Parameter estimation for stochastic diffusion process with drift proportional to Weibull density function

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Abstract. In the present paper we propose a new stochastic diffusion process with drift proportional to the Weibull density function defined as

\[ X_\varepsilon = x, \quad dX_t = \left( \gamma t (1 - t^{\gamma+1}) - t^\gamma \right) X_t dt + \sigma X_t dB_t, \quad t > 0, \]

with parameters \( \gamma > 0 \) and \( \sigma > 0 \), where \( B \) is a standard Brownian motion and \( \varepsilon \) is a time near to zero. First we interested to probabilistic solution of this process as the explicit expression of this process. By using the maximum likelihood method and by considering a discrete sampling of the sample of the new process we estimate the parameters \( \gamma \) and \( \sigma \).

Keyword : Maximum Likelihood, Itô’s formula, Weibull density, stochastic diffusion process, parameter estimation

1 Introduction

In the present paper we propose a new stochastic Weibull process \( X = \{X_t, t > 0\} \) given by the following linear stochastic differential equation

\[ X_\varepsilon = x > 0; \quad dX_t = \left( \frac{\gamma}{t} (1 - t^{\gamma+1}) - t^\gamma \right) X_t dt + \sigma X_t dB_t, \quad t > 0 \]

where \( \mu(t, X_t; \gamma) = \frac{\gamma}{t}(1 - t^{\gamma+1})X_t - t^\gamma X_t \) and \( g(X_t; \sigma) = \sigma X_t \). We denote by \( B \) a standard Brownian motion and \( \gamma \) and \( \sigma \) are unknown parameters. An interesting problem is to estimate the parameters \( \gamma \) and \( \sigma \) are time independent reel parameters to be estimate when one observes the whole trajectory of \( X \).

The estimation for diffusion processes by discrete observation has been studied by several authors (see for example [6], [7], [1], [2], [5]) and its references. Prakasa Rao [7] treats this problem and shows that the least square estimator is asymptotically normal and efficient under the assumption \( h\sqrt{N} \rightarrow 0 \), the condition for “rapidly increasing experimental design” [7].

The organization of our paper is as follows. Section 2 contains the presentation of the basic tools that we will need throughout the paper: basic properties of standard Brownian motion and Itô's formula. The aim of section 3 is twofold. Firstly, we prove the close formula of SDE under the conditions (1) - (3). Secondly, we investigate the mean of \( X_t \) an explicit solution of SDE and we prove that the drift is proportionnel to Weibull density. The section 4 is devoted to estimate a parameters by using Maximum likelihood. In the last section we present a numerical test.
Given the general one-dimensional time-homogeneous SDE
\[ X_t = x > 0; \quad dX_t = \mu(X_t; \gamma)dt + g(X_t; \sigma)dB_t, \quad t > 0. \] (1.1)
where \( \mu(t, X_t; \gamma) = \frac{2}{\gamma}(1 - t^{\gamma + 1})X_t - t^{\gamma}X_t \) and \( g(X_t; \sigma) = \sigma X_t \). The following conditions are assumed in this article:

1. There is a constant \( L > 0 \) such that
   \[ |\mu(t, x; \gamma)| + |g(x; \sigma)| \leq L(1 + |x|) \]
2. There is a constant \( L > 0 \) such that
   \[ |\mu(t, x; \gamma) - \mu(t, y; \gamma)| + |g(x; \sigma) - g(y; \sigma)| \leq L|x - y| \]
3. For each \( q > 0 \), \( \sup_t \mathbb{E}(|X_t|^q) < \infty \).

2 Preliminaries

In this section we start by recalling the definition of Brownian motion, which is a fundamental example of a stochastic process. The underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\) of Brownian motion can be constructed on the space \( \Omega = C_0(\mathbb{R}^+) \) of continuous real-valued functions on \( \mathbb{R}^+ \) started at 0. For more complete presentation on the subject, see [4], [3].

**Definition 2.1.** The standard Brownian motion is a stochastic process \((B_t, t \geq 0)\) such that

(a) \( B_0 \) almost surely;

(b) With probability one \( t \to B_t \) is continuous.

(c) For any finite sequence of times \( 0 \leq t_0 < t_1 < \cdots < t_n \) the increments
   \[ B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}} \]
   are independent.

(d) for any given times \( 0 \leq s < t \), \( B_t - B_s \) has the Gaussian distribution \( \mathcal{N}(0, t-s) \) with mean zero and variance \( t - s \).

We refer to Theorem 10.8 of [4] and to Chapter 5 of [4] for the proof of the existence of Brownian motion as a stochastic process \((B_t, t \geq 0)\) satisfying the above properties \((a)-(d)\). In the sequel the filtration \((\mathcal{F}_t)_{t \geq 0}\) will be generated by the Brownian paths up to time \( t \), in other words we write

\[ \mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t), \quad t \geq 0. \]

we give a basic properties of Brownian motion and extensions ([?]):

- The crucial fact about Brownian motion, which we need is \((dB)^2 = dt\);
- For every \( 0 \leq s \leq t \), \( B_t - B_s \) is independent of \( \{B_u, u \leq s\} \) and has a \( \mathcal{N}(0, t-s) \).
- Brownian motion \((B_t)\) is a process Markov property;
- \((-B_t)_{t \geq 0}\) is a Brownian motion;
- \(X_t = X_0 + \mu t + \sigma B_t\) is a Brownian motion with drift with mean equal to \(X_0 + \mu t\);
- \(X_t = X_0 \exp(\mu t + \sigma B_t)\) is a Geometric Brownian motion with mean equal to \(X_0 \exp(\mu t + \sigma^2 / 2)\).

We introduce the following two version of Itô’s formula

**Theorem 2.2.** (Itô’s formula v.1) Let \(f \in C^2(\mathbb{R})\). Then for \(a < t\),

\[
\frac{f(B_t) - f(B_a)}{t - a} = f'(B_a) + \int_a^t f''(B_s) \, ds
\]

**Theorem 2.3.** (Itô’s formula v.2) Let \(f(t, x)\) be a continuous in \([a, b] \times \mathbb{R}\) with \(f_t, f_x, f_{xx}\) continuos \((a, b) \times \mathbb{R}\). Then for \(a < t < b\),

\[
\frac{f(t, B_t) - f(a, B_a)}{t - a} = \int_a^t f_x(s, B_s) \, dB_s + \int_a^t \left( f_t(s, B_s) + \frac{1}{2} f_{xx}(s, B_s) \right) \, ds.
\]

### 3 Closed formula and mean of \(X_t\)

By curiosity, we focus on the explicit formula for the SDE to find the mean and variance of \(X_t\). By using the Itô’s formula to \(Y_t = \ln(X_t)\) we obtain

\[
dY_t = \left( \frac{\gamma t \ln(\gamma) - \gamma^\gamma - \frac{\sigma^2}{2}}{t} \right) dt + \sigma dB_t, \quad Y_\varepsilon = \ln x.
\]

By integrating between \(\varepsilon\) and \(t\) it follows

\[
Y_t = \ln x + \gamma \ln \left( \frac{t}{\varepsilon} \right) - \left( t^{\gamma+1} - \varepsilon^{\gamma+1} \right) - \frac{\sigma^2}{2} \left( t - \varepsilon \right) + \sigma(B_t - B_\varepsilon).
\]

Then the explicit solution of SDE is given by

\[
X_t = x \left( \frac{t}{\varepsilon} \right) ^\gamma \exp \left( -\left( t^{\gamma+1} - \varepsilon^{\gamma+1} \right) - \frac{\sigma^2}{2} \left( t - \varepsilon \right) + \sigma(B_t - B_\varepsilon) \right), \quad \forall t > 0.
\]

We intersted now to mean of \(X_t\). By using the conditional expectation or the Geomtric Brownian we prove that the trend of the process \(X_t\) is given by

\[
\mathbb{E}(X_t) = x \left( \frac{t}{\varepsilon} \right) ^\gamma \exp \left( \varepsilon^{\gamma+1} - t^{\gamma+1} \right),
\]

then it follows that the trend of \(X_t\) is proportional to Weibul density.
4 Maximum likelihood estimators for the parameters of SDE

A formal statement of the parameter estimation problem to be addressed is as follows. Given the general one-dimensional time-homogeneous SDE

\[ X_t = x > 0; \quad dX_t = \mu(X_t; \gamma)dt + g(X_t; \sigma)dB_t, \quad t > 0, \] (4.1)

where \( \mu(t, X_t; \gamma) = \frac{\gamma}{2}(1 - t^{\gamma+1})X_t - t^\gamma X_t \) and \( g(X_t; \sigma) = \sigma X_t \).

The task is to estimate the parameters \( \theta = (\gamma, \sigma^2) \) of this SDE from a sample of \( N + 1 \) observations \( X_1, X_2, \cdots, X_N \) of the stochastic process at known times \( t_1, \cdots, t_N \). In the statement of equation (4.1), \( dB \) is the differential of the Brownian motion and the instantaneous drift \( \mu(t, x; \gamma) \) and instantaneous diffusion \( g(x; \sigma) \). Assuming that \( \mathbb{P}(X_{t_1} = x) = p \neq 0 \), for the sake of simplicity, let us assume that \( p = 1 \).

The ML estimate of \( \theta \) is generated by minimising the negative log-likelihood function of the observed sample, namely

\[ LL(X_1, X_2, \cdots, X_n; \theta) = \log f_1(X_1|\theta) - \sum_{k=2}^{N} \log f(X_{k+1}|X_k; \theta) \] (4.2)

with respect to the parameters \( \theta = (\gamma, \sigma^2) \). In this expression, \( f_1(X_1|\theta) \) is the density of the initial state and \( f(X_{k+1}|X_k; \theta) \equiv f((X_{k+1}, t_{k+1})|(X_k, t_k); \theta) \) is the value of the transitional PDF at \((X_{k+1}, t_{k+1})\) for a process starting at \((X_k, t_k)\) and evolving to \((X_{k+1}, t_{k+1})\) in accordance with equation (4.1). Note that the Markovian property of equation (4.1) ensures that the transitional density of \( X_{k+1} \) at time \( t_{k+1} \) depends on \( X_k \) alone.

ML estimation relies on the fact that the transitional PDF, \( f(x, t) \), is the solution of the Fokker-Planck equation

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{\partial (g(x; \sigma)f)}{\partial x} - \mu(x; \gamma)f \right) \] (4.3)

satisfying a suitable initial condition and boundary conditions. Suppose, furthermore, that the state space of the problem is \([a, b]\) and the process starts at \( x = X_k \) at time \( t_k \). In the absence of measurement error, the initial condition is

\[ f(x, t_k) = \delta(x - X_k) \] (4.4)

where \( \delta \) is the Dirac delta function, and the boundary conditions required to conserve unit density within this interval are

\[ \lim_{x \to a^+} \left( \frac{1}{2} \frac{\partial (g(x; \sigma)f)}{\partial x} - \mu(x; \gamma)f \right) = 0, \quad \lim_{x \to b^-} \left( \frac{1}{2} \frac{\partial (g(x; \sigma)f)}{\partial x} - \mu(x; \gamma)f \right) = 0. \]

By using Itô formula to \( Y_t = \ln(X_t) \) we have the linear diffusion

\[ dY_t = \left( \frac{\gamma t}{2} (1 - t^{\gamma+1}) - \gamma - \frac{\sigma^2}{2} \right) dt + \sigma dB_t, \] (4.5)

By integrating between \( t_k \) and \( t_{k+1} \), the exact discret model correspond to (4.1) is given by

\[ \ln(X_{k+1}) = \ln(X_k) + \gamma \ln \left( \frac{t_{k+1}}{t_k} \right) - (t_{k+1}^{\gamma+1} - t_k^{\gamma+1}) - \frac{\sigma^2}{2} (t_{k+1} - t_k) + \sigma (B_{t_{k+1}} - B_{t_k}), \] (4.6)
From the above, the variance and esperance of $\ln(X_{k+1})$ is given by

$$
\mathbb{E}(\ln(X_{k+1})|X_k) = \ln(X_k) + \gamma \ln \left( \frac{t_{k+1}}{t_k} \right) - (t_{k+1}^\gamma - t_k^\gamma) - \frac{\sigma^2}{2}(t_{k+1} - t_k),
$$
$$
V(\ln(X_{k+1})|X_k) = \sigma^2(t_{k+1} - t_k),
$$

then the transitional probability density function (PDF) for SDE has the following closed form expression:

$$
X_{k+1}|X_k \sim \mathcal{N} \left( \ln(X_k) + \gamma \ln \left( \frac{t_{k+1}}{t_k} \right) - (t_{k+1}^\gamma - t_k^\gamma) - \frac{\sigma^2}{2}(t_{k+1} - t_k), \sigma^2(t_{k+1} - t_k) \right)
$$

(4.7)

Now, the classical approach to the ML method requires the computation of the first-order partial derivatives of the log-likelihood function with respect to each of its parameters, equating them equal to zero and then solving the resulting system of equations. So, the first-order partial derivatives are obtained as follows:

$$
\frac{\partial LL}{\partial \gamma} = \frac{1}{2\sigma^2} \sum_{k=2}^{N} (\sigma^2 + 2A_k(X_k, t_k)) \left( \ln \left( \frac{t_k}{t_{k-1}} \right) - (\gamma + 1)(t_k^\gamma - t_{k-1}^\gamma) \right) = 0,
$$

(4.8)

$$
\frac{\partial LL}{\partial \sigma^2} = -\frac{n - 1}{2\sigma^2} + \frac{1}{8\sigma^4} \sum_{k=2}^{N} (\sigma^2 + 2A_k(X_k, t_k))^2 - \frac{1}{4\sigma^2} \sum_{k=2}^{N} (\sigma^2 + 2A_k(X_k, t_k)) = 0
$$

(4.9)

with $A_k(X_k, t_k) = \ln \left( \frac{X_k}{X_{k-1}} \right) - \gamma \ln \left( \frac{t_k}{t_{k-1}} \right) + t_k^\gamma - t_{k-1}^\gamma$.

From the equation (4.9) we obtain

$$
4(N-1)\hat{\sigma}^2 - (N-1)\hat{\sigma}^4 = 4 \sum_{k=2}^{N} \hat{A}_k^2(X_k, t_k)
$$

(4.10)

By using the positivity of $\hat{\sigma}^2$ we have the following expression of estimator $\hat{\sigma}^2$

$$
\hat{\sigma}^2 = \left( 4 - \frac{4}{N-1} \sum_{k=2}^{N} \hat{A}_k^2(X_k, t_k) \right)^{1/2} - 2.
$$

(4.11)

Consequently, we prove the non-linear expression of estimator $\hat{\gamma}$ by replacing $\hat{\sigma}^2$ in (4.8):

$$
\sum_{k=2}^{N} (\hat{\sigma}^2 + 2\hat{A}_k(X_k, t_k)) \left( \ln \left( \frac{t_k}{t_{k-1}} \right) - (\hat{\gamma} + 1)(t_k^{\hat{\gamma}} - t_{k-1}^{\hat{\gamma}}) \right) = 0
$$

(4.12)

where $\hat{A}_k(X_k, t_k) = \ln \left( \frac{X_k}{X_{k-1}} \right) - \hat{\gamma} \ln \left( \frac{t_k}{t_{k-1}} \right) + t_k^{\hat{\gamma}} - t_{k-1}^{\hat{\gamma}}$.

It is assumed that the observation from the realization consists of $X_{tk}$, $t_1 = \varepsilon$, $t_k = kh$ with $h > 0$, $k = 2, 3, \ldots, N$. We define the new estimator of $\sigma^2$:

$$
\hat{\sigma}^2 = \left( 4 - \frac{4}{N-1} \sum_{k=2}^{N} \left( \ln \left( \frac{X_k}{X_{k-1}} \right) - \hat{\gamma} \ln \left( \frac{k}{k-1} \right) + h^{\hat{\gamma}+1}(k^{\hat{\gamma}+1} - (k-1)^{\hat{\gamma}+1}) \right)^2 \right)^{1/2} - 2.
$$

(4.13)

Assuming that $\gamma \approx \hat{\gamma}$. The following proposition is more or less well known.
Proposition 4.1. If condition (1) − (3) hold, then
\[
\mathbb{E} (\hat{\sigma}^2 - \sigma^2) \leq C \left( \frac{1}{N-1} + h \right).
\]

**Proof.** By using integrating the following equation between \( t_{k-1} = (k-1)h \) and \( t_k = kh \):
\[
dY_t = \left( \frac{\gamma}{t} (1 - t^{\gamma+1}) - t^{\gamma} - \frac{\sigma^2}{2} \right) dt + \sigma dB_t,
\]
(4.14)

By summing between \( k = 2 \) and \( N \) and using \( \mathbb{E}(B_{t_k} - B_{t_{k-1}})^2 = h \). Then we prove a result. \( \square \)

5 Numerical test

We present numerical results in the software R for the following SDE:
\[
X_t = 10, \quad dX_t = \left( \frac{\gamma}{t} (1 - t^{\gamma+1}) - t^{\gamma} - \frac{\sigma^2}{2} \right) dt + \sigma X_t dB_t, \quad t > 0,
\]
with: and we generate sampled data \( X_t \) with \( \gamma = 1, \sigma = 0.2 \) and time step \( \Delta = 10^{-3} \) as following:

- \( \text{mu} <- \text{expression(} \frac{(1./t)- 2*t}{x} \text{)} \)
- \( \text{g} <- \text{expression(} 0.2 \times x \text{)} \)
- \( \text{Simulation} <- \text{SNSDE(drift=} \text{mu, diffus=} \text{g, N=} 1000, \text{Dt=} 0.001, x0=10 \)\)
- \( \text{MyData} <- \text{Simulation.X} \)
- \( \text{mux} <- \text{expression(} gama \times x./t - (gama + 1) \times t^{(gama+1)} \times x \text{)} \)
- \( \text{gx} <- \text{expression(} sigma \times x \text{)} \)
- \( \text{Model} <- \text{FDE(data=} \text{MyData, drift=} \text{mux, diffus=} \text{gx, start} = \text{list(gama=} 1, \text{sigma=} 0.2), ”euler” \)\)
- \( \text{summary(Model)} \)

Result :

Pseudo maximum likelihood estimation

Method: Euler

Call:
\[
\text{FSDE(data=} \text{MyData, drift=} \text{mux, diffus=} \text{gx, start} = \text{list(gama=} 1, \text{sigma=} 1), ”euler” \)
\]

The following table prove the estimate coefficients and standard error with \(-2 \log L : 8794.337\) by unsig our methods:

| \text{Estimate coefficient} | \text{Standard error} |
|-----------------------------|------------------------|
| \( \gamma \)               | 0.9856006              | 0.006651509            |
| \( \sigma \)               | 0.2310434              | 0.005168557            |

Figure 1: Estimate coefficients and Standard Error

By using the command in the R software ” confint(Model, level= 0.9)” , we obtain the following table. It shows the confidence interval for the estimated variables \( \gamma \) and \( \sigma \).
|     | 5%       | 15%     | 75%     | 95%     |
|-----|----------|---------|---------|---------|
| $\gamma$ | 0.9711487 | 1.0000124 | 1.0003527 | 1.0012685 |
| $\sigma$ | 0.2225419 | 0.2238546 | 0.2310215 | 0.2395449 |

Figure 2: Confidence interval

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