A bilinear identification-modeling framework from time domain data

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An ever-increasing need for improving the accuracy includes more involved and detailed features, thus inevitably leading to larger-scale dynamical systems [1]. To overcome this problem, efficient finite methods heavily rely on model reduction. One of the main approaches to model reduction of both linear and nonlinear systems is by means of interpolation. The Loewner framework is a direct data-driven method able to identify and reduce models derived directly from measurements. For measured data in the frequency domain, the Loewner framework is well established in linear case [2] while it has already extended to nonlinear [6]. On the other hand in the case of time domain data, the Loewner framework was already applied for approximating linear models [3]. In this study, an algorithm which uses time domain data for nonlinear (bilinear) system reduction and identification is presented.

1 Introduction

Evolutionary phenomena (e.g. heat diffusion, wave propagation) can be modeled as dynamical systems. The mathematical description of such systems uses the concept of DAEs (Differentiable Algebraic Equations), which can be used to simulate the input-output relation. Causal systems are characterized by outputs that depend on past and current inputs. The evolutionary information for a stationary system is captured in the differential part with the underlying functional (f) remaining invariant with respect in time. If (f) fulfills the superposition and scaling principle then it describes a linear map. Otherwise the system is nonlinear.

Below, the general linear and a mildly nonlinear one (bilinear) are presented in the SISO (Single Input Single Output) case.

**Linear system**

\[
\begin{align*}
E \mathbf{x}(t) &= A \mathbf{x}(t) + B \mathbf{u}(t), \\
\mathbf{y}(t) &= C \mathbf{x}(t) + D \mathbf{u}(t).
\end{align*}
\]

Next, zero initial conditions are assumed, i.e. \( \mathbf{x}(0) = 0 \). Additionally, \( D = 0 \), and \( E, A, N \in \mathbb{C}^{N \times N} \) with \( B, C^T \in \mathbb{C}^{N \times 1} \).

**Nonlinear (bilinear) system**

\[
\begin{align*}
E \mathbf{x}(t) &= A \mathbf{x}(t) + N \mathbf{x}(t) \mathbf{u}(t) + B \mathbf{u}(t), \\
\mathbf{y}(t) &= C \mathbf{x}(t) + D \mathbf{u}(t).
\end{align*}
\]

2 Harmonic probing method

A single sinusoidal input \( \mathbf{u}(t) = \alpha \cos(\omega t) = \frac{\alpha}{2} e^{j\omega t} + \frac{\alpha}{2} e^{-j\omega t} \), is used. The \( n \)th Volterra term and the \( n \)th harmonic are defined respectively as:1

\[
y_{n}(t) = \left( \frac{\alpha}{2} \right)^n \sum_{p+q=n} C_{n} H_{n}^{p,q}(j\omega) e^{j\omega_p t}, \quad y_{x,n}(t) = \sum_{i=1}^{\infty} \left( \frac{\alpha}{2} \right)^{n+2i - 2} c_{i-1}^{n+1-i-1}(j\omega) e^{j\omega t}.
\]

With Fourier Transform (FT) we can derive harmonics in the frequency domain as:

\[
Y_{1}(j\omega) = H_{1}(j\omega) U + 0.75 H_{2}(j\omega, j\omega, j\omega) U^2 + \cdots, \quad Y_{11}(2j\omega) = 0.5 H_{2}(j\omega, j\omega) U^2 + 0.5 H_{4}(j\omega, j\omega, j\omega, j\omega) U^3 + \cdots.
\]

By exciting the system with a small amplitude (\( \alpha < 1 \)) we can neglect the higher order terms of the power series. So, by keeping only the order of magnitude up to \( \alpha^2 \), the approximation follows as: \( H_{1}(j\omega) \approx Y_{1}(j\omega) / U \), \( H_{2}(j\omega, j\omega) \approx 2Y_{11}(2j\omega) / U^2 \). At this point, the procedure of estimating samples of GFRFs from time domain data is described. In the case of bilinear systems,

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the growing exponential approach provides a way of explicitly deriving GFRFs similar to [6]. Assuming that the solution $x$, can be written as: $x(t) = \sum_{p,q} G^p_q (j\omega)^{p+q} e^{j\omega t}$, and by substituting in the differential equation of the bilinear model, we can derive the following symmetric GFRFs by denoting the resolvent $\Phi(j\omega) = (j\omega E - A)^{-1} \in \mathbb{C}^{N \times N}$ as:

$$H^r_n(j\omega, ..., j\omega_n) = C\Phi(nj\omega)N\Phi((n-1)j\omega)\cdots N\Phi(j\omega)B, \; n \geq 1.$$ 

Algorithm 1: Time domain bilinear system identification (2nd-stage approximation)

Input: Apply signals $u(t) = \alpha_x \cos(\omega_k t)$ with driving frequency $\omega_k$, $k = 1, \ldots, n$, and collect snapshots $y(t)$ in steady state ($\alpha_x \sim$ small).

1. Apply FT and measure $U(j\omega_k)$, $Y_j(j\omega_k)$, $Y_{j1}(j\omega_k)$, $\ldots$

2. $H^r_1(j\omega_k) = C(j\omega_k E - A)^{-1}B \approx \frac{Y_1(j\omega_k)}{U(j\omega_k)} \approx H^r_1(j\omega_k) \rightarrow \{E_r, A_r, B_r, C_r\}$, linear Loewner framework [2, 4].

3. $H^r_2(j\omega_k, j\omega_l) = C_r(2j\omega_k E_r - A_r)^{-1}N_r(j\omega_k E_r - A_r)^{-1}B^2 \approx \frac{2Y_{j1}(j\omega_k)}{U(j\omega_k)^2}$

4. Identify the matrix $N_r$ by solving the system: $H^r_2(j\omega_i, j\omega_i) = O_{i,i}N_r, R_{i,i} \Rightarrow H^r_2(j\omega_i, j\omega_i) = [O_{i,i} \otimes R_{i,i}] \text{vec}(N_r), i = 1, \ldots, n$.

3 Numerical example

The following 2D heat transfer model from [5] simulates the heat diffusion over a square domain with Robin boundary conditions. The semi-discrete model is bilinear. By equating the input control signals $u_1 = u_2$, we enforce the SISO case.

\[
x_i = \Delta x \frac{\partial^2}{\partial x^2} \; \text{with boundary conditions:} \nonumber\]

\[
\begin{align*}
n \nabla x &= u_1(x - 1), \; \Gamma_1 := \{0\} \times [0, 1], \\
\n \nabla x &= u_2(x - 1), \; \Gamma_2 := [0, 1] \times \{0\}, \\
x &= 0, \; \Gamma_3 := \{1\} \times [0, 1], \; \Gamma_4 := [0, 1] \times \{1\}.
\end{align*}
\]

Data collection: By exciting the system with input signals as $u(t) = 0.01 \cos(kt)$, $k = 1, \ldots, 100$, and simulating the output with Euler method (dt=0.001), we have access to time domain snapshots in steady state. Here we simulate the output for validation reasons while in general, our data-driven approach acquires measurements (time or frequency) directly from experimental procedures.

Results: By applying the proposed algorithm 1, with only one amplitude, we are able to achieve up to 2nd stage approximation. In Fig. 1, it is shown the singular value decay of the Loewner matrices which provides a trade-off between accuracy of fit and complexity of the reduced model. The order of the reduced model is chosen $r = 3$ with $2\overline{r}$ less than $O(dt)$. In Fig. 2, the first three identified GFRFs with the corresponding error to be proportional to $O(dt)$ are presented. In Fig. 3, the evaluation of the identified bilinear model using a sawtooth input is depicted. It is observed that the bilinear reduced model improves the accuracy 2 orders of magnitude from the linear one.

Fig. 1: The singular value decay of the Loewner matrices.
Fig. 2: The first two GFRFs and the corresponding error plots.
Fig. 3: Linear & bilinear models (dim $r = 3$) compared with original (dim $N = 100$) for a sawtooth input.

4 Conclusion

In this study, we presented a method that is able to identify and reduce bilinear systems from time domain measurements. Using the Loewner framework, it is feasible to identify the linear part $(E, A, B, C)$ and by solving an overdetermined linear system of equations, we are able to identify the matrix $N$. An adaptive algorithm for which the level of approximation can be increased (i.e. by varying the amplitude $\alpha_x$ of the input signals) relies also on algorithm 1. Finally, a multi-tone input would allow evaluating the off-diagonal frequency response functions and will hence extend the identification method over the whole complex plane.

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