Cosmological contraction of the atomic space-time scale

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Quantum mechanical effects related to the recently developed projective theory of relativity are considered. It is shown that at cosmological time intervals the light velocity increases and the atomic units of length and time are shrinking. A new derivation of the Hubble Law is presented.

1 Introduction

The modern relativistic theory was initially based on the two postulates put forward by Einstein [1], namely, the principles of relativity and constancy of light velocity. However, shortly after the Einstein’s theory was developed, it was shown [2] that these postulates are not independent. Moreover, it turned out that the number of axioms underlying the relativity theory is actually less than that of the classical mechanics.

This property of the relativistic theory is shared by other fundamental physical theories too. The principle of “parametric incompleteness” was formulated in Ref. [4], which states that fundamental physical constants like c and ℏ appear in the theory as a result of reducing the number of axioms of classical mechanics. This principle leads to an heuristic method for creating new physical theories: If it is possible to reduce the number of axioms of a theory in such a way that all the functional relations (theorems) within this theory can be derived, one obtains a new, more general physical theory. New physical constants appear in this new theory as a consequence of reducing the information content of the initial set of axioms. Classical mechanics is a complete theory and contains no fundamental physical constants. Relativistic and quantum theories can be derived from classical mechanics if one rejects some of its axioms, and so they are parametrically incomplete theories depending on the constants c and ℏ, the values of which cannot be inferred using only those theories’ intrinsic means.

Recently this method was applied to constructing a generalization of the special theory of relativity [3]-[5], which was named the projective theory of relativity [6]. Formally, this theory incorporates two parts, the generalized Lorenz transformations (between two inertial reference frames) [3],[4], and the generalized translational transformations (between observers situated at different places in the same inertial frame) [5]. Together they compose a six-parametric group of linear fractional transformations of the projective geometry. In addition to the fundamental constant velocity c, the formulae of the projective relativity contain a new constant of inverse-length dimension, κ. It was denoted differently when first introduced, namely κ = −1/R in Ref. [3], and κ = −λc in Ref. [4],
We will show in the present paper that the constant $R (\lambda)$ is in fact negative, which makes the new notation more natural.

When $\kappa$ is small enough, in the limit $\kappa \to 0$ the convenient relativistic theory is restored, and corrections are of any significance only for events that are far distant in time and/or space from the observer. Therefore, we see that the projective relativity is a cosmological theory.

The pivotal property of the discussed theory is that the maximal possible speed at a given point of space is a function of both the temporal and spatial coordinates:

$$\vec{C}(t, \vec{x}) = c\vec{n} - \kappa \vec{x} / (1 - \kappa ct),$$

where $c$, $\kappa$ are fundamental constants, $\vec{n}$ is a unit vector along the flux of the electromagnetic wave and $\vec{x}$ is the distance to the observer. We will identify $\vec{C}(t, \vec{x})$ with the physical velocity of light. At present, in the vicinity of the observer $\vec{C}(0, 0) = c = 299 792 458 \text{ m/s}$. It is important that while being a function of coordinates $C(t, \vec{x})$, the light speed is nevertheless invariant under the group of transformations of the theory. Moreover, it does not change along the trajectory of a light pulse [5]. It follows from Eq. (1) that the speed of light increases with time ($\kappa > 0$). This fact could find its application to recently developed models relying upon a time-dependent light velocity [7]-[15]. In particular, the projective theory of relativity could reconcile the principle of relativity and the speed of light varying with time [14].

As the light velocity is coordinate- and time-dependent, the frequency of signals emitted by a fixed remote atomic source experiences a redshift, which is a growing function of the distance to the observer:

$$\omega / \omega_0 = 1 + z = \sqrt{1 + \kappa R / (1 - \kappa R)},$$

where $\omega$ is the frequency of the signal at the moment of detection, $\omega_0$ - the frequency of an identical atom on Earth, $R$ - the distance between the source and the observer. Equation (2) remarkable coincides with the observed Hubble redshift law. Therefore, if the considered generalization of the relativistic theory is indeed realized in our Universe, the redshift in radiation of distant objects can be, completely or in part, of a non-Doppler nature. It was this analysis of the redshift that earlier motivated us to consider $\kappa$ positive.

In the present paper we consider the properties of wave and quantum processes within the framework of the projective theory of relativity. In Sec. 2, the basic relations of the theory are formulated. In Sec. 3, the Lagrangian equations of motion are derived. The next section deals with propagation of electromagnetic waves, and the transformation rules for frequency and wave vector. Elements of quantum mechanics are considered in the projective relativity context in Sec. 5. The sixth section contains the derivation of the redshift in radiation from remote sources. Finally, in the last section we elaborate on the effect of contraction of the atomic space and time scales.
2 The Projective Theory of Relativity

To obtain the space and time transformations between two inertial observers (the Galilean transformations) one needs a certain subset of the classical mechanical set of axioms. If one leaves out the one that renders the time absolute, one obtains the generalization of the Galilean transformations - the Lorentz transformations and the fundamental velocity constant, \(c\). If one goes further and drops the speed absoluteness axiom, (which requires that if the speeds of two particles are equal for an observer, they are equal for any other inertial observer), one arrives at so called projective Lorenz transformations [3], [4]:

\[
\begin{align*}
\frac{x'}{1 - \kappa ct'} &= \gamma \frac{x - vt}{1 - \kappa ct}, \\
\frac{y'}{1 - \kappa ct'} &= \frac{y}{1 - \kappa ct}, \\
\frac{t'}{1 - \kappa ct'} &= \frac{\gamma (t - vx/c^2)}{1 - \kappa ct},
\end{align*}
\]

(3)

where \(v\) is the relative speed of the observers situated at the origins of two inertial reference frames, \(\gamma = 1/\sqrt{1 - v^2/c^2}\) is the Lorentz factor, and \(\kappa\) is a new fundamental constant.

If one considers two motionless observers, one at the origin \(\bar{x} = 0\) of an inertial frame measuring distances and times \((\bar{x}, t)\), and the other in the same frame at the point \((x = R, y = 0)\), measuring \((\bar{X}, T)\), the generalized translation transformations between them are as follows [3]:

\[
\begin{align*}
X &= \frac{x - R}{1 - \kappa^2 Rx}, \\
Y &= \frac{\sqrt{1 - (\kappa R)^2}}{1 - \kappa^2 Rx} y.
\end{align*}
\]

(5)

\[
1 - \kappa cT = \frac{\sqrt{1 - (\kappa R)^2}}{1 - \kappa^2 Rx} (1 - \kappa ct).
\]

(6)

Equations (5) have the same structure as the velocity transformations in the conventional relativity theory. Thus, the components of \(\bar{x}\) form a Beltrami set of coordinates in the Lobachevsky space. Besides, neither time nor relative speeds are absolute for the two fixed observers. For instance, the velocities of a particle as measured by each of these observers, \(\bar{u} = d\bar{x}/dt\) and \(\bar{U} = d\bar{X}/dT\) respectively, are related as:

\[
\begin{align*}
U_X &= \frac{u_x \sqrt{1 - (\kappa R)^2}}{1 - \kappa R u_x / c - \kappa^2 R (x - u_x t)}, \\
U_Y &= \frac{u_y + \kappa^2 R (y u_x - x u_y)}{1 - \kappa R u_x / c - \kappa^2 R (x - u_x t)}.
\end{align*}
\]

(7)

Only the objects with a zero speed possess the same speed from the point of view of both observers at rest.

The maximal velocity \(\parallel\) is invariant under the transformations (3) - (6). For example, the same function of space and time stands in the right- and left-hand sides of Eq. (7).

Transformations (3) - (6) also leave invariant the following metric

\[
ds^2 = \frac{1 - (\kappa \bar{x})^2}{(1 - \kappa ct)^4} c^2 dt^2 - \frac{2 \kappa \bar{x} d\bar{x} d\bar{t}}{(1 - \kappa ct)^3} - \frac{d\bar{x}^2}{(1 - \kappa ct)^2}
\]

(8)

and form a six-parametric group with generators \(\bar{v}, \bar{R}\) (see Appendix 1).
The expressions for the energy and momentum of a particle, moving at a speed $\vec{u}$ at a distance $\vec{x}$ from the observer, were suggested in Ref. [3]:

$$ E = \frac{mc^2}{\sqrt{1 - \left[ (1 - \kappa ct) \frac{\vec{u}}{c} + \kappa \vec{x} \right]^2}} $$
$$ \vec{p} = \frac{m [\vec{u}(1 - \kappa ct) + \kappa \vec{x}]}{\sqrt{1 - \left[ (1 - \kappa ct) \frac{\vec{u}}{c} + \kappa \vec{x} \right]^2}}. $$

They possess a number of interesting properties:

1. Both energy and momentum become infinite when the particle’s speed approaches the speed of light $\vec{u} \rightarrow \vec{C}(t, \vec{x})$. Therefore, $\vec{C}(t, \vec{x})$ is the maximal possible speed at a given point in space and time.

2. Despite time and space dependence, the energy and momentum of an uniformly moving particle are constant, $d\vec{p}/dt = dE/dt = 0$. Any closed system (e.g. an atom) that is resting at the origin of the reference frame $\vec{x} = \vec{u} = 0$ has constant energy (due to non-zero mass) $E = mc^2$ and zero momentum $\vec{p} = 0$.

3. The conventional relation holds between energy, momentum and mass:

$$ \frac{E^2}{c^2} - \vec{p}^2 = (mc)^2. $$

4. The relation between energy, momentum and speed reads

$$ \frac{\vec{p}}{E} c = \frac{\vec{u}}{c} (1 - \kappa ct) + \kappa \vec{x}, $$

thus, for photons $\vec{u} = \vec{C}(t, \vec{x})$ and

$$ \vec{p} c = \vec{n} E, $$

where $c$ is the fundamental speed constant, which is different from the physical light velocity $\vec{C}(t, \vec{x})$. Therefore, the photon momentum, as well as that of any massive particle, is not in general collinear to its speed. Moreover, for a particle remote from the observer, the momentum is not zero even when the particle is motionless.

5. One can easily see that for two observers in different frames the energy-momentum transformations are given by the usual formulae:

$$ p'_{x} = \gamma \left( p_{x} - \frac{v}{c^2} E \right), \quad p'_{y} = p_{y}, $$
$$ E' = \gamma \left( E - v p_{x} \right). $$

For two rest observers in the same frame $E$ and $p$ are transformed as

$$ P_{x} = \sigma \left( p_{x} - \frac{\kappa R}{c} \varepsilon \right), \quad P'_{y} = p_{y}, $$
$$ E = \sigma \left( \varepsilon - c \kappa R p_{x} \right) $$

where $(\varepsilon, \vec{p})$ are energy and momentum measured by the observer at $(t, \vec{x})$, and $(E, \vec{P})$ - by the one at $(T, \vec{X})$, and $\sigma = 1/\sqrt{1 - (\kappa R)^2}$. We would like to stress the symmetry of transformations (13),(14) and (15),(16) under the substitution $v/c \rightarrow \kappa R$.

We see that the transformations (13)-(16) are linear, and this ensures that if the energy and momentum of a non-interacting particle system are conserved in some reference frame, they will be conserved for any other inertial observer too.
3 The Lagrangian Formulation

The requirement that Lagrangian density should be invariant under the transformations of the model leads to the following expression for $L$:

$$L(\vec{u}, \vec{x}, t) = -mc \frac{d\vec{s}}{dt} = -\frac{mc^2}{(1 - \kappa ct)^2} \sqrt{1 - \left(\frac{\vec{u}}{c}(1 - \kappa ct) + \kappa \vec{x}\right)^2} \quad (17)$$

Since the Lagrangian is coordinate dependent, and, so, not invariant under translations $\vec{x} \to \vec{x} + \vec{a}$, one can see that a conveniently defined momentum is not an integral of motion either. However, the Lagrangian is invariant under the transformations:

$$\vec{x} \to \vec{x}' = \vec{x} + (1 - \kappa ct) \vec{a}, \quad (18)$$

where $\vec{a}$ is an arbitrary constant vector. Thus, an infinitesimal variation of the Lagrangian with respect to $\vec{a}$ produces a conserved quantity, which we will call momentum:

$$\vec{p} = (1 - \kappa ct) \frac{\partial L}{\partial \vec{u}} = \frac{m [\vec{u}(1 - \kappa ct) + c\kappa \vec{x}]}{\sqrt{1 - \left((1 - \kappa ct)\frac{\vec{u}}{c} + \kappa \vec{x}\right)^2}}. \quad (19)$$

Energy conservation is related to time-invariance; the projective theory of relativity is invariant under the transformations of the form:

$$\frac{t'}{1 - \kappa ct'} = \frac{t}{1 - \kappa ct} + t_0, \quad \frac{x'}{1 - \kappa ct'} = \frac{x}{1 - \kappa ct}, \quad (20)$$

where $t_0$ is an arbitrary time shift. This invariance leads to the conserved energy:

$$E = \vec{p}[\vec{u}(1 - \kappa ct) + \kappa \vec{x}] - (1 - \kappa ct)^2 L. \quad (21)$$

Upon substitution of the Lagrangian density into this equation, the expression for the energy (9) is exactly reproduced.

Let us note that transformations (20) also leave unchanged the following modified Lagrangian:

$$L = L_0 - \frac{V(x/(1 - \kappa ct))}{(1 - \kappa ct)^2}, \quad (22)$$

where $L_0$ is the free Lagrangian. Thus, the energy of the particle in the external field is conserved, provided that:

$$E = \sqrt{m^2c^4 + c^2\vec{p}^2} + V\left(\frac{\vec{x}}{1 - \kappa ct}\right). \quad (23)$$

We will show below that a point charge, for instance, creates field potential of this kind.

Let us finally note that in the central field there is another integral of motion

$$\vec{M} = \frac{\vec{x} \times \vec{p}}{1 - \kappa ct}, \quad (24)$$

which is the projective relativistic generalization of the angular momentum.
4 Electromagnetic Wave Propagation

Let us consider a simple thought experiment. The observer situated at the frame origin \( x = 0 \) sends out a series of equally separated (by time intervals \( \tau \)) light pulses along the \( x \) axis. According to (1), the speed of these signals increases: \( C(t, 0) = c/(1 - \kappa ct) \). The trajectories of two impulses emitted at the moments \( t_0 \) and \( t_0 + \tau \) are, respectively:

\[
\begin{align*}
x_1(t) &= C(t_0, 0)(t - t_0) \\
x_2(t) &= C(t_0 + \tau, 0)(t - t_0 - \tau)
\end{align*}
\]

Consider now a fixed point \( x \) and measure the time \( \Delta t \) between these two signals arrive at it. One can see that the frequency \( \nu_t = 1/\Delta t \) grows with time:

\[
(1 - \kappa ct)\nu_t = (1 - \kappa ct_0)\nu_0,
\]

where \( \nu_0 = 1/\tau \).

The corresponding “wavelength”, that is, the distance between two consequent impulses at the moment \( t \), is:

\[
\lambda_t = x_2(t) - x_1(t) = \frac{C(t_0 + \tau, 0)}{\nu_t}.
\]

Taking the limit \( \tau \to 0 \) and recalling that the light speed is constant along the trajectory of the signal \( C(t_0, 0) = C(t, x) \), we obtain the relation between frequency and wavelength:

\[
\lambda(t, x)\nu(t, x) = C(t, x).
\]

Therefore, in the projective theory of relativity not only the light speed is time-space dependent, so are both the frequency and wavelength of propagating light.

Examine now a more general case of electromagnetic wave propagation. The velocity of light \( \vec{C}(t, \vec{x}) \) is a function of space and time. It is naturally to assume that the field strength in the traveling wave is periodic in both time and space, and its phase is given by

\[
\phi(t, \vec{x}) = \vec{a} \cdot (\vec{x} - \vec{C}(t, \vec{x})t) = \vec{a} \cdot \frac{\vec{x} - c\vec{n}t}{1 - \kappa ct},
\]

where \( \vec{a} = a\vec{n} \) is an arbitrary constant vector, connected to the frequency and wave vector. In other words, we assume that \( \vec{C}(t, \vec{x}) \) is the phase velocity of the wave.

We proceed to determine how the phase (30) is affected by the transformations (3) - (6). It turns out that, under the projective Lorentz transformations (3)-(4), the phase will only then be invariant

\[
\phi(t', \vec{x}') = \phi(t, \vec{x}),
\]

if the quantities \( \vec{n} \) and \( a \) obey the following transformation laws:

\[
\begin{align*}
\vec{n}' &= \frac{1}{\gamma} \left( \vec{n} + (\gamma - 1)\vec{v}(\vec{n}\vec{v})/\nu^2 - \gamma\vec{v}/c \right), \\
a' &= \frac{1 - \vec{v}\vec{n}/c}{\sqrt{1 - \nu^2/c^2}} a,
\end{align*}
\]

where we have used the vector form of the projective Lorentz transformations, given in Appendix 1.
An analogous calculation shows that the wave phase is shifted by a constant under the projective translations:

$$\phi(T, \vec{X}) = \phi(t, \vec{x}) + \phi_0$$  \hspace{1cm} (34)

if the following conditions are satisfied:

$$\vec{N} = \frac{1}{\sigma} \vec{n} + (\sigma - 1) \vec{R}((\vec{n} \vec{R})/R^2 - \sigma \kappa \vec{R})$$  \hspace{1cm} (35)

$$\vec{A} = \frac{1 - \kappa \vec{R} \vec{n}}{\sqrt{1 - (\kappa R)^2}} a,$$  \hspace{1cm} (36)

where $\sigma = 1/\sqrt{1 - (\kappa R)^2}$. Note once again the symmetry of Eqs. (32,33) and (35,36) revealed by the substitution $\vec{v}/c \rightarrow \kappa \vec{R}$. The relation (35) provides the vector form for the light aberration formulae obtained in Ref. [5].

Evidently, the constant vector $\vec{a}$ is related to the wave vector and frequency of the propagating electromagnetic wave. Since the phase $\phi(t, \vec{x})$ depends on the space-time coordinates in a non-linear fashion, we define the wave vector $\vec{k}$ and frequency $\omega$ as follows:

$$\vec{k} = \frac{\partial \phi(t, \vec{x})}{\partial \vec{x}} = \frac{\vec{a}}{1 - \kappa c T},$$  \hspace{1cm} (37)

$$\omega = -\frac{\partial \phi(t, \vec{x})}{\partial t} = ac \frac{1 - \kappa \vec{C} \vec{n}}{(1 - \kappa c T)^2}.$$  \hspace{1cm} (38)

They are interconnected by the relation:

$$\omega(t, \vec{x}) = \vec{k}(t, \vec{x}) \vec{C}(t, \vec{x}).$$  \hspace{1cm} (39)

It follows from Eq. (38) that $\omega = a \vec{n} \vec{C}(t, \vec{x})/(1 - \kappa c T)$ and, taking into account the light speed constancy along the trajectory, we are again led to the conclusion that the speed of light increases with time (27).

Using the equations (32)-(36) one can obtain the transformation rules for frequency and wave vector. For two observers at rest in the same reference frame, space-time translations lead to:

$$\Omega = \frac{1 - \kappa^2 \vec{x} \vec{R}}{\sqrt{1 - (\kappa R)^2}} \omega,$$  \hspace{1cm} (40)

$$(1 - \kappa c T) \vec{K} = (1 - \kappa c T) \left[ \vec{k} + (\sigma - 1) \frac{\vec{R}(\vec{k} \vec{R})}{R^2} - \sigma \kappa \vec{R} \vec{n} \vec{k} \right],$$  \hspace{1cm} (41)

where $(\omega, \vec{k})$ are the frequency and the wave vector measured by the observer at the origin, and $(\Omega, \vec{K})$ - those measured by the one at $\vec{x} = \vec{R}$. It is possible to derive the transformation law for the frequency directly from the coordinate transformations, by setting $\omega = 2\pi/\Delta t$ and assuming that both $\vec{x}$ and $\vec{X}$ are unchanged.

The properties of $\omega$ and $\vec{k}$ under the projective Lorentz transformations can be obtained from Eqs. (40,41) simply by the substitution $\kappa \vec{R} \rightarrow \vec{v}/c$. 


5 The Quantum Theory

Constructing a consistent quantum theory requires a careful consideration of its axiomatic foundation and of the amendments that are necessary within the context of projective relativity. Here, we undertake a more modest task—to consider some heuristic conjectures, which will allow us to obtain quantum mechanical relations that could be expected to hold in the complete theory too.

Let us find what changes are required to the Planck—Einstein relations for the light quanta. To this end, we will proceed from the requirement of relativistic invariance, in analogy with the well-known argumentation by de Broglie. We write the transformations for the momentum (15) in the standard vector form (Appendix 1) and taking into account that Eq. (12) gives \( \varepsilon = c(p\vec{n}) \) when applied to light, we obtain the following relation between the values of the photon momentum, as measured by two observers in the same inertial reference frame:

\[
P = \vec{p} + (\sigma - 1) \frac{\vec{R}(\vec{p}\vec{R})}{R^2} - \sigma \kappa \vec{R}(\vec{n}\vec{p}).
\]  

(42)

Comparing this expression with that for the wave vector (41), we can conclude that the simplest covariant connection between them can be written as

\[
\vec{p} = (1 - \kappa ct)\hbar\vec{k} = \hbar\vec{a}.
\]  

(43)

Note that the momentum of the quantum particle is constant, as it is for the classical one.

Substituting the wave vector (43) expressed in terms of the particle momentum into Eq. (39) and using (12) we obtain a generalized Plank formula:

\[
E = \frac{(1 - \kappa ct)^2}{1 - \kappa \vec{x}\vec{n}} \hbar \omega = \hbar a.
\]  

(44)

The relations (43), (44) are “cosmological” generalizations of the standard quantum-mechanical expressions \( E = \hbar \omega, \vec{p} = \hbar\vec{k} \).

One could also derive these relations within the framework of conventional canonical quantization. Indeed, it follows from Eq. (19), that the canonical momentum is related to the wave vector in a usual way:

\[
\pi = \frac{\partial L}{\partial \dot{u}} = \frac{\vec{p}}{1 - \kappa ct} = \hbar\vec{k}.
\]  

(45)

Besides, the Hamiltonian (which, unlike the energy, is not conserved!) is conveniently connected to the frequency of light emitted during a quantum transition:

\[
H = \pi \dot{u} - L = \frac{E - \kappa \vec{c}\vec{x}\vec{p}}{(1 - \kappa ct)^2} = \hbar \omega.
\]  

(46)

One can speculate that the standard commutation rule holds for the canonical momentum operator:

\[
[\pi_\alpha, x_\beta] = \frac{\hbar}{i} \delta_{\alpha\beta},
\]  

(47)

and, thus, the momentum operator \( p \) introduced above has coordinate representation

\[
\vec{p} = (1 - \kappa ct)\frac{\hbar}{i} \vec{\nabla}.
\]  

(48)
In this case, the phase of the free particle wave function is analogous to the phase of the electromagnetic wave

$$\psi_p \sim \exp \frac{i}{\hbar} \frac{\vec{p} \vec{x} - Et}{1 - \kappa ct}. \quad (49)$$

In Sect. 7 below, we will consider the quantization of a hydrogen atom, and some consequences of space-time scale contraction, which are relevant to this problem.

6 The Redshift

The most spectacular result of the projective theory of relativity is the fact that the frequency of the radiation changes, and this frequency shift increases with the distance from the source. We will rederive this result here, slightly changing the approach pursued in Ref. [5].

Let us place the Earth laboratory at the frame origin $\vec{x} = 0$, and the remote source at a distance $R$ from it ($\vec{x} = \vec{R}$). At the moment when the light signal is emitted by the source, the frequency measured by local and remote observers, $\omega_1$ and $\Omega_1$, are related, according to Eq. (40), by the expression:

$$\Omega_1 = \sqrt{1 - (\kappa R)^2} \omega_1. \quad (50)$$

The frequency of the propagating electromagnetic wave increases (27) and, when the signal reaches the laboratory observer at time $t_2$, becomes $\omega_2$:

$$(1 - \kappa c t_2) \omega_2 = (1 - \kappa c t_1) \omega_1. \quad (51)$$

The relation between the time $t_1$ when the signal was emitted (according to our watch) and the time $t_2$ when it arrived can be easily found if one recalls that the light speed is constant along the trajectory $C(t_1, R) = C(t_2, 0)$, $\vec{n} = -\vec{R}/R$:

$$1 - \kappa c t_1 = (1 + \kappa R)(1 - \kappa c t_2). \quad (52)$$

Now we can obtain the expression linking the frequency $\Omega_1$ measured by the remote observer and the frequency $\omega_2$ of the signal that reaches Earth:

$$\frac{\Omega_1}{\omega_2} = \sqrt{\frac{1 - \kappa R}{1 + \kappa R}}. \quad (53)$$

It was the intention to reproduce the observed Hubble effect that motivated the choice of the negative sign for $\kappa$ in Refs. [3]–[5].

This derivation, however, should be further continued to incorporate the quantum mechanical relations obtained above. When one studies the redshift in the spectra of remote galaxies, one compares the observed frequency $\omega_2$ with the spectral frequencies $\omega_0$ of the identical atoms on the Earth, and not with $\Omega_1$. We thus look at the Universe at a certain moment in the past; the time of emission by the local watch $T_1$ is connected with the present time on the Earth (6), (52) by the formula

$$1 - \kappa c T_1 = \frac{1 - \kappa c t_1}{\sqrt{1 - (\kappa R)^2}} = \sqrt{\frac{1 + \kappa R}{1 - \kappa R}} (1 - \kappa c t_2). \quad (54)$$

Since the frequency of the radiation changes with time, we see that the frequency $\Omega_1$ is lower than that of the same atom on Earth at present:

$$\Omega_1 = \frac{E/\hbar}{(1 - \kappa c T_1)^2}, \quad \omega_0 = \frac{E/\hbar}{(1 - \kappa c t_2)^2}. \quad (55)$$
where $E$ is the radiation energy of one of the two identical motionless atoms in the vicinity of the observer. Therefore,

$$\frac{\Omega_1}{\omega_0} = \frac{1 - \kappa R}{1 + \kappa R}. \quad (56)$$

Substituting this relation into Eq. (53) we finally obtain the correct expression for the redshift in the atomic spectra:

$$\frac{\omega_0}{\omega_2} = \sqrt{1 + \kappa R} - \kappa R. \quad (57)$$

One can see that this formula conforms with the redshift observed in experiment ($\omega_2 > \omega_0$) if one chooses the constant $\kappa$ to be positive, $\kappa > 0$.

If one assumes that the cosmological redshift is completely accounted for by the effects of the projective theory of relativity, the fundamental constant $\kappa$ is directly related to the Hubble constant:

$$\kappa c = H = 65 \text{ km/sec} \text{ Mps} = 6.7 \times 10^{-11} \text{ year}. \quad (58)$$

As we noted above, the velocity of light increases with time and at the moment $t_0 = 1/\kappa c$ it will become infinite. This future event is the singularity of the considered theory. There is no matter singularity in the Universe described by the projective relativity. The Universe existed infinitely long in the past; the gradual change of the light velocity from its initial zero value was accompanied by various evolutionary processes.

A question arises about the behaviour of matter in this Universe. Gravitational forces can contribute to the redshift due to the convenient cosmological model mechanism; this contribution can be of either sign. There is, however, another possibility. Under the influence of gravity, the matter gradually concentrates around initial inhomogeneities, whereas remaining on average at rest at large scales. This process, becoming increasingly faster as the speed of light grows, leads to the formation of stars, galaxies, etc.

### 7 Expanding Universe vs Contracting Space-Time Scales

Let us consider quantization of a hydrogen atom within the framework of the projective theory of relativity. The electromagnetic field action invariant under the projective transformations can be written as

$$S = -mc \int ds - \frac{e}{c} \int A_\nu dx^\nu - \frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} \, d^4x. \quad (59)$$

The components of the metric tensor are, from Eq. (8)

$$g_{00} = \frac{1 - (\kappa \vec{x})^2}{(1 - \kappa ct)^4}, \quad g_{0i} = -\frac{\kappa x^i}{(1 - \kappa ct)^3}, \quad g_{ij} = -\frac{\delta_{ij}}{(1 - \kappa ct)^2} \quad (60)$$

and, so, the invariant four-volume is $d^4x \sqrt{-g} = d^4x/(1 - \kappa)^5$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The field equation describing a set of point charges has the usual form [11]:

$$\partial_\alpha (\sqrt{-g} F^{\alpha\beta}) = 4\pi \sum e_a \delta(x - \vec{x}_a) \frac{dx^3}{dx^3}. \quad (61)$$

When the magnetic field is absent, $A_i = 0$, Eq. (61) is reduced to the Poisson type equation:

$$(1 - \kappa ct) \Delta A_0 = -4\pi e_a \delta(\vec{x}). \quad (62)$$
So, the contribution to the Lagrangian from the interaction between a charged particle and the field of a point charge,

\[ L_{\text{int}} = \frac{e}{c} A_\nu \frac{dx^\nu}{dt} = \frac{e^2}{(1 - \kappa ct) r} \]  

in accordance with (23) results in the conserved energy

\[ E = \sqrt{m^2 c^4 + c^2 \vec{p}^2} - \frac{(1 - \kappa ct) e^2}{r}. \]  

One can note a formal coincidence between the obtained “time-dependence” of charge and Gamow’s hypothesis [17].

The operator expression for the energy eigenvalues equation has the form:

\[ \left[ \sqrt{m^2 c^4 - c^2 \hbar^2 (1 - \kappa ct)^2 \Delta} - \frac{(1 - \kappa ct) e^2}{r} \right] \psi_n = E_n \psi_n. \]  

Performing the substitution \( x_i \rightarrow x_i (1 - \kappa ct) \) one can easily see that energy levels are stationary and given by the standard formula

\[ E_n = \frac{m e^4}{\hbar^2} f_n \left( \frac{e^2}{\hbar c} \right), \]  

while the eigenfunctions are time-dependent. In particular, the mean value of electron position operator in a central field decreases with time:

\[ \langle r \rangle = \langle r \rangle_0 (1 - \kappa ct). \]  

This result is in fact quite general for the closed systems with conserved energy (23). Therefore, all length units that are based upon properties of atomic objects tied by the electromagnetic forces, also decrease with time [18].

If we take the wavelength and frequency of the atomic spectra as the standard length and time, we would have

\[ \Delta \tau(t) = \frac{1}{\omega} = \frac{(1 - \kappa ct)^2}{E/\hbar}, \quad \Delta l(t) = \frac{\lambda}{2\pi} = \frac{1 - \kappa ct}{E/\hbar c}. \]  

and the variation in the light velocity would be non-observable [3]. A note is in order here that, since the standard second determined from frequency shortens with time, the future singularity at \( t = 1/\kappa c \) cannot be reached if we measure time with the clock [18].

On the atomic scale, one could change the notation using the coordinates introduced in Ref. [3]:

\[ \tilde{t} = \frac{t}{1 - \kappa ct}, \quad \tilde{x} = \frac{x}{1 - \kappa ct}. \]  

In terms of these “atomic variables” \( (\tilde{x}, \tilde{t}) \), the transformations between two inertial frames (3), (4) become the conventional Lorentz transformations. The translational transformations between two spatially separated rest observers (5), (6) can also be written in the Lorenz form:

\[ \tilde{X} = \frac{\tilde{x} - R - v \tilde{t}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \tilde{Y} = \tilde{y}, \]  

\[ (70) \]
\[ \hat{T} = \frac{\tilde{t} - \tilde{t}_0 - v\tilde{x}/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

(71)

However, the formal velocity parameter entering these formulae is actually the distance between the observers: \( v/c = \kappa R \), \( \tilde{t}_0 = (1 - \sqrt{1 - (\kappa R)^2})/\kappa c \). Thus, in terms of \((\tilde{x}, \tilde{t})\) the translational transformations describe two observers that are moving away from each other at a speed proportional to the distance between them. In contrast to the original transformations (3 - 6), the interpretation of Eq. (70,71) is not immediately obvious. The fact that the two observers in (3 - 6) have zero relative velocity is verified by repeated measurements and distinguish these observers from the others. The origin of the redshift lies in the time transformation between them. On the other hand, when expressed through variables \((\tilde{x}, \tilde{t})\), this effect has a Doppler nature and is intrinsic to Eqs. (70,71).

One can treat the transformations (70),(71) as the basis of the theory, provided that an additional postulate is introduced: \( v/c = \kappa R \). Indeed, assuming that the Universe is governed by the standard theory of relativity, save that any two observers are moving away from each other at a speed proportional to the distance between them, we obtain the Eq. (70),(71). Let us note that the parameter \( \tilde{t}_0 \) necessarily arises in the described theory. It must be present if we require that the transformations compose a group. Due to its presence, all the observers are equal, and the Big Bang occurs at the same moment according to all local atomic clocks \( \tilde{T}_{BB} = \tilde{t}_{BB} = -1/\kappa c \). In terms of \((x,t)\) this singularity is absent, because it corresponds to the infinitely distant moment in the past.

**Conclusion**

The initial motivation for developing the projective theory of relativity was the desire to build a parametric generalization of special relativity. The basic relations of the theory (3 - 6) were obtained by using only universal arguments and without introducing any additional axioms (the principle of parametric incompleteness). The variability of the light velocity and the cosmological redshift arose within this model as quite unexpected consequences. It was shown in the present paper that, under the plausible assumptions, the sizes of atomic systems decrease with time.

We thus face the well known dilemma of the expanding Universe: either the Universe is unchanged and our standard rules are contracting, or the units of length are constant and the Universe is expanding. Whichever point of view one prefers, there is a redshift in the spectra of distant objects.

The question as to what is the “true” constant length, cannot be unambiguously answered at present. It could be the “radius of the Universe” \( 1/\kappa \), and could be the cesium atomic wavelength. If the space-time is indeed quantized, the size of this quantum would play this role and provide the solution to the outlined problem.

If the projective relativity is indeed realized in the world we observe, the cosmological redshift is as fundamental as the existence of the highest possible speed. In particular, it does not depend on the matter density and the initial conditions in the Universe, and is an intrinsic property of the space itself.

**Acknowledgement**

I would like to thank Prof. Manida for the incisive comments he made in the communication with the author [3]. Some of the result reported here were also independently obtained by Prof. Manida.
Appendix 1.
Some vector relations of the Projective STR

The projective Lorentz transformations can be written in the standard vector form:
\[
\frac{\vec{x}'}{1 - \kappa c t'} = \frac{1}{1 - \kappa c t} \left[ \vec{x} + (\gamma - 1) \frac{\vec{x}(\vec{x} \vec{u})}{v^2} - \gamma \vec{v} t' \right],
\]
\[
\frac{t'}{1 - \kappa c t'} = \gamma \frac{t - \vec{x} \vec{u}/c^2}{1 - \kappa c t}.
\]

The projective translational transformations between two observers at rest are:
\[
\vec{X} = \vec{x} - \vec{R} + \left(1 - \sqrt{1 - (\kappa R)^2} \right) \frac{\vec{R} \times (\vec{R} \times \vec{x})}{R^2},
\]
\[
1 - \kappa c T = \frac{\sqrt{1 - (\kappa R)^2}}{1 - \kappa^2 x R} (1 - \kappa c t).
\]

Under the projective transformations, Manida’s 4-vector
\[
d\tilde{x}^\mu = d \left( \frac{ct}{1 - \kappa c t}, \frac{\vec{x}}{1 - \kappa c t} \right) = \left( \frac{cdt}{(1 - \kappa c t)^2}, \frac{d\vec{x}(1 - \kappa c t) + \kappa c \vec{x} dt}{(1 - \kappa c t)^2} \right)
\]
is transformed by the usual linear Lorenz rules. For the rest observers the transformations have the form:
\[
d\tilde{X}^0 = \sigma \left( d\tilde{x}^0 - \kappa \vec{R} d\vec{x} \right),
\]
\[
d\tilde{\vec{X}} = d\tilde{x} + (\sigma - 1) \frac{\vec{R} (\vec{R} d\vec{x})}{R^2} - \sigma \kappa \vec{R} d\tilde{x}^0,
\]
where \( \sigma = 1/\sqrt{1 - (\kappa R)^2} \).

The transformational rules for the energy and momentum \( p^\nu = mc d\vec{x}/ds = (\varepsilon, \vec{p}) \) under space-time translations can be written as:
\[
E = \sigma \left( \varepsilon - c \kappa \vec{R} p \right),
\]
\[
\vec{P} = \vec{p} + (\sigma - 1) \frac{\vec{R} (\vec{R} d\vec{x})}{R^2} - \sigma \kappa \vec{R} \frac{\varepsilon}{c}.
\]

One can derive the Lorenz rules for \( (\varepsilon, \vec{p}) \) from the Eqs. (73), (80) by substituting \( \kappa \vec{R} \to \vec{v}/c \).

Let us note also the following invariant of translational transformations (74), (75):
\[
\frac{1 - \kappa c T}{\sqrt{1 - (\kappa \vec{X})^2}} = \frac{1 - \kappa c t}{\sqrt{1 - (\kappa \vec{x})^2}} = inv.
\]
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