ABSTRACT

I study phase transitions occurring in noncollinear magnets by means of a self-consistent screening approximation. The Ginzburg-Landau theory involves two N-component vector fields with two independent quartic couplings allowing a symmetry-breaking scheme which is $SO(N) \times SO(2) \rightarrow SO(N-2) \times SO(2)_{\text{diag}}$. I find that there is a second-order phase transition in the physical cases $N=2,3$, $D=3$ and that there is no fluctuation-induced first-order transition. This is very similar to the case of the normal-to-superconducting phase transition as recently found by Radzihovsky. The exponents are $\eta(N = 3, D = 3) \approx 0.11$, $\eta(N = 2, D = 3) \approx 0.15$ and go smoothly to the large-N limit.
Many magnetic systems have a low-temperature ordered phase that breaks completely the rotation invariance due to a noncollinear pattern of the magnetic moments\(^1\). Examples includes the rare-earths Ho, Dy and Tb. This is generic for vector-spin models with strong enough competing interactions. The phase transition associated with this ordering does not belong \textit{a priori} to the well-studied universality classes corresponding to the collinear ordering with symmetry breaking \(O(N) \rightarrow O(N-1)\). Several groups have investigated the critical behaviour of the three-dimensional stacked triangular (STA) Heisenberg antiferromagnet which is a simple example of commensurate noncollinear ordering and there is general agreement\(^2,3,4\) that this system has a second-order phase transition with exponents\(^3\) \(\nu = 0.585(9), \gamma/\nu = 2.011(14)\) that do not correspond to the \(O(N)\) exponents. This peculiar set of exponents is apparently associated with a new universality class: they appear also in the body-centered tetragonal antiferromagnet\(^5\). This class should also include the transition\(^1\) from the \(^3\)He liquid to the A-phase, Josephson-junction arrays in a transverse field as well as the fully frustrated bipartite lattice (Villain model). In the XY case there is also a new set of exponents\(^1,2\): \(\nu = 0.54(2), \gamma = 1.13(5)\)

The renormalization group has been applied to the Ginzburg-Landau theory of such systems by Garel and Pfeuty\(^6\). They used the expansion in \(\epsilon = 4 - D\), where \(D\) is the dimension, for any number \(N\) of components of the vector spins. In this framework, there is a Heisenberg \(O(2N)\) fixed point which is always unstable. In the physical case \(N=3\) there is no new fixed point and this runaway is interpreted as indicating a fluctuation-induced first-order phase transition. This is curiously similar to the normal-superconducting (NS) phase transition as revealed by Halperin, Lubensky and Ma\(^7\).

Recently, the NS phase transition has been studied by Radzihovsky\(^8\) who used the self-consistent screening approximation (SCSA) due to Bray\(^9\). This approximation includes
an infinite subset of the $1/N$ expansion by means of a simple self-consistency condition on the propagator. It allows to obtain the exponent $\eta$ as a function of $N$ and $D$. In the NS case the SCSA leads to a stable fixed point in the physical case $D=3$, in agreement with Monte-Carlo findings and other theoretical arguments\textsuperscript{10}. In the neighborhood of $D=4$, the corresponding fixed point survives for all values of the number of components of the order parameter, contrary to the prediction of the $4-\epsilon$ expansion, suggesting that the fluctuation-induced first-order transition is an artifact of the $\epsilon$ expansion. It is the purpose of this Letter to apply this very same method to the Ginzburg-Landau theory of helimagnets. I find that there is a non-trivial fixed point that survives in the whole domain $2 \leq D < 4$ and $2 \leq N \leq \infty$ which is different from the $O(2N)$ Heisenberg fixed point. In the physical cases of interest, I find $\eta(N = 3, D = 3) \approx 0.11$ and $\eta(N = 2, D = 3) \approx 0.15$. This fixed point goes smoothly to the $N = \infty$ result and there is no fluctuation-induced first-order phase transition. This is very close to the analysis of the NS transition of Ref.8. The mere existence of this fixed point for $N = 2, 3, D = 3$ provides a natural explanation to the Monte-Carlo results and, possibly, of some experimental results.

The Ginzburg-Landau theory for a generic Heisenberg helimagnet involves two vector fields that correspond to the Fourier modes of the magnetization near the ordering wavevectors $\pm \vec{Q}$. The effective action contains two quartic invariants:

\begin{equation}
A = \frac{1}{2} \left( (\nabla \vec{\phi}_1)^2 + (\nabla \vec{\phi}_2)^2 \right) + r \left( \vec{\phi}_1^2 + \vec{\phi}_2^2 \right) + u \left( (\vec{\phi}_1 + \vec{\phi}_2)^2 \right) + v \left( (\vec{\phi}_1 \cdot \vec{\phi}_2)^2 - \vec{\phi}_1^2 \vec{\phi}_2^2 \right).
\end{equation}

This free energy has a global symmetry $O(N) \times O(2)$. When the coefficient $v$ is positive, the ground state consists of orthogonal vectors and the residual symmetry is $O(N-2) \times O(2)_{\text{diag}}$.
where the subgroup $O(2)_{\text{diag}}$ acts diagonally on vector indices and internal (1,2) indices$^{11}$. The global symmetry allows only the two quartic invariants in Eq.(1). A detailed study of this model has been performed by H. Kawamura$^{12}$. In the $D = 4 - \epsilon$ calculation, one finds the $O(2N)$ Wilson-Fisher fixed point on the line $v = 0$ at a distance $\epsilon$ from the origin, in the $(u, v)$ plane. The operator $[\vec{\phi}_1^2 \vec{\phi}_2^2 - (\vec{\phi}_1 \cdot \vec{\phi}_2)^2]$ opens a direction of instability for all values of $N$. However, if $N$ is greater than $N_c(D) = 21.8 - 23.4\epsilon + O(\epsilon^2)$ (obtained from a two-loop computation$^{12}$), there is an additional stable fixed point with $u^* \neq 0, v^* \neq 0$. In the neighborhood of the upper critical dimension, there is thus a dividing line $N_c(D)$ in the $(N, D)$ plane above which one has a second-order phase transition and below which one conjectures a fluctuation-induced first-order phase transition. This is similar to the normal to superconductor transition$^{7,8}$ except for numerical factors: there the transition is second-order only when the number of complex components of the order parameter is larger than $\approx 183$.

The self-consistent screening approximation is an improvement over the large-$N$ expansion which has been applied with success to several physical problems$^{8,9,13}$. In the leading large-$N$ expression for the self-energy of the basic field containing the geometric sum of bubbles, one uses renormalized propagators everywhere: this leads to a self-consistency equation that goes beyond the simple result $N \to \infty$. An interesting byproduct is that this approximation gives back the $N \to \infty$ result automatically. Its validity extends to all $N$ and $D$ values but, of course, this is not a systematic procedure and we do not expect to get precise numerical values for the exponent $\eta$. The propagator of the theory (1) is defined as $G_{11}(k) = \langle \phi_1^\alpha(k)\phi_1^\alpha(-k) \rangle = \langle \phi_2^\alpha(k)\phi_2^\alpha(-k) \rangle = G_{22}(k) \equiv G(k)$. The self-energy is defined by $G^{-1}(k) = k^2 + r + \Sigma(k)$. The Dyson equation is then:

$$\Sigma(k) - \Sigma(0) = \int \frac{d^D p}{(2\pi)^D} (G(k + p) - G(p))[4\hat{u}(p) + \hat{v}(p)], \quad (2)$$
where the free propagator is \( G^{-1}_0(k) = k^2 + r \) and the dressed vertices \( \hat{u}(p) \), \( \hat{v}(p) \) are given by series involving only powers of the polarization bubble \( \Pi(k) = \int \frac{d^Dp}{(2\pi)^D} G(p)G(p+k) \):

\[
\hat{v}(p) = \frac{v}{1 + vN\Pi(p)}; \quad \hat{u}(p) = \frac{u + v(2u - v/2)N\Pi(p)}{1 + 4uN\Pi(p) + v(4u - v)N^2\Pi(p)^2}.
\]

(3)

Note that it is the full propagators that enter the quantities \( \Pi(p) \), \( \hat{u}(p) \), \( \hat{v}(p) \) and thus Eq. (2) is a self-consistency condition. The corresponding equation in the large-N limit is obtained by using the free propagator \( G_0 \) in the right-hand side of Eq. (2).

The SCSA strategy\(^9\) amounts to considering criticality i.e. \( r = 0 \) and writing the scaling form of the propagator \( G^{-1}(k) = k_c^\eta k^{2-\eta} \) since there is no longer any correlation length. With this scaling form for \( G \) one can compute the polarization bubble following Bray\(^9\):

\[
\Pi(p) = \frac{\Gamma(2 - D/2 - \eta)\Gamma^2(D/2 + \eta/2 - 1)}{(4\pi)^{D/2}\Gamma(D - 2 + \eta)\Gamma^2(1 - \eta/2)} \times k_c^{-2\eta}p^{D-4+2\eta}.
\]

(4)

The Heisenberg O(2N) fixed point can be found by setting \( v = 0 \). If we suppose that \( \Pi \) blows up for small momenta then \( \hat{u}(p) \approx 1/4N\Pi \). One can now match the powers of \( k^{-2\eta} \) in Eq. (2): this fixes the value of \( \eta \) through the condition:

\[
N = \frac{\Gamma(\eta/2 - 1)\Gamma(2 - \eta)\Gamma(D - 2 + \eta)\Gamma(1 - \eta/2)}{\Gamma(D/2 + \eta - 2)\Gamma(D/2 + \eta/2 - 1)\Gamma(D/2 - \eta/2 + 1)\Gamma(2 - \eta - D/2)}.
\]

(5)

This is precisely the result of the SCSA for the \( O(2N) \)-vector model as expected. As long as \( D \leq 4 \), one can check the assumption that \( \eta \) leads to \( \Pi >> 1 \) at low momenta (when \( D > 4 \), then \( \Pi \) is negligible and one encounters only the Gaussian fixed point).

The other fixed point corresponds to \( v \neq 0 \): assuming also \( \Pi >> 1 \) we use the asymptotic behaviour \( \hat{v}(p) \approx 1/N\Pi \) and \( \hat{u}(p) \approx 1/2N\Pi \), deduced from the series (3). These leading terms add up to give a factor of \( 3/N \) in the right-hand side of the Dyson
equation and thus the only modification with respect to the pure Heisenberg case is that in Eq(5) the factor $N$ should be replaced by $N/3$:

$$\frac{N}{3} = \frac{\Gamma(\frac{\eta}{2} - 1) \Gamma(2 - \eta) \Gamma(D - 2 + \eta) \Gamma(1 - \eta/2)}{\Gamma(D/2 + \eta - 2) \Gamma(D/2 + \eta/2 - 1) \Gamma(D/2 - \eta/2 + 1) \Gamma(2 - \eta - D/2)}.$$  \hfill (6)

This equation leads to a $\eta_{SCSA}$ which is well-behaved in the physical case $N=2,3, D=3$. It can be solved now numerically or expanded in various limits. Numerically $\eta_{SCSA}(N = 3, D = 3) \approx 0.11$ and $\eta_{SCSA}(N = 2, D = 3) \approx 0.15$, a perfectly sensible result. Present Monte-Carlo estimates\(^3\) favor a smaller $\eta$ for $N=3$ since $\gamma/\nu = 2.011(14)$. The SCSA should not be expected to be quantitative: this is the mere existence of the fixed point which is the relevant information.

The large-$N$ limit of the SCSA agrees by construction with the result of the direct large-$N$ result\(^{12}\): one recovers the result $\eta = 6[4/D - 1]S_D/N$ where $S_D = \sin[\pi(D - 2)/2]\Gamma(D - 1)/(2\pi\Gamma^2(D/2))$. In the neighborhood of $D=4$, one finds $\eta_{SCSA} = 3\epsilon^2/4N + O(\epsilon^3)$. This $\epsilon$-expansion is well-behaved for any $N$: in the SCSA there is no hint of a fluctuation-induced first-order transition. This limiting case matches the $\epsilon$-expansion only if one takes also the limit $N \to \infty$. In fact, $\eta_{4-\epsilon} = f(N)\epsilon^2 + O(\epsilon^3)$ with $f(N)$ a complicated function of $N$ which has the asymptotic behaviour $f(N) \approx 3/4N$ as $N \to \infty$.

In the case of the NS transition, Radzihovsky has shown\(^8\) that $\eta_{SCSA}$ is well-behaved near $D=4$ but has a singularity in the $\epsilon$-expansion. This has led him to propose that the lack of stable fixed point seen in the $\epsilon$-expansion studies\(^7\) is due to a breakdown of the expansion itself instead of a fluctuation-induced first-order phase transition. Since it is difficult to estimate the validity of the SCSA itself, it may be that the SCSA is not accurate enough to capture the fluctuation-induced transition. In the present context of
noncollinear magnets, the situation is slightly different: $\eta_{SCSA}$ is well-behaved near D=4 and has a regular $\epsilon$-expansion for any N.

In the neighborhood of the lower critical dimension, the exponent $\eta$ can be expanded in D=2+$\epsilon$ with the result $\eta_{SCSA} = 3\epsilon/2(N - 3) + O(\epsilon^2)$. There is a singularity at N=3 in the $\epsilon$-expansion due to the fact that $\eta$ no longer vanishes (in the SCSA) at D=2 when $N < 3$. For N=3, one has from the SCSA (equation 6) $\eta=0$ at D=2 which is coherent with the fact that there is no phase transition. This behaviour appears of course in the SCSA treatment of the O(N)-vector model since Eqs 5 and 6 differ by the substitution $N \rightarrow N/3$.

However in the O(N) case, $\eta_{SCSA} = \epsilon/(N - 2)$ near D=2 and $\eta_{SCSA}$ is nonzero at D=2 when $N < 2$. Since we expect in a N-vector model that in D=2 $\eta$ vanishes for $N > 2$, this means that the N-dependence from SCSA cannot be blindly believed near N=3.

Noncollinear magnets have been studied near two dimensions in a sigma model approach\textsuperscript{11}. The symmetry breaking pattern defines a homogeneous non-symmetric manifold $G/H = O(N) \times O(2)/O(N - 2) \times O(2)_{diag}$ that specifies uniquely a nonlinear sigma model. For any finite N there is a fixed point in D=2+$\epsilon$ which merges smoothly with the fixed point obtained in the large-N limit of the linear theory. The exponent $\eta$ is given by\textsuperscript{14,11}:

$$\eta_{2+\epsilon} = \frac{3N^2 - 10N + 9}{2(N - 2)^3} \epsilon + O(\epsilon^2).$$

(7)

For large N, this agrees with $\eta_{SCSA} \approx 3\epsilon/2N$. In addition the formula (7) has the correct feature that it blows up for N=2 instead of N=3 as found in the SCSA. For N=3, the sigma model has a peculiarity: since $O(3) \times O(3) \equiv O(4)$, it describes O(4) critical behaviour. This is incompatible with numerical simulations\textsuperscript{3} in D=3. The reason for this failure is not known at the present time. A view has been reported\textsuperscript{14} that the inability of the
ε = D − 2 expansion to detect the nontrivial topological structure of the order-parameter space might be the reason for this failure. It is clear that the SCSA cannot shed any light on this problem since its functional dependence upon N is too approximate. In the SCSA there is no hint of any symmetry enhancement since O(4) critical behaviour cannot be reached by the linear theory (1).

In this Letter, I have studied the phase transition occurring in noncollinear magnets that break the full rotation group at low temperatures. A self-consistent screening approximation valid for all dimensions and any number of components of the order parameter leads to a second-order phase transition in the physical case of Heisenberg and XY systems in D=3. There is no sign of fluctuation-induced first-order phase transition contrary to the result of the standard 4 − ε-expansion. This is in agreement with Monte-Carlo results on model systems that possess the correct symmetry-breaking pattern. There are also some experimental results\(^{11}\) that may be explained by the existence of the corresponding universality class. It is worth pointing out that the SCSA approximation is not a controlled approximation with a systematic expansion parameter and thus it is difficult to estimate its validity. The whole picture is close to the normal-to-superconducting phase transition as studied by Radzihovsky: here the SCSA leads also to a second order transition in the whole (N, D)-plane instead of the first-order transition predicted by the 4 − ε-expansion. There is however a difference: in the NS case \(\eta_{SCSA}\) is well-behaved but has a singularity when expanded in \(\varepsilon\) while in helical magnets even the expansion is regular. In the two cases (NS and helical magnets) the transition is continuous near two dimensions\(^{11,15}\) and in D=3: this is reproduced by the SCSA calculation.

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A SELF-CONSISTENT THEORY OF PHASE TRANSITIONS
IN NONCOLLINEAR MAGNETS

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