonal to \psi_{k0} for k such that \(\mu_k \geq \mu_j - \ell\). We may then replace \psi_j by \psi_j - \chi. The proof is completed by ‘reverse’ induction on \(n = \mu_j - \ell\) beginning with \(n = \tilde{\mu}_1 - 1\), the above being both the first and general steps.

Let now \(Y\) be a compact manifold and \(E\) a complex vector bundle over \(Y\). We fix a Hermitian metric on \(E\) and a Riemannian metric on \(E\) with respect to which we define the Sobolev spaces \(H^s(Y; E)\). Let \(\mathcal{P}(\sigma) : H^m(Y; E) \rightarrow L^2(Y; E)\) be a holomorphic family of elliptic operators of order \(m\) defined for \(\sigma\) near \(\sigma_0\) in \(\mathbb{C}\). Suppose \(\mathcal{P}(\sigma)\) is invertible for \(\sigma \neq \sigma_0\) but \(\mathcal{P}(\sigma_0)\) is not invertible. Let \(K = \ker \mathcal{P}(\sigma_0), \ R = \operatorname{rg} \mathcal{P}(\sigma_0)\). Then \(K\) and \(R^\perp\) are finite dimensional of the same dimension, say \(d\), and consist of smooth sections of \(E\). Regard \(\mathcal{P}(\sigma)\) as an operator

\[
\begin{pmatrix}
\mathcal{P}_{11}(\sigma) & \mathcal{P}_{12}(\sigma) \\
\mathcal{P}_{21}(\sigma) & \mathcal{P}_{22}(\sigma)
\end{pmatrix} : K \oplus R^\perp \rightarrow K^\perp \oplus R
\]

in the usual way. All the \(\mathcal{P}_{ij}\) are holomorphic, and the operator \(\mathcal{P}_{22}(\sigma)\) is invertible for \(\sigma\) close to \(\sigma_0\). Thus \(\mathcal{P}_{11} - \mathcal{P}_{12} \mathcal{P}_{22}^{-1} \mathcal{P}_{21} : K \rightarrow R^\perp\) depends holomorphically on \(\sigma \in U\) and is invertible for \(\sigma \neq \sigma_0\). We can then find \(\hat{\psi}_1, \ldots, \hat{\psi}_d \in \mathcal{M}_{\sigma_0}(K)\) such that \((\mathcal{P}_{11} - \mathcal{P}_{12} \mathcal{P}_{22}^{-1} \mathcal{P}_{21})\hat{\psi}_j \in \mathfrak{so}\mathfrak{t}_{\sigma_0}(R^\perp)\), as in Lemma 5.2. Let \(\psi_j\) be the singular part of \(\hat{\psi}_j - \mathcal{P}_{22}^{-1} \mathcal{P}_{21} \psi_j\). In this last function, \(\mathcal{P}_{22}^{-1} \mathcal{P}_{21} \psi_j\) has values in \(K^\perp\), while \(\hat{\psi}_j\) has values in \(K\), so the order of the pole of \(\psi_j\) is the same as that of \(\hat{\psi}_j\) (there is no cancellation). Note that furthermore the order of the pole of \(\mathcal{P}_{22}^{-1} \mathcal{P}_{21} \psi_j\) is lower than that of \(\psi_j\) because \(\mathcal{P}_{22}^{-1} \mathcal{P}_{21}\) vanishes at \(\sigma = \sigma_0\). Let \(\mu_j\) be the order of the pole of \(\psi_j\).

**Proposition 5.9.** Let \(\hat{u}\) be an \(H^m(Y; E)\)-valued meromorphic function with pole at 0. Then \(\mathcal{P}(\sigma)(\hat{u}(\sigma))\) is holomorphic if and only if there are \(\mathbb{C}\)-valued polynomials \(p_j(\sigma)\) of degree \(\mu_j - 1\) such that \(\hat{u} - \sum_{j=1}^d p_j(\sigma)\psi_j(\sigma)\) is holomorphic. Thus if \(\hat{f}\) is holomorphic and \(\hat{u} = \mathcal{P}(\sigma)^{-1}(\hat{f})\), then \(\hat{u}\) is meromorphic with singularity of the form \(\sum_{j=1}^d p_j(\sigma)\psi_j(\sigma)\).

**Proof.** Suppose \(\hat{f} = f \oplus g\) is a holomorphic function with values in \(R^\perp \oplus R\) and let \(\hat{u} = u \oplus v = \mathcal{P}(\sigma)^{-1}(f \oplus g), \ \sigma \neq \sigma_0\), decomposed according to \(K \oplus K^\perp\), so

\[
\begin{align*}
\mathcal{P}_{11}u + \mathcal{P}_{12}v &= f \\
\mathcal{P}_{21}u + \mathcal{P}_{22}v &= g
\end{align*}
\]

From the second equation, \(v = \mathcal{P}_{22}^{-1}(g - \mathcal{P}_{21}u)\), which replaced in the first gives

\(\mathcal{P}_{11} - \mathcal{P}_{12} \mathcal{P}_{22}^{-1} \mathcal{P}_{21}u = f - \mathcal{P}_{12} \mathcal{P}_{22}^{-1} g\).

Since the \(\hat{\beta}_j = (\mathcal{P}_{11} - \mathcal{P}_{12} \mathcal{P}_{22}^{-1} \mathcal{P}_{21})\hat{\psi}_j\) are holomorphic near \(\sigma_0\) and independent at \(\sigma_0\), there are \(f_j, q_j \in \mathfrak{so}\mathfrak{t}_0(\mathbb{C})\) such that \(f = \sum f_j \hat{\beta}_j, \ \mathcal{P}_{12} \mathcal{P}_{22}^{-1} g = \sum q_j \hat{\beta}_j\) (the \(q_j\) vanish at \(\sigma_0\) because \(\mathcal{P}_{12}\) does). Then \(u = \sum_j (f_j - q_j)\hat{\psi}_j\). Replacing this in the expression for \(v\) gives

\[v = \mathcal{P}_{22}^{-1} g + \sum_j (f_j - q_j) \mathcal{P}_{22}^{-1} \mathcal{P}_{21} \hat{\psi}_j\]
so
\[
    u + v = \sum (f_j - q_j)(\tilde{\psi}_j - P_{22}^{-1}P_{21}\tilde{\psi}_j) + P_{22}^{-1}g
\]
\[
    = \sum p_j(\sigma)\psi_j + h,
\]
where the \( p_j \) are polynomials and \( h \) is holomorphic.

Note that each \( \psi_j \) can be written as
\[
    \psi_j(\sigma, y) = \sum_{\ell=0}^{\mu_j - 1} \frac{1}{\sigma^{\mu_j - \ell}} \psi_{j\ell}(y),
\]
with smooth sections \( \psi_{j\ell} \) of \( E \to Y \).

**Appendix: Saturated domains**

Let \( H \) be a Hilbert space, \( \Omega \subset \mathbb{C} \) open, and \( S \subset \Omega \) a finite set. Let \( \mathcal{M}_{\Omega,S}(H) \) be the space of meromorphic \( H \)-valued functions on \( \Omega \) with poles in \( S \), and let \( \mathcal{H} \mathcal{O}_{\Omega}(H) \) be the subspace of holomorphic elements. Multiplication by a holomorphic function \( f(\sigma) \) defines an operator
\[
    f(\sigma) : \mathcal{M}_{\Omega,S}(H) \to \mathcal{M}_{\Omega,S}(H)
\]
such that \( f(\sigma)\mathcal{H} \mathcal{O}_{\Omega}(H) \subset \mathcal{H} \mathcal{O}_{\Omega}(H) \), so it induces an operator on \( \mathcal{M}_{\Omega,S}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \) also denoted \( f(\sigma) \).

**Definition 5.10.** A subspace of \( \mathcal{M}_{\Omega,S}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \) which is invariant under multiplication by \( f(\sigma) = \sigma \) will be called saturated.

A saturated subspace \( \hat{\mathcal{E}} \) is thus a module over the ring \( \mathbb{C}[\sigma] \).

**Lemma 5.11.** Let \( S = \{s_1, \ldots, s_s\} \). If \( \hat{\mathcal{E}} \subset \mathcal{M}_{\Omega,S}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \) is saturated, then there are saturated spaces \( \hat{\mathcal{E}}_j \subset \mathcal{M}_{\Omega,\{s_j\}}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \subset \mathcal{M}_{\Omega,S}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \) such that
\[
    \hat{\mathcal{E}} = \hat{\mathcal{E}}_1 \oplus \cdots \oplus \hat{\mathcal{E}}_s.
\]

**Proof.** For every \( j \) there is a polynomial \( q_j(\sigma) \) such that \( q_j(\sigma_j) = 1 \) and \( q_j(\sigma_k) = 0 \) for \( k \neq j \), with equalities satisfied to a sufficiently high order. Take \( \hat{\mathcal{E}}_j = q_j(\sigma)\hat{\mathcal{E}} \), which is saturated because \( \sigma q_j(\sigma) = q_j(\sigma)\sigma \). Finally, note that \( q_1 + \cdots + q_s = 1 \) to high order at each \( \sigma_j \). For more details see the proof of the next lemma.

**Lemma 5.13.** A finite dimensional space \( \hat{\mathcal{E}} \subset \mathcal{M}_{\Omega,S}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \) is saturated if and only if it is invariant under multiplication by \( \tau^\sigma \) for \( \tau > 0 \).

**Proof.** Suppose first that \( \hat{\mathcal{E}} \) is saturated, so by the previous lemma, \( \hat{\mathcal{E}} = \hat{\mathcal{E}}_1 \oplus \cdots \oplus \hat{\mathcal{E}}_s \) with saturated spaces \( \hat{\mathcal{E}}_j \subset \mathcal{M}_{\Omega,\{s_j\}}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \). It is then enough to prove that each \( \hat{\mathcal{E}}_j \) is invariant under multiplication by \( \tau^\sigma \). But this is clear, since \( \tau^\sigma \) is a polynomial plus an entire function vanishing to high order at \( \sigma_j \).

Suppose now that \( \hat{\mathcal{E}} \) is invariant under multiplication by \( \tau^\sigma \) for any \( \tau > 0 \). We will first reduce the problem to the situation where \( \hat{\mathcal{E}} \subset \mathcal{M}_{\Omega,\{s_0\}}(H)/\mathcal{H} \mathcal{O}_{\Omega}(H) \).

Let the integers \( \mu_j \) be chosen so that for any representative \( \psi \) of an element of \( \hat{\mathcal{E}} \), \( (\sigma - \sigma_j)^{\mu_j} \) is regular at \( \sigma_0 \). It is possible to find such numbers because \( \hat{\mathcal{E}} \) is finite dimensional. For any \( (\xi_1, \ldots, \xi_s) \in \mathbb{C}^s \) with \( \xi_j \neq \xi_k \) if \( j \neq k \), let \( \varphi_{jk}(\xi) = \left[ \frac{\xi_j - \xi_k}{(\xi_j - \xi_k)^{\mu_k}} \right] \mu_k \), let \( p_j(\xi; \xi_1, \ldots, \xi_s) = \prod_{k \neq j} \varphi_{jk} \). Then \( p_j \) vanishes to order \( \mu_k \) at \( \xi_k \), \( k \neq j \), and
has value 1 at $\zeta_j$, so $p_j = 1 + (\zeta - \zeta_j)a_j$. The $a_j$ are polynomials in $\zeta$ whose coefficients depend holomorphically on the $\zeta_j$. Let $b_j = \sum_{\ell=0}^{\mu_j-1}(-1)^\ell[(\zeta - \zeta_j)a_j]^{\ell}$. This is again a polynomial in $\zeta$, and so is $q_j = b_j p_j$. Thus there are polynomials in $\zeta$ with coefficients depending on the $\zeta_j$, $h_{j,k}(\zeta; \zeta_1, \ldots, \zeta_s)$, $j, k = 1, \ldots, s$, such that

$$Q_j(\zeta; \zeta_1, \ldots, \zeta_s) = \delta_{jk} + (\zeta - \zeta_k)^{\mu_k} h_{j,k}(\zeta; \zeta_1, \ldots, \zeta_s)$$

Let now $q_j(\sigma, \tau) = Q_j(\tau^{i\sigma}; \tau^{i\sigma_1}, \ldots, \tau^{i\sigma_s})$, defined for those $\tau$ for which the numbers $\tau^{i\sigma_j}$ are distinct. Since this is a polynomial in $\tau^{i\sigma}$, and since $\hat{E}$ is invariant under multiplication by $\tau^{i\sigma}$, each $q_j$ defines a linear map $\pi_j : \hat{E} \to \hat{E}$. Since $q_j(\sigma, \tau) = \delta_{jk} + (\sigma - \tau)^{\mu_k} h_{j,k}(\sigma, \tau)$, $\pi_j \psi_j \in \mathcal{M}_{\Omega, i\sigma_j}(H)/\mathcal{F}_\Omega(H)$ and $\pi_j \circ \pi_k = \delta_{jk} \pi_j$, and since $\sum_{j=1}^{s} q_j = 1$, $\sum_{j} \pi_j = I$. Thus with $\hat{E}_j = \pi_j(\hat{E})$ we get

$$\hat{E} = \hat{E}_1 \oplus \cdots \oplus \hat{E}_s$$

The spaces $\hat{E}_j$ are invariant under multiplication by $\tau^{i\sigma}$ because $\tau^{i\sigma} q_j = q_j \tau^{i\sigma}$. The $\pi_j$ are independent of $\tau$.

Suppose now that $\hat{E} \subset \mathcal{M}_{\Omega, i\sigma_0}(H)/\mathcal{F}_\Omega(H)$ is invariant under multiplication by $\tau^{i\sigma}$ for all $\tau$ ($\tau = e^{it}$ suffices). Let $\lambda(w) = \sum_{\ell=1}^{N} (-1)^{\ell+1} w^\ell$, so that $\lambda(e^{\zeta} - 1) = \zeta + \zeta^\ell h(\zeta)$, with $h(\zeta)$ entire, and some integer $\mu$ depending on $N$ which may be assumed as large as desired by taking $N$ large enough. Let $q(\sigma) = \lambda(\tau^{i\sigma} \tau^{-i\sigma_0} - 1)$. Then

$$q(\sigma) = i(\sigma - \sigma_0) \log \tau + (\sigma - \sigma_0)^\mu h(\sigma, \tau)$$

with $h(\sigma, \tau)$ holomorphic in $\sigma$. If $\psi \in \hat{E}$, then $\psi \in \hat{E}$ because $q$ is a polynomial in $\tau^{i\sigma}$. But with $\mu$ large enough, $q \psi = (\sigma - \sigma_0) \log \tau \psi$. Thus $\hat{E}$ is saturated.

One can also give a proof using that if $\hat{E}$ is invariant under multiplication by $e^{it}(\sigma) = e^{i\sigma t}$ for any $t$ then $e^{it}$ defines a one parameter group on $\hat{E}$. This approach involves the topology of $\mathcal{M}_{\Omega, i\sigma_j}(H)$ (to prove that the group is continuous, hence differentiable). The proof given is better because it is elementary.

6. Canonical Pairing

Suppose that $K$ and $R^\perp$ are hermitian finite dimensional vector spaces. Define a pairing

$$\iota_{\sigma_0, K} : \mathcal{M}_{\sigma_0}(K) \times \mathcal{M}_{\sigma_0}(K) \to \mathcal{M}_{\sigma_0}(C)$$

by

$$\mathcal{M}_{\sigma_0}(K) \times \mathcal{M}_{\sigma_0}(K) \ni (u, v) \mapsto \iota_{\sigma_0, K}(u, v) = (u(\sigma), v(\overline{\sigma})) \in \mathcal{M}_{\sigma_0}(C).$$

and likewise a pairing $\iota_{\sigma_0, R^\perp}$ associated with $R^\perp$. Let $\mathcal{P}(\sigma) : K \to R^\perp$ be defined and holomorphic in a neighborhood of $\sigma_0 \in C$. Define $P^*(\sigma) = \mathcal{P}(\sigma)^*$, where $\ast$ denotes the pointwise adjoint of $\mathcal{P} : K \to R^\perp$. $P^*$ is holomorphic in a neighborhood of $\mathcal{F}_0$. Then with the induced map $\mathcal{P}^* : \mathcal{M}_{\sigma_0}(R^\perp) \to \mathcal{M}_{\sigma_0}(K)$ we have

$$\iota_{\sigma_0, R^\perp}(\mathcal{P}(\sigma) u(\sigma), v(\sigma)) = \iota_{\sigma_0, K}(u(\sigma), P^*(\sigma) v(\sigma)).$$

Furthermore, define $\Theta : \mathcal{M}_{\sigma_0}(C) \to \mathcal{M}_{\sigma_0}(C)$ by $\Theta(f)(\sigma) = \overline{f(\overline{\sigma})}$, and likewise $\Theta : \mathcal{M}_{\sigma_0}(C) \to \mathcal{M}_{\sigma_0}(C)$. 
Lemma 6.1. Let $\beta_1, \ldots, \beta_d \in \mathfrak{hol}_{\sigma_0}(R^+) \setminus \{0\}$ be such that the $\beta_j(\sigma_0)$ are independent. Then there are $\tilde{\beta}_1, \ldots, \tilde{\beta}_d \in \mathfrak{hol}_{\sigma_0}(R^+) \setminus \{0\}$ such that

$$t_{\sigma_0,R^+}(\beta_i, \tilde{\beta}_j) = \delta_{ij}.$$ 

If $\sigma_0$ is real, then

$$t_{\sigma_0,R^+}(\sigma, \tilde{\beta}) = \delta_{ij}.$$ 

Proof. Let $b_i = \beta_i(\sigma_0)$ and write $\beta_i = b_i + (\sigma - \sigma_0)\sum_k a_{ik}b_k$ with $a_{ik} \in \mathfrak{hol}_{\sigma_0}(\mathbb{C})$. We seek $\tilde{\beta}_j = \sum h_{ij}b_i$ with $h_{ij} \in \mathfrak{hol}_{\sigma_0}(\mathbb{C})$. We need

$$t_{\sigma_0,R^+}(\beta_i, \tilde{\beta}_j) = \sum \Theta(h_{ij})(b_i, b_\ell) + (\sigma - \sigma_0)\sum a_{ik}\Theta(h_{ij})(b_k, b_\ell) = \delta_{ij},$$

or, with the matrices $H = [h_{ij}], A = [a_{ik}], B = [(b_k, b_\ell)],$

$$B\Theta(H) + (\sigma - \sigma_0)AB\Theta(H) = I$$

so set

$$\Theta(H) = [I + (\sigma - \sigma_0)B^{-1}AB]^{-1}B^{-1}.$$ 

Lemma 6.2. Let $P(\sigma) : K \to R^+$ be defined and holomorphic near $\sigma_0$, invertible for $\sigma \neq \sigma_0$, $P(\sigma_0) = 0$. Let $\psi_j \in \mathfrak{M}_{\sigma_0}(K)$ have independent leading coefficients and be such that $\beta_j = P\psi_j \in \mathfrak{hol}_{\sigma_0}(R^+) \setminus \{0\}$, with the $\beta_j(\sigma_0)$ linearly independent. Let $\tilde{\beta}_j \in \mathfrak{hol}_{\sigma_0}(R^+) \setminus \{0\}$ be such that

$$t_{\sigma_0,R^+}(\beta_i, \tilde{\beta}_j) = \delta_{ij}.$$ 

Let $\mu_j$ be the order of the pole of $\psi_j$, let $\beta^*_j = (\sigma - \sigma_0)^{\mu_j}\psi_j$, so $\tilde{\beta}^*_j \in \mathfrak{hol}_{\sigma_0}(K)$ and the $\beta^*_j(\sigma_0)$ are independent. Let $\tilde{\beta}^*_j \in \mathfrak{hol}_{\sigma_0}(K)$ be such that

$$t_{\sigma_0,K}(\tilde{\beta}^*_j, \tilde{\beta}^*_j) = \delta_{ij}.$$ 

Then $P^*\tilde{\beta}_j = (\sigma - \sigma_0)^{\mu_j}\beta^*_j$ so with

$$\psi_j^* = \frac{1}{(\sigma - \sigma_0)^{\mu_j}}\tilde{\beta}_j$$

we have

$$P^*(\psi_j^*) = \beta^*_j.$$ 

Clearly the leading coefficients of the $\psi_j^*$ are independent, as are the $\tilde{\beta}_j(\sigma_0)$, thus the multiplicities $\mu_j^*$ for $P^*$ are the same as those for $\beta$, $\mu_j^* = \mu_j$.

Proof. We have $P\beta^*_j = (\sigma - \sigma_0)^{\mu_j}\beta_j$, so

$$t_{\sigma_0,R^+}(P\beta^*_j, \tilde{\beta}_k) = (\sigma - \sigma_0)^{\mu_k}\delta_{jk}$$

$$= (\sigma - \sigma_0)^{\mu_k}t_{\sigma_0,K}(\tilde{\beta}^*_j, \tilde{\beta}^*_k)$$

$$= t_{\sigma_0,K}(\tilde{\beta}^*_j, (\sigma - \sigma_0)^{\mu_k}\tilde{\beta}^*_k)$$

The last expression must be $t_{\sigma_0,K}(\tilde{\beta}^*_j, P^*\tilde{\beta}_k)$, so $P^*\tilde{\beta}_k = (\sigma - \sigma_0)^{\mu_k}\beta^*_k$. 

\qed
With the assumptions on $\mathcal{P}(\sigma)$ as in the previous lemma let

$$u \in \mathcal{P}^{-1}(\mathfrak{hol}_{\sigma_0}(K)) \quad \text{and} \quad v \in (\mathcal{P}^*)^{-1}(\mathfrak{hol}_{\sigma_0}(R^\perp)).$$

With suitable small $\varepsilon$ (depending on representatives of $u$ and $v$) we get a number

$$[u, v]_{\mathcal{P}, \sigma_0} = \frac{1}{2\pi} \oint_{|\sigma - \sigma_0| = \varepsilon} \iota_{\sigma_0, R^\perp}(\mathcal{P} u, v) \, d\sigma.$$

The circle of integration is oriented counterclockwise. If $v$ is holomorphic, then $[u, v]_{\mathcal{P}, \sigma_0} = 0$ because $\mathcal{P} u$ is holomorphic. If $u$ is holomorphic, then also $[u, v]_{\mathcal{P}, \sigma_0} = 0$, since

$$\frac{1}{2\pi} \oint_{|\sigma - \sigma_0| = \varepsilon} \iota_{\sigma_0, R^\perp}(\mathcal{P} u, v) \, d\sigma = \frac{1}{2\pi} \oint_{|\sigma - \sigma_0| = \varepsilon} \iota_{\sigma_0, R^\perp}(u, \mathcal{P}^* v) \, d\sigma.$$

Thus $[\cdot, \cdot]_{\mathcal{P}, \sigma_0}$ defines a pairing

$$[\cdot, \cdot]_{\mathcal{P}, \sigma_0} : \mathcal{E}_{\sigma_0} \times \mathcal{E}^*_{\sigma_0} \to \mathbb{C},$$

where

$$\mathcal{E}_{\sigma_0} = \mathcal{P}^{-1}(\mathfrak{hol}_{\sigma_0}(K))/\mathfrak{hol}_{\sigma_0}(K),$$

$$\mathcal{E}^*_{\sigma_0} = (\mathcal{P}^*)^{-1}(\mathfrak{hol}_{\sigma_0}(R^\perp))/\mathfrak{hol}_{\sigma_0}(R^\perp).$$

**Theorem 6.4.** $[\cdot, \cdot]_{\mathcal{P}, \sigma_0}$ is a nonsingular paring of vector spaces.

**Proof.** Pick $\psi_j \in \mathfrak{M}_{\sigma_0}(K)$ such that $\mathcal{P} \psi_j = \beta_j \in \mathfrak{hol}_{\sigma_0}(R^\perp)$, with the leading coefficients of the $\psi_j$ forming a basis of $K$ and with the $\beta_j(\sigma_0)$ forming a basis of $R^\perp$. Let $\beta_j$, $\beta_j^*$, and $\beta_j^\ast$ be as in Lemma 5.2. According to the proof of Lemma 5.3, if $u \in \mathcal{E}_{\sigma_0}$ and $v \in \mathcal{E}^*_{\sigma_0}$, then $u$ and $v$ are represented by

$$u = \sum_{i=1}^{d} \sum_{k=0}^{\mu_i - 1} (\sigma - \sigma_0)^k u_{ik} \psi_i \quad \text{and} \quad v = \sum_{j=1}^{d} \sum_{\ell=0}^{\mu_j - 1} (\sigma - \sigma_0)^\ell v_{j\ell} \psi_j^*$$

with constant $u_{ik}$ and $v_{j\ell}$. Now,

$$\mathcal{P}(u) = \sum_{i=1}^{d} \sum_{k=0}^{\mu_i - 1} (\sigma - \sigma_0)^k u_{ik} \beta_i$$

so

$$\iota_{\sigma_0, R^\perp}(\mathcal{P} u, v) = \sum_{i,j=1}^{d} \sum_{k,\ell=0}^{\mu_i - 1} (\sigma - \sigma_0)^{k+\ell} u_{ik} \psi_i \psi_j^* \iota_{\sigma_0, R^\perp}(\beta_i, \psi_j^*)$$

$$= \sum_{i,j=1}^{d} \sum_{k,\ell=0}^{\mu_i - 1} (\sigma - \sigma_0)^{k+\ell-\mu_j} u_{ik} \psi_j^* \iota_{\sigma_0, R^\perp}(\beta_i, \beta_j^*)$$

$$= \sum_{j=1}^{d} \sum_{k,\ell=0}^{\mu_j - 1} (\sigma - \sigma_0)^{k+\ell-\mu_j} u_{jk} \psi_j^* \iota_{\sigma_0, R^\perp}(\beta_j, \beta_j^\ast)$$

Thus

$$[u, v]_{\mathcal{P}, \sigma_0}^\ast = i \sum_{j=1}^{d} \sum_{k+\ell-\mu_j = -1}^{\mu_j - 1} u_{jk} \psi_j^* \iota_{\sigma_0, R^\perp}(\beta_j, \beta_j^\ast)$$

$$= i \sum_{j=1}^{d} \sum_{k=0}^{\mu_j - 1} u_{jk} \psi_j^* \iota_{\sigma_0, R^\perp}(\beta_j, \beta_j^\ast).$$
If \([u, v]_\mathcal{P}, \sigma_0 = 0\) for all \(v\), pick \(v\) as above so that \(v_j = u_j, \mu_j+\ell-1, j = 1, \ldots, d, \ell = 0, \ldots, \mu_j - 1\). Then
\[
[u, v]_\mathcal{P}, \sigma_0 = i \sum_{j=1}^{d} \sum_{k=0}^{\mu_j - 1} u_j k \mathcal{P}_{jk} = 0
\]
implies \(u = 0\). \(\square\)

**Remark 6.7.** In the situation of the proposition and with the notation in the proof, suppose that all \(\mu_j\) are even. Let
\[
U = \{ \sum_{j=1}^{d} \sum_{\ell=\mu_j/2}^{\mu_j - 1} (\sigma - \sigma_0)^{\ell} u_j \psi_j | u_j \in \mathbb{C} \}
\]
which can be regarded as a subspace of \(\check{\mathcal{E}}_{\sigma_0}\). Likewise, since the \(\mu_j^*\) associated with \(\mathcal{P}\) are equal to the \(\mu_j\)'s, define
\[
V = \{ \sum_{j=1}^{d} \sum_{\ell=\mu_j/2}^{\mu_j - 1} (\sigma - \sigma_0)^{\ell} v_j \psi_j^* | v_j \in \mathbb{C} \}
\]
which again can be regarded as a subspace of \(\check{\mathcal{E}}_{\sigma_0}\). It follows from (6.6) that
\[
[u, v]_A = 0 \quad \text{if} \quad u \in U, v \in V
\]
and therefore, by dimensional considerations and the proposition itself, the orthogonal of \(U\) in \(\check{\mathcal{E}}_{\sigma_0}\) is \(V\).

The spaces \(U\) and \(V\) are independent of the \(\psi_j\) used to represent them, as long as these functions are chosen according to Lemma 5.2. Indeed, if \(\{\psi_i^j\}_{i=1}^d\) is another such choice, then according to Lemma 5.7, possibly after reordering, we can write
\[
\psi_i^j = \sum_{\{j \mid \mu_j < \mu_i\}} (\sigma - \sigma_0)^{\mu_j - \mu_i} f_{ij} \psi_i + \sum_{\{j \mid \mu_j \geq \mu_i\}} f_{ij} \psi_i
\]
and
\[
(\sigma - \sigma_0)^{\mu_j/2} \psi_i^j = \sum_{\{j \mid \mu_j < \mu_i\}} (\sigma - \sigma_0)^{\mu_j - \mu_i/2} f_{ij} (\sigma - \sigma_0)^{\mu_i/2} \psi_i + \sum_{\{j \mid \mu_j \geq \mu_i\}} (\sigma - \sigma_0)^{\mu_j - \mu_i/2} f_{ij} (\sigma - \sigma_0)^{\mu_i/2} \psi_i
\]

**Lemma 6.8.** Suppose \(\sigma_0\) is real, let \(\psi_1, \ldots, \psi_d \in \mathcal{M}_{\sigma_0}(K)\) have independent leading coefficients and be such that their orders \(\mu_j\) form a nonincreasing sequence. Then there are holomorphic functions \(f_{ij}\) with \(f_{ij} = 0\) if \(i > j\) such that
\[
(6.9) \quad \langle \sigma - \sigma_0 \rangle^{\mu_k - \mu_i} f_{ik} \psi_k, \sum_{\ell=1}^{j} (\sigma - \sigma_0)^{\mu_{\ell} - \mu_j} f_{\ell j} \psi_{\ell} \rangle = (\sigma - \sigma_0)^{-\mu_j - \mu_k} \delta_{ij}
\]

This lemma combined with Lemma 5.7 says that when \(\sigma_0\) is real, the \(\check{\mathcal{P}}_i^*\) in Lemma 6.2 can be assumed to form an "orthonormal" system: \(\langle \sigma - \sigma_0 \rangle (\check{\mathcal{P}}_i^*, \check{\mathcal{P}}_j^*) = \delta_{ij}\).
Proof. We apply the Gram-Schmidt orthonormalization process. There is no loss of generality if we assume \( \sigma_0 = 0 \). Write \( \beta_j^* = \sigma_j^* \psi_j \). The holomorphic function \( h(\sigma) = \omega_0, K(\beta_j^*, \beta_k^*) \) is positive when \( \sigma \) is real, so there is \( k(\sigma) \) positive defined for \( \sigma \) real (close to 0) such that \( k = h \). Since \( k \) is real analytic, it has a holomorphic extension to a neighborhood of 0. Since \( k(\sigma)|k(\sigma)| = h(\sigma) \) holds when \( \sigma \) is real, equality holds also for complex \( \sigma \) near 0. Let \( f_{11} = 1/k \). Then (6.9) holds for \( i, j = 1 \), and we replace \( \psi_1 \) with \( k^{-1}\psi_1 \) and each \( \psi_i, i > 1 \), by \( \psi_i - (\sigma - \sigma_0)^{\mu_j - \mu_i} t_{\sigma_0, K}(\psi_i, \psi_1) h^{-1} \psi_1 \). Now repeat the process with \( \psi_2 \) and \( \psi_i \) with \( i > 2 \).

Appendix: Selfadjoint subspaces

Suppose for the rest of this section that \( R^\perp = K \) and \( \sigma_0 \in \mathbb{R} \). Motivated by Proposition 5.13 a subspace \( \mathcal{E}_{\sigma_0}^o \) of \( \mathcal{E}_{\sigma_0} \) will be called \( \mathcal{P} \)-selfadjoint (or just selfadjoint if there is no risk of confusion) if

\[
\mathcal{E}_{\sigma_0}^o = \{ u \in \mathcal{E}_{\sigma_0} \mid [u, v]_{\mathcal{P}, \sigma_0} = 0 \text{ for all } v \in \mathcal{E}_{\sigma_0}^o \}.
\]

In other words, \( \mathcal{E}_{\sigma_0}^o \) is selfadjoint if \( \mathcal{E}_{\sigma_0}^o = (\mathcal{E}_{\sigma_0}^o)^\perp \) with respect to \([\cdot, \cdot]_{\mathcal{P}, \sigma_0} \).

Let \( \mathcal{P}(\sigma) \) be defined and holomorphic near \( \sigma_0 \) (real). We call \( \mathcal{P} \) selfadjoint if \( \mathcal{P}^*(\sigma) = \mathcal{P}(\sigma) \) near \( \sigma_0 \). If \( \mathcal{P} \) is selfadjoint we say that \( \mathcal{P} \) is positive if for each real \( \sigma \neq \sigma_0 \) close to \( \sigma_0 \), \( \mathcal{P}(\sigma) : K \rightarrow K \) is nonnegative.

Lemma 6.10. Let \( \mathcal{P} \) be defined near \( \sigma_0 \in \mathbb{R} \), selfadjoint, positive, with \( \mathcal{P}(\sigma_0) = 0 \). Let \( \psi_1, \ldots, \psi_d \in \mathcal{P}^{-1}(\delta\mathfrak{m}_{\sigma_0}(K)) \) be chosen as in Proposition 5.8. Then the numbers \( \mu_j \) are even.

Proof. Without loss of generality we assume that \( \sigma_0 = 0 \). We use the notation of Proposition 5.8 and assume that the \( \psi_j \) are ordered so that the \( \mu_j \) are nonincreasing in \( j \) and define \( \tilde{\mu}_i \) and \( s_i \) as in (5.4). We replace the \( \psi_j \) by suitable linear combinations of themselves to arrange that with \( \tilde{\beta}_j^* = \sigma_j^* \psi_j \) we have \( t_{\sigma_0, K}(\tilde{\beta}_j^*, \tilde{\beta}_k^*) = \delta_{jk} \). This does not change the multiplicities \( \mu_j \). Thus \( \tilde{\beta}_j^* \) is holomorphic and the \( \tilde{\beta}_j^*(0) \) form a basis of \( K \) (orthonormal), and if \( \mathcal{P}\psi_j = \beta_j \), then \( \beta_j = \sum p_{j\ell} \tilde{\beta}_j^* \) for some holomorphic functions \( p_{j\ell} \). Thus

\[
\mathcal{P}\tilde{\beta}_j^* = \sigma_j^* \sum p_{j\ell} \tilde{\beta}_j^*.
\]

Since both the vectors \( \beta_j(0) \) and the \( \tilde{\beta}_j^* \) give bases of \( K \), the matrix \( [p_{j\ell}(0)] \) is nonsingular. We have

\[
t_{0, K}(\mathcal{P}\tilde{\beta}_j^*, \tilde{\beta}_k^*) = \sigma_j^* p_{j\ell}(\sigma)
\]

and

\[
t_{0, K}(\tilde{\beta}_j^*, \mathcal{P}\tilde{\beta}_k^*) = \sigma_k^* p_{k\ell}(\sigma).
\]

Since \( \mathcal{P} \) is selfadjoint, these two functions are equal. Consequently, if \( \mu_k \geq \mu_j \), then \( p_{j\ell}(\sigma) = \sigma_k^* \sigma_j^* p_{k\ell}(\sigma) \). Thus \( [p_{j\ell}(0)] \) is an upper-triangular block matrix, the \( i \)-th diagonal block corresponding to the indices \( j \) such that \( \mu_j = \mu_i \) \((i = 1, \ldots, L) \). Moreover, these diagonal blocks are selfadjoint matrices. It follows that for each \( i \) one can replace the \( \psi_j \) \((j \text{ such that } \mu_j = \tilde{\mu}_i) \) by linear combinations of themselves with constant coefficients, and assume that the diagonal blocks of
Since both \( \tilde{\beta}^* \) and \( \beta^* \) are diagonal themselves. Let \( p_{jj}(0) = \lambda_j \). Since \( [p_{jk}(0)] \) is nonsingular, all \( \lambda_j \) are different from 0. Now,

\[
t_0,K(\mathcal{P} \tilde{\beta}^*, \beta^*) = \sigma^{\mu_j} (p_{jj} + \sigma \sum \ell p_{\ell j} h_{\ell j}) = \sigma^{\mu_j} (\lambda_j + \sigma f_j)
\]

for some holomorphic function \( f_j \). Since \( \mathcal{P} \) is positive, \( \sigma^{\mu_j} (\lambda_j + \sigma f_j) \) is nonnegative for real \( \sigma \), and then necessarily \( \mu_j \) is even (and \( \lambda_j > 0 \)).

If \( \mathcal{E}_{\sigma_0} \) is a saturated subspace of \( \mathcal{M}_{\sigma_0}(K)/\mathcal{H}\sigma_{\sigma_0}(K) \) (see the appendix of Section 5 for the definition) then there is a set of elements \( \psi_j \in \mathcal{E}_{\sigma_0} \), \( j = 1 \ldots d \), such that the products \( (\sigma - \sigma_0)^\ell \psi_j \), \( \ell = 0, \ldots, \mu_j - 1 \) form a basis of \( \mathcal{E}_{\sigma_0} \). The proof of this is that of Lemma 5.2 where only the saturation property was used. We may further assume, as in that lemma, that if the \( \psi_j \) are represented as in (5.3), then the \( \psi_{j_0} \) are independent and the \( \mu_j \) form a nonincreasing sequence.

**Proposition 6.11.** Let \( \mathcal{P} \) be defined near \( \sigma_0 \in \mathbb{R} \), selfadjoint, positive and such that \( \mathcal{P}(\sigma_0) = 0 \), and let \( \mathcal{E}'_{\sigma_0} \) be a selfadjoint saturated subspace of \( \mathcal{E}_{\sigma_0} \). Then every \( u \in \mathcal{E}'_{\sigma_0} \) can be represented as

\[
u = \sum_{j=1}^{d} \sum_{\ell=\mu_j/2}^{\mu_j-1} (\sigma - \sigma_0)^\ell u_{j\ell} \psi_j
\]

with constant \( u_{j\ell} \), where the \( \psi_1, \ldots, \psi_d \in \mathcal{P}^{-1}(\mathcal{H}\sigma_{\sigma_0}(K)) \) are chosen as in Proposition 5.8 and the \( \mu_j \) are the respective multiplicities, which by Lemma 6.10 are even. The space of such elements will be denoted \( \mathcal{E}_{\sigma_0} \).

**Proof.** We may assume that \( \sigma_0 = 0 \), without loss of generality. By Remark 6.7, if we show that for some choice of \( \psi_j \) as in the statement the elements of \( \mathcal{E}'_{\sigma_0} \) can be represented as stated, then for any such choice of \( \psi_j \) they are represented as stated. We then take advantage of Lemma 6.8 and assume that if \( \tilde{\beta}^* = \sigma^{\mu_j} \psi_j \) then \( \iota_{\sigma_0,K}(\tilde{\beta}^*, \beta^*) = \delta_{ij} \). In the notation of Lemma 6.2 we then have \( \beta^* = \tilde{\beta}^* \) (\( \sigma_0 \) is real). As in that lemma let \( \beta_j = \mathcal{P} \psi_j \) and let \( \tilde{\beta}_j \in \mathcal{H}\sigma_0(K) \) be such that \( \iota_{0,K}(\beta_j, \tilde{\beta}_j) = \delta_{ij} \). By Lemma 6.2, \( \mathcal{P} \sigma^{-\mu_j} \beta_j = \beta_j^* \), but now the latter is equal to \( \tilde{\beta}^* \), and \( \mathcal{P} \) is selfadjoint, so

\[
\mathcal{P}(\sigma^{-\mu_j} \tilde{\beta}_j) = \tilde{\beta}_j^*
\]

Since both the \( \tilde{\beta}_j(0) \) and the \( \tilde{\beta}^*_j(0) \) form bases of \( K \), the functions \( \psi_j^* = \sigma^{-\mu_j} \tilde{\beta}_j \) satisfy the conditions in Lemma 5.3, so they are related as stated there, and from Remark 6.7 we get that if

\[
u = \sum_{j=1}^{d} \sum_{\ell=\mu_j/2}^{\mu_j-1} \sigma^\ell u_{j\ell} \psi_j, \quad v = \sum_{j=1}^{d} \sum_{\ell=\mu_j/2}^{\mu_j-1} \sigma^\ell v_{j\ell} \psi_j
\]

then

\[
[u, v]_{\sigma_0, \mathcal{P}} = 0.
\]
This said, we now show that if there is an element in $\mathcal{E}_{\sigma_0}'$ represented by

$$u = \sum_{j=1}^{d} \sum_{\ell=\ell_j}^{\mu_j-1} \sigma^j u_{j,\ell} \psi_j \in \hat{\mathcal{E}}_{\sigma_0}$$

($\ell_j \geq 0$) where for some $j$, $\ell_j < \mu_j/2$ with $u_{j,\ell_j} \neq 0$, then $\hat{\mathcal{E}}_{\sigma_0}$ is not selfadjoint. Thus let $\delta_j = \mu_j/2 - \ell_j$, let $\delta = \max_j \delta_j$ and suppose $\delta \geq 1$. Since $\hat{\mathcal{E}}_{\sigma_0}$ is saturated both $\sigma^{\delta-1} u$ and $\sigma^\delta u$ represent elements in $\hat{\mathcal{E}}_{\sigma_0}$, and we will show that $[\sigma^{\delta-1} u, \sigma^\delta u]_\mathcal{P} \neq 0$.

Let $J = \{ j : \delta_j = \delta \}$. Then

$$\sigma^{\delta-1} u \equiv \sum_{j \in J} u_{j,\ell_j} \sigma^{\mu_j/2-1} \psi_j + \sum_{j=1}^{d} \sum_{\ell=\ell_j}^{\mu_j-1} \sigma^j \tilde{u}_{j,\ell} \psi_j \mod \delta \sigma_0(K)$$

with some $\tilde{u}_{j,\ell}$. Write this as $u_0 + \tilde{u}_1$. Then

$$[u_0 + u_1, \sigma u_0 + \sigma u_1]_\mathcal{P},\sigma_0 = [u_0, \sigma u_0 + \sigma u_1]_\mathcal{P},\sigma_0$$

because of Remark 3.7.

$$[u_0, \sigma u_0 + \sigma u_1]_\mathcal{P},\sigma_0 = [\sigma u_0, u_0 + u_1]_\mathcal{P},\sigma_0$$

because multiplication by $\sigma$ is selfadjoint, and finally,

$$[\sigma u_0, u_0 + u_1]_\mathcal{P},\sigma_0 = [\sigma u_0, u_0]_\mathcal{P},\sigma_0$$

again by Remark 3.7. Thus $[\sigma^{\delta-1} u, \sigma^\delta u]_\mathcal{P},\sigma_0 = [u_0, \sigma u_0]_\mathcal{P},\sigma_0$. As in the proof of Proposition 1.11 $\mathcal{P} \beta_j^* = \sigma^{\mu_j} \sum_{k=1}^{d} p_{jk} \beta_k^*$ with holomorphic $p_{jk}$ which because of the selfadjointess are such that $p_{jk}(\sigma) = \sigma^{\mu_k-\mu_j} p_{kj}(\sigma)$ if $\mu_k \geq \mu_j$. We thus have (since $\psi_j = \sigma^{-\mu_j} \beta_j^*$)

$$\mathcal{P} u_0 = \sum_{j \in J} \sum_{k} u_{j,\ell_j} p_{jk} \sigma^{\mu_j/2-1} \beta_k^*$$

and so

$$t_{\sigma_0,K}(\mathcal{P} u_0, \sigma u_0) = \sum_{j,j' \in J} \sum_{k=1}^{d} p_{jk} u_{j,\ell_j} \sigma^{\mu_j/2-1} u_{j',\ell_{j'}} t_{\sigma_0,K}(\beta_k, \sigma^{-\mu_j/2} \psi_{j'})$$

$$= \sum_{j,j' \in J} p_{jj'} \sigma^{\mu_j/2-\mu_{j'}/2-1} u_{j,\ell_j} u_{j',\ell_{j'}}$$

It is the residue of this what we will show is nonzero. Terms with $\mu_{j'} < \mu_j$ clearly do not contribute to the residue. For terms with $\mu_{j'} > \mu_j$ we have $p_{jj'}(\sigma) = \sigma^{\mu_j-\mu_{j'}} p_{jj'}(\sigma)$; such terms again contribute nothing, and we conclude that

$$[u_0, \sigma u_0]_\mathcal{P},\sigma_0 = i \sum_{j,j' \in J} p_{jj'}(0) u_{j,\ell_j} \pi_{j',\ell_{j'}}$$

The positivity of $\mathcal{P}$ now enters again: as pointed out at the end of the proof of Lemma 6.10, the selfadjoint matrices $[p_{jj'}]$ with $j, j'$ such that $\mu_j = \mu_{j'}$ are positive definite. Thus $[u_0, \sigma u_0]_\mathcal{P},\sigma_0 \neq 0$ since $u_{j,\ell_j} \neq 0$ for at least for one $j$. \qed
Remark 6.12. Note that when the $\mu_j$ are even numbers the space
\[
\hat{E}_{\sigma_0, \frac{1}{4}} = \text{span}_C\{(\sigma - \sigma_0)\ell \psi_j \mid j = 0, \ldots, d; \ell = \mu_j/2, \ldots, \mu_j - 1\}
\]
is a canonical saturated subspace of $E_{\sigma_0}$, even if the operator $P$ is not selfadjoint. The same holds for $E_{\sigma_0, \frac{1}{4}}^\ast$ and we have $(E_{\sigma_0, \frac{1}{4}})^\perp = E_{\sigma_0, \frac{1}{4}}^\ast$.

7. Structure of the Adjoint Pairing

Let now $E \to M$ be a vector bundle, $H^s_b(M; E)$ be the totally characteristic Sobolev space of order $s$, defined as usual. Suppose that $A = x^{-\nu}P \in x^{-\nu}\text{Diff}^m_b(M; E)$, $\nu > 0$, is a $b$-elliptic cone operator considered initially as a densely defined unbounded operator
\[
A : C_c^\infty(\hat{M}; E) \subset x^{-\nu/2}L^2_0(M; E) \to x^{-\nu/2}L^2_0(M; E),
\]
and define $D_{\min} = D_{\min}(A), D_{\max} = D_{\max}(A)$ as in Section 3. It is well known from the proof of the existence of asymptotic expansions of solutions of $Pu = 0$ (cf. [3], [10], [14]) that if $u \in x^{-\nu/2}L^2_0(M; E)$ and $Au \in x^{-\nu/2}L^2_0(M; E)$, then $u \in D_{\max}$, then $u$ is meromorphic in $\Im \sigma > -\nu/2$ with values in $H^m(\partial M; E_{|\partial M})$ and poles contained in
\[
\Sigma(A) = (\text{spec}_b(A) - i\mathbb{N}_0) \cap \{\sigma \mid -\nu/2 < \Im \sigma < \nu/2\}. 
\]
Moreover, if for $u$ as above, $\hat{u}$ is holomorphic in $\Im \sigma > -\nu/2$, then in fact $u \in D_{\max} \cap x^{\nu/2-\varepsilon}H^m_b(M; E)$ for any $\varepsilon > 0$, so by Proposition 3.14, $u \in D_{\min}(A)$. Thus $E(A) = D_{\max}/D_{\min}$ is isomorphic to a certain subspace $\hat{E}(A)$ of
\[
\mathfrak{M}_{\Omega, \Sigma}(H^m(\partial M; E_{|\partial M}))/\mathfrak{H}o\mathfrak{L}i(H^m(\partial M; E_{|\partial M}))
\]
where $\mathfrak{M}_{\Omega, \Sigma}(H^m(\partial M; E_{|\partial M}))$ is the space of meromorphic $H^m(\partial M; E_{|\partial M})$-valued functions on $\Omega = \{\sigma \mid -\nu/2 < \Im \sigma\}$ and $\mathfrak{H}o\mathfrak{L}i(H^m(\partial M; E_{|\partial M}))$ is the subspace of holomorphic elements. It is clear that the space in (7.2) is localizable on $\text{spec}_b(A)$ in the sense that it is isomorphic to the direct sum
\[
\bigoplus_{\Sigma(A)} \mathfrak{M}_{\Omega, \Sigma}(H^m(\partial M; E_{|\partial M}))/\mathfrak{H}o\mathfrak{L}i(H^m(\partial M; E_{|\partial M}))
\]
where
\[
\Sigma(A) = \{\sigma \in \text{spec}_b(A) \mid -\nu/2 < \Im \sigma < \nu/2\}
\]
and
\[
\Sigma_\sigma' = \{\sigma - i\ell \mid \ell \in \mathbb{N}_0, \ell < \Im \sigma + \nu/2\}.
\]
It is also the case that $\hat{E}(A)$ is localizable: we will show that $\hat{E}(A)$ is the direct sum of the spaces
\[
\hat{E}_\sigma(A) = \hat{E}(A) \cap [\mathfrak{M}_{\Omega, \Sigma}(H^m(\partial M; E_{|\partial M}))/\mathfrak{H}o\mathfrak{L}i(H^m(\partial M; E_{|\partial M}))].
\]
To see this, begin by writing
\[
P = \sum_{k=0}^N P_kx^k + \hat{P}_Nx^N
\]
where the $P_k$ have coefficients independent of $x$ near $\partial M$ and $N = \min\{k \in \mathbb{N} \mid \nu \leq k\}$. Let $\sigma_0 \in \Sigma(A) = \text{spec}_b(A) \cap \{\sigma \mid -\nu/2 < \Im \sigma < \nu/2\}$. By the discussion immediately preceding Proposition 3.9 and the proposition itself with $P = \hat{P}_0$.
at \( \sigma_0 \), there are elements \( \psi_{\sigma_0,j,0,0} \in \mathcal{M}_{\sigma_0}(C^\infty(\partial M; E_{\partial M})) \) and positive integers \( \mu_{\sigma_0,j} \) (forming a nonincreasing sequence), \( j = 1 \ldots, d_{\sigma_0} \), such that for any \( u \in \mathcal{D}_{\max} \), if \( \hat{P}_0(\sigma) \hat{u}(\sigma) \) is holomorphic at \( \sigma_0 \), then there are polynomials \( p_j \) such that

\[
\hat{u} - \sum_{j=1}^{d_{\sigma_0}} p_j(\sigma) \psi_{\sigma_0,j,0,0}(\sigma) \text{ is holomorphic at } \sigma_0.
\]

If \( \Im \sigma_0 - \vartheta > -\nu/2 \) define \( \psi_{\sigma_0,j,0,0,\vartheta} \) inductively as the singular part at \( \sigma_0 - i\vartheta \) of

\[
-\hat{P}_0^{-1}(\sigma) \sum_{\zeta=0}^{\vartheta-1} \hat{P}_0^{-\zeta}(\sigma) \psi_{\sigma_0,j,0,0,\zeta}(\sigma + i(\vartheta - \zeta)).
\]

and for convenience define \( \psi_{\sigma_0,j,0,0,\vartheta} = 0 \) if \( \Im \sigma_0 - \vartheta \leq -\nu/2 \). The largest index \( \theta \) such that \( \Im \sigma_0 - \vartheta > -\nu/2 \) will be denoted by \( N(\sigma_0) \). Define also \( \psi_{\sigma_0,j,\ell,\vartheta} \) (for \( \ell = 0 \ldots, \mu_j - 1 \)) to be the principal part of \( (\sigma + i\vartheta)^{\ell} \psi_{\sigma_0,j,0,0,\vartheta} \) (at \( \sigma_0 - i\vartheta \)), and finally, let

\[
\Psi_{\sigma_0,j,\ell} = \sum_{\vartheta \geq 0} \psi_{\sigma_0,j,\ell,\vartheta}.
\]

Then

\[
\sum_{k=0}^{N} P_k(\sigma) \Psi_{\sigma_0,j,\ell}(\sigma + ik)
\]

is holomorphic in \( \Im \sigma > -\nu/2 - \varepsilon \) for any sufficiently small \( \varepsilon > 0 \). The claim is now that the images in \( \hat{\mathcal{E}}(A) \) of the \( \Psi_{\sigma_0,j,\ell} \) form a basis (over \( \mathbb{C} \)). This is easy to prove beginning with the fact that the \( \psi_{\sigma_0,j,\ell,0} \) form a basis of \( \hat{P}_0(\sigma)^{-1} \mathcal{H}_{\sigma_0} \). Once this is proved, the assertion about \( \hat{\mathcal{E}}(A) \) being localizable is clear.

Note that one may multiply each \( \psi_{\sigma_0,j,\ell,\vartheta} \) by a suitable entire function which is equal to 1 to high order at \( \sigma_0 - i\vartheta \) so that the resulting function is, modulo an entire function, the Mellin transform of an element in \( x^{-\nu/2} H^\infty(M; E) \). This will not change the fact that the images in \( \hat{\mathcal{E}}(A) \) of the modified \( \Psi_{\sigma_0,j,\ell} \) form a basis. An immediate convenient consequence is

**Lemma 7.6.** For each \( u \in \mathcal{D}_{\max}(A) \) there is \( u_0 \in \mathcal{D}_{\min}(A) \) such that \( (u - u_0)^{-} \) is meromorphic on \( \mathbb{C} \) with poles only in \( \Sigma'(A) \).

**Definition 7.7.** For \( \sigma_0 \in \Sigma(A) \), \( \mathcal{D}_{\sigma_0}(A) \) is the space of elements \( u \in \mathcal{D}_{\max}(A) \) such that \( \hat{u}(\sigma) \) represents an element in \( \hat{\mathcal{E}}_{\sigma_0}(A) \), that is, \( \hat{u} \) has poles at most at \( \sigma_0 - i\vartheta \) for \( \vartheta = 0 \ldots, N(\sigma_0) \). Further,

\[
\mathcal{E}_{\sigma_0}(A) = \mathcal{D}_{\sigma_0}(A)/\mathcal{D}_{\min}(A).
\]

For \( \sigma_0 \in \text{spec}_{\mathbb{C}}(A) \cap \{ \Im \sigma = 0 \} \), and if all the multiplicities \( \mu_{\sigma_0,j} \) associated with \( \sigma_0 \) are even, we let \( \mathcal{D}_{\sigma_0,1}(A) \) be the space of elements \( u \in \mathcal{D}_{\sigma_0}(A) \) such that

\[
\hat{u} \mod \mathcal{H}_{\sigma_0}(\Im \sigma > -\varepsilon) \text{ belongs to } \hat{\mathcal{E}}_{\sigma_0,1}.
\]

for any small \( \varepsilon > 0 \). The space \( \hat{\mathcal{E}}_{\sigma_0,1} \) is the one defined in Proposition 7.11, now with \( P = \hat{P}_0 \).

Thus, modulo \( \mathcal{H}_{\sigma_0}(\Im \sigma > -\nu) \), the Mellin transform of an element \( u \in \mathcal{D}_{\sigma_0}(A) \) can be written as

\[
\hat{u}(\sigma) = \sum_{\vartheta = 0}^{N(\sigma_0)} \psi_{\vartheta}(\sigma) \text{ where the } \psi_{\vartheta}(\sigma) \text{ have poles only at } \sigma_0 - i\vartheta,
\]
and from the fact that \( \hat{P}u(\sigma) = \sum_{\nu, \delta} \hat{P}_\nu(\sigma)\psi_\delta(\sigma + i\delta) \) is holomorphic in \( \Im \sigma > -\nu/2 \) one deduces that

\[
(7.8) \quad \sum_{\nu=0}^\ell P_{\nu-\nu}(\sigma)\psi_\delta(\sigma + i(\ell - \delta)) \text{ is holomorphic if } \ell \leq N(\sigma_0).
\]

Likewise, if \( v \in \mathcal{D}_{\sigma_0}^*(A^*) \) then \( \hat{v}(\sigma) = \sum_{\nu=0}^N(\sigma_0^*) \psi_\delta^*(\sigma) \) where now

\[
(7.9) \quad \sum_{\nu=0}^\ell P_{\nu-\nu}(\sigma + i(\ell - \delta))\psi_\delta^*(\sigma + i(\ell - \delta)) \text{ is holomorphic if } \ell \leq N(\sigma_0^*)
\]

since from (7.5),

\[
(7.10) \quad P^* = \sum_{\ell=0}^{N-1} x^\ell P_\ell^* + x^N \hat{P}_N^*.
\]

The closed extensions of \( A : \mathcal{D}_{\min}(A) \subset x^{-\nu/2}L^2_\delta(M ; E) \to x^{-\nu/2}L^2_\delta(M ; E) \) are in one to one correspondence with the subspaces of \( \mathcal{E}(A) \), therefore with the subspaces of \( \hat{\mathcal{E}}(A) \). Since we are interested in duality and selfadjoint extensions, we will now turn our attention towards understanding the pairing \( \langle u, v \rangle_A \) for \( u \in \mathcal{D}_{\max}(A) \) and \( v \in \mathcal{D}_{\max}(A^*) \) as a pairing of elements of \( \hat{\mathcal{E}}(A) \). In the following theorem and its proof, the pairing in the integrands is that of \( L^2(\partial M ; E|_{\partial M}) \).

**Theorem 7.11.** Let \( A = x^{-\nu}P \in x^{-\nu}\text{Diff}^m_\delta(M ; E) \) be b-elliptic, let \( \sigma_0 \in \Sigma(A) \) and \( \sigma_0^* \in \Sigma(A^*) \), suppose \( u \in \mathcal{D}_{\max}(A) \) and \( v \in \mathcal{D}_{\max}(A^*) \) are such that

\[
\hat{u} = \sum_{\nu=0}^{N(\sigma_0)} \psi_\delta \quad \text{and} \quad \hat{v} = \sum_{\nu=0}^{N(\sigma_0^*)} \psi_\delta^* \mod \mathfrak{H}o(\Im \sigma > -\nu/2),
\]

where \( \psi_\delta \) has a pole only at \( \sigma_0 - i\delta \), and \( \psi_\delta^* \) only at \( \sigma_0^* - i\delta \), in other words, \( u \in \mathcal{D}_{\sigma_0}(A) \) and \( v \in \mathcal{D}_{\sigma_0^*}(A^*) \). If \( \sigma_0 \) is not of the form \( \sigma_0^* + i\tau \) with \( \tau \in \mathbb{N}_0 \), then \( \langle u, v \rangle_A = 0 \). Otherwise, if \( \sigma_0 = \sigma_0^* + i\tau \) for \( \tau \in \mathbb{N}_0 \), then

\[
\langle u, v \rangle_A = \frac{1}{2\pi} \sum_{\nu=0}^{\tau} \int_{\gamma_\nu} (\psi_{\tau - \nu}(\sigma), \sum_{\nu=0}^{\tau} \hat{P}_{\nu - \nu}(\sigma - i(\delta - \delta'))\psi_{\delta'}^*(\sigma - i(\delta - \delta'))) d\sigma,
\]

where \( \gamma_\nu = \gamma_0 + i\delta \) and \( \gamma_0 \) is a positively oriented simple closed curve surrounding \( \sigma_0^* \). In particular, if \( \tau = 0 \), i.e., \( \sigma_0 = \sigma_0^* \), then

\[
(7.12) \quad \langle u, v \rangle_A = \frac{1}{2\pi} \int_{\gamma_0} (\psi_0(\sigma), \hat{P}_0^*(\sigma)\psi_0^*(\sigma)) d\sigma.
\]

**Proof.** For general \( u \in \mathcal{D}_{\max}(A) \) and \( v \in \mathcal{D}_{\max}(A^*) \) and \( \omega \in C^\infty(M) \) supported near the boundary and equal to 1 near the boundary one has

\[
\langle u, v \rangle_A = \langle [\omega u, \omega v] \rangle_A
\]

because \( (1 - \omega)u \in \mathcal{D}_{\min}(A) \) if \( u \in \mathcal{D}_{\max}(A) \), and analogously for \( (1 - \omega)v \). Recall that the Mellin transform was defined using a cut-off function like \( \omega \). Suppose \( P \) is written as in (7.3) where the \( P_\ell \) have coefficients independent of \( x \) near the
boundary, say, in a neighborhood of the closure of the support of \( \omega \). Then, using the expression for \( A^* \) obtained from (7.10), we have

\[
[wu, wv]_A = (x^{-\nu} \sum_{\ell=0}^{N-1} P_{\ell} x^\ell \omega u, \omega v)_{x^{\nu/2}L^2} - (wu, x^{-\nu} \sum_{\ell=0}^{N-1} x^\ell P_{\ell}^* \omega v)_{x^{\nu/2}L^2}
\]

\[
+ (x^{-\nu} \tilde{P}_N x^N \omega u, \omega v)_{x^{\nu/2}L^2} - (wu, x^N \tilde{P}_N x^{-\nu} \omega v)_{x^{\nu/2}L^2}.
\]

The last two terms cancel out since \( x^N u \in x^{\nu/2}H^m \) can be approximated from \( C^\infty(M; E) \) in \( x^{\nu/2}H^m \) norm. Thus

\[
[wu, wv]_A = (x^{-\nu} \sum_{\ell=0}^{N-1} P_{\ell} x^\ell \omega u, \omega v)_{x^{\nu/2}L^2} - (wu, x^{-\nu} \sum_{\ell=0}^{N-1} x^\ell P_{\ell}^* \omega v)_{x^{\nu/2}L^2}.
\]

It is always the case that \( \hat{u} \) has no poles on \( \Im \sigma = \nu/2 \), and adding a suitable element of \( D_{\text{min}}(A) \) to \( u \) we may assume that \( \hat{u} \) has no poles on \( \Im \sigma = -\nu/2 \). A similar remark applies to \( \hat{v} \), and we may and will assume that neither \( \hat{u} \) nor \( \hat{v} \) has poles on \( \Im \sigma = -\nu/2 \). For \( \varepsilon > 0 \) let

\[
(7.13) \quad \beta_0 = -\frac{\nu}{2} + \varepsilon \quad \text{and} \quad \beta_k = \frac{\nu}{2} - \varepsilon - N + k \quad \text{for} \quad k = 1, \ldots, N.
\]

We take \( \varepsilon > 0 \) sufficiently small so that \( \beta_0 < \beta_1 \). There is \( \varepsilon_0 > 0 \) such that for any \( \varepsilon < \varepsilon_0 \), if \( \sigma_0 \in \text{spec}_b(A) \cup \text{spec}_b(A^*) \) and \( -\nu/2 < \Im \sigma < \nu/2 \), then for any \( \ell \in \mathbb{N}_0 \), the point \( \sigma - i\ell \) does not lie on a line \( \Im \sigma = \beta_k \). That is, no \( u \in D_{\text{max}}(A) \) or \( v \in D_{\text{max}}(A^*) \) has poles on a line \( \Im \sigma = \beta_k \). Fix one such \( \varepsilon \) and let

\[
S_k = \{ \sigma \in \mathbb{C} \mid \beta_k \leq \Im \sigma \leq \beta_{k+1} \}.
\]

These strips partition \( \{ \sigma \in \mathbb{C} \mid -\nu + \varepsilon \leq \Im \sigma < \nu/2 - \varepsilon \} \). We now show that

\[
(7.14) \quad [u, v]_A = \sum_{k=0}^{N-1} \int_{\beta_k}^{\beta_{k+1}} \sum_{\ell=0}^{N-k-1} \hat{P}_\ell(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma) \rangle d\sigma
\]

For any \( \ell < N \) and \( \varepsilon \) small, we have, on the one hand,

\[
\frac{1}{x^{\nu}} P_{\ell} x^\ell u, v \rangle = (x^{-\nu} P_{\ell} x^\ell u, x^{-\nu} v)
\]

\[
= \frac{1}{2\pi} \int_{\Im \sigma = \frac{\varepsilon}{2}} (\hat{P}_\ell(\sigma + i(\varepsilon - \nu)) \hat{u}(\sigma + i(\varepsilon - \nu + \ell)), \hat{v}(\sigma - i\varepsilon)) d\sigma
\]

\[
= \frac{1}{2\pi} \int_{\Im \sigma = -\frac{\varepsilon}{2}} (\hat{P}_\ell(\sigma + i\varepsilon) \hat{u}(\sigma + i(\varepsilon + \ell)), \hat{v}(\sigma + i\varepsilon)) d\sigma
\]

\[
= \frac{1}{2\pi} \int_{\Im \sigma = \beta_0} (\hat{P}_\ell(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma)) d\sigma
\]
and
\[
(u, \frac{1}{x^\nu} x^\ell P_k^* v) = (x^{-\nu} u, x^{\nu-\nu} x^\ell P_k^* v)
\]
\[
= \frac{1}{2\pi} \int_{\partial S_k} (\hat{u}(\sigma - i\varepsilon), \hat{P}_k^*(\sigma + i(\varepsilon - \nu + \ell))) v(\sigma + i(\varepsilon - \nu + \ell)) d\sigma
\]
\[
= \frac{1}{2\pi} \int_{\partial S_k} (\hat{u}(\sigma - i\varepsilon + i\ell), \hat{P}_k^*(\sigma - i\varepsilon) v(\sigma - i\varepsilon)) d\sigma
\]
\[
= \frac{1}{2\pi} \int_{\partial S_k} (\hat{P}_k(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma)) d\sigma
\]

so
\[
[u, v]_A = \sum_{\ell=0}^{N} \frac{1}{2\pi} \int_{\partial S_k} \hat{P}_k(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma)) d\sigma
\]
\[
= \sum_{\ell=0}^{N} \sum_{k=0}^{N-\ell-1} \hat{P}_k(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma)) d\sigma
\]
\[
= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-k-1} \hat{P}_k(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma)) d\sigma
\]

Let \(\sigma_0 \in \Sigma\) and suppose \(\hat{u} = \sum_{\vartheta=0}^{\infty} \psi_\vartheta\), where \(\psi_\vartheta\) has a pole only at \(\sigma_0 - i\vartheta\) and \(\psi_0 = 0\) if \(\vartheta > N(\sigma_0)\), where \(N(\sigma)\) be the number \(k\) such that \(\sigma \in S_k\). Thus \(\hat{P}_k(\sigma) \psi_\vartheta(\sigma + i\ell)\) has a pole at \(\sigma_0 - i(\vartheta + \ell)\) if at all, and the poles of
\[
(7.15) \quad \sum_{\ell=0}^{N-k-1} \hat{P}_k(\sigma) \psi_\vartheta(\sigma + i\ell) = \sum_{\ell=0}^{N-k-1} \sum_{\vartheta=0}^{\infty} \hat{P}_k(\sigma) \psi_\vartheta(\sigma + i\ell),
\]
if any, that lie in \(\beta_k < \Im \sigma < \beta_{k+1}\), come from indices \(\vartheta, \ell\) with
\[
\beta_k < \Im \sigma_0 - (\vartheta + \ell) < \beta_{k+1},
\]
that is, \(\vartheta + \ell = N(\sigma_0) - k\). So in (7.15) only the terms
\[
\sum_{\vartheta=0}^{N(\sigma_0)-k} \hat{P}_N(\sigma_0-k-\vartheta) \psi_\vartheta(\sigma + i(N(\sigma_0) - k - \vartheta))
\]
may produce poles in \(S_k\). If \(k > N(\sigma_0)\) there are no poles. If \(k \leq N(\sigma_0)\), this is
\[
(7.16) \quad [u, v]_A = \frac{1}{2\pi} \sum_{k=0}^{N-1} \sum_{\vartheta=0}^{N-k-1} \oint_{\partial S_k} \hat{P}_k(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma)) d\sigma,
\]
where the $\gamma_{s,k}$ are simple closed positively oriented curves surrounding (and separating) the conjugates of the poles of $\hat{v}(\sigma)$ in the strip $S_k$.

Suppose now that also $\hat{v} = \sum_{\vartheta=0}^{\vartheta_n} \psi_\vartheta^*$, where $\psi_\vartheta^*$ has a pole only at $\sigma_0^* - i\vartheta$. Here $\sigma_0^* \in \Sigma(A^*) = \Sigma(A)$ and as before, $\psi_\vartheta^* = 0$ if $\vartheta > N(\sigma_0^*)$. Thus

$$\sigma \mapsto (\sum_{\ell=0}^{N-k-1} \hat{P}_\ell(\sigma) \hat{u}(\sigma + i\ell), \hat{v}(\sigma))$$

has poles at $\overline{\sigma_0^* + i\vartheta}, \vartheta = 0, \ldots, N(\sigma_0^*)$, and the pole in $S_k$ satisfies $\beta_k < \Im\overline{\sigma_0^* + \vartheta} < \beta_{k+1}$, that is, $k = N(\sigma_0^*) + \vartheta$. In particular, there are poles only in the strips with $k$ satisfying $N(\overline{\sigma_0^*}) \leq k \leq N - 1$.

Pick a positively oriented simple closed curve $\gamma_0$ surrounding $\overline{\sigma_0^*}$, let $\gamma_\vartheta = \gamma_0 + i\vartheta$. Using this, the right hand side of (7.16) becomes

$$\frac{1}{2\pi} \sum_{k=N(\overline{\sigma_0^*})}^{N-1} \mathcal{F}_{\gamma_{k-N(\overline{\sigma_0^*)}}}^{\gamma_0}(\hat{P}_\ell(\sigma - i\ell) \hat{u}(\sigma), \hat{v}(\sigma - i\ell)) \, d\sigma$$

Now replace $\sigma + i\ell$ by $\tilde{\sigma}$. The resulting expression is (dropping the tilde)

$$\frac{1}{2\pi} \sum_{k=N(\overline{\sigma_0^*})}^{N-1} \sum_{\ell=0}^{N-k-1} \int_{\gamma_{k-N(\overline{\sigma_0^*)}}}^{\gamma_0} (\hat{P}_\ell(\sigma - i\ell) \hat{u}(\sigma), \hat{v}(\sigma - i\ell)) \, d\sigma$$

after reorganizing (notice that $N - N(\overline{\sigma_0^*}) - 1 \leq N(\sigma_0^*)$). Now, if $\hat{v}(\sigma) = \sum_{\vartheta=0}^{\infty} \psi_\vartheta^*$ is as above, then for any given $\vartheta$ the only terms in

$$\sum_{k=0}^{N(\sigma_0^*)} \hat{P}_{\sigma - \vartheta}^*(\sigma + i(\vartheta - k)) \sum_{\vartheta'=0}^{N(\sigma_0^*)} \psi_{\vartheta'}^*(\sigma + i(\vartheta - k))$$

which may contribute to the integral along $\gamma_\vartheta$ are those which in principle have poles at $\overline{\sigma_0^* - i\vartheta}$, namely those in the sum

$$\sum_{\vartheta'=0}^{N(\sigma_0^*)} \hat{P}_{\sigma - \vartheta'}^*(\sigma + i(\vartheta - \vartheta')) \psi_{\vartheta'}^*(\sigma + i(\vartheta - \vartheta'))$$

But according to (7.9), this has no poles at $\overline{\sigma_0^* - i\vartheta}$ (or anywhere else, for that matter). So, if $\hat{u}$ has no poles in $\{\overline{\sigma_0^* + i\vartheta} | \vartheta = 0, \ldots, N(\sigma_0^*)\}$ then $[u, v]_A = 0$.

The only case where $[u, v]_A$ may be different from 0 occurs when there are integers $\vartheta, \vartheta' \geq 0$ such that $\sigma_0 - i\vartheta = \overline{\sigma_0^* + \vartheta'}$, that is, if $\sigma_0 = \sigma_0^* + i\tau$ for some nonnegative integer $\tau$, in which case

$$[u, v]_A = \frac{1}{2\pi} \sum_{\vartheta=0}^{\tau} \int_{\gamma_\vartheta} (\psi_{\tau - \vartheta}(\sigma), \sum_{\vartheta'=0}^{\infty} \hat{P}_{\sigma - \vartheta'}^*(\overline{\sigma - i(\vartheta - \vartheta')}) \psi_{\vartheta'}^*(\sigma - i(\vartheta - \vartheta'))) \, d\sigma$$

as claimed in the theorem."
If $\sigma_0, \sigma_1 = \overline{\sigma_0} \in \Sigma(A)$ are such that $\sigma_0 = \sigma_1 + i\tau$ with some integer $\tau > 0$, then if $u \in D_{\sigma_0}(A)$ and $v \in D_{\sigma_1}(A^*)$ it may happen that $[u, v]_A \neq 0$.

Since $[u, v]_A = 0$ if $u \in D_{\min}(A)$ and $v \in D_{\max}(A^*)$, or if $u \in D_{\max}(A)$ and $v \in D_{\min}(A^*)$, there is a well defined pairing of elements of $\mathcal{E}(A)$ and $\mathcal{E}(A^*)$.

**Theorem 7.17.** $[\cdot, \cdot]^b_A$ is a nonsingular pairing of $\mathcal{E}_{\sigma_0}(A)$ and $\mathcal{E}_{\sigma_0}(A^*)$.

**Proof.** We work with (7.12). $\hat{P}_0(\sigma)$ is a closed operator $L^2(Y) \to L^2(Y)$ with domain $H^m = H^m(Y; E)$. Let $K = \ker \hat{P}_0(\sigma_0)$, $R = \hat{P}_0(\sigma_0)(H^m(Y))$. Decompose $P(\sigma)$ as

$$
\begin{bmatrix}
\hat{P}_{11}(\sigma) & \hat{P}_{12}(\sigma) \\
\hat{P}_{21}(\sigma) & \hat{P}_{22}(\sigma)
\end{bmatrix} = \begin{bmatrix} K \\ K \cap H^m \end{bmatrix} \oplus \begin{bmatrix} R \\ R \cap H^m \end{bmatrix}
$$

for $\sigma$ near $\sigma_0$. Here $K$, $R$ are computed in $L^2(Y)$. Since $\hat{P}_0(\sigma_0)$ is Fredholm, $\hat{P}_0(\sigma_0)(H^m) = K^\perp$ and the analogous decomposition for $\hat{P}(\sigma)^* = \hat{P}(\sigma)$ is

$$
\begin{bmatrix}
\hat{P}^*_{11}(\sigma) & \hat{P}^*_{12}(\sigma) \\
\hat{P}^*_{12}(\sigma) & \hat{P}^*_{22}(\sigma)
\end{bmatrix} = \begin{bmatrix} R \\ R \cap H^m \end{bmatrix} \oplus \begin{bmatrix} K \\ K \cap H^m \end{bmatrix}
$$

near $\sigma_0$. Since $\text{ind} \hat{P}(\sigma) = 0$, $\dim K = \dim R^\perp$.

Let $u \in \mathcal{E}_{\sigma_0}(A)$ represent an element in $D_{\sigma_0}(A)$, let $\psi$ be the Mellin transform of $\phi u$. The principal part of $\psi$ at $\sigma_0$ is the principal part $\psi_0$ at $\sigma_0$ of a germ of the form

$$
\tilde{\psi} = \hat{P}^{-1}_{22}\hat{P}_{21}\tilde{\psi}
$$

where $\tilde{\psi} \in \mathfrak{M}_{\sigma_0}(K)$ is such that

$$
(\hat{P}_{11} - \hat{P}_{12}\hat{P}^{-1}_{22}\hat{P}_{21})\tilde{\psi} = \beta
$$

is holomorphic near $\sigma_0$. Likewise let $u^* \in \mathcal{E}_{\sigma_0}(A^*)$ represent an element in $D_{\sigma_0}(A^*)$, $\psi^*$ the Mellin transform of $\phi u^*$. Again the principal part $\psi^*_0$ of $\psi^*$ at $\sigma_0$ is the principal part at $\sigma_0$ of germ of the form

$$
\tilde{\psi}^* = (\hat{P}^{-1}_{22})^*\hat{P}^*_{12}\tilde{\psi}^*
$$

where $\tilde{\psi} \in \mathfrak{M}_{\sigma_0}(R^\perp)$ is such that

$$
(\hat{P}^*_{11} - \hat{P}^*_{21}(\hat{P}^{-1}_{22})^*\hat{P}^*_{12})\tilde{\psi}^* = \beta^*
$$

is holomorphic, near $\sigma_0$. Let

$$
\mathcal{P} = \hat{P}_{11} - \hat{P}_{12}\hat{P}^{-1}_{22}\hat{P}_{21}
$$

$$
\mathcal{P}^* = \hat{P}^*_{11} - \hat{P}^*_{21}(\hat{P}^{-1}_{22})^*\hat{P}^*_{12}
$$

Then as discussed before the lemma,

$$
[u, v]_A = \frac{1}{2\pi} \oint_{\gamma_0} (\psi(\sigma), \hat{P}^*_0(\sigma)\psi^*(\sigma))d\sigma
$$

with a positively oriented curve $\gamma_0$ surrounding $\sigma_0$ and no other pole. Since
\[ \tilde{P}_0^*(\tilde{\psi}^* - (\tilde{P}_2^*)^* \tilde{P}_{12}^* \tilde{\psi}^*) \]
\[ = \tilde{P}_{11}^* \tilde{\psi}^* - \tilde{P}_{21}^* (\tilde{P}_2^*)^* \tilde{P}_{12}^* \tilde{\psi}^* + \tilde{P}_{12}^* (\tilde{P}_2^*)^* \tilde{P}_{12}^* \tilde{\psi}^* \]
\[ = P^* \tilde{\psi}^* \]

and
\[ (\psi(\sigma), P^*(\sigma) \tilde{\psi}^*(\sigma)) = (\tilde{\psi}(\sigma) - \tilde{P}_{22}(\sigma^{-1} \tilde{P}_{21}(\sigma) \tilde{\psi}(\sigma), P^*(\sigma) \tilde{\psi}^*(\sigma)) \]
\[ = (\tilde{\psi}(\sigma), P^*(\sigma) \tilde{\psi}^*(\sigma)) \]

we have
\[ [u, v]_A = \frac{1}{2\pi} \int_0^\nu (\tilde{\psi}(\sigma), P^*(\sigma) \tilde{\psi}^*(\sigma))d\sigma. \]

Thus the pairing of \( E_{\sigma_0}(A) \) and \( E_{\sigma_0}(A^*) \) is the pairing of the spaces associated to \( P \) at \( \sigma_0 \) and \( P^* \) at \( \sigma_0 \) which Theorem 6.4 asserts is nonsingular.

8. FRIEDRICHSS EXTENSION

Suppose \( A \in x^{-\nu}\, \text{Diff}^m_b(M; E), \) \( A = x^{-\nu}P, \) is \( b \)-elliptic, symmetric and bounded from below by some \( c_0 \leq 0, \) as an operator \( C_\infty^r(M; E) \subset x^{-\nu/2}L_b^2(M; E) \rightarrow x^{-\nu/2}L_b^2(M; E). \) The domain of the Friedrichs extension is denoted \( D_F(A). \) Recall (Definition \( \text{[6.3]} \) that we denote by \( D_{\sigma_0}(A) \) the space of functions \( u \in D_{\max}(A) \) such that \( \tilde{u}(\sigma) \) has poles at most at \( \sigma_0 - i\theta \) for \( \theta = 0, 1, \ldots, \) by \( E_{\sigma_0} \) the quotient \( D_{\sigma_0}(A)/D_{\min}(A) \) and by \( [u, v]_A = (Au, v) - (u, A^*v), \) as introduced in \( \text{[3.12]}. \)

Lemma 8.1. \( D_F(A) \) contains all \( u \in D_{\max}(A) \) such that \( \tilde{u}(\sigma) \) has no poles in \( \{3\sigma \geq 0\}. \) That is, \( D_{\max}(A) \cap H_b^m(M; E) \subset D_F(A). \)

Proof. We will show that if \( u \in D_{\max}(A) \cap H_b^m(M; E), \) there is a sequence \( \{u_n\}_{n \in \mathbb{N}} \) in \( C_\infty^r(M; E) \) such that
\[ c \|u - u_n\|^2_{x^{-\nu/2}L_b^2} + (A(u - u_n), u - u_n)_{x^{-\nu/2}L_b^2} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

This will imply \( u \in D_F(A) \) by Lemma \( \text{[6.3]} \). Consider \( P = x^\nu A^* \) as an unbounded operator on \( L_b^2(M; E). \) Since it is \( b \)-elliptic, we have \( H_b^m(M; E) \subset D_{\min}(P). \) Therefore, if \( u \in D_{\max}(A) \cap H_b^m(M; E), \) there is a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset C_\infty^r(M; E) \) such that
\[ \|u - u_n\|_{L_b^2(M; E)} \rightarrow 0 \text{ and } \|P(u - u_n)\|_{L_b^2(M; E)} \rightarrow 0 \text{ as } n \rightarrow \infty. \]

With this sequence we have
\[ \|u - u_n\|^2_{x^{-\nu/2}L_b^2} \rightarrow 0 \text{ as } n \rightarrow \infty \]
since \( L_b^2(M; E) \nrightarrow x^{-\nu/2}L_b^2(M; E). \) Also,
\[ (A(u - u_n), u - u_n)_{x^{-\nu/2}L_b^2} = (P(u - u_n), u - u_n)_{L_b^2} \]
\[ \leq \|P(u - u_n)\|_{L_b^2}\|u - u_n\|_{L_b^2} \]
so
\[ (A(u - u_n), u - u_n)_{x^{-\nu/2}L_b^2} \rightarrow 0 \text{ as } n \rightarrow \infty \]
and the proof is complete.
Lemma 8.2. \( \mathcal{D}_F(A) \) contains no \( u \in \mathcal{D}_{\max}(A) \) such that \( \hat{u}(\sigma) \) has a pole in \( \{3\sigma > 0\} \). Thus \( \mathcal{D}_F(A) \subset \mathcal{D}_{\max}(A) \cap x^{-\varepsilon}H_6^m(M; E) \) for any \( \varepsilon > 0 \).

Proof. Let \( \sigma_0 \in \text{spec}_b(A) \) be such that \( 3\sigma_0 > 0 \). Suppose that \( u \in \mathcal{D}_{\sigma_0} \cap \mathcal{D}_F \). In particular, \( [u, v]_A = 0 \) for all \( v \in \mathcal{D}_F(A) \) since \( A_F \) is selfadjoint. From the previous lemma we know that \( \mathcal{D}_{\sigma_0}(A) \subset \mathcal{D}_F(A) \), hence \( [u, v]_A = 0 \) for all \( v \in \mathcal{D}_{\sigma_0}(A) \), hence \( u = 0 \) since by Theorem 7.17 the induced pairing \( [\cdot, \cdot]_A \) of \( \mathcal{E}_{\sigma_0} \) and \( \mathcal{E}_{\sigma_0} \) is nonsingular.

As a consequence of these two lemmas we get

Theorem 8.3. Suppose \( A \in x^{-\nu} \text{Diff}_b^m(M; E) \) is \( b \)-elliptic and semibounded. If

\[ \text{spec}_b(A) \cap \{3\sigma = 0\} = \emptyset, \]

then the domain of the Friedrichs extension of \( A \) is

\[ \mathcal{D}_F(A) = \sum_{\sigma \in \text{spec}_b(A)} \mathcal{D}_\sigma(A) \]

That is, \( \mathcal{D}_F(A) = \mathcal{D}_{\max}(A) \cap H_6^m(M; E) \).

This finishes the discussion of the Friedrichs extension of \( A \) when \( \text{spec}_b(A) \cap \{3\sigma = 0\} \). In order to determine the domain of the Friedrichs extension if \( \text{spec}_b(A) \) does contain real elements, we need two more ingredients. The first is an invariance property, under certain circumstances (small \( \nu \)), of \( \mathcal{D}_F(A) \), which translates into the saturation property on the Mellin transform side. The second is the positivity of the conormal symbol of \( A \) when \( A \) is bounded from below.

Let \( \omega \in C_0^\infty(\mathbb{R}) \) be a function with sufficiently small support, and equal 1 near the origin. Let \( \phi \) be the flow of \( X \), which we shall write multiplicatively: the integral curve of \( X \) through \( p \) is \( t \mapsto \phi_{\tau}(p) \). We can write

\[ \phi_{\tau/2}^*m = c_\tau^2m, \quad \phi_{\tau/2}^*x = \tau^{-1}\xi_\tau^2x \]

with smooth positive functions \( c_\tau \) and \( \xi_\tau \) which are equal to 1 if \( \tau = 1 \), or if \( x \) is close to \( \partial M \) (how close depends on \( \tau \)), or in complement of the support of \( \omega \). Thus we have

\[ \int \phi_{\tau}^*f m = \int \phi_{\tau}^*f \phi_{\tau}^*(\phi_{1/\tau}^*m) = \int f c_\tau^2m \]

Recall that for sections \( u \) of \( E \), \( \phi_{\tau}^*u \) is the section whose value at \( p \) is the result of parallel transport of \( u(\phi_{\tau}(p)) \) to \( p \) along the curve through \( p \). The connection is compatible with the hermitian form on \( E \), so for sections \( u, v \) of \( E \),

\[ (\phi_{\tau}^*u, \phi_{\tau}^*v)_p = (u, v)_{\phi_{\tau}(p)}. \]

Let \( \gamma_{\tau} = \phi_{\tau}^*\frac{1}{\xi_\tau^2c_\tau} \), define

\[ \kappa_{\tau}u = \tau^{\nu/2}\gamma_{\tau}\phi_{\tau}^*u, \quad u \in C_\infty(M; E). \]

Then \( \kappa_{\tau} \) defines an isometry

\[ x^{-\nu/2}L_2^2(M; E) \to x^{-\nu/2}L_6^2(M; E) \]

and \( \kappa_{\tau}^* = \kappa_{1/\tau} \). On functions \( f \), \( \kappa_{\tau} \) is defined as \( \kappa_{\tau}f = \phi_{\tau}^*f \), so that if \( f \) is a function and \( u \) a section of \( E \) then \( \kappa_{\tau}(fu) = \kappa_{\tau}(f)\kappa_{\tau}(u) \).
A subspace $\mathcal{D} \subset x^{-\nu/2}L^2_b(M;E)$ is $\kappa$-invariant if
$$\kappa \tau u \in \mathcal{D} \text{ for every } u \in \mathcal{D} \text{ and } \tau > 0.$$ For example $C^\infty(M;E)$ is $\kappa$-invariant. Let $A \in x^{-\nu} \text{Diff}_b^m(M;E)$ be arbitrary, write $A = \sum_{k=0}^{N-1} x^k A_k + x^N \hat{A}_N$ where $x^\nu A_k \in \text{Diff}_b^m(M;E)$ has coefficients independent of $x$ near $\partial M$, and $\hat{A}_N \in x^{-\nu} \text{Diff}_b^m(M;E)$. Then
$$\kappa \tau \kappa^{-1} = \sum_{k=0}^{N-1} \tau^{\nu-k} x^k A_k + \tau^{\nu-N} x^N \hat{A}_{N,\tau}$$ for some $\hat{A}_{N,\tau} \in x^{-\nu} \text{Diff}_b^m(M;E)$. In particular, for $x$ near $\partial M$ and $\tau$ smaller than some $\tau_0$ (depending on $x$),
$$\kappa \tau \kappa^{-1} \tau^-\kappa = \tau^{-\nu} A_0.$$

This identity and the $\kappa$-invariance of $C^\infty_c(M;E)$ easily imply that the canonical domains $\mathcal{D}_{\min}(A_0), \mathcal{D}_{\max}(A_0)$ and $\mathcal{D}_{\max}(A_0) \cap x^\nu \mathcal{H}^m_b(M;E)$ are $\kappa$-invariant. If $A_0$ is symmetric and bounded from below, then using Lemma 8.3 one also proves easily that the domain $\mathcal{D}_F(A_0)$ of the Friedrichs extension is also $\kappa$-invariant.

**Lemma 8.4.** Let $A \in x^{-\nu} \text{Diff}_b^m(M;E)$ be b-elliptic. Let $\mathcal{D} \subset \mathcal{D}_{\max}(A)$ be a domain on which $A$ is closed. $\mathcal{D}$ is $\kappa$-invariant if and only if the finite dimensional space
$$\hat{\mathcal{D}} = \{ \hat{u} : u \in \mathcal{D} \}/\mathfrak{hol}(\Re \sigma > -\nu/2)$$ is a saturated space.

**Proof.** If $u$ is a smooth function on $M$, then for $\tau > 0$
$$\kappa \tau \omega(x) u(\tau x, y) \frac{dx}{x} = \tau^{i\sigma} \int x^{-i\sigma} \omega(\tau^{-1} x) u(x, y) \frac{dx}{x} = \tau^{i\sigma} \hat{u}(\sigma) + \tau^{i\sigma} \hat{w}_r(\sigma)$$ for $w_r = (\omega(\tau^{-1} x) - \omega(x)) u$. Now, since $\omega(\tau^{-1} x) - \omega(x)$ is a smooth function supported in the interior of $M$, then $\hat{w}_r$ is an entire function, that is, $\kappa \tau \omega(\sigma) - \tau^{i\sigma} \hat{u}(\sigma)$ is entire. The same conclusion holds when $u$ is a smooth section of $E$:
$$\kappa \tau \omega(\sigma) = \tau^{i\sigma} \hat{u}(\sigma) \mod \mathfrak{hol}(\Re \sigma > -\nu/2)$$

This proves that $\kappa \tau \omega \mod \mathfrak{hol}(\Re \sigma > -\nu/2)$ is an element of $\hat{\mathcal{D}}$ if and only if $\hat{\mathcal{D}}$ is invariant under multiplication by $\tau^{i\sigma}$, i.e., if and only if $\hat{\mathcal{D}}$ is saturated, due to Lemma 5.13. The assertion thus follows from the isomorphism between $\mathcal{D}/\mathcal{D}_{\min}$ and $\hat{\mathcal{D}}$ given by the Mellin transform.

The second ingredient we need to determine the Friedrichs extension is the positivity of the conormal symbols of operators bounded from below. This is standard but we provide a proof.

**Lemma 8.5.** Let $A \in x^{-\nu} \text{Diff}_b^m(M;E)$ be b-elliptic, symmetric and bounded from below. For every $\sigma \in \mathbb{R}$ the conormal symbol $P_0(\sigma)$ of $A$ is nonnegative.
Proof. Suppose \( v \in C^\infty(\partial M; E) \). Let \( \phi \in C_c^\infty(0,1) \) be such that \( \int |\phi(x)|^2 \frac{dx}{x} = 1 \), and let

\[
\phi_n(x) = \frac{1}{n^{1/2}} \phi(x^{1/n}), \quad n \in \mathbb{N}.
\]

Then \( \phi_n v \in \mathcal{D}_{\min}(A) \) since it is smooth and supported in the interior of \( M \). It is easy to prove that for real \( \sigma \) one has

\[
(A(x^{i\sigma} \phi_n v), x^{-\nu/2} L^2_\nu(M; E)) \to (\tilde{P}_0(\sigma) v, v)_{L^2(\partial M; E|_{\partial M})} \quad \text{as} \quad n \to \infty.
\]

Pick \( c \) real such that \( A - cI \geq 0 \). The conormal symbol of \( A - cI \) is then the same as that of \( A \). Thus

\[
0 \leq ((A - cI)x^{i\sigma} \phi_n v, x^{-\nu/2} L^2_\nu(M; E)) \to (\tilde{P}_0(\sigma) v, v)_{L^2(\partial M; E|_{\partial M})}.
\]

From Lemma 6.11, the multiplicities associated to each of the points of \( \text{spec}(A) \) lying on the real line are even, and the last part of Definition 7.7 makes sense: there are well defined spaces \( \mathcal{D}_{\sigma, \pm}(A) \) for each \( \sigma \in \text{spec}_b(A) \cap \{ \exists \sigma = 0 \} \).

**Theorem 8.6.** Let \( A \in x^{-\nu} \text{Diff}^m_b(M; E) \) be h-elliptic, symmetric, bounded from below, and such that \( P \) has coefficients independent of \( x \) for \( x \) small. Suppose \( \sigma \in \text{spec}_b(A) \) \( \Rightarrow \exists \sigma = 0 \) or \( |\exists \sigma| > \nu/2 \). Then the domain of the Friedrichs extension of \( A \) is given by

\[
\mathcal{D}_F(A) = \sum_{\exists \sigma = 0} \mathcal{D}_{\sigma, \pm}(A).
\]

In the situation of the theorem, the spaces \( \mathcal{D}_{\sigma, \pm}(A)/\mathcal{D}_{\min}(A) \) agree via the Mellin transform, with those defined in Proposition 6.11 since \( P \) has coefficients independent of \( x \) near the boundary.

Proof. With the hypotheses of the proposition,

\[
\mathcal{E}(A) = \bigoplus_{\exists \sigma = 0} \mathcal{E}_\sigma(A),
\]

where \( \mathcal{E}_\sigma(A) \) was defined in 7.7. Let

\[
\mathcal{E}_F(A) = \mathcal{D}_F(A)/\mathcal{D}_{\min}(A),
\]

a subspace of \( \mathcal{E}(A) \). Passing to the Mellin transform side, \( \hat{\mathcal{E}}_F \) is saturated since \( \mathcal{D}_F \) and \( \mathcal{D}_{\min}(A) \) are \( \kappa \)-invariant, and selfadjoint in \( \hat{\mathcal{E}}(A) \) in the sense of the appendix of Section 8. Since \( A \) with domain \( \mathcal{D}_F \) is selfadjoint. Since \( \hat{\mathcal{E}}_F \) is saturated, by Lemma 6.11 there are saturated subspaces \( \hat{\mathcal{E}}_{\sigma_j,F} \subset \hat{\mathcal{E}}_{\sigma_j}(A) \) such that

\[
\hat{\mathcal{E}}_F = \bigoplus_{\sigma_j \in S} \hat{\mathcal{E}}_{\sigma_j,F}.
\]

Since \( \hat{\mathcal{E}}_F \) is selfadjoint in \( \hat{\mathcal{E}} \), since \([\cdot, \cdot]_A^b \) is nondegenerate, and since \([\hat{u}, \hat{v}]_A^b = 0 \) if \( \hat{u} \in \hat{\mathcal{E}}_{\sigma_j}(A) \) and \( \hat{v} \in \hat{\mathcal{E}}_{\sigma_k}(A) \) with \( \sigma_j \neq \sigma_k \) (Proposition 7.11), each \( \hat{\mathcal{E}}_{\sigma_j,F} \) is selfadjoint in \( \hat{\mathcal{E}}_{\sigma_j} \). Moreover, because \( A \) is bounded from below, \( P = \tilde{P}_0 \) is nonnegative by
Lemma 8.5, and so, by Lemma 6.10 the $\mu_j$ are even. Now Proposition 4.11 applies and we deduce
\[ \xi_{\sigma,j} = \xi_{\sigma,j}^+ \]
But by definition, $D_{\sigma,j}^+$ is the space of elements on $D_{\text{max}}$ such that $\xi$ represents an element in $\xi_{\sigma,j}^+$.
\[
\square
\]

**Lemma 8.7.** Let $A \in x^{-\nu} \text{Diff}^m_0(M; E)$ be $b$-elliptic, symmetric and bounded from below, and let $P = x^{\nu}A$. Then
\[ D_{\text{max}}(A) \cap D_F(x^{-\nu}P) \subset D_F(A) \text{ for any positive } \varepsilon < \nu. \]

**Proof.** Note that $x$ is symmetric in $v$ because $\omega$ shows, in particular, that $A$ is elliptic in the interior and therefore $P$ is also bounded from below because
\[
(x^{-\nu}P)_{|_{\partial M}} = \tilde{x}^{-\varepsilon/2}L^2_0(M; E) \to x^{-\nu/2}L^2_0(M; E).
\]
Then $(x^{-\nu}P(u - u_n), u - u_n)_{x^{-\nu/2}L^2_0} \to 0$ because of (8.8). Since also $\|u - u_n\|_{x^{-\nu/2}L^2_0} \to 0$ because of (8.8), the lemma is proved.
\[ \square \]

**Lemma 8.10.** Let $A = x^{-\nu}P$ be as in Lemma 8.7. Then $P$ can be written as
\[ P = P_0 + xP_1 \]
with $P_0, P_1 \in \text{Diff}^m_0(M; E)$ such that $x^{-\nu}P_0$ is $b$-elliptic, symmetric, bounded from below, and has coefficients independent of $x$ for $x$ small.

**Proof.** Near the boundary $\partial M$, $P$ can be written as $P = \tilde{P}_0 + x\tilde{P}_1$, where $\tilde{P}_0$ has coefficients independent of $x$. Let $\omega \in C^\infty_c(\mathbb{R})$ be equal to 1 near $\partial M$, define
\[ P_0 = \omega\tilde{P}_0 \omega + (1 - \omega)P(1 - \omega). \]
Clearly $(1 - \omega)P(1 - \omega)$ is symmetric and bounded from below. As the conormal symbol of $P$, $\tilde{P} = \tilde{P}_0$ is a selfadjoint holomorphic family in the sense that $P(\sigma)^* = P(\overline{\sigma})$ on $H^m(\partial M; E_{\text{bIM}})$, and positive by Lemma 8.6. From the Mellin transform version of Plancherel’s identity it follows that $\omega P_0 \omega$ is also bounded from below if $\omega$ has sufficiently small support. Evidently, if the support of $\omega$ is small enough, then $P_0$ is elliptic in the interior and therefore $b$-elliptic.
\[ \square \]

**Lemma 8.11.** Let $A = x^{-\nu}P$ with $P = P_0 + xP_1$ as in Lemma 8.10. Then for $0 < \varepsilon < \nu$, $x^{-\varepsilon}P$ and $x^{-\varepsilon}P_0$ are both $b$-elliptic, symmetric and bounded from below as operators on $x^{-\varepsilon/2}L^2_0(M; E)$, and for small $\varepsilon$, $D_F(x^{-\varepsilon}P) = D_F(x^{-\varepsilon}P_0)$.

The first statement follows from Lemma 8.10 and the proof of Lemma 8.7, and the equality of the Friedrichs domains is a consequence of part 3 of Proposition 4.1.
Theorem 8.12. Let $A \in x^{-\nu} \text{Diff}_b^m(M; E)$ be $b$-elliptic, symmetric and bounded from below. Then the domain of the Friedrichs extension of $A$ is

$$\mathcal{D}_F(A) = \sum_{-\nu/2 < \sigma < 0} \mathcal{D}_\sigma(A) + \sum_{\sigma, 0 \in \text{spec}_b(A)} \mathcal{D}_{\sigma}^{+}(A).$$

Proof. Let $\mathcal{D}'_F(A)$ be the space on the right in the statement. Then $A$ with domain $\mathcal{D}'_F(A)$ is selfadjoint, so we only need to prove that $\mathcal{D}'_F(A) \subset \mathcal{D}_F(A)$, and we proceed to do so. Write $x^\nu A = P_0 + xP_1$ as in Lemma 8.10. From Lemmas 8.7 and 8.11 we get that if $\varepsilon > 0$ is small enough then $\mathcal{D}_\text{max}(A) \cap \mathcal{D}'_F(x^{-\varepsilon} P_0) \subset \mathcal{D}_F(A)$. We may apply Theorem 8.6 to $x^{-\varepsilon} P_0$ and deduce that

$$\mathcal{D}_F(x^{-\varepsilon} P_0) = \sum_{\sigma \in \text{spec}_b(A)} \mathcal{D}_{\sigma}^{+}(x^{-\varepsilon} P_0).$$

But the intersection of this space and $\mathcal{D}_\text{max}(A)$ is $\mathcal{D}'_F(A)$. Thus $\mathcal{D}'_F(A) \subset \mathcal{D}_F(A)$ and therefore $\mathcal{D}'_F(A) = \mathcal{D}_F(A)$. \hfill $\square$

Together with Theorem 8.3 we in particular obtain

$$\mathcal{D}_F = \mathcal{D}_\text{max}(A) \cap \mathcal{H}_b^m(M; E)$$

if and only if $\text{spec}_b(A) \cap \{3\sigma = 0\} = \emptyset$.

The following corollary improving part [4] of Proposition [4] is an immediate consequence of the theorem, since the hypothesis implies that for $-\nu \leq 3\sigma \leq 0$ the spaces $\mathcal{D}_\sigma$ for both operators are equal:

Corollary 8.13. Suppose $A_0, A_1 \in x^{-\nu} \text{Diff}_b^m(M; E)$ are $b$-elliptic, symmetric and bounded from below. If $A_0 - A_1$ vanishes to order $k$, $k > \nu/2$, then $\mathcal{D}_F(A_0) = \mathcal{D}_F(A_1)$.

9. Applications and Examples

Let $A \in x^{-\nu} \text{Diff}_b^m(M; E)$ be $b$-elliptic and assume that for any two distinct $\sigma_0$ and $\sigma_1$ in $\Sigma(A)$, $3(\sigma_0 - \sigma_1) \notin \mathbb{Z}$. Using that $[\cdot, \cdot]_A$ pairs $\mathcal{E}_{\sigma_0}(A)$ and $\mathcal{E}_{\sigma_1}(A^*)$ nonsingularity, one can extend the examples to the excluded case. Recall that $A^*$ denotes the formal adjoint of $A$, and $\Sigma(A) = \text{spec}_b(A) \cap \{-\nu/2 < 3\sigma < \nu/2\}$.

Example 9.1. Regard $A$ with domain $\mathcal{D}_{\sigma_0}(A)$ for some $\sigma_0 \in \Sigma(A)$. Then,

$$\mathcal{D}(A^*) = \sum_{\sigma \in \Sigma(A) \setminus 3\sigma \neq \sigma_0} \mathcal{D}_\sigma(A^*),$$

where $A^*$ denotes the Hilbert space adjoint of $A$. If there are one or more $\sigma_j \in \Sigma(A)$ such that $\sigma_0 - \sigma_j = i\tau$ with $\tau \in \mathbb{N}$, then in place of the $\mathcal{D}_{\sigma_j}(A^*)$ one must use certain subspaces.

Example 9.2. If $A$ is given the domain

$$\sum_{\sigma \in \Sigma(A) \setminus 3\sigma < 0} \mathcal{D}_\sigma(A),$$

then the domain of the adjoint is

$$\mathcal{D}(A^*) = \sum_{\sigma \in \Sigma(A) \setminus 3\sigma \geq 0} \mathcal{D}_\sigma(A^*).$$
Note that the poles of the Mellin transforms of elements in this space are on or below the real axis.

**Example 9.3.** Suppose \(\text{spec}_b(A)\) does contain points on \(\Im \sigma = 0\), but that all the partial multiplicities \(\mu_{\sigma,j}\) of each such point are even. Referring to Definition 7.2 for the notation, let \(A\) have domain
\[
\sum_{\sigma \in \Sigma(A), \Im \sigma < 0} D_\sigma(A) + \sum_{\sigma \in \Sigma(A), \Im \sigma = 0} D_{\sigma, \frac{1}{2}}(A)
\]
Then,
\[
D(A^*) = \sum_{\sigma \in \Sigma(A), \Im \sigma > 0} D_{\sigma}(A^*) + \sum_{\sigma \in \Sigma(A), \Im \sigma = 0} D_{\sigma, \frac{1}{2}}(A^*).
\]
Thus if \(A\) is symmetric then \(A\) with the given domain is selfadjoint. In Section 8 we proved that if \(A\) is symmetric and bounded from below then this is the domain of the Friedrichs extension.

In some cases, in particular geometric problems, one encounters operators of the form \(B^*B\) and \(BB^*\). Below we discuss some aspects of \(B^*B\) using some of the results of this paper. The operator \(BB^*\) can be treated in the same manner.

Assume that \(B \in x^{-\nu} \text{Diff}^m_b(M)\) is \(b\)-elliptic, \(B = x^{-\nu}Q\). Let \(D(B) \subset D_{\text{max}}(B)\) be such that \(B : D(B) \to x^{-\nu/2}L^2_b(M)\) is closed. Then \(B^*B\) is a selfadjoint extension of the symmetric operator \(B^*B\), considered as an unbounded operator on \(x^{-\nu/2}L^2_b(M)\). Recall that
\[
D(B^*B) = \{u \in D(B) \mid Bu \in D(B^*)\}
\]
Since \(B^*\) is a closed extension of the formal adjoint \(B^*\), \(B^*B\) is indeed a closed extension of \(B^*B\) with
\[
D_{\text{min}}(B^*B) \subset D(B^*B) \subset D_{\text{max}}(B^*B).
\]
Note that if \(u \in D_{\text{max}}(B^*B)\), then \(\hat{u}\) is meromorphic in \(\{\Im \sigma > -3\nu/2\}\) with poles on the strip \(\{\nu/2 > \Im \sigma > -3\nu/2\}\). Write \(B^*B = x^{-2\nu}P\) with \(P \in \text{Diff}^m_b(M)\). The conormal symbol of \(B^*B\) is then given by
\[
\hat{P}_0(\sigma) = \hat{Q}_0(\sigma - i\nu) \circ \hat{Q}_0(\sigma),
\]
where \(\hat{Q}_0(\sigma)\) is the conormal symbol of \(B\). Thus, the boundary spectrum of \(B^*B\) contains \(\text{spec}_b(B)\) and its reflection with respect to \(\{\Im \sigma = -\nu/2\}\) (line of symmetry). In particular, every \(\sigma \in \text{spec}_b(B^*B) \cap \{\Im \sigma = -\nu/2\}\) has even multiplicities. For \(\sigma_0 \in \text{spec}_b(B)\) define \(D_{\sigma_0}(B)\) as the space of elements \(u \in D_{\text{max}}(B)\) such that \(Bu \in D_{\text{max}}(B^*)\), and such that \(\hat{u}\) is meromorphic in \(C\) with poles at most at \(\sigma = -i\vartheta\) for \(\vartheta = 0, \ldots, [\nu]\). Thus, for \(\Im \sigma_0 = -\nu/2\), we have \(D_{\sigma_0}(B) \subset D_{\text{min}}(B)\).

**Lemma 9.4.** If \(u \in D_{\sigma_0}(B)\) and \(\Im \sigma_0 = -\nu/2\), then \(Bu \in D_{\text{min}}(B^*)\).

**Proof.** Let \(\sigma_0 \in \text{spec}_b(B)\) be such that \(\Im \sigma_0 = -\nu/2\). If \(u \in D_{\sigma_0}(B)\), then \(Bu \in x^{-\nu/2}L^2_b\) and \(\hat{u}\) is meromorphic with poles at \(\sigma_0 - i\vartheta\) for \(\vartheta = 0, \ldots, [\nu]\). It follows that \(\hat{B}u\) is then holomorphic in \(\Im \sigma > -\nu/2\) which implies \(Bu \in D_{\text{min}}(B^*)\).

Since \(D_{\text{min}}(B) \subset D(B)\) and \(D_{\text{min}}(B^*) \subset D(B^*)\), the previous lemma implies

**Lemma 9.5.** If \(\Im \sigma_0 = -\nu/2\) then \(D_{\sigma_0}(B) \subset D(B^*B)\).
Note that $\mathcal{D}_{\sigma_0}(B) = \mathcal{D}_{\sigma_0}(B^*)$.

**Example 9.6.** If $\mathcal{D}(B) = \mathcal{D}_{\min}(B)$, $B^*B$ is the Friedrichs extension of $B^*B$ and (9.7) $\mathcal{D}(B^*) = \{u \in \mathcal{D}_{\min}(B) \mid Bu \in \mathcal{D}_{\max}(B^*)\}$.

Denote the set on the right by $\mathcal{D}_F$. Since $\mathcal{D}(B^*B) \subset \mathcal{D}_F$, then $\mathcal{D}_F \cap \mathcal{D}(B^*B)^{\perp} = \mathcal{D}(B^*)$, where $\perp$ means the orthogonal in $\mathcal{D}_{\max}(B^*)$ with respect to $[,]_{B^*B}$. Now let $u \in \mathcal{D}(B^*)$, so $Bu \in \mathcal{D}(B^*)$. Then, for every $v \in \mathcal{D}_F$

$$0 = [v, Bu]_B = (Bv, Bu) - (v, B^*Bu) = (B^*Bv, u) - (v, B^*Bu) = [v, u]_{B^*B}.$$

This implies $u \in \mathcal{D}_F$ and we get (9.7). In this case, the Mellin transform $\hat{u}$ of an element $u \in \mathcal{D}(B^*)$ is holomorphic in $\{\Re \sigma > -\nu/2\}$ and meromorphic in $\{\Re \sigma > -3\nu/2\}$ with poles on $\{\sigma - i\nu \mid \sigma \in \text{spec}_b(B)\}$.

If $\mathcal{D}(B) = \mathcal{D}_{\max}(B)$, then

$$\mathcal{D}(B^*) = \{u \in \mathcal{D}_{\max}(B) \mid Bu \in \mathcal{D}_{\min}(B^*)\}$$

and for $u \in \mathcal{D}(B^*)$, $\hat{u}$ has poles at most on $\text{spec}_b(B^*) \cap \{-\nu/2 \leq \Re \sigma < \nu/2\}$.

If $\mathcal{D}(B) = \mathcal{D}_{\sigma_0}(B)$ for some $\sigma_0 \in \Sigma(B) = \text{spec}_b(B) \cap \{-\nu/2 < 3\Re \sigma < \nu/2\}$, then

$$\mathcal{D}(B^*) = \{u \in \mathcal{D}_{\sigma_0}(B) \mid Bu \in \mathcal{D}(\sigma_0(B^*)) \text{ for } \sigma \in \Sigma(B^*) \text{, } \sigma \neq \sigma_0\}$$

and for $u \in \mathcal{D}(B^*)$, $\hat{u}$ has poles at $\sigma_0$ and on $\{\sigma - i\nu \mid \sigma \in \text{spec}_b(B), \sigma \neq \sigma_0\}$.

We finish this section with some concrete examples. Assume now that, near the boundary, $M$ is of the form $[0,1) \times S^n$ with $\partial M = \{0\} \times S^n$. Let $D_y^2$ be the Laplacian on $S^n$, let $x$ be the variable in $[0,1)$. For $0 < \nu \leq 2$ let $A = x^{-\nu}P$ with (9.8)

$$P = (xD_x)^2 + a^2D_y^2 + \beta b^2,$$

$\beta = \pm 1$ and nonnegative constants $a$ and $b$. They arise, for instance, as Laplace-Beltrami operators on scalar functions associated to metrics of the form

$$g = dx^2/x^{2-\nu} + x^\nu dy^2.$$

Compare Brüning and Seeley [5], Lesch [4], and especially Mooers [14] on $k$-forms.

The conormal symbol of $A$ is defined to be

$$\tilde{P}_0(\sigma) = \sigma^2 + a^2D_y^2 + \beta b^2,$$

and the boundary spectrum $\text{spec}_b(A)$ is the set of points in $\mathbb{C}$ such that $\tilde{P}(\sigma)$ is not invertible. Since the set of eigenvalues of $D_y^2$ is $\{k(k + n - 1) \mid k \in \mathbb{N}_0\}$, then

$$\text{spec}_b(A) = \{\sigma \in \mathbb{C} \mid \sigma^2 + a^2k(k + n - 1) + \beta b^2 = 0 \text{ for some } k \in \mathbb{N}_0\}$$

Consider $A : C^\infty_c(M) \subset x^{-\nu/2}H_0^2(M) \rightarrow x^{-\nu/2}H_0^2(M)$ as an unbounded operator. Then $\Sigma = \text{spec}_b(A) \cap \{-\nu/2 \leq \Re \sigma \leq \nu/2\}$ is the only set that matters when looking for the closed extensions of $A$. We look separately at the cases $b = 0$ and $b \neq 0$.

**Example 9.9.** Let $b = 0$ and $a > 0$ in (9.8). In this case, $A$ is symmetric and bounded from below. Note that $a^2 + a^2k(k + n - 1)$ has simple roots in $\Sigma$ when $k \neq 0$, and a root of order 2 when $k = 0$. For illustration purposes it is enough to look at the case when $\nu = 2$ and $n = 1$. Thus $\Sigma = \{\pm ika \mid ka \leq 1, k \in \mathbb{N}_0\}$.

Suppose $a > 1$ so that the only point in $\Sigma$ is 0. Then $\mathcal{D}_{\min}(A) = xH_0^2(M)$ and

$$\mathcal{E}(A) = \mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A) = \text{span}\{\omega, \omega x\}$$

for some $\omega \in C^\infty_c(\mathbb{R})$ such that $\omega(x) = 1$ near the origin.
Let $\psi_0 = \omega$ and $\psi_1 = \omega \log x$. Then,
\[
\hat{\psi}_0(\sigma) = \frac{\Phi(\sigma)}{\sigma} \quad \text{and} \quad \hat{\psi}_1(\sigma) = \frac{\Phi(\sigma)}{\sigma^2} - \frac{\Phi'(\sigma)}{\sigma}
\]
with
\[
\Phi(\sigma) = \int_{\mathbb{R}^+} x^{-i\sigma} D_x\omega(x) \, dx.
\]
Let $u = u_0\psi_0 + u_1\psi_1$ and $v = v_0\psi_0 + v_1\psi_1$ be elements of $\mathcal{E}(A)$. Then, with $\gamma$ a positively oriented simple closed curve in $\mathbb{C}$ surrounding 0,
\[
[u, v]_A = \frac{1}{2\pi} \int_{\gamma} \sigma^2 \hat{u}(\sigma) \hat{\bar{v}}(\sigma) \, d\sigma = \frac{1}{2\pi} \int_{\gamma} \left( \frac{u_0 \bar{v}_1 + u_1 \bar{v}_0}{\sigma} \right) \Phi(\sigma) \bar{\Phi}(\sigma) \, d\sigma = i(u_0 \bar{v}_1 + u_1 \bar{v}_0).
\]
If $\mathcal{D}$ is a domain on which $A$ is selfadjoint, then $\mathcal{D} = \mathcal{D}_{\min}(A) \oplus \text{span} \, u$ with some $u$ as above such that $[u, u]_A = 0$. Thus $u_0 \bar{v}_1 + u_1 \bar{v}_0 = 0$ which implies $u_0 \bar{v}_1 \in \mathbb{R}$, so span $u$ is one dimensional. Moreover, all selfadjoint extensions of $A$ are of the form $A_{\alpha} : \mathcal{D}^\lambda \to x^{-1}H^1_{b_{\alpha}}(M)$ with
\[
\mathcal{D}^\lambda/\mathcal{D}_{\min}(A) = \text{span}\{e^{i\lambda} + 1\omega + (e^{i\lambda} - 1)\omega \log x\}.
\]
In particular, $\mathcal{D}^0$ is precisely the domain of the Friedrichs extension of $A$. If $\frac{1}{a < b = 1}{\ell} \in \mathbb{N}$, then $\Sigma = \{\pm ika \mid k = 0, 1, \ldots, \ell\}$ and we have
\[
\mathcal{D}_F(A) = \mathcal{D}^0 \oplus \sum_{k=1}^\ell \mathcal{D}_{\{ika\}}(A).
\]
Note that the partial multiplicities of the poles $ika, k \neq 0$ are equal to 1, but the total multiplicity is 2.

**Example 9.10.** Let $\alpha \in C_c^\infty(\mathbb{R})$ such that $\alpha(0) > 1$. Then
\[
A = x^{-2} [(xD_x)^2 + \alpha(x)2D^2_{y_{\alpha}}]
\]
is symmetric and bounded from below. Moreover, 0 is the only point of the boundary spectrum in $\{-1 \leq \Im \sigma \leq 1\}$. If $A_0 = x^{-2}[(xD_x)^2 + \alpha(0)2D^2_{y_0}]$, then
\[
\mathcal{D}_{\min}(A) = \mathcal{D}_{\min}(A_0), \quad \mathcal{D}_{\max}(A) = \mathcal{D}_{\max}(A_0), \quad \text{and} \quad \mathcal{D}_F(A) = \mathcal{D}_F(A_0).
\]

**Example 9.11.** Let $b \neq 0$. If $\beta = 1$ in $[0, 1]$, then $A = x^{-2}P$ behaves similarly but ‘nicer’ than the operator in Example 9.9 since $\sigma^2 + \alpha^2k(k + n - 1) + b^2$ has only simple roots, and $\text{spec}_s(A) \cap \{\Im \sigma = 0\} = \emptyset$.

If $\beta = -1$, the situation is different. In this case, the conormal symbol of $A$
\[
\hat{P}(\sigma) = \sigma^2 + \alpha^2D^2_y - b^2
\]
fails to be nonnegative for real $\sigma$ and therefore $A$ is not bounded from below. Moreover, $\sigma^2 + \alpha^2k(k + n - 1) - b^2$ has two simple real roots, $-b$ and $b$. We assume $b \leq 1 < a$, so $\text{spec}_s(A) \cap \{-1 \leq \Im \sigma \leq 1\}$ contains only $-b$ and $b$. Thus
\[
\mathcal{E}(A) = \mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A) = \text{span}\{\omega x^{ib}, \omega x^{-ib}\}
\]
for some $\omega \in C_c^\infty(\mathbb{R})$ such that $\omega(x) = 1$ near the origin.
Let $\psi_+ = \omega x^{ib}$ and $\psi_- = \omega x^{-ib}$. Then, with $\Phi$ as above,

$$\hat{\psi}_\pm(\sigma) = \frac{\Phi(\sigma \mp b)}{\sigma \mp b}.$$ 

Let $u = u_+\psi_+ + u_-\psi_-$, $v = v_+\psi_+ + v_-\psi_-$. Then, with $\gamma$ a positively oriented simple curve in $\mathbb{C}$ surrounding $b$ and $-b$,

$$[u, v]_A = \frac{1}{2\pi} \oint_\gamma (\sigma^2 - b^2) \hat{u}(\sigma) \overline{\hat{v}(\sigma)} d\sigma$$

$$= \frac{1}{2\pi} \oint_\gamma \left\{ \left( \frac{\sigma + b}{\sigma - b} \right) u_+ \overline{v}_+ \Phi(\sigma - b) \overline{\Phi(\sigma - b)} + \left( \frac{\sigma - b}{\sigma + b} \right) u_- \overline{v}_- \Phi(\sigma + b) \overline{\Phi(\sigma + b)} \right\} d\sigma$$

$$= 2bi(u_+ \overline{v}_- - u_- \overline{v}_+)$$

Thus $[u, u]_A = 2bi(|u_+|^2 - |u_-|^2)$ and if $A$ with domain $\mathcal{D}_{\min}(A) \oplus \text{span } u$ is self-adjoint then $|u_+| = |u_-|$. Thus span $u$ is one dimensional and all selfadjoint extensions of $A$ are of the form $A_\lambda : D_\lambda \to x^{-1}H^2(M)$ with

$$D_\lambda / \mathcal{D}_{\min}(A) = \text{span}\{\omega x^{ib} + e^{i\lambda} \omega x^{-ib}\}.$$

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