Asymptotic behavior of blowing-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids

Ahmed Bachir, Jacques Giacomoni and Guillaume Warnault

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Abstract

In this paper, we deal with the following quasilinear elliptic system involving gradient terms in the form:

\[
\begin{align*}
\Delta_p u &= v^m |\nabla u|^\alpha & \text{in } \Omega \\
\Delta_p v &= v^\beta |\nabla u|^q & \text{in } \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is either equal to \( \mathbb{R}^N \) or equal to a ball \( B_R \) centered at the origin and having radius \( R > 0 \), \( 1 < p < \infty \), \( m, q > 0 \), \( \alpha \geq 0 \), \( 0 \leq \beta \leq m \) and \( \delta := (p - 1 - \alpha)(p - 1 - \beta) - mq \neq 0 \).

Our aim is to establish the asymptotics of the blowing-up radial solutions to the above system. Precisely, we provide the accurate asymptotic behavior at the boundary for such blowing-up radial solutions. For that, we prove a strong maximal principle for the problem of independent interest and study an auxiliary asymptotically autonomous system in \( \mathbb{R}^3 \).

1 Introduction and main results

This paper deals with a class of quasilinear elliptic system of the following type:

\[
\begin{align*}
\Delta_p u &= v^m |\nabla u|^\alpha & \text{in } \Omega \\
\Delta_p v &= v^\beta |\nabla u|^q & \text{in } \Omega,
\end{align*}
\]

(1.1)

Here the operator \( \Delta_p \) represents the standard \( p \)-Laplace operator, and \( \Omega \subset \mathbb{R}^N (N \geq 2) \) is either equal to \( \mathbb{R}^N \) if the solution is global or equal to a ball \( B_R \) if the solution blows up at some \( \hat{R} \geq R \) with \( \hat{R} < \infty \), and the parameters involved in (1.1) satisfy

\[
1 < p < \infty, \quad m, q > 0, \quad \alpha \geq 0, \quad 0 \leq \beta \leq m.
\]

The global homogeneity parameter related to the system (1.1), \( \delta \), verifies additionally

\[
\delta := (p - 1 - \alpha)(p - 1 - \beta) - mq \neq 0.
\]

(1.2)

Our goal is to determine the behaviour of radial and positive blowing-up solutions at boundary. In this regard, the present paper can be considered as the subsequent continuation of [8], where several results about existence, global behavior and uniqueness are obtained for global solutions to the system (1.2) (see also [25] and [7] for the particular case \( p = 2 \)). The semilinear case \( p = 2, \alpha = \beta = 0, m = 1, q = 2 \) of problem (1.1) was considered in details by Diaz, Lazzo and Schmidt [6] as a model arising from the study of dynamics of a viscous, heat-conducting fluid. Considering a unidirectional flow, independent of distance in the flow direction under the Boussinesq approximation, the velocity \( u \) and the temperature \( \theta \) satisfy the following system

\[
\begin{align*}
u_t - \Delta u &= \theta & \text{in } \Omega \times (0, T), \\
\theta_t - \Delta \theta &= |\nabla u|^2 & \text{in } \Omega \times (0, T).
\end{align*}
\]

(1.3)
Theorem 1.1. Assume that \( \Omega = B_R, 1 < p < \infty, m, q > 0, 0 \leq \alpha < p - 1, 0 \leq \beta \leq m \) and \( \delta \neq 0 \). Then,

(i) There are no positive radial solutions \((u, v)\) with \( u(R^+) = \infty \) and \( v(R^+) < \infty \).

(ii) All positive radial solutions of (1.1) are bounded i.i.f. \( \delta > 0 \).

(iii) There are positive radial solutions \((u, v)\) of (1.1) with \( u(R^+) < \infty \) and \( v(R^+) = \infty \) i.i.f. \( \delta < -mp - (p - 1 - \beta) \).

(iv) There are positive radial solutions \((u, v)\) of (1.1) with \( u(R^-) = v(R^-) = \infty \) i.i.f. \( \delta \in [-mp - (p - 1 - \beta), 0) \).

From the seminal works of [2], Keller-Osserman condition was investigated for a very large class of elliptic equations, systems and inequalities. We refer to [9] and [23] for a survey of the corresponding results. The second result in [8] concerns the existence of nonconstant global positive radial solutions of (1.1), which shows that Theorem (1.1)-(ii) is sharp regarding the existence of global solutions:

Theorem 1.2. Assume that \( \Omega = \mathbb{R}^N, p > 1, m, q > 0, \alpha \geq 0, 0 \leq \beta \leq m \) and \( \delta \neq 0 \). Then, (1.1) admits nonconstant global positive radial solutions if and only if

\[
0 \leq \alpha < p - 1 \quad \text{and} \quad \delta > 0.
\]

In [8], the asymptotic behavior and the uniqueness of global solutions are also investigated. As it is observed in [8], by taking advantage of the homogeneity of system (1.1), we have that any nonconstant positive radial solution \((u, v)\) of (1.1) in a ball \( B_R \) satisfies the following system

\[
\begin{align*}
\left( (u')^{p-1} \right)' + \frac{N-1}{r} u^{p-1} = v^m (u')^\alpha & \quad \text{in } (0, R), \\
\left( (v')^{p-1} \right)' + \frac{N-1}{r} v^{p-1} = v^q (v')^\beta & \quad \text{in } (0, R),
\end{align*}
\]

and \( u' \) and \( v' \) are increasing on \((0, R)\). In the present paper, to complement the above results, we focus on the asymptotic behaviour of blowing-up solutions. Thanks to the homogeneity property of the system, (1.6) can be reduced to the problem

\[
\begin{align*}
\left( (u')^{p-1-\alpha} \right)' + \frac{\gamma}{r} (u')^{p-1-\alpha} = \frac{\gamma}{N-1} v^m & \quad \text{in } (0, R), \\
\left( (v')^{p-1} \right)' + \frac{N-1}{r} (v')^{p-1} = v^\beta (v')^q & \quad \text{in } (0, R),
\end{align*}
\]

and

\[
\begin{align*}
u'(0) = v'(0) = 0, \quad u, v > 0 \quad \text{in } (0, R),
\end{align*}
\]
where

\[
(1.8) \quad \gamma = \frac{(N - 1)(p - 1 - \alpha)}{p - 1}.
\]

Let \((u, v)\) be a nonconstant positive radially symmetric solution of (1.7) in the ball \(B_R\), then the pair

\[
(1.9) \quad (u_\lambda, v_\lambda) = \left(\lambda^{1-\alpha_0}u\left(\frac{r}{\lambda}\right), \lambda^{-\beta_0}v\left(\frac{r}{\lambda}\right)\right)
\]

where \(\alpha_0 = \frac{1 + \beta - p(m + 1)}{\delta}\) and \(\beta_0 = -\frac{p(p - 1 - \alpha) + q}{\delta}\), provides a nonconstant positive radially symmetric solution of (1.7) in the ball \(B_{\lambda R}\). This guarantees that in any ball of positive radius there is nonconstant positive radially symmetric solution of (1.7). Hence, thanks to this scaling property, \(R\) can be arbitrary, then without loss of generality we fix in the sequel \(R = 1\).

Considering \(R = 1\), the main result of the present paper yields the asymptotic behavior of the blowing-up solutions of (1.1) namely when (iii) or (iv) holds.

**Theorem 1.3.** Under the conditions of Theorem 1.2 assume \(\delta < 0\) and \(\alpha_0 > 0\). Then,

\[
u'(r) \sim \frac{\lambda}{(1-r)^{\alpha_0}} \text{ and } v(r) \sim \frac{\mu}{(1-r)^{\beta_0}} \text{ as } r \to 1^-
\]

where \(\lambda\) and \(\mu\) are positive constants given in (3.8).

**Remark 1.1.** From Theorem 1.3 we get the asymptotics of \(u\) when \(\alpha_0 \in [1, \infty]\), that is as \(r \to 1^-\), \(u(r) \sim \frac{\lambda}{\beta_0 - 1}(1-r)^{-\alpha_0 - 1}\) if \(\alpha_0 > 1\) and \(u(r) \sim -\lambda \ln(1-r)\) if \(\alpha_0 = 1\). \(\alpha_0 < 1\) holds when \(\delta < -mp - (p - 1 - \beta)\) and then one falls in the case of (iii) of Theorem 1.1 whereas assertion (iv) is verified as \(\alpha_0 \geq 1\).

To prove Theorem 1.3 we perform several changes of variables that reduce (1.6) to an asymptotically autonomous dynamical system for which we appeal the rich theory of cooperative systems (see [11], [15], [12] for further details on the subject). We also use chain-recurrence properties of asymptotically-autonomous flows (see [20]) to identify the \(\omega\)-limit set of the non autonomous dynamical system. We also need properties concerning equation (1.6) or (1.7) as comparison principle and continuity with respect the initial data. Section 2 is dedicated to prove these tools whereas Section 3 deals the proof of Theorem 1.3. We also give at the end of Section 3 some numerical simulations associated to asymptotics proved in Theorem 1.3.

## 2 Technical results

More precisely, we study in this section the pairs of positive functions \((U, v)\) which satisfy, for some \(R > 0\)

\[
(2.1) \quad \begin{cases}
(U^{p-1-\alpha})' + \frac{\gamma}{N - 1}U^{p-1-\alpha} = \frac{\gamma}{N - 1}v^m & \text{in } (0, R), \\
(v^{p-1})' + \frac{\gamma}{r} - 1(v^{p-1}) = \nu^q & \text{in } (0, R), \\
v(0) = U(0) = 0 \text{ and } v \to +\infty \text{ as } r \to R.
\end{cases}
\]

We start with a comparison principle satisfied by (2.1).

**Proposition 2.1.** Let \((U, v)\) and \((\hat{U}, \hat{v})\) be positive functions which satisfy

\[
(2.2) \quad \begin{cases}
(r^\gamma U^{p-1-\alpha})' \geq \frac{\gamma}{N - 1}r^\gamma v^m & \text{for any } r \in (0, R), \\
(r^N v^{p-1})' \geq r^{N-1}v^{p-1} & \text{for any } r \in (0, R), \\
v(0) = v_0 \text{ and } v'(0) = U(0) = 0 \\
U > 0, \; v' > 0 \text{ in } (0, R)
\end{cases}
\]

\[\text{Uncomment unique, remoove uniq, uniq remove uniq, uniq remove uniq} \]
for some $R, \hat{R} > 0$.

If $\hat{v}(0) < v(0)$, then $\hat{v} < v$ and $\hat{U} < U$ in $(0, \min(R, \hat{R}))$.

In addition, assuming that $\hat{v}$ blows up at $\hat{R}$, we have $R < \hat{R}$.

**Proof.** Define $\rho^* = \sup \{ \rho > 0 \mid \hat{v}(r) < v(r) \text{ on } (0, \rho) \}$. Since $\hat{v}(0) < v(0)$, $\rho^* > 0$ hence from the first equations of (2.2)-(2.3), we deduce $U > \hat{U}$ on $(0, \rho^*)$. Moreover using the second equations of (2.2)-(2.3), we get $v' > \hat{v}'$ on $(0, \rho^*)$. Thus the mapping $v - \hat{v}$ is increasing $(0, \rho^*)$ and we deduce $\rho^* = \min(R, \hat{R})$ and $\hat{v} < v$ on $(0, \rho^*)$.

Now, assume that $\hat{v}$ blows up at $\hat{R}$. Since $\hat{v} < v$ on $(0, \min(R, \hat{R}))$, we have $R \leq \hat{R}$.

Consider the pair $(U_\lambda, v_\lambda)$ where $U_\lambda = \lambda^{-\alpha_0} U(\frac{\lambda}{\alpha})$ and $v_\lambda$ is defined in (2.1). Thus $(U_\lambda, v_\lambda)$ satisfies (2.2) in $(0, \lambda R)$.

Choosing $\lambda$ such that $\lambda(0) \leq \lambda^{-\beta_0} < 1$ then $v_\lambda(0) > \hat{v}(0)$ and from the first part, we deduce $v_\lambda > \hat{v}$ on $(0, \min(\lambda R, \hat{R})$ and $\lambda R \leq \hat{R}$.

\[\square\]

**Proposition 2.2.** The mapping $\Phi : v_0 \to R$, where $(0, R)$ is the maximal interval of a solution $(U, v)$ of (2.1) with $v(0) = v_0$, is continuous on $(0, +\infty)$.

**Proof.** Let $(v_{0,n}) \subset \mathbb{R}$ such that $v_{0,n} \to v_0$ and . We define $(U, v)$ and $R$ (respectively $(U_n, v_n)$ and $R_n$) solutions of (2.1) with $v(0) = v_0$ defined on the maximal interval $(0, R)$ (respectively with $v_n(0) = v_{0,n}$) defined on the maximal interval $(0, R_n)$).

Let $\varepsilon > 0$. Then, we define $\lambda_\varepsilon = \left(1 + \frac{\varepsilon}{v_0}\right)^{-\frac{1}{\beta_0}}$

and the pair $(U_{\lambda_\varepsilon}, v_{\lambda_\varepsilon}) = \left(\lambda^{-\alpha_0} U(\frac{\lambda_\varepsilon}{\alpha}), \lambda^{-\beta_0} v(\frac{\lambda_\varepsilon}{\alpha})\right)$ is solution of (2.1) on $(0, \lambda_\varepsilon R)$ with $v_{\lambda_\varepsilon}(0) = v_0 \pm \varepsilon$. For $n$ large enough, $v_{0,n} \in (v_0 - \varepsilon, v_0 + \varepsilon)$, hence from Lemma 2.1 we have $\lambda_\varepsilon R < R_n < \lambda_\varepsilon R$ and we deduce $R_n \to R$ as $n \to +\infty$.

\[\square\]

**Corollary 2.1.** For any $R > 0$, there exist an unique $v_0 > 0$ and an unique pair of positive functions $(U, v)$ solution to (2.1) such that $v(0) = v_0$.

**Proof.** Lemma 2.1,2.2 imply that the mapping $\Phi$ is decreasing and continuous on $(0, +\infty)$. Thus $\Phi$ is invertible on $(0, +\infty)$ and we deduce the uniqueness.

\[\square\]

**Remark 2.1.** Using the scaling property and the previous results, we have, for any $R > 0$, $\Phi^{-1}(R) = \Phi^{-1}(1) R^{-\beta_0}$ and the unique solution $(U_R, v_R)$ on $(0, R)$ of (2.1) is given by

$U_R = R^{-\alpha_0} U_1 \left(\frac{R}{R}\right)$ and $v_R = R^{-\beta_0} v_1 \left(\frac{R}{R}\right)$

where $(U_1, v_1)$ is the unique solution of (2.1) on $(0, 1)$.

**Remark 2.2.** Let $\varepsilon > 0$ and let $(U, v)$ and $(U_\varepsilon, v_\varepsilon)$ two pairs of solutions of (2.1) such that $v(0) - \hat{v}(0) = \pm \varepsilon$. From Remark 2.1, there exists $\lambda_\varepsilon$ such that $\lambda_\varepsilon \to 1$ as $\varepsilon \to 0$, $\hat{v}_\varepsilon = \lambda_\varepsilon^{-\beta_0} v(\frac{\lambda_\varepsilon}{\alpha})$ and $U_\varepsilon = \lambda_\varepsilon^{-\alpha_0} U(\frac{\lambda_\varepsilon}{\alpha})$.

By the proof of Lemma 2.1, the sequences $(v_\varepsilon)_\varepsilon$, $(U_\varepsilon)_\varepsilon$ and $(v'_\varepsilon)_\varepsilon$ are monotone and converge everywhere to $v$, $U$ and $v'$ on the compact sets of $[0, R)$. By Dini’s Theorem, we deduce the uniform convergence on the compact sets of $[0, R)$. 4
\section{Asymptotic behavior and proof of Theorem 1.3}

Through this section, we only consider $\delta < 0$, $\alpha_0 > 0$ and $R = 1$. Moreover we have $\alpha_0(p - 1 - \alpha) + 1$, $\beta_0$ and $\gamma_0$ are positive. At the end of the section, we give the proof of Theorem 1.3. Let $(u, v)$ be a pair of nonconstant positive radial solutions of (1.1) given by Theorem 1.1. Then, setting $U = u'$ and $V = v'$ in (1.7), $U$ and $V$ are positive, increasing on $(0, 1)$ and satisfy

\begin{equation}
\begin{cases}
(U^{p-1-\alpha})' + \frac{2}{p}U^{p-1-\alpha} = \frac{\gamma}{N-1}v^m & \text{for } r \in (0, 1), \\
(V^{p-1})' + \frac{N-1}{r}V^{p-1} = v^q & \text{for } r \in (0, 1), \\
U(0) = V(0) = 0.
\end{cases}
\end{equation}

Now, set

$$U(r) = \frac{\lambda}{(1-r)^{\alpha_0}}a(r), \quad v(r) = \frac{\mu}{(1-r)^{\beta_0}}b(r) \quad \text{and} \quad V(r) = \frac{\nu}{(1-r)^{\gamma_0}}c(r)$$

where $\lambda, \mu, \nu > 0$ to be fixed later. Thus, (3.1) leads to

\begin{equation}
\begin{cases}
\left(\left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}}\right)^{p-1-\alpha}\right)' + \frac{2}{p} \left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}}\right)^{p-1-\alpha} = \frac{\nu^{1-\alpha}}{p-1} \left(\frac{\mu b(r)}{(1-r)^{\beta_0}}\right)^m & \text{for } r \in (0, 1), \\
\left(\left(\frac{\nu c(r)}{(1-r)^{\gamma_0}}\right)^{p-1}\right)' + \frac{N-1}{r} \left(\frac{\nu c(r)}{(1-r)^{\gamma_0}}\right)^{p-1} = \left(\frac{\mu b(r)}{(1-r)^{\beta_0}}\right)^\beta \left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}}\right)^\gamma & \text{for } r \in (0, 1).
\end{cases}
\end{equation}

Simple computations give $\alpha_0(p - 1 - \alpha) + 1 = m\beta_0$ and

$$\left(\left(\frac{\lambda a(r)}{(1-r)^{\alpha_0}}\right)^{p-1-\alpha}\right)' = (p - 1 - \alpha)\lambda^{p-1-\alpha}(1-r)d'(r) + \alpha_0 a(r)(1-r)^{m-2-\alpha}\mu^m.$$  

Thus, multiplying the first equation of (3.2) by $(1-r)^m\beta_0$ choosing $\lambda$ and $\mu$ such that $\lambda^{p-1-\alpha}\alpha_0(1-r)^{-2}a(r)$, this yields for any $r \in (0, 1)$

\begin{equation}
(1-r)a^{p-\alpha-2}(p - 1 - \alpha)a' + \frac{7}{7}a = \alpha_0(p - 1 - \alpha)(b^m - a^{p-1-\alpha}).
\end{equation}

In the same way, noting that $\gamma_0(p - 1 + 1 = \beta_0 + \alpha_0 q$ and multiplying the second equation of (3.2) by $(1-r)^{\beta_0 + \alpha_0 q}$, we get

\begin{equation}
(1-r)c^{p-2}(p - 1)c' + \frac{N-1}{r}c = \gamma_0(p - 1)(b^q a^q - c^{p-1})
\end{equation}

where $\lambda, \mu$ and $\nu$ satisfy $\mu^\lambda \lambda^\gamma = \gamma_0(p - 1)\nu^{p-1}$.

Moreover, since $V = v'$, we deduce that

\begin{equation}
\mu(b'(r)(1-r) + \beta_0 b(r)) = \nu c(r).
\end{equation}

Choosing $\mu$ and $\nu$ such that $\beta_0 \mu = \nu$, (3.5) becomes

\begin{equation}
b'(r)(1-r) = \beta_0 (c(r) - b(r)).
\end{equation}

Hence, we deduce that the triplet $(a, b, c)$ satisfies

\begin{equation}
\begin{cases}
(1-r)a^{p-\alpha-2}(p - 1 - \alpha)a' + \frac{7}{7}a = \alpha_0(p - 1 - \alpha)(b^m - a^{p-1-\alpha}), \\
(1-r)c^{p-2}(p - 1)c' + \frac{N-1}{r}c = \gamma_0(p - 1)(b^q a^q - c^{p-1}), \\
(1-r)b' = \beta_0 (c - b)
\end{cases}
\end{equation}

under the following relations

$$\lambda^{p-1-\alpha}\alpha_0 = \frac{\mu^m}{p-1}, \quad \mu^\lambda \lambda^\gamma = \gamma_0(p - 1)\nu^{p-1}, \quad \beta_0 \mu = \nu.$$
which leads to
\[
\begin{align*}
\lambda &= \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{p-1}{p-\alpha}}\right)^{-\frac{\alpha}{p-\alpha}}, \\
\mu &= \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{p-\alpha}{p-\alpha}}\right)^{-\frac{\alpha}{p-\alpha}}, \\
\nu &= \beta_0 \left(\gamma_0(p-1)\beta_0^{p-1}(\alpha_0(p-1))^{\frac{p-\alpha}{p-\alpha}}\right)^{-\frac{\alpha}{p-\alpha}}.
\end{align*}
\]
\[eq13\]

Consider now the following change of variable \( r = 1 - e^{-t} \) i.e. \( t = - \ln(1-r) \) and we define \( X, Y \) and \( Z \) as follows
\[X(t) = e^{p-1-a}(r), \quad Y(t) = b(r) \text{ and } Z(t) = e^{p-1}(r) .\]

Hence, \((X, Y, Z)\) are positive functions satisfying the following \( C^1 \)-asymptotically autonomous system on \([t_0, +\infty)\) for some \( t_0 > 0 \):
\[\begin{align*}
X'(t) + \frac{\gamma}{e^t - 1} X(t) &= \alpha_0(p-1-\alpha)|Y|^{p-1}(t)Y(t) - X(t) \\
Y'(t) &= \beta_0(|Z|^{\frac{1}{p-\alpha}}(t)Z(t) - Y(t)) \\
Z'(t) + \frac{N-1}{e^t - 1} Z(t) &= \gamma_0(p-1)|Y|^{\beta-1}(t)Y(t)|X|^{\frac{p-\alpha}{p-\alpha}}(t)X(t) - Z(t)
\end{align*}\]
\[eq14\]

for some \((X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}^3\). The asymptotic dynamical system associated to (3.9) is given by:
\[\begin{align*}
X'(t) &= \alpha_0(p-1-\alpha)|Y|^{p-1}Y - X \\
Y'(t) &= \beta_0(|Z|^{\frac{1}{p-\alpha}}Z - Y) \\
Z'(t) &= \gamma_0(p-1)|Y|^{\beta-1}Y|X|^{\frac{p-\alpha}{p-\alpha}}X - Z
\end{align*}\]
\[eq10\]

The system (3.10) can be rewritten as \( \zeta'(t) = g(\zeta), \zeta(t_0) = (X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}^3 \) where
\[\zeta(t) = \begin{pmatrix} X(t) \\ Y(t) \\ Z(t) \end{pmatrix} \text{ and } g(\zeta) = \begin{pmatrix} \alpha_0(p-1-\alpha)|Y|^{p-1}Y - X \\ \beta_0(|Z|^{\frac{1}{p-\alpha}}Z - Y) \\ \gamma_0(p-1)|Y|^{\beta-1}Y|X|^{\frac{p-\alpha}{p-\alpha}}X - Z \end{pmatrix} .\]
\[eq11\]

The system (3.10) is cooperative, \( \text{div } g < 0 \) and we have the following properties.

**Proposition 3.1.** The following assertions hold:

1. \((0,0,0)\) and \((1,1,1)\) are the only equilibrium points of the system (3.10).
2. \((0,0,0)\) is asymptotically stable (and then a sink).
3. \((1,1,1)\) is an hyperbolic saddle point.

**Proof.** Let \( P = (X_e, Y_e, Z_e) \) be an equilibrium point of the system (3.10), then \( g(P) = 0 \) which implies that
\[\begin{align*}
|Y_e|^{\alpha-1}Y_e &= X_e, \\
|Z_e|^{\frac{1}{p-\alpha}}Z_e &= Y_e, \\
|Y_e|^{\beta-1}|X_e|^{\frac{p-\alpha}{p-\alpha}}Y_eX_e &= Z_e.
\end{align*}\]
\[eq12\]

Proof of 1.: Obviously, \( Y_e = 0 \) yields \( X_e = Z_e = 0 \) and solve the system (3.12).

If \( Y_e \neq 0 \), from (3.12), we deduce that \( X_e, Y_e \) and \( Z_e \) have the same sign and that \( |Y_e|^{p-2-\beta}X_e = |Y_e|^{p-2-\beta}Y_e \). Thus, the first equation of (3.12) implies \( Y_e^\frac{2}{p-\alpha} = 1 \). Hence, we get \( Y_e = X_e = Z_e = 1 \).

Proof of 2.: The assertion follows directly looking the linearized matrix at \((0,0,0)\) given by:
\[M_0 = \begin{pmatrix} -\alpha_0(p-1-\alpha) & 0 & 0 \\ 0 & -\beta_0 & 0 \\ 0 & 0 & -\gamma_0(p-1) \end{pmatrix} .\]
Proof of 3.: The linearized matrix at \((1,1,1)\) is given by
\[
M_1 = \begin{pmatrix}
-a_0(p-1-\alpha) & ma_0(p-1-\alpha) & 0 \\
0 & -b_0 & 0 \\
qu_0(p-1) & -\beta_0 & -\gamma_0(p-1)
\end{pmatrix}.
\]
Thus, \(\det(\lambda I - M_1) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3\) where
\[
\begin{align*}
C_1 &= a_0(p-1-\alpha) + b_0 + \gamma_0(p-1), \\
C_2 &= b_0\gamma_0(p-1 - \beta) + a_0(p-1-\alpha)(b_0 + \gamma_0(p-1)), \\
C_3 &= a_0 b_0 \gamma_0 \delta.
\end{align*}
\]
Since \(\beta_0 + 1 = \gamma_0, a_0(p-1-\alpha) + 1 = b_0 m\) and \(\gamma_0(p-1) + 1 = \beta_0 b_0 + a_0 q\), we get \(C_1 + C_2 + C_3 = -1\). Hence \(\lambda_1 = 1\) is an eigenvalue of \(M_1\).

Let us call \(\lambda_2, \lambda_3\) the two other eigenvalues of \(M_1\). Then we have \(\lambda_2 + \lambda_3 = -C_1 - 1 < 0\) and \(\lambda_2\lambda_3 = -C_3 = -a_0 b_0 \gamma_0 \delta > 0\).

Thus, \(Re(\lambda_2), Re(\lambda_3) < 0\) and we deduce that \((1,1,1)\) is hyperbolic point with 2-dimensional stable manifold. \(\square\)

We have the following properties about the dynamical system \((3.9)\):

**Proposition 3.2.** Let \(T > 0\) and \(t_0 \in (0, T]\). Assume \((X,Y,Z)\) a triplet of functions satisfying \((3.9)\) on \([t_0, T)\) with \((X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}_+^3\). Then, for any \(t \in [t_0, T), (X(t), Y(t), Z(t)) \in \mathbb{R}_+^3\).

**Proof.** Define \(T^* = \sup\{t > t_0 \mid X(s) > 0, Y(s) > 0 \text{ and } Z(s) > 0 \text{ for any } s \in (0, t)\}\) and assume that \(T^* < T\).

From the second equation of \((3.9)\), we deduce that for any \(t \in [t_0, T^*), (e^{\beta_0 t} Y(t))' > 0\) and hence \(e^{\beta_0 t} Y(t) < e^{\beta_0 t_0} Y(t_0) > 0\). This implies that \(Y(T^*) > 0\).

Assuming \(X(T^*) = 0\), the first equation of \((3.9)\) yields for \(t = T^*\), \((X'(T^*)) = a_0(p-1-\alpha) Y^m(T^*) > 0\). This implies there exists a small neighbourhood \((T^* - \eta, T^* + \eta)\) of \(T^*\) such that \(X' > 0\) on \((T^* - \eta, T^* + \eta)\). Since \(X(T^*) = 0\), we deduce \(X(t) < 0\) on \((T^* - \eta, T^*)\) which contradicts the definition of \(T^*\).

In the same, using the third equation of \((3.9)\), we can not have \(Z(T^*) = 0\) and hence we conclude \(T^* = T\). \(\square\)

**Remark 3.1.** Proposition 3.2 holds considering a triplet of functions satisfying the autonomous system \((3.10)\).

**Proposition 3.3.** Let \(T > 0\) and \(t_0 \in (0, T]\). Let \((X,Y,Z)\) be a triplet of functions satisfying \((3.9)\) on \([t_0, T)\) with \((X(t_0), Y(t_0), Z(t_0)) \in \mathbb{R}_+^3\). Assume \(X(t_0) < 1, Y(t_0) < 1 \text{ and } Z(t_0) < 1\). Then, for any \(t \in [t_0, T), (X(t), Y(t), Z(t)) \in \mathbb{R}_+^3\), and hence we deduce
\[
\begin{align*}
(X - 1)'(t) &< a_0(p-1-\alpha)(Y^m - 1 - (X - 1)), \\
(Y - 1)'(t) &< b_0 \left(Z^\frac{1}{1-p} - 1 - (Y - 1)\right), \\
(Z - 1)'(t) &< c_0(p-1) \left(Y^\beta X^\frac{2}{1-p} - 1 - (Z - 1)\right)
\end{align*}
\]
and we get
\[
\begin{align*}
e^{-a_0(p-1-\alpha)t} (e^{a_0(p-1-\alpha)t}(X - 1))' &< a_0(p-1-\alpha)(Y^m - 1), \\
e^{-\beta_0 t} (e^{\beta_0 t}(Y - 1))' &< b_0 \left(Z^\frac{1}{1-p} - 1\right), \\
e^{-c_0(p-1)t} (e^{c_0(p-1)t}(Z - 1))' &< c_0(p-1) \left(Y^\beta X^\frac{2}{1-p} - 1\right).
\end{align*}
\]

Define \(\tau = \sup\{t > t_0 \mid X(t) < 1, Y(t) < 1 \text{ and } Z(t) < 1 \text{ on } (t_0, t)\}\).

Assume \(\tau < T\) then \(X(\tau) = 1\) or \(Y(\tau) = 1\) or \(Z(\tau) = 1\).

In any cases, the third equation of \((3.13)\) at \(t = \tau\) implies that \(Z(t) < 1\) on the interval \([\tau, \tau + \eta]\) for
some $\eta > 0$. Hence, if $Z(\tau) = 1$, we get a contradiction. Otherwise, using the previous estimate in the second equation of (3.13), we get $Y(\tau) < 1$ which would contradict $Y(\tau) = 1$. Finally, using $Y(\tau) < 1$ in the first equation of (3.13), this yields $X(\tau) < 1$ and a contradiction with $X(\tau) = 1$. $\square$

For $t_0$ small enough, $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}^+$, hence the triplet $(X, Y, Z)$ satisfies the $C^1$-asymptotically autonomous system

\begin{equation}
\begin{aligned}
X'(t) + \frac{\gamma}{c^2 - 1} X(t) &= \alpha_0(p - 1 - \alpha)(Y^m(t) - X(t)) &\text{on } [t_0, \infty) \\
Y'(t) &= \beta_0(Z^m(t)Z(t) - Y(t)) &\text{on } [t_0, \infty) \\
Z'(t) + \frac{N - 1}{c^2 - 1} Z(t) &= \gamma_0(p - 1)(Y^\beta(t)X^{\frac{q}{\beta + q}}(t) - Z(t)) &\text{on } [t_0, \infty)
\end{aligned}
\end{equation}

with $(X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}^3$. Hence we get the following asymptotic dynamical system:

\begin{equation}
\begin{aligned}
X' &= \alpha_0(p - 1 - \alpha)(Y^m - X) \\
Y' &= \beta_0(Z^m - Y) \\
Z' &= \gamma_0(p - 1)(Y^\beta X^{\frac{q}{\beta + q}} - Z).
\end{aligned}
\end{equation}

Now, we study the trajectories $(X, Y, Z)$. The first result is the relatively compactness:

**Proposition 3.4.** The solution $(X, Y, Z)$ is bounded as $t \to +\infty$.

**Proof.** Firstly, assume that $Y$ is bounded, then we have from the first equation of (3.14)

$$X'(t) + \alpha_0(p - 1 - \alpha)X \leq C,$$

which is equivalent to

$$\left(e^{\alpha_0(p - 1 - \alpha)t}X\right)' \leq C e^{\alpha_0(p - 1 - \alpha)t}.$$ 

Integrating this last inequality from $t_0$ to $t$ leads to

$$e^{\alpha_0(p - 1 - \alpha)t}X(t) - e^{\alpha_0(p - 1 - \alpha)t_0}X(t_0) \leq \frac{C}{\alpha_0(p - 1 - \alpha)} \left(e^{\alpha_0(p - 1 - \alpha)t} - e^{\alpha_0(p - 1 - \alpha)t_0}\right).$$

Since $X$ is positive, this implies that $X$ is bounded. A similar argument implies that $Z$ is bounded.

It remains to prove that $Y$ is actually bounded and for this, we proceed by contradiction. Consider $(\tilde{u}, \tilde{v})$ the solution of (1.7) with $\tilde{u}(0) = u(0)$ and $\tilde{v}(0) > v(0)$ defined in $(0, \tilde{R})$. From Proposition 2.1, it follows that $\tilde{R} < 1$. We define $(\tilde{X}, \tilde{Y}, \tilde{Z})$ on $(0, -\ln(1 - \tilde{R}))$ the transform of $(\tilde{u}, \tilde{v})$ constructed as previously. We also define $\tilde{X}(t) = X(t + T_0)$, $\tilde{Y}(t) = Y(t + T_0)$ and $\tilde{Z}(t) = Z(t + T_0)$ with $T_0$ large enough such that $Y(T_0) > \tilde{Y}(0)$ since $Y$ is unbounded. Then $(\tilde{u}, \tilde{v})$ the reverse transform of $(\tilde{X}, \tilde{Y}, \tilde{Z})$ satisfies

$$\begin{aligned}
&\left((\tilde{u}')^{p - 1 - \alpha}\right)' + \frac{\gamma}{r + r_0} (\tilde{u}')^{p - 1 - \alpha} = \frac{\gamma}{N - 1} \tilde{v}^m &\text{in } (0, 1), \\
&\left((\tilde{v}')^{p - 1}\right)' + \frac{N - 1}{r + r_0} (\tilde{v}')^{p - 1} = v^\beta (\tilde{u}')^q &\text{in } (0, 1), \\
&\tilde{u}'(0) \geq 0, \tilde{v}'(0) \geq 0, \tilde{u}, \tilde{v} > 0 &\text{in } (0, 1),
\end{aligned}$$

where $r_0 = \frac{\tilde{R}}{1 - \tilde{R}}$ with $r_0 = 1 - e^{-T_0}$.

Noting $(\tilde{u}', \tilde{v}')$ satisfies (2.2) and $\tilde{v}(0) > \tilde{v}(0)$, we apply Proposition 2.1 with $(\tilde{u}', \tilde{v})$, we deduce $\tilde{v} < \tilde{v}$ in $(0, \min(1, \tilde{R})) = (0, \tilde{R})$. Since $\tilde{v}$ blows up at $\tilde{R}$, we get the contradiction. $\square$

From proposition 3.4, we deduce the convergence of the trajectories $(X, Y, Z)$:

**Proposition 3.5.** We have

$$\lim_{t \to +\infty} (X(t), Y(t), Z(t)) = (1, 1, 1).$$

For the proof of Proposition 3.5, we need to recall two notions of dynamical systems:
Definition 3.1. A circuit is a finite sequence of equilibria $z_1, z_2, ..., z_K = z_1$, ($K \geq 2$) such that $W^u(z_i) \cap W^s(z_{i+1}) \neq \emptyset$ where $W^u, W^s$ denote the stable and unstable manifolds of each equilibrium.

Definition 3.2. Let $X \subset \mathbb{R}^3$ be a nonempty positively invariant subset for an autonomous semiflow $\phi$ and $x, y \in X$.

(i) For $\epsilon > 0$ and $t > 0$, an $(\epsilon, t)$-chain from $x \in X$ to $y \in X$ is a sequence of points in $X$, $x = x_1, x_2, ..., x_n, x_{n+1} = y$ and of times $t_1, t_2, ..., t_n \geq t$ such that $|\phi(t_i, x_i) - x_{i+1}| < \epsilon$.

(ii) A point $x \in X$ is called chain recurrent if for every $\epsilon > 0$, $t > 0$ there is an $(\epsilon, t)$-chain from $x$ to $x$ in $X$.

(iii) The set $X$ is said to be chain recurrent if every point $x \in X$ is chain recurrent in $X$.

Proof. Setting $\omega = (X(t_0), Y(t_0), Z(t_0)) \in \text{Int } \mathbb{R}_+^3$, we define the $\omega$-limit set $\omega = \omega(t_0, \zeta_0) \subset \mathbb{R}^3_{\omega}$ of $\phi$ the semiflow of the asymptotically autonomous system (3.34).

From Proposition 3.4, $\omega$ is bounded. Thus Theorem 1.8 in [20] implies $\omega$ is nonempty, compact and connected. Moreover $\omega$ is invariant for the semiflow denoted by $\phi_A$ associated to the asymptotic autonomous system (3.18), i.e. for any $t \geq 0$

$$\phi_A(t, \omega) = \omega$$

and $\omega$ is chain recurrent for $\phi_A$.

Finally, $\omega$ satisfies $\text{dist}(\phi(t, t_0, \zeta_0), \omega) \to 0$ as $t \to +\infty$.

**Step 1:** $\omega_0 \subset \{(0, 0, 0); (1, 1, 1)\}$

For that we argue by contradiction: assume that there exists $P_0 \in \omega_0 \setminus \{(0, 0, 0), (1, 1, 1)\}$.

Then, $\phi_A(t, P_0) \in \omega_0$ for any $t \geq 0$ thus the trajectory $t \to \phi_A(t, P_0)$ is bounded as $t \to +\infty$. From Proposition 1.2 in [20], we have $\omega_{\phi_A}(P_0)$ is invariant. Hence, we deduce for any $t < 0$:

$$\omega_{\phi_A}(P_0) = \phi_A(0, \omega_{\phi_A}(P_0)) = \phi_A(t, \phi_A(-t, \omega_{\phi_A}(P_0))) = \phi_A(t, \omega_{\phi_A}(P_0)).$$

Thus we deduce that $t \to \phi_A(t, P_0)$ is bounded as $t \to -\infty$ and the limit sets of $P_0$ associated to $\phi_A$ is bounded.

Then, applying Theorem 10 in [12] together with $\text{div} g < 0$ and Proposition 3.1, we infer that

$$\lim_{|t| \to +\infty} \phi_A(t, P_0) \in \{(0, 0, 0), (1, 1, 1)\}.$$ (3.16)

Now, since $(0, 0, 0)$ is asymptotically stable, this implies $\phi_A(t, P_0) \to (1, 1, 1)$ as $t \to -\infty$ and we have two following cases:

**Case 1:** heteroclinic orbit i.e $\phi_A(t, P_0) \to (0, 0, 0)$ as $t \to +\infty$

or

**Case 2:** homoclinic orbit i.e $\phi_A(t, P_0) \to (1, 1, 1)$ as $t \to +\infty$.

Since $\omega_0$ is invariant by the flow $\phi_A$, both cases imply that $\omega(\zeta_0)$ contains either a connecting orbit between the two different equilibria ($P_0 \in W_u(1, 1, 1)$) or a cycle with respect to $\phi_A$.

In the first case, the heteroclinic orbit included in $\omega_0$ is not chain recurrent since $(0, 0, 0)$ is asymptotically stable which contradicts $\omega_0$ is chain recurrent.

For the second case, $(1, 1, 1)$ is hyperbolic with transverse unstable and stable manifolds, there can not be any circuit (see for instance [11] page 1228 or [12] pages 1677-1678) and we have a contradiction.

**Step 2:** $\omega_0 = (1, 1, 1)$

Since $\omega_0$ is connected then either $\omega_0 = \{(0, 0, 0)\}$ or $\omega_0 = \{(1, 1, 1)\}$.

The first possibility does not hold. Indeed arguing by contradiction, then there exists $T_0 > 0$ such that $X(T_0) < 1, Y(T_0) < 1$ and $Z(T_0) < 1$.

Consider $(\tilde{u}, \tilde{v})$ a couple of blow-up solutions of (1.7) with the initial data $\tilde{v}(0) = v(0) + \epsilon$ on $(0, R_t)$ where $R_t < 1$ by Proposition 2.1. Moreover from Proposition 2.2, we have $R_t$ goes to 1 as $\epsilon \to 0$.

Let $R \in (0, 1)$, we define $T_1, T_2$ and $T_3$ from $C([0, R])$ to $C([-\ln(1-R), 0])$ three linear and continuous operators as follows: for any $w \in C([0, R])$, for any $t \in [0, T] = [0, -\ln(1-R)]$,

$$T_1(w)(t) = \frac{e^{-\alpha t}}{\lambda} w(1-e^{-t}), \ T_2(w)(t) = \frac{e^{-\beta t}}{\mu} w(1-e^{-t}) \text{ and } T_3(w)(t) = \frac{e^{-\gamma t}}{\nu} w(1-e^{-t}).$$
By Remark 2.2, we get as $\varepsilon \to 0$:

$$\sup_{[0,T]} |X - \tilde{X}| = \sup_{[0,T]} |T_1(u') - \tilde{T}_1(\tilde{u}')| \leq C \sup_{[0,T]} |u - \tilde{u}'| \to 0,$$

$$\sup_{[0,T]} |Y - \tilde{Y}| = \sup_{[0,T]} |T_2(v) - \tilde{T}_2(\tilde{v})| \leq C \sup_{[0,R]} |v - \tilde{v}| \to 0$$

and

$$\sup_{[0,T]} |Z - \tilde{Z}| = \sup_{[0,T]} |T_3(v') - \tilde{T}_3(\tilde{v}')| \leq C \sup_{[0,R]} |v' - \tilde{v}'| \to 0.$$ 

We deduce, for $\varepsilon$ small enough that then there exists $R \in (1 - e^{-T_0}, R_\varepsilon)$,

$$\sup_{[0,T]} |X - \tilde{X}| < 1 - X(T_0), \quad \sup_{[0,T]} |Y - \tilde{Y}| < 1 - Y(T_0) \quad \text{and} \quad \sup_{[0,T]} |Z - \tilde{Z}| < 1 - Z(T_0).$$

Hence $\tilde{X}(T_0) < 1$, $\tilde{Y}(T_0) < 1$ and $\tilde{Z}(T_0) < 1$.

From Proposition 3.3, we deduce that, for any $t \in (T_0, -\ln(1 - R_\varepsilon))$, $\tilde{X}(t) < 1$, $\tilde{Y}(t) < 1$ and $\tilde{Z}(t) < 1$.

Finally, by construction of $\tilde{Y}$, we have $\tilde{Y}(t) \to +\infty$ as $t \to -\ln(1 - R_\varepsilon)$ and we obtain a contradiction.

**Remark 3.2.** Figures 1-3 give examples of convergence of the triplet functions $(a, b, c)$ to $(1, 1, 1)$ as $r \to 1$ where $(a, b, c)$ satisfies (3.7). More precisely, we have $\alpha_0 < 1$ in Figure 1 and $\alpha_0 \geq 1$ in Figure 2. We give particular examples for the conditions of Theorem 1.3 in Figure 3: in the left-hand graph, we choose $p > 2$ and $q < p - 1 - \alpha$ and in the right-hand graph, $\beta < 1$ and $m < 1$. For every graph, $\delta < 0$, $b_0 > 0$, $\alpha_0 > 0$ and $N = 3$. 

![Figure 1: $\alpha_0 < 1$](fig1)

![Figure 2: $\alpha_0 \geq 1$](fig2)
Proof of Theorem 3.
We get from Proposition 3.3 that, as $r \to 1^-$
\[ a(r) = \frac{U(r)}{\lambda} (1 - r)^{\alpha_0} \to 1, \quad b(r) = \frac{v(r)}{\mu} (1 - r)^{\beta_0} \to 1 \quad \text{and} \quad c(r) = \frac{V(r)}{\nu} (1 - r)^{\gamma_0} \to 1 \]

hence
\[ u'(r) \sim \frac{\lambda}{(1 - r)^{\alpha_0}} \quad \text{and} \quad v(r) \sim \frac{\mu}{(1 - r)^{\beta_0}} \quad \text{as} \quad r \to 1^- \]

Since $\beta_0 > 0$, we have obviously $v(1) = \infty$. Finally we deduce
\[ u(r) \sim \frac{\lambda}{(\alpha_0 - 1)(1 - r)^{\alpha_0 - 1}} \quad \text{as} \quad r \to 1^- \]
if $\alpha_0 \neq 1$ and
\[ u(r) \sim \lambda \ln(1 - r) \quad \text{as} \quad r \to 1^- \]
if $\alpha_0 = 1$.

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