How to aggregate Top-lists: Approximation algorithms via scores and average ranks

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Abstract
A top-list is a possibly incomplete ranking of elements: only a subset of the elements are ranked, with all unranked elements tied for last. Top-list aggregation, a generalization of the well-known rank aggregation problem, takes as input a collection of top-lists and aggregates them into a single complete ranking, aiming to minimize the number of upsets (pairs ranked in opposite order in the input and in the output). In this paper, we give simple approximation algorithms for top-list aggregation.

• We generalize the footrule algorithm for rank aggregation (which minimizes Spearman’s footrule distance), yielding a simple 2-approximation algorithm for top-list aggregation.

• Ailon’s RepeatChoice algorithm for bucket-orders aggregation yields a 2-approximation algorithm for top-list aggregation. Using inspiration from approval voting, we define the score of an element as the frequency with which it is ranked, i.e., appears in an input top-list. We reinterpret RepeatChoice for top-list aggregation as a randomized algorithm using variables whose expectations correspond to score and to the average rank of an element given that it is ranked.

• Using average ranks, we generalize and analyze Borda’s algorithm for rank aggregation. We observe that the natural generalization is not a constant approximation.

• We design a simple 2-phase variant of the Generalized Borda’s algorithm, roughly sorting by scores and breaking ties by average ranks, yielding another simple constant-approximation algorithm for top-list aggregation.

• We then design another 2-phase variant in which in order to break ties we use, as a black box, the Mathieu-Schudy PTAS for rank aggregation, yielding a PTAS for top-list aggregation. This solves an open problem posed by Ailon.

• Finally, in the special case in which all input lists have length at most k, we design another simple 2-phase algorithm based on sorting by scores, and prove that it is an EPTAS – the complexity is \( O(n \log n) \) when \( k = o(\log n) \).

1 Introduction
1.1 Context. Rank aggregation is a classical problem in combinatorial optimization, where the goal is to take elements from a ground set (candidates) and find a ranking which is “closest” to a set of input rankings (voting profile). Rank aggregation comes up in machine learning [CSS98], natural language processing [Li14], bio-informatics [LWX17], and is relevant in the field of information retrieval (meta search and spam reduction [DKNS01], similarity search [FKS03c] and more). Historically, rank aggregation was first studied in social choice theory, where the underlying properties of a ranking method are of critical importance [Bor81, dCS85, Arr51]. In this paper, we use terminology (candidates, votes, voting profile, ...) derived from social choice theory.

1.2 Rank aggregation. There are several ways to measure how close the output ranking is to the input rankings: the most popular is Kendall’s tau distance, that has several satisfying structural properties [Kem59, Kem62, YL78]. In this paper, we focus on Kendall’s tau distance, that counts the number of pairs of candidates that are ranked in reverse order in the two rankings. Rank aggregation is NP-hard [BTT89, DKNS01], but one of the simplest randomized algorithm yields a constant factor approximation: algorithm RANDOM simply takes a random input ranking and declares it to be the output. Many other constant factor approximation algorithms are known: the Footrule algorithm [DKNS01]; the randomized KwikSort algorithm and variants [ACN05, ACN08], derandomized in [vZW09]; Borda’s method [CFR10]; Copeland’s method, the median rank algorithm and more [FKM+16]. There is even a polynomial-time approximation scheme [MS07, MS09], but work on constant factor approximations nevertheless continued: they are popular because of their simplicity. Experimental studies can be found e.g., in [CW09] (algorithms inspired by standard sorting algorithms, and local search algorithms); [SZ09] (Footrule, Markov chain algorithms, sorting algorithms, local search algorithms, hybrid algorithms, and more); [AM12] (additionally includes exact LP-based and branch-and-bound algorithms as well as various heuristics, with a focus on social choice theory).
1.3 From full-ranking to top-lists. A meta search engine aggregates information from different search engines to answer users’ requests with a ranked selection of web-pages. In such settings, a useful extension of the rank aggregation problem is to deal with incomplete data, where each vote provides, not a full-ranking of all candidates, but an ordered selection of his preferred candidates, the remaining ones being implicitly tied at the end. Such a partial ranking is called a top-list. Incomplete rankings were studied in [DKNS01], but for the most part without the assumption that candidates that do not appear are implicitly ranked after the candidates that appear in the list, so the input model is different. Top-list aggregation also comes up in bio-informatics: “In previous research, attention has been focused on aggregating full lists. However, partial and/or top ranked lists are prevalent because of the great heterogeneity of genomic studies and limited resources for follow-up investigation.” [LWX17]. There is an extensive discussion about distances between top-lists in [FKS03a, FKS03b] (and more generally between bucket-orders in [FKM+04, FKM+06]), and they propose several aggregation problems. Experimental studies (for a different, related objective) can be found for example in [CSS98] for top-30-lists.

1.4 Top-list aggregation. In this paper we study the top-list aggregation problem (Top-Agg) that takes top-lists as input. The goal is to find an full-ranking that minimizes the average distance to a top-list from the input. We use a natural generalization of Kendall’s tau distance: we still count the number of pairs of candidates that are ranked in reverse order in the two rankings, without counting pairs of candidates that are tied in one ranking. This problem was defined by Ailon in [Ail07, Ail10], where he showed that Top-Agg is an NP-Hard even if each top-list rank exactly two candidates. Some approximation algorithm for full-ranking aggregation extend to top-list aggregation: Algorithm RepeatChoice (Ailon’s generalization of algorithm Random) is a 2-approximation. Algorithm KwikSort (introduced in [ACN05, ACN08], improved in [Ail07, Ail10], and determinized in [vZW09]) also extends to top-list aggregation, and one of its variants yields a 3/2 approximation algorithm: candidates are ranked using a quick-sort like approach and a randomized rounding of the relaxation of an integer-LP.

1.5 Our results. We study whether other approximation algorithms for rank aggregation can be extended to aggregate top-lists. The Footrule Algorithm [DKNS01] is an intuitive way to aggregate full-rankings into a single full-ranking, using Spearman’s footrule distance (which approximates Kendall’s tau distance, and is much easier to minimize). We use a natural generalization of the footrule distance: the generalized footrule distance between two partial orders is the minimum footrule distance between their linear extensions (similar distance are discussed in [FKM+06]). This enables us to extend the result from [DKNS01].

Theorem 1.1. Algorithm Footrule+ is a 2-approximation for Top-Agg. Its running time is linear in the size of the input, and cubic in the number of candidates.

Since Ailon’s work [Ail10], a simple 2-approximation for Top-Agg was already known: algorithm RepeatChoice is a randomized algorithm that was designed in the more general setting of bucket-orders. A close-up look at RepeatChoice in the context of Top-Agg reveals that it can be reinterpreted to use random variables whose averages are related to the scores and average ranks of the candidates. The score of a candidate is the frequency with which he is ranked in the input, and his average rank is the average value of his rank when he is ranked.

Theorem 1.2. Algorithm RandomSort (the specialization of Algorithm RepeatChoice from [Ail10] to Top-Agg) is a randomized 2-approximation algorithm. Its running time is quasi-linear in the size of the input.

In the context of full-ranking, sorting candidates by average rank is precisely Borda’s voting method [Bor81], another simple and popular algorithm, known to be a 5-approximation [CFR10]. This leads us to analyze the generalization of Borda’s algorithm to Top-Agg, where a preprocessing step eliminates all candidates with zero scores such that the average rank of each candidate is well defined.

Theorem 1.3. Algorithm Borda+ is a \((4\alpha + 2)\)-approximation algorithm for Top-Agg, where \(\alpha\) is the ratio between the maximum and minimum scores of candidates (assuming that all scores are non-zero). Its running time is quasi-linear in the size of the input.

Unfortunately, Algorithm Borda+ is not an \(O(1)\)-approximation for Top-Agg (see Section 5 for a counterexample). This indicates that the score of a candidate is of primary importance. We therefore design
a slightly less simple, but still elementary 2-phase algorithm, that first roughly sorts by scores, putting into buckets candidates that have similar scores, then refines the ordering using average rank, yielding a new constant factor approximation.

**Theorem 1.4.** Algorithm **Score-then-Borda** is a randomized (8\(\varepsilon + 4\))-approximation algorithm, which only uses the scores and average ranks of the candidates. Its running time quasi-linear in the number of candidates.

The proof of Theorem 6.1 relies on a critical lemma (Lemma 6.1) proving that there exists a near-optimal full-ranking that respects the rough ordering by scores. As for aggregation ranking, not only are there \(O(1)\)-approximations, but there also exists a polynomial time approximation scheme [MS09]. Building on the intuition acquired so far, it is now easy to generalize that result, designing a 2-phase algorithm, that first roughly sorts by scores, then refines the ordering using the full-ranking aggregation PTAS, yielding a PTAS for Top-Agg.

**Theorem 1.5.** (PTAS for Top-Agg) For all fixed \(\varepsilon > 0\), Algorithm **Score-then-PTAS** is a randomized \((1 + \varepsilon)\)-approximation algorithm for Top-Agg. Its time complexity is \(O\left(1 + \varepsilon\right)\cdot n^3\log n + n \exp(\exp(O(1)))\), the algorithm can be derandomized with an additional cost of \(n \exp(\exp(O(1)))\).

To summarize, several criteria come into play when designing an algorithm for Top-Agg: approximation, running time, and simplicity, and there is a trade-off between those. Our contribution is to explore the spectrum of existing approximation algorithms for rank aggregation, and generalize them to map out possible approximation algorithms for Top-Agg.

Now, remembering our initial motivations (applications to information retrieval), one might notice that in some practical cases the total number of candidates is several order of magnitude above the number of candidates ranked in each input top-list. So far, we have focused on simplicity and on quality of approximation. For the Top-k-Agg problem (when each input top-lists ranks \(k\) candidates), we can actually get both at the same time: we design a very simple algorithm that is an efficient PTAS. Intuitively, when \(k\) is constant, for candidates that will be ranked quite far, the average rank matters little and the score is most important. Therefore sorting by score produces a near-optimal ranking, except for the first few candidates, hence Algorithm **Score-then-Adjust**.

**Theorem 1.6.** (EPTAS for Top-k-Agg) For all fixed \(\varepsilon > 0\), Algorithm **Score-then-Adjust** is a \((1 + \varepsilon)\)-approximation algorithm for Top-k-Agg. Its time complexity is \(O(n \log n + m \cdot 2^m)\), with \(m := \lceil (1 + \frac{1}{2})(k - 1) \rceil\).

Thus, in addition to a variety of simple approximation algorithms, we provide two approximation scheme, both solving an open problem stated in [Ail10].

### 1.6 Bucket orders.

A further generalization of rank aggregation is obtained by letting the input consist of bucket-orders (where a bucket-order is an ordered partition of candidates into equivalence classes). This was considered by [FKM+06] and studied by Ailon [Ail10] who gave two approximation algorithms: RepeatChoice and KwikSort. The scenery of potential constant factor approximations for the problem still remains to be done, and we leave the existence of an approximation scheme for that generalization as an outstanding open problem.

### 2 Definitions

Let \([n] := \{1, \ldots, n\}\) be the set of candidates.

**Definition 2.1.** (Full-ranking, Top-k-List, Top-list)

Let \(k \in [n]\). A top-k-list \(\pi\), to each candidate \(i \in [n]\), assigns a rank \(\pi_i \in [k] \cup \{\infty\}\) such that there is exactly one candidate of each rank \(1, 2, \ldots, k\). A top-list is a top-k-list for some \(k\).

The set of top-k-lists is denoted \(T_n^k\) and the set of top-lists is denoted \(T_n\).

For \(k = n\), a top-n-list is also called a full-ranking and the set is denoted \(S_n\) (also called the set of permutations over \([n]\)).

A top candidate is a candidate \(i\) such that \(\pi_i < \infty\).

For example, if we have \(n = 8\) candidates and \(k = 3\) ranks with the gold, silver and bronze medals given to candidates 2, 5 and 1 respectively, the corresponding top-3-list is written \(\pi = [2, 5, 1; \ldots]\). This top-list can be represented as in Figure 1, with candidates listed by order of rank, and candidates with rank \(\infty\) listed in arbitrary order.

\[
\begin{align*}
\pi_1 &= 3 & \pi_5 &= 3 & \pi_4 &= 2 & \pi_6 &= 3 & \pi_7 &= \infty & \pi_8 &= \infty \\
2 & 5 & 1 & 3 & 4 & 6 & 7 & 8
\end{align*}
\]

Figure 1: Representation of a top-3-list \(\pi = [2, 5, 1; \ldots]\)
Definition 2.2. (Kendall’s tau distance) The generalized Kendall’s tau distance \( K(\sigma, \pi) \) between a full-ranking \( \sigma \) and a top-list \( \pi \) is the number of pairs of candidates that are ranked in reverse order in \( \sigma \) and in \( \pi \), i.e.

\[
K(\sigma, \pi) := \sum_{i \in [n]} \sum_{j \in [n]} 1_{\sigma_i < \sigma_j} \cdot 1_{\pi_i > \pi_j},
\]

where \( 1_p \) denotes the indicator function. When no candidate has rank \( \infty \) (i.e. when \( \pi \) is a full-ranking), this definition coincides with Kendall’s tau distance between two full-rankings.

A pair \( \{i, j\} \) of candidates that are tied in \( \pi \) does not contribute to \( K(\sigma, \pi) \). Thus, considering the full-ranking \( \tau \) which is a linear extension of \( \pi \) where ties are broken according to \( \sigma \), the generalized Kendall’s tau distance between \( \sigma \) and \( \pi \) is exactly Kendall’s tau distance between \( \sigma \) and \( \tau \). (We note that this is different from the distances discussed in [FKS03a, FKS03b, FKM04], where breaking ties incurs a non-zero cost.)

![Figure 2: Representation of the generalized Kendall’s tau distance between \( \pi \) and \( \sigma \). Here \( K(\sigma, \pi) = 8 \), and the eight pairs that contribute to the cost are materialized by the eight circles.](image)

Using the graphical representation of a top-list where candidates with rank \( \infty \) are listed using their order in \( \sigma \), we can represent \( K(\sigma, \pi) \) as in Figure 2. Each candidate \( i \) is associated to a line segment connecting the position of \( i \) in the representation of \( \pi \) and of \( \sigma \), and each crossing pair \( \{i, j\} \) that contributes towards \( K(\sigma, \pi) \) is marked by a small circle at the intersection of the two corresponding line segments.

Definition 2.3. (Voting profile) A voting profile is a distribution \( p \) over top-lists. The distance between a full-ranking \( \sigma \) and a voting profile \( p \) is the average distance between \( \sigma \) and a top-list sampled from \( p \).

\[
K(\sigma, p) := \sum_{\pi \in \mathcal{T}_n} p(\pi) \cdot K(\sigma, \pi) = \sum_{i \in [n]} \sum_{j \in [n]} 1_{\sigma_i < \sigma_j} \cdot p(\pi_i < \pi_j)
\]

We denote by \( p(E) = \sum_{\pi \in E} p(\pi) \) the probability of an event \( E \subseteq \mathcal{T}_n \). We also use the notation \( p(\text{Property on } \pi) := p(\{\pi \in \mathcal{T}_n \mid \text{Property on } \pi\}) \).

Equivalently, the reader may consider that a voting profile is a set of top-lists with weights. The size of a voting profile is the sum of sizes of the top-lists in its support.

Definition 2.4. (Top-Agg problem) The top-list aggregation problem Top-Agg takes as input a set of candidates \( [n] \) and a voting profile \( p \), and outputs a full-ranking \( \sigma \) of the \( n \) candidates. The goal is to minimize the distance \( K(\sigma, p) \): the weighted average value of the generalized Kendall’s tau distance between \( \sigma \) and a top-list \( \pi \) from \( p \).

Let \( p \) be a voting profile such that:

\[
\pi_1 = [3, 5, 1, 7; \ldots] \quad p(\pi_1) = 1/10
\]
\[
\pi_2 = [3, 1, 4, 5; \ldots] \quad p(\pi_2) = 2/10
\]
\[
\pi_3 = [4, 1, 5, 2; \ldots] \quad p(\pi_3) = 3/10
\]
\[
\pi_4 = [6, 1, 2, 3; \ldots] \quad p(\pi_4) = 4/10
\]

The optimal solution is \( \sigma^* = [1, 2, 3, 4, 5, 6, 7, 8] \):

\[
K(\sigma^*, p) = \frac{1}{10} \cdot 8 + \frac{2}{10} \cdot 4 + \frac{3}{10} \cdot 5 + \frac{4}{10} \cdot 5 = 5.1
\]

Figure 3: Example of instance of Top-\(k\)-Agg with \( k = 4 \) and \( n = 8 \).

In Figure 3 we give an instance of Top-Agg that will be reused in the next sections. Observe that \( K(\sigma, \pi_1) = 8 \) is represented in Figure 2. The optimal solution ranks candidate 1 first, since he is preferred to every other candidate (this property is known as Condorcet’s criterion).

In the input top-lists of Figure 3, observe that candidate 8 is never a top candidate, so the optimal solution ranks it last. In the upcoming algorithms, we are often going to assume without loss of generality that no such candidates exist, since they may be eliminated in a preprocessing step.

3 Generalized footrule algorithm

Spearman’s footrule distance between two full-rankings \( \sigma \) and \( \tau \) is the sum of displacement of each candidate: \( F(\sigma, \tau) = \sum_{i=1}^{n} |\sigma_i - \tau_i| \). Diaconis and Graham showed in [DG77] that distances \( K \) and \( F \) are always within a constant factor of each other: \( K(\sigma, \tau) \leq F(\sigma, \tau) \leq 2K(\sigma, \tau) \). Thus, approximating with respect to one distance also yields an approximation with respect to the other distance.

Dwor, Kumar, Naor and Sivakumar noticed this fact in [DKNS01], and proved that minimizing \( F \) can be done in polynomial; which yields a 2-approximation for
full-ranking aggregation with $K$. The algorithm computes the cost induced by ranking candidate $i$ at rank $j$, then uses a minimum-cost-perfect-matching algorithm to assign candidates to ranks. Algorithm Footrule+ is a generalization of the approach.

Using Algorithm Footrule+ on the instance from Figure 3, we obtain a full-ranking $\sigma = [4, 1, 2, 3, 5, 6, 7, 8]$ which is at a distance $K(\sigma, p) = 5.8$ from $p$. Observe that candidate 1 is ranked second instead of first, which would have been optimal with respect to $K$.

**Theorem 3.1.** Algorithm Footrule+ is a 2-approximation for Top-Agg. Its running time is linear in the size of the input, and cubic in the number of candidates.

**Proof.** Let $(n, p)$ be an instance of Top-Agg, and let $\sigma$ be the output of Algorithm Footrule+.

To define a generalized version of Spearman’s footrule between a full-ranking $\sigma$ and a top-list $\pi$, we use the linear extension $\tau$ of $\pi$ in which ties are broken according to $\sigma$: $F(\sigma, \pi) := F(\sigma, \tau) = \sum_{i=1}^{n} |\sigma_i - \tau_i|$. As noticed in section 2, we also have $K(\sigma, \pi) = K(\sigma, \tau)$. Thus the property of full-rankings from [DG77] still holds for $\pi$ a top-list: $K(\sigma, \pi) \leq F(\sigma, \pi) \leq 2K(\sigma, \pi)$. Letting $F(\sigma, p) := \sum_{\pi} p(\pi) F(\sigma, \pi)$ we have: $K(\sigma, p) \leq F(\sigma, p) \leq 2F(\sigma, p)$.

![Figure 4: Representation of the generalized Spearman’s footrule distance between $\pi$ and $\sigma$. Here $F(\sigma, \pi) = 16$, and the three candidates that contribute to the cost are materialized by three arrows.](image)

To generalize the footrule algorithm from [DKNS01], we need to express $F(\sigma, \pi)$ as a sum over $i \in [n]$ of the cost of putting candidate $i$ at rank $\sigma_i$. We first notice that the sum of displacements in one direction is equal to the sum of displacements in the other, thus $F(\sigma, \tau) = 2\sum_{i=1}^{n} (\sigma_i - \tau_i) \cdot 1_{\tau_i < \sigma_i}$. Note that if $\pi_i < \infty$ then $\tau_i = \pi_i$; and if $\pi_i = \infty$, then $\tau_i \geq \sigma_i$. Thus $1_{\tau_i < \sigma_i} = 1_{\sigma_i, < \tau_i}$ and $F(\sigma, \pi) = 2\sum_{i=1}^{n} (\sigma_i - \pi_i) \cdot 1_{\sigma_i, < \tau_i}$.

**Hence:**

$$F(\sigma, p) = \sum_{\pi \in [n]} p(\pi) F(\sigma, \pi) = 2 \sum_{i \in [n]} \sum_{\pi \in [n]} p(\pi) \cdot (\sigma_i - \pi_i) \cdot 1_{\pi_i < \sigma_i} = 2 \sum_{i \in [n]} \sum_{k=1}^{n} p(\pi_i = k) \cdot (\sigma_i - k)$$

Because of that, Algorithm Footrule+ is able to optimize $F(\sigma, p)$ by solving a min-cost-perfect-matching problem. For any full-ranking $\sigma^*$ we have $F(\sigma, p) \leq F(\sigma^*, p) \leq 2K(\sigma^*, p)$. Hence, Algorithm Footrule+ is a 2-approximation for Top-Agg. The time complexity is the time complexity of the Hungarian algorithm, which computes a minimum-weight-perfect-matching.

**4 Scores and average ranks**

In this section we introduce the scores and average ranks of candidates. Those two parameters are central to the problem of top-list aggregation.

When aggregating full-rankings, it is folklore that outputting a full-ranking randomly sampled from the input gives an expected 2-approximation. Ailon generalized this into design algorithm RepeatChoice [Ail10], which is a 2-approximation in the more general setting of bucket-order aggregation. Algorithm RandomSort below is algorithm RepeatChoice specialized to Top-Agg and reinterpreted using exponential random variables.

For example, if we take the instance from Figure 3, Algorithm RandomSort randomly orders the top-lists $\pi_1$, $\pi_2$, $\pi_3$ and $\pi_4$, by sorting top-lists by increasing order of their values $X_\pi$. With probability $4/35 = 3/10 \cdot 4/7 \cdot 2/3$, the ordering is $x_{\pi_4} < x_{\pi_3} < x_{\pi_2} < x_{\pi_1}$. Observe that sorting candidates (by the values of their tuples) is equivalent to processing the top-lists in order, appending candidates sequentially: from $\pi_4$, we append candidates 4, 1, 5, 2; then from $\pi_4$ we append candidates 6, 3; then from $\pi_2$ we append no candidate; then from $\pi_1$ we append candidate 7; then we append 8 who is the only remaining candidate. The resulting full-ranking $\sigma = [4, 1, 5, 2, 6, 3, 7, 8]$ is at a distance $K(\sigma, p) = 5.9$ from $p$. This algorithm is a 2-approximation, but
Algorithm RandomSort

Input: instance (\(n, p\)) of Top-Agg
For each top-lists \(\pi\) of the voting profile \(p\):
\(\) Draw a real value \(X_\pi\) from an exponential distribution of parameter \(p(\pi)\).
For each candidate \(i\) in \([n]\):
\(\) Consider tuples \((X_\pi, \pi_i)\) with \(\pi\) such that \(i\) is a top candidate.
\(\) Choose \(t_i\) to be the one with smallest value of \(X_\pi\).
Build a full ranking \(\sigma\), sorting the candidates using the lexicographical order over the \(t_i\)'s.
Output \(\sigma\).

Observe that candidate 1 is never ranked first (even thought that would have been optimal).

Theorem 4.1. Algorithm RandomSort (the specialization of Algorithm RepeatChoice from [Ai10] to Top-Agg) is a randomized 2-approximation algorithm. Its running time is quasi-linear in the size of the input.

Proof. The time complexity is studied in the standard randomized real RAM model. Let \(i\) and \(j\) be two distinct candidates, each appearing at least once as a top candidate in the input voting profile. We compute the probability (over the values of the \(X_\pi\)) that \(\sigma_i > \sigma_j\). Let \(I = \min\{X_\pi : \pi_i < \pi_j\}\) and \(J = \min\{X_\pi : \pi_i < \pi_i\}\) be the minimum values of the exponential random variables over the sets of top-lists which respectively prefers \(i\) to \(j\) and \(j\) to \(i\). Observe that \(\sigma_i > \sigma_j\) if and only if \(I > J\). As the minimum of several exponential random variables is an exponential random variable with a parameter equal to the sum of parameters, \(I\) and \(J\) are two independent exponential random variables of parameters \(p(\pi_i < \pi_j)\) and \(p(\pi_j < \pi_i)\). Thus the probability that \(\sigma_i > \sigma_j\) is \(\mathbb{P}(\sigma_i > \sigma_j) = p(\pi_i > \pi_j)/p(\pi_i \neq \pi_j)\). We now compute the expected cost of the output \(\sigma\).

\[
\mathbb{E}(K(\sigma, p)) = \sum_{(i, j) \in [n]^2} p(\pi_i < \pi_j) \cdot \mathbb{E}(1_{\sigma_i > \sigma_j})
\leq 2 \sum_{(i, j) \in [n]} \min\left\{ p(\pi_i < \pi_j), p(\pi_i > \pi_j) \right\}
\]

Let \(\sigma^*\) denote the optimal solution. For all distinct \(i, j \in [n]\), \(\sigma^*\) must rank \(i\) before \(j\) or \(j\) before \(i\), which costs at least the minimum between \(p(\pi_i < \pi_j)\) and \(p(\pi_i > \pi_j)\). Therefore \(\mathbb{E}(K(\sigma, p)) \leq 2K(\sigma^*, p)\). \(\square\)

Observe that in Algorithm RandomSort for any candidate \(i\), the expected value of his tuple can be computed easily. Indeed, the first coordinate of his tuple is the minimum of several exponential random variables (all \(X_\pi\) such that \(i\) is a top candidate in \(\pi\)); thus it is an exponential random variable whose parameter is \(p(\pi_i < \infty)\). As for the second coordinate, we can easily compute the probability that an exponential random variable \(X_\pi\) is smaller than all the exponential random variables of top-lists having \(i\) as a top candidate: this probability is \(p(\pi)/p(\pi_i < \infty)\), which directly gives the expected value of the second coordinate.

\[
\mathbb{E}(t_i) = \left(\frac{1}{p(\pi_i < \infty)}, \sum_{r=1}^n \frac{p(\pi_i = r)}{p(\pi_i < \infty)} \cdot r\right)
\]

From this observation we define the score and average rank of a candidate. The score is known in the literature as the approval score under a voting profile that ignores the ordering between top candidates (in the setting where, instead of ranking top candidates, each voter gives a subset of approved candidates).

Definition 4.1. (Score, Average rank) Given a voting profile \(p\), the score of a candidate \(i \in [n]\) is the probability that she is a top candidate: \(Score_i := p(\pi_i < \infty)\). Assuming that each candidate appears at least once as a top candidate in the input, the average rank of a candidate is her expected rank, conditioning on her being a top candidate:

\[
Rank_i := \sum_{r=1}^n \frac{p(\pi_i = r)}{Score_i} \cdot r.
\]

Figure 5: Scores and average ranks of candidates in the instance from Figure 3.

5 Generalized Borda’s algorithm

In this section, we draw inspiration from two noteworthy papers that study the approximation ratios of simple algorithms for full-ranking aggregation. In [CFR10], Coppersmith, Fleischer and Rudra proved that Borda’s method is a 5-approximation. In [FKM+16], Fagin, Kumar, Mahdian, Sivakumar and Vee designed a general framework to prove constant factor approximation bounds.

In Borda’s method for full-ranking aggregation, a candidate ranked in \(r\)-th position by an input ranking
Algorithm \textbf{Borda+} (Generalization of Borda) \vspace{1mm}

**Input:** instance \((n, p)\) of TOP-AGG
For each candidate \(i \in [n]\), compute \(\text{Rank}_i \leftarrow \sum_{r=1}^{n} \frac{p(\pi_i=r)}{p(\pi_i<\infty)} \cdot r\)
Sort candidates by increasing value of \(\text{Rank}_i\).

**Output** the resulting full-ranking \(\alpha\).

gets \(n - r\) points, and then candidates are sorted by total number of points. This is equivalent to sorting candidates by increasing average ranks. Thus Algorithm \textbf{Borda+} can be seen as a generalization of Borda’s method to TOP-AGG, where the analysis uses insights from [CFR10] to bound the approximation ratio when the scores of candidates are within a constant factor of each other.

Let us give an example of execution of Algorithm \textbf{Borda+}, using the instance from Figure 3. (Candidate 8 never appears as a top candidate so it is ranked last in the output). Sorting the instance from Figure 3 by average ranks produces the full-ranking \([6, 4, 1, 3, 5, 2, 7, 8]\), which is at distance \(K(\sigma, p) = 6.3\) from \(p\).

**Theorem 5.1.** Algorithm \textbf{Borda+} is a \((4\alpha + 2)\)-approximation algorithm for TOP-AGG, where \(\alpha\) is the ratio between the maximum and minimum scores of candidates (assuming that all scores are non-zero). Its running time is quasi-linear in the size of the input.

**Proof.** Let \((n, p)\) be an instance of TOP-AGG, \(\sigma\) be the output of Algorithm \textbf{Borda+}, and let \(\sigma^*\) be the full-ranking minimizing \(K(\sigma^*, p)\). We define \(F(\sigma, p)\) as in the proof of Theorem 3.1. To simplify notations, we also define the positive part function \(x \mapsto x^+ = x \cdot \mathbb{1}_{x > 0}\).

\[
F(\sigma, p) = 2 \sum_{i=1}^{n} \sum_{k=1}^{n} p(\pi_i = k) \cdot (\sigma_i - k)^+
\]

We have the triangle inequality: for all \(x, y \in \mathbb{R}\), \((x+y)^+ \leq x^+ + y^+\). Thus for all \(i, r\) we have \((\sigma_i - r)^+ \leq (\sigma_i^* - r)^+ + (\sigma_i - \text{Rank}_i)^+ + (\text{Rank}_i - \sigma_i^*)^+\). Recalling that \(Score_i = \sum_{r=1}^{n} p(\pi_i = r)\), we obtain an upper bound on \(F(\sigma, p)\):

\[
F(\sigma, p) \leq F(\sigma^*, p) + 2 \sum_{i=1}^{n} Score_i \cdot (\sigma_i - \text{Rank}_i)^+
+ 2 \sum_{i=1}^{n} Score_i \cdot (\text{Rank}_i - \sigma_i^*)^+
\]

One can prove (e.g. Lemma 3.5 from [CFR10]) that sorting by increasing average rank minimizes \(\sum_{i=1}^{n} (\sigma_i - \text{Rank}_i)\). Using the fact that the scores are all within a factor \(\alpha\) of each other, we have:

\[
\sum_{i=1}^{n} Score_i \cdot (\sigma_i - \text{Rank}_i)^+
\leq \left(\max_{i \in [n]} Score_i\right) \cdot \sum_{i=1}^{n} (\sigma_i - \text{Rank}_i)^+
\leq \alpha \sum_{i=1}^{n} (\sigma_i^* - \text{Rank}_i)^+
\]

Using this inequality to bound \(F(\sigma, p)\), we obtain:

\[
F(\sigma, p) \leq F(\sigma^*, p) + 2 \alpha \sum_{i=1}^{n} Score_i \cdot |\sigma_i^* - \text{Rank}_i|
\]

For all \(i \in [n]\), we use the convexity of \(x \mapsto |\sigma_i^* - x|\) and the definition \(\text{Rank}_i = \sum_{k=1}^{n} \frac{p(\pi_i=k)}{Score_i} \cdot k\).

\[
\sum_{i=1}^{n} Score_i \cdot |\sigma_i^* - \text{Rank}_i| \leq \sum_{i=1}^{n} \sum_{k=1}^{n} p(\pi_i = k) \cdot |\sigma_i^* - k|
\leq F(\sigma^*, p)
\]

Combining the last two inequalities, we obtain \(F(\sigma, p) \leq (1 + 2\alpha)F(\sigma^*, p)\). Using the relation between \(F\) and \(K\), we conclude with \(K(\sigma, p) \leq F(\sigma, p) \leq (1 + 2\alpha)F(\sigma^*, p) \leq (2 + 4\alpha)K(\sigma^*, p)\).

**Tightness.** We notice that \textbf{Borda+} is an \(\Omega(\alpha)\) approximation in the worst case: let \(n = 2\) and let \(p\) such that \(p([1; \ldots]) = 0.999\) and \(p([2; 1; \ldots]) = 0.001\); the optimal solution is \([1, 2]\) and costs 0.001 whereas sorting by average ranks produces \([2, 1]\) which costs 0.999. Thus, in general, \textbf{Borda+} is not an \(O(1)\)-approximation algorithm.

Observe that sorting by decreasing scores is not a \(O(1)\)-approximation algorithm either: let \(n = 2\) and let \(p\) such that \(p([1; 2; \ldots]) = 0.999\) and \(p([2; 2; \ldots]) = 0.001\); the optimal solution is \([1, 2]\) and costs 0.001 whereas sorting by scores produces \([2, 1]\) which costs 0.999.

However, in the next section we show that sorting first by decreasing scores, then by increasing average ranks, yields an \(O(1)\)-approximation algorithm
Algorithm Score-then-Borda+

Input: an instance \((n, p)\) of Top-AGG

Step 1, partition candidates into intervals:

\[ u \leftarrow \text{uniformly random value on } [0, 1). \]

for all candidate \(i \in [n]\) do

Compute \(Score_i \leftarrow p(\pi_i < \infty)\).
Set \(t \leftarrow \lfloor u - \ln(Score_i) \rfloor\) and put candidate \(i\) in interval \(E_t\).

Step 2, solve the problem in each interval:

for all \(t \in \mathbb{N} \cup \{\infty\}\) such that \(E_t\) is non-empty do

Order \(E_t\) sorting candidates \(i\) by average rank \(Rank_i \leftarrow \sum_{r=1}^{n} \frac{p(\pi_r \in [\pi_i, \infty))}{p(\pi_r < \infty)} \cdot r\).

Concatenate the ranking of \(E_0\), ranking of \(E_1, \ldots\), and ranking of \(E_{\infty}\).

Output resulting full-ranking.

6 Combining approval and Borda’s methods

In the previous section, we saw that when all scores are within a constant factor of each other, then sorting by average rank yields a constant factor approximation. In this section we argue that we can always do an approximate sort of the candidates using rough scores, and then obtain a constant factor approximation. This statement is made more precise in Lemma 6.1, and used in Theorem 6.1 to prove that Algorithm Score-then-Borda+ is a constant factor approximation.

Let us give an example of the execution of Algorithm Score-then-Borda+, using the instance from Figure 3. In the first step, we sample a random value \(u\) from \([0, 1)\), for example \(u = 0.4\), and use this value to define thresholds on the scores:

- a candidate \(i\) such that \(0.55 \approx \exp(u - 1) \leq Score_i\) will go in interval \(E_0\);
- a candidate \(i\) such that \(0.20 \approx \exp(u - 2) \leq Score_i \leq \exp(u - 1) \approx 0.55\) will go in interval \(E_1\);
- a candidate \(i\) such that \(0.07 \approx \exp(u - 3) \leq Score_i \leq \exp(u - 2) \approx 0.20\) will go in interval \(E_2\);
- and a candidate \(i\) such that \(Score_i = 0\) will go in interval \(E_{\infty}\).

At the end of the first step we have \(E_0 = \{1, 2, 3, 5\}\), \(E_1 = \{4, 6\}\), \(E_2 = \{7\}\) and \(E_{\infty} = \{8\}\). In the second step, we reorder candidates by increasing average ranks: the ordering of \(E_1\) is \([1, 3, 5, 2]\); the ordering of \(E_1\) is \([6, 4]\); the ordering of \(E_2\) is \([7]\); the ordering of \(E_{\infty}\) is \([8]\). Finally, we concatenate the rankings of \(E_0, E_1, E_2\) and \(E_{\infty}\). We obtain a full-ranking \(\sigma = [1, 3, 5, 2, 6, 4, 7, 8]\) which is at a distance \(K(\sigma, p) = 5.8\) from \(p\).

Theorem 6.1. Algorithm Score-then-Borda+ is a randomized \((8\epsilon + 4)\)-approximation algorithm, which only uses the scores and average ranks of the candidates. Its running time quasi-linear in the number of candidates.

Lemma 6.1. Consider a constant \(\eta > 0\) and an instance \((n, p)\) of Top-AGG. Sample a random variable \(u\) uniformly at random from \([0, 1)\). Define a partition function \(f : s \mapsto [u - \eta \ln(s)]\). A full-ranking \(\sigma\) respects the partition if for any two candidates \(i, j\), having \(f(Score_i) < f(Score_j)\) implies that \(\sigma_i < \sigma_j\). The expected cost of the best full-ranking that respects the partition is at most \((1 + \eta)\) times the cost of the overall optimal full-ranking.

Proof. For all \(t \in \mathbb{N}\) we define \(E_t\) to be the set of candidates that are sent in the \(t\)-th interval by the partition function. Let \(\sigma^*\) be an optimal solution and let \(\sigma'\) be the full-ranking which is closest to \(\sigma^*\) and respects the partition. More precisely, for all \(t \in \mathbb{N}\), the full-ranking \(\sigma'\) induces an ordering of the candidates from \(E_t\); we build \(\sigma'\) as a concatenation of those rankings. The cost of the best full-ranking that respects the partition is smaller than \(K(\sigma', p)\). From the definition of cost, we have:

\[
K(\sigma', p) - K(\sigma^*, p) = \sum_{i \in [n]} \sum_{j \in [n]} 1_{\sigma'_i > \sigma'_j} \cdot 1_{\sigma'_i < \sigma'_j} \cdot (p(\pi_i < \pi_j) - p(\pi_j < \pi_i))
\]

Let \(i, j\) be two candidates such that \(\sigma^*_i < \sigma^*_j\). Observe that having \(\sigma'_i > \sigma'_j\) implies that \(Score_{\sigma'_i} < Score_{\sigma'_j}\); thus we assume the later. We are going to compute the probability (over the randomness \(u\)) that \(\sigma'_i > \sigma'_j\). Candidates \(i\) and \(j\) are not in the same interval if and only if

\[
\exists t \in \mathbb{N}, \quad t + \eta \ln(Score_i) \leq u < t + \eta \ln(Score_j)
\]

This happens with probability at most \(\eta \ln(Score_j / Score_i) \leq \eta (Score_j / Score_i - 1)\).
Hence:

\[
\mathbb{E}_u[K(\sigma', p)] - K(\sigma^*, p) \\
\leq \sum_{i \in [n]} \sum_{j \in [n]} 1_{\sigma_i' < \sigma_j'} \cdot \mathbb{E}_u[1_{\sigma_i' > \sigma_j'} \cdot \text{Score}_i] \\
\leq \eta \sum_{i \in [n]} \sum_{j \in [n]} 1_{\sigma_i' < \sigma_j'} \cdot (\text{Score}_j - \text{Score}_i) + \\
\leq \eta K(\sigma^*, p)
\]

Observe that \(\text{Score}_j - \text{Score}_i\) is a lower bound on the weight of top-lists for which \(j\) is a top candidate but \(i\) is not. We recognize a lower bound on the cost of \(\sigma^*\), thus \(\mathbb{E}_u[K(\sigma', p)] \leq (1 + \eta)K(\sigma^*, p)\).

Proof. (Theorem 6.1) Let \((n, p)\) be an instance of Top-Agg, let \(\sigma^*\) be an optimal solution and let \(\sigma\) be the output of Algorithm \text{SCORE-THEN-BORDA+}. The proof of this theorem is in two parts, corresponding to the two steps of the algorithm.

Firstly, let \(u\) be the random variable sampled during the first step, and let \(\sigma'\) be the best full-ranking that respects the partition. From Lemma 6.1 with \(\eta = 1\), we have \(\mathbb{E}_u[K(\sigma', p)] \leq 2K(\sigma^*, p)\).

Secondly, we reuse the proof of Theorem 5.1, with some additional details: every full-ranking that we consider needs to respect the partition (hence we replace every instance of \(\sigma^*\) by \(\sigma'\)). On every interval, the ratio between the largest and smallest score is upper-bounded by \(\alpha = \epsilon\); thus we have \(K(\sigma, p) \leq (4\epsilon + 2)K(\sigma', p)\).

Combining both parts, we obtain that Algorithm \text{SCORE-THEN-BORDA+} is a randomized \((8\epsilon + 4)\) approximation. Note that we did not try to optimize the approximation ratio.

7 PTAS for top-list aggregation

In the case of full-ranking aggregation, [MS07, MS09] show that there is a PTAS. The approximation scheme with the best running time is algorithm \text{FASTER-Scheme} from [MS09]. Rephrasing it to the setting of top-list aggregation, it requires that all candidates are compared a similar number of times (Theorem 7.2 makes this statement more precise). We notice that this condition is equivalent with having all the scores within a constant factor of each other; therefore we can use the techniques from the previous section.

At a high level, Algorithm \text{SCORE-THEN-PTAS} starts by fixing thresholds on the scores of candidates (exactly as Algorithm \text{SCORE-THEN-BORDA+} does), to partition candidates into intervals. Then it uses \text{FASTER-Scheme} as a black-box, to find a nearly-optimal solution on each interval. We show in Theorem 7.1 that Algorithm \text{SCORE-THEN-PTAS} is a PTAS for top-list aggregation.

Let us give an example of execution of Algorithm \text{SCORE-THEN-PTAS} with \(\epsilon = 3\), using the instance from Figure 3. In the first step, we sample a random value \(u\) from \([0,1]\), for example \(u = 0.4\). Observe that we chose \(\epsilon\) and \(u\) such that the partition in interval is the same as in Algorithm \text{SCORE-THEN-BORDA+}: \(E_0 = \{1, 2, 3, 5\}, E_1 = \{4, 6\}, E_2 = \{7\}\) and \(E_\infty = \{8\}\).

In the second step, we find an approximate solution of the optimal ordering of every non-empty interval. For \(E_0\), we build the restriction \(p_0\) of the voting profile \(p\) on candidates from \(E_0\); here \(\pi_1 = [3, 5, 1; \ldots]\), \(\pi_2 = [3, 1, 5; \ldots]\), \(\pi_3 = [1, 5, 2; \ldots]\) and \(\pi_4 = [1, 2, 3; \ldots]\).

Then we use the algorithm \text{FASTER-Scheme} to find a \((1 + \epsilon/3)\) approximation of the optimal solution for \(p_0\), and get (for example) the ranking \([1, 2, 3, 5]\). We do the same for \(E_1\), \(E_2\) and \(E_\infty\) and get (for example) the rankings \([4, 6], [7]\) and \([8]\). Finally, we concatenate the rankings of \(E_0, E_1, E_2\) and \(E_\infty\). We obtain a full-ranking \(\sigma = [1, 2, 3, 5, 4, 6, 7, 8]\) which is a distance \(K(\sigma, p) = 5.5\) from \(p\).

Theorem 7.1. (PTAS for Top-Agg) For all fixed \(\epsilon > 0\), Algorithm \text{SCORE-THEN-PTAS} is a randomized \((1 + \epsilon)\)-approximation algorithm for Top-Agg. Its time complexity is \(O\left(\frac{1}{\epsilon} \cdot n^3 \log n + n \exp(\exp(O(\frac{1}{\epsilon})))\right)\), the algorithm can be derandomized with an additional cost of \(\exp(\log n \exp(\exp(O(\frac{1}{\epsilon})))\).

Theorem 7.2. (from [MS09] Theorem 1.2) Let \(b \in (0, 1]\) be a parameter. There exists a randomized polynomial time approximation scheme (called \text{FASTER-Scheme} in [MS09]) for the special case of Top-Agg such that the input \((n, p)\) satisfies

\[
\min_{i, j \in [n], i \neq j} p(\pi_i \neq \pi_j) \geq b \cdot \max_{i, j \in [n], i \neq j} p(\pi_i \neq \pi_j)
\]

The running time\(^3\) is \(O((\log(\frac{1}{b}) + \frac{1}{\epsilon}) \cdot n^3 \log n) + n \exp(\exp(O(\frac{1}{\epsilon}))\) is added to the running time.

Proof. (Theorem 7.1) Let \(0 < \epsilon \leq 3\), let \((n, p)\) be an instance of Top-Agg, let \(\sigma^*\) be an optimal solution, and let \(\sigma\) be the output of Algorithm \text{SCORE-THEN-PTAS} on \((n, p)\) with error parameter \(\epsilon\). The proof of this theorem is in two parts, corresponding to the two steps of Algorithm \text{SCORE-THEN-PTAS}.

Firstly, we prove that there exists a ranking \(\sigma'\) whose expected cost (over the randomness of \(u\)) is nearly-optimal and that respects the partition \((E_i)\), in the sense that all candidates of \(E_i\) precede all candidates

\(^3\text{Recall that } f(x) = \tilde{O}(g(x)) \text{ if there is a constant } \ell \text{ such that } f(x) = O(g(x) \log^\ell(g(x)))\)
Algorithm Score-then-PTAS with error parameter $\varepsilon > 0$.

**Input:** an instance $(n, p)$ of Top-AGG

**Step 1.** Partition candidates into intervals:

- $u \leftarrow$ uniformly random value on $[0, 1)$.
- for all candidate $i \in [n]$ do
  - Compute $\text{Score}_i \leftarrow p(\pi_i < \infty)$.
  - Set $t \leftarrow \lceil u - (\varepsilon/3) \ln(\text{Score}_i) \rceil$ and put candidate $i$ in interval $E_t$.

**Step 2.** Solve the problem in each interval:

- for all $t \in \mathbb{N} \cup \{\infty\}$ such that $E_t$ is non-empty do
  - $p_t \leftarrow$ restriction of input top-lists to $E_t$.
  - Order $E_t$ using FASTER-Scheme [MS09] on instance $p_t$ with error parameter $\varepsilon/3$.

Concatenate the ranking of $E_0$, ranking of $E_1$, . . . , and ranking of $E_\infty$.

**Output** resulting full-ranking.

8 EPTAS for top-$k$-list aggregation.

In this section we study Top-$k$-AGG, the special case of Top-AGG when all top-lists in the input voting profile have exactly $k$ top candidates. The main result of this section is Theorem 8.1, which proves that Algorithm Score-then-Adjust is an EPTAS.

Let us give an example of execution of Algorithm Score-then-Adjust with $n = 8$, $k = 4$ and $\varepsilon = 3$, using the instance from Figure 3. First, we sort candidates by non-increasing scores, which (for example) gives us the full-ranking $\sigma^* = [1, 3, 2, 5, 4, 6, 7, 8]$. Then consider the first $m = \lceil (1 + \frac{1}{6})(k - 1) \rceil$ = 4 candidates of $\sigma^*$, who are $1, 3, 2$ and $5$. To compute the cost of a reordering of those candidates, we just need to consider a restricted instance on those $m$ candidates: here $\pi_1 = [3, 5, 1; \ldots]$, $\pi_2 = [3, 1, 5; \ldots]$, $\pi_3 = [1, 5, 2; \ldots]$ and $\pi_4 = [1, 2, 3; \ldots]$. To find the optimal solution, one can enumerate the $m!$ possible full-rankings, or use a dynamic programming approach and compute the optimal ordering of each of the $2^m$ subsets of $\{1, 3, 2, 5\}$. With both methods we find that the optimal reordering is $[1, 2, 3, 5]$. Hence we have $\sigma = [1, 2, 3, 5, 4, 6, 7, 8]$, which is at distance $K(\sigma, p) = 5.5$ from $p$.

**Theorem 8.1.** (EPTAS for Top-$k$-AGG) For all fixed $\varepsilon > 0$, Algorithm Score-then-Adjust is a $(1 + \varepsilon)$-approximation algorithm for Top-$k$-AGG. Its time complexity is $O(n \log n + m \cdot 2^m)$, with $m := \lceil (1 + \frac{1}{6})(k - 1) \rceil$.

We begin the analysis with a simple observation. Consider a full-ranking $\sigma$ and let $i$ such that $\sigma_i = n$. If an input top-$k$-list $\pi$ ranks $i$ among its top $k$ elements, then at least $n - k$ pairs $\{i, j\}$ are ranked in reverse order in $\pi$ and in $\sigma$, so $K(\sigma, \pi) \geq n - k$. Thus

---

3 In this new instance, top-lists of the voting profile does not necessarily have all the same size.

5 One could use a polynomial time approximation algorithm here, but the resulting algorithm will not be a PTAS.
This concludes the proof.

**Algorithm Score-then-Adjust**

**Input:** Instance \((n, p)\) of Top-\(k\)-Agg.
- For each candidate \(i \in [n]\), compute \(\text{Score}_i \leftarrow p(\pi_i < \infty)\).
- Define \(\sigma^*\), the full-ranking obtained by sorting candidates by non-increasing scores.
- By permuting the first \([1 + \frac{1}{e}](k - 1)\] candidates of \(\sigma^*\), choose \(\sigma\) which minimizes \(K(\sigma, p)\).

**Output:** \(\sigma\).

\[
K(\sigma, p) \geq (n - k) \cdot p(\pi_i < \infty) = (n - k) \cdot \text{Score}(i).
\]
This observation can be generalized, leading to the statement of Lemma 8.1.

**Lemma 8.1.** Let \((n, p)\) be an instance of Top-\(k\)-Agg.
For any full-ranking \(\sigma^*\), we have a lower bound on the objective function: \(K(\sigma^*, p) \geq \sum_{i : \sigma^*_i \geq k} (\sigma^*_i - k) \cdot \text{Score}_i\).

**Proof.** We write

\[
K(\sigma^*, p) = \sum_{i \in [n]} \sum_{\pi \in \mathbb{T}^k_n} \left\{ j \in [n] \mid \pi_j < \pi_i \right\} \cdot p(\pi)
\]

Let \(i \in [n]\) be a candidate and \(\pi \in \mathbb{T}^k_n\) be a top-list in which \(i\) is a top candidate:

\[
\left| \left\{ j \mid \pi_j < \pi_i \right\} \right| = \left| \left\{ j \mid \sigma^*_j < \sigma^*_i \right\} \right| - \left| \left\{ j \mid \pi_j \leq \pi_i \right\} \right| \geq \sigma^*_i - k.
\]

Thus, summing only over \(i \in [n]\) such that \(\sigma^*_i > k\) and over \(\pi \in \mathbb{T}^k_n\) for which \(i\) is a top candidate:

\[
K(\sigma^*, p) \geq \sum_{i : \sigma^*_i \geq k} \sum_{\pi} (\sigma^*_i - k) \cdot p(\pi) \cdot \mathbb{1}_{\pi_i < \infty}
\]

\[
\geq \sum_{i : \sigma^*_i \geq k} (\sigma^*_i - k) \cdot \text{Score}_i
\]

This concludes the proof.

**Proof.** (Theorem 8.1) Let \(\sigma\) be a full-ranking and let \(\pi\) a top-list. The distance between \(\sigma\) and \(\pi\) can be split into two: \(K(\sigma, \pi) = K_{top}(\sigma, \pi) + K_{score}(\sigma, \pi)\).

\[
K_{top}(\sigma, \pi) := \sum_{i \in [n]} \sum_{j \in [n]} \mathbb{1}_{\sigma_j > \sigma_i} \cdot \mathbb{1}_{\pi_i < \pi_j < \infty}
\]

\[
K_{score}(\sigma, \pi) := \sum_{i \in [n]} \sum_{j \in [n]} \mathbb{1}_{\sigma_j > \sigma_i} \cdot \mathbb{1}_{\pi_i, \pi_j = \infty}
\]

The value \(K_{top}(\sigma, \pi)\) is the number of inversions of top candidates of \(\pi\), between \(\sigma\) and \(\pi\); whereas \(K_{score}(\sigma, \pi)\) can be seen as the distance between \(\sigma\) and a bucket-order with two buckets: a partial order where all candidates of the first bucket (top candidates of \(\pi\)) are ranked before candidates of the second bucket.

We now define \(K_{top}(\sigma, p)\) and \(K_{score}(\sigma, p)\) as weighted averages of \(K_{top}(\sigma, \pi)\) and \(K_{score}(\sigma, \pi)\), over all top-lists \(\pi\) of the voting profile \(p\). We have \(K(\sigma, p) = K_{top}(\sigma, p) + K_{score}(\sigma, p)\). One can observe that sorting candidates by decreasing score actually minimizes \(K_{score}(\sigma, p)\); this result is proved in Theorem 3 of [All10]. More precisely, whenever there are two candidates \(i\) and \(j\) such that \(\sigma_i = \sigma_j + 1\) and \(\text{Score}_i > \text{Score}_j\), the pair \((i, j)\) costs \(p(\pi_i < \pi_j = \infty)\) in \(K_{score}(\sigma, p)\).

However, by definition of score:

\[
0 < \text{Score}_i - \text{Score}_j = p(\pi_j < \pi_i < \infty) + p(\pi_i < \pi_j < \infty) + p(\pi_i < \pi_j = \infty) - p(\pi_i < \pi_j < \infty) - p(\pi_j < \pi_i < \infty) - p(\pi_j < \pi_i = \infty)
\]

Therefore swapping candidates \(i\) and \(j\) strictly decreases \(K_{score}(\sigma, p)\).

Let \(m := \lceil (1 + \frac{1}{e})(k - 1) \rceil\). Let \(\sigma''\) denote the full-ranking obtained from \(\sigma'\) by reordering the first \(m\) candidates according to their relative order in the (unknown) optimal order \(\sigma^*\). The algorithm outputs a full-ranking \(\sigma\) such that \(K(\sigma, p) \leq K(\sigma'', p)\).

**Figure 6:** Graphical representation of full-rankings \(\sigma^*\), \(\sigma''\) and \(\sigma'\). Elements from \(S\) are represented with light circles.

Letting \(S\) denote the set of candidates of rank greater than \(m\) in \(\sigma'\), observe that \(\sigma''\) can also be defined from \(\sigma^*\) by doing a partial bubble sort, repeatedly swapping adjacent elements whenever their scores are out of order and at least one of the two is in \(S\); thus \(K_{score}(\sigma'', p) \leq K_{score}(\sigma^*, p)\). Moreover, as \(\sigma''\) and \(\sigma^*\) can disagree on the relative order of two candidates only if at least one of the two is in \(S\):

\[
K_{top}(\sigma'', p) - K_{top}(\sigma', p) = \sum_{i \in [n]} \mathbb{1}_{\sigma''_i > \sigma'_i} \cdot \left( p(\pi_i < \pi_j < \infty) - p(\pi_j < \pi_i < \infty) \right) < p(\pi_i < \pi_j < \infty) = \sum_{s \in S} \sum_{t \neq s} p(\pi_s < \infty \text{ and } \pi_t < \infty).
\]
Since the input consists of top-$k$-lists\footnote{Actually the Theorem also holds when the input top-lists have ties among the top $k$ candidates; the only thing that matters is that at least $n-k$ candidates are such that $\pi(i) = \infty$.}, for all $s \in S$ we have
\[
\sum_{t: t \neq s} p(\pi_s < \infty \text{ and } \pi_t < \infty) = \sum_{t: t \neq s} p(\pi_s < \infty) \cdot p(\pi_t < \infty | \pi_s < \infty) \leq (k - 1) \cdot p(\pi_s < \infty) = (k - 1) \cdot \text{Score}_s.
\]
Thus $K_{top}(\sigma'', p) - K_{top}(\sigma^*, p) \leq (k - 1) \sum_{s \in S} \text{Score}_s$. Since $S$ is also the set of $n-m$ elements with the smallest scores, we have $\sum_{s \in S} \text{Score}_s \leq \sum_{i: \sigma_i > m} \text{Score}_i$. In summary, we proved that:
\[
K(\sigma'', p) - K(\sigma^*, p) \leq (k - 1) \sum_{i: \sigma_i > m} \text{Score}_i
\]
Applying Lemma 8.1, with the fact that we always have $m \geq k$,
\[
\sum_{i: \sigma_i > m} \text{Score}_i \leq \sum_{i: \sigma_i > m} \frac{(\sigma_i - k) \cdot \text{Score}_i}{m + 1 - k} \leq \frac{K(\sigma^*, p)}{m + 1 - k}
\]
Recalling that $m = \lceil (1 + \frac{1}{\varepsilon})(k - 1) \rceil \geq k - 1 + (k - 1)/\varepsilon$
we finally obtain
\[
K(\sigma'', p) \leq \left(1 + \frac{k - 1}{m + 1 - k}\right) \cdot K(\sigma^*, p).
\]
To achieve the claimed running time, we see that computing $\sigma'$ takes time $O(n \log n)$. To find the optimal reordering of the first $m$ candidates, we first precompute the values of $p(\pi_i < \pi_j)$ for all $i, j$ such that $\sigma_i' \leq m$ and $\sigma_j' \leq m$. Then, we use dynamic programming: for each subset of the first $m$ candidates of $\sigma'$ we try all possibilities for the candidate that will be ranked first, and store this candidate together with the cost of the associated solution. Time complexity is $O(m \cdot 2^m)$. Space complexity is exponential; if someone needs to be memory efficient the exhaustive search approach might be preferable. 

We remark that the running time of Algorithm Score-then-Adjust is quasi-linear as long as $k = o(\log n)$, and polynomial as long as $k = \Theta(\log n)$.

References

- Nir Ailon, Moses Charikar, and Alantha Newman. Aggregating inconsistent information: ranking and clustering. In Proceedings of the 37th Annual ACM Symposium on Theory of Computing (STOC). ACM, 2005.
- Nir Ailon, Moses Charikar, and Alantha Newman. Aggregating inconsistent information: ranking and clustering. Journal of the ACM (JACM), 55(5), 2008.
- Nir Ailon. Aggregation of partial rankings, p-ratings and top-m lists. In Proceedings of the 18th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA). SIAM, 2007.
- Ahur Ali and Marina Meilă. Experiments with Kemeny ranking: What works when? Mathematical Social Sciences, 64(1), 2012.
- Kenneth J. Arrow. Social Choice and Individual Values. 1951.
- Jean-Charles de Borda. Mémoire sur les élections au scrutin. 1781.
- John Bartholdi, Craig A. Tovey, and Michael A. Trick. Voting schemes for which it can be difficult to tell who won the election. Social Choice and Welfare, 6(2), 1989.
- Don Coppersmith, Lisa K. Fleischer, and Atri Rurda. Ordering by weighted number of the 37th Annual ACM Symposium on Theory of Computing (STOC). ACM, 2005.
- William W Cohen, Robert E Schapire, and Yoram Singer. Learning to order things. In Advances in Neural Information Processing Systems (NIPS), 1998.
- Tom Coleman and Anthony Wirth. Ranking tournaments: Local search and a new algorithm. ACM Journal of Experimental Algorithmics (JEA), 14, 2009.
- Marquis de Condorcet. Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix. 1785.
Persi Diaconis and Ronald L. Graham. Spearman’s footrule as a measure of disarray. *Journal of the Royal Statistical Society. Series B (Methodological)*, 1977.

Cynthia Dwork, Ravi Kumar, Moni Naor, and D. Sivakumar. Rank aggregation methods for the web. In *Proceedings of the 10th International Conference on World Wide Web (WWW)*. ACM, 2001.

Ronald Fagin, Ravi Kumar, Mohammad Mahdian, D. Sivakumar, and Erik Vee. Comparing and aggregating rankings with ties. In *Proceedings of the 23rd ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems (PODS)*. ACM, 2004.

Ronald Fagin, Ravi Kumar, Mohammad Mahdian, D. Sivakumar, and Erik Vee. Comparing partial rankings. *SIAM Journal on Discrete Mathematics (SIDMA)*, 20(3), 2006.

Ronald Fagin, Ravi Kumar, Mohammad Mahdian, D. Sivakumar, and Erik Vee. An algorithmic view of voting. *SIAM Journal on Discrete Mathematics (SIDMA)*, 30(4), 2016.

Ronald Fagin, Ravi Kumar, and D. Sivakumar. Comparing top k lists. In *Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2003.

Ronald Fagin, Ravi Kumar, and D. Sivakumar. Comparing top k lists. *SIAM Journal on Discrete Mathematics (SIDMA)*, 17(1), 2003.

Ronald Fagin, Ravi Kumar, and Dandapani Sivakumar. Efficient similarity search and classification via rank aggregation. In *Proceedings of the 2003 International Conference on Management of data (MOD)*. ACM, 2003.

John G. Kemeny. Mathematics without numbers. *Daedalus*, 88(4), 1959.

John G. Kemeny. *Mathematical Models in the Social Sciences*. 1962.

Hang Li. Learning to rank for information retrieval and natural language processing. *Synthesis Lectures on Human Language Technologies*, 7(3), 2014.

Xue Li, Xinlei Wang, and Guanghua Xiao. A comparative study of rank aggregation methods for partial and top ranked lists in genomic applications. *Briefings in bioinformatics*, 2017.

Claire Kenyon Mathieu and Warren Schudy. How to rank with few errors. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC)*. ACM, 2007.

Claire Mathieu and Warren Schudy. How to rank with fewer errors. 2009. Unpublished, http://cs.brown.edu/people/wschudy/papers/fast_journal.pdf.

Frans Schalekamp and Anke van Zuylen. Rank aggregation: together we’re strong. In *Proceedings of the 11th Workshop on Algorithm Engineering and Experiments (ALENEX)*. SIAM, 2009.

Anke van Zuylen and David P Williamson. Deterministic pivoting algorithms for constrained ranking and clustering problems. *Mathematics of Operations Research*, 34(3), 2009.

Hobart Peyton Young and Arthur Levenglick. A consistent extension of Condorcet’s election principle. *SIAM Journal on Applied Mathematics (SIAP)*, 35(2), 1978.