The external fundamental group of an algebraic number field

T.M. Gendron

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Av. Universidad s/n, Lomas de Chamilpa, Cuernavaca CP 62210, Morelos, Mexico

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Abstract

We associate to every algebraic number field $K/Q$ a hyperbolic surface lamination and an external fundamental group $\hat{\Gamma}_K$: a generalization of the fundamental germ construction of Gendron that necessarily contains external (not first order definable) elements. The external fundamental group $\hat{\Gamma}_Q$ is an extension of the absolute Galois group $\hat{\Gamma}_Q$, that conjecturally contains a subgroup whose abelianization is isomorphic to the idèle class group. 

Résumé

Le groupe fondamental externe d’un corps de nombres algébriques. On associe à chaque corps de nombres algébriques $K/Q$ une lamination en surfaces hyperboliques et un groupe fondamental externe $\hat{\Gamma}_K$: une généralisation de la construction du germe fondamental de Gendron, qui contient nécessairement des éléments externes (non définissables au premier ordre). Le groupe fondamental externe $\hat{\Gamma}_Q$ est une extension décomposée du groupe de Galois absolu $\hat{\Gamma}_Q$, qui contient d’après une conjecture un sous groupe avec une « abélianisation » isomorphe au groupe de classes des idèles. Pour citer cet article: T.M. Gendron, C. R. Acad. Sci. Paris, Ser. I 346 (2008).

1. Introduction

The search for a geometrization of an algebraic number field $K/Q$ has been one of the longstanding ambitions of algebraic number theory: indeed, it could be said that the specter of such a geometrization haunts some of its most celebrated enterprises viz. the Riemann hypothesis, non-Abelian class field theory, Grothendieck–Teichmüller theory. One phenomenon which could achieve structural clarity via geometrization is the isomorphism of class field theory $C_Q \cong \mathbb{R}_+^\times \times \hat{\mathbb{Z}}$, where $C_Q$ is the idèle class group of $Q$. Since $\hat{\mathbb{Z}} \cong \hat{\Gamma}_Q$, where $\hat{\Gamma}_Q = \text{Gal}(\bar{Q}/Q)$, it has been suggested by a number of authors [6,1,2] that the factor $\mathbb{R}_+^\times$ ought to have also a Galois interpretation. Formally, one seeks an extension $\hat{\Gamma}_Q \to \hat{\Gamma}_Q$ in which $\hat{\Gamma}_Q$ has arithmetic meaning (a “cosmic Galois group” [1]), and for which

E-mail address: tim@matcuer.unam.mx.

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\( \tilde{\pi}_Q^{ab} \cong C_Q \). In this Note, we shall construct a candidate for \( \tilde{\pi}_Q \), defined as the external fundamental group of a geometrization of \( Q \) by a hyperbolic surface lamination.

2. Internal fundamental group

Let \( M \) be a compact \( n \)-manifold, \( p : \tilde{M} \to M \) a universal cover and write \( \pi = \pi_1(M) \). Fix an ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \) whose elements are of infinite cardinality. Denote by \( \pi^* \) the ultraproduct of \( \pi \) with respect to \( \mathcal{U} \). Note that there is a monomorphism \( c : \pi \to \pi^* \) given by the constant sequences, and we identify \( \pi \) with its image. The ultraproduct \( \pi^* \) is an example of a non-standard model of \( \pi \) [5].

Suppose that \( M \) is Riemannian, and equip \( \tilde{M} \) with the pull-back metric so that \( \pi \) acts by isometries on \( \tilde{M} \). Let \( \pi^* \)

be the quotient of \( \pi^*(\tilde{M}) \) obtained by identifying sequence classes that are asymptotic. There is a canonical surjective map \( \pi^* \to \tilde{M} \) which associates to each class \( \pi^* \tilde{x} \) the limit of \( \{ \tilde{x}_i \} \) where \( \{ \tilde{x}_i \} \in \pi^* \tilde{x} \) is any representative sequence for which \( p(\tilde{x}_i) \) converges. Note that \( \pi^* \) acts on the left on \( \pi^* \tilde{M} \).

We may view \( \pi^* \tilde{M} \) as a lamination with discrete transversals: the leaf containing \( \pi^* \tilde{x} \in \pi^* \tilde{M} \) consists of those sequence classes of bounded distance from \( \pi^* \tilde{x} \), itself a Riemannian manifold. In fact, \( \pi^* \tilde{M} \) may be identified with the suspension of the inclusion \( c \), i.e. \((\tilde{M} \times \pi)/\pi\), where \((\tilde{x}, \pi^* \alpha) \cdot \gamma = (\gamma \cdot \tilde{x}, (\pi^* \alpha) \gamma^{-1}) \) for all \( \gamma \in \pi \). In the suspension description, the action of \( \pi^* \) is induced by \((\tilde{x}, \pi^* \alpha) \to (\tilde{x}, \pi^* \gamma \alpha), \pi^* \gamma \in \pi^* \) and can be seen to be by leaf-wise isometries. This discussion applies to any group extension \( \pi \subset G \) (particularly, when \( G \) is a non-standard model of \( \pi \)), the appropriate universal covering space being the suspension of the inclusion \( \pi \hookrightarrow G \).

We now indicate how \( \pi^* \) codifies laminated coverings of \( M \). For simplicity, we shall restrict ourselves to suspensions over \( M \). Let \( G \) be a compact topological group and let \( p : \pi \to G \) be a representation. The suspension of \( \rho \), denoted \( M(\rho) \), is a principal \( G \)-bundle as well as a lamination over \( M \), minimal if and only if \( \rho \) has dense image, with simply connected leaves if and only if \( \text{Ker}(\rho) = 1 \). Three examples:

a. If \( G = 1 \) then \( M(\rho) \approx M \).

b. Let \( G = \tilde{\pi} = \) the profinite completion of \( \pi \), \( \rho \) the canonical map. Then \( M(\rho) \approx \tilde{M} = \) the algebraic universal cover of \( M \), a \( \tilde{\pi} \)-principal bundle over \( M \) e.g. \( \tilde{\pi} \backslash \tilde{M} \approx M \). It is classical that \( M \) and \( \tilde{\pi} \) are the appropriate notions of universal cover and fundamental group for \( M \) within the étale category.

c. Let \( M = G = S^1 = \mathbb{R}/\mathbb{Z} \), and for \( r \in \mathbb{R} - \mathbb{Q} \), define \( \rho \) by \( \rho(n) = m \) = the image of \( nr \) in \( S^1 \). Then \( M(\rho) = F_r = \) the irrational foliation of the 2-torus by lines of slope \( r \).

An analogue of the fundamental group for \( M(\rho) \) is given by the fundamental germ \( [\pi] = [\pi^*] \rho M(\rho) \), [3,4]. In the case when the suspension \( M(\rho) \) is minimal, it has the following description. Since \( \rho \) has dense image, the ‘standard part’ map \( \text{std}(\rho) : *\pi \to G \), defined by taking a sequence class to the unique limit in \( G \) of its image by \( \rho \), is onto. We define \( [\pi] : = \text{Ker}(\text{std}(\rho)) \) and refer to \( 1 \to [\pi] \to *\pi \to G \to 1 \) as the standardization exact sequence. For the three examples above we have:

a. \( [\pi^*] M = *\pi \).

b. \( [\pi^*] \tilde{M} = \bigcap *H \) where \( H < \pi \) runs through the subgroups of finite index. This is a non-trivial subgroup of \( *\pi \) even when \( \pi \) is residually finite i.e when \( \bigcap H \) is trivial (for example, when \( M \) is a compact surface).

c. We say that a sequence class \( *\epsilon \in *\mathbb{R} \) is an infinitesimal if it contains a sequence converging to 0. Then we may identify \( [\pi^*] F_r \) with the subgroup of \( *n \in *\mathbb{Z} \) for which \( r^n + *m \) is an infinitesimal for some \( *m \in *\mathbb{Z} \); in other words, \( [\pi^*] F_r \) is the group of Diophantine approximations of \( r \).

We now discuss covering space theory. Let \( M(\rho) \) be as above, assumed for simplicity to be minimal with simply connected leaves. Assume also that \( M \) has been equipped with a Riemannian metric, so that \( M(\rho) \) has a leaf-wise Riemannian metric. There is a canonical map \( \tilde{M} \to M(\rho) \), induced by \( \tilde{M} \times \{1\} \to \tilde{M} \times G \). The image of this map is a leaf \( L_0 \) called the canonical leaf. There is a surjective map \( \pi^* \tilde{M} \to M(\rho) \) assigning to a sequence class \( \pi^* \tilde{x} \) the limit of its image via \( \tilde{M} \to M(\rho) \) – which is a local isometry along the leaves. Any continuous self-map of \( M(\rho) \) preserving \( L_0 \) lifts uniquely to a self-map of \( \pi^* \tilde{M} \). The natural action of \( [\pi^*] \) on \( \pi^* \tilde{M} \) has quotient \( [\pi^*] / \pi^* \tilde{M} \) which is in canonical bijection with \( M(\rho) \). For example, when \( M = \mathbb{H}^2 / \Gamma \) is a closed hyperbolic surface, we may identify \( [\pi^*] M(\rho) \) with a ‘Fuchsian germ’ \( [\Gamma^*] < \text{PSL}(2, *\mathbb{R}) \) and \( [\Gamma] \) \( \times \mathbb{H}^2 \) is in bijection with \( M(\rho) \).
It is possible to endow \( \hat{M} \) with a non-trivial transverse topology in such a way that \( \| \pi \| \) acts by homeomorphisms and so that the quotient \( \| \pi \| \backslash \hat{M} \) is homeomorphic to \( M(\rho) \). To do this, we choose a set-theoretic section \( s : G \to \pi \) of \( \text{std}(\rho) \), so that \( s(\rho(\gamma)) = \gamma \) for all \( \gamma \in \pi \), and for which \( s(G) \) is a right \( \pi \)-set. Then if we give \( \pi \) the topology: (topology of \( G \times \) (discrete), this gives a topology on \( M \times \pi \) invariant by the action of \( \pi \), hence inducing a topology on \( \hat{M} \). The left multiplication action by elements of \( \| \pi \| \) permutes the “cosets” \([x]s(G)\), \([x] \in \| \pi \|\), hence \( \| \pi \| \) acts by homeomorphisms, and with the quotient topology, the bijection between \( \text{Out}^F \) of \( G \) and \( \hat{M} \) is borne out by the following: (N.B. We may even choose the section \( s \) in order that any leaf of \( \hat{M} \) intersects a given \( s(G) \)-transversal no more than once: so that \( \hat{M} \) is a lamination with no non-trivial holonomy.)

3. External fundamental group

Let \( F \) be the free group on two generators, \( \hat{F} \) its profinite completion and consider the standardization sequence

\[ 1 \to \| F \| \to \pi \to \hat{F} \to 1. \]

Neither \( \pi \) nor \( \hat{F} \) are free groups in the discrete (combinatorial) sense. Let \( \hat{F} \) be the free group generated by \( \hat{F} \) (viewed as a set), which has cardinality of the continuum. By universality, there is a canonical epimorphism \( \hat{\rho} : \hat{F} \to \hat{F} \). If \( \sigma : F \to \pi \) is a set-theoretic section of the standardization sequence whose image contains a generating set of \( \*F \), then the induced map \( \*\rho : \hat{F} \to \pi \) is an epimorphism, and \( \hat{\rho} = \text{std} \circ \rho \) (by the uniqueness part of universality). If \( \hat{K}, K \) are the kernels of \( \hat{\rho}, \rho \) then \( \*K < K \).

Denote by \( \text{Aut}(\hat{F}) \) the group of bicontinuous automorphisms of \( \hat{F} \), and by \( \*\text{Aut}(F) \) the subgroup of \( \text{Aut}(\hat{F}) \) of automorphisms which induce elements of \( \*\text{Aut}(F) \) of automorphisms which induce elements of \( \text{Aut}(\hat{F}) \) i.e. automorphisms which stabilize \( \| F \| \) and induce bicontinuous automorphisms of \( \hat{F} \). Note that \( \text{Aut}(\hat{F}) \) as well as \( \*\text{Aut}(F) \) include canonically in \( \*\text{Aut}(F) \). Indeed, if \( \*A \in \*\text{Aut}(F) \) and \( \*x \in \| F \| \), then \( \*A(\*x) \) is represented by a sequence \( \{A_i(x_i)\} \), and \( A_i(x_i) \) is in a subgroup of index \( N_i \to \infty \) if and only if \( x_i \) is.

**Theorem 3.1.** The canonical homomorphism \( \*\text{Aut}(F) \to \text{Aut}(\hat{F}) \) is surjective.

The theorem is proved as follows: note first that any element \( \alpha \in \text{Aut}(\hat{F}) \) defines a bijection of the generating set of \( \hat{F} \), hence an automorphism \( \alpha \) of the latter. As such, \( \alpha \) necessarily stabilizes \( \hat{K} \); we may arrange that it also stabilizes \( \*K \) by composing, if necessary, with a suitable automorphism covering the identity of \( \hat{F} \). The result descends to an automorphism \( \*\alpha \) of \( \*F \). The association \( \alpha \mapsto \*\alpha \) evidently defines a (set-theoretic) section.

Denote by \( \*\text{Inn}(F) \) those elements of \( \*\text{Aut}(F) \) which map to inner automorphisms of \( \hat{F} \). (N.B. \( *F \), acting innerly, is a subgroup of \( \*\text{Inn}(F) \)). If we denote by \( \*\text{Out}(F) \) the quotient of \( \*\text{Aut}(F) \) by \( \*\text{Inn}(F) \), we obtain an exact sequence

\[ 1 \to \| F \| \to \*\text{Out}(F) \to \text{Out}(\hat{F}) \to 1. \]

It is important to note that \( \*\text{Out}(F) \) contains as a proper subgroup the ultraproduct \( \*\text{Out}(F) \cong \*\text{GL}(2, \mathbb{Z}) \cong \text{GL}(2, \mathbb{Z}) \). The latter is called the group of internal outer automorphisms of \( \*F \), and elements of \( \*\text{Out}(F) \) which are not internal are called external. That we cannot replace \( \*\text{Out}(F) \) by \( \*\text{Out}(F) \) is borne out by the following:

**Fact 1.** Although \( F \) is dense in \( \hat{F} \), \( \*\text{Out}(F) \) is not dense in \( \text{Out}(\hat{F}) \), hence \( \text{Out}(\hat{F}) \) is not the profinite completion of \( \*\text{Out}(F) \cong \text{GL}(2, \mathbb{Z}) \). Thus, \( \*\text{Out}(F) \) does not map epimorphically onto \( \text{Out}(\hat{F}) \).

Recall that the theory of a group \( G \) is the collection \( \text{Th}(G) \) of all first order sentences which are true in \( G \). We say \( G' \) is a non-standard model of \( G \) if \( \text{Th}(G') = \text{Th}(G) \) but \( G' \not\cong G \). For example, the ultrapower \( \*G \) is a non-standard model of \( G \).

**Question 1.** Is \( \*\text{Out}(F) \) a non-standard model of \( \text{Out}(F) \)?

In what follows \( K/\mathbb{Q} \) is an arbitrary algebraic number field and \( \hat{F}_K \) is its absolute Galois group. Recall the Belyi monomorphism \( \beta : \Gamma_K \to \hat{F}_K \to \text{Out}(\hat{F}) \). We will not distinguish between \( \hat{F}_K \) and its image in \( \text{Out}(\hat{F}) \).

Let \( \text{SL}(2, \mathbb{Z}) \cong \text{Out}_+(F) \to \text{Out}(\hat{F}) \) be the canonical inclusion. Define \( \Sigma = \Sigma_K \) as the suspension \( \Pi^2 \times \text{Out}(\hat{F})/\text{SL}(2, \mathbb{Z}) \), where the action of \( A \in \text{SL}(2, \mathbb{Z}) \) is defined \( A(z, f) = (Az, AfA^{-1}) \). Then \( \Sigma \) is a non-minimal solenoid by hyperbolic surface orbifolds that covers the modular orbifold \( \text{SL}(2, \mathbb{Z}) \backslash \Pi^2 \). The action of \( \hat{F}_K \) on the product \( \Pi^2 \times \text{Out}(\hat{F}) \), \( \hat{\sigma}(z, f) = (z, \hat{\sigma} f) \), descends to an action on \( \Sigma \) by leaf-wise isometries. Since \( \hat{F}_K \) is a closed
subgroup of $\text{Out}(\hat{F})$, the quotient $\hat{\Sigma}_K = \hat{\Sigma}_K \backslash \hat{\Sigma}$ is also a lamination by hyperbolic surface orbifolds. By construction, the association $K \mapsto \hat{\Sigma}_K$ is Galois natural.

Denote by $\circlearrowleft \Gamma_K$ the pre-image of $\hat{\Gamma}_K$ in $\circlearrowleft \text{Out}(F)$ so that $[\Gamma]$ is the kernel of the standardization epimorphism $\circlearrowleft \Gamma_K \to \hat{\Gamma}_K$. We have $[\Gamma] = \cap \circlearrowleft \Gamma_K$. Recall that there is a canonical inclusion $\text{SL}(2, \mathbb{Z}) \cong \text{Out}_+(F) \hookrightarrow \circlearrowleft \text{Out}(F)$. By suspending this inclusion with respect to the action of $\text{SL}(2, \mathbb{Z})$ on $\mathbb{H}^2$, we obtain a trivial lamination which we denote $\circlearrowleft \mathbb{H}^2$. We note that the quotient of $\circlearrowleft \mathbb{H}^2$ by the left action of $\circlearrowleft \text{Out}(F)$ is isometric to the modular orbifold.

We topologize $\circlearrowleft \text{Out}(F)$ by choosing a set-theoretic section of $\circlearrowleft \text{Out}(F) \to \text{Out}(\hat{F})$ whose image is a right $\text{SL}(2, \mathbb{Z})$-set and which maps $\text{SL}(2, \mathbb{Z})$ to itself (as we did at the end of the last section). This induces a topology on $\circlearrowleft \mathbb{H}^2$ making it a solenoid by hyperbolic surface orbifolds, with respect to which the action by $\circlearrowleft \text{Out}(F)$ is by homeomorphisms which are isometries along the leaves. The quotient by $[\Gamma]$ can be identified with $\hat{\Sigma} = \hat{\Sigma}_Q$ and in addition $\hat{\Sigma}_K \cong \circlearrowleft \Gamma_K \backslash \circlearrowleft \mathbb{H}^2 \cong \hat{\Sigma}_K \backslash \hat{\Sigma}$. This justifies viewing $\circlearrowleft \Gamma_K$ as a fundamental group, in a way which generalizes the internal fundamental group defined in §2.

Conjecture 3.2. There is a subgroup $\hat{\Gamma}_Q < \circlearrowleft \Gamma_Q$ which is an extension of $\hat{\Gamma}_Q$ and for which $\hat{\Gamma}_Q^{ab} \cong C_Q$.

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