Low energy Quantum Gravity from the Effective Average Action

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Abstract

Within the effective average action approach to quantum gravity, we recover the low energy effective action as derived in the effective field theory framework, by studying the flow of possibly non-local form factors that appear in the curvature expansion of the effective average action. We restrict to the one-loop flow where progress can be made with the aid of the non-local heat kernel expansion. We discuss the possible physical implications of the scale dependent low energy effective action through the analysis of the quantum corrections to the Newtonian potential.

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1 Introduction

General relativity and quantum mechanics are not yet unified in a coherent theory as we do not have yet a fully successful theory of quantum gravity. But this does not mean that we lack any kind of quantum gravitational predictions: at least at low energy, quantum gravity can be described by an effective field theory based on metric degrees of freedom, as was first shown by Donoghue and others [1, 2].

At scales much smaller than the characteristic scale, which is here the Planck mass, effective field theory predictions are possible and calculable. Examples of this kind are the calculation of the first quantum corrections to the gravitational interaction potential between two masses [1, 3], and the low-energy graviton scattering cross-section [4].

The important point about these predictions is that no matter which theory actually describes high-energy quantum gravity - a string theory, a spin foam model, or other approaches - in the infrared (IR) limit any physically valid theory must reproduce the results found in the effective field theory framework.

In the last years, the hypothesis that the high energy completion of quantum gravity can still be described using the metric as fundamental degrees of freedom has gained some new support. In particular the possibility that there exists a non-trivial ultraviolet (UV) fixed point of the renormalization group (RG) flow, with a finite dimensional UV critical surface, has been investigated within what is called the “Asymptotic Safety” scenario [5, 6, 7, 8, 9]. Progress in this direction has been possible thanks to the development of a powerful functional RG framework based on the construction of a scale dependent effective action, called effective average action [10, 11, 12]. For a general discussion of the relation of this approach to perturbative theory, see [13].

In this paper we show how the low energy effective field theory predictions naturally arise in the effective average action approach to quantum gravity. In this way we start to delineate a picture able to describe gravitational phenomena at all scales: from the UV physics of the non-trivial fixed point down to the IR physics of the low energy effective action.

The basic technical tool at our disposal is the exact RG flow equation that the effective average action satisfy [10]. The solution of this integro-differential equation poses challenging technical difficulties. As of now there are no completely adequate techniques able to fully extract all the non-perturbative information which is in principle contained in the RG flow of the effective average action. In this work we will focus on a truncation ansatz where we consider an expansion of the effective average action in powers of the curvature and where the scale dependence is encoded in the RG running of (possibly nonlocal) form factors. In
this way we project the flow from the full theory space onto a subspace which is still infinite dimensional: the functional space of the running form factors. It was shown in [14] that to recover sensible results in the IR limit this is the minimal type of truncation ansatz we have to consider. Also, we will consider only the one-loop flow, driven by the Einstein-Hilbert operator \( \int \sqrt{g} R \), of the effective average action, where progress can be made with the aid of the non-local heat kernel expansion [14].

The paper is organized as follows. In Section 2 we shortly review the construction of the effective average action for quantum gravity, we introduce the truncation ansatz we will consider, and we show how the one-loop flow equations in \( d \)-dimensions are obtained using the non-local heat kernel expansion exposed in Appendix A. In Section 3 we integrate the flow equations for the form factors in \( d = 4 \) from the ultraviolet (UV) scale to the infrared (IR) scale recovering in this limit the effective field theory results. In Section 4 we discuss how the quantum corrections to the Newtonian potential arise in our approach. Finally, Section 5 contains our concluding remarks. In the Appendices we give an overview of the non-local heat kernel expansion and include some details of the calculations.

2 Effective average action in quantum gravity: curvature expansion

The effective average action is a coarse-grained effective action depending on the IR cutoff scale \( k \). It interpolates smoothly between the bare action in the UV, for \( k \to \Lambda_1 \), and the full effective action in the IR, for \( k \to 0 \) [10, 11, 12]. In the gravitational context it is constructed using the background field formalism [6, 8], and is a functional \( \Gamma_k[h, \bar{C}, C; \bar{g}] \) depending on the average metric fluctuation \( h_{\mu\nu} \), on the average ghost fields \( \bar{C}_\mu, C^\mu \) and on the background metric \( \bar{g}_{\mu\nu} \). The full quantum field is recovered in the sum \( g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \). The background metric is used to implement in a gauge invariant way the coarse-graining procedure. A cutoff operator constructed with the background metric - generally, a Laplacian type operator - is used to divide the slow field modes from the fast field modes, which are then integrated out. Actually, \( \Gamma_k[h, \bar{C}, C; \bar{g}] \) is invariant only under combined physical plus background gauge transformations; in its most general form can be written as

\[
\Gamma_k[h, \bar{C}, C; \bar{g}] = \bar{\Gamma}_k[\bar{g} + h] + \hat{\Gamma}_k[h, \bar{C}, C; \bar{g}],
\]

\(^{1}\)Here \( \Lambda \) is the UV scale, not to be confused with the running cosmological constant \( \Lambda_k \).
where the functionals introduced are defined by the relations \( \hat{\Gamma}_k[0, 0; \bar{g}] = 0 \) and \( \bar{\Gamma}_k[\bar{g}] = \Gamma_k[0, 0; \bar{g}] \). It follows that the functional \( \hat{\Gamma}_k[g] \) is invariant under physical gauge transformations while the functional \( \bar{\Gamma}_k[h, \bar{C}, C; \bar{g}] \) is invariant under physical plus background gauge transformations and can be interpreted as a generalized background gauge-fixing and ghost action. In this paper we will restrict ourselves to a commonly used truncation where \( \hat{\Gamma}_k[h, \bar{C}, C; \bar{g}] \) is approximated by the classical background gauge-fixing and ghost actions:

\[
\hat{\Gamma}_k[h, \bar{C}, C; \bar{g}] = S_{gf}[h; \bar{g}] + S_{gh}[h, \bar{C}, C; \bar{g}].
\]

Here the background gauge-fixing condition is \( f_\mu[h, \bar{g}] = 0 \), with

\[
f_\mu[h, \bar{g}] = \left( \delta_\mu^\alpha \nabla_\beta - \frac{\beta}{2} \bar{g}^{\alpha\beta} \nabla_\mu \right) h_{\alpha\beta};
\]

the background gauge-fixing action is

\[
S_{gf}[h; \bar{g}] = \frac{1}{2\alpha} \int d^d x \sqrt{\bar{g}} g^{\mu\nu} f_\mu[h, \bar{g}] f_\nu[h, \bar{g}].
\]

\( \alpha \) and \( \beta \) are gauge-fixing parameters that we will fix to \( \alpha = \beta = 1 \). The ghost action related to the background gauge-fixing condition is

\[
S_{gh}[h, \bar{C}, C; \bar{g}] = - \int d^d x \sqrt{\bar{g}} \bar{C}^\mu \left( \nabla_\mu g_{\nu\alpha} \nabla_\nu + \bar{\nabla}_\mu g_{\nu\alpha} \nabla_\nu - \beta \nabla_\mu g_{\nu\alpha} \nabla_\nu \right) C^\nu.
\]

With this choice for \( \hat{\Gamma}_k[\varphi; \bar{g}] \) all the scale dependence is in the functional \( \bar{\Gamma}_k[g] \). Usually (e.g. \([8, 9]\)) truncations of \( \bar{\Gamma}_k[g] \) have been chosen to be a sum of local operators with \( k \)-dependent coupling constants as coefficients. Truncations involving functions of the Ricci scalar as integrand have been considered in \([15, 16]\). Here we introduce a general truncation scheme, which may be called “curvature expansion”, where we expand \( \bar{\Gamma}_k[g] \) in powers of the curvatures, retaining all possible terms involving the D’Alambertian operator \( \Box = \nabla_\mu \nabla^\mu \) by the way of form factors. To order \( R^2 \), we consider\(^1\)

\[
\bar{\Gamma}_k[g] = \int d^d x \sqrt{g} \left[ \frac{1}{16\pi G_k} (2\Lambda_k - R) + RF_{1,k} (-\Box) R + R_{\mu\nu} F_{2,k} (-\Box) R^{\mu\nu} \right] + O \left( R^3 \right).
\]

This is the kind of general action that is postulated in effective field theory in a regime where \( R^2 \ll \nabla \nabla \mathcal{R} \) (see \([17, 18]\)); it is, in fact, the most general action quadratic in the curvature.

\(^{1}\mathcal{R} \) being any curvature.
Note that the form factors $F_{i,k}(x)$ can make the effective average action \((4)\) non-local.

It is possible to derive an exact RG flow equation describing the dependence of $\bar{\Gamma}_k[g]$ on the RG parameter $t = \log k/k_0$, where $k_0$ is an arbitrary reference scale. This reads \((5)\):

$$
\partial_t \bar{\Gamma}_k[g] = \frac{1}{2} \text{Tr}_2 \frac{\partial_t R_k(\Delta_2)}{\Gamma_k^{(2,0,0;0)}[0,0,0;g] + R_k(\Delta_2)} - \text{Tr}_1 \frac{\partial_t R_k(\Delta_1)}{\Gamma_k^{(0,1,1;0)}[0,0,0;g] + R_k(\Delta_1)},
$$

\(R_k(\Delta_2)\) and $R_k(\Delta_1)$ are, respectively, the cutoff kernels of spin-two gravitons and spin-one ghosts; they are functions of the cutoff operators $\Delta_2$ and $\Delta_1$ (to be chosen in a moment) used to separate the slow field modes from the fast ones. The functional form $R_k(z)$ is arbitrary except for the requirements that it should be a monotonically decreasing function in both $z$ and $k$, that $R_k(z) \to 0$ for $z \gg k^2$ and that $R_k(z) \to k^2$ for $z \ll k^2$. The notation $\Gamma_k^{(2,0,0;0)}[0,0,0;g]$ and $\Gamma_k^{(0,1,1;0)}[0,0,0;g]$ is used for the Hessians of the functional $\Gamma_k[h, \bar{C}, C; \bar{g}]$.

Since to insert in the flow equation \((5)\) the full truncation \((4)\) is technically a very difficult task, in this paper we will adopt a one-loop approximation: when computing the Hessians in the right hand side of \((5)\), we drop the contributions from the form factors, we set $\Lambda_k = 0$, and we disregard the running of the wave-functions renormalization of the metric fluctuation and of the ghosts that are contained in the cutoff kernels. Computing the traces and integrating the flow will then give us the one-loop running of the couplings $G_k, \Lambda_k$ and of the form factors $F_{i,k}(x)$ induced by the operator $\int \sqrt{g}R$ in the Einstein-Hilbert part of the ansatz \((4)\).

Within this approximation the flow equation \((5)\), after calculating the Hessians, becomes:

$$
\partial_t \bar{\Gamma}_k[g] = \frac{1}{2} \text{Tr}_2 \frac{\partial_t R_k(\Delta_2)}{\Delta_2 + R_k(\Delta_2)} - \text{Tr}_1 \frac{\partial_t R_k(\Delta_1)}{\Delta_1 + R_k(\Delta_1)}.
$$

We have chosen the cutoff operators $\Delta_2$ and $\Delta_1$ to be the inverse graviton and ghost propagators; this corresponds to a type II cutoff in the nomenclature of \([9]\). These inverse propagators read:

$$
(\Delta_2)^{\alpha\beta}_{\rho\sigma} = -\Box \delta^{\alpha\beta}_{\rho\sigma} + R \left( \delta^{\alpha\beta}_{\rho\sigma} - \frac{1}{2} g^{\alpha\beta} g_{\rho\sigma} \right) + \left( g^{\alpha\beta} R_{\rho\sigma} + R^{\alpha\beta} g_{\rho\sigma} \right) + \frac{1}{2} \left( \delta^{\gamma}_{\rho} R^\beta_{\gamma\sigma} + \delta^\alpha_{\rho} R^\beta_{\gamma\sigma} + \delta_{\rho}^\alpha R_{\gamma\sigma}^{\beta} + R_{\rho}^{\alpha} \delta_{\gamma\sigma}^\beta \right) - \left( R_{\rho}^{\beta} \delta_{\gamma\sigma}^\alpha + R_{\gamma\sigma}^{\alpha} \delta_{\rho}^\beta \right) + \frac{d-4}{2(d-2)} \left( R g^{\alpha\beta} g_{\rho\sigma} + g^{\alpha\beta} R_{\rho\sigma} + R_{\rho\sigma} g^{\alpha\beta} \right),
$$

$$
(\Delta_1)_{\nu}^{\mu} = -\Box \delta^{\mu}_{\nu} - R_{\nu}^{\mu}.
$$

(7)

The running of the zeroth and first order terms in the curvature, in other words of the running
of the cosmological constant $\Lambda_k$ and of Newton’s constant $G_k$, can be computed without need
of the one-loop approximation. This computation has been done, within a type II cutoff here
considered, in [9], so we will limit ourselves to noting the one-loop approximation of the
result, which is:

$$
\begin{align*}
\partial_t \tilde{\Lambda}_k &= -2\tilde{\Lambda}_k + \frac{8\pi(d-3)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \tilde{G}_k \\
\partial_t \tilde{G}_k &= (d-2)\tilde{G}_k - \frac{4\pi(5d^2 - 3d + 24)}{3(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)} \tilde{G}_k^2.
\end{align*}
$$

(8)

where $\tilde{\Lambda}_k = k^{-2}\Lambda_k$ and $\tilde{G}_k = k^{d-2}G_k$ are the dimensionless couplings. In [8] the optimized
cutoff defined later in (18) was used. Already at the one-loop level here considered, the beta
functions (8) exhibit, in $d = 4$, a non-trivial UV attractive fixed-point at the couplings values
$\tilde{\Lambda}_* = \frac{3}{46} \simeq 0.07$ and $\tilde{G}_* = \frac{6\pi^2}{23} \simeq 0.82$. For more details on the Asymptotic Scenario see [9].

The novel contribution in this paper lies in the running of the form factors in $\tilde{\Gamma}_k[g]$. Their
running is obtained expanding the trace on the right hand side of the flow equation (6) in a
curvature power series and computing the $O(R^2)$ terms. This can be done with the non-local
heat kernel expansion methods detailed in Appendix A. We find:

$$
\partial_t \tilde{\Gamma}_k[g]\big|_{R^2} = \frac{1}{(4\pi)^{d/2}} \int d^dx\sqrt{g} \left\{ R \left[ \int_0^\infty ds \tilde{h}_k(s) s^{2-\frac{d}{2}} f_1(-s\Box) \right] R \\
+ R_{\mu\nu} \left[ \int_0^\infty ds \tilde{h}_k(s) s^{2-\frac{d}{2}} f_2(-s\Box) \right] R_{\mu\nu} \right\}.
$$

(9)

Here we defined the function $h_k(z) = \frac{\partial R_k(z)}{z + R_k(z)}$, of which $\tilde{h}_k(s)$ is the anti-Laplace transform.
The functions $f_1(x)$ and $f_2(x)$ in (9) are derived combining the non-local heat kernel functions
for the operators (7) and expressing them in the $\{R^2, R_{\mu\nu}R_{\mu\nu}\}$ basis, as further explained in
Appendix A; they read:

$$
\begin{align*}
\left. f_1(x) \right|_{R^2} &= \frac{9d^3 - 61d^2 - 10d + 320}{128(d-2)} f(x) - \frac{3d^2 + 7d + 16}{32x} f(x) \\
&\quad + \frac{17d^2 + 45d + 96}{192x} - \frac{d(d-3)}{32x^2} [f(x) - 1], \\
\left. f_2(x) \right|_{R^2} &= \frac{d^2 + 2d - 16}{4(d-2)} f(x) + \frac{d}{x} f(x) - \frac{(27 - d)d}{24x} f(x) + \frac{d(d-3)}{4x^2} [f(x) - 1],
\end{align*}
$$

(10)
where the basic heat kernel non-local form factor \( f(x) \) is defined by

\[
f(x) = \int_0^1 d\xi e^{-\xi(1-\xi)x}.
\]  

From (9) we can extract the running of the form factors:

\[
\partial_t F_{i,k} = \frac{1}{(4\pi)^{d/2}} \int_0^\infty ds \tilde{h}_k(s) s^{2-d/2} f_i(-s\Box),
\]  

for \( i = 1, 2 \). For each \( i \) this can be rewritten in terms of a combination of \( Q \)-functionals within parameter integrals. Inserting (10) in (12) and using the definitions of the \( Q \)-functionals from Appendix B, we find the form:

\[
(4\pi)^{d/2} \partial_t F_{1,k}(x) = \frac{9d^3 - 61d^2 - 10d + 320}{128(d-2)} \int_0^1 d\xi \, Q_{\frac{d}{2}-2}[h_k(z + x\xi(1-\xi))]
- \frac{3d^2 + 7d + 16}{32x} \int_0^1 d\xi \, Q_{\frac{d}{2}-1}[h_k(z + x\xi(1-\xi))]
+ \frac{17d^2 + 45d + 96}{192x} Q_{\frac{d}{2}-1}[h_k]
- \frac{d(d-3)}{32x^2} \left\{ \int_0^1 d\xi \, Q_{\frac{d}{2}}[h_k(z + x\xi(1-\xi))] - Q_{\frac{d}{2}}[h_k] \right\}
\]  

and

\[
(4\pi)^{d/2} \partial_t F_{2,k}(x) = \frac{d^2 + 2d - 16}{4(d-2)} \int_0^1 d\xi \, Q_{\frac{d}{2}-2}[h_k(z + x\xi(1-\xi))]
+ \frac{d}{x} \int_0^1 d\xi \, Q_{\frac{d}{2}-1}[h_k(z + x\xi(1-\xi))] - \frac{(27 - d)d}{24x} Q_{\frac{d}{2}-1}[h_k]
+ \frac{d(d-3)}{4x^2} \left\{ \int_0^1 d\xi \, Q_{\frac{d}{2}}[h_k(z + x\xi(1-\xi))] - Q_{\frac{d}{2}}[h_k] \right\},
\]  

where \( x \) stands for \(-\Box\).

Equations (13) and (14) are the contributions induced by the operator \( \int \sqrt{g}R \), within a type II cutoff, to the flow of the curvature squared form factors in truncation (4). They are valid in general dimension and for arbitrary cutoff shape function. These are the main results of this section.
3 Non-local effective average action in four dimensions: flow and renormalization

We now study the flow (13) and (14) of the form factors in the physical dimension \( d = 4 \). We find:

\[
(4\pi)^2 \partial_t F_{1,k}(x) = -\frac{15}{32} \int_0^1 \! \! d\xi Q_0 [h_k (z + x\xi(1 - \xi))] - \frac{23}{8x} \int_0^1 \! \! d\xi Q_1 [h_k (z + x\xi(1 - \xi))] + \\
+ \frac{137}{48x} Q_1[h_k] - \frac{1}{8x^2} \left\{ \int_0^1 \! \! d\xi Q_2 [h_k (z + x\xi(1 - \xi))] - Q_2[h_k] \right\} \tag{15}
\]

and

\[
(4\pi)^2 \partial_t F_{2,k}(x) = \int_0^1 \! \! d\xi Q_0 [h_k (z + x\xi(1 - \xi))] + \frac{4}{x} \int_0^1 \! \! d\xi Q_1 [h_k (z + x\xi(1 - \xi))] + \\
- \frac{23}{6x} Q_1[h_k] + \frac{1}{x^2} \left\{ \int_0^1 \! \! d\xi Q_2 [h_k (z + x\xi(1 - \xi))] - Q_2[h_k] \right\} . \tag{16}
\]

The previous equations can be rewritten as:

\[
\partial_t F_{i,k}(x) = \frac{1}{(4\pi)^2 g_i \left( \frac{x}{k^2} \right)} , \tag{17}
\]

where the functions \( g_i(u) \) can be calculated once a cutoff shape function has been chosen. Note that, in \( d = 4 \), all the \( k \)-dependence is through the \( x/k^2 \) dependence of the functions \( g_i(u) \). If we employ the optimized cutoff shape function [19, 20]:

\[
R_k(z) = (k^2 - z) \theta(k^2 - z) , \tag{18}
\]

we find (see the Appendix B for the relevant integrals used):

\[
g_1(u) = \frac{1}{60} + \left( -\frac{1}{60} + \frac{19}{5u} + \frac{1}{15u^2} \right) \sqrt{1 - \frac{4}{u} \theta(u - 4)} \ tag{19}
\]

\[
g_2(u) = \frac{7}{10} - \left( \frac{7}{10} + \frac{76}{15u} + \frac{8}{15u^2} \right) \sqrt{1 - \frac{4}{u} \theta(u - 4)} . \tag{20}
\]

These are the beta functions for the non-local form factors. They are plotted in Figure 1.

The functions (19) and (20) are constant in a neighborhood of the origin: if instead of considering the full functional dependence of the form factors \( F_{i,k}(x) \) on the D’Alambertian,
Figure 1: The functions $g_1(u) - \frac{1}{60}$ (upper curve) and $g_2(u) - \frac{7}{10}$ (lower curve), representing the flow of the form factors in (17) after we imposed the UV boundary conditions (25), as a function of $u$. Note that the flow stops for $x < 4k^2$: only IR or slow modes contribute effectively to the RG running of the form factors.

we had considered only a local expansion to a polynomial, we would had found zero beta functions for the running couplings of all derivative terms of the form $\int \sqrt{g} R_{\mu\nu}(-\Box)^n R^{\mu\nu}$ or $\int \sqrt{g} R (-\Box)^n R$. As already noticed [14], this shows that truncations to a finite number of local terms are generally not powerful enough to describe correctly IR physics. Truncations of the effective average action need at least to project the flow onto an infinite dimensional subspace of theory space, as here the one of the functions $F_{i,k}(x)$, to correctly recover the effective action for $k = 0$.

We now integrate the flow equations (17) from a UV scale $\Lambda$ down to a generic IR scale $k$. It is expected that for $\Lambda \to \infty$ we will encounter the usual ultraviolet divergences that are found in one-loop effective theory. Using $\partial_t = k\partial_k$ in (17), we find

$$F_{i,\Lambda}(x) - F_{i,k}(x) = \frac{1}{(4\pi)^2} \int_k^\Lambda \frac{dk'}{k'} g_i \left( \frac{x}{k'^2} \right),$$

(21)

and after going to the variable $y = x/k^2$ (with $dk/k = -dy/2y$) we get

$$F_{i,\Lambda}(x) - F_{i,k}(x) = \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{dy}{2y} g_i(y).$$

(22)

The constant terms in the flow functions (19) and (20) make the integrals in (22) logarithmically divergent at the lower limit when $\Lambda \to \infty$. We can isolate this divergences in the
following way:

\[ F_{1,\Lambda}(x) - F_{1,k}(x) = \frac{1}{(4\pi)^2} \left[ \frac{1}{60} \left( \frac{\log \Lambda}{k_0} + \frac{\log k_0}{k} \right) + \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{dy}{2y} \left[ g_1(y) - \frac{1}{60} \right] \right], \tag{23} \]

\[ F_{2,\Lambda}(x) - F_{2,k}(x) = \frac{7}{10} \frac{1}{(4\pi)^2} \left[ \frac{1}{60} \left( \frac{\log \Lambda}{k_0} + \frac{\log k_0}{k} \right) + \frac{1}{(4\pi)^2} \int_{x/\Lambda^2}^{x/k^2} \frac{dy}{2y} \left[ g_2(y) - \frac{7}{10} \right] \right], \tag{24} \]

where \( k_0 \) is an arbitrary reference scale which plays a role akin to \( \mu \) in effective field theory. The logarithmic \( \Lambda \) terms in (23) and (24) correspond to the UV divergences first calculated in [21]. We can renormalize the theory imposing the UV boundary conditions:

\[ F_{1,\Lambda}(x) = \frac{1}{(4\pi)^2} \left[ \frac{1}{60} \left( \frac{\log \Lambda}{k_0} + c_1 \right) \right], \]

\[ F_{2,\Lambda}(x) = \frac{7}{10} \frac{1}{(4\pi)^2} \left[ \frac{1}{60} \left( \frac{\log \Lambda}{k_0} + c_2 \right) \right], \tag{25} \]

where the \( c_i \) are possible finite renormalizations[3]. The important point in the equations (23) and (24) is that the integrals are now convergent in the lower limit when we take \( \Lambda \to \infty \).

The scale-dependent form factors at the scale \( k \) turn out to be:

\[ F_{1,k}(x) = \frac{1}{32\pi^2} \left\{ \frac{1}{30} \log \left( \frac{1 + \sqrt{1 - \frac{4k^2}{x}}}{2} \right) + \left( -\frac{601}{900} + \frac{1139k^2}{450x} + \frac{2k^4}{75x^2} \right) \right\} \theta(x - 4k^2) + \frac{1}{60} \log \left( \frac{k^2}{k_0^2} \right) \theta(4k^2 - x) \]

\[ F_{2,k}(x) = \frac{1}{32\pi^2} \left\{ \frac{7}{5} \log \left( \frac{1 + \sqrt{1 - \frac{4k^2}{x}}}{2} \right) - \left( \frac{41}{75} + \frac{84k^2}{25x} + \frac{16k^4}{75x^2} \right) \right\} \theta(x - 4k^2) + \frac{7}{10} \log \left( \frac{k^2}{k_0^2} \right) \theta(4k^2 - x) \right\} . \tag{26} \]

These results are now to be reinserted in the \( O(R^2) \) part of the truncation ansatz [4]:

\[ \bar{\Gamma}_k[g]_{R^2} = \int d^4x \sqrt{g} \left[ RF_{1,k}(-\Box) R + R_{\mu\nu} F_{2,k}(-\Box) R^{\mu\nu} \right]. \tag{27} \]

[3] We stress that there is no nontrivial assumption in setting these boundary conditions: the functions \( F_{i,k} \) diverge logarithmically for large \( k \) irrespectively of this choice, which is just a technical convenience for eliminating \( \Lambda \). This is a one-loop result and does not conflict with previous results about the non-perturbative fixed point for the curvature squared couplings [22].
This is the result we were looking for. Note that the form factors are continuous at $x = 4k^2$, and that for $k \rightarrow 0$ we obtain a well-defined limit, namely the action:

$$\bar{\Gamma}_0[g] |_{\mathbb{R}^2} = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \left[ \frac{1}{60} R \log \left( \frac{-\Box}{k_0^2} \right) R + \frac{7}{10} R_{\mu\nu} \log \left( \frac{-\Box}{k_0^2} \right) R^{\mu\nu} \right], \quad (28)$$

where we fixed the finite renormalizations in $(25)$ to $c_1 = -\frac{1}{16}\frac{\pi^2}{1800}$ and $c_2 = -\frac{1}{(4\pi)^2}\frac{41}{150}$. The resulting non-local terms equal the part of standard one-loop quantum gravity effective action that arises solely from graviton and ghosts vacuum polarization [23].

In summary, we have obtained a non-local effective average action, equations (26) and (27), that flows from an ultraviolet scale $k = \Lambda$ to the infrared limit $k = 0$, and in the latter limit equals the expected effective field theory result. In the next section we will discuss possible physical effects arising from this effective average action.

4 Quantum Newtonian potential from the effective average action

In this section we will compute the quantum corrections to the Newtonian potential stemming from our effective action. We couple the gravity effective action to a classical matter source:

$$\bar{\Gamma}_k[g] = \frac{1}{16\pi G_k} \int d^4x \sqrt{g} \left( 2\Lambda_k - R \right) + \int d^4x \sqrt{g} \left[ RF_{1,k}(-\Box) R + R_{\mu\nu} F_{2,k}(\Box) R^{\mu\nu} \right] + \int \sqrt{g} \mathcal{L}_{\text{mat}}. \quad (29)$$

Note that the matter action is taken to be scale independent. From this action we will derive the equations of motion under the assumption that the gravitational field is weak throughout space in addition to static. Before presenting the explicit calculations, we would like to clarify two important points. Firstly, we will switch to work in a 3+1 static spacetime, analytically continuing our Euclidean expressions for the form factors into the Lorentzian sector. With the usual definition of the “in-out” effective action, this would imply a replacement of the Euclidean propagators by the corresponding Feynman propagators in the form factors, and the resulting field equations would be neither real nor causal. In order to get real and causal equations, one can introduce a “Closed Time Path” (CTP) or “in-in” effective action [24]. As shown in [25], when the quantum fluctuations are in the vacuum state, the CTP procedure is equivalent to the replacement of the Euclidean by the retarded propagators in the form...
factors appearing in the field equations. Due to the staticity assumption, this is equivalent to the replacement $\Box \rightarrow \nabla^2$ (the 3-Laplacian) in the form factors. Secondly, as pointed out in [23], the solutions of the field equations derived from the effective action will depend on the gauge fixing parameters, and therefore they are not physical. Indeed, for our calculations we considered the particular values $\alpha = \beta = 1$ in the gauge fixing condition (3). In the general case, the effective action and the quantum corrections to the metric will depend explicitly on $\alpha$ and $\beta$. In order to obtain physical results, it is necessary to define an observable from the quantum corrected metric, as proposed in [23]. Though important, this issue will not be relevant in the discussion that follows.

We write $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ with $h_{\mu\nu}$ small everywhere, and we will proceed assuming $k$ to be a fixed parameter, ignoring for the moment the possibility (discussed e.g. in [26, 27, 28, 29], and also later in this paper) that it should depend on position. The equations of motion are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \Lambda_k g_{\mu\nu} = 8\pi G_k \left( T^{\text{mat}}_{\mu\nu} - F_{1,k}(-\nabla^2)H^{(1)}_{\mu\nu} - F_{2,k}(-\nabla^2)H^{(2)}_{\mu\nu} \right)$$

(30)

where $H^{(1)}_{\mu\nu} = 4\nabla_\mu \nabla_\nu R - 4\eta_{\mu\nu} \nabla^2 R$ and $H^{(2)}_{\mu\nu} = 2\nabla_\mu \nabla_\nu R - \eta_{\mu\nu} \nabla^2 R - 2\nabla^2 R_{\mu\nu}$ are the variations of the squared curvature scalar and squared Ricci tensor. All curvature tensors are evaluated at first order in $h_{\mu\nu}$. We will assume in what follows that we are in a non-cosmological regime where $\Lambda_k$ can be neglected.

We choose the classical matter to be static and nonrelativistic so that there exists a quasi-Cartesian coordinate system in which

$$T^{\text{mat}}_{\mu\nu} = \text{diag}(\rho(x), 0, 0, 0).$$

(31)

We write for the metric perturbation $h_{\mu\nu} = h^c_{\mu\nu} + h^q_{\mu\nu}$ where $h^c_{\mu\nu}$ solves the classical equations of motion and $h^q_{\mu\nu}$ is $O(\hbar)$; also, we write $G_k = G_0(1 + \delta G_k)$ with $G_0$ being the experimental value of $G$, assumed to be measured at $k = 0$, and $\delta G_k$ being $O(\hbar)$. Expanding to the first order in $\hbar$ and working in the Lorentz gauge, the equations for the classical and quantum parts of the metric read:

$$\nabla^2 h^c_{\mu\nu} = -16\pi G_0(T^{\text{mat}}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\lambda\kappa} T^{\text{mat}}_{\lambda\kappa}) = -8\pi G_0 \rho \text{diag}(1, 1, 1, 1)$$

(32a)

$$\nabla^2 h^q_{\mu\nu} = 16\pi G_0 \left[ \left( 4F_{1,k}(-\nabla^2) + 2F_{2,k}(-\nabla^2) \right) \partial_\mu \partial_\nu R + \eta_{\mu\nu} \left( 2F_{1,k}(-\nabla^2) + F_{2,k}(-\nabla^2) \right) \nabla^2 R - 2F_{2,k}(-\nabla^2) \nabla^2 R_{\mu\nu} - \frac{1}{2} \delta G_k \rho \text{diag}(1, 1, 1, 1) \right]$$

(32b)
In (32b) the Ricci tensor and its trace are understood to be computed from $h^c_{\mu\nu}$ exclusively. The classical Newtonian potential $\phi(x)$ is equal to $-\frac{1}{2}h^c_{00}$ and, per (32a), is found solving Poisson’s equation as usual. Its quantum correction, bearing the same relation to $h^q_{00}$, will be found from

$$h^q_{00}(x) = \delta G h^c_{00}(x) - 256\pi^2 G_0^2 \left[ F_{1,k}(-\nabla^2) + F_{2,k}(-\nabla^2) \right] \rho(x),$$

(33)

which is obtained replacing in (32b) $R_{\mu\nu}$ by $-\frac{1}{2} \nabla^2 h^c_{\mu\nu}$, using (32a), and cancelling Laplacians. Therefore, the quantum correction to the Newtonian potential consists of two terms: a trivial shift due to renormalization of $G$, plus a nontrivial part that is found by direct application of the nonlocal form factor $F_1 + F_2$ to the classical matter distribution. From now on we take $h^q_{00}$ to refer only to the nontrivial part, absorbing the first term in a redefinition of $h^c_{00}$.

Since the Laplacian in all the preceding expressions is flat (to keep only the first order in the metric perturbation), the action of the form factors in (33) can be computed with an ordinary Fourier transform:

$$h^q_{00}(x) = -256\pi^2 G_0^2 \left( \frac{1}{(2\pi)^3} \right) \int d^3x' \rho(x') \int d^3p \left[ F_{1,k}(p^2) + F_{2,k}(p^2) \right] e^{-ip(x-x')}$$

$$= -\frac{4G_0^2}{\pi^2} \int d^3x' \frac{\rho(x')}{|x-x'|} \int_0^\infty dp \, pF_k(p) \sin(p|x-x'|),$$

(34)

where we have defined

$$F_k(p) = 32\pi^2(F_{1,k}(p^2) + F_{2,k}(p^2))$$

$$= \frac{43}{60} \log \left( \frac{p^2 + (k^2 - p^2)\theta(4k^2 - p^2)}{k_0^2} \right) + \left[ \left( -\frac{1093}{900} - \frac{373k^2}{450p^2} - \frac{14k^4}{75p^4} \right) \sqrt{1 - \frac{4k^2}{p^2}} \right]$$

$$+ \frac{43}{30} \log \left( \frac{1 + \sqrt{1 - \frac{4k^2}{p^2}}}{2} \right) \theta(p^2 - 4k^2).$$

(35)

Equations (34) and (35) comprise the result we wanted for the quantum correction. For further analysis we call $I_1$ the term of the $p$-integral in (34) that comes from the first term of (35), and $I_2$ the $p$-integral of the remaining terms. $I_1$ can be evaluated exactly as a combination of elementary integrals and distributional Fourier transforms, whereas in $I_2$ no
such closed form can be found. We have for $I_1$:

$$I_1 = \frac{43}{60} \left[ \log \left( \frac{k^2}{k_0^2} \right) \int_0^{2k} dp \, p \sin(p |x - x'|) + \int_{2k}^{\infty} dp \, p \log \left( \frac{p^2}{k_0^2} \right) \sin(p |x - x'|) \right]$$

$$= \frac{43}{60} \left[ \int_0^{\infty} dp \, p \log \left( \frac{p^2}{k_0^2} \right) \sin(p |x - x'|) - \int_{2k}^{\infty} dp \, p \log \left( \frac{p^2}{k_0^2} \right) \sin(p |x - x'|) \right]$$

$$= 43 \frac{60}{60} \left[ -\pi + \log(k_0^2) \frac{\delta'(X)}{2} + \log(4) \left( \frac{\sin(2kX)}{X^2} - \frac{2k \cos(2kX)}{X} \right) 
+ \frac{2 \text{Si}(2kX)}{X^2} - \frac{2 \sin(2kX)}{X^2} \right],$$

(36)

The first integral, though divergent, is well-defined as the Fourier transform of a generalized function and can be computed from the formulas in [30]; the second one can be computed analytically. Thus:

$$I_1 = \frac{43}{60} \left[ -\frac{\pi}{X^2} + \log(k_0^2) \frac{\delta'(X)}{2} + \log(4) \left( \frac{\sin(2kX)}{X^2} - \frac{2k \cos(2kX)}{X} \right) 
+ \frac{2 \text{Si}(2kX)}{X^2} - \frac{2 \sin(2kX)}{X^2} \right],$$

(37)

where we defined $X = |x - x'|$.

The remaining terms, comprising $I_2$, can also be rewritten as a combination of distributional Fourier transforms and convergent integrals, but for them the convergent part cannot usually be computed in closed form (though it can be investigated numerically if so desired). For this reason, we restrict ourselves to evaluating the large $X$ asymptotic expansion of the result. If the matter distribution is a point source, i.e. $\rho(x) = M \delta^3(x)$, this will give us the long-distance quantum corrections to the Newtonian potential of a point source in an asymptotic series. However, note that we will later call into question the physical validity of such an asymptotic expansion, so the following calculation needs to be taken with a grain of salt.

The large frequency asymptotic form of the Fourier transform of a generalized function of the kind we have can be found using Theorem 19 from reference [30]. Applying it gives us that for large $X$ we have:

$$I_2 = -\frac{43}{30} \log(4) \frac{k \cos(2kX)}{X} - \frac{7}{200} \sqrt{\pi} k \frac{\sin \left( \frac{2kX + \frac{3\pi}{4}}{X^{3/2}} \right)}{X^{3/2}} 
+ \frac{43}{30} \left( 1 + \log(2) \right) \frac{\sin(2kX)}{X^2} + o \left( \frac{1}{X^2} \right).$$

(38)

Joining this with the asymptotic expansion for the result we obtained for $I_1$, we conclude that the quantum correction for the Newtonian potential of a point source of mass $M$ is given
The full Newtonian potential of the point source would therefore be given asymptotically by:

\[
\phi(r) = - \frac{MG_0}{r} \left[ 1 + \frac{43G_0k}{15\pi r^2} + \frac{7G_0\sqrt{k} \sin(2kr + 3\pi/4)}{100\pi^{3/2} r^{3/2}} + \frac{43G_0}{30\pi^2} \frac{(2\cos(2kr) + \pi)}{r^2} + o \left( \frac{1}{r^2} \right) \right].
\]  

(40)

One might find this asymptotic result peculiar in several respects: the leading order correction is dominant over the EFT ones (\(k/r^2\) versus \(1/r^3\)), and the subleading terms are oscillatory, which is difficult to interpret physically. We suggest that this is due to the large \(X\) expansion being unphysical for the problem under consideration. Recall that the equations of motion derived from the effective action predict the expectation values of quantities such as the metric. From the effective average action, which is defined by a path integral with an infrared cutoff at scale \(k\), we would expect to obtain, if \(k\) is treated as fixed, equations of motion for “approximate expectation values”, that do not include the information about low frequency field modes. These might be a good approximation in the \(kr \ll 1\) regime (where \(r\) is the radial coordinate for us, or more in general the greatest physical length scale involved in the problem) but we would expect the results to be incorrect for \(kr \gg 1\). If this reasoning is accepted, the result in (40) is not physical, since it is obtained as a large \(r\) expansion for fixed \(k\); the physical quantum correction to the potential is to be obtained instead from the \(k \to 0\) limit at fixed \(r\).

Taking the limit \(k \to 0\) in (34) in \(I_2\) gives zero up to a purely local term involving \(\delta'(r)\). In \(I_1\), this results in a similar local term plus \(43\pi/(60r^2)\). This latter term implies a nontrivial long-distance correction for the Newtonian potential

\[
\phi(r) = - \frac{MG_0}{r} \left[ 1 + \frac{43G_0}{30\pi r^2} \right]
\]  

(41)

(because \(\delta G_k\), obtained by integrating the flow (8), is \(\sim G_0k^2\) and thus vanishes at \(k = 0\)). This is the same result obtained in effective quantum gravity for the contribution of the graviton and ghost vacuum polarization diagrams to \(\phi\) \([1, 23]\). Since we have treated the matter source in a purely classical way, we are not obtaining the terms due to vertex...
corrections. This confirms that the effective field theory quantum corrections to Newton’s law are indeed recovered in the right limit from the effective average action.

Our remarks above imply a general philosophy of interpreting the effective average action as a useful device to follow the flow of the renormalization group (and hopefully discover a UV fixed point) but not as an action from which physics can be directly extracted by solving equations of motion: these, predicting expectation values of observables, should be computed from the $k = 0$ usual effective action. On the other hand, there have been several attempts [26, 27, 28, 29] to extract physics from the effective average action itself by identifying $k$ with an inverse distance scale of the spacetime under consideration, instead of as a fixed parameter. For a static and spherically symmetric situation like the one we are considering (with just a point source) this proposal means $k = \zeta/r$ with $\zeta$ a numerical constant. (Note that above we argued that the effective average action with cutoff $k$ is to be trusted only for $k \ll r^{-1}$, whereas this approach conjectures that it can be trusted only for $k = \zeta r^{-1}$.) If we do this replacement before computing $I_1$ and $I_2$, we see that in the case of a point source the variable $x$ is just $2\zeta$, and so all the functions of it are numerical constants. The Newtonian potential would therefore be

$$\phi(r) = -\frac{MG_0}{r} \left[ 1 + \frac{\zeta G_0}{r^2} \right]$$

with $\zeta$ a numerical constant (the $\delta G_k$ term gives a similar contribution, as has often been noted). This agrees with the general form of the effective field theory correction discussed above. However, this agreement in form is rather trivial, following just from dimensional analysis and the lack of other length scales in the problem once we make the conjectured identification for $k$. A more interesting test of the conjectured identification is in the case of an extended source with spherical symmetry, such as a spherical star. Here a naive application of this conjecture that equates $k$ with the inverse distance to the center of the star, at the level of (34), leads to

$$h_{00}^i(x) = -\frac{4\zeta^2 G_0^2}{\pi^2} \int \frac{d^3x'}{|x - x'| r^2 f\left(\frac{\zeta |x - x'|}{r}\right)}$$

which is obtained from (34) by making the replacement $k = \zeta/r$, changing variables from $p$ to $v = pr$, and defining $f$ as the result of the $v$-integrals $I_1 + I_2$ with $1/r^2$ taken out. The

\footnote{A local term with $\delta'(r)$ is also found in effective quantum gravity, with an arbitrary constant involving the renormalization scale $\mu$.}
other hand, the result obtained from effective theory (and from our formulas as $k \to 0$) is

$$h_{00}^q(x) = \frac{43G_0^2}{15\pi} \int d^3x' \frac{\rho(x')}{|x - x'|^3}. \quad (44)$$

These two formulas agree asymptotically for large $r$ (up to a factor related to $\zeta$), but will in general be different at closer distances where results depend on new length scales implicit in the function $\rho(r)$ for the density within the star (see [31] for a discussion of how (44) depends on this).

It is not surprising that the direct identification $k = \zeta/r$ fails, even in spherically symmetric spacetimes, when there is more than one length scale of interest. $k$ is a momentum scale, and matching it with a spacetime scale, as its inverse, should be possible unambiguously when there is only one spacetime scale to be used (such as $r$ for a point source). But when there are more length scales involved, such as the radius of a star or other parameters relevant to its internal structure, there is no a priori reason to expect the simple matching of $k$ with inverse radial distance to be correct. It is noteworthy that the qualitative agreement with effective field theory for a general source could be obtained if (instead of $r^{-1}$) we identified $k$ with $|x - x'|^{-1}$ before integrating over $x'$, in effect defining a $k(x, x')$ as the inverse distance to each point source $x'$ instead of the inverse distance to the global center of symmetry of the spacetime. The plausible reason why this works is because for each point source the direct identification with inverse radial distance is valid, and within our weak field approximation linear superposition of sources is valid; we make no claim as to its validity in a more general situation, a matter which deserves further research. The alternative position that we are suggesting is that the equations of motion derived from the effective average action are to be trusted only in a small $k$ regime, and ultimately are only correct when $k \to 0$ and we recover the usual effective action.

Conclusions

The effective average action and its renormalization flow are promising research avenues for developing a theory of quantum gravity. In this paper we have sought to accomplish two related goals. The first one is to take the first steps towards investigating the flow of non-local terms in the action. We have been able to compute and exhibit the solution for the non-local form factors in the second-order term in a general curvature expansion for the action, within the one-loop approximation where the flow is generated only by the operator $\int \sqrt{g} R$.
Equations (26) and (27) summarize this result.

It would obviously be desirable to go beyond our approximations. One-loop computations of the flow of higher order non-local terms seems possible in principle by extension of the methods of this paper, though computationally very intensive. On the other hand, going beyond the one-loop approximation and obtaining information about the non-perturbative flow is a difficult and open research question. A first step towards it would be to improve the running of the form factors (17) by computing the beta function not from the bare action but from an action with running $G_k$ and $\Lambda_k$, which can be found from the beta functions (8).

Another question is how our results generalize to $d$ dimensions; so far, the only similar case that has been investigated is the $d = 2$ computation for the Polyakov action in [14]. Local truncations in extra dimensions were explored in [32].

Our second goal has been to clarify the relationship between the effective average action formalism and the usual perturbative effective field theory. These goals are related in that the one-loop effective action for quantum gravity is non-local. We have shown how this action is recovered in the $k \to 0$ limit of our flow, and how the long-distance quantum corrections it induces on the Newtonian potential are also recovered in the same limit. We have also exhibited a computation of the quantum correction from the $k$-dependent average action, and we have argued that it cannot be trusted except in the $k \to 0$ limit, and in particular not in the $kr \gg 1$ regime. It has been shown in passing that the often-suggested prescription of identifying $k$ with the inverse radial distance does not recover effective theory results in general, beyond the simplest case of a point source. However, more sophisticated relationships of $k$ with inverse distance scales (such as the one considered in the last paragraph of the previous section) are not ruled out and require further research. Our results might also be of interest in applications of the effective average action to cosmology, where similar identifications of $k$ with inverse time scales have been suggested [33, 34].

A central question faced by the effective average action formalism is how to extract physical predictions from the flow. In our opinion, our results lend credence to the position that the relevant physics is to be extracted from the full effective action at $k \to 0$. $\Gamma_k$ would then be considered a useful device for studying the flow of the theory and unveiling a UV fixed point, but not to compute observables, which should in principle be computed (at all scales) from the effective action $\Gamma_0$. This is to be taken only as a tentative position, and more work is needed on the foundational conceptual issues of the effective average action approach to settle these questions definitely.
A  Non-local heat kernel expansion and computation of the beta functions

Our problem is to compute functional traces of the form $\text{Tr}[W(\Delta)]$ where $W$ for gravitons and for ghosts is given in the two terms of (6) and $\Delta$ for each is given in (9). Following standard practice, e.g. [9], we use $K(s)$, the heat kernel of the operator $\Delta$:

$$\text{Tr}[W(\Delta)] = \int_0^\infty ds \text{Tr}[K(s)] \tilde{W}(s)$$

where $\tilde{W}$ is the Laplace anti-transform of $W$. A general expression for the trace of the heat kernel of operators of the form (9), in the form of a curvature expansion with non-local form factors, is obtained in [17, 35]. Using a notation in which $\Delta = -\Box - \hat{P} + \frac{1}{6} \hat{R} \hat{1}$, it reads

$$\text{Tr}[K(s)] = \frac{1}{(4\pi s)^{d/2}} \int d^d x \sqrt{-g} \text{tr} \left\{ 1 + s\hat{P} + s^2 \left[ R_{\mu\nu}\phi_1(-s\Box)R^{\mu\nu}\hat{1} + R\phi_2(-s\Box)R\hat{1} + \hat{P}\phi_3(-s\Box)R + \hat{P}\phi_4(-s\Box)\hat{P} + \hat{R}_{\mu\nu}\phi_5(-s\Box)\hat{R}^{\mu\nu} \right] \right\} + O(\mathcal{R}^3),$$

where $\hat{R}_{\mu\nu}$ is the curvature of the covariant derivative acting on the space where the operator $\Delta$ acts. In our case we have $\hat{R}_{\mu\nu} = -2R^{\gamma\alpha\beta\mu\nu} \delta_\gamma^\sigma$ for the graviton case, and $\hat{R}_{\mu\nu} = R_{\beta\mu\nu}$ in the ghost case. The functions $\phi_i(x)$ take the form:

$$\phi_1(x) = \frac{f(x) - 1 + \frac{1}{6}x}{x^2},$$

$$\phi_2(x) = \frac{1}{8} \left[ \frac{1}{36} f(x) + \frac{f(x) - 1}{3x} - \frac{f(x) - 1 + \frac{1}{6}x}{x^2} \right],$$

$$\phi_3(x) = \frac{1}{12} f(x) + \frac{f(x) - 1}{x},$$

$$\phi_4(x) = \frac{1}{2} f(x),$$

$$\phi_5(x) = -\frac{1}{2} \frac{f(x) - 1}{x},$$

where $f(x)$ is defined above in (11). To obtain our claimed result (10), all we need to do is reexpress the five second-order terms in (46) in terms of the basis $R_{\mu\nu}R^{\mu\nu}$, $R^2$. This is done by expressing $\hat{R}_{\mu\nu}$ in terms of $R_{\alpha\beta\mu\nu}$ as defined above, expressing $\hat{P}$ in terms of $R_{\alpha\beta\mu\nu}$, $R_{\mu\nu}$ and $R$ by using the definition (7) for the $\Delta$ operators, and eliminating $R_{\alpha\beta\mu\nu}$ in terms of $R_{\mu\nu}$.
and $R$ by means of the identity

$$R_{\alpha\beta\mu\nu} = \nabla_\mu \nabla_\alpha \frac{1}{R_{\nu\beta}} - \nabla_\nu \nabla_\alpha \frac{1}{R_{\mu\beta}} - \nabla_\mu \nabla_\beta \frac{1}{R_{\nu\alpha}} + \nabla_\nu \nabla_\beta \frac{1}{R_{\mu\alpha}} + O(R^2), \quad (48)$$

proven in [17]. Note that to the order we are working we can freely commute covariant derivates as well as functions of them, throwing away terms of higher order in the curvature arising from commutators. Doing the described operations for the graviton part and the ghost part of the trace and combining them we obtain result [10].

**B The $Q$-functionals and their integrals**

The $Q$-functionals of a function $f(z)$ are defined by:

$$Q_n[f] = \int_0^\infty ds \tilde{f}(s)s^{-n}, \quad (49)$$

where $\tilde{f}(s)$ is the anti-Laplace transform of $f(z)$. They have an equivalent expression in terms of $f(z)$ as:

$$Q_n[f] = \left\{ \begin{array}{ll}
\frac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} f(z) & n > 0 \\
(-1)^n f^{(n)}(0) & n \leq 0
\end{array} \right. \quad (50)$$

In the case an additional factor $e^{-as}$ is present in (49) the functions $f(z)$ in (50) are to be evaluated at the shifted point $z + a$.

Our expressions for the traces can be rewritten in terms of $Q$-functionals with $n = 0, 1, 2$. We encounter $Q$-functionals inside parameter integrals of the form:

$$\int_0^1 d\xi Q_n[h_k(z + x\xi(1 - \xi))] \quad (51)$$

where recall that $h_k(z) = \frac{\partial R_k(z)}{z + R_k(z)}$. These integrals can be calculated analytically if we employ the optimized cutoff (18). The results are:

$$\int_0^1 d\xi Q_0[h_k(z + x\xi(1 - \xi))] = 2 \left[ 1 - \sqrt{1 - \frac{4}{u}\theta(u - 4)} \right], \quad (52)$$

$$\int_0^1 d\xi Q_1[h_k(z + x\xi(1 - \xi))] = 2k^2 \left[ 1 - \frac{u}{6} + \frac{u}{6} \left( 1 - \frac{4}{u} \right)^\frac{3}{2} \theta(u - 4) \right], \quad (53)$$
\[ \int_0^1 d\xi Q_2 [h_k (z + x\xi (1 - \xi))] = 2k^4 \left[ \frac{1}{2} - \frac{u}{6} + \frac{u^2}{60} - \frac{u^2}{60} \left( 1 - \frac{4}{u} \right)^{\frac{5}{2}} \theta(u - 4) \right], \quad (54) \]

where \( u = x/k^2 \). Using these results in (16) leads to the beta functions (19) and (20).

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