Switching \( m \)-edge-coloured graphs using non-Abelian groups

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Abstract

Let \( G \) be a graph whose edges are each assigned one of the \( m \)-colours \( 1, 2, \ldots, m \), and let \( \Gamma \) be a subgroup of \( S_m \). The operation of switching at a vertex \( x \) with respect to \( \pi \in \Gamma \) permutes the colours of the edges incident with \( x \) according to \( \pi \). There is a well-developed theory of switching when \( \Gamma \) is Abelian. Much less is known for non-Abelian groups. In this paper we consider switching with respect to non-Abelian groups including symmetric, alternating and dihedral groups. We first consider the question of whether there is a sequence of switches using elements of \( \Gamma \) that transforms an \( m \)-edge-coloured graph \( G \) to an \( m \)-edge coloured graph \( H \). Necessary and sufficient conditions for the existence of such a sequence are given for each of the groups being considered. We then consider the question of whether an \( m \)-edge coloured graph can be switched using elements of \( \Gamma \) so that the transformed \( m \)-edge coloured graph has a vertex \( k \)-colouring, or a homomorphism to a fixed \( m \)-edge coloured graph \( H \). For the groups just mentioned we establish dichotomy theorems for the complexity of these decision problems. These are the first dichotomy theorems to be established for colouring or homomorphism problems and switching with respect to any group other than \( S_2 \).

1 Introduction and Definitions

An \( m \)-edge-coloured graph is an ordered pair \( G = (H, \Sigma) \), where \( H \) is a graph and \( \Sigma : E(H) \to \{1, 2, \ldots, m\} \) is its signature. The graph \( H \) is the underlying graph of \( G \), and may also be referred to as \( \text{underlying}(G) \). The vertices of \( G \) are the vertices of \( H \). The edges of \( G \) are coloured edges of \( H \), that is, edges \( e \in E(H) \) together with their signature (or colour) \( \Sigma(e) \). We use \( E_i(G) \) to denote the set of edges of \( G \) with colour \( i \). An \( m \)-edge-coloured graph is monochromatic of colour \( j \) if all of its edges have colour \( j \).

Let \( G \) be an \( m \)-edge-coloured graph and let \( \Gamma \) be a subgroup of \( S_m \). For \( x \in V \) and \( \pi \in \Gamma \), the operation of switching at \( x \) with respect to \( \pi \) transforms \( G \) into the \( m \)-edge-coloured graph \( G^{(x, \pi)} \) that has the same underlying graph as \( G \) and with the colours of the edges incident with \( x \) permuted according to \( \pi \), that is, if \( \Sigma(G)(xy) = i \), then \( \Sigma(G^{(x, \pi)})(xy) = \pi(i) \).
Let \( S = (x_1, \pi_1), (x_2, \pi_2), \ldots, (x_t, \pi_t) \) be a sequence of elements of \( V(G) \times \Gamma \). Recursively define

\[
G^S = G^{(x_1, \pi_1)}, (x_2, \pi_2), \ldots, (x_t, \pi_t) = \left( G^{(x_1, \pi_1)} \right)^{(x_2, \pi_2)} \ldots (x_t, \pi_t)
\]

We call the sequence \( S \) a \( \Gamma \)-switching sequence, and say that it transforms \( G \) into \( G^S \). Two \( m \)-edge-coloured graphs \( G \) and \( H \) are called \( \Gamma \)-switch equivalent if there exists a \( \Gamma \)-switching sequence \( S \) such that \( G^S \cong H \). In other words, \( G \) and \( H \) are \( \Gamma \)-switch equivalent if there exists a \( \Gamma \)-switching sequence that transforms \( G \) into an \( m \)-edge coloured graph that is isomorphic to \( H \). It is easy to see that \( \Gamma \)-switch equivalence defines an equivalence relation on the set of all \( m \)-edge coloured graphs. The equivalence class of the \( m \)-edge-coloured graph \( G \) is denoted by \( [G]_{\Gamma} \).

Switching 2-edge coloured graphs with respect to \( S_2 \) first appears in the work of Abelson and Rosenberg in the context of behavioural science [1]. Switching 2-edge coloured graphs in which the colours are \( \{+1, -1\} \) is integral to the study of signed graphs. These are different than 2-edge coloured graphs because the product of colours on each cycle is invariant under switching, which leads to the fundamental concept of balance of a cycle. Signed graphs have been extensively studied by Zaslavsky; for example see [13, 14]. The related concept of pushing vertices in oriented graphs is considered in [8]. Switching \( m \)-edge coloured graphs with respect to cyclic groups was first studied by Brewster and Graves [2]. Their results are extended to all Abelian groups in [9].

After noting some preliminary information, in the first part of this paper we consider the question of when two \( m \)-edge-coloured graphs \( G \) and \( H \) are \( \Gamma \)-switch equivalent when \( \Gamma \) is a symmetric, alternating or dihedral group, or belongs to a family of other groups. In each case we give necessary and sufficient conditions for two \( m \)-edge coloured graphs \( G \) and \( H \) to be \( \Gamma \)-switch equivalent. We believe these to be the first results on switch equivalence with respect to non-Abelian groups. For an Abelian group \( \Gamma \) and an \( m \)-edge coloured graph \( G \) there is a graph \( P_\Gamma(G) \) such that \( G \) and \( H \) are switch equivalent if and only if \( P_\Gamma(G) \cong P_\Gamma(H) \) [9] (also see [2, 11], and [8] for similar results in the context of oriented graphs), if and only if \( H \) is a special type of subgraph of \( P_\Gamma(G) \) [9] (the result is implicit in [2]). No similar results are known to hold when \( \Gamma \) is non-Abelian. For switching with respect to \( S_2 \), another different necessary and sufficient condition has been given by Zaslavsky (see Corollary 3.5, where a similar condition is shown to hold for \( D_m \), \( m \) even).

In the last part of the paper we consider colourings and homomorphisms of \( m \)-edge coloured graphs. We are interested in the complexity of deciding whether a given \( m \)-edge coloured graph \( G \) can be switched so it has a vertex \( k \)-colouring or a homomorphism to a fixed \( m \)-edge coloured graph \( H \). We are able to give dichotomy theorems for these problems with respect to the groups we consider. A dichotomy theorem for \( \Gamma \)-switchable \( k \)-colouring when \( \Gamma \) is Abelian appears in [9]. Kidner has proved that for all groups \( \Gamma \) the problem of deciding whether an \( m \)-edge coloured graph \( G \) has a \( \Gamma \)-switchable \( k \)-colouring is solvable in polynomial time when \( k \leq 2 \) and is NP-hard when \( k \geq 3 \) [7] (also see [11, 9]). The dichotomy theorems for the homomorphism problem generalize the fundamental result of Hell and Nešetřil, and the dichotomy theorem for \( S_2 \) switchable homomorphism due to Brewster et al. [3]. Related
results for oriented graphs appear in \[8\].

We note that, for any group \(\Gamma\), two \(m\)-edge-coloured graphs which are both monochromatic of colour \(j\) are \(\Gamma\)-switch equivalent if and only if their underlying graphs are isomorphic. Hence deciding whether two \(m\)-edge coloured graphs are switch equivalent is at least as hard as deciding whether they are isomorphic.

We now describe a way to determine whether two \(m\)-edge coloured graphs \(G\) and \(H\) are \(\Gamma\)-switch equivalent with respect to any group \(\Gamma\). One can construct an auxiliary graph with vertex set equal to the set of all \(\frac{m!}{|\text{Aut}(G)| |E(G)|^m}\) labelled \(m\)-edge coloured graphs, and an edge from \(F\) to \(F'\) if there exists a vertex \(x\) of \(F\) and a permutation \(\pi\) \(\in\ \Gamma\) such that \(F^{(x,\pi)} = F'\). Two (labelled) \(m\)-edge coloured graphs are \(\Gamma\)-switch equivalent if and only if they belong to the same component of the auxiliary graph. Determining whether \(G\) and \(H\) are \(\Gamma\)-switch equivalent using this procedure involves considering \(\Gamma\)-switching sequences of length at most \(\frac{m!}{|\text{Aut}(G)| |E(G)|^m}\).

Suppose \(\Gamma\) is Abelian. Then the same transformed graph arises from any rearrangement of a given \(\Gamma\)-switching sequence. Since there is a rearrangement so that the switches at each vertex occur consecutively, and the result of switching with respect to \((x, \alpha_1), (x, \alpha_2), \ldots, (x, \alpha_k)\) is the same as the result of switching with respect to \((x, \alpha_1\alpha_2\ldots \alpha_k)\), it suffices to consider \(\Gamma\)-switching sequences in which there is at most one switch at each vertex. Such a sequence has length at most \(|V(G)|\).

To see that the order of switches can matter when \(\Gamma\) is non-Abelian, consider a 3-edge-coloured graph \(G\), the group \(\Gamma = S_3\), and an edge \(xy\) of colour 1. Let \(\alpha = (1 2)\) and \(\beta = (2 3)\). For the switching sequence \(S = (x, \alpha)(y, \beta)(x, \alpha^{-1})\) we have \(\Sigma(G^S)(xy) = 3\), whereas for the switching sequence \(S' = (x, \alpha)(x, \alpha^{-1})(y, \beta)\) we have \(\Sigma(G^{S'})(xy) = 1\).

## 2 Property \(T_j\)

In this section we introduce a group property which we call property \(T_j\). If the group \(\Gamma\) has property \(T_j\) then for any edge \(xy\) in an \(m\)-edge coloured graph \(G\) there exists a \(\Gamma\)-switching sequence such that \(xy\) is of colour \(j\), and the colour of every other edge of \(G\) is unchanged. It follows that \(G\) can be transformed to be monochromatic of colour \(j\) by changing the colour of one edge at a time.

We motivate the definition of property \(T_j\) by considering switching with respect to \(S_m\), \(m \geq 3\).

**Proposition 2.1.** Let \(G\) be a \(m\)-edge-coloured graph, where \(m \geq 3\), and let \(i, j \in \{1, 2, \ldots, m\}\) be such that \(i \neq j\). If \(xy \in E_i(G)\), then there exists \(G' \in [G]_{S_m}\) such that \(xy \in E_j(G')\) and \(G - xy = G' - xy\).

*Proof.* Since \(m \geq 3\), for any \(k \in \{1, 2, \ldots, m\}\) there exists a transposition \((i j)\) with \(i, j \in \{1, 2, \ldots, m\} \setminus \{k\}\). Let \(\alpha = (i j)\) and \(\beta = (j k)\). Consider the \(S_m\)-switching sequence \((x, \alpha), \ldots, (x, \alpha^{m-1})\). Then for any \(k \neq i, j\) we have \(\Sigma(G^{m!(x, \alpha)^k})(xy) = 3\), whereas for \(k = i\) we have \(\Sigma(G^{m!(x, \alpha)^k})(xy) = 1\). Hence, \(G - xy = G' - xy\), where \(G' = G^{m!(x, \alpha)^k}\).\(\square\)
\((y, \beta), (x, \alpha), (y, \beta)\). This transforms \(G\) into \(G'\).

The only edges which change colour in the transformation are incident with \(x\) or \(y\). It is given that the edge \(xy\) has colour \(i\) in \(G\). After the first, second, third and fourth switch, the edge \(xy\) has colour \(j, k, k, j\), respectively, in the transformed graph. Any edge \(e\) incident with \(x\) and not \(y\) changes from its colour, \(c_e\), to \(\alpha(c_e)\) and then back to \(c_e\). Similarly, any edge incident with \(y\) has the same colour as in \(G\) after switching. The result follows.

**Corollary 2.2.** For \(m \geq 3\), two \(m\)-edge-coloured graphs \(G\) and \(G'\) are \(S_m\)-switch equivalent if and only if \(\text{underlying}(G) \cong \text{underlying}(G')\).

Let \(\Gamma\) be a subgroup of \(S_m\). For \(i, j \in \{1, 2, \ldots, m\}\), we say \(\Gamma\) has property \(T_{i,j}\) if there exist permutations \(\alpha, \beta \in \Gamma\) such that \(\alpha\) maps \(i\) to \(j\) and fixes some element \(k\), and \(\beta\) maps \(j\) to \(k\).

**Proposition 2.3.** Let \(\Gamma\) be a subgroup of \(S_m\) with property \(T_{i,j}\). If \(G\) is a \(m\)-edge-coloured graph, where \(m \geq 3\), and \(xy \in E_i(G)\), then there exists \(G' \in [G]_\Gamma\) such that \(xy \in E_j(G')\) and \(G - xy = G' - xy\).

**Proof.** The switching sequence \((x, \alpha), (y, \beta), (x, \alpha^{-1}), (y, \beta^{-1})\) transforms \(G\) to \(G'\). \(\square\)

If there exists \(j \in \{1, 2, \ldots, m\}\) such that \(\Gamma\) has Property \(T_{i,j}\) for all \(i \in \{1, 2, \ldots, m\}\), then we say it has Property \(\mathcal{T}\).

**Corollary 2.4.** Let \(\Gamma\) be a subgroup of \(S_m\) with property \(T_j\). Then for any \(m\)-edge coloured graph \(G\) and any colour \(j\) there exists a switching sequence \(S\) such that \(G^S\) is monochromatic of colour \(j\).

**Corollary 2.5.** Let \(\Gamma\) be a subgroup of \(S_m\) with property \(T_j\). Then two \(m\)-edge-coloured graphs \(G_1\) and \(G_2\) are \(\Gamma\)-switch equivalent if and only if \(\text{underlying}(G_1) \cong \text{underlying}(G_2)\).

**Proof.** By definition, two \(m\)-edge coloured graphs which are \(\Gamma\)-switch equivalent have isomorphic underlying graphs.

Now suppose \(\text{underlying}(G_1) \cong \text{underlying}(G_2)\). By Corollary 2.4, both \(G_1\) and \(G_2\) are \(\Gamma\)-switch equivalent to an \(m\)-edge-coloured graph which is monochromatic of colour \(j\). Since \(\Gamma\)-switch equivalence is an equivalence relation, \(G_1\) and \(G_2\) are \(\Gamma\)-switch equivalent. \(\square\)

It is easy to see that when \(m \geq 4\) the alternating group \(A_m\) has property \(T_j\).

**Corollary 2.6.** For \(m \geq 4\), two \(m\)-edge-coloured graphs \(G\) and \(G'\) are \(A_m\)-switch equivalent if and only if \(\text{underlying}(G) \cong \text{underlying}(G')\).
3 The dihedral group

For $m \geq 3$ we denote by $D_m$ the group of permutations of $\{1, 2, \ldots, m\}$ corresponding to symmetries of the regular $m$-gon with vertices $1, 2, \ldots, m$ in cyclic order. The cases $m$ odd and $m$ even are different. We consider the case of odd $m$ first.

Proposition 3.1. For any odd integer $m \geq 3$ and any $j \in \{1, 2, \ldots, m\}$, the group $D_m$ has property $T_j$.

Proof. Let $i, j \in \{1, 2, \ldots, m\}$. We show $D_m$ has property $T_{i,j}$. There is nothing to prove if $i = j$, so assume $i \neq j$.

Since $m$ is odd, either the least residue of $i - j$ modulo $m$ is even, or the least residue of $j - i$ modulo $m$ is even. Without loss of generality, the latter holds. Then there exists $k \in \{1, 2, \ldots, m-1\}$ such that $j - i \equiv 2k \pmod{m}$, so that $j - k \equiv k - i \pmod{m}$. Let $\alpha$ be the permutation of $\{1, 2, \ldots, m\}$ which corresponds to flipping the $m$-gon over while fixing vertex $k$. Then $\alpha$ maps $i$ to $j$ and fixes $k$, so $D_m$ has property $T_{i,j}$. This completes the proof. \qed

Corollary 3.2. For odd $m \geq 3$, two $m$-edge-coloured graphs $G$ and $G'$ are $D_m$-switch equivalent if and only if underlying $(G) \cong$ underlying $(G')$.

We now consider the case of switching with respect to $D_m$ when $m$ is even. The following basic facts from group theory will be used.

Observation 3.3. Suppose $m \geq 2$ is even. Let $\mathcal{E} = \{2, 4, \ldots, m\}$ and $\mathcal{O} = \{1, 3, \ldots, m-1\}$. Then,

1. $\{\mathcal{O}, \mathcal{E}\}$ is a block system for the action of $D_m$ on $\{1, 2, \ldots, m\}$.
2. $\text{Stabilizer}(\mathcal{E}) = \text{Stabilizer}(\mathcal{O})$ is a normal subgroup of $D_m$,
3. $D_m/\text{Stabilizer}(\mathcal{E}) \cong D_m/\text{Stabilizer}(\mathcal{O}) \cong S_2$, and
4. $\text{Stabilizer}(\mathcal{E})$ has Property $T_{i,j}$ for all $i, j \in \mathcal{E}$, and
5. $\text{Stabilizer}(\mathcal{O})$ has Property $T_{i,j}$ for all $i, j \in \mathcal{O}$.

Let $G$ be an $m$-edge-coloured graph, where $m \geq 2$ is an even integer. The 2-edge-coloured graph $G_2$ is obtained from $\text{underlying}(G)$ by assigning each edge $e$ colour 1 if $\Sigma(G)(e) \in \mathcal{O}$, and colour 2 if $\Sigma(G)(e) \in \mathcal{E}$. Notice that this is equivalent to regarding the edge colours of $G_2$ to be $\mathcal{E}$ and $\mathcal{O}$, with the colour of an edge of $G_2$ being the name of the block containing the colour of the corresponding edge in $G$. The colours of the edges of $G_2$ are naturally permuted by $D_m/\text{Stabilizer}(\mathcal{E}) \cong S_2$. 

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Theorem 3.4. Let $G$ and $H$ be $m$-edge-coloured graphs, where $m \geq 2$ is an even integer. Then $G$ and $H$ are switch equivalent with respect to $D_m$ if and only if $G_2$ and $H_2$ are switch equivalent with respect to $S_2$.

Proof. Suppose $G$ and $H$ are switch equivalent with respect to $D_m$. Then there is a $D_m$-switching sequence $S = (x_1, \pi_1), (x_2, \pi_2), \ldots, (x_t, \pi_t)$ that transforms $G$ to $H$. By Observation 3.3, each permutation $\pi_i \in D_m$ either maps $E$ to $E$ and $O$ to $O$, or maps $E$ to $O$ and vice-versa. Let $S'$ be the subsequence of $S$ consisting of the permutations that map $E$ to $O$. Replacing each of the permutations in this subsequence by the transposition $(1 \ 2)$ gives a $S_2$-switching sequence that transforms $G_2$ so it is isomorphic to $H_2$.

Now suppose $G_2$ and $H_2$ are switch equivalent with respect to $S_2$. Then, underlying($G_2$) $\cong$ underlying($H_2$). Without loss of generality, underlying($G_2$) $= $ underlying($H_2$). Let $A = (x_1, \sigma_1), (x_2, \sigma_2), \ldots, (x_p, \sigma_p)$ be a $S_2$-switching sequence that transforms $G_2$ to $H_2$. Replacing each permutation $\sigma_i \in S_2$ by the $m$-cycle $(1 \ 2 \ \cdots \ m) \in D_m$ gives a $D_m$-switching sequence that transforms $G$ to a graph $G'$ in which the colour of edge belongs to the same block as the corresponding edge $G$. Since Stabilizer($E$) has Property $T_j$ for all $j \in E$, and Stabilizer($O$) has Property $T_j$ for all $j \in O$, the $m$-edge-coloured graph $G'$ is $D_m$-switch equivalent to $H$ (as in the proof of Proposition 3.3 edges other then the one whose colour is intended to change switches from their colour then back again).

Zaslavsky proved that the 2-edge-coloured graphs $G$ and $H$ with the same underlying graph are switch equivalent with respect to $S_2$ if and only if they have the same collection of cycles for which the number of edges whose colour is in $E_2$ is odd [13]. Together with Theorem 3.4, this yields a similar result for $D_m$, where $m \geq 2$ is even.

Corollary 3.5. Suppose $m \geq 2$ is even. Two $m$-edge-coloured graphs $G$ and $H$ with the same underlying graph are switch equivalent with respect to $D_m$ if and only if they have the same collection of cycles for which the number of edges whose colour is in $E$ is odd.

4 Colourings and Homomorphisms

Recall that a $k$-colouring of a graph $G$ is a function $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that if $xy \in E(G)$ then $c(x) \neq c(y)$. A homomorphism from a graph $G$ to a graph $H$ is a function $f : V(G) \rightarrow V(H)$ such that if $xy \in E(G)$, then $f(x)f(y) \in E(H)$.

Our goal in this section is to present analogues of Theorems 4.1 and 4.2 below for $\Gamma$-switchable colourings and homomorphisms when $\Gamma$ is a group with property $T_j$ for some $j \in \{1, 2, \ldots, m\}$, or an even order dihedral group. Theorems such as these are known as dichotomy theorems because they exhibit a dichotomy for the complexity of a particular decision problem.

Theorem 4.1 ([5]). For an integer $k \geq 1$, the problem of deciding whether a given graph $G$ has a $k$-colouring is solvable in polynomial time when $k \leq 2$, and is NP-complete if $k \geq 3$. 


Theorem 4.2 ([6]). If \( H \) is a fixed graph then the problem of deciding whether a given graph \( G \) has a homomorphism to \( H \) is solvable in polynomial time if \( H \) is bipartite, and is NP-complete if \( H \) is not bipartite.

Before introducing colourings and homomorphisms of \( m \)-edge coloured graphs, we review a connection between these concepts for graphs (without loops). Suppose there is a homomorphism, \( f \), of the graph \( G \) to a graph \( H \) on \( k \) vertices. If the vertices of \( H \) are regarded as colours, then \( f \) is an assignment of these colours to the vertices of \( G \) such that adjacent vertices in \( G \) are assigned adjacent (hence different) colours. Thus a \( k \)-colouring of a graph \( G \) can equivalently be defined as a homomorphism of \( G \) to some graph \( H \) on \( k \) vertices. Defining a \( k \)-colouring in this way allows the idea of a (vertex) \( k \)-colouring to be extended to \( m \)-edge coloured graphs (see [10]), oriented graphs [12], and other types of graphs (see [10]).

Let \( G \) and \( H \) are \( m \)-edge coloured graphs. A homomorphism of \( G \) to \( H \) is a function \( f : V(G) \to V(H) \) such that, for all \( i \in \{1, 2, \ldots, m\} \), if \( xy \in E_i(G) \) then \( f(x)f(y) \in E_i(H) \). For an integer \( k \geq 1 \), a vertex \( k \)-colouring of an \( m \)-edge coloured graph \( G \) is a homomorphism of \( G \) to some \( m \)-edge coloured graph on \( k \) vertices.

Let \( \Gamma \) be a subgroup of \( S_m \). A \( m \)-edge coloured graph \( G \) has a \( \Gamma \)-switchable homomorphism to an \( m \)-edge coloured graph \( H \) if some \( G' \in [G]_\Gamma \) has a homomorphism to \( H \), that is, if \( G \) can be \( \Gamma \)-switched so that the transformed graph has a homomorphism to \( H \). For an integer \( k \geq 1 \), an \( m \)-edge coloured graph \( G \) has a \( \Gamma \)-switchable \( k \)-colouring if it has a \( \Gamma \)-switchable homomorphism to some \( m \)-edge coloured graph on \( k \) vertices.

If follows from the definition that if there exists a \( \Gamma \)-switchable homomorphism of an \( m \)-edge coloured graph \( G \) to an \( m \)-edge coloured graph \( H \), then there is a homomorphism from underlying\((G)\) to underlying\((H)\). To see that the converse is false, let \( \Gamma = S_2 \), let \( G \) be the 2-edge coloured \( K_3 \) with two edges of colour 1 and one edge of colour 2 and let \( H \) be the 2-edge coloured \( K_3 \) with two edges of colour 2 and one edge of colour 1. There is a homomorphism of underlying\((G)\) to underlying\((H)\) but no \( S_2 \)-switchable homomorphism of \( G \) to \( H \).

The following theorem from [9] is useful because it transforms the problem of deciding whether there is a \( \Gamma \)-switchable homomorphism of \( G \) to \( H \) to the problem of deciding the existence of a homomorphism (with no switching) to any \( m \)-edge coloured graph \( \Gamma \) switch equivalent to \( H \).

Theorem 4.3 ([9]). Let \( G \) and \( H \) be \( m \)-edge coloured graphs and let \( \Gamma \) be a subgroup of \( S_m \). Then there is a \( \Gamma \)-switchable homomorphism of \( G \) to \( H \) if and only if, for all \( H' \in [H]_\Gamma \) there exists \( G' \in [G]_\Gamma \) such that there is a homomorphism of \( G' \) to \( H' \).

Theorem 4.4. Let \( \Gamma \) be a subgroup of \( S_m \) that has Property \( T_j \) for some \( j \in \{1, 2, \ldots, m\} \), and let \( k \geq 1 \) be an integer. If \( k \leq 2 \) then the problem of deciding whether a given \( m \)-edge coloured graph has a \( \Gamma \)-switchable \( k \)-colouring is solvable in polynomial time. If \( k \geq 3 \) then the problem of deciding whether a given \( m \)-edge coloured graph has a \( \Gamma \)-switchable \( k \)-colouring is NP-complete.
Proof. By Corollary 2.5 every $m$-edge-coloured graph $F$ is $\Gamma$-switch equivalent to an $m$-edge-coloured graph $F'$ which is monochromatic of colour $j$. Thus by Theorem 4.3 there is a $\Gamma$-switchable homomorphism of $G$ to an $m$-edge-coloured graph $H$ on $k$ vertices if and only if there is a homomorphism of $G'$ to $H'$, if and only if there is a homomorphism of $\text{underlying}(G)$ to $\text{underlying}(H)$, if and only if $\text{underlying}(G)$ has a $k$-colouring. The result now follows from from Theorem 4.1.

The proof of the corresponding result for $\Gamma$-switchable homomorphisms is virtually identical.

**Theorem 4.5.** Let $\Gamma$ be a subgroup of $S_m$ that has Property $T_j$ for some $j \in \{1, 2, \ldots, m\}$. Let $H$ be a fixed $m$-edge-coloured graph. If $H$ is bipartite the problem of deciding whether a given $m$-edge-coloured graph has a $\Gamma$-switchable homomorphism to $H$ is solvable in polynomial time. If $H$ is not bipartite then the problem of deciding whether a given $m$-edge-coloured graph has a $\Gamma$-switchable homomorphism to $H$ is NP-complete.

Proof. By Corollary 2.5 the $m$-edge-coloured graphs $G$ and $H$ are $\Gamma$-switch equivalent to $m$-edge-coloured graphs $G'$ and $H'$ which are monochromatic of colour $j$. It follows from Theorem 4.3 that there is a $\Gamma$-switchable homomorphism of $G$ to $H$ if and only if there is a homomorphism of $G'$ to $H'$, if and only if there is a homomorphism of $\text{underlying}(G)$ to $\text{underlying}(H)$. The result now follows from Theorem 4.2.

We have found dichotomy theorems for the complexity of the $\Gamma$-switchable $k$-colouring problem and the problem of deciding whether there exists a $\Gamma$-switchable homomorphism to a fixed $m$-edge coloured graph $H$ when $\Gamma$ is one of $S_m, m \geq 3$; $A_m, m \geq 4$; $D_m, m \geq 2$ and odd; any other group with property $T_j$ for some $j$. Finally, we consider dihedral groups of even order. The following theorem will be useful.

**Theorem 4.6 ([3]).** Let $H$ be a 2-edge-coloured graph. If there is a $S_2$-switchable homomorphism of $H$ to a monochromatic $K_2$, then the problem of deciding whether a given 2-edge-coloured graph $G$ has a $S_2$-switchable homomorphism to $H$ is solvable in polynomial time. If there is no $S_2$-switchable homomorphism of $H$ to a monochromatic $K_2$, then the problem of deciding whether a given 2-edge-coloured graph $G$ has a $S_2$-switchable homomorphism to $H$ is NP-complete.

As an aside, we note that it is easy to test whether a 2-edge coloured graph is $S_2$-switch equivalent to a monochromatic $K_2$. By [2] such a $S_2$-switchable homomorphism exists if and only if there is a homomorphism (without switching) of $H$ to a 4-cycle where the edge colours alternate. The latter condition can be tested in polynomial time. Without loss of generality $H$ is connected. By symmetry the image of any vertex can be chosen without loss of generality. Since each vertex is adjacent with exactly one edge of each colour, there is only one choice to extend the mapping to a neighbouring vertex. Successively doing so either leads to the desired homomorphism or to a contradiction in which some vertex is forced to have two different images.
Theorem 4.7. Let $k \geq 1$ be an integer, and let $m \geq 2$ be an even integer. The problem of deciding whether a given $m$-edge-coloured graph has a $D_m$-switchable $k$-colouring is solvable in polynomial time if $k \leq 2$ and is NP-complete if $k \geq 3$.

Proof. Let $G$ be an $m$-edge coloured graph. It is clear that $G$ has a $D_m$-switchable 1-colouring if and only if it has no edges.

Suppose that $k = 2$. By definition, $G$ has a $D_m$-switchable 2-colouring if and only if there exists $j$ such that $G$ has a $D_m$-switchable homomorphism to a $K_2$ of colour $j$. Thus, by Theorem 4.3, $G$ has a $D_m$-switchable 2-colouring if and only if it is bipartite and there exists $G' \in [G]_{D_m}$ such that $G'$ is monochromatic of colour $j$.

Without loss of generality $j$ is odd. By Theorem 3.4 the $m$-edge coloured graph $G'$ exists if and only if $G_2$ (as in Theorem 3.4) is $S_2$-switch equivalent to $G'_2$. Since $G'_2$ is bipartite and switchable homomorphisms compose [9], this is equivalent to $G_2$ having a $S_2$-switchable homomorphism to a $K_2$ of colour 1, which is decidable in polynomial time by Theorem 4.6.

Now suppose $k \geq 3$. The transformation is from the problem of deciding whether a given graph $G$ has a $k$-colouring. Suppose a graph $G$ is given. We claim that $G$ has a $k$-colouring if and only if the $m$-edge-coloured graph $G'$ with underlying $(G') = G \cup K_k$ and which is monochromatic of colour $j$ has a $D_m$-switchable $k$-colouring. Clearly $G'$ can be constructed in polynomial time.

We now show that $G$ has a $k$-colouring if and only if $G'$ has a $D_m$-switchable $k$-colouring.

Suppose $G'$ has a $D_m$-switchable $k$-colouring. By definition, such a mapping is a $k$-colouring of underlying $(G')$. Since $G$ is a subgraph of underlying $(G')$, it follows that $G$ is $k$-colourable.

Now suppose that $G$ has a $k$-colouring. Then there is a homomorphism of $G \cup K_k$ to $K_k$. Therefore there is a homomorphism of $G'$ to a $K_k$ which is monochromatic of colour $j$. Thus $G'$ has a $\Gamma$-switchable $k$-colouring.

It now follows that $\Gamma$-switchable $k$-colouring is NP-complete.

The proof of the following lemma is very similar to the proof of Theorem 3.4.

Lemma 4.8. Let $G$ and $H$ be $m$-edge coloured graphs, and $m \geq 2$ be an even integer. There is a $D_m$-switchable homomorphism of $G$ to $H$ if and only if there is a $S_2$-switchable homomorphism of $G_2$ to $S_2$.

Proof. Suppose first that there is a $D_m$-switchable homomorphism of $G$ to $H$. Then there is a $D_m$-switching sequence $S = (x_1, \pi_1), (x_2, \pi_2), \ldots, (x_t, \pi_t)$ that transforms $G$ to $G' \in [G]_{D_m}$ for which there is a homomorphism of $G'$ to $H$. By Observation 3.3 each permutation $\pi_i \in D_m$ either maps $E$ to $E$ and $O$ to $O$, or maps $E$ to $O$ and vice-versa. Let $S'$ be the subsequence of $S$ consisting of the permutations that map $E$ to $O$. Replacing each of the permutations in this subsequence by the transposition $(1 \ 2)$ gives a $S_2$-switching sequence that transforms
$G_2$ to a 2-edge coloured graph $G'_2$ that has a homomorphism to $H_2$. Therefore there is a $S_2$-switchable homomorphism of $G_2$ to $H_2$.

Now suppose there is a $S_2$-switchable homomorphism of $G_2$ to $H_2$.

We claim that $G$ and $H$ are $D_m$-switch equivalent to $G_2$ and $H_2$ (considered as $m$-edge coloured graphs), respectively. Since $\text{Stabilizer}(\mathcal{E})$ has property $T_j$ for all $j \in \mathcal{E}$, any edge of $G$ whose colour is in $\mathcal{E}$ can be $D_m$-switched to have colour 2. As in the proof of Proposition 2.3, edges other than the one whose colour is intended to change switch from their colour then back again. Similarly, any edge of $G$ whose colour is in $\mathcal{O}$ can be $D_m$-switched to have colour 1. Thus $G$ is $D_m$-switch equivalent to $G_2$. Similarly $H$ is $D_m$-switch equivalent to $H_2$, and the claim is proved.

The same function which is a homomorphism of the 2-edge coloured graph $G_2$ to the 2-edge coloured graph $H_2$ is also a homomorphism of the $m$-edge coloured graph $G_2$ to the $m$-edge coloured graph $H_2$. It now follows from Theorem 4.3 that there is a $D_m$-switchable homomorphism of $G$ to $H$.

**Theorem 4.9.** Let $H$ be an $m$-edge coloured graph, and $m \geq 2$ be an even integer. If there is a homomorphism of $H_2$ to a monochromatic $K_2$, then the problem of deciding whether a given $m$-edge-coloured graph $G$ has a $D_m$-switchable homomorphism to $H$ is solvable in polynomial time. If there is no homomorphism of $H_2$ to a monochromatic $K_2$, the problem of deciding whether a given $m$-edge-coloured graph $G$ has a $D_m$-switchable homomorphism to $H$ is NP-complete.

**Proof.** Let $H$ be an $m$-edge coloured graph, and $m \geq 2$ be an even integer.

Suppose first there is a homomorphism of (the 2-edge coloured graph) $H_2$ to a monochromatic $K_2$. Let $G$ be an $m$-edge coloured graph. Then, by Theorem 4.3, it can be decided in polynomial time whether $G_2$ has a $S_2$-switchable homomorphism to $H_2$. Since $G_2$ can be constructed in polynomial time, Lemma 4.8 implies it can be decided in polynomial time whether $G$ has a $D_m$-switchable homomorphism to $H$.

Now suppose there is no homomorphism of (the 2-edge coloured graph) $H_2$ to a monochromatic $K_2$. We want to show the problem of deciding whether a given $m$-edge coloured graph has a $D_m$-switchable homomorphism to $H$ is NP-complete. The transformation is from the problem of deciding whether a given 2-edge coloured graph $F$ has a $S_2$-switchable homomorphism to $H_2$, which is NP-complete by Theorem 4.6.

Suppose such a 2-edge-coloured graph $F$ is given. The transformed instance the problem is the $m$-edge coloured graph $F'$ in which every edge has the same colour as in $F$. Then $F'_2 = F$, and the result follows from Lemma 4.8. 

\[\square\]
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