Dedicated to the memory of Uffe V. Haagerup

C*-DYNAMICAL RAPID DECAY

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Abstract. Some well known results by Haagerup, Jolissaint and de la Harpe may be extended to the setting of a reduced crossed product of a C*-algebra $A$ by a discrete group $G$. We show that for many discrete groups, which include Gromov’s hyperbolic groups and finitely generated discrete groups of polynomial growth, an inequality of the form

$$\|X\| \leq C \sqrt{\sum_{g \in G} (1 + |g|)^4 \|X_g\|^2}$$

holds for any finitely supported operator $X$ in the reduced crossed product.

1. Introduction

Any result in classical harmonic analysis naturally raises the question, does this extend to a non commutative setting? In the situation of a discrete dynamical system, you may work with the group of integers $\mathbb{Z}$ which acts on a compact space as the group of homeomorphisms generated by a single homeomorphism. By Gelfand’s fundamental theorem we know that the set of compact topological spaces correspond to unital commutative C*-algebras, and then the classical discrete dynamical system as described above can be made non commutative in 2 ways, either by studying a general non commutative C*-algebra equipped with a single *-automorphism or by investigating properties of an action of a general discrete group of homeomorphisms on a compact space. The construction named reduced crossed product of a C*-algebra by a discrete group encodes a set up for the study of the action of a general discrete group on a C*-algebra.

Here we will extend results, by Haagerup from [8] on free non abelian groups $\mathbb{F}_d$, by Jolissaint [11] on the concept named rapid decay and by de la Harpe [10] on hyperbolic groups, from the setting of a reduced group C*-algebra to the setting of a reduced crossed product.
The C*-algebra $\mathcal{A}$ upon which the discrete group $G$ acts by *-automorphisms $\alpha_g$ may be abelian or non abelian, and the case where the algebra is the trivial $\mathcal{A} = \mathbb{C}I$ is the reduced group C*-algebra case.

Uffe Haagerup’s article [8] has been very influential in the study of discrete groups and von Neumann algebras of type II$_1$, and his name and results are linked to many fundamental mathematical properties such as the Haagerup approximation property [3], [9], the completely bounded approximation property [5] and the Haagerup tensor product [16]. We will not go into the study of any of these aspects but concentrate on an extension of the first basic result from [8] and lift it from the reduced group case to the reduced crossed product case for discrete hyperbolic groups acting on general C*-algebras. In this setting we prove that there exists an $C > 0$, depending on the group only, such that for any linear combination $X = \sum L_gX_g$ in the algebraic reduced crossed product we have the following inequality

$$\|X\| \leq C \sqrt{\sum_{g \in G} (1 + |g|)^4 \|X_g\|^2}$$

which is a direct generalization of Haagerup’s estimate in the group algebra case, since for these groups $C < 2$. In the article [11] Jolissaint showed that, in the group algebra case, inequalities like (1.1) may be obtained for many other discrete groups, which are not free, and he introduced the property named rapid decay for a discrete group $G$ with a length function $|g|$, if there exist positive constants $C, r$ such that for any $x = \sum x_g\lambda_g$ in $\mathbb{C}[G]$ we have the following inequality

$$\|x\|_{op} \leq C \sqrt{\sum (1 + |g|)^{2r} |x_g|^2}.$$

Shortly after Jolissaint had obtained his results they were extended to the setting of hyperbolic groups by de la Harpe [10], and it turns out that it is possible to extend both Jolissaint’s and de la Harpe’s results to the crossed product setting. The basic insight which makes this possible was formulated by Jollisaint in his lemmas 3.2.2 and 3.2.3. Then de la Harpe proved that these lemmas are true in the setting of discrete finitely generated hyperbolic groups, so Jolissaint’s findings could be extended. Here we have collected the statements from Jollissaint’s basic lemmas into a property (J) which a discrete group with a length function may have, and then we prove that the inequality (1.1) holds, in any reduced crossed product of a C*-algebra and a group satisfying property (J). It seems possible to us that property (J) implies
hyperbolicity, but we have very little experience in dealing with such a question.

We have also considered reduced crossed products of C*-algebras by discrete finitely generated groups which have polynomial growth. In this case it is possible to get the following - much better - result. If a discrete finitely generated group has polynomial growth then there exist $C > 0$, $s > 0$ such that for any finitely supported operator $X = \sum L_gX_g$ in a reduced crossed product $\mathcal{C}_r^*(\mathcal{A} \rtimes_\alpha G)$ we have

\[ \|X\| \leq C \left\| \sum_{g \in G} (1 + |g|)^{2s}L_gX_gX_g^* \right\|^{\frac{1}{2}} \]

The advantage of (1.3) over (1.2) is that usually

\[ \left( \sum_{g \in \mathcal{C}_k} \|X_g^*X_g\| \right)^{\frac{1}{2}} < \left( \sum_{g \in \mathcal{C}_k} \|X_g^*X_g\| \right)^{\frac{1}{2}}. \]

2. Notation and norms

Most of the content of this section is well known in the setting of a reduced group C*-algebra, but here we deal with a reduced crossed product C*-algebra. In order to be able to generalize the methods from Haagerup's paper [8], we need a characterization of the operator norm in a reduced crossed product of a C*-algebra by a discrete group, and this result, which we think might be known but unpublished, is presented in Proposition 2.4 in a self contained frame. We start with a well known operator theoretical version of Cauchy-Schwarz' inequality, [16].

**Definition 2.1.** Let $H$, $K$ be Hilbert spaces, $J$ be an index set and $(a_\iota)_{\iota \in J}$ a family of bounded operators in $B(H,K)$.

(i) If the sum $\sum_\iota a_\iota^*a_\iota$ is ultrastrongly convergent in $B(H)$ we say that the family $(a_\iota)$ is **column bounded** with column norm $\|(a_\iota)\|_c := \left\| \sum a_\iota^*a_\iota \right\|^{\frac{1}{2}}$.

(ii) If the sum $\sum_\iota a_\iota a_\iota^*$ is ultrastrongly convergent in $B(K)$ we say that the family $(a_\iota)$ is **row bounded** with row norm $\|(a_\iota)\|_r := \left\| \sum a_\iota a_\iota^* \right\|^{\frac{1}{2}}$.

**Proposition 2.2.** Let $H$, $K$ be Hilbert spaces, $J$ be an index set and $(a_\iota)_{\iota \in J}$, $(b_\iota)_{\iota \in J}$ be column bounded families of operators in $B(H,K)$.

(i) The sum $\sum_\iota a_\iota^*b_\iota$ is ultrastrongly convergent in $B(H)$ and the operator norm of the sum satisfies $\left\| \sum_\iota a_\iota^*b_\iota \right\| \leq \|(a_\iota)\|_c \|(b_\iota)\|_c$. 

There exists a column bounded family \((e_i)_{i \in I}\) of column norm at most 1 such that for the positive bounded operator \(h\) on \(H\) defined by 
\[
h := \left( \sum_{i} a_i^* a_i \right)^{1/2}
\]
we have for each \(i \in J,\ a_i = e_i h\) and the sum \(\sum_i e_i^* e_i\) equals the range projection of \(h\).

(ii) Let \(k := \left( \sum_i b_i^* b_i \right)^{1/2}\), then there exists a contraction \(c\) in \(B(H)\) such that 
\[
\sum_i b_i^* a_i = k c h.
\]

Proof. The families \((a_i)\) and \((b_i)\) represent bounded column operators in the operator space \(M_{(J,1)}(B(H,K))\) and the statement (i) follows from properties of the operator product.

The statement (ii) follows from the polar decomposition applied to the column operator \((a_i)\).

The result in statement (ii) may be applied to the column operator \((b_i)\) such that each \(b_i = f_i k\), we can then define a contraction \(c\) in \(B(H)\) by 
\[
c := \sum_i f_i^* e_i\]
and statement (iii) follows.

The rest of this article takes place in the setting of the reduced crossed product of a \(C^*\)-algebra \(A\) by a discrete group \(G\), which acts on \(A\) by the \(*\)-automorphisms \(\alpha_g\). We made a study of the properties of this crossed product in the article [4] and we will use most of the notation and several of the results of that article below. A basic point of view in Section 2 of [4] is that there are many facts related to properties of the coefficients of a Fourier series which generalize to properties of the coefficients of an element in a reduced discrete \(C^*\)-crossed product.

We recall that any element \(X\) in \(C := C^*_r(A \rtimes_{\alpha} G)\) has a Fourier series expansion \(X \sim \sum_{g \in G} L_g X_g\), where the sum is convergent in the norm \(\|\cdot\|_\pi\) described in Proposition 2.2 of [4]. A simple computation shows that for \(X \sim \sum L_g X_g\) we have that the column and row norms of the family \((L_g X_g)_{g \in G}\) may be computed as

\[
\| (L_g X_g)_{g \in G} \|_c^2 = \| \pi(X^* X) \| = \| (X^* X)_e \|
\]

\[
\| (L_g X_g)_{g \in G} \|_r^2 = \| \pi(X X^*) \| = \| (X X^*)_e \|.
\]

In particular we notice the following proposition.

**Proposition 2.3.** Let \(X \sim \sum_{g} L_g X_g\) be an element in \(C\) then the sum converges in the column norm.

We will use the \(\pi\)-norm or column norm to estimate the operator norm in the computations to come, so we need the following proposition. It may be known to several people, but may be in a slightly different setting. We are not aware of an explicit formulation as the one we present, but the results of [17] have a similar flavour. We do
also think that people who prefer to look at completely positive mappings as correspondences or operator bimodules may know the result, but still we are missing a reference.

**Proposition 2.4.** Let $\mathcal{B}$ be a $C^*$-algebra, $H$ a Hilbert space and $\pi: \mathcal{B} \to B(H)$ a completely positive and faithful mapping then

$$\forall b \in \mathcal{B} : \|b\| = \sqrt{\sup\{\|\pi(x^*b^*bx)\| : \|\pi(x^*x)\| \leq 1\}}.$$ 

**Proof.** We may suppose that $\mathcal{B}$ is a subalgebra of $B(K)$ for some Hilbert space $K$, then since $\pi$ is completely positive and faithful there exists by Stinespring’s result [18] a faithful representation $\rho$ of $\mathcal{B}$ on $H$ and a bounded operator $C$ in $B(K, H)$ such that

$$\forall b \in \mathcal{B} : \pi(b) = C^* \rho(b) C.$$ 

We may define a semi norm, say $n$, on $\mathcal{B}$ by

$$\forall b \in \mathcal{B} : n(b) := \sup\{\|\rho(bx)C\| : \|\rho(x)C\| \leq 1\}. \tag{2.2}$$

Since $\|C^* \rho(y^*y)C\| = \|\rho(y)C\|^2$ for any $y$ in $\mathcal{B}$ and $\pi$ is faithful we get that $n$ is a norm and that

$$\forall b \in \mathcal{B} : n(b) = \sqrt{\sup\{\|\pi(x^*b^*bx)\| : \|\pi(x^*x)\| \leq 1\}}.$$ 

On the other hand the definition (2.2) implies that for $b, d$ in $\mathcal{B}$ we have $n(bd) \leq n(b)n(d)$ so $n$ is an algebra norm and $n(b) \leq \|b\|$. For any pair $b, x$ in $\mathcal{B}$ with $\|\rho(x)C\|^2 = \|\pi(x^*x)\| \leq 1$ we have

$$n(b^*b) \geq \|\rho(b^*bx)C\| \geq \|C^* \rho(x^*b^*bx)C\| = \|\rho(bx)C\|^2,$$

so

$$n(b^*)n(b) \geq n(b^*b) \geq n(b)^2.$$

From here it follows that $n(b) = n(b^*)$ and then $n(b^*b) = n(b)^2$, and the completion say $\hat{\mathcal{B}}$ of $\mathcal{B}$ with respect to the $C^*$-norm $n$ becomes a $C^*$-algebra such that the inclusion of $\mathcal{B}$ in $\hat{\mathcal{B}}$ is a contractive faithful $*$-homomorphism and hence an isometry. The proposition follows. \qed

In Haagerup’s and Jolissaint’s articles, [8], [11] they use the symbol $\ast$ to denote the operator product in a group algebra, since this is really a convolution product. In our article on crossed product $C^*$-algebras [4] the operator product is an invisible dot and the $\ast$ is used to denote the Hadamard product, which in the notation from above takes the form

$$(\sum_g L_g X_g) \ast (\sum_h L_h Y_h) := \sum_f L_f X_f Y_f.$$ 

We will use this convention here, too.
3. The property (J)

The basic ideas in the arguments to come are taken from Haagerup’s article, and in the setting of a non commutative free group it is clear that for two group elements \( x, y \) with reduced words \( x = x_1 \ldots x_k \) and \( y = y_1 \ldots y_l \) the number of cancellations, say \( p \), needed to spell \( xy \) gives the spelling of \( xy \) directly as \( xy = x_1 \ldots x_{(k-p)}y_{(p+1)} \ldots y_l \).

In Jolissaint’s article he uses this observation in a very clever original way, and he shows that this idea may be generalized to work up to a controllable error in some groups of isometries on a Riemannian manifold with bounded strictly negative sectional curvature. Then de la Harpe showed that Jolissaint’s method of dealing with cancellations works in any finitely generated discrete group which is hyperbolic, as defined by Gromov, [7] and [6]. Here we will instead take these results as the basis for the definition of a property we have named (J).

We will now define the setting in which we will use Haagerup’s, Jolissaint’s and de la Harpe’s ideas. We define the cancellation number in a general group with a length function as follows.

**Definition 3.1.** Let \( G \) be a group with a length function \( g \to |g| \). For \( g, h \in G \) the cancellation number \( c(g, h) \) of the pair is defined as the non negative integer \( p(g, h) \) which satisfies

\[
2p(g, h) \leq |g| + |h| - |gh| < 2p(g, h) + 2.
\]

It follows from the properties of a length function that \( 0 \leq p(g, h) \leq \min\{|g|, |h|\} \), and the cancellation number divides the cartesian product \( G \times G \) into a sequence of disjoint subsets \( (P_p)_{p \in \mathbb{N}_0} \) defined by \( P_p := \{(g, h) \in G \times G : p(g, h) = p\} \).

Following Jolissaint we define certain subsets of a group \( G \) with a length function \( |g| \) as follows

**Definition 3.2.**

(i) \( \forall r \geq 0 : \quad B_r := \{g \in G : |g| \leq r\} \),

(ii) \( \forall k \geq 0 : \quad C_k := \{g \in G : k - 1 < |g| \leq k\} \),

(iii) \( \forall k \geq 0 \forall \alpha \geq 0 : \quad C_{k, \alpha} := \{g \in G : k - \alpha \leq |g| \leq k + \alpha\} \).

We can now collect the sufficient conditions, we have have dragged out of [11], into a property we name (J). We will not focus on which groups that may satisfy the property (J), but we think that the survey article on rapid decay [2] by Chatterji will show that many groups do have property (J). On the other hand we will sketch arguments which show that Jolissaint’s and de la Harpe’s examples do have property (J). It is easy to see that free non abelian groups do have property (J).
with the extra very nice properties that the constants of the definition satisfy \( \alpha = \beta = \gamma = 0 \) and \( N = 1 \).

**Definition 3.3.** Let \( G \) be a discrete group with a length function \( g \to |g| \). The pair \((G, | \cdot |)\) has property, (J) if

\[
(3.1)
\]

\[\exists \alpha > 0, \beta > 0, \gamma > 0, N \in \mathbb{N} : \exists u_{(g,s)} \in C_{s, \alpha} \]

\[\forall g \in G, \forall 0 \leq s < |g| + 1, \exists u_{(g,s)} \in C_{|g|+1-s, \beta} \]

\[\forall p \in \mathbb{N}_0, \forall (a, b) \in P_p \exists c(a, b) \in C_{p, \gamma} s.t. \]

\[a = u_{(ab,(|a| - p))}c(a, b), b = c(a, b)^{-1}v_{(ab,(|a| - p))}. \]

\[
(3.2)
\]

\[\forall \mu > 0, \nu > 0 \exists N \in \mathbb{N} : \]

\[\forall b \in G \forall 0 \leq p \leq |b| : |\{(c, v) \in C_{p, \mu} \times C_{(|b| - p), \nu} : c^{-1}v = b\}| \leq N. \]

**Remark 3.4.** It is important for the following proofs in the next section to notice that the group element \( u_{(g,s)} \) is determined uniquely by \( g \) and \( s \), and then \( v_{(g,s)} = u_{(g,s)}^{-1}g \) is also determined by \( g \) and \( s \) only.

The interesting thing about the factors \( c(a, b) \) is that they always approximately satisfy \( |c(a, b)| = p \).

In Jolissaint’s proof the group element \( u_{(g,s)} \) is chosen geometrically, as we sketch now. On the geodesic which connects a point \( m \) in the manifold with its image \( g(m) \), one chooses the point \( n_s \) which has the distance \( s \) to \( g(m) \). Then \( u_{(g,s)} \) is chosen such that the distance between \( n_s \) and \( u_{(g,s)}^{-1}(g(m)) \) is minimal among the distances between \( n_s \) and the set \( \{u(m) : u \in G\} \). We will not continue to quote Jolissaint’s proof, but show that a group \( G \), which is hyperbolic with respect to a given word length, has the property (J). The proof follows that of de la Harpe [10], but in order to explicitly establish the property (J) from the definition above, we repeat part of it here. We will use the following notation. For a real number \( s \), the expression \([s]\) means the largest integer dominated by \( s \).

**Lemma 3.5.** Let \( G \) be a finitely generated discrete group which is hyperbolic with respect to word length. Then \( G \) has the property (J).

**Proof.** For a group element \( g \) written in reduced form as \( g = g_1 \ldots g_{|g|} \) and a real \( s, 0 \leq s < |g| + 1 \) we define

\[
u_{(g,s)} := \begin{cases} e & \text{if } 0 \leq s < 1 \\
g_1 \ldots g_{[s]} & \text{if } 1 \leq s < |g| + 1, \end{cases}
\]
and we find that \( u_{(g,s)} \in C_{s,1} \) so \( \alpha = 1 \) may be used. Similarly we find

\[
v_{(g,s)} = \begin{cases} 
g([s]+1) \cdots g\lfloor g \rfloor & \text{if } 0 \leq s < |g| \\
e & \text{if } |g| \leq s < g + 1,
\end{cases}
\]

and we get \( v_{(g,s)} \in C_{(|g|+1-s),1} \), so \( \beta = 1 \) is possible.

Given a non negative integer \( p \) and a pair of group elements \((a,b) \in P_p\) with \( ab = g \), then for \( k := |a|, l := |b| \) there exists \( c \in \{0, 1\} \) such that \( |g| = k + l - 2p - c \). Then for \( u_{(g,k-p)} \) we may apply the lemma at the bottom of page 771 in \([10]\), to see that there exists an \( M \geq 0 \), independent of \( k, l, p, c \) such that for \( c(a,b) := u_{(g,k-p)}^{-1}a \) the following inequalities hold

\begin{equation}
\tag{3.3}
p \leq |c(a,b)| \leq p + M,
\end{equation}

so \( c(a,b) \in C_{p,M} \).

In order to establish the property (3.2) we remark, that in our case we have \( v_{(g,(k-p))} \in C_{(|g|+1-k+p),1} = C_{(l-p-c+1),1} \), which means

\[
c(a,b) \in C_{p,M}, \quad v_{g,(k-p)} \in C_{(|b|,2)} \quad \text{and} \quad c(a,b)v_{(g,(k-p))} = b.
\]

The result then follows from item (ii) in the lemma of \([10]\).

\[\square\]

4. Rapid Decay

The most basic example of the phenomenon named rapid decay by Paul Jolissaint in \([11]\) is presented quite early in many courses on Fourier series. The example tells, that if \( f(t) \) is a differentiable complex \( 2\pi \)-periodic function on \( \mathbb{R} \), then its Fourier series is uniformly convergent. This is proven via the following argument based on the Cauchy-Schwarz inequality, as follows. Let \( f(t) \) have the Fourier series \( f(t) \sim \sum_{n} c_n e^{int} \), then the derivative \( f'(t) \) has the Fourier series \( f'(t) \sim \sum_{n} in c_n e^{int} \), and the sequence of complex numbers \( (nc_n)_{n \in \mathbb{Z}} \) is in \( \ell^2(\mathbb{Z}) \). Since for \( n \neq 0 \) we may write \( c_n = \frac{1}{n}(nc_n) \), we get that the sequence \( (c_n)_{n \in \mathbb{Z}} \) is in \( \ell^1(\mathbb{Z}) \) with \( \|(c_n)\|_1 \leq |c_0| + \sum_{n} \| (nc_n) \|_2 \). If we translate this to the setting of the discrete group \( \mathbb{Z} \) equipped with the natural length function \( |n| \), we find that the group algebra \( C^*_r(\mathbb{Z}) \) may be identified with the complex continuous \( 2\pi \)-periodic functions on the real axis and an element \( x \) in the group algebra which correspond to a differentiable function has a presentation as a uniformly convergent sum \( x = \sum_{n \in \mathbb{Z}} x_n \lambda_n \). This example may be generalized to the setting of a discrete group with a length function when the content of the example is formulated as in the following proposition.
**Proposition 4.1.** Let \( x \sim \sum \lambda_n x_n \) be an operator in \( C^*_r(\mathbb{Z}) \). If the sequence \( ((1 + |n|)x_n)_{n \in \mathbb{Z}} \) is in \( \ell^2(\mathbb{Z}) \), then the series is uniformly convergent and
\[
\|x\| \leq \left( \frac{\pi^2}{3} - 1 \right)^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} |x_n|^2 (1 + |n|)^2 \right)^{\frac{1}{2}}.
\]

We recall from Chatterji’s survey article [2], Definition 2.9, in a modified form.

**Definition 4.2.** Let \( G \) be a discrete group with a length function \( |g| \), then \( G \) has the **rapid decay property**, with respect to \( |g| \) if there exists positive constants \( C, s \) such that for any operator \( x = \sum_g x_g \lambda_g \) in the reduced group \( C^* \)-algebra and with finite support we have
\[
\|x\| \leq C \sqrt{\sum_{g \in G} |x_g|^2 (1 + |g|)^{2s}}.
\]

It is immediate that this definition may be extended to the setting of a reduced crossed product of a \( C^* \)-algebra by a discrete group in several ways. We have played with 3 possibilities of definition, but only been able to obtain results for the 2 of them, which we define below. The third possibility is mentioned after the definition.

**Definition 4.3.** Let \( G \) be a discrete group with a length function \( g \to |g| \), such that \( G \) acts on a \( C^* \)-algebra \( A \) via a group of \( * \)-automorphisms \( \alpha_g \). The reduced crossed product \( \mathcal{C} := C^*_r(A \rtimes_h G) \) has **rapid decay** of operator type or scalar type if there exist positive constants \( C, s \) such that for any operator \( X = \sum L_g X_g \) with finite support:

- **operator type:**
  \[
  \|X\| \leq C \left\| \sum_{g \in G} (1 + |g|)^{2s} (L_g X_g X_g^* L_g^* + X_g^* X_g) \right\|^{\frac{1}{2}}
  \]

- **scalar type:**
  \[
  \|X\| \leq C \left( \sum_{g \in G} (1 + |g|)^{2s} \|X_g\|^2 \right)^{\frac{1}{2}}
  \]

If one of the properties above holds for any \( C^* \)-algebra \( A \) carrying an action \( \alpha_g \) of \( G \) we say that \( G \) possesses **complete rapid decay** of respectively operator type and scalar type.

In the very first version of the article we thought that we could prove that free non abelian groups do have a sort of mixed rapid decay defined as
mixed type:
\[ \|X\| \leq C\left( \sum_{k=0}^{\infty} (1 + |k|)^{2s} \| \sum_{g \in C_k} (L_g X_g X_g^* L_g^* + X_g^* X_g) \| \right)^{\frac{1}{2}}. \]

Unfortunately we were wrong, but it may still be that some discrete groups with non polynomial growth satisfy such a condition, and that would be very helpful in the study of multipliers of the form \(M\varphi\), as it follows from the proof of Proposition 7.1.

The complete operator type of rapid decay may be established for finitely generated groups with polynomial growth by a simple imitation of the proofs from the group algebra case. For other groups it seems impossible to us to establish the operator type of rapid decay outside the reduced group algebra case. We establish the complete scalar type of rapid decay for discrete groups which possess property (J) with respect to a length function.

5. COMPLETE OPERATOR RAPID DECAY IN A REDUCED CROSSED PRODUCT OF A \(C^*\)-ALGEBRA BY A FINITELY GENERATED GROUP WITH POLYNOMIAL GROWTH

Here we modify some of Jolissaint’s results from his section 3.1 of [11] to cover our situation.

**Theorem 5.1.** Let \(A\) be a \(C^*\)-algebra, \(G\) a finitely generated discrete group with polynomial growth which has an action \(\alpha_g\) on \(A\) as a group of \(*\)-automorphisms. There exists positive reals \(M, s\) such that for any finitely supported operator \(X = \sum L_g X_g\) in \(C := C_r^*(A \rtimes_\alpha G)\):

\[ \|X\| \leq M \| \sum_{g \in G} (1 + |g|)^{s+2} L_g X_g X_g^* L_g^* \|^\frac{1}{2}. \]
\[ \|X\| \leq M \| \sum_{g \in G} (1 + |g|)^{s+2} X_g^* X_g \|^\frac{1}{2}. \]

**Proof.** The polynomial growth implies that there exists positive reals \(C, s\) such that \(|C_k| \leq C(1 + k)^s\), then the following manipulations are standard, and the proof follows from Proposition 2.2 as follows.
\[ X = \sum_{k=0}^{\infty} \sum_{g \in C_k} \frac{1}{(1+k)|C_k|^{\frac{1}{2}}} ((1+k)|C_k|^{\frac{1}{2}} L_g X_g) \]

\[ \leq \sqrt{C} \sum_{k=0}^{\infty} \sum_{g \in C_k} \frac{1}{(1+k)|C_k|^{\frac{1}{2}}} ((1+k)^{1+\frac{s}{2}} \frac{1}{2} L_g X_g) \]

\[ \|X\| \leq \frac{\pi \sqrt{C}}{\sqrt{6}} \| \sum_{k=0}^{\infty} \sum_{g \in C_k} (1+k)^{(2+s)} X_g^* X_g \|^{\frac{1}{2}} \]

\[ \leq \frac{\pi \sqrt{C}}{\sqrt{6}} \sqrt{2} \| \sum_{g \in G} (1+|g|)^{(2+s)} X_g^* X_g \|^{\frac{1}{2}} \]

so \( M := \frac{\pi \sqrt{C}}{\sqrt{6}} \), may be used. The inequality involving \( L_g X_g X_g^* L_g^* \) follows in the same way. \( \square \)

6. Complete scalar rapid decay for discrete groups with property (J)

The basic result in this section is the proposition just below, and this is a combined extension of some of the first results in [8].

**Proposition 6.1.** Let \((G, \cdot, \cdot)\) be a discrete group with a length function satisfying the property (J). There exists a positive constant \( N \) such that for any action of \( G \) as a group of \(^*\)-automorphisms \( \alpha \) on a C*-algebra \( A \), any non negative integer \( k \) and any operator \( X = \sum_{g \in G} L_g X_g \) in \( C := C^*_r(A \rtimes_{\alpha} G) \) with finite support in \( C_k \):

\[ \|X\| \leq N (1+k) \sqrt{\sum \|X_g\|^2}. \]

**Proof.** We will let \( Y \) denote any finitely supported element in \( C \) with column or \( \pi \)-norm at most 1. By Proposition 2.4 it is sufficient to bound the \( \pi \)-norm of \( XY \) in order to bound the operator norm \( \|X\| \). Using the cancellation numbers, the operator \( XY \) may be written as a sum of \( k+1 \) summands \( S_p \) defined as follows

\[ XY = \sum_{a \in C_k} \sum_{b \in G} L_a X_a L_b Y_b \]

\[ = \sum_{p=0}^{k} \sum_{\{(a,b) \in F_p : a \in C_k\}} L_a X_a L_b Y_b \]

\[ = \sum_{p=0}^{k} S_p \]
Then let us fix a \( p \) in the set \( \{0, 1, \ldots, k\} \). By the property (J) there exists \( \alpha > 0, \beta > 0, M > 0 \) such that to each group element \( g \), each \( p \) and each pair of group elements \( (a, b) \in P_p \) with \( ab = g \) and \( a \in C_k \) there exist group elements \( u_{(g,(k-p))}, v_{(g,(k-p))}, c(a, b) \) such that

\[
(6.2) \quad u_{(g,(k-p))} \in C_{(k-p),\alpha}, \quad v_{(g,(k-p))} \in C_{(|g|+1-k+p),\beta}, \quad c(a, b) \in C_{p,M},
\]

\[
a = u_{(g,(k-p))}c(a, b), \quad b = c(a, b)^{-1}v_{(g,(k-p))}.
\]

We have \( |g| = |a| + |b| - 2p - c \) with \( c \in \{0, 1\} \), hence since \( |a| = k \) we get \( |g| + 1 - k + p = |b| - p + 1 - c \) so for \( \beta := \hat{\beta} + 1 \) we have

\[
(6.3) \quad v_{(g,(k-p))} \in C_{(|b|-p),\beta}.
\]

In Haagerup’s case with free groups we get \( u_{(g,(k-p))} = a_1 \ldots a_{(k-p)} \), \( c(a, b) = a_{(k-p+1)} \ldots a_k \), \( v_{(g,(k-p))} = b_{(p+1)} \ldots b_{|b|} \), with \( u_{(g,(k-p))} \in C_{(k-p)}, c(a, b) \in C_p, v_{(g,(k-p))} \in C_{(|b|-k)} \). In the case of a free non commutative group the 2 first elements \( u, c \) are determined by \( a, p \) and the last 2 elements \( c, v \) are determined by \( b, p \), but in the general case \( a, p \) does not determine the pair \( u_{(g,(k-p))}, c(a, b) \) nor does the pair \( b, p \) determines the pair \( c(a, b), v_{(g,(k-p))} \). The condition (3.2) is designed to deal with this problem.

We can now start the estimation, and we will use the results of Proposition 2.2, so for a given \( g \) we define a positive operator \( Q_{u_{(g,(k-p))}} \) by

\[
(6.4) \quad Q_{u_{(g,(k-p))}}^2 := \sum_{\{(a,b)\in P_p:a\in C_k, ab=g\}} L_a X_a X_a^* L_a^*.
\]

To each pair \( (a, b) \) in \( P_p \) with \( a \) in \( C_k \), and \( ab = g \) there exists a contraction \( q(a, b) \) such that \( L_a X_a = Q_{u_{(g,(k-p))}} \) \( q(a, b) \) with

\[
\sum_{\{(a,b)\in P_p:a\in C_k, ab=g\}} q(a, b)q(a, b)^* \leq I.
\]

Analogously we define \( R_{v_{(g,(k-p))}}^2 \) as the positive operator which is given by

\[
(6.5) \quad R_{v_{(g,(k-p))}}^2 := \sum_{\{(a,b)\in P_p:a\in C_k, ab=g\}} Y_b Y_b^*
\]

To each group element \( g \) and each pair \( (a, b) \) in \( P_p \) with \( g = ab \) and \( a \) in \( C_k \), there exists a contraction \( r(a, b) \) such that \( L_b Y_b = r(a, b) R_{v_{(g,(k-p))}} \) with

\[
\sum_{\{(a,b)\in P_p:a\in C_k, ab=g\}} r(a, b)^* r(a, b) \leq I,
\]
and according to Proposition 2.2 we may define a contraction operator $m_g$ by

$$(6.6) \quad \forall g \in G : \quad m_g := \sum_{\{ (a, b) \in P_p : a \in C_k, ab = g \}} q(a, b) r(a, b).$$

When combining these equations we get

$$(6.7) \quad S_p(g) = Q_{u(g, (k-p))} m_g R_{v(g, (k-p))}$$

and then

$$(6.8) \quad \sum_{g \in G} S_p(g)^* S_p(g) = \sum_{g \in G} R_{v(g, (k-p))} m_g^* Q_{u(g, (k-p))}^2 m_g R_{v(g, (k-p))} \, \text{by } \|m_g\| \leq 1$$

$$\leq \sum_{g \in G} \|Q_{u(g, (k-p))}^2 R_{v(g, (k-p))}^2 \| \, \text{by (6.4) and (6.5)}$$

$$\leq \sum_{g \in G} \left( \sum_{(a, b) \in P_p : ab = g} \|X_{u(g, (k-p))} c(a, b)\|^2 \right).$$

$$(\sum_{(e, f) \in P_p : ef = g} Y_{c(e, f)}^{*} Y_{c(e, f)}^{-1}) \text{ split to sum over } h, g$$

$$\leq \left( \sum_{g \in G} \sum_{(a, b) \in P_p : ab = g} \|X_{u(g, (k-p))} c(a, b)\|^2 \right).$$

$$(\sum_{h \in G} \sum_{(e, f) \in P_p : ef = h} Y_{c(e, f)}^{*} Y_{c(e, f)}^{-1})$$

In the last 2 sums depending on $g$ and $h$ respectively, one can via the property (3.2) get an upper bound on the number of times each element of the form $\|X_a\|^2$ or $Y_f^* Y_f$ appears in the sum. Let $a$ in $C_k$ be given, then the number of solutions to the equation

$$u \in C_{(k-p), \alpha}, \ c \in C_{p, M} : \quad uc = a$$

is at most $N$. Similarly for each $f$ the number of solutions to the equation

$$v \in C_{(|f|-p), \beta}, \ c \in C_{p, M} : \quad c^{-1} v = f$$

is at most $N$, and hence by (6.8)
\[(6.9) \quad \|S_p\|_\pi \leq N\left( \sum_{a \in C_k} \|X_a\|^2 \right)^{\frac{1}{2}} \|Y\|_\pi \text{ so}
\]
\[
\|XY\|_\pi \leq (k + 1)N\left( \sum_{a \in C_k} \|X_a\|^2 \right)^{\frac{1}{2}} \|Y\|_\pi \text{ by Proposition 2.4}
\]
\[
\|X\| \leq (k + 1)N\left( \sum_{a \in C_k} \|X_a\|^2 \right)^{\frac{1}{2}},
\]

and the proposition follows \( \square \)

In the case of a free non abelian group \( F_d \) with any set, finite or infinite, of generators the proposition above holds with \( N = 1 \). The reason is that for a pair \( (a, b) \) in \( P_p \) with \( a \in C_k \) the decompositions \( a = u_{(g, (k-p))}c(a, b) \) and \( b = c(a, b)^{-1}v_{(g, (k-p))} \) are described in an exact form just below the equation (6.3), such that the constants \( \alpha, \beta, \gamma \) in Definition 3.3 may be used with value 0. The exact decomposition also shows that in this case there will only be one solution to the equation (3.2) so we get \( N = 1 \) in this case, and we may note the following corollary.

**Corollary 6.2.** Let \( \mathcal{A} \) be a \( C^* \)-algebra with an action \( \alpha_g \) of \( F_d \), the free non commutative group with \( d \) generators, then for any \( k \in \mathbb{N}_0 \) and any \( X \) in \( C^*_r(\mathcal{A}) \) with finite support in \( C_k \):

\[
\|X\| \leq (k + 1)\left( \sum_{a \in C_k} \|X_a\|^2 \right)^{\frac{1}{2}}.
\]

It is worth to remark that the result in the corollary above in the case of the trivial \( C^* \)-algebra \( \mathcal{A} = \mathbb{C}I \) gives exactly the content of Lemma 1.5 in [8]. The content of Lemma 1.3 in [8] is named the **Haagerup property** in [15]. We may also here add a corollary describing the form the **Haagerup property** takes in the setting of a reduced crossed product of a \( C^* \)-algebra by a discrete group with the property (J) This is the natural extension of Jolissaint’s Proposition 3.2.4 in [11].

**Corollary 6.3.** For \( k, l, m \) non negative integers and any pair of finitely supported elements \( X = \sum_g L_g X_g \) with support in \( C_k \) and \( Y = \sum_g L_g Y_g \) with support in \( C_l \) :
\[ \|M_{\chi_m} \ast (XY)\| \leq N \left( \sum_{a \in C_k} \|X_a\|^2 \right)^{\frac{1}{2}} \sum_{b \in C_l} \|Y_b^* Y_b\|^{\frac{1}{2}} \]

\[ \|M_{\chi_m} \ast (XY)\| \leq N \left\| \sum_{a \in C_k} L_a X_a X_a^* L_a^* \right\|^{\frac{1}{2}} \left( \sum_{b \in C_l} \|Y_b\|^2 \right)^{\frac{1}{2}} \]

When \( G \) is a free non abelian group \( N = 1 \).

**Proof.** Let \( a \in C_k \) and \( b \in C_l \) be such that \( ab \in C_m \) then there exists uniquely determined \( c \in \{0, 1\} \) and \( p \) in \( \mathbb{N}_0 \) such that

\[ k + l - m = 2p + c. \]

In particular at most one \( S_p \neq 0 \), and the first inequality of the corollary follows from the proof of the theorem. The second follows from the first when applied to \( (Y^* X^*) \). The free group statement follows from the corollary just above. \( \square \)

The fact that there are 2 inequalities above indicates to us that there might be a hope for the desired inequality below to be true for a finitely supported \( X \) with support in \( C_k \) and a finitely supported \( Y \) with support in \( C_l \) we hope that

desired inequality

\[ \|M_{\chi_m} \ast (XY)\| \leq N \left\| \sum_{a \in C_k} L_a X_a X_a^* L_a^* \right\|^{\frac{1}{2}} \left( \sum_{b \in C_l} \|Y_b\|^2 \right)^{\frac{1}{2}} \]

We may then continue and consider general elements \( X \) with finite support.

**Theorem 6.4.** Let \( G \) be a discrete group with a length function such that \( G \) satisfies the condition (J). There exists an \( M > 0 \) such that for any action \( \alpha_g \) of \( G \) on a \( C^* \)-algebra \( \mathcal{A} \) and any \( X = \sum_{g \in G} L_g X_g \) of finite support in \( C_r^*(\mathcal{A} \rtimes \alpha G) \):

\[ \|X\| \leq M \sqrt{\sum_{g \in G} (1 + |g|)^4 \|X_g\|^2}. \]
Proof. We may proceed as in the proof of Lemma 1.5 in [8], so
\[ \|X\| \leq \sum_{k=0}^{\infty} \| \sum_{g \in C_k} X_g \| \]
\[ \leq N \sum_{k=0}^{\infty} (k + 1)^{-1} \left( (k + 1)^2 \left( \sum_{g \in C_k} \|X_g\|^2 \right)^{\frac{1}{2}} \right) \]
\[ \leq N \frac{\pi}{\sqrt{6}} \left( \sum_{k=0}^{\infty} (k + 1)^4 \sum_{g \in C_k} \|X_g\|^2 \right)^{\frac{1}{2}} \]
\[ \leq N \frac{\pi}{\sqrt{6}} \sqrt{2} \left( \sum_{g \in G} (1 + |g|)^4 \|X_g\|^2 \right)^{\frac{1}{2}} \]
and the theorem follows, since for \( g \in C_k \) we have \( k - 1 < |g| \leq k \). \( \square \)

Again there is a sharper estimate in the case of a free non abelian group.

**Corollary 6.5.** If \( G \) is a free non abelian group the constant \( M \) may be chosen as \( M = 2 \).

### 7. Applications

The theory of *rapid decay* for group \( C^* \)-algebras has been applied to various types of approximation properties for operator algebras [2], [3] [5], [9], and many research articles are based on, or inspired by these works.

Jolissaint realized from the beginning [12] that the rapid decay property makes it possible to base some K-theoretical computations on a subalgebra of *rapidly decreasing operators*, and this was then used by Lafforgue [14] in his fundamental work on the Baum-Connes conjecture.

The construction of a spectral triple for a reduced group \( C^* \)-algebra of a discrete group, which occurs in Connes’ non commutative geometry, has an obvious candidate if the group has a length function. It seems natural to use the property *rapid decay* to get some information on the properties of this spectral triple, and such attempts have appeared in [1] and [15]. It is interesting to see that the so-called *Haagerup* condition of [15] is the content of the very basic Lemma 1.3 of [8] and of Proposition 3.2.4 of [11]. Here it is contained in the corollary 6.3.

It is not our intent to pursue possible extensions of the results based on *rapid decay* from the group algebra case to the crossed product
setting, but we have made one easy observation, which may be applied
to a possible extension of some of the approximation properties.

In Haagerup’s first article [8] he shows in Lemma 1.7 that for a func-
tion $\varphi$ on a free non abelian group $G$ the multiplier $M_{\varphi}$ is bounded if
\[
\sup \{|\varphi(g)|(1+|g|)^2 : g \in G\} \text{ is finite and } \|M_{\varphi}\| \leq 2 \sup \{|\varphi(g)|(1+|g|)^2 : g \in G\}.
\]
This result makes it possible for him to cut the completely
positive multiplier of norm 1 given by $M_{\varphi \lambda} \lambda$ with
$\varphi \lambda(g) := e^{-\lambda|g|}$ to
the subsets $B_n$, and in this way he obtains a bounded approximate
multiplier unit consisting of functions with finite support. We can not
obtain such a nice result here because Haagerup’s estimate is based on
the fact that in the group algebra case we have $\|\lambda(f)\| \geq \|f\|_2$, and
the analogous statement for crossed products is not true. We can get a
result which is is similar to Haagerup’s Lemma 1.7 for a group action
which has operator rapid decay.

**Proposition 7.1.** Let $A$ be a C*-algebra, $G$ a discrete group with a
length function and $\alpha_g$ an action of the group on $A$ such that the reduced
crossed product has operator rapid decay with coefficients $C, s$. If a com-
plex function $\varphi$ on the group satisfies $m := \sup \{|\varphi(g)|(2 + |g|)^{(s+1)} : g \in G\} < \infty$ then the multiplier $M_{\varphi}$ on $C^*_r(A \rtimes G)$ is bounded and
satisfies $\|M_{\varphi}\| \leq 2Cm$. If the action of $\alpha_g$ has complete operator rapid
decay, then $M_{\varphi}$ is completely bounded with $\|M_{\varphi}\|_{cb} \leq 4Cm$.

**Proof.** Suppose $\varphi$ is given with $m$ finite then for any $X = \sum L_g X_g$ with
finite support

\begin{align}
\|M_{\varphi} \ast X\|^2 &\leq C^2 \| \sum_{g \in G} (1 + |g|)^2 |\varphi(g)|^2 (L_g X_g X_g^* L_g^* + X_g^* X_g) \| \\
&\leq C^2 \| \sum_{g \in G} (1 + |g|)^{-2} m^2 (L_g X_g X_g^* L_g^* + X_g^* X_g) \| \\
&\leq 4C^2 m^2 \sum_{k=0}^{\infty} (1 + k)^{-2} \| \sum_{g \in C_k} L_g X_g X_g^* L_g^* + X_g^* X_g \| \\
&\leq 8C^2 m^2 \frac{\pi^2}{6} \| X \|^2 \\
&\leq 16C^2 m^2 \| X \|^2.
\end{align}

If $G$ possesses complete operator rapid decay, the action $\alpha_g$ on $A$ may
be lifted to actions on $M_n(A)$, which all have operator rapid decay with
coefficients $C, s$ and the result follows. \qed
References

[1] C. Antonescu, E. Christensen, Metrics on group $C^*$-algebras and a non-commutative Arzelà-Ascoli theorem, J Funct. Anal. 214 (2004), 247–259.

[2] I. Chatterji, Introduction to the rapid decay property. Around Langlands correspondences, Contemp. Math. Amer. Math. Soc. 691 (2017), 53 – 72.

[3] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, A. Valette, Groups with the Haagerup property, Progress in Mathematics, 197 (2001), BirkhäuserVerlag, Basel.

[4] E. Christensen, The block Schur product is a Hadamard product, Math. Scand. 126 (2020), 603–616.

[5] M. G. Cowling, U. Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one, Invent. Math. 96 (1989), 507–549.

[6] E. Ghys, P. de la Harpe, Sur les groupes hyperboliques d’après Mikhael Gromov, ed., Birkhäuser Boston Inc., Boston, MA, 1990.

[7] M. Gromov, Hyperbolic groups, Essays in group theory, S. M. Gersten, ed., Springer, 1987.

[8] U. Haagerup, An example of a non nuclear $C^*$-Algebra, which has the metric approximation property, Inv. Math. 50 (1979), 279–293

[9] U. Haagerup, J. Kraus Approximation properties for group $C^*$-Algebras and group von Neumann algebras, Transactions Amer. Math. Soc. 334 (1994), 667–699

[10] P. de la Harpe, Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint, C. R. Acad. Sci. Paris, 307 1988, 771–774.

[11] P. Jolissaint, Rapidly decreasing functions in reduced $C^*$-algebras of groups, Trans. Amer. Math. Soc. 317 (1990), 167 – 196.

[12] P. Jolissaint, K-Theory of reduced $C^*$-Algebras and rapidly decreasing functions on groups, K-Theory 2 (1989), 723–735.

[13] R. V. Kadison, J. R. Ringrose, Fundamentals of the theory of operator algebras, Academic Press, 1986.

[14] V. Lafforgue, KK-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, Invent. Math. 149 (2002), 1–95.

[15] N. Ozawa, M. A. Rieffel, Hyperbolic group $C^*$-algebras and free-product $C^*$-algebras as compact quantum metric spaces, Canad. J. Math. 57 (2005), 1056–1079.

[16] V. Paulsen, Completely bounded maps and operator algebras, Cambridge Univ. Press, Cambridge, 2002.

[17] F. Pop, A. M. Sinclair, R. R. Smith, Norming $C^*$-algebras by $C^*$-subalgebras, J. Funct. Anal. 175 (2000), 168–196.

[18] W. F. Stinespring, Positive functions on $C^*$-algebras, Proc. Amer. Math. Soc. 6 (1955), 211–216.

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