Higher order fully implicit cell-centered finite volume method for simulation of oil-water displacement in porous medium

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Abstract. In this paper a fully implicit cell-centered finite volume method with Higher-Order Upwind scheme (HOU) for convective flow approximation is presented for numerical solution of oil-water displacement problem. Main advantages of this scheme are the robustness, accuracy and rapid convergence of Newton technique, used for linearization obtained discrete equations. Using of HOU scheme allows receive second order approximation of flux in the regions of slow saturation changing and stable solution near to the shocks. Efficiency of proposed scheme is verified by numerical tests.

1. Introduction

Modern problems of numerical simulation of multi-phase underground flows require the design of accurate schemes, which can be used for fluids with wide range of changing of properties. The increase of accuracy for numerical solutions with preserving rapid rate of convergence for iterative methods is a great challenge for researchers in the field of mathematical modeling. The first step for solving this problem is the choice of method for discretization of governing equations, which allow find reliable higher order scheme for flux approximation. In the present time various modifications of finite volume approaches are employed more often in a practice.

There are two main strategies for finite volume method, common used for simulation multiphase flow in reservoirs. The first one is a vertex-centered scheme. Lamine and Edwards [4] developed high order cell-based finite volume approach, in which they implemented Control-Volume Distributed (CVD) or, in other words, Multi-Point Flux Approximation scheme (MPFA). It allowed reduce directional and cross-wind diffusion for convective flow approximation. The second strategy is a cell-centered scheme. Xie and Edwards [5] presented two different formulations of higher order cell-centered finite volume method with CVD-MPFA scheme for incompressible two-phase flow. They demonstrated benefit of higher resolution total velocity approach compared to the first-order method. Contreras et al. [6] proposed cell-centered MPFA method with a diamond type stencil coupled with one of the Monotone Upstream Schemes for Conservation Laws (MUSCL). Numerical tests showed advantages of this
method in robustness and accuracy compared to the other finite volume approaches. For this reason analogous method was chosen as a basement of numerical algorithm in this work.

The second step for solving the formulated problem is the choice of time discretization scheme. There are two main approaches to obtaining a numerical solution for coupled system of governing equations, describing two-phase flows in porous medium. The first one is an IMPlicit Pressure Explicit Saturation (IMPES) scheme. Different formulations of this procedure were used in all above mentioned works excepting spectral methods. This approach allows avoid solving of nonlinear discrete equations, but it have significant restrictions. The most serious one is a requirement of constant density for each phase. The second one is a restriction on size of time step. Fully Implicit (FI) scheme hasn’t such disadvantages. Recently Luo et al. [7] presented FI discontinuous Galerkin method for two-phase flow in porous medium. Bubenchikov et al. [8] developed simple FI finite difference approach for simulation methane flow in coal bed when filled with water. Hamon and Mallison [9] proposed FI finite volume metod, in which they implemented multidimensional hybrid upwind scheme that is more accurate than first-order upwind schemes. In this paper, classical cell-centered FI finite volume method is coupled with higher order monotone scheme for numerical simulation of oil-water displacement in porous medium. Main purpose of this work is design of accurate scheme, which can be used for FI approach and will not decrease rate of convergence for Newton method, used for linearization discrete equations.

2. Mathematical model

Isothermal, incompressible and immiscible two-phase flow will be considered in this work. Mass conservation law for water (w) and oil (o) will be the following:

\[
\frac{\partial s_w}{\partial t} + \nabla \cdot u_w = Q_w \tag{1}
\]

\[
-\frac{\partial s_o}{\partial t} + \nabla \cdot u_o = Q_o \tag{2}
\]

where \( \phi \) is the porosity, \( s \) is the water saturation, \( u_\alpha \) is the velocity of phase \( \alpha \), \( Q_\alpha = q_\alpha / \rho_\alpha \) denote injection or production specific rate for each phase ( \( \rho_\alpha \) is the density). Flow velocity for each fluid is defined by Darcy’s law:

\[
uw = -\lambda_w KV (p - p_c + \rho_w g z) \tag{3}
\]

\[
uo = -\lambda_o KV (p + \rho_o g z) \tag{4}
\]

where \( \lambda_\alpha = k_{\alpha_\alpha} (s) / \mu_\alpha \) is the mobility of phase \( \alpha \), \( k_{\alpha_\alpha} (s) \) and \( \mu_\alpha \) are the relative permeability and viscosity of fluid \( \alpha \), respectively. The absolute permeability of porous medium is given by \( K \), \( p \) is the oil pressure, \( p_c \) is the capillary pressure which depends on water saturation, \( g \) it the gravitational constant and \( z \) is the vertical component of the displacement vector.

The relative permeability for each phase and capillary pressure are determined as follows:

\[ k_{w_\alpha} (s) = \left( \frac{s - s_r}{1 - s_r} \right)^{l_w} \]

\[ k_{o_\alpha} (s) = \left( \frac{1 - s - s_{o_r}}{1 - s_{o_r}} \right)^{m} (1 + a s) \]

\[ p_c = 2 \rho_{o_w} \frac{(1 - s^2)^{1/2}}{s^2} \]

where \( s_r \) and \( s_{o_r} \) are the residual saturations for water and oil, respectively. Parameters \( l, m \) and \( a \) can be obtained from experimental measurements.
The one-dimensional reservoir with length $L$ is considered as a flow domain in this work. The initial and boundary conditions are given by:

$$s(x,0) = s_r, \ p(x,0) = p_o, \ 0 < x \leq L$$
$$s(0,t) = 1 - s_w, \ p(0,t) = p_{\text{max}}$$

3. Numerical method

Cell-centered fully implicit finite volume scheme will be used for discretization governing equations. Consider a regular uniform grid consisting of $N$ cells discretizing the computational domain. After substituting velocities (3) and (4) into equations (1) and (2) and integrating over finite volume, these equations are written as:

$$V_i \phi \left( s_i^n - s_i^{n-1} \right) / \Delta t - \sum_{j=1}^{nf} \lambda_n (S_{i,j}^n) K \left( p_i^n - p_s(s_i^n) + \rho_o g z_j - p_j^n + p_s(s_j^n) - \rho_o g z_i S_j \right) = 0$$

$$-V_i \phi \left( s_i^n - s_i^{n-1} \right) / \Delta t - \sum_{j=1}^{nf} \lambda_n (S_{i,j}^n) K \left( p_j^n + \rho_o g z_i - p_i^n - \rho_o g z_j S_j \right) = 0$$

where $V_i$ is the volume of cell with number $i$, $n$ is the number of time step, $j$ denote number of adjacent volume, $\Delta x_j$ and $S_j$ are the distance between centers and area of common face for $i$ and $j$ cells, respectively. The water saturation on this face for $\alpha$ phase $s_{i,j}^n$ is calculated by higher order monotone upwind scheme. The system of nonlinear equations (5) and (6) is solved by the Newton method.

3.1. Higher-resolution reconstruction

In order to obtain the saturation for face between $i$ and $j$ volumes a simple stencil is used (Figure 1).

![Figure 1. Stencil used to obtain reconstructed saturation.](image)

Initially $s_{i,j}^n$ can be determined as follows:

$$s_{i,j}^n = \begin{cases} s_j + \frac{1}{2} \left( v_j(s_j - s_j) + (1 - v_j)(s_j - s_j) \right) & \text{if } u_{\alpha} \cdot n_j < 0 \\ s_j + \frac{1}{2} \left( v_j(s_j - s_j) + (1 - v_j)(s_j - s_j) \right) & \text{if } u_{\alpha} \cdot n_j \geq 0 \end{cases}$$

where parameter $v_k$ ($k = i, j$) used to determine the degree of the approximation, to control convergence of Newton’s method and to limit slope of reconstructed gradients of saturation simultaneously. For the first case (when $u_{\alpha} \cdot n_j < 0$) it can be computed as:

$$v_j = \begin{cases} 3/4 & \text{if } |s_i - s_j| \leq |s_j - s_j| \\ 0 & \text{if } |s_i - s_j| > |s_j - s_j| \end{cases}$$

For the second case it calculated in the same way. Value 3/4 represents second order method. Numerical experiments showed that other values significantly decrease rate of convergence for Newton technique.
Using of parameter (7) allows guarantee robustness for this method and receive higher order resolution for two-phase flow problem.

4. Numerical results
In this section, two numerical test problems are analyzed. The first one is the classical Buckley-Leverett problem. In the second one, 1D oil-water displacement including gravitational and capillary pressure effects is considered. In both examples porosity and absolute permeability of porous medium equal 0.03 and 50 mD, respectively, length of the computational domain is given by $L = 1000$ m, the residual saturations for oil and water are determined as 0.15 and 0.2, respectively. The pressure maximum is given by $p_{\text{max}} = 200$ bar, initial pressure equal 1 bar, density and viscosity are determined as 860 kg/m$^3$ and 0.005 Pa∙s for oil and 1000 kg/m$^3$ and 0.001 Pa∙s for water.

4.1. Buckley-Leverett problem
To evaluate computational effectiveness of proposed scheme (HOU) comparison with other methods and semi analytical solution was performed. The first one is the standard First Order Upwind (FOU) method. The second one is the High Order MUSCL-type scheme (HOM) proposed by Contreras et al. [6]. In Figure 2, the saturation profiles obtained using all this methods for mesh size 100 are presented.

![Figure 2](image.png)

**Figure 2.** Saturation profiles for the FOU, HOM and HOU methods at $t = 720$ days.

As it can be seen, high order methods allow receive much more accurate solutions for this problem. Such error in the $L_1$ norm [6] consist 0.0088879 for FOU method, 0.0027921 for HOM and 0.0028558 for HOU schemes. Difference between errors for high order methods is only 2.23 % against 213.46 % for the FOU scheme. On the other hand, HOM method increase rate of convergence for Newton’s method. As a result HOU method presented in this paper approximately three times faster.

4.2. Two-phase flow problem with capillary and gravity effects
In this subsection, one-dimensional two-phase flow in undeformable porous medium with gravitational and capillary pressure effects is considered. In this case the angle between direction $x$ and horizontal doesn’t equals zero. For this case it given by $\theta = 60^\circ$. Solution profiles for FOU and HOU methods are shown in Figure 3.

As it can be seen, the gravitational effect causes smaller displacement of saturation front than in previous problem. But both effects not change difference between solutions obtained by the first and high order methods.
5. Conclusions
A high order fully implicit cell-centered finite volume method was successfully implemented for 1D oil-water displacement in undeformable porous medium. Numerical tests showed the effectiveness of this method for receiving more accurate solution in comparison with first order method without increasing of Newton’s method convergence rate. Proposed method can be useful for numerical simulation any multiphase flows in the reservoirs, where using fully implicit formulation is necessary.

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