BÖCHER-TYPE RESULTS FOR THE FOURTH AND HIGHER
ORDER EQUATIONS ON SINGULAR MANIFOLDS WITH
CONICAL METRICS

FANGSHU WAN

School of Mathematical Sciences
University of Science and Technology of China
Hefei, 230026, China

Abstract. We obtain the Böcher-type theorems and present the sharp characteriza-
tion of the asymptotic behavior at the isolated singularities of solutions of some fourth and higher order equations on singular manifolds with conical metrics. It is seen that the equations on singular manifolds with conical metrics are equivalent to weighted elliptic equations in $B \setminus \{0\}$, where $B \subset \mathbb{R}^N$ is the unit ball. The weights can be singular at $x = 0$. We present the sharp asymptotic behavior of nonnegative solutions of the weighted elliptic equations near $x = 0$ and the Liouville-type results for the degenerate elliptic equations in $\mathbb{R}^N \setminus \{0\}$.

1. Introduction. Let $M^0$ be an open manifold of dimension $N$ ($N \geq 2$) which is a compact manifold $M$ with a finite set of points removed, that is, $M^0 = M \setminus \{p_1, p_2, \ldots, p_k\}$. At each puncture point $p_i$ ($i = 1, 2, \ldots, k$), we choose a neighborhood $U_i$ of $p_i$ such that $U_i \cap U_j = \emptyset$ ($i \neq j$) and $\varphi_i : U_i \to B$ is a diffeomorphism with $\varphi_i(p_i) = 0$ where and in the following $B$ is the unit ball of $\mathbb{R}^N$. We introduce the Riemannian metric in $U_i^* = U_i \setminus \{p_i\}$,

$$d s^2 = |x|^{2 \beta_i} d s_0^2, \quad \beta_i > -1,$$

(1.1)

where $x \in B^* = B \setminus \{0\}$ and $d s_0^2$ is the Euclidean metric. Then we extend this metric to $M^0$ and denote it by $g$ which is called a conical metric. Obviously, the manifold is noncomplete with a finite volume.

We will study the asymptotic behavior near each singular point $p_i$ of the nonnegative solutions for the homogenous bi-Laplace equation with the conical metric:

$$\Delta^2_g u = 0 \quad \text{on } M^0 \quad (P)$$

and generalize the classical Böcher-type theorem to the bi-Laplace equation with the conical metric and the higher order equations with the conical metric:

$$\Delta^m_g u = 0 \quad \text{on } M^0 \quad (Q)$$

where $m \geq 2$ is a positive integer.

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At each singular point $p_i$, the Laplace-Beltrami operator $\Delta_g$ induced by the conical metric can be written as
\[
\Delta_g = \frac{1}{|x|^{N\beta}} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \left(|x|^{(N-2)\beta} \frac{\partial}{\partial x_j}\right),
\]
in the neighborhood $U_i \setminus \{p_i\}$. Therefore, the homogenous Laplace equation $\Delta_g u = 0$ in the neighborhood $U_i \setminus \{p_i\}$ is equivalent to the weighted elliptic equation
\[
\text{div}(|x|^{\theta} \nabla u) = 0 \quad \text{in} \quad B \setminus \{0\},
\]
where $\theta = (N-2)\beta_i$. Since the weight $|x|^{\theta}$ can be singular at $x = 0$, the standard arguments, for example, the moving-plane method, can not be easily used to deal with (1.3). In a recent paper [30], we generalize the classical Bôcher-type theorem to (1.3). It is clear that (1.3) has only one weight and the form of the equation is quite simple.

In this paper, we will continue the study. We consider the fourth and higher order equations with the conical metric. We see that the homogenous bi-Laplace equation $\Delta_g^2 u = 0$ in the neighborhood $U_i \setminus \{p_i\}$ is equivalent to the weighted elliptic equation
\[
\text{div}(|x|^{\theta} \nabla (|x|^{-\tau} \text{div}(|x|^{\theta} \nabla u))) = 0 \quad \text{in} \quad B \setminus \{0\}
\]
where $\theta = (N-2)\beta_i$, $l = N\beta_i$. Note that $N + \theta = N + (N-2)\beta_i \geq 2$ and $\tau = l - \theta = N\beta_i - (N-2)\beta_i = 2\beta_i > -2$. It is clear that (1.4) has two weights and both of them can be singular at $x = 0$. The form of (1.4) is more complicated than that of (1.3). If we define $v = -|x|^{-\tau} \text{div}(|x|^{\theta} \nabla u)$, then the equation (1.4) can be written to the following system of equations:
\[
\begin{cases}
-\text{div}(|x|^{\theta} \nabla u) = |x|^\nu v & \text{in} \quad B \setminus \{0\}, \\
-\text{div}(|x|^{\theta} \nabla v) = 0 & \text{in} \quad B \setminus \{0\}.
\end{cases}
\]

To obtain the Bôcher-type theorem for (P) on the singular manifold $M^0$, we only need to establish the Bôcher-type theorem for the equation (1.4) or the system (1.5). However, to make our study can be continued, we need to put a crucial assumption on $v$:
\[
v \geq 0 \quad \text{in} \quad B \setminus \{0\}.
\]
We do not know how to establish the Bôcher-type theorem for (1.4) if the assumption (1.6) does not hold.

The main results of this paper are the following theorems.

**Theorem 1.1.** Let $u$ be a nonnegative solution of the equation
\[
\Delta_g^2 u = 0 \quad \text{on} \quad M^0
\]
with
\[
v := -\Delta_g u \geq 0 \quad \text{in} \quad \text{every} \quad U_i, \quad i = 1, 2, \ldots, k.
\]
Then at every singular point $p_i$, there exists a neighborhood $V_i(p_i) \subset U_i$ such that
\[
u(x) = \begin{cases}
a_i|x-p_i|((2-N)(1+\beta_i) + c_i|x-p_i|^{(4-N)(1+\beta_i)} + h(x) & \text{if} \quad \beta_i > -1 \quad \text{&} \quad N \geq 3, \\
-a_i \ln |x-p_i| + c_i |x-p_i|^{(1+\beta_i) \ln |x-p_i|} + h(x) & \text{if} \quad \beta_i > -1 \quad \text{&} \quad N = 2, \\
-a_i (\ln |x-p_i|)^3 + c_i (\ln |x-p_i|)^2 + d_i \ln |x-p_i| + h(x) & \text{if} \quad \beta_i = -1 \quad \text{&} \quad N \geq 2
\end{cases}
\]
and
\[ v(x) = \begin{cases} 
    a_i|x - p_i|^{(2-N)(1+\beta_i)} + k(x) & \text{if } \beta_i > -1 \text{ and } N \geq 3, \\
    -a_i \ln |x - p_i| + k(x) & \text{if } \beta_i = -1 \text{ or } N = 2
\end{cases} \tag{1.10} \]

where \( a_i \geq 0, \ c_i, \ d_i \) are constants, and \( h(x), \ k(x) \) are Hölder continuous at \( p_i \). Consequently,
\[ \Delta^2 u = \tau_i \delta_{p_i} \text{ in } V_i(p_i) \tag{1.11} \]
and
\[ -\Delta v = \rho_i \delta_{p_i} \text{ in } V_i(p_i) \tag{1.12} \]
for some \( \tau_i, \rho_i \geq 0 \). Moreover, when \( N \geq 4 \) and \( \beta_i \geq -1, \ h(x) \) is a weak bi-harmonic function with the equation \( (1.7) \) in the whole neighborhood \( V_i(p_i) \), and when \( N \geq 2 \) and \( \beta_i \geq -1, \ k(x) \) is a weak harmonic function with the equation
\[ -\Delta v = 0 \text{ in } M^0 \]
in the whole neighborhood \( V_i(p_i) \).

**Theorem 1.2.** Let \( u \in C^4(\mathbb{R}^N \setminus \{0\}) \) be a solution of the equation
\[ \text{div}(|x|^\theta \nabla (|x|^{-\theta} \text{div}(|x|^\theta \nabla u))) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}. \]
Suppose \( u \) and \( v := -|x|^{-\theta} \text{div}(|x|^\theta \nabla u) \) are bounded from above or from below in a neighborhood of \( 0 \) as well as in a neighborhood of infinity. Then there are constants \( a, c, d, e \) such that
\[
u(x) = \begin{cases} 
    a|x|^{-N'+2} + c|x|^{-N'+4-\tau} + d|x|^{2+\tau} + e & \text{if } N' > 2 \text{ and } \tau > -2, \\
    a|x|^{-N'} \ln |x| + c|x|^{-N'-2} + d \ln |x| + e & \text{if } N' > 2 \text{ and } \tau = -2, \\
    a \ln |x| + c|x|^{2+\tau} \ln |x| + d|x|^{2+\tau} + e & \text{if } N' = 2 \text{ and } \tau > -2, \\
    a(\ln |x|)^3 + c(\ln |x|)^2 + d \ln |x| + e & \text{if } N' = 2 \text{ and } \tau = -2, \\
    a|x|^{2+\tau} + c|x|^{4+\tau-N'} + d|x|^{2-N'} + e & \text{if } N' < 2 \text{ and } \tau > -2, \\
    a|x|^{2-N'} \ln |x| + c|x|^{2-N'} + d \ln |x| + e & \text{if } N' < 2 \text{ and } \tau = -2.
\end{cases} \tag{1.13} \]
and
\[ v(x) = \begin{cases} 
    a|x|^{-N'} + c & \text{if } N' > 2, \\
    a \ln |x| + c & \text{if } N' = 2, \\
    a|x|^{2-N'} + c & \text{if } N' < 2.
\end{cases} \tag{1.14} \]

In particular, \( a = 0 \) when \( N' = 2 \).

Theorem 1.2 can be seen as a Liouville-type result for (1.4) in \( \mathbb{R}^N \setminus \{0\} \). To obtain Theorem 1.1, we need to establish the Böcher-type results for (1.4) firstly. The fourth order elliptic equations as (1.4) have been studied by many authors recently, see, for example, [2, 4, 9, 10, 11, 12, 13, 15, 17, 18, 19, 22, 24, 25, 26, 27, 32, 37]. Meanwhile, the equations with weights have also been studied by many authors, see, for example, [6, 7, 8, 20, 21, 23, 28, 33, 34, 35, 36]. We will also establish the Böcher-type results for higher order equation (Q) with the conical metric and the related 2m-order elliptic equation in \( B \setminus \{0\} \) with \( m \geq 2 \).

In [5], Böcher showed a classical theorem which asserts that any nonnegative harmonic function in the punctured ball \( B \setminus \{0\} \subset \mathbb{R}^N \) can be expressed as the sum of a constant multiple of the fundamental solution of the Laplace equation and a harmonic function in the whole unit ball \( B \). Gilbarg and Serrin [16] studied fundamental solutions and isolated singularities of the Laplace equation in view of modern theories. In recent years, extensions of Böcher-type theorem have also been
established for fully nonlinear uniformly elliptic equations in [3, 14, 29]. The authors in [31] gave a generalization to a class of conformally invariant fully nonlinear degenerate elliptic equations.

In section 2, we establish the Böcher-type results for nonnegative solutions of (1.4) with the crucial assumption (1.6). In section 3, we present the proof of our main theorems and in the final section we generalize the Böcher-type results to the higher order equations.

2. The Böcher-type theorem for the equation (1.4). In this section, we study the asymptotic behavior of the nonnegative solution for the equation (1.4) as $|x| \to 0$ and obtain the generalized Böcher-type theorem in $B$.

**Theorem 2.1.** Let $N' := N + \theta \geq 2$, $\tau := l - \theta \geq -2$, $u \in C^4(B\{0\})$ be a nonnegative solution of the equation (1.4) with $v = -|x|^{-l}\text{div}(|x|^\theta \nabla u) \geq 0$ in $B\{0\}$. Then there exist constants $a \geq 0$ and $c, d$ such that

$$u(x) = \begin{cases} a|x|^{-(N'-2)} + c|x|^{-(N'-4-\tau)} + h(x) & \text{if } N' > 2 \& \tau > -2, \\ -a|x|^{-(N'-2)} \ln |x| + c|x|^{-(N'-2)} + d\ln |x| + h(x) & \text{if } N' > 2 \& \tau = -2, \\ -a \ln |x| + c|x|^{2+\tau} \ln |x| + h(x) & \text{if } N' = 2 \& \tau > -2, \\ -a(\ln |x|)^3 + c(\ln |x|)^2 + d\ln |x| + h(x) & \text{if } N' = 2 \& \tau = -2, \\ \end{cases}$$

and

$$v(x) = \begin{cases} a|x|^{2-N'} + k(x) & \text{if } N' > 2, \\ -a \ln |x| + k(x) & \text{if } N' = 2 \end{cases}$$

where $h(x), k(x)$ are Hölder continuous at $x = 0$. Moreover, when $N' > 2$, $\tau \geq -2$ and $N' - \tau \geq 4$: $N' = 2$, $\tau > -2$ and $2 + \tau \leq \sqrt{N-1}$; $N' = 2$ and $\tau = -2$, $h$ satisfies the equation

$$\text{div}(|x|^\theta \nabla (|x|^{-l}\text{div}(|x|^\theta \nabla h))) = 0 \text{ in } B$$

in the weak sense and when $N' \geq 2$, $\tau \geq -2$, $k(x)$ satisfies the equation

$$-\text{div}(|x|^\theta \nabla v) = 0 \text{ in } B$$

in the weak sense.

If $u \in C^4(B\{0\})$ is a nonnegative solution of (1.4) with $v = -|x|^{-l}\text{div}(|x|^\theta \nabla u) \geq 0$ in $B\{0\}$ and $u = v = 0$ on $\partial B$, then $u$ and $v$ are radial functions.

**Proof.** Since $v := -|x|^{-l}\text{div}(|x|^\theta \nabla u) \geq 0$ with $v \in C^2(B\{0\})$ satisfies the equation

$$-\text{div}(|x|^\theta \nabla v) = 0 \text{ in } B\{0\},$$

it is known from Theorem 2.1 of [30] that there exist constant $a \geq 0$ and $c$ such that

$$v(x) = \begin{cases} c|x|^{2-N'} + k(x) & \text{if } N' < 2, \\ -a \ln |x| + k(x) & \text{if } N' = 2, \\ a|x|^{2-N'} + k(x) & \text{if } N' > 2 \end{cases}$$

where $k(x)$ is Hölder continuous at $x = 0$. Moreover, if $N' \geq 2$, $k$ satisfies (2.4) in the weak sense. If $N' < 2$, $\lim \limits_{|x| \to 0} v(x) = \rho_0$ for some $\rho_0 \in \mathbb{R}$.

In the following, we consider $u$. Instead of considering the equation (1.4), we consider the system (1.5).
Define \( r = |x|, \omega = \frac{x}{|x|} \) and
\[
\begin{align*}
  w(t, \omega) &= r^{N'-2} u(r, \omega), \quad z(t, \omega) = r^{\frac{N'-2}{2}} v(r, \omega), \quad t = \ln r \in (\infty, 0).
\end{align*}
\] (2.5)

Then, \((w, z)\) satisfies the problem:
\[
\begin{align*}
\partial_t w + \Delta_{SN-1} w - \frac{(N'-2)^2}{4} w + e^{(2+\tau)t} z &= 0 \quad (t, \omega) \in (-\infty, 0) \times SN^{-1}, \\
\partial_t z + \Delta_{SN-1} z - \frac{(N'-2)^2}{4} z &= 0 \quad (t, \omega) \in (-\infty, 0) \times SN^{-1}.
\end{align*}
\] (2.6)

Since \( 2 + \tau \geq 0 \), we see that \( e^{(2+\tau)t} \leq 1 \) for \( t \in (-\infty, 0) \). By using the Harnack’s inequality (see Theorem 2.1 of [1]) of system of equations over \([L - 1, L] \times SN^{-1}\) with \( L \leq \ln \sigma \) and \( 0 < \sigma < 1/10 \) sufficiently small, we have that there exists \( C > 0 \) independent of \( t \in (-\infty, \ln \sigma) \) such that
\[
\max_{SN^{-1}} \left\{ w(t, \cdot), z(t, \cdot) \right\} \leq C \min_{SN^{-1}} \left\{ w(t, \cdot), z(t, \cdot) \right\} \quad \text{for} \quad t \leq \ln \sigma.
\] (2.7)

Let \( \overline{w}(t) = \frac{1}{|S_{N^{-1}}} \int_{S_{N^{-1}}} w(t, \omega) d\omega \) and \( \overline{z}(t) = \frac{1}{|S_{N^{-1}}} \int_{S_{N^{-1}}} z(t, \omega) d\omega \). It follows from (2.7) that there exist \( 0 < c_1 < c_2 \) and \( 0 < c_3 < c_4 \) such that
\[
\begin{align*}
  c_1 \overline{w}(t) &\leq w(t, \omega) \leq c_2 \overline{w}(t) \quad \text{for} \quad (t, \omega) \in (-\infty, \ln \sigma) \times SN^{-1}, \\
  c_3 \overline{z}(t) &\leq z(t, \omega) \leq c_4 \overline{z}(t) \quad \text{for} \quad (t, \omega) \in (-\infty, \ln \sigma) \times SN^{-1}.
\end{align*}
\] (2.8)

It is easily seen from (2.6) that \((\overline{w}, \overline{z})\) satisfies the system
\[
\begin{align*}
\overline{w}' - \frac{(N'-2)^2}{4} \overline{w} + e^{(2+\tau)t} \overline{z} &= 0 \quad t \in (-\infty, \ln \sigma), \\
\overline{z}' - \frac{(N'-2)^2}{4} \overline{z} &= 0 \quad t \in (-\infty, \ln \sigma).
\end{align*}
\] (2.10)

We consider four cases here:
(i) \( N' > 2 \) and \( \tau > -2 \), (ii) \( N' > 2 \) and \( \tau = -2 \), (iii) \( N' = 2 \) and \( \tau > -2 \), (iv) \( N' = 2 \) and \( \tau = -2 \).

For the case (i), it follows from the second equation of (2.10) that
\[
\overline{z}(t) = A e^{\frac{N' - 2}{2} t} + B e^{-\frac{N' - 2}{2} t} \quad \text{for} \quad t \in (-\infty, \ln \sigma),
\] (2.11)

where \( A \) and \( B \) are arbitrary constants. This implies that
\[
\overline{z}(t) = O \left( e^{-\frac{(N' - 2)}{2} t} \right) \quad \text{for} \quad t \rightarrow -\infty.
\] (2.12)

This also implies that
\[
e^{(2+\tau)t} \overline{z}(t) = O \left( e^{-\frac{N' - 6 - 2\tau}{2} t} \right) \quad \text{for} \quad t \rightarrow -\infty.
\] (2.13)

Note that \( e^{-\frac{N' - 2}{2} t} \geq e^{-\frac{N' - 6 - 2\tau}{2} t} \) since \( N' - 2 - (N' - 6 - 2\tau) = 4 + 2\tau > 0 \). We now obtain from the first equation in (2.10) and the fact
\[
\begin{align*}
  \overline{w}(t) &= M_1 e^{\frac{N' - 2}{2} t} + M_2 e^{-\frac{N' - 2}{2} t} + B_1 \int_T^t e^{\frac{N' - 2}{2} (t-s)} O \left( e^{-\frac{(N' - 6 - 2\tau)}{2} s} \right) ds \\
  & \quad - B_2 \int_{-\infty}^t e^{-\frac{N' - 2}{2} (t-s)} O \left( e^{-\frac{(N' - 6 - 2\tau)}{2} s} \right) ds \\
  &= \hat{M}_1 e^{\frac{N' - 2}{2} t} + \hat{M}_2 e^{-\frac{N' - 2}{2} t} + \hat{B}_3 e^{-\frac{N' - 6 - 2\tau}{2} t}
\end{align*}
\]
for some \( -\infty < T < \ln \sigma \) that
\[
\overline{w}(t) = O \left( e^{-\frac{(N' - 2)}{2} t} \right) \quad \text{for} \quad t \rightarrow -\infty.
\] (2.14)
It follows from (2.8), (2.9), (2.12) and (2.14) that
\[ w(t, \omega) = O\left( e^{-\frac{N'-2}{2}t} \right), \quad z(t, \omega) = O\left( e^{-\frac{N'-2}{2}t} \right) \quad \text{for} \quad (t, \omega) \in (-\infty, \ln \sigma_0) \times S^{N-1}, \]
(2.15)
where \( 0 < \sigma_0 < \sigma ).

For the case (ii), it follows from the second equation of (2.10) that
\[ z(t) = A e^{\frac{N'-2}{2}t} + B e^{-\frac{N'-2}{2}t} \quad \text{for} \quad t \in (-\infty, \ln \sigma), \]
(2.16)
where \( A \) and \( B \) are arbitrary constants. We now can obtain from the first equation in (2.10) and the fact
\[ w(t) = M_1 e^{\frac{N'-2}{2}t} + M_2 e^{-\frac{N'-2}{2}t} + B_1 t e^{\frac{N'-2}{2}t} + B_2 t e^{-\frac{N'-2}{2}t} \]
that
\[ w(t) = O\left( |t| e^{\frac{N'-2}{2}t} \right) \quad \text{for} \quad t \text{ near } -\infty. \]
(2.17)

It follows from (2.8), (2.9), (2.16) and (2.17) that
\[ w(t, \omega) = O\left( |t| e^{\frac{N'-2}{2}t} \right), \quad z(t, \omega) = O\left( e^{-\frac{N'-2}{2}t} \right) \quad \text{for} \quad (t, \omega) \in (-\infty, \ln \sigma_0) \times S^{N-1}, \]
(2.18)
where \( 0 < \sigma_0 < \sigma ).

For the case (iii), it follows from the second equation of (2.10) that
\[ z(t) = A + B t \quad \text{for} \quad t \in (-\infty, \ln \sigma), \]
(2.19)
where \( A \) and \( B \) are arbitrary constants. This also implies that
\[ e^{(2+\tau)t} z(t) = O\left( |t| e^{(2+\tau)t} \right) \quad \text{for} \quad t \text{ near } -\infty. \]
(2.20)

We now obtain from the first equation in (2.10) and the fact
\[ w(t) = M_1 + M_2 t + B_1 t e^{2+\tau} + B_2 t e^{2+\tau} \]
that
\[ w(t) = O(|t|) \quad \text{for} \quad t \text{ near } -\infty. \]
(2.21)

It follows from (2.8), (2.9), (2.19) and (2.21) that
\[ w(t, \omega) = O(|t|), \quad z(t, \omega) = O(|t|) \quad \text{for} \quad (t, \omega) \in (-\infty, \ln \sigma_0) \times S^{N-1}, \]
(2.22)
where \( 0 < \sigma_0 < \sigma ).

For the case (iv), it follows from the second equation of (2.10) that
\[ z(t) = A + B t \quad \text{for} \quad t \in (-\infty, \ln \sigma), \]
(2.23)
where \( A \) and \( B \) are arbitrary constants. We now obtain from the first equation in (2.10) (note that \( 2 + \tau = 0 \)) and the fact
\[ w(t) = M_1 + M_2 t + B_1 t^2 + B_2 t^3 \]
that
\[ w(t) = O(|t|^3) \quad \text{for} \quad t \text{ near } -\infty. \]
(2.24)

It follows from (2.8), (2.9), (2.23) and (2.24) that
\[ w(t, \omega) = O(|t|^3), \quad z(t, \omega) = O(|t|) \quad \text{for} \quad (t, \omega) \in (-\infty, \ln \sigma_0) \times S^{N-1}, \]
(2.25)
where \( 0 < \sigma_0 < \sigma ).
On the other hand, we see that
\[ w(t, \omega) = \sum_{k=0}^{\infty} f_k(t)Q_k(\omega), \quad z(t, \omega) = \sum_{k=0}^{\infty} g_k(t)Q_k(\omega), \quad (t, \omega) \in (-\infty, 0) \times S^{N-1}, \]
where \( \{Q_k(\omega)\} \) are the eigenfunctions of \(-\Delta_{S^{N-1}}\), i.e.
\[ -\Delta_{S^{N-1}}Q_k(\omega) = \lambda_kQ_k(\omega) \]
where \( \lambda_0 = 0, Q_0(\omega) \equiv 1 \) and \( \lambda_1 = \ldots = \lambda_N = N-1 \). Obviously, \( \{Q_k(\omega)\} \) forms an orthonormal basis of \( L^2(S^{N-1}) \). Since \((w, z)\) satisfies (2.6), we see that for any \( k \geq 0 \), \((f_k(t), g_k(t))\) satisfies
\[
\begin{cases}
    f''_k(t) - \left[ \lambda_k + \frac{(N' - 2)^2}{4} \right] f_k(t) + e^{(2+\tau)t} g_k(t) = 0, \\
    g''_k(t) - \left[ \lambda_k + \frac{(N' - 2)^2}{4} \right] g_k(t) = 0.
\end{cases} \tag{2.26}
\]
It is known from the second equation of (2.26) that
\[ g_k(t) = C_ke^{\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t} + D_ke^{-\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t}, \]
and
\[ z(t, \omega) = \sum_{k=0}^{\infty} \left( C_ke^{\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t} + D_ke^{-\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t} \right) Q_k(\omega). \]
Since \( \lambda_0 = 0, Q_0(\omega) = 1 \), we see that
\[ z(t, \omega) = C_0e^{\frac{N' - 2}{2} t} + D_0e^{-\frac{N' - 2}{2} t} + \sum_{k=1}^{\infty} \left( C_ke^{\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t} + D_ke^{-\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t} \right) Q_k(\omega). \tag{2.27} \]
For the case (i), it follows from (2.15) that
\[ D_k = 0 \quad \text{for} \quad k = 1, 2, \ldots \]
and
\[ z(t, \omega) = D_0e^{-\frac{N' - 2}{2} t} + C_0e^{\frac{N' - 2}{2} t} + \sum_{k=1}^{\infty} C_ke^{\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} t} Q_k(\omega), \tag{2.28} \]
where \( D_0 \geq 0 \).
For the case (ii), we also have (2.28).
For the case (iii), it is known from the second equation of (2.26) that
\[ g_k(t) = C_ke^{\sqrt{\lambda_k} t} + D_ke^{-\sqrt{\lambda_k} t} \quad \text{for} \quad k = 1, 2, \ldots \]
and
\[ g_0(t) = C_0 + D_0t. \]
Therefore,
\[ z(t, \omega) = C_0 + D_0t + \sum_{k=1}^{\infty} \left( C_ke^{\sqrt{\lambda_k} t} + D_ke^{-\sqrt{\lambda_k} t} \right) Q_k(\omega). \tag{2.29} \]
It follows from (2.22) that
\[ D_k = 0 \quad \text{for} \quad k = 1, 2, \ldots \]
and
\[ z(t, \omega) = C_0 + D_0 t + \sum_{k=1}^{\infty} C_k e^{\sqrt{\lambda_k} t} Q_k(\omega), \quad (2.30) \]

where \( D_0 \leq 0 \).

For the case (iv), it is known from the second equation of (2.26) that
\[ g_k(t) = C_k e^{\sqrt{\lambda_k} t} + D_k e^{-\sqrt{\lambda_k} t} \quad \text{for } k = 1, 2, \ldots \]

and
\[ g_0(t) = C_0 + D_0 t. \]

Therefore,
\[ z(t, \omega) = C_0 + D_0 t + \sum_{k=1}^{\infty} \left( C_k e^{\sqrt{\lambda_k} t} + D_k e^{-\sqrt{\lambda_k} t} \right) Q_k(\omega). \quad (2.31) \]

It follows from the second identity in (2.25) that
\[ D_k = 0 \quad \text{for } k = 1, 2, \ldots \]

and
\[ z(t, \omega) = C_0 + D_0 t + \sum_{k=1}^{\infty} C_k e^{\sqrt{\lambda_k} t} Q_k(\omega), \quad (2.32) \]

where \( D_0 \leq 0 \).

We assume \( z(0, \omega) \equiv 0 \). Then, for the case (i), it is seen from (2.28) that
\[ D_0 + C_0 = 0, \quad C_k = 0 \quad \text{for } k = 1, 2, \ldots. \quad (2.33) \]

Therefore, in this case
\[ z(t, \omega) = D_0 \left( e^{-\frac{N'-2}{2} t} - e^{\frac{N'-2}{2} t} \right). \quad (2.34) \]

Since \( v(r, \omega) = e^{-\frac{N'-2}{2} t} z(t, \omega) \), we see that
\[ v(r, \omega) = D_0 \left( r^{-\left(N'-2\right)} - 1 \right). \quad (2.35) \]

This implies that \( v \) is a radial function.

For the case (ii), we can also obtain (2.35) and \( v \) is a radial function.

For the case (iii), we can obtain from (2.30) that
\[ C_k = 0 \quad \text{for } k = 0, 1, 2, \ldots \]

and
\[ z(t, \omega) = D_0 t. \quad (2.36) \]

Since \( v(r, \omega) = z(t, \omega) \), we see that
\[ v(r, \omega) = D_0 \ln r \quad (2.37) \]

This implies that \( v \) is a radial function.

For the case (iv), we can also obtain (2.37) and hence \( v \) is a radial function.

We now consider the general case.

For the case (i), we see from (2.28) that
\[ g_0(t) = D_0 e^{-\frac{N'-2}{2} t} + C_0 e^{\frac{N'-2}{2} t} \]

and
\[ g_k(t) = C_k e^{\sqrt{\frac{\lambda_k + \left(N'-2\right)^2}{4}} t} \quad \text{for } k = 1, 2, \ldots. \]
From this and the first equation of (2.26), we have

\[ f_k(t) = A_k^{(1)} e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} t} + A_k^{(2)} e^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} t} + B_k^{(1)} C_k e^{\left[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau)\right] t}, \]

where \( B_k^{(1)} \) depends only on \( \sqrt{\lambda_k + \frac{(N'-2)^2}{4}}, -\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} \) and \( 2 + \tau \) for \( k = 1, 2, \ldots, \) and

\[ f_0(t) = A_0^{(1)} e^{-\frac{N'-2}{2} t} + A_0^{(2)} e^{-\frac{N'-2}{2} t} + B_0^{(2)} D_0 e^{-\frac{N'-6-2\tau}{2} t} + B_0^{(1)} C_0 e^{\frac{N'+2+2\tau}{2} t}. \]

Since \( w(t, \omega) = \sum_{k=0}^{\infty} f_k(t) Q_k(\omega) \), it follows from (2.15) that

\[ A_k^{(2)} = 0 \quad \text{for} \quad k = 1, 2, \ldots \]

and

\[ f_k(t) = A_k^{(1)} e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} t} + B_k^{(1)} C_k e^{\left[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau)\right] t} \quad \text{for} \quad k = 1, 2, \ldots \] (2.38)

and

\[ w(t, \omega) = A_0^{(2)} e^{-\frac{N'-2}{2} t} + A_0^{(1)} e^{-\frac{N'-2}{2} t} + B_0^{(2)} D_0 e^{-\frac{N'-6-2\tau}{2} t} + B_0^{(1)} C_0 e^{\frac{N'+2+2\tau}{2} t} + \sum_{k=1}^{\infty} \left[ A_k^{(1)} e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} t} + B_k^{(1)} C_k e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau) - \frac{N'-2}{2}} \right] Q_k(\omega), \] (2.39)

where \( A_0^{(2)} \geq 0. \) Therefore,

\[ u(r, \omega) = A_0^{(2)} r^{-(N'-2)} + B_0^{(2)} D_0 r^{-(N'-4-\tau)} + A_0^{(1)} + B_0^{(1)} C_0 r^{2+\tau} + \sum_{k=1}^{\infty} \left[ A_k^{(1)} r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} t} + B_k^{(1)} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau) - \frac{N'-2}{2}} \right] Q_k(\omega), \] (2.40)

We now show that \( h(x) \) is a Hölder continuous function at \( x = 0. \) We see

\[ h(x) = A_0^{(1)} + B_0^{(1)} C_0 r^{2+\tau} + \sum_{k=1}^{\infty} \left[ A_k^{(1)} r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} t} + B_k^{(1)} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau) - \frac{N'-2}{2}} \right] Q_k(\omega). \]

Let

\[ h_1(x) = \sum_{k=1}^{\infty} \left[ A_k^{(1)} r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} t} + B_k^{(1)} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau) - \frac{N'-2}{2}} \right] Q_k(\omega). \]

Then

\[ ||h_1(r, \omega)||_{L^2(SN-1)}^2 = \sum_{k=1}^{\infty} \left[ A_k^{(1)} r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} t} + B_k^{(1)} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau) - \frac{N'-2}{2}} \right]^2. \]

Note that \( ||Q_k||_{L^2(SN-1)} = 1. \) Since \( ||h_1(r, \omega)||_{L^2(SN-1)} \leq C \) for \( r \in [\frac{1}{2}, \frac{3}{4}], \) we see that

\[ \left| A_k^{(1)} r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} t} + B_k^{(1)} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} + (2+\tau) - \frac{N'-2}{2}} \right| \leq C \]
Therefore, for $r \in \left[\frac{1}{2}, \frac{3}{4}\right]$ and all $k = 1, 2, \ldots$
and hence
\[
\left|A_k^{(1)} + B_k^{(1)} C_k r^{2+r}\right| \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} 
\] (2.41)
for $r \in \left[\frac{1}{2}, \frac{3}{4}\right]$ and all $k = 1, 2, \ldots$ We can use contradiction arguments to show that
\[
|A_k^{(1)}| \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}, \quad |B_k^{(1)} C_k| \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}},
\] (2.42)
for all $k = 1, 2, \ldots$. Suppose for contradiction that $|A_k^{(1)}| \geq c_k 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}$ with $c_k \to +\infty$ as $k \to +\infty$ and $|B_k^{(1)} C_k| \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}$. We have
\[
|A_k^{(1)} + B_k^{(1)} C_k| \geq (c_k - C) 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}},
\]
which is a contradiction to (2.41). Similarly, suppose that
\[
|B_k^{(1)} C_k| \geq c_k 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \quad \text{and} \quad |A_k^{(1)}| \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}},
\]
there is again a contradiction to (2.41). Suppose now that
\[
|A_k^{(1)}| \geq c_k 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}, \quad |B_k^{(1)} C_k| \geq c_k 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}.
\]
We have
\[
A_k^{(1)} + B_k^{(1)} C_k \left(\frac{3}{2} + r\right) \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \quad \text{for all } k = 1, 2, \ldots \text{ due to (2.41).}
\]
This is a contradiction to $|A_k^{(1)} + B_k^{(1)} C_k \left(\frac{3}{2} + r\right)| \leq C 2 \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}$ for all $k = 1, 2, \ldots$. Therefore, for $r \in \left[0, \frac{1}{2}\right]$,
\[
|h_1(x)| \leq \sum_{k=1}^{\infty} \left[ C(2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2} + C(2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} r^{2+r} \right] |Q_k(\omega)|
\]
\[
\leq \sum_{k=1}^{\infty} C(2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2} |Q_k(\omega)|
\]
\[
\leq \sum_{k=1}^{\infty} C(2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2} \lambda_k^\sigma N}
\]
where $\sigma_N$ is a positive constant which depends on $N$ only and
\[
\max_{S^N-1} |Q_k(\omega)| \leq C \lambda_k^{-\sigma_N}, \quad \max_{S^N-1} |(Q_k)_\omega(\omega)| \leq C \lambda_k^{-\sigma_N}.
\]
Therefore, for $r \in \left[0, \frac{1}{2}\right]$,
\[
|h_1(x)| \leq C(2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \sum_{k=1}^{\infty} (2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \lambda_k} \lambda_k^\sigma N.
\]
The expressions of $\lambda_k$ imply that
\[
\lim_{k \to \infty} \left(\frac{\lambda_k + 1}{\lambda_k}\right)^{\sigma_N} (2r) \sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \lambda_k} \lambda_k^\sigma N \leq \frac{1}{2}
\]
for $r \in \left[0, \frac{1}{2}\right]$. Therefore,
\[
|h_1(x)| \leq C r \sqrt{\lambda_1 + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \quad \text{for } r \in \left[0, \frac{1}{8}\right] \quad (2.43)
\]
and
\[
|h(x) - h(0)| \leq C \left(r^{2+r} + r \sqrt{\lambda_1 + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \right) \leq C r^\sigma \quad \text{for } r \in \left[0, \frac{1}{8}\right], \quad (2.44)
\]
where \( s = \min \left\{ 2 + \tau, \sqrt{\lambda_1 + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \right\} \). Hence, \( h \) is Hölder continuous at \( x = 0 \). Moreover, we can also obtain by the similar arguments that

\[
|\nabla h(x)| \leq Cr^{s-1} \quad \text{for} \quad r \in [0, \frac{1}{8}].
\]  

(2.45)

Note that \( |\nabla h|^2 = (h_r)^2 + \frac{1}{r^2} (h_\omega)^2 \).

For the case (ii), we see from (2.28) that

\[
g_0(t) = D_0e^{-\frac{N'-2}{2}t} + C_0e^\frac{N'-2}{2}t
\]

and

\[
g_k(t) = C_k e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t \quad \text{for} \quad k = 1, 2, \ldots.
\]

From this and the first equation of (2.26), we have that

\[
f_k(t) = \left[ A_k^{(1)} + B_k^{(1)} C_k \right] e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t + \left[ A_k^{(2)} + B_k^{(2)} C_k \right] e^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t
\]

\[
+ B_k^{(3)} C_k t e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t
\]

for \( k = 1, 2, \ldots \), where \( B_k^{(1)}, B_k^{(2)} \) and \( B_k^{(3)} \) depend only on \( \sqrt{\lambda_k + \frac{(N'-2)^2}{4}} \) and \( -\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} \), and

\[
f_0(t) = \left[ A_0^{(1)} + B_0^{(1)} C_0 \right] e^{\frac{N'-2}{2} t} + \left[ A_0^{(2)} + B_0^{(2)} D_0 \right] e^{-\frac{N'-2}{2} t}
\]

\[
+ B_0^{(3)} C_0 t e^{\frac{N'-2}{2} t} + B_0^{(4)} D_0 t e^{-\frac{N'-2}{2} t}
\]

Since \( w(t, \omega) = \sum_{k=0}^{\infty} f_k(t) Q_k(\omega) \), it follows from (2.18) that

\[
A_k^{(2)} + B_k^{(2)} C_k = 0 \quad \text{for} \quad k = 1, 2, \ldots
\]

and

\[
f_k(t) = \left[ A_k^{(1)} + B_k^{(1)} C_k \right] e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t + B_k^{(3)} C_k t e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t
\]

for \( k = 1, 2, \ldots \)

(2.46)

and

\[
w(t, \omega) = \hat{A}_0^{(2)} e^{-\frac{N'-2}{2} t} + \hat{A}_0^{(1)} e^{\frac{N'-2}{2} t} + B_0^{(3)} C_0 t e^{\frac{N'-2}{2} t} + B_0^{(4)} D_0 t e^{-\frac{N'-2}{2} t}
\]

\[
+ \sum_{k=1}^{\infty} \left[ (A_k^{(1)} + B_k^{(1)} C_k) e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t + B_k^{(3)} C_k t e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}} t \right] Q_k(\omega)
\]

(2.47)

where \( B_0^{(4)} D_0 \leq 0 \). Therefore,

\[
u(r, \omega) = \hat{A}_0^{(2)} r^{-(N'-2)} + \hat{A}_0^{(1)} + B_0^{(3)} C_0 \ln r + B_0^{(4)} D_0 r^{-(N'-2)} \ln r
\]

\[
+ \sum_{k=1}^{\infty} \left[ (A_k^{(1)} + B_k^{(1)} C_k) r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}}} + B_k^{(3)} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4} - \frac{N'-2}{2}} \ln r} \right] Q_k(\omega)
\]

\[
= B_0^{(4)} D_0 r^{-(N'-2)} \ln r + \hat{A}_0^{(2)} r^{-(N'-2)} + B_0^{(3)} C_0 \ln r + h(x).
\]

(2.48)

Arguments similar to those in the proof of (2.44) imply that \( h(x) \) is a Hölder continuous function at \( x = 0 \).
For the case (iii), it is known from the second equation of (2.26) that
\[ g_k(t) = C_k e^{\sqrt{x_k} t} \quad \text{for } k = 1, 2, \ldots \]
and
\[ g_0(t) = C_0 + D_0 t. \]
From this and the first equation of (2.26), we have that for \( k = 1, 2, \ldots \),
\[ f_k(t) = A_k^{(1)} e^{\sqrt{x_k} t} + A_k^{(2)} e^{-\sqrt{x_k} t} + B_k^{(1)} C_k e^{\sqrt{x_k} t + (2+\tau) t}, \]
and
\[ f_0(t) = A_0^{(1)} + A_0^{(2)} t + B_0^{(1)} e^{(2+\tau) t} + B_0^{(2)} D_0 t e^{(2+\tau) t}. \]
Since \( w(t, \omega) = \sum_{k=0}^{\infty} f_k(t) Q_k(\omega) \), it follows from (2.22) that
\[ A_k^{(2)} = 0 \quad \text{for } k = 1, 2, \ldots \]
and
\[ f_k(t) = A_k^{(1)} e^{\sqrt{x_k} t} + B_k^{(1)} C_k e^{\sqrt{x_k} t + (2+\tau) t} \quad \text{for } k = 1, 2, \ldots \]
\[ (2.49) \]
and
\[ w(t, \omega) = A_0^{(2)} t + A_0^{(1)} + B_0^{(1)} e^{(2+\tau) t} + B_0^{(2)} D_0 t e^{(2+\tau) t} \]
\[ + \sum_{k=1}^{\infty} \left[ A_k^{(1)} e^{\sqrt{x_k} t} + B_k^{(1)} C_k e^{\sqrt{x_k} t + (2+\tau) t} \right] Q_k(\omega), \]
\[ (2.50) \]
where \( A_0^{(2)} \leq 0 \). Therefore,
\[ u(r, \omega) = A_0^{(2)} \ln r + A_0^{(1)} + B_0^{(1)} r^{2+\tau} + B_0^{(2)} D_0 r^{2+\tau} \ln r \]
\[ + \sum_{k=1}^{\infty} \left[ A_k^{(1)} e^{\sqrt{x_k} t} + B_k^{(1)} C_k e^{\sqrt{x_k} t + (2+\tau) t} \right] Q_k(\omega) \]
\[ = A_0^{(2)} \ln |x| + B_0^{(2)} D_0 |x|^{2+\tau} \ln |x| + h(x). \]
\[ (2.51) \]
Arguments similar to those in the proof of (2.44) imply that \( h(x) \) is a Hölder continuous function at \( x = 0 \).

For the case (iv), it is known from the second equation of (2.26) that
\[ g_k(t) = C_k e^{\sqrt{x_k} t} \quad \text{for } k = 1, 2, \ldots \]
and
\[ g_0(t) = C_0 + D_0 t. \]
From this and the first equation of (2.26), we have that for \( k = 1, 2, \ldots \),
\[ f_k(t) = (A_k^{(1)} + B_k^{(1)} C_k)e^{\sqrt{x_k} t} + (A_k^{(2)} + B_k^{(2)} C_k)e^{-\sqrt{x_k} t} + B_k^{(3)} C_k t e^{\sqrt{x_k} t}, \]
and
\[ f_0(t) = A_0^{(1)} + A_0^{(2)} t + B_0^{(1)} C_0 t^2 + B_0^{(2)} D_0 t^3. \]
Since \( w(t, \omega) = \sum_{k=0}^{\infty} f_k(t) Q_k(\omega) \), it follows from the first identity in (2.25) that
\[ A_k^{(2)} + B_k^{(2)} C_k = 0 \quad \text{for } k = 1, 2, \ldots \]
and
\[ f_k(t) = (A_k^{(1)} + B_k^{(1)} C_k)e^{\sqrt{x_k} t} + B_k^{(3)} C_k t e^{\sqrt{x_k} t} \quad \text{for } k = 1, 2, \ldots \]
\[ (2.52) \]
and
\[ w(t, \omega) = B_0^{(2)} D_0 t^3 + B_0^{(1)} C_0 t^2 + A_0^{(2)} t + A_0^{(1)} \]
\[ + \sum_{k=1}^{\infty} \left[ (A_k^{(1)} + B_k^{(1)} C_k) e^{\sqrt{\lambda} t} + B_k^{(3)} C_k t e^{\sqrt{\lambda} t} \right] Q_k(\omega), \]
(2.53)
where \( B_0^{(2)} D_0 \leq 0 \). Therefore,
\[ u(r, \omega) = B_0^{(2)} D_0 (\ln r)^3 + B_0^{(1)} C_0 (\ln r)^2 + A_0^{(2)} \ln r + A_0^{(1)} \]
\[ + \sum_{k=1}^{\infty} \left[ (A_k^{(1)} + B_k^{(1)} C_k r^{\sqrt{\lambda}} + B_k^{(3)} C_k r^{\sqrt{\lambda}} \ln r \right] Q_k(\omega) \]
\[ = B_0^{(2)} D_0 (\ln r)^3 + B_0^{(1)} C_0 (\ln r)^2 + A_0^{(2)} \ln r + h(x). \]
(2.54)
Arguments similar to those in the proof of (2.44) imply that \( h(x) \) is a Hölder continuous function at \( x = 0 \).

We assume \( z(0, \omega) \equiv 0 \) and \( w(0, \omega) \equiv 0 \).
For the case (i), we see from (2.33) that \( D_0 = -C_0, \ C_k = 0 \) for \( k = 1, 2, \ldots \).
Therefore, we obtain from (2.39) that
\[ A_k^{(1)} = 0 \text{ for } k = 1, 2, \ldots, \ A_0^{(2)} + A_0^{(1)} + B_0^{(2)} D_0 - B_0^{(1)} D_0 = 0 \]
and
\[ w(t, \omega) = A_0^{(2)} e^{-N' t} + A_0^{(1)} e^{-N' t} + B_0^{(2)} D_0 e^{-N' t} \]
\[ - (A_0^{(2)} + A_0^{(1)} + B_0^{(2)} D_0) e^{-\frac{N' t}{2} + \frac{1}{2} t}. \]
(2.55)
Since \( u(r, \omega) = e^{-N' t} w(t, \omega) \), we see that
\[ u(r, \omega) = A_0^{(2)} r^{-(N'-2)} + B_0^{(2)} D_0 r^{-(N'-4-\tau)} \]
\[ + A_0^{(1)} - (A_0^{(2)} + A_0^{(1)} + B_0^{(2)} D_0) r^{(2+\tau)}, \]
(2.56)
where \( A_0^{(2)} \geq 0 \). This implies that under the boundary conditions: \( u = v = 0 \) on \( \partial B \), \( u, v \) is a radial solution of the system (1.5).
For the case (ii), we also see from (2.33) that \( D_0 = -C_0, \ C_k = 0 \) for \( k = 1, 2, \ldots \).
Then we obtain from (2.47) that
\[ A_k^{(1)} = 0 \text{ for } k = 1, 2, \ldots, \ A_0^{(2)} + A_0^{(1)} = 0 \]
and
\[ w(t, \omega) = A_0^{(2)} e^{-N' t} - A_0^{(2)} e^{-\frac{N' t}{2} + \frac{1}{2} t} + B_0^{(4)} D_0 t e^{-\frac{N' t}{2} + \frac{1}{2} t} - B_0^{(3)} D_0 t e^{-\frac{N' t}{2} + \frac{1}{2} t}. \]
(2.57)
Since \( u(r, \omega) = e^{-\frac{N' t}{2} + \frac{1}{2} t} w(t, \omega) \), we see that
\[ u(r, \omega) = B_0^{(4)} D_0 r^{-(N'-2)} \ln r + A_0^{(2)} r^{-(N'-2)} - B_0^{(3)} D_0 \ln r - \tilde{A}_0^{(2)}, \]
(2.58)
where \( B_0^{(4)} D_0 \leq 0 \). Hence under the boundary conditions: \( u = v = 0 \) on \( \partial B \), \( u, v \) is a radial solution of the system (1.5).
For the case (iii), we can obtain from (2.30) that
\[ C_k = 0 \text{ for } k = 0, 1, 2, \ldots. \]
Moreover, it follows from (2.50) that
\[ A^{(1)}_0 + B^{(1)}_0 = 0, \quad A^{(1)}_k = 0 \text{ for } k = 1, 2, \ldots. \]
Therefore, we see from (2.50) that
\[ w(t, \omega) = A^{(2)}_0 t + A^{(1)}_0 - A^{(1)}_0 e^{(2+\tau)t} + B^{(2)}_0 D_0 t e^{(2+\tau)t} \]
and hence
\[ u(r, \omega) = A^{(2)}_0 \ln r + A^{(1)}_0 - A^{(1)}_0 r^{2+\tau} + B^{(2)}_0 D_0 r^{2+\tau} \ln r, \]
where \( A^{(2)}_0 \leq 0 \). Therefore, under the boundary conditions: \( u = v = 0 \) on \( \partial B \), \((u, v)\) is a radial solution of the system (1.5).

For the case (iv), we can obtain from (2.32) that
\[ C_k = 0 \text{ for } k = 0, 1, 2, \ldots. \]
Moreover, it follows from (2.53) that
\[ A^{(1)}_k = 0 \text{ for } k = 0, 1, 2, \ldots. \]
Therefore, we see from (2.53) that
\[ w(t, \omega) = B^{(2)}_0 D_0 t^3 + B^{(1)}_0 C_0 t^2 + A^{(2)}_0 t \]
and hence
\[ u(r, \omega) = B^{(2)}_0 D_0 (\ln r)^3 + B^{(1)}_0 C_0 (\ln r)^2 + A^{(2)}_0 \ln r, \]
where \( B^{(2)}_0 D_0 \leq 0 \). Therefore, under the boundary conditions: \( u = v = 0 \) on \( \partial B \), \((u, v)\) is a radial solution of the system (1.5).

To complete the proof of this theorem, we only need to show that \( h \) satisfies (2.3)
in the weak sense. The facts that \( k \) is Hölder continuous at \( x = 0 \) and satisfies (2.4) in the weak sense have been known from [30].

We can assume \( h \in C^s(B) \) for some \( s \in (0, 1) \). The calculations above imply that \( h \) satisfies (1.4) in \( B \setminus \{0\} \). For \( N' \geq 2 \) and \( \tau \geq -2 \), we will prove \( h \) is a weak solution of (1.4) in the whole ball \( B \) in the following.

**Definition 2.2.** We say that \( u \) is a weak solution of (1.4) in \( B \), if
\[ \int_B |x|^{-1} \text{div}(|x|^\theta \nabla u) \text{div}(|x|^\theta \nabla \phi) dx = 0 \quad \forall \phi \in \mathcal{G} \]
where
\[ \mathcal{G} = \{ \phi \in C^\infty_0(B) : \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial B \} \]
and \( \nu \) is the unit outward normal vector of \( \partial B \).

We consider the cut-off function
\[ \eta(x) = \begin{cases} 1 & \text{in } B^{C_2}_r, \\ 0 & \text{in } B_r \end{cases} \tag{2.59} \]
with \( |\nabla \eta| \leq \frac{C}{r} \) and \( |\Delta \eta| \leq \frac{C}{r^2} \), \( 2r < \frac{1}{2} \). Hence, we have that for \( \phi \in \mathcal{G} \),
\[ \eta \phi \in C^\infty_0(B \setminus \{0\}), \quad \frac{\partial (\eta \phi)}{\partial \nu} = 0 \text{ on } \partial B, \]
and
\[ 0 = \int_B |x|^{-1} \text{div}(|x|^\theta \nabla h) \text{div}(|x|^\theta \nabla (\eta \phi)) dx \]
Therefore, for \( N' \geq 2 \),
\[
|I_{12}| = \int_{B_{2r}\setminus B_r} k(x)|x|^{\theta-1}h_{rr} + \frac{N' - 1}{r}h_r + \frac{1}{r^2} \Delta \nu h dx
\leq C r^{N' + \theta - 1} \to 0 \text{ as } r \to 0.
\]
This implies that
\[
I_1 \to \int_B |x|^{-l} \text{div}(|x|^{\theta} \nabla h) \text{div}(|x|^{\theta} \nabla \phi) dx \text{ as } r \to 0.
\]
On the other hand, using \( k \in C^{s_1}_0(B) \) and arguments similar to those in the proof of (2.45), we have that there exists \( C > 0 \) such that
\[
|\nabla k(x)| \leq C |x|^{s_1 - 1} \forall x \in \overline{B_{\frac{r}{2}}}.
\]
Therefore,
\[
|I_2| = \int_B k(x) \text{div}(|x|^{\theta} \nabla \phi) dx = \int_{B_{2r}\setminus B_r} |x|^{\theta} \phi(x) \nabla \eta \nabla k dx
\]
\[
\leq C \int_r^{2r} \rho^{s_1-1} \rho^\delta \rho^{N-1} dr = C \int_r^{2r} \rho^{N'+s_1-3} dr \to 0 \quad \text{as } r \to 0
\]
since \(N' + s_1 - 2 > 0\). It follows from (2.60) that
\[
\int_B |x|^{-\ell} \text{div}(|x|^\delta \nabla h) \text{div}(|x|^\delta \nabla \phi) dx = 0.
\]
Therefore, \(h\) is a weak solution of (1.4) in \(B\).

**Corollary 1.** Let \(N' := N + \theta < 2\), \(\tau := l - \theta \geq -2\), \(u \in C^4(B \setminus \{0\})\) be a nonnegative solution of the equation (1.4) with \(v = -|x|^{-l} \text{div}(|x|^\delta \nabla u) \geq 0\) in \(B \setminus \{0\}\). Then there exist constants \(a \geq 0\) and \(c, d, e\) such that
\[
u(x) = \begin{cases} 
  c|x|^{2-N'} + d|x|^{2+\tau} + e|x|(4+\tau-N') + h(x) & \text{if } N' < 2 \text{ & } \tau > -2, \\
-a \ln |x| + c|x|^{2-N'} \ln |x| + d|x|^{2-N'} + h(x) & \text{if } N' < 2 \text{ & } \tau = -2,
\end{cases}
\]
and
\[
v(x) = c|x|^{2-N'} + k(x) \quad \text{if } N' < 2
\]
where \(h(x)\) and \(k(x)\) are Hölder continuous at \(x = 0\). Moreover,
\[
\lim_{|x| \to 0} u(x) = \tau_0, \quad \text{if } N' < 2 \text{ & } \tau > -2,
\]
and
\[
\lim_{|x| \to 0} v(x) = \rho_0 \quad \text{if } N' < 2
\]
for some \(\tau_0, \rho_0 \in \mathbb{R}\).

**Proof.** The proof of this corollary can be done by arguments similar to those in the proof of Theorem 2.1. Note 4 + \(\tau - N' = (2 + \tau) + (2 - N') > 2 + \tau \geq 0\) in this case.

**Corollary 2.** Let \(\tau := l - \theta \leq -2\) and \(u \in C^4(\mathbb{R}^N \setminus \bar{B})\) be a nonnegative solution of the equation (1.4) with \(v = -|x|^{-l} \text{div}(|x|^\delta \nabla u) \geq 0\) in \(\mathbb{R}^N \setminus \bar{B}\). Then there exist constants \(a \geq 0\) and \(c, d, e\) such that
\[
u(x) = \begin{cases} 
  a|x|^{2-N'} + c|x|^{4+\tau-N'} + h(x) & \text{if } N' < 2 \text{ & } \tau < -2, \\
-a \ln |x| + c|x|^{2-N'} \ln |x| + h(x) & \text{if } N' < 2 \text{ & } \tau = -2,
\end{cases}
\]
and
\[
u(x) = \begin{cases} 
  a|x|^{2-N'} + k(x) & \text{if } N' < 2, \\
-a \ln |x| + k(x) & \text{if } N' = 2,
\end{cases}
\]
where \(h(x), k(x) \in C^\infty(\mathbb{R}^N \setminus \bar{B})\) and both of them satisfy
\[
|m(x) - m(y)| \leq CR^{-s} \quad \text{for any } x, y \in \mathbb{R}^N \setminus \bar{B}
\]
for some \(s \in (0, 1)\) and any \(R > 1\). Moreover, if \(N' > 2\) and \(\tau < -2\), then \(u(x) \to \tau_0\), and if \(N' > 2\), \(v(x) \to \rho_0\) as \(|x| \to \infty\) for some \(\tau_0, \rho_0 \in \mathbb{R}\).
Proof. The conclusions in \( \mathbb{R}^N \setminus \overline{B} \) can be obtained by arguments similar to those in the proof of Theorem 2.1. The only difference is that we define \( t = \ln r \in (0, \infty) \) and \( r = |x| \in (1, \infty) \). By Kelvin transformation:

\[
w(y) = |x|^{N' - 2}u(x), \quad z(y) = |x|^{N' - 2}v(x), \quad y = \frac{x}{|x|^2}
\]

for \( x \in \mathbb{R}^N \setminus \overline{B} \), we see that \((w(y), z(y))\) satisfies the system of equations:

\[
\begin{aligned}
-\text{div}(|y|^\theta \nabla w) &= |y|^{l'} z & \text{in } B \setminus \{0\}, \\
-\text{div}(|y|^\theta \nabla z) &= 0 & \text{in } B \setminus \{0\}
\end{aligned}
\]  

(2.66)

where \( l' = -(4 + l - 2\theta) \). We now have \( \tau' := l' - \theta = -(4 + l - \theta) = -(4 + \tau) \geq -2 \) since \( 2 + \tau \leq 0 \). We can obtain the conclusions as in Theorem 2.1 for (2.66). Then we derive the conclusions in this corollary by using the Kelvin transformation. \( \square \)

3. Proof of the main results. In this section we present the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. At every singular point \( p_i \), since \( v = -\Delta_g u \geq 0 \), we see that \( -\Delta_g v = 0 \) in the neighborhood \( U_i \setminus \{p_i\} \), which is equivalent to the weighted elliptic equation

\[
-\text{div}(|x|^\theta \nabla v) = 0 \quad \text{in } B \setminus \{0\}.
\]

It is known from Theorem 1.1 of [30] that at every singular point \( p_i \), there exists a neighborhood \( V_i(p_i) \subset U_i \) such that

\[
v(x) = \begin{cases} 
  a_i |x - p_i|^{(2 - N)(1 + \beta_i)} + k(x) & \text{if } \beta_i > -1 \text{ and } N \geq 3, \\
  -a_i \ln |x - p_i| + k(x) & \text{if } \beta_i = -1 \text{ or } N = 2
\end{cases}
\]  

(3.1)

where \( a_i \geq 0, k(x) \) is a weak harmonic function with the equation

\[
-\Delta_g v = 0 \quad \text{in } M_0
\]

in the whole neighborhood \( V_i(p_i) \). Moreover,

\[
-\Delta_g v = \rho_i \delta_{p_i} \quad \text{in } V_i(p_i)
\]  

(3.2)

for some \( \rho_i \geq 0 \).

At every singular point \( p_i \), the homogeneous bi-Laplace equation \( \Delta_g^2 u = 0 \) in the neighborhood \( U_i \setminus \{p_i\} \) is equivalent to the weighted elliptic equation (1.4) in \( B \setminus \{0\} \). Note that \( N' = N + (N - 2)\beta_i \geq 2 \) and \( \tau = N\beta_i - (N - 2)\beta_i = 2\beta_i \geq -2 \) if \( \beta_i \geq -1 \). By Theorem 2.1, we can obtain the conclusions of Theorem 1.1. In fact, when \( N \geq 3 \) and \( \beta_i > -1 \), we have \( N' > 2 \) and \( \tau > -2 \). Then

\[
u(x) = a_i |x - p_i|^{(2 - N)(1 + \beta_i)} + c_i |x - p_i|^{(4 - N)(1 + \beta_i)} + h(x).
\]  

(3.3)

When \( N = 2 \) and \( \beta_i > -1 \), we have \( N' = 2 \) and \( \tau > -2 \). Then

\[
u(x) = -a_i \ln |x - p_i| + c_i |x - p_i|^{2(1 + \beta_i)} \ln |x - p_i| + h(x).
\]  

(3.4)

When \( N \geq 2 \) and \( \beta_i = -1 \), we have \( N' = 2 \) and \( \tau = -2 \). Then

\[
u(x) = -a_i (\ln |x - p_i|)^3 + c_i (\ln |x - p_i|)^2 + d_i \ln |x - p_i| + h(x)
\]  

(3.5)

In all the cases, \( h \) is Hölder continuous at \( p_i \). Moreover, when \( N \geq 4 \) and \( \beta_i \geq -1 \), \( h \) is a global weak solution of \( \Delta_g^2 h = 0 \) in a neighborhood \( V_i(p_i) \subset U_i \). We have

\[
\int_{V_i(p_i)} \Delta_g h \Delta_g \phi dV_g = 0
\]  

(3.6)
that is
\[
\int_{B_r} \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla h)))\phi(x)dx = 0, \quad (3.7)
\]
for any \( \phi \in \mathcal{G} = \{ \phi \in C_0^\infty (B) : \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial B \} \).

To complete the proof, it remains to show (1.11) in \( V_i(p_i) \). By the equivalence of (P) and (1.4), we need to show that
\[
\int_B \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla u)))\varphi(x)dx = \tau_i \varphi(0) \quad \forall \varphi \in C_0^\infty (B) \quad (3.8)
\]
for some \( \tau_i \geq 0 \). We only show (3.8) in the case (3.3), the other cases can be treated similarly. It follow from (3.3) that
\[
\int_B \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla u)))\varphi(x)dx = \int_{B \setminus B_r} \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla u)))\varphi(x)dx
\]
\[+ \int_{B_r} \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla u)))\varphi(x)dx = \int_{B_r} \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla u)))\varphi(x)dx
\]
\[= - \int_{B_r} \text{div}(|x|^\theta \nabla u(x))\varphi(x)dx
\]
where \( v = -|x|^{-1} \text{div}(|x|^\theta \nabla u) \). Since
\[
v(x) = \begin{cases} 
  a_i \Phi(x) + k(x) & \text{if } N' > 2, \\
  -a_i \ln |x| + k(x) & \text{if } N' = 2,
\end{cases}
\]
where \( \Phi(x) = |x|^{2-N'} \) and \( k(x) \) is Hölder continuous at \( x = 0 \) and that \( k(x) \) is a global weak solution of the equation \( \text{div}(|x|^\theta \nabla k) = 0 \) in \( B_r \) (see [30]), that is
\[
\int_{B_r} \text{div}(|x|^\theta \nabla k)\varphi(x)dx = 0, \quad (3.9)
\]
we have
\[
\int_B \text{div}(|x|^\theta \nabla (|x|^{-1} \text{div}(|x|^\theta \nabla u)))\varphi(x)dx
\]
\[= - \int_{B_r} \text{div}(|x|^\theta \nabla u(x))\varphi(x)dx = - \int_{B_r} \text{div}(|x|^\theta \nabla (a_i \Phi + k))\varphi(x)dx
\]
\[= - \int_{B_r} a_i \text{div}(|x|^\theta \nabla \Phi)\varphi(x)dx - \int_{B_r} \text{div}(|x|^\theta \nabla k)\varphi(x)dx
\]
\[= -a_i \int_{B_r} \text{div}(|x|^\theta \nabla \Phi)\varphi(x)dx
\]
\[= a_i \left\{ - \int_{\partial B_r} |x|^\theta \phi(x) \frac{\partial \Phi}{\partial x_i} \frac{x_i}{|x|} dS_x + \int_{B_r} |x|^\theta \nabla \Phi \nabla \varphi dx \right\}
\]
\[:= J_1 + J_2.
\]
For \( J_1 \), we have
\[
J_1 = a_i (N' - 2) \int_{\partial B_r} |x|^{-N+1} \varphi(x) dS_x = a_i (N' - 2) \frac{1}{\epsilon^{N-1}} \int_{\partial B_r} \varphi(x) dS_x
\]
bounded from above in a neighborhood of $0$. \( \tau_i = a_i(N'-2)\omega_{N-1} \) as $\epsilon \to 0$, and $\omega_{N-1} \geq 0$, where $\omega_{N-1}$ is the volume of $S^{N-1}$.

Obviously,

$$J_2 = a_i \int_{B_r} |x|^\theta \Phi \nabla \varphi \, dx$$

$$= a_i \left\{ \int_{\partial B_r} \frac{\partial \varphi}{\partial \nu_x} |x|^\theta \Phi \, d\Sigma_x - \int_{B_r} \nabla |x|^\theta \Phi \varphi \, dx - \int_{B_r} |x|^\theta \Phi \Delta \varphi \, dx \right\} \to 0 \quad \text{as } \epsilon \to 0.$$

This completes the proof of Theorem 1.1.

\( \square \)

**Remark 1.** (i) We can define the fundamental solution $\Gamma(x)$:

$$\Gamma(x) = \begin{cases} a_i|x-p_i|^{(2-N)(1+\beta_i)} + c_i|x-p_i|^{(4-N)(1+\beta_i)} & \text{if } \beta_i > -1 \text{ & } N \geq 3, \\ -a_i \ln|x-p_i| + c_i|x-p_i|^{1(1+\beta_i)} \ln|x-p_i| & \text{if } \beta_i > 1 \text{ & } N = 2, \\ -a_i(\ln|x-p_i|)^3 + c_i(\ln|x-p_i|)^2 + d_i \ln(|x-p_i|) & \text{if } \beta_i = 1 \text{ & } N \geq 2, \\ \end{cases}$$

of the bi-Laplace $\Delta^2$ induced by the conical metric in $V(p_i)$. By Theorem 1.1, we see that $\Delta^2 \Gamma$ is interpreted as the Dirac mass at the singularity $p_i$.

(ii) When $N = 2$ and $-1 < \beta_i \leq -\frac{1}{2}$, $h(x)$ is a weak bi-harmonic function of the equation (1.7) in the whole neighborhood $V(p_i)$. Moreover, for the special case $N \geq 2$ and $\beta_i = -1$, $h(x)$ also satisfies the equation (1.7) in the whole neighborhood $V(p_i)$ in the weak sense.

**Proof of Theorem 1.2.** Similar to the proof of Theorem 2.1 and Corollary 1, we make the special transformation:

$$w(t, \omega) = r^{\frac{N'-2}{2}} u(r, \omega), \quad z(t, \omega) = r^{\frac{N'-2}{2}} v(r, \omega), \quad t = \ln r \in (-\infty, \infty). \quad (3.11)$$

For $N' > 2$, we have

$$z(t, \omega) = C_0 e^{\frac{N'-2}{2} t} + D_0 e^{-\frac{N'-2}{2} t} + \sum_{k=1}^{\infty} \left( C_k e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} t} + D_k e^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} t} \right) Q_k(\omega). \quad (3.12)$$

and

$$v(r, \omega) = C_0 + D_0 r^{-(N'-2)} + \sum_{k=1}^{\infty} \left( C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}} + D_k r^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}} \right) Q_k(\omega). \quad (3.13)$$

Since $v$ is bounded from above in a neighborhood of $\infty$ as well as in a neighborhood of 0, we can see that for any $K \geq 1$,

$$v_1(r, \omega) := \sum_{k=1}^{\infty} C_k r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}} Q_k(\omega)$$

is bounded from above in a neighborhood of $\infty$ and

$$v_2(r, \omega) := \sum_{k=1}^{\infty} D_k r^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}} Q_k(\omega)$$

is bounded from above in a neighborhood of 0.
We now show
\[ C_k = 0 \quad \text{for } k = 1, 2, \ldots \] (3.14)
Otherwise, there is some \( k \) such that \( C_k > 0 \). We can choose \( \omega \) such that \( Q_k(\omega) > 0 \), then we have that \( v(r, \omega) \to \infty \) as \( r \to \infty \), which contradicts to the fact that \( v \) is bounded from above in a neighborhood of \( \infty \). On the other hand, if there is some \( k \) such that \( C_k < 0 \), we can choose \( \omega \) such that \( Q_k(\omega) < 0 \), then we have that \( v(r, \omega) \to \infty \) as \( r \to \infty \), which contradicts to the fact that \( v \) is bounded from above in a neighborhood of \( \infty \). Therefore, (3.14) holds. Similarly, we can obtain
\[ D_k = 0 \quad \text{for } k = 1, 2, \ldots \] (3.15)
by using the fact that \( v \) is bounded from above in a neighborhood of \( 0 \). So, (3.13) yields
\[ v(r, \omega) = D_0r^{-(N'-2)} + C_0 \]
and \( \infty \) is the removable singularity of solutions of the equation
\[ \text{div}(|x|^p \nabla v) = 0 \quad \text{in } \mathbb{R}^N \backslash \{0\}. \] (3.16)

We can also obtain (3.14) and (3.15) if \( v \) is bounded from below in a neighborhood of \( \infty \) as well as in a neighborhood of \( 0 \). If \( v \) is bounded from below in a neighborhood of \( \infty \) as well as in a neighborhood of \( 0 \), then \(-v\) is bounded from above in a neighborhood of \( \infty \) as well as in a neighborhood of \( 0 \). This implies that (3.14) and (3.15) hold.

For \( N' < 2 \), we can obtain
\[ v(r, \omega) = C_0 + D_0r^{2-N'} \]
similarly and see that \( x = 0 \) is the removable singularity of solutions of (3.16).

For \( N' = 2 \), the proof is similar. In this case, we have
\[ v(r, \omega) = a \ln r + b \]
for some constants \( a \) and \( b \). But we can easily see that \( a = 0 \) by the boundedness on one side in both a neighborhood of the origin and a neighborhood of \( \infty \).

To complete the proof of Theorem 1.2, we consider \( u(r, \omega) \). We only consider the case \( N' > 2 \) and \( \tau > -2 \). The other cases can be studied similarly.

Similar to the proof of Theorem 2.1 and Corollary 1, we see that \( g_k(t) \equiv 0 \) for \( k = 1, 2, \ldots \), since (3.14) and (3.15) hold. Then
\[ f_k(t) = A_k^{(1)} e^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}t} + A_k^{(2)} e^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}}t} \quad \text{for } k = 1, 2, \ldots \]
and
\[ f_0(t) = A_0^{(1)} e^{\frac{N'-2}{2}t} + A_0^{(2)} e^{-\frac{N'-2}{2}t} + B_0^{(2)} D_0 e^{-\frac{N'-4-2\tau}{2}t} + B_0^{(1)} C_0 e^{\frac{N'+2+2\tau}{2}t}. \]

Therefore,
\[ u(r, \omega) = A_0^{(2)} r^{-(N'-2)} + B_0^{(1)} D_0 r^{-(N'-4-\tau)} + A_0^{(1)} + B_0^{(2)} C_0 r^{2+\tau} \]
\[ + \sum_{k=1}^{\infty} \left[ A_k^{(1)} r^{\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}} + A_k^{(2)} r^{-\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}} \right] Q_k(\omega). \] (3.17)

Arguments similar to those in the proof of the case of \( v(r, \omega) \) imply that
\[ A_k^{(1)} = 0, \quad A_k^{(2)} = 0 \quad \text{for } k = 1, 2, \ldots \]
and
\[ u(r, \omega) = A_0^{(2)} r^{-(N'-2)} + B_0^{(1)} D_0 r^{-(N'-4-\tau)} + A_0^{(1)} + B_0^{(2)} C_0 r^{2+\tau} \] (3.18)
if $u$ is bounded on one side in both a neighborhood of the origin and a neighborhood of $\infty$. The proof of Theorem 1.2 is complete.

4. The Bôcher-type results for higher order equations on singular manifolds with conical metrics. In this section we will generalize the results of the fourth order equation to higher order equations

$$\Delta^m_g u = 0 \quad \text{on } M_0,$$

where $m \geq 2$ is a positive integer. We define the operators:

$$L \cdot = -|x|^{-l} \text{div}(|x|^{\theta} \nabla \cdot), \quad L = -\text{div}(|x|^\theta \nabla \cdot).$$

Then the equation (4.1) is equivalent to the equation

$$L \circ L^{m-1}(u) = 0.$$

Let

$$v_1 = L^{m-1}(u), \quad v_2 = L^{m-2}(u), \ldots, v_m = u.$$

Then equation (4.2) is equivalent to the system of equations

\begin{align*}
-\text{div}(|x|^{\theta} \nabla v_1) &= 0, \\
-\text{div}(|x|^{\theta} \nabla v_2) &= |x|^l v_1, \\
&\quad \ldots \ldots \ldots \\
-\text{div}(|x|^{\theta} \nabla v_m) &= |x|^l v_{m-1}.
\end{align*}

**Theorem 4.1.** Let $u \in C^{2m}(B\setminus\{0\})$ be a nonnegative solution of (4.2) in $B\setminus\{0\}$ with

$$v_{m-1} := L(u) \geq 0, \quad v_{m-2} := L^2(u) \geq 0, \quad \ldots, \quad v_1 := L^{m-1}(u) \geq 0 \quad \text{in } B\setminus\{0\}. \quad (4.4)$$

Then there exist constants $a \geq 0$, $c_0, c_1, c_2, \ldots, c_{2(m-1)}$ and $d_1, d_2, \ldots, d_{m-1}$ such that

\begin{align*}
u(x) = \begin{cases}
a|x|^{-(N'-2)} + \sum_{q=1}^{m-1} c_q |x|^{-(N'-2(q+1)-q\tau)} + h(x) \quad &\text{if } N' > 2 & \& \tau > -2, \\
-a(ln |x|)^{m-1-|x|^{-(N'-2)}} + \sum_{q=0}^{m-2} c_q (ln |x|)^q |x|^{-(N'-2)} \quad &\text{if } N' > 2 & \& \tau = -2, \\
-a ln |x| + ln |x| \sum_{q=1}^{m-1} c_q |x|^{q(2+\tau)} \\
+ \sum_{q=1}^{m-1} d_q |x|^{q(2+\tau)} + h(x) \quad &\text{if } N' = 2 & \& \tau > -2, \\
-a(ln |x|)^{2m-1} + \sum_{q=1}^{2(m-1)} c_q (ln |x|)^{2m-(q+1)} + h(x) \quad &\text{if } N' = 2 & \& \tau = -2,
\end{cases}
\end{align*}

and for any integer $1 \leq j < m$, 

\begin{align*}
u(x) = \begin{cases}
&\text{if } N' > 2 \\
&\text{if } N' = 2
\end{cases}
\end{align*}
\[ v_j(x) = \begin{cases} 
  a|x|^{-(N'-2)} + \sum_{q=1}^{j-1} c_q |x|^{-(N'-2(q+1)-\tau)} + k_j(x) & \text{if } N' > 2 \land \tau > -2, \\
  -a(\ln |x|)^{j-1} |x|^{-(N'-2)} + \sum_{q=0}^{j-2} c_q (\ln |x|)^q |x|^{-(N'-2)} \\
  + \sum_{q=1}^{j-1} d_q (\ln |x|)^q + k_j(x) & \text{if } N' > 2 \land \tau = -2, \\
  -a \ln |x| + \ln |x| \sum_{q=1}^{j-1} c_q |x|^{q(2+\tau)} \\
  + \sum_{q=1}^{j-1} d_q |x|^{q(2+\tau)} + k_j(x) & \text{if } N' = 2 \land \tau > -2, \\
  -a(\ln |x|)^{2j-1} + \sum_{q=1}^{2(j-1)} c_q (\ln |x|)^{2j-(q+1)} + k_j(x) & \text{if } N' = 2 \land \tau = -2 
\end{cases} \] (4.6)

where \( h(x) \) and \( k_j(x) \) are H"older continuous at \( x = 0 \). Moreover, when \( N' > 2 \), \( \tau \geq -2 \) and \( N' - (m-1)\tau \geq 2m \); \( N' = 2 \), \( \tau > -2 \) and \( (m-1)(2+\tau) \leq \sqrt{N-1} \); \( N' = 2 \) and \( \tau = -2 \), \( h \) satisfies equation (4.2) in \( B \) in the weak sense, and when \( N' > 2 \), \( \tau \geq -2 \) and \( N' - (j-1)\tau \geq 2j \); \( N' = 2 \), \( \tau > -2 \) and \( (j-1)(2+\tau) \leq \sqrt{N-1} \); \( N' = 2 \) and \( \tau = -2 \), \( k_j \) satisfies the equation

\[ \mathcal{L} \circ \mathcal{L}^{-1}(v_j) = 0 \] (4.7)

in \( B \) in the weak sense. If \( u \in C^4(B \setminus \{0\}) \) is a nonnegative solution of (4.2) satisfying (4.4) and \( u = v_1 = \ldots = v_{m-1} = 0 \) on \( \partial B \), then \( u \) and \( v_1, v_2, \ldots, v_{m-1} \) are radial functions.

**Theorem 4.2.** Let \( u \) be a nonnegative solution of the equation

\[ \Delta_g^m u = 0 \text{ on } M_0 \] (4.8)

with

\[ v_{m-1} := -\Delta_g u \geq 0, \ v_{m-2} := (-\Delta_g)^2 u \geq 0, \ \ldots, \ v_1 := (-\Delta_g)^{m-1} u \geq 0 \] (4.9)

in every \( U_i, \ i = 1, 2, \ldots, k \). Then at every singular point \( p_i \), there exists a neighborhood \( V_i(p_i) \subset U_i \) such that

\[ u(x) = \begin{cases} 
  a_i |x - p_i|^{\alpha_i} + \sum_{q=0}^{m-1} c_q^i |x - p_i|^\beta_i + h(x) & \text{if } \beta_i > -1 \land N \geq 3, \\
  -a_i \ln |x - p_i| + \ln |x - p_i| \sum_{q=0}^{m-1} c_q^i |x - p_i|^{2q(1+\beta_i)} \\
  + \sum_{q=1}^{m-1} d_q^i |x - p_i|^{2q(1+\beta_i)} + h(x) & \text{if } \beta_i > -1 \land N = 2, \\
  h(x) - a_i (\ln |x - p_i|)^{2m-1} + \sum_{q=0}^{2(m-1)} c_q^i (\ln |x - p_i|)^{2m-(q+1)} & \text{if } \beta_i = -1 \land N \geq 2 
\end{cases} \] (4.10)
and for any integer \(1 \leq j < m\),
\[
v_j(x) = \begin{cases} 
  a_i |x - p_i|^{\alpha_i} + \sum_{q=0}^{j-1} c_q^i |x - p_i|^{\theta_q^i} + k_j(x) & \text{if } \beta_i > -1 \text{ & } N \geq 3, \\
  -a_i \ln |x - p_i| + \ln |x - p_i| \sum_{q=0}^{j-1} c_q^i |x - p_i|^{2q(1+\beta_i)} & \text{if } \beta_i > -1 \text{ & } N = 2, \\
  -a_i \ln |x - p_i| + \sum_{q=1}^{j-1} d_q^i |x - p_i|^{2q(1+\beta_i)} + k_j(x) & \text{if } \beta_i = -1 \text{ & } N \geq 2 \\
\end{cases}
\]
\[
(4.11)
\]
where \(\alpha_i = (2 - N)(1 + \beta_i)\), \(\theta_q^i = (2 + 2q - N)(1 + \beta_i)\), \(\tau_q^j = 2j - (q + 1)\), \(\rho^i = 2j - 1\), \(a_i \geq 0\), \(c_0^i = 0\), \(c_1^i, \ldots, c_{2(m-1)}^i\) and \(d_1^i, \ldots, d_{m-1}^i\) are constants, \(h(x)\) and \(k_j(x)\) are Hölder continuous at \(p_i\). Consequently,
\[
(-\Delta_g)^m u = \tau_i \delta_{p_i} \text{ in } V_i(p_i) \quad (4.12)
\]
and
\[
(-\Delta_g)^j v_j = \rho_j^i \delta_{p_i} \text{ in } V_i(p_i) \quad (4.13)
\]
for some \(\tau_i \geq 0\) and \(\rho_j^i \geq 0\). Moreover, when \(N \geq 2m\) and \(\beta_i \geq -1\), \(h(x)\) and \(k_j(x)\) are weak solutions of \((-\Delta_g)^m z = 0\) and
\[
(-\Delta_g)^j z = 0 \quad (4.14)
\]
respectively in the whole neighborhood \(V_i(p_i)\).

We will show Theorem 4.1. The proof of Theorem 4.2 is similar to that of Theorem 4.1.

Proof of Theorem 4.1. We consider the system (4.3). Similar to the proof of Theorem 2.1, we introduce the transformation:
\[
 z_j(t, \omega) = r^{N/2} v_j(r, \omega) \quad j = 1, 2, \ldots, m - 1, m, \quad t = \ln r \in (-\infty, 0).
\]
Then \((z_1, z_2, \ldots, z_m)\) satisfies the system
\[
\begin{align*}
 \partial_t z_1 + \Delta_{S^{N-1}} z_1 - \frac{(N^2 - 2)^2}{4} z_1 &= 0 \quad (t, \omega) \in (-\infty, 0) \times S^{N-1}, \\
 \partial_t z_2 + \Delta_{S^{N-1}} z_2 - \frac{(N^2 - 2)^2}{4} z_2 + e^{(2+\tau)t} z_1 &= 0 \quad (t, \omega) \in (-\infty, 0) \times S^{N-1}, \\
 \quad & \quad \ldots \ldots \\
 \partial_t z_m + \Delta_{S^{N-1}} z_m - \frac{(N^2 - 2)^2}{4} z_m + e^{(2+\tau)t} z_{m-1} &= 0 \quad (t, \omega) \in (-\infty, 0) \times S^{N-1}.
\end{align*}
\]
\[
(4.14)
\]
Since \(2 + \tau \geq 0\), we see that \(e^{(2+\tau)t} \leq 1\) for \(t \in (-\infty, 0)\). By using the Harnack's inequality (see Theorem 2.1 of [1]) of system of equations over \([L - 1, L] \times S^{N-1}\) with \(L < \ln \sigma\) and \(0 < \sigma < 1/10\) sufficiently small, we have that there exists \(C > 0\) independent of \(t \in (-\infty, \ln \sigma)\) such that
\[
\max_{S^{N-1}} \{z_j(t, \cdot)\} \leq C \max_{1 \leq \omega \leq m} \min_{S^{N-1}} \{z_j(t, \cdot)\} \quad \text{for } t \leq \ln \sigma, \ \forall j \in \{1, 2, \ldots, m\}. \quad (4.15)
\]
Let \(\overline{z}_j(t) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} z_j(t, \omega) d\omega\) for \(j = 1, 2, \ldots, m\). It follows from (4.15) that there exist \(0 < c^1_j < c^2_j\) such that
\[
c^1_j \overline{z}_j(t) \leq z_j(t, \omega) \leq c^2_j \overline{z}_j(t) \quad \text{for } (t, \omega) \in (-\infty, \ln \sigma) \times S^{N-1}. \quad (4.16)
\]
It is easily seen from (4.14) that \((\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_m(t))\) satisfies the system
\[
\begin{cases}
\bar{z}_1'' - \frac{(N' - 2)^2}{4}\bar{z}_1 = 0 & \quad t \in (-\infty, \ln \sigma), \\
\bar{z}_2'' - \frac{(N' - 2)^2}{4}\bar{z}_2 + e^{(2+\tau)t}\bar{z}_1 = 0 & \quad t \in (-\infty, \ln \sigma), \\
\vdots & \\
\bar{z}_m'' - \frac{(N' - 2)^2}{4}\bar{z}_m + e^{(2+\tau)t}\bar{z}_{m-1} = 0 & \quad t \in (-\infty, \ln \sigma).
\end{cases}
(4.17)
\]

Let
\[
z_j(t, \omega) = \sum_{k=0}^{\infty} f^j_k(t)Q_k(\omega), \quad (t, \omega) \in (-\infty, 0) \times S^{N-1}.
\]

Then \((f^1_k(t), f^2_k(t), \ldots, f^m_k(t))\) satisfies
\[
\begin{cases}
(f^1_k)'(t) = 0, \\
(f^2_k)'(t) - \frac{(N' - 2)^2}{4} f^1_k(t) + e^{(2+\tau)t} f^1_k(t) = 0, \\
\vdots \\
f^m_k'(t) - \frac{(N' - 2)^2}{4} f^{m-1}_k(t) + e^{(2+\tau)t} f^{m-1}_k(t) = 0.
\end{cases}
(4.18)
\]

We only show the case \(N' > 2\) and \(\tau > -2\). The other cases can be done similarly. Arguments similar to those in the proof of Theorem 2.1 imply that for every \(j \in \{1, 2, \ldots, m\}\),
\[
z_j(t) = O(e^{-\frac{N'-2}{2}t}) \quad \text{for } t \text{ near } -\infty
\]
and
\[
z_j(t, \omega) = O(e^{-\frac{N'-2}{2}t}) \quad \text{for } (t, \omega) \in (-\infty, \ln \sigma_0) \times S^{N-1},
(4.20)
\]
via (4.16), where \(0 < \sigma_0 < \sigma\).

Using the theory of ODE, (4.20) and arguments similar to those in the proof of Theorem 2.1, we obtain that
\[
f^j_k(t) = \sum_{q=0}^{j-1} A^j_{k,q} e^{\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} + q(2+\tau)t} \quad \text{for } k = 1, 2, \ldots,
(4.21)
\]
where
\[
A^j_{k,q} = M^j_{k,q} A^{j-1}_{k,q-1}, \quad q \neq 0
(4.22)
\]
and \(M^j_{k,q}\) is a generic constant and
\[
f^j_0(t) = \sum_{q=0}^{j-1} \left[ A^j_{0,q} e^{\sqrt{q(2+\tau) + \frac{N'-2}{2}}} + B^j_{0,q} e^{\sqrt{(2+\tau) - \frac{N'-2}{2}}} \right].
(4.23)
\]

Therefore,
\[
z_j(t, \omega) = \sum_{q=0}^{j-1} \left[ A^j_{0,q} e^{\sqrt{q(2+\tau) + \frac{N'-2}{2}}} + B^j_{0,q} e^{\sqrt{(2+\tau) - \frac{N'-2}{2}}} \right] + \sum_{k=1}^{\infty} \sum_{q=0}^{j-1} A^j_{k,q} e^{\sqrt{\lambda_k + \frac{(N' - 2)^2}{4}} + q(2+\tau)t} Q_k(\omega)
(4.24)
and

\[ v_j(r, \omega) = \sum_{q=0}^{j-1} \left[ A_{0,q}^j r^{[q(2+\tau) + (N'-2)]} + B_{0,q}^j r^{[q(2+\tau)-(N'-2)]} \right] \\
+ \sum_{k=1}^{\infty} \sum_{q=0}^{j-1} A_{k,q}^j r^{[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} + q(2+\tau)]} Q_k(\omega) \\
= a|x|^{-(N'-2)} + \sum_{q=1}^{j-1} c_q |x|^{-(N'-2(q+1)-q\tau)} + k_j(x) \tag{4.25} \]

with \( a := B_{0,0}^j \geq 0 \) and

\[ k_j(x) = \sum_{q=0}^{j-1} A_{0,q}^j r^{[q(2+\tau)]} + \sum_{k=1}^{\infty} \sum_{q=0}^{j-1} A_{k,q}^j r^{[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} + q(2+\tau)]} Q_k(\omega). \tag{4.26} \]

Arguments similar to those in the proof of Theorem 2.1 imply that \( k_j(x) \) is Hölder continuous at \( x = 0 \). This also implies that

\[ u(r, \omega) = \sum_{q=0}^{m-1} \left[ A_{0,q}^m r^{[q(2+\tau)]} + B_{0,q}^m r^{[q(2+\tau)-(N'-2)]} \right] \\
+ \sum_{k=1}^{\infty} \sum_{q=0}^{m-1} A_{k,q}^m r^{[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} + q(2+\tau)]} Q_k(\omega) \\
= a|x|^{-(N'-2)} + \sum_{q=1}^{m-1} c_q |x|^{-(N'-2(q+1)-q\tau)} + h(x), \tag{4.27} \]

with \( a = B_{0,0}^m \geq 0 \), where

\[ h(x) = \sum_{q=0}^{m-1} A_{0,q}^m r^{[q(2+\tau)]} + \sum_{k=1}^{\infty} \sum_{q=0}^{m-1} A_{k,q}^m r^{[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2} + q(2+\tau)]} Q_k(\omega) \]

is Hölder continuous at \( x = 0 \).

If \( v_1 = v_2 = \ldots = v_{j-1} = 0 \) on \( \partial B \), we see from (4.22) and (4.25) that

\[ A_{k,q}^j = 0 \text{ for } q = 1, 2, \ldots, j-1 \text{ & } k = 1, 2, \ldots \]

and

\[ v_j(r, \omega) = \sum_{q=0}^{j-1} \left[ A_{0,q}^j r^{[q(2+\tau)]} + B_{0,q}^j r^{[q(2+\tau)-(N'-2)]} \right] \\
+ \sum_{k=1}^{\infty} A_{k,0}^j r^{[\sqrt{\lambda_k + \frac{(N'-2)^2}{4}} - \frac{N'-2}{2}] Q_k(\omega).} \tag{4.28} \]

If \( v_j = 0 \) on \( \partial B \), we see that

\[ A_{k,0}^j = 0, \text{ for } k = 1, 2, \ldots \tag{4.29} \]

and

\[ \sum_{q=0}^{j-1} [A_{0,q}^j + B_{0,q}^j] = 0. \tag{4.30} \]
Therefore,

\[ v_j(r, \omega) = \sum_{q=0}^{j-1} \left[ A^j_{0,q} r^{[q(2+r)]} + B^j_{0,q} r^{[q(2+r)-(N'-2)]} \right] \quad (4.31) \]

with \( B^j_{0,0} \geq 0 \). Therefore, \( v_j \) is a radial function.

If \( u := v_m = v_{m-1} = \ldots = v_{1} = 0 \) on \( \partial B \),

\[ u(r, \omega) = \sum_{q=0}^{m-1} \left[ A^m_{0,q} r^{[q(2+r)]} + B^m_{0,q} r^{[q(2+r)-(N'-2)]} \right] \quad (4.32) \]

which is a radial function.

We now show that for each \( j \in \{2, \ldots, m\} \), \( k_j \) satisfies the equation

\[ \mathcal{L} \circ \mathcal{L}^{-1}(k_j) = 0 \quad (4.33) \]

in \( B \) in the weak sense. It is easily seen from the expression of \( k_j \) that \( k_j \) satisfies (4.33) in \( B \setminus \{0\} \).

To show (4.33), we firstly present the following definition.

**Definition 4.3.** We say that \( u \) is a weak solution of (4.33) in \( B \), if \( j \geq 2 \) and

\[ \int_B |x|^j \nabla [\mathcal{L}^n(u)] \nabla \mathcal{L}^n(\phi) \, dx = 0 \quad \forall \phi \in \mathcal{G} \quad \text{and} \quad j = 2n+1, \]

where

\[ \mathcal{G} = \{ \phi \in C^\infty_0(B) : \phi = \mathcal{L}(\phi) = \ldots \mathcal{L}^n(\phi) = 0, \phi_\nu = (\mathcal{L}(\phi))_\nu = \ldots = (\mathcal{L}^{n-1}(\phi))_\nu = 0 \text{ on } \partial B \} \]

and

\[ \int_B |x|^{-j} \mathcal{L} \circ \mathcal{L}^{-1}(u) \mathcal{L} \circ \mathcal{L}^{-1}(\phi) \, dx = 0 \quad \forall \phi \in \mathcal{G} \quad \text{and} \quad j = 2n, \]

where

\[ \mathcal{G} = \{ \phi \in C^\infty_0(B) : \phi = \mathcal{L}(\phi) = \ldots \mathcal{L}^{n-1}(\phi) = 0, \phi_\nu = (\mathcal{L}(\phi))_\nu = \ldots = (\mathcal{L}^{n-1}(\phi))_\nu = 0 \text{ on } \partial B \} \]

and \( \nu \) is the unit outward normal vector of \( \partial B \).

For \( j = 2 \), it is known from Theorem 2.1 that \( k_2 \) satisfies \( \mathcal{L} \circ \mathcal{L}(k_2) = 0 \) in \( B \) in the weak sense, when \( N' - \tau \geq 4 \). We now show \( k_3 \) satisfies \( \mathcal{L} \circ \mathcal{L}^2(k_3) = 0 \) in \( B \) in the weak sense, i.e.,

\[ \int_B |x|^j \nabla [\mathcal{L}(k_3)] \nabla \mathcal{L}(\phi) \, dx = 0 \quad \forall \phi \in \mathcal{G} \quad (4.34) \]

where

\[ \mathcal{G} = \{ \phi \in C^\infty_0(B) : \phi = \mathcal{L}(\phi) = 0, \phi_\nu = 0 \text{ on } \partial B \}. \]

Let \( \eta(x) \) be the cut-off function in (2.59). Then for \( \eta \phi \in C^\infty_0(B \setminus \{0\}) \),

\[ 0 = \int_B |x|^j \nabla [\mathcal{L}(k_3)] \nabla \mathcal{L}(\eta \phi) \, dx = -\int_B \mathcal{L}^2(k_3) \mathcal{L}(\eta \phi) \, dx \]

Setting \( \bar{k}(x) = \mathcal{L}^2(k_3) \), we see that \( \bar{k} \) satisfies the equation

\[ \mathcal{L}(\bar{k}) = 0, \text{ i.e. } -\text{div}(|x|^j \nabla \bar{k}) = 0 \text{ in } B \setminus \{0\}. \]

We can use the arguments showing (2.3) in the proof of Theorem 2.1 to see that \( \bar{k} \) is Hölder continuous at \( x = 0 \) provided \( N' - 2\tau \geq 6 \) and to obtain (4.34). Hence
Proof of Theorem 4.2. We only need to show (4.12) and (4.13). Others can be obtained from Theorem 4.1. For any \( j \in \{2, 3, \ldots, m\} \), when \( N' - (j - 1) \tau \geq 2j \), the proof of Theorem 4.1 is complete.

Proof of Theorem 4.2. We only need to show (4.12) and (4.13). Others can be obtained from Theorem 4.1. For any \( j \in \{2, 3, \ldots, m\} \), we see that \( k_j \) is Hölder continuous at \( p_i \). Moreover, when \( N \geq 2m \) and \( \beta_i \geq -1 \), it is a global weak solution of \((-\Delta_g)^j k_j = 0\) in a neighborhood \( V_i(p_i) \subset U_i \). We have

\[
\begin{align*}
\int_{V_i(p_i)} (-\Delta_g)^n k_j \cdot (-\Delta_g)^n \phi dV_g &= 0 \quad \text{if } j = 2n, \\
\int_{V_i(p_i)} \nabla_g (-\Delta_g)^n k_j \cdot \nabla_g (-\Delta_g)^n \phi dV_g &= 0 \quad \text{if } j = 2n + 1,
\end{align*}
\]

that is

\[
\int_{B_i} \mathcal{L} \circ \mathcal{L}^{j-1}(k_j)\phi(x)dx = 0 \quad \text{for any } \phi \in \mathcal{G}.
\]

To show (4.13) in \( V_i(p_i) \), we need to show that

\[
\int_B \mathcal{L} \circ \mathcal{L}^{j-1}(v_j)(x)\varphi(x)dx = \rho_j^i \varphi(0) \quad \forall \varphi \in C_c^\infty(B)
\]

for some \( \rho_j^i \geq 0 \). Since \( v \equiv \mathcal{L}^{j-1}(v_j)(x) \) satisfies \( \mathcal{L}(v)(x) = 0 \) in \( B \setminus \{0\} \), i.e. \( -\text{div}(|x|^\beta \nabla v) = 0 \) in \( B \setminus \{0\} \), we see that

\[
v(x) = \begin{cases} 
\alpha_i \Phi(x) + k(x) & \text{if } N' > 2, \\
-\alpha_i \ln |x| + k(x) & \text{if } N' = 2,
\end{cases}
\]

where \( \Phi(x) = |x|^{2-N'} \) and \( k(x) \) is Hölder continuous at \( x = 0 \) and that \( k(x) \) is a global weak solution of the equation \( \text{div}(|x|^\beta \nabla k) = 0 \) in \( B \) (see [30]). Arguments similar to those to obtain (1.11) in the proof of Theorem 1.1 imply that (4.37) holds and thus (4.13) holds. This also implies that (4.12) holds. The proof of Theorem 4.2 is complete.

Remark 2. When \( N = 2 \) and \( -1 < \beta_i \leq \frac{1}{2(m-1)} - 1 \), \( h \) is a weak solution of \((-\Delta_g)^m z = 0\) in the whole neighborhood \( V_i(p_i) \). Similarly, in the special case \( N \geq 2 \) and \( \beta_i = -1 \), we also have that \( h \) satisfies equation \((-\Delta_g)^m z = 0\) in the neighborhood \( V_i(p_i) \) in the weak sense.

Using the arguments similar to those in the proof of Theorem 4.1, we can obtain the following corollary.

Corollary 3. Let \( u \in C^{2m}(B\setminus\{0\}) \) be a nonnegative solution of (4.2) in \( B \setminus \{0\} \) with

\[
\begin{align*}
v_{m-1} &:= \mathcal{L}(u) \geq 0, \\
v_{m-2} &:= \mathcal{L}^2(u) \geq 0, \\
&\quad \vdots, \\
v_1 &:= \mathcal{L}^{m-1}(u) \geq 0
\end{align*}
\]
Then there exist constants $a \geq 0, c_0, c_1, \ldots, c_{m-1}$ and $d_0, d_1, \ldots, d_{m-1}$ such that

$$u(x) = \begin{cases} 
\sum_{q=0}^{m-1} c_q |x|^{(2q+1)+q\tau} - N' + h(x) & \text{if } N' < 2 \& \tau > -2, \\
\alpha |\ln |x||^m - 1 + \sum_{q=1}^{m-2} c_q (\ln |x|)^q \\
+ \sum_{q=0}^{m-1} d_q (\ln |x|)^q |x|^{2-N'} + h(x) & \text{if } N' < 2 \& \tau = -2 
\end{cases}$$

(4.39)

and for any integer $1 \leq j < m$,

$$v_j(x) = \begin{cases} 
\sum_{q=0}^{j-1} c_q |x|^{(2q+1)+q\tau} - N' + k_j(x) & \text{if } N' < 2 \& \tau > -2, \\
\alpha |\ln |x||^j - 1 + \sum_{q=1}^{j-2} c_q (\ln |x|)^q \\
+ \sum_{q=0}^{j-1} d_q (\ln |x|)^q |x|^{2-N'} + k_j(x) & \text{if } N' < 2 \& \tau = -2 
\end{cases}$$

(4.40)

where $h(x)$ and $k_j(x)$ are Hölder continuous at $x = 0$. Moreover, when $N' < 2$ and $\tau > -2$,

$$\lim_{|x| \to 0} u(x) = \tau,$$

(4.41)

$$\lim_{|x| \to 0} v_j(x) = \rho_j$$

(4.42)

for some $\tau, \rho_j \in \mathbb{R}$.

Using the arguments similar to those in the proof of Theorems 4.1 and Theorem 1.2, we can also obtain the following Liouville-type result.

**Corollary 4.** Let $u \in C^2(m)(\mathbb{R}^N \setminus \{0\})$ be a solution of the equation (4.2) in $\mathbb{R}^N \setminus \{0\}$. Suppose $u$ and $v_1, v_2, \ldots, v_{m-1}$ are bounded from above or from below in a neighborhood of $0$ as well as in a neighborhood of infinity. Then there exist constants $c_0, c_1, c_2, \ldots, c_{2m-1}$ and $d_0, d_1, \ldots, d_{m-1}$ such that

$$u(x) = \begin{cases} 
\sum_{q=0}^{m-1} c_q |x|^{-(N' - 2q + 1) - q\tau} + d_q |x|^{q(2+\tau)} & \text{if } N' > 2 \& \tau > -2, \\
\sum_{q=0}^{m-1} c_q (\ln |x|)^q |x|^{-(N' - 2)} + d_q (\ln |x|)^q \\
+ \sum_{q=0}^{m-1} c_q |x|^{2m-(q+1)} + \sum_{q=0}^{m-2} d_q |x|^{q(2+\tau)} & \text{if } N' = 2 \& \tau > -2, \\
\sum_{q=0}^{2m-1} c_q (\ln |x|)^{2m-(q+1)} & \text{if } N' = 2 \& \tau = -2, \\
\sum_{q=0}^{m-1} c_q |x|^{(2q+1)+q\tau} - N' + d_q |x|^{q(2+\tau)} & \text{if } N' < 2 \& \tau > -2, \\
\sum_{q=0}^{m-1} c_q (\ln |x|)^q |x|^{2-N'} + d_q (\ln |x|)^q & \text{if } N' < 2 \& \tau = -2 
\end{cases}$$

(4.43)
and for any integer $1 \leq j < m$,

\[
v_j(x) = \begin{cases} 
  \sum_{q=0}^{j-1} c_q |x|^{-N_q - 2(q+1) - \tau} + d_q |x|^{q(2+\tau)} & \text{if } N' > 2 \text{ and } \tau > -2, \\
  \sum_{q=0}^{j-1} c_q \ln |x| |x|^{-N_q - 2} + d_q \ln |x|^{q} & \text{if } N' > 2 \text{ and } \tau = -2, \\
  2^{-1} \sum_{q=0}^{j-1} c_q \ln |x| |x|^{2j-(q+1)} + d_q & \text{if } N' = 2 \text{ and } \tau > -2, \\
  \sum_{q=0}^{j-1} c_q |x|^{(2(q+1)+\tau)-N'} + d_q |x|^{q(2+\tau)} & \text{if } N' < 2 \text{ and } \tau > -2, \\
  \sum_{q=0}^{j-1} c_q \ln |x|^{q} |x|^{2-N'} + d_q \ln |x|^{q} & \text{if } N' < 2 \text{ and } \tau = -2.
\end{cases}
\]

(4.44)

In particular, $c_0 = 0$ when $N' = 2$.

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E-mail address: vfangshu@mail.ustc.edu.cn