1. INTRODUCTION

$B$ and $D$ meson hadronic matrix elements play an important role in determinations of the CKM matrix elements and overconstraining the unitarity triangle of the Standard Model. Form factors, decay constants and bag parameters are determined from these hadronic matrix elements which can be calculated directly on the lattice. $B$ factories (BaBar and Belle) and Charm factories (CLEO-c) will reduce experimental uncertainties in measurements of decay process, leaving the theoretical calculations as the dominant uncertainty. Although the focus of lattice heavy quark calculations has traditionally been on $B$ physics, new experiments such as CLEO-c will determine $f_D(s)$ and semileptonic branching fractions to a few percent, testing lattice results for these quantities.

In this talk I will report on progress made in the last year in calculations of heavy quark quantities. Many quantities have now been determined by different groups using different heavy quark methods. Progress in determinations of the $B$ parameters, $B_{B_{d,s}(q)}$, which together with $f_{B_{d,s}}$ are relevant to determinations of $|V_{td}|$ and $|V_{ts}|$ and to $B_s$ width and $b$ hadron lifetime measurements, is then reviewed. Turning to semileptonic decays, I will focus on $B \to \pi l \nu$ and a new method to determine the zero-recoil form factor of $B \to D^* l \nu$. A new calculation of the $b$ quark mass and an outstanding puzzle in lattice calculations of spin splittings are briefly discussed.

1.1. Heavy quark methods

The calculations described here employ three different heavy quark methods, each of which has different systematic errors. The UKQCD and APE groups use a nonperturbatively $O(a)$-improved Sheikholeslami-Wohlert (SW) action (NPSW) at quark masses about that of charm. Results are extrapolated to the $b$ quark guided by heavy quark effective theory (HQET). A large source of uncertainty comes from this extrapolation. It is well known that the nonperturbative $O(a)$-improvement scheme breaks down in the limit $m_Q \to \infty$. This was discussed last year by Bernard [1] and an explicit example was given at this conference by Kurth and Sommer [2]. They have shown that the continuum limit ($a \to 0$) must be taken before an extrapolation in heavy mass is attempted.

In nonrelativistic QCD (NRQCD) [3] the heavy quark is assumed to be nonrelativistic ($v/c \approx (0.3\, \text{GeV}/5.0\, \text{GeV})^2$). Relativistic momenta are excluded by introducing a finite cut-off such that $p \approx m_Q v \ll m_Q$. Then, $|p|/m_Q \ll 1$ and the QCD Lagrangian can be expanded in powers of $1/m_Q$. This has been successful for $B$ physics, where the quark mass is large, so the expansion is convergent. The theory is nonrenormalisable but at finite lattice spacing, lattice artefacts can be removed by including higher orders in $1/m_Q$ and $a$.

The Fermilab approach (FNAL) [4], as used in current heavy quark calculations, is a reinterpretation of the SW action which identifies and correctly renormalises nonrelativistic operators present in the SW action. Discretisation errors are then $O(a\Lambda_{QCD})$ and not $O(am_Q)$. The
existence of a continuum limit means a continuum extrapolation is possible. An implementation of the full approach, for $O(a)$-improvement, has been done by the Fermilab group [3].

2. LEPTONIC DECAY CONSTANTS

The $B$ meson decay constant is an input in determinations of the CKM matrix element $|V_{td}|$ through $B^0_d - \bar{B}^0_d$ mixing. The mixing, $\Delta m_d$, is known quite precisely and the dominant uncertainty comes from $f_B\sqrt{B_{B_d}}$.

2.1. The quenched approximation

A compilation of recent determinations of $f_B$ and $f_D$ is shown in Figs 1 and 2. The calculations are ordered by the heavy quark action used. The solid bands indicate my average for the particular heavy quark action used and values of $f_B$ obtained for a given heavy quark action can be seen to be in good agreement. In addition there is a clear overlap between results from the different heavy quark actions.

The most recent calculations of $f_D$ in Fig 2 also show there is broad general agreement between groups and between the two treatments of heavy quarks.

In 2001 the CP-PACS and UKQCD collaborations both reported updates to their quenched determinations of $f_B$, in Refs [13] and [16,15] respectively. In this CP-PACS calculation an NRQCD action, corrected to $O(1/M)$, is used and $f_B$ is determined in both the quenched approximation and using unquenched gauge configurations. The unquenched result is unchanged and is included in Table 1. The central value in the quenched approximation is also unchanged from that discussed at Lattice’00 by Bernard [1]. However, the systematic error has been revised upwards to reflect the observed large discretisation effects. The estimate of the continuum value is now $f_B = 191 \pm 4 \pm 27$ MeV and $f_{B_s} = 220 \pm 4 \pm 31$ MeV, where the first error is statistical and the second is the uncertainty due to discretisation. There are additional uncertainties in excess of 30% from setting the lattice scale and 3% in $f_{B_s}$ from ambiguities in determining the strange quark mass.

Although the central value is in reasonable agreement with results from other groups such large discretisation errors have not been observed previously. This scaling analysis is done using fairly coarse lattices, $1.017\text{GeV} \leq a_{\pi}^{-1} \leq 1.743\text{GeV}$ but, nonetheless, both GLOK and JLQCD who also use an NRQCD action corrected
to $O(1/M)$ and take operator mixing fully into account, as in the CP-PACS calculation, see no such scaling violations. The GLOK group were the first to show that including the $O(a, \alpha_s)$ discretisation correction to the heavy-light axial current leads to a significant decrease in scaling violations of $f_B$. Indeed both GLOK and JLQCD find quite mild dependence on the lattice spacing, in contrast to CP-PACS. The calculations differ in the choice of gauge action. CP-PACS use an Iwasaki RGI action which although it has been shown to have small scaling violations, the decay constants seem to be an exception.

It is important to understand the reason for the observed scaling violations, especially in a quenched calculation, where a motivation for the approximation is that it allows us to control and understand other systematic uncertainties.

The systematic errors in the calculation of decay constants by Lellouch and Lin in Ref [16] have been revised to include the effects of using different physical quantities to determine $a_m$. The changes involved are very small.

The results of the UKQCD Collaboration, reported in 1999 in Ref [19] and discussed last year in Ref [20] have been substantially revised. The new analysis was described by Maynard at this conference [8]. UKQCD determine $f_B$ using a non-perturbatively $O(a)$-improved action, extrapolated from charm to the bottom quark mass. The first change is in the current and mass improvement and renormalisation coefficients. A consistent set of nonperturbative coefficients, as determined by Bhattacharya et al. [14] is used. Secondly, the scale is set by the pion decay constant rather than $\rho_0$ as previously. This is a better choice since $\rho_0$ is poorly determined experimentally. Finally, an extended discussion and analysis of the systematic errors and in particular the error in the heavy mass extrapolation is included.

The mass-dependent normalisations proposed in the Fermilab formalism [4] and by Bernard [2] are compared to the nonperturbative normalisation which is used to determine the central value. The effect of $O(a^2 m_Q^2)$ lattice artefacts at the charm scale on the extrapolation to the bottom quark mass is studied by simultaneously fitting data from two lattice spacings to

$$
\Phi(M, a) = \gamma \left( 1 + \frac{\delta}{M} + \frac{\eta}{M^2} + \epsilon (aM)^2 + \xi (aM)^3 \right).
$$

This is the usual HQET scaling relation with additional mass-dependent terms. From the fit a so-called “quasi-continuum” result is determined and compared to fits at individual lattice spacings, the difference being taken as a measure of the effect of $O(a^2 m_Q^2)$ lattice artefacts. While the method proposed here is a reasonable one, the size of the error envelope at the $B$ meson mass from a fit to Eqn [1] would be a more realistic estimate of the uncertainty in this extrapolation. This would almost certainly be larger than the error currently quoted by UKQCD.

I conclude this section with a description of a calculation by Davies et al. [21] of $f_B$ at rest and at non-zero momentum, using NRQCD. The motivation for such a study comes from the need to understand momentum-dependent errors in semileptonic decays, for which $f_B$ offers an easy place to start. The matrix element $\langle 0 | A_\mu | B(\rho) \rangle = f_B p_\mu$ is studied with temporal, $A_0$ and spatial, $A_k$ currents. A number of different smearings are investigated and narrow smearing of heavy quarks is found to be optimal for moving $B$ mesons. Constrained curve fitting [22] gives reliable results at momenta up to $(p_\mu)^2 = 16$, in units $(2\pi/L)$ and good agreement is found for $f_B$ at zero and non-zero momentum. Finally, the discrepancy discussed in Ref [23] between $f_B$ from $A_0$ and $A_k$ is resolved by including a correct power counting and appropriate normalisation of the different $A_\mu$ in the matrix element.

### 2.2. Unquenching

The last year has seen some new calculations of decay constants with dynamical quarks. These and other recent calculations are summarised in Table [3]. As in the quenched case these calculations are performed with a number of heavy quark actions. I will discuss the results marked as “new” in Table [3]. Details of the other calculations can be found in the references quoted and in Ref [4].

Yamada [24], for the JLQCD Collaboration, presented preliminary results from a calculation...
Table 1

Leptonic decay constants with $N_f = 2$. A * indicates a new result.

| Group                | $f_B$ (MeV) | $f_B^{N_f=2}$/$f_B^{N_f=0}$ | $f_D$ (MeV) | $f_D^{N_f=2}$/$f_D^{N_f=0}$ |
|----------------------|-------------|-----------------------------|-------------|-----------------------------|
| Collins 99           | 186(5)(25)(+$0$) | $1.11(6)$                   | 215(5)(+$17$)(+$8$) | $1.03(6)$                   |
| MILC’00              | 191(6)(+$24$)(+$11$) | $1.03(6)$                   | 225(14)(40) | $1.03(6)$                   |
| MILC’01* ($N_f = 2 + 1$) | 190(14)(7) | $1.23(11)$                   | 225(14)(40) | $1.03(6)$                   |
| CP-PACS’00(FNAL)     | 208(10)(29) | $1.11(6)$                   | 225(14)(40) | $1.03(6)$                   |
| CP-PACS’00(NR)       | 204(8)(29) | $1.10(5)$                   | 225(14)(40) | $1.03(6)$                   |
| JLQCD*               | 190(14)(7) | $1.14$                       | 225(14)(40) | $1.03(6)$                   |

of $f_{B(a)}$ using NRQCD, including all $1/M$ corrections. A nonperturbatively $O(a)$-improved Wilson action for the sea and valence light quarks and a plaquette gauge action are used. They compare values of $f_B$ in the quenched approximation using a perturbative (tadpole-improved) and nonperturbative value of $c_{SW}$. The difference is found to be negligible. Using nonperturbative improvement coefficients, the quenched value of $f_B$ is compared to the two flavour result ($N_f = 2$). $f_B$ is larger in the unquenched theory but the size of the difference depends on the type of valence chiral extrapolation used. Linear, quadratic and log fits were compared and the latter two yield good $\chi^2/N_{df}$. The log fits are motivated by chiral perturbation theory and to my knowledge this fit form has not been explored by other groups, for this particular quantity. The results from JLQCD indicate a careful analysis of the chiral behaviour is warranted. For their preliminary result, in Table 1 the central value is an average of the quadratic and log fits and the second error is the difference from the linear extrapolation. They are gathering more statistics for a detailed study of the chiral extrapolation. The group also presented results for the $B$ parameters which I will discuss below.

The “MILC’00” result was discussed in some detail in Ref [3]. I refer the reader there for more details and turn to the newest result from MILC. Preliminary results for $f_B$ and ratios were presented at this conference by Bernard. An improved Kogut-Susskind action (“Asqtad”) with $N_f = 2 + 1$ was used. For heavy quarks they use a tadpole-improved SW action with the Fermilab nonrelativistic interpretation. The dependence of $f_B$ on dynamical $(m_\pi/m_\rho)^2$ is investigated with constant and linear fits. The valence chiral extrapolation is seen to be well controlled with the difference between linear and quadratic fits being quite small, in contrast to the JLQCD result. Since both calculations are preliminary, definite conclusions are premature and further investigation is required. The currents have not been normalised and so I do not include the value of $f_B$ in Table 1. In a ratio of decay constants the normalisation factors largely cancel, as do many systematic uncertainties. MILC find a mild dependence on both $N_f$ and $a^{-1}$ in the ratio $f_B/f_B$.

Unquenched simulations (with $N_f = 2$) of $D$ meson decay constants have been done by CP-PACS and MILC as reported in Table 1. Both groups use the Fermilab nonrelativistic interpretation for the heavy quarks and an improved Wilson action for light quarks, although with different gauge actions. The results agree within the quoted uncertainties. The results at three lattice spacings do not show scaling and CP-PACS note that $f_D$ may in fact be smaller than their reported value due to discretisation effects. They are able to conclude however that the dynamical effect for $D$ mesons is smaller than for $B$ mesons. This is supported by the MILC data.

It is a difficult task to combine results given the very different systematic errors in different calculations. Nonetheless, I believe the estimates below represent a summary of the current status.

Quenched

|                | $f_B$ (MeV) | $f_B$ (MeV) |
|----------------|-------------|-------------|
| $f_B$          | 173(23)     | 198(30)     |
| $f_{B_0}$      | 200(20)     | 230(30)     |
| $f_D$          | 203(14)     | 226(15)     |
| $f_{D_0}$      | 230(14)     | 250(30)     |

Unquenched

|                | $f_B$ (MeV) | $f_B$ (MeV) |
|----------------|-------------|-------------|
| $f_B$          | 173(23)     | 198(30)     |
| $f_{B_0}$      | 200(20)     | 230(30)     |
| $f_D$          | 203(14)     | 226(15)     |
| $f_{D_0}$      | 230(14)     | 250(30)     |
\[ f_{B_s}/f_B = 1.15(3) \quad f_{B_s}/f_B = 1.16(5) \]
\[ f_{D_s}/f_D = 1.12(2) \quad f_{D_s}/f_D = 1.12(4) \]

Of these, only \( f_{D_s} \) has been measured experimentally. A recent average is \( f_{D_s} = 280 \pm 48 \) MeV \[28\], in agreement with the average above.

3. \textbf{B PARAMETERS}

In comparison to \( f_B \) lattice calculations of \( B \) parameters have not received as much attention. There is however, progress to report this year.

A new calculation of \( \Delta B = 0 \) operators by APE is described in Ref \[29\]. These matrix elements are important in lifetime ratios of \( B \) mesons. Reyes \[30\] presented details of a quenched calculation by the APE collaboration of the matrix elements of \( \Delta B = 2 \) operators. A more detailed description has recently appeared in Ref \[31\]. Results from a relativistic simulation (at the charm quark mass) are combined with results in the static limit and interpolated to the \( b \) quark mass. The operators in the relativistic and static theories are matched at NLO, including the anomalous dimension matrix up to two loops and the one-loop matching for the \( \Delta B = 2 \) operators. This is the first time such a consistent matching has been done and since the \( \Delta B = 2 \) operators are determined by interpolation rather than extrapolation the systematic error is significantly reduced. The result is \( B_{B_s}(m_b) = 0.87(4)(3)(0) \left( \pm \frac{3}{4} \right) \), where the errors are statistical, systematic errors in the static calculation, error in the renormalisation in QCD and a combined error due to the uncertainty in \( a^{-1} \) and in the improvement coefficient of the axial current. Not included here is a full estimate of the discretisation error although it is discussed in the text. The operators are not \( O(a) \)-improved and the static and SW calculations are done at different lattice spacings. The discretisation error due to the unimproved current is included in the third error bar, any remaining \( a \)-dependence is difficult to estimate with just one lattice spacing.

JLQCD \[25\] use NRQCD determine \( B \) parameters from the quenched and unquenched data sets, already described in Section \[2.2\]. They find a negligible difference in the two results. The matrix elements are renormalised at the one-loop level in perturbation theory. The light quark mass dependence is mild and no significant dependence on the choice of fit function is observed. The dependence on the quantity used to set the scale and the determination of the strange quark mass are included in the error budget. Their result, with \( N_f = 2 \) is \( B_{B_s}(m_h) = 0.87(4)(3)(0) \). The errors are statistical, the difference from four methods to determine \( B_B \) and the chiral extrapolation. Power counting leads to an additional 8% systematic error. Sharpe has estimated \[32\] the effect of the quenched approximation, from quenched chiral perturbation theory, to be \( \approx 10\% \). This calculation suggests that this 10% may be an upper bound on the effect of unquenching but as this is a preliminary result I think it is best to leave the 10% uncertainty and await the final result and subsequent confirmation by other groups.

Taking these two new (albeit preliminary) results into account I estimate the renormalisation group invariant \( B \) parameters to be

\[ \hat{B}_{B_s} = 1.30(12)(<10\%) \]
\[ \hat{B}_{B_s} = 1.34(10)(<10\%) \]
\[ \hat{B}_{B_s}/B_{B_s} = 1.03(3) \]
\[ \xi = f_{B_s} \sqrt{\hat{B}_{B_s}}/f_B \sqrt{\hat{B}_{B_s}} = 1.15(6) \left( \pm \frac{7}{6} \right) \]

The second, asymmetric error on \( \xi \) takes into account the preliminary result from JLQCD. This suggests that in the unquenched calculation of \( f_B \) chiral logs are important in the chiral extrapolation. \( f_{B_s} \) is unaffected so the value of \( \xi \) in future may be larger than presently quoted.

4. \textbf{SEMIlectedonic DECAYs}

A determination of \( |V_{ub}| \) and \( |V_{cb}| \) can be made by combining experimental measurements of exclusive branching fractions of \( B \to \pi, \rho \nu \) and \( B \to D^{(*)} \nu \) with lattice calculations of the hadronic matrix elements (or corresponding form factors). In practice, lattice calculations of these matrix elements, with experimentally favoured kinematics present various problems. I will report on some recent progress in calculations of \( B(D) \to \pi, \rho \nu \) and \( B \to D^{(*)} \nu \). Preliminary results, from UKQCD for the decay \( B \to \rho \nu \) were
presented at this conference by Gill [37].

5. \( B \to \pi l\nu \)

The hadronic matrix elements parameterising the decay of heavy-light to light mesons are not protected by flavour symmetries and hence \(|V_{ub}|\) is dominated by theoretical uncertainties.

\(|V_{ub}|\) is determined from the decay rate

\[
\frac{d\Gamma}{dp} = \frac{G_F^2 |V_{ub}|^2 2m_B p^4 |f_+(E)|^2}{24\pi^3} E ,
\]

where the form factors, \( f_+(E) \) and \( f_0(E) \) parameterise the matrix elements in the standard way

\[
\langle \pi(p_\pi)|V^{\mu}|B(p_B)\rangle = f_+(E) [p_B + p_\pi
\]

\[\frac{m_B^2 - m_\pi^2}{q^2} q^\mu + f_0(E) \frac{m_B^2 - m_\pi^2}{q^2} q^\mu \]

I will focus on four recent calculations by UKQCD [33], APE [42], FNAL [35] and JLQCD [36]. UKQCD and APE extrapolate from quark masses around charm to the bottom quark mass. The Fermilab group use the Fermilab formalism for heavy quarks and JLQCD use NRQCD. Thus it is possible to investigate the agreement of different analyses and treatments of heavy quarks. Table 6 lists some details of these analyses. A number of improvements, compared to previous calculations are included. All groups make the chiral extrapolation and the renormalisation of matrix elements is done nonperturbatively or at one loop in perturbation theory. The lattice spacing dependence is studied by the Fermilab group who also simulate directly at the lattice spacing. Alternatively new experiments may be able to extend the current kinematic range for this decay.

5.1. Fermilab and JLQCD

Rather than use the parameterisation of Eqn 3 both Fermilab and JLQCD use forms motivated by HQET.

\[
\langle \pi(p_\pi)|V^{\mu}|B(p_B)\rangle = \sqrt{2m_B} \left[ v^\mu f_1(E) + p_\perp^\mu f_\perp(E) \right] = 2 \left[ f_1(E) v^\mu + f_2(E) p_\perp^\mu \right] ,
\]

where \( v = p_B/m_B \) and \( p_\perp = p_\pi - E v \). The first parameterisation is used by Fermilab and the second by JLQCD. The traditional form factors, \( f_+ \) and \( f_0 \) are easily obtained from \( f_\parallel \) and \( f_\perp \) or \( f_1 \) and \( f_2 \). These are useful quantities to focus on. Considering the \( m_\pi, E \to 0 \) limit yields

\[
f_\parallel = \frac{f_B \sqrt{m_B}}{\sqrt{2f_\pi}} \frac{1}{\sqrt{2f_\pi}} \frac{m_B^2 - q^2}{2m_B} \frac{1}{m_B - q^2} \]

\[
f_\perp = \frac{f_B \sqrt{m_B}}{\sqrt{2f_\pi}} \frac{1}{\sqrt{2f_\pi}} \frac{2m_B}{m_B - q^2} \]

Figure 3. Comparison of differential decay rates.
Table 2
Overview of the $B \to \pi l \nu$ analyses. PT is perturbation theory and NP is nonperturbative.

|                | APE | UKQCD | FNAL | JLQCD |
|----------------|-----|-------|------|-------|
| simulate at $m_b$ | no  | no    | yes  | yes   |
| quark mass       | yes | yes   | yes  | yes   |
| chiral extrapol. | 1   | 1     | 3    | 1     |
| lattice spacings | NP at $m_q = 0$ | NP at $m_q = 0$ | mostly NP: $\sqrt{Z_{V_{uu}} Z_{V_{bb}}}$ | PT |
| matching         | includes KLM term | includes KLM term | PT: radiative corrections | 1-loop |

In addition, these form factors have simple descriptions in HQET and so are natural in the Fermilab and NRQCD approaches. Finally, they emerge directly from the lattice calculation and can be analysed directly, forming $f_{l,+0}$ at the end.

In Ref [36] a discrepancy between one of the Fermilab and JLQCD form factors was discussed. Fig 4 shows this discrepancy in $f_1 + f_2$. This form factor yields $f_0$ which does not contribute to the rate. However, one would expect the NRQCD and Fermilab approaches to be in agreement especially as the simulation parameters are also similar. In fact the functional form of the chiral extrapolation, the ($\pm 14\%$) uncertainty due to chiral extrapolation assigned to $d\Gamma/dp$ in the Fermilab analysis seems prudent. It is in fact the largest systematic error in that analysis. Lighter quark masses and higher statistics would certainly help to reduce this uncertainty.

5.2. Heavy mass dependence
Turning now to the heavy mass dependence. This has been studied by JLQCD and APE although not in the same region of heavy quark mass. JLQCD simulate at four heavy quark masses around the $b$ quark. Plotting the quantities $\Phi_+ \equiv (\alpha_s(M_P)/\alpha_s(M_B))^{-2/11} f_+/\sqrt{M_P}$ and $\Phi_0 \equiv (\alpha_s(M_P)/\alpha_s(M_B))^{-2/11} f_0/\sqrt{M_P}$ at fixed $E$ they see no dependence on the heavy mass. This analysis is restricted to the bottom quark regime.

On the other hand, APE find a negative slope when plotting the same quantities versus $1/M_P$ with heavy quarks at around the charm mass. These differences in slope can be seen in Figure 5. The Fermilab nonrelativistic approach is valid at arbitrary quark mass. Fig 5 compares the JLQCD, APE and Fermilab data where only Fermilab have simulated directly at both the charm and bottom quark masses. The figure reveals that the Fermilab data agree with JLQCD at $M_B$ but show slope as $1/M_P \to 1/M_D$. The slope of the mass dependence is gentler than that observed by APE agreeing more with a linear extrapolation of the APE data to $M_B$.

I make some final remarks in this section about...
the soft pion theorem (SPT). The relation applies in the limit $m_\pi, E \to 0$ as given by Eqn 4. The first remark is that the limit in which this relation holds implies $f_0$ should be evaluated at $m_B^2$ rather than $q_{\text{max}}^2 = m_B^2 - m_\pi^2$.

The question of whether or not lattice calculations satisfy this relation has been discussed in previous reviews [1, 23]. In the analyses discussed here both APE and UKQCD report that the relation in Eqn 4 holds. In the JLQCD and Fermilab results this relation does not hold. It is not at all clear what the reasons for this are. The APE group use the soft pion theorem as a criterion for choosing the functional form (quadratic rather than linear) of the heavy mass extrapolation so it is perhaps not surprising then that the relation holds. UKQCD also choose a quadratic form for the heavy mass extrapolation and there is a dependence on the pole form introduced when interpolation to a common set of $q^2$. JLQCD observe that a $\sqrt{m_\pi}$ term in the chiral extrapolation raises the value of the soft pion limit. Thus the problem is dependent on the form of the extrapolation and the interpolation to fixed $E$. In addition, Onogi has pointed out [23] that the perturbative uncertainty in the renormalisation coefficients of the heavy-light currents may be contributing to the SPT discrepancy. This year Onogi [40] reported on a nonperturbative determination of the ratio, $Z_A/Z_V$ with static heavy quarks and SW light quarks. The result brings the two sides of Eqn 4 into better agreement, but does not explain completely the difference.

Finally, the Fermilab group note that while the lattice data for $f_0$ do not satisfy Eqn 4 the experimental rate at this point goes to zero and the lattice systematics (mostly due to chiral extrapolation) increase at the soft pion limit. For these reasons cuts are imposed on the range of energy considered such that $E \geq 0.424\text{GeV}$. Although the phenomenologically interesting region is far from $q_{\text{max}}^2$ the reasons for the violations of the SPT are important. It is a good indication of the control of systematic errors.

The sources and severity of the different systematic errors in the calculation of differential decay rates and their contribution to the theoretical uncertainty in $|V_{ub}|$ were tabulated in the Fermilab analysis. The details are in Table 3. $T_B$ is defined from Eqn 3 as $T_B = 2m_B p^4 f_+ (E)^2 / E$ and similarly for $T_D$. The $D \to \pi l \nu$ calculation is discussed in Section 5.

### Table 3

| Error                  | $T_B$ | $|V_{ub}|$ | $T_D$ | $|V_{cd}|$ |
|------------------------|-------|-----------|-------|-----------|
| statistical            | +27   | +14       | +17   | +9        |
| excited state          | -9    | -5        | -8    | -4        |
| $\vec{p}$ extrapolation| $\pm$10 | $\pm$5 | $\pm$9 | $\pm$5 |
| $m_q$ extrapolation    | $\pm$16 | $\pm$8 | $\pm$3 | $\pm$2 |
| adjusting $m_Q$        | $\pm$6   | $\pm$3 | $\pm$6 | $\pm$3 |
| HQET matching          | $\pm$10 | $\pm$5 | $\pm$10 | $\pm$5 |
| $a$ dependence         | +16   | +8        | +23   | +11       |
| definition of $a$      | -3    | -2        | -6    | -3        |
| Total systematic       | 30    | 15        | 28    | 14        |
| Total (stat$\otimes$sys) | $\pm$40 | $\pm$20 | $\pm$32 | $\pm$16 |

6. $D \to \pi(K) l \nu$

This decay has traditionally received less attention than the $B \to \pi l \nu$. With the planned charm factories at CLEO-c and elsewhere it is increasingly important to focus on $D$ meson decays. This
is a region of quark mass where Fermilab’s nonrelativistic interpretation and the $O(a)$-improved approach should be in agreement. A nice advantage in this decay is that lattice calculations can reach the entire kinematical range with no extrapolation in $q^2$. This removes what is one of the major problems with the $B$ meson decay.

Both APE and Fermilab have reported results for the decay $D \to \pi l\nu$. In addition the APE group presented their results for $D \to K l\nu$. Preliminary results for this transition were presented by the Fermilab group in Ref [41].

As already indicated by Fig 3 the results of the Fermilab and APE analysis do not agree at the $D$ meson. The difference is even greater when one looks at the differential decay rate since $(d\Gamma/dy^2) \propto |f_+(E)|^2$. There are a number of differences in both the action and improvement coefficients and the analysis of the groups. The Fermilab group define the quark mass from the kinetic mass rather than the other choice. In the Fermilab formalism (as well, of course, as in NRQCD). However, at masses around that of charm the difference between rest and kinetic mass is not as great and the choice of one definition rather than the other cannot, I believe, explain the lack of agreement. Further differences arise in the current renormalization in the Fermilab scheme the bulk of this is done nonperturbatively in a fully mass-dependent way. The remaining matching from lattice HQET to continuum HQET is done at one-loop level in perturbation theory. APE use a nonperturbative determination of the matching coefficients for massless quarks with the so-called KLM term to correct for $O(ma)$ effects. In addition there are differences such as the use of a quadratic (Fermilab) and a linear (APE) chiral extrapolation. It may be that these differences combine to produce the large discrepancy seen in the form factors. With experiments soon to test lattice predictions it is important to understand what is going on. I would also like to note that the UKQCD collaboration do have preliminary results. Since they are not final I have not included them in this discussion although interestingly they lie between the Fermilab and APE results.

7. $B \to D^{(*)} l\nu$

New results were presented in two talks at this conference. Simone [12] reported on work by the Fermilab group to measure the form factors for this decay at zero recoil. Lacagnina [13] reported preliminary results from UKQCD for the shape of the Isgur-Wise function determined from $B \to D^{*} l\nu$ and $B \to D l\nu$. I will focus the Fermilab results since they were (almost) final at the time of the conference and the calculation has recently been completed and appeared in Ref [14].

At the zero recoil point the simple relation $\mathcal{F}_{B \to D^{(*)}}(1) = h_{A1}(1)$ holds and heavy quark symmetry constrains $h_{A1}$ to have the heavy quark expansion

$$h_{A1} = \eta_A \left( 1 - \frac{l_V}{(2m_c)^2} + \frac{2l_A}{2m_c 2m_b} - \frac{l_P}{(2m_b)^2} \right) . (6)$$

In Eqn 6 the radiative correction, $\eta_A$ is known to two-loop level from a calculation by Czarnecki and Melnikov [15]. The three $l$s are hadronic matrix elements of the HQET and are calculable in lattice QCD.

To determine these matrix elements, for the first time in a lattice calculation, a new method is introduced. It exploits an idea developed in a previous paper where double ratios of matrix elements were used to determine the form factors of $B \to D l\nu$ at zero recoil [16]. Three such double ratios are employed to determine $l_V, l_P$ and $l_A$.

$$\mathcal{R}_{+-} = \frac{\langle D | \bar{c} \gamma^4 b | \bar{B} \rangle \langle \bar{B} | \bar{b} \gamma^4 c | D \rangle}{\langle D | \bar{c} \gamma^4 c | D \rangle \langle \bar{B} | \bar{b} \gamma^4 b | B \rangle} = |h_{A1}(1)|^2$$
$$\mathcal{R}_1 = \frac{\langle D^* | \bar{c} \gamma^4 b | \bar{B}^* \rangle \langle \bar{B}^* | \bar{b} \gamma^4 c | D^* \rangle}{\langle D^* | \bar{c} \gamma^4 c | D^* \rangle \langle \bar{B}^* | \bar{b} \gamma^4 b | B^* \rangle} = |h_{A1}(1)|^2$$
$$\mathcal{R}_{A1} = \frac{\langle D^* | \bar{c} \gamma_j \gamma_5 b | \bar{B} \rangle \langle \bar{B} | \bar{b} \gamma_j \gamma_5 c | D \rangle}{\langle D^* | \bar{c} \gamma_j \gamma_5 c | D \rangle \langle \bar{B} | \bar{b} \gamma_j \gamma_5 b | B \rangle} = |h_{A1}(1)|^2$$

where $|h_{A1}(1)|^2$ is defined by $\frac{h_{A1}^{B \to D^{(*)}}(1)}{h_{A1}^{B \to D^{(*)} \to D^{(*)}}(1)}$. The matrix elements $l_P, l_V$ and $l_A$ are obtained from the heavy quark expansions of $h_{A1}(1)$ and $h_{A1}(1)$ respectively. The required form
factor, $h_{A_1}(1)$ can then be reconstructed using Eqn (1). The Fermilab group report a final result
\[ F_{B \to D^* l \nu}(1) = 0.9130^{+0.0238+0.0171}_{-0.0173-0.0302} \] where the first error is statistical and fitting and the second systematic, resulting from matching lattice gauge theory and HQET to QCD, lattice spacing dependence, light quark mass dependence and the quenched approximation.

The bulk of the matching cancels in the double ratios and the remaining short-distance coefficients that match lattice gauge theory to QCD and HQET to QCD are small and are calculated at one-loop level in perturbation theory. Lattice spacing dependence is studied using three lattices in the range $5.7 \leq \beta \leq 6.1$. $h_{A_1}(1)$ is extrapolated linearly in $m_{\pi}^2$ to the chiral limit, showing a downward trend. The quoted uncertainty is a result of including nonlinear terms present in the chiral expansion. The chiral extrapolation is in fact the largest source of uncertainty. Finally the error due to quenching is estimated by summing the effect on the running coupling, $\alpha_s$ and allowing an additional 10% uncertainty.

This is a precision calculation of a phenomenologically interesting quantity. The result agrees well with those from other methods (non-relativistic quark models \cite{57} and a zero-recoil sum rule \cite{58}). Combining the result with measurements by CLEO \cite{59}, LEP \cite{51} and Belle \cite{52} implies
\[
10^3 |V_{cb}| = \begin{cases} 
45.9 \pm 2.4^{+1.8}_{-1.4} \\
38.7 \pm 1.8^{+1.5}_{-1.2} \\
39.3 \pm 2.5^{+1.6}_{-1.3}
\end{cases}
\]

8. $m_b$

A new method to determine the $b$ quark mass with nonperturbative accuracy was presented by Sommer \cite{57}. This method has the advantage of avoiding the perturbative subtraction of power law divergences in the relation $m_b(\mu) = Z_{cont}(\mu)M_{pole}$ and therefore, taking the continuum limit is possible with this method. Fig 6 summarises the current status, including quenched and unquenched calculations and Sommer’s preliminary result. This is in good agreement with previous determinations. I estimate an average of these calculations to be $m_b(m_b) = 4.30(10)$ GeV.

9. SPECTROSCOPY

In this section I will focus on one quantity: the hyperfine splitting in both heavy-light and quarkonia systems. Lattice calculations of spectroscopic quantities are generally better controlled than matrix element calculations of similar scope. A major advantage is that no renormalisation is required. However, the hyperfine splitting (HFS) remains an exception to this rule.

In the heavy-light sector it is experimentally observed that the value of the HFS remains constant for all flavours. In practice, in the meson sector, lattice results are suppressed relative to experiment by as much as 20%(40%) for $D(B)$ mesons. At this conference Tsutsui \cite{59} described a calculation of heavy quark expansion parameters using NRQCD which includes results for the heavy-light meson and baryon splittings. The meson splitting is $\simeq 30\%$ below the experimental value as in other calculations while this discrepancy does not exist in the baryon splittings.

It is argued that the discrepancy is a quenching
effect since the meson splitting is proportional to the wavefunction at the origin which is suppressed in the quenched approximation and to the strong coupling which runs differently in the full and quenched theories. This however, has not been verified by an unquenched calculation of these splittings. Lewis \[61\] presented results for the charmed baryon spectrum (both $QQq$ and $Qqq$) using anisotropic and NRQCD actions. Despite larger statistical errors than in the meson spectrum they see no suppression of the HFS. In fact there may be evidence that it is overestimated although better data are required to make a conclusive statement.

Turning to the charmonium and bottomonium spectra. Garcia-Perez reported preliminary results on behalf of the QCDTARO collaboration \[62\]. They have calculated the HFS on a very fine lattice, $\beta = 6.6$, using the SW action. They take the continuum limit using UKQCD data at $\beta = 6.0, 6.2$ and observe significant lattice spacing dependence. They are currently producing their own configurations at a range of $\beta$ values to make a more consistent continuum extrapolation.

New results for the bottomonium spectrum and HFS were presented by Manke \[62\]. The calculation is done with an anisotropic relativistic heavy quark action. Two levels of anisotropy are used in the quenched approximation. Two levels of anisotropy are used in the quenched approximation. They also find significant scaling violations.

10. CONCLUSIONS

Lattice calculations offer the prospect of model-independent determinations of hadronic matrix elements. The effect of unquenching in calculations of leptonic decay constants and their ratios is becoming more precise.

For the semileptonic decay $B \to \pi l \nu$ control of the systematic errors in the quenched approximation has improved. Problems remain however, including simulations with pion momentum above 1GeV, the chiral extrapolation and unquenching. An important development is the lattice determination of the zero-recoil form factor for $B \to D^* l \nu$. The calculation is systematically improvable and with this method it is conceivable that the error on $\mathcal{F}_{B\to D^*l\nu}(1)$ can be lowered to under 1%.

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