Geometric and Combinatorial Properties of Well-Centered Triangulations in Three and Higher Dimensions

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Abstract

An \(n\)-simplex is said to be \(n\)-well-centered if its circumcenter lies in its interior. We introduce several other geometric conditions and an algebraic condition that can be used to determine whether a simplex is \(n\)-well-centered. These conditions, together with some other observations, are used to describe restrictions on the local combinatorial structure of simplicial meshes in which every simplex is well-centered. In particular, it is shown that in a 3-well-centered (2-well-centered) tetrahedral mesh there are at least 7 (9) edges incident to each interior vertex, and these bounds are sharp. Moreover, it is shown that, in stark contrast to the 2-dimensional analog, where there are exactly two vertex links that prevent a well-centered triangle mesh in \(\mathbb{R}^2\), there are infinitely many vertex links that prohibit a well-centered tetrahedral mesh in \(\mathbb{R}^3\).

1 Introduction

An \(n\)-dimensional simplex is \(n\)-well-centered if its circumcenter lies in its interior. More generally, it is \(k\)-well-centered if each of its \(k\)-dimensional faces is \(k\)-well-centered. It is completely well-centered if it is \(k\)-well-centered for each \(k\), \(1 \leq k \leq n\) [15]. Several authors have noted the possible application of well-centered meshes to particular problems [15]. Among these are Nicolaides [10] and Sazonov et al. [11, 12], who have discussed the covolume method and its application in electromagnetics simulations. Also, Kimmel and Sethian [5] described an algorithm for numerically solving the Eikonal equation on triangulated domains. Their algorithm, which can be used to compute geodesic paths on triangulated surfaces, is described first for acute triangulations (i.e., 2-well-centered triangulations) and requires additional work for triangulations that are nonacute. Ungör and Sheffer [13] used acute planar triangulations when they introduced the tent-pitching algorithm for space-time meshing. Well-centered meshes also find application within Discrete Exterior Calculus (DEC), a framework for designing numerical methods for partial differential equations [4, 3]. In DEC, a sufficient (but not necessary) condition for discretizing the Hodge star operator so that it is represented by a diagonal matrix is to use a well-centered mesh. The diagonal matrix leads to efficient numerical solution.

Constructing well-centered meshes is a nontrivial task, so researchers have put effort into finding ways to work around an algorithmic requirement for well-centered meshes. Recently, however, there has been progress towards making well-centered meshes through mesh optimization. In [16] we described a heuristic for obtaining well-centered meshes of planar domains by starting with a given triangle mesh of the domain and relocating the interior vertices of the mesh to minimize a cost function defined on the coordinates of those vertices. We later generalized the cost function, making it possible to optimize meshes of any dimension [17]. In [15] we give a variety of examples of tetrahedra that are and are not well-centered, and we show that it is possible to mesh many simple shapes in \(\mathbb{R}^3\) with completely well-centered tetrahedra. In many cases, the completely well-centered tetrahedral meshes of [15] were obtained by creating initial relatively high-quality meshes by hand and applying the optimization method to the initial meshes.
Not all simplicial meshes can be made well-centered by optimizing the cost functions of \[16\] and \[17\]. In some cases the combinatorial properties of the mesh prevent the mesh from becoming well-centered. In such cases the mesh will not be well-centered for any choice of vertex coordinates. This paper develops theory and intuition related to the geometry of well-centered simplices and applies the geometric conditions to investigate combinatorial properties of well-centered meshes.

Before discussing the specific results of this paper, a few comments about terminology are in order. The term well-centered may be used without a qualifying dimension to refer to the general concept, or may refer to one of the more precise terms if the context makes clear which more precise term is appropriate. When speaking of a triangle, for example, well-centered simultaneously means 2-well-centered and completely well-centered, since every simplex is trivially 1-well-centered. All of these definitions can be applied to simplicial meshes or simplicial complexes embedded in Euclidean space. Thus if a simplicial mesh is said to be \(k\)-well-centered, this means that every \(k\)-dimensional simplex appearing in the mesh properly contains its circumcenter.

## 2 Results

After giving definitions and notation in Sec. 3, we introduce in Sec. 4 several geometric and algebraic conditions for an \(n\)-simplex to be well-centered. Sections 5 and 6 apply the theory developed in Sec. 4 to establish conditions on the combinatorial structure of the neighborhood of a vertex in a well-centered tetrahedral mesh. Finally, Sec. 7 records some observations specific to constructing well-centered meshes of the cube, and Sec. 8 offers some concluding thoughts.

We enumerate the contributions of this paper in more detail in the following paragraphs.

In Sec. 4 we prove three new conditions on when an \(n\)-simplex is \(n\)-well-centered, each phrased in terms of the location of a vertex \(v_i\) given the facet of the simplex opposite \(v_i\). These conditions are (a) a necessary condition expressed in terms of geometry (Prop. 3—the Cylinder Condition), (b) a sufficient condition expressed geometrically (Prop. 8—the Prism Condition), and (c) a both necessary and sufficient condition expressed in terms of cubic polynomial inequalities (Prop. 11). The two geometric conditions are generalizations to higher dimensions of conditions in \(\mathbb{R}^2\).

Section 5 investigates combinatorial properties that follow from the results in Sec. 4. (d) We prove a new combinatorial condition that must be satisfied by the link of an interior vertex in an \(n\)-well-centered mesh in \(\mathbb{R}^n\) (Thm. 13). (e) As an easy corollary we show that in a 3-well-centered tetrahedral mesh in \(\mathbb{R}^3\), every interior vertex has at least seven incident edges (Cor. 14). (f) We show that, in stark contrast to the analogous case in \(\mathbb{R}^2\), where there are only two vertex links that cannot appear in a 2-well-centered mesh, there are infinitely many vertex links that cannot appear in a 3-well-centered mesh in \(\mathbb{R}^3\) (Cor. 15). (g) We also construct an infinite family of vertex links that can appear in a completely well-centered tetrahedral mesh in \(\mathbb{R}^3\). (h) The section closes by showing that if a vertex link can appear in a 3-well-centered mesh and the vertex link satisfies some minor additional conditions, then degree three vertices can be successively inserted into the vertex link to create an infinite family of vertex links that can appear in a 3-well-centered mesh (Prop. 18).

Section 6 develops combinatorial conditions that 2-well-centered tetrahedral meshes in \(\mathbb{R}^3\) must satisfy. (i) We prove in Thm. 22 that no triangulation of \(S^2\) on \(m\) vertices with a vertex of degree at least \(m - 3\) can appear in a 2-well-centered tetrahedral mesh in \(\mathbb{R}^3\). (j) It follows that in a 2-well-centered tetrahedral mesh in \(\mathbb{R}^3\), every interior vertex has at least nine incident edges (Cor. 23). (k) We show that vertices of degree three can be inserted into or deleted from triangulations that permit 2-well-centered neighborhoods to create other triangulations that permit 2-well-centered neighborhoods (Prop. 24). (l) Vertices of degree four can also be added to such triangulations (Prop. 25).

At several points in the paper, we make claims about additional results beyond what is actually proved in this paper. Further details about some of these claims appear in the first author’s dissertation [14].
3 Definitions and Notation

We begin by introducing some definitions and notation that will be used throughout the paper. A simplex is referred to with a Greek letter, usually σ or τ. A superscript for a simplex indicates the dimension, so, for example, σ^n is an n-simplex. The notation σ^n = [v_0 v_1 ... v_n] is used to indicates that σ^n is the convex hull of the n + 1 vertices v_0, v_1, ..., v_n. It is assumed that the vertices of a simplex are in general position, i.e., that the vertices are affinely independent, so σ^n is fully n-dimensional. The circumcenter of a simplex σ is denoted c(σ). For an n-simplex σ^n embedded in \( \mathbb{R}^m \), c(σ^n) is the unique point which has the same distance from every vertex of σ^n. When σ^n is embedded in \( \mathbb{R}^m \) for \( m > n \), c(σ^n) is the unique point that among all points equidistant from the vertices of σ^n minimizes the distance to the vertices of σ^n. The circumradius of a simplex σ, i.e., the distance from c(σ) to the vertices of σ, is denoted R(σ).

We also use the cone operation of algebraic topology [9], writing \( u * σ^n \) to indicate the simplex formed by taking the convex hull of a vertex u together with the n-dimensional simplex σ^n to form a simplex of dimension \( n + 1 \). This notation may also be used for a set K of simplices; \( u * K \) is the set of simplices \{ u * σ : σ ∈ K \}. The affine hull of a set \( S ⊂ \mathbb{R}^m \), which we denote aff(S), is the smallest affine space that contains S. For a simplex σ^n = [v_0 ... v_n] we can define it as

\[
\text{aff}([v_0 ... v_n]) = \left\{ \sum_{i=0}^{n} λ_i v_i : \sum_{i=0}^{n} λ_i = 1, \ -∞ < λ_i < ∞ \text{ for } i = 0, ..., n \right\}.
\]

The affine hull of a simplex σ may also be called the plane of σ.

When referring to a simplex σ, the boundary of the simplex, denoted Bd(σ), is the union of the set of proper faces of σ, i.e., the set of all faces of σ other than σ itself. The interior of the simplex, denoted by Int(σ), is defined as σ - Bd(σ). More generally, for a set S, we use Int(S) to refer to the interior of S taken with respect to the usual topology of aff(S). For the closure of a set S we use the notation Cl(S).

For a vertex u of a simplicial complex we define St u, the star of the vertex, to be the union of the interiors of all the simplices for which u is a vertex. The closure of the star, or the closed star, Cl(St u), is the union of all simplices incident to u. The link of a vertex u is defined by Lk u = Cl(St u) - St u. Many of the terms briefly defined here are defined and discussed at more length in [9].

We wish to avoid any ambiguity about the dimension of the circumsphere or circumball of a simplex σ. Throughout this paper the objects circumsphere and circumball always live in aff(σ). Thus the circumsphere of a triangle is always a copy of \( S^1 \), even when the triangle is embedded in \( \mathbb{R}^3 \) as the facet of a tetrahedron. The equatorial ball of a simplex σ, sometimes denoted B(σ), is a ball of radius R(σ) centered at c(σ), but distinguished from the circumball of σ by the fact that the equatorial ball is considered in a higher-dimensional space. For example, the equatorial ball of a triangle τ considered in \( \mathbb{R}^3 \) is the unique 3-dimensional ball that has the circumcircle of τ as an equator (see Fig. 1). The ambient higher-dimensional space that contains the equatorial ball should be made clear wherever the term is used. In this paper, wherever the term equatorial ball is used and a higher-dimensional space is not explicitly specified, the term appears in the context of a simplex \( σ^n = u * τ^{n-1} \), and B(τ^{n-1}) is an n-dimensional subset of aff(σ^n).

We frequently discuss the facets of a simplex σ^n = [v_0 ... v_n]. As a matter of convention, the facets usually are denoted \( τ_0^{n-1}, ..., τ_m^{n-1} \) with the understanding that the facet \( τ_i^{n-1} \) is the facet opposite vertex \( v_i \). Thus σ^n = v_i * τ_i^{n-1} for each \( i = 0, 1, ..., n \).

4 Characterizing the Well-Centered Simplex

We now investigate geometric properties of an n-well-centered n-simplex. The context for this discussion is a simplex \( σ^n = u * τ^{n-1} \) with facet \( τ^{n-1} \) given. The vertex u is free to move, and we wish to determine whether \( σ^n \) is n-well-centered based on the position of u relative to the fixed vertices of \( τ^{n-1} \).
Figure 1: Four views of the same 3-well-centered tetrahedron $\sigma^3$ in the same orientation. From left to right the views show $\sigma^3$ with the equatorial balls of its bottom, right, left, and rear facets. For each facet $\tau^2_i$ of $\sigma^3$, the circumsphere (i.e., circumcircle) of $\tau^2_i$, which is an equator of the equatorial ball $B(\tau^2_i)$, is shown. Because the tetrahedron is 3-well-centered, the vertex $v_i$ opposite facet $\tau^2_i$ lies outside of $B(\tau^2_i)$; an $n$-simplex is $n$-well-centered if and only if for each vertex $v_i$, $v_i$ lies outside of the equatorial ball of the facet $\tau^{n-1}_i$ opposite $v_i$. For the bottom facet and rear facet views in the figure, the reader may need to look closely to see that the edges incident to $v_i$ do pierce $B(\tau^2_i)$, and $v_i$ lies outside $B(\tau^2_i)$.

We first recall an alternate geometric characterization of the $n$-well-centered $n$-simplex and state its consequences in this context. The alternate characterization is stated in terms of equatorial balls. Using the result, which adopts the notational convention that $\sigma^n = v_i \ast \tau^{n-1}_i$ for each $i = 0, 1, \ldots, n$, one can determine whether an $n$-simplex is $n$-well-centered without explicitly computing $c(\sigma^n)$.

**Theorem 1** (Equatorial Balls Condition). The simplex $\sigma^n$ is $n$-well-centered if and only if vertex $v_i$ lies strictly outside $B(\tau^{n-1}_i)$ for each $i = 0, 1, \ldots, n$.

**Proof.** See [17].

Figure 1 illustrates Theorem 1 as it applies to a tetrahedron. For each vertex $v_i$ of the tetrahedron, Fig. 1 shows the equatorial ball $B(\tau_i)$ of the facet $\tau_i$ opposite $v_i$, emphasizing in a darker color the circumcircle of $\tau_i$ (which is an equator of $B(\tau_i)$). The figure shows that in each case $v_i$ is outside the equatorial ball of $\tau_i$, so we can conclude that the tetrahedron is 3-well-centered. Moreover, this same condition is satisfied by every 3-well-centered tetrahedron.

In the context of an $n$-simplex $\sigma^n$ with a free vertex $u$ opposite a fixed facet $\tau^{n-1}$, Theorem 1 becomes a necessary condition that vertex $u$ must satisfy if $\sigma^n$ is to be $n$-well-centered.

**Corollary 2** (One-Facet Equatorial Ball Condition). Let $\sigma^n = u \ast \tau^{n-1}$. If the simplex $\sigma^n$ is $n$-well-centered, then $u$ lies strictly outside of $B(\tau^{n-1})$.

To introduce the remaining results of this section and get a somewhat different perspective on Corollary 2, we consider the sketch in Fig. 2. In the sketch, $\tau$ is a given triangle in $\mathbb{R}^3$ with fixed vertices. Triangle $\tau$ represents a facet of a tetrahedron $\sigma$ whose fourth vertex $u$ has not yet been determined. (Thus tetrahedron $\sigma$ does not appear in Fig. 2.) We suppose, however, that the circumcenter $c(\sigma)$ of the tetrahedron is known, so $u$ is constrained to lie on the circumsphere of $\sigma$. The two sides of Fig. 2 show two different cases; on the left $\tau$ is not 2-well-centered, and on the right $\tau$ is 2-well-centered.

Now $\tau$ lies in the plane $\text{aff}(\tau)$, and the reflections of $\tau$ and $\text{aff}(\tau)$ through $c(\sigma)$ are $\tau'$ and $\text{aff}(\tau')$ respectively. The plane $\text{aff}(\tau)$ intersects the circumsphere to determine a lower spherical
cup $C$, and $\text{aff}(\tau')$ determines an upper spherical cup $C'$. The necessary condition of Corollary 2 says that when $\sigma$ is 3-well-centered $u$ does not lie in the lower spherical cup $C$. From the geometry one can see that, in fact, if $\sigma$ is to be 3-well-centered, then $u$ must lie strictly inside the spherical triangle determined by the intersection of $C'$, the upper cup of the circumsphere of $\sigma$, with the geometric cone on $\tau'$ with apex $c(\sigma)$. (This spherical triangle is drawn in the case on the right, but not in the case on the left.)

In particular, there is a necessary condition that $u$ must lie strictly in the upper cup $C'$ of the sphere. Thus, speaking with regard to the orthogonal projection into $\text{aff}(\tau)$, $u$ projects (vertically in the figure) to the interior of the circumball of $\tau$. Moreover, if $\tau$ is 2-well-centered, as on the right, then the projection of the spherical triangle into $\text{aff}(\tau)$ contains the projection of $\tau'$ into $\text{aff}(\tau)$. Thus if the projection of $u$ into $\text{aff}(\tau)$ lies inside the projection of $\tau'$ into $\text{aff}(\tau)$, this is sufficient to establish that $\sigma$ is 3-well-centered. These conditions and their generalizations into higher dimensions are the first two conditions discussed in this section. The geometric intuition developed here is formalized and proved algebraically in Propositions 3 and 8.

Finally, we consider varying the position of $c(\sigma)$. Notice that as $c(\sigma)$ moves in Fig. 2 from the circumcenter of $\tau$ upward along a line orthogonal to $\text{aff}(\tau)$, the spherical triangle of $u$-positions that produce a 3-well-centered tetrahedron with circumcenter $c(\sigma)$ sweeps out a solid 3-dimensional region. Tetrahedron $\sigma$ will be 3-well-centered if and only if $u$ lies in this region. The section closes by describing this region for arbitrary dimensions in terms of polynomial inequalities. (See Fig. 9.)

Now we state a proposition that gives a necessary condition for an $n$-simplex $\sigma^n$ to be $n$-well-centered. See Fig. 3.

**Proposition 3** (Cylinder Condition). Let $\sigma^n$ be an $n$-well-centered $n$-simplex in $\mathbb{R}^n$ with $u$ a vertex of $\sigma^n$ and $\tau^{n-1}$ the facet of $\sigma^n$ opposite $u$. That is, let $\sigma^n = u \ast \tau^{n-1}$. Let $P$ be the orthogonal projection $P : \mathbb{R}^n \to \text{aff}(\tau^{n-1})$. Then $\|P(u) - c(\tau^{n-1})\| < R(\tau^{n-1})$, i.e., $u$ projects to the interior of the circumball of $\tau^{n-1}$.

**Proof.** Consider the coordinate system on $\mathbb{R}^n$ such that $c(\sigma^n)$ is the origin, and $\text{aff}(\tau^{n-1}) = \{x \in \mathbb{R}^n : x_n = k\}$ for some constant $k \leq 0$. In this coordinate system, $P$ is the projection

$$P : (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, k).$$
Let \( u = (x_1, \ldots, x_n) \) in this coordinate system. We have assumed that \( \sigma^\circ \) is \( n \)-well-centered, so \( c(\sigma^\circ) \) (the origin) is strictly interior to \( \sigma^\circ \). It follows that \( k < 0 \) and \( x_n > 0 \).

Consider the line segment \( \ell \) from \( u \) to \( -u \). Observe that \( \ell \) is a diameter of the circumsphere of \( \sigma^\circ \). Moreover, \( \ell \cap \text{Int}(\sigma^\circ) \neq \emptyset \). This follows from the fact that \( \sigma^\circ \) is \( n \)-well-centered; we have \( \sigma^\circ = u \star \tau^{n-1} \) and \( c(\sigma^\circ) \in \text{Int}(\sigma^\circ) \), so there must be some point \( w \in \text{Int}(\sigma^\circ) \) such that \( c(\sigma^\circ) \) lies on \( uw \subseteq \ell \). We notice, then, that the point \( -u \) lies below \( \text{aff}(\tau^{n-1}) \) and conclude that \( x_n > -k \).

By the Pythagorean theorem,

\[
R(\tau^{n-1})^2 + k^2 = R(\sigma^\circ)^2.
\]

We also have

\[
\sum_{i=1}^{n} x_i^2 = R(\sigma^\circ)^2,
\]

since \( u \) lies on the circumsphere of \( \sigma^\circ \). It follows that

\[
\|P(u) - c(\tau^{n-1})\|^2 = \sum_{i=1}^{n-1} x_i^2 = R(\sigma^\circ)^2 - x_n^2 < R(\sigma^\circ)^2 - k^2 = R(\tau^{n-1})^2.
\]

The statement is not limited to \( \sigma^\circ \subset \mathbb{R}^n \), of course. For a simplex \( \sigma^\circ \subset \mathbb{R}^m \) with \( m > n \), there exists a coordinate system such that

\[
\text{aff}(\sigma^\circ) = \{ x \in \mathbb{R}^m : x_i = 0 \text{ for } i = n+1, \ldots, m \},
\]

and the same proof applies.

**Remark 4.** Given a particular simplex \( \tau^{n-1} \subset \mathbb{R}^n \), Proposition 3 provides a geometric necessary condition on the location of vertex \( u \) to create an \( n \)-well-centered simplex \( \sigma^\circ = u \star \tau^{n-1} \). Vertex \( u \) must lie within a solid right spherical cylinder over the circumsphere of \( \tau^{n-1} \) if \( \sigma^\circ \) is to be \( n \)-well-centered. Figure 4 illustrates the condition in 2D and 3D, making it clear how this condition generalizes from the familiar 2-D case into higher dimensions. In each case the vertices of the base simplex \( \tau^{n-1} \), as well as the circumcenter \( c(\tau^{n-1}) \), are marked by small dark-colored balls. If \( u \star \tau^{n-1} \) is \( n \)-well-centered, then the vertex \( u \) must lie inside the gray cylinder over the circumsphere of \( \tau^{n-1} \). In the notation of Fig. 2, where the circumcenter of \( \sigma \) is known, the Cylinder Condition says that \( u \) must lie either in the upper cup \( C' \) or the lower cup \( C \).

**Remark 5.** The One-Facet Equatorial Ball Condition (Corollary 2) is also a necessary condition on the location of vertex \( u \). In \( \mathbb{R}^2 \), the combination of Corollary 2 and the Cylinder Condition
is sufficient to guarantee that a triangle (a 2-simplex) is acute (is 2-well-centered). In \( \mathbb{R}^n \) for \( n \geq 3 \), however, an \( n \)-simplex \( u \ast \tau^{n-1} \) for which \( u \) satisfies both of these necessary conditions might not be \( n \)-well-centered.

**Example 6.** For example, consider the tetrahedron \( \sigma = \sigma^3 \) with vertices \((-0.152, 0.864, -0.48), (-0.64, -0.6, -0.48), (0.6, -0.64, -0.48), \) and \((-0.192, -0.64, 0.744) \), whose circumcenter lies at the origin. For three of the four facets \( \tau^2_i \) of \( \sigma^3 \), vertex \( v_i \) satisfies both necessary conditions with respect to \( \tau^2_i \), and for the fourth facet \( v_i \) satisfies the Cylinder Condition, but not the One-Facet Equatorial Ball Condition. The tetrahedron \( \sigma^3 \) is not 3-well-centered.

Figure 5 shows several different views of \( \sigma^3 \). The large view at left shows that \( \sigma^3 \) is not 3-well-centered; the circumcenter of \( \sigma^3 \), marked by a small axes indicator, lies outside the tetrahedron. The four small views on the right side of Fig. 5 are views directly down onto the facets \( \tau^2_i \) of \( \sigma^3 \). In each case the circumcircle of \( \tau^2_i \) is rendered in a darker color, and one can see that the vertex above the facet projects to the interior of the circumball of the facet, i.e., that vertex \( v_i \) satisfies the Cylinder Condition with respect to \( \tau^2_i \). In three of the four cases — all except the case at lower left — the vertex also satisfies the One-Facet Equatorial Ball Condition. The particular example in Fig. 5 is also mentioned in [15], which gives some additional statistics on the tetrahedron.

**Example 7.** The tetrahedron with vertices at \((-0.01, -0.01, -0.01), (1, 0, 0), (0, 1, 0), \) and \((0, 0, 1) \) is another tetrahedron that is not 3-well-centered. It also has three vertices that satisfy the One-Facet Equatorial Ball Condition and four vertices that satisfy the Cylinder Condition. This example is dihedral acute, in contrast to the previous example.

The above examples illustrate that the One-Facet Equatorial Ball Condition and the Cylinder Condition are not enough to establish that the \( n \)-simplex \( u \ast \tau^{n-1} \) is \( n \)-well-centered. However, the following proposition does provide sufficient conditions that \( u \ast \tau^{n-1} \) is \( n \)-well-centered. See also Fig. 6.

**Proposition 8 (Prism Condition).** Let \( \tau^{n-1} \) be an \((n-1)\)-well-centered simplex in \( \mathbb{R}^n \) and \( \sigma^n = u \ast \tau^{n-1} \). If \( u \) lies outside the equatorial ball \( B(\tau^{n-1}) \) and the reflection of \( P(u) \) through \( c(\tau^{n-1}) \) is interior to \( \tau^{n-1} \), then \( \sigma^n \) is \( n \)-well-centered.

**Proof.** We assume the stated hypothesis and take the same coordinate system that was used in the proof of Proposition 3. Observe that if \( u \) were on the equatorial ball of \( \tau^{n-1} \), then \( c(\sigma^n) \) would lie in \( \tau^{n-1} \), coinciding with \( c(\tau^{n-1}) \). Because \( u \) lies outside the equatorial ball of \( \tau^{n-1} \),
A tetrahedron that is not 3-well-centered, even though every vertex satisfies the necessary condition of Proposition 3. Three of the vertices also lie outside the equatorial balls of their respective opposite facets.

\[ \text{aff}(\tau_{n-1}) \]

\[ u \rightarrow \ell \rightarrow c(\sigma^n) = (0, 0, 0) \]

\[ P(u) \rightarrow P(-u) \rightarrow c(\tau_{n-1}) \]

\[ \text{aff}(\tau_{n-1}) \]

\[ -u \]

Figure 5: Because \( P(-u) \) and \( c(\tau_{n-1}) = P(c(\sigma^n)) \) are both interior to \( \tau_{n-1} \), we know that the tetrahedron \( \sigma^n = u \ast \tau_{n-1} \) is 3-well-centered.

\[ c(\sigma^n) \text{ lies interior to the same halfspace as } u \text{ with respect to } \text{aff}(\tau_{n-1}). \]

It follows that \( k < 0 \) and \( x_n > k \).

Observe that, as shown in Fig. 6, the reflection of \( P(u) \) through \( c(\tau_{n-1}) \) is \( P(-u) \). By the hypothesis, \( P(-u) \) is interior to \( \tau_{n-1} \). Thus \( P(-u) \) is interior to the circumball of \( \sigma^n \) and

\[
\|P(u)\|^2 = \|P(-u)\|^2 = k^2 + \sum_{i=1}^{n-1} x_i^2 < R(\sigma^n)^2 = \sum_{i=1}^{n} x_i^2
\]

It follows that \( |x_n| > |k| = -k \). Since we know that \( x_n > k \), we conclude that \( x_n > -k > 0 \).

Let \( \ell \) be the line segment from \( u \) to \( -u \). We will show that \( \ell \) intersects the interior of \( \tau_{n-1} \). Then, because \( \sigma^n = u \ast \tau_{n-1} \) and \( k < 0 < x_n \) (so that \( c(\sigma^n) \in \ell \) is above \( \tau_{n-1} \) and below \( u \)), we will be able to conclude that \( c(\sigma^n) \) is interior to \( \sigma^n \). We know that \( P(c(\sigma^n)) = c(\tau_{n-1}) \) is interior to \( \tau_{n-1} \) because \( \tau_{n-1} \) is \((n-1)\)-well-centered. Since \( P(-u) \) and \( P(c(\sigma^n)) \) are both interior to \( \tau_{n-1} \), the line segment from \( c(\sigma^n) \) to \( -u \), which is contained in \( \ell \), is interior to the (convex) infinite prism \( \tau_{n-1} \times \mathbb{R} \). Moreover, \( 0 > k > -x_n \) (i.e., \( c(\sigma^n) \) is above \( \tau_{n-1} \) and \( -u \) is below \( \tau_{n-1} \)), so this part of segment \( \ell \) intersects the interior of \( \tau_{n-1} \).
As was the case for Proposition 3, Proposition 8 is not limited to \( \sigma^n \subset \mathbb{R}^n \); in higher-dimensional spaces \( \mathbb{R}^m \) there is a coordinate system such that

\[
\text{aff}(\sigma^n) = \{ x \in \mathbb{R}^m : x_i = 0 \text{ for } i = n+1, \ldots, m \},
\]

and the same proof applies.

After reading Proposition 8, one might ask whether the requirement that the facet \( \tau^{n-1} \) be \((n-1)\)-well-centered can be removed from the proposition. It may already be clear from the discussion of Fig. 2 that the answer to this question is no. The tetrahedron in Fig. 7 is an explicit example that confirms the requirement cannot be removed.

**Example 9.** The tetrahedron in Fig. 7 is the convex hull of vertices \( v_0 = (0.224, -0.768, -0.6) \), \( v_1 = (0.8, 0, -0.6) \), \( v_2 = (0.224, 0.768, -0.6) \), and \( v_3 = (-0.28, 0, 0.96) \). The bottom facet in Fig. 7, which is the triangle \( \tau_2^2 \) \( = [v_0v_1v_2] \), lies in the plane \( x_3 = -0.6 \) and is an obtuse triangle. The obtuse angle is at vertex \( v_2 \), the rightmost vertex in Fig. 7. Taking this bottom facet to be \( \tau^2 \) as in Proposition 8, and the top vertex to be \( u = v_3 \), we satisfy the conditions that \( u \) lie outside the equatorial ball of \( \tau^2 \) and that the reflection of \( P(u) \) through \( c(\tau^2) \) be interior to \( \tau^2 \). Indeed, \( c(\tau^2) = (0, 0, -0.6) \) and \( R(\tau^2) = 0.8 \) with \( \| u - c(\tau^2) \| = \sqrt{2.512} \approx 1.58 \), so \( u \) is outside \( B(\tau^2) \), and \( P(-u) = (0.28, 0, -0.6) \) lies inside \( \tau^2 \). Thus we satisfy all of the Prism Condition except the requirement that \( \tau^2 \) be 2-well-centered. It is clear from Fig. 7 that this is not sufficient; the circumcenter of tetrahedron \( u * \tau^2 \), marked by a small axes indicator, lies outside the tetrahedron, so the tetrahedron is not 3-well-centered.

**Remark 10.** Like the condition of Proposition 3, the condition of Proposition 8 has a nice geometric interpretation. Given an \((n-1)\)-well-centered facet \( \tau^{n-1} \), if the vertex \( u \) opposite \( \tau^{n-1} \) lies outside \( B(\tau^{n-1}) \) and within an infinite prism (a right cylinder) over the reflection of \( \tau^{n-1} \) through its circumcenter, then \( \sigma^n = u * \tau^{n-1} \) is \( n \)-well-centered. Figure 8 portrays the region defined by the Prism Condition for specific examples in 2 and 3 dimensions. In each case the base simplex \( \tau^{n-1} \) is shown in dark colors and solid lines, and its reflection is outlined with lighter colors and dashed lines. In the figure, each \( \tau^{n-1} \) is \((n-1)\)-well-centered, so for a vertex \( u \) lying inside the prism over the reflection of \( \tau^{n-1} \) through \( c(\tau^{n-1}) \) and outside the equatorial ball of \( \tau^{n-1} \), i.e., for a vertex \( u \) lying in the gray region shown in Fig. 8, the simplex \( u * \tau^{n-1} \) will be \( n \)-well-centered. Note that on the left in Fig. 8 the base simplex and its reflection should actually lie on top of each other, but are set slightly apart in the drawing so the reader can distinguish them from each other.

We have now established two different conditions for an \( n \)-simplex to be \( n \)-well-centered. One condition is a necessary condition, and the other condition is a sufficient condition. Both conditions are stated in terms of the location of a vertex \( u \) relative to the facet \( \tau^{n-1} \) opposite \( u \).
Figure 8: If the base simplex $\tau^{n-1}$ is $(n-1)$-well-centered and vertex $u$ is both outside $B(\tau^{n-1})$ and inside the infinite prism over the reflection of $\tau^{n-1}$ through $c(\tau^{n-1})$, then the simplex $u * \tau^{n-1}$ is $n$-well-centered.

The regions defined by the necessary condition and the sufficient condition may be quite different from each other. For example, in the 3-D portions of Figs. 4 and 8 the same base simplex $\tau^{n-1}$ yields rather different regions for the two conditions. It is natural to seek a precise description of the region where the vertex $u$ will produce an $n$-well-centered $n$-simplex $u * \tau^{n-1}$. The following discussion develops just such a set of conditions on the location of $u$. The conditions take the form of a system of cubic polynomial inequalities in the coordinates of $u$. The simplex $u * \tau^{n-1}$ will be $n$-well-centered if and only if the coordinates of $u$ satisfy the polynomial inequalities.

The inequalities are derived from a linear system of equations discussed in [1]. This linear system, which provides one way to compute the circumcenter of a simplex $\sigma^n$ embedded in $\mathbb{R}^m$ for $m \geq n$, is briefly reviewed here. We may write the circumcenter $c$ of a simplex $\sigma^n = [v_0 v_1 \ldots v_n]$ as a linear combination of the vertices $v_i \in \mathbb{R}^m$,

$$c = \alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_n v_n,$$

with the coefficients $\alpha_i$ satisfying $\sum_{i=0}^{n} \alpha_i = 1$. The coefficients $\alpha_i$ are known as the barycentric coordinates of the circumcenter. The condition that $\sigma^n$ be $n$-well-centered is the same as the condition that $0 < \alpha_i$ for every $\alpha_i$, i.e., the condition that the circumcenter be a convex combination of the vertices of $\sigma^n$ with strictly positive coefficients.

Suppose we are given the coordinates of the vertices $v_i$ of $\sigma^n$. We know that

$$\langle c - v_i, c - v_i \rangle = \|c - v_i\|^2 = R^2$$

for each vertex $v_i$. Introducing the variable $\lambda = R^2 - \|c\|^2$, we obtain the $n + 1$ equations $2\langle c, v_i \rangle + \lambda = \|v_i\|^2$. Since the vertices $v_i$ are known, each equation is a linear equation in the $n + 2$ unknowns $\alpha_0, \alpha_1, \ldots, \alpha_n, \lambda$. The final equation of the system is $\sum_{i=0}^{n} \alpha_i = 1$, which forces the $\alpha_i$ to be barycentric coordinates. As long as this linear system of $n + 2$ equations in $n + 2$ unknowns is nonsingular, we can solve for the barycentric coordinates. If the simplex is nondegenerate, i.e., if the $n + 1$ vertices are affinely independent, then the simplex has a unique, finite circumcenter, which has unique barycentric coordinates. It follows that the linear system has a unique solution; hence the matrix is nonsingular.
Let $A$ be the matrix of this linear system and $b$ the right-hand side,

$$A = \begin{pmatrix} 2(v_0, v_0) & 2(v_0, v_1) & \cdots & 2(v_0, v_n) & 1 \\ 2(v_1, v_0) & 2(v_1, v_1) & \cdots & 2(v_1, v_n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2(v_n, v_0) & 2(v_n, v_1) & \cdots & 2(v_n, v_n) & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} (v_0, v_0) \\ (v_1, v_1) \\ \vdots \\ (v_n, v_n) \\ 1 \end{pmatrix}.$$ 

For $i = 0, 1, \ldots, n$ we let $A_i$ be the matrix $A$ with column $i + 1$ replaced by $b$. Cramer's rule tells us that $\alpha_i = \det(A_i)/\det(A)$. If we consider vertices $v_0, \ldots, v_{n-1}$ to be the vertices of some given $\tau^{n-1}$ and $v_n$ to be a free vertex $u$, then the barycentric coordinates $\alpha_i$ are rational functions of the coordinates of $u$. Thus the conditions $\alpha_i > 0$ become algebraic inequalities in the coordinates of $u$.

To simplify matrix $A$ a little, we translate each vertex of the simplex by $-v_0$. The translation may change the value of $\lambda$ in the solution vector — in fact, $\lambda = 0$ always holds for the translated system — but the barycentric coordinates of the circumcenter are not changed by translating the vertices of the simplex. If $m > n$ we make one further simplification. In the translated coordinate system, we rotate the simplex about the origin $v_0$ to obtain a simplex for which vector $v_i - v_0 \in \{x : x_{n+1} = \cdots = x_m = 0\}$ for each $i = 1, \ldots, n$. Rotation about the origin is an orthogonal transformation, so it does not change any of the entries of the linear system and does not affect the barycentric coordinates.

If we restrict our attention to one of the open halfspaces bounded by $\text{aff}(\tau^{n-1})$, we have either $\det(A) > 0$ or $\det(A) < 0$ throughout the halfspace, because $\det(A)$ is a continuous function of the entries in $A$ and $A$ is singular only when $u \in \text{aff}(\tau^{n-1})$. We will see that, in fact, $\det(A) \leq 0$ holds everywhere, so $\det(A) < 0$ throughout the halfspace.

The first row and the first column of $A$ in the simplified linear system are all zeroes except for the last entry, which is 1 in both cases. Computing the determinant of $A$ by first expanding across the first row and then expanding down the first column (one with an odd number of entries and the other with an even number of entries) we find that $\det(A) = -\det(B)$ where $B$ is the submatrix of $A$ spanning rows 2 to $n + 1$ and columns 2 to $n + 1$. The $n \times n$ submatrix $B$ has the form $2V^T V$, where $V$ is the $m \times n$ matrix

$$V = (v_1 - v_0 \ v_2 - v_0 \ \cdots \ v_n - v_0).$$

Because of the earlier rotation of the simplex, the last $m - n$ coordinates of each vector $v_i - v_0$ are zeroes, and if we take $\tilde{V}$ to be the first $n$ rows of $V$, then $\tilde{V}$ is an $n \times n$ matrix that satisfies $V^T V = \tilde{V}^T \tilde{V}$. It follows that $B = 2\tilde{V}^T \tilde{V}$. Thus $\det(B) = 2^n \det(\tilde{V})^2 \geq 0$. Observing that $\det(\tilde{V})$ is the signed volume of the parallelepiped spanned by the vectors that form the columns of $\tilde{V}$, we note that $\det(B) > 0$ holds when the columns of $\tilde{V}$ are linearly independent, i.e. when the vertices of the original simplex are affinely independent.

Thus with the assumption that $\tau^{n-1}$ is a fully $(n-1)$-dimensional simplex, we know that $\det(A) < 0$ when the vertex $u$ lies in either of the open halfspaces bounded by $\text{aff}(\tau^{n-1})$. For $u$ outside $\text{aff}(\tau^{n-1})$, then, we conclude that $\alpha_i = \det(A_i)/\det(A) > 0$ if and only if $\det(A_i) < 0$. Hence the simplex $u \ast \tau^{n-1}$ will be $n$-well-centered if and only if the coordinates of $u$ satisfy the polynomial inequality $\det(A_i) < 0$.

It remains to show that the equation $\det(A_i) = 0$ is a polynomial in the coordinates of $u$ of degree at most 3. To do this we examine the entries of $A_i$ that depend on $u$. All of these entries appear in row $n + 1$ or in column $n + 1$. At most two of these entries are quadratic in the coordinates of $u$—the entry at position $(n + 1, n + 1)$ and the entry at $(n + 1, i + 1)$. (Only one entry is quadratic in the coordinates of $u$ when $i = n$.) Every other entry that depends on $u$ is linear in the coordinates of $u$. Using $S_n$ to denote the group of permutations on $n$ letters, the determinant of an $n \times n$ matrix $M$ can be written as

$$\det(M) = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{j=1}^{n} M_{j\pi(j)},$$
Figure 9: Given a facet $\tau^{n-1}$, the region where the vertex $u$ may lie to produce an $n$-well-centered simplex $u + \tau^{n-1}$ is defined by a system of polynomial inequalities. When $\tau^{n-1}$ is $(n-1)$-well-centered, so that there are regions related to both the necessary and sufficient conditions, the actual region where $u$ may lie is somewhere in between the regions defined by the necessary Cylinder Condition (Fig. 4) and the sufficient Prism Condition (Fig. 8).

where $M_{jk}$ stands for the entry in row $j$, column $k$ of matrix $M$, and $\text{sgn}(\pi)$ is the signum function applied to the permutation. Considering the structure of matrix $A_i$, we observe that each product in this definition of $\det(A_i)$ involves at most two terms that depend on $u$, and at most one of these—the entry selected from row $n+1$—is quadratic in the coordinates of $u$. Thus the determinant is a summation of terms that are polynomial in the coordinates of $u$ and have degree at most 3.

We can also explain this from the perspective of computing the determinant by expanding it along a row or column. We will consider a specific example with $i = 2$ arising from a tetrahedron (dimension $n = 3$), but the discussion applies to the general case. We have

$$A_2 = \begin{pmatrix}
2\langle v_0, v_0 \rangle & 2\langle v_0, v_1 \rangle & \langle v_0, v_0 \rangle & 2\langle v_0, u \rangle & 1 \\
2\langle v_1, v_0 \rangle & 2\langle v_1, v_1 \rangle & \langle v_1, v_1 \rangle & 2\langle v_1, u \rangle & 1 \\
2\langle v_2, v_0 \rangle & 2\langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & 2\langle v_2, u \rangle & 1 \\
2\langle u, v_0 \rangle & 2\langle u, v_1 \rangle & \langle u, u \rangle & 2\langle u, u \rangle & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}$$

for this particular example. If we compute $\det(A_i)$ by expanding down column $n+1$ (column 4, in this case), we find that term $n+1$ of the summation is a quadratic function of the coordinates of $u$ multiplied by the determinant of a submatrix that is constant with respect to $u$. In our example, this is the fourth term in the summation,

$$2\langle u, u \rangle \cdot \det \begin{pmatrix}
2\langle v_0, v_0 \rangle & 2\langle v_0, v_1 \rangle & \langle v_0, v_0 \rangle & 1 \\
2\langle v_1, v_0 \rangle & 2\langle v_1, v_1 \rangle & \langle v_1, v_1 \rangle & 1 \\
2\langle v_2, v_0 \rangle & 2\langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.$$

The remaining $n+1$ of the terms in the summation are linear (or constant) functions of $u$ multiplied by a determinant of some other submatrix of $A_i$ that is not constant with respect to $u$. For our example, the first term of the summation is

$$-2\langle v_0, u \rangle \cdot \det \begin{pmatrix}
2\langle v_1, v_0 \rangle & 2\langle v_1, v_1 \rangle & \langle v_1, v_1 \rangle & 1 \\
2\langle v_2, v_0 \rangle & 2\langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & 1 \\
2\langle u, v_0 \rangle & 2\langle u, v_1 \rangle & \langle u, u \rangle & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.$$
Expanding the appropriate row (usually row \(n\)) of each of these submatrices in similar fashion, we obtain a summation of terms that are either linear or quadratic in the coordinates of \(u\) (at most one term is quadratic), each multiplied by the determinant of a smaller submatrix that is constant with respect to \(u\).

We state the conclusions of the foregoing discussion as a formal proposition.

**Proposition 11.** Let \(\sigma^n = u \ast \tau^{n-1}\) for a fixed facet \(\tau^{n-1}\). The \(n\)-simplex \(\sigma^n\) is \(n\)-well-centered if and only if the coordinates of vertex \(u\) satisfy the inequalities \(\det(A_i) < 0\), which are cubic polynomial inequalities in the coordinates of \(u\).

Figure 9 gives a graphical representation of the precise region where the vertex \(u\) may be placed to produce a 3-well-centered tetrahedron \(u \ast \tau^{n-1}\). The facet \(\tau^{n-1}\) used in Fig. 9 is the same facet used to illustrate the necessary condition for a tetrahedron in Fig. 4 and the sufficient condition for a tetrahedron in Fig. 8, so readers can see for this specific case how the full region compares to the regions defined by the necessary condition and the sufficient condition. The facet \(\tau^{n-1}\) along with its circumcircle and the reflection of \(\tau^{n-1}\) through \(c(\tau^{n-1})\) are shown in Fig. 9 to aid this comparison. It should also be noted that Fig. 9 was generated using MATLAB’s isosurface function and evaluations of the polynomial inequalities on a finite grid, so the graphical representation has some slight imperfections. For instance, the entire circumcircle of \(\tau^{n-1}\) lies in the boundary of the region even though in Fig. 9 it appears that there is a small gap above and below \(\text{aff}(\tau^{n-1})\).

## 5 Local Combinatorial Properties of 3-Well-Centered Tetrahedral Meshes

The geometric properties of the \(n\)-well-centered \(n\)-simplex discussed in Sec. 4 have implications for the combinatorial properties of well-centered meshes. As a simple motivating example we consider the 2-dimensional case of a triangle mesh in the plane. If \(v\) is a vertex interior to this mesh and there are fewer than five edges incident to \(v\), then some angle incident to \(v\) has measure \(\pi/2\) radians or larger. Thus the mesh has a nonacute triangle. This geometric observation can be restated as a combinatorial property of 2-well-centered (i.e., acute) triangle meshes. Namely, there are at least five edges incident to every interior vertex of an acute triangle mesh in \(\mathbb{R}^2\). This well-known fact is a key ingredient in the generation of 2-well-centered triangle meshes through optimization of vertex coordinates; the mesh must satisfy this combinatorial condition at every interior vertex if optimizing the vertex coordinates is to have any hope of finding an acute mesh.

Similarly, tetrahedral meshes in \(\mathbb{R}^3\) that are 2-well-centered or 3-well-centered must satisfy certain local mesh connectivity conditions. These combinatorial conditions, which are key to creating well-centered tetrahedral meshes, are analyzed in the next two sections. This section develops some of the combinatorial properties of 3-well-centered tetrahedral meshes, and the next section examines combinatorial properties of 2-well-centered tetrahedral meshes.

The combinatorial properties of tetrahedral meshes in \(\mathbb{R}^3\) are more complex than the analogous properties for triangle meshes in \(\mathbb{R}^2\). In a triangle mesh in \(\mathbb{R}^2\), the link of an interior vertex is a set of edges that form a cycle around the vertex, i.e., a triangulation of a topological circle (\(S^1\)). The number of edges incident to the interior vertex, which is the number of vertices on the cycle, completely characterizes the neighborhood of the vertex. In tetrahedral meshes in \(\mathbb{R}^3\), on the other hand, the link of an interior vertex is a triangulation of a topological sphere \(S^2\). Thus the number of edges incident to the vertex does not completely characterize the neighborhood of the vertex. We do, however, prove necessary conditions on the number of edges that must be incident to an interior vertex in a tetrahedral mesh in \(\mathbb{R}^3\) in order for the mesh to be 3-well-centered, 2-well-centered, or completely well-centered. We also show that there is no sufficient condition in terms of the number of edges incident to an interior vertex.
Much of the discussion in Secs. 5 and 6, then, is phrased in terms of the link of an interior vertex. For a tetrahedral mesh in $\mathbb{R}^3$, this is a triangulation of $S^2$, which corresponds to a planar triangulation in a graph theoretic sense. We try to avoid the term planar triangulation to prevent possible confusion with triangle meshes in $\mathbb{R}^2$.

We begin with two results that apply in arbitrary dimension. The first lemma generalizes the following statement about planar triangle meshes, using the Cylinder Condition (Proposition 3) to relate geometry to combinatorics. If a planar triangle $\sigma^2 = [abc]$ is subdivided into three triangles by adding a vertex $u$ interior to $\sigma^2$ and adding edges $[ua], [ub], \text{ and } [uc]$ to obtain $u * \text{Bd}(\sigma^2)$, then at most one of the three triangles $[au], [bcu], \text{ and } [cau]$ in $u * \text{Bd}(\sigma^2)$ is an acute triangle. The main idea of the proof of Lemma 12 is illustrated by Fig. 10.

**Lemma 12.** For $n \geq 2$, let $\sigma^n = [v_0v_1 \ldots v_n]$ have facets $\tau_0^{n-1}, \tau_1^{n-1}, \ldots, \tau_n^{n-1}$. If $u$ is a point lying in $\sigma^n$, then at most one of the n-simplices of $u * \text{Bd}(\sigma^n)$, i.e., at most one of the simplices $u * \tau_0^{n-1}, u * \tau_1^{n-1}, \ldots, u * \tau_n^{n-1}$, is an n-well-centered n-simplex.

**Proof.** It suffices to prove the statement when $u$ is in the interior of $\sigma^n$. Indeed, if $u$ is on the boundary of $\sigma^n$ and two or more of the simplices $u * \tau_i^{n-1}$ are n-well-centered, then we can slightly perturb $u$ into the interior and obtain a point $u \in \text{Int}(\sigma^n)$ with at least two n-well-centered n-simplices. Thus we assume that $u \in \text{Int}(\sigma^n)$.

Let $\tau_i^{n-1}$ and $\tau_j^{n-1}$ be two distinct facets of $\sigma^n$. Then $u * \tau_i^{n-1}$ and $u * \tau_j^{n-1}$ are n-simplices, and $\tau_i^{n-1} \cap \tau_j^{n-1}$ is an $(n-2)$-dimensional face of $\sigma^n$. The face $\tau_i^{n-1} \cap \tau_j^{n-1}$ is incident to all but two of the vertices of $\sigma^n$, the two vertices $v_i$ and $v_j$. (Recall that $v_i$ is opposite $\tau_i^{n-1}$ and $v_j$ is opposite $\tau_j^{n-1}$.) Notice that $u * \tau_i^{n-1}$ and $u * \tau_j^{n-1}$ have a common facet, the $(n-1)$-simplex $\tau_u^{n-1} \coloneqq u * (\tau_i^{n-1} \cap \tau_j^{n-1})$. We let $T \subset \text{aff}(\sigma^n)$ be the solid right spherical cylinder over the circumball of $\tau_u^{n-1}$.

Assume towards contradiction that $u * \tau_i^{n-1}$ and $u * \tau_j^{n-1}$ are both n-well-centered. By the Cylinder Condition (Proposition 3), both $v_i$ and $v_j$ lie in $\text{Int}(T)$. Now $T$ is a convex set, and all the vertices of $\sigma^n$ lie in $T$, so $\sigma^n \subset T$. On the other hand, $u$ lies on the circumsphere of $\tau_u^{n-1}$, so $u \in \text{Bd}(T)$. Thus $u \notin \text{Int}(\sigma^n) \subset \text{Int}(T)$, contradicting the assumption we made in the first paragraph of the proof. We conclude that at most one of $u * \tau_i^{n-1}, u * \tau_j^{n-1}$ is n-well-centered. \qed

The next theorem shows that Lemma 12 has implications for the local combinatorial properties of n-well-centered meshes. The theorem is stated using the language of simplicial complexes. We say that a vertex $u$ is an *interior vertex* in an $n$-dimensional simplicial complex embedded
in \( \mathbb{R}^n \) if (the underlying space of) \( \text{Lk} \ u \) is homeomorphic to \( S^{n-1} \), the sphere of dimension \( n-1 \). Thus the closed star of \( u \) is homeomorphic to an \( n \)-dimensional ball in \( \mathbb{R}^n \), and the point \( v \) lies in the interior of the ball in the standard topology on \( \mathbb{R}^n \).

When we speak of an abstract simplicial complex \( K \) we make an important distinction between an embedding of \( K \) and a geometric realization of \( K \). An embedding of \( K \) is an assignment of coordinates in \( \mathbb{R}^n \) to the vertices of \( K \) such that \( K \) is a simplicial complex in \( \mathbb{R}^n \) with vertices at the specified locations. By a geometric realization of \( K \) we mean merely some assignment of coordinates in \( \mathbb{R}^n \) to the vertices of \( K \). Thus in a geometric realization of \( K \) in \( \mathbb{R}^n \), it is possible for \( K \) to have self-intersections. Figure 11, which is related to the proof of Theorem 13, illustrates the distinction between these two terms.

**Theorem 13** (One-Ring Necessary Condition). Let \( u \) be an interior vertex of an \( n \)-dimensional simplicial complex \( M \) (e.g., a mesh) embedded in \( \mathbb{R}^n \), and set \( L = \text{Lk} \ u \). If there exists an abstract finite \( n \)-dimensional simplicial complex \( K \) such that

(i) \( K \) is an \( n \)-manifold complex (with boundary)

(ii) \( \text{Bd} (K) \) is isomorphic to \( L \), and

(iii) for every \( n \)-simplex \( \sigma^n \in K \), there are at least two \( (n-1) \)-simplices in \( \text{Bd}(\sigma^n) \cap L \),

then \( u \ast L \) is not \( n \)-well-centered.

**Proof.** We first observe that every vertex of \( K \) must also be a vertex of \( L \). By assumption (i), every simplex of \( K \) is a face of some \( n \)-simplex of \( K \), so if \( K \) had a vertex \( v \) not in \( L \), then there would be some \( n \)-dimensional simplex \( \sigma^n \in K \) incident to \( v \), and \( \sigma^n \) would have only one \((n-1)\)-dimensional face not incident to \( v \). Since \( v \notin L \), it follows that \( \text{Bd}(\sigma^n) \cap L \) would contain at most one \((n-1)\)-simplex, and (iii) would not be satisfied.

The embedding of \( M \) in \( \mathbb{R}^n \) includes an embedding of \( u \ast L \) in \( \mathbb{R}^n \). Since every vertex of \( K \) is a vertex of \( L \), this embedding of \( u \ast L \) in \( \mathbb{R}^n \) induces a geometric realization of \( K \) into \( \mathbb{R}^n \). (As shown in Fig. 11, the geometric realization might not be an embedding.)

We have an embedding of the simplicial complex \( u \ast L \) in \( \mathbb{R}^n \). Since it is an embedding, each \( n \)-dimensional simplex is a fully \( n \)-dimensional geometric object, and we have consistent orientation. Moreover, \( L \) is star-shaped with respect to \( u \). We claim that by (i) and (ii) this implies that there is some simplex in the induced geometric realization of \( K \) that contains the
point \( u \) (possibly on its boundary). We return to this claim in a moment, but first we show how this completes the proof.

Fix a simplex \( \sigma^n \in K \) that contains \( u \). Now consider the \( n \)-simplices of \( u \ast \text{Bd}(\sigma^n) \). By assumption (iii) of the hypothesis, at least two of these simplices have a facet in \( L \). Each simplex of \( u \ast \text{Bd}(\sigma^n) \) with a facet in \( L \) is a member of \( u \ast L \), and by Lemma 12 at most one of these simplices is \( n \)-well-centered. We conclude that at least one of the simplices of \( u \ast L \subseteq M \) is not \( n \)-well-centered.

Now we prove the claim that there is a simplex of the geometric realization of \( K \) that contains the point \( u \). Choose a line \( \ell \) through \( u \) in general position. General position here means that \( \ell \) does not intersect any face of \( K \) of dimension less than \( n - 1 \). Such an \( \ell \) can be chosen unless \( u \) itself lies on a simplex \( \rho^k \) of \( K \) of dimension \( k < n - 1 \), and in that case we are done, since there is some \( \sigma^n \supset \rho^k \) that contains \( u \).

Since \( u \ast L \) is a simplicial complex and \( L \) is star-shaped with respect to \( u \), \( \ell \) intersects exactly two simplices of \( L \), each of dimension \( n - 1 \), and the intersection points are in opposite directions from \( u \) along \( \ell \). For reference, we designate a + and a − direction and name facet \( \tau_+^{n-1} \) (resp. \( \tau_-^{n-1} \)) as the facet of \( L \) intersected by \( \ell \) in the + (−) direction from \( u \). Starting from \( \tau_+^{n-1} \) we describe a walk along \( \ell \) through \( n \)-simplices and \((n-1)\)-simplices of the geometric realization of \( K \) that ends at \( \tau_-^{n-1} \). By continuity of this walk and \( \tau_+^{n-1}, \tau_-^{n-1} \) in opposite directions from \( u \), there must be some \( \sigma^n \supset \rho^k \) in the geometric realization of \( K \) that contains \( u \).

The walk is as follows. Since \( K \) is a manifold with a boundary and \( \tau_+^{n-1} \) is on the boundary, there is a unique \( \sigma^n_0 \) incident to \( \tau_0^{n-1} := \tau_+^{n-1} \). Then for a given \( \sigma^n_i \), the walk is on \( \ell \) at \( \tau_i^{n-1} \), and \( \ell \) intersects some unique second facet of \( \sigma^n_i \), which we name \( \tau_i^{n-1} \). As long as \( \tau_i^{n-1} \neq \tau_-^{n-1} \), we are not on the boundary of \( K \), so (since \( K \) is manifold) there are exactly two \( n \)-dimensional simplices incident to \( \tau_i^{n-1} \). One of these is \( \sigma^n_i \), and the other we name \( \sigma^n_{i+1} \). Since \( K \) is a manifold complex, the sequence \( \tau_i^{n-1} \) has no repetitions and must eventually end at \( \tau_-^{n-1} \). (The \( \sigma^n_i \) in the sequence may flip back and forth in orientation, which corresponds to the walk going back and forth along \( \ell \).

It is worth noting that the existence of the abstract simplicial complex \( K \) has no dependence on the particular embedding of \( M \) in \( \mathbb{R}^n \). Theorem 13 is really a combinatorial statement, and we can use it to show that a particular abstract simplicial complex \( L = \text{Bd}(K) \) cannot appear as the link of an interior vertex in an \( n \)-well-centered mesh embedded in \( \mathbb{R}^n \).

The case \( n = 3 \) is of particular interest. Using the One-Ring Necessary Condition of Theorem 13 it is fairly easy to establish a tight lower bound on the number of edges incident to a vertex in a 3-well-centered tetrahedral mesh embedded in \( \mathbb{R}^3 \).

**Corollary 14.** Let \( M \) be a 3-well-centered tetrahedral mesh embedded in \( \mathbb{R}^3 \). For every vertex \( u \) interior to \( M \), at least 7 edges of \( M \) are incident to \( u \).

**Proof.** Britton and Dunitz have assembled a catalog of all polyhedra with at most 8 vertices, which includes all the triangulations of \( S^2 \) with at most 8 vertices [2]. By Theorem 13 it suffices to show that each such triangulation \( L \) of \( S^2 \) with at most 6 vertices has a corresponding tetrahedral complex \( K \) such that each tetrahedron of \( K \) has at least two facets in common with \( L \).

There is only one triangulation of \( S^2 \) with 4 vertices—the boundary of a tetrahedron. The corresponding tetrahedral complex is that single tetrahedron.

There is also only one triangulation of \( S^2 \) with 5 vertices. This triangulation is shown in Fig. 12 along with two corresponding tetrahedral complexes. Either complex certifies that the triangulation cannot be the link of any vertex in a 3-well-centered mesh.

For six vertices there are two nonisomorphic triangulations of \( S^2 \). The first is shown in Fig. 13 along with its corresponding tetrahedral complex. The second is drawn in Fig. 14 along with its corresponding tetrahedral complex.
Figure 12: There is only one triangulation of $S^2$ with 5 vertices, and it has two corresponding tetrahedral complexes such that each tetrahedron has at least two facets in common with the triangulation.

Figure 13: For one of the triangulations of $S^2$ with 6 vertices, each vertex has exactly four neighbors. There is a tetrahedral complex consisting of four tetrahedra such that each tetrahedron has two facets in common with this triangulation of $S^2$.

Figure 14: In the other triangulation of $S^2$ with 6 vertices, the degree list is $(5, 5, 4, 4, 3, 3)$. This triangulation of $S^2$ also has a corresponding tetrahedral complex such that each tetrahedron has at least two facets in common with the triangulation.
Figure 15: A 3-well-centered mesh with an interior vertex $u$ such that $L_k u$ has seven vertices and degree list $(5, 5, 5, 4, 4, 4, 3)$. The vertex coordinates are listed in the table at right; vertex $u$ is at the origin.

| $x$  | $y$  | $z$  |
|------|------|------|
| 0    | 0    | 0    |
| 0    | 0    | 1    |
| -0.1041 | -0.0601 | 0.0117 |
| 0.1041 | -0.0601 | 0.0117 |
| 0    | 0.1202 | 0.0117 |
| 0    | -0.3622 | -0.8656 |
| 0.3137 | 0.1811 | -0.8656 |
| -0.3137 | 0.1811 | -0.8656 |

Figure 16: A 3-well-centered mesh with an interior vertex $u$ such that $L_k u$ has seven vertices and degree list $(6, 5, 5, 5, 5, 3, 3, 3)$. The vertex coordinates are listed in the table at right; vertex $u$ is at the origin.

| $x$  | $y$  | $z$  |
|------|------|------|
| 0    | 0    | 0    |
| 0    | 0    | 1    |
| 0    | 0.8334 | -0.8588 |
| -0.7217 | -0.4167 | -0.8588 |
| 0.7217 | -0.4167 | -0.8588 |
| 0    | -5.0494 | 1.0696 |
| 4.3729 | 2.5247 | 1.0696 |
| -4.3729 | 2.5247 | 1.0696 |

When there are $m \geq 7$ vertices, there exist triangulations $L$ of $S^2$ with $m$ vertices such that there is no tetrahedral complex $K$ satisfying both $\text{Bd}(K) = L$ and the condition that every tetrahedron of $K$ have at least two facets in $L$. In particular, the triangulations of $S^2$ with 7 vertices and degree lists $(5, 5, 5, 4, 4, 4, 3)$ and $(6, 5, 5, 5, 3, 3, 3)$, i.e., polyhedra 7–1 and 7–4 in the catalog of Britton and Dunitz, both can appear as the link of a vertex in a 3-well-centered mesh. Figure 15 shows an example of a 3-well-centered mesh in $\mathbb{R}^3$ consisting of a single vertex $u$ and its neighborhood $\text{Cl}(\text{St} u)$ such that $L_k u$ is a triangulation with degree list $(5, 5, 5, 4, 4, 4, 3)$. Figure 16 shows a similar example for the degree list $(6, 5, 5, 5, 3, 3, 3)$. There are three other triangulations of $S^2$ with 7 vertices. Each has a corresponding tetrahedral complex $K$ satisfying the requirements of the One-Ring Necessary Condition (Theorem 13), so none of these triangulations can appear as the link of a vertex in a 3-well-centered mesh.

There are 14 nonisomorphic triangulations of $S^2$ with 8 vertices. Of these, 5 have tetrahedral complexes $K$ that certify they cannot be the link of a vertex in a 3-well-centered tetrahedral mesh in $\mathbb{R}^3$. Each of the other 9 triangulations can appear as the link of a vertex in a 3-well-centered tetrahedral mesh in $\mathbb{R}^3$. (We mention these results without proof here.) For $m \leq 8$ vertices, then, the necessary condition of Theorem 13 completely characterizes which triangulations can and cannot be made 3-well-centered. We leave open the question of whether the One-Ring Necessary Condition stated in Theorem 13 is a complete characterization for $m > 8$ vertices in 3 dimensions or for $n$-well-centeredness in dimensions $n \geq 4$.

The triangulations on 8 vertices that cannot be made 3-well-centered are polyhedra 8–4, 8–5, 8–6, 8–7, and 8–13 in the catalog [2] of Britton and Dunitz. It is interesting to note that the degree list of 8–7, which cannot be made 3-well-centered, is the same as the degree list of 8–8,
which can be made 3-well-centered [14]. Thus the degree list of a triangulation does not provide
enough information to determine whether the triangulation can be the link of a vertex in a 3-well-
centered tetrahedral mesh in $\mathbb{R}^3$. There are 50 nonisomorphic triangulations with 9 vertices and
an exponentially growing number of triangulations with more vertices [8], so although making
a catalog for 9 or 10 vertices might be somewhat interesting, something more abstract will be
necessary to definitively characterize which triangulations can be made 3-well-centered.

In the rest of this section we discuss some more general results in the direction of char-
acterizing which triangulations of $S^2$ necessary to definitively characterize which triangulations can be made 3-well-centered.

Corollary 15. For any integer $m \geq 4$ there is a triangulation of $S^2$ with $m$ vertices that cannot
appear as the link of a vertex in a 3-well-centered mesh.

Proof. We have already proved that this holds for $4 \leq m \leq 6$.

For $m \geq 7$ we note that the tetrahedral complexes shown on the right hand sides of Figs. 12 and
13 can be generalized. Consider a tetrahedral complex $K$ consisting of a set of $m - 2$ tetrahedra that close around a common edge. The complex $K$ satisfies the conditions of Theorem 13, so $\text{Bd}(K)$ cannot appear as the link of a vertex in a 3-well-centered mesh. $\text{Bd}(K)$ is a triangulation of $S^2$ on $m$ vertices with degree list $(m - 2, m - 2, 4, \ldots, 4)$.

We note that by removing a single tetrahedron from the example complex $K$ of Corollary 15,
we obtain another infinite family of triangulations of $S^2$ that cannot appear as the link of a vertex
in a 3-well-centered mesh. Each member of this family is a triangulation on $m$ vertices with
degree list $(m - 1, m - 1, 4, \ldots, 4, 3, 3)$. This family generalizes the tetrahedral complexes shown
on the left hand side of Fig. 12 and the right hand side of Fig. 14.

Corollary 15 is one instance that shows how substantial the difference is between tetrahedral and
triangle meshes. In the case of triangle meshes in $\mathbb{R}^2$, where we consider triangulations of $S^1$ as the link of a vertex, the only two triangulations that cannot appear as the link of a vertex are the 3-cycle and the 4-cycle. In contrast, there are infinitely many triangulations of $S^2$ that cannot be the link of a vertex in a 3-well-centered mesh in $\mathbb{R}^3$. One may wonder whether there are still infinitely many triangulations of $S^2$ that can appear as the link of a vertex in a 3-well-centered mesh in $\mathbb{R}^3$. The answer is yes. One way to prove this is to explicitly construct an infinite family of 3-well-centered meshes with different vertex links. We will do exactly that in a moment, with the help of the following lemma, which we prove using the Prism Condition (Proposition 8).

Lemma 16. Let $S^{n-1}_0$ be a unit $(n-1)$-sphere centered at a point $u$. If $\tau^{n-1}$ is an $(n-1)$-
well-centered $(n-1)$-simplex whose vertices lie on $S^{n-1}_0$, and the distance from $u$ to $\text{aff}(\tau^{n-1})$
is greater than $1/\sqrt{2}$, then $\sigma^n := u * \tau^{n-1}$ is an $n$-well-centered $n$-simplex.

Proof. Suppose that $\tau^{n-1}$ is an $(n-1)$-well-centered simplex meeting the conditions specified
in the hypothesis. Let $S^{n-2}_0$ be the circumsphere of $\tau^{n-1}$. $S^{n-2}_0$ is the intersection of $\text{aff}(\tau^{n-1})$
with $S^{n-1}_0$, i.e., an $(n-2)$-sphere lying in $S^{n-1}_0$. The orthogonal projection of $u$ into $\text{aff}(\tau^{n-1})$,
which we denote by $P(u)$, is the center of $S^{n-2}_0$, i.e., the circumcenter $c(\tau^{n-1})$ of $\tau^{n-1}$.

Since $\tau^{n-1}$ is $(n-1)$-well-centered, it contains the point $c(\tau^{n-1})$. Thus $\tau^{n-1}$ contains the
reflection of $P(u)$ through $c(\tau^{n-1})$. The circumradius of $\tau^{n-1}$ satisfies $R(\tau^{n-1})^2 + z^2 = 1$, where
$z$ is the distance from $u$ to $\text{aff}(\tau^{n-1})$, so because $z^2 > 1/2$, we have $R(\tau^{n-1}) < 1/\sqrt{2}$, and $u$ lies
outside the equatorial ball of $\tau^{n-1}$. By the Prism Condition, $\sigma^n$ is $n$-well-centered.

\[ \square \]
Figure 17: For $k \geq 4$ we can create an acute triangulation of the unit sphere from a set of vertices consisting of the north and south poles and two out-of-phase regular $k$-gons. Coning such a triangulation to the origin produces a completely well-centered tetrahedral mesh. The figure shows the tetrahedral mesh obtained for $k = 7$. It is relatively straightforward to prove the converse as well. For $\sigma^n = u \ast \tau^{n-1}$ with the vertices of $\tau^{n-1}$ lying on a sphere $S_0^{n-1}$ centered at $u$, if $\tau^{n-1}$ is not $(n-1)$-well-centered or the distance $z$ from $u$ to $\text{aff}(\tau^{n-1})$ satisfies $z \leq 1/\sqrt{2}$, then $\sigma^n$ is not $n$-well-centered. This proof is left to the reader; the result is not needed in this paper.

The simplex $\sigma^n = u \ast \tau^{n-1}$ in Lemma 16 is an isosceles simplex with all vertices of $\tau^{n-1}$ equidistant from the apex vertex $u$. When $n = 2$, Lemma 16 reduces to the statement that an isosceles triangle is acute if the apex angle is acute. In higher dimensions Lemma 16 tells us when an isosceles simplex is $n$-well-centered. Note that in an isosceles simplex all of the faces incident to the apex vertex $u$ are isosceles; the plane of each such face intersects the sphere $S^{n-1}$ in some lower-dimensional sphere centered at $u$, and Lemma 16 can be applied to these isosceles faces. It follows that $\sigma^n$ will be completely well-centered if $\tau^{n-1}$ is completely well-centered and $z > 1/\sqrt{2}$. In particular, for the case $n = 3$, an isosceles tetrahedron with an acute triangle facet opposite the apex vertex is a completely well-centered tetrahedron.

Thus from any triangulation of a unit sphere $S^2$ with sufficiently small acute triangles we can create a completely well-centered tetrahedral mesh in $\mathbb{R}^3$ by taking the cone $u \ast \tau^2$ of each acute triangle $\tau^2$ with the center of the sphere $u$. Figure 17 shows a completely well-centered tetrahedral mesh constructed in this fashion. The boundary of the mesh in Fig. 17 is an acute triangulation of $S^2$ selected from an infinite family of acute triangulations of $S^2$. The next two paragraphs describe this family.

Consider the set of vertices consisting of the north pole $(0, 0, 1)$, the south pole $(0, 0, -1)$, and the vertices of two regular $k$-gons, one in the plane $z = 0.352$ and the other in the plane $z = -0.352$. We set the polygons exactly off phase from each other. For instance, let the coordinates of the polygon vertices be

$$(0.936 \cos \left( \frac{2i\pi}{k} \right), 0.936 \sin \left( \frac{2i\pi}{k} \right), 0.352), \quad i = 0, 1, \ldots, k - 1,$$

$$(0.936 \cos \left( \frac{(2i + 1)\pi}{k} \right), 0.936 \sin \left( \frac{(2i + 1)\pi}{k} \right), -0.352), \quad i = 0, 1, \ldots, k - 1.$$

Let each pole vertex be adjacent to all of the vertices of the closer regular polygon. This constructs $k$ isosceles triangles incident to each pole. We take each vertex of a regular polygon to be adjacent to the closer pole, the two neighbors on its own regular polygon, and two vertices from the other regular polygon. Triangles formed entirely from vertices of the two regular polygons are also isosceles. The example in Fig. 17 uses the result of this construction for the case $k = 7$.

We claim that if $k \geq 4$, then each triangle $\tau^2$ of this construction is acute and satisfies the condition that the distance from the origin to $\tau^2$ is greater than $1/\sqrt{2}$. Since $k \geq 4$ it is clear that the apex angles of the isosceles triangles incident to the poles are acute angles.
Verifying that the other triangles are acute and that the triangles are far enough from the origin is straightforward and we omit the details. Lemma 16 applies, and as an immediate consequence we have the following.

**Proposition 17.** There are infinitely many triangulations of $S^2$ that can appear as the link of a vertex in a completely well-centered mesh.

For large enough $k$, this construction of completely well-centered neighborhoods of a vertex using acute triangulations of a unit sphere $S^2$ can be generalized. One can use more than two regular $k$-gons, alternating the phase between each successive $k$-gon.

We have seen that there are infinitely many triangulations of $S^2$ that cannot appear and infinitely many that can appear as a link of a vertex in a 3-well-centered mesh. The authors suspect that for $m \geq 8$ vertices the majority of triangulations of $S^2$ on $m$ vertices are triangulations that can appear as a link of a vertex in a 3-well-centered mesh. We do not formally prove that conjecture in this paper, but in light of the the next proposition, it is highly likely; Proposition 18 provides a method for constructing new triangulations that can appear as the link of a vertex in a 3-well-centered tetrahedral mesh in $\mathbb{R}^3$.

In Proposition 18 we consider a triangulation $G$ of $S^2$ with a vertex of degree 3. In this context, the notation $G - v_1$ refers to the triangulation of $S^2$ obtained by deleting vertex $v_1$ and all faces incident to $v_1$, replacing them with the face $[v_2v_3v_4]$, where $v_2, v_3, v_4$ are the neighbors of $v_1$ in $G$, ordered to keep the orientation consistent.

**Proposition 18.** Let $G$ be a triangulation of $S^2$ with a vertex $v_1$ of degree 3, and let $v_2, v_3, v_4$ be the neighbors of $v_1$ in $G$. Let $M$ be a tetrahedral mesh in $\mathbb{R}^3$ consisting of a vertex $u$ and its closed neighborhood $\text{Cl}(St u)$, with $\text{Lk} u$ isomorphic to $G - v_1$. If

(i) $M$ is 3-well-centered

(ii) face angle $\angle uv_i v_j$ is acute for each $i, j \in \{2, 3, 4\}, i \neq j$,

then there exists a tetrahedral mesh $\tilde{M}$ in $\mathbb{R}^3$ and a vertex $u$ of $\tilde{M}$ such that

(i) $\text{Lk} u$ is isomorphic to $G$

(ii) $\tilde{M}$ is 3-well-centered

(iii) face angle $\angle uv_i v_j$ is acute for each $i, j \in \{1, 2, 3, 4\}, i \neq j$.

**Proof.** Figure 18 accompanies this proof and may help the reader understand the geometric constructions discussed in the proof. Consider a particular tetrahedral mesh that satisfies the conditions of the hypothesis. In this mesh the tetrahedron $\sigma = \sigma^3 = [uv_2v_3v_4]$ is 3-well-centered, so $c(\sigma)$ is interior to $\sigma$.

Let $\ell$ be the line through $u$ and $c(\sigma)$. Line $\ell$ intersects the circumsphere of $\sigma$ at two points. One of these is $u$, and the other we name $u'$. We define

$$u'_\varepsilon = (1 - \varepsilon)u' + \varepsilon u,$$

a point lying on $\ell$. Because $\sigma$ is 3-well-centered, we know that segment $uu'$ intersects triangle $[v_2v_3v_4]$ at some point $u'_\varepsilon$, with $1/2 > \varepsilon_0 > 0$. We can cut $\sigma$ into the three tetrahedra $[uv_2v_3u'_\varepsilon]$, $[uv_3v_4u'_\varepsilon]$, and $[uv_4v_2u'_\varepsilon]$.

For $\varepsilon_0 > \varepsilon > 0$ we consider the three tetrahedra $[uv_2v_3u'_\varepsilon]$, $[uv_3v_4u'_\varepsilon]$, and $[uv_4v_2u'_\varepsilon]$. We claim that for sufficiently small $\varepsilon > 0$ these three tetrahedra are 3-well-centered and the face angles $\angle uv_i' v_1$, $\angle uv_i u'_\varepsilon$ are acute for $i = 2, 3, 4$.

Examining the face angles first, we note that at $\varepsilon = 0$ the circumcenters of the facets $[uv_i u'_\varepsilon]$ coincide with $c(\sigma)$ and with each other. Indeed, each of these facets is a right triangle with its circumcenter lying on the hypotenuse $uu'_\varepsilon$. As $\varepsilon$ increases, $\angle v_i u u'_\varepsilon$ does not change, $\angle uv_i u'_\varepsilon$
Figure 18: Given a 3-well-centered tetrahedron $\sigma = [uv_2v_3v_4]$ with acute angles $\angle uv_1v_4$, one can construct three tetrahedra $[uv_2v_3v_1], [uv_3v_4v_1]$, and $[uv_4v_2v_1]$ by adding a new vertex $v_1 = u'$ along the line $\ell$ through $u$ and $c(\sigma)$. The circumcenters of the constructed tetrahedra lie along lines connecting $c(\sigma)$ to the circumcenters $c(\tau_i)$ of the $[uv_i]_{v_j}$ facets of $\sigma$. As discussed in Proposition 18, when $v_1$ is close enough to $u'$—the reflection of $u$ through $c(\sigma)$—the constructed tetrahedra will be 3-well-centered and the angles $\angle uv_1v_i, \angle uv_i;v_4$ will be acute. The angles $\angle v_1uv_j$ do not need to be acute for this construction. For example, $\angle v_2w_3$ is not an acute angle in this figure.

decreases, becoming smaller than $\pi/2$, and $\angle uu'v_i$ increases but remains less than $\pi/2$ for sufficiently small $\varepsilon$.

Turning to the tetrahedra, then, we will argue that the specific tetrahedron $[uv_2v_3u'_3]$ is 3-well-centered for sufficiently small $\varepsilon$. An argument identical except for changed labels applies to the other two tetrahedra, so this will complete the proof. We know that, regardless of the value of $\varepsilon$, the circumcenter of $[uv_2v_3u'_3]$ lies on the line orthogonal to $\text{aff}(\{uv_2v_3\}) = \text{aff}(\tau_4)$ passing through $c(\tau_4)$; this line is the locus of points equidistant from $u$, $v_2$, and $v_3$. The location of $c([uv_2v_3u'_3])$ varies continuously with $\varepsilon$. At $\varepsilon = 0$, the circumcenter of tetrahedron $[uv_2v_3u'_3]$ coincides with $c(\sigma)$, and as $\varepsilon$ increases from 0 towards $\varepsilon_0$, $c([uv_2v_3u'_3])$ moves in the direction of vector $c(\tau_4) - c(\sigma)$. Because $\angle uv_2v_3$ and $\angle uv_3v_2$ are acute, we know that $c(\tau_4)$ lies in the sector of $\text{aff}(\tau_4)$ interior to angle $\angle uv_2v_3$. Thus segment $c(\sigma)c(\tau_4) \cap \sigma$ is contained in $[uv_2v_3u'_3]$, and for sufficiently small $\varepsilon > 0$, tetrahedron $[uv_2v_3u'_3]$ is 3-well-centered.

Because the face angles $\angle uv_1v_1, \angle uv_3v_1$ are acute in the construction of Proposition 18, the construction can be iterated. If a triangulation $G$ of $S^2$ satisfies the conditions of Proposition 18, then a degree 3 vertex $v_1$ can be inserted into face $[v_2v_3v_4]$. In the new triangulation of $S^2$, the three new faces incident to $v_1$ satisfy the conditions of Proposition 18, so a degree 3 vertex can be inserted into any one of those three faces, and so on. In particular, starting from any completely well-centered mesh constructed from an acute triangulation of a unit sphere $S^2$, one can successively insert vertices of degree 3 to create an infinite family of triangulations that can appear as the link of a vertex in a 3-well-centered mesh.

It is also worth mentioning that each triangulation of a topological $S^2$ with 8 vertices $v_1, \ldots, v_8$ that can appear as $Lk u$ for a vertex $u$ in a 3-well-centered mesh in $\mathbb{R}^3$ has an embedding into $\mathbb{R}^3$ for which all of the face angles $\angle uv_iv_j$ are acute for $i, j \in \{1, \ldots, 8\}, i \neq j$, where $v_iv_j$ is an edge in the triangulation. Recall that there are 50 nonisomorphic triangulations of $S^2$ with 9 vertices [8]. Using Proposition 18 to add vertices of degree 3 to the various faces of triangulations of $S^2$ with 8 vertices, one can show that at least 34 of these 50 triangulations of $S^2$ with 9 vertices can appear as the link of a vertex in a 3-well-centered tetrahedral mesh embedded in $\mathbb{R}^3$. 

22
6 Local Combinatorial Properties of 2-Well-Centered Tetrahedral Meshes

Corollary 14 shows that in a 3-well-centered mesh there are at least 7 edges incident to each vertex. In the following discussion we will see that the combinatorial constraints for a mesh to be 2-well-centered are quite different from the constraints for a mesh to be 3-well-centered, and in terms of the minimum number of edges incident to a vertex, are more stringent. As in Sec. 5, the discussion focuses on $Lk_u$ where $u$ is a vertex interior to a tetrahedral mesh in $\mathbb{R}^3$.

**Definition.** We say that a particular triangulation $G$ of $S^2$ permits a 2-well-centered neighborhood of a vertex $u$ if there exists a tetrahedral mesh $M$ in $\mathbb{R}^3$ such that $u$ is an interior vertex of $M$, $Lk_u$ is isomorphic to $G$ (as a simplicial complex), and all facets of $M$ incident to $u$ are 2-well-centered. (A facet means a 2-simplex in this context—a face of dimension $n-1$.)

It should be noted that this definition does not directly address the question of whether the tetrahedra incident to $u$ are 2-well-centered, since each tetrahedron incident to $u$ has one facet lying on $Lk_u$, and that facet is not incident to $u$. We shall see, however, that for tetrahedral meshes in $\mathbb{R}^3$, the smallest triangulation that permits a 2-well-centered neighborhood in the sense of this definition can, in fact, appear as the link of a vertex in a completely well-centered mesh. Finally, note that phrasing the problem in terms of the facets of $M$ incident to $u$ actually reduces the problem to determining whether the face angles at $u$ are acute, because if there is an arrangement of rays at $u$ such that all of the face angles formed at $u$ by these rays are acute, then we can place the neighbors of $u$ at the points where these rays intersect a unit sphere centered at $u$. This will create a neighborhood of $u$ in which every 2-dimensional face incident to $u$ is an isosceles triangle with an acute apex angle at $u$.

The first result of this section is a simple observation that forms the foundation for the theory developed in the rest of the section.

**Lemma 19.** Let $u$ and $v_1$ be adjacent vertices in a tetrahedral mesh $M$ embedded in $\mathbb{R}^3$ and let $v_i$ be a vertex of $Lk_u$ that is adjacent to $v_1$. The angle $\angle v_1uv_i$ is acute if and only if $v_i \in H_1$, where $H_1$ is the open halfspace that contains $v_1$ and is bounded by the plane through $u$ orthogonal to the vector $v_1 - u$.

**Proof.** The angle $\angle v_1uv_i$ is acute if and only if $\langle v_1 - u, v_i - u \rangle > 0$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^3$, and this holds if and only if $v_i$ lies in $H_1$. $\square$

The next two technical lemmas are based on Lemma 19. They lead to the proof of the main result of this section. In both lemmas and in the subsequent theorem we use the following notation. We denote by $u$ a vertex in a tetrahedral mesh in $\mathbb{R}^3$, and the $m$ vertices of $Lk_u$ are labeled $v_1, \ldots, v_m$. For each vertex $v_i$, the plane through $u$ orthogonal to $v_i - u$ is denoted $P_i$, and the open halfspace bounded by $P_i$ that contains $v_i$ is denoted $H_i$. The other halfspace bounded by $P_i$ will be called $H_i'$, and we take this to be a closed halfspace, which contains its boundary $P_i$. The orthogonal projection of a vertex $v_j$ into $P_i$ will be denoted $Pr_i(v_j)$.

**Lemma 20.** Let $v_1$ and $v_2$ be nonadjacent vertices of $Lk_u$, with $v_2 \in H_i'$. If $v_1$ is a vertex adjacent to both $v_1$ and $v_2$ such that $\angle v_1uv_i$ and $\angle v_2uv_i$ are both acute angles, then the orthogonal projection of $v_i$ into $P_i$ lies in $P_i \cap H_2$.

**Proof.** The sketch in Fig. 19 illustrates this result. For an algebraic proof we assign a coordinate system with $u$ as the origin and $v_1$ lying on the positive $z$ axis. Using coordinates $(x_i, y_i, z_i)$ for vertex $v_i$, the condition $v_2 \in H_i'$ means that $z_2 \leq 0$. Since the angle $\angle v_1uv_i$ is acute, Lemma 19 implies that $v_i$ must lie in $H_1$, and since the angle $\angle v_2uv_i$ is acute, Lemma 19 implies that $v_i$ must lie in $H_2$. Thus $v_i$ lies in $H_1 \cap H_2$. Since $H_1 \cap H_2$ would be empty if $v_2$ had coordinates...
Figure 19: If a 2-well-centered mesh contains two vertices $v_1$ and $v_2$ that both lie in $Lk(u)$, are not adjacent to each other, and have a common neighbor $v_i$ and if $v_2$ lies in $H_1$, then the orthogonal projection of $v_i$ into $P_1$, i.e., the point $Pr_1(v_i)$, must lie in $P_1 \cap H_2$.

Figure 20: Let $u$ be a vertex of a tetrahedral mesh embedded in $\mathbb{R}^3$, and let $v_1$, $v_2$, $v_i$, $v_j$ be vertices of $Lk(u)$ with adjacencies as shown. If the face angles at $u$ between adjacent vertices of $Lk(u)$ are all acute angles, but $\angle v_1uv_2$ is nonacute, then the projection of facet $[v_i v_2 v_j]$ into $P_1$ lies in $P_1 \cap H_2$.

(0, 0, z_2), we can conclude that $v_2$ does not lie on the z-axis. With the remaining freedom in defining a coordinate system we specify that $v_2$ has coordinates $(x_2, 0, z_2)$ with $x_2 < 0$.

Now since $v_i \in H_1$, we know that $z_i \geq 0$. We also know that $(v_i, v_2) = x_1 x_2 + z_1 z_2 > 0$, because $v_j \in H_2$. We have established that $z_i z_2 \leq 0$ and that $x_2 < 0$. It follows that $x_i < 0$. The projection $Pr_1(v_i)$ has coordinates $(x_i, y_i, 0)$ and is interior to $P_1 \cap H_2 = \{(x, y, 0) : x < 0\}$.

**Lemma 21.** Let $v_1$ and $v_2$ be nonadjacent vertices of $Lk(u)$, with $v_2 \in H_1$. If $[v_i v_2 v_j]$ is a 2-simplex of $Lk(u)$, such that $v_i, v_j$ are both adjacent to $v_1$ and the face angles $\angle v_1 uv_i, \angle v_1 uv_j, \angle v_2 uv_i, \angle v_2 uv_j$, are all acute angles, then $Pr_1([v_i v_2 v_j]) \subset P_1 \cap H_2$, i.e., the orthogonal projection of the entire facet $[v_i v_2 v_j]$ into $P_1$ lies in $H_2$.

**Proof.** See the sketch in Fig. 20. From the given hypotheses we can conclude by Lemma 20 that $Pr_1(v_i)$ and $Pr_1(v_j)$ both lie in $P_1 \cap H_2$. Using the same coordinate system defined in the proof of Lemma 20, the point $Pr_1(v_2)$ has coordinates $(x_2, 0, 0)$ with $x_2 < 0$, thus it lies in $P_1 \cap H_2$ as well. It follows that the orthogonal projection of the facet $[v_i v_2 v_j]$ into $P_1$, which is the convex hull of $Pr_1(v_i), Pr_1(v_2)$, and $Pr_1(v_j)$, lies entirely in the convex set $P_1 \cap H_2$.

Applying the above two lemmas, we obtain a combinatorial necessary condition on the neighborhood of an interior vertex in a 2-well-centered mesh.

**Theorem 22.** Let $G$ be a triangulation of $S^2$ with $m$ vertices. If $G$ contains a vertex $v_1$ of degree $d(v_1) \geq m - 3$, then $G$ does not permit a 2-well-centered neighborhood.
Figure 21: When $L_k u$ has $m$ vertices and one of the vertices $v_1$ has degree $d(v_1) \geq m - 3$, any geometric realization of $\text{Cl}(\text{St} u)$ in $\mathbb{R}^3$ with all face angles at $u$ acute is not an embedding. Theorem 22 shows that if we consider such a geometric realization and project every facet that intersects $H'_1$ into $P_1$, then the union of the projected facets does not contain $u$. The sketch at left shows an example of a geometric realization of a tetrahedral mesh $\text{Cl}(\text{St} u)$ in $\mathbb{R}^3$ such that every face angle at vertex $u$ is acute. In the sketch, $v_1$ has degree $d(v_1) = 6 = m - 3$ in $L_k u$. The sketch at right shows the result of taking the geometric realization on the left and projecting each facet that intersects $H'_1$ into $P_1$.

Proof. We consider a vertex $u$ such that $L_k u$ is isomorphic to $G$ where $G$ has a vertex of degree at least $m - 3$ and consider a geometric realization of $\text{Cl}(\text{St} u)$ in $\mathbb{R}^3$. Label the vertices of $L_k u$ with the labels $v_1, v_2, \ldots, v_m$ such that $v_1$ is a vertex of maximum degree and the (at most two) vertices not adjacent to $v_1$ are listed immediately after $v_1$ (e.g., labeled $v_2, v_3$ if there are two of them). We choose a coordinate system on $\mathbb{R}^3$ such that $u$ is at the origin and $v_1$ lies on the positive $z$-axis.

Assume that all of the face angles $\angle v_i u v_j$ are acute. We claim this implies that for any facet $[v_i v_j v_k]$ with at least one vertex in $H'_1$, the orthogonal projection of the facet into $P_1$, i.e., $\text{Pr}_1([v_i v_j v_k])$, does not contain vertex $u$. Assuming this claim for the moment, we see that $u$ lies outside the (solid) polyhedron bounded by $L_k u$. (See Fig. 21.) Since $u$ is outside this polyhedron, some 3-simplex incident to $u$ must be inverted. Thus the geometric realization of $\text{Cl}(\text{St} u)$ is not an embedding, and the claim completes the proof.

We proceed to prove the claim. Noting that $v_1 \in H_1$ by our definition of $H_1$, we observe that for $i \geq 4$, vertex $v_i$ must lie in $H_1$ because $v_i$ is adjacent to $v_1$. (This follows from Lemma 19.) Thus there are only two types of facets that may have nonempty intersection with $H'_1$. The first type is $[v_1 v_2 v_3]$ or $[v_1 v_3 v_j]$ where $v_1$ and $v_j$ both are adjacent to $v_1$, and the second type is $[v_2 v_3 v_j]$ for $j \geq 4$. Consider, then, the first type of facet, taking the specific notation $[v_1 v_2 v_j]$. (The same argument applies to $[v_1 v_3 v_j]$.) If $v_2$ lies in $H_1$, we are done; the facet does not intersect $H'_1$. Otherwise $v_2$ lies in $H'_1$. Hence $\angle v_1 u v_2$ is nonacute, and $v_2$ is not adjacent to $v_1$. Lemma 21 applies.

The proof for facets of the second type is more complicated. If both $v_2$ and $v_3$ lie in $H_1$, we are done. If one vertex lies in $H_1 \cup P_1$ and the other lies in $H'_1$, we assume without loss of generality that $z_2 \leq 0$ and $z_3 \geq 0$.

Then $v_2$ is not adjacent to $v_1$. If $v_3$ is adjacent to $v_1$, then Lemma 21 applies directly with $v_3$ functioning as $v_i$. On the other hand, even if $v_3$ is not adjacent to $v_1$, the arguments of Lemmas 20 and 21 can be applied with $v_3$ functioning as $v_i$. (In the proofs of Lemmas 20 and 21 we used $v_i$ adjacent to $v_1$ to establish only that $z_i \geq 0$ and that $v_2$ does not lie on the $z$-axis. The latter holds in this case because $v_2$ and $v_1$ have common neighbor $v_j \neq v_3$.)

This leaves the case $z_2 < 0$ and $z_3 < 0$. As noted above, $v_2$ does not lie on the $z$-axis. We choose the coordinate system with $v_2 = (x_2, 0, z_2)$, $x_2 < 0$. We also assume without loss of generality that $y_3 \geq 0$. (We can reflect through the plane $y = 0$ if $y_3 < 0$.) See Fig. 22 for sketches related to this case.
Proof. Let $G = \text{Lk} u$ for some interior vertex $u$ of a 2-well-centered mesh $M$, and let $m$ be the number of edges incident to $u$, i.e., the number of vertices of $G$. Consider the possibility $m = 8$.
Euler’s formula shows that for \( m = 8 \) we have \( \sum_i d(v_i) = 36 \) so the average vertex degree is 4.5, and there must be at least one vertex of degree at least \( 5 = m - 3 \). By Theorem 22, this cannot occur, for such a graph \( G \) would not permit a 2-well-centered neighborhood of \( u \). Similarly, if \( m = 7 \) the average degree is \( 30/7 > 4 \) and there must be a vertex of degree at least \( m - 2 \). For \( m = 6 \) the average degree is 4 and there must be a vertex of degree at least \( m - 2 \). In each of the cases \( m = 5 \) and \( m = 4 \), there is only one triangulation, and this triangulation has a vertex of degree \( m - 1 \). 

When \( m = 9 \), the average degree is \( 4\frac{2}{3} \), and there is a triangulation of \( S^2 \) with degree list \( (5, 5, 5, 5, 5, 4, 4, 4) \) that permits a completely well-centered neighborhood. Figure 23 shows a figure of a completely-well-centered mesh that has a single interior vertex \( u \) such that \( \text{Lk} u \) is a 9-vertex triangulation of \( S^2 \) with the specified degree list.

We have already seen that there are infinitely many triangulations of \( S^2 \) that can appear as the link of an interior vertex in a 2-well-centered mesh (Proposition 17). In the spirit of Proposition 18, we now discuss some ways to use an existing triangulation that permits a 2-well-centered neighborhood to construct new triangulations that permit a 2-well-centered neighborhood. The next two propositions show that one can add vertices of degree 3, subtract vertices of degree 3, or add vertices of degree 4 to obtain new triangulations that permit a 2-well-centered neighborhood. In Proposition 24 we again use the notation \( G - v_1 \) used in Proposition 18.

**Proposition 24.** A triangulation \( G \) of \( S^2 \) that contains a vertex \( v_1 \) of degree three permits a 2-well-centered neighborhood if and only if the triangulation \( G - v_1 \) permits a 2-well-centered neighborhood.

**Proof.** First we suppose that \( G \) permits a 2-well-centered neighborhood. Then consider some tetrahedral mesh embedded in \( \mathbb{R}^3 \) that contains a vertex \( u \) with \( \text{Lk} u \) isomorphic to \( G \) and all face angles \( \angle v_j w_1 \) acute. We choose a coordinate system on \( \mathbb{R}^3 \) such that \( u \) lies at the origin and identify each vertex \( v_i \) of \( \text{Lk} u \) with the vector originating at the origin and terminating at \( v_i \). Now vector \( v_1 \) makes an acute face angle for each of the three facets that are incident to the edge \( [w_1] \). Deleting \( v_1 \) from \( \text{Lk} u \) removes the three facets that are incident to edge \( [w_1] \), but has no effect on the other facets incident to \( u \) or face angles at \( u \). Thus all facets incident to \( u \) remain acute after removing \( v_1 \), and the modified neighborhood of \( u \) is a mesh embedded in \( \mathbb{R}^3 \) that certifies that \( G - v_1 \) permits a 2-well-centered neighborhood.
On the other hand, if we suppose that $G - v_1$ permits a 2-well-centered neighborhood, we will be able to add vertex $v_1$ and still have all face angles at $u$ acute. Consider some specific tetrahedral mesh embedded in $\mathbb{R}^3$ containing a vertex $u$ such that $Lk_u$ is isomorphic to $G - v_1$. Let $v_2, v_3,$ and $v_4$ be the three vertices of $Lk_u$ that are adjacent to $v_1$ in $G$. Then the mesh contains facets $[v_2uv_3]$, $[v_3uv_4]$, and $[v_4uv_2]$. Moreover, since the face angles at $u$ are acute, we have $\langle v_i, v_j \rangle > 0$ for each $(i, j) \in \{2, 3, 4\} \times \{2, 3, 4\}$. It follows that if we insert vertex $v_1$ such that all face angles at $u$ are acute, and similarly $d(v_i) = 1$, then for $i = 2, 3, 4$ we have

$$\langle v_1, v_i \rangle = \lambda_2v_2 + \lambda_3v_3 + \lambda_4v_4$$

In other words, as long as $v_1$ lies interior to the cone at $u$ bounded by vectors $v_2$, $v_3$, and $v_4$, it will make acute face angles with each of $v_2$, $v_3$, and $v_4$.

Notice that Proposition 24 also implies that adding or deleting a degree three vertex from a triangulation of $S^2$ that does not permit a 2-well-centered neighborhood creates another triangulation of $S^2$ that does not permit a 2-well-centered neighborhood. In particular, this means that Theorem 22 does not characterize the triangulations of $S^2$ that cannot appear as the link of a vertex in a 2-well-centered tetrahedral mesh in $\mathbb{R}^3$.

In the next proposition, we consider the case of a triangulation $G$ of $S^2$ with a vertex $v_1$ such that $d(v_1) = 4$. To talk about removing vertex $v_1$ from $G$ in this case, we need to specify an edge to add after removing the vertex. Let $v_2, v_3, v_4,$ and $v_5$ be the neighbors of $v_1$, listing in cyclic order. Then $(G - v_1) \cup [v_2v_4]$ is the triangulation of $S^2$ obtained from $G$ by removing vertex $v_1$ along with the four edges and triangles incident to $v_1$ and adding the edge $[v_2v_4]$ along with the two triangles $[v_2v_3v_4]$ and $[v_2v_4v_5]$.

**Proposition 25.** Consider a triangulation $G$ of $S^2$ that contains a vertex $v_1$ of degree four with neighbors $v_2, v_3, v_4, v_5$ (listed in clockwise order). If $(G - v_1) \cup [v_2v_4]$ or $(G - v_1) \cup [v_3v_5]$ permits a 2-well-centered neighborhood, then $G$ permits a 2-well-centered neighborhood.

**Proof.** Suppose without loss of generality that $(G - v_1) \cup [v_2v_4]$ permits a 2-well-centered neighborhood. Let $u$ be a vertex for which $Lk_u$ is isomorphic to $(G - v_1) \cup [v_2v_4]$ and consider some embedding of $u \ast Lk_u$ into $\mathbb{R}^3$ such that all face angles at $u$ are acute. We choose a coordinate system on $\mathbb{R}^3$ such that $u$ lies at the origin and identify each vertex of $Lk_u$ with the vector originating at the origin $u$ and terminating at the vertex. We know that $\langle v_2, v_3 \rangle > 0$, $\langle v_3, v_4 \rangle > 0$, $\langle v_4, v_5 \rangle > 0$, $\langle v_5, v_2 \rangle > 0$, and $\langle v_2, v_4 \rangle > 0$, because each pair of vectors bounds a face with an acute face angle at $u$.

Now let $v_1 = (v_2 + v_4)/2$ and, deleting the facet $[v_2v_4]$, add the four facets $[v_1uv_i]$ for $i = 2, 3, 4, 5$. The new facets $[v_1uv_2]$ and $[v_1uv_4]$ have face angles at $u$ that are smaller than the face angle $\angle_{v_2v_4}$ was, so they are acute. The facets $[v_1uv_3]$ and $[v_1uv_5]$ also have acute face angles at $u$ because

$$\langle v_1, v_3 \rangle = \frac{1}{2} \langle v_2, v_3 \rangle + \frac{1}{2} \langle v_4, v_3 \rangle > 0$$

and similarly

$$\langle v_1, v_5 \rangle = \frac{1}{2} \langle v_2, v_5 \rangle + \frac{1}{2} \langle v_4, v_5 \rangle > 0.$$ 

We see that adding $v_1 = (v_2 + v_4)/2$ has created a new mesh for which all face angles at $u$ are acute. Thus $G$ permits a 2-well-centered neighborhood.
7 Applications to the Cube

The theoretical results presented in this paper are useful for creating well-centered meshes of specific regions in $\mathbb{R}^3$. In particular, one might design a tetrahedral mesh of a volume so that it meets all of the combinatorial conditions discussed in Secs. 5 and 6. Then applying the optimization procedure discussed in [17], one may hope to obtain a well-centered mesh of the domain. This technique was successfully used in [15] to create well-centered meshes of several domains in $\mathbb{R}^3$, including the cube.

The theory developed in this paper has several obvious implications for the combinatorial properties of a well-centered triangulation of the cube. For example, no cube corner tetrahedron, e.g., the tetrahedron shown in Fig. 25, can be 3-well-centered; considering the bottom facet to be a given facet, we see that the fourth vertex of the tetrahedron projects onto (not inside) the circumcircle of the given facet, violating the necessary Cylinder Condition of Proposition 3.

It follows that in a 3-well-centered mesh of the cube there must be at least two tetrahedra incident to each corner of the cube. Indeed, there must be at least three tetrahedra incident to each corner of the cube in a 3-well-centered mesh. In the case of two tetrahedra incident to a corner vertex there must be exactly four edges incident to the corner vertex, of which three are in the directions of the coordinate axes. The fourth edge must lie in a face, and both tetrahedra are incident to the axis orthogonal to the face containing the fourth edge. The Cylinder Condition applies again, and we see that the mesh cannot be 3-well-centered.

Ad hoc arguments from basic Euclidean geometry provide more restrictions on well-centered triangulations of the solid cube. For instance, in any 3-well-centered mesh of the cube, no face of the cube is triangulated as shown in Fig. 24, with two right triangles meeting along the hypotenuse. The two right triangles have the same circumcenter, which lies at the midpoint of the common hypotenuse of the triangles — the center of the face of the cube. For either triangle, a tetrahedron having that triangle as a facet must have its circumcenter on a line $\ell$ perpendicular to the face of the cube that meets the cube face at its center. Considering two tetrahedra $\sigma_1$ and $\sigma_2$, each having one of the right triangles as a face, it can be shown that at most one of $\sigma_1, \sigma_2$ can be 3-well-centered. There is a plane that contains the hypotenuse of the right triangles and divides $\mathbb{R}^3$ into two open halfspaces $H_1$ and $H_2$ such that $\sigma_1 \in \text{Cl}(H_1)$ and $\sigma_2 \in \text{Cl}(H_2)$. If $\sigma_1$ and $\sigma_2$ share a common face, the plane must be $\text{aff}(\sigma_1 \cap \sigma_2)$, but otherwise there is some flexibility in the choice of the plane. The portion of $\ell$ interior to the cube is either in the boundary between $H_1$ and $H_2$ or without loss of generality can be assumed to lie entirely in $H_1$. In either case, the circumcenter of $\sigma_2$ is not strictly interior to $\sigma_2$, so $\sigma_2$ is not 3-well-centered.

The paper [15], along with discussing some well-centered meshes of the cube, raises the
question of the minimum number of tetrahedra needed to create a well-centered triangulation of the cube. We can use the statements above to derive some simple lower bounds on the number of tetrahedra in a well-centered mesh of the 3-cube. In a triangulation of the cube, the number of tetrahedra incident to the surface of the cube is a lower bound on the total number of tetrahedra, so one can obtain a lower bound on the number of tetrahedra by counting the number of triangular facets in a surface triangulation. The number of facets is not a direct lower bound, since there may be a single tetrahedron with multiple facets in the surface of the cube. Because there are at least three distinct tetrahedra incident to a cube corner in a 3-well-centered triangulation of the cube, a tetrahedron cannot be counted more than twice in counting the number of surface facets of a 3-well-centered triangulation of the cube. The same holds true for 2-well-centered triangulations of the cube, since three of the facets of a cube corner tetrahedron are right triangles.

Noting, then, that each face of the cube must contain at least 3 triangles in a 3-well-centered mesh of the cube and at least 8 triangles in a 2-well-centered mesh of the cube, we easily obtain a lower bound of 9 tetrahedra for a 3-well-centered triangulation of the cube, and 24 tetrahedra for a 2-well-centered triangulation of the cube. (These lower bounds are mentioned in [15] without the details of the geometric or combinatorial arguments.) It should be possible to improve both of these bounds, but these relatively simple bounds help demonstrate a possible application of this paper’s theory and are a starting place for a more careful analysis.

8 Conclusions

In this paper we introduced several geometric propositions related to $n$-well-centered simplices and gave an algebraic characterization of an $n$-well-centered simplex in terms of cubic polynomial inequalities. We applied the geometric propositions to the study of the combinatorial properties of well-centered meshes, especially well-centered tetrahedral meshes.

We considered triangulations of topological $S^2$ and showed that the set of such triangulations that cannot appear as the link of a vertex in a 3-well-centered (or 2-well-centered or completely well-centered) tetrahedral mesh embedded in $\mathbb{R}^3$ is an infinite set, contrasting this to the analogous question for triangle meshes in $\mathbb{R}^2$. We showed also that the set of triangulations of $S^2$ that do appear as the link of a vertex in some completely well-centered (or 3-well-centered or 2-well-centered) tetrahedral mesh embedded in $\mathbb{R}^3$ is an infinite set. We proved several results in the direction of classifying which triangulations of $S^2$ can appear as the link of a vertex in a 2-well-centered or 3-well-centered tetrahedral mesh embedded in $\mathbb{R}^3$.

The work on combinatorial properties of well-centered meshes leads to some interesting open questions. Is there a compact way to express a complete characterization of which triangulations of $S^2$ can appear in a 3-well-centered (or 2-well-centered or completely well-centered) mesh embedded in $\mathbb{R}^3$? Is the necessary condition described in Theorem 13 a complete characterization for vertex links in 3-well-centered tetrahedral meshes in $\mathbb{R}^3$? If a triangulation of $S^2$ permits a 2-well-centered neighborhood in the sense defined in this paper, does this imply that it can appear as the link of a vertex in a 2-well-centered tetrahedral mesh embedded in $\mathbb{R}^3$? If a triangulation of $S^2$ can appear as the link of a vertex in both a 2-well-centered tetrahedral mesh in $\mathbb{R}^3$ and a 3-well-centered tetrahedral mesh in $\mathbb{R}^3$, does this guarantee that it can appear as the link of a vertex in a completely well-centered mesh?

Beyond the questions about tetrahedral meshes there are questions about higher dimensions. Is it possible to extend the results of Sec. 6 to say something about 2-well-centered meshes in higher dimensions? Certainly Lemmas 19, 20, and 21 can be generalized to higher dimensions. Is it the case that for each $n$ there exists a completely well-centered $n$-simplicial neighborhood of a vertex embedded in $\mathbb{R}^n$? If so, Lemma 16 may provide a way of constructing such neighborhoods. We note, however, that for $n \geq 5$ there is no dihedral acute $n$-simplicial neighborhood of a vertex embedded in $\mathbb{R}^n$ [6] [7].

This work also leads to some interesting practical questions about creating well-centered tetrahedral meshes in $\mathbb{R}^3$. In particular, now that there is some understanding of which tri-
angulations can appear as the link of a vertex in a well-centered mesh, we hope that practical methods can be developed for improving the local mesh connectivity of tetrahedral meshes. An algorithm for improving mesh connectivity of triangle meshes in $\mathbb{R}^2$ appears in [16], but it is not obvious how to formulate that type of algorithm for tetrahedral meshes in $\mathbb{R}^3$. It is also worth noting that a triangulation of $S^2$ that can theoretically appear as the link of a vertex in a well-centered mesh might be a poor neighborhood for a vertex in a practical setting. For instance, in a triangle mesh in $\mathbb{R}^2$, a vertex link with 100 vertices can appear in a 2-well-centered triangle mesh embedded in $\mathbb{R}^2$, but a mesh containing such a vertex link would be considered poor quality in almost any application. Is there a good way to rate vertex links in tetrahedral meshes according to their applicability in a practical setting?

We hope that our results will motivate others to investigate these interesting and important questions.

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