Supersymmetric D3 brane action in AdS$_5 \times $S$^5$

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Abstract

We find the space-time supersymmetric and $\kappa$-invariant action for a D3-brane propagating in the AdS$_5 \times $S$^5$ background. As in the previous construction of the fundamental string action in this maximally supersymmetric string vacuum the starting point is the corresponding superalgebra $su(2,2\mid4)$. We comment on the super Yang-Mills interpretation of the gauge-fixed form of the action.
1 Introduction

The action of a D3-brane probe propagating in a curved type IIB supergravity background is described by a combination of a Born-Infeld-type term and a Wess-Zumino type term (see, e.g., [1, 2, 3, 4, 5]). The supersymmetric and $\kappa$-invariant expressions for the D3-brane action in flat space and general type IIB backgrounds were constructed in [6, 7, 8].

Type IIB supergravity has two maximally supersymmetric vacua: flat space and $AdS_5 \times S^5$ background [9]. In the case of the flat space the D3-brane action computed in the static gauge and $\kappa$-symmetry gauge is a non-linear generalisation of the abelian $\mathcal{N} = 4$ supersymmetric Yang-Mills action which has 16 linearly and 16 non-linearly realised supersymmetries [10]. Since $AdS_5 \times S^5$ is the large charge or near-horizon limit [10] of the D3-brane solution [11, 12], the corresponding action may be interpreted as describing a D3-brane probe propagating near the core of a D3-brane source.

In string theory, a collection of $N$ parallel D3-branes is described by connecting open strings [1], or (at low energies) by $U(N)$ SYM theory [13]. The action for a single D3-brane separated from $N - 1$ others is obtained by integrating out massive open strings stretched between the ‘probe’ and the ‘source’ (see, e.g., [14]). In the large $N$ (large charge) limit the BI-type D3-brane action should essentially coincide with the leading IR part of the quantum $\mathcal{N} = 4$ SYM effective action obtained by keeping the $U(1)$ $\mathcal{N} = 4$ vector multiplet as an external background and integrating out massive SYM fields [13, 15, 17, 18]. Since the quantum $\mathcal{N} = 4$, $D = 4$ SYM theory is conformally invariant, the resulting action should also have (spontaneously broken by scalar field background and thus non-linearly realised) conformal symmetry. The non-linear conformal invariance of the bosonic part of the static-gauge D3-brane action in $AdS_5 \times S^5$ was demonstrated in [17, 19, 20].

Our aim will be to find the complete supersymmetric (invariant under the 32 global supersymmetries of the $AdS_5 \times S^5$ vacuum) and $\kappa$-invariant form of the D3-brane action in $AdS_5 \times S^5$ space. After fixing the static gauge and $\kappa$-symmetry gauge the action will have a ‘SYM effective action’ interpretation. Like the flat space action, it will have 16 linear and 16 non-linear (conformal) supersymmetries. Its conformal invariance is a consequence of the $SO(4, 2) \times SO(6)$ isometry of the $AdS_5 \times S^5$ metric and is manifest before the static gauge fixing.

The knowledge of such supersymmetric action is quite non-trivial in view of its non-polynomiality and the absence of the manifestly supersymmetric $\mathcal{N} = 4$ (superfield) formalism: one effectively determines the exact supersymmetry transformation laws to all orders in low-energy expansion. The fermionic structure of the action is of interest also in connection with recent discussions of the SYM–supergravity correspondence (see, e.g., [21, 22, 17, 23, 24, 25, 26, 27]).

One possible approach to the construction of the action is to start with the general type IIB background actions in [6, 7, 8] and plug in the type IIB superfields representing the $AdS_5 \times S^5$ vacuum (the corresponding supergeometry was recently discussed in [28, 30]). This approach, however, is indirect and somewhat complicated as it does not make
explicit use of the basic symmetries of the problem.

Our strategy will be instead to use directly the supergroup $SU(2, 2|4)$ which is the underlying symmetry of the $AdS_5 \times S^5$ string vacuum. As in the previous construction of the type IIB string action in $AdS_5 \times S^5$, we shall obtain the space-time supersymmetric and $\kappa$-symmetric D3-brane action in terms of the invariant Cartan one-forms defined on the coset superspace $SU(2, 2|4)/[SO(4, 1) \otimes SO(5)]$. Our method is conceptually very close to the one used in [7] to find the action of a D3-brane propagating in flat space as a $D = 4$ ‘Born-Infeld plus Wess-Zumino’ type model on the flat coset superspace $(D = 10$ super Poincare group)/$(D = 10$ Lorentz group).

As in [29], our starting point will be the superalgebra $su(2, 2|4)$ containing the two pairs of translations and rotations, $(P_a, J_{ab})$ for $AdS_5$ and $(P^{\prime}_a, J_{a'b'})$ for $S^5$, and the supersymmetry generators (32 odd translations) which form the two $D = 10$ Majorana-Weyl spinors $Q^{a \alpha'}_1$ of the same chirality. Our notation and conventions will be close to those of [29] and are explained in Appendix A where we also write down the commutation relations for $su(2, 2|4)$. Appendix B contains some basic relations for Cartan forms $L^A$ on the coset superspace.

## 2 Cartan forms

To find the manifestly super-invariant and $\kappa$-invariant D3-brane action we will use the formalism of Cartan forms on the coset superspace $SU(2, 2|4)/[SO(4, 1) \otimes SO(5)]$. The left-invariant Cartan 1-forms

$$L^A = dX^M L^A_M, \quad X^M = (x, \theta),$$

are defined by

$$G^{-1}dG = L^{\dot{a}} P_{\dot{a}} + \frac{1}{2} L^{\dot{a} \dot{b}} J_{\dot{a} \dot{b}} + L^{\dot{a}} Q_{\dot{a}}, \quad (2.1)$$

where $G = G(x, \theta)$ is a coset representative in $SU(2, 2|4)$. $L^a$ and $L^{\dot{a}}$ are the 5-beins, $L^{\dot{a}1,2}$ are the two Majorana-Weyl spinors and $L^{ab}$ and $L^{a'b'}$ ($L^{a'a'} = 0$) are the Cartan connections (for their detailed form see [24] and Appendix B). They satisfy the Maurer-Cartan equations (B.1)–(B.4) implied by the structure of the $su(2, 2|4)$ superalgebra.

A specific choice of $G(x, \theta)$ which we shall use in this paper is

$$G = g(x)e^{\theta Q}, \quad (2.2)$$

where $g(x)$ is a coset representative of $[SO(4, 2) \otimes SO(6)]/[SO(4, 1) \otimes SO(5)]$, i.e. $x = (x^\mu, x^{\mu'})$ provides a certain parametrization of $AdS_5 \times S^5$ which may be kept arbitrary.

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1Since the $U(1)$ symmetry relating the two supercharges of IIB supergravity is not a symmetry of type IIB string theory [22, 23], one is to omit this $U(1)$ factor from the $U(2, 2|4)$ symmetry [21] of the supergravity vacuum. The corresponding superalgebra we use in this paper does not contain the $U(1)$ and central charge generators. In mathematics literature this superalgebra is denoted by $psu(2, 2|4)$ [34], but we will use its common ‘simplified’ notation.

2The use of a concrete parametrization for $\theta$ is needed, however, to find the representation for the 2-form $F$ which enters the BI action (see below). As in the flat space case [7], $F$ cannot be expressed in terms of the Cartan forms only.
Then
\[ G^{-1}dG = e^{-\theta Q}De^{\theta Q}, \] (2.3)
where \( D \) is the closed covariant differential
\[ D = d + \frac{1}{2} \omega^{\hat{a}\hat{b}} F_{\hat{a}\hat{b}} + e^{\hat{a}} P_{\hat{a}}, \quad D^2 = 0. \] (2.4)
The explicit form of \( D\theta \) is
\[ D\theta = (d + \frac{1}{4} \omega^{\hat{a}\hat{b}} \Gamma_{\hat{a}\hat{b}} - \frac{i}{2} \mathcal{E} e^{\hat{a}} \sigma_{\pm} \Gamma^{\hat{a}})\theta, \] (2.5)
where the matrices \( \sigma_{\pm} \) and \( \mathcal{E} \) are defined in Appendix A.

Let us note that in the 32-component spinor notation used in the present paper the superstring action in \( AdS_5 \times S^5 \) background found in [29] has the following form valid in an arbitrary parametrisation (\( \partial M_3 = M_2 \))
\[ I = -\frac{1}{2} \int_{M_2} d^2 \sigma \sqrt{-g} \ g^{ij} L_i^\hat{a} L_j^\hat{a} + \int_{M_3} H_3, \] (2.6)
\[ H_3 = i L_i^\hat{a} \wedge \bar{L}_j^\hat{a} \wedge \mathcal{K} L. \]
In the explicit parametrisation (2.2)
\[ I = -\frac{1}{2} \int_{M_2} d^2 \sigma \left[ \sqrt{-g} \ g^{ij} L_i^\hat{a} L_j^\hat{a} + 4i \epsilon^{ij} \int_0^1 dt \ L_i^\hat{a} \ L_j^\hat{a} \ \bar{\theta} \Gamma^{\hat{a}} \mathcal{K} L \right], \] (2.7)
where the notation are explained in (A.6), (B.9).

### 3 D3-brane action

The D3-brane action depends on the coset superspace coordinates \( X^M = (x^\hat{m}, \theta) \) and vector field strength \( \partial_i A_j - \partial_j A_i \). As in [3, 6], it is given by the sum of the BI and WZ terms
\[ S = S_{BI} + S_{WZ}, \] (3.1)
where (we set the 3-brane tension to be 1 and \( M_4 = \partial M_5 \))
\[ S_{BI} = -\int_{M_4} d^4 \sigma \sqrt{-\det(G_{ij} + F_{ij})}, \] (3.2)
\[ S_{WZ} = \int_{M_4} \Omega_4 = \int_{M_5} H_5, \quad H_5 = d\Omega_4. \] (3.3)
The induced world-volume metric \( G_{ij} \) is (\( i, j = 0, 1, 2, 3 \))
\[ G_{ij} = L_i^\hat{a} L_j^\hat{a} = \partial_i X^M L_M^\hat{a} \partial_j X^N L_N^\hat{a}, \quad L_i^\hat{a}(X(\sigma)) = d\sigma^i L_i^\hat{a}. \] (3.4)
The supersymmetric extension \( F = \frac{1}{2} F_{ij} d\sigma^i \wedge d\sigma^j \) of the world-volume gauge field strength 2-form \( dA \) is found to be

\[
F = dA + 2i \int_0^1 dt \; L_t^\hat{a} \wedge \bar{\theta} \Gamma^\hat{a} KL_t ,
\]

where \( L_t^\hat{a}(x, \theta) \equiv L_t^\hat{a}(x, t\theta), \; L_t(x, \theta) \equiv L(x, t\theta) \). The \( \theta \)-dependent correction term in (3.5) given by the integral over the auxiliary parameter \( t \) is exactly the same 2-form as in the string action (2.7). This representation corresponds to the specific choice of coset representative made above. Note that while \( F \) is not expressible in terms of Cartan forms only, its exterior derivative is

\[
dF = i \bar{\theta} L \wedge \hat{\imath} KL , \quad \hat{\imath} L \equiv L^\hat{a} \Gamma^\hat{a} .
\]

This important formula can be proved by making use of the Maurer-Cartan equations and equations (B.11)–(B.13) from Appendix B.

As a result, \( dF \) is manifestly invariant under supersymmetry, and then so is \( F \), provided one defines appropriately the transformation of \( A \) to cancel the exact variation of the second (string WZ) term in (3.5) (cf. [6, 7]). Note that this is the same transformation that is needed to make the superstring action (2.7) defined on a disc and coupled to \( A \) at the boundary invariant under supersymmetry.

As in flat space [6, 7], the super-invariance of \( S_{WZ} \) follows from supersymmetry of the closed 5-form \( H_5 = d \Omega_4 \). We shall determine the supersymmetric \( H_5 \) from the requirement of \( \kappa \)-symmetry of the full action \( S \) which fixes this 5-form uniquely. The \( \kappa \)-transformations are defined by (see (B.8))

\[
\delta_\kappa x^\hat{a} = 0 , \quad \delta_\kappa \theta = \kappa ,
\]

where the transformation parameter satisfies the constraint

\[
\Gamma \kappa = \kappa , \quad \Gamma^2 = 1 .
\]

\( \Gamma \) is given by

\[
\Gamma = \frac{e^{i_1 \ldots i_4}}{\sqrt{-\det(G_{ij} + F_{ij})}} \left( \frac{1}{4!} \Gamma_{i_1 \ldots i_4} \mathcal{E} + \frac{1}{4} \Gamma_{i_1 i_2} F_{i_3 i_4} \mathcal{J} + \frac{1}{8} F_{i_1 i_2} F_{i_3 i_4} \mathcal{E} \right) ,
\]

where

\[
\Gamma_{i_1 \ldots i_n} \equiv \hat{\imath} L_{[i_1} \ldots \hat{\imath} L_{i_n]} , \quad \hat{\imath} L_i \equiv L^\hat{a}_i \Gamma^\hat{a} .
\]

The corresponding variation of the metric \( G_{ij} \) is

\[
\delta_\kappa G_{ij} = -2i \delta_\kappa \bar{\theta} (\hat{\imath} L_i L_j + \hat{\imath} L_j L_i) .
\]

The variation of \( F \) is given by

\[
\delta_\kappa F = 2i \delta_\kappa \bar{\theta} \hat{\imath} L \wedge \mathcal{K} L ,
\]

(3.12)
or, in components,\[ \delta_\kappa F_{ij} = 2i \delta_\kappa \theta (\bar{L}_i \kappa L_j - \bar{L}_j \kappa L_i). \] (3.13)

Our main statement is that the D3-brane action \( S \) in (3.1) is \( \kappa \)-invariant provided the 5-form \( H_5 \) is given by
\[
H_5 = i L \wedge \left( \frac{1}{6} \bar{L} \wedge \bar{L} \mathcal{E} + F \wedge \bar{L} \mathcal{F} \right) \wedge L \\
+ \frac{1}{30} \left( e^{a_1 \ldots a_5} L^{a_1} \wedge \ldots \wedge L^{a_5} + e^{a'_1 \ldots a'_5} L^{a'_1} \wedge \ldots \wedge L^{a'_5} \right). \] (3.14)

It is possible to check (using Maurer-Cartan equations and Fierz identities) that \( H_5 \) is closed, i.e. the equation (3.3) is consistent and thus determines \( S_{\text{WZ}} \).

The important fact is that \( H_5 \) is expressed in terms of the Cartan 1-forms and super-invariant \( F \) only. This implies that \( H_5 \) is invariant under space-time supersymmetry. Then from (3.3) we conclude that \( \delta_{\text{susy}} \Omega_4 \) is exact, so that the WZ term (3.3), like the BI term (3.2), is supersymmetry invariant.

To put the fermionic part of the WZ term in the action in a more explicit form let us make a rescaling \( \theta \rightarrow t \theta \) and define
\[
H_{5t} \equiv H_5|_{\theta \rightarrow t \theta}, \quad F_t \equiv F|_{\theta \rightarrow t \theta}. \] (3.15)

Since \( L(x, t \theta) = L_t(x, t \theta) = L_{tt'}(x, \theta) \) one can show that (cf. (3.3), (3.4))
\[
F_t = dA + 2i \int_0^t dt' \bar{\theta} \bar{L}_{tt'} \wedge \kappa L_{tt'}, \quad \partial_t F_t = 2i \bar{\theta} \bar{L}_t \wedge \kappa L_t. \] (3.16)

Then using the defining equations for the Cartan forms (B.11)–(B.13) one finds from (3.14) the following differential equation
\[
\partial_t H_{5t} = d \left[ 2i \left( \frac{1}{6} \bar{\theta} \bar{L}_t \wedge \bar{L}_t \wedge \bar{L}_t \mathcal{E} L_t + \bar{\theta} \bar{L}_t \wedge F_t \wedge \mathcal{J} L_t \right) \right], \] (3.17)

which determines the \( \theta \)-dependence of \( H_5 \). With the initial condition
\[
(H_{5t})_{t=0} = H_5|_{\theta=0} = H_5^{(\text{base})} = \frac{1}{30} \left( e^{a_1 \ldots a_5} e^{a_1} \wedge \ldots \wedge e^{a_5} + e^{a'_1 \ldots a'_5} e^{a'_1} \wedge \ldots \wedge e^{a'_5} \right), \] (3.18)

where \( e^a \) and \( e^{a'} \) are (pull-backs of) the vielbein forms of \( AdS_5 \) and \( S^5 \), we conclude that \( H_5 = (H_{5t})_{t=1} \) is given by \( d\Omega_4 \) (see (3.3)) where
\[
\Omega_4 = 2i \int_0^1 dt \left( \frac{1}{6} \bar{\theta} \bar{L}_t \wedge \bar{L}_t \wedge \bar{L}_t \mathcal{E} L_t + \bar{\theta} \bar{L}_t \wedge F_t \wedge \mathcal{J} L_t \right) + \Omega_4^{(\text{base})}. \] (3.19)

The explicit form of the \( \theta \)-independent part \( \Omega_4^{(\text{base})} \) (satisfying \( d\Omega_4^{(\text{base})} = H_5^{(\text{base})} \)) depends on a particular choice of coordinates on \( AdS_5 \times S^5 \). Thus the WZ term in (3.3) can be written as (cf. (2.7))
\[
S_{\text{WZ}} = 2i \int_{M_4} \int_0^1 dt \left( \frac{1}{6} \bar{\theta} \bar{L}_t \wedge \bar{L}_t \wedge \bar{L}_t \mathcal{E} L_t + \bar{\theta} \bar{L}_t \wedge F_t \wedge \mathcal{J} L_t \right) + \int_{M_5} H_5^{(\text{base})}. \] (3.20)
Using (B.14),(B.15) one can then find the expansion of $S_{WZ}$ in powers of $\theta$.

It is useful to recall that the only non-trivial background fields in $AdS_5 \times S^5$ vacuum are the space-time metric and the self-dual RR 5-form. The bosonic parts of the last two terms in $H_5$ (3.14) represent, indeed, the standard bosonic couplings of the D3-brane to the 5-form background (their explicit coordinate form can be found, e.g., in [20]). The action we have obtained contains also the fermionic terms required to make this coupling supersymmetric and $\kappa$-invariant.

Let us stress again that we have started with the BI action expressed in terms of the Cartan 1-forms and the 2-form in (2.7),(3.3) (determined in [29]) as implied by the structure of $AdS_5 \times S^5$ space or the $su(2,2|4)$ superalgebra. We then fixed the form of $H_5$ from the requirement of $\kappa$-symmetry of the full action. The fact that we have reproduced the bosonic part of the self-dual 5-form is in perfect agreement with the result of [6, 8] that the D3-brane action is $\kappa$-invariant only in a background which is a solution of IIB supergravity (for $AdS_5 \times S^5$ space this implies the presence of the non-trivial self-dual 5-form field [9]).

4 Remarks on gauge fixing

To summarize, we have found the supersymmetric action for D3-brane probe propagating in $AdS_5 \times S^5$ background. The action is given by (3.1)–(3.5),(3.20), with the closed 5-form defining the WZ term given in (3.14).

This action is world-volume reparametrisation invariant and $\kappa$-invariant. Its advantage is that it is manifestly invariant under the symmetries of $AdS_5 \times S^5$ vacuum: bosonic isometries $SO(4,2) \times SO(6)$ and 32 supersymmetries. However, it does not have a particularly simple form when written in terms of the coordinates $(x, \theta)$, even using the closed expressions for the Cartan forms in terms of $\theta$ [31], given in (B.14),(B.15).

To put the action in a more explicit form and also to establish a connection to the SYM theory discussed in the Introduction we need to (i) choose special bosonic coordinates in $AdS_5 \times S^5$, (ii) fix the static gauge so that the D3-probe is oriented parallel to the D3-source, and (iii) fix the $\kappa$-symmetry gauge in a way that simplifies the fermionic part of the action. After fixing the local symmetry gauges only the $ISO(3,1) \otimes SO(6)$ and 16 supersymmetries part of the original symmetry will remain manifest, while the superconformal symmetry will be realised non-linearly.

One standard choice of the bosonic coordinates in $AdS_5 \times S^5$ is such that $ds^2 = \frac{x^2}{R^2}dx_idx_i + \frac{R^2}{x^2}dx_sdx_s$, where $x^2 \equiv x_s x_s$, $s = 1, ... , 6$, and $R$ is the radius (which is set equal to 1 in the rest of the paper). The static gauge choice is then $x_i = \sigma_i$.\footnote{The flat-space limit (which is possible to take in the D3-brane action before the static gauge choice) is obtained by changing the coordinates so that $x^2 = R^2 e^{-2z/R}$, etc., and taking the limit $R \to \infty$.}

Both D3-brane actions – in flat space and in $AdS_5 \times S^5$ space – inherit the full set of the 32 supersymmetries of the corresponding type IIB supergravity vacua. Their gauge-fixed forms, however, have only 16 linearly realised supersymmetries. The interpretation of the remaining 16 supersymmetries as conformal ones is possible only in the $AdS_5 \times S^5$ case.
Moreover, this interpretation seems to depend on a proper choice the \(\kappa\)-symmetry gauge which should be different, e.g., from the \(\theta_1 = 0\) choice in \cite{7}. How to fix \(\kappa\)-symmetry gauge in the \(AdS_5 \times S^5\) action case in the most natural way is an important and open question. The difference between the two actions is related to the fact that while the flat space action has explicit scale \(\alpha'\), the role of such scale in the \(AdS_5 \times S^5\) action is played by the modulus of the scalar field.

A more complicated open problem is a non-abelian generalisation of the abelian D3-brane action we have found. As the \(AdS_5 \times S^5\) action is different from the flat space one this problem may have a different solution compared to the one proposed in \cite{34}. One obvious suggestion – to replace the fields \((x, \theta, A)\) by \(U(N)\) matrices and to add the overall symmetrised trace – may not work as the trace structure of the quantum SYM action appears to be more involved (cf. \cite{15}).

Acknowledgments

We are grateful to R. Kallosh for useful discussions. This work was supported in part by PPARC, the European Commission TMR programme grant ERBFMRX-CT96-0045, the INTAS grant No.96-538, the Russian Foundation for Basic Research Grant No.96-01-01144 and the Royal Society visiting grant.

Appendix A Notation and conventions

We use the following conventions for indices:

\[ a, b, c = 0, 1, \ldots, 4 \quad \text{so}(4, 1) \text{ vector indices (} AdS_5 \text{ tangent space}) \]
\[ a', b', c' = 5, \ldots, 9 \quad \text{so}(5) \text{ vector indices (} S^5 \text{ tangent space}) \]
\[ \hat{a}, \hat{b}, \hat{c} = 0, 1, \ldots, 9 \quad \text{combination of } (a, a'), (b, b'), (c, c') (D = 10 \text{ vector indices}) \]
\[ \alpha, \beta, \gamma, \delta = 1, \ldots, 4 \quad \text{so}(4, 1) \text{ spinor indices (} AdS_5 \text{)} \]
\[ \alpha', \beta', \gamma', \delta' = 1, \ldots, 4 \quad \text{so}(5) \text{ spinor indices (} S^5 \text{)} \]
\[ \hat{\alpha}, \hat{\beta}, \hat{\gamma} = 1, \ldots, 32 \quad D = 10 \text{ Majorana-Weyl spinor indices} \]

The commutation relations of the even part of \(su(2, 2|4)\) which is \(so(4, 2) \oplus so(6)\) are

\[
[P_a, P_b] = J_{ab}, \quad [P_{a'}, P_{b'}] = -J_{a'b'},
\]
\[
[P_a, J_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b, \quad [P_{a'}, J_{b'c'}] = \eta_{a'b'} P_{c'} - \eta_{a'c'} P_{b'},
\]
\[
[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + 3 \text{ terms}, \quad [J_{a'b'}, J_{c'd'}] = \eta_{b'c'} J_{a'd'} + 3 \text{ terms}.
\]

\(^4\)Let us note also that in contrast to the \(AdS_5 \times S^5\) one, the flat-space BI-type D3-brane action is not related to quantum SYM theory – the higher-order terms in it may be interpreted as tree-level string-theory \(\alpha'\) corrections.
In [29] the commutation relations for the odd part were written in terms of 16-component spinor notation. It turns out, however, that calculations are simplified if one uses the 32-component notation. We shall use the following representation for $32 \times 32$ Γ-matrices

\[ \Gamma^a = \gamma^a \otimes 1 \otimes \sigma_1, \quad \Gamma^{a'} = 1 \otimes \gamma^{a'} \otimes \sigma_2, \quad C = C \otimes C' \otimes i\sigma_2, \quad (A.1) \]

where $C$, $C'$ and $C''$ are the charge conjugation matrices for $so(9,1)$, $so(4,1)$ and $so(5)$ Clifford algebras respectively. The Majorana condition is $\bar{\Psi} = \Psi^\dagger \Gamma^0 = \Psi^T \mathcal{C}$. The 5d Dirac matrices satisfy $\gamma^{(a \gamma^b)} = \eta^{ab} = (-+++)$, $\gamma^{(a' \gamma^{b'})} = \eta^{a'b'} = (++++)$ and $\gamma^{a_1...a_5} = i\epsilon^{a_1...a_5}$, $\gamma^{a'_1...a'_5} = \epsilon^{a'_1...a'_5}$.

We shall use the following representation for each of the two 32-component Majorana-Weyl (negative chirality) supergenerators

\[ Q^\hat{\alpha} = \begin{pmatrix} 0 \\ -Q^{\alpha \alpha'} \end{pmatrix}, \quad Q_{\hat{\alpha}} = Q^{\hat{\beta}} \mathcal{C}_{\hat{\beta} \hat{\alpha}}, \quad (A.2) \]

where $Q^{\alpha \hat{\alpha}}$ is a 16-component spinor. The two supergenerators $(Q_1^\hat{\alpha}, Q_2^\hat{\alpha})$ of $su(2,2|4)$ will be combined into a 2-vector

\[ Q = \begin{pmatrix} Q_1^\hat{\alpha} \\ Q_2^\hat{\alpha} \end{pmatrix}. \quad (A.3) \]

The commutation relations for the odd part of the superalgebra in [29] can be rewritten as

\[ [Q, P_\hat{a}] = \frac{i}{2} Q \mathcal{E} \sigma_+ \Gamma_\hat{a}, \]

\[ \{Q_\hat{\alpha}, Q_\beta\} = -2i(C \Gamma^{\hat{\alpha}} \pi_+)_{\hat{a} \hat{b}} P_\hat{a} + \mathcal{E} \left[ (C \Gamma^{ab} \sigma_-)_{\hat{a} \hat{b}} J_{ab} - (C \Gamma^{a'b'} \sigma_-)_{\hat{a}' \hat{b}'} J_{a'b'} \right], \quad (A.4) \]

where $\pi_+$ and $\sigma_\pm$ stand for $32 \times 32$ matrices $1 \times 1 \times \pi_+$ and $1 \times 1 \times \sigma_\pm$,

\[ \pi_+ = \frac{1}{2}(1 + \sigma_3), \quad \sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \quad \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2), \quad (A.5) \]

and $\sigma_i$ are the usual Pauli matrices. We shall also use the following $2 \times 2$ matrices

\[ \mathcal{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (A.6) \]

which will act on the two internal indices of $Q$ in (A.3).

The chirality of the spinor Cartan 1-forms $L^{\hat{a}_1,2}$ and odd coordinates $\theta^{\hat{a}_1,2}$ is opposite to that of $Q^{\hat{a}}$, i.e. in the 32-component notation $L^{\hat{a}} = \begin{pmatrix} L^{a \alpha'} \\ 0 \end{pmatrix}$. The $L^{1,2}$ and $\theta^{1,2}$ can be combined into 2-vectors like (A.3) and $L$ and $\theta$ in the text stand for such vectors.
Appendix B Basic relations for Cartan forms on coset superspace

The Cartan forms satisfy the Maurer-Cartan equations implied by the $su(2,2|4)$ superalgebra

\begin{align}
    dL^\hat{a} &= -L^\hat{a} \wedge L^\hat{b} - i\bar{L}\Gamma^\hat{a} \wedge L, \\
    dL &= \frac{i}{2} \sigma_+ L^\hat{a} \Gamma^\hat{a} \wedge \mathcal{E}L - \frac{1}{4} L^\hat{a} \delta x^\hat{a} \wedge \mathcal{E}L, \\
    d\bar{L} &= \frac{i}{2} \bar{L}\mathcal{E} \wedge \Gamma^\hat{a} L^\hat{a} \sigma_+ - \frac{1}{4} \bar{L}\Gamma^\hat{a} \wedge L^\hat{b}, \\
    dL^{ab} &= -L^a \wedge L^b - L^{ac} \wedge L^{cb} + \bar{L}\Gamma^{ab}\sigma_+ \wedge \mathcal{E}L, \\
    d\bar{L}^{a'b'} &= L^{a'} \wedge L^{b'} - L^{a'c'} \wedge L^{c'b'} - \bar{L}\Gamma^{a'b'}\sigma_+ \wedge \mathcal{E}L.
\end{align}

As in [29] we set the radii of $AdS_5$ and $S^5$ to be 1. It is often useful to use the following expressions for the variations of Cartan forms which are also implied by the structure of the $su(2,2|4)$

\begin{align}
    \delta L^\hat{a} &= d\delta x^\hat{a} + L^\hat{b} d\delta x^\hat{b} + L^\hat{b} d\delta x^\hat{a} + 2i\bar{L}\Gamma^\hat{a} \delta \theta, \\
    \delta L &= d\delta \theta - \frac{i}{2} \sigma_+ L^\hat{a} \Gamma^\hat{a} \delta \theta + \frac{i}{4} L^\hat{a} \delta x^\hat{a} \Gamma^\hat{a} \mathcal{E}L - \frac{1}{4} \delta x^\hat{a} \Gamma^\hat{a} L, \\
    \delta \bar{L} &= d\delta \bar{\theta} + \frac{i}{2} \delta \bar{\theta} \mathcal{E} \Gamma^\hat{a} L^\hat{a} \sigma_+ - \frac{1}{4} \delta \bar{\theta} \Gamma^\hat{a} L^\hat{b} - \frac{i}{4} \bar{L}\mathcal{E} \Gamma^\hat{a} \delta x^\hat{a} \sigma_+ + \frac{1}{4} \bar{L}\Gamma^\hat{a} \delta x^\hat{b},
\end{align}

where

\begin{align}
    \delta x^\hat{a} &\equiv \delta X^M L^\hat{a}_M, & \delta \bar{x}^\hat{a} &\equiv \delta X^M L^\hat{b}_M, & \delta \theta &\equiv \delta X^M L^\hat{b}_M,
\end{align}

Let us make the rescaling $\theta \rightarrow t\theta$ and introduce

\begin{align}
    L^\hat{a}_t(x,\theta) &\equiv L^\hat{a}(x,t\theta), & L^\hat{b}_t(x,\theta) &\equiv L^\hat{b}(x,t\theta), & L_t(x,\theta) &\equiv L(x,t\theta),
\end{align}

with the initial condition

\begin{align}
    L^\hat{a}_{t=0} = e^\hat{a}, & \quad L^\hat{b}_{t=0} = \omega^\hat{a} & \quad L_{t=0} = 0,
\end{align}

where $e^\hat{a}$, $\omega^\hat{a}$ are the 5-beins and the Lorentz connections for $AdS_5 \times S^5$. Then the defining equations for the Cartan 1-forms are (see eqs. (A.10)–(A.12) and (B.2)–(B.4) in [29] for details)

\begin{align}
    \partial_t L^\hat{a} &= -2i\bar{\theta} \Gamma^\hat{a} L_t, \\
    \partial_t L_t &= d\theta - \frac{i}{2} \sigma_+ \Gamma^\hat{a} \mathcal{E} \partial L^\hat{a} + \frac{1}{4} \delta x^\hat{a} \sigma_+ L^\hat{a}, \\
    \partial_t L^{ab} &= 2\bar{\theta} \mathcal{E} \Gamma^{ab} \sigma_+ L_t, & \partial_t L^{a'b'}_t &= -2\bar{\theta} \mathcal{E} \Gamma^{a'b'} \sigma_+ L_t.
\end{align}

One can find a closed solution to these equations [30] (we set $t = 1$)

\begin{align}
    L = V(\theta) D \theta, & \quad L^\hat{a} = e^\hat{a} - 2i\bar{\theta} \Gamma^\hat{a} W(\theta) D \theta,
\end{align}
\[ L^{ab} = \omega^{ab} + 2\bar{\theta}\xi \Gamma^{ab} \sigma_- W(\theta) D\theta, \quad L^{a'b'} = \omega^{a'b'} - 2\bar{\theta}\xi \Gamma^{a'b'} \sigma_- W(\theta) D\theta, \quad (B.15) \]

where the matrices \( V \) and \( W \) are defined by

\[ V \equiv \frac{\sinh \sqrt{m}}{\sqrt{m}} = 1 + \frac{1}{3!} m + \frac{1}{5!} m^2 + \ldots, \quad (B.16) \]
\[ W \equiv \frac{\cosh \sqrt{m}}{m} - \frac{1}{2} + \frac{1}{4!} m + \frac{1}{6!} m^2 + \ldots, \quad (B.17) \]

and \( m \) is a matrix quadratic in \( \theta \)

\[ m = -\sigma_- \Gamma^{a} \xi \theta \tilde{\theta} \Gamma^{a} + \frac{1}{2} \Gamma^{a} \theta \tilde{\theta} \xi \Gamma^{a} \sigma_- - \frac{1}{2} \Gamma^{a} \theta \tilde{\theta} \xi \Gamma^{a} \sigma_- . \quad (B.18) \]

While the relations (B.1)–(B.7) are valid in an arbitrary parametrisation of the coset superspace, (B.11)–(B.18) apply only in the parametrisation of (2.2).

Note that in many formal calculations it is more convenient to use directly the defining equations (B.11)–(B.13) rather than the solution (B.14), the explicit expressions (B.14),(B.15) may be useful in discussion of \( \kappa \)-symmetry gauge fixing and related applications. As is well known, the analogs of the expressions (B.14),(B.15) can be written down for the Cartan forms corresponding to a general symmetric space (see, e.g., [36]).

References

[1] J. Polchinski, TASI lectures on D-branes, [hep-th/9611050].
[2] R.G. Leigh, Dirac-Born-Infeld action from Dirichlet sigma model, Mod. Phys. Lett. A4 (1989) 2767.
[3] M. Douglas, Branes within branes, [hep-th/9512071].
[4] C. Schmidhuber, D-brane actions, Nucl. Phys. B467 (1996) 146, [hep-th/9601003].
[5] A.A. Tseytlin, Self-duality of Born-Infeld action and Dirichlet 3-brane of type IIB superstring theory, Nucl. Phys. B469 (1996) 51, [hep-th/96020064].
[6] M. Cederwall, A. von Gussich, B.E.W. Nilsson and A. Westemberg, The Dirichlet super-three-brane in ten-dimensional type IIB supergravity, Nucl. Phys. B490 (1997) 163, [hep-th/9610148].
[7] M. Aganagic, C. Popescu and J.H. Schwarz, D-brane actions with local kappa-symmetry, Phys. Lett. B393 (1997) 311, [hep-th/9610249].
[8] E. Bergshoeff and P.K. Townsend, Super D-branes, Nucl. Phys. B490 (1997) 145, [hep-th/9611173].
[9] J.H. Schwarz, Covariant field equations of chiral N=2, D=10 supergravity, Nucl. Phys. B226 (1983) 269.
[10] G.W. Gibbons and P.K. Townsend, Vacuum interpolation in supergravity via super p-branes, Phys. Rev. Lett. 71 (1993) 3754, hep-th/9307049.
[11] G.T. Horowitz and A. Strominger, Black strings and p-branes, Nucl. Phys. B360 (1991) 197.
[12] M.J. Duff and J.X. Lu, The selfdual type IIB superthreebrane, Phys. Lett. B273 (1991) 409.
[13] E. Witten, Bound states of strings and p-branes, Nucl. Phys. B460 (1996) 335, hep-th/9510135.
[14] M.R. Douglas and M. Li, D-Brane realization of N=2 super Yang-Mills theory in four dimensions, hep-th/9604041.
[15] I. Chepelev and A.A. Tseytlin, Long-distance interactions of branes: correspondence between supergravity and super Yang-Mills descriptions, Nucl. Phys. B515 (1998) 73, hep-th/9709087.
[16] J.M. Maldacena, Branes probing black holes, hep-th/9709099.
[17] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, hep-th/9711200.
[18] E. Keski-Vakkuri and P. Kraus, Born-Infeld actions from matrix theory, Nucl. Phys. B518 (1998) 212, hep-th/9709122. S. de Alwis, Matrix models and string world sheet duality, Phys. Lett. B423 (1998) 59, hep-th/9710219. V. Balasubramanian, R. Gopakumar and F. Larsen, Gauge theory, geometry and the large N limit, hep-th/9712077.
[19] R. Kallosh, J. Kumar and A. Rajaraman, Special conformal symmetry of world-volume actions, Phys. Rev. D57 (1998) 6452, hep-th/9712073.
[20] P. Claus, R. Kallosh, J. Kumar, P. Townsend and A. van Proeyen, Conformal Theory of M2, D3, M5 and D1-Branes+D5-branes, hep-th/9801206.
[21] I.R. Klebanov, World volume approach to absorption by nondilatonic branes, Nucl. Phys. B496 (1997) 231, hep-th/9702070.
[22] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, String theory and classical absorption by three-branes, Nucl. Phys. B499 (1997) 217, hep-th/9703040.
[23] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Gauge Theory Correlators from Non-Critical String Theory, hep-th/9802109.
[24] E. Witten, Anti-de Sitter space and holography, hep-th/9802150.
[25] S.R. Das and S.T. Trivedi, Three-Brane action and the correspondence between N=4 Yang-Mills theory and Anti-de Sitter space, hep-th/9804149.
[26] S. Ferrara, M. A. Lledo and A. Zaffaroni, Born-Infeld corrections to D3 brane action in $AdS_5 \times S^5$ and N=4, d=4 primary superfields, hep-th/9805082.
[27] S.P. de Alwis, Supergravity the DBI Action and Black Hole Physics, hep-th/9804019.
[28] R. Kallosh and A. Rajaraman, Vacua of M-theory and string theory, hep-th/9805041.
[29] R.R. Metsaev and A.A. Tseytlin, Type IIB superstring action in $AdS_5 \times S^5$ background, hep-th/9805028.
[30] R. Kallosh, J. Rahmfeld and A. Rajaraman, Near Horizon Superspace, hep-th/9805217.
[31] M. G"unaydin and N. Marcus, The spectrum of the $S^5$ compactification of the chiral $N=2$, $D = 10$ supergravity and the unitary supermultiplets of $U(2,2|4)$, Class. Quantum Grav. 2 (1985) L11.
[32] M.B. Green and J.H. Schwarz, Covariant description of superstrings, Phys. Lett. B136 (1984) 367.
[33] M.T. Grisaru, P.S. Howe, L. Mezincescu, B.E.W. Nilsson and P.K. Townsend, $N=2$ superstrings in a supergravity background, Phys. Lett. B162 (1985) 116.
[34] P.G.O. Freund and I. Kaplansky, Simple supersymmetries, J. Math. Phys. 17 (1976) 228; B. Binegar, Conformal superalgebras, massless representations, and hidden symmetries, Phys. Rev. D34 (1986) 525.
[35] A.A. Tseytlin, On non-abelian generalisation of Born-Infeld action in string theory, Nucl. Phys. B501 (1997) 41, hep-th/9701125.
[36] D.G. Boulware and L.S. Brown, Symmetric space scalar field theory, Ann. of Phys. 138 (1982) 392.