Quasi-streamwise vortices and enhanced dissipation for the incompressible 3D Navier-Stokes equations

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Abstract

We consider the 3D incompressible Navier-Stokes equations under the following 2+\(\frac{1}{2}\)-dimensional situation: vertical vortex blob (quasi-streamwise vortices) being stretched by two-dimensional shear flow. We prove enhanced dissipation induced by such quasi-streamwise vortices.

1 Introduction

One of the most important laws in the study of developed turbulence is Kolmogorov’s 4/5-law, and in the derivation of this law, a significant assumption is the zeroth law (see Frisch [1] for example). The zeroth law of turbulence states that, in the limit of vanishing viscosity, the rate of kinetic energy dissipation for solutions to the incompressible Navier-Stokes equations becomes nonzero. To formulate this law, let us recall the 3D incompressible Navier-Stokes equations on \(T^3 := (\mathbb{R}/\mathbb{Z})^3\):

\[
\begin{aligned}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla p &= \nu \Delta u^\nu + f, \\
\nabla \cdot u^\nu &= 0, \\
u(t=0) &= u^\nu_0,
\end{aligned}
\]

(1.1)

where \(\nu > 0\) is the viscosity and \(u^\nu : T^3 \to \mathbb{R}^3, p : T^3 \to \mathbb{R}\) denote the velocity and pressure of the fluid, respectively. Here \(f : T^3 \to \mathbb{R}^3\) is some external force. Assuming that the solution is sufficiently smooth, taking the dot product of the equation with \(u^\nu\) and integrating over the domain \(T^3\) gives the energy balance:

\[
\frac{d}{dt} \frac{1}{2} \|u^\nu(t)\|_{L^2}^2 = \int_{T^3} f(t) \cdot u^\nu(t) dx - \nu \|\nabla u^\nu(t)\|_{L^2}^2.
\]

(1.2)
The zeroth law then postulates that, under the normalization $\|u_0\|_{L^2} = 1$, the mean energy dissipation rate does not vanish as $\nu \to 0^+$:

$$\liminf_{\nu \to 0^+} \nu \langle |\nabla u_\nu|^2 \rangle > 0,$$

where $\langle \cdot \rangle$ usually denotes some ensemble or long-time, space averages. Laboratory experiments and numerical simulations of turbulence both confirm the above zeroth law [6]. Keeping this classical physical phenomena in mind, in this paper, we compare the Navier-Stokes flow and the heat flow with the same initial data, and show that vortex stretching is enhancing the viscosity dissipation. This could be a good example clarifying a mechanism of the actual zeroth law. Here, by the heat flow, we shall mean the unique $L^2$ solution of

$$\begin{aligned}
\partial_t u^{\nu,\text{heat}} &= \nu \Delta u^{\nu,\text{heat}}, \\
u^{\nu,\text{heat}}(t=0) &= u_0^\nu
\end{aligned} \tag{1.3}$$

in $(t, x) \in \mathbb{R}_+ \times T^3$.

We now state our main result.

**Theorem 1.1.** For any $\alpha \in (0, 3/4)$, there is a sequence of smooth initial data $\{u_\nu^n\}_{n \geq 1}$ on $T^3$ with $\nu_n = 2^{-2n}$ and $\|u_0^n\|_{L^2} = 1$ such that, for the solution $u^{\nu_n}$ to the 3D incompressible Navier-Stokes equations (1.1) with zero external force, we have

$$\liminf_{n \to \infty} \nu_n \int_0^{2n/3} \|\nabla u^{\nu_n}(t)\|^2_{L^2} dt \geq c \tag{1.4}$$

for some absolute constant $c > 0$, whereas for the solution to the heat equation (1.3), we have

$$\nu_n \int_0^{2n/3} \|\nabla u^{\nu_n,\text{heat}}(t)\|^2_{L^2} dt \leq C\nu_n^{2\alpha/3}(1 + \nu_n^{-2\max\{1/2, \alpha\}}) \tag{1.5}$$

with some absolute constant $C > 0$, so that the right hand side vanishes as $n \to \infty$.

Recalling the energy balance (1.2), (1.4) implies that at least a fixed portion of the initial $L^2$ energy for $u^{\nu_n}$ is lost in the $O(\nu_n^{2\alpha/3})$-timescale. Moreover, the comparison between (1.3) and (1.5) simply tells us that viscosity alone is not enough to obtain such an enhanced dissipation.

**Remark 1.2.** With a different construction, the same authors considered the zeroth law under the $2 + \frac{1}{2}$-dimensional Navier-Stokes flow: small-scale horizontal vortex blob being stretched by large-scale, anti-parallel pairs of vertical vortex tubes [3]. It is inspired by the direct numerical simulations of the 3D Navier-Stokes equations by Goto, Saito and Kawahara [2]. They have found that sustained turbulence in a periodic box consists of a hierarchy of antiparallel pairs of vortex tubes.
Remark 1.3. In this paper, we set a sequence of initial data tending to highly oscillating two-dimensional shear flow, adding oscillating vorticity to the vertical direction. Let us mention a recent numerical simulation which have inspired this construction of the initial data. Recently, using direct numerical simulations of the 3D Navier-Stokes equations, Motoori and Goto [5] considered the generation mechanism of turbulent boundary layer. They found that small-scale vortices in the log layer are generated predominantly by the stretching in a strain-rate field at larger scale rather than by the mean-flow stretching. On the other hand, large-scale vortices, namely, vortices as large as the height of the 99% boundary layer thickness are stretched and amplified directly by the mean flow (we call “quasi-streamwise vortices”). In particular large-scale vortices tend to align with the stretching direction of the mean flow, which is inclined at nearly $\pi/4$ from the streamwise direction. We are inspired by the generation mechanism of this quasi-streamwise vortices.

Notations. We write $A \lesssim B$ if there exists some constant $C > 0$ independent of $\nu$ such that $A \leq CB$. Then, we say $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Finally we write $A \simeq B$ if $A \approx B$ and $B \approx A$. We use the notation $\langle \cdot, \cdot \rangle$ for the standard $L^2$ inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}^3} f(x)g(x)\, dx.$$  

Here $f, g$ are real-valued functions. We also have $\|f\|_{L^2}^2 = \langle f, f \rangle$.

2 Proof of the main theorem

We consider solutions of the following form

$$u^\nu(t, x_1, x_2, x_3) = u^{L,\nu}(t, x_2) + u^{S,\nu}(t, x_1, x_2)$$

where $u^{L,\nu}$ and $u^{S,\nu}$ only have nontrivial first and third components, respectively. Furthermore, the solution depends only on $x_1$ and $x_2$. It is well known that these assumptions (so-called $2 + \frac{1}{2}$ dimensional flow) propagate by the Navier-Stokes equations (cf. [4]). With some abuse of notation, we shall identify $u^{L,\nu}$ and $u^{S,\nu}$ with their single nontrivial component, and drop the dependence of the solution on $\nu$. Then, it is not difficult to see that the 3D Navier-Stokes equations reduce to two scalar equations

$$\partial_t u^L = \nu \partial_{x_2}^2 u^L$$

and

$$\partial_t u^S + u^L \partial_{x_1} u^S = \nu(\partial_{x_1}^2 + \partial_{x_2}^2) u^S.$$
2.1 Choice of $u^L$

We take $u_0^L$ to be a rescaling of smoothed triangular wave

$$u_0^L(x_2) = T(\nu^{-\frac{1}{2}}x_2). \quad (2.1)$$

To be precise, we take $T$ to be a $C^\infty(T)$-smooth function satisfying

$$T(z) = z \quad \text{for} \quad z \in \left[ -\frac{1}{4}, \frac{1}{4} \right],$$

and $T(z) = -T(1/2 + z)$. It is not difficult to see that

$$\|u_0^L\|_{L^2} \approx 1.$$  

We may expand $T$ in sine series: we have

$$T(z) = \sum_{i \geq 1} a_i \sin(2(2i - 1)\pi z),$$

where $\{a_i\}_i$ satisfies $|a_i| \lesssim |i|^{-\delta}$ for any $\delta \geq 0$. Here, it is assumed that $0 < \nu \leq 1$ is given in a way that $\nu^{-\frac{1}{2}}$ is an integer. Then, $u^L(t, x_2)$ is given by the solution to the one dimensional heat equation with initial data $u_0^L$:

$$\partial_t u^L = \nu \partial_{x_2}^2 u^L.$$  

From the explicit formula for the initial data in sine series, we have that the solution is

$$u^L(t, x_2) = \sum_{i \geq 1} a_i \sin \left( 2(2i - 1)\pi \nu^{-\frac{1}{2}}x_2 \right) \exp \left( -4\pi^2(2i - 1)^2t \right).$$

We shall compare it with

$$\bar{u}^L(t, x_2) := u_0^L(x_2) \exp \left( -4\pi^2t \right).$$

Then by direct computation (using $\{a_i\}_i$ is rapidly decaying) we see that

$$\|u^L(t) - \bar{u}^L(t)\|_{L^\infty} \leq Ct,$$  

for $t \geq 0$ with $C > 0$ independent of $\nu > 0$. Indeed,

$$u^L(t, x_2) - \bar{u}^L(t, x_2) = \sum_{i \geq 2} a_i \sin \left( 2(2i - 1)\pi \nu^{-\frac{1}{2}}x_2 \right) \left( \exp \left( -4\pi^2(2i - 1)^2t \right) - \exp \left( -4\pi^2t \right) \right)$$

so that

$$|u^L(t, x_2) - \bar{u}^L(t, x_2)| \leq C \sum_{i \geq 2} |a_i|((2i - 1)^2 - 1)t \leq Ct.$$  

for any $x_2$. The advantage of $\bar{u}^L$ over $u^L$ is that $\bar{u}^L$ is exactly linear in some region of $T$, which makes it easy to analyze the corresponding transport operator.
Remark 2.1. We see that
\[ \| \nabla u^{L,\text{heat}}(t) \|_{L^2}^2 \lesssim \| \nabla u_0^L \|_{L^2}^2 \lesssim \nu^{-1}. \]

Thus
\[ \nu \int_0^{\nu^{2\alpha/3}} \| \nabla u^{L,\text{heat}}(t) \|_{L^2}^2 dt \lesssim \nu^{2\alpha/3}. \]

2.2 Choice of \( u^S \)

For \( \alpha \in (0, 3/4) \), we assume for simplicity that \( \nu^{-\alpha} \) is an integer (otherwise, we can simply replace \( \nu^{-\alpha} \) with its integer part in the following argument). Then we set
\[ u_0^S(x_1, x_2) = \sin(\nu^{-\alpha}x_1) \phi(\nu^{-\frac{1}{2}}x_2). \] \( (2.3) \)

Here \( \phi \geq 0 \) is a smooth function supported in \((0, 1/4)\) with \( \phi = 1 \) in \((1/8, 3/16)\).

Note that
\[ \| u_0^S \|_{L^2} \approx 1. \]

Remark 2.2. We see that
\[ \| \nabla u^{S,\text{heat}}(t) \|_{L^2}^2 \lesssim \| \nabla u_0^S \|_{L^2}^2 \lesssim \nu^{-2 \max\{1/2, \alpha\}}. \]

Thus
\[ \nu \int_0^{\nu^{2\alpha/3}} \| \nabla u^{S,\text{heat}}(t) \|_{L^2}^2 dt \lesssim \nu^{1+2\alpha/3-2 \max\{1/2, \alpha\}}. \]

We first consider the linear advection-diffusion equation by rescaled and smoothed triangular wave, possibly with an additional term \( f \):
\[ \begin{align*}
\partial_t \pi^S + \pi^L \cdot \nabla \pi^S &= \nu \Delta \pi^S + f, \\
\pi^S(t = 0) &= u_0^S.
\end{align*} \] \( (2.4) \)

Explicitly, the transport operator is given by
\[ \pi^L \cdot \nabla = e^{-c_0 t} T(\nu^{-\frac{1}{2}}x_2) \partial_{x_1}, \quad c_0 = 4 \pi^2. \]

Let us consider the ansatz
\[ \pi^S(t, x_1, x_2) = e^{-a(t)} \sin \left( \nu^{-\alpha} \left( x_1 - \frac{1 - e^{-c_0 t}}{c_0} T(\nu^{-\frac{1}{2}}x_2) \right) \phi(\nu^{-\frac{1}{2}}x_2) \right). \] \( (2.5) \)

Here, \( a(t) \) is a continuous function of time which will be determined below. We shall now see that under an appropriate choice of \( a(t) \), \( (2.5) \) provides an
approximate solution to \( (2.4) \) with \( f = 0 \). Let us compute the error. Taking a
time derivative to \( (2.5) \),
\[
\partial_t \pi^S = -a'(t)\pi^S - e^{-a(t)-c_0 t}\nu^{-\alpha}T(\nu^{-\frac{1}{2}}x_2) \cos \left( \nu^{-\alpha}(x_1 - \frac{1-e^{-c_0 t}}{c_0}T(\nu^{-\frac{1}{2}}x_2)) \right) \phi(\nu^{-\frac{1}{2}}x_2)
\]
so that
\[
\partial_t \pi^S + \pi^S \cdot \nabla \pi^S = -a'(t)\pi^S.
\]
On the other hand, applying spatial derivatives to \( (2.5) \), we have
\[
\partial_{x_1} \pi^S = -\nu^{-2\alpha} \pi^S,
\]
\[
\partial_{x_2} \pi^S = e^{-a(t)} \cos \left( \nu^{-\alpha}(x_1 - \frac{1-e^{-c_0 t}}{c_0}T(\nu^{-\frac{1}{2}}x_2)) \right) \times
\left( -\frac{1}{c_0} - \nu^{-\frac{1}{2}}\nu^{-\alpha}T'(\nu^{-\frac{1}{2}}x_2) \right) \phi(\nu^{-\frac{1}{2}}x_2)
\]
\[
+ e^{-a(t)} \sin \left( \nu^{-\alpha}(x_1 - \frac{1-e^{-c_0 t}}{c_0}T(\nu^{-\frac{1}{2}}x_2)) \right) \nu^{-\frac{1}{2}}\phi'(\nu^{-\frac{1}{2}}x_2)
\]
and
\[
\partial_{x_2} \pi^S = -\pi^S \times \left( -\frac{1}{c_0} - \nu^{-\frac{1}{2}}\nu^{-\alpha}T'(\nu^{-\frac{1}{2}}x_2) \right)^2
\]
\[
+ e^{-a(t)} \cos \left( \nu^{-\alpha}(x_1 - \frac{1-e^{-c_0 t}}{c_0}T(\nu^{-\frac{1}{2}}x_2)) \right) \times
\left( -\frac{1}{c_0} - \nu^{-\frac{1}{2}}\nu^{-\alpha}T''(\nu^{-\frac{1}{2}}x_2) \right) \phi(\nu^{-\frac{1}{2}}x_2)
\]
\[
+ 2e^{-a(t)} \sin \left( \nu^{-\alpha}(x_1 - \frac{1-e^{-c_0 t}}{c_0}T(\nu^{-\frac{1}{2}}x_2)) \right) \times
\left( -\frac{1}{c_0} - \nu^{-\frac{1}{2}}\nu^{-\alpha}T'(\nu^{-\frac{1}{2}}x_2) \right) \nu^{-\frac{1}{2}}\phi'(\nu^{-\frac{1}{2}}x_2)
\]
\[
+ e^{-a(t)} \sin \left( \nu^{-\alpha}(x_1 - \frac{1-e^{-c_0 t}}{c_0}T(\nu^{-\frac{1}{2}}x_2)) \right) \nu^{-1}\phi''(\nu^{-\frac{1}{2}}x_2).
\]
Recalling that
\[
T'(z) = 1 \quad \text{for} \quad z \in \left[ -\frac{1}{4}, \frac{1}{4} \right],
\]
we have that \( T' = 1 \) and \( T'' = 0 \) on the support of \( \phi \). With these observations,
\( \partial_{x_2x_2} \bar{u}^S \) is simply
\[
\partial_{x_2x_2} \bar{u}^S = -\bar{u}^S \left( 1 - \frac{e^{-c_0 t}}{c_0^2} \right)^2 \nu^{-1-2\alpha} - 2nu^{-\alpha - 1}e^{-a(t)} \cos \left( \nu^{-\alpha} (x_1 - \frac{1 - e^{-c_0 t}}{c_0} T(\nu^{-\alpha} x_2)) \right) \times \\
\frac{1 - e^{-c_0 t}}{c_0} \phi'(\nu^{-\alpha} x_2) + \nu^{-1} e^{-a(t)} \sin \left( \nu^{-\alpha} (x_1 - \frac{1 - e^{-c_0 t}}{c_0} T(\nu^{-\alpha} x_2)) \right) \phi''(\nu^{-\alpha} x_2).
\]

Therefore,
\[
\Delta \bar{u}^S = (\partial_{x_1x_1} + \partial_{x_2x_2}) \bar{u}^S = -\left( \nu^{-2\alpha} + \frac{(1 - e^{-c_0 t})^2}{c_0^2} \nu^{-1-2\alpha} \right) \bar{u}^S + g,
\]
where
\[
g = -2nu^{-\alpha - 1}e^{-a(t)} \cos \left( \nu^{-\alpha} (x_1 - \frac{1 - e^{-c_0 t}}{c_0} T(\nu^{-\alpha} x_2)) \right) \times \\
\frac{1 - e^{-c_0 t}}{c_0} \phi'(\nu^{-\alpha} x_2) + \nu^{-1} e^{-a(t)} \sin \left( \nu^{-\alpha} (x_1 - \frac{1 - e^{-c_0 t}}{c_0} T(\nu^{-\alpha} x_2)) \right) \phi''(\nu^{-\alpha} x_2).
\]

Hence, defining \( a(t) \) to be the solution to the ODE
\[
a'(t) = \nu^{1-2\alpha} + \frac{(1 - e^{-c_0 t})^2}{c_0^2} \nu^{-2\alpha},
\]
\( a(0) = 1 \),
we have that (2.5) is a solution to
\[
\partial_t \bar{u}^S + \bar{u}^L \cdot \nabla \bar{u}^S = \nu \Delta \bar{u}^S - \nu g. \tag{2.6}
\]

**Remark 2.3.** Observe that for \( O(1) \)-small \( t > 0 \), we have
\[
a'(t) \simeq \nu^{1-2\alpha} + t^2 \nu^{-2\alpha}
\]
and hence
\[
a(t) \simeq 1 + \nu^{1-2\alpha} t + \frac{t^3}{3} \nu^{-2\alpha}.
\]

This shows that for \( \bar{u}^S \), loss of energy occurs within an \( O(\nu^{2\alpha}) \)-timescale. Here, it is required that \( \nu^{1-2\alpha} t \lesssim 1 \) for \( t = \nu^{2\alpha} \), which is equivalent to \( \alpha < \frac{3}{4} \). Indeed, when \( 0 < \alpha < \frac{3}{4} \), we have that \( \exp(-a(t)) \gtrsim 1 \) for \( 0 \leq t \leq \nu^{2\alpha/3} \), and a direct computation gives
\[
\nu \int_0^{\nu^{2\alpha/3}} \| \partial_{x_2} \bar{u}^S(t) \|_{L_2}^2 dt \gtrsim \nu \int_0^{\nu^{2\alpha/3}} t^2 \nu^{-1-2\alpha} dt \approx 1. \tag{2.7}
\]
2.3 Difference estimate

We consider the solution of
\[
\partial_t u^S + u^L \cdot \nabla u^S = \nu \Delta u^S \tag{2.8}
\]
with initial data \( u^S_0(x_1, x_2) = \sin(\nu^{-\alpha}x_1)\phi(\nu^{-\frac{1}{4}}x_2) = \overline{u}_0^S(x_1, x_2) \) and compare it with \( \overline{u}^S \) from (2.3): the difference \( \epsilon = \overline{u}^S - u^S \) satisfies the equation
\[
\partial_t \epsilon + \overline{u}^L \cdot \nabla \epsilon + (u^L - \overline{u}^L) \cdot \nabla u^S = \nu \Delta \epsilon - \nu g. \tag{2.9}
\]
Taking the \( L^2 \) inner product with \( \epsilon \),
\[
\frac{1}{2} \frac{d}{dt} \| \epsilon \|^2_{L^2} + \nu \| \nabla \epsilon \|^2_{L^2} = -\langle \nu g, \epsilon \rangle - \langle (u^L - \overline{u}^L) \cdot \nabla u^S, \epsilon \rangle.
\]
We easily estimate
\[
\left| \langle (u^L - \overline{u}^L) \cdot \nabla u^S, \epsilon \rangle \right| \lesssim \| u^L - \overline{u}^L \|_{L^\infty} \| \partial_x u^S \|_{L^2} \| \epsilon \|_{L^2}
\lesssim t \| \partial_x u^S \|_{L^2} \| \epsilon \|_{L^2}
\lesssim t^\nu^{-\alpha} \| \epsilon \|_{L^2}.
\]
Moreover, observing that \( \| \nu g \|_{L^2} \lesssim 1 + t \nu^{-\alpha} \), we have
\[
\left| \frac{1}{2} \frac{d}{dt} \| \epsilon \|^2_{L^2} + \nu \| \nabla \epsilon \|^2_{L^2} \right| \lesssim (1 + t \nu^{-\alpha}) \| \epsilon \|_{L^2}. \tag{2.10}
\]
Therefore,
\[
\frac{d}{dt} \| \epsilon \|_{L^2} \lesssim 1 + t \nu^{-\alpha},
\]
and
\[
\| \epsilon(t) \|_{L^2} \lesssim 1 + t^2 \nu^{-\alpha}. \tag{2.11}
\]
Finally, we have after integrating (2.10) in \( t \in [0, \nu^{2\alpha/3}] \) and applying (2.11),
\[
\nu \int_0^{\nu^{2\alpha/3}} \| \nabla \epsilon(t) \|_{L^2}^2 dt \lesssim \int_0^{\nu^{2\alpha/3}} \left| \langle \nu g, \epsilon \rangle + \left| \langle (u^L - \overline{u}^L) \cdot \nabla u^S, \epsilon \rangle \right| \right| dt
\lesssim \int_0^{\nu^{2\alpha/3}} t(1 + t \nu^{-\alpha})^2 dt \lesssim \nu^{4\alpha/3} + \nu^{2\alpha/3}. \tag{2.12}
\]

2.4 Completion of the proof

We are now in a position to complete the proof. Let us briefly note that strictly speaking, we only have \( \| u^\nu \|_{L^2} \approx 1 \) while the statement of the Theorem requires \( \| u^\nu \|_{L^2} = 1 \). This can be fixed with a simple rescaling of \( u^\nu \).
By (2.7) from Remark 2.3 and (2.12), we have

\[ \nu \int_0^{\nu^{2n/3}} \| \nabla u^\nu(t) \|^2_{L^2} dt \geq \nu \int_0^{\nu^{2n/3}} \| \nabla u^{S,\nu}(t) \|^2_{L^2} dt \]

\[ \geq \nu \int_0^{\nu^{2n/3}} \| \partial_{x_2} u^{S,\nu}(t) \|^2_{L^2} dt \]

\[ \gtrsim \nu \int_0^{\nu^{2n/3}} \| \partial_{x_2} \bar{u}^{S,\nu}(t) \|^2_{L^2} dt - \nu \int_0^{\nu^{2n/3}} \| \partial_{x_2} \epsilon(t) \|^2_{L^2} dt \]

\[ \gtrsim 1 - \nu^{2n/3}. \]

This gives the enhanced dissipation statement (1.4). For the heat flow case, combining Remark 2.1 and Remark 2.2, we immediately obtain the desired estimate.

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