ON THE CONNECTION BETWEEN FUNDAMENTAL GROUPS AND PENCILS WITH MULTIPLE FIBERS

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INTRODUCTION

The study of the topology of complex projective (or quasiprojective) smooth varieties depends strongly on the knowledge of the topology of the complement of hypersurfaces in a projective space. Considering a projection, any smooth projective variety is a covering of a projective space of the same dimension ramified along a hypersurface. These coverings are measured by (finite index subgroups of) the fundamental group of the complement of the hypersurface. Using Lefschetz-Zariski theory, if we take a generic plane section the fundamental group of the complement does not change. As a consequence, for fundamental group purposes, one can restrict their attention to the study of complements of curves in the projective plane, as stated in the foundational paper by O. Zariski [26].

The richness of coverings for a space depends on its fundamental group. This is why we are mostly interested in curves $C \subset \mathbb{P}^2$ whose $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian. The first known example is probably the curve formed by three lines $C := L_1 \cup L_2 \cup L_3$ intersecting at one point $P$. There is an easy way to compute this fundamental group; the pencil of lines through $P$ is parametrized by $\mathbb{P}^1$; this pencil induces an epimorphism of $\mathbb{P}^2 \setminus C$ onto $\mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$ (the punctures corresponding to the three lines). Moreover, this map is a locally trivial fibration (with fiber isomorphic to $\mathbb{C}$) and hence $\pi_1(\mathbb{P}^2 \setminus C) \cong \pi_1(\mathbb{P}^1 \setminus \{p_1, p_2, p_3\})$, which is a free group of two generators.

The first known examples of irreducible curves whose fundamental groups are known to be non-abelian appeared in [26]. The first one corresponds to a hexacuspidal sextic, with its six cusps on a conic; the equation of such a curve is of the form $f_3^2 + f_3^2 = 0$, where $f_j$ is a homogeneous polynomial of degree $j$. Its fundamental group is $\mathbb{Z}/2 \ast \mathbb{Z}/3$; in [2] we will see the relation between this group and the pencil generated by $f_3^2 = 0$ and $f_3^2 = 0$. This kind of examples have been generalized by various authors replacing $(2, 3)$ by $(p, q)$. In the same paper, Zariski found the irreducible curve with smallest possible degree having a non-abelian fundamental group: the tricuspidal quartic. This example and many others appearing in the literature are also connected with pencils.

The precise connection with pencils can be stated as follows: a pencil defines a dominant morphism to a quasi-projective curve, inducing an epimorphism at the level of fundamental groups. The multiplicities of the fibers of the pencil induce an orbifold structure on the quasi-projective group, and the map defines an epimorphism onto the orbifold fundamental group. When such an orbifold fundamental group is non-abelian, then the original fundamental group has a surjection onto a non-abelian group. Such surjections coming from dominant maps will be referred to as geometric surjections.

The tricuspidal quartic is the only irreducible curve of degree 4 with a non-abelian fundamental group. The degree-five case was studied by A. Degtyarev [9]; he found exactly two irreducible
quintics with non-abelian fundamental groups. One of them, also studied by the first author \[2\], has an infinite fundamental group. In \[2\] we will study its relationship with a pencil. The question whether or not all non-abelian fundamental groups have a geometric surjection onto an orbifold group naturally arises. A positive result in this direction is given in \[8\] for certain roots of the Alexander polynomial. In addition, all the examples studied, up to now, supported an affirmative answer to this question.

In this paper we will show an explicit example of a non-abelian fundamental group whose complement admits no geometric surjections. This curve is one of the quintics referred to in the previous paragraph, which will be called the \textit{projective Degtyarev curve} throughout this text. As a brief description, the projective Degtyarev curve has exactly three singular points of type $A_4$; its fundamental group is finite and non-abelian. In Proposition 4.4, we prove that this group admits no geometric surjections. Once the group is computed, the proof is rather straightforward; it depends on the orders of the group and its abelianization and on the properties of orbifold groups.

If we add a tangent line to one of the singular points of the projective Degtyarev curve, the complement of the union in $\mathbb{P}^2$ is the complement of an affine curve, which will be called the \textit{affine Degtyarev curve}. This affine curve has an infinite non-abelian fundamental group and non-trivial characteristic varieties (see §1 for the definition). Extending results of Arapura and others, it is known that irreducible components of positive dimension (for the fundamental group of a quasiprojective variety) are obtained as pull-back of irreducible components of characteristic varieties of orbifolds. A natural question arises: Is it also true for irreducible components of dimension 0 (isolated points)? Plenty of computations supported a positive answer: most quasiprojective groups satisfy the property for irreducible components of any dimension (see §2 for examples). The main Theorem 4.5 of this paper shows that the fundamental group of the complement of the affine Degtyarev curve does not satisfy this property. This is the only known example, up to now.

The paper is organized as follows. In §1 the concepts of orbifold and characteristic varieties are recalled, also some orbifold groups are studied. In §2 we relate non-abelian fundamental groups of the complements of curves (which are known in the literature) with orbifold morphisms (via pencils of curves). In §3 we describe Degtyarev curves and, in order to obtain a presentation for their fundamental groups, we compute a special braid monodromy. The fundamental groups are obtained in §4, where also the main results of the paper are stated and proved. Finally, further properties of the affine Degtyarev curve are sketched in §5.

1. Orbifold groups and characteristic varieties

The fundamental groups of oriented Riemann surfaces have been extensively studied. The fundamental group of a compact Riemann surface of genus $g$ is

$$\pi_g := \left\langle a_i, b_i, 1 \leq i \leq g \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} \right\rangle.$$ 

If $C$ is a surface with genus $g$ and $k > 0$ punctures then its fundamental group is free of rank $2g + k - 1$. We are going to extend this family by considering orbifold groups.

In this paper, we will refer to an orbifold $X_\varphi$ as an orbifold Riemann surface, that is, a quasiprojective Riemann surface $X$ with a function $\varphi : X \to \mathbb{N}$ with value 1 outside a finite number of points. The finite set $M_\varphi = \{x \in X \mid \varphi(x) > 1\}$ will be called the set of orbifold points and $\varphi(x)$ is the orbifold index of $x \in M_\varphi$. 
We may think that a neighborhood of a point \( P \in X_\varphi \) such that \( \varphi(P) = n \) is the quotient of a disk (centered at \( P \)) by a rotation of angle \( \frac{2\pi}{n} \). We will consider that a loop around \( P \) is trivial if its lifting bounds a disk. Following this idea, we define orbifold fundamental groups.

**Definition 1.1.** For an orbifold \( X_\varphi \), let \( p_1, \ldots, p_n \) the points such that \( m_j := \varphi(p_j) > 1 \). Then, the *orbifold fundamental group* of \( X_\varphi \) is

\[
\pi_1^{\text{orb}}(X_\varphi) := \pi_1(X \setminus \{p_1, \ldots, p_n\})/\langle \mu_j^{m_j} = 1 \rangle
\]

where \( \mu_j \) is a meridian of \( p_j \). We denote \( X_\varphi \) by \( X_{m_1, \ldots, m_n} \).

**Example 1.2.** If \( X \) is a compact surface of genus \( g \) and type \( X_{m_1, \ldots, m_n} \), then

\[
\pi_1^{\text{orb}}(X_\varphi) = \left\{ a_1, \ldots, a_r, b_1, \ldots, b_g, \mu_1, \ldots, \mu_n \mid \prod_{i=1}^g [a_i, b_i] = \prod_{j=1}^n \mu_j, \mu_j^{m_j} = 1 \text{ (} j = 1, \ldots, n \text{)} \right\},
\]

where products are supposed to respect the order. If \( X \) is not compact and \( \pi_1(X) \) is free of rank \( r \), then

\[
\pi_1^{\text{orb}}(X_\varphi) = \langle a_1, \ldots, a_r, \mu_1, \ldots, \mu_n \mid \mu_j^{m_j} = 1 \text{ (} j = 1, \ldots, n \text{)} \rangle.
\]

**Definition 1.3.** A dominant algebraic morphism \( \rho : Y \rightarrow X \) between an algebraic manifold \( Y \) and a Riemann surface \( X \) defines an *orbifold morphism* \( Y \rightarrow X_\varphi \) if for all \( p \in X \), the divisor \( \rho^*(p) \) has multiplicity \( \varphi(p) \), that is, \( \rho^*(p) = \varphi(p)D \), where \( D \) is a (possibly non-reduced) divisor in \( Y \).

**Proposition 1.4.** Let \( \rho : Y \rightarrow X \) define an orbifold morphism \( Y \rightarrow X_\varphi \). Then \( \rho \) induces a homomorphism \( \rho_* : \pi_1(Y) \rightarrow \pi_1^{\text{orb}}(X_\varphi) \). Moreover, if the generic fiber is connected, then \( \rho_* \) is surjective.

**Proof.** Let \( M_\varphi := \{ x \in X \mid \varphi(x) > 1 \} \); we consider the restriction mapping \( \tilde{\rho} := \rho_1 : Y \setminus \rho^{-1}(M_\varphi) \rightarrow X \setminus M_\varphi \). This map induces a morphism \( \tilde{\rho}_* : \pi_1(Y \setminus \rho^{-1}(M_\varphi)) \rightarrow \pi_1(X \setminus M_\varphi) \) fitting in the following commutative diagram:

\[
\pi_1(Y \setminus \rho^{-1}(M_\varphi)) \xrightarrow{\tilde{\rho}_*} \pi_1(X \setminus M_\varphi) \xrightarrow{\rho_*} \pi_1(X) \xrightarrow{j_*} \pi_1(X).
\]

The vertical mappings are induced by the inclusions. They are both surjective; the kernel of \( j_* \) is generated by the meridians of the points in \( M_\varphi \) while the kernel of \( \tilde{\rho}_* \) is generated by the meridians of the irreducible components of \( \rho^{-1}(M_\varphi) \), i.e., the components of the pull-back divisor \( \rho^*(M_\varphi) \).

Let us consider an irreducible component \( D \) of \( \rho^*(M_\varphi) \) such that \( \rho(D) =: x \in M_\varphi \). Let \( n := \varphi(x) \); note that the multiplicity \( m_D \) of \( D \) in \( \rho^*(M_\varphi) \) is a multiple of \( n \). We can interpret \( m_D \) as follows. If \( \mu_D \) denotes a meridian of \( D \), then there is a meridian \( \mu_x \) of \( x \) such that \( \tilde{\rho}_*(\mu_D) = (\mu_x)^{m_D} \). Following Definition 1.1, it is easily seen that \( \tilde{\rho}_* \) factorizes through a morphism (also called \( \rho_* \)) \( \pi_1(Y) \rightarrow \pi_1^{\text{orb}}(X_\varphi) \).

The above argument also works if one replaces \( M_\varphi \) by a finite set \( M \supseteq M_\varphi \). In particular, one can choose \( M \) to be the bifurcation locus of \( \rho \), i.e., the mapping is a differentiable locally trivial fibration outside \( M \). If the fiber is generically connected, the long exact homotopy sequence of this fibration implies the surjectivity of \( \rho_* \) (for \( M \)). The result follows.

**Definition 1.5.** A fundamental group \( G := \pi_1(Y) \) of an algebraic manifold is said to pass a **geometric surjection** if \( Y \) possesses an orbifold morphism \( Y \rightarrow X_\varphi \) whose generic fiber is connected, and such that \( \pi_1^{\text{orb}}(X_\varphi) \) is non-abelian.
We recall the notion of characteristic varieties and its relationship with orbifolds. We focus our attention on the characteristic varieties of quasiprojective manifolds, though they can be defined in general and depend only on the fundamental group. Let $X$ be a connected topological space $X$, having the homotopy type of a finite $CW$-complex, and let $G := \pi_1(X, x_0)$, $x_0 \in X$ which will be omitted if it is not necessary. Recall that the space of characters of $G$ is

\[ H^1(X; \mathbb{C}^*) \cong \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{C}^*) = \text{Hom}(\pi_1(X), \mathbb{C}^*) =: T_G. \]

**Remark 1.6.** Since $G$ is finitely generated, then it is also the case for $H_1(X; \mathbb{Z})$. Let $n := \text{rk} H_1(X; \mathbb{Z})$ and $\text{Tors}_G$ be the torsion subgroup of $H_1(X; \mathbb{Z})$. Then $T_G$ is an abelian complex Lie group with $|\text{Tors}_G|$ connected components (each one is isomorphic to $(\mathbb{C}^*)^n$) satisfying the following exact sequence:

\[ 1 \to T^1_G \to T_G \to \text{Tors}_G \to 1, \]

where $T^1_G$ is the connected component containing the trivial character $1$.

For a character $\xi \in T_G$, we can construct a local system of coefficients $\mathbb{C}_\xi$ over $X$.

**Definition 1.7.** The $k$-th characteristic variety of $X$ is the subvariety of $T_G$, defined by:

\[ V_k(X) = \{ \xi \in T_G \mid \dim H^1(X, \mathbb{C}_\xi) \geq k \}, \]

where $H^1(X, \mathbb{C}_\xi)$ is the cohomology with coefficients in the local system $\xi$. In some cases we will use the notation $V_k(G)$ since it is independent of $X$ as far as $\pi_1(X) \cong G$. The definition also applies to orbifolds replacing $\pi_1$ by $\pi_1^{\text{orb}}$.

The following result is straightforward.

**Proposition 1.8.** Let $\varphi : G \to H$ be a group epimorphism. Then $\varphi^*$ induces injections $T_H \cong \varphi^* T_G \hookrightarrow T_G$ and $V_j(H) \cong \varphi^* V_j(G) \hookrightarrow V_j(G)$.

**Remark 1.9.** Let us explain how to compute these invariants. For the sake of simplicity, the twisted homology, instead of the cohomology, will be computed. Let us consider a finite $CW$-complex homotopy equivalent to $X$; for the sake of simplicity the $CW$-complex will be denoted $X$. Let $\pi : \tilde{X} \to X$ be the maximal abelian covering. Note that $\tilde{X}$ inherits a $CW$-complex structure. The group of automorphisms of $\pi$ is $H_1(X; \mathbb{Z})$. The action of this Abelian group endows the chain complex $C_*(\tilde{X}; \mathbb{C})$ with a module structure over the ring $\Lambda := \mathbb{Z}[H_1(X; \mathbb{Z})]$. The differentials of the complex are $\Lambda$-homomorphisms. Moreover, $C_*(\tilde{X}; \mathbb{C})$ is a free $\Lambda$-module of finite rank. If we fix a character $\xi$, $\mathbb{C}$ has a natural $\Lambda$-module structure which is denoted by $\mathbb{C}_\xi$ (as the local system of coefficients). The twisted homology of $X$ is the homology of the $C_*(\tilde{X}; \mathbb{C})^\Lambda := C_*(\tilde{X}; \mathbb{C}) \otimes_\Lambda \mathbb{C}_\xi$. Following this interpretation, it is not difficult to prove that the characteristic varieties are algebraic subvarieties of $T_G$, defined with integer equations.

This $i$-th jumping loci of $C_*(X; \mathbb{C})$ with respect to $\otimes_\Lambda \mathbb{C}_\xi$ can also be viewed as the zero locus of the $i$-th Fitting ideal of $H_1(\tilde{X}; \mathbb{C})$ or, analogously, the support of the module $\wedge^i H_1(\tilde{X}; \mathbb{C})$ over the ring $\Lambda$ (see [17]).

Following the theory developed by various authors (Beauville [6], Arapura [1], Simpson [22], Budur [7], Delzant [11], Dimca [13]), the structure of characteristic varieties for quasiprojective manifolds can be stated as follows.

**Theorem 1.10** ([4]). Let $\Sigma$ be an irreducible component of $V_k(G)$, $k \geq 1$. Then one of the two following statements holds:

- There exists a surjective orbifold morphism $\rho : X \to C_{\varphi}$ and an irreducible component $\Sigma_1 \subset V_k(\pi_1^{\text{orb}}(C_{\varphi}))$ such that $\Sigma = \rho^*(\Sigma_1)$.
- $\Sigma$ is an isolated torsion point.
Proposition 1.13. Let \( G \) be the orbifold group of \( \mathbb{P}^2_{2,5,10} \). Then \( G \) is a semidirect product of the fundamental group of a compact surface of genus 2 and \( \mathbb{Z}/10\mathbb{Z} \). The torus \( T_G \) is \( \mu_{10} \), the group of 10-th roots of unity, \( V_1(G) \) consists of the primitive 10-th roots of unity and \( V_2(G) = \emptyset \).

Proof. Let us consider the short exact sequence associated with the abelianization map \( (\mathbb{Z}/10 : = \langle t \mid t^{10} = 1 \rangle) \) is \( G/G' \). This sequence corresponds to an orbifold morphism, which is a ramified cyclic covering of degree 10 of \( \mathbb{P}^1 \). The ramification points correspond with the orbifold points whose ramification indices equal the orbifold index. Using Riemann-Hurwitz one checks that the covering space is a compact Riemann surface of genus 2. Since the meridian of an orbifold point of index 10 is of order 10 in \( G \), then the exact sequence splits and we have a semidirect-group structure.

In order to compute \( V_1(G) \) we follow the construction outlined in Remark 1.9, applied to the CW-complex associated with the presentation of \( G \) given by \( \langle x, y \mid x^2 = y^5 = (xy)^{10} = 1 \rangle \). Let us denote \( p \) the unique 0-cell, \( x, y \) the 1-cells and \( A, B, C \) the 2-cells (corresponding to the relations in the given order). Let us fix a character \( \xi \in T_G \). It is clear that \( 1 \notin V_1(G) \). We can assume that \( \zeta := \xi(t) \neq 1 \). The complex \( C_\ast(X; \mathbb{C})^\xi \) is given by

\[
0 \to \mathbb{C}^3 \xrightarrow{\partial_2} \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C} \to 0.
\]

The matrix for \( \partial_1 \) is \( \begin{pmatrix} \zeta^5 - 1 & \zeta^2 - 1 \end{pmatrix} \). In particular, \( \dim \ker \partial_1 = 1 \). The matrix for \( \partial_2 \) equals

\[
\begin{pmatrix}
\zeta^5 + 1 & 0 & \zeta^{10} - 1 \\
0 & \zeta^8 - \zeta^6 + \zeta^4 - \zeta^2 + 1 & \zeta^6 \zeta^{10} - 1 \\
0 & \zeta^{10} - 1 & \zeta - 1
\end{pmatrix}
\]

In order to have non-trivial homology, this matrix must vanish and this happens only when \( \zeta \) is a primitive 10-th root of unity.

Proposition 1.14. Let \( G \) be the orbifold group of \( \mathbb{P}^2_{2,2,5,5} \). Then \( G \) is an extension of \( \mathbb{Z}/10\mathbb{Z} \) by the fundamental group of a compact surface of genus 4. The torus \( T_G \) is \( \mu_{10} \), the group of 10-th roots of unity, and both \( V_1(G) \) and \( V_2(G) \) consist of the primitive 10-th roots of unity.

Proof. The short exact sequence associated with the abelianization map \( (G/G' = \mathbb{Z}/10) \) corresponds to a covering of the orbifold as in the proof of Proposition 1.13 and using Riemann-Hurwitz one obtains that the covering space is a compact Riemann surface of genus 4.

We compute the characteristic varieties as in the proof of Proposition 1.13 for the presentation of \( G \) given by \( \langle x, y, z \mid x^5 = y^5 = z^2 = (xyz)^2 = 1 \rangle \). Let us denote \( p \) the unique 0-cell, \( x, y, z \) the 1-cells and \( A, B, C, D \) the 2-cells (corresponding to the relations in the given order). Let us fix a character \( \xi \in T_G \). It is clear that \( 1 \notin V_1(G) \). We can assume that \( \zeta := \xi(t) \neq 1 \). The complex \( C_\ast(X; \mathbb{C})^\xi \) is given by

\[
0 \to \mathbb{C}^3 \xrightarrow{\partial_2} \mathbb{C}^3 \xrightarrow{\partial_1} \mathbb{C} \to 0.
\]
The matrix for $\partial_1$ is $(\zeta^2 - 1 \quad \zeta^2 - 1 \quad \zeta^5 - 1)$. In particular, $\dim \ker \partial_1 = 2$. The matrix for $\partial_2$ equals
\[
\begin{pmatrix}
\zeta^8 - \zeta^6 + \zeta^4 - \zeta^2 + 1 & 0 & 0 & \zeta^5 + 1 \\
0 & \zeta^8 - \zeta^6 + \zeta^4 - \zeta^2 + 1 & 0 & \zeta^5 + 1 \\
0 & 0 & \zeta^5 + 1 & \zeta^5 + 1
\end{pmatrix}
\]
In order to have non-trivial homology, this matrix must have rank less than 2 and this happens only when $\zeta$ is a primitive 10-th root of unity. Moreover, in that case, the matrix vanishes. \hfill \Box

2. Examples

In this section, we will present a collection of examples of curves with non-abelian fundamental groups and geometric surjections and its relationship with characteristic varieties.

Remark 2.1. If $Y := \mathbb{P}^2 \setminus C$ admits an orbifold morphism $Y \to X$, then the non-singular compactification $\bar{X}$ of $X$ is $\mathbb{P}^1$.

Remark 2.2. The easiest examples of curves with non-abelian fundamental groups and geometric surjections come from hyperplane (or line) arrangements. If a line arrangement $\mathcal{A}$ has a point $P$ of multiplicity $k \geq 3$, then the pencil of lines through $P$ defines a morphism $\rho : \mathbb{P}^2 \setminus \mathcal{A} \to X$, where $X$ is a $k$-punctured projective line. We have an epimorphism $\rho_* : \pi_1(\mathbb{P}^2 \setminus \mathcal{A}) \to \pi_1(X)$ and the latter is a free group of rank $k - 1$ (hence non abelian).

The following result is well known for specialists.

Proposition 2.3. The following three assertions are equivalent:

1. The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ is non abelian,
2. The arrangement $\mathcal{A}$ has a point of multiplicity at least 3,
3. The group $\pi_1(\mathbb{P}^2 \setminus \mathcal{A})$ has a geometric surjection.

Proof. By the remark above, it is obvious that (2) implies (1) and (3). Also, by definition, (3) implies (1). Hence it is enough to prove that (1) implies (2). Note that, if (2) does not hold, then $\mathcal{A}$ is an arrangement in general position. Either we choose a particular example (e.g. a real arrangement) and a braid monodromy argument implies immediately the abelianity or we use Hattori’s topological description of arrangements of hyperplanes in general position [16]. It is also the starting point of Zariski’s proof of Zariski’s conjecture in [26] (we thank the referee for pointing this out to us). \hfill \Box

The argument used in Remark 2.2 can be easily generalized when, instead of considering three (or more) incident lines, one considers three (or more) fibers of any pencil of curves in $\mathbb{P}^2$. Of course, any such example corresponds to curves with at least three irreducible components. The notion of orbifold allows for wider generalizations of this concept to curves with any number of irreducible components (for example to irreducible curves).

As stated in the Introduction, the first example of this kind is rather old, see [26]. Let us consider a conic $C_2$ of equation $f_2 = 0$ and a cubic $C_3$ of equation $f_3 = 0$. Let us assume that they do not have common components and they are not multiple lines. Let $C$ be a curve of equation $f_2^3 - f_3^2$. Note that the mapping $\rho : \mathbb{P}^2 \setminus C \to \mathbb{P}^1 \setminus \{[1 : 1]\}$ given by $[x : y : z] \mapsto [f_2(x, y, z)^3 : f_3(x, y, z)^2]$ is well defined (all the base points of the pencil are in $C$) and surjective. This mapping induces an orbifold map onto a 1-punctured Riemann sphere with two orbifold points of multiplicities 2 and 3 (at $[0 : 1]$ and $[1 : 0]$ respectively). Thus according to Proposition 1.4, one obtains an epimorphism $\pi_1(\mathbb{P}^2 \setminus C)$ onto $\mathbb{Z}/2 * \mathbb{Z}/3$. 

Proposition 2.4. Let $G$ be the orbifold fundamental group of $C_{2,3}$. Then, $\mathbb{T}_G = \mu_6$, $\mathcal{V}_1(G)$ consists of the 6-th primitive roots of unity and $\mathcal{V}_2(G) = \emptyset$. In particular, any curve with equation $f_2^3 - f_3^2 = 0$ has non-trivial characteristic varieties.

The proof of this Proposition follows easily from the above arguments.

Remark 2.5. For generic choices of $f_2$ and $f_3$ this epimorphism is in fact an isomorphism (this is actually the case originally considered by Zariski in [26]). However, this is not the case, for instance, when $C$ is reducible (since $b_1(\mathbb{P}^2 \setminus C) > 1$). Even if $C$ is irreducible one may not necessarily have an isomorphism for several reasons: either there are few non-generic fibers in the pencil (e.g., a sextic with six cusps and four ordinary nodes) or there are several pencils (a sextic with nine cusps).

These examples can be generalized if we replace $(2, 3)$ by any coprimes $(p, q)$, see Oka [21], Némethi [20] and Dimca [12]. In such cases, the fundamental group of a generic curve with equation $f_p^3 + f_q^3 = 0$ is $\mathbb{Z}/p \ast \mathbb{Z}/q$. Also Zariski [20] considered another interesting example where the target orbifold is compact.

Let us consider the tricuspidal quartic $C_4$ with equation $f_4 = 0$. It is not hard to prove that we can choose

$$f_4 := x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z).$$

and $\text{Sing}(C_4) = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. The curve $C_4$ is parametrized by

$$[t : s] \mapsto [t^2s^2 : (t - s)s^2 : t^2(t - s)^2]$$

and its singular points correspond to $[t : s] = [0 : 1], [1 : 1]$, and $[1 : 0]$. Let $P \in C_4$ be a smooth point with parameter $\alpha \equiv [\alpha : 1]$ and let $L_4$ be the tangent line to $C_4$ at $P$, with equation $f_1 = 0$, where

$$f_1 := (\alpha - 1)^3x - \alpha^3y + z.$$

Let $C_2$ be the conic passing through the singular points of $C_4$ and tangent to $C_4$ at $P$. Since five (non-degenerate) conditions are imposed, such a conic is unique. As before, let $f_2 = 0$ be the equation of $C_2$, where

$$f_2 := \alpha(\alpha - 1)xy - (\alpha - 1)xz + \alpha yz.$$

We consider now a cubic $C_3$ having a nodal point at $P$ (one of the branches tangent to $C_4$ at $P$) and tangent to $C_4$ at the three cuspidal points. Counting the conditions it is easy to prove that only one such cubic exists, with equation $f_3 = 0$, where

$$f_3 := -(\alpha - 2)(2\alpha - 1)(\alpha + 1)xyz - \alpha^3xy^2 - xz^2 - (\alpha - 1)^3x^2y + yz^2 + (\alpha - 1)^3x^2z + \alpha^3y^2z.$$

Lemma 2.6. $f_if_j^3 = f_3^3 - 4f_2^3$.

A straightforward computation provides a proof of this Lemma, which easily results in the following:

Proposition 2.7. The fundamental group of $\mathbb{P}^2 \setminus C_4$ possesses a geometric surjection onto $\mathbb{P}^1_{2,2,3}$. 

Remark 2.8. Zariski proved in [26] that $\pi_1(\mathbb{P}^2 \setminus C_4)$ is a non-abelian group of order 12. The above mapping induces a central extension of $\mathbb{B}_6$ (dihedral group of order 6) whose kernel is cyclic of order 2. Note that there is an epimorphism from $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_4))$ onto the orbifold group of a 1-punctured Riemann sphere with two multiple points $(2, 3)$. For a generic $P$ it is possible to prove that $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_4))$ equals $\mathbb{B}_3$. There is a particular case corresponding to the bitangent line $L_6$. In this case there are two such mappings and $\pi_1(\mathbb{P}^2 \setminus (C_4 \cup L_6))$ is the Tits-Artin group of a triangle.
In [9], Degtyarev proved that only two irreducible curves of degree 5 have non-abelian fundamental groups. One of them is extensively studied in §3. The other one was also studied by the first author in [2]. It is a rigid curve with one point of type $A_6$ and three cuspidal points (it is the dual curve of the quartic with one $A_6$). Let $C_5$ be this curve (with equation $f_5 = 0$). In [2] this group was shown to be non-abelian by finding an epimorphism from an actual presentation of $\pi_1(\mathbb{P}^2 \setminus C_5)$ onto the triangle group of type $2, 3, 7$, which is the orbifold group of $\mathbb{P}_\varphi^1$ with three multiple points of these orders. In fact, one has the following:

**Proposition 2.9.** The fundamental group of $\mathbb{P}^2 \setminus C_5$ possesses a geometric surjection onto $\mathbb{P}_2, 3, 7$.

**Proof.** The three summands of Lemma 2.6, which are polynomials of degree six, obviously belong to a pencil of sextics and, hence, they define a map outside the base points. For a particular parameter (a primitive 6-th root of unity, $L_t = L_b$ is the bitangent of $C_4$. We are going to consider the Cremona transformation $\rho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ associated with the net of conics having three infinitely near points in common with $C_5$ at $P$, the singular point of type $A_6$. Let us describe this transformation. After blowing up these three infinitely near points one obtains a rational surface $X$ with a morphism $\sigma_1 : X \to \mathbb{P}^2$. Let us denote the three exceptional components (in order of appearance) by $E_1, E_2$, and $T$, and finally the tangent line of $C$ at $P$ by $L$ (see Figure 1).

**Convention 2.10.** For birational morphisms, we keep the notation of a curve for its strict transform unless otherwise stated.

In $X$ one has $E_1 \cdot E_1 = -2$ and $E_1 \cdot E_1 = T \cdot T = E \cdot L = -1$. Since $L$ and $T$ are **combinatorially equivalent**, one can consider the birational morphism $\sigma_2 : X \to \mathbb{P}^2$ obtained as the composition of the contractions of $L, E_2$ and $E_1$. The resulting surface is rational with Euler characteristic 3 and hence it is a projective plane. It is not hard to prove that $\rho = \sigma_2^{-1} \circ \sigma_1$. Let us denote $C := \rho(C)$. Note that $C$ is a tricuspidal quartic and $T$ is its unique bitangent line $L_b$, one point $P$ comes from the infinitely near point of $C$ at $P$ and the other one $Q$ comes from the other intersection point of $C$ and $L$.

We consider the pencil defined by the orbifold map of Proposition 2.7 where the base point is $P$. Let $C_3$ be the cubic of equation (2.4) such that $2C_3$ is in the pencil. Following $C_3$ by $\sigma_2$ and $\sigma_1$, $C_6 := \rho^*(C_3)$ is a sextic with only one singular point at $P$ (with two branches, one of type $A_6$ and a smooth branch with maximal contact with the singular branch). With the same ideas, if $C_2$ is the conic of equation (2.5) such that $3C_2$ is in the pencil, then $C_4 := \rho^*(C_2)$ is a quartic with an $A_6$ singular point at $P$.

Finally $\rho^*(C + 2T) = C + 7L$. We have a pencil of degree 12 containing the fibers $2C_6, 3C_4$ and $C + 7L$. This pencil produces the desired morphism. \qed
One can find more examples in the literature: Degtyarev [9], Flenner-Zaïdenberg [14], and Tono [24]. In what follows, the last two families will be described. We start with some definitions.

**Definition 2.11.** A Hirzebruch surface is a rational surface $X$ with a morphism $\pi : X \to \mathbb{P}^1$ which is a holomorphic (or algebraic) fibration with fiber $\mathbb{P}^1$. Such a surface is either $\Sigma_0 := \mathbb{P}^1 \times \mathbb{P}^1$ or it has a unique section $S_0$ with negative self-intersection $-n$, $n > 0$; in that case $\pi$ is unique and $X$ is denoted by $\Sigma_n$ ($\Sigma_1$ is the blowing-up of one point in $\mathbb{P}^2$).

For any Hirzebruch surface $X$ there is a family of birational maps which are called elementary Nagata transformations. They are obtained as follows. Let us consider $\pi : X \to \mathbb{P}^1$, $P \in X$ and $F := \pi^{-1}(\pi(P))$; we consider the blowing-up $\sigma : \tilde{X} \to X$ of $P$, with exceptional component $F$. Since $(F \cdot F)_X = 0$, we have that $(F \cdot F)_\tilde{X} = -1$. By Castelnuovo criterion, we can blow down $F$ and we obtain a new Hirzebruch surface $\tilde{X}$ where $\tilde{F}$ is a fiber.

**Definition 2.12.** An elementary Nagata transformation is said to be positive (resp. negative) if $P$ belongs (resp. does not belong) to a section with non-positive self-intersection. For a positive one, one goes from $\Sigma_n$ to $\Sigma_{n+1}$; for a negative one, from $\Sigma_n$ to $\Sigma_{n-1}$.

In [3], the first author computed the fundamental group of Flenner-Zaïdenberg curves and showed when it is non-abelian using orbifold groups. We show here that this can also be geometrically proved. In order to construct these curves, we start with a smooth conic $C$ with two tangent lines $L_1$ and $L_2$, intersecting at some point $P$. After blowing up $P$ one obtains $\pi : \Sigma_1 \to \mathbb{P}^1$ with exceptional component $E$. Let $L_3$ be another line in the pencil through $P$ which intersects $C$ at two points $Q_1$ and $Q_2$. Let us fix two positive integers $a$, $b$. After performing $a$ positive elementary Nagata transformations at the point corresponding to the fiber of $L_1$ and $b$ at the point corresponding to the fiber of $L_2$ one obtains a Hirzebruch surface $\Sigma_{a+b+1}$. One can then perform $a + b$ negative elementary Nagata transformations on the fiber corresponding to $L_3$ and based at a point in $C$ (say $Q_2$ for the first one). After this process, $E$ can be blown down which turns our surface into $\mathbb{P}^2$. The curve $C_{a,b}$ obtained has degree $d := a + b + 2$ and three singular points of type $\mathbb{A}_{2a}$, $\mathbb{A}_{2b}$, and a third one with local equation $u^{d-2} = v^{d-1}$.

**Proposition 2.13.** The fundamental group of $\mathbb{P}^2 \setminus C_{a,b}$ possesses a geometric surjection onto $\mathbb{P}_{2,a,b,c}^1$, where $c := \gcd(2a + 1, 2b + 1)$.

**Proof.** It is enough to follow the pencil of conics generated by $L_1 + L_2$ and $C$ through the above transformations. We obtain a pencil of curves of degree $2(d - 1)$, where one fiber is $(2a + 1)L_1 + (2b + 1)L_2$ (they are the lines corresponding to the fibers of $L_1$ and $L_2$). The fiber containing $C_{a,b}$ is of the form $C_{a,b} + (d - 2)L_3$. Finally the double line in the pencil becomes a double curve of degree $d - 1$. \qed

In [24], K. Tono describes all rational unicuspidal curves such that its complement in $\mathbb{P}^2$ has logarithmic Kodaira dimension $1$. The construction given in [24] Theorem 1] shows that the complement of these curves have non-abelian fundamental group. Any other known rational unicuspidal curve has abelian fundamental group (for the complement).

**Proposition 2.14.** For any Tono’s curve $C$ their fundamental group possesses a geometric surjection onto $\mathbb{P}_{\mu_A, \mu_G, n(C)}^1$, where $\mu_A, \mu_G \geq 2$ and the number $n(C)$ is the opposite of the self-intersection of the strict transform of $C$ after the minimal embedded resolution of its unique singular point. This number is at least 2.

**Proof.** It is enough to consider the construction of [24] Theorem 1] where a pencil is obtained with two multiple fibers $\mu_A A$ and $\mu_G G$ and a reducible fiber of the form $C + n(C)B$, where $B$ is either a line (type I) or a smooth conic (type II). \qed
Example 2.15. The curves of type I are parametrized by two integers \( n, s \geq 2 \). The curve \( C \) has degree \( (n + 1)^2(s - 1) + 1 \), where \( n(C) = n, \mu_A = n + 1 \) and \( \mu_L = (n + 1)(s - 1) + 1 \). For \( n = s = 2 \), we obtain the multiplicities 2, 3, 4; in fact, one can compute that this group is finite.

3. Degtyarev curves

Let us consider a projective Degtyarev curve, i.e., a plane projective curve of degree 5 such that \( \text{Sing}(C) \) consists of three points, and for each point \( P \in \text{Sing}(C) \) the germ \( (C, P) \) is topologically equivalent to an \( \mathbb{A}^3 \)-singularity, i.e. with local equation \( v^2 - u^3 = 0 \); note that in this case, the germs are also analytically equivalent.

Most of the following properties appear in [9] and [19], but we include for the sake of completeness.

Properties 3.1. Let \( C \subset \mathbb{P}^2 \) be a projective Degtyarev curve. Then:

(D1) The curve \( C \) is irreducible.
(D2) The tangent line \( L \) of \( C \) at a singular point \( P \) satisfies \( (L \cdot C)_P = 4 \).
(D3) Two Degtyarev projective curves are projectively equivalent.
(D4) The subgroup of projective transformations preserving \( C \) is cyclic of order 3.
(D5) The curve \( C \) is autodual.

Proof. Since the three singular points are locally irreducible, (D1) is true. For (D2) note that \( 4 \leq (L \cdot C)_P \leq 5 \). Let us assume that \( (L \cdot C)_P = 5 \); considering \( L \) as the line at infinity, \( C \setminus L \) is an affine curve homeomorphic to \( \mathbb{C} \). This case is discarded using Zaidenberg-Lin Theorem [26] and results.

In order to prove (D3), there are two approaches. The direct approach consists of computing the equations of the curve \( C \) fixing the position of the singular points and some of their tangent lines. The second method is quite simple and worth describing here: Let \( C_1, C_2 \) be two projective Degtyarev curves. By Bézout’s Theorem, the singular points are not aligned; and hence, after a projective transformation, one may assume that \( \text{Sing}(C_1) = \text{Sing}(C_2) =: S \). Assuming that \( S := \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\} \), one can perform a standard Cremona transformation \( \psi : \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) based on the three singular points and defined by \( \psi([x : y : z]) = [yz : xz : xy] \).

Geometrically, this rational map is obtained by blowing-up the three vertices of \( S \) (obtaining a rational surface \( X' \)) and then blowing down the strict transforms of the lines joining the points of \( S \) (which have self-intersection \( -1 \) in \( X' \)). One can easily compute that \( \hat{C}_1 := \psi(C_1) \) is a tricuspidal quartic. It is well known that there is only one tricuspidal quartic, up to projective transformation, therefore, after a suitable change of coordinates, one may assume \( \hat{C}_1 = \hat{C}_2 := \hat{C} \), where \( \hat{C} \) is the curve with equation given in (2.1). The tricuspidal quartic satisfies the following properties. Let \( \text{Sing}(\hat{C}) = \{P_1, P_2, P_3\} \); there are three points \( Q'_1, Q'_2, Q'_3 \in \hat{C}, \ell = 1, 2 \) such that \( P_1, Q'_j, Q'_k \) are aligned for all the possibilities with \( \#\{i, j, k\} = 3 \). Let \( \mathcal{A}_\ell \) be the arrangements of curves given by \( \hat{C} \) and the lines joining \( Q'_1, Q'_2 \).

The curve \( \hat{C} \) is parametrized as in (2.2) and the singular points \( P_1 = [0 : 1 : 0], P_2 = [1 : 0 : 0], \) and \( P_3 = [0 : 0 : 1] \) correspond to \([t : s] = [0 : 1], [1 : 1], \) and \([1 : 0] \). It is not hard to check that \( \mathcal{A}_\ell := (\alpha_\ell, 2 + \alpha_\ell, -\alpha_\ell) \) are affine parameters of \( (Q'_1, Q'_2, Q'_3) \). The last condition implies that \( \alpha_\ell^2 + \alpha_\ell - 1 = 0 \). If \( \alpha_1 = \alpha_2 \) then \( \mathcal{A}_1 = \mathcal{A}_2 \).

The group of projective transformations fixing \( \hat{C} \) is the group of the permutation of the coordinates. The mapping \([x : y : z] \mapsto [x : z : y] \) induces \([t : s] \mapsto [s : t] \) in the parametrization, and \([x : y : z] \mapsto [y : z : x] \) induces \([t : s] \mapsto [s : s - t] \). Let us assume that \( \alpha_1 \neq \alpha_2 \) Applying the projective transformation \( \sigma \), results in two operations on \( \mathcal{A}_\ell \) : the permutation \((1, 3)\), and the change of parameters. Thus, \( \sigma(\mathcal{A}_1) = (-\alpha_1^{-1}, \alpha_1 + 2)^{-1}, \alpha_1^{-1}) = \mathcal{A}_2 \), which implies \( \sigma(\mathcal{A}_1) = \mathcal{A}_2 \).
Note that any projective transformation sending \( A_1 \) to \( A_2 \) lifts to an isomorphism \( X_1 \to X_2 \) and this isomorphism induces a projective transformation of the source \( \mathbb{P}^2 \), hence \([D3]\) results.

In order to prove \([D4]\), one can use a similar argument on the projective transformations fixing \( C \) (this last property was communicated to the authors by C.T.C. Wall).

The property \([D5]\) follows from \([D2]\) and Plücker generalized formulæ, see \([19]\). More precisely, given a curve \( D \) and a point \( P \in D \), the order of the curve is the degree of its dual curve of \( D \):

\[
\deg(D) = \deg(D) + (\deg(D) - 1) - \sum_{P \in D} (\mu(C, P) - 1 + m(C, P)).
\]

This formula implies that \( \deg(D) = 5 \). The dual of a singular point of type \( A_4 \) is either of the same type or of type \( E_8 \) (in case the tangent line has multiplicity of intersection 5 with the curve at the singular point). Thus \([D5]\) holds.

**Remark 3.2.** Note that any two projective Degtyarev curves are isotopic. Using the direct approach, we can give a symmetric equation:

\[
7 + 3 \sqrt{3} (x^3 y^2 + x^2 y^3 + y^2 z^3) + (2 \sqrt{3} + 6) (x^3 y z + x y^3 z + x y z^3) + \\
2(x^3 y^2 + x^2 y z + y^2 z^2) + (33 + 11 \sqrt{3}) (x y z^2 + x^2 y^2 z + x y^2 z^2) = 0.
\]

Note that the permutation of two variables comes from the Galois transformation in \( \mathbb{Q}(\sqrt{3}) \). The curve also admits an equation with rational coefficients; in that case one of the singular points has rational coordinates but the other two are conjugate in \( \mathbb{Q}(\sqrt{3}) \):

\[
z^2 y^3 - z(33 x z + 2 x^2 + 8 z^2) y^2 + (21 z^2 + 21 x z - x^2) (z^2 + 11 x z - x^2) y + (x - 18 z) (z^2 + 11 x z - x^2)^2 = 0
\]

Properties \([3.1]\) imply that the affine Degtyarev curve is also rigid, i.e. any two affine Degtyarev curves are projectively equivalent, and in particular, they are isotopic. In order to study its complement, it is convenient to assume that the line corresponds to the line at infinity and hence it is enough to consider the complement of the affine curve whose equation is obtained from \([3.1]\) by taking \( z = 1 \).

The fundamental group of the projective Degtyarev curve was computed in \([9]\). Here we will compute the fundamental group of the affine curve and also show how to recover the group of the projective curve. In order to compute the group we will use the braid monodromy associated with the projection \( (x, y) \mapsto x \). Note that the discriminant of the equation \([3.1]\) (with \( z = 1 \)) is (up to a constant) \( x(x^2 - 11 x - 1)^5 \). Since the three roots are real and the projection is 3 : 1 with enough real roots, the real picture in Figure 2 contains all the required information to obtain the braid monodromy (the dotted lines represent the real part of the complex conjugate roots).

The braid monodromy is defined as a representation \( \nabla_0 : \pi_1(\mathbb{C} \setminus \{0, a_+, a_-\}; x_0) \to \mathbb{B}_3 \). The source is a free group of rank three generated by:

\[
\mu_+ := \alpha_+ \cdot \beta_+ \cdot \gamma_+ \cdot \alpha_+^{-1}, \quad \mu_0 := \alpha_+ \cdot \beta_+ \cdot \alpha_0 \cdot \beta_0 \cdot \gamma_0 \cdot \alpha_0^{-1} \cdot \beta_0^{-1} \cdot \alpha_+^{-1} \quad \text{and}
\]

\[
\mu_- := \alpha_+ \cdot \beta_+ \cdot \alpha_0 \cdot \beta_0 \cdot \alpha_- \cdot \beta_- \cdot \gamma_- \cdot \alpha_-^{-1} \cdot \beta_-^{-1} \cdot \alpha_0^{-1} \cdot \beta_0^{-1} \cdot \alpha_+^{-1}.
\]

Figure 3 shows a geometric basis of \( \pi_1(\mathbb{C} \setminus \{0, a_+, a_-\}; x_0) \). The braids are obtained by considering the way the roots with respect to \( y \) move when the parameters move along \( x \). We follow these conventions:

- \([B1]\) In order to draw the braids we consider the projection onto the real axis.
- \([B2]\) When two points have the same real part, we perturb the projection such that positive imaginary parts go to the right and negative imaginary parts go to the left.
- \([B3]\) Roots will be numbered from right to left.
\[ x = \frac{11 - 5\sqrt{5}}{2} = a_- \quad \text{or} \quad x = \frac{11 + 5\sqrt{5}}{2} = a_+ \]

**Figure 2.** Real picture of the affine Degtyarev curve

**Figure 3.** Paths in \( \mathbb{C} \setminus \{0, a_+, a_-\} \)

| Paths | Braids |
|-------|--------|
| \( \alpha_+ \) | 1 |
| \( \beta_+ \) | \( \sigma_2 \) |
| \( \gamma_+ \) | \( \sigma_1 \) |
| \( \alpha_0 \) | \( \sigma_1^{-1} \sigma_2 \) |
| \( \beta_0 \) | 1 |
| \( \gamma_0 \) | \( \sigma_1 \) |
| \( \alpha_- \) | 1 |
| \( \beta_- \) | \( \sigma_2 \) |
| \( \gamma_- \) | \( \sigma_2 \) |

**Table 1.** Braids

\[(\mathbb{B}4)\) The above conventions give a canonical way to identify open braids with closed braids.

Using the standard Artin generators of the braid groups, the braids obtained from following the paths in \( \mathbb{C} \setminus \{0, a_+, a_-\} \) shown in Figure 3 are presented in Table 1.

**Proposition 3.3.** The braid monodromy for the chosen projection of the affine Degtyarev curve is given by:

\[ \nabla_0(\mu_+) = \sigma_2^5, \quad \nabla_0(\mu_0) = (\sigma_2^5 \sigma_1^{-1} \sigma_2) \ast \sigma_1, \quad \nabla_0(\mu_-) = (\sigma_2^5 \sigma_1^{-1} \sigma_2 \sigma_1) \ast \sigma_2^5 = \sigma_2^5 \ast \sigma_1^5, \]

where \( a \ast b := aba^{-1} \).
4. Groups of Degtyarev curves

In order to compute the fundamental groups we apply the Zariski-van Kampen method. Let us consider the vertical line $F$ of equation $x = x_0$. The set $F \setminus C$ is of the form $\{x_0\} \times \mathbb{C} \setminus \{y_1, y_2, y_3\}$, where $y_1, y_2, y_3 \in \mathbb{R}$. We choose a big real number $y_0$ in order to fix $(x_0, y_0) =: p_0$ as the base point. The free group $\pi_1(F \setminus C; p_0)$ has a free basis $g_1, g_2, g_3$ constructed as in Figure 3. The natural action of $\mathbb{B}_3$ on the free group $F_3$ is expressed in this case as

\[
\begin{align*}
\sigma_i^j &:= \begin{cases} 
g_{i+1} & \text{if } i = j, 
g_{i+1} \ast g_i & \text{if } i = j + 1, 
g_i & \text{if } i \neq j, j + 1. 
\end{cases}
\end{align*}
\]

Proposition 4.1. The fundamental group of the affine Degtyarev curve has a presentation

\[
\langle g_1, g_2, g_3 \mid g_i^{\nabla_0(\mu_i)} = g_i, i = 1, 2, 3, j = -, 0, + \rangle.
\]

In this presentation, a meridian of the line at infinity is (up to conjugation) $(g_1(g_2g_1)^2)^{-1}$. In particular, a presentation for the projective Degtyarev curve is

\[
\langle g_1, g_2, g_3 \mid (4.2), g_3 = (g_2g_1)^{-2} \rangle.
\]

Proof. The first presentation is a consequence of the Zariski-van Kampen method by means of the braid monodromy. In order to prove the second one may consider a small deformation of the vertical line $F$. It will intersect the curve at five points. Three of them are close to $(x_0, y_i)$, $i = 1, 2, 3$, and the other two ones lie in the real branches which go faster to infinity. The boundary of a big disk in this line is the inverse of a meridian of the line at infinity. \qed

Remark 4.2. Proposition 4.1 provides right presentations of the group, but they may be quite cumbersome to work with by hand. Even if one wants to work with them with computer programs, like GAP [15], the presentations could be intractable. There are several ways around this problem

(P1) The presentation $\langle 4.2 \rangle$ works if we replace the braid monodromy $\nabla_0$ for a conjugate. For example, conjugating the braids in Proposition 3.3 by $\sigma_2^2$ produces simpler braids and hence a simpler presentation of the group.

(P2) Instead of finding a good braid to perform the conjugation in [P1] by inspection, one can try to interpret this conjugation in a geometric way. Changing the base point in $\mathbb{C} \setminus \{0, a_+, a_-\}$ might produce simpler braids. For example choosing a real number $y_0 \in (a_-, 0)$ as a base point, one obtains the following as braid monodromy (for the new generators of the group):

\[
\tilde{\mu}_+ \mapsto (\sigma_2^{-1}\sigma_1) \ast \sigma_2^2, \quad \tilde{\mu}_0 \mapsto \sigma_1, \quad \tilde{\mu}_- \mapsto \sigma_2^2.
\]

These braids have been obtained by conjugation of the ones in Proposition 3.3 by $\sigma_2^2\sigma_1\sigma_2^{-1}$.

(P3) If $g$ is a meridian of the line at infinity obtained using a braid monodromy $\nabla_0$, then, for a braid monodromy $(\nabla_0)^\tau := \tau^{-1}\nabla_0\tau = (\tau^{-1}) \ast \nabla_0$, a meridian of the line at infinity is $g^\tau$.

(P4) There is another geometric way to reduce the presentation. Note that among the relations $(g_j)^{\sigma_2^2} = g_j, j = 1, 2, 3$, one only needs to keep the relation given by $j = 2$. First of all, the relation for $j = 1$ is trivial; secondly $(g_3g_2)^\tau = g_3g_2$ and hence one of them is redundant. In the general case, this can be summarized as follows:
• Let us consider the action (4.1) (replacing 3 by $n$) of $\mathbb{B}_n$ on the free group with basis $g_1, \ldots, g_n$: let us consider a braid $\tau \in \mathbb{B}_n$ which can be decomposed as $\tau = \tau_1 \cdots \tau_r$, where $\tau_j$ involves only a set of $n_j$ consecutive strings and $n = \sum_{j=1}^r n_j$. Then, among the relations $g_j^2 = g_j$, we only need to consider $s := \sum_{j=1}^r (n_j - 1) = n - r$, disregarding one for each block of strings. Let $J_r$ be the chosen subset of indices.

• If $\beta = (\tau)^{r}$, and $\tau$ can be decomposed as above, then the set of relations $g_j^3 = g_j$, $j = 1, \ldots, n$, is equivalent to $(g_j^3)^{r} = g_j^r$, $j \in J_r$.

For example, in our case the presentation (4.2) can be reduced to have 3 relators.

Proposition 4.3. The group $G$ of the affine Degtyarev curve has a presentation:

\[
x, y \mid x y x y x = y x y x, [x, y x y^{-1} x y x^{-1} x y] = 1
\]

A presentation of the group $G_p$ of the projective Degtyarev curve is obtained from (4.5) by adding $x^5 = 1$. It turns out that $G_p$ is a group of order 320 with the following properties:

(G$_p$1) $G_p/G'_p$ is cyclic of order 5.

(G$_p$2) The center $Z(G_p)$ is the Klein group of order 4.

(G$_p$3) The group $G/Z(G_p)$ is a semidirect product of $(\mathbb{Z}/2)^4$ by $\mathbb{Z}^5$, where the action of a generator of $\mathbb{Z}^5$ cyclically permutes a generator system $h_1, \ldots, h_5$ of order 2 elements of $(\mathbb{Z}/2)^4$ satisfying $\sum h_i \equiv 0$.

Proof. The presentation of $G$ is obtained using the braid monodromy [4.3] and Remark [4.4], where $x = g_1$, $g_2$ and $y = g_3$; note that $x$ and $y$ are conjugate. In order to obtain the presentation of $G_p$ the relation of the line at infinity needs to be added. This is a complicated product of five conjugates of $x$. If one types this presentation in GAP, the output is that $G_p$ has order 320 and that $x$ is an element of order 5. Also according to GAP, the order of the quotient of $G$ obtained by adding the relation $x^5 = 1$ is 320. These facts give the presentation of the statement. The properties of $G_p$ are either trivial or easily computed using GAP.

Proposition 4.4. The group $G_p$ possesses no geometric surjections.

Proof. The only properties needed for this are the size of both the group $G_p$ and its abelianization. Let us assume that $G_p$ possesses a geometric surjection. Since it is finite, the orbifold group must be finite. The only orbifolds having a finite non-abelian fundamental group are those of type $\mathbb{P}^n_{a, b, c}$, with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1$ (the so-called spherical orbifolds): either $\mathbb{P}^n_{2, 2, n}$, $n \geq 3$, or $\mathbb{P}^n_{2, 3, m}$, $m = 3, 4, 5$. Since the order of the orbifold group must divide 320, the only possibilities are $(2, 2, n)$, where $n|160$. The group is dihedral and its abelianization is either $\mathbb{Z}/2$ or $(\mathbb{Z}/2)^2$. Since the abelianization of $G_p$ is of order 5, the result follows.

We finish this section with the main result of this paper. We are going to compute the characteristic varieties of the complement of the affine Degtyarev curve and we will prove that these components cannot come from the characteristic varieties of an orbifold.

Theorem 4.5. Let $T_G = \mathbb{C}^*$ be the character torus of $G$. Then $V_1(G)$ is the set containing 1 and the 10-th primitive roots of unity, whereas $V_2(G) = \emptyset$. Therefore there is no geometric surjection of $G$ onto an infinite orbifold group.

Since finite group orbifolds do not have characteristic varieties, the following Corollary holds.

Corollary 4.6. No irreducible component of $V_1(G)$ is obtained as the pull-back of an irreducible component of the $V_1(\Gamma)$ where $\Gamma$ is an orbifold group.
Proof of Theorem 4.5 We are going to change the presentation (4.5), by taking a new generator \( t \) satisfying \( y = xt \):

\[
\langle x, t \mid xt^2tx = tx^2tx^2, [x, txt^{-1}xt^{-1}tx] = 1 \rangle
\]

It is clear that \( 1 \in \mathcal{V}_1(G) \setminus \mathcal{V}_2(G) \) since the non-twisted homology has rank 1. Let us consider a non-trivial character \( \xi \in \mathbb{V} \) which is identified by the image \( 1 \neq \zeta \) of a positive generator of \( \mathbb{Z} \).

One can associate a CW-complex with the presentation (4.6) with one 0-cell \( p \), two 1-cells \( x, t \) and two 2-cells \( A, B \) (corresponding to the relations). Then, the complex \( C_\ast(X; \mathbb{C})^\xi \) with which to compute the twisted homology is

\[
0 \longrightarrow \mathbb{C}^2 \xrightarrow{\partial_1} \mathbb{C}^2 \xrightarrow{\partial_2} \mathbb{C} \longrightarrow 0.
\]

The matrix for \( \partial_1 \) is \((\zeta - 1 \ 0)\). In particular, \( \dim \ker \partial_1 = 1 \) and hence \( \mathcal{V}_2(G) = 0 \). The matrix for \( \partial_2 \) equals

\[
\begin{pmatrix}
0 \\
1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 \\
(1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4)(\zeta - 1)
\end{pmatrix}.
\]

The homology is non-trivial if and only if the matrix vanishes and hence \( \mathcal{V}_1(G) \) is as in the statement.

Since we are working with the complement of an affine (hence projective) curve, if \( G \) admits a geometric surjection onto an infinite orbifold group, the orbifold must be over a rational curve. Since the abelianization has rank 1, the rational curve must be either \( \mathbb{C} \) or \( \mathbb{P}^1 \). Any dominant morphism with target \( \mathbb{C} \) can be considered as dominant on \( \mathbb{P}^1 \) and we treat only this case.

One needs to consider only orbifolds over \( \mathbb{P}^1 \) whose fundamental groups are infinite, have cyclic abelianizations and admit the 10-th primitive roots of unity in their characteristic varieties. In particular, the abelianization must be of the type \( \mathbb{Z}/n\mathbb{Z} \), where 10 divides \( n \).

Let us prove that any such orbifolds admit dominant morphisms in \( \mathbb{P}^1_{2,5,10} \) and \( \mathbb{P}^1_{2,2,5,5} \). It is not hard to prove (see, e.g., [1]) for details) that for a prime \( p \), the abelianization of \( G \) has non-trivial \( p \)-factors if at least two orbifold points have indices divisible by \( p \). Using the identity mapping, we obtain dominant morphisms in either the above orbifolds or \( \mathbb{P}^1_{10,10} \). We need to exclude the case where only a dominant morphism in \( \mathbb{P}^1_{10,10} \) exists. In this case, \( \mathbb{P}^1_{10n_1,10n_2,n_3,...,n_r} \), \( \gcd(n_j, 10) = 1 \), \( j = 1, \ldots, r \). We proved in [3] that no element of order 10 is in the characteristic varieties of this orbifold, and hence, these orbifolds do not satisfy the claim of the statement.

The properties of \( \mathcal{V}_2 \) allow us to discard \( \mathbb{P}^1_{2,2,5,5} \), see Proposition 1.14. Let us assume that there is a geometric surjection onto the orbifold \( \mathbb{P}^1_{2,5,10} \). Proposition 1.13 does not provide a direct obstruction in terms of \( \mathcal{V}_1 \). Moreover, the kernel of the abelianization map is the fundamental group \( K_2 \) of a compact Riemann surface of genus 2, see Proposition 1.13.

Note that \((xy)^5 = (x^2t)^5\) is a central element and the group \( K \) generated by this element defines an injection in \( G/G' \). Following [10], if \( G_0 := G/K \), the groups \( G_0' \) and \( G' \) are isomorphic and hence \( G' \) is finitely presented. Using the Reidemeister-Schreier method, we find the following presentation:

\[
G' = \langle t_0, t_1, t_2, t_3, t_4 \mid t_{n+1}t_{n+3} = t_nt_{n+2}t_{n+4}, B_n = B_{n+1} \rangle,
\]

where \( B_n := t_nt_{n+1}^{-1}t_{n+2}^{-1}t_{n+3}^{-1}t_{n+4} \) and \( x \ast t_n = t_{n+1} \). Note that \( x^{10} \ast t_n = t_{n+10} = A \ast t_n \), where \( A := t_{n}t_{n+2}t_{n+4}t_{n+6}t_{n+8} \) for any \( n \). This guarantees that the above presentation is finite. Summarizing, one can deduce that the kernel \( K_1 \) of the epimorphism onto \( \mathbb{Z}/10 \) equals \( \mathbb{Z} \times G' \).

Note that the rank of \( K_1 \) equals 5 and the rank of \( K_2 \) equals 4, so no contradiction arises.

According to \textsc{GAP} the next quotients of the lower central series have ranks 5 and 16 for \( K_2 \), and 2 and 0 (order 5) for \( K_1 \) and hence such an epimorphism cannot exist. \( \square \)
5. Further properties of the affine Degtyarev curve

The affine Degtyarev curve is related with elliptic fibrations as follows. In order to work in a projective setting, one can first consider the projective Degtyarev curve, and fix a singular point $P$. We will denote by $L$ the tangent line of $C$ at $P$, and the remaining singular points by $P_3$. Let $\sigma : \Sigma_1 \to \mathbb{P}^2$ be the blow-up of $P$ where $E$ denotes the exceptional component. Strict transforms will follow Convention 2.10.

Each generic fiber of $\Sigma$ intersects $C$ at three points. There are four exceptions; three of them can be seen in Figure 2, and they are denoted by $F_+$, $F_0$, and $F_-$. The fourth one is $L$, which intersects $C$ at two points: one is smooth and transversal and the other one is the infinitely near point of $P$ in $E$, which is of type $A_2$. In order to separate $C$ and $E$ we perform a positive elementary Nagata transformation $\rho : \Sigma_1 \dashrightarrow \Sigma_2$ on the fiber corresponding to $L$. The fiber which replaces $L$ is denoted by $F_\infty$. Note that $F_\infty$ intersects $C$ at two points: one of them corresponds to the blow-down of $L$ and the other one is a point with a generic tangency. In particular, the combinatorics of the intersections at $F_0$ and $F_\infty$ coincides.

Remark 5.1. Properties [3,1] imply the rigidity of this arrangement of curves in $\Sigma_2$. In particular, once the four fibers are ordered the cross-ratio of their images in $\mathbb{P}^1$ provides an invariant of the arrangement. The existence of an automorphism of $\Sigma_2$ preserving $C$ and exchanging the two fibers containing the singular points can be easily checked. As a consequence of the cross-ratio argument, the two tangent fibers must also be exchanged. This automorphism defines a birational map of $\mathbb{P}^2$ which is related to the two solutions in $\mathbb{Q}(\sqrt{5})$ exhibited in the proof of Property [3,1,13].

Let us consider the minimal resolution $Z$ of the double covering of $\Sigma_2$ ramified at $C + E$.

The ruling of $\Sigma_2$ induces a morphism $\rho : Z \to \mathbb{P}^1$ such that the generic fiber is elliptic. The only singular fibers are the preimages of $F_+$, $F_-$ (of type $I_5$ in Kodaira notation), $F_0$, and $F_\infty$ (of type $I_1$). These elliptic fibrations have been extensively studied in [18]. Once a section is fixed (e.g. the preimage of $E$), the set of sections has an abelian group structure (inherited by the structure on the fibers) which is called the Mordell-Weil group. Note that the involution associated with the double covering is defined by taking the opposite. It is known that the Mordell-Weil group of $Z$ is cyclic of order 5.

Let us consider a conic $C_1$ tangent to $C$ both at $P$ and at another singular point and transversal to the third singular point. The preimage of $C_1$ by the double covering has two irreducible components which are denoted by $E_1$ and $-E_1$: they are opposite sections in the Mordell-Weil group. Interchanging the two singular points, one obtains the remaining two sections $E_2$ and $-E_2$ of $Z$.

Let us recall that $G$ denotes the fundamental group of the complement of the affine Degtyarev curve, i.e. \( \mathbb{P}^2 \setminus (C \cup L) = \Sigma_2 \setminus (C \cup E \cup L_\infty) \).

Remark 5.2. Despite Proposition 4.4 note that its affine version, \( G = \pi_1(\mathbb{P}^2 \setminus (C \cup L)) \) does possess a geometric surjection onto the orbifold over $\mathbb{P}^2_{2,2,5}$, since $G$ admits an epimorphism onto the dihedral group of order 10, see for instance [5].

In order to construct this morphism, we may use the ideas in [23]. The mapping is obtained by a pencil of rational curves of degree 10, with the following non-reduced fibers:

- A smooth conic $C_2$ of multiplicity 5 such that \((C \cdot C_2)_{P_+} = 2\), \((C \cdot C_2)_{P_-} = 4\) and \((C \cdot C_2)_P = 4\).
- A quintic $C_5$ of multiplicity 2 such that \((C \cdot C_5)_{P_+} = 5\) (\(P_+\) is a smooth point of $C_5$), \((C \cdot C_5)_{P_-} = 10\) (\(P_+\) is a singular point of $C_5$ of type $A_4$), and \((C \cdot C_2)_P = 10\) (\(P\) is a singular point of $C_5$ of type $D_6$).
The curve \( C + L + 2D_2 \) where \( D_2 \) is a smooth conic such that \((C \cdot D_2)_{P_+} = 0, (C \cdot D_2)_{P_-} = 5, \) and \((C \cdot D_2)_{P} = 4\).

We finish this section by describing some properties of the group \( G \). For a point \( Q \in C \), the local fundamental group \( \pi_1^\text{loc}(C, Q) \) of \( C \) at \( Q \) is \( \pi_1(\mathbb{B}_Q \setminus C) \), where \( \mathbb{B}_Q \) is a Milnor ball. The inclusion \( \mathbb{B}_Q \setminus C \to \mathbb{C}^2 \setminus C \) induces a conjugacy class of subgroups (since the base point is not fixed) which will be called the image of the local fundamental group.

**Proposition 5.3.** Let \( P_{\pm} \) be the two singular points of the affine Degtyarev curve.

(a) The images of the local fundamental groups at \( P_+ \) and \( P_- \) are the whole group \( G \).

(b) The center of \( G \) contains an abelian free subgroup of rank 2.

**Proof.** The property about the image of the local fundamental group at \( P_- \) is obvious from the presentation \( \{15\} \). For \( P_+ \) it can be deduced using GAP. As a consequence we obtain two central elements (the images of the central elements of the local fundamental groups). The last property can be deduced by studying some quotients of subgroups of \( G \).

\[ \square \]

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