Dark matter explanation from quasi-metric gravity

Dag Østvang

Department of Physics, Norwegian University of Science and Technology, Høgskoleringen 5, N-7491 Trondheim, Norway

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Abstract. The gravitational field of an isolated, axisymmetric flat disk of spinning dust is calculated approximatively in the weak-field limit of quasi-metric gravity. Boundary conditions single out the exponential disk as a “preferred” physical surface density profile. Moreover, collective properties of the disk, in the form of an extra “induced associated” surface density playing the role of “dark matter” also emerge. Taken as an idealized model of spiral galaxy thin disks, it is shown that including this “dark matter” into the model as a gravitating source yields asymptotically flat rotation curves and a correspondence with MOND.

1 Introduction

The concept of dark matter (DM), without which mainstream astrophysics and cosmology would not be observationally viable, is an acknowledged part of the modern scientific worldview. Yet all the observational evidence in favour of its existence is based on interpretations of astronomical data coming from distance scales much larger than the solar system. In addition, despite its rather flexible nature, the DM proposal faces some real observational challenges. In particular galactic phenomenology, including spiral galaxy rotation curve shapes and the seemingly existence of a fixed acceleration scale, seems difficult to understand in terms of DM.

This motivates the alternative approach of trying to explain anomalous galactic observations from modified gravity. The most famous of these approaches is the proposal known as MOND; i.e., a specific modification of Newtonian dynamics whenever the gravitational acceleration falls below a critical value $a_0$. The fact is that MOND does a very good job of modelling rotation curve shapes of spiral galaxies just from their visible matter content, using only one freely variable parameter $a_0$. That this feat is possible at all makes the DM approach look suspicious in comparison since no $a$ priori reason is given for why a much greater variety of rotation curve shapes is not observed. However, MOND does not work so well on larger distance scales such as clusters of galaxies. Besides, relativistic extensions of MOND involve arbitrary extra fields reducing their predictive powers and thus their advantages vis-à-vis DM.

Rather than trying to modify gravity by introducing extra fields tailored to fit galactic phenomenology, a less contrived approach would be to adopt a general alternative framework of relativistic gravity and see if galactic phenomena can be correctly predicted from first principles. Such an approach can hardly be made to work within the traditional framework of metric theories of gravity [1]. However, another possibility is the so-called quasi-metric framework (QMF) published some time ago [2,3]. Quasi-metric gravity is rather arcane and it has not yet been shown to be viable. On the other hand, nor is it obvious how conflicts with observations, even if it would seem so at first glance. The QMF is also radical inasmuch as the role of the cosmic expansion is described very differently from its counterpart in metric theories of gravity. Specifically, the QMF predicts quantitatively, and from first principles, how the cosmic expansion influences gravitationally bound systems. In particular, the QMF predicts how the cosmic expansion should affect the solar system and that these effects should be observable. A number of observed solar system phenomena have been reinterpreted and shown to be in good agreement with these predictions as long as no extra theory-dependent assumptions are made [4]. Unfortunately, these results indicate that general relativity (GR) is fundamentally flawed and that interpretations of some solar system observations are crucially theory-dependent. Well-known results based on traditional interpretations of such observations may therefore be unreliable. Moreover, questioning traditional interpretations is probably why quasi-metric gravity is able to explain galactic phenomena from first principles as well, this will contribute to undercutting the relevance of the DM concept. In that case the legitimacy of mainstream astrophysics will be weakened even further.

\[ \text{e-mail: dagost@tf.phys.ntnu.no} \]
The goal of showing the compatibility of the QMF with galactic phenomena, without the assumption of DM, will be partially fulfilled in this paper. That is, we calculate the gravitational field of a very thin, rotating disk of dust in the nonrelativistic limit of quasi-metric gravity. Taking the spinning disk as an idealised model of a spiral galaxy, we indicate a solution of some galactic observations without the need of DM. The correspondence of this solution to that of MOND is also discussed. But first, in the next section, a brief introduction to quasi-metric gravity will be presented.

2 Basic quasi-metric gravity

The QMF has been described in detail elsewhere [2–4]. Here we include only a minimum of motivation and general formulae necessary to do the calculations presented in later sections.

The basic motivation for introducing the QMF is of a very general philosophical nature. That is, traditional field theories consist of two independent parts; field equations and initial conditions. This general form ensures that field theories can in principle be applied to all physical systems within their domain of validity. But for cosmology this general flexibility is a liability; since the Universe is unique from an observational point of view, it is in principle impossible to have observational knowledge of alternatives to cosmic initial conditions, global evolution and structure. Any diversity of such possibilities represents a serious limitation to what can be known in principle, and should be avoided if possible. That is, since the Universe is observationally unique, so should the nature of its global evolution be.

It turns out that, to construct a general mathematical framework fulfilling this requirement, one is pretty much led to the geometrical structure of the QMF. Moreover, the QMF accomplishes this requirement by describing the global cosmic expansion as an absolute, prior-geometric phenomenon not being part of space-time's causal structure. In this way, the cosmic expansion does not depend on gravitational field equations or initial conditions, meaning that the Universe is not described as a purely dynamical system. That is, the QMF describes the cosmic expansion as “new physics” with no correspondence (in any limit) to the standard Lorentzian space-time framework.

Similarly to the Robertson-Walker (RW) manifolds in GR, the cosmic expansion in the QMF is defined by means of a family of “preferred” observers, the so-called fundamental observers (FOs). A further similarity with the RW manifolds is the existence of a global time function \( t \), such that \( t \) splits up space-time into a “distinguished” set of spatial hypersurfaces, the so-called fundamental hypersurfaces (FHSs). But since the cosmic expansion in the QMF by hypothesis is not part of space-time’s causal structure, \( t \) cannot be an ordinary time coordinate on a Lorentzian manifold. Rather, \( t \) should play the role of an independent evolution parameter parametrizing any change in the space-time geometry that has to do with the cosmic expansion. On the other hand, space-time must also be equipped with a causal structure in the form of a Lorentzian manifold. This Lorentzian manifold must also accommodate the FOs and the FHSs, which means that its topology should allow the existence of a global ordinary time coordinate \( x^0 \).

(Note that, to ensure the uniqueness of this construction, the FHSs must be compact.)

Taking into account the above considerations, the geometrical basis of the QMF can now be defined. That is, the basic geometrical structure underlying the QMF consists of a 5-dimensional differentiable manifold with topology \( M \times R_1 \), where \( M = S \times R_2 \) is a Lorentzian space-time manifold, \( R_1 \) and \( R_2 \) both denote the real line and \( S \) is a compact 3-dimensional manifold (without boundaries). This means that, in addition to the usual time dimension and 3 space dimensions, there is an extra time dimension represented by the global time function \( t \). Moreover, the manifold \( M \times R_1 \) is equipped with two degenerate 5-dimensional (covariant) metrics \( \bar{g}_l \) and \( g_l \), where the degeneracies are determined by the conditions \( \bar{g}_l (\frac{\partial}{\partial t}) \equiv 0 \) and \( g_l (\frac{\partial}{\partial t}) \equiv 0 \), respectively. The metric \( \bar{g}_l \) is directly coupled to matter fields via gravitational field equations, whereas the “physical” metric \( g_l \) can be constructed from \( \bar{g}_l \) in a way described in refs. [2, 3]. (See also sect. 6.) Note that \( \bar{g}_l \) and \( g_l \) have the property that the FOs always move orthogonally to the FHSs.

To reduce space-time to 4 dimensions, one obtains the quasi-metric space-time manifold \( N \) by slicing the submanifold determined by the equation \( x^0 = ct \) out of the 5-dimensional differentiable manifold \( M \times R_1 \). It is essential that this slicing is unique since the two global time coordinates should be physically equivalent; the only reason to separate between them is that they are designed to parameterize fundamentally different physical phenomena. Since the geometric structure on \( N \) is inherited from that on \( M \times R_1 \) just by restricting the fields to \( N \) (no projections), the 5-dimensional degenerate metric fields \( \bar{g}_l \) and \( g_l \) may be regarded as one-parameter families of Lorentzian 4-metrics on \( N \) (this terminology is merely a matter of semantics). Note that the existence of a “preferred frame” is an intrinsic, global geometric property of quasi-metric space-time. (For an isolated system, the preferred frame is the one where the system is at rest, see ref. [5].) Furthermore, there exists a set of particular coordinate systems especially well adapted to the geometrical structure of quasi-metric space-time, the global time coordinate systems (GTCSs). A coordinate system is a GTCS iff the time coordinate \( x^0 \) is related to \( t \) via \( x^0 = ct \) in \( N \).

Since the role of \( t \) is to describe how the cosmic expansion directly influences space-time geometry, \( t \) should enter \( \bar{g}_l \) and \( g_l \) explicitly as a scale factor. However, unlike its counterpart in the RW models, this scale factor cannot be calculated from gravitational field equations, but must be an “absolute” quantity. Since the form of the scale factor should not introduce any extra arbitrary scale or parameter, the only possible option for a scale factor with the dimension of length is to set it equal to \( ct \). This scale factor may be multiplied with a second, dimensionless scale...
factor taking into the account the effects of gravity. But since the geometry of the FHSs in \((\mathcal{N}, \bar{g}_t)\) is postulated to represent a measure of gravitational scales in terms of atomic units [2,3], any extra dimensionless scale factor should enter \(\bar{g}_t\) as a conformal factor.

Furthermore, since there is no reason to introduce any nontrivial spatial topology, the global basic geometry of the FHSs (neglecting the effects of gravity) should be that of the 3-sphere \(S^3\). To fulfill all said requirements and to avoid that this fixation of the spatial geometry interferes with the dynamics of \(\bar{g}_t\), it should take a restricted form. It then turns out that the most general form of \(\bar{g}_t\) (expressed in a GTCS) can be written as a family of line elements (using Einstein’s summation convention)

\[
\overline{ds}^2 = \left[ \bar{N}^t \bar{N}_a - \bar{N}^2_t \right] (dx^0)^2 + 2 \left( \frac{t}{t_0} \bar{N}_i dx^i dx^0 + \frac{t^2}{t_0^2} \bar{N}^2_t S_{ik} dx^i dx^k, \right. \tag{1}
\]

where \(t_0\) is some arbitrary reference epoch setting the scale of the spatial coordinates, \(\bar{N}_t\) is the family of lapse functions of the FOs and \(\bar{N}^k_t\) are the components of the shift vector family of the FOs in \((\mathcal{N}, \bar{g}_t)\). Moreover \(S_{ik} dx^i dx^k\) is the metric of \(S^3\) (with radius equal to \(ct_0\)) and \(\bar{N}_t \equiv \bar{N}^2_t S_{ik} \bar{N}^k_t\).

The affine structure constructed on \(\mathcal{M} \times \mathbb{R}_\gamma\) (see below) limits any possible \(t\) dependence of the quantities present in eq. (1). Specifically, \(\bar{N}_t\) may depend explicitly on \(t\), whereas \(\bar{N}^i\) not. Note that the form (1) of \(\bar{g}_t\) is strictly preserved only under coordinate transformations between GTCSs where the spatial coordinates do not depend on \(t\) (but some exceptions to this rule exist). Also note that, since the 5-dimensional metrics are degenerate, there are no components of lapse and shift in the \(t\)-direction (i.e., there is no motion and proper time does not elapse along the \(t\)-direction).

Next, \((\mathcal{N}, \bar{g}_t)\) and \((\mathcal{N}, g_t)\) are equipped with linear and symmetric connections \(\hat{\nabla}\) and \(\hat{\nabla}\), respectively. These connections are identified with the usual Levi-Civita connection for constant \(t\), yielding the standard connection coefficients, while the rest of the connection coefficients are determined by the requirements

\[
\hat{\nabla}_{\mathcal{N}} \bar{g}_t = 0, \quad \hat{\nabla}_n \bar{g}_t = 0, \quad \hat{\nabla}_n g_t = 0, \quad \hat{\nabla}_n n_t = 0, \tag{2}
\]

where \(n_t\) and \(n_s\) are families of unit normal vector fields in \((\mathcal{N}, \bar{g}_t)\) and \((\mathcal{N}, g_t)\), respectively. The requirements shown in eq. (2) yield some extra, potentially nonzero connection coefficients; see refs. [2,3] for these.

The restricted form (1) of \(\bar{g}_t\) implies that a full coupling of the active stress-energy tensor \(T_t\) [2,3] to space-time curvature cannot exist. Moreover, since the restrictions involve the spatial geometry only and since these restrictions should not affect the dynamics of \(\bar{g}_t\), the intrinsic geometry of the FHSs cannot couple explicitly to matter sources. Besides, only that part of the space-time curvature obtained by holding \(t\) constant should couple to matter. Fortunately, it turns out that a subset of the Einstein field equations can be tailored to \(\bar{g}_t\), so that a partial coupling exists [2,3], having the desired properties. The field equations then read (expressed in a GTCS)

\[
2\bar{R}_{(t)\perp\perp} = 2(\bar{e}^{-2} a^2 x_{(i)} + \bar{e}^{-4} a_{(i)j} a_{(j)}^2) - K^{(t)ik} \bar{K}^{(t)k} + L_{\bar{L}} \bar{K}^{(t)l} \tag{3}
\]

\[
\bar{R}_{(t)\perp} = K^{(t)ij} - L_{\bar{L}} T_{(t)\perp} = \kappa T_{(t)\perp}, \tag{4}
\]

Here \(\bar{R}_t\) is the Ricci tensor family corresponding to the metric family \(\bar{g}_t\) and the symbol “\(\perp\)” denotes a scalar product with \(-\bar{n}_t\). Moreover, \(L_{\bar{L}}\) denotes the Lie derivative in the direction normal to the FHSs, \(\bar{K}_t\) denotes the extrinsic curvature tensor family (with trace \(\bar{K}_t\)) of the FHSs, a “hat” denotes an object projected into the FHSs and the symbol “\(\perp\)" denotes spatial covariant derivation. Finally \(\kappa \equiv 8\pi G/c^4\), where the value of \(G\) is by convention chosen as that measured in a (hypothetical) local gravitational experiment in an empty universe at epoch \(t_0\).

An explicit coordinate expression for \(\bar{K}_t\) may be calculated from eq. (1). This expression reads (using a GTCS)

\[
\bar{K}^{(t)ij} = \frac{t}{2t_0 N_t} \left( \bar{N}^{(i)}_{(j)} + \bar{N}^{(j)}_{(i)} \right) + \left( \bar{N}^{(i)}_{L} - \frac{t_0}{t} \bar{e}^{-2} a_{(i)k} \bar{N}^{(k)}_{L} / \bar{N}_t \right) \frac{t^2}{t_0^2} \bar{N}^2_t S_{ij}. \tag{5}
\]

As should be clear by now, the most characteristic feature of the QMF is the existence of a nonmetric sector of quasi-geometric space-time, describing the cosmic expansion as a general physical phenomenon not depending on space-time’s causal structure. This feature makes the QMF unique and distinguishes it from all other, more “standard” theories of modified gravity where it is assumed that space-time is still modelled as a Lorentzian manifold, but with field equations derived from some alternative action. But even within its metric sector, the QMF differs from other modified gravity theories by the existence of an indirectly coupled dynamical degree of freedom represented by the transformation \(\bar{g}_t \rightarrow g_t\) [2,3]. This transformation is of a purely geometrical nature and its effects are at the post-Newtonian level, so it does not have much relevance for the results presented in this paper (but see sect. 6).
On the other hand, the transformation $\bar{g}_t \rightarrow g_t$ is of crucial importance for the QMF to pass the solar system tests [4]. To have a weak-field approximation scheme valid for the QMF, the general weak-field form of $g_t$ should be worked out in an appropriate formulation. This can be done by first analyzing the field equations (3) and (4) within the parametrized post-Newtonian (PPN) formalism to give the general weak-field form of $\bar{g}_t$. For example, the PPN parameters $\gamma$ and $\beta$ corresponding to the metric sector of $\bar{g}_t$ can easily be calculated yielding $\gamma = -1, \beta = 0$ [2, 3]. Second, one must work out the weak-field effect of the transformation $\bar{g}_t \rightarrow g_t$. This transformation changes the geometry in a way not covered by the PPN formalism. However, for the special case of a spherically symmetric vacuum, the transformed geometry corresponds to effective parameter values $\gamma = 1, \beta = 1$, i.e., identical to the GR values. On the other hand, for deviations from spherical symmetry, the weak-field approximation schemes for the two theories will be different.

The Sun is very nearly spherically symmetric since the largest deviation from spherical symmetry is given from its quadrupole moment $J_2$ (estimated to be not larger than $\sim 10^{-5}$). Moreover, the effect of a nonzero $J_2$ on the transformation $\bar{g}_t \rightarrow g_t$ is at the post-post-Newtonian level [5]. This means that the difference in the effective PPN parameters $\gamma$ and $\beta$ between the QMF and GR for observations in the solar system should be unmeasurably small, so that such a correspondence could occur for special cases (not covered by the PPN formalism), even if no general weak-field correspondence exists. Indeed, it is possible since the shapes of orbits are unaffected by the cosmic expansion while orbital sizes, orbital periods and active gravitational masses increase according to the Hubble law [4]. The Schapiro delay test is also passed since any extra delay due to the cosmic expansion is insignificant for these tests.) Also the geodetic and Lense-Thirring effects are the same as in GR within the current experimental accuracy [5].

At this point, it should be clear that the basic structure of the QMF has nothing whatsoever in common with that of MOND or any of its relativistic extensions, and in particular not in the weak-field approximation. How then is it possible to get a correspondence with MOND at all? The answer to this is that such a correspondence occurs as a nonlocal, collective phenomenon for the case of a flat disk. And as we shall see, the combination of a flat matter distribution and a Universe with spherical spatial geometry is what makes this correspondence possible.

3 Axisymmetric, metrically stationary, flat systems

3.1 Real weak-field solution

The most general form of $\bar{g}_t$ for an isolated, metrically stationary, axially symmetric system can be found from eq. (1). Introducing a spherically symmetric GTCS $\{x^0, \rho, \theta, \phi\}$, where $\rho$ is an isotropic radial coordinate, $N_t$ and $N_\phi$ do not depend on $\phi$ and eq. (1) takes the form [5]

$$\bar{d}s_t^2 = \bar{B} \left[ - (1 - \bar{V}^2 \rho^2 \sin^2 \theta) (dx^0)^2 + 2 \frac{t}{t_0} \bar{V} \rho^2 \sin^2 \theta d\phi dx^0 + \frac{t^2}{t_0^2} \left( \frac{d\rho^2}{\rho^2} + \rho^2 d\Omega^2 \right) \right],$$  (6)

where $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2$, $\Xi_0 \equiv ct_0$, $\bar{B} \equiv N_t^2$ and $\bar{V} \equiv \frac{N_\phi}{\bar{B} \rho^2 \sin \theta}$. The field equations (3) and (4) exterior to the source then read [5]

$$\left( 1 - \frac{\rho^2}{\Xi_0^2} \right) B_{\rho\rho} + \frac{1}{\rho^2} B_{\rho\theta} + \frac{2}{\rho} \left( 1 - \frac{3 \rho^2}{\Xi_0^2} \right) B_{\theta\theta} + \csc \theta \frac{\partial}{\partial \rho} \left( \hat{B}_{\rho} \right) = \frac{\rho^2}{\Xi_0^2} \left( (V_{,\rho})^2 + \frac{1}{\rho^2} (V_{,\theta})^2 \right),$$  (7)

$$\left( 1 - \frac{\rho^2}{\Xi_0^2} \right) V_{,\rho\rho} + \frac{1}{\rho^2} V_{,\rho\theta} + \frac{4}{\rho} - \frac{5 \rho}{\Xi_0^2} + \left( 1 - \frac{\rho^2}{\Xi_0^2} \right) \frac{\hat{B}_{\rho}}{\bar{B}} V_{,\theta} + \left( 3 \cot \theta + \frac{\hat{B}_{,\theta}}{\bar{B}} \right) \frac{1}{\rho^2} V_{,\theta} = 0.$$  (8)

For flat systems, it is convenient to switch to a cylindrical GTCS $(x^0, \xi, z, \phi)$, where $\xi \equiv \rho \sin \theta$, $z \equiv \rho \cos \theta$. Then eq. (6) becomes

$$\bar{d}s_t^2 = \bar{B} \left[ - (1 - \bar{V}^2 \xi^2)(dx^0)^2 + 2 \frac{t}{t_0} \bar{V} \xi^2 d\phi dx^0 + \frac{t^2}{t_0^2} \left( \frac{d\xi^2}{\xi^2} + \frac{dz^2}{\Xi_0^2} + \frac{2 \xi}{\Xi_0^2} d\xi dz + \xi^2 d\phi^2 \right) \right],$$  (9)
and the field equations (7) and (8) read
\[
\left(1 - \frac{\xi^2}{\Xi^2}\right) B_{,\xi\xi} + \left(1 - \frac{z^2}{\Xi^2}\right) B_{,zz} = -2\frac{\xi z}{\Xi^2} \frac{\partial}{\partial z} B_{,\xi z} + \frac{1}{\xi} \left(1 - 3\frac{\xi^2}{\Xi^2}\right) B_{,\xi\xi} - \frac{3z}{\Xi^2} B_{,zz} =
\]
\[
\left(1 - \frac{\xi^2}{\Xi^2}\right) \nabla_{\xi} + \left(1 - \frac{z^2}{\Xi^2}\right) \nabla_{zz} = -2\frac{\xi z}{\Xi^2} \frac{\partial}{\partial z} \nabla_{\xi z} + \frac{1}{\xi} \left(1 - 3\frac{\xi^2}{\Xi^2}\right) \nabla_{\xi\xi} - \frac{3z}{\Xi^2} \nabla_{zz} =
\]
\[
\left(1 - \frac{\xi^2}{\Xi^2}\right) \nabla_{\xi\xi} + \left(1 - \frac{z^2}{\Xi^2}\right) \nabla_{zz} = \left(1 - \frac{\xi z}{\Xi^2}\right) \nabla_{\xi z} + \left(1 - 3\frac{\xi^2}{\Xi^2}\right) \nabla_{\xi\xi} - \frac{3z}{\Xi^2} \nabla_{zz} = 0.
\]
(10)
\(\tag{10}
\)

We now assume that the gravitational field is weak, so that we may set \(\bar{V} = 0\). This implies that eq. (11) becomes vacuous and that the right-hand side of eq. (10) vanishes. To find \(\bar{B}(\xi, z)\), we are thus left to solve eq. (10) with the right-hand side equal to zero. Note that this is equivalent to solving the Laplace equation on a subset of the 3-sphere.

Unfortunately, eq. (10) is a nonseparable partial differential equation (PDE). This means that there is not much hope of finding exact solutions. However, one may try to find approximate series solutions for \(|z| \ll \Xi_0\). The series expansions in terms of \(|z|\) should then take the same form as for the corresponding Newtonian problem, recovered by letting \(\Xi_0 \to \infty\) in eq. (10). One then gets a separable PDE in the Newtonian potential. That problem was solved many years ago [6,7], yielding a continuous spectrum of solutions \(\Phi_k(\xi, z) \propto J_0(k\xi) \exp(-k|z|)\), where \(J_0(k\xi)\) is a Bessel function of the first kind. We are thus led, via correspondence with the Newtonian case, to try solutions \(\bar{B}(\xi, z) = 1 + \frac{2}{z^2} \Phi(\xi, z)\) built from mode solutions of the form
\[
\Phi_k(\xi, z) = \Phi_k(\xi) \left(1 - k|z| + \frac{1}{2!} \alpha_k(k|z|)^2 + \ldots\right),
\]
(12)

where the \(\alpha_k\) are constants (but higher-order coefficients will in general depend on \(\xi\)). The reason why the \(\alpha_k\) do not depend on \(\xi\), is that any deviation from Euclidean space for small \(|z|\) occurs at order \(|z|^3\) and higher. That is, by integrating the spatial line element a distance \(|z| \ll \Xi_0\) in the \(z\)-direction we find (for \(B \approx 1\))
\[
\sqrt{1 - \frac{\xi^2}{\Xi^2}} \int_{\Xi_0}^{\Xi} \frac{dz'}{\sqrt{1 - \frac{\xi^2 + 2z' \xi}{\Xi^2}}} = z \left(1 + \frac{1}{6(1 - \frac{\xi^2}{\Xi^2})} \frac{z^2}{\Xi^2} + \ldots\right).
\]
(13)

We now insert eq. (12) into eq. (10) (with the right-hand side set to zero) and collect terms to get separate equations for each power of \(|z|\). To lowest order, the terms independent of \(|z|\) yield an equation for \(\Phi_k(\xi)\), i.e.
\[
\left(1 - \frac{\xi^2}{\Xi^2}\right) \Phi_{k,\xi\xi} + \frac{1}{\xi} \left(1 - 3\frac{\xi^2}{\Xi^2}\right) \Phi_{k,\xi} + \alpha_k k^2 \Phi_k = 0.
\]
(14)

Similarly, collecting terms of first order in \(|z|\) yields an equation determining the third order coefficient of the series expansion in eq. (12), and so on for each order in \(|z|\).

Now, since \(k\) is interpreted as a wavenumber on the 3-sphere, it must have a minimum value \(k_0 = \frac{1}{\Xi_0}\) corresponding to the maximum value \(\xi_{\text{max}} = \Xi_0\). This indicates that, rather than the continuous spectrum of solutions found for the Newtonian solution, the solution of eq. (14) should involve a discrete spectrum of solutions \(\Phi_n(\xi) \equiv \Phi_{k_n}(\xi)\). In fact the general solution of eq. (14) is (with \(\alpha_n \equiv \alpha_{k_n}\))
\[
c^{-2} \Phi_n(\xi) = -C_n P_n(u) - C_0 Q_n(u), \quad n = \left(\sqrt{\alpha_n k_n^2 \Xi_0^2 + 1} - 1 \right) / 2, \quad u \equiv 1 - 2 \frac{\xi^2}{\Xi^2},
\]
(15)

where \(C_n\) and \(C_0\) are (dimensionless) constants and \(P_n(u), Q_n(u)\) are Legendre functions of the first and second kind, respectively. (Notice that the mode solutions \(C_n Q_n(u)\) diverge logarithmically when \(u \to \pm 1\).) From eq. (15) we see that we get a discrete spectrum of solutions if \(n\) is required to be a nonnegative integer. Moreover, we require that \(k_n\) should be an integer multiple of \(k_0\), and that in the continuum limit \(\lim_{n \to \infty} \alpha_n = 1\), so we must choose
\[
\alpha_n = 1 - \frac{1}{k_n^2 \Xi_0^2}, \quad \Rightarrow \quad k_n = \frac{2n + 1}{\Xi_0}, \quad n = 0, 1, 2, \ldots,
\]
(16)

and this choice will also be consistent with the indicated value of \(k_0\). Note that, to get the correspondence with the Newtonian case, one may choose \(C_0 = 0\) and \(C_n = e^{-s} k_n^2 / n!\) where \(s\) is a nonnegative real number. Summing over \(n\) and
using the generating function
\[ g(u, s) = \exp(s u)J_0 \left( s \sqrt{1-u^2} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} P_n(u) s^n, \] (17)
for Legendre polynomials [8], we get the continuous spectrum of solutions
\[ \Phi_s(u) \propto \exp(-s) \sum_{n=0}^{\infty} \frac{1}{n!} P_n(u) s^n = \exp \left( -2s \frac{\xi^2}{2 \xi^2} \right) J_0 \left( 2s \frac{\xi^2}{2 \xi^2} \sqrt{1 - \frac{\xi^2}{2 \xi^2}} \right). \] (18)

Setting \( k \equiv \frac{2 \xi}{\sqrt{2}} \) and then taking the limit \( \Xi \to \infty \) in eq. (18), we get back the Newtonian case.

To find the mode surface densities \( \Sigma_n(u) \) and \( \Sigma_n'(u) \) corresponding to the specific mode solutions \( -C_n P_n(u) \) and \( -C_n' Q_n(u) \), respectively, we lay a Gauss surface around the disk and use Gauss’ theorem across it. This procedure is exactly similar to the Newtonian case treated in [7]. Assuming a weak field \((B_n \approx 1)\) we find
\[ \Sigma_n(u) = \frac{c^2 C_n}{2\pi G \Xi_0^2} (2n+1) P_n(u), \quad \Sigma_n'(u) = \frac{c^2 C_n}{2\pi G \Xi_0^2} (2n+1) Q_n(u). \] (19)

Due to the fact that the set of Legendre polynomials \( P_n(u) \), \( u \in (-1, 1) \), is complete and orthogonal [9], it is possible to expand any real surface density \( \Sigma(u) \) in terms of the real surface densities \( \Sigma_n(u) \). Setting \( C_n \equiv C \Sigma_n \) (where \( C \) is some nonzero constant) and summing over \( n \), we find
\[ \Sigma(u) = \sum_{n=0}^{\infty} \Sigma_n(u) = \frac{c^2 C}{\pi G \Xi_0^2} \sum_{n=0}^{\infty} \frac{1}{2} (2n+1) S_n P_n(u). \] (20)

But this means that \( \Sigma(u) \) is expressed as a Legendre Fourier series [9], and that its inverse \( S_n \) is the finite Legendre transform [9] of \( \frac{\pi G \Xi_0}{c^2} \Sigma(u) \), i.e.
\[ S_n = \frac{\pi G \Xi_0}{c^2 C} \int_{-1}^{1} \Sigma(u') P_n(u') du', \] (21)
so that
\[ \Sigma(u) = \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) P_n(u) \int_{-1}^{1} \Sigma(u') P_n(u') du'. \] (22)

The real solution \( \Phi(u) \) corresponding to the given surface density \( \Sigma(u) \) is then obtained by summing up the mode solutions \( -C_n P_n(u) \), i.e.
\[ \Phi(u) = -c^2 C \sum_{n=0}^{\infty} S_n P_n(u) = -\pi G \Xi_0 \sum_{n=0}^{\infty} P_n(u) \int_{-1}^{1} \Sigma(u') P_n(u') du'. \] (23)

The corresponding series solution \( \tilde{B}_{\text{real}}(u, z) \) is then found by combining eqs. (12), (16), (22) and (23). The result is
\[ \tilde{B}_{\text{real}}(u, z) = 1 + \frac{2}{c^2} \Phi(u) + \frac{4\pi G}{c^2} |z| \Sigma(u) \\
- \frac{4\pi G}{c^2 \Xi_0^2} \sum_{n=0}^{\infty} n(n+1) P_n(u) \int_{-1}^{1} \Sigma(u') P_n(u') du' + \ldots. \] (24)

The circular speed \( \tilde{w}_{\text{real}} \) due to the solution (23) as function of radius of the disk can now be found from the usual nonrelativistic formula (this is justified for a weak field \( B_{\text{real}} \approx 1 \), see [4]). Expressed as a function of \( u \) we find
\[ \tilde{w}^2_{\text{real}} = 2\pi G \Xi_0 (1-u) \sum_{n=0}^{\infty} \frac{\partial P_n(u)}{\partial u} \int_{-1}^{1} \Sigma(u') P_n(u') du' \\
= \frac{2\pi G \Xi_0}{1+u} \sum_{n=0}^{\infty} (n+1)(u P_n(u) - P_{n+1}(u)) \int_{-1}^{1} \Sigma(u') P_n(u') du'. \] (25)

It cannot be expected that rotation curves found from eq. (25) should deviate significantly from their Newtonian counterparts (they do not). Thus it would seem that the need for galactic DM is the same as for the standard model. However, here we have not taken into account possible solutions constructed from the specific mode solutions \( -C_n' Q_n(u) \). As we shall see later, if we do this, new possibilities of getting rid of galactic DM open up. But first, in the next section, we need to consider restrictions coming from boundary conditions.
3.2 Boundary conditions

So far we have implicitly assumed that there are no particular preferences regarding the form of $\Sigma(u)$ as long as it is physically reasonable and the resulting $\Phi(u)$ is small everywhere. But is this true? To answer that question, we first notice from eq. (23) that the real potential at the center of the disk is given by

$$
\Phi(1) = -\pi G \Xi_0 \sum_{n=0}^{\infty} \int_{-1}^{1} \Sigma(u') P_n(u') du' = -\frac{\pi G \Xi_0}{\sqrt{2}} \int_{-1}^{1} \sqrt{1-u'} \Sigma(u') du',
$$

where the last expression follows from the generating function [8]

$$
\sum_{n=0}^{\infty} P_n(u) s^n = \frac{1}{\sqrt{1 - 2us + s^2}}, \quad -1 < s < 1,
$$

for the borderline case $s = 1$. Moreover, eq. (26) may be written in the form

$$
\Phi(1) = -c^2 \frac{\bar{\Sigma}^+}{\Sigma^*}, \quad \bar{\Sigma}^+ \equiv \frac{1}{2} \int_{-1}^{1} \frac{\Sigma(u') du'}{\sqrt{1-u'}}, \quad \Sigma^* \equiv \frac{e^2}{2\pi G \Xi_0},
$$

where $\bar{\Sigma}_+ = 1$ is a weighted average, and where the constant $\bar{\Sigma}_+$ sets a specific surface density scale depending on the finite size of space. Besides, a second weighted average surface density $\bar{\Sigma}_-$, related to the total (active) mass $M_{t0}$ of the disk, can be defined by

$$
\bar{\Sigma}_- \equiv \frac{1}{2\sqrt{2}} \int_{-1}^{1} \frac{\Sigma(u') du'}{\sqrt{1+u'}}, \quad \Rightarrow \quad \Phi(-1) = -c^2 \frac{\bar{\Sigma}_-}{\Sigma^*} = -\frac{GM_{t0}}{2\pi \Xi_0},
$$

where $\Phi(-1)$ is found from an expression similar to eq. (26) by using eq. (27) for the borderline case $s = -1$. We note that $\Sigma_+$ is a purely geometric quantity, whereas $\Sigma_+$ and $\Sigma_-$ depend on the real surface density profile.

Next we note that $\sqrt{\Phi(-1)}$ represents a specific (nonvanishing) velocity scale. This indicates the possibility of defining some quantity $\sqrt{\Phi(-1)}$ with the property that it relates $\Phi(1)$ to $\sqrt{\Phi(-1)}$ via a definition similar to eq. (28). By combining eqs. (28) and (29) such a relationship may readily be found. That is, we may define

$$
\Phi(1) = -c \sqrt{\Phi(-1)} \frac{\Sigma_+}{\Sigma^*}, \quad \Phi(-1) \equiv -c \sqrt{\Phi(-1)} \frac{\pi}{\sqrt{2}} \frac{|\bar{\sigma}(1) - \bar{\sigma}(1)|}{\Sigma^*},
$$

Notice that eq. (30) involves the geometric quantity $\Sigma_+$, but since $\Phi(1) = \frac{\Sigma_+}{\Sigma^*} \Phi(-1)$, the analogous relationship between $\Phi(1)$ and $\Phi(-1)$ does not. Also notice that the factor $\pi/\sqrt{2}$ is included into the definition (30) since $\bar{\sigma}(1) - \bar{\sigma}(1)$ should be more similar to a mode surface density (see eq. (19)) rather than to a weighted average like $\Sigma_+$.

A definition similar to eq. (30) may be made for the contribution $\Phi_{\geq u}(1)$ to $\Phi(1)$ from the part of the disk interior to some arbitrary coordinate $u$. The purpose of such a definition is to construct a new "associated" surface density $\bar{\sigma}(u)$. That is, we may define

$$
\Phi_{\geq u}(1) \equiv -c \sqrt{\Phi(-1)} \frac{\pi}{\sqrt{2}} \frac{|\bar{\sigma}(u) - \bar{\sigma}(1)|}{\Sigma^*}, \quad \Phi_{\leq u}(1) \equiv -c \sqrt{\Phi(-1)} \frac{\pi}{\sqrt{2}} \int_{-1}^{1} \frac{\Sigma(u') du'}{\sqrt{1-u'}},
$$

or, equivalently (where the constant $\bar{\sigma}(1)$ must be determined separately, see below)

$$
\bar{\sigma}(u) \equiv \bar{\sigma}(1) + \frac{1}{2\pi} \sqrt{\Sigma_+} \int_{-1}^{1} \frac{\Sigma(u') du'}{\sqrt{1-u'}}, \quad \bar{\sigma}(1) \equiv \bar{\sigma}(1) + \frac{\sqrt{2}}{\pi} \sqrt{\Sigma_+ \Sigma_-}.
$$

We see from the definition (32) that $\bar{\sigma}(u)$ is increasing from the center of the disk and outwards. Thus, $\bar{\sigma}(u)$ could be interpreted as some kind of "inverted" surface density. This means that $\bar{\sigma}(u)$ should not be considered as an independent, gravitating source. Rather, $\bar{\sigma}(u)$ should give some restrictions on the possible forms of $\Sigma(u)$. Such restrictions can be found by requiring that $\bar{\sigma}(u)$ be linearly related to $\Sigma(u)$, so that the corresponding potential can be written as a linear combination of $\Phi(u)$ and some constant potential. Then $\bar{\sigma}(u)$ is not independent. A "preferred" form of $\bar{\sigma}(u)$ can thus be found by requiring that $\bar{\sigma}(u) - \bar{\sigma}(1) + \lambda \Sigma(u) - \bar{\sigma}(1)$, where $\lambda$ is a constant, or equivalently

$$
\bar{\sigma}(u) = \bar{\sigma}(1) - \frac{\bar{\sigma}(1)}{\Sigma(1)} \left[ (\bar{\sigma}(u) - \bar{\sigma}(1)) \right], \quad \Rightarrow \quad \bar{\sigma}(1) = \Sigma(1) \frac{\bar{\sigma}(1)}{\Sigma(1)}.
$$
(We see that in this case, since $\Sigma(-1)$ should be negligible, $\sigma(1)$ should be also.) Equation (33) is then an integral equation determining the most basic form of the real surface density $\Sigma(u)$ for an isolated disk. To find exactly what this form is, it is convenient to turn eq. (33) into a first order separable differential equation by taking the derivative w.r.t. $u$ at both sides of it. Solving this equation is straightforward, and the result is an exponential disk, i.e.,

$$\Sigma(u) = \Sigma(1) \exp \left[ - \left( \frac{\Sigma(1)}{\Sigma_+} - \frac{\Sigma(-1)}{\Sigma_+} \right) \sqrt{(1-u)/2} \right] \equiv \Sigma(1) \exp \left[ - \frac{\Sigma_0}{\xi_d} \sqrt{(1-u)/2} \right],$$

(34)

where $\xi_d \equiv \Sigma_0 \Sigma_+ / [\Sigma(1) - \Sigma(-1)]$ is the disk length (at epoch $t_0$). This result answers the question we posed at the beginning of this section: the simple requirement that $\Sigma(u)$ and $\sigma(u)$ should be linearly related implies that there is a particular preference regarding the form of $\Sigma(u)$. That is, it would seem that the exponential disk should represent a preferred surface density profile among all the possibilities that might exist. This is confirmed observationally, since an exponential surface density is the hallmark density profile of the outer regions of spiral galaxies. We will return to the exponential disk in sect. 4.

3.3 The induced solution

Contrary to the Legendre polynomials $P_n(u)$, the functions $Q_n(u)$ are not polynomials, and nor do they constitute an orthogonal set for $u \in (-1, 1)$. Rather, the functions $Q_n(u)$ can be separated into two subsets depending on whether $n$ is even or odd. That is, each function with odd $n$ is orthogonal to every function with even $n$ and vice versa. On the other hand, functions within each subset are linearly dependent. This can be easily seen from the formulae [8]

$$\int_{-1}^{1} Q_n(u)Q_n(u)du = \frac{\pi^2}{2} + 2 \sum_{k=1}^{\infty} \frac{1}{2n + 1} ,$$

$$\int_{-1}^{1} Q_n(u)Q_m(u)du = \frac{1 + (-1)^{n-m}}{(m-n)(m+n+1)} \left( \sum_{k=1}^{n} - \sum_{k=1}^{m} \right) \frac{1}{k!}, \quad n \neq m. \quad (35)$$

The problem now is to construct a solution $\Phi(u)$ from the mode solutions $-C_n^i Q_n(u)$ such that $\Phi(u)$ and $\Phi^i(u)$ are linearly independent, i.e., we require that $\int_{-1}^{1} \Phi^i(u)\Phi(u)du = 0$. However, since there are only two sets of linearly independent mode solutions, it must be possible to find many such solutions by summing over different numbers of mode solutions (in general, at least two mode solutions must be included, one from each linearly independent set). This means that, merely requiring linear independence is not sufficient to arrive at a unique solution $\Phi(u)$. However, a unique linearly independent solution $\Phi^i(u)$ can indeed be found by summing over all the mode solutions. We will call this solution the induced solution, since it is found indirectly by summing up all the mode solutions $C_n^i Q_n(u)$ such that every term in the mode sum has a linearly independent counterpart $C_n^i P_n(u)$ from eq. (23). Moreover, the requirement of linear independence is not trivial since it forces the constants $C_n^i$ to be dependent on $S_n$ and thus the real surface density. The corresponding surface density $\Sigma^i(u)$ will be called the induced density. The induced density is not real, but could still have physical consequences indirectly.

The induced solution $\Phi^i(u)$ obtained by summing over all the mode solutions can be found by assuming that the coefficients $C_n^i$ can be written in the form $C_n^i = C^i S_n$, where $C^i$ is a normalisation constant. Then, using the formulae [8]

$$\int_{-1}^{1} P_n(u)Q_m(u)du = \frac{1 - (-1)^{n+m}}{(n-m)(m+n+1)}, \quad n \neq m, \quad (36)$$

it is easy to see that

$$\int_{-1}^{1} \Phi(u)\Phi(u)du \propto \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} S_n S_m \frac{1 - (-1)^{n+m}}{(n-m)(m+n+1)} = 0, \quad (37)$$

from the obvious antisymmetry obtained by permuting the summation indices. To completely specify the solution $\Phi^i(u)$, it remains to specify the normalisation constant $C^i$. But the only natural choice is really to set $C^i = C$. We thus have that

$$C_n^i = C_n = CS_n = \frac{\pi G \Sigma_0}{c^2} \int_{-1}^{1} \Sigma(u') P_n(u')du'. \quad (38)$$

The solution $\Phi^i(u)$ is now completely specified, and we have

$$\Phi^i(u) = -i^2 C \sum_{n=0}^{\infty} S_n Q_n(u) = -\pi G \Sigma_0 \sum_{n=0}^{\infty} \int_{-1}^{1} \Sigma(u') P_n(u')du' Q_n(u). \quad (39)$$
Equation (39) may be written in a more convenient form by using the identity [8] (the integral being defined by its Cauchy principal value)

\[ Q_n(u) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(s)ds}{u - s}, \]

so that by using eq. (23) we find that (again using Cauchy principal values)

\[ \Phi^i(u) = \frac{1}{2} \int_{-1}^{1} \Phi(s)ds. \]

From eqs. (19), (22), (38) and (40) we can also find the induced density \( \bar{\Sigma}^i(u) \) corresponding to the induced solution \( \Phi^i(u) \), i.e.,

\[ \bar{\Sigma}^i(u) = \sum_{n=0}^{\infty} \frac{c^2 C_0}{2\pi G} (2n + 1)Q_n(u) = \frac{1}{2} \int_{-1}^{1} \bar{\Sigma}(s)ds. \]

However, there is a fundamental problem with the induced quantities. That is, both \( \Phi^i(u) \) and \( \bar{\Sigma}^i(u) \) will in general contain generic logarithmic divergences (typically located at \( u = \pm 1 \)) inherited from the mode solutions \( Q_n(u) \). This means that \( \Phi^i(u) \) and \( \bar{\Sigma}^i(u) \) cannot be used directly as physical quantities. But \( \bar{\Sigma}^i(u) \) may be used indirectly, since it is possible to construct a new, geometric, singularity-free surface density from it. We will show how to do this in the next section.

### 3.4 The induced associated potential

Although \( \bar{\Sigma}^i(u) \) cannot be used directly, it still may have physical significance. The reason for this is that it is possible to construct a new surface density \( \bar{\sigma}^i(u) \) from \( \bar{\Sigma}^i(u) \) using the results found in sect. 3.2. There, the definition (32) of \( \bar{\sigma}(u) \) was motivated from the possibility of restricting possible forms of \( \bar{\Sigma}(u) \) due to boundary conditions. However, since \( \bar{\sigma}(u) \) is by construction unphysical, there never was any reason to interpret it as an independent gravitating source. On the other hand, there is always the possibility that a definition similar to (32), but with \( \bar{\Sigma}^i(u) \) substituted for \( \bar{\Sigma}(u) \), may have the properties that makes it possible to use it as a gravitating, geometric source. That is, we may construct the so-called induced associated surface density \( \bar{\sigma}^i(u) \), defined by

\[ \bar{\sigma}^i(u) = \frac{1}{2\pi} \sqrt{\frac{\Sigma}{\Sigma - \int_{-1}^{1} \bar{\Sigma}^i(u')du'}} = 0, \quad \bar{\sigma}^i(1) = \frac{c^2 \Xi_0}{M_\odot G}. \]

Note that, by integrating over \( \bar{\Sigma}^i(u') \), its generic logarithmic divergences disappear, so that \( \bar{\sigma}^i(u) \) does not contain such singularities. Moreover, since it is possible that \( \bar{\Sigma}^i(u) \) may change sign somewhere in the interval \( u \in (-1, 1) \), \( \bar{\sigma}^i(u) \) may take a form similar to some physical surface density profile. Finally, unlike \( \bar{\sigma}(u) \), \( \bar{\sigma}^i(u) \) can certainly not be algebraically related to \( \bar{\Sigma}(u) \). Thus \( \bar{\sigma}^i(u) \) may be considered as a physical, independent gravitating geometric quantity, but not as a material density.

We may now use \( \bar{\sigma}^i(u) \) as a source in eq. (23) to get a new potential, i.e., the so-called induced associated potential \( \bar{\Psi}(u) \) defined by

\[ \bar{\Psi}(u) = -\pi G\Xi_0 \int_{-1}^{1} \bar{\sigma}^i(u')P_n(u')du'. \]

Since \( \bar{\sigma}^i(u) \) does not contain generic divergences, we see from eq. (44) that \( \bar{\Psi}(u) \) should be nonsingular everywhere, so it may be accepted as a physical quantity. Thus to any real surface density profile \( \bar{\Sigma}(u) \) and its corresponding real potential \( \Phi(u) \), there is always associated a surface density \( \bar{\sigma}^i(u) \) (playing the role of galactic “dark matter”), and its corresponding potential \( \bar{\Psi}(u) \). These quantities should always be considered together when making predictions.

The series solution \( B(u, z) \) containing both real and induced associated contributions is then given by

\[ B(u, z) = 1 + \frac{2}{c^2} \left( \Phi(u) + \bar{\Psi}(u) \right) + \frac{4\pi G}{c^2} |z| \left( \bar{\Sigma}(u) + \bar{\sigma}^i(u) \right) \]

\[ -\frac{4\pi G}{c^2 \Xi_0} \sum_{n=0}^{\infty} P_n(u) \int_{-1}^{1} \left( \bar{\Sigma}(u') + \bar{\sigma}^i(u') \right)P_n(u')du' + \ldots. \]

Note that the second-order term in eq. (45) is divergent for \( u = 1, z \neq 0 \). See the next section for more comments. Moreover, the (nonrelativistic) circular speed \( \bar{w}_{\text{circ}} \) calculated from eq. (45) is due to both real matter and induced
associated “phantom” matter, and similar to eq. (25) we find that

\[ \bar{u}_{\text{circ}}^2(u) = \frac{2\pi G \Xi_0}{1 + u} \sum_{n=0}^{\infty} (n + 1) \left( u P_n(u) - P_{n+1}(u) \right) \int_{-1}^{1} (\bar{\Sigma}(u') + \bar{\sigma}^i(u')) P_n(u') du'. \]  

(46)

We see from eqs. (41) and (42) that the induced solutions and densities are obtained by integrating real solutions and densities over the whole disk; thus these quantities describe nonlocal, collective properties of the system. This is also true for \( \bar{\sigma}^i(u) \). That is why any extra gravitational acceleration obtained from \( \Psi(u) \) should not be interpreted as a fundamental modification of the Newtonian force law such as in MOND – rather the extra acceleration should be seen as an emergent property of the whole system.

### 3.5 An important transformation

Given as input the real surface density \( \bar{\Sigma}(u) \), one should now in principle be able to calculate the real potential \( \Phi(u) \), the real series solution \( B_{\text{real}}(u, z) \), the corresponding induced quantities \( \bar{\Sigma}^i(u) \) and \( \bar{\Phi}^i(u) \), the induced associated quantities \( \bar{\sigma}^i(u) \) and \( \bar{\Psi}(u) \), the total series solution \( B(u, z) \), and finally the rotation curve from eq. (46). However, as seen from eqs. (23) and (44), the expressions for \( \Phi(u) \) and \( \Psi(u) \) contain an infinite sum over (a product of) Legendre polynomials. Since this sum will in general converge slowly, its presence makes numerical calculations quite awkward. Fortunately, it is possible to rewrite this infinite sum in terms of an elliptic integral, making numerical calculations much easier. The key to this important transformation is using the generating function [10] \( \sum_{n=0}^{\infty} P_n(u) P_n(u') s^n = \frac{1}{\pi} \int_{0}^{\pi} \frac{d\omega}{\sqrt{1 - 2s(uu' + \sqrt{(1 - u^2)(1 - u'^2)}cos\omega) + s^2}} \).

(47)

for the borderline case \( s = 1 \). (Note that, for the special case \( u' = 1 \), we get back eq. (27).)

We now use eq. (47) to rewrite eqs. (23) and (44). Interchanging the sum and the integral in these equations, adding them and then using eq. (47), we get

\[
\Phi(u) + \Psi(u) = -\frac{G \Xi_0}{\sqrt{2}} \int_{-1}^{1} \frac{\bar{\Sigma}(u') + \bar{\sigma}^i(u')}{\sqrt{1 - uu' - \sqrt{(1 - u^2)(1 - u'^2)}cos\omega}} du' d\omega
\]

\[
= -\sqrt{2}G \Xi_0 \int_{-1}^{1} \frac{\bar{\Sigma}(u') + \bar{\sigma}^i(u') K\left( \sqrt{\frac{2\sqrt{(1 - u^2)(1 - u'^2)}}{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}}} \right)}{\sqrt{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}}} du'
\]

\[
= -2\sqrt{2}G \Xi_0 \int_{-1}^{1} \frac{\bar{\Sigma}(u') + \bar{\sigma}^i(u') K\left( \sqrt{\frac{(1 - u^2)(1 - u'^2)}{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}}} \right)}{\sqrt{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}} + \sqrt{1 - uu' - \sqrt{(1 - u^2)(1 - u'^2)}}} du'.
\]  

(48)

where \( K(k) \equiv \int_{0}^{\pi/2} d\theta / \sqrt{1 - k^2 \sin^2 \theta} \) is the complete elliptic integral of the first kind [8]. The last form (see [10]) of eq. (48) is a little better to use for numerical purposes.

Similarly, it may also be tempting to interchange the infinite sum and the integral present in the second-order term of eq. (45), and then transform the infinite sum into an integral. However, as we shall see, this procedure does not work for higher-order terms, since all will be divergent. To illustrate this, a somewhat lengthy calculation using eq. (47) yields

\[
\sum_{m=0}^{\infty} m(m+1) P_m(u) P_m(u') = \lim_{s \to 1} \frac{\partial}{\partial s} s^2 \frac{\partial}{\partial s} \sum_{m=0}^{\infty} P_m(u) P_m(u') s^m
\]

\[
= E(k) - \left( 1 - uu' - \sqrt{(1 - u^2)(1 - u'^2)} \right) K(k)
\]

\[
= 2\sqrt{2\pi |u - u'| \sqrt{1 - uu' - \sqrt{(1 - u^2)(1 - u'^2)}}}
\]

\[
k^2 = \frac{2\sqrt{(1 - u^2)(1 - u'^2)}}{1 - uu' + \sqrt{(1 - u^2)(1 - u'^2)}}.
\]  

(49)
where \( E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta \) is the complete elliptic integral of the second kind [8]. To get an expression for \( \dot{B}(u, z) \) (for small \(|z|\)) more suitable for numerical calculations, one may now insert eq. (48) into eq. (45). However, if one tries to insert eq. (49) into the quadratic term, interchanging the infinite sum and the integral, it is straightforward to see that this term will diverge. This behaviour is quite similar to the Newtonian case (where the potential takes the form of a double improper integral [7]) if one tries to expand the Newtonian potential in powers of \(|z|\); interchanging the two improper integrals implies that all terms with power \( \geq 2 \) will diverge, yet if all the terms are included, the resulting exact result is finite. But only including the linear term in the series expansion approximates the exact result well for small enough \(|z|\). One expects that this is valid for eq. (45) also, so that skipping terms with power \( \geq 2 \) is justified for small \(|z|\). Note that in eq. (45) (also just as for its Newtonian counterpart), all terms of order \( \geq 2 \) will diverge on the \( z \)-axis for \( z \neq 0 \).

Of course, if \(|z|\) is not small enough, the second-order term (and possibly higher-order terms) must be calculated in some admissible way and included. That is, it would be preferable to find some other way of rewriting the series expansion in \(|z|\) (i.e., without using eq. (49)), such that all terms can be expressed in a form suitable for numerical calculations. Fortunately, such a form for the second-order term can be found from eqs. (45) and (10) (with \( V = 0 \)). From these equations we find that (for small \(|z|\))

\[
\dot{B}_{zz} = -\frac{2}{c^2} \left[ \left( 1 - \frac{\xi^2}{\xi_0^2} \right) (\Phi_{,\xi\xi} + \Psi_{,\xi\xi}) + \frac{1}{\xi} \left( 1 - \frac{3\xi^2}{\xi_0^2} \right) (\Phi_{,\xi} + \Psi_{,\xi}) \right] + O(|z|)
\]

so that the first few terms of the rewritten series expansion read

\[
\dot{B}(u,z) = 1 + \frac{2}{c^2} (\Phi(u) + \Psi(u)) + \frac{4\pi G}{c^2 c_0} |z| \left( \dot{\Sigma}(u) + \dot{\sigma}^i(u) \right)
\]

\[
- \frac{4\pi^2}{c^2 c_0} \left[ (1 + u)(\Phi_{,uu} + \Psi_{,uu}) - 2u(\Phi_{,u} + \Psi_{,u}) \right] + O(|z|^2).
\]

Note that the second-order and higher-order terms are still expected to diverge in the limit \( u \to 1 \), \( z \neq 0 \). This means that, to do calculations to a given accuracy at some fixed \( z \neq 0 \), one needs to include ever more higher-order terms into the series expansion when moving towards the \( z \)-axis.

Finally, interchanging the sum and the integral in eq. (46), yields, after some tedious calculations, that

\[
\dot{\omega}_{\text{circ}}^2(u) = \sqrt{2G\xi_0} \left\{ \frac{u}{1 + u} \int_{-1}^{1} \left[ \Sigma(u') + \dot{\sigma}^i(u') \right] K \left( \frac{2\sqrt{(1-u')^2(1-u'^2)}}{1-uu'+\sqrt{1-u'^2}(1-u'^2)} \right) \frac{du'}{\sqrt{1-uu'+\sqrt{1-u'^2}(1-u'^2)}} + \frac{1}{1 + u} \int_{-1}^{1} \left( \frac{u - u'}{1 - uu' - \sqrt{(1-u'^2)(1-u'^2)}} \right) K \left( \frac{2\sqrt{(1-u')^2(1-u'^2)}}{1-uu'+\sqrt{1-u'^2}(1-u'^2)} \right) \frac{du'}{\sqrt{1-uu'+\sqrt{1-u'^2}(1-u'^2)}} \right\}.
\]

Inserting any given surface density \( \Sigma(u) \) and its induced associated surface density \( \dot{\sigma}^i(u) \) into eq. (52) now yields the full rotation curve of the disk.

\section{The exponential disk}

\subsection{Approximate solutions}

For spiral galaxies, the observed general trend is that surface brightness (and thus the luminosity due to stars in the disk) falls off exponentially from the center and outwards. Therefore, to explain spiral galaxy rotational curves without dark matter, one is required to assume that surface density profiles of stars are proportional to luminosity profiles. This means that, in the general and idealised case of a truncated disk of extension \( \xi_0 \), we must have that

\[
\Sigma(\xi) = \begin{cases} 
\Sigma_c \exp(-\xi/\xi_d) & \xi \leq \xi_0, \\
0 & \xi > \xi_0,
\end{cases} \\
\Sigma(u) = \begin{cases} 
\Sigma_c \exp\left(-\frac{u}{\xi_0} \sqrt{\frac{1-u}{2}}\right) & 1 \geq u \geq u_0, \\
0 & u < u_0,
\end{cases}
\]

(53)
where \( \Sigma_c \) is the central surface density and \( \xi_d \) is the disk length (both taken at epoch \( t_0 \)). Note that, since \( \xi_d \ll \Xi_0 \), \( \Sigma^i(\xi) \) falls off so fast that we are justified in treating spiral galaxies as truncated disks for sufficiently large \( \xi \), i.e., for \( \xi_0 \gg \xi_d \). For computational purposes, serious errors are hardly made if we choose \( \Xi_0 \) to lie somewhere in the interval 20 \( \xi_d \leq \Xi_0 \ll \Xi_0 \). On the other hand, truncation inevitably yields truncation singularities. However, their contributions to calculations are usually negligible. In what follows, we will assume that the disk is not truncated, but the results are not significantly affected if it is.

Now we may calculate the induced surface density \( \Sigma^i(\xi) \) associated with the exponential disk. To do that, we compute the integral

\[
\int_{-1}^{1} (u - s)^{-1} \exp\left( - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - s}{2}} \right) ds = \exp\left( - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right) \left\{ Ei\left[ - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 + u}{2}} \right] - Ei\left[ - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right] \right\} + \exp\left( - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right) \left\{ Ei\left[ - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right] - Ei\left[ - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right] \right\} 
\]

\[
\approx - \exp\left( - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right) Ei\left[ - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right] - \exp\left( - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right) Ei\left[ - \frac{\Xi_0}{\xi_d} \sqrt{\frac{1 - u}{2}} \right].
\]

(54)

where the approximation holds as long as \( u \) is not very close to \(-1\), and if \( \xi_d \ll \Xi_0 \). Here \( Ei(x) \) is the exponential integral defined by [8]

\[
Ei(x) \equiv - \int_{-\infty}^{\infty} \frac{\exp(-s)}{s} ds = \int_{-\infty}^{x} \frac{\exp(s)}{s} ds, \quad Ei(-x) \equiv - \int_{x}^{\infty} \frac{\exp(-s)}{s} ds, \quad x > 0.
\]

(55)

We then find from eq. (42) that

\[
\Sigma^i(\xi) \approx - \frac{\Sigma_c}{2} \left\{ \exp(\xi/\xi_d) Ei(-\xi/\xi_d) + \exp(-\xi/\xi_d) Ei(\xi/\xi_d) \right\}, \quad 0 < \xi < \Xi_0.
\]

(56)

Note that for (moderately) large distances, unlike \( \Sigma(\xi) \), \( \Sigma^i(\xi) \) tends slowly towards zero (from below) with increasing \( \xi \). Besides, note that \( \Sigma^i(\xi) \) is positive for small \( \xi \); and that it diverges logarithmically towards the origin. (As can be seen from eq. (54), the exact expression for \( \Sigma^i(\xi) \) also has a similar singularity when \( \xi \to \Xi_0 \).)

The induced associated surface density \( \bar{\sigma}^i(\xi) \) is now straightforwardly found from eqs. (43) and (56). Neglecting the very small quantity \( \bar{\sigma}^i(0) \), we find that (setting \( M_{t_0} \approx 2\pi \Sigma_c \xi_d^2 \), i.e., neglecting corrections coming from the finite size of space)

\[
\bar{\sigma}^i(\xi) \approx \frac{c}{2\sqrt{2\pi^2 \xi_d}} \sqrt{\frac{M_{t_0}}{G \Xi_0}} \left\{ \exp(-\xi/\xi_d) Ei(\xi/\xi_d) - \exp(\xi/\xi_d) Ei(-\xi/\xi_d) \right\}, \quad 0 \leq \xi \leq \Xi_0.
\]

(57)

Note that \( \bar{\sigma}^i(\xi) \) (even using the exact expression for \( \Sigma^i(\xi) \) found from eq. (54)) is nonsingular and nonnegative everywhere. Thus \( \bar{\sigma}^i(\xi) \) may formally play the role of DM.

Next we find an approximative expression (valid if \( \xi \ll \Xi_0 \)) for the real potential \( \Phi(\xi) \) from eq. (48) (omitting \( \bar{\sigma}^i(u) \)). Splitting the integral up into two improper parts, we find that

\[
\Phi(\xi) \approx - 4G \Sigma_c \xi_d \left[ \int_{0}^{\xi_0/\xi_d} + \int_{\xi_0/\xi_d}^{\infty} \right] x \exp(-x) K_0 \left( \frac{2\sqrt{x/\xi_d}}{x + \xi/\xi_d} \right) dx, \quad \xi \ll \Xi_0.
\]

(58)

Note that, as can be readily verified numerically, \( \Phi(\xi) \) as calculated from eq. (58) is essentially identical to its Newtonian counterpart \( \Phi_N(\xi) \) (valid for an infinite exponential disk), given by [7]

\[
\Phi_N(\xi) = - \pi G \Sigma_c \xi \left[ I_0(\xi/2\xi_d) K_1(\xi/2\xi_d) - I_1(\xi/2\xi_d) K_0(\xi/2\xi_d) \right],
\]

(59)

where \( I_\nu(x) \) and \( K_\nu(x) \) are modified Bessel functions [8]. Similarly, the corresponding circular speed \( \bar{w}_{\text{real}} \) can be found approximately from eq. (52) (omitting \( \bar{\sigma}^i(u) \)), and this is also numerically very close to its Newtonian counterpart. That is, again splitting up integrals into improper parts, we find that for \( \xi \ll \Xi_0 \),

\[
\bar{w}_{\text{real}}^2(\xi) \approx 2G \Sigma_c \xi_d \left[ \int_{0}^{\xi_0/\xi_d} + \int_{\xi_0/\xi_d}^{\infty} \right] x \exp(-x) \left[ K_0 \left( \frac{2\sqrt{x/\xi_d}}{x + \xi/\xi_d} \right) - E_0 \left( \frac{2\sqrt{x/\xi_d}}{x - \xi/\xi_d} \right) \right] dx.
\]

(60)
Note that the last term in eq. (60) diverges for each integral, but that added together, the sum converges (taking its Cauchy principal value).

An approximate expression for the induced associated potential $\Psi(\xi)$ is given by

$$\Psi(\xi) \approx -\frac{\sqrt{2}e}{\pi^2} \sqrt{\frac{M_{\odot} G}{\Xi_0}} \left[ \int_0^{\xi/\xi_d} + \int_{-\xi/\xi_d}^{0} \right] x \left\{ e^{-x} Ei(x) - e^x Ei(-x) \right\} K(\frac{2\sqrt{x \xi_d}}{\pi}) dx,$$

for $\xi \ll \Xi_0$. Note that the magnitude of $\Psi(\xi)$ depends critically on the upper limit of integration (if $\xi \sim \Xi_0$, a more accurate expression for $\Psi(\xi)$ may be found from eq. (48)). Also note that $\Psi(\xi)$ is nonsingular everywhere, and that if $\Sigma_c$ is small enough, $|\Psi(\xi)|$ may in principle dominate over $|\Phi(\xi)|$ for the whole disk. Finally, the full rotational curve may be found approximately from the formula ($\xi \ll \Xi_0$)

$$\bar{w}_{\text{circ}}^2(\xi) \approx \tilde{w}_{\text{real}}^2(\xi) + \frac{c}{\sqrt{2\pi^2}} \sqrt{\frac{M_{\odot} G}{\Xi_0}} \left[ \int_0^{\xi/\xi_d} + \int_{-\xi/\xi_d}^{0} \right] x \left\{ \exp(-x) Ei(x) - \exp(x) Ei(-x) \right\} \times K\left(\frac{2\sqrt{x \xi_d}}{\pi \xi / \xi_d}\right) \left(\frac{x + \xi / \xi_d}{x - \xi / \xi_d}\right) dx.$$

(62)

Numerically, it is found that with increasing $\xi$, the integrals converge rather quickly towards a constant factor of $2\pi$, so for $\xi$ above about 8–10 disk lengths the induced associated contribution changes very little. This means that, for large distances, the expression for $\bar{w}_{\text{circ}}^2(\xi)$ does not depend significantly on the upper limit of integration, and that one automatically gets an asymptotically flat rotation curve. Also notice that for small distances, the integrals yield a negative contribution to $\bar{w}_{\text{circ}}^2(\xi)$.

### 4.2 The correspondence with MOND

The most basic feature of MOND is the postulation of a fundamental acceleration scale $a_0$, observationally estimated to $a_0 \sim (1 - 2) \times 10^{-10} \text{m/s}^2$. That is, for Keplerian accelerations well below $a_0$, the Newtonian acceleration $a_N = GM/r^2$ in a spherically symmetric system is replaced by the MOND acceleration $a_M = \sqrt{a_N a_0}$. With the help of results derived in the preceding section we may now easily find a correspondence with MOND and actually calculate $a_0$ in the case of an exponential disk. From eq. (62) we find that the asymptotic rotational speed and the corresponding Keplerian acceleration are given by

$$\bar{w}_\infty = \frac{c}{\pi} \sqrt{\frac{2M_{\odot} G}{\Xi_0}} \Rightarrow \bar{a}(\xi) = \frac{c}{\pi} \sqrt{\frac{2M_{\odot} G}{\Xi_0}} \frac{1}{\xi} \quad \xi \gg \xi_d.$$

(63)

We are now able to compare eq. (63) to the MOND acceleration. We then get a correspondence if

$$a_0 = \frac{2c^2}{\pi^2 \Xi_0} = \frac{2}{\pi^2} c H_0 \approx 1.5 \times 10^{-10} \text{m/s}^2,$$

(64)

where the expression is valid for an exponential disk at epoch $t_0$ (with $H_0$ as the corresponding Hubble parameter), and where the final, numerical result is valid for the present epoch. (Note that $a_0$ is predicted to decrease with time, since the Hubble parameter decreases as the inverse of cosmic epoch.) This means that there is a correspondence between the model of a thin disk presented in this paper and MOND for the asymptotically flat part of the rotation curve. In particular, the fact that MOND successfully fits the observed baryonic Tully-Fisher relation [11] means that quasi-metric gravity does as well. That is, from eq. (63) we get that

$$\bar{w}_\infty^4 = \frac{2M_{\odot} G c^2}{\pi^2 \Xi_0} = \frac{2M_{\odot} G}{\pi^2} c H_0, \quad \Rightarrow \quad M_{t_0} = \frac{\pi^2 \Xi_0}{2Gc^2} \bar{w}_\infty^4 = \frac{\pi^2}{2GcH_0} \bar{w}_\infty^4,$$

(65)

and since the constant of proportionality in the relation $M_{t_0,\infty} \bar{w}_\infty^4$ is measured to be about $50 M_{\odot} s^2/km^4$ [11], it is straightforward to check that this result is in very good agreement with eq. (65) (for the present epoch). This justifies the definition of $\sigma(-1)$ made in eq. (32). Note that, what enters into eq. (65) is the active gravitational mass $M_t$ (at epoch $t_0$); this increases linearly with cosmic epoch. Moreover, since the Hubble parameter decreases as the inverse of cosmic epoch, this means that $\bar{w}_\infty$ is constant; i.e., that it does not evolve with epoch. In other words, the quasi-metric model predicts that the sizes of metrically stationary galactic disks increase with epoch, but such that rotational speeds are unaffected.
Moreover, a FO at an arbitrary epoch \( t_1 \), using his locally measured values \( \frac{1}{2} G \) and \( \frac{1}{2} H_0 \) of the gravitational “constant” and the Hubble parameter, respectively, is predicted to measure the same slope of the local baryonic Tully-Fisher relation as will a FO at epoch \( t_0 \). This means that the slope of the baryonic Tully-Fisher relation is predicted not to depend on epoch, and this seems to be confirmed by observations [12,13]. (The analyses presented in these papers are based on the standard cosmological framework, so some inconsistency with the QMF might be expected. But as long as theory-dependent gravitational physics (i.e., inconsistent with the QMF) is not used to infer masses, this should not matter too much.)

5 The general weak-field, axisymmetric case

Having analysed a flat disk, the question now is if the results of the previous sections can be extended to the general, weak-field axisymmetric case. To answer that question, we will follow the standard method of breaking up an arbitrary axisymmetric matter distribution into a series of concentric, spherical shells (of negligible thickness) with corresponding surface densities. The total potential at a given point will then be the sum of the potentials due to the collection of shells. Except for the limitation to axial symmetry, this procedure is the counterpart to the derivation of a general multipole expansion for the Newtonian case as given in [7], which we will follow closely.

The mathematical task is then to solve eq. (7) (with \( V = 0 \)) interior, respectively, exterior to a given isolated shell, subject to suitable boundary conditions. Since eq. (7) becomes separable we may write solutions in the form \( B(\rho, \theta) = 1 + 2c^{-2}\phi(\rho, \theta) \), were \( c^{-2}\phi(\rho, \theta) \equiv -P(\rho)G \). General mode solutions \( F_{\beta \rho}(\rho), \tilde{G}_{\beta \rho}(\theta) \) are given by eqs. (A.3), (A.4) in appendix A. We will select specific mode solutions \( \phi(\rho, \theta) \). The mathematical task is then to solve eq. (7) (with \( V = 0 \)) interior, respectively, exterior to a given isolated shell, subject to suitable boundary conditions. Since eq. (7) becomes separable we may write solutions in the form \( B(\rho, \theta) = 1 + 2c^{-2}\phi(\rho, \theta) \), were \( c^{-2}\phi(\rho, \theta) \equiv -P(\rho)G \). General mode solutions \( F_{\beta \rho}(\rho), \tilde{G}_{\beta \rho}(\theta) \) are given by eqs. (A.3), (A.4) in appendix A. We will select specific mode solutions \( \phi(\rho, \theta) \).

The exterior solution is found by requiring that \( \phi(\rho, \theta) \) should be regular at the center of the shell we find that

\[
c^{-2}\phi_{\text{ext}}(\rho, \theta) = -\sqrt{\frac{\Xi_0}{\rho}} \sum_{n=0}^{\infty} A_n P_{\frac{1}{2}}^{n-\frac{1}{2}}(1 - \frac{\rho^2}{\Xi_0}) P_n(\cos \theta), \quad \rho > \rho_s. \tag{67}
\]

The exterior solution is found by requiring that \( \phi_{\text{ext}}(\Xi_0, \theta) \) must vanish. So, for \( \rho > \rho_s \),

\[
c^{-2}\phi_{\text{ext}}(\rho, \theta) = -\sqrt{\frac{\Xi_0}{\rho}} \sum_{n=0}^{\infty} D_n \rho P_{\frac{1}{2}}^{n-\frac{1}{2}}(1 - \frac{\rho^2}{\Xi_0}) P_n(\cos \theta), \quad \rho > \rho_s. \tag{68}
\]

where the constants \( D_n \) are given by the expression \( D_n = -B_n P_{\frac{1}{2}}^{n+\frac{1}{2}}(0)/P_{\frac{1}{2}}^{n-\frac{1}{2}}(0), \) which vanishes for all even \( n \) (see eq. (A.5)).

Now the surface density \( \Sigma_s(\rho_s, \theta) \) of the shell can be expressed as a sum of modes; this yields counterparts to eqs. (20)–(22) valid for a thin disk. We thus have

\[
\Sigma_s(\rho_s, \theta) = \frac{1}{2} \sum_{n=0}^{\infty} (2n + 1) \Sigma_{\text{int}}(\rho_s, \theta) P_n(\cos \theta), \quad \Sigma_{\text{ext}}(\rho_s, \theta) = \int_0^\pi \Sigma_s(\rho_s, \theta') P_n(\cos \theta') \sin \theta' d\theta'. \tag{69}
\]

To determine the constants \( A_n \) and \( B_n \), we now apply Gauss’ theorem across the shell. Assuming a weak gravitational field \( (B=1) \) we then have

\[
\int \left[ \frac{\partial \phi_{\text{int}}}{\partial \rho} \right]_{\rho = \rho_s} - \frac{\partial \phi_{\text{ext}}}{\partial \rho} \right]_{\rho = \rho_s} = 4\pi G \Sigma_s(\rho_s, \theta). \tag{70}
\]

Since the potential must be continuous over the shell, we have that \( \phi_{\text{ext}}(\rho_s, \theta) = \phi_{\text{int}}(\rho_s, \theta) \). Furthermore, due to the orthogonality of the Legendre polynomials, this means that

\[
(A_n + \frac{P_{\frac{1}{2}}^{n+\frac{1}{2}}(0)}{P_{\frac{1}{2}}^{n-\frac{1}{2}}(0)} B_n) P_{\frac{1}{2}}^{n-\frac{1}{2}}(1 - \frac{\rho_s^2}{\Xi_0}) = B_n P_{\frac{1}{2}}^{n+\frac{1}{2}}(1 - \frac{\rho_s^2}{\Xi_0}). \tag{71}
\]
A second equation relating the constants $A_n$ and $B_n$ can be found from eq. (70) by inserting eq. (69). Calculating the derivatives and using recurrence formulae valid for Legendre functions, one finds that (since the Legendre polynomials are orthogonal)

\[
(1 - n^2) P_n^{s+\frac{1}{2}} \left( \frac{1 - \rho^2}{\Xi_0} \right) B_n + \left( A_n + \frac{P_n^{s+\frac{1}{2}}(0)}{P_n^{s-\frac{1}{2}}(0)} B_n \right) P_n^{s+\frac{1}{2}} \left( \frac{1 - \rho^2}{\Xi_0} \right) = (2n + 1) \sqrt{\frac{\rho_s}{\Xi_0}} \frac{\Sigma_{sn}(\rho_s)}{\Sigma_s}.
\]  

(72)

One may find explicit expressions for $A_n$ and $B_n$ from eqs. (71) and (72). We get

\[
A_n = (2n + 1) \sqrt{\frac{\rho_s}{\Xi_0}} \frac{\Sigma_{sn}(\rho_s)}{\Sigma_s} g_n(\rho_s), \quad B_n = (2n + 1) \sqrt{\frac{\rho_s}{\Xi_0}} \frac{\Sigma_{sn}(\rho_s)}{\Sigma_s} f_n(\rho_s),
\]

(73)

\[
f_n(\rho_s) \equiv \frac{P_n^{s+\frac{1}{2}}(y_s)}{(1 - n^2) P_n^{s-\frac{1}{2}}(y_s) P_n^{\frac{s+1}{2}}(y_s) + P_n^{s+\frac{1}{2}}(y_s) P_n^{s-\frac{1}{2}}(y_s)}.
\]

(74)

\[
y_n(\rho_s) \equiv \left[ \frac{P_n^{s+\frac{1}{2}}(y_s)}{P_n^{s-\frac{1}{2}}(y_s)} - \frac{P_n^{s+\frac{1}{2}}(0)}{P_n^{s-\frac{1}{2}}(0)} \right] f_n(\rho_s).
\]

(75)

Equations (73)–(75) may now be inserted into eqs. (67), (68) to get complete expressions for $\Phi_{int}(\rho, \theta)$ and $\Phi_{ext}(\rho, \theta)$. This yields the potential generated by a single, infinitesimally thin shell.

To evaluate the potential generated by an entire collection of shells filling space, we let $\delta \Sigma_s(\rho_s)$ and $\delta \Sigma_{sn}(\rho_s)$ denote the relevant quantities for a shell lying between $\rho_s$ and $\rho_s + d\rho_s$. From eq. (69) we then have (inserting $\delta \Sigma_s(\rho_s, \theta') = \tilde{\varrho}_m(\rho_s, \theta') d\rho_s / \sqrt{1 - \rho_s^2}$)

\[
\delta \Sigma_{sn}(\rho_s) = \int_0^\pi \tilde{\varrho}_m(\rho_s, \theta') P_n(\cos \theta') \sin \theta' d\theta' \frac{\delta \rho_s}{\sqrt{1 - \rho_s^2}} \equiv \frac{2 \tilde{\varrho}_m(\rho_s) d\rho_s}{\sqrt{1 - \rho_s^2}},
\]

(76)

where $\tilde{\varrho}_m(\rho_s, \theta)$ is the coordinate volume density of (active) mass [4,5]. Substituting eq. (76) into the complete expressions for the corresponding potentials $\delta \Phi_{int}(\rho, \theta)$ and $\delta \Phi_{ext}(\rho, \theta)$ and integrating over $\rho_s$, we find that (with $x_s \equiv \frac{\rho_s}{\Xi_0}, y \equiv \sqrt{1 - \rho_s^2}$ and $\tilde{\varrho}_m(\rho_s, \theta) \equiv \frac{\rho_s}{\Xi_0}$)

\[
c^{-2} \Phi(\rho, \theta) = c^{-2} \sum_{\rho_s = 0}^\rho \delta \Phi_{ext}(\rho, \theta) + c^{-2} \sum_{\rho_s = 0}^\rho \delta \Phi_{int}(\rho, \theta)
\]

\[
= -\frac{\Xi_0}{\rho} \int_0^\infty \rho P_n(\cos \theta) \left[ \left( P_n^{s+\frac{1}{2}}(y) - \frac{P_n^{s+\frac{1}{2}}(0)}{P_n^{s-\frac{1}{2}}(0)} P_n^{s-\frac{1}{2}}(y) \right) g_n(\rho_s) \right] dx_s \sqrt{x_s g_n(x_s) g_m(x_s)} dx_s
\]

\[
\times \int_0^\infty \frac{1}{\sqrt{1 - x_s^2}} \frac{\sqrt{x_s g_n(x_s) g_m(x_s)} dx_s}{\sqrt{1 - x_s^2}} + \frac{P_n^{s-\frac{1}{2}}(y)}{\sqrt{1 - x_s^2}} \left[ \int_0^1 \sqrt{x_s g_n(x_s) g_m(x_s)} dx_s \right].
\]

(77)

Equation (77) is the axisymmetric counterpart to the similar general multipole expansion formula in Newtonian theory; this is given explicitly in [7]. These two expressions have a correspondence in the limit $\Xi_0 \to \infty$. For example, the potential at $\rho = 0$ is given by $c^{-2}\Phi(0) = -\frac{\Xi_0}{\rho} \int_0^\infty x_s \tilde{\varrho}_m(x_s) dx_s$, the same as the Newtonian expression in said limit. This means that as expected, eq. (77) cannot describe DM effects.

Moreover, due to the boundary condition at $\rho = \Xi_0$, there is no obvious way to define an associated (volume) density as a counterpart to eq. (32). But if no associated density can be defined, one cannot motivate a definition similar to eq. (43) for the induced associated (volume) density. So it would seem that, the explanation of DM effects presented in this paper should exclusively be connected to disks or other structures that do not satisfy the boundary condition $\Phi(\Xi_0) = 0$, answering the question asked at the beginning of this section.

However, DM effects are seen in galaxies other than spiral galaxies. In particular, observations indicate that dwarf spheroidal galaxies are the most DM-dominated systems ever found [14]. Yet the general observational status for the existence of DM in elliptical galaxies is more complicated than for spiral galaxies, since some ordinary elliptical galaxies apparently lack significant amounts of it [15], while others seem to have plenty [16]. It still remains the challenging task of explaining these observations without DM.
6 Gravitational lensing

For a sufficiently weak gravitational field, rotation curves can be calculated accurately enough using the auxiliary metric family $\bar{g}_s$ rather than the full “physical” metric family $g_s$. However, when calculating deflection of light, or gravitational lensing, it is not sufficient to know $\bar{g}_s$, even if the gravitational field is weak. But to calculate gravitational lensing, it is fortunately not necessary to know $g_s$ in full; as we shall see, a suitable approximation will be sufficient.

The general formulae describing the transformation $g_s \rightarrow g_t$ are given by $[2,3]$

$$g(t)_{00} = \left(1 - \frac{v^2}{c^2}\right)^2 \bar{g}(t)_{00},$$  \hspace{1cm} (78)  

$$g(t)_{0j} = \left(1 - \frac{v^2}{c^2}\right) \left[\bar{g}(t)_{0j} + \frac{t}{t_0} \frac{\bar{g}(t)_{0j}}{1 - \frac{v^2}{c^2}} (e^i_t N_i) e^j_t\right],$$  \hspace{1cm} (79)  

$$g(t)_{ij} = \bar{g}(t)_{ij} + \frac{t^2}{t_0^2} \frac{4 \bar{e}_b^i \bar{e}_b^j}{(1 - \frac{v^2}{c^2})^2} e^b_t e^b_t,$$  \hspace{1cm} (80)  

where $\bar{e}_b^i \equiv \frac{\partial}{\partial x^i} \bar{g}^{ij}$ and $e^b_t \equiv \frac{1}{t_0} e^i_t \mathrm{d}x^i$ are unit vector and covector fields, respectively, along the 3-vector field $b_F$ found from the set of linear algebraic equations

$$\left[\bar{a}^k_{Fij} + c^{-2} \bar{a}_{Fk} \bar{a}^k_F\right] b^j_F = - \left[\bar{a}^j_{Fik} + c^{-2} \bar{a}_{Fk} \bar{a}^j_F\right] b^k_F - 2 \bar{a}^j_F = 0,$$  \hspace{1cm} (81)  

and where $v \equiv c^{-1} \sqrt{\bar{a}_{Fk} \bar{a}^k_F b_F, b^2_F}$. For a weak gravitational field and for distances much smaller than $\Xi_0$ (at epoch $t_0$), we have that $v \ll c$, so to a good approximation we may neglect terms of order 2 or higher in the small quantity $v/c$. This means that we may set (assuming $N_i \approx 0$)

$$g(t)_{00} \approx \bar{g}(t)_{00}, \hspace{1cm} g(t)_{0j} \approx 0, \hspace{1cm} g(t)_{ij} \approx \bar{g}(t)_{ij} + \frac{t^2}{t_0^2} \frac{4 v^2}{c^2} e^b_t e^b_t,$$  \hspace{1cm} (82)  

To get explicit formulae, it is easiest to solve the set of equations (81) using standard methods, in spherical coordinates. One may then transform to cylindrical coordinates. If one additionally assumes a weak field in vacuum, and neglects all terms proportional to $\Xi_0^{-2}$, one may give $b^\xi_F$ and $b^\eta_F$ as series expansions in $z$. Terminating the series after the linear term, yields (using the fact that in vacuum, $B_{zz} \approx -B_{\xi\xi} = -\frac{1}{\xi} B_{\xi}$ from eq. (10))

$$b^\xi_F \approx -2 \left[\frac{(B_{\xi\xi} + \frac{1}{2} B_{\xi}) B_{\xi\xi} + B_{\xi\xi} \xi B_{\xi}}{B_{\xi\xi} + \frac{1}{2} B_{\xi}}\right] + \frac{z}{\xi} b^\eta_F + O(z^2),$$  \hspace{1cm} (83)  

$$b^\eta_F \approx 2 \left[\frac{B_{\xi\xi} B_{\xi} - B_{\xi\xi} B_{\xi}}{B_{\xi\xi} + \frac{1}{2} B_{\xi}}\right] + O(z^2),$$  \hspace{1cm} (84)  

where the derivatives may be calculated approximately to first order in $|z|$ from the expression

$$\bar{B}(\xi, z) \approx 1 + \frac{2}{c^2} \left(\Phi(\xi) + \Psi(\xi)\right) + \frac{4 \pi G}{c^2} \left(\Sigma(\xi) + \sigma^1(\xi)\right) |z| - \frac{1}{c^2} \left[\Phi_{\xi\xi} + \frac{1}{\xi} \Phi_{\xi\xi} + \Psi_{\xi\xi} + \frac{1}{\xi} \Psi_{\xi\xi}\right] z^2 + O(|z|^3),$$  \hspace{1cm} (85)  

obtained from eq. (51) and the approximation for $\bar{B}_{\xi\xi}$ shown above. Furthermore, since the quantities $\bar{e}^\rho_F$ and $\bar{e}^\eta_F$ may be found to the relevant accuracy from the definition

$$e^\rho_t = \bar{h}_{(t)ij} e^\rho_b \equiv \bar{h}_{(t)ij} \frac{b^\rho_F}{|b_F|} b^\eta_F |b_F| \equiv \sqrt{\bar{a}_{Fk} b_{Fk}},$$  \hspace{1cm} (86)  

we may also calculate the relevant approximation of the desired quantities $g_{(t)ij}$ from eq. (82), i.e.,

$$g_{(t)ij} \approx \bar{g}_{(t)ij} + \frac{t^2}{t_0^2} \frac{v}{c} e^b_t e^b_t \bar{g}_{(t)ij} + \frac{t^2}{t_0^2} \left[\frac{B_{\xi\xi} B_{\xi}}{b_F, b^\eta_F} b_F b_F, b^\eta_F \right].$$  \hspace{1cm} (87)
From the geodesic equation, we may now calculate the gravitational bending of a light ray grazing the plane of the disk, in this case, $|z|$ is small enough so that the approximation given in eq. (85) is valid. To deal with the opposite situation, where the light path is nearly orthogonal to the disk plane, the approximations given in eqs. (83)–(85) may not be sufficient; then one must include terms of higher order in $|z|$.

Finally, we note that in GR, the weak field form of the metric outside the disk is assumed to take the form

$$ds^2 = -\left(1 + \frac{2\Phi}{c^2}\right)(dx^0)^2 + \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2),$$

where $\Phi$ is the Newtonian potential for the sum of visible and DM. But in general, the quantities $g_{(ij)}$ as found from eq. (87) will not correspond to their counterparts given by eq. (88). This means that any observationally based mapping of DM distributions using gravitational lensing, assuming the weak field approximation obtained from GR, is explicitly theory-dependent and may give misleading results.

### 7 Discussion

For many years, it has been known that galactic dynamics is incompatible with a straightforward application of Newtonian theory to visible matter. However, the most glaring discrepancies between observations and theory can be removed by assuming the existence of galactic DM. Since the introduction of DM can be done without making radical changes to the standard theoretical framework underlying mainstream astrophysics, this is currently the preferred approach. On the other hand, MOND interpreted as an empirical recipe, has an impressive successful record when predicting galactic phenomena. But the connection between MOND and fundamental physics has been unclear so far.

Contrary to other approaches, the explanation of some galactic phenomena given in this paper has not assumed any empirical aspect of galactic dynamics as input to the model. Rather, while formally belonging to the “modified gravity” category, the model comes directly from the weak field approximation of the QMF, without any extra modifications of the theory. (The only extra assumption made, is that the induced associated surface density $\bar{\sigma}(u)$ should be treated as a gravitating source in the field equations.) The main reason why this is possible is that according to the QMF, the Universe is finite and “small”, so that boundary conditions depend crucially on the shape of the matter distribution. (A finite and “small” Universe is incompatible with cosmological data as interpreted within the standard framework, therefore the DM explanation given here is compatible with the weak field limit of the QMF but not with the weak field limit of standard cosmology.) In particular, for a flat disk we found that $\Phi(u = -1) \neq 0$ in eq. (29), defining a specific velocity scale dependent on the total mass of the disk. To be able to define the associated surface density $\bar{\sigma}(u)$ in eq. (32), it is essential that this velocity scale does not vanish. However, by construction, it does vanish for the general axisymmetric matter distribution considered in sect. 5, so flat disks seem to be an exceptional case. Anyway, that case should apply to all thin disks (even if they are not exactly flat).

The other crucial feature is the existence of an induced matter surface density $\Sigma(u)$ directly dependent on the real matter surface density as shown in eq. (42). This, together with the existence of $\bar{\sigma}(u)$, is sufficient to define the induced associated surface density $\bar{\sigma}(u)$ playing the role of DM. The introduction of $\bar{\sigma}(u)$ may seem “contrived” to some people. However, the fact that the induced associated surface density corresponding to an exponential disk automatically yields an asymptotically flat rotation curve and a correspondence with MOND are calculated results, and not put in by hand a priori. It was not at all obvious that these results would be possible.

It has been claimed [17] that observations of gravitational lensing in the colliding clusters 1E0657-56 (the Bullet cluster) represent a “direct” detection of DM, since these observations indicate that the DM is associated with the regions containing the field galaxies rather than with the regions containing the more massive gas making up the bulk of the cluster. However, as we have seen, in the QMF, the existence of any sort of “phantom” matter density similar to $\bar{\sigma}(u)$, playing the role of DM, is crucially dependent on the shape of the matter distribution. This means that, e.g., a large nearly spherical or spheroidal mass distribution of gas, should not necessarily be associated with much DM as inferred from gravitational lensing. So, since the colliding gas clouds in the Bullet cluster shown in [17] do not seem to have shapes that could in any way resemble disks, this might be a natural explanation of why they do not seem to be associated with much DM. But of course, further justification of this explanation will be necessary, together with an explanation of why dwarf spheroidals and some elliptic galaxies seem to be DM-dominated. Anyway, the mere existence of such an explanation shows that the interpretation of the Bullet cluster observations is not theory-independent, so citing them as definite evidence of the existence of DM is unjustified.

In light of the results found in this paper, it seems that some lines of argument favouring DM over modified gravity have been shown to be invalid. First, there now exists a natural correspondence between MOND and fundamental physics. This indicates that at least some galactic phenomenology has its basis in geometry rather than in the properties of some unknown exotic particle. Second, while the correspondence with MOND works for spiral galaxies, this does not imply that such a correspondence is necessarily valid for other types of galaxies or galaxy clusters. This means that it may be possible to share MOND’s successes but not necessarily its failures. (Further work should be done to see if
there is a serious lack of critical assessment of basic assumptions underlying mainstream knowledge. One may hope even if alternative interpretations are not ruled out [4]. So it would seem that, for at least some parts of astrophysics, interpretations of some indirect observations made in the solar system that are also presented as indisputable facts, attitude, since interpretations of indirect observations are often crucially theory-dependent. Another example of this ad hoc out of the question, regardless of how contrived and astrophysicists, taking the focus off DM and recognising the merits of modified gravity, is a radical move that is opinion that indirect observations are sufficient to rule out any alternative explanations. This means that for many astrophysicists, the analysis of the cosmic microwave background, has reinforced this consensus. That is, many astrophysicists are of the in the early Universe, in order to have standard cosmology agree with primordial nucleosynthesis and a standard values of the corresponding complex-valued quantities undefined, that expression cannot be considered at all.

In this appendix, we list mode solutions of the vacuum field equations for axially symmetric, metrically static, isolated sources with various “pure” values of their multipole moments. As we shall see, these solutions can be classified into two groups; those that admit the boundary condition \( B(\Xi_0) = 1 \) and those that do not.

Starting with eq. (7), we may set \( \dot{V} = 0 \) for a nonrotating source (this is a good approximation for slowly rotating sources and weak gravitational fields also). Equation (7) then becomes separable, i.e., solutions of it can be written in the form \( \dot{B}(\rho, \theta) = 1 - 2\dot{F}(\rho)\dot{G}(\theta) \). The new functions \( \dot{F} \) and \( \dot{G} \) must then satisfy the ordinary differential equations

\[
\left(1 - \frac{\rho^2}{\Xi_0^2}\right) \frac{d^2\dot{F}}{d\rho^2} + \frac{2}{\rho} \left(1 - \frac{3\rho^2}{2\Xi_0^2}\right) \frac{d\dot{F}}{d\rho} - \frac{\beta}{\rho^2}\dot{F} = 0, \\
\frac{d^2\dot{G}}{d\theta^2} + \cot\theta \frac{d\dot{G}}{d\theta} + \beta \dot{G} = 0,
\]

where \( \beta \) is some complex-valued constant. Restricting \( \beta \) to be real and requiring that \( \beta \geq -1/4 \), the general solutions of eqs. (A.1) and (A.2) may be written in the form

\[
\dot{F}_{\beta \pm}(\rho) = \sqrt{\frac{\Xi_0}{\rho}} \left[ c_{\beta \pm}\rho^\frac{1}{4}\sqrt{\beta + \frac{1}{2}} \left(1 - \frac{\rho^2}{\Xi_0^2}\right)^\frac{1}{4} + c_{\beta \pm}Q_{\frac{1}{4}\beta}^\pm \sqrt{\beta + \frac{1}{2}} \left(1 - \frac{\rho^2}{\Xi_0^2}\right)^\frac{1}{4}\right],
\]

\[
\dot{G}_{\beta \pm}(\theta) = C_{\beta \pm}\rho^\frac{1}{4}\sqrt{\beta + \frac{1}{2}} \left(\cos\theta\right) + C_{\beta \pm}Q_{\frac{1}{4}\beta}^\pm \sqrt{\beta + \frac{1}{2}} \left(\cos\theta\right),
\]

where \( P_{\nu}^\mu(\rho) \), \( Q_{\nu}^\mu(\rho) \) are the usual associated Legendre functions of the first and second kind, respectively, and where \( c_{\beta \pm}, C_{\beta \pm}, C_{\beta \pm} \) (dimensionless) constants. Note that the solutions (A.3) and (A.4) are real-valued functions, and so are \( P_{\nu}^\mu(\rho) \), \( Q_{\nu}^\mu(\rho) \) (for real \( \mu, \nu \)) since they are defined on the cut \((-1, 1)\) by averaging the relevant limiting values of the corresponding complex-valued quantities \( P_{\nu}^\mu(z) \), \( Q_{\nu}^\mu(z) \) [8].

We will now find solutions (A.3) admitting the boundary condition \( \dot{F}_{\beta \pm}(\Xi_0) = 0 \). To do that, we first notice that, since \( P_{\frac{1}{2}}^\frac{3}{2}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = \sqrt{\frac{2\Xi_0(1 - \frac{\rho^2}{\Xi_0^2})}{\pi \rho}} \) and since only the trivial constant solution is obtained from \( P_{\frac{1}{2}}^\frac{3}{2}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = \sqrt{\frac{2\Xi_0}{\pi \rho}}, \) choosing the + -sign, with \( C_{0+} = 1 \) and an appropriate choice of \( c_{0+} \), the value \( \beta = 0 \) corresponds to the unique spherically symmetric solution found in ref. [4]. That solution is unique since the function \( Q_{\frac{1}{2}}^\frac{3}{2}(x) \) differs from \( P_{\frac{1}{2}}^\frac{3}{2}(x) \) only by a numerical factor, and since \( Q_{\frac{1}{2}}^\frac{3}{2}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) \) again only yields the trivial constant solution. Besides, since the function \( Q_0(\cos\theta) \) has singularities whenever \( \theta = 0 \) or \( \theta = \pi \), the corresponding solution cannot be physical (but might be considered as a mode solution). Moreover, \( P_{-1}(\cos\theta) = P_0(\cos\theta) \) gives nothing new, and since \( Q_{-1}(\cos\theta) \) is undefined, that expression cannot be considered at all.

Secondly, since choosing \( \beta = 2 \) yields \( P_{\frac{3}{2}}^\frac{3}{2}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = -\sqrt{\frac{2\Xi_0(1 - \frac{\rho^2}{\Xi_0^2})}{\pi \rho}} \) and \( P_{\frac{1}{2}}^\frac{3}{2}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) = \sqrt{\frac{2\Xi_0}{\pi \rho}} \left(\arccos\left(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}\right) - \frac{\rho}{\Xi_0 \sqrt{1 - \frac{\rho^2}{\Xi_0^2}}\right) \right) \) which do not fulfil the required boundary condition, and since \( Q_{\frac{3}{2}}^\pm(x) \equiv 0 \) on the cut \( Q_{\frac{3}{2}}^\pm(x) \) is undefined, the next suitable solution is found by choosing \( \beta = 6 \). This is so, since even though \( P_{\frac{3}{2}}^\frac{3}{2}(\sqrt{1 - \frac{\rho^2}{\Xi_0^2}}) \) does not fulfil the
required boundary condition, \( P_{2n}^1(x) = 3(1-x^2) \), obviously does. This solution was found in ref. [5] and corresponds to a pure quadrupole field since it involves the Legendre polynomial \( P_{-3}(\cos \theta) = P_2(\cos \theta) \) (also, \( Q_2(\cos \theta) \) is singular and the corresponding mode solution thus unphysical, and \( Q_{-3}(\cos \theta) \) is undefined). Moreover, given the required boundary condition, this solution is also unique since \( Q_{-3}^1(x) \equiv 0 \) on the cut (\( Q_{-3}^1(x) \) is undefined).

Similarly, since the subsequent suitable solution is found by choosing \( \beta = 20 \) and the + sign, corresponding to a pure octopole field involving the Legendre polynomial \( P_{-5}(\cos \theta) = P_4(\cos \theta) \), it would seem that all suitable solutions are given by choosing the + sign and \( \beta = 2n(2n+1) \), \( n = 0, 1, 2, \ldots \), corresponding to pure even multipole fields involving the Legendre polynomials \( P_{2n-1}(\cos \theta) = P_{2n}(\cos \theta) \). With the required boundary condition, these solutions are also unique, since the functions \( P_{2n-1}^m(\sqrt[3]{-x^2}) \), \( m = 0, 1, 2, \ldots \), do not admit it, and since \( Q_{2n}^m(x) \equiv 0 \) on the cut (found by using the recurrence relation \( Q_{n+2}(x) + 2(\mu+1)x(1-x^2)^{-1/2}Q_n(x) + (\nu - \mu)(\nu + \mu + 1)Q_n(x) = 0 \). Also, \( Q_{-2n+1}^m(x) \) and \( Q_{-2n-1}(\cos \theta) \) are all undefined. Finally, since \( Q_{2n}(\cos \theta) \) is singular for all \( n \), the corresponding mode solutions are unphysical.

That the above extrapolation is correct, can be checked by using the formulae [8]

\[
P_{2\pm}^1(x) = 2^{\mp 1/2} \pi^{-1/2} \cos \left[ \frac{\pi}{2} \left( \frac{1}{2} + \frac{1}{2} \sqrt{\beta + \frac{1}{4}} \right) \right] \frac{\Gamma \left( \frac{3}{2} + \frac{1}{2} \sqrt{\beta + \frac{1}{4}} \right)}{\Gamma \left( \frac{5}{2} + \frac{1}{2} \sqrt{\beta + \frac{1}{4}} \right)},
\]

\[
Q_{2\pm}^1(x) = -2^{\mp 1/2} \pi^{-1/2} \sin \left[ \frac{\pi}{2} \left( \frac{1}{2} + \frac{1}{2} \sqrt{\beta + \frac{1}{4}} \right) \right] \frac{\Gamma \left( \frac{3}{2} + \frac{1}{2} \sqrt{\beta + \frac{1}{4}} \right)}{\Gamma \left( \frac{5}{2} + \frac{1}{2} \sqrt{\beta + \frac{1}{4}} \right)}.
\]

Since the reciprocal gamma function \( 1/\Gamma(x) \) possesses simple zeros at \( x = 0, -1, -2, \ldots \), [8], these expressions vanish for the chosen values \( \beta = 2n(2n+1) \), \( n = 1, 2, 3, \ldots \), and using the + sign (the special case \( \beta = 0 \) was addressed previously). Moreover, no other values of \( \beta \) will do (since the other possible values \( \beta = 2n(2n-1) \) yield functions of the form \( Q_{2n-1}^m(x) \equiv 0 \)), so we have really found all relevant solutions satisfying the boundary condition \( F_{\beta \pm}(\Xi_0) = 0 \) (using the + sign rather than the + sign yields no further zeros).

So, to summarise, said boundary condition is fulfilled by choosing the functions \( P_{2n+1}^{\pm 1}(\sqrt[3]{-x^2}) \) in eq. (A.3); all other choices fail (or are irrelevant).

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