Goldstone Boson’s Valence-Quark Distribution*

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Dynamical chiral symmetry breaking (DCSB) is one of the keystones of low-energy hadronic phenomena. Dyson-Schwinger equations provide a model-independent quark-level understanding and correlate that with the behaviour of the pion’s Bethe-Salpeter amplitude. This amplitude is a core element in the calculation of pion observables and combined with the dressed-quark Schwinger function required by DCSB it yields a valence-quark distribution function for the pion that behaves as $(1 - x)^2$ for $x \sim 1$, in accordance with perturbative analyses. This behaviour can be verified at contemporary experimental facilities.

1. DCSB and the Gap Equation

Dynamical chiral symmetry breaking (DCSB) is a signature characteristic of the strong interaction spectrum and it is a feature of QCD that can only be understood via nonperturbative analysis. In this connection one particularly insightful tool is the QCD gap equation; i.e., the Dyson-Schwinger equation (DSE) \[ \square \] for the dressed-quark self-energy:

\[
S(p)^{-1} = Z_2 (i\gamma \cdot p + m_{\text{bare}}) + Z_1 \int_q g^2 D_{\mu \nu}(p-q) \frac{\lambda^a}{2} \gamma_\mu S(q) \Gamma^a_\nu(q,p). \tag{1}
\]

In Eq. (1), $D_{\mu \nu}(k)$ is the renormalised dressed-gluon propagator; $\Gamma^a_\nu(q;p)$ is the renormalised dressed-quark-gluon vertex; $m_{\text{bare}}$ is the $\Lambda$-dependent current-quark bare mass that appears in the Lagrangian; and $\int_q^A := \int_q^A d^4q/(2\pi)^4$ represents mnemonically a translationally-invariant regularisation of the integral, with $\Lambda$ the regularisation mass-scale. Also, $Z_1(\zeta^2, \Lambda^2)$ and $Z_2(\zeta^2, \Lambda^2)$ are the quark-gluon-vertex and quark wave function renormalisation constants, which depend on the renormalisation point, $\zeta$, and the regularisation mass-scale, as does the mass renormalisation constant:

\[
Z_m(\zeta^2, \Lambda^2) = Z_A(\zeta^2, \Lambda^2)/Z_2(\zeta^2, \Lambda^2), \tag{2}
\]

with the renormalised mass given by $m(\zeta) := m_{\text{bare}}(\Lambda)/Z_m(\zeta^2, \Lambda^2)$.

The solution of Eq. (1) has the form

\[
S(p)^{-1} = i\gamma \cdot p A(p^2, \zeta^2) + B(p^2, \zeta^2) = \frac{1}{Z(p^2, \zeta^2)} [i\gamma \cdot p + M(p^2, \zeta^2)], \tag{4}
\]

where the functions $A(p^2, \zeta^2)$, $B(p^2, \zeta^2)$ express the effects of dressing induced by the quark’s interaction with its own gluon field. Equation (4) must be solved subject to a renormalisation condition, and in QCD it is practical to impose the requirement that at a large spacelike $\zeta^2$

\[
S(p)^{-1}|_{p^2 = \zeta^2} = i\gamma \cdot p + m(\zeta). \tag{5}
\]

A weak coupling expansion of the DSEs reproduces every diagram in perturbation theory and in connection with Eq. (1) this means that at large spacelike-$p^2$ the solution for massive quarks is, in Landau gauge,

\[
M(p^2) = \frac{\bar{m}}{\left(\frac{1}{2} \ln \frac{p^2/\Lambda_{\text{QCD}}}{\bar{m}}\right)^\gamma}, \tag{6}
\]

\[\star\]

These observations are also true of confinement but model-independent statements harder to make.

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\( \gamma_m = \frac{12}{(33 - 2N_f)} \), where \( \tilde{m} \) is the renormalisation-point-independent current-quark mass, and the mass renormalisation constant, Eq. (B), is \( Z_m(\zeta^2, \Lambda^2) = \left[ \alpha(\Lambda^2)/\alpha(\zeta^2) \right]^{\gamma_m} \). At one-loop \( Z_2 \equiv 1 \equiv Z(p^2) \). In perturbation theory each contribution to the dressed-quark mass function is proportional to the current-quark mass and hence \( M(p^2) \equiv 0 \) in the chiral limit, which is unambiguously defined by \( \tilde{m} = 0 \).

Dynamical chiral symmetry breaking is the appearance of a \( B(p^2, \zeta^2) \neq 0 \) solution of Eq. (4) in the chiral limit. This guarantees a nonzero value of the vacuum quark condensate [3].

\[-\langle \bar{q}q \rangle^0 = Z_1(\zeta^2, \Lambda^2) N_c \text{tr}_D \int_q^\Lambda S^0(q, \zeta), \quad (7)\]

where \( \text{tr}_D \) identifies a trace over Dirac indices only and the superscript "0" indicates the quantity was calculated in the chiral limit. (NB. The factor of \( Z_1 \) here guarantees that the condensate is gauge-parameter- and cutoff-independent. Its omission yields a formula that is incorrect.)

As I have just observed, \( B(p^2, \zeta^2) \neq 0 \) is impossible in perturbation theory. Hence a nonperturbative analysis of QCD’s gap equation is required to explore this possibility. To arrive at model-independent conclusions a systematic, symmetry-preserving truncation scheme for the \( n \)-point functions in the gap equation must be used.

1.1. Goldstone’s Theorem

At least one such scheme is known [3] and it has been used to good effect. For example, the pion’s Bethe-Salpeter amplitude has the general form

\[ \Gamma_\pi^j(k; P) = \tau^j \gamma_5 \left[ iE_\pi(k; P) + \gamma \cdot PF_\pi(k; P) + \gamma \cdot k k \cdot P G_\pi(k; P) + \sigma_{\mu\nu} k_\mu P_\nu H_\pi(k; P) \right], \quad (8)\]

where \( P \) is the pion’s total momentum and \( k \) is the relative momentum between the bound state’s constituents. The scheme of Ref. [3] guarantees that the axial-vector Ward-Takahashi identity (here written for \( \tilde{m} = 0, k_\pm = k \pm P/2)\):

\[-iP_\mu \Gamma^{\mu}_\pi(k; P) = S^{-1}(k_+) \gamma_5 \frac{\tau^j}{2} + \gamma_5 \frac{\tau^j}{2} S^{-1}(k_-), \quad (9)\]

where

\[ \Gamma^{\mu}_\pi(k; P) = \frac{\gamma^j}{2} \gamma_5 \gamma_\mu F_R(k; P) + \gamma \cdot k \sigma_{\mu\nu} k_\nu H_R(k; P), \quad (10)\]

with \( F_R(k; P), G_R(k; P), H_R(k; P) \) regular at \( P^2 = 0 \), is satisfied at every order. This identity, which is fundamental to the successful application of chiral perturbation theory, leads to the following quark-level Goldberger-Treiman relations [3], which are exact in QCD:

\[ f^0 \bar{E}_\pi(k; 0) = B(k^2), \quad (11)\]

\[ f^0 \bar{F}_\pi(k; 0) = A(k^2), \quad (12)\]

\[ f^0 \bar{G}_\pi(k; 0) = 2A'(k^2), \quad (13)\]

\[ f^0 \bar{H}_\pi(k; 0) = 0, \quad (14)\]

where \( f^0 \) is the pion’s chiral-limit leptonic decay constant, obtained in general from

\[ \delta^{ij} f_\pi^j P_\mu = \int_q^\Lambda \text{tr} \left[ \frac{\tau^i}{2} \gamma_5 \gamma_\mu S(q, \zeta) \Gamma^j_\pi(q; P) S(q_-) \right], \quad (15)\]

with the factor of \( Z_2 \) on the right-hand-side guaranteeing that \( f_\pi \) is independent of the renormalisation point, cutoff and gauge parameter. (A formula without \( Z_2 \) is incorrect.) Equations (11)–(14) form an essential part of the proof [3] that, if, and only if, a theory exhibits DCSB then the homogeneous, isovector, pseudoscalar Bethe-Salpeter equation has a massless, \( P^2 = 0 \), solution and the axial-vector vertex is dominated by the pion pole for \( P^2 \simeq 0 \); i.e., a proof of Goldstone’s theorem in QCD.

1.2. A Pseudoscalar Meson Mass Formula

Similar analysis of the \( \tilde{m} \neq 0 \) axial-vector Ward-Takahashi identity yields the following mass formula for flavour nonsinglet pseudoscalar mesons:

\[ f_H^2 \bar{m}_H^2 = -\langle \bar{q}q \rangle^H \mathcal{M}_H^H, \quad (16)\]

where: \( \mathcal{M}_H^H = m_{q_1}^2 + m_{q_2}^2 \) is the sum of the current-quark masses of the meson’s constituents; \( f_H \) is the meson’s leptonic decay constant, obtained via obvious analogy with Eq. (8); and the
in-meson condensate is defined via

$$i \frac{1}{f_H} \langle \bar{q} q \rangle^H =$$

$$Z_4 \int_q \Lambda \left[ (\gamma^\mu \gamma^\nu \sigma_{\mu \nu}) (q_{\perp}) \right],$$

(17)

with, e.g., $T^{a+} = (\lambda^a + i\lambda^3)/2$, where $\{\lambda^a, a = 1, \ldots, 8\}$ are the Gell-Mann matrices, and $\langle . \rangle^H$ denoting matrix transpose.

A number of observations are now in order. Equation (14) makes plain that in the presence of DCSB the magnitude of $f_\pi$ is set by the constituent-quark mass: $M(p^2 \approx 0)$. The same is true of $\langle \bar{q} q \rangle^H$, which, in the chiral limit, is identical to the vacuum quark condensate, Eq. (7). This hints at a corollary of Eq. (16): in the chiral limit it yields the Gell-Mann–Oakes–Renner relation, a primary element in the application of chiral perturbation theory to light-quark mesons. Furthermore, while it is not readily apparent from the material presented heretofore, the DSE derivation of Eq. (16) assumes nothing about the magnitude of $m_{q_{\perp}} \approx 1$ GeV, tuned to fit experimental values of $m_\pi$, $f_\pi$, $m_K$. This fit yields: the value of $\sigma_{\Delta} \approx 70 \Lambda^2_{QCD}$, Eq. (20): $m^{1\text{GeV}}_u = 5.5 \text{MeV}$, $m^{1\text{GeV}}_s = 124 \text{MeV}$; and a value of the vacuum quark condensate:

$$-\langle \bar{q} q \rangle^0 = (0.242 \text{GeV})^3,$$

(22)

for which a comparison with the calculated value

$$-\langle \bar{q} q \rangle^1 = (0.245 \text{GeV})^3$$

(23)

emphasises just how close one is to the chiral limit when dealing with pion observables. In addition, the essential character of the dressed-quark
propagator predicted by the model, which underlies these condensate values, has recently been confirmed in numerical simulations of lattice-QCD [13].

Fixing the single parameter at this fitted value, many observables are predicted and the model achieves a r.m.s. error over predicted quantities of $\lesssim 4\%$ [15]. Furthermore, it is the only model to predict a behaviour for the pion’s electromagnetic form factor that agrees with the results of a recent Hall C experiment [16]. The large-$Q^2$ behaviour of the form factor can be obtained algebraically and one finds $Q^2 F_\pi(Q^2) = \text{const.}$, up to logarithmic corrections, in agreement with the perturbative-QCD expectation. This result relies on the presence of pseudovector components in the pion’s Bethe-Salpeter amplitude, which is guaranteed by the quark-level Goldberger-Treiman relations in Eqs. (12), (13).

The unification of light- and heavy-meson masses via the mass formula in Eq. (16) has also been quantitatively explored using the model of Refs. [4,7–9]. This is illustrated in Fig. 1 wherein the calculated mass of a $u\bar{q}$ pseudoscalar meson is plotted as a function of $m_\zeta$, with $m_\zeta$ fixed to a value corresponding to that in Eq. (21). The fitted curve is, in MeV [13]:

$$m_H = 83 + 500\sqrt{X} + 310X, \quad X = m_\zeta^2/\Lambda_{\text{QCD}}, (24)$$

with the renormalisation mass-scale $\zeta = 19$ GeV and $\Lambda_{\text{QCD}} = 0.234$ GeV.

In this figure the curvature appears slight but that is misleading: the nonlinear term in Eq. (24) accounts for almost all of $m_\pi$ (the Gell-Mann–Oakes-Renner relation is nearly exact for the pion) and 80% of $m_K$. NB. The dashed line in Fig. 1 fits the $K$, $D$, $B$ subset of the data exactly. It is drawn to illustrate how easily one can be misled: without careful calculation one might infer from this apparent agreement that the large-$m_q$ limit of Eq. (16) is already manifest at the $s$-quark mass. However, in reality, the linear term only becomes dominant for $m_q \gtrsim 1$ GeV, providing 50% of $m_D$ and 67% of $m_B$. The model predicts $m^{1\text{GeV}}_s = 1.1$ GeV and $m^{1\text{GeV}}_b = 4.2$ GeV, values that are typical of Poincaré covariant treatments of heavy-meson systems [8].

A similar analysis of pseudoscalar mesons with equally massive constituents has also been performed [13] and this predicts [13]:

$$\frac{m_{H_{m=2m}}}{m_{H_{m=m_*}}} = 2.2, (25)$$

in agreement with a result obtained in recent quenched lattice simulations [20]. The model calculation provides an understanding of the lattice result. It shows that the persistent dominance by the term nonlinear in the current-quark mass owes itself to a large value of the in-meson condensates for light-quark mesons; e.g., $-\langle \bar{q}q \rangle_{1\text{GeV}} = \langle 0.32 \text{ GeV} \rangle^2$, and thereby provides a confirmation of the large condensate version of chiral perturbation theory.
the functions: they are only weakly sensitive to
the behaviour of the distribution functions on the
valence-quark domain, $x \gtrsim 0.5$.

The DSE model used in illustrating the framework’s ability to unify the small- and large-$Q^2$
behaviour of the pion’s form factors was used to
calculate $u_\pi^v(x)$ [24]. This calculation of the
appropriate “handbag diagrams” indicates that at
a resolving scale $q_0 = 0.54$ GeV = 1/(0.37 fm) valence
quarks, with a mass of 0.3 GeV, carry 71% of the
pion’s momentum, and yields the distribution function depicted in Fig. 2.

Over the entire range of $q_0$ considered, the
calculated distribution function is precisely fitted by a
MSR form

$$
u_\pi^v(x) = A_u x^{\eta_1-1} (1-x)^{\eta_2} (1-\epsilon_u \sqrt{x} + \gamma_u x),$$

with exemplary, calculated parameter values:

$$
\begin{array}{lcccc}
 q_0 & A_u & \eta_1 & \eta_2 & \epsilon_u & \gamma_u \\

data & 0.54 & 11.24 & 1.43 & 1.90 & 2.44 & 2.54 \\

data & 2.0 & 4.25 & 0.97 & 2.43 & 1.82 & 2.46 \\

data & 4.05 & 3.56 & 0.89 & 2.61 & 1.62 & 2.30 \\
\end{array}
$$

At $q_0 = 2.0$ GeV two low moments of the distribution
function are [21]:

$$
\begin{array}{lcccc}
 & (x^2) & (x^3) \\
\text{Calc.} & 0.098 & 0.049 \\
\text{Exp.} & 0.10 \pm 0.01 & 0.058 \pm 0.004 \\
\text{Latt.} & 0.11 \pm 0.3 & 0.048 \pm 0.020 \\
\end{array}
$$

with the experimental results from Ref. [22] and
the lattice results from Ref. [24]. NB. Given the
evident disagreement between the DSE calculation and the data, the agreement between the
calculated moments emphasises the insensitivity of
low moments to the $x$-dependence of the distribution
function on the valence-quark domain.

A material feature of the DSE result is the
value of $\eta_2 \simeq 2$ because while perturbative QCD
cannot be used to obtain the pointwise dependence
of the distribution functions it does give a prediction for the power-law dependence at $x \simeq 1$.
That prediction is [23]:

$$
pQCD: \quad u_\pi^v(x) \overset{x\to 1}{\sim} (1-x)^2,
$$

in agreement with the DSE result. However, as
will have been anticipated from Fig. 2, this pre-
Drell Yan: $u_\pi^V(x) \approx (1 - x)$.

The disagreement is very disturbing because a verification of this experimental result would present a profound threat to QCD, even challenging the assumed vector-exchange nature of the force underlying the strong interaction.

The DSE study [21] has refocused attention on this disagreement, and is the catalyst for a resurgence of interest in $u_\pi^V(x)$ and proposals for its remeasurement. One proposal that could use existing facilities would employ the (Sullivan) process depicted in Fig. 3 at JLAB [23], with the anticipated accuracy illustrated in Fig. 2. This process could also be used efficaciously at a future electron-proton collider to accurately probe $u_\pi^V(x)$ on the valence-quark domain, as emphasised by Fig. 4 [26].

4. Epilogue

Dynamical chiral symmetry breaking (DCSB) is a keystone of hadron physics. It is the effect responsible for turning perturbative current-quark masses into nonperturbative constituent-quark masses, and for ensuring that the pion is light while the pion’s electroweak decay constant nevertheless sets a large mass-scale: $4\pi f_\pi \sim 1$ GeV. It is thereby the foundation for the successful application of chiral perturbation theory to low-energy hadronic phenomena.

The QCD gap equation supplies a quark-level explanation of DCSB and, as one of the tower of Dyson-Schwinger equations (DSEs), unifies that with a Poincaré covariant understanding of the structure of QCD’s bound states and their interactions. In this way one finds, for example, that DCSB is also the main reason for the large $\pi - \rho$ and $\rho - a_1$ mass-splittings.

The fact that perturbation theory is recovered via a weak coupling expansion of the DSEs provides a tight constraint on the ultraviolet behaviour of the calculated Schwinger functions that describe hadronic interactions. Therefore their use in predicting and correlating observables, which necessarily sees the introduction of some model dependence, provides a means by which experimental results can be used to probe the infrared (long-range) behaviour of the quark-quark interaction; i.e., of exploring the mechanism of confinement. In this way the framework’s adaptability to modelling is a material asset.

As reviewed in Refs. [15,27], modern applications of the DSEs have met with substantial success, even at nonzero temperature and chemical potential, but challenges remain, of course. The gap equation makes clear that the existence of DCSB signals a significant enhancement of the strong coupling over the perturbative expectation on the infrared domain: $k^2 \lesssim 1$ GeV$^2$. However, the mechanism in QCD that supplies that enhancement is yet to be conclusively identified, and is being explored using DSE and lattice methods.

A direct bound state treatment of the scalar meson sector is also wanting. However, at least one now understands why the lowest order non-
perturbative truncation of the kernels in the relevant integral equations (rainbow-ladder), so successful for pseudoscalar and vector mesons, fails for the scalars \cite{28}. Improvements, systematic and/or imaginative, are being explored.

This, after all, is a core issue in DSE studies and the primary point of criticism: the system of integral equations must be truncated and how does one judge, \textit{a priori}, the fidelity of a given procedure? Addressing this open question is a key focus of contemporary research but the question’s existence does not itself diminish the efficacy nor the value of modern DSE phenomenology.

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