Invariant Measures for Dissipative Dynamical Systems: Abstract Results and Applications

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1 Introduction

Essential characteristics of dynamical systems are described by invariant probability measures. These measures identify statistical equilibria and can provide important information about the long time behavior of the dynamics. It is therefore of paramount interest both in theory and applications to better understand this class of probability measures.

Frequently however, statistically steady states represented by invariant measures are difficult to determine. In practice physicists and engineers compute approximations of invariant probability measures by taking averages of time series (observables) associated to the system and invoking an ‘ergodicity assumption’; they posit an equivalence between these temporal averages and averages against the unknown invariant measure which they are trying to identify. Mathematically speaking, the complete and rigorous justification of such an ‘ergodic hypothesis’ seems unreachable for non-specific classes of dynamical systems and remains a challenging problem even for specific examples. With this backdrop in mind, our aim in this and future works ([CGHb, CGHa]) is to further the development of a particular mathematical framework coming from [FMRT01], which establishes a weak link between ensemble averages and temporal averages based on generalized limits but which we show is in fact applicable to a wide class of dissipative dynamical systems.\(^1\)

For the study of infinite dimensional evolution equations a number of different mathematical approaches to ergodicity have been developed, each relevant to different situations. One approach has been to focus on some classes of simple linear or semi-linear first order partial differential equations. See, for example, [Las79, Daw83, BK84, Rud88, Rud04] and references therein. While these works are significant for the fact that they provide examples of linear systems where solutions exhibit chaotic or ‘turbulent’ behavior, the methods developed are limited to a very specific class of equations. Another, different, approach involves the study of stochastic partial differential equations (and other related infinite dimensional stochastic systems) where ergodicity is defined in terms of the Markov semigroup generated by the stochastic semiflow. See e.g. [DPZ96, FM95, KS01, DPD03, HM06, HM11, Deb11]. The methods developed in these works apply to a wide variety of stochastically perturbed dissipative linear and nonlinear systems, for instance the Navier-Stokes equations, Reaction-Diffusion equations, Delay Equations, weakly damped nonlinear Schrödinger equations, complex Ginzburg-Landau equations. On the other hand, this approach relies in an essential way on an underlying mechanism of stochasticity, one in which the noise injected into the system has to take a very specific form. Furthermore, this approach identifies invariant measures of the Markov semigroup which are deterministic probability measures and thus are not carried by the global random attractor.\(^2\)

In this work we follow a different approach, pioneered in [FT75] and developed in [BCFM95, FMRT01]. These works link ensemble and temporal averages via the notion of the so called ‘generalized Banach limit’, a linear functional acting over the space of bounded continuous functions which in particular associates those elements converging at \(+\infty\) with their classical limit; cf. Definition 2.1 below. While the ideas in these works were developed in the specific setting of the 2D Navier-Stokes equations on a bounded domain some essential aspects of the framework generalize in a straightforward way to any compact semigroup \(\{S(t)\}_{t \geq 0}\) acting on a complete, separable metric space \((X,d)\). On the other hand, in the important case of non-compact semigroups (see below for an extended discussion of examples of such systems) certain difficulties appear in generalizing this approach since the construction, as described in [FT75, BCFM95, FMRT01], seems to rely in an essential way on the existence of a compact absorbing set.

More recent works, [Wan09, LRR11], have been able to remove this compactness assumption by restricting \(X\) to be a reflexive Banach space and by requiring an additional ‘weak-to-weak’ continuity assumption on \(\{S(t)\}_{t \geq 0}\). In essence, these works make use of the fact that bounded sets are weakly compact so that the arguments employed rely on the structure of reflexive Banach spaces and their associated weak topologies. While these requirements may be met in certain examples, the theory necessitates the verification of this additional weak-to-weak assumption which can be involved in practice (see e.g. [Ros98]). Moreover, there

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\(^1\) In this article, dissipative dynamical systems are defined as those which possess a bounded absorbing set, cf. [Tem97]. Note that other authors, e.g. [Hal88], have referred to dynamical systems with this property as having bounded dissipation.

\(^2\) On other hand these invariant measures are linked to the global random attractor; see [CSG11] for a discussion of such relationships in the context of stochastic differential equations (in finite dimensions).
are other interesting classes of dissipative dynamical systems which do not fall into this category. We develop some examples in detail below in this connection.

We show here that these cumbersome assumptions, imposed in [Wan09, LRR11], are in fact totally unnecessary. We demonstrate that the corresponding results in [FMRT01] extend to any continuous semigroup \( \{S(t)\}_{t \geq 0} \), evolving on any complete, separable metric space, which possesses a global attractor. Moreover, our method of proof is more elementary in character; we extend the methods of [FMRT01] via a simple topological observation which limits the growth of continuous functions in a neighborhood of any compact subset of a metric space. See Lemma 3.1 below.

With this background in mind we may describe the main abstract results in this work as follows: Suppose that \((X, d)\) is any complete, separable metric space which is acted on by a continuous semigroup \( \{S(t)\}_{t \geq 0} \) possessing a global attractor \( \mathcal{A} \). We show that for any probability distribution \( m_0 \) of initial conditions over \( X \), that there is an invariant probability measure \( m \) for \( \{S(t)\}_{t \geq 0} \), which has its support contained in the global attractor \( \mathcal{A} \), such that

\[
\int_X \varphi(x) dm(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_X \varphi(S(t)x) dm_0(x) dt,
\]

for all real valued continuous bounded observables \( \varphi \). Note that, as in [LRR11], we show here that (1.1) holds for an arbitrary probability distribution of initial conditions and not just for individual trajectories emanating from some \( x_0 \in X \); but again we do so without any restrictions on \( X \) beyond that it be a complete and separable metric space. On the other hand, in the case when \( m_0 \) is a Dirac measure supported on some point \( x_0 \in X \) (i.e. when \( m_0 = \delta_{x_0} \)) we show that (1.1) holds for any continuous real valued observable \( \varphi \). In any case, as with the previous results in this direction, (1.1) may be seen as establishing a kind of weak notion of ergodicity; by replacing the operation \( \lim_{T \to \infty} \) with the operation \( \lim_{T \to \infty} \), we show that the equality of time averages and ensemble averages can be obtained. Note finally that our results generalize naturally to a non-autonomous or a stochastic setting but this comes at the cost of significant additional technicalities. We refer the interested reader to [CGHb, CGHa] where these generalizations will be carried out in the non-autonomous and stochastic cases respectively.

The initial motivations that led us to discover the results appearing herein arose in ongoing work on nonlinear partial differential equations of parabolic type with memory effects added in the diffusion terms (see [CDPGHP11, CGHb, CGHa]). Such systems have a ‘hyperbolic character’ in comparison to their more classical cousins; they provide an interesting class of examples which do not possess in general a compact absorbing set for the associated semigroup. Of course there are many other important evolution equations which exhibit non-compact semigroups; for example, the Navier-Stokes equations on unbounded (i.e. domains where the Poincaré inequality holds) [Ros98], retarded equations with infinite delay [HL93], neutral functional differential equations [HL93], certain partial functional differential equations [Wu96], the linearly damped nonlinear wave equation as well as other equations of (partially) hyperbolic type [Lad91, Tem97]. Note that restricting consideration to semigroups evolving on reflexive Banach spaces is too stringent for many of the above cited equations. For instance in the case of neutral functional differential equations, cf. [HL93], the phase-space is typically \( C([-\tau, 0], \mathbb{R}^n) \) for some delay time \( \tau > 0 \).

Here, in order to illustrate the flexibility of our main abstract results, we will study two dynamical systems in detail. In each case we consider systems with memory which generate non-compact semigroups. We first consider a model for a viscoelastic fluids similar to the 2D Navier-Stokes equations but incorporating non-local, integro-differential diffusive terms that depend on the past history of the flow. See e.g. [Jos90, AS98, Orl99, GGP05] and below for further mathematical and physical background. We then turn to a class of neutral delay differential equations (NDDEs). These are functional differential equations with dependence on the past values of the solution and its time derivatives. Such systems arose in relation to the study of certain line transmission problems modeled by the telegrapher’s equation [BM64], and have been since encountered in various engineering and physical applications involving other hyperbolic PDEs; see e.g. [KH10] and Remark 4.5 below for connections to systems in geophysical fluid dynamics. For further background on the mathematical theory of NDDEs see e.g. [HL93].
The manuscript is organized as follows: We begin by briefly reviewing some mathematical generalities and setting notations that will be employed throughout the rest of the work. The main results are then given in precise terms in Theorems 2.1 and 2.2 below. Complete, self-contained, proofs of both results are next given in Section 3. Finally in Section 4, we turn to the study of the two concrete examples, the Navier-Stokes equations with memory and certain classes of NDDEs, establishing novel results linking invariant measures to temporal averages for these systems in Theorem 4.1 and Theorems 4.2, 4.3 respectively.

2 Notations, mathematical preliminaries and statement of the main results

Before stating the main abstract results of the work in precise terms (Theorems 2.1, 2.2) we first review some basic definitions and other essential mathematical preliminaries setting notations that will be used below. Throughout the rest of the article we will always take \((X, d)\) to be an arbitrary complete, separable metric space and consider a continuous semigroup \(\{S(t)\}_{t \geq 0}\) on \(X\); more precisely we assume that \(S(0) = \text{Id}_X\), \(S(t + s) = S(t)S(s)\) for all \(t, s \in \mathbb{R}^+\), and that \(S : \mathbb{R}^+ \times X \to X\) is separately continuous.

Recall that a global attractor \(A\) is a compact subset of \(X\) that is invariant under \(S(t)\), i.e. such that \(S(t)A = A\) for all \(t \geq 0\) and which attracts all bounded subsets of \(X\) viz.

\[
\lim_{t \to \infty} d_H(S(t)B, A) = 0, \quad \text{for all } B \subset X, B \text{ bounded},
\]

where \(d_H\) is the Hausdorff semi-distance

\[
d_H(E, F) := \sup_{x \in E} \inf_{y \in F} d(x, y) \text{ for any } E, F \subset X.
\]

The study of attractors is an extensive and well-developed subject see e.g. [Tem97, Hal88, Lad91, Chu99, Rob01, CV02, MZ08]. For the abstract results, Theorems 2.1, 2.2, we will assume that \(\{S(t)\}_{t \geq 0}\) possesses a global attractor \(A\). On the other hand, for the concrete examples considered in Section 4, we employ the following useful and rather general sufficient condition for the existence of a global attractor. Note also that this result follows immediately as a special case of the characterizations of semigroups possessing a global attractor appearing in e.g. [MWZ02, Theorem 3.8] or [CCP, Theorem 11].

**Proposition 2.1.** Let \(H\) be a Banach Space (with an associated norm \(\| \cdot \|\)) and consider \(\{S(t)\}_{t \geq 0}\) a continuous semigroup acting on \(H\). Suppose that

(i) there exists a bounded set \(\mathfrak{B} \subset H\) such that for every \(B \subset H, B\) bounded, there exists \(t^* = t^*(B) > 0\) such that \(S(t)B \subset \mathfrak{B}\) for every \(t \geq t^*\).

(ii) For \(t \geq 0\) we may split \(S(t)\) as \(S(t) = S_1(t) + S_2(t)\) such that, for every \(K > 0\),

\[
\sup_{x \in H: \|x\| \leq K} \|S_1(t)x\| \xrightarrow{t \to \infty} 0
\]

and for every bounded set \(B\) and every \(t > 0\), \(S_2(t)B\) is a precompact subset of \(H\).

Then \(\{S(t)\}_{t \geq 0}\) has a (connected) global attractor \(A\) which is the omega limit set of \(\mathfrak{B}\) i.e.

\[
A = \omega(\mathfrak{B}) := \bigcap_{t \geq 0} \bigcup_{s \geq t} S(s)\mathfrak{B}.
\]

We next introduce some notations and recall some basic properties associated with probability measures defined on the general metric space \(X\). Let \(\text{Pr}(X)\) be the set of all Borel probability measures on \(X\) with \(\mathcal{B}(X)\) the associated collection of Borel measurable sets. For \(\mathfrak{m} \in \text{Pr}(X)\), we define \(\text{supp}(\mathfrak{m})\) to be the
smallest closed set \( E \) such that \( m(E) = 1 \).
See e.g. [Rud87].
Take \( C(X) \) (resp. \( C_b(X) \)) to be the collection of real-valued continuous
(resp. bounded continuous) functions defined on \( X \).
A measure \( m \in Pr(X) \) is said to be invariant (relative to \( \{S(t)\}_{t \geq 0} \)) if
\[
m(E) = m(S(t)^{-1}E), \quad \text{for all } t \geq 0 \text{ and every } E \in B(X),
\]
or equivalently if
\[
\int_X \varphi(x)dm(x) = \int_X \varphi(S(t)x)dm(x), \quad \text{for all } t \geq 0 \text{ and every } \varphi \in C_b(X).
\]
Note that, for every invariant measure \( m \), \( \text{supp}(m) \) is contained in the global attractor \( A \).
For completeness we recall the proof of this fact, which is elementary, in an Appendix.
See Lemma 4.7 below.

Recall that a sequence \( \{m_n\}_{n \geq 0} \subset Pr(X) \) is said to converge weakly to a measure \( m \) if \( \lim_{n \to \infty} \int_X \varphi dm_n = \int_X \varphi dm \),
for every \( \varphi \in C_b(X) \). As such \( \{m_n\}_{n \geq 0} \) may be said to be weakly compact
if we can extract from \( \{m_n\}_{n \geq 0} \) a weakly convergent subsequence.
On the other hand we say such a collection \( \{m_n\}_{n \geq 0} \) is tight
if, for every \( \epsilon > 0 \) there is a corresponding compact set \( K_\epsilon \subset X \) so that \( m_n(K_\epsilon) \geq 1 - \epsilon \), for every \( n \).
Classically these two notions, tightness and weak compactness, are equivalent,
a result usually referred to as Prokhorov's theorem.
See e.g. [Bill99].

Let us also recall a special case of the classical Kakutani-Riesz Representation theorem,
as suits for our purposes below. See e.g. [Rud87] for further details.

**Lemma 2.1.** Let \( K \) be a compact Hausdorff space and suppose that \( \mathcal{L} \) is a positive linear functional
on \( C(K) \) (the continuous real valued functions on \( K \) with the usual sup norm).
Then there exists a unique positive Borel measure \( m \) on \( K \) such that, for every \( \varphi \in C(K) \),
\[ \mathcal{L}(\varphi) = \int_K \varphi(x)dm(x). \]

Finally we turn to the notion of a generalized Banach limit which is defined as follows:

**Definition 2.1.** Consider the collection \( B_+ \) of all bounded real-valued functions on \([0, \infty)\) endowed with sup norm.
A generalized Banach limit, which we denote by \( \text{LIM} \), is any linear functional on \( B_+ \) such that

\[ \text{(a)} \quad \text{LIM} g(t) \geq 0 \quad \text{for all } g \in B_+ \text{ with } g(s) \geq 0, \quad \text{for all } s \geq 0. \]

\[ \text{(b)} \quad \text{LIM} g(t) = \lim_{t \to \infty} g(t) \quad \text{for all } g \in B_+ \text{ for which the usual limit exists.} \]

It is not hard to establish the existence of such a positive linear functional as a consequence of the Hahn-Banach theorem.
Note also that, for any such \( \text{LIM} \), it may be shown that
\[
\left| \text{LIM} g(T) \right| \leq \lim_{T \to \infty} \sup_{T \to \infty} |g(T)|, \quad (2.3)
\]
for any \( g \in B_+ \). We will use this observation frequently below. See e.g. [Lax02] for further background and properties.

With these preliminaries in hand we now state the first main result which shows that (1.1) holds for any continuous observable
in the case when \( m_0 \) is a Dirac measure.

**Theorem 2.1.** Suppose that \((X,d)\) is a complete, separable metric space and \( \{S(t)\}_{t \geq 0} \) is a continuous
semigroup on \( X \) that possesses a global attractor \( A \).
Fix a generalized Banach limit \( \text{LIM} \). Then, for any \( x_0 \in X \), there exists a unique invariant measure \( m \in Pr(X) \) for \( \{S(t)\}_{t \geq 0} \) such that
\[
\int_X \varphi(x)dm(x) = \text{LIM} \frac{1}{T} \int_0^T \varphi(S(t)x_0)dt, \quad \text{for any } \varphi \in C(X), \quad (2.4)
\]
and such that \( \text{supp}(m) \subseteq A \).

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3 Note that a related, but purely measure theoretic notion of a ‘carrier’ is also sometimes used in this connection. However,
both as it concerns us here and in its usage in previous related works, cf. [FMRT01, BCFM99], the two notions can be seen to be applied
in a completely equivalent fashion. We will make this equivalence precise after the statement of the main Theorems 2.1.2.2 in Remark 2.1, (iii) below.
The second result establishes (1.1) for any initial probability measure $m_0$ but requires further in this generality that the observable be both continuous and bounded.

**Theorem 2.2.** Suppose that $(X,d)$ is a complete, separable metric space and $\{S(t)\}_{t \geq 0}$ is a continuous semigroup on $X$ that possesses a global attractor $A$. Fix a generalized Banach Limit $\lim_{T \to \infty}$. Then, for any $m_0 \in Pr(X)$, there exists a unique invariant measure $m \in Pr(X)$ for $\{S(t)\}_{t \geq 0}$ such that

$$\int_X \varphi(x)dm(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_X \varphi(S(t)x)dm_0(x)dt, \quad \text{for any } \varphi \in C_b(X),$$

and such that $\text{supp}(m) \subseteq A$.

We conclude this section with some remarks concerning Theorems 2.1, 2.2.

**Remark 2.1.**

(i) The uniqueness of the invariant measures $m$ satisfying (2.4) or (2.5) follows as a direct consequence of the regularity of borel probability measures on metric spaces; see e.g. [Bil99]. We therefore need only to establish the existence of such measures in the proofs below.

(ii) By considering, for $x_0 \in X$, the Dirac measure $m_0 = \delta_{x_0}$ we partially recover Proposition 2.1 from Proposition 2.2. Notwithstanding, we separate the two results since, in the former case of Proposition 2.1, we are able to establish (2.4) in the larger class of test functions $C(X)$. Note that the use of the smaller collection $C_b(X)$ for the space of test functions in Proposition 2.2 allows us in particular to work in the topology of weak convergence of measures. This is needed to be able to pass to a limit in a sequence of approximating measures via the Prokhorov Theorem. See Step 2 of the proof of Proposition 2.2 below.

(iii) Theorem 2.1, in the given generality of complete separable metric spaces acted on by arbitrary semigroups possessing a global attractor, also appears in [LRR11]. We give a different proof of Theorem 2.1 below, based on Lemma 3.1. Lemma 3.1 is also used in an essential way in the proof of Theorem 2.2. In contrast to Theorem 2.1, Theorem 2.2 establishes results analogous to those appearing in previous works in a much greater generality and is thus new.

(iv) In other related works, e.g. [FMRT01, BCFM95], the notion of a carrier is sometimes used as an alternative to the support of a measure. Recall that an element $m \in Pr(X)$ is said to be carried by a set $E \in B(X)$ (or that $E$ is a carrier for $m$) if $m(E) = 1$. Of course, $m$ is carried by $\text{supp}(m)$ (the smallest closed set of full measure), but this particular carrier is not unique in general (and rarely would be in practice). In any case, since the global attractor $A$ is closed, when say in Theorems 2.1, 2.2 that ‘$\text{supp}(m) \subseteq A$’ we may equivalently state that ‘$m$ is carried by $A$’.

(v) The so called Krylov–Bogoliubov procedure provides another means of associating invariant measures with time averages starting from a given fixed initial measure $\mu_0$. In [LRR11] it was shown that this procedure could be used to establish the existence of invariant measures for the class of dynamical systems considered in, for example, Theorems 2.1, 2.2.

### 3 Proof of abstract results

We turn in this section to the proof of Theorem 2.1, 2.2. Both proofs lean heavily on the following general topological lemma which is established along elementary lines.

**Lemma 3.1.** Let $(X,d)$ be a metric space and consider any compact subset $K \subseteq X$. For $\epsilon > 0$ we let, $K_\epsilon := \{x \in X : \inf_{y \in K} d(x,y) < \epsilon\}$. Then the following properties hold:
(a) For every \( \varphi \in C(X) \), there exists \( \epsilon > 0 \) so that \( \sup_{x \in K} |\varphi(x)| < \infty \).

(b) Suppose \( \varphi, \psi \in C(X) \) are such that \( \varphi(x) = \psi(x) \) for every \( x \in K \), then for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, \( \sup_{x \in K_\delta} |\varphi(x) - \psi(x)| < \epsilon \).

Proof. For the first item, (a), fix \( \varphi \in C(X) \). For every \( x \in K \) we may choose \( \delta = \delta_x > 0 \) so that, for every \( y \in B(\varphi, \delta_x) := \{ y : d(\varphi, y) < \delta_x \} \), \( |\varphi(x) - \varphi(y)| < 1 \). Choosing numbers \( \delta_x > 0 \) in this manner we may form the open cover \( C = \{ B(\varphi, \delta_x/3) : x \in K \} \) for \( K \). Since \( K \) is compact, we can extract from this cover a finite sub-cover \( C' = \{ B(x_1, \delta_{x_1}/3), \ldots, B(x_n, \delta_{x_n}/3) \} \). Take \( \epsilon = \frac{\min(\delta_{x_1}, \ldots, \delta_{x_n})}{3} \) and let \( M = 1 + \max_{j=1, \ldots, n} |\varphi(x_j)| \).

Given any \( x \in K \), we may choose \( y \in K \) such that \( d(x, y) < 2\epsilon \). Since \( C' \) covers \( K \) we may pick \( x_j \) such that \( d(y, x_j) < \delta_{x_j}/3 \). Combining these two observations we conclude perforce that \( d(x, x_j) < 2\epsilon + \delta_{x_j}/3 \leq \delta_{x_j} \) so that \( |\varphi(x)| \leq M \). Since \( x \in K \) was arbitrary to begin with this gives (a).

We turn to the second item (b). Fix \( \epsilon > 0 \). For every \( x \in K \) we choose \( \gamma_x > 0 \) so that \( |\varphi(x) - \varphi(y)| + |\psi(x) - \psi(y)| < \epsilon \) whenever \( y \in B(x, \gamma_x) \). Again, due to the compactness of \( K \), we may cover \( K \) with a finite collection \( \{ B(x_1, \gamma_{x_1}/3), \ldots, B(x_n, \gamma_{x_n}/3) \} \). Take \( \delta = \frac{\min(\gamma_{x_1}, \ldots, \gamma_{x_n})}{3} \). Similarly to the previous case we observe that \( K_\delta \subseteq \bigcup_{k=1}^{n} B(x_k, \gamma_k) \). Fix arbitrary \( y \in K_\delta \) and choose \( k \) so that \( y \in B(x_k, \gamma_k) \). Noting that \( \varphi(x_k) = \psi(x_k) \), this implies \( |\varphi(y) - \psi(y)| = |\varphi(y) - \varphi(x_k) + \psi(x_k) - \psi(y)| \leq |\varphi(y) - \varphi(x_k)| + |\psi(x_k) - \psi(y)| < \epsilon \), as needed for (b). The proof is therefore complete. \( \square \)

Proof of Theorem 2.1:

Fix any \( x_0 \in X \). We proceed in steps. First we show that the generalized Banach limit on the right hand side of (2.4) exists. We then show that the resulting functional is uniquely defined by the restriction of \( \varphi \) to the global attractor \( A \). Since \( A \) is compact, this is sufficient to infer the needed \( m \) via an application of the Kakutani-Riesz theorem. In the final step we show that the \( m \) that we have found is indeed invariant.

Step 1: Existence of the positive linear functional. To carry out this first step, we show that the map defined by \( T \mapsto \frac{1}{T} \int_0^T \varphi(S(t)x_0)dt \) is well defined as a positive linear functional on \( C(X) \) (cf. Definition 2.1). Since, by assumption \( \{S(t)\}_{t \geq 0} \) possesses a global attractor \( A \), we have that,

\[
\mathfrak{L}_{x_0}(\varphi) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(S(t)x_0)dt, \quad \text{for } \varphi \in C(X),
\]  

(3.1)

is well defined as a positive linear functional on \( C(X) \) (cf. Definition 2.1). Since, by assumption \( \{S(t)\}_{t \geq 0} \) possesses a global attractor \( A \), we have that,

\[
\text{for every } \epsilon > 0, \text{ there exists a time } T_\epsilon \geq 0 \text{ such that } S(t)x_0 \in A_\epsilon \text{ for every } t \geq T_\epsilon,
\]  

(3.2)

where here and below, \( A_\epsilon = \{ x \in X : \inf_{y \in A} d(x, y) < \epsilon \} \). Applying Lemma 3.1, (a) to \( A \) and \( \varphi \) we may choose \( \epsilon > 0 \) so that \( K_A := \sup_{x \in \mathcal{A}} |\varphi(x)| < \infty \). Taking \( T_\epsilon \) as required in (3.2) for this value of \( \epsilon \) we find also that \( K_I := \sup_{t \in [0, T]} |\varphi(S(t)x_0)| < \infty \) owing to the fact that the interval \( [0, T] \) is compact and that \( t \mapsto S(t) \) is continuous. By now taking \( K = \max\{ K_I, K_A \} \), we infer, for every \( T > 0 \), that

\[
\frac{1}{T} \int_0^T \varphi(S(t)x_0)dt \leq K < \infty
\]

as needed to finally justify the definition of \( \mathfrak{L}_{x_0} \) given in (3.1).

Step 2: Restriction to \( A \) and the application of the Kakutani-Riesz theorem. The next step will be to show that \( \mathfrak{L}_{x_0}(\varphi) \) depends only on the values of \( \varphi \) on \( A \). More precisely, we establish that if \( \varphi(x) = \tilde{\varphi}(x) \) for every \( x \in A \) then \( \mathfrak{L}_{x_0}(\varphi) = \mathfrak{L}_{x_0}(\tilde{\varphi}) \). To this end we fix any \( \epsilon > 0 \) and according to Lemma 3.1, (b) choose a corresponding \( \delta > 0 \) such that \( \sup_{x \in A_\delta} |\varphi(x) - \tilde{\varphi}(x)| < \epsilon \). We now take \( T_\delta \) so that \( S(t)x_0 \in A_\delta \) for each \( t \geq T_\delta \)

and let \( K_\delta := \sup_{t \in [0, T_\delta]} (|\varphi(S(t)x_0)| + |\tilde{\varphi}(S(t)x_0)|) \). For the reasons noted for \( K_I \) in the previous step above,
\( \hat{K}_\delta \) is finite. Using the basic properties of \( \operatorname{LIM}_{T \to \infty} \), (2.3), we estimate

\[
|\Sigma_x (\varphi - \hat{\varphi})| = \left| \operatorname{LIM}_{T \to \infty} \frac{1}{T} \int_0^T \varphi(S(t)x_0) - \hat{\varphi}(S(t)x_0)dt \right| \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^T \varphi(S(t)x_0) - \hat{\varphi}(S(t)x_0)dt \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^{T_2} |\varphi(S(t)x_0) - \hat{\varphi}(S(t)x_0)|dt + \limsup_{T \to \infty} \frac{1}{T} \int_{T_2}^T |\varphi(S(t)x_0) - \hat{\varphi}(S(t)x_0)|dt \leq \limsup_{T \to \infty} \frac{T_2 \hat{K}_\delta}{T} + \limsup_{T \to \infty} \frac{1}{T} \int_{T_2}^T |\varphi(S(t)x_0) - \hat{\varphi}(S(t)x_0)|dt \leq \limsup_{T \to \infty} \frac{T_2 \hat{K}_\delta}{T} + \limsup_{T \to \infty} \frac{(T - T_\delta)\epsilon}{T} \leq \epsilon.
\]

Since, to begin with, the choice of \( \epsilon > 0 \) was arbitrary we conclude perforce that \( \Sigma_x (\varphi - \hat{\varphi}) = 0 \) as desired.

With this in hand we may now unambiguously define

\[
\mathfrak{G}(\psi) = \Sigma_x (\ell(\psi)), \quad \text{for } \psi \in C(\mathcal{A}),
\]

where we take \( \ell : C(\mathcal{A}) \to C(X) \) to be an extension operator, \( \ell(\psi)(x) = \psi(x) \) for \( x \in \mathcal{A} \), such that

\[
\inf_{x \in X} \ell(\varphi) = \inf_{x \in \mathcal{A}} \varphi(x), \quad \sup_{x \in X} \ell(\varphi) = \sup_{x \in \mathcal{A}} \varphi(x).
\]

The existence of such an extension operator is guaranteed by the Dugundji extension theorem, [Dug51, Theorem 4.1]. One may readily verify that \( \mathfrak{G} \) is linear. With (3.4), (3.1) and Definition 2.1, (a) we see that \( \mathfrak{G} \) is also positive. Thus, by the Kakutani-Riesz representation theorem, recalled above in Lemma 2.1, there exists a unique positive, finite, Borel measure \( m \) on \( \mathcal{A} \) such that

\[
\mathfrak{G}(\psi) = \int_{\mathcal{A}} \psi(x)dm(x), \quad \text{for } \psi \in C(\mathcal{A}).
\]

Abusing notation slightly, we extend \( m \) to a Borel measure on all of \( X \) by taking \( m(E) := m(E \cap \mathcal{A}) \), \( E \in B(X) \). Clearly, \( m(\mathcal{A}^c) = 0 \), and so with (3.1), (3.3), (3.5) it follows that, for every \( \varphi \in C(X) \), \( \Sigma_x (\varphi) = \int_{\mathcal{A}} \varphi(x)dm(x) = \int_X \varphi(x)dm(x) \), which is (2.4). To see that \( m \) lies in \( \operatorname{Pr}(X) \) we simply insert \( \varphi = 1 \) into (2.4).

**Step 3. The invariance of \( m \).** For the final step we show that the \( m \) we have found in Step 2 is indeed invariant by establishing (2.2). To this end fix any \( t^* \geq 0 \) and any \( \varphi \in C_b(X) \). We compute:

\[
\int_X \varphi(S(t^*)x)dm(x) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \varphi(S(t^*)S(t)x_0)dt = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \varphi(S(t + t^*)x_0)dt = \limsup_{T \to \infty} \frac{1}{T} \int_0^{T + t^*} \varphi(S(t)x_0)dt \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^{T + t^*} \varphi(S(t)x_0)dt - \liminfsup_{T \to \infty} \frac{1}{T} \int_0^T \varphi(S(t)x_0)dt = \int_X \varphi(x)dm(x).
\]

Note that the first equality is justified since \( \psi(\cdot) = \varphi(S(t^*)\cdot) \in C_b(X) \). For the final equality above we note that both \( \limsup_{T \to \infty} \frac{1}{T} \int_0^{T + t^*} \varphi(S(t)x_0)dt \) and \( \liminfsup_{T \to \infty} \frac{1}{T} \int_0^{T + t^*} \varphi(S(t)x_0)dt \) are zero due to the boundedness of \( \varphi \) and the basic properties of \( \operatorname{LIM}_{T \to \infty} \), Definition 2.1, (b). We have thus completed the proof of Theorem 2.1.
Remark 3.1. For the proof of Theorem 2.1, the fact that \( \text{supp}(m) \subseteq A \) follows directly from the application of the Kakutani-Riesz theorem on the global attractor \( A \). The situation is different in e.g. [FMRT01] where Kakutani-Riesz theorem is applied on a compact absorbing set. Here one needs to rely on the general result, recalled in Lemma 4.7 and cf. [FMRT01, Chapter 4, Theorem 4.1], that the support of an invariant measure is always contained in the global attractor \( A \). On the other hand we use Lemma 4.7 in Step 2 of the proof of Theorem 2.2 for the general case of an \( m_0 \in Pr(X) \) with an unbounded support.

Proof of Theorem 2.2:

Let \( m_0 \in Pr(X) \) be given. Since we are assuming here that \( \varphi \) is bounded, in contrast to the proof of Proposition 2.1 it is trivial to define:

\[
\mathcal{L}_{m_0}(\varphi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_X \varphi(S(t)x) dm_0(x) dt, \quad \text{for } \varphi \in C_b(X).
\]

(3.7)

As in [LRR11], we now proceed to establish (2.5) in two steps. Initially we assume that \( m_0 \) has a bounded support so that this support is attracted to \( A \). At the second step we drop this assumption and pass to the general case of any \( m_0 \in Pr(X) \) with compactness arguments involving the Prokhorov theorem and a suitable sequence of probability measures \( m_0^n \) approximating \( m_0 \).

Step 1: \( m_0 \) with bounded support. Assuming that \( \text{supp}(m_0) \) is a bounded subset of \( X \), we fix any \( \varphi, \tilde{\varphi} \in C_b(X) \) with \( \varphi(x) = \tilde{\varphi}(x) \) for all \( x \in A \). As in Step 2 of the proof of Proposition 2.1 we would like to show that \( \mathcal{L}_{m_0}(\varphi) = \mathcal{L}_{m_0}(\tilde{\varphi}) \) so that we may once again apply the Kakutani-Riesz theorem. For this purpose fix \( \epsilon > 0 \). Invoking, as above, Lemma 3.1, (b) we choose a corresponding \( \delta > 0 \) such that \( \sup_{x \in A_t} |\varphi(x) - \tilde{\varphi}(x)| < \epsilon \). Since \( A \) is attracting and \( \text{supp}(m_0) \) is assumed to be bounded, we may choose then a time \( t^* > 0 \) such that \( S(t)(\text{supp}(m_0)) \subseteq A_s \) for all \( t \geq t^* \). Using basic properties of \( \lim_{T \to \infty} \), (2.3), we estimate

\[
|\mathcal{L}_{m_0}(\varphi - \tilde{\varphi})| \leq \limsup_{T \to \infty} \frac{1}{T} \int_0^{t^*} \int_X |\varphi(S(t)x) - \tilde{\varphi}(S(t)x)| dm_0(x) dt
\]

\[+ \limsup_{T \to \infty} \frac{1}{T} \int_{t^*}^T \int_{\text{supp}(m_0)} |\varphi(S(t)x) - \tilde{\varphi}(S(t)x)| dm_0(x) dt
\]

\[\leq \limsup_{T \to \infty} \frac{t^*}{T} \sup_{x \in X} \left( |\varphi(x)| + |\tilde{\varphi}(x)| \right) + \limsup_{T \to \infty} \frac{(T - t^*) \epsilon}{T} = \epsilon.
\]

Since \( \epsilon \) was arbitrary to begin with we infer that \( \mathcal{L}_{m_0}(\varphi) = \mathcal{L}_{m_0}(\tilde{\varphi}) \) for all \( \varphi, \tilde{\varphi} \in C_b(X) \) with \( \varphi |_A = \tilde{\varphi} |_A \).

With this in hand we are able to define a functional \( \mathcal{G} \) as in (3.3) and obtain an associated \( m \) with \( m(A^c) = 0 \). With this measure \( m \), (2.5) now follows exactly as in Step 1 of the proof of Proposition 2.1. To show that \( m \in Pr(X) \), we take \( \varphi \equiv 1 \) in (2.5) and use that \( m_0 \in Pr(X) \).

Step 2: passage to the general case via the Prokhorov theorem. Now suppose that \( m_0 \in Pr(X) \) is arbitrary. Since \( m_0 \) is a Borel probability measure it is tight; we may choose a sequence of compact sets \( K_n \subseteq X \) such that \( m_0(K_n) \geq 1 - 1/n \) (see e.g. [Bil99]). Accordingly we now define a sequence of measures \( m^n_0 \in Pr(X) \) via

\[
m^n_0(E) = \frac{m_0(E \cap K_n)}{m_0(K_n)}.
\]

Clearly each \( m^n_0 \) has a bounded support. Invoking Step 1, we obtain, for each \( n \), a corresponding invariant measure \( m^n \) such that \( m^n_0 \) and \( m^n \) satisfy (2.5). Note also that, for each \( n \supp(m^n) \subseteq A \). Hence, since \( A \) is by definition a compact set, by the Prokhorov theorem (see e.g. [Bil99]), we infer that \( \{m^n\}_{n \geq 1} \) must

\footnote{Note that, due to (3.4), the extension operator \( \ell \) is chosen to map \( C(A) \) into \( C_b(X) \).}
have a weakly convergent subsequence. Thinning this sequence of \(m^n\)s as necessary we find that, for each \(\varphi \in C_b(X)\),
\[
\int_X \varphi(x) dm(x) = \lim_{n \to \infty} \int_X \varphi(x) dm^n(x) = \lim_{n \to \infty} \frac{1}{m_0(K_n)} \lim_{T \to \infty} \int_0^T \int_{K_n} \varphi(S(t)x) dm_0(x) dt
\]
\[
= \lim_{n \to \infty} \frac{1}{m_0(K_n)} \left( \lim_{T \to \infty} \int_0^T \int_X \varphi(S(t)x) dm_0(x) dt - \lim_{T \to \infty} \int_0^T \int_{X \setminus K_n} \varphi(S(t)x) dm_0(x) dt \right)
\]
\[
= \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_{X \setminus K_n} \varphi(S(t)x) dm_0(x) dt,
\]
which is (2.5). Note that we may justify the last equality above by observing that
\[
\left| \frac{1}{T} \int_0^T \int_{X \setminus K_n} \varphi(S(t)x) dm_0(x) dt \right| \leq \sup_{x \in X} |\varphi(x)| \frac{m_0(X \setminus K_n)}{n} \leq \frac{\sup_{x \in X} |\varphi(x)|}{n}.
\]

We show that \(m \in Pr(X)\) exactly as in the previous step by inserting \(\varphi \equiv 1\) into (2.5). The invariance of \(m\) (for the case of a \(m_0\) with bounded support or otherwise) follows from a computation like (3.6). Having shown that \(m\) is an invariant probability measure it follows that \(\text{supp}(m) \subseteq A\). Indeed this is a general property of invariant measures that we restate and prove for the sake of completeness in Lemma 4.7 below. The proof of Theorem 2.2 is now complete.

4 Application to evolution equations with memory

In this section we consider two concrete dynamical systems which are non-compact but which nevertheless possess a global attractor. The common theme between of these examples is the presence of memory terms in the governing equations; we suppose that the evolution of the state variables depends on both past and current states of the system. As noted above in the introduction, each of these examples are not covered under the previous results appearing in [Wan09], [LRR11]. In the first section we study a variation on the Navier-Stokes equations, often referred to as the Jeffery’s Model, which has diffusive memory terms. Further on in Section 4.2, we consider a class of neutral differential equations, i.e. differential equations involving time derivatives of the state variable at lagged times. Here, in addition to the fact that the semigroup is non-compact, the underlying phase space \(X\) is a non-reflexive Banach space.

4.1 A viscoelastic fluids model: The Navier-Stokes equations with memory

For the first example we consider a variation on the Navier-Stokes equations, the so called Jeffery’s model, which incorporates the past history of the flow through additional diffusive, integro-differential ‘memory’ terms. See (4.1) – (4.4) below for the precise formulation of this model. Such equations arise in the study of certain viscoelastic fluids; for example to model dilute solutions of polymers or bubbly liquids. See [Jos90, GGP05].

Physically speaking, viscoelastic materials exhibit effects of both elasticity and viscosity. For such materials, the stress is typically a functional of the past history of the strain, instead of being a function of the present strain value (elastic) or of the present value of the time derivative of the strain (viscous). When an integral term is used for the history dependence, the dynamical equations become partial integrodifferential equations with a character somewhere between hyperbolic (elastic) and parabolic (viscous); see e.g. [FS86, RHN87]. Thus, it is not surprising that the Jeffery’s model has interesting hyperbolic properties in comparison to the classical Navier-Stokes equations, as we will see below.

Numerous models coming from mathematical physics or mathematical ecology incorporate such memory terms. For example one may add memory to the damped or strongly damped wave equation for the study of
wave propagation in materials with memory, or to nonlinear parabolic equations that result in "reaction-diffusion systems" that take non-Fickian diffusive effects into account. Such parabolic equations are used in the modeling of heat propagation in certain materials \cite{CG67, GP68} or more recently have been advocated in the study of population dynamics in situations where the individuals spread according to environmental feedbacks, see e.g. \cite{Pao07, CG06}.

In this type of "reaction-diffusion system" with non-Fickian diffusive effects, the contribution of the medium to the change in mobile concentration is modeled by a linear density exchange process which may be viewed as a source term with respect to the memory-less (i.e. Fickian) reaction-diffusion system. This source term can be expressed as the convolution product of a time-dependent function \( \kappa \), called the memory kernel, and some spatial variation of the mobile concentration \( u \). For instance this might lead to a diffusive memory term of the form \( \int_{-\infty}^{t} \kappa(t-s) \Delta u(s)ds \). In any case, main feature of this formulation is that the memory kernel depends only on the properties of the medium, and is therefore an intrinsic characteristic of the system. See \cite{GMLB*08} for other types of convolution products between the time derivative of the mobile concentration and the memory kernel as arising in non-Fickian dispersion in porous media and other 'real-world' situations.

The mathematical study of hyperbolic and parabolic systems modified by the kind of diffusive memory terms discussed above is now extensive. See e.g. \cite{Daf70, GP06, CDPGHP11, PZ01, GMPZ08, GP05, GGMP06, DPPZ08, DPPO8, CP06, CG*06, GGP99, GP02, GP68}. The Jeffery’s model we will consider here was previously studied in \cite{AS98, Orl99, GGP05}. Since we are interested in the dynamical properties of this system, we will treat \eqref{eq:j} - \eqref{eq:jeffery} as an autonomous dynamical system within the extended phase space formalism developed in \cite{Daf70} and used subsequently in many of the works just cited. While the existence of an attractor for \eqref{eq:j} - \eqref{eq:jeffery} in this extended phase space setting was studied in \cite{GGP05} we will establish such results here under a larger class of conditions on the memory decay kernel \( \kappa \) than was possible in \cite{GGP05}. We are able to accomplish this feat by making use of an appropriate energy functional (see Lemma 4.2 and also \cite{CDPGHP11, GMPZ08}). Note that while this more general condition, \eqref{eq:decay}, is much more challenging technically, it allows for the interesting case of a memory term \( \kappa \) exhibiting a linear decay as we describe below in Remark 4.1. Having established that \eqref{eq:j} - \eqref{eq:jeffery} possesses a global attractor within this framework the existence of invariant measures naturally follow from Theorem 2.1 and Theorem 2.2. In any case, to the best of our knowledge, no one has previously considered the existence of invariant measures for the Jeffery’s model, or for that matter any other parabolic or hyperbolic systems with such diffusive memory terms.

Note that the time asymptotic dynamics of \eqref{eq:j}–\eqref{eq:jeffery} can be quantified in the cases of a time dependent or a stochastic, white noise type forcing. This will be carried out elsewhere in \cite{CGHb, CGHa}. Of course, we could formally consider a three dimensional version of \eqref{eq:j}–\eqref{eq:jeffery}. However, as with the classical three dimensional Navier-Stokes equations, the existence and uniqueness of strong solutions is unknown. The three dimensional system is thus far from the reach of the theory we have developed here.

### 4.1.1 The Jeffery’s model: governing equations and basic assumptions

We introduce the model as follows. Fix a bounded open domain \( D \subset \mathbb{R}^2 \) with smooth boundary \( \partial D \). On \( D \) we consider the following integro-differential equation

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= \nu \Delta u + \int_{-\infty}^{t} \kappa(t-s) \Delta u(s)ds + \nabla p + f, \\
\nabla \cdot u &= 0.
\end{align*}
\]

(4.1)

Here \( u = (u_1, u_2) \), \( p \) represent the flow field and the pressure of a viscous incompressible fluid filling \( D \). The function \( \kappa \) determines the dependence on the past history of the flow through the integro-differential 'memory' term \( \int_{-\infty}^{t} \kappa(t-s) \Delta u(s)ds \) and distinguishes \eqref{eq:j}–\eqref{eq:jeffery} from the classical Navier-Stokes equations. As such, \( \kappa \) is assumed to be positive and decreasing. Further decay conditions for \( \kappa \) are be imposed below in \eqref{eq:dec1}–\eqref{eq:dec3}. We will assume throughout what follows that the external body forcing \( f \) is time independent and that \( f \in L^2(D) \).
Of course (4.1)–(4.2) is supplemented with initial and boundary conditions. We impose the no-slip (Dirichlet) boundary condition
\[ u|_{\partial D} = 0. \] (4.3)
Observe that, in view of the history term \( \int_{t-\infty}^t \kappa(t-s) u(s) ds \), the divergence-free vector field \( u \) must be known for all \( t \leq 0 \) in order to make sense of (4.1). Accordingly, the boundary-value problem is supplemented with the ‘initial condition’, or more precisely the initial past history:
\[ u(t) = u_0(t), \quad t \leq 0. \] (4.4)
Of course for each \( t \leq 0 \), \( u_0(t) \) respects the conditions (4.2), (4.3).

Concerning the memory terms in (4.1), we assume that \( \kappa \) may be written in the form:
\[ \kappa(s) := \int_0^\infty \mu(\sigma) d\sigma := \kappa_0 - \int_s^0 \mu(\sigma) d\sigma, \] (4.5)
for some nonnegative, nonincreasing function \( \mu \in L^1(\mathbb{R}^+) \), so that \( \kappa \) is nonincreasing and nonnegative. We will sometimes refer to this \( \mu \) as the memory kernel as well, when no confusion is possible with \( \kappa \). Note that \( \kappa(0) = \kappa_0 \) is the ‘total mass of \( \mu \)’ i.e.
\[ \int_0^\infty \mu(s) ds = \kappa_0 \] and that \( \kappa' = -\mu \) for almost every \( t \geq 0 \). Throughout the following, we impose the decay condition on \( \mu \):
\[ \mu(s + \sigma) \leq K e^{-\delta \sigma} \mu(s), \quad a.e. \ s, \sigma \geq 0, \] (4.6)
for any fixed \( K \geq 1, \delta > 0 \) desired. This is equivalent to requiring that
\[ \kappa(s) \leq \beta \mu(s), \quad a.e. \ s \geq 0, \] (4.7)
where we may take \( \beta = \frac{K}{\delta} \). As a byproduct of (4.7), \( \kappa \in L^1(\mathbb{R}^+) \). Using additionally that \(-\kappa' = \mu\) we have
\[ \kappa(s) \leq \kappa_0 e^{-s/\beta} = \kappa_0 e^{-(s\delta)/K} \text{ for every } s > 0. \]

Remark 4.1. The condition (4.6) is more general in comparison to many previous works including \([GGP05]\) where it was required that \( K = 1 \). The ‘physical’ significance of (4.6) is that it allows the treatment of a memory kernel \( \mu \) that has ‘flat zones’, i.e. (4.6) allows for the consideration of a \( \kappa \) with an linear decay profile. Indeed, take
\[ \mu(s) = \begin{cases} 
\mu_0 & \text{for } s \leq t^*, \\
0 & \text{for } s > t^*,
\end{cases} \]
where \( t^*, \mu_0 > 0 \) are fixed constants. Then, according to (4.5), \( \kappa_0 = \mu_0 t^* \) and
\[ \kappa(s) = \begin{cases} 
(1 - s/t^*) \kappa_0 & \text{for } s \leq t^*, \\
0 & \text{for } s > t^*.
\end{cases} \]
Observe that (4.6) is satisfied by any \( \delta > 0 \) by taking \( K = e^{\delta t^*} > 1 \). On the other hand (4.6) is clearly violated for \( K = 1, \) regardless of the choice of \( \delta > 0 \).

4.1.2 The extended phase space and its functional setting

In order to treat (4.1) - (4.4) as an autonomous dynamical system we follow \([Daf70]\) (and see above for extensive further references) and introduce an additional ‘memory’ variable \( \eta \), which is defined according to
\[ \eta(s) := \int_0^s u(t-\sigma) d\sigma = \int_1^t u(\sigma) d\sigma, \quad \text{for } t \geq 0, \ s \geq 0. \] (4.8)
Arguing formally, we obtain from (4.1) - (4.4) the following coupled system of equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= \nu \Delta u - \int_0^\infty \mu(s) \Delta \eta^t(s) \, ds + \nabla p + f \\
\partial_t \eta^t(s) &= -\partial_s \eta^t(s) + u(t), \\
\nabla \cdot u &= \nabla \cdot \eta = 0, \\
u(0) &= u_0(0), \quad \eta^0(s) = \int_{-s}^0 u_0(\sigma) \, d\sigma, \\
u_{\partial \Omega} &= 0, \quad \eta_{\partial \Omega} = 0
\end{align*}
\]  

(4.9a) - (4.9e)

We recall this derivation in detail [CGHb].

In order to place (4.9a)-(4.9e) in a rigorous functional framework we next recall some classical spaces in the mathematical theory of the Navier-Stokes equations. See e.g. [Tem01] for further background. Further spaces needed for the memory variable \( \eta \) will be recalled further on below.

Take \( \mathcal{U} := \{ \phi \in (C_0^\infty(D))^2 : \nabla \cdot \phi = 0 \} \) and define \( H := cl_{L^2(D)} \mathcal{U} = \{ u \in L^2(D)^2 : \nabla \cdot u = 0, u \cdot n = 0 \} \). Here \( n \) is the outer pointing normal to \( \partial D \). On \( H \) we take the \( L^2 \) inner product \( (u, v) := \int_D u \cdot v dD \) and associated norm \( |u| := \sqrt{(u, u)} \). The Leray-Hopf projector, \( P_H \), is defined as the orthogonal projection of \( L^2(D)^2 \) onto \( H \). At the next order let \( V := cl_{H^1(D)} \mathcal{U} = \{ u \in H^1_0(D)^2 : \nabla \cdot u = 0 \} \). On this set we use the \( H^1 \) inner product \((u, v) := \int_D \nabla u \cdot \nabla v dD \) and norm \( ||u|| := \sqrt{(u, u)} \). Note that due to the Dirichlet boundary condition the Poincaré inequality \( |u| \leq \lambda_1^{-1} ||u|| \) holds for all \( u \in V \). Here, the constant \( \lambda_1 \) is the first eigenvalue of the stokes operator \( A \) defined in the next paragraph. This justifies taking \( ||.|| \) as a norm for \( V \). We take \( V' \) to be the dual of \( V \), relative to \( H \) with the pairing notated by \( \langle, \cdot \rangle \).

We next define the Stokes operator \( A \) which is understood as a bounded linear map from \( V \) to \( V' \) via:

\[
\langle Au, v \rangle = ((u, v)) \quad u, v \in V.
\]

(4.10)

\( A \) can be extended to an unbounded operator from \( H \) to \( H \) according to \( Au = -P_H \Delta u \) with the domain \( D(A) = H^2(D)^2 \cap V \). By applying the theory of symmetric, compact, operators for \( A^{-1} \), one can establish the existence of an orthonormal basis \( \{ e_k \}_{k \geq 1} \) for \( H \) of eigenfunctions of \( A \). The associated eigenvalues \( \{ \lambda_k \}_{k \geq 1} \) form an unbounded, increasing sequence viz. \( 0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots \). We shall also make use of the fractional powers of \( A \). For \( u \in H \), we denote \( u_k = (u, e_k) \). Given \( m \geq 0 \) take \( D(A^{m/2}) = \{ u \in H : \sum_k \lambda_k^m |u_k|^2 < \infty \} \) and define \( A^{m/2}u = \sum_k \lambda_k^{m/2} u_k e_k , u \in D(A^m) \). We equip \( D(A^{m/2}) \) with the norm \( |u|_m := |A^{m/2}u| = (\sum_k \lambda_k^m |u_k|^2)^{1/2} \). Note that \( V = D(A^{1/2}) \) and that for \( u \in D(A^{1/2}) \), \( ||u|| = ||u||_1 \).

The nonlinear portion and part of the pressure gradient in (4.9) is captured in the bilinear form:

\[
B(u, v) := \langle P_H(u \cdot \nabla)v, v \rangle = \sum_{j=1}^2 P_H(u_j \partial_j v) \quad u, v \in V, \quad v \in D(A).
\]

(4.11)

For notational convenience we will sometimes write \( B(u) := B(u, u) \). Note that \( B \) is also well defined as a continuous map from \( V \times V \) to \( V' \) according to \( \langle B(u, v), w \rangle := \int_D (u \cdot \nabla)v \cdot \nabla w dD = \sum_{j,k=1}^2 \int_D u_j \partial_j v_k w_k dD \). It is easy to show that

\[
\langle B(u, v), v \rangle = 0, \quad \text{for all } u, v \in V.
\]

Classically, with Hölder’s inequality and Sobolev embeddings we have the estimate

\[
|\langle B(u, v), w \rangle| \leq C ||u||^{1/2} ||v||^{1/2} ||w||^{1/2} ||v||^{1/2}, \quad \text{for all } u, v, w \in V.
\]

(4.12)

On the other hand, slightly different estimates along the same lines lead to,

\[
|\langle B(u, v), w \rangle| \leq C ||u||^{1/2} ||v||^{1/2} ||w||^{1/2} ||Av||^{1/2} ||w||, \quad \text{for every } u, v \in D(A), \quad \text{and } w \in H.
\]

(4.13)

See e.g. [Tem01] for further details.
We now summarize some basic properties of $T$ where $A$ denotes the Stokes operator introduced above according to (4.10) along with its corresponding fractional powers recalled there. For notational convenience and in accordance with the classical notations for the Navier-Stokes equations we shall frequently denote $\cdot$ and in view of term involving $\int_0^\infty \mu(s)\eta(s)^{m/2}ds < \infty$.

For any $m \geq 0$ we define the Hilbert spaces

$$\mathcal{M}_m := L^2(\mathbb{R}^+; D(A^{m/2})) = \left\{ \eta : \eta \text{ is } (\mathbb{R}^+, D(A^{m/2})) \text{ measurable, } \int_0^\infty \mu(s)|\eta(s)|^2_m ds < \infty \right\},$$

which are equipped with the inner products,

$$[\eta, \rho]_m := \int_0^\infty \mu(s)(\eta, \rho)_m ds = \int_0^\infty \mu(s)\int_D A^{m/2}\eta \cdot A^{m/2}\rho dx ds$$

where $A$ denotes the Stokes operator introduced above according to (4.10) along with its corresponding fractional powers recalled there. For notational convenience and in accordance with the classical notations for the Navier-Stokes equations we shall frequently denote $\cdot$ and in view of term involving $\int_0^\infty \mu(s)\eta(s)^{m/2}ds < \infty$.

(ii) Suppose that $\eta_0 \in M$ and that $\xi \in L^1_{\text{loc}}([0, \infty); D(A^{m/2}))$. Then there exists a unique mild solution $\eta \in C([0, \infty); M_m)$ of the system:

$$\partial_t \eta^f = T\eta^f + \xi(t), \quad \eta^0 = \eta_0. \quad (4.19)$$

Moreover $\eta$ has explicit representation (cf. (8.4)):

$$\eta^f(s) = \begin{cases} \int_0^s \xi(t-s)ds, & \text{if } 0 < s \leq t, \\ \eta_0(s-t) + \int_0^t \xi(s)ds, & \text{if } s > t, \end{cases} \quad (4.20)$$

which is valid for any $t > 0$.

**Remark 4.2.** In the case where $K = 1$ in (4.6) it is not hard to show that $[T\eta, \eta] \leq -\delta/2$. This significantly simplifies the estimates below in e.g. (4.26), (4.36). Indeed, it is only with the functional described in Lemma 4.2 (see [GMPZ08]) that the needed dissipativity properties for the memory variable $\eta$ can be achieved.

We next define the product spaces which will serve as the phase spaces associated with the system (4.9). For $m \geq 0$ let $\mathcal{H}_m := D(A^{m/2}) \times \mathcal{M}_{m+1}$ which we endow with the product norm: $\|(u, \eta)\|^2_{\mathcal{H}_m} := \|u\|^2_m + \|\eta\|^2_{m+1}$. We will sometimes write $x = (u, \eta) \in \mathcal{H}_m$ and denote the underlying projection operators by $Px = u$ and $Qx = \eta$. In view of the above conventions, to simplify notations we will use often $\mathcal{H} := \mathcal{H}_0$ and $\mathcal{V} := \mathcal{H}_1$ and let

$$\|(u, \eta)\|^2_0 = \|u\|^2 + \|\eta\|^2,$$

and $\|(u, \eta)\|^2_1 = \|u\|^2 + \|\eta\|^2.$
Remark 4.3. The additional degree of ‘regularity’ in the memory space is dictated by basic the structure of system (4.9). See, for example, the estimates (4.25), (4.39) in the proof of Proposition 4.1 below.

For the each of the spaces $M_m$ we do not have the compact embedding of $M_{m+1}$ into $M_m$. Thus, in contrast to $H$ and $V$, $V \not\in H$. Of course this introduces additional complications for the proof of the existence of a global attractor in the extended phase for (4.9) and leads to the introduction of still further spaces. See Lemma 4.3 below.

With these definitions in hand we may now capture the memory term in (4.9) a bounded operator $M : M_m \rightarrow D(A^{m/2})$ defined, for $m \geq 0$ via,

$$M(\eta) := \int_0^\infty \mu(s)\eta(s)ds, \quad \eta \in M_m.$$  \hfill (4.22)

This definition is justified since, cf. (4.6), $\left|\int_0^\infty \mu(s)\eta(s)ds\right|_{H_k} \leq \int_0^\infty \mu(s)^{1/2}\mu(s)^{1/2} |\eta(s)|_k ds \leq C_{\mu}[\eta]_k$.

With all of these preliminaries in hand, using (4.10), (4.11), (4.17), (4.19) we now rewrite the system (4.9) in the abstract form

$$\partial_t u + \nu Au + M(\eta) + B(u) = F$$
$$\partial_t \eta = T\eta + u$$
$$\{u(0), \eta^0\} = (u_0, \eta_0)$$

where $F = P_H f$. As in [GGP05] we may now associate a dynamical system $\{S(t)\}_{t \geq 0}$ with (4.23) as follows:

Proposition 4.1. Suppose that $F \in V'$ and $x_0 = (u_0, \eta_0) \in H$. Then there exists a unique $x = (u, \eta) \in C([0, \infty), H)$ with $x(0) = x_0$, such that $P x = u \in L^2_{loc}([0, \infty), V)$ and satisfying (4.23) for every $t > 0$ (in the appropriate weak sense). By taking, for any $t \geq 0$, $S(t)x_0 = x(t) = (u(t), \eta^t)$ we may thus define a continuous semigroup $\{S(t)\}_{t \geq 0}$ on the phase space $X = H$ (endowed with its usual topology).

This Proposition may be established along classical lines with a Galerkin approximation. See e.g. [Tem01] and additionally [GGP99, GGP04] for further details on Galerkin schemes in the history space context.

4.1.3 The global attractor, invariant measures and related estimates

With this basic framework now in place we proceed to prove that the dynamical system $\{S(t)\}_{t \geq 0}$ defined by (4.23), in the sense made precise by Proposition 4.1, possesses a global attractor so that we may with no further efforts apply the results in Section 2 to (4.23). More precisely we prove:

Theorem 4.1. Suppose in addition to the conditions imposed in Proposition 4.1 that $F \in H$. Then the dynamical system $\{S(t)\}_{t \geq 0}$ defined on $H$ by (4.23) according to Proposition 4.1 has a connected global attractor $A$ and hence the results in Theorems 2.1, 2.2 hold for $\{S(t)\}_{t \geq 0}$ with $(X, d) = (H, \| \cdot \|)$.

The rest of this section is devoted to the proof of Theorem 4.1. The proof consists in verifying each of the two conditions imposed by Proposition 2.1. With the sufficient condition supplied by this Proposition in mind, we proceed in two step. At the first step we use energy type estimates to establish the existence of an absorbing set $\mathcal{B}$. Note that, in view of (4.18), the needed dissipation in variable $\eta$ does not follow directly from these standard estimates. As such we employ an additional functional as dictated by Lemma 4.2 introduced below. For the second step we achieve the splitting suggested by Proposition 2.1 by considering solution operators $S_1$ associated with no external forcing $F$ and $S_2$ with zero initial data. See (4.34), (4.35) respectively. Since, as mentioned in Remark 4.3, $V$ is not compactly embedded in $H$, we are required to introduce still further spaces at achieve the compactness desired for $S_2$. This is the significance of the space $E$ in (4.42) and Lemma 4.3 immediately following. Note that the formal estimates that follow may be rigorously justified in the context of a suitable Galerkin approximation scheme for (4.23).
Step 1: Energy estimates and dissipativity. By multiplying the first equation in (4.23) with \( u \) and making use of the cancellation properties of \( B \) we infer
\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \nu \|u\|^2 + [\eta, u] = \langle F, u \rangle. \tag{4.24}
\]
We next multiply the second equation in (4.23) by \( A\eta \) and deduce:
\[
\frac{1}{2} \frac{d}{dt} [\eta]^2 = [T\eta, \eta] + [\eta, u]. \tag{4.25}
\]
Adding (4.24) and (4.25), using (4.18) and making an obvious application of Young’s inequality we have
\[
\frac{d}{dt} (|u|^2 + [\eta]^2) + \nu \|u\|^2 \leq C \nu |F|_{V'}^2. \tag{4.26}
\]
This inequality does not evidence any dissipation in the ‘memory variable’ \( \eta \). We are therefore unable to directly invoke the Poincaré inequality and the Gronwall lemma in order to conclude the existence of a bounded absorbing set as we would for e.g. the classical 2-D Navier-Stokes equations. See however Remark 4.2 above.

To overcome this difficulty we recall the following Lemma (see [GMPZ08] and also [CDPGHP11, CGHb]) which we will use at this step and later on in Step 2 when we introduce a splitting of \( S(t) \) (cf. Proposition 2.1, above).

**Lemma 4.2.** Suppose that \( \mu \) and \( \kappa \) satisfy the conditions imposed in (4.5), (4.6). Fix any \( m \geq 0 \) and assume that \( \eta_0 \in \mathcal{M}_m, \xi \in L^1_{loc}([0, \infty); D(A^{m/2})) \) and that \( \eta \) is the unique solution of (4.19) corresponding to \( \xi \) and \( \eta_0 \) as guaranteed by Lemma 4.1, (ii). Define, the functional
\[
\Gamma_m(t) = \Gamma_m(A\eta^t) := \int_0^\infty \kappa(s) |\eta^t(s)|_m^2 ds. \tag{4.27}
\]
Then \( \Gamma_m(t) \) satisfies the following differential inequality,
\[
\frac{d\Gamma_m(t)}{dt} + \frac{1}{4\beta} (\Gamma_m(t) + \beta |\eta|^2_m) \leq 2\beta^2 \|\mu\|_{L^1(\mathbb{R}^+)} \|\xi(t)\|_m^2, \tag{4.28}
\]
where \( \beta \) is the constant arising in (4.7).

We now introduce, for each \( \Lambda > 0 \), the following functional,
\[
\Phi_\Lambda^0(t) := \frac{1}{2} (|u|^2 + [\eta]^2) + \Lambda \|u\|^2. \tag{4.29}
\]
Applying Lemma 4.1 with \( k = 1 \) to the second equation in (4.23) we find that
\[
\frac{d}{dt} \Gamma_1 + \frac{1}{4\beta} (\Gamma_1 + \beta |\eta|^2) \leq C_\mu \|u\|^2, \tag{4.28}
\]
where \( C_\mu := 2\beta^2 \|\mu\|_{L^1(\mathbb{R}^+)} \), as arising in (4.28). Combining this observation with (4.26), we estimate
\[
\frac{d}{dt} \Phi_\Lambda^0 + \frac{1}{4\beta} (\Gamma_1 + \beta |\eta|^2) \leq (C_\mu - \Lambda \nu) \|u\|^2 + \Lambda C \nu |F|_{V'}^2. \tag{4.30}
\]
Take
\[
\Lambda := \frac{1}{4\lambda_1 \nu} + \frac{C_\mu}{\nu} = \frac{1}{4\lambda_1 \nu} + \frac{2\beta^2 \|\mu\|_{L^1(\mathbb{R}^+)} \nu}{\nu}, \quad \gamma := \frac{1}{4\max \{\beta, \Lambda\}} \tag{4.31}
\]
where we recall that $\lambda_1$ the constant in the Poincaré inequality and $\beta$ comes from (4.7). With this choice of $\Lambda$ and associated $\gamma$, we observe that
\[
\frac{1}{4\beta} \Phi_0^0 \geq \gamma \Phi_0^0.
\]
and so with (4.30) and (4.31)
\[
\frac{d}{dt} \Phi_0^0 + \gamma \Phi_0^0 \leq C_{\nu, \lambda_1, \mu} |F|^2_{V'}.
\]
With the semigroup notation $x(t) = S(t)x_0$, $x_0 = (u_0, \eta_0)$ we therefore estimate using the Grönwall inequality that
\[
\|x(t)\|_0^2 \leq C_{\nu, \lambda_1, \mu} \left( e^{-\gamma t} \Phi_0^0(0) + \int_0^t e^{-\gamma (t-s)} |F|^2_{V'} \right) \leq C_{\nu, \lambda_1, \mu} (e^{-\gamma t}\|x_0\|^2_0 + |F|^2_{V'}).
\]
(4.32)

Note that we have used (4.7) with (4.29) to infer the last inequality. We may hence infer the existence of an absorbing ball $B$ in $H$ whose radius depend only on $|F|_{V'}$ and the universal $C = C_{\nu, \lambda_1, \mu}$ in the final inequality above. More explicitly, (4.32) shows that, given any $K > 0$, there exists a time $t_1^* = t_1^*(K)$ such that for $t > t_1^*$:
\[
S(t)\{x_0 \in H : \|x_0\| \leq K\} \subset B.
\]
(4.33)
This is the first item required for Proposition 2.1. We turn to the second step.

**Step 2: The splitting property.** We next exhibit a splitting of $S(t) = S_1(t) + S_2(t)$ taking the form required by Proposition 2.1. Given arbitrary initial data $x_0 = (u_0, \eta_0) \in H$ we define $(u_1(t), \eta_1^0) := S_1(t)x_0$ to be the solution at time $t$ of:
\[
\begin{align*}
\partial_t u_1 + \nu Au_1 + M(A\eta_1) &= -B(u, u_1), \\
\partial_t \eta_1 &= T\eta_1 + u_1,
\end{align*}
\]
(4.34)

We take $(u_2(t), \eta_1^0) = S_2(t)x_0$ to be the solution of:
\[
\begin{align*}
\partial_t u_2 + \nu Au_2 + M(A\eta_2) &= F - B(u, u_2), \\
\partial_t \eta_2 &= T\eta_2 + u_2,
\end{align*}
\]
(4.35)

Note that the $u$ appearing in the nonlinear terms of both (4.34), (4.35) is the first component of the solution of (4.23); in other words $u = PS(t)x_0$. Clearly, given any $x_0 \in H$, we have $S(t)x_0 = S_1(t)x_0 + S_2(t)x_0$.

The estimates for $S_1$, in view of Proposition 2.1, (ii) are carried in $H$ and are very similar to those in Step 1 for $S$. Indeed following the arguments leading up to (4.26) we obtain
\[
\frac{d}{dt} (\|u_1\|^2 + \|\eta_1\|^2) + \nu \|u_1\|^2 \leq 0.
\]
(4.36)

We now make a second application of Lemma 4.2. Defining, for $\Lambda > 0$, $\Phi_1^0(t) := \Gamma_1(\eta_1^0(t)) + \Lambda (\|u_1(t)\|^2 + \|\eta_1\|^2)$, arguing as above in (4.30) and tuning $\Lambda$ and $\gamma$ as in (4.31) we infer
\[
\|x_1(t)\|_0^2 \leq C_{\nu, \lambda_1, \mu} (e^{-\gamma t}\|x_0\|^2_0).
\]
(4.37)

where $x_1(t) = S_1(t)x_0$. We infer that for every $K > 0$:
\[
\sup_{\|x_0\| \leq K} \|S_1(t)x_0\|_0 \overset{t \to \infty}{\to} 0.
\]
(4.38)

so that indeed $S_1$ plays the desired role required for Proposition 2.1, (ii).

In remains to establish the requirement on $S_2$ given in Proposition 2.1, (ii): we must show that, for each $t > 0$, and each bound subset $B$ of $H$ that $S_2(t)B$ is a precompact subset of $H$. To this end we first show
that \(S(t)B\) is a bounded subset of \(\mathcal{V}\) and then, owing to the complication that \(\mathcal{V}\) is not compactly embedded in \(\mathcal{H}\), we must take further steps. See Remark 4.3 and Lemma 4.3 below.

In order to have an equation for \(t \mapsto \|S_2(t)x_0\|_1\), we multiply the first equation of (4.35) by \(Au_2\) and the second equation by \(A^2\eta_2\). With (4.18) one finds that

\[
\frac{d}{dt} ([u_2]^2 + [\eta_2]^2) + \nu|Au_2|^2 \leq 2(|B(u, u_2), Au_2)| + C|F|^2.
\]

By applying standard estimates on the nonlinear terms, (4.13), we find:

\[
2(|B(u, u_2), Au_2)| \leq C|u|^{1/2}||u||^{1/2}||u_2||^{1/2}|Au_2|^{3/2} \leq C\nu||u||^2||u_2||^2 + \nu|Au_2|^2.
\]

Hence:

\[
\frac{d}{dt} ([u_2]^2 + [\eta_2]^2) \leq C\nu([u]^2||u||^2||u_2||^2 + |F|^2) \leq C\nu([u]^2||u||^2||u_2||^2 + [\eta_2]^2) + |F|^2.
\] (4.39)

Let \(\Upsilon(t) = C\nu \int_0^t |u|^2|u|^2 dt\) where \(C\nu\) is precisely the constant appearing in (4.39). Returning to (4.26) we may observe that for any \(x_0 = (u_0, \eta_0)\), we have \(\sup_{s \leq t} |u|^2 + \nu \int_0^t |u|^2 dt \leq |u_0|^2 + tC\nu|F|_{V'}\), so that clearly \(e^\Upsilon(t) \leq \infty\) for each \(t > 0\). Applying this functional to (4.39), integrating in time appropriately, and noting that \(S_2(0)x_0 \equiv 0\) we find that

\[
\sup_{0 \leq r \leq t} \|S_2(t)x_0\|_1 = \sup_{0 \leq r \leq t} ([u_2(r)]^2 + [\eta_2(r)]^2) \leq C\nu \int_0^t e^{\Upsilon(t) - \Upsilon(s)} |F|^2 ds \leq C\nu t |F|^2.
\] (4.40)

This shows that, for every bounded set \(B \subset \mathcal{H}\) and for every \(t \geq 0\) that \(S(t)B\) is a bounded subset of \(\mathcal{V}\). However, as described above in Remark 4.3, \(\mathcal{V}\) is not compactly embedded in \(\mathcal{H}\) and further steps are therefore required to established the desired compactness of \(S_2\).

To compensate for this difficulty we follow previous works (see e.g. [GGP05, CPS06]) and introduce some additional spaces. On \(\mathcal{M}_1\) we define ‘tail functional’

\[
\mathcal{T}_\mu(\eta) = \sup_{\sigma \geq 1} \sigma \cdot \int_{(0,1/\sigma) \cup (\sigma, \infty)} \mu(s)||\eta(s)||^2 ds,
\] (4.41)

and consider the subspace

\[
\mathcal{E} = \{\eta \in \mathcal{N}_2 : \mathcal{T}_\mu(\eta) < \infty\} \subset \mathcal{N}_2,
\] (4.42)

which we endow with the norm, \(|\eta|_\mathcal{E}^2 : = [\|\eta\|^2 + [T\eta]^2 + \mathcal{T}_\mu(\eta)]\). Under these definitions \(\mathcal{E}\) may be shown to be a closed subset of \(\mathcal{M}_2\) (relative to \(\|\cdot\|_{\mathcal{E}}\)) and hence is a Banach space; see [PZ01]. We thus define the product space \(\mathcal{V} = V \times \mathcal{E} \subset \mathcal{V}\) and endow \(\mathcal{V}\) with the norm,

\[
\|(u, \eta)\|_{\mathcal{V}}^2 = ||u||^2 + |\eta|_\mathcal{E}^2 = ||(u, \eta)||_\mathcal{V}^2 + [T\eta]^2 + \mathcal{T}_\mu(\eta).
\] (4.43)

We have the following compactness results and related estimates on the evolution equation for the history variable. See e.g. [GGP05, CPS06, CGH]B.

**Lemma 4.3.** Assume that the memory kernel \(\mu \in L^1(\mathbb{R}^+)\) is nonnegative, non-increasing and satisfies (4.6).

(i) \(\mathcal{E}\) is compactly embedded in \(\mathcal{M}_1\) and hence \(\mathcal{V}\) is compactly embedded in \(\mathcal{H}\).

(ii) Suppose that \(\eta_0 = 0\)\(^5\), and that \(\xi \in L^\infty_{\text{loc}}(0, \infty); \mathcal{V}\). Let \(\eta^t\) be the unique mild solution in \(C([0, \infty), \mathcal{M}_1)\) of

\[
\partial_t \eta^t = T\eta^t + \xi(t), \quad \eta^0 = 0.
\] (4.44)

\(^5\)With some modifications we may assume merely that \(\eta_0 \in \mathcal{E}\). Since this is unneeded for our current purposes we do not state the Lemma in this greater generality here.
guaranteed by Lemma 4.1, (iii). Then there exists a universal positive constant $C = C_\mu$ (depending on $\mu$ but independent of $t$ and the data) such that

$$\|T\eta^t\|^2 + \exists \mu(\eta^t) \leq C \sup_{s \in [0,t]} \|\xi(s)\|^2 < \infty.$$  \hspace{1cm} (4.45)

With Lemma 4.3 in hand and having already established (4.40) we finally proceed to show that $S_2$ is compact. Fix any $x_0 = (u_0, \eta_0) \in \mathcal{H}$. Observe that, due to (4.40) we may infer that

$$\sup_{0 \leq \tau \leq t} \|PS_2(t)x_0\|^2 \leq \sup_{0 \leq \tau \leq t} \|S_2(t)x_0\|^2 \triangleq J(t) < \infty$$  \hspace{1cm} (4.46)

where we note that $J(t)$ is independent of $x_0$. We now apply Lemma 4.3, (ii) with $\xi = PS_2(t)x_0$ in (4.44). From (4.35), (4.45) and (4.46) we infer

$$\|TQS_2(t)x_0\|^2 + \exists \mu(QS_2(t)x_0) < CJ(t)$$  \hspace{1cm} (4.47)

where $C$ is the constant appearing in (4.45) and is independent of $x_0$. Combining (4.46), (4.47) and cf. (4.33)

$$\sup_{\|x_0\| \leq K} \|S_2(t)x_0\|_{\mathcal{V}}^2 \leq CJ(t) < \infty,$$

for any $K > 0$. In other words we infer that for every bounded set $B \subset \mathcal{H}$ and for every $t \geq 0$ that $S_2(t)B$ is a bounded subset of $\mathcal{V} \subset \mathcal{V}$ and so conclude from Lemma 4.3, (i) that $S_2$ in compact. The second requirement for Proposition 2.1 has therefore been fulfilled and the proof of Proposition 4.1 is finally complete.

### 4.2 Neutral delay differential equations

A retarded functional differential equation (RFDE) describes a system where the rate of change of its state is determined by the present and the past states of the system. If, additionally, the rate of change of the state depends on the rate of change of the state in the past, the system is called a neutral functional differential equation (NFDE). When only discrete values of the past have influence on the present rate of change of the system’s state, the corresponding mathematical model is either a delay differential equation (DDE) or a neutral differential equation (NDDE). The theory of RFDEs and NFDEs is both of theoretical and practical interest, as these types of models provide a powerful framework used in the study of many phenomena in the applied sciences; for example in physics, biology, economics, and control theory to name a few.

The initial development of neutral functional differential equations has been intimately related to some particular transmission line problems modeled by hyperbolic PDEs such as the telegrapher’s equations. In the seminal works [AM60], [BM64], it was observed that certain linear hyperbolic PDEs with \textit{nonlinear boundary conditions} were equivalent to an equation of the form $\dot{x}_t - L\dot{x}_t = f(x_t)$ where $L$ is an operator which does not depend upon the values of $x(t)$ but only on values in the past. In other words the evolution equation of the system may be characterized by the simultaneous presence of delayed and non-delayed derivatives. We will explore this relationship in further detail for the specific case of the telegrapher’s equation, with simplified, linear, boundary conditions (4.55), below in subsection 4.2.1.

The linear boundary conditions (4.55) have been chosen here merely to give a simple illustration of how NDDEs may arise in problems modeled by hyperbolic PDEs. More complex, physically realistic boundary conditions, such as nonlinear or dynamic boundary conditions, lead to many other (nonlinear) NDDEs. See e.g. [HL93, BC04, BKW07]. In [BC04] for example, a nonlinear NDDE model derived from the telegrapher’s equation, but with nonlinear boundary conditions, was employed to model an actual electronic device. Here the delay may be seen to arise partially from the signal passage time through the transmission line itself. The presence of rich, chaotic delay-induced dynamics, was observed experimentally and shown to be in good agreement with the numerical simulation of their nonlinear NDDE model. More precisely it was shown that, in the appropriate regime, the attractor determined by this NDDE presented a similar coarse-grained structure in comparison to the attractor reconstructed from the empirical observation of the electronic device.
they were studying. This particular NDDE has been further analyzed in [BKW07]. Here it was shown that homoclinic bifurcations were at the origin of the complicated dynamics observed in [BC04].

Of course, to determine if such complicated dynamics may be associated with the existence of a “complicated” invariant probability measure is out of reach due to the infinite dimensional character of the phase space $X = C([-\tau,0], \mathbb{R}^n)$. Indeed, in view of the rich chaotic dynamics observed in [BC04, BKW07], even the question of the existence of an invariant probability measure for such systems seems to be nontrivial and, to the best of the authors’ knowledge, this question seems to have never been previously addressed in the literature.

Theorem 4.2 below yields a general, abstract result concerning the existence of invariant probability measures of NFDEs and furthermore associates these measures with certain temporal averages. For each initial Borel probability measure $m_0$, we obtain an invariant measure $m$ as a generalized Banach limit of $m_0$ through the time average of the semigroup generated by the NDDE. Note that, by our construction, the support of any invariant measures obtained by Theorem 4.2 (see also Theorem 4.3) is furthermore contained in $\mathcal{A}$. This does not preclude the possibility that the invariant measure $m$ could be supported by the whole of the global attractor with its complex geometry. Our results apply to any NDDE possessing a global attractor $\mathcal{A}$; cf. Theorem 2.1, 2.2 and Definition 2.1 above. Note that, in this case, such a semigroup has no smoothing effects in finite time (i.e. the semigroup does not have a compact absorbing set) and acts on $X = C([-\tau,0]; \mathbb{R}^n)$ which is a non-reflexive Banach space. As such, the results in [LRR11] do not apply for such equations. To provide concrete situations where NDDEs possess a global attractor, Theorem 4.3 provides a novel sufficient and “checkable” condition which yields a broad class of interesting examples.

The rest of this section is organized as follows. We begin by further detailing the relationship between NDDEs and hyperbolic PDEs with an example involving the telegrapher’s equation (with simplified boundary conditions). In subsection 4.2.2 we briefly recall some preliminaries from the general theory of NDDEs which allow us to establish the main abstract result, Theorem 4.2, which establishes the existence of invariant probability measures for a broad class of NFDEs. In subsection 4.2.4 we formulate and prove a result of a more practical nature, Theorem 4.3, which gives concrete sufficient conditions for the results in Theorem 4.2. The final subsection applies Theorem 4.3 to some examples related to the classical NDDE model proposed in [BM64].

### 4.2.1 NDDEs and hyperbolic PDEs

We can get some idea of how NDDEs (and also difference equations with retarded arguments may) be derived from certain classes of hyperbolic PDEs by considering an example involving the telegrapher’s equation with particular linear boundary conditions in some details. We shall see that the type of retarded system derived depends on the boundary conditions. For this purpose, we recall the classical problem considered in [BM64] of the determination of the current $I$ and voltage $V$ in a transmission line terminated at each end by linear resistors or nonlinear circuit elements such as diodes. Note that we adopt here a different approach than the one classically presented in [HL93]. The reasons of this, which are related to stability and regularity issues, will be clarified in the course of the forthcoming discussions. In particular, the exposition adopted here allows us to illustrate an important relationship between difference operators and NDDEs. As we shall see, cf. Lemma 4.4 below, the structure of a certain difference operator associated with the type of NDDEs (or NFDEs) we consider here determines the non-compact nature of the associated semigroup.

Let us take the $x-$axis in the direction of the line, with ends at $x = 0$ and $x = 1$. It is assumed that the voltage $V$ and current $I$ at each point along the line are governed by the following telegrapher’s equations

\[
\begin{align*}
\partial_x V &= -L\partial_t I, \\
\partial_x I &= -C\partial_t V,
\end{align*}
\]

\[\text{(4.48)}\]

\[\text{This question seems intractable since there is no equivalent notion of the Lebesgue measure to employ as a universal reference measure } m_0. \text{ We thus have no means to declare that a measure } m \text{ to be “complicated” in terms of a lack of absolute continuity with respect to a reference measure.}\]
Similarly we initially determine \( \psi \) now since which allows one to determine \( \phi \) where \( I \) is a solution of the increasing (resp. decreasing) characteristic \( \frac{d\psi}{dt} = c^{-1} \) (resp. \( \frac{d\psi}{dt} = -c^{-1} \)); see e.g. [ZT86].

Let us consider first the following linear boundary conditions

\[
V(0, t) + R_0 I(0, t) = E, \quad I(1, t) = 0, \quad \text{for } t \geq 0,
\]

where \( E \) is a voltage which is assumed to be constant for simplicity. Physically, this corresponds to a transmission line that is “disconnected” from the nonlinear circuit element (diode) at the right hand boundary, i.e. \( I(1, t) = 0 \), with some resistor of resistance \( R_0 \) at the left boundary. Such boundary conditions are expressed in terms of \( \phi \) and \( \psi \) as

\[
(1 + R_0 \sqrt{C/L})\phi(t) + (1 - R_0 \sqrt{C/L})\psi(t) = 2E, \\
\psi(t + \sqrt{LC}) = \phi(t - \sqrt{LC}).
\]

Consider the Cauchy problem associated with (4.48) – (4.50) and given initial data \( V(0, \cdot) = V_0(\cdot) \) and \( I(0, \cdot) = I_0(\cdot) \). From the theory of first order PDEs, we know that this Cauchy problem is well-posed provided that \( (x, t) \) belongs to the triangular domain \( \mathcal{T} \) determined by the characteristics according to,

\[
\mathcal{T} := \left\{ (x, t) \in [0, 1] \times \mathbb{R}^+ : \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \right\}
\]

Now, in order to determine \( \phi \) and \( \psi \) from the initial data and boundary conditions, we observe that since \( x \in [0, 1] \), \( \phi \) can be determined from \( V_0 \) and \( I_0 \) only on \( [-\sqrt{LC}, 0] \) according to \( \phi(-\hat{\tau}) = V_0(x) + \sqrt{LC}I_0(x) \).

Similarly we initially determine \( \psi \) only on \( [0, \sqrt{LC}] \) via \( \psi(\hat{\tau}) = V_0(x) - \sqrt{LC}I_0(x) \). It is then equation (4.51) which allows one to determine \( \phi \) and \( \psi \) for all of \( \mathbb{R} \). To be more precise, let us introduce \( \hat{\psi}(t) = \psi(t + \sqrt{LC}) \); now since \( \psi \) is known on \( [0, \sqrt{LC}] \), \( \hat{\psi} \) is determined on \( [-\sqrt{LC}, 0] \), as \( \phi \) is. We can therefore rewrite the system (4.51) as the following difference equation in continuous time:

\[
\begin{bmatrix} \phi(t) \\ \hat{\psi}(t) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1 - R_0 \sqrt{C/L}}{1 + R_0 \sqrt{C/L}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi(t - \tau) \\ \hat{\psi}(t - \tau) \end{bmatrix} + \begin{bmatrix} 2E \\ 0 \end{bmatrix},
\]

where \( \tau = \sqrt{LC} \). This delay \( \tau \) may therefore be interpreted as the characteristic time associated with wave propagation; it represents the transmission time through the transmission line itself.

The system (4.52) may be written in a compact form as

\[
x(t) - Bx(t - \tau) = f,
\]

where \( x(t) = [\phi(t) \ \hat{\psi}(t)]^T \), \( B \) is the matrix arising in (4.52) and \( f \) denotes the constant vector in (4.52). By introducing the operator \( D_0 \Phi := \Phi(0) - B\Phi(-\tau) \) acting on functions \( \Phi \in X := C([-\tau, 0], \mathbb{R}^2) \) this equation takes the functional representation

\[
D_0 x_t = f,
\]
where \( x_t \in X, \) for all \( t \) and we take \( x_t(\theta) = x(t + \theta) \) for all \( \theta \in [-\tau, 0] \). When supplemented with the initial data \( x_0 = [\phi, \psi]^T, \) where \( \phi(\theta) = V_0(-\theta) + \sqrt{L/C}I_0(-\theta) \) and \( \psi(\theta) = V_0(\theta + 1) - \sqrt{L/C}I_0(\theta + 1), \) with \( \theta \in [-\tau, 0], \) equation (4.54)(along with (4.49)) gives thus another representation of the Cauchy problem associated with the original boundary value problem (BVP) (4.48)-(4.50).

**Remark 4.4.** Let \( x_0 : [-\tau, 0] \to \mathbb{R}^2 \) be some initial condition in \( X \) and define \( x(t), \) starting from \( x_0 \) according to the so-called method of steps (see e.g. [Sm11]); having determined \( x(t) \) on the interval \([-\tau, (k - 1)\tau]\) for \( k \geq 1, \) we determine \( x(t) \) on \([(k - 1)\tau, k\tau]\) using (4.53). Then \( x(0^+) - x(0^-) = Bx_0(-\tau) + f - x_0(0) \) with \( x(0^+) = \lim_{t \to 0^+} x(t); \) unless \( x_0 \) satisfies (4.53) at \( t = 0, \) we obtain a discontinuity in \( x \) at \( t = 0, \) i.e. \( x(0^+) - x(0^-) \neq 0, \) that will propagate to each \( t = k\tau, k \in \mathbb{N}. \) To avoid such a phenomenon we need to assume that \( Bx_0(-\tau) + f = x_0(0) \) which translates to \( V(0, 0) + R_0I(0, 0) = V_0(0) + R_0I_0(0) = E \) and \( I(1, 0) = I_0(1) = 0 \) in the original problem BVP (4.48)-(4.50), requiring in other words that the initial condition \( I_0, V_0 \) must satisfy the boundary conditions (4.50). This phenomenon is reminiscent of the well-known phenomena of singularity propagation along the characteristics, typical of hyperbolic PDEs, see e.g. [ZT86].

Now, if we consider instead of (4.50) the following linear boundary conditions

\[
\frac{dc(0, t)}{dt} + R_0 \frac{di(0, t)}{dt} = E, \quad \frac{di(1, t)}{dt} = 0, \tag{4.55}
\]

similar computations lead us to the following functional equation which is now an NDDE of the form

\[
\frac{d}{dt} D_0x_t = f, \tag{4.56}
\]

giving another formulation of the Cauchy problem associated with the boundary value problem (BVP) (4.48)-(4.55). Note that, here the operator \( D_0 \) is still defined as \( D_0\Phi := \Phi(0) - B\Phi(-\tau) \) acting on functions \( \Phi \in \tilde{X} := C([-\tau, 0], \mathbb{R}^2). \) From this introductory example, we can see how the difference operator \( D_0 \) arises naturally in the NDDE formulation. Further details concerning the relationship between difference equations and NDDEs, will be given in the next section.

**Remark 4.5.** As mentioned above, the telegrapher’s equations are not the only examples of hyperbolic PDEs which can be reformulated as systems involving retarded arguments of similar nature to NDDEs. For instance, in modeling of the El Niño-Southern Oscillation (ENSO), the authors in [GT00] have proposed such a neutral formulation derived from a model introduced in [JN93a, JN93b]. In these later works the ENSO dynamics were described by a linearized shallow-water (hyperbolic) model on an equatorial \( \beta \)-plane coupled with an advection equation describing the sea surface temperature changes at the earth’s equator. In [GT00], by dropping the spatial dependence in the advection equation, the authors reformulated the resulting PDE-ODE model as a system made up of difference equations coupled with ODEs. Note that, coupled systems of ODEs with difference equations arise in other physical applications dealing with wave propagations phenomena, and are strongly related to the theory of NDDEs, see [Nic01].

### 4.2.2 Existence of invariant measures for NFDEs: preliminaries and abstract result

We recall here some preliminaries concerning neutral functional differential equations used in the statement and the proof of Theorem 4.2 below concerning the existence of global attractors and hence of invariant measures for the semigroup generated by such equations. In the next subsection (see Theorem 4.3 below) we will provide concrete conditions which allow us to apply this abstract result to a broad class of NDDEs. While, as we said above, the existence of invariant measures for NFDEs is new, the exposition here largely follows the framework presented in [HL93], to which we refer the reader for further details on the extensive general theory.

Throughout what follows we take \( \langle \cdot, \cdot \rangle \) be the inner product on \( \mathbb{R}^n, \) with \( |\cdot| \) the corresponding norm. Let \( \tau > 0, \) and take \( X := C([-\tau, 0], \mathbb{R}^n) \) be the space of continuous functions taking \([-\tau, 0]\) into \( \mathbb{R}^n \) which will
be endowed in what follows with the topology induced by the supremum norm viz.

$$|\phi|_\infty := \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|, \text{ for } \phi \in X.$$  

We denote by $\mathcal{M}_n(\mathbb{R})$, the space of $n \times n$ matrix with real coefficients which we endow with the natural norm i.e. $\|M\| := \sup\{|Mx| : x \in \mathbb{R}^n, |x| = 1\}$ for any $M \in \mathcal{M}_n(\mathbb{R})$.

Throughout this section we consider a fixed map $M$ taking $[-\tau, 0]$ into $\mathcal{M}_n(\mathbb{R})$ which is of bounded variations in the sense that,

$$\text{Var}(M) := \sup_{[-\tau, 0]} \sum_i \|M(\theta_i) - M(\theta_{i+1})\| < \infty,$$

the supremum being taken over all finite partitions $-\tau = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_k = 0$ of the interval $[-\tau, 0]$. We assume furthermore that $\gamma$ a continuous nondecreasing scalar function for $s \geq 0$, such that $\gamma(0) = 0$. In language of measure theory, this last requirement implies that the vector measure $M$ (after identifying $M_n$ with $\mathbb{R}^n$) is nonatomic at zero, i.e. $M(0) - M(0^-) = 0$. In other words $M$ does not attribute a non-zero measure to $\{0\}$. This assumption about $M$ being non-atomic at zero gives a sufficient condition to develop a theory of existence and uniqueness of solutions of NFDEs of the form $(4.58)$; see [HL93] and Proposition 4.2 below.

With this $M$ in hand we define a bounded linear operator $D : X \to \mathbb{R}^n$ according to

$$D\phi := \phi(0) - \int_{-\tau}^0 [dM(\theta)]\phi(\theta). \quad (4.57)$$

In $(4.57)$, the notation $dM(\theta)$ before the integrand $\phi(\theta)$ emphasizes that $M(\theta)$ is a matrix and $\phi(\theta)$ is a column vector so that the integral is column vector-valued.

**Remark 4.6.** A classical and fundamental variety of examples of such operators $D$ arises when we take $M$ to be a step function. This leads to

$$D\phi = \phi(0) - \sum_{k=1}^N B_k \phi(-\tau_k),$$

where the $B_k$’s are $n \times n$ matrices and the delays satisfy $0 < \tau_1 < \cdots < \tau_N \leq \tau$. This type of operator $D$ is very often encountered in the applications, see e.g. [MN07] and the references therein.

Let $x$ be an $\mathbb{R}^n$ valued, continuous function defined on an interval $[-\tau, T]$ for some $T > 0$. For $t \in [0, T]$, we define $x_t \in X$ as being the “copy” of $x$ over the time interval $[t - \tau, t]$ shifted down to $[-\tau, 0]$, i.e. for each $t \in [0, T]$, we define $x_t \in X$ via $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-\tau, 0]$. For simplicity, let us assume that $f : X \to \mathbb{R}^n$ is as regular as is needed for a moment. A neutral functional differential equation (NFDE) is then given by the following relation:

$$\frac{d}{dt} Dx_t = f(x_t), \quad (4.58)$$

where $\frac{d}{dt}$ denotes the right hand side derivative at $t$. The initial data is given as element in $X$ to make sense to $(4.58)$ it is required that $t \to Dx_t$ is $C^1$ as a map with values in $\mathbb{R}^n$; cf. [HL93]. More precisely we have the following definition of solutions of $(4.58)$:

Note that, by application of the Bartle-Dunford-Schwartz theorem (see [BDS55, Theorem 3.2]), any bounded linear operator $L$ from $X$ to $\mathbb{R}^n$ may be represented as Stieltjes integral $\int_{-\tau}^0 [dN(\theta)]\phi(\theta)$ where $N$ is an $\mathbb{R}^n$ valued measure on Borel sets of $[-\tau, 0]$ which is of bounded variations. Operators having the form of $(4.57)$ may therefore be interpreted as those which have the representation corresponding to $N = \delta_0 - M$ with $M$ being non-atomic at zero and $\delta_0$ being the Dirac measure at 0.
Definition 4.1. For a given \( \phi \in X \), we say that \( x_t(\cdot;\phi) \) is a solution of (4.58) on the interval \([0,\alpha_\phi)\), \( \alpha_\phi > 0 \), with initial data \( \phi \) at \( t = 0 \), if \( x_0(\cdot;\phi) = \phi(\cdot) \), \( x_t(\cdot;\phi) \in X \), for all \( t \in [0,\alpha_\phi) \), the map \( t \to Dx_t(\cdot;\phi) \) is in \( C^1([0,\alpha_\phi),\mathbb{R}^n) \), and \( x_t(\cdot;\phi) \) satisfies (4.58) for all \( t \in [0,\alpha_\phi) \).

Remark 4.7.

(i) Note that, since \( M \) is assumed to be non atomic at zero, we have that \( \int_\beta^\alpha [dM(\theta)]\phi(\theta) \xrightarrow{\tau \to 0} 0 \) for any \( \phi \in X \). In this limiting case \( Dx_t = x(t) \) for all \( t \geq 0 \), which means that the the case of ODEs corresponds formally to \( \tau = 0 \) with the well-posed problem of \( \dot{x} = f(x) \) reducing from Definition 4.1 to the classical sense of Hadamard.

(ii) Similarly to the case of difference equations, starting from \( C^0 \) initial data, singularities may propagate but at the level of the derivatives (cf. Remark 4.4) and we do not have in general that \( t \to x_t \) is \( C^1 \) but in such a case, (4.58) reduces to \( D\dot{x}_t = f(x_t) \), an equation in which the derivative occur with delayed arguments; see [HL93] for further details.

We recall now the main results on existence, uniqueness and continuous dependence of the solutions of (4.58), and we refer to [HL93] for a proof of these properties.

Proposition 4.2. Let \( D \) be as given by (4.57) and consider any \( f \) be in \( C^1(X,\mathbb{R}^n) \). Then, for any \( \phi \in X \), there exists a unique solution \( x_t(\cdot;\phi) \) of (4.58) through \( \phi \) at \( t = 0 \) defined on a maximal interval \([0,\alpha_\phi)\), such that if \( \alpha_\phi < \infty \) then \( \lim_{\tau \to \alpha_\phi} \|x_t(\cdot;\phi)\|_\infty = \infty \). Furthermore, the map \( (t,\phi) \to x_t(\cdot;\phi) \) is continuous from the set \([0,\alpha_\phi)\times X \) into \( X \). If we assume that all solutions of (4.58) are global in time, i.e. we assume that \( \alpha_\phi = \infty \) for all \( \phi \in X \), and we introduce \( S_{D,f}(t)\phi := x_t(\cdot;\phi) \), then \( \{S_{D,f}(t)\}_{t \geq 0} \) is a continuous semigroup acting on \( X \).

Hereafter in this section, we will assume that \( D \) as given by (4.57) and \( f \) in \( C^1(X,\mathbb{R}^n) \) are such that all solutions of (4.58) are global in time.\(^9\)

We next describe a decomposition of the semigroup \( S_{D,f}(t) \), given in precise form in Lemma 4.4 below, which splits \( S_{D,f}(t) \) into two semigroups, one of which is compact and the other of which decays exponentially towards zero. As we shall see, this decomposition will depend on the asymptotic stability of the zero solution of a certain difference equation associated with \( D \); cf. (4.59), (4.60) below. Note that, in view of Proposition 2.1, Lemma 4.4 will provide a sufficient condition for the existence of a global attractor.

To formulate this result we will assume that the operator \( D \) is given by,

\[
D\phi = D_0\phi - \int_{-\tau}^0 M(\theta)\phi(\theta)d\theta, \quad \text{with},
\]

\[
D_0\phi = \phi(0) - \sum_{k=1}^N B_k\phi(-\tau_k) \quad \text{for} \quad \phi \in X,
\]

(4.59)

where the \( B_k \)'s are \( n \times n \) matrices with the delays of the form \( 0 < \tau_1 < \cdots < \tau_N \leq \tau \), and where the family of matrices \( \{M(\theta)\}_{\theta \in [-\tau,0]} \) satisfies \( \int_{-\tau}^0 \|M(\theta)\|d\theta < \infty \). We now consider a linear difference equation associated with the operator \( D_0 \) given by:

\[
D_0y_t = y(t) - \sum_{k=1}^N B_ky(t-\tau_k) = 0, \quad \text{for} \quad t \geq 0.
\]

(4.60)

Clearly, any initial value \( y_0 \) of this equation must live in the null space \( \mathcal{N}(D_0) \), so we can restrict our attention to \( X_{D_0} \), the closed subset of \( X \) defined according to:

\[
X_{D_0} := X \cap \mathcal{N}(D_0) = \{\phi \in X : D_0\phi = 0\}.
\]

\(^8\)In the sense recalled at the beginning of Section 2.

\(^9\)See Theorem 2.1 in [HC69] and references therein, for classical conditions ensuring existence of global solutions in time. See also Lemma 4.5 proved below.
This implies that the translation along the solutions of (4.60),

$$S_{D_0}(t) \phi = y_t(\cdot; \phi)$$

defines a $C^0$ semigroup on $X_{D_0}$. As shown in [Hal77, Theorem 3.3, p. 284], the infinitesimal generator $A_0$ of this semigroup is given by

$$A_0 \phi = \dot{\phi} \text{ for } \phi \in \mathcal{D}(A_0) = \{ \phi \in X_{D_0} : \dot{\phi} \in X_{D_0} \},$$

and the spectrum of $A_0$, $\sigma(A_0)$, is then given by

$$\sigma(A_0) = \{ \lambda \in \mathbb{C} : \det(H(\lambda)) = 0 \}$$

where,

$$H(\lambda) = I - \sum_{k=1}^{N} B_k e^{-\lambda r_k}.$$  

This last relationship can be heuristically understood by looking for nontrivial solutions of (4.60) of the form $y_t(\theta) = e^{\lambda(t+\theta)} v_\lambda$, $\theta \in [-\tau, 0]$ where $v_\lambda$ is some vector living in $\ker(H(\lambda))$.

As we already mentioned, Lemma 4.4 below is conditioned to the stability of the zero solution of (4.60). This stability is related to the location of the spectrum of $A_0$ with respect to the imaginary axis, which is itself determined by the roots of the characteristic equation $\det(H(\lambda)) = 0$; see e.g. [MN07]. According to [Hal77, Theorem 4.1, p. 287, Lemma 3.3, p. 284], we deduce that if the real part of the rightmost eigenvalue of $A_0$ is located in the left half-complex plane, i.e. if

$$\Re A_0 := \sup \left\{ \Re(\lambda) : \det \left[ I - \sum_{k=1}^{N} B_k e^{-\lambda r_k} \right] = 0 \right\} < 0,$$  \hspace{1cm} (4.61)

then 0 is exponentially stable in $X_{D_0}$ for $S_{D_0}(t)$ in the sense that there exist $C > 0$ such that for all $t \geq 0$ and all $\phi \in X_{D_0}$ we have $|S_{D_0}(t)\phi|_{\infty} \leq C e^{-\alpha t} |\phi|_{\infty}$, with $\alpha = \Re A_0$. In what follows we will summarize this property by just saying that $D_0$ is stable.

With this background now in place, the desired decomposition lemma may now be stated as follows:

**Lemma 4.4.** Consider $D$, and $f$ such that (4.58) is globally well posed in the sense of Proposition 4.2. Let $\{ S_{D,f}(t) \}_{t \geq 0}$ be the continuous semigroup generated by (4.58) with $D$ satisfying (4.59) such that the difference operator $D_0$ is stable in the sense that $\Re A_0$ as given by (4.61) is strictly less than zero. Then there exists a (time-independent) linear bounded operator $\Psi : X \to X_{D_0}$ such that,

$$S_{D,f}(t) = S_{D_0}(t) \circ \Psi + U_D(t), \quad t \geq 0,$$  \hspace{1cm} (4.62)

where $\{ U_D(t) \}_{t \geq 0}$ is a compact semigroup on $X$, and for every $K > 0$,

$$\sup_{|\phi|_{\infty} \leq K} |S_{D_0}(t) \circ \Psi(\phi)|_{\infty} \xrightarrow{t \to \infty} 0 \text{ exponentially with uniform decay rate } \Re A_0.$$

Proofs of this lemma may be found in [HC69, CK70]; cf. also [HL93] for a unified treatment. Note that the noncompact part of $S_{D,f}(t)$ comes from the semigroup $S_{D_0}(t)$ which is associated with the difference equation (4.60) for which $y_t = 0$ is solution. The linear bounded operator $\Psi$ arises essentially in order to map $X$ into $X_{D_0}$, the domain of $S_{D_0}(t)$. Consider, for instance, the case $M = 0$, and $B_k = 0$ for all $k \in \{1, \ldots, N \}$ in (4.59), i.e. when the NFDE (4.58) becomes a standard RFDE with no retarded arguments.

---

\[\text{Note that such a } y_t \text{ can be derived from } \phi \in X_{D_0} \text{ using the method of steps; cf. Remark 4.4, above.}\]
on the derivative. Here, $\Psi$ is simply defined as the shift $\Psi(\phi) = \phi - \phi(0)$ so that $\Psi(\phi) \in X_{D_0}$ where, in this case, $X_{D_0} = \{\phi \in X : \phi(0) = 0\}$. In the general case, $\Psi = \text{Id}_X - \Phi \circ D_0$ where $\Phi : \mathbb{R}^n \to X_{D_0}$ is the right inverse of $D_0$, i.e. $D_0 \circ \Phi = \text{Id}_{\mathbb{R}^n}$. Indeed, for such a $\Psi$, we get $\Psi(\phi) = \phi - \Phi \circ D_0(\phi)$ for all $\phi \in X$ which gives 0 when we compose to the left by $D_0$ so that, as desired, $\Psi(\phi) \in X_{D_0}$ for all $\phi \in X$. The existence of such a right inverse $\Phi$ is established in [Hal77] for operators $D$ of the form (4.57) with $M$ nonatomic and in particular for operators such as $D_0$ given in (4.59).

Combining Lemma 4.4 and Proposition 2.1, we conclude the following general result concerning the existence of a global attractor for a wide class of NFDEs. The existence of invariant probability measures, associated with temporal averages via generalized limits, then follow for such NFDEs in view of Theorem 2.1 and Theorem 2.2.

**Theorem 4.2.** Let $X := C([-\tau, 0], \mathbb{R}^n)$. Assume that $D$ satisfies (4.59) with $D_0$ stable in the sense that $\Re_{A_0} < 0$, where $\Re_{A_0}$ is defined according to (4.61). Suppose that for this $D$ and some given $f \in C^1(X, \mathbb{R}^n)$ that all of the solutions of the associated system (4.58) are well-defined for all $t \geq 0$ in the sense of Definition 4.1. Let $\{S_{D,f}(t)\}_{t \geq 0}$ be the continuous semigroup generated by (4.58) on $X$. If there exists an absorbing set $\mathcal{B} \subset X$, for $S_{D,f}$ then the omega-limit set, $\omega(\mathcal{B})$, is the global attractor of $S_{D,f}$. Moreover, for any given generalized Banach Limit $\lim_{T \to \infty}$, and for any $m_0 \in \text{Pr}(X)$ there exists an invariant measure $m \in \text{Pr}(X)$ for $\{S_{D,f}(t)\}_{t \geq 0}$ whose support is contained in $\mathcal{A}$ and such that

$$\int_X \varphi(u)dm(u) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_X \varphi(S_{D,f}(t)u)dm_0(u)dt, \text{ for any } \varphi \in C_b(X). \quad (4.63)$$

Furthermore, if $m_0 = \delta_0$, the Dirac measures for some $\phi \in X$, then (4.63) holds for any $\varphi \in C(X)$.

**Proof.** The result follow as an obvious consequence of Lemma 4.4, Proposition 2.1, Theorem 2.1 and Theorem 2.2.

### 4.2.3 Invariant measures for NDDEs

In view of its generality, Theorem 4.2 provides a powerful tool that may be employed to establish the existence of a global attractor and hence of invariant probability measures for particular NDDEs.

Of course, in practice, we need to verify the condition $\Re_{A_0} < 0$, which ensures the splitting of the semigroup generated by the NDDE of interest, and furthermore to establish the existence of a bounded absorbing set. We next provide below a useful criterion for the verification of this latter dissipativity condition. This criteria is satisfied for a special but still rather broad class of NDDEs. The verification of $\Re_{A_0} < 0$ has been and is still an intensive topic of research. Many criterion exist depending on the situation of interest; see e.g. [MN07]. In what follows we will need only the Schur-Cohn criterion, see e.g. [Nic01].

**Lemma 4.5.** Let $\tau$ be a positive constant, and $B$ a $n \times n$ matrix with real entries. Let $g$ be in $C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Consider the neutral delay differential equation given by

$$\frac{d}{dt}(x(t) - Bx(t - \tau)) = g(x(t), x(t - \tau)), \text{ for } t \geq 0, \quad (4.64)$$

on the phase space $X := C([-\tau, 0], \mathbb{R}^n)$.

Assume that there exist $\alpha, \beta > 0$, $\rho \in \mathbb{R}$ such that

$$\langle u - Bv, g(u, v) \rangle \leq -\gamma - \alpha |u|^2 + \beta |v|^2, \forall u, v \in \mathbb{R}^n, \quad (4.65)$$

and that,

$$\mathcal{E} := \|B\| + \sqrt{(1 + \|B\|)^2 e^{-\alpha \tau} + 2(\beta + \alpha \|B\|^2) \frac{1 - e^{-\alpha \tau}}{\alpha}} < 1. \quad (4.66)$$

Then the solutions of (4.64) are global in time, and the ball $B(0, 2r \sqrt{\sum_{k=0}^{\infty} \mathcal{E}^k})$ in $X = C([-\tau, 0], \mathbb{R}^n)$ with $r = \sqrt{\frac{2 \sqrt{(1 - e^{-\alpha \tau})/\alpha}}{\alpha}}$, is absorbing for the continuous semigroup generated by (4.64) on $C([-\tau, 0], \mathbb{R}^n)$.
Proof. Equation (4.64) may be written in the functional form $\frac{d}{dt}D_0x_t = f(x_t)$ with $D_0\phi := \phi(0) - B\phi(-\tau)$ and $f(\phi) := g(\phi(0), \phi(-\tau))$ for any $\phi \in X$, and since $g \in C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, we have that $f \in C^1(X, \mathbb{R}^n)$. The local existence and uniqueness of solutions up to a maximal time $\alpha_\phi$ (4.64) is ensured by Proposition 4.2. From the same proposition, in order to conclude to the global existence of solutions of (4.64), i.e. to show that $\alpha_\phi = \infty$ for every $\phi \in X$, it suffices to show that for any $\phi$ and for any $T > 0$, $\sup_{t \in [0, \min(T, \alpha_\phi)]} |x_t(\cdot, \phi)|_\infty < \infty$. The estimates provided below for the existence of an absorbing ball contain such an argument implicitly and existence of a continuous semigroup associated to (4.64) follows from Proposition 4.2. We turn next to these estimates.

We first show that, for any $0 \leq t_1 < t_2$,

$$|x(t_2) - Bx(t_2 - \tau)|^2 \leq e^{-\alpha(t_2 - t_1)}|x(t_1) - Bx(t_1 - \tau)|^2 + 2\frac{\gamma}{\alpha}(1 - e^{-\alpha(t_2 - t_1)}) + 2(\beta + \alpha\|B\|^2)\sup_{s \in [t_1 - \tau, t_2 - \tau]} |x(s)|^2 (1 - e^{-\alpha(t_2 - t_1)})/\alpha. \quad (4.67)$$

This inequality may be derived by considering the quantity $u(t) := |x(t) - Bx(t - \tau)|^2 = (x(t) - Bx(t - \tau), x(t) - Bx(t - \tau))$. Since $D_0x \in C^1([0, \infty), \mathbb{R}^n)$, $u$ is differentiable we find with (4.66) that

$$u'(t) \leq 2(\gamma - \alpha |x(t)|^2 + \beta |x(t - \tau)|^2),$$

Now since $2|x(t)|^2 \geq u(t) - 2|Bx(t - \tau)|^2$, we infer

$$u'(t) \leq 2(\gamma - \alpha u(t) + 2(\beta + \alpha\|B\|^2)|x(t - \tau)|^2), \quad \text{for } t > 0.$$

The inequality (4.67) is then easily derived by multiplying this last inequality by $e^{\alpha t}$ and integrating between $t_1$ and $t_2$.

Let us take now $t$ to be in $(0, \tau)$. We use (4.67) with $t_1 = 0$ and $t_2 = t$, it follows that,

$$|x(t) - Bx(t - \tau)|^2 \leq e^{-\alpha t}|x(0) - Bx(-\tau)|^2 + 2\frac{\gamma}{\alpha}(1 - e^{-\alpha t}) + 2(\beta + \alpha\|B\|^2)|\phi|^2_\infty \frac{(1 - e^{-\alpha t})}{\alpha},$$

and since $|x(0) - Bx(-\tau)|^2 \leq (1 + \|B\|^2)|\phi|^2_\infty$, we get

$$|x(t) - Bx(t - \tau)|^2 \leq \left[(1 + \|B\|^2)2(\beta + \alpha\|B\|^2)\frac{(1 - e^{-\alpha t})}{\alpha}\right]|\phi|^2_\infty + 2\frac{\gamma}{\alpha}(1 - e^{-\alpha t}),$$

which gives finally,

$$|x(t)| \leq \mathcal{C}_0|\phi|_\infty + r, \quad \text{for } t \in (0, \tau], \quad (4.68)$$

with $\mathcal{C}_0 := \sqrt{(1 + \|B\|^2)2(\beta + \alpha\|B\|^2)\frac{(1 - e^{-\alpha t})}{\alpha} + \|B\|}$, and $r = \sqrt{2\frac{\gamma}{\alpha}(1 - e^{-\alpha t})}$. For $t > \tau$ we take $t_1 = t - \tau$ and $t_2 = t$ in (4.67). By arguing in similar manner to the previous case we obtain:

$$|x(t) - Bx(t - \tau)|^2 \leq \left[(1 + \|B\|^2)e^{-\alpha t} + 2(\beta + \alpha\|B\|^2)\frac{(1 - e^{-\alpha t})}{\alpha}\right]|x_{t - \tau}|^2_\infty + r^2, \quad (4.69)$$

which leads to,

$$|x(t)| \leq \mathcal{C}|x_{t - \tau}|_\infty + r, \quad \text{for } t > \tau, \quad (4.70)$$

with $\mathcal{C}$ given in (4.66). We infer from (4.68), (4.69) and a simple induction that, for every $k \geq 0$,

$$|x(t)| \leq \mathcal{C}^k \mathcal{C}_0|\phi|_\infty + r \sum_{j=0}^{k} \mathcal{C}^j, \quad \text{for } t \in (k\tau, (k + 1)\tau]. \quad (4.71)$$

The existence of the bounded absorbing set for the semigroup generated by (4.64) therefore follows, completing the proof.
Note that the dissipation condition (4.66) is valid for “large” delays provided that $2\beta < \alpha$ and $\|B\|$ is appropriately chosen with respect to $\frac{\alpha}{\beta}$. More precisely we have the following:

**Lemma 4.6.** (Dissipation for large delays) Suppose that $2\beta < \alpha$ and $\|B\| < \sqrt{2(1 - \frac{\beta}{\alpha})} - 1$. Then (4.66) holds for all $\tau > \tau^*$, with $\tau^* = -\frac{1}{\alpha} \log \left( \frac{2(1 - \frac{\beta}{\alpha}) - \|B\| + 1}{2(1 - \frac{\beta}{\alpha}) - \|B\| - 1} \right)$.

**Proof.** Direct manipulations show that (4.66) is equivalent to:

$$P(\|B\|)e^{-\alpha \tau} < P(\|B\| + 2),$$

where $P$ is the polynomial function given by $P(x) = -[(x - 1)^2 + 2(\frac{\beta}{\alpha} - 1)]$. Under the assumed conditions on $\beta$ and $\alpha$,

$$\sqrt{2 \left( 1 - \frac{\beta}{\alpha} \right)} > 1$$

and $P(x + 2)$ possesses two real distinct roots, namely $x_1 = -1 - \sqrt{2(1 - \frac{\beta}{\alpha})} < 0$, $x_2 = -1 + \sqrt{2(1 - \frac{\beta}{\alpha})} > 0$ and reaches its maximum at $-1$. As such, for all $x \in [0, \sqrt{2(1 - \frac{\beta}{\alpha})} - 1]$, it is clear that $P(x), P(x + 2) > 0$ and also that $P(x) - P(x + 2)$ is strictly increasing and non-negative. Since, by assumption, $\|B\|$ falls in this range, (4.72) holds for all $\tau > \tau^*$, completing the proof. $\square$

We are now in position to prove our main theorem about the existence of invariant probability measures for NDDEs.

**Theorem 4.3.** Let $\tau$ be a positive constant, and $B$ a $n \times n$ matrix with real entries and let $g$ be in $C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Consider the neutral delay differential equation given by

$$\frac{d}{dt} \left(x(t) - Bx(t - \tau)\right) = g(x(t), x(t - \tau)), \text{ for } t \geq 0,$$

on the phase space $X := C([-\tau, 0], \mathbb{R}^n)$. Assume that the dissipation conditions (4.65) and (4.66) are satisfied. Then, the continuous semigroup $\{S(t)\}_{t \geq 0}$ acting on $X$ which is generated by (4.64) possesses a global attractor $A$ and the results in Theorems 2.1, 2.2 hold for $\{S(t)\}_{t \geq 0}$.

**Proof.** Keeping the previous notations, we note that for all $\phi \in X$, $D_0 \phi = \phi(0) - B\phi(-\tau)$ in this case. By Lemma 4.5, we infer the existence of an absorbing set for $\{S(t)\}_{t \geq 0}$, and since (4.64) is a particular form of the type of NFDEs handled by Lemma 4.4, to apply Theorem 4.2 and hence to infer all of the desired results we need now only check that $\Re A_0 < 0$, where $\Re A_0$ reduces simply here to $\text{sup } \{\Re (\lambda) : \det \left[ I - Be^{-\lambda \tau} \right] = 0 \}$. It is well known that in such a case $D_0$ is stable (independently of the delay $\tau$) if and only if $\rho(B) < 1$ where $\rho(B)$ is the spectral radius of $B$. This is the so-called Schur-Cohn condition; see [Nic01]. This last condition is guaranteed by the imposed dissipation condition (4.66) since $\|B\| < 1$ necessarily, and $\rho(B) \leq \|B\|$ trivially. The proof is therefore complete. $\square$

**Remark 4.8.** Note that this theorem may be extended to NDDEs of the form $\frac{d}{dt} D_0 x_t = G(x_t)$ with $D_0$ as given in (4.59) and $G$ being the functional representation associated to $g$ in (4.64). The dissipation estimates and the stability criteria are however more involved. We leave such a generalization of Theorem 4.3 to the interested reader.
4.2.4 Application to a nonlinear Brayton-Miranker-like model

We now return to a particular system of NDDEs, (4.76), arising from the transmission line problem discussed above in Section 4.2.1. Our goal is to provide conditions under which (4.76) exhibits a global attractor and hence to infer the existence of invariant measures for this example. In view of Theorem 4.3 we are left with the verification of the dissipation conditions (4.65) and (4.66) for this system which we establish below.

Let $0 < m, q < 1$; $p \in \mathbb{R}$, and let $b$ and $c$ be strictly positive real numbers. Let us introduce the notations $\Phi(t) := (\phi_1(t), \phi_2(t))^T$, and $\nu = (\nu_1, \nu_2)$ and $u = (u_1, u_2)$. We use here bold typeface to distinguish vectors in $\mathbb{R}^2$ from real numbers. We warn the reader that the symbol $\| \cdot \|$ will be used both for the absolute value of real numbers and for the Euclidean vector norm in $\mathbb{R}^2$.

Let us consider $F_1, F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ two $C^1$ functions which satisfy respectively that there exist $\alpha' > 0$, $\gamma_1 \geq 0$, $M'_1 \geq 0$ and $M_1 > 0$ such that for all $(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$
\begin{align*}
    u_1 F_1(u, v) &\leq -\alpha' u_1^2 + \gamma_1, \\
    |v_2 F_1(u, v)| &\leq M_1 u_1 + M'_1 v_2,
\end{align*}
$$

and that there exist $\gamma_2 \geq 0$, $M'_2 \geq 0$ and $M_2 > 0$ such that for all $(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$
\begin{align*}
    u_2 F_2(u, v) &\leq -\alpha' u_2^2 + \gamma_2, \\
    |v_1 F_2(u, v)| &\leq M_2 u_2 + M'_2 v_1,
\end{align*}
$$

For instance $F_i(u, v) = -\alpha_i u_i (1 + |v|^2)^{-1}$ verifies the conditions (4.74) for $i = 1$ and (4.75) for $i = 2$.

We consider now the following system of NDDEs:

$$
\frac{d}{dt} \begin{pmatrix}
    \phi_1(t) \\
    \phi_2(t)
\end{pmatrix} = \begin{pmatrix}
    p & q \\
    m & 0
\end{pmatrix} \begin{pmatrix}
    \phi_1(t) \\
    \phi_2(t)
\end{pmatrix} - \begin{pmatrix}
    b \phi_1(t) \\
    c \phi_2(t)
\end{pmatrix} + \begin{pmatrix}
    F_1(\Phi(t), \Phi(t - \tau)) \\
    F_2(\Phi(t), \Phi(t - \tau))
\end{pmatrix},
$$

We introduce furthermore $k := \max(m, q)$, which is the norm of the matrix arising in the LHS of (4.76) which has to be strictly less than 1 in view of Lemma 4.6, which explains the constraints imposed on $m$ and $q$.

The condition (4.65) to satisfy here can be written as:

$$
N(u, v) := (u_1 - q v_2) \left( p - bu_1 + F_1(u, v) \right) + (u_2 - mv_1) \left( -cu_2 + F_2(u, v) \right) \leq \gamma - \alpha |u|^2 + \beta |v|^2,
$$

with $\alpha > 0, \beta \in \mathbb{R}$ and $\gamma > 0$ to find independently of $u, v \in \mathbb{R}^2$.

An easy computation shows that,

$$
N(u, v) \leq \gamma_1 + \gamma_2 + pu_1 + |pqv_2| - bu_1^2 - cu_2^2 - \alpha' |u|^2 + M_1 u_1 + M'_1 v_1 + M_2 u_2 + M'_2 v_2 + mv_1 u_2 + bu_1 v_2,
$$

now by applying the Young inequality on the two last terms and the $\epsilon$-Young inequality appropriately on the rest of the terms concerned, it is easy to show that for any $\epsilon > 0$ there exists $\gamma_\epsilon > 0$ such that

$$
N(u, v) \leq \gamma_\epsilon - \frac{1}{2} \min(b, c) + \alpha' - \epsilon |u|^2 + \frac{1}{2} \max(b, c) + \epsilon |v|^2,
$$

which shows that (4.65) is satisfied with $B = \begin{pmatrix}
    0 & q \\
    m & 0
\end{pmatrix}$ and $g(u, v) = (p - bu_1 + F_1(u, v), -cu_2 + F_2(u, v))^T$.

Let us introduce $\alpha_\epsilon := \frac{1}{2} \min(b, c) + \alpha' - \epsilon$ and $\beta_\epsilon := \frac{1}{2} \max(b, c) + \epsilon$. Assume now that $\alpha'$ is such that $\max(b, c) < \frac{1}{2} \min(b, c) + \alpha'$. Let $\epsilon > 0$ be fixed sufficiently small such that $2\beta_\epsilon < \alpha_\epsilon$. Now if we assume furthermore that $k < -1 + \sqrt{2(1 - \frac{\beta_\epsilon}{\alpha_\epsilon})}$, then from Lemma 4.6 we deduce that there exists $\tau^*$, such that (4.66) is satisfied for $\tau > \tau^*$.

By using now Theorem 4.3 we have thus proved the following proposition which constitutes, to the best of the authors’ knowledge, the first result regarding existence of invariant measures for Brayton-Miranker-like models such as (4.76).
Proposition 4.3. Assume that $\alpha'$ arising in (4.74) and (4.75) is such that $\max(b, c) < \left(\frac{1}{2} \min(b, c) + \alpha'\right)$. For some $\varepsilon > 0$ such that $2\beta_\varepsilon < \alpha_\varepsilon$, assume furthermore that the coefficients $q$ and $m$ of the matrix arising in (4.76) satisfy:

$$\max(q, m) < -1 + \sqrt{2 \left(1 - \frac{\beta_\varepsilon}{\alpha_\varepsilon}\right)}$$

where $\alpha_\varepsilon := \frac{1}{2} \min(b, c) + \alpha' - \varepsilon$ and $\beta_\varepsilon := \frac{1}{2} \max(b, c) + \varepsilon$. Let $\tau^* = -\frac{1}{\alpha_\varepsilon} \log \left(P(\max(q, m) + 2)/P(\max(q, m))\right)$ with $P(x) := -[(x - 1)^2 + 2(\frac{\beta_\varepsilon}{\alpha_\varepsilon} - 1)]$. Then for all $\tau > \tau^*$ the continuous semigroup generated by (4.76) possesses a global attractor $A$ living in $C([-\tau, 0], \mathbb{R}^2)$. This semigroup possesses furthermore invariant probability measures whose support is contained in $A$ and which satisfy the “weak ergodic” property (4.63).

Appendix: invariant measures and the global attractor

For the sake of completeness we recall briefly a proof of the classical fact that invariant probability measures must always have their support contained in the global attractor; see e.g. [BCFM95, Lemma 4.2] or [FMRT01] given in the concrete case of the 2D Navier-Stokes equations.

Lemma 4.7. Let $\{S(t)\}_{t \geq 0}$ be a continuous semigroup defined on a metric space $(X, d)$. Suppose that $\{S(t)\}_{t \geq 0}$ possesses a global attractor $A$. Then any invariant borel probability measure $m$ (relative to $\{S(t)\}_{t \geq 0}$) has its support contained in $A$ so that, in particular, $m(A) = 1$.

Proof. For $\delta > 0$ form the sets $A_\delta = \{y : \inf_{x \in A} d(x, y) < \delta\}$. In view of the basic continuity properties of measures, it is sufficient to show that, $m(A_\delta) = 1$, for every $\delta > 0$, since, evidently, $A = \cap_{\delta > 0} A_\delta$.

To this end fix $\delta > 0$. Since $A$ is attracting, for every $R > 0$ and each $x \in X$ we may select $t_R > 0$ such that $S(t_R)B_R(x) \subset A_\delta$, where $B_R(x)$ is the ball of radius $R$ around the point $x$. This implies, $B_R(x) \subset S(t_R)^{-1}S(t_R)B_R(x) \subset S(t_R)^{-1}A_\delta$. Thus, due to the invariance of $m$, we have, for every $R > 0$ that $m(B_R(x)) \leq m(S(t_R)^{-1}A_\delta) = m(A_\delta)$. Since for every $x$ the collection $\{B_R(x)\}_{R > 0}$ is a nested collection of sets with $\bigcup_{R > 0} B_R(x) = X$ we now infer $m(A_\delta) = 1$ by again invoking basic continuity properties of measures. The proof is therefore complete.

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