A Twistor Space Action for Yang-Mills Theory

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Abstract
We consider the twistor space $\mathcal{P}^6 \cong \mathbb{R}^4 \times \mathbb{C}P^1$ of $\mathbb{R}^4$ with a non-integrable almost complex structure $J$ such that the canonical bundle of the almost complex manifold $(\mathcal{P}^6, J)$ is trivial. It is shown that $J$-holomorphic Chern-Simons theory on a real $(6|2)$-dimensional graded extension $\mathcal{P}^{6|2}$ of the twistor space $\mathcal{P}^6$ is equivalent to self-dual Yang-Mills theory on Euclidean space $\mathbb{R}^4$ with Lorentz invariant action. It is also shown that adding a local term to a Chern-Simons-type action on $\mathcal{P}^{6|2}$, one can extend it to a twistor action describing full Yang-Mills theory.
1 Introduction

Let $M^4$ be an oriented real four-manifold with a Riemannian metric and $P(M^4, SO(4))$ the principal bundle of orthonormal frames over $M^4$. The twistor space $\text{Tw}(M^4)$ of $M^4$ can be defined as an associated bundle \[^1\]

$$\text{Tw}(M^4) = P \times_{SO(4)} SO(4)/U(2) \quad (1.1)$$

with the canonical projection

$$\pi : \text{Tw}(M^4) \rightarrow M^4. \quad (1.2)$$

Fibres of this bundle are two-spheres $S_x^2 \cong SO(4)/U(2)$ which parametrize complex structures $J_x$ on the tangent space $T_xM^4$ at $x \in M^4$ compatible with a Euclidean metric and orientation of $M^4$. It means that $J_x \in \text{End}(T_xM^4)$ with $J_x^2 = -\text{Id}$ and $J_x$ is an isometry of $T_xM^4$ preserving orientation.

An almost complex structure $J$ on $M^4$ is a global section of the bundle \[(1.2).\] Note that while a manifold $M^4$ admits in general no almost complex structure (e.g. four-sphere $S^4$), its twistor space $\text{Tw}(M^4)$ can always be equipped with two natural almost complex structures. The first, $J = J_+$, introduced in \[^1\], is integrable if and only if the Weyl tensor of Riemannian metric on $M^4$ is self-dual, while the second, $J = J_-$, introduced in \[^2\], is non-integrable (and never integrable), i.e. the Nijenhuis tensor of $J$ does not vanish.

Twistor space $\mathcal{P}^6 = \text{Tw}(\mathbb{R}^4) \cong \mathbb{R}^4 \times S^2$ of $\mathbb{R}^4$ with an almost complex structure $J$ is a particular case of almost complex six-manifolds to be discussed in this paper. Twistor space $(\mathcal{P}^6, J)$ is a complex manifold $\mathcal{P}^3_C$ for integrable $J$ and it is an almost complex manifold with an $SU(3)$-structure and non-vanishing torsion for non-integrable $J$. Twistor literature focuses on complex twistor space $\mathcal{P}^3_C$ (see e.g. \[^3, 4, 5\]) and very rarely on the non-integrable case (see e.g. \[^2, 6, 7\]).

The goal of twistor theory is to take some unconstraint analytic object on $\text{Tw}(M^4)$ (e.g. Dolbeault cohomology classes) and transform them to objects on $M^4$ which will be constrained by some differential equations \[^3, 4\]. In particular, the self-dual Yang-Mills (SDYM) equations on Euclidean space $\mathbb{R}^4$ can be described as field equations of holomorphic Chern-Simons theory defining holomorphic bundles on the complex twistor space $\mathcal{P}^3_C$ via the Penrose-Ward correspondence \[^3, 4, 5\]. This correspondence can be extended to the non-integrable case (see e.g. \[^6, 7\]).

The field equations of $J$-holomorphic Chern-Simons ($J$-hCS) theory on $(\mathcal{P}^6, J)$ read

$$\mathcal{F}^{0,2} = P^{0,1} P^{0,1} \mathcal{F} = (d\mathcal{A} + \mathcal{A} \wedge \mathcal{A})^{0,2} = 0,$$

where $P^{0,1} = \frac{1}{2}(\text{Id} + iJ)$ is the projector onto $(0,1)$-part of one-forms, $\mathcal{A}$ is a connection one-form on a complex vector bundle $\mathcal{E}$ over $(\mathcal{P}^6, J)$ and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature of $\mathcal{A}$. One can expect that equations \[^1, 3\] are obtained by variation of the action functional

$$S = \frac{i}{8} \int_{\mathcal{P}^6} \Omega \wedge \text{CS}(\mathcal{A})^{0,3} = \frac{i}{8} \int_{\mathcal{P}^6} \Omega \wedge \text{tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})^{0,3}, \quad (1.4)$$

where $\Omega$ is a $(3,0)$-form w.r.t. $J$ on $(\mathcal{P}^6, J)$, i.e. $\Omega$ is a global section of the canonical bundle of $(\mathcal{P}^6, J)$. However, the canonical bundle of $\mathcal{P}^3_C \cong \mathbb{C}P^3 \setminus \mathbb{C}P^1$ is the non-trivial holomorphic line bundle $\mathcal{O}(-4)$ with the first Chern class -4. Hence, there is no non-singular holomorphic volume form $\Omega$ on $\mathcal{P}^3_C$. Thus, the functional \[^1, 4\] is not defined on $\mathcal{P}^3_C$. 


The triviality of the canonical bundle can be restored if instead of $\mathcal{P}_C^3$ one considers the supertwistor space $\mathcal{P}_C^{3|4} \cong \mathbb{C}P^{3|4} \setminus \mathbb{C}P^{1|4}$ with four holomorphic fermionic dimensions, each of type $\Pi \mathcal{O}(1)$ bundle, where the operator $\Pi$ inverts the Grassmann parity of fibre coordinates. The canonical bundle of $\mathcal{P}_C^{3|4}$ is trivial and hence there is a holomorphic volume form $\Omega$ on $\mathcal{P}_C^{3|4}$. This fact was used by Witten for introducing twistor string theory and holomorphic Chern-Simons theory (hCS) on $\mathcal{P}_C^{3|4}$ [8]. The action of hCS theory on $\mathcal{P}_C^{3|4}$ can be written in the form (1.4) after substituting $\Omega$ instead of $\Omega$ and integrating over $\mathcal{P}_C^{3|4}$. The field equations will be (1.3) with $A^{0,1} = P^{0,1}A$ depending on four Grassmann variables taking values in the bundle $\Pi \mathcal{O}(1) \otimes \mathbb{C}^4$ over $\mathcal{P}_C^3$. This hCS theory on $\mathcal{P}_C^{3|4}$ in turn is equivalent [8] to self-dual subsector of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$ (see e.g. [9, 10, 11] for reviews and references) in the form of Chalmers and Siegel [12]. The $\mathcal{N}=4$ SDYM equations can be truncated to the bosonic SDYM equations [12] and on the twistor level this was discussed e.g. in [13, 14, 15].

Despite the success of the supertwistor description of supersymmetric Yang-Mills theories, there was a desire to get a twistor description of pure bosonic SDYM theory. Recently, it was proposed by Costello to work with hCS theory on the bosonic twistor space $\mathcal{P}_C^3$ by allowing $\Omega$ in (1.3) to be meromorphic instead of holomorphic [16]. After choosing a meromorphic form $\Omega$ on $\mathcal{P}_C^3$ and imposing some boundary conditions on fields at poles of $\Omega$, one can reduce the action (1.4) to the 4d action for SDYM theory as it was demonstrated in [16, 17]. Depending on the gauge choice, the twistor action is reduced to the action for group-valued fields [18, 19] or to the action for Lie-algebra valued fields [20, 21], both of which are well known in the literature. However, the choice of (3,0)-form $\Omega$ and of its singularities is not unique and different choices lead to a range of actions on $\mathbb{R}^4$, not all of which have equations of motion equivalent to the SDYM equations [17].

All the above-mentioned actions break Lorentz invariance. The actions [18–21] for the SDYM equations were discussed long time ago by Chalmers and Siegel in [12], where it was shown that these 4d actions at more than one loop generate diagrams that do not relate to quantum Yang-Mills theory. These flaws are absent for the Chalmers-Siegel 4d action which is a truncation (a limit of small coupling constant) of the standard Yang-Mills action. We want to obtain this 4d action in the framework of twistor approach. We show that this is possible by using a non-integrable almost complex structure $\mathcal{J}$ on the twistor space $\mathcal{P}^6$ such that the canonical bundle becomes trivial and hence there exists a globally defined (3,0)-form $\Omega$ on $(\mathcal{P}^6, \mathcal{J})$ which can be used in (1.3).

The action [12] contains gauge field coupled with a propagating anti-self-dual auxiliary field $G_{\hat{\alpha}\hat{\beta}} = \epsilon^{\hat{\alpha}\hat{\beta}} G_{a\hat{\alpha},\hat{\beta}}$ with $\hat{\alpha}, \hat{\beta} = 1, 2$. The field $G_{\hat{\alpha}\hat{\beta}}$ corresponds to additional degrees of freedom parametrized by some cohomology groups on the complex twistor space $\mathcal{P}_C^3$ [22, 11] and can be obtained from the component $A^{0,1}$ along $\mathbb{C}P^1 \hookrightarrow \mathcal{P}_C^{3|4}$ in hCS theory on the supertwistor space (see e.g. [11] and references therein). This $G_{\hat{\alpha}\hat{\beta}}$ enters into the $\mathcal{N}=4$ SDYM supermultiplet $(f_{\hat{\alpha}\hat{\beta}}, \chi^{\hat{\alpha}}, \phi^{\hat{\beta}}, \tilde{\chi}^{\hat{\alpha}}, G_{\hat{\alpha}\hat{\beta}})$, where the fields have helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$, $i = 1, \ldots, 4$. Truncations of the self-dual $\mathcal{N}=4$ super-Yang-Mills to the case $\mathcal{N}<4$, including the bosonic case $\mathcal{N}=0$, can be obtained by considering weighted projective supertwistor space [14, 10] or exotic supertwistor space [15, 9]. The approach similar to that in [14, 15] can be used in the case of twistor space $(\mathcal{P}^6, \mathcal{J})$ with non-integrable almost complex structure $\mathcal{J}$ on $\mathcal{P}^6$. We will show that the 4d Chalmers-Siegel action [12] can be obtained from an action functional for $\mathcal{J}$-hCS theory on a graded twistor space $\mathcal{P}^{6|2}$ with two real fermionic directions, each parametrizing trivial real line bundle over $(\mathcal{P}^6, \mathcal{J})$. The Chern-Simons type action on $\mathcal{P}^{6|2}$ is introduced by using globally defined form $\tilde{\Omega} = \Omega \wedge d\eta_1 \wedge d\eta_2$ on $\mathcal{P}^{6|2}$, where $\Omega$ is a global section of the trivial canonical bundle of $\mathcal{P}^6$. Compo-
ments of gauge potential $A$ in this theory take values in the Grassmann algebra $\Lambda(\mathbb{R}^2)$ generated by two real scalars $\eta_1, \eta_2$. We also show that this action can be extended to a twistor action describing full Yang-Mills theory on $\mathbb{R}^4$ after adding some local terms to $J$-hCS Lagrangian on the twistor space $\mathcal{P}^6$.

2 Self-dual Yang-Mills and twistors

Almost complex structures on $\text{Tw}(M^4)$. We defined the twistor space $\text{Tw}(M^4)$ of a Riemannian manifold $M^4$ as the associated bundle $\text{Tw}(M^4)$ of complex structures $J_x$ on tangent spaces $T_x M^4$. Global sections of the projection $\pi$ are identified, if such sections exist, with almost complex structures $J$ on $M^4$, i.e. with tensors $J = (J^\mu_\nu) \in \text{End}(TM^4)$ such that $J^\mu_\nu J^\nu_\mu = -\delta^\mu_\nu$, $\mu, \nu = 1, \ldots, 4$.

While a manifold $M^4$ has in general no almost complex structures, its twistor space $Q^6 := \text{Tw}(M^4)$ can be always provided in a natural way with an almost complex structure $\mathcal{J}$, a tensor on $Q^6$ with $\mathcal{J}^2 = -\text{Id}$. In fact, the Levi-Civita connection on $M^4$ generates the splitting of the tangent bundle $TQ^6$ into the direct sum

$$TQ^6 = V \oplus H \tag{2.1}$$

of vertical and horizontal subbundles of $TQ^6$. The space $V_q$ in $q \in Q^6$ is tangent to the fibre $\pi^{-1}(\pi(q))$ over $x = \pi(q) \in M^4$ of the projection $\pi : Q^6 \to M^4$. Recall that the fibre over $x = \pi(q)$ is identified with $S_2^2 \cong \text{SO}(4)/\text{U}(2)$ and so it has a natural complex structure $J^q$. Hence, we can define an almost complex structure $\mathcal{J}$ on $Q^6$ using the decomposition (2.1) by setting

$$\mathcal{J} = \mathcal{J}^{\text{int}} = \mathcal{J}^v \oplus \mathcal{J}^h \tag{2.2}$$

where $\mathcal{J}^h$ is an almost complex structure equal in the point $q \in Q^6$ to the complex structure $\mathcal{J}^h_q$ on $H_q \cong T_{\pi(q)} M^4 = T_x M^4$. Thus, the twistor space $Q^6$ has a natural almost complex structure $\mathcal{J}$.

It was shown in [1] that if the Weyl tensor of $M^4$ is self-dual then the almost complex structure (2.2) on $Q^6$ is integrable and $(Q^6, \mathcal{J}^{\text{int}})$ inherits the structure of a complex analytic 3-manifold $Q^3_C$. It was also shown in [2] that

$$\mathcal{J} = \mathcal{J}^{\text{non}} = \mathcal{J}^v \oplus (-\mathcal{J}^h) \tag{2.3}$$

is an almost complex structure on $Q^6$ which is never integrable. These structures differ in sign along $M^4$.

Twistor correspondence. Let $E$ be a rank $k$ complex vector bundle over $M^4$ and $A$ a connection one-form (gauge potential) on $E$ with the curvature $F = dA + A \wedge A$ (gauge field). The gauge field $F$ is called self-dual if it satisfies the equations

$$* F = F \quad \Leftrightarrow \quad \frac{1}{2} \varepsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma} = F_{\mu\nu}, \tag{2.4}$$

where $*$ denotes the Hodge star operator, $\varepsilon_{\mu\nu\lambda\sigma}$ is the completely antisymmetric tensor on $M^4$ with $\varepsilon_{1234} = 1$ in the Riemannian metric $ds^2 = \delta_{\mu\nu} e^\mu e^\nu$ for an orthonormal basis $\{e^\mu\}$ on $T^* M^4$.

Bundles $E$ with self-dual connections $A$ are called self-dual. It was proven in [1] that the self-dual bundle $E$ over self-dual manifold $M^4$ lifts to a holomorphic bundle $\mathcal{E}$ over the complex twistor space $Q^3_C = (\text{Tw}(M^4), \mathcal{J}^{\text{int}})$ and $\mathcal{E}$ is holomorphically trivial on fibres $\mathbb{C}P_x^1$ of projections $\pi : Q^6 \to M^4$ for each $x \in M^4$. The bundle $\mathcal{E} = \pi^* E$ is defined by the connection $\mathcal{A} = \pi^* A$ such that its curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ satisfies the equations (1.3) and $\mathcal{F} = \pi^* F$ is the pull-back to
\( \mathcal{E} \) of self-dual gauge field \( F \) on \( E \to M^4 \). Vice versa, solutions to the holomorphic Chern-Simons field equations (1.3) on the twistor space \( Q^3_C \), with \( \mathcal{F}_{C|P^1} = 0 \) for any \( x \in M^4 \), give solutions to the SDYM equations (2.4) on \( M^4 \). The map between solutions to the SDYM equations on \( M^4 \) and solutions to the hCS field equations on \( Q^3_C = (\text{Tw}(M^4), \mathcal{J}^\text{int}) \) is called the Penrose-Ward transform.

For non-integrable almost complex structure (2.3) on \( Q^6 \) the manifold \( (Q^6, \mathcal{J}^{\text{non}}) \) is not complex. However, on \( (Q^6, \mathcal{J}^{\text{non}}) \) one can introduce bundles with \( \mathcal{J} \)-holomorphic structure (pseudoholomorphic bundles) [23]. Let \( E \) be a complex rank \( k \) vector bundle over \( Q^6 \) endowed with a connection \( A \). According to Bryant [23], a connection \( A \) on \( E \) is said to define a \( \mathcal{J} \)-holomorphic structure if it has curvature \( F \) of type (1,1) w.r.t. \( \mathcal{J} \), i.e.

\[
F^{0,2} = 0
\]

(2.5)

It is not difficult to show that twistor correspondence between solutions of SDYM equations (2.4) on \( M^4 \) and solutions of \( \mathcal{J} \)-hCS equations (2.5) on the almost complex twistor space \( (Q^6, \mathcal{J}) \) still persists (see e.g. [7]). This will be discussed in more details later for the case of flat Euclidean space \( M^4 = \mathbb{R}^4 \).

3 Twistor space of \( \mathbb{R}^4 \)

According to the definition (1.1), twistor space of \( \mathbb{R}^4 \) is \( \mathcal{P}^6 = \text{Tw}(\mathbb{R}^4) \cong \mathbb{R}^4 \times S^2 \). Due to diffeomorphism with \( \mathbb{R}^4 \times S^2 \), the manifold \( \mathcal{P}^6 \) is fibred not only over \( \mathbb{R}^4 \),

\[
\pi : \mathcal{P}^6 \xrightarrow{\mathbb{R}^4} S^2,
\]

(3.1)

but also over \( S^2 \),

\[
\mathcal{P}^6 \xrightarrow{\mathbb{R}^4} S^2,
\]

(3.2)

with spaces \( \mathbb{R}^4 \) as fibres.

**Almost complex structures \( \mathcal{J} \).** In section 2 we described generic construction of an almost complex structure \( \mathcal{J} \) on a twistor space \( \text{Tw}(M^4) \). Here, we give explicit form of \( \mathcal{J} \) for the case \( M^4 = \mathbb{R}^4 \).

Recall that a complex structure \( J \) on \( \mathbb{R}^4 \) is a tensor \( J = (J^\nu_\mu) \) such that \( J^\nu_\mu J^\mu_\nu = -\delta^\nu_\mu \). All constant complex structures on \( \mathbb{R}^4 \) are parametrized by the two-sphere \( S^2 \cong \text{SO}(4)/\text{U}(2) \cong \text{SU}(2)/\text{U}(1) \) defined by the equation

\[
\delta_{ab} s^a s^b = 1
\]

(3.3)

for \( s^a \in \mathbb{R}^3 \), \( a, b = 1, 2, 3 \). One can choose generic \( J \) in the form

\[
J^\nu_\mu = s_a \bar{\eta}^a_\mu \delta^\nu_\mu,
\]

(3.4)

where

\[
\bar{\eta}^a_\mu = \{ \varepsilon^a_{bc}, \mu = b, \nu = c; -\delta^a_\mu, \nu = 4; \delta^a_\nu, \mu = 4 \}
\]

(3.5)

are antisymmetric 't Hooft tensors, \( \mu, \nu = 1, \ldots, 4 \). Using the identities

\[
\bar{\eta}^a_\mu \bar{\eta}^b_\nu = -\delta^{ab} \delta_\mu \nu - \varepsilon^{abc} \delta^c_\mu \bar{\eta}^a_\nu,
\]

(3.6)
one can show that \( J^2 = -\text{Id} \). Here, we consider \( \mathbb{R}^4 \) as a space with the metric \( ds_{\mathbb{R}^4}^2 = \delta_{\mu\nu} dx^\mu dx^\nu \), where \( x^\mu \) are coordinates on \( \mathbb{R}^4 \).

Let \( \{e^\alpha\} \) represents an orthonormal coframe on \( S^2 \), i.e.
\[
ds_{S^2}^2 = \delta_{\alpha\beta} e^\alpha e^\beta
\]
for \( \alpha, \beta = 1, 2 \). The canonical form of complex structure \( j \) on \( S^2 \) is
\[
j = (i_\alpha^\beta) \quad \text{with} \quad i_1^2 = -i_2^1 = 1 \quad \Rightarrow \quad i_\alpha^\alpha j^\beta = -\delta_\alpha^\beta.
\]
It is obvious that both
\[
\mathcal{J} = \mathcal{J}^{\text{int}} = (J, j)
\]
and
\[
\mathcal{J} = \mathcal{J}^{\text{non}} = (-J, j)
\]
are almost complex structures on the twistor space \( P^6 \) of \( \mathbb{R}^4 \). Complex twistor space \( P^3_C = (P^6, \mathcal{J}) \) with integrable almost complex structure \( \mathcal{J} = \mathcal{J}^{\text{int}} \) has been studied a lot in the literature and in the following we will focus on non-integrable almost complex structure \( \mathcal{J} = \mathcal{J}^{\text{non}} \).

**Complex coordinates for \( \mathcal{J} = \mathcal{J}^{\text{int}} \).** The two-sphere \( S^2 \), global coordinates \( s^\alpha \) on which are used in (3.4), is conformally equivalent to \( \mathbb{R}^2 \). One can cover \( S^2 \) by two patches \( U_\pm \cong \mathbb{R}^2 \) with local coordinates
\[
v_+^\alpha = \frac{s_+^\alpha}{1 + s_+^3} \quad \text{on} \quad U_+ \quad \text{and} \quad v_-^\alpha = \frac{s_-^\alpha}{1 - s_-^3} \quad \text{on} \quad U_-,
\]
in which the metric on \( S^2 \) is conformally flat,
\[
ds_{S^2}^2|_{U_\pm} = \delta_{\alpha\beta} e^\alpha_\pm e^\beta_\pm = \frac{4\delta_{\alpha\beta} dv_\pm^\alpha dv_\pm^\beta}{(1 + \rho_\pm^2)^2} \quad \text{with} \quad \rho_\pm^2 = \delta_{\alpha\beta} v_\pm^\alpha v_\pm^\beta.
\]
On the intersection of two patches we have
\[
v_+^\alpha = \rho_+^2 v_-^\alpha,
\]
where \( \alpha, \beta = 1, 2 \).

On \( S^2 \) one can introduce vector fields of type (1,0) and (0,1) w.r.t. \( j \) from (3.8),
\[
\frac{\partial}{\partial \lambda_\pm} \quad \text{and} \quad \frac{\partial}{\partial \bar{\lambda}_\pm}, \quad j(\partial_{\lambda_\pm}) = i\partial_{\lambda_\pm} \quad \text{and} \quad j(\partial_{\bar{\lambda}_\pm}) = -i\partial_{\bar{\lambda}_\pm},
\]
where
\[
\lambda_\pm = v_+^1 + iv_+^2 \quad \text{and} \quad \bar{\lambda}_+ = \lambda_-^{-1} \quad \text{on} \quad U_+ \cap U_-.
\]
are complex coordinates on \( U_\pm \subset S^2 \). One-forms, dual to the vector fields (3.14), are \( d\lambda_\pm \) and \( d\bar{\lambda}_\pm \). Sphere \( (S^2, i) \) with the coordinates (3.15) can be identified with the Riemann sphere \( \mathbb{C}P^1 \).

By using the complex structure (3.4) on \( \mathbb{R}^4 \), one can introduce a \( \mathbb{C}P^1 \)-family of complex coordinates on \( \mathbb{R}^4 \) given by formulæ
\[
w_+^1 = y^1 + \lambda_+ \bar{y}^2 \quad \text{and} \quad w_+^2 = y^2 - \lambda_+ \bar{y}^1,
\]
where
\[ y^1 = x^1 + ix^2, \quad y^2 = x^3 - ix^4, \quad \tilde{y}^1 = x^1 - ix^2 \quad \text{and} \quad \tilde{y}^2 = x^3 + ix^4. \]
The coordinates (3.15) together with (3.16) provide complex coordinates on \( \mathcal{P}^6 \) given by
\[ w_+^1, \quad w_+^2 \quad \text{and} \quad w_+^3 = \lambda_+ \quad \text{on} \quad \mathcal{U}_+ = U_+ \times \mathbb{R}^4 \subset \mathcal{P}^6 \]
and
\[ w_-^1 = \lambda_- y^1 + \tilde{y}^2, \quad w_-^2 = \lambda_- y^2 - \tilde{y}^1 \quad \text{and} \quad w_-^3 = \lambda_- \quad \text{on} \quad \mathcal{U}_- = U_- \times \mathbb{R}^4 \subset \mathcal{P}^6. \]
On the intersection \( \mathcal{U}_+ \cap \mathcal{U}_- \) of patches \( \mathcal{U}_\pm \subset \mathcal{P}^6 \) these coordinates are related by formulae
\[ w_+^\alpha = w_+^3 w_-^\alpha \quad \text{and} \quad w_+^3 = \frac{1}{w_-^3} \quad \text{on} \quad \mathcal{U}_+ \cap \mathcal{U}_-. \]
Hence, the transition functions relating \( w_\pm^a \) and \( w_\pm^\alpha \) are holomorphic functions on \( \mathcal{U}_+ \cap \mathcal{U}_-, \ a = 1,2,3 \). This means that \( \mathcal{J}^{\text{int}} \) is an integrable almost complex structure and \( \mathcal{P}^3_\mathcal{C} = (\mathcal{P}^6, \mathcal{J}^{\text{int}}) \) is a complex 3-manifold. From (3.16) - (3.18) it follows that the manifold \( \mathcal{P}^3_\mathcal{C} \) can be identified with the total space of the holomorphic vector bundle over \( \mathbb{C}P^1 \),
\[ \mathcal{P}^3_\mathcal{C} = \mathcal{O}(1) \oplus \mathcal{O}(1) \quad \longrightarrow \quad \mathbb{C}P^1, \]
with coordinates \( w_\pm^a \) on fibres \( \mathbb{C}^2_j \) over points \( j \in \mathbb{C}P^1 \) parametrized by \( \lambda_\pm \subset U_\pm \subset \mathbb{C}P^1 \).

**Complex coordinates for \( \mathcal{J} = \mathcal{J}^{\text{non}} \).** By using the almost complex structure (3.10), we can introduce complex coordinates
\[ z_+^1 = w_+^1 = \tilde{y}^1 + \lambda_+ y^2, \quad z_+^2 = w_+^2 = \tilde{y}^2 - \lambda_+ y^1, \quad z_+^3 = w_+^3 = \lambda_+ \quad \text{on} \quad \mathcal{U}_+ \subset \mathcal{P}^6 \]
and
\[ z_-^1 = w_-^1 = \lambda_- y^1 + y^2, \quad z_-^2 = w_-^2 = \lambda_- y^2 - y^1, \quad z_-^3 = w_-^3 = \lambda_- \quad \text{on} \quad \mathcal{U}_- \subset \mathcal{P}^6. \]
On the intersection \( \mathcal{U}_+ \cap \mathcal{U}_- \) of two coordinate patches \( \mathcal{U}_\pm \subset \mathcal{P}^6 = \mathcal{U}_+ \cup \mathcal{U}_- \) we have
\[ z_+^\alpha = z_+^3 z_-^\alpha \quad \text{and} \quad z_+^3 = \frac{1}{z_-^3}. \]
From (3.22) we see that the transition functions on \( \mathcal{U}_+ \cap \mathcal{U}_- \) are not holomorphic. This reflects non-integrability of the almost complex structure (3.10). From (3.22) it follows that the manifold \( (\mathcal{P}^6, \mathcal{J}) \) with \( \mathcal{J} = \mathcal{J}^{\text{non}} \) can be identified with the total space of the anti-holomorphic vector bundle
\[ \mathcal{O}(1) \oplus \mathcal{O}(1) \quad \longrightarrow \quad \mathbb{C}P^1 \]
over \( \mathbb{C}P^1 \). Both base and fibres \( \mathbb{C}^2_j \) of this bundle are complex spaces but they do not glue into a complex manifold for \( \mathcal{J} \) given by (3.10).

**Spinor notation.** The rotation group \( \text{SO}(4) \) of space \( \mathbb{R}^4 \) is locally isomorphic to the group \( \text{SU}(2) \times \text{SU}(2) \), where both groups \( \text{SU}(2) \) have two-dimensional fundamental (spinor) representations
\[ \mu = (\mu_\alpha) \quad \text{and} \quad \lambda = (\lambda_\dot{\alpha}). \]
Commuting components \( \lambda_\alpha \) of the spinor \( \lambda \) are homogeneous coordinates on the Riemannian sphere \( \mathbb{C}P^1 \) such that
\[
\frac{\lambda_2}{\lambda_1} =: \lambda_+ \quad \text{on} \quad U_+ \subset \mathbb{C}P^1 \quad \text{and} \quad \frac{\lambda_1}{\lambda_2} =: \lambda_- \quad \text{on} \quad U_- \subset \mathbb{C}P^1 .
\]
(3.25)

Obviously, \( \lambda_+ = \lambda_-^{-1} \) if \( \lambda_1 \neq 0 \) and \( \lambda_2 \neq 0 \).

Isomorphism \( \text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2) \) allows also to introduce spinor notation for complex coordinates on \( \mathbb{R}^4 \) by formula
\[
(x^{\alpha\dot{\alpha}}) = \begin{pmatrix} x^{11} \\ x^{21} \\ x^{12} \\ x^{22} \end{pmatrix} = \begin{pmatrix} y^1 + iy^2 \\ y^2 - iy^1 \end{pmatrix} = \begin{pmatrix} x^1 + ix^2 \\ x^1 - ix^2 \end{pmatrix} .
\]
(3.26)

From (3.26) it follows that
\[
x^{11} = x^{22} \quad \text{and} \quad x^{12} = -x^{21} ,
\]
(3.27)

where the overbar denotes complex conjugation. By using (3.26), one can rewrite (3.16) and (3.20) as follows
\[
w^\alpha_+ = x^\alpha \lambda^\alpha_+ \quad \text{and} \quad z^\alpha_+ = -j^{\alpha}_\beta x^{\beta\dot{\lambda}}_\dot{\lambda} \lambda^\dot{\lambda}_,
\]
(3.28)

where
\[
(\lambda^\dot{\alpha}) = \frac{1}{\lambda_1} (\lambda_\dot{\alpha}) = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \text{and} \quad (\lambda^\dot{\lambda}_+) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\lambda^\dot{\lambda}_+) = \begin{pmatrix} -\lambda_+ \\ 1 \end{pmatrix} .
\]
(3.29)

By definition, we have \( \lambda_\dot{\alpha} = \lambda_\dot{\alpha}^{-1} \lambda^\dot{\alpha}_+ \) and \( \lambda^\dot{\lambda}_+ = \lambda^\dot{\lambda}_+^{-1} \lambda^\dot{\lambda}_+ \).

Vector fields and one-forms. On the twistor space \((\mathcal{P}^6, \mathcal{J})\) with \( \mathcal{J} \) from (3.10) we have the natural basis \( \left\{ \frac{\partial}{\partial z^\alpha_+} \right\} \) for the space of \((1,0)\) vector fields. On the intersection we have
\[
\frac{\partial}{\partial z^\alpha_+} = z^3 \frac{\partial}{\partial z^\alpha_-} \quad \text{and} \quad \frac{\partial}{\partial z^\alpha_-} = -(z^3_-)^2 \frac{\partial}{\partial z^\alpha_-} = z^3 z^\alpha \frac{\partial}{\partial z^\alpha_-} .
\]
(3.30)

Using formulæ (3.28), we can express these vector fields in terms of coordinates \((x^{\alpha1}, \lambda_\pm)\) and their complex conjugates according to
\[
\frac{\partial}{\partial x^{\alpha}_\pm} = -\gamma_+ i^{\beta}_\alpha \lambda_{\dot{\alpha}}^\beta \frac{\partial}{\partial x^{\beta\dot{\lambda}}} =: -i^{\beta}_\alpha V^\pm_\beta ,
\]
\[
\frac{\partial}{\partial z^\alpha_+} = \frac{\partial}{\partial \lambda_+} \gamma_+ i^{\beta}_\alpha x^{\alpha i} V^+_\beta , \quad \frac{\partial}{\partial z^\alpha_-} = \frac{\partial}{\partial \lambda_-} - \gamma_- i^{\beta}_\alpha x^{\alpha2} V^-_\beta ,
\]
(3.31)

where we have used
\[
\lambda_{\dot{\alpha}}^\dot{\lambda} = e^{\dot{\alpha}}_{\dot{\beta}} \lambda^\pm_{\dot{\beta}} \quad \text{with} \quad e^{\dot{1}2} = -e^{\dot{2}1} = 1 \quad \text{and} \quad \gamma_\pm = \frac{1}{1 + \lambda_\pm \lambda_\pm} = \frac{1}{\lambda_\pm^2 \lambda^\pm_\dot{\alpha}}
\]
(3.32)

together with the convention \( e^{\dot{1}2} = -e^{\dot{2}1} = -1 \), which implies \( e^{\dot{\alpha}}_{\dot{\beta}} e^{\dot{\gamma}}_{\dot{\delta}} = \delta^\dot{\gamma}_{\dot{\delta}} \). Thus, the vector fields
\[
V^\pm_\alpha = \gamma_+ \lambda^3 \lambda_{\dot{\alpha}} \partial_{\dot{\alpha}} , \quad V^+_3 = \gamma_+ \lambda_3 \partial_{\lambda_+} \quad \text{and} \quad V^-_3 = \lambda^-_3 \lambda^+_3.
\]
(3.33)
can be chosen as a basis of vector fields of type (1,0) on \( U_{\pm} \subset \mathcal{P}^6 \) in the coordinates \((x^{\alpha\dot{\alpha}}, \lambda_{\pm}, \bar{\lambda}_{\pm})\).

Complex conjugate of (3.33) provide us with the vector fields
\[
\dot{V}_a^{\pm} = \gamma_{\pm}^{\dot{\alpha}} \lambda_{\dot{\alpha}} \partial_{\alpha}, \quad \bar{V}_3^{\pm} = \gamma_{\pm}^{-2} \partial_{\lambda_{\pm}} \quad \text{and} \quad \bar{V}_3^{-} = \lambda_{\pm}^{-2} \bar{V}_3^{+}
\] (3.34)
which form a basis of vector fields of type (0,1) on \( U_{\pm} \subset \mathcal{P}^6 \).

It is easy to check that the basis of (1,0)- and (0,1)-forms on \( U_{\pm} \), which are dual to the vector fields (3.33) and (3.34), is given by forms
\[
E_{\pm}^a = -(d x^{\alpha\dot{\alpha}}) \lambda_{\dot{\alpha}} \partial_{\alpha}, \quad \bar{E}_3^{a} = \gamma_{\pm}^{\dot{\alpha}} \partial_{\bar{\lambda}_{\pm}} \quad \text{and} \quad \bar{E}_3^{a} = \lambda_{\pm}^{-2} \bar{E}_3^{+}.
\] (3.35)

One can easily verify that
\[
du_{\pm} = dz^{\pm}_{\pm} \frac{\partial}{\partial z^{\pm}_{\pm}} + dz^{\pm}_{\pm} \frac{\partial}{\partial \bar{z}^{\pm}_{\pm}} = E_{\pm}^{a} V_{a}^{\pm} + \bar{E}_{\pm}^{a} \bar{V}_{a}^{\pm}.
\] (3.36)

**Geometry of \((\mathcal{P}^6, \mathcal{J})\).** We consider the twistor space \((\mathcal{P}^6, \mathcal{J})\) with \( \mathcal{J} \) from (3.11) and coordinates \( \{ z^{a}_{\pm} \} \) on \( U_{\pm} \subset \mathcal{P}^6 \) given by (3.20)-(3.22). In the following we often omit the signs \( \pm \) in coordinates, vector fields, one-forms etc. by considering all formulæ on the patch \( U_{\pm} \subset \mathcal{P}^6 \).

By direct calculations we obtain that nonzero commutators of vector fields (3.33) and (3.34) are
\[
[V_3, V_a] = -\gamma^{-1} i_{\alpha}^{\dot{\alpha}} \bar{V}_{\beta} , \quad [V_3, \bar{V}_{\dot{a}}] = -\bar{\lambda} \gamma^{-1} \bar{V}_{\dot{a}} , \quad [V_3, \bar{V}_3] = 2\gamma (\bar{\lambda} \bar{V}_3 - \lambda V_3) ,
\] (3.37)
\[
[V_3, \bar{V}_{a}] = -\gamma^{-1} i_{\alpha}^{\dot{\alpha}} V_{\beta} \quad \text{and} \quad [V_3, \bar{V}_3] = -\lambda \gamma^{-1} V_{a} ,
\] (3.38)
where we used the formulæ
\[
\partial_{\lambda}(\gamma \lambda^{\dot{\alpha}}) = \gamma^{2} \dot{\lambda}^{\dot{\alpha}} \quad \text{and} \quad \partial_{\bar{\lambda}}(\gamma \bar{\lambda}^{\dot{\alpha}}) = -\gamma^{2} \bar{\lambda}^{\dot{\alpha}}.
\] (3.39)

To prove integrability of an almost complex structure \( \mathcal{J} \) one has to show that commutators of vector fields of type (0,1) w.r.t. \( \mathcal{J} \) will again be vector fields of type (0,1) [24]. From (3.37) we see that this is not the case and therefore \( \mathcal{J} \) is not integrable. For one-forms (3.35) we have
\[
dE_{1} = \lambda \gamma^{-1} \bar{E}^{3} \land E^{1} + \gamma^{-1} \bar{E}^{2} \land E^{3} ,
\]
\[
dE_{2} = \lambda \gamma^{-1} \bar{E}^{3} \land E^{2} + \gamma^{-1} \bar{E}^{3} \land E^{1} ,
\]
\[
dE_{3} = -2\lambda \gamma^{-1} \bar{E}^{3} \land E^{3} ,
\] (3.40)
and complex conjugate formulæ. The first terms in (3.40) define a torsionful connection on \( \mathcal{P}^6 \) with values in \( u(1) \subset su(3) \) and the last terms define the Nijenhuis tensor (torsion) with non-vanishing components
\[
N_{23}^{1} = \gamma^{-1} , \quad N_{31}^{2} = \gamma^{-1}
\] (3.41)
plus their complex conjugate \( N_{23}^{1} = N_{31}^{2} = \gamma^{-1} \). From (3.40) we again see that \((\mathcal{P}^6, \mathcal{J})\) is not a complex manifold but the total space (3.23) of the anti-holomorphic bundle over \( \mathbb{C}P^1 \). Furthermore, from (3.40) we see that \((\mathcal{P}^6, \mathcal{J})\) has an SU(3)-structure and the globally defined (3,0)-form \( \Omega \) with
\[
\Omega = E_{1}^{1} \land E_{1}^{2} \land E_{3}^{3} = E_{1}^{2} \land E_{2}^{2} \land E_{3}^{3} \quad \text{on} \quad U_{+} \cap U_{-}
\] (3.42)
since
\[ E^a_i = \tilde{\lambda}_i E^a , \quad E^3_i = \tilde{\lambda}_i^2 E^3 . \]  
(3.43)

Hence, the canonical bundle of \((P^6, J)\) is trivial. From (3.30) it follows that
\[ d(\text{Im}\Omega) = 0 , \]  
(3.44)
\[ d(\text{Re}\Omega) = -\gamma^{-1}(E^1 \wedge E^1 + E^2 \wedge \bar{E}^2) \wedge E^3 \wedge \bar{E}^3 , \]  
(3.45)
i.e. the real part of \(\Omega\) is not closed. For the volume form on \(P^6\) we have
\[ \text{Vol}_6 = \frac{i}{8} \Omega \wedge \bar{\Omega} = -\frac{i}{2} d^4 x \wedge \frac{d\lambda \wedge d\bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} , \]  
(3.46)
where \(d^4 x = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4\) in the coordinates (3.26).

**Twistor correspondence.** To conclude this section we describe a twistor correspondence between the SDYM model on \(\mathbb{R}^4\) and \(J\)-hCS theory on \((P^6, J)\).

Consider a complex vector bundle \(E\) over \(\mathbb{R}^4\) with a connection \(A = A_{\alpha \dot{\alpha}} dx^\alpha \bar{\alpha}\) and the covariant derivative \(\nabla = dx^\alpha (\partial_{\alpha \dot{\alpha}} + A_{\alpha \dot{\alpha}})\). Using the projection \(\pi : P^6 \rightarrow \mathbb{R}^4\) from (3.1), we can pull back the bundle \(E\) to a bundle \(\pi^* E\) with the pulled back connection \(\bar{A} = \pi^* A\) and the covariant derivative \(\nabla = \pi^* \nabla\), whose \((0,1)\)-component is
\[ \nabla_{0,1} = \bar{E}^\alpha (\bar{V}_\alpha + \gamma \bar{j}_\alpha \bar{\lambda} \bar{A}_{\bar{\beta} \bar{\beta}}) + \bar{E}^{\bar{\beta}} \bar{V}_{\bar{\beta}} . \]  
(3.47)
Equations (2.5) of \(J\)-holomorphic Chern-Simons theory on \((P^6, J)\) read
\[ [\nabla_{0,1}^a , \nabla_{0,1}^b] - \nabla_{0,1}^{[a \bar{V}_b]} = 0 , \]  
(3.48)
where \(a = (\alpha, 3) = 1, 2, 3\). Substituting (3.7) into (3.48) with
\[ \bar{A}_\alpha = \gamma \bar{j}_\alpha \bar{\lambda} \bar{A}_{\bar{\beta} \bar{\beta}} , \quad \bar{A}_3 = 0 , \quad A_\alpha = \gamma \bar{\lambda} \bar{A}_{\alpha \dot{\alpha}} \quad \text{and} \quad A_3 = 0 , \]  
(3.49)
we see that (3.48) are equivalent to the equations
\[ \bar{\lambda}_a \bar{\lambda}_b \left[ \partial_{\alpha \dot{\alpha}} + A_{\alpha \dot{\alpha}} , \ partial_{\beta \dot{\beta}} + A_{\beta \dot{\beta}} \right] = \bar{\lambda}_a \bar{\lambda}_b F_{a \dot{\alpha} \beta \dot{\beta}} = 0 , \]  
(3.50)
where \(F = dA + A \wedge A\) is the curvature of \(A\). Recall that in the spinor notation \(F\) has the components
\[ F_{a \dot{\alpha} \beta \dot{\beta}} = \epsilon_{\dot{a} \dot{\alpha}} \dot{f}_{\alpha \beta} + \epsilon_{\alpha \beta} \dot{f}_{\dot{a} \dot{\beta}} , \]  
(3.51)
where symmetric tensors
\[ \dot{f}_{\alpha \beta} = \frac{i}{2} \epsilon_{\dot{a} \dot{\alpha}} \dot{f}_{a \dot{a} \beta \dot{\beta}} \quad \text{and} \quad \dot{f}_{\alpha \beta} = \frac{i}{2} \epsilon_{\alpha \beta} \dot{f}_{a \dot{a} \beta \dot{\beta}} \]  
(3.52)
represent self-dual \(F^+\) and anti-self-dual \(F^-\) parts of the curvature \(F = F^+ + F^-\). Hence, the \(J\)-hCS equations (3.48) on \((P^6, J)\) with \(F(V_3, \bar{V}_3) = 0\) are equivalent to the SDYM equations on \(\mathbb{R}^4\),
\[ F^- = 0 \iff \epsilon_{\alpha \beta} F_{a \dot{a} \beta \dot{\beta}} = 0 , \]  
(3.53)
and any solution \(A\) of the SDYM equations (3.53) defines a solution of the \(J\)-hCS equations (3.48) and vice versa.

\[ ^1\text{We are working on the patch } U_+ = U_+ \times \mathbb{R}^4 \subset P^6 \text{ and omit subscript } \& \text{ superscript } "+" \text{ in formulae.} \]
4 Twistor actions for Yang-Mills theory

Graded twistor space $\mathcal{P}^6[2]$. Recall that on $\mathcal{P}^6$ there are globally defined $(3,0)$-form $\Omega$ given by (3.32) and its complex conjugate $(0,3)$-form $\bar{\Omega}$. Hence, the $\mathcal{J}$-hCS action functional (4.4) is well defined on $(\mathcal{P}^6, \mathcal{J})$. However, if we substitute (3.49) into (1.4) then we obtain $S = 0$ since $(0,3)$-part of Chern-Simons form $\text{CS}(\mathcal{A})$ on $(\mathcal{P}^6, \mathcal{J})$ vanishes if $A_3 = \bar{A}_3 = 0$. To obtain a nontrivial Lagrangian, one can perform a gauge transformation, which will give some non-vanishing term as it was done in [16, 17]. We will not follow this path here because this way we can at best get the actions [18–21] which have various limitations in comparison with the Chalmers-Siegel action [12].

The action [12] cannot be obtained without introducing additional degrees of freedom since it contains an extra propagating field $G_{\alpha\beta}$. One of the possibilities for introducing additional fields is to consider vector bundles $\mathcal{E}$ over $\mathcal{P}^6$ that are not trivial after restriction to $\mathbb{C}P^1 \hookrightarrow \mathcal{P}^6$. Another possibility is to consider a graded extension of the twistor space $(\mathcal{P}^6, \mathcal{J})$ similar to the cases considered by Wolf [10, 14] and Sämann [9, 15] for the complex twistor space $\mathcal{P}_c^6$. We will use the second option and introduce a graded twistor space $\mathcal{P}^{6[2]}$.

The space $\mathcal{P}^{6[2]}$ is parametrized by bosonic coordinates on $\mathcal{P}^6$ and by two anticommuting (fermionic) coordinates $\eta_i$,  
$$ \eta_1 \eta_2 + \eta_2 \eta_1 = 0, \quad (4.1) $$
generating the Grassmann algebra  
$$ \Lambda(\mathbb{R}^2) = \Lambda^0(\mathbb{R}^2) \oplus \Lambda^1(\mathbb{R}^2) \oplus \Lambda^2(\mathbb{R}^2), \quad (4.2) $$
where  
$$ 1 \cdot \mathbb{R} \in \Lambda^0(\mathbb{R}^2), \quad \eta_i \in \Lambda^1(\mathbb{R}^2), \quad i = 1, 2 \quad \text{and} \quad \eta := \eta_1 \eta_2 \in \Lambda^2(\mathbb{R}^2). \quad (4.3) $$
In the algebra (4.2) one may introduce $\mathbb{Z}_2$-grading,  
$$ \Lambda(\mathbb{R}^2) = \Lambda_0(\mathbb{R}^2) \oplus \Lambda_1(\mathbb{R}^2), \quad (4.4) $$
where  
$$ \Lambda_0(\mathbb{R}^2) = \Lambda^0(\mathbb{R}^2) \oplus \Lambda^2(\mathbb{R}^2) \quad \text{and} \quad \Lambda_1(\mathbb{R}^2) = \Lambda^1(\mathbb{R}^2). \quad (4.5) $$
We set $\text{gr} f = 0$ if $f \in \Lambda_0(\mathbb{R}^2)$ and $\text{gr} f = 1$ if $f \in \Lambda_1(\mathbb{R}^2)$, $\text{gr} f$ is the Grassmann parity of $f$.

On the space $\mathcal{P}^6$ we consider the space $\text{Gr}_{P^6}$ of locally defined functions (a sheaf) with values in the Grassmann algebra $\Lambda(\mathbb{R}^2)$. A manifold $\mathcal{P}^{6[2]}$ with the sheaf $\text{Gr}_{P^{6[2]}}$ is a graded manifold $\mathcal{P}^{6[2]} = (\mathcal{P}^6, \text{Gr}_{P^{6[2]}})$ [26, 27] that can be viewed as the trivial bundle $\mathcal{P}^6 \times \Lambda_1(\mathbb{R}^2) \to \mathcal{P}^6$. Tangent spaces of $\mathcal{P}^{6[2]}$ are defined by the even vector fields (3.33), (3.34) together with the odd vector fields  
$$ \partial^i := \frac{\partial}{\partial \eta_i} \quad \text{such that} \quad \frac{\partial}{\partial \eta_1} \frac{\partial}{\partial \eta_2} + \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_1} = 0, \quad (4.6) $$
commuting with the even vector fields on $\mathcal{P}^6$. Respectively, the space of differential forms on $\mathcal{P}^{6[2]}$ has the local basis $\{E^a, \bar{E}^a, d\eta_i\}$ with commutation relations  
$$ d\eta_1 \wedge d\eta_2 = d\eta_2 \wedge d\eta_1, \quad E^a \wedge d\eta_i = d\eta_i \wedge E^a \quad \text{and} \quad \bar{E}^a \wedge d\eta_i = d\eta_i \wedge \bar{E}^a \quad (4.7) $$

\footnote{Chern-Simons term $\text{CS}(\mathcal{A})$ is not invariant under gauge transformations.}
where \{E^a, \bar{E}^a\} are given in (3.35).

Recall that on \((\mathcal{P}^6, \mathcal{J})\) there are globally defined forms \(\Omega\) and \(\bar{\Omega}\). Hence, on \(\mathcal{P}^6|2\) we can introduce a closed \((3|2)\)-form

\[
\text{Im} \Omega \wedge d\eta_1 \wedge d\eta_2
\]

and the volume form

\[
\frac{i}{8} \Omega \wedge \bar{\Omega} \wedge d\eta = -\frac{i}{2} d^4x \wedge \frac{d\lambda \wedge d\bar{\lambda}}{(1 + \lambda \bar{\lambda})^2} \wedge d\eta,
\]

where \(d\eta = d\eta_1 \wedge d\eta_2\).

**Chern-Simons type theory on \(\mathcal{P}^6|2\).** Let \(\mathcal{E}\) be a trivial rank \(k\) complex vector bundle over \(\mathcal{P}^6|2\) and \(\mathcal{A}\) a connection one-form on \(\mathcal{E}\). We choose the connection \(\mathcal{A}\) depending on all coordinates on \(\mathcal{P}^6|2\) and having no components along the Grassmann directions. The curvature \(\mathcal{F}\) of such \(\mathcal{A}\) is

\[
\mathcal{F} = \mathcal{F}^B + \mathcal{F}^F = d^B \mathcal{A} + \mathcal{A} \wedge \mathcal{A} + d^F \mathcal{A},
\]

where \(d^B\) is the bosonic part (3.36) of the exterior derivative \(d = d^B + d^F\) and

\[
d^F = d\eta_i \partial^i \quad \text{for} \quad \partial^i = \frac{\partial}{\partial \eta_i}
\]

is the fermionic part of \(d\).

Consider the action functional

\[
S = \int_{\mathcal{P}^6|2} \text{Im} \Omega \wedge d\eta \wedge \text{CS}(\mathcal{A}),
\]

where

\[
\text{CS}(\mathcal{A}) = \text{tr}(\mathcal{A} \wedge d^B \mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A})
\]

is the Chern-Simons 3-form. Field equations following from (4.12) read

\[
\text{Im} \Omega \wedge \mathcal{F}^B = 0,
\]

where \(\mathcal{F}^B\) is defined in (4.10). From (4.14) it follows that

\[
\text{Re} \Omega \wedge \mathcal{F}^B = 0,
\]

since \(\Omega\) is a \((3,0)\)-form w.r.t. \(\mathcal{J}\),

\[
\mathcal{J} \Omega = i \Omega \quad \Leftrightarrow \quad \mathcal{J} \text{Im} \Omega = \text{Re} \Omega.
\]

Combining (4.14) and (4.15), we obtain

\[
\Omega \wedge \mathcal{F}^{0,2}_B = 0 \quad \Leftrightarrow \quad \mathcal{F}^{0,2}_B = 0.
\]

Note that from (3.45) and (4.17) it follows that \([28, 29]\)

\[
\mathcal{F}^B(V_1, \bar{V}_1) + \mathcal{F}^B(V_2, \bar{V}_2) = 0.
\]

The action functional (4.12) and solution to the equations (4.14)-(4.17) were considered in \([7, 28, 29]\).
Field equations on $\mathcal{P}^{6|2}$. Having given necessary ingredients, we may now consider $J$-hCS field equations (4.17). These equations on the patch $\mathcal{U}_+ = \mathcal{U}_+ \times \Lambda_1(\mathbb{R}^2)$ of $\mathcal{P}^{6|2}$ read

$$
\tilde{V}_\alpha \tilde{A}_\beta - \tilde{V}_\beta \tilde{A}_\alpha + [\tilde{A}_\alpha, \tilde{A}_\beta] = 0, \quad \tilde{V}_3 \tilde{A}_\alpha - \tilde{V}_\alpha \tilde{A}_3 + [\tilde{A}_3, \tilde{A}_\alpha] - [\tilde{V}_3, \tilde{V}_\alpha] \lambda A = 0, \quad (4.18)
$$

where “$\cdot$” denotes the interior product of vector fields with differential forms. Here we used components of $A$ in the expansion

$$
A = A_\alpha E^\alpha + \bar{A}_\alpha \bar{E}^\alpha = A_\alpha E^\alpha + \bar{A}_3 E^3 + \bar{A}_\alpha \bar{E}^\alpha + \bar{A}_3 \bar{E}^3. \quad (4.19)
$$

As usual in the twistor approach, we work in a gauge in which $\bar{A}_3 \neq 0$ but the bosonic part of $\bar{A}_3$ is zero. Note that in general the gauge potential $A$ in (4.18) and (4.19) can be expanded in the odd coordinates $\eta$ as

$$
A = A + \eta_i \psi^i + \eta_{12} G. \quad (4.20)
$$

For simplicity and more clarity we first consider the truncated case $\psi^i = 0$ and discuss the case $\psi^i \neq 0$ afterwards.

Remark. The connection (4.20) on the vector bundle $\mathcal{E}$ over $\mathcal{P}^{6|2} \cong \mathcal{P}^6 \times \Lambda_1(\mathbb{R}^2)$ takes values in the Lie algebra $\mathfrak{g}$ of a semi-simple Lie group $\mathfrak{G}$. Note that maps from the space $\Lambda_1(\mathbb{R}^2)$ in (4.15) to the group $\mathfrak{G}$ form a supergroup super-$T \mathfrak{G}$ [30], where $T \mathfrak{G} = \mathfrak{G} \times \mathfrak{g}$ is the semi-direct product of $\mathfrak{G}$ and $\mathfrak{g}$. That is why the field $A$ in (4.20) can be considered as a connection on a super-$T \mathfrak{G}$ bundle $\mathcal{E}'$ over the bosonic twistor space $\mathcal{P}^6$. This kind of correspondence was found by Witten when studying Chern-Simons theories on 3-manifolds [30].

From (3.33)-(3.35) one concludes that components $A_\alpha$ and $\bar{A}_3$ take values in the bundles $\mathcal{O}(-1)$ and $\mathcal{O}(2)$ over $\mathbb{C}P^1$ and $A_\alpha$ and $\bar{A}_3$ take values in the complex conjugate bundles. This fixes the dependence of $A$ on $\lambda$ and $\bar{\lambda}$ up to a gauge transformations (cf. [9] [10] [11] [14] [15]). Namely, we obtain

$$
A_\alpha = \gamma \left\{ \lambda^{\hat{\alpha}} A_{a\hat{\alpha}} + \eta (\lambda^{\hat{\alpha}} \sigma_{a\hat{\alpha}} + \gamma \lambda^{\hat{\alpha}} \bar{\lambda}^{\hat{\bar{\beta}}} \lambda^{3} G_{a\hat{\alpha}a\hat{\beta}a\hat{\gamma}}) \right\},
$$

$$
\bar{A}_a = \gamma \left\{ \bar{\lambda}^{\hat{\alpha}} A_{\bar{\alpha}a\hat{\alpha}} + \eta (\bar{\lambda}^{\hat{\alpha}} \bar{\sigma}_{a\hat{\alpha}} + \gamma \bar{\lambda}^{\hat{\alpha}} \lambda^{\hat{\beta}} \bar{\lambda}^{\hat{\bar{\gamma}}} \lambda^{3} G_{a\hat{\alpha}a\hat{\beta}a\hat{\gamma}}) \right\},
$$

$$
A_3 = \eta \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} \lambda^{\hat{\gamma}} G_{\hat{\alpha}a\hat{\beta}a\hat{\gamma}} \quad \text{and} \quad \bar{A}_3 = -\eta \lambda^{\hat{\alpha}} \lambda^{\hat{\beta}} \lambda^{\hat{\gamma}} G_{a\hat{\alpha}a\hat{\beta}a\hat{\gamma}}, \quad (4.21)
$$

where

$$
B_{a\hat{\alpha}} := G_{a\hat{\alpha}} - \frac{1}{3} \varepsilon^{\hat{\beta} \hat{\gamma}} G_{a\hat{\alpha}a\hat{\beta}a\hat{\gamma}} \quad (4.22)
$$

and the coefficient fields $A_{a\hat{\alpha}}$, $G_{a\hat{\alpha}}$ and $G_{a\hat{\alpha}a\hat{\beta}a\hat{\gamma}}$ do only depend on $x^{a\hat{\alpha}} \in \mathbb{R}^4$. Here $\lambda^{\hat{\alpha}}$, $\bar{\lambda}^{\hat{\alpha}}$ are given in (3.29) and (3.32), $\eta = \eta_1 \eta_2$, and parentheses denote normalized symmetrization with respect to the enclosed indices.

Substituting (4.21) into (4.18), we obtain the equations

$$
G_{a(\hat{\alpha} \hat{\beta} \hat{\gamma})} = \nabla_a (\sigma_{\hat{\alpha} \hat{\beta}} G_{\hat{\beta} \hat{\gamma}}), \quad (4.23)
$$

showing that $G_{a(\hat{\alpha} \hat{\beta} \hat{\gamma})}$ are composite fields describing no independent degrees of freedom. Other nontrivial equations following from (4.18) after substituting (4.21) read

$$
\varepsilon^{\hat{\alpha} \hat{\beta}} \left[ \partial_{a\hat{\alpha}} + A_{a\hat{\alpha}}, \partial_{a\hat{\beta}} \right] = \varepsilon^{\hat{\alpha} \hat{\beta}} F_{a\hat{\alpha}a\hat{\beta}} = 0, \quad (4.24)
$$

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\( \varepsilon^{\alpha\beta} \nabla_{\alpha\dot{\alpha}} B_{\beta\dot{\beta}} = 0 \), \hfill (4.25) \\
\( \varepsilon^{\dot{\alpha}\dot{\beta}} \nabla_{\alpha\dot{\alpha}} G_{\beta\dot{\beta}} = 0 \). \hfill (4.26)

We see that (4.24) coincide with the SDYM equations on \( \mathbb{R}^4 \) and (4.25) are the linearized SDYM equations for 
\( \delta A_{\beta\dot{\beta}} = B_{\beta\dot{\beta}} \). \hfill (4.27)

Hence, \( B_{\alpha\dot{\alpha}} \) is a tangent vector at \( A_{\alpha\dot{\alpha}} \) to the solution space of the SDYM equations. It is a secondary field (a symmetry) depending on \( A_{\alpha\dot{\alpha}} \) and for simplicity we neglect it by choosing \( B_{\alpha\dot{\alpha}} = 0 \). The rest equations (4.21) and (4.26) are the Chalmers-Siegel equations describing the self-dual gauge potential \( A_{\alpha\dot{\alpha}} \) and the anti-self-dual field \( G_{\alpha\dot{\alpha},\beta\dot{\beta}} = \varepsilon_{\alpha\beta} G_{\alpha\dot{\beta}} \) propagating in the self-dual background.

The action functional associated with (4.24) and (4.26) is given by 
\[ S_{sd} = 2 \int_{\mathbb{R}^4} d^4x \, \text{tr}(G^{\dot{\alpha}\dot{\beta}} f_{\dot{\alpha}\dot{\beta}}) \] \hfill (4.28)

with \( f_{\dot{\alpha}\dot{\beta}} \) given by (3.52). This action can be obtained from (4.12) after splitting, 
\[ A = X + \eta Y \] \hfill (4.29)

into ordinary bosonic and even nilpotent parts, using the formula\(^3\)
\[ \text{CS}(X + \eta Y) = \text{CS}(X) + 2\eta \text{tr}(Y \wedge F(X)) - \eta d^B(\text{tr}(X \wedge Y)) \] \hfill (4.30)

and integrating over the nilpotent coordinate \( \eta \) and over \( \mathbb{C}P^1 \hookrightarrow \mathbb{P}^4 \).

**Extra terms.** As we mentioned earlier, the general expansion (4.20) of connection \( A \) in odd coordinates \( \eta_i \) contains fermionic fields \( \psi^i(x^{\alpha\dot{\alpha}}) \) which we consider now. Expansion (4.20) can be written in components as 
\[ A_\alpha = \gamma_\dot{\alpha} A_{\alpha\dot{\alpha}}(\eta_1, \eta_2) \quad \text{and} \quad A_3 = \hat{\lambda}_\dot{\alpha}\hat{\lambda}_{\dot{\beta}} G_{\dot{\alpha}\dot{\beta}}(\eta_1, \eta_2) \] \hfill (4.31)

where 
\[ A_{\alpha\dot{\alpha}}(\eta_1, \eta_2) = A_{\alpha\dot{\alpha}} + \eta_i(\psi^i + \gamma_\dot{\alpha}\lambda^\dot{\gamma} \psi^i_{\alpha(\dot{\alpha})\dot{\gamma}}) + \eta_1 \eta_2 (B_{\alpha\dot{\alpha}} + \gamma_\dot{\alpha}\hat{\lambda}_{\dot{\beta}} G_{\alpha(\dot{\alpha})\dot{\beta}}) \] \hfill (4.32)

\[ G_{\dot{\alpha}\dot{\beta}}(\eta_1, \eta_2) = \eta_i \psi^i_{\dot{\alpha}\dot{\beta}} + \eta_1 \eta_2 G_{\dot{\alpha}\dot{\beta}} \] \hfill (4.33)

For \( A_\alpha \) and \( A_3 \) we have 
\[ \bar{A}_\alpha = \gamma\hat{\lambda}_\dot{\alpha}\hat{\lambda}_{\dot{\beta}} A_{\beta\dot{\beta}}(\eta_1, \eta_2) \quad \text{and} \quad \bar{A}_3 = -\hat{\lambda}_\dot{\alpha}\hat{\lambda}_{\dot{\beta}} G_{\dot{\alpha}\dot{\beta}}(\eta_1, \eta_2) \] \hfill (4.34)

Substituting (4.31)-(4.34) into the equations (4.18), we obtain the equations (4.23)-(4.27) and additional equations 
\[ \varepsilon^{\alpha\beta} \nabla_{\alpha\dot{\alpha}} \psi^1_{\beta\dot{\beta}} = 0 \quad \text{and} \quad \psi^2_{\beta\dot{\beta}} = 0 \] \hfill (4.35)

\[ \varepsilon^{\dot{\alpha}\dot{\beta}} \nabla_{\alpha\dot{\alpha}} \psi^1_{\beta\dot{\beta}} = 0 \quad \text{and} \quad \psi^1_{\alpha(\dot{\alpha})\dot{\gamma}} = \nabla_{\alpha(\dot{\alpha})\dot{\beta}} \psi^1_{\beta\dot{\beta}} \] \hfill (4.36)

\(^3\)Recall that in all formulae here \( d^B \) is the ordinary bosonic exterior derivative.
so that $F$ do not vanish. In particular, for restriction of Full Yang-Mills.

Thus, the general form (4.20) of components along the Grassmann directions but the mixed components of the curvature, $g$ is known that the action (4.28) is a limit of the full Yang-Mills action for small coupling constant can be obtained from the Chern-Simons type action (4.12) on the graded twistor space equations and $\psi$ is even and odd tangent vectors to the solution space.

Substituting (4.40) back into (4.39), we obtain

$$S_\varepsilon = -\varepsilon^2 \int_{\mathbb{R}^4} d^4x \text{tr}(G^{\hat{\alpha}\hat{\beta}} G_{\hat{\alpha}\hat{\beta}}) ,$$

so that

$$S_{\text{tot}} = S_{\text{sd}} + S_\varepsilon = 2 \int_{\mathbb{R}^4} d^4x \text{tr}(G^{\hat{\alpha}\hat{\beta}} f_{\hat{\alpha}\hat{\beta}} - \frac{1}{2} \varepsilon^2 G^{\hat{\alpha}\hat{\beta}} G_{\hat{\alpha}\hat{\beta}}) .$$

Here $\varepsilon$ is some small parameter. Variation of $S_{\text{tot}}$ w.r.t. $G_{\hat{\alpha}\hat{\beta}}$ gives

$$G_{\hat{\alpha}\hat{\beta}} = \frac{1}{\varepsilon^2} f_{\hat{\alpha}\hat{\beta}} .$$

Substituting (4.40) back into (4.39), we obtain

$$S_{\text{tot}} = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^4} d^4x \text{tr}(f_{\hat{\alpha}\hat{\beta}} f^{\hat{\alpha}\hat{\beta}}) = \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^4} \text{tr}(F^- \wedge F^-)$$

$$= -\frac{1}{4\varepsilon^2} \int_{\mathbb{R}^4} \text{tr}(F \wedge \ast F) + \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^4} \text{tr}(F \wedge F) .$$

Hence, the action (4.41) is equivalent to the standard Yang-Mills action

$$S_{\text{YM}} = -\frac{1}{4\varepsilon g_{\text{YM}}^2} \int_{\mathbb{R}^4} \text{tr}(F \wedge \ast F) ,$$

with the coupling constant $g_{\text{YM}} = \varepsilon$, plus the topological term. Therefore, for obtaining the Yang-Mills action (4.42) we should derive the term (4.38) from the twistor space.

**Twistor action for full Yang-Mills.** Recall that $\eta = \eta_1 \eta_2$, where $\eta_1$ and $\eta_2$ are real Grassmann variables. Consider a connection $A$ depending on $\eta$ as written in (4.21)\footnote{We do not consider the more general dependence (4.31)-(4.34) since we only want to show that one can obtain the action (4.42) from the twistor space. Consideration of (4.31)-(4.34) will give the Yang-Mills theory with its infinitesimal symmetries as we saw in the case of the SDYM equations.} It does not have components along the Grassmann directions but the mixed components of the curvature,

$$\mathcal{F}^F = d^F A = (\partial^\alpha A_\alpha) d\eta_\alpha \wedge E^\alpha + (\partial^\alpha \bar{A}_\alpha) d\eta_\alpha \wedge \bar{E}^\alpha = \mathcal{F}_\alpha^i d\eta_\alpha \wedge E^\alpha + \mathcal{F}_\alpha^{\bar{i}} d\eta_\alpha \wedge \bar{E}^\alpha ,$$

do not vanish. In particular, for restriction of $\mathcal{F}^F$ to $\mathbb{C}P^{1|2} \hookrightarrow \mathcal{P}^{6|2}$ we have

$$\mathcal{F}^F_{|\mathbb{C}P^{1|2}} = \mathcal{F}_\lambda^\alpha d\eta_\alpha \wedge d\lambda + \mathcal{F}_\lambda^{\bar{\alpha}} d\eta_\alpha \wedge d\bar{\lambda} ,$$

(4.44)
where
\[ F^i_\lambda = -\varepsilon^{ij} \eta^2 \hat{\lambda}^\alpha \hat{\lambda}^\beta G_{\alpha\beta} \quad \text{and} \quad F^{\bar{i}}_{\bar{\lambda}} = \varepsilon^{ij} \eta^j \gamma^2 \lambda^\alpha \lambda^\beta G_{\alpha\beta}. \] (4.45)

Using (4.45), we can introduce the gauge invariant functional
\[ \frac{\varepsilon^2}{8} \int_{\mathbb{P}^6|2} \Omega \wedge \bar{\Omega} \wedge d\eta_1 \wedge d\eta_2 \varepsilon_{ij} g^{i\lambda} \text{tr}(F^i_\lambda F^{\bar{i}}_{\bar{\lambda}}), \] (4.46)

where
\[ ds^2_{\mathbb{C}P^1} = g_{\lambda\bar{\lambda}} E^\lambda \otimes E^{\bar{\lambda}} \iff g_{\lambda\bar{\lambda}} = \gamma^2 \quad \text{and} \quad g^{\lambda\bar{\lambda}} = \gamma^{-2}. \] (4.47)

Integrating \( \text{tr}(F^i_\lambda F^{\bar{i}}_{\bar{\lambda}}) \) in (4.46) over fermionic coordinates and over \( \mathbb{C}P^1 \hookrightarrow \mathbb{P}^6|2 \), we obtain the functional \( S_\varepsilon \) given by (4.38). Hence, adding the local term given by (4.46) to the Chern-Simons type Lagrangian in (4.12), we obtain the full Yang-Mills action (4.42).

5 Conclusions

In this paper we considered graded twistor space \( \mathbb{P}^6|2 \) with a non-integrable almost complex structure \( J \) and \( J \)-holomorphic Chern-Simons theory on \( \mathbb{P}^6|2 \). It was shown that under some assumptions this theory is equivalent to self-dual Yang-Mills theory on \( \mathbb{R}^4 \). In our discussion we tried to be close to the consideration of the papers [14, 15], where \( \mathcal{N} < 4 \) SDYM theories were derived from holomorphic Chern-Simons theory on complex supertwistor spaces. We have also shown that the full Yang-Mills action in \( \mathbb{R}^4 \) can be obtained from a twistor action on \( \mathbb{P}^6|2 \) with a locally defined Lagrangian. We did not pursue the goal of studying all these tasks in full generality. We wanted to show the principal possibility of obtaining actions for Yang-Mills and its self-dual subsector from a twistor action. Examining all aspects of the model requires additional efforts.

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