A MANIN-MUMFORD THEOREM FOR THE MAXIMAL COMPACT SUBGROUP OF A UNIVERSAL VECTORIAL EXTENSION OF A PRODUCT OF ELLIPTIC CURVES

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Abstract. We study the intersection of an algebraic variety with the maximal compact subgroup of a universal vectorial extension of a product of elliptic curves. For this intersection we show a Manin-Mumford type statement. This answers some questions posed by Corvaja-Masser-Zannier which arose in connection with their investigation of the intersection of a curve with real analytic subgroups of various algebraic groups. They prove finiteness in the situation of a single elliptic curve. Using Khovanskii’s zero-estimates combined with a stratification result of Gabrielov-Vorobjov and recent work of the authors we obtain effective bounds for this intersection that only depend on the degree of the algebraic variety, and the dimension of the group. This seems new even if restricted to the classical Manin-Mumford statement.

1. Introduction

The Manin-Mumford conjecture predicts that the intersection of a subvariety of a commutative algebraic group with the torsion points of the group is, in a precise sense, controlled by the group structure. This conjecture, in its original form, was proved by Raynaud [19], [20], and the general form was proved by Hindry [10]. There are now several proofs, see for instance [11], [18]. More recently, Corvaja, Masser and Zannier investigated the intersection of a subvariety with the euclidean closure of the torsion points in a commutative algebraic group over the complex numbers. This euclidean closure forms the maximal compact subgroup of the algebraic group. They focused in particular on the case of additive extensions of elliptic curves.

For an abelian variety $\mathcal{A}$ over $\mathbb{C}$ of dimension $g$, there is an extension $\mathcal{U}$ of $\mathcal{A}$ by the vector group $\mathbb{G}_a^g$ such that every other extension of $\mathcal{A}$ by a vector group is a pushout of $\mathcal{U}$ [5, Prop. 2.3]. This extension is unique up to isomorphism and called the universal vectorial (or additive) extension of $\mathcal{A}$. For each abelian subvariety $\mathcal{A}'$ of $\mathcal{A}$ there exists a universal vectorial extension $\mathcal{U}'$ of $\mathcal{A}'$ that is contained in $\mathcal{U}$. This $\mathcal{U}'$ is then unique. These groups $\mathcal{U}'$ together with their torsion

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translates are closed under intersections. We will focus on the case that our abelian variety is $E = E_1^{g_1} \times \cdots \times E_n^{g_n}$, a product of elliptic curves. Let $g = g_1 + \ldots + g_n$. We write $G$ for the universal vectorial extension of $E$ and write $C$ for the maximal compact subgroup of this extension. Throughout this article all algebraic varieties are defined over the complex numbers. Our main result is an extension of a result due to Corvaja, Masser and Zannier.

**Theorem 1.1.** Let $g$, $G$ and $C$ be as above. Further let $V \subset G$ be an algebraic subvariety of dimension at most $g$. Then there exists a natural number $N$ such that

$$V \cap C \subset \bigcup_{i=1}^{N} (t_i + H_i)$$

where the $H_i$ are universal vectorial extensions of abelian subvarieties of $E$ contained in $G$ and $t_i \in C$.

Consider the case $g = 1$. Then we have $G$ the universal vectorial extension of an elliptic curve $E$, and $V \subseteq G$ a curve. Then the theorem says that there are at most finitely many points on $V$ lying in the maximal compact subgroup of $G$. This is the result of Corvaja, Masser and Zannier [6, Theorem 1] mentioned above. Their paper was the main motivation for our work.

We could also ask similar questions about other groups, for instance $G_m^2$. In fact Corvaja, Masser and Zannier already pointed out that there are examples of curves in $G_m^2$ which have infinite intersection with $S^1 \times S^1$ but which are not contained in a translate of an algebraic subgroup. In this connection it is perhaps worth noting that the maximal compact subgroup of $G_m^2$ is semi-algebraic, whereas in the case of our $G$ above, the maximal compact is rather transcendental.

The requirement in our theorem that $\dim V$ be at most $g$ appears at first sight undesirable. But this condition is necessary. This can be seen as follows. Let $\pi$ be the projection of $G$ to $E$ and let $V^\pi$ be an irreducible curve in $E$. The inverse image $V = \pi^{-1}(V^\pi)$ is an algebraic variety of dimension $g + 1$ and for each point $p$ of $V^\pi$ we can find a point in $C$ that projects down to $p$. Thus the intersection of $V$ with $C$ is infinite and if $V^\pi$ does not lie in a translate of an abelian subvariety the intersection $V \cap C$ is also not contained in a finite union of translates of universal vectorial extensions strictly contained in $G$.

To state our second result we need to fix a model for $G$. For each elliptic curve $E$ over the complexes, we fix a Weierstrass model with invariants $g_2$ and $g_3$. There is a corresponding model of the universal extension $G$ of $E$, described on page 245 of [6]. It is given by $G = \overline{G} \setminus L$, with $\overline{G}$ the projective surface defined by

$$X_0X_2^2 = 4X_1^3 - g_2X_0^2X_1 - g_3X_0^3, \quad X_0X_4 - X_2X_3 = 2X_1^2,$$
and \( L \) the line defined by \( X_0 = X_1 = X_2 = 0 \). Using this embedding we can embed \( G \) in multiprojective space \( \mathbb{P}^g \). This embedding comes with a notion of degree. We use a rather simple minded definition. For an algebraic variety \( V \) in \( G \) we define its degree of definition to be the minimal number \( \delta \) such that \( V \) is defined by multiprojective polynomials of degree at most \( \delta \). Corvaja, Masser and Zannier showed \([6, \text{ theorem } 6]\) that, with \( E, G \) as above and \( V \) a curve in \( G \), the bound on the cardinality of the intersection of \( V \) can be taken to depend only on \( G \) and the degree of \( V \). They also asked if this result could be made effective. Here, using our recent work on pfaffian definitions of elliptic functions, we answer this positively and also show that the bound can be taken independently of \( G \). In fact, we obtain an effective uniform version of Theorem 1.1.

**Theorem 1.2.** Let \( V \) be as in Theorem 1.1 with degree \( \delta \). Then there exist effectively computable constants \( c_1, c_2 \) depending only on \( g \) such that for \( N \) in Theorem 1.1 holds,

\[
N \leq c_1 \delta^{c_2}.
\]

If a translate \( t_i + H_i \) in Theorem 1.1 contains a torsion point then we can choose \( t_i \) to be a torsion point. So Theorem 1.2 provides an entirely uniform and effective bound for Manin-Mumford in that setting. This is somewhat similar to the results obtained by Hrushovski and Pillay \([12]\), which were strengthened by Binyamini \([3]\). They obtain explicit uniform bounds when intersecting subvarieties of semiabelian varieties (all over the algebraic numbers) with finite rank subgroups. Their bound only applies to transcendental points. Our bound applies to all points, including those over \( \overline{\mathbb{Q}} \), but we consider additive extensions of elliptic curves, rather than semiabelian varieties, and have no analogue of finite rank subgroups.

If we restrict our attention to isolated points in the intersection \( V \cap C \) we can avoid the use of the stratification theorem of Gabrielov-Vorobjov and obtain an explicit bound.

**Theorem 1.3.** Let \( V, G \) be as in Theorem 1.1. The number \( N_{iso} \) of isolated points in \( V \cap C \) is bounded by

\[
N_{iso} \leq 2^{42g^2+126g} g^{30g} \max\{3, \delta\}^{21g}.
\]

We will later show (see Theorem 6.1) that for \( V \) an irreducible surface \( N_{iso} \) is also a bound for \( N \) in Theorem 1.1. And for \( V \) a curve all points in \( V \cap C \) are isolated by \([6, \text{ Theorem } 1]\). This has some concrete consequences for families of polynomial Pell equations. If we consider \( D = X^3Q \) where \( Q \in \mathbb{C}(T)[X] \) is a polynomial of degree 3 with coefficients in the function field of a complex curve \( T \) then we can think of the equation...
\[
A^2 - D_t B^2 = 1; \quad A, B \in \mathbb{C}[X], B \neq 0, t \in T(\mathbb{C}), \quad (*)
\]

where \(D_t\) is the specialization of \(D\) at (a suitable) \(t\), as a family of Pellian equations parametrized by \(T\). Families of this kind were studied extensively by Masser and Zannier (see for example [17]) and they introduced a method to study the qualitative behaviour of such families of equations. Now, if the family of elliptic curves defined by \(Y^2 = Q\) is isotrivial and \((*)\) does not have a generic solution then Theorem 1.3 provides an explicit bound, depending only on the degree of \(T\), for the number of \(t \in T(\mathbb{C})\) such that \((*)\) has a solution (see [21, Theorem 3]).

We expect our method to extend to certain real subgroups of universal vectorial extensions of abelian varieties. This will be carried out in later work. However the effectivity is then not so clear. We also expect the method to work in other situation such as intersections of algebraic varieties with certain real subtori of abelian varieties.

We remark that for algebraic \(g_2, g_3\) the only algebraic points on \(C\) are torsion points. This was shown by Bertrand for real points [1, Théorème 3.1] and by Bost, Künnemann [4, Theorem 3.1.2] in general.

This is how the rest of this article is organized. In the next section we prepare the setting for the proof of Theorem 1.1 and reduce it to a proposition. We also recall an Ax-Schanuel statement for universal vectorial extension due to Bertrand. In section 3 we recall some basic facts about endomorphisms and isogenies between elliptic curves and their universal extensions. Then in section 4 we prove the Lemmas necessary to apply the Ax-Schanuel statement from section 2. In section 5 we carry out the proof of Theorem 1.1 with the results from section 3 and 4. Finally in section 6 we show how to derive the explicit bound for \(N_{iso}\) in Theorem 1.1 from [13] and Khovanskii’s zero-estimates [9, Corollary 3.3] and the effective bound for \(N\) using also the stratification theorem of Gabrielov-Vorobjov.

2. Setting

In this section we prepare the setting for the proof of Theorem 1.1. We will use several properties of the exponential map. For the model given by (1) we can take

\[
\exp_G : \mathbb{C}^2 \rightarrow G
\]

given by

\[
(z, w) \rightarrow (\wp(z), \wp'(z), \zeta(z) + w, \wp'(z)(\zeta(z) + w) + 2\wp(z)^2)
\]
for $z$ not in the lattice of $\wp$, on the affine chart $\hat{G}$ defined by $X_0 \neq 0$ (see [6, p.251 (3.11)]). Here $\wp$ and $\zeta$ are the standard Weierstrass functions associated to the lattice given by $g_2$ and $g_3$. This map $\exp_G$ has kernel equal to $P\mathbb{Z}^2$ were

$$P = \begin{pmatrix} \omega_1 & \omega_2 \\ -\eta_1 & -\eta_2 \end{pmatrix}$$

is a period matrix of $G$. The pair $(\omega_1, \omega_2)$ is a basis for the lattice of $E$ while $\eta_1$ and $\eta_2$ are the associated quasi-periods as defined in [15, p.6].

We note that

$$\exp_G(P\mathbb{R}^2) = C$$

and we see that $C$ is a real analytic manifold of dimension 2. In contrast to the situation in $G_m^2$, it can be shown, using a theorem of Ax mentioned below, that $C$ is not semialgebraic.

The statement of the theorem is isogeny invariant, so we can assume that no pair of $E_i, E_j$ with $i \neq j$ are isogenous and that $\mathcal{G} = G_1^{q_1} \times \cdots \times G_n^{q_n}$ with $G_i$ the universal extension of $E_i$ for $i = 1, \ldots, n$ given by the model [11]. Here we used the fact that the product of two universal extensions is the universal extension of the product of the corresponding abelian varieties. This follows from [5, Prop 2.3]. We only have to note that the product of two anti-affine varieties is again anti-affine where a variety is anti-affine if the only regular functions of that variety are the constant functions.

We denote by $\omega_{1k}, \omega_{2k}, \eta_{1k}, \eta_{2k}$ the entries of a (from this point on fixed) period matrix as in (3) of the $k$-th universal vectorial extension in the product $\mathcal{G}$, for $k = 1, \ldots, g$. The choice of the particular model is not important for the proof of Theorem [11.3] but is necessary to formulate Theorem [11.2].

By compactness the intersection $V \cap C$ has only finitely many connected components. We will show that each such component is contained in a finite union of translates of universal vectorial extensions that is strictly contained in $\mathcal{G}$.

For points this is trivial. So it is enough to consider a component of $V \cap C$ of dimension $d \geq 1$. We pick a smooth point of this component and $U_0 \subset \mathbb{C}^{2g}$ such that $\exp(U_0) = U$ is an open neighbourhood of our smooth point in $V \cap C$. We can assume that $U_0$ is parametrized by real analytic functions

$$(z_1, w_1, \ldots, z_g, w_g) : B_0 \to \mathbb{C}^{2g}$$
where $B_0$ is an open ball in $\mathbb{R}^d$. We define $p_1, q_1, \ldots, p_g, q_g$ by

$$
(z_k, w_k) = p_k(\omega_{1k}, -\eta_{1k}) + q_k(\omega_{2k}, -\eta_{2k}), \quad k = 1, \ldots, g
$$

These functions are real-analytic on $B_0$. And they take real values, as $U$ is contained in $\mathcal{C}$. We call these functions the Betti coordinates of $U_0$. For a study of similar Betti maps in a different context, and some discussion of the terminology and of related concepts, see [7].

We write $\bar{\cdot}$ for complex conjugation. And we extend this coordinate-wise, without changing notation, to subsets of complex affine and projective spaces. As our set $U$ is contained in $V \cap \mathcal{C}$ we have $\overline{U} \subset \overline{V} \cap \overline{\mathcal{C}}$. Complex conjugation is a continuous isomorphism between the (real) Lie groups $\mathcal{G}$ and $\overline{\mathcal{G}}$ and so $\overline{\mathcal{C}}$ is the maximal compact subgroup of the group $\overline{\mathcal{G}}$. For the exponential maps we have

$$
\exp_{\overline{\mathcal{G}}}(z, w) = \exp_{\overline{\mathcal{G}}}(\overline{z}, \overline{w}).
$$

Of course $\overline{U_0}$ is parametrized by $\overline{z}_1, \overline{w}_1, \ldots, \overline{z}_g, \overline{w}_g$ with

$$
(\overline{z}_k, \overline{w}_k) = p_k(\overline{\omega_{1k}}, -\overline{\eta_{1k}}) + q_k(\overline{\omega_{2k}}, -\overline{\eta_{2k}}), \quad k = 1, \ldots, g;
$$
as on $B_0$, the functions $p_1, q_1, \ldots, p_g, q_g$ are real. This reality property of maximal compact subgroups will be exploited in our proof.

We pause to note that in what follows we will always work with the fibred product $U_0 \times_{\text{conj}} \overline{U_0} \times U \times_{\text{conj}} \overline{U} \subset U_0 \times \overline{U_0} \times \exp_{\overline{\mathcal{G}}} \times_{\overline{\mathcal{G}}}(U_0 \times \overline{U_0})$ where $\text{conj}$ is the complex conjugation map. This will be clear throughout as we work with a fixed parametrization (namely by the real ball $B_0$ above) of our sets.

Now suppose that $U$ is contained in a universal vectorial extension strictly contained in $\mathcal{G}$. If $M$ is a connected analytic manifold containing $U$ and of the same dimension as $U$, then $M$ too is contained in the same universal vectorial extension, by analytic continuation. By analytic cell-decomposition in the structure $\mathbb{R}_{an}$, the intersection $V \cap \mathcal{C}$ is a finite union of analytic cells. So, in order to establish Theorem 1.1, it is sufficient to show the following.

**Proposition 2.1.** The set $U$ is contained in a universal vectorial extension strictly contained in $\mathcal{G}$.

We will need the following Ax-Schanuel statement of Bertrand [2, Proposition 1.b] which he attributes to Ax.

**Theorem 2.1.** (Ax, Bertrand) Let $\mathcal{U}$ be the universal vectorial extension of an abelian variety of dimension $g$ defined over $\mathbb{C}$. Further let

$$
\exp : \mathbb{C}^{2g} \to \mathcal{U}
$$

be its exponential map. Let $x = (x_1, \ldots, x_{2g})$ be a $2g$-tuple of analytic functions (in several variables). We assume that some $x_i$ is not constant. Further let $y = \exp(x)$. Now let $U_y$ be the smallest lift of
a translate of an abelian subvariety of $E$ to a translate of a universal vectorial extension contained in $U$ that contains $y$. Then

$$\text{trdeg}_C(x, y) \geq \dim(U_y) + 1$$

where $\text{trdeg}_C$ denotes the transcendence degree over $\mathbb{C}$ and $\dim$ the dimension as an algebraic variety.

Bertrand’s version concerns more general fields but the above is enough for our purposes. He deduces the above statement from Ax’s theorem. The key point in his proof is that the only algebraic subgroup of $G^a_2 \times U$ that project surjectively to both factors is $G^a_2 \times U$ itself.

Now complexify the functions $p_1, q_1, \ldots, p_g, q_g$. We apply Theorem 2.1 with $U = \mathcal{G} \times \overline{\mathcal{G}}$, and

$$x = (z_1, w_1, \ldots, z_g, w_g, \tilde{z}_1, \tilde{w}_1, \ldots, \tilde{z}_g, \tilde{w}_g), \quad y = \exp_{\mathcal{G} \times \overline{\mathcal{G}}}(x).$$

Note that by (1) and (5) we have

$$\text{trdeg}_C(x, y) \leq 2g + 2 \dim(V).$$

The dimension of a translate $G_y$ containing $y$ is always even so, with our assumption $\dim(V) \leq g$, we deduce from the theorem of Bertrand that

$$\dim(G_y) \leq 4g - 2.$$ (7)

It follows that $U \times_{\text{conj}} \overline{U}$ is contained in a translate of a universal vectorial extension strictly contained in $\mathcal{G} \times \overline{\mathcal{G}}$. By translating $V$ by a point in $\mathcal{C}$ we can assume that $U$ contains the origin. So in what follows $U \times_{\text{conj}} \overline{U}$ is contained in a universal vectorial extension $H$ in $\mathcal{G} \times \overline{\mathcal{G}}$ of positive codimension and we assume that $H$ is minimal with this property. We note here that if there is no elliptic curve in the product defining $E$ that is isogenous to the complex conjugate of another elliptic curve in the product, Theorem 1.1 for $G$ follows already from the considerations above. We will assume that $U$ is not contained in a universal extension in $\mathcal{G}$ of positive codimension. The same then holds for $\overline{U}$ in $\overline{\mathcal{G}}$.

To ease notation we will abbreviate universal vectorial extension by universal extension and a universal vectorial extension that is strictly contained in another by universal subextension.

3. ISOCENICIES AND ENDMORPHISMS

In this section we recall some facts about universal vectorial extensions and elliptic curves.
We recall that the universal extension \( U \) of an abelian variety \( A \) of dimension \( g \) defined over \( \mathbb{C} \) is a vector extension of \( A \) by \( \mathbb{G}_m^g \) such that for each vectorial extension \( U' \) of \( A \) there exist unique maps \( \gamma_1, \gamma_2 \) that make the following diagram commutative

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_m^g & \overset{\phi}{\longrightarrow} & U & \overset{\pi}{\longrightarrow} & A & \longrightarrow & 0 \\
& & \downarrow{\gamma_1} & & \downarrow{\gamma_2} & & \downarrow{id} & & \\
0 & \longrightarrow & V & \longrightarrow & U' & \longrightarrow & A & \longrightarrow & 0
\end{array}
\]

This universal property characterizes \( U \) up to isomorphism [5, Prop 2.1].

If we have a diagram as above but where the identity map \( id \) is replaced with an isogeny \( \varphi \) (and \( A \) by an isogenous abelian variety) we refer to the map between \( U \) and \( U' \) as a lift of \( \varphi \).

Now let \( A \) and \( A' \) be abelian varieties with universal extensions \( U \) and \( U' \) respectively. Each isogney between \( A \) and \( A' \) lifts to an isogeny between \( U \) and \( U' \). For a proof of this, see for instance [21, p.1345]. In fact, this lift is unique. We have now been able to find a reference for this, so we give the proof here. Let \( \tilde{\varphi} \) be a lift of an isogeny \( \varphi \) between \( A \) and \( A' \). By quotienting out by the kernel \( K \) of \( \varphi \) and the kernel of \( \tilde{\varphi} \) we obtain an isomorphism between \( A/K \) and \( A' \) as well as a lift of this isomorphism to a map between \( U/K \) and \( U' \). Now assume that for two isogenies \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) we have \( \tilde{\varphi}_1 = \tilde{\varphi}_2 \mod \ker \tilde{\varphi} \). Then \( \tilde{\varphi}_1 - \tilde{\varphi}_2 \) is a continuous homomorphism whose image lies in a finite group. As \( U \) is connected it follows that this map sends everything to the identity element, so that \( \tilde{\varphi}_1 = \tilde{\varphi}_2 \).

Thus it is sufficient to show that if \( i \) is an isomorphism between \( A \) and \( A' \) and we fix the map between the vector groups corresponding to \( U \) and \( U' \) to be the identity then there is a unique isomorphism \( \tilde{i} \) between \( U \) and \( U' \) that lifts \( i \). To see this, take any two isomorphisms \( \tilde{i}_1 \) and \( \tilde{i}_2 \) lifting \( i \). Then \( \tilde{i}_1^{-1} \circ \tilde{i}_2 \) is an automorphism of \( U \) that lifts the identity map on \( U \). It then follows from the universal property of \( U \) that \( \tilde{i}_1^{-1} \circ \tilde{i}_2 \) is the identity map, and thus the required uniqueness holds.

We now return to our \( G \), the universal extension of a product \( \mathcal{E} \) of elliptic curves. Suppose that \( A \) is a connected algebraic subgroup of \( \mathcal{E} \). By [14], the subgroup \( A \) is defined by the vanishing of a homomorphism \( \phi : \mathcal{E} \to \mathcal{E} \). With the fact proven above one can show that there exists a canonical lift \( \hat{\phi} : G \to \hat{G} \) such that the connected component of its kernel is an algebraic subgroup \( U \) of \( G \) that projects down to \( A \). Further it is clear that the ring of regular functions on \( U \) consists of the constant functions. This is because \( U \) is isogenous to a product of
universal extensions of elliptic curves. Thus $U$ is anti-affine and has twice the dimension of $A$ so we may deduce from [5, Prop. 2.3] that it is a universal extension of $A$.

Fix two isogenous elliptic curves $E$ and $\tilde{E}$ with universal extensions $G$ and $\tilde{G}$, and let $P$ and $\tilde{P}$ be their period matrices as in [5]. Let $(\alpha)$ be an isogeny from $\tilde{E}$ to $E$ given by multiplication by $\alpha \in \mathbb{C}^*$ on the tangent space. As mentioned above there exists a unique lift of $(\alpha)$ to an isogeny $[\alpha]$ between $\tilde{G}$ and $G$ and the map that sends $(\alpha)$ to $[\alpha]$ is an isomorphism between the isogenies from $\tilde{E}$ to $E$ and the isogenies from $\tilde{G}$ and $G$. The isogeny $[\alpha]$ acts on the tangent space by multiplication by a matrix $L(\alpha)$ from the left and on $P\mathbb{Z}^2$ by multiplication of an integer matrix from the right. We define a map $B$ from the isogenies from $\tilde{E}$ to $E$ to $GL_2(\mathbb{Q})$ as follows.

**Definition 3.1.** For each isogeny $(\alpha)$ from $\tilde{E}$ to $E$ we define $B(\alpha)$ by

$$L(\alpha)\tilde{P} = PB(\alpha).$$

where $P$ and $\tilde{P}$ are the period matrices as above.

If $E = \tilde{E}$ then we consider the endomorphisms of $E$ and define the map $A$ as follows.

**Definition 3.2.** For each endomorphism $(\alpha)$ of $E$ we set $A(\alpha)$ to be

$$L(\alpha)P = PA(\alpha)$$

where $P$ is the period matrix from above.

The relations displayed in Definition 3.1 and 3.2 are a key part of the proof of Theorem 1.1 as they allow us to switch the action of the isogeny respectively endomorphism from the left to the right. Note that for endomorphisms $(\alpha)$ with $\alpha \notin \mathbb{Z}$ the matrix $L(\alpha)$ still has entries in the field $\mathbb{Q}(\tau, g_2, g_3)$ where $\tau$ is quadratic imaginary. The diagonal entries are given by $\alpha$ and its complex conjugate while the lower left entry is an algebraic number closely related to non-holomorphic modular forms as studied in [16] or more recently in [22].

If $E$ has complex multiplication and we set $A(0) = 0$ this map $A$ induces an embedding of the field $\mathbb{Q}(\tau)$ into $GL_2^+(\mathbb{Q}) \cup \{0\}$. We choose $P$ such that

\begin{equation}
A_D = A(\sqrt{D}) = \begin{pmatrix} 0 & D \\ 1 & 0 \end{pmatrix}
\end{equation}

where $D$ is the discriminant of the field $\mathbb{Q}(\tau)$. This then fixes the embedding and we will always assume it is so given.
4. Transcendence degree

Recall that we assume that $G = G_1^{g_1} \times \cdots \times G_n^{g_n}$ where the factors are pairwise non-isogenous given by a Weierstrass model as described in the introduction.

For an elliptic curve $E$ with complex multiplication we set $K = \mathbb{Q}(\tau)$ and $R = A(K)$ with $A$ given as described at the end of the last section. If $E$ does not have complex multiplication we set $K = \mathbb{Q}(\sqrt{D})$ for some negative integer $D$ and extend the map $A$ that embeds $\mathbb{Q}$ canonically into $GL_2^+(\mathbb{Q})$ to $K$ by (8) and again set $R = A(K)$. The following lemma is central to the proof of Theorem 1.1.

Lemma 4.1. Let $M$ and $\tilde{M}$ be two $e \times r$ matrices with entries $m_{st}$ and $\tilde{m}_{st}$ in $K$, respectively, where $s = 1, \ldots, e$ and $t = 1, \ldots, r$. Suppose that $M$ has rank $r$. If $B$ be an integer $2 \times 2$ matrix with negative determinant then the $2e \times 2r$ integer matrix $\hat{M}$ defined block-wise by

$$\hat{M}_{st} = (A(m_{st}) + A(\tilde{m}_{st})B)$$

has rank at least $r$.

Proof. Before we begin the proof we point out that all the names of the indices in this proof are forgotten after the proof.

For the proof itself, we first note that the (abelian) group $R + RB$ is a direct sum of $R$ and $RB$. For suppose that $A_1, A_2 \in R$ and $A_1 + A_2B = 0$. Then $\det(A_1) = \det(A_2) \det(B)$. But $\det(B) < 0$ so that $\det(A_1) = \det(A_2) = 0$ and $A_1 = A_2 = 0$. By slight abuse of notation we will write $AB(\alpha) = A(\alpha)B$ and extend the maps $A$ and $AB$ to vectors by entry-wise application.

We write $M_t$ and $\tilde{M}_t$ for the $t$-th row of $M$ and $\tilde{M}$ respectively and write

$$\hat{M}^{(t)} = A(M_t) + AB(\tilde{M}_t).$$

We note that the $\hat{M}^{(t)}$ span a vector space over $K$ via the maps $A$ and $AB$. To begin, we show that this space has dimension $r$. If not then there would exist $\alpha_1, \ldots, \alpha_r \in K$ not all zero such that $A_1 = A(\alpha_1), \ldots, A_r = A(\alpha_r)$ satisfy

$$\sum_t A_t \hat{M}^{(t)} = \sum_t A(\alpha_t M_t) + \sum_t AB(\alpha_t \tilde{M}_t) = 0.$$

From our comments above we deduce that

$$\sum_t \alpha_t M_t = 0,$$

contradicting our assumption on $M$. 

Now let \( \omega_t \) and \( \epsilon_t \) be the odd and even rows of \( \hat{M} \), respectively. Here \( t = 1, \ldots, r \). And let \( V_\omega \) and \( V_\epsilon \) be the vector spaces over \( \mathbb{Q} \) generated by the \( \omega_t \) and the \( \epsilon_t \). We set \( d_\epsilon = \dim(V_\epsilon), d_\omega = \dim(V_\omega) \) and \( \delta = \dim(V_\epsilon \cap V_\omega) \). As the dimension of the vector space spanned by the rows of \( \hat{M} \) is equal to \( d_\epsilon + d_\omega - \delta \), in order to prove the lemma, it is enough to prove that

\[
d_\epsilon + d_\omega \geq r + \delta,
\]

which we now do. There exist linearly independent integer vectors \( a_i = (a_{i1}, \ldots, a_{ir}), i = 1, \ldots, r - d_\omega \) and \( b_j = (b_{j1}, \ldots, b_{j\ell}), j = 1, \ldots, r - d_\epsilon \) such that

\[
\sum_{t=1}^{r} a_{it} \omega_t = \sum_{t=1}^{r} b_{jt} \epsilon_t = 0
\]

for all \( i \) and \( j \).

Then

\[
\sum_{t=1}^{r} a_{it} \hat{M}(t) = \begin{pmatrix} 0 \\ v_i \end{pmatrix}, \quad A_D \sum_{t=1}^{r} b_{jt} \hat{M}(t) = \begin{pmatrix} 0 \\ w_j \end{pmatrix}
\]

for some integer vectors \( v_i \), for \( i = 1, \ldots, r - d_\omega \) and \( w_j \) for \( j = 1, \ldots, d_\epsilon \). Since \( \hat{M}(1), \ldots, \hat{M}(r) \) are linearly independent over \( K \) the integer vectors \( v_i, w_j \) are also linearly independent. Further there are linearly independent vectors \( c_k = (c_{k1}, \ldots, c_{kr}), \) for \( k = 1, \ldots, \delta \) and \( d_k = (d_{k1}, \ldots, d_{kr}), k = 1, \ldots, \delta \) such that

\[
\sum_{t} c_{kt} \omega_t + D \sum_{t} d_{kt} \epsilon_t = 0
\]

and such that both \( \sum_{t} c_{kt} \omega_t \) and \( \sum_{t} d_{kt} \epsilon_t \) are linearly independent. Thus we get

\[
\sum_{t} (c_{kt} + d_{kt} A_D) \hat{M}(t) = \begin{pmatrix} 0 \\ x_k \end{pmatrix}, \quad k = 1, \ldots, \delta.
\]

Now if \( d_\epsilon + d_\omega < r + \delta \) then there exists a linear relation between \( v_i, w_j, x_k \). As these have entries in \( \mathbb{Z} \) the linear relation will also be defined over \( \mathbb{Z} \). From the fact that the vectors \( \hat{M}(t) \) are linearly independent over \( K \) we deduce that there exist integers \( \lambda_i, \mu_j, \kappa_k \) not all zero such that

\[
\sum_{i} \lambda_i a_i = \sum_{k} \kappa_k c_k, \quad \sum_{j} \mu_j b_j = \sum_{k} \kappa_k d_k.
\]

It follows that

\[
\sum_{i} \lambda_i \sum_{t} a_{it} \omega_t = \sum_{k} \kappa_k \sum_{t} c_{kt} \omega_t
\]
and as \( \sum_i a_i \omega_i = 0 \) for all \( i \) it follows that \( \kappa_k = 0 \) for all \( k \). Thus
\[
\sum_i \lambda_i a_i = \sum_j \mu_j b_j = 0
\]
and so \( \lambda_i = \mu_j = 0 \) for all \( i \) and \( j \). This is a contradiction and we obtain the inequality (9).

We will use Lemma 4.1 to lower the transcendence degree of \( x \). We first consider the projections \( \hat{H} \) of \( H \) to products \( G_i \times \overline{G_i} \) such that \( G_i \) and \( \overline{G_i} \) are isogenous and \( \hat{H} \) is a universal subextension of \( G_i \times \overline{G_i} \). In order to ease notation we set \( E_i = E \), \( G_i = G \) and \( g_i = e \). We further denote by \( P \) the period matrix of \( E \). By abuse of notation we will renumber \( p_1, q_1, \ldots, p_e, q_e \) to be the Betti coordinates of the projection of \( U_0 \) to the tangent space of \( G_i \).

Lemma 4.2. Let \( \hat{x} = (p_1, q_1, \ldots, p_e, q_e) \) and \( \hat{H} \) be as described above. Then
\[
\text{trdeg}_C(\hat{x}) \leq \frac{1}{2} \dim(\hat{H}).
\]

Proof. Let \( r = 2e - \frac{1}{2} \dim(\hat{H}) \) and let \( (Q_1, \ldots, Q_e, \tilde{Q}_1, \ldots, \tilde{Q}_e) \) be the coordinate functions of the projection of \( U \times_{\text{conj}} U \) to \( G^e \times \overline{G}^e \). As \( U \) is not contained in a universal subextension of \( G \), the group \( \hat{H} \) is defined by relations of the following form
\[
\sum_{s=1}^e \left[ m_{st} \right] Q_s = -\sum_{s=1}^e \left[ \tilde{m}_{st} \right] (\left[ \alpha \right] \tilde{Q}_s), \quad t = 1, \ldots, r, \quad m_{st}, \tilde{m}_{st} \in K
\]
where \( [\alpha] \) is an isogeny from \( \overline{G} \) to \( G \) and \( M = (m_{st}) \) and \( \tilde{M} = (\tilde{m}_{st}) \) are matrices of rank \( r \) with entries in \( K \). This is because if the rank of either \( M \) or \( \tilde{M} \) is less than \( r \) we find in both cases that the projection of \( H \) to \( G^e \) is contained in a strict subgroup of \( G^e \). This in turn contradicts our assumption on \( H \). The isogeny \( [\alpha] \) descends to an isogeny \( (\alpha) \) between \( E \) and \( \overline{E} \) and it acts by
\[
\alpha(\omega_1, \omega_2) = (\omega_1, \omega_2)B
\]
for an integer matrix \( B \) with negative determinant. So \( B = B(\alpha) \) as in Definition 3.1 where \( \overline{P} \) is replaced by \( \overline{P} \). So we have
\[
L(\alpha)\overline{P} = PB.
\]

By taking the logarithm, one can check that on \( U \times_{\text{conj}} U \) the relation (10) translates to a relation on \( U_0 \times_{\text{conj}} U_0 \) of the following form
\[
\sum_{s=1}^e \left( L(m_{st})P \begin{pmatrix} p_s \\ q_s \end{pmatrix} + L(\tilde{m}_{st})L(\alpha)\overline{P} \begin{pmatrix} p_s \\ q_s \end{pmatrix} \right) = 0 \pmod{\mathbb{C}^2}.
\]
Recalling Definition 3.2 and using (11) we finally end up with
\[ \sum_{s=1}^{e} (A(m_{st}) + A(\tilde{m}_{st})B) \left( \begin{array}{c} p_s \\ q_s \end{array} \right) = 0 \mod \mathbb{C}^2 \]
for \( t = 1, \ldots, r \). By Lemma 4.1 this implies that \( \text{trdeg}_{\mathbb{C}}(p_1, q_1, \ldots, p_e, q_e) \leq 2e - r \).

Before we begin the proof of Theorem 1.1 we need another Lemma, similar to Lemma 4.1. We have not found a reference for it so we supply a proof here.

**Lemma 4.3.** Let \( M \) be a matrix of rank \( r \) with entries \( m_{st} \in K, s = 1, \ldots, e, t = 1, \ldots, r \). Then the \( 2e \times 2r \) integer matrix \( \hat{M} \) defined block-wise by
\[
\hat{M} = (A(m_{st}))_{st}
\]
has rank \( 2r \).

**Proof.** We can write \( A(m_{st}) = a_{st} \text{Id} + b_{st}A_D \) for rational numbers \( a_{st}, b_{st} \), where \( \text{Id} \) is the identity matrix. We denote by \( a_t, b_t \) the \( t \)-th row of the matrices \( (a_{st}), (b_{st}) \) respectively. Now we can check that if the rows of \( \hat{M} \) are linearly dependent we can find integers \( \lambda_t, \mu_t \) not all zero such that
\[
\sum_t \lambda_t a_t + \sum_t \mu_t b_t = 0.
\]
But then
\[
\sum_t (\sqrt{D} \lambda_t + \mu_t)(a_t + \sqrt{D} b_t) = D \sum_t \lambda_t b_t + \sum_t \mu_t a_t = 0
\]
which contradicts our assumption on \( M \). \( \square \)

We need one further Lemma to treat the projections of \( H \) to \( \hat{H} \) in products \( G_i^{g_i} \times G_j^{\tilde{g}_j}, i \neq j \) for which \( G_i \) and \( G_j \) are isogenous. We set \( e = g_i, \tilde{e} = g_j \) and \( G_i = G_i, G_j = \tilde{G}_j \). We further denote by \((p_1, q_1, \ldots, p_e, q_e)\) respectively \((\tilde{p}_1, \tilde{q}_1, \ldots, \tilde{p}_\tilde{e}, \tilde{q}_\tilde{e})\) the Betti coordinates of the projection of \( U \) to the tangent space of \( G_i^{g_i} \) respectively that of \( G_j^{\tilde{g}_j} \).

**Lemma 4.4.** Let \( \hat{x} = (p_1, q_1, \ldots, p_e, q_e, \tilde{p}_1, \tilde{q}_1, \ldots, \tilde{p}_\tilde{e}, \tilde{q}_\tilde{e}) \) and \( \hat{H} \) be as above. Then
\[
\text{trdeg}_{\mathbb{C}}(\hat{x}) \leq \dim(\hat{H}).
\]

**Proof.** Similarly as for Lemma 4.2 we set \( r = e + \tilde{e} - \frac{1}{2} \dim(\hat{H}) \). Let \((Q_1, \ldots, Q_e, \tilde{Q}_1, \ldots, \tilde{Q}_{\tilde{e}})\) be the coordinate functions of the projection of \( U \times_{\text{conj}} \bigcup \) to \( G_j^{g_j} \times G_i^{\tilde{g}_j} \). They satisfy relations of the following form
\[
(12) \sum_{s=1}^{e} [m_{st}]Q_s = - \sum_{\tilde{s}=1}^{\tilde{e}} [\tilde{m}_{\tilde{st}}][\tilde{Q}_{\tilde{s}}], \ t = 1, \ldots, r, \ m_{st}, \tilde{m}_{\tilde{st}} \in K.
\]
where \([\alpha]\) is an isogeny from \(\tilde{G}\) to \(G\). We note as in the proof of Lemma 4.2 that both matrices \((m_{st})\) and \((\tilde{m}_{st})\) have rank \(r\) because of our assumption on \(H\). As in Lemma 4.2 this translates to a relation on \(U_0 \times \tilde{U}_0\) of the following form
\[
\sum_{s=1}^{e} A(m_{st}) \begin{pmatrix} p_s \\ \tilde{q}_s \end{pmatrix} + \sum_{\tilde{s}=1}^{\tilde{e}} A(\tilde{m}_{\tilde{st}})B \begin{pmatrix} \tilde{p}_\tilde{s} \\ \tilde{q}_\tilde{s} \end{pmatrix} = 0 \mod \mathbb{C}^2
\]
for some integer matrix \(B\). As \((m_{st})\) has rank \(r\) it follows from Lemma 4.3 that \(\text{trdeg}_\mathbb{C}(\hat{x}) \leq 2(e + \tilde{e}) - 2r\).
\[\square\]

5. Proof of Theorem 1.1

Proof. We will prove Theorem 1.1 by induction on \(g + \dim(V)\). If \(\dim(V) = 0\), Theorem 1.1 is trivial. So Theorem 1.1 holds for \(g + \dim(V) = 1\).

For our induction argument we may assume that \(\dim(V) \geq 1\) and that \(U, U_0, x, y\) and \(H\) are as in section 2. Recall that we assume that \(U\) is not contained in a universal subextension. We show that this leads to a contradiction. For each elliptic curve \(E\) there is up to isogeny exactly one elliptic curve \(E'\) such that \(E\) is isogenous to \(E'\). So we may permute the set \(\{1, \ldots, n\}\) such that for odd \(i \leq 2m\), the curve \(E_i\) is isogenous to \(E_{i+1}\) while for \(2m < i \leq 2m + m'\) the curve \(E_i\) is isogenous to \(E_i\) and the rest are not isogenous to any complex conjugate of any of \(E_1, \ldots, E_n\).

We then write
\[
G \times \overline{G} = G_1 \times \cdots \times G_{2m+m'} \times R \tag{13}
\]
with \(G_i = G_i^{g_i} \times \overline{G}_j^{g_j}\) where we set \(j = i+1\) for odd \(i \leq 2m\) and \(j = i-1\) for even \(i \leq 2m\) and finally for \(2m < i \leq 2m + m'\) we set \(j = i\). We denote by \(\pi_G\) the projection to a subgroup \(G\) appearing as a factor in the product (13) of \(G \times \overline{G}\) and we use the same notation to mean the projection to the corresponding tangent spaces.

We first investigate the transcendence degree of \(x\) over \(\mathbb{C}\). Note that \(H = \tilde{H} \times R\). This is the case since the projection of \(R\) to \(E\) is a product of elliptic curves such that no elliptic curve in it is isogenous to the complex conjugate of any other in \(E\). So if the projection of \(H\) to \(R\) were strictly contained in \(R\) then the projection of \(H\) to \(G\) would be too, contradicting our assumption. It follows that
\[
\dim(H) = \sum_{i=1}^{2m+m'} \dim(\pi_G(H)) + \dim(R). \tag{14}
\]
Applying Lemma 4.4, we see that
\[ \text{trdeg}_C(\pi_G \cdot (x)) \leq \dim(\pi_G(H)). \]
for \( i \leq 2m \). But we also note that for these \( i \) the conjugate \( \overline{G}_i \) also turns up in the product \( G_1 \times \cdots \times G_{2m} \) and that
\[ \text{trdeg}_C(\pi_{G_i} \times \overline{G}_i(x)) \leq \frac{1}{2} \dim(\pi_{G_i} \times \overline{G}_i(H)). \]
And by Lemma 4.2 we have
\[ \text{trdeg}_C(\pi_{G_i}(x)) \leq \frac{1}{2} \dim(\pi_{G_i}(H)) \]
for \( 2m < i \leq 2m + m' \). The conjugate of each factor of \( R \) also appears in \( R \) and so
\[ \text{trdeg}_C(\pi_R(x)) \leq \frac{1}{2} \dim(R). \]
Thus we deduce that
\[ \text{trdeg}_C(x) \leq \frac{1}{2} \dim(H). \]
If \( \dim(H) < 2g \) then \( \dim(\pi_G(H)) < 2g \) and so \( U \) is contained in a universal subextension. So we assume now that \( \dim(H) \geq 2g \).

Let \( r' = 2g - \frac{1}{2} \dim(H) \). We denote by \( Q_1, \ldots, Q_g \) the coordinate functions of \( U \) and by \( \tilde{Q}_1, \ldots, \tilde{Q}_g \) the coordinate functions of \( \overline{U} \). There are \( r' \) sums appearing on the left of the relations (10), (12) defining \( H \). These sums give rise to a surjective group homomorphism
\[ \mathcal{L} : G \rightarrow G' \]
where \( G' \) is a universal extension of \( r' \) elliptic curves. Let \( V' \) be the Zariski-closure of \( \mathcal{L}(U) \) in \( G' \). We distinguish between two cases. We first assume that
\[ \dim(V') < r'. \]
Let \( C' \) be the maximal compact subgroup of \( G \). The map \( \mathcal{L} \) sends \( U \) to a component of \( C' \cap V' \). As \( \dim(H) \geq 2g \) it holds that \( r' \leq g \) so it follows by induction that \( V' \cap C' \) is contained in a finite union of universal extensions properly contained in \( G' \). It follows that \( U \) is contained in a proper universal sub-extension of \( G \) and we derive a contradiction. It remains to consider the case that
\[ \dim(V') \geq r'. \]
Clearly \( \mathbb{C}(\mathcal{L}) = \mathbb{C}(\mathcal{L}(Q_1, \ldots, Q_g)) \subset \mathbb{C}(Q_1, \ldots, Q_g) \cap \mathbb{C}(\tilde{Q}_1, \ldots, \tilde{Q}_g) \) and by (15), \( \mathbb{C}(\mathcal{L}) \) has transcendence degree at least \( r' \) over \( \mathbb{C} \) and both \( \mathbb{C}(Q_1, \ldots, Q_g) \) and \( \mathbb{C}(\tilde{Q}_1, \ldots, \tilde{Q}_g) \) have transcendence degree at most \( g \) over \( \mathbb{C} \). It follows that
\[ \text{trdeg}_\mathbb{C}(Q_1, \ldots, Q_g, \tilde{Q}_1, \ldots, \tilde{Q}_g) \leq 2g - r' = \frac{1}{2} \dim(H). \]
Hence
\[ \text{trdeg}_C(x, y) \leq \dim(H). \]

As we assumed that the dimension of \( U \) is at least 1 and that \( H \) is minimal we deduce a contradiction from the inequality (9) in Theorem 2.1. Thus \( U \) has to be contained in a proper universal extension of \( G \).

6. Effectivity and a refinement of Theorem 1.1

To get the explicit bound in Theorem 1.3, we will use our earlier work [13]. Here are the details.

Proof of Theorem 1.3. For a \( G \) with exponential as in (2) we fix periods \( \omega_1, \omega_2 \) in (3) such that \( \omega_2/\omega_1 \) lies in the standard fundamental domain in the upper half plane and let \( F^0 \) be the fundamental parallelogram spanned by \( \omega_1, \omega_2 \) but deprived of 0. We will work with the affine chart \( G^0 \) defined by \( X_0 \neq 0 \). The exponential of \( G \) restricted to \( F^0 \times \mathbb{C} \) maps onto \( G^0 \), as described in (2).

By the proof of [13, Theorem 11, page 17] the graph \( \text{graph}(\varphi, \zeta) \) of \( (\varphi, \zeta) \) where \( \varphi, \zeta \) are restricted to \( F^0 \) is a piecewise semi-pfaffian set of format \( (9, 9, 3, 12, 144503, 1) \) (for the definition see the beginning of the introduction of [13]). We define \( b_1, b_2 \) to be the Betti coordinates of \( z \) which are degree 1 polynomials in the real and imaginary part of \( z \). We identify \( \mathbb{C} \) and \( \mathbb{R}^2 \) (as we shall do throughout this proof). Then using the piecewise semi-pfaffian description of \( \varphi \) and \( \zeta \) mentioned above, we see that

\[
X_G = \left\{ (z, w, X_\varphi, Y_\varphi, X_\zeta, Y_\zeta) \in \mathbb{C}^6 : (z, X_\varphi, X_\zeta - w) \in \text{graph}(\varphi, \zeta), Y_\varphi^2 - 4X^3_\varphi - g_2X_\varphi - g_3 = 0, Y_\zeta - X_\zeta Y_\varphi + 2X^2_\varphi = 0, w - b_1 \eta_1 - b_2 \eta_2 = 0 \right\}
\]

is a piecewise sub-pfaffian set. And \( X_G \) has format \( (9, 9, 3, 12, 144503, 10) \) as we have added 6 additional equations that involve polynomials of degree at most 3 to the definition of \( \text{graph}(\varphi, \zeta) \).

We embed \( G = \prod_{k=1}^g G_k \) into multiprojective space \( \mathbb{P}_4^g \) by embedding each \( G_k \) into \( \mathbb{P}_4 \) as in (11). We denote by \( X_{0k}, X_{1k}, X_{2k}, X_{3k}, X_{4k} \) the projective coordinates of \( G_k \) and set \( G_k^0 \) to be the affine chart of \( G_k \) defined by \( X_{0k} \neq 0 \). The projection of each point in the intersection of \( V \cap C \) to \( G_k \) is either equal to the identity element or lies on the chart defined by \( X_0 \neq 0 \). Thus the intersection \( V \cap C \) lies in the union of the \( 2^g \) sets defined by these conditions. The intersection of \( V \) with such a set is given by, for each \( k \), either specializing \( X_{0k} = 1 \) or specializing \( X_{0k}, X_{1k}, X_{2k}, X_{3k}, X_{4k} \) to the identity element of \( G_k \) in the equations defining \( V \).
For each $k = 1, \ldots, g$ we form the set $X_{G_k}$ as in (16) with $G = G_k$. For each subset $S$ of $\{1, \ldots, g\}$ we form the cartesian product $X_S = \prod_{k \in S} X_{G_k}$. These products are piecewise semi-pfaffian sets of format $(9g', 9g', 3, 12g', 144503g', 10g')$ with $g' \leq g$. For each such $S$ we consider the real and imaginary part of the polynomials defining $V$ and set $X_0 k = 1$ for $k \in S$ and $X_0 k, X_1 k, X_2 k, X_3 k, X_4 k$ equal to the identity element otherwise. The resulting polynomials have total degree bounded by $\delta$. We add those equations to definition of $\prod_{k \in S} X_{G_k}$ and obtain a set $V_S$ that is contained in the union of $144503g$ zero sets of pfaffian functions of order $9g$ and degree $(9g, \max\{3, \delta\})$. Each isolated point of $V \cap \mathcal{C}$ lies in the projection of exactly one connected component of such a set $V_S$ to $\mathcal{G}$. The number $N_{\text{iso}}$ of isolated point in $V \cap \mathcal{C}$ is thus bounded by $2^g$ times a bound for the number of connected components of $V_S$. Using Khovanskii’s bounds [9, Corollary 3.3] we obtain

$$N_{\text{iso}} \leq 2^{42g^2 + 126g} g^{30g} \max\{3, \delta\}^{21g}.$$ 

Now we quickly show that Theorem 1.3 implies an explicit bound for $N$ if $V$ has dimension at most 2.

**Theorem 6.1.** For $V$ an irreducible surface and $g \geq 2$ the intersection $V \cap \mathcal{C}$ is finite or $V$ is equal to a translate of a universal subextension.

**Proof.** We can argue by contradiction. Suppose that $V$ is not a translate and that there is a positive dimensional component $C$ of $V \cap \mathcal{C}$. By Theorem 1.1 $C$ is contained in a finite union of translates of universal subextensions. If the dimension of the intersection of $V$ with a translate is smaller than 2 then the intersection is finite. So there has to be a translate $c + H$ for which the dimension is 2. As $V$ is irreducible, $V$ is contained in that translate. Thus $V - c$ is contained in $H$ and $C - c$ is a positive dimensional component of $V$ intersected with the maximal compact of $H$. We can now argue by induction on $g$. □

Finally we prove the effective version of Theorem 1.1. We will again use our earlier work but we will also need the stratification theorem of Gabrielov-Vorobjov. We assume that the reader is familiar with the definitions from [8].

**Proof of theorem 1.2.** Here and also below all constants depend only on $g$, and can be computed from $g$.

Let $V_S$ be as in the proof of Theorem 1.3. As we mentioned there it is a piecewise semi-pfaffian set and the entries of its format are bounded by $c_1$, with the exception of the $\beta$-entry, which is bounded by $c_1 \delta$.

We want to write the intersection $V \cap \mathcal{C}$ as a union of some number, $N'$, of connected analytic manifolds. It follows from Proposition 2.1 and the comments preceding it that this $N'$ will then be a bound for the $N$ in the statement. Because the projection from $X_S$ to the maximal
compact subgroup is an analytic isomorphism, it is enough to show that 
$V_S$ is the union of some number, $N''$, of connected analytic manifolds, 
and to give a bound on $N''$. (To get a bound for $N'$ we then have to 
multiply $N''$ by $2^g$.)

Since $V_S$ is a piecewise semi-pfaffian set, with the bounds on format 
mentioned above, it is a union of $c_1$ elementary semi-pfaffian sets, $X_\nu$ 
say. These elementary semi-pfaffian sets have format entries bounded 
by $c_1$, again expect the $\beta$-entry which is bounded by $c_1\delta$. It is enough 
to show that each of these elementary semi-pfaffian sets can be written 
as a union of a controlled number of connected analytic manifolds. 
This we will do using the Gabrielov-Vorobjov stratification theorem 
([8, Theorem 2, page 82]. We apply this theorem to each $X_\nu$ in turn. 
For each $\nu$ the theorem gives a stratification of $X_\nu$ into at most $c_2\delta^{c_3}$ 
smooth strata. Moreover, each of these strata has format bounded 
(entrywise) by $(c_6\delta^{c_7}, c_5, c_6, c_6\delta^{c_7})$. (This means that the number of 
equations and the degrees of the polynomials in the pfaffian functions 
are bounded by $c_6\delta^{c_7}$, everything else by $c_5$). To finish it is enough 
to bound the number of connected components of these strata. To do 
this, we add extra variables to remove the inequalities involved in the 
definitions of the strata. So, for each inequality of the form $g(x) > 0$ we 
add variables $y, z$ and equations $g(x) - y^2 = 0, yz = 1$. So we add $2 \cdot c_5$ 
variables, and $2 \cdot c_5$ equations, and write our stratum as the projection 
of a zero set of set of pfaffian functions in the higher-dimensional space 
given by the extra variables. By Khovanski’s theorem (see [9, Corollary 
3.3]), this pfaffian set will have at most $c_8\delta^{c_9}$ connected components. 
Combining the estimates gives our bound:

$$N \leq c_1 \cdot c_2 \cdot c_8 \cdot \delta^{c_3+c_9}.$$  

□

If the computation is carried out using the more explicit bounds given 
by Khovanski, and by Gabrielov and Vorobjov, then it can be shown that 
both the constants in the exponent of $\delta$ have the form $(cg)^{c'}$ for 
absolute effective constants $c$ and $c'$.

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