A Vector Girsanov Result and its Applications to Conditional Measures via the Birkhoff Integrability

Domenico Candeloro, Anna Rita Sambucini and Luca Trastulli

In memory of Domenico Candeloro who is for us, Master, Mentor and Friend, † May 3° 2019.

Abstract. Some integration techniques for real-valued functions with respect to vector measures with values in Banach spaces (and vice versa) are investigated to establish abstract versions of classical theorems of probability and stochastic processes. In particular, the Girsanov Theorem is extended and used with the treated methods.

Mathematics Subject Classification. 28B20, 58C05, 28B05, 46B42, 46G10, 18B15.

Keywords. Birkhoff integral, vector measure, Girsanov Theorem.

1. Introduction

The theory of stochastic processes plays a very important role in modelling various phenomena, in a large class of disciplines such as physics, economics, statistics, finance, biology and chemistry. Then it is crucial to extend as much as possible the tools and results regarding this theory, to make them available even in abstract and general contexts. A very important tool in measure theory is the Girsanov Theorem, strictly linked to the well-known Wiener stochastic process called the standard Brownian motion \((w_t)_{t \in [0, \infty)}\), defined on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\). A classic formulation of this result in the real case can be found in [31] and it allows to change the underlying probability measure \(\mathbb{P}\), through the definition of Radon–Nikodým derivative, to obtain an equivalent measure \(Q\). This turns out to be useful, for example, in mathematical finance when a neutral risk measure must be determined, in the Black–Scholes model.
Here we generalize the Girsanov Theorem to the case of vector measure spaces following the idea formulated in [31] for the real Brownian motion and using the Birkhoff vector integral studied in [12,14,16,19,20,22–25,30,35–39], in [17] for non-additive-measures and in [3–9,13,18,21] for the multivalued integration. Other results on the Brownian motion subject are given also in [10,26,29,32,40]. This paper is inspired by [11,41], in particular some of the results were announced at the ICSSA 2018 conference.

Now, we give a plan of the paper. In Sect. 2, after an introduction of the Birkhoff integrals (Definitions 2.2 and 2.3) the properties of such integrals are studied together with a link between them (Theorem 2.7). Moreover, the notions of conditional expectation in this framework is given together with a tower property for martingales and Theorem 2.13. In Sect. 3, the main result: a vector version of the Girsanov result (Theorem 3.6) is presented after having introduced the equivalent martingale measure and under Assumptions A1 and A2. At the end of this section, an example is given satisfying Theorem 3.6. Section 4 is devoted to applications of Theorem 3.6 such as conditional measures (Proposition 4.1 for $C([0,T])$-valued measures) and to extensions of the Itô representation of stochastic $X$-valued processes, using vector-stochastic integral and the classical Itô formulas. In particular, in Theorem 4.8, a process of the type

$$C_t = (B_{i,t}) \int_0^t \Psi(s)ds + (B_{i,t}^0) \int_0^t \Phi(s)dw_s, \quad (\text{under } \mathbb{P}), \quad t \in [0,T]$$

is considered and it is proved that it is possible to eliminate the drift term to obtain a local martingale.

2. The Birkhoff Integrals and Their Properties

Let $(\Omega, \mathcal{A}, \nu)$ denote a measure space and $(X, \|\cdot\|)$ a Banach space. From now on, with the letters $\mu, \nu$ we refer to scalar measures, that is $\nu : \mathcal{A} \rightarrow \mathbb{R}_0^+$ while we use letters as $N, M$ to denote vector measures, that is $N : \mathcal{A} \rightarrow X$. We also use capital letters like $X, Y$ to denote arbitrary Banach spaces, $X^*, Y^*$ to denote their dual spaces and $x^*, y^*$ the elements of the dual spaces. With letters like $\phi, \psi$ we refer to scalar functions, and with $\Phi, \Psi$ to vector-valued functions, defined on the measure space $(\Omega, \mathcal{A}, \nu)$. We also denote with $\mathcal{B}(\mathbb{R}), (\mathcal{B}(I))$ as usual the Borel $\sigma$-algebra on the real line (on the interval $I = [0,T], T > 0$) and with $\mathcal{B}(X)$ the Borel $\sigma$-algebra on $X$. Finally, we shall denote by $a_t, b_t, z_t$, scalar-valued stochastic processes, while $A_t, B_t, Z_t$ denote $X$-valued stochastic processes. We recall that an $X$-valued stochastic process is a strongly measurable function $Z : (I \times \Omega, \mathcal{B}(I) \otimes \mathcal{A}) \rightarrow (X, \mathcal{B}(X))$. There are many different versions of the Birkhoff integral (see [1]). We shall use the following one that provides two different kinds of integrals. To do this, we recall some basic definitions.

**Definition 2.1.** We say that a finite or countable family of non-empty measurable sets $P := (U_j)_{j \in J}$ is a partition of $\Omega$ if $U_j$ and $U_k$ are pairwise disjoint and they cover $\Omega$, that is $\bigcup_{j \in J} U_j = \Omega$. We denote by $\mathcal{P}$ the set of
all partitions of the set $\Omega$. Given two different partitions $P, P'$ we say that the partition $P$ is finer than $P'$ if for every $U \in P$ there exists a $U' \in P'$ such that $U \subset U'$.

We distinguish between two different kinds of Birkhoff integral. The first one is about the integration of an $X$-valued function with respect to a scalar measure. The second one is the notion of integral of a scalar function with respect to a vector measure.

**Definition 2.2** (First type Birkhoff integral). Let $\Phi : \Omega \to X$ be a vector space-valued function and $\nu : A \to R_0^+$ be a scalar, countably additive measure. Then $\Phi$ is said to be first type Birkhoff integrable with respect to $\nu$, briefly $\Phi \in Bi_1(\Omega, \nu)$ (or simply $Bi_1$ if there is no ambiguity about the space and the measure), if there exists $I \in X$ such that for all $\varepsilon > 0$ there exists a partition of $\Omega$, $P_\varepsilon$, such that, for every countable partition $(U_n)_{n \in \mathbb{N}}$ of $\Omega$, finer than $P_\varepsilon$ and for all $\omega_n \in U_n$, it is $\limsup_n \| (\sum_{k=1}^n \Phi(\omega_k) \nu(U_k) - I) \| < \varepsilon$. We call $I \in X$ the Birkhoff integral of $\Phi$ with respect to $\nu$, and we denote it by $(Bi_1) \int_\Omega \Phi d\nu$.

Now we are going to define the second type of Birkhoff integral.

**Definition 2.3.** Let $\phi : \Omega \to \mathbb{R}$ and $N : A \to X$ be a countably additive measure. Then $\phi$ is said to be second-type Birkhoff integrable with respect to $N$, briefly $\phi \in Bi_2(\Omega, N)$ (or simply $Bi_2$), if there exists $I \in X$ such that for all $\varepsilon > 0$ there exists a partition of $\Omega$, $P_\varepsilon$, such that, for every countable partition $(U_n)_{n \in \mathbb{N}}$ of $\Omega$, finer than $P_\varepsilon$ and for all $\omega_n \in U_n$, it is $\limsup_n \| (\sum_{k=1}^n \phi(\omega_k) N(U_k) - I) \| < \varepsilon$. We call $I \in X$ the Birkhoff integral of $\phi$ with respect to $N$ and we denote it by $(Bi_2) \int_\Omega \phi dN$.

The first (second)-type Birkhoff integrability/integral of a function on a set $A \in A$ is defined in the usual manner since, thanks to a Cauchy criterion, the integrability of the function restricted to $A$ follows immediately.

**Remark 2.4.** If $\mu$ is $\sigma$-finite then the $Bi_1$ integrability is correspondent to the classic Birkhoff integrability for Banach space-valued mappings (see also [5, Theorem 3.18]). Moreover, since the Birkhoff integral is stronger than the Pettis integral, it is clear that, as soon as $F$ is first-type Birkhoff integrable with respect to $m$, the mapping $M := A \mapsto (Bi_1) \int_A F dm$ is a countably additive measure.

For the second type, the mapping $M := A \mapsto (Bi_2) \int_A f dN$ is weakly countably additive since for each $x^*$ in the dual space $X^*$ the scalar mapping $f$ is integrable with respect to the scalar measure $x^*(N)$ (and to its variations). Then, thanks to the Orlicz–Pettis Theorem (see [34]), $M$ turns out to be also strongly countably additive.

We want to show a link between these two types of Birkhoff integral. First, we recall a result concerning the first-type Birkhoff integrability.
Theorem 2.5 [7, Th 3.14]. Let $\Phi$ be a strongly measurable and $\text{Bi}_1(\Omega, \nu)$-integrable function. Then for every $\varepsilon > 0$ there exists a countable partition $P^* := \{U_n, \ n \in \mathbb{N}\}$ of measurable subsets of $\Omega$, such that, for every finer partition $P' := \{V_m, \ m \in \mathbb{N}\}$ of $P^*$ and for every $\omega_m \in V_m$, we have

$$\sum_m \left\| \Phi(\omega_m)\nu(V_m) - (\text{Bi}_1) \int_{V_m} \Phi d\nu \right\| \leq \varepsilon.$$ 

Lemma 2.6. If $\phi$ is a scalar measurable function and $\Phi$ is an $X$-valued strongly measurable function in $\text{Bi}_1(\Omega, \nu)$ then, given

$$(G_n)_n := (\{\omega \in \Omega : n - 1 \leq |\phi(\omega)| \leq n\})_n \in \mathcal{A},$$

for every $\varepsilon > 0$ there exists a measurable countable partition $\{U_n^j, \ j \in \mathbb{N}\}$ of $G_n$ such that for every finer partition $\{V_n^j, \ j \in \mathbb{N}\}$ and for every $\omega_n^j \in V_n^j$,

$$\sum_n \sum_j \left\| \Phi(\omega_n^j)\phi(\omega_n^j)\nu(V_n^j) - \phi(\omega_n^j)N(V_n^j) \right\| \leq 2\varepsilon.$$  

Proof. By hypothesis the product function $\phi(\omega)\Phi(\omega)$ is strongly measurable. Then, thanks to Theorem 2.5, for every $\varepsilon > 0$ and for every $n$ there exists a measurable countable partition $\{U_n^j, \ j \in \mathbb{N}\}$ of $G_n$ such that, for every finer partition $\{V_n^j, \ j \in \mathbb{N}\}$ and for every $\omega_n^j \in V_n^j$, we obtain

$$\sum_j \left\| \Phi(\omega_n^j)\phi(\omega_n^j)\nu(V_n^j) - \int_{V_n^j} \Phi d\nu \right\| \leq \frac{\varepsilon}{n2^n}.$$ 

Then it follows that

$$\sum_n \sum_j \left\| \Phi(\omega_n^j)\phi(\omega_n^j)\nu(V_n^j) - \phi(\omega_n^j)N(V_n^j) \right\| \leq 2\varepsilon.$$ 

Theorem 2.7. Let $\phi : \Omega \to \mathbb{R}$ be a measurable function and $\Phi \in \text{Bi}_1(\Omega, \nu)$ be a vector valued strongly measurable function. We denote by $N(A) = (\text{Bi}_1) \int_A \Phi d\nu$. Then $\phi(\cdot)\Phi(\cdot) \in \text{Bi}_1(\Omega, \nu) \iff \phi(\cdot) \in \text{Bi}_2(\Omega, N)$ and

$$(\text{Bi}_1) \int_{\Omega} \phi(\omega)\Phi(\omega) d\nu = (\text{Bi}_2) \int_{\Omega} \phi(\omega) dN.$$  

Proof. Suppose that $\phi(\omega)\Phi(\omega) \in B_1(\Omega, \nu)$ and let $J = (\text{Bi}_1) \int_{\Omega} \phi(\omega)\Phi(\omega) d\nu$. Then, fixed arbitrarily $\varepsilon > 0$, we can find a measurable partition $P^* := \{U_n : n \in \mathbb{N}\}$ of $\Omega$ such that

$$\sum_n \left\| \Phi(\omega_n)\phi(\omega_n)\nu(U_n') - (\text{Bi}_1) \int_{U_n'} \Phi(\omega)\phi(\omega) d\nu \right\| \leq \varepsilon$$ 

for every finer partition $\{U_n', n \in \mathbb{N}\}$ and for every $\omega_n \in U_n'$. So, taking a partition $\{V_m, m \in \mathbb{N}\}$ finer than $P^*$ and a partition $\{V_m^k, m, k \in \mathbb{N}\}$ given by Lemma 2.6, using (1) we infer that

$$\left\| \sum_m \phi(\omega_m)N(V_m) - J \right\| = \left\| \sum_m \phi(\omega_m)N(V_m) - \sum_m (\text{Bi}_1) \int_{V_m} \Phi(\omega)\phi(\omega) d\nu \right\|.$$
Definition 2.8. Let $\omega \in \mathcal{P}$ be a measurable partition. We define the measure induced by $\phi$ as $\nu = \mu(\Phi)_\omega$ for every choice of $\Phi$. Then, for every $\epsilon > 0$ there exists a countable measurable partition $P^\# := \{U_n : n \in \mathbb{N}\}$ such that

$$\limsup_n \left\| \sum_{i=1}^n \phi(\omega_i) N(U_i) - (B_{i+1})_\phi \int_{\Omega} \phi d\nu \right\| \leq \epsilon.$$ 

So, considering $\{V_k, k \in \mathbb{N}\}$ a finer partition than $P^\#$ and $\{V_j^k, j, k \in \mathbb{N}\}$, as in Lemma 2.6, we get, using (1),

$$\limsup_n \left\| \sum_{k=1}^n \phi(\omega_k) \Phi(\omega_k) \nu(V_k) - (B_{i+1})_\phi \int_{\Omega} \phi d\nu \right\| \leq \epsilon.$$ 

This shows that $\phi(\cdot) \Phi(\cdot) \in \mathcal{B}_1(\Omega, \nu)$ and the equality (2) is true. \hfill \Box

To find applications in stochastic processes, we need to extend notions like distribution of a function with respect to a vector measure.

**Definition 2.8.** Let $\phi : \Omega \to \mathbb{R}$ be a measurable function and $N : \mathcal{A} \to X$ be a countably additive measure. We define the measure induced by $\phi$ as $N_\phi(B) = N(\phi^{-1}(B))$ for every $B \in \mathcal{B}(\mathbb{R})$. This measure is countably additive and we call it the distribution of $\phi$ with respect to $N$.

We give now an integration by substitution result for the $(B_{i+1})_\phi$ integral.

**Theorem 2.9.** Given two measurable functions $\phi : \Omega \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$, then the following relation holds: $(B_{i+1})_\phi \int_{\Omega} \psi(\phi) dN = (B_{i+1})_\phi \int_{\mathbb{R}} \psi(\phi) dN_\phi$ under the assumption that the integrals involved exist.

**Proof.** Suppose that the two integrals above exist as second type Birkhoff integrals. Now if we fix $x^* \in X^*$, we can consider the real measures $x^*(N)$ and $x^*(N_\phi) = (x^*(N))_\phi$. Then $\psi(\phi)$ is integrable with respect to $x^*(N)$ and $\psi$ is integrable with respect to $(x^*(N))_\phi$. So we obtain that

$$x^* \left( (B_{i+1})_\phi \int_{\Omega} \psi(\phi) dN \right) = \int_{\Omega} \psi(\phi) dx^*(N) = \int_{\mathbb{R}} \psi(\phi) dx^*(N)_\phi.$$
By the arbitrariness of $x^*$, the assertion follows. □

To introduce in this setting the definition of a martingale the notions of conditional expectation and filtration are needed.

**Definition 2.10.** Let $\phi : \Omega \to \mathbb{R}$ be a scalar function and $\text{Bi}_2(\Omega, N)$ integrable. We denote with $\sigma_{\phi}$ the sub-$\sigma$-algebra of $A$, obtained by taking all pre-images $\phi^{-1}(B)$, for all $B \in \mathcal{B}(\mathbb{R})$. In the special case of $\phi = z_t$, with $t \in [0, T]$ fixed, then we define the natural filtration of $z_t$, denoted by $\mathcal{F}_t = \sigma_{z_t}$ for every $t$.

Now we give the notion of conditional expectation.

**Remark 2.11.** Given a sub-$\sigma$-algebra $\mathcal{F}$ of $A$, we say that an $X$-valued function is $\mathcal{F}$-measurable if it is strongly measurable as a function from the measurable space $(\Omega, \mathcal{F})$ to the measurable Banach space $(X, \mathcal{B}(X))$.

**Definition 2.12.** Let $\Phi \in \text{Bi}_1(\Omega, \nu)$ and $\mathcal{F}$ be a sub-$\sigma$-algebra of $A$. We define, provided that it exists, the conditional expectation of $\Phi$ with respect to $\mathcal{F}$, indicated by $E_{\nu}(\Phi | \mathcal{F})$ if there is no ambiguity about the measure, as the strongly $\mathcal{F}$-measurable function $\Psi$ such that $\Psi \in \text{Bi}_1(\Omega, \nu)$ and for every $E \in \mathcal{F}$ it holds

$$(\text{Bi}_1) \int_E \Phi d\nu = (\text{Bi}_1) \int_E \Psi d\nu.$$  \hspace{1cm} (4)

From this, the classic tower property follows, i.e. for every sub-$\sigma$-algebras of $A$, such that $\mathcal{F} \subset \mathcal{G} \subset A$, we have that

$$E(\Phi | \mathcal{F}) = E(E(\Phi | \mathcal{G}) | \mathcal{F}).$$  \hspace{1cm} (3)

Another important property of the conditional expectation that we can extend is the following.

**Theorem 2.13.** Let $\Phi : \Omega \to X$ be a strongly measurable function and $\mathcal{F}$ be a sub-$\sigma$-algebra of $A$, having conditional expectation $E(\Phi | \mathcal{F})$. Then given a $\mathcal{F}$ measurable function $\phi : \Omega \to \mathbb{R}$ so that the product function $\Phi(\cdot)\phi(\cdot) \in \text{Bi}_1(\Omega, \nu)$, it holds

$$E(\Phi(\omega)\phi(\omega) | \mathcal{F}) = \phi(\omega)E(\Phi | \mathcal{F}).$$

**Proof.** We claim that, for every $E \in \mathcal{F}$, the following relation is satisfied:

$$(\text{Bi}_1) \int_E \Phi(\omega)\phi(\omega) d\nu = (\text{Bi}_1) \int_E \phi(\omega)E(\Phi(\omega) | \mathcal{F})(\omega) d\nu.$$  \hspace{1cm} (4)

For every $x^* \in X^*$, it holds $x^* (E(\Phi | \mathcal{F})) = E(x^* (\Phi) | \mathcal{F})$, then we obtain that

$$x^* \left( (\text{Bi}_1) \int_E \Phi(\omega)\phi(\omega) d\nu \right) = \int_E x^* (\Phi(\omega)) \phi(\omega) d\nu = \int_E x^* (E(\Phi | \mathcal{F})) \phi(\omega) d\nu = x^* \left( (\text{Bi}_1) \int_E E(\Phi | \mathcal{F}) \phi(\omega) d\nu \right).$$
So the function $\omega \mapsto E(\Phi|F)\phi(\omega)$ is Pettis integrable with respect to $\mu$; since the product is strongly measurable, this means that the product is $B_1(\Omega,\mu)$-integrable. Moreover, by the Hahn–Banach theorem, the formula (4) follows.

\[
\square
\]

Theorems 2.7, 2.9 and 2.13 of this section were announced in [11, Theorems 2,3,4], respectively, without any proof.

3. The Girsanov Theorem for Vector Measures

To the aim of extending the Girsanov Theorem to the Banach-valued measures, we shall find the distribution of a scalar-valued stochastic process under a vector measure $N$, make a transformation of this process, compute its new distribution and then define a new measure using these two density functions.

First, we need to define the concept of martingale when we use a vector measure $N$. Let a scalar measure $\nu : \mathcal{A} \to \mathbb{R}_0^+$ be fixed and $\Phi : \mathcal{A} \to X$ be an $X$-valued function strongly measurable and $B_1(\Omega,\nu)$ integrable. We define the vector measure $N : \mathcal{A} \to X$ as follows:

$$N(A) := (B_i) \int_A \Phi d\nu.$$ 

A measure $Q$ is equivalent to a measure $N$ $(Q \sim N)$ if there exists a $B_2(\Omega,N)$ integrable and positive function $\phi$ such that $dQ/dN = \phi$. So the definition of martingale for a scalar process $(z_t)_t$ can be given.

Definition 3.1. Let $(z_t : \Omega \to \mathbb{R})_{t \in [0,T]}$ be a scalar stochastic process on the probability space $(\Omega,\mathcal{A},N)$, where $N$ is as above. We say that $z_t$ is a $N$-martingale in itself, that is a martingale with respect to its natural filtration $\mathcal{F}_z = (\mathcal{F}_t)_t$, if for every $s < t, s, t \in [0,T]$, we have that $E_N(z_t|\mathcal{F}_s) = z_s$.

Remark 3.2. The identity $E_N(z_t|\mathcal{F}_s) = z_s$ in terms of integrals means that, for every $E \in \mathcal{F}_s$, it is $(B_i) \int_E z_t dN = (B_i) \int_E z_s dN$.

Definition 3.3. Let $(z_t : \Omega \to \mathbb{R})_{t \in [0,T]}$ be a scalar stochastic process on the probability space $(\Omega,\mathcal{A},\mathbb{P},\mathcal{F})$ which is adapted to $\mathcal{F}$. A vector-valued measure $Q$ is called an equivalent martingale measure for $(z_t)_t$ with respect to $\mathbb{P}$ if $(z_t)_t$ is $B_2$-integrable, it is a martingale with respect to $Q$ and $Q \sim \mathbb{P}$.

In financial market the equivalent martingale measure is called also the risk-neutral measure. Now we give some basic assumptions on stochastic processes.

Assumption A1. Let us assume that $(z_t)_{t \in [0,T]}$ is a stochastic scalar process defined as usual on the space $(\Omega,\mathcal{A},N)$ such that these conditions are satisfied:

A1.a The function $\omega \mapsto z(t,\omega)$ belongs to the space $B_2(\Omega,N)$, for every $t \in [0,T]$, with null integral and admits a density function, that means that its distribution under the vector measure $N$, denoted by $N_t(B) := N_{z_t}(B) = N(z_t^{-1}(B))$, for every $B \in \mathcal{B}(\mathbb{R})$ could
be written as a Birkhoff first type integral of some vector function \( F_t : \mathbb{R} \to X \in \text{Bi}_1(\Omega, \lambda) \) (where \( \lambda \) is the Lebesgue measure), namely

\[
N_t(B) = (\text{Bi}_1) \int_B F_t(x)dx.
\]

A1.b Let \( \tilde{z}_t = z_t + \theta(t) \), where \( \theta : [0, T] \to \mathbb{R} \) is a measurable function.

Suppose that for every \( t \in [0, T] \) there exists a scalar positive function \( g_t(x) \) such that the following factorization holds

\[
F_t(x) = g_t(x) \tilde{F}_t(x) = g_t(x)F_t(x - \theta(t)).
\]

A1.c The process defined as \( y_t = (g_t(\tilde{z}_t))_t \) is a \( N \)-martingale in itself.

Remark 3.4. Conditions A1.a and A1.b allow us to work with the distributions of \( z_t \) and \( \tilde{z}_t \) under the measure \( N \). We note that by A1.a it follows that the process \( \tilde{z}_t \) defines again a \( \text{Bi}_2(\Omega, N) \) random variable, for every \( t \in [0, T] \) and it admits a density function too, of type \( \tilde{F}_t := F_t(x - \theta(t)) \).

Finally, we recall that, when \( \{z_t\}_t \) is the classical (scalar) Brownian Motion, then it turns out that \( g_t(x) = \exp(-qx + \frac{1}{2}q^2t) \) and so the process

\[
\{g_t(\tilde{z}_t)\}_t = \left\{ \exp(-q\tilde{z}_t - \frac{1}{2}q^2t) \right\}_t
\]

is a martingale with respect to the natural filtration of \( \{w_t\}_t \) (see, e.g. [31, (4.20)], [33]).

So, we define the change of measure setting the new vector measure \( Q : A \to X \) as follows:

\[
Q(A) = (\text{Bi}_2) \int_A y_t dN = (\text{Bi}_1) \int_A y_T \Phi d\nu,
\]

for every \( A \in A \). Observe that \( Q \sim N \). Under Assumption A1, the marginal distribution of the stochastic process \( z_t \) is preserved when we change the underlying measure. This means that the distribution of \( z_t \) under \( N \) is the same of the process \( \tilde{z}_t \) under the new measure \( Q \).

**Theorem 3.5.** Let \( N, Q, z_t, \tilde{z}_t \) be the vector measures and the scalar stochastic processes defined before. Under Assumption A1, the marginal distributions of these two processes are preserved under the change of measure, that is, for every \( t \in [0, T] \), \( N_{z_t} = Q_{\tilde{z}_t} \).
Proof. We fix $B \in \mathcal{B}(\mathbb{R})$ and $t \in [0, T]$. By the Assumptions A1.a and A1.b, we have that $Q_\tilde{z}_t(B) = Q((\tilde{z}_t)^{-1}(B)) = (B_{i_2}) \int_{(\tilde{z}_t)^{-1}(B)} y_t dN$. Now, using the fact that $y_t$ is a martingale, that is, the Assumption A1.c with respect to $N$, and by Theorem 2.9, we write
\[ (B_{i_2}) \int_{(\tilde{z}_t)^{-1}(B)} y_t dN = (B_{i_2}) \int_{(\tilde{z}_t)^{-1}(B)} g_t(\tilde{z}_t) dN = (B_{i_2}) \int_B g_t(x + \theta(t)) dN_{\tilde{z}_t}. \]

Since this holds for every $B \in \mathcal{B}(\mathbb{R})$, the proof is complete. \[ \square \]

Our aim is to prove that $Q$ is an equivalent martingale measure for $(\tilde{z}_t)_t$ with respect to $N$. To do this, we assume

**Assumption A2.** The product process $(\tilde{z}_t y_t)_t$ is a martingale in itself with respect to $N$.

Then we are ready to formulate the main theorem, that is, the Girsanov Theorem for vector measures.

**Theorem 3.6 (Girsanov Theorem).** Under Assumptions A1 and A2 the process $(\tilde{z}_t)_t$ is a $Q$-martingale in itself, where $Q$ is the vector measure, defined in (6).

Proof. By Assumptions A1.a and A1.b, we can define the positive real process $y_t$ and by means of Assumption A1.c, we have that $Q \sim N$. Then it is
\[ (B_{i_2}) \int_{E} \tilde{z}_t dQ = (B_{i_2}) \int_{E} \tilde{z}_t y_t dN. \]

Now we fix $s, t \in [0, T]$, with $s \leq t$ and $E \in \mathcal{F}_s$. Using the martingale property of the process $y_t$, we have
\[ (B_{i_2}) \int_{E} \tilde{z}_t dQ = (B_{i_2}) \int_{E} \tilde{z}_t y_t dN = (B_{i_2}) \int_{E} z_t y_t dN + (B_{i_2}) \int_{E} \theta(t) y_t dN = (B_{i_2}) \int_{E} z_t y_t dN + \theta(t) \int_{E} y_s dN = (B_{i_2}) \int_{E} z_t y_t dN + \theta(t) Q(E). \]

Now, from Assumptions A1.c and A2, we deduce
\[ (B_{i_2}) \int_{E} \tilde{z}_t dQ = (B_{i_2}) \int_{E} (z_s y_s + \theta(s) y_s) dN = (B_{i_2}) \int_{E} z_s y_s dN + \theta(s) (B_{i_2}) \int_{E} y_s dN \]

but using the tower property (3), we have
\[ (B_{i_2}) \int_{E} z_s y_t dN = (B_{i_2}) \int_{E} z_s y_s dN, \]
and then we get
\[(Bi_2) \int_E \tilde{z}_t dQ = (Bi_2) \int_E z_s y_s dN + \theta(s)(Bi_2) \int_E y_s dN\]
\[= (Bi_2) \int_E z_s y_T dN + \theta(s)(Bi_2) \int_E y_T dN\]
\[= (Bi_2) \int_E (z_s + \theta(s)) dQ = (Bi_2) \int_E \tilde{z}_s dQ.\]

The last relation is the martingale property of the process \(\tilde{z}_t\) and the proof is completed. □

Theorems 3.5 and 3.6 were announced in [11, Theorems 5, 6]; the last one with also a brief sketch of the proof.

If we consider, for example, a Brownian motion \((w_t)_{t \geq 0}\), we know that this process is a martingale on this space with respect to this filtration \(\mathcal{F}\). However, if we condition it on an expiration time \(T > 0\) fixed, the distribution of this process changes and in general, it does not preserve some of its properties, such as the martingale property (see, for example, [15, Theorem 5.4]).

Let
\[N := \mathbb{P} (\cdot | w_T) : A \rightarrow L^1 (\Omega)\] (i.e. \(N(A) = \mathbb{P}(A|w_T) = E(1_A|w_T)\)). If we study the future time process \((w_{t+T})_{t > 0}\) under this new measure, we can observe that the stochastic process \(\tilde{w}_t = w_{t+T} + qt\) is an \(N\)-martingale. In fact, since \(N(A) = \mathbb{P}(A|w_T) = E(1_A|w_T)\), the last term is a \(L^1\) random variable, depending on \(w_T\) and it shows that \(N\) is a vector \(L^1\) measure. We observe that the process \(w_{t+T}\) has the same distribution of \(w_t + w_T\), where here we consider \(w_t\) as independent of \(w_T\), and, under the measure \(N\), it has a Gaussian distribution of parameters \(N(w_T, t)\). Then it admits the following density function:
\[f_t(x) = \frac{1}{\sqrt{2\pi t}} \exp \left( - \frac{(x - w_T)^2}{2t} \right).\]

Now, if we consider the transformed process \((\tilde{w}_t)_t = (w_t + w_T + qt)_t\), its marginal distribution admits a conditional density given by
\[\tilde{f}_t(x) = \frac{1}{\sqrt{2\pi t}} \exp \left( - \frac{(x - qt - w_T)^2}{2t} \right).\]

Their quotient is
\[g_t(x) = \frac{f_t(x)}{\tilde{f}_t(x)} = \exp \left( \frac{q^2 t}{2} - q(x - w_T) \right).\]

Then evaluating \(g_t(\tilde{w}_{t+T})\), we obtain the process
\[y_t := g_t(\tilde{w}_t) = \exp \left( - \frac{q^2 t}{2} - qw_t \right),\]
again considering \(w_t\), and then the process \(y_t\), independent of \(w_T\). Then the process \(y_T\), that we know to be a martingale under \(\mathbb{P}\) (by [31, (4.20)]) since it
is independent by $w_T$, is a martingale even under the conditional probability $N = \mathbb{P}|w_T$. Furthermore, we have that the process

$$\tilde{w}_t Y_t = (w_t + qt) g_t(\tilde{w}_t) + w_T g_t(\tilde{w}_t)$$

is the sum of two martingales under the conditional probability $N$, again considering $w_t$ as independent by $w_T$, so it is a martingale too. Then the process $(\tilde{w}_t)_t > 0$, under the probability $N$ obtained conditioning with respect to $w_T$, satisfies all the condition of the Girsanov Theorem 3.6 and then the new process $(\tilde{w}_t)_t$ is a martingale under $N = \mathbb{P}|w_T$.

4. Some Applications

Since conditional measures can be seen as vector measures, using the tools obtained previously about the Birkhoff integral, we give some examples of applications of the Girsanov Theorem 3.6. This could be useful for example when conditioning of random variables to future (or past times) is considered. Now consider a $C([0,T])$-valued stochastic process $\Phi$ defined as

$$\Phi(\omega, t) = \exp\left\{ -w_\tau - w_t + w_{t\wedge \tau} - \frac{t + \tau - \tau \wedge t}{2} \right\}$$

$$\quad = \begin{cases} \exp\{-w_\tau - \tau/2\} & \text{if } t \leq \tau \\ \exp\{-w_t - t/2\} & \text{if } t > \tau \end{cases} \quad (8)$$

with $\tau \in [0, T]$ and

$$N^\circ (A) = \int_A \Phi(\omega, T) d\mathbb{P} \quad (9)$$

be a $C([0,T])$-valued measure.

**Proposition 4.1.** Let $(w_t)_{t \in [0,T]}$ be a Brownian motion on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$. Let $F_t$ and $\tilde{F}_t$ be the density functions of $w_t$ and $\tilde{w}_t = w_t + t$ under $N^\circ$, respectively. Then for every $t \in [0,T]$ there exists a real function $g_t(x)$ such that $F_t(x) = g_t(x)\tilde{F}_t(x)$ for every $x \in \mathbb{R}$ and the vector measure $Q$, defined by $dQ/dN^\circ = g_t(\tilde{w}_t)$, is an equivalent martingale measure for the process $\tilde{w}_t$.

**Proof.** We are going to prove that for every $s \leq t$ in $[0,T]$

$$\mathbb{E}(\Phi(\omega, t)|\mathcal{F}_s) = \exp\left\{ -w_s - \frac{1}{2} s \right\}, \quad (10)$$

namely it is independent of $\tau$. We distinguish three cases:

$(\tau \leq s \leq t)$ in this case, we have $\Phi(\cdot, t) = \exp\{ -w_t - \frac{1}{2} t \}$ so, being a martingale process, it is trivial to deduce that $\mathbb{E}(\Phi(\omega, t)|\mathcal{F}_s) = \exp\left\{ -w_s - \frac{1}{2} s \right\}$. 


(s ≤ τ ≤ t) again Φ(ω, t) = exp \(-w_t - \frac{1}{2} t\), then, as before, we get

\[ \mathbb{E}(\Phi(\omega, t)|\mathcal{F}_s) = \exp \left\{ -w_s - \frac{1}{2} s \right\} \] .

(s ≤ t ≤ τ) in this case Φ(ω, t) = exp \(-w_\tau - \frac{1}{2} \tau\), and since it is a martingale, it follows: \( \mathbb{E}(\Phi(\omega, t)|\mathcal{F}_s) = \exp \left\{ -w_s - \frac{1}{2} s \right\} \).

Now, considering the vector measure \( N^\circ \), it is

\[ \int_E w_t dN^\circ = \int_E w_t \Phi(\omega, T) d\mathbb{P} = \int_E w_t \exp \left\{ -w_t - \frac{1}{2} t \right\} d\mathbb{P}, \]

for every \( E \in \mathcal{F}_s \), where we have used the tower property (3), that is,

\[ \mathbb{E}^\mathbb{P}(w_t \Phi(\omega, T)|\mathcal{F}_s) = \mathbb{E}^\mathbb{P}(\mathbb{E}^\mathbb{P}(w_t \Phi(\omega, T)|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}^\mathbb{P}(w_t \exp \left\{ -w_t - \frac{1}{2} t \right\} |\mathcal{F}_s). \]

This shows that \( w_t \Phi(\omega, t) \) is not a martingale under \( \mathbb{P} \) and this is equivalent to say that it is not a martingale under \( N^\circ \). Since the marginal distributions of the processes \( w_t \) and \( \tilde{w}_t \) under \( \mathbb{P} \) are Gaussian with parameters, respectively \( \mathcal{N}(0, t) \) and \( \mathcal{N}(t, t) \), the marginal distributions under \( N^\circ \) admit a density function obtained in this way:

\[ N^\circ_t(B) = \int_{(w_t)^{-1}(B)} dN^\circ = \int_{(w_t)^{-1}(B)} \Phi(\omega, T) d\mathbb{P} \]

\[ = \int_B \mathbb{E}(\Phi(\omega, T)|\mathcal{F}_t) d\mathbb{P}_{w_t} \]

\[ = \int_B \exp \left\{ -x - \frac{1}{2} t \right\} \frac{1}{\sqrt{2\pi t}} \exp \left\{ -\frac{x^2}{2t} \right\} dx. \]

Then we have the following densities function under \( N^\circ \):

\[ F_t(x) = \frac{1}{\sqrt{2\pi t}} \exp \left\{ -x - \frac{1}{2} t - \frac{x^2}{2t} \right\}, \]

\[ \tilde{F}_t(x) = \tilde{F}_t(x - t) = \frac{1}{\sqrt{2\pi t}} \exp \left\{ -x - t - \frac{1}{2} t - \frac{(x - t)^2}{2t} \right\}, \]

and then it is easy to see that the function \( g_t \) we are looking for is given by

\[ \frac{f_t(x)}{\tilde{f}_t(x)} = g_t(x) = \exp \left\{ -\frac{1}{2} t - x \right\}. \]

Then we have that \( y_t = g_t(w_t + t) = \exp \left\{ -\frac{3}{2} t - w_t \right\} \) and we want to prove that it is a martingale under the measure \( N^\circ \), which is equivalent to prove that the vector process \( \Phi(\omega, T)y_t \) is a \( \mathbb{P} \)-martingale. Again, using the tower property (3), we find that

\[ \mathbb{E}^\mathbb{P}(y_t \Phi(\omega, T)|\mathcal{F}_s) = \mathbb{E}^\mathbb{P}(\mathbb{E}^\mathbb{P}(y_t \Phi(\omega, T)|\mathcal{F}_t)|\mathcal{F}_s) \]
\[= \mathbb{E}^P (y_t \mathbb{E}^P (\Phi(\omega, T) | \mathcal{F}_t) | \mathcal{F}_s) \]
\[= \mathbb{E}^P \left(y_t \exp \left\{-w_t - \frac{1}{2} t\right\} | \mathcal{F}_s\right) \]
\[= \mathbb{E}^P (\exp \{-2w_t - 2t\} | \mathcal{F}_s) \]
\[= \exp\{-2w_s - 2s\} = \mathbb{E}^P (y_s \Phi(\omega, T) | \mathcal{F}_s) \]

because \(\exp\{-2w_s - 2s\}\) is a martingale under \(\mathbb{P}\). Since \(y_t\) is a positive \(\mathbb{N}^0\)-martingale satisfying Assumptions A1 with respect to \(\mathbb{N}^0\) and with \(\theta(t) = t\), we define a new vector measure as \(Q(A) := \int_A y_T \, d\mathbb{N}^0\), \(A \in \mathcal{A}\). Now, recalling the increment invariance of the Brownian motion and the expected value of the log normal random variable, we prove that

\[\mathbb{E}^Q (w_t + t | \mathcal{F}_s) = \mathbb{E}^{\mathbb{N}^0} (w_t y_t | \mathcal{F}_s) + t \mathbb{E}^{\mathbb{N}^0} (y_t | \mathcal{F}_s) \]
\[= \mathbb{E}^P (w_t y_t \Phi(\omega, T) | \mathcal{F}_s) + t y_s = \mathbb{E}^P (w_t \exp\{-2w_t - 2t\} | \mathcal{F}_s) + t y_s \]
\[= \mathbb{E}^P ((w_t - w_s) \exp\{-2w_t + 2w_s\} \exp\{-2w_s - 2t\}) + t y_s \]
\[+ w_s \exp\{-2w_s - 2t\} \exp\{-2w_t + 2w_s\} \exp\{-2w_s - 2t\} \exp\{-2w_s - 2t\} + (t + w_s) y_s \]
\[= (s - t) y_s + (t + w_s) y_s = (w_s + s) y_s = \bar{w}_s y_s. \]

This means that the process \(\bar{w}_s y_t\) is an \(\mathbb{N}^0\)-martingale and this proves Assumption A2 with respect \(\mathbb{N}^0\). So, using the Girsanov Theorem 3.6, the process \(\bar{w}_t\) is a \(Q\)-martingale and the proof is completed. \(\square\)

### A Birkhoff Integral Representation

Using the classical Girsanov Theorem, it is possible to change the drift term of an Itô integral and to obtain a local martingale. We want to prove an analogous result for vector processes that have an integral representation in terms of Birkhoff integrals (Theorem 4.8). The main problem is to adapt to this theory the expression

\[A_t = (B_{t1}) \int_0^t \Psi(s)ds + (\cdot) \int_0^t \Phi(s)dw_s, \quad \text{(under } \mathbb{P}), \quad (11)\]

where \(\mathbb{P}\) is the underlying probability. The first integral could be seen as a Birkhoff integral of first type. We can observe that the function \(\Phi\) integrated with respect to the Brownian motion is a vector one and the Brownian motion could also be seen as a vector-valued function, taking values for example in \(L^1(\Omega)\). Then the second integral in (11) cannot be defined as a first-type nor as a second-type Birkhoff integral. To solve this problem, we recall the Itô formula. If we consider as usual the standard Brownian motion \(w_t\) and the stochastic process \(z_t = h(t, w_t)\), where the function \(h : [0, T] \times \mathbb{R} \to \mathbb{R}\) is in the class \(C^2([0, T] \times \mathbb{R})\), then it is

\[d(z_t) = d(h(t, w_t)) = h'_t(t, w_t)dt + h'_x(t, w_t)dw_t + \frac{1}{2} h''_{xx}(t, w_t)dt. \]

Obviously, in the special case that the function \(h\) is linear with respect to \(w_t\) and it is of type \(h(t, x) = xr(t)\), we get \(h''_{xx} = 0, h'_t = r'(t)x\) and \(h'_x = r(t)\).
Then we obtain 
\[ d \left( h(t, w_t) \right) = w_t r'(t) dt + r(t) dw_t. \]
This is equivalent to the following integral relation:
\[ \int_0^t r(s) dw_s = r(t) w_t - \int_0^t r'(s) w_s ds. \] (12)

It is interesting to note that, if \( X = \mathbb{R} \), then the Eq. (12) can be deduced simply applying the classic Itô formula. If we focus on the expression in (12), we observe that it should be used as definition of a stochastic integral in the Birkhoff sense. In fact, for every fixed \( t \in [0, T] \), the Brownian motion \( w_t : \Omega \rightarrow \mathbb{R} \) is a scalar random variable and given a vector function \( \Phi : [0, T] \rightarrow X \), we have that \( \Phi(t) w_t : \Omega \rightarrow X \) is an \( X \)-valued random variable. Thus, the first term on the right side of the Eq. (12) is an \( X \)-valued random variable.

For the second one we need to recall the definition of Fréchet derivative, for other definitions see for example [2,27,28].

**Definition 4.2.** A function \( \Phi : [0, T] \rightarrow X \) is said to be differentiable at a point \( t \in [0, T] \) if exists \( \Phi'_t \in L(\mathbb{R}, X) \), (where \( L(\mathbb{R}, X) \) denotes the space of all continuous and linear operators from \( \mathbb{R} \) to \( X \)) such that
\[ \lim_{h \to 0} \frac{\| \Phi(t + h) - \Phi(t) - \Phi'_t(h) \|_X}{|h|} = 0. \]

\( \Phi'_t \) is said to be the Fréchet differential of \( \Phi \) at \( t \in [0, T] \). We say that \( \Phi \) is differentiable on \( [0, T] \) if it is differentiable at every \( t \in [0, T] \).

**Remark 4.3.** Now, if \( \Phi \) is differentiable, we consider the \( X \)-valued function \( t \mapsto \Phi'_t(t) := \Phi'_t(1) \). Let \((w_s)_s\) be a Brownian process. If \( \Phi' w \) it is first type Birkhoff integrable with respect to the Lebesgue measure, we can denote, with the integral notation
\[ \left( (Bi_1) \int_0^t \Phi'(s) w_s ds \right) (\omega) = (Bi_1) \int_0^t \Phi'(s) (w(s, \omega)) ds. \]

Now we are ready to define the stochastic integral of the second summand in (11).

**Definition 4.4.** A function \( \Phi : \mathbb{R} \rightarrow X \) is said to be stochastic Bi1-integrable ((Bi1) for short) with respect to a Brownian motion \( w_t \) on the filtered space, \( (\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F}) \) if
- \( \Phi \) is differentiable on \([0, T] \) (\( \Phi' \) being its differential);
- for every fixed \( \omega \in \Omega \) the function \( \Phi'(\cdot) w(\cdot)(\omega) : [0, T] \rightarrow X \) is Bi1-integrable with respect to the Lebesgue measure.

In this case, for every \([a, b] \subset [0, T] \), we define
\[ (Bi_1) \int_a^b \Phi(r) dw_r = \Phi(b) w_b - \Phi(a) w_a - (Bi_1) \int_{[a,b]} \Phi'(r) w_r dr. \]
Assumption A3. Suppose that $\Phi$ is strongly measurable and differentiable on $[0, T]$ and such that for every $x^* \in X^*$, the function $\langle x^*, \Phi(\cdot) \rangle \in L^2(\Omega)$ and satisfies
\[
\frac{d}{dt} \langle x^*, \Phi(t) \rangle = \langle x^*, \Phi'(t) \rangle. \tag{13}
\]

Remark 4.5. When $X = \mathbb{R}$, Definition 4.4 agrees with the classic Itô formula. In fact, given a scalar function $r(t)$, it is $\int_0^t r(s)dw_s = r(t)w_t - \int_0^t r'(s)w_sds$. Note that in this case, the Fréchet differential of $r$ is the classic differential given by $D_x r(t) = r'(t)t$, then $D_x r(1) = r'(t)$ and Assumption A3 hold.

From now on, we suppose that $\Phi, \Psi$ are $\text{Bi}_1^*$-integrable and satisfy Assumptions A3. So, we get the following

Proposition 4.6. For every $x^* \in X^*$, the process $\langle x^*, \Phi \rangle$ is integrable with respect to the Brownian motion $(w_t)_t$ and, moreover,
\[
\left\langle x^*, (\text{Bi}_1^*)^t \right\rangle = \int_0^t \langle x^*, \Phi(s) \rangle dw_s. \tag{14}
\]

Proof. Fixed $x^* \in X^*$, it is
\[
\left\langle x^*, (\text{Bi}_1^*)^t \right\rangle = \langle x^*, \Phi(t)w_t \rangle - \left\langle x^*, (\text{Bi}_1^*)^t \left( \Phi'(r)w_r \right) dr \right\rangle
= \langle x^*, \Phi(t) \rangle w_t - \int_0^t \langle x^*, \Phi'(r)w_r \rangle dr
= \langle x^*, \Phi(t) \rangle w_t - \int_0^t \frac{d}{dt} \langle x^*, \Phi(t) \rangle w_r dr
= \int_0^t \langle x^*, \Phi(r) \rangle dw_r,
\]

where, in the last equalities, we have used the Assumption A3 and the formula (12) applied to $h(t, w_t) = w_t \langle x^*, \Phi(t) \rangle$. Then the assertion follows. \qed

Moreover, the next result holds.

Theorem 4.7. The process
\[
(A_t)_{t \in [0, T]} := \left( (\text{Bi}_1^*)^t \right)_{t \in [0, T]}
\]
is a martingale with respect to the natural filtration of the Brownian motion $(w_t)_t$.

Proof. Let $s \leq t$ in $[0, T]$ and fix $x^* \in X^*$. By Proposition 4.6 we have that $\langle x^*, \Phi(r) \rangle$ is integrable with respect to the Brownian motion $(w_t)_t$ and
\[
\left\langle x^*, \mathbb{E} \left( (\text{Bi}_1^*)^t \Phi(r)dw_r | \mathcal{F}_s \right) \right\rangle = \mathbb{E} \left( \int_0^t \langle x^*, \Phi(r) \rangle dw_r | \mathcal{F}_s \right)
= \mathbb{E} \left( \int_0^s \langle x^*, \Phi(r) \rangle dw_r \right) = \langle x^*, (\text{Bi}_1^*)^s \Phi(r) \rangle dw_r.
\]
So, by arbitrariness of $x^*$, we have

$$
\mathbb{E} \left( (B_{i_1}) \int_0^t \Phi(r)dw_r | \mathcal{F}_s \right) = (B_{i_1}) \int_0^s \Phi(r)dw_r
$$

so the proof is completed. □

Now, we consider a process as follows:

$$
C_t = (B_{i_1}) \int_0^t \Psi(s)ds + (B_{i_1}) \int_0^t \Phi(s)dw_s, \quad ( \text{under } \mathbb{P}), \quad t \in [0, T]
$$

(in the scalar case, it is an Itô process). We would like to eliminate the drift term so that we obtain a local martingale. To obtain it, we need a hypothesis of connection between the drift term $\Psi$ and the diffusion term $\Phi$, namely we need to know if there exists a stochastic process $r(t)$ such that $\Psi(t) = r(t)\Phi(t)$, for every $t \in [0, T]$ (and this is the process that we use to define the change of measure). The main peculiarity of this kind of result is that we could change the drift term, without changing the diffusion term.

**Theorem 4.8** (Change of a drift term). Let $(w_s)_s$ be a Brownian motion on $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$, $\Phi, \Psi$ be two processes that are $B_{i_1}^*$-integrable and satisfy Assumptions A3. Let $C_t : [0, T] \times \Omega \to X$ be the stochastic process defined by

$$
C_t = (B_{i_1}) \int_0^t \Psi(s)ds + (B_{i_1}^*) \int_0^t \Phi(s)dw_s, \quad ( \text{under } \mathbb{P}).
$$

If there exists $r : [0, T] \to \mathbb{R}$ be in $L^2([0, T])$ such that $\Psi(t) = r(t)\Phi(t)$ and the process $y_t = \exp \left\{ -\frac{1}{2} \int_0^t r^2(s)ds - \int_0^t r(s)dw_s \right\}$ is a martingale with respect to $\mathcal{F}$, denoted by $Q^o = (B_{i_1}) \int y_t d\mathbb{P}$ and $\tilde{w}_t = w_t + \int_0^t r(s)ds$, then

$$
C_t = (B_{i_1}^*) \int_0^t \Phi(s)dw_s, \quad ( \text{under } Q^o).
$$

Therefore, the process $C_t$ is a martingale under $Q^o$.

**Proof.** By construction and using the classical Girsanov Theorem the process $\tilde{w}_t$ is a Brownian motion under the new probability $Q^o$. Now we claim that

$$
(B_{i_1}) \int_0^t \Phi(s)dw_s = (B_{i_1}^*) \int_0^t \Phi(s)dw_s + (B_{i_1}) \int_0^t r(s)\Phi(s)ds \quad (15)
$$

as a vector equivalence between the Birkhoff stochastic integral and the Birkhoff integral. To prove this we consider, for every $x^* \in X^*$,

$$
\left\langle x^*, (B_{i_1}) \int_0^t \Phi(s)dw_s \right\rangle = \int_0^t \left\langle x^*, \Phi(s) \right\rangle d\tilde{w}_s
$$

$$
= \int_0^t \left\langle x^*, \Phi(s) \right\rangle (dw_s + r(s)ds)
$$

$$
= \int_0^t \left\langle x^*, \Phi(s) \right\rangle dw_s + \int_0^t \left\langle x^*, \Phi(s) \right\rangle r(s)ds
$$
\[ \begin{align*}
\langle x^*, (B_i^*) \int_0^t \Phi(s) dw_s \rangle &+ \langle x^*, (B_i^*) \int_0^t \Phi(s) r(s) ds \rangle \\
= \langle x^*, (B_i^*) \int_0^t \Phi(s) dw_s + (B_i^*) \int_0^t \Phi(s) r(s) ds \rangle .
\end{align*} \]

So, by the arbitrariness of \( x^* \in X^* \), the Eq. (15) holds. Thus,

\[ C_t = (B_i^*) \int_0^t \Psi(s) ds + (B_i^*) \int_0^t \Phi(s) dw_s \]
\[ = (B_i^*) \int_0^t \Psi(s) ds + (B_i^*) \int_0^t \Phi(s) d\tilde{w}_s - (B_i^*) \int_0^t r(s) \Phi(s) ds \]
\[ = (B_i^*) \int_0^t \Phi(s) d\tilde{w}_s , \]

and then we have that

\[ C_t = (B_i^*) \int_0^t \Psi(s) d\tilde{w}_s \quad \text{(under \( Q^\circ \))} \]

and it turns out to be a martingale, thanks to Theorem 4.7. \( \square \)

Acknowledgements
The Fondo Ricerca di Base 2018 University of Perugia—and the GNAMPA—INDAM (Italy) Project “Metodi di Analisi Reale per l’Approccimazione attraverso operatori discreti e applicazioni” (2019) supported this research.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References
[1] Birkhoff, G.: Integration of functions with values in a Banach space. Trans. Am. Math. Soc. 38(2), 357–378 (1935)
[2] Boccuto, A., Candeloro, D.: Differential calculus in Riesz spaces and applications to g-calculus. Med. J. Math. 8(3), 315–329 (2011). https://doi.org/10.1007/s00009-010-0072-x
[3] Boccuto, A., Candeloro, D., Sambucini, A.R.: Henstock multivalued integrability in Banach lattices with respect to pointwise non atomic measures. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26(4), 363–383 (2015). https://doi.org/10.4171/RLM/710
[4] Boccuto, A., Sambucini, A.R.: A note on comparison between Birkhoff and Mc Shane integrals for multifunctions. Real Anal. Exchange 37(2), 3–15 (2012). https://doi.org/10.14321/realanalexch.37.2.0315
[5] Candeloro, D., Croitoru, A., Gavriliţ, A., Sambucini, A.R.: An extension of the Birkhoff integrability for multifunctions. Mediterr. J. Math. 13(5), 2551–2575 (2016). https://doi.org/10.1007/s00009-015-0639-7
[6] Candeloro, D., Di Piazza, L., Musial, K., Sambucini, A.R.: Relations among gauge and Pettis integrals for multifunctions with weakly compact convex values. Annali di Matematica 197(1), 171–183 (2018). https://doi.org/10.1007/s10231-017-0674-z

[7] Candeloro, D., di Piazza, L., Musial, K., Sambucini, A.R.: Some new result on the integration for multifunctions. Ricerche di Matematica 67(2), 361–372 (2018). https://doi.org/10.1007/s11587-018-0376-x

[8] Candeloro, D., Sambucini, A.R.: Order-type Henstock and Mc Shane integrals in Banach lattice setting. In: Proceedings of the SISY 2014—IEEE 12th International Symposium on Intelligent Systems and Informatics, pp. 55–59. ISBN 978-1-4799-5995-2 (2014). https://doi.org/10.1109/SISY.2014.6923557

[9] Candeloro, D., Sambucini, A.R.: Comparison between some norm and order gauge integrals in Banach lattices. PanAm. Math. J. 25(3), 1–16 (2015)

[10] Candeloro, D., Labuschagne, C.C.A., Marraffa, V., Sambucini, A.R.: Set-valued Brownian motion. Ricerche di Matematica 67(2), 347–360 (2018). https://doi.org/10.1007/s11587-018-0372-1

[11] Candeloro, D., Sambucini, A. R.: A Girsanov result through birkhoff integral. In: Gervasi, O. et al. (eds.) Computational Science and its Applications ICCSA 2018, LNCS 10960, pp. 676–683 (2018). https://doi.org/10.1007/978-3-319-95162-1_47

[12] Caponetti, D., Marraffa, V., Naralenkov, K.: On the integration of Riemann-measurable vector-valued functions. Monatsh. Math. 182, 513–536 (2017). https://doi.org/10.1007/s00605-016-0923-z

[13] Cascales, B., Rodríguez, J.: Birkhoff integral for multi-valued functions. Special issue dedicated to John Horváth. J. Math. Anal. Appl. 297(2), 540–560 (2004)

[14] Cascales, B., Rodríguez, J.: The Birkhoff integral and the property of Bourgain. Math. Ann. 331(2), 259–279 (2005)

[15] Chang, J.: Stochastic Processes (1999). https://iid.yale.edu/sites/default/files/files/chang-notes.pdf

[16] Cichoń, K., Cichoń, M.: Some applications of nonabsolute integrals in the theory of differential inclusions in Banach spaces. In: Curbera, G.P., Mockenhaupt, G., Rickéer, W.J. (eds.) Vector Measures, Integration and Related Topics, Operator Theory: Advances and Applications, vol. 201, pp. 115–124. Birkhäuser, ISBN: 978-3-0346-0210-5 (2010)

[17] Croitoru, A., Mastorakis, N.: Estimations, convergences and comparisons on fuzzy integrals of Sugeno, Choquet and Gould type. In: IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), pp. 1205–1212 (2014). https://doi.org/10.1109/FUZZ-IEEE.2014.6891590

[18] Croitoru, A., Gavriluţ, A., Iosif, A.E.: Birkhoff weak integrability of multifunctions. Int. J. Pure Math. 2, 47–54 (2015)

[19] Croitoru, A., Gavriluţ, A.: Comparison between Birkhoff integral and Gould integral. Mediterr. J. Math. 12(2), 329–347 (2015)

[20] Croitoru, A., Iosif, A., Mastorakis, N., Gavriluţ, A.: Fuzzy multimeasures in Birkhoff weak set-valued integrability. In: 2016 Third International Conference on Mathematics and Computers in Sciences and in Industry (MCSI), Chania, pp. 128–135 (2016). https://doi.org/10.1109/MCSI.2016.034

[21] Di Piazza, L., Marraffa e B.Satco, V.: Set valued integrability in non separable Fréchet spaces and applications. Math. Slovaca 66(5), 1119–1138 (2016)
[22] Fernández, A., Mayoral, F., Naranjo, F., Rodríguez, J.: On Birkhoff integrability for scalar functions and vector measures. Monatsh. Math. 157(2), 131–142 (2009)

[23] Fremlin, D.H.: Integration of vector-valued functions. Atti Semin. Mat. Fis. Univ. Modena 42, 205–211 (1994)

[24] Fremlin, D.H., Mendoza, J.: On the integration of vector-valued functions Illinois. J. Math. 38(1), 127–147 (1994)

[25] Fremlin, D.H.: The McShane and Birkhoff integrals of vector-valued functions, University of Essex Mathematics Department Research Report 92-10, version of 13.10.04. http://www.essex.ac.uk/math8/staff/fremlin/preprints.htm

[26] Grobler, J.J., Labuschagne, C.C.A.: Girsanov’s theorem in vector lattices. Positivity 23(5), 1065–1099 (2019)

[27] Kaliaj, S.B.: Some full characterizations of differentiable functions. Mediterr. J. Math. 12, 639–646 (2015). https://doi.org/10.1007/s00009-014-0458-20378-620X/15/030639-8

[28] Kaliaj, S.B.: Differentiability and weak differentiability. Mediterr. J. Math. 13, 2801–2811 (2016). https://doi.org/10.1007/s00009-015-0656-61660-5446/16/052801-11

[29] Labuschagne, C.C.A., Marraffa, V.: On set-valued cone absolutely summing map. Cent. Eur. J. Math. 8(1), 148–157 (2010)

[30] Marraffa, V.: A Birkhoff type integral and the Bourgain property in a locally convex space. Real Anal. Exchange 32(2), 409–428 (2006–2007). https://doi.org/10.14321/realanallexch.32.2.0409

[31] Mikosch, T.: Elementary Stochastic Calculus (with finance in view). World Scientific Publishing Co., Inc., River Edge, NJ (1998). ISBN: 981-02-3543-7 (source MathScinet)

[32] Mushambi, N., Watson, B.A., Zinsou, B.: Generalization of the theorems of Barndorff-Nielsen and Balakrishnan-Stepanov to Riesz spaces. Positivity (2019). https://doi.org/10.1007/s11117-019-00705-0

[33] Novikov, A.: A certain identity for stochastic integrals. Theory Probab. Appl. 17(4), 717–720 (1972)

[34] Pettis, B.J.: On the integration in vector spaces. Trans. Am. Math. Soc. 44, 277–304 (1938)

[35] Potyrala, M.: Some remarks about Birkhoff and Riemann-Lebesgue integrability of vector valued function. Tatra Mt. Math Publ. 35, 97–106 (2007)

[36] Potyrala, M.: The Birkhoff and variational McShane integrals of vector valued functions. Folia Mathematica 13(1), 31–39 (2006)

[37] Rodríguez, J.: Absolutely summing operators and integration of vector-valued functions. J. Math. Anal. Appl. 316(2), 579–600 (2006)

[38] Rodríguez, J.: Pointwise limits of Birkhoff integrable functions. Proc. Am. Math. Soc. 137(1), 235–245 (2009)

[39] Rodríguez, J.: Universal Birkhoff integrability in dual Banach spaces. Quaest. Math. 28(4), 525–536 (2005)

[40] Stoica, G.: Limit laws for martingales in vector lattices. J. Math. Anal. Appl. 476(2), 715–719 (2019)

[41] Trastulli, L.: Methods of integration with respect to non additive measures and some applications. Tesi Magistrale (2018)
