UNIVERSAL COEFFICIENT THEOREMS FOR $C^*$-ALGEBRAS
OVER FINITE TOPOLOGICAL SPACES

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Abstract. We determine the class of finite $T_0$-spaces allowing for a universal
coefficient theorem computing equivariant KK-theory by filtrated K-theory.

1. Introduction

The universal coefficient theorem of Rosenberg and Schochet [11] states that for
separable $C^*$-algebras $A$ and $B$ with $A$ being in a certain bootstrap class there is a
short exact sequence of $\mathbb{Z}/2$-graded Abelian groups

$$\text{Ext}^1(K_{*+1}(A), K_*(B)) \rightarrow KK_*(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B)).$$

Apart from being very useful for computations of KK-groups, it plays an important
role in the classification of $C^*$-algebras by K-theoretic invariants.

The corresponding sequence for $A = B$ is an extension of rings with the product
in $\text{Ext}^1(K_{*+1}(A), K_*(A))$ being zero. Therefore $KK_*(A, A)$ is a nilpotent extension
of $\text{Hom}(K_*(A), K_*(A))$; this shows that isomorphisms in K-theory lift to isomorph-
isms in KK-theory. Results by Kirchberg [4] and Phillips [10] then show that
every KK-equivalence between $A$ and $B$ is induced by an actual $^*$-isomorphism of
$C^*$-algebras, provided that $A$ and $B$ are stable, nuclear, separable, purely infinite and
simple. Both facts together give the following strong classification result:

$C^*$-algebras $A$ with the above-mentioned properties are completely classified by the
$\mathbb{Z}/2$-graded Abelian group $K_*(A)$.

It is interesting to extend this result to the non-simple case. In [4], Eberhard
Kirchberg indicated a construction of an equivariant version $KK(X)$ of bivariant
K-theory for $C^*$-algebras over a given $T_0$-space $X$ and proved a corresponding
classification result: a $KK(X)$-equivalence between two $C^*$-algebras over $X$ lifts
to an equivariant $^*$-isomorphism if both $C^*$-algebras are stable, nuclear, separable, purely infinite and tight—the notion of tightness generalises simplicity; its name
was coined in [8].

Our aim is therefore to compute $KK_*(X; A, B)$ for a topological space $X$ and
$C^*$-algebras $A$ and $B$ over $X$ by a universal coefficient theorem, that is, by an exact
sequence of the form

$$\text{Ext}^1_{\mathcal{C}}(H_{*+1}(A), H_*(B)) \rightarrow KK_*(X; A, B) \rightarrow \text{Hom}_{\mathcal{C}}(H_*(A), H_*(B))$$

for some reasonably tractable homology theory $H_*$ for $C^*$-algebras over $X$, taking
values in some Abelian category $\mathcal{C}$. Here $A$ is assumed to belong to the bootstrap
class $\mathcal{B}(X)$ introduced in [8]. As in the non-equivariant case, a universal coefficient
theorem of this form allows to lift an isomorphism $H_*(A) \cong H_*(B)$ in $\mathcal{C}$ to a
KK($X$)-equivalence $A \simeq B$ if both $A$ and $B$ belong to the bootstrap class $\mathcal{B}(X)$.  

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In [7], Ralf Meyer and Ryszard Nest applied their machinery of homological algebra in triangulated categories developed in [6,9] with the aim of deriving a UCT short exact sequence which computes $\text{KK}(X; A, B)$ for a finite $T_0$-space $X$ by filtrated $K$-theory (in the following denoted by FK). They obtain the desired short exact sequence in the case of the totally ordered space $O_n$ with $n$ points, that is,
\[
O_n = \{1, 2, \ldots, n\}, \quad \tau_{O_n} = \{\emptyset, \{n\}, \{n, n-1\}, \ldots, X\}.
\]
A $C^*$-algebra $A$ over this space is essentially the same as a $C^*$-algebra $A$ together with a finite increasing chain of ideals
\[
\{0\} \triangleleft I_0 \triangleleft I_1 \triangleleft \cdots \triangleleft I_n = A.
\]

On the other hand, Meyer and Nest give an example of a finite $T_0$-space $Y$ for which the following strong non-UCT statement holds: There are objects $A$ and $B$ in $\mathcal{B}(Y)$ with isomorphic filtrated $K$-theory which are not $\text{KK}(Y)$-equivalent.

The aim of this article is to give a complete answer to the following question: given a finite $T_0$-space $X$, is there a UCT short exact sequence which computes $\text{KK}(X; A, B)$ by filtrated $K$-theory? The assumption of the separation axiom $T_0$ is not a loss of generality here, since all that matters is the lattice of open subsets of $X$ (see [3, §2.5]).

In order to describe the general form of those finite $T_0$-spaces for which there is such a UCT short exact sequence, we have to introduce some notation. For topological spaces $X$ and $Y$ and $x \in X$, $y \in Y$, let us denote by $Y \setminus x \equiv Y$ the quotient space of the disjoint union $X \sqcup Y$ by the equivalence relation generated by $x \sim y$.

**Definition 1.1.** Let $X$ be finite $T_0$-space. We say that $X$ is of type $(A)$—A for accordion—if $X$ is of the form
\[
O_{n_1} \bigvee_{n_1 = n_2} O_{n_2} \bigvee_{1=1} O_{n_3} \cdots O_{n_{m-1}} \bigvee_{n_{m-1} = n_m} O_{n_m}
\]
for $m \in 2\mathbb{N}_{>0}$, $n_i \in \mathbb{N}_{>0}$ and $n_i > 1$ for $2 \leq i \leq m - 1$.

To get an alternative description of type $(A)$ spaces recall from [6] how finite spaces can be visualised as directed graphs:

**Definition 1.2.** Let $X$ be a finite $T_0$-space. Define $\Gamma(X) = (V, E)$ by $V := X$, and $(x, y) \in E$ if and only if $x \neq y$, $x \in \{y\}$ and $(x \in \{z\}, z \in \{y\}) \Rightarrow z = x$ or $z = y$.

The graph of a space of type $(A)$ looks as follows (see also Figure 1 on page 23):\[
\bullet \leftarrow \cdots \leftarrow \bullet \rightarrow \cdots \rightarrow \bullet \leftarrow \cdots \leftarrow \bullet \rightarrow \cdots \rightarrow \bullet.
\]

In particular, every connected $T_0$-space with at most three points is of type $(A)$.

A characterisation of type $(A)$ spaces in terms of unoriented edge degrees of the associated graphs is given in Lemma [5,3].

The main result of this paper now reads as follows; it is a consequence of Theorem [5,9] see also [11]

**Theorem 1.3.** Let $X$ be finite $T_0$-space. The following statements are equivalent:

(i) $X$ is a disjoint union of spaces of type $(A)$.

(ii) Let $A, B \in \mathcal{B}(X)$. Then $\text{FK}(A) \cong \text{FK}(B)$ implies that $A$ is $\text{KK}(X)$-equivalent to $B$.

(iii) Let $A$ and $B$ be separable $C^*$-algebras over $X$. Suppose $A \in \mathcal{B}(X)$. Then there is a natural short exact UCT sequence
\[
\text{Ext}_{\mathcal{N}^T(X)}^1(\text{FK}(A)[1], \text{FK}(B)) \rightarrow \text{KK}_{\tau}(X; A, B) \rightarrow \text{Hom}_{\mathcal{N}^T(X)}(\text{FK}(A), \text{FK}(B)).
\]

Here the subscript $\mathcal{N}^T(X)$ denotes that $\text{Ext}^1$ and $\text{Hom}$ are taken in the category $\text{Mod}(\mathcal{N}^T(X))_{/\tau}$, the target category of $\text{FK}$.
2. **C*-algebras over topological spaces**

Throughout this article, $X$ denotes a finite $T_0$-space. In the following, we introduce $C^*$-algebras over $X$ along the lines of \[8\]. The definition of a $C^*$-algebra over a topological space actually works in greater generality.

2.1. **Basic notions.** For a $C^*$-algebra $A$ denote by $\text{Prim}(A)$ its primitive ideal space. A $C^*$-algebra over $X$ is a pair $(A, \psi)$ consisting of a $C^*$-algebra $A$ and a continuous map $\psi : \text{Prim}(A) \to X$.

Let $\mathcal{O}(X)$ denote the set of open subsets of $X$, partially ordered by $\subseteq$ and $\mathcal{I}(A)$ the set of closed $^*$-ideals in $A$, partially ordered by $\subseteq$. The partially ordered sets $(\mathcal{O}(X), \subseteq)$ and $(\mathcal{I}(A), \subseteq)$ are complete lattices, that is, any subset has both an infimum and a supremum. A continuous map $\psi : \text{Prim}(A) \to X$ induces a map $\psi^* : \mathcal{O}(X) \to \mathcal{I}(A)$ which commutes with infima and suprema. By \[8\] Lemma 2.25, this correspondence gives an equivalent description of a $C^*$-algebra over $X$ as a pair $(A, \psi^*)$ where $\psi^* : \mathcal{O}(X) \to \mathcal{I}(A)$, $U \mapsto A(U)$ commutes with infima and suprema.

A $^*$-homomorphism $f : A \to B$ between two $C^*$-algebras over $X$ is $X$-equivariant if $f(A(U)) \subseteq B(U)$ for all $U \in \mathcal{O}(X)$. The category of $C^*$-algebras over $X$ with $X$-equivariant $^*$-homomorphisms is denoted by $C^*\text{alg}(X)$, its full subcategory consisting of all separable $C^*$-algebras over $X$ is denoted by $C^*\text{sep}(X)$.

A subset $Y \subseteq X$ is locally closed if and only if $Y = U \setminus V$ for open subsets $V, U \in \mathcal{O}(X)$ with $V \subseteq U$. Then we define $A(Y) := A(U)/A(V)$ for a $C^*$-algebra $A$ over $X$; this does not depend on the choice of $U$ and $V$ by \[8\] Lemma 2.16. We write $\mathcal{LC}(X)$ for the set of locally closed subsets of $X$. By $\mathcal{LC}(X)^*$ we denote the set of connected, non-empty locally closed subsets of $X$.

We write $x \in \mathcal{C}$ for objects of a category $\mathcal{C}$ as opposed to morphisms.

2.2. **Functoriality.** A continuous map $f : X \to Y$ induces a functor $f_* : C^*\text{alg}(X) \to C^*\text{alg}(Y)$ which is given by $(A, \psi) \mapsto (A, f \circ \psi)$. We have $g_* f_* = (gf)_*$ for composable continuous maps $f$ and $g$.

If $f : X \to Y$ is the embedding of a subset with the subspace topology, we also write $i^X_Y$ instead of $f_*$ and call it extension.

A locally closed subset $Y \in \mathcal{LC}(X)$ induces the restriction functor $r^X_Y : C^*\text{alg}(X) \to C^*\text{alg}(Y)$ given by $(r^X_Y B)(Z) := B(Z)$ for all $Z \in \mathcal{LC}(Y) \subseteq \mathcal{LC}(X)$. We have $r^Y_Z \circ r^X_Y = r^Y_Z$ if $Z \subseteq Y \subseteq X$ and $r^X_X = \text{id}$.

Induction and restriction are related by $r^X_Y \circ i^X_Y = \text{id}$ and various adjointness relations; see \[8\] Definition 2.19 and Lemma 2.20 for a discussion.

2.3. **Specialisation preorder.** There is the specialisation preorder on $X$, defined by $x \preceq y \iff \{x\} \subseteq \{y\}$. A subset $Y \subseteq X$ is locally closed if and only it is convex with respect to $\preceq$, that is, if and only if $x \preceq y \preceq z$ and $x, z \in Y$ implies $y \in Y$ for all $x, y, z \in X$. A subset $Y \subseteq X$ has a locally closed hull $\text{LC}(Y)$ defined as $\text{LC}(Y) := \{x \in X \mid \exists y_1, y_2 \in Y : y_1 \preceq x \preceq y_2\}$.

**Lemma 2.1.** We have $\text{LC}(\text{LC}(Y)) = \text{LC}(Y)$. Moreover, $\text{LC}(Y)$ is the smallest locally closed subset of $X$ containing $Y$. 

Proof. Obviously \( Y \subseteq \text{LC}(Y) \). Let \( y \in \text{LC}(\text{LC}(Y)) \). Then there are \( y_1, y_2 \in \text{LC}(Y) \) such that \( y_1 \preceq y \preceq y_2 \). By definition there are \( z_1, z_2, z_3, z_4 \in Y \) such that \( z_1 \preceq y_1 \preceq z_2, z_3 \preceq y_2 \preceq z_4 \). Hence \( z_1 \preceq y_1 \preceq y \preceq y_2 \preceq z_4 \) and therefore \( y \in \text{LC}(Y) \). Using the characterization of locally closed subsets as convex subsets, the second statement is obvious. \( \square \)

A map \( f : X_1 \to X_2 \) between two finite topological spaces is continuous if and only if it is monotone with respect to \( \preceq \), that is, if \( x \preceq y \Rightarrow f(x) \preceq f(y) \).

Note that \( \preceq \) is a partial order if and only if \( X \) is \( T_0 \). By [3] Corollary 2.33, this yields a bijection of \( T_0 \)-topologies and partial orders on a given finite set. The preimage of a partial order \( \preceq \) is called the Alexandrov topology associated to \( \preceq \) and denoted by \( \tau_{\preceq} \).

2.4. Representation as finite directed graphs. We describe a well-known way to represent finite \( T_0 \)-spaces via finite directed acyclic graphs. Several examples can be found in [3] §2.8.

To establish notation, first collect a few elementary notions of graph theory: a directed graph is a tuple \( \Gamma = (V, E) \), where \( V \) is a set and \( E \subseteq (V \times V) \setminus \Delta(V) \); elements of \( V \) are called vertices and elements of \( E \) are called edges. We will also write \( E(\Gamma) \) and \( V(\Gamma) \) to denote the edges and vertices associated to \( \Gamma \). Hence we do neither allow loops nor multiple edges to exist. A graph \((V', E')\) is a subgraph of \((V, E)\) if and only if \( V' \subseteq V \) and \( E' = \{(a, b) \in E \mid a, b \in V'\} \).

A directed path \( \rho \) is a sequence \( \rho = (v_i)_{i=0,\ldots,n} \) such that \( (v_i, v_{i+1}) \in E \) for \( i = 1, \ldots, n \) with all \( (v_i) = 1, \ldots, n \) being pairwise distinct. The length of \( \rho = (v_i)_{i=0,\ldots,n} \) is \( n \). We say that \( \rho \) is a path from \( a \) to \( b \) if \( v_0 = a \) and \( v_n = b \).

A directed cycle is a directed path of length greater than 1 such that \( v_0 = v_n \). For two paths \( \rho_1 = (v_i)_{i=0,\ldots,n} \) and \( \rho_2 = (w_i)_{i=0,\ldots,m} \) we define sets

\[
\rho_1 \cap \rho_2 := \{v_i \mid i = 0, \ldots, n\} \cap \{w_i \mid i = 0, \ldots, m\}
\]

and

\[
\rho_1 \cup \rho_2 := \{v_i \mid i = 0, \ldots, n\} \cup \{w_i \mid i = 0, \ldots, m\}.
\]

An edge \((v_0, v_1)\) is called outgoing edge of \( v_0 \) and incoming edge of \( v_1 \). The unoriented degree \( d(v) \) of \( v \in V \) is defined as

\[
d(v) := \#\{e \in E \mid e \text{ outgoing edge of } v\} + \#\{e \in E \mid e \text{ incoming edge of } v\},
\]

while the oriented degree \( d_o(v) \) of \( v \in V \) is defined as

\[
d_o(v) := \#\{e \in E \mid e \text{ outgoing edge of } v\} - \#\{e \in E \mid e \text{ incoming edge of } v\}.
\]

An undirected path is a sequence \((v_i)_{i=0,\ldots,n}\) such that for \( i = 1, \ldots, n \) either \((v_i, v_{i+1}) \in E \) or \((v_{i+1}, v_i) \in E \) with all \((v_i) = 1, \ldots, n\) being pairwise distinct. We say that \( \rho \) is an undirected path from \( a \) to \( b \) if \( v_0 = a \) and \( v_n = b \).

A cycle is an undirected path \( \rho = (v_i)_{i=0,\ldots,n} \) of length greater than 0 such that \( v_0 = v_n \). A directed graph is called acyclic if it has no cycles.

To a partial order \( \preceq \) on \( X \), we associate a finite directed acyclic graph \( \Gamma(X) \) as follows. We write \( x \prec y \) to denote that \( x \preceq y \) and \( x \neq y \).

**Definition 2.2.** Let \( \Gamma(X) \) be the directed graph with vertex set \( X \) and with an edge \( x \prec y \) if and only if \( x \prec y \) and there is no \( z \in X \) with \( x \prec z \prec y \).

In other words, \( \Gamma(X) \) is the Hasse diagram corresponding to the specialisation order on \( X \).

We can recover the partial order from this graph by letting \( x \preceq y \) if and only if the graph contains a directed path from \( y \) to \( x \). This is the reachability relation on the vertex set of \( \Gamma(X) \), which makes sense for every finite directed acyclic graph.
Note that we cannot obtain every finite acyclic directed graph in this way. In fact, a finite directed acyclic graph is of the form $\Gamma(X)$ for some $T_0$-space $X$ if and only if it is transitively reduced, that is, if it is (isomorphic to) the graph associated to its reachability relation (see, for instance, [3]).

For later reference, we list restrictions on $\Gamma(X)$ in the following lemma which follows directly from the definitions.

**Lemma 2.3.** The directed graph $\Gamma(X)$ is acyclic. Let $x, y$ be vertices in $\Gamma(X)$. If $\rho_1$ and $\rho_2$ are two distinct directed paths from $x$ to $y$. Then $\rho_1$ and $\rho_2$ have length at least 2.

Let $S$ be a finite set. If $\Gamma$ is a directed graph with vertex set $S$, then we can define a preorder on $S$ by setting $s_1 \leq_{\Gamma} s_2$ if and only if there is a directed path from $s_2$ to $s_1$. Note that $\leq_{\Gamma}$ is a partial order if and only if $\Gamma$ is acyclic. Let $E(S)$ be the set of acyclic directed graphs with vertex set $S$ having the following property: if $\rho_1$ and $\rho_2$ are two distinct directed paths in $\Gamma$ from $x$ to $y$, then $\rho_1$ and $\rho_2$ have length at least 2. It is easy to check that $\leq_{\Gamma} \rightarrow \Gamma(S, \tau_{\leq_{\Gamma}})$ and $\Gamma \mapsto \leq_{\Gamma}$ yield inverse bijections between the set of partial orders on $S$ and the set $E(S)$.

**Lemma 2.4.** A finite $T_0$-space $X$ is connected if and only if $\Gamma(X)$ is connected as an undirected graph.

**Proof.** Assume first that $X$ is connected. Let $x_0 \in X$ and set

$$X_1 := \{x \in X \mid \text{there is an undirected path from } x_0 \text{ to } x \text{ in } \Gamma(X)\}.$$ 

Note that if $y \in \overline{x}$ then there is an undirected path from $x$ to $y$. Hence, if $x \in X_1$, then $\overline{x} \subseteq X_1$, therefore $\bigcup_{x \in X_1} \{x\} = X_1$ and $X_1$ is closed. On the other hand, if $x \not\in X_1$, then $\overline{x} \subset X \setminus X_1$, hence $X_1 = \bigcap_{x \not\in X_1} X \setminus \{x\}$ is open. Since $X$ is connected and $X_1$ is nonempty, we have $X = X_1$.

Now assume that $\Gamma(X)$ is connected as a graph and that $X = X_1 \cup X_2$ can be written as a disjoint union of nonempty clopen subsets $X_1$ and $X_2$. Let $x_i \in X_i$, $i = 1, 2$, and let $\rho$ be an undirected path from $x_1$ to $x_2$. We find neighbouring vertices $y_1$ and $y_2$ on the path $\rho$ such that $y_i \in X_i$ for $i = 1, 2$. Without loss of generality we may assume that $y_2 \in \{y_1\} \subseteq X_1$ which is a contradiction. \qed

3. FILTRATED K-theory

3.1. Equivariant KK-theory. As explained in [8] §3.1, there is a version of bivariant K-theory for $C^*$-algebras over $X$. Let $A, B \in \mathcal{C}^*\text{-}\text{sep}(X)$. A cycle in $KK_*(X; A, B)$ is given by a cycle $(E, T)$ for $KK_*(A, B)$ which is $X$-equivariant, that is, $A(U) \cdot E \subseteq E \cdot B(U)$ for all $U \in \mathcal{O}(X)$. There is also a Kasparov product

$$KK_*(X; A, B) \otimes KK_*(X; B, C) \to KK_*(X; A, C).$$

Thus we may define the category $\mathcal{R}(X)$ whose objects are separable $C^*$-algebras over $X$ and morphisms from $A$ to $B$ are given by $KK_0(X; A, B)$. As shown in [8] §3.2, $\mathcal{R}(X)$ carries all basic structures we would expect from a bivariant $K$-theory. In particular, it is additive, has countable coproducts, exterior products, satisfies Bott periodicity and has six-term exact sequences for semi-split extensions of $C^*$-algebras over $X$.

Moreover, $\mathcal{R}(X)$ carries the structure of the triangulated category ([8] §3.3); the suspension functor is given by the exterior product with $C_0(\mathbb{R})$ and a sequence $SB \to C \to A \to B$ is an exact triangle if and only if it is isomorphic to a mapping cone triangle $SB' \to C_\phi \to A' \to B'$ for some $X$-equivariant $*$-homomorphism $\phi: A' \to B'$.
The bootstrap class \( B(X) \) defined in [8, §4] is the localising subcategory of \( \mathfrak{R}(X) \) generated by the objects \( i_x C \) for all \( x \in X \). That is, it is the smallest class of objects containing these generators that is closed under suspensions, \( \text{KK}(X) \)-equivalence, semi-split extensions and countable direct sums. Here \( i_x C := i_x^X C \), where \( C \) is regarded as a \( C^* \)-algebra over the one-point space in the obvious way.

### 3.2. The definition of filtrated K-theory

We recall the definition of filtrated K-theory from [21, §4]. For each locally closed subset \( Y \subseteq X \), one defines a functor

\[
\text{FK}(X)_Y : \mathfrak{R}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2}, \quad \text{FK}(X)_Y(A) := K_*(A(Y)).
\]

These functors are stable and homological, that is, they intertwine the suspension on \( \mathfrak{R}(X) \) with the translation functor on \( \mathfrak{Ab}^{\mathbb{Z}/2} \) and they map exact triangles to long exact sequences.

Let \( \mathcal{N}(X) \) be the \( \mathbb{Z}/2 \)-graded category whose object set is \( \mathcal{LC}(X) \) and whose morphism space \( Y \to Z \) is \( \mathcal{N}(X)(Y,Z) \), the \( \mathbb{Z}/2 \)-graded Abelian group of all natural transformations \( \text{FK}_Y \Rightarrow \text{FK}_Z \).

A module over \( \mathcal{N}(X) \) is an additive, grading preserving functor \( G : \mathcal{N}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2} \). Let \( \mathfrak{Mod}(\mathcal{N}(X)) \) be the category of \( \mathcal{N}(X) \)-modules. The morphisms in \( \mathfrak{Mod}(\mathcal{N}(X)) \) are the natural transformations of functors or, equivalently, families of grading preserving group homomorphisms \( G_Y \to G'_Y \) that commute with the action of \( \mathcal{N}(X) \).

Let \( \mathfrak{Mod}(\mathcal{N}(X))_c \) be the full subcategory of countable modules.

Filtrated K-theory is the functor

\[
\text{FK}(X) = (\text{FK}(X)_Y)_{Y \in \mathcal{LC}(X)} : \mathfrak{R}(X) \to \mathfrak{Mod}(\mathcal{N}(X))_c, \quad A \mapsto \left( K_*(A(Y)) \right)_{Y \in \mathcal{LC}(X)}.
\]

To keep notation short, we often write \( \mathcal{N} \) for \( \mathcal{N}(X) \) and \( \text{FK} \) for \( \text{FK}(X) \).

**Remark 3.1.** Restriction to connected, non-empty locally closed subsets of \( X \) does not lose any relevant information: since \( X \) is finite, every subset of \( X \) is the finite union of its connected components. Moreover, this decomposition \( Y = \bigsqcup_{i \in \pi_0(Y)} Y_i \) into connected components corresponds to a biproduct decomposition \( Y \cong \bigoplus_{i \in \pi_0(Y)} Y_i \) in \( \mathcal{N} \) yielding a canonical isomorphism

\[
G(Y) \cong \bigoplus_{i \in \pi_0(Y)} G(Y_i)
\]

for all \( Y \in \mathcal{LC}(X) \) and \( G \in \mathfrak{Mod}(\mathcal{N}(X))_c \).

In particular, the empty subset of \( X \) is a zero object in \( \mathcal{N} \).

It follows that, denoting by \( \mathcal{N}^* \) the full subcategory of \( \mathcal{N} \) consisting of connected, non-empty locally closed subsets of \( X \), we have a canonical equivalence of categories

\[
\Upsilon : \mathfrak{Mod}(\mathcal{N}(X))_c \to \mathfrak{Mod}(\mathcal{N}^*)_c,
\]

which is just given by composing an \( \mathcal{N}(X) \)-module \( M : \mathcal{N}(X) \to \mathfrak{Ab}^{\mathbb{Z}/2} \) with the inclusion \( \mathcal{N}^* \hookrightarrow \mathcal{N}(X) \). A pseudo-inverse \( \Upsilon^{-1} \) is given by taking direct sums over connected components of objects, that is, by \( \Upsilon^{-1}(G)(Y) := \bigoplus_{i \in \pi_0(Y)} G(Y_i) \) on objects \( Y \in \mathcal{LC}(X) \), and a similar direct sum operation on morphisms. Hence, we can minimise our calculations by replacing filtrated K-theory with the reduced version \( \text{FK}^* := \Upsilon \circ \text{FK} \).

### 3.3. Functoriality

The canonical functor \( C^*_{\text{sep}}(X) \to \mathfrak{R}(X) \) is the universal split-exact, \( C^* \)-stable functor ([8 Theorem 3.7]). Using this universal property, we may extend the functoriality results for \( C^*_{\text{alg}}(X) \) in the space variable to \( \mathfrak{R}(X) \): a continuous map \( f : X \to Y \) induces a functor \( f_* : \mathfrak{R}(X) \to \mathfrak{R}(Y) \), in particular this yields an extension functor \( i_X^Y \) for a subspace \( X \subseteq Y \). Similarly, for \( Y \in \mathcal{LC}(X) \) the restriction functor descends to a functor \( r_Y^X : \mathfrak{R}(X) \to \mathfrak{R}(Y) \).
Our next aim is to construct an algebraic variant of $f_*$, that is, a functor

$$f_* : \text{Mod}(\mathcal{N}(X)) \rightarrow \text{Mod}(\mathcal{N}(Y))$$

such that

$$\mathcal{A}(X) \xrightarrow{\phi} f_* \xrightarrow{\mathcal{A}(Y)} \text{Mod}(\mathcal{N}(X))$$

commutes. Let us do so by first constructing a functor $f^* : \mathcal{N}(Y) \rightarrow \mathcal{N}(X)$.

For $Z \in \mathcal{N}(Y) = \mathcal{L}(Y)$ set $f^*(Z) = f^{-1}(Z)$. A morphism $\tau \in \mathcal{N}(Y)(Z, Z')$ is a natural transformation $\tau : \text{FK}(Y)_Z \rightarrow \text{FK}(Y)_{Z'}$, that is, a collection

$$\{\tau_A\}_{A \in \mathcal{A}(Y)}$$

of morphisms of Abelian groups

$$\tau_A : \text{FK}(Y)_Z(A) = K_*(A(Z)) \rightarrow K_*(A(Z')) = \text{FK}(Y)_{Z'}(A)$$

that is natural with respect to morphisms in $C^*\text{alg}(Y)$. For $B \in \mathcal{A}(X)$ and $Z \in \mathcal{L}(Y)$ we have

$$\text{FK}(Y)_Z(f_*B) = K_*(B(f^{-1}(Z))) = \text{FK}(X)_{f^{-1}(Z)}(B).$$

Hence $f_*B$ is also a morphism from $\text{FK}(X)_{f^{-1}(Z)}(B)$ to $\text{FK}(X)_{f^{-1}(Z')}(B)$ and it makes sense to define

$$f^*(\tau) := \{\tau_{f_*B}\}_{B \in \mathcal{A}(X)}.$$

We therefore have constructed an additive, grading preserving functor

$$f^* : \mathcal{N}(Y) \rightarrow \mathcal{N}(X).$$

This gives rise to an additive, grading preserving functor

$$f_* : \text{Mod}(\mathcal{N}(X)) \rightarrow \text{Mod}(\mathcal{N}(Y)),$$

$$f_*(M) := M \circ f^*.$$

**Lemma 3.3.** Let $X$, $Y$, $f$ and $f_*$ be as above. The diagram \((3.2)\) commutes.

**Proof.** Recall that there is a canonical functor $\text{KK}(X) : C^*\text{alg}(X) \rightarrow \mathcal{A}(X)$. By the universal property of $\text{KK}(X)$ (see [8, Theorem 3.7]) we see that it suffices to check that

$$f_* \circ \text{FK}(X) \circ \text{KK}(X) = \text{FK}^Y \circ f_* \circ \text{KK}(X).$$

On objects there is no difference anyway: let $A \in \mathcal{A}(X)$ and $Z \in \mathcal{L}(Y)$. Then

$$f_*(\text{FK}(X)(A)(Z)) = K_*(A(f^{-1}(Z))) = \text{FK}^Y \circ f_*(A)(Z).$$

Let $\phi : A \rightarrow B$ be a morphism of $C^*$-algebras over $X$ and $Z \in \mathcal{L}(Y)$. Passing to subquotients, $\phi$ induces a $^*$-homomorphism $\phi(Z') : A(Z') \rightarrow B(Z')$ for all $Z' \in \mathcal{L}(X)$.

The push-forward $f_*(\phi) : f_*^*(A) \rightarrow f_*^*(B)$ is a morphism of $C^*$-algebras over $Y$ which is given by $\phi$ as a $^*$-homomorphism from $A$ to $B$ if we forget the structure over $X$ (or $Y$). Note that $f_*(\phi)(Z) = \phi(f^{-1}(Z))$ as $^*$-homomorphisms. Now the equalities

$$f_*(\text{FK}(X)(\phi))(Z) = f_* \circ \text{FK}(X)([\phi])(Z)$$

$$= \text{FK}(X)([\phi])(f^{-1}(Z)) = K_*(\phi(f^{-1}(Z)))$$
and
\[FK(Y) \circ f_* \circ KK(X)(\phi)(Z) = FK(Y)([f_*(\phi)])(Z) = K_*(f_*(\phi)(Z)) = K_*(\phi(f^{-1}(Z)))\]
give the desired result. \(\square\)

3.4. Canonical transformations and relations. In this section we describe certain canonical elements and relations in the category \(\mathcal{NT}\). Moreover, we provide indecomposability criteria for these canonical transformations assuming that there are no “hidden” relations. The results we establish will be used for concrete computations in later chapters.

**Proposition 3.4.** Let \(U\) be a relatively open subset of a locally closed subset \(Y\) of \(X\). Then there are the following natural transformations:

(i) an even transformation
\[i^Y_U: FK_U \Rightarrow FK_Y\]
induced by the inclusion \(A(U) \hookrightarrow A(Y)\);

(ii) an even transformation
\[r^Y_{Y \setminus U}: FK_Y \Rightarrow FK_{Y \setminus U}\]
induced by the projection \(A(Y) \twoheadrightarrow A(Y \setminus U)\);

(iii) an odd transformation
\[\delta^U_{Y \setminus U}: FK_{Y \setminus U} \Rightarrow FK_U\]
defined as the six-term sequence boundary map
\[K_*(A(Y \setminus U)) \to K_{*+1}(A(U)).\]

**Proof.** This is a consequence of the naturality of the six-term sequence in K-theory associated to the ideal \(A(U) \triangleleft A(Y)\). \(\square\)

**Definition 3.5.** The natural transformations introduced in Proposition 3.4 are called canonical transformations in \(\mathcal{NT}\). We call \(i^Y_U\) an extension transformation, \(r^Y_{Y \setminus U}\) a restriction transformation and \(\delta^U_{Y \setminus U}\) a boundary transformation.

In all cases that have been investigated so far, the category \(\mathcal{NT}\) is generated by these canonical transformations. The absence of a general proof for this phenomenon motivates the following definition.

**Definition 3.6.** Let \(\mathcal{NT}_{6\text{-term}}\) be the subcategory of \(\mathcal{NT}\) generated by all canonical transformations, that is, by the set of morphisms
\[\bigcup_{Y \subset X \text{ locally closed}, \quad U \subset Y \text{ relatively open}} \{i^Y_U, r^Y_{Y \setminus U}, \delta^U_{Y \setminus U}\}.
\]
Let \(\mathcal{NT}_{\text{even 6-term}}\) be the subcategory of \(\mathcal{NT}_{6\text{-term}}\) generated by all even canonical transformations, that is, by the set of morphisms
\[\bigcup_{Y \subset X \text{ locally closed}, \quad U \subset Y \text{ relatively open}} \{i^Y_U, r^Y_{Y \setminus U}\}.
\]
According to our previous convention, the respective full subcategories with object set \(\mathcal{LC}(X)^*\) are denoted by \(\mathcal{NT}_{6\text{-term}}^*\) and \(\mathcal{NT}_{\text{even 6-term}}^*\). Similarly, \(\mathcal{NT}_{\text{even}}^*\) is the subcategory of \(\mathcal{NT}^*\) generated by even transformations.
Warning 3.7. The subcategory $\mathcal{N}T_{\text{even 6-term}}$ of $\mathcal{N}T_{\text{6-term}}$ need not exhaust the whole even part of $\mathcal{N}T_{\text{6-term}}$. However, this is true if any product of two odd natural transformations vanishes. This fails to be true for the four-point space $S$ defined in [7] which was investigated in [2, §6.2].

The manifest elements of $\mathcal{N}T$ we have just discussed fulfill some canonical relations, which we present in this section. The following proposition investigates compositions of even six-term sequence maps, that is, compositions in $\mathcal{N}T_{\text{even 6-term}}$.

**Proposition 3.8.** Let $Y$ be a locally closed subset of $X$.

(i) Let $U$ be a relatively open subset of $Y$ and let $V$ be a relatively open subset of $U$. Then $V$ is relatively open in $Y$ and

$$i_Y^V \circ i_U^V = i_Y^V.$$ 

Moreover, $i_Y^Y = \text{id}_Y$.

(ii) Let $C$ be a relatively closed subset of $Y$ and let $D$ be a relatively closed subset of $C$. Then $D$ is relatively closed in $Y$ and

$$r_C^D \circ r_C^Y = r_D^U.$$ 

Moreover, $r_Y^Y = \text{id}_Y$.

(iii) Let $U$ be a relatively open subset of $Y$ and let $C$ be a relatively closed subset of $Y$. Then $U \cap C$ is relatively closed in $U$ and relatively open in $C$, and

$$r_C^Y \circ i_Y^U = i_{U \cap C}^C \circ r_{U \cap C}^U = \sum_{Z \in \pi(U \cap C)} i_C^Z \circ r_Z^U.$$

In particular, if $U$ and $C$ are disjoint, then $i_C^Y \circ r_Y^U = 0$.

**Proof.** Assertions (i) and (ii) and the first equation in (iii) follow from respective identities on the $C^*$-algebraic level. The remaining assertions in (iii) follow from the biproduct decomposition in Remark 3.1 and the fact that the empty set is a zero object.

**Definition 3.9.** A morphism $Y \to Z$ in a category $\mathfrak{C}$ is called **indecomposable** if it cannot be written as a composite $Y \to W \to Z$ except for the trivial ways involving identity morphisms.

**Definition 3.10.** Let $Y \subset X$ be a subset. Since $X$ is finite there is a smallest open subset $\tilde{Y}$ of $X$ containing $Y$. This set is given by the intersection of all open subsets of $X$ containing $Y$.

We define the boundary operations corresponding to the usual and to the above closure operation by

$$\partial Y := Y \setminus Y \quad \text{and} \quad \tilde{\partial} Y := \tilde{Y} \setminus Y.$$ 

**Proposition 3.11.** Let $Y$ be a connected, locally closed subset of $X$. Suppose that the relations in $\mathcal{N}T_{\text{even 6-term}}$ are spanned by the canonical ones listed in Proposition 3.8.

(i) The natural transformation $i_Y^U$ for an open subset $U$ of $Y$ is indecomposable in $\mathcal{N}T_{\text{even 6-term}}$ if and only if $Y$ is of the form

$$U \cup y := U \cup \{ x \in X \mid x \succeq y, \text{ but } x \neq u \text{ for all } u \in U \}$$

for a maximal element $y$ of $\partial U$.

(ii) The natural transformation $r_C^Y$ for a closed subset $C$ of $Y$ is indecomposable in $\mathcal{N}T_{\text{even 6-term}}$ if and only if $Y$ is of the form

$$C \cup y := C \cup \{ x \in X \mid x \preceq y, \text{ but } x \neq c \text{ for all } c \in C \}$$

for a minimal element $y$ of $\tilde{\partial} C$. 

Proof. We prove (i). The second assertion follows in an analogous manner or by considering the dual partially ordered set of $X$.

Suppose that $i_U^Y$ is indecomposable in $\mathcal{NT}_{even}^*$-6-term. Then $U$ is a maximal connected proper open subset of $Y$ because otherwise $i_U^Y$ could be written as a composition of two proper extension transformations, that is, extension transformations which are not identity transformations.

We choose a minimal element $y$ of $\overline{U} \cap Y$. We may assume $y \in \partial U$ because otherwise $U$ were a proper clopen subset of $Y$ contradicting connectedness of $Y$. Moreover, $y$ is a maximal element of $\partial U$. To see this, assume that there is $z \in \partial U$ with $z \succ y$. Then $z \in Y$ because $Y$ is locally closed. Hence $U \cup \{z\} \cap Y$ is a proper connected open subset of $Y$ containing $U$ as a proper subset. This contradicts our previous observation that $U$ is a maximal connected proper open subset of $Y$.

We claim that $y$ is a least element of $Y$. Assume, conversely, that there is $w \in Y$ with $w \not\succ y$. Then $U \cup (\{y\} \cap Y)$ is a proper connected open subset of $Y$ containing $U$ as a proper subset, which again yields a contradiction. For this reason and since $Y$ contains $U$ as an open subset, we have $Y \subseteq U \cup y$.

Now we observe that $Y$ is closed in $U \cup y$—this holds for every connected locally closed subset of $U \cup y$ containing $U$. Hence $\overline{i_U^Y} = r_{U \cup y} \circ i_{U \cup y}^Y$, and the indecomposability of $i_U^Y$ implies $Y = U \cup y$.

For the converse implication, let $Y = U \cup y$ for a maximal element $y$ of $\partial U$. Then $U$ is a maximal connected proper open subset of $Y$ and hence $i_U^Y$ does not decompose as the composite of two proper extension transformations. On the other hand, $i_U^Y$ does not decompose as $r_W \circ i_U^Y$ with $Y \subseteq W$ either. To see this, we assume the opposite: let $W$ be a connected locally closed subset of $X$ containing $Y$ as a proper closed subset. Since $Y$ cannot be open in $W$, there are $w \in W \setminus Y$ and $y' \in Y$ with $w \succ y'$. Consequently, we either have $w \succ u$ for some $u \in U$, or $w \succ y$. But, since $w \not\in Y = U \cup y$, the inequality $w \succ y$ implies $w \succ u$ for some $u \in U$ as well. This follows from the definition of $U \cup y$. Thus $U$ is not open in $W$—a contradiction.

In the following, we examine the category $\mathcal{NT}_{6-term}$, so that boundary transformations come into play.

Definition 3.12. A boundary pair in $\mathcal{NT}$ is a pair $(U, C)$ of disjoint subsets $U, C \in \text{LC}(X)$ such that

- $U \cup C$ is locally closed,
- $U$ is relatively open in $U \cup C$,
- $C$ is relatively closed in $U \cup C$.

The third condition is of course redundant since it is equivalent to the second one. Since local closedness is preserved under finite intersections, $U$ and $C$ are locally closed. For each boundary pair we have the natural transformation $\delta_C^U$ defined in Proposition 3.11.

We begin by investigating compositions of boundary transformations with even six-term sequence transformations.

Proposition 3.13. Let $(U, C)$ be a boundary pair in $\mathcal{NT}$ and define $Y = U \cup C$.

(i) Let $C' \subseteq C$ be a relatively open subset. Then $U \cup C'$ is relatively open in $U \cup C$, the set $C'$ is relatively closed in $U \cup C'$, and we have

$$\delta_C^U \circ i_{C'}^U = \delta_C^{U'}.$$

(ii) Let $U' \subseteq U$ be a relatively closed subset. Then $U' \cup C$ is relatively closed in $U \cup C$, the set $U'$ is relatively open in $U' \cup C$, and

$$r_U^{U'} \circ \delta_C^U = \delta_C^{U'}.$$
Then in particular, by Proposition 3.13(ii) we have
\[ i_U^U \circ \delta_C = \delta_C^U. \]

Proof. This follows from the fact that K-theoretic boundary maps are natural with respect to morphisms of extensions. \( \square \)

It is, however, not true that every morphism of extensions decomposes as a composition of pullbacks and pushouts as above. To see this, consider the morphism
\[
\begin{array}{ccc}
A(U) & \to & A(Y) \to A(C) \\
\downarrow & & \downarrow \\
A(U') & \to & A(Y) \to A(C')
\end{array}
\]
for appropriate boundary pairs \((U, C)\) and \((U', C')\) in \(X\). This morphism need not split into pullbacks and pushouts because \(U \cup C\) need not be locally closed, and the union \(U' \cup C\) need not be disjoint. We phrase the relation corresponding to the above morphism in the following proposition.

**Proposition 3.15.** Let \((U, C)\) and \((U', C')\) be boundary pairs in \(N^T\) with \(U \cup C = U' \cup C'\), and such that \(U\) is an open subset of \(U'\) and \(C'\) is a closed subset of \(C\). Then
\[ i_U^U \circ \delta_C = \delta_C^U \circ r_C^U. \]
In particular, \(i_U^U \circ \delta_C = 0\) and \(\delta_C^U \circ r_C^U\) for every boundary pair \((U, C)\) in \(N^T\).

Proof. The first assertion follows from the naturality of the boundary map in K-theory with respect to the morphism of extensions (3.14). For the second assertion, consider the boundary pairs \((U \cup C, \emptyset)\) and \((\emptyset, U \cup C)\) and use that \(\emptyset\) is a zero object. \( \square \)

**Corollary 3.16.** Let \(Y, Z \in \mathcal{L}(X)\) such that \(W := Y \cap Z\) is open in \(Y\) and closed in \(Z\). If \(Y \cap Z\) is locally closed then
\[ \delta_W^Z \circ \delta_W = 0. \]

Proof. By Proposition 3.15(iii) we have \(\delta_W^Z = r_W^Z \circ \delta_W^Z\). Hence, by Proposition 3.15,
\[ \delta_W^Z \circ \delta_W = \left(\delta_W^Z \circ r_W^Z\right) \circ \delta_W^Z = 0. \] \( \square \)

**Proposition 3.17.** Let \(Y, Z \in \mathcal{L}(X)\).

(i) Let \(Y\) be a proper open subset of \(Y\). Let \(C_1, \ldots, C_k\) be the connected components of \(Y \setminus Z\). Then
\[ \sum_{j=1}^k \delta_C^W \circ r_C = 0. \]

(ii) Let \(Y\) be a proper closed subset of \(Z\). Let \(C_1, \ldots, C_k\) be the connected components of \(Z \setminus Y\). Then
\[ \sum_{j=1}^k \delta_C^W \circ r_C = 0. \]
Proof. Let $C := Y \setminus Z$. Then $A(C) = \prod_{j=1}^{k} A(C_j)$ for every $C^*$-algebra $A$ over $X$ and we get
\[ \sum_{j=1}^{k} \delta_Z^{C_j} \circ r_V^{C_j} = \delta_Z^{C} \circ \sum_{j=1}^{k} r_V^{C_j} \circ \delta_Z^{C_j} = \delta_Z^{C} \circ r_V^{C} = 0 \]
using Proposition 3.15. The second assertion follows analogously. \qed

**Definition 3.18.** Restricting to connected, non-empty locally closed closed subsets of $X$, Propositions 3.8, 3.13, 3.15 and 3.17 establish various relations in the category $\mathcal{NT}^*$. These are referred to as canonical relations in $\mathcal{NT}^*$.

**Remark 3.19.** Since we have ruled out the empty set as an object of the category $\mathcal{NT}^*$, the last relations of Propositions 3.8 and 3.15, respectively, become independent from the other relations in these propositions.

**Remark 3.20.** In §6 we will see that, for a finite space $W$ of type (A), all relations in the category $\mathcal{NT}^*(W)$ follow from these canonical relations. This is also true for all four-point spaces considered in §7.

In the following we make some definitions in order to describe the boundary pairs $(U, C)$ that correspond to indecomposable boundary transformations $\delta_U^{C}$ in $\mathcal{NT}^*$-term.

**Definition 3.21.** A boundary pair in $\mathcal{NT}^*$ is a boundary pair $(U, C)$ in $\mathcal{NT}$ such that $U$, $C$ and $U \cup C$ are connected.

**Definition 3.22.** For two boundary pairs $(U, C)$ and $(U', C')$ in $\mathcal{NT}^*$, we say that $(U', C')$ is an extension of $(U, C)$ if
- $U$ is a relatively closed subset of $U'$,
- $C$ is a relatively open subset of $C'$.

**Example 3.23.** Let $X = \{1, 2, 3, 4\}$ with Alexandrov topology given by the partial order generated by $1 \prec 2 \prec 3 \prec 4$:

\[
\begin{array}{cccc}
4 & 3 & 2 & 1
\end{array}
\]

Then $(\{3, 4\}, \{1, 2\})$ is an extension of $(\{3\}, \{2\})$.

**Lemma 3.24.** For an extension $(U', C')$ of $(U, C)$ we have the relation
\[ \delta_U^{C} = r_{U'}^{C'} \circ \delta_{U'}^{C'} \circ r_{C'}^{C} . \]

**Proof.** This follows immediately from Proposition 3.12(i) and (ii). \qed

**Definition 3.25.** A boundary pair in $\mathcal{NT}^*$ is called complete if it has no proper extension in $\mathcal{NT}^*$.

**Proposition 3.26.** A boundary pair $(U, C)$ in $\mathcal{NT}^*$ is complete if and only if $U$ is open and $C$ is closed.

**Proof.** Suppose that $U$ is open and $C$ is closed. Let $(V, D)$ be an extension of $(U, C)$. Then $U$ is clopen in $V$ and $C$ is clopen in $D$. Since $V$ and $D$ are connected we get $U = V$ and $C = D$.

Conversely, let $(U, C)$ be complete. Assume that $U$ is not open, so that there is $b \in \partial U$. Define $Y := U \cup C$ and $U' := U \cup (\{b\} \cap \partial Y) \supseteq U$. We show that $(U', C)$ is an extension of $(U, C)$.

Recall that a subset of $X$ is locally closed if and only if it is convex with respect to the specialisation preorder. The union $U' \cup C = Y \cup (\{b\} \cap \partial Y)$ is convex because $Y$ and $\{b\} \cap \partial Y$ are convex, and if $Y \ni y \prec x \prec z \in \{b\} \cap \partial Y$ for some
\[ x \in X \text{ then } y \prec x \prec b \text{ and thus } x \in \overline{\{b\}} \cap \tilde{Y} \subset U' \cup C. \] Note that the situation \( Y \ni y \succ x \succ z \in \overline{\{b\}} \cap \partial Y \) is impossible because \( Y \) is convex.

The subset \( C \subset U' \cup C \) is closed. Otherwise, there were \( c \in C \) and \( z \in \overline{\{b\}} \cap \partial Y \) with \( c \succ z \). Since \( z \in \partial Y \) there were \( y \in Y \) with \( z \succ y \), and we get the contradiction \( c \succ y \).

Up to now we have shown that \( (U',C) \) is a boundary pair in \( X \). It remains to show that \( U \) is closed in \( U' \). This is equivalent to \( \overline{\{b\}} \cap \partial Y \) being open in \( U' \). To see this, consider \( z \in \overline{\{b\}} \cap \partial Y \) and \( w \in U' \) with \( w \succ z \). Since \( z \in \partial Y \) there is \( y \in Y \) with \( y \succ z \). Now \( w \succ z \succ y \) implies \( w \not\in Y \) since \( Y \) is convex. Consequently \( w \in \overline{\{b\}} \cap \partial Y \).

This proves that \( (U',C) \) is an extension of \( (U,C) \). Finally, if \( C \) is not closed, we can construct an extension \( (U,C') \) of \( (U,C) \) in a similar fashion. \( \square \)

**Definition 3.27.** For two boundary pairs \((U,C)\) and \((U',C')\) in \( \mathcal{NT}^+ \) we say that \((U',C')\) is a sub-boundary pair of \((U,C)\) if

- \( U' \) is a (relatively open) subset of \( U \),
- \( C' \) is a (relatively closed) subset of \( C \),
- \( U' \cup C \) is relatively open in \( U \cup C \),
- \( U \cup C' \) is relatively closed in \( U \cup C \).

In fact, the assumptions that \( U' \) be relatively open in \( U \) and that \( C' \) be relatively closed in \( C \) are redundant.

**Example 3.28.** Let \( X = \{1, 2, 3, 4\} \) with Alexandrov topology given by the partial order generated by \( 1 \prec 3, 1 \prec 4, 2 \prec 4 \):

\[
\begin{array}{c}
3 \\
1 \\
4 \\
2
\end{array}
\]

Then \( \{4\}, \{1\} \) is a sub-boundary pair of \( \{2, 4\}, \{1, 3\} \).

**Lemma 3.29.** For a sub-boundary pair \((U',C')\) in \((U,C)\) we have the relation

\[
\delta_U^{C'} = \iota_U^{U'} \circ \delta_{U'}^{C'} \circ \iota_C^{U'}.
\]

**Proof.** By assumption \( U' \cup C \) is open in \( U \cup C \). This implies that \( U' \cup C' \) is open in \( U \cup C \) and thus \( \delta_U^{C'} = \iota_U^{U'} \circ \delta_{U'}^{C'} \circ \iota_C^{U'} \) by Proposition 3.13(iii). The second step follows from Proposition 3.13(iv). \( \square \)

**Definition 3.30.** A boundary pair in \( \mathcal{NT}^+ \) is called reduced if it has no proper sub-boundary pair in \( \mathcal{NT}^+ \).

**Proposition 3.31.** A boundary pair \((U,C)\) in \( \mathcal{NT}^+ \) is reduced if and only if \( \overline{U} \supset C \) and \( \tilde{C} \supset U \).

**Proof.** Suppose that \( \overline{U} \supset C \) and \( \tilde{C} \supset U \), and let \((V,D)\) be a sub-boundary pair of \( (U,C) \). Set \( Y := U \cup C \). Then, by definition, \( V \cup C \) is open in \( Y \) and hence \( V \cup C \supset \tilde{C} \cap Y \supset U \). This shows \( V = U \). Analogously, \( U \cup D \supset \overline{U} \cap Y \supset C \), so that \( C = D \).

Conversely, let \((U,C)\) be reduced. Assume that \( \tilde{C} \not\supset U \). Define \( U' := U \cap \tilde{C} \). Then \( \emptyset \neq U' \subset U \). We will show that \((U',C)\) is a sub-boundary pair of \((U,C)\)—this yields a contradiction to the reducedness of \((U,C)\).

The set \( U' \cup C = Y \cap \tilde{C} \) is locally closed as a finite intersection of locally closed subsets and connected because \( C \) is connected and \( C \subset Y \). Since \( C \) is closed in \( Y \) it is also closed in the subset \( U' \cup C \). This shows that \((U',C)\) is a boundary pair.

The subset \( U' \cup C = Y \cap \tilde{C} \) is open in \( Y \) because \( \tilde{C} \) is open in \( X \). Hence \((U',C)\) is a sub-boundary pair of \((U,C)\).

Assuming, on the other hand, that \( \overline{U} \not\supset C \), we find the sub-boundary pair \((U,C \cap \overline{U})\) of \((U,C)\). \( \square \)
Corollary 3.32. Let $(U, C)$ be a boundary pair in $\mathcal{NT}^*$. Suppose that the relations in $\mathcal{NT}^*_{6\text{-term}}$ are spanned by the canonical ones listed in Definition 3.18. Then the natural transformation $\delta^U_C$ is indecomposable in $\mathcal{NT}^*_{6\text{-term}}$ if and only if $U$ is open, $C$ is closed, $U \supset C$ and $C \supset U$.

Proof. Under the assumption that the relations in $\mathcal{NT}^*_{6\text{-term}}$ are spanned by the canonical ones, the natural transformation $\delta^U_C$ is indecomposable if and only if the boundary pair $(U, C)$ is complete and reduced. Hence the assertion follows from Propositions 3.26 and 3.31. Notice that the relations in Propositions 3.15 and 3.17 cannot be used to decompose the boundary transformation corresponding to a boundary pair. □

3.5. The representability theorem and its consequences. The representability theorem is a powerful tool. It enables us to describe the category $\mathcal{NT}$ by computing plain topological $K$-groups. We follow [7, §2.1].

Theorem 3.33 ([Representability Theorem [7, Theorem 2.5]). Let $Y$ be a locally closed subset of $X$. The functor $FK_Y$ is representable; more precisely, there are a separable $C^*$-algebra over $X$ such that $R_Y(Y)$ is unital and a natural isomorphism

$$KK_*(X; R_Y, \cdot) \cong FK_Y$$

given by

$$KK_*(X; R_Y, A) \to FK_Y(A), \quad f \mapsto f_*([1_{R_Y(Y)}])$$

for all $A \in \mathcal{R}(X)$. Here $[1_{R_Y(Y)}]$ is the class of the unit element $1_{R_Y(Y)}$ of $R_Y(Y)$ in $FK_Y(R_Y)$, and $f_* = FK_Y(f)$.

Let $\text{Ch}(X)$ denote the order complex corresponding to the specialisation preorder on $X$ as defined in [7, §2]. This order complex comes with two functions $m, M : \text{Ch}(X) \to X$ with the property that the map

$$(m, M) : \text{Ch}(X) \to X^{\text{op}} \times X$$

is continuous. Here $X^{\text{op}}$ denotes the topological space whose underlying set is $X$ and whose open subsets are the closed subsets of $X$.

The primitive ideal space of the commutative $C^*$-algebra

$$\mathcal{R} := C(\text{Ch}(X))$$

is $\text{Ch}(X)$. Hence the map $(m, M)$ turns $\mathcal{R}$ into a $C^*$-algebra over $X^{\text{op}} \times X$. For locally closed subsets $Y, Z$ of $X$, we define

$$S(Y, Z) := m^{-1}(Y) \cap M^{-1}(Z) \subset \text{Ch}(X).$$

This is a locally closed subset of $\text{Ch}(X)$.

Definition 3.34. Let $Y$ be a locally closed subset of $X$. We define $R_Y$ to be the restriction of $\mathcal{R}$ to $Y^{\text{op}} \times X$, regarded as a $C^*$-algebra over $X$ via the coordinate projection $Y^{\text{op}} \times X \to X$. More explicitly, we have

$$(3.35) \quad R_Y(Z) = R(Y^{\text{op}} \times Z) = C_0(S(Y, Z)).$$

Lemma 3.36 ([7, Lemma 2.14]). If $Y, Z \in \mathcal{L}(X)$, then

$$S(Y, Z) = \text{Ch}(\overline{Y} \cap \overline{Z}) \setminus \left(\text{Ch}(\overline{Y} \cap \overline{Z}) \cup \text{Ch}(\partial Y \cap \overline{Z})\right).$$

An application of the Yoneda Lemma yields graded Abelian group isomorphisms

$$\mathcal{NT}_*(Y, Z) \cong KK_*(X; R_Z, R_Y) \cong FK_Z(R_Y) \cong KK_*(R_Y(Z)) \cong K^*(S(Y, Z)).$$

However, it is not obvious how to express the composition of natural transformations

$$\mathcal{NT}_*(Y, Z) \times \mathcal{NT}_*(W, Y) \to \mathcal{NT}_*(W, Z)$$
directly in terms of these topological K-groups. In principle, it is of course always possible to lift elements back to the respective KK-groups and then compose them.

We have identified natural transformations $F_K Y \Rightarrow F_K Z$ with $KK(X)$-morphisms $\mathcal{R}_Z \Rightarrow \mathcal{R}_Y$ and with classes of vector bundles over the topological space $S(Y, Z)$. Now we explicitly describe the $F_K$- and $KK(X)$-elements corresponding under the above identifications to compositions of the natural transformations introduced in Proposition 3.3.

Let $Y \subseteq \mathcal{L}(X)$ and let $U \subseteq Y$ be an open subset. Then $U \times Z$ is a closed subset of $Y \times Z$ and $(Y \backslash U) \times Z$ is an open subset of $Y \times Z$ for every $Z \in \mathcal{L}(X)$. By (3.35), we have an extension of $C^*$-algebras $\mathcal{R}_{Y \backslash \cup}(Z) \Rightarrow \mathcal{R}_Y \Rightarrow \mathcal{R}_U(Z)$ for every $Z \in \mathcal{L}(X)$. This, in turn, is nothing but an extension $\mathcal{R}_{Y \backslash \cup} \Rightarrow \mathcal{R}_Y \Rightarrow \mathcal{R}_U$ of $C^*$-algebras over $X$. Since $\mathcal{R}_Y$ is commutative and therefore nuclear, this extension is semi-split and hence has a class in $KK_1(X; R_U, R_{Y \backslash \cup})$ which produces an exact triangle

\[
\begin{array}{c}
\Sigma \mathcal{R}_U \\
\mathcal{R}_{Y \backslash \cup} \\
\mathcal{R}_Y \\
\mathcal{R}_U
\end{array}
\]

in $\mathfrak{R}(X)$.

We recall a lemma from [7] which we will refine in the remainder of this section.

**Lemma 3.38** ([7] Lemma 2.19). Let $Y \subseteq \mathcal{L}(X)$, let $U \subseteq \mathcal{O}(Y)$, and set $C := Y \backslash U$. In the notation of Proposition 3.3 and within the meaning of the above correspondences,

(i) the transformation $i_U^Y : F_K U \Rightarrow F_K Y$ corresponds to the class of

$\mathcal{R}_Y \Rightarrow \mathcal{R}_U$

in $KK_0(X; \mathcal{R}_Y, \mathcal{R}_U)$ and to the class of the trivial rank-one vector bundle in $K^0(S(U, Y)) = K^0(Ch(U));$

(ii) the transformation $r_C^Y : F_K Y \Rightarrow F_K C$ corresponds to the class of

$\mathcal{R}_C \Rightarrow \mathcal{R}_Y$

in $KK_0(X; \mathcal{R}_C, \mathcal{R}_Y)$ and to the class of the trivial rank-one vector bundle in $K^0(S(Y, C)) = K^0(Ch(C));$

(iii) the transformation $\delta_C : F_K C \Rightarrow F_K U$ corresponds to the class of the extension

$\mathcal{R}_C \Rightarrow \mathcal{R}_Y \Rightarrow \mathcal{R}_U$

in $KK_1(X; \mathcal{R}_U, \mathcal{R}_C)$ and to the class $f^*(v) \in K^1(S(C, U)) = K^1(Ch(Y) \backslash (Ch(U) \cup Ch(C))$, where $v$ denotes a generator of the group $K^1((0, 1)) \cong \mathbb{Z}$ and $f$ is a continuous map $Ch(Y) \to [0, 1]$ with $f^{-1}(0) = Ch(U)$ and $f^{-1}(1) = Ch(C)$.

**Lemma 3.39.** Let $Y$ and $Z$ be locally closed subsets of $X$, and let $C \subseteq Y \cap Z$ be closed in $Y$ and open in $Z$. The transformation $i_C^Z \circ r_C^Z : F_K Y \Rightarrow F_K Z$ corresponds to the class of the composition $\mathcal{R}_Z \Rightarrow \mathcal{R}_C \Rightarrow \mathcal{R}_Y$ in $KK_0(X; \mathcal{R}_Z, \mathcal{R}_Y)$ and to the class $[\xi_C]$ in $K^0(S(Y, Z))$ which is induced by rank-one trivial vector bundle $\xi_C$ on the compact open subspace $Ch(C) \subseteq S(Y, Z)$.

**Proof.** It is a consequence of Lemma 3.38 that $i_C^Z \circ r_C^Z$ corresponds to the composition $\mathcal{R}_Z \Rightarrow \mathcal{R}_C \Rightarrow \mathcal{R}_Y$. Since $(r_C^Y)_{\mathcal{R}_Y} : \mathcal{R}_Y(Y) \to \mathcal{R}_Y(C)$ is the restriction $\mathcal{C}(Ch(Y)) \to \mathcal{C}(Ch(C))$ and $(i_C^Z)_{\mathcal{R}_Y} : \mathcal{R}_Y(C) \to \mathcal{R}_Y(Z)$ is the embedding $\mathcal{C}(Ch(C)) \to \mathcal{C}_0(S(Y, Z))$, the trivial rank-one bundle on $Ch(Y)$ is restricted to $Ch(C)$ and then extended by 0 to $S(Y, Z)$.

\[\square\]
Let $Y$ and $Z$ be locally closed subsets of $X$. Since the property of being relatively closed in $Y$ and relatively open in $Z$ is preserved under finite unions, there is a maximal subset $R(Y,Z)$ of $Y \cap Z$ with this property.

**Corollary 3.40.** The monomials $i_Z^\ast \circ r_Y^\ast$, where $Y$ and $Z$ are locally closed subsets of $X$, and $C$ is a connected component of $R(Y,Z)$, form a $\mathbb{Z}$-basis of the category $\mathcal{N}_\text{even}$. 

**Proof.** Every morphism in $\mathcal{N}_\text{even}$ is a $\mathbb{Z}$-linear combination of monomials in composable extension and restriction transformations. The relations given in Proposition 3.3.8 show that such a monomial can be rewritten as $i_Z^\ast \circ r_Y^\ast$ for locally closed subsets $D$, $Y$ and $Z$ of $X$, such that $D$ is a closed subset of $Y$ and an open subset of $Z$. In this case, $D$ is a clopen subset of $Y \cap Z$, and therefore a union of connected components of $R(Y,Z)$. Hence $i_Z^\ast \circ r_Y^\ast$ is the sum of the transformations $i_Z^\ast \circ r_Y^\ast$, where $C$ runs through the connected components of $R(Y,Z)$ contained in $D$.

The independence of the above-mentioned transformations is a consequence of Lemma 3.39 and the existence of a natural surjective homomorphism from $K^0$ onto the zeroth cohomology group with compact support, carrying the trivial rank-one bundles on compact connected components to independent generators: the dimension function.

**Corollary 3.41.** Let $Y$ and $Z$ be locally closed subsets of $X$, let $Y \cap Z$ be closed in $Y$ and open in $Z$, and let $n$ be the number of connected components of $Y \cap Z$. If $K^0(S(Y,Z)) \cong \mathbb{Z}^n$ then $\mathcal{N}_\text{even}(Y,Z)$ is generated by the natural transformations $i_Z^\ast \circ r_Y^\ast$ with $C \in \pi_0(Y \cap Z)$.

**Proof.** This follows from the observation that, in the above situation, the group $K^0(S(Y,Z)) = K^0(\text{Ch}(Y \cap Z))$ is generated by the classes of the trivial rank-one bundles $\xi_C$ on $\text{Ch}(C) \subset \text{Ch}(Y \cap Z)$ with $C \in \pi_0(Y \cap Z)$.

**Warning.** It is in general not true that the group $\mathcal{N}_\text{even}(C,U)$ for a boundary pair $(U,C)$ is generated by $\delta_U^\ast$ once it is isomorphic to $\mathbb{Z}$; a counterexample is given in [2, 3.3.19].

**Lemma 3.43.** Let $(U,C)$ be a boundary pair in $\mathcal{N}_\text{even}$, and let $U', C' \in \mathcal{LC}(X)$ such that $U$ is an open subset of $U'$ and $C$ is a closed subset of $C'$. The transformation $i_U^\ast \circ \delta_U^\ast \circ r_{C'}^\ast: FK_{C'} \Rightarrow FK_{U'}$ corresponds to the composition

\[
\mathcal{R}_{U'} \longrightarrow \mathcal{R}_U \longrightarrow \mathcal{R}_{C'} \longrightarrow \mathcal{R}_C
\]

and to the class $\left( i_{S(C')U'}^\ast \circ f \right)^\ast (v)$ in $K^1(S(C',U'))$, where $f$ is defined as in Lemma 3.3.8.

**Proof.** The first assertion follows from Lemma 3.3.8. For the second assertion, note that $S(C,U)$ is open in $S(C',U)$. This follows from the definition $S(Y,Z) := m^{-1}(Y) \cap M^{-1}(Z)$ because $m$ is continuous as a map from $\text{Ch}(X)$ to $X^{\text{op}}$. We get the following commutative diagram:

\[
\begin{array}{ccc}
K^0(S(C,C)) & \xrightarrow{\delta} & K^1(S(C,C)) \\
\downarrow & & \downarrow i \\
K^0(S(C',C')) & \xrightarrow{\iota} & K^0(S(C',C)) & \xrightarrow{\delta} & K^1(S(C',C)) & \xrightarrow{i} & K^1(S(C',U')).
\end{array}
\]

The class $[C_C] \in K^0(S(C',C'))$ is mapped to $[C_C] \in K^0(S(C,C))$, to $f^\ast(v) \in K^1(S(C,C))$ and, finally, to $\left( i_{S(C')U'}^\ast \circ f \right)^\ast (v) \in K^1(S(C',U'))$. □
Corollary 3.44. Let \((U, C)\) be a boundary pair in \(\mathcal{N}T\), and let \(U', C' \in \mathcal{L}C(X)\) such that \(U\) is an open subset of \(U'\) and \(C\) is a closed subset of \(C'\). Assume that \(K^1(S(C', U')) \cong \mathbb{Z}\) and further that the composition

\[
K^0(S(C, C)) \xrightarrow{\delta} K^1(S(C, U)) \xrightarrow{i} K^1(S(C', U'))
\]

maps the class of the trivial rank-one bundle in \(K^0(S(C, C))\) to a generator of \(K^1(S(C', U'))\). Then \(\mathcal{N}T_1(C', U')\) is generated by the composition \(i_{U'}^U \circ \delta_C^U \circ r_C\).

The above condition is fulfilled, in particular, when \(K^0(S(C, C))\) is isomorphic to \(\mathbb{Z}\) and the groups \(K^1(S(C, C) \cup S(C, U))\) and \(K^1(S(C', U') \setminus S(C, U))\) vanish.

Proof. The main assertion is a direct consequence of Lemma 3.43. The addendum follows from the exactness of the six-term sequence in K-theory. \(\square\)

4. The UCT criterion

Theorem 4.8 in [7] shows what is actually needed to obtain a UCT short exact sequence which computes \(KK(X, \mathbb{C})\) in terms of filtrated K-theory:

**Theorem 4.1.** Let \(A, B \in \mathcal{K} \mathcal{R}(X)\). Suppose that \(FK(X)(A) \in \mathcal{M}od(\mathcal{N}T(X))\) has a projective resolution of length 1 and that \(A \in \mathcal{B}(X)\). Then there are natural short exact sequences

\[
\text{Ext}_{\mathcal{N}T(X)}^j(FK(X)(A)[j + 1], FK(B)) \to KK_j(X; A, B)
\]

\[
\to \text{Hom}_{\mathcal{N}T(X)}(FK(X)(A)[j], FK(X)(B))
\]

for \(j \in \mathbb{Z}/2\), where \(\text{Hom}_{\mathcal{N}T(X)}\) and \(\text{Ext}_{\mathcal{N}T(X)}^j\) denote the morphism and extension groups in the Abelian category \(\mathcal{M}od(\mathcal{N}T(X))_C\).

Since we are interested in determining the class of spaces that allow for a UCT short exact sequence for filtrated K-theory, it makes sense to view the crucial assumption in the theorem above as a property of the space \(X\).

**Definition 4.2.** Let \(X\) be a finite \(T_0\)-space. We say that \(UCT(X)\) holds if for all \(A \in \mathcal{B}(X)\), \(FK(X)(A) \in \mathcal{M}od(\mathcal{N}T(X))_C\) has a projective resolution of length 1.

Let us also mention an important conclusion which can be drawn from the existence of a UCT short exact sequence.

**Corollary 4.3** ([7 Corollary 4.9]). Let \(A, B \in \mathcal{B}(X)\) and suppose that both \(FK(A)\) and \(FK(B)\) have projective resolutions of length 1 in \(\mathcal{M}od(\mathcal{N}T)_C\). Then any homomorphism \(FK(A) \to FK(B)\) in \(\mathcal{M}od(\mathcal{N}T)_C\) lifts to an element in \(KK_0(X; A, B)\), and an isomorphism \(FK(A) \cong FK(B)\) lifts to an isomorphism in \(\mathcal{B}(X)\).

As indicated in the introduction, the possibility of lifting isomorphisms in filtrated K-theory to isomorphisms in \(\mathcal{K} \mathcal{R}(X)\) is one of the main reasons why one is interested in a UCT short exact sequence. On the other hand, the impossibility of lifting isomorphisms in \(FK(X)\) can of course be viewed as an obstruction to the existence of a UCT short exact sequence.

**Definition 4.4.** Let \(X\) be a finite \(T_0\)-space. We say that \(\neg UCT(X)\) holds if there are \(A, B \in \mathcal{B}(X)\) such that \(A \not\cong B\) in \(\mathcal{K} \mathcal{R}(X)\) and \(FK(X)(A) \cong FK(X)(B)\) in \(\mathcal{M}od(\mathcal{N}T(X))_C\).

It is clear that there is no finite \(T_0\)-space such that both \(UCT(X)\) and \(\neg UCT(X)\) hold. Moreover, as suggested by the notation, we will show that for every such \(X\) either \(UCT(X)\) or \(\neg UCT(X)\) holds. We may restrict attention to connected spaces:
Lemma 4.5. Let $X$ be a finite $T_0$-space which is a disjoint union of topological spaces $X_1, \ldots, X_n$. Then $\text{UCT}(X)$ holds if and only if $\text{UCT}(X_i)$ holds for $i = 1, \ldots, n$. Similarly, $\neg\text{UCT}(X)$ holds if and only if $\neg\text{UCT}(X_i)$ holds for all $X_i$.

Proof. This is a consequence of the natural product decompositions $\mathfrak{R}(X) \cong \prod_{i=1}^{n} \mathfrak{R}(X_i)$ and $\mathcal{Mod}(\mathcal{N}^T(X)) \cong \prod_{i=1}^{n} \mathcal{Mod}(\mathcal{N}^T(X_i))$, which are compatible with filtrated K-theory. □

The next proposition roughly tells us that, if $X$ has a subspace for which there is no UCT, then there cannot exist a UCT for $X$ either.

Proposition 4.6. Let $X$ be a finite $T_0$-space.

(i) Let $Y \in \mathbb{L}(\mathbb{C}(X))$ such that $\neg\text{UCT}(Y)$ holds. Then $\neg\text{UCT}(X)$ holds as well.

(ii) Let $Y$ be a finite $T_0$-space and $f : X \to Y$, $g : Y \to X$ continuous maps with $f \circ g = \text{id}_Y$. Suppose that $\neg\text{UCT}(Y)$ holds. Then $\neg\text{UCT}(X)$ holds as well.

Proof. By assumption there are $A, B \in \mathcal{B}(Y)$ such that $A \not\subseteq B$ in $\mathfrak{R}(Y)$ and $\text{FK}(Y)(A) \cong \text{FK}(Y)(B)$. As already noted above, we have $\iota_X^X \circ \iota_X^X = \text{id}_X$ (see also [8] Lemma 2.20(c)); therefore $\iota_X^X(A) \not\subseteq \iota_X^X(B)$ in $\mathfrak{R}(X)$. Recall that $\iota_X^X$ is just $\iota_X$ for the embedding $\iota : Y \hookrightarrow X$. Hence $\text{FK}(X)(\iota_X^X(A)) = \iota_X(\text{FK}(Y)(A)) \cong \iota_X(\text{FK}(Y)(B)) = \text{FK}(X)(\iota_X^X(B))$. The bootstrap $\mathcal{B}(Y)$ is generated by $i_y^y \mathbb{C}, y \in Y$, and $\iota_y^y \circ i_y^y \mathbb{C} = i_y^y \mathbb{C}$; therefore $\iota_y^y \mathbb{B}(Y) \subseteq \mathcal{B}(X)$. This shows the first statement.

To prove the second statement let $A, B \in \mathcal{B}(Y)$ such that $A \not\subseteq B$ in $\mathfrak{R}(Y)$ and $\text{FK}(Y)(A) \cong \text{FK}(Y)(B)$. Since $f \circ g = \text{id}_{\mathfrak{R}(Y)}$ we have that $g_*(A) \not\subseteq g_*(B)$. $g_\ast \circ \text{FK}(Y) = \text{FK}(X) \circ g_\ast$ implies $\text{FK}(X)(g_*(A)) \cong \text{FK}(X)(g_*(B))$. Since $g_* i_y^y \mathbb{C} = i_{g(y)}^y \mathbb{C}$, we have $g_\ast \mathbb{B}(Y) \subseteq \mathcal{B}(X)$. □

5. Positive results

Let $W$ be a finite $T_0$-space of type (A). In this section we show that $\text{UCT}(W)$ holds, that is, we prove that the filtrated K-module $\text{FK}(A)$ has a projective resolution of length 1 in $\mathcal{Mod}(\mathcal{N}^T(W))$, for every $A \in \mathcal{B}(W)$. This proof was given in [2]; it relies on methods developed in [7]; our argument is based on a comparison of the category $\mathcal{N}^T(W)$ and the category $\mathcal{N}^T(\mathcal{O}_n)$ associated to the totally ordered space $\mathcal{O}_n$ of the same cardinality as $W$ (see Theorem 5.12).

Definition 5.1. For $Y \in \mathbb{L}(\mathbb{C}(W))^*$ we define the free $\mathcal{N}^T(W)$-module on $Y$ by $P_Y(Z) := \mathcal{N}T_*(Y, Z)$ for all $Z \in \mathbb{L}(\mathbb{C}(W))^*$.

An $\mathcal{N}^T(W)$-module is called free if it is isomorphic to a direct sum of degree-shifted free modules $P_Y[j], j \in \mathbb{Z}/2$.

Definition 5.2. An $\mathcal{N}^T(W)$-module $M$ is called exact if the $\mathbb{Z}/2$-graded chain complexes

$$
\cdots \to M(U) \xrightarrow{\delta^U_Y} M(Y) \xrightarrow{\iota_Y^Y/V} M(Y \setminus U) \xrightarrow{\delta^U_Y} M(U)[1] \to \cdots
$$

are exact for all $U, Y \in \mathbb{L}(\mathbb{C}(W))$ with $U$ open in $W$.

An $\mathcal{N}^T(W)$-module $M$ is called exact if the corresponding $\mathcal{N}^T(W)$-module $\mathcal{Y}^{-1}(M)$ is exact (see Remark 5.1).

Definition 5.3. Let $\mathcal{N}T_{\text{all}} \subset \mathcal{N}^T$ be the ideal generated by all natural transformations between different objects.

Let $\mathcal{N}T_{\text{all}} \subset \mathcal{N}^T(W)$ be the subgroup spanned by all identity transformations $\text{id}_Y$, that is, $\mathcal{N}T_{\text{all}} := \bigoplus_{Y \in \mathbb{L}(\mathbb{C}(W))} \mathbb{Z} \cdot \text{id}_Y$. This is, in fact, a semi-simple subring of $\mathcal{N}^T$. 

Definition 5.4. Let $M$ be an $\mathcal{N}^*\mathcal{T}$-module. We define

$$\mathcal{N}^*\mathcal{T}_{\text{nil}} \cdot M := \{ x \cdot m \mid x \in \mathcal{N}^*\mathcal{T}_{\text{nil}}, m \in M \}, \quad M_{\text{ss}} := M / \mathcal{N}^*\mathcal{T}_{\text{nil}} \cdot M.$$ 

An $\mathcal{N}^*\mathcal{T}$-module is called entry-free if $M(Y)$ is a free Abelian group for all $Y \in \mathcal{L}\mathcal{C}(W)^*.$

Lemma 5.5. Let $M$ be an $\mathcal{N}^*\mathcal{T}(W)$-module. The following assertions are equivalent:

1. $M$ is a free $\mathcal{N}^*\mathcal{T}(W)$-module.
2. $M$ is a projective $\mathcal{N}^*\mathcal{T}(W)$-module.
3. $M_{\text{ss}}(Y)$ is a free Abelian group for all $Y \in \mathcal{L}\mathcal{C}(W)^*$ and

$$\text{T}or^1_{\mathcal{N}^*\mathcal{T}(W)}(\mathcal{N}^*\mathcal{T}_{\text{ss}}, M) = 0.$$ 

4. $M$ is entry-free and exact.

Lemma 5.6. Let $M$ be a countable $\mathcal{N}^*\mathcal{T}(W)$-module. The following assertions are equivalent:

1. $M = \text{FK}^*(A)$ for some $A \in \mathcal{R}\mathcal{R}(W)$.
2. $M$ is exact.
3. $\text{T}or^2_{\mathcal{N}^*\mathcal{T}(W)}(\mathcal{N}^*\mathcal{T}_{\text{ss}}, M) = 0$ and $\text{T}or^1_{\mathcal{N}^*\mathcal{T}(W)}(\mathcal{N}^*\mathcal{T}_{\text{ss}}, M) = 0.$
4. $\text{T}or^2_{\mathcal{N}^*\mathcal{T}(W)}(\mathcal{N}^*\mathcal{T}_{\text{ss}}, M) = 0$ and $\text{T}or^1_{\mathcal{N}^*\mathcal{T}(W)}(\mathcal{N}^*\mathcal{T}_{\text{ss}}, M)$ is a free Abelian group.

5. $M$ has a free resolution of length 1 in $\mathcal{M}\mathcal{O}\mathcal{D}(\mathcal{N}^*\mathcal{T}(W))_c.$

6. $M$ has a projective resolution of length 1 in $\mathcal{M}\mathcal{O}\mathcal{D}(\mathcal{N}^*\mathcal{T}(W))_c.$

7. $M$ has a projective resolution of finite length in $\mathcal{M}\mathcal{O}\mathcal{D}(\mathcal{N}^*\mathcal{T}(W))_c.$

Remark 5.7. In [7], Meyer and Nest prove these lemmas for the special case of the totally ordered space $W = O_n.$ In Lemma 5.5, we have replaced condition (3) from [7, Theorem 4.14] by two conditions which we are able to prove equivalent to the rest. We remark that (4) and (5) in Lemma 5.6 are equivalent even for underlying spaces that only have Property 1 below.

An investigation of the proofs in [7] shows that the only properties of the category $\mathcal{N}^*(O_n)$ Meyer and Nest actually use are the following (we formulate these properties for our general type $(A)$ space $W$ as underlying space, because we will show in Theorem 5.7, 12 that they are indeed present in this generality):

Property 1. The ideal $\mathcal{N}^*\mathcal{T}_{\text{nil}}$ is nilpotent and the ring $\mathcal{N}^*\mathcal{T}(W)$ decomposes as the semi-direct product

$$\mathcal{N}^*\mathcal{T}(W) = \mathcal{N}^*\mathcal{T}_{\text{nil}} \rtimes \mathcal{N}^*\mathcal{T}_{\text{ss}}.$$ 

This semi-direct product decomposition just means that $\mathcal{N}^*\mathcal{T}_{\text{nil}}$ is an ideal, $\mathcal{N}^*\mathcal{T}_{\text{ss}}$ is a subring, and $\mathcal{N}^*\mathcal{T}(W) = \mathcal{N}^*\mathcal{T}_{\text{nil}} \oplus \mathcal{N}^*\mathcal{T}_{\text{ss}}$ as Abelian groups. Notice that in this case we have $M_{\text{ss}} = \mathcal{N}^*\mathcal{T}_{\text{ss}} \otimes \mathcal{N}^*\mathcal{T}(W) M.$

Property 2. The Abelian group $\mathcal{N}^*\mathcal{T}(W)(Y, Z)$ is free for all $Y, Z \in \mathcal{L}\mathcal{C}(W)^*.$

Property 3. For every $Y \in \mathcal{L}\mathcal{C}(W)^*$ there is $Z \in \mathcal{L}\mathcal{C}(W)$ and a natural transformation $\nu \in \mathcal{N}^*\mathcal{T}(W)(Y, Z)$ such that

$$\langle \mathcal{N}^*\mathcal{T}_{\text{nil}} \cdot M \rangle(Y) = \ker(\nu: M(Y) \to M(Z))$$ 

holds for every exact $\mathcal{N}^*\mathcal{T}(W)$-module $M.$

Here, $M$ is regarded as an $\mathcal{N}^*\mathcal{T}(W)$-module in the canonical way described in Remark 5.1, so that the action of $\nu \in \mathcal{N}^*\mathcal{T}(W)(Y, Z)$ is well-defined also if $Z$ is not connected.
In the following, we prove Lemma 5.5 and Lemma 5.6 using only the properties listed above. Afterwards we will see that the category $\mathcal{N}T^*$ has these properties if $W$ is of type (A). We often abbreviate $\mathcal{N}T^*(W)$ by $\mathcal{N}T^*$ in the following.

Proof of Lemma 5.5 using Properties 1 2 3 Let $Y \in \mathcal{L}\mathcal{C}(W)^*$. Yoneda’s Lemma implies $\text{Hom}_{\mathcal{N}T^*}(P_Y, M) \cong M(Y)$ for all $\mathcal{N}T^*$-modules $M$. This shows that the functor $\text{Hom}_{\mathcal{N}T^*}(P_Y, \_)$ is exact, which means that $P_Y$ is projective. Since projectivity is preserved by direct sums, every free $\mathcal{N}T^*$-module is projective, that is, (1)$\implies$(2).

If $M$ is a projective $\mathcal{N}T^*$-module, then $M_{ss} = \mathcal{N}T_{ss} \otimes_{\mathcal{N}T^*} M$ is a projective $\mathcal{N}T_{ss}$-module. Since $\mathcal{N}T_{ss} \cong \mathcal{Z}_{\mathcal{L}\mathcal{C}(W)^*}$, this shows that $M_{ss}(Y)$ is a projective and thus free Abelian group for every $Y \in \mathcal{L}\mathcal{C}(W)^*$. We have $\text{Tor}_1^{\mathcal{N}T^*}(\mathcal{N}T_{ss}, M) = 0$ because $M$ is projective. Altogether, we get (2)$\implies$(3).

Now we prove (3)$\implies$(1). For this we need the following proposition.

Proposition 5.8. In the presence of Property 1 let $M$ be an $\mathcal{N}T^*$-module with $M_{ss} = 0$. Then $M = 0$.

Proof. If $M_{ss} = 0$ then $M = \mathcal{N}T_{nil}(\cdot) \cdot M$ and hence $M = \mathcal{N}T_{j}\cdot M$ for all $j \in \mathbb{N}$. This implies $M = 0$ since $\mathcal{N}T_{nil}$ is nilpotent.

The module $M_{ss}$ is free over $\mathcal{N}T_{ss} \cong \mathcal{Z}_{\mathcal{L}\mathcal{C}(W)^*}$ because $M_{ss}(Y)$ is free for all $Y \in \mathcal{L}\mathcal{C}(W)^*$. Hence $P := \mathcal{N}T \otimes_{\mathcal{N}T_{ss}} M_{ss}$ is a free $\mathcal{N}T^*$-module. Since $M_{ss}$ is free over $\mathcal{N}T_{ss}$, the projection $M \twoheadrightarrow M_{ss} = M/\mathcal{N}T_{nil} \cdot M$ splits by an $\mathcal{N}T_{ss}$-module homomorphism. This induces an $\mathcal{N}T^*$-module homomorphism $f : P \to M$ (by tensoring over $\mathcal{N}T_{ss}$ with the identity on $\mathcal{N}T^*$ and composing with the multiplication map from $\mathcal{N}T^* \otimes_{\mathcal{N}T_{ss}} M$ to $M$). We will show that $f$ is invertible, which implies that $M \cong P$ is free over $\mathcal{N}T^*$.

We have an isomorphism $P_{ss} \cong \mathcal{N}T_{ss} \otimes_{\mathcal{N}T^*} M \cong M_{ss}$, which is induced by $f : P \to M$. Using the right-exactness of the functor $M \mapsto M_{ss} = \mathcal{N}T_{ss} \otimes_{\mathcal{N}T^*} M$, we find $\text{coker}(f)_{ss} = \text{coker}(f_{ss}) = 0$ and hence $\text{coker}(f) = 0$ by Proposition 5.8. Therefore, $f$ is surjective. Since $P$ is projective the extension $\text{ker}(f) \hookrightarrow P \twoheadrightarrow M$ induces the following long exact Tor-sequence:

\[(5.9) \quad 0 \to \text{Tor}_1^{\mathcal{N}T^*}(\mathcal{N}T_{ss}, M) \to \text{ker}(f)_{ss} \to P_{ss} \cong M_{ss} \to 0.\]

Notice that $\text{Tor}_1^{\mathcal{N}T^*}(\mathcal{N}T_{ss}, P) = 0$ since $P$ is projective. Therefore, the assumption $\text{Tor}_1^{\mathcal{N}T^*}(\mathcal{N}T_{ss}, M) = 0$ implies $\text{ker}(f)_{ss} = 0$, and hence $\text{ker}(f) = 0$ by Proposition 5.8. Therefore, $f$ is invertible.

Up to now we have shown the equivalence of the first three conditions using only Property 1. The implication (1)$\implies$(4) follows from Property 2 and from the fact that free modules are exact. This can be seen as follows: let $U$ be an open subset of a locally closed subset $Y$ of $W$. We have the exact triangle 3.37

\[\Sigma\mathcal{R}_U \longrightarrow \mathcal{R}_{Y \setminus U} \longrightarrow \mathcal{R}_Y \longrightarrow \mathcal{R}_U,\]

which induces the long exact sequence

\[\cdots \to \text{KK}_s(W; \mathcal{R}_U, A) \to \text{KK}_s(W; \mathcal{R}_Y, A) \to \cdots\]

for all $A \in \mathcal{R}(W)$. In particular, when $A = \mathcal{R}_V$ for some $V \in \mathcal{L}\mathcal{C}(W)^*$, by the Representability Theorem 5.33 and Yoneda’s Lemma this sequence translates to the sequence

\[\cdots \to \mathcal{N}T_*(V, U) \to \mathcal{N}T_*(V, Y) \to \mathcal{N}T_*(V, Y \setminus U) \to \mathcal{N}T_{s+1}(V, U) \to \cdots,\]
proving the desired exactness. Notice that exactness is preserved by direct sums and degree-shifting, so that indeed every free \(N'T^*\)-module is exact.

We complete the proof by showing (4)\(\implies\)(3). By Property \[\ref{item:1} \text{ and } \ref{item:2}\] \(M_{\text{ss}}(Y)\) is isomorphic to a subgroup of \(M(Z)\) for some \(Z \in \mathbb{L}C(W)\) and hence a free Abelian group by assumption because \(M(Z) = \bigoplus_{C \in \pi_0(Z)} M(C)\). The assertion now follows from Proposition \[\ref{proposition:5\text{-}10}\].

**Proposition 5.10.** Let \(M\) be an exact \(N'T^*\)-module. If Properties \[\ref{item:1} \text{ and } \ref{item:2}\] are fulfilled, then \(\text{Tor}_1^{N'T^*}(N'T_{\text{ss}}, M) = 0\).

**Proof.** Choose an epimorphism \(f : P \rightarrow M\) with a projective \(N'T^*\)-module \(P\). We get the long exact sequence (5.9). We have seen that any projective \(N'T^*\)-module is free and thus exact. Hence \(P\) is exact. By the two-out-of-three property, \(\ker(f)\) is exact as well. Using Property \[\ref{item:3}\] we identify \(\ker(f)_{\text{ss}}(Y)\) and \(P_{\text{ss}}(Y)\) with subgroups of \(\ker(f)(Z)\) and \(P(Z)\) for some \(Z \in \mathbb{L}C(W)\). Therefore, the injectivity of the map \(\ker(f)(Z) \rightarrow P(Z)\) implies the injectivity of the map \(\ker(f)_{\text{ss}}(Y) \rightarrow P_{\text{ss}}(Y)\). This shows that \(\ker(f)_{\text{ss}} \rightarrow P_{\text{ss}}\) is a monomorphism and hence that \(\text{Tor}_1^{N'T^*}(N'T_{\text{ss}}, M) = 0\) by (5.9). \(\square\)

**Proof of Lemma \[\ref{lemma:5\text{-}11}\] using Lemma \[\ref{lemma:5\text{-}9}\] and Properties \[\ref{item:1} \text{ and } \ref{item:2}\] The exactness of the six-term sequence yields (1)\(\implies\)(2). The implication (5)\(\implies\)(1) follows from [7, Theorem 4.11], and the implications (3)\(\implies\)(4) and (5)\(\implies\)(6)\(\implies\)(7) are trivial. We will complete the proof by showing (7)\(\implies\)(2), (2)\(\implies\)(5), and (4)\(\implies\)(5)\(\implies\)(3).

For (7)\(\implies\)(2), let \(0 \rightarrow P_m \rightarrow \cdots \rightarrow P_0 \rightarrow M\) be a projective resolution. Define \(Z_j := \ker(P_j \rightarrow P_{j-1}) = \text{im}(P_{j+1} \rightarrow P_j)\). Then \(P_j/Z_j \cong \text{im}(P_j \rightarrow P_{j-1}) = Z_{j-1}\), yielding the short exact sequences \(Z_j \rightarrow P_j \rightarrow Z_{j-1}\). Starting with \(Z_m = 0\), the two-out-of-three property applied to the extensions \(Z_j \rightarrow P_j \rightarrow Z_{j-1}\) inductively implies the exactness of \(Z_j\) for \(j = m-1, m-2, \ldots, 0\). Hence \(M \cong P_0/Z_0\) is exact as well.

In order to prove (2)\(\implies\)(5), we choose an epimorphism \(P \rightarrow M\) with a countable free \(N'T^*\)-module \(P\), and set \(K := \ker(P \rightarrow M)\). By the two-out-of-three property, \(K\) is exact. Since \(P\) is a free \(N'T^*\)-module, its entries are free Abelian groups by Lemma \[\ref{lemma:5\text{-}9}\]. This property is inherited by the submodule \(K\). Hence \(K\) is free, again by Lemma \[\ref{lemma:5\text{-}9}\] and \(0 \rightarrow K \rightarrow P \rightarrow M\) is a free resolution of length 1.

Now we show (4)\(\implies\)(5). Choose an epimorphism \(P \rightarrow M\) with a countable free \(N'T^*\)-module \(P\), and set \(K := \ker(P \rightarrow M)\). Since \(K\) is a first syzygy of \(M\), the assumption \(\text{Tor}_2^{N'T^*}(N'T_{\text{ss}}, M) = 0\) implies \(\text{Tor}_2^{N'T^*}(N'T_{\text{ss}}, K) = 0\). The long exact sequence (5.10) shows that \(K_{\text{ss}}\) is an extension of free Abelian groups and thus has free entries itself. By Lemma \[\ref{lemma:5\text{-}9}\] the \(N'T^*\)-module \(K\) is free, and \(0 \rightarrow K \rightarrow P \rightarrow M\) is a free resolution of length 1.

Finally, we prove (5)\(\implies\)(3). We have already established the implication (5)\(\implies\)(2). Hence \(M\) is exact, and Proposition \[\ref{proposition:5\text{-}10}\] shows that \(\text{Tor}_1^{N'T^*}(N'T_{\text{ss}}, M) = 0\). The \(N'T^*\)-module \(\text{Tor}_2^{N'T^*}(N'T_{\text{ss}}, M)\) also vanishes because, by (4), the flat dimension of \(M\) is at most 1. \(\square\)

We now introduce ungraded \(N'T\)-modules and use them to formulate a central observation.

**Definition 5.11.** Let \(\text{Mod}^{\text{ungr}}(N'T^*(W))_c\) denote the category of ungraded, countable \(N'T^*(W)\)-modules, that is, of additive functors from \(N'T^*(W)\) to the category of countable Abelian groups. As in Remark \[\ref{remark:5\text{-}11}\] there is a forgetful functor

\[\mathcal{T} : \text{Mod}^{\text{ungr}}(N'T(W))_c \rightarrow \text{Mod}^{\text{ungr}}(N'T^*(W))_c\]
and a pseudo-inverse $\Upsilon^{-1}$. An ungraded module $M \in \text{Mod}^{\text{ungr}}(\mathcal{NT}(W))_c$ is called exact if the chain complexes
\[
\cdots \to M(U) \overset{\iota^U}{\to} M(Y) \overset{\delta^U}{\to} M(Y \setminus U) \overset{\delta^U}{\to} M(U) \to \cdots
\]
are exact for all $U, Y \in \mathcal{L}(W)$ with $U$ open in $Y$.

An ungraded module $M \in \text{Mod}^{\text{ungr}}(\mathcal{NT}^*(W))_c$ is called exact if $\Upsilon^{-1}(M)$ is an exact $\mathcal{NT}(W)$-module.

As mentioned above, Meyer and Nest verified Properties 1, 2 and 3 for the special case of the totally ordered space $O_n$. The key observation made in Propositions 5.12 and 1.1 allowing to generalise this to a general space of type (A) is the following:

**Theorem 5.12.** Let $W$ be a finite $T_0$-space of type $(A)$. Let $n$ be the number of points in $W$, and let $O_n$ denote the totally ordered space with $n$ points. There is an (ungraded) isomorphism $\Phi \colon \mathcal{NT}^*(W) \to \mathcal{NT}^*(O_n)$, and $\Phi^* \colon \text{Mod}^{\text{ungr}}(\mathcal{NT}^*(O_n))_c \to \text{Mod}^{\text{ungr}}(\mathcal{NT}^*(W))_c$ restricts to a bijective correspondence between exact ungraded $\mathcal{NT}^*(O_n)$-modules and exact ungraded $\mathcal{NT}^*(W)$-modules. Moreover, the isomorphism $\Phi$ restricts to isomorphisms from $\mathcal{NT}_{ss}(W)$ onto $\mathcal{NT}_{ss}(O_n)$ and from $\mathcal{NT}_{nil}(W)$ onto $\mathcal{NT}_{nil}(O_n)$.

We postpone the proof of Theorem 5.12 to §6. Combining Theorem 4.1 and Lemma 5.6 we obtain the desired UCT:

**Theorem 5.13.** Let $W$ be a finite $T_0$-space of type $(A)$. Then $UCT(W)$ holds.

**Proof.** It remains to verify Properties 1, 2 and 3 for the category $\mathcal{NT}^*(W)$. This follows from Theorem 5.12 once it is done for $\mathcal{NT}^*(O_n)$—this has been accomplished in [7].

6. **Proof of Theorem 5.12**

We introduce a more explicit notation for the type $(A)$ space $W$ which involves certain parameters, namely, an even natural number $m$ and natural numbers $n_1, \ldots, n_m$. We number the underlying set of $W$ as

\[
\begin{align*}
\{ & 1^0 = 1, 2^1, \ldots, (n_1 - 1)^1, n_1^1 = n_2^2, (n_2 - 1)^2, \ldots, 2^2, 1^2 = 1^3, \\
& 2^3, \ldots, (n_3 - 1)^3, n_3^3 = n_4^4, (n_4 - 1)^4, \ldots, 2^4, 1^4 = 1^5, \\
& \vdots \\
& 2^{m-1}, \ldots, (n_{m-1} - 1)^{m-1}, n_{m-1}^{m-1} = n_m^m, \\
& (n_m - 1)^m, \ldots, 2^m, 1^m = 1^{m+1} \},
\end{align*}
\]

such that the specialisation order corresponding to the topology on $W$ is generated by the relations

\[
\begin{align*}
1^1 & \prec 2^1 \prec \ldots \prec (n_1 - 1)^1 \prec n_1^1 = n_2^2 \prec (n_2 - 1)^2 \prec \ldots \prec 2^2 \prec 1^2 = 1^3, \\
1^3 & \prec 2^3 \prec \ldots \prec (n_3 - 1)^3 \prec n_3^3 = n_4^4 \prec (n_4 - 1)^4 \prec \ldots \prec 2^4 \prec 1^4 = 1^5, \\
& \vdots \\
1^{m-1} & \prec 2^{m-1} \prec \ldots \prec (n_{m-1} - 1)^{m-1} \prec n_{m-1}^{m-1} = n_m^m, \\
& n_m^m \prec (n_m - 1)^m \prec \ldots \prec 2^m \prec 1^m = 1^{m+1}.
\end{align*}
\]

Without loss of generality, we can assume that the numbers $n_2, \ldots, n_{m-1}$ are larger than 1. This makes the description of the space $W$ by the parameters $m$ and $n_1, \ldots, n_m$ easier.
n_1, \ldots, n_m$ unique up to reversion of the order of the superscripts. The total number of points in $W$ is $n := \sum_{i=1}^{m} n_i - (m - 1)$.

The specialisation order on the topological space $W$ corresponds to the directed graph displayed in Figure 1.

**Figure 1.** Directed graph corresponding to the type (A) space $W$

6.1. **Computation of the groups of natural transformations.** The order complex $\text{Ch}(W)$ is a union of simplices $\Delta_k$, $k = 1, \ldots, m$, of dimensions $n_k - 1$. For $i < j$ the intersection $\Delta_i \cap \Delta_j$ is a point if $i + 1 = j$, and otherwise is empty.

The connected, locally closed subsets of $W$ are exactly the “chain-like” subsets. In order to define them, we introduce a total order $\leq$ on $W$:

$$a^i \leq b^j :\iff \begin{cases} \{i < j\} & \text{or} \\ \{i = j \text{ is odd and } a \preceq b\} & \text{or} \\ \{i = j \text{ is even and } a \succeq b\}. \end{cases}$$

This means, that $x \preceq y$ exactly if $y$ is “further down” in Figure 1 than $x$. 
We observe that the number of elements in $\mathcal{LC}(W)^*$ is \( \frac{n(n+1)}{2} \). The next step is to compute for a connected locally closed subset $Y = \langle a^i, b^j \rangle$ the two closures $\tilde{Y}$ and $\overline{Y}$ and the corresponding boundaries defined in Definition 3.10. For this computation we do a case differentiation with respect to the parity of the numbers $i$ and $j$. The result is given in Table 1.

| $\langle a^i, b^j \rangle$ | $i$ and $j$ odd, $a \neq 1, b \neq n_j$ | $i$ odd, $j$ even, $a \neq 1, b \neq 1$ | $i$ even, $j$ odd, $a \neq n_i, b \neq n_j$ | $i$ and $j$ even, $a \neq n_i, b \neq 1$ |
|--------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\langle a^i, b^j \rangle$ | $\langle 1^i, b^j \rangle$ | $\langle 1^i, 1^j \rangle$ | $\langle a^i, b^j \rangle$ | $\langle a^i, 1^j \rangle$ |
| $\overline{\langle a^i, b^j \rangle}$ | $\langle 1^i, a^j \rangle$ | $\langle 1^i, a^j \cup b^j, 1^j \rangle$ | $\emptyset$ | $\langle b^j, 1^j \rangle$ |
| $\overline{\langle a^i, b^j \rangle}$ | $\langle a^i, n_j^i \rangle$ | $\langle a^i, b^j \rangle$ | $\langle n_i^j, n_j^i \rangle$ | $\langle n_i^j, b^j \rangle$ |
| $\partial \overline{\langle a^i, b^j \rangle}$ | $\langle 1^i, a^j \rangle$ | $\emptyset$ | $\langle a^i, a^j \cup b^j, n_j^i \rangle$ | $\langle n_i^j, a^j \rangle$ |

Table 1. Closures and boundaries of locally closed subsets of the space $W$

Now let $Y = \langle a_1^i, b_1^j \rangle$ and $Z = \langle a_2^k, b_2^l \rangle$ be connected, locally closed subsets of $W$. We calculate $S(Y, Z) = \mathrm{Ch}(\tilde{Y} \cap Z) \setminus \left( \mathrm{Ch}(Y \cap Z) \cup \mathrm{Ch}(\partial Y \cap Z) \right)$ and the associated K-groups (which describe the category $\mathcal{NT}$) by distinguishing six cases concerning the order of the points $a_1^i, b_1^j, a_2^k$ and $b_2^l$ with respect to $\leq$, and subcases concerning the parity of the numbers $i$, $j$, $k$, and $l$.

The cases 1b, 2b, 3b are very similar to the cases 1a, 2a, and 3a, respectively. Therefore, we only give the results for them without repeating the arguments. For the sake of clarity, we provide small sketches of the relative location of the sets $Y$ and $Z$.

**Case 1a** $a_1^i \leq a_2^k \leq b_1^j \leq b_2^l$

(i) Let $j$ and $k$ be even, $b_1 \neq 1$ and $a_2 \neq n_k$.

Then $S(Y, Z) = \mathrm{Ch}\left( \langle a_2^k, b_1^j \rangle \right)$ is contractible. Thus $K^*(S(Y, Z)) \cong \mathbb{Z}[0]$. 
(ii) Let $j$ be even, $k$ odd, $b_1 \neq 1$ and $a_2 \neq 1$.

\[
\begin{array}{c}
Y \\
\searrow \\
Z
\end{array}
\]

- For $a_1 = a_2^k$ the space $S(Y, Z) = \text{Ch}(Y)$ is contractible and thus $K^*(S(Y, Z)) \cong \mathbb{Z}[0]$.
- If $a_1^j < a_2^k$ then
  \[S(Y, Z) = \text{Ch}((1^k, b_1^j)) \setminus \text{Ch}((1^k, a_2^k))\]
  for $a_1^j \leq 1^k$, and
  \[S(Y, Z) = \text{Ch}((a_1^j, b_1^j)) \setminus \text{Ch}((a_1^j, a_2^k))\]
  otherwise. This is the difference of a contractible compact pair and we have $K^*(S(Y, Z)) = 0$.

(iii) Let $j$ be odd, $k$ even, $b_1 \neq n_j$ and $a_2 \neq n_k$.

\[
\begin{array}{c}
Y \\
\searrow \\
Z
\end{array}
\]

Analogously to (ii), we obtain $K^*(S(Y, Z)) \cong \mathbb{Z}[0]$ for $b_1 = b_2^k$, and $K^*(S(Y, Z)) = 0$ for $b_1^k < b_2^k$.

(iv) Let $j$ and $k$ be odd, $b_1 \neq n_j$ and $a_2 \neq 1$.

\[
\begin{array}{c}
Y \\
\searrow \\
Z
\end{array}
\]

- If $a_1^j = a_2^k$ and $b_1^j = b_2^k$ then $S(Y, Z) = \text{Ch}(Y)$ is contractible, so $K^*(S(Y, Z)) \cong \mathbb{Z}[0]$.
- For $a_1^j < a_2^k$ and $b_1^j = b_2^k$ we have
  \[S(Y, Z) = \text{Ch}((1^k, b_1^j)) \setminus \text{Ch}((1^k, a_2^k))\]
  for $a_1^j \leq 1^k$, and
  \[S(Y, Z) = \text{Ch}((a_1^j, b_1^j)) \setminus \text{Ch}((a_1^j, a_2^k))\]
  otherwise. This is the difference of a contractible compact pair. Hence $K^*(S(Y, Z)) = 0$.
- Analogously, $K^*(S(Y, Z)) = 0$ for $a_1^j = a_2^k$ and $b_1^j < b_2^k$.
- Finally, in the case $a_1^j < a_2^k$, $b_1^j < b_2^k$, the space is the difference of a compact pair $(K, L)$ with $K$ contractible and $L$ the disjoint union of two contractible subspaces. Hence that $K^*(S(Y, Z)) \cong \mathbb{Z}[1]$.

**Case 1b** $a_1^j \leq a_1^j \leq b_2^j \leq b_1^j$

Proceeding as in case 1a we obtain the following results:

(i) If $i$ and $l$ are odd, $a_1 \neq 1$, and $b_2 \neq n_l$, then $K^*(S(Y, Z)) = \mathbb{Z}[0]$.
(ii) If $i$ is odd, $l$ is even, $a_1 \neq 1$, $b_2 \neq 1$ and $b_2^j = b_1^j$, then $K^*(S(Y, Z)) = \mathbb{Z}[0]$.
(iii) If $i$ is even, $l$ is odd, $a_1 \neq n_l$, $b_2 \neq n_l$ and $a_2^j = a_1^j$, then $K^*(S(Y, Z)) = \mathbb{Z}[0]$. 
(iv) If $i$ and $l$ are even, $a_1 \neq n_i$, $b_2 \neq 1$, then
  - $K^*(S(Y, Z)) = \mathbb{Z}[0]$, when $a_k^2 = a_1^i$ and $b_l^2 = b_1^l$;
  - $K^*(S(Y, Z)) = \mathbb{Z}[1]$, when $a_k^2 < a_1^i$ and $b_l^2 < b_1^l$.
(v) In all other cases $K^*(S(Y, Z)) = 0$.

**Case 2a** $a_1^i \leq b_1^l < a_2^k \leq b_2^j$

(i) Let $j$ and $k$ be even, $b_1 \neq 1$ and $a_2 \neq n_k$.

$$
\begin{array}{c}
Y \\
\downarrow \\
Z
\end{array}
$$

Then $S(Y, Z) = \emptyset$ and we get $K^*(S(Y, Z)) = 0$.

(ii) Let $j$ be even, $k$ odd, and $b_1 \neq 1$.

$$
\begin{array}{c}
Y \\
\downarrow \\
Z
\end{array}
$$

Then $S(Y, Z)$ is again empty and $K^*(S(Y, Z)) = 0$.

(iii) Let $j$ be odd, $k$ even, and $a_2 \neq n_k$.

$$
\begin{array}{c}
Y \\
\downarrow \\
Z
\end{array}
$$

Once more, $S(Y, Z) = \emptyset$ and $K^*(S(Y, Z)) = 0$.

(iv) Let $j$ and $k$ be odd.

$$
\begin{array}{c}
Y \\
\downarrow \\
Z
\end{array}
$$

- If $j < k$ then $S(Y, Z) = \emptyset$ and thus $K^*(S(Y, Z)) = 0$.
- If $j = k$ and $b_1 + 1 < a_2$ then $S(Y, Z)$ is the difference of a contractible compact pair and thus $K^*(S(Y, Z)) = 0$.
- However, if $j = k$ and $b_1 + 1 = a_2$ then $S(Y, Z)$ is the difference of a compact pair $(K, L)$ as in Case 1a(iv), and we get $K^*(S(Y, Z)) = \mathbb{Z}[1]$.

**Case 2b** $a_2^k \leq b_2^j < a_1^i \leq b_1^l$

Similarly to Case 2a we get:

(i) If $l = i$ are even and $b_2^k + 1 = a_1^i$, then $K^*(S(Y, Z)) = \mathbb{Z}[1]$.

(ii) In all other cases $K^*(S(Y, Z)) = 0$.

**Case 3a** $a_2^k < a_1^i \leq b_1^l < b_2^j$

(i) Let $i$ be odd, $j$ even, $a_1 \neq 1$ and $b_1 \neq 1$.

$$
\begin{array}{c}
Y \\
\downarrow \\
Z
\end{array}
$$

Then $S(Y, Z)$ is contractible and thus $K^*(S(Y, Z)) = \mathbb{Z}[0]$. 
(ii) Let $i$ and $j$ be even, $a_1 \neq n_i$ and $b_1 \neq 1$.

Then $S(Y, Z)$ is the difference of a contractible compact pair and therefore $K^*(S(Y, Z)) = 0$.

(iii) Let $i$ and $j$ be odd, $a_1 \neq 1$ and $b_1 \neq n_j$.

Again, $S(Y, Z)$ is the difference of a contractible compact pair and we get $K^*(S(Y, Z)) = 0$.

(iv) Let $i$ be odd, $j$ even, $a_1 \neq n_i$ and $b_1 \neq n_j$.

In this case $S(Y, Z)$ is the difference of a compact pair $(K, L)$ as in Case 1a(iv) and thus $K^*(S(Y, Z)) = \mathbb{Z}[1]$.

**Case 3b** $a_1^i < a_2^i \leq b_2^i < b_1^i$

For this constellation we find:

(i) If $k$ is even, $l$ odd, $a_2 \neq n_k$ and $b_2 \neq n_l$, then $K^*(S(Y, Z)) = \mathbb{Z}[0]$.

(ii) If $k$ is odd, $l$ even, $a_2 \neq 1$ and $b_2 \neq 1$, then $K^*(S(Y, Z)) = \mathbb{Z}[1]$.

(iii) In all other cases $K^*(S(Y, Z)) = 0$.

The computations concerning $\mathcal{N}T_*(Y, Z) \cong K^*(S(Y, Z))$ carried out in this section are summarised in the following observation; recall that we abbreviate $\mathcal{N}T(W)$ by $\mathcal{N}T$.

**Observation 6.1.** Let $Y, Z \in \mathcal{LC}(W)^*$.

(i) $\mathcal{N}T_*(Y, Z) \cong \mathbb{Z}[0]$ if and only if $Y \cap Z$ is non-empty, closed in $Y$ and open in $Z$.

(ii) $\mathcal{N}T_*(Y, Z) \cong \mathbb{Z}[1]$ if and only if

either: $Y \cup Z$ is connected, and $Y \cap Z$ is a proper open subset of $Y$ and a proper closed subset of $Z$;

or: $Z$ is a proper open subset of $Y$ and $Y \setminus Z$ has two connected components;

or: $Y$ is a proper closed subset of $Z$ and $Z \setminus Y$ has two connected components.

(iii) $\mathcal{N}T_*(Y, Z) = 0$ in all other cases.

**6.2. Generators and relations of natural transformations.** We determine generators of the groups of natural transformations listed in Observation 6.1 and their relations in the category $\mathcal{N}T^*$.

In case (i), we have the grading-preserving natural transformation $\mu_{Y}^Z := i_{Y \cap Z}^Z \circ r_{Y \cap Z}^{Y \cap Z}$ induced by the natural non-zero $^*$-homomorphism

$$A(Y) \to A(Y \cap Z) \to A(Z).$$

In fact, by Corollary 3.31 the natural transformation $\mu_{Y}^Z$ generates the group $\mathcal{N}T_0(Y, Z) \cong \mathbb{Z}$. 
Lemma 6.2. Let $Y, Z, V \in \mathbb{LC}(W)^*$ such that $V \cap Y$ is non-empty, closed in $V$ and open in $Y$, and such that $Y \cap Z$ is non-empty, closed in $Y$ and open in $Z$. With the above convention, we have $\mu^Z_Y \circ \mu^Z_Y = \mu^Z_Y$ if $V \cap Z$ is non-empty, closed in $V$ and open in $Z$. Otherwise, we have $\mu^Z_Y \circ \mu^Z_Y = 0$.

Proof. Proposition 6.3 yields the commutative diagram in $\mathcal{N}T$

$\begin{array}{ccc}
V \cap Y & \xrightarrow{i} & Y \\
\downarrow{r} & & \downarrow{r} \\
V \cup Y \cap Z & \xrightarrow{i} & Y \cap Z
\end{array}$

Since $V \cap Y$ is closed in $V$ and $Y \cap Z$ is open in $Z$, the subset $V \cap Y \cap Z$ is clopen in $V \cap Z$. Thus we have either $V \cap Y \cap Z = \emptyset$ or $V \cap Y \cap Z = V \cap Z$ because $V \cap Z$ is connected—it is a specific property of the space $W$ that the intersection of two connected subsets is again connected.

In the case $V \cap Y \cap Z = \emptyset$, we get $\mu^Z_Y \circ \mu^Z_Y = 0$. However, as $V \cap Y \neq \emptyset$ and $Y \cap Z \neq \emptyset$, the constellation $V \cap Y \cap Z = \emptyset$ can only occur if $V \cap Z = \emptyset$. This is because $V$, $Y$, and $Z$ are intervals with respect to the total order $\leq$ on $W$. Hence we are in the second case, and the proclaimed relation for this case holds.

For $V \cap Y \cap Z = V \cap Z$ the above diagram shows that $\mu^Z_Y \circ \mu^Z_Y = \mu^Z_Y$. Hence the desired relation for the first case holds as well.$\square$

Corollary 6.3. The category $\mathcal{N}T_0$ of grading-preserving natural transformations $FK_Y \Rightarrow FK_Z$ for $Y, Z \in \mathbb{LC}(W)^*$ is the pre-additive category generated by natural transformations $\mu^Y_Z$ for all $Y, Z \in \mathbb{LC}(W)^*$ such that $Y \cap Z$ is non-empty, closed in $Y$ and open in $Z$, whose relations are generated by the following:

- $\mu^Z_Y \circ \mu^Z_Y = \mu^Z_Y$ for $Y, Z, V \in \mathbb{LC}(W)^*$ such that $V \cap Y$ is non-empty, closed in $V$ and open in $Y$, and such that $Y \cap Z$ is non-empty, closed in $Y$ and open in $Z$;

- $\mu^Z_Y \circ \mu^Z_Y = 0$ otherwise.

Proof. We have verified the relations above in Lemma 6.2. Computing the morphism groups for the universal pre-additive category $\mathcal{U}$ with generators and relations as above yields precisely the groups $\mathcal{N}T_0(Y, Z)$ as in Observation 6.1. This shows that the canonical functor $\mathcal{U} \rightarrow \mathcal{N}T_0$ is an isomorphism.$\square$

The list of generators can of course be shortened by restricting to indecomposable transformations. These are discussed in the next section.

Now we incorporate the odd natural transformations into our investigation. Observation 6.1(iii) describes the three (disjoint) cases in which an odd transformation from $Y$ to $Z$ occurs.

In the first case, $Y \cup Z$ is connected, and $Y \cap Z$ is a proper open subset of $Y$ and a proper closed subset of $Z$. Under these assumptions, $Z$ is open in $Y \cup Z$ and we have the odd transformation

$\delta^Z_Y : Y \xrightarrow{r} Y \setminus (Y \cap Z) \xrightarrow{r} Z.$

In the second case, $Z$ is a proper open subset of $Y$ and $Y \setminus Z$ has two connected components. We define $Y^< \subset Y$ to be the smaller component with respect to $\leq$, and $Y^>$ to be the greater component. Then $Z$ is open in $Z \cup Y^<$ and in $Z \cup Y^>$ and we have two odd transformations

$\delta^Z_Y^< : Y \xrightarrow{r} Y^< \xrightarrow{r} Z,$

$\delta^Z_Y^> : Y \xrightarrow{r} Y^> \xrightarrow{r} Z.$

By Proposition 6.17 we have $(\delta^Z_Y)^< = - (\delta^Z_Y)^>$. We define $\delta^Z_Y := (\delta^Z_Y)^<$. 
In the third case, we similarly define $\delta_Y^Z$ as the composite
\[
\delta_Y^Z := (\delta_Y^Z)^{<}: Y \xrightarrow{\delta_Y} Z^< \xrightarrow{i} Z,
\]
where $Z^<$ is the component of $Z \setminus Y$ which is smaller with respect to the total order $\leq$.

**Lemma 6.4.** Let $Y, Z \in \mathbb{LC}(W)^*$ as in Observation 6.1 (ii). The natural transformation $\delta_Y^Z$ generates the group $\mathcal{N}T_1^*(Y, Z) \cong \mathbb{Z}$.

**Proof.** We begin with the first case. Then $Y \cup Z$ is connected, and $Y \cap Z$ is a proper open subset of $Y$ and a proper closed subset of $Z$. Let $C := Y \setminus (Y \cap Z)$. By Corollary 5.4 it suffices to check that $K^1(S(C, C) \cup S(C, Z)) = 0$ and $K^1(S(Y, Z) \setminus S(C, Z)) = 0$. These $K^1$-groups vanish because both $S(C, C) \cup S(C, Z)$ and $S(Y, Z) \setminus S(C, Z)$ are a difference of a contractible compact pair.

Now we turn to the second case. Then $Z$ is a proper open subset of $Y$ and $Y \setminus Z$ has two connected components $Y^<$ and $Y^>$. As in the first case the assertion follows from $K^1(S(Y^<, Y^<)) \cong \mathbb{Z}$ and $K^1(S(Y^<, Z)) \cong \mathbb{Z}$, together with $K^1(S(Y^<, Y^<) \cup S(Y^<, Z)) = 0$ and $K^1(S(Y, Z) \setminus S(Y^<, Z)) = 0$. The proof for the third case in analogous. \[\square\]

**Lemma 6.5.** The composition of any two odd natural transformations in $\mathcal{N}T^*$ vanishes.

**Proof.** An application of Proposition 4.13 (i) and (ii) shows that it suffices to consider a composition of two boundary transformations coming from boundary pairs. The assertion for this special case follows from Corollary 5.10 because the union of two connected, locally closed subsets of $W$ with non-empty intersection is again locally closed. \[\square\]

Our investigations above show that all morphisms in the category $\mathcal{N}T^*$ arise as compositions of six-term sequence transformations associated to open inclusions of locally closed subsets of $X$, that is, $\mathcal{N}T^* = \mathcal{N}T_{6\text{-term}}^*$.

By Lemma 6.5 the category $\mathcal{N}T^*$ is a split extension of its even subcategory $\mathcal{N}T_{6\text{-term}}^*$ by the bimodule $\mathcal{N}T_{6\text{-term}}^*$ of odd transformations. Propositions 5.13 [5.13] and 5.17 show that the bimodule structure is as follows: a product $\mu_W^{\delta_Y^Z} \circ \delta_Y^Z$ or $\delta_Y^Z \circ \mu_W^*$ is equal to $\delta_Y^Z$ or $-\delta_Y^Z$, whenever all three natural transformations are defined, and zero otherwise. The occurrence of the minus sign is due to the non-canonical definition of $\delta_Y^Z$.

An argument as in the proof of Corollary 5.9 now shows that the relations in $\mathcal{N}T^*$ are generated by the canonical ones listed in Definition 5.18. The above description of $\mathcal{N}T^*$ as a split extension was given in [7] for the category of natural transformations corresponding to the totally ordered space.

We now apply our indecomposability criteria established in 5.4 Using that compositions of odd transformations vanish, we find that the characterisations of even and odd indecomposable transformations in Propositions 5.11 and Corollary 5.12 are valid in the category $\mathcal{N}T^*$. We obtain a complete list of indecomposable natural transformations in the category $\mathcal{N}T^*$ consisting of essentially only five different types.

**Observation 6.6.** The category $\mathcal{N}T^*$ is generated by the following indecomposable natural transformations:

1. an extension $\langle (a + 1)^i, b^i \rangle \rightarrow \langle a^i, b^i \rangle$ whenever $i$ is odd, $a \neq 1$, $a \neq n_i$, and $a^i \neq b^i$;
2. an extension $\langle 2i+1, b^i \rangle \rightarrow \langle n_i, b^i \rangle$ whenever $i$ is even and $b^i > 1^{i+1}$.
(3) an extension $\langle a^i, (b+1)^j \rangle \rightarrow \langle a^i, b^j \rangle$ whenever $j$ is even, $b \neq 1$, $b \neq n_j$, and $a^i \neq b^j$;
(4) an extension $\langle a^i, 2^{i-1} \rangle \rightarrow \langle a^i, n_j^i \rangle$ whenever $j$ is odd and $a^{i} < 2^{i-1}$;
(5) a restriction $\langle (a+1)^i, b^j \rangle \rightarrow \langle a^i, b^j \rangle$ whenever $i$ is even, $a \neq n_i$ and $a \neq n_i - 1$;
(6) a restriction $\langle 1^{i-1}, b^j \rangle \rightarrow \langle (n_i - 1)^i, b^j \rangle$ whenever $i$ is even;
(7) a restriction $\langle a^i, (b+1)^j \rangle \rightarrow \langle a^i, b^j \rangle$ whenever $j$ is odd, $b \neq n_j$ and $b \neq n_j - 1$;
(8) a restriction $\langle a^i, 1^{j+1} \rangle \rightarrow \langle a^i, (n_j - 1)^j \rangle$ whenever $j$ is odd;
(9) a boundary transformation $\langle 1^i, (a-1)^j \rangle \rightarrow \langle a^i, n_j^i \rangle$ whenever $i$ is odd and $a \neq 1$;
(10) a boundary transformation $\langle (b-1)^j, 1^i \rangle \rightarrow \langle n_j^i, b^j \rangle$ whenever $j$ is even and $b \neq 1$.

We make two further observations which are direct consequences of Observation 6.6 and which will be used in the next section.

**Observation 6.7.** There are precisely $n + 1$ sets $C \in \mathcal{LC}(W)^*$ with the property that there is only one indecomposable transformation to $C$, namely:
- the singletons $\{1^2\}, \{1^m\}$ and $\{a^i\}$ with $i \in \{1, \ldots, m\}$ and $a \not\in \{1, n_i\}$;
- the maximal totally ordered subsets $\{1^2, 2^i, \ldots, n_i^j\}$ for $i \in \{1, \ldots, m\}$.
Moreover, these are precisely the sets $C \in \mathcal{LC}(W)^*$ such that there is only one indecomposable transformation out of $C$. We call these sets singular subsets of $W$.

For all other subsets $D \in \mathcal{LC}(W)^*$ there are precisely two indecomposable transformations to $D$ and precisely two indecomposable transformations out of $D$. Altogether, the category $\mathcal{NT}^*$ is thus generated by $n^2 - 1$ indecomposable transformations.

As we have seen, every indecomposable natural transformations $\mu$ in $\mathcal{NT}^*$ is a six-term sequence transformation. We therefore have an associated subsequent six-term sequence transformation $\eta$. That is, if $\mu$ is an extension transformation $i_Y^U$, then $\eta = \eta_Y^{U \setminus Y}$; similarly for restriction and boundary transformations.

**Observation 6.8.** Let $Y$ be a singular subset and $\mu : Y \rightarrow Z$ an indecomposable transformations. Let $\eta : Z \rightarrow V$ be the subsequent six-term sequence transformation. Then $\eta$ is indecomposable and $V$ is singular.

### 6.3 Construction of an isomorphism of categories

In this section, we show that the category $\mathcal{NT}^*$ essentially depends only on the number $n$, the total number of points in $W$. We illustrate our approach by an example first.

**Example 6.9.** As an example, we compare the two categories $\mathcal{NT}^*(O_4)$ and $\mathcal{NT}^*(W_4)$ for the topological spaces $O_4$ and $W_4$ which correspond to the partial orders $1 \prec 2 \prec 3 \prec 4$ and $1 \prec 2 \prec 3 \prec 4$ on the set $\{1, 2, 3, 4\}$, respectively. The indecomposable transformations in these categories are displayed in Figure 2 and 3, where we use the abbreviation $234 := \{2, 3, 4\}$, and so on. In Figure 2 all squares are commutative. This is also true for Figure 3 except for the single square

$\begin{array}{ccc}
\text{1234} & \overset{r}{\longrightarrow} & 4 \\
\downarrow & & \downarrow \\
12 & \overset{\delta}{\longrightarrow} & 3,
\end{array}$

which anti-commutes. Moreover, the compositions of indecomposable transformations of the form $S \rightarrow S' \rightarrow S''$ for singular subsets $S$, $S'$ all vanish as part of exact six-term sequences. For a proof of these relations, see [44]. Arguing as in the
proof of Corollary 6.3, we see that the relations above generate all relations in the category $\mathcal{N} T^* (O_4)$. By replacing the generator $\delta^4_4$ with its additive inverse, we can make all squares in Figure 3 commute. Now it can be verified by a direct check that the obvious bijection between the chosen sets of generators of the two categories extends to an isomorphism of categories. This isomorphism is not grading-preserving. However, it has the following property: a subset $\{U,Y,C\} \subset \mathcal{L}C(O_4)^*$ consisting of a boundary pair $(U,C)$ and its union $Y = U \cup C$ is mapped to a subset of $\mathcal{L}C(W_4)^*$ of the same kind, though the roles of each particular set may be interchanged. This shows that the isomorphism respects exactness of modules.

Now we generalise the observations in Example 6.9 to the general situation. We begin with describing certain chains of indecomposable natural transformations connecting two singular subsets of $W$. Every chain consists of $n - 1$ transformations.

Starting with the point $1^1$, we have the chain

$$\{1^1\} \xrightarrow{\delta} \langle 2^1, n_1^1 \rangle \xrightarrow{i} \langle 2^1, (n_2 - 1)^2 \rangle \xrightarrow{i} \cdots \xrightarrow{i} \langle 2^1, 2^2 \rangle$$

$$\xrightarrow{i} \langle 2^1, n_3^3 \rangle \xrightarrow{i} \langle 2^1, (n_4 - 1)^4 \rangle \xrightarrow{i} \cdots \xrightarrow{i} \langle 2^1, 2^4 \rangle$$

$$\vdots$$

$$\xrightarrow{i} \langle 2^1, n_{m-1}^{m-1} \rangle \xrightarrow{i} \langle 2^1, (n_m - 1)^{m} \rangle \xrightarrow{i} \cdots \xrightarrow{i} \langle 2^1, 1^m \rangle$$

$$\xrightarrow{r} \langle 2^1, (n_{m-1} - 1)^{m-1} \rangle \xrightarrow{r} \cdots \xrightarrow{r} \langle 2^1, 1^{m-1} \rangle$$

$$\vdots$$

$$\xrightarrow{r} \langle 2^1, (n_1 - 1)^1 \rangle \xrightarrow{r} \cdots \xrightarrow{r} \{2^1\}$$

Figure 2. Diagram of indecomposable natural transformations in $\mathcal{N} T^*(O_4)$

Figure 3. Diagram of indecomposable natural transformations in $\mathcal{N} T^*(W_4)$
from \(\{1^1\} \) to \(\{2^1\}\), which we denote by \(\{1^1\} \implies \{2^1\}\).

In the following, we make the underlying rule for this procedure precise. Fix an indecomposable transformation \(\nu: Y \to Z\). We distinguish two cases:

If \(Z\) is a singular subset, then there is precisely one indecomposable transformation \(S(\nu)\) out of \(Z\). If, on the other hand, \(Z\) is a non-singular subset, then there are precisely two indecomposable transformations out of \(Z\) (cf. Observation 6.7), and we want to choose the “right” one.

The following lemma describes the indecomposable transformations out of a non-singular subset \(Z\) with respect to an indecomposable transformation into \(Z\). It provides us with a way to define the successor of an indecomposable transformation into a non-singular subset.

**Lemma 6.10.** Let \(Z\) be non-singular subset. Let \(\nu: Y \to Z\) be an indecomposable transformation.

(i) If \(Y\) is singular, then there is precisely one of the two indecomposable transformations out of \(Z\)—denoted by \(S(\nu)\)—which is not the subsequent transformation in the six-term sequence of \(\nu\).

(ii) If \(Y\) is non-singular, then there is precisely one of the two indecomposable transformations out of \(Z\)—denoted by \(S(\nu)\)—such that the composition \(S(\nu) \circ \nu\) cannot be factorised into a product of two other indecomposable transformations.

The underlying rule for our chains of indecomposable transformations is now simply as follows. The well-definition of this rule is a consequence of Lemma 6.10.

**Definition 6.11.** The successor of an indecomposable transformation \(\nu\) is the indecomposable transformation \(S(\nu)\).

**Proof of Lemma 6.10.** The first assertion is a consequence of Observation 6.8. The second assertion can be checked by a case differentiation using the list in Observation 6.6. By symmetry considerations, it suffices to check the cases (1), (2), (5), (6), and (9) from that list. As an example, we discuss case (1) here. In the thus remaining four cases, the assertion can be verified in the straightforward but lengthy manner outlined below.

Consider the indecomposable extension \(i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j}\) with \(i\) is odd, \(a \neq 1\), \(a \neq n_s\), and \(a' \neq b'\).

The set \(\langle (a+1)\rangle^i, b^j\) is non-singular if and only if \((a+1)^i \succ b^j\). In this case, the indecomposable transformations out of \(\langle a', b' \rangle\) are

\[
\begin{cases}
   i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j} & \text{if } a > 2, \\
i_{\langle a', b' \rangle}^{\langle n_s-1, b^j \rangle} & \text{if } a = 2,
\end{cases}
\]

\[
\begin{cases}
   i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j} & \text{if } j \text{ odd, } b \neq 1, \\
i_{\langle a', b' \rangle}^{\langle a', (n_s-1)j^{-1} \rangle} & \text{if } j \text{ even, } b = 1, \\
i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j} & \text{if } j \text{ even, } b \neq 1, 2, \\
i_{\langle a', b' \rangle}^{\langle a', n_s^{-1} \rangle} & \text{if } j \text{ even, } b = 2.
\end{cases}
\]

While \(i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j} \circ i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j} = i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j}\) and \(i_{\langle a', b' \rangle}^{\langle n_s-1, b^j \rangle} \circ i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j} = i_{\langle a', b' \rangle}^{\langle (a+1)\rangle^i, b^j}\) do not factorise in a non-trivial way different from the given one (which may also be read from the list in Observation 6.6 since we know how these generators multiply),
we have

\[
\begin{align*}
& r_{a', b'}(a', (b-1)^i) \circ r_{a', b'}(a', b') = r_{a', (b-1)^i} \circ r_{a', (b-1)^i}, \\
& r_{a', b'}(n_{j-1}^{-1}j^j, a', b') = r_{a', (n_{j-1}^{-1}j^j, a', b')}, \\
& l_{n_{j-1}^{-1}j^j} \circ l_{n_{j-1}^{-1}j^j} = l_{n_{j-1}^{-1}j^j} \circ l_{n_{j-1}^{-1}j^j}.
\end{align*}
\]

}\[0.5cm]\text{This is followed by the chains}\[0.5cm]\text{respectively.}

\[\square\]

In addition to the previously described chain of indecomposable transformations from \(\{1^i\}\) to \(\{2^i\}\), we obtain the following chains of indecomposable transformations between singular subsets when applying the rule from Definition 6.11.

If \(n_1 > 2\), we have a chain \(\{2^1\} \implies \{3^1\}\), namely

\[
\begin{align*}
\{2^1\} \xrightarrow{i} \langle 2^1, 1^1 \rangle \xrightarrow{i} \langle 3^1, n_1^1 \rangle \xrightarrow{i} \langle 3^1, (n_2 - 1)^2 \rangle \xrightarrow{i} \ldots & \xrightarrow{i} \langle 3^1, 2^n \rangle \\
& \xrightarrow{i} \langle 3^1, n_3^3 \rangle \xrightarrow{i} \langle 3^1, (n_4 - 1)^4 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 3^1, 2^{n^i} \rangle \\
& \xrightarrow{i} \langle 3^1, n_{m-1}^{n-1} \rangle \xrightarrow{i} \langle 3^1, (n_m - 1)^m \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 3^1, 1^m \rangle \\
& \xrightarrow{i} \langle 3^1, (n_{m-1} - 1)^{m-1} \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 3^1, 1^{m-2} \rangle \\
& \xrightarrow{i} \langle 3^1, (n_1 - 1)^{1^1} \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 3^1, 1^{1^1} \rangle.
\end{align*}
\]

In the same way, we obtain chains of indecomposable transformations \(\{3^1\} \implies \{4^1\} \implies \ldots \implies \{n_1 - 1^1\}\). This is followed by the chains

\[
\begin{align*}
\{(n_1 - 1)^1\} \xrightarrow{i} \langle (n_1 - 2)^1, (n_1 - 1)^1 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 1^1, (n_1 - 1)^1 \rangle \\
& \xrightarrow{i} \langle n_1^1 \rangle \xrightarrow{i} \langle n_1^1, (n_2 - 1)^2 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle n_1^1, 2^n \rangle \\
& \xrightarrow{i} \langle n_1^1, n_3^3 \rangle \xrightarrow{i} \langle n_1^1, (n_4 - 1)^4 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle n_1^1, 2^{n^i} \rangle \\
& \xrightarrow{i} \langle n_1^1, n_{m-1}^{n-1} \rangle \xrightarrow{i} \langle n_1^1, (n_m - 1)^m \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle n_1^1, 1^m \rangle \\
& \xrightarrow{i} \langle n_1^1, (n_{m-1} - 1)^{m-1} \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle n_1^1, 1^{m-2} \rangle \\
& \xrightarrow{i} \langle n_1^1, (n_3 - 1)^3 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle n_1^1, 1^2 \rangle
\end{align*}
\]

from \(\{(n_1 - 1)^1\}\) to \(\langle n_1^1, 1^2 \rangle\), and

\[
\begin{align*}
\langle n_1^1, 2^1 \rangle \xrightarrow{i} \langle (n_1 - 1)^1, 2^1 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 1^1, 2^1 \rangle \\
& \xrightarrow{i} \langle (n_2 - 1)^2, 1^2 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 1^1, 2^2 \rangle \\
& \xrightarrow{i} \langle 2^2, n_3^3 \rangle \xrightarrow{i} \langle 2^2, (n_4 - 1)^4 \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 2^2, 1^m \rangle \\
& \xrightarrow{i} \langle 2^2, (n_{m-1} - 1)^{m-1} \rangle \xrightarrow{i} \ldots \xrightarrow{i} \langle 2^2 \rangle\]
\]
The diagram is periodic in the horizontal direction; the dashed arrows indicate that at each intersection point of two chains, a non-singular subset of

\[ \{1^1\} \Rightarrow \{2^1\} \Rightarrow \cdots \Rightarrow \{(n_1 - 1)^1\} \Rightarrow \{n_1^1, 1^2\} \]

\[ \Rightarrow \{2^3\} \Rightarrow \cdots \Rightarrow \{(n_3 - 1)^3\} \Rightarrow \{n_3^3, 1^4\} \]

\[ \cdots \]

\[ (6.12) \]

\[ \Rightarrow \{2^{m-1}\} \Rightarrow \cdots \Rightarrow \{(n_{m-1} - 1)^{m-1}\} \Rightarrow \{n_{m-1}^{m-1}, 1^m\} \]

\[ \Rightarrow \{1^m\} \Rightarrow \{2^m\} \Rightarrow \cdots \Rightarrow \{(n_m - 1)^m\} \Rightarrow \{1^{m-1}, n_m^m\} \]

\[ \Rightarrow \{2^{m-2}\} \Rightarrow \cdots \Rightarrow \{(n_{m-2} - 1)^{m-2}\} \Rightarrow \{1^{m-3}, n_{m-2}^{m-2}\} \]

\[ \cdots \]

\[ \Rightarrow \{2^2\} \Rightarrow \cdots \Rightarrow \{(n_2 - 1)^2\} \Rightarrow \{1^1, n_1^1\} \Rightarrow \{1^1\} . \]

This long chain is the composition of \( n + 1 \) of the previously described chains, each of them connecting two singular subsets of \( W \). We obtain an enumeration (without repetitions) of the singular subsets of \( W \). We denote the so enumerated singular subsets of \( W \) by \( S_i \) with \( i \in \{1, \ldots, n + 1\} \).

In fact, each of the \( n^2 - 1 \) indecomposable transformations in \( \mathcal{NT}^\ast \) occurs precisely once in the above long cyclic chain. This is simply because this long chain consists of \( (n + 1)(n - 1) = n^2 - 1 \) indecomposable transformations and non of them occurs more than once. To see this, observe that we return to the singular subset \( \{1^1\} \) only after \( n^2 - 1 \) steps, and that the succeeding transformations is well-defined. Hence, we obtain an enumeration of the indecomposable natural transformations in \( \mathcal{NT}^\ast \) as well.

Each non-singular subset of \( W \) is listed precisely twice in (6.12). Figures 4 and 5 indicate a way in which the long chain (6.12) can be entangled in order to list each element of \( \text{LC}(W)^\ast \) only once. The singular subsets of \( W \) and the chains of indecomposable transformations between them are indicated explicitly. At each intersection point of two chains, a non-singular subset of \( W \) is situated.

The diagram is periodic in the horizontal direction; the dashed arrows indicate that the vertical order of the repeating objects is reversed after one period.

**Figure 4.** Diagram of indecomposable natural transformations in \( \mathcal{NT}^\ast \) for the space \( W \) with odd number of points.
As remarked in the previous subsection, the relations in the category $\mathcal{NT}^*(W)$ are generated by the canonical ones from Definition 3.18. In the following, we shall spell these out more explicitly for the case at hand.

The canonical relations translate to the vanishing of all compositions of consecutive six-term sequence transformations together with commutativity relations for all squares in the above diagram; more precisely, all squares either commute or anti-commute. The only squares that anti-commute are those of the form

$$\delta \ U \ C \ \delta \ \delta \ U \ C$$

or

$$Y \ C \ Y \ U.$$ (6.13)

To see that the sum-relations in Propositions 3.8 and 3.17 do not contribute anything beyond the anti-commutativity relations (6.13), notice that intersections of connected subsets of $W$ are again connected, and that complements of connected subsets of $W$ have at most two connected components.

We can make all the anti-commuting squares in (6.13) commute by replacing the boundary transformations $\delta_U$ for all boundary pairs $(U, C)$ with $C < U$ by their additive inverses. Recall that $\leq$ denotes the total order on $W$ defined in §6.1. This change in the choice of generators does not affect the commutativity of the remaining squares because each of them contains either no boundary transformations or two boundary transformations with the same orientation concerning the order $\leq$.

By Observation 6.8, the above relations imply that the compositions

$$S_i \rightarrow S_{i-1}$$

vanish for all $i \in \{1, \ldots, n+1\}$; here we set $S_0 := S_{n+1}$, and $\sharp$ denotes the unique object sitting between $S_i$ and $S_{i-1}$ in the above diagram of indecomposable transformations.

\[\text{Figure 5. Diagram of indecomposable natural transformations in } \mathcal{NT}^* \text{ for the space } W \text{ with even number of points}\]
Lemma 6.15. The relations in $\mathcal{NT}^*(W)$ are generated by the commutativity relations for all squares and the vanishing of the compositions $[6.14]$.

Proof. By the list of generating relations given above, it suffices to show that, for an arbitrary boundary pair $(U, C)$ in $W$, all three compositions of two successive transformations in the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\eta} & U \cup C \\
\delta & & \downarrow r \\
C & \xrightarrow{\mu} & C
\end{array}
\]

vanish. Let $u \in U$ and $c \in C$ be the unique elements such that $\{u, c\}$ is connected. Let $F(u)$ denote $\overline{\{u\}} \setminus \{c\}$ and let $F(c)$ denote $\overline{\{c\}} \setminus \{u\}$. Then $F(u)$ and $F(c)$ are either non-closed and non-open singletons or maximal totally ordered subsets of $W$. In either case, $F(u)$ and $F(c)$ are singular subsets of $W$, and the composition $U \xrightarrow{\eta} U \cup C \xrightarrow{\mu} C$ factors as

\[U \xrightarrow{\eta} F(u) \xrightarrow{\eta'} F(u) \cup F(c) \xrightarrow{\mu} F(c) \xrightarrow{\eta} C.\]

The transformations $U \xrightarrow{\eta} F(u)$ and $F(c) \xrightarrow{\eta} C$ are either extensions or restrictions, depending on the form of $F(u)$ and $F(c)$. Notice that the transformations $F(u) \xrightarrow{\eta'} F(u) \cup F(c)$ and $F(u) \cup F(c) \xrightarrow{\mu} F(c)$ are indecomposable. We have thus verified that the vanishing of the first composition follows from the given relations; the other two compositions can be proven to vanish similarly. 

The above description shows that the ungraded isomorphism class of the category $\mathcal{NT}^*(W)$ depends only on the number $n$. More precisely, let $O_n$ denote the totally ordered space with $n$ points. Forming the long chains $[6.12]$ for both $W$ and $O_n$, we obtain a bijection between a set of generators of $\mathcal{NT}^*(W)$ and a set of generators of $\mathcal{NT}^*(O_n)$. The foregoing arguments on relations in the two categories show that this bijection extends to an isomorphism $\Phi$ between the (ungraded) categories $\mathcal{NT}^*(W)$ and $\mathcal{NT}^*(O_n)$.

Finally, we convince ourselves that $\Phi$ is compatible with the notion of exactness of modules. Of course it is in general not true that $\Phi$ maps boundary pairs to boundary pairs.

Lemma 6.16. Let $V \xrightarrow{\eta} Y$ be a natural transformation in $\mathcal{NT}_*(V, Y)$ coming from a six-term exact sequence, and let $Y \xrightarrow{\eta'} Z$ be the subsequent natural transformation in this six-term exact sequence. Then every natural transformation $Y \xrightarrow{\eta'} Z'$ with $\eta' \circ \mu = 0$ factors through $\eta$.

Proof. Consider the exact sequence $\mathcal{NT}_*(Z, Z') \xrightarrow{\eta'} \mathcal{NT}_*(Y, Z') \xrightarrow{\mu^*} \mathcal{NT}_*(V, Z')$. We have $\mu^*(\eta') = \eta' \circ \mu = 0$ and thus $\eta' \in \text{im}(\eta^*)$. 

In other words, the transformation $\eta$ is the universal transformation out of $Y$ with $\eta \circ \mu = 0$. It is uniquely determined up to sign by this property. This is because the only isomorphisms in the category $\mathcal{NT}^*(W)$ are automorphisms of the form $\pm \text{id}_Z$ for some object $Z$.

Corollary 6.17. A composite $V \xrightarrow{\eta} Y \xrightarrow{\eta'} Z$ in $\mathcal{NT}^*$ is (up to sign) part of a six-term exact sequence (in the sense of two successive transformations) if and only if $\mu$ is a six-term exact sequence transformation, $\eta \circ \mu = 0$, and every transformation $\eta'$ out of $Y$ with $\eta' \circ \mu = 0$ factors through $\eta$. 

The characterisation in Corollary \ref{cor:exactness} shows that the isomorphism $\Phi$ and its inverse respect the property “being part of a six-term exact sequence” for pairs of composable natural transformations. Hence an ungraded $\mathcal{N}T^*(O_n)$-module $M$ is exact if and only if $\Phi^*(M)$ is an exact ungraded $\mathcal{N}T^*(W)$-module.

The obtained facts are summarised in Theorem \ref{thm:counterexamples}. The last assertion in this theorem follows from the fact that $\Phi$ maps identities to identities and morphisms between different objects to morphisms between different objects.

7. Counterexamples

In this section we discuss several examples of finite $T_0$-spaces for which $\sim UCT(X)$ holds. First we describe a general approach to obtain counterexamples. The ideas are due to Meyer and Nest \cite{MeyerNest}.

If our method for finding resolutions of length 1 described in \S 5 fails, we would like to find counterexamples to Lemmas \ref{lem:resolution} and \ref{lem:resolution_2}, and to the hypothesis that $C^*$-algebras over $X$ in the bootstrap class are classified up to $KK(X)$-equivalence by filtrated $K$-theory.

The general procedure is as follows. If, while trying to show that a given space $X$ has Property \ref{prop:property}, we encounter a subset $Y \subseteq \mathcal{LC}(X)^*$ for which this is impossible, then we consider the $\mathcal{N}T^*$-module homomorphism $j: P_Y \to P^0 := \bigoplus \{P_Z \mid \text{there is an indecomposable transformation } Z \to Y\}$ induced by all indecomposable transformations $Z \to Y$ in $\mathcal{N}T^*$.

If this homomorphism happens to be injective, then the module $M := P^0/j(P_Y)$ has the projective resolution

$$0 \to P_Y \xrightarrow{j} P^0 \to M.$$

If, moreover, this resolution does not split—for instance, when there is no non-zero homomorphism from $P^0$ to $P_Y$—then the module $M$ is not projective. However, it is always exact by the two-out-of-three property, and in all cases we will consider it happens to have free entries. In this situation the module $M$ yields a counterexample to Lemma \ref{lem:resolution}.

We then go on and define the $\mathcal{N}T^*$-module $M_k := M/k \cdot M$ for some natural number $k \in \mathbb{N}_{\geq 2}$. This module is exact and has the following projective resolution of length 2:

$$0 \to P_Y \xrightarrow{(-k,j)} P_Y \oplus P^0 \xrightarrow{(j,k)} P^0 \to M_k.$$

Under the above assumption that there is no non-zero-homomorphism from $P^0$ to $P_Y$ we can therefore compute

$$\text{Ext}^2_{\mathcal{N}T^*}(M_k, P_Y) \cong \text{Hom}_{\mathcal{N}T^*}(P_Y, P_Y)/(-k,j)^*(\text{Hom}_{\mathcal{N}T^*}(P_Y \oplus P^0, P_Y))$$

$$\cong \text{Hom}_{\mathcal{N}T^*}(P_Y, P_Y)/k \cdot \text{Hom}_{\mathcal{N}T^*}(P_Y, P_Y)$$

$$\neq 0,$$

which shows that $M_k$ has projective dimension 2 and provides a counterexample to Lemma \ref{lem:resolution_2}. The above term never vanishes because $\text{Hom}_{\mathcal{N}T^*}(P_Y, P_Y) \cong \mathcal{N}T_*(Y, Y) \cong K^*(\text{Ch}(Y))$ is a finitely generated Abelian group containing at least one free summand.

By Lemma \ref{lem:resolution_2} there is a $C^*$-algebra $A$ for which $FK^*(A)$ is isomorphic to $M$. The Künneth Theorem for the $K$-theory of tensor products \cite[V.1.5.10]{Blackadar} shows that the filtrated $K$-theory of the tensor product $A_k := A \otimes \mathcal{O}_{k+1}$ with the Cuntz algebra $\mathcal{O}_{k+1}$ is isomorphic to $M_k$. This is because $FK^*(A) \cong M$ is torsion-free.

**Theorem 7.1.** In the above situation, the $C^*$-algebra $A_k$ is not $\ker(FK^*)^2$-projective.
Theorem 7.3. If \( 7.4 \) Remark \([7, \text{Theorem 4.10}]\) these assumptions have to be checked in each particular case. The following theorem from \([7]\) then provides two non-isomorphic \( C^* \)-algebras over \( X \) in the bootstrap class with isomorphic filtrated K-theory.

**Theorem 7.2 (\([7, \text{Theorem 4.10}]\)).** Let \( \mathcal{I} \) be a homological ideal in a triangulated category \( \Sigma \) with enough projective objects. Let \( F : \Sigma \to \mathcal{A}_\mathcal{I} \Sigma \) be a universal \( \mathcal{I} \)-exact stable homological functor. Suppose that \( \mathcal{I}^2 \neq 0 \). Then there exist non-isomorphic objects \( B, D \in \Sigma \) for which \( F(B) \cong F(D) \) in \( \mathcal{A}_\mathcal{I} \Sigma \).

The objects \( B \) and \( D \) can be obtained as follows: Choose a non-\( \mathcal{I}^2 \)-projective object \( A \in \Sigma \) and embed it into an exact triangle

\[
\Sigma N_2 \longrightarrow \tilde{A}_2 \longrightarrow A \xrightarrow{\imath_2} N_2
\]

with \( \imath_2 \in \mathcal{I}^2 \) and an \( \mathcal{I}^2 \)-projective object \( \tilde{A}_2 \). Then \( F(\tilde{A}_2) \cong F(A) \oplus F(N_2)[1] \) whereas \( \tilde{A}_2 \not\approx A \oplus N_2[1] \).

Now we apply the above procedure to certain explicit examples which will be used in the next chapter. If \( X \) is a space, let \( X^{\text{op}} \) denote its dual space, that is, \( X^{\text{op}} = X \) as a set and the open sets in \( X^{\text{op}} \) are exactly the closed sets in \( X \). Let us define the following spaces:

1. \( X_1 = \{1, 2, 3, 4\} \), a basis of the topology \( \tau_{X_1} \), is given by
   \[
   \emptyset, \{1\}, \{2\}, \{3\};
   \]

2. \( X_2 = X_1^{\text{op}} \);
3. \( X_3 = \{1, 2, 3, 4\}, \quad \tau_{X_3} = \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}; \)
4. \( X_4 = X_3^{\text{op}} \);
5. \( S = \{1, 2, 3, 4\}, \quad \tau_S = \emptyset, S, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\} \)
6. \( C_n = \{1, 2\} \times \mathbb{Z}_n, \ n \geq 2; \) a subbasis of the topology \( \tau_{C_n} \) is given by
   \[
   \{(2, k), (1, k)(2, k + [1])\}_{k \in \mathbb{Z}_n}.
   \]

Here \( \mathbb{Z}_n \) denotes the set \( \{0, 1, 2, \ldots, n-1\} \). In the following we write elements of \( C_n = \{1, 2\} \times \mathbb{Z}_n \) in the form \( a^k \) instead of \( (a, k) \). The directed graphs corresponding to these topological spaces are displayed in Figure 6.

**Theorem 7.3.** If \( X \in \{X_1, X_2, X_3, X_4, S\} \cup \{C_n \mid n \geq 2\} \), then \( \neg \text{UCT}(X) \) holds.

More precisely, our procedure provides the desired counterexamples for all spaces in the above list. For the space \( X_2 \) this was shown in \([7]\), and for \( X_4, S \) and all \( C_n \) it was verified in \([2]\). Hence we investigate the spaces \( X_1 \) and \( X_3 \) here. We also include the discussion of \( C_n \) from \([2]\).

**Remark 7.4.** The investigations cited above and those that follow at this place show that the categories \( \mathcal{N}^\mathcal{T}(X) \) for \( X \in \{X_1, X_2, X_3, X_4, S\} \) are all isomorphic in the same sense as described in Theorem 5.12.
7.1. Discussion of the space $X_3$. We begin with the case of the space $X_3$ which we describe in most detail. The specialisation order on $X_3 = \{1, 2, 3, 4\}$ is generated by the relations $1 \succ 3$, $2 \succ 3$, and $3 \succ 4$. We use abbreviatory notation like $134 := \{1, 3, 4\}$, and similarly. By [8, Lemma 2.35], a $C^*$-algebra over $X_3$ is a $C^*$-algebra $A$ with three distinguished ideals $I_1 := A(1)$, $I_2 := A(2)$, $I_3 := A(123)$,
subject to the conditions $I_1 \cup I_2 \subset I_3$ and $I_1 \cap I_2 = \{0\}$.

The connected, nonempty, locally closed subsets of $X_3$ are $\mathcal{LC}(X_3)^* = \{4, 34, 134, 234, 13, 23, 123, 1, 2\}$.

The top-dimensional simplices of the geometric realisation of the order complex $Ch(X_3)$ are the two 2-simplices corresponding to the triples $(1, 3, 4)$ and $(2, 3, 4)$. Their intersection is the 1-simplex corresponding to the pair $(3, 4)$.

Table 2 contains the isomorphism classes of the groups $K^\ast(S(Y, Z)) \cong \mathcal{NT}(Y, Z)$ for $Y, Z \in \mathcal{LC}(X_3)^*$. The determination of the spaces $S(Y, Z)$ is straight-forward from their definition, and the computation of the K-groups is elementary as well.

Using the general results from §3.5 one can simply determine generators of the groups $\mathcal{NT}_*(Y, Z) \cong K^\ast(S(Y, Z))$ computed above.

For instance, for all pairs $(Y, Z)$ of subsets $Y, Z \in \mathcal{LC}(X_3)^*$ with $\mathcal{NT}_*(Y, Z) \cong \mathbb{Z}$, the intersection $Y \cap Z$ is non-empty, closed in $Y$ and open in $Z$. Thus, by Corollary 3.41 $\mathcal{NT}_*(Y, Z)$ is generated by $\mu_Z^Y := r_{Y \cap Z}^Z \circ r_Y^{Y \cap Z}$.
Similarly, all odd natural transformations arise by composing the transformations $\mu^* Z$ induced by natural $\ast$-homomorphisms with boundary transformations in K-theory exact six-term sequences. For example, the group $\mathcal{NT}(34, 123) \cong \mathbb{Z}$ is generated by the two transformations $i_1 123 \circ \delta_{134}$ and $i_2 123 \circ \delta_{34}$. The corresponding generators in $K^\ast \left(\text{S}(34, 123) \setminus \{1, 2, 4\} \right)$ can be written as $f^\ast (v)$ and $g^\ast (v)$, where $v$ is a generator of $K^\ast (\{0, 1\})$, and $f, g : \text{Ch}(X_3) \Rightarrow \{0, 1\}$ are continuous maps defined similarly to the map in Lemma 3.38(iii) with the property that

$$f^{-1}(0) = \{1\}, \quad f^{-1}(1) = \{4\}$$

and

$$g^{-1}(0) = \{2\}, \quad g^{-1}(1) = \{4\}.$$  

We have now shown that the category $\mathcal{NT}^\ast$ is generated by transformations coming from natural six-term sequences. With respect to the canonical relations established in §3.4 we obtain the indecomposable transformations indicated in the following diagram:

$$\begin{array}{c}
134 \rightarrow 34 \rightarrow 1, \quad 234 \rightarrow 34 \rightarrow 2, \quad 3 \rightarrow 34 \rightarrow 4, \\
1 \rightarrow 123 \rightarrow 23, \quad 2 \rightarrow 123 \rightarrow 13, \quad 4 \rightarrow 123 \rightarrow 1234.
\end{array}$$

Table 2. Groups $\mathcal{NT}(Y, Z)$ of natural transformations for $X_3$
Proceeding as in the proof of Corollary 6.3, we find that $N^r_T(X_3)$ is the universal pre-additive category with these generators and relations, because the morphism groups of the universal pre-additive category are precisely those in Table 2.

**Lemma 7.6.** The ideal $NT_{nil}$ is nilpotent and the category $N^r_T$ decomposes as the semi-direct product $NT_{nil} \rtimes NT_{ss}$.

**Proof.** By the computations above, we have

$$NT_{nil} = \bigoplus_{Y \neq Z \in \text{LC}(X_3)^*} NT_*(Y, Z).$$

Hence $N^r_T = NT_{nil} \oplus NT_{ss}$ as Abelian groups. This implies the semi-direct product decomposition. The fact that $NT_{nil}$ is nilpotent follows immediately from the characterisation of the composition in $N^r_T$ provided in the previous section. □

Therefore, Properties 1 and 2 are fulfilled. However, Property 3 does not hold for the locally closed subset $34 \in \text{LC}(X)_*$: we have

$$(NT_{nil} \cdot M)(34) = \text{range}(r^{34}_{134}) + \text{range}(r^{34}_{234}) + \text{range}(i^{34}_3) = \ker(i^{1234}_1 \circ \delta^{34}_3) + \text{range}(i^{34}_3).$$

For a further simplification we would need an exact sequence containing the map $\delta^{1234}_3 := i^{1234}_1 \circ \delta^3_3$ which does not exist.

**7.2. The counterexamples for $X_3$.** In the previous section our classification method broke down because there is no exact sequence with connecting map $\delta^{1234}_3 = i^{1234}_1 \circ \delta^3_3$. In fact, the desired classification is wrong. In this section we exhibit

(1) an exact, entry-free module $M$ which is not projective,
(2) an exact module that has no projective resolution of length one,
(3) two non-isomorphic objects in the bootstrap class $B(X_3)$ with isomorphic filtrated $K$-theory.

The non-projective, exact, entry-free module. For $Y \in \text{LC}(X_3)^*$ we have defined the free $N^r_T$-module on $Y$ in Definition 5.1. The three transformations $134, 3, 234 \to 34$ in (7.5) induce a module homomorphism

$$j: P_{34} \to P^0 := P_{134} \oplus P_{1} \oplus P_{234}.$$

**Lemma 7.7.** The homomorphism $j$ is injective.

**Proof.** The longest transformations out of 34 are those to 13, 1234 and 23. With this we mean that every transformation out of 34 is a sum of transformations each factoring one of the three transformations above and that the list of these three transformations is minimal with this property. Therefore, it suffices to check that the maps

$$P_{34}(13) \to P^0(13), \quad P_{34}(1234) \to P^0(1234), \quad \text{and} \quad P_{34}(23) \to P^0(23)$$

are injective. This is true because the maps

$$N^r_T(34, 13) \to N^r_T(234, 13), \quad N^r_T(34, 1234) \to N^r_T(3, 1234),$$

$$\text{and} \quad N^r_T(34, 23) \to N^r_T(134, 23)$$

are isomorphisms of free cyclic Abelian groups. This, in turn, follows from the exactness of free modules and the vanishing of the groups $N^r_T(2, 13), N^r_T(4, 1234),$ and $N^r_T(1, 23)$. □
Since $j$ is a monomorphism, we can easily compute the cokernel

$$M := \text{coker}(j: P_{34} \to P^0).$$

We get the following values $M(Y)$ for $Y \in \mathcal{LC}(X_3)^*$:

$$(7.8) \quad \begin{array}{c}
\xymatrix{ & Z[1] & \\
0 \ar[r] & Z \ar[r]^-i & Z[1] \\
0 \ar[r] & Z \ar[r]^-i & Z[1].}
\end{array}$$

As a quotient of two exact modules, the module $M$ is exact by the two-out-of-three property. Therefore, the extension maps $i_1^{23}$ and $i_2^{23}$, and the boundary map $\delta_1^{23}$ act by isomorphisms on $M$. The other maps can be described in the following way: write $M(34)$ as $\mathbb{Z}^3/(1, 1, 1)$ and $M(4)$, $M(2)$, $M(1)$ as $\mathbb{Z}^2/(1, 1)$. Then the three maps $\mathbb{Z} \to \mathbb{Z}^2$ correspond to the three coordinate embeddings $\mathbb{Z} \hookrightarrow \mathbb{Z}^3$, and the maps $\mathbb{Z}^2 \to \mathbb{Z}$ correspond to the three projections $\mathbb{Z}^3 \twoheadrightarrow \mathbb{Z}^2$ onto coordinate hyperplanes.

**Proposition 7.9.** The module $M$ is exact and entry-free, but it is not projective.

**Proof.** We have already seen that $M$ is exact and entry-free. The projective resolution $P_{34} \to P^0 \to M$ does not split because there is no non-zero module homomorphism $P^0 \to P_{34}$ since $K^*\{S(34, 134)\} \cong K^*\{S(34, 3)\} \cong K^*(S(34, 234)) \cong 0$ by Table 2. This shows that $M$ is not projective. \qed

The exact module with projective dimension 2. For $k \in \mathbb{N}_{\geq 2}$ we define $M_k := M/k \cdot M$. This module is exact by the two-out-of-three property and it has the following projective resolution of length 2:

$$(7.10) \quad 0 \to P_{123} \xrightarrow{(−k,j)} P_{1323} \oplus P^0 \xrightarrow{(j,k)} P^0 \twoheadrightarrow M_k.$$  

From this resolution we compute

$$\text{Ext}^2_{\mathcal{N}T^*}(M_k, P_{123}) \cong \text{Hom}_{\mathcal{N}T^*}(P_{123}, P_{123})/(−k,j)^*(\text{Hom}_{\mathcal{N}T^*}(P_{123} \oplus P^0, P_{123}))$$

$$\cong \mathbb{Z}/k \cdot \mathbb{Z},$$

because $\text{Hom}_{\mathcal{N}T^*}(P_{123}, P_{123}) \cong \mathbb{Z}$ and $\text{Hom}_{\mathcal{N}T^*}(P^0, P_{123}) = 0$. This shows that the projective dimension of $M_k$ is 2.

Non-isomorphic objects in $\mathcal{B}(X_3)$ with isomorphic filtrated K-theory. As described in the beginning of this section we can find a $C^*$-algebra $A_k$ with $\text{FK}^*(A_k) \cong M_k$. Theorem 7.1 shows that $A_k$ is not $\mathcal{J}^2$-projective, and Theorem 7.2 yields the desired counterexample:

**Theorem 7.11.** There exist $C^*$-algebras $B$ and $D$ in the bootstrap class $\mathcal{B}(X_3)$ that are not $\text{KK}(X_3)$-equivalent but have isomorphic filtrated K-theory.

### 7.3. Counterexamples for $X_1$

The computations for the space $X_1$ are very similar to those for $X_3$. We will therefore only state the key results used for the construction of counterexamples on the different levels. The non-empty, connected, locally closed subsets are

$$\mathcal{LC}(X_1)^* = \{1234, 124, 134, 234, 34, 24, 14, 4, 1, 2, 3, 1234\}.$$  

The computation of the groups $\mathcal{N}T(Y, Z) \cong K^*(S(Y, Z))$ is summarised in Table 3.
Table 3. Groups $\mathcal{NT}(Y, Z)$ of natural transformations for $X_1$

| $Y \setminus Z$ | 1234 | 124 | 134 | 234 | 34 | 24 | 14 | 1 | 2 | 3 |
|-----------------|------|-----|-----|-----|----|----|----|---|---|---|
| 1234            | Z    | Z   | Z   | Z   | Z  | Z  | 0  | 0 | 0 |   |
| 124             | 0    | Z   | 0   | 0   | Z  | Z  | Z  | 0 | 0 | Z[1]|
| 134             | 0    | 0   | Z   | 0   | Z  | 0  | Z  | 0 | Z[1]|   |
| 234             | 0    | 0   | 0   | Z   | Z  | 0  | Z  | 0 | Z[1]|   |
| 34              | Z[1] | Z[1]| 0   | 0   | Z  | 0  | Z  | 0 | Z[1]| Z[1]| |
| 24              | Z[1]| 0   | Z[1]| 0   | 0  | Z  | 0  | Z[1]| 0 | Z[1]| Z[1]| |
| 14              | Z[1]| 0   | 0   | Z[1]| 0 | 0 | Z  | 0 | Z[1]| Z[1]|   |
| 4               | Z[1]| 2   | Z[1]| Z[1]| Z[1]| 0 | 0 | Z[1]| Z[1]| Z[1]|   |
| 1               | Z    | Z   | Z   | 0   | 0 | Z  | 0  | Z  | 0  |   |
| 2               | Z    | Z   | 0   | Z   | 0 | Z  | Z  | 0 | 0 | Z  |
| 3               | Z    | 0   | Z   | Z   | Z | Z  | 0  | 0 | 0 | Z  |

Again, it turns out that the category $\mathcal{NT}$ is generated by the canonical morphisms and relations discussed in §3.3. The indecomposable morphisms in $\mathcal{NT}$ are displayed in the following diagram.

As in the previous example, we construct a non-projective, exact, entry-free module

$$M := \text{coker}(P_4 \twoheadrightarrow P_{14} \oplus P_{24} \oplus P_{34}),$$

given by the cokernel of the monomorphisms induced by the natural transformations $r_{14}, r_{24}$, and $r_{34}$. The remaining counterexamples—the exact module with projective dimension 2 and the non-isomorphic objects in the bootstrap class $\mathcal{B}(X_1)$ with isomorphic filtrated K-theory—can now be obtained as described in the beginning of §7.

### 7.4. Counterexamples for the space $C_n$.

We apply our method described above for constructing counterexamples for the space $C_n$. We adopt the notation

$$C_n = \{1^0, 2^0, 1^1, 2^1, 1^2, \ldots, 1^{n-1}, 2^{n-1}, 1^n = 1^0\}$$

with the partial order given by the relations

$$1^0 < 2^0 > 1^1 < 2^1 < 1^2 < \ldots < 1^{n-1} < 2^{n-1} > 1^0.$$  

We define

$$F := C_n \setminus \{2^{n-1}, 1^0, 2^0\} = \{1^1, 2^1, 1^2, \ldots, 2^{n-2}, 1^{n-1}\}.$$  

**Definition 7.12.** In the following proofs we will say that a topological space is of *type H* if it is the difference of a contractible compact pair. We will say that it is of *type O* if it is the difference of a compact pair $(Z, W)$, where $Z$ is a contractible space and $W$ is the (topologically) disjoint union of two contractible subspaces.
Lemma 7.13. The indecomposable natural transformations in $\mathcal{N}T^*$ into $F$ are the two restriction transformations from $F^0 := \{1^0, 2^0\} \cup F$ and $F^n := F \cup \{2^{n-1}, 1^0\}$ to $F$.

Proof. For a start, $S(F, F) = \text{Ch}(F)$ and $\mathcal{N}T_\ast(F, F) \cong \mathbb{Z}$ is generated by the identity transformation. We have $S(F^0, F) = S(F^n, F) = \text{Ch}(F)$, so that $\mathcal{N}T_\ast(F^0, F) \cong \mathbb{Z}$ and $\mathcal{N}T_\ast(F^n, F) \cong \mathbb{Z}$. Corollary 3.41 implies that these groups are generated by the natural transformations $r^F_{F_0}$ and $r^F_{F^n}$, respectively. In the following we will determine generators of all further groups $\mathcal{N}T_\ast(Y, F)$ with $Y \in \mathcal{L}(C_n)^\ast$, $Y \neq F$, and verify that each of them factors through one of the two transformations $r^F_{F_0}$ and $r^F_{F^n}$.

We begin with supersets of $F$. Since $S(C_n, F) = \text{Ch}(F)$ is contractible, we have $\mathcal{N}T_\ast(C_n, F) \cong \mathbb{Z}$. By Corollary 3.41 this group is generated by $r^F_{C_n} = r^F_{F_0} \circ r^F_{F^n}$. Similarly, $S(F \cup \{2^0\}, F) = \text{Ch}(F)$, so that $\mathcal{N}T_\ast(F \cup \{2^0\}, F) \cong \mathbb{Z}$ is generated by the transformation $r^F_{F \cup \{2^0\}} = r^F_{F_0} \circ i^F_{F \cup \{2^0\}}$. The same reasoning applies to the set $F \cup \{2^{n-1}\}$.

Now we consider proper subsets of $F$. Let $Y = \{1^k, 2^k, \ldots, l\}$ with $1 < k \leq l < n - 1$. Then $S(Y, F)$ is of type O and hence $\mathcal{N}T_\ast(Y, F) \cong \mathbb{Z}[1]$. We claim that this group is generated by the transformation $i^F_Y \circ \delta^F_Y$, where $D = \{2^1, 1^{i+1} \ldots, 1^{n-1}\}$ is one of the two connected components of $F \setminus Y$. This follows from Corollary 3.44 because the spaces $S(Y, D) \cup S(Y, Y)$ and $S(Y, F) \setminus S(Y, D)$ have trivial K-theory. We have $i^F_D \circ \delta^F_Y = r^F_{F_0} \circ i^F_{Y \cup D} \circ \delta^F_Y$.

Let $Y$ be of one of the forms
\[
\{2^k, 1^{k+1}, \ldots, l\}, \quad \{1^k, 2^1, \ldots, l\}, \quad \{2^k, 1^{k+1}, \ldots, 1^{n-1}\}
\]
for $1 \leq k < l < n - 1$. Then $S(Y, F) = \text{Ch}(Y)$ and $\mathcal{N}T_\ast(Y, F) \cong \mathbb{Z}$ is generated by the transformation $i^F_Y$ which can either be written as $r^F_{F_0} \circ i^F_Y$ or as $r^F_{F_0} \circ i^F_Y^{-1}$.

For $Y = \{1^k, 2^k, \ldots, l\}$ with $1 < k \leq l < n - 1$ we have $\mathcal{N}T_\ast(Y, Z) = 0$ because $S(Y, F)$ is of type H. The same holds for $Y = \{2^k, 1^{k+1}, \ldots, l\}$ with $1 \leq k < l < n - 1$.

Finally, we investigate the sets $Y \in \mathcal{L}(C_n)^\ast$ that are neither supersets nor subsets of $F$. For $Y = \{1^k, 2^k, \ldots, b\}$ with $k > 1$ and $b \in \{2^{n-1}, 1^0, 2^0\}$ the space $S(Y, F)$ is of type H, so that $\mathcal{N}T_\ast(Y, F) = 0$.

However, if $Y = \{2^k, 1^{k+1}, \ldots, b\}$ with $k \geq 1$ and $b \in \{2^{n-1}, 1^0, 2^0\}$, we get $S(Y, F) = \text{Ch}(Y \cap F)$ and find that $\mathcal{N}T_\ast(Y, F) \cong \mathbb{Z}$ is generated by the natural transformation $i^F_{Y \cup D} \circ \delta^F_{Y \cup F}$. We have already seen that the transformation $r^F_{Y \cup F}$ factors through $r^F_{F_0}$. Analogous reasonings can be performed, respectively, for sets of the form $\{a, \ldots, 1^k\}$ or $\{a, \ldots, 2^k\}$ with $k \leq n - 2$ and $a \in \{2^{n-1}, 1^0, 2^0\}$.

The last remaining kind of connected, locally closed subsets of $C_n$ are those with non-connected intersection with $F$. Let
\[
(7.14) \quad Y = \{2^k, 1^{k+1}, \ldots, 2^{n-1}, 1^0, 2^0, \ldots, l\}
\]
with $1 \leq l < k < n - 1$. Then $S(Y, F)$ is the disjoint union of two contractible sets. Thus $\mathcal{N}T_\ast(Y, F) \cong \mathbb{Z}^2$. Two generators of this group are given by the transformations $i^F_Y \circ r^F_Y$, where $D_i$ with $i \in \{1, 2\}$ denote the two connected components of $Y \cap F$. Notice that $i^F_{D_i}$ factors through one of the two transformations $r^F_{F_0}$ and $r^F_{F^n}$.

Given the form (7.14) for $Y$ with $l < k - 1$, adding each of the points $1^k$ and $1^{k+1}$ turns one of the components of $S(Y, F)$ into a type H space whose K-theory vanishes, and thus removes one of the above generators. The description of the respective remaining one does not change.

This completes the list of locally closed, connected subsets of $C_n$. \qed
Lemma 7.15. The longest natural transformations in $N\mathcal{T}^*$ out of $F$ are the boundary transformations $\delta^0_F(1^n, Z^0)$, $\delta^1_F(2^{n-1}, I^0)$, and $\delta^0_{C^n} := i^{C^n}_{(2^n)} \circ \delta^0_F$.

Proof. The space $S(F, \{1^0, 2^0\})$ is homeomorphic to the open interval. This shows that $N\mathcal{T}_s(F, \{1^0, 2^0\}) \cong \mathbb{Z}[1]$. By Corollary 3.14, this group is generated by the natural transformation $\delta^1_F(1^n, Z^0)$ because the $K^1$-group of $S(F, F) \cup S(F, \{1^0, 2^0\})$ is trivial. Symmetrically, $N\mathcal{T}_s(F, \{2^{n-1}, 1^0\}) \cong \mathbb{Z}[1]$ is generated by the transformation $\delta^0_F(2^{n-1}, 1^0)$.

The space $S(F, C_n)$ is of type O as well, and, by Corollary 3.14, we find that $N\mathcal{T}_s(F, C_n) \cong \mathbb{Z}[1]$ is generated by $\delta^0_{C^n}$ as defined above because the spaces $S(F, F) \cup S(F, \{2^n\})$ and $S(F, C_n) \setminus S(F, \{2^n\})$ have vanishing K-theory. In the following we will determine generators of all further groups $N\mathcal{T}_s(F, Z)$ with $Z \in LC(C_n)^* \setminus F$, and verify that each of them factors one of the three transformations $\delta^1_F(1^n, Z^0)$, $\delta^1_F(2^{n-1}, I^0)$, and $\delta^0_{C^n}$. In fact, we will find that all transformations out of $F$ factor the transformation $\delta^0_F$ (except for $\delta^0_F(1^n, Z)$ and $\delta^0_F(2^{n-1}, I^0)$, of course).

We will not explicitly cite the theorems used for this each time.

We begin with the supersets of $F$ again. Since $S(F, F^0)$ is of type H, we get $N\mathcal{T}_s(F, F^0) = 0$ and, symmetrically, $N\mathcal{T}_s(F, F^n) = 0$. The same holds for the sets $F \cup \{2^n\}$ and $F \cup \{2^{n-1}\}$. For $Z = F \cup \{2^0, 2^{n-1}\}$, however, the space $S(F, Z)$ is of type O so that $N\mathcal{T}_s(F, Z) \cong \mathbb{Z}[1]$. A generator of this group is given by the composition $i_{Z(2^n)} \circ \delta^0_F$. We have $i_{Z(2^n)} \circ \delta^0_F = \delta^0_F$, which proves that $i_{Z(2^n)} \circ \delta^0_F$ factors the transformation $\delta^0_F$.

Now we examine proper subsets of $F$. Let $Z = \{1^k, 2^k, \ldots, 1^l\}$ with $1 \leq k \leq l \leq n - 1$. Then $S(F, Z)$ is contractible and $N\mathcal{T}_s(Y, F) \cong Z$ is generated by the restriction $r^\delta_F$. We have $\delta^0_{Z^n} \circ r^\delta_F = \pm \delta^0_Z$, where $\delta^0_Z$ denotes the composition $i_{Z(2^n-1)} \circ \delta^0_F(Z^n-1)$. Replacing $Z$ as above by $Z \setminus \{1^k\}$ or $Z \setminus \{1^l\}$ yields a trivial group of natural transformations. For $Z' = \{2^k, 1^{k+1}, \ldots, 2^{l-1}\}$ with $1 \leq k < l \leq n - 1$ we get $N\mathcal{T}_s(F, Z') \cong \mathbb{Z}[1]$ and find the generator $\delta^0_{Z'} \circ r^\delta_F$, where $D = \{1^k, 2^k, \ldots, 1^{n-1}\}$ is one of the two components of $F \setminus Z'$. We have $i_{Z'} \circ \delta^0_{Z'} \circ r^\delta_F = \delta_{F}$. For $Z = \{1^k, 2^k, \ldots, b\}$ with $k > 1$ and $b \in \{2^{n-1}, 1^0\}$, the space $S(F, Z)$ is of type H, so that $N\mathcal{T}_s(F, Z) = 0$. Yet if $b = 2^0$, then $S(F, Z)$ is of type H $\cup O$, that is, it is the disjoint union of a space of type H and a space of type O, and $N\mathcal{T}_s(F, Z) \cong \mathbb{Z}[1]$ is generated by $\delta^0_{Z'} \circ r^\delta_F$. Notice that $i_{Z'} \circ \delta^0_{Z'} \circ r^\delta_F = \delta_{F}$. Symmetrical results hold if $Z'$ is of the form $\{a, \ldots, 1^k\}$ with $k < n - 1$ and $a \in \{2^{n-1}, 1^0, 2^0\}$.

Now let $Z = \{2^k, 1^{k+1}, \ldots, b\}$ with $b \geq 1$ and $b \in \{2^{n-1}, 1^0\}$. Then $S(F, Z)$ is of type O and $N\mathcal{T}_s(F, Z) \cong \mathbb{Z}[1]$ is generated by $\delta^0_{Z'} \circ r^\delta_F$ and we have $i_{Z'} \circ \delta^0_{Z'} \circ r^\delta_F = \pm \delta^0_{Z'}$. For $Z$ as above, but with $b = 2^0$, the space $S(F, Z)$ is of type O $\cup$ O. Hence $N\mathcal{T}_s(F, Z) \cong \mathbb{Z}[1]^2$. Two generators are given by $\delta^0_{Z'} \circ r^\delta_F$ and $\delta^0_{Z'} \circ r^\delta_F$ if $k > 1$, and by $i_{Z'} \circ \delta^0_{Z'} \circ r^\delta_F$ and $i_{Z'} \circ \delta^0_{Z'} \circ r^\delta_F$ for $k = 1$. These can be seen to factor the transformation $\delta^0_{Z^n}$ as before. Again, symmetrical arguments apply to sets $Z$ of the form $\{a, \ldots, 2^k\}$ with $k < n - 1$ and $a \in \{2^{n-1}, 1^0, 2^0\}$.

Finally, let

$$Z = \{2^k, 1^{k+1}, \ldots, 2^{n-1}, 1^0, 2^0, 2^0, \ldots, 2^k\}$$

with $1 \leq l < k < n - 1$. Generators of the group $N\mathcal{T}_s(F, Z) \cong \mathbb{Z}[1]$ can be described as in the previous paragraph, including the factorisation of $\delta^0_{Z^n}$. Adding
The points $1^k$ and $1^{k+1}$ to $Z$ as in (7.10) with $l < k - 1$ removes respectively one of the afore-stated generators, not violating the desired characterisation.

The two restriction transformations from $F^0$ and $F^n$ to $F$ induce a module homomorphism
\[ j : P_F \rightarrow P^0 := P_{F^0} \oplus P_{F^n}. \]

**Lemma 7.17.** The homomorphism $j$ is injective.

**Proof.** By Lemma 7.15 if suffices to show that the maps
\[
\begin{align*}
P_F(\{1^0, 2^0\}) & \rightarrow P^0(\{1^0, 2^0\}), \\
P_F(\{2^{n-1}, 1^0\}) & \rightarrow P^0(\{2^{n-1}, 1^0\}), \\
P_F(C_n) & \rightarrow P^0(C_n)
\end{align*}
\]
are injective. This follows from the injectivity of the maps
\[
\begin{align*}
\mathcal{N}T(F, \{1^0, 2^0\}) & \rightarrow \mathcal{N}T(F^n, \{1^0, 2^0\}), \\
\mathcal{N}T(F, \{2^{n-1}, 1^0\}) & \rightarrow \mathcal{N}T(F^0, \{2^{n-1}, 1^0\}), \\
\mathcal{N}T(F, C_n) & \rightarrow \mathcal{N}T(F^0, C_n),
\end{align*}
\]
which we obtain from the vanishing of the groups
\[
\begin{align*}
\mathcal{N}T(\{2^{n-1}, 1^0\}, \{1^0, 2^0\}), & \mathcal{N}T(\{1^0, 2^0\}, \{2^{n-1}, 1^0\}) \\
\mathcal{N}T(\{1^0, 2^0\}, C_n) & \mathcal{N}T(\{1^0, 2^0\}, C_n) \mathcal{N}T(\{1^0, 2^0\}, C_n)
\end{align*}
\]
and $\mathcal{N}T(\{1^0, 2^0\}, C_n)$. \[ \square \]

**Proposition 7.18.** The module $M := \text{coker}(j : P_F \rightarrow P^0)$ is exact and entry-free, but it is not projective.

**Proof.** The module $M$ is exact by the two-out-of-three property. The fact that $M$ is entry-free follows from a direct investigation of the map $j$ via generators of the Abelian groups involved. This is particularly easy when $P_F(Z)$ is of rank 1. As an example for one of the more complicated cases, we discuss the case $Z = \{2^{n-1}, 1^0, 2^0\} = C_n \setminus F$. In the proof of Lemma 7.15, we found that $P_F(Z) = \mathcal{N}T_+(F, Z) \cong \mathbb{Z}[1]_2.$ is generated by $\delta_{\{1^1\}}^F \circ r_{F^n}^{(1)}$ and $\delta_{\{1^2\}}^F \circ r_{F^n}^{(1)}$. Similarly, $P_F^n(Z) = \mathcal{N}T_+(F^n, Z)$ is generated by $\delta_{\{1^2\}}^F \circ r_{F^n}^{(1)} = \delta_{\{1^1\}}^F \circ r_{F^n}^{(1)} = \delta_{\{1^2\}}^F \circ r_{F^n}^{(1)} \circ r_{F^n}$ and $\delta_{\{1^1\}}^F \circ r_{F^n}^{(1)} = \delta_{\{1^2\}}^F \circ r_{F^n}^{(1)} \circ r_{F^n}$, and $P_F^n(Z) = \mathcal{N}T_+(F^n, Z)$ is generated by $\delta_{\{1^2\}}^F \circ r_{F^n}^{(1)} = \delta_{\{1^1\}}^F \circ r_{F^n}^{(1)} \circ r_{F^n}$ and $\delta_{\{1^2\}}^F \circ r_{F^n}^{(1)} = \delta_{\{1^1\}}^F \circ r_{F^n}^{(1)} \circ r_{F^n}$. Hence the map $j(Z) : P_F(Z) \rightarrow P^0(Z)$ can be identified with the map
\[
\mathbb{Z}^2 \rightarrow \mathbb{Z}^4, \quad (a, b) \mapsto (a, b, a, b)
\]
whose cokernel is entry-free. The computations for all other subsets $Z$ in $\text{LC}(C_n)^*$ are similar. The projective resolution $0 \rightarrow P_F \rightarrow P^0 \rightarrow M$ does not split because there is no non-zero homomorphism from $P^0$ to $P_F$. This follows from $\mathcal{N}T(F, F^0) = 0$ and $\mathcal{N}T(F, F^n) = 0$. \[ \square \]

This provides the counterexample on the level of projective modules. The counterexamples on the two deeper levels now follow as described in the beginning of this section. The three assumptions listed at the beginning of 7.7 have all been verified and we obtain the desired result.

**Theorem 7.19.** There exist $C^*$-algebras $B$ and $D$ in the bootstrap class $B(C_n)$ that are not $\text{KK}(C_n)$-equivalent but have isomorphic filtrated $K$-theory.
8. The complete description

We already know that, if $X$ is of type (A), then $UCT(X)$ holds. The aim of this section is to prove the converse implication. We want to show that, if $X$ is not of type (A), then we can “embed” one of the counterexamples from §7 into $X$. Knowing that $\neg UCT$ holds for the counterexample, we will use the results from §7 to conclude that $\neg UCT(X)$ holds.

**Definition 8.1.** A topological subspace $X'$ of a finite $T_0$-space $X$ is **tight** if
\[
y \to x \text{ in } X' \iff y \to x \text{ in } X,
\]
that is, there is a directed edge from $y$ to $x$ in $\Gamma(X')$ if and only if there is a directed edge from $y$ to $x$ in $\Gamma(X)$ (see Definition 2.2).

So, if $X'$ is a topological subspace of $X$, then $X'$ is tight in $X$ if and only if $\Gamma(X')$ is a subgraph of $\Gamma(X)$. If $\Gamma'$ is another finite $T_0$-space such that there exists an embedding $\Gamma(\Gamma') \hookrightarrow \Gamma(X)$ as directed graphs, then $\Gamma'$ may be viewed as a tight subspace of $X$.

**Lemma 8.2.** Let $X$ be a finite $T_0$-space such that $\Gamma(X)$ contains either $\Gamma(X_1)$ or $\Gamma(X_2)$ as a subgraph. Then $\neg UCT(X)$ holds.

**Proof.** $\Gamma(X_1) \subseteq \Gamma(X)$ allows us to view $X_1$ as a tight subspace of $X$. Let $y \in LC(X_1)$ then there are $x_1, x_2 \in X_2$ such that $x_1 \geq y \geq x_2$. Without loss of generality we may assume that $x_1 = 1$ and $x_2 = 4$. Since $1 \to 4$ we have $y = 1$ or $y = 4$ by Lemma 2.3. Therefore $X_1$ is locally closed in $X$, similarly we see that $X_2$ is locally closed in $X$ if $\Gamma(X_2) \subseteq \Gamma(X)$. Therefore $\neg UCT(X)$ holds by Theorem 7.3 and Proposition 4.6(ii).

**Proposition 8.3.** Let $X$ be a finite $T_0$-space such that $\Gamma(X)$ contains $\Gamma(X_3)$ as a subgraph. Define
\[
\pi_3: LC(X_3) \to X_3, \quad \pi_3(x) = \begin{cases} x & \text{if } x \in X_3, \\ 3 & \text{else.} \end{cases}
\]
Then $\pi_3$ is continuous.

**Proof.** Let us first show the following claim:

If $x \in LC(X_3) \setminus X_3$, then $x \not\geq 4$, $x \not\leq 3$, $x \not\leq 1$, $x \not\geq 2$.

Let $x \in LC(X_3) \setminus X_3$. Then there are $x_1, x_2 \in X_3$ such that $x_1 \prec x \prec x_2$. Since $1 \to 3$, $2 \to 3$, $3 \to 4$ Lemma 2.3 shows that $x_1 \geq 4$ and $x_2 \in \{1, 2\}$. Without loss of generality we may assume that $x_2 = 1$. This implies of course that $x \not\leq 1$ and $x \not\geq 4$. Assume $x \geq 2$, then $1 \succ x \succ 2 \succ 3$ this is a contradiction to $1 \to 3$. By the same argument $x \geq 3$ leads to a contradiction. Assume $x \leq 3$, then $4 \prec x \prec 3$. This is a contradiction to $3 \to 4$. This shows the claim.

To check that $\pi_3$ is continuous, we have to check that it is monotone. Let $x, y \in LC(X_3)$, if $x, y \in X_3$ then $x \preceq y$ clearly implies $\pi_3(x) \preceq \pi_3(y)$. If $x, y \in LC(X_3) \setminus X_3$ then $\pi_3(x) = 3 = \pi_3(y)$. If $x \in LC(X_3) \setminus X_3$, $y \in X_3$ and $y \prec x$, then $y = 4$ by Claim #1. Therefore $\pi_3(4) = 4 \prec 3 = \pi_3(x)$. If $y \in X_3$, $x \in LC(X_3) \setminus X_3$ and $y \succ x$, then either $y = 1$ or $y = 2$ by Claim #1, and in both cases $\pi_3(y) = y \succ 3 = \pi_3(x)$. This shows that $\pi_3$ is continuous.

**Proposition 8.4.** Let $X$ be a finite $T_0$-space such that $\Gamma(X)$ contains $\Gamma(X_4)$ as a subgraph. Define
\[
\pi_4: LC(X_4) \to X_4, \quad \pi_4(x) = \begin{cases} x & \text{if } x \in X_4, \\ 3 & \text{else.} \end{cases}
\]
Then $\pi_4$ is continuous.
These maps are continuous since they are monotone. It is clear that

\[ \therefore \]

This means that there are as many vertices with oriented degree

\[ n \]

group of order

\[ n \]

case; only notation becomes a bit more complicated. Let

\[ X \]

Γ(\(X\))

48 RASMUS BENTMANN AND MANUEL KÖHLER

Proof. Assume \(\Gamma(X_3) \subseteq \Gamma(X)\) and let \(Y = LC(X_3)\). There is an inclusion \(\iota_3: X_3 \hookrightarrow LC(X_3)\) and \(\pi_3: LC(X_3) \to X_3\) from Proposition \(8.3\). We clearly have \(\pi_3 \circ \iota_3 = id_{X_3}\). This shows that \(\neg UCT(LC(X_3))\) holds by Proposition \(1.9(i)\) and therefore \(\neg UCT(X)\) holds by Proposition \(1.6(i)\). The same arguments using \(\iota_4: X_4 \hookrightarrow LC(X_4)\) and \(\pi_4\) from Proposition \(8.4\) show the corresponding statement for \(X_4\).

\[ \square \]

Corollary 8.5. Let \(X\) be a finite \(T_0\)-space such that \(\Gamma(X)\) contains either \(\Gamma(X_3)\) or \(\Gamma(X_4)\) as a subgraph. Then \(\neg UCT(X)\) holds.

Proof. \(\Gamma(X)\) must contain either \(\Gamma(X_1), \Gamma(X_2), \Gamma(X_3)\) or \(\Gamma(X_4)\) as a subgraph.

\[ \square \]

Proposition 8.7. Let \(X\) be a finite connected \(T_0\)-space such that every vertex of \(\Gamma(X)\) has (unoriented) unoriented degree 2. Then \(\neg UCT(X)\) holds.

Proof. The assumption means that \(\Gamma(X)\) as an undirected graph consists of a cycle. By the definition of the oriented degree \(d_o\) from §2.4 we have \(d_o(x) \in \{-2, 0, 2\}\) for every \(x \in X\) and

\[ \sum_{x \in X} d_o(x) = 0. \]

This means that there are as many vertices with oriented degree 2 as vertices with oriented degree \(-2\). Let \(n\) be the number of vertices with oriented degree 2. Since \(\Gamma(X)\) cannot be a directed circle, \(n\) is at least 1.

We first consider the case \(n = 1\). Then there is exactly one vertex \(a\) with oriented degree 2, one vertex \(b\) with oriented degree \(-2\) and two directed paths \(\rho = (v_1)i=0,...,n\) and \(\sigma = (w_i)i=0,...,m\) from \(a\) to \(b\) such that

\[ \rho \cap \sigma = \{a, b\}, \quad \rho \cup \sigma = X. \]

Define maps \(f: X \to S\) and \(g: S \to X\) via

\[ f(x) = \begin{cases} 1 & \text{if } x = a, \\ 2 & \text{if } x = v_i \text{ for } i = 1, \ldots, n - 1, \\ 3 & \text{if } x = w_i \text{ for } i = 1, \ldots, m - 1, \\ 4 & \text{if } x = b, \end{cases} \]

and \(g(s) = \begin{cases} a & \text{if } s = 1, \\ v_1 & \text{if } s = 2, \\ w_1 & \text{if } s = 3, \\ b & \text{if } s = 4. \end{cases} \)

These maps are continuous since they are monotone. It is clear that \(f \circ g = id_S\). Therefore \(\neg UCT(X)\) holds by Theorem \(2.3\) and Proposition \(4.6(ii)\).

Now we investigate the case \(n > 1\). We will basically proceed as in the previous case; only notation becomes a bit more complicated. Let \(C(n)\) denote the cyclic group of order \(n\). Ordering the vertices of oriented degree 2 and \(-2\) clockwise, we obtain sequences \((a_k)\) and \((b_k)\) in \(X\) such that \(d_o(a_k) = 2\) and \(d_o(b_k) = -2\) for all \(k \in C(n)\). Analogously to the previous case, there is a sequence of directed paths \((\rho^k = (v^k_i)i=1,...,m_k)\) from \(a_k\) to \(b_k\) and a sequence of directed paths \((\sigma^k = (w^k_i)i=1,...,m_k)\) from \(a_k\) to \(b_k-1\) such that

\[ \rho^k \cap \rho^l = \sigma^k \cap \sigma^l = \emptyset \text{ if } k \neq l, \quad \rho^k \cap \sigma^l = \begin{cases} a_k & \text{if } k = l, \\ b_k & \text{if } k = l - 1, \\ \emptyset & \text{else,} \end{cases} \]

and

\[ \bigcup_{k \in C(k)} \rho^k \cup \bigcup_{k \in C(k)} \sigma^k = X. \]
Define maps \( f: X \to C_\infty \) and \( g: C_\infty \to X \) via

\[
f(x) = \begin{cases} 
(k, a) & \text{if } x = a, \\
(k, b) & \text{if } x = v_k^i \text{ for } i = 1, \ldots, n_k, \\
(1, b) & \text{if } x = w_k^i \text{ for } i = 1, \ldots, m_k - 1,
\end{cases}
\]

and

\[
g((k, y)) = \begin{cases} 
 a & \text{if } y = a, \\
b & \text{if } y = b.
\end{cases}
\]

The maps \( f \) and \( g \) are monotone and thus continuous. Clearly, \( f \circ g = \text{id}_{C_\infty} \).

Therefore \( \neg \text{UCT}(X) \) holds by Theorem 7.3 and Proposition 4.6(ii).

The following lemma provides an alternative characterization of type (A) spaces.

**Lemma 8.8.** Let \( X \) be a finite connected \( T_0 \)-space with more than one point. The following statements are equivalent:

1. \( X \) is of type (A);
2. there are exactly two vertices in \( X \) with unoriented degree 1, all other vertices have unoriented degree 2.

**Proof.** The direction (1) \( \Rightarrow \) (2) is obvious (see Figure II on page 23). For the converse direction, notice that (2) implies that the graph \( \Gamma(X) \) corresponding to the specialisation preorder on \( X \) is isomorphic as an undirected graph to the graph corresponding to the specialisation preorder on the totally ordered space with the same number of points. This shows that \( \Gamma(X) \) is isomorphic as a directed graph to the graph corresponding to the specialisation preorder on some type (A) space as displayed in Figure II. Since we are dealing with \( T_0 \)-spaces this implies that \( X \) is homeomorphic to that type (A) space.

**Theorem 8.9.** Let \( X \) be a finite \( T_0 \)-space. Then \( \text{UCT}(X) \) holds if and only if \( X \) is a disjoint union of spaces of type (A). Otherwise, \( \neg \text{UCT}(X) \) holds.

**Proof.** That \( \text{UCT}(X) \) holds if \( X \) is a disjoint union of spaces of type (A) follows from Theorem 5.13 and Lemma 4.5. Now let \( X \) be a space such that \( \neg \text{UCT}(X) \) does not hold. By Lemma 4.5 it suffices to show that \( X \) is of type (A) under the assumption that \( X \) is connected (and hence, by Lemma 4.4, that \( \Gamma(X) \) is connected as an undirected graph). By Corollary 8.6 all vertices \( x \) of \( \Gamma(X) \) have unoriented degree less than 3. By the last remark and Proposition 8.7 there is at least one vertex of unoriented degree less than 2. Since \( \Gamma(X) \) is connected as an undirected graph and finite, there are exactly two vertices of unoriented degree 1 and all other vertices have unoriented degree 2. Thus \( X \) is of type (A) by Lemma 8.8 as claimed.

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