Finite-size scaling for two-flavor QCD and comparison with $O(N)$ models

Tereza Mendes\textsuperscript{a}MCSD\textsuperscript{a}IFSC–USP, Caixa Postal 369, 13560-970 São Carlos SP, Brazil
\textsuperscript{a}[]

The chiral transition for two-flavor QCD is predicted to be in the same universality class as the $3d\ O(4)$ model. This prediction is verified in the Wilson case, but not for the staggered-fermion case. The comparison is usually done assuming infinite-volume behavior. Here we make an analysis of existing staggered-fermion data using finite-size scaling and normalizing the QCD data. We find better agreement for larger quark masses.

In order to check if two systems belong to the same universality class, one looks at universal quantities, such as the critical exponents. More generally, one compares scaling functions for both systems. The scaling ansatz implies that the magnetization $M$ of a system be described by a universal function

$$M/h^{1/\delta} = f_G(t/h^{1/\beta})$$

where $t \equiv (T-T_c)/T_0$ and $h \equiv H/H_0$ are the reduced temperature and magnetic field, respectively. Thus, once the non-universal normalization constants $T_0$ and $H_0$ are determined for a given system in the universality class, its order parameter $M$ scales according to the same scaling function $f_G$ for all systems in this class. For comparisons with QCD, an important region is the pseudo-critical line, defined by the points where the susceptibility $\chi$ shows a (finite) peak. This corresponds to the rounding of the divergence observed for $H=0, T=T_c$. The susceptibility scales as $\chi = \partial M/\partial H = (1/H_0) h^{1/\beta-1} g(t/h^{1/\delta})$ where $g(z) = 1/\delta [f_G(z) - z/\beta f_G'(z)]$. At each fixed $h$ the peak in $\chi$ is given by $t_p = z_p h^{1/\delta}$ and we have $M_p = h^{1/\delta} f_G(z_p), H_0 \chi_p = h^{1/\beta-1} g(z_p)$. Thus, the behavior along the pseudo-critical line is determined by the universal constants $z_p, f_G(z_p), g(z_p)$.

In addition to these infinite-volume scaling laws we may also consider finite-size-scaling functions. In fact, the scaling ansatz also implies

$$M = L^{-\beta/\nu} Q_z(h L^{\beta/\nu})$$

where $L$ is the linear size of the system and we consider fixed values of the ratio $t/h^{1/\beta} \equiv z$ (e.g. $z = 0$ as in the critical isotherm, or $z_p$ as along the pseudo-critical line). Thus, $M$ can be described by a universal finite-size-scaling (FSS) function of one variable. We note that in order to recover the infinite-volume expression $M = h^{1/\delta} f_G(z)$ as $L \to \infty$, we must have $Q_z(u) \to f_G(z) u^{1/\delta}$ for large $u$. Thus, in this limit, the FSS functions are given simply in terms of the scaling function $f_G(z)$. Working with the FSS functions $Q_z$ instead of the infinite-volume scaling function $f_G$ has the disadvantage that one must consider $z$ fixed (thus restricting the regions to be compared in parameter space) but the advantage that a comparison can be made already at finite values of $L$. 

Critical exponents and the scaling function $f_G$ are well-known for the 3d $O(4)$ model, whereas the pseudo-critical line and FSS functions have been recently studied in [1]. We find that the peak susceptibilities are given by $z_p = 1.33(5)$ and that the FSS functions $Q_z(u)$ approach their asymptotic values already at $u \sim 2$. For QCD, the order parameter is given by the chiral condensate $\langle \bar{\psi} \psi \rangle$, the analogue of the magnetic field is the quark mass $m_q$, and $t$ is proportional to $6/g^2 - 6/g^2_c(0)$. Thus, the chiral susceptibility peaks at $t_p \sim m_q^{1/\beta^G}$. Studies at $N_\tau = 4$ by the Bielefeld, JLQCD and MILC groups [2] show that the peak positions scale with the predicted exponents, but one sees no agreement with the $O(N)$ scaling function. This comparison was done up to multiplicative constants in the fields $t$ and $h$. We redo it below, performing a normalization of the QCD data.

Let $h \equiv N_\tau m_q/H_0$, $t \equiv (6/g^2 - 6/g^2_c(0))/T_0$. From the observed scaling along the pseudo-critical line and the universal quantities $z_p, f_G(z_p)$ from the $O(4)$ model, we determine the normalization constants $H_0$ and $T_0$. This allows an unambiguous comparison of the QCD data to $f_G$, as shown in Fig. [1]. We then do a comparison with the FSS functions $Q_z(h L^{\beta^G/\nu})$ using the data normalized above. We see that along the pseudo-critical line (JLQCD and Bielefeld data) FSS works well: the points are already in the asymptotic region and scale with the predicted exponent $\delta$ — as shown in [1] — and with the correct multiplicative constant, as expected from our normalization. In fact, all QCD data are asymptotic in $h L^{\beta^G/\nu}$, and we can thus rescale all data by their corresponding $f_G(z)$ factors and plot them together, as shown in Fig. [2]. We see a better agreement overall, although most MILC points are several deviations away from the predicted curve. As in Fig. [1], this happens for points outside the pseudo-critical line. Among these points, the ones with larger quark masses come closer to the curve. In particular, for the points with $m_q = 0.025$ we see reasonable agreement, suggesting that after finite-size effects have been taken into account one may see universal scaling even away from the pseudo-critical line.

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