1 Introduction. Let $X$ be an irreducible smooth projective curve of genus $g$ over an algebraically closed field $k$ of characteristic $p > 0$, and $F : X \to X$ the absolute Frobenius morphism on $X$. It is known that pulling back a stable vector bundle on $X$ by $F$ may destroy stability. One may measure the failure of (semi-)stability by the Harder-Narasimhan polygons of vector bundles.

In more formal language, let $n \geq 2$ be an integer, $\mathcal{M}$ the coarse moduli space of stable vector bundles of rank $n$ and a fixed degree on $X$. Applying a theorem of Shatz (an analogue of the Grothendieck specialization theorem for $F$-isocrystals) to the pull-back by $F$ of the universal bundle (assuming the existence) on $\mathcal{M}$, we see that $\mathcal{M}$ has a canonical stratification by Harder-Narasimhan polygons ([S]). This interesting extra structure on $\mathcal{M}$ is a feature of characteristic $p$. However, very little is known about these strata. Scattered constructions of points outside of the largest (semi-stable) stratum can be found in [G], [RR], [R], and [JX].

This paper deals exclusively with $p = 2$ and $n = 2$. On any curve $X$ of genus $\geq 2$, we provide a complete classification of rank-2 semi-stable vector bundles $V$ with $F^*V$ not semi-stable. We also obtain fairly good information about the locus destabilized by Frobenius in the moduli space, including the irreducibility and the dimension of each non-empty Harder-Narasimhan stratum. This shows that the bound in [Su, Theorem 3.1] is sharp. A very interesting consequence of our classification is that high unstability of $F^*V$ implies high stability of $V$.

We conclude this introduction by remarking that the problem studied here can be cast in the generality of principal $G$-bundles over $X$, where $G$ is a connected reductive group over $k$. More precisely, consider the pull-back by $F$ of the universal object on the moduli stack of semi-stable principal $G$-bundles on $X$. Atiyah-Bott’s generalization of the Harder-Narasimhan filtration should then give a canonical stratification of the moduli stack ([AB], see also [C]). In this context, our paper treats the case of $p = 2$, $G = \text{GL}_2$.

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2 A measure of stability. In this paper, “a vector bundle” always means “a vector bundle over $X$”. Following [LN], for a rank-$2$ vector bundle $V$, we put

$$s(V) = \deg(V) - 2 \max\{|\deg(L) : L \hookrightarrow V|\},$$

where the maximum is taken over all rank-$1$ sub-module of $V$. By definition, $s(V) > 0$ (resp. $s(V) \geq 0$) if and only if $V$ is stable (resp. semi-stable). When $s(V) \leq 0$, the information of $(s(V), \deg(V))$ is the same as that of the Harder-Narasimhan polygon of $V$. Therefore, one may regard $s$ as a measure of stability extrapolating the Harder-Narasimhan polygons, though it is only for the rank-$2$ case (for possible variants for the higher rank case, see [BL]; for general reductive group, see [HN]).

3 Raynaud’s distinguished theta characteristic. From now on, $p = 2$ and $g \geq 2$. Following Raynaud [R, §4], we have a distinguished line bundle $B$ on $X$ defined by the exact sequence

$$0 \to \mathcal{O}_X \to F_s \mathcal{O}_X \to B \to 0.$$ 

Raynaud shows that $B^2 \simeq \Omega_X$, i.e. $B$ is a theta characteristic.

Proposition. Let $\xi$ be a line bundle on $X$ and put $V = F_s(\xi \otimes B^{-1})$. Then $V$ is a vector bundle of rank $2$ such that $\det(V) = \xi$, $V$ is stable, and $F^sV$ is not semi-stable. In fact,

$$s(V) \geq g - 1 \text{ and } s(F^sV) = -(2g - 2).$$

If $M$ is the sub-bundle of $F^sV$ of rank $1$ such that $\deg M > \deg(F^sV)/2$, then $M \simeq \xi B$.

Proof. Write $L = \xi \otimes B^{-1}$. On an affine open set $U$ on which $F_s\mathcal{O}_X$, $B$, $L$ are trivial, choose a section $s \in (F_s\mathcal{O}_X)(U)$ such that the image of $s$ generates $B = F_s(\mathcal{O}_X)/\mathcal{O}_X$ on $U$, and a section $t \in L(U)$ generating $L|U$. Then $(t, st)$ generates $(F_sL)|U$ and $t \wedge (st) \mapsto t \otimes s$ is an isomorphism $\det(F_sL)|U \to (L \otimes B)|U$. One can check that this isomorphism is independent of the choices of $s, t$; hence, we obtain an isomorphism $\det(F_sL) \to L \otimes B$ by gluing these isomorphisms over various $U$’s.

Write $L = \xi \otimes B^{-1}$ and $d = \deg L$. Notice that $\deg V = d + g - 1$. Suppose that $M \hookrightarrow V$ is a sub-bundle of rank $1$. By adjunction, there is a non-zero morphism $F^sM \to L$. Therefore, $\deg(F^sM) \leq \deg(L)$. Thus $\deg M \leq d/2 < (d + g - 1)/2 = \deg(V)/2$. Therefore, $V$ is stable and $s(V) \geq g - 1$.

Now consider the identity morphism $F_sL \to F_sL$. By adjunction, this gives a non-zero morphism $F^sV \to L$, which is surjective by a local calculation. The kernel of this morphism is a line bundle of degree $2(d + g - 1) - d = d + 2g - 2 > d + g - 1 = \deg(F^sV)/2$. So $F^sV$ is not semi-stable and $s(F^sV) = -(2g - 2)$. 

Remark. Let $V = F_s(\xi \otimes B^{-1})$. The extension

$$(*) \quad 0 \to \xi \otimes B \to F^sV \to \xi \otimes B^{-1} \to 0$$

2
defines a class in \( \operatorname{Ext}^1(\xi \otimes B^{-1}, \xi \otimes B) \simeq H^1(X, B^2) \simeq k \). This class is trivial precisely when \( \deg(\xi \otimes B^{-1}) \) is even.

**Proof.** Suppose that \( \deg(\xi \otimes B^{-1}) \) is even. Then we can write \( L = \xi \otimes B^{-1} = M^2 \). By [JX, §2], there is an exact sequence \( 0 \to M \to V \to M \otimes B \to 0 \). Pulling back by \( F \), we get \( 0 \to L \to F^*V \to L \otimes B^2 \to 0 \). This shows that \((*)\) is split.

Suppose that \( L = \xi \otimes B^{-1} \) has odd degree \( 2n+1 \). By a theorem of Nagata ([LN], Cf. 8) and the above proposition, there is an exact sequence \( 0 \to M_1 \to V \to M_2 \to 0 \), where \( M_1, M_2 \) are line bundles with degrees \( n \) and \( n+g \) respectively. From the exact sequence \( 0 \to M_1 \to F^*V \to M_2 \to 0 \), we deduce that \( \dim \operatorname{Hom}(L, F^*V) \leq \dim \operatorname{Hom}(L, M_1) + \dim \operatorname{Hom}(L, M_2) = 0 + g = g \) by the Riemann-Roch formula. Since \( \operatorname{Hom}(L, \xi \otimes B) = H^0(X, B^2) \) has dimension \( g \), any morphism \( L \to F^*V \) factors through the sub-module \( \xi \otimes B \) in (\( *)\). Therefore, (\( *\)) is not split. [\( \blacksquare \)]

4 **The basic construction.** Henceforth, fix an integer \( d \). For an injection \( V' \hookrightarrow V'' \) of vector bundles of the same rank, define the \emph{co-length} \( l \) of \( V' \) in \( V'' \) to be the length of the torsion \( \mathcal{O}_X \)-module \( V''/V' \). Clearly, \( s(V') \geq s(V'') - l \).

We now give a basic construction of stable vector bundles \( V \) of rank \( 2 \) with \( F^*V \) not semi-stable. Let \( l \leq g - 2 \) be a non-negative integer, \( L \) a line bundle of degree \( d - 1 - (g - 2 - l) \), and \( V \) a sub-module of \( F_2L \) of co-length \( l \), then \( \deg V = d \) and \( s(V) \geq (g - 1) - l > 0 \) by Proposition 3. Therefore, \( V \) is stable.

On the other hand, by adjunction, there is a morphism \( F^*V \to L \), and the kernel is a line bundle of degree \( g - 2 + (g - 2 - l) > d = \deg(F^*V)/2 \). Therefore, \( F^*V \) is not semi-stable.

5 **Exhaustion.** Suppose that \( V \) is semi-stable of rank \( 2 \) and \( F^*V \) is not semi-stable.

Let \( \xi = \det(V) \) and \( d = \deg \xi = \deg V \). Since \( F^*V \) is not semi-stable and of degree \( 2d \), there are line bundles \( L, L' \) and an exact sequence \( 0 \to L' \to F^*V \to L \to 0 \) with \( \deg L' \geq d + 1 \), \( \deg L \leq d - 1 \). By adjunction, this provides a non-zero morphism \( V \to F_2L \).

If the image is a line bundle \( M \), we have \( \deg M \geq d/2 \) by semi-stability of \( V \), and \( \deg M \leq (d - 1 + g - 1)/2 - (g - 1)/2 = (d - 1)/2 \) by Proposition 3. This is a contradiction.

Thus the image is of rank \( 2 \). Since \( \deg V = d \) and \( \deg(F_2L) \leq d + (g - 2) \), \( V \) is a sub-module of \( F_2L \) of co-length \( l \leq g - 2 \), and \( \deg L = d - 1 - (g - 2 - l) \).

Thus the basic construction yields all semi-stable vector bundles \( V \) of rank \( 2 \), with \( F^*V \) not semi-stable.

5.1 **Corollary.** If \( V \) is semi-stable of rank \( 2 \) with \( F^*V \) not semi-stable, then \( V \) is actually stable. [\( \blacksquare \)]

5.2 **Corollary.** The basic construction with \( l = g - 2 \) already yields all semi-stable vector bundles \( V \) of rank \( 2 \), with \( F^*V \) not semi-stable.
6 Classification. Let \( L \) be a line bundle and let \( Q = Q_1 = Q_{1,L} = \text{Quot}_t(F_s L/X/k) \) be the scheme classifying sub-modules of \( F_s L \) of co-length \( l \). Thus \( V \) arises from the basic construction with \((l', L')\) playing the role of \((l, L)\).

**Proposition.** The basic construction gives a bijection
\[
\bigcup_{0 \leq l \leq g - 2} Q_{l,L}(k) \rightarrow \mathcal{M}_1(k),
\]
where the disjoint union is taken over all \( l \in [0, g - 2] \) and a set of representatives of all isomorphism classes of line bundles \( L \) of degree \( d - 1 - (g - 2 - l) \).

**Proof.** By 5, the map is a surjection. Now suppose that \((l, L, V \subset F_s L)\) and \((l', L', V' \subset F_s L')\) give the same point in \( \mathcal{M}_1(k) \), i.e. \( V \simeq V' \). Since the unstable bundle \( F^*V \) has a unique quotient line bundle of degree \( < \deg(V)/2 \) (i.e. the second graded piece of the Harder-Narasimhan filtration), which is isomorphic to \( L \), we must have \( L = L' \). Consider the diagram
\[
\begin{array}{ccc}
F^*V & \longrightarrow & L \\
\downarrow & & \downarrow \\
F^*V' & \longrightarrow & L',
\end{array}
\]
where the vertical arrow is induced from an isomorphism \( V \simeq V' \) and the horizontal arrows are the unique quotient maps. This diagram is commutative up to a multiplicative scalar in \( k^* \).

By adjunction, \( V \hookrightarrow F_s L \) and \( V' \hookrightarrow F_s L \) have the same image. In other words, \( V = V' \) as sub-modules of \( F_s L \). This proves the injectivity of the map. \( \blacksquare \)
7 Moduli space. To ease the notation, let \( d_i = d - 1 - (g - 2 - l) \). Let \( \text{Pic}^d_i X \) be the moduli space of line bundles of degree \( d_i \) on \( X \), and \( L \to \text{Pic}^d_i(X) \times X \) the universal line bundle.

By [FGA, 3.2], there is a scheme \( \Omega = \Omega_l = \text{Quot}_l((\text{id} \times F)_* L / (\text{Pic}^d_i(X) \times X) / \text{Pic}^d_i X) \to \text{Pic}^d_i X \) such that \( \Omega_x \) (the fiber at \( x \)) is \( Q_{\mathcal{L}_x} \) for all \( x \in (\text{Pic}^d_i(X))(k) \). By the same argument as before, there is an open sub-scheme \( \Omega^* \subset \Omega \) such that \( \Omega^*_x = Q_{\mathcal{L}_x}^\prime \) for all \( x \in \text{Pic}^d_i(X)(k) \). The scheme \( \Omega \) is projective over \( \text{Pic}^d_i(X) \) ([FGA, 3.2]), hence is proper over \( k \). By checking the condition of formal smoothness (cf. [L, 8.2.1]), it can be shown that \( \Omega \) is smooth over \( \text{Pic}^d_i(X) \), hence is smooth over \( k \).

The coarse moduli scheme \( \overline{M} \) is canonically stratified by Harder-Narasimhan polygons. Concretely, for \( j \geq 0 \), let \( P_j \) be the polygon from \( (0, 0) \) to \( (1, d + j) \) to \( (0, 2d) \). Let \( \overline{M}_0 = \overline{M} \), and for \( j \geq 1 \), let \( \overline{M}_j(k) \) be the subset of \( \overline{M}(k) \) parametrizing those \( V \)'s such that the Harder-Narasimhan polygons ([S]) of \( F^* V \) lie above or are equal to \( P_j \). Notice that \( \overline{M}_1(k) \) agrees with the one defined in 6.

As mentioned in the introduction, the existence of a universal bundle on \( \overline{M} \) would imply that each \( \overline{M}_j(k) \) is Zariski closed by Shat's theorem [S]. In general, one can show that \( \overline{M}_j(k) \) is closed in \( \text{GIT} \) (geometric invariant theory) construction of \( \overline{M} \). This fact also follows from our basic construction:

**Theorem.** The subset \( \overline{M}_j(k) \) is Zariski closed in \( \overline{M}(k) \), hence underlies a reduced closed sub-scheme \( \overline{M}_j \) of \( \overline{M} \). The scheme \( \overline{M}_j \) is proper. The Harder-Narasimhan stratum \( \overline{M}_j \setminus \overline{M}_{j+1} \) is non-empty precisely when \( 0 \leq j \leq g - 1 \). For \( 1 \leq j \leq g - 1 \), write \( l = g - 1 - j \). Then there is a canonical morphism

\[
\Omega_l \to \overline{M}
\]

which has scheme-theoretic image \( \overline{M}_j \) and induces a bijection from \( \Omega^*_l(k) \) to \( \overline{M}_j(k) \setminus \overline{M}_{j+1}(k) \).

**Proof.** Suppose \( 0 \leq l \leq g - 2 \) and \( j + l = g - 1 \). The universal object \( V \to \Omega_l \times X \) is a family of stable vector bundles on \( X \). This induces a canonical morphism \( \Omega_l \to \overline{M} \). The image of \( \Omega_j(k) \) is precisely \( \overline{M}_j(k) \) by (the proof of) Corollary 5.2. Since \( \Omega_l \) is proper, \( \overline{M}_j \) is proper and closed in \( \overline{M} \). The rest of the proposition follows from 6 and 5, and the fact that \( \Omega^*_l(k) \) is non-empty for \( 0 \leq l \leq g - 2 \) (see Lemma 9.3).

8 Remark. By a theorem of Nagata ([LN], [HN]), \( s(V) \leq g \) for all \( V \). Therefore, \( s(V) \leq g \) if \( \deg V \equiv g \) (mod 2), and \( s(V) \leq g - 1 \) if \( \deg V \not\equiv g \) (mod 2). By Proposition 3, \( V = F_s L \) achieves the maximum value of \( s \) among rank-2 vector bundles of the same degree.

By the preceding theorem, vector bundles of the form \( V = F_s L \) are precisely members of the smallest non-empty Harder-Narasimhan stratum \( \text{M}_{g-1} \). Therefore, in a sense \( V \) is most stable yet \( F^* V \) is most unstable. More generally, for \( 1 \leq j \leq g - 1 \), we have (from 4)

\[
s(M_j(k)) \geq \begin{cases} j & \text{if } d \equiv j \pmod{2}, \\ j + 1 & \text{if } d \not\equiv j \pmod{2}. \end{cases}
\]
Therefore, high unstability of $F^*V$ implies high stability of $V$.

9 Irreducibility. We will make use of the following simple lemma.

9.1 Lemma. Let $Y$ be a proper scheme over $k$, $S$ an irreducible scheme of finite type over $k$ of dimension $s$, $r$ an integer $\geq 0$, and $f : Y \to S$ a surjective morphism. Suppose that all fibers of $f$ are irreducible of dimension $r$. Then $Y$ is irreducible of dimension $s + r$. ■

9.2 Lemma. The scheme $Q = Q_l$ is irreducible of dimension $2l + g$.

PROOF. There is a surjective morphism ([FGA, §6])

$$\delta : Q \to \text{Div}^l(X) = \text{Sym}^l(X), \quad q \mapsto \sum_{P \in X/k} \text{length}_{\mathcal{O}_P}((F_*\mathcal{L}_{\mathcal{P}(q)})/\mathcal{V}_q) \cdot P.$$ 

The morphism $Q \to \text{Div}^l(X) \times \text{Pic}^{d_1}(X)$ is again a surjection. The fibers are irreducible schemes of dimension $l$ according to the last lemma of [MX]. Since $Q$ is proper, the result follows from Lemma 9.1. ■

9.3 Lemma. $Q^*$ is open and dense in $Q$.

PROOF. By the construction in 6 and 7, $Q^*$ is open in $Q$. Since $Q$ is irreducible of dimension $2l + g$, it suffices to show that $Q^*$ is non-empty. We will do more by exhibiting an open subset of $Q^*$ of dimension $2l + g$.

Indeed, let $B(X, l) \subset \text{Div}^l(X)$ be the open sub-scheme parametrizing multiplicity-free divisors of degree $l$, also known as the configuration space of unordered $l$ points in $X$. Let $U$ be the inverse image of $B(X, l) \times \text{Pic}^{d_1}(X)$ under $Q^* \to \text{Div}^l(X) \times \text{Pic}^{d_1}(X)$. A quick calculation shows that each fiber of $U \to B(X, l) \times \text{Pic}^{d_1}(X)$ is isomorphic to $\mathbb{A}^l$. Therefore, $U$ is an open subset of $Q^*$ of dimension $2l + g$. ■

9.4 Theorem. For $1 \leq j \leq g - 1$, $M_j$ is proper, irreducible, and of dimension $g + 2(g - 1 - j)$. In particular, $M_1$ is irreducible and of dimension $3g - 4$. ■

10 Fixing the determinant. Fix a line bundle $\xi$ of degree $d$. Let $\overline{M}(\xi) \subset \overline{M}$ be the closed sub-scheme of $\overline{M}$ parametrizing those $V$’s with $\text{det}(V) = \xi$. Let $M_j(\xi) = \overline{M}(\xi) \cap M_j$ for $j \geq 0$.

Remark. For $1 \leq j \leq g - 1$, $\dim M_j(\xi) = 2(g - 1 - j)$. In particular, $\dim M_1(\xi) = 2(g - 2)$.

PROOF. Since $M_j(\xi)$ is nothing but the fiber of the surjective morphism $\text{det} : M_j \to \text{Pic}^d(X)$, it has dimension $2(g - 1 - j)$ for a dense open set of $\xi \in \text{Pic}^d(X)(k)$. However, $M_j(\xi_1)$ is isomorphic to $M_j(\xi_2)$ for all $\xi_1, \xi_2 \in \text{Pic}^d(X)(k)$, via $V \mapsto V \otimes L$, where $L^2 \simeq \xi_2 \otimes \xi_1^{-1}$. Thus the remark is clear. ■
A slight variation of the above argument shows that $M_j(\xi)$ is irreducible. Alternatively, assume $1 \leq j \leq g - 1$. Let $l = g - 1 - j$ and let $Q(\xi) = Q_l(\xi)$ be the inverse image of $\xi$ under $Q \to \text{Pic}^d(X)$, $q \mapsto \det(V_q)$. Since $\det(V_q) = B \otimes L_{\pi(q)} \otimes \mathcal{O}(\delta(q))$, the morphism $\det: Q \to \text{Pic}^d(X)$ factors as

$$Q \to \text{Div}^l(X) \times \text{Pic}^{d_l}(X) \xrightarrow{\psi} \text{Pic}^d(X),$$

where $\psi$ is $(D, L) \mapsto B \otimes L \otimes \mathcal{O}(-D)$. It is clear that $\psi^{-1}(\xi)$ is isomorphic to $\text{Div}^l(X)$, and hence is an irreducible variety.

The fibers of $Q(\xi) \to \psi^{-1}(\xi)$ are just some fibers of $Q \to \text{Div}^l(X) \times \text{Pic}^{d_l}(X)$; hence they are irreducible of dimension $l$ as in the proof of Lemma 9.2. Being a closed sub-scheme of $Q$, $Q(\xi)$ is proper, thus, irreducible by Lemma 9.1. Now it is easy to deduce

**Theorem.** The scheme $\overline{M}(\xi)$ admits a canonical stratification by Harder-Narasimhan polygons

$$\emptyset = M_g(\xi) \subset M_{g-1}(\xi) \subset \cdots \subset M_0(\xi) = \overline{M}(\xi),$$

with $M_j(\xi)$ non-empty, proper, irreducible, and of dimension $2(g - 1 - j)$ for $1 \leq j \leq g - 1$. $lacksquare$

**11 A variant.** Let $M'(k)$ be the subset of $\overline{M}(k)$ consisting of those $V$ such that $F^*V$ is not stable. Clearly, $M'(k) \supset M_1(k)$.

By Corollary 5.1, the closed subset $M_{sa}(k) = \overline{M}(k) \setminus M(k)$ is contained in $M'(k) \setminus M_1(k)$. On the other hand, if $V \in M'(k) \setminus M_{sa}(k)$, the argument of 5 shows that there is a line bundle $L$ of degree $d$ such that $V \hookrightarrow F_*L$ is a sub-module of co-length $\leq g - 1$. Conversely, the argument of 4 shows that if $V$ is of co-length $\leq g - 1$ in $F_*L$ for some $L$ of degree $d$, then $V \in M'(k)$.

Thus we conclude that $M'(k)$ is the union of $M_{sa}(k)$ and the image $M'_0(k)$ of $Q_{g-1}(k)$ for a suitable morphism $Q_{g-1} \to \overline{M}$, where $Q_{g-1}$ is defined in 7. It follows that $M'_0(k)$ and $M'(k)$ are Zariski closed in $\overline{M}(k)$, hence are sets of $k$-points of reduced closed sub-scheme $M'_0$ and $M'$ of $\overline{M}$.

**Theorem.** The scheme $M'_0$ is irreducible of dimension $3g - 2$. It contains two disjoint closed subsets: $M'_0 \cap M_{sa}$, which is irreducible of dimension $2g - 1$ when $d$ is even and empty when $d$ is odd, and $M_1$, which is irreducible of dimension $3g - 4$.

**Remark.** $M' \setminus M_1$ is the first stratum in the $s$-stratification ([LN]) which is not a Harder-Narasimhan stratum. The other $s$-stratas are more complicated and not pursued here.

**Proof.** Since $Q_{g-1}$ is irreducible, $M'_0$ is irreducible. We now analyze $M'_0 \cap M_{sa}$. Suppose that $V \in M'_0(k) \cap M_{sa}(k)$. Then $d = \deg V$ is even and there exists $L$ of degree $d$ such that $V$ is a sub-module of $F_*L$ of co-length $g - 1$. By assumption, there is a sub-bundle $M$ of $V$ of degree $d/2$. Adjunction applied to the composition $M \hookrightarrow V \hookrightarrow F_*L$ provides a non-zero
morphism $F^*(M) = M^2 \to L$. This implies that $M^2 \simeq L$. We may assume that $L = M^2$. Since there is only one (modulo $k^*$) non-zero morphism $M^2 \to L$, there is only one non-zero morphism $M \to F_*(F^*M)$. By [JX, §2], this morphism is part of an exact sequence $0 \to M \to F_*(F^*M) \to M \otimes B \to 0$. Thus to have $V$ is to have a sub-module of $M \otimes B$ of co-length $g-1$. Conversely, starting with a sub-module of $M \otimes B$ of co-length $g-1$, we obtain a vector bundle $V \in \mathcal{M}_g(k) \cap \mathcal{M}^h(k)$ as the inverse image of that sub-module in $F_*(F^*M)$.

The sub-modules of $M \otimes B$ of co-length $g-1$ are of the form $M \otimes B \otimes \mathcal{O}(-D)$ for $D \in \text{Div}^{g-1}(X)(k)$. Thus there is a morphism $\pi' : \mathcal{Q} = \text{Div}^{g-1}(X) \times \text{Pic}^{d/2}(X) \to \overline{M}$ inducing a surjection $\mathcal{Q}(k) \to \mathcal{M}_g(k) \cap \mathcal{M}^h(k)$. We claim that this morphism is generically finite of separable degree at most 2. This claim implies that $\mathcal{M}_g \cap \mathcal{M}^h$ is irreducible of dimension $2g-1$.

Indeed, there is an open subset $U$ of $\text{Div}^{g-1}(X)(k)$ such that if $D, D' \in U$ are distinct, then $D \neq D'$. We now show that $\pi'(U \times \text{Pic}^{d/2}(X)(k))$ is at most 2-to-1. Suppose that $D \in U$, $M \in \text{Pic}^{d/2}(X)(k)$, and $\pi'(D, M) = V$. Then $V$ has at most two isomorphism classes of rank-1 sub-bundles of degree $d/2$, and $M$ is one of them. After obtaining $M$, one can determine $D$ uniquely by the condition $\det(V) \simeq M^2 \otimes B \otimes \mathcal{O}(-D)$. This proves the claim.

Next, we consider the morphism $\mathcal{Q}_{g-1} \to \mathcal{M}_g$. It induces a surjection $\mathcal{Q}_{g-1}^*(k) \to \mathcal{M}_g(k) \cap \mathcal{M}_1(k)$. Again the claim is that the morphism is generically finite of separable degree at most 2. This claim implies that $\mathcal{M}_g$ is irreducible of dimension $3g-2$.

Indeed, let $U$ be the open subset of $\mathcal{Q}_{g-1}^*(k)$ consisting of those $q$’s such that $\mathcal{O}(2\delta(q)) \not\simeq \Omega_X$. Now assume that $q \in U$ gives rise to $V \in \mathcal{M}_g(k)$. Then there is an exact sequence $0 \to L \otimes B^2 \otimes \mathcal{O}(-2\delta(q)) \to F^*V \to L \to 0$, where $L = L_{\pi(q)}$. The assumption on $q$ implies that $F^*V$ has at most 2 quotient line bundles of degree $d$, say $F^*V \to L_1$ and $F^*V \to L_2$. Then $q$ must be one of the two data $V \leftrightarrow F_2L_1$ or $V \leftrightarrow F_2L_2$ provided by adjunction. This proves the claim.

12 Example. When $g = 2$, $\mathcal{M}_1(\xi)$ is a single point, corresponding to the vector bundle $F_*(\xi \otimes B^{-1})$.

When $\xi = B$, this refines a result of Joshi and one of us [JX, 1.1], which says that $\mathcal{M}_1(\xi)$ is a single $\text{Pic}(X)[2]$-orbit.

When $\xi = \mathcal{O}_X$, this extends a theorem of Mehta [JX, 3.2], which states that there are only finitely many rank-2 semi-stable vector bundles $V$’s on $X$ with $\det(V) = \mathcal{O}_X$ and $F^*V$ not semi-stable when $p \geq 3, g = 2$. We now have this result for $p = 2, g = 2$ with the stronger conclusion of uniqueness.

13 Erratum for [JX]. We correct a minor error in the statement of [JX, Theorem 1.1]. The expression “$V_1 \in \text{Ext}^1(L_\theta, \mathcal{O}_X)$” should be replaced by “$V_1 \in S_\theta$” (the original version is valid when $L_\theta = B$). Also, $\Omega$ should be replaced by $L_\theta$.

14 References.

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