Polygons in hyperbolic geometry using Beltrami-Klein models

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Abstract. Hyperbolic geometry was a geometry based on Hyperbolic Parallel Postulate. The purpose of this research was to describe the model of basic objects, concepts of triangles and polygons, as well explain types of polygons in hyperbolic geometry used the Beltrami-Klein model. The research method was literature study, it studied the definition and axiom were related to Euclid geometry and hyperbolic geometry such as point, line, triangle, and polygon. The result of this study was: 1) The plane model for Beltrami-Klein model is a circle. The object of geometry such as Klein points was expressed as a dot, Klein lines was expressed as part of Euclid lines that contained in a circle, Klein distance and Klein angles has the same presentation as Euclid distances and Euclid angles. 2) The Klein triangles have different concepts from the Euclid triangles. 3) The Klein polygons were defined as a combination of Klein segments, Klein rays, or Klein lines that was foreign to each other and have n different Klein points that was not collinearities. 4) The Klein polygons were divided into two types, the ordinary Klein polygons and the asymptotic Klein polygons.

1. Introduction

Euclidean geometry is first appeared geometry. Euclides, one of the Mathematic figure of Alexandria, made a book called “The Element” that discussed about Euclid geometry. Euclid geometry has a weakness for the fifth postulate that is “If a straight line intersects two straight lines and make unilateral inner angles less than two right angles, if both lines are extended indefinitely, then it will meet on both sides where the two inner angles are less than two right angles”[1]. This postulate states that the angular conditions formed by a transversal, that is the number of angles in a unilateral angle are less than the two right angles shows that the two lines are not parallel [2]. The term parallel lines are not contained in this postulate, but this postulate is often referred as Euclid Parallel Postulate. Mathematicians consider this postulate as a theorem derived from the four previous postulates and needs to be proven, but Euclid did not mention it as theorem, Euclid call it a postulate that no longer needs to be verified [3].

Euclid fifth postulates, as noted above, became a disagreement between mathematicians at that time. The postulate is said to be Euclid Parallel Postulate, but the fact has contradiction in its visualization, that is unilateral inner angles formed by a transversal less than two right angles stating that the two lines are not parallel [2]. In the visualization it is said that it is not parallel, but the postulate is called the Euclid Parallel Postulate. This inconsistency led to the idea of a new geometry system, that is the Hyperbolic Geometry System.

Nikolai Lobachevsky (1792-1856) is one of the mathematician that find new ideas about Euclid Parallel Postulate, where that idea said that “For each line l and for each point P not on l, there are at least two lines m and n so that P lies on both and both are parallel to l” [2]. Lobachevsky’s statement is known as the Lobachevsky Parallel Postulate which is the basic of Lobachevsky Geometry or Hyperbolic Geometry. This geometry has the opposite concept to the Parallel Postulate.
Euclid geometry has a different base postulate than hyperbolic geometry. Euclid geometry has Euclid Parallel Postulate and hyperbolic geometry has Lobachevsky Parallel Postulate. The axioms and theorems that apply due to differences in postulates in these two geometry systems result in different plane models. As a result, visualization to object to geometry is also might be different. Polygons are object of Euclid geometry. If there are polygons in Euclid geometry, it is also possible for hyperbolic geometries to have polygons. Polygon types of hyperbolic geometry maybe have differences in polygon types of Euclid geometry. The possibility of different types of polygons in two geometries is influenced by differences in the postulates that apply to these two geometry systems. Plane models commonly used to express Euclid geometry objects are parallelogram. On the other hand, plane models used to express hyperbolic geometry objects have several plane models, that is upper half plane model, the Beltrami-Klein model, the Poincare Disc model, and the hyperboloids model [4].

Geometry objects presented by different plane models allow different visualizations. The use of a model can facilitate the illustration of objects and concepts in hyperbolic geometry. The results of previous studies have used a model, that is Poincare Disc model, by Lestari [5] and no one has used another model to present hyperbolic geometry, specifically to present hyperbolic polygons. Therefore, polygons in hyperbolic geometry need to be studied using a plane model to find out the illustration of objects and concepts related to hyperbolic polygons in different plane models. The plane model that will be used is the Beltrami-Klein model. The Beltrami-Klein model is used because it is very simple in describing points and lines [2]. So, this paper will discuss about the presentation of the basic objects of hyperbolic geometry using Beltrami-Klein models, triangular model on hyperbolic geometry using Beltrami-Klein models, polygon model in hyperbolic geometry using Beltrami-Klein models, and types of polygons in hyperbolic geometry using Beltrami-Klein models. The research method used is the literature study. The study was conducted by examining the definitions, axioms, postulates, and theorems related to Euclid geometry and hyperbolic geometry used to determine the types of hyperbolic polygons in the Beltrami-Klein model from the triangle model to the polygon model.

2. Discussion
Hyperbolic geometry is a geometry system that was born in the inconsistency of fifth Euclid Postulates. This geometry has several models for the presentation of its objects. One of the model is the Beltrami-Klein model. To form a polygon, basic objects such as points, lines, and angles are needed. Therefore, it will be discussed in advance regarding the presentation of basic geometrical objects using the Beltrami-Klein model.

2.1 Presentation of the basic objects of Hyperbolic Geometry using Beltrami-Klein Models
The basic object of hyperbolic geometry requires a plane model as a plane for its presentation. Therefore, it will be discussed in advance about the plane model used to present the basic objects of hyperbolic geometry. Eugenio Beltrami developed the first detailed model of hyperbolic geometry in 1868 [6]. In addition, another mathematician who also developed this model was Felix Klein in 1849-1925 [2]. Therefore, this hyperbolic geometry presentation model is called Beltrami-Klein model. The unit circles \( \gamma \) and its interior is the hyperbolic plane of the Beltrami-Klein model [7].

The point in the Beltrami-Klein model is the Euclid point which is located inside the unit \( \gamma \) circle [2]. The point referred here is Klein ordinary point. Furthermore, also known the Klein ideal point and the ultraideal Klein point. Point that are located on the edge of the hyperbolic plane are called ideal point [2]. The point outside the circle \( \gamma \) is called the ultraideal point [7]. The line of the Beltrami-Klein model is part of the Euclid line which is located inside the unit \( \gamma \) circle [2]. The lines of the Beltrami-Klein model are represented by line segments of the Euclid line [7]. In addition to lines, this model also has line segments and line rays. The line segment of the Beltrami-Klein model is part of the Klein line [2]. The Klein ray is part of the Klein line which has a starting point and end point in Klein ideal point or ultraideal Klein point. Distance pairs each point in a line of a non-negative real number. Hyperbolic distance which is the distance from the \( \gamma \) circles as the Beltrami-Klein plane model can be called the Klein distance. Klein distance is a non-negative real number, as the Distance Postulate says. The Klein angle is a combination of two Klein line rays allied to the base. The notation for Klein angle is same as notation for Euclid angle, that is \( \angle \).

Klein perpendicular is divided into two cases. Case 1: One of \( l \) and \( m \) is the diameter of a \( \gamma \) circle [7]. Case 2: None of these \( l \) or \( m \) is the diameter of the \( \gamma \) circle [7]. The two Klein lines are said to be
asymptotically parallel if they contain asymptotic rays where the asymptotic rays are hyperbolic line rays of different end points and will not intersect in the same direction. Two Klein lines are said to be ultraparallel if hyperbolic parallel lines have a perpendicular line coalition [2]. Some of these Klein lines are said to be multiparallel if some Klein lines that are parallel to a Klein line that is known through a Klein ordinary point not on the line that is known.

Figure 1. Illustration for basic objects in Hyperbolic Geometry and relation of Klein lines using Beltrami-Klein model

![Figure 1](image)

Figure 1. be an illustration for a hyperbolic geometry object of the Beltrami-Klein model. The Beltrami-Klein plane model is shown with a γ circle centered on point O. Ordinary Klein points are shown by points A, B, C, and O. Ideal Klein points are shown by points Ω3, Ω4, Ω5, and Ω6. Ultraideal Klein points are shown by points P(l), Ω4 and Ω2. The Klein line is shown by Ω3Ω4 (r), q, p, Ω4Ω6 (n), l, o, and m. The Klein line segment is shown by ŌC. The Klein rays are shown by OΩ3, OΩ4, OΩ5, and OΩ6. One of example Klein distance is shown by d(OC) which is also an example of the Klein line segment. Klein angle is shown by ∠Ω3Ω4Ω5, ∠Ω4Ω5Ω6, ∠Ω5Ω6Ω3, etc. The Klein perpendicular relation in case 1 are shown by the line n which is perpendicular to the line m because they are perpendicular to each other in Euclid. The Klein perpendicular relation in case 2 are shown by the line l which is perpendicular to m even though they are not the diameter of the γ circle because if the pole line l is made in P(l), then the extension of m will also pass the P(l). Two Klein line that asymptotic parallel is shown by the line m which is asymptotic parallel to the line o because both of them toward to an infinite point outside the γ circle. Two Klein line is said to be ultraparallel is shown by line n which is ultraparallel with l because it has a perpendicular line coalition, that is line m. Some of these Klein lines are said to be multiparallel are shown by lines m that is mutually parallel with lines p, q, and r where the line p, q, and r are Klein lines that pass through point o not on line m. Geometry objects require a plane for their presentation. As the description above, the Beltrami-Klein plane model is a γ circle because to show its objects, a γ circles is needed. For example, to show Klein ordinary point, Klein ideal point and Klein ultraideal point need a plane model. To indicate that there is an ultraideal Klein point that is infinitely further than Klein ideal point, a γ circle is needed. Besides, to show two lines are perpendicular to each other, a tangent to a circle is needed to obtain a line pole. Likewise to show two ultraparallel lines that also need tangents to a circle. This reason reinforces that the Beltrami-Klein plane model is a γ circle.

2.2 Triangular model on Hyperbolic Geometry using Beltrami-Klein Models

Triangles are geometric objects of the form of the simplest flat shapes. Triangles on hyperbolic geometry have different concept than triangles on Euclid geometry. Triangles on hyperbolic geometry, especially on Beltrami-Klein model are an area bound by three sides, where the three sides can be combination of Klein line segments, Klein line rays, or Klein lines. In general, the triangle in hyperbolic geometry is divided into two types, that is ordinary triangles and asymptotic triangles. The ordinary Klein triangles are three non-colliner points and each pair of two adjacent points is connected by a line segment so that it has a finite side length. An asymptotic Klein triangle is a triangle with an infinite length and contains the ideal Klein point or ultraideal Klein point. Asymptotic Klein triangles are divided into three types based on the number of ideal Klein points and ultraideal Klein points contained on it. The asymptotic Klein triangle with one ideal Klein point or one ultraideal Klein point as its vertex is called the single asymptotic Klein triangle. The second types of asymptotic Klein triangle are the asymptotic Klein triangle with two ideal Klein points or two ultraideal Klein points or one of ideal Klein point and one of ultraideal Klein point as its vertex. That is called double asymptotic Klein triangle. The third types
of asymptotic Klein triangle are asymptotic Klein triangle with three ideal Klein points, or three ultraideal Klein points, or both types of points as its vertex. That is called the treble asymptotic Klein triangle. Illustration for Klein triangle is on the Figure 2. Point (a) is illustration for ordinary Klein triangle, point (b) is illustration for single asymptotic Klein triangle, point (c) is illustration for double asymptotic Klein triangle, and point (d) is illustration for treble asymptotic Klein triangle. The Klein triangle model has been explained on description above with its examples. Next will be discussed about the Klein polygon model.

2.3 Polygon Model in Hyperbolic Geometry using Beltrami-Klein Models
The Klein polygon can be defined as a combination of the Klein line segment, the Klein lines rays, or the Klein line which are foreign to each other or have the same endpoints and have n different Klein points that are not collinearities. Figure 3 is an illustration for one of the Klein polygons. There are Klein polygons P1P2P3P4P5P6 ... Pn or can be wrote by P1P2P3P4P5P6 ... Pn = P1P2P3P4P5P6 ... Pn which is a combination of Klein line segments P1P2, P2P3, P3P4, P4P5, P5P6, ..., PnP1 which are foreign to each other and have n distinct Klein ordinary points that are P1, P2, P3, P4, P5, P6, ..., Pn which are not collinear. Klein polygon P1P2P3P4P5P6 ... Pn has Klein angles as many as n, each of which is not zero because all vertex are ordinary Klein points. Polygons are combination of several regions of triangles, or commonly called triangulation.

Based on Figure 3, the Klein triangulation where Klein polygons are present in Klein polygon P1P2P3P4P5P6 ... Pn which is a combination of the Klein ordinary triangle ΔP1OP2, ΔP2OP3, ΔP3OP4, ΔP4OP5, ΔP5OP6, ΔP6OP1, dan ΔPnP1.

\[ R = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6 \cup ... \cup T_n \]

\[ R = \Delta P_1OP_2 \cup \Delta P_2OP_3 \cup \Delta P_3OP_4 \cup \Delta P_4OP_5 \cup \Delta P_5OP_6 \cup \Delta P_6OP_1 \]

The polygon model in Klein has been explained above with its star triangulation examples. Next will be explained about the types of Klein polygons.

2.4 Types of Polygons in Hyperbolic Geometry using Beltrami-Klein Models
The *Klein* polygon is a convex polygon in *Klein* where each pair of *Klein* adjacent vertex $P_i, P_{i+1}$ and each other *Klein* vertex on the *Klein* polygon is one sided with the *Klein* line $PP_{i+1}$ with $p$ is the vertex of the *Klein* polygon. Based on the many *ideal Klein* points and *ultraideal Klein* points contained in a polygon as their vertex, *Klein* polygons are divided into two types, namely ordinary *Klein* polygons and asymptotic *Klein* polygons. Ordinary *Klein* polygons are *Klein* polygons that do not contain *ideal Klein* points and also do not contain *ultraideal Klein* points. This ordinary *Klein* polygon is bounded by $n$ sides of the form of *Klein* line segments. Figure 4 is shown an example of the ordinary *Klein* polygon $P_1P_2P_3P_...P_{n-1}P_n$ with $n$ ordinary *Klein* points $P_1, P_2, P_3, P_..., P_{n-1}, P_n$. This ordinary *Klein* polygon does not contain both *ideal Klein* points or *ultraideal Klein* points so that each angle is not zero. Polygons are combination of several regions of triangles, or commonly called triangulation. The ordinary *Klein* polygon $P_1P_2P_3P_...P_{n-1}P_n$ also combination of several regions of triangles. One of example of triangulation to ordinary *Klein* polygon $P_1P_2P_3P_...P_{n-1}P_n$ is shown on Figure 4. Figure 4 is an illustration for the star triangulation of ordinary *Klein* polygon $P_1P_2P_3P_...P_{n-1}P_n$ which is a combination of the ordinary *Klein* triangle $\Delta P_1OP_1$, $\Delta P_1OP_2$, $\Delta P_2OP_3$, $\Delta P_3OP_{n-1}$, and $\Delta P_{n-1}OP_n$.

\[
R = T_1 \cup T_2 \cup T_3 \cup T_\ldots \cup T_n
\]

\[
R = \Delta P_1OP_1 \cup \Delta P_1OP_2 \cup \Delta P_2OP_3 \cup \Delta P_3OP_{n-1} \cup \Delta P_{n-1}OP_n
\]

![Figure 4](https://example.com/figure4.png)

**Figure 4.** Illustration for ordinary *Klein* Polygon in Hyperbolic Geometry

The asymptotic *Klein* polygon is a *Klein* polygon containing *ideal Klein* point or *ultraideal Klein* point which represents a point at infinity so that if *ideal Klein* point or *ultraideal Klein* point as an vertex, the angle is considered to be a zero. Based on the type of points contained, the *Klein* polygon is divided into three conditions, namely the asymptotic *Klein* polygon containing only the *ideal Klein* point, containing only the *ultraideal Klein* point, and containing both. The number of ordinary *Klein* points expressed by $i$, the number of *ideal Klein* points expressed by $j$, and the number of *ultraideal Klein* points expressed by $k$.

The asymptotic *Klein* polygon containing only *ideal Klein* point is the asymptotic *Klein* polygon containing $j = 1,2,\ldots,n$ *ideal Klein* points. Polygons of this type are grouped in two cases, that are when $j < n$ and when $j = n$. Between the *ideal Klein* points on the asymptotic *Klein* polygon that containing only *ideal Klein* point can be adjacent and can not be adjacent. Illustration for these types of asymptotic *Klein* polygon is on Figure 5. Figure 5 on point (a) is shown an example for the asymptotic *Klein* polygon containing only *ideal Klein* point when $j < n$ $\Omega_1,\Omega_2,\Omega_3,\ldots,\Omega_{n-1},\Omega_n$, point (b) is shown an example for the asymptotic *Klein* polygon containing only *ideal Klein* point when $j < n$ $\Omega_1,\Omega_2,\Omega_3,\ldots,\Omega_{n-1},\Omega_n$, and point (c) is shown an example for the asymptotic *Klein* polygon containing only *ideal Klein* point when $j = n$ $\Omega_1,\Omega_2,\Omega_3,\ldots,\Omega_{n-1},\Omega_n$. Polygons are combination of several regions of triangles, or commonly called triangulation. The illustration of star triangulation for those example are on Figure 5 below.
The asymptotic Klein polygons containing only ultraideal Klein points are asymptotic Klein polygons containing \( k = 1, 2, \ldots, n \) ultraideal Klein points. Polygons of this type are grouped in two cases, that are when \( k < n \) and when \( k = n \). Between the ultraideal Klein points on the asymptotic Klein polygon that containing only ultraideal Klein point can be adjacent and cannot be adjacent. Illustration for these types of asymptotic Klein polygon is on Figure 6. Figure 6 on point (a) is shown an example for the asymptotic Klein polygon containing only ultraideal Klein point when \( k < n \) and point (b) is shown an example for the asymptotic Klein polygon containing only ultraideal Klein point when \( k = n \). Polygons are combination of several regions of triangles, or commonly called triangulation. The illustration of star triangulation for those example are on Figure 6 below.
Figure 6 on point (a) is an illustration for the star triangulation of Klein asymptotic polygon \( \Omega_1 \Omega_2 P_1 P_2 P_\ldots P_{n-1} P_n \).

\[
R = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_\ldots \cup T_n
\]

\[
R = \Delta P_n O \Omega_1 \cup \Delta \Omega_1 O \Omega_2 \cup \Delta \Omega_2 O P_1 \cup \Delta P_1 O P_2 \cup \Delta P_2 O P_\ldots \cup \Delta P_{n-1} O P_n
\]

Figure 6 on point (b) is an illustration for the star triangulation of Klein asymptotic polygon \( \Omega_1 \Omega_2 P_1 P_2 P_\ldots P_{n-1} P_n \).

\[
R = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_\ldots \cup T_n
\]

\[
R = \Delta P_n O \Omega_1 \cup \Delta \Omega_1 O \Omega_2 \cup \Delta \Omega_2 O P_1 \cup \Delta P_1 O P_2 \cup \Delta P_2 O P_\ldots \cup \Delta P_{n-1} O P_n
\]

Figure 6 on point (c) is an illustration for the star triangulation of Klein asymptotic polygon \( \Omega_1 \Omega_2 \Omega_3 \ldots \Omega_{n-1} \Omega_n \).

\[
R = T_1 \cup T_2 \cup T_3 \cup T_\ldots \cup T_n
\]

\[
R = \Delta \Omega_1 O \Omega_2 \cup \Delta \Omega_2 O \Omega_3 \cup \Delta \Omega_3 O \ldots \cup \Delta \Omega_{n-1} O \Omega_{n-1} \cup \Delta \Omega_n O \Omega_{n-1} \cup \Delta P_{n-1} O P_n
\]

The asymptotic Klein polygons containing ideal Klein points and ultraideal Klein points together are asymptotic Klein polygons containing \( j = 1, 2, \ldots, n \) ideal Klein points and \( k = 1, 2, \ldots, n \) ultraideal Klein points. Polygons of this type are grouped in three cases, that are when \( j, k < n \) are adjacent, when \( j, k < n \) are not adjacent, and when \( j = k = \left(\frac{1}{2}\right)n \). Illustration for these types of asymptotic Klein polygon is on Figure 7. Figure 7 on point (a) is shown an example for the asymptotic Klein polygon containing ideal Klein points and ultraideal Klein point when \( j, k < n \) are adjacent \( \Omega_1 \Omega_2 P_1 P_2 P_\ldots P_{n-1} P_n \), point (b) is shown an example for the asymptotic Klein polygon containing ideal Klein points and ultraideal Klein point when \( j, k < n \) are not adjacent \( \Omega_1 \Omega_2 P_1 P_2 P_\ldots P_{n-1} P_n \), and point (c) is shown an example for the asymptotic Klein polygon containing ideal Klein points and ultraideal Klein point when \( j = k = \left(\frac{1}{2}\right)n \) \( \Omega_1 \Omega_2 \Omega_3 \Omega_4 \Omega_5 \Omega_6 \). Polygons are combination of several regions of triangles, or commonly called triangulation. The illustration of star triangulation for those example are on Figure 7.

![Figure 7](image-url)

**Figure 7.** Illustration for asymptotic Klein polygon containing Ideal and Ultraideal Klein points

Figure 7 on point (a) is an illustration for the star triangulation of Klein asymptotic polygon \( \Omega_1 \Omega_2 P_1 P_2 P_\ldots P_{n-1} P_n \).

\[
R = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_\ldots \cup T_n
\]

\[
R = \Delta P_n O \Omega_1 \cup \Delta \Omega_1 O \Omega_2 \cup \Delta \Omega_2 O P_1 \cup \Delta P_1 O P_2 \cup \Delta P_2 O P_\ldots \cup \Delta P_{n-1} O P_n
\]

Figure 7 on point (b) is an illustration for the star triangulation of Klein asymptotic polygon \( \Omega_1 \Omega_2 P_1 P_2 P_\ldots P_{n-1} P_n \).
$$R = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_n$$

$$R = \Delta P_n O \Omega_1 \cup \Delta \Omega_1 O P_1 \cup \Delta P_1 O \Omega_2 \cup \Delta \Omega_2 O P_2 \cup \Delta P_2 O \Omega_3 \ldots \Delta P_{n-1} O \Omega_n \cup \Delta \Omega_{n-1} O P_n$$

Figure 7 on point (c) is an illustration for the star triangulation of Klein asymptotic polygon $\Omega_1 \Omega_2 \Omega_3 \Omega_4 \Omega_5 \Omega_6$.

$$R = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6$$

$$R = \Delta \Omega_1 O \Omega_2 \cup \Delta \Omega_2 O \Omega_3 \cup \Delta \Omega_3 O \Omega_4 \cup \Delta \Omega_4 O \Omega_5 \cup \Delta \Omega_5 O \Omega_6 \cup \Delta \Omega_6 O \Omega_1$$

3. Conclusion

Based on the discussion, the following conclusions can be drawn. The plane model for Beltrami-Klein model is a $\gamma$ circle. The basic object of hyperbolic geometry using Beltrami-Klein model in the form of the Klein point is expressed as a dot, the Klein line is expressed as part of the Euclid line contained on circle $\gamma$, the distance and angle of Klein have the same representation as Euclid distance and angle. The triangle models on Klein has a different concept from the Euclid triangle. The sides of the Klein triangle can be either the Klein line segments, the Klein line rays, or the Klein lines. The Klein triangle is divided into two types, that are the ordinary Klein triangle and the asymptotic Klein triangle. The polygon model on Klein has a different concept from the concept of Euclid polygon. The sides of the Klein polygon can be either the Klein line segment, the Klein line rays, or the Klein line. The Klein polygon is divided into two types, that are ordinary Klein polygon that does not contain ideal Klein point or ultraideal Klein point as its vertex and the asymptotic Klein polygon that contains ideal Klein point, ultraideal Klein point, or both points as their vertex.

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