NONISOSPECTRAL SYMMETRIES OF THE KDV EQUATION
AND THE CORRESPONDING SYMMETRIES
OF THE WHITHAM EQUATIONS. [1]

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0. INTRODUCTION.

In our paper we construct a new infinite family of symmetries of the Whitham
equations (averaged Korteveg-de-Vries equation). In contrast with the ordinary hydro-
dynamic-type flows these symmetries are nonhomogeneous (i.e. they act nontrivially at
the constant solutions), are nonlocal, explicitly depend upon space and time coordinates
and form a noncommutative algebra, isomorphic to the algebra of the polynomial vector
fields in the complex plane.

We will study the averaged Korteveg-de-Vries (KdV) equation. But the main results
of our paper can be easily extended to other averaged systems, associated with the
integrable by the inverse scattering transform 1 + 1 equations.

We will assume original KdV to be normalized as:

\[ u_t = \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x. \]  \hspace{1cm} (0.1)

We will also need a 2 + 1 - dimensional generalization of KdV - the Kadomtsev -
Petviashvili (KP) equation

\[ (u_t - \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x)_x = 3 \alpha^2 u_{yy}, \; \alpha^2 = \pm 1. \]  \hspace{1cm} (0.2)

Let us recall the definition of the averaged KdV equations in the classical one-phase
case (see [1] for more details). Equation (0.1) possesses a three-parametric family of

[1] This text was published in the book “Singular Limits of Dispersive Waves” eds.
N.M.Ercolany, I.R.Gabitov, C.D.Levermore, D.Serre, Plenum Press, NY, 1994.
hydrodynamical-type equations: $u(x,t) = 2\varphi(x-vt|g_2, g_3) - v/6.$ (0.3)

$(\varphi(x,|g_2, g_3)$ denotes the Weierstrass $\varphi$ - function). We may try to construct asymptotical KdV solutions of the modulated wave type

\[ u(x,t) = 2\varphi(x-v(X,T)t + \phi(X,T)|g_2(X,T), g_3(X,T)) - v(X,T)/6 + \epsilon u_1, \] (0.4)

where $X = \epsilon x$, $T = \epsilon t$, $\epsilon$ is a small parameter ($X$ and $T$ are called "slow" variables). If the correcting term $u_1$ is a bounded function as $x \sim 1/\epsilon, t \sim 1/\epsilon$ then it can be proved (see [1], [3]) that the functions $g_2(X,T), g_3(X,T), v(X,T)$ satisfy some hydrodynamical-type equations:

\[ \frac{\partial}{\partial T} \begin{pmatrix} g_2(X,T) \\ g_3(X,T) \\ v(X,T) \end{pmatrix} = \hat{V}(g_2, g_3, v) \cdot \frac{\partial}{\partial X} \begin{pmatrix} g_2(X,T) \\ g_3(X,T) \\ v(X,T) \end{pmatrix}, \] (0.5)

where the matrix $\hat{V}(g_2, g_3, v)$ can be expressed via elliptic functions.

The averaged KdV (0.5) may be transformed to the Riemann diagonal form

\[ \frac{\partial}{\partial T} \begin{pmatrix} r_1(X,T) \\ r_2(X,T) \\ r_3(X,T) \end{pmatrix} = \hat{V}(1) \begin{pmatrix} v_1(\vec{r}) & 0 & 0 \\ 0 & v_2(\vec{r}) & 0 \\ 0 & 0 & v_3(\vec{r}) \end{pmatrix} \frac{\partial}{\partial X} \begin{pmatrix} r_1(X,T) \\ r_2(X,T) \\ r_3(X,T) \end{pmatrix}, \] (0.6)

The diagonalizing change of variables (Whitham [1]) is the following: the Riemann invariants $r_i, i = 1, 2, 3$ are the roots of the polynomial $R(\lambda) = 4(\lambda + v/6)^3 - g_2(\lambda + v/6) - g_3, r_1 \leq r_2 \leq r_3$. The matrix $\hat{V}$ in the diagonal form reads as:

\[ v_1(\vec{r}) = \frac{r_1 + r_2 + r_3}{3} - \frac{2}{3}(r_2 - r_1) \frac{K(s)}{K(s) - E(s)}, \]
\[ v_2(\vec{r}) = \frac{r_1 + r_2 + r_3}{3} - \frac{2}{3}(r_2 - r_1) \frac{(1 - s^2)K(s)}{E(s) - (1 - s^2)K(s)}, \]
\[ v_3(\vec{r}) = \frac{r_1 + r_2 + r_3}{3} + \frac{2}{3}(r_3 - r_1) \frac{(1 - s^2)K(s)}{E(s)} \] (0.7)

where $s^2 = (r_2 - r_1)/(r_3 - r_1), E(s), K(s)$ are elliptic integrals.

Averaging procedure can be treated as a nonlinear analog of the WKB method. It can also be applied to the multiphase solutions of soliton equations (nonlinear superpositions of moving waves) (Flaschka - Forest - McLaughlin [2], Lax - Levermore [3]). Averaged KdV equations are known as Whitham equations. Averaged equations associated with soliton systems are also known as equations of slow modulations or as equations of the soliton lattice dynamics.

It is well-known that the original KdV equation is an integrable in the Liouville sense hamiltonian system. For averaging (at least in the one-phase case) the integrability is not necessary but additional structures of the original equations are usually inherited in the associated averaged equations. (The connection between the original structures and the averaged ones may be very nontrivial). The Whitham equations, associated with the KdV solutions with arbitrary number of phases have the following properties.

1) They can be presented in the hamiltonian form (Dubrovin - Novikov [4]).
2) They can be written in the Riemann diagonal form (Flaschka - Forest - McLaughlin [2]). For the hydrodynamical-type systems with more than 2 components the existence of the Riemann invariants is a very nontrivial property.

3) They have an infinite number of conservation laws in involution (i.e. an infinite number of mutually commuting symmetries). The full set of integrals consist of \(2g + 1\) infinite families where \(g\) is the number of phases. One of these families is formed by the averaged higher KdV flows, the other \(2g\) have no direct analogs in the KdV theory. These \(2g\) additional families were constructed by S.P.Tsarev [5] for \(g = 1\) and by B.A.Dubrovin [6] for all \(g\).

4) They can be integrated by the generalized hodograph transform (Tsarev [7], [5]). In fact Tsarev proved that any hamiltonian hydrodynamical-type 1 + 1 system possessing Riemann invariants is integrable. Algebro-geometrical interpretation of the hodograph transform was suggested by I.M.Krichever [8].

The fact that the KdV equation has an infinite set of commuting symmetries (higher KdV) is well-known. They can be expressed via recursion operator \(\Lambda\)

\[
\frac{\partial u}{\partial t_{2n+1}} = \frac{d}{dx} \left( -\frac{\Lambda}{4} \right)^n u = -2 \frac{d}{dx} \left( \frac{\Lambda}{4} \right)^{n+1} \cdot 1. \tag{0.8}
\]

where \(\Lambda = -\partial_x^2 + 2\partial_x^{-1}u\partial_x + 2u\), \(x = t_1\), \(t = t_3\). But it is less known that the group of symmetries of KdV is much wider (the same is valid for all integrable by the inverse scattering transform systems). In our paper the averaged KdV symmetries, associated with the following KdV symmetries will be studied

\[
\frac{\partial u}{\partial \beta_{2n}} = -2 \frac{d}{dx} \left( -\frac{\Lambda}{4} \right)^{n+1} \left( \sum_{k=0}^{\infty} ((2k + 1)t_{2k+1} \left( -\frac{\Lambda}{4} \right)^k \right) \cdot 1. \tag{0.9}
\]

Let all the the higher times \(t_k\), \(k > 3\) be equal to zero. Then for \(n = -2, 0, 2\) we have:

\[
\frac{\partial u}{\partial \beta_{-2}} = 3t u_x - 2 \quad \text{(Galilean).} \tag{0.10}
\]

\[
\frac{\partial u}{\partial \beta_0} = 3t \left( \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x \right) + xu_x + 2u \quad \text{(Scaling).} \tag{0.11}
\]

\[
\frac{\partial u}{\partial \beta_2} = 3t \left( \frac{1}{16} u^5 - \frac{5}{8} uu_{xxx} - \frac{5}{4} u_x u_{xx} + \frac{15}{8} u^2 u_x \right) + x \left( \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x \right) + u_{xx} - 2u^2 - \frac{1}{2} u_x \partial_x^{-1} u. \tag{0.12}
\]

We call the symmetries (0.9) nonisospectral because they change the spectrum of the auxiliary linear problem (see section 2) or Virasoro because they form a noncommutative algebra, isomorphic to the algebra of the polynomial vector fields in the complex plane (the full Virasoro algebra does emerge in the KP theory). The ordinary higher KdV are isospectral in this sense.

First examples of the nonisospectral integrable equations (without connection with the symmetry problem) were constructed by F.Calogero - A.Degasperis [9], V.A.Belinskii - V.E.Zakharov [10], Maison [11], G.Barucchy - T.Regge [12]. The fact that we can generate new KdV symmetries by applying the recursion operator to the scaling transformation was pointed out by N.H.Ibragimov - A.B.Shabat in [13], but they did not pay much attention to this fact because they studied in [13] only local symmetries. Some nonisospectral KP symmetries were constructed by H.H.Chen, Y.C.Lee, J.E.Lin [14],
F.J.Schwarz [15]. A general approach based on the so-called mastersymmetries was developed by B.Fuchssteiner and W.Oevel [16]. A very convenient method of constructing nonisospectral symmetries for the integrable by the inverse scattering transform equations was developed by A.Yu.Orlov and E.I.Schulman [17]. An analog of formula (0.9) for the KP equation was found by A.S.Fokas and P.M.Santini [18].

All these papers were dedicated to the algebraic theory of the nonisospectral symmetries or to the vanishing in the infinity case. The periodic (finite-gap) theory needs a special consideration. I.M.Krichever and S.P.Novikov calculated the action of a part of such symmetries on the finite-gap KP solutions in [19] (their subalgebra was associated with a given KP solution and did not vary the spectral curve). They posed also the problem for the full KP analog of the algebra (0.9).

The action of the symmetries (0.9) on the finite-gap KdV solutions was calculated by A.Yu.Orlov and the author in [20], [21]. Similar results were also obtained in [20], [21] for the KP equation. It was shown that this action has a natural geometrical interpretation (see paragraph 1.3).

Late 80’ies a number of links between the nonlinear integerable equations and the conformal field theory was discovered. For example the finite-gap \( \tau \)-function in the KP theory coincides with the determinant of the \( \partial \) operator on the appropriate bundle (see, for example [20]). This determinant plays the central role in the string theory. The matrix models of the two-dimensional gravity give us another important example [22]. The partition function for the one-matrix model in the double scale limit coincides with the \( \tau \)-function, corresponding to a special KdV solution, determined by the constraint ([22]):

\[
\frac{\partial u}{\partial \beta_{-2}} = 0. \tag{0.13}
\]

In the conformal field theory the algebra of the vector fields in the circle and its central extension - the Virasoro algebra play the key role. A subalgebra of the algebra of nonisospectral symmetries corresponds to them in the KP theory (the central extension does emerge if we calculate the action of these symmetries at the \( \tau \)-function) (see [34], [20]).

Similar results for the two-dimensional topological quantum theory were obtained by I.M.Krichever [23]. He proved that the averaged KP equations are connected with the topological models and calculated the action of the Virasoro symmetries on these equations in terms of the averaged \( \tau \)-function. Applications of the Whitham equations to the topological models were also considered by B.A.Dubrovin [6].

In our paper we use the results of [23], [20], [21] to calculate the action of nonisospectral symmetries on the averaged KdV in terms of the Riemann invariants. We will assume the averaged KdV to be written in the Flaschka - Forest - McLaughlin form. The plan of our parer in the following. In the first section we recall the necessary definitions and results from the Riemann surfaces theory, including the deformations of the complex structures via the vector fields action. In the section 2 we recall the necessary results from the periodic KdV and KP theory including the action of the nonisospectral symmetries on the finite-gap solutions. The section 3 is dedicated to the averaged KdV equations. We recall the construction of the Abel Whitham hierarchy and present a new noncommutative set of symmetries.
1. THE RIEMANN SURFACES THEORY. SOME DEFINITIONS AND RESULTS.

In this section we recall the constructions from the Riemann surfaces theory we will use later ([24] - [26], [6], [20], [33]).

In our paper the words "Riemann surface" will always denote a compact nondegenerate Riemann surface. Such surfaces may be characterized as close Riemann surfaces of finite genus or as algebraic Riemann surfaces. We will consider two main classes - the general finite-gap surfaces (they correspond to the KP quasiperiodic solutions) and the hyperelliptic Riemann surfaces with a branch point in the infinity (they correspond to the quasiperiodic KdV solutions).

A Riemann surface \( \Gamma \) is called hyperelliptic if there exists a meromorphic function \( E \) on \( \Gamma \) with two simple poles or with one second-order pole. This function maps \( \Gamma \) to the complex plane, the covering \( E: \Gamma \to \mathbb{C} \) is two-sheeted so \( \Gamma \) is isomorphic to

\[
a) \quad Y^2 = (E - E_1) \cdots (E - E_{2g+2}) \quad \text{or} \quad (1.1.a)
b) \quad Y^2 = (E - E_1) \cdots (E - E_{2g+1}) \quad (1.1.b)
\]

respectively (here \( g \) is the genus of \( \Gamma \)). The branch points of \( \Gamma \) over \( E \) are \( E_1, \ldots, E_{2g+2} \) in the case a or \( E_1, \ldots, E_{2g+1}, \infty \) in the case b. In our paper only the hyperelliptic surfaces of the type b will be considered because only they are related to the KdV theory.

1.1. Cycles On The Riemann Surfaces.

Any compact nondegenerate Riemann surface is topologically equivalent to the sphere with a finite number of handles. This number is called genus, we will denote it \( g \).

The canonical basis of 1-cycles in \( \Gamma \) consists of \( 2g \) elements \( a_1, \ldots, a_g, b_1, \ldots, b_g \) such that

\[
a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}. \quad (1.2)
\]

where \( \circ \) denotes the intersection number. (Of course there are infinitely many nonhomotopical choices of canonical bases.) The surface \( \Gamma \) can be described as the result of gluing the sides of a \( 4g \)-sided polygon (the picture corresponds to \( g = 2 \))

in the following way: \( a_i^+ \leftrightarrow a_i^-, \quad b_i^+ \leftrightarrow b_i^- \). (All the vertices are glued together). The sides \( a_i^\pm, b_i^\pm \) correspond to the cycles \( a_i, b_i \) respectively.

Let \( \Gamma \) be a hyperelliptic surface - a two-sheeted covering over the \( E \)-plane with the branch points \( \infty, E_1, \ldots, E_{2g+1} \) and the cuts \( (-\infty, E_1), (E_2, E_3), \ldots, (E_{2g}, E_{2g+1}) \). Then we can choose the following canonical basis
where the point $E_0$ correspond to the vertices of the polygon.

1.2. Differentials On The Riemann Surfaces.

Let us have a marked point $\infty$ in our Riemann surface $\Gamma$ with a local parameter $z$. We use the notation $\infty$ because this point corresponds to the infinite value of energy in the spectral theory. The local parameter $z$ is a function defined in the neighbourhood of $\infty$ such that $z(\infty) = 0$, $dz(\gamma) |_{\gamma=\infty} \neq 0$. For convenience we introduce an additional function $\lambda(\gamma) = 1/z(\gamma)$. For the hyperelliptic surfaces we will always assume $E(\infty) = \infty$ and $E = -\lambda^2$. All the $\lambda$-expansions will be defined in the neighbourhood of the point $\infty$.

For us the following objects in $\Gamma$ will be necessary.

1) Canonical basis of holomorphic differentials $\omega_1, \ldots, \omega_g$. Differentials $\omega_j$ are determined by the normalization conditions

$$\oint_{a_j} \omega_i = \delta_{ij}. \quad (1.3)$$

The matrix $(B_{ij})$

$$B_{ij} = \oint_{b_j} \omega_i. \quad (1.4)$$

is called Riemann matrix. It is symmetrical ($B_{ij} = B_{ji}$), the imaginary part of the Riemann matrix $\text{Im} B_{ij}$ is positive definite. The coefficients of the Taylor series for $\omega_j$ in $\infty$ we will denote $q_{jk}^H$:

$$\omega_j = \sum_{m \geq 1} q_{jm}^H \lambda^{-m-1} d\lambda \quad (1.5)$$

2) Meromorphic differentials $\Omega_j$ with the only pole in the point $\infty$ such, that

$$\Omega_k = d(\lambda^k) + O(1), \quad \oint_{a_i} \Omega_k = 0. \quad (1.6)$$

We denote

$$\oint_{b_j} \Omega_k = (U_k)_j, \quad \bar{U}_k = ((U_k)_1, \ldots, (U_k)_g)$$

$$\Omega_k = d(\lambda^k) + \sum_{m \geq 1} Q_{km} \lambda^{-m-1} d\lambda. \quad (1.7)$$
The Riemann bilinear relations (see (1.16) below) result in

\[(U_j)_k = -2\pi i q^h_{kj}, \quad Q_{ij} = Q_{ji}.\]  

(1.8)

If \(\Gamma\) is a hyperelliptic curve then \(\Omega_{2k} = d((-E)^k), \ (U_{2k})_l = 0, \ Q_{kl} = 0\) if at least one of the indexes \(k, \ l\) is even.

3) Multivalued holomorphic differentials \(\omega^i_k, \sigma^i_k\) (they are defined only in the hyperelliptic Riemann surfaces). These differentials are determined by the following properties

\[
\Delta_{a_i} \omega^i_k = \delta_{ij}d(E^k), \quad \Delta_{b_j} \omega^i_k = 0,
\]

(1.9)

\[
\Delta_{b_j} \sigma^i_k = -\delta_{ij}d(E^k), \quad \Delta_{a_i} \sigma^i_k = 0,
\]

(1.10)

\[
\oint_{a_j} \omega^i_k = \oint_{b_j} \sigma^i_k = 0.
\]

(1.11)

Here \(\Delta_{a_i}\) and \(\Delta_{b_i}\) mean the increment of the differentials when going along the cycles \(a_i\) and \(b_i\) respectively. We will denote

\[
\oint_{b_j} \omega^i_k = (U^i_k)_j, \quad \oint_{b_j} \sigma^i_k = (V^i_k)_j
\]

(1.12)

\[
\omega^i_k = \sum_{m \geq 1} Q^{i}_{km} \lambda^{-m-1}d\lambda, \quad \sigma^i_k = \sum_{m \geq 1} R^{i}_{km} \lambda^{-m-1}d\lambda,
\]

(1.13)

4) Quasimomentum 1-differential \(dp\), meromorphic in \(\Gamma\) with the only pole in the point \(\infty\) such, that

\[
dp = -id(\lambda) + O(1), \quad \text{Im} \oint_{a_j} dp = \text{Im} \oint_{b_j} dp = 0.
\]

5) Algebra of the holomorphic vector fields in the punctured neighbourhood of \(\infty\) with the basis

\[
l_i = \lambda^{i+1}\partial\lambda, \quad -\infty < i < \infty.
\]

(1.14)

These fields have the following commutators

\[
[l_i, l_j] = (j - i)l_{i+j}.
\]

(1.15)

In the hyperelliptic case we will consider a subalgebra, generated by the elements with even indexes. All elements of this subalgebra are single-valued in the \(E\)-plane (in the neighbourhood of infinity).

\[
L_i = l_{2i} = 2(-1)^i E^{i+1}\partial E.
\]

The deformations of \(\Gamma\), generated by these elements preserve the hyperelliptic structure. (The action of the vector fields on the Riemann surfaces will be described below in the section 1.3).

6) Holomorphic 2-differentials in \(\Gamma\). These differentials can be written in the local coordinates as \(\omega^{(2)} = \omega^{(2)}(z)(dz)^2\), the space of such differentials is \(3g - 3\) - dimensional as \(g > 1\), \(1\) - dimensional as \(g = 1\) and empty as \(g = 0\). We have a standard overdetermined full set in this space \(\omega^{(2)}_{ij} = \omega_i \omega_j\) where \(\omega_i\) are holomorphic 1-differentials.

7) Riemann bilinear relations.
Let \( \Omega_1, \Omega_2 \) be multivalued meromorphic differentials in \( \Gamma \) such that
a) \( \Omega_1, \Omega_2 \) have no branch points and are locally single-valued.
b) All the residues of \( \Omega_1, \Omega_2 \) are equal to 0.
c) \( d^{-1}\Delta_a \Omega_j, d^{-1}\Delta_b \Omega_j \) are single-valued meromorphic functions in \( \Gamma \) without singularities on the cycles. We assume all these functions to be equal to 0 in the point of intersection of all cycles (this point correspond to the vertices of the polygon).

Then we have the following relation:

\[
2\pi i \sum \text{res}(d^{-1}\Omega_1)\Omega_2 = \sum_i \left\{ \oint_{a_i} \Omega_1 \oint_{b_i} \Omega_2 - \oint_{a_i} \Omega_2 \oint_{b_i} \Omega_1 + \oint_{a_i} (d^{-1}\Delta_b \Omega_2)\Omega_1 \right\} + \\
+ \sum_i \left\{ - \oint_{a_i} (d^{-1}\Delta_b \Omega_1)\Omega_2 + \oint_{b_i} (d^{-1}\Delta_a \Omega_1)\Omega_2 - \oint_{b_i} (d^{-1}\Delta_a \Omega_2)\Omega_1 \right\} + \\
+ \sum_i \left\{ - \oint_{a_i} (d^{-1}\Delta_b \Omega_1)(\Delta_b \Omega_2) - \oint_{b_i} (d^{-1}\Delta_a \Omega_2)(\Delta_a \Omega_1) \right\}.
\] (1.16)

All the terms in the right-hand side of (1.16) are correctly defined because of c). The property b) guaranties the correctness of the left-hand side.

Remark. All the residues of \( \Omega_1, \Omega_2 \) are equal to 0 so we have

\[
\sum \text{res}(d^{-1}\Omega_1)\Omega_2 = - \sum \text{res}(d^{-1}\Omega_2)\Omega_1
\] (1.17)

8) We will use the following scalar product, introduced in [27] and generalized in [23], [6]. Let the 1-forms \( \Omega_1, \Omega_2 \) be linear combinations of differentials, defined in the points 1-3. Then

\[
V_{\Omega_1 \Omega_2} = \text{res } (d^{-1}\Omega_1)\Omega_2 - \frac{1}{2\pi i} \sum_k \oint_{a_k} \Omega_1 \oint_{b_k} \Omega_2 + \\
+ \frac{1}{2\pi i} \sum_k \oint_{a_k} (d^{-1}\Delta_{b_k} \Omega_1)\Omega_2 - \frac{1}{2\pi i} \sum_k \oint_{b_k} (d^{-1}\Delta_{a_k} \Omega_2)\Omega_2.
\] (1.18)

(We have only one singular point in (1.18) so we omit the \( \sum \) sign before the residue). The brackets \( (\ )_+ \) in (1.18) denote the singular part of a function

\[
\left( \sum_{m=-\infty}^{m=+\infty} a_m \lambda^m \right)_+ = \sum_{m=1}^{m=+\infty} a_m \lambda^m
\] (1.19)

Comparing (1.17) with the Riemann relations (1.16) we see that the product (1.18) is symmetrical

\[
V_{\Omega_1 \Omega_2} = V_{\Omega_2 \Omega_1}.
\] (1.20)

( The last two terms in (1.16) are equal to 0. )

Simple direct calculations result in:

\[
V_{\omega_{k} \Omega_l} = Q_{kl} = Q_{lk}
\]

\[
V_{\omega_{k} \Omega_l} = q^H_{kl}
\]

\[
V_{\omega_{k} \Omega_l} = Q_{\alpha k} = - \frac{1}{2\pi i} \oint_{b_\alpha} (E^k - E^\alpha_0)\Omega_l
\]
\[
V_{\omega_k} = -\frac{1}{2\pi i} \oint \omega \frac{(E^k - E_0^k)\partial_k}{\partial\omega}, \quad V_{\sigma_k} = -\frac{1}{2\pi i} \oint \frac{(E^k - E_0^k)\partial_k}{\partial\sigma_k}
\]

\[
\color{black} \omega_k = \frac{\left(\sum_{l=1}^{g} A_{kl} E_l^{-1}\right) dE}{\sqrt{-4(E - E_1) \cdots (E - E_{2g+1})}}, \quad \Omega_k = \frac{\left(-k(E)\sum_{l=1}^{g} C_{kl} E_l^{-1}\right) dE}{\sqrt{-4(E - E_1) \cdots (E - E_{2g+1})}}, \quad k = 2n + 1, n \geq 0.
\]

The coefficients \(A_{kl}, C_{kl}\) are uniquely determined by the normalization conditions

\[
\oint_a \omega_i = \delta_{ij}, \quad \oint_a \Omega_i = 0. \quad (1.23)
\]

Equations (1.23) are equivalent to a linear system on \(A_{kl}, C_{kl}\). All the coefficients of this system are hyperelliptic integrals, the system is non-degenerate (see [25]) and we can solve it. The coefficients of the Taylor series in \(\infty\) for \(\omega_k, \Omega_k\) can be algebraically expressed via \(A_{kl}, C_{kl}, E_j\). The remark that all the integrals

\[
\oint (E^k - E_0^k)\omega_l, \quad \oint (E^k - E_0^k)\Omega_l
\]

are hyperelliptic completes the proof.

1.3. Deformations Of The Riemann Surfaces And The Riemann Problem. The Moduli Space.
All the surfaces of genus $g$ are topologically equivalent. But if $g \geq 1$ then they may be different as complex manifolds. Consider the simplest case $g = 1$ of complex tori. There exists an unique holomorphic differential $\omega_1$ such that

$$\oint_a \omega_1 = 1.$$  \hfill (1.25)

The parameter

$$\tau = \oint_b \omega_1, \quad \text{Im} \ \tau > 0$$  \hfill (1.26)

is correctly defined if we have a fixed basis of 1-cycles $a, b$ in $\Gamma$. The surface $\Gamma$ can be described as a factor of the complex plane by the group of shifts, generated by:

$$z \to z + 1, z \to z + \tau.$$  \hfill (1.27)

Let $\Gamma, \Gamma'$ be complex tori ($g=1$) with the bases of 1-cycles $a, b$ and $a', b'$ respectively. Then a complex map $f : \Gamma \to \Gamma'$ such that $f(a) = a', f(b) = b'$ exists if and only if $\tau = \tau'$.

Parameter $\tau$ depends on the choice of basic cycles. Let us have two bases $a, b$ and $\tilde{a}, \tilde{b}$ respectively. Then there exists a matrix $\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{sl}(2, \mathbb{Z})$ such that

$$\left( \begin{array}{c} \tilde{a} \\ \tilde{b} \end{array} \right) = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \left( \begin{array}{c} a \\ b \end{array} \right), \quad \tilde{\tau} = \frac{\gamma + \delta \tau}{\alpha + \beta \tau}.$$  \hfill (1.28)

The group $\text{sl}(2, \mathbb{Z})$ consists of all $2 \times 2$ integer matrices such that $\alpha \delta - \beta \gamma = 1$.

Summing all of this we have:

Two complex (one-dimensional in the complex sense) tori with the parameters $\tau$ and $\tau'$ are isomorphic if and only if there exists a matrix

$$\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \text{sl}(2, \mathbb{Z}) \quad \text{such, that} \quad \tau' = \frac{\gamma + \delta \tau}{\alpha + \beta \tau}.$$  \hfill (1.29)

The space of all 1-dimensional complex tori is called the moduli space for $g = 1$ or the space of all complex structures on the surfaces of genus $g = 1$. We have proved that the moduli space for $g = 1$ is the factor of the complex upper-plane $\text{Im} \ \tau > 0$ by the group $\text{sl}(2, \mathbb{Z})$.

It is possible to consider the moduli space of all Riemann surfaces with a set of marked points and the moduli space of the Riemann surfaces with a set of marked points and a local parameter in one of them. The last space is infinite-dimensional.

The moduli space of the Riemann surfaces with $g > 1$ is a $3g - 3$-dimensional complex manifold. We do not want to discuss its properties in details. But the following construction is very important for us.

Let $\Gamma$ be a Riemann surface with a marked point $\infty$, a local parameter $z = 1/\lambda$ in the neighbourhood of $\infty$, $S$ be a small contour surrounding $\infty$, $v = v(\lambda)d\lambda$ be a holomorphic vector field in the vicinity of the contour $U(S)$. Then we can construct a family of new Riemann surfaces $\Gamma_\alpha$ depending on a parameter $\alpha$ (see [25]). We will assume $\alpha$ to be sufficiently small.

The contour $S$ splits $\Gamma$ to a small disk $D$, containing the point $\infty$ and an open Riemann surface $\Gamma \setminus D$ ( $\Gamma \setminus D$ denotes the set of point $\gamma$ such, that $\gamma \in \Gamma, \gamma \notin D$). We can cover $\Gamma$ by two regions $\Gamma_+, \Gamma_-$ such, that:
1) \( D \subset \Gamma_- \).
2) \( \Gamma \setminus D \subset \Gamma_+ \).
3) \( \Gamma_+ \cap \Gamma_- = U(S) \).

If \( \gamma \) is a point of \( U(S) \) then we will denote the correspondent points in \( \Gamma_+ \) and \( \Gamma_- \) \( \gamma_+ \) and \( \gamma_- \) respectively. \( \Gamma \) may be treated as the result of gluing \( \Gamma_+ \) to \( \Gamma_- \) \( f : \Gamma_+ \to \Gamma_- \)
\( f(\gamma_+) = \gamma_- \). Let us introduce a new gluing function \( f_\alpha \) by formula

\[
\begin{equation}
f_\alpha(\gamma_+) = \exp(\alpha v) \gamma_-,
\end{equation}
\]

where \( \exp(\alpha v) \gamma_- \) denotes the shift of the point \( \gamma_- \) via the vector field \( v \) after the lapse of the time \( \alpha \). Then we can define a new Riemann surface \( \Gamma_\alpha \) as the result of gluing \( \Gamma_+ \) to \( \Gamma_- \) via the function \( f_\alpha \).

The Riemann surfaces \( \Gamma \) and \( \Gamma_\alpha \) are constructed from the same parts \( \Gamma_+ \) and \( \Gamma_- \). Thus we have a map \( E : \Gamma \to \Gamma_\alpha \), coinciding with identical maps \( \Gamma_+ \to \Gamma_+ \) on \( \Gamma \setminus D \) and \( \Gamma_- \to \Gamma_- \) on \( D \). Map \( E \) has a jump on the contour \( S \). If we have some marked points in \( \Gamma \) or local parameters in some of them then we will map these objects by \( E \).

Let us have an infinitesimal transformation \( (\alpha \) is infinitely small) and \( \Delta \) be a holomorphic tensor field in \( \Gamma_\alpha \). The map \( E \) carries \( \Delta \) from \( \Gamma_\alpha \) to \( \Gamma \) so we can treat \( \Delta \) as a tensor field in \( \Gamma \) with a jump on the contour \( S \). The boundary values of \( \Delta \) on \( S \) \( \Delta_+ \) and \( \Delta_- \) satisfy the following relation

\[
\begin{equation}
\Delta_+ - \Delta_- = \alpha L_v \Delta,
\end{equation}
\]

where \( L_v \) denotes the Lie derivative

\[
\begin{equation}
L_v(\lambda) \partial_\lambda g(\lambda)(d\lambda)^\alpha = (v(\lambda)g'(\lambda) + \alpha v'(\lambda)g(\lambda)) (d\lambda)^\alpha
\end{equation}
\]

(\( \alpha \) is infinitely small so we can write \( \Delta_+ \) in the right-hand side of (1.31) as well as \( \Delta_- \)).

Formula (1.31) gives us a very convenient method for calculation the variations of tensor object via complex structure variations: we calculate the right-hand side of (1.31) and solve the Riemann problem in the appropriate functional class. Riemann problem is one of the basic objects in the soliton theory so this approach allows us to connect these deformations of the Riemann surfaces with the non-isospectral symmetries (see section 2.2).

In the hyperelliptic case our construction has the following interpretation. Let \( \Gamma \) be a hyperelliptic surface over the \( E \)-plane with the branch points \( \infty, E_1, \ldots, E_{2g+1} \), \( v = v(E) \partial_{E} \) be a vector field in the \( E \)-Plane, nonsingular for all \( E \neq \infty \), the local parameter in the point \( \infty \) be \( z = 1/\lambda, \lambda = \sqrt{-E} \).

Then \( \Gamma_\alpha \) is a hyperelliptic Riemann surface with the branch points \( \infty, \exp(\alpha v)E_1, \ldots, \exp(\alpha v)E_{2g+1} \) and the local parameter \( z = 1/\lambda, \lambda = \sqrt{-E}_\pm \). The map \( E \) carries the point \( E_{\pm} \) in \( \Gamma \) to the point \( \exp(\alpha v)E_{\pm} \) in \( \Gamma_\alpha \) if the point \( E_{\pm} \) is located outside the neighbourhood of \( \infty \) surrounded by a small contour \( S \) and carries the point \( E_{\pm} \) in \( \Gamma \) to the point \( E_{\pm} \) in \( \Gamma_\alpha \) if the point \( E_{\pm} \) is located inside the neighbourhood of \( \infty \) (the sign \( \pm \) means the upper or lower sheet).

If we have a family of deformed hyperelliptic Riemann surfaces \( \Gamma_\alpha \) and a function on this family \( f(E_{\pm}, \alpha) \) then we have two differential operators

\[
\partial_\alpha f = \frac{\partial f}{\partial \alpha}
\]
and

\[ D_\alpha f = \frac{\partial f}{\partial \alpha} + L_v f, \tag{1.34} \]

where \( L_v \) is the Lie derivative (1.32) in the first argument. In contrast with \( \partial_\alpha \) the operator \( D_\alpha \) can be generalized to arbitrary Riemann surfaces and can be treated as a connection generated by the map \( E \).

The following property of the connection \( D_\alpha \) is very important for us:

**Lemma 1.2.** Let \( \Delta(E_\pm, \alpha) \) be a holomorphic tensor field in \( \Gamma_\alpha \setminus \infty \) for all \( \alpha \). Then \( D_\alpha \Delta(E_\pm, \alpha) \) is nonsingular outside the neighbourhood of \( \infty \) (for \( \partial_\alpha \) in is not valid in the branch points).

While calculating the nonisospectral symmetries we will need the derivatives of the holomorphic differentials by the complex structures. In the hyperelliptic case we have:

**Lemma 1.3.** Let \( \Gamma \) be a hyperelliptic Riemann surface, \( L_n = l_{2n} = 2(-1)^n E^{n+1} \partial_E \) be a holomorphic vector field in the \( E \)-plane, \( i \geq -1 \), \( \Gamma, \alpha \) be the deformation of \( \Gamma \) via the vector field \( v \). Then

\[ D_\alpha \Omega_k = k\Omega_{2n+k} + \sum_{m=1}^{2n-1} Q_{km} \Omega_{2n-m}, \tag{1.35} \]

\[ D_\alpha \omega = \sum_{m=1}^{2n-1} q_{km}^H \Omega_{2n-m}, \tag{1.36} \]

\[ D_\alpha \omega^\alpha_k = 2(-1)^n k \omega^\alpha_{n+k} \sum_{m=1}^{2n-1} Q_{km}^\alpha \Omega_{2n-m}, \tag{1.37} \]

\[ D_\alpha \sigma^\alpha_k = 2(-1)^n k \sigma^\alpha_{n+k} \sum_{m=1}^{2n-1} R_{km}^\alpha \Omega_{2n-m}, \tag{1.38} \]

Proof of the Lemma 1.3. In accord with the rule described above we have to solve the following Riemann problem:

Let \( \Delta \) be one of the differentials \( \Omega_k \), \( \omega_k \), \( \omega^\alpha_k \), \( \sigma^\alpha_k \). Then we have to construct a pair of differentials \( \Delta_+ \), \( \Delta_- \) such that

1) \( \Delta_+ - \Delta_- = L_v \Delta \).

2) \( \Delta_- \) is defined and nonsingular in the neighbourhood of \( \infty \).

3) \( \Delta_+ \) is nonsingular for all \( E_\pm \neq \infty \).

4) \( \oint_a \Delta_+ = 0 \).

5) \( \Delta_+ \) is single-valued as \( \Delta = \Omega_k \) or \( \Delta = \omega_k \),

\[ \Delta_a \Delta_+ = 2(-1)^n k \Delta_a \omega^\alpha_{n+k}, \Delta_a \Delta_+ = 0 \text{ as } \Delta_+ = \omega^\alpha_k \]

\[ \Delta_a \Delta_+ = 2(-1)^n k \Delta_a \sigma^\alpha_{n+k}, \Delta_a \Delta_+ = 0 \text{ as } \Delta_+ = \sigma^\alpha_k \]

Then \( D_\alpha \Delta = \Delta_+ \).

In fact we are looking for a differential with the properties 3-5 in the finite part of \( \Gamma \) and the prescribed singularity \( L_v \Delta \) in \( \infty \). But such differential can be easily constructed as a linear combination of \( \Omega_k \), \( \omega_k \), \( \omega^\alpha_k \), \( \sigma^\alpha_k \). This completes the proof.

**Corollary 1.** ([25]).

\[ \frac{\partial B_{ij}}{\partial \alpha} = \oint_S v_j \omega_i \omega_j. \tag{1.39} \]
where the product $v(\lambda) \partial_\lambda \omega_i(\lambda)d\lambda \omega_j(\lambda)d\lambda$ is the 1-form $v(\lambda)\omega_i(\lambda)\omega_j(\lambda)d\lambda$. This formula is valid for general Riemann surfaces as well as for hyperelliptic ones. We assume that the contour $S$ goes around the point $\infty$ counterclockwise.

**Corollary 2.** The vector field $v$ do not vary the complex structure of $\Gamma$ if and only if all the integrals

$$\oint_S v\omega_i^{(2)}$$

is are equal to 0. Here $\omega_i^{(2)}$ is the basis of holomorphic 2-forms. It proves that the moduli space is $3g - 3$ dimensional as $g > 1$ and 1-dimensional as $g = 1$.

### 1.4. Riemann Theta-Functions.

Let $b_{ij}$ be a complex symmetrical $g \times g$ matrix such, that $\text{Re} b_{ij}$ is negative definite, $\vec{z}$ be a complex $g$-component vector. Then the Riemann theta-function can be defined as an infinite sum ([26])

$$\theta(\vec{z} | b_{ij}) = \sum_{m_1, \ldots, m_g} \exp \left\{ \frac{1}{2} \sum_{kj} b_{kj} m_km_j + \sum_k z_km_k \right\},$$

(1.40)

where $m_k, k = 1, \ldots, g$ are arbitrary integers. This sum converges for all $\vec{z}$. The theta function has the following periodicity properties

$$\theta(z_1, z_2, \ldots, z_k + 2\pi i, \ldots, z_g | b_{ij}) = \theta(z_1, z_2, \ldots, z_k, \ldots, z_g | b_{ij}).$$

(1.41)

$$\theta(z_1 + b_{1k}, z_2 + b_{2k}, \ldots, z_g + b_{gk} | b_{ij}) = \theta(z_1, z_2, \ldots, z_g | b_{ij}) \exp \left\{ -b_{kk}/2 - z_k \right\}. \quad (1.42)$$

The zeros of the theta function are described by the following Lemma (see [26])

**Lemma 1.4.** Let $\Gamma$ be a Riemann surface with a marked point $\gamma_0$, $B_{ij}$ be the matrix of periods (1.4), $b_{ji} = 2\pi i B_{ji}$, $\vec{A}(\gamma)$ be the Abel transform, i.e. $\vec{A}(\gamma)$ is a multivalued map $\Gamma \rightarrow \mathbb{C}^g$ determined by the formula

$$(\vec{A}(\gamma))_k = 2\pi i \int_{\gamma_0}^{\gamma} \omega_k, \quad k = 1, \ldots, g.$$

(1.43)

Then there exists a $g$-dimensional complex vector $\vec{K} = \vec{K}(\Gamma, \gamma_0)$ such that the function

$$\varphi(\gamma) = \theta(\vec{A}(\gamma) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_g) + \vec{K} | b_{ij}), \quad \gamma \in \Gamma.$$

(1.44)

has one of the following properties:

1) $\varphi(\gamma) \equiv 0$ or

2) $\varphi(\gamma)$ has exactly $g$ zeros in the points $\gamma_1, \ldots, \gamma_g$.

$\vec{K}$ is called Riemann constants vector.

### 2. PERIODIC THEORY OF THE KORTEVEG - DE - VRIES AND KADOMTSEV - PETVIASHVILI EQUATIONS.

In this section we will recall some facts from the KdV and KP theory. Two topics are the most interesting for us: periodic (quasiperiodic) finite-gap theory and the action of the nonisospectral symmetries on the finite-gap solutions.
Finite-gap KdV solutions can be treated as nonlinear superpositions of the moving waves. Such solutions are the basic objects for the averaging procedure. They have been constructed in the papers of S.P. Novikov, B.A. Dubrovin, V.B. Matveev, A.R. Its, P. Lax, H. McKean and P. van Moerbeke (see book [28] for more detailed description and references). Finite-gap KP solutions were constructed by I.M. Krichever [29]. The direct periodic problem for KP is more complicated (the results, obtained by I.M. Krichever can be found in [29]).

The first step in constructing averaged nonisospectral symmetries is the following: the action of these symmetries on the finite-gap solution is calculated. This problem was solved by A.Yu. Orlov and the author in [20], [21]. We recall some results of these papers. We use the representation for nonisospectral symmetries, suggested by A.Yu. Orlov and E.I. Schulman [17] as the most convenient for us.

2.1. KdV And KP Theory. Integration And Isospectral Symmetries. Periodic Theory. Baker-Akhiezer Function And Cauchy Kernel.

The theory of the KdV equation

$$u_t = \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x. \tag{2.1}$$

is based upon the existence of the following representation. Let

$$L = -\partial_x^2 + u(x, t), \quad A = \partial_x^2 - \frac{3}{4} (u \partial_x + \partial_x u) \tag{2.2}$$

be ordinary differential operators depending on an extra parameter $t$. Then the function $u(x, T)$ satisfy (2.1) if and only if the following relation takes place

$$\partial L/\partial t = [L, A]. \tag{2.3}$$

Representation (2.3) is called Lax pair or $L - A$ pair for KdV.

One of the first results of the soliton theory was the existence of infinitely many mutually commuting KdV symmetries. They can be written via the recursion operator

$$\frac{\partial u}{\partial t_{2n+1}} = \frac{d}{dx} \left( -\frac{\Lambda}{4} \right)^n u = -2 \frac{d}{dx} \left( -\frac{\Lambda}{4} \right)^{n+1} \cdot 1, \tag{2.4}$$

where

$$\Lambda = -\partial_x^2 + 2\partial_x^{-1} u \partial_x + 2u \tag{2.5}$$

or in the Lax form

$$\partial L/\partial t_{2n+1} = [L, A_{2n+1}], \tag{2.6}$$

where

$$A_{2n+1} = \left\{ (-L)^{\frac{2n+1}{2}} \right\}_+, \tag{2.7}$$

$(-L)^{\frac{2n+1}{2}}$ denotes a formal pseudodifferential operator, i.e a series in $\partial_x$ with finite number of positive terms and infinite number of negative, $\{ \}_+$ denotes the differential part (see [31]). We mark the times by odd indexes to have unified notations for KdV and KP. Here $t_1 = x$, $t_3 = t$. 
Equations (2.3), (2.6) result in the following property: the spectrum of $L$ does not depend upon the times $t_3, t_5, \ldots, t_{2k+1}, \ldots$. This is the reason why we call these symmetries isospectral.

KdV equation is integrated by the inverse scattering transform, i.e. we consider the "scattering data" for $L$ as a new variable instead of $u(x)$. (We write "scattering data" in quotation marks to stress that these data coincides with the physical scattering data only for some functional classes of potentials). Because of the isospectral property the evolution law for the "scattering data" is very simple (see [28]). The map from $u$ to the scattering data for the vanishing in the infinity potentials in the small amplitude limit coincides with the Fourier transform and can be treated as its nonlinear analog.

The realization of this scheme depends on the functional class of the potential. Let us recall the scheme for the periodic case

$$u(x + \Pi, t) = u(x, t),$$

(2.8)

where $u(x, t)$ is a real nonsingular potential.

We consider the following spectral problems for $L$:

a) Main problem $L\psi = E\psi$, $\psi$ is bounded in $x$.

b) Auxiliary problem $L\psi_n = E\psi_n$, $\psi_n(0) = \psi_n(\Pi) = 0$.

The spectrum of the main problem consists of a set of intervals $[E_1, E_2], [E_3, E_4], \ldots, [E_{2n-1}, E_{2n}], \ldots$ where $E_1 < E_2 < E_3 < E_4 \leq E_5 \ldots, E_{2n+1} - E_{2n} \to 0$ as $n \to \infty$.

The spectrum of the auxiliary problem consists of an infinite number of points $d_1 < d_2 < d_3 < \ldots$ located in the gaps $d_1 \in [E_2, E_3], d_2 \in [E_4, E_5], d_3 \in [E_6, E_7], \ldots$.

The Bloch eigenfunction $\psi(x, E)$ normalized by the conditions $\psi(x + \Pi, E) = \exp(\Pi ip(E)) \psi(x, E)$ and $\psi(0, E) = 1$ is meromorphic on a two-sheeted Riemann surface $\Gamma$ over the $E$-plane with branch points $E_1, E_2, \ldots, \infty$, and has simple poles in the points $\gamma_1, \gamma_2, \ldots$, such that the projection of $\gamma_n$ to the $E$-plane coincides with $d_n$.

The function $p(E_\pm)$ is defined in $\Gamma$ and is called quasimomentum.

The spectrum corresponding to a general potential has infinite number of gaps. But for us the so-called finite-gap case when $E_{2n} = E_{2n+1}$ for all $n > g$ is the most important. (The infinite-genus case can also be studied [32] but the answers are much more complicated). Finite-gap potentials are dense in the space of all periodic potentials.

The inverse problem data in the finite-gap case is the following:

1) $2g + 1$ real numbers $E_1, E_2, \ldots, E_{2g+1}, E_1 < E_2 < \ldots < E_{2g+1}$. The points $E_k$ are the boundary point of the spectrum.

2) $g$ points $\gamma_1, \ldots, \gamma_g$ in a hyperelliptic Riemann surface $\Gamma$ with the branch points $\infty, E_1, E_2, \ldots, E_{2g+1}$, such, that $E(\gamma_k) \in [E_{2k}, E_{2k+1}]$ where $E$ is the projection to the $E$-plane.

This data uniquely determines the potential $u(x)$.

The flows (2.3), (2.6) do not move the branch points $E_k$ but they shift the divisor $\gamma_k$. The evolution of the points $\gamma_k$ can be described in terms of ordinary differential equations, derived by B.A.Dubrovin (see [28]). This system is nonlinear, but it has a very nice explicit solution.

**Lemma 2.1.** (see [28]). Let $\tilde{A}(\gamma)$ be the Abel transform defined in the paragraph 1.4. Then

$$\tilde{A}(\gamma_1) + \tilde{A}(\gamma_2) + \ldots + \tilde{A}(\gamma_g) = \tilde{A}_0 + x\tilde{U}_1 + t\tilde{U}_3 + t_5\tilde{U}_5 + \ldots,$$

(2.9)

where $\tilde{A}_0$ is some constant vector, the vectors $U_k$ are defined by (1.7).
Inverse transform to (2.9) can be written in terms of the theta-functions. The answer is given by the A.R.Its - V.B.Matveev formula (see [28], [33]).

\[ u(x, t_3, t_5, \ldots) = -2\partial_x^2 \log \theta(V_0(\gamma_1, \ldots, \gamma_g) + x\tilde{U}_1 + t\tilde{U}_3 + t_5\tilde{U}_5 + \ldots |b_{ij}) + C(\Gamma), \quad (2.10) \]

where \( V_0(\gamma_1, \ldots, \gamma_g) \), \( C(\Gamma) \) are some constants, \( b_{ij} = 2\pi iB_{ij} \), \( B_{ij} \) is the Riemann matrix (1.4).

General finite-gap solutions are quasiperiodic. The characterization of periodic solutions in terms of the inverse data is a complicated problem.

A slightly different approach to the inverse problem is more convenient in some situations (see [28]). Instead of normalizing \( \psi(E, 0) = 1 \) for all \( t \) we consider a function \( \Psi(\gamma, \vec{t}), \gamma \in \Gamma; \vec{t} = (t_1, t_3, t_5, \ldots), t_1 = x, t_3 = t \) such that

1) \( L\Psi(\gamma, \vec{t}) = E(\gamma)\Psi(\gamma, \vec{t}) \).
2) \( \Psi(\gamma, x + \Pi, t_3, t_5, \ldots) = \exp(i\Pi p(\gamma))\Psi(\gamma, x, t_3, t_5, \ldots) \)
3) \( \Psi(\gamma, 0, 0, 0, \ldots) = 1 \)
4) \( (\partial_{t_{2n+1}} - A_{2n+1})\Psi(\gamma, \vec{t}) = 0 \). (Symmetries (2.6) is mutually commuting so the condition 4 is self-consistent).

\( \Psi(\gamma, \vec{t}) \) is called Baker-Akhiezer function. It has the following analytical properties:

Pr.1) \( \Psi(\gamma, \vec{t}) \) is meromorphic in \( \Gamma \setminus \infty \).

Pr.2) \( \Psi(\gamma, \vec{t}) \) has simple poles in the points \( \gamma_1, \ldots, \gamma_g \) and no other singularities in \( \Gamma \setminus \infty \).

Pr.3) \( \Psi(\gamma, \vec{t}) \) has an essential singularity as \( \gamma \to \infty \)

\[ \Psi(\lambda, \vec{t}) = \exp(\Theta(\gamma, \vec{t})) [1 + \chi_1(\vec{t})/\lambda + \chi_2(\vec{t})/\lambda^2 + \ldots] \quad (2.11) \]

where in the correspondent to KdV hyperelliptic case \( \lambda = \sqrt{-E}, \Theta(\gamma, \vec{t}) = \lambda x + \lambda^3 t + \lambda^5 t_5 + \ldots \)

**Lemma 2.2.** The properties Pr.1 - Pr.3 uniquely determined the function \( \Psi(\gamma, \vec{t}) \). It can be expressed in terms of the theta-functions (see, for example review [33])

\[ \Psi(\gamma, \vec{t}) = \frac{\theta(\sum t_k \tilde{U}_k + \vec{A}(\gamma) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_g) + \vec{K})\theta(-\vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_g) + \vec{K})}{\theta(\sum t_k \tilde{U}_k - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_g) + \vec{K})\theta(\vec{A}(\gamma) - \vec{A}(\gamma_1) - \ldots - \vec{A}(\gamma_g) + \vec{K})} \cdot \exp(\sum t_k \int^\gamma \Omega_k). \quad (2.12) \]

Here \( \vec{K} \) is the vector of Riemann constants (see paragraph 1.4), the integrals are normalized by \( \int^\gamma \Omega_k = \lambda^k + o(1) \).

Comparing (2.12) with

\[ u(\vec{t}) = 2\partial_x \chi_1(\vec{t}) \quad (2.13) \]

we can easily derive (2.12).

Now we will recall the finite-gap inverse scattering transform for KP [29].

\[ \left( u_x - \frac{1}{4} u_{xxx} + \frac{3}{2} u_x \right)_x = 3u_{yy}. \quad (2.14) \]

Let \( \Gamma \) be arbitrary Riemann surface of genus \( g \) with a marked point \( \infty \), a local parameter \( 1/\lambda \) in \( \infty \) and \( g \) marked points \( \gamma_1, \ldots, \gamma_g \). Then for the data of general position there exists a unique function \( \Psi(\gamma, \vec{t}), \gamma \in \Gamma; \vec{t} = (t_1, t_2, t_3, t_4, t_5, \ldots), t_1 = x, t_2 = y, t_3 = t \) with the analytic properties Pr.1 - Pr.3 (for KP \( \Theta(\gamma, \vec{t}) = \lambda x + \lambda^2 y + \lambda^3 t + \lambda^4 t_4 + \lambda^5 t_5 + \ldots \)
(2.15) The compatibility conditions

\[ [(\partial_{t_2} - \tilde{A}_2), (\partial_{t_k} - \tilde{A}_k)] = 0 \] (2.16)
give the Lax pair for KP as \( k = 2 \) and isospectral symmetries as \( k > 3 \). The potential \( u(\vec{t}) \) is given by slightly changed (2.10) - the sum is over all indexes - odd and even. \( y \)-independent KP solutions satisfy KdV.

We need also the Baker-Akhiezer conjugate differential \( \Psi^+(\gamma, \vec{t}) \). Its analytical properties are the following

- **Pr.1’** \( \Psi^+(\gamma, \vec{t}) \) is a holomorphic in \( \Gamma \setminus \infty \) 1-differential.
- **Pr.2’** \( \Psi^+(\gamma, \vec{t}) \) has simple zeros in the points \( \gamma_1, \ldots, \gamma_g \).
- **Pr.3’** \( \Psi^+(\gamma, \vec{t}) \) has an essential singularity as \( \gamma \to \infty \)

\[ \Psi(\lambda, \vec{t}) = d\lambda \exp(-\Theta(\gamma, \vec{t}))[1 + \chi_1^+(\vec{t})/\lambda + \chi_2^+(\vec{t})/\lambda^2 + \ldots] \] (2.17)

where \( \Theta(\gamma, \vec{t}) = \lambda x + \lambda^2 y + \lambda^3 t + \lambda^4 t_4 + \lambda^5 t_5 + \ldots \)

\[ (\partial_{t_n} + \tilde{A}_n^+)\Psi^+(\gamma, \vec{t}) = 0, \] (2.18)

where \( \tilde{A}_n^+ \) is the formal conjugate to \( \tilde{A}_n \), i.e. \( (a(x)(\partial_x)^n)^+ = (-\partial_x)^n a(x) \). The following ortogonality properties are important:

**Lemma 2.3.** 1) Let \( S \) be a small contour surrounding the point \( \infty \). Then

\[ \oint_S \Psi(\gamma, \vec{t})\Psi^+(\gamma, \vec{t}') = 0 \] (2.19)

for all \( \vec{t}, \vec{t}' \).

2) Let \( p(\gamma) = \int' dp \) where \( dp \) is the quasimomentum differential defined in the section 1, \( G(\gamma) \) be a contour in \( \Gamma \) consisting of all points \( \gamma' \) such that \( \text{Im} p(\gamma') = \text{Im} p(\gamma) \). Then in the contour \( G(\gamma) \) the following relation is valid

\[ \int_{-\infty}^{+\infty} \Psi(\gamma, \vec{t})\Psi^+(\gamma', \vec{t'}) = 2\pi i \delta(\gamma - \gamma'). \] (2.20)

We have to calculate deformations of the Baker - Akhiezer function. For this purpose we need an appropriate Cauchy kernel.

**Lemma 2.4.** ([20]). The Cauchy - Baker - Akhiezer kernel \( \omega(\gamma, \gamma', \vec{t}) \) with the following properties:

- **1)** \( \omega(\gamma, \gamma', \vec{t}) \) is a function in \( \gamma \) and a 1-form in \( \gamma' \).
- **2)** \( \omega(\gamma, \gamma', \vec{t}) \) is meromorphic function of \( \gamma \) in \( \Gamma \setminus \infty \) with simple poles \( \gamma_1, \ldots, \gamma_g, \gamma' \).
- **3)** As a function of \( \gamma' \) \( \omega(\gamma, \gamma', \vec{t}) \) is meromorphic in \( \Gamma \setminus \infty \) with one pole \( \gamma \) and zeros in the points \( \gamma_1, \ldots, \gamma_g \).
- **4)** \( \omega(\gamma, \gamma', \vec{t}) = o(\exp(\Theta(\gamma, \vec{t}))) \) as \( \gamma \to \infty \).
- **5)** \( \omega(\gamma, \gamma', \vec{t}) = o(\exp(-\Theta(\gamma', \vec{t})))d\lambda \) as \( \gamma' \to \infty \).
- **6)** \( \omega(\gamma, \gamma', \vec{t}) \sim \frac{d\lambda'}{2\pi i (\lambda' - \lambda)} \) as \( \lambda \to \lambda' \).
can be written in the following form

\[ \omega(\gamma', x, y, t, \ldots) = \frac{i}{2\pi} \int_{\pm\infty}^{x} \Psi(\gamma, x', y, t, \ldots)\Psi^+(\gamma', x', y, t, \ldots)dx'. \] (2.21)

If \( \gamma' \not\in G(\gamma) \) the sing in limit of integration in (2.21) is uniquely determined by the convergence condition. For \( \gamma' \in G(\gamma) \) the property (2.20) guaranties correctness.

First formula similar to (2.21) was obtained by I.M.Krichever and S.P.Novikov in [19] for systems with discrete \( x \).

### 2.2. Nonisospectral Symmetries. Action On The Finite-Gap Solutions.

We have pointed out in the introduction that KdV equation possesses the following set of symmetries

\[ \frac{\partial u}{\partial \tau_{2n}} = -2 \frac{d}{dx} \left( -\frac{\Lambda}{4} \right)^{n+1} \left( \sum_{k=0}^{\infty} ((2k+1)t_{2k+1} \left( -\frac{\Lambda}{4} \right)^k \right) \cdot 1. \] (2.22)

They can be written in much more convenient form suggested by A.Yu.Orlov and E.I.Schulman [17].

\[ \frac{\partial u}{\partial \tau_{2n}} = -2 \frac{d}{dx} \text{res} |_{\lambda=\infty} \lambda^{2n+1} \Psi(\lambda, \vec{t})\Psi^+(\lambda, \vec{t}), \] (2.23)

Ordinary higher KdV equations (2.4) have similar representation

\[ \frac{\partial u}{\partial \tau_{2n+1}} = -2 \frac{d}{dx} \text{res} |_{\lambda=\infty} \lambda^{2n+1} \Psi(\lambda, \vec{t})\Psi^+(\lambda, \vec{t}). \] (2.24)

The formulas (2.22), (2.24) can be treated in the following way. Let us substitute the asymptotical expansions (2.11), (2.17) to the linear problem

\[ L\Psi(\lambda, \vec{t}) = -\lambda^2 \Psi(\lambda, \vec{t}), \quad L\Psi^+(\lambda, \vec{t}) = -\lambda^2 \Psi^+(\lambda, \vec{t}). \] (2.25)

From (2.25) we can calculate all the coefficients \( \chi_k(\vec{t}), \chi_k^+(\vec{t}) \) via \( u(\vec{t}) \) (in nonlocal form). Then we substitute them to (2.23), (2.25). All the exponents in \( \Psi^+ \) are reduced and the residue can be explicitly calculated. We obtain a close system on \( u(\vec{t}) \) (may be nonlocal).

**Lemma 2.5.** Equations (2.23), (2.25) coincides with (2.22), (2.4) respectively.

Proof of the Lemma. For small \( n \) we can check it by direct calculations. Then we apply the following identity

\[ \Lambda(\Psi(\lambda, \vec{t})\Psi^+(\lambda, \vec{t})) = -4\lambda^2(\Psi(\lambda, \vec{t})\Psi^+(\lambda, \vec{t})). \] (2.26)

(it is a direct consequence of (2.25)).

For the KP equation we can write two-parametric set of symmetries ([17])

\[ \frac{\partial u}{\partial \tau_{nm}} = -2 \frac{d}{dx} \text{res} |_{\lambda=\infty} \lambda^n \partial^m_{\lambda}\Psi(\lambda, \vec{t})\Psi^+(\lambda, \vec{t}) \] (2.27)

but only the symmetries with \( m = 0, 1 \) are compatible with the finite-gap structure.
Theorem 2.1. Let $\Gamma$ be a Riemann surface with a marked point $\infty$, a local parameter $1/\lambda$ in $\infty$, a set of points $\gamma_1, \ldots, \gamma_g$ in $\Gamma$ where $g$ is the genus of $\Gamma$, $l_n = \lambda^{n+1} \partial_\lambda$ be a vector field in the punctured neighbourhood of $\infty$. Consider the deformation of $\Gamma$ generated by the field $l_n$ (it was described in the paragraph 1.3). Then the correspondent variation of the KP solution, constructed by this data reads as

$$\frac{\partial u}{\partial \tau_n} = -2 \frac{d}{dx} \text{res} \mid_{\lambda=\infty} (\lambda^{n+1} \partial_\lambda \Psi(\lambda, \vec{t})) \Psi^+(\lambda, \vec{t})$$

(2.28)

i.e. it coincides with a symmetry (2.27) such that $m = 1$.

Proof of the theorem. In the paragraph 1.3 we have explained that the calculation of the Baker-Akhiezer function variation is equivalent to the following Riemann problem on the contour $S$, surrounding the point $\infty$

$$(\delta \Psi(\lambda, \vec{t}))_+ - (\delta \Psi(\lambda, \vec{t}))_- = \lambda^{n+1} \partial_\lambda \Psi(\lambda, \vec{t}).$$

(2.29)

Solution of (2.29) reads as

$$\delta \Psi(\lambda, \vec{t}) = \oint_S \omega(\lambda, \mu, \vec{t}) \mu^{n+1} \partial_\mu \Psi(\mu, \vec{t}).$$

(2.30)

where $\omega(\lambda, \mu, \vec{t})$ is the Cauchy - Baker - Akhiezer kernel defined in the Lemma 2.4. Expanding (2.30) as $\lambda \to \infty$ and using (2.20) we obtain (2.28).

Corollary 1. Symmetries (2.23) act on the finite-gap KdV solutions as

$$\partial E_s / \partial \tau_{2n} = 2(-1)^n E_s^{n+1}, \partial E(\gamma_k) / \partial \tau_{2n} = 2(-1)^n E^{n+1}(\gamma_k).$$

(2.31)

i.e. all the spectral data is shifted via the vector field $2(-1)^n E^{n+1} \partial / \partial E$ (see paragraph 1.3).

3. WHITHEAM EQUATIONS. THE FULL ABEL HIERARCHY AND NONISOSPECTRAL SYMMETRIES.

The KdV equation (2.3) and the symmetries (2.6) (they are called higher KdV) form a commutative set of flows. This set is called KdV hierarchy.

Averaged KdV hierarchy was constructed in [2], [3]. We will not discuss how the averaged KdV equations can be derived from the original ones and so we will only recall the answer.

The starting point for the averaging procedure is the space of all $g$-gap KdV solutions. Such solutions are parameterized by the branch points $E_1, \ldots, E_{2g+1}$ and the points $\gamma_1, \ldots, \gamma_g$ in $\Gamma$. The branch points $E_k$ are integrals of motion and the points $\gamma_k$ play the role of phases (see for example [28] and references therein).

If we consider a slow modulated wave-type solution then the points $E_k$ slowly depend on coordinate and times $E_k = E_k(X, T, T_5, \ldots)$ where $X = \epsilon x, T = \epsilon t, T_5 = \epsilon t_5, \ldots, \epsilon \ll 1$. Functions $X, T, T_5, \ldots$ are called slow variables.

For averaging any full set of integrals can be used. But direct calculations for $g = 1$ (see [1]) shows that the variables $E_k$ result in the simplest form of the averaged equations so it is very natural to use them for higher genera.
The averaged KdV hierarchy can be written in the following form, suggested by Flaschka - Forest - McLaughlin [2]

\[
\frac{\partial E_k}{\partial T_{2n+1}} = w_{2n+1}^k(E_1, \ldots, E_{2g+1}) \frac{\partial E_k}{\partial X}, \tag{3.1}
\]

where

\[
w_{2n+1}^k(E_1, \ldots, E_{2g+1}) = \frac{\Omega_{2n+1}(E_k)}{\Omega_1(E_k)}, \tag{3.2}
\]

\( \Omega_k \) are the differentials defined in the paragraph 1.2. We see that the flows (3.1) have the Riemann diagonal form.

The averaged KdV equations have wider symmetry group then the original KdV. For example the scaling transform \( X \to \alpha X, \ T \to \alpha T \) has no analogs in original equations.

From the results of S.P. Tsarev [5], [7] it was known that the Whitham equations have \( 2g + 1 \) infinite series of symmetries but the averaged KdV hierarchy gives us only one of them. The full set of symmetries was constructed by S.P. Tsarev in [5] for \( g = 1 \) and B.A. Dubrovin for all \( g \). It contains \( 2g + 1 \) infinite series plus one finite and can be written as

\[
\begin{align*}
\frac{\partial E_k}{\partial T_{n+1}^a} &= \frac{\omega_{n}(E_k) \partial E_k}{\Omega_1(E_k) \partial X}, \\
\frac{\partial E_k}{\partial T_{n+1}^b} &= \frac{\sigma_{n}(E_k) \partial E_k}{\Omega_1(E_k) \partial X}, \\
\frac{\partial E_k}{\partial T_{n+1}^R} &= \frac{\omega_{i}(E_k) \partial E_k}{\Omega_1(E_k) \partial X}, \tag{3.3}
\end{align*}
\]

where \( i = 1, \ldots, g, \ n = 1, 2, \ldots \). Let us denote

\[
w_{k}^{a,n} = \frac{\omega_{n}(E_k)}{\Omega_1(E_k)}, w_{k}^{b,n} = \frac{\omega_{n}(E_k)}{\Omega_1(E_k)}, w_{k}^{H,n} = \frac{\omega_{i}(E_k)}{\Omega_1(E_k)}. \tag{3.4}
\]

(see the definitions in the paragraph 1.2).

We have four families of differentials, times and velocities. To avoid too long notations we will use the following agreement:

\( \Omega_{\alpha}^F \) may denote any of the differentials \( \Omega_k, \omega_i, \omega_k^i, \sigma_k^i, w_s^F, \) and \( T_{\alpha}^F \) are the correspondent velocities and times, \( Q_{\alpha,\beta}^F = V_{\Omega_\alpha^F \Omega_\beta^F} \) where \( V \) is the scalar product defined in the paragraph 1.2.

The following statements are important for us.

**Lemma 3.1** ([2], [6]). The flows (3.1), (3.3) can be written as

\[
\frac{\partial \Omega_1}{\partial T_{\alpha}^F} = \frac{\partial \Omega_1^F}{\partial X}, \tag{3.5}
\]

Here we always assume

\[
\frac{\partial}{\partial T_{\alpha}^F} = \left. \frac{\partial}{\partial T_{\alpha}^F} \right|_{E=\text{const}}.
\]

Proof of the Lemma. The differentials in the both sides of (3.5) have the following properties.
1) They are single-valued in $\Gamma$.
2) Their integrals by $a_j$-cycles are equal to 0.
3) They have second-order poles in the branch points $E_1, \ldots, E_{2g+1}$ and no other singularities. All their residues are equal to zero.
4) Let $z = \sqrt{E - E_k}$ be local parameter in the neighbourhood of the branch point $E_k$. Then the singular part of both differentials is equal to $\Omega_\alpha^F(z)/2z^2$.

Properties 1-4 uniquely determined a differential so the left-hand side is equal to the right-hand one.

**Lemma 3.2** ([2], [6]). All the symmetries (3.1), (3.3) are mutually commuting.

Proof of the Lemma. Consider the differentials $\partial \Omega_\alpha^F / \partial T_\beta^F$, $\partial \Omega_\beta^F / \partial T_\alpha^F$. They satisfy the properties 1-3 of the Lemma 3.1 and have the same singularities in the branch point so they are equal

$$\frac{\partial \Omega_\alpha^F}{\partial T_\beta^F} = \frac{\partial \Omega_\beta^F}{\partial T_\alpha^F},$$

and we have

$$\frac{\partial}{\partial T_\alpha^F} \frac{\partial}{\partial T_\beta^F} \Omega_1^F = \frac{\partial}{\partial X} \frac{\partial \Omega_\beta^F}{\partial T_\alpha^F} = \frac{\partial}{\partial T_\beta^F} \frac{\partial \Omega_\alpha^F}{\partial T_\alpha^F} \Omega_1^F.$$

**Lemma 3.3** ([8], [23], [6]). The scalar products $Q_{\alpha\beta}^F$ satisfy the following relation

$$\frac{\partial Q_{\alpha\beta}^F}{\partial T_\gamma^F} = \frac{\partial Q_{\alpha\gamma}^F}{\partial T_\beta^F}. \tag{3.6}$$

Thus we can define functions

$$Q_\alpha^F(\vec{T}^F) = \int_0^{\vec{T}^F} \sum_\beta Q_{\alpha\beta}^F dT_\beta^F. \tag{3.7}$$

Here the vector $\vec{T}^F$ contains all the times.

Proof of the Lemma. From the formulas (1.21) we see that the coefficients $\partial Q_{\alpha\beta}^F / \partial T_\gamma^F$ can be expressed as integrals or expansion coefficients for the differential $\partial \Omega_\gamma^F / \partial T_\gamma^F$. Applying the Lemma 3.2 we complete the proof.

**Lemma 3.4** Let us shift the branch points $E_s$ of the surface $\Gamma$ and the normalization point $E_0$ via a vector field $l_{2k} = 2(-1)^kE_{k+1}\partial E$

$$\frac{\partial E_s}{\partial \tau_{2n}} = 2(-1)^nE_s^{n+1}, \ s = 0, 1, \ldots, 2g + 1. \tag{3.8}$$

Then we have

$$\frac{\partial w_{s}^{2k+1}}{\partial \tau_{2n}} = (2k+1)w_{s}^{2k+2n+1} - w_{s}^{2n+1}w_{s}^{2k+1} + \sum_{m=0}^{n-1} \left[ \left( \frac{\partial}{\partial T_{2k+1}^F} - w_{s}^{2k+1} \frac{\partial}{\partial X} \right) Q_{2m+1} \right] w_{s}^{2n-2m-1}. \tag{3.9}$$

$$\frac{\partial w_{s}^{Hk}}{\partial \tau_{2n}} = -w_{s}^{2n+1}w_{s}^{Hk} + \sum_{m=0}^{n-1} \left[ \left( \frac{\partial}{\partial T_{2k}^F} - w_{s}^{Hk} \frac{\partial}{\partial X} \right) Q_{2m+1} \right] w_{s}^{2n-2m-1}. \tag{3.10}$$

$$\frac{\partial w_{s}^{a_{ik}}}{\partial \tau_{2n}} = 2(-1)^n k w_{s}^{a_{i+1}k} - w_{s}^{2n+1}w_{s}^{a_{ik}} + \sum_{m=0}^{n-1} \left[ \left( \frac{\partial}{\partial T_{2k}^F} - w_{s}^{a_{ik}} \frac{\partial}{\partial X} \right) Q_{2m+1} \right] w_{s}^{2n-2m-1}. \tag{3.11}$$
\[ \frac{\partial w^{b_i k}}{\partial \tau_{2n}} = 2(-1)^n k w^{b_i k+n}_s - w^{2n+1}_s w^{b_i k}_s + \sum_{m=0}^{n-1} \left[ \left( \frac{\partial}{\partial T^b_k} - w^{b_i k}_s \frac{\partial}{\partial X} \right) Q_{2m+1} \right] w^{2n-2m-1}_s. \] (3.12)

(The functions \(Q_j\) were defined in Lemma 3.3).

The Lemma is proved by direct calculations using (1.35) - (1.38) and (3.2), (3.4).

**Lemma 3.5.** Let us have a pair of flows

\[ \frac{\partial E_s}{\partial \tau} = R_s(E_s) + \left( \sum_{\alpha} f_\alpha(\vec{T}) w^{F\alpha}_s \right) \frac{\partial E_s}{\partial X}, \] (3.13)

\[ \frac{\partial E_s}{\partial T^F_\gamma} = w^{F\gamma}_s \frac{\partial E_s}{\partial X}. \] (3.14)

Then the flows (3.13) and (3.14) commute if and only if the following compatibility condition holds:

\[ \frac{\partial w^{F\gamma}_s}{\partial \tau} = \sum_{\alpha} \left[ \left( \frac{\partial}{\partial T^F_\gamma} - w^{F\gamma}_s \frac{\partial}{\partial X} \right) f_\alpha(\vec{T}) \right] w^{F\alpha}_s \] (3.15)

where \(\partial w^{F\gamma}_s / \partial \tau\) denotes the variations of velocities via the shift

\[ \frac{\partial E_s}{\partial \tau} = R_s(E_s). \] (3.16)

This Lemma is proved by simple direct calculation.

Now we have prepared everything to formulate and prove our main result.

**Theorem 3.1.** The flows

\[ \frac{\partial E_s}{\partial \tau_{2n}} = 2(-1)^n E^{n+1}_s + \left\{ \sum_{k\geq0} (2k+1) T_{2k+1} w^{2k+2n+1}_s \right\} \frac{\partial E_s}{\partial X} + \] \[ + \left\{ 2(-1)^n \sum_{k\geq0} k(T^a_k w^{a_i k+n}_s + T^b_k w^{b_i k+n}_s) + \sum_{0 \leq k < n} Q_{2k+1} w^{2n-2k-1}_s \right\} \frac{\partial E_s}{\partial X}. \] (3.17)

commute with the whole Whitham hierarchy (3.1), (3.3). The nonlocal functions \(Q_j\) were defined in the Lemma 3.3. All the partial derivatives of \(Q_j\) can be expressed via branch points \(E_s\) and normalization point \(E_0\) in terms of hyperelliptic integrals (see Lemma 1.1).

Comparing the Lemmas 3.4 and 3.5 we prove this theorem.

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