Dynamics of a self-gravitating thin cosmic string

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Abstract

We assume that a self-gravitating thin string can be locally described by what we shall call a *smoothed cone*. If we impose a specific constraint on the model of the string, then its central line obeys the Nambu-Goto equations. If no constraint is added, then the worldsheet of the central line is a totally geodesic surface.

*PCAS numbers : 04.20.-q, 11.27.+d*
I Introduction

In the framework of general relativity, two theoretical aspects of cosmic strings have been mainly studied. The dynamical one in which the equations of motion of the infinitely thin cosmic string are governed by the Nambu-Goto action and secondly the self-gravitating one in which the straight cosmic string in particular is considered as the source of a gravitational field. In this case one finds that the asymptotical metric generated by the straight cosmic string is a conical metric [1] and the case of a singular line is obtained in a limit process.

The study of line sources in gravitation can be originated from a paper of Israel [2] in which the author concluded that there existed “no simple general prescription [...] for obtaining the physical characteristics of an arbitrary line source”. However, Vilenkin [3] gave the physical meaning of the conical singularity as being a cosmic string. Attention has been recently devoted in understanding the dynamics of a conical-type line source of the Einstein equations [4, 5, 6, 7]. The conclusions summarised in [8] are as follows. A two-dimensional timelike worldsheet whose points are conical singularities of the 4-geometry cannot be in general the dynamical evolution of a Nambu-Goto string of arbitrary initial shape. The conical singularity requires that the worldsheet is totally geodesic. This restraints the initial shape of the string as well as the evolution of each of its points to be a geodesic of the 4-geometry.

In this paper, we take up again the question of the dynamics but for a self-gravitating extended string. For a straight cosmic string, it is known that the exterior metric may be matched with an interior metric having as source an energy-momentum tensor with suitable properties. We consider a
string of arbitrary shape but we restrict ourselves to the case of a thin tube of matter adopting the assumption that the exterior metric is locally the one describing a straight cosmic string. The interior metric is constrained from this and the energy-momentum tensor could be derived from the metric. The aim of the present work is to find the equations of motion of the central line of this thin tube of matter in the limit where its radius tends to zero.

We emphasize that we do not consider directly a self-gravitating line. In our method, the conical points are smoothed out on a scale comparable to the radius of the string. The central line of this tube sweeps a timelike 2-worldsheet whose points are perfectly regular. A local coordinate system can then be attached to our spacetime by taking the two parameters of the worldsheet as the first two coordinates and the other two as geodesic coordinates pointing in a direction orthogonal to the worldsheet. This local coordinate system affixed to the worldsheet allows the natural introduction of the extrinsic curvature and other geometric parameters of the worldsheet.

The interior metric of the string is essentially characterised by a function \( f(l) = \epsilon h(l/\epsilon) \) in which \( l \) is the radial coordinate whose origin lies on the worldsheet and \( \epsilon \) is a length typical of the thickness of the string. The function \( h \) is arbitrary, only submitted to certain conditions expressing the smoothness of the spacetime on the worldsheet, as well as the matching conditions on the boundary between the interior of the string and the vacuum.

If no other conditions are imposed, we shall prove that the worldsheet tends to a totally geodesic surface, \( i.e. \) of vanishing extrinsic curvature, when \( \epsilon \) goes to zero. However we have found another possibility. If we impose on the function \( h \) a specific supplementary constraint then the extrinsic curvature no longer tends to zero. Nevertheless the mean curvature tends to
zero when $\epsilon$ does and thus the worldsheet tends to be locally extremal which is precisely the behavior of the Nambu-Goto string. Since the function $h$ is a characteristic of the metric in the interior of the string thus the constraint imposed is interpreted as a specification of the matter of the string.

The paper is organised in the following manner. We recall in Sec. II the basic results on self-gravitating straight strings with some thickness and we describe the formalism that we shall use. In Sec. III, a self-gravitating string of arbitrary shape is introduced and the coordinate system adopted to its study is introduced. In Sec. IV, we expand in powers of $1/\epsilon$ the geometrical quantities appearing in the Einstein equations. In Sec. V, taking the limit of $\epsilon$ going to zero of the Einstein equations on the boundary between the string and the vacuum, we obtain the constraints that the worldsheet swept by the central line of the string must satisfy. Finally in Sec. VI, we find the Nambu-Goto energy-momentum tensor in the zero limit of $\epsilon$, confirming the choice of the definitions adopted in Sec. III.

II Straight strings as smoothed cones

In a general relativistic context, it is possible to portray a straight string as a thin cylinder of matter [9, 10, 11, 12]. We give some basic features of a self-gravitating straight string since they will be used later on in the general case of strings of arbitrary shape.

In the coordinate system $(t, z, l, \phi)$ with $l \geq 0$ and $0 \leq \phi < 2\pi$, the energy-momentum tensor has the form

$$T^t_t = T^z_z = -\sigma(l) \quad T^l_l = T^\phi_\phi = 0 \quad 0 \leq l < l_0$$

where $l_0$ is the given radius of the cylinder. The energy density $\sigma$ is a posi-
tive regular function of $\mathbb{R}^2$. Taking into account the Einstein equations, the interior metric can be written as

$$ds^2_{\text{INT}} = -dt^2 + dz^2 + dl^2 + f^2(l)d\phi^2, \quad 0 \leq l < l_0$$  \hfill (2)

where the positive function $f$ determines $\sigma$ by the formula,

$$\sigma = -\frac{1}{8\pi G} \frac{f''}{f},$$  \hfill (3)

$G$ being the Newtonian gravitational constant. In order to ensure a regular behavior of metric (2) at $l = 0$, one must choose $f$ such that

$$f(l) \sim l + a_3l^3 + a_5l^5 + \cdots \quad \text{as} \quad l \to 0.$$  \hfill (4)

We can also require that the function $f$ is increasing. The exterior metric which can be matched to the interior metric (2) can be expressed in the form

$$ds^2_{\text{EXT}} = -dt^2 + dz^2 + dl^2 + \sin^2 \alpha (l - \tilde{l}_0)^2d\phi^2, \quad l > l_0$$  \hfill (5)

where the constants $\alpha$ and $\tilde{l}_0$ are determined from the following matching conditions of the metric and its first derivatives at $l = l_0$

$$f(l_0) = \sin \alpha(l_0 - \tilde{l}_0) \quad \text{and} \quad f'(l_0) = \sin \alpha.$$  \hfill (6)

The linear mass density $\mu$ of the straight string is defined by

$$\mu = \int_{l<l_0} \sigma f \, dl \, d\phi.$$  \hfill (7)

By using (3), (4) and (6), it is easy to see that it is related to the angular deficit $\Delta$ of metric (3) by

$$\Delta = 2\pi(1 - \sin \alpha) = 8\pi G\mu.$$  \hfill (8)
From (8), we note that the exterior metric (5) is independent of the details of the internal string as well as of its radius.

At this point, it is enlightening to recall the solution for a straight string with $\sigma =$constant. The interior metric is characterised by

$$f(l) = \epsilon \sin \frac{l}{\epsilon} \quad \text{and} \quad \sigma = \frac{1}{8\pi G\epsilon^2}$$

and the exterior metric is again (5). The geometrical interpretation of the parameter $\epsilon$ will be given below. The matching conditions (6) give the relation $l_0/\epsilon = \pi/2 - \alpha$ and thereby

$$\epsilon = (l_0 - \bar{l}_0)\frac{\sin \alpha}{\cos \alpha}.$$ \hspace{1cm} (10)

We point out that $l_0/\epsilon$ depends only on $\alpha$.

Returning to the general case this particular solution suggests to impose in the generic interior metric (2) the following form

$$f(l) = \epsilon h\left(\frac{l}{\epsilon}\right)$$ \hspace{1cm} (11)

where $h$ is a smooth function and $\epsilon$ is a parameter which takes again the value (10). Now the matching conditions (3) yield simply

$$h\left(\frac{l_0}{\epsilon}\right) = \cos \alpha \quad \text{and} \quad h'\left(\frac{l_0}{\epsilon}\right) = \sin \alpha.$$ \hspace{1cm} (12)

For a given function $h$, the quotient $l_0/\epsilon$ depends only on the constant angle $\alpha$, i.e. on the linear mass density $\mu$ by virtue of relation (8).

One can give a geometrical interpretation. By performing the change of coordinate $\rho = l - \bar{l}_0$ in metric (5), we recognise the so called conical metric representing a cone of half angle $\alpha$. It is usefull to give a representation in a Euclidean 3-space of the 2-surface $t =$ constant and $z =$ constant for
the interior and exterior metrics (cf. fig. 1). It can be visualised as a cone whose top is cut out at a distance \( \rho_0 = l_0 - \tilde{l}_0 \) of the vortex and replaced by an axisymmetric cap which joins the cone tangentially along the circle of junction of radius \( \rho_0 \sin \alpha \). The coordinate \( l \) represents the length of the radial geodesic of the 2-surface originated at the top of the cap. It takes the value \( l_0 \) at the boundary between the smooth cap and the cone and then it continues along a generatrix of the cone. The coordinate \( \phi \) is the azimuthal angle. Referring to fig. 1, we see that \( \epsilon = \rho_0 \sin \alpha / \cos \alpha \), which represents the distance of the junction to the central axis, coincides with choice (10). For the particular solution (9) we have a spherical cap of radius \( \epsilon \).

In the generic case, \( h \) is characteristic of the form of the cap and \( \epsilon \) of its size. This explains the choice of form (11) of \( f \) which allows us to wedge in the cone a cap of a given form of any size. In short, the 2-surface of the Euclidean 3-space will be called a smoothed cone of half angle \( \alpha \). The corresponding spacetime will be said to have a smooth conical point. We emphasize that we can choose \( l_0 \) and \( \epsilon \) as little as we want thus making the smooth cone as sharp as we want, but keeping the quotient \( l_0 / \epsilon \) fixed.

Finally, we shall also require the continuity of the second derivatives of the metric at the junction. This is simply achieved by putting

\[
f''(l_0) = 0 \quad \text{or} \quad h''\left(\frac{l_0}{\epsilon}\right) = 0. \quad (13)
\]

As a consequence the Ricci tensor is continuous at the junction. Let us note that the supplementary condition (13) is not verified by the particular solution (9). This was obvious from the beginning since the energy density \( \sigma \) is constant in the interior of a straight string and decreases sharply to zero at the boundary. The supplementary condition (13) is physically natural for a thick relativistic cosmic string but not absolutely necessary. Its real
Figure 1: Smoothed cone
usefulness is that the continuity of the Ricci tensor makes easier further calculations in the neighborhood of the boundary between the cap and the cone.

III Self-gravitating string of arbitrary shape and coordinate system

In the previous section we described a straight self-gravitating string with some thickness. The aim of this section is to extend this construction to a string of arbitrary shape. If the string thickness is sufficiently small and if the central line of the string spans a sufficiently smooth world sheet, we can suppose that the string approaches a straight one and thus it can be locally characterised by a smooth conical point. This suggests to determine the spacetime in the neighborhood of a small portion of the string in the following way:

- the interior metric given by

\[ ds^2_{\text{INT}} = g_{AB}(\tau^A, l, \phi)d\tau^A d\tau^B + dl^2 + f^2(l)d\phi^2 \quad 0 \leq l \leq l_0 \]  

where \( f(l) = \epsilon h(l/\epsilon) \) was specified in Sec. II and the metric components \( g_{AB} \) \((A, B = 0, 3)\) are smooth.

- in the vicinity of the string, the exterior metric given by

\[ ds^2_{\text{EXT}} = g_{AB}(\tau^A, l, \phi)d\tau^A d\tau^B + dl^2 + \sin^2 \alpha (l - l_0)^2 d\phi^2 \quad l > l_0. \]  

We note that \textit{a priori} we have omitted cross terms in (14) and (15) in order to simplify the calculus. Metric (14) is regular but the coordinate system breaks down at \( l = 0 \). We introduce coordinates \( \rho^a \) \((a = 1, 2)\) which are well
defined
\[ \rho^1 = l \cos \phi \quad \text{and} \quad \rho^2 = l \sin \phi. \quad (16) \]

Then, metrics (14) and (15) become
\[ ds^2 = g_{AB}(\tau^A, \rho^a) d\tau^A d\tau^B + g_{ab}(\rho^a) d\rho^a d\rho^b. \quad (17) \]

Let us introduce the function
\[ r(l) = \frac{f^2(l)}{l^4} - \frac{1}{l^2} \quad \text{with} \quad l = \sqrt{(\rho^1)^2 + (\rho^2)^2} \quad (18) \]
which is well defined and smooth in the interval \( 0 \leq l \leq l_0 \) from (4). It has null odd derivatives at \( l = 0 \). The components \( g_{ab} \) have the simple expression
\[ g_{11} = 1 + r(l)(\rho^2)^2, \quad g_{12} = -r(l)(\rho^1 \rho^2), \quad g_{22} = 1 + r(l)(\rho^1)^2 \quad (19) \]
in the interval \( 0 \leq l \leq l_0 \). It is interesting to note that \( g_{ab} = \delta_{ab} + O(\rho^2) \).

There will be no need in the following to express explicitly \( g_{ab} \) for \( l > l_0 \).

The central line of the string defined by \( l = 0 \) spans a timelike worldsheet parametrised by \( \tau^A \) whose induced metric is simply
\[ \gamma_{AB}(\tau^A) = g_{AB}(\tau^A, \rho^a = 0). \quad (20) \]

A radial geodesic on the 2-surface \( \tau^A = \text{constant} \), is also a geodesic of the spacetime, normal to the worldsheet at the point \( P(\tau^A) \). The coordinate \( l \) represents the length along this geodesic measured from the point \( P(\tau^A) \). So, the 2-surface \( \tau^A = \text{constant} \) is generated by the spacetime geodesics tangent at \( P(\tau^A) \) to a 2-plane orthogonal to the worldsheet. Hence we recognise in the coordinate system \( (\tau^A, \rho^a) \) a known system (see for example [13]) where \( \rho^a \) are geodesic coordinates and from which it is easy to extract the extrinsic curvature \( K_{aAB} \) of the worldsheet by expanding the metric components
\[ g_{AB}(\tau^A, \rho^a) = \gamma_{AB}(\tau^A) + 2K_{aAB}(\tau^A) \rho^a + O(\rho^2). \quad (21) \]
As we shall see in Sec. V, the extrinsic curvature is implicated in the dynamics of a self-gravitating string.

IV Expansion of geometrical quantities

The aim of this section is to expand in powers of $1/\epsilon$ geometrical quantities such as the metric, connection and Ricci tensor when $\epsilon$, that is a measure of the thickness of the string, is small but not null. Let us point out that all length parameters $\rho_0^a, l_0, \epsilon$ are of the same order since $l_0/\epsilon$ is constant.

In the interval $0 \leq l \leq l_0$ all these quantities depend on the function $r(l)$ through the metric components $g_{ab}(\rho^a)$. However since $r(l_0) = O(1/\epsilon^2)$ it is preferable to substitute $r(l)$ by a function of $l/\epsilon$

$$q\left(\frac{l}{\epsilon}\right) = \epsilon^2 r(l) = \frac{\epsilon^4}{l^4} h^2 \left(\frac{l}{\epsilon}\right) - \frac{\epsilon^2}{l^2}.$$ (22)

It is evident that $q(l_0/\epsilon)$ is fixed and depends only of the angle $\alpha$. The same holds for its first and second derivatives $q'(l_0/\epsilon)$ and $q''(l_0/\epsilon)$.

The metric components (19), their inverse, and the determinant can now be written

$$g_{ab} = \delta_{ab} + q\left(\frac{l}{\epsilon}\right) \epsilon^a_c \epsilon^b_d \frac{\rho_c \rho_d}{\epsilon^2},$$ (23)

$$g^{ab} = \frac{\delta^{ab} + q\left(\frac{l}{\epsilon}\right) \epsilon^a_c \epsilon^b_d}{1 + \frac{l^2}{\epsilon^2} q\left(\frac{l}{\epsilon}\right)},$$ (24)

$$\hat{g} = \det(g_{ab}) = 1 + \frac{l^2}{\epsilon^2} q\left(\frac{l}{\epsilon}\right).$$ (25)

where $\epsilon^a_c$ is the totally antisymmetric Levi-Civita symbol and $\hat{()}$ stands for the induced geometrical quantities of the 2-dimensional smoothed cone.
Then one immediately gets

\[(g_{ab})_{l=l_0} = O(1), \quad (g^{ab})_{l=l_0} = O(1), \quad (\dot{g})_{l=l_0} = O(1). \]

(26)

By direct calculation of the first derivative one obtains

\[(g_{ab,c})_{l=l_0} = O(1/\epsilon), \quad (g^{ab}_{,c})_{l=l_0} = O(1/\epsilon). \]

(27)

It is also easy to see that

\[(g_{AB})_{l=l_0} = O(1), \quad (g_{AB,d})_{l=l_0} = O(1), \quad (g_{AB,C})_{l=l_0} = O(1). \]

(28)

Now we can calculate the order of the non null connection coefficients for \(l = l_0\)

\[\Gamma^A_{BC} = \frac{1}{2} g^{AD} (g_{DB,C} + g_{DC,B} - g_{BC,D}) = O(1), \]

(29)

\[\Gamma^D_{ab} = \frac{1}{2} g^{AD} g_{AB,a} = O(1), \]

(30)

\[\Gamma^a_{AB} = -\frac{1}{2} g^{ab} g_{AB,b} = O(1), \]

(31)

\[\Gamma^c_{ab} = \frac{1}{2} g^{cd} (g_{da,b} + g_{db,a} - g_{ab,d}) = O(1/\epsilon). \]

(32)

We can now deal with the Ricci tensor for \(l = l_0\). The derivatives of (31) will give terms of the order of \(1/\epsilon\), whereas the derivatives of (27) and (32) will give terms of the order \(1/\epsilon^2\). In obvious notations we obtain

\[R_{AB} = R_{AB}(\frac{1}{\epsilon}) + R_{AB}(1) \]

(33)

in which we have

\[R_{AB}(\frac{1}{\epsilon}) = \partial_a \Gamma^a_{AB} + \Gamma^d_{ad} \Gamma^a_{AB}, \]

(34)

\[R_{AB}(1) = \tilde{*} R_{AB} + \Gamma^D_{aD} \Gamma^a_{AB} - \Gamma^C_{aB} \Gamma^a_{AC} - \Gamma^D_{BD} \Gamma^a_{AB} \]

(35)

with \( \tilde{*} R_{AB} = \partial_D \Gamma^D_{AB} - \partial_B \Gamma^D_{AD} + \Gamma^D_{CD} \Gamma^C_{AB} - \Gamma^C_{BD} \Gamma^D_{AC} \).
For the small case indices we get

\[ R_{ab} = R_{ab}\left(\frac{1}{\epsilon^2}\right) + R_{ab}\left(\frac{1}{\epsilon}\right) + R_{ab}(1) \tag{36} \]

in which we have

\[ R_{ab}\left(\frac{1}{\epsilon^2}\right) = \hat{R}_{ab} \tag{37} \]

where \( \hat{R}_{ab} \) is the Ricci tensor associated with the connection \( \mathcal{B} \) of the smoothed cone,

\[ R_{ab}\left(\frac{1}{\epsilon}\right) = \Gamma^{D}_{\epsilon} \Gamma^{d}_{ab} \tag{38} \]

and

\[ R_{ab}(1) = -\partial_{b} \Gamma^{D}_{Da} - \Gamma^{C}_{bD} \Gamma^{D}_{aC} \tag{39} \]

Finally for the mixed indices we get

\[ R_{aB} = R_{aB}(1) = \partial_{A} \Gamma^{A}_{aB} - \partial_{B} \Gamma^{D}_{Da} + \Gamma^{C}_{D} \Gamma^{D}_{aB} - \Gamma^{D}_{BC} \Gamma^{C}_{aD}. \tag{40} \]

The curvature scalar \( R \) for \( l = l_0 \) is given by the equation

\[ R = R\left(\frac{1}{\epsilon^2}\right) + R\left(\frac{1}{\epsilon}\right) + R(1) \tag{41} \]

where

\[ R\left(\frac{1}{\epsilon^2}\right) = \hat{R}_{ab} \ g^{ab} = \hat{R}, \tag{42} \]

\[ R\left(\frac{1}{\epsilon}\right) = \partial_{a} \Gamma^{a}_{AB} g^{AB} + \Gamma^{a}_{ad} \Gamma^{d}_{AB} g^{AB} + \Gamma^{d}_{ab} \Gamma^{D}_{Dd} g^{ab}, \tag{43} \]

\[ R(1) = \hat{R} + \Gamma^{D}_{ad} \Gamma^{a}_{AB} g^{AB} - \Gamma^{C}_{aB} \Gamma^{a}_{AC} g^{AB} - \Gamma^{a}_{BD} \Gamma^{D}_{Aa} g^{AB} - \partial_{a} \Gamma^{D}_{Da} g^{ab} - \Gamma^{D}_{bD} \Gamma^{D}_{Ca} g^{ab} \tag{44} \]

with \( \hat{R} = R_{AB} \ g^{AB} \).

We have calculated the above quantities for \( l = l_0 \) since we need these estimations precisely at the boundary in order to obtain the equations of motion in the following section.
V Equations of motion of the string

We now focus attention on the self-gravitating string of arbitrary shape described by metric (17) in the coordinate system $(\tau^A, \rho^a)$ where the function $h$ is specified in Sec. II. We suppose that the exterior spacetime is a vacuum solution: $R_{\alpha\beta} = 0$, $(\alpha = A, a)$ for $l > l_0$.

The energy-momentum tensor $T_{\alpha\beta}$ of the extended string is the source of the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi GT_{\alpha\beta}, \quad 0 \leq l \leq l_0.$$  (45)

As a consequence of the matching conditions adopted, the Ricci tensor must be continuous at the junction $l = l_0$. So, the interior Einstein equations (45) coincide with the vacuum Einstein equations at $l = l_0$

$$R_{\alpha\beta} = 0.$$  (46)

These boundary conditions impose some constraints on the timelike worldsheet swept by the central line of the string. We shall examine (46) when the parameter $\epsilon$ is arbitrarily small.

According to (36)-(39), the components $(a, b)$ of equations (46) are written

$$\left(R_{ab}\right)_{l_0} = \left(\Gamma^D_{dD}\right)_{l_0} \left(\Gamma^d_{ab}\right)_{l_0} + \left(R_{ab}(1)\right)_{l_0} = 0,$$  (47)

equation (47) being obtained by noting from (37) that $\left[R_{ab}(\frac{1}{\epsilon^2})\right]_{l_0} = \left(\hat{R}_{ab}\right)_{l_0} = 0$ since the junction $l = l_0$ belongs to the cone.

We need to calculate the limit of the connection coefficient $\Gamma^D_{dD}$ appearing in equation (47). From (30), (20), (21) and since $l_0/\epsilon$ is constant we obtain

$$\lim_{\epsilon \to 0}(\Gamma^D_{dD})_{l_0} = \lim_{l_0 \to 0}(\Gamma^D_{Dd})_{l_0} = \gamma^{AB}K_{dAB} = K_d$$  (48)
where $K_d$ is the mean curvature of the timelike worldsheet. On the other hand we know from (32) that $(\Gamma^d_{ab})_{l0}$ is of order $1/\epsilon$. Thus, the limit $\epsilon \to 0$ of equation (47) multiplied by $\epsilon$ gives

$$F^d_{ab}(k^a)K_d = 0 \quad \text{with} \quad k^a = \frac{\rho_0^a}{l_0}$$

where

$$F^d_{ab}(k^a) = \lim_{\epsilon \to 0} [\epsilon(\Gamma^d_{ab})_{l0}]$$

is finite. Equation (49) is independent of $\epsilon$ or $l_0$ and depends only on the azimuthal angle $\phi$ on the boundary or equivalently as indicated of the unitary 2-vector $k^a$.

In the same way using (33)-(35) along with (31),(32), we obtain the zero limit of $\epsilon$ for the $(A, B)$ components of equations (46)

$$F^b_{kAB}(k^a)K_{bAB} = 0 \quad \text{with} \quad k^a = \frac{\rho_0^a}{l_0}$$

where

$$F^b_{kAB}(k^a) = \lim_{\epsilon \to 0} [\epsilon(\partial_0 g^{ab} + \Gamma^a_{ad} g^{db})_{l0}]$$

is finite. By introducing the quantities independent of $\epsilon$

$$A = q\left(\frac{l_0}{\epsilon}\right)\frac{l_0^2}{\epsilon^2} \quad \text{and} \quad B = q'\left(\frac{l_0}{\epsilon}\right)\frac{l_0^3}{\epsilon^3};$$

after some algebra we can express $F^b_{kAB}(k^a)$ in the following way

$$F^b = \frac{\epsilon}{l_0} \frac{1}{1 + A} [2A + \frac{1}{2}B]k^b.$$

Let us note that the mixed components $(a, A)$ of (46) give no constraint.

If the coefficients $F^d_{ab}(k^a)$ and $F^b_{kAB}(k^a)$ are non vanishing, which is the general case, equations (49) and (51) yield respectively

$$K_d = 0,$$

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\[ K_{\alpha \beta} = 0. \]  

Equation (56) means that, at the zero limit of \( \epsilon \), the worldsheet swept by the central line of the string is totally geodesic. This of course implies (55). We recover the situation described in the literature [4, 5, 6, 7, 8] when the string is a self-gravitating singular line.

However we see from (54) that if we take the condition

\[ B = -4A, \]  

then one annihilates the extrinsic curvature coefficient and thus constraint (56) disappears. Hence we are left with constraint (55) which expresses that in the zero limit of \( \epsilon \) the world sheet is minimal. In other words the worldsheet has the same evolution as a Nambu-Goto string.

We must verify however that when condition (57) is applied then equation (59) does not also disappear. Equations (59) are explicitly expressed as

\[
\begin{align*}
  k^2[P + Q(k^1)^2]K_1 + k^1[P + Q(k^2)^2]K_2 &= 0, \\
  -k^1Q(k^2)^2K_1 - k^2[2P + Q(k^2)^2]K_2 &= 0, \\
  -k^1[2P + Q(k^1)^2]K_1 - k^2Q(k^1)^2K_2 &= 0
\end{align*}
\]

for all \( k^\alpha \), where \( P = B + 2A \) and \( Q = -B + AB + 4A^2 \). If \( A \) and \( B \) verify condition (54), then the three equations (58) are reduced to only one: \( k^2K_1 - k^1K_2 = 0 \) for all \( k^\alpha \). Hence (53) is again verified.

By their definition (53), the quantities \( A \) and \( B \) are independent of the length parameters and depend only on the angle \( \alpha \). That is, we can express condition (57) in terms of the angle \( \alpha \) by using (12) and (22). We obtain the simple expression

\[ \sin \alpha \cos \alpha = \frac{l_0}{\epsilon}. \]

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This relation is a supplementary constraint on $h$ as it will be explained in the conclusion.

VI  **Energy-momentum tensor of the string**

The energy-momentum tensor $S_{\alpha\beta}$ of the string is obtained by an integration of $T_{\alpha\beta}$ on the section $\tau^A = \text{constant}$ of the string.

$$S_{\alpha\beta} = \int_{l \leq l_0} T_{\alpha\beta} \sqrt{\hat{g}} \, d\rho^1 \, d\rho^2$$  \hfill (60)

where now $\epsilon$ and thus $l_0$ are small but fixed.

From the Einstein equations (45) and the algebraic expressions of the Ricci tensor given in Sec. IV, we can write in obvious notations

$$T_{\alpha\beta}(\tau^A, \rho^a) = T_{\alpha\beta}(\frac{1}{\epsilon^2}) + T_{\alpha\beta}(\frac{1}{\epsilon}) + T_{\alpha\beta}(1) \quad l \leq l_0.$$  \hfill (61)

We first apply formula (60) for the $(A, B)$ components. Only the first term of (61) will yield a non null finite integral when $\epsilon$ tends to 0 since the volume element of the smoothed cone is of the order $l_0^2$.

$$S_{AB} = \int_{l \leq l_0} T_{AB}(\frac{1}{\epsilon^2}) \sqrt{\hat{g}} \, d\rho^1 \, d\rho^2 + O(\epsilon).$$  \hfill (62)

Now we have

$$T_{AB}(\frac{1}{\epsilon^2}) = -\frac{1}{16\pi G} g_{AB} \hat{R} = -\frac{1}{8\pi G} g_{AB} K$$  \hfill (63)

where $K$ is the Gauss curvature, therefore

$$S_{AB} = -\frac{1}{8\pi G} \int_{l \leq l_0} g_{AB} K \sqrt{\hat{g}} \, d\rho^1 \, d\rho^2 + O(\epsilon).$$  \hfill (64)

Since $l_0$ is small we can also approximate the metric $g_{AB}$ by development (21). We finally obtain

$$S_{AB} = -\frac{1}{8\pi G} \gamma_{AB} \int_{l \leq l_0} K \sqrt{\hat{g}} \, d\rho^1 \, d\rho^2 + O(\epsilon) = -\mu \gamma_{AB} + O(\epsilon)$$  \hfill (65)
where $\mu$ is the linear mass density. The Gauss-Bonnet formula gives
\[
\int_{\ell \leq \ell_0} K \sqrt{\hat{g}} \, d\ell_1 \, d\rho_2 = 2\pi (1 - \sin \alpha) \tag{66}
\]
and consequently $\mu$ is related to $\alpha$ by formula (8). This result is not obvious a priori. It originates specifically in the choice of the form of the metric (14) where $f$ is given by (11). A general discussion of the problem of obtaining the energy momentum tensor for a concentrate distribution of matter can be found in [14].

The integral of the $(a, A)$ components obviously vanishes as $\epsilon^2$. For the integral of the $(a, b)$ components the non null finite part vanishes since $\hat{R}_{ab} - \frac{1}{2} g_{ab} \hat{R} = 0$ for a 2-dimensional manifold. Thus the integral vanishes as $\epsilon$.

Discarding terms in $\epsilon$, $S_{\alpha \beta}$ reduces to $S_{\alpha \beta}$ given by (65) which is the energy-momentum tensor of the Nambu-Goto string. This last result which is proved independently of the equations of motion is in accordance with them. It can justify a posteriori the definition of a self-gravitating string as a smooth cone with metric (14) and (15).

VII Conclusion

In this paper we have investigated the dynamics of a self-gravitating string in the limit where the thickness $\epsilon$ becomes negligible. In the generic case the worldsheet swept by the string (more precisely by the central line of the string) is a totally geodesic surface (the extrinsic curvature of the worldsheet is null). This result could have been somehow guessed since the strings defined as singular lines of conical points have precisely this property [4, 5, 6, 7].
However we have found another pleasant possibility: by imposing relation (59) the extrinsic curvature is no more relevant and only the mean curvature is null. This expresses that the worldsheet is extremal which is the behaviour of the Nambu-Goto string. If we suppose that the linear mass density $\mu$ is given, then the angle $\alpha$ is fixed by equation (66). It is easily seen that relation (59) is a constraint on the function $h$; it can be rewritten

$$h\left(\frac{l_0}{\epsilon}\right)h'\left(\frac{l_0}{\epsilon}\right) = \frac{l_0}{\epsilon}.$$  

(67)

The above equation is an algebraic relation taken at the fixed point $l_0/\epsilon$ which is added to the matching conditions (12) and (13). It is easily shown that such a function $h$ can be found in a polynomial form. The simplest solution is an odd polynomial (cf. (4)) of order seven:

$$h\left(\frac{l_0}{\epsilon}\right) = \frac{l_0}{\epsilon} + b_3\left(\frac{l_0}{\epsilon}\right)^3 + b_5\left(\frac{l_0}{\epsilon}\right)^5 + b_7\left(\frac{l_0}{\epsilon}\right)^7.$$

The four unknown quantities $l_0/\epsilon$, $b_3$, $b_5$ and $b_7$ can be determined as functions of the angle $\alpha$ (or equivalently as a function of the linear mass density $\mu$ of the string) by the four equations (12), (13) and (67). Of course by taking a polynomial of larger order one can get an infinity of solutions. It would probably be interesting to investigate the physical meaning of the constraint (67) on the matter of the string.

By using the method described in Sec. IV, we can evaluate the Riemann tensor $(R_{\alpha\beta\gamma\delta})_{l_0}$ at the junction $l = l_0$ or equivalently the Weyl tensor since the Ricci tensor (46) vanishes. This gives the magnitude of the gravitational field near the thin cosmic string. In the generic case where the extrinsic curvature $K_{aBC}$ vanishes, the Riemann tensor is bounded as the radius $l_0$ becomes arbitrarily small. On the contrary when only the mean curvature $K_a$ is null, the Riemann tensor components $(R_{aBCd})_{l_0}$ blow up as $1/\epsilon$, i.e.
$1/l_0$, when the radius $l_0$ tends to zero. We see that the gravitational aspect is completely different in the latter case. Let us add that if we require that the Riemann tensor remains bounded then we necessarily fall in the generic case.
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