A LIOUVILLE’S THEOREM FOR SOME MONGE-AMPERE TYPE EQUATIONS

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Abstract. We study a Monge-Ampère type equation that interpolates the classical $\sigma_2$-Yamabe equation in conformal geometry and the 2-Hessian equation in dimension 4.

1. Introduction

In this paper, we study some Monge-Ampère type equation on $\mathbb{R}^4$. Let $u(x) \in C^2(\mathbb{R}^4)$, $\rho \in \mathbb{R}$, define the following symmetric 2 tensor

$$A(\rho, u) = -u_{ij} + \rho u_i u_j - \frac{\rho}{2} |\nabla u|^2 \delta_{ij} \in \text{Sym}^2(\mathbb{R}^4),$$

where we use $\nabla_i u = u_i$ and $\nabla_{ij} u = u_{ij}$ to denote partial derivatives of $u$ with respect to the coordinate system $\{x^i\}$ of $\mathbb{R}^4$. Note that when $\rho = 0$, $A_0$ is just the Hessian matrix of the function $u$. While $\rho = 1$, up to a multiplying factor, $A_1$ is the Schouten curvature of the metric $\exp(2u) g_E$. We define $\sigma_k(A_\rho)$ to be the degree $k$ symmetric polynomial of all eigenvalues of $A_\rho$.

In this paper, we consider fully non-linear problems

$$(1.1) \quad \sigma_2(A(\rho, u(x))) = f(u(x)), \quad x \in \Omega \subset \mathbb{R}^4,$$

and

$$(1.2) \quad \sigma_2(A(\rho, u(x))) \geq f(u(x)), \quad x \in \Omega \subset \mathbb{R}^4,$$

where $f(u) > 0$.

Let

$$\Gamma_2^+ = \{ A \in \text{Sym}^2(\mathbb{R}^4), \quad \sigma_1(A) > 0, \sigma_2(A) > 0 \},$$

$$\Gamma_2^- = \{ A \in \text{Sym}^2(\mathbb{R}^4), \quad \sigma_1(A) < 0, \sigma_2(A) > 0 \}.$$ 

We define positive and negative cones as

$$C_2^+ := \{ u \in C^2(\Omega), \quad A(\rho, u(x)) \in \Gamma_2^+, \quad x \in \Omega \},$$

$$C_2^- := \{ u \in C^2(\Omega), \quad A(\rho, u(x)) \in \Gamma_2^-, \quad x \in \Omega \}.$$ 

It is known that when $u$ satisfies (1.1) or (1.2) in a connected domain, then $u$ belongs to either $C_2^+$ or $C_2^-$. However, solutions in different cones can behave very differently.

In [21, 20], the first-named author and the third-named author have studied the $\sigma_2$-curvature in both the positive cone and the negative cone case, where special functions are chosen for the right hand side of (1.1) and (1.2). A major technique in [21, 20] is to construct monotonic “quasi-local mass” type quantities that lead to interesting geometric conclusions. In this paper, we would like to further explore this construction, state the most general results, and apply them to study solutions
Assume (1.4) holds, or super-solutions of related partial differential equations. In particular, we have established the following:

**Theorem 1.1.** Consider a bounded domain \( \Omega \subset \mathbb{R}^4 \) with almost \( C^2 \)-boundary. Assume \( u \in C^2_{+} \cap C^3(\Omega) \) satisfying differential inequality (1.3) and \( u|_{\partial \Omega} = \tau \). Define the level set \( L_t = \{ x : u(x) = t \} \) and \( \Omega_t = \{ x : u(x) > t \} \). Let \( F(t) \) be an antiderivative of \( f(t) \), and \( |S^3| \) be the area of a unit 3-sphere. Then, the quasi-local mass

\[
M(t) := -2 \left( \frac{1}{|S^3|} \int_{\Omega_t} \sigma_2(A(\rho, u(x)))dx \right) \left( \frac{1}{|S^3|} \int_{\Omega_t} \text{div}(|\nabla u|^2 \nabla u)dx \right)^\frac{1}{2}
+ \frac{\rho}{8} \left( \frac{1}{|S^3|} \int_{\Omega_t} \text{div}(|\nabla u|^2 \nabla u)dx \right)^\frac{3}{2} - \frac{12}{|S^3|} \int_{\Omega_t} \left( F(t) - F(u(x)) \right) dx
\]

is well defined for \( t \in (\tau, \max_{\Omega} u) \) and is monotonically increasing.

Suppose that \( u \in C^2_{+} \cap C^3(\Omega) \cap C^2(\Omega) \) and further satisfies the equality (1.1). Then, for almost all \( t \in (\tau, \max_{\Omega} u) \),

\[
M(t) = \frac{9 \rho}{8} \left( \frac{1}{|S^3|} \int_{L_t} |\nabla u|^3 dl \right)^\frac{4}{3} + \frac{1}{|S^3|} \int_{L_t} |\nabla u|^3 (x, \nabla u) dl
+ \frac{1}{|S^3|} \int_{L_t} H(x)|\nabla u|^2 \left( \frac{-1}{|S^3|} \int_{L_t} |\nabla u|^3 dl \right)^\frac{4}{3} - \langle x, \nabla u \rangle \right) dl,
\]

where \( H \) is the mean curvature of \( L_t \).

\( M(t) \) in (1.3) works for super-solutions satisfying (1.2) and the monotonicity is independent of \( \rho \). It covers different types of mass quantities that appeared in [21, 20] and can be applied to 2-Hessian equations. For solutions to (1.5), the expression of \( M(t) \) in (1.4) solely depends on the geometry of the level set hypersurfaces and the gradient of \( u \). It is a major improvement compared to [21, 20]. (1.3) is achieved by establishing a Pohozaev type identity which can be stated in a more general form. See the appendix for details.

An immediate application of (1.4) is the following:

**Theorem 1.2.** Let \( \Omega \) be a round ball in \( \mathbb{R}^4 \). Assume that for \( \rho \leq 0 \), \( u \in C^2_{+} \) satisfies (1.2), \( f(u) \) is integrable, and \( u|_{\Omega_1} = \tau \). Then \( u \) is rotationally symmetric in \( \Omega \).

We point out that our method extends that of Lions in [46] for the Laplacian in dimension 2 and we have not assumed the regularity of function \( f \) of (1.1). When \( f \) has higher regularity, regardless of the sign of \( \rho \), it is possible to establish the result of Theorem 1.2 using the moving plane method. The approach here does not rely on the regularity of \( f \); however, it works only for \( \rho \) being non-positive.

As a special case of (1.1), consider the following PDE:

\[
\begin{aligned}
\sigma_2(A(\rho, u)) &= \exp(\beta u) p(u), \quad x \in \mathbb{R}^4 \\
\sigma_1(A(\rho, u)) &> 0,
\end{aligned}
\]

where \( \beta > 0 \) and \( p \) is a smooth positive polynomial-like function satisfying (6.2) and (6.3). In particular, \( p \) may be chosen as a positive polynomial or simply a positive number. Noting that when \( \rho = 1, p = \frac{4}{3} \) and \( \beta = 4 \), (1.5) is the so-called \( \sigma_2 \)-Yamabe problem for the round sphere, which is well studied in conformal geometry. It is also worth pointing out that there is a natural transformation of a problem in the negative cone into a problem in the positive cone. See Section 2 for more details.
Especially, when $\rho = -1$, $p = \frac{3}{2}$ and $\beta = -4$, (1.5) is equivalent to the $\sigma_2$-Yamabe problem for the hyperbolic 4-space, which has been studied in [20]. When $\rho = 0$, (1.5) represents the 2-Hessian equation of critical degree.

We use the mass monotonicity formula (1.4) together with careful analysis of the geometry of level sets to establish a Liouville type theorem for equation (1.5). We now state our main theorem as follows:

**Theorem 1.3.** For $\rho > 0$, if $u \in C^2_+\mathbb{R}^n$ is a $C^2_+$-solution of (1.5), then $u$ is rotationally symmetric with respect to a point.

**Remark 1.4.** By rescaling, we may assume $\beta = 4$. See Section 2, Remark 2.1. Also, if $\rho \geq 2$, there is no entire solution to (1.5) on the whole space $\mathbb{R}^4$. See Corollary 7.7. When $p$ is a positive constant and $\rho \in (0,2)$, rotationally symmetric solutions of (1.5) exist. See Section 9 for details.

Our study and results are influenced by the studies of $\sigma_k$-curvature in conformal geometry and $k$-Hessian equations in the theory of non-linear differential equations, which have been developed with intertwining themes in the past few decades.

In the context of conformal geometry, $\sigma_k$-curvature was first studied by Viaclovsky [60] as a natural extension of the scalar curvature. Let $(M^n,g)$ be a Riemannian manifold and

$$A_g := \frac{1}{n-2} \left( \text{Ric}_g - \frac{R_g}{2(n-1)} \cdot g \right)$$

be its Schouten tensor, where $\text{Ric}_g$ is the Ricci curvature, and $R_g$ is the scalar curvature. $\sigma_k$-curvature, denoted as $\sigma_k(g^{-1}A_g)$, is the degree $k$ elementary symmetric polynomial of the eigenvalues of $g^{-1}A_g$. $\sigma_k$-Yamabe problems are analog to the classic Yamabe problem in which we seek a conformal metric $g_u = \exp(2u)g$ such that

$$\sigma_k(g_u^{-1}A_{g_u}) = \text{const},$$

$$\sigma_1(g_u^{-1}A_{g_u}), \ldots, \sigma_k(g_u^{-1}A_{g_u}) > 0.$$  

Equation (1.6) has significant geometric applications. See [60, 61, 62, 63, 9, 12, 27, 23, 13, 24, 21] for some incomplete references in this field. In particular, the work of Chang-Gursky-Yang [10, 11] explores properties of $\sigma_2$-curvature in closed 4-manifolds and gives a conformal characterization of 4-spheres. The majority of geometric applications in this direction require a positive cone condition. In [20], a special negative cone case is studied. In particular, the standard hyperbolic 4-space can be characterized as the sharp case of the quasi-local mass inequality.

When $n \geq 3, k \geq 1$, Liouville type theorems on $\mathbb{R}^n$ for the $\sigma_k$-Yamabe equations have been established. A powerful analytic device is the moving plane/sphere method. Let $g_u = u^{\frac{4-n}{n-2}}|dx|^2$ be a solution to (1.6). For $k = 1$, it is proved in the celebrated paper [7] by Caffarelli-Gidas-Spruck that $u$ is rotationally symmetric. See also Li-Zhang [45] for an alternative proof. Under the assumption that $u(x) = O(|x|^{4-n})$, Viaclovsky in [61] proved the Liouville theorem for $k \geq 2$. Li-Li in [35, 36] demonstrated the Liouville type theorems for a class of conformally invariant equations including $\sigma_2$-Yamabe equation (1.6).

If $k = 2$ and $n \geq 4$, Chang-Gursky-Yang [9, 12] classified the solution to (1.6) without using the moving plane/sphere method. The classification in [9,12] requires an additional integrability condition for $n \geq 6$. However, the integrability condition is not required for $n = 4,5$. 

For \( n = 2, k = 1 \), instead of (1.6), one considers the problem of prescribing constant Gaussian curvature with the equation:

\[
- \Delta u = e^u \text{ on } \mathbb{R}^2.
\]

Chen-Li in [17] proved that under the integrability assumption \( \int e^u \, dx < \infty \), the only \( C^2 \) solution to (1.7) are rotationally symmetric. In a recent paper [41], Li-Lu-Lu extended Chen-Li’s results to Möbius invariant equations in \( \mathbb{R}^2 \) without the integrability assumption similar to the case in [9, 12].

\( k \)-Hessian equations are intermediate cases between Laplacian equations and Monge-Ampère equations, with many applications in geometric analysis and other fields. Consider the positive solution to the equation

\[
\sigma_k(-\nabla^2 u) = u^p; \quad \sigma_1(-\nabla^2 u), \ldots, \sigma_k(-\nabla^2 u) > 0.
\]

Tso in [58] indicated that \( k^* = \frac{(n+2k)}{n-2k} \) for \( k < \frac{n}{2} \) is the critical exponent and proved a non-existence theorem for supercritical exponent \( p > k^* \) on star-shaped domains. The critical exponents are related to the Sobolev inequality of \( k \)-Hessian equations, which was studied by Wang in [65]. In the dimensional critical case \( n = 2k \), Tian-Wang in [57] proved a Moser-Trudinger type inequality. Notably, sharp constants of the Sobolev and Moser-Trudinger type inequalities in [65, 57] are attained for entire functions on \( \mathbb{R}^n \). See the lecture notes of Wang [67]. Compared to classical Sobolev inequalities, Liouville type theorems for \( k \)-Hessian equations are expected.

When \( k = 1 \), it is a classical result of Gidas-Spruck [22] that (1.8) only admits zero solutions when \( p < k^* \). Using the nonlinear potential theory, Phuc-Verbitsky [50] extended Gidas-Spruck’s result for \( p \leq \frac{nk}{n-2k} < k^* \). See also an alternative proof by Ou [49] using methods developed by González [24, 25]. For \( \frac{nk}{n-2k} < p \leq k^* \), the classification of positive solutions to \( \sigma_k(-\nabla^2 u) = u^p \) in \( \mathbb{R}^n \) remains open. The classification of radial solutions in the entire space can be found in [48, 68] and references therein.

For constant \( k \)-Hessian equations, Liouville type theorems (also called Bernstein type theorems in the literature) have been established in various settings. Consider

\[
\sigma_k(-\nabla^2 u) = 1; \quad \sigma_1(-\nabla^2 u), \ldots, \sigma_k(-\nabla^2 u) > 0.
\]

One may expect that (1.9) only admits solutions of quadratic functions and it has been confirmed in many cases: the classic Jörgens-Calabi-Pogorelov theorem [33, 8, 51] for Monge-Ampère equation; [1] on convex solutions of (1.9); \( n = 3, k = 2 \) with a quadratic growth condition by Warren-Yuan [70]; \( k = 2 \) by Chang-Yuan [14] and Shankar-Yuan [74] with semi-convexity assumptions; Li-Ren-Wang [38] with \( k + 1 \)-convexity and quadratic growth conditions; \( k = 2 \) by Chen-Xiang in [15] assuming a quadratic growth condition and a lower bound of \( \sigma_3(-\nabla^2 u) \). The Liouville type theorem to (1.9) in general does not hold without additional convexity or growth conditions, since there are some non-polynomial-like solutions to (1.9) demonstrated by Warren [69] and Li [57].

Though leading terms of (1.6), (1.8), and (1.9) look similar, each of these equations has unique features. In the case of \( \sigma_k \)-Yamabe equation (1.6) on \( \mathbb{R}^n \), the eigenvalues of the Schouten tensor are invariant under the Möbius group. A special element in the Möbius group is the inversion map, which reduces the study of asymptotic behavior of solutions at infinity to that of solutions with an isolated point singularity. Once an estimate near the point singularity is established, by the inversion map, we will obtain the information of solutions near infinity to
initiate the moving plane/sphere method. See [7, 34, 35, 36, 32, 29] for details. On the other hand, the M"obius group, in general, does not preserve the $k$-Hessian equations, which causes some difficulties to understand the asymptotic behavior of corresponding solutions. When $\rho \neq 1$ in equation (1.5), the M"obius invariance is also not available as the $k$-Hessian equations. It is worth noting that the affine invariance of Monge-Amp"ere equations simplifies the proof of the corresponding Liouville theorems, c.f. [6].

Another feature differing $\sigma_k$-Yamabe equations from $k$-Hessian equations is the formula of local $C^1, C^2$ estimates. Instead of dealing with the eigenvalues of $\nabla^2 u$, with the help of some lower order terms, one magically obtains certain a priori $C^1$ and $C^2$ estimates independent of the lower bound of $u$. Such estimates have been established by Chen [16] for a class of fully nonlinear equations including $\sigma_k$-Yamabe equations and our equation (1.5). See Section 6 for details. See also Guan-Wang [28], Wang [66], Li [40] for gradient estimates of equations similar to $\sigma_k$-Yamabe equations.

We briefly explain our approach to prove Theorem 1.3. Our method needs 3 crucial estimates: a total integral upper bound, a local $C^1$ and $C^2$ estimate, and a decay estimate from an $\epsilon$-regularity type argument. These estimates are well known for experts for $\sigma_2$-Yamabe equations and one of our key observations is that when $\rho$ is positive, they are applicable to equation (1.5). For the total integral bound, we follow methods from Chang-Gursky-Yang [12]; for the local $C^1$ and $C^2$ estimate, we use theorems of Chen [16] directly. Both estimates essentially depend on the positivity of $\rho$. Combining the first 2 estimates, we generalize an $\epsilon$-regularity estimate by Guan-Wang [28] for some general function $f(u)$ in the form of (1.5), which leads to the decay estimate. See Corollary 6.4 for details.

The main technical contribution of this paper is to combine the above mentioned estimates with our mass formula in Theorem 1.1 to establish the uniqueness and the symmetry of solutions to (1.5). In fact, with the above mentioned estimates, and some techniques in Li [40] and Li-Nugyen [43], we can prove that the level set after proper rescaling converges to sphere in Gromov-Hausdorff sense. We then evaluate the corresponding mass quantity for a sequence of enlarging level sets and shows that it has to be constant, which implies the rotational symmetry. Overall, our mass inequality gives a direct way to compare geometric and analytical information at infinity and at local maximal points of $u$, which leads to our theorem.

Our approach has several advantages. First, only information of a sequence of level sets near infinity is needed to obtain the desired comparison result. See details in sections 7 and 8. Second, the mass quantity (1.3) only involves boundary integrals and is invariant under rescaling. Note that we do not use the M"obius transform, which is essential in earlier works of Yamabe type equations in conformal geometry. Third, our approach is center-point free since the symmetry of solutions can be deduced from the rigidity of the isoperimetric inequality. Last, the mass quantity works for Hessian equations, which will be the focus of our future work.

Naturally, one of our next goals is to consider the dimension 4 case of the following conjecture for $k$-Hessian equations, which is well known among experts in the field:
Conjecture 1.5. For $\rho = 0$ and $n = 2k$, the equation (1.5), re-written as
\[
\begin{align*}
\sigma_k(-\nabla^2 u) &= e^{nu}, \\
\sigma_1(-\nabla^2 u), \ldots, \sigma_k(-\nabla^2 u) > 0,
\end{align*}
\]
has only rotationally symmetric solutions on $\mathbb{R}^n$.

Note that Theorem 1.3 indicates a possible approach to solve Conjecture 1.5 in dimension 4, though several of our key estimates fail when $\rho = 0$. Apart from Conjecture 1.5 when $\rho > 0$, $n = 2k > 4$, the Liouville theorem similar to Theorem 1.3 is unknown. It is interesting to see if our current approach may be extended to general dimensions. If $n > 2k$, Liouville type theorems for (1.8) are also expected but yet to be established. The $k$-Hessian equation in Conjecture 1.5 is a limiting case for either $\rho > 0$ or $\rho < 0$.

When $\rho < 0$, $A(\rho, u) \in \Gamma^+_2$ is equivalent to the negative cone condition: $A(-\rho, -u) \in \Gamma^-_2$. See Section 2 for more details. For the $\sigma_2$-Yamabe equation in dimension 4, mass monotonicity formulae have been established by Fang-Wei [21, 20] in both positive and negative cone. It is reasonable to find some parallel theories for general negative $\rho$. We also expect that the mass, defined in [20], of the asymptotically hyperbolic metric for $\sigma_2$ in 4-manifolds holds in multiply connected domain. This is indicated by the work of Shen-Wang [55], which has explored a mass type quantity for $-\Delta u = e^u$ by complex analysis in a multiply connected domain in dimension 2.

The rest of this paper is organized as follows. In Section 2, we state some preliminary facts regarding our symmetric tensor $A(\rho, u)$ and the corresponding $\sigma_k$ quantity. From now on, we assume that $n = 4$.

Remark 2.1. Direct computation shows the following
\[
A(-\rho, -u) = -A(\rho, u).
\]

Thus, all results regarding the positive cone case can be stated for the negative cone case, with a sign change of $\rho$ and vice versa. For simplicity, we only discuss the positive cone case in this paper. We state some scaling properties of $A$ here for future use:
\[
\begin{align*}
A(\rho, u(ax) + b) &= a^2 A(\rho, u), \\
A(\rho, u) &= \frac{1}{\rho} A(1, \rho u),
\end{align*}
\]
for $\rho \neq 0$.

The divergence structure of $A(\rho, u)$ is of crucial importance in our proof. It is easy to check that

$$
\sigma_2(A(\rho, u)) = \sigma_2(\nabla^2 u) + \frac{\rho}{2} \partial_i(\|\nabla u\|^2 u_i)
$$

(2.1)

$$
= -\frac{1}{2} \partial_i[(-\Delta u \delta_{ij} + u_{ij} - \rho\|\nabla u\|^2 \delta_{ij})u_j].
$$

In [43], Li-Nguyen have studied the $\sigma_k$-Yamabe problem and shown that $A(1, u(\rho(r))) \in \overline{\Gamma^+_2}$ in viscosity sense. Following [40, 43], we define the corresponding viscosity solution for (1.5):

**Definition 2.2 (Viscosity solutions).** Let $w$ be a lower semi-continuous (respectively, upper-semi-continuous) function in $\Omega$. We say that $A(\rho, w) \in \overline{\Gamma^+_2}$, (resp. $A(\rho, w) \in \text{Sym}^2(\mathbb{R}^4) \setminus \overline{\Gamma^+_2}$), in the viscosity sense if for all $x_0 \in \Omega$, $\varphi \in C^2(\Omega)$, $w - \varphi(x_0) = 0$ and $w - \varphi \geq 0$ (resp. $w - \varphi \leq 0$) near $x_0$, it holds

$$
A(\rho, \varphi)|_{x_0} \in \overline{\Gamma^+_2}, \quad (\text{resp. } A(\rho, \varphi) \in \text{Sym}^2(\mathbb{R}^4) \setminus \overline{\Gamma^+_2}).
$$

For a continuous function $w$ in $\Omega$, we say $A(\rho, w) \in \partial\overline{\Gamma^+_2}$ in the viscosity sense if $A(\rho, w) \in \overline{\Gamma^+_2} \setminus \overline{\Gamma^+_2}$. If $A(\rho, w) \in \overline{\Gamma^+_2}$, we call $w$ a viscosity super-solution to the equation $A(\rho, w) \in \partial\overline{\Gamma^+_2}$.

It is straightforward to check that a $C^2$ function satisfying relations mentioned above in the viscosity sense satisfies the corresponding relations in the classic sense. Let

$$
u(r) := \min_{|x|=r} u(x).
$$

Since super-solutions are preserved when taking minimum (see [5], chapter 2), a straightforward argument shows the following:

**Lemma 2.3.** $A(\rho, \nu(|x|)) \in \overline{\Gamma^+_2}$ in the viscosity sense.

We now state an important monotonicity formula for $\nu(r)$ following a construction of Li-Nguyen [43]:

**Theorem 2.4 ([43], Lemma 2.10).** For $r > r_0 > 0$,

$$
(2.2)
$$

$$
\frac{\nu(r) - \nu(r_0)}{\ln r - \ln r_0}
$$

is non-increasing. In other words, $\nu \circ \exp : (-\infty, \infty) \rightarrow \mathbb{R}$ is concave.

The proof of Theorem 2.4 is almost the same as that appeared in [43], Lemma 2.10. We omit it here.

### 3. Mass monotonicity formulae

In this section, we generalize some results from [21] regarding the construction of a monotonic quantity along level sets. We assume that $u \in C^+_2$ satisfies (1.2). Note that some of the notations that we use here are different from those in [21].
Remark 3.1. If \( u \in C^2_2 \), then \( \sigma_1(A(\rho,u)) = -\Delta u - \rho|\nabla u|^2 > 0 \), which, by the maximum principle, indicates that \( u \) has no interior minimums; similarly, if \( u \in C^2_2 \), then \( u \) has no interior maximums.

Let \( \Omega \) be either \( \mathbb{R}^4 \) or a bounded domain in \( \mathbb{R}^4 \), and assume that \( u \) satisfies (1.2) in \( \Omega \). When \( \Omega \) is bounded, we further assume that \( u|_{\partial \Omega} = \tau \). For \( t \in \mathbb{R} \), define the following sets

\[
L_t := \{ x \in \Omega : u(x) = t \}, \quad \Omega_t := \{ x \in \Omega : u(x) > t \}.
\]

We note that when \( \Omega = \mathbb{R}^4 \), if \( f(u) \) is integrable in \( \mathbb{R}^4 \), a similar argument of Proposition 3.6 in Guan-Wang [28] shows that \( \lim_{x \to \infty} u(x) = -\infty \). See Corollary [6.4] for details. Thus by maximum principle, without loss of generality, we make the following assumptions:

1. \( \tau' := \max_{\Omega} u > \tau \) is attained inside \( \Omega \). Define
   \[
   \mathcal{I} = \begin{cases} 
   (-\infty, \tau'), & \text{if } \Omega = \mathbb{R}^4, \\
   (\tau, \tau'), & \text{if } \Omega \text{ is bounded}.
   \end{cases}
   \]

2. \( \Omega_t \) and \( L_t \) are non-empty for \( t \in \mathcal{I} \).
3. \( \Omega_t \) and \( L_t \) are bounded sets in \( \Omega \).

From Lemma 10 in [21], we have

Lemma 3.2. Let

\[
S := \{ x \in \mathbb{R}^4 : \nabla u(x) = 0 \}.
\]

If \( \sigma_2(A(\rho,u)) \neq 0 \) in \( \Omega \), \( S \cap \Omega \) has Hausdorff dimension at most 2.

Locally, \( L_t \setminus S \) is a regular \( C^2 \) hypersurface. As \( u \) is continuous, \( \partial \Omega_t \subset L_t \) and \( L_t = \partial \Omega_t \cup (S \cap L_t) \). By Lemma 3.2, \( L_t \cap S \) has co-dimension at least 1 in \( L_t \) and

\[
0 = \mathcal{H}^3(L_t \cap S) = \mathcal{H}^3(L_t \setminus \partial \Omega_t),
\]

where \( \mathcal{H}^3 \) is the 3-dimensional Hausdorff measure. Therefore, the singular part of \( \partial \Omega_t \) has zero \( \mathcal{H}^3 \)-measure, implying \( \Omega_t \) has almost \( C^2 \) boundary.

We use \( dl \) to denote the \( \mathcal{H}^3 \)-measure on \( L_t \) and it agrees with the volume element on \( L_t \setminus S \). By (3.1), we may use \( \partial \Omega_t, L_t, L_t \setminus S \) interchangeably when taking integrals of measurable functions with respect to \( dl \).

In \( L_t \setminus S \), let \( \nu \) be the outward normal vector of \( L_t \), then by Remark 3.1

\[
(3.2) \quad \nu = -\frac{\nabla u}{|\nabla u|}.
\]

Fix a regular point \( x \in L_t \setminus S \). In a small open neighborhood \( U \) of \( x \), we define, for any \( x' \in U \)

\[
y^4(x') := -\text{sgn}(u(x') - t)\text{dist}(x', L_t).
\]

We may choose a normal coordinate \( y^1, y^2, y^3 \) on \( L_t \) near \( x \) and extend them in \( U \) so that \( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \) \((x) = \delta_{ij} \) and \( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \) \( u = \delta_{ij} \). In particular, by the choice of \( y^4 \), we have \( \frac{\partial}{\partial y^4} = \nu \). Let \( h_{\alpha\beta} \) be the second fundamental form of the level set \( L_t \) with respect to \( -\nu \). Using the Gauss-Weingarten formula in our coordinate system.
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\{y^i\}, we obtain

(3.3)

\[
A(\rho, u) = \begin{pmatrix}
    h_{\alpha\beta} |\nabla u| - \frac{\rho}{2} |\nabla u|^2 \delta_{\alpha\beta} & -\nabla_{41} u & -\nabla_{42} u & -\nabla_{43} u & -\nabla_{44} u + \frac{\rho}{2} |\nabla u|^2 \\
    -\nabla_{41} u & -\nabla_{42} u & -\nabla_{43} u & -\nabla_{44} u + \frac{\rho}{2} |\nabla u|^2 \\
\end{pmatrix},
\]

and

(3.4)

\[
-\Delta u = H |\nabla u| - u_{44}.
\]

Here we use \(\nabla_{ij} u = u_{ij}\) to denote the covariant derivatives of \(u\) with respect to our chosen coordinate system. For future use, define a symmetric \(3 \times 3\) matrix as

\[
\tilde{A}(x) := (h_{\alpha\beta} |\nabla u| - \frac{\rho}{2} |\nabla u|^2 \delta_{\alpha\beta}).
\]

With (3.3), we prove a useful lemma.

**Lemma 3.3.** If \(\rho \geq 0\) and \(\sigma_2(A(\rho, u)) > 0\), then in \(\mathbb{R}^4 \setminus S\),

(3.5)

\[
\text{div}(|\nabla u|^2 \nabla u) = 3 |\nabla u|^2 \left( u_{44} - \frac{1}{3} H |\nabla u| \right) < 0,
\]

(3.6)

\[
\sigma_2(A(\rho, u)) \leq \sigma_1(\tilde{A}) \left( \frac{1}{3} H |\nabla u| - u_{44} \right).
\]

**Proof.** Since \(u \in C^+_2\),

(3.7)

\[
\sigma_1(\tilde{A}) = H |\nabla u| - \frac{3\rho}{2} |\nabla u|^2 > 0.
\]

From (3.3) and Newton-MacLaurin inequality,

\[
\sigma_2(A(\rho, u)) = \sigma_1(\tilde{A}) (-u_{44} + \frac{\rho}{2} |\nabla u|^2) + \sigma_2(\tilde{A}) - \sum_{1 \leq i < 4} u_{i4}^2
\]

\[
\leq \sigma_1(\tilde{A}) \left( -u_{44} + \frac{\rho}{2} |\nabla u|^2 \right) + \sigma_2(\tilde{A})
\]

\[
\leq \sigma_1(\tilde{A}) \left( -u_{44} + \frac{\rho}{2} |\nabla u|^2 \right) + \frac{1}{3} \sigma_1^2(\tilde{A})
\]

\[
= \sigma_1(\tilde{A}) \left( -u_{44} + \frac{1}{3} H |\nabla u| \right).
\]

Since \(\sigma_2(A(\rho, u)) > 0\) and \(\sigma_1(\tilde{A}) > 0\), by (3.6), we have \(-u_{44} + \frac{1}{3} H |\nabla u| > 0\). Then,

\[
\text{div}(|\nabla u|^2 \nabla u) = 3 |\nabla u|^2 \left( u_{44} - \frac{1}{3} H |\nabla u| \right) < 0.
\]

\[\square\]
We choose \( F(t) \) such that \( F'(t) = f(t) \) and define the following quantities:

\[
N(t) := \frac{1}{|S^3|} \int_{\Omega_t} \sigma_2(A(x,u)) \, dx, \\
P(t) := \frac{1}{|S^3|} \left( F(t)|\Omega_t| - \int_{\Omega_t} F(u(x)) \, dx \right) = \frac{1}{|S^3|} \int_{\Omega_t} [F(t) - F(u(x))] \, dx, \\
Q(t) := - \left( \frac{1}{|S^3|} \int_{\Omega_t} \text{div}(\nabla u^2 \nabla u) \, dx \right)^{\frac{1}{2}}, \\
V(t) := \frac{1}{|S^3|} \int_{\Omega_t} dx = \frac{1}{|S^3|} |\Omega_t|.
\]

We show that these quantities are absolutely continuous with respect to \( t \). In the following, \( a.e. \) means almost everywhere with respect to the Lebesgue measure on \( I \) unless otherwise mentioned.

**Lemma 3.4.** Let \( \phi \) be a function such that \( \|\phi\|_{L^\infty(\Omega_t)} < C \) for \( t \in [t_0, t_1] \subset I \). Then,

\[
m_\phi(t) := \int_{\Omega_t} \phi \, dx
\]

is an absolutely continuous function defined on \([t_0, t_1]\). Moreover,

\[
m'_\phi(t) = - \int_{L_t} \frac{\phi}{|\nabla u|} \, d\mathcal{H}^3 \quad a.e.
\]

**Proof.** Since \( \|\phi\|_{L^\infty(\Omega_t)} < \infty \), and \( \Omega_t \) is bounded, \( m_\phi(t) \) is well defined. We first assume \( \phi \geq 0 \). By Lemma 3.2 and the co-area formula (Proposition 2.1 in [4]),

\[
(3.9) \quad m_\phi(t_0) - m_\phi(t) = \int_{\partial \Omega_t \cap u^{-1}((t_0, t))} \phi \, d\mathcal{H}^2 + \int_{t_0}^t \int_{L_r} \frac{\phi}{|\nabla u|} \, d\mathcal{H}^3 \, dr
\]

Thus, \( m_\phi \) is absolutely continuous. Since the left hand side of (3.9) is finite, we have \( \int_{L_t} \frac{\phi}{|\nabla u|} \, d\mathcal{H}^3 < \infty \) \( a.e. \), and

\[
(3.10) \quad m'_\phi(t) = - \int_{L_t} \frac{\phi}{|\nabla u|} \, d\mathcal{H}^3 \quad a.e.
\]

If \( \phi \) has negative parts, we let \( \phi^+ = \max\{\phi, 0\}, \phi^- = \max\{-\phi, 0\} \). Applying (3.9) to \( \phi^+ \) and \( \phi^- \) shows that \( m_{\phi^+}(t), m_{\phi^-}(t) \) are both finite and absolutely continuous with derivatives \( a.e. \) in the form of (3.10). We then conclude the Lemma since \( m_\phi = m_{\phi^+} - m_{\phi^-} \). \( \square \)

**Corollary 3.5.** \( N, P, Q, V \) are absolutely continuous functions in \( t \). Furthermore, \( \mathcal{H}^3(L_t) \) is positive and finite for almost all \( t \) in \( I \).

**Proof.** \( N, P, Q, V \) are all integrals of continuous functions on \( \Omega_t \). So we can apply Lemma 3.4 directly. By Lemma 3.3 and Lemma 3.2, \( \text{div}(\nabla u^2 \nabla u) < 0 \) \( \mathcal{H}^4\text{-}a.e. \) in \( \Omega \). If \( Q^3(t') = 0 \) for some \( t' \), then \( |\nabla u| \) vanishes identically on \( \Omega_t \), contradicted to Lemma 3.2. Hence, \( Q^3(t) < 0 \) in any closed interval in \( I \). Since \( Q^3(t) \) stays away from 0, \( Q(t) \) is also absolutely continuous.

The second statement is a direct application of Lemma 3.4 to \( m_{|\nabla u|}(t) = \int_{\Omega_t} |\nabla u| \, dx \).

Then, \( m'_{|\nabla u|}(t) = \int_{L_t} 1 \, d\mathcal{H}^3 < \infty \text{ a.e.} \) Hence, \( \mathcal{H}^3(L_t) < \infty \text{ a.e.} \) By De Giorgi’s
structure theorem (Theorem 15.9 in [47]) and Lemma 11, the perimeter of \( \Omega_t \), 
\[ P(\Omega_t) = H^3(L_t) \text{ a.e.} \]
Then, \( H^3(L_t) > 0 \text{ a.e.} \), and the positive of \( H^3(L_t) \) is due to
isoperimetric inequality (Theorem 14.1 in [47]). □

As all functions defined in (3.8) are absolutely continuous, we can compute their
derivatives.

**Lemma 3.6.** For almost all \( t \) in \( I \), we have
\[
Q(t) = -\left( \frac{1}{|S^3|} \int_{L_t} |\nabla u|^3 \, dl \right)^{\frac{1}{3}},
\]
\[
N(t) = \frac{1}{2|S^3|} \int_{L_t} (H|\nabla u|^2 - \rho|\nabla u|^3) \, dl,
\]
\[
N'(t) = -\frac{1}{|S^3|} \int_{L_t} \frac{\sigma_2(A(\rho,u))}{|\nabla u|} \, dl,
\]
\[
V'(t) = -\frac{1}{|S^3|} \int_{L_t} \frac{1}{|\nabla u|} \, dl,
\]
\[
P'(t) = \frac{1}{|S^3|} f(t)V(t),
\]
\[
Q'(t) = \frac{1}{3Q^2} \cdot \frac{1}{|S^3|} \int_{L_t} (H|\nabla u| - 3u_{44})|\nabla u| \, dl.
\]

Here \( H \) is the mean curvature of regular points on \( L_t \).

**Proof.** Since \( H^3(L_t) < \infty \) for almost all \( t \) in \( I \), we can apply the generalized diver-
gence theorem in \( \Omega_t \) (see [19], Section 5.8, Theorem 1). Then, (3.11) follows
from the generalized divergence theorem and (3.2). (3.12) is a direct application of
the generalized divergence theorem, (2.1), and (3.4). (3.13) - (3.16) follow from the
co-area formula and direct computation. □

We are now ready to define the monotonic quantity
\[
M(t) = 2N(t)Q(t) + \frac{Q^4(t)}{8} - 12P(t).
\]

**Remark 3.7.** The definition of \( M(t) \) is a generalization of a similar construction in
[21]. We choose a different scaling factor for convenience.

The following theorem has been proved for solutions to (1.5) with \( \rho = 1 \). For
general \( \rho \), we prove it for completeness with minor changes.

**Theorem 3.8.** Assume that on an open domain \( \Omega, u \in C^2(\Omega) \) satisfies
\[
\begin{cases}
\sigma_2(A(\rho,u)) \geq f(u) > 0, \\
A(\rho,u) \in \Gamma_+^2.
\end{cases}
\]

If \( \Omega \neq \mathbb{R}^4 \), we assume that \( \Omega \) is bounded and \( u|_{\partial \Omega} = \tau \). Then for \( t \in (\max_\Omega u, \tau) \),
\( M(t) \) is non-decreasing and absolutely continuous with respect to \( t \).

Furthermore, if \( M(t) \) is constant, then \( u \) is a radial solution.

**Proof.** \( M(t) \) is absolutely continuous by Corollary 3.5. From Lemma 3.3, it holds
on \( L_t \) \( H^3 \)-a.e. that
\[
f(u) \leq \sigma_2(A(\rho,u)) \leq \sigma_1(A)|\frac{1}{3}H|\nabla u| - u_{44}|
\]
Using the Hölder inequality and the isoperimetric inequality,
\[
\left(\frac{1}{|S^3|} \int_{L_t} f(t) |\nabla u| dl\right)^2 \cdot \left(\frac{1}{|S^3|} \int_{L_t} \sigma_1(\tilde{A}) |\nabla u| dl\right) \cdot \left(\frac{1}{|S^3|} \int_{L_t} (\frac{H}{3} |\nabla u| - u_{44}) |\nabla u| dl\right)
\]
(3.19)
\[
\geq f(t)^3 |L_t|^4 \frac{1}{|S^3|^4} \geq f(t)^3 4^3 |\Omega|^3 \frac{1}{|S^3|^3}.
\]
By Lemma 3.6,
\[
\frac{1}{|S^3|} \int_{L_t} \sigma_1(\tilde{A}) |\nabla u| dl = 2N(t) + \frac{\rho}{2} Q^3(t).
\]
We use Lemma 3.6 to rewrite (3.19) as
\[
4^3 P^3(t) \leq N^3(t)(2N(t) + \frac{\rho}{2} Q^3(t))Q'(t)Q^2
\]
which, by using the inequality of arithmetic and geometric means, leads to
\[
4P' \leq \frac{2}{3} (N'Q) + \frac{1}{3} (2N + \frac{\rho}{2} Q^3)Q' = \frac{1}{3} (2NQ + \frac{\rho}{8} Q^4)'.
\]
Checking with the definition of $M(t)$, we have proved that $M'(t) \geq 0$ a.e. $t \in \mathcal{L}$.

If $M'(t')$ is well defined for some $t'$ and $M'(t') = 0$, then all inequalities are identities in the previous computations, implying the level set $L_{t'}$ is a round 3-sphere. Thus, the solution $u$ is radial if $M(t)$ is constant for all $t$.

Remark 3.9. By rescaling the section 2 Remark 2.1 and Theorem 3.8, it is easy to recover different types of mass inequalities that appeared in [21, 20]. Rather than the solutions of (1.1), our construction in (1.3) works for super-solutions satisfying (1.4). Furthermore, the monotonicity of $M(t)$ is independent of $\rho$ and it can be applied to 2-Hessian equations.

Finally, we remark that similar to the $\sigma_k$-Yamabe problem in conformal geometry, a Pohozaev type identity regarding our problem can be established. See Theorem A.5. We include a proof in the Appendix.

We apply Theorem A.5 to revise $M(t)$, making it dependent only on quantities on $L_t$:

**Corollary 3.10.** Let $u$ be a solution to
\[
\begin{cases}
\sigma_2(A(\rho, u)) = f(u), \\
A(\rho, u) \in \Gamma_2^+.
\end{cases}
\]
Assume that $\partial \Omega_t$ is smooth and $u \in C^2(\Omega_t)$; or $\partial \Omega_t$ is almost $C^2$ and $u \in C^3(\Omega_t) \cap C^2(\Omega_t)$. Then for almost all $t$, we have
\[
\int_{\Omega_t} 8[F(u(x)) - F(t)] dx = \int_{\partial \Omega_t} \left(-\frac{3}{4} |\nabla u|^4(x, \nu) + \frac{2}{3} H |\nabla u|^3(x, \nu)\right) dl.
\]
(3.21)
\[
M(t) = \frac{9\rho}{8} \left(Q^4(t) + \frac{1}{|S^3|} \int_{L_t} |\nabla u|^3(x, \nabla u) dl\right)
\]
\[
+ \frac{1}{|S^3|} \int_{L_t} H(x) |\nabla u|^2(Q(t) - \langle x, \nabla u \rangle) dl.
\]
Notice that all terms appeared in $M(t)$ can be expressed as some integrals on $L_t$. 

4. Dirichlet Problem for Balls

In this section, we prove the symmetry of the solutions to the following Dirichlet problem \((4.1)\) by the mass formula and the Pohozaev identity.

By Remark 2.1, for simplicity, we may assume that \(\Omega = \{ |x| < 1 \} \subset \mathbb{R}^4\) and \(u\) is a \(C^2(\Omega)\) solution to
\[
\begin{aligned}
\sigma_2(A(\rho, u)) &= f(u) > 0, \quad \Omega, \\
A(\rho, u) &\in \Gamma_2^+, \\
u = \tau, \quad \partial\Omega.
\end{aligned}
\]

By Theorem A.5 and Theorem 3.8, we prove the following theorem.

**Theorem 4.1.** If \(\rho \leq 0\), then every \(C^2\) solution \(u\) to equation \((4.1)\) with integrable \(f(u)\) is radial.

**Proof.** Let \(\tau' = \max_{\Omega} u\). By the maximum principle, it is clear that \(\tau' \geq \tau\), and \(M(\tau') = 0\).

We compute \(M(\tau)\). Note that on \(\partial\Omega = S^3\), \(H = 3\) and \(\nu = x\). By \((3.21)\) and \((3.17)\)
\[
M(\tau) = \frac{\rho}{8}Q^4 + Q \frac{1}{|S^3|} \int_{L(t)} (3|\nabla u|^2 - \rho|\nabla u|^3) - \frac{3}{2} \int_{S^3} \left( \frac{3}{4} \rho|\nabla u|^4 - 2|\nabla u|^3 \right) dl \\
= \frac{9\rho}{8}Q^4 + \frac{3}{|S^3|} \int_{L(t)} |\nabla u|^2 \right) Q - \frac{1}{|S^3|} \int_{\partial\Omega} \left( \frac{9\rho}{8} |\nabla u|^4 - 3|\nabla u|^3 \right) dl \\
= \frac{9\rho}{8} \left[ Q^4 - \frac{1}{|S^3|} \int_{\partial\Omega} |\nabla u|^4 \right] + 3Q \left[ \frac{1}{|S^3|} \int_{\partial\Omega} |\nabla u|^2 dl - Q^2 \right].
\]

It is easy to see that \(Q(\tau) \leq 0\),
\[
Q^4(\tau) = \left( \frac{1}{|S^3|} \int_{\partial\Omega} |\nabla u|^3 dl \right)^{\frac{4}{3}} \leq \frac{1}{|S^3|} \int_{\partial\Omega} |\nabla u|^4 dl,
\]
and
\[
\frac{1}{|S^3|} \int_{\partial\Omega} |\nabla u|^2 dl \leq \left( \frac{1}{|S^3|} \int_{\partial\Omega} |\nabla u|^3 \right)^{\frac{4}{3}}.
\]

We conclude that \(M(\tau) \geq 0 = M(\tau')\). By Theorem 3.8, we obtain \(M(t) \equiv 0\) for all \(t \in [\tau, \tau']\) and hence \(u\) is rotationally symmetric. \(\square\)

**Remark 4.2.** By Remark 2.1, same results hold for \(A(\rho, u) \in \Gamma_2^-\) and \(\rho \geq 0\).

We point out here that on annular domains, under some regularity assumptions of \(f\), Wang-Bao in [64] proved the radial symmetric property of solutions to \(k\)-Hessian equation by the moving plane method.

5. Bounded Integral of \(\sigma_2(A(\rho, u))\)

In this section, we establish an upper bound of \(\int_{\mathbb{R}^4} \sigma_2(A(\rho, u)) dx\), assuming \(A(\rho, u) \in \Gamma_2^+\). The argument is modified from a similar one of Chang-Gurksy-Yang [12]. We note that the positivity of \(\rho\) here is essential. The main result of this section is the following
**Theorem 5.1.** If \( A(\rho, u) \in \Gamma_+^2 \), then
\[
\int_{\mathbb{R}^4} \sigma_2(A(\rho, u)) \, dx < \frac{C}{\rho^2} < \infty,
\]
for some constant \( C \).

The proof uses the divergence structure of \( \sigma_2(A(\rho, u)) \) to perform integration by parts with a cut-off function. Let \( \eta \) to be a smooth function such that
\[
\eta(x) = \begin{cases} 
1 & x \in B_R, \\
0 & x \in \mathbb{R}^4 \setminus B_{2R},
\end{cases}
\]

and
\[
|\nabla \eta|^2 + |\nabla^2 \eta| < \frac{C_0}{R^2},
\]
for some fixed constant \( C_0 \). We first state two lemmas.

**Lemma 5.2.** It holds
\[
2 \int_{\mathbb{R}^4} \sigma_2(A(\rho, u)) \eta^4 \, dx = \int_{\mathbb{R}^4} \left( (\nabla^2 \eta^4) u_k u_j - |\nabla u|^2 \Delta \eta^4 \right) \, dx \\
+ \int_{\mathbb{R}^4} \rho \eta^4 \text{div}(|\nabla u|^2 \nabla u) \, dx.
\]

**Proof.** Recall the divergence structure of \( \sigma_2(A(\rho, u)) \),
\[
\sigma_2(A(\rho, u)) = -\frac{1}{2} \partial_j \left( (-\Delta u \delta_{ij} + u_{ij}) u_i - \rho |\nabla u|^2 u_j \right).
\]

We compute
\[
- \int_{\mathbb{R}^4} \partial_j \left( (-\Delta u \delta_{ij} + u_{ij}) u_i \right) \eta^4 \, dx = \int_{\mathbb{R}^4} (\nabla^2 \eta^4) u_k u_j \, dx.
\]

Note
\[
\int_{\mathbb{R}^4} \nabla_j \eta^4 u_k u_j \, dx = -\int_{\mathbb{R}^4} (\nabla^2 \eta^4 u_k u_j + \nabla_j \eta^4 u_k u_j) \, dx \\
= -\int_{\mathbb{R}^4} \nabla^2 \eta^4 u_k u_j \, dx + \int_{\mathbb{R}^4} \Delta \eta^4 u_k u_k \, dx + \int_{\mathbb{R}^4} \nabla_j \eta^4 u_k u_j \, dx.
\]
Hence,
\[
\int_{\mathbb{R}^4} \nabla_j \eta^4 (u_{ij} u_i - u_{kk} u_j) \, dx = \int_{\mathbb{R}^4} \nabla^2 \eta^4 u_k u_j \, dx - \int_{\mathbb{R}^4} \Delta \eta^4 u_k u_k \, dx.
\]
Thus, by (5.2-5.4),
\[
2 \int_{\mathbb{R}^4} \eta^4 \sigma_2(A(\rho, u)) = \int_{\mathbb{R}^4} (\nabla^2 \eta^4 u_k u_j - |\nabla u|^2 \Delta \eta^4) \, dx \\
+ \int_{\mathbb{R}^4} \rho \eta^4 \text{div}(|\nabla u|^2 \nabla u) \, dx.
\]

**Lemma 5.3.** Notations as above. We have
\[
\int_{\mathbb{R}^4} |\nabla^2 \eta^4 u_k u_j - |\nabla u|^2 \Delta \eta^4| \, dx \leq \frac{C_1}{\rho^2},
\]
where \( C_1 \) is a constant independent of \( R \) and \( u \).
Proof. Notice that, since $A(\rho, u) \in \Gamma^+_2$, 

$$0 < \sigma_1(A(\rho, u)) = -\Delta u - \rho|\nabla u|^2.$$ 

It implies

$$\int_{\mathbb{R}^4} \rho|\nabla u|^2 \eta^2 \, dx \leq -\int_{\mathbb{R}^4} (\Delta u) \eta^2 \, dx + \int_{\mathbb{R}^4} \nabla u \cdot \nabla \eta^2 \, dx \leq 2 \int_{\mathbb{R}^4} |\nabla u| \cdot |\nabla \eta| \eta \, dx \leq \frac{\rho}{2} \int_{\mathbb{R}^4} |\nabla u|^2 \eta^2 \, dx + \frac{2}{\rho} \int_{\mathbb{R}^4} |\nabla \eta|^2 \, dx.$$ 

Hence,

$$\int_{\mathbb{R}^4} |\nabla u|^2 \eta^2 \, dx \leq \frac{4}{\rho^2} \int_{\mathbb{R}^4} |\nabla \eta|^2 \, dx < \frac{4C_2}{\rho^2} R^2.$$ 

By (5.1) and (5.5),

$$\int_{\mathbb{R}^4} |\nabla^2 \eta^4| u_k u_j \, dx \leq C_3 \int_{B_{2R}(0)} (|\nabla^2 \eta| + |\nabla \eta|^2) |\nabla u|^2 \eta^2 \, dx \leq C_4 \frac{1}{R^2} \int_{\mathbb{R}^4} \eta^2 |\nabla u|^2 \, dx \leq \frac{C_5}{\rho^2}.$$ 

Similarly,

$$\int_{\mathbb{R}^4} |\Delta \eta^4| \cdot |\nabla u|^2 \eta \, dx \leq \frac{4C_6}{\rho^2}.$$ 

Hence we have proved this lemma. \[\square\]

Combining Lemmas 5.2 and 5.3 we prove Theorem 5.1.

**Proof of Theorem 5.1.** By Lemma 3.3, div($|\nabla u|^2 \nabla u$) < 0. From Lemma 5.2, since $\rho > 0$,

$$2 \int_{\mathbb{R}^4} \sigma_2(A(\rho, u)) \eta^4 \, dx = \int_{\mathbb{R}^4} \left( (\nabla^2 \eta^4) u_k u_j - |\nabla u|^2 \Delta \eta^4 \right) \, dx + \int_{\mathbb{R}^4} \rho \eta^4 \text{div}(|\nabla u|^2 \nabla u) \, dx \leq \int_{\mathbb{R}^4} \left( (\nabla^2 \eta^4) u_k u_j - |\nabla u|^2 \Delta \eta^4 \right) \, dx.$$ 

By Lemma 5.3,

$$2 \int_{\mathbb{R}^4} \sigma_2(A(\rho, u)) \eta^4 \, dx \leq \frac{C_1}{\rho^2}. \quad (5.6)$$ 

As the right-hand side of (5.6) is independent of $R$, let $R \to \infty$, it holds

$$\int_{\mathbb{R}^4} \sigma_2(A(\rho, u)) \, dx < \frac{C_1}{2\rho^2} < \infty.$$ 

We have finished the proof. \[\square\]
6. Local estimates

In this section, following the work of Chen [16] and Guan-Wang [28], we establish local estimates for the equation (6.1).

Let \( \Omega \subset \mathbb{R}^4 \) be an open domain. Consider the equation in \( \Omega \):

\[
\begin{cases}
\sigma_2(A(\rho, u)) = f(u) = \exp(4u)p(u), \\
\sigma_1(A(\rho, u)) > 0.
\end{cases}
\]

Here \( p : \mathbb{R} \to \mathbb{R} \) is a smooth positive polynomial-like function such that for some \( m \geq 0 \), \( C_p > 0 \), and any \( a \in \mathbb{R} \), it holds

\[
p(a) + |p'(a)| (1 + |a|) + |p''(a)| (1 + |a|)^2 \leq C_p (1 + |a|)^{4m},
\]

and

\[
p(a) > C_p^{-1} (1 + |a|)^{4m}.
\]

In particular, \( p \) can be a positive constant, or any positive polynomial.

We note that comparing to (1.5), we have fixed the positive constant \( \beta \) as 4 in (6.1). This does not however reduce the generality due to Remark 2.1.

First, we state the following local \( C^1 \) and \( C^2 \) estimate, which is a key ingredient for (6.1).

**Theorem 6.1** ([16], Theorem 1.1). Let \( u \) be a \( C^4 \)-solution to (6.1) for \( \rho > 0 \) in \( B_{2R}(0) \). Assume that \( p \) satisfies (6.2). Then, we have for \( x \in B_R(0) \),

\[
\rho |\nabla^2 u| + \rho^2 |\nabla u|^2 \leq C \left( \frac{1}{|R|^2} + c_{\sup}(f) \right),
\]

for some universal constant \( C \), and

\[
c_{\sup}(f) = \sup_{x \in B_R} \left( |f(u)|^\frac{1}{\delta} + |f(u)|^\frac{1}{2} + |f(u)|^\frac{1}{2} + |f(u)|^{\frac{1}{2}} \right).
\]

**Remark 6.2.** If we start with a \( C^2 \) solution \( u \) of (6.1) in \( B_R(0) \), by our assumption of \( f(u) \) in (6.1), and the fact that \( \sigma_2^\frac{1}{2} \) is concave in \( \Gamma_2^+ \), we may apply Evans-Krylov theorem and linear elliptic theory to conclude that \( u \in C^4(B_{R/2}(0)) \). See [5], chapter 8 and 9. In particular, by Chen’s estimate (6.4), all higher order estimates only depend on \( R, \sup |f(u)|^\frac{1}{2} \), and bounds of derivatives of \( f^\frac{1}{2} \). Without no confusion, the solutions in the following sections are at least \( C^4 \) smooth.

Second, we establish the following improved estimate, which is a modification of Proposition 3.6 in [28] to our settings, also c.f [30] [44]. From now on, we fix a \( \rho > 0 \).

**Theorem 6.3.** Assume that \( p \) satisfies (6.2) and (6.3). Let \( u \) be a solution to (6.1) for \( \rho > 0 \) in \( B_R(0) \) for some \( R > 1 \). For any \( \delta > 0 \), there exists an \( \epsilon_0 = \epsilon_0(\delta, p, m, \rho) > 0 \) independent of \( R \) such that if

\[
\int_{B_R(0)} f(u) dx < \epsilon_0,
\]

then

\[
\sup_{B_{\frac{1}{4}R}(0)} \frac{1}{4} \ln f(u) + \ln R < \ln \delta.
\]
Using Theorems 6.1 and 6.3 we obtain the key estimates:

**Corollary 6.4.** Let \( u \) be a solution to (1.5) for \( \rho > 0 \). Assume that \( p \) satisfies (6.2) and (6.3). Then,

\[
\lim_{x \to \infty} u(x) + \ln |x| = -\infty.
\]

For \( |x| > 0 \), there is a universal constant \( c \) such that

\[
|\nabla u| \leq \frac{c}{\rho |x|}, \quad |\nabla^2 u| \leq \frac{c}{\rho |x|^2}.
\]

**Proof.** Fix \( \delta > 0 \) and \( \epsilon = \epsilon_0(\delta) \) in Theorem 6.3. By Theorem 5.1, a solution to (1.5) satisfies \( \int_{B_R} f(u) < \infty \). For \( \epsilon \), we can find a big \( \tilde{R} \) such that \( \int_{B_{\tilde{R}}(0)} f(u) dx < \epsilon \).

Then, for any \( x \in \mathbb{R}^4 \setminus B_{\tilde{R}}(0) \), we have \( B_1(x) \subset \mathbb{R}^4 \setminus B_{\tilde{R}}(0) \), and thereby

\[
\sup_{B_1(x)} \frac{1}{4} \ln f(u) < -\ln \frac{|x|}{2} + \ln \delta,
\]

which implies (6.6). By (6.4),

\[
\rho |\nabla^2 u| + \rho^2 |\nabla u|^2 \leq \frac{c}{|x|^2} + \frac{4c}{|x|^2},
\]

implying (6.7) for \( |x| > 2\tilde{R} \). We obtain (6.7) in \( B_{2\tilde{R}}(0) \) with a proper choice of constant \( c \). \( \square \)

In the rest of this section, we prove Theorem 6.3. We begin with a technical lemma.

**Lemma 6.5.** Let \( p : \mathbb{R} \to \mathbb{R}^+ \) be a smooth positive function.

1. If \( p \) satisfies (6.2) and for some \( b \)

\[
e^{4a} p(a + b) \geq c_0 > 0,
\]

then \( a > C_0(c_0, p, m) \) independent of \( b \).

2. If \( p \) satisfies (6.3) and for some \( b \)

\[
e^{4a} p(a + b) \leq c_1,
\]

then \( a < C_1(c_1, p, m) \) independent of \( b \).

**Proof.** If \( p \) satisfies (6.2) and (6.8),

\[
c_0 < \frac{e^{4a} p(a + b)}{(1 + |b|)^{4m}} < \frac{e^{4a} C_p (1 + |a + b|)^{4m}}{(1 + |b|)^{4m}} \leq e^{4a} C_2 (1 + |a|)^{4m},
\]

for some positive \( C_2 = C_2(p, m) \). Thus,

\[
e^{4a} (1 + |a|)^{4m} \geq \frac{c_0}{C_2}.
\]

Hence, \( a \) is bounded from below by some properly chosen constant \( C_0(c_0, p, m) \).
Next, we prove the second statement. Suppose that \( p \) satisfies (6.3) and (6.9). If \( a < 0 \), the statement already holds. We may assume that \( a > 0 \). Note by (6.3),

\[
C_p^{-1} e^{4a} \frac{1}{(1 + |b|)^{4m}} < e^{4a} p(a + b) \frac{(a + b)}{(1 + |b|)^{4m}} < c_1.
\]

Thus,

\[
(6.10) \quad a < \frac{1}{4} \ln (c_1 C_p) + m \ln(1 + |b|).
\]

Pick a positive constant \( C_3 = C_3(c, p, m) \) so that

\[
\frac{1}{2} |b| > \frac{1}{4} \ln (c_1 C_p) + m \ln(1 + |b|)
\]

whenever \( |b| > C_3 \). We discuss the following 3 cases.

If \( |b| \leq C_3 \), then by (6.10),

\[
a \leq \frac{1}{4} \ln (c_1 C_p) + m \ln(1 + |b|).
\]

If \( b > 0 \), by (6.3) and the fact that \( a > 0 \),

\[
C_p^{-1} e^{4a} < e^{4a} \frac{(1 + a + b)}{(1 + |b|)^{4m}} \leq c_1.
\]

Thus, we get \( a < \frac{1}{4} \ln (c_1 C_p) \).

If \( b < -C_3 \), then

\[
-b - a \geq |b| - \frac{1}{4} \ln (c_1 C_p) - m \ln(1 + |b|) > \frac{|b|}{2} > 0.
\]

Thus, by (6.3),

\[
C_p^{-1} \left( \frac{1}{2} \right)^{4m} e^{4a} \leq e^{4a} C_p^{-1} \left( \frac{1 + |b|}{2} \right)^{4m} < e^{4a} p(a + b) < c_1.
\]

Hence,

\[
a < m \ln 2 + \frac{1}{4} \ln (c_1 C_p).
\]

Combining all 3 cases, we have finished the proof. \( \square \)

Finally, we are ready to prove Theorem 6.3.

**Proof.** (of Theorem 6.3) Let \( \delta > 0 \). Suppose that

\[
\sup_{B_{\frac{R}{2}}(0)} \left( \ln f(u) + 4 \ln R \right) > 4 \ln \delta.
\]

We only need to show that

\[
(6.12) \quad \int_{B_R(0)} f(u) dx > \epsilon_0,
\]

for some \( \epsilon_0 = \epsilon_0(\delta, p, m, \rho) \). Define

\[
(6.13) \quad \phi : \lambda \mapsto \left( \frac{3}{4} - \lambda \right)^4 \sup_{B_{3R/4}(0)} f(u)
\]

for \( \lambda \in (0, \frac{3}{4}) \). Then, there exist \( z \in B_{3R/4}(0) \) and \( \lambda_0 \in (0, 3/4) \) such that

\[
(6.14) \quad \phi(\lambda_0) = \sup_{\lambda_0 \in (0, \frac{3}{4})} \phi(\lambda_0), \quad f(u(z)) = \sup_{x \in B_{\lambda_0 R}(0)} f(u(x)).
\]
Pick \( s = \frac{1}{2} \left( \frac{3}{4} - \lambda_0 \right) \). We have \( B_{sR}(z) \subset B_R(0) \) and

\[
\sup_{B_{sR}(z)} f(u) \leq \sup_{B_{(\lambda_0 + s)R}(0)} f(u).
\]

Denote

\[
\mu := f(u(z))^{\frac{1}{4}}.
\]

Since

\[
\left( \frac{3}{4} - s - \lambda_0 \right)^4 \sup_{B_{(\lambda_0 + s)R}(0)} f(u) \leq \left( \frac{3}{4} - \lambda_0 \right)^4 \sup_{B_{\lambda_0 R}(0)} f(u) = \left( \frac{3}{4} - \lambda_0 \right)^4 \mu^4,
\]

(6.15) \[
\sup_{B_{sR}(z)} f(u) \leq \sup_{B_{(\lambda_0 + s)R}(0)} f(u) \leq 2^4 \mu^4.
\]

Define the rescaled function \( v \) as

\[
v(x) := u(z + \mu^{-1} x) - \ln \mu',
\]

where \( \mu' \) satisfies

\[
\mu'(1 + |\ln \mu'|)^m = \mu.
\]

Let \( \tilde{p}(v) := \frac{p(v + \ln \mu')}{(1 + |\ln \mu'|)^m} \). Then, \( v \) satisfies

\[
\sigma_2(A(\rho, v)) = \frac{f(v + \ln \mu')}{(1 + |\ln \mu'|)^{4m}} = e^{4v} \tilde{p}(v).
\]

From (6.11), we have

\[
\phi \left( \frac{1}{2} \right) R^4 = \left( \frac{1}{4} \right)^4 \sup_{B_{\frac{1}{2}R}(0)} e^{\ln f(u) + 4 \ln R} > (\frac{1}{4})^4 \delta^4.
\]

Then, by (6.13) and (6.14),

\[
s^4 \mu^4 R^4 = \left( \frac{3}{4} - \lambda_0 \right)^4 f(u(z)) R^4
\]

\[
\geq 2^{-4} \phi(\lambda_0) R^4
\]

\[
\geq 2^{-4} \phi \left( \frac{1}{2} \right) R^4
\]

\[
> 2^{-4} 4^{-4} \delta^4.
\]

Hence, \( B_{s\mu R}(0) \) contains a fixed ball \( B_{R'}(0) \), if we denote \( R' := 8^{-1} \delta \).

We claim that \( v \) is uniformly bounded in \( B_{R'}(0) \). Since

\[
1 = e^{4v(0)} \tilde{p}(v(0)) = e^{4v(0)} \frac{p(v(0) + \ln \mu')}{(1 + |\ln \mu'|)^{4m}},
\]

we have \( v(0) > C_0(1, p, m) \) by Lemma 6.5. By (6.13),

\[
\sup_{x \in B_{\mu R}(0)} \frac{e^{4v} p(v + \ln \mu')}{(1 + |\ln \mu'|)^{4m}} = \sup_{x \in B_{sR}(z)} \frac{f(u)}{\mu^4} \leq 2^4,
\]

(6.15) \[
\sup_{B_{sR}(z)} f(u) \leq \sup_{B_{(\lambda_0 + s)R}(0)} f(u) \leq 2^4 \mu^4.
\]
and by Lemma 6.5, we have \( \sup_{B_{R}(0)} v < C_{1}(2^{4}, p, m) \). Since \( \tilde{p}(v) \) also satisfies growth condition (6.2), and \( v \) is bounded from above by \( C_{1}(2^{4}, p, m) \), it holds that
\[
\begin{align*}
\csc_{\sup}(e^{2v}p^{2}(v)) & \leq C_{4}e^{2v}\left(\frac{1}{R^{2}} + c_{\sup}(e^{2v}p^{2}(v))\right) \\
& \leq C_{5}e^{2v}(1 + |v|^{2m}) \\
& < C_{6},
\end{align*}
\]
where \( C_{4}, C_{5}, C_{6} \) are constants only depending on \( p \) and \( m \).

From (6.16) and Theorem 6.1,
\[
\sup_{B_{R}(0)} |\nabla v|^{2} < C_{7},
\]
for some constant \( C_{7} = C_{7}(R', C_{6}, p, m, \rho) = C_{7}(p, m, \rho) \).

However, if \( v \) is uniformly bounded in \( B_{R'}(0) \), then
\[
\int_{B_{R'}(0)} e^{4\tilde{p}(v)}dx > C_{8}(\delta, p, m, \rho) > 0.
\]

Let \( \epsilon_{0} = C_{8} \) and we obtain (6.12). Thus, we have finished the proof. \( \square \)

### 7. Asymptotic Behavior

In this section, we use the existing estimate of Corollary 6.4 to describe the asymptotic behavior of our solution \( u \) near infinity. As a consequence, we establish the crucial fact that the shape of level sets of \( u \), after some proper re-scaling, is asymptotically spherical. The main result in this section is Theorem 7.9.

First, we set some notations for our analysis. Throughout this section, \( u \) is a solution of (1.5) with \( \beta = 4 \) and \( \rho > 0 \).

**Definition 7.1.** We define
\[
\begin{align*}
\bar{u}(r) := & \max_{|x|=r} u(x), \quad \underline{u}(r) := \min_{|x|=r} u(x); \\
\bar{\sigma}(t) := & \max\{|x| : u(x) = t\}, \quad \underline{\sigma}(t) := \min\{|x| : u(x) = t\}.
\end{align*}
\]

Second, we list some simple facts.

**Lemma 7.2.** For some constant \( C = C(\rho) \), we have
\[
\bar{u}(r) - u(r) \leq C.
\]

**Proof.** This is a direct consequence of the gradient estimate (6.14). \( \square \)

**Lemma 7.3.** \( \bar{u}(r) \) is non-increasing.

**Proof.** Since \( u \in C_{2}^{+} \),
\[
\sigma_{1}(A(\rho, u)) = -\Delta u - \rho|\nabla u|^{2} > 0.
\]
We verify the claim by the maximum principle. \( \square \)

**Lemma 7.4.** We have
\[
\begin{align*}
\bar{u}(\bar{\sigma}(t)) = t, \quad \bar{\sigma}(\bar{\sigma}(t)) = t, \quad \bar{u}(\bar{\sigma}(r)) = r, \quad \bar{\sigma}(\bar{\sigma}(r)) = r.
\end{align*}
\]
Proof. For a given \( t \), there exists \( x_0 \) such that \( |x_0| = r(t) \) and \( u(x_0) = t \). Then,
\[
    u(r(t)) \leq u(x_0) = t.
\]
By the maximum principle,
\[
    r(t) = \min \{|x| : u(x) = t\} = \min \{|x| : u(x) \leq t\}.
\]
Hence, \( u \geq t \) in \( B_{r(t)}(0) \). Therefore, \( u(r(t)) = t \).

The proof for other identities (7.2) is similar so we omit it here. \( \square \)

Third, we consider the growth rate of \( u(r) \) and \( u(x) \).

**Lemma 7.5.** Let \( s = \log r \). Denote \( \frac{d}{ds}(u(e^s))^+ \) to be the right derivative with respect to \( s \). Then,
\[
    \lim_{s \to \infty} \frac{d}{ds}(u(e^s))^+ \geq -\frac{2}{\rho}.
\]

**Proof.** By Lemma 7.3 and Theorem 2.4, \( u \) is non-increasing and concave as a function of \( s \). Thus, the right derivative of \( u(e^s) \) exists and is monotonic. By the classic Alexandrov’s theorem, the first and second derivatives \( u_s, u_{ss} \) exist almost everywhere and
\[
    u_{ss} \leq 0 \quad a.e.
\]
In particular, the limit appeared in (7.3) exists.

By Definition 2.2, for almost all \( s \),
\[
    3u_s (u_s + \frac{\rho}{2} u_{ss}^2) > 0.
\]
We combine (7.4) and (7.3) to prove (7.3). \( \square \)

**Lemma 7.6.** If \( u \) is a solution to (1.5), then there exists \( \alpha \in [-\frac{2}{\rho}, -1) \) such that
\[
    \lim_{|x| \to \infty} \frac{u(x)}{\ln |x|} = \alpha.
\]

**Proof.** Define
\[
    \alpha := \lim_{s \to \infty} \frac{u(e^s)}{s}.
\]
By the concavity of \( u(e^s) \) and Lemma 7.5, \( \alpha \) is well defined and is bounded below by \( -\frac{2}{\rho} \). By Corollary 6.4, \( \alpha \leq -1 \). We claim that \( \alpha < -1 \).

If not, then we assume that \( \alpha = -1 \). We use Theorem 2.4 to argue that for \( s_0 < s_1 \),
\[
    \frac{u(e^{s_1}) - u(e^{s_0})}{s_1 - s_0} \geq \lim_{s \to \infty} \frac{u(e^s) - u(e^{s_0})}{s - s_0} = \alpha,
\]
\[
    0 \leq \frac{u(e^{s_1}) - u(e^{s_0})}{s_1 - s_0} - \alpha = \frac{u(e^{s_1}) - \alpha s_1 - (u(e^{s_0}) - \alpha s_0)}{s_1 - s_0}.
\]
Hence, \( u(e^s) - \alpha s \) is increasing. For \( s > 0 \),
\[
    u(e^s) - \alpha s \geq u(1) - 0 > -\infty,
\]
which contradicts with (6.6) since \( \alpha = -1 \). We have thus proved that \( \alpha < -1 \).

In addition, by (7.1),
\[
    \lim_{r \to \infty} \frac{\bar{u}(r)}{\ln r} = \lim_{r \to \infty} \frac{u(r)}{\ln r} = \alpha.
\]
Since \( u(|x|) \leq u(x) \leq \bar{u}(|x|) \), we have thus proved the lemma.

An immediate consequence of Lemma 7.6 and Lemma 7.6 is the following non-existence result for \( \rho \geq 2 \).

**Corollary 7.7.** If \( \rho \geq 2 \), then there is no solution to (1.6).

Fourth, we use Lemma 7.6 to control the shape of level sets of \( u \). The next proposition shows that \( \bar{r} \) and \( \bar{u} \) cannot grow disproportionately as \( t \to -\infty \).

**Proposition 7.8.** There exists some uniform constant \( C = C(\rho) \) such that

\[
\limsup_{t \to -\infty} \frac{\bar{r}(t)}{\bar{u}(t)} < C.
\]

**Proof.** For any \( t \), define \( t' = \bar{u}(\bar{r}(t)) \). By (7.2), \( t' = \bar{r}(t) \) and \( t = \bar{u}(\bar{r}(t)) \). Thus, by (7.1),

\[
|t' - t| = |u(\bar{r}(t)) - \bar{u}(\bar{r}(t))| \leq C.
\]

We prove the proposition by contradiction. If there is a sequence \( \{t_i\} \) such that \( \lim_{i \to \infty} \frac{u(t_i)}{\bar{u}(t_i)} = \infty \), then with (7.8), and the obvious fact that \( \bar{r}(t_i) \to \infty \) as \( i \to \infty \), we get

\[
\lim_{i \to \infty} \frac{t_i' - t_i}{\ln \bar{r}(t_i) - \ln \bar{r}(t_i)} = \lim_{i \to \infty} \frac{t_i' - t_i}{\ln \bar{r}(t_i) - \ln \bar{r}(t_i)} = 0,
\]

where \( t_i' = u(t_i') \). Let \( r_i = \bar{r}(t_i) \) and \( r_i' = \bar{r}(t_i') \). Since \( \lim_{i \to \infty} \frac{u(r_i)}{\bar{u}(r_i)} = \alpha < -1 \) by Lemma 7.6, replacing by a subsequence, we may assume that

\[
\lim_{i \to \infty} \frac{t_i - t_{i-1}}{\ln r_i - \ln r_{i-1}} = \alpha.
\]

Since \( t_i' = u(r_i') \), \( r_i' = \bar{r}(t_i) = \bar{r}(t_i') \), after further replacing by a subsequence, we may assume:

\[
r_1 < r_1' < r_2 < r_2' \cdots < r_n < r_n' < \cdots,
\]

and

\[
t_1 > t_1' > t_2 > t_2' \cdots > t_n > t_n' > \cdots.
\]

By Theorem 2.4 we have

\[
\frac{t_i - t_{i-1}}{\ln r_i - \ln r_{i-1}} \geq \frac{t_i' - t_i}{\ln r_i' - \ln r_i},
\]

However, by (7.9) and (7.10)

\[
-1 > \alpha = \lim_{i \to \infty} \frac{t_i - t_{i-1}}{\ln r_i - \ln r_{i-1}} \geq \lim_{i \to \infty} \frac{t_i' - t_i}{\ln r_i' - \ln r_i} = 0,
\]

which is a contradiction. We have thus established our claim.

Finally, we present our blow-down analysis. Consider a monotone decreasing sequence \( \{t_i\} \) such that \( t_i \to -\infty \) and the level sets

\[
L_{t_i} = \{x : u(x) = t_i\}.
\]

Define the following blow-down sequence of functions

\[
u_i(x) := u(\bar{r}(t_i)x) - t_i.
\]
The re-scaled level sets are defined by
\[ \hat{L}_i := \frac{L_i}{\rho(t_i)} = \{ x : u_i(x) = 0 \}. \]
We present the main result in this section.

**Theorem 7.9.** For any compact subset \( E \subset \mathbb{R}^4 \backslash \{0\} \), after replacing by a subsequence,
\[ \lim_{i \to \infty} \| u_i - \alpha \ln |x| \|_{C^{1,\gamma}(E)} = 0, \]
for any \( \gamma \in (0,1) \).

**Proof.** By the definition of \( \rho(t_i) \), we have
\[ (7.12) \quad \inf_{B_i \backslash \{0\}} u_i = \min_{\partial B_i(0)} u_i = 0. \]
From (6.7), for some uniform constant \( C = C(\rho) \),
\[ |\nabla u_i(x)| = \rho(t_i)|\nabla u(\rho(t_i)x)| \leq \rho(t_i) \left( \frac{C}{\rho(t_i)|x|} \right) \leq \frac{C}{|x|}, \]
\[ |\nabla^2 u_i(x)| = \rho^2(t_i)|\nabla^2 u(\rho(t_i)x)| \leq \rho(t_i)^2 \left( \frac{C}{\rho(t_i)^2|x|^2} \right) \leq \frac{C}{|x|^2}. \]
Thus, \( u_i \) is uniformly bounded in any compact set \( E \subset \mathbb{R}^4 \backslash \{0\} \) in \( C^2(E) \) norm. By Arzela-Ascoli theorem, \( u_i \to u_\infty \) subsequentially in \( C^{1,\gamma}(E) \) for any \( \gamma \in (0,1) \) and \( u_\infty \in C^{1,1}(\mathbb{R}^4 \backslash \{0\}) \).

Notice that \( u_i \) satisfies the equation
\[ \sigma_2(A(\rho, u_i)) = (\rho(t_i))^4 \exp(p(u_i + t_i) e^{4u_i}). \]
By Lemma 7.6
\[ (\rho(t_i))^4 \exp(p(u_i + t_i) e^{4u_i}) \to 0, \]
uniformly in any compact subsets of \( \mathbb{R}^4 \backslash \{0\} \). Therefore, by \( C^{1,\gamma}_{loc} \) convergence, \( u_\infty \) satisfies \( A(\rho, u_\infty) \in \partial \Gamma^*_\rho \) in viscosity sense (see [18] section 6 , [12] chapter 2). Since \( A(\rho, u_\infty) = \rho A(1, \frac{1}{\rho} u_\infty) \), from Theorem 1.18 in [30] or Theorem 1.5 in [39],
\[ (7.13) \quad u_\infty = c \ln |x| + c', \]
where \( c, c' \) are constants. In particular, by (7.12), \( \min_{|x|=1} u_\infty(x) = 0 \), and then \( c' = 0 \).

Next, we prove that \( u_\infty(x) = \alpha \ln |x| \). Fix \( x \) such that \( |x| > 1 \). By Theorem 2.4, the concavity of \( u \circ \exp \) implies that
\[ u(r|x|) \ln r - \ln 1 + u(1) \ln(r|x|) - \ln 1 \leq u(r). \]
Hence,
\[ (7.14) \quad \frac{u(r|x|) - u(r)}{\ln(r|x|) - \ln 1} \leq \frac{u(r|x|) - u(1)}{\ln(r|x|) - \ln 1}. \]
By Lemma 7.6
\[ (7.15) \quad \lim_{r \to \infty} \frac{u(r|x|) - u(1)}{\ln(r|x|) - \ln 1} = \alpha. \]
Thus, for any \( \epsilon > 0 \), by (7.14) and (7.15), there exists a \( N \in \mathbb{N} \) such that for all \( i > N \),
\[ u(\rho(t_i)|x|) - u(r) < (\alpha + \epsilon) \ln(|x|). \]
Noticing that \( t_i = u(y(t_i)) \) by (7.2), we use (7.1) to conclude that for any \( i > N \),
\[
    (7.16) \quad u_i(x) = u(y(t_i)x) - t_i
    \leq u(y(t_i)|x|) + C - u(y(t_i))
    \leq (\alpha + \epsilon) |x| + C.
\]
Thus, \( \limsup_{i \to \infty} \frac{u_i(x)}{\ln |x|} \leq \alpha \). On the other hand, by (7.6),
\[
    \frac{u(r|x|) - \alpha \ln (r|y|) - (u(r) - \alpha \ln r)}{\ln |x|} \geq 0.
\]
Hence, a similar argument shows that \( \liminf_{i \to \infty} \frac{u_i(x)}{\ln |x|} \geq \alpha \). We have thus proved that \( c = \lim_{i \to \infty} \frac{u_i(x)}{\ln |x|} = \alpha \).

We finish this section by stating the following corollary, which will be useful later.

**Corollary 7.10.** For a sequence \( t_i \to -\infty \), \( u_i \) is given as in (7.11). After passing to a possible subsequence, we have
\[
    \lim_{i \to \infty} \| \frac{\nabla u_i(y)}{|\nabla u_i(y)|} - \frac{y}{|y|} \|_{L^\infty(L_i)} = 0,
\]
\[
    \lim_{i \to \infty} \| y \cdot \nabla u_i(y) - \alpha \|_{L^\infty(L_i)} = 0.
\]

**Proof.** By Proposition 7.8 we may pick a \( R \in \mathbb{R} \) such that \( \tilde{L}_i = L_i / y(t_i) \subset E = BR(0) \setminus B_R(0) \). Thus, for any \( y \in E \), \( | \nabla \ln |y|| = \frac{1}{|y|} \in [1/R, R] \). By Theorem 7.9 there exists a large \( N \in \mathbb{N} \) such that for all \( i > N \), and \( y \in \tilde{L}_i \),
\[
    \frac{1}{2R} < |\nabla u_i(y)| < \frac{2}{R}.
\]
Thus, by Theorem 7.9 again, for \( y \in \tilde{L}_i \subset E \)
\[
    \lim_{i \to \infty} \| \frac{\nabla u_i(y)}{|\nabla u_i(y)|} - \frac{y}{|y|} \|_{L^\infty(L_i)} = 0,
\]
\[
    \lim_{i \to \infty} \| y \cdot \nabla u_i(y) - \alpha \|_{L^\infty(L_i)} = 0.
\]
Here the second identity is due to the fact that \( y \cdot \nabla \ln |y| = 1 \). \( \square \)

8. **Proof of main theorems**

In this section, we prove Theorem 1.3. Using our Pohozaev identity, the quasi-local mass \( M(t_i) \) can be expressed by terms defined on \( L_i \). We argue that \( L_i \) sub-converges to the standard round sphere in the Gromov-Hausdorff sense. Once we find the limits of the related terms on \( L_i \), we can argue that \( M(t_i) \) converges to \( 0 \) as \( i \to \infty \). Thus, by the monotonicity, \( M(t) \equiv 0 \), and we can establish Theorem 1.3 using the rigidity result of \( M(t) \).

By Remark 2.1 without loss of generality, we may assume that \( \beta = 4 \) and we use the same notation as in Section 7.

By Theorem 7.9 Corollary 7.10 and Proposition 7.8 after passing to a subsequence, we may assume the following conditions:

i) For some fixed \( R > 1 \), we assume \( L_i = \{ u_i = 0 \} \subset E = BR(0) \setminus B_{1/R}(0) \) for all \( i \in \mathbb{N} \);
ii) For some $\gamma \in (0, 1)$, and $y \in E$,
\begin{equation}
\|u_i(y) - \alpha \ln |y|\|_{C^{1, \gamma}(E)} \to 0,
\end{equation}
\begin{equation}
\lim_{i \to \infty} |y \cdot \nabla u_i(y) - \alpha| = 0;
\end{equation}

iii) In particular, we assume that for all $i$,
\begin{equation}
|y \cdot \nabla u_i(y)| \geq |\alpha|/2;
\end{equation}

iv) We assume that $t_i < u(1)$.

We set up some notations. We use polar coordinate $(r, \theta)$ for all $y \in E$, where $r = |y|$ and $\theta = \frac{y}{|y|} \in S^3$. We define the projection $\pi : E \to S^3 : y \mapsto \frac{y}{|y|}$, and $\pi_i = \pi|_{L_i}$. Let $dl_0$ be the standard volume form on the unit sphere $S^3$. Let $d\tilde{l}_i$ be the volume element for $\tilde{L}_i$.

**Proposition 8.1.** We use notations as listed above. Then, for any $i$ we have the following claims:

1. $\tilde{L}_i$ is a regular $C^2$ hyper-surface;
2. $\pi_i : \tilde{L}_i \to S^3$ is a $C^{1, \gamma}$ diffeomorphism; In particular, $\tilde{L}_i$ is star-shaped;
3. $L_i$ converges to $S^3$ in the Gromov-Hausdorff sense;
4. For any $\theta \in S^3$, we have
\begin{equation}
\lim_{i \to \infty} ((\pi_i^{-1})^* d\tilde{l}_i)(\theta) = dl_0(\theta).
\end{equation}

**Proof.** Claim 1 is trivial due to (8.3).

For Claim 2, we first prove that $\pi_i$ is both surjective and injective. Since $t_i < u(1)$, $\tilde{L}_i$ is closed, compact, and enclosing $0$, $\pi_i$ is a surjection. If $\pi_i$ is not injective, then there exists $\theta_0 \in S^3$ and $0 < \psi_1 < \psi_2$ such that $\psi_1 \theta_0, \psi_2 \theta_0 \in \tilde{L} \subset E$ and $u_i(\psi_1 \theta_0) = u_i(\psi_2 \theta_0) = 0$. By the mean value theorem, there exists $\psi_0 \in (\psi_1, \psi_2)$ such that
\begin{equation}
\nabla u_i(\psi_0 \theta_0) \cdot \theta_0 = 0.
\end{equation}

However, by (8.3)
\begin{equation}
|\nabla u_i(\psi_0 \theta_0) \cdot \theta_0| \geq \frac{|\alpha|}{\psi_0} \geq \frac{|\alpha|}{2R},
\end{equation}
which is contradicted to (8.4). Thus, $\pi_i$ is bijective.

Next, we prove that $\pi_i$ is $C^{1, \gamma}$ regular. Let $y = r\theta \in \tilde{L}_i$ and $u_i(r\theta) = 0$.

By (8.3) and $\tilde{L}_i$,
\begin{equation}
|\frac{\partial u_i}{\partial r}(y)| = \frac{|y \cdot \nabla u_i|}{|y|} \geq \frac{|\alpha|}{2R} > 0.
\end{equation}

By the implicit function theorem, we may define $|y| = r_i(\theta)$ as a local $C^{1, \gamma}$ function. In addition, in a local coordinate $(\theta^1, \theta^2, \theta^3)$ on $S^3$,
\begin{equation}
\frac{\partial r_i}{\partial \theta^k} = -\frac{\partial u_i}{\partial \theta^k} \cdot \frac{\partial u_i}{\partial r}.
\end{equation}

Since $\pi_i$ is a bijective, $r_i(\theta)$ is a globally defined $C^{1, \gamma}$ function, which leads to the conclusion of Claim 2.
To prove Claim 3, it is sufficient to show that
\[
\lim_{i \to \infty} \max_{y \in L_i} ||y| - 1| = 0.
\]
Since \(u_i(y) = 0\) for any \(y \in \bar{L}_i\), we get
\[
|\alpha \ln |y|| \leq ||u_i - \alpha \ln |y||_{C^{1,\gamma}(E)}.
\]
Hence, by the mean value theorem,
\[
\max_{y \in L_i} ||y| - 1| \leq \frac{||u_i(y) - \alpha \ln |y||_{C^{1,\gamma}(E)}}{|\alpha| \min_{y \in E} \frac{1}{|y|}} \leq \frac{R}{|\alpha|} ||u_i - \alpha \ln |y||_{C^{1,\gamma}(E)},
\]
which leads to (8.7). In addition, by (8.2) (8.6), we have
\[
\frac{\partial u_i(y)}{\partial r} = \frac{\omega}{|y|} \cdot \nabla u_i(y) \to \alpha
\]
and
\[
\lim_{i \to \infty} \nabla \rho r_i(\theta) = 0
\]
uniformly in \(C^\gamma(S^3)\).

For Claim 4, noting that \(dl_0\) is the standard volume form on \(S^3\), we follow the standard volume form computation for star-shaped hypersurface to get
\[
((\pi_i^{-1})^* dl_i)(\theta) = \sqrt{\det (\pi_i^2 Id + \nabla \rho r_i \otimes \nabla \rho r_i)} dl_0(\theta)
\]
\[
= r_i^2 \sqrt{r_i^2 + |\nabla \rho r_i|^2} dl_0(\theta).
\]
We use (8.7) (8.8) and (8.9) to prove Claim 4. \(\square\)

Remark 8.2. For a continuous function \(\psi_i\) defined on \(E\), if \(\psi_i \to \psi\) uniformly, then \((\pi_i^{-1})^* \psi_i \to \psi|_{S^3}\) in \(C^0(S^3)\). Hence, by Proposition 8.1,
\[
\int_{L_i} \psi_i dl_i = \int_{S^3} ((\pi_i^{-1})^* \psi_i dl_i) \to \int_{S^3} \psi|_{S^3} dl_0.
\]

Finally, we are ready to prove our main theorem.

Proof of Theorem 1.3. We use notations as above. Let \(\Omega_i = \{x : u(x) > t\}\). By (3.17), we consider the sequence
\[
M(t_i) = 2N(t_i)Q(t_i) + \frac{\rho}{8}Q^4(t_i) - 12P(t_i).
\]
Since \(Q(t)\) and \(N(t)\) are scaling invariant, we have
\[
Q(t_i) = -\left(\frac{1}{|S^3|} \int_{L_i} |\nabla u_i|^3 dl_i\right)^{\frac{2}{3}}, \quad N(t_i) = \frac{1}{2|S^3|} \int_{L_i} \left(\tilde{H}(y)|\nabla u_i|^2 - \rho|\nabla u_i|^3\right) dl_i,
\]
where \(\tilde{H}\) is the mean curvature of \(\bar{L}_i\). From Corollary 3.10,
\[
12P(t_i) = \frac{3}{2|S^3|} \int_{L_i} \left(\frac{3}{4}\rho|\nabla u_i|^4(x,\nu) + \frac{2}{3}\tilde{H}|\nabla u_i|^3(x,\nu)\right) dl(x)
\]
\[
= \frac{3}{2|S^3|} \int_{L_i} \left(\frac{3}{4}\rho|\nabla u_i|^4(y,\nu) + \frac{2}{3}\tilde{H}|\nabla u_i|^3(y,\nu)\right) dl_i(y).
\]
Thus,
\begin{equation}
M(t_i) = \frac{\rho}{8} Q^4(t_i) + \frac{3}{2 |S^3|} \int_{L_i} \left( -\frac{3}{4} \rho |\nabla u_i|^4 \langle y, \nu \rangle + \frac{2}{3} \tilde{H} |\nabla u_i|^3 \langle y, \nu \rangle \right) d\tilde{L}_i \\
+ 2Q(t_i) \left( \frac{1}{2 |S^3|} \int_{L_i} \tilde{H}(y) |\nabla u_i|^2 - \rho |\nabla u_i|^3 d\tilde{L}_i(y) \right) \\
= \frac{9\rho}{8} Q^4(t_i) + \frac{1}{|S^3|} \int_{L_i} |\nabla u_i|^3 \langle y, \nabla u_i \rangle d\tilde{L}_i \\
+ \frac{1}{|S^3|} \int_{L_i} \tilde{H}(y) |\nabla u_i|^2 \left( Q(t_i) - \langle y, \nabla u_i \rangle \right) d\tilde{L}_i.
\end{equation}

We have used the fact that \( |\nabla u_i| \) is uniformly bounded. We have thus proved (8.15). By (8.11), (8.14) and (8.15),
\begin{equation}
\lim_{i \to \infty} M(t_i) = 0.
\end{equation}

Now take the limits of two terms on the right-hand side of (8.11) separately.
As \( i \to \infty \), from Theorem 7.9, \( |\nabla u_i| \to -\alpha, y \cdot \nabla u_i(y) \to \alpha \) on \( E \) uniformly. By (8.10) and Corollary 7.10 as \( i \to \infty \),
\begin{equation}
Q(t_i) = - \left( \frac{1}{|S^3|} \int_{L_i} |\nabla u_i(y)|^3 d\tilde{L}_i(y) \right) \to -\alpha,
\end{equation}
and
\begin{equation}
\frac{1}{|S^3|} \int_{L_i} |\nabla u_i|^3 \langle y, \nabla u_i \rangle d\tilde{L}_i(y) \to -\alpha^4.
\end{equation}

In addition, as \( i \to \infty \),
\begin{equation}
Q(t_i) - \langle y, \nabla u_i \rangle \to 0 \quad \text{uniformly} \quad \text{on} \quad \tilde{L}_i.
\end{equation}

By (8.12), as \( i \to \infty \),
\begin{equation}
Q^4(t_i) + \frac{1}{|S^3|} \int_{L_i} |\nabla u_i|^3 \langle y, \nabla u_i \rangle d\tilde{L}_i \to |S^3| (\alpha^4 - \alpha^4) = 0.
\end{equation}

Next, we claim that
\begin{equation}
\int_{L_i} \tilde{H}(y) |\nabla u_i|^2 \left( Q(t_i) - \langle y, \nabla u_i \rangle \right) d\tilde{L}_i \to 0 \quad \text{as} \quad i \to \infty.
\end{equation}

Since by (8.13),
\begin{equation}
\left| \int_{L_i} \tilde{H} |\nabla u_i|^2 \left( Q(t_i) - \langle y, \nabla u_i \rangle \right) d\tilde{L}_i \right| \leq \max_{y \in L_i} |Q(t_i) - \langle y, \nabla u_i \rangle| \cdot \int_{L_i} |\tilde{H}| \cdot |\nabla u_i|^2 d\tilde{L}_i,
\end{equation}
it is sufficient to prove that \( \int_{L_i} \tilde{H} |\nabla u_i|^2 d\tilde{L}_i \) is uniformly bounded. We notice that \( H \), the mean curvature of \( L_i \), is positive by (3.7). By (8.12), \( Q(t_i) \) is uniformly bounded and
\begin{equation}
\frac{1}{|S^3|} \int_{L_i} |\tilde{H}| \cdot |\nabla u_i|^2 d\tilde{L}_i = \frac{1}{|S^3|} \int_{L_i} H |\nabla u_i|^2 d\tilde{L}_i \\
= 2N(t_i) + \rho |Q(t_i)|^3 \\
\leq \frac{1}{|S^3|} \int_{\mathbb{R}^4} f(u) dx + \rho |Q(t_i)|^3
\end{equation}
is uniformly bounded. We have thus proved (8.15). By (8.11), (8.14) and (8.15),
\begin{equation}
\lim_{i \to \infty} M(t_i) = 0.
\end{equation}
It is clear that \( M(\tau') = 0 \), where \( \tau' = \max_{\mathbb{R}^n} u \). By Theorem 3.8 we conclude that \( M(t) \equiv 0 \) and then \( u \) is rotationally symmetric.

9. Radial solutions

In this section, we study the radial solution \( u(|x|) \) of (1.5) where \( f(u) = \frac{3}{2} e^{4u} \).

Denote \( r = |x| \). By (1.5), \( u(r) \) satisfies the following differential equation:

\[
(3u'' + \frac{3u'}{r}) \left( \frac{u'}{r} + \frac{\rho (u')^2}{2} \right) = \frac{3}{2} e^{4u}, \quad u'(0) = 0.
\]

Let \( s = \ln r \). We denote \( u_s := \frac{du(\rho e^s)}{ds} = ru' \). Then, from (9.1) we have

\[
3u_{ss} \left( u_s + \frac{\rho}{2} (u_s)^2 \right) - \frac{3}{2} e^{4(u+s)} = 0, \quad u_s(-\infty) = 0.
\]

The condition that \( \lambda(A(\rho, u)) \in \Gamma_2^{+} \) implies that

\[
u_s + \frac{\rho}{2} (u_s)^2 < 0, \quad u_{ss} < 0,
\]

and it also implies \(- \frac{2}{\rho} < u_s < 0 \) for \( s \in (-\infty, \infty) \). Note (9.2) is invariant if we rescale \( u \) by

\[
(\tilde{u}(e^s)) = u(e^{s+c}) + c.
\]

Lemma 9.1. (9.2) admits a first integral:

\[
(u_s)^2 \left( \frac{\rho}{4} u_s^2 + \frac{2 + \rho}{3} u_s + 1 \right) - \frac{1}{4} e^{4(u+s)} \equiv 0.
\]

Proof. Multiply \( (u_s + 1) \) to both sides of (9.2) and get

\[
0 = 3u_{ss}(u_s + 1) \left( u_s + \frac{\rho}{2} (u_s)^2 \right) - \frac{3}{2} e^{4(u+s)} (u_s + 1)
\]

\[
= \frac{3}{2} \left[ (u_s)^2 \left( \frac{\rho}{4} u_s^2 + \frac{2 + \rho}{3} u_s + 1 \right) - \frac{1}{4} e^{4(u+s)} \right] .
\]

Thus, we obtain a first integral of (9.2):

\[
(u_s)^2 \left( \frac{\rho}{4} u_s^2 + \frac{2 + \rho}{3} u_s + 1 \right) - \frac{1}{4} e^{4(u+s)} \equiv c_0,
\]

for some constant \( c_0 \). As \( s \to -\infty \), we have \( \lim_{s \to -\infty} u_s = \lim_{r \to 0} ru' = 0 \). Hence, \( c_0 = 0 \) and

\[
(u_s)^2 \left( \frac{\rho}{4} u_s^2 + \frac{2 + \rho}{3} u_s + 1 \right) - \frac{1}{4} e^{4(u+s)} = 0.
\]

\[
\square
\]

Theorem 9.2. For \( \rho \in [0, 2) \), (9.7) admits a unique radial solution up to a rescaling.

Proof. To fix the gauge, we may assume that \( u_s|_{s=0} = -\epsilon \) for some \( \epsilon > 0 \) after a rescaling in (9.4), and consider

\[
3u_{ss} \left( u_s + \frac{\rho}{2} (u_s)^2 \right) - \frac{3}{2} e^{4(u+s)} = 0, \quad u_s(-\infty) = 0, \quad u_s|_{s=0} = -\epsilon.
\]
By (9.5) and (9.8)
\[(u_s)^2 \left( \frac{\rho}{4} u_s^2 + \frac{2 + \rho}{3} u_s + 1 \right) = \frac{1}{2} u_{ss} \left( u_s + \frac{\rho}{2} (u_s)^2 \right).\]

Then
\[(9.9) \int_{-\epsilon}^{u_s} \frac{1}{2x} \frac{(1 + \frac{\rho}{2} x)}{(\frac{\rho}{2} x^2 + \frac{2 + \rho}{3} x + 1)} dx = \int_0^s ds.

Case 1. If \(\rho = 0\), then
\[\int_{-\epsilon}^{u_s} \frac{1}{2x} \frac{(1 + \frac{\rho}{2} x)}{x + 1} dx = \int_0^s ds.

Thus,
\[\frac{1}{2} \ln \left( \left| \frac{u_s}{3 + 2u_s} \right| \right) - \frac{1}{2} \ln \left( \frac{\epsilon}{3 - 2\epsilon} \right) = s.

We obtain that
\[u_s = -\frac{3 - 2e^{2s}}{e^{2s} + \frac{1}{2} \left( \frac{3 - 2\epsilon}{\epsilon} \right)}, \quad u = -\frac{3}{4} \ln \left( e^{2s} + \frac{1}{2} \left( \frac{3 - 2\epsilon}{\epsilon} \right) \right).

Since \(u \sim -\frac{3}{4} \ln r\) as \(r \to \infty\), the total integral \(\int_{\mathbb{R}^4} e^{4u} < \infty\).

Case 2. Suppose \(\rho \in (0, 2)\). Now, \(x \left( \frac{\rho}{4} x^2 + \frac{2 + \rho}{3} x + 1 \right) = 0\) has 3 roots
\[x_0 = 0, \quad x_1 = -\frac{(\rho + 2) - \sqrt{\rho^2 - 5\rho + 4}}{3\rho/4}, \quad x_2 = -\frac{(\rho + 2) + \sqrt{\rho^2 - 5\rho + 4}}{3\rho/4}.

Simple calculation shows \(-1 > x_1 > -\frac{2}{\rho}\). Thus, \(\frac{(1 + \frac{\rho}{2} x)}{2x(\frac{\rho}{2} x^2 + \frac{2 + \rho}{3} x + 1)} < 0\) for \(x \in (x_1, 0)\), and \(u_s\) defined by (9.9) is decreasing in \(s\). From (9.9), \(u_s \to x_0 = 0\), as \(s \to -\infty\), and \(u_s \to x_1\), as \(s \to \infty\). From the first integral (9.5), the solution is uniquely determined. Since \(\lim_{s \to -\infty} u_s = x_1 < -1\), the total integral
\[\int_{\mathbb{R}^4} e^{4u(|x|)} dx = |S^3| \int_{-\infty}^{\infty} e^{4u(e^s)} e^{4s} ds < \infty.

\[\square\]

Remark 9.3. The existence of radial solutions to (1.5) for general \(f(u) = e^{4u} p(u)\) cannot be achieved through the previous argument. It will be an interesting question to find such solutions and see if the polynomial term \(p(u)\) affects the asymptotic behavior. We leave the question for future studies.

Appendix A. Pohozaev identity

In this appendix, we prove a Pohozaev identity, which is stated for a slightly more general form of PDEs. For \(k\)-Hessian equation, Tso in [55] and Brandolini-Nitsch-Salani-Trombetti [2] have given the Pohozaev identity by the variational method of Pucci-Serrin [52]. Schoen in [53] mentioned that Kazdan-Warner identity is equivalent to Pohozaev identity. Deducing Pohozaev identity from Kazdan-Warner identity is not obvious and Gover-Ørsted in [26] provided a proof for this fact. For \(\sigma_k\)-Yamabe equation, the Kazdan-Warner identity was established by Han [31] and Viaclovsky [59] respectively. We will adopt the argument of [52] to deduce the Pohozaev identity for the more general form.
For a fixed constant $\tau$, consider the following Dirichlet problem on a bounded open domain $\Omega$:

\begin{equation}
\begin{aligned}
\sigma_2(A(\rho, u)) &= K(x)f(u), &\quad &\Omega \subset \mathbb{R}^4, \\
A(\rho, u) &\in \Gamma_2^+, &\quad &u = \tau, \\
\end{aligned}
\tag{A.1}
\end{equation}

We assume that $\Omega$ has finite perimeter so that we can use generalized divergence theorem (\cite{19}, Section 5.8, Theorem 1). Note that $\Omega = \{u > \tau\}$. For simplicity, we fix a function $F(t)$ such that $F(\tau) = 0$ and $F'(t) = f(t)$. We define a vector field $X = \sum x^i \frac{\partial}{\partial x^i}$. We first prove that

**Lemma A.1.** Let $u \in C^2(\overline{\Omega})$ be a solution to (A.1) and use notations as above. Then the following identity holds

$$
\int_\Omega X(u) \cdot \sigma_2(A(\rho, u)) dx = -\int_\Omega (4K(x) + X(K(x)))F(u) dx.
$$

**Proof.** Note that, since $F' = f$ and (A.1),

$$
\text{div}(F(u(x)))K(x)X = (\text{div}X)F(u(x))K(x) + X(F(u(x)))K(x)
$$

$$
= (\text{div}X)F(u(x))K(x) + f(u)X(u(x))K(x) + F(u)X(K(x)).
$$

(A.2)

We integrate (A.2). Using the divergence theorem and the fact that $F(u)|_{\partial \Omega} = 0$, we have proved the lemma. \hfill \square

Let

$$
T_{ij}^1 := (-\Delta u)\delta_{ij} + u_{ij}, \quad T_{ij}^2 := u_{ik}T_{kj}^1 + \sigma_2(-\nabla^2 u)\delta_{ij}
$$

be the Newton tensors of $\sigma_2(-\nabla^2 u)$ and $\sigma_3(-\nabla^2 u)$. It is well known that $-T_{ij}^1 u_{ij} = 2\sigma_2(-\nabla^2 u)$. Also, it is easy to verify that if $u$ is in $C^3(\Omega)$, then

$$
\frac{\partial}{\partial x^j}T_{ij}^1 = 0, \quad \frac{\partial}{\partial x^j}T_{ij}^2 = 0.
$$

(A.3)

We claim the following:

**Lemma A.2.** Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$. Then,

$$
-2 \int_\Omega \langle x, \nabla u \rangle \sigma_2(A(\rho, u)) dx = -\frac{2}{3} \int_{\partial \Omega} \left( -x^i u_{ij} T_{ij}^1 u_j + u_j T_{ij}^1 (u - \tau) + x^i T_{ij}^2 (u - \tau) \right) \nu_k dl.
$$

**Proof.** Similar computations have been carried out for the 2-Hessian equations and the $\sigma_2$-Yamabe equation in conformal geometry, which we follow closely.

We claim that

$$
\int_\Omega \langle x, \nabla u \rangle \frac{\partial}{\partial x^i} (|\nabla u|^2 u_i) \frac{\partial}{\partial x^j} (|\nabla u|^2 u_j) dx = \frac{3}{4} \int_{\partial \Omega} |\nabla u|^4 \langle x, \nu \rangle dl,
$$

(A.4)

and

$$
\int_\Omega \langle x, \nabla u \rangle \sigma_2(\nabla^2 u) dx = \frac{1}{5} \int_{\partial \Omega} x^i u_{ij} H |\nabla u|^2 dl.
$$

(A.5)

Since $2\sigma_2(A(\rho, u)) = 2\sigma_2(-\nabla^2 u) + \rho \frac{\partial}{\partial x^i} (|\nabla u|^2 u_i)$, (A.11) follows immediately.
To prove (A.4), we compute directly.

(A.6) \[ x^j u_j \partial \left( |\nabla u|^2 u_i \right) = \partial \left( x^j u_j |\nabla u|^2 u_i \right) - (u_i + x^j u_{ji}) |\nabla u|^2 u_i \]

\[ = \frac{\partial (x^j u_j |\nabla u|^2 u_i)}{\partial x^j} - \left( |\nabla u|^4 + \frac{1}{4} x^j \frac{\partial}{\partial x^j} |\nabla u|^4 \right) \]

\[ = \frac{\partial (x^j u_j |\nabla u|^2 u_i)}{\partial x^j} - \left( |\nabla u|^4 + \frac{1}{4} \frac{\partial (x^j |\nabla u|^4)}{\partial x^j} - |\nabla u|^4 \right) \]

\[ = \frac{\partial}{\partial x^j} \left( x^j u_j |\nabla u|^2 u_i - \frac{1}{4} x^i |\nabla u|^4 \right). \]

By (A.6), (3.2), and the divergence theorem,

\[ \int_{\Omega} x^j u_j \frac{\partial (|\nabla u|^2 u_i)}{\partial x^j} \, dx = \frac{3}{4} \int_{\partial \Omega} |\nabla u|^4 x^i u_i \, dl. \]

Thus, we have proved the first claim.

Next, we prove (A.5). Recall in (A.3) that tensors $T^1$ and $T^2$ are divergence free. Also, $-T^1_{ij} u_{ij} = 2\sigma_2(-\nabla^2 u)$. From (A.3), $2\sigma_2(-\nabla^2 u) = -\frac{\partial}{\partial x^\nu} (T^1_{ij} u_j)$. We have

(A.7) \[ 2 \int_{\Omega} x^i u_i \sigma_2(-\nabla^2 u) \, dx = - \int_{\Omega} \frac{\partial}{\partial x^j} (x^i u_i T^1_{jk} u_k) \, dx + \int_{\Omega} \frac{\partial}{\partial x^j} (x^i u_i) T^1_{jk} u_k \, dx. \]

The following holds in $\Omega$:

(A.8) \[ \frac{\partial (x^i u_i)}{\partial x^j} T^1_{jk} u_k \, dx = u_j T^1_{jk} u_k + (x^i u_{ij}) \, T^1_{jk} u_k \]

\[ = \frac{\partial}{\partial x^k} \left( u_j T^1_{jk} (u - \tau) \right) - u_j T^1_{jk} (u - \tau) + x^i u_{ij} T^1_{jk} u_k \]

\[ = \frac{\partial}{\partial x^k} \left( u_j T^1_{jk} (u - \tau) \right) + 2\sigma_2(-\nabla^2 u)(u - \tau) + x^i u_{ij} T^1_{jk} u_k; \]

(A.9) \[ x^i u_{ij} T^1_{jk} u_k = x^i \left( T^2_{ik} - \sigma_2(-\nabla^2 u) \delta_{ik} \right) \, u_k \]

\[ = \frac{\partial}{\partial x^k} \left( x^i T^2_{ik} (u - \tau) \right) - \delta^i_k T^2_{ik} (u - \tau) - x^k u_k \sigma_2(-\nabla^2 u) \]

\[ = \frac{\partial}{\partial x^k} \left( x^i T^2_{ik} (u - \tau) \right) - 2\sigma_2(-\nabla^2 u)(u - \tau) - x^k u_k \sigma_2(-\nabla^2 u). \]

We have used the fact that $T^2_{ii} = 2\sigma_2(-\nabla^2 u)$ in the last equality. By (A.7), (A.8), (A.9), and the divergence theorem, we have

(A.10) \[ 3 \int_{\Omega} x^i u_i \sigma_2(-\nabla^2 u) \, dx = \int_{\Omega} \left( \frac{\partial}{\partial x^k} \left( -x^i u_i T^1_{kj} u_j + u_j T^1_{jk} (u - \tau) + x^i T^2_{ik} (u - \tau) \right) \right) \, dx \]

\[ = \int_{\partial \Omega} \left( -x^i u_i T^1_{kj} u_j + u_j T^1_{jk} (u - \tau) + x^i T^2_{ik} (u - \tau) \right) u_k \, dl. \]
Lemma A.3. Assume that $\Omega$ has almost $C^2$ boundary, i.e. the singular part of $\partial \Omega$ has zero $\mathcal{H}^3$-measure. If $u \in C^3(\Omega) \cap C^2(\bar{\Omega})$ and $u|_{\partial \Omega} = \tau$, then we have

(A.11) \(-2 \int_{\Omega} \langle x, \nabla u \rangle \sigma_2(A(\rho, u))dx = \int_{\partial \Omega} \left( -\frac{3}{4} \rho |\nabla u|^4(x, \nu) + \frac{2}{3} H |\nabla u|^3(x, \nu) \right) dl.\)

Proof. As $\Omega$ has almost $C^2$ boundary, the regular part of $\partial \Omega$ is a $C^2$ hypersurface. The singular part has zero $\mathcal{H}^3$-measure. As $u \in C^2(\bar{\Omega})$ and Lemma A.2, the integral on the singular part of $\partial \Omega$ is zero. Using $u|_{\partial \Omega} = \tau$, and the fact that

$$u_j T_{jk}^1 \nu_k|_M = (-u_\nu \Delta u + u_{\nu\nu} u_\nu) = H |\nabla u| u_\nu = (-H |\nabla u|^2)|_M,$$

we conclude from (A.10) that

$$\int_{\Omega} x^i u_i \sigma_2(-\nabla^2 u)dx = \frac{1}{3} \int_{\partial \Omega} x^i u_i T_{jk}^1 u_j \nu_k dl = \frac{1}{3} \int_{\partial \Omega} x^i u_i H |\nabla u|^2 dl.$$

\[\square\]

Lemma A.4. Assume that $\partial \Omega$ is smooth. If $u \in C^3(\bar{\Omega})$ and $u|_{\partial \Omega} = \tau$, then we have

(A.12) \(-2 \int_{\Omega} \langle x, \nabla u \rangle \sigma_2(A(\rho, u))dx = \int_{\partial \Omega} \left( -\frac{3}{4} \rho |\nabla u|^4(x, \nu) + \frac{2}{3} H |\nabla u|^3(x, \nu) \right) dl.\)

Proof. Since $\partial \Omega$ is smooth, we may extend $u$ to be a $C^2$ function on $\mathbb{R}^4$. Then, we can find a sequence of $C^3$ functions $u_p$ such that $||u_p - u||_{C^2(\mathbb{R}^4)} \to 0$ for $p \to \infty$. From Lemma A.2, we know that

$$2 \int_{\Omega} \langle x, \nabla u_p \rangle \sigma_2(A(\rho, u_p))dx = \frac{2}{3} \int_{\partial \Omega} \nu_k \left( -x^i u_{p,i} T_{jk}^1 u_{p,j} + u_{p,j} T_{jk}^1 (u_p - \tau) + x^i T_{jk}^2 (u_p - \tau) \right) dl,$$

where $T_{ij}^1 = (-\Delta u_p) \delta_{ij} + u_{p,j}$, and $T_{ij}^2 = u_{p,ik} T_{jk}^1 + \sigma_2(-\nabla^2 u_p) \delta_{ij}$. By $||u_p - u||_{C^2(\mathbb{R}^4)} \to 0$ for $p \to \infty$ and $u = \tau|_{\partial \Omega}$, we have

$$-2 \int_{\Omega} \langle x, \nabla u \rangle \sigma_2(A(\rho, u))dx = \frac{2}{3} \int_{\partial \Omega} x^i u_i T_{jk}^1 u_j \nu_k dl.$$

Similar to the argument in Lemma A.3, we have proved this Lemma. \[\square\]

Combining Lemmas A.1, A.3 and A.4, we have the following Pohozaev type identity:

Theorem A.5. Suppose either of the following conditions holds.

1. $\Omega$ has almost $C^2$ boundary, and $u$ is a $C^3(\Omega) \cap C^2(\bar{\Omega})$ solution to (A.1);

2. $\Omega$ has smooth boundary, and $u$ is a $C^2(\bar{\Omega})$ solution to (A.1).

Then, we have

$$\int_{\Omega} 8(K(x) + \frac{1}{4} (x, \nabla K)) F(u)dx = \int_{\partial \Omega} \left( -\frac{3}{4} \rho |\nabla u|^4(x, \nu) + \frac{2}{3} H |\nabla u|^3(x, \nu) \right) dl,$$

where $\nu$ is the unit outward normal vector of $\partial \Omega$. 


Theorem A.5 can also be established using a classical variational identity of Pucci-Serrin [52]. We present this approach for comparison.

We set up some notations first. Let
\[ F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}(\mathbb{R}^{n \times n}) \to \mathbb{R} \]
\[ (x, u, p, r) \mapsto F(x, u, p, r) \]
be a smooth function where \( p = (p_1, \cdots, p_n) \in \mathbb{R}^n \), \( r = (r_{ij})_{n \times n} \). Consider the variational problem on an open domain \( \Omega \subset \mathbb{R}^n \)
\[ \delta \int_{\Omega^n} F(x, u, \nabla u, \nabla^2 u) \, dx = 0. \]
(A.14)

Denote \( F_u = \frac{\partial F}{\partial u} \), \( F_{p_i} = \frac{\partial F}{\partial p_i} \), \( F_{r_{ij}} = \frac{\partial F}{\partial r_{ij}} \), and \( F_{x^i} = \frac{\partial F}{\partial x^i} \). The corresponding Euler-Lagrange equation for (A.14) is
\[ \partial_2 F_{r_{ij}} \partial x^i \partial x^j - \partial F_{p_i} \partial x^i + F_u = 0. \]
(A.15)

Then, the following identity holds.

**Theorem A.6** ([52], Proposition 3). Let \( u \in C^4(\Omega) \) be a solution of (A.15). Then
\[ \partial \left[ x^i F - x^k \frac{\partial u}{\partial x^k} (F_{p_i} - \frac{\partial F}{\partial x^j} F_{r_{ij}}) - F_{r_{ij}} x^k \frac{\partial F_u}{\partial x^k} \right] \]
\[ = n F + x^i F_{x^i} - \frac{\partial u}{\partial x^i} F_{p_i} - 2 \frac{\partial^2 u}{\partial x^i \partial x^j} F_{r_{ij}}. \]
\[ \text{(A.16)} \]

For (A.14), we pick the following
\[ F(x, u, \nabla u, \nabla^2 u) = -2 \frac{(u - \tau)\sigma_2(\nabla^2 u)}{3} + \frac{\rho}{4} |\nabla u|^4 + 2K(x)F(u). \]
\[ \text{(A.17)} \]

The partial derivatives of \( F \) are given by
\[ F_u = -2\sigma_2(\nabla^2 u) \frac{3}{3} + 2K(x)f(u), \quad F_{x^i} = 2(x, \nabla K)F(u), \]
\[ \text{(A.18)} \]
\[ F_{p_i} = \rho |\nabla u|^2 u_i, \quad F_{r_{ij}} = -\frac{2}{3} (u - \tau) \frac{\partial \sigma_2}{\partial r_{ij}} = \frac{2}{3} (u - \tau) (\delta_{ij} \Delta u - u_{ij}). \]
\[ \text{(A.19)} \]

**Remark A.7.** Notice that (A.3) holds if we assume \( u \in C^3(\Omega) \). Thus,
\[ \frac{\partial}{\partial x^j} F_{r_{ij}} = -\frac{2}{3} u_j (\delta_{ij} \Delta u - u_{ij}) \]
which only needs derivatives up to 2nd order. Similarly, \( \frac{\partial^2}{\partial x^i \partial x^j} F_{r_{ij}} \) in (A.15) just needs derivatives up to order 2. Hence, for our functional (A.17), Theorem A.6 holds for \( u \in C^3(\Omega) \cap C^2(\Omega) \).

The Euler-Lagrange equation (A.15) in \( \Omega \) is
\[ \sigma_2(A(\rho, u)) - K(x)f(u) = 0. \]
\[ \text{See [3] the variational function for } \sigma_2(A(1, u)) \text{ on closed manifolds.} \]

Now, we present the proof of Theorem A.5 using (A.16).
Another proof of Theorem A.5

We first assume that \(u\) is a solution in \(C^3(\Omega) \cap C^2(\bar{\Omega})\) on a domain \(\Omega\) with almost \(C^2\) boundary. By (A.18) and (A.19),

\[
4F + x^i F_{x^i} - \frac{\partial u}{\partial x^i} F_{p_i} - 2 \frac{\partial^2 u}{\partial x^i \partial x^j} F_{r_{ij}} = 8F(u)K(x) + 2\langle x, \nabla K \rangle F(u).
\]

By Remark A.7, we can apply (A.16). Integrate both sides of (A.16) to get

\[
\int_{\partial \Omega} \left( F(x, \nu) - \langle x, \nabla u \rangle \left( F_{p_i} - \frac{\partial}{\partial x^i} F_{r_{ij}} \right) \nu_i - \frac{\partial (x, \nabla u)}{\partial x^i} F_{r_{ij}} \nu_i \right) dl
\]

\[
= \int_\Omega 8(K(x) + \frac{1}{4} \langle x, \nabla K \rangle F(u) dx.
\]

We simplify the left hand side of (A.20) by facts that \(u|_{\partial \Omega} = \tau\), \(F|_{\partial \Omega} = \rho\), \(|\nabla u|_4\), \(\nu = -\frac{\nabla u}{|\nabla u|}\), and (3.4) to get

\[
\langle F_{p_i} - \frac{\partial}{\partial x^i} F_{r_{ij}}, \nu_i \rangle = \left( \frac{\rho}{4} |\nabla u|^4 u_i + \frac{2}{3} u_j (\delta_{ij} \Delta u - u_{ij}) \right) \nu_i \\
= -\rho |\nabla u|^3 + \frac{2}{3} H|\nabla u|^2.
\]

We have thus proved our theorem for \(u \in C^3(\Omega) \cap C^2(\bar{\Omega})\).

If the boundary of \(\Omega\) is smooth, we may first extend \(u\) to be a \(C^2\) function on \(\mathbb{R}^4\). We can find a sequence of \(C^3\) functions \(u_p\) such that \(||u_p - u||_{C^2(\bar{\Omega})} \to 0\). By \(C^2\) convergence, (A.13) holds for \(u\). We finish the proof.

\[\square\]

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