Perturbations of Admissibility, Exact Controllability, Exact Observability and Regularity *†

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Abstract

This paper is concerned with the notions of admissibility, exact controllability, exact observability and regularity of linear systems in the Banach space setting. It is proved that admissible controllability, exact controllability, admissible observation, exact observability and regularity are invariant under some regular perturbations of the generators, such results are generalizations of some previous references. Moreover, the related boundary linear systems and some illustrative examples are presented.

Key words: Admissibility; Exact controllability; Exactly observability; Regular linear systems; Boundary linear systems.

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1 Introduction

In the theory of finite dimensional linear control system, the final state and output are continuously depended on the initial state and input. Observe that such continuous dependence is the essential property in system theory. Motivated by this, Salamon [34] introduce the class of well-posed linear systems by continuous dependence in Hilbert space setting. Later, Weiss [39, 40, 41] simplified Salamon’s theory; he described well-posed linear system equivalently by using four algebraic equations (see the description in Section 2). In the functional analysis frame, the control operators and observation operators of well-posed linear system may be unbounded, which allow ones to study partial differential equations with boundary control and boundary observation. Over the last decades there has been a growing interest in well-posedness of partial differential equations with control and observation on the boundary, and it has been proved that many partial differential equations can be formulated as well-posed linear systems [1, 11, 22, 24, 25, 34, 35]. Regular linear systems, introduced by Weiss [41], are among the well-posed systems whose output function corresponding to a step input function and zero initial state is not very discontinuous at zero (see the definition in Section 2). Many well-posed physical systems are also regular, see [2, 5, 6, 7, 9, 10, 15, 17, 45]. Regular linear systems have a convenient representation, similar to that of finite dimensional systems. Concretely, Weiss showed in [41] that regular linear systems with unbounded control and observation operators can be simply represented by

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = C_{\Lambda}x(t) + Du(t),
\]

where \(C_{\Lambda}\) is the \(\Lambda\)-extension of the observation operator \(C\) with respect to system operator \(A\) (see Section 2). In this sense, an infinite-dimensional regular linear systems have the characteristic of “finite-dimensional systems”. Many references were concerned with abstract control theory under the frame of regular linear systems.

In order to obtain a well-posed and regular linear system, the control and observation operators should be admissible for the system operator (see, e.g., [34, 35, 41, 42, 43]). Hence the the concepts of admissibility of control and observation operators have been discussed by many references, most of which are interested in proving or disproving Weiss’
conjecture (see, e.g., [12, 19, 47, 48]). Here, we mention an important work due to Zwart [48]; he proved that the Weiss conjecture almost holds in Hilbert spaces.

For admissible control and admissible observation system, one can consider the notion of exact controllability and exact observability, because the enters into the study of many other important concepts. For instance, exact controllability is closely related to stabilizability and optimizability, while exact controllability is closely related to detectability and estimatability [4, 24, 44]. Exact controllability and exact observability have received considerable attention in the functional analysis frame (see e.g. [13, 18, 20, 23, 31, 32, 33, 46]), where some necessary and/or sufficient conditions have been given.

Generally, it is not an easy task to verify the admissibility, exact controllability, exact observability and regularity for a specific linear system with boundary control and/or boundary observation. Due to the difficulties of direct proving the admissibility and regularity, perturbation method has been successfully used to study the admissibility and regularity. Weiss [40] discussed the admissibility of observation system under bounded perturbation of the system operator, namely, $C$ being admissible observable operator for $A$ implies that $C$ is admissible for $A + P$, provided $P$ is an bounded linear operator on the state space. In [43], Weiss showed that the closed-loop system of well-posed linear system preserves the admissibility, exact controllability and exact observability. Moreover, the closed-loop system of regular linear system preserve the regularity. Hadd [14] proved that both $B$ and $\Delta A$ are $p$-admissible controllable operators for $A$ imply that $B$ is $p$-admissible for $(A + \Delta A)|_X$; $((A + \Delta A)|_X, B)$ is exactly controllable provided $(A, B)$ is exactly $p$-controllable and $\Delta A$ is “small” enough. In their paper [15], Hadd showed that $C$ and $\Delta A$ being $p$-admissible observable operators for $A$ implies $C$ is $p$-admissible for $A + \Delta A$. Moreover, Tucsnak and Weiss [38] proved that if $(A, C)$ is exactly observable and $\Delta A$ is “small” enough, then $(A + \Delta A, C)$ is exactly observable. Later, Mei and Peng [28, 29] weakened the condition of [14, 16, 38] that $\Delta A$ is $p$-admissible controllable (observable) operator to $\Delta A$ being $q$-admissible controllable (observable) operator. Mei and Peng [26] proved that the admissibility, exact controllability and exact observation are preserved under cross perturbations, that is, $(A, B, \Delta A)$ is a regular linear system,
then \( B \) is admissible for \( A + \Delta A \) and \((A + \Delta A, B)\) is exactly controllable provided \((A, B)\) is exactly controllable and \( \Delta A \) is “small” enough; \((A, \Delta A, C)\) is a regular linear system, then \( C^A \) is admissible for \((A^{-1} + \Delta A)|_X\) and \(((A^{-1} + \Delta A)|_X, C^A)\) is exactly observable provided \((A, C)\) is exactly observable and \( \Delta A \) is “small” enough. In their paper [27], Mei and Peng proved that \((A, B, \Delta A)\) and \((A, B, C)\) generating regular linear systems imply that \((A + \Delta A, B, C^A)\) generates a regular linear system; \((A, \Delta A, C)\) and \((A, B, C)\) generating regular linear systems imply that \(((A^{-1} + \Delta A)|_X, B, C)\) generates a regular linear system.

The aim of this paper is to study some general perturbation theorems of admissibilities, exact controllabilities, exact observations and regularities. Apart from the introduction, our arrangement is as follows. In Section 2, we introduce some basic notions and properties related to regular linear systems and boundary systems; the notions of exact controllability and exact observation are also be introduced. Section 3 is to give our main results. Concretely, we obtain admissible controllability, admissible observation, exact controllability, exact observation and regularity under some regular perturbations. Moreover, all the perturbation results are used to solve the corresponding boundary systems. The systems governed by specific partial differential equations are presented to illustrate our results.

2 Preliminaries

In this section, we recall some definitions related to regular linear systems and boundary linear systems. As stated in the introduction, Weiss has showed that the continuous dependence of state and output on the initial state and input can be simplified to four algebraic equations. We adopted Weiss’ definition for well-posed linear system [41].

Definition 2.1 A quadruple \( \Sigma = (T, \Phi, \Psi, F) \) is said to be a well-posed linear system on \((X, U, Y)\), if the following four conditions are satisfied:

(i) \( T = \{T(t)\}_{t \geq 0} \) is a \( C_0 \)-semigroup generated by \( A \) on \( X \);
(ii) \( \Phi = \{\Phi(t)\}_{t \geq 0} \) is a family of bounded linear operators, called input maps, from
\( L^p(\mathbb{R}^+ , U) \) to \( X \) such that

\[
\Phi(t + \tau)u = T(t)\Phi(\tau)u + \Phi(t)u(\cdot + \tau), \quad \forall u \in L^p(\mathbb{R}^+, U), \tau \geq 0, t \geq 0,
\]

we call \((T, \Phi)\) an abstract linear control system;

(iii) \( \Psi = \{\Psi(t)\}_{t \geq 0} \) is a family of bounded linear operators, called output maps, from \( X \) to \( L^p(\mathbb{R}^+, Y) \) such that

\[
(\Psi(t + \tau)x)(s) = (\Psi(t)T(\tau)x)(s - \tau), \quad \forall x \in X, t + \tau \geq s \geq \tau \geq 0, t \geq 0,
\]

we call \((T, \Psi)\) an abstract linear observation system;

(iv) \( F = \{F(t)\}_{t \geq 0} \) is a family of bounded linear operators, called input-output map, from \( L^p(\mathbb{R}^+, U) \) to \( L^p(\mathbb{R}^+, Y) \) such that

\[
(F(t + \tau)u)(s) = (\Psi(t)\Phi(\tau)u + F(t)u(\cdot + \tau))(s - \tau), \quad \forall u \in L^p(\mathbb{R}^+, U), t + \tau \geq s \geq \tau \geq 0, t \geq 0.
\]

By a representation theorem due to Salamon [35] (see also Weiss [39]), corresponding to abstract linear control system \((T, \Phi)\), there is a unique control operator \( B \in L(U, X_{-1}) \), called admissible control operator (also \( p \)-admissible control operator), satisfying

\[
\Phi(t)u = \int_0^t T_{-1}(t - s)Bu(s)ds \in X, \forall u \in L^p(\mathbb{R}^+, U), t \geq 0.
\]

Here \( T_{-1} \) is the extrapolation semigroup, which is the continuous extension of \( T \) to the extrapolation space \( X_{-1}^A \) defined by the completion of \( X \) under the norm \( \|R(\lambda_0, A)\cdot\| \) with \( R(\lambda_0, A) \) being the resolvent of \( A \) and \( \lambda \) belonging the resolvent set of \( A \). The generator of \( T_{-1} \) is the continuous extension of \( A \) to \( X \) and is denoted by \( A_{-1} \). In this case, we also say \((A, B)\) generates an abstract linear control system and denote \( \Phi_{A,B} \). Moreover, if \( \Phi_{A,B}(\tau) \) is surjective, we call \((A, B)\) to be exactly controllable (also exactly \( p \)-controllable) at \( \tau \).

It follows from Salamon [35] or Weiss [40] that an abstract linear observation system \((T, \Psi)\) corresponds a unique operator, called admissible observation operator (also \( p \)-admissible observation operator) \( C \in L(D(A), Y) \) satisfying

\[
\int_0^t \|CT(t)x\|^p dt \leq c(t_0)\|x\|^p, \forall x \in D(A)
\]

such that \((\Psi(t)x)(\tau) = CT(\tau)x, \forall x \in D(A), \tau \leq t \). In this case, we also say \((A, C)\) generates an abstract linear control system and denote \( \Psi_{A,C} \). Moreover, \((A, C)\) is called
to be exactly observable (also exactly $p$-observable) at $\tau$, provided there exists a constant $k > 0$ such that $\|\Psi_{A,C}(\tau)x\| \geq k\|x\|$, $x \in X$.

Let $\Sigma = (T, \Phi, \Psi, F)$ be well-posed linear system. For any $x(0) \in X$, $u \in L^p_{\text{loc}}(R^+, U)$, $x(t) = T(t)x(0) + \Phi(t)u$ is the solution of equation $\dot{x}(t) = A_{-1}x(t) + Bu(t)$. Define output $y = \Psi(\infty)x_0 + F(\infty)u$, where $\Psi(\infty) : X \to L^2_{\text{loc}}(R^+, Y)$ and $F(\infty) : L^p_{\text{loc}}(R^+, U) \to L^p_{\text{loc}}(R^+, Y)$ are the extended output map defined by the strong limit of $\Phi(\tau)$ and $F(\tau)$ as $\tau \to +\infty$, respectively (see [40, 41]). In the special case $u = 0$, it follows from [40, Theorem 4.5 and Proposition 4.7] that for any $x(0) \in X$, $y(t) = C^A_T(t)x(0)$ a.e. $t \geq 0$, where $C^A_{\Lambda}$ defined by

$$C^A_{\Lambda}x = \lim_{\lambda \to \infty} CLR(\lambda, A)x, \quad x \in D(C^A_{\Lambda}) = \{x \in X : \text{this above limit exists in } Y\} \tag{2.1}$$

is called $\Lambda$-extension of $C$ with respect to $A$. By [36], it follows that the output $y(t)$ can be expressed by

$$y(t) = C^A_{\Lambda}[x(t) - (\lambda - A_{-1})^{-1}B]u(t) + G(\lambda)u(t),$$

a.e. $t \geq 0$, where $G(\lambda)$ is the transform function. It is not hard to see that Definition [2.1] implies continuous dependence, that is, there exist positive function $m$ and $n$ on $R^+$ such that

$$\|x(t)\| + \|y\|_{L^p([0, t], Y)} \leq m(t)\|x(0)\| + n(t)\|u\|_{L^p([0, t], U)}, \quad t \geq 0.$$

The well-posed linear system $\Sigma$ is called a regular linear system if, there exists a bounded operator $D$, called feedthrough operator, such that the limit

$$\lim_{s \to 0} \frac{1}{s} \int_{0}^{s} (F(t)u_0)(\sigma)d\sigma = Dz$$

exists in $Y$ for the constant input $u_0(t) = z$, $z \in U$, $t \geq 0$. Weiss showed that well-posed linear system $\Sigma$ is regular if and only if $G(\lambda)$ strongly converges to $D$ as $\lambda \to +\infty$, that is,

$$\lim_{\lambda \to +\infty} G(\lambda)u = Du, \quad u \in U.$$

The regular linear system is described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = C^A_{\Lambda}x(t) + Du(t).$$
**Definition 2.2** An operator $\Gamma \in L(Y, U)$ is called an admissible feedback operator for $\Sigma = (T, \Phi, \Psi, F)$ if $I - F(t)\Gamma$ is invertible for some $t \geq 0$ (hence any $t \geq 0$).

**Theorem 2.3** Let $(A, B, C, D)$ be the generator of regular linear system $\Sigma = (T, \Phi, \Psi, F)$ on $(X, U, Y)$ with admissible feedback operator $\Gamma \in L(Y, U)$. Suppose that $I - D\Gamma$ is invertible. Then the feedback system $\Sigma_{\Gamma}$ is a well-posed linear system generated by $(A^{\Gamma}, B^{\Gamma}, C^{\Gamma})$:

$$A^{\Gamma} = (A_{-1} + B\Gamma(I - D\Gamma)^{-1}C_{\Lambda})|_{X},$$

$$D(A^{\Gamma}) := \{z \in D(C_{\Lambda}^{A}) : (A_{-1} + B\Gamma(I - D\Gamma)^{-1}C_{\Lambda})z \in X\},$$

$$C^{\Gamma} = (I - D\Gamma)^{-1}C_{\Lambda}^{A} \text{ restricted to } D(A^{\Gamma}) \text{ and } B^{\Gamma} = J^{A^{\Gamma}}(I - D\Gamma)^{-1}B,$$

where $J^{A^{\Gamma}}x = \lim_{\lambda \to \infty}(\lambda - A_{-1})^{-1}x \text{ (in } X_{A}^{\Gamma}) \text{ with } D(J^{A^{\Gamma}}) = \{x \in X_{A}^{\Gamma} : \text{the limit } \lim_{\lambda \to \infty}(\lambda - A_{-1})^{-1}x \text{ exists }\}$.

In the rest of this section, we introduce some notions related to linear boundary system described in the abstract frame as follows [25, 33].

$$\begin{cases}
    \dot{z}(t) = Lz(t), \\
    Gz(t) = u(t), \\
    y(t) = Kz(t),
\end{cases}$$

(2.2)

where $\begin{bmatrix} L \\ G \end{bmatrix}$ is closed linear operators from $D(L)$ to space $X \times U \times Y$; $D(L)$ is continuously embedded in $X$; $G$ is surjection and $\text{Ker}\{G\} := \{z \in Z : Gz = 0\}$ is dense in $X$; $L|_{\text{Ker}\{G\}}$ generates a $C_{0}$-semigroup on $X$. We denote system (2.2) by $(L, G, K)$ for brief.

Denote $A = L|_{G}, \ C = K|_{D(A)}$. By [3], $D(L)$ can be decomposed to direct sum $D(L) = D(A) \oplus \text{Ker}\{\lambda - L\}$ and the operator $G$ is bijective from $\text{Ker}\{\lambda - L\}$ onto $U$, where $\lambda$ is any component of resolvent set $\rho(A)$ of $A$. Hence we can denote $D_{\lambda,L,G}$ by the solution operator from $z$ to $u$ of the following function

$$\begin{cases}
    (\lambda - L)z = 0, \\
    Gz = u,
\end{cases}$$
that is $z = D_{\lambda,L,G}u$. By [25, 31], it follows that boundary control system

$$
\begin{cases}
\dot{z}(t) = Lz(t), \\
Gz(t) = u(t),
\end{cases}
$$

is equivalent to system

$$
\dot{z}(t) = A_{-1}z(t) + Bu(t)
$$

in the sense of classical solution, where $B$ is given by $B = (\lambda - A_{-1})D_{\lambda,L,G} \in L(U, X_{-1})$. Concretely, $z(t)$ and $u(t)$ satisfying $Gz(t) = u(t)$ is equivalent to $A_{-1}z(t) + Bu(t) \in X$; if $z(0) \in X$, $u \in W^{2,p}(R^+, U)$ satisfying $A_{-1}z(0) + Bu(0) \in X$, we have $\dot{z}(t) = Lz(t) = A_{-1}x(t) + Bu(t)$. Moreover the initial condition implies that $y(t) = Kz(t) = C(x(t) - (\lambda - A_{-1})^{-1}Bu(t) + K(\lambda - A_{-1})^{-1}Bu(t)$. Hence $C$ is the observation operator of the boundary system $(L, G, K)$ and the corresponding transform function is $K(\lambda - A_{-1})^{-1}B$, $\lambda \in \rho(A)$.

Boundary system $(L, G, K)$ is well-posed if there exist positive function $m$ and $n$ on $R^+$ such that

$$
\|x(t)\| + \|y\|_{L^p([0, t], Y)} \leq m(t)\|x(0)\| + n(t)\|u\|_{L^p([0, t], U)}, \quad t \geq 0.
$$

It is regular if it is well-posed and the strong limit of the transform function exists, that is, $\lim_{\lambda \to +\infty}KD_{\lambda,L,G}u$ exists for any $u \in U$. In this case, denote by $K_{A,B}$ the corresponding feedthrough operator, which means $K_{A,B}u := \lim_{\lambda \to +\infty}D_{\lambda,L,G}u$, $u \in U$. Then boundary system $(L, G, K)$ is regular with generator $(A, B, C, K_{A,B})$. We also say that the generator is $(A, B, K, K_{A,B})$ and denote $K_{A,\lambda}$ by $C_{\lambda}^A = (K_{D(A)})_{\lambda}^A$.

The following two lemmas will be used in the next section.

**Lemma 2.4** [30] Assume that the boundary control system

$$(BCS) \begin{cases}
\dot{z}(t) = Lz(t) \\
Gz(t) = u(t)
\end{cases}$$

is an abstract linear control system generated by $(A, B)$. Then the boundary system $(L, G, Q)$ is a regular linear system on $(X, U, Y)$ if and only if $(A, B, Q)$ generates a regular linear system. In this case, for any $z \in Z$, we have

$$Qz = Q_{A,\lambda}^\Lambda z + QGz.$$
Lemma 2.5 \[27\] Assume that the boundary system \((L, G, Q)\) is a regular linear system generated by \((A, B, Q, Q_{A,B})\) on \((X, U, X)\) with admissible feedback operator \(I\). Then the system

\[
(OS) \begin{cases}
\dot{z}(t) = Lz(t) \\
Gz(t) = Qz(t) + v(t)
\end{cases}
\]

is an abstract linear control system generated by \((A^I, B^I)\).

3 Main Results

In this section, we shall obtain the admissible controllability, exact controllability, admissible observation, exact observation and regularity under some regular perturbations. Moreover, all the perturbation results are used to solve the corresponding boundary systems. The systems governed by specific partial differential equations are presented after every perturbation result to illustrate our results.

We first consider the admissible controllability and exactly controllability under associated perturbation. To prove the robustness of exact controllability, we introduce the following important lemma related to radius of surjectivity.

Lemma 3.1 \[24\] page 227\] Let \(E\) and \(F\) be Banach spaces. Then, \(\mathcal{S}(E,F) := \{\Xi \in L(E,F) : \Xi \text{ is surjective}\}\) is an open set in \(L(E,F)\), i.e., given \(\Pi \in \mathcal{S}(E,F)\), there exists \(\alpha > 0\) such that

\[
\{\Xi \in L(E,F) : \|\Pi - \Xi\| < \alpha\} \subset \mathcal{S}(E,F).
\]

The constant \(\alpha\) is called radius of surjectivity of \(\Pi\).

Theorem 3.2 Let \((A, B, C, D)\) generate a regular linear system with admissible feedback operator \(I\) on \((X, Y, Y)\), \(I - D\) is invertible, and \((A, \Delta B, C, P)\) generate a regular linear system on \((X, U, Y)\). Then \((A^I, J^A^I, B(I - D)^{-1}P + J^A^I \Delta B, (I - D)^{-1}C^A_I)\) generates a regular linear system, and there holds

\[
\Phi_{A^I, J^A^I, B(I-D)^{-1}P+J^A^I \Delta B} = \Phi_{A, B}(I - F_{A,B,C,D})^{-1}F_{A, \Delta B, C, P} + \Phi_{A, \Delta B},
\]
where $A^I = (A + B(I - D)^{-1}C_A^A)|_X$. Moreover, if $(A, \Delta B)$ is exactly controllable at $t_0 > 0$, then there exists $k_0 > 0$ such that $\left( A^I(k), J_{A^I(k)}B(I - kD)^{-1}kP + J_{A^I(k)}\Delta B \right)$ is also exactly controllable at $t_0$ whenever $k < k_0$, where $A^I(k) = (A + B(I - kD)^{-1}kC_A^A)|_X$.

**Proof.** We consider the operators $\tilde{B} := (B, \Delta B) : Y \times U \rightarrow X_{-1}$, $\tilde{C} = \begin{pmatrix} C \\ 0 \end{pmatrix} : X_1 \rightarrow Y \times U$, $\tilde{D} = \begin{pmatrix} D & P \\ 0 & 0 \end{pmatrix} : Y \times U \rightarrow Y \times U$.

Since $(A, B, C, D)$ and $(A, \Delta B, C, P)$ generate regular linear systems, it is easy to verify that $(A, \tilde{B}, \tilde{C}, \tilde{D})$ generates a regular linear system given by

\[
\Sigma_{A,B,C,D} := \begin{pmatrix} T \\ (\Phi_{A,B}, \Phi_{A,\Delta B}) \\ 0 \\ F_{A,B,C,D} \\ F_{A,\Delta B,C,P} \end{pmatrix}
\]

Observe that $I$ is an admissible feedback operator for $\Sigma_{A,B,C,D}$. We have that

\[
I_{Y \times U} - \begin{pmatrix} F_{A,B,C,D} & F_{A,\Delta B,C,P} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I - F_{A,B,C,D} & -F_{A,\Delta B,C,P} \\ 0 & I \end{pmatrix}
\]

is invertible and

\[
\left( I_{Y \times U} - \begin{pmatrix} F_{A,B,C,D} & F_{A,\Delta B,C,P} \\ 0 & 0 \end{pmatrix} \right)^{-1} = \begin{pmatrix} (I - F_{A,B,C,D})^{-1} & (I - F_{A,B,C,D})^{-1}F_{A,\Delta B,C,P} \\ 0 & I \end{pmatrix},
\]

that is, $I_{X \times U}$ is an admissible feedback operator for $\Sigma_{A,B,\tilde{C},\tilde{D}}$. It follows from Theorem 2.3 that $A^{I|_{X \times U}} = (A_{-1} + \tilde{B}(I - \tilde{D})\tilde{C}^A_A)|_X = (A_{-1} + B(I - D)^{-1}C^A_A)|_X = A^I$, $\tilde{B}^{I|_{X \times U}} = J_{A^I}\tilde{B}(I - \tilde{D})^{-1} = \begin{pmatrix} J_{A^I}B(I - D)^{-1} & J_{A^I}B(I - D)^{-1}P + J_{A^I}\Delta B \end{pmatrix}$, $\tilde{C}^I_{X \times U} = (I-$
\[
(\tilde{D})^{-1}\tilde{C}_\Lambda = \begin{pmatrix} (I - D)^{-1}C^A_{\Lambda} & 0 \\ 0 & 0 \end{pmatrix}
\]
and

\[
\Phi_{A', J^A, A' B(I-D)^{-1}P + J^A, A' \Delta B}
\]

\[
= \Phi_{A', \tilde{B}'x_U} \begin{pmatrix} 0 \\ I \end{pmatrix}
\]

\[
= \Phi_{A, \tilde{B}}(I - F_{A, \tilde{B}, \tilde{C}, \tilde{D}})^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}
\]

\[
= (\Phi_{A, B}, \Phi_{A, \Delta B}) \begin{pmatrix} (I - F_{A, B, C, D})^{-1} (I - F_{A, B, C, D})^{-1} F_{A, \Delta B, C, P} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix}
\]

\[
= \Phi_{A, B}(I - F_{A, B, C, D})^{-1} F_{A, \Delta B, C, P} + \Phi_{A, \Delta B}.
\]

Moreover, it is not hard to see that

\[
(A^I, J^A, A' B(I-D)^{-1}P + J^A, A' \Delta B, (I-D)^{-1}C^A_{\Lambda}) = \left( A^I, \tilde{B}'x_U \begin{pmatrix} 0 \\ I \end{pmatrix}, \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{C}'x_U \right)
\]

is a regular linear system.

Below we prove the robustness of exact controllability. Observe that \(I - kD\) is invertible for any \(k < \frac{1}{\|D\|}\) (if \(\|D\| = 0\), \(\|D\| = +\infty\)). By \[3\] Proposition 3.12 and Proposition 4.10, \(kI\) is admissible feedback for \((A, B, C, D)\) whenever for \(k < \frac{1}{\|D\|}\), which indicates that \(I\) is admissible feedback for \((A, B, kC, kD)\). Since \((A, \Delta B)\) is exactly controllable at \(t_0 > 0\), \(\Phi_{A, \Delta B}(t_0)\) is surjective. Let \(s_0\) be the radius of surjectivity of \(\Phi_{A, \Delta B}(t_0)\). It follows from the above proof that

\[
\|\Phi_{A' B(I-kD)^{-1}P + J^A, A' \Delta B}(t_0) - \Phi_{A, \Delta B}(t_0)\|
\]

\[
= \|\Phi_{A, B}(t_0)(I - F_{A, B, kC, kP}(t_0))^{-1} F_{A, \Delta B, kC, kP}(t_0)\|
\]

\[
\leq k \|\Phi_{A, B}(t_0)(I - kF_{A, B, C, D}(t_0))^{-1} F_{A, \Delta B, C, P}(t_0)\|.
\]

Let

\[
k_0 = \min\{ \frac{1}{\|D\|}, \frac{1}{\|F_{A, B, C, D}(t_0)\|}, \frac{1}{\|\Phi_{A, B}\|}, \|F_{A, B, C, P}(t_0)\| + s_0 \|F_{A, B, C, D}(t_0)\| \}.
\]

11
Then \( \|\Phi_{A^I, J^A A^I(k)} B(I-kD)^{-1} k P + J^A A^I(k) \Delta B(t_0) - \Phi_{A, \Delta B}(t_0)\| < s_0 \) whenever \( k < k_0 \). It follows from Lemma [3.1] that \( \Phi_{A^I, J^A A^I(k)} B(I-kD)^{-1} k P + J^A A^I(k) \Delta B(t_0) \) is surjective. This implies that \((A^I(k), J^A A^I(k) B(I-kD)^{-1} k P + J^A A^I(k) \Delta B)\) is exactly controllability at \( t_0 \). The proof is therefore completed.

**Remark 3.3** In the special case that \( Y = X, D = 0, P = 0 \) and \( C = I \), the above theorem says that both \( B \) and \( \Delta B \) being admissible for \( A \) implies that \( J^{A, (A-1+B)} X \Delta B \) is admissible for \((A-1 + B)|_X\), such result has been proved by Hadd [10], as mentioned in the introduction section. If \( Y = X, D = 0, P = 0 \) and \( B = I \), Theorem 3.2 tells that \((A, \Delta B, C)\) generating a regular linear system implies that \((A + C, J^{A,A+C} \Delta B, C)\) generates a regular linear system, particularly, \( J^{A,A+C} \Delta B \) is admissible for \( A + C \), such result has been proved by Mei and Peng [27]. This means that our result is a generalization of [10] and [27].

**Theorem 3.4** Assume that the boundary system \[
\begin{cases}
\dot{z}(t) & = Lz(t) \\
G_1 z(t) & = u(t) \\
G_2 z(t) & = 0 \\
y(t) & = Kz(t)
\end{cases}
\]
is a regular linear system generated by \((A, B_1, K, \overline{K}_{A,B_1})\) on \((X, U, U)\) with \( I \) being admissible feedback operator. Suppose that \[
\begin{cases}
\dot{z}(t) & = Lz(t) \\
G_1 z(t) & = 0 \\
G_2 z(t) & = v(t) \\
y(t) & = Kz(t)
\end{cases}
\]
is regular linear system on \((X, V, U)\) with control operator \( B_2 \). Then \[
\begin{cases}
G_1 z(t) & = Kz(t) \\
G_2 z(t) & = v(t)
\end{cases}
\]
is an abstract linear control system generated by \((A^I, J^A A^I B_1(I - \overline{K}_{A,B_1})^{-1} \overline{K}_{A,B_2} + J^A A^I B_2)\). If \[
\begin{cases}
\dot{z}(t) & = Lz(t) \\
G_1 z(t) & = 0 \\
G_2 z(t) & = v(t)
\end{cases}
\]
is exactly controllable at \( t_0 \), there exists \( k_0 > 0 \) such that \[
\begin{cases}
\dot{z}(t) & = Lz(t) \\
G_1 z(t) & = kKz(t) \\
G_2 z(t) & = v(t)
\end{cases}
\]
is exactly controllable at \( t_0 \).
$t_0$ whenever $k < k_0$.

**Proof.** By the assumption, $G_1 : D(L) \cap \text{Ker}G_2 \to U$ and $G_2 : D(L) \cap \text{Ker}G_1 \to V$ are surjectives. This implies that for any $u \in U$ and $v \in V$, there exist $z_1 \in D(L) \cap \text{Ker}G_2$ and $z_2 \in D(L) \cap \text{Ker}G_1$ such that $G_1z_1 = u$, $G_2z_2 = v$. Hence, $z_1 + z_2 \in D(L)$ and

$$
\begin{pmatrix}
G_1 \\
G_2
\end{pmatrix}
\begin{pmatrix}
z_1 + z_2
\end{pmatrix} =
\begin{pmatrix}
u \\
v
\end{pmatrix},
$$

that is, $egin{pmatrix} G_1 \\ G_2 \end{pmatrix}$ are surjective. Moreover, $B_1 = (\lambda - A_{-1})D_{1,\lambda}$ and $B_1 = (\lambda - A_{-1})D_{2,\lambda}$ are indicated by the assumption. Here $D_{1,\lambda}u$ and $D_{1,\lambda}u$ are the solution of the equations

$$
\begin{cases}
\lambda z = Lz, \\
G_1z = u, \\
G_2z = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\lambda z = Lz, \\
G_1z = 0, \\
G_2z = v
\end{cases},
$$

respectively. Hence, the solution

$$
D_{\lambda} \begin{pmatrix} u \\ v \end{pmatrix}
$$

of

$$
\begin{cases}
\lambda z = Lz, \\
G_1z = u, \\
G_2z = v
\end{cases}
$$

satisfies

$$
D_{\lambda} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D_{1,\lambda} & D_{1,\lambda} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
$$

So the control operator of boundary system

$$
\begin{cases}
\dot{z}(t) = Lz \\
G_1z = u(t) \\
G_2z = v(t)
\end{cases}
$$

is $B = (\lambda - A_{-1})D_{\lambda} = (\lambda - A_{-1}) \begin{pmatrix} B_1 & B_2 \end{pmatrix}$. Let $Y(t) = \begin{pmatrix} K \\ 0 \end{pmatrix} z(t)$. Since $(A, B_1, K)$ and $(A, B_2, K)$ are regular linear system, we obtain that $(A, B, \begin{pmatrix} K \\ 0 \end{pmatrix})$ is a regular linear system. Then

$$
\begin{cases}
\lambda z = Lz, \\
G_1z = u, \\
G_2z = v
\end{cases}
$$

with output $Y(t)$ is a regular linear system, the feedthrough...
operator \( D_0 \) is computed by

\[
D_0 \begin{pmatrix} p \\ q \end{pmatrix} = \lim_{\lambda \to +\infty} \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} (\lambda - A)^{-1} B \begin{pmatrix} p \\ q \end{pmatrix}
\]

\[
= \begin{pmatrix} \lim_{\lambda \to +\infty} K(\lambda - A)^{-1} B_1 p + \lim_{\lambda \to +\infty} K(\lambda - A)^{-1} B_2 q \\ 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} K_{A,B_1} & K_{A,B_2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \forall \begin{pmatrix} p \\ q \end{pmatrix} \in U \times V.
\]

Sine \( I \) is admissible feedback operator of regular linear system \((A, B_1, K, K_{A,B_1})\), \( I \) is admissible feedback operator for \((A, B_1, K, K_{A,B_1}, K_{A,B_2})\). By Lemma 2.5, it follows that

\[
\begin{aligned}
\dot{z}(t) &= Lz(t) \\
G_1 z(t) &= K z(t) + u(t) \\
G_2 z(t) &= v(t)
\end{aligned}
\]

is an abstract linear control system with generator \((A^I, J^{A,A^I} B \begin{pmatrix} (I - K_{A,B_1})^{-1} (I - K_{A,B_1})^{-1} K_{A,B_2} \\ 0 \end{pmatrix})\). Hence

\[
\begin{aligned}
\dot{z}(t) &= Lz(t) \\
G_1 z(t) &= K z(t) \\
G_2 z(t) &= v(t)
\end{aligned}
\]

is an abstract linear control system with generator \((A^I, J^{A,A^I} B_1(I - K_{A,B_1})^{-1} K_{A,B_2} + J^{A,A^I} B_2)\). The rest result is obtained directly from Theorem 3.2. This completes the proof. ■

Example 3.5 We consider Schrödinger equation equation with Dirichlet boundary control and observation described by

\[
\begin{aligned}
\begin{cases}
&w_{tt}(x, t) = \Delta w(x, t), \quad x \in \Omega, t > 0, \\
&w(x, t) = -\frac{\partial (\Delta^{-1} w)}{\partial \nu}, \quad x \in \Gamma_1, t \geq 0, \\
&w(x, t) = u(x, t), \quad x \in \Gamma_0, t \geq 0,
\end{cases}
\end{aligned}
\]

(3.1)

where \( \Omega \subset \mathbb{R}^n, n \geq 2 \) is an open bounded region with smooth \( C^3 \)-boundary \( \partial \Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1} \). \( \Gamma_0, \Gamma_1 \) are disjoint parts of the boundary relatively open in \( \partial \Omega \), \( \text{int}(\Gamma_1) \neq \emptyset \) and \( \text{int}(\Gamma_0) \neq \emptyset \), \( \nu \) is the unit normal vector of \( \Gamma_0 \) pointing towards the exterior of \( \Omega \), \( u \) is the input function (or control) and \( y \) is the output function (or output).
Let $H = H^{-1}(\Omega)$ be the state space and $U = L^2(\partial\Omega)$ be the control (input) or observation (output) space. It has been proved in [5] that
\[
\begin{array}{l}
\begin{cases}
w_t(x,t) = -i\Delta w(x,t), & x \in \Omega, t > 0, \\
w(x,t) = v(x,t), & x \in \Gamma_1, t \geq 0, \\
w(x,t) = 0, & x \in \Gamma_0, t \geq 0, \\
y(x,t) = -i\partial(\Delta^{-1}w) / \partial\nu, & x \in \Gamma_1, t \geq 0
\end{cases}
\end{array}
\]
is regular linear systems with feedthrough operator zero and $I$ being admissible feedback operator. Similarly, one can obtain that
\[
\begin{array}{l}
\begin{cases}
w_t(x,t) = -i\Delta w(x,t), & x \in \Omega, t > 0, \\
w(x,t) = 0, & x \in \Gamma_1, t \geq 0, \\
w(x,t) = u(x,t), & x \in \Gamma_0, t \geq 0, \\
y(x,t) = -i\partial(\Delta^{-1}w) / \partial\nu, & x \in \Gamma_1, t \geq 0
\end{cases}
\end{array}
\]
is regular linear systems with feedthrough operator zero. The combinations of [1] and [4] implies that system
\[
\begin{array}{l}
\begin{cases}
w_t(x,t) = -i\Delta w(x,t), & x \in \Omega, t > 0, \\
w(x,t) = 0, & x \in \Gamma_1, t \geq 0, \\
w(x,t) = u(x,t), & x \in \Gamma_0, t \geq 0, \\
y(x,t) = -i\partial((\Delta)^{-1}w) / \partial\nu, & x \in \Gamma_1, t \geq 0
\end{cases}
\end{array}
\]
is exactly controllable at some $t_0 > 0$. By Theorem 3.4, it follows that (3.1) is an abstract linear control system. Moreover, there exists a constant $k_0 > 0$ such that
\[
\begin{array}{l}
\begin{cases}
w_t(x,t) = -i\Delta w(x,t), & x \in \Omega, t > 0, \\
w(x,t) = -ki\partial((\Delta)^{-1}w) / \partial\nu, & x \in \Gamma_1, t \geq 0, \\
w(x,t) = u(x,t), & x \in \Gamma_0, t \geq 0, \\
y(x,t) = -i\partial((\Delta)^{-1}w) / \partial\nu, & x \in \Gamma_1, t \geq 0
\end{cases}
\end{array}
\]
is exactly controllable at $t_0 > 0$ whenever $k < k_0$.

Next, we are concerned with admissible observation and exactly observation under some regularity perturbation.
Theorem 3.6  Let \((A, B, C, D)\) generate a regular linear system with admissible feedback operator \(I\) on \((X, U, U)\), and \((A, B, \Delta C, P)\) generate a regular linear system on \((X, U, Y)\). Then \((A^I, B^I, P(I - D)^{-1}C_A^I + \Delta C_A^I)\) generates a regular linear system, and there holds

\[
\Psi_{A^I, B^I, P(I - D)^{-1}C_A^I + \Delta C_A^I} = F_{A, B, \Delta C, P}(I - F_{A, B, C, D})^{-1}\Psi_{A, C} + \Psi_{A, \Delta C},
\]

where \(A^I = (A + B(I - D)^{-1}C_A^I)|_X\) and \(B^I = J^{A, I}B\). Moreover, if \((A, \Delta C)\) is exactly observable at \(t_0 > 0\), then there exists \(k_0 > 0\) such that \((A^I(k), kP(I - kD)^{-1}C_A^I + \Delta C_A^I)\) is also exactly observable at \(t_0\) whenever \(k < k_0\), where \(A^I(k) = (A + B(I - kD)^{-1}kC_A^I)|_X\).

Proof. Similar to the proof of [30, Lemma 4.3], let \(\tilde{B} = (B, 0)\), \(\tilde{C} = \begin{pmatrix} C \\ \Delta C \end{pmatrix}\),

\[
\tilde{D} = \begin{pmatrix} D & 0 \\ P & 0 \end{pmatrix},
\]

we obtain that \((A^I, B^I, P(I - D)^{-1}C_A^I + \Delta C_A^I)\) generates a regular linear system and

\[
\Psi_{A^I, B^I, P(I - D)^{-1}C_A^I + \Delta C_A^I} = (0, I)\Psi_{A^I, B^I, \tilde{C}, \tilde{D}} = (0, I)(I - F_{A, B, \tilde{C}, \tilde{D}})^{-1}\Psi_{A, C} = (0, I)\begin{pmatrix} (I - F_{A, B, C, D})^{-1} & 0 \\ F_{A, B, \Delta C, P}(I - F_{A, B, C, D})^{-1} & I \end{pmatrix} \begin{pmatrix} \Psi_{A, C} \\ \Psi_{A, \Delta C} \end{pmatrix} = F_{A, B, \Delta C, P}(I - F_{A, B, C, D})^{-1}\Psi_{A, C} + \Psi_{A, \Delta C}.
\]

Below we prove the robustness of exactly observability. As stated in the proof of Theorem 3.2 \(kI\) is admissible feedback for \((A, B, C, D)\) whenever for \(k < \frac{1}{\|D\|}\), which indicates that \(I\) is admissible feedback for \((A, B, kC, kD)\). Since \((A, \Delta C)\) is exactly observable at \(t_0\), there exists a constant \(k_0 > 0\) such that \(
\|\Psi_{A, \Delta C}(t_0)x\| \geq k_0\|x\|, \ x \in X
\). It follows from the above proof that

\[
\|\Psi_{A^I, kP(I - kC)}^{-1}C_A^I + \Delta C_A^I(t_0)x\| \\
\geq\|\Psi_{A, \Delta C}(t_0)x\| - \|\Psi_{A^I, kP(I - kC)^{-1}C_A^I + \Delta C_A^I(t_0)x} - \Psi_{A, \Delta C}(t_0)x\| \\
\geq k_0\|x\| - \|F_{A, B, \Delta C, P}(t_0)(I - F_{A, B, kC, kD}(t_0))^{-1}\Psi_{A, kC}(t_0)x\| \\
= k_0\|x\| - \|F_{A, B, \Delta C, P}(t_0)(I - kF_{A, B, C, D}(t_0))^{-1}k\Psi_{A, C}(t_0)x\|.
\]
Let \( \alpha_0 \in (0, k_0) \) and

\[
\theta_0 = \min \left\{ \frac{1}{\|D\|}, \frac{1}{\|F_{A,B,C,D}(t_0)\|}, \frac{k_0 - \alpha_0}{\|F_{A,B,C,D}(t_0)\| + \|F_{A,B,C,D}(t_0)\|, k_0 - \alpha_0} \right\}.
\]

Then

\[
\|\Psi_{A^I,kP(I-kD)^{-1}C^A+\Delta C^A}(t_0)x\| > \alpha_0 \|x\|
\]

whenever \( k < \theta_0 \). The proof is therefore completed.

**Remark 3.7** In the special case that \( Y = X \) and \( B = I \), the above theorem says that both \( P \) and \( C \) being admissible for \( A \) implies that \( C \) is admissible for \( A + C \), such result has been proved by Hadd [16]. If \( Y = X \) and \( C = I \), theorem tells that \((A, B, P)\) generating a regular linear system implies that \((A_{-1} + B)|_X, J^{A_{-1}+B}|_X, P_A\) generates a regular linear system, particularly, \(J^{A_{-1}+B}|_X B\) is admissible for \( A + P \). This means that our result is a generalization of [16].

**Theorem 3.8** Assume that the boundary system \((L, G, Q)\) is a regular linear system generated by \((A, B, G, G_A)\) on \((X, U, U)\) with admissible feedback operator \( I \). Suppose that boundary system \((L, G, K)\) is a regular linear system on \((X, U, Y)\). Then the system

\[
\begin{align*}
\dot{z}(t) &= Lz(t) \\
Gz(t) &= Qz(t) \\
y(t) &= Kz(t)
\end{align*}
\]  

(3.2)

is an abstract linear observation system generated by \((A^I, K)\). If, in addition, system

\[
\begin{align*}
\dot{z}(t) &= Lz(t) \\
Gz(t) &= 0 \\
y(t) &= Kz(t)
\end{align*}
\]

is exactly observable at some \( t_0 > 0 \), there exists a constant \( \theta_0 > 0 \) such that system

\[
\begin{align*}
\dot{z}(t) &= Lz(t) \\
Gz(t) &= kQz(t) \\
y(t) &= Kz(t)
\end{align*}
\]

is exactly observable at \( t_0 > 0 \) whenever \( k < \theta_0 \).

**Proof.** Since boundary system \((L, G, K)\) is a regular linear system with admissible feedback operator \( I \), it follows from Lemma 2.4 that

\[
Kz = K_A^A z + K_{A,B}Gz, \quad z \in Z,
\]

(3.3)
and

\[ Gz = Qz, \ z \in D(A^I) \subset D(L). \]  (3.4)

The assumption \((L, G, Q)\) is a regular linear system implies

\[ Qz = Q_A^A z + \overline{Q}_{A,B} Gz, \ z \in D(L). \]  (3.5)

Observe that \(I - \overline{Q}_{A,B}\) is invertible. The combination of (3.4) and (3.5) implies that

\[ Qz = (I - \overline{Q}_{A,B})^{-1} Q_A^A z, \ z \in D(A^I), \]

substituted which into 3.3 to get

\[ Kz = K_A^A z + \overline{K}_{A,B} (I - \overline{Q}_{A,B})^{-1} Q_A^A z, \ z \in D(A^I). \]

By Theorem 3.6, (3.2) is an abstract linear observation system generated by \((A^I, K)\). Furthermore, the rest result is obtained directly from Theorem 3.6. This completes the proof.

**Example 3.9** Consider the following one-dimensional Euler-Bernoulli beam equation

\[
\begin{aligned}
& w_{tt}(x, t) + w_{xxxx}(t, x) = 0, \quad x \in (0, 1) \\
& w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, \ w_{xxx}(1, t) = w_t(1, t), \\
& y(t) = w_x(1, t).
\end{aligned}
\]  (3.6)

It follows from [4] that

\[
\begin{aligned}
& w_{tt}(x, t) + w_{xxxx}(t, x) = 0, \quad x \in (0, 1) \\
& w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, \ w_{xxx}(1, t) = u(t), \\
& y(t) = w_t(1, t).
\end{aligned}
\]  (3.7)

is a regular linear system with admissible feedback operator \(I\) and the corresponding feedback operator is zero. By Theorem 3.4, to obtain that (3.6) is an abstract linear observation system, we only have to prove that

\[
\begin{aligned}
& w_{tt}(x, t) + w_{xxxx}(t, x) = 0, \quad x \in (0, 1) \\
& w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, \ w_{xxx}(1, t) = u(t), \\
& y(t) = w_x(1, t),
\end{aligned}
\]  (3.8)

is a regular linear systems. We divide the rest proof into three steps.
Step 1. Boundary observation system

\[
\begin{align*}
\begin{cases}
  w_{tt}(x,t) + w_{xxxx}(t,x) &= 0, & x \in (0, 1) \\
  w(0,t) = w_x(0,t) = w_{xx}(1,t) = 0, & w_{xxxx}(1,t) = 0, \\
  y(t) = w_x(1,t),
\end{cases}
\end{align*}
\]

is an abstract linear observation system. To this end, we let

\[ F(t) = \frac{1}{2} \int_0^1 [w_t^2(x,t) + w_{xx}^2(x,t)] dx. \]

It is not hard to see that \( \dot{F}(t) = 0 \) thereby \( F(t) = F(0), \ t \geq 0 \). Set

\[ \rho(t) = \int_0^1 x(x - 1)w_t(x,t)w_x(x,t) dx. \]

We obtain \(|\rho(t)| \leq F(t) = F(0)\). Take the derivative with respect to the time on both sides to get

\[ \dot{\rho}(t) = \int_0^1 x(x - 1)w_{tt}(x,t)w_x(x,t) dx + \int_0^1 x(x - 1)w_t(x,t)w_{x}(x,t) dx \]

\[ = - \int_0^1 x(x - 1)w_{xxxx}(x,t)w_x(x,t) dx + \frac{1}{2} \int_0^1 x(x - 1)\frac{\partial}{\partial x}w_t^2(x,t) dx \]

\[ = - x(x - 1)w_{xx}(x,t)w_x(x,t)\big|_{x=0}^1 + \int_0^1 x(x - 1)w_{xxx}(x,t)w_{xx}(x,t) dx \]

\[ + \int_0^1 (2x - 1)w_{xx}(x,t)w_x(x,t) dx + \frac{1}{2} \int_0^1 x(x - 1)\frac{\partial}{\partial x}w_t^2(x,t) dx \]

\[ = \frac{1}{2} \int_0^1 x(x - 1)\frac{\partial}{\partial x}w_{xx}^2(x,t) dx + \frac{1}{2} \int_0^1 x(x - 1)\frac{\partial}{\partial x}w_t^2(x,t) dx \]

\[ + (2x - 1)w_{xx}(x,t)w_x(x,t)\big|_{x=0}^1 - \int_0^1 w_{xx}(x,t)[2w_x(x,t) + (2x - 1)w_{xx}(x,t)] dx \]

\[ = \frac{1}{2} x(x - 1)[w_t^2(x,t) + w_{xx}^2(x,t)]\big|_{x=0}^1 - \frac{1}{2} \int_0^1 (2x - 1)[w_t^2(x,t) + 3w_{xx}^2(x,t)] dx \]

\[ - \int_0^1 2w_{xx}(x,t)w_x(x,t) dx \]

\[ = - \frac{1}{2} \int_0^1 (2x - 1)[w_t^2(x,t) + 3w_{xx}^2(x,t)] dx - w_x^2(1,t). \]

Integrate from 0 to \( T \) with respect to \( t \) to derive

\[ \int_0^T w_x^2(1,t) dt = \int_0^T \left[ - \frac{1}{2} \int_0^1 (2x - 1)[w_t^2(x,t) + 3w_{xx}^2(x,t)] dx \right] dt + \rho(0) - \rho(T) \]

\[ \leq (3T + 2)F(0). \]
**Step 2.** Boundary system (3.8) is a well-posed. We consider the boundary system under the zero initial condition: \( w(x,0) = w_t(x,0) = 0 \). Define \( F(t) \) and \( \rho \) as the same in Step 1. By [8], it follows that

\[
F(t) \leq C_{\delta,T} \int_0^T u^2(t)dt, \quad \forall t \in [0,T],
\]

where \( \delta \in (0, \frac{1}{1+4T}) \) and \( C_{\delta,T} = \frac{1+\delta+4T}{2(1-(1+4T)^0)} + \frac{1}{2T} \).

Observe that \( |\rho(t)| \leq F(t) \) holds. Take the derivative with respect to the time on both sides to get

\[
\dot{\rho}(t) = -\frac{1}{2} \int_0^1 (2x-1)[w_t^2(x,t) + 3w_{xx}^2(x,t)]dx - w_x^2(1,t).
\]

Integrate from 0 to \( T \) with respect to \( t \) to derive

\[
\int_0^T w_x^2(1,t)dt = \int_0^T \left[ -\frac{1}{2} \int_0^1 (2x-1)[w_t^2(x,t) + 3w_{xx}^2(x,t)]dx \right] dt - \rho(T)
\leq 3 \int_0^T F(t)dt + F(T)
\leq (1 + 3T)C_{\delta,T} \int_0^T u^2(t)dt.
\]

**Step 3.** Boundary system (3.8) is regular. Denote by \( \hat{w}(x,s) \) the Laplace transform of \( w(x,s) \) with respect to \( t \), that is, \( \hat{w}(x,s) = \int_0^\infty w(x,t)e^{-st}dt \). Similarly, the Laplace transform \( \hat{u}(s) \) of \( u(t) \) with respect to \( t \) is \( \hat{u}(s) = \int_0^\infty u(t)e^{-st}dt \). For the zero initial condition \( w(x,0) = w_t(x,0) = 0 \), we get

\[
\begin{aligned}
s^2\hat{w}(x,s) + \hat{w}_{xxxx}(x,s) & = 0, \quad x \in (0,1) \\
\hat{w}(0,s) = \hat{w}_x(0,s) = \hat{w}_x(1,s) = 0, \quad \hat{w}_{xx}(1,s) = \hat{u}(s), \\
\hat{y}(s) = \hat{w}_x(1,t).
\end{aligned}
\]

Denote by \( H(s) \) the corresponding transform function. Since the system is well-posed, then we have that \( H(s) \) satisfies that \( \hat{y}(s) = H(s)\hat{u}(s) \) and it is bounded on some right half plane. In order derive the regularity, we only need to show that the limit of transfer function exists as \( s \to +\infty \). So we can set \( s > 0 \) and \( t = \sqrt{\frac{s}{2}} \). The first equation implies that

\[
\hat{w}(x,s) = ach(tx)cos(tx) + bch(tx)sin(tx) + csh(tx)cos(tx) + dsh(tx)sin(tx)
\]
with $a, b, c$ and $d$ being to be determined. Use $\hat{w}(0, s) = 0$ to get $a = 0$. We obtain

$$\hat{w}_x(x, s) = t[b\text{sh}(tx)\sin(tx) + (b + c)\text{ch}(tx)\cos(tx) - c\text{sh}(tx)\sin(tx)]$$

$$+ d\text{ch}(tx)\sin(tx) + d\text{sh}(tx)\cos(tx)].$$

Use $\hat{w}_x(0, s) = 0$ to get $b + c = 0$. We obtain

$$\hat{w}_{xx}(x, s) = 2t^2[b\text{ch}(tx)\sin(tx) + b\text{sh}(tx)\cos(tx) + d\text{ch}(tx)\cos(tx)]$$

and

$$\hat{w}_{xxx}(x, s) = 2t^3[2b\text{ch}(tx)\cos(tx) + d\text{ch}(tx)\cos(tx) - d\text{ch}(tx)\sin(tx)].$$

Use $\hat{w}_{xx}(1, s) = 0$ and $\hat{w}_{xxx}(1, s) = \hat{u}(s)$ to get

$$\left\{ \begin{array}{l}
    b = \frac{\text{cht} \text{cost}}{2t^3(\text{ch}^2 t + \cos^2 t)} \hat{u}(s), \\
    d = -\frac{\text{cht} \text{sint} + \text{shcost}}{2t^3(\text{ch}^2 t + \cos^2 t)} \hat{u}(s).
\end{array} \right.$$

Hence we obtain that

$$H(s) = -\frac{s\text{hch}\text{sintcost} - (\text{cht} \text{sint} + \text{shcost})^2}{2t^2(\text{ch}^2 t + \cos^2 t)}.$$

Observe that

$$|H(s)| \leq \frac{\text{cht}^2 t + (2\text{cht})^2}{2t^2 \text{ch}^2 t} = \frac{5}{2t^2} = \frac{5}{s}.$$

Hence $H(s) \to 0$ as $s \to +\infty$. The regularity of (3.8) is therefore proved. This completes the proof.

**Example 3.10** Consider one-dimensional Euler-Bernoulli beam equation

$$\begin{cases}
   w_{tt}(x, t) + w_{xxxx}(t, x) = 0, & x \in (0, 1) \\
   w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, & w_{xxx}(1, t) = w_t(1, t), \\
   y(t) = w_{xx}(0, t).
\end{cases} \tag{3.9}$$

Guo, Wang and Yang showed in [8] that

$$\begin{cases}
   w_{tt}(x, t) + w_{xxxx}(t, x) = 0, & x \in (0, 1) \\
   w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, & w_{xxx}(1, t) = w(1, t), \\
   y(t) = w_{xx}(0, t).
\end{cases} \tag{3.10}$$
is a well-posed linear system. From Step 3 of the above Example 3.9, we obtain that the transform function of system (3.10) is presented on $R^+$ by

$$H_1(s) = \frac{\hat{w}_{xx}(0, s)}{\hat{u}(s)} = \frac{2t^2d}{u(s)} = \frac{ch\sin t + sh\cos t}{t(ch^2 + cos^2 t)}$$

with $t = \sqrt{\frac{s}{2}}$. Then

$$|H_1(s)| \leq \frac{cht + sh}{tch^2} \leq \frac{2cht}{tch} \to 0,$$

as $s \to +\infty$, that is, system (3.10) is regular. Observe that system (3.7) is regular. By Theorem 3.8, system (3.9) is an abstract linear observation system.

Next, we show that system

$$\begin{cases}
  w_{tt}(x, t) + w_{xxxx}(t, x) = 0, & x \in (0, 1) \\
  w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, & w_{xxx}(1, t) = 0, \\
  y(t) = w_{xx}(0, t),
\end{cases} \quad (3.11)$$

is exactly observable. Let

$$F(t) = \frac{1}{2} \int_0^1 [w_x^2(x, t) + w_{xx}^2(x, t)]dx.$$ 

We have $\dot{F}(t) = 0$ thereby $F(t) = F(0), \ t \geq 0$. Set

$$\rho_1(t) = \int_0^1 (x - 1)w_t(x, t)w_x(x, t)dx.$$ 

Obviously, $|\rho_1(t)| \leq F(t) = F(0)$. We compute

$$\dot{\rho}_1(t) = \frac{1}{2}w_{xx}^2(0, t) - \frac{1}{2} \int_0^1 (w_t^2(x, t) + 3w_{xx}^2(x, t))dx.$$ 

Integrate from 0 to $T$ with respect to $t$ to get

$$\int_0^T w_{xx}^2(0, t)dt = \frac{1}{2} \int_0^t \int_0^1 (w_t^2(x, t) + 3w_{xx}^2(x, t))dxdt + \rho_1(0) - \rho_1(T) \geq (T - 2)E(0).$$

This indicates that system (3.11) is exactly observable at any $T > 2$. Then, by Theorem 3.8 system

$$\begin{cases}
  w_{tt}(x, t) + w_{xxxx}(t, x) = 0, & x \in (0, 1) \\
  w(0, t) = w_x(0, t) = w_{xx}(1, t) = 0, & w_{xxx}(1, t) = w_t(1, t), \\
  y(t) = w_{xx}(0, t),
\end{cases}$$

is exactly observable at $T > 2$ whenever $k$ is small enough.
In the rest of this section, we are concerned with the regularity under perturbations.

**Theorem 3.11** Assume that \((A, B, C, D)\) generates a regular linear system with admissible feedback operator \(I\). Suppose \((A, \Delta B, C)\), \((A, B, \Delta C)\) and \((A, \Delta B, \Delta C)\) generates regular linear systems. Then \((A^I, J^{A^I} \Delta B, \Delta C^A)\) generates a regular linear system.

**Proof.** By Theorem 3.2 it follows that \((A^I, J^{A^I} \Delta B)\) generates an abstract linear control system with

\[
\Phi_{A^I, J^{A^I} \Delta B} = \Phi_{A, B} (I - F_{A, B, C, D})^{-1} F_{A, \Delta B, C} + \Phi_{A, \Delta B}.
\]

Theorem 3.6 implies that \((A^I, \Delta C^A)\) generates an abstract linear observation system with

\[
\Psi_{A^I, \Delta C^A} = F_{A, B, \Delta C} (I - F_{A, B, C, D})^{-1} \Psi_{A, C} + \Psi_{A, \Delta C}.
\]

Since \((A, B, \Delta C)\) and \((A, \Delta B, \Delta C)\) are regular linear systems, we define \(F = F_{A, B, \Delta C} (I - F_{A, B, C, D})^{-1} F_{A, \Delta B, C} + F_{A, \Delta B, \Delta C}\). Then it is not hard to verify that \((T_{A^I}, \Phi_{A^I, J^{A^I} \Delta B}, \Psi_{A^I, \Delta C^A}, F)\) is a regular linear system generated by \((A^I, J^{A^I} \Delta B, \Delta C^A)\). The proof is therefore completed. 

**Remark 3.12** In the special case that \(Y = X\) and \(C = I\), the above theorem says that both \((A, B, \Delta C)\) and \((A, \Delta B, \Delta C)\) being regular linear system implies that \(((A_{-1} + B)|_X, J^{A_{-1} + B}|_X \Delta B, \Delta C^A)\) generates a regular linear system, such result has been proved by Hadd [16]; If \(Y = X\) and \(B = I\), the above theorem says that both \((A, \Delta B, C)\) and \((A, \Delta B, \Delta C)\) being regular linear system implies that \((A + C, J^{A + C} \Delta B, \Delta C)\) generates a regular linear system. This means that our result is a generalization of [19].
is a regular linear system generated by \((A^I, J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2, W, \overline{W}_{A,B_1}(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + \overline{W}_{A,B_2})\).

**Proof.** By Theorem 3.11 it follows that \[
\begin{align*}
\begin{cases}
\dot{z}(t) &= Lz(t) \\
G_1z(t) &= Kz(t) \\
G_2z(t) &= v(t) \\
y(t) &= Wz(t)
\end{cases}
\] (3.12)
is an abstract linear control system with generator \((A^I, J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2)\). It follows from Theorem 3.8 that \[
\begin{align*}
\begin{cases}
\dot{z}(t) &= Lz(t) \\
G_1z(t) &= Kz(t) \\
G_2z(t) &= 0 \\
y(t) &= Wz(t)
\end{cases}
\]
is an abstract linear observation system with generator \((A^I, W)\) and the restriction of \(W\) to \(D(A^I)\) is equal to \(W_A^A + \overline{W}_{A,B_1}(I - \overline{K}_{A,B_1})^{-1}K_A^A\). By Theorem 3.11 our assumptions imply that \((A^I, J^{A^I}B_2, W_A^A)\) generates a regular linear system. Combining this with the boundedness of operator \(J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2}\) implies that \((A^I, J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2, W_A^A)\) generates a regular linear system. By Theorem 3.2 we obtain that \((A^I, J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2, (I - \overline{K}_{A,B_1})^{-1}K_A^A)\) generates a regular linear system. Since \(\overline{W}_{A,B_1}\) is bounded, \((A^I, J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2, \overline{W}_{A,B_1}(I - \overline{K}_{A,B_1})^{-1}K_A^A)\) generates a regular linear system. Therefore, \((A^I, J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2, W)\) is a regular linear system. Hence the regularity of system (3.12) is obtained by Lemma 2.4.

Next, we shall compute the feedthrough operator. By Theorem 3.2 for any enough big \(Re(\lambda)\), we have
\[
(\lambda - (A^I)_{-1})^{-1}\left(J^{A^I}B_1(I - \overline{K}_{A,B_1})^{-1}\overline{K}_{A,B_2} + J^{A^I}B_2\right) = (\lambda - A_{-1})^{-1}B_1(I - G_{A,B_1,K,A,B_1}(\lambda))^{-1}G_{A,B_2,K,A,B_2}(\lambda) + (\lambda - A_{-1})^{-1}B_2,}
\]
and the transform function of (3.12) is given by
\[ W(\lambda - A_{-1})^{-1}B_1(I - G_{A,B_1,K,K_{A,B_1}}(\lambda))^{-1}G_{A,B_2,K,K_{A,B_2}}(\lambda) + W(\lambda - A_{-1})^{-1}B_2. \]

Observe that the assumption implies that the strong limit
\[ \lim_{\lambda \to +\infty} \left( W(\lambda - A_{-1})^{-1}B_1(I - G_{A,B_1,K,K_{A,B_1}}(\lambda))^{-1}G_{A,B_2,K,K_{A,B_2}}(\lambda) + W(\lambda - A_{-1})^{-1}B_2 \right) \]
\[ = \lim_{\lambda \to +\infty} W(\lambda - A_{-1})^{-1}B_1(I - G_{A,B_1,K,K_{A,B_1}}(\lambda))^{-1}G_{A,B_2,K,K_{A,B_2}}(\lambda) \]
\[ + \lim_{\lambda \to +\infty} W(\lambda - A_{-1})^{-1}B_2 \]
\[ = \left( W_{A,B_1}(I - K_{A,B_1})^{-1}K_{A,B_2} + W_{A,B_2} \right) \]
hold. Therefore the feedthrough operator of (3.12) is
\[ W_{A,B_1}(I - K_{A,B_1})^{-1}K_{A,B_2} + W_{A,B_2}. \]
The proof is therefore completed. ■

**Example 3.14** Consider the following boundary system governed by wave equations
\[
\begin{cases}
  w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, t > 0, \\
  w(x, t) = -\frac{\partial (A^{-1}w)}{\partial \nu}, & x \in \Gamma_1, t \geq 0, \\
  w(x, t) = u(x, t), & x \in \Gamma_0, t \geq 0, \\
  y(x, t) = -\frac{\partial (A^{-1}w)}{\partial \nu}, & x \in \Gamma_0, t \geq 0
\end{cases}
\]
(3.13)

where \( \Omega \subset \mathbb{R}^n, n \geq 2 \) is an open bounded region with smooth \( C^3 \)-boundary \( \partial \Omega = \Gamma_0 \cup \Gamma_1 \). \( \Gamma_0, \Gamma_1 \) are disjoint parts of the boundary relatively open in \( \partial \Omega \), \( \text{int}(\Gamma_1) \neq \emptyset \) and \( \text{int}(\Gamma_0) \neq \emptyset \), \( \nu \) is the unit normal vector of \( \Gamma_0 \) pointing towards the exterior of \( \Omega \), \( u \) is the input function (or control) and \( y \) is the output function (or output).

Let \( H = L^2(\Omega) \times H^{-1}(\Omega) \) be the state space and \( U = L^2(\partial \Gamma_0), V = L^2(\partial \Gamma_1) \) be the control (input) or observation (output) space. Guo and Zhang [9] proved that system
\[
\begin{cases}
  w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, t > 0, \\
  w(x, t) = v(x, t), & x \in \Gamma_1, t \geq 0, \\
  w(x, t) = 0, & x \in \Gamma_0, t \geq 0, \\
  y(x, t) = -\frac{\partial (A^{-1}w)}{\partial \nu}, & x \in \Gamma_1, t \geq 0
\end{cases}
\]
is a regular linear system with feedthrough operator $I$ and with admissible feedback operator $I$. Moreover,

$$
\begin{cases}
    w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, t > 0, \\
    w(x, t) = 0, & x \in \Gamma_1, t \geq 0, \\
    w(x, t) = u(x, t), & x \in \Gamma_0, t \geq 0, \\
    y(x, t) = -\frac{\partial (A^{-1}w)}{\partial \nu}, & x \in \Gamma_1, t \geq 0
\end{cases}
$$

is a regular linear system. By the same procedure, one can verify that

$$
\begin{cases}
    w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, t > 0, \\
    w(x, t) = v(x, t), & x \in \Gamma_1, t \geq 0, \\
    w(x, t) = 0, & x \in \Gamma_0, t \geq 0, \\
    y(x, t) = -\frac{\partial (A^{-1}w)}{\partial \nu}, & x \in \Gamma_0, t \geq 0
\end{cases}
$$

and

$$
\begin{cases}
    w_{tt}(x, t) = \Delta w(x, t), & x \in \Omega, t > 0, \\
    w(x, t) = 0, & x \in \Gamma_1, t \geq 0, \\
    w(x, t) = u(x, t), & x \in \Gamma_0, t \geq 0, \\
    y(x, t) = -\frac{\partial (A^{-1}w)}{\partial \nu}, & x \in \Gamma_1, t \geq 0
\end{cases}
$$

are regular linear systems. Then we claim by Theorem 3.13 that system (3.13) is regular with feedthrough operator $I$.

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