Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces

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Abstract. In this paper, we achieve the general solution and the generalized Hyers-Ulam-Rassias stability of the following functional equation

\[ f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2) f(x) \]

for fixed integers \( k \) with \( k \neq 0, \pm 1 \) in the quasi-Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [20] in 1940, concerning the stability of group homomorphisms. Let \((G_1, \cdot)\) be a group and let \((G_2, \ast)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(x \cdot y), h(x) \ast h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)? In the other words, Under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers [9] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \( f : E \to E' \) be a mapping between Banach spaces such that

\[ \| f(x + y) - f(x) - f(y) \| \leq \delta \]

for all \( x, y \in E \), and for some \( \delta > 0 \). Then there exists a unique additive mapping \( T : E \to E' \) such that

\[ \| f(x) - T(x) \| \leq \delta \]

for all \( x \in E \). Moreover if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in E \), then \( T \) is linear.

In 1978, Th. M. Rassias [16] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.

The functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (1.1) \]

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is related to symmetric bi-additive function \([1,2,10,13]\). It is natural that this equation is
called a quadratic functional equation. In particular, every solution of the quadratic equation
(1.1) is said to be a quadratic function. It is well known that a function \(f\) between real vector
spaces is quadratic if and only if there exits a unique symmetric bi-additive function \(B\) such
that \(f(x) = B(x,x)\) for all \(x\) (see \([1,13]\)). The bi-additive function \(B\) is given by
\[
B(x,y) = \frac{1}{4}(f(x+y) - f(x-y)).
\]

A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.1) was
proved by Skof for functions \(f : A \rightarrow \mathbb{R}\), where \(A\) is normed space and \(B\) Banach space
(see \([18]\)). Cholewa [4] noticed that the Theorem of Skof is still true if relevant domain \(A\) is
replaced an abelian group. In the paper \([5]\), Czerwik proved the Hyers-Ulam-Rassias
stability of the equation (1.1). Grabiec [8] has generalized these result mentioned above.
Jun and Kim [11] introduced the following cubic functional equation
\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x),
\]
and they established the general solution and the generalized Hyers-Ulam-Rassias stability
for the functional equation (1.3). The \(f(x) = x^3\) satisfies the functional equation (1.3), which
is called a cubic functional equation. Every solution of the cubic functional equation is said
to be a cubic function.

Jun and Kim proved that a function \(f\) between real vector spaces \(X\) and \(Y\) is a solution
of (1.3) if and only if there exits a unique function \(C : X \times X \times X \rightarrow Y\) such that
\(f(x) = C(x, x, x)\) for all \(x \in X\), and \(C\) is symmetric for each fixed one variable and is
additive for fixed two variables.

K. Jun and H. Kim [12], have obtained the generalized Hyers-Ulam stability for a mixed type
of cubic and additive functional equation. In addition the generalized Hyers-Ulam-Rassias
for a mixed type of quadratic and additive functional equation in quasi-Banach spaces have
been investigated by A. Najati and M. B. Moghimi [14]. Also A. Najati and G. Zamani
Eskandani [15] introduced the following functional equation
\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2f(2x) - 4f(x),
\]
with \(f(0) = 0\). It is easy to see that the mapping \(f(x) = ax^3 + bx\) is a solution of the
functional equation (1.4). They established the general solution and the generalized Hyers-
Ulam-Rassias stability for the functional equation (1.4) whenever \(f\) is a mapping between
two quasi-Banach spaces. Now, we introduce the following functional equation for fixed
integers \(k\) with \(k \neq 0, \pm 1:\)
\[
f(x + ky) + f(x - ky) = k^2f(x + y) + k^2f(x - y) + 2(1 - k^2)f(x),
\]
with \(f(0) = 0\). It is easy to see that the function \(f(x) = ax^2 + bx^2 + cx\) is a solution of the
functional equation (1.5). In the present paper we investigate the general solution of
functional equation (1.5) when \(f\) is a mapping between vector spaces, and we establish the
generalized Hyers-Ulam-Rassias stability of the functional equation (1.5) whenever \(f\) is a
function between two quasi-Banach spaces.

We recall some basic facts concerning quasi-Banach space and some preliminary results.

**Definition 1.1.** (See \([3,17]\).) Let \(X\) be a real linear space. A quasi-norm is a real-valued
function on \(X\) satisfying the following:

1. \(\|x\| \geq 0\) for all \(x \in X\) and \(\|x\| = 0\) if and only if \(x = 0\).
2. \(\|\lambda x\| = |\lambda|\|x\|\) for all \(\lambda \in \mathbb{R}\) and all \(x \in X\).
3. There is a constant \(M \geq 1\) such that \(\|x + y\| \leq M(\|x\| + \|y\|)\) for all \(x, y \in X\).
It follows from condition (3) that
\[
\| \sum_{i=1}^{2n} x_i \| \leq M^n \sum_{i=1}^{2n} \| x_i \|, \quad \| \sum_{i=1}^{2n+1} x_i \| \leq M^{n+1} \sum_{i=1}^{2n+1} \| x_i \|
\]
for all \( n \geq 1 \) and all \( x_1, x_2, ..., x_{2^{n+1}} \in X \).

The pair \( (X, \| . \|) \) is called a quasi-normed space if \( \| . \| \) is a quasi-norm on \( X \). The smallest possible \( M \) is called the modulus of concavity of \( \| . \| \). A quasi-Banach space is a complete quasi-normed space.

A quasi-norm \( \| . \| \) is called a p-norm \((0 < p \leq 1)\) if
\[
\| x + y \|^p \leq \| x \|^p + \| y \|^p,
\]
for all \( x, y \in X \). In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula \( d(x, y) := \| x - y \|^p \) gives us a translation invariant metric on \( X \). By the Aoki-Rolewicz Theorem [17] (see also [3]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms, henceforth we restrict our attention mainly to p-norms. More over in [19], J. Tabor has investigated a version of Hyers-Rassias-Gajda Theorem (see [6,16]) in quasi-Banach spaces.

2. General solution

Throughout this section, \( X \) and \( Y \) will be real vector spaces. Before proceeding the proof of Theorem 2.3 which is the main result in this section, we shall need the following two Lemmas.

**Lemma 2.1.** If an even function \( f : X \rightarrow Y \) with \( f(0) = 0 \) satisfies (1.5), then \( f \) is quadratic.

**Proof.** Setting \( x = 0 \) in (1.5), by evenness of \( f \), we obtain \( f(kx) = k^2 f(x) \). Replacing \( x \) by \( kx \) in (1.5) and then using the identity \( f(kx) = k^2 f(x) \), we lead to
\[
f(kx + y) + f(kx - y) = f(x + y) + f(x - y) + 2(k^2 - 1)f(x) \quad (2.1)
\]
for all \( x, y \in X \). Interchange \( x \) with \( y \) in (1.5), gives
\[
f(y + kx) + f(y - kx) = k^2 f(y + x) + k^2 f(y - x) + 2(1 - k^2)f(y) \quad (2.2)
\]
for all \( x, y \in X \). By evenness of \( f \), it follows from (2.2) that
\[
f(kx + y) + f(kx - y) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(y) \quad (2.3)
\]
for all \( x, y \in X \). But, \( k \neq 0, \pm 1 \) so from (2.1) and (2.3), we obtain
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
for all \( x, y \in X \). This shows that \( f \) is quadratic, which completes the proof of Lemma. \( \square \)

**Lemma 2.2.** If an odd function \( f : X \rightarrow Y \) satisfies (1.5), then \( f \) is a cubic-additive.

**Proof.** Letting \( y = x \) in (1.5), we get by oddness of \( f \),
\[
f((k+1)x) = f((k-1)x) + k^2 f(2x) + 2(1-k^2)f(x) \quad (2.4)
\]
for all \( x, y \in X \). Replacing \( x \) by \( (k-1)x \) in (1.5), gives
\[
f((k-1)x + ky) + f((k-1)x - ky)
\]
\[
= k^2 f((k-1)x + y) + k^2 f((k-1)x - y) + 2(1-k^2)f((k-1)x) \quad (2.5)
\]
for all $x, y \in X$. Now, if we replacing $x$ by $(k + 1)x$ in (1.5) and using (2.4), we see that
\[
\begin{align*}
    f((k + 1)x + ky) + f((k + 1)x - ky) &= k^2 f((k + 1)x + y) + k^2 f((k + 1)x - y) + 2(1 - k^2) f((k - 1)x) \\
    &+ 2k^2(1 - k^2) f(2x) + 4(1 - k^2)^2 f(x)
\end{align*}
\] (2.6)
for all $x, y \in X$. We substitute $x = x + y$ in (1.5) and then $x = x - y$ in (1.5) to obtain that
\[
    f(x + (k + 1)y) + f(x - (k - 1)y) = k^2 f(x + 2y) + 2(1 - k^2) f(x + y) + k^2 f(x)
\] (2.7)
and
\[
    f(x - (k + 1)y) + f(x + (k - 1)y) = k^2 f(x - 2y) + 2(1 - k^2) f(x - y) + k^2 f(x)
\] (2.8)
for all $x, y \in X$. If we subtract (2.8) from (2.7), we have
\[
\begin{align*}
    f(x + (k + 1)y) - f(x - (k + 1)y) &= k^2 f(x + 2y) - k^2 f(x - 2y) + f(x + (k - 1)y) - f(x - (k - 1)y) \\
    &+ 2(1 - k^2) f(x + y) - 2(1 - k^2) f(x - y)
\end{align*}
\] (2.9)
for all $x, y \in X$. Interchange $x$ with $y$ in (2.9) and using oddness of $f$, we get the relation
\[
\begin{align*}
    f(x + (k + 1)x + y) + f((k + 1)x - y) &= k^2 f(2x + y) + k^2 f(2x - y) + f((k - 1)x + y) + f((k - 1)x - y) \\
    &+ 2(1 - k^2) f(x + y) + 2(1 - k^2) f(x - y)
\end{align*}
\] (2.10)
for all $x, y \in X$. It follows from (2.6) and (2.10) that
\[
\begin{align*}
    f((k + 1)x + ky) + f((k + 1)x - ky) &= k^2 f((k + 1)x + y) + k^2 f((k + 1)x - y) + k^4 f(2x - y) \\
    &+ 2k^2(1 - k^2) f(x + y) + 2k^2(1 - k^2) f(x - y) \\
    &+ 2(1 - k^2) f((k - 1)x) + 2k^2(1 - k^2) f(2x) + 4(1 - k^2)^2 f(x)
\end{align*}
\] (2.11)
for all $x, y \in X$. We substitute $y = x + y$ in (1.5) and then $y = x - y$ in (1.5), we get by the oddness of $f$,
\[
    f((k + 1)x + ky) - f((k - 1)x + ky) = k^2 f(2x + y) + k^2 f(-y) + 2(1 - k^2) f(x)
\] (2.12)
and
\[
    f((k + 1)x - ky) - f((k - 1)x - ky) = k^2 f(2x - y) + k^2 f(y) + 2(1 - k^2) f(x)
\] (2.13)
for all $x, y \in X$. Then, by adding (2.12) to (2.13) and then using (2.5), we lead to
\[
\begin{align*}
    f((k + 1)x + ky) + f((k + 1)x - ky) &= k^2 f((k + 1)x + y) + k^2 f((k + 1)x - y) \\
    &+ k^2 f(2x + y) + k^2 f(2x - y) + 4(1 - k^2) f(x)
\end{align*}
\] (2.14)
for all $x, y \in X$. Finally, if we compare (2.11) with (2.14), then we conclude that
\[
    f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 2(f(2x) - 2f(x))
\]
for all $x, y \in X$. Hence, $f$ is cubic-additive function (see[15]). This completes the proof of Lemma.
Theorem 2.3. A function \( f : X \to Y \) with \( f(0) = 0 \) satisfies (1.5) for all \( x, y \in X \) if and only if there exist functions \( C : X \times X \times X \to Y \) and \( B : X \times X \to Y \) and \( A : X \to Y \), such that \( f(x) = C(x, x, x) + B(x, x) + A(x) \) for all \( x \in X \), where the function \( C \) is symmetric for each fixed one variable and is additive for fixed two variables and \( B \) is symmetric bi-additive and \( A \) is additive.

Proof. Let \( f \) with \( f(0) = 0 \) satisfies (1.5). We decompose \( f \) into the even part and odd part by putting

\[
\begin{align*}
\hat{f}_e(x) &= \frac{1}{2}(f(x) + f(-x)), \\
\hat{f}_o(x) &= \frac{1}{2}(f(x) - f(-x)),
\end{align*}
\]

for all \( x \in X \). It is clear that \( f(x) = f_e(x) + f_o(x) \) for all \( x \in X \). It is easy to show that the functions \( \hat{f}_e \) and \( \hat{f}_o \) satisfy (1.5). Hence by Lemmas 2.1 and 2.2, we achieve that the functions \( \hat{f}_e \) and \( \hat{f}_o \) are quadratic and cubic-additive, respectively, thus there exist a symmetric bi-additive function \( B : X \times X \to Y \) such that \( f_e(x) = B(x, x) \) for all \( x \in X \), and the function \( C : X \times X \times X \to Y \) and additive function \( A : X \to Y \) such that \( f_o(x) = C(x, x, x) + A(x) \), for all \( x \in X \), where the function \( C \) is symmetric for each fixed one variable and is additive for fixed two variables. Hence, we get \( f(x) = C(x, x, x) + B(x, x) + A(x) \), for all \( x \in X \).

Conversely, let \( f(x) = C(x, x, x) + B(x, x) + A(x) \) for all \( x \in X \), where the function \( C \) is symmetric for each fixed one variable and is additive for fixed two variables and \( B \) is bi-additive and \( A \) is additive. By a simple computation one can show that the functions \( x \mapsto C(x, x, x) \) and \( x \mapsto B(x, x) \) and \( A \) satisfy the functional equation (1.5). So the function \( f \) satisfies (1.5).

\[\square\]

3. Stability

Throughout this section, assume that \( X \) quasi-Banach space with quasi-norm \( \|\cdot\|_X \) and that \( Y \) is a p-Banach space with p-norm \( \|\cdot\|_Y \). Let \( M \) be the modulus of concavity of \( \|\cdot\|_Y \).

In this section, using an idea of Gǎvruta [7] we prove the stability of Eq.(1.5) in the spirit of Hyers, Ulam and Rassias. We need the following Lemma in the main Theorems. Now before taking up the main subject, given \( f : X \to Y \), we define the difference operator \( D_f : X \times X \to Y \) by

\[
D_f(x, y) = f(x + ky) + f(x - ky) - k^2 f(x + y) - k^2 f(x - y) - 2(1 - k^2) f(x)
\]

for all \( x, y \in X \).

Lemma 3.1. (see [14]) Let \( 0 < p \leq 1 \) and let \( x_1, x_2, \ldots, x_n \) be non-negative real numbers. Then

\[
\left( \sum_{i=1}^{n} x_i \right)^p \leq \sum_{i=1}^{n} x_i^p.
\]

Theorem 3.2. Let \( j \in \{-1, 1\} \) be fixed and let \( \varphi : X \times X \to [0, \infty) \) be a function such that

\[
\lim_{n \to \infty} k^{2nj} \varphi(x/y) = 0
\]

for all \( x, y \in X \) and

\[
\psi(x) := \sum_{i=1}^{\infty} k^{2ipj} \varphi(x/y) < \infty
\]

for all \( x \in X \). Suppose that an even function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|D_f(x, y)\|_Y \leq \varphi(x, y)
\]

(3.3)
for all $x, y \in X$. Then the limit

$$Q(x) := \lim_{n \to \infty} k^{2n+1} f\left(\frac{x}{k^n}\right)$$

exists for all $x \in X$ and $Q : X \to Y$ is a unique quadratic function satisfying

$$\|f(x) - Q(x)\|_Y \leq \frac{M}{2k^2} |\psi_n(x)|^p$$

(3.5)

for all $x \in X$.

**Proof.** Let $j = 1$. By putting $x = 0$ in (3.3), we get

$$\|2f(ky) - 2k^2 f(y)\|_Y \leq \varphi(0, y)$$

(3.6)

for all $y \in Y$. If we replace $y$ in (3.6) by $x$, and divide both sides of (3.6) by 2, we get

$$\|f(kx) - k^2 f(x)\|_Y \leq \frac{1}{2} \varphi(0, x)$$

(3.7)

for all $x \in X$. Let $\psi_n(x) = \frac{1}{2^n} \varphi(0, x)$ for all $x \in X$, then by (3.7), we get

$$\|f(kx) - k^2 f(x)\|_Y \leq \psi_n(x)$$

(3.8)

for all $x \in X$. If we replace $x$ in (3.8) by $\frac{x}{k^{n+1}}$ and multiply both sides of (3.8) by $k^{2n}$, then we have

$$\|k^{2(n+1)} f\left(\frac{x}{k^{n+1}}\right) - k^{2n} f\left(\frac{x}{k^n}\right)\|_Y \leq M k^{2n} \psi_n\left(\frac{x}{k^{n+1}}\right)$$

(3.9)

for all $x \in X$ and all non-negative integers $n$. Since $Y$ is $p$-Banach space, then by (3.9) gives

$$\|k^{2(n+1)} f\left(\frac{x}{k^{n+1}}\right) - k^{2n} f\left(\frac{x}{k^n}\right)\|_Y \leq \sum_{i=m}^{n} \|k^{2(i+1)} f\left(\frac{x}{k^{i+1}}\right) - k^{2i} f\left(\frac{x}{k^i}\right)\|_Y$$

$$\leq M^p \sum_{i=m}^{n} k^{2ip} \psi_{ip}\left(\frac{x}{k^{i+1}}\right)$$

(3.10)

for all non-negative integers $n$ and $m$ with $n \geq m$ and all $x \in X$. Since $\psi_{ip}(x) = \frac{1}{2^p} \varphi_p(0, x)$ for all $x \in X$, therefore by (3.2) we have

$$\sum_{i=1}^{\infty} k^{2ip} \psi_{ip}\left(\frac{x}{k^i}\right) < \infty$$

(3.11)

for all $x \in X$. Therefore we conclude from (3.10) and (3.11) that the sequence $\{k^{2n} f\left(\frac{x}{k^n}\right)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{k^{2n} f\left(\frac{x}{k^n}\right)\}$ converges for all $x \in X$. So one can define the function $Q : X \to Y$ by (3.4) for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (3.10), we get

$$\|f(x) - Q(x)\|_Y \leq M^p \sum_{i=0}^{\infty} k^{2ip} \psi_{ip}\left(\frac{x}{k^{i+1}}\right) = \frac{M^p}{k^p} \sum_{i=1}^{\infty} k^{2ip} \psi_p\left(\frac{x}{k^i}\right)$$

(3.12)

for all $x \in X$. Therefore (3.5) follows from (3.2) and (3.12). Now we show that $Q$ is quadratic. It follows from (3.1), (3.3) and (3.4)

$$\|DQ(x, y)\|_Y = \lim_{n \to \infty} k^{2n} \|Df\left(\frac{x}{k^n}, \frac{y}{k^n}\right)\|_Y \leq \lim_{n \to \infty} k^{2n} \varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) = 0$$

for all $x, y \in X$. Therefore the function $Q : X \to Y$ satisfies (1.5). Since $f$ is an even function, then (3.4) implies that the function $Q : X \to Y$ is even. Therefore by Lemma 2.1, we get that the function $Q : X \to Y$ is quadratic.
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To prove the uniqueness of \( Q \), let \( Q' : X \to Y \) be another quadratic function satisfying (3.5). Since

\[
\lim_{n \to \infty} k^{2np} \sum_{i=1}^{\infty} k^{2ip} \varphi^p(0, \frac{x}{k^n}) = \lim_{n \to \infty} \sum_{i=n+1}^{\infty} k^{2ip} \varphi^p(0, \frac{x}{k^n}) = 0
\]

for all \( x \in X \), then

\[
\lim_{n \to \infty} k^{2np} \tilde{\psi}_{\varepsilon}(\frac{x}{k^n}) = 0
\]

for all \( x \in X \). Therefore it follows from (3.5) and the last equation that

\[
\|Q(x) - Q'(x)\|_Y^p = \lim_{n \to \infty} k^{2np} \|f(\frac{x}{k^n}) - Q'(\frac{x}{k^n})\|_Y^p \leq \frac{M^p}{2^k 2^p} \lim_{n \to \infty} k^{2np} \tilde{\psi}_{\varepsilon}(\frac{x}{k^n}) = 0
\]

for all \( x \in X \). Hence \( Q = Q' \).

For \( j = -1 \), we can prove the Theorem by a similar technique.

**Corollary 3.3.** Let \( \theta, r, s \) be non-negative real numbers such that \( r, s > 2 \) or \( 0 \leq r, s < 2 \). Suppose that an even function \( f : X \to Y \) with \( f(0) = 0 \) satisfies the inequality

\[
\|Df(x, y)\|_Y \leq \theta(\|x\|_X^r + \|y\|_X^s), \quad (3.13)
\]

for all \( x, y \in X \). Then there exists a unique quadratic function \( Q : X \to Y \) satisfies

\[
\|f(x) - Q(x)\|_Y \leq \frac{M \theta}{2} (\frac{1}{|k^{2p} - k^p|}) \|x\|_X^r \|^p \]

for all \( x \in X \).

**Proof.** It follows from Theorem 3.2 by putting \( \varphi(x, y) := \theta(\|x\|_X^r + \|y\|_X^s) \) for all \( x, y \in X \). □

**Theorem 3.4.** Let \( j \in \{-1, 1\} \) be fixed and let \( \varphi_a : X \times X \to [0, \infty) \) be a function such that

\[
\lim_{n \to \infty} 2^{nj} \varphi_a(\frac{x}{2^n}, \frac{y}{2^n}) = 0 \quad (3.14)
\]

for all \( x, y \in X \) and

\[
\sum_{n=0}^{\infty} 2^{nj} \varphi_a(\frac{x}{2^n}, \frac{y}{2^n}) < \infty \quad (3.15)
\]

for all \( x \in X \) and for all \( y \in \{x, 2x, 3x\} \). Suppose that an odd function \( f : X \to Y \) satisfies the inequality

\[
\|Df(x, y)\|_Y \leq \varphi_a(x, y) \quad (3.16)
\]

for all \( x, y \in X \). Then the limit

\[
A(x) := \lim_{n \to \infty} 2^{nj}[f(\frac{x}{2^n}) - 8f(\frac{x}{2^n})] \quad (3.17)
\]

eexists for all \( x \in X \) and \( A : X \to Y \) is a unique additive function satisfying

\[
\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{M^5}{2} [\tilde{\psi}_{\varepsilon}(x)]^p \quad (3.18)
\]
for all $x \in X$, where
\[
\tilde{\varphi}_a(x) := \sum_{i=0}^{\infty} 2^{ipj} \left\{ \frac{1}{k^{ip}(1-k^j)^p} \left[ (5 - 4k^2)p \varphi_a^p \left( \frac{x}{2^j} \right) + k^{2p} \varphi_a^p \left( \frac{2x}{2^j} \right) \right] + (2k^2) \varphi_a^p \left( \frac{2x}{2^j}, \frac{x}{2^j} \right) + \varphi_a^p \left( \frac{3x}{2^j}, \frac{x}{2^j} \right) + (4 - 2k^2) \varphi_a^p \left( \frac{x}{2^j}, \frac{2x}{2^j} \right) \\
+ 2^p \varphi_a^p \left( 1+ \frac{kx}{2^j}, \frac{x}{2^j} \right) + 2^p \varphi_a^p \left( \frac{1-kx}{2^j}, \frac{1-kx}{2^j} \right) + \varphi_a^p \left( 1- \frac{2kx}{2^j}, \frac{x}{2^j} \right) \right\}.
\]

(3.19)

Proof. Let $j = 1$. By replacing $y$ by $x$ in (3.16), we have
\[
\| f((1 + k)x) + f((1 - k)x) - k^2 f(2x) - 2(1-k^2) f(x) \| \leq \varphi_a(x, x)
\]
for all $x \in X$. It follows from (3.20) that
\[
\| f((1+k)x) + f((1-k)x) - k^2 f(2x) - 2(1-k^2) f(2x) \| \leq \varphi_a(2x, 2x)
\]
for all $x \in X$. Replacing $x$ and $y$ by $2x$ and $x$ in (3.16), respectively, we get
\[
\| f((2 + k)x) + f((2 - k)x) - k^2 f(3x) - 2(1-k^2) f(2x) - k^2 f(x) \| \leq \varphi_a(2x, x)
\]
for all $x \in X$. Letting $y$ by $2x$ in (3.16) gives
\[
\| f((1+2k)x) + f((1-2k)x) - k^2 f(3x) - k^2 f(-x) - 2(1-k^2) f(x) \| \leq \varphi_a(x, 2x)
\]
for all $x \in X$. Putting $y$ by $3x$ in (3.16), we obtain
\[
\| f((1+3k)x) + f((1-3k)x) - k^2 f(4x) - k^2 f(-2x) - 2(1-k^2) f(2x) \| \leq \varphi_a(x, 3x)
\]
for all $x \in X$. Replacing $x$ and $y$ by $(1+k)x$ and $x$ in (3.16), respectively, we get
\[
\| f((1+2k)x) + f(x) - k^2 f((2 + k)x) - k^2 f(kx) - 2(1-k^2) f((1+k)x) \| \leq \varphi_a((1+k)x, x)
\]
for all $x \in X$. Replacing $x$ and $y$ by $(1-k)x$ and $x$ in (3.16), respectively, one gets
\[
\| f((1-2k)x) + f(x) - k^2 f((2 - k)x) - k^2 f(-kx) - 2(1-k^2) f((1-k)x) \| \leq \varphi_a((1-k)x, x)
\]
for all $x \in X$. Replacing $x$ and $y$ by $(1+2k)x$ and $x$ in (3.16), respectively, we obtain
\[
\| f((1+3k)x) + f((1+k)x) - k^2 f(2(1+k)x) - k^2 f(2kx) - 2(1-k^2) f((1+2k)x) \| \leq \varphi_a((1+2k)x, x)
\]
for all $x \in X$. Replacing $x$ and $y$ by $(1-2k)x$ and $x$ in (3.16), respectively, we have
\[
\| f((1-3k)x) + f((1-k)x) - k^2 f(2(1-k)x) - k^2 f(-2kx) - 2(1-k^2) f((1-2k)x) \| \leq \varphi_a((1-2k)x, x)
\]
for all $x \in X$. It follows from (3.25), (3.26) and oddness $f$ that
\[
\| f((1+2k)x) + f((1-2k)x) + 2f(x) - k^2 f((2 + k)x) - k^2 f((2 - k)x) - 2(1-k^2) f((1-k)x) \| \leq M(\varphi_a((1+k)x, x) + \varphi_a((1-k)x, x))
\]
(3.29)
for all $x \in X$. Now, from (3.20), (3.22), (3.23) and (3.29), we conclude that
\[
\|f(3x) - 4f(2x) + 5f(x)\| \leq \frac{M^3}{k^2(1 - k^2)} \left[ 2(1 - k^2)\varphi_a(x, x) + k^2\varphi_a(2x, x) + \varphi_a(x, 2x) + \varphi_a((1 + k)x, x) + \varphi_a((1 - k)x, x) \right] \tag{3.30}
\]
for all $x \in X$. On the other hand it follows from (3.27), (3.28) and oddness $f$ that
\[
\|f((1 + 3k)x) + f((1 - 3k)x) + f((1 + k)x) + f((1 - k)x) - k^2f(2(1 + k)x) - k^2f(2(1 - k)x) - 2(1 - k^2)f((1 + 2k)x) - 2(1 - k^2)f((1 - 2k)x)|x\| \leq M(\varphi_a((1 + 2k)x, x) + \varphi_a((1 - 2k)x, x)) \tag{3.31}
\]
for all $x \in X$. Also, from (3.20), (3.21), (3.23), (3.24) and (3.31), we lead to
\[
\|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \leq \frac{M^3}{k^2(1 - k^2)} \left[ \varphi_a(x, x) + k^2\varphi_a(2x, 2x) + 2(1 - k^2)\varphi_a(x, 2x) + \varphi_a(x, 3x) + 2\varphi_a((1 + k)x, x) + \varphi_a((1 + k)x, x) + \varphi_a((1 - k)x, x) + \varphi_a((1 - k)x, x) \right] \tag{3.32}
\]
for all $x \in X$. Finally, by using (3.30) and (3.32), we obtain that
\[
\|f(4x) - 10f(2x) + 16f(x)\| \leq \frac{M^3}{k^2(1 - k^2)} \left[ (5 - 4k^2)\varphi_a(x, x) + k^2\varphi_a(2x, 2x) + 2k^2\varphi_a(2x, x) + (4 - 2k^2)\varphi_a(x, 2x) + \varphi_a(x, 3x) + 2\varphi_a((1 + k)x, x) + 2\varphi_a((1 - k)x, x) + \varphi_a((1 - k)x, x) \right] \tag{3.33}
\]
for all $x \in X$, and let
\[
\psi_a(x) := \frac{1}{k^2(1 - k^2)} \left[ (5 - 4k^2)\varphi_a(x, x) + k^2\varphi_a(2x, 2x) + 2k^2\varphi_a(2x, x) + (4 - 2k^2)\varphi_a(x, 2x) + \varphi_a(x, 3x) + 2\varphi_a((1 + k)x, x) + 2\varphi_a((1 - k)x, x) + \varphi_a((1 - k)x, x) \right] \tag{3.34}
\]
for all $x \in X$. Therefore (3.33) means that
\[
\|f(4x) - 10f(2x) + 16f(x)\| \leq M^3\psi_a(x) \tag{3.35}
\]
for all $x \in X$. Letting $g : X \rightarrow Y$ be a function defined by $g(x) := f(2x) - 8f(x)$ then, we conclude that
\[
\|g(2x) - 2g(x)\| \leq M^3\psi_a(x) \tag{3.36}
\]
for all $x \in X$. If we replace $x$ in (3.36) by $\frac{x}{2^n}$ and multiply both sides of (3.36) by $2^n$, we get
\[
\|2^{n+1}g(\frac{x}{2^{n+1}}) - 2^ng(\frac{x}{2^n})\|_Y \leq M^32^n\psi_a(\frac{x}{2^n+1}) \tag{3.37}
\]
for all $x \in X$ and all non-negative integers $n$. Since $Y$ is p-Banach space, therefore by inequality (3.37), gives
\[
\|2^{n+1}g(\frac{x}{2^{n+1}}) - 2^ng(\frac{x}{2^n})\|_p^p \leq \sum_{i=m}^n \|2^{i+1}g(\frac{x}{2^{i+1}}) - 2^ig(\frac{x}{2^n})\|_Y^p \leq M^{3p}\sum_{i=m}^n 2^{ip}\psi_a(\frac{x}{2^{i+1}+1}) \tag{3.38}
\]
for all non-negative integers \( n \) and \( m \) with \( n \geq m \) and all \( x \in X \). Since \( 0 < p \leq 1 \), then by Lemma 3.1, we get from (3.34),
\[
\psi_a^p(x) \leq \frac{1}{k^p(1-k^2)^p} \left[ (5-4k^2)^p \varphi_a^p(x,x) + k^2 \varphi_a^p(2x,2x) + (2k^2)^p \varphi_a^p(2x,2x) + (4-2k^2)^p \varphi_a^p(x,3x) + 2^p \varphi_a^p((1+k)x,x) + 2^p \varphi_a^p((1-k)x,x) + \varphi_a^p((1+2k)x,x) + \varphi_a^p((1-2k)x,x) \right],
\tag{3.39}
\]
for all \( x \in X \). Therefore it follows from (3.15) and (3.39) that
\[
\sum_{i=1}^{\infty} 2^n \psi_a^p \left( \frac{x}{2^n} \right) < \infty
\tag{3.40}
\]
for all \( x \in X \). Therefore we conclude from (3.38) and (3.40) that the sequence \( \{ 2^n g(\frac{x}{2^n}) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ 2^n g(\frac{x}{2^n}) \} \) converges for all \( x \in X \). So one can define the mapping \( A : X \rightarrow Y \) by
\[
A(x) = \lim_{n \rightarrow \infty} 2^n g \left( \frac{x}{2^n} \right)
\tag{3.41}
\]
for all \( x \in X \). Letting \( m = 0 \) and passing the limit \( n \rightarrow \infty \) in (3.38), we get
\[
\| g(x) - A(x) \|_Y \leq M^p \sum_{i=0}^{\infty} 2^i \psi_a^p \left( \frac{x}{2^{i+1}} \right) = M^p \sum_{i=1}^{\infty} 2^n \psi_a^p \left( \frac{x}{2^n} \right)
\tag{3.42}
\]
for all \( x \in X \). Therefore (3.18) follows from (3.15) and (3.42). Now we show that \( A \) is additive. It follows from (3.14), (3.37) and (3.41) that
\[
\| A(2x) - 2A(x) \|_Y = \lim_{n \rightarrow \infty} \| 2^n g \left( \frac{x}{2^{n-1}} \right) - 2^{n+1} g \left( \frac{x}{2^n} \right) \|_Y
\]
\[
= 2 \lim_{n \rightarrow \infty} \| 2^n g \left( \frac{x}{2^{n-1}} \right) - 2^{n} g \left( \frac{x}{2^n} \right) \|_Y
\]
\[
\leq M^p \lim_{n \rightarrow \infty} 2^n \psi_a \left( \frac{x}{2^n} \right) = 0
\]
for all \( x \in X \). So
\[
A(2x) = 2A(x)
\tag{3.43}
\]
for all \( x \in X \). On the other hand it follows from (3.14), (3.16) and (3.17) that
\[
\| D_A(x,y) \|_Y = \lim_{n \rightarrow \infty} 2^n \| D_y \left( \frac{x}{2^n} \frac{y}{2^n} \right) \|_Y = \lim_{n \rightarrow \infty} 2^n \| D_f \left( \frac{x}{2^n-1} \frac{y}{2^n-1} \right) - 8D_f \left( \frac{x}{2^{n-1}} \frac{y}{2^{n-1}} \right) \|_Y
\]
\[
\leq M^5 \lim_{n \rightarrow \infty} 2^n \left\{ \| D_f \left( \frac{x}{2^n-1} \frac{y}{2^n-1} \right) \|_Y + 8\| D_f \left( \frac{x}{2^n} \frac{y}{2^n} \right) \|_Y \right\}
\leq M^5 \lim_{n \rightarrow \infty} 2^n \left\{ \varphi_a \left( \frac{x}{2^n-1} \frac{y}{2^n-1} \right) + 8\varphi_a \left( \frac{x}{2^n} \frac{y}{2^n} \right) \right\} = 0
\]
for all \( x,y \in X \). Hence the function \( A \) satisfies (1.5). By Lemma 2.2, the function \( x \mapsto \lambda \circ A(x) \) is additive. Hence, (3.43) implies that the function \( A \) is additive.
To prove the uniqueness property of \( A \), let \( A' : X \rightarrow Y \) be another additive function satisfying (3.18). Since
\[
\lim_{n \rightarrow \infty} 2^n \sum_{i=1}^{\infty} 2^i \psi_a^p \left( \frac{x}{2^{i+1}} \frac{y}{2^{i+1}} \right) = \lim_{n \rightarrow \infty} 2^n \sum_{i=0}^{\infty} 2^i \psi_a^p \left( \frac{x}{2^n} \frac{y}{2^n} \right) = 0
\]
for all \( x \in X \) and for all \( y \in \{ x, 2x, 3x \} \), then
\[
\lim_{n \rightarrow \infty} 2^n \psi_a \left( \frac{x}{2^n} \right) = 0
\tag{3.44}
for all \( x \in X \). It follows from (3.18) and (3.44) that
\[
\|A(x) - A'(x)\|_Y = \lim_{n \to \infty} 2^{np} \|g(\frac{x}{2^n}) - A'(\frac{x}{2^n})\|_Y \leq \frac{M^{2p}}{2^p} \lim_{n \to \infty} 2^{np} \tilde{\psi}_n(\frac{x}{2^n}) = 0
\]
for all \( x \in X \). So \( A = A' \).

For \( j = -1 \), we can prove the Theorem by a similar technique. \(\square\)

**Corollary 3.5.** Let \( \theta, r, s \) be non-negative real numbers such that \( r, s > 1 \) or \( 0 \leq r, s < 1 \). Suppose that an odd function \( f : X \to Y \) satisfies the inequality
\[
\|Df(x, y)\| \leq \begin{cases} 
\theta, & r = s = 0; \\
\|x\|_X, & r > 0, s = 0; \\
\|y\|_X, & r = 0, s > 0; \\
\theta(\|x\|_X + \|y\|_X), & r, s > 0.
\end{cases}
\]
(3.45)
for all \( x, y \in X \). Then there exists a unique additive function \( A : X \to Y \) satisfying

\[
\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{M^2\theta}{k^2(1 + k^2)} \left\{ \begin{array}{ll}
\delta_a, & r = s = 0; \\
\alpha_a \|x\|_X, & r > 0, s = 0; \\
\beta_a \|x\|_X, & r = 0, s > 0; \\
(\alpha^p_a \|x\|_X^p + \beta^p_a \|y\|_X^p)^{\frac{1}{p}}, & r, s > 0.
\end{array} \right.
\]

for all \( x \in X \), where
\[
\delta_a = \left\{ \frac{1}{2^p - 1} \left[ (5 - 4k^2)^p + (4 - 2k^2)^p + k^2p(2^p) + (1 + 2^p + 3) \right] \right\}^{\frac{1}{p}},
\]
\[
\alpha_a = \left\{ \frac{1}{2^p - 2^p} \left[ (5 - 4k^2)^p + (4 - 2k^2)^p + (1 + 2k)^p + (1 - 2k)^p + 2^p(1 + k)^p + 2^p(1 - k)^p + 2^p k^2p(2^p + 1) + 1 \right] \right\}^{\frac{1}{p}},
\]
\[
\beta_a = \left\{ \frac{1}{2^p - 2^p} \left[ (5 - 4k^2)^p + 2^p(4 - 2k^2)^p + k^2p(2^p + 2^p) + 3^p + 2^p + 1 + 2 \right] \right\}^{\frac{1}{p}}.
\]

**Proof.** It follows from Theorem 3.4 by putting \( \varphi_a(x, y) := \theta(\|x\|_X + \|y\|_X) \) for all \( x, y \in X \). \(\square\)

**Corollary 3.6.** Let \( \theta \geq 0 \) and \( r, s > 0 \) be non-negative real numbers such that \( \lambda := r + s \neq 1 \). Suppose that an odd function \( f : X \to Y \) satisfies the inequality
\[
\|Df(x, y)\| \leq \theta\|x\|_X \|y\|_X,
\]
(3.46)
for all \( x, y \in X \). Then there exists a unique additive function \( A : X \to Y \) satisfying

\[
\|f(2x) - 8f(x) - A(x)\|_Y \leq \frac{M^2\theta}{k^2(1 - k^2)} \varepsilon_a \|x\|_X^\lambda,
\]
for all \( x \in X \), where
\[
\varepsilon_a = \left\{ \frac{1}{2^p - 2^p} \left[ (5 - 4k^2)^p + 2^p(4 - 2k^2)^p + (1 + 2k)^p + (1 - 2k)^p + 2^p(1 + k)^p + 2^p(1 - k)^p + k^2p(2^p + 2^p + 2^p + 2^p + 1 + 2) \right] \right\}^{\frac{1}{p}}.
\]

for all \( x \in X \).

**Proof.** It follows from Theorem 3.4 by putting \( \varphi_a(x, y) := \theta\|x\|_X \|y\|_X \) for all \( x, y \in X \). \(\square\)
**Theorem 3.7.** Let \( j \in \{-1,1\} \) be fixed and let \( \varphi_c : X \times X \to [0,\infty) \) be a function such that
\[
\lim_{n \to \infty} 8^n j \varphi_c(\frac{x}{2^n}, \frac{y}{2^n}) = 0
\] (3.47)
for all \( x, y \in X \) and
\[
\sum_{i=\pm j}^\infty 8^n j \varphi_c^p(\frac{x}{2^n}, \frac{y}{2^n}) < \infty
\] (3.48)
for all \( x \in X \) and for all \( y \in \{x, 2x, 3x\} \). Suppose that an odd function \( f : X \to Y \) satisfies the inequality
\[
\|D_f(x,y)\|_Y \leq \varphi_c(x,y)
\] (3.49)
for all \( x, y \in X \). Then the limit
\[
C(x) := \lim_{n \to \infty} 8^n j [f(\frac{x}{2^n}) - 2f(\frac{x}{2^n})]
\] (3.50)
exists for all \( x \in X \) and \( C : X \to Y \) is a unique cubic function satisfying
\[
\|f(2x) - 2f(x) - C(x)\|_Y \leq \frac{M^5}{8} \varphi_c(x)
\] (3.51)
for all \( x \in X \), where
\[
\tilde{\psi}_c(x) := \sum_{i=\pm j}^\infty 8^n j \{ \frac{1}{k^2 p(1-k^2)} \left[ (5-4k^2)p \varphi_c^p(\frac{x}{2^n}, \frac{x}{2^n}) + k^2 p \varphi_c^p(\frac{2x}{2^n}, \frac{2x}{2^n}) \\
+ (2k^2)p \varphi_c^p(\frac{2x}{2^n}, \frac{x}{2^n}) + \varphi_c^p(\frac{3x}{2^n}, \frac{3x}{2^n}) + (4-2k^2)p \varphi_c^p(\frac{x}{2^n}, \frac{2x}{2^n}) \\
+ 2p \varphi_c^p(\frac{1+k}{2^n}, \frac{x}{2^n}) + 2p \varphi_c^p(\frac{1-k}{2^n}, \frac{x}{2^n}) \\
+ \varphi_c^p(\frac{1+2k}{2^n}, \frac{x}{2^n}) + \varphi_c^p(\frac{1-2k}{2^n}, \frac{x}{2^n}) \right] \}
\] (3.52)

**Proof.** Let \( j = 1 \). Similar to the proof of Theorem 3.4, we have
\[
\|f(4x) - 10f(2x) + 16f(x)\| \leq M^5 \tilde{\psi}_c(x),
\] (3.53)
for all \( x \in X \), where
\[
\psi_c(x) = \frac{1}{k^2(1-k^2)} \{ (5-4k^2)\varphi_c(x,x) + k^2 \varphi_c(2x,2x) \\
+ 2k^2 \varphi_c(2x,2x) + (4-2k^2)\varphi_c(x,2x) + \varphi_c(x,3x) + 2\varphi_c((1+k)x,x) \\
+ 2\varphi_c((1-k)x,x) + \varphi_c((1+2k)x,x) + \varphi_c((1-2k)x,x) \},
\] (3.54)
for all \( x \in X \). Letting \( h : X \to Y \) be a function defined by \( h(x) := f(2x) - 2f(x) \). Then, we conclude that
\[
\|h(2x) - 8h(x)\| \leq M^5 \psi_c(x)
\] (3.55)
for all \( x \in X \). If we replace \( x \) in (3.55) \( \frac{x}{2^n} \) and multiply both sides of (3.55) by \( 8^n \), we get
\[
\|8^{n+1} h(\frac{x}{2^{n+1}}) - 8^n h(\frac{x}{2^n})\|_Y \leq M^5 8^n \psi_c(\frac{x}{2^{n+1}})
\] (3.56)
for all \( x \in X \).
for all $x \in X$ and all non-negative integers $n$. Since $Y$ is $p$-Banach space, then by (3.56), we have

$$
\|s^{n+1}h(x/2^{n+1}) - s^nh(x/2^n)\|^p_Y \leq \sum_{i=m}^{n} \|s^{i+1}h(x/2^{i+1}) - s^ih(x/2^i)\|^p_Y 
\leq M^{p} \sum_{i=m}^{n} s^{i}p\phi_p(x/2^{i+1})
$$

(3.57)

for all non-negative integers $n$ and $m$ with $n \geq m$ and all $x \in X$. Since $0 < p \leq 1$, then by Lemma 3.1, we get from (3.54),

$$
\psi_p(x) \leq \frac{1}{k^{2p}[1 - k^2]^p} [l(4 - 2k^2)p\varphi_p(x, x) + k^{2p}\varphi_p(2x, 2x) 
+ (2k^2)p\varphi_p(2x, x) + (4 - 2k^2)p\varphi_p(x, 2x) + \varphi_p(x, 3x) + 2p\varphi_p((1 + k)x, x) 
+ 2p\varphi_p((1 - k)x, x) + \varphi_p((1 + 2k)x, x) + \varphi_p((1 - 2k)x, x)]
$$

(3.58)

for all $x \in X$. Therefore it follows from (3.48) and (3.58) that

$$
\sum_{i=1}^{\infty} s^{i}p\psi_p(x/2^{i}) < \infty
$$

(3.59)

for all $x \in X$. Therefore we conclude from (3.57) and (3.59) that the sequence $\{s^n h(x/2^n)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{s^n h(x/2^n)\}$ converges for all $x \in X$. So one can define the function $C : X \rightarrow Y$ by

$$
C(x) = \lim_{n \rightarrow \infty} s^n h(x/2^n)
$$

(3.60)

for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3.57), we get

$$
\|h(x) - C(x)\|^p_Y \leq M^{p} \sum_{i=0}^{\infty} s^{i}p\psi_p(x/2^{i+1}) = \frac{M^{p}}{s^p} \sum_{i=1}^{\infty} s^{i}p\psi_p(x/2^{i})
$$

(3.61)

for all $x \in X$. Therefore, (3.51) follows from (3.48) and (3.61). Now we show that $C$ is cubic. It follows from (3.47), (3.56) and (3.60) that

$$
\|C(2x) - 8C(x)\|^p_Y = \lim_{n \rightarrow \infty} \|s^n h(x/2^{n-1}) - s^{n+1} h(x/2^n)\|^p_Y 
= 8 \lim_{n \rightarrow \infty} \|s^{n-1} h(x/2^{n-1}) - s^n h(x/2^n)\|^p_Y 
\leq M^{5} \lim_{n \rightarrow \infty} s^n \psi(x/2^n) = 0
$$

for all $x \in X$. So

$$
C(2x) = 8C(x)
$$

(3.62)

for all $x \in X$. On the other hand it follows from (3.47) , (3.49) and (3.50) that

$$
\|D_C(x, y)\|^p_Y \leq \lim_{n \rightarrow \infty} s^n \|D_h(x/2^{n-1}, y/2^{n-1}) - 2D_f(x/2^n, y/2^n)\|^p_Y 
\leq M^{3} \lim_{n \rightarrow \infty} s^n \{ \|D_f(x/2^{n-1}, y/2^{n-1})\|^p_Y + 2\|D_f(x/2^n, y/2^n)\|^p_Y \} 
\leq M^{3} \lim_{n \rightarrow \infty} s^n \{ \varphi(x/2^{n-1}, y/2^{n-1}) + 2\varphi(x/2^n, y/2^n) \} = 0
$$

for all $x, y \in X$. Hence the function $C$ satisfies (1.5). By Lemma 2.2, the function $x \mapsto C(2x) - 8C(x)$ is additive. Hence, (3.62) implies that function $C$ is cubic.
To prove the uniqueness of $C$, let $C': X \to Y$ be another additive function satisfying (3.51). Since
\[
\lim_{n \to \infty} 8^np \sum_{i=1}^{\infty} 8^p \varphi_{i}^{p} \left( \frac{x}{2^{n+2}}, \frac{x}{2^{n+2}} \right) = \lim_{n \to \infty} 8^p \varphi_{c}^{p} \left( \frac{x}{2^n}, \frac{x}{2^n} \right) = 0
\]
for all $x \in X$ and for all $y \in \{x, 2x, 3x\}$, then
\[
\lim_{n \to \infty} 8^p \tilde{\psi}_{c} \left( \frac{x}{2^n} \right) = 0
\]
(3.63)
for all $x \in X$. It follows from (3.51) and (3.63) that
\[
\|C(x) - C'(x)\|_{Y} = \lim_{n \to \infty} 8^p \|h \left( \frac{x}{2^n} \right) - C' \left( \frac{x}{2^n} \right)\|_{Y}^{p} \leq \frac{M^p}{8^n} \lim_{n \to \infty} 8^p \tilde{\psi}_{c} \left( \frac{x}{2^n} \right) = 0
\]
for all $x \in X$. So $C = C'$. For $j = -1$, we can prove the Theorem by a similar technique. 

**Corollary 3.8.** Let $r, s$ be non-negative real numbers such that $r, s > 3$ or $0 \leq r, s < 3$. Suppose that an odd function $f: X \to Y$ satisfies the inequality (3.45) for all $x, y \in X$. Then there exists a unique cubic function $C: X \to Y$ satisfying
\[
\|f(2x) - 2f(x) - C(x)\|_{Y} \leq \frac{M^{2} \theta}{k^{2}(1-k^{2})} \left\{ \begin{array}{ll}
\delta_{c}, & r = s = 0; \\
\alpha_{c} \|x\|_{X}, & r > 0, s = 0; \\
\beta_{c} \|x\|_{X}, & r = 0, s > 0; \\
(\alpha_{c}^{p} \|x\|_{X}^{p} + \beta_{c}^{p} \|x\|_{X}^{p})^{\frac{1}{p}}, & r, s > 0.
\end{array} \right.
\]
for all $x \in X$, where
\[
\delta_{c} = \{ \frac{1}{8^{p} - 1} \left[ (5 - 4k^{2})^{p} + (4 - 2k^{2})^{p} + k^{2p}(2^{p} + 1) + 2^{p+1} + 3 \right] \}^{\frac{1}{p}},
\]
\[
\alpha_{c} = \{ \frac{1}{8^{p} - 2^{p}} \left[ (5 - 4k^{2})^{p} + (4 - 2k^{2})^{p} + (1 + 2k)^{p} + (1 - 2k)^{p} + 2^{p}(1 + k)^{p} + 2^{p}(1 - k)^{p} + 2^{p}k^{2p}(2^{p} + 1) + 1 \right] \}^{\frac{1}{p}},
\]
\[
\beta_{c} = \{ \frac{1}{8^{p} - 2^{p}} \left[ (5 - 4k^{2})^{p} + 2^{p}(4 - 2k^{2})^{p} + k^{2p}(2^{2p} + 2^{2p} + 3^{p} + 2^{p+1}) \right] \}^{\frac{1}{p}}.
\]

**Proof.** In Theorem 3.7, let $\varphi_{c}(x, y) := \theta(\|x\|_{X} + \|y\|_{X})$ for all $x, y \in X$. 

**Corollary 3.9.** Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \neq 3$. Suppose that an odd function $f: X \to Y$ satisfies the inequality (3.46) for all $x, y \in X$. Then there exists a unique cubic function $C: X \to Y$ satisfying
\[
\|f(2x) - 2f(x) - C(x)\|_{Y} \leq \frac{M^{2} \theta}{k^{2}(1-k^{2})} \varepsilon_{c} \|x\|_{X},
\]
for all $x \in X$, where
\[
\varepsilon_{c} = \{ \frac{1}{8^{p} - 2^{p}} \left[ (5 - 4k^{2})^{p} + 2^{p}(4 - 2k^{2})^{p} + (1 + 2k)^{p} + (1 - 2k)^{p} + 2^{p}(1 + k)^{p} + 2^{p}(1 - k)^{p} + k^{2p}(2^{2p} + 2^{2p} + 3^{p} + 2^{p+1}) \right] \}^{\frac{1}{p}}.
\]

**Proof.** In Theorem 3.7, let $\varphi_{c}(x, y) := \theta(\|x\|_{X} + \|y\|_{X})$ for all $x, y \in X$. 

Theorem 3.10. Let $j \in \{-1, 1\}$ be fixed and let $\varphi : X \times X \to [0, \infty)$ be a function such that
\[
\lim_{n \to \infty} \left\{ \left( \frac{1 + j}{2} \right)^{2n} \varphi\left( \frac{x}{2^n}, \frac{y}{2^n} \right) + \left( \frac{1 - j}{2} \right)^{2n} \varphi\left( \frac{x}{2^n}, \frac{y}{2^n} \right) \right\} = 0
\]
for all $x, y \in X$ and
\[
\sum_{i=1}^{\infty} \left\{ \left( \frac{1 + j}{2} \right)^{2i} \varphi^p\left( \frac{x}{2^i}, \frac{y}{2^i} \right) + \left( \frac{1 - j}{2} \right)^{2i} \varphi^p\left( \frac{x}{2^i}, \frac{y}{2^i} \right) \right\} < \infty,
\]
for all $x \in X$ and for all $y \in \{x, 2x, 3x\}$. Suppose that an odd function $f : X \to Y$ satisfies the inequality
\[
|D_f(x, y)| < \varphi(x, y),
\]
for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that
\[
\|f(x) - A(x) - C(x)\|_Y \leq \frac{M^6}{48} \left( 4[\tilde{\psi}_u(x)]^\frac{1}{2} + [\tilde{\psi}_c(x)]^\frac{1}{2} \right)
\]
for all $x \in X$, where $\tilde{\psi}_u(x)$ and $\tilde{\psi}_c(x)$ has been defined in (3.19) and (3.52), respectively, for all $x \in X$.

Proof. Let $j = 1$. By Theorem 3.4 and 3.7, there exist an additive function $A_0 : X \to Y$ and a cubic function $C_0 : X \to Y$ such that
\[
\|f(2x) - 8f(x) - A_0(x)\|_Y \leq \frac{M^5}{2} [\tilde{\psi}_u(x)]^\frac{1}{2}, \quad \|f(2x) - 2f(x) - C_0(x)\|_Y \leq \frac{M^5}{8} [\tilde{\psi}_c(x)]^\frac{1}{2}
\]
for all $x \in X$. Therefore, it follows from the last inequality that
\[
\|f(x) + \frac{1}{6} A_0(x) - \frac{1}{6} C_0(x)\|_Y \leq \frac{M^6}{48} \left( 4[\tilde{\psi}_u(x)]^\frac{1}{2} + [\tilde{\psi}_c(x)]^\frac{1}{2} \right)
\]
for all $x \in X$. So we obtain (3.67) by letting $A(x) = -\frac{1}{6} A_0(x)$ and $C(x) = \frac{1}{6} C_0(x)$ for all $x \in X$. To prove the uniqueness property of $A$ and $C$, let $A_1, C_1 : X \to Y$ be another additive and cubic functions satisfying (3.67). Let $A' = A - A_1$ and $C' = C - C_1$. So
\[
\|A' (x) + C' (x)\|_Y \leq M \left\{ \|f(x) - A(x) - C(x)\|_Y + \|f(x) - A_1(x) - C_1(x)\|_Y \right\}
\]
\[
\leq \frac{M^7}{24} \left( 4[\tilde{\psi}_u(x)]^\frac{1}{2} + [\tilde{\psi}_c(x)]^\frac{1}{2} \right)
\]
(3.68)
for all $x \in X$. Since
\[
\lim_{n \to \infty} 2^n \tilde{\psi}_u\left( \frac{x}{2^n} \right) = \lim_{n \to \infty} 8^n \tilde{\psi}_c\left( \frac{x}{2^n} \right) = 0
\]
for all $x \in X$. Then (3.68) implies that
\[
\lim_{n \to \infty} 8^n \|A' \left( \frac{x}{2^n} \right) + C' \left( \frac{x}{2^n} \right)\|_Y = 0
\]
for all $x \in X$. Therefore $C' = 0$. So it follows from (3.68) that
\[
\|A' (x)\|_Y \leq \frac{5M^7}{24} [\tilde{\psi}_u(x)]^\frac{1}{2}
\]
Corollary 3.11. Let $\theta, r, s$ be non-negative real numbers such that $r, s > 3$ or $1 < r, s < 3$ or $0 \leq r, s < 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.45) for all $x, y \in X$. Then there exists a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{M^6 \theta}{6k^2(1 - k^2)} \begin{cases} \delta_y + \delta_c, & r = s = 0; \\ (\alpha_{a} + \alpha_{c}) \|x\|_X^r, & r > 0, s = 0; \\ (\beta_{a} + \beta_{c}) \|x\|_X^r, & r = 0, s > 0; \\ \gamma_a(x) + \gamma_c(x), & r, s > 0. \end{cases}$$

for all $x \in X$, where $\delta_a, \delta_c, \alpha_a, \alpha_c, \beta_a$ and $\beta_c$ are defined as in Corollaries 3.5 and 3.8 and

$$\gamma_a(x) = \{\alpha_a \|x\|_X^r + \beta_a \|x\|_X^s\}^\frac{1}{2}, \quad \gamma_c(x) = \{\alpha_c \|x\|_X^r + \beta_c \|x\|_X^s\}^\frac{1}{2}$$

for all $x \in X$.

Corollary 3.12. Let $\theta \geq 0$ and $r, s > 0$ be non-negative real numbers such that $\lambda := r + s \in (0, 1) \cup (1, 3) \cup (3, \infty)$. Suppose that an odd function $f : X \to Y$ satisfies the inequality (3.46) for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - C(x)\|_Y \leq \frac{M^6 \theta}{6k^2(1 - k^2)} (\varepsilon_a + \varepsilon_c) \|x\|_X^\lambda,$$

for all $x \in X$, where $\varepsilon_a$ and $\varepsilon_c$ are defined as in Corollaries 3.6 and 3.9.

Theorem 3.13. Let $\varphi : X \times X \to [0, \infty)$ be a function which satisfies (3.1) for all $x, y \in X$ and (3.2) for all $x \in X$ or $y \in X$ satisfies (3.64) for all $x, y \in X$ and (3.65) for all $x \in X$ and for all $y \in \{x, 2x, 3x\}$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.3) for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$, a unique quadratic function $Q : X \to Y$, and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - A(x) - Q(x) - C(x)\|_Y \leq \frac{M^8}{96} \{4\|\tilde{\psi}_a(x) + \tilde{\psi}_a(-x)\|_Y^\frac{1}{2} + \|\tilde{\psi}_c(x) + \tilde{\psi}_c(-x)\|_Y^\frac{1}{2}\}$$

$$+ \frac{M^3}{4k^2} \{\|\tilde{\psi}_c(x) + \tilde{\psi}_c(-x)\|_Y^\frac{1}{2}\}$$

(3.69)

for all $x \in X$, where $\tilde{\psi}_c(x), \tilde{\psi}_a(x)$ and $\tilde{\psi}_c(x)$ have been defined in (3.2), (3.19) and (3.52), respectively, for all $x \in X$.

Proof. Assume that $\varphi : X \times X \to [0, \infty)$ satisfies (3.1) for all $x, y \in X$ and (3.2) for all $x \in X$. Let $f_c(x) = \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $f_c(0) = 0$, $f_c(-x) = f_c(x)$ and

$$\|D_{f_c}(x, y)\| \leq \frac{M}{2}\|\varphi(x, y) + \varphi(-x, -y)\|$$

for all $x, y \in X$. Hence, from Theorem 3.2, there exists a unique quadratic function $Q : X \to Y$ satisfying

$$\|f_c(x) - Q(x)\|_Y \leq \frac{M}{2k^2} \|\tilde{\psi}_c(x)\|_Y^\frac{1}{2}. \quad (3.70)$$

for all $x \in X$. It is clear that

$$\tilde{\psi}_c(x) \leq \frac{M^p}{2p} \|\tilde{\psi}_c(x) + \tilde{\psi}_c(-x)\|,$$
for all \(x \in X\). Therefore it follows from (3.70) that
\[
\|f_\alpha(x) - Q(x)\|_Y \leq \frac{M^2}{2k^2} (\|\tilde{\varphi}(x) + \tilde{c}(x)\|_X)^\frac{1}{p}.
\]
(3.71)
Let \(f_\alpha(x) = \frac{1}{2}(f(x) - f(-x))\) for all \(x \in X\). Then \(f_\alpha(0) = 0\), \(f_\alpha(-x) = -f_\alpha(x)\) and
\[
\|Df_\alpha(x, y)\| \leq \frac{M}{2} |\varphi(x, y) + \varphi(-x, -y)|
\]
for all \(x, y \in X\). By Theorem 3.10, there exist a unique additive function \(A : X \rightarrow Y\) and a unique cubic function \(Q : X \rightarrow Y\) satisfy
\[
\|f_\alpha(x) - A(x) - C(x)\|_Y \leq \frac{M^6}{48} \left(4[\tilde{\varphi}(x)]^\frac{1}{p} + [\tilde{c}(x)]^\frac{1}{p}\right)
\]
for all \(x \in X\). Since
\[
\tilde{\varphi}(x) \leq \frac{M^p}{2p} [\tilde{\varphi}(x) + \tilde{c}(x)], \quad \tilde{c}(x) \leq \frac{M^p}{2p} [\tilde{c}(x) + \tilde{c}(-x)]
\]
for all \(x \in X\). Therefore it follows from (3.72) that
\[
\|f_\alpha(x) - A(x) - C(x)\|_Y \leq \frac{M^7}{96} \left\{4[\tilde{\varphi}(x) + \tilde{c}(x)]^\frac{1}{p} + [\tilde{c}(x) + \tilde{c}(-x)]^\frac{1}{p}\right\}
\]
(3.73)
for all \(x \in X\). Hence (3.69) follows from (3.71) and (3.73). Now, if \(\varphi : X \times X \rightarrow [0, \infty)\) satisfies (3.64) for all \(x, y \in X\) and (3.65) for all \(x \in X\) and for all \(y \in \{x, 2x, 3x\}\), we can prove the theorem by a similar technique. □

**Corollary 3.14.** Let \(\theta, r, s\) be non-negative real numbers such that \(r, s > 3\) or \(2 < r, s < 3\) or \(1 < r, s < 2\) or \(0 < r, s < 1\). Suppose that a function \(f : X \rightarrow Y\) with \(f(0) = 0\) satisfies the inequality (3.13) for all \(x, y \in X\). Then there exist a unique additive function \(A : X \rightarrow Y\) and a unique quadratic function \(Q : X \rightarrow Y\) and a unique cubic function \(C : X \rightarrow Y\) such that
\[
\|f(x) - A(x) - Q(x) - C(x)\|_Y \leq \frac{M^{3\theta}}{6k^2(1 - k^2)} \left\{\left|\alpha_x^p \|x\|_X^p + \beta_x^p \|x\|_X^p\right|^\frac{1}{p} \right\}
\]
\[
+ \left[\alpha_x^p \|x\|_X^p + \beta_x^p \|x\|_X^p\right]^\frac{1}{p}
\]
\[
\frac{M^3}{2} \left[\frac{1}{(k^2 - k^2)} \|x\|_X^p\right]^\frac{1}{p}
\]
for all \(x \in X\), where \(\alpha_x, \alpha_x^p, \beta_x^p\) and \(\beta_x\) are defined as in Corollaries 3.5 and 3.8.

**Proof.** Put \(\varphi(x, y) := \theta \|x\|_X^p + \|y\|_X^p\), since \(\|Df_\alpha(x, y)\| \leq M\varphi(x, y)\), then \(\|Df_\alpha(x, y)\| \leq M\varphi(x, y)\)
for all \(x, y \in X\). Thus the result follows from Corollaries 3.3 and 3.11. □

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