NONPARAMETRIC ESTIMATION OF MULTIVARIATE SCALE MIXTURES OF UNIFORM DENSITIES

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Suppose that $U = (U_1, \ldots, U_d)$ has a Uniform([0, 1]$^d$) distribution, that $Y = (Y_1, \ldots, Y_d)$ has the distribution $G$ on $\mathbb{R}_+^d$, and let $X = (X_1, \ldots, X_d) = (U_1 Y_1, \ldots, U_d Y_d)$. The resulting class of distributions of $X$ (as $G$ varies over all distributions on $\mathbb{R}_+^d$) is called the Scale Mixture of Uniforms class of distributions, and the corresponding class of densities on $\mathbb{R}_+^d$ is denoted by $F_{\text{SMU}}(d)$. We study maximum likelihood estimation in the family $F_{\text{SMU}}(d)$. We prove existence of the MLE, establish Fenchel characterizations, and prove strong consistency of the almost surely unique maximum likelihood estimator (MLE) in $F_{\text{SMU}}(d)$. We also provide an asymptotic minimax lower bound for estimating the functional $f \mapsto f(x)$ under reasonable differentiability assumptions on $f \in F_{\text{SMU}}(d)$ in a neighborhood of $x$. We conclude the paper with discussion, conjectures and open problems pertaining to global and local rates of convergence of the MLE.

1. Introduction and summary. Fix a non-negative integer $k$, and suppose that $X_1, \ldots, X_n$ are i.i.d. random variables distributed according to a density in the convex family of $k$-monotone densities (with respect to Lebesgue measure) on $(0, \infty)$:

$$
F_k := \left\{ f_{k,G}(\cdot) \equiv \int_0^\infty k \frac{(y-x)^{k-1}}{y^k} dG(y) \mid G \in \mathcal{G}_1 \right\},
$$

where $\mathcal{G}_1$ will denote the set of all distribution functions on $(0, \infty)$ grounded at 0. Here, we use the notation $x_+ \equiv x \cdot 1_{\{x \geq 0\}}$ for any $x \in \mathbb{R}$. It has been shown by Williamson [1956] that the family $F_k$ is identifiably indexed by $\mathcal{G}_1$. In other words, if $G_1, G_2$ are distinct elements in $\mathcal{G}_1$, then $f_{k,G_1}(\cdot)$

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and $f_k, G_2(\cdot)$ differ on a Lebesgue non-null set. Note that $\mathcal{F}_k$ is exactly the collection of all scale mixtures of Beta$(1, k)$ densities.

The Beta$(1, 1)$ distribution is the standard uniform distribution, $U(0, 1)$. Therefore, the class $\mathcal{F}_1$ coincides with the class of all scale mixtures of uniform densities on $(0, \infty)$. A well-known theorem by Khintchine (see, e.g., Feller [1971, p.158]) asserts that the class of densities on $(0, \infty)$ with concave distribution functions is one and the same with our class $\mathcal{F}_1$. It can be seen that $\mathcal{F}_1$ is also the class of all upper semi-continuous, non-increasing densities on $(0, \infty)$. This class is induced by order restrictions, a term we use to explicitly mean that there exists a partial ordering ($\prec$) on the common support $\mathcal{X}$ of the densities in $\mathcal{F}_1$ such that $f \in \mathcal{F}_1$ if and only if $f$ is isotone with respect to this ordering; i.e., $f \in \mathcal{F}_1$ if and only if $f(x) \leq f(y)$ whenever $x, y \in \mathcal{X}$ such that $x \prec y$. In this case, ($\prec$) is the natural partial ordering, ($\geq$), on $(0, \infty)$.

Non-increasing, upper semi-continuous densities (in short, monotone densities) arise naturally via connections with renewal theory and uniform mixing (see, e.g., Woodroofe and Sun [1993].) Maximum likelihood estimation of monotone densities on $(0, \infty)$ was initiated by Grenander [1956a,b], with related work by Ayer et al. [1955], Brunk [1958] and van Eeden [1956a,b,c, 1957a,b]. Asymptotic theory of the MLE in $\mathcal{F}_1$ (the Grenander estimator) was developed by Prakasa Rao [1969] with later contributions by Groeneboom [1985, 1989], Birgé [1987, 1989] and Kim and Pollard [1990]. See Balabdaoui et al. [2010] for descriptions of the behavior of the Grenander estimator at zero.

Nonparametric estimation in families of densities described by order restrictions goes back at least to the work of Grenander [1956a,b], Brunk [1958, 1970] and Robertson [1967], with further development by Wegman [1969, 1970a,b] and Sager [1979, 1982]. Also see the books by Barlow et al. [1972] and by Robertson et al. [1988]. Polonik [1995a,b, 1997, 1998] addressed estimation in various order restricted classes of multivariate densities from the perspective of the excess mass approach studied previously by e.g., Sager [1979, 1982] and Müller and Sawitzki [1991]. Polonik shows that (under reasonable assumptions) the MLE in such classes exists and coincides with an estimator he constructs and calls the silhouette. Forcing the elements of the class to be upper semi-continuous, the MLE is seen to be unique. Brunk [1958] also gives a graphical construction of the maximum likelihood estimator, and establishes $L_1$-consistency of the MLE.

In this paper our goal is to extend the notion of “monotone densities” to higher dimensions; i.e., to densities on $(0, \infty)^d$ with $d > 1$. Such an extension is not unique: For example, we may consider the family, $\mathcal{F}_{BDD}(d)$, of “block-decreasing densities” (a term coined by Biau and Devroye [2003]) that con-
tains all upper-semicontinuous densities on \((0, \infty)^d\) that are non-increasing in each coordinate, while keeping all other coordinates fixed. This class was perhaps first introduced by Robertson [1967]. The particular proper subclass of \(\mathcal{F}_{\text{BDD}}(d)\) studied here is the family \(\mathcal{F}_{\text{SMU}}(d)\) of all multivariate scale mixtures of uniform densities; i.e. the family of upper semi-continuous densities on \((0, \infty)^d\) of the form

\[
 f_G(x) = \int_{(0, \infty)^d} \left( \frac{1}{|y|} 1_{(0,y]}(x) \right) \ dG(y), \quad x \in (0, \infty)^d
\]

for some \(G \in \mathcal{G}_d\), the set of all distribution functions on \((0, \infty)^d\) that grounded (zero) at \(0\); here we use the notation \(|y| = \prod_{i=1}^d y_i\) for \(y = (y_1, \ldots, y_d)^t \in (0, \infty)^d\). For any fixed \(G \in \mathcal{G}_d\), it is clear that if \(Y = (Y_1, \ldots, Y_d)^t\) is distributed according to \(G\) on \((0, \infty)^d\) and if \(U_1, \ldots, U_d\) are i.i.d. \(U(0,1)\) (and independent of \(Y\)), then the vector \(X := (U_1 Y_1, \ldots, U_d Y_d)\) is distributed according to \(f_G(\cdot)\) on \((0, \infty)^d\).

Whereas the family \(\mathcal{F}_{\text{BDD}}(d)\) is characterized by order restrictions (and thus the results by Polonik apply), its subclass \(\mathcal{F}_{\text{SMU}}\) is not; as will be made more explicit in section 2, densities in the class \(\mathcal{F}_{\text{SMU}}\) also satisfy non-negativity restrictions on their \(d\)-dimensional differences around all rectangles. Because of this additional shape restriction, estimation in this family requires separate treatment.

A univariate parallelism to the latter point would be to consider the family \(\mathcal{F}_2\) in (1.1), induced by mixtures of triangular densities; this class can easily be seen to be exactly the class of all non-increasing, convex (and hence continuous) densities on \((0, \infty)\). Thus \(\mathcal{F}_2 \subset \mathcal{F}_1\) is not an order-constrained class of densities, in contrast to its superclass \(\mathcal{F}_1\). Convex densities arise in connection with Poisson process models for bird migration and scale mixtures of triangular densities (see, e.g., Hampel [1987], Anevski [2003] and Lavee et al. [1991]). Estimation of non-increasing, convex densities on \((0, \infty)\) was apparently initiated by Anevski [1994] and was further pursued by Wang [1994], Jongbloed [1995] and Anevski [2003]. The asymptotic distribution theory and further characterizations of the nonparametric MLE of such a density and its first derivative at a fixed point (both under reasonable assumptions) was obtained by Groeneboom et al. [2001a, b]. These authors show that the local rate of convergence of the MLE of the functional \(f \mapsto f(x)\) is of the order \(n^{2/5}\), whereas the Grenander estimator (the MLE in \(\mathcal{F}_1\)) converges locally at the rate of only \(n^{1/3}\).

Here is an outline of the remainder of the present paper: In Section 2 we provide characterizations of the family \(\mathcal{F}_{\text{SMU}}(d)\) that will prove useful in the sequel. Section 3 addresses existence, strong, pointwise consistency as
well as $L_1$ and Hellinger consistency of a sequence of maximum likelihood estimators in $F_{\text{SMU}}(d)$. In Section 4 we derive a local asymptotic minimax lower bound for estimation of $f(x)$ at a fixed point $x$ under for which $f$ satisfies $\partial^d f(x)/(\partial x_1 \cdots \partial x_d) \neq 0$. The lower bound entails a rate of convergence of $n^{1/3}$ for all dimensions $d$ and yields a constant depending on $f$ which reduces to the known lower bound constant for $d = 1$. The paper concludes in Section 5 with a discussion of conjectures and open problems related with both the local (pointwise) and the global ($L_1$ and Hellinger) rates of convergence of the MLE in $F_{\text{SMU}}(d)$.

2. Properties of the Scale Mixtures of Uniform family of densities.

2.1. Properties of $F_{\text{SMU}}(d)$. A density function, $f$, on $(0, \infty)^d$ will be called a (multivariate) Scale Mixture of Uniform densities if there exists a distribution function, $G$, on $(0, \infty)^d$ such that

$$f(x) = f_G(x) = \int_{(0,\infty)^d} \frac{1}{|v|} 1_{[0,v]}(x) \ dG(v) \quad (2.1)$$

$$= \int_{v \geq x} \frac{1}{|v|} \ dG(v) \quad \text{for all } x \in (0,\infty)^d. \quad (2.2)$$

It is clear from (2.2) that a SMU density is also a block-decreasing density: $f_G(\cdot)$ is non-increasing in each coordinate, while keeping all other coordinates fixed. Also, the map $G \mapsto f_G$ is identifiable in the following sense: if $G_1 \neq G_2$, then $f_{G_1} \neq f_{G_2}$ on a set of positive Lebesgue measure; also see Theorem 2.3 below. The following lemma gives a formal statement and proof of a slightly more general result.

**Lemma 2.1.** Two upper semi-continuous and block-decreasing functions $f$ and $g$ on $\mathbb{R}^d$ differ nowhere in the interior of their support or else on a Lebesgue non-negligible set.

**Proof.** Assume that $x$ is in the interior of the support of both $f$ and $g$ and that $f(x) \neq g(x)$. Without loss of generality, assume that $f(x) > g(x)$. Since $g$ is upper semi-continuous and $x$ is an element of the $\| \cdot \|_2$-open set $\{y \mid g(y) < f(y)\}$, we have that there exists an $\epsilon > 0$ such that the $\| \cdot \|_2$-ball of radius $\epsilon$ around $x$, $B_{\| \cdot \|_2}(x, \epsilon)$, be a subset of $\{y \mid g(y) < f(y)\}$. In fact, we have that $f$ and $g$ differ on the Lebesgue non-null set $A \equiv \{y \leq x \mid \|x - y\|_2 < \epsilon\}$ since $y \in A$ implies that $g(y) < f(x) \leq f(y)$ and subsequently that $g(y) < f(y)$ – where here we have also used the fact that $f$ is block-decreasing. The proof is complete. ■
The distribution function $F_G$ corresponding to $X \sim f_G$ is given by

$$F_G(x) = \int_{(0,\infty)^d} |x \wedge v| dG(v),$$

where $\leq$ denotes the natural partial ordering on $\mathbb{R}^d$, while

$$x \wedge v \equiv (x_1, \ldots, x_d) \wedge (v_1, \ldots, v_d) = (\min\{x_1, v_1\}, \ldots, \min\{x_d, v_d\}),$$

and

$$x \vee v \equiv (x_1, \ldots, x_d) \vee (v_1, \ldots, v_d) = (\max\{x_1, v_1\}, \ldots, \max\{x_d, v_d\}).$$

The distribution function $F_G$ of $X \sim f_G$ is generally not concave when $d > 1$, unlike the case when $d = 1$. A SMU density (and a block-decreasing density, in general) can possibly diverge at the origin, whereas the pointwise bound $f(x) \leq 1/|x|$ holds since, for $x \in (0, \infty)^d$ we have

$$1 = \int_{(0,\infty)^d} f(y) \, dy \geq \int_{(0,x]} f(y) \, dy \geq |x|f(x).$$

Further, a $d-$dimensional analogue of the proof of Devroye [1986, Theorem 6.2, p. 173] can be used to show that

$$\lim_{|x| \to \infty} \{ |x|f(x) \} = \lim_{x \to 0} \{ |x|f(x) \} = 0,$$

whenever $f$ is a block-decreasing density on $(0,\infty)^d$.

For any two points $x, y \in [0,\infty)^d$, such that $x \leq y$, we write $[x, y] \equiv [x_1, y_1] \times \cdots \times [x_d, y_d]$, $[x, y] \equiv (x_1, y_1) \times \cdots \times (x_d, y_d)$, $(x, y) \equiv (x_1, y_1) \times \cdots \times (x_d, y_d)$ for the natural closed, lower-closed upper open, lower open upper closed, and open rectangles respectively. Note that the closed rectangle $[x, y]$ has (at most) $2^d$ vertices, the points $u = (u_1, \ldots, u_d)$ where each $u_i$ is either $x_i$ or $y_i$. Following Billingsley [1995], we write $\text{sgn}_{[x,y]}(u) \in \{-1,1\}$, the signum of the vertex $u$, according as the number of $i$, $1 \leq i \leq d$, satisfying $u_i = x_i$ is odd or even respectively.

Thus any two vertices defining an edge of the rectangle have alternating signs. Then, if $u = (u_1, \ldots, u_d)$ is some vertex of $[x,y]$ and $\delta \in \{-1,+1\}$ is its signum, then $(\delta, u)$ is an element of the set

$$\Delta_d[x, y] = \left\{ \left( (-1)^{\sum_{i=1}^d 1[u_i = z_i]} , \ u \right) \mid u \in \{x_1, y_1\} \times \cdots \times \{x_d, y_d\} \right\}.$$
Definition 2.1. For an upper semicontinuous and coordinatewise decreasing function \( g : (0, \infty)^d \to [0, \infty) \) define the \( g \)-volume of a (possibly degenerate) rectangle \([x, y]\) by:

\[(2.5) \quad V_g[x, y] = \sum_{(\delta, u) \in \Delta_d[x, y]} \{\delta g(u)\},\]

provided that \( g \) is defined and is finite for all \( u \) in the summand. Correspondingly, for an upper semicontinuous and coordinatewise increasing function \( g : (0, \infty)^d \to [0, \infty) \), we define the \( g \)-volume of a rectangle \((x, y)\) by the sum on the right side of (2.5).

It is easily seen that for a SMU density, \( f_G \), the \( f_G \)-volume of any rectangle \([x, y]\) is always of the sign \((-1)^d\): Indeed, consider (2.2) and observe that

\[(2.6) \quad (-1)^d V_{f_G}[x, y] = \int_{[x, y]} \frac{1}{|v|} dG(v) \geq 0.\]

From (2.6), or, alternatively, from the fact that the class of sets \([x, y]\) is a \( \pi \)-system which generates the Borel \( \sigma \)-field of subsets of \([0, \infty)^d\) and then extending as in Billingsley [1995], it is clear that \((-1)^d V_f\) extends uniquely to a (non-negative) measure on the Borel \( \sigma \)-field \( \mathcal{B}_+^d = \mathcal{B}^d \cap [0, \infty)^d \) given by

\[(-1)^d V_f(A) = \int_A \frac{1}{|v|} dG(v) \quad \text{for} \quad A \in \mathcal{B}_+^d;\]

in particular,

\[(-1)^d V_f(x, y) = \int_{[x, y]} \frac{1}{|v|} dG(v).\]

The following lemma extends this argument to an arbitrary upper semicontinuous function \( g \) with the \((-1)^d g\)-volumes of all rectangles \([x, y]\) non-negative.

Lemma 2.2. Suppose that \( g \) is a non-negative, upper semi-continuous function satisfying \((-1)^d V_g[x, y] \geq 0\) for all lower-closed upper open rectangles \([x, y]\), and vanishing if any coordinate tends to \( \infty \). Then \((-1)^d V_g\) can be extended to a countably additive measure on \( \mathcal{B}_+^d\).

Proof. Since the class of all rectangles of the form \([x, y]\) is a \( \pi \)-system which generates \( \mathcal{B}_+^d\), this follows immediately from the analogue of Billingsley [1995] with obvious modifications (replace Billingsley’s sets \( A \) with our sets \([x, y]\) and \( F \) with \( \bar{F}(x) = V_g[x, \infty) \) continuous from below).
Of course it is easy to exhibit a block-decreasing density that is not a SMU density: consider the uniform density on the closed triangle in \( \mathbb{R}^2_+ \) with vertices \((0,0), (0,1)\) and \((1,0)\). Then,

\[
(-1)^2V_f((1/8,1/8),(1/2,3/4)) = -2 < 0,
\]

showing that this density is not a SMU density, even though it is block-decreasing.

The following theorem establishes identifiability of the mixing distribution \(G\) as well as providing a useful characterization of SMU densities.
Theorem 2.3.

(a) For the class of SMU densities \( F_{\text{SMU}}(d) = \{ f_G : G \in \mathcal{G}_d \} \) with \( f_G \) as given in (2.1), \( f \in F_{\text{SMU}}(d) \) if and only if \( f \equiv f_G \), where \( G \in \mathcal{G}_d \) is given by

\[
G(x) = \int_{(0,\infty)^d} (-1)^d V_f(u,x) \cdot \mathbb{1}_{u \leq x} \, du .
\]

Thus there is a one-to-one correspondence between \( G \in \mathcal{G}_d \) and \( f_G \in F_{\text{SMU}}(d) \).

(b) Suppose that the Lebesgue density \( f \) on \((0,\infty)^d\) is such that it converges to zero in each coordinate, while keeping all other coordinates fixed. Then, \( f \) is a SMU density if and only if \((-1)^d V_f(x,y) \geq 0\) for all \( 0 \leq x \leq y \).

Proof. (a) Suppose that \( f \equiv f_G \), for \( G \in \mathcal{G}_d \) (recall that this implies that \( G(0) = 0 \)), is a SMU density evaluated at an arbitrary \( x \in (0,\infty)^d \) as:

\[
f(x) = \int_{(0,\infty)^d} \frac{1}{|y|} \mathbb{1}_{(0,x]}(y) \, dG(y) = \int_{y_1 \geq x_1} \cdots \int_{y_d \geq x_d} \frac{1}{|y|} \, dG(y) ,
\]

so that \( df(x) = (-1)^d |x|^{-1} \, dG(x) \) and thus,

\[
G(x) = \int_{(0,\infty)^d} \mathbb{1}_{(0,x]}(y) |y| \, d\{(-1)^d f(y)\} = \int_{(0,\infty)^d} \mathbb{1}_{(0,x]} \int_{(0,\infty)^d} \mathbb{1}_{(0,y]}(u) \, du \, d\{(-1)^d f(y)\} = \int_{(0,\infty)^d} \int_{y \in (u,x]} \, d\{(-1)^d f(y)\} \, du = \int_{(0,\infty)^d} (-1)^d V_f(u,x) \, du ,
\]

where the second to last equality follows by Fubini-Tonelli.

We will now show that \( G \) is unique: Suppose that (2.8) above holds for \( G = G_i \in \mathcal{G}_d \) and \( i = 1, 2 \). Recall that this implies that \( G_1(0) = G_2(0) = 0 \) and, thus, \( G_0(\cdot) \equiv G_1(\cdot) - G_2(\cdot) \) is such that \( G_0(0) = 0, \int_{(0,\infty)^d} G_0(x) \, dx = 0 \) and

\[
0 = \int_{(0,\infty)^d} \frac{1}{|y|} \mathbb{1}_{(0,x]}(y) \, dG_0(y) = \int_{(0,\infty)^d} \frac{1}{|y|} dG_0(y) .
\]
holds for all \( x \in (0, \infty)^d \) and, thus, necessarily \( G_0(x) \) has to be independent of \( x \) and therefore everywhere equal to its value at 0: \( G_0(0) = 0 \). This completes the assertion of uniqueness, since \( G_1 \equiv G_2 \).

(b) If \( f \) is in \( \mathcal{F}_{SMU} \), there exists \( G \in \mathcal{G}_d \) such that

\[
f(x) = \int_{(0,\infty)^d} \frac{1}{|y|} \mathbf{1}_{[0,\infty]}(x) \, dG(y) = \int_{y \geq x} \frac{1}{|y|} \, dG(y),
\]

so that it is easily seen that \((-1)^d V_f[x, y] = \int_{x, y} |y|^{-1} \, dG(y) \geq 0 \) holds true for all \( 0 \leq x \leq y \).

On the other hand, assume that the Lebesgue density \( f \) is such that it converges to zero in each coordinate, while keeping all other coordinates fixed, and satisfies \((-1)^d V_f[x, y] \geq 0 \) for all \( 0 \leq x \leq y \). First, observe that, by Lemma 2.2, this implies that for \( x_1 \leq x_2 \leq x \), elements of \((0, \infty)^d\), we have

\[
(-1)^d V_f[x_1, x] \geq (-1)^d V_f[x_2, x]
\]

and, letting \( x \to \infty \), this yields \( f(x_1) \leq f(x_2) \) because we assumed that \( f \) vanishes as any one of its coordinates diverges to infinity, so that \( V_f[x_1, x] \to (-1)^d f(x_i) \) for \( i \in \{1, 2\} \). Thus, \( f \) is block-decreasing.

Hence, by appealing to part (i), it thus suffices to show that \( G \), as defined on \((0, \infty)^d\) by (2.7) is a valid distribution function.

(i) \( G \) is grounded at 0 trivially by inspection: \( G(0) = 0 \).

(ii) Notice that

\[
\lim_{x_1, \ldots, x_d \to \infty} G(x_1, \ldots, x_d) = \lim_{n \to \infty} \{G(n1)\} = \lim_{n \to \infty} \int_{(0,\infty)^d} (-1)^d V_f(u, n1) \mathbf{1}_{[u \leq n1]} \, du
\]

\[
= (-1)^d \int_{(0,\infty)^d} \lim_{n \to \infty} \{V_f(u, n1)\} \lim_{n \to \infty} \{\mathbf{1}_{[u \leq n1]}\} \, du
\]

\[
= (-1)^d \int_{(0,\infty)^d} (-1)^d f(u+) \, du = \int_{(0,\infty)^d} f(u) \, du = 1,
\]

where in the steps above we have used the fact that for each fixed \( u \in (0, \infty)^d \), the sequence \( X_n(u) := V_f(u, n1) \mathbf{1}_{[u \leq n1]} \) is increasing in \( n \in \mathbb{N} \) and we applied the monotone convergence theorem, and noted that \( \lim_{n \to \infty} \{\mathbf{1}_{[u \leq n1]}\} = 1 \) for any fixed \( u \in (0, \infty)^d \), and that

\[
\lim_{n \to \infty} \{V_f(u, n1)\} = \lim_{n \to \infty} \sum_{(\delta, v) \in \Delta_d[u, n1]} \delta f(v) = (-1)^d f(u+).\]
because

\[ 0 \leq \lim_{|x| \to \infty} f(x) \leq \lim_{|x| \to \infty} \{1/|x|\} = 0, \]

since \( f \) is block-decreasing. Finally, the proof is complete as soon as we observe that \((-1)^{2d} = 1\) and that \( \int_{(0,\infty)^d} f(u) \, du = 1 \), since \( f \) is a density.

(iii) Now, fix \( 0 \leq x \leq y \) and note that (since \( G \) is an increasing upper-semicontinuous function)

\[
V_G(x, y) = \sum_{(\delta, v) \in \Delta_d[x, y]} \{\delta G(v)\} \\
= (-1)^d \int_{(0,\infty)^d} \sum_{(\delta, v) \in \Delta_d[x, y]} \{\delta V_f(u, v) \mathbb{1}_{u \leq v}\} \, du \\
= \int_{(0,y]} (-1)^d V_f(u \lor x, y) \, du \geq 0,
\]

by geometric inspection and Lemma 2.2.

2.2. Lebesgue measurability of block-decreasing functions. Now we establish a technical fact concerning the (Lebesgue) measurability of block-decreasing functions which will be needed in our proofs in Section 3.2. We begin with a definition and then a lemma.

**Definition 2.2.** We call a subset \( C \) of \( \mathbb{R}^d \) a “defective rectangle” if and only if there exist real numbers \( a_i < b_i \) for \( i = 1, 2, \ldots, d \), such that

\[
(a_1, b_1) \times \cdots \times (a_d, b_d) \subseteq C \subseteq [a_1, b_1] \times \cdots \times [a_d, b_d].
\]

Thus, by definition, a defective rectangle is a compact rectangle in \( \mathbb{R}^d \) minus a potentially non-void subset of its boundary. In our definition, a defective rectangle is taken to be both bounded and non-degenerate.

**Lemma 2.4.** Any union of defective rectangles in \( \mathbb{R}^d \) is a Lebesgue set.

**Proof.** Let \( C = \{C_j \mid j \in J\} \) be a family of defective rectangles in \( \mathbb{R}^d \), indexed by some set \( J \). For each \( j \in J \) let the real numbers \( a_{i,j} < b_{i,j} \), for \( i \in \{1, 2, \ldots, d\} \), be uniquely determined by

\[
(a_{1,j}, b_{1,j}) \times \cdots \times (a_{d,j}, b_{d,j}) \subseteq C_j \subseteq [a_{1,j}, b_{1,j}] \times \cdots \times [a_{d,j}, b_{d,j}].
\]

For any \( x \in \mathbb{R}^d \) and \( \epsilon > 0 \) let \( B(x, \epsilon) \) denote the open \( \| \cdot \|_2 \)-ball centered at \( x \) and with radius less than \( \epsilon \). Let also \( \lambda^s \) denote outer-Lebesgue measure on \( \mathbb{R}^d \) and \( \lambda \) its restriction on the Lebesgue sets.
Let $\Delta \equiv \bigcup_{j \in J} C_j$ denote the union of the elements in $\mathcal{C}$ and notice that the interior subset of $\Delta$ is the set

$$\text{int}(\Delta) = \bigcup_{j \in J} (a_{1,j}, b_{1,j}) \times \cdots \times (a_{d,j}, b_{d,j}),$$

exactly because $\text{int}(C_j) = (a_{1,j}, b_{1,j}) \times \cdots \times (a_{d,j}, b_{d,j})$ for each $j \in J$ and because an arbitrary union of open sets is open. Since $\text{int}(\Delta)$ is an open set, to show that $\Delta$ is a Lebesgue set, it suffices to show that $\lambda^*(\Delta \setminus \text{int}(\Delta)) = 0$, from which one concludes that $\Gamma \equiv \Delta \setminus \text{int}(\Delta)$ is a Lebesgue-null set and hence $\Delta$ a Lebesgue set also.

Notice that if $\Gamma = \emptyset$ there is nothing to show. Now, given $\Gamma \neq \emptyset$, fix an arbitrary element $y \in \Gamma$ and observe that there exists an index $k \in J$ such that $y$ lies on the boundary of $C_k$; i.e., $y \in \partial \text{cl}(C_k)$ where $\lambda(\text{cl}(C_k)) = \prod_{i=1}^d (b_{i,k} - a_{i,k}) > 0$. Letting $V_{C_k} \equiv \{a_{1,k}, b_{1,k}\} \times \cdots \times \{a_{d,k}, b_{d,k}\}$ denote the $2^d$ vertices of $\text{cl}(C_k)$ we have that

$$\frac{\lambda(\text{int}(C_k) \cap B(y, \epsilon))}{\lambda(B(y, \epsilon))} \geq \left(\frac{1}{2}\right)^d$$

holds true for all $0 < \epsilon < \min\{\|y - z\|_2 \mid z \in V_{C_k} \setminus \{y\}\}$. This observation, in conjunction with the fact that $\text{int}(C_k) \subseteq \Gamma^c$, immediately yield

$$\lim_{\epsilon \downarrow 0} \left\{ \frac{\lambda^*(\Gamma \cap B(y, \epsilon))}{\lambda(B(y, \epsilon))} \right\} \leq 1 - \left(\frac{1}{2}\right)^d < 1 .$$

The last inequality, and the fact that $y \in \Gamma$ was arbitrary, show (by appealing to the Lebesgue density theorem, see e.g. Cohn [1980, Corollary 6.2.6, pg. 184]) that $\Gamma$ contains no density points and is consequently a Lebesgue-null set.

With this lemma at hand we are ready to prove Lebesgue measurability of non-negative, block-decreasing functions that vanish at infinity.

**Proposition 2.5.** Let $f$ be a real-valued, non-negative function on $(0, \infty)^d$ that is non-increasing and convergent to zero in each coordinate $x_j$, keeping all other coordinates fixed, as $x_j$ coordinate tends to $\infty$. Then:

(a) $f$ is Lebesgue-measurable.

(b) There exists such a function $f$ that is not Borel-measurable. Such an $f$ exists with $f$ also satisfying $\sup\{f(x) \mid x \in (0,\infty)^d\} < \infty$. 


Proof. Proposition 2.5 follows from Theorem 3 of Lang [1986], but for completeness we give another proof here. (a) Note that $[f \geq 0] \equiv [x \in (0, \infty)^d \mid f(x) \geq 0]$, the support of $f$, is the closure of $[x \in (0, \infty)^d \mid f(x) > 0]$, and thus a Borel set; hence it is also a Lebesgue set.

Fix $t > 0$; since $f$ is non-negative, block-decreasing and vanishes at infinity, $[f \geq t] \equiv [x \in (0, \infty)^d \mid f(x) \geq t]$ has the form

$$[f \geq t] = \bigcup_{x \in A_t} C_x$$

for some (non-unique) subset $A_t$ of $(0, \infty)^d$, where

$$C_x \in \{(0, x], (0, x) \setminus \{x\}\}$$

is a defective rectangle (by Definition 2.2), for each $x \in A_t$. Hence it follows by Lemma 2.4 that $[f \geq t]$ is a Lebesgue set. Since the argument above holds for all $t > 0$, the proof of Lebesgue-measurability of $f$ is complete since the class of sets $\{[t, \infty) \mid t \in \mathbb{R}\}$ generates the Borel $\sigma$-field.

(b) We shall provide a counter-example in two dimensions, $d = 2$. For higher dimensions, analogous counter-examples can be constructed. As soon as we convince ourselves that a non-Borel subset, $A$, of $\Delta \equiv \{(x, 1 - x) \in (0, 1)^2 \mid 0 < x < 1\}$ exists, we construct $f$ on $(0, \infty)^2$, satisfying $\sup\{f(x) \mid x \in (0, \infty)^2\} < \infty$, by $f(\cdot) \equiv 1_{\tilde{A}}(\cdot)$ where

$$\tilde{A} \equiv \bigcup_{(x,y) \in A} (0, x] \times (0, y] .$$

Notice then that $[f \geq 1] = \tilde{A}$ is not a Borel set as $A$ is taken to be a non-Borel subset of $\Delta$ and it is an easy task to verify that $\Delta \cap \tilde{A} = A$. Indeed, on one hand $A \subseteq \Delta \cap \tilde{A}$ follows directly from $A \subseteq \tilde{A}$ and $A \subseteq \Delta$. On the other hand, if $(x, y) \in \Delta \cap \tilde{A}$ we have that there exists an $(x_0, y_0) \in A$ such that

$$0 < x, x_0, y, y_0 < 1 ,$$

$$x + y = x_0 + y_0 = 1 ,$$

$$x \leq x_0 \text{ and } y \leq y_0 .$$

Combining the above relationships we conclude that necessarily $(x, y) = (x_0, y_0) \in A$ and the proof of $\Delta \cap \tilde{A} = A$ is complete.

To conclude this counter-example we elaborate briefly on the existence of a non-Borel subset $A$ of $\Delta$. In doing so, we follow steps as in Shorack [2000]. Let $D$ be a subset of $(0, 1)$ that is not a Lebesgue set – the existence of
which is guaranteed by Proposition 1.2.2 in Shorack [2000]. As in Example 7.1.1 of Shorack [2000], let \( F \) be the Lebesgue singular distribution function that gives mass 1 and is 1–1 on the Cantor set, \( C \). Let \( B = F^{-1}(D) \) so that \( B \) be a subset of the Cantor set, \( C \), and a Lebesgue-null set as \( B \subseteq C \) and \( \lambda(C) = 0 \). Let also \( A \equiv \{(x, 1 - x) \mid x \in B\} \). We argue that \( A \) so constructed is not a Borel subset of \( \mathbb{R}^2 \). Assume the contrary, i.e. assume that \( A \) is in fact a Borel set. Since the vector-valued function \( x \mapsto (x, 1 - x) \) is a one-to-one, (Borel)-measurable mapping on \((0, 1)\) we have immediately that \( B \) must also be a Borel set in \( \mathbb{R} \). But then, since \( F \) is non-decreasing, we have that \( F(B) \) is also a Borel set. In addition, since \( F \) is one-to-one on \( C \), we have that \( D = F(B) \) and thus that \( D \) is a Borel and hence a Lebesgue set. This is a contradiction, because \( D \) was taken to be a non-Lebesgue set, by definition. This contradiction yields that \( A \), so constructed, is indeed a non-Borel subset of \( \mathbb{R}^2 \). \( \blacksquare \)

3. Existence and Consistency of the MLE. Let \( X_1, \ldots, X_n \) be i.i.d. random vectors distributed according to some density \( f_0 = f_{G_0} \in \mathcal{F}_{SMU}(d) \) where \( f_0 \) is unknown. Our goal is to estimate the unknown SMU density, \( f_0 \), based on \( X_1, \ldots, X_n \). We will be interested in maximizing the likelihood function \( f \mapsto \prod_{i=1}^n f(X_i) \), or, equivalently, the log-likelihood function \( f \mapsto n \mathbb{P}_n \log \{f(X)\} \) over \( f \in \mathcal{F}_{SMU}(d) \) where \( \mathbb{P}_n = n^{-1} \sum_{i=1}^n \delta_{X_i} \) is the empirical measure of the data. Any such maximizer, \( \hat{f}_n \in \mathcal{F}_{SMU}(d) \), should exist, will be called a (nonparametric) maximum likelihood estimator of \( f_0 \), based on \( X_1, \ldots, X_n \). Since \( f_0 = f_{G_0} \) is given by (2.1) it follows from Theorem 2.3 that estimation of \( f_0 \in \mathcal{F}_{SMU} \) is equivalent to estimation of \( G_0 \).

3.1. On existence and uniqueness of an MLE. We begin with a definition followed by the main theorem of this subsection.

**Definition 3.1.** [Rectangular grid generated by data] Suppose that \( x_1, \ldots, x_n \) are (fixed or random) elements in \((0, \infty)^d\) and suppose that \( x_i = (x_{i1}, \ldots, x_{id})' \) where \( i = 1, 2, \ldots, n \). Define the matrix \( A = [x_{ij}] \in M_{n \times d}((0, \infty)) \) whose \( i \)th row is exactly \( x_i' \), for \( i \in \{1, 2, \ldots, n\} \). Also let

\[
A^2 = \{ (x_{i1}, x_{i2}, \ldots, x_{id}) \mid i_1, \ldots, i_d \in \{1, 2, \ldots, n\} \}
\]

denote the rectangular grid generated by \( A \), where \( x_{i(j)} \) denotes the \( i \)th smallest element among \( x_{i1}, \ldots, x_{ij} \) where \( i \in \{1, 2, \ldots, n\} \) and \( j \in \{1, 2, \ldots, d\} \). In particular, \( x_s = (x_{(1)}, x_{(2)}, \ldots, x_{(d)}) \) and \( x^* = (x_{(1)}, x_{(2)}, \ldots, x_{(d)}) \) denote the element-wise minimum and maximum of \( x_1, \ldots, x_n \), respectively. For each fixed \( j \in \{1, 2, \ldots, d\} \), let
\(n_j(A) := \text{card}\{x_{i,j} \mid i = 1,2,\ldots,n\}\), and notice that we have: \(\text{card}(A^d) = \prod_{j=1}^d n_j(A) \equiv N \leq n^d\).

**Theorem 3.1.** [Existence and characterization of an MLE in \(F_{\text{SMU}}(d)\)]

(a) A maximum likelihood estimator (MLE), \(\hat{f}_n = f_{\hat{G}_n} \in F_{\text{SMU}}(d)\) of \(f_0 = f_{G_0} \in F_{\text{SMU}}(d)\) almost surely exists, where \(\hat{G}_n \in \mathcal{G}_d\) is a purely-atomic probability measure, with at most \(n\) atoms, all of which are concentrated on \(A^d\) – the rectangular grid generated by the data \(X_1,\ldots,X_n\).

(b) For almost all \(\omega\), the unique MLE, \(\hat{f}_n = f_{\hat{G}_n} \in F_{\text{SMU}}(d)\), is completely characterized by the following Fenchel conditions:

\[
P_n \left\{ \frac{1}{f_n(X)} \leq |x| \right\} \leq |x|; \quad \text{for all } x \in (0,\infty)^d,
\]

\[
\text{and } P_n \left\{ \frac{1}{f_n(X)} = |y| \right\} = |y|; \quad \text{if and only if}
\]

\[
y \in (0,\infty)^d \text{ satisfies } \hat{G}_n(\{y\}) > 0; \text{ or, equivalently,}
\]

\[
(-1)^d \lim_{\epsilon \downarrow 0} \left\{ V_{\hat{f}_n}[y, y + \epsilon 1] \right\} > 0.
\]

Maximum likelihood estimation in mixture models has been studied in general by Lindsay [1983], and this material is nicely summarized in Lindsay [1995, Chapter 5]. To prove the present theorem, we will therefore appeal to the results in Lindsay [1995, Chapter 5] and Rockafellar [1970]. We begin with three lemmas.

**Lemma 3.2.** The support set of the mixing measure \(\hat{G}_n\) of any MLE \(\hat{f}_n\) is contained in the grid \(A^\# \subset (0,\infty)^d\) generated by the observed data \(X_1,\ldots,X_n\); i.e. \(\text{supp}(\hat{G}_n) \subset A^\#\).

**Proof.** First we show that \(\mathcal{Y} \subset (0,X^\star]\) where \(X^\star \equiv X_1 \vee \cdots \vee X_n\) and the maximums are taken coordinatewise. If \(\hat{f}_n\) maximizes \(L_n(f) = n\mathbb{P}_n \log f(X)\) over \(f \in F_{\text{SMU}}(d)\) and there is some \(y \in (0,\infty)^d \setminus (0,X^\star]\) with \(y \in \mathcal{Y}\), then \(\hat{f}_n(y) > 0\). Since \(\hat{f}_n\) is block decreasing, this implies that \(0 < \int_{(0,X^\star]} \hat{f}_n(x) dx \equiv \beta < 1\). Then consider \(\tilde{f}(x) \equiv (\hat{f}_n(x)/\beta)1_{(0,X^\star]}(x)\); it is easily seen that \(\tilde{f} \in F_{\text{SMU}}(d)\) and has greater likelihood than \(\hat{f}_n\), contradicting the assumption that \(\hat{f}_n\) maximizes the likelihood. Thus \(\mathcal{Y} \subset (0,X^\star]\).
and we may restrict attention to the class of estimators with support contained in \((0, X^*)\), say \(\mathcal{K}^*(d)\). Suppose that \(\hat{f}_n \in \mathcal{K}^*(d)\). Consider the mixing measure \(\tilde{G}_n\) defined by
\[
\tilde{G}_n \equiv \sum_{j : \mathbf{W}_j \in A^\#} \pi_j \delta_{\mathbf{W}_j} \bigg/ \sum_{j : \mathbf{W}_j \in A^\#} \pi_j \equiv C \sum_{j : \mathbf{W}_j \in A^\#} \pi_j \delta_{\mathbf{W}_j}
\]
where
\[
\pi_j \equiv (-1)^d V_{\hat{f}_n}[\mathbf{W}_j, \mathbf{W}_j^+ \cdot |\mathbf{W}_j|, \text{ for } \mathbf{W}_j \in A^\#}
\]
where \(\mathbf{W}_j^+ \in A^\#\) defines the smallest rectangle above and right of \(\mathbf{W}_j\) in the partition of \([0, X^*]\) defined by the data. Then it is easy to see that
\[
\tilde{f}(\mathbf{x}) = \int_{(0, \infty)^d} \frac{1}{|\mathbf{u}|} 1_{(0, \mathbf{u})}(|\mathbf{x}|) d\tilde{G}_n(\mathbf{u})
\]
satisfies
\[
\tilde{f}(\mathbf{W}_j) = C \sum_{k : \mathbf{W}_k \geq \mathbf{W}_j} \frac{\pi_j}{|\mathbf{W}_j|} = C \sum_{k : \mathbf{W}_k \geq \mathbf{W}_j} \{(-1)^d V_{\hat{f}_n}[\mathbf{W}_j, \mathbf{W}_k]\}
\]
\[
= C(-1)^d V_{\hat{f}_n}[\mathbf{W}_j, 2X^*] = C\hat{f}_n(X_j),
\]
and this implies that
\[
\tilde{f}(\mathbf{x}) = C \sum_{j : \mathbf{W}_j \in A^\#} 1_{(\mathbf{W}_j^-, \mathbf{W}_j)}(\mathbf{x})
\]
where \(\mathbf{W}_j^-\) defines the smallest rectangle below and to the left of \(\mathbf{W}_j\) in the partition of \([0, X^*]\) defined by the data. If \(\hat{f}_n \neq \tilde{f}\), then there exists \(\mathbf{y} \in (\mathbf{W}_j^-, \mathbf{W}_j]\) for some \(\mathbf{W}_j \in A^\#\) such that \(\hat{f}_n(\mathbf{y}) \neq \tilde{f}(\mathbf{y})\), and then necessarily \(\hat{f}_n(\mathbf{y}) > \tilde{f}(\mathbf{y}) = \tilde{f}(\mathbf{W}_j)\). This yields, since \(\hat{f}_n \in \mathcal{K}^*(d)\),
\[
1 = \int_{(0, X^*)} \hat{f}(\mathbf{x}) d\mathbf{x} = C \sum_{j : \mathbf{W}_j \in A^\#} \left\{ \hat{f}_n(\mathbf{W}_j) \int_{(\mathbf{W}_j^-, \mathbf{W}_j]} d\mathbf{x} \right\}
\]
\[
< C \sum_{j : \mathbf{W}_j \in A^\#} \int_{(\mathbf{W}_j^-, \mathbf{W}_j]} \hat{f}_n(\mathbf{x}) d\mathbf{x} = C \int_{(0, X^*)} \hat{f}_n(\mathbf{x}) d\mathbf{x} = C
\]
since \(f \in \mathcal{K}^*(d)\). Thus \(\tilde{f}\) has a greater log-likelihood than \(\hat{f}_n\), and it follows that \(\text{supp}(\tilde{G}_n) \subset A^\#\). 
\[\blacksquare\]
Now we can prove uniqueness of the MLEs $\hat{f}_n$ and $\hat{G}_n$.

**Lemma 3.3.** There exists a set of points $\mathcal{Y} = \{y_1, \ldots, y_m\} \subset (0, \infty)^d$ with $m \leq n$ such that a $F_{\text{SMU}}(d)$ density $\hat{f}_n$ with corresponding mixing measure $\hat{G}_n$ is the MLE only if $\text{supp}(\hat{G}_n) \subset \mathcal{Y}$. Thus any MLE has the form

$$\hat{f}_n(x) = \sum_{j=1}^{m} \pi_j \frac{1}{|y_j|} 1_{(0,y_j)}(x)$$

where $\pi_j \geq 0$, $\sum_{j=1}^{m} \pi_j = 1$. Moreover, the vector $(\hat{f}_n(X_i))_{i=1}^{n}$ is unique.

**Proof.** As in Lindsay [1983, 1995], define $\Gamma(u) \in (0, \infty)^n$ by

$$\Gamma(u) := \left( \frac{1}{|u|} 1_{(0,u)}(X_1), \ldots, \frac{1}{|u|} 1_{(0,u)}(X_n) \right),$$

and define the set $\Gamma \equiv \{ \Gamma(u) \mid u \in (0, \infty)^d \}$. Then $\Gamma$ is a closed and bounded, hence compact, subset of $[0, \infty)^n$. Thus by Rockafellar [1970, Theorem 17.2] $\text{conv}(\Gamma) = \text{conv}(\Gamma) = \text{conv}(\Gamma)$. The continuous function $\prod_{i=1}^{n} z_i$ attains its supremum on $\text{conv}(\Gamma)$. Let $S = \arg\max_{z \in \text{conv}(\Gamma)} \sum_{i=1}^{n} \log z_i$. Since the intersection of $\Gamma$ and the interior of $[0, \infty)^n$ is not empty, we have $S \subset (0, \infty)^n$. Since $\sum_{i=1}^{n} \log z_i$ is strictly concave, $S$ consists of a single point, $\mathbf{f} = (\hat{f}_i)_{i=1}^{n} > \mathbf{0}$. Therefore for any MLE $\hat{f}_n$ it follows that the vector $(\hat{f}_n(X_i))_{i=1}^{n}$ is unique. Note that the gradient of $\sum_{i=1}^{n} \log z_i$ at $\mathbf{f}$ is proportional to $1/\mathbf{f} \equiv (1/\hat{f}_i)_{i=1}^{n}$.

Now $\dim(\text{conv}(\Gamma)) = n$; if we consider the $n$ points $\mathbf{u}_i = X_i$, then the $n$ vectors $\Gamma(\mathbf{u}_i) = (1_{(0,u_i)}(X_1), \ldots, 1_{(0,u_i)}(X_n))/|X_i|$, $i = 1, \ldots, n$, are almost surely linearly independent. (In fact, the matrix $M$ with rows $|X_i|\Gamma(X_i)$, $i = 1, \ldots, n$ has $\det(M) = 1$ a.s. if the $X_i$’s are i.i.d. with any density $f$.) By Rockafellar [1970, Theorem 27.4] the vector $1/\mathbf{f}$ belongs to the normal cone of $\text{conv}(\Gamma)$ at $\mathbf{f}$. Since $1/\mathbf{f} > 0$ we have $\mathbf{f} \in \partial(\text{conv}(\Gamma))$ and the plane $\tau$ defined by $\sum_{i=1}^{n} z_i/\hat{f}_i = n$ is a support plane of $\text{conv}(\Gamma)$ at $\mathbf{f}$. Thus for $v_i = 1/(\hat{f}_i)$, $i = 1, \ldots, n$, it follows that

$$q(u) \equiv |u| - \sum_{i=1}^{n} v_i 1_{(0,u)}(X_i) \geq 0$$

for all $u \in [0, \infty)^d$ and $q(u) = 0$ if $u = 0$ or $\Gamma(u) \in \tau$. We let $\mathcal{Y}$ denote the set of vectors $u$ such that $\Gamma(u) \in \tau$; i.e. $\Gamma(\mathcal{Y}) = \tau \cap \Gamma$.

The intersection $\tau \cap \text{conv}(\Gamma)$ is an exposed face of $\text{conv}(\Gamma)$; see e.g. Rockafellar [1970, p. 162]. By Rockafellar [1970, Theorem 18.3], $\tau \cap \text{conv}(\Gamma) = \text{conv}(\Gamma(\mathcal{Y}))$. 


and by Theorem 18.1, supp($\hat{G}_n \subset Y$). This implies that for any MLE $\hat{f}_n$, the support of the corresponding mixing measure $\hat{G}_n$ is a subset of $Y$, and thus any MLE has the form (3.3) with $y_j \in Y$ for $j = 1, \ldots, m$. To see that $m \leq n$, note that $y_j \in Y \subset A^\#$ satisfy

\[(3.4) \quad |y_j| = \sum_{i=1}^n v_i 1_{(0,y_j)}(X_i) = \langle v, |y_j|\Gamma(y_j) \rangle, \quad j = 1, \ldots, m.\]

Suppose that the vectors $\{|y_j|\Gamma(y_j)\}_{j=1}^m$ are linearly dependent; i.e.

\[\sum_{j=1}^m b_j |y_j|\Gamma(y_j) = 0\]

in $\mathbb{R}^n$ for some $b_j, j = 1, \ldots, m$. Since all the coordinates of the $|y_j|\Gamma(y_j)$ vectors take values in $\{0, 1\}$, this system of equations is algebraically equivalent to the same system in which all the $b_j$'s take only integer values, i.e. $b_j \in \mathbb{Z}$ for $j = 1, \ldots, m$.

Then it follows on the one hand that

\[
\sum_{j=1}^m b_j \langle v, |y_j|\Gamma(y_j) \rangle = \sum_{j=1}^m b_j \sum_{i=1}^n v_i 1_{(0,y_j)}(X_i) = \langle v, \sum_{j=1}^m b_j |y_j|\Gamma(y_j) \rangle = \langle v, 0 \rangle = 0,
\]

and hence, by (3.4), $\sum_{j=1}^m b_j |y_j| = 0$, or, since $y_j = W_{ij} \in A^\#$ for some $i_j$,

\[
\sum_{j=1}^m b_j |W_{ij}| = 0
\]

with all $b_j \in \mathbb{Z}$. But this equation has at most countably many solutions $\{|W_{ij}, j = 1, \ldots, m\}$, and hence occurs with $P_0^n$-probability 0. That is, for any fixed vector $b = (b_j)_{j=1}^k$ with all $b_j \in \mathbb{Z}$, the function $f_b(X_1, \ldots, X_n) = \sum_{j=1}^k b_j |W_{ij}|$ has at most a finite number of zeros, so $P_0^n(f_b(X_1, \ldots, X_n) = 0) = 0$, and since $\mathbb{Z}$ is countable $P_0^n(\cup_{b \in \mathbb{Z}^k} \{f_b(X_1, \ldots, X_n) = 0\}) = 0$. Thus $P_0^n(\cap_{b \in \mathbb{Z}^k} \{f_b(X_1, \ldots, X_n) \neq 0\}) = 1$. Hence it follows that the linear dependence condition only holds on an event with probability 0.

Thus the vectors $|y_j|\Gamma(y_j), j = 1, \ldots, m$ are linearly independent almost surely $P_0^n$, and hence $m \leq n (P_0^n$ - almost surely).
Lemma 3.4. The discrete mixing measure $\hat{G}_n$ which defines an MLE is $P_0^n$-almost surely unique.

Proof. Suppose that there exist two different MLE’s $\hat{f}_1^n$ and $\hat{f}_2^n$, then

$$\hat{f}_l^n(x) = \sum_{j=1}^{m} \pi_j^l \frac{1}{|y_j|} 1_{(0,y_j]}(x), \quad l = 1, 2,$$

where $\pi_j^l \geq 0$ and $\sum_{j=1}^{m} \pi_j^l = 1$ for $l = 1, 2$. Therefore

$$\delta_n(x) \equiv \hat{f}_1^n(x) - \hat{f}_2^n(x) = \sum_{j=1}^{m} r_j \frac{1}{|y_j|} 1_{(0,y_j]}(x)$$

where $r_j \equiv \pi_j^1 - \pi_j^2$ has at least $n$ zeros (since we know that

$$(\hat{f}_1^n(X_i))_{i=1}^n = (\hat{f}_2^n(X_i))_{i=1}^n = (\hat{f}_n(X_i))_{i=1}^n$$

is unique). So, uniqueness holds if the vectors

$$(1_{(0,y_j]}(X_i))_{i=1}^n \in \{0,1\}^n, \quad \text{for } j = 1, \ldots, m \leq n$$

are (almost surely) linearly independent. But this follows from the proof of Lemma 3.3. □

Theorem 3.1 does not assert that the MLE is always unique. A MLE is $P_0^n$ almost surely unique, but we now present an example in which there exist an infinite number of MLE’s.

Example 3.1. [A MLE in $\mathcal{F}_{\text{SMU}}$ is not always unique] To be able to graphically illustrate the set $\Gamma$, in the proof of Theorem 3.1, we need to restrict consideration to $n = 2$ and in order that we be able to graphically illustrate the MLE(s) we need to restrict consideration to $d = 2$. Suppose that $X_1 = (1,3)$ and $X_2 = (3,2)$ are the observation points. The set

$$\Gamma \equiv \left\{ \frac{1}{u_1 u_2} (1_{(0,u_1]}(X_1), 1_{(0,u_2]}(X_2)) \mid (u_1, u_2) \in (0,\infty)^2 \right\}$$

and its convex hull, $\text{Conv}(\Gamma)$, are illustrated in Figure 1.

Using Lindsay [1995, Theorem 22, pg. 118], it follows that any MLE, $\hat{f}_2$, will have a unique value for $\hat{f} \equiv (\hat{f}_2(X_1), \hat{f}_2(X_2))$ that is given by...
The union of the bold lines represents the set $\Gamma$. The shaded area represents the set, Conv($\Gamma$).

\[ \hat{f} = (\tilde{w}_1^{-1}, \tilde{w}_2^{-1}) \] where $\tilde{w} = (\tilde{w}_1, \tilde{w}_2)$ maximizes the function $(w_1, w_2) \mapsto \log(w_1w_2)$ on the set

\[ \left\{(w_1, w_2) \in (0, \infty)^2 \left| \frac{w_1}{3} \leq 2 \text{ and } \frac{w_2}{6} \leq 2 \right. \right\} . \]

It is immediate that $\tilde{w} = (6, 12)$ from which we conclude that $\hat{f} = (1/6, 1/12)$ has exactly two representations as a convex combination of extreme elements in Conv($\Gamma$) (see Figure 1(b) again):

\[ \left( \frac{1}{6}, \frac{1}{12} \right) = \frac{1}{2} \left( 0, \frac{1}{6} \right) + \frac{1}{2} \left( \frac{1}{3}, 0 \right), \]

and

\[ \left( \frac{1}{6}, \frac{1}{12} \right) = \frac{1}{4} \left( \frac{1}{3}, 0 \right) + \frac{3}{4} \left( \frac{1}{9}, \frac{1}{9} \right) . \]

These two convex combinations yield two different maximum likelihood estimators, as shown in Figures 2(a) and 2(b).

It should be noted however that infinitely many maximum likelihood estimators exist in this case: Observe that the hyperplane that passes through $\hat{f}$ intersects Conv($\Gamma$) on the line segment joining the points $(0, 1/6)$ and $(1/6, 1/12)$. 

Fig 1. The sets $\Gamma$ and Conv($\Gamma$) based on two observations: $X_1 = (1, 3)$ and $X_2 = (3, 2)$. 

\[ \hat{f} = (\tilde{w}_1^{-1}, \tilde{w}_2^{-1}) \]
Fig 2. Two maximum likelihood estimators in $\mathcal{F}_{SMU}(2)$, supported on the grid generated by the data: $X_1 = (1, 3)$ and $X_2 = (3, 2)$. The two figures show the contour/level plots of the respective maximum likelihood densities.
(1/3, 0). Then \( \hat{f} \) can be written in infinitely many ways as a convex combination of points on this line segment. However, the corresponding MLEs will no longer be supported solely on the grid generated by the data. ■

3.2. Strong pointwise consistency of the MLE. Let \( X_1, X_2, \ldots, X_n, \ldots \) be the coordinate random elements on the (completed) infinite product space \((\Omega^\infty, A^\infty, P^\infty)\) such that these coordinates are i.i.d. according to \( f_0 \equiv f_{G_0} \) on \((0, \infty)^d\). Let \( A \in A^\infty \) be the event (with \( P^\infty \)-probability one) that for each \( n \in \mathbb{N} \) there exists a unique SMU density, \( \hat{f}_n \equiv f_{\hat{G}_n} \), maximizing the log-likelihood.

From Theorem 2.3 we have that for each \( n \in \mathbb{N} \) and a fixed \( \omega \in A \), there exists a unique Borel probability measure, \( \hat{G}_n \) on \(((0, \infty)^d, \| \cdot \|_2)\), such that

\[
\hat{f}_n(x) = \int_{(0, \infty)^d} \frac{1}{|u|} \mathbb{1}_{(0,u]}(x) \, d\hat{G}_n(u)
\]

holds true for all \( x \in (0, \infty)^d \). We are ready to formulate and prove the following proposition.

**Proposition 3.5.** [Strong Consistency of the MLE in \( F_{\text{SMU}} \)]

(a) (i) The sequence of maximum likelihood mixing distributions \( \{\hat{G}_n\}_{n=1}^\infty \) converges weakly to \( G_0 \) as \( n \to \infty \), \( P^\infty \)-almost surely.

(ii) In addition, for Lebesgue almost all \( x \in (0, \infty)^d \), \( \hat{f}_n(x) \to_{a.s.} f_0(x) \) as \( n \to \infty \). In particular, if \( f_0 \) is continuous at \( x \in (0, \infty)^d \), then

\[
\left| \hat{f}_n(x) - f_0(x) \right| \to_{a.s.} 0 \quad \text{as } n \to \infty.
\]

(b) The sequence of maximum likelihood estimators, \( \{\hat{f}_n\}_{n=1}^\infty \), is strongly consistent in the total variation (or \( L_1 \)) and in the Hellinger metrics. That is,

\[
\int_{(0, \infty)^d} \left| \hat{f}_n(x) - f_0(x) \right| \, dx \to_{a.s.} 0 \quad \text{as } n \to \infty,
\]

and, with \( h^2(p, q) = (1/2) \int \{\sqrt{p(x)} - \sqrt{q(x)}\}^2 \, dx \),

\[
h \left( \hat{f}_n, f_0 \right) \to_{a.s.} 0 \quad \text{as } n \to \infty.
\]
Proof. (a) (i) To be able to apply Theorems 3.4, 3.5 and 3.7 of Pfanzagl [1988], with the refinement on page 143 of the same article, we need to provide the relevant setup as well as establish the assumptions of Pfanzagl’s theorems. We do this below.

Let $C_0(0, \infty)^d \cdot \| \cdot \|_2$ denote the set of all real-valued, continuous functions on $(0, \infty)^d$ that vanish at $\infty$. Let $\Theta_*$ denote the set of all Borel sub-probability measures on $(0, \infty)^d$, equipped with the vague topology, $\tau$, which makes the space a compact, metrizable, topological space – and thus with a countable base. It is also a convex subset of the linear space of all finite, signed, Borel measures on $((0, \infty)^d, \| \cdot \|_2)$. For clarity, the vague topology is the smallest topology that makes the functions $\mu \mapsto \int_{(0, \infty)^d} g(x) \, d\mu(x)$

continuous, for each $g \in C_0((0, \infty)^d, \| \cdot \|_2)$. By metrizability, the topology $\tau$ is completely characterized by convergent sequences, $\theta_n \Rightarrow \theta$ as $n \to \infty$, on $(\Theta_*, \tau)$.

Let also $\Theta \subseteq \Theta_*$ be the set of all Borel probability measures on $(0, \infty)^d$, and notice that $\mu \in \Theta$. Also, for each $\theta_* \in \Theta_*$ there exists a unique $c \in [0, 1]$ and a unique $\theta \in \Theta$, such that $\theta_* = c\theta$. Further, notice that letting $m(\nu, \cdot) \equiv f_{\nu}(\cdot)$, for each $\nu \in \Theta_*$, and $M_n(\cdot) \equiv \mathbb{P}_n \log \{ m(\cdot, X) \}$, we have

$M_n(\theta_*) = \log \{ c \} + M_n(\theta) \leq M_n(\theta)$, since $c \in [0, 1]$,

whence, $\sup_{\theta \in \Theta_*} (M_n(\theta)) = \sup_{\theta \in \Theta} (M_n(\theta))$.

With reference measure the Lebesgue measure $\lambda \equiv Q$ and for each $\nu \in \Theta_*$, let $P_\nu \in \Theta_*$ be the sub-probability, Borel measure on $((0, \infty)^d, \| \cdot \|_2)$ with Radon-Nikodym derivative with respect to $\lambda$ being $f_{\nu}$, Lebesgue almost surely. Then by virtue of Fubini-Tonelli, $P_\nu \in \Theta$ when and only when $\nu \in \Theta$. Also, notice that for each fixed $x \in (0, \infty)^d$, the functional $\nu \mapsto f_{\nu}(x)$ is not vaguely continuous at any $\nu \in \Theta_*$ with a discontinuity point on the boundary of $[x, \infty)$. However, since for a fixed $x \in (0, \infty)^d$, the function $y \mapsto 1_{[x, \infty)}(y)/|y|$ is easily seen to be an upper semi-continuous function on $(0, \infty)^d$ – vanishing at $\infty$, Doob [1994], Theorem 10, p. 138, applies and asserts that the function $\nu \mapsto f_{\nu}(x)$ on $(\Theta_*, \tau)$ is itself (vaguely) upper semi-continuous. Since this holds for all $x \in (0, \infty)^d$, it holds almost-surely. Also, the mapping $\nu \mapsto f_{\nu}(x)$ is affine on $\Theta_*$ (and hence concave also.)

It remains to establish that for each fixed $\tau$-open subset $U$ of $\Theta_*$, the
real-valued function $T_U(\cdot)$ on $(0, \infty)^d$ defined by
\begin{align*}
T_U(x) = \sup_{\nu \in U} \left\{ \int_{(0, \infty)^d} \frac{1}{|u|} 1_{(0,u)}(x) \, d\nu(u) \right\}
\end{align*}
is a $\mathcal{A}$-measurable function. We can choose to take $A$ to be the Lebesgue $\sigma$-field, in which case measurability follows by observing that $T_U(\cdot)$ is a block-decreasing function and appeal to Proposition 2.5.

We now apply our setup to Theorem 3.4 of Pfanzagl [1988] and further appeal to the fact that a vaguely convergent sequence of probability measures with limit a probability measure, is, in fact, weakly convergent. This gives the desired conclusion: the random sequence of maximum likelihood mixing probability measures $\{\hat{G}_n\}_{n=1}^\infty$ converges weakly to $G_0$ as $n \to \infty$, $P^\infty$-almost surely.

(ii) Combining the fact that, for each fixed $x \in (0, \infty)^d$, $\nu \mapsto f_\nu(x)$ is vaguely upper semi-continuous on $\Theta^*$ with the conclusion of part (a)(i), we get
\begin{align}
\lim_{n \to \infty} \left\{ f_{\hat{G}_n}(x) \right\} \leq f_0(x); \quad P^\infty\text{-a.s. for all } x \in (0, \infty)^d.
\end{align}

Let
\begin{align*}
F_{G_0}(\cdot) = \int_{(0, \infty)^d} \frac{|\cdot \wedge u|}{|u|} \, dG_0(u)
\end{align*}
and
\begin{align*}
F_{\hat{G}_n}(\cdot) = \int_{(0, \infty)^d} \frac{|\cdot \wedge u|}{|u|} \, d\hat{G}_n(u)
\end{align*}
be the distribution functions corresponding to the densities $f_0(\cdot)$ and $\hat{f}_n(\cdot)$, respectively, $n \in \mathbb{N}$. These distribution functions are everywhere continuous on the Euclidean set $(0, \infty)^d$. In fact, since for each fixed $x \in (0, \infty)^d$, the function $u \mapsto |x \wedge u|/|u|$ is bounded (by 1) and continuous on $(0, \infty)^d$, we then have that
\begin{align}
F_{\hat{G}_n}(x) \to_{a.s.} F_{G_0}(x) \text{ for all } x \in (0, \infty)^d
\end{align}
follows directly by the definition of almost sure weak convergence of the mixing random measures $\{\hat{G}_n\}_{n=1}^\infty$ to $G_0$, established in part (a)(i).

Let $B$ be the set of points on $(0, \infty)^d$ at which $f_0$ is continuous. Then $B^c$ has Lebesgue measure zero, $\lambda(B^c) = 0$, exactly because $f_0$ is discontinuous on the boundary $\partial[x_0, \infty)$ for a (possibly non-existent) $x_0 \in (0, \infty)^d$ where
$P_0$ is discontinuous (i.e. such that $P_0(\{x_0\}) > 0$). Since $P_0$ can have at most countably many discontinuity points $x_0 \in (0, \infty)^d$ and since $\lambda(\partial [x_0, \infty)) = 0$, we get by countable subadditivity of $\lambda$ that indeed $\lambda(B^c) = 0$.

Fix arbitrary $x \in B$ and $\epsilon > 0$. Then, since $f_0$ is lower semi-continuous at $x$, there exists an open neighborhood $U_{x,\epsilon}$ of $x$ such that for every $y \in U_{x,\epsilon}$ we have that $f_0(y) > f_0(x) - \epsilon$. In particular, there exists an $U_{x,\epsilon} \supset x, > x$ satisfying $f_0(x, > x) > f_0(x) - \epsilon$. Since $f_0$ is block-decreasing, we have:

\begin{equation}
V_{\hat{F}_G_n}(x, x_\epsilon) = \int_{[x, x_\epsilon]} \{f_\hat{G}_n(y)\} \frac{dy}{\lambda((x, x_\epsilon))} \geq f_\hat{G}_n(x, x_\epsilon) - f_0(x) - \epsilon.
\end{equation}

Equation (3.7) further implies that

\begin{equation}
V_{\hat{F}_G_n}(x, x_\epsilon) \rightarrow V_{\hat{F}_{G_0}}(x, x_\epsilon), \quad \text{as } n \rightarrow \infty.\end{equation}

Combining equations (3.8)–(3.11) and the fact that $\epsilon > 0$ was arbitrary, we get

\begin{equation}
\lim_{n \rightarrow \infty} \left\{f_\hat{G}_n(x)\right\} \geq f_0(x); \quad P^\infty\text{-a.s. for } x \in B.
\end{equation}

Equations (3.6) and (3.12) yield the assertion: for Lebesgue almost all $x \in (0, \infty)^d$ (and, in particular, at the points of continuity of $f$), $f_\hat{G}_n(x) \rightarrow a.s.\ f_0(x)$ as $n \rightarrow \infty$ holds.

(b) Showing consistency in the $L_1$ (total-variation) norm is a direct consequence of part (a) (ii) and Glick’s Theorem, Glick [1974]); see also Devroye [1987], p. 25.

Convergence in the Hellinger metric follows from the following well-known inequalities of Le Cam [1986, p.46]:

\begin{equation}
h^2(P, Q) \leq \frac{1}{2} \|P - Q\|_{L_1} \leq h(P, Q)\{2 - h^2(P, Q)\}^{\frac{1}{2}},
\end{equation}

where $h^2(P, Q) = 2^{-1} \int \left(\sqrt{dP} - \sqrt{dQ}\right)^2$ is the squared Hellinger metric and $\| \cdot \|_{L_1}$ is the $L_1$-norm.\]

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4. A local asymptotic minimax lower bound. Let $X_i := (X_{i,1}, \ldots, X_{i,d})'$ for $i = 1, 2, \ldots, n$ be i.i.d. random vectors from density $f \in F_{SMU}(d)$. For a fixed $x_0 \equiv (x_{0,1}, \ldots, x_{0,d})' \in (0, \infty)^d$, we want to estimate the functional $T(f) := f(x_0)$ on the basis of $X_1, \ldots, X_n$. We shall make the following assumption:

**Assumption 4.1.** Suppose that $f \in F_{SMU}$ is continuously differentiable at $x_0$, $f(x_0) > 0$, and, in particular, there exists an open ball $A(x_0)$ around $x_0$ such that $f$ is everywhere strictly positive on $A(x_0)$ and where $(\partial/\partial x_j)f(x_0) < 0$ exist for all $j \in \{1, 2, \ldots, d\}$ and are continuous on $A(x_0) \subseteq (0, \infty)^d$. Further, we assume that the full mixed derivative of $f$ exists, is continuous on $A(x_0)$, and satisfies

\[
(-1)^d \left. \frac{\partial^d f}{\partial x_1 \cdots \partial x_d}(x) \right|_{x=y} > 0 \quad \text{for all } y \in A(x_0).
\]

**Proposition 4.1.** Suppose that $f \in F_{SMU}$ satisfies Assumption 4.1 at the fixed point $x_0 \in (0, \infty)^d$. Then there is a sequence $\{f_n\} \subset F_{SMU}$ such that any estimator sequence $\{T_n\}$ of $f(x_0)$ satisfies

\[
\lim_{n \to \infty} \left\{ E_{f_n} \left\{ n^{\frac{d}{2}} |T_n - f_n(x_0)| \right\} \right\} \leq \frac{e^{-\frac{1}{2} \frac{1}{d} (3^{d-1})}}{\sqrt{d}} (-1)^d \left. \frac{\partial^d f(x)}{\partial x_1 \cdots \partial x_d} \right|_{x=x_0} \cdot f(x_0) \left[ \frac{1}{2} \right].
\]

**Remark.** The lower bound in Proposition 4.1 should be contrasted to a similar lower bound for estimation of $f(x_0)$ for $f \in F_{BDD}$ which is derived by Pavlides [2009]. In that case the natural hypothesis is $\partial f(x_0)/\partial x_i < 0$ for $i = 1, \ldots, d$, and the resulting rate of convergence is $n^{1/(d+2)}$.

To prove Proposition 4.1 we will make use of the following lemma. It was established in the form presented here by Groeneboom and Jongbloed [1995]; see also Groeneboom [1996] and Jongbloed [2000].

**Lemma 4.2.** Let $F$ be a class of densities on a measurable space $(\mathcal{X}, \mathcal{A})$ and $f$ a fixed element of $F$. Let $F_f$ denote any open Hellinger ball with center $f \in F$. Assume that there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq F$ such that

\[
\lim_{n \to \infty} \left\{ \sqrt{n} h(f_n, f) \right\} = \alpha
\]
and
\begin{equation}
\lim_{n \to \infty} |T(f_n) - T(f)| = \beta
\end{equation}
both hold for some constants $0 < \alpha, \beta < \infty$, and where $T$ is a functional on $\mathcal{F}$. Here, $h^2(f_n, f) \equiv 2^{-1} \int \{ \sqrt{f_n(x)} - \sqrt{f(x)} \}^2 \, d\mu(x)$, is the Hellinger distance between the $\mu$-densities $f_n$ and $f$. Let $l(\cdot)$ be a convex function, symmetric about zero, which is non-decreasing on $[0, \infty)$.

Then, it holds that
\begin{equation}
\lim_{n \to \infty} \{ R_n,l(F_f) \} \geq \frac{1}{4} \beta e^{-2\alpha^2}
\end{equation}

where $R_n,l(F_f) \equiv \inf_{T_n} \sup_{g \in \mathcal{F}} E_g \{ l(T_n - T(g)) \}$ is the minimax risk for estimating the functional $T(f)$ based on $n$ i.i.d observations from $\mathcal{F}$.

In particular, for the loss $l(x) = |x|$ on we have
\begin{equation}
\lim_{n \to \infty} \{ R_n,|\cdot|(F_f) \} \geq \frac{1}{4} \beta e^{-2\alpha^2}.
\end{equation}

Hereafter, fix an otherwise arbitrary vector $h := (h_1, \ldots, h_d) \in (0, \infty)^d$, and define $H := \text{diag}(h) \in M_{d \times d}((0, \infty))$. For each $k \in \mathbb{N}$, consider the perturbation rectangle
\[ I_n(k) := \bigotimes_{i=1}^d \left[ x_{0,i} - n^{-\frac{1}{d}} h_i, x_{0,i} + n^{-\frac{1}{d}} h_i \right]. \]

only for those positive integers $n \geq n_0(k, x_0, h)$ for which $I_n(k) \subseteq A(x_0)$ for all $n \geq n_0$. The two-dimensional case, $d = 2$, is illustrated in Figure 3.

Recall Assumption 4.1. Let $b := (\partial^d / \partial x_1 \cdots \partial x_d) f(x) |_{x=x_0}$ and observe that $(-1)^d b > 0$. Finally, define the functions $h_n$ on $I_n(3d)$ as follows:
\[ h_n(y_1, \ldots, y_d) := (-1)^d \prod_{i=1}^d \left\{ 1 \left[ x_{0,i},x_{0,i} + n^{-\frac{1}{d}} h_i \right] (y_i) - 1 \left[ x_{0,i} - n^{-\frac{1}{d}} h_i, x_{0,i} \right] (y_i) \right\}, \]

and
\[ g_n(y) := b \int_{u \succeq y} \{ 1_{I_n(3d)}(u) \cdot h_n(u) \} \, du, \]

where we observe that $g_n(y) \geq 0$ for all $y \in I_n(3d)$, since $x_0$ is the center of the rectangle $I_n(3d)$. In fact, consideration of the geometry of the definition
of $g_n(\cdot)$ reveals that, for $y \in I_n$, $g_n(y)$ is equal to $(-1)^d b > 0$ times the volume of the rectangle $[v_n(y) \land y, v_n(y) \lor y]$, where $v_n(y)$ is defined as that vertex of $I_n$ that is closest in $L_2$-distance from $y \in I_n$. Since $I_n$ is a decreasing sequence of compact sets, it is then immediately clear that $g_n(y)$ is (pointwise) non-increasing in $n \in \mathbb{N}$, for each fixed $y \in (0, \infty)^d$.

Assume that $f \in \mathcal{F}_{\text{SMU}}$, and for fixed vectors $x_0, h \in (0, \infty)^d$ we further assume that $f$ satisfies Assumption 4.1. For $n \geq n_0(3d, x_0, h)$, define the perturbed density, $f_n$ of $f$ at $x_0$, by

$$f_n(x) = \begin{cases} \frac{f(x) + \theta g_n(x)}{d_n} & : \text{if } x \in I_n(3d) \\ \frac{f(x)}{d_n} & : \text{if } x \in I_n^c(3d) \end{cases}$$

for some arbitrary but fixed $\theta \in (0, 1)$ and where $d_n$ is the normalizing constant for $f_n$, uniquely determined by $\int_{(0, \infty)^d} f_n(x) \, dx = 1$. We will see the importance of the value of $b$ and the fact that $0 < \theta < 1$ in the following proposition that establishes that $\{f_n\}_{n \geq n_1} \subseteq \mathcal{F}_{\text{SMU}}(d)$ for a sufficiently large $n_1 \in \mathbb{N}$.

**Proposition 4.3.** There exists a positive integer $n_1 := n_1(d, x_0, h) \geq n_0(3d, x_0, h)$ such that $f_n \in \mathcal{F}_{\text{SMU}}$ for all $n \geq n_1$.

**Proof.** Since $f \in \mathcal{F}_{\text{SMU}}(d)$, we get from Theorem 2.3 that

$$V_f[x, y] \geq 0, \quad \text{for all } d\text{-boxes } [x, y].$$
From the definition of \( g_n(\cdot) \), we see that its full mixed partial derivative exists in a neighborhood of \( x_0 \). Hence, by definition and the fact that \((-1)^d b > 0\) and \( \theta \in (0,1) \), we have that

\[
(-1)^d \frac{\partial^df_n}{\partial x_1 \cdots \partial x_d}(x) \bigg|_{x=y} \geq (-1)^d \frac{\partial^df}{\partial x_1 \cdots \partial x_d}(x) \bigg|_{x=y} - (-1)^d b \theta \\
= \left[ (-1)^d \frac{\partial^df}{\partial x_1 \cdots \partial x_d}(x) \bigg|_{x=y} - (-1)^d b \right] + (1-\theta)(-1)^d b
\]

(4.8) \[ \geq 2^{-1}(1-\theta)(-1)^d b > 0 , \]

where the second to last inequality follows from Assumption 4.1 that the full mixed partial derivative of \( f \) exists and is continuous at \( x_0 \) from which we get, by definition of continuity, that there exists a large enough positive integer \( n_1 := n_1(d, x_0, h) \geq n_0(3d, x_0, h) \) such that

\[
(-1)^d \frac{\partial^df}{\partial x_1 \cdots \partial x_d}(x) \bigg|_{x=y} - (-1)^d b \geq -2^{-1}(1-\theta)(-1)^d b
\]

holds true for all \( y \in I_n(3d) \) and \( n \geq n_1 \). The result in (4.8) suggests that

\[
(-1)^d V_{f_n}[x,y] \equiv (-1)^d \int_{[x,y]} \left\{ \frac{\partial^df}{\partial w_1 \cdots \partial w_n}(w) \bigg|_{w=u} \right\} \, du \geq 0
\]

holds true for all \( d \)-boxes \( [x,y] \) with \( x,y \in I_n(3d) \) and \( n \geq n_1 \).

The last case not considered is the one that exactly one between \( x \) and \( y \), in the \( d \)-box \( [x,y] \), is an element of \( I_n(3d) \). See also Figure 4. For this case, we can appeal to Lemma 2.2 by setting \([x_0, y_0] := [x,y] \cap I_n(3d)\) – the latter being well-defined as the intersection of two rectangles is itself an rectangle. Then, from Lemma 2.2 and (4.7), we have,

\[
(-1)^d V_{f_n}[x,y] = (-1)^d V_{f_n}[x_0,y_0] + (-1)^d \sum_{i=1}^{m} \{V_{f_n}[x_i, y_i]\} \geq 0 + 0 = 0 ,
\]

exactly since \([x_i, y_i] \subseteq I_n(3d)\) for all \( i \in \{1, 2, \ldots, m\}\) (where \( m \) is as defined in Lemma 2.2). For completeness, notice that we were not concerned above with end-point discontinuities of \( f \) (or \( f_n \)) on the entailed rectangle, subsets of \( I_n(3d) \), as, in fact, \( f \) (and \( f_n \)) is (are) continuous there for \( n \geq n_1 \), by Assumption 4.1.

All these observations finally yield that \((-1)^d V_{f_n}[x,y] \geq 0\) holds true for all \( d \)-boxes \([x,y]\) and thus Theorem 2.3 asserts that \( f_n \in F_{SMU} \) for all \( n \geq n_1 \).
We are ready to prove the main proposition of this section.

**Proof.** Recall Proposition 4.3. First, we establish that

\[
\int_{I_n} g_n(x) \, dx = (-1)^d b \prod_{i=1}^{d} \{h_i^2\} \cdot n^{-\frac{d}{2}},
\]

where, hereafter, \(I_n\) will be the short-hand form for \(I_n(3d)\). By definition,
notice that,

\[
\frac{1}{b} \int_{I_n} g_n(x) \, dx = \int_{I_n} \int_{I_n} \prod_{i=1}^{d} \{1_{[x_i \leq u_i]} \} \, h_n(u) \, du \, dx
\]

\[
= \int_{I_n} h_n(u) \left\{ \int_{I_n} 1_{(0,u]}(x) \, dx \right\} \, du
\]

\[
= \int_{I_n} \prod_{i=1}^{d} \left\{ u_i - \left(x_{0_i} - h_in^{-\frac{1}{3d}}\right) \right\} \, h_n(u) \, du
\]

\[
= \prod_{i=1}^{d} \left\{ \int_{x_{0_i} - h_in^{-\frac{1}{3d}}}^{x_{0_i}} \left[ u_i - \left(x_{0_i} - h_in^{-\frac{1}{3d}}\right) \right] \, du_i + \\
- \int_{x_{0_i}}^{x_{0_i} + h_in^{-\frac{1}{3d}}} \left[ u_i - \left(x_{0_i} - h_in^{-\frac{1}{3d}}\right) \right] \, du_i \right\}
\]

\[
= \prod_{i=1}^{d} \left\{ \int_{0}^{h_in^{-\frac{1}{3d}}} \left(-2y \right) \, dy \right\} = (-1)^d \prod_{i=1}^{d} \left\{ h_i^2 \cdot n^{-\frac{2}{3}} \right\} = (-1)^d \prod_{i=1}^{d} \left\{ h_i^2 \right\} \cdot n^{-\frac{2}{3}},
\]

thus yielding (4.9).

We next derive another equality, the most important fact about it being the factor \(n^{-1}\) on the right hand side:

(4.10) \[
\int_{I_n} g_0^2(x) \, dx = \left(\frac{8}{3}\right)^d b^2 \prod_{i=1}^{d} \{ h_i^3 \} \cdot n^{-1}.
\]

Before we start deriving (4.10), let us first define four rectangles \(R_j^i\) with \(j = 1, 2, 3, 4\) for each \(i \in \{1, 2, \ldots, d\}\):

(i) \( R_1^i = \left[ x_{0_i} - h_in^{-\frac{1}{3d}}, x_{0_i} \right] \times \left[ x_{0_i} - h_in^{-\frac{1}{3d}}, x_{0_i} \right] \),

(ii) \( R_2^i = \left[ x_{0_i} - h_in^{-\frac{1}{3d}}, x_{0_i} \right] \times \left[ x_{0_i}, x_{0_i} + h_in^{-\frac{1}{3d}} \right] \),
(iii) \( R^3_i = \left( x_{0i}, x_{0i} + h_i n^{-\frac{1}{3d}} \right) \times \left[ x_{0i} - h_i n^{-\frac{1}{3d}}, x_{0i} \right], \)

(iv) \( R^4_i = \left( x_{0i}, x_{0i} + h_i n^{-\frac{1}{3d}} \right) \times \left( x_{0i}, x_{0i} + h_i n^{-\frac{1}{3d}} \right). \)

Then, by definition:

\[
\frac{1}{b^2} \int_{I_n} g_n^2(x) \, dx = \int_{I_n} \left\{ \int_{I_n} h_n(u) 1_{[x \leq u]} \, du \right\}^2 \, dx
\]
\[
= \int_{I_n} \int_{I_n} \int_{I_n} h_n(u) h_n(v) 1_{[x \leq u \wedge v]} \, dv \, du \, dx
\]
\[
= \int_{I_n} \int_{I_n} \left\{ \prod_{i=1}^d [(u_i \wedge v_i) - (x_{0i} - h_i n^{-\frac{1}{3d}})] \times h_n(u_i) h_n(v_i) \right\} \, dv \, du
\]
\[
= \prod_{i=1}^d \left\{ \int_{R^1_i + R^3_i} [(u \wedge v) - (x_{0i} - h_i n^{-\frac{1}{3d}})] \, dv \, du + \right.
\]
\[
\left. - 2 \int_{R^2_i} [(u \wedge v) - (x_{0i} - h_i n^{-\frac{1}{3d}})] \, dv \, du \right\}
\]

(4.11) \( = 2^d \prod_{i=1}^d \{S_{1i} + S_{2i} - S_{3i}\}, \)

where the last equality follows by symmetry and Fubini-Tonelli and the integrals in the braces are to be evaluated below:

\[
S_{1i} \equiv \int_{x_{0i} - h_i n^{-\frac{1}{3d}}}^{x_{0i}} \int_{x_{0i} - h_i n^{-\frac{1}{3d}}}^{x_{0i}} \{ v - (v - h_i n^{-\frac{1}{3d}}) \} \, dv \, du
\]
\[
= \int_{x_{0i} - h_i n^{-\frac{1}{3d}}}^{x_{0i}} \{ (x_{0i} - v) (v - x_{0i} + h_i n^{-\frac{1}{3d}}) \} \, dv
\]
\[
= \int_{x_{0i} + h_i n^{-\frac{1}{3d}}}^{x_{0i} - h_i n^{-\frac{1}{3d}}} \{ y (y + h_i n^{-\frac{1}{3d}}) \} \, dy \quad \text{[change of variable]}
\]
while, again, by a change of variable argument:

\[ S_{2i} \equiv \int_{x_{0i}}^{x_{0i} + h_i n^{-\frac{1}{3d}}} \int_v^v \left\{ v - \left( x_{0i} - h_i n^{-\frac{1}{3d}} \right) \right\} \, dudv \]

\[ = \int_{x_{0i}}^{x_{0i} + h_i n^{-\frac{1}{3d}}} \left\{ \left( x_{0i} - v \right) + h_i n^{-\frac{1}{3d}} \right\} \left( v - x_{0i} \right) + h_i n^{-\frac{1}{3d}} \right\} \, dv \]

\[ = \int_0^{h_i n^{-\frac{1}{3d}}} \left\{ -y + h_i n^{-\frac{1}{3d}} \right\} \left( y + h_i n^{-\frac{1}{3d}} \right) \right\} \, dy , \]

and similarly:

\[ S_{3i} \equiv \int_{x_{0i} - h_i n^{-\frac{1}{3d}}}^{x_{0i}} \left\{ h_i n^{-\frac{1}{3d}} \left( v - x_{0i} + h_i n^{-\frac{1}{3d}} \right) \right\} \, dv \]

\[ = h_i n^{-\frac{1}{3d}} \int_0^{h_i n^{-\frac{1}{3d}}} \left\{ h_i n^{-\frac{1}{3d}} - y \right\} \, dy . \]

Let now \( q_i := h_i n^{-1/3d} \), for \( i \in \{1, 2, \ldots, d\} \), and observe that

\[ S_{1i} + S_{2i} - S_{3i} = \int_0^{q_i} \left\{ y(q_i - y) + q_i^2 - y^2 + q_i^2 - q_i y \right\} \, dy = \cdots = \frac{4}{3} h_i^3 n^{-\frac{1}{3d}} , \]

so that plugging all these in (4.11) yields the desired (4.10).

Now, recall from the definition of \( f_n \) that \( \theta \in (0, 1) \) was arbitrary but fixed. Also, from \( \int_{(0,\infty)^d} f_n(x) \, dx = 1 \) we can get an explicit expression for the normalizing constant \( d_n \):

\[ d_n = \int_{I_n} f(x) \, dx + \int_{I_n} f(x) \, dx + \theta \int_{I_n} g_n(x) \, dx \]

\[ = 1 + \theta \int_{I_n} g_n(x) \, dx = 1 + (-1)^d \theta b \prod_{i=1}^d \{ h_i \} \cdot n^{-\frac{2}{3}} , \]

where the second to last equality follows from \( \int_{(0,\infty)^d} f(x) \, dx = 1 \), while the last equality follows from (4.9). Notice from (4.12) that \( d_n \downarrow 1 \) as \( n \uparrow \infty \). Also, from the easily verifiable identity \( g_n(x_0) = (-1)^d b \prod_{i=1}^d \{ h_i \} n^{-1/3} \), we
have
\[
\begin{align*}
\frac{n^\frac{1}{3}}{2} |f_n(x_0) - f(x_0)| &= \frac{n^\frac{1}{3}}{2} \left| f(x_0) + (-1)^d b \prod_{i=1}^d \{ h_i \} n^{-\frac{1}{3}} - f(x_0) \right| \\
&= \left| n^\frac{1}{3} \left\{ \frac{1}{d_n} - 1 \right\} f(x_0) + (-1)^d b \prod_{i=1}^d \{ h_i \} \right|
\end{align*}
\]
\quad \rightarrow \quad (-1)^d b \prod_{i=1}^d \{ h_i \} \; (> 0), \quad \text{as } n \to \infty.
\]

Also,
\[
\begin{align*}
2nh^2(f_n, f) &= n \int_{I_n} \left\{ \sqrt{f_n(x)} - \sqrt{f(x)} \right\}^2 \, dx + n \int_{I_n} \left\{ \sqrt{f_n(x)} - \sqrt{f(x)} \right\}^2 \, dx \\
&= n \int_{I_n} \left\{ \frac{f_n(x) - f(x)}{\sqrt{f_n(x)} + \sqrt{f(x)}} \right\}^2 \, dx + \delta_n^2 \int_{I_n} f(x) \, dx,
\end{align*}
\]
where,
\[
\delta_n \equiv \sqrt{n} \left\{ 1 - \frac{1}{\sqrt{d_n}} \right\} = \sqrt{n} \left\{ \sqrt{d_n} - 1 \right\} \\
= \sqrt{n} \left\{ \sqrt{1 + O} \left( n^{-\frac{2}{3}} \right) \right\} - 1 \to 0, \quad \text{as } n \to \infty,
\]
with the convergence on the last display following from (4.12). Applying this to (4.14), we have:
\[
\begin{align*}
(4.15) \quad 2nh^2(f_n, f) &= n \int_{I_n} \left\{ \frac{f_n(x) - f(x)}{\sqrt{f_n(x)} + \sqrt{f(x)}} \right\}^2 \, dx + o(1)
\end{align*}
\]
as \( n \to \infty \), because \( 0 \leq \int_{I_n} f(x) \, dx \leq 1 \).

For fixed \( n \in \mathbb{N} \), such that \( f \) and \( g_n \) be continuous and strictly positive on \( I_n \), let \( x_{(n)} \) and \( x^{(n)} \) denote, respectively, a minimizer and a maximizer of \( f \) on the compact set \( I_n \). Let also \( y_{(n)} \) and \( y^{(n)} \) denote, respectively, a minimizer and a maximizer of \( g_n \) on the compact set \( I_0 \). Observe that, since \( I_n \) is a decreasing sequence of compact sets converging to \( \{ x_0 \} \), all of \( x_{(n)} \),
\(x^{(n)}, y^{(n)}\) and \(y^{(n)}\) converge to \(x_0\) as \(n \to \infty\). Also,

\[
\sup_{x \in I_n} \left| \frac{f_n(x) - f(x)}{f(x)} \right| = \sup_{x \in I_n} \left| \left( \frac{1}{d_n} - 1 \right) + \frac{\theta g_n(x)}{d_n f(x)} \right| \\
\leq \left( 1 - \frac{1}{d_n} \right) + \frac{\theta \sup_{x \in I_n} \{g_n(x)\}}{d_n \inf_{x \in I_n} \{f(x)\}} \\
\to 0, \quad \text{as} \quad n \to \infty,
\]

(4.16)

because \(g_n\) is pointwise non-increasing in \(n \in \mathbb{N}\), \(g_n(x_0) = O(n^{-1/3})\) and \(f(x_0) > 0\).

Also,

\[
D_1(n) \equiv \int_{I_n} \{f_n(x) - f(x)\}^2 \, dx \\
= \frac{1}{d_n^2} \int_{I_n} \left\{ \theta^2 g_n^2(x) - O(n^{-\frac{2}{3}}) f(x) g_n(x) + O(n^{-\frac{2}{3}}) f^2(x) \right\} \, dx
\]

and noticing that

\[
0 \leq \int_{I_n} \{g_n(x) f(x)\} \, dx \leq f \left( x^{(n)} \right) \int_{I_n} \{g_n(x)\} \, dx = O \left( n^{-\frac{2}{3}} \right),
\]

so that,

\[
nD_1(n) = \frac{n}{d_n^2} \left\{ \left( \frac{8}{3} \right)^d \theta^2 b^2 \prod_{i=1}^d \{h_i^3\} \cdot n^{-1} + o \left( n^{-\frac{4}{3}} \right) \right\} \\
\to \left( \frac{8}{3} \right)^d \theta^2 b^2 \prod_{i=1}^d \{h_i^3\}, \quad \text{as} \quad n \to \infty.
\]

(4.17)

Now, since \(f\) is block-decreasing, we have,

\[0 < f \left( x_0 + n^{-\frac{1}{d}} I_d h \right) \leq f(x) \leq f \left( x_0 - n^{-\frac{1}{d}} I_d h \right)\]

for all \(x \in I_n\) and \(n \geq n_1\). Hence,

\[
nD_1(n) \leq \int_{I_n} \frac{\{f_n(x) - f(x)\}^2}{f(x)} \, dx \leq \frac{nD_1(n)}{f \left( x_0 - n^{-\frac{1}{d}} I_d h \right)}
\]

which, ahead with (4.17) and sandwich, yields

\[
n \int_{I_n} \frac{\{f_n(x) - f(x)\}^2}{f(x)} \, dx \to \left( \frac{8}{3} \right)^d \theta^2 b^2 \cdot \prod_{i=1}^d \{h_i^3\} \cdot f(x_0), \quad \text{as} \quad n \to \infty.
\]
Applying all of the above to (4.15), and appealing to Lemma 2 of Jongbloed [2000], we get

\begin{equation}
(4.18) \quad nh^2(f_n, f) = \frac{1}{8} \int_{I_n} \frac{\{f_n(x) - f(x)\}^2}{f(x)} \, dx + o(1)
\end{equation}

\begin{equation}
(4.19) \quad \Rightarrow \quad \frac{8^{d-1}}{3^d f(x_0)} \theta^2 b^2 \prod_{i=1}^d \{h_i^3\}
\end{equation}

as \( n \to \infty \), so that by applying (4.13) and (4.19) to Lemma 4.2, we get

\[
\lim_{n \to \infty} \inf \max \left\{ E_{f_n} \left\{ n_\theta \left| T_n - f_n(x_0) \right| \right\}, E_f \left\{ n_\theta \left| T_n - f(x_0) \right| \right\} \right\} \\
\geq \frac{1}{4} \left\{ (-1)^d b \right\} \theta c \exp \left\{ -\frac{\theta^3 d - 2}{3^d f(x_0)} \theta^2 b^2 c^3 \right\} =: G_{f,x_0}(c, \theta)
\]

where \( c \equiv \prod_{i=1}^d \{h_i\} \). For a fixed \( \theta \in (0, 1) \) the maximum of \( G_{f,x_0}(c, \theta) \) is attained at

\[
c(\theta) = \left\{ \frac{3^{d-1} f(x_0)}{2^{d-2} \theta^2 b^2} \right\}^{\frac{1}{3}}
\]

and is equal to

\[
G_f(c(\theta), \theta) = \frac{e^{-\frac{1}{3}}}{2^d} \left\{ 3^{d-1} \theta \right\} \left\{ (-1)^d \frac{\partial^d f(x)}{\partial x_1 \cdots \partial x_d} \big|_{x=x_0} f(x_0) \right\}^{\frac{1}{3}},
\]

the latter being an increasing function of \( \theta \in (0, 1) \).

This suggests that

\[
\lim_{n \to \infty} \inf \max \left\{ E_{f_n} \left\{ n_\theta \left| T_n - f_n(x_0) \right| \right\}, E_f \left\{ n_\theta \left| T_n - f(x_0) \right| \right\} \right\} \\
\geq \frac{e^{-\frac{1}{3}}}{2^d} \left\{ \theta \cdot 3^{d-1} \right\} \left\{ (-1)^d \frac{\partial^d f(x)}{\partial x_1 \cdots \partial x_d} \big|_{x=x_0} \cdot f(x_0) \right\}^{\frac{1}{3}}.
\]

Overall, we are allowed to take \( \theta \uparrow 1 \) in the above display, even if \( \theta = 1 \) is not a valid configuration, yielding the lower bound in the wording of the proposition. The proof is thus complete.■

5. Discussion and open problems. Once consistency has been established, interest focuses on rates of convergence of the MLE and other properties, including the behavior of \( \hat{f}_n \) at zero and pointwise limiting distributions. We have the following conjectures concerning the MLE \( \hat{f}_n \) for the
class $\mathcal{F}_{\text{SMU}}(d)$. Work is currently underway on all of these further problems.

**Conjecture 1.** If $f_0(0) < \infty$, then we conjecture that

$$P_0(\hat{f}_n(0) \leq M(\log n)^{d-1}) \rightarrow 1$$

for some $M > 0$.

**Conjecture 2.** If $f_0(0) < \infty$ and $f_0$ is concentrated on $[0, M]$ for some $0 < M < \infty$, then $h(\hat{f}_n, f_0) = O_p(n^{-1/3}(\log n)^{\gamma})$ for some $\gamma$ depending only on $d$.

Concerning rates of convergence of the estimators at a fixed point, we do not yet have any upper bound results to accompany the lower bound results of Proposition 4.1. Thus there remain the following two possibilities: (a) the pointwise rate of convergence under Assumption 4.1 is $n^{1/3}$, and we expect convergence in distribution with the rate $n^{1/3}$; or, (b) the lower bound given in Proposition 4.1 is not yet sharp, and we should expect log terms in the rate (as might be expected from the covering number results of Blei et al. [2007]). Our corresponding conjectures for these two possible scenarios are given below as Conjectures 3a and 3b respectively.

**Conjecture 3a.** Suppose that $f_0$ has $\partial^d f_0(x)/\partial x_1 \cdots \partial x_d$ continuous in a neighborhood of $x_0$ with

$$\partial^d f_0(x_0) \equiv \frac{\partial^d f_0(x)}{\partial x_1 \cdots \partial x_d} \bigg|_{x=x_0} \neq 0.$$ 

Let $\{W(t) : t \in \mathbb{R}^d\}$ be a $2d$-sided Brownian sheet process on $\mathbb{R}^d$ and let

$$\mathbb{Y}(t) \equiv \sqrt{f_0(x_0)} W(t) + \frac{(-1)^d}{2^d} (-1)^d \partial^d f_0(x_0) |t|^2.$$ 

Then, in keeping with our lower bound results of Section 4, we conjecture that

$$n^{1/3}(\hat{f}_n(x_0) - f_0(x_0)) \rightarrow_d \partial^d \mathbb{H}(t)|t=0$$

where the process $\mathbb{H}$ is determined by

(i) $\mathbb{H}(t) \geq \mathbb{Y}(t)$ for all $t \in \mathbb{R}^d$,

(ii) $\int_{\mathbb{R}^d} (\mathbb{H}(t) - \mathbb{Y}(t)) d(\partial^d \mathbb{H}(t)) = 0$, and

(iii) $V_{\partial^d \mathbb{H}}[u, v] \geq 0$ for all $u \leq v \in \mathbb{R}^d$.

Partial results concerning Conjecture 3a were obtained in Pavlides [2008].
Conjecture 3b. As suggested in part by the covering number results of Blei, Gao and Li [2007], the pointwise rate of convergence is \( (n/(\log n)^{d-1/2})^{1/3} \). This would entail an improved version of Proposition 4.1. In this case we do not yet have conjectures concerning the limiting distribution.

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