Equiconvergence in Summation Associated with Elliptic Polynomial

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Abstract. In this paper we prove a precise equiconvergence relation between index of the Bochner-Riesz means of the expansions and power of the singularity of the distributions with compact support in summation associated with the elliptic operator.

1. Introduction
Localization principle for the Fourier series at certain point means dependence of the convergence or divergence only from the small neighbourhood of that point [1]. Equiconvergence of the Fourier series and the Fourier integral means convergence of both at the same and at the term. Same for equisummability when summation is going by regular method. Here in this paper we consider the Reisz method of summation.

Note that, in N dimensional case, when N > 1, equiconvergence and localization principles for the Fourier series and the integral is not valid by the Pringsheim convergence [2]. Equiconvergence of a spectral expansion corresponding to a schrodinger operator with summable potential, with Fourier integral is studied in [3]. A comparison theorem on equiconvergence of the Fourier Jacobi series with certain trigonometric Fourier series is proved [4]. In [5] it is showed that uniform equiconvergence of the expansions of the integrable functions in eigenfunctions of the Sturm-Liouville operator. And the more general expanded expansions of distributions is studied in [6]-[12].

In this paper we discussed about equiconvergence in summation of the Fourier series and integral of the linear continuous functional which associated with an elliptic polynomial.

2. Preliminaries
Let the space of infinitely differentiable function φ: T^N → C is defined by ε(T^N). The locally convex topology formed from the system of semi norms

P_{k,γ}(φ) = \sup_{x∈K} |D^γ \phi(x)|,

where, K is a compact subset of T^N = [−π, π]^N, γ = (γ_1, γ_2, ..., γ_N) is N dimensional vector with the non-negative integer components γ_j (j = 1, 2, ..., N). By |γ| = (γ_1 + γ_2 + ... + γ_N) we denote a
length of the multi-index $\gamma$. For instance, For instance, $D_j = D_1^{i\xi_j} \cdots D_N^{i\xi_N}$, where $\frac{\partial}{\partial \xi_j} = D_j$. Consider the

The conjugate space $e^o(T^N)$ to the local convex topological space $e(T^N)$ is the set of all distributions with the compact support in $T^N$. Any functional $f \in e^o(T^N)$ can be written as

$$f = (2\pi)^{-1} \sum_{n \in \mathbb{Z}^N} f_n \exp(inx),$$

where $\mathbb{Z}^N$ is the set of all vectors with integer components, $f_n$ is the Fourier coefficient which is defined as the value of $f$ on the test function on $f = (2\pi)^{-N/2} \exp(-inx)$ and $x \in T^N$. Consider the following elliptic polynomial:

$$A(n) = \left( \sum_{j=1}^{r} n_j \right)^2 + \left( \sum_{j=r+2}^{N} n_j \right)^2 \left( \sum_{j=1}^{r} n_j \right)^2,$$

where $n = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$. $m$ is a positive integer number, and $r = 0, 1, 2, \ldots, N-1$.

The Riesz means of order $s$ ($s$ is non-negative real number) of the Fourier series (1) is define as

$$\sigma_s^f(x) = (2\pi)^{-1} \sum_{A(n)<\lambda} \left( 1 - \frac{A(n)}{\lambda} \right)^s f_n \exp(inx).$$

Now, we extend a distribution $f$ from $N$-dimensional torus $T^N$ to the whole space $\mathbb{R}^N$ by zero. For the extended distribution use again symbol $f$. Then the Bochner-Riesz means of order $s$ of the Fourier integral of $f$ is,

$$R_f^s(x) = (2\pi)^{-1} \int_{A(\xi) < \lambda} \left( 1 - \frac{A(\xi)}{\lambda} \right)^s \hat{f}(y) \exp(iA(\xi)x) d\xi,$$

where, $\hat{f}(y) = \langle f, (2\pi)^{-N/2} \exp(-iA(\xi)x) \rangle$ is the Fourier transformation of the extended functional $f$ and it acts on, $(2\pi)^{-N/2} \exp(-iA(\xi)x)$ via $x$.

Alimov [13] has showed the sufficient condition for localization in the Liouville space $L^p_2(T^N)$ for arbitrary self-adjoint elliptic operators. He showed that for Reisz means of order $s$ of multiple Fourier series and integrals the localization is valid under the conditions,

$$l + s \geq \max \left\{ \frac{N-1}{2}, \frac{N-1}{p} \right\}, \quad 1 \leq p \leq \infty.$$

Il'in [14] is proved that this condition cannot be improved even for the Laplace operator. Actually, the sufficient localization condition for any elliptic operator is $l + s \geq \frac{N-1}{p}$ [see in 15]. Ashurov [16] is
proved that the conditions \( l + s \geq \frac{N}{p} - r \left( \frac{1}{p} - \frac{1}{2} \right) \), \( 1 \leq p \leq 2 \) are sufficient for the expansion for localization of multiple Fourier integrals.

3. Main Theorem

Let \( l \) be any real number and \( L^l_2(T^N) \) denote the Liouville space of distributions

\[
L^l_2(T^N) = \{ f \in \mathcal{E}' : \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^l f_n^2 < \infty \}.
\]

3.1. Theorem 1

Let \( l > 0 \). Then for any \( f \in L^l_2(T^N) \)

\[
\sigma^l_\lambda f(x) = R^l_\lambda f(x) + O(1) \|f\|_l,
\]

where \( \|\cdot\|_l \) is a norm in \( L^l_2(T^N) \) :

\[
\|f\|_l = (2\pi)^{N} \left( \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^l f_n^2 \right)^{1/2}.
\]

4. Estimation of the Regularized Direchlet Kernel

The regularised Dirichlet kernel \( D^l_\lambda(x) \) is the Riesz means of the partial sums of the Fourier series of the Dirac delta function can be written as:

\[
D^l_\lambda(x) = (2\pi)^{-N} \sum_{A(n) < \lambda} \left( 1 - \frac{A(n)}{\lambda} \right)^s f_n \exp(inx).
\]

From (4) it follows that for any linear continuous functional \( f \) with the compact support formula (2) can be stated as

\[
\sigma^l_\lambda f(x) = \langle f, D^l_\lambda(x-y) \rangle,
\]

where \( f \) acts to the regularized kernel \( D^l_\lambda(x-y) \) with the respect to the variable \( y \). In the same way, for the Fourier integral (3) can be written as

\[
R^l_\lambda f(x) = \langle f, \Theta^l_\lambda(x-y) \rangle,
\]

where \( \Theta^l_\lambda(x) \) is the Bochner-Riesz means of the Fourier integral of the Dirac delta function can be expressed as:

\[
\Theta^l_\lambda f(x) = (2\pi)^{-N/2} \int_{A(\xi) < \lambda} \left( 1 - \frac{A(\xi)}{\lambda} \right)^s e^{iA(\xi)x} d\xi = c \mu^N \cos \left( \mu |x| + \left( \frac{r}{2} - s \right) \frac{\pi}{2} \right) \left( \mu |x| \right)^{s+2} \mu^{s(N-1-r)/2m} \times \left( 1 + O\left( \frac{1}{\mu |x|} + O(\mu^{-1/2m}) \right) \right),
\]

\( c \) is a constant depending on \( N, r, s \), and \( \mu \) is a parameter.

\[ n \in \mathbb{Z}^N, \quad r \in \mathbb{Z}, \quad s \in [0, r], \quad \lambda > 0 \]
where \( x' \) and \( x'' \) are the vectors of the spaces \( R^{r+1} \) and \( R^{N-r-1} \) and here \( \mu = \lambda^{1/2(m+1)} \).

Let \( \hat{\Theta}_s^\prime (\xi) \) be the Fourier transformation of the Bochner- Riesz kernel (7). Then

\[
\left| \hat{\Theta}_s^\prime (\xi) \right| \leq \text{cons}(1 + |\xi|)^{N-d}, \tag{8}
\]

\[
\left| \Theta_s^\prime (x) \right| \leq \text{cons}(1 + |x|)^{N-d}, \tag{9}
\]

From the definition of the kernel \( \Theta_s^\prime (x) \), it is apparent that

\[
\hat{\Theta}_s^\prime (\xi) = \begin{cases} 
\left(1 - \frac{A(\xi)}{\lambda}\right)^s, & \text{if } A(\xi) < \lambda, \\
0, & \text{otherwise}.
\end{cases} \tag{10}
\]

The Poisson summation formula is valid if the function \( g(x) \) and its Fourier transformation \( \hat{g}(\xi) \) satisfy the inequality (8) and (9). Then

\[
\sum_{n \in \mathbb{Z}^N} g(x + 2\pi n) = (2\pi)^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^N} \hat{g}(n) \exp(\imath nx). \tag{11}
\]

Applying (11) for the function \( g(x) = \Theta_s^\prime (x) \) we obtain

\[
\sum_{n \in \mathbb{Z}^N} \Theta_s^\prime (x + 2\pi n) = (2\pi)^{\frac{1}{2}} \sum_{n \in \mathbb{Z}^N} \hat{\Theta}_s^\prime (n) \exp(\imath nx). \tag{12}
\]

Further from (7) and (12) we have

\[
D_s^\prime (x) = \sum_{n \in \mathbb{Z}^N} \Theta_s^\prime (x + 2\pi n) \tag{13}
\]

Separating the right hand side of (13) by \( n = 0 \), we obtain

\[
D_s^\prime (x) = \Theta_s^\prime (x) + \Theta_{s,0}^\prime (x), \tag{14}
\]

where the function \( \Theta_{s,0}^\prime (x) \) defined as

\[
\Theta_{s,0}^\prime (x) = \sum_{n \in \mathbb{Z}^N, n \neq 0} \Theta_s^\prime (x + 2\pi n). \tag{15}
\]

5. Proof of Theorem 1

Now, from formula (15) we can write

\[
\sigma_s^\prime f(x) - R_s^\prime f(x) = \langle f, \Theta_{s,0}^\prime (x-y) \rangle. \tag{16}
\]
Equality (16) and the bellowing lemma ensue the assertion of the Theorem 1.

**Lemma 1.** Let $l > 0$, $f \in L_2^{-l}(T^N) \cap C(T^N)$ and let $\sup f < \Omega \subset T^N$.

Then

$$< f, \Theta^*_{l,j} (x-y) > = O(1) \| f \|_{-l},$$

uniformly in any compact set $K \subset T^N \setminus \Omega$.

**Proof.** Let $\Omega_0$ is a proper sub domain of the domain $\Omega$. Then

$$< f, \Theta^*_{l,j} (x-y) > \leq \| f \|_{-l} \| \Theta^*_{l,j} (x-y) \|_{l,0},$$

(17)

where $\| \cdot \|_{l,0}$ is a norm of in the space $L_2^l(\Omega_0)$ via the variable $y \in \Omega_0$.

When $|x-y| > c$, we have [17]

$$\| \Theta^*_{l,j} (x-y) \|_{l,0} = O(A^{1/2,1/2+(N-1-1/2)-(N-1-1/2)/2m}),$$

(18)

where $\| \cdot \|_0$ is a norm in $L_2^l(\Omega_0)$.

Then Lemma 1 follows from (17) and

$$\| \Theta^*_{l,j} (x-y) \|_{l,0} = O(A^{1/2,1/2+(N-1-1/2)-(N-1-1/2)/2m}) \| \Theta^*_{l,j} (x-y) \|_{l,0}.$$

(19)

6. **Summary**

In this paper the uniform convergence of eigenfunction expansions of the distributions from the generalized Sobolev spaces. Note, that in [18] it is studied localization of the Riesz mean of the Fourier trigonometric series in summations associated with the elliptic polynomials. The results of the current paper generalized the results of the papers [19]-[21] where equisummability studied for the spherical summations. In [22] the uniform convergence of the Fourier series studied on a closed domain.

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