Saturation of a large scale instability and non linear structures in a rotating stratified flow

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Abstract

In this letter, we present non linear structures resulting from the saturation of a large scale instability in rotating stratified fluids with small scale forced turbulence which we found in [1]. These structures are of helical kinks type. The different solutions that we plotted correspond to solutions linking two different stationary points for different values of the Rayleigh and the Taylor numbers.

Keywords: Large scale instability, Coriolis force, buoyancy, instability saturation, non linear structures

1 Introduction

Large scale instabilities are of a great interest in fluid dynamics. They generate vortices which play a fundamental role in turbulence and in transport processes. The generation of this kind of instability has been widely studied, for example in [3], [5], [6], [7]. Direct numerical simulations of Boussinesq Equation confirmed the existence of large scale vortex generation in stratified and rotating flows [2]. Many papers as well as the results of numerical modelling are described in detail in review [6]. Several works have shown that a necessary condition for these instabilities to appear is the lack of reflection invariance or parity invariance [3], [4], [5]. In fluid dynamics, the widespread mechanism used to break this symmetry is the helicity \( \vec{v} \cdot \vec{\omega} \vec{v} \neq 0 \). However, it has been shown that the helicity itself is not enough to generate these large scale structure, and others factors are necessary like for example temperature gradient [5], [7]. It has also been shown in a previous work [7] that the method of multi-scale development can lead to the occurrence of large scale instabilities in helical stratified turbulence.

In the work [1], we found a new large scale instability in rotating stratified fluids with small scale forced turbulence. The helicity can be used in an explicit way [7] as well as in an internal way [8]. In this previous paper, we used the Coriolis force and the buoyancy to create naturally the internal helicity which is required to break the symmetry of the flow. That is what allowed us to use a force which does not have any particular properties (especially it is nonhelical and it does not lack parity invariance). The force only maintains turbulent fluctuations.

In this letter, we complete this previous study by solving the non linear instability equations. As a result of saturation, the non linear structures appearing are rotational kinks. Our letter is organized as follows: In section 2 we remind the basics equations of the problem, in section 3 we briefly remind the scheme of asymptotic development and write down the main equations for large scale instability which was found in [1]. In section 4, we expose the expression of the external force and the Reynolds stresses calculated in
[1], in section 5 we discuss the instability conditions. Section 6 is devoted to the calculation of the non linear saturated instability, a brief discussion of the result is given as a conclusion in section 7.

2 The main equations and formulation of the problem

Let’s consider the equations for the motion of an incompressible fluid with a constant temperature gradient in the Boussinesq approximation:

\[ \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} + 2 \Omega \times \vec{V} = -\frac{1}{\rho_0} \nabla P + \nu \Delta \vec{V} + g \beta T \vec{l} + \vec{F}_0 \]  

\[ \frac{\partial T}{\partial t} + (\vec{V} \cdot \nabla) T = \chi \Delta T - V_z A. \]  

\[ \nabla \cdot \vec{V} = 0 \]  

Where \( \vec{l} = (0, 0, 1) \), \( \beta \) is the thermal expansion coefficient, \( A = \frac{dT_0}{dz} \) is the constant equilibrium gradient of the temperature, \( \rho_0 = \text{Const.} \), and \( \nabla T_0 = A \vec{l} \).

Using dimensionless variables, we can rewrite the previous equation as:

\[ \frac{\partial \vec{V}}{\partial t} + R(\vec{V} \cdot \nabla) \vec{V} - \Delta \vec{V} + D \vec{l} \times \vec{V} = -\nabla P + Ra T \vec{l} + \vec{F}_0 \]  

\[ \left( \frac{\partial T}{\partial t} - \Delta T \right) = -V_z - R(\vec{V} \cdot \nabla) T \]  

\[ \nabla \cdot \vec{V} = 0 \]  

where \( R \) and \( D \) are respectively the Reynolds number and the scare root of the Taylor numbers. \( Pr \) represents the Prandtl number and \( Ra \) is the Rayleigh number. We also introduce a dimensionless temperature \( T \).

We will consider as a small parameter of an asymptotic development the Reynolds number \( R \). Let us denote the small scale variables by \( \vec{x}_0 = (\vec{x}_0, t_0) \), and the large scale ones by \( \vec{X} \) and \( T \). The small scale partial derivative operation \( \frac{\partial }{\partial x_0^i}, \frac{\partial }{\partial x_0^i} \), and the large scale ones \( \frac{\partial }{\partial X}, \frac{\partial }{\partial T} \) are written, respectively, as \( \partial_i, \partial_t, \nabla_i \) and \( \partial_T \). To construct a multi-scale asymptotic development we follow the method which is proposed in [3].

3 The multi-scale asymptotic development

Following [1], let us look for the solutions to Equations (4) and (5) in the following form:

\[ \vec{V}(\vec{x}, t) = \frac{1}{R} \vec{W}_{-1}(X) + \vec{v}_0(x_0) + R \vec{v}_1 + R^2 \vec{v}_2 + R^3 \vec{v}_3 + \cdots \]  

\[ T(\vec{x}, t) = \frac{1}{R} T_{-1}(X) + T_0(x_0) + RT_1 + R^2 T_2 + R^3 T_3 + \cdots \]  

\[ P(\vec{x}, t) = \frac{1}{R^3} P_{-3}(X) + \frac{1}{R^2} P_{-2}(X) + \frac{1}{R} P_{-1}(X) + P_0(x_0) + R(P_1 + \overline{P}_1(X)) + R^2 P_2 + R^3 P_3 + \cdots \]  

where \( \vec{X} = R^2 \vec{x}_0 \) and \( T = R^4 t_0 \).

We remind that the scale relation is the following: \( \vec{X} = R^2 \vec{x}_0 \) and \( T = R^4 t_0 \).

From this we get the main secular equation at order \( O(R^3) \):

\[ \partial_T W_{-1} + \Delta W_{-1} + \nabla_k (v_{0k}^i v_{0i}) = -\nabla_i \overline{P}_1, \]  

\[ \partial_T T_{-1} + \Delta T_{-1} + \nabla_k (v_{0k} T_0) = 0. \]
4 Calculations of the Reynolds stresses

The essential equation for finding the nonlinear alpha-effect is equation (9). In order to obtain these equations in closed form, we need to calculate the Reynolds stresses $\nabla_k (v_k v_0)$. The external force can be chosen in a general 3D form like for example:

$$\vec{F}_0 = f_0 \left( i \cos \varphi_1 + j \cos \varphi_2 + k \cos \varphi_2 \right), \tag{11}$$

where

$$\varphi_1 = k_0 z - \omega_0 t, \varphi_2 = k_0 x - \omega_0 t, \tag{12}$$

However, it can be shown that only one component of this force is responsible for displaying large scale instability, which is the $(x,z)$ plan component. The force can then be reduced to:

$$\vec{F}_0 = f_0 \left( i \cos \varphi_1 + k \cos \varphi_2 \right), \tag{13}$$

Below are the expression of the Reynolds stresses as calculated in [1]:

$$T_{31}^{(1)} = \frac{D^2[2 + Ra - 2(1 - W_1)^2]}{\Xi_{(1)}},$$

$$T_{31}^{(2)} = \frac{Ra[2 + D^2 - 2(1 - W_2)^2]}{\Xi_{(2)}},$$

$$T_{32}^{(1)} = \frac{-D^3[Ra + 2[1 - (1 - W_1)^2]]}{\Xi_{(1)}},$$

$$T_{32}^{(2)} = \frac{-DRa[2 + D^2 - 6(1 - W_2)^2]}{\Xi_{(2)}},$$

where

$$\Xi_{(1),(2)} = 2[D^4 + Ra^2 + 2D^2Ra + 4[1 + (1 - W_{1,2})^2]^2 + (2D^2 + 2Ra)[2 - 2(1 - W_{1,2})^2][1 + (1 - W_{1,2})^2].$$

5 Large scale instability

Let us write down in the explicit form the equations for nonlinear instability:

$$\partial_t W_1 + \nabla_Z T_{31}^{(1)} + \nabla_Z T_{31}^{(2)} = \Delta W_1, \tag{14}$$

$$\partial_t W_2 + \nabla_Z T_{32}^{(1)} + \nabla_Z T_{32}^{(2)} = \Delta W_2, \tag{15}$$

where the components $T_{31}^{(1)}, T_{31}^{(2)}, T_{32}^{(1)}$ and $T_{32}^{(2)}$ of the Reynolds stress tensor are as defined in the previous section.

We shew in a previous paper [1] that the linear instability can appear for specific values of $D$ and $Ra$. We show below two figures representing the area (in grey) of the plane $(D, Ra)$ for which the discriminant is negative, this means that an instability can appear. The first plot shows the conditions for a negative temperature gradient and the second plot, for a positive one.
6 Instability saturation and non linear structures

The increase of $W_1$ and $W_2$ leads to the saturation of the instability. As a result of the development and stabilization of the instability, non linear structures appear. The study of these structures is of interest. Then, we take back equations (14) and (15) allowing that $\partial_T W_1 = \partial_T W_2 = 0$. Integrating these equations over $z$, we obtain the following system:

\[ \nabla_Z W_1 = T_{(1)}^{31} + T_{(2)}^{31} + C_1, \quad (16) \]
\[ \nabla_Z W_2 = T_{(1)}^{32} + T_{(2)}^{32} + C_2, \quad (17) \]

Where $C_1$ and $C_2$ are integration constants. Let introduce the new variables $X = 1 - W_1$, $P = 1 - W_2$. Then the equations (16) and (17) can be written:

\[ \nabla_Z X = \frac{-D^2[Ra + 2(1 - X^2)]}{\Sigma_{(1)}} - \frac{Ra(2 + D^2 - 2P^2)}{\Sigma_{(2)}} + C_1, \quad (18) \]
\[ \nabla_Z P = \frac{D^3[Ra + 2(1 - X^2)]}{\Sigma_{(1)}} + \frac{DRa(2 + D^2 - 6P^2)}{\Sigma_{(2)}} + C_2, \quad (19) \]

Where $\Sigma_{(1)}$ and $\Sigma_{(2)}$ are defined as below:

\[ \Sigma_{(1)} = 2[D^4 + Ra^2 + 2D^2Ra + 4(1 + X^2)^2 + 2(2D^2 + 2Ra)(2 - 2X^2)](1 + X^2). \quad (20) \]
We have similar formulae for $\Sigma_{(2)}$ after having replaced $X$ with $P$. Integrating this system using simple numerical tools allow us to display non linear structures. Below is the phase portrait of this integrated system of equations:

![Phase portrait](image-url)

**Figure 3: Phase plane with stationnary points**

The phase portrait is unusual and doesn’t contain any elliptic point, which means there is no periodical solutions. The only solutions we get which link different stationary points are helical kinks, namely, the velocity has a given constant value until a specific value of $z$, then it rotates helically and becomes constant again but with a different value of velocity. Below are some figures of these kinks:

![Helical Kinks](image-url)

**Figure 4: Helical Kinks between two nodes, with $D = 1$, $Ra = 1$, $C_1 = 0, 13$, $C_2 = -0, 1$.**
Figure 5: Helical Kinks between a node and a hyperbolic point, with $D = 1$, $Ra = 1$, $C_1 = 0.13$, $C_2 = -0.1$.

Figure 6: Helical Kinks between two nodes, with $D = 1.5$, $Ra = 1.9$, $C_1 = 0.05$, $C_2 = 0.01$.

Figure 7: Helical Kinks between a node and a hyperbolic point, with $D = 1.5$, $Ra = 1.9$, $C_1 = 0.05$, $C_2 = 0.01$.

7 Conclusions and discussion of the results

In this letter, we first briefly reviewed the main results of large scale instability in rotating and stratified flow found in [1], which was dedicated to the linear stage. Then we shew non linear large scale structures displayed by such a flow, which completes the previous study. These solutions show that in the non linear stage, the instability saturation leads to specific velocity profiles (helical kinks) for which the velocity tends to be constant for large values of $z$. These structures are of helical type and are the result of the saturation of the instability found in [1]. Since the phase portrait doesn’t contain any elliptic stationary point there is no periodical solution but only rotational kink solutions.
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