INDEPENDENCE OF THE TOTAL REFLEXIVITY CONDITIONS FOR MODULES

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Abstract. We show that the conditions defining total reflexivity for modules are independent. In particular, we construct a commutative Noetherian local ring $R$ and a reflexive $R$-module $M$ such that $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$, but $\text{Ext}_R^i(M^*, R) \neq 0$ for all $i > 0$.

INTRODUCTION

Let $R$ be a commutative Noetherian ring. For any $R$-module $M$ we set $M^* = \text{Hom}_R(M, R)$. An $R$-module $M$ is said to be reflexive if it is finite and the canonical map $M \to M^{**}$ is bijective. A finite $R$-module $M$ is said to be totally reflexive if it satisfies the following conditions:

(i) $M$ is reflexive
(ii) $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$
(iii) $\text{Ext}_R^i(M^*, R) = 0$ for all $i > 0$.

This notion is due to Auslander and Bridger [1]: the totally reflexive modules are precisely the modules of $G$-dimension zero. The $G$-dimension of a module is one of the best studied non-classical homological dimensions, and is defined in terms of the length of a resolution of the module by totally reflexive modules.

Given any homological dimension, a serious concern is whether its defining conditions can be verified effectively. For example, the projective dimension of a finite $R$-module $M$ is zero if and only if $\text{Ext}_R^1(M, N) = 0$ for all finite $R$-modules $N$. However, when $R$ is local with maximal ideal $m$, one only needs to check vanishing for $N = R/m$. In the same spirit, it is natural to ask whether the set of conditions defining total reflexivity is overdetermined (cf. [4, §2]) and in particular, whether total reflexivity for a module can be established by verifying vanishing of only finitely many $\text{Ext}$ modules.

When $R$ is a local Gorenstein ring, (ii) implies the other two conditions above, and it is equivalent to $M$ being maximal Cohen-Macaulay. Recently, Yoshino [9] studied other situations when (ii) alone implies total reflexivity, and raised the question whether this is always the case.

In the present paper we give an example of a local Artinian ring $R$ which admits modules whose total reflexivity conditions are independent, in that (ii) implies neither (i) nor (iii); (i) and (ii) do not imply (iii), equivalently, (i) and (iii) do not imply (ii). More precisely, we prove the following result as Theorem 1.7.

Theorem. There exists a local Artinian ring $R$, and a family $\{M_s\}_{s \geq 1}$ of reflexive $R$-modules such that

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Most conditions we will assume tacitly that

\[ \sum_{i \geq 0} \text{rank}_{R}(R_i)t^i = 1 + 4t + 3t^2. \]

The ring \( R \) is thus local, and its maximal ideal \( m \) satisfies \( m^3 = 0 \). Note that our example is minimal in the following sense: if \( m^2 = 0 \), then any finite \( R \)-module \( M \) which satisfies \( \text{Ext}^i_R(M, R) = 0 \) for some \( i > 1 \) is totally reflexive, hence (ii) alone implies total reflexivity. (See 1.1 below.)

Our construction involves a minimal acyclic complex \( C \) of finite free \( R \)-modules such that the sequence \( \{\text{rank}_{R}(C_i)\}_{i \geq 0} \) is strictly increasing and has exponential growth, while the sequence \( \{\text{rank}_{R}(C_{-i})\}_{i \geq 0} \) is constant. In the last section we raise several related questions.

1. Independence

Let \( R \) be a Noetherian commutative ring and \( M \) a finite \( R \)-module. Suppose that \( \phi : G \to F \) is a homomorphism of finite free \( R \)-modules with \( M = \text{Coker} \phi \). The \( R \)-module \( \text{Tr}(M) := \text{Coker} \phi^* \) is called the transpose of \( M \), and it is unique up to projective equivalence; it is thus well-defined in the stable category of \( R \). The \( R \)-modules \( \text{Tr}(\text{Tr}(M)) \) and \( M \) are isomorphic up to projective summands and there is an exact sequence

\[ 0 \to \text{Ext}^1_R(\text{Tr}(M), R) \to M \to M^{**} \to \text{Ext}^2_R(\text{Tr}(M), R) \to 0 \]

where the map in the middle is the natural map. In particular, \( M \) is reflexive if and only if \( \text{Ext}^i_R(\text{Tr}(M), R) = 0 \) for \( i = 1, 2 \). Also note that \( M^* \) is a second syzygy of \( \text{Tr}(M) \), hence \( \text{Ext}_R^2(M^*, R) \cong \text{Ext}_R^{i+2}(\text{Tr}(M), R) \) for all \( i > 0 \). Thus the definition of \( M \) being totally reflexive can be recast as follows:

\[ \text{Ext}_R^i(M, R) = 0 = \text{Ext}_R^i(\text{Tr}(M), R) \]

for all \( i > 0 \).

It is convenient to have a uniform notation for the conditions above: let \( i \in \mathbb{Z} \), \( i \neq 0 \). We say that \( M \) satisfies condition \((\text{TR}_i)\) provided

\[
(\text{TR}_i) : \begin{cases} 
\text{Ext}^i_R(M, R) = 0 & \text{if } i \geq 1 \\
\text{Ext}^{-i}_R(\text{Tr}(M), R) = 0 & \text{if } i \leq -1.
\end{cases}
\]

Thus \( M \) is totally reflexive if and only if \( M \) satisfies \((\text{TR}_i)\) for all \( i \neq 0 \).

Note that we did not define a condition \((\text{TR}_i)\) for \( i = 0 \). When referring to these conditions we will assume tacitly that \( i \neq 0 \).
1.1. Remarks. (1) Assume that $R$ is local Gorenstein. The module $M$ is then totally reflexive if and only if $(TR_i)$ is satisfied for all $i$ with $0 < i < \dim R$.

(2) Assume that $R$ is local, with maximal ideal $m$ and residue field $k$. If $m^2 = 0$, then the first syzygy $N$ in a minimal free resolution of $M$ has $mN = 0$, hence it is a finite dimensional $k$-vector space. If $\Ext^i_R(M, R) = 0$ for some $i > 1$ then either $M$ is free or $\Ext^{i-1}_R(k, R) = 0$. Thus, when $m^2 = 0$ we obtain: $M$ is totally reflexive if and only if $(TR_i)$ is satisfied for some $i$ with $i > 1$ or $i < -1$.

(3) Assume that $R$ is a commutative Noetherian ring. Yoshino proved in [9] that if the full subcategory of $R$-modules $M$ with $\Ext^i_R(M, R) = 0$ for all $i > 0$ is of finite type, then a module $M$ is totally reflexive if and only if it is an object of this subcategory.

The remarks above lead to the question: How many conditions $(TR_i)$ does one need to check for total reflexivity, without placing extra assumptions on the ring? Is it possible that only finitely many suffice? Is it enough to check $(TR_i)$ for all $i > 0$, or, more generally, for all $i > s$ for some integer $s$?

Theorem 1.7 (stated also in the introduction) provides negative answers to these questions: in general, one needs to check the conditions $(TR_i)$ for infinitely many positive values of $i$ and infinitely many negative values of $i$.

We now describe the ring of Theorem 1.7. Related rings were used in [6] and then in [7] to disprove various conjectures.

Let $k$ be a field which is not algebraic over a finite field and let $\alpha \in k$ be an element of infinite multiplicative order. For the remainder of this section we assume the ring $R$ to be defined as follows.

1.2. Consider the polynomial ring $k[V, X, Y, Z]$ in four variables (each of degree one) and set

$$R = k[V, X, Y, Z]/I,$$

where $I$ is the ideal generated by the following quadratic relations:

$$V^2, Z^2, XY, VX + \alpha XZ, VY + YZ, VX + Y^2, VY - X^2.$$

As a vector space over $k$, it has a basis consisting of the following 8 elements:

$$1, v, x, y, z, vx, vy, vz,$$

where $v, x, y, z$ denote the residue classes of the variables modulo $I$. In particular, $R$ has Hilbert series $H_R(t) = 1 + 4t + 3t^2$.

1.3. Remark. One may check that the generators for $I$ listed above are a Gröbner basis for $I$. Therefore by [5] Section 4, the ring $R$ is Koszul, and it follows that the Poincaré series of its residue field $k$ is $\sum_i \rank_k \Tor_i^R(k, k)t^i = (1 - 4t + 3t^2)^{-1}$.

For each integer $i \leq 0$ we let $d_i : R^2 \to R^2$ denote the map given with respect to the standard basis of $R^2$ by the matrix

$$\begin{pmatrix} v & \alpha^{-i}x \\ y & z \end{pmatrix}.$$

Also, let $d_1 : R^3 \to R^2$ denote the map represented by the matrix

$$\begin{pmatrix} v & \alpha^{-1}x & yz \\ y & z & 0 \end{pmatrix}.$$
Consider a minimal free resolution of $\text{Coker } d_2$ with $d_2$ as the first differential:
\[ \cdots \to R^{b_3} \xrightarrow{d_3} R^{b_2} \xrightarrow{d_2} R^3 \xrightarrow{d_1} R^2 \xrightarrow{d_0} R^1 \xrightarrow{d_{-1}} R^0 \xrightarrow{d_{-2}} R \to \cdots, \]
where for each $i \geq 3$ the map $d_i : R^i \to R^{i-1}$ denotes the $(i-1)$st differential in this resolution.

1.4. Lemma. The sequence of homomorphisms:
\[ C : \cdots \to R^{b_3} \xrightarrow{d_3} R^2 \xrightarrow{d_2} R \to \cdots \]

is a doubly infinite complex with $H_i(C) = 0$ for all $i \in \mathbb{Z}$.

**Proof.** (1). The defining equations of $R$ guarantee that $d_i d_i = 0$ for all $i \leq 2$. For $i \geq 2$ the maps $d_i$ are differentials in a free resolution, hence the equality holds for all $i \geq 3$, as well. We conclude that $C$ is a complex and $H_i(C) = 0$ for all $i \geq 2$.

We let $(a, b)$ denote an element of $R^2$ written in the standard basis of $R^2$ as a free $R$-module. For each $i \leq 0$ the $k$-vector space $\text{Im } d_i$ is generated by the elements:
\[
\begin{align*}
&d_i(1, 0) = (v, y) & d_i(z, 0) = (vz, -vy) \\
&d_i(0, 1) = (\alpha^{-1} x, z) & d_i(0, v) = (\alpha^{-1} vx, vz) \\
&d_i(v, 0) = (0, vy) & d_i(0, x) = (\alpha^{-1} vy, -\alpha^{-1} vx) \\
&d_i(x, 0) = (vx, 0) & d_i(0, y) = (0, -vy) \\
&d_i(y, 0) = (vy, -vx) & d_i(0, z) = (-\alpha^{-1} vx, 0)
\end{align*}
\]

It is clear that $\text{rank}_k(\text{Im } d_i) = 8$ for all $i \leq 0$. Since $\text{rank}_k(R^2) = 16$, this implies that $\text{rank}_k(\text{Ker } d_i) = 8$ for all $i \leq 0$, showing that $H_i(C) = 0$ for all $i \leq -1$.

Notice that for $i = 1$ the elements above give 7 linearly independent elements in $\text{Im } d_1$, and the 8th can be taken to be $z(0, 0, 1) = (yz, 0)$. (Here $(a, b, c)$ denotes an element of $R^3$ in its standard basis as a free $R$-module.) Thus $\text{rank}_k(\text{Im } d_1) \geq 8$, and so $\text{rank}_k(\text{Ker } d_1) \leq 16$. In particular, we obtain $H_0(C) = 0$.

To prove $H_1(C) = 0$ we need to show that $\text{rank}_k(\text{Im } d_2) \geq 16$. Indeed, the following elements in $\text{Im } d_2$ are linearly independent:
\[
\begin{align*}
&d_2(e_1) = (v, y, 0) & d_2(ve_4) = (0, 0, vy) \\
&d_2(e_2) = (\alpha^{-2} x, z, 0) & d_2(ve_4) = (0, 0, vz) \\
&d_2(e_3) = (-y, ax, 0) & d_2(xe_1) = (vx, 0, 0) \\
&d_2(e_4) = (0, 0, v) & d_2(ve_1) = (vy, -vx, 0) \\
&d_2(e_5) = (0, 0, x) & d_2(ve_1) = (vx, -vy, 0) \\
&d_2(e_6) = (0, 0, y) & d_2(ve_2) = (0, vy, 0) \\
&d_2(e_7) = (0, 0, z) & d_2(ve_2) = (\alpha^{-2} vx, vz, 0) \\
&d_2(xe_4) = (0, 0, vx) & d_2(xe_2) = (\alpha^{-2} vy, -\alpha^{-1} vx, 0)
\end{align*}
\]

where $e_1, \ldots, e_7$ denote the elements comprising the standard basis of $R^7$ as a free $R$-module. \(\square\)
If \( f: M \to N \) is a homomorphism of \( R \)-modules, we let \( f^* \) represent the induced map \( \text{Hom}_R(f, R): \text{Hom}_R(M, R) \to \text{Hom}_R(N, R) \). If \( (D, \delta) \) is a complex of \( R \)-modules, then the complex \( (D^*, \delta^*) \) has \( (D^*)_i = (D_{-i})^* \) and differentials \( (\delta^*)_i = (\delta_{-i})^* \). We write \( \delta^*_i \) for \( (\delta^*)_i \).

Note that, upon identification of \( R^* \) with \( R \), the map \( d^*_i: R^2 \to R^2 \) for \( i \geq 0 \) is given in the standard basis of \( R^2 \) by the matrix

\[
\begin{pmatrix} v & y \\ \alpha^i x & z \end{pmatrix}.
\]

Similarly, the maps \( d^*_{-1} \) and \( d^*_{-2} \) are given by the transposes of the matrices defining \( d_1 \) and \( d_2 \), respectively.

1.5. Lemma. The complex

\[
C^*: \cdots \to R^2 \xrightarrow{d_2^*} R^2 \xrightarrow{d_1^*} R^2 \xrightarrow{d_0^*} R^2 \xrightarrow{d_{-1}^*} R^2 \xrightarrow{d_{-2}^*} R^2 \xrightarrow{d_{-3}^*} R^2 \xrightarrow{d_{-4}^*} \cdots
\]

satisfies \( H_i(C^*) = 0 \) if and only if \( i \geq 1 \).

Proof. As a \( k \)-vector space, \( \text{Im} d_i^* \) for \( i \geq 0 \) is generated by the following elements:

\[
\begin{align*}
&d_1^*(1, 0) = (v, \alpha^i x) \quad d_1^*(z, 0) = (vz, -\alpha^{-i} vz) \\
&d_1^*(0, 1) = (y, z) \quad d_1^*(0, v) = (vy, vz) \\
&d_1^*(v, 0) = (0, \alpha^i vx) \quad d_1^*(0, x) = (0, -\alpha^{-i} vx) \\
&d_1^*(x, 0) = (vx, \alpha^i vy) \quad d_1^*(0, y) = (-vx, -vy) \\
&d_1^*(y, 0) = (vy, 0) \\
&d_1^*(y, 0) = (-vy, 0)
\end{align*}
\]

One can see therefore that \( \operatorname{rank}_k(\text{Im} d_1^*) = 8 \) if \( i \geq 1 \) and \( \operatorname{rank}_k(\text{Im} d_0^*) = 7 \). This implies that \( \operatorname{rank}_k(\text{Ker} d_1^*) = 8 \) if \( i \geq 1 \) and \( \operatorname{rank}_k(\text{Ker} d_0^*) = 9 \), and it follows that \( H_i(C^*) = 0 \) for all \( i \geq 1 \) and \( H_0(C^*) \neq 0 \).

For the proof that \( H_i(C^*) \neq 0 \) for \( i = -1, -2 \), note that the image of \( d_{-1}^* \) is generated as a \( k \)-vector space by the following elements.

\[
\begin{align*}
&d_{-1}^*(1, 0) = (v, \alpha^{-i} x, yz) \quad d_{-1}^*(z, 0) = (vz, -\alpha^{-2i} vz, 0) \\
&d_{-1}^*(0, 1) = (y, z, 0) \quad d_{-1}^*(0, v) = (vy, vz, 0) \\
&d_{-1}^*(v, 0) = (0, \alpha^{-i} vx, 0) \quad d_{-1}^*(0, x) = (0, -\alpha^{-i} vx, 0) \\
&d_{-1}^*(x, 0) = (vx, \alpha^{-i} vy, 0) \quad d_{-1}^*(0, y) = (-vx, -vy, 0) \\
&d_{-1}^*(y, 0) = (vy, 0, 0) \quad d_{-1}^*(0, z) = (-vy, 0, 0)
\end{align*}
\]

One sees easily that \( \operatorname{rank}_k(\text{Im} d_{-1}^*) \leq 8 \). Therefore \( \operatorname{rank}_k(\text{Ker} d_{-1}^*) \geq 8 \), and so \( H_{-1}(C^*) \neq 0 \). Clearly \( \operatorname{rank}_k(\text{Ker} d_{-2}^*) \) consists of at least nine linearly independent elements, namely the nine quadric elements in \( R_2^3 \). This shows that \( H_{-2}(C^*) \neq 0 \).

Finally, we note from the matrix representing \( d_2 \) that \( \operatorname{Coker} d_2 \cong N \oplus k \), for some finite \( R \)-module \( N \). Therefore \( H_i(C^*) \cong \operatorname{Ext}_R^{i-2}(N \oplus k, R) \neq 0 \) for all \( i \leq -3 \), since \( R \) is not Gorenstein.

1.6. For each integer \( s \geq 1 \) let \( M_s \) be the cokernel of the map \( d_{-s}: R^2 \to R^2 \). Using Lemma 1.4, note that

\[
M_s = \operatorname{Coker}(d_{-s}) \cong \text{Im}(d_{-s-1}) = \text{Ker}(d_{-s-2})
\]
and a truncation of the complex \( C \) gives the beginning a minimal free resolution of the \( R \)-module \( M_s \):
\[
\cdots \to R^7 \overset{d_7}{\to} R^6 \overset{d_6}{\to} R^5 \overset{d_5}{\to} R^4 \overset{d_4}{\to} R^3 \overset{d_3}{\to} R^2 \to M_s \to 0.
\]
The proof of Lemma \ref{lem:tr} shows that \( M_s \) has Hilbert series \( H_{M_s}(t) = 2t + 6t^2 \).

We are now ready to state our main theorem:

\section{1.7. Theorem} For the family of \( R \)-modules \( \{ M_s \}_{s \geq 1} \) defined above, we have:

1. \( M_s \) satisfies \((\text{TR}_i)\) if and only if \( i < s \).
2. \( \text{Tr}(M_s) \) satisfies \((\text{TR}_i)\) if and only if \( i > -s \).

Note that this contains the Theorem stated in the introduction: indeed, one can take there the modules \( M_s \) to be as above and \( L = \text{Tr}(M_1) \).

\textbf{Proof of Theorem \ref{thm:main}} The second part of the theorem follows from the first part and the simple fact that a finite \( R \)-module \( N \) satisfies the condition \((\text{TR}_i)\) if and only if \( \text{Tr}(N) \) satisfies \((\text{TR}_{-i})\).

To compute \( \text{Ext}^i_R(N, R) \) for an \( R \)-module \( N \) we take a minimal free resolution of \( N \), we apply \((-)^*\) to it, and then compute homology of the resulting complex.

Applying \((-)^*\) to the minimal free resolution of \( M_s \) given in \ref{lem:tr} and identifying \( R \) with \( R^s \), one obtains the complex
\[
\cdots \to R^2 \overset{d^*_7}{\to} R^2 \overset{d^*_6}{\to} R^2 \to \cdots \to R^2 \overset{d^*_1}{\to} R^2 \overset{d^*_0}{\to} R^2 \overset{d^*_2}{\to} R^3 \overset{d^*_3}{\to} R^2 \overset{d^*_4}{\to} R^2 \to \cdots
\]
Lemma \ref{lem:tr} shows that \( \text{Ext}^i_R(M_s, R) = 0 \) for all \( 1 \leq i \leq s - 1 \), and \( \text{Ext}^i_R(M_s, R) \neq 0 \) for \( i \geq s \).

A minimal free resolution of \( \text{Tr}(M_s) \) is given by
\[
\cdots \to R^2 \overset{d^*_{s+1}}{\to} R^2 \overset{d^*_s}{\to} R^2,
\]
and applying \((-)^*\) we get
\[
R^2 \overset{d^{-s}}{\to} R^2 \overset{d^{-s-1}}{\to} R^2 \to \cdots.
\]
Lemma \ref{lem:tr} shows that \( \text{Ext}^i_R(\text{Tr}(M_s), R) = 0 \) for all \( i \geq 0 \). This establishes (1), and hence the entire theorem. \hfill \Box

\section{2. Dependence}

Theorem \ref{thm:main} shows that the conditions \((\text{TR}_i)\) are, to a large extent, independent. However, in the Artinian graded case, there is some overlap between these conditions.

Let \( R \) be a standard graded ring and \( M \) a finite graded \( R \)-module. As noted by Avramov and Martsinkovsky in \ref{avramov-martsinkovsky}, the module \( M \) is totally reflexive if and only if it satisfies \((\text{TR}_i)\) for all \( i \neq -1 \) if and only if it satisfies \((\text{TR}_i)\) for all \( i \neq -2 \). Thus, the condition \((\text{TR}_i)\) for \( i = -1 \) is a consequence of the condition \((\text{TR}_i)\) for all \( i \neq -1 \), and the condition \((\text{TR}_i)\) for \( i = -2 \) is a consequence of the condition \((\text{TR}_i)\) for all \( i \neq -2 \).

The result above is based on a formula obtained by Avramov, Buchweitz and Sally in \ref{avramov-buchweitz-sally}. Buchweitz pointed out to us that the same formula also yields the following.
2.1. Proposition. Assume that $R$ is an Artinian standard graded ring, and $M$ is a finitely generated graded $R$-module. Let $A$ be a finite set of integers of the same parity. If $M$ satisfies $(\text{TR}_i)$ for all $i \in \mathbb{Z} \setminus A$, then the module $M$ is totally reflexive.

Proof. We may assume $0 \notin A$. Suppose that $M$ satisfies $(\text{TR}_i)$ for all $i \in \mathbb{Z} \setminus A$. Since $A$ is a finite set, we conclude that $M$ satisfies $(\text{TR}_i)$ for all $i$ with $|i| \geq 0$. This allows us to use the main formula in [3] which asserts equalities of rational functions

$$\sum_{n \in \{0\} \cup A} (-1)^n H_{\text{Ext}_R^n(M,R)}(t) = \frac{H_M(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} \quad (1)$$

$$\sum_{-n \in \{0\} \cup A} (-1)^n H_{\text{Ext}_R^n(M^*,R)}(t) = \frac{H_{M^*}(t^{-1}) \cdot H_R(t)}{H_R(t^{-1})} \quad (2)$$

We set $P(t) = \sum_{n \in A} H_{\text{Ext}_R^n(M,R)}(t)$ and $Q(t) = \sum_{-n \in A} H_{\text{Ext}_R^n(M^*,R)}(t)$.

Let $\sigma = 0$ if $A$ contains only odd integers and $\sigma = 1$ if $A$ contains only even integers. The formulas (1) and (2) give then

$$H_{M^*}(t) = \frac{H_{M}(t^{-1}) \cdot H_{R}(t)}{H_{R}(t^{-1})} + (-1)^\sigma P(t)$$

$$H_{M^{**}}(t) = \frac{H_{M^*}(t^{-1}) \cdot H_{R}(t)}{H_{R}(t^{-1})} + (-1)^\sigma Q(t)$$

Substituting the formula for $H_{M^*}(t^{-1})$ given by the first equation into the second equation, we obtain:

$$H_{M^{**}}(t) = H_{M}(t) + \frac{H_{R}(t)}{H_{R}(t^{-1})} \cdot (-1)^\sigma P(t^{-1}) + (-1)^\sigma Q(t)$$

and it follows that

$$H_{R}(t) \cdot P(t^{-1}) + H_{R}(t^{-1}) \cdot Q(t) = (-1)^\sigma H_{R}(t^{-1})(H_{M^{**}}(t) - H_{M}(t)).$$

Assume that $A$ contains only odd integers. We have then $-2 \notin A$, hence $(\text{TR}_i)$ is satisfied for $i = -2$. The map $M \to M^{**}$ is then surjective, implying a coefficientwise inequality $H_{M^{**}}(t) \leq H_{M}(t)$. Since $\sigma = 0$ in this case and $H_{R}(t^{-1})$ has positive coefficients, it follows that the Laurent polynomial on the right has nonpositive coefficients.

Both terms of the left-hand side sum are Laurent polynomials with nonnegative coefficients, and it follows that $P(t) = 0$ and $Q(t) = 0$, implying that $\text{Ext}_R^n(M,R) = 0$ for all $n \in A$ and $\text{Ext}_R^n(M^*,R) = 0$ for all $n$ with $-n \in A$. In conclusion, $(\text{TR}_i)$ is satisfied for all $i \neq -1$. Furthermore, we conclude from the formula above that $H_{M^{**}}(t) = H_{M}(t)$. Since the map $M \to M^{**}$ is surjective, it follows that it is an isomorphism, hence $(\text{TR}_i)$ is satisfied for $i = -1$ as well.

Proceed similarly when $A$ contains only even integers.  

Proposition 2.1 leads to the following question:

Question. Let $R$ be a commutative (local) Artinian ring. If a finite $R$-module $M$ satisfies $(\text{TR}_i)$ for all but finitely many values of $i$, does it follow that $M$ is totally reflexive?
3. Minimal acyclic complexes of free modules

Let $S$ be a commutative Noetherian local ring with maximal ideal $n$. A complex $F$ of free $S$-modules

$$
\cdots \rightarrow F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \rightarrow \cdots
$$

is said to be minimal if $\phi_i(F_i) \subseteq nF_{i-1}$ for all $i$. The complex $F$ is acyclic if $H_i(F) = 0$ for all $i \in \mathbb{Z}$. For any minimal acyclic complex of finite free $S$-modules $F$ we can consider two sequences:

$$
\beta^+_F := \{\text{rank}_R F_i\}_{i \geq 0} \quad \text{and} \quad \beta^-_F := \{\text{rank}_R F_{i-1}\}_{i \geq 0}
$$

Assuming that $F_i \neq 0$ for all $i$, it is then natural to ask whether these two sequences have similar asymptotic behavior.

A sequence $\{\beta_i\}_{i \geq 0}$ is said to have exponential growth if there exist numbers $1 < A \leq B$ such that inequalities $A^i \leq \beta_i \leq B^i$ hold for all $i \gg 0$.

When the maximal ideal of $S$ satisfies $n^3 = 0$, Lescot [8] proved that the Betti numbers of a finitely generated $S$-module $N$ are either eventually stationary, or they have exponential growth; in the last case they are eventually strictly increasing. It is clear from the Poincaré series given in 1.2 that the Betti numbers of $k$ over our ring $R$ have exponential growth. Furthermore, with $d_2$ as defined there, since $\text{Coker}(d_2)$ has a copy of $k$ as a direct summand, its Betti numbers have exponential growth and are eventually strictly increasing.

In conclusion, the complex $C$ of Lemma 1.4 has the following properties:

(a) $\beta^+_C$ has exponential growth and is eventually strictly increasing.

(b) $\beta^-_C$ is constant (nonzero).

Several questions arise:

Question. Does there exist a ring $S$ as above and a minimal acyclic complex of free nonzero $S$-modules $F$ such that $\beta^-_F$ has exponential growth (or is eventually strictly increasing) and $\beta^+_F$ is eventually constant?

Question. Do there exist examples of different asymptotic behavior for $\beta^-_F$ and $\beta^+_F$ if we also require $H(F^* ) = 0$? Can such examples exist over a Gorenstein ring?

The last question is equivalent to asking whether the Betti numbers of $M$ and $M^*$ can have different asymptotic behavior when $M$ is totally reflexive, and, in particular, when $S$ is Gorenstein. Theorem 5.6 of [2] shows that the answer to this question is “no” when $S$ is a complete intersection.

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