Comment on “On the Origin of the OZI Rule in QCD”, by N. Isgur and H. B. Thacker

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We comment on the recent paper (hep-lat/0005006) by Isgur and Thacker on the origin of the OZI rule in QCD. We show that instantons explain the sign and magnitude of the observed OZI-violating amplitude in all mesonic channels, not just in the \( \eta' \) channel. We comment on the role of instantons in hadronic spectroscopy and the relation between instantons and the large \( N_c \) limit of QCD.

\section{I. INTRODUCTION}

In their recent paper \cite{1}, Isgur and Thacker discuss an issue of paramount importance for mesonic spectroscopy, the nature of the large OZI-violating amplitude observed in the pseudoscalar nonet. The \( \pi - \eta' \) splitting is the largest mass splitting among light mesons, and understanding its physical origin is clearly a key issue.

In order to clarify the physical origin of the large \( \eta' \) mass Isgur and Thacker studied its analogue in the scalar \( 0^{++} \), vector \( 1^{-} \), and axialvector \( 1^{++} \) channels. They observe that the OZI-violating amplitude \( A^{0^{++}} \) in the scalar channel is as large as the one in the pseudoscalar channel. In the first version of their paper, Isgur and Thacker claimed that the signs of \( A^{0^{++}} \) and \( A^{0^{++}} \) are the same. After being alerted to an error in their work by us and other, the second version concludes that the two amplitudes are \emph{opposite} in sign. As we discuss below, this sign is crucial for phenomenology, and it is indeed what instantons predict it to be. Nevertheless, Isgur and Thacker still conclude that “... \emph{our result favors the large} \( N_c \) \emph{and not the instanton interpretation of the solution of the \( \eta' \) mass problem}”. To us, this appears to be a significant misunderstanding, and we would like to clarify the issue in this comment.

We emphasize that the issue of the sign of the \( A^{0^{++}} \) amplitude is also connected with an old misunderstanding concerning the scalar isoscalar (sigma) meson channel. Because the sigma resonance around 600 MeV is so broad and has not always been included in the Particle Data Table, it is sometimes assumed that the lightest state in this channel is located around 1.5 GeV or higher, and that the interaction must therefore be strongly repulsive. But we know that, fundamentally, the interaction in the \( 0^{++} \) channel must be very attractive, how else could chiral symmetry breaking take place? Using models of QCD it is hard to see how, after the vacuum is rearranged and chiral symmetry breaking has taken place, the \( \sigma \) state could be pushed up to a mass much beyond 600 MeV. One might argue that there is an attractive interaction in the \( 0^{++} \) channel, but that it respects the OZI rule. But in this case one is faced with the problem that no low mass strength is seen in the isovector channel, so the \( I=1 \) \( 0^{++} \) channel is indeed repulsive. This is clearly seen in lattice calculations of the scalar isovector correlation function.

We would also like to discuss a somewhat secondary point. Isgur and Thacker emphasize that their results are in agreement with \( 1/N_c \) arguments, even though the \( 1/N_c \) expansion makes no prediction concerning the sign of the amplitude. They also emphasize that \( 1/N_c \) arguments are generally incompatible with instantons. This is an issue on which there is a lot of confusion in the literature, and we shall comment on it below.

A general issue worth discussing in this note is the question of “conspiracies” among different hadronic amplitudes, which Isgur and Thacker mention in connection with the OZI rule in the vector channel. On this point we completely agree with Isgur and Thacker. We discuss other examples of similar conspiracies that have appeared in the study of hadronic correlation functions. Despite our criticism, we consider the paper by Isgur and Thacker to be a positive step. The problems related to OZI-violating amplitudes, chiral symmetry breaking, the \( U(1)_A \) anomaly, and their relation to the foundations of the constituent quark model are rarely discussed in the literature on hadronic spectroscopy.

\section{II. THE SIGN OF THE OZI-VIOLATING AMPLITUDE: SCALARS VS PSEUDOSCALARS}

Let us first recall the setting. Isgur and Thacker consider QCD with two flavors and introduce \( 4 \times 4 \) mass matrices in the basis of \( ud, d\bar{u}, u\bar{d}, dd \) states of the form

\begin{equation}
\mathbf{T} = \begin{bmatrix}
S & 0 & 0 & 0 \\
0 & S & 0 & 0 \\
0 & 0 & S + A & A \\
0 & 0 & A & S + A
\end{bmatrix}.
\end{equation}

Here, we ignore the effects of quark masses, and the eigenstates have to fall into isospin multiplets. It is then sufficient to consider the lower \( 2 \times 2 \) block of the mass matrix

\begin{equation}
\mathbf{T} \sim \begin{bmatrix}
D & A \\
A & D
\end{bmatrix}.
\end{equation}

where \( D = S + A \) in the notation of Isgur and Thacker. So there are two amplitudes, \( D, A \), and two different eigen-
states in every $J^{PC}$ channel. In the pseudoscalar channel the physical states are the $\eta' \sim (\bar{u}i\gamma_5u + \bar{d}i\gamma_5d)$ and $\pi^0 \sim (\bar{u}i\gamma_5u - \bar{d}i\gamma_5d)$, with masses (or mass squared) $D - A$ and $D + A$. We emphasize that both $S$ and $A$ depend on the quantum numbers of the current. In the case $\Gamma = i\gamma_5$ there is no question about the sign of $A$: we want $m_{\eta'} > m_\pi$ and thus $A > 0$.

Similarly, there are two independent scalar channels, traditionally called $\sigma \sim (\bar{u}u + \bar{d}d)$ and $\delta^0 \sim (\bar{u}u - \bar{d}d)$ ($f_0$ and $a_0$ in modern notation). The issue at hand is the sign of the corresponding amplitude $A^{0++}$.

Before we go into phenomenology, let us explain the instanton and perturbative QCD predictions. As discovered in the classical paper by t’Hooft [2] the effect of instantons on fermionic correlation functions can be summarized in terms of an effective interaction

$$L = G \left[ (\bar{\psi}\tau_a\psi)^2 - (\bar{\psi}\psi)^2 - (\bar{\psi}i\gamma_5\tau_a\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right],$$

(3)

where $G$ is an effective coupling (that depends on the instanton amplitude) and $\tau$ is an isospin matrix. We can directly read off the interaction in the channel characterized by the current $\bar{\psi}\Gamma \psi$. The interaction is attractive in the pion $\bar{\psi}i\gamma_5\bar{\tau}\psi$ and sigma $\bar{\psi}\psi$ channels, and repulsive in the eta prime $\bar{\psi}i\gamma_5\bar{\tau}\psi$ and delta $\bar{\psi}\bar{\tau}\psi$ channels. So, to first order, the instanton-induced interaction corresponds to $A^{0++} = -A^{0++}$.

![Figure 1](image)

**FIG. 1.** Instanton (left panel) and perturbative (right panel) contribution to OZI-violating correlation functions.

It is worth repeating why this is so [3]. The instanton interaction corresponds to the contribution of fermionic zero modes to the quark propagator. Since there is exactly one zero mode for every flavor, there are no diagonal $(\bar{u}u)(\bar{u}u)$ or $(\bar{d}d)(\bar{d}d)$ interactions. Second, since the fermion zero modes for quarks and anti-quarks have opposite chirality, the interaction is also off-diagonal in the basis spanned by right and left-handed fermions $q_R, q_L$.

$$T \sim \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}.$$  

(4)

This means that the instanton-induced amplitude follows very simple rules: (i) the sign flips in going from scalar to pseudoscalar, (ii) the sign flips in going from $I = 0$ to $I = 1$ states, (iii) to leading order there is no interaction in vector channels. This means that the OZI-violating amplitude in the vector channels receives no direct instanton contribution and is expected to be small. To summarize, we have the following prediction for the signs of the instanton contribution

$$\begin{pmatrix} \eta' & \pi \\ \sigma & \delta \end{pmatrix}.$$  

(5)

So, to first order in the interaction, the $\pi$ and $\sigma$ form a light multiplet, and the $\eta'$ and $\delta$ a heavy multiplet. The degeneracy is due to $SU(2)$ chiral symmetry, and the splitting between the multiplets is a manifestation of $U(1)_A$ violation. Of course, because the interaction in the scalar channel is so strong, the vacuum is rearranged and chiral symmetry is broken spontaneously. The pion becomes a Goldstone boson and is exactly massless in the chiral limit. The sigma corresponds to the massive excitation of the quark condensate and is pushed up to $\sim 500 - 600$ MeV. But it is clear that the interaction at short distances remains attractive, so the sigma will always be lighter than the delta.

Short distance perturbative interactions cannot account for OZI-violation in the scalar channels. The reason is that chirality is strictly conserved in perturbative QCD. The only possible annihilation channels are $q_Lq_L$ or $\bar{q}_R\bar{q}_R$, which do not contribute to the scalar or pseudoscalar channel. Perturbative effects do contribute to OZI violation in the vector channel. The OZI violating amplitudes in the vector channel (which are suppressed by more than an order of magnitude) then provide an estimate of the relative role of instanton/pQCD effects.

Phenomenologically, the situation in the scalar $0^{++}$ channel appears confused, because of the difficulties in classifying the observed scalar mesons. But what is important here is not whether some state is too broad to be considered a “true” resonance, or mixes strongly with $\pi\pi$ or $\bar{K}K$, etc. In fact it is pretty clear that there is strong evidence for substantial strength in the $\sigma$ channel at energies below 800 MeV. This strength is seen experimentally as the strong rise in the $I = 0 \pi\pi$ phase shift and the “$\sigma$” meson of nuclear physics.

### III. HOW TO MEASURE THE OZI-VIOLATING AMPLITUDE

In order to measure the OZI-violating amplitudes we have to replace the mass matrices introduced above by objects more directly related to field theory. In particular, we shall consider correlation functions involving the scalar and pseudoscalar currents introduced above. The correlators are defined by
The correlation function.

where $S(x, y)$ is the fermion propagator and $\langle \rangle$ denotes the average over all gauge configurations. The OZI-violating difference between the $\pi, \eta'$ and $\sigma, \delta$ channels is determined by “disconnected” (or double hairpin) contributions to the correlation functions. Note that it is important to correctly define the gamma matrices in the correlation functions in order to ensure positivity and the existence of a spectral representation. In particular, we have to use $\Gamma = i\gamma_5$ in the pseudoscalar channel in order to guarantee $\Pi_\pi(x, y) > 0$. If this is taken care of, we can directly determine the sign of the OZI-violating amplitude from the sign of the disconnected correlation function. There is a subtlety in the scalar $\sigma$ channel, because we need to subtract the constant $\langle \bar{\psi}\psi \rangle^2$ term from the correlation function.

Isgur and Thacker find, in the revised version of their paper, that the disconnected correlation function is large and negative in the pseudoscalar channel, large and positive in the scalar channel, and very small in the vector channel. In Figs. 1 and 2 we compare these results to correlation functions obtained in unquenched instanton simulations. The correlation functions clearly show exactly the same pattern. There are some technical differences as compared to the work of Isgur and Thacker. We measure point-to-point rather than point-to-plane correlators, and the simulations are unquenched rather than quenched. This means that there is no need to extract amputated matrix elements, one can just measure the masses directly. We find a heavy $\eta'$ and $\delta$, $m_\delta \simeq m_{\eta'} \geq 1$ GeV, and a light sigma meson, $m_\sigma \simeq 600$ MeV.

IV. THE LARGE $\mathcal{N}_c$ LIMIT

The basis of the large $\mathcal{N}_c$ approach is the assumption that $\mathcal{N}_c = 3$ QCD is similar to QCD in the limit $\mathcal{N}_c = \infty$. In particular, it is assumed that there are no phase transitions as we go from $\mathcal{N}_c = 3$ to $\mathcal{N}_c \to \infty$. Currently, the status of these assumptions is not clear, because not much is known about QCD($\mathcal{N}_c = \infty$).

Let us start with what is firmly known. In the large $\mathcal{N}_c$ limit gauge invariant quantities do not fluctuate. This means that all physical quantities can be obtained from a classical master field. For many years, there was no progress in obtaining the master field from QCD, except in the case of zero dimensions, or $SU(N)$ matrix models. Recently, Maldacena discovered a master field for $N = 4$ supersymmetric QCD. This master field was described as a certain gravitational metric, together with a set of rules that relate gauge theory to supergravity observables. Amazingly, the same correlation functions can be obtained from an instanton master field. This configuration is a coherent superposition of many instantons with different colors orientations but the same size and position, held together by fermion exchanges. Progress was also made in understanding $N = 2$ SUSY QCD. Seiberg and Witten determined the low energy effective action of this theory. In the semi-classical limit, their result can be expressed as a one-loop contribution plus an infinite series on $n$-instanton corrections, and nothing else. Witten and Seiberg’s result was generalized to arbitrary $\mathcal{N}_c$ by Douglas and Shenker. They identified a special form of the large $\mathcal{N}_c$ limit in which instantons (and monopoles) survive.

In practice the large $\mathcal{N}_c$ expansion is used to determine the relative importance of certain classes of diagrams. In order to have a sensible large $\mathcal{N}_c$ limit one requires $g^2 N_c = \text{const.}$ We emphasize that the results are based...
on the usual perturbative expansion around the trivial vacuum. Instantons, and other non-perturbative effects, can spoil large $N_c$ counting rules. Applying large $N_c$ rules to hadronic correlation functions leads to the usual predictions

\begin{align}
M_{\text{generic meson}} &\sim N_c^0 M, \\
M_{\text{generic baryon}} &\sim N_c^1 M, \\
M_{\eta'} &\sim N_c^{-1} M,
\end{align}

where $M$ is some mass scale of order $\Lambda_{QCD}$. This prediction is a little bit of an embarrassment, because in the real world the nucleon and the $\eta'$ are very close in mass. Of course, one can argue that the numerical coefficient in the different channels could be very different. Nevertheless, the large mass scale that appears in the $\eta'$ channel shows that large $N_c$ rules do not work equally well in all channels. This observation repeats itself in the scalar $0^++$ channel, where deviations from large $N_c$ counting are unusually charge. On the other hand, large $N_c$ arguments have proved to be useful in analyzing many properties of octet and decuplet baryons.

So what is the $N_c$ dependence of the instanton effects? In his well known paper \cite{7}, Witten argued that one has to choose between instantons and the large $N_c$ explanations of the $U(1)_A$ problem, because instanton effects scale as $\exp(-N_c)$. However, twenty years later the dilemma does not seem that clear cut. New arguments have appeared (many by Witten himself), and instantons and the large $N_c$ limit may well be reconciled one day.

(i) The suppression of instantons in the large $N_c$ limit seems to follow from the instanton amplitude $\exp(-8\pi^2/g^2)$, together with the 't Hooft scaling $g^2 \sim 1/N_c$. But this applies to small instantons only. For large instantons $\rho \sim \Lambda^{-1}$ the action is $O(1)$, and there is no suppression. This is the scenario that was suggested for the $CP^N$ model: For small $N$ instantons are small and semi-classical, but for large $N$ instantons become strongly overlapping. The question of the instanton size distribution in QCD is a very non-trivial dynamical question. From lattice calculations we only know that the typical instanton size is about the same in $N_c = 2$ and $N_c = 3$ QCD, $\rho \sim 1/3$ fm.

(ii) Collecting all factors of $N_c$ in the one-loop instanton amplitude one finds that they all exponentiate, giving $dN/d\rho \sim \exp(N_c F(\rho))$ \cite{8}. The function $F(\rho)$ has a non-trivial zero, so the instanton density may scale like a power of $N_c$, rather than an exponential, provided the instanton distribution becomes a delta function $\delta(\rho - \rho_0)$ where $\rho_0$ is the zero of $F(\rho)$.

(iii) The second statement in Witten’s paper is that, even in the absence of instantons, the $\eta'$ mass can be related to the topological susceptibility $\chi_{\text{top}}$ measured in pure gauge theory

\begin{equation}
\frac{f_2^2}{2N_f}(m_{\eta'}^2 + m_{\eta''}^2 - 2m_K^2) = \chi_{\text{top}}.
\end{equation}

This prediction has been checked on the lattice and it works quite well. After much effort, the lattice measurements of $\chi_{\text{top}}$ are stable, and it has become clear that the topological susceptibility is dominated by instantons, and not by some mysterious fluctuation.

(iv) As we emphasized above the large $N_c$ limit of both $N = 2$ and $N = 4$ SUSY QCD is consistent with the survival of instantons in this limit. The $N = 2$ case is particularly interesting, because there are two different ways to take the large $N_c$ limit. In the “naive” limit, $W$ bosons have masses of $O(1)$, monopoles have masses $O(N_c)$, and instantons have action $O(N_c)$ and are not important. In this case, the one loop perturbative result is exact and nothing interesting happens. A different large $N_c$ limit is possible near the singular points on the space of vacua where monopoles condense. In this regime one has to tune the Higgs VEVs such that $|v_i - v_j| \sim 1/N_c$. Then monopole masses are $O(1)$, and instantons are not suppressed. In this case the naive scaling relation $g^2 \sim 1/N_c$ is not satisfied.

(v) The large $N_c$ prediction for pure gluodynamics $\chi_{\text{top}} \sim \Lambda_{QCD}^{-4} N_c^{-2} \epsilon_{\text{vac}}$ was recently verified in a string theory setting \cite{9}. This result is consistent with a number of scenarios for the instanton liquid. For example, the instanton ensemble might evolve from a random gas of density $n \sim \chi_{\text{top}}$ at small $N_c$ to a strongly correlated liquid with large density $(N/V) \sim \epsilon_{\text{vac}}$ but very small fluctuations $\chi_{\text{top}} = \langle (N_I - N_A)^2 \rangle / V \sim N_c^{-2} (N/V)$.

(vi) Instantons are not incompatible with the success of large $N_c$ arguments in the baryon sector. The instanton induced interaction between quarks is $1/N_c$ suppressed, just like one gluon exchange. Also, the topological soliton model can be derived from instantons in the large $N_c$ limit \cite{10}. This model automatically incorporates all the large $N_c$ predictions.

\section{V. SMALLNESS OF OZI-VIOLATING AMPLITUDES IN CHANNELS OTHER THAN $O^2$: CONSPIRACY BETWEEN HADRONIC AMPLITUDES}

We completely agree with Isgur and Thacker on this point. We would like to make two comments. First of all, instantons are compatible with the smallness of the OZI violating amplitude in the vector channel. The direct instanton contribution to the disconnected correlator vanishes. The next correction is suppressed also. Correlated instanton-anti-instanton pairs contribute to connected, but not to disconnected vector correlators \cite{11}.

Second, we would like to point out that we have compiled an extensive phenomenological analysis of hadronic correlation functions \cite{12}. This compilation contains a number of examples for cancellations between a large number of hadronic amplitudes. An example is the strik-
The full correlation function remains close to free field behavior for distances as large as $|x - y| < 1.3$ fm. In terms of the spectral representation, this behavior requires remarkable fine tuning between different hadronic amplitudes. This result can be directly verified from data, using the cross section for $e^+e^-$ annihilation into hadrons.

Another important conclusion from this study is the fact that OZI violation in the scalar and pseudoscalar channel appears at very short distances. This is one more argument in favor of instantons rather than confinement as the source of OZI violation. While instantons appear at short distances, confinement is a long distance effect.

VI. CONCLUSIONS

In summary we showed that instantons explain the sign and magnitude of the OZI violating amplitude in all mesonic channels, not just for the $\eta'$. In addition to that, instantons provide a mechanism for chiral symmetry breaking, and a successful description of light hadron spectroscopy. The role of instantons in the large $N_c$ limit is a complicated issue, but instantons and the large $N_c$ need not be incompatible. Nevertheless, naive large $N_c$ counting rules may not work in all channels. Instantons provide an explanation why large $N_c$ predictions work well in some cases, but fail badly in others.

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