$\omega = -1$ crossing in quintessence models in Lyra’s geometry

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Abstract

We study the cosmology of quintessence models in an extended theory of gravity in Lyra’s geometry. By analyzing the possible interactions between the quintessence scalar and the intrinsic displacement field in Lyra’s geometry, we obtain the closed form solutions of the modified Friedmann equations for four classes of quintessence models. Though the presence of the geometrical displacement field promises the possibility for the effective equation of state $\omega$ of the quintessence-displacement mixture crossing the cosmological constant boundary, the reliable quintessence scenarios in Lyra’s geometry with stable perturbation modes are still those in which $-1 \leq \omega \leq 1$. 
INTRODUCTION

Recent cosmic observations have indicated that our Universe is undergoing an accelerated expansion at the present epoch. The cause for such a cosmic acceleration is attributed to an unknown dominant energy component, dubbed dark energy, with negative pressure generating thus repulsive gravitational forces that counteract the attractive forces produced by radiation, baryons and the cold dark matter. However, the exact nature of dark energy is currently a significant part of the realm of speculations. Some believe that dark energy is the energy of the quantum vacuum, modelled by the cosmological constant $\Lambda$ of general relativity. Interpreting dark energy as a cosmological constant means that the density of dark energy is uniform throughout the universe and invariable in time. This is the simplest explanation for dark energy, which was introduced by Einstein for building a static universe but has a good fit with the available data of the current cosmological observations. If dark energy takes this form, it is a fundamental property of the universe. The cosmological constant thus faces two fundamental problems in physics, namely the fine-tuning and coincidence puzzles.

The late-time cosmic acceleration may alternatively be driven by a dynamic dark energy which could be a time evolving and spatially dependent scalar field. Lots of such dynamic dark energy models have been proposed, which are roughly classified into three categories: quintessence, phantom and quintom. In quintessence models, a scalar field $\varphi$ with a canonical kinetic energy and a self-interaction potential energy $V(\varphi)$ is supposed to be minimally coupled to Einstein gravity. In a flat Robertson-Walker background the quintessence scalar behaves as a perfect fluid with an evolving equation-of-state (EoS) parameter $\omega = p/\rho$ lying in the range $-1 \leq \omega \leq 1$. In phantom models the quintessence is replaced by a ghost scalar of which the kinetic energy is negative and $\omega < -1$. Due to the no-go theorem proposed in Ref. [31, 32, 37–40], the model buildings of quintom dark energy are generally very complicated [36]. The simplest quintom model is composed of two scalar fields, one is a quintessence scalar and another a phantom [29, 30]. Quintom models characterize themselves by the property that the effective EoS parameter can cross the cosmological constant boundary $\omega = -1$, which makes them to fit the observational data better [36].

Crossing the $\omega = -1$ divide in a dynamic dark energy model is bewitching. However, the
emergence of a phantom mode with negative kinetic energy in quintom models brings about great embarrassment in understanding it. The consistence coming from the Null Energy Condition in physics requires the kinetic energy of a normal scalar field not to be negative, otherwise the theory might be unstable and unbounded. Therefore, it is worthwhile to study the mechanism of removing the phantom field from the quintom models. In fact, there has lots of such attempts to investigate the possibility of $\omega = -1$ crossing in quintessence like models $[35, 41, 43]$. It has been empirically realized that to cross the $\omega = -1$ barrier and remove ghost mode at the same time, the model building should be involved in either modifying the general theory of Einstein’s relativity or introducing some higher derivative terms for the scalar fields. For example, In the so-called Galileon cosmology $[41, 43]$ of a scalar field, the higher derivatives of operators are introduced into the Lagrangian but the equation of motion of the scalar remains of the second order. The Galileon models can have $\omega = -1$ crossing without ghost modes involved. It goes without saying, however, that these models are generally very complicated to deal with.

Of the modification attempts beyond Einstein’s general theory of relativity, there is an extended theory of gravity (ETG) based on the so-called Lyra’s geometry $[44–48]$. As is well known, the general theory of relativity is a theory of gravity built on (pseudo-) Riemannian geometry. Lyra’s geometry is a modification of Riemannian geometry where a gauge function is introduced. Due to the presence of this gauge function on the structure-less manifold, an extra geometrical ingredient, i.e., the displacement vector $\beta_\mu$ arises in Lyra’s geometry. It is remarkable that the connection in both Riemannian and Lyra’s geometries are metric preserving. The extended theory of gravity in Lyra’s geometry is much motivated by the fact that it could predict the same effects as Einstein’s general relativity within observations limits, as far as the classical Solar System, as well as tests based on the linearised form of the field equations $[48–59]$. The extended theory of gravity on Lyra’s geometry distinguishes itself by the fact that it is a scalar-tensor theory of gravity, where the scalar field is not alien, but intrinsic to the geometry $[45, 47]$. Moreover, in the so-called normal gauge $[60–64]$, a constant displacement vector can play the role of a positive cosmological constant (in the presence of other cosmic matter ingredients) $[65–67]$, which is in contrast to general relativity where the cosmological constant must be added in an ad hoc manner into the gravitational field equations.

In this paper, we study the $\omega = -1$ crossing possibility in some quintessence models in
the framework of the ETG in Lyra’s geometry. Despite the impossibility for a canonical quintessence model to cross the phantom divide $\omega = -1$ in Einstein’s gravity in pseudo-Riemannian geometry \cite{31, 32, 37}, the existence of a displacement field in the ETG in Lyra’s geometry does probably modify the effective distribution of the cosmic fluids so that the EoS parameter of the quintessence scalar may cross this boundary. Aimed at finding the exact solutions of the modified Friedmann equations in a flat Robertson-Walker background, we propose several candidate interactions between the quintessence scalar and the displacement field which have simple mathematical expressions. For some of these possible interactions, crossing $\omega = -1$ barrier for quintessence models in Lyra’s geometry is available. Unfortunately, crossing this phantom divide in these models will, without any exception, give rise to the instability of the relevant perturbations. The reliable quintessence scenarios in Lyra’s geometry are still those in which $-1 \leq \omega \leq 1$.

The paper is organized as follows. Section II begins with a brief introduction to modified Einstein equations in the considered ETG and its application to the cosmology of a quintessence model in a flat Robertson-Walker background spacetime. By analyzing the equation of motion of the quintessence scalar, we determine phenomenologically several candidate interactions between this scalar and the displacement field. In Section III we study the quintessence cosmology for each of the candidate interaction terms. The self-interaction potential of the quintessence scalar is not given a prior, which is defined during the process solving the modified Friedmann equations, motivated by the requirement to have closed form solutions to these equations. Among the four quintessence models proposed, three of them naively allow $\omega = -1$ crossing. By requiring the squared sound speed of the quintessence scalar preserves finite and non-negative during its evolution, the possibility for $\omega$ crossing the phantom divide is excluded. It turns out that the reliable quintessence models in Lyra’s geometry are also characterized by inequalities $-1 \leq \omega \leq 1$, similar to those in pseudo-Riemannian geometry. We conclude in Section IV with a summary of the results and some remarks. For simplicity we work in the Planck units $c = \hbar = \kappa^2 = 1$ throughout the paper.

QUINTESSENCE AND ACCELERATED EXPANSION

The quintessence models in an extended theory of gravity (ETG) in Lyra’s geometry, in the so-called normal gauge, is described by the following modified Einstein gravitational
field equations [47]:
\[ G_{\mu\nu} = T_{\mu\nu}^\varphi + T_{\mu\nu}, \] (1)
where,
\[ T_{\mu\nu} = -\frac{3}{2} \left( \beta_\mu \beta_\nu - \frac{1}{2} g_{\mu\nu} \beta_\lambda \beta^\lambda \right), \] (2)
is an intrinsic geometrical stress tensor, corresponding to the existence of the displacement vector \( \beta_\mu \) which emerges from the integrability condition of length of a vector under parallel transport [44, 47]. \( T_{\mu\nu}^\varphi \) is the stress tensor of the quintessence scalar \( \varphi \) which is assumed to have a self-interaction potential \( V(\varphi) \) and be canonically coupled to gravity,
\[ T_{\mu\nu}^\varphi = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \varphi \nabla^\lambda \varphi - g_{\mu\nu} V(\varphi) \] (3)
We further assume that, at present epoch, the quintessence scalar \( \varphi \) dominates over other cosmic fluids such as baryonic dust and radiation. The displacement vector \( \beta_\mu \) is allowed to be a time-like 4-vector field [47, 60–62],
\[ \beta_\mu = (\beta(t), 0, 0, 0) \] (4)
Its unique non-vanishing component \( \beta(t) \) can either be a constant or time-dependent.

In a flat Friedmann-Lemaître-Robertson-Walker background \( ds^2 = -dt^2 + a^2 dx^2 \), the modified Einstein equations given in Eq.(1) become:
\[ 3H^2 = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) - \frac{3}{4} \beta^2(t) \] (5)
\[ 2\dot{H} + 3H^2 = -\frac{1}{2} \dot{\varphi}^2 + V(\varphi) + \frac{3}{4} \beta^2(t) \] (6)
where an overdot denotes derivative with respect to the cosmic time \( t \). The equation of motion of the quintessence scalar, which comes from the Bianchi identities of modified Einstein equations, i.e., from the compatibility of Eq.(5) with Eq.(6), reads,
\[ \varphi (\ddot{\varphi} + 3H \dot{\varphi} + V_\varphi) = \frac{3}{4} \left( \dot{\theta} + 6H \theta \right), \] (7)
In Eq.(7) \( V_\varphi = \frac{dV}{d\varphi} \) and \( \theta \equiv \beta^2 \) (\( \theta \) is also referred to as the displacement field). Eq.(7) implies that the displacement field and the quintessence scalar \( \varphi \) interact as the universe evolves.

The mixture of the quintessence scalar \( \varphi \) and the displacement field \( \theta \) is conventionally viewed as a perfect fluid, whose energy density and pressure are defined by,
\[ \rho = \frac{1}{2} \dot{\varphi}^2 + V(\varphi) - \frac{3}{4} \theta(t) \] (8)
\[ p = \frac{1}{2} \dot{\varphi}^2 - V(\varphi) - \frac{3}{4} \theta(t) \] (9)
where \(\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)\) and \(p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)\) are respectively the energy density and pressure of the quintessence scalar \(\varphi\). The real displacement vector \(\beta_\mu\) (or positive \(\theta\)) that is a necessary geometrical ingredient in Lyra’s geometry, nevertheless, behaves as an exotic cosmic matter with negative energy density and negative pressure, \(\rho_\theta = p_\theta = \frac{-3}{4}\theta\). In the absence of quintessence scalar, the effective EoS parameter of the displacement field is equal to \(\omega_\theta = 1\), which corresponds to the so-called stiff fluid [66]. When the quintessence scalar exists, the displacement field would probably play the role of a phantom [22–24], so that the mixture probably behaves as an effective quintom [29, 30] to cause the late time accelerated expansion of our universe.

The effective EoS parameter of the mixed fluid is,

\[
\omega := \frac{p}{\rho} = \frac{2\dot{\varphi}^2 - 4V(\varphi) - 3\theta(t)}{2\dot{\varphi}^2 + 4V(\varphi) - 3\theta(t)}
\]

(10)

If there were no displacement field \(\theta\) (as in general relativity based on pseudo-Riemannian geometry), the EoS parameter \(\omega\) would only include the contribution of quintessence scalar \(\varphi\), and \(-1 \leq \omega \leq 1\). This is not the case in the ETG in Lyra’s geometry [66, 67]. In a Lyra manifold with the positive displacement field \(\theta\), the universe evolves between \(\omega = 1\) (stiff matter), where either the kinetic term dominates or both kinetic term and the displacement field dominate, and a phantom regime [22, 24], where \(\omega \leq -1\), provided that the kinetic term of the scalar field is negligible. The interesting possibility for \(\omega\) crossing the cosmological constant boundary seems plausible. To clarify such a possibility, the mechanism describing the possible interactions between the quintessence scalar and the displacement field is required. Owing to the lack of such a mechanism, in this paper, we determine the relevant interaction terms by a simple dimensional analysis. The interaction strength is inevitably encoded into a few phenomenological parameters. Following other works [68, 69], we interpret Eq. (7) as an effective energy conservation equation, and recast it as:

\[
\dot{\varphi} (\ddot{\varphi} + 3H\dot{\varphi} + V_\varphi) = \mathcal{Q},
\]

(11)

\[
\frac{3}{4} \left( \dot{\theta} + 6H\theta \right) = \mathcal{Q},
\]

(12)

where \(\mathcal{Q}\) stands for the required interaction. By dimensional analysis, \(\mathcal{Q}\) is generally of the form \(\mathcal{Q} = \alpha_1 \rho_1 + \alpha_2 \rho_2 + \cdots\), with \(\rho_i\) the different energy components. The coupling parameters \(\alpha_i\) have the dimension of Hubble parameter \(H\) or \(\dot{\varphi}\), and should not be simultaneously set to zero. In the models under consideration the possible candidates of \(\rho_i\) are \(\dot{\varphi}^2\), \(V(\varphi)\), etc.
and $\theta$. We will simply assume that,

$$\mathcal{L} = 3cH\dot{\varphi}^2 + \frac{3b}{2\sqrt{6}}\dot{\varphi}\theta + \frac{9}{2}bH\theta + 6fHV(\varphi)$$  \hspace{1cm} (13)$$

where $c$, $b$, $\tilde{b}$ and $f$ are some dimensionless coupling constants. The choice for the interaction terms in Eq.(13) allows us, not only to explain the present accelerated expansion of our universe, but to have closed form solutions of the modified Friedmann equations also.

We now proceed to study the cosmological consequences emerging from each of these interactions. Instead of $t$, we will use the e-folding number $x = \ln a$ as the time variable for convenience, with $x = 0$ representing the present time ($a(0) = 1$). The modified Friedmann equations in Eqs.(5) and (6) are recast as:

$$\frac{dH^2}{dx} + 6H^2 = 2V$$  \hspace{1cm} (14)$$

$$\theta = \frac{2}{3}H^2\left(\frac{d\varphi}{dx}\right)^2 + \frac{4}{3}V - 4H^2$$  \hspace{1cm} (15)$$

Similarly, Eqs.(11) and (12) that can be viewed as the equations of motion of quintessence scalar and displacement field become:

$$\frac{d\varphi}{dx}\left[H^2\frac{d^2\varphi}{dx^2} + \left(3H^2 + \frac{1}{2}dH^2\right)\frac{d\varphi}{dx} + V,\varphi\right] = \mathcal{\tilde{L}}$$  \hspace{1cm} (16)$$

$$\frac{3}{4}\left(\frac{d\theta}{dx} + 6\theta\right) = \mathcal{\tilde{L}}$$  \hspace{1cm} (17)$$

where,

$$\mathcal{\tilde{L}} = 3cH^2\left(\frac{d\varphi}{dx}\right)^2 + \frac{3b}{2\sqrt{6}}\left(\frac{d\varphi}{dx}\right)\theta + \frac{9}{2}b\theta + 6fV(\varphi)$$  \hspace{1cm} (18)$$

Relying on the fact that Eq.(7) is the Bianchi identity of the modified Friedmann equations (5) and (6), any non-degenerate combination of three of equations (14), (15), (16) and (17) will be mathematically equivalent.

**Case 1** $c \neq 0$ but $b = \tilde{b} = f = 0$:

In this case, Eq.(16) pretends to decouple from the displacement field $\theta$ and becomes an effective equation of motion of quintessence scalar $\varphi$,

$$H^2\frac{d^2\varphi}{dx^2} + \left[3(1 - c)H^2 + \frac{1}{2}\frac{dH^2}{dx}\right]\frac{d\varphi}{dx} + V,\varphi = 0$$  \hspace{1cm} (19)$$
Equivalently,

\[ H^2 \left( \frac{d^2 \varphi}{dx^2} - 3c \frac{d\varphi}{dx} \right) + \left( V_{,\varphi} + V \frac{d\varphi}{dx} \right) = 0 \] (20)

when Eq. (14) is taken into account. In view of the mathematical structure of Eq. (20), we assume that the self-interaction quintessence potential \( V(\varphi) \) is defined by,

\[ V_{,\varphi} + V \frac{d\varphi}{dx} = 0 \] (21)

in the models under consideration. The evolution of quintessence scalar in terms of e-folding time \( x \) turns out to be independent of the Hubble parameter,

\[ \frac{d^2 \varphi}{dx^2} - 3c \frac{d\varphi}{dx} = 0 \] (22)

By assigning the initial conditions \( \varphi(0) = p \) and \( d\varphi(0)/dx = q \), with \( p, q \) two outstanding constants, we can solve Eq. (22) analytically. The solution reads,

\[ \varphi(x) = p + q(e^{3cx} - 1) \] (23)

The constant \( p \) does not matter whether the scalar field \( \varphi \) is a quintessence or a phantom, because it is irrelevant to the kinetic energy of the field. So, we set \( p = 0 \) for simplicity from now on. Substitution of Eq. (23) into Eq. (21) leads to:

\[ V(x) = r H_0^2 \exp \left[ -\frac{3}{2} c q^2 (e^{6cx} - 1) \right] \] (24)

In Eq. (24) we have used a constant \( r \) labeling the present value \( r H_0^2 \) of the quintessence potential, with \( H_0 \) the Hubble parameter at present epoch. The potential can be expressed as a closed form function of the quintessence scalar \( \varphi \) itself,

\[ V(\varphi) = r H_0^2 \exp \left[ -\frac{3}{2} c \varphi (\varphi + 2q) \right] \] (25)

If \( c < 0 \) and \( q = 0 \), such a \( V(\varphi) \) could be interpreted as the tachyon potential describing the excitation of massive scalar fields on the anti-D branes [70, 71], with \( r H_0^2 \) the brane tension and \(-3c\) the mass squared of the field \( \varphi \) in the Planckian units. However, \( q = 0 \) is forbidden in our case. The physics behind the proposed potential (25) is an open issue.

With Eq. (25), we can obtain the evolution of Hubble parameter \( H \) by solving Eq. (14). The result is,

\[ H^2(x) = H_0^2 e^{-6x} \left\{ 1 + \frac{r}{3c} \sqrt{\frac{2}{3cq^2}} e^{3cq^2/2} \left[ \Gamma \left( \frac{1}{c} \cdot \frac{3cq^2}{2} \right) - \Gamma \left( \frac{1}{c} \cdot \frac{3cq^2}{2} e^{6cx} \right) \right] \right\} \] (26)
where $\Gamma(\tau, z)$ is the upper incomplete Gamma function \[72\],

$$\Gamma(\tau, z) = \int_z^{+\infty} d\zeta \, \zeta^{\tau-1} e^{-\zeta}$$

(27)

Furthermore, employment of Eqs. (25) and (26) in Eq. (15) yields,

$$\begin{aligned}
\theta(x) &= \frac{4}{3} r H_0^2 \exp \left[ \frac{3}{2} c q^2 (1 - e^{6c x}) \right] + 2 H_0^2 (3c^2 q^2 e^{6c x} - 2) e^{-6x} \\
&+ \frac{2r}{3c} H_0^2 (3c^2 q^2 e^{6c x} - 2) \sqrt{\frac{2}{3c q^2}} \left[ \Gamma\left(\frac{1}{c}, \frac{3c q^2}{2} e^{6c x} \right) - \Gamma\left(\frac{1}{c}, \frac{3c q^2}{2} \right) \right] \exp \left( \frac{3}{2} c q^2 - 6x \right)
\end{aligned}$$

(28)

The cosmology of the mixed fluid, at the background level, is determined by its EoS parameter $\omega$ (See Eq. (10)). For the models under consideration,

$$\omega(x) = 1 - \frac{2c r \exp \left[ 6x - \frac{3}{2} c q^2 (e^{6c x} - 1) \right]}{3c + r \sqrt{\frac{2}{3c q^2} e^{3c q^2/2}} \left[ \Gamma\left(\frac{1}{c}, \frac{3c q^2}{2} e^{6c x} \right) - \Gamma\left(\frac{1}{c}, \frac{3c q^2}{2} \right) \right]}$$

(29)

Although the evolution of EoS parameter $\omega$ depends upon three parameters, i.e., $c$, $q$ and $r$, its value $\omega_0$ at the present epoch is completely given by the dimensionless parameter $r$ which represents the present-epoch value of the quintessence self-interaction potential,

$$\omega_0 = 1 - \frac{2r}{3}$$

(30)

Provided $r > 2$, $\omega_0 < -1/3$, the late-time accelerated expansion occurs. In particular, the mixed fluid in ETG of Lyra’s geometry will respectively mimic the quintessence, cosmological constant and phantom in the Einstein’s general relativity if the parameter $r$ takes its value in the regions $2 < r < 3$, $r = 3$ and $r > 3$. However, for $r \geq 3$, the EoS parameter $\omega$ given in Eq. (29) increases monotonically near $x = 0$ for real parameters $c$ and $q$, as seen from the asymptotic expansion of Eq. (29) at small $x$,

$$\omega \approx 1 - \frac{2r}{3} + \frac{2r}{3} (9e^2 q^2 + 2r - 6)x$$

(31)

Such a $\omega$ conflicts with our common sense about the universe evolution. We choose to abandon this possibility. The parameter $r$ is consequently restricted within the region $2 < r < 3$ for these models, in turn the aspired $\omega = -1$ crossing is unavailable.

It is interesting to study the dependence of the evolution of $\omega$ upon the magnitude of the dimensionless coupling parameter $c$. To this end we plot Eq. (29) in FIG. 1. for three
different choices of coupling \( c \), \textit{i.e.}, \( c = 2 \), \( c = 1.5 \) and \( c = 1 \), and the parameters \( q \) and \( r \) are fixed at \( q = 0.05 \) and \( r = 2.75 \), respectively. It is manifest that the evolution of the universe from matter dominant era to the present acceleration phase depends weakly upon what the coupling constant \( c \) is. However, the value of \( c \) will strongly influence the universe evolution in the future. The larger the value of \( c \) is, the earlier will the universe exit from its accelerated expansion.

![Graph](image)

FIG. 1. Evolution of \( \omega \) versus \( x \) for different coupling parameter \( c \) for interaction term \( \mathcal{Q} = 3cH\dot{\varphi}^2 \). Here we take \( q = 0.05 \) and \( r = 2.75 \) (So \( \omega_0 \approx -0.83 \)). The solid, dashed and dotted curves correspond to \( c = 2 \), \( c = 1.5 \) and \( c = 1 \) respectively.

The above is the cosmological implications of the background dynamics of the proposed model in Lyra’s geometry. The concordance cosmology is a science based on precise observations of which lots are tightly connected to the growth of perturbations. Thus we must examine the stability issue of the perturbation modes in the model under consideration. According to the linear perturbation theory, the stability of the linear perturbation modes during their evolution requires \( c_{s,i}^2 \geq 0 \) for each component fluid \([31, 69, 73]\), where \( c_{s,i}^2 \equiv \partial p_i / \partial \rho_i \) is its squared sound speed at the background level. In Lyra’s geometry, the squared sound speed of the displacement field is always definitely positive. In fact, \( c_{s,\theta}^2 = \omega_\theta = 1 \). On the other hand,

\[
c_{s,\varphi}^2 = 1 + \frac{2r \exp \left[ 6x - \frac{3}{2} cq^2 (e^{6cx} - 1) \right]}{(c - 1) \left\{ 3 + \frac{r}{c} \sqrt{\frac{2}{3cq^2}} e^{3cq^2/2} \left[ \Gamma \left( \frac{1}{c}, \frac{3}{2}cq^2 \right) - \Gamma \left( \frac{1}{c}, \frac{3}{2}cq^2 e^{6cx} \right) \right] \right\}} \tag{32}
\]
The squared sound speed $c_{s,\varphi}^2$ varies continuously with respect to e-folding time unless $c = 1$. For a positive $c$ ($c \neq 1$), $c_{s,\varphi}^2 \rightarrow 1$ when $x \rightarrow \pm \infty$, while,

$$c_{s,\varphi}^2 \approx 1 + \frac{2r [1 + (6 - 2r - 9c^2q^2)x]}{3(c - 1)}$$  \hspace{1cm} (33)

at $x \approx 0$. Recall that $2 < r < 3$, the squared sound speed $c_{s,\varphi}^2$ of the quintessence scalar diverges or takes negative values for $0 < c \leq 1$, the corresponding perturbation modes are violently unstable and do not have any physical significance. The stability of perturbation modes is also sensitive to the initial velocity $q$ of the quintessence scalar. To ensure a finite and non-negative $c_{s,\varphi}^2$, the coupling constant $c$ should be restricted to the region $c > 1$, and at the same time the coefficient of $x$ in the RHS of Eq.\((33)\) should be set to zero. Therefore, the model is physical acceptable only if $c > 1$, $q = \pm \frac{1}{3r} \sqrt{6 - 2r}$ and $2 < r < 3$.

**Case 2** $b \neq 0$ but $c = \tilde{b} = f = 0$:

In this case, Eq.\((16)\) is translated into,

$$\left\[ \frac{d^2 \varphi}{dx^2} - b \left( \frac{d\varphi}{dx} \right)^2 + \sqrt{6b} \right\] H^2 + \left\[ V_{\varphi} + \left( \frac{d\varphi}{dx} \right) V - \frac{\sqrt{6b}}{3} V \right\] = 0$$ \hspace{1cm} (34)

Similar to Case 1, we further assume that the self-interaction quintessence potential $V(\varphi)$ satisfies the constraint condition,

$$V_{\varphi} + \left( \frac{d\varphi}{dx} \right) V - \frac{\sqrt{6b}}{3} V = 0$$ \hspace{1cm} (35)

Consequently, the evolution of quintessence scalar in the models under consideration is also fictitiously independent of the evolution of Hubble parameter,

$$\frac{d^2 \varphi}{dx^2} - b \left( \frac{d\varphi}{dx} \right)^2 + \sqrt{6b} = 0$$ \hspace{1cm} (36)

The solution of Eq.\((36)\) which satisfies the initial conditions $\varphi(0) = 0$ and $d\varphi(0)/dx = \sqrt{6}q$ reads,

$$\varphi(x) = -\frac{\sqrt{6}}{b} \ln [\cosh(bx) - q \sinh(bx)]$$ \hspace{1cm} (37)

Plugging Eq.\((37)\) into Eq.\((35)\) yields,

$$V(x) = \frac{r H_0^2}{[\cosh(bx) - q \sinh(bx)]^2} \exp \left[ -6x + \frac{6(1 - q^2)}{b} \frac{\sinh(bx)}{\cosh(bx) - q \sinh(bx)} \right]$$ \hspace{1cm} (38)
where \( r \) is an integration constant which is, as before, used to specify the present-epoch value of quintessence potential, \( V(0) = rH^2_0 \). Different from Case 1, for the current models, it is difficult to express the quintessence potential as a closed form function \( V(\varphi) \). Fortunately, this does not affect our investigation to cosmology. With (38), we can easily solve Eq.(14) and obtain the evolution of Hubble parameter in these models,

\[
H^2(x) = H^2_0 e^{-6x} \left[ 1 - \frac{r}{3(1-q^2)} + \frac{r}{3(1-q^2)} \exp \left( \frac{6(1-q^2)}{b} \frac{\sinh(bx)}{\cosh(bx) - q \sinh(bx)} \right) \right]
\]

and then,

\[
\theta(x) = 4H^2_0 e^{-6x} \frac{(3q^2 + r - 3)}{3 \left( \cosh(bx) - q \sinh(bx) \right)^2}
\]

Therefore,

\[
\omega = 1 - \frac{2(1-q^2) r \exp \left( \frac{6(1-q^2)}{b} \frac{\sinh(bx)}{\cosh(bx) - q \sinh(bx)} \right)}{\left[ \cosh(bx) - q \sinh(bx) \right]^2} \cdot \frac{3(1-q^2) - r + r \exp \left( \frac{6(1-q^2)}{b} \frac{\sinh(bx)}{\cosh(bx) - q \sinh(bx)} \right)}{3(1-q^2) - r + r \exp \left( \frac{6(1-q^2)}{b} \frac{\sinh(bx)}{\cosh(bx) - q \sinh(bx)} \right)}
\]

The evolution of the effective EoS parameter depends upon three parameters, i.e., \( b \) and \( q \). However, as in Case 1, its present value \( \omega_0 \) is only relevant to \( r \),

\[
\omega_0 = 1 - \frac{2r}{3}
\]

Provided \( r > 2 \), \( \omega_0 < -1/3 \), the late-time accelerated expansion occurs. Notice that the asymptotic expansion of Eq.(41) at small \( x \) is,

\[
\omega \approx 1 - \frac{2r}{3} - 4rx \left[ (1-q^2) - \frac{1}{3}(r-bq) \right]
\]

So long as \( r < bq + 3(1-q^2) \), the EoS parameter \( \omega \) will decrease monotonically near \( x = 0 \) for real parameters \( b \) and \( q \), implying that the \( w = -1 \) crossing is possible. For a given coupling constant \( b \), the initial "velocity" \( q \) of the quintessence scalar have to take its value in the region \( (b - \sqrt{b^2 + 12})/6 < q < (b + \sqrt{b^2 + 12})/6 \) to guarantee the inequality \( 2 < r < bq + 3(1-q^2) \). In FIG. 2, we plot the evolution of EoS parameter \( \omega \) in Eq.(41) for three different choices of coupling \( b \), i.e., \( b = 2.6 \), \( b = 3 \) and \( b = 3.6 \), and the parameters \( q \) and \( r \) are fixed at \( q = 1 \) and \( r = 2.05 \), respectively. The remarkable difference between the present case and Case 1 is that the EoS in the present models can cross the phantom divide \( \omega = -1 \). Is this quintom scenario reliable?
FIG. 2. Evolution of $\omega$ versus $x$ for different coupling parameter $b$ for interaction term $Q = 3b \dot{\varphi}/2\sqrt{6}$. Here we take $q = 1$ and $r = 2.05$. The solid, dashed and dotted curves correspond to $b = 2.6$, $b = 3$ and $b = 3.6$ respectively.

It has been pointed out [31, 36] that a viable quintom scenario can not be realized only by virtue of the parameterization of EoS. The stability of the relevant perturbation modes must be ensured also. In other words, we have to guarantee $c_{s,\varphi}^2 \geq 0$ at background level. For simplicity we only consider a special case $q = 1$. In this case, the solution to the background dynamics reduces to:

$$\varphi(x) = \varphi(0) + \sqrt{6}x$$  \hspace{1cm} (44)

$$V(x) = rH_0^2e^{2(b-3)x}$$ \hspace{1cm} (45)

$$H^2 = H_0^2e^{-6x} \left[1 + \frac{r}{b}(e^{2bx} - 1)\right]$$ \hspace{1cm} (46)

$$\theta(x) = \frac{4}{3}rH_0^2e^{2(b-3)x}$$ \hspace{1cm} (47)

$$\omega = 1 - \frac{2br e^{2bx}}{3[b + r(e^{2bx} - 1)]}$$ \hspace{1cm} (48)

The energy density and pressure of the quintessence scalar read,

$$\rho_\varphi = \frac{H_0^2}{b}e^{-6x}[3(b - r) + (3 + b)re^{2bx}]$$, \hspace{0.5cm} $p_\varphi = \frac{H_0^2}{b}e^{-6x}[3(b - r) + (3 - b)re^{2bx}]$  \hspace{1cm} (49)

Hence,

$$c_{s,\varphi}^2 = \frac{\partial p_\varphi}{\partial \rho_\varphi} = \frac{(3 - b)^2re^{2bx} + 9(b - r)}{(9 - b^2)re^{2bx} + 9(b - r)}$$ \hspace{1cm} (50)
The physical acceptance requires \( \omega < -1/3 \) for \( x \geq 0 \) and \( c_{s,\phi}^2 \geq 0 \). Obviously, both inequality can be satisfied if \( 2 < r < b \leq 3 \). With respect to such a parameter constraint, however, \( 1 \geq \omega \geq 1 - 2b/3 \), the quintom scenario where the EoS of mixed fluid can cross the cosmological constant boundary is forbidden. It seems that the no-go theorem \([31, 32, 37-40]\) is valid also for a generic \( q \) so that a reasonable quintom scenario is unavailable in the present model.

![Graph](image)

**FIG. 3.** Evolution of \( c_{s,\phi}^2 \) versus \( x \) for different coupling parameter \( b \) for interaction term \( \mathcal{Q} = 3b\dot{\theta}\dot{\phi}/2\sqrt{6} \). Here we take \( q = 1 \) and \( r = 2.05 \). The solid, dashed and dotted curves correspond to \( b = 2.6, b = 3 \) and \( b = 3.6 \) respectively. In the first two cases, \( c_{s,\phi}^2 \) is finite and positive. In the last case, \( c_{s,\phi}^2 \) diverges during its evolution, implying the instability of the relevant perturbation modes.

**Case 3 \( \tilde{b} \neq 0 \) but \( c = b = f = 0 \):**

In this case the interaction between the quintessence scalar and displacement field is \( \mathcal{Q} = \frac{9}{2}\tilde{b}H\theta \). We choose Eqs. (14), (16) and (17) to form the set of independent equations. The latter two can be recast as,

\[
H^2 \left[ \frac{d^2 \varphi}{dx^2} + \left( 3 + \frac{1}{2} \frac{dH^2}{dx} \right) \frac{d\varphi}{dx} \right] + \frac{dV}{dx} - \frac{9}{2} \tilde{b}\theta = 0 \quad (51)
\]

\[
\frac{d\theta}{dx} + 6(1 - \tilde{b})\theta = 0 \quad (52)
\]
among which, Eq. (52) is easily to solve. By assigning the initial condition \( \theta(0) = 4(\tilde{b} - 1)sH_0^2 \), with \( s \) a dimensionless constant, we have the solution of Eq. (52) as follow,

\[
\theta(x) = 4(\tilde{b} - 1)sH_0^2e^{6(\tilde{b} - 1)x} \quad (53)
\]

We further assume that in the models under consideration the self-interaction potential of the quintessence scalar possesses property,

\[
\frac{dV}{dx} - \frac{9}{2}b\theta = 0 \quad (54)
\]

Under this assumption, Eq. (51) is reduced to,

\[
\frac{d^2\varphi}{dx^2} + \left( 3 + \frac{1}{2H^2} \frac{dH^2}{dx} \right) \frac{d\varphi}{dx} = 0 \quad (55)
\]

Substitution of Eq. (53) into (54) yields,

\[
V(x) = H_0^2 \left[ r + 3\tilde{b}s(e^{6(\tilde{b} - 1)x} - 1) \right] \quad (56)
\]

where \( r \) is an integration constant. As in the previous cases, this parameter characterizes the present-epoch value of the quintessence potential. With Eq. (56), we can obtain the evolution of the Hubble parameter by solving Eq. (14). The result is,

\[
H^2 = H_0^2 \left[ (1 + \tilde{b}s - s)e^{-6x} + se^{6(\tilde{b} - 1)x} - \tilde{b}s \right] + \frac{r}{3}H_0^2(1 - e^{-6x}) \quad (57)
\]

Plugging Eq. (57) into (55) gives,

\[
\frac{d\varphi}{dx} = \frac{q}{\sqrt{3 + (r - 3\tilde{b}s)(e^{6x} - 1) + 3s(e^{6bx} - 1)}} \quad (58)
\]

where the integration constant \( q \) is related to the initial ”velocity” of the quintessence scalar by \( d\varphi(0)/dx = q/\sqrt{3} \). The cosmology of the considered models at the background level is described by the following effective EoS parameter:

\[
\omega = 1 - 2e^{6x} \left[ r + 3\tilde{b}s(1 - e^{6(\tilde{b} - 1)x}) \right] \left[ 3 + (3\tilde{b}s - r)(1 - e^{6x}) - 3s(1 - e^{6bx}) \right] \quad (59)
\]

The EoS parameter seems not to depend upon the choice of the initial ”velocity” \( q \) of the quintessence scalar, but upon the initial value \( s \) of the displacement field instead. This is, however, merely an optical illusion. The consistence of the above results with Eq. (15) requires,

\[
(\tilde{b} - 1)s = \frac{q^2 + 6(r - 3)}{18} \quad (60)
\]
The present-epoch value of EoS parameter is still given by the same formula as either Eq. (31) or (42), which depends only upon the parameter \( r \),

\[
\omega_0 = 1 - \frac{2r}{3} \quad (61)
\]

Provided \( r > 2 \), the universe is destined to enter a late-time acceleration phase.

To examine the stability of the relevant perturbation modes, we calculate the squared sound speed of the quintessence scalar. The result is,

\[
c_{s,\phi}^2 = \frac{3 - r + 3(\tilde{b} - 1)s + 3\tilde{b}(\tilde{b} - 1)se^{6\lambda}}{3 - r + 3(\tilde{b} - 1)s - 3\tilde{b}(\tilde{b} - 1)se^{6\lambda}} \quad (62)
\]

As long as \((\tilde{b} - 1)s \neq 0\), \( c_{s,\phi}^2 \) either diverges or becomes negative during its evolution, which will give rise to the unstable and then the unacceptable perturbation modes. On the other hand, when \((\tilde{b} - 1)s = 0\), which occurs for either \( \tilde{b} = 1 \) or \( s = 0 \), \( c_{s,\phi}^2 = 1 \), the relevant perturbation modes might evolve stably. For \( \tilde{b} = 1 \), the solution to the background dynamics reduces to:

\[
\varphi(x) = -\sqrt{\frac{2}{3}} \tanh^{-1} \sqrt{\frac{3 - r + re^{6\lambda}}{3 - r}} \quad (63)
\]

\[
V(x) = rH_0^2 \quad (64)
\]

\[
H^2 = \frac{1}{3}H_0^2 \left[ r + (3 - r)e^{-6\lambda} \right] \quad (65)
\]

\[
\theta(x) = 0 \quad (66)
\]

and in particular,

\[
\omega = -1 + \frac{2(3 - r)}{3 - r + re^{6\lambda}} \quad (67)
\]

In this case, the displacement field is effectively absent, and the self-interaction potential of the quintessence scalar plays the role of the cosmological constant. The potential parameter \( r \) must be restricted to the region \( 2 < r \leq 3 \), otherwise \( \omega \) will diverge during its evolution. Consequently, crossing the phantom divide \( \omega = -1 \) in the present model is practically impossible. Fig. 4. shows the evolution of EoS parameter in Eq. (67) for three choices of the potential parameter \( r \), i.e., \( r = 2.6 \), \( r = 2.9 \) and \( r = 3 \), with the coupling constant fixed at \( \tilde{b} = 1 \). It appears that the accelerated expansion in these models will last for a very long time. Besides, the larger the parameter \( r \) is, the more closely the quintessence scalar resembles the cosmological constant.
FIG. 4. Evolution of $\omega$ versus $x$ for different potential parameter $r$ if the interaction term is $\mathcal{Q} = \frac{9}{2} bH\theta$ with $\tilde{b} = 1$. The solid, dashed and dotted curves correspond to $r = 2.6$, $r = 2.9$ and $r = 3$, respectively.

Case 4 $f \neq 0$ but $b = \tilde{b} = c = 0$:

In this case, the interaction between the quintessence scalar and the displacement field is assumed to be proportional to the quintessence self-interaction potential, $\mathcal{Q} = 6fHV(\varphi)$. Under such an assumption, Eqs.(16) and (17) become,

$$
\frac{d\varphi}{dx} \left[ H^2 \frac{d^2 \varphi}{dx^2} + \left( 3H^2 + \frac{1}{2} \frac{dH^2}{dx} \right) \frac{d\varphi}{dx} \right] + \frac{dV}{dx} - 6fV = 0 \quad (68)
$$

$$
\frac{d\theta}{dx} + 6\theta - 8fV = 0 \quad (69)
$$

As before, we further assume that the quintessence potential is defined by condition,

$$
\frac{dV}{dx} - 6fV = 0 \quad (70)
$$

This implies that the quintessence self-interaction potential in the considered models is of the form,

$$
V(x) = r H_0^2 e^{6fx} \quad (71)
$$

where a real parameter $r$ is used to characterize the present-epoch value of the potential and $H_0$ stands for the present value of the Hubble parameter. Substitution of Eq.(71) into Eq.(14) yields,

$$
H^2 = H_0^2 e^{-6x} \left[ 1 + \frac{r}{3(f+1)} (e^{6(f+1)x} - 1) \right] \quad (72)
$$
Furthermore, we can obtain the evolution of the quintessence scalar by plugging Eqs. (71) and (72) into Eq. (68),

\[
\varphi(x) = \frac{q}{\sqrt{3(f + 1)(3 + 3f - r)}} \left[ \tanh^{-1} \sqrt{\frac{3(f + 1)}{3 + 3f - r}} - \tanh^{-1} \sqrt{\frac{3(f + 1) + r(e^{6(f+1)x} - 1)}{3 + 3f - r}} \right]
\]

(73)

where the parameter \(q\) is the integration constant which can be interpreted as the initial velocity of the quintessence scalar, \(d\varphi(0)/dx = q\). Finally, the evolution of the displacement field is obtained from Eqs. (15), (71), (72) and (73),

\[
\theta(x) = \frac{2}{3} H_0^2 e^{-6x} \left[ q^2 - 6 + 2r \frac{f e^{6(f+1)x} + 1}{f + 1} \right]
\]

(74)

The effective EoS parameter of the quintessence scalar and the displacement field in the models under consideration reads,

\[
\omega = 1 - \frac{2(f + 1) r e^{6(f+1)x}}{3(f + 1) + r(e^{6(f+1)x} - 1)}
\]

(75)

The present-epoch EoS parameter takes the same formula as those in the previous three cases,

\[
\omega_0 = 1 - \frac{2r}{3}
\]

(76)

So the late-time cosmological acceleration is available in these models if \(r > 2\). Different from the EoS parameter in the present case which depends only upon the coupling constant \(f\) and the potential parameter \(r\), the squared sound speed of the quintessence scalar depends also upon the initial velocity \(q\) of the quintessence scalar,

\[
c_{s,\varphi}^2 = \frac{q^2 + 2r e^{6(f+1)x}}{q^2 - 2r e^{6(f+1)x}}
\]

(77)

Stability condition \(c_{s,\varphi}^2 \geq 0\) requires \(f = 0\) and \(q \neq 0\). Therefore, the model is physically acceptable only if there is no interaction between the quintessence scalar and the displacement field. When \(f = 0\), \(c_{s,\varphi}^2 = 1\), the solution to the background dynamics reduces to:

\[
\frac{d\varphi}{dx} = \frac{\sqrt{3}q}{\sqrt{3 - r + re^{6x}}}
\]

\[
V(x) = r H_0^2
\]

\[
H^2 = \frac{1}{3} H_0^2 \left[ r + (3 - r) e^{-6x} \right]
\]

\[
\theta(x) = \frac{2}{3} H_0^2 e^{-6x} (q^2 - 2r - 6)
\]
Because of $\theta(x) \geq 0$, the value of parameter $q$ is restricted to either $q \geq \sqrt{2(3-r)}$ or $q \leq -\sqrt{2(3-r)}$. For $f = 0$, Eq.(75) reduces to:

$$\omega = 1 - \frac{2r e^{6x}}{3 - r + r e^{6x}}$$

(82)

To have $\omega < -1/3$ at $x = 0$, $r > 2$. To avoid the possible divergence of $\omega$ during its evolution, $r \leq 3$. FIG. 5. illustrates the dependence of the EoS parameter in Eq.(82) upon the potential parameter $r$. Obviously, $-1 \leq \omega \leq 1$, crossing the phantom divide $\omega = -1$ is prohibited once more in the quintessence model in Lyra’s geometry.

![Graph](image)

FIG. 5. Evolution of $\omega$ versus $x$ for different potential parameter $r$ if there is no interaction between the quintessence scalar and the displacement field. The solid, dashed and dotted curves correspond to $r = 2.1$, $r = 2.9$ and $r = 3$, respectively.

CONCLUSION

In this work, we have established four classes of the quintessence models in ETG in Lyra’s geometry. The classification of these models depends upon how the quintessence scalar $\phi$ interacts with the geometrical displacement field $\theta$ (or $\beta_\mu$). The mixture of interacting quintessence scalar and the displacement field supplies as a cosmological perfect fluid which can cause late time accelerated expansion of our universe. Owing to the subtle choices of the quintessence self-interaction potential, all quantities relevant to the study of
cosmology, including the Hubble parameter, the displacement field, the time derivatives of the quintessence scalar and the potential itself, are expressed as closed form functions of e-folding time $x = \ln(a)$, and so is the effective EoS parameter $\omega$ of the mixed fluid. The evolution of $\omega$ is different for the quintessence models of different classes, which depends also upon the present values of the time derivative of quintessence scalar, its potential and what the coupling constant is. However, today’s $\omega$ does only depend upon the present-epoch value of the quintessence potential. The appearance of the displacement field in Lyra’s geometry improves greatly the late-time evolution of the quintessence-displacement mixture, however, crossing the phantom divide $\omega = -1$ is still forbidden in these models by the necessary condition $c_{s,\varphi}^2 \geq 0$ to ensure the stability of the cosmological perturbations. Establishing a reliable quintom scenario remains a challenge in ETG in Lyra’s geometry.

In the proposed quintessence models in ETG in Lyra’s geometry, we have defined the quintessence scalar by making some careful choices for its self-interaction potential and interaction terms with the geometrical displacement. These choices, in this paper, are mainly motivated by the requirement to obtain the analytical solutions of the modified Friedmann equations. More important issue that remains unsolved is to investigate the physics behind these choices. Why the EoS parameter of the quintessence scalar at present epoch depends only upon the parameter $r$ for all four kinds of models is also a mystery. In addition, the characteristic behaviour that the effective EoS parameter $\omega$ decrease from $\omega \approx 1$ in the past conflicts with the well-known Big-Bang diagram. In our $\omega \sim x$ figures the dust-dominant phase does not form an expected plateau. To be more realistic, the pressureless cold dark matter components have to be introduced into the model buildings also.

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