THEOREM OF ITERATES FOR ELLIPTIC
AND NON-ELLIPTIC OPERATORS

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Abstract. We introduce a new approach for the study of the Problem of Iterates using the theory on
general ultradifferentiable structures developed in the last years. Our framework generalizes many of
the previous settings including the Gevrey case and enables us, for the first time, to prove non-analytic
Theorems of Iterates for non-elliptic differential operators. In particular, by generalizing a Theorem of
Baouendi and Metivier we obtain the Theorem of Iterates for analytic hypoelliptic operators of principal
type with respect to several non-analytic ultradifferentiable structures.

1. Introduction

In recent years there has been renewed interest in the Problem of Iterates, i.e. the study of vectors of
differential operators, we mention in particular [7], [8], [9], [10], [18], [20], [23], [24], [25], [26], [27], [33]
and [63]. For the history of the problem we refer to the survey [14].

The aim of this paper is to present a new approach to the Problem of Iterates using the ultradif-
ferentiable structures introduced in [55] and [56], which generalizes and unifies many of the previous
cases.

In our context an ultradifferentiable structure $U$ is a subalgebra of smooth functions which is defined
by estimates on the derivatives of its elements. Well-known ultradifferentiable structures include the
Denjoy-Carleman classes which are given by weight sequences and the Braun-Meise-Taylor classes whose
defining data are weight functions. The latter were originally introduced by [4] and [5], but the modern
formulation of these classes was given in [17]. The classes discussed in [55], which are determined by
weight matrices, i.e. families of weight sequences, encompass both Denjoy-Carleman classes and Braun-
Meise-Taylor classes. Other examples of ultradifferentiable spaces are the Gelfand-Shilov classes, cf. [30]
and the recently introduced $L^p$-ultradifferentiable classes, see [32].

Then ultradifferentiable vectors of some operator $P$ associated to the structure $U$ are those functions
(or distributions) which satisfy the defining estimates of $U$ for the iterates $P^k$ of $P$. Thus the Problem
of Iterates in its general form can rather casually be formulated as the following question:

Given an operator $P$ suppose that a function (or distribution) $u$ satisfies the defining
estimates of an ultradifferentiable structure $U$ for the iterates $P^k$ of $P$. Can we conclude
that $u$ satisfies these estimates for all derivatives?

Or more concisely, are the ultradifferentiable vectors of $P$ with respect to $U$ already ultradifferentiable
functions of class $U$? If the answer to this question is ”yes” then we say that the Theorem of Iterates
holds for the operator $P$ and the structure $U$.

Our main goal is to develop a unified approach to the problem of iterates using the recent development
of the theory of general ultradifferentiable classes given in [55], [56] and in particular the microlocal theory
in [29]. This approach allows us not only to unify and generalize previously known results but also to
treat cases which have not been available in the literature up to now. In particular, in the case of principal
type operators we are able to use the technical estimate in [3] to infer the Theorem of Iterates for a wide
variety of ultradifferentiable classes, which include quasianalytic and non-quasianalytic classes. We note
that, to our knowledge, this is the first time the Theorem of Iterates is proven for a non-elliptic operator
and a non-analytic ultradifferentiable structure.

In the case of Braun-Meise-Taylor classes our main Theorem takes a relatively concise form. However,
in order to formulate it correctly, we need to recapitulate some notions: We say that a differential

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operator $P$ defined on some open set $U \subseteq \mathbb{R}^n$ is of principal type\footnote{We follow here the classic definition, see e.g. [35] and the references therein. It sometimes does not agree with the definition of principal type operators given in modern treatises, for example in [35, Chapter 26].} or that $P$ is an operator with simple real characteristics if the principal symbol $p_d$ of $P$ satisfies

$$|p_d(x, \xi)| + \sum_{j=1}^n |\partial_\xi_j p_d(x, \xi)| \neq 0$$

for all $(x, \xi) \in U \times \mathbb{R}^n \setminus \{0\}$.

A weight function in the sense of [17] is a continuous and increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ which satisfies

$$\omega(2t) = O(\omega(t)) \quad \text{as } t \to \infty, \quad (\alpha)$$

$$\log t = o(\omega(t)) \quad \text{as } t \to \infty, \quad (\beta)$$

$$\varphi_\omega = \omega \circ \exp \quad \text{is convex.} \quad (\gamma)$$

We set

$$\|f\|_{V,\omega,h} = \sup \nolimits_{x \in V} |D^\alpha f(x)| e^{-\frac{1}{h} \varphi_\omega^*(h|\alpha|)},$$

where $V \subseteq U$ is a relatively compact subset of $U$, $f \in \mathcal{E}(U)$ is a smooth function, $h > 0$ and $\varphi_\omega^*(t) := \sup_{s \geq 0}(st - \varphi_\omega(s))$ is the conjugate function of $\varphi_\omega$. The Roumieu class (of ultradifferentiable functions) associated with $\omega$ is given by

$$\mathcal{E}^{(\omega)}(U) = \left\{ f \in \mathcal{E}(U) : \forall V \subseteq U \exists h > 0 \quad \|f\|_{V,\omega,h} < \infty \right\}$$

and the Beurling class associated to $\omega$ is

$$\mathcal{E}^{(\omega)}(U) = \left\{ f \in \mathcal{E}(U) : \forall V \subseteq U \forall h > 0 \quad \|f\|_{V,\omega,h} < \infty \right\}.$$

Similarly, for a partial differential operator $P$ of order $d$ with analytic coefficients we set

$$\mathcal{E}^{(\omega)}(U; P) = \left\{ u \in D'(U) : \forall V \subseteq U \exists h > 0 \quad \|u\|_{V,\omega,h}^P < \infty \right\}$$

and

$$\mathcal{E}^{(\omega)}(U; P) = \left\{ u \in D'(U) : \forall V \subseteq U \forall h > 0 \quad \|u\|_{V,\omega,h}^P < \infty \right\},$$

where

$$\|u\|_{V,\omega,h}^P = \sup \nolimits_{k \in \mathbb{N}_0} \|p_k u\|_{L^2(V)} e^{-\frac{1}{h} \varphi_\omega^*(h|\alpha|)}.$$

Our main result in the case of weight functions is:

**Theorem 1.1.** Let $U \subseteq \mathbb{R}^n$ be an open set and $P$ a hypoelliptic operator of principal type with analytic coefficients in $U$. Furthermore assume that $\omega$ is a weight function satisfying

$$\exists H > 0 : \quad \omega(t^2) = O(\omega(Ht)) \quad \text{as } t \to \infty. \quad (\Xi)$$

Then

$$\mathcal{E}^{(\omega)}(U; P) = \mathcal{E}^{(\omega)}(U),$$

$$\mathcal{E}^{(\omega)}(U; P) = \mathcal{E}^{(\omega)}(U).$$

We may note that the condition $\Xi$ has appeared in various applications of Braun-Meise-Taylor classes, e.g. in the study of global pseudodifferential operators in [2].
1A. Preliminaries. We denote by \( \mathbb{N} = \{1, 2, \ldots \} \) the set of positive integers and by \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) the set of non-negative integers. Furthermore \( U \subseteq \mathbb{R}^n \) is always an open set. In this paper we focus on linear differential operators with analytic coefficients, i.e.

\[
P(x, D) = \sum_{|\alpha| \leq d} a_\alpha(x)D^\alpha
\]

with \( a_\alpha \in \mathcal{A}(U) \). We use here the convention \( D_j = -i\partial_{x_j} \). Then the symbol of \( P \) is

\[
p(x, \xi) = \sum_{|\alpha| \leq d} a_\alpha(x)\xi^\alpha
\]

and

\[
p_d(x, \xi) = \sum_{|\alpha| = d} a_\alpha(x)\xi^\alpha
\]

is the principal symbol of \( P \). The characteristic set of \( P \) is given by

\[
\text{Char} P = \{(x, \xi) \in U \times \mathbb{R}^n \setminus \{0\} : p_d(x, \xi) = 0\}.
\]

Hence \( \text{Char}(P) = \emptyset \) if and only if \( P \) is elliptic.

We say that a distribution \( u \in \mathcal{D}'(U) \) is an analytic vector of the operator \( P \) if for any \( V \subseteq U \) there are constants \( C, h > 0 \) such that

\[
\|P^k u\|_{L^2(V)} \leq Ch^k k!
\]

for all \( k \in \mathbb{N}_0 \). We write \( \mathcal{A}(U; P) \) for the space of analytic vectors of \( P \). In [13] and [16] it was shown separately that if \( P \) is elliptic then \( \mathcal{A}(U; P) = \mathcal{A}(U) \). A similar result was proven in [54] for elliptic systems of vector fields.

We can consider this problem in a more general setting, if we replace the factor \( k! \) in the estimate above by \( \text{e.g. } (k!)^s \). Recall that a smooth function \( f \in \mathcal{E}(U) \) is an \( s \)-Gevrey function, \( s \geq 1 \), if for all \( V \subseteq U \) there are constants \( C, h > 0 \) such that

\[
\sup_{x \in V} |D^\alpha f(x)| \leq Ch^{|\alpha|}(|\alpha|!)^s, \quad \forall \alpha \in \mathbb{N}_0^n.
\]

The space of \( s \)-Gevrey functions on \( U \) is denoted by \( \mathcal{G}^s(U) \). Analogously, an \( s \)-Gevrey vector \( u \) of \( P \) is a distribution \( u \in \mathcal{D}'(U) \) which satisfies the estimate

\[
\|P^k u\|_{L^2(V)} \leq Ch^k (k!)^s.
\]

We denote the space of \( s \)-Gevrey vectors of \( P \) by \( \mathcal{G}^s(U; P) \) and if \( P \) is elliptic then \( \mathcal{G}^s(U; P) = \mathcal{G}^s(U) \) for all \( s \geq 1 \) according to [11].

In fact, Métivier [52, Theorem 1.2] showed that the ellipticity of an analytic differential operator \( P \) can be characterized by the regularity of its non-analytic Gevrey vectors: If \( s > 1 \) then \( P \) is elliptic if and only if \( \mathcal{G}^s(U; P) = \mathcal{G}^s(U) \).

Clearly the Problem of Iterates is closely related to other regularity questions of the operator \( P \), see e.g. [14]. This connection has been extensively studied for operators with constant coefficients, see for example [6], [8], [10], [12] and [54]. However, in the wake of Métivier’s Theorem the study of vectors of a differential operator with variable coefficients has mainly split into two directions:

- If \( P \) is elliptic then the Theorem of Iterates has been proven for a large class of ultradifferentiable structures: e.g. for Denjoy-Carleman classes in [13], for Braun-Meiss-Taylor classes in [9], for Gelfand-Shilov classes in [19] and for \( L^p \)-ultradifferentiable functions in [53]. In particular, in [13] a microlocal elliptic Theorem of Iterates for Denjoy-Carleman classes is proven: If \( u \) is an ultradifferentiable vector of \( P \) with respect to a weight sequence \( \mathbf{M} \) then \( \text{WF}_{\mathbf{M}} u \subseteq \text{Char} P \), where \( \text{WF}_{\mathbf{M}} u \) denotes the ultradifferentiable wavefront set of \( u \) with respect to \( \mathbf{M} \) introduced by [53].

- If \( P \) is non-elliptic then it might still be possible to show that analytic vectors are analytic, cf. the surveys [14] and [23]. For non-analytic Gevrey vectors one tries to determine the loss of regularity in terms of the Gevrey scale \( (\mathcal{G}^{s'})_s \). More precisely, we want to find for each \( s > 1 \) some \( s' > s \) such that every \( s \)-Gevrey vector is an \( s' \)-Gevrey function. This approach was used for example in [3], [12], [13] and [23].
The simplest class of non-elliptic operators with variable coefficients are the operators of principal type. The main result on Gevrey vectors of principal type operators is the following result of Baouendi and Mérivier \cite{[55], Theorem 1.3}: If \( P \) is a hypoelliptic operator of principal type with analytic coefficients in \( U \subset \mathbb{R}^n \) then for each \( V \in U \) there is some \( \delta > 0 \) such that for all \( s \geq 1 \) we have that every \( s \)-Gevrey vector \( u \) of \( P \) in \( U \) is an \( s' \)-Gevrey function in \( V \) where \( s' = (ds - \delta)/(d - \delta) \).

In this paper we are going to generalize the result of Baouendi and Mérivier using the new theory on ultradifferentiable structures defined by weight matrices introduced in \cite{[55]} which in turn will yield the Theorem of Iterates for hypoelliptic operators of principal type with respect to ultradifferentiable structures given by certain weight matrices. For example, the following observation was the starting point of this paper: The prototypical example of a nontrivial weight matrix is the Gevrey matrix
\[
\mathfrak{G} = \{ G^s = ((k!)^s)_k : s > 1 \}.
\]

For a discussion of the properties of \( \mathfrak{G} \) we refer to \cite{[55], Section 5}. The Roumieu classes of ultradifferentiable functions and vectors associated to \( \mathfrak{G} \) are
\[
\mathcal{E}^{(\mathfrak{G})}(U) = \{ f \in \mathcal{E}(U) : \forall V \in U \exists s > 1 : f \in G^s(V) \}
\]
and
\[
\mathcal{E}^{(\mathfrak{G})}(U; P) = \{ u \in \mathcal{D}'(U) : \forall V \in U \exists s > 1 : u \in G^s(V; P) \},
\]
respectively, whereas the Beurling classes are given by
\[
\mathcal{E}^{(\mathfrak{G})}(U) = \{ f \in \mathcal{E}(U) : \forall V \in U \forall s > 1 : f \in G^s(V) \}
\]
\[
= \bigcap_{s>1} G^s(U)
\]
and
\[
\mathcal{E}^{(\mathfrak{G})}(U; P) = \{ u \in \mathcal{D}'(U) : \forall V \in U \forall s > 1 : u \in G^s(V; P) \}
\]
\[
= \bigcap_{s>1} G^s(U; P),
\]
respectively.

**Proposition 1.2.** Let \( P \) be a hypoelliptic partial differential operator of principal type with analytic coefficients on an open set \( U \subset \mathbb{R}^n \). Then
\[
\mathcal{E}^{(\mathfrak{G})}(U; P) = \mathcal{E}^{(\mathfrak{G})}(U).
\]

**Notation 1.3.** Throughout the article, we are going to use the convention that \([*]=\{*,(*)&\) where \(*=M,\mathfrak{G},\omega\).

**Proof of Proposition 1.2.** Since \( G^s(U) \subset G^s(U; P) \) for all \( s \geq 1 \), cf. \cite{[14]}, it is enough to show \( \mathcal{E}^{(\mathfrak{G})}(U; P) \subset \mathcal{E}^{(\mathfrak{G})}(U) \). If \( u \in \mathcal{E}^{(\mathfrak{G})}(U; P) \) then \( u \in G^{1+s}(U; P) \) for all \( s > 0 \). For \( V \in U \) we have that \( u|_V \in G^{1+s}(V) \) by \cite{[55], Theorem 1.3} where \( s' = (d(\sigma + 1) - \delta)/(d - \delta) - 1 = d\sigma/(d - \delta) \) for some \( \delta \geq 0 \) depending on the operator and \( V \). Thence \( u|_V \in \bigcap_{s>0} G^{s+s'}(V) = \mathcal{E}^{(\mathfrak{G})}(U) \) for all \( V \in U \) and therefore \( u \in \mathcal{E}^{(\mathfrak{G})}(U) \).

If \( u \in \mathcal{E}^{(\mathfrak{G})}(U; P) \) then for every \( V \in U \) there is some \( s > 1 \) such that \( u|_V \in G^s(V; P) \). \cite{[55], Theorem 1.3} implies that for every \( W \subset V \) there is some \( s' > 1 \) such that \( u|_W \in G^{s'}(W) \). It follows that \( u \in \mathcal{E}^{(\mathfrak{G})}(U) \).

The concept of weight matrix was introduced in \cite{[55]} in order to deal simultaneously with Denjoy-Carleman classes and Braun-Meise-Taylor classes. It is well known that the Gevrey classes can be realized as Denjoy-Carleman classes or as Braun-Meise-Taylor classes, but in general weight sequences and weight functions describe different classes, cf. \cite{[16]}.

The theory of weight matrices allows us to deal with countable intersections and also countable unions (in the sense of germs) of Denjoy-Carleman classes, which will be of some importance in our considerations. For example, \( \mathcal{E}^{(\mathfrak{G})}(U) \) can neither be described as Denjoy-Carleman classes nor as Braun-Meise-Taylor classes, cf. \cite{[55], Theorem 5.22].

Weight matrices have been used to generalize and unify results regarding ultradifferentiable classes in various areas, see e.g. \cite{[39], [57], [58]} or \cite{[59]}. In particular, in \cite{[29]} we defined the ultradifferentiable wavefront set associated with classes given by weight matrices and generalized and unified results on the
wavefront set for Denjoy-Carleman classes proved in \[28\] and \[35\] and for Braun-Meise-Taylor classes in \[1\].

As we have seen, we can associate to each weight matrix (or weight sequence or weight function) two different ultradifferentiable classes, the Roumieu class and the Beurling class, respectively. Since the Gevrey classes are Roumieu classes, such spaces have been mainly studied as for example in \[13\]. But when both Roumieu and Beurling classes have been considered, there seems to be no much difference regarding the results obtained, see e.g. \[7\], \[10\] or also Theorem 1.1 above. Nevertheless, we will notice that in the case of weight matrices there is occasionally a difference between the Beurling and the Roumieu case when we regard vectors of a non-elliptic operator.

1B. Outline of the paper. We want to present in this paper a throughout introduction to the theory of ultradifferentiable vectors associated to weight matrices. In Section 2 we recall for the convenience of the reader the definitions and facts from the theory of weight matrices we are going to need, including some statements concerning the ultradifferentiable wavefront set, which have not been explicitly stated in \[21\]. Then we show in Section 3 that the microlocal theory in \[13\] can be extended to classes given by weight matrices. In particular we prove the elliptic Theorem of Iterates for these classes. We should note that the restriction to analytic operators allows us to work with rather weak conditions on the weight matrices. In fact, we require only that the associated classes are invariant under the action of analytic differential operators and under the composition with analytic diffeomorphisms.

Next we want to generalize Proposition 1.2 to other weight matrices. In order to do so we introduce in Section 4 the notion of ultradifferentiable scales, which can be considered as a special kind of weight matrices. This allows us to extend \[3\] Theorem 1.3 \(\text{i.e.} \) Theorem 4.5 and Proposition 1.2 \(\text{cf.} \) Theorem 4.7 \(\text{to} \) ultradifferentiable scales and their associated weight matrices, respectively. We will see that many families of weight sequences, which have been studied previously in the literature, constitute ultradifferentiable scales, including the scale \((N^q)_{q \geq 1}\) of \(q\)-Gevrey sequences which are given by \(N^q = q^k\) and the scale \((B^q)_{q \geq 0}\) given by \(B^q = k!(\log(k+e))^{2k}\).

In Section 5 the proof of Theorem 1.1 and especially condition (Ξ) are discussed. Furthermore, we discuss in the second part of this section how the exact definition of ultradifferentiable scales is tied to the rather precise estimates obtained in \[3\] and how to modify it for the study of vectors of other operators.

In the final section we have included some selected topics. In Subsection 6A we observe that the theory of ultradifferentiable scales developed in Section 4 can also be applied to generalize the results of \[21\]. Subsection 6B explores for which weight sequences \(M\) the associated weight function \(\omega_M\) satisfies (Ξ). In the next subsection we take a first look at vectors determined by a family of weight functions. We close the paper with the proof of the following variant of \[22\] Theorem 1.2, where \(E^{(N^q)}(U)\) and \(E^{(B^q)}(U;P)\) are the Roumieu class and the space of Roumieu vectors of \(P\) associated with the weight sequence \(N^q\), respectively.

**Theorem 1.4.** Let \(P\) be a differential operator with analytic coefficients in \(U\) and \(q > 1\). Then the following statements are equivalent:

1. \(P\) is elliptic.
2. \(E^{(N^q)}(U;P) = E^{(N^q)}(U)\).

2. ULTRADIFFERENTIABLE CLASSES

2A. Weight matrices. A sequence \(M = (M_k)_{k \in \mathbb{N}_0}\) of positive numbers is a weight sequence if it is normalized, i.e. \(M_0 = 1\), \(\lim_{k \to \infty} (M_k)^{1/k} = \infty\) and logarithmically convex, i.e.

\[
(M_k)^2 \leq M_{k-1}M_{k+1}
\]

for all \(k \in \mathbb{N}\). Note that for any such weight sequence \(M\) we have

\[
M_jM_k \leq M_{j+k}, \quad j, k \in \mathbb{N}_0.
\]

We are also going to use frequently the sequence \(m_k = M_k/k!\).

For a weight sequence \(M\), a bounded open set \(V \subset \mathbb{R}^n\) and a constant \(h > 0\) we set

\[
\|f\|_{V,M,h} = \sup_{x \in V} \max_{a \in \mathbb{N}_0^n} \frac{|D^a f(x)|}{h^{|a|}M_{|a|}}, \quad f \in E(V).
\]

We define the Roumieu class (over an open set \(U \subset \mathbb{R}^n\)) associated to \(M\) as

\[
E^{(M)}(U) = \left\{ f \in E(U) : \forall V \subset U \exists h > 0 : \|f\|_{V,M,h} < \infty \right\}
\]
whereas the Beurling class associated with $M$ is

$$\mathcal{E}^{(M)}(U) = \left\{ f \in \mathcal{E}(U) : \forall V \Subset U \forall h > 0 : \| f \|_{V,M,h} < \infty \right\}.$$  

Clearly, the vector space $\mathcal{E}^{(M)}(U)$ is an algebra with respect to the pointwise multiplication, due to (2.2).

Recall that a subspace $E \subseteq \mathcal{E}(U)$ is said to be quasianalytic if $E$ contains no non-trivial functions of compact support, i.e. $E \cap \mathcal{D}(U) = \{0\}$. In the case of Denjoy-Carleman classes $\mathcal{E}^{(M)}(U)$ quasianalyticity is characterized by the Denjoy-Carleman theorem (see e.g. [30]):

**Theorem 2.1.** Let $M$ be a weight sequence. The space $\mathcal{E}^{(M)}(U)$ is quasianalytic if and only if

$$\sum_{k=0}^{\infty} \frac{M_k}{M_{k+1}^k} = \infty. \quad (2.3)$$

We say that the sequence $M$ is quasianalytic if it satisfies (2.3). Otherwise $M$ is non-quasianalytic.

If $M$ and $N$ are two sequences we write

$$M \leq N :\iff \forall k \in N_0 : M_k \leq N_k,$$

$$M \preceq N :\iff \left(\frac{M}{N}\right)^{1/k} \text{ is bounded for } k \to \infty,$$

$$M \prec N :\iff \left(\frac{M}{N}\right)^{1/k} \to 0 \text{ if } k \to \infty,$$

and $M \approx N$ when $M \preceq N$ and $N \preceq M$. We recall that $\mathcal{E}^{(M)}(U) \subseteq \mathcal{E}^{(N)}(U)$ if $M \preceq N$ and $\mathcal{E}^{(M)}(U) \subseteq \mathcal{E}^{(N)}(U)$ when $M \prec N$.

For later use we note the following result in the spirit of [45, Lemma 6]. Throughout the paper, if not indicated otherwise, we are going to consider the constants appearing in the proofs to be generic, that is they may change their value from line to line.

**Lemma 2.2.** Let $M$ be a weight sequence and $L'$ be a sequence with $L'_0 = 1$ and $G^1 \preceq L' \preceq M$. Then there is a weight sequence $N$ such that

$$L' \preceq N \prec M.$$  

**Proof.** For each $h > 0$ we denote by $C_h$ the smallest constant $C > 0$ such that

$$L_k \leq C h^k M_k$$

holds for all $k \in N_0$. We define a new sequence $L$ by setting

$$L_k := \inf_{h > 0} C_h h^k M_k.$$  

Clearly $L' \preceq L$. If we put $\mu_k = M_k / M_{k-1}$ and $\lambda_k = L_k / L_{k-1}$ for $k \in N$ then we recall from [45, Lemma 6] that $\mu_k / \lambda_k$ is increasing and unbounded.

Set

$$\nu_k = \max\left\{ \sqrt[\mu_k]{\max_{1 \leq j \leq k} \lambda_j} \right\}$$

for $k \in N$ and define the sequence $N$ by $N_0 = 1$ and

$$N_k = \prod_{j=1}^{k} \nu_j$$

if $k \in N$. The sequence $\nu_k$ is increasing since $\mu_k$ is increasing, thence $N$ satisfies (2.1). It is easy to see that $L \preceq N$ and therefore $k \leq C \sqrt[\nu_k]{k}$ for some constant $C > 0$ independent of $k \in N$. It follows that $N$ is a weight sequence since $(N_k)^{1/k} \geq (M_k)^{1/2k}$.

It remains to prove $N \prec M$. For this it is enough to show

$$\lim_{k \to \infty} \frac{\nu_k}{\mu_k} = 0.$$  

We have

$$\frac{\nu_k}{\mu_k} = \max\left\{ \left(\mu_k\right)^{-1/2}, \left(\mu_k\right)^{-1} \max_{1 \leq j \leq k} \lambda_j \right\}$$

for all $k \in N$. For each $\varepsilon > 0$ there has to exist $k_{\varepsilon} \in N$ such that $\lambda_k / \mu_k \leq \varepsilon$ for all $k \geq k_{\varepsilon}$. Hence

$$\frac{\nu_k}{\mu_k} \leq \max\left\{ \left(\mu_k\right)^{-1/2}, \varepsilon, \left(\mu_k\right)^{-1} \max_{1 \leq j \leq k_{\varepsilon}} \lambda_j \right\}$$

and thus $\frac{\nu_k}{\mu_k} \leq \varepsilon$ for large enough $k$. \qed
Following [29] we say that a weight sequence $M$ is semiregular if
\[
\lim_{k \to \infty} \sqrt[k]{m_k} = \infty \quad (2.4)
\]
\[
\exists C > 0 \ \forall k \in \mathbb{N}_0 : \ M_{k+1} \leq C^{k+1}M_k. \quad (2.5)
\]
Observe that (2.4) implies that for all $\gamma > 0$ there is some constant $C > 0$ such that
\[
k^k \leq C \gamma^k M_k. \quad (2.6)
\]
Remark 2.3. If $M$ is a weight sequence then $A(U) \subseteq E^{|M|}(U)$ if and only if $M$ satisfies (2.4). On the other hand, if $M$ satisfies (2.5) then $E^{|M|}(U)$ is closed under derivation, i.e. if $f \in E^{|M|}(U)$ then also $\partial_j f \in E^{|M|}(U)$ for all $1 \leq j \leq n$. We may also note that (2.5) is equivalent to
\[
\exists C > 0 \ \forall k \in \mathbb{N}_0 : \ m_{k+1} \leq C^{k+1}m_k.
\]
If $M$ is semiregular then $E^{|M|}$ is closed under composition with analytic mappings, that is, if $\Phi : U \to V$ is an analytic mapping between two open sets $U \subset \mathbb{R}^r$ and $V \subset \mathbb{R}^s$ then for all $f \in E^{|M|}(U)$ we have $f \circ \Phi \in E^{|M|}(U)$, cf. [39] and [29], respectively.

Example 2.4. We present some examples of weight sequences, which will appear throughout the paper.

1. The Gevrey class of order $s > 1$ is defined by the semiregular non-quasianalytic weight sequence $G^s = (G^s_k) = (k!)^s$. Note that $G^s < G^r$ if and only if $s < t$.

2. Let $q,r > 1$ be two parameters. The weight sequence $L^q,r = (L^q,r_k)$ defined by $L^q,r_k = k^{1/q}r^k$ is semiregular if and only if $r \leq 2$. Observe that for all $q,r,s > 1$ we have $G^s < L^q,r$. Furthermore $L^{q_0,r_0} \leq L^{q_1,r_1}$ if $q_0 < q_1$ and $q_1 > 1$ arbitrary or if $q_0 = q_1$ and $1 < q_0 < q_1$.

3. Let $\sigma > 0$. The semiregular weight sequence $B^\sigma = (B^\sigma_k)$ given by $B^\sigma_k = k!(\log(k+\epsilon))^\sigma$ is quasianalytic if and only if $\sigma \leq 1$, cf. [64].

4. We can generalize the previous example, cf. [17]: For $j \in \mathbb{N}$ we define the function $\log^{(j)}$ recursively by
\[
\log^{(1)}(t) = \log t, \quad \log^{(j+1)}(t) = \log(\log^{(j)}(t)), \quad t \text{ large enough}.
\]
Furthermore set $e^{(1)} = e$ and $e^{(j+1)} = e^{(j+1)}$.

We consider the 2-parameter family of semiregular weight sequences $B^{j,\sigma}, j \in \mathbb{N}, \sigma > 0$, given by
\[
B^{j,\sigma}_k = k!(\log^{(j)}(k+\epsilon))^\sigma.
\]
We have that $B^{1,\sigma} = B^\sigma$ and $B^{j,\sigma}$ is quasianalytic when $j \geq 2$. If $j_1 < j_2$ then $B^{j_2,\sigma} \leq B^{j_1,\sigma}$ for any $\sigma > 0$ and $B^{j,\sigma} < B^{j',\sigma}$ for $\sigma < \tau$.

A weight matrix $M$ is a family of weight sequences such that for each pair $M,N \in M$ we have either $M \leq N$ or $N \leq M$. The Roumieu class associated with the weight matrix $M$ is
\[
E^{(M)}(U) = \left\{ f \in E(U) : \ \forall V \subseteq U \ \exists M \in M \ \exists h > 0 : \ |f|_{V,M,h} < \infty \right\}
\]
and the corresponding Beurling class is defined by
\[
E^{(M)}(U) = \left\{ f \in E(U) : \ \forall V \subseteq U \ \forall M \in M \ \forall h > 0 : \ |f|_{V,M,h} < \infty \right\}.
\]
Observe that $E^{(M)}(U) = \bigcap_{M \in M} E^{|M|}(U)$ and $E^{(M)} = \bigcup_{M \in M} E^{|M|}$ in the sense of germs. It follows that $E^{(M)}(U)$ is an algebra due to the definition of the weight matrix.

Let $M$ and $\bar{M}$ be two weight matrices. We define
\[
M[M] \iff \forall M \in M \ \exists N \in \bar{M} : M \leq N,
\]
\[
M[M] \iff \forall N \in \bar{M} \ \exists M \in M : M \leq N.
\]
Furthermore we write $M[\approx]M$ if $M[M]\approx M[M]\approx M[M]$. It follows that $E^{(M)}(U) \subseteq E^{(M)}(U)$ if $M[\leq]M$ and $E^{(M)}(U) \subseteq E^{(M)}(U)$ if $M[\leq]M$.

For each weight matrix $M$ there exists a countable weight matrix $\Sigma$ by [29] Lemma 2.5] such that $M[\leq]M$, i.e. $E^{(M)}(U) = E^{(\Sigma)}(U)$. It follows that $E^{(M)}(U)$ is non-quasianalytic if and only if there is some non-quasianalytic sequence $M \in M$. On the other hand $E^{(M)}(U)$ is non-quasianalytic if and only if all sequences $M \in M$ are non-quasianalytic, see [61] Sect. 4. We should note that in the last statement in
particular the fact that \( \mathfrak{M} \) is equivalent to a countable weight matrix is important: The intersection of uncountably many non-quasianalytic Denjoy-Carleman classes might be quasianalytic, cf. [15].

Combined with Remark 2.3 we moreover conclude that \( \mathcal{A}(U) \subseteq \mathcal{E}^{(\mathfrak{M})}(U) \) when
\[
\forall M \in \mathfrak{M} : \lim_{k \to \infty} \sqrt[k]{m_k} = 0.
\]

A weight matrix \( \mathfrak{M} \) is called \( R \)-semiregular if \( \mathfrak{M} \) satisfies (2.7) and
\[
\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} \exists C > 0 : M_{k+1} \leq C^k N_k, \quad k \in \mathbb{N},
\]
and \( B \)-semiregular if (2.7) and
\[
\forall N \in \mathfrak{M} \exists M \in \mathfrak{M} \exists C > 0 : M_{k+1} \leq C^k N_k, \quad k \in \mathbb{N},
\]
hold. We say that \( \mathfrak{M} \) is semiregular if the matrix is \( R \)- and \( B \)-semiregular. We also write \( \text{[semiregular]} \) when we mean \( R \)-semiregular or \( B \)-semiregular depending on the case considered.

**Remark 2.5.** We list here some consequences of the conditions above.

1. Let \( \mathfrak{M} \) be a weight matrix. The spaces \( \mathcal{E}^{(\mathfrak{M})}(U) \) and \( \mathcal{E}^{(\mathfrak{M})}(U) \) are closed under derivation when \( \mathfrak{M} \) satisfies (2.8) or (2.9), respectively. If \( \mathfrak{M} \) is \( \text{[semiregular]} \) then \( \mathcal{E}^{(\mathfrak{M})} \) is closed under composition with analytic mappings, see [29, Theorem 2.9]. Hence, if \( X \) is an analytic manifold, then \( \mathcal{E}^{(\mathfrak{M})}(X) \) is well defined.

2. If \( \ell \in \mathbb{N} \) is fixed and \( M, N \) are two weight sequences satisfying \( M_{k+\ell} \leq \gamma^{k+\ell} N_k \) for some constant \( \gamma > 0 \) independent of \( k \in \mathbb{N} \) then there are constants \( C, h > 0 \) such that
\[
(M_k)^{\ell} \leq C \gamma^k N_k, \quad k \in \mathbb{N}.
\]
Indeed, it follows from (2.11) that the sequence \( ((L_k)^{1/k}) \) is increasing for any weight sequence \( L \). Thus we have
\[
(M_k)^{1/k} \leq (M_{k+\ell})^{1/(k+\ell)} \leq \gamma (N_k)^{1/(k+\ell)}
\]
for all \( k \in \mathbb{N} \).

Hence if \( \mathfrak{M} \) is a weight matrix satisfying (2.8) then by iterating (2.8) we obtain for each \( k \in \mathbb{N} \) and \( \mathfrak{M} \) in \( \mathfrak{M} \) and \( C, h > 0 \) satisfying (2.10).

3. An equivalent condition to (2.8) is
\[
\forall M \in \mathfrak{M} \exists N \in \mathfrak{M} \exists C > 0 : m_{k+1} \leq C^{k+1} n_k \quad k \in \mathbb{N}.
\]
Similarly, we can without loss of generality replace in (2.9) \( M_{k+1} \) and \( N_k \) by \( m_{k+1} \) and \( n_k \), respectively.

**Example 2.6.** Here are some families of weight matrices which will play a prominent role later on.

1. The Gevrey matrix \( \mathfrak{G} = \{ G^s : s > 1 \} \) is semiregular. Both spaces \( \mathcal{E}^{(\mathfrak{G})}(U) \) are non-quasianalytic and furthermore we have the identity \( \mathcal{E}^{(\mathfrak{G})}(U) = \mathcal{E}^{(\mathfrak{G^{-1}})}(U) = \mathcal{E}^{(\mathfrak{G}^*)}(U) \) since \( G^s \preceq G^t \) for \( s < t \), cf. [55].

2. Using the sequences \( L^{s,r} \) we can define two families of semiregular matrices:
\[
\mathcal{L}^{s} = \{ L^{s,r} : q > 1 \} \quad r > 1,
\]
\[
\mathcal{N}^{s} = \{ L^{r,s} : r > 1 \} \quad q > 1.
\]
Since \( L^{s_1,s_2} < L^{s_1,s_2} \) for \( r_1 > r_2 \) and all \( q_1, q_2 > 1 \) we have that \( \mathcal{N}^{s} \preceq \mathcal{N}^{r} \) for any \( q > 1 \). Hence if we set \( \mathfrak{K} = \mathcal{N}^{s} \) then \( \mathcal{E}^{(\mathfrak{K})}(U) = \mathcal{E}^{(\mathfrak{N})}(U) \) for all \( q > 1 \). Furthermore \( \mathcal{L}^{s} \preceq \mathcal{L}^{r} \) for all \( r < r' \) and \( \mathcal{N}^{s} \preceq \mathcal{N}^{r} \) and \( \mathcal{L}^{s} \preceq \mathcal{L}^{r} \) for all \( r > 1 \). We note finally that \( \mathfrak{G} \preceq \mathfrak{K} \).

3. For \( j \in \mathbb{N} \) consider the semiregular weight matrix \( \mathfrak{B}^j = \{ B^{j,\sigma} : \sigma > 0 \} \). Then the spaces \( \mathcal{E}^{(\mathfrak{B}^j)}(U) \) and \( \mathcal{E}^{(\mathfrak{B}^j)}(U) \), \( j \geq 2 \), are quasianalytic and \( \mathcal{E}^{(\mathfrak{B}^j)}(U) \) is non-quasianalytic.

Analogously to above we set
\[
\mathcal{J}^\sigma = \{ B^{j,\sigma} : j \in \mathbb{N} \}, \quad \sigma > 0.
\]
Then \( \mathcal{J}^\sigma \preceq \mathcal{B}^j \) for all \( j \in \mathbb{N} \) and \( \sigma > 0 \). Clearly we have also that \( \mathcal{J}^{(\sigma)} \preceq \mathcal{J}^{(\sigma)} \) for all \( \sigma_1, \sigma_2 > 0 \) and finally \( \mathcal{J}^{(\sigma)} \preceq \mathcal{B}^{(\sigma)} \) for any \( \sigma > 0 \).

**Notation 2.7.** We say that \( f \) is an ultradifferentiable function of class \( [s] \) in \( U \), \( * = \mathfrak{M}, \mathfrak{G}, \omega \), if \( f \in \mathcal{E}^{[*]}(U) \).
2B. Weight functions. In this section we discuss briefly the relationship between weight functions and weight matrices, as described in [55].

Recall that a weight function \( \omega : [0, \infty) \to [0, \infty) \) in the sense of [17] is continuous, increasing, \( \omega(0) = 0 \), \( \omega(t) \to \infty \) and satisfies the conditions (\( \square \)), (\( \square \)) and \( \square \). If \( \sigma, \tau \) are two weight functions then we write

\[
\sigma \preceq \tau : \iff \tau(t) = O(\sigma(t)) \quad \text{if} \quad t \to \infty, \\
\sigma \preceq \tau : \iff \tau(t) = o(\sigma(t)) \quad \text{if} \quad t \to \infty.
\]

It follows that \( \mathcal{E}^{(\sigma)}(U) \subseteq \mathcal{E}^{(\tau)}(U) \) if \( \sigma \preceq \tau \) and \( \sigma \prec \tau \) implies \( \mathcal{E}^{(\sigma)}(U) \subseteq \mathcal{E}^{(\tau)}(U) \). We write \( \omega \sim \sigma \) when \( \omega \preceq \sigma \) and \( \sigma \preceq \omega \).

Remark 2.8. (1) It is well known that the weight function \( t^{1/s} \) generates the Gevrey class of order \( s \), i.e. \( \mathcal{G}^s(U) = \mathcal{E}^{(t^{1/s})}(U) \) and in particular, for \( s = 1 \), \( \mathcal{A}(U) = \mathcal{G}^1(U) = \mathcal{E}^{(t)}(U) \) is the space of analytic functions. It follows that if \( \omega \) is a weight function such that \( \omega(t) = o(t^\alpha) \) for some \( 0 < \alpha \leq 1 \) then

\[
\mathcal{G}^{1/\alpha}(U) \subseteq \mathcal{E}^{(\omega)}(U).
\]

(2) In general, weight sequences and weight functions describe distinct spaces, see [16].

(3) The space \( \mathcal{E}^{(\omega)}(U) \) is quasianalytic if and only if

\[
\int_1^\infty \frac{\omega(t)}{t^2} \, dt = \infty. \tag{2.11}
\]

We say that a weight function \( \omega \) is quasianalytic if \( \omega \) satisfies (2.11) and non-quasianalytic otherwise. If \( \omega \) is non-quasianalytic then \( \omega(t) = o(t) \) for \( t \to \infty \).

According to [17] we can without loss of generality assume that \( \omega \) vanishes on \( [0, 1] \). Then the Young conjugate \( \varphi_\omega^*(t) = \sup_{s \geq 0} (st - \varphi_\omega(s)) \) of \( \varphi_\omega = \omega \circ \exp \) is convex, increasing, \( \varphi_\omega^*(0) = 0 \) and \( \varphi_\omega^{**} = \varphi_\omega \).

Furthermore both functions

\[
t \mapsto \frac{\varphi_\omega(t)}{t} \quad \text{and} \quad t \mapsto \frac{\varphi_\omega^*(t)}{t}
\]

are increasing on \( [0, \infty) \).

Definition 2.9. Let \( \omega \) be a weight function such that \( \omega|_{[0,1]} = 0 \). The associated weight matrix \( \mathfrak{W} = \{ W^\lambda = (W^\lambda_k)_k : \lambda > 0 \} \) to \( \omega \) is given by

\[
W^\lambda_k = \exp \left[ \lambda^{-1} \varphi_\omega^*(\lambda k) \right]. \tag{2.12}
\]

We summarize the basic properties of \( \mathfrak{W} \) from [55, Section 5]: First, each sequence \( W^\lambda \) is a weight sequence and \( \mathfrak{W} \) satisfies

\[
W^\lambda_{j+k} \leq W^\lambda_j W^\lambda_k. \tag{2.13}
\]

for all \( j, k \in \mathbb{N}_0 \) and all \( \lambda > 0 \). We note that (2.13) implies (2.8) and (2.9). Furthermore we have

\[
\forall h \geq 1 \, \exists A \geq 1 \, \forall t > 0 \, \exists D \geq 1 \, \forall j \in \mathbb{N}_0 : \quad h^2 W^\lambda_j \leq DW_j^A. \tag{2.14}
\]

From (2.14) we obtain that

\[
\mathcal{E}^{(\omega)}(U) = \mathcal{E}^{([\mathfrak{W}])}(U)
\]

as topological vector spaces. Finally, \( \omega(t) = o(t) \) if and only if

\[
\lim_{k \to \infty} \left( W^\lambda_k \right)^{1/k} = \infty
\]

for all \( \lambda > 0 \), where \( W^\lambda_k = W^\lambda_k / k! \).

Proposition 2.10. (1) Let \( \omega \) be a weight function with \( \omega(t) = o(t) \) when \( t \to \infty \). Then the associated weight matrix is semiregular.

(2) If \( \sigma, \tau \) are two weight functions with associated weight matrices \( \mathcal{G}, \mathcal{I} \) then \( \sigma \sim \tau \) if and only if \( \mathcal{G}(\approx) \mathcal{I} \) and \( \mathcal{I}(\approx) \mathcal{G} \).

Example 2.11. Let \( s > 1 \). If we consider the associated weight matrix \( \mathfrak{W}^s = \{ W^\lambda : \lambda > 0 \} \) of the weight function \( \omega^s(t) = (\max\{0, \log t\})^s \) then we observe that after a reparametrization of the matrix we have \( W^\lambda_k = e^{\lambda k^r} \) where \( r = s / (s - 1) \), i.e. the parameters \( s \) and \( r \) are conjugated: \( \frac{1}{s} + \frac{1}{r} = 1 \), see [57].
subsection 5.5. If \( q = \epsilon^3 \) then clearly \( W^{[\lambda,s]} \subseteq L^{q,r} \) and it is easy to see that \( L^{q,r} \ll W^{[\lambda,s]} \) when \( \lambda < \lambda' \). It follows that \( \mathcal{M}^{[\lambda,s]} \). Hence

\[
E^{[\lambda,s]}(U) = E^{[\lambda']}(U), \quad \frac{1}{\lambda} + \frac{1}{\lambda'} = 1.
\]

2C. The ultradifferentiable wavefront set. The ultradifferentiable wavefront set for Roumieu classes given by weight sequences was introduced in [35]. In [1] the wavefront set was defined in the case of weight functions. These definitions have been generalized by [29] to the category of classes given by weight matrices.

For the convenience of the reader we recall from [29] the definition of the wavefront set associated to classes given by weight matrices. We give also a summary of the results we need later on, observing in particular that, in analogy to the results of [35] in the case of a single weight sequence, semiregularity of the weight matrix is sufficient for the ultradifferentiable microlocal elliptic regularity Theorem to hold for operators with analytic coefficients; a fact that was not explicitly stated in [29] because in that paper we worked in a more general setting.

We define the Fourier transform of a distribution \( u \in \mathcal{E}'(U) \) to be

\[
\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \langle u(x), e^{i\xi x} \rangle_x,
\]

where the bracket on the right-hand side denotes the distributional action.

**Definition 2.12.** Let \( \mathcal{M} \) be a weight matrix, \( u \in \mathcal{D}'(U) \) and \( (x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\} \). Then

1. \( (x_0, \xi_0) \notin WF(\mathcal{M}) u \) iff there exist a neighborhood \( V \) of \( x_0 \), a conic neighborhood \( \Gamma \) of \( \xi_0 \), and a bounded sequence \( (u_k)_k \subseteq \mathcal{E}'(U) \) with \( u_k|_V = u|_V \) such that for some \( \mathbf{M} \in \mathcal{M} \) and a constant \( h > 0 \) we have

\[
\sup_{\xi \in \Gamma} \sup_{k \in \mathbb{N}} \frac{|\xi|^k \hat{u}_k(\xi)|}{h^k M_k} < \infty,
\]

(2.15)

2. \( (x_0, \xi_0) \notin WF(\mathcal{M}) u \) iff there exist a neighborhood \( V \) of \( x_0 \), a conic neighborhood \( \Gamma \) of \( \xi_0 \), and a bounded sequence \( (u_k)_k \subseteq \mathcal{E}'(U) \) with \( u_k|_V = u|_V \) such that (2.15) is satisfied for all \( \mathbf{M} \in \mathcal{M} \) and all \( h > 0 \).

The basic properties of the ultradifferentiable wavefront set are summarized in the following Proposition.

**Proposition 2.13 ([29] Proposition 5.4(1)-(4)).** Let \( \mathcal{M}, \mathcal{N} \) be weight matrices and \( u \in \mathcal{D}'(U) \). Then the following statements hold:

1. \( WF(\mathcal{M}) u \) is a closed subset of \( U \times \mathbb{R}^n \setminus \{0\} \) which is conic in the second variable.
2. \( WF(\mathcal{M}) u \subseteq WF(\mathcal{N}) u \).
3. \( WF(\mathcal{M}) u \subseteq WF(\mathcal{N}) u \) iff \( \mathcal{M} \preceq \mathcal{N} \).
4. \( WF(\mathcal{M}) u \subseteq WF(\mathcal{N}) u \) if \( \mathcal{M} \sim \mathcal{N} \).

If we assume that \( \mathcal{M} \) satisfies additional conditions then we can show more properties of \( WF(\mathcal{M}) u \):

**Proposition 2.14 ([29] Proposition 5.6(1)).** Let \( \mathcal{M} \) be a weight matrix satisfying (2.7) and \( u \in \mathcal{D}'(U) \). We have

\[
WF(\mathcal{M}) u = \bigcap_{\mathbf{M} \in \mathcal{M}} WF(\mathbf{M}) u \quad \text{and} \quad WF(\mathcal{N}) u = \bigcup_{\mathbf{M} \in \mathcal{M}} WF(\mathbf{M}) u.
\]

Similar to the smooth category we define the \( \mathcal{M} \)-singular support \( \text{sing supp}(\mathcal{M}) u \) of a distribution \( u \in \mathcal{D}'(U) \) as the complement of the largest subset \( V \subseteq U \) such that \( u|_V \in \mathcal{E}(\mathcal{M}) \). We have

**Proposition 2.15 ([29] Proposition 5.4(6)).** Let \( \mathcal{M} \) be a \{semiregular\} weight matrix and \( u \in \mathcal{D}'(U) \). Then

\[
\pi_1 \left( WF(\mathcal{M}) u \right) = \text{sing supp}(\mathcal{M}) u
\]

where \( \pi_1 : U \times \mathbb{R}^n \setminus \{0\} \to U \) is the projection to the first variable.

It is possible to choose the distributions \( u_k \) in Definition 2.12 in a special manner. For our purpose a simplified version of [29] Lemma 5.3 is sufficient:

**Lemma 2.16.** Let \( \mathcal{M} \) be \{semiregular\}, \( u \in \mathcal{D}'(U) \), \( K \subseteq U \) a compact subset and \( F \subseteq \mathbb{R}^n \setminus \{0\} \) a closed cone such that

\[
WF(\mathcal{M}) u \cap K \times F = \emptyset.
\]
Furthermore assume that \( \chi_k \in \mathcal{D}(U) \) is a sequence of functions with common support in \( K \) and for all \( \alpha \in \mathbb{N}_0^n \) there are constants \( C_{\alpha, h_0} > 0 \) such that

\[
|D^{\alpha + \beta} \chi_k| \leq C_{\alpha, h_0} |\beta|^k, \quad |\beta| \leq k.
\]

If \( \mu \) is the order of \( u \) near \( K \) then the sequence \( (\chi_k u)_k \) is bounded in \( \mathcal{E}^{\mu}(K) \) and

1. In the Roumieu case we have

\[
\sup_{\xi} |\xi|^k |\mathcal{F}(\chi_k u)(\xi)| \leq C h^k M_k
\]

for some \( M_k \in \mathfrak{M} \) and \( C, h > 0 \).

2. In the Beurling case for all \( M_k \in \mathfrak{M} \) and all \( h > 0 \) there is \( C = C_{M, h} \) such that the estimate \( 2.10 \) holds.

Lemma 2.16 allows us directly to generalize the proof of [35, Theorem 5.4] in order to show the microlocal elliptic regularity theorem for classes given by weight matrices and operators with analytic coefficients. For a more general version see [29, Theorem 7.1].

**Theorem 2.17.** Let \( P \) be a differential operator with analytic coefficients on \( U \) and \( \mathfrak{M} \) be a [semiregular] weight matrix. Then we have

\[ \WF(\mathfrak{M}) P u \subseteq \WF(\mathfrak{M}) u \subseteq \WF(\mathfrak{M})(Pu) \cup \Char P \]

for all \( u \in \mathcal{D}'(U) \).

3. Ultradifferentiable vectors

3A. Microlocal theory. The aim of this section is to generalize the microlocal theory presented in [13] to the setting of weight matrices. In order to accomplish this we have to use a more generalized notion of vectors than the one from Section 1. For this we need to recall some notions.

Let \( \sigma \in \mathbb{R} \). We denote the Sobolev space of order \( \sigma \) by \( H^\sigma(\mathbb{R}^n) \), which is equipped with the norm

\[
\| g \|_\sigma = \left( \int (1 + |\xi|^2)^{2\sigma} |\hat{g}(\xi)|^2 \, d\xi \right)^{1/2}.
\]

The localized Sobolev space \( H^\sigma_{loc}(U) \) consists of those distributions \( g \in \mathcal{D}'(U) \) which satisfy \( \varphi \cdot g \in H^\sigma(\mathbb{R}^n) \) for all \( \varphi \in \mathcal{D}(U) \). It is a locally convex space whose topology is given by the seminorms

\[
g \mapsto \| \varphi g \|_\sigma.
\]

A different seminorm on \( H^\sigma(\mathbb{R}^n) \) is

\[
\| g \|_{H^\sigma(V)} = \inf \{ \| G \|_\sigma : G \in H^\sigma(\mathbb{R}^n), G|_V = g \}
\]

where \( V \subseteq U \).

**Definition 3.1.** Let \( \mathfrak{M} \) be a weight matrix, \( P = \{P_1, \ldots, P_{\ell}\} \) a system of differential operators of order \( d_j, \ j = 1, \ldots, \ell \), with analytic coefficients in the open set \( U \subseteq \mathbb{R}^n \) and \( \sigma \in \mathbb{R} \). If \( V \subseteq U \), \( \mathfrak{M} \in \mathfrak{M} \) and \( h > 0 \) then we set

\[
\| u \|_{V; \mathfrak{M}, h} = \sup_{\alpha \in \{1, \ldots, \ell\}^k} \frac{\| P^{\alpha} u \|_{H^\sigma(V)} \} h^{d_{\alpha}} M_{\alpha}}{\langle \alpha \rangle !}
\]

where \( P^{\alpha} = P_{\alpha_1} \ldots P_{\alpha_k}, d_{\alpha} = d_{\alpha_1} + \cdots + d_{\alpha_k} \) and \( u \in \mathcal{D}'(U) \) such that \( P^{\alpha} u \in H^\sigma_{loc}(U) \) for all \( \alpha \in \{1, \ldots, \ell\}^k \) and \( k \in \mathbb{N}_0 \). In the case \( k = 0 \) we use the convention \( \{1, \ldots, \ell\}^0 = \emptyset \) and \( d_{\emptyset} = 0 \). We set

\[
\mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) = \left\{ u \in \mathcal{D}'(U) : \forall V \subseteq U \quad \exists M \in \mathfrak{M} \quad \exists h > 0 : \| u \|_{V; \mathfrak{M}, h} < \infty \right\},
\]

\[
\mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) = \left\{ u \in \mathcal{D}'(U) : \forall V \subseteq U \forall M \in \mathfrak{M} \forall h > 0 : \| u \|_{V; \mathfrak{M}, h} < \infty \right\}.
\]

An element of \( \mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) \) is called an ultradifferentiable vector of class \( \mathfrak{M} \) (or an \( \mathfrak{M} \)-vector) of the system \( P \). We also define \( \mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) \) to be the space of those distributions \( u \in \mathcal{D}'(U) \) such that for all \( x_0 \in U \) there is a neighborhood \( V \) of \( x_0 \) such that \( u|_V \in \mathcal{E}^{(\mathfrak{M})}_\sigma(V; P) \). If \( P = \{P\} \) consists of a single operator then we write \( \mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) = \mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) \).

**Proposition 3.2.** Let \( \mathfrak{M}, \mathfrak{N} \) be weight matrices and \( P \) be a system of analytic differential operators.

Then the following holds:

1. If \( \mathfrak{M} \leq \mathfrak{N} \) then \( \mathcal{E}^{(\mathfrak{M})}_\sigma(U; P) \subseteq \mathcal{E}^{(\mathfrak{N})}_\sigma(U; P) \).
(2) If \( \mathfrak{M} \triangleleft \mathfrak{N} \) then \( E^{(\mathfrak{M})}(U; P) \subseteq E^{(\mathfrak{N})}(U; P) \).

(3) If \( \mathfrak{M} \) satisfies (2.7) then \( E^{(\mathfrak{M})}(U) \subseteq E^{(\mathfrak{M})}(U; P) \).

Proof. If \( \mathfrak{M} \triangleleft \mathfrak{N} \) and \( V \subseteq U \) are given then for all \( M \in \mathfrak{M} \) and all \( h > 0 \) there are \( N \in \mathfrak{R} \) and \( h', C > 0 \) such that

\[
\|u\|_{\mathcal{L}^r(N, h')} \leq C\|u\|_{\mathcal{L}^r(M, h')}
\]

for \( u \in \mathcal{D}'(U) \). Hence (1) holds in the Roumieu case. The proofs of the other statements in (1) and (2) are similar.

In order to show (3), recall that (2.7) implies that for all \( M \in \mathfrak{M} \) and all \( \gamma > 0 \) there is a constant \( C > 0 \) such that

\[
k! \leq C\gamma M_k, \quad k \in \mathbb{N}_0.
\]

We may also note that if \( Q = \sum_{|\beta| \leq d} a_\beta(x)D^\beta \) is an operator with analytic coefficients in \( U \) then for each \( V \subseteq U \) we can find constants \( C, r > 0 \) such that

\[
|D^\alpha a_\beta(x)| \leq C\{\alpha, |\alpha|!\}, \quad x \in V, \alpha \in \mathbb{N}_0.
\]

This means that for \( f \in E^{(\mathfrak{M})}(U) \), and \( \alpha \in \mathbb{N}_0 \) we can estimate in \( V \subseteq U \) that

\[
\sup_{x \in V} |D^\alpha Qf(x)| \leq C \sum_{|\beta| \leq d} \sum_{|\alpha'| \leq \alpha} \left( \frac{\alpha}{\alpha'} \right)^{\alpha - \alpha'} |\alpha - \alpha'| \|h|\alpha' + \beta\| \|M| \alpha' + \beta\|
\]

\[
\leq C \sum_{|\beta| \leq d} \sum_{|\alpha'| \leq \alpha} \left( \frac{\alpha}{\alpha'} \right)^{\alpha - \alpha'} h|\alpha - \alpha'| M|\alpha' + \beta\|
\]

\[
\leq C2\{\alpha\} \sum_{|\beta| \leq d} h|\alpha + |\beta\| M|\alpha + |\beta\|
\]

\[
\leq C(2\delta)^{|\alpha| + d} M|\alpha| + d
\]

for some weight sequence \( M \in \mathfrak{M} \) and \( h \geq 1 \). Iterating this argument we conclude that there are weight sequences \( M \in \mathfrak{M} \) and constants \( C, h > 0 \) such that, when \( \alpha \in \{1, \ldots, \ell\}^k \), \( \ell, k \in \mathbb{N} \), we have

\[
\sup_{x \in V} |D^\alpha f(x)| \leq Ch^{d_\alpha}M_{d_\alpha},
\]

where \( P^\alpha \) and \( d_\alpha \) are defined as in Definition 3.1. Therefore

\[
\|P^\alpha f\|_{\mathcal{L}^r(V)} \leq Ch^{d_\alpha}M_{d_\alpha}
\]

for some constants \( C, h > 0 \) independent of \( k \in \mathbb{N} \) and \( \alpha \in \{1, \ldots, \ell\}^k \) and hence \( f \in E^{(\mathfrak{M})}(U; P) \).

If \( f \in E^{(\mathfrak{M})}(U) \) and \( V \subseteq U \) then we define a sequence \( L' \triangleleft M \) by setting

\[
L'_k = \max \left\{ k!, \sup_{x \in V, |\alpha| \leq k} |D^\alpha f(x)| \right\}.
\]

According to Lemma 2.2 for each \( M \in \mathfrak{M} \) there is a weight sequence \( N \) such that \( G^1 \leq L' \leq N \triangleleft M \) and by construction we have that there are constants \( \gamma > 0 \) and \( C > 0 \) such that

\[
|D^\alpha f(x)| \leq C\gamma^{\alpha}N_{\alpha}, \quad \alpha \in \mathbb{N}_0,
\]

for \( x \in V \). We obtain

\[
\sup_{x \in V} |D^\alpha Qf(x)| \leq C \sum_{|\beta| \leq d} \sum_{|\alpha'| \leq \alpha} \left( \frac{\alpha}{\alpha'} \right)^{\alpha - \alpha'} |\alpha - \alpha'| \|h|\alpha' + \beta\| N_{\alpha' + \beta}\]

\[
\leq C \sum_{|\beta| \leq d} \sum_{|\alpha'| \leq \alpha} \left( \frac{\alpha}{\alpha'} \right)^{\alpha - \alpha'} N_{\alpha - \alpha'} |\alpha' - \beta| N_{\alpha' + \beta}\]

\[
\leq C \sum_{|\beta| \leq d} h|\alpha + |\beta\| N_{|\alpha + |\beta\|}\]

\[
\leq C h|\alpha + d N_{|\alpha| + d}.
\]

Thence for each \( M \in \mathfrak{M} \) and \( h > 0 \) there is a constant \( C > 0 \) such that

\[
\sup_{x \in V} |D^\alpha Qf(x)| \leq C h^{\alpha + d} M_{\alpha + d}.
\]

From this estimate it follows in the same manner as in the Roumieu case that \( f \in E^{(\mathfrak{M})}(U; P) \).
Remark 3.3. Traditionally, the $L^2$-norm is mainly used in the definition of vectors, but in the literature the norm in the definition of vectors is chosen according to the techniques used in the paper in question, see e.g. the discussion in [13]. We have already mentioned that Definition 3.4 is more general than the definition of vectors used in Section 1, because, as we will see in a moment, Definition 3.1 is microlocalizable, cf. [13] and [12].

However, cf. [14], if the system $P = \{P_1, \ldots, P_l\}$ is subelliptic, that is for each $V \Subset U$ there is $\varepsilon > 0$ such that for all $\alpha \in \mathbb{R}$ the estimate

$$
\| \varphi \|_{s + \varepsilon} \leq C \left( \sum_{j=1}^l \| P_j \varphi \|_{\alpha} + \| \varphi \|_{\alpha} \right), \quad \varphi \in \mathcal{D}(V),
$$

holds for some $C > 0$, then we obtain that

$$
\mathcal{E}_{\alpha}^{(\mathfrak{M})}(U; P) = \mathcal{E}_{\alpha}^{(\mathfrak{M})}(U; P)
$$

for all $\alpha \in \mathbb{R}$ when $\mathfrak{M}$ is [semiregular].

Indeed, if $u \in \mathcal{E}_{\alpha}^{(\mathfrak{M})}(U; P)$ then by definition $P^\alpha u \in H^{s}_{loc}(U)$ for all $\alpha$. It is well known that this implies therefore $u \in E(U) = H^{s}_{loc}(U)$, see e.g. [37] or [66]. Furthermore, $\mathcal{E}_{\alpha}^{(\mathfrak{M})}(U; P) \subseteq \mathcal{E}_{\gamma}^{(\mathfrak{M})}(U; P)$ for $\tau \leq \sigma$ since $|g|, \leq |g|_{\sigma}$ for all $g \in H^{\tau}(\mathbb{R}^n)$.

If now $V$ and $W$ are two open sets with $V \Subset W \Subset U$ then (3.2) implies that

$$
\| f \|_{\mathcal{H}^{s+\varepsilon}(V)} \leq C \left( \sum_{j=1}^l \| P_j f \|_{\mathcal{H}^{s}(V)} + \| f \|_{\mathcal{H}^{s}(V)} \right), \quad f \in \mathcal{E}(U),
$$

(3.2')

where $\varepsilon$ is the subellipticity index of $W$, see [13].

We suppose for a moment that $M, N \in \mathfrak{M}$ are two weight sequences for which there exists a constant $\gamma \geq 1$ such that

$$
M_{k+1} \leq \gamma^{k+1}N_k, \quad k \in \mathbb{N}_0.
$$

(3.3)

If we combine (3.2) with (3.3) we conclude that

$$
\| u \|_{\mathcal{E}_{\gamma}^{(\mathfrak{M})},h} \leq C \| u \|_{\mathcal{E}_{\gamma}^{(\mathfrak{M},\gamma)/\gamma}}
$$

for $u \in \mathcal{E}(U)$.

If $\mathfrak{M}$ is $B$-semiregular and $u \in \mathcal{E}_{\gamma}^{(\mathfrak{M})}(U; P)$ then by definition

$$
\| u \|_{\mathcal{E}_{\gamma}^{(\mathfrak{M},h)}}, < \infty
$$

for all $V \Subset U$, all $M \in \mathfrak{M}$ and all $h > 0$. Hence, by the above arguments we can conclude that actually

$$
\| u \|_{\mathcal{E}_{\gamma}^{(\mathfrak{M},h)}}, < \infty
$$

for all $V \Subset U$, all $M \in \mathfrak{M}$, all $h > 0$ and every $\sigma \in \mathbb{R}$, that is

$$
\mathcal{E}_{\sigma}^{(\mathfrak{M})}(U; P) = \mathcal{E}_{\sigma}^{(\mathfrak{M})}(U; P)
$$

for all $\sigma, \tau \in \mathbb{R}$. The Roumieu case follows similarly.

We are now able to begin to extend the microlocal theory developed in [13] for Roumieu vectors given by a semiregular weight sequence of an operator with analytic coefficients to vectors associated to a [semiregular] weight matrix. We follow mainly the presentation given in [12]. We start with a characterization of the property of being a vector by the Fourier transform.

Theorem 3.4. Let $P$ be a differential operator of order $d$ with analytic coefficients in $U$, $u \in \mathcal{D}'(U)$, $x_0 \in U$ and $\mathfrak{M}$ be a weight matrix. Then

1. $u \in \mathcal{E}_{\sigma}^{(\mathfrak{M})}(V; P)$ for some neighborhood $V$ of $x_0$ if and only if there are a neighborhood $W$ of $x_0$ and a sequence $f_k \in \mathcal{E}'(U)$ such that $f_k|_W = (P_k u)|_W$ and

$$
| \hat{f}_k(\xi) | \leq C h^k M_{dk}(1 + |\xi|)^\nu, \quad \forall \xi \in \mathbb{R}^n,
$$

(3.4)

for a sequence $M \in \mathfrak{M}$ and some constants $C, h > 0$ and $\nu \in \mathbb{R}$.

2. $u \in \mathcal{E}_{\sigma}^{(\mathfrak{M})}(V; P)$ for some neighborhood $V$ of $x_0$ if and only if there are a neighborhood $W$ of $x_0$, a sequence $f_k \in \mathcal{E}'(U)$ and a constant $\nu \in \mathbb{R}$ such that $f_k|_W = (P_k u)|_W$ and for all $M \in \mathfrak{M}$ and every $h > 0$ there is some $C > 0$ so (3.4) is satisfied.
Proof. We begin with the Roumieu case. Hence suppose that $u \in E^\infty(V; P)$ for some neighborhood $V$ of $x_0$ and $\sigma \in \mathbb{R}$. Following [12] let $W_2 \subseteq W_1 \subseteq V$ be two neighborhoods of $x_0$ and choose $\varphi, \psi \in \mathcal{D}(W_1)$ with $\psi \varphi = \varphi$ and $\varphi = 1$ in $W_2$. If we set $f_k = \varphi P^k u$ then $f_k \in \mathcal{E}'(V)$ and $f_k = P^k u$ in $W_2$. Furthermore

$$
\left| \hat{f}_k(\xi) \right| = \left| \mathcal{F}(\psi \varphi P^k u)(\xi) \right| = \left| \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\eta|)^{-\sigma} \hat{\varphi}(\xi - \eta)(1 + |\eta|)^{\sigma} \mathcal{F}(\psi P^k u)(\eta) d\eta \right|
$$

$$
\leq C \|\psi P^k u\|_{L^\infty(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} (1 + |\eta|)^{-2\sigma} |\hat{\varphi}(\xi - \eta)|^2 d\eta \right)^{1/2}
$$

$$
\leq C \|P^k u\|_{L^\infty(W_2)} (1 + |\xi|)^{-\sigma} \left( \int_{\mathbb{R}^n} (1 + |\xi - \eta|)^{2\sigma} |\hat{\varphi}(\xi - \eta)|^2 d\eta \right)^{1/2}
$$

$$
\leq Ch^k M_{dk} (1 + |\xi|)^{-\sigma} \|\varphi\|_{L^\infty(\mathbb{R}^n)}
$$

$$
\leq Ch^k M_{dk} (1 + |\xi|)\nu
$$

for some $M \in \mathbb{R}$ and some constants $h > 0$ and $\nu = -\sigma$.

On the other hand assume that there is a sequence $f_k \in \mathcal{E}'(U)$ and a neighborhood $V$ of $x_0$ such that $f_k|_V = P^k u|_V$ and (3.4) holds for some $M \in \mathbb{R}$ and constants $C, h, \nu > 0$. Now let $\sigma \leq -\nu - (n + 1)/2$. Then we obtain for every $W \subseteq V$ that

$$
\|P^k u\|_{L^\infty(W)} \leq \|f_k\|_{L^\infty(W)}
$$

$$
= \left( \int_{\mathbb{R}^n} (1 + |\xi|)^{2\sigma} |\hat{f}_k(\xi)|^2 d\xi \right)^{1/2}
$$

$$
\leq Ch^k M_{dk} \left( \int_{\mathbb{R}^n} (1 + |\xi|)^{2(\sigma + \nu)} d\xi \right)^{1/2}
$$

$$
\leq C'h^k M_{dk}
$$

for some $C' > 0$ since $\sigma$ was chosen appropriately.

The Beurling case follows in a similar manner. \hfill \Box

In the definition of the wavefront set of iterates the estimate $\ref{6.1}$ will correspond to $\ref{2.4}$. The following statement is going to provide a correspondence of the boundedness of the sequence $u_k$ in Definition $\ref{2.1}$.

**Proposition 3.5** ([12] Proposition 1.6]). Let $u \in \mathcal{D}'(U)$, $P$ be an analytic partial differential operator of order $d$ and $K \subseteq U$ be a compact set. Furthermore assume that $\chi_k \in \mathcal{D}(U)$ is a sequence of functions with common support in $K$ satisfying

$$
|D^\alpha \chi_k(x)| \leq C(Ck)^{1/\alpha}
$$

for $|\alpha| \leq k \in \mathbb{N}_0$ and some constant $C > 0$.

If $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ then the sequence $f_k = \chi_{pd+q} u$ obeys the estimate

$$
\left| \hat{f}_k(\xi) \right| \leq C'(C'\nu + |\xi|)^{dk+\nu} \quad \xi \in \mathbb{R}^n, \quad k \in \mathbb{N}_0,
$$

for some constants $C', \nu > 0$.

**Definition 3.6.** Let $P$ be a differential operator with analytic coefficients of order $d$, $M$ a weight matrix, $u \in \mathcal{D}'(U)$ and $(x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}$. Then we say that

1. $(x_0, \xi_0) \notin \text{WF}_M[u; P]$ if there is a neighborhood $V$ of $x_0$, a conic neighborhood $\Gamma$ of $\xi_0$ and a sequence $f_k \in \mathcal{E}'(V)$ satisfying $f_k|_V = (P^k u)|_V$ and there are a sequence $M \in \mathbb{R}$ and constants $C, h > 0$ and $\nu \in \mathbb{R}$ such that

$$
\left| \hat{f}_k(\xi) \right| \leq Ch^k \left[ (M_{dk})^{1/\nu} + |\xi| \right]^{\nu + dk} \quad \forall k \in \mathbb{N}, \forall \xi \in \mathbb{R}^n \quad (3.5)
$$

$$
\left| \hat{f}_k(\xi) \right| \leq C M_{dk} (1 + |\xi|)^\nu \quad \forall k \in \mathbb{N}, \forall \xi \in \Gamma. \quad (3.6)
$$
Lemma 3.8. \( (x_0, \xi_0) \notin \text{WF}(\mathcal{M})(u; P) \) if there is a neighborhood \( V \) of \( x_0 \), a conic neighborhood \( \Gamma \) of \( \xi_0 \) and a sequence \( f_k \in \mathcal{E}'(U) \) with \( f_k|_V = (P^k u)|_V \) and there exists some \( \nu \in \mathbb{R} \) such that for all \( M \in \mathfrak{M} \) and all \( h > 0 \) there is a constant \( C > 0 \) for which the estimates (3.5) and (3.6) are satisfied.

It is easy to see that \( \text{WF}(\mathcal{M})(u; P) \) satisfies the same basic properties as \( \text{WF}(\mathcal{M})u \), cf. Proposition 2.14.

Proposition 3.7. Let \( \mathfrak{M} \) and \( \mathfrak{N} \) be two weight matrices and \( u \in \mathcal{D}'(U) \). Then:

1. \( \text{WF}(\mathfrak{M})(u; P) \) is a closed, conic in the second variable, subset of \( U \times \mathbb{R}^n \setminus \{0\} \).
2. \( \text{WF}(\mathfrak{M})(u; P) \subseteq \text{WF}(\mathfrak{N})(u; P) \).
3. If \( \mathfrak{M} \subseteq \mathfrak{N} \) then \( \text{WF}(\mathfrak{M})(u; P) \subseteq \text{WF}(\mathfrak{N})(u; P) \) for all \( u \in \mathcal{D}'(U) \).
4. If \( \mathfrak{M} \subset \mathfrak{N} \) then \( \text{WF}(\mathfrak{M})(u; P) \subseteq \text{WF}(\mathfrak{N})(u; P) \).

We have also a variant of Lemma 2.16.

Lemma 3.8. Let \( \mathfrak{M} \) be [semiregular], \( u \in \mathcal{D}'(U) \), and \( K \subseteq U \) be a compact subset, \( F \subseteq \mathbb{R}^n \) a closed cone and \( \chi_k \in \mathcal{D}(U) \) a sequence of functions with support in \( K \) such that for all \( \alpha \in \mathbb{N}_0^\mathbb{N} \) there are constants \( C_{\alpha, h} > 0 \) with

\[
|D^{\alpha} \chi_k| \leq C_{\alpha, h} |k|^{|\beta|} \quad \text{for all } k \in \mathbb{N}. \tag{3.7}
\]

1. If \( \text{WF}(\mathfrak{M})(u; P) \cap K \times F = \emptyset \) then there are a sequence \( M \in \mathfrak{M} \) and constants \( C, h > 0 \) and \( \nu \in \mathbb{R} \) such that the sequence \( \chi_k P^k u \) satisfies (3.5) for \( \xi \in F \).
2. If \( \text{WF}(\mathfrak{M})(u; P) \cap K \times F = \emptyset \) then there is some \( \nu \in \mathbb{R} \) such that for all \( h > 0 \) and all \( M \in \mathfrak{M} \) the estimate (3.6) holds for the sequence \( \chi_k P^k u \) in \( F \).

Proof. First we prove the Roumieu case. Let \( x_0 \in K \) and \( \xi_0 \in F \). Then \( (x_0, \xi_0) \notin \text{WF}(\mathfrak{M})(u; P) \) and we choose \( V, \Gamma \) and \( f_k \) according to Definition 3.6. If \( \supp \chi_k \subseteq V \) then \( \chi_k P^k u = \chi_k f_k \) and therefore

\[
(2\pi)^n \mathcal{F}(\chi_k P^k u)(\xi) = \int \chi_k(\xi - \eta) f_k(\eta) d\eta.
\]

Note that without loss of generality we can always assume \( \nu \geq 0 \). We observe that (3.7) gives

\[
|\eta^{\alpha} \chi_k(\eta)| \leq C_{\alpha, h} |k|^{|\beta|}, \quad \alpha, \beta \in \mathbb{N}_0^\mathbb{N}, \quad |\beta| \leq k \in \mathbb{N},
\]

for some \( C_{\alpha, h} > 0 \). It follows that there are constants \( C, h > 0 \) such that

\[
|\chi_k(\eta)| \leq C h^k (1 + |\eta|)^{-n-1-\nu}. \tag{3.8}
\]

For \( \ell, j \geq 0 \) we have, (cf. 29, p. 26)

\[
|\eta|^{\ell+j} \leq \sum_{|\gamma|=\ell+j} \binom{\ell+j}{\gamma} |\eta|^\gamma.
\]

If \( j \leq k \) then

\[
|\eta|^{\ell+j} |\chi_k(\eta)| \leq \sum_{|\gamma|=\ell+j} \binom{\ell+j}{\gamma} |\eta|^\gamma |\chi_k(\eta)| \leq n^{\ell+j} \sum_{|\alpha|\leq \ell, |\beta|=j} |\eta^{\alpha+\beta} \chi_k(\eta)| \leq C h^k k^j
\]

for some \( C, h > 0 \). For \( M \in \mathfrak{M} \) we obtain

\[
(M_k)^{1/k} |\chi_k(\eta)| \leq (M_k)^{k+n+1} |\chi_k(\eta)| \leq C h^k (M_k)^{k+n+1} |\chi_k(\eta)| \leq C h^k |\chi_k(\eta)|
\]

Since \( \mathfrak{M} \) is R-semiregular it follows from Remark 2.2(2) that for each \( M \in \mathfrak{M} \) there are \( N \in \mathfrak{M}, C, h > 0 \) such that

\[
|\chi_k(\eta)| \leq C h^k N_k \left((M_k)^{1/k} + |\eta|\right)^{-k-\nu-n-1}. \tag{3.9}
\]
The estimate (3.8) implies
\[ \int_{\Gamma} |\hat{\chi}k(\xi) - \hat{\chi}(\eta)| \, d\eta \leq Ch^k Mdk \int_{\mathbb{R}^n} (1 + |\xi - \eta|)^{-n-1} (1 + |\eta|)^{\nu} \, d\eta \]
\[ \leq Ch^k Mdk (1 + |\xi|)^{\nu}. \]

On the other hand choose a closed cone \( \Gamma_1 \subseteq \Gamma \cup \{0\} \) with \( \xi_0 \in \Gamma_1 \). Then there is a constant \( c > 0 \) such that
\[ |\xi - \eta| \geq c(|\xi| + |\eta|) \]
for all \( \xi \in \Gamma_1 \) and \( \eta \notin \Gamma \). If we also use (3.9) and set \( \bar{c} = \min\{1, c\} \) then it follows that for each \( M \in \mathfrak{M} \) there is some \( N \in \mathfrak{M} \) such that
\[ \int_{\mathbb{R}^n \setminus \Gamma} |\hat{\chi}k(\xi) - \hat{\chi}(\eta)| \, d\eta \leq Ch^k Ndk \int_{\mathbb{R}^n} (Mdk)^{\frac{1}{\nu}} + |\xi - \eta|^{-dk - \nu - n - 1} \]
\[ \times (Mdk)^{\frac{1}{\nu}} + |\eta|^{-dk + \nu} \, d\eta \]
\[ \leq Ch^k Ndk \int_{\mathbb{R}^n} (Mdk)^{\frac{1}{\nu}}(\hat{\chi}(\xi) + \bar{c}|\eta|)^{-n-1} \, d\eta \]
\[ \leq Ch^k Ndk. \]

We have shown that if \( \text{supp} \, \chi_k \subseteq U \) and \( \xi_0 \in F \setminus \{0\} \) there is a closed conic neighborhood \( \Gamma' \) of \( \xi_0 \) such that
\[ |F (\hat{\chi}k - \hat{\chi}) (\xi)| \leq Ch^k Mdk (1 + |\xi|)^{\nu_0}, \quad \xi \in \Gamma', \]
(3.10)
for some \( C_0, h_0 > 0, M \in \mathfrak{M} \) and \( \nu_0 \in \mathbb{R} \). Since \( \xi_0 \in F \setminus \{0\} \) was chosen arbitrarily, note that \( F \) can be covered by a finite number of cones like \( \Gamma' \) and therefore (3.10) holds in \( F \) for some constants \( C, h \) and \( \nu_0 \) as long as \( \text{supp} \, \chi_k \subseteq U \) is a small enough neighborhood of \( x_0 \). But \( K \) is compact hence we can argue as in the proof of (3.9) Lemma 8.4.4. There is a finite number of such open sets \( U_j \) that cover \( K \) and we can choose a partition of unity \( \chi_{j,k} \in \mathcal{D}(U_j) \) such that \( (\chi_{j,k})_k \) satisfies (3.4) for each \( j \). Then the same is true for \( \chi_{j,k} \chi_k \) and we conclude from above that (3.10) holds for \( \chi_{j,k} \chi_k \). Since \( \sum \chi_{j,k} \chi_k \chi_k \) we have proven (3.10) in the general case.

The proof of the estimate in the Beurling category is analogous. Just note that if \( \mathfrak{M} \) is \( B \)-semiregular then Remark 2.5(2) implies that for all \( N \in \mathfrak{M} \) there are \( M \in \mathfrak{M}, C, h > 0 \) such that (3.9) holds. \( \square \)

Lemma 3.8 allows us to prove an analogue of Proposition 2.15.

**Theorem 3.9.** If \( \mathfrak{M} \) is \( [\text{semiregular}] \) and \( u \in \mathcal{D}(U) \) then \( U_0 = U \setminus \pi_1(\WF_{\mathfrak{M}}(u; P)) \) is the greatest open set such that \( u \in \mathcal{C}^0_{loc}(U_0; P) \).

**Proof.** Let \( U_1 \subseteq U \) be an open set such that \( u \in \mathcal{C}^0_{loc} (U_1; P) \). If \( x \in U_1 \) then by Theorem 3.3 (and Proposition 3.5) it follows that \( (x, \xi) \notin \WF_{\mathfrak{M}}(u; P) \) for all \( (x, \xi) \in U_1 \times \mathbb{R}^n \setminus \{0\} \).

On the other hand if \( x \in U \) is such that \( (x, \xi) \notin \WF_{\mathfrak{M}}(u; P) \) for all \( \xi \in \mathbb{R}^n \setminus \{0\} \) then we can find a compact neighborhood \( K \) of \( x \) such that \( K \times \mathbb{R}^n \cap \WF_{\mathfrak{M}}(u; P) = \emptyset \). If we choose functions \( \chi_k \in \mathcal{D}(K) \) satisfying (3.7) which equal 1 in some neighborhood \( V \) of \( x \), which is possible due to (3.9) Theorem 1.4.2], then Lemma 3.8 implies that \( f_k = \chi_{dk+q} P_k u \) satisfies (3.3). Hence by Theorem 3.4 \( u \in \mathcal{C}^0_{loc}(V; P) \). \( \square \)

**3B. Invariance under analytic mappings.** The aim of this section is to prove the invariance of the definition of \( \WF_{\mathfrak{M}}(u; P) \). We begin by recalling two results from [35], see also [14].

**Lemma 3.10 ([35] Lemma 3.6).** Let \( U_1 \subseteq \mathbb{R}^n \) and \( U_2 \subseteq \mathbb{R}^{n_2} \) be two open sets, \( a \in \mathcal{A}(U_1) \) and \( f : U_1 \to U_2 \) be an analytic mapping. Furthermore assume that \( \chi_k \in \mathcal{D}(U_2) \) is a sequence of functions with support in the same fixed compact set and there are constants \( C, h > 0 \) such that
\[ |D^\alpha \chi_k(x)| \leq C(hk)^{|\alpha|}, \quad |\alpha| \leq k. \]
Then the sequence \( \chi_k' = a(\chi_k \circ f) \) has the same properties with different constants \( C, h \).

**Lemma 3.11 ([35] Lemma 3.7).** Let \( F \) be a compact family of analytic real-valued functions on \( U \) which do not have a critical point in \( x_0 \in U \). Further suppose \( \chi_k \in \mathcal{D}(U) \) is a sequence of functions with support in the same small enough neighborhood of \( x_0 \) which satisfies
\[ |D^\alpha \chi_k(x)| \leq C(hk)^{|\alpha|}, \quad |\alpha| \leq k, \]
Then there exist constants \( C', h' > 0 \) such that for all \( t \in \mathbb{R} \) and \( f \in F \) we have
\[
\left| \int \chi_k(x) e^{-itf(x)} \, dx \right| \leq C' (h')^k (k + |t|)^{-k}, \quad k \in \mathbb{N}.
\]

**Theorem 3.12.** Let \( x_0 \in U, u \in D'(U), P \) be a differential operator of order \( d \) with analytic coefficients in \( U \) and \( F \) be a compact family of analytic real-valued functions. Assume also that \( \chi_k \in D(U) \) is a sequence of functions satisfying
\[
|D^\alpha \chi_k| \leq Ch|\alpha| k^{i|\alpha|}, \quad |\alpha| \leq k,
\]
with supports inside of the same small enough neighborhood \( W \) of \( x_0 \). Then the following holds:

1. If \( M \) is an \( R \)-semi-regular weight matrix and \((x_0, df(x_0)) \notin WF_{\{(\mathbb{R})\}}(u; P) \cup \{0\}\) for all \( f \in F \) then there is a sequence \( M \in \mathfrak{M} \), constants \( C, h > 0 \) and \( \gamma \) such that for some sufficiently small \( \varepsilon > 0 \) we have
\[
\left| \langle \chi_{dk+q} P^k u, e^{-itf} \rangle \right| \leq C h^{k} M dk^h, \quad k \in \mathbb{N}, t \geq 1.
\]

2. If \( M \) is \( B \)-semi-regular and \((x_0, df(x_0)) \notin WF_{\{(\mathbb{R})\}}(u; P) \cup \{0\}\) for all \( f \in F \) then there is some \( C > 0 \) satisfying (3.11).

**Proof.** Note first that the set \( F = \{ tdf(x_0) : t > 0, f \in F \} \) is a closed cone in \( \mathbb{R}^n \setminus \{0\} \). Since by Proposition 3.7(1) \( WF_{\{(\mathbb{R})\}}(u; P) \) is a closed subset of \( U \times \mathbb{R}^n \setminus \{0\} \) which is conic in the second variable, there has to be a neighborhood \( V \) of \( x_0 \) and an open conic neighborhood \( \Gamma \subseteq \mathbb{R}^n \setminus \{0\} \) of \( F \) such that \( WF_{\{(\mathbb{R})\}} u \cap \Gamma \times \mathbb{R}^n = \emptyset \). Then Lemma 3.8 implies that we can find a sequence \( f_k \in \mathcal{E}'(U) \) and \( \nu \in \mathbb{R} \) such that the following holds. First, \( f_k | V = (P^k u) | \) and the Fourier transforms of the \( f_k \) either satisfy
\[
\left| \hat{f}_k(\xi) \right| \leq C h^{k} \left( M dk \right) \frac{1}{\xi + dk} \quad \forall k \in \mathbb{N}, \forall \xi \in \mathbb{R}^n, \tag{3.12}
\]
\[
\left| \hat{f}_k(\xi) \right| \leq C h^{k} M dk (1 + |\xi|)^\nu \quad \forall k \in \mathbb{N}, \forall \xi \in \Gamma \tag{3.13}
\]
in the Roumieu case, for some constants \( C, h \) and \( M \in \mathfrak{M} \) or, in the Beurling case, for all \( M \in \mathfrak{M} \) and \( h > 0 \) there is some \( C > 0 \) such that (3.12) and (3.13) hold.

We assume for the moment that (3.12) and (3.13) holds for some fixed \( M \in \mathfrak{M} \) and some constants \( C, h > 0 \). We can further suppose that supp \( \chi_k \subseteq W = V \). Moreover, we set \( v_{k,t} = \chi_{dk+q} e^{-itf} \) for some fixed integer \( q \geq n + 1 + \nu \). We conclude that
\[
\langle \chi_{dk+q} P^k u, e^{-itf} \rangle = \frac{1}{(2\pi)^n} \int \hat{f}_k(\xi) \tilde{v}_{k,t}(\xi) \, d\xi
\]
where
\[
\tilde{v}_{k,t}(\xi) = \int \chi_{dk+q} e^{i(x \xi - t f(x))} \, dx.
\]

The normalized functions
\[
x \mapsto \frac{x \xi - tf(x)}{|t| + |\xi|}
\]
with \( f \in F \) and \( t > 0 \) form a compact family of analytic functions without a critical point in \( x_0 \) as long as \( \xi \notin \Gamma \) or \( \xi \in \Gamma \) and \( \min(|t|/|\xi|, |\xi|/|t|) < \varepsilon \) for some sufficiently small \( \varepsilon > 0 \).

If the supports of the \( \chi_k \) are sufficiently small around \( x_0 \) Lemma 3.11 allows us to estimate \( \tilde{v}_{k,t}(\xi) \). In fact, there exist constant \( C', h' > 0 \) such that
\[
|\tilde{v}_{k,t}(\xi)| \leq C' (h' (dk + q))^{dk+q} (dk + q + |t| + |\xi|)^{-dk-q}, \quad k \in \mathbb{N}, \tag{3.15}
\]
for \( f \in F, t > 0, \xi \notin \Gamma \) or \( \xi \in \Gamma \) and \( \min(|t|/|\xi|, |\xi|/|t|) < \varepsilon \). Note that the right-hand side of (3.15) can be bounded by \( C' (h')^{dk+q} \).

Now recall that (3.11) implies that for all \( M \in \mathfrak{M} \) there is some \( \gamma > 0 \) such that
\[
k \leq \gamma \left( M_k \right)^{\frac{1}{k}}, \quad k \in \mathbb{N}.
\]
From this we obtain, with the same constant \( \gamma \),
\[
\frac{k}{k + \tau} \leq \frac{\gamma \left( M_k \right)^{\frac{1}{k}}}{\gamma \left( M_k \right)^{\frac{1}{k + \tau}}}
\]
for all \( k \in \mathbb{N} \) and all \( \tau > 0 \). Hence we obtain from (3.12), (3.13), (3.14) and (3.15) the following estimate
\[
\left| \langle \chi_{dk+q} P^k u, e^{-i t f} \rangle \right| = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}_k(\xi) \hat{\psi}_{k,t}(\xi) d\xi + \frac{1}{(2\pi)^n} \int_{\Gamma} \hat{f}_k(\xi) \hat{\psi}_{k,t}(\xi) d\xi \\
\leq C \left[ \int_{\mathbb{R}^n} h^k \left( (M_{dk})^{\frac{1}{dk+q}} + |\xi| \right)^{dk+\nu} \right] \\
\times \langle h^{\gamma} \rangle \left( \gamma (M_{dk+q})^{\frac{1}{dk+q}} + |t| + |\xi| \right)^{dk+q} d\xi \\
+ \int_{\xi \leq |\xi| \\leq t/\epsilon} \langle h^{\gamma} \rangle h^k M_{dk} \left( 1 + |\xi| \right)^{\nu} d\xi.
\]
Note that if \( 0 < \gamma \leq 1 \) then
\[
\gamma (M_{dk+q})^{\frac{1}{dk+q}} + |t| + |\xi| \geq \gamma \left( (M_{dk+q})^{\frac{1}{dk+q}} + |t| + |\xi| \right).
\]
On the other hand, for \( \gamma > 1 \) we have the trivial estimate \( (M_k)^{1/k} \leq \gamma (M_k)^{1/h} \). Hence, since \( t \geq 1 \), the first integrand in the right-hand side above can be bounded by
\[
C_1 h^k M_{dk+q} t^{-q+\nu} \left( 1 + \frac{|\xi|}{t} \right)^{\nu-q}
\]
with \( h_1 \) being a multiple of \( h, h' \) and possibly \( \gamma \).

Following iterated application of (2.8) we can conclude that there are constants \( C, h > 0 \) and a weight sequence \( M' \in \mathcal{W} \) such that
\[
\left| \langle \chi_{dk+q} P^k u, e^{-i t f} \rangle \right| \leq C h^k M'_{dk} \left( t^{-q+\nu+n} + (1 + t/\epsilon)^{\nu} t^n \right)
\]
and we have proven the theorem in the Roumieu case.

It is easy to see that the same proof holds also in the Beurling category. \( \square \)

**Theorem 3.13.** If \( \mathcal{W} \) is [semiregular] then \( \text{WF}_{\mathcal{W}}(u; P) \) is invariant under analytic changes of coordinates.

**Proof.** Let \( F : U \to U' \) be an analytic diffeomorphism from \( U \) onto an open subset \( U' \subseteq \mathbb{R}^n \) which transforms the operator \( P \) into the operator \( P_F \) defined by
\[
P_F \psi = P(\psi \circ F) \circ F^{-1}, \quad \psi \in \mathcal{D}(U).
\]
Then
\[
(P_F)^k \psi = P^k (\psi \circ F) \circ F^{-1}
\]
for all \( k \in \mathbb{N}_0 \). We set \( y = F(x) \) and \( u = v \circ F \). We are going to show that, if \( (x_0, \xi_0) \notin \text{WF}_{\mathcal{W}}(u; P) \) then \( (y_0, \eta_0) \notin \text{WF}_{\mathcal{W}}(v; P) \) where \( y_0 = F(x_0) \) and \( \xi_0 = F'(x_0)^T \eta_0 \).

Let \( \chi_k \in \mathcal{D}(U) \) be a sequence of functions with supports in a small enough neighborhood of \( x_0 \) and which are equal to 1 near \( x_0 \) and satisfy \( |D^\alpha \chi_k| \leq C (hk)^{\alpha} \) when \( |\alpha| \leq k \). If \( \Gamma \) is the cone associated to \( \xi_0 \) in Definition 3.6 then \( (F'(x_0)^T)^{-1} \Gamma \) is an open conic neighborhood of \( \eta_0 \). It follows that the family
\[
F_\eta : x \mapsto \frac{1}{1 + |\eta|}(F(x), \eta), \quad \eta \in (F'(x_0)^T)^{-1} \Gamma
\]
is a compact set of real-valued analytic functions with \( (x_0, dF_\eta(x_0)) \notin \text{WF}_{\mathcal{W}}(u; P) \cup \{0\} \) since \( (x_0, \xi_0) \notin \text{WF}_{\mathcal{W}}(u; P) \).

According to Lemma 3.10 we have that
\[
|D^\alpha \left( (F'(x)^T) \chi_k(x) \right) | \leq C (hk)^{\alpha}, \quad |\alpha| \leq k \in \mathbb{N}_0, \quad x \in U,
\]
for some constants \( C, h > 0 \). In the Roumieu case Theorem 3.12 implies that there are constants \( C, h > 0, \nu' \in \mathbb{R} \) and \( q \in \mathbb{N} \) such that
\[
\left| \langle F'(x)^T \chi_{dk+q} P^k u, e^{-iF'(x)^T \eta} \rangle \right| \leq Ch^k M_{dk} \left( 1 + |\eta| \right)^{\nu'}.
\]
If we define \( \varphi_k = \chi_k \circ F^{-1} \) and \( g_k = \varphi_{dk+q} P^k u \) then we obtain
\[
|\varphi_k(\eta)| \leq Ch^k M_{dk} \left( 1 + |\eta| \right)^{\nu'}, \quad k \in \mathbb{N}_0, \quad \eta \in (F'(x_0)^T)^T \Gamma.
\]
Furthermore, by Lemma 3.10 the functions $\varphi_k$ satisfy

$$|D^\alpha \varphi_k| \leq C(hk)^{|\alpha|}, \quad |\alpha| \leq k \in \mathbb{N}_0,$$

for some constants $C, h > 0$. Hence, by Proposition 3.3 the estimate (3.10) holds for the sequence $g_k$ too. Since $g_k|_V = P^k v$ in some neighborhood $V \subseteq U'$ of $y_0$ we have therefore shown that $(y_0, \eta_0) \notin \text{WF}_{[\alpha], v}^\infty(\varphi_k, P_F)$.

Virtually the same proof gives us also the result in the Beurling case. □

3C. The elliptic Theorem of Iterates. We are now in the position to prove the microlocal elliptic Theorem of Iterates for $[\alpha]$-vectors. We want to begin by showing that $\text{WF}_{[\alpha]}(u; P)$ is in fact a refinement of $\text{WF}_{[\alpha]}(u)$, but to this end we need a variant of Lemma 2.16.

Lemma 3.14. Let $K \subseteq U$ be compact, $F \subseteq \mathbb{R}^n \setminus \{0\}$ be a closed cone, $u \in \mathcal{D}'(U)$, $P$ be an analytic differential operator and $\varphi_k(x, \xi)$ be a sequence of smooth functions on $U \times F$ with $\text{supp} \varphi_k(\cdot, \xi) \subseteq K$ for all $k \in \mathbb{N}_0$ and $\xi \in F$ for which there are constants $C, h > 0$ such that

$$|D^\alpha \varphi_k(x, \xi)| \leq C(hk)^{|\alpha|}, \quad |\alpha| \leq k, x \in K, \xi \in F, |\xi| > k,$$

for all $k \in \mathbb{N}_0$. Furthermore assume that $\mathcal{R}$ is a [semiregular] weight matrix and let $\mu$ be the order of $u$ near $K$. Then the following holds:

1. If $\text{WF}_{[\alpha]}(u) \cap (K \times F) = \emptyset$ then there are $M \in \mathcal{R}$ and constants $C, h > 0$ (resp. for all $M \in \mathcal{R}$ and $h > 0$ there exists a constant $C > 0$) such that

$$|\hat{\varphi}_k(\eta)| \leq Ch^k M_{k-\mu-n} |\eta|^{\mu-n-k}, \quad \eta \in F, \quad |\eta| > k, \quad k \geq \mu + n.$$

2. If $\text{WF}_{[\alpha]}(u; P) \cap (K \times F) = \emptyset$ then there are $M \in \mathcal{R}$ constants $\nu \geq 0$ and $h, C > 0$ (resp. there is some $\nu \geq 0$ such that for all $M \in \mathcal{R}$ and $h > 0$ there exists some $C > 0$) satisfying

$$|F(\hat{\varphi}_k u^{p_k})| \leq C h^k M_{dk(1 + |\xi|)^\nu}, \quad \xi \in F, \quad |\xi| > dk + q, \quad q \geq n + \nu + 1.$$

Proof. We begin with the proof of (1) in the Roumieu category. Due to Lemma 2.10 there is a bounded sequence $u_k \in \mathcal{D}'(U)$ such that $u_k|_W = u|_W$ in some neighborhood $W$ of $K$ and

$$|\hat{u}_k(\eta)| \leq Ch^k M_k |\eta|^{-k}, \quad \eta \in \Gamma$$

for some $C, h > 0$ and $M \in \mathcal{R}$ where $\Gamma$ is an open conic neighborhood of $F$. Clearly $\varphi_k u = \varphi_k u^{p_k}$, $k' = k - \mu - n$. The estimate (3.10) gives us

$$|\hat{\varphi}_k(\eta, \xi)| \leq Ch^k \left(\frac{k}{k + |\eta|}\right)^k, \quad \eta \in \mathbb{R}^n, \quad \xi \in F, \quad |\xi| > k,$$

(3.17)

where $\hat{\varphi}_k(\eta, \xi) = \int e^{-ix\eta} \varphi_k(x, \xi) \, dx$ is the partial Fourier transform of $\varphi_k$. Furthermore if $\xi \in F$ we can choose $0 < c < 1$ such that $\eta \in \Gamma$ when $|\xi - \eta| \leq c|\xi|$. [33] equation (8.1.3)] states that

$$(2\pi)^n |\hat{\varphi}_k u(\xi)| \leq \|\hat{\varphi}_k (\cdot, \xi)\|_{L^1(\Gamma)} \sup_{|\eta - \xi| \leq c|\xi|} |\hat{u}_{k'}(\eta)|$$

$$+ C (1 + c^{-1})^\mu \int_{|\eta| > c|\xi|} |\hat{\varphi}_k(\eta, \xi)| (1 + |\eta|)^\mu \, d\eta$$

for some $C > 0$. We have, if $k > \mu + n + d$,

$$\|\varphi_k u\|_{L^1(\Gamma)} \leq Ch^{k+n}$$

for some $C, h > 0$. Since $|\eta| \leq (1 - c)|\xi|$ we conclude that

$$|\xi|^{k'} |\hat{\varphi}_k u(\xi)| \leq Ch^k \left(k^{\mu+n}(1-c)^{-k'} \sup_{\eta \in \Gamma} |\hat{u}_{k'}(\eta)| |\eta|^{k'}

+(1+c^{-1})^\mu k^{k} \int_{|\eta| > c|\xi|} |\eta|^{\mu-k} \right).$$

Hence there are some $C, h > 0$ such that

$$|\hat{\varphi}_k u(\xi)| \leq Ch^k M_k |\xi|^{-k'}$$

for $\xi \in F, |\xi| > k$ and $k > \mu + n$. 19
We now turn to the proof of (2). In the Roumieu case Lemma 3.15 and Proposition 3.16 imply that there are a neighborhood $W$ of $K$, an open conic neighborhood $\Gamma$ of $F$ and a sequence $E(U)$ such that $f_k = P_k u$ in $W$ and

$$|f_k(\xi)| \leq Ch^k \left( \frac{M}{dk} + |\xi|^\nu \right)^{\nu + dk} \quad \forall k \in \mathbb{N}, \forall \xi \in \mathbb{R}^n,$$

$$|\hat{f}_k(\xi)| \leq Ch^k M_{dk} (1 + |\xi|)^\nu \quad \forall k \in \mathbb{N}, \forall \xi \in \Gamma,$$

for some $M \in \mathbb{R}$ and constants $\nu \in \mathbb{R}$ and $C, h > 0$. Similarly to above we have

$$(2\pi)^n F(\varphi_{dk+q} P_k u)(\xi) = \int \varphi_{dk+q}(\xi - \eta) \hat{f}_k(\eta) d\eta.$$ 

Without loss of generality we may assume that $\nu \geq 0$. By (3.16) we have that there are constants $C, h > 0$ such that

$$|\varphi_k(\eta, \xi)| \leq Ch^k (1 + |\eta|)^{\nu - \eta - n - 1}$$

for $|\xi| > k, k \geq \nu + n + 1$. It follows that

$$\int_\Gamma |\varphi_{dk+q}(\xi - \eta)| \hat{f}_k(\eta) d\eta \leq Ch^k M_{dk} \int_{\mathbb{R}^n} (1 + |\xi - \eta|)^{-n - \nu - 1} (1 + |\eta|)^\nu d\eta$$

$$\leq Ch^k M_{dk} (1 + |\xi|)^n + \nu + 1$$

for $\xi \in F, |\xi| > dk + q$. Moreover, there is a constant $\kappa > 0$ such that if $\xi \in F$ and $\eta \notin \Gamma$ then $|\xi - \eta| \geq \kappa(|\xi| + |\eta|)$. Hence, by (3.17) we have

$$\int_{\mathbb{R}^n \setminus \Gamma} |\varphi_{dk+q}(\xi - \eta)| \hat{f}_k(\eta) d\eta \leq \int_{|\xi - \eta| \geq \kappa(|\xi| + |\eta|)} |\varphi_{dk+q}(\xi - \eta)| \hat{f}_k(\eta) d\eta$$

$$\leq Ch^k M_{dk+q}$$

$$\int_{|\xi - \eta| \geq \kappa(|\xi| + |\eta|)} \left( \frac{M_{dk+q}}{dk} + \xi - \eta \right)^{-dk-q} \times$$

$$\times \left( \frac{M_{dk}}{dk} \right)^{\nu + dk} d\eta.$$ 

Thus there exists some $M' \in \mathbb{R}$ and constants $C, h > 0$ such that

$$|F(\varphi_{dk+q} P_k u)(\xi)| \leq Ch^k M_{dk} (1 + |\xi|)^{\nu + n + 1}$$

for $\xi \in F, |\xi| > dk + q$. \hfill \Box

**Theorem 3.15.** Let $P$ be a differential operator with analytic coefficients on $U$ and $\mathcal{M}$ be a semiregular weight matrix. Then

$$WF_{[\mathcal{M}]}(u; P) \subseteq WF_{[\mathcal{M}]}(Pu) \subseteq WF_{[\mathcal{M}]} u$$

for $u \in \mathcal{D}'(U)$.

**Proof.** It is enough to prove

$$WF_{[\mathcal{M}]}(u; P) \subseteq WF_{[\mathcal{M}]} u.$$

Indeed, the semiregularity gives $WF_{[\mathcal{M}]}(u; P) = WF_{[\mathcal{M}]}(Pu; P)$ and $WF_{[\mathcal{M}]} Pu \subseteq WF_{[\mathcal{M}]} u$ by Theorem 2.17.

Now assume that $(x_0, \xi_0) \notin WF_{[\mathcal{M}]} u$. Then there are a neighborhood $V$ of $x_0$, a conic neighborhood $\Gamma$ of $\xi_0$ and a bounded sequence $u_k \in \mathcal{E}'(U)$ with $u_k|_V = u|_V$ such that

$$|\xi|^k |u_k| \leq Ch^k M_k \quad \forall \xi \in \Gamma, \forall k \in \mathbb{N}_0$$

for some $M \in \mathbb{R}$ and some constants $C, h > 0$.

Let $W \subseteq V$ be a neighborhood of $x_0$ and $F \subseteq \Gamma \cup \{0\}$ a closed conic neighborhood of $\xi_0$. Choose a sequence $\chi_k \in \mathcal{D}(V)$ with $\chi|_W = 1$ and $|D^\alpha \chi_k(x)| \leq Ch^{|\alpha|} |x|^{|\alpha|}$ for $|\alpha| \leq k$. We set $f_k = \chi_{2dk} P_k u$. It follows that

$$\hat{f}_k(\xi) = \langle \chi_{2dk} P_k u, e^{-i\xi \cdot \xi} \rangle = \langle u, Q_k (e^{-i\xi \cdot \chi_{2dk}}) \rangle$$

for some $M \in \mathbb{R}$ and some constants $C, h > 0$. Indeed, the semiregularity gives $WF_{[\mathcal{M}]} Pu \subseteq WF_{[\mathcal{M}]} u$ by Theorem 2.17.
where $Q$ denotes the formal adjoint of $P$ given by $(Q\phi, \psi) = (\phi, P\psi)$ with $\phi, \psi \in D$. Hence if $P = \sum_{|\alpha| \leq d} p_{\alpha}(x)D^\alpha$ then $Qg = \sum_{|\alpha| \leq d} (-D)^\alpha(p_{\alpha}g) = \sum_{|\alpha| \leq d} q_{\alpha}D^\alpha g$. We define a new differential operator $R$ by setting

$$Q \left( e^{-ix \xi}2d_{\alpha} \right) = e^{-ix \xi}\left| \xi \right|^{d_{\alpha}}R_{\alpha}2d_{\alpha}.$$ 

It follows that $R = R_1 + \cdots + R_d$, where $R_j = R_j(x, \xi, D)$ is a differential operator of order $\leq j$ with analytic coefficients which are homogeneous of degree $-j$ with respect to $\xi$. More precisely,

$$R_j(x, \xi, D) = \sum_{|\alpha| \leq d} \sum_{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} q_{\alpha}(x) \frac{\xi^\beta}{\left| \xi \right|^d} \frac{D^\alpha - \beta}{\alpha - \beta}.$$ 

It follows that

$$Q^k \left( e^{-ix \xi}2d_{\alpha} \right) = e^{-ix \xi}\left| \xi \right|^{d_{\alpha}}R^k_{\alpha}2d_{\alpha} = e^{-ix \xi}\left| \xi \right|^{d_{\alpha}} \sum_{0 \leq \mu \leq d_{\alpha}} R_{\alpha \mu}.$$ 

By [35, Lemma 5.2] we have, for $|\beta| + j \leq 2d_{\alpha}$ and $j = j_1 + \cdots + j_k$,

$$\left| D^\beta R_{\alpha j_1} \cdots R_{\alpha j_k} \right| \leq C h^k |\beta|^{j_1 + j_2} \left| \xi \right|^{-j}$$

for some constants $C, h > 0$. Hence if $|\xi| \geq d_{\alpha}$ then

$$\left| D^\beta R_{\alpha j_1} \cdots R_{\alpha j_k} \right| \leq Ch^k |\beta|^j.$$ 

We conclude that

$$\left| D^\beta R^k_{\alpha} \right| \leq Ch^k (d\beta)$$

when $|\xi| \geq d_{\alpha}$ and $|\beta| \leq d_{\alpha}$.

Lemma [3.14] gives that there is some $M \in \mathfrak{M}$ such that

$$\left| \hat{\psi}(\xi) \right| = |\xi|^{d_{\alpha}} |\mathcal{F}(u R^k_{\alpha} (\xi))| \leq Ch^k M_{d_{\alpha} - d - \mu}$$

for $\xi \in F$, $|\xi| > d_{\alpha}$ where $\mu$ is the order of $u$ near $\overline{W}$. If we set $g_k = f_{k + d + \mu}$ then since $\mathfrak{M}$ is $R$-semiregular we obtain that there is some $M' \in \mathfrak{M}$ such that

$$|\hat{\psi}(\xi)| \leq Ch^k M_{d_{\alpha} + (d - 1)(d + \mu)} \leq Ch^k M''_{d_{\alpha}}$$

for $\xi \in F$, $|\xi| > d_{\alpha}$.

Proposition [3.5] implies that there is some $\nu$ such that

$$|\hat{\psi}(\xi)| = |\mathcal{F}(\chi^{d_{\alpha} + 2d + 2\mu} u)(\xi)| \leq Ch^k (d\xi |\xi|)^{d\nu + \nu}$$

$$\leq Ch^k \left( (d\xi)^{\frac{1}{d\xi} + \xi} \right)^{d\nu + \nu}, \quad \xi \in \mathbb{R}^n, \, k \in \mathbb{N} \quad (3.18)$$

for any $M \in \mathfrak{M}$ and hence $(\hat{\psi}_k)_k$ satisfies [3.5].

On the other hand, if $|\xi| < d_{\alpha}$ then by [3.18] we obtain

$$|\hat{\psi}(\xi)| \leq Ch^k \left( (d\xi)^{\frac{1}{d\xi} + d} \right)^{d\nu + \nu}$$

$$\leq Ch^k M_{d_{\alpha}} \left( \frac{d\xi}{d\xi} + d \right)^{d\nu + \nu}$$

$$\leq Ch^k M''_{d_{\alpha}}$$

for some $M'' \in \mathfrak{M}$. Hence if we choose $M''' = \max\{M', M'' \}$ then $f_k$ satisfies [3.5] and [3.6] for $M'''$.

A close inspection of the proof in the Roumieu case reveals that a few obvious modifications allow us also to show that $(x_0, \xi_0) \not\in WF_{(\mathfrak{M})} (u; P)$. 

Theorem 3.16. Let $P$ be a differential operator with analytic coefficients on $U$ and $\mathfrak{M}$ be a [semiregular] weight matrix. Then

$$WF_{(\mathfrak{M})} u \subseteq WF_{(\mathfrak{M})} (u; P) \cup WF_{(\mathfrak{M})} (u; P).$$

for $u \in \mathcal{D}'(U)$. 

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for $u \in \mathcal{D}'(U)$.
Proof. As in [13] for the Denjoy-Carleman case the proof follows closely the pattern used in [35] to show the elliptic regularity theorem, see also [1] and [29].

Let \((x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}\) be such that \((x_0, \xi_0) \notin WF_{[0]}(U; P)\) and \(p_d(x_0, \xi_0) \neq 0\). Hence there exist a conic neighborhood \(V \times \Gamma\) of \((x_0, \xi_0)\) and a sequence \(f_k \in \mathcal{E}'(U)\) with \(f_k|_V = p_k|_V\) which satisfies (3.5) and (3.6). Moreover there are a compact neighborhood \(K\) of \(x_0\) and a conic neighborhood \(F\) of \(\xi_0\), closed in \(\mathbb{R}^n \setminus \{0\}\), such that \(p_d(x, \xi) \neq 0\) for \((x, \xi) \in K \times F\). W.l.o.g. we can assume that \(K \times F \subseteq V \times \Gamma\).

Suppose that \(\chi_k \in D(K)\) is a sequence with

\[
|D^\alpha \chi_k| \leq C(hk)^{|\alpha|}, \quad |\alpha| \leq k,
\]

for some constants \(C, h\) independent of \(k\).

We set \(u_k = \chi_{3d^{2}k}u\) and thus have

\[
\hat{u}_k(\xi) = \langle u, \chi_{3d^{2}k}e^{-i\xi}\rangle.
\]

If \(Q\) is the adjoint of \(P\) then we want to construct a solution \(v\) of the equation

\[
Q^k(x, D)v(x) = \chi_{3d^{2}k}e^{-i\xi}.
\]

We define a differential operator \(R(x, \xi, D)\) on \(K \times F\) by

\[
Q \left( e^{-i\xi} \frac{g}{p_d(x, \xi)} \right) = e^{-i\xi}(I - R)g.
\]

Then \(R = R_1 + \cdots + R_d\) where \(R_j = R_j(x, D)\) is a differential operator of order \(\leq j\) with analytic coefficients in \(x\) which are homogeneous of degree \(-j\) in \(\xi\). By recurrence we obtain for \(k \in \mathbb{N}\) that

\[
Q^k \left( e^{-i\xi} \frac{g}{p_d(x, \xi)} \right) = e^{-i\xi}(I - R)^{k}p_d(x, \xi)^{k}(p_d^{-k}g).
\]

If we set in (3.19)

\[
v = e^{-i\xi} \frac{w}{p_d^{k}(x, \xi)}
\]

then \(w\) satisfies the equation

\[
((I - R)p_d)^{k} \frac{w}{p_d^{k}} = \chi_{3d^{2}k}(x).
\]

A formal solution of the above equation would be

\[
w = p_d^k(x, \xi) \left[ \frac{1}{p_d(x, \xi)} \sum_{\ell=0}^{\infty} R^{\ell} \right] \chi_{3d^{2}k}.
\]

However, we cannot estimate arbitrary high derivatives of \(\chi_{3d^{2}k}\), hence we consider the following approximate solution of (3.20)

\[
w_k = p_d^{k} \sum_{\ell_1 + \cdots + \ell_k \leq d} R^{\ell_1} \cdots R^{\ell_k} \chi_{3d^{2}k}.
\]

Then we obtain

\[
((I - R)p_d)^{k} \frac{w_k}{p_d^{k}} = \chi_{3d^{2}k} - e_k
\]

where

\[
e_k = \sum_{j=1}^{k} ((I - R)p_d)^{k-j} \sum_{\ell_j + \cdots + \ell_k \leq d} R^{\ell_j} \cdots R^{\ell_k} \chi_{3d^{2}k}.
\]

Inserting (3.21) in (3.19) gives

\[
Q^k \left( e^{-i\xi} \frac{w_k}{p_d^{k}} \right) = e^{-i\xi}(\chi_{3d^{2}k} - e_k).
\]

Hence we obtain the following representation for \(\hat{u}_k\), i.e.

\[
\hat{u}_k(\xi) = \langle u, e_k(x, \xi)e^{-i\xi}\rangle + \left\langle f_k, e^{-i\xi} \frac{w_k(x, \xi)}{p_d^{k}(x, \xi)} \right\rangle
\]

for \(\xi \in F\).
Since $p_d^{-1}$ is real analytic in a neighborhood of $K$ and homogeneous of degree $-d$ in $\xi \in F$ we can apply the proof of [23, Lemma 5.2] in order to obtain that there are constants $C, h > 0$ such that
\begin{align}
|D^{\beta}w_k(x, \xi)| &\leq Ch^{k}(dk)^{|\beta|} \\
|D^{\beta}e_k(x, \xi)| &\leq Ch^{k}(dk)^{|\beta|+dk}|\xi|^{-dk}
\end{align}
for $|\beta| \leq dk$, $|\xi| \geq dk$, $\xi \in F$ and $x \in K$. 

If $\tau$ is the order of $u$ near $K$ then we can estimate the first term on the right-hand side of (3.22) by
\begin{align}
|\langle u, e_k(x, \xi)e^{-ix\xi}\rangle| &\leq C \sum_{|\alpha| \leq \tau} \sup_{x \in K} |D^{\alpha}(e_k(x, \xi)e^{-ix\xi})| \\
&\leq C \sum_{|\alpha| \leq \tau} |\xi|^{-|\alpha|} |D^{\alpha}e_k(x, \xi)|
\end{align}
for $\xi \in F$, $|\xi| \geq 1$. If $k \geq \tau/d$ then (3.24) gives that
\begin{align}
|\langle u, e_k(x, \xi)e^{-ix\xi}\rangle| &\leq Ch^{k}(dk)^{\tau+dk}|\xi|^{-dk}
\end{align}
for $\xi \in F$, $|\xi| > dk$.

Since $w_k$ satisfies (3.23) we obtain that
\begin{align}
|D^{\beta}\left(\frac{w_k(x, \xi)}{p_d^k(x, \xi)}\right)| &\leq Ch^{k}(dk)^{|\beta|} |\xi|^{-dk}
\end{align}
for $|\beta| \leq dk$, $\xi \in F$, $|\xi| > dk$ and $x \in K$. Thus, if $(x_0, \xi_0) \notin WF_{\mathfrak{R}}(u; \mathcal{P})$ then Lemma 3.14(2) implies that there exist constants $C, h, \nu > 0$ and a weight sequence $M \in \mathfrak{M}$ such that
\begin{align}
\left|\left\langle f_k, e^{-ix\xi}w_k(x, \xi)\right\rangle\right| &\leq Ch^{k}M_{dk}\left|\frac{M_{dk}}{|\xi|^{\nu k}}(1+|\xi|)^{\nu}\right|
\end{align}
when $\xi \in F$, $|\xi| > dk$ and $k > (n + \nu + 1)/d$.

If $\mu = \max\{\tau, \nu\}$ then we conclude that
\begin{align}
|\tilde{u}_k(\xi)| &\leq Ch^{k}(M_{dk})^{\frac{dk+\mu}{d}}|\xi|^{|\mu|-dk}
\end{align}
for $\xi \in F$, $|\xi| > dk$ and $k > (n + \mu + 1)/d$. We set
\begin{align}
\tilde{v}_k = u_{\lfloor k/d \rfloor}
\end{align}
where $\lfloor y \rfloor$ denotes the largest integer $\leq y \in \mathbb{R}$. Hence, due to (3.25) and (2.8), there are constants $C, h > 0$ and a weight sequence $M \in \mathfrak{M}$ such that
\begin{align}
|\mathcal{F}(\tilde{v}_k)(\xi)| &\leq Ch^{k}M_{k}|\xi|^{|\mu|-k}
\end{align}
for $\xi \in F$, $|\xi| > dk$ and $k > n + \mu + 1$. If we put $v_k = \tilde{v}_k(x+n+\mu+1)$ then there exist $C, h > 0$ and $M \in \mathfrak{M}$ such that
\begin{align}
|\hat{v}_k(\xi)| &\leq Ch^{k}M_{k}|\xi|^{-k}
\end{align}
when $\xi \in F$ and $|\xi| > dk$.

Since $u$ is of order $\tau$ near $K$ it follows that the sequence $v_k$ is bounded in $\mathcal{E}'^{\tau}(K)$. Thus we have
\begin{align}
|\hat{v}_k(\xi)| &\leq C(1+|\xi|)\tau
\end{align}
and therefore, for $|\xi| < dk$,
\begin{align}
|\xi|^\tau |\hat{v}_k(\xi)| &\leq C(dk)^{k+\tau}
\end{align}
and since $\mathfrak{M}$ is $R$-semiregular we obtain that there are $C, h > 0$ and $M \in \mathfrak{M}$ such that
\begin{align}
|\hat{v}_k(\xi)| &\leq Ch^{k}M_{k}|\xi|^{-k}
\end{align}
for $|\xi| \leq dk$. We conclude that $(x_0, \xi_0) \notin WF_{\mathfrak{M}}(u)$.

If $\mathfrak{M}$ is $B$-semiregular and $(x_0, \xi_0) \notin WF_{\mathfrak{M}}(u; \mathcal{P}) \cap \text{Char } \mathcal{P}$ then we can argue similarly in order to conclude that $(x_0, \xi_0) \notin WF_{\mathfrak{M}}(u)$. \hfill \Box

Recall that a system $\{P_1, \ldots, P_\ell\}$ of differential operators defined on $U$ is said to be elliptic if
\begin{align}
\bigcap_{j=1}^{\ell} \text{Char } P_j = \emptyset.
\end{align}
Corollary 3.17. Let $P = \{P_1, \ldots, P_\ell\}$ be an elliptic system of analytic differential operators and $\mathfrak{M}$ be a [semiregular] weight matrix. Then
\[
\bigcap_{j=1}^\ell \mathcal{E}^{[\mathfrak{M}]}(U; P_j) = \mathcal{E}^{[\mathfrak{M}]}(U).
\]
In particular
\[
\mathcal{E}^{[\mathfrak{M}]}(U; P) = \mathcal{E}^{[\mathfrak{M}]}(U).
\]

Proof. We have only to prove $\bigcap \mathcal{E}^{[\mathfrak{M}]}(U; P_j) \subseteq \mathcal{E}^{[\mathfrak{M}]}(U)$. Assume that $u \in \bigcap \mathcal{E}^{[\mathfrak{M}]}(U; P_j)$. Then $WF_{[\mathfrak{M}]}(u; P_j) = \emptyset$ for all $j = 1, \ldots, \ell$. Hence by Theorem 3.16
\[
WF_{[\mathfrak{M}]} u \subseteq \bigcap_{j=1}^\ell \text{Char } P_j = \emptyset.
\]
We conclude that $u \in \mathcal{E}^{[\mathfrak{M}]}(U)$, cf. Proposition 2.15. Therefore we have obtained
\[
\mathcal{E}^{[\mathfrak{M}]}(U) \subseteq \mathcal{E}^{[\mathfrak{M}]}(U; P) \subseteq \bigcap_{j=1}^\ell \mathcal{E}^{[\mathfrak{M}]}(U; P_j) \subseteq \mathcal{E}^{[\mathfrak{M}]}(U),
\]
cf. Proposition 3.2(3). □

Remark 3.18. Clearly the correspondence between weight functions and their associated weight matrices as described in Subsection 2B yields instantly the transfer of all results in this section to structures given by weight functions. Thus we have in particular generalized the results of [7] to operators with analytic coefficients. We note here only the version of Corollary 3.17.

Corollary 3.19. Let $P = \{P_1, \ldots, P_\ell\}$ be an elliptic system of analytic differential operators and $\omega$ be a weight function such that $\omega(t) = o(t)$ for $t \to \infty$. Then
\[
\bigcap_{j=1}^\ell \mathcal{E}^{[\omega]}(U; P_j) = \mathcal{E}^{[\omega]}(U).
\]

Here we have to generalize the definition of $\mathcal{E}^{[\omega]}(U; P_j)$ from Section 1 in analogy to Definition 3.1. However, note that by Remark 3.3 the two definitions agree for subelliptic systems of operators. The proof of Corollary 3.19 follows then immediately from Corollary 3.17 if we recall that $\mathfrak{M}$ satisfies (2.14). We leave the details to the reader.

4. ULTRADIFFERENTIABLE SCALES

In this section we introduce the notion of ultradifferentiable scales and apply them to the Problem of Iterates of analytic differential operators of principal type.

4A. Definition. Let $\Lambda$ be a totally ordered set. We call a map
\[
\zeta : \Lambda \times [0, \infty) \to [0, \infty)
\]
a generating function if for each $\lambda \in \Lambda$ the function $\zeta_\lambda = \zeta(\lambda, \cdot)$ is continuous, increasing and satisfies the following conditions:
\[
\zeta_\lambda(0) = 0,
\]
the mapping $k \mapsto \log k + \zeta_\lambda(k) - \zeta_\lambda(k-1)$ is increasing,
\[
\lim_{t \to \infty} \frac{\zeta_\lambda(t)}{t} = \infty.
\]
For $\lambda \leq \lambda'$ we also assume that $\zeta_\lambda(t) \leq \zeta_{\lambda'}(t)$ when $t \in [1, \infty)$.

To each such $\zeta$ we can associate a weight matrix $\mathfrak{M}_\zeta = \{M^\lambda = M^\lambda_\zeta : \lambda \in \Lambda\}$ by setting
\[
M^\lambda_k = k!e^{\zeta_\lambda(k)}.
\]
More precisely, $M^\lambda$ is a weight sequence satisfying (2.1) for each $\lambda \in \Lambda$ and $M^\lambda \preceq M^{\lambda'}$ when $\lambda \preceq \lambda'$, by definition. Hence every sequence $M^\lambda$ is semiregular if and only if
\[
\forall \lambda \in \Lambda \exists \gamma > 0 : \zeta_\lambda(p+1) - \zeta_\lambda(p) \leq \gamma(p+1) \quad \forall p \in \mathbb{N}_0.
\]
On the other hand the matrix $\mathfrak{M}_\lambda$ is $R$-semiregular if and only if
\begin{equation}
\forall \lambda \in \Lambda \ \exists \sigma \in \Lambda \ \exists \gamma > 0 : \ \zeta_\lambda(p+1) - \zeta_\sigma(p) \leq \gamma(p+1) \ \forall p \in \mathbb{N}_0
\end{equation}
and $B$-semiregular if and only if $\zeta$ satisfies
\begin{equation}
\forall \lambda \in \Lambda \ \exists \sigma \in \Lambda \ \exists \gamma > 0 : \ \zeta_\sigma(p+1) - \zeta_\lambda(p) \leq \gamma(p+1) \ \forall p \in \mathbb{N}_0.
\end{equation}

For a generating function $\zeta$ we call the ordered family of weight sequences $(M^\lambda)_{\lambda}$ the ultradifferentiable scale generated by $\zeta$. We also say that $\mathfrak{M}$ is the weight matrix associated to the scale $(M^\lambda)_{\lambda}$.

To each ultradifferentiable scale $(M^\lambda)_{\lambda}$ we can associate two scales of ultradifferentiable classes, namely
\begin{equation}
\left( \mathcal{E}(\mathcal{M}^\lambda)(U) \right)_\lambda \quad \text{and} \quad \left( \mathcal{E}^*(\mathcal{M}^\lambda)(U) \right)_\lambda,
\end{equation}
the scale of Roumieu classes and of Beurling classes, respectively. Clearly, $\mathcal{E}(\mathcal{M}^\lambda)(U) \subseteq \mathcal{E}(\mathcal{M}^\sigma)(U)$ when $\lambda \leq \sigma$ and $\mathcal{E}^*(\mathcal{M}^\lambda)(U) \subseteq \mathcal{E}^*(\mathcal{M}^\sigma)(U)$ for all $\lambda \in \Lambda$.

We say that an ultradifferentiable scale $(M^\lambda)_{\lambda}$ with generating function $\zeta$ is fitting if $\zeta$ satisfies $[\Box]$ and
\begin{equation}
\forall \lambda \in \Lambda \ \forall \alpha > 1 \ \exists \lambda^* \geq \lambda \ \exists \gamma > 0 : \ \zeta_\lambda(at) \leq \zeta_{\lambda^*}(t) + \gamma(t+1) \ \forall t \in [1, \infty).
\end{equation}

On the other hand, the scale $(M^\lambda)_{\lambda}$ is apposite if the generating function $\zeta$ obeys $[\Box]$ and
\begin{equation}
\forall \lambda^* \in \Lambda \ \forall \alpha > 1 \ \exists \lambda \leq \lambda^* \ \exists \gamma > 0 : \ \zeta_\lambda(at) \leq \zeta_{\lambda^*}(t) + \gamma(t+1) \ \forall t \in [1, \infty).
\end{equation}

Furthermore, a scale $(M^\lambda)_{\lambda}$ is $R$-admissible if $[\Box]$ and $[\Box]$ hold for $\zeta$ and $B$-admissible if $[\Box]$ and $[\Box]$ are satisfied. We use the notation $[\Box]$ if the scale is either $R$- or $B$-admissible, depending on the context. Furthermore we say that a scale is admissible if it is $R$- and $B$-admissible. We observe that a fitting scale is also $R$-admissible and an apposite scale is $B$-admissible but the other implications do not hold in general.

If $\Lambda \subseteq V$ is the open positive cone of a totally ordered vector space $V$, we say that $\zeta$ is pseudo-homogeneous if
\begin{equation}
\forall \lambda \in \Lambda \ \forall \alpha > 1 \ \exists \gamma, c, q > 0 : \ \zeta_\lambda(at) \leq \zeta_{c \alpha \lambda}(t) + \gamma(t+1) \ \forall t \in [1, \infty).
\end{equation}

If $\zeta$ is pseudo-homogeneous then $\zeta$ satisfies both $[\Box]$ and $[\Box]$.

**Example 4.1.** The families of weight sequences from Example 2.4 are ultradifferentiable scales with pseudo-homogeneous generating functions:

1. The Gevrey scale $(G^{1+\lambda})_{\lambda>0}$ is generated by the function $\zeta(\lambda, t) = \lambda t \log t$ if $t > 1$ and $\zeta(\lambda, t) = 0$ for $0 \leq t \leq 1$. Since the sequence $G^{1+\lambda}$ is semiregular for all $\lambda > 0$ we know that $[\Box]$ holds. For $\alpha > 1$ and $t \geq 1$ we have that
\[\zeta(\lambda, at) = \lambda at \log(at) = (\alpha \lambda)t \log(\alpha + \log t) \leq (\alpha \lambda)t \log t + \gamma(t+1) = \zeta(\alpha \lambda, t) + \gamma(t+1).\]

Hence $\zeta$ is pseudo-homogeneous and therefore $(G^{1+\lambda})_{\lambda}$ is a fitting and apposite scale.

2. Let $r > 1$. The scale $(L^{\theta r})_{q>1}$ is generated by $\zeta^r(\lambda, t) = \theta^r \lambda$ where $\lambda = \log q$. We have that
\[\zeta^r(\lambda, at) = (at)^r \lambda = t^r(\alpha^r \lambda) = \zeta^r(\alpha^r \lambda, t).\]

It follows that the scale $(L^{\theta r})_{q}$ is admissible. It is fitting and apposite if and only if $r \leq 2$. 

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(3) The generating function for the scale $(B^{1,\lambda} = B^{\lambda})_{\lambda > 0}$ is $\zeta(\lambda, t) = \lambda t \log(t)$. For $\alpha > 1$ and $t \geq 1$ we conclude:

$$
\zeta(\lambda, at) = \lambda at \log(at + e) \\
\leq (\alpha \lambda)t \log\left(\frac{\log(at + e)}{\log(t + e)}\right) \\
= (\alpha \lambda)t \left(1 + \frac{\log\alpha}{\log(t + e)}\right) \\
\leq (\alpha \lambda)t \left(\log(at + e) + \log(1 + \log\alpha)\right) \\
\leq (\alpha \lambda)t \log(t + e) + \gamma_{\alpha, \lambda}(t + 1) \\
= \zeta(\alpha \lambda, t) + \gamma_{\alpha, \lambda}(t + 1).
$$

Hence the scale $(B^{\lambda})_{\lambda}$ is fitting and apposite.

(4) Generally, the scale $(B^{j,\lambda})_{\lambda, j \in \mathbb{N}}$, is generated by $\zeta^j(\lambda, t) = \lambda t \log^{(j+1)}(t + e^{(j)})$. If $\alpha > 1$ and $t \geq 1$ we can argue analogously to above and obtain:

$$
\zeta^j(\lambda, at) = \lambda at \log^{(j+1)}(at + e^{(j)}) \\
\leq (\alpha \lambda)t \left(\log^{(j+1)}(t + e^{(j)}) + \alpha^{[j]}\right) \\
\leq \zeta^j(\alpha \lambda, t) + \gamma_{\alpha, \lambda}(t + 1),
$$

where $\alpha^{[j]}$ is defined recursively by $\alpha^{[1]} = \log(1 + \log\alpha)$ and $\alpha^{[j+1]} = \log(1 + \log\alpha^{[j]})$. Therefore $(B^{j,\lambda})_{\lambda}$ is a fitting and apposite scale.

4B. Vectors of operators of principal type. If $P$ is an operator of principal type with analytic coefficients in $U \subseteq \mathbb{R}^n$ and $(x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}$ then we say following [68] that $P$ satisfies Condition $C_{x_0, \xi_0}$ if either $p_t(x_0, \xi_0) \neq 0$ or $p_d(x_0, \xi_0) = 0$ and for all $z \in \mathbb{C}$ with $d_t \text{Re}(zp_d(x_0, \xi_0)) \neq 0$ we have that the function $\text{Im}(zp_d)$, restricted to the bicharacteristic strip of $\text{Re}(zp_d)$ through $(x_0, \xi_0)$, has a zero of finite even order. We recall

**Theorem 4.2** ([68] Theorem II). Let $P$ be an analytic differential operator of principal type. The following statements are equivalent:

1. $P$ is hypoelliptic.
2. $P$ is subelliptic.
3. $P$ satisfies Condition $C_{x_0, \xi_0}$ for all $(x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}$.

Since $P$ is subelliptic we have by Remark 5.3 that $u \in \mathcal{E}^{(3\mathbb{N})}(U; P)$ if and only if for every $V \subseteq U$ there are $M \in \mathfrak{M}$ and constants $h, C > 0$ such that $P^k u \in L^2(V)$ and

$$
\|P^k u\|_{L^2(V)} \leq C h^k M_{dk}
$$

for all $k \in \mathbb{N}_0$. On the other hand $u \in \mathcal{E}^{(3\mathbb{N})}(U; P)$ if and only if $P^k u \in L^2_{\text{loc}}(U)$ and for all $V \subseteq U$, all $M \in \mathfrak{M}$ and all $h > 0$ there is some $C > 0$ such that (4.2) is satisfied for all $k$.

The main technical result of [3] is the following theorem:

**Theorem 4.3** ([3] Theorem 1.2]). Let $P$ be a differential operator of order $d$ with analytic coefficients in $U \subseteq \mathbb{R}^n$. Let $(x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}$ and assume that there is a conic neighborhood $W_0 \times \Gamma_0$ of $(x_0, \xi_0)$ such that $P$ is of principal type in $W_0 \times \Gamma_0$ and Condition $C_{x, \xi}$ is satisfied for all $(x, \xi) \in W_0 \times \Gamma_0$.

Then there are neighborhoods $W' \subseteq W \cdot W_0$ of $x_0$, a conical neighborhood $\Gamma \subseteq \Gamma_0$ of $\xi_0$, $C > 0$, $0 \leq \delta < 1$ and a sequence of functions $(\psi_k)_k \subseteq D(W)$ satisfying $0 \leq \psi_k \leq 1$ and $\psi_k \equiv 1$ on $W'$ such that the following holds: For every $k \in \mathbb{N}$ and $u \in L^2(W)$ with $P^k u \in L^2(W)$ we have

$$
\left|\xi\right|^{(d-\delta)k} \psi_k u(\xi) \leq \frac{C^{k+1}}{(k!)^d} \left(\|P^k u\|_{L^2(W)} + (k!)^d \|u\|_{L^2(W)}\right)
$$

if $\xi \in \Gamma$.

**Remark 4.4.** According to [3] Remark 1.2 the number $\delta$ in Theorem 4.3 can be chosen to be $0$ if $P$ is elliptic at $(x_0, \xi_0)$. When $P$ is non-elliptic at $(x_0, \xi_0)$ then we can take $\delta = 2k/(2k + 1)$ where $2k$ is the maximum order of vanishing of $\text{Im}(zp_d)$ mentioned in Condition $C_{x, \xi}$, for $(x, \xi)$ in a compact neighborhood of $(x_0, \xi_0)$ and $z \in \mathbb{C}$.
Hence, if $V \subseteq U$ then we set
\[
\delta = \delta(V) = \frac{2k}{2k+1}
\]
where now $2k$ is the maximum order of vanishing of $\text{Im}(z\rho_d)$ in Condition $C_{x,\xi}$ for $(x, \xi) \in V \times \mathbb{R}^n \setminus \{0\}$.

Note that $\delta(V)$ is closely related to the subellipticity of $P$: For $V \subseteq U$ we can choose in (4.2) $\varepsilon = d - \delta(V)$, see [67].

Now suppose that $P$ is a hypoelliptic operator of principal type with analytic coefficients in $U$ and that $(M^\lambda_k)$ is a fitting ultradifferentiable scale with generating function $\zeta$. Recall that Theorem 4.4 implies that Condition $C_{x,\xi}$ holds for all $(x, \xi) \in U \times \mathbb{R}^n \setminus \{0\}$. Furthermore let $u \in \mathcal{D}'(U)$ be an $(M^\lambda)$-vector of $P$ for some $\lambda \in \Lambda$ and $(x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}$. Applying Theorem 4.3 we conclude that there are neighborhoods $W' \subseteq W \subseteq U$ of $x_0$, a conical neighborhood $\Gamma$ of $\xi_0$, $0 \leq \delta < 1$ and a bounded sequence $u_k \in \mathcal{E}'(W)$ such that $u|_{W'} = u_k|_{W'}$ and
\[
\|\xi|^{(d-\delta)\ell}\hat{u}_k(\xi)\| \leq \frac{C_{k+1}}{(k!)^\ell} (Ckh_0^\ell M_k^\lambda + (k!)^d)
\]
for $\xi \in \Gamma$, where $C, C_0, h_0 > 0$ are independent of $k \in \mathbb{N}$ and $\delta = \delta(V)$ is defined in Remark 4.4. Now, since (2.4) is satisfied for all $M^\lambda_k$, we have that for each $\rho > 0$ there exists $C_\rho > 0$ such that $1 \leq C_\rho \rho^k m_k^{\lambda_0}$ for all $k \in \mathbb{N}_0$. Applying also Stirling’s formula we obtain that there are constants $h > 0$ and $C > 0$ such that
\[
|\xi|^{(\delta-\delta)\ell}\hat{u}_k(\xi) \leq Ch^k (d-\delta)\ell \hat{\epsilon}^\lambda(\ell), \quad k \in \mathbb{N}.
\]
If we denote by $[y]$ the smallest integer $\geq y \in \mathbb{R}$ then we choose for $\ell \in \mathbb{N}$ an integer $k_\ell$ in the following way
\[
k_\ell = \left[ \frac{\ell}{d-\delta} \right] \leq \frac{\ell}{d-\delta} + 1 \leq \frac{\ell + d}{d-\delta}
\]
and therefore $\ell \leq (d-\delta)k_\ell$. Note that if $\delta \geq 1$ then $\delta^{k_\ell} \leq \delta^{(\ell + d)/(d-\delta)}$ and on the other hand $0 < \delta < 1$ implies that $\delta^{k_\ell} \leq \delta^{\ell/(d-\delta)}$. Thus, if we set $v_\ell = u_{k_\ell}$ then we have that
\[
|\xi|^{\ell} |v_\ell(\xi)| \leq \left| |\xi|^{(d-\delta)\ell}\hat{u}_{k_\ell}(\xi)\right|
\leq Ch_{k_\ell} (d-\delta)\ell \hat{\epsilon}^\lambda(\ell) \exp \left[ \zeta_\lambda(\ell) \right]
\leq Ch_{k_\ell} (d-\delta)^{-\ell - d(\ell + d)} \exp \left[ \zeta_\lambda \left( \frac{\ell}{d-\delta} \right) \right]
\leq C \left( \frac{\zeta_\lambda}{d-\delta} \right)^\ell \exp \left[ \zeta_\lambda(\ell + d) \right]
\]
for $\xi \in \Gamma$ with $|\xi| \geq 1$ and some $\lambda^* \not\in \Lambda$ according to (3). Then (3) and the Stirling formula imply that
\[
|\xi|^{\ell} |\hat{v}_\ell(\xi)| \leq Ch^\ell \hat{\epsilon}^\lambda(\ell) = Ch^\ell M^\lambda_\ell, \quad \ell \in \mathbb{N},
\]
for some constants $C, h > 0$. Hence $(x_0, \xi_0) \not\in \text{WF}_{(M^\lambda_\ell)} u$.

If $u \in \mathcal{D}'(U)$ is a $(M^\lambda)$-vector of $P$ for some $\lambda$, then we have by essentially the same arguments that for every $(x_0, \xi_0) \in U \times \mathbb{R}^n \setminus \{0\}$ there is some $\lambda^* \in \Lambda$ such that $(x_0, \xi_0) \not\in \text{WF}_{(M^\lambda_\ell)} u$.

In fact, we have obtained the following theorem.

**Theorem 4.5.** Let $P$ be a hypoelliptic differential operator of principal type with analytic coefficients in $U \subseteq \mathbb{R}^n$ and $(M^\lambda_\ell)$ be an ultradifferentiable scale. Then the following holds:

1. If $(M^\lambda_\ell)_\lambda$ is fitting then for all $V \subseteq U$ and all $\lambda \in \Lambda$ there is some $\lambda^* \in \Lambda$ such that every $(M^\lambda_\ell)$-vector of $P$ in $U$ is of class $[M^\lambda_\ell]$ in $V$.

2. If the scale $(M^\lambda_\ell)_\lambda$ is opposite then for all $V \subseteq U$ and all $\lambda^* \in \Lambda$ there exists $\lambda \in \Lambda$ such that every $u \in \mathcal{E}[M^\lambda_\ell](U; P)$ is of class $[M^\lambda_\ell]$ in $V$.

**Proof.** Note first that by Remark 4.4 for every $V \subseteq U$ there is some $\delta(V) \in [0, 1)$ such that (4.3) holds with $\delta = \delta(V)$ for all $(x_0, \xi_0) \in V \times \mathbb{R}^n \setminus \{0\}$. Condition (4.4) implies that for every $\lambda$ there is some $\lambda^*$ such that
\[
\zeta_\lambda \left( \frac{d}{d - \delta(V)} \right) \leq \zeta_\lambda^*(t) + C(t + 1)
\]
for all $t \in [1, \infty)$ and some $C > 0$. Thus the above arguments give
\[
WF_{(M^\lambda_\ell)} u \cap \{V \times \mathbb{R}^n \setminus \{0\}\} = \emptyset.
\]
Hence \( u \) is of class \( [M^{\lambda^*}] \) in \( V \) by Proposition \text{2.15} which proves (1).

On the other hand, by (2) we obtain that for every \( \lambda^* \) there is some \( \lambda \) such that (3.3) holds for \( t \in [1, \infty) \) and some \( C > 0 \). Adapt the arguments above we then conclude that \( \text{WF}_{[M^{\lambda^*}]} u \cap (V \times \mathbb{R}^n \setminus \{0\}) = \emptyset \) for all \( u \in \mathcal{E}([M^{\lambda^*}](U; P)) \) and thus Proposition \text{2.15} implies again that \( u \) is of class \( [M^{\lambda^*}] \) in \( V \). \( \Box \)

For special scales, like the Gevrey scale, cf. [3 Theorem 1.3], we may obtain rather precise information about the loss of regularity of vectors. For example, for the other scales in Example 4.4 we have

Corollary 4.6. Let \( P \) be as in Theorem 4.3. \( V \subseteq U, \delta = \delta(V) \) be as defined in Remark 4.4 and \( u \in \mathcal{D}'(U) \).

1. If \( u \) is an \([L^{q,r}]\)-vector of \( P \) for some \( q > 1 \) and \( 1 < r \leq 2 \) then \( u \) is of class \([L^{q,r}]\) in \( V \), where \( q' = q^{r/(d-\delta r)} \).

2. If \( u \) is a \([B^{j,\lambda}]\)-vector for some \( j \in \mathbb{N} \) and \( \lambda > 0 \), then \( u \) is of class \([B^{j,\lambda}]\) in \( V \) where

\[
\lambda' = \frac{d}{d-\delta(V)} \lambda.
\]

Theorem 4.7. Let \( P \) be as in Theorem 4.3, \( (M^\lambda)_\lambda \) be an \([\text{admissible}]\) ultradifferentiable scale and \( \mathcal{W}_\xi \) the associated weight matrix. Then

\[
\mathcal{E}([\mathcal{W}_\xi])(U; P) = \mathcal{E}([\mathcal{W}_\xi])(U).
\]

Proof. We begin with the Roumieu case. If \( u \in \mathcal{E}([\mathcal{W}_\xi])(U; P) \) then for every \( V \subseteq U \) there are \( \lambda \in \Lambda \) and \( C, h > 0 \) such that

\[
\|P^k u\|_{L^2(V)} \leq Ch^k M^\lambda_{dk}, \quad k \in \mathbb{N}_0.
\]

Suppose that \( (x_0, \xi_0) \in V \times \mathbb{R}^n \setminus \{0\} \). As above we obtain from Theorem 4.3 and (1) that there is a bounded sequence \( u_k \in \mathcal{E}(V) \) such that \( u_k|_W = u|_W \) for some neighborhood \( W \subseteq V \) of \( x_0 \) and

\[
|\lambda|^{(d-\delta(W))k} \|u_k(\xi_0)\| \leq Ch^k k^{(d-\delta)k} e^{\zeta_0(\delta k)}, \quad \xi_0 \in \Gamma, \ |\xi_0| \geq 1,
\]

where \( C > 0, h > 0, \Gamma \) is a conic neighborhood of \( \xi_0 \) and \( \delta = \delta(V) \), depending only on the operator \( P \) and \( V \), as is as Remark 4.4. If we choose \( k_0, \ell \in \mathbb{N} \), as before and set \( v_\ell = u_{k_0} \) then we can conclude in the same manner from (2) that

\[
|\xi|^{\ell} \|\hat{v}_\ell(\xi)| \leq Ch^\ell (\ell + d)! \exp [\zeta_\ell (\ell + d)]
\]

for some \( \lambda^* \in \Lambda \). Hence (3) gives

\[
|\lambda|^{\ell} \|\hat{v}_\ell(\xi)| \leq Ch^\ell \ell! \exp [\zeta_\ell (\ell)] \leq Ch^\ell M^\lambda_{\lambda^*}
\]

for some constants \( C, h > 0 \) and \( \lambda^* \in \Lambda \) independent of \( \ell \). Therefore, since \( (x_0, \xi_0) \in V \times \mathbb{R}^n \setminus \{0\} \) was chosen arbitrarily,

\[
\text{WF}_{[M^{\lambda^*}]} u \cap (V \times \mathbb{R}^n \setminus \{0\}) = \emptyset
\]

and by Theorem 2.13

\[
\text{WF}_{[\mathcal{W}_\xi]} u \cap (V \times \mathbb{R}^n \setminus \{0\}) = \emptyset.
\]

Since this holds for all \( V \subseteq U \) it follows that \( \text{WF}_{[\mathcal{W}_\xi]} u = \emptyset \). Hence \( u \in \mathcal{E}([\mathcal{W}_\xi])(U) \) by Proposition 2.15.

If \( u \in \mathcal{E}([\mathcal{W}_\xi])(U; P) \) then for all \( V \subseteq U \), \( \lambda \in \Lambda \) and \( h > 0 \) there is a constant \( C > 0 \) such that

\[
\|P^k u\|_{L^2(V)} \leq Ch^{\lambda h} M_{\lambda^*}^{\lambda h}, \quad k \in \mathbb{N}_0.
\]

If \( (x_0, \xi_0) \in V \times \mathbb{R}^n \setminus \{0\} \) then Theorem 4.3 gives that there is a bounded sequence \( u_k \in \mathcal{E}(V) \) with \( u_k|_W = u|_W \) in some neighborhood \( W \subseteq V \) of \( x_0 \). Furthermore there is a conic neighborhood \( \Gamma \) of \( \xi_0 \) such that for all \( \lambda \in \Lambda \) and all \( h > 0 \) there exists a constant \( C > 0 \) such that

\[
|\lambda|^{(d-\delta(W))k} u_k(\xi_0)\| \leq Ch^k k^{(d-\delta)k} e^{\zeta_0(\delta k)}, \quad \xi_0 \in \Gamma, \ |\xi_0| \geq 1.
\]

If \( k_\ell \) for \( \ell \in \mathbb{N} \) is defined as before then it is easy to see that (3) implies that for all \( \lambda^* \) and \( h > 0 \) there is a constant \( C > 0 \) such that

\[
|\lambda|^\ell \|\hat{v}_\ell(\xi)| \leq Ch^\ell (\ell + d)! \exp [\zeta_\ell (\ell + d)].
\]

It follows from (2) that for all \( \lambda^* \in \Lambda \) and \( h > 0 \) there is some \( C > 0 \) such that for all \( \ell \in \mathbb{N}_0 \) we have

\[
|\lambda|^\ell \|\hat{v}_\ell(\xi)| \leq Ch^\ell \ell! \exp [\zeta_\ell (\ell)] = Ch^\ell M^\lambda_{\lambda^*}.
\]
Hence  
\[ \text{WF}_{(M_{\lambda'})} u \cap (V \times \mathbb{R}^n \setminus \{0\}) = \emptyset \]
for all \( \lambda' \in \Lambda \) and therefore by Proposition 2.14  
\[ \text{WF}_{(M_{\lambda})} u \cap (V \times \mathbb{R}^n \setminus \{0\}) = \emptyset. \]
This means that \( \text{WF}_{(M_{\lambda})} u = \emptyset \) and consequently \( u \in \mathcal{E}^{(\mathfrak{M}_{\lambda})}(U) \).

**Corollary 4.8.** Let \( P \) be as in Theorem 4.7. Then  
\[ \mathcal{E}^{[\mathfrak{Q}]}(U; P) = \mathcal{E}^{[\mathfrak{Q}]}(U), \quad r > 1, \]
and  
\[ \mathcal{E}^{[\mathfrak{B}]}(U; P) = \mathcal{E}^{[\mathfrak{B}]}(U), \quad j \in \mathbb{N}. \]

**Example 4.9.** Let \( P \) be as in Theorem 4.5.

1. If we consider the scale \( (\mathfrak{L}^{c,r})_r \) with generating function \( \zeta(r,t) = t^r \) and associated weight matrix \( \mathfrak{R} \) then we have also that  
\[ \mathcal{E}^{[\mathfrak{R}]}(U; P) = \mathcal{E}^{[\mathfrak{R}]}(U). \]
Indeed, \( \mathfrak{L}^{c,r_1} \subseteq \mathfrak{L}^{c,r_2} \) for all \( q > 1 \) and \( r_1 < r_2 \), we obtain that for all \( \alpha > 1 \) and \( r_1 < r_2 \) there is a constant \( \gamma > 0 \) such that  
\[ \zeta(r_1, \alpha t) = \alpha^r t^r \leq \zeta(r_2, t) + \gamma(t + 1), \quad t \geq 1. \]

2. We can also show that  
\[ \mathcal{E}^{(3)}(U; P) = \mathcal{E}^{(3)}(U) \]
where \( \mathfrak{J} = \mathfrak{J}^1 = \{ \mathfrak{B}^{j,1} : j \in \mathbb{N} \} \). Indeed, \( \mathfrak{J} \) is associated to the scale \( (\mathfrak{B}^{j,1})_{j \in \mathbb{N}} \), which is generated by \( \zeta(j,t) = t \log^{(j+1)}(t + e^{(j)}) \). Here we consider \( \Lambda = (\mathbb{N}, \preceq) \) with the inverse order \( \preceq \) defined by  
\[ j \preceq k : \iff k \leq j \]
for \( j, k \in \mathbb{N} \). More generally, the function \( \zeta_j(j,t) = \alpha \zeta(j,t) \) generates the ultradifferentiable scale \( (\mathfrak{B}^{j,\sigma})_j \) for \( \sigma > 0 \). If \( \alpha > 0 \) then we compute  
\[ \zeta(j, \alpha t) = (\alpha t) \log^{(j+1)}(\alpha t + e^{(j)}) \leq (\alpha t) \log^{(j+1)}(t + e^{(j)}) + \gamma_{j,\alpha}(t + 1) \]

\[ \leq \alpha t \log^{(j)}(t + e^{(j-1)}) + \gamma_{j,\alpha}(t + 1) = \zeta_0(j - 1, t) + \gamma_{j,\alpha}(t + 1) \]
for \( t \geq 1 \) since \( \log^{(j+1)}(t + e^{(j)}) \leq \log^{(j)}(t + e^{(j-1)}) \) when \( t \geq 1 \). Since \( \mathfrak{J}^\sigma \) is the weight matrix associated to \( (\mathfrak{B}^{j,\sigma})_j \) we obtain by arguing as in the proof of Theorem 4.7 that for \( V \subseteq U \) all \( (\mathfrak{J}) \)-vectors of \( P \) are of class \( \mathcal{E}^{(3)} \) in \( V \), where \( \alpha = d/(d - \delta(V)) \) and \( \delta(V) \) as in Remark 4.4. We have proven the claim because \( \mathcal{E}^{(3)}(V) = \mathcal{E}^{(3)}(V) \) for all \( \sigma > 0 \), cf. Example 2.6(3).

**Remark 4.10.** In the last example the estimate \( \frac{4.6}{4.6} \) involved two different scales in a “mixed” version of \( \frac{4.4}{4.4} \). We can use “mixed” versions of \( \frac{2.3}{2.3} \) and \( \frac{4.5}{4.5} \) to obtain results similar to Theorem 4.5 in the case of weight matrices. More precisely, let \( P \) be a hypoelliptic analytic differential operator of principal type on \( U \subseteq \mathbb{R}^n \) with analytic coefficients, \( V \subseteq U \) and \( \delta(V) \) as in Remark 4.4. In the Roumieu case we consider two ultradifferentiable scales \( (\mathfrak{M}_{\lambda})_{\lambda \in \Lambda} \) and \( (\mathfrak{N}_{\mu})_{\mu \in \mathfrak{Y}} \) with generating functions \( \zeta : \Lambda \times [0, \infty) \to [0, \infty) \) and \( \eta : \mathfrak{Y} \times [0, \infty) \to [0, \infty) \) which satisfy both \( \frac{4.5}{4.5} \) and

\[ \forall \lambda \in \Lambda \ \exists v \in \mathfrak{Y} \ \exists \gamma > 0 : \zeta_{\lambda}(at) \leq \eta_v(t) + \gamma(t + 1) \quad \forall t \in [1, \infty) \]

where \( \alpha = d/(d - \delta(V)) \) and \( d \) denotes the order of the operator. Then every \( u \in \mathcal{E}^{(\mathfrak{M}_{\lambda})}(U; P) \) is of class \( \mathfrak{M}_{\mu} \) in \( V \), and \( \mathfrak{M}_{\lambda} \) and \( \mathfrak{N}_{\mu} \) are the weight matrices associated to the scales \( (\mathfrak{M}_{\lambda})_{\lambda} \) and \( (\mathfrak{N}_{\mu})_{\mu} \), respectively. In the Beurling setting we assume that the generating functions \( \zeta \) and \( \eta \) satisfy both \( \frac{4.6}{4.6} \) and

\[ \forall v \in \mathfrak{Y} \ \exists \lambda \in \Lambda \ \exists \gamma > 0 : \zeta_{\lambda}(at) \leq \eta_v(t) + \gamma(t + 1) \quad \forall t \in [1, \infty). \]

Then any \( (\mathfrak{M}_{\lambda}) \)-vector \( u \in \mathcal{D}'(U) \) is of class \( \mathfrak{M}_{\mu} \) in \( V \).
Remark 4.11. Another important fact in Example 4.3(2) was that the weight matrices \( \mathcal{J}^\sigma \) associated to the scales \( (B^{\beta,\sigma})_\alpha \) satisfy \( \mathcal{J}^\sigma(\infty) = \mathcal{J}^\sigma \) for all \( \rho, \sigma \). Of course, we can express this property in terms of the generating functions of the scales.

Assume, again, that two ultradifferentiable scales \( (M^\lambda_\zeta)_{\alpha \in \Lambda} \) and \( (N^\nu_\eta)_{v \in \Upsilon} \) with generating functions \( \zeta : \Lambda \times [0, \infty) \to [0, \infty) \) and \( \eta : \Upsilon \times [0, \infty) \to [0, \infty) \), respectively, are given. For such a pair of generating functions we define an auxiliary function \( \Phi^\zeta_\eta : \Lambda \times \Upsilon \times [0, \infty) \to \mathbb{R} \) by

\[
\Phi^\zeta_\eta(\lambda, v; t) = \frac{\zeta(\lambda) - \eta(v)}{t}.
\]

It is clear that \( M^\lambda \leq N^\nu \) if \( \limsup_{t \to \infty} \Phi^\zeta_\eta(\lambda, v; t) < \infty \). We can distinguish the following cases:

1. We have that \( \mathcal{M}_\zeta \{ \leq \} \mathcal{N}_\eta \) when
   \[
   \forall \lambda \in \Lambda \ \exists v \in \Upsilon : \limsup_{t \to \infty} \Phi^\zeta_\eta(\lambda, v; t) < \infty.
   \]

2. On the other hand \( \mathcal{M}_\zeta \{ < \} \mathcal{N}_\eta \) if
   \[
   \forall v \in \Upsilon \ \exists \lambda \in \Lambda : \limsup_{t \to \infty} \Phi^\zeta_\eta(\lambda, v; t) < \infty.
   \]

3. Finally \( \mathcal{M}_\zeta \{ \prec \} \mathcal{N}_\eta \) when
   \[
   \forall \lambda \in \Lambda \ \forall v \in \Upsilon : \lim_{t \to \infty} \Phi^\zeta_\eta(\lambda, v; t) = -\infty.
   \]

We might also ask ourselves, when do two ultradifferentiable scales generate the same scales of Denjoy-Carleman classes? In order to give an answer to this question, we say that two generating functions \( \zeta \) and \( \eta \) are comparable if there is a bijective mapping \( \chi : \Lambda \to \Upsilon \) such that for each \( \lambda \in \Lambda \) we have

\[
-\infty < \liminf_{t \to \infty} \Phi^\zeta_\eta(\lambda, \chi(\lambda); t) \leq \limsup_{t \to \infty} \Phi^\zeta_\eta(\lambda, \chi(\lambda); t) < +\infty.
\]

If \( \zeta \) and \( \eta \) are comparable then \( M^\lambda_\zeta \approx N^\nu_\eta \) and thus \( E^{[M^\lambda_\zeta]}(U) = E^{[N^\nu_\eta]}(U) \) and \( E^{(M^\lambda_\zeta)}(U; P) = E^{(N^\nu_\eta)}(U; P) \) for all \( \lambda \in \Lambda \) and all systems \( P \) of differential operators.

5. Scales induced by weight functions

5A. Condition \([\mathfrak{E}]\). In this section we are going to prove Theorem 1.1 but first we need to analyze condition \([\mathfrak{E}]\). It is useful for our deliberations to set

\[
\mathcal{W}_0 = \{ \omega \in C([0, \infty); \mathbb{R}) : \omega(t) \to \infty \text{ is increasing,} \}
\]

\[
\omega|_{[0,1]} \equiv 0 \text{ and } \omega \text{ satisfies } [\mathfrak{E}] \text{ and } [\mathfrak{F}] \}
\]

since we have the following Lemma.

Lemma 5.1. If \( \omega \in \mathcal{W}_0 \) satisfies \([\mathfrak{E}]\) then \( \omega \) is a weight function. Furthermore there is some \( 0 < \alpha < 1 \) such that \( \omega = O(t^\alpha) \).

Proof. It is easy to see that \([\mathfrak{E}]\) implies \([\mathfrak{E}]\). On the other hand, by [40] Lemma A.1 and [39] Lemma 4.3 we know that \( \omega \) satisfies the strong non-quasianalyticity condition:

\[
\exists C > 0 : \forall y > 0 : \int_1^\infty \frac{\omega(yt)}{t^2} \leq C\omega(y) + C.
\]

Hence [50] Corollary 4.3] states that there has to be some \( 0 < \alpha < 1 \) such that \( \omega(t) = O(t^\alpha) \) for \( t \to \infty \). \( \square \)

We continue by recalling from [40] Lemma A.1, Remark A.2, cf. also Example 2.11 the following fact.

Proposition 5.2. Let \( \omega \) be a weight function which satisfies \([\mathfrak{E}]\) and denote its associated weight matrix by \( \mathfrak{W} = \{ W^\lambda : \lambda > 0 \} \). If we define another weight matrix \( \hat{\mathfrak{W}} \) by

\[
\hat{\mathfrak{W}} := \{ (k!W^\lambda)_k : \lambda > 0, \ W^\lambda \in \mathfrak{W} \}
\]

then \( \mathfrak{W} \{ \approx \} \hat{\mathfrak{W}} \) and \( \mathfrak{W} \{ \approx \} \hat{\mathfrak{W}} \).

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Lemma 5.3. Let \( \omega \in \mathcal{W}_0 \) and \( \varphi_{\lambda,\omega}^*(t) := \frac{1}{\lambda} \varphi_{\lambda}^*(\lambda t) \) for \( \lambda > 0 \). Then we have
\[
\varphi_{\lambda,\omega}^*(t+1) \leq \varphi_{2\lambda,\omega}^*(t) + \varphi_{2\lambda,\omega}^*(1), \quad t \geq 0,
\]
for all \( \lambda > 0 \).

Proof. The argument is similar to the one in the proof of (2.13), cf. [55]. We include the proof for the convenience of the reader.

Let \( \lambda > 0 \) be arbitrary but fixed. The convexity of \( \varphi_{\lambda}^* \) implies that \( \varphi_{\lambda}^*((t+s)/2) \leq \frac{1}{2} \varphi_{\lambda}^*(t) + \frac{1}{2} \varphi_{\lambda}^*(s) \) for all \( s,t \geq 0 \). Hence the choices \( t' := \frac{1}{2\lambda} \) and \( s := 2\lambda t' \geq 2\lambda \varphi_{2\lambda,\omega}^*(1) \) for all \( t' \geq 0 \) that
\[
\frac{1}{\lambda} \varphi_{\lambda}^*(\lambda(t'+1)) \leq \frac{1}{2\lambda} \varphi_{\lambda}^*(2\lambda t') + \frac{1}{2\lambda} \varphi_{\lambda}^*(2\lambda).
\]

Lemma 5.4. Let \( \omega \in \mathcal{W}_0 \) and \( \varphi_{\lambda,\omega}^*(t) := \frac{1}{\lambda} \varphi_{\lambda}^*(\lambda t) \) for \( \lambda > 0 \). Then we have
\[
\varphi_{\lambda,\omega}^*(t+1) \leq \varphi_{2\lambda,\omega}^*(t) + \varphi_{2\lambda,\omega}^*(1), \quad t \geq 0,
\]
for all \( \lambda > 0 \).

Proof. The argument is similar to the one in the proof of (2.13), cf. [55]. We include the proof for the convenience of the reader.

Let \( \lambda > 0 \) be arbitrary but fixed. The convexity of \( \varphi_{\lambda}^* \) implies that \( \varphi_{\lambda}^*((t+s)/2) \leq \frac{1}{2} \varphi_{\lambda}^*(t) + \frac{1}{2} \varphi_{\lambda}^*(s) \) for all \( s,t \geq 0 \). Hence the choices \( t' := \frac{1}{2\lambda} \) and \( s := 2\lambda t' \geq 2\lambda \varphi_{2\lambda,\omega}^*(1) \) for all \( t' \geq 0 \) that
\[
\frac{1}{\lambda} \varphi_{\lambda}^*(\lambda(t'+1)) \leq \frac{1}{2\lambda} \varphi_{\lambda}^*(2\lambda t') + \frac{1}{2\lambda} \varphi_{\lambda}^*(2\lambda).
\]

Lemma 5.5. Let \( \omega \in \mathcal{W}_0 \). The following statements are equivalent:

1. \( \omega \) satisfies (10).
2. For all \( \gamma > 1 \) there is a constant \( C > 0 \) such that
\[
\omega(t^2) \leq C(\omega(t+1)), \quad t \geq 0.
\]

Proof. Condition (2) is equivalent to the existence of constants \( C, H > 0 \) such that
\[
\omega(t^2) \leq C(\omega(Ht+1)), \quad t \geq 0.
\]

Hence (2) implies (1).

For \( \gamma > 1 \) fixed choose \( j \in \mathbb{N} \) such that \( \gamma \leq 2^j \). If we iterate (2) we conclude that
\[
\omega(t^2) \leq \omega(t^{2^j}) \leq C(\omega(t+1)), \quad t \geq 0,
\]
for some constant \( C > 0 \), since \( \omega \) is increasing and (1) implies (2).

Lemma 5.6. Let \( \alpha > 1 \) and \( \omega, \sigma \in \mathcal{W}_0 \). Then the following are equivalent:

1. \( \exists \alpha \geq 1 \exists C > 0 : \omega(t^\alpha) \leq C(\sigma(Ht+1)), \quad t \geq 0, \)
2. \( \exists \alpha \geq 1 \forall \lambda > 0 \exists D > 0 : \varphi_{\lambda,\omega}^*(at) \leq \varphi_{\lambda,\omega}^*(t) + D(t+1), \quad t \geq 0, \)
3. \( \exists \alpha \geq 1 \exists \lambda > 0 \exists D > 0 : \varphi_{\lambda,\omega}^*(at) \leq \varphi_{\lambda,\omega}^*(t) + D(t+1), \quad t \geq 0. \)

Proof. The implication (2) \( \Rightarrow \) (3) is trivial. If (3) holds then we have for some \( \lambda > 0 \)
\[
\varphi_{\alpha,\omega}^*(y) = \sup_{x \geq 0} [xy - \varphi_{\alpha}^*(x)] = \sup_{x \geq 0} \{\lambda \alpha x - \varphi_{\alpha}^*(\lambda x')\}
\geq \sup_{x' \geq 0} \{\lambda \alpha x - A^{-1} \varphi_{\alpha}^*(A \lambda x') - \lambda D(x' + 1)\}
= A^{-1} \sup_{w \geq 0} [\alpha yw - \varphi_{\alpha}^*(w) - Dw - D\lambda]
= A^{-1} \varphi_{\alpha}^*(\alpha y - D) - D\lambda.
\]

Since \( \varphi_{\alpha}^* = \varphi_{\alpha} \) for any \( \alpha \in \mathcal{W}_0 \), we conclude that
\[
\omega(t^\alpha) \leq A\sigma(e^{D/\alpha}t) + D\lambda, \quad t \geq 0.
\]

Hence we have proven (1) with \( H = e^{D/\alpha} \) and \( C = \max\{A, D\lambda\} \).

On the other hand, if (1) holds then there are constants \( C, h > 0 \) such that
\[
\varphi_{\omega}(at) \leq C\varphi_{\sigma}(t + h) + C, \quad t \geq 0.
\]
Thus for $t \geq 0$ we can compute that
\[
\varphi_\omega(t) = \sup_{s \geq 0} [ast - \varphi_\omega(\alpha s)] \\
\geq \sup_{s \in \mathbb{R}} [ast - C\varphi_\sigma(s + h)] - C \\
\geq C \sup_{u \in \mathbb{R}} \left[ \frac{t}{C} au - \varphi_\varphi(u) \right] - hat - C \\
= C\varphi^*_\sigma(\alpha C^{-1}t) - hat - C
\]
where we have $\varphi_\varphi(u) = 0$ for $u < 0$ by normalization. Hence for all $\lambda > 0$ and $t \geq 0$ we have
\[
\frac{1}{\lambda} \varphi^*_\sigma(\lambda at) \leq \frac{1}{C\lambda} \varphi^*_\varphi(C\lambda t) + hat + \frac{1}{\lambda}.
\]
Thus $(2)$ is verified with the constants $A := C$ and $D := \max\{ha, \varphi^{-1}\}$. Observe that $A$ does not depend on $\lambda$. \hfill \□

An immediate consequence of Lemma \ref{lemma} is

Corollary 5.6. If $\omega \in W_0$ then the following are equivalent:
1. For all $\alpha > 1$ there exists $\sigma \in W_0$ and $L \geq 1$ such that
   \[\omega(t^\alpha) \leq L(\sigma(Lt) + 1), \quad t \geq 0.\]
2. For all $\alpha > 1$ there exists $\sigma \in W_0$ such that
   \[\exists A \geq 1 \forall \lambda > 0 \exists D > 0 : \varphi^*_\lambda,\sigma(\alpha t) \leq \varphi^*_\lambda,\omega(t) + D(t + 1), \quad t \geq 0.\]

Hence, if we combine Corollary \ref{corollary} with Lemma \ref{lemma} we obtain

Corollary 5.7. Let $\omega \in W_0$. The following two conditions are equivalent:
1. $\omega$ satisfies $\mathbb{E}$.
2. The function $\zeta_\omega(\lambda, t) = \varphi^*_\lambda,\omega(t)$ satisfies
   \[\forall \alpha > 1 \exists A \geq 1 \forall \lambda > 0 \exists D > 0 : \varphi^*_\lambda,\omega(\alpha t) \leq \varphi^*_\lambda,\omega(t) + D(t + 1), \quad t \geq 0.\]

If we summarize we have proven

Proposition 5.8. Let $\omega$ be a weight function such that $\mathbb{E}$ holds. Then the scale $(M^\lambda_\omega)_{\lambda > 0}$ generated by $\zeta(\lambda, t) = \zeta_\omega(\lambda, t) = \varphi^*_\lambda,\omega(t)$ is admissible. Furthermore, if $\mathbb{M} = \mathbb{M}_\zeta$ denotes the weight matrix associated to the scale $(M^\lambda_\omega)_{\lambda}$ then
\[E^{(\omega)}(U) = E^{(\mathbb{M})}(U), \quad E^{(\omega)}(U; P) = E^{(\mathbb{M})}(U; P)\]
for any differential operator $P$ with analytic coefficients.

Proof of Theorem 1.1. Combine Theorem 1.7 with Proposition 5.8. \hfill \□

Remark 5.9. It is clear that Theorem 1.1 cannot hold for general weight functions. For example, if $s > 1$ then $E^{(t^{1/s})}(U) \nsubseteq E^{(t^{1/s})}(U; P)$ for all non-elliptic operators $P$ by \ref{theorem} Theorem 1.3. Using the proof of \ref{abstract} Theorem 1.2 Boiti and Jornet \ref{example} Example 3.1 showed that if $P$ is not elliptic then there is a weight function $\omega_P$ which is not equivalent to any Gevrey weight function $t^{1/s}$ such that $E^{(\omega_P)}(U) \nsubseteq E^{(\omega_P)}(U; P)$. This example does not contradict Theorem 1.1 since $\omega_P$ does not satisfy $\mathbb{E}$. In fact, for each $\omega_P$ there exist $1 < s < s'$ by construction such that $G^s(U) \nsubseteq E^{(\omega_P)}(U) \subseteq G^{s'}(U)$, but the class associated with a weight function satisfying $\mathbb{E}$ is not contained in any Gevrey class as the following result shows.

Proposition 5.10. Let $\omega \in W_0$ be such that $\mathbb{E}$ holds. Then $E^{(\omega)}(\mathbb{R}) \nsubseteq E^{(t^{1/s})}(\mathbb{R})$ for all $s > 1$.

Proof. Suppose that $E^{(\omega)}(\mathbb{R}) \subseteq E^{(t^{1/s})}(\mathbb{R})$ for some $s > 1$. Then according to \ref{abstract} Corollary 5.17(i) we obtain that $\omega \preceq t^{1/s}$, i.e. there is some $B > 0$ such that
\[t^{1/s} \leq B(\omega(t) + 1), \quad t \geq 0,
\]
and therefore by Lemma \ref{lemma} for all $\alpha > 1$ we can find a constant $B_1 > 0$ such that
\[t^\alpha \preceq B_1(\omega(t^\alpha) + 1) \leq B_1(\omega(t) + 1), \quad t \geq 0.
\]
Hence if we choose \( \alpha = s \) then \( \omega \leq t \), which means that \( \mathcal{E}^{[\omega]}(\mathbb{R}) \) is contained in the space of analytic functions on \( \mathbb{R} \).

However, by Lemma 5.4 there is some \( 0 < \gamma < 1 \) such that \( t^\gamma \leq \omega \), which in particular implies that the space of analytic functions is strictly contained in \( \mathcal{E}^{[\omega]}(\mathbb{R}) \).

5B. Some remarks. We can use the “mixed” conditions of Corollary 5.6 to obtain results like Theorem 4.5 cf. also Remark 4.10 for weight functions. In fact, the conditions in Corollary 5.6 seem to be similar to those in Remark 4.10. However, arguing absolutely analogously to Section 4, we would not obtain results for some weight functions \( \omega \) and \( \sigma \) and their associated weight matrices \( W \) and \( \mathcal{S} \) but for the weight matrices \( \hat{W} \) and \( \hat{\mathcal{S}} \), cf. Proposition 5.2. As we have seen that does not matter if \( \omega = \sigma \) satisfies \( \mathcal{E} \).

But for the “mixed” setting note first that we can drop \( (k!)^{-d} \) in (4.3) since \( (k!)^d \geq 1 \) for all \( k \in \mathbb{N}_0 \) and \( \delta > 0 \). The other estimates before Theorem 4.5 remain also valid if we drop the “factorial” factors of the form \( k^{(d-\delta)} \). We obtain therefore the following Theorem, but we need to discuss subsequently how it fits in the theory presented in Section 4.

**Theorem 5.11.** Let \( P \) be a hypoelliptic operator of principal type with analytic coefficients in \( U \subseteq \mathbb{R}^n \), \( V \subseteq U \) and \( \delta = \delta(V) \) as in Remark 4.7. Furthermore suppose that \( \omega \) and \( \sigma \) are two weight functions satisfying

\[
\omega(t^\alpha) = O(\sigma(\theta t)), \quad t \to \infty,
\]

where \( H \geq 1 \) and \( \alpha = d/(d-\delta) \). Then every \( [\sigma]\)-vector of \( P \) is an ultradifferentiable function of class \([\omega]\) in \( V \).

**Proof.** We denote by \( W = \{W^\lambda : \lambda > 0\} \), \( W^\lambda = \varphi_{\lambda,\omega}(k) \), \( \mathcal{S} = \{S^\lambda : \lambda > 0\} \), \( S^\lambda = \varphi_{\lambda,\sigma}(k) \), the weight matrices associated to \( \omega \) and \( \sigma \), respectively. According to Corollary 5.6 there is a constant \( A > 0 \) such that for every \( \lambda > 0 \) we have

\[
\varphi_{\lambda,\sigma}(a t) \leq \varphi_{\lambda,\omega}(D(t) + 1)
\]

(5.2)

for some constant \( D > 0 \).

If \( u \in \mathcal{E}^{[\sigma]}(U; P) \) then there exist \( \lambda > 0 \), \( h > 0 \) and \( C > 0 \) such that

\[
\|P^k u\|_{L^1(V)} \leq C h^k S^\lambda
\]

for all \( k \in \mathbb{N}_0 \). Now (4.3) and (5.2) imply similarly to the argument before Theorem 4.5 that

\[
WF_{\{W^\lambda\}} u \cap V \times \mathbb{R}^n \setminus \{0\} = \emptyset.
\]

Hence \( u|_V \in \mathcal{E}^{[\mathcal{W}]}(V) = \mathcal{E}^{[\omega]}(V) \) by Proposition 2.14 and Proposition 2.15.

The Beurling case follows analogously.

**Remark 5.12.** If we set in Theorem 5.11 \( \omega(t) = t^{l/(\cos s)} \) and \( \sigma(t) = t^{l/s} \) then we obtain that any \( s \)-Gevrey vector is a \( \alpha,s \)-Gevrey function in \( V \). But this is a weaker result than [3, Theorem 1.3]. In particular by Theorem 5.11 we would only obtain that an analytic vector is an \( \alpha \)-Gevrey function in \( V \).

This reflects the difference in the definition of the ultradifferentiable scales: In section 4 we have defined the weight sequences \( M^\lambda \) of the scale generated by \( \zeta \) by \( m^k = \exp \circ \zeta_\lambda(k) \), i.e. \( M^\lambda \approx k!( \exp \circ \zeta_\lambda(k) ) \), whereas the definition of the scale associated to a weight function in this section corresponds to \( M^\lambda = \exp \circ \zeta_\lambda(k) \). By Proposition 5.2 the two definitions are essentially equivalent when the weight function satisfies \( \mathcal{E} \). For the moment we may call a scale \((M^\lambda)_\lambda \) of irregular weight sequences weak if it is defined via the sequences \( M^\lambda = \exp \circ \zeta_\lambda(k) \). On the other hand, we might say that the scales from Section 4 are those that are represented by \( k! \exp \circ \zeta_\lambda(k) \) are strong.

We observe that Theorem 5.11 shows that it would make a big difference if we would have used weak scales in Section 4. As we pointed out above, for the Gevrey scale it would mean that we could only prove a weaker version of [3, Theorem 1.3], and we would prove the Roumieu version of Proposition 1.2 but not the Beurling version. We note also that in this situation the scales \((B^{\lambda\sigma})_\sigma \) are not recognized under the framework of weak scales, as the fact from above that analytic vectors might only be Gevrey functions indicates.

---

2In order to guarantee that the sequences \( M^\lambda \) of a weak scale are irregular weight sequences we need to change the definition of generating functions in Subsection 4A a little bit. For example, instead of demanding only that \( \log k + \zeta_\lambda(k) - \zeta_\lambda(k-1) \) is increasing in \( k \) for fixed \( k \) we need that the sequence \( \zeta_\lambda(k) - \zeta_\lambda(k-1) \) is increasing. Furthermore we replace \( \lim(t^{-1}\zeta_\lambda(t)) = \infty \) for each \( \lambda \) with \( \lim(t^{-1}\zeta_\lambda(t) - \log(t)) = \infty \). In fact, these are the only changes necessary.
On the other hand, if we consider scales that are larger than the Gevrey scales there is not much difference. In the case of the scales \((L^{0,r})_q\) we have already noted that for the proof of Theorem 4.4 for the matrices \(\Omega^r\) (and therefore for the weight matrix \(R\)) there is no real difference if we use the scale \((L^{0,r})_q\) or the scale \((N^{0,r})_q\) given by \(N^q_k = q^k\). In fact, we have the following variant of Corollary 4.6:

**Corollary 5.13.** Let \(q > 1, 1 < r \leq 2\) and \(P\) be as in Theorem 4.4 and suppose that \(u\) is an \([N^{0,r}]\)-vector. If \(V \subseteq U\) then \(u\) is of class \([N^{0,r}]\) in \(V\), where \(q^r\) is as in Corollary 4.6(1).

Remark 5.14. In order to decide which kind of scales should be used for studying the regularity of vectors of a given operator, one can, in the case of operators which have been already studied, look at the regularity of Gevrey vectors. The technical reason why strong scales are advantageous for the study of vectors of operators of principal type is the factor \((k!)^{-\delta}\) in the main estimate (4.3), cf. the estimates before Theorem 4.5. For another example using the definition from section 6A see Subsection 6A.

On the other hand, in the case of Hörmander’s sum of squares operators, introduced in [34], there are analytic vectors which are not analytic functions, see [52] and also [18]. The analytic vectors from Subsection 6A do). In contrast, the sums of squares operator of Hörmander are subelliptic but seen that hypoelliptic operators of principal type satisfy both conditions (as the systems of vector fields two properties play an important role: subellipticity and that analytic vectors are analytic. We have kind of scales to use for the study of vectors of a given operator or system of operators. It seems that results are strict, see [52], [18] and also the survey in [23]. A similar result was obtained for some class of locally integrable structures of corank one in [20]. Hence in these two instances weak scales are more appropriate for the study of ultradifferentiable vectors.

We can try to analyze these examples to find some general conditions which can help to decide which kind of scales to use for the study of vectors of a given operator or system of operators. It seems that two properties play an important role: subellipticity and that analytic vectors are analytic. We have seen that hypoelliptic operators of principal type satisfy both conditions (as the systems of vector fields from Subsection 6A do). In contrast, the sums of squares operator of Hörmander are subelliptic but there are analytic vectors which are not analytic functions, see [52] and also [18]. The analytic vectors of the locally integrable structures considered in [20] are analytic but locally integrable structures are in general not subelliptic, cf. [11]. In this case we refer also to the discussion in [20] Section 10.

## 6. Miscellanea

### 6A. Systems of vector fields

Let \(X_1, \ldots, X_\ell\) be smooth real-valued vector fields on \(U \subseteq \mathbb{R}^n\). We say that the family \(X = \{X_1, \ldots, X_\ell\}\) is of finite type of order \(\nu \in \mathbb{N}\) if the tangent space \(T_xU\) at each point \(x \in U\) is generated by the iterated Lie brackets of order \(\nu\)

\[
X_I = [X_{j_1}, \ldots, [X_{j_k-1}, X_{j_k}]] \cdots, \quad I = (j_1, \ldots, j_k) \in \{1, \ldots, \ell\}^k, \quad k \leq \nu.
\]

The main result of [31] is that if the system \(X = \{X_1, \ldots, X_\ell\}\) is of finite type then \(X\) is hypoelliptic. In the case of analytic vector fields [22] showed that the finite type condition is also necessary for smooth hypoellipticity.

In [31] it was proven that if the family \(X = \{X_1, \ldots, X_\ell\}\) is analytic and of finite type then

\[
\bigcap_{j=1}^\ell \mathcal{A}(U; X_j) = \mathcal{A}(U).
\]

For Gevrey vectors [21] showed that

\[
G^{1+\sigma}(U; X) \subseteq G^{1+\sigma}(U), \quad \sigma \geq 0,
\]

if the collection of analytic \(X_1, \ldots, X_\ell\) is of finite type of order \(\nu\) and generates a stratified nilpotent Lie algebra \(G\) of rank \(\nu\), i.e.

\[
G = G_1 \oplus \cdots \oplus G_\nu
\]

with

\[
[G_1, G_j] = G_{j+1}, \quad 1 \leq j < \nu,
\]

\[
[G_1, G_\nu] = 0.
\]

The theory of ultradifferentiable scales from Section 4 allows us to generalize the result of [21]:

**Theorem 6.1.** Let \(X = \{X_1, \ldots, X_\ell\}\) be a family of analytic, real-valued vector fields on \(U \subseteq \mathbb{R}^n\) that is of finite type of order \(\nu\) and generates a stratified Lie algebra of rank \(\nu\).

1. If \((M^\lambda_\nu)\) is a fitting scale then for every \(\lambda \in \Lambda\) there is some \(\lambda^* \in \Lambda\) such that \(E^{(M^\lambda_\nu)}(U; X) \subseteq E^{(M^\lambda^*_\nu)}(U)\).
Lemma 6.3. Let $(M_k^\lambda)_\lambda$ be a weight sequence in the sense of [17] by Lemma 5.1. We also recall that we will also assume in this section that $M$ is a weight sequence, associated weight functions and condition [17]. However, if $\lambda^\ast$ may not satisfy (6.1) then we obtain that

$$\frac{X_k^\lambda}{k!} = \sum_{k_1+\cdots+k_\nu=k} \frac{a_1^{k_1} \cdots a_N^{k_\nu}}{k_1! \cdots k_\nu!} X_{j(1)}^{k_1} \cdots X_{j(\nu)}^{k_\nu}.$$ 

Hence if $u \in \mathcal{E}^{(M^\lambda)}(U;X)$ and $|I| = \mu$ then for $V \subset U$ there are constants $C, h > 0$ such that

$$\frac{\|X_k^I u\|_{L^2(V)}}{k!} \leq C h^{\mu k} \exp(\zeta_\lambda(\mu)) \sum_{k_1+\cdots+k_\nu=\mu k} (\mu k)! |a_1|^{k_1} \cdots |a_N|^{k_\nu}.$$ 

If we apply (6.1) then we obtain that

$$\frac{\|X_k^I u\|_{L^2(V)}}{k!} \leq C \gamma (\gamma h^{\mu} (|a_1| + \cdots + |a_N|)^\mu) \exp(\zeta_\lambda(\mu))$$

for some $\lambda_\mu$. In other words

$$\frac{\|X_k^I u\|_{L^2(V)}}{k!} \leq C_1 h_k^{\lambda_\nu} M_k^\lambda$$

for some constants $C_1, h_1 > 0$ and therefore $u \in \mathcal{E}^{(M^\lambda)}(U;X)$. If $\lambda^\ast = \max \{\lambda_1, \ldots, \lambda_\nu\}$ then

$$u \in \bigcap_{|I| \leq \nu} \mathcal{E}^{(M^\lambda)}(U;X).$$

Now Corollary 3.17 implies that $u \in \mathcal{E}^{(M^\lambda)}(U)$.

The rest of the theorem can be shown in a similar manner.

Corollary 6.2. Let $X$ be as in Theorem 6.1 and assume that $\omega$ is a weight function which satisfies (6.1). Then

$$\mathcal{E}^{[\omega]}(U;X) = \mathcal{E}^{[\omega]}(U).$$

6B. Weight sequences, associated weight functions and condition [17]. If $M$ is a weight sequence then the associated weight function

$$\omega_M(t) = \sup_{k \in \mathbb{N}_0} \log \frac{t^k}{M_k}$$

may not satisfy (6.1) in general, cf. [17, Theorem 3.1]. Therefore $\omega_M$ is not necessarily a weight function in the sense of [17]. However, $\omega_M \in W_0$, see e.g. [44] or [49]. Thus, if $\omega_M$ satisfies (6.1) then $\omega_M$ is a weight function in the sense of [17] by Lemma 5.1. We also recall that

$$M_k = \sup_{t > 0} \exp(\omega_M(t))$$

for all $k \in \mathbb{N}_0$.

We want to characterize those weight sequences $M$ for which $\omega_M$ satisfies (6.1). For technical reasons we will also assume in this section that $M_1 \geq 1$ for all weight sequences $M$.

We begin with an analogue of Lemma 5.3.

Lemma 6.5. Let $M, N$ be two weight sequences. The following statements are equivalent:

1. $\exists H \geq 1, \exists C > 0$ : $\omega_N(t^2) \leq C (\omega_M(H t) + 1)$, $t \geq 0$,
2. $\exists q \in \mathbb{N}, \exists A, \gamma > 0$ : $M_k^{2q} \leq A \gamma_k N_{qk}$, $k \in \mathbb{N}_0$. 

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Proof. If (1) holds then we can without loss of generality assume that $C \in \mathbb{N}$. Hence (6.1) gives

$$M_{2k} = \sup_{t > 0} \frac{(Ht)^{2k}}{\exp(\omega_M(Ht))} \leq eH^{2k} \sup_{t > 0} \frac{t^{2k}}{\exp(C^{-1}\omega_N(t^2))} = eH^{2k} \left( \sup_{s > 0} \frac{s^{Ck}}{\exp(\omega_N(s))} \right)^{1/C} = eH^{2k} N^{1/C}_k.$$ 

Since $M_{k}^2 \leq M_{2k}$ by (6.1) we have proven (2) for $q = C$, $A = eC$ and $\gamma = H^{2C}$.

On the other hand (2) implies that

$$M_{k}^2 \leq A^{1/q} \gamma^{-k/q} N^{1/q}_k = A_1 \gamma^k N^{1/q}_k.$$ 

We denote by $\mathfrak{M} = \{ M^{(\lambda)} : \lambda > 0 \}$ resp. $\mathfrak{N} = \{ N^{(\lambda)} : \lambda > 0 \}$ the weight matrix associated to $\omega_M$ resp. $\omega_N$. It is easy to show that $M = M^{(1)}$ (see for example the proof of [61, Theorem 6.4]) and observe also that

$$N^{(1)}_k = \exp \left[ q^{-1} \omega^*_N(qk) \right] = \left( N^{(1)}_{qk} \right)^{1/q} = N^{1/q}_k$$

for all $q \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Therefore from (2) we obtain

$$\exists q \in \mathbb{N} \quad \exists A_1, \gamma_1 > 0 : \quad M_{k}^2 \leq A_1 \gamma_1^k N^{(q)}_k, \quad k \in \mathbb{N}_0,$$

and thus

$$\log \left( \frac{t^k}{N^{(q)}_k} \right) \leq \log \left( \frac{(\gamma_1 t)^k}{M_{k}^2} \right) + \log A_1 = 2 \log \left( \frac{(\gamma_1 t)^{k/2}}{M_{k}^2} \right) + \log A_1$$

for all $t > 0$ and $k \in \mathbb{N}_0$. Hence by definition

$$\omega_N^{(q)}(t^2) \leq 2q \omega_M (\sqrt{\gamma_1} t) + \log A_1, \quad t \geq 0.$$ 

We recall that $\omega_N^{(1)}(t^2) \sim \omega_N$, more precisely we have

$$\forall \lambda > 0 \exists D_\lambda > 0 : \lambda \omega_N^{(\lambda)}(t) \leq \omega_N(t) \leq 2\lambda \omega_N^{(\lambda)}(t) + D_\lambda, \quad t \geq 0,$$

cf. [55, Section 5] or [39, Lemma 2.5].

Combining the last two estimates together we conclude that

$$\omega_N(t^2) \leq 4q \omega_M (\sqrt{\gamma_1} t) + 2q \log A_1 + D_q, \quad t \geq 0.$$ 

Hence (1) is proven with $H = \sqrt{\gamma_1}$ and $C = \max \{ 4q, 2q \log(A_1) + D_q \}$. \hfill \Box

**Corollary 6.4.** Let $M$ be a weight sequence. Then the following are equivalent:

1. The associated weight function $\omega_M$ satisfies (3).
2. There is a positive integer $p \in \mathbb{N}$ and constants $A, B > 0$ such that

$$\left( M_k \right)^{2p} \leq AB^k M_{pk}$$

holds for all $k \in \mathbb{N}_0$.

Note that in (6.2) we can assume that $p \geq 2$, because $p = 1$ would yield that $\sup_k M_{k}^{1/k} < \infty$.

It is a natural question to ask if there is a weight sequence $M$ such that (6.2) and $\mathcal{E}(M^U) = \mathcal{E}(\omega_M(U))$. However, according to [16], a necessary condition for the last identity is for $M$ to be of moderate growth, i.e. there is a constant $\gamma > 0$ such that

$$M_{j+k} \leq \gamma^{j+k} M_j M_k$$

for all $j, k \in \mathbb{N}_0$.

**Lemma 6.5.** A weight sequence $M$ does not satisfy simultaneously (6.2) and (6.3).

*Proof.* Assume that both (6.2) and (6.3) hold for $M$. Then (6.3) implies that

$$M_{pk} \leq \gamma^{p^2k} (M_k)^p, \quad k \in \mathbb{N}_0,$$

where $p$ is the integer from (6.2). Hence we have, if we combine the estimate above with (6.2), that

$$\left( M_k \right)^{2p} \leq AB^k M_{pk} \leq AB^k \gamma^{p^2k} (M_k)^p.$$
It follows that $\sup_k (M_k)^{1/k} < \infty$ and therefore $M$ is not a weight sequence. □

**Example 6.6.** (1) The sequences $N^{q, r} = q^k$ satisfy (6.2) for $p \geq 2^{1/(r-1)}$: Then $2p \leq p'$ and therefore

$$(N_k^{q, r})^{2p} = q^{2pk} \leq q^{(pk)'^r} = N_k^{q, r'}.$$ 

(2) The sequence $M$ given by $M_0 = 1$ and $M_k = e^{e_k}$, $k \in \mathbb{N}$, satisfies (6.2) with $p = 8$ because we have the estimate

$$(e^{e_k})^{16} = e^{16e_k} \leq e^{e_k},$$

since $4 + k \leq 8k$ for all $k \in \mathbb{N}$.

6C. **Families of weight functions: An example.** Let $P$ be again a hypoelliptic operator of principal type with analytic coefficients in an open set $U \subseteq \mathbb{R}^n$ and consider $\Omega = \{\omega_s : s > 0\}$, where $\omega_s(t) = (\max\{0, \log(t)\})^s$ is the weight function from Example 2.11. Then by Theorem 1.1 we know that $\mathcal{E}^{[\omega_s]}(U; \mathcal{P}) = \mathcal{E}^{[\omega_s]}(U)$, but analogously to the case of weight matrices, i.e. families of weight sequences, we can also consider the spaces associated to $\Omega$, i.e. we define

$$\mathcal{E}^{[\Omega]}(U) = \left\{ f \in \mathcal{E}(U) : \forall V \subseteq U \exists s > 1 \exists h > 0 \parallel f \parallel_{V, \omega_s, h} < \infty \right\},$$

and also

$$\mathcal{E}^{[\Omega]}(U; P) = \left\{ u \in \mathcal{E}'(U) : \forall V \subseteq U \exists s > 1 \exists h > 0 \parallel u \parallel_{P, V, \omega_s, h} < \infty \right\},$$

We have

**Proposition 6.7.** If $P$ is a hypoelliptic analytic operator of principal type then $\mathcal{E}^{[\Omega]}(U; P) = \mathcal{E}^{[\Omega]}(U)$.

**Proof.** Observe in the Beurling case that

$$\mathcal{E}^{[\Omega]}(U; P) = \bigcap_{s > 1} \mathcal{E}^{[\omega_s]}(U; P) = \bigcap_{s > 1} \mathcal{E}^{[\omega_s]}(U) = \mathcal{E}^{[\Omega]}(U).$$

On the other hand, if $u \in \mathcal{E}^{[\Omega]}(U; P)$ then for all $V \subseteq U$ there is some $s > 1$ such that

$$u|_V \in \mathcal{E}^{[\omega_s]}(V; P) = \mathcal{E}^{[\omega_s]}(V) \subseteq \mathcal{E}^{[\Omega]}(V).$$

Hence $u \in \mathcal{E}^{[\Omega]}(U)$.

For the other direction, recall that $\mathcal{E}^{[\omega_s]}(U) \subseteq \mathcal{E}^{[\omega_s]}(U; P)$ for all $s > 0$ by Proposition 6.2. Arguing analogously to above gives the desired inclusion. □

However, it turns out that we have already encountered the spaces $\mathcal{E}^{[\Omega]}$:

**Theorem 6.8.** Let $P$ as above and $\mathcal{R}$ be as in Example 2.6(2). Then

$$\mathcal{E}^{[\Omega]}(U) = \mathcal{E}^{[\mathcal{R}]}(U),$$

$$\mathcal{E}^{[\Omega]}(U; P) = \mathcal{E}^{[\mathcal{R}]}(U; P).$$

The first equality follows from a more general theorem in [61]. In order to state that theorem we need to recall some notations. If $\mathcal{R}$ is a weight matrix we denote by $\Omega_{\mathcal{R}} = \{\omega_M : M \in \mathcal{R}\}$ the family of weight functions associated to $\mathcal{R}$. Similarly to above we can define the spaces of ultradifferentiable functions associated to $\Omega_{\mathcal{R}}$:

$$\mathcal{E}^{(\Omega_{\mathcal{R}})}(U) = \left\{ f \in \mathcal{E}(U) : \forall V \subseteq U \exists M \in \mathcal{R} \exists h > 0 \parallel f \parallel_{V, \omega_M, h} < \infty \right\},$$

$$\mathcal{E}^{(\Omega_{\mathcal{R}})}(U; P) = \left\{ u \in \mathcal{E}'(U) : \forall V \subseteq U \forall M \in \mathcal{R} \forall h > 0 \parallel u \parallel_{P, V, \omega_M, h} < \infty \right\}.$$
We consider the following conditions

\[
\forall M \in \mathfrak{M} \ \exists N \in \mathfrak{M} \ \exists C > 0, \ \forall j, k \in \mathbb{N}_0: \ M_{j+k} \leq C^{j+k} N_j N_k, \tag{6.4}
\]

\[
\forall N \in \mathfrak{M} \ \exists M \in \mathfrak{M} \ \exists C > 0, \ \forall j, k \in \mathbb{N}_0: \ M_{j+k} \leq C^{j+k} N_j N_k, \tag{6.5}
\]

\[
\forall M \in \mathfrak{M} \ \forall h > 0 \ \exists N \in \mathfrak{M} \ \exists D > 0 \ \forall k \in \mathbb{N}_0: \ h^k M_k \leq D N_k, \tag{6.6}
\]

\[
\forall N \in \mathfrak{M} \ \forall h > 0 \ \exists M \in \mathfrak{M} \ \exists D > 0 \ \forall k \in \mathbb{N}_0: \ h^k M_k \leq D N_k. \tag{6.7}
\]

In [38] the following result was shown:

**Theorem 6.9.** Let \( \mathfrak{M} \) be a weight matrix. Then we have

1. If \( \mathfrak{M} \) satisfies (6.4) and (6.6) then \( \mathcal{E}^{[\mathfrak{M}]}(U) = \mathcal{E}^{[\mathfrak{M}]}(U) \).
2. If \( \mathfrak{M} \) satisfies (6.5) and (6.7) then \( \mathcal{E}^{[\mathfrak{M}]}(U) = \mathcal{E}^{[\mathfrak{M}]}(U) \).

**Proof of Theorem 6.9.** We recall from Example 2.14 that the weight matrix associated to \( \omega_s, s > 1 \), is \( \mathfrak{M}_s = \{N_q^r: q > 1\} \), where \( r = s/(s-1) \) and \( N^r_q = q^r \). Then \( \omega_s \sim \omega_s \) for all \( s > 0 \) and \( q > 0 \) by [55] Lemma 5.7. Note also that by Proposition 5.2 we have \( \mathfrak{M}_s [\mathbb{N}^1_\omega \mathcal{D}^\omega] \). Thus \( \mathcal{E}^{[\mathfrak{M}]}(U) = \mathcal{E}^{[\mathfrak{M}]}(U) \) and \( \mathcal{E}^{[\mathfrak{M}]}(U; P) = \mathcal{E}^{[\mathfrak{M}]}(U; P) \).

For given \( r' > r > 1 \) choose \( A > 0 \) large enough such that

\[
2^{r'-r} \leq k^{r'-r} + k^{1-r} \frac{\log A}{\log 2}
\]

for all \( k \in \mathbb{N} \). We conclude that

\[
N_{2k}^{2,r} \leq A^k \left( N_k^{r'} \right)^2
\]

which implies (6.3) and (6.5) by [60] Theorem 9.5.1 and Theorem 9.5.3.

On the other hand, for any \( h > 0 \) and \( r > 1 \) we can choose \( r' > 1 \) and \( D > 0 \) large enough such that

\[
k \log h \leq \log 2 \left( k^{r'} - k^r \right) + \log D
\]

for all \( k \) which gives

\[
h^k N_{2k}^{2,r} \leq D N_{2,k}^{2,r'}, \quad k \in \mathbb{N}_0.
\]

It follows that \( \mathfrak{M} \) satisfies (6.5) and (6.7).

Hence \( \mathcal{E}^{[\mathfrak{M}]}(U) = \mathcal{E}^{[\mathfrak{M}]}(U) \) by Theorem 6.9. A close inspection shows that the proof of Theorem 6.9 in [61] applies also to the spaces \( \mathcal{E}^{[\mathfrak{M}]}(U; P) \).

Therefore

\[
\mathcal{E}^{[\mathfrak{M}]}(U; P) = \mathcal{E}^{[\mathfrak{M}]}(U; P) = \mathcal{E}^{[\mathfrak{M}]}(U) = \mathcal{E}^{[\mathfrak{M}]}(U).
\]

\( \square \)

6D. A characterization of ellipticity by non-Gevrey vectors. The aim of this section is to prove Theorem 1.4. We begin with noticing two easy observations, which we will need later on:

**Lemma 6.10.** Let \( M \) be a weight sequence and \( \rho, R \geq 1 \). Then

\[
\rho^j M_{k+\ell} R^\ell \leq \rho^{j+\ell} M_k + M_{j+k+\ell} R^{j+\ell}
\]

for all \( j, k, \ell \in \mathbb{N}_0 \).

**Proof.** If \( \mu_k = M_k/M_{k-1} \) then (2.14) implies that the sequence \( (\mu_k)_k \) is increasing. For \( \rho \geq \mu_{k+\ell} R \) we obtain that

\[
M_{k+\ell} R^\ell = M_k \mu_{k+1} R \ldots \mu_{k+\ell} R \leq \rho^\ell M_k.
\]

If \( \rho \leq \mu_{k+\ell} R \) then

\[
\rho^\ell \leq \mu_{k+\ell+1} R \ldots \mu_{k+\ell+\ell} R \leq \frac{M_{j+k+\ell} R^\ell}{M_{k+\ell} R^\ell}.
\]

\( \square \)

**Lemma 6.11** (cf. [55] Lemma 5.7). Let \( \omega \in \mathcal{W}_0 \) and \( \mathfrak{M} = \{W^\rho: \rho > 0\} \) be the weight matrix associated to \( \omega \). If \( \omega_\rho \) is the weight function associated to \( W^\rho \) then

\[
\omega_\rho(t) \leq \frac{\omega(t)}{\rho}
\]

for all \( t > 0 \).
Proof. Since \( W_\delta^\rho = \exp \left[ \rho^{-1} \varphi_\rho^\delta (\rho k) \right] \) we obtain
\[
\omega_\rho(t) = \sup_{k \in \mathbb{N}_0} \left[ k \log t - \rho^{-1} \varphi_\rho^\delta (\rho k) \right] \leq \sup_{s \geq 0} \left[ s \log t - \rho^{-1} \varphi_\rho^\delta (\rho s) \right] = \frac{1}{\rho} \omega(t).
\]

Before we can begin with the proof of Theorem 1.4 we also need to take a closer look at the scale \((\mathbb{N}^\lambda)_\delta\), given by \( N_k^\delta = q^{k^2} \), specifically. Recall that \( \mathfrak{R} = \{\mathbb{N}^\rho : \rho > 1\} \) is the weight matrix associated to \( \omega(t) = (\max(0; \log t))^2 \). More precisely, \( \varphi_\omega(t) = \omega \circ \exp(t) = t^2 \) and \( \varphi_\omega^\delta(t) = t^{2/4} \). Hence the canonical weight matrix \( W_\delta^\rho = \{W_\rho^k : \rho > 0\} \) associated to \( \omega_\rho \) is given by \( W_\delta^\rho = \exp(\rho k^2/4) \), cf. [57, Section 5.5]. Thus it is convenient to set \( \lambda = \log q \) and to write in a slight abuse of notation \( \mathbb{N}^\lambda = \mathbb{N}^\rho \). It follows that \( \mathbb{N}^\lambda = \mathcal{W}^{2,4\lambda} \) and therefore Lemma 6.11 implies that
\[
\omega_{\mathbb{N}^\lambda}(t) \leq \frac{(\log t)^2}{4\lambda}, \quad t \geq 1. \tag{6.8}
\]

Now observe that \( (\mathbb{N}^\lambda)_\lambda \) is the weak scale associated to the generating function \( \zeta(t, \lambda) = \lambda^2 \), which clearly can be extended to an entire function \( \zeta(z, \lambda) \) in the first variable. Hence \( \theta(z, \lambda) = \exp_{\zeta}(z, \lambda) \) is holomorphic in \( z \) and when \( \lambda \) is fixed we have that for every strip \( G = \{w = u + iv \in \mathbb{C} : a < u < b\} \) there is a constant \( C > 0 \) such that \( |\theta(z, \lambda)| \leq Ce^{-|\text{Im} z|^2} \). It follows that \( \theta( , \lambda) \) is the Mellin transform of the function
\[
\Theta(t, \lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\sigma \log t} e^{-\lambda \sigma^2} \, d\sigma,
\]
that is
\[
\theta(z, \lambda) = \int_0^\infty t^{z-1} \Theta(t, \lambda) \, dt,
\]
see e.g. [55] or [51]. In particular
\[
\mathbb{N}_k^\lambda = \int_0^\infty t^{k-1} \Theta(t, \lambda) \, dt. \tag{6.9}
\]

If we set \( t = e^s \) we can compute
\[
\Theta(e^s, \lambda) = \frac{1}{2\pi} \int_\mathbb{R} e^{-i\sigma s} e^{-\lambda \sigma^2} \, d\sigma = \frac{1}{\sqrt{4\pi \lambda}} e^{-s^2/4\lambda}
\]
and therefore
\[
\Theta(t, \lambda) = \frac{1}{\sqrt{4\pi \lambda}} \exp \left[ -\frac{(\log t)^2}{4\lambda} \right].
\]

In order to prove Theorem 1.4 it is enough to show the following statement.

**Theorem 6.12.** Let \( P \) be a differential operator with analytic coefficients on \( U \) which is not elliptic at some point \( x_0 \in U \). Then we have
\[
\mathcal{E}(\mathbb{N}^\lambda) \subset \mathcal{E}(\mathbb{N}^\lambda)(U; P)
\]
for all \( \lambda > 0 \).

**Proof.** It is sufficient for given \( \lambda > 0 \) to construct a function \( u \) which is an \( \{\mathbb{N}^\lambda\} \)-vector of \( P \) but is not in \( \mathcal{E}(\mathbb{N}^\lambda)(U) \). In order to do so we shall try to follow the pattern of the proof of [52, Theorem 2.3]. From now on let \( \lambda > 0 \) be arbitrary but fixed and choose parameters \( \varepsilon, \lambda' > 0 \) and \( 0 < \lambda_0 < \lambda \) depending on \( \lambda \) which will later be specified. Since \( P \) is not elliptic at \( x_0 \) there exists some \( \xi_0 \in S^{n-1} \) such that
\[
p_d(x_0, \xi_0) = 0. \tag{6.10}
\]

Let \( \delta > 0 \) be such that \( B_0 = \{x \in \mathbb{R}^n : |x - x_0| < 2\delta\} \in U \) and let \( \psi \in \mathcal{E}(\mathbb{N}^{\lambda_0})(\mathbb{R}^n) \) be such that \( \text{supp} \psi \subseteq \{x \in \mathbb{R}^n : |x| < 2\delta\} \) and \( \psi(x) = 1 \) for \( |x| \leq \delta \). Thus there are constants \( C_0, h_0 > 0 \) such that
\[
|D^\nu \psi(x)| \leq C_0 h_0^{|\nu|} N_{\lambda_0}^{|\nu|} \tag{6.11}
\]
for all \( x \in \mathbb{R}^n \). This is possible since \( \mathbb{N}^\tau \) is non-quasianalytic for any \( \tau > 0 \).
Then we define the function $u$ to be of the form

$$
u(x) = \int_{\nu}^{\infty} \psi(t^\varepsilon(x-x_0)) \Theta(t, \lambda') e^{it(x-x_0)x_0} dt.$$ 

It follows that

$$D_{\xi_0}^k u(x_0) = \int_{\nu}^{\infty} t^k \Theta(t, \lambda') dt,$$

where $D_{\xi_0} = -i \frac{\partial}{\partial \xi_0}$ is the directional derivative in direction $\xi_0$. Thence \(6.9\) implies that

$$D_{\xi_0}^k u(x_0) = N_{k+1}^{\lambda'} - \int_{\nu}^{\infty} t^k \Theta(t, \lambda') dt.$$ 

Since $\int_{\nu}^{\infty} t^k \Theta(t, \lambda') dt \to 0$ when $k \to \infty$ we have shown that $u$ cannot be of class $\{N^\varepsilon\}$ in any neighborhood of $x_0$ for all $\tau < \lambda'$.

On the other hand, it is easy to see that

$$P_k^g u(x) = \int_{\nu}^{\infty} Q_k(x, t) \Theta(t, \lambda') e^{it(x-x_0)x_0} dt$$

where $Q_k$ is defined recursively by

$$Q_0(x, t) = \psi(t^\varepsilon(x-x_0))$$

and

$$Q_{k+1}(x, t) = \sum_{|\alpha| \leq d} \frac{1}{|\alpha|!} \frac{\partial^\alpha}{\partial \xi^\alpha} \Theta(t, \lambda') D_{\xi_0}^\alpha Q_k(x, t).$$

Since $P$ is analytic in $U$ we have that there is a constant $H > 0$ such that for all $\nu, \alpha \in \mathbb{N}_0$ with $|\alpha| \leq d$, all $x \in B_2(s(x_0))$ and all $t \geq 1$:

$$|D_{\xi_0}^\nu Q_k(x, t)| \leq C_0 |\nu| |t^d|.|\alpha|,$$

and due to \(6.10\) for all $0 < \varepsilon < 1$ there is $C_1 > 0$ such that for all $t > 0$ and all $x \in U$ with $|x-x_0| \leq 2\delta t^{-\varepsilon}$:

$$|p(x, t\xi_0)| \leq C_1 t^{d-\varepsilon}.$$ 

Using the above estimates \(6.12\) and \(6.13\) together with Lemma \(6.10\), it is easy to see that we can adapt the proof of \(6.2\), Lemma 2.1] and therefore obtain the following statement.

**Lemma 6.13.** There exists a constant $A > 0$ such that for all $k \in \mathbb{N}_0$, all $\nu \in \mathbb{N}_0^d$, all $x \in B_0$ and all $t \geq 1$ we have

$$|D_{\xi_0}^\nu Q_k(x, t)| \leq C_0 A^k \left( t^{(d-\varepsilon)k} N_{\nu}^{\lambda_0} + t^{(2d-1)k} N_{\nu}^{\lambda_0} \right).$$ 

If we set $\nu = 0$ in \(6.14\) then we get

$$|Q_k(x, t)| \leq C_0 A^k \left( t^{(d-\varepsilon)k} + t^{(2d-1)k} N_{\nu}^{\lambda_0} \right).$$

When we set $\rho = t^{1-\varepsilon/d}$ and $R = t^{(2-1/d)}$ then $\rho^d = t^{(d-\varepsilon)k}$ and $R^d = t^{(2d-1)k}$, respectively. Hence, if $\lambda_1 = \lambda - \lambda_0$ then we have

$$\rho^d \leq N_{\nu}^{\lambda_0} e^{-\varepsilon N_{\nu}^{\lambda} \rho} \leq N_{\nu}^{\lambda} \exp \left[ \frac{(log t)^2}{4 \lambda_1 (d-\varepsilon)^2} \right],$$

$$R^d \leq N_{\nu}^{\lambda_1} \exp \left[ \frac{(log t)^2}{4 \lambda_1 (d-\varepsilon)^2} \right],$$

by \(6.8\) and thus

$$|Q_k(x, t)| \leq C_0 A^k N_{\nu}^{\lambda_0} \exp \left[ \frac{(log t)^2}{4} \left( \frac{(d-\varepsilon)^2}{\lambda d^2} + \frac{\varepsilon^2 (2d-1)^2}{\lambda_1 d^2} \right) \right].$$
If for fixed $\lambda > 0$ we choose the parameters $0 < \lambda_0 < \lambda$ and $\varepsilon$ such that

$$0 < \varepsilon \leq \frac{d\sqrt{\lambda - \lambda_0}}{\sqrt{\lambda - \lambda_0} + \sqrt{\lambda(2d - 1)}} < \frac{1}{2},$$

then

$$\frac{\varepsilon^2(2d - 1)^2}{(\lambda - \lambda_0)d^2} \leq \frac{(d - \varepsilon)^2}{\lambda d^2}.$$

It follows that

$$\|P_k u(x)\| \leq C_0 A_k N_d k \int_1^\infty \exp \left[ \frac{(d - \varepsilon)^2 - \frac{1}{4}}{4} \left( \log t \right)^2 \right] dt.$$ 

The integral converges as long as

$$\lambda' < \frac{d^2}{(d - \varepsilon)^2} \frac{\lambda}{2}.$$

The proof of Theorem 6.12 is complete if we put additionally $\lambda' > \lambda$. \hfill $\square$

**Appendix A. Subelliptic estimates**

The aim of this appendix is to indicate how (3.2) implies (3.2'). Following [48] we introduce the local Sobolev space $H^\sigma(V)$, $\sigma \in \mathbb{R}$, over an arbitrary open set $V \subseteq \mathbb{R}^n$ as the quotient space $H^\sigma(V) = \mathcal{H}^\sigma(\mathbb{R}^n)/F^\sigma(V)$, where $F^\sigma(V)$ is the space of all functions $f \in H^\sigma(\mathbb{R}^n)$ which vanish on $V$. Clearly $F^\sigma(V)$ is a closed subspace of $H^\sigma(\mathbb{R}^n)$ hence $H^\sigma(V)$ is a Hilbert space with the structure inherited from $H^\sigma(\mathbb{R}^n)$.

**Remark A.1.** It is easy to see that $H^0(V) = L^2(V)$ for all open sets $V \subseteq \mathbb{R}^n$. However, if we consider the classical Sobolev space

$$W^k(V) = \{ f \in L^2(V) : \partial^\alpha f \in L^2(V) \ \forall |\alpha| \leq k \}$$

then we cannot conclude in general that $W^k(V) = H^k(V)$ for $k \in \mathbb{N}$, unless $V$ is a relatively compact set in $\mathbb{R}^n$ with smooth boundary.

We denote the (quotient) norm of $H^\sigma(V)$ by $\| \cdot \|_{H^\sigma(V)}$. For $f \in H^\infty_{\text{loc}}$ this agrees with the previous definition of $\|f\|_{H^\sigma(V)}$ in (3.1). More precisely, if $U$ is an open set in $\mathbb{R}^n$, $V \subseteq U$ and $\iota$ is the natural inclusion map of $H^\infty(U)$ into $H^\sigma(V)$ then $\|f\|_{H^\sigma(V)} = \|\iota(f)\|_{H^\sigma(V)}$ for all $f \in H^\infty_{\text{loc}}(U)$.

Now suppose that $U$ and $V$ are given open sets in $\mathbb{R}^n$ such that $V \subseteq U \subseteq \mathbb{R}^n$ and $\mathcal{P} = \{ P_1, \ldots, P_k \}$ is a family of analytic partial differential operators of orders $d_j$ on $U$ satisfying

$$\|\varphi\|_{\sigma + \varepsilon} \leq C \left( \sum_{j=1}^k \|P_j \varphi\|_{\sigma} + \|\varphi\|_{\sigma} \right)$$

(A.1)

for all $\varphi \in \mathcal{D}(V)$ and some constant $C > 0$ independent of $\varphi$. If we multiply all of the coefficients of the operator $P_j$ with a test function $\chi \in \mathcal{D}(U)$ satisfying $\chi|_V = 1$ we may assume that the operator $P_j$ is a continuous mapping from the space $H^\sigma(\mathbb{R}^n)$ into $H^{\sigma - d_j}(\mathbb{R}^n)$ for all $\sigma$. This clearly does not change the value of $\|P_j \varphi\|_{\sigma}$ when $\varphi \in \mathcal{D}(V)$ or of $\|P_j g\|_{H^\sigma(V)}$ when $g \in H^\sigma_{\text{loc}}(U)$. Therefore the mapping $P_j : H^{\infty}(\mathbb{R}^n) \to H^{\infty}(\mathbb{R}^n)$, where $H^{\infty}(\mathbb{R}^n) = \text{proj}_\sigma H^{\sigma}(\mathbb{R}^n)$, is also continuous. Moreover, observe that $F^{\infty}(V) = \bigcap_\sigma F^{\sigma}(V)$ is closed in $H^{\infty}(\mathbb{R}^n)$. Similarly, $F^{\infty}(V) = \bigcap_\sigma F^{\sigma}(V)$ is a Fréchet space and $P_j$ is a continuous automorphism on $H^{\infty}(V)$ since $P_j$ is a local operator.

We recall that we want to show that (A.1) implies

$$\|g\|_{H^{\sigma + \varepsilon}(V)} \leq C \left( \sum_{j=1}^k \|P_j g\|_{H^\sigma(V)} + \|g\|_{H^\sigma(V)} \right)$$

(A.2)

for all $g \in \mathcal{E}(U)$.

For $V \subseteq U$ given we denote by $\iota_V : H^\infty_{\text{loc}}(U) \to H^\infty(V)$ the canonical inclusion mapping. Due to the continuity of the operators $P_j$ we would be done if we could show that $\iota_V(H^\infty_{\text{loc}}(U)) \subseteq \mathcal{D}(V)$, where $\mathcal{D}(V)$ is the closure of $\mathcal{D}(V)$ in the topology of $H^{\infty}(V)$. If $V = B$ we have the following result:

**Theorem A.2.** Let $U \subseteq \mathbb{R}^n$ be an open set and $B$ be an open ball such that $B \subseteq U$. Then we have $\iota_B(H^\infty_{\text{loc}}(U)) \subseteq \mathcal{D}(B)$. 41
Proof. For each \( g \in H_{100}^∞(U) = \mathcal{E}(U) \) we have to find a sequence \( \varphi_j \in \mathcal{D}(B) \) such that \( \hat{\varphi}_j = \varphi_j + F^∞(B) \) converges to \( \hat{g} = i_B(g) \) in \( H^∞(B) \). A representative of \( \hat{g} \) is given by \( \chi \in \mathcal{D}(U) \) with \( \chi|_B = 1 \).

We choose two sequences \( (K_j)_j, (L_j)_j \) of compact subsets of \( U \) with the following properties:

- \( K_j \subseteq B \) and dist\((K_j, U \setminus B) → 0 \) when \( j → \infty \).
- \( \overline{B} \subseteq L_j \) and dist\((\overline{B}, \partial L_j) → 0 \) if \( j → \infty \).

For each \( j \in \mathbb{N} \) choose test functions \( \psi_j \in \mathcal{D}(B) \) and \( \lambda_j \in \mathcal{D}(U) \) such that \( 0 ≤ \psi_j, \lambda_j ≤ 1, \psi_j|_{K_j} = 1, \lambda_j|_{\text{supp}\chi \setminus L_j} = 1 \) and \( \text{supp} \lambda_j \subseteq U \setminus \overline{B} \). We set \( \varphi_j = \psi_j g \in \mathcal{D}(B) \) and \( h_j = \lambda_j \chi g \in \mathcal{D}(U) \). Then \( \text{supp}\chi \varphi_j - h_j \subseteq L_j \setminus K_j \) and

\[
\left| \int_{L_j \setminus K_j} (\chi(x)g(x) - \varphi_j(x) - h_j(x))\Phi(x) \, dx \right| ≤ \sup\{|\chi g - \varphi_j - h_j|\} \cdot \frac{|L_j \setminus K_j|}{|\partial B| = 0}
\]

for \( \Phi \in \mathcal{E}(U) \), i.e. \( \chi g - \varphi_j - h_j \rightarrow 0 \) in \( \mathcal{S}'(\mathbb{R}^n) \) and thus \( \mathcal{F}(\chi g - \varphi_j - h_j) \rightarrow 0 \) in \( \mathcal{S}'(\mathbb{R}^n) \) since \( \mathcal{F} : \mathcal{S}'(\mathbb{R}^n) → \mathcal{S}'(\mathbb{R}^n) \) is continuous. But this means that \( \mathcal{F}(\chi g - \varphi_j - h_j)(\xi) → 0 \) almost everywhere. Note also that since \( \chi g - \varphi_j - h_j \in \mathcal{D}(\text{supp} \chi \setminus K_j) \)

\[
|\chi g - \varphi_j - h_j| ≤ \sup |\chi g|
\]

we have by the Paley-Wiener Theorem, see [36, page 181], that for each \( N \in \mathbb{N} \) there is a constant \( C_N \) depending on \( N, g \) and \( \chi \) such that

\[
|\mathcal{F}(\chi g - \varphi_j - h_j)(\xi)| ≤ C_N(1 + |\xi|)^{-N}
\]

for all \( \xi \in \mathbb{R}^n \). Hence the dominated convergence theorem implies that

\[
\|\chi g - \varphi_j - h_j\|^2 = \int (1 + |\xi|)^2 |\mathcal{F}(\chi g - \varphi_j - h_j)(\xi)|^2 \, d\xi → 0
\]

for all \( \sigma \in \mathbb{R} \). It follows that \( \varphi_j → \hat{g} \) in \( H^∞(B) \). \( \square \)

For the proof of (A.2) we also need the following fact.

Proposition A.3. Let \( E \) be a Hilbert space and \( \{M_j : j \in I\} \) a family of closed subspaces of \( E \). If \( M = \bigcap_{j \in I} M_j \) and if \( \| \cdot \|_M \) and \( \| \cdot \|_{M_j} \) are the quotient norms of \( E/M \) and \( E/M_j \), respectively, then

\[
\|x\|_M = \sup_{j \in I} \|f_j(x)\|_{M_j}, \quad x \in E,
\]

where the \( f_j \) are the induced canonical projections \( E/M → E/M_j \) given by \( x + M → x + M_j \).

Proof. Let denote the inner product on \( E \) by \( \langle \cdot, \cdot \rangle_E \) and

\[
M_j^⊥ = \{ x \in E : \langle x, y \rangle_E = 0 \ \forall y \in M_j \}
\]

\[
M^⊥ = \{ x \in E : \langle x, y \rangle_E = 0 \ \forall y \in M \}
\]

be the orthogonal complements of \( M_j \) and \( M \), respectively. It is well-known that \( M_j^⊥ \) and \( M_j \) are Hilbert spaces as closed subspaces of \( E \). Using the canonical Hilbert space isomorphisms \( M_j^⊥ ∼= E/M_j \), and \( M^⊥ ∼= E/M \) we can identify \( f_j \) with the canonical projection \( M^⊥ → M_j^⊥ \). Since \( M = \bigcap_{j \in I} M_j \) we have that \( \bigcap \ker f_j = \{0\} \). It is easy to see that the topology on \( M^⊥ \) is equivalent to the initial topology with respect to the mappings \( f_j \). Indeed, the closed subsets of \( M^⊥ \) and \( M_j^⊥ \) are exactly the sets of the form \( V = A \cap M^⊥ \) and \( V_j = A \cap M_j^⊥ \), respectively, where \( A \subseteq E \) is closed. Clearly, the canonical topology on \( M^⊥ \) is finer than the initial topology induced by the \( f_j \)’s which is generated by

\[
f_j^{-1}(V_j) = A \cap M^⊥ + \ker f_j.
\]

It follows that

\[
\bigcap_{j \in I} f^{-1}(V_j) = A \cap M^⊥ + \bigcap_{j \in I} \ker f_j = A \cap M^⊥.
\]

Now set \( \varphi_j(x) = \|x\|_{M_j}, x \in E/M ≃ M^⊥ \). Obviously \( \varphi_j \) is a seminorm on \( E/M ≃ M^⊥ \). The same is true for

\[
\Phi(x) = \sup_{j \in I} \varphi_j(x), \quad x \in E/M.
\]
In fact $\Phi$ is a norm. Suppose $\Phi(x) = 0$ for some $x \in E/M$, thus $\varphi_j(x) = 0$ for all $j \in I$. Hence $f_j(x) = 0$ for all $j \in I$, since $\| \cdot \|_{M_j}$ is a norm. We conclude that

$$x \in \bigcap_{j \in I} \ker f_j = \{0\}.$$  

If $B$ is the closed unit ball in $E$ then $B_0 = B \cap M^\perp$ and $B_j = B \cap M_j^\perp$ are the unit balls in $M^\perp$ and $M_j^\perp$, respectively. By the above we know that

$$\bigcap_{j \in I} f_j^{-1}(B_j) = B_0$$

and furthermore

$$B_\Phi = \{x \in M^\perp \mid \Phi(x) \leq 1\} = \{x \in M_j^\perp \mid \|x\|_{M_j} \leq 1, \ \forall j \in I\} = \bigcap_{j \in I} f_j^{-1}(B_j)$$

is the closed unit ball for the norm $\Phi$. Since both norms have the same closed unit ball, they have to agree everywhere. \qed

Now there are at most countable many open balls $B_j$, $j \in J$, such that $V = \bigcup_{j \in J} B_j$. In particular, $F^\sigma(V) = \bigcap_{j \in J} F^\sigma(B_j)$ for all $\sigma \in \mathbb{R}$. Hence Proposition A.3 implies that

$$\| \cdot \|_{H^\sigma(V)} = \sup_{j \in I} \| \cdot \|_{H^\sigma(B_j)}. \tag{A.3}$$

As indicated above, Theorem A.2 shows A.2 if $V$ is a ball. The general case follows from A.3.

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