The unity of dark matter and dark energy in a curvaton scenario

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According to the whole evolution of curvaton, we systematically investigate a broad new class of curvaton scenarios supplying a unified framework for dark matter and dark energy, in which the effective mass of curvaton is running (its potential consists of coupling part and exponential part). In this scenario, the coupling is between the curvaton and inflaton. Consequently, we adopt the narrow and broad resonance to study the production of curvation in terms of preheating, whose criteria comes via the spectral index of curvaton. While the inflaton decays, the effective mass varies which plays a role of a free parameter as showing in Ref. [1], moreover one always finds the satisfied combinations of the Hubble parameter and the mass of curvaton. Therefore, it naturally explains the abundance of dark matter. At the very late time of Universe, the inflaton and the curvaton almost decay, the exponential potential of curvaton will be approaching a constant of order of dark energy. As a consequence, it naturally explains the dark energy from the perspective of phenomenology. In order to elaborate this scenario, we implement the $\delta N$ formalism to calculate the local Non-Gaussianity parameter $f_{NL}$ and the power spectrum $P_\zeta$ of curvaton. Remarkably, our calculation shows that these two observables are independent of potential of inflaton. Once the constraints are given, these two observables are consistent with the observations.

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I. INTRODUCTION

In the mainstream of inflationary theory, the origin of curvature perturbation is generated by the inflationary perturbation seeding the CMB temperature fluctuations. However, this broad kind of theories highly depends on the shape of inflationary potential. In order to relax this strict condition, many alternatives are proposed. Such kind of models called curvaton mechanism are given, in which the curvaton field is an extra scalar field and is subdominant during inflation. After inflation, the density of curvaton will be more and more significant which will generate the curvature perturbation (much larger than the curvature perturbation from inflaton) [2–4].

All of the particles of the standard model (SM) come from preheating or reheating period [5–11]. Especially for the narrow resonance, Ref. [7] gives the general discussion under the framework of the WKB approximation. Since the reheating temperature is not sufficient for the coupling between the inflaton and other particles of SM, the preheating process is mandatory [12]. Once adding the curvaton decay after inflation, most of them decay perturbatively [14]. It also has been envisaged that curvaton could non-perturbatively decay [13]. Furthermore, the curvaton can be dubbed as a source to generate amount of gravitational waves (GW) during preheating [15]. Due to the rich phenomenon of curvaton, it could be embedded into multi-field framework [16].

If the curvaton lives longer, it could couple to the Higgs field in which case the mass of the curvaton can vary significantly [18–19]. In this paper, we propose the similar idea, which the coupling is between the inflaton and curvaton, to vary the mass of curvaton. Generally, curvaton decay occurs when the curvaton decay rate $\Gamma$ is comparable to Hubble parameter $H$. Upon relaxing the condition that the condensate of curvaton dominates over its perturbations, it could yield large local non-Gaussianities [27]. Current observations [22] severely constrain these models however, as (local) non-Gaussianity $f_{NL}$ cannot be too large ($|f_{NL}| < 10$), it rules out curvaton models that produce large non-Gaussianities. However, this local type of Non-Gaussianity can be suppressed by quadratic potential plus quartic potential [24] and also in string axionic potential [25–26]. Once taking the local Non-Gaussianity into account, we will see that it will strongly constrain the decay of convaton, precisely for the so-called fraction of curvaton’s energy density among radiation period, the newest investigations can be found in [28].
Curvaton field is generally deemed as an independent and extra scalar field comparing to inflaton. Thereby, curvaton could play various roles in particles, such as axion [29], in order to account for dark matter (DM). Due to the role of axion, curvaton could also produce the axionic-primordial blackhole [30, 31], in some sense it explains the DM. Furthermore, primordial blackhole as DM could be generated by curvaton and inflaton mixed model [32].

Moreover, the origin of dark energy (DE) from curvaton is still missing. One purpose of this paper is to account for the DE from the perspective of phenomenology. From a thorough investigation of curvaton (from its production to decay), we firstly supply for a unified framework of DM and DE in curvaton scenario. By implementing the idea of [1], we consider that DM comes via the quantum fluctuations of curvaton field. As for DE, it can be dubbed as the relic of exponential potential of curvaton when curvaton almost vanish (decay into other particles). Therefore, we give a fully analysis of curvation from its generation in an inflationary period to the very late Universe (up to the present dark energy epoch).

This paper is organized as following. In section II we introduce our inflationary model with two scalars, one being inflaton and the other is curvaton, whose potential (curvaton’s potential) contains two part: coupling part (between the curvaton and inflaton), exponential part. In section III the production of curvation comes via the decay of inflaton in terms of parametric resonance preheating. In section IV the detailed calculation of power spectrum and its corresponding local Non-Gaussianity are given by $\delta N$ formalism. In section V we introduce the dark matter comes from the quantum fluctuations of curvaton in light of [1]. In section VII we illustrate that the dark energy comes from the exponential potential of curvaton. Finally, section VIII gives the conclusion.

We work in natural units in which $c = 1 = \hbar$, but retain the Newton constant $G$.

II. THE MODEL

In a traditional scenario of curvaton, it encounters with a fatal problem since the reheating temperature is too low to reheat the Universe [12]. In this picture, this coupling is proportional to the mass of curvaton. If the effective mass of curvaton is tiny enough, one can obtain the efficient temperature to reheat the Universe. One mechanism is that the effective mass of curvaton is proportional to potential of inflaton, namely there is coupling between the curvaton and inflaton. Subsequently, in order to account for the origin of dark
energy, we assume that the second part of the potential for curvaton is of exponential form. Armed with these assumptions, the total action can be constructed as following,

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{M_P^2}{2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - V(\phi) - \frac{g_0}{M_P^2} \chi^2 V(\phi) - \lambda_0 \exp\left[ -\lambda_1 \frac{\chi}{M_P} \right] \right\} \]  

(1)

where \( \chi \) and \( \phi \) denote curvaton and inflaton, respectively. \( R \) presents the Ricci scalar and \( g \) is the determinant of \( g_{\mu\nu} \). \( g_0, \lambda_0 \) and \( \lambda_1 \) are the dimensionless parameters determined by observations in Lagrangian. In order to get more understanding of this scenario, we elaborate action (1) for more details. In some sense, the curvaton could produce via inflaton decay \[33\] albeit the branch of decay cannot be large. Therefore, we deem that our model is one field theory at the very beginning. Since there is a coupling between the inflaton and curvaton, we consider the curvaton is from energy transferring of inflaton borrowing the spirit of parametric resonance of field, in which this parametric resonance is creating the curvaton. Another perspective is that this energy transferring process is an highly non-equilibrium process. In the following section, we will show that this process is outlined in terms of parametric resonance.

III. PREHEATING OF PRODUCTION OF CURVATON

Strictly speaking, the energy transferring is an highly non-equilibrium process since the temperature changes rapidly when inflation happens. Naturally, it should be treated by the parametric resonance. Armed with concept, we will proceed the whole paper according to time-line of evolution for the Universe. In this paper, we assume that there is only inflaton at the very beginning whose energy can be transferred towards curvaton.

Following the standard procedure \[6, 10\], first what we need is the equation of motion (EOM) of curvaton field \( \chi \), which can be varied with action (1) and transformed into momentum space, one can obtain EOM of \( \chi \) field,

\[ (\ddot{\chi}_k + 3 \frac{\dot{a}}{a} \dot{\chi}_k) + \left( \frac{k^2}{a^2} \chi_k + m^2 \frac{g_0}{M_P^2} \phi^2 \chi_k - \frac{\lambda_0 \lambda_1}{M_P} \chi_k \right) \approx 0, \]  

(2)

where \( \dot{\chi} = \frac{d\chi}{dt} \), the subscript \( k \) denotes the momentum space and we assume that \( \lambda_1 \frac{\chi}{M_P} \ll 1 \). Changing variable \( \tilde{\chi}_k = a^{3/2} \chi_k \), subsequently Eq. (2) becomes,

\[ \ddot{\tilde{\chi}}_k - \left( \frac{9}{4} H^2 \tilde{\chi}_k + \frac{3}{2} \dot{H} \tilde{\chi}_k \right) + \left( \frac{k^2}{a^2} \tilde{\chi}_k + \frac{g_0}{M_P^2} m^2 \phi^2 \tilde{\chi}_k - \frac{\lambda_0 \lambda_1}{M_P} \tilde{\chi}_k \right) \approx 0. \]  

(3)
After inflation ends, the contribution of $\frac{9}{4}H^2\ddot{x}_k + \frac{3}{2}\dot{H}\dot{x}_k$ becomes less and less significant, therefore it can be neglected. Finally, Eq. (3) will be simplified into

$$\ddot{x}_k + \left(\frac{k^2}{a^2}\ddot{x}_k + \frac{g_0}{M_P^2}m^2\phi^2\ddot{x}_k - \frac{\lambda_0\lambda_1}{M_p^2} [1 - \lambda_1/\ddot{x}_k] \right) \approx 0. \tag{4}$$

In order to solve Eq. (4), the solution of background for $\phi$ is mandatory. Its corresponding EOM is derived by,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0 \tag{5}$$

where $\partial_\phi V = \partial_\phi(\frac{1}{2}m^2\phi^2 + \frac{g_0}{M_P^2}/2\chi^2 m^2\phi^2)$. Here we define the effective mass of inflaton $m_{eff}^2 = m^2 + \frac{g_0}{M_P^2}/2\chi^2 m^2$, it leads to $m_{eff} \approx m, \partial_\phi V = m_{eff}^2 \phi$ with $\frac{g_0}{M_P^2}/2\chi^2 \ll 1$ illustrating in figure (1).

Like deriving Eq. (4) and neglecting $\frac{9}{4}H^2\ddot{x}_k + \frac{3}{2}\dot{H}\dot{x}_k$. In the later investigation, we will show that this is a rational assumption. Meanwhile, changing variable as $\tilde{\phi}(t) = a^{3/2}\phi(t)$ where $a$ is scale factor and implementing the same trick adopted for deriving Eq. (4), Eq. (5) finally becomes,

$$\ddot{\tilde{\phi}} + m_{eff}^2 \tilde{\phi} = 0, \tag{6}$$

its solution is

$$\phi(t) = a^{-3/2} \cos(m_{eff}t). \tag{7}$$

Using solution (7) to plug into Eq. (4) and rearranging this equation with some efforts, this equation becomes,

$$\ddot{x}_k + \left(\frac{k^2}{m_{eff}^2a^2}\ddot{x}_k + \frac{g_0}{2M_P^2}\phi_0^2\ddot{x}_k a^{-3} + \frac{m^2g_0}{2M_P^2}\phi_0^2 a^{-3} \cos(2m_{eff}t) + \frac{\lambda_0\lambda_1^2}{M_p^2} \ddot{x}_k \right) = \lambda_0\lambda_1/M_p, \tag{8}$$

Thereafter, one can set $z = m_{eff}t$, Eq. (8) turns into,

$$\ddot{x}_k' + \left(\frac{k^2}{m_{eff}^2a^2}\ddot{x}_k + \frac{g_0}{2M_P^2}\phi_0^2\ddot{x}_k a^{-3} + \frac{g_0}{2M_P^2}\phi_0^2 a^{-3} \cos(2z) + \frac{\lambda_0\lambda_1^2}{m_{eff}^2M_p^2} \ddot{x}_k \right) = \lambda_0\lambda_1/(m_{eff}^2M_p), \tag{9}$$

where we have used the approximation $m_{eff} \approx m$ and $\ddot{x}_k' = \frac{dx_k}{dz}$. Next, we will follow the notation of Ref. [7], first the standard form can be written as

$$\ddot{x}_k'' + (A_k + 2q_k \cos(2z))\ddot{x}_k = c \tag{10}$$

where $c = \lambda_0\lambda_1/(m_{eff}^2M_p)$, subsequently, one can find the correspondence as follows,

$$A_k = \frac{k^2}{m_{eff}^2a^2} + \frac{g_0\phi_0^2}{2M_P^2}a^{-3} + \frac{\lambda_0\lambda_1^2}{m_{eff}^2M_p^2}, \tag{11}$$

$$q_k = \frac{\phi_0^2g_0}{2M_P^2}a^{-3}. \tag{12}$$
As mentioned in section (II), the role of exponential potential $\lambda_0 \exp(-\lambda_1 \chi \bar{M_P})$ is mimicking the evolution of dark energy, thus $\chi$ field will be approached zero and $\lambda_0$ can be determined as the same order with dark energy. Comparing to other terms in Eq. (10), one can approximately set $c \approx 0$. In order to achieve the range of $q$, we analyse the spectral index of curvaton whose detailed calculation is showing in appendix. The formula of spectral index of curvaton is as follows,

$$n_\chi = -2\epsilon_1 - \eta_c$$

(13)

where the definition of $\epsilon_1$ and $\eta_c$ are Eqs. (56,59) and spectral index is denoted in terms of leading order of slow-roll parameters. Assuming that the slow-roll conditions are applicable for inflaton and curvaton. In curvaton scenario, the energy density of inflation is dominant before it decays into curvaton which is also an assumption for curvaton mechanism. Consequently, the value of Hubble parameter is almost determined by inflaton. Afterwards, one adopts the slow-roll approximation of inflation, we can derive the $\epsilon_1 = \epsilon_V = \frac{M_P^2}{2 \left( \frac{V'}{V} \right)^2}$ where $V' = \frac{\partial V}{\partial \phi}$. In light of $N \approx \frac{\phi^2}{3M_P^2}$, one easily derives $\epsilon_1 = \frac{1}{N}$. As for $\eta = -\frac{2}{3} \frac{V''(\chi)}{H^2}$, we also implement the slow-roll approximation and pay attention $V''(\chi) = \frac{\partial^2 V(\chi)}{\partial \phi^2}$. Subsequently, we remarkably find $\eta_c = -4g_0$ which is independent of the model of inflation. Combine these two parts, finally we obtain that

$$n_\chi = -\frac{1}{N} + 4g_0,$$

(14)

where $N$ is the e-folding number. According to this equation, we can see the varying trends of $n_\chi$ from figure (69), from lasted observational constraint for $n_\chi \approx 0.035$ from [23], the possible value of $g_0 \approx 0.013$ being of order $10^{-2}$. The most essential part of spectral index is that it is almost independent of the potential of inflation, however the difference comes via the coefficient of inflationary potential. And it indicates that the value of $g_0$ can almost be determined by the spectral index of curvaton not only giving its range. Therefore, our model will give strong predictions of observations. Although we have given the constraint for $g_0$, we still cannot determine the value of $\phi_0$ since it could be large if the initial value still locates during inflationary period.

In light of $q_k \ll 1$ and $q_k \geq 1$, equation (10) can be treated as narrow resonance and broad resonance, respectively.
FIG. 1: Density plot of spectral index \([14]\): The horizontal line corresponds to e-folding number \(N\) whose range is \(50 \leq N \leq 60\). The vertical line denotes the value of \(g_0\) whose range locates from 0.000001 to 0.023. The right panel shows that the value of \(n_\chi\) matching its corresponding color.

A. Narrow resonance

In this case, it corresponds to \(q_k \ll 1\) which means that \(g_0 \phi_0^2 \ll 2M_P^2\). Ref. [7] provides a general framework of narrow parametric resonance. The most essential physical quantities are decay rate \(\Gamma_\chi\) and number density of resonance \(N_{res}\). \(\Gamma_\chi\) depicts how much energy has been transferred to curvaton. As for \(N_{res}\), it represents the number density of curvaton.

From Ref. [7], one can directly find the formulas of \(\Gamma_\chi\) and \(N_{res}\):

\[
N_{res} = \sinh^2 \left( \frac{\pi g_n^2}{2 \Gamma_\chi \omega_{res}^3} \right),
\]

\[
\Gamma_\chi = \frac{\pi g_n^2}{\omega_{res}^3} \ln^{-1} \left( \frac{32\pi^2 \rho}{\omega_{res}^4} \right),
\]

where \(\rho = \frac{1}{2} m^2 \phi^2\) and \(n\) denotes the \(n\)th band of periodic function, here there is only \(g_1 = g_1 = \frac{g_0}{2M_P^2}\) and \(\omega_{res} = \frac{1}{2} m_{eff} = \frac{1}{2} \omega\). In order to get efficient production of curvaton, it is requiring that \(N_{res} \gg 1\). As a consequence, it is simply concluded that \(32\pi^2 \rho \gg \omega_{res}^4\) which means that the initial value of \(\phi\) is much more lager than its mass (exactly the effective mass) which is consistent with results of chaotic inflation [20]. Namely, the efficient production requires that inflaton is a large field.
B. Broad resonance

In this case, the theoretical framework [6, 10] provides a general method for studying the broad resonance corresponding to \( q_k \geq 1 \). Firstly, the standard equation can be written as,

\[ \ddot{\chi}_k + \omega^2 \dot{\chi}_k = 0, \quad (17) \]

where

\[ \omega^2 = \left( \frac{k^2}{a^2} \right) + \frac{g_0 m^2}{2 M_p^2} \phi_0^2 a^{-3} + m^2 g_0 \phi_0^2 a^{-3} \cos(2 m_{\text{eff}} t) + \frac{\lambda_0 \lambda_1^2}{M_p^2}, \quad (18) \]

and \( \dot{\chi} = \frac{d\chi}{dt} \). Although the production of particles are highly non-equilibrium process, one can still use the WKB approximation to analytically solve Eq. (18), in which if we presume that it is an adiabatic process of a time interval from \( t_i \) to \( t_{i+1} \) (a very short time interval). Its general solution under WKB approximation,

\[ \tilde{\chi}_k^{wkb} \approx \frac{\alpha_k}{\sqrt{2\omega}} e^{-i \int \omega dt} + \frac{\beta_k}{\sqrt{2\omega}} e^{i \int \omega dt} \quad (19) \]

where \( \alpha_k \) and \( \beta_k \) are constants with adiabatic condition in a short time interval and \( |\alpha_k|^2 - |\beta_k|^2 = 1 \). In this short time interval, we can also make an analytical approximation of \( \frac{g_0^2 \phi_0^2}{M_p} \approx \frac{m_{\text{eff}} g_0}{2 M_p^2} \phi_0^2 a^{-3} (t_{i+1} - t_i)^2 \) and define new variables,

\[ \tau^2 = \frac{m_{\text{eff}} g_0}{2 M_p^2} \phi_0^2 a^{-3} (t - t_j)^2, \quad (20) \]
\[ \kappa^2 = \frac{k^2}{a^2} + \frac{\lambda_0 \lambda_1^2}{M_p^2}, \quad (21) \]
\[ k^2_* = \frac{m_{\text{eff}} g_0}{2 M_p^2} \phi_0^2 a^{-3}. \quad (22) \]

Be armed with these variables, Eq. (19) can be rearranged as

\[ \frac{d^2 \chi_k}{d\tau^2} + (\kappa^2 + \tau^2) \dot{\chi_k} = 0. \quad (23) \]

There is a Bogoliubov transformation for coefficients \( \alpha_k \) and \( \beta_k \) of time interval from \( t_i \) to \( t_{i+1} \).

\[
\begin{pmatrix}
\alpha_k^{i+1} \\
\beta_k^{i+1}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{1 + e^{-\pi \kappa^2} e^{i \varphi_k}} & ie^{-\frac{\pi}{2} \kappa^2 + 2 i \theta_k} \\
-ie^{-\frac{\pi}{2} \kappa^2 - 2 i \theta_k} & \sqrt{1 + e^{-\pi \kappa^2} e^{-i \varphi_k}}
\end{pmatrix}
\begin{pmatrix}
\alpha_k^i \\
\beta_k^i
\end{pmatrix},
\]
\[
\varphi_k = \arg\{\Gamma\left(\frac{1+in^2}{2}\right)\} + \frac{\kappa^2}{2}(1+\ln \frac{k}{\kappa}) \quad (\Gamma \text{ is special function}).
\]

Combine with \(|\alpha_k|^2 - |\beta_k|^2 = 1\) and \(n_i = |\beta_k|^2\), one can derive this,

\[
n_{i+1}^k = e^{-\pi\kappa^2} + (1 + 2e^{-\pi\kappa^2})n_i^k - 2e^{-\frac{\pi}{2}\kappa^2}\sqrt{1 + e^{-\pi\kappa^2}}\sqrt{n_i^k(1+n_i^k)} \sin \theta_{tot}^i
\]

(25)

\[
\theta_{tot}^i = 2\theta_k^i - \varphi_k + \arg(\alpha_k^i) - \arg(\beta_k^i).
\]

(26)

In order to obtain the continuous production, namely the value of \(n_{i+1}\) is enhanced comparing to \(n_i\). As taking the limit of \(i \rightarrow \infty\), we acquire that \(n_k \gg 1\) satisfied with \(\pi\kappa_n^2 \leq 1\) which equals to

\[
\frac{k^2}{a^2} + \frac{\lambda_0}{M_P^2} \leq \frac{m_{eff}^2g_0\phi_0^2a^{-3}}{\pi},
\]

(27)

where \(k\) is the momentum after inflation ends, as a rational approximation one can set \(k = aH_*\) in which \(H_*\) denotes that the value of Hubble parameter at the frozen time, whose value can be \(3 \times 10^{-5}\) according to slow-roll conditions of inflation and COBE normalization [23]. In light of this observational constraints, the lower bound can be given by (setting \(a = 1\) and \(M_P = 1\),

\[
\frac{m_{eff}^2g_0\phi_0^2}{M_P^2} \geq 3.0 \times 10^{-9}.
\]

(28)

In this section, the production of curvaton is investigated in terms of parametric resonance consisting of narrow resonance and broad resonance. For narrow resonance, it has shown that inflaton field is a large field whose value is much larger than its mass. We have given the lower bound of \(\frac{m_{eff}^2g_0\phi_0^2}{M_P^2}\) in case of broad resonance. In the following section, more precise constraints of these parameters will be obtained based on power spectrum and local non-Gaussianity.

### IV. POWER SPECTRUM AND NON-GAUSSIANITY

We work in conformally flat cosmological space-times whose metric is a conformal rescaling of Minkowski metric,

\[
g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(-, +, +, +),
\]

(29)

where \(\tau\) is conformal time. In order to obtain the curvaton power spectrum, one ought to solve the operator equation of motion for the curvaton, which follows from action [1],

\[
\left[\partial_0^2 + 2\mathcal{H}\partial_0 - \nabla^2\right]\chi(x) + a^2V_\chi(\phi, \dot{\chi}) = 0,
\]

(30)
where $\hat{V}_\chi \equiv \partial\hat{V}(\hat{\phi}, \hat{\chi})/\partial\hat{\chi}$, $\mathcal{H} = a'/a$ ($a' = \partial_\theta a$) is the conformal Hubble rate, $\nabla^2 \equiv \sum_{i=1}^3 \partial_i^2$ and we have neglected the curvaton coupling to gravitational perturbations, which is for most purposes justified, as the curvaton is to a good approximation spectator field during inflation.

The field $\hat{\chi}$ in (30) satisfies standard canonical quantization relations,

$$\left[\hat{\chi}(\tau, \vec{x}), \hat{\pi}_\chi(\tau, \vec{x}')\right] = i(2\pi)^3 \delta^3(\vec{x} - \vec{x}'), \quad \left[\hat{\chi}(\tau, \vec{x}), \hat{\chi}(\tau, \vec{x}')\right] = 0, \quad \left[\hat{\pi}_\chi(\tau, \vec{x}), \hat{\pi}_\chi(\tau, \vec{x}')\right] = 0,$$

(31)

where $\pi_\chi = a^2 \chi'$ ($\chi' = \partial_\theta \chi$) denotes the curvaton canonical momentum. Since we are here primarily interested in the curvaton spectrum of free theory, it suffices to linearise (30) in small perturbations around the curvaton condensate, $\langle \hat{\chi} \rangle \equiv \bar{\chi}(\eta)$. The procedure of studying the dynamics of linear curvaton perturbations is standard, details of which can be found in the Appendix.

By using the $\delta N$ formalism [34–38], the power spectrum can be given by [39, 40]

$$P_{\zeta^*} = \left(\frac{\partial N}{\partial \chi} \frac{H_*}{2\pi}\right)^2,$$

(32)

where $\partial N/\partial \chi$ is given by

$$\frac{\partial N}{\partial \chi} = \frac{1}{3} r_{\text{decay}} \frac{1}{1 - X(\chi_{\text{osc}})} \left[\frac{V'(\chi_{\text{osc}})}{V(\chi_{\text{osc}})} - \frac{3X(\chi_{\text{osc}})}{\chi_{\text{osc}}} \right] \frac{V'(\chi_{\text{osc}})}{V(\chi_{\text{osc}})},$$

(33)

with $\chi_{\text{osc}}$ and $\chi_*$ being (Einstein frame) field values at the time of the onset of the oscillation and the horizon exit during inflation. The time of the onset of the curvaton oscillation can be evaluated by

$$\left|\frac{\dot{\chi}}{H\chi}\right| = 1,$$

(34)

which can also be written as [39, 40]

$$H_{\text{osc}}^2 = \frac{V'(\chi_{\text{osc}})}{c\chi_{\text{osc}}},$$

(35)

where $c$ is given by $9/2$ and $5$ when the curvaton begins to oscillate during MD and RD, respectively. $X(\chi_{\text{osc}})$ represents the perturbation generated from the non-uniform onset of the oscillation of the curvaton field, which is written as

$$X(\chi_{\text{osc}}) = \frac{1}{2(c - 3)} \left(\frac{\chi_{\text{osc}} V''(\chi_{\text{osc}})}{V'(\chi_{\text{osc}})} - 1\right).$$

(36)
When the potential is quadratic, $X(\chi_{\text{osc}})$ vanishes. $r_{\text{decay}}$ roughly corresponds to the fraction of the curvaton energy density to the total one, which is defined as

$$r_{\text{decay}} = \frac{3\bar{\rho}_X}{3\bar{\rho}_X + 4\bar{\rho}_{\text{rad}}} = \frac{3\Omega_X}{3\Omega_X + 4\Omega_{\text{rad}}}, \quad \Omega_X = \frac{\bar{\rho}_X}{\bar{\rho}_X + \bar{\rho}_{\text{rad}}}, \quad \Omega_{\text{rad}} = \frac{\bar{\rho}_{\text{rad}}}{\bar{\rho}_X + \bar{\rho}_{\text{rad}}}. \quad (37)$$

Non-linearity parameter $f_{\text{NL}}$ is given by

$$f_{\text{NL}} = -\frac{5}{6}r_{\text{decay}} - \frac{5}{3} + \frac{5}{2r_{\text{decay}}} (1 + A), \quad (38)$$

where $A$ is given by

$$A = \left[ \frac{V''(\chi_{\text{osc}})}{V'(\chi_{\text{osc}})} - \frac{3X(\chi_{\text{osc}})}{\chi_{\text{osc}}} \right]^{-1} \left[ \frac{X'(\chi_{\text{osc}})}{1 - X(\chi_{\text{osc}})} + \frac{V''(\chi_{\text{osc}})}{V'(\chi_{\text{osc}})} - (1 - X(\chi_{\text{osc}})) \frac{V''(\chi_s)}{V'(\chi_{\text{osc}})} \right]$$

$$+ \left[ \frac{V''(\chi_{\text{osc}})}{V'(\chi_{\text{osc}})} - \frac{3X(\chi_{\text{osc}})}{\chi_{\text{osc}}} \right]^{-2} \left[ \frac{V''(\chi_{\text{osc}})}{V'(\chi_{\text{osc}})} - \left( \frac{V''(\chi_{\text{osc}})}{V'(\chi_{\text{osc}})} \right)^2 - \frac{3X'(\chi_{\text{osc}})}{\chi_{\text{osc}}} + \frac{3X(\chi_{\text{osc}})}{\chi_{\text{osc}}^2} \right]. \quad (39)$$

In what follows, we will analyse it in our framework showing that the power spectrum $P_\zeta$ and local non-Gaussianity $f_{\text{NL}}$ are independent of the potential of inflation.

In order to obtain $P_\zeta$ and $f_{\text{NL}}$, the key step is to find the relation between $V(\chi_{\text{osc}})$ and $V(\chi_s)$, namely we need the relation between $\chi_{\text{osc}}$ and $\chi_s$. Through modified Klein-Gordon (KG) equation of $\chi$,

$$\dot{\chi} = -\frac{1}{cH} \frac{\partial V(\chi)}{\partial \chi}, \quad (40)$$

where $\chi \equiv \chi(t)$ denotes the background field of curvaton. Combine with the potential of curvaton and the definition of $dN = Hdt \rightarrow dt = \frac{dN}{H}$ and integrate both sides of Eq (40), then one can obtain,

$$M_p^2 \log \left( \frac{g_0 V_{\chi_{\text{osc}}} - \lambda_0 \lambda_1 M_P + \lambda_0 \lambda_1^2 \chi_{\text{osc}}}{g_0 V + \lambda_0 \lambda_1^2} \right) = -\left( \int_{N_{\text{end}}=0}^{N_{\text{osc}}} \frac{dN}{3H^2} + \int_{N_{\text{end}}}^{N_{\text{osc}}} \frac{dN}{cH^2} \right). \quad (41)$$

So as to achieve the analytic relation between $\chi_{\text{osc}}$ and $\chi_s$, in which the contribution of term consisting of $\lambda_0$ can be neglected since its value being of order of dark energy ($\lambda_0 \approx 10^{-120}$ as setting $M_P = 1$). In light of slow roll conditions $3M_p^2H^2 = V$, consequently one derives the relation as follows,

$$\chi_{\text{osc}} = \chi_s \exp \left[ -6g \left( \frac{1}{c} N_{\text{osc}} - \frac{1}{3} N_s \right) \right], \quad (42)$$

where $N_{\text{osc}}$ and $N_s$ represent the e-folding number during the time of curvaton oscillation and the horizon exit, respectively.
According to COBE normalization \( [23] \), we set \( H_I = 3.0 \times 10^{(-5)} \) and find \( A_s = 2.1 \times 10^{-9} \) corresponding to the purple plate in this figure, where \( r = r_{\text{decay}} \).

Eq. (42) tells us that the curvaton approximately decays during inflation and starts to oscillate in RD. Hence, it is a good proxy to the value of curvature during onset of curvaton oscillation, which determines the relation between the \( V(\chi_{\text{osc}}) \) and \( V(\chi_*) \). Armed with this relation, we can obtain the statistical properties of curvaton via Eq. (32) and Eq. (38).

Combine with definition of e-folding number \( N = \int dt H \) and slow roll conditions \( 3M_P^2 = V \), in which the potential of inflation is dominant, then one simply derives as \( \Delta N = \frac{\phi_*^2}{4M_P^2} - \frac{1}{2} \approx \frac{\phi_*^2}{4M_P^2} \) since the first term is dominant comparing to \( \frac{1}{2} \). According to these approximations and \( \lambda_0 \approx 0 \) in Planck units, we remarkably find that

\[
P_\zeta = \frac{H_* r^2}{9\pi^2 \chi_*^2},
\]

\[
f_{NL} = -\frac{5r}{6} + \frac{5}{4r} - \frac{5}{3},
\]

which are independent of potential of inflation. In this approximation of \( \lambda_0 \approx 0 \), it corresponds to the case of \( A = \frac{1}{2} \) of \([39, 40]\), even the relation \( (42) \) between \( \chi_{\text{osc}} \) and \( \chi_* \) is not mandatory. From Eq. (44) shows that the coupling \( g_0 \) does not significantly affect the power spectrum and non local Gaussianity. In order to elaborate this scenario, we plot \( f_{NL} \) and \( P_\zeta \).

From figure \( (2) \), it tells that the trend of power spectrum with \( r_{\text{decay}} \) and \( \chi_* \), in which it clearly indicates that power spectrum decreases as \( \chi_* \) enhances. However, we cannot figure
FIG. 3: Plot of $f_{NL}$: According to the Planck collaboration constraint on $f_{NL}$ from [22], we show the upper and lower bound of $f_{NL}$, in which we are able to find the lower bound of $r_{\text{decay}} \approx 0.12$.

out the range of fraction rate of curvaton $r_{\text{decay}}$.

In order to compensate this range, the plot of $f_{NL}$ is necessary. From figure (3), it posits the range of $r_{\text{decay}}$ of our model, in which it indicates that the upper bound of $r_{\text{decay}} \approx 0.12$. Namely, following the standard procedure, a completely and fully decay of curvaton cannot occur.

In this section, we premeditate that power spectrum of and non-local Gaussianity under the framework of $\delta N$ formalism. Taking the appropriate approximation for the calculation, the results imply that it is independent of potential of inflation. From the figure (3, 2), the constraint of $r_{\text{decay}}$ can be found, precisely speaking for its lower bound. Once the transferring of energy from inflaton to curvaton occurs, the generation of curvature is a natural production. Following the Ref. [1], we will consider the dark matter comes via the quantum fluctuation of curvaton.

V. DARK MATTER FROM THE QUANTUM FLUCTUATION OF CURVATON

We will see that how the idea of Ref. [1] can be embedded into our scenario. Recall that Ref. [1] proposed a simple mechanism for explaining the origin of dark matter, in which the dark matter comes via the quantum fluctuations of some scalar field (i.e. inflaton, Higgs
field, etc.). Ref. [1] considers the most simple action of some scalar field,
\[ \mathcal{L}_{DM} = \frac{1}{2} \partial^\mu \chi \partial_\mu \chi - \frac{1}{2} m_\chi^2 \chi^2. \]  
(45)

In this simple action, this scalar field is assumed to be tiny during inflation. Thus, for the first impression, it can be dubbed as curvaton comparing to inflaton. Subsequently, author derives the EOM of background of curvaton. From Eq. (6) via Ref. [1],
\[ \frac{\Omega_\chi h^2}{0.12} = 3.5 \times 10^{17} g_*(H_{osc})^{1/4} \left( \frac{\chi}{M_p} \right) \sqrt{\frac{m_\chi}{GeV}}, \]  
(46)

where \( M_p \) is the reduced Planck mass and \( g_*(H_{osc}) \) is the effective number of entropic degrees of freedom at the moment of \( H_{osc} \). Author claims that some combinations of \( H_*, \chi_* \) (where * presents the moment of field frozen) and \( m_\chi \) could account for the abundance of cold dark matter. However, one cannot determine what energy scale belongs to this massive parameter. In our model, it naturally repeats his analysis since the mass of curvaton is running proportional to potential of inflation, whose range locates from inflationary energy scale to be almost vanishing (since \( r_{\text{decay}} \) consists of lower bound). Therefore, some values of curvaton mass satisfy with these combinations.

Due to the existence of decay rate \( r_{\text{decay}} \), it would generate the isocurvature perturbation. Following [1], we postulate that curvaton and inflaton are decoupled from radiation for simplicity. Then, one derives the same formula for power spectrum of isocurvature as [1] and its corresponding spectral index. In particular, the spectral index \( n_\chi \) can be blue tilted when the mass of curvaton is large, in which it corresponds to case of inflaton just starts to decay into curvaton. Furthermore, the new result of this curvaton scenario is that the mass is running which means that the spectral index of isocurvature covers the range from blue tilted to red tilted. According to [23], the order of spectrum of isocurvature is \( 10^{-10} \). Once improving the accuracy of telescope, one may test the validity of our model in light of varying spectral index of isocurvature. Its detailed observational effects in various inflationary potentials will be left for the future work. [1] also derives the lower bound of \( \frac{\chi}{H} \) whose value is \( 7.0 \times 10^4 \), in which this lower bound could avoid the large isocurvature perturbation. Notice that this happens at the moment of inflation ends which is also consistent with our model since we have assumed that the preheating of inflaton is to curvaton. While it almost finished this physical process, it naturally generates the large initial value of curvaton. Meanwhile, in order to obtain the accurate abundance of DM, we should need
the large variety of masses and initial field values, which can be achieved due to the running mass of curvaton and preheating of inflaton to curvaton.

Finally, we will analyze Eq. (46), especially for the analysis of $\chi_*$, $H_{osc}$ and $m_\phi$. While the inflation ends, the inflaton starts to oscillate its energy will be transferring to curvaton (namely the production of curavton). Strictly speaking, $H_{osc}$ will be determined by curvaton as energy transferring occurs. Our inflationary potential is adopted by the framework of chaotic infaltion [20], which is the same as [41, 42] showing that the field distribution in patches with size the horizon at the end of inflation is Guassian and meanwhile the variance is defined by Eq. (15) via [1]. And the field values can be determine by the the equilibrium distribution regarding of e-folding number $N_*$ [43]. As mentioned, $\chi_*$ is not a free parameter any more as evaluating the DM abundance. According to figure (2) through [1], we see that the allowed range of $H_*$, also including the $H_{osc}$ (the red line of Figure 2 in [1]). Consequently, there is one unknown cosmological parameter once determining $H_{osc}$ or $\chi_*$. And our model recovers the range of $m_\chi$ automatically thanks to the running mass.

In this section, we have shown that how the mechanism of generating DM from the quantum fluctuations of curvaton in light of [1]. The key step is postulated that the effective mass of curvaton is running proportional to potential of inflation. In order to accurately account for the abundance of DM, we have shown that it can be illustrated in our scenario since the detail is repeated by assuming that the dynamic of curvaton is dominant after inflation. Once the $H_{osc}$ and $\chi_*$ are determined, then only mass is a free parameter perfectly matching our assumption.

VI. REHEATING THE UNIVERSE

In the beginning, the process of preheating has been applied for the production of curvaton and meanwhile the Universe has experienced an exponential expansion. As a consequence, the Universe would become cold whose temperature proportional to $a^{-1}$ and the energy scale is not sufficient for generating the standard model’s (SM) particles (i.e. Higg particle, photon, $W^\pm$, etc). Therefore, the appearance of mechanism for reheating the Universe is a natural process. However, the traditional paradigm cannot obtain the efficient temperature for reheating the Universe [12].

This fatal problem can be cured by introducing the coupling between curvaton and infla-
ton. Then we will proceed this step by step. It is postulated that the curvaton decays into Higg particles. Consequently, there is also a coupling between the curvaton and Higgs field where only the simplest case will be considered, namely only the condensation of Higgs field using its amplitude, following Ref. [6], the action can read as

$$L_{\text{int}} = \int d^4x \sqrt{-g} \left\{ -g_{\text{int}} \chi^2 h^2 \right\} \quad (47)$$

where $h$ denotes the amplitude of Higgs field condensation and $g_{\text{int}}$ is the coupling constant between the curvaton field and Higgs field. In light of QFT, the decay rate Higgs field can be derived by

$$\Gamma_H = \frac{g_{\text{int}} \chi^2}{8 \pi m_\chi}, \quad (48)$$

where $m_{\text{eff}}$ is the effective mass of curvaton defined by

$$m_\chi^2 = \partial^2_V \chi \frac{g_0}{M_P^2} V(\phi) + \frac{\lambda_0 \lambda_1^2}{M_P^2} \exp(-\lambda_1 \chi M_P). \quad (49)$$

The reheating temperature is defined by

$$T_{\text{re}} \sim \left( \Gamma_H M_P \right)^{1/2}. \quad (50)$$

In the traditional scenario of reheating period, the coupling constant is too tiny to yield the sufficient temperature. Customarily, the mass of curvaton is premeditated as a constant. Therefore, the preheating is necessary. However, this fatal problem is solved due to the running mass of curvaton. Observing that the first contribution is almost vanishing since the inflaton approximately decays into curvaton, then only the second part is left being of order dark energy due to $\lambda_0$. Consequently, it is sufficiently to obtain the reheating temperature.

In this section, we only adopt the reheating mechanism to acquire the adequate reheating temperature by introducing the coupling $g_0$ between the curvaton and inflaton. From Eq. (49), it obviously shows that $m_\chi$ will be tiny due to the vanishing of inflationary field which naturally generates the enough reheating temperature via Eqs. (50, 48).

**VII. DARK ENERGY EPOCH**

Currently, it is dark energy epoch which responses the accelerated expansion of Universe. However, its origin is still mysterious. During many optional mechanisms, the cosmological
constant is the simplest explanation of dark energy firstly proposed by Einstein [44]. It plays a role of dark energy coming from the contribution of James Peebles e.t.c [45].

In this scenario, the dark energy will be mimicked by the exponential potential of curvaton in action [1]. From this action, it obviously shows that the effective mass becomes very tiny due the vanishing of inflationary field. Consequently, only the exponential potential of curvaton is dominant, being playing a role of cosmological constant. Current stage is dark epoch dominated by dark energy and the curvaton field is also almost vanishing since it should decay into other particles, especially for the particles of SM. From lower bound of $\frac{\chi_e}{H_c} = 7.0 \times 10^4$ via [1] and a good proxy of $H_I = H_* = 3.0 \times 10^{-5}$ in Planck units, one easily obtains that $\chi_* = 2.1$. Then using relation (42) between $\chi_{osc}$ and $\chi_*$, and relation can approximated to $\chi_{osc} \approx \chi_* \exp(-2g\Delta N)$ where we have used $c \approx 3$ and $\Delta N$ is the variance of e folding number, thus $\Delta N > 60$ and here we set $\Delta N \approx 100$. Meanwhile, the $g_0$ is approximately constraint to 0.01 from figure [1], it yields $\chi_{osc} \approx \chi_* \exp(-2) \approx 0.28$ whose value is located at which the curvaton field starts to oscillate, even it could be smaller due to the decay of curvaton until to being almost vanishing (cannot completely decay via the constraint from lower bound of $r_{decay}$). From another perspective, there is an extra parameter $\lambda_1$ not determined by observations. Due to the smallness of $\chi_e$ denoting the final value after decaying, we have lots of freedom to choose the range of $\lambda_1$ only keeping the $\lambda_1 \frac{\chi_e}{M_p} \ll 1$, in which it means that exponential potential is almost to be a constant $\lambda_0$. Afterwards, we retain $\lambda_0 \approx 10^{-120}$ in Planck units. This potential naturally plays a role of cosmological constant. Finally, the Universe undergoes the dark epoch.

VIII. CONCLUSION AND OUTLOOK

In this paper, we have constructed a model describing the whole evolution of curvaton, which supplied a unified framework of dark matter and dark energy. The effective mass of curvaton was proportional to inflationary potential as showing in action [1]. Due to this coupling, we can solve various unsolved problems from the perspective of phenomenology.

Firstly, there was only one field (inflaton) at the very beginning of Universe. Subsequently, energy was transferring from inflaton to curvaton via preheating-like process. Through section [III], it divided into two cases: narrow resonance and broad resonance. For narrow resonance, using the constraints from the number density of curvaton and decay rate of
inflaton (energy transferring from inflaton to curvaton), one naturally concluded that inflationary field was large field (its initial value much larger than its effective mass). As for the broad resonance, the lower bound of $m_{\text{eff}}^2 g_0 \phi_0^2 M_P^2 \geq 3.0 \times 10^{-9}$ was given. Once generating the curvaton, we calculated the power spectrum and local Non-Gaussianity. Our results agreed with observational constraints via figure 2. Remarkably, we found these results were independent of the inflationary potential albeit effective mass of curvaton was proportional to inflationary potential. This provides us huge freedom to construct the inflationary part.

Secondly, the quantum fluctuations of curvaton could account for the origin of dark matter thanks to the running mass. In order to obtain the accurate abundance of DM $H_{osc}$ and $m_{\chi}$, however one cannot get the precise value of mass for curvaton since the mass parameter is varying. This mechanism was naturally embedded into our scenario by introducing the coupling between the curvaton and inflaton. Even according to discussions of $H_{osc}$, the spectral index of isocurvature depended on the mass of curvaton. In our model, the spectral index varied from blue tiled to red tiled also due to the running properties of curvaton. This new feature could test the validity of our model upon improving the accuracy of telescope.

Thirdly, even for the production of SM’s particles ($W^\pm$, $W^0$, Higgs particles, $e\tau$), via the decay of curvaton, we could see that the reheating temperature was still sufficient in this scenario. From Eq. (50), it clearly told that the reheating temperature would be dramatically enhanced due to the almost vanishing of effective mass of curvaton, although the coupling between the curvaton and Higgs field was tiny (the tininess would not yield the sufficient temperature).

Finally, as the decay of curvaton occurred, its field value would be smaller and smaller. From section (VII), we have discussed that the exponential potential would be approaching to be a constant $\lambda_0$ of order of cosmological constant, although the curvaton field cannot completely decay into the radiation (constraint from $r_{\text{decay}}$). Therefore, this relic of exponential potential would play a role of cosmological constant solving the dark energy problem from the perspective of phenomenology.

In this work, the exponential potential was introduced by hands. Its origin needs more understanding left for the future work. We adopted the chaotic inflationary framework whose potential directly couples to curvaton. However, one could discover that the introduction of coupling was not natural. In order to adequately obtain this coupling, we
should use the spontaneously broken of symmetry. As Ref. [5] mentioned, before a certain moment $t_1$, the expectation value of inflationary field is zero, namely, $\langle \phi^2 \rangle = 0$. Up to the occurrence of phase transition, $\langle \phi^2 \rangle$ becomes non-vanishing, thus the interaction of between $\chi$ and $\phi$ is no longer non-zero, which means that it would generate the mass of curvaton naturally due to the symmetry breaking. Thus, it is automatically consider inflationary field as Higgs field in light of [18, 19]. In our curvaton scenario, we could construct the Higgs field to response the inflation and curvaton is generating the curvature perturbations. Furthermore, the one loop correction can be taken account into considerations under the framework of finite temperature field theory, particularly the effects via temperature to observable. In a near future, one could also use the asymptotic safety to construct the inflationary part [17], even we are also interested in study of dark matter constraint via framework of brane world [47–49].

Acknowledgements

LH is grateful for Ai-Chen Li and Hai-qing Zhang of the fruitful discussions and comments for this manuscripts, and thanks for hospitality of Institute of Theoretical Physics in Beijing University of Technology and Beihang University when we begun this project. LH is very precious for his Ph.D supervisor, Prof. Tomislav Prokopec helps the calculation of appendix (forever thanks for his guidance and endless discussions during his whole Phd period). LH is funded by initial started funding of Jishou University. WL is funded by NSFC 1175012.

Appendix: Linearized curvaton perturbations

In this appendix we recall how to calculate the spectrum of curvaton perturbations during inflation in the simplest, tree level (linearized) approximation. On a fixed cosmological gravitational background the curvaton dynamics is governed by Eq. (30), which is valid provided the curvaton can be regarded as a spectator field, i.e. if its energy density is subdominant during inflation. Assuming this is true and moreover the curvaton perturbations are small, one can linearize (30) around the background field values, $\bar{\chi}_E(t) = \langle \hat{\chi}_E \rangle$, $\bar{\phi}_E(t) = \langle \hat{\phi}_E \rangle$ such that, upon a convenient rescaling and linearization, Eq. (30) simplifies
to,

\[ \left( \partial^2_0 - \nabla^2 + a^2 V''_E - \frac{a''}{a} \right) (a \delta \hat{\chi}_E) = 0, \]

(51)

where \( \delta \hat{\chi}_E = \hat{\chi}_E - \langle \hat{\chi}_E \rangle \), \( V''_E = \partial^2 V_E(\bar{\chi}_E, \bar{\phi}_E) / \partial^2 \bar{\chi}_E \), and \( a'' = d^2 a / d\tau^2 \). Since the background (29) is invariant under spatial translations, it is natural to assume that the state respects the same symmetry. In that case one can expand \( \delta \hat{\chi}_E(x) \) in terms of mode functions \( \chi_E(\tau, k) \) and \( \chi^*_E(\tau, k) \) as follows,

\[ \delta \hat{\chi}_E(\tau, \vec{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{x}} \left[ \chi_E(\tau, k) \hat{a}(\vec{k}) + \chi^*_E(\tau, k) \hat{a}^\dagger(-\vec{k}) \right], \]

(52)

where \( k = \| \vec{k} \| \), \( \hat{a}(\vec{k}) \) is the particle annihilation operator that annihilates the vacuum \( |\Omega\rangle \), \( \hat{a}(\vec{k}) |\Omega\rangle = 0 \), and \( \hat{a}^\dagger(\vec{k}) \) is the particle creation operator that creates one quantum of momentum \( \vec{k} \). These operators obey,

\[ [\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}'')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}''), \quad [\hat{a}(\tau, \vec{k}), \hat{a}(\tau, \vec{k}'')] = 0, \quad [\hat{a}^\dagger(\tau, \vec{k}), \hat{a}^\dagger(\tau, \vec{k}'')] = 0. \]

(53)

From (51) one can see that the mode function \( \chi_E(\tau, k) \) satisfies the following differential equation,

\[ \left( \frac{d^2}{d\tau^2} + k^2 + a^2 V''_E - \frac{a''}{a} \right) [a \chi_E(\tau, k)] = 0. \]

(54)

The last term in (54) can be written as,

\[ \frac{a''}{a} = \mathcal{H}^2_E \left( 1 + \frac{\mathcal{H}'_E}{\mathcal{H}^2_E} \right) = \mathcal{H}^2_E \left( 2 - \epsilon_1 \right) \]

(55)

\[ \epsilon_1 = -\frac{\dot{\mathcal{H}}_E}{\mathcal{H}^2_E} = 1 - \frac{\mathcal{H}'_E}{\mathcal{H}^2_E} \]

(56)

is the principal slow roll parameter. The conformal Hubble parameter \( \mathcal{H}_E \) can be expressed in terms of conformal time and a power series in slow roll parameters as follows,

\[ \mathcal{H}_E = -\frac{1}{\tau} \left[ 1 + \epsilon_1 + \epsilon_1(\epsilon_1 + \epsilon_2) + \mathcal{O}(\epsilon^3_i) \right]. \]

(57)

Taking account of these relations and calculating to the second order in slow roll parameters, Eq. (54) becomes

\[ \left( \frac{d^2}{d\tau^2} + k^2 - \frac{1}{\tau^2} \left[ 2 + 3\epsilon_1 + 4\epsilon_1(\epsilon_1 + \epsilon_2) + \frac{3}{2}(1 + 2\epsilon_1) \eta_c + \mathcal{O}(\epsilon^3_i, \eta_c \epsilon^2_j) \right] \right) [a \chi_E(\tau, k)] = 0, \]

(58)

where we have introduced the principal curvaton slow roll parameter \( \eta_c \) and the second slow roll parameter \( \epsilon_2 \) as,

\[ \eta_c = -\frac{2}{3} \frac{V''_E}{\mathcal{H}^2_E}, \quad \epsilon_2 = \frac{\dot{\epsilon}_1}{\epsilon_1 H}. \]

(59)
Assuming the term in (58) that multiplies $1/\tau^2$ varies adiabatically in time, Eq. (58) can be solved in terms of Hankel functions. The fundamental solutions are given by,

$$\psi(\tau, k) = \frac{1}{a} \sqrt{-\frac{\pi \tau}{4}} H_\nu^1(-k \tau), \quad \psi^*(\tau, k) = \frac{1}{a} \sqrt{-\frac{\pi \tau}{4}} H_\nu^2(-k \tau),$$

their Wronskian normalization is,

$$W[\psi(\tau, k), \psi^*(\tau, k)] = \frac{i}{a^2}.$$  \hspace{1cm} (60)

and the index reads,

$$\nu^2 = \frac{9}{4} + 3\epsilon_1 + \frac{3}{2} \eta_c + \epsilon_1 (4\epsilon_1 + 4\epsilon_2 + 3\eta_c) \implies \nu = \frac{3}{2} + \epsilon_1 + \frac{1}{2} \eta_c + \frac{1}{3} \epsilon_1 (3\epsilon_1 + 4\epsilon_2 + 3\eta_c) + \mathcal{O}(\epsilon_1^3, \eta_c \epsilon_2).$$ \hspace{1cm} (61)

The general mode consistent with spatial homogeneity and isotropy is then,

$$\chi_E(\tau, k) = \alpha(k) \psi(\tau, k) + \beta(k) \psi^*(\tau, k), \quad |\alpha(k)|^2 - |\beta(k)|^2 = 1.$$ \hspace{1cm} (62)

A standard Bunch-Davies choice of the vacuum amounts to $\alpha(k) = 1$ and $\beta(k) = 0$, which is what we assume throughout this work.

The corresponding power spectrum and spectral index are defined by,

$$P_{\chi_E}(\tau, k) = \frac{k^3}{2\pi^2} |\chi_E|^2 = P_{\chi_E^*} \left( \frac{k}{k_\nu} \right)^{n_{\chi_E}}.$$ \hspace{1cm} (63)

By making use of (63) and (60) in (64) one obtains,

$$P_{\chi_E}(k, \tau) = \frac{1}{a^2} \frac{k^3 |\tau|}{8\pi} |H_\nu^1(-k \tau)|^2.$$ \hspace{1cm} (64)

We are particularly interested in super-Hubble scales, where $|k \tau| \ll 1$, and the Hankel functions of the first kind,

$$H_\nu^1(-k \tau) = \frac{1}{\sin(\pi \nu)} \left[ e^{i\pi \nu} J_\nu(-k \tau) - i J_{-\nu}(-k \tau) \right] \quad (|\arg[-k \tau]| < \pi)$$ \hspace{1cm} (65)

can be expanded as,

$$H_\nu^1(-k \tau) = \frac{1}{\pi} \left[ -e^{i\pi \nu} \Gamma(-\nu) \left(\frac{-k \tau}{2}\right)^\nu - i \Gamma(\nu) \left(\frac{-k \tau}{2}\right)^{-\nu} \right] + \mathcal{O} (|k \tau|^\nu + 2, |k \tau|^{-\nu + 2}).$$ \hspace{1cm} (66)

Since $\nu > 0$, the second term of Eq. (67) dominates and we arrive at the curvaton power spectrum on super-Hubble scales,

$$P_{\chi_E}(\tau, k) = \frac{H_E^2 \Gamma^2(\nu)}{\pi^3 [1 + \epsilon_1 + \epsilon_1 (\epsilon_1 + \epsilon_2)]^2} \left[ k [1 + \epsilon_1 + \epsilon_1 (\epsilon_1 + \epsilon_2)] / 2 H_E a \right]^{n_{\chi}},$$ \hspace{1cm} (68)
with $\nu$ given in (62),

\[
n_\chi = 3 - 2\nu = -2\epsilon_1 - \eta_c - \frac{2}{3}\epsilon_1 (3\epsilon_1 + 4\epsilon_2 + 3\eta_c) + \mathcal{O}(\epsilon_1^2, \eta_c \epsilon_1^2) \tag{69}
\]

where we have used, $-k\tau \approx k[1 + \epsilon_1 + \epsilon_1(\epsilon_1 + \epsilon_2)]/(H_E a)$, see (57). From (68) one can easily read off the spectrum amplitude $P_{\chi E^*}$. To linear order in the slow roll parameters it reads,

\[
P_{\chi E^*}(t, k_*) = \frac{H_E^2}{4\pi^2} \left[ 1 - 2\epsilon_1 + \frac{2}{3}(2\epsilon_1 + \eta_c)\psi(3/2) \right] \exp \left[ -n_\chi \left( N + \ln \frac{2H_E(1-\epsilon_1)}{k_*} \right) \right], \tag{70}
\]

where $\psi(3/2) = 2 - \gamma_E - 2\ln(2) \approx 0.0367$ is the di-gamma function of $3/2$ and $H^2 \simeq H_0^2 e^{-2\epsilon_1 N}$. We see that the amplitude $P_{\chi E^*}$ depends weekly on time. For example, for a red-tilted spectrum for which $n_\chi < 0$, $P_{\chi E^*}$ grows exponentially with the number of e-foldings $N = \ln(a)$, e.g. for $n_\chi \approx -0.04$ and $\epsilon_1 = 0.01$, $P_{\chi E^*}$ grows about 2% per e-folding.

References

[1] T. Tenkanen, Phys. Rev. Lett. 123 (2019) no.6, 061302 doi:10.1103/PhysRevLett.123.061302 [arXiv:1905.01214 [astro-ph.CO]].

[2] K. Enqvist and M. S. Sloth, “Adiabatic CMB perturbations in pre - big bang string cosmology,” Nucl. Phys. B 626 (2002) 395 doi:10.1016/S0550-3213(02)00043-3 [hep-ph/0109214].

[3] D. H. Lyth and D. Wands, “Generating the curvature perturbation without an inflaton,” Phys. Lett. B 524 (2002) 5 doi:10.1016/S0370-2693(01)01366-1 [hep-ph/0110002].

[4] T. Moroi and T. Takahashi, “Effects of cosmological moduli fields on cosmic microwave background,” Phys. Lett. B 522 (2001) 215 Erratum: [Phys. Lett. B 539 (2002) 303] doi:10.1016/S0370-2693(02)02070-1, 10.1016/S0370-2693(01)01295-3 [hep-ph/0110096].

[5] J. H. Traschen and R. H. Brandenberger, “Particle Production During Out-of-equilibrium Phase Transitions,” Phys. Rev. D 42 (1990) 2491. doi:10.1103/PhysRevD.42.2491

[6] L. Kofman, A. D. Linde and A. A. Starobinsky, “Reheating after inflation,” Phys. Rev. Lett. 73 (1994) 3195 doi:10.1103/PhysRevLett.73.3195 [hep-th/9405187].

[7] Y. Shtanov, J. H. Traschen and R. H. Brandenberger, “Universe reheating after inflation,” Phys. Rev. D 51 (1995) 5438 doi:10.1103/PhysRevD.51.5438 [hep-ph/9407247].
[8] T. Prokopec and T. G. Roos, “Lattice study of classical inflaton decay,” Phys. Rev. D 55 (1997) 3768 doi:10.1103/PhysRevD.55.3768 [hep-ph/9610400].

[9] B. R. Greene, T. Prokopec and T. G. Roos, “Inflaton decay and heavy particle production with negative coupling,” Phys. Rev. D 56 (1997) 6484 doi:10.1103/PhysRevD.56.6484 [hep-ph/9705357].

[10] L. Kofman, A. D. Linde and A. A. Starobinsky, “Towards the theory of reheating after inflation,” Phys. Rev. D 56 (1997) 3258 doi:10.1103/PhysRevD.56.3258 [hep-ph/9704452].

[11] P. B. Greene, L. Kofman, A. D. Linde and A. A. Starobinsky, “Structure of resonance in preheating after inflation,” Phys. Rev. D 56 (1997) 6175 doi:10.1103/PhysRevD.56.6175 [hep-ph/9705347].

[12] R. Allahverdi, R. Brandenberger, F. Y. Cyr-Racine and A. Mazumdar, Ann. Rev. Nucl. Part. Sci. 60 (2010) 27 doi:10.1146/annurev.nucl.012809.104511 [arXiv:1001.2600 [hep-th]].

[13] K. Enqvist, S. Nurmi and G. I. Rigopoulos, “Parametric Decay of the Curvaton,” JCAP 0810 (2008) 013 doi:10.1088/1475-7516/2008/10/013 [arXiv:0807.0382 [astro-ph]].

[14] N. Bartolo and A. R. Liddle, “The Simplest curvaton model,” Phys. Rev. D 65 (2002) 121301 doi:10.1103/PhysRevD.65.121301 [astro-ph/0203076].

[15] D. G. Figueroa and F. Torrenti, “Gravitational wave production from preheating: parameter dependence,” JCAP 1710 (2017) no.10, 057 doi:10.1088/1475-7516/2017/10/057 [arXiv:1707.04533 [astro-ph.CO]].

[16] L. H. Liu, “Exploring new windows into fundamental physics at high energies,” ISBN: 9789463803120.

[17] L. H. Liu, T. Prokopec and A. A. Starobinsky, “Inflation in an effective gravitational model and asymptotic safety,” Phys. Rev. D 98 (2018) no.4, 043505 doi:10.1103/PhysRevD.98.043505 [arXiv:1806.05407 [gr-qc]].

[18] K. Enqvist, R. N. Lerner and T. Takahashi, “The minimal curvaton-higgs model,” JCAP 1401 (2014) 006 doi:10.1088/1475-7516/2014/01/006 [arXiv:1310.1374 [astro-ph.CO]].

[19] K. Enqvist, D. G. Figueroa and R. N. Lerner, “Curvaton Decay by Resonant Production of the Standard Model Higgs,” JCAP 1301 (2013) 040 doi:10.1088/1475-7516/2013/01/040 [arXiv:1211.5028 [astro-ph.CO]].

[20] A. D. Linde, Phys. Lett. 129B (1983) 177. doi:10.1016/0370-2693(83)90837-7

[21] D. H. Lyth, C. Ungarelli and D. Wands, “The Primordial density perturbation in the cur-
vation scenario,” Phys. Rev. D 67 (2003) 023503 doi:10.1103/PhysRevD.67.023503 [astro-ph/0208055].

[22] P. A. R. Ade et al. [Planck Collaboration], “Planck 2015 results. XVII. Constraints on primordial non-Gaussianity,” Astron. Astrophys. 594 (2016) A17 doi:10.1051/0004-6361/201525836 [arXiv:1502.01592 [astro-ph.CO]].

[23] Y. Akrami et al. [Planck Collaboration], “Planck 2018 results. X. Constraints on inflation,” arXiv:1807.06211 [astro-ph.CO].

[24] K. Mukaida, K. Nakayama and M. Takimoto, Phys. Rev. D 89 (2014) no.12, 123515 doi:10.1103/PhysRevD.89.123515 [arXiv:1402.1856 [astro-ph.CO]].

[25] K. Dimopoulos, K. Kohri, D. H. Lyth and T. Matsuda, JCAP 1203 (2012) 022 doi:10.1088/1475-7516/2012/03/022 [arXiv:1110.2951 [astro-ph.CO]].

[26] M. Kawasaki, T. Kobayashi and F. Takahashi, JCAP 1303 (2013) 016 doi:10.1088/1475-7516/2013/03/016 [arXiv:1210.6595 [astro-ph.CO]].

[27] D. H. Lyth, C. Ungarelli and D. Wands, Phys. Rev. D 67 (2003) 023503 doi:10.1103/PhysRevD.67.023503 [astro-ph/0208055].

[28] M. K. Sharma, K. Myrzakulov and M. A. Ajmi, arXiv:1905.12433 [gr-qc].

[29] J. O. Gong, N. Kitajima and T. Terada, JCAP 1703 (2017) 053 doi:10.1088/1475-7516/2017/03/053 [arXiv:1611.08975 [astro-ph.CO]].

[30] M. Kawasaki, N. Kitajima and T. T. Yanagida, Phys. Rev. D 87 (2013) no.6, 063519 doi:10.1103/PhysRevD.87.063519 [arXiv:1207.2550 [hep-ph]].

[31] K. Ando, M. Kawasaki and H. Nakatsuka, Phys. Rev. D 98 (2018) no.8, 083508 doi:10.1103/PhysRevD.98.083508 [arXiv:1805.07757 [astro-ph.CO]].

[32] C. Chen and Y. F. Cai, JCAP 1910 (2019) no.10, 068 doi:10.1088/1475-7516/2019/10/068 [arXiv:1908.03942 [astro-ph.CO]].

[33] C. T. Byrnes, M. Corts and A. R. Liddle, Phys. Rev. D 94 (2016) no.6, 063525 doi:10.1103/PhysRevD.94.063525 [arXiv:1608.02162 [astro-ph.CO]].

[34] A. A. Starobinsky, JETP Lett. 42, 152 (1985) [Pisma Zh. Eksp. Teor. Fiz. 42, 124 (1985)].

[35] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. 95, 71 (1996) doi:10.1143/PTP.95.71 [astro-ph/9507001].

[36] D. Wands, K. A. Malik, D. H. Lyth and A. R. Liddle, Phys. Rev. D 62, 043527 (2000) doi:10.1103/PhysRevD.62.043527 [astro-ph/0003278].
[37] D. H. Lyth, K. A. Malik and M. Sasaki, JCAP 0505, 004 (2005) doi:10.1088/1475-7516/2005/05/004 [astro-ph/0411220].

[38] M. Sasaki, J. Valiviita and D. Wands, “Non-Gaussianity of the primordial perturbation in the curvaton model,” Phys. Rev. D 74 (2006) 103003 doi:10.1103/PhysRevD.74.103003 [astro-ph/0607627].

[39] M. Kawasaki, T. Kobayashi and F. Takahashi, Phys. Rev. D 84, 123506 (2011) doi:10.1103/PhysRevD.84.123506, 10.1103/PhysRevD.85.029905 [arXiv:1107.6011 [astro-ph.CO]].

[40] T. Kobayashi and T. Takahashi, JCAP 1206, 004 (2012) doi:10.1088/1475-7516/2012/06/004 [arXiv:1203.3011 [astro-ph.CO]].

[41] T. Markkanen, A. Rajantie, S. Stopyra and T. Tenkanen, “Scalar correlation functions in de Sitter space from the stochastic spectral expansion,” JCAP 1908 (2019) 001 doi:10.1088/1475-7516/2019/08/001 [arXiv:1904.11917 [gr-qc]].

[42] A. A. Starobinsky and J. Yokoyama, “Equilibrium state of a selfinteracting scalar field in the De Sitter background,” Phys. Rev. D 50 (1994) 6357 doi:10.1103/PhysRevD.50.6357 [astro-ph/9407016].

[43] K. Enqvist, R. N. Lerner, O. Taanila, and A. Tranberg, JCAP 1210, 052 (2012), arXiv:1205.5446 [astro-ph.CO].

[44] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1917 (1917) 142.

[45] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75 (2003) 559 doi:10.1103/RevModPhys.75.559 [astro-ph/0207347].

[46] W. L. Xu and Y. C. Huang, Int. J. Mod. Phys. A 34 (2019) no.23, 1950132 doi:10.1142/S0217751X1950132X [arXiv:1905.03688 [gr-qc]].

[47] W. L. Xu, A. C. Li and Y. C. Huang, arXiv:1901.02155 [gr-qc].

[48] A. C. Li, W. L. Xu and D. F. Zeng, JCAP 1903 (2019) 016 doi:10.1088/1475-7516/2019/03/016 [arXiv:1812.07224 [hep-th]].

[49] A. C. Li, H. q. Shi and D. f. Zeng, Phys. Rev. D 97 (2018) no.2, 026014 doi:10.1103/PhysRevD.97.026014 [arXiv:1711.04613 [hep-th]].