THE SPACES OF LAURENT POLYNOMIALS,
GROMOV-WITTEN THEORY OF \( \mathbb{P}^1 \)-ORBIFOLDS, AND
INTEGRABLE HIERARCHIES

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Abstract. Let \( M_{k,m} \) be the space of Laurent polynomials in one variable
\( x^k + t_1 x^{k-1} + \ldots + t_{k+m} x^{-m} \), where \( k, m \geq 1 \) are fixed integers and \( t_{k+m} \neq 0 \).
According to B. Dubrovin [11], \( M_{k,m} \) can be equipped with a semi-simple
Frobenius structure. In this paper we prove that the corresponding de-
scentendent and ancestor potentials of \( M_{k,m} \) (defined as in [16]) satisfy Hirota
quadratic equations (HQE for short).

Let \( C_{k,m} \) be the orbifold obtained from \( \mathbb{P}^1 \) by cutting small discs \( D_1 \sim \{ |z| \leq \epsilon \} \) and
\( D_2 \sim \{ |z|^{-1} \leq \epsilon \} \) around \( z = 0 \) and \( z = \infty \) and gluing back
the orbifolds \( D_1/\mathbb{Z}_k \) and \( D_2/\mathbb{Z}_m \) in the obvious way. We show that the
orbifold quantum cohomology of \( C_{k,m} \) coincides with \( M_{k,m} \) as Frobenius
manifolds. Modulo some yet-to-be-clarified details, this implies that t he
descendent (respectively the ancestor) potential of \( M_{k,m} \) is a generating
function for the descendent (respectively ancestor) orbifold Gromov–Witten
invariants of \( C_{k,m} \).

There is a certain similarity between our HQE and the Lax operators
of the Extended bi-graded Toda hierarchy, introduced by G. Carlet in [7].
Therefore, it is plausible that our HQE characterize the tau-functions of
this hierarchy and we expect that the Extended bi-graded Toda hierarchy
governs the Gromov–Witten theory of \( C_{k,m} \).

1. Introduction

1.1. Background. By definition (see [11] or [26]), a Frobenius structure on a
manifold \( M \) is a collection of a flat metric \( g \) on \( M \), a multiplication \( \bullet \) in each
tangent space \( T_t M \), depending smoothly on \( t \) and satisfying the Frobenius
property \( g(X \bullet Y, Z) = g(Y, X \bullet Z) \), and a flat vector field \( e \) which is a unity
with respect to \( \bullet \), such that certain integrability conditions are satisfied. For
example, if \( X \) is a compact symplectic manifold then the cohomology algebra
\( H^*(X) \) is naturally equipped with a Frobenius structure where the metric is
given by the Poincaré pairing and the multiplication by the quantum cup
product, see e.g. [18] for more details.

Date: February 12, 2022.
2000 Math. Subj. Class. 14N35, 17B69, 32S30.
Key words and phrases. oscillating integrals, Frobenius structure, orbifold quantum co-
homology, bosonic Fock space, vertex operators, Hirota quadratic (bilinear) equations.

1
A Frobenius manifold $M$ is called (generically) semi-simple if there exists a point $t \in M$ such that the corresponding tangent space $T_t M$ is a semi-simple algebra, i.e., it has no nilpotents. For semi-simple $M$, A. Givental [16] introduced the so-called total descendent and total ancestor potentials, denoted respectively by $D^M_t$ and $A^M_t$, where $t \in M$ is a semi-simple point. They belong to the Fock space $B$, which is an infinite dimensional vector space described as follows: if we pick a trivialization of the tangent bundle $TM \cong M \times H$, corresponding to a choice of a flat coordinate system on $M$ (here $H$ is an arbitrary fixed tangent space of $M$), then $B$ is a certain completion of the space of functions on $H_+ := H[z]$. Moreover, A. Givental conjectured that if $M$ is a Frobenius manifold coming from the quantum cohomology theory of a compact Kähler manifold $X$ then $D^M_t$ (respectively $A^M_t$) are generating functions for the descendent (respectively ancestor) Gromov–Witten invariants of $X$. This conjecture is proven for toric manifolds (see [18] for Toric Fano case and [22] for general toric case), Flag manifolds [24], and Grassmannians [5]. Recently, C. Teleman announced a classification of semi-simple cohomological field theories. Together with some yet-to-be-clarified technical details, this implies Givental’s conjecture in general.

Let $q_n^a, 1 \leq a \leq N, n = 0, 1, 2, \ldots$ be a set of formal variables. By fixing a basis in $H_+$, we identify the Fock space $B$ with the space of formal series on the variables $q_n^a$ with complex coefficients. Given $\tau \in B$, we will refer to the coefficients in the corresponding formal series as Fourier coefficients. In this paper we prove that if $M = M_{k,m}$ is the space of Laurent polynomials in one variable, then the Fourier coefficients of $D^M_t$ and $A^M_t$ satisfy an infinite system of quadratic relations. Alternatively, these quadratic relations can be written as an infinite system of PDE’s which involve quadratic expressions of $\tau$, its partial derivatives, and its translation. We refer to such a system of PDEs as Hirota Quadratic Equations (HQE for short). Recently, G. Carlet [7] associated an integrable hierarchy to $M_{k,m}$ which fits in the general framework of [15]. We expect that our HQE give a description of Carlet’s hierarchies in terms of HQEs and tau-functions. We also prove that the Frobenius manifold $M_{k,m}$ is isomorphic to the orbifold quantum cohomology of $C_{k,m}$.

Frobenius manifolds and integrable systems are closely related (c.f. [15]). Some classes of integrable systems can be described in terms of $\tau$-functions and Hirota quadratic equations (also known as Hirota bi-linear equations). Examples include KdV, KP, and Toda lattice hierarchies. Here is, to the best of our knowledge, a complete list of pairs consisting of a semi-simple Frobenius manifold $M$ and an integrable hierarchy for which it is known that the potential $D^M_t$ is a tau-function of the corresponding hierarchy:

— $M$ is the space of miniversal deformations of $A$, $D$, or $E$ type singularity AND the Kac–Wakimoto hierarchies corresponding to the Coxeter
transformation in the Weyl algebra of the simple Lie algebras of A, D, or E type ([17], [19]).
— Quantum cohomology of a point (which coincides with the miniversal deformation of $A_1$ singularity) AND the KdV hierarchy ([35], [25]).
— Quantum cohomology of $\mathbb{CP}^1$ AND the Extended Toda hierarchy ([27], [30]).
— Equivariant quantum cohomology of $\mathbb{CP}^1$ AND the 2-Toda hierarchy ([29], [30]).
— Orbifold quantum cohomology of the classifying stack $BG$ of a finite group $G$ AND $|\text{Conj}(G)|$ commuting copies of the KdV hierarchies, where $\text{Conj}(G)$ is the set of conjugacy classes of $G$ ([23]).

The results of this paper suggest that we can add one more pair to the above list:
— Orbifold quantum cohomology of $\mathcal{C}_{k,m}$ AND the Extended bi-graded Toda hierarchy.

To complete this, it remains to clarify the following details: the functions satisfying our HQEs are tau-functions of the Extended bi-graded Toda hierarchy, and the potential $\mathcal{D}^M$ (respectively $\mathcal{A}^M_i$) is a generating function for descendant (respectively ancestor) orbifold Gromov–Witten invariants of $\mathcal{C}_{k,m}$.
The solutions to these two problems should not be very difficult: for the first one we need to generalize the techniques from [28], and the second one follows either from Teleman’s work [32], or alternatively can be proven by virtual localization (see [34] for details).

1.2. Summary of results. Given positive integers $k$ and $m$ and a non-zero complex number $Q$, we denote by $M$ the space of Laurent polynomials

\[ f = x^k + \sum_{i=1}^k t_i x^{k-i} + \sum_{j=1}^{m-1} t_{k+j} \left(Q e^{tN}/x\right)^j + \left(Q e^{tN}/x\right)^m, \]

where $N = k + m$, i.e., $M \cong \mathbb{C}^{N-1} \times \mathbb{C}^*$. Each tangent space $T_f M$ is naturally identified with the local algebra $\mathbb{C}[x, x^{-1}]/\langle \partial_x f \rangle$: the vector field $\partial/\partial t_i$ corresponds to the projection of $\partial f/\partial t_i$ in $\mathbb{C}[x, x^{-1}]/\langle \partial_x f \rangle$. Via this identification the product in the local algebra defines an associative, commutative product $\bullet_f$ on the tangent space $T_f M$ with unity $e = \partial/\partial t_k$.

Furthermore, let $\omega = dx/x$ be the standard volume form on $\mathbb{C}^*$. Then we equip each tangent space with a residue pairing

\[ (\partial_{t_i}, \partial_{t_j})_f = - (\text{res}_{x=0} + \text{res}_{x=\infty}) \frac{(\partial_{t_i} f \omega) (\partial_{t_j} f \omega)}{df}. \]

Finally, we assign degrees to $x$ and $t_i$ such that $f$ becomes a homogeneous polynomial of degree 1. In order to keep track of the homogeneity properties
of functions on \( M \), we introduce the following Euler vector field:

\[
E = \sum_{i=1}^{k} \frac{i}{k} t_i \partial_i + \sum_{j=1}^{m-1} \left( 1 - \frac{j}{m} \right) t_{k+j} \partial_{k+j} + \left( \frac{1}{k} + \frac{1}{m} \right) \partial_N.
\]

The data introduced here satisfy an integrability condition: for each \( z \in \mathbb{C}^* \)

\[
\nabla = \nabla^{L.C.} - z^{-1} \sum_{i=1}^{N} (\partial_i \bullet) dt_i,
\]

(1.2)

is a flat connection (i.e. \( \nabla^2 = 0 \)) on \( TM \), where \( \nabla^{L.C.} \) is the Levi-Civita connection of the residue metric. In particular, \( \nabla^{L.C.} \) is flat as well.

Let \( A^p_{ij} \) be the structure constants of \( \bullet \) and \( g_{ij} \) is the tensor of the residue metric, i.e., \( \partial/\partial t_i \bullet \partial/\partial t_j = \sum_p A^p_{ij}(f) \partial/\partial t_p \) and \( g_{ij}(f) := (\partial/\partial t_i, \partial/\partial t_j) f \). Note that \( A^p_{ij} \) and \( g_{ij} \) are polynomials in \( t_1, \ldots, t_{N-1}, Q e^{\tau N} \), thus taking the corresponding free terms yields (in each tangent space \( T_f M \)) an associative, commutative multiplication, which will be called cup product or classical multiplication, and a non-degenerate bilinear pairing. The corresponding algebra structure on \( T_f M \) can be described explicitly as follows: under the map

\[
\partial/\partial t_{k-i} \mapsto \phi_i := X^i, \partial/\partial t_{k+j} \mapsto \phi_{k+j} := Y^j, \partial/\partial t_N \mapsto m \phi_N := m Y^m,
\]

where \( 1 \leq i \leq k-1, 0 \leq j \leq m-1 \), the cup product corresponds to the multiplication in the algebra \( H := \mathbb{C}[X, Y]/\langle kX^k - m Y^m, XY \rangle \) and the free terms of \( g_{ij} \) induce a non-degenerate bilinear pairing on \( H : (\phi_i, \phi_{k-j}) = 1/k, (\phi_{k+j}, \phi_{k+m-j}) = 1/m, \) and all other pairs of vectors are orthogonal. Moreover, using the Levi–Civita connection we can choose a flat coordinate system \( \tau_1, \ldots, \tau_{N-1}, e^{\tau N} \) on \( M \) such that the map \( \partial/\partial \tau_i \rightarrow \phi_i, 1 \leq i \leq N, \) gives a trivialization of the tangent bundle \( TM \cong M \times H \) under which the cup product and the residue pairing correspond respectively to the multiplication and the bilinear pairing of \( H \). Such flat coordinates will be constructed explicitly in Section 3. We will denote by \( \partial_i \) the vector field \( \partial/\partial \tau_i \) and by \( f_r \) the Laurent polynomial in \( M \) corresponding to \( \tau = (\tau_1, \ldots, \tau_N) \).

A direct computation of the orbifold cohomology \( H^*_{orb}(C_{k,m}; \mathbb{C}) \) shows that \( H^*_{orb}(C_{k,m}; \mathbb{C}) \cong H \) where the isomorphism is given by \( X = \text{P.D.}([B \mathbb{Z}_k]) \) and \( Y = \text{P.D.}([B \mathbb{Z}_m]) \). Our first result is:

**Theorem 1.1.** \( M_{k,m} \) is isomorphic to the Frobenius manifold corresponding to the big orbifold quantum cohomology of \( C_{k,m} \).

In other words \( M_{k,m} \) is the full mirror model of \( C_{k,m} \).

Details of the proof of Theorem 1.1 and some background on orbifolds and their quantum cohomologies will be given in Section 4.
Let $\mathcal{H} = H((z^{-1}))$ be the space of formal Laurent series in $z^{-1}$ with vector coefficients equipped with a symplectic structure,

$$\Omega(f, g) := \frac{1}{2\pi i} \oint (f(-z), g(z)) \, dz, \quad f, g \in \mathcal{H}.$$ 

The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, defined by the Lagrangian subspaces $\mathcal{H}_+ = H[z]$ and $\mathcal{H}_- = z^{-1}H[[z^{-1}]]$, identifies $\mathcal{H}$ with the cotangent bundle $T^* \mathcal{H}_+$.

Let $\epsilon$ be a formal variable – the genus parameter in Gromov–Witten theory. By definition, the (Bosonic) Fock space $B_H$ is the vector space of functions on $\mathcal{H}_+$, completed in a certain way. Namely, if we let $q(z) = \sum_{k \geq 0} q_k z^k \in \mathcal{H}_+$ then $B_H$ is the space of formal series in the sequence of vector variables $q_0, q_1 + 1, q_2, \ldots$, whose coefficients are formal Laurent series in $\epsilon$. We construct a representation of the Heisenberg Lie algebra generated by the linear Hamiltonians on the Fock space $B_H$. Let $\{\phi^i\}$ be a basis of $H$ dual to $\{\phi_i\}$ with respect to the residue pairing. Then the linear functions on $\mathcal{H}$ defined by $p_{k,i} = \Omega(\cdot, \phi_i z^k)$ and $q^k_i = \Omega(\phi_i (-z)^{-k-1}, \cdot)$ form a Darboux coordinate system on $\mathcal{H}$. Thus the formulas

\begin{align}
\widehat{q}_k^i := q_k^i / \epsilon, \quad \widehat{p}_{k,i} := \epsilon \partial / \partial q_k^i,
\end{align}

define a representation on $B_H$. Given a vector $f \in \mathcal{H}$ we define a vertex operator acting on $B_H$: $e^{\tau \cdot} := (e^{\tau \cdot})^\circ := e^{\tau \cdot} e^{\widehat{\tau} \cdot}$, where $f_{\pm}$ is the projection of $f$ on $\mathcal{H}_\pm$ and $f_\pm$ is identified with the linear Hamiltonian $\Omega(\cdot, f_\pm)$.

A fundamental solution to the system of differential equations corresponding to the flat connection (1.2) has two singularities – at $z = 0$ and $z = \infty$. The information about these singular points is encoded in two vectors $D^M, A^M_\tau \in B_H$, called total descendent and total ancestor potentials of $M$, where $\tau$ is a semi-simple point, i.e., for $\tau'$ in a neighborhood of $\tau$ the critical values of $f_{\tau'}$ form a local coordinate system on $M$. In other words, $f_\tau$ has only Morse type critical points, see Section 3.5 for precise definitions.

Let $\Delta \subset M \times \mathbb{C}$ be the set of pairs $(\tau, \lambda)$ such that the equation $f_\tau(x) = \lambda$ has less than $N$ solutions. Then the space $(M \times \mathbb{C}) \setminus \Delta$ admits an $N$-fold covering $V$ : the fiber over $(\tau, \lambda)$ is $V_{\tau,\lambda} := f_\tau^{-1}(\lambda)$. The relative homology groups $H_1(\mathbb{C}\setminus, V_{\tau,\lambda}; \mathbb{C})$ vary naturally with respect to $(\tau, \lambda)$, so they define a vector bundle on $(M \times \mathbb{C}) \setminus \Delta$. Moreover, this bundle is equipped with a connection, called Gauss-Manin connection: given a path $C$ from $(\tau_1, \lambda_1)$ to $(\tau_2, \lambda_2)$ there is a natural identification (since $V$ is a fibration) between the corresponding relative homology groups.

Fix an arbitrary reference point $(\tau_0, \lambda_0) \in (M \times \mathbb{C}) \setminus \Delta$. For each $\beta \in H_1(\mathbb{C}\setminus, V_{\tau_0,0}; \mathbb{C})$ and $n \in \mathbb{Z}$ we define multivalued period mappings $I_{\beta}^{(n)} :$
\((M \times \mathbb{C}) \setminus \Delta \to H\) as follows:

\[
(I^{(-p)}_\beta)(\tau, \lambda, \partial_i) = -\partial_i \int_{\beta(\tau, \lambda)} \frac{(\lambda - f_\tau)^p}{p!} \omega, \quad 1 \leq i \leq N
\]

\[
I^{(p)}_\beta(\tau, \lambda) = \partial^* \Gamma^{(0)}_\beta(\tau, \lambda),
\]

where \(p\) is a non-negative integer and \(\beta(\tau, \lambda) \in H_1(\mathbb{C}^*, f^{-1}_\tau(\lambda); \mathbb{Z})\) is a cycle obtained from \(\beta\) via a parallel transport along a path \(C\) connecting \((\tau_0, \lambda_0)\) and \((\tau, \lambda)\). Note that the value of \(I^{(n)}_\beta\) depends on the choice of the path \(C\). Finally, put

\[
f_\tau^\beta(\lambda) = \sum_{n \in \mathbb{Z}} I^{(n)}_\beta(\tau, \lambda)(-z)^n, \quad \Gamma^{\beta}_\tau = e^{f_\tau^\beta}.
\]

Let \(x^0_1, \ldots, x^0_N\) be the solutions to \(f_{\tau_0}(x) = \lambda_0\). For each \(1 \leq a \leq N\) choose a path in \(\mathbb{C}^*\) from 1 to \(x^0_a\), i.e., fix a value of \(\log x^0_a\). We introduce vertex operators \(\Gamma^{n}_\tau, \ a = 1, 2, \ldots, N\) corresponding to the one-point cycles \([x^0_a] \in H_0(f_{\tau_0}^{-1}(\lambda_0); \mathbb{Z})\), as follows. Given an integer \(n\), we define a multivalued period mapping \(I^{(n)}_a: (M \times \mathbb{C}) \setminus \Delta \to H\) by

\[
(I^{(-p)}_a)(\tau, \lambda, \partial_i) = -\partial_i \int_{[x_a]} d^{-1} \left( \frac{1}{p!} (\lambda - f_\tau)^p \omega \right),
\]

\[
I^{(p)}_a = \partial^* \Gamma^{(0)}_a,
\]

where \(p \geq 0\) and \(d^{-1}\) is a linear operator acting on the space of volume forms on \(\mathbb{C}^*\) according to the rule \(d^{-1}(x^k dx) = x^{k+1}/(k + 1)\) if \(k \neq -1\) and \(d^{-1}(dx/x) = \log x\). The periods \(I^{(n)}_a\) are multi-valued: the values of \(x_a = x_a(\tau, \lambda)\) and \(\log x_a\) depend on the choice of a path (avoiding \(\Delta\)) from \((\tau_0, \lambda_0)\) to \((\tau, \lambda)\). Finally, put

\[
f^{\tau}_a(\lambda) = \sum_{n \in \mathbb{Z}} I^{(n)}_a(\tau, \lambda)(-z)^n, \quad \Gamma^{a}_\tau = e^{f^{\tau}_a}.
\]

Note that if \(\beta \in H_1(\mathbb{C}^*, f^{-1}_{\tau_0}(\lambda_0); \mathbb{Z})\) is a relative cycle represented by the composition of the two paths \(x^0_a\) to 1 and 1 to \(x^0_b\) – the same ones which specify the branch of \(\log x^0_a\) and \(\log x^0_b\), then a simple application of the Stokes’ formula implies: \(f^{\tau}_a = f^\beta_a - f^{\tau}_a\).

The vertex operators \(\Gamma^{n}_\tau, \ a = 1, 2, \ldots, N\) depend on the choice of \(\log x^0_a\) as follows. Let \(\phi \in H_1(\mathbb{C}^*, (2\pi i)^{-1}\mathbb{Z})\) be a cycle normalized by \(\int_{\phi} \omega = 1\). Then changing the value from \(\log x^0_a\) to \(\log x^{0+a}_a + r_a, \ r_a \in 2\pi i\mathbb{Z}\), transforms the vertex operators

\[
\Gamma^{a}_\tau \otimes \Gamma^{-a}_\tau \quad \text{into} \quad \left( \Gamma^{a+\phi}_\tau \otimes \Gamma^{-a\phi}_\tau \right) \left( \Gamma^{a}_\tau \otimes \Gamma^{-a}_\tau \right).
\]

To offset this ambiguity, we allow vertex operators acting on a larger Fock space \(B_H := \mathcal{A} \otimes \mathcal{C} B_H\). Here \(\mathcal{A}\) is the algebra of differential operators \(\sum_{0 \leq k \leq N} a_k(x; \epsilon) \partial_x^k\),
where each $a_k$ is a formal Laurent series in $\epsilon$ with coefficients smooth functions in $x$. We equip $\mathcal{A}$ with an anti-involution $\#$ defined by its action on the generators $x$ and $\epsilon \partial_x$ of $\mathcal{A}$:

$$(\epsilon \partial_x)^\# = - \epsilon \partial_x, \quad x^\# = x.$$ 

Let $w_\infty = -d \tau_k z^{-1}$ and $w_\infty = \partial_k$. There are unique vectors $w_\tau, v_\tau \in \mathcal{H}$ such that they are horizontal sections of $\nabla$ (see (1.2)), depend polynomially on $\tau_1, \ldots, \tau_N, Q e^{\tau N}$ and their free terms are respectively $w_\infty$ and $v_\infty$. Introduce a vertex operator (acting on $x$).

$$\Gamma^\delta := \exp \left( (f^\delta - w_\tau) \epsilon \partial_x \right) \exp \left( x v_\tau / \epsilon \right).$$

It has the following crucial property:

$$(\Gamma^\delta_{\tau} \otimes \Gamma^\delta_{\tau}) \left( \Gamma^\phi_{\tau} \otimes \Gamma^{-\phi}_{\tau} \right) = e^{(\hat{w}_\tau \otimes 1 - 1 \otimes \hat{w}_\tau') r} \Gamma^\delta_{\tau} \otimes \Gamma^\delta_{\tau}.$$ (1.5)

Finally, for each $i$ with $1 \leq i \leq N$, define $c_i^\tau(\lambda) = 1/f'_\tau(x_i)$, where $x_i = x_i(\tau, \lambda)$ is a solution to $f_\tau(x) = \lambda$ and $'$ is the derivative with respect to $x$.

**Definition 1.2.** We say that $\mathcal{T} \in B_H$ satisfies the HQE (1.6) if the 1-form

$$(\Gamma^\delta_{\tau} \otimes \Gamma^\delta_{\tau}) \left( \sum_{i=1}^N c_i^\tau \Gamma^\iota_{\tau} \otimes \Gamma^{-i}_{\tau} \right) (\mathcal{T} \otimes \mathcal{T}) \ d\lambda,$$

computed at $q'$ and $q''$ such that $\hat{w}'_\tau - \hat{w}''_\tau = r$, is regular in $\lambda$ for each $r \in \mathbb{Z}$.

Here $\mathcal{T} \otimes \mathcal{T}$ means the function $\mathcal{T}(q')\mathcal{T}(q'')$ on the two copies of the variable $q = \{q_k^a | 1 \leq a \leq N, k \geq 0\}$ and the vertex operators in $\Gamma^\tau_{\tau} \otimes \Gamma^{-\tau}_{\tau}$ preceding (respectively following) $\otimes$ act on $q'$ (respectively on $q''$). Furthermore, $\hat{w}_\tau$ is identified with a linear function in $q$ via the symplectic form, i.e., $\hat{w}_\tau(q) = e^{-1}Q(q, w_\tau)$, where $q := \sum q_k^a \phi_a z^k$. The expression (1.6) is interpreted as taking values in the vector space $B_H \otimes \mathcal{A} B_H$. Thanks to (1.5), when $\hat{w}'_\tau - \hat{w}''_\tau = r \in \mathbb{Z}$ the expression (1.6) is single-valued near $\lambda = \infty$.

After the change $y = (q' - q'')/(2\epsilon), x = (q' + q'')/2$ and the substitution

$$y_0^a = -r/2 + \ldots,$$

where the dots stand for a linear combination of $y_i^a, i \geq 1, 1 \leq a \leq N$, it expands (for each integer $r \in \mathbb{Z}$) as a power series in $y$ (with $y_0^a$ excluded) with coefficients which are Laurent series in $\lambda^{-1}$ (whose coefficients are differential operators in $x$ depending on $x$ via $\mathcal{T}$, its translations and partial derivatives). The regularity condition means that all coefficients in front of the negative powers of $\lambda$ vanish, i.e., the Laurent series are polynomials in $\lambda$.

**Theorem 1.3.** Let $\tau \in M$ be a semi-simple point. Then the total ancestor potential $\mathcal{A}_M$ satisfies the HQE (1.6)

$${}^1\text{Note that } w_\tau = dt_k (-z)^{-1} + O(z^{-2}). \text{ Also, } \hat{w}_\tau(y) = \hat{w}_\tau(q' - q'')/(2\epsilon) = (\hat{w}'_\tau - \hat{w}''_\tau)/(2\epsilon) = r/(2\epsilon). \text{ Thus we can express } y_0^a \text{ as a linear combination of } y_i^a, i \geq 1, 1 \leq a \leq N.$$
The HQE \((1.6)\) admit some kind of a classical limit. More precisely, the map
\[ \tau_i \mapsto T_i, \quad 1 \leq i \leq N, \quad Qe^{\tau_N} \mapsto T_{N+1}, \]
identifies the rings \(\mathbb{C}[\tau_1, \ldots, \tau_N, Qe^{\tau_N}]\) and \(\mathbb{C}[T_1, \ldots, T_{N+1}]\). Given an element \(f \in \mathbb{C}[\tau_1, \ldots, \tau_{N-1}, Qe^{\tau_N}]\), we define the classical limit of \(f\) to be \(f(0)\) where we are identifying \(f\) with a polynomial in \(\mathbb{C}[T_1, \ldots, T_{N+1}]\) and then we are setting \(T_1 = \ldots = T_{N+1} = 0\). Slightly abusing the notations we will also say that we are setting \(\tau_1 = \ldots = \tau_N = Qe^{\tau_N} = 0\).

The coefficients of the vertex operators \(\Gamma^a\) and \(\Gamma^b\) depend polynomially on \(\tau_1, \ldots, \tau_N, Qe^{\tau_N}\). After taking the classical limit we obtain another set of Hirota quadratic equations. For more details, see Sections 5.3 and 8. Here we summarize the answer. Put
\[
(1.7) \quad f^a_\infty = \sum_{n \geq 0} \frac{\lambda^n}{n!} d\tau_k(-z)^{-n-1}.
\]
Denote by Log \(\lambda\) a branch of the logarithmic function near \(\lambda = \infty\). For each \(1 \leq a \leq k\), we introduce a vector in \(\mathcal{H}\):
\[
(1.8) \quad f^a_\infty = \frac{1}{k} g^a_\infty + \sum_{i=1}^{k-1} \sum_{n \in \mathbb{Z}} \prod_{l=-\infty}^{n} (i/k - l) \lambda^{i/k-n-1} \partial_i (-z)^n,
\]
where
\[
\sum_{n \geq 0} \frac{\lambda^n}{n!} (\log \lambda - C_n) d\tau_k(-z)^{-n-1} + \sum_{n \geq 0} n! \lambda^{-n-1} d\tau_k z^n,
\]
and \(C_0 := 0, \quad C_n := 1 + 1/2 + \ldots + 1/n\) are the harmonic numbers. In the formulas above \(a\) parametrizes different choices of \(k\)-th root of 1:
\[
\log \lambda = \log \lambda + 2\pi i (a - 1), \quad \lambda^{1/k} = \exp \left( \frac{1}{k} \log \lambda \right).
\]
We also introduce a vector in \(\mathcal{H}\) for each \(b\) with \(k+1 \leq b \leq k+m\):
\[
(1.9) \quad f^b_\infty = -\frac{1}{m} g^b_\infty - \sum_{j=1}^{m} \sum_{n \in \mathbb{Z}} \prod_{l=-\infty}^{n} (j/m - l) \lambda^{j/m-n-1} \partial_{k+m-j} (-z)^n,
\]
where
\[
\sum_{n \geq 0} \frac{\lambda^n}{n!} \left[ \log(\lambda Q^{-m}) - C_n \right] d\tau_k(-z)^{-n-1} + \sum_{n \geq 0} n! \lambda^{-n-1} d\tau_k z^n.
\]
Just like above, \(b\) parametrizes different choices of \(m\)-th root of 1:
\[
\log \lambda = \log \lambda + 2\pi i (b - k - 1), \quad \lambda^{1/m} = \exp \left( \frac{1}{m} \log \lambda \right).
\]
Furthermore, introduce a vertex operator (acting on \(\mathcal{B}_H\)):
\[
\Gamma^\delta = \exp \left( (f^\delta_\infty - w_\infty) \epsilon \partial_x \right) \exp (x u_\infty / \epsilon) \sim.
\]
Finally, put
\[ c_a = \frac{1}{k} \lambda^{(1-k)/k}, \quad 1 \leq a \leq k, \quad c_b = -\frac{1}{m} Q \lambda^{-(1+m)/m}, \quad k+1 \leq b \leq k+m. \]

The limit of (1.6) has the following form.

**Definition 1.4.** We say that \( T \in B_H \) satisfies the HQE (1.10) if the 1-form
\[
(\Gamma_{\infty}^a \otimes \Gamma_{\infty}^{-a}) \left( \sum_a c_a \Gamma_{\infty}^a \otimes \Gamma_{\infty}^{-a} + \sum_b c_b \Gamma_{\infty}^b \otimes \Gamma_{\infty}^{-b} \right) (\mathcal{T} \otimes \mathcal{T}) \, d\lambda
\]
computed at \( q' \) and \( q'' \) such that \( \hat{w}'_\infty - \hat{w}''_\infty = r \), is regular in \( \lambda \) for each \( r \in \mathbb{Z} \).

We remark that (1.5) holds with \( \tau = \infty \), which implies that the expression (1.10) is single-valued near \( \lambda = \infty \) and independent of our choice of the branch Log \( \lambda \). The regularity condition is interpreted as before.

**Theorem 1.5.** The total descendent potential \( D^{\mathcal{M}} \) satisfies the HQE (1.10).

**Acknowledgments.** We thank Y. Ruan for his interests in this work. Part of this work was pursued in the Mathematical Sciences Research Institute where the second author held a postdoctoral fellowship in the spring of 2006. It is a pleasure to acknowledge its hospitality and support. Finally, we want to thank the referee for pointing out some inaccuracies and fixing a numerous number of misprints.

2. **Hirota Quadratic Equations and \( A_1 \) singularity**

According to the main result in [17], Theorems 1.3 and 1.5 have analogues for the Frobenius manifold \( M_{A_1} \) corresponding to \( A_1 \) singularity. Here we review the results. \( M_{A_1} \simeq \mathbb{C} \) is the space of quadratic polynomials \( f = x^2/2 + u, \ u \in \mathbb{C} \). The metric and the Frobenius multiplication are the standard ones of \( \mathbb{C} \). The construction of a symplectic loop space \( \mathcal{H} \) and corresponding Bosonic Fock space \( B \) depends only on a vector space equipped with a non-degenerate bi-linear form. Let \( \mathcal{H}_\mathbb{C} \) and \( B_\mathbb{C} \) be the corresponding objects for \( \mathbb{C} \) equipped with the standard pairing.

Let \( x_\pm = \pm \sqrt{2(\lambda - u)} \) be the solutions of \( f(x) = \lambda \). Then we define period vectors
\[
I^{(-n)}_\pm(u, \lambda) := -\partial_u \frac{1}{n!} \int_{x_\pm} d^{-1} ((\lambda - f)^n \, dx) = \pm \frac{(2(\lambda - u))^{n-1/2}}{(2n-1)!!},
\]
\[
I^{(n)}_\pm(u, \lambda) := \partial_\lambda I^{(0)}_\pm(u, \lambda) = \pm (-1)^n \frac{(2n-1)!!}{(2(\lambda - u))^{n+1/2}}.
\]

Let \( \Gamma^\pm_u \) be the vertex operator (acting on \( B_\mathbb{C} \)) corresponding to the vector
\[
\Gamma^\pm_u = \sum_{n \in \mathbb{Z}} I^{(n)}_\pm(u, \lambda) (-z)^n.
\]
It is known that the potentials $\mathcal{D}_{\text{pt}}^A$ and $\mathcal{A}_u^A$ coincide with the so called Witten–Kontsevich tau-function:

$$\mathcal{D}_{\text{pt}}(t) = \exp \left( \sum_{g,n} \frac{e^{2g-2}}{n!} \int_{\mathcal{M}_{g,n}} t_1 \cdots t_n \right),$$

where $t = (t_0, t_1, \ldots)$ is a sequence of formal variables, $\mathcal{M}_{g,n}$ is the moduli space of stable Riemann surfaces of genus $g$ with $n$ marked points, $\psi_i$ is the first Chern class of the $i$-th universal cotangent line bundle on $\mathcal{M}_{g,n}$, $t_i(\psi_i) = \sum t_i \psi_i^i$, and the sum is over all $g$ and $n$ with the convention that the integral is 0 if $\mathcal{M}_{g,n}$ is empty. $\mathcal{D}_{\text{pt}}$ is identified with an element of the Fock space $B_C$ via the dilaton shift $t_i(z) = q(z) + z$.

According to [17], corollary of Proposition 2, the Witten–Kontsevich tau-function satisfies the following HQE: the 1-form

$$(2.11) \left( \frac{1}{\sqrt{2(\lambda - u)}} \Gamma_u^+ \otimes \Gamma_u^- - \frac{1}{\sqrt{2(\lambda - u)}} \Gamma_u^- \otimes \Gamma_u^+ \right) (\mathcal{D}_{\text{pt}} \otimes \mathcal{D}_{\text{pt}}) d\lambda.$$

is regular in $\lambda$ (in the sense explained in the Introduction).

We make several remarks. First of all the coefficients in front of the vertex operators are precisely $c_j^\pm(\lambda) = 1/f'_x(x_\pm)$ which agrees with the formula for $c_j^\pm(\lambda)$ in the case $f \in M$ – the space of Laurent polynomials. Second, when $u = 0$, (2.11) is precisely the Witten’s conjecture [35], proved by Kontsevich [25]. Finally, the proof of (2.11) for $u \neq 0$ follows from the case $u = 0$ and the string equation.

3. Frobenius structure on the space of Laurent polynomials

3.1. Flat structure. In this section, following [14], we will show that the residue metric (1.1) is flat.

Let $t = (t_1, t_2, \ldots, t_N)$ and $f_t$ be the corresponding Laurent polynomial in $M$. In a neighborhood of $\lambda = \infty$ the equation $f_t(x) = \lambda$ has two types of solutions depending on whether $x$ is close to $\infty$ or to 0. Let $x_a$, $a = 1, 2, \ldots, k$ be the solutions close to $\infty$ and $x_b$, $b = k + 1, k + 2, \ldots, k + m$ the solutions close to 0. They expand as series in $\lambda$ as follows:

$$\log x_a = \frac{1}{k} \left[ \log \lambda - \tau_1 \lambda^{-1/k} - \cdots - \tau_{k-1} \lambda^{-(k-1)/k} - \tau_k \lambda^{-1} \right] + O \left( \lambda^{-1/k} \right),$$

$$\log x_b = \frac{1}{m} \left[ - \log \frac{\lambda}{Q^m} + \tau_{k+m} + \tau_{k+m-1} \lambda^{-1/m} + \cdots + \tau_k \lambda^{-1} \right] + O \left( \lambda^{-1/m} \right),$$

where indices $a$ and $b$ have the same ranges as above. They parametrize different choices of $k$-th and $m$-th root of 1 respectively. The coefficients $\tau_i$,
\( i = 1, 2, \ldots, k + m \) can be expressed in terms of \( t_1, \ldots, t_N \) as follows:

\[
\tau_i = \frac{k}{i} \operatorname{Res}_{x=\infty} f_i(x)^{i/k} \omega, \quad 1 \leq i \leq k - 1
\]

\[
\tau_{k+m-j} = \frac{m}{j} \operatorname{Res}_{x=0} f_i(x)^{j/m} \omega, \quad 1 \leq j \leq m - 1
\]

\[
\tau_N = mt_N, \quad \tau_k = t_k.
\]

Using these formulas we get

\[
(3.12) \quad t_i = \tau_i + f_i(\tau_1, \ldots, \tau_{i-1}), \quad 1 \leq i \leq k - 1
\]

\[
(3.13) \quad t_{k+j} = \tau_{k+j} + h_j(\tau_{k+j+1}, \ldots, \tau_{k+m-1}), \quad 1 \leq j \leq m - 1
\]

where \( h_j \) and \( f_i \) are certain polynomials of degrees at least 2. Thus the corresponding Jacobian is non-degenerate and the functions \( \tau_1, \ldots, \tau_{N-1}, e^{\tau_N} \) give a coordinate system on \( M \). Moreover, according to [14], in such coordinates the residue metric has the form

\[
(\partial/\partial \tau_i, \partial/\partial \tau_{k-i}) = 1/k, \quad i = 1, 2, \ldots, k - 1
\]

\[
(\partial/\partial \tau_{k+j}, \partial/\partial \tau_{k-j+m}) = 1/m, \quad j = 0, 1, \ldots, m,
\]

and all other pairings between \( \partial/\partial \tau_i \), \( 1 \leq i \leq N \) are 0.

For the sake of completeness let us show how to compute \( (\partial/\partial \tau_i, \partial/\partial \tau_{k-i}) \). Put \( \xi = \lambda^{1/k} \) and identify \( \xi \) with a new coordinate on \( \mathbb{C}^* \) near \( x = \infty \), related to \( x \) via \( f(x) = \xi^k \). By chain rule we have \( \partial_{\tau_i} f + \partial_x f \partial_{\tau_i} x = 0 \), which implies that \( \partial_{\tau_i} f \partial_x \omega = -(\partial_{\tau_i} \log x) df_x \). Using the expansion of \( \log x \) from above, we find

\[
(\partial/\partial \tau_i)(f \partial_x) = k \xi^k[k^{-1} \xi^{-i} + O(\xi^{-k})] d\xi/\xi.
\]

Note that in the residue pairing only the residue at \( x = \infty \) contributes:

\[
(\partial_{\tau_i}, \partial_{\tau_j}) = -\operatorname{Res}_{x=\infty} \left( \frac{1}{k} \xi^{k-1} \right) d\xi/\xi = \frac{1}{k} \delta_{i+j, k}.
\]

In flat coordinates the Euler vector field takes on the form:

\[
E = \tau_k \partial_{\tau_k} + \sum_{i=1}^{k-1} \frac{i}{k} \tau_i \partial_{\tau_i} + \sum_{j=1}^{m-1} \left( 1 - \frac{j}{m} \right) \tau_{k+j} \partial_{\tau_{k+j}} + \left( \frac{1}{k} + \frac{1}{m} \right) m \partial_{\tau_N}.
\]

3.2. Oscillating integrals. Let \( \{\tau_i\}_{i=1}^N \) be the flat coordinates introduced above. Denote the corresponding coordinate vector fields by \( \partial_i := \partial/\partial \tau_i \). The following lemma is crucial for our construction. Probably it could be derived from [12, 13] or [4]. However we prefer to give a direct proof.

**Lemma 3.1.** For each \( i \) and \( j \), \( 1 \leq i, j \leq N \) there is a Laurent polynomial \( G_{ij}(\tau, x) \) in \( x \) such that

\[
(3.14) \quad \frac{\partial^2 f_x}{\partial \tau_i \partial \tau_j} \omega = dG_{ij}, \quad \left( \partial_i f_x \right) \left( \partial_j f_x \right) - \sum_p A_{ij}^p \partial_p f_x \right) \omega = G_{ij} df_x,
\]

where \( d \) is the De Rham differential on \( \mathbb{C}^* \).
**Proof.** From the definition of $\bullet$, it follows that the second equality holds for a uniquely determined Laurent polynomial

$$G_{ij} = \sum_{a=-m}^{k-1} G_{ij}^{(a)} (\tau) x^a.$$ 

Write the polynomial $f_\tau$ as $x^k + T_1 x^{k-1} + \ldots + T_N x^{-m}$. Then we need to show that

$$\frac{\partial^2 T_a}{\partial \tau_i \partial \tau_j} = (k-a) G_{ij}^{(k-a)}, \quad 1 \leq a \leq N.$$

Assume first that $1 \leq a \leq k$. Let $\xi$ be a new coordinate in a neighborhood of $x = \infty$ defined by $f_\tau(x) = \xi^k$, and so $\log x$ is expressed in terms of $\xi$ according to the expansions defining the flat coordinates $\tau_i$, $1 \leq i \leq k$, except that we need to put $\lambda = \xi^k$. Then we have

$$T_a = - \text{Res}_{x=\infty} f_\tau(x) x^{a-k-1} dx = - \frac{1}{a-k} \text{Res}_{\xi=\infty} \xi^k d(x^{a-k}) = \frac{k}{a-k} \text{Res}_{\xi=\infty} x^{a-k} \xi^{k-1} d\xi,$$

where $x = \xi(1 + O(\xi^{-1}))$. Using this formula we compute $\partial_i \partial_j T_a$ :

$$\frac{\partial^2 T_a}{\partial \tau_i \partial \tau_j} = k \text{Res}_{\xi=\infty} \left[ (a-k-1) x^{a-k-2} (\partial_i x)(\partial_j x) + x^{a-k-1} \partial_i \partial_j x \right] \xi^{k-1} d\xi = k \text{Res}_{\xi=\infty} \left[ (a-k) x^{a-k} (\partial_i \log x)(\partial_j \log x) + x^{a-k} \partial_i \partial_j \log x \right] \xi^{k-1} d\xi.$$

Notice that the last term in the square brackets does not contribute to the residue because the highest possible power of $\xi$ is $a-k-k-1$. Therefore, after passing back to the old coordinate $x$ and using that $(\partial_i \log x) df = -\partial_i f \omega$, we get

$$\frac{\partial^2 T_a}{\partial \tau_i \partial \tau_j} = (a-k) \text{Res}_{x=\infty} \left[ x^{a-k-1} (\partial_i f_\tau)(\partial_j f_\tau) \frac{\omega}{f_\tau^p} \right] = (a-k) \text{Res}_{x=\infty} \left[ x^{a-k} G_{ij} + x^{a-k-1} \left( \sum_p A_{ij}^p \partial_p f_\tau \right) \frac{1}{f_\tau^p} \right] \omega.$$

Again, the second term in the square brackets does not contribute to the residue because the highest possible power of $x$ is $a-k-1 \leq -1$. The residue of the first term is clearly $- G_{ij}^{(k-a)}$.

In the case when $k+1 \leq a \leq N$, we pass to a new coordinate $\eta$ near $x = 0$ via $f_\tau(x) = T_N \eta^{-m}$. Then proceed by a similar argument. \hfill \Box

In particular, the oscillating integrals

$$\mathcal{J}_B(\tau, z) = (-2\pi z)^{-1/2} \int_B e^{f_\tau/z} \omega \quad (3.15)$$
satisfy the following differential equations

\[(3.16) \quad z \frac{\partial^2 J_B}{\partial \tau_i \partial \tau_j} = \sum_{p=1}^{N} A_{ij}^p \frac{\partial J_B}{\partial \tau_p},\]

where the integration cycle \(B\) is an element of the relative homology group

\[(3.17) \quad \lim_{M \to \infty} H_1(\mathbb{C}^*, \{x \in \mathbb{C}^* \mid \Re(f_\tau/z) < -M\}; \mathbb{Z}) \cong \mathbb{Z}^N.\]

The oscillating integral \(J_B\) also satisfies some homogeneity conditions due to the fact that \(f_\tau\) and \(\omega\) are homogeneous:

\[(3.18) \quad (z \partial_z + E + 1/2) J_B = 0.\]

Let \(J_B\) be a vector field on \(M\) defined by

\[ (J_B(\tau, z), \partial_i) = z \partial_i J_B. \]

Then equations (3.16) and (3.18) are equivalent to

\[(3.19) \quad z \partial_i J_B = (\partial_i \bullet) J_B, \quad (z \partial_z + E) J_B = \mu J_B, \]

where \(\mu\) is the Hodge grading operator:

\[ \mu(\partial_i) = \left( \frac{i}{k} - \frac{1}{2} \right) \partial_i, \quad \mu(\partial_{k+j}) = \left( \frac{1}{2} - \frac{j}{m} \right) \partial_{k+j}, \quad 1 \leq i \leq k, 0 \leq j \leq m. \]

Let \(B_i\) be a basis of cycles in the relative homology group (3.17). Then the matrix \(J\) with columns \(J_{B_i}\) is a fundamental solution to the system (3.19). This means that we can extend the connection \(\nabla\) defined in (1.2) to a connection on the trivial bundle on \(M \times \mathbb{C}^*\) with fiber \(H\) by setting

\[ \nabla_{\partial/\partial z} := \partial/\partial z + (E \bullet) z^{-2} - \mu z^{-1}. \]

The extended connection is flat because the corresponding system of differential equations admits a fundamental solution.

For each \(\tau \in M\), the \(z\)-direction of \(\nabla\) defines a connection on \(\mathbb{CP}^1\) which has an irregular singular point at \(z = 0\) and a regular singular point at \(z = \infty\). At \(z = 0\) the fundamental solution of \(\nabla_{\partial/\partial z} \Phi = 0\) admits a certain asymptotic and at \(z = \infty\), \(\nabla_{\partial/\partial z}\) can be transformed via a gauge transformation into a canonical form. These two ingredients, the asymptotic and the gauge transformation, contain the essential information about the Frobenius structure. They will be used to define the total descendant and the total ancestor potentials of \(M\).
3.3. Stationary phase asymptotic. Let $\tau \in M$ be a semi-simple point, i.e., $f_\tau$ has only Morse type critical points $q_i$, $1 \leq i \leq N$. Denote the corresponding critical values by $u_i$. They form a coordinate system called canonical coordinate system. Let $\Delta_i$ be the Hessians of $f_\tau$ at $x_i$ with respect to the volume form $\omega$. Then the linear map $f$ is an isomorphism of Frobenius algebras. Here $e_i$ satisfies the symplectic condition $R_{ij} = 1$, Proposition, part (d), such a solution is unique and it automatically satisfies the symplectic condition $R^*(-z)R(z) = 1$.

3.4. Calibration of $M$.

**Proposition 3.2.** There exist a gauge transformation $S_\tau = 1 + S_1z^{-1} + \ldots$, $S_p \in \text{End}(H)$ satisfying the symplectic condition $S_\tau^*(-z)S_\tau(z) = 1$, such that

$$z\partial_i S_\tau = (\partial_i \bullet) S_\tau, \quad z\partial_z S_\tau = [\mu, S_\tau] - ((E \bullet) S_\tau - S_\tau \rho) z^{-1},$$

where $\mu$ is the Hodge grading operator and $\rho$ is the cup product multiplication by $(1/k + 1/m)m\partial_N$.

**Proof.** The first equation in (3.20) gives us the following recursive relation: $\partial_i S_p = (\partial_i \bullet) S_{p-1}$. The multiplication operator $\partial_i \bullet$ depends polynomially on $\tau_1, \ldots, \tau_N$, and $Qe^{\tau N}$. Thus starting from $S_0 = 1$ we can recover uniquely all other $S_p$, $p \geq 1$ by integrating the recursive relations and requiring that $S_p$ vanishes when $\tau_1, \ldots, \tau_N, Qe^{\tau N}$ are set to 0.

We claim that the so constructed series $S$ automatically satisfies the second equation of (3.20) and the symplectic condition. Identify $S(\tau, z)$ with a section of the bundle $\pi^*(\text{EndTM})$, where $\pi : M \times \mathbb{C}^* \to M$ is the projection. Then (3.20) means that $S$ is a horizontal section of the following connection

$$\nabla^\text{End} = d - \sum_{i=1}^N (\partial_i \bullet) z^{-1} d\tau_i - (\text{ad}(\mu) z^{-1} - (E \bullet - \rho^R) z^{-2}) dz,$$
where $\rho^R$ is the classical multiplication by $(1/k+1/m)m\partial_N$ from the right. The flatness of $\nabla$ implies the flatness of $\nabla^{\text{End}}$. In particular, if we set $\Phi(\tau, z) = \nabla^{\text{End}}_{\partial^j/\partial \tau^i}S$ then $\nabla^{\text{End}}_{\partial^j/\partial \tau^i}\Phi = 0$. However, $\Phi$ is a power series in $z^{-1}$ with coefficients depending polynomially on $\tau_1, \ldots, \tau_N, Qe^{\tau N}$, and $\Phi$ vanishes when those variables are set to 0. Thus $\Phi = 0$, which is precisely the second equation in (3.20).

Let us prove that $S$ satisfies the symplectic condition $S^*(-z)S(z) = 1$. Differentiate with respect to $\partial_i$ and use the first equation in (3.20) and the fact that the operators $\partial_i \bullet$ are self adjoint, we get that $\partial_i (S^*(-z)S(z)) = 0$, i.e., $S^*(-z)S(z)$ is a constant independent of $\tau$. Set $\tau_1,\ldots,\tau_N, Qe^{\tau N}$ to 0, then $S = 1$ by construction. Thus $S^*(-z)S(z) = 1$. \hfill $\square$

The choice of a solution to (3.20) is called a calibration. We choose a calibration of $M$ as follows: each coefficient $S_p$ depends polynomially on $\tau_1,\ldots,\tau_N$, and $Qe^{\tau N}$. We require that $S_\tau = 1$ when the variables $\tau_1,\ldots,\tau_N, Qe^{\tau N}$ are set to 0.

**3.5. Descendants and ancestors.** By definition, the twisted loop group is

$$\mathcal{L}^{(2)}\text{GL}(H) = \{ M(z) \in \mathcal{L}\text{GL}(H) \mid M^*(-z)M(z) = 1 \},$$

where $*$ means the transposition with respect to the bilinear pairing. The elements of the twisted loop group of the form $M = 1 + M_1 z + M_2 z^2 + \ldots$ (respectively $M = 1 + M_1 z^{-1} + M_2 z^{-2} + \ldots$) are called upper-triangular (respectively lower-triangular) linear transformations. They can be quantized as follows: write $A = \log M$, then $A(z)$ is an infinitesimal symplectic transformation. We define $\hat{M} = \exp \hat{A}$, where $A$ is identified with the quadratic Hamiltonian $\Omega(Af, f)/2$ and on the space of quadratic Hamiltonians the quantization rule $\hat{}$ is defined by:

$$\hat{(q_{k,i}q_{l,j})} := \frac{q_{k,i}q_{l,j}}{\epsilon^2}, \quad \hat{(q_{k,i}p_{l,j})} := q_{k,i} \frac{\partial}{\partial q_{l,j}}, \quad \hat{(p_{k,i}p_{l,j})} := \epsilon^2 \frac{\partial^2}{\partial q_{k,i}\partial q_{l,j}}.$$ 

We remark that $\hat{}$ defines only a projective representation of the lower- and upper-triangular subgroups.

Motivated by Gromov–Witten theory, A. Givental [16] introduced the so-called total ancestor and total descendant potentials of a semi-simple Frobenius manifold. The total ancestor potential $A_\tau$ of $M$ is defined for any semi-simple point $\tau \in M$ as follows:

$$A_\tau = \hat{\Psi} \hat{R} \left( e^{U/z} \right)^\tau \prod_{i=1}^N \mathcal{D}^{(i)}_{pt} \Delta_i^{-1/48},$$

where $\mathcal{D}_{pt}$ is the Witten–Kontsevich tau-function, $\mathcal{D}^{(i)}_{pt}$ is a vector in the Fock space $\mathcal{B}_C$, defined by $\mathcal{D}^{(i)}_{pt}(\mathbf{q}) = \mathcal{D}_{pt}(\mathbf{q}^i)$ with $\mathbf{q} = \sum q^i e_i \in \mathbb{C}[z]$. The linear operators $R$ and $e^{U/z}$ are elements of the upper- and lower-triangular twisted
loop subgroups respectively. They act on $B_{CN}$ according to the above quantization rules. Finally, $\tilde{\Psi}$ is just the identification between the Fock spaces $B_H$ and $B_{CN}$, i.e., $(\tilde{\Psi}^{\mathcal{G}})(q) = \mathcal{G}(\Psi^{-1}q)$.

The total descendent potential is defined by

$$D = C(\tau)\hat{S}_r^{-1}A_{r}, \quad C(\tau) = \exp \left( \frac{1}{2} \int^{\tau} R_{1}^{ii}du_{i} \right),$$

where $R_{1}^{ii}$ are the diagonal entries of $R_{1} \in \text{GL}(C_{N})$. The constant $C$ is chosen in such a way that $D$ is independent of $\tau$.

4. $\mathbb{P}^1$-ORBIFOLDS

In this Section we discuss Gromov-Witten theory of the orbifold $C_{k,m}$ - an orbifold obtained from $\mathbb{P}^1$ by cutting two small discs $D_1 = \{ |z| \leq \epsilon \}$ and $D_2 = \{ |z^{-1}| \leq \epsilon \}$ respectively near $z = 0$ and $z = \infty$ and gluing back the orbifolds $D_1/\Z_k$ and $D_2/\Z_m$ in the obvious way. The main goal is to compare the Frobenius manifold $M_{k,m}$ and a Frobenius manifold corresponding to the orbifold quantum cohomology of $C_{k,m}$. Our approach is to compute the small orbifold quantum cohomology of $C_{k,m}$ and then use a reconstruction result, see Theorem 4.2.

4.1. Reconstruction theorem. Let $M$ be a small ball centered at 0 in $C_{N}$. Assume that $g$ is a non-degenerate bi-linear pairing on $TM$, $A$ is a holomorphic section of $T^*M \otimes TM$, i.e., the tangent spaces $T_tM$ are equipped with a multiplication $\bullet_t$ which depends holomorphically on $t \in M$, $e$ is a vector field on $M$ such that its restriction to $T_tM$ is a unity with respect to $\bullet_t$, and finally $E$ is a vector field on $M$.

Definition 4.1. The data $(M, g, A, e, E)$ form a Frobenius structure on $M$ if the following conditions are satisfied.

1. $g$ and $\bullet$ satisfy the Frobenius property: $g(X \bullet Y_1, Y_2) = g(Y_1, X \bullet Y_2)$,
2. The one-parameter group corresponding to $E$ acts on $M$ by conformal transformations of $g$, i.e., $\mathcal{L}_E g = Dg$, for some constant $D \in \C$,
3. $e$ is a flat vector field: $\nabla^{L.C.} e = 0$, where $\nabla^{L.C.}$ is the Levi-Civitá connection of $g$,
4. The connection operator

$$\nabla = \nabla^{L.C.} - z^{-1} \sum_{i=1}^{N} \left( \frac{\partial}{\partial t_i} \bullet_t \right) dt_i + (z^{-2}(E \bullet_t) - z^{-1}\mu) dz,$$

where $\mu := \nabla^{L.C.}(E) - (D/2)\text{Id} : TM \to TM$ is the Hodge grading operator, is flat, i.e., $\nabla^2 = 0$. 
Here \( \{ t_i \} \) are arbitrary coordinates on \( M \), \( \partial_{t_i} \cdot \) (respectively \( E \cdot \)) is the \( \cdot \)-multiplication by the vector field \( \partial_{t_i} \) (respectively \( E \)), and \( \nabla \) is a connection on the bundle \( \pi^*(TM) \) with base \( M \times \mathbb{C}^* \), where \( \pi : M \times \mathbb{C}^* \to M \) is the projection.

Let us assume that \( 0 \in M \) is a semi-simple point, i.e., the Frobenius algebra \( T_0M \) is diagonalizable. Equivalently, there are local coordinates \( u^i, 1 \leq i \leq N \), called canonical coordinates which diagonalize the metric \( g \) and the multiplication \( \cdot \):

\[
\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i}, \quad g(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}) = \delta_{ij} \theta_j, \quad 1 \leq i, j \leq N,
\]

where \( \theta_j \) are some holomorphic functions on \( M \). Moreover, from the flatness of the connection operator (4.21), it follows that the coordinates \( u^i \) could be chosen such that the Euler vector field assumes the form \( E = \sum_i u^i \frac{\partial}{\partial u^i} \) (see [11], Lemma 3.5). The goal in this subsection is to prove the following theorem:

**Theorem 4.2.** Let \( M \) be a holomorphic Frobenius manifold and \( 0 \in M \) is a semi-simple point such that the following conditions are satisfied:

1. The restriction \( E_0 \) of the Euler vector field to \( 0 \) has a \( k \)-th root \( v \in T_0M \) (i.e. \( E_0 = v^k \)) such that \( v \) is invertible and \( v \) generates the Frobenius algebra \( T_0M \).
2. Let \( \mu \) and \( \text{Diag} \left( u_{(0)}^1, \ldots, u_{(0)}^N \right) \) be the matrices respectively of the Hodge grading operator \( \mu \) and the operator of multiplication by the Euler vector \( E_0 \), in a basis of \( T_0M \) which diagonalizes \( \cdot_0 \). If \( (i, j) \) is a pair of indices such that \( u_{(0)}^i = u_{(0)}^j \) then the \( (i, j) \)-th entry of \( \mu \) is zero.

Then the Frobenius structure on \( M \) is uniquely determined from the Frobenius algebra \( T_0M \) and the Hodge grading operator \( \mu \).

The idea of the proof is to reconstruct successively the terms of the Taylor’s expansion of \( u^i \), \( 1 \leq i \leq N \). We use certain recursive relations, constructed from condition (1) and the flatness of (4.21). Condition (2) guarantees that the recursive relations can be solved. Our argument was inspired by the proof of Lemma 2.9 in [21].

**Proof.** Condition (4) in Definition 4.1 implies that \( \nabla^{L.C.} \) is a flat connection and that the multiplication \( \cdot_\tau \) is associative and commutative. Let us identify \( T_0M \) with \( \mathbb{C}^N \) by fixing a basis \( e_1, e_2 := v, e_3, \ldots, e_N \) of \( T_0M \). Using the flat connection \( \nabla^{L.C.} \) we extend \( \phi_a, 1 \leq a \leq N \) to vector fields \( \partial_a := \partial/\partial \tau^a \) on \( M \), where \( \tau = (\tau^1, \ldots, \tau^N) \) is a flat coordinate system on \( M \), and so all other tangent spaces \( T_\tau M \) are canonically identified with \( \mathbb{C}^N \) as well.

Let \( U := E \cdot \), and \( F_a := \partial_a \cdot \), \( 1 \leq a \leq N \) be the linear operators of multiplication by the corresponding vector fields. In view of the above identifications, we may regard \( U \) and \( F_a \) as \( N \times N \)-matrices, whose entries are holomorphic
functions in $\tau$. We define a grading on the space of holomorphic $N \times N$ matrices by assigning degree 1 to each of the coordinate functions $\tau^1, \ldots, \tau^N$. If $A(\tau)$ is a holomorphic matrix then we denote by $A^{(0)} + A^{(1)} + A^{(2)} + \ldots$ its homogeneous decomposition, i.e., $A^{(n)}$ is a finite sum of matrices whose entries are monomials of degree $n$. For some matrices, in order to avoid cumbersome notations, we write $A_{(n)}$ instead of $A^{(n)}$. Also, we denote by $A^{(\geq n)}$ (resp. $A^{(\leq n)}$) the matrix obtained from $A$ by truncating all terms of degree $< n$ (resp. $> n$).

Let us denote by $\Psi$ the matrix whose $(i,a)$-entry (i.e. $i$-th row and $a$-th column) is given by: $\Psi_{ia} = \partial_a u_i$. It is easy to see that:

\begin{equation}
\Psi F_a \Psi^{-1} = \text{Diag} \left( \partial_a u^1, \ldots, \partial_a u^N \right) =: D_a, \quad 1 \leq a \leq N
\end{equation}

and

\[ \Psi U \Psi^{-1} = \text{Diag}(u^1, \ldots, u^N) =: D. \]

We know $U^{(0)}$ and $F_a^{(0)}$, $1 \leq a \leq N$ and we want to reconstruct $U^{(n)}$ and $F_a^{(n)}$ for $n > 0$.

Note that the matrix $U$ admits a holomorphic $k$-th root $V$. Indeed, we have $U = \Psi^{-1} D \Psi$ and $D$ is diagonal with entries $u^i(\tau) = u^i_{(0)} + u^i_{(\geq 1)}$. On the other hand $u^i_{(0)}$ is non-zero, because it is a $k$-th power of an eigenvalue of $v \bullet e$ and the later is an invertible matrix by definition. Therefore we may define (by using the binomial formula):

\[(u^i)^{1/k} := \left( u^i_{(0)} \right)^{1/k} \left( 1 + \sum_{j=1}^{\infty} \binom{1/k}{j} \left( u^i_{(\geq 1)}/u^i_{(0)} \right)^j \right).\]

Therefore $V := U^{1/k} := \Psi D^{1/k} \Psi^{-1}$ is a $k$-th root of $U$. Moreover, without any restrictions we may assume that $V_{(0)} = v \bullet e$. Since $v$ generates the Frobenius algebra $T_0 M$, we can find polynomials $f_a(x)$, $1 \leq a \leq N$ such that $\phi_a = f_a(V_{(0)})e$, where $e \in H$ is the unity, or equivalently $F_a^{(0)} = f_a(V_{(0)})$.

Assume that we have determined the matrices $U^{(i)}$, $V^{(i)}$, $F_a^{(i)}$ for all $a = 1, 2, \ldots, N$ and all $i = 0, 1, \ldots, n - 1$. We want to prove that the matrices for $i = n$ are uniquely determined as well. Let us remark that there is a small difference between the cases $n = 1$ and $n > 1$ which however appears only in the proof of Lemma [14] part b), below.

From the flatness of the connection operators (4.21) we have $[\nabla_{\partial_a}, \nabla_{\partial_{a'}}] = 0$. Comparing the terms of degree $n - 1$ we get:

\begin{equation}
\partial_a U^{(n)} = F_a^{(n-1)} + [\mu, F_a^{(n-1)}].
\end{equation}

A direct corollary of this equation is that $U^{(n)}$ is uniquely determined from $F_a^{(n-1)}$, $1 \leq a \leq N$. 

Put $P := \Psi^{(0)}$. The entries of this matrix can be determined as follows. We know that $P$ diagonalizes the Frobenius product $\bullet_0$. Therefore we have:

$$PV^{(0)}P^{-1} = \text{Diag}(\lambda_1, \ldots, \lambda_N)$$

and

$$PF^{(0)}_aP^{-1} = P f_a(V^{(0)})P^{-1} = \text{Diag}(f_a(\lambda_1), \ldots, f_a(\lambda_N)), \quad 1 \leq a \leq N.$$ 

Comparing with (4.22) we get that the $(i,a)$-entry of $P$ is given by:

$$P_{ia} = f_a(\lambda_i) =: \lambda_{a,i}.$$ 

Let us remark that the eigenvalues $\lambda_{a,i}$ have the following two properties. First, according to our choice of a basis of $T^*_0 M$ we have $\phi_1 = e$ and $\phi_2 = v$, therefore $\lambda_{1,i} = 1$ and $\lambda_{2,i} = \lambda_i$. Second, the eigenvalues $\lambda_i, 1 \leq i \leq N$ are pairwise different. Indeed, if this is not the case then, there exists a non-diagonal matrix $A \neq 0$ that commutes with $V^{(0)}$, and hence it commutes with $F^{(0)}_a = f_a(V^{(0)}), 1 \leq a \leq N$. This is impossible because, $A, F_a, 1 \leq a \leq N$ are linearly independent and a maximal abelian Lie subalgebra of $\mathfrak{gl}(N, \mathbb{C})$ has dimension $N$.

Given a $N \times N$ matrix $A$ we put $\overline{A} = PAP^{-1}$. Let us compare the degree $n$ terms in the equation $U = V^k$. We get

$$U^{(n)} = \sum_{s=0}^{k-1} V^{(0)}_s V^{(n)} V^{(0)}_{k-1-s} + \ldots,$$

where the dots stand for terms which depend on $V^{(i)}$ with $i \leq n - 1$. Note that the $(i,j)$ entry of the matrix sum from above is

$$[\overline{V}^{(n)}]_{ij} (\lambda_{1,i}^{k-1} + \lambda_{1,j}^{k-2} \lambda_{2,j} + \ldots + \lambda_{2,j}^{k-1}).$$

The above sum of $\lambda$’s is zero precisely when the pair $(i,j)$ is such that $\lambda_i \neq \lambda_j$ but $\lambda_i^k = \lambda_j^k$. We call such a pair exceptional and the entries in a $N \times N$ matrix corresponding to an exceptional pair are called exceptional as well. From (4.24) we deduce that all non-exceptional entries of $\overline{V}^{(n)}$ are uniquely determined from the lower degree terms.

**Lemma 4.3.** If $i \neq j$ then the entries of $F^{(n)}_a$ satisfy the following equalities:

$$[F^{(n)}_a]_{ij} = [\overline{V}^{(n)}]_{ij} \frac{\lambda_{a,i} - \lambda_{a,j}}{\lambda_i - \lambda_j} + \ldots,$$

where the dots stand for terms depending only on $V^{(n')}, n' < n$.

**Proof.** We use Taylor’s theorem for matrices:

$$f(X + Y) = f(X) + d_X f(Y) + \ldots,$$

where the dots stand for at least quadratic terms in $Y$. Assume now that $X = \text{Diag}(x_1, \ldots, x_N)$ is a diagonal matrix and that $f(x)$ is an arbitrary
polyomial. Let us compute $d_X f(Y)$. First, if $f(x) = x^n$ then $d_X f(Y) = X^{n-1}Y + X^{n-2}YX + \ldots + YX^{n-1}$, i.e.

$$
(4.26) \quad [d_X f(Y)]_{ij} = \begin{cases} 
Y_{ij} \frac{f(x_i) - f(x_j)}{x_i - x_j}, & \text{if } i \neq j \\
Y_{ii} f'(x_i), & \text{if } i = j.
\end{cases}
$$

By linearity we get that the above formula holds for all polynomials $f$.

Note that if $f(x)$ is an arbitrary polynomial then

$$
(4.27) \quad f(V) = f(V(0)) + dV(0)f(V(n)) + \ldots,
$$

where the dots stand for terms of degree either greater than $n$ or terms of degree not exceeding $n$ but depending only on $V_{(n')}$, $n' < n$. On the other hand we have:

$$
F_a \partial_b = F_a f_b(V(0)) e = f_b(V(0)) F_a e + [F_a, f_b(V(0))] e,
$$

where $e$ is the unity. Note that $F_a e = F_a^{(0)} e$. Therefore, by comparing the degree $n$ terms in the above equality and by using (4.27) together with the fact that $f_b(V)$ and $F_a$ commute we get:

$$
F_a^{(n)} \partial_b = [dV(0)f_b(V(n)), F_a^{(0)}] e + \ldots
$$

where the dots stand for terms depending only on $V_{(n')}$, $n' < n$. We multiply both sides of the above equality by $P$ from the left:

$$
F_a^{(n)} P \partial_b = [dV(0)f_b(V(n)), F_a^{(0)}] P e + \ldots.
$$

Since both $V(0)$ and $F_a^{(0)}$ are diagonal matrices we can easily get (see (4.26))

$$
\sum_{s=1}^N [F_a^{(n)}]_{is} P_{sb} = \sum_{s=1}^N [V(n)]_{is} \frac{f_b(\lambda_i) - f_b(\lambda_s)}{\lambda_i - \lambda_s} (\lambda_{a,s} - \lambda_{a,i}) P_{s1} + \ldots.
$$

On the other hand we know that $P_{sb} = \lambda_{b,s} = f_b(\lambda_s)$ and $P_{s1} = \lambda_{1,s} = 1$. Multiply the above equality by $[P^{-1}]_{bj}$ and sum over all $b = 1, 2, \ldots, N$:

$$
[F_a^{(n)}]_{ij} = \sum_{s=1}^N [V(n)]_{is} \frac{\delta_{i,j} - \delta_{s,j}}{\lambda_i - \lambda_s} (\lambda_{a,s} - \lambda_{a,i}) + \ldots.
$$

We are given that $i \neq j$ thus $\delta_{i,j} = 0$. The only non-zero term in the above sum is the one corresponding to $s = j$. The lemma follows. \qed

**Lemma 4.4.** a) The diagonal entries of $\nabla (1)$ are given by

$$
[\nabla (1)]_{ii} = \sum_{a=1}^N \frac{\lambda_{a,i}}{k \lambda_i^{k-1}} r_a.
$$
b) Assume that \((i,j)\) is an exceptional pair of indices. Then

\[
[U^{(n+1)}]_{ij} = \left( \sum_{a=1}^{N} \frac{\lambda_{a,i} - \lambda_{a,j}}{\lambda_i - \lambda_j} \tau_a \right) [V_{(n)}]_{ij} + \ldots ,
\]

where the dots stand for terms depending on \(V_{(n')}, \ n' < n\), and the non-exceptional entries of \(V_{(n)}\).

**Proof.** a) We have the following equations:

\[
(U^{(1)}) = V_{(0)}^{k-1} V_{(1)} + V_{(0)}^{k-2} V_{(1)} V_{(0)} + \ldots + V_{(1)} V_{(0)}^{k-1}
\]

and \(\partial_a U^{(1)} = F^{(0)}_a + [\pi, F^{(0)}_a]\). On the other hand \(V_{(0)}\) and \(F^{(0)}_a\) are diagonal matrices and the \(i\)-th diagonal entries are respectively \(\lambda_i\) and \(\lambda_{a,i}\). Part a) follows.

b) Assume that \(n > 1\). Let us compare the \((i,j)\)-th, degree \(n + 1\) entries in the equality \(U = V^k\):

\[
[U^{(n+1)}]_{ij} = \sum_{a_1, a_2, a_3 \leq k-2, a_1 + a_2 + a_3 = k-2} [V_{(0)}^{a_1} V_{(1)}^{a_2} V_{(n)}^{a_3}] + \sum_{a_1, a_2, a_3 \leq k-2, a_1 + a_2 + a_3 = k-2} [V_{(0)}^{a_1} V_{(0)}^{a_2} V_{(n)}^{a_3} + V_{(1)} V_{(0)}^{a_1} V_{(1)}^{a_2} V_{(0)}^{a_3}]_{ij} + \ldots ,
\]

where the dots stand for terms which depend only on \(V_{(n')}, \ n' < n\). Note that if \(n = 1\) then the second summand should be removed, or equivalently we have to divide the sum by 2.

The sum is easy to simplify because \(V_{(0)}\) is a diagonal matrix. We get

\[
(4.28) \sum_{s=1}^{N} \left( \sum_{0 \leq a_1, a_2, a_3 \leq k-2, a_1 + a_2 + a_3 = k-2} \lambda_i^{a_1} \lambda_s^{a_2} \lambda_j^{a_3} \right) ([V_{(1)}]_{is} [V_{(n)}]_{sj} + [V_{(n)}]_{is} [V_{(1)}]_{sj}) ,
\]

We want to compute this sum up to terms independent of the exceptional entries of \(V_{(n)}\). There are three cases.

**Case 1:** If \(s \neq i\) and \(s \neq j\). Then pairs \((s,i)\) and \((s,j)\) are either both exceptional or both non-exceptional, because \((i,j)\) is an exceptional pair, (i.e., \(\lambda_i \neq \lambda_j\) but \(\lambda_i^k = \lambda_j^k\)). We can assume that \((s,i)\) and \((s,j)\) are exceptional pairs. In particular, \(\lambda_i^k = \lambda_j^k = \lambda_k^k\). On the other hand, the sum of the \(\lambda_i\)'s in \(4.28\) is

\[
\frac{1}{\lambda_i - \lambda_s} \left( \frac{\lambda_i^k - \lambda_j^k}{\lambda_i - \lambda_j} - \frac{\lambda_k^k - \lambda_j^k}{\lambda_s - \lambda_j} \right).
\]

The above sum is 0. Hence in case \(s \neq i\) and \(s \neq j\) there is no contributions.

**Case 2:** \(s = j\). Then the sum of the \(\lambda_i\)'s in \(4.28\) is \(-k\lambda_j^{k-1}/(\lambda_i - \lambda_j)\). On the other hand, since the entry \([V_{(n)}]_{sj}\) is not exceptional and \([V_{(1)}]_{is}\) is already
determined we get that the contribution we are interested in is

$$-\frac{k\lambda_i^{k-1}}{\lambda_i - \lambda_j}[V(n)]_{ij}[V(1)]_{jj} = \left(-\sum_{a=1}^{N} \frac{\lambda_{a,i}}{\lambda_i - \lambda_j} \tau_a\right)[V(n)]_{ij},$$

where in the first equality we used part a).

**Case 3:** \(s = i\). Just like in the second case we get that the contribution is

$$-\left(-\sum_{a=1}^{N} \frac{\lambda_{a,i}}{\lambda_j - \lambda_i} \tau_a\right)[V(n)]_{ij}.$$  

We sum up the contributions from the three cases and then part b) follows.

If \(n = 1\), then the contributions from cases 2 and 3 should be doubled, because we have

$$[V(1)]_{is}[V(n)]_{sj} + [V(n)]_{is}[V(1)]_{sj} = 2[V(1)]_{is}[V(1)]_{sj}. $$

However this additional factor of 2 is compensated by an earlier division by 2 as it was already explained in the beginning of our proof of part b).

Now we are ready to finish the proof of the reconstruction theorem. We need to prove that the exceptional entries \([\overline{V}(n)]_{ij}\) are uniquely determined in terms of \(V(n')\), \(n' < n\) and the non-exceptional entries of \(\overline{V}(n)\). This follows from the equation

\[(4.29) \quad \partial_a U^{(n+1)} = F_a^{(n)} + [\overline{\mu}, F_a^{(n)}].\]

Let \((i, j)\) be an exceptional pair and let us compare the \((i, j)\)-th entries in \[(4.29)\]. We claim that the \((i, j)\)-th entry of \([\overline{\mu}, F_a^{(n)}]\) is independent of the exceptional entries of \(\overline{V}(n)\). Indeed, the entry is given by

$$\sum_{s=1}^{n} [\overline{\mu}]_{is}[F_a^{(n)}]_{sj} - [F_a^{(n)}]_{is} [\overline{\mu}]_{sj}. $$

If \(s\) equals \(i\) or \(j\) then \(\lambda^k_i = \lambda^k_j = \lambda^k_s\), where the first equality holds because \((i, j)\) is an exceptional pair. On the other hand, since \(U^{(0)} = V_k\), we have \(u^i_{ij} = \lambda^k_i\) for all \(i = 1, \ldots, N\). Recalling the second condition of our theorem we get: \([\overline{\mu}]_{is} = [\overline{\mu}]_{sj} = 0\). If \(s\) is different from both \(i\) and \(j\) then we can assume also that \(\lambda^k_i \neq \lambda^k_j\) and \(\lambda^k_i \neq \lambda^k_s\), otherwise respectively \([\overline{\mu}]_{is} = 0\) and \([\overline{\mu}]_{sj} = 0\). In other words \((s, i)\) and \((s, j)\) are not exceptional pairs. According to Lemma \textbf{4.3}\( [\overline{F}_a^{(n)}]_{sj}\) and \([\overline{F}_a^{(n)}]_{is}\) depend only on \(V(n')\), \(n' < n\) and the non-exceptional entries of \(\overline{V}(n)\).
To finish the proof it remains only to recall Lemma 4.3 part b) of Lemma 4.4 and (4.29). We get
\[
\left( \sum_{a=1}^{N} \frac{\lambda_{a,i} - \lambda_{a,j}}{\lambda_i - \lambda_j} \right) \partial_{\tau_a}[V(n)]_{ij} = \text{known terms}.
\]
Notice that the above sum is non-zero because the coefficient in front of $\tau_2$ is 1.

4.2. Twisted curves. In this Section every scheme is over $\mathbb{C}$, and the terms “orbifold” and “smooth Deligne-Mumford stack” are used interchangeably. Let $C$ be a smooth curve, $p_1, \ldots, p_n \in C$ distinct points, and $k_1, \ldots, k_n$ positive integers. Given these data, we consider the stack $C[(p_1, k_1), \ldots, (p_n, k_n)]$, which is constructed as the stack of roots of line bundles on $C$. More precisely, the stack $C[(p_1, k_1), \ldots, (p_n, k_n)]$ is the fiber product $\sqrt[k_n]{(\mathcal{O}_{p_1}, \sigma_1)/\mathcal{C}} \times_C \ldots \times_C \sqrt[k_1]{(\mathcal{O}_{p_1}, \sigma_1)/\mathcal{C}}$. The stack $\sqrt[k_i]{(\mathcal{O}_{p_i}, \sigma_i)/\mathcal{C}}$ is the stack of $k_i$-th root of the line bundle $\mathcal{O}(p_i)$ with the canonical section $\sigma_i : \mathcal{O} \to \mathcal{O}(p_i)$. An object of $\sqrt[k_i]{(\mathcal{O}_{p_i}, \sigma_i)/\mathcal{C}}$ over a $\mathbb{C}$-scheme $T$ is
- a line bundle $M$;
- an isomorphism $\phi$ of $M^{\otimes k_i}$ with the pullback of $\mathcal{O}(p_i)$ via $T \to C$;
- a section $\tau$ of $M$ such that $\phi(\tau^{k_i}) = \sigma_i$.

More details of this construction can be found in [2] and [3]. An alternative description of $C[(p_1, k_1), \ldots, (p_n, k_n)]$ using log geometry may be found in [31]. Étale locally near a point $p \in C \setminus \{p_1, \ldots, p_n\}$, the stack $C[(p_1, k_1), \ldots, (p_n, k_n)]$ is isomorphic to the curve $C$. Étale locally near the point $p_i$, the stack $C[(p_1, k_1), \ldots, (p_n, k_n)]$ is isomorphic to the stack quotient $[\text{Spec } \mathbb{C}[x]/\mu_{k_i}]$ where the group $\mu_{k_i}$ acts via $x \mapsto \zeta x$ for $\zeta \in \mu_{k_i}$. The natural projection
\[
C[(p_1, k_1), \ldots, (p_n, k_n)] \to C
\]
exhibits $C$ as its coarse moduli space.

4.3. Orbifold quantum cohomology. Our focus is a simple case of this construction, namely
\[
\mathcal{C}_{k,m} := \mathbb{P}^1[(0, k), (\infty, m)],
\]
for integers $k, m \geq 1$. We call $\mathcal{C}_{k,m}$ a 2-pointed $\mathbb{P}^1$-orbifold. Roughly speaking, this is the curve $\mathbb{P}^1$ with orbifold points $BZ_k$ and $BZ_m$ at 0 and $\infty$ respectively. Note that for $k, m$ coprime, the orbifold $\mathbb{P}^1[(0, k), (\infty, m)]$ is isomorphic to the weighted projective line $\mathbb{P}^1(k, m)$. If $k$ and $m$ are not coprime, then the weighted projective line $\mathbb{P}^1(k, m)$ has nontrivial generic stabilizer. But $\mathbb{P}^1[(0, k), (\infty, m)]$ always has trivial generic stabilizers, thus it is not isomorphic to $\mathbb{P}^1(k, m)$. Also, it is obvious that $\mathbb{P}^1[(0, k), (\infty, m)] \simeq \mathbb{P}^1[(0, m), (\infty, k)]$.

\footnote{Of course placing the two orbifold points elsewhere on $\mathbb{P}^1$ results isomorphic orbifolds.}
To each orbifold one can associate another orbifold, the so-called inertia orbifold, which plays a key role in orbifold Gromov-Witten theory. By definition, the inertia orbifold of a given orbifold $\mathcal{X}$ is defined to be the fiber product (in the 2-category of stacks) $I\mathcal{X} := \mathcal{X} \times_{\Delta \times \mathcal{X} \times \Delta} \mathcal{X}$, where $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is the diagonal morphism. In categorical terms, the objects of $I\mathcal{X}$ are:

$$Ob(I\mathcal{X}) := \{(x, g)\mid x \in Ob(\mathcal{X}), g \in Aut(x)\} = \{(x, H, g)\mid x \in Ob(\mathcal{X}), H \subset Aut(x), g \text{ a generator of } H\}.$$ 

There are two natural maps: $q : I\mathcal{X} \rightarrow \mathcal{X}$ given by forgetting the choice of $g \in Aut(x)$, and $I : I\mathcal{X} \rightarrow I\mathcal{X}$ given by $g \mapsto g^{-1}$. The inertia orbifold $I\mathcal{X}$ is disconnected (unless $\mathcal{X}$ is a connected manifold). Write $I\mathcal{X} = \coprod_{i \in \mathbb{Z}} \mathcal{X}_i$ for the decomposition into connected components. On each connected component $\mathcal{X}_i$ there is a trivial action of the cyclic group $\mathbb{Z}_{r_i}$. Thus $\mathbb{Z}_{r_i}$ acts on the vector bundle $q^* T\mathcal{X}$. This yields a decomposition $q^* T\mathcal{X} = \oplus_j E_{ij}$ into eigen-bundles, where $\mathbb{Z}_{r_i}$ acts on $E_{ij}$ via multiplication by $\exp(2\pi \sqrt{-1} k_j/r_i)$ with $0 \leq k_j < r_i$.

The age associated to the component $\mathcal{X}_i$ is defined to be $age(\mathcal{X}_i) := \sum_j k_j/r_i$.

In our case, it is easy to see that

$$IC_{k,m} \simeq C_{k,m} \cup \bigcup_{1 \leq i \leq k-1} B\mu_k(i) \cup \bigcup_{1 \leq j \leq m-1} B\mu_m(j).$$

Here for each $i, j$ we have $B\mu_k(i) \simeq B\mu_k$ and $B\mu_m(j) \simeq B\mu_m$. The age associated to the component $C_{k,m}$ is 0, the age associated to $B\mu_k(i)$ is $i/k$, the age associated to $B\mu_m(j)$ is $j/m$.

Next we turn to orbifold cohomology. As a graded vector space, the orbifold cohomology of $\mathcal{X}$ (with complex coefficients) is defined to be $H^*(I\mathcal{X})$ with the grading defined as follows: a class $a \in H^p(\mathcal{X}_i)$ is assigned the degree $p + 2age(\mathcal{X}_i)$. The orbifold cohomology of $C_{k,m}$, as a vector space, is given by

$$H^*_\text{orb}(C_{k,m}) = H^0(C_{k,m}) \oplus H^2(C_{k,m}) \oplus \bigoplus_{1 \leq i \leq k-1} H^0(B\mu_k(i)) \oplus \bigoplus_{1 \leq j \leq m-1} H^0(B\mu_m(j)).$$

An element in $H^0(B\mu_k(i))$ is assigned degree $2i/k$, and an element in $H^0(B\mu_m(j))$ is assigned degree $2j/m$. This gives $H^*_\text{orb}(C_{k,m})$ the structure of a graded vector space.

We fix some notations. Let $1 \in H^0(C_{k,m})$ be the Poincaré dual of the fundamental class, $p \in H^2(C_{k,m})$ the Poincaré dual of a point, $x_i \in H^0(B\mu_k(i))$ the Poincaré dual of the fundamental class for each $i$, and $y_i \in H^0(B\mu_m(j))$ the Poincaré dual of the fundamental class for each $j$.

In general the orbifold cohomology space $H^*_\text{orb}(\mathcal{X})$ carries a non-degenerate pairing $\langle \ , \ \rangle_{\text{orb}}$ called orbifold Poincaré pairing. It is defined as follows: for $a, b \in H^*(I\mathcal{X})$, define $\langle a, b \rangle_{\text{orb}} := \int_{I\mathcal{X}} a \wedge I^*b$. In our case this pairing is given as follows:

$$\langle x_i, x_{k-i} \rangle_{\text{orb}} = 1/k, \langle y_j, y_{m-j} \rangle_{\text{orb}} = 1/m, \langle 1, p \rangle_{\text{orb}} = 1 = \langle p, 1 \rangle_{\text{orb}}; \text{ and } 0 \text{ otherwise}.$$
A recent advance in the study of orbifolds is that orbifold cohomology \( H^*_{\text{orb}}(X) \) carries a nontrivial ring structure called orbifold cup product, see [8] and [1]. We briefly recall its definition. The geometric object central to the construction of this ring structure, as well as orbifold Gromov-Witten theory, is the notion of orbifold stable maps. An orbifold stable map \( f : \mathcal{C} \to X \) is a representable map from a nodal curve \( \mathcal{C} \), possibly having orbifold structures at marked points and nodes, to the orbifold \( X \). We may fix discrete invariants and consider moduli spaces \( \overline{M}_{g,n}(X,d) \) parametrizing \( n \)-pointed orbifold stable maps of genus \( g \) and degree \( d \in H_2(X,\mathbb{Z}) \). These moduli spaces come with two kinds of maps: the evaluation map at the \( i \)-th marked point \( ev_i : \overline{M}_{g,n}(X,d) \to \mathcal{I}X \); the map \( \pi : \overline{M}_{g,n}(X,d) \to \overline{M}_{g,n}(X,d) \) given by passing to coarse moduli spaces.

Deformation theory of orbifold stable maps yields a perfect obstruction theory on \( \overline{M}_{g,n}(X,d) \), from which one can construct a virtual fundamental class \( [\overline{M}_{0,3}(X,0)]^\text{vir} \), see [2].

Now we can define orbifold cup products: for \( a,b \in H^*_{\text{orb}}(X) \), define
\[
a \cdot b := (I \circ ev_3)_* (ev_1^*a \cup ev_2^*b \cap [\overline{M}_{0,3}(X,0)]^\text{vir}).
\]

This product respects gradings, making \( (H^*_{\text{orb}}(X),\cdot) \) a graded commutative associative \( \mathbb{C} \)-algebra.

We now describe the orbifold cohomology ring structure of \( H^*_{\text{orb}}(\mathcal{C}_{k,m}) \) as explained above. By definition, classes in \( H^*(\mathcal{C}_{k,m}) = H^*(\mathbb{P}^1) \) multiply as usual. Furthermore, using only the definition, we have
\[
\begin{align*}
x_i \cdot y_j &= 0 \text{ for every } i,j; \\
x_{i_1} \cdot x_{i_2} &= x_{i_1+i_2} \text{ if } i_1 + i_2 \leq k - 1; \\
y_{j_1} \cdot y_{j_2} &= y_{j_1+j_2} \text{ if } j_1 + j_2 \leq m - 1; \\
k x_1^k &= m y_1^m = p.
\end{align*}
\]

It follows that, as rings,
\[
H^*_{\text{orb}}(\mathcal{C}_{k,m}) \simeq \mathbb{C}[x,y]/(k x^k - m y^m, x y),
\]
where we identify \( x_1 = x \) and \( y_1 = y \).

**Remark 4.5.** The calculation of orbifold Poincaré pairing and orbifold cup product for \( \mathcal{C}_{k,m} \) can be easily generalized to the more general twisted curve \( C[(p_1,k_1),...,(p_n,k_n)] \). We won’t need this here.

The definition of orbifold cup product involves only degree 0 orbifold stable maps. Intersection numbers on moduli spaces of orbifold stable maps of nonzero degrees can be packaged to give a deformation of the orbifold cup product, which we now describe.

---

3A technical point: we consider here orbifold stable maps with sections to all gerbes. Our notation here agrees with that in [1] and [33].
For classes $a_1, \ldots, a_n \in H_{orb}^*(\mathcal{X})$, define the genus zero primary orbifold Gromov-Witten invariant $\langle a_1, \ldots, a_n \rangle_{0,n,d}$ to be the integral

$$\int_{\mathcal{M}_{0,n}(\mathcal{X},d)^{vir}} ev_1^*a_1 \wedge \ldots \wedge ev_n^*a_n.$$ 

Fix an additive basis $\{\phi_\alpha\}$ of $H_{orb}^*(\mathcal{X})$ and write $\{\phi^\alpha\}$ for its dual basis. For classes $a, b \in H_{orb}^*(\mathcal{I}\mathcal{X})$, the formula

$$a \ast_t b := \sum_{n,d} \frac{Q^d}{n!} \langle a, b, \phi_\alpha, t, \ldots, t \rangle_{0,n+3,d} \phi^\alpha,$$

defines a ring structure on $H_{orb}^*(\mathcal{X})$ with coefficient ring enlarged to the Novikov ring $\mathbb{C}[\![H_2(\mathcal{X})]\!]$. This product $\ast_t$ depends on a parameter $t \in H_{orb}^*(\mathcal{X})$. Associativity of $\ast_t$ is nontrivial. The ring $BQH_{orb}^*(\mathcal{X}) := (H_{orb}^*(\mathcal{X}), \ast_t)$ is called the big orbifold quantum cohomology ring. The variables $Q$ in the Novikov ring are assigned degrees so that $deg(Q^d) = 2 \int_d c_1(T_{\mathcal{X}})$. The product $\ast_t$ respects degrees.

When restricting to $t \in H^2(\mathcal{X})$, an easy application of the divisor equation shows that $\ast_t$ may be identified with the following product:

$$a \ast_t b := \sum_d (Qe^d)^d(I \circ ev_3)_*(ev_1^*a \cup ev_2^*b \cap [\overline{\mathcal{M}}_{0,3}(\mathcal{X}, d)]^{vir}).$$

We call $QH_{orb}^*(\mathcal{X}) := (H_{orb}^*(\mathcal{X}), \ast_t)$ the small orbifold quantum cohomology ring.

We now describe the small orbifold quantum cohomology ring $QH_{orb}^*(\mathcal{C}_{k,m})$. First note that the Picard group $Pic(\mathcal{C}_{k,m})$ is generated by two line bundles $L_0, L_\infty$ such that $L_0^\otimes k \simeq L_\infty^\otimes m$ and both are isomorphic to the pull-back of $O_{\mathbb{P}^1}(1)$. We have $deg L_0 = 1/k, deg L_\infty = 1/m$. The canonical line bundle $K_{\mathcal{C}_{k,m}} \simeq L_0^\vee \otimes L_\infty^\vee$ has degree $-1/k - 1/m$. Additively we have $QH_{orb}^*(\mathcal{C}_{k,m}) = H_{orb}^*(\mathcal{C}_{k,m}) \otimes_{\mathbb{C}} \mathbb{C}[r,q]$, where the variable $q := Qe^t, t \in H^2(\mathcal{C}_{k,m})$ is assigned degree $2/k + 2/m$. The product structure of $QH_{orb}^*(\mathcal{C}_{k,m})$ is a deformation of that on $H_{orb}^*(\mathcal{C}_{k,m})$. So we only have to analyze how to deform the relations in $H_{orb}^*(\mathcal{C}_{k,m})$. By degree consideration, the relation $xy = 0$ can only be deformed to $xy = cq$. Here by definition $c$ is the orbifold Gromov-Witten invariant $\langle x, y, p \rangle_{0,3,1}$, which is clearly equal to 1.

The relation $kx^k - my^m = 0$ remains unchanged. This is easily seen by degree consideration if $k, m$ are co-prime. In general, it follows from the fact that the classes $kx^k, my^m$ in $H_{orb}^*(\mathcal{C}_{k,m})$ are both equal to $p$, and the following

Lemma 4.6.

1. The product $x^{a_1}$ of $a$ copies of $x$ is equal to $x^a$ if $1 \leq a \leq k$.
2. The product $y^{b_1}$ of $b$ copies of $y$ is equal to $y^b$ if $1 \leq b \leq m$. 


Proof. We only prove the statement about $x^{*a}$, an analogous argument proves the statement about $y^{*b}$.

We need the following non-vanishing conditions:

\[(4.30)\quad \text{If } \langle x^i, x^j, x^l \rangle_{0,3,d} \neq 0, \text{ then } i + j + l \equiv d \pmod{k}, \text{ and } d \equiv 0 \pmod{m}.\]

\[(4.31)\quad \text{If } \langle x^i, x^j, y^l \rangle_{0,3,d} \neq 0, \text{ then } i + j + l \equiv d \pmod{k}, \text{ and } d \equiv 0 \pmod{m}.\]

We first prove the statement about $x^{*a}$ assuming (4.30), (4.31). We proceed by induction on $a$. Clearly $x^{*1} = x$. Suppose that $x^{*a} = x^{a}$ for some $a \leq k-1$.

We may write

\[x^{*a+1} = x^a \star t \cdot x = x^{a+1} + \sum_{d>0} q^d \left( a_{0,q} 1 + b_{0,q} + \sum_{i=1}^{k-1} a_i d x^i + \sum_{j=1}^{m-1} b_j d y^j \right).\]

Note that the right side of the equation above should be homogenous of degree $2(a+1)/k$.

Suppose that $a_{i,d} \neq 0$ for some $1 \leq i \leq k-1$. Then by definition we have $\langle x^a, x^{k-i} \rangle_{0,3,d} \neq 0$. By (4.30), we have $a + 1 \equiv i + d \pmod{k} \text{ and } d \equiv 0 \pmod{m}$. In particular $d/m \geq 1$. Now by comparing degrees, we find

\[1 \geq \frac{a + 1}{k} = \frac{i}{k} + d \left( \frac{1}{k} + \frac{1}{m} \right) = \frac{i + d}{k} + \frac{d}{m} > 1,\]

which is a contradiction. The same argument proves that $a_{0,d} = 0$.

Suppose that $b_{j,d} \neq 0$ for some $1 \leq j \leq m-1$. Then by definition we have $\langle x^a, y^{m-j} \rangle_{0,3,d} \neq 0$. By (4.31), we have $a + 1 \equiv d \pmod{k} \text{ and } m - j \equiv d \pmod{m}$. In particular $(d + j)/m \geq 1$. Again by comparing degrees, we find

\[1 \geq \frac{a + 1}{k} = \frac{j}{m} + d \left( \frac{1}{k} + \frac{1}{m} \right) = \frac{d}{k} + \frac{d + j}{m} > 1,\]

which is a contradiction.

Finally, $b_{0,d} = 0$ because the degree of $p \cdot q^d$ is $2(1 + d(1/k + 1/m)) > 2$, while the degree of $x^a \star t \cdot x$ is at most 2.

Now we prove (4.30). If $\langle x^i, x^j, x^l \rangle_{0,3,d} \neq 0$, then the relevant moduli space must be non-empty. So there exists a three-pointed, degree $d$ orbifold stable map $f : \mathcal{C} \to \mathcal{C}_{k,m}$ with stack structures on $\mathcal{C}$ prescribed by the insertions. The holomorphic Euler characteristics $\chi(\mathcal{C}, f^*L_0)$ and $\chi(\mathcal{C}, f^*L_{\infty})$ are integers. By Riemann-Roch, we find

\[\chi(\mathcal{C}, f^*L_0) = 1 + \frac{d}{k} - \frac{i}{k} - \frac{j}{k} - \frac{l}{k},\]

\[\chi(\mathcal{C}, f^*L_{\infty}) = 1 + \frac{d}{m}.\]
The result follows. The proof of (4.31) is similar: if $\langle x^i, x^j, y^l \rangle_{0,3,d} \neq 0$, then the relevant moduli space is not empty. So there exists a three-pointed, degree $d$ orbifold stable map $f : \mathcal{C} \to \mathcal{C}_{k,m}$ with stack structures on $\mathcal{C}$ prescribed by the insertions. One calculates by Riemann-Roch that, in this case,
\[
\chi(\mathcal{C}, f^*L_0) = 1 + \frac{d}{k} - \frac{i}{k} - \frac{j}{k},
\]
\[
\chi(\mathcal{C}, f^*L_\infty) = 1 + \frac{d}{m} - \frac{l}{m}.
\]
The result follows by integrality of $\chi(\mathcal{C}, f^*L_0)$ and $\chi(\mathcal{C}, f^*L_\infty)$. □

Hence we obtain the following presentation of the small orbifold quantum cohomology ring:
\[
(4.32) \quad QH^*_\text{orb}(\mathcal{C}_{k,m}) \simeq \mathbb{C}[[q]][x, y]/(kx^k - my^m, xy - q).
\]

This presentation allows us to set $Q$ to any nonzero complex number. We do so from now on.

Note that in case of $k, m$ coprime, (4.32) coincides with the calculations in [2] for weighted projective lines $\mathbb{P}^1(k, m)$.

4.4. Frobenius structure. It is known that genus zero Gromov-Witten theory provides a natural Frobenius structure on the cohomology of the target space. The same is true for orbifolds: orbifold cohomology $H^*_\text{orb}(\mathcal{X})$ of an orbifold $\mathcal{X}$ carries a natural Frobenius structure arising from genus zero orbifold Gromov-Witten invariants. The ingredients of this Frobenius structure are summarized as follows.

- The space on which the Frobenius structure is based: the orbifold cohomology $H^*_\text{orb}(\mathcal{X}) \otimes \mathbb{C}\{H_2(\mathcal{X})\}$;
- the flat metric is given by the orbifold Poincaré pairing $(\ , )_\text{orb}$;
- the product structure is given by the orbifold big quantum product $\star_t$.

We now turn to the special case $BQH^*_\text{orb}(\mathcal{C}_{k,m})$. Consider the following homogeneous additive basis of $H^*_\text{orb}(\mathcal{C}_{k,m})$,
\[
(4.33) \quad x_{k-1}, \ldots, x_1, 1, y_1, \ldots, y_{m-1}, p.
\]

Note the ordering of these classes. In this basis, we may write a class in $H^*_\text{orb}(\mathcal{C}_{k,m})$ as
\[
\sum_{i=1}^{k-1} s_i x_{k-i} + s_k 1 + \sum_{j=1}^{m-1} s_{k+j} y_j + s_N p,
\]
where $N = k + m$. 
By expressing multiplications by \( x \) and \( y \) in the presentation (4.32) as matrices using the (ordered) basis (4.33), it is easy to show that the Frobenius manifold \( BQH^*_{orb}(C_{k,m}) \) is semi-simple along \( H^2(C_{k,m}) \).

The coordinates \( \{s_1, ..., s_N\} \) are flat coordinates of \( BQH^*_{orb}(C_{k,m}) \). In these coordinates, the Euler vector field reads

\[
s_k \partial_{s_k} + \sum_{i=1}^{k-1} \left( \frac{i}{k} \right) s_i \partial_{s_i} + \sum_{j=1}^{m-1} \left( 1 - \frac{j}{m} \right) s_{k+j} \partial_{s_{k+j}} + \left( \frac{1}{k} + \frac{1}{m} \right) \partial_{s_N}.
\]

**Proof of Theorem 1.1.** Using Theorem 4.2, we prove that the following map (4.34)

\[
\begin{array}{ccc}
s_i & \mapsto & \tau_i, \\
& \text{for } i = 1, ..., N-1, & s_N \mapsto \tau_N/m,
\end{array}
\]

is an isomorphism between the Frobenius structures respectively on the big quantum cohomology \( BQH^*_{orb}(C_{k,m}) \) and on \( M_{k,m} \).

The map (4.34) identifies \( M_{k,m} \) and \( BQH^*_{orb}(C_{k,m}) \) as complex manifolds. It also identifies the corresponding flat metrics, unity vector fields and Euler vector fields. It remains only to verify that both Frobenius structures satisfy the conditions of Theorem 1.2.

Note that at the point \( \tau_1 = ... = \tau_N = 0 \) we have isomorphisms of Frobenius algebras \( T_0 M_{k,m} \simeq T_0 BQH^*_{orb}(C_{k,m}) \cong \mathbb{C}[x,y]/\langle kx^k - my^m, xy - Q \rangle \), by (4.32). Up to a scalar, the \( k \)-th root of the restriction of the Euler vector field to \( T_0 M_{k,m} \) is given by \( x \). From the above presentation of \( T_0 M_{k,m} \), it follows that \( x \) is an invertible generator of the Frobenius algebra and that the point \( \tau = 0 \) is semisimple, i.e., the first condition in Theorem 1.2 is satisfied. It remains only to verify the second one.

Pick the following basis of \( T_0 M_{k,m} \):

\[
\phi_1 = x^{k-1}, \ldots, \phi_k = 1, \phi_{k+1} = \frac{Q}{x}, \ldots, \phi_{k+m} = \left( \frac{Q}{x} \right)^m.
\]

It is easy to see that the eigenvalues of \( x \bullet_0 \) are given by:

\[
\lambda_i = \left( \frac{mQ^m/k}{\lambda_i} \right)^{1/k} \exp \left( 2\pi \sqrt{-1} i/N \right), \quad 1 \leq i \leq N.
\]

On the other hand, the quantum product \( \bullet_0 \) is diagonalized by the matrix \( P \) whose \( i \)-th row is given by \( \left( \lambda_i^{k-1}, \ldots, 1, Q/\lambda_i, \ldots, (Q/\lambda_i)^m \right) \). The \( i \)-th column of the matrix inverse to \( P \) is given by: \( \frac{1}{N} \left( \lambda_i^{(k-1)}, \ldots, 1, \lambda_i/Q, \ldots, (\lambda_i/Q)^m \right) \) (recall that by definition \( N = k + m \)). We need to prove that if \( i \) and \( j \) are such that \( \lambda_i^k = \lambda_j^k \) then the \((i, j)\)-th entry of \( \overline{P} := P \mu P^{-1} \) is 0. On the other hand in the above basis \( \mu \) is represented by the diagonal matrix

\[
\text{Diag} \left( i/k - 1/2, 1 \leq i \leq k-1, 1/2 - j/m, 0 \leq j \leq m \right).
\]
Therefore we have to verify that
\[
(4.35) \quad \sum_{s=1}^{k-1} P_{is} \left( \frac{s}{k} - \frac{1}{2} \right) [P^{-1}]_{sj} + \sum_{s=0}^{m} P_{i,k+s} \left( \frac{1}{2} - \frac{s}{m} \right) [P^{-1}]_{k+s,j} = 0.
\]

Note that \( P_{is} [P^{-1}]_{sj} = (\lambda_i/\lambda_j)^{k-s}/N \) = \((\lambda_j/\lambda_i)^{s}/N\), where for the second equality we used that \( \lambda_k^i = \lambda_k^j \). Putting \( x = \lambda_j/\lambda_i \), we get that the first sum in (4.35) equals \( \frac{1}{N} \left( \frac{1}{2} + \frac{1}{x-1} \right) \). For the second sum we have \( P_{i,k+s} [P^{-1}]_{k+s,j} = (\lambda_j/\lambda_i)^{s}/N \). Note that the summands corresponding to \( s = 0 \) and \( s = m \) cancel each other, so we may assume that the summation range is from 1 to \( m - 1 \). Also, we have that \( x^m = 1 \), because \( N = k + m \), \( x^k = 1 \) and \( x^N = 1 \). So the second sum simplifies to \( -\frac{1}{N} \left( \frac{1}{2} + \frac{1}{x-1} \right) \).

4.5. Descendent potential. We recall the definition of the descendent orbifold Gromov-Witten invariants, which plays an important role in orbifold Gromov-Witten theory.

Recall that on the moduli space \( \overline{M}_{g,n}((X,d) \to \text{coarse moduli space } X \text{ there are } n \text{ line bundles } L_1, \ldots, L_n \text{ associated to the marked points. The fiber of } L_i \text{ at a moduli point } (f : (C,p_1,\ldots,p_n) \to X) \text{ is the cotangent space } T^*_p C. \text{ Consider the pullback line bundles } \pi^* L_i. \text{ The descendent classes in orbifold Gromov-Witten theory of } \mathcal{X} \text{ are defined to be } \tilde{\psi}_i := c_1(\pi^* L_i). \text{ The totality of descendent orbifold Gromov-Witten invariants can be packaged in a generating function, called the total descendent potential of } \mathcal{X}, \text{ which is defined as follows:}
\[
\mathcal{D}_X := \exp \left( \sum_{g \geq 0} \epsilon^{2g-2} \sum_{n,d} \frac{Q^d}{n!} \int_{[\overline{M}_{g,n}(X,d)]^{vir}} \prod_{i=1}^{n} \sum_{k=0}^{\infty} e^{t_i k} t_i \tilde{\psi}_i^k \right).
\]

The total descendent potential \( \mathcal{D}_X \) is viewed as a function of \( t := \sum_{k \geq 0} t_k z^k \) and, via the dilaton shift \( q = t - 1z \), as an element in the Fock space—the space of functions on \( H^*_\text{orb}(\mathcal{X})[z] \). Assume that \( \mathcal{X} \) has semi-simple orbifold quantum cohomology and denote by \( \mathcal{D}^{BQH(\mathcal{X})} \) the descendent potential corresponding to the semi-simple Frobenius structure ([16]). Then we have the orbifold version of Givental’s conjectural formula: \( \mathcal{D}_X = \mathcal{D}^{BQH(\mathcal{X})} \). Recent work of C. Teleman ([32]) is very close to providing a proof of this. In our case (i.e., \( \mathcal{X} = \mathcal{C}_{k,m} \)) this formula can be proven by virtual localization. Details will be given in ([34]).

In conclusion, we formulate the following conjecture:
Conjecture 4.7. The total descendent potential of $C_{k,m}$ is a tau-function of the extended bi-graded Toda hierarchy corresponding to $M_{k,m}$ (7).

Once the HQE (1.10) is shown to describe the extended bigraded Toda hierarchy, Conjecture 4.7 will follow from results in this paper and Givental’s formula.

We remark that the extended bigraded Toda hierarchy with $k = m = 1$ coincides with the extended Toda hierarchy. However, the HQE (1.10) specialized to $k = m = 1$ are different from the HQE for extended Toda hierarchy given in [28]. Therefore, the case $k = m = 1$ of Conjecture 4.7 provides yet another formulation of the Toda conjecture about Gromov–Witten invariants of $\mathbb{CP}^1$.

5. Period vectors and vertex operators

5.1. Period vectors near a critical value. Let $\tau \in M$ be a semi-simple point, i.e., $f_\tau$ is a Morse function. A relative cycle $\beta \in H_1(C^*, V_{\tau, \lambda_0}; \mathbb{Z})$ is called a Lefschetz thimble corresponding to a path $C_i$ from $\lambda_0$ to a critical value $u_i$ of $f_\tau$ if $\beta$ is represented by the two components in $f_\tau^{-1}(C_i)$ which meet at the critical point above $u_i$.

Lemma 5.1. Let $C_\infty$ be a path from $\lambda_0 = \infty$ to $\lambda = -z \cdot (+\infty)$ and denote by

$$B \in \lim_{M \to \infty} H_1(C^*, \{x \in C^* : \text{Re}(f_\tau/z) < -M\}; \mathbb{Z})$$

the cycle obtained from $\beta$ by a parallel transport along $C_\infty$. Then

$$J_B(\tau, z) = (2\pi z)^{-1/2} \int_{u_i}^{-z(+\infty)} e^{\lambda/z} I^{(0)}(0) \beta(\tau, \lambda) d\lambda.$$

Proof. The oscillating integral $(-2\pi z)^{1/2} J_B$ can be transformed as follows.

$$\int_B e^{f_\tau/z} \omega = \int_{u_i}^{-z(+\infty)} e^{\lambda/z} \left( \int_{\partial \beta} \frac{\omega}{d f_\tau} \right) d\lambda = \int_{u_i}^{-z(+\infty)} e^{\lambda/z} \partial_\lambda \left( \int_{\partial \beta} d^{-1} \omega \right) d\lambda$$

where in the last equality we applied integration by parts and the Stokes’ formula. The lemma follows because, by definition,

$$(J_B, \partial_i) = z \partial_i J_B \quad \text{and} \quad (I^{(0)}_\beta, \partial_i) = -\partial_i \int_\beta \omega.$$

Lemma 5.2. Let $\xi$ be sufficiently close to $u_i$ Then

$$(5.36) I^{(0)}_\beta(\tau, \xi) = \frac{2}{\sqrt{2(\xi - u_i)}} \left( 1_i + A_{i,1}[2(\xi - u_i)] + A_{i,2}[2(\xi - u_i)]^2 + \ldots \right),$$
where the path $C'_i$ specifying $\beta(\tau, \xi)$ is the same as $C_i$ except for the end where the two paths split: $C_i$ leads to $u_i$ and $C'_i$ leads to $\xi$.

Proof. We follow [3], chapter 3, section 12, Lemma 2. In a neighborhood of the critical point above $u_i$, we choose a unimodular coordinate $y$ for the volume form $\omega$ i.e., $\omega = dy$. The Taylor’s expansion of $f_\tau$ is

$$f_\tau(y) = u_i + \frac{\Delta_i}{2}(y - y_i)^2 + \ldots,$$

where $y_i$ is the $y$-coordinate of the critical point $q_i$ corresponding to $u_i$. From this expansion we find that the equation $f_\tau(y) = \xi$ has two solutions in a neighborhood of $y = q_i$:

$$y_\pm = y_i \pm \frac{1}{\sqrt{\Delta_i}}\sqrt{2}(\xi - u_i) + \text{h.o.t.},$$

where h.o.t. means higher order terms. Thus the integral in the definition of $I^{(0)}_{\beta}(\tau, \xi)$ has the following expansion:

$$\int_{\beta(\tau, \xi)} \omega = y_+(\tau, \xi) - y_-(\tau, \xi) = \frac{2}{\sqrt{\Delta_i}}\sqrt{2}(\xi - u_i) + \text{h.o.t.},$$

which yields

$$(I^{(0)}_{\beta}(\tau, \xi), \partial_j) = \frac{2}{\sqrt{2}(\xi - u_i)} \frac{1}{\sqrt{\Delta_i}}\partial_j u_i + \text{h.o.t.}.$$}

The lemma follows because

$$1_i = \sqrt{\Delta_i} \partial u_i = \frac{1}{\sqrt{\Delta_i}}du_i = \sum_{j=1}^N \frac{1}{\sqrt{\Delta_i}}(\partial_j u_i) d\tau_j.$$

\[\square\]

**Lemma 5.3.** Let $\lambda$ be close to $u_i$. Then

$$(5.37) \quad f_r^{\beta/2}(\lambda) = \Psi R \sum_{n \in \mathbb{Z}} (-z \partial_\lambda)^n \frac{e_i}{\sqrt{2(\lambda - u_i)}}.$$

where $e_i \in \mathbb{C}^N \cong T_\tau M$ corresponds to $1_i$.

Proof. Using Lemma 5.1 we will compute the stationary phase asymptotic of $J_B$. The computation is the same as in the proof of Theorem 3 in [17].

Near the critical value the period $I^{(0)}_{\beta}(\tau, \lambda)$ has the expansion (5.36). Using the change of variables $\lambda - u_i = -zx^2/2$ we compute

$$\frac{2}{\sqrt{-2\pi z}} \int_{u_i}^{-(+\infty)} e^{\lambda/z}[2(\lambda - u_i)]^{-1/2}d\lambda = (-z)^k \frac{e_i}{\sqrt{2}\pi} \int_{-\infty}^{\infty} e^{-x^2/2}x^{2k}dx = e^{u_i/z}(-z)^k(2k - 1)!!.$$
Thus $J_B$ has the following asymptotic:

$$J_B \sim \left( \sum_{k=0}^{\infty} (2k-1)!! A_{i,k}(-z)^k \right) e^{u_i/z}.$$  

Since by definition the asymptotic of $J_B$ is $\Psi Re^{U/z} e_i$, we get $(-1)^k(2k-1)!! A_{i,k} = \Psi R_{k} e_i$. Thus

$$f_{\tau}^{a}(\lambda) = \sum_{n \in \mathbb{Z}} (-z \partial_{\lambda})^n I^{(0)}_{\beta}(\tau, \lambda)$$

$$= 2 \sum_{n \in \mathbb{Z}} \sum_{k=0}^{\infty} (-z \partial_{\lambda})^n (-1)^k \Psi R_k e_i \frac{[2(\lambda - u_i)]^{k-1/2}}{(2k-1)!!}$$

$$= 2 \sum_{n \in \mathbb{Z}} \sum_{k=0}^{\infty} (-z \partial_{\lambda})^n \Psi R_k (-\partial_{\lambda})^{-k} \frac{e_i}{\sqrt{2(\lambda - u_i)}}$$

$$= 2 (\Psi R) \sum_{n \in \mathbb{Z}} (-z \partial_{\lambda})^n \frac{e_i}{\sqrt{2(\lambda - u_i)}}.$$  

□

5.2. Differential equations. Let $(\tau, \lambda) \in (M \times \mathbb{C})\setminus \Delta$. We will show that $f_{\tau}^{a}(\lambda)$, where $*$ is either a relative cycle or a one-point cycle, satisfies a certain system of differential equations which, in some sense, uniquely determines the corresponding periods.

Note that

$$\frac{1}{n!} d^{\frac{1}{n}} ((\lambda - f_{\tau})^n \omega) = \Phi_{\tau,\lambda}^{(n)} + \frac{1}{n!} \left( \int_{\phi} (\lambda - f_{\tau})^n \omega \right) \log x,$$

where $\Phi_{\tau,\lambda}^{(n)}$ is a Laurent polynomial in $x$. Moreover, $\Phi_{\tau,\lambda}^{(n)}$ is homogeneous of degree $n$ with respect to our grading conventions.

Lemma 5.4. The following differential equations hold:

$$z \partial_{\tau} f_{\tau}^{a} = (\partial_{\tau} \bullet) f_{\tau}^{a}, \quad \partial_{\lambda} f_{\tau}^{a} = -z^{-1} f_{\tau}^{a}, \quad (z \partial_{z} + \lambda \partial_{\lambda} + E) f_{\tau}^{a} = (\mu - 1/2) f_{\tau}^{a} + \frac{1}{k} f_{\tau}^{0}.$$  

Proof. We refer to these three equations as the first, the second, and the last.

The second equation is equivalent to $\partial_{\lambda} f_{\tau}^{(n)} = f_{\tau}^{(n+1)}$. For $n \geq 0$ this equation holds by definition. Assume that $n = -p < 0$. Look at the definition (1.4).

The derivative with respect to $\lambda$ of

$$\frac{1}{p!} \int_{[x_a]} d^{\frac{1}{n}} ((\lambda - f_{\tau})^p \omega) =: \int_{[x_a]} \omega_p$$
is a sum of two terms: \( \int_{[x_a]} d\omega_p/df_\tau + \int_{[x_a]} \partial_\lambda \omega_p \). The first term vanishes because \( d\omega_p \) contains a factor of \( (\lambda - f_\tau)^p \) and \( f_\tau(x_a) = \lambda \). The second term is precisely \( \int_{[x_a]} \omega_{p-1} \). The second equation is proved.

For the last equation we will prove that the homogeneity properties of the Frobenius structure implies the equality between the coefficients in front of \( z^{-n} \), \( n \geq 0 \). Then the equality between the positive powers of \( z \) follows easily from the second equation. We have

\[
(\lambda \partial_\lambda + E) \int_{[x_a]} \Phi^{(n)}_{\tau,\lambda} = n \int_{[x_a]} \Phi^{(n)}_{\tau,\lambda}, \quad (\lambda \partial_\lambda + E) \log x_a = 1/k,
\]

because \( \Phi^{(n)}_{\tau,\lambda}(x_a) \) and \( x_a \) are homogeneous in \( \lambda \) and \( \tau \) of degrees \( n \) and \( 1/k \) respectively. Now, to prove that the coefficients in front of \( z^{-n} \) are equal, we only need to use that \( \tau_i, 1 \leq i \leq k \) has degree \( i/k \), and \( \mu(d\tau_i) = (1/2 - i/k)d\tau_i \), and \( \tau_{k+j}, 1 \leq j \leq m \) has degree \( 1 - j/m \) and \( \mu(d\tau_{k+j}) = (-1/2 + j/m)d\tau_{k+j} \).

The first equation is equivalent to \( \partial_\lambda I_{-p} = - (\partial_i \bullet) \partial_\lambda I_{-p}^{(n)} \). It is enough to show that the later equation holds for all \( n = -p, p \geq 2 \) because for other \( n \) we just need to recall the second equation. Recall the definition (1.4) of \( I_{-p} \). We need to prove that for each \( i \) and \( j \) the differential operators \( \partial_i \partial_j + \sum_l A_{ij}^l \partial_\lambda \) (\( A_{ij}^l \) are the structure constants of \( \bullet \)) annihilate the function (5.38). Note that

\[
\left( \partial_i \partial_j + \sum_l A_{ij}^l \partial_\lambda \right) \int_{[x_a]} \omega_p = \int_{[x_a]} \left( \partial_i \partial_j + \sum_l A_{ij}^l \partial_\lambda \right) \omega_p.
\]

In general the above formula will have more terms. However, in our case, the integrands of those terms contain factors of \( (\lambda - f_\tau)^p \) or \( (\lambda - f_\tau)^{p-1} \) and so they vanish on the one point cycle \([x_a]\). On the other hand we have

\[
\left( \partial_i \partial_j + \sum_l A_{ij}^l \partial_\lambda \right) \frac{1}{p!} (\lambda - f_\tau)^p \omega = \left[ -\frac{(\lambda - f_\tau)^{p-1}}{(p-1)!} \frac{\partial^2 f_\tau}{\partial \tau_i \partial \tau_j} + \frac{(\lambda - f_\tau)^{p-2}}{(p-2)!} (\partial_i f_\tau)(\partial_j f_\tau) - \sum_l A_{ij}^l \frac{(\lambda - f_\tau)^{p-2}}{(p-2)!} \partial_\lambda f_\tau \right] \omega.
\]

Now we use that (3.14) holds. Thus the RHS of the last equation can be transformed into

\[
-d \left( G_{ij} \frac{(\lambda - f_\tau)^{p-1}}{(p-1)!} \right).
\]

Now it is easy to finish the proof:

\[
\left( \partial_i \partial_j + \sum_l A_{ij}^l \partial_\lambda \right) \int_{[x_a]} \omega_p = - \int_{[x_a]} G_{ij} \frac{(\lambda - f_\tau)^{p-1}}{(p-1)!} = 0.
\]
Corollary 5.5. Let $\beta \in H_1(\mathbb{C}^*, V_{\tau_0, \lambda_0}; Q)$ be any cycle. Then
\[ z\partial f_\phi^\beta = (\partial \tau) f_\phi^\beta, \quad \partial_\lambda f_\phi^\beta = -z^{-1} f_\phi^\beta, \quad (z\partial_z + \lambda \partial_\lambda + E) f_\phi^\beta = (\mu - 1/2) f_\phi^\beta. \]

Proof. Let $(\tau, \lambda) \in (M \times \mathbb{C}) \setminus \Delta$. Using the Stokes’ formula it is easy to see that there are one point cycles $x_a(\tau, \lambda)$ and $x_b(\tau, \lambda)$ such that $f_\phi^\beta = f_\tau^a - f_\tau^b$, where the paths specifying the values respectively of $f_\phi^\beta$ and $f_\tau^a$ are possibly different from the path specifying the value of $f_\tau^\beta$. \hfill \square

5.3. Period vectors near $\lambda = \infty$. According to Lemma 5.4, $f_\phi^a$ and $S_\tau$ satisfy the same differential equations with respect to $\tau$. In this section we will prove that in a neighborhood of $\lambda = \infty$ the series $f_\tau^a$ can be expanded as follows:

$\lambda_a(\lambda) = S_\tau f_\infty^a(\lambda)$, where $f_\infty^a = \sum_n I_a^{(n)}(\infty, \lambda)(-z)^n$ is independent of $\tau$ and each mode is a series of the following type,

\[ (5.39) \quad \left( \sum_{i=0}^{N_1} A_i \lambda^i \right) \log \lambda + \sum_{i \geq N_2} B_{i,1} \lambda^{-i/k} + B_{i,2} \lambda^{-i/m}, \]

where $A_i$, $B_{i,1}$, and $B_{i,2}$ are certain vectors in $H$.

We show that each period vector $I_a^{(n)}(\tau, \lambda)$ expands in a neighborhood of $\lambda = \infty$ as a series of type (5.39) with coefficients in $H[\tau_1, \ldots, \tau_N, Q e^{\tau N}]$. Then, to obtain $I_a^{(n)}(\infty, \lambda)$, we just need to let $\tau_1 = \ldots = \tau_N = Q e^{\tau N} = 0$.

By definition,

\[ \left( I_\phi^{(-1)}(\tau, \lambda), \partial_\tau \right) = -\partial_\tau \int_\phi (\lambda - f_\tau) \omega = \partial_\tau t_k \int_\phi \omega = \delta_{i,k}. \]

For the second equality we used that only the free term in $\lambda - f_\tau$ contributes to $\int_\phi$ and for the last one note that $t_k = \tau_k$. Thus $I_\phi^{(-1)} = d\tau_k$. The rest of the periods are uniquely determined from the differential equations in Corollary 5.3. Indeed, the second equation implies that $\partial_\lambda I_\phi^{(n)} = I_\phi^{(n+1)}$, thus $I_\phi^{(n)} = 0$ for $n \geq 0$. Using the first equation we can express the differentiations $\partial_\tau$ in terms of multiplication operators $\partial_i \cdot$. Thus the last equation assumes the following form

\[ (\lambda - E \cdot) I_\phi^{(n+1)} = (\mu + n - 1/2) I_\phi^{(-n)}. \]

In the basis $\partial_1, \ldots, \partial_N$ the Hodge grading operator $\mu$ is diagonal with entries

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
& & & & & & & \\
\end{array}
\]

Thus $\mu + n - 1/2$ is invertible for $n > 1$, i.e., $I_\phi^{(-n)}$ is uniquely determined from $I_\phi^{(-n+1)}$. Moreover, it is clear that all modes $I_\phi^{(n)}$ will depend polynomially on $\lambda$, $\tau$ and $Q e^{\tau N}$.
Similarly, the differential equations in Lemma 5.4 imply that the one-point periods satisfy the following recursive relation,

\[(\lambda - E \bullet) I_a^{(-n+1)} = (\mu + n - 1/2) I_a^{(-n)} + \frac{1}{k} I_a^{(-n)}.\]

Thus it suffices to prove that \(I_a^{(-1)}(\tau, \lambda)\) has an expansion of type (5.39). This is obvious because \(I_a^{(-1)}(\tau, \lambda)\) are obtained by integrating the 0-form (5.42)

\[
\omega_1 = -\frac{x^k}{k} - \sum_{j=1}^{k-1} \frac{t_{k-j} x^j}{j} + (\lambda - t_k) \log x + \sum_{j=1}^{m-1} \frac{t_{k+j}}{j} \left( \frac{Q e^{t_N}}{x} \right)^j + \frac{1}{m} \left( \frac{Q e^{t_N}}{x} \right)^m,
\]

over one-point cycles \(x_a(\tau, \lambda)\) — the solutions of \(f_\tau(x) = \lambda\). Near \(\lambda = \infty\), the one-point cycle \(x_a\) expands as a series in \(\lambda^{-1/k}\) or \(\lambda^{-1/m}\), depending on whether \(1 \leq a \leq k\) or \(k + 1 \leq a \leq k + m\), whose coefficients depend polynomially on \(\tau\) and \(Q e^{t_N}\).

Now our goal is to compute \(I_a^{(n)}(\infty, \lambda)\) explicitly. Assume that \(\lambda\) is sufficiently large and that the path specifying the corresponding branch of \(I_a^{(-1)}(\tau, \lambda)\) is such that the one-point cycles (i.e. the solutions to \(f_\tau(x) = \lambda\)) split into two groups: \(x_1, \ldots, x_k\) close to \(\infty\) and \(x_{k+1}, \ldots, x_{k+m}\) close to \(0\), and so the expansions of \(\log x_i\) coincide with the ones given in Section 3.1.

**Lemma 5.6.** The vector-valued functions \(f^a_\infty, f^a_\tau,\) and \(f^b_\tau\) are given respectively by formulas (1.7), (1.8), and (1.9).

**Proof.** We prove that \(f^b_\infty\) is given by (1.9). The other two cases are similar.

The series \(f^b_\infty\) is obtained from \(f^b_\tau\) by letting \(\tau_1 = \ldots = \tau_N = Q e^{t_N} = 0\). As explained above, the equations \(-z \partial_x f^b_\tau = f^b_\tau\) and (5.41) determine \(f^b_\tau\) uniquely from \(I_b^{(-1)}\). Thus the same is true for \(f^b_\infty\), i.e., it is uniquely determined from \(I_b^{(-1)}(\infty, \lambda)\) and the differential equations (the second of the following two equations is obtained from (5.41) by letting \(\tau_1 = \ldots = \tau_N = Q e^{t_N} = 0\)):

\[(-z \partial_x) f^b_\infty = f^b_\infty, \quad (z \partial_z + (\lambda - \rho) \partial_\lambda) f^b_\infty = (\mu - 1/2) f^b_\infty + \frac{1}{k} f^b_\infty,
\]

where \(\rho\) is the classical multiplication by \((1/k + 1/m)\partial_N\). It is straightforward to check that the RHS of (1.9) satisfies the above differential equations. Thus it remains to prove that

\[(5.43) I_b^{(-1)}(\infty, \lambda) = -\log(\lambda^{1/m} Q^{-1}) d\tau_k - \sum_{j=1}^{m} \frac{\lambda^{j/m}}{j/m} \partial_{k+m-j}.\]

By definition,

\[(5.44) (I_b^{(-1)}(\tau, \lambda), \partial_\lambda) = -\partial_1 \int_{[x_1]} \omega_1 = \int_{[x_1]} \partial_1 F \frac{d\omega_1}{dF} - \int_{[x_1]} \partial_1 \omega_1.
\]
The first integral vanishes because the integrand contains a factor of \((\lambda - f_\tau)\). The second one can be expanded into a series in \(\log \lambda \) and \(\lambda^{-1/m}\). To obtain \(I^{(-1)}_b(\infty, \lambda)\) we need to ignore all terms which depend on \(\tau\). Note that \(x_b = Qe^{tN} \lambda^{-1/m} + \text{(terms that depend on } \tau)\). Thus our integral will have terms independent of \(\tau\) only if \(i = k + j, \ 0 \leq j \leq m\). When \(i = k\) we get

\[- \int _{[x_b]} \partial_k \omega_1 = \log x_b = \log (Q \lambda^{-1/m}) + \ldots,\]

where the dots indicate terms depending on \(\tau\). Thus we obtain the logarithmic term in (5.43). Furthermore, let \(i = k + j, \ 1 \leq j \leq m - 1\). Then, using the change of the variables (3.12) and (3.13), we get (recall also (5.42))

\[- \int _{[x_b]} \partial_{k+j} \omega_1 = \sum _{p=1}^{m-1} \frac{1}{p}(\partial_{k+j} t_{k+p}) \left( \frac{Qe^{tN}}{x_b} \right)^p + \ldots = -\frac{1}{j} \lambda^j/m + \ldots,\]

where the dots stand for terms which depend on \(\tau\). Finally, let \(i = k + m = N\). Recall that \(\tau_N = mt_N\). Thus we have

\[- \int _{[x_b]} \partial_N \omega_1 = - \int _{[x_b]} \frac{1}{m} \partial_{t_N} \omega_1 = \frac{1}{m} \left( \frac{Qe^{tN}}{x_b} \right)^m + \ldots = -\frac{1}{m} \lambda + \ldots.\]

The lemma is proved.

\[\square\]

6. Phase factors

The proof of Theorems 1.5 and 1.3 amounts to conjugating the vertex operators \(\Gamma^a_{\infty}\) and \(\Gamma^a_\tau\) respectively with the symplectic transformations \(S\) and \(\Psi R\).

According to [17], formula (17), conjugation of vertex operators by lower-triangular transformations \(S \in \mathcal{L}^{(2)}\text{GL}(H)\) is given by the following formula:

\[\hat{S} e^{\hat{f} \hat{S}^{-1}} = e^{W(f_+ f_+) / 2} e^{(Sf)}\]

where \(+\) means truncating the terms corresponding to the negative powers of \(z\) and the quadratic form \(W(q, q) = \sum (W_{kl} q_l q_k)\) is defined by

\[\sum _{k,l \geq 0} W_{kl} w^{-k} z^{-l} = \frac{S^*(w) S(z) - 1}{w^{-1} + z^{-1}}.\]

Similarly, for upper-triangular transformations \(R \in \mathcal{L}^{(2)}\text{GL}(H)\) we have ([17], section 7):

\[\hat{R}^{-1} e^{\hat{f} \hat{R}} = e^{\sqrt{f^2} / 2} \left( e^R \right)^\wedge,
\]

where \(-\) means truncating the non-negative powers of \(z\),

\[f_- = \sum _{k \geq 0} (-1)^{-1-k} (f_{-1-k}, q_k)\]
is interpreted (via the symplectic form) as a linear function in $q$, and $V(\partial, \partial) = \sum (V_{kl} \phi^a, \phi^b) \partial_{q_l, a} \partial_{q_k, b}$ is a second order differential operator whose coefficients are defined by

\begin{equation}
\sum_{k,l \geq 0} V_{kl} w^k z^l = 1 - R(w) R^*(z) w + z.
\end{equation}

In this section we will compute the phase factors $W(f_+, f_+)$ and $Vf^2$ respectively for the vertex operators $\Gamma^{a\infty}$ and $\Gamma^{b\tau}$, where $\beta$ is a Lefschetz thimble.

6.1. The phase form. The phase factors can be conveniently expressed in terms of the so-called phase form $W_{a,b}$ – a multi-valued meromorphic section of $\pi^* (T^* M)$ (here $\pi : M \times \mathbb{C} \to M$ is the projection) with poles along the discriminant $\Delta$, defined by

\[ W_{a,b}(\tau, \lambda) = \sum_{i=1}^N (I_{a}^{(0)}(\tau, \lambda), \partial_i \cdot I_{b}^{(0)}(\tau, \lambda)) d\tau_i = I_{a}^{(0)}(\tau, \lambda) \cdot I_{b}^{(0)}(\tau, \lambda). \]

In the last equality vectors and covectors are identified via the residue pairing.

**Lemma 6.1.** Let $\tau \in M$ be a semi-simple point, and $x_a, x_b \in V_{\tau, \lambda}$ two one-point cycles. Then

\[ W_{a,b}(\tau, \lambda) = \begin{cases} 
  d^M \log (x_a - x_b) & \text{if } a \neq b, \\
  -d^M \log (f'_\tau(x_a)) & \text{if } a = b,
\end{cases} \]

where $d^M$ is the de Rham differential on $M$ and $'$ means derivative with respect to $x$.

**Proof.** The idea is to express both sides in terms of canonical coordinates. Let $q_1, \ldots, q_N$ be the critical points of $f_\tau$ and $u_i$ are the corresponding critical values, i.e.,

\[ x^{m+1} f'_\tau(x) = k(x - q_1) \ldots (x - q_N), \quad u_i = f_\tau(q_i). \]

That $\tau$ is a semi-simple point means that the critical values $u_i$ form a coordinate system on $M$ in a neighborhood of $\tau$. Moreover, in that coordinate system the product and the metric assume the following forms:

\[ \partial / \partial u_i \cdot \partial / \partial u_j = \delta_{ij} \partial / \partial u_i, \quad (\partial / \partial u_i, \partial / \partial u_i) = \delta_{ij} \Delta_i, \]

where $\Delta_i$ is the Hessian of $f_\tau$ with respect to the volume form $\omega$ at the critical point $q_i$, i.e., $\Delta_i = (\partial^2_{xy} f_\tau)(q_i)$ where $y := \log x$ is a unimodular coordinate on a neighborhood of $q_i$ in $\mathbb{C}^*$. Furthermore, introduce an auxiliary function

\[ \Phi : M \times \mathbb{C}^* \to \mathbb{C}, \quad \Phi(\tau, x) = f_\tau(x). \]
Then the period vector \( I_a^{(0)} \) can be written as follows

\[
I_a^{(0)} = -d^M \int_{[x_a]} d^{-1}\omega = \sum_{i=1}^N \int_{[x_a]} \partial_{u_i} \Phi \frac{\omega}{\partial \Phi} du_i = \sum_{i=1}^N \int_{[x_a]} \frac{\partial_{u_i} \Phi }{\partial y} du_i,
\]

where \( d \) is the de Rham differential on \( \mathbb{C}^* \).

Borrowing an argument from the proof of Lemma 4.5 in [11], we will express the partial derivative \( \partial_{u_i} \Phi \) in terms of partial derivatives of \( \Phi \) with respect to \( y \). By definition \( u_i(\tau) = \Phi(\tau, q_i) \). Applying \( \partial/\partial u \) to both sides and using the chain rule, we get

\[
\delta_{ij} = \partial_{u_i} \Phi(q_i) + (\partial_x \Phi)(q_i) \partial_{u_j} q_i = \partial_{u_i} \Phi(q_i) + f'_\tau(q_i) \partial_{u_j} q_i = \partial_{u_i} \Phi(q_i),
\]

where we suppressed the dependence of \( \Phi \) on \( \tau \).

On the other hand \( x^m \partial_{u_j} \Phi \) is a polynomial in \( x \) of degree \( N-1 \) and the above formula shows that the value of this polynomial at \( q_i, 1 \leq i \leq N \) is \( q_i^m \delta_{ij} \). Therefore the Lagrange interpolation formula yields

\[
\partial_{u_j} \Phi(\tau, x) = \frac{q_j}{x - q_j} \partial_y \Phi(\tau, x) = \frac{q_j}{\Delta_j(x - q_j)} \partial_y \Phi(\tau, x).
\]

Hence formula (6.49) transforms into

\[
I_a^{(0)}(\tau, \lambda) = \sum_{i=1}^N \frac{q_i}{\Delta_i(x_a - q_i)} du_i.
\]

Using that \( du_i \cdot du_j = \delta_{ij} \Delta_i du_j \) we find that in canonical coordinates the phase form is given by the following formula

\[
\mathcal{W}_{a,b} = \sum_{i=1}^N \frac{q_i^2}{\Delta_i(x_a - q_i)(x_b - q_i)} du_i.
\]

Assume that \( x_a \neq x_b \). Then

\[
d^M \log(x_a - x_b) = \sum_{i=1}^N \frac{\partial_{u_i} x_a - \partial_{u_i} x_b}{x_a - x_b} du_i.
\]

By definition \( \lambda = \Phi(\tau, x_a) \). Apply \( \partial/\partial u \) to both sides and solve the resulting identity with respect to \( \partial_{u_i} x_a \), we find:

\[
\partial_{u_i} x_a = -\frac{\partial_{u_i} \Phi(x_a)}{\partial \Phi(x_a)} = \frac{x_a \partial_{u_i} \Phi(x_a)}{\partial_y \Phi(x_a)} = -\frac{x_a q_i}{\Delta_i(x_a - q_i)}.
\]

Substitute this in (6.50). After a short simplification we get precisely the formula for \( \mathcal{W}_{a,b} \).
The case when \( a = b \) is similar. We have

\[
-d^M \log(\partial_x \Phi(x_a)) = -\sum_{i=1}^{N} \frac{1}{\partial_x \Phi(x_a)} \left[ (\partial_x \partial_{u_i} \Phi)(x_a) + (\partial_x^2 \Phi)(x_a)\partial_{u_i} x_a \right] du_i.
\]

The first term in the brackets equals to

\[
\left( \partial_x \frac{q_i}{\Delta_i(x - q_i)} \partial_y \Phi \right)(x_a) = -\frac{q_i x_a}{\Delta_i(x_a - q_i)^2} \partial_x \Phi(x_a) + \frac{q_i}{\Delta_i(x_a - q_i)} (\partial_x \partial_y \Phi)(x_a).
\]

The second one equals to

\[
-(\partial_x^2 \Phi)(x_a) \frac{q_i x_a}{\Delta_i(x_a - q_i)}.
\]

Using that \((\partial_x \partial_y \Phi)(x_a) = \partial_x \Phi(x_a) + x_a \partial_x^2 \Phi(x_a)\) we find that the sum of the two terms is

\[
-\frac{q_i x_a}{\Delta_i(x_a - q_i)^2} \partial_x \Phi(x_a) + \frac{q_i}{\Delta_i(x_a - q_i)} (\partial_x \partial_y \Phi)(x_a) = -\frac{q_i^2}{\Delta_i(x_a - q_i)^2} \partial_x \Phi(x_a).
\]

Therefore \(d^M \log(\partial_x \Phi(x_a)) = W_{a,a} \). □

**Corollary 6.2.** Assume the notations from Lemma 6.1. Then

\[
\left( I_{a}^{(0)}(\tau, \lambda), I_{b}^{(0)}(\tau, \lambda) \right) = \begin{cases} -\partial_\lambda \log(x_a - x_b) & \text{if } a \neq b, \\ \partial_\lambda \log (f'_a(x_a)) & \text{if } a = b. \end{cases}
\]

**Proof.** It follows from the definitions that the one-point cycles \( x_a(\tau, \lambda) \) depend only on the difference \( t_k - \lambda \), i.e., \( \partial_\lambda x_a = -\partial_k x_a \). Recall Lemma 6.1 and pair the differential 1-forms with the vector field \( \partial_k \). □

### 6.2. Phase factors near \( \infty \).

Assume that \( \lambda \) is sufficiently large and that the path specifying the branches of the period vectors \( I_i^{(0)} \) is such that \( \log x_i \) have the same expansions as in Section 3.1. For each \( i, 1 \leq i \leq N \) put \( f_i^* = f_i^\infty \) to avoid cumbersome notations. Using definition (6.46) of the quadratic form \( W_{\tau} \) and that \( \partial_\lambda I_{i}^{(n)} = I_{i}^{(n+1)} \) we get

\[
\frac{d}{d\xi} W_{\tau}(f_i^*(\xi), f_i^*(\xi)) = -\sum_{k,l \geq 0} \left( (W_{k,l-1} + W_{k-1,l})(-1)^l I_i^{(l)}(\xi), (-1)^k I_i^{(k)}(\xi) \right) + \left( I_i^{(0)}(\xi), I_i^{(0)}(\xi) \right)
\]

\[
= -\sum_{k,l \geq 0} \left( S_l(-1)^l I_i^{(l)}(\xi), S_k(-1)^k I_i^{(k)}(\xi) \right) + \left( I_i^{(0)}(\xi), I_i^{(0)}(\xi) \right)
\]

\[
= -\left( I_i^{(0)}(\tau, \xi), I_i^{(0)}(\tau, \xi) \right) + \left( I_i^{(0)}(\xi), I_i^{(0)}(\xi) \right).
\]

There are two cases: \( i = a, 1 \leq a \leq k \) or \( i = b, k + 1 \leq b \leq k + m \) which correspond respectively to \( x_i \) being close to \( x = \infty \) or \( x = 0 \). In the first case...
According to Lemma 5.6, where the integration path is a ray starting at \( \lambda \) and approaching \( \lambda = \infty \). Thus we get the following formula:

\[
W_\tau(f^a_+, f^a_-) = \int_\lambda^\infty \left[ (I_a^{(0)}(\tau, \xi), I_a^{(0)}(\tau, \xi)) - (I_a^{(0)}(\infty, \xi), I_a^{(0)}(\infty, \xi)) \right] d\xi,
\]

where the integration path is the same as above. According to Lemma 5.6, the one-point cycles are represented by some \( x \). On the other hand, the quadratic form \( f \) is a Morse function. Assume that \( f \) is close to a critical value \( u_i \), \( 1 \leq i \leq N \), of \( f \) and that \( \beta \in H_1(\mathbb{C}^*, V_{\tau,\lambda}; \mathbb{Z}) \) is a Lefschetz thimble corresponding to the path \( C_i \), where \( C_i \) is the straight segment from \( \lambda \) to \( u_i \). Let \( \partial \beta = [x_a] - [x_b] \), where the one-point cycles are represented by some \( x_a, x_b \in V_{\tau,\lambda} \).

**Lemma 6.3.** Let \( V \) be the quadratic form (6.48). Then

\[
Vf^2 = -\lim_{\epsilon \to 0} \int_\lambda^{u_i+\epsilon} \left( (I_{\beta/2}^{(0)}(\tau, \xi), I_{\beta/2}^{(0)}(\tau, \xi)) - \frac{1}{2(\xi - u_i)} \right) d\xi,
\]

where \( f := f^{\beta/2} \) and the limit is taken along the path \( C_i \).
Proof. The proof is taken from [17], page 490. When \( f = \sum_{k \in \mathbb{Z}} I_{\beta/2}^{(k)}(-z)^k \), we have \( \partial_{x}Vf^2 = \frac{1}{4} \sum_{k,l \geq 0} \partial_{x} \left( V_{k,l} I_{\beta}^{(-1-k)}, I_{\beta}^{(-1-k)} \right) = \frac{1}{4} \sum_{k,l \geq 0} \left( [V_{k-1,l} + V_{k,l-1}] I_{\beta}^{(-l)}, I_{\beta}^{(-k)} \right) \)

\[
\partial_{x}Vf^2 = \frac{1}{4} \left( I_{\beta}^{(0)}, I_{\beta}^{(0)} \right) - \frac{1}{4} \left( \sum_{l \geq 0} R_{k,l} I_{\beta}^{(-l)}, \sum_{k \geq 0} R_{k,l} I_{\beta}^{(-k)} \right).
\]

On the other hand, \( R_{k,l}^*(z) = R_{k,l}^{-1}(-z) \) because \( R \in \mathcal{L}^{(2)} \text{GL}(H) \). Thanks to Lemma 5.3, \( \sum_{k,l \geq 0} R_{k,l}^* I_{\beta}^{(-k)} = 21_i/\sqrt{2(\lambda - u_i)} \). Also \( Vf^2 = 0 \) at \( \lambda = u_i \) because \( I_{\beta}^{(-1-k)} = 2[2(\lambda - u_i)]^{k+1/2}(1_i + \ldots) \) vanish at \( \lambda = u_i \). The lemma follows. \( \square \)

**Lemma 6.4.** Let \( \alpha = s([x_a] + [x_b]), \ s \in \mathbb{Q} \). Then the period vectors \( I_{\alpha}^{(n)} := s(I_{\alpha}^{(n)} + I_{\beta}^{(n)}) \) are holomorphic in a neighborhood of \( \lambda = u_i \) for each \( n \in \mathbb{Z} \).

**Proof.** The function \( f_{\tau} \) is Morse in a neighborhood of \( \lambda = u_i \). Let \( y \) be a Morse coordinate, i.e.,

\[
(6.54) \quad f_{\tau} = u_i + \frac{y^2}{2}, \quad x = q_i(\tau) + a_1(\tau)y + a_2(\tau)y^2 + \ldots,
\]

where \( q_i \) is a critical point of \( f_{\tau} \) corresponding to the critical value \( u_i \). Therefore, in the Morse coordinate, the one-point cycles \( x_a(\tau, \lambda) \) and \( x_b(\tau, \lambda) \) are given by \( y_{a/b} = \pm \sqrt{2(\lambda - u_i)} \). The last formula shows that the period vectors \( I_{\alpha}^{(n)} \) (which by definition are integrals of certain 0-forms over the one-point cycles \( [x_a] + [x_b] \)) are single-valued near \( \lambda = u_i \). Moreover, they are obtained from \( I_{\alpha}^{(0)} \) by differentiating and antiderivating with respect to \( \lambda \). Thus it is enough to show that \( I_{\alpha}^{(0)} \) is holomorphic near \( \lambda = u_i \).

On the other hand \( I_{\alpha}^{(0)} = -s d^{M} \log x_{a} + \log x_{b} \). The last expression is holomorphic near \( \lambda = u_i \) because it is singlevalued (the analytical continuation around \( u_i \) transforms \( x_{a/b} \) into \( x_{b/a} \)) and \( x_a, x_b \to q_i \) when \( \lambda \to u_i \) and \( q_i \neq 0 \). \( \square \)

**Lemma 6.5.** Let \( \alpha = s([x_a] + [x_b]). \) Then the following formula holds:

\[
\Gamma_{\alpha \beta}^{r/2} = \exp \left( \int_{u}^{\lambda} \left( I_{\alpha}^{(0)}(\tau, \xi), I_{\beta}^{(0)}(\tau, \xi) \right) d\xi \right) \Gamma_{\alpha}^{r/2} \Gamma_{\beta}^{r/2},
\]

where \( r \in \mathbb{Q} \) is an arbitrary number.

**Proof.** See [17], Proposition 4. \( \square \)
7. From ancestors to KdV

Let \( \tau \in M \) be an arbitrary semi-simple point. We want to show that the ancestor potential \( A^M_\tau \) satisfies the HQE (1.6). A priori the vertex operators in (1.6) depend on the choice of a path \( C \) in \( \mathbb{C} \setminus \{ u_1, \ldots, u_N \} \) from \( \lambda_0 \) to \( \lambda \). We begin by showing that the HQE are independent of \( C \). Assume that \( \lambda_0 \) is another path from \( \lambda_0 \) to \( \lambda \), denote the corresponding one-point cycles by \( I^{(n)}_\lambda \). For some permutation \( \sigma \) of \( 1, 2, \ldots, N \). Using the definition of the period vectors we find

\[
(I^{(n)}_\lambda - I^{(n)}_{\sigma(a)}) = -\partial_i \int_{[x_a^n] - [x_{\sigma(a)}]} d^{-1} \left( \frac{1}{n!} (\lambda - f)^n \omega \right) - \partial_i \int_{\delta f} \frac{1}{n!} (\lambda - f)^n \omega,
\]

for some \( r \in 2\pi i \mathbb{Z} \). Thus \( f^{(a)}_\lambda = f^{(a)}_{\tau} + rf^{(a)}_{\delta} \). Using that \( I^{(n)}_{\delta}(\tau, \lambda) = 0 \) for \( n \geq 0 \) we get \( \Gamma^{\sigma}_{\tau} = \Gamma^{\phi}_{\delta} \Gamma^{\sigma}_{\tau} \). It is obvious that \( c_{\alpha'} = c_{\sigma(a)} \), therefore changing the path \( C \) transforms the HQE (1.6) into

\[
(\Gamma^{\delta}_{\tau} \otimes \Gamma^\delta) \left( \sum_{i=1}^{N} c^{(i)}_\tau (\Gamma^{\phi}_{\tau} \otimes \Gamma^{\sigma}_{\tau}) (\Gamma^{\sigma}_{\tau} \otimes \Gamma^{\phi}_{\tau}) \right) (T \otimes T) d\lambda.
\]

The later expression, when computed at \( q', q'' \) such that \( \hat{w}' - \hat{w}'' \in \mathbb{Z} \), coincides with (1.6) thanks to the following Lemma.

**Lemma 7.1.** Let \( r \in \mathbb{C} \). The following identity between operators acting on \( \mathcal{B}_H \otimes \mathcal{A} \mathcal{B}_H \) holds:

\[
(\Gamma^{\delta}_{\tau} \otimes \Gamma^\delta) \left( \Gamma^{\phi}_{\tau} \otimes \Gamma^{\sigma}_{\tau} \right) = e^{(\hat{w}_{\tau} \otimes 1 - 1 \otimes \hat{w}_{\tau})} r (\Gamma^{\delta}_{\tau} \otimes \Gamma^\delta).
\]

**Proof.** By definition

\[
f^{\phi}_{\tau}(\lambda) = d\tau_k(-z)\lambda - \sum_{n \geq 1} I^{(-1-n)}_{\lambda}(\tau, \lambda)(-z)^{-k-1}.
\]

Since \( w_{\infty} = -d\tau_k z^{-1} \) and \( w_{\tau} = S_{\tau} w_{\infty} \), we get \( f^{\phi}_{\tau}(\lambda) \in w_{\tau} + z^{-1} \mathcal{H}_{\tau} \). Thus \( \Omega(v_{\tau}, f^{\phi}_{\tau}(\lambda)) = \Omega(v_{\tau}, w_{\tau}) = -1, \quad \Omega(w_{\tau}, f^{\phi}_{\tau}(\lambda)) = 0 \).

For all \( f, g \in \mathcal{H} \) we have:

\[
e^{j \partial} e^{\delta} = e^{[j, \delta]} e^{\delta} e^{j} = e^{\Omega(j, \delta)} e^{\delta} e^{j},
\]

because for linear Hamiltonians the quantization is a representation of Lie algebras. Thus

\[
\Gamma^{\delta}_{\tau} = \exp \left( (\hat{f}_{\tau} - \hat{w}_{\tau})(\epsilon \partial_z) \right) \exp \left( \frac{1}{\epsilon} \hat{w}_{\tau} \right) \exp \left( r \hat{f}_{\tau} \right)
\]
\[ \exp\left( rT^\phi \right) \exp \left( \hat{\mathcal{F}}^\phi - \hat{\omega}_r (\epsilon \partial_x) \right) \exp \left( (rx/\epsilon) \Omega(v_r, \mathcal{F}^\phi_r) \right) \exp \left( \frac{x}{\epsilon} \hat{\nu}_r \right) \]

Similarly, \( \Gamma^\phi \Gamma_r - \hat{\omega}_r r \Gamma^\phi \Gamma_r e^{x_r/\epsilon} \). The Lemma follows. \( \square \)

7.2. Tame asymptotical functions. The total ancestor potential has some special property which makes the expression (1.6) a formal series with coefficients meromorphic functions in \( \lambda \).

**Definition 7.2 ([17]).** An asymptotical function is by definition an expression

\[ T = \exp \left( \sum_{g=0}^{\infty} \frac{2g-2}{g-1} T^{(g)}(t; Q) \right), \]

where \( T^{(g)} \) are formal series in the sequence of vector variables \( t_0, t_1, t_2, \ldots \) with coefficients in the Novikov ring \( \mathbb{C}[\![Q]\!] \). Furthermore, \( T \) is called tame if

\[ \left. \frac{\partial}{\partial t_{k_1}^{\alpha_1}} \cdots \frac{\partial}{\partial t_{k_r}^{\alpha_r}} \right|_{t=0} T^{(g)} = 0 \quad \text{whenever} \quad k_1 + k_2 + \ldots + k_r > 3g - 3 + r, \]

where \( t_k^a \) are the coordinates of \( t_k \) with respect to \( \{\phi_a\}, 1 \leq a \leq N \).

According to [17], Proposition 5, the total ancestor potential \( A^M_T \) is a tame asymptotical function.

Let \( T \) be a tame asymptotical function. The dilaton shift \( t(z) = q(z) + z \) identifies \( T \) with an asymptotical element of the Fock space \( B_H \). Let \( \Gamma_i^\pm = \exp \left( \pm \sum I_i^{(n)}(\lambda)(-z)^n \right) \) be a finite set of vertex operators, where \( I_i^{(n)} \) are meromorphic functions. Consider the expression

\[ \sum_i c^i(\lambda) \left( \Gamma_i^+ \otimes \Gamma_i^- \right) (T(q') \otimes T(q'')) d\lambda, \]  

where \( c^i \) are meromorphic functions. According to [17], Proposition 6, the tameness of \( T \) implies that (7.55), after the substitution \( q' = x + \epsilon y, \ q'' = x - \epsilon y \) and dividing by \( \exp \left( 2T^{(0)}(x)/\epsilon^2 \right) \), expands into a power series in \( \epsilon, x, \) and \( y \) whose coefficients depend polynomially on finitely many \( I_i^{(n)} \).

In particular, (1.6) can be interpreted as a formal series in \( \epsilon, x, \) and \( y \) (with \( y_k^0 \) excluded) with coefficients meromorphic functions in \( \lambda \). The vertex operators could have poles only at the critical values \( u_i, 1 \leq i \leq N \). Thus the regularity property of (1.6) follows if we prove that there are no poles at \( \lambda = u_i \).
7.3. Proof of Theorem 1.3. Fix an arbitrary critical value $u_i$. We need to show that (1.6) does not have a pole at $\lambda = u_i$. Assume that $\lambda$ is sufficiently close to $u_i$ and let $C_i$ be the straight segment from $\lambda$ to $u_i$. Then there are exactly two one-point cycles $x_a(\tau, \lambda)$ and $x_b(\tau, \lambda)$ which will coincide after transported along $C_i$ towards the critical point. Note that all period vectors $I_i^{(n)}$, $i \neq a$ or $b$, are holomorphic functions in a neighborhood of $\lambda = u_i$. Also, the vertex operator $\Gamma^\tau_\delta$ is holomorphic (even polynomial) in $\lambda$. Thus we need to show that

$$\left(1 \frac{f_\tau'(x_a)}{f_\tau'(x_b)} (\Gamma^a_\tau \otimes \Gamma^{-a}_\tau + \frac{1}{f_\tau'(x_b)} \Gamma^b_\tau \otimes \Gamma^{-b}_\tau) \right) (A_\tau \otimes A_\tau) d\lambda$$

has no poles in a neighborhood of $u_i$.

The 1-point cycle $x_a$ can be splitted into $x_a = (x_a - x_b)/2 + (x_a + x_b)/2$. Using Lemma 6.5 we get

$$\Gamma^a_\tau \otimes \Gamma^{-a}_\tau = e^{K_a} \left( \Gamma^\alpha_\tau \otimes \Gamma^{-\alpha}_\tau \right) \left( \Gamma^\beta/2_\tau \otimes \Gamma^{-\beta/2}_\tau \right),$$

where $\alpha = (x_a + x_b)/2$, $\beta$ is the Lefschetz thimble corresponding to the critical point $u_i$, and

$$K_a = \int_{u_i}^\lambda \left( I_i^{(0)}(\alpha) \right) d\xi.$$ 

Similarly,

$$\Gamma^b_\tau \otimes \Gamma^{-b}_\tau = e^{K_b} \left( \Gamma^\alpha_\tau \otimes \Gamma^{-\alpha}_\tau \right) \left( \Gamma^{-\beta/2}_\tau \otimes \Gamma^{\beta/2}_\tau \right),$$

where $K_b = -K_a$.

Furthermore, we recall formula (6.47):

$$\left( \Gamma^{\pm \beta/2}_\tau \otimes \Gamma^{-\beta/2}_\tau \right) \left( \widehat{\Psi} \widehat{R} \otimes \widehat{\Psi} \widehat{R} \right) = e^{W} \left( \widehat{\Psi} \widehat{R} \otimes \widehat{\Psi} \widehat{R} \right) \left( \Gamma^{\pm}_i \otimes \Gamma^{-i}_i \right),$$

where, according to Lemma 6.3

$$W = \lim_{\epsilon \to 0} \int_{u_i + \epsilon}^\lambda \left( \left( I_i^{(0)}(\beta/2, \xi), I_i^{(0)}(\beta/2, \xi) \right) - \frac{1}{2(\xi - u_i)} \right) d\xi,$$

and according to Lemma 5.3

$$\Gamma_i^\pm = \exp \left( \pm \sum_{n \in \mathbb{Z}} (-z \partial_\lambda)^n \frac{e_i}{\sqrt{2(\lambda - u_i)}} \right).$$

Finally, the periods $I_i^{(n)}$, $n \in \mathbb{Z}$, are holomorphic near $\lambda = u_i$ (see Lemma 6.4). Therefore, to prove that (7.56) is holomorphic near $\lambda = u_i$, it is enough to show that the 1-form

$$\left( C_a \Gamma_{u_i}^+ \otimes \Gamma_{u_i}^- + C_b \Gamma_{u_i}^- \otimes \Gamma_{u_i}^+ \right) (D_{pt} \otimes D_{pt}) d\lambda,$$

is holomorphic near $\lambda = u_i$. From the above calculations, it follows that (7.56) is holomorphic near $\lambda = u_i$. Therefore, the above calculations hold for any critical value $u_i$. Thus, we have shown that (1.6) does not have a pole at $\lambda = u_i$.
is analytic in $\lambda$. Here

$$C_{a/b} = \exp \left( K_{a/b} + W - \log f'_\tau(x_{a/b}) \right),$$

i.e.,

$$\log C_{a/b} = -\log f'_\tau(x_{a/b}) + \lim_{\epsilon \to 0} \int_{u_i+\epsilon}^\lambda \left( \pm (I^{(0)}_\alpha, I^{(0)}_\beta) + (I^{(0)}_{\beta/2}, I^{(0)}_{\beta/2}) - \frac{1}{2(\xi - u_i)} \right) d\xi,$$

where the periods in the integrand are computed at the point $(\tau, \xi)$. If we change $C_{a/b}$ by adding $(I^{(0)}_\alpha, I^{(0)}_\alpha)$ to the integrand of the RHS, then the expression (7.57) will change by an invertible holomorphic factor. Thus we can assume that the integrand is given by

$$(I^{(0)}_\alpha, I^{(0)}_\alpha) \pm (I^{(0)}_\alpha, I^{(0)}_\beta) + (I^{(0)}_{\beta/2}, I^{(0)}_{\beta/2}) - \frac{1}{2(\xi - u_i)} = (I^{(0)}_{a/b}, I^{(0)}_{a/b}) - \frac{1}{2(\xi - u_i)}.$$

Now recall Corollary 6.2

$$\log C_{a/b} = -\log f'_\tau(x_{a/b}) + \lim_{\xi \to u_i} \log \frac{f'_\tau(x_{a/b})}{\sqrt{2(\xi - u_i)}},$$

where, in the last expression, the one-point cycles are computed at the point $(\tau, \xi)$. We will show that the above limit is $\log(\pm \sqrt{f''_\tau(q_i)})$, where $q_i$ is a critical point of $f_\tau$ with critical value $u_i$. Indeed, let us expand $f_\tau$ in a Taylor’s series about $x = q_i$:

$$f_\tau(x) = u_i + \frac{1}{2!} f''_\tau(q_i) (x - q_i)^2 + \ldots.$$

From this we find that $x_{a/b}(\tau, \xi)$ expands into a series in $\xi - u_i$:

$$x_{a/b} = q_i \pm \frac{1}{\sqrt{f''_\tau(q_i)}} \sqrt{2(\xi - u_i)} + \ldots,$$

where the dots stand for higher order terms in $\xi - u_i$. Moreover, differentiating the Taylor’s expansion in $x$ and then substituting $x = x_{a/b}$ yield

$$f'_\tau(x_{a/b}) = \pm \sqrt{f''_\tau(q_i)} \sqrt{2(\xi - u_i)} + \ldots.$$ 

Note that $\log C_a - \log C_b = -\log(-1)$, i.e., $C_a = -C_b$. Therefore the expression (7.57), up to an invertible holomorphic factor, equals to

$$\left( \frac{1}{\sqrt{2(\lambda - u_i)}} \Gamma^{+}_{u_i} \otimes \Gamma^{-}_{u_i} - \frac{1}{\sqrt{2(\lambda - u_i)}} \Gamma^{-}_{u_i} \otimes \Gamma^{+}_{u_i} \right) (D_{pt} \otimes D_{pt}).$$

This is precisely (2.11) and is therefore holomorphic at $\lambda = u_i$. \qed
8. From ancestors to descendents

In this section we prove Theorem 1.5. Recall that the descendent and the ancestor potentials are related by $D^M = C_\tau \hat{S}_\tau^{-1} A_\tau$, where $C_\tau$ is some constant and $S_\tau$ is the calibration of $M$. The action of $\hat{S}_\tau^{-1}$ on the Fock space $B_H$ is given by the following formula (18, Proposition 5.3):

$$\left(\hat{S}_\tau^{-1} G\right)(q) = e^{W_\tau(q,q)/2}\mathcal{G}([S_\tau q]_+),$$

where $W_\tau$ is a quadratic form defined by (6.46) and $[\cdot]_+$ means truncating the terms with negative powers of $z$.

On the other hand $S_\tau$ acts on the set of vertex operators $\Gamma^i_\infty$, $1 \leq i \leq N$, and $\Gamma^i_\infty$ by conjugation.

Lemma 8.1. The following formula holds:

$$c_\infty^i (\Gamma^i_\infty \otimes \Gamma^i_\infty) \left(\hat{S}_\tau^{-1} \otimes \hat{S}_\tau^{-1}\right) = \left(\hat{S}_\tau^{-1} \otimes \hat{S}_\tau^{-1}\right) c_\infty^i \left(\Gamma^i_\tau \otimes \Gamma^i_\tau\right).$$

Proof. We prove the case $1 \leq i \leq k$. The other case $k+1 \leq i \leq N$ is similar.

According to formulas (6.45) and (6.51) we have

$$c_\infty^i (\Gamma^i_\infty \otimes \Gamma^i_\infty) \left(\hat{S}_\tau^{-1} \otimes \hat{S}_\tau^{-1}\right) = C_i \left(\hat{S}_\tau^{-1} \otimes \hat{S}_\tau^{-1}\right) \left(\Gamma^i_\tau \otimes \Gamma^i_\tau\right),$$

where

$$\log C_i = \frac{1-k}{k} \log \lambda - \log k + \int_\lambda^\infty \left[\left(I^i_\tau^0(\tau,\xi), I^i_\tau^0(\tau,\xi)\right) - \frac{k-1}{k} \xi^{-1}\right] d\xi.$$

On the other hand, using Corollary 6.2 we find that the above integral equals

$$\lim_{\xi \to \infty} \left(\log f'_i(x_i(\xi))\right) - \log f'_i(x_i) + \frac{k-1}{k} \log \lambda.$$

Thus it remains to show that the above limit is $\log k$. Indeed, near $\xi = \infty$, we have $x_i(\xi) = \xi^{1/k} + \ldots$, where here and further the dots stand for lower order terms. Hence $f'_i(x_i) = kx_i^{k-1} + \ldots = k\xi^{(k-1)/k} + \ldots$. The lemma follows.

Lemma 8.2. The following formula holds:

$$(\Gamma^\delta_\infty \otimes \Gamma^\delta_\infty) \left(\hat{S}_\tau^{-1} \otimes \hat{S}_\tau^{-1}\right) = e^{t_N x^2/(2\epsilon)} \left(\hat{S}_\tau^{-1} \otimes \hat{S}_\tau^{-1}\right) \left(\Gamma^\delta_\tau \otimes \Gamma^\delta_\tau\right) e^{t_N x^2/(2\epsilon)}.$$
Denote the 1-form (1.10) corresponding to $\mathcal{T} = \mathcal{D}^M$ by $\Omega_\infty$ and the 1-form (1.6) corresponding to $\mathcal{T} = \mathcal{A}_\tau^M$ by $\Omega_\tau$.

**Lemma 8.3.** Let $r \in \mathbb{Z}$, $q'$ and $q''$ be such that $\hat{\omega}'_\infty - \hat{\omega}''_\infty = r$. Then up to factors independent of $\lambda$ the 1-forms $\Omega_\infty(q', q'')$ and $\Omega_\tau([S_\tau q']_+, [S_\tau q'']_+)$ coincide.

**Proof.** We compute $\left(\tilde{S}_\tau^{-1} \otimes \tilde{S}_\tau^{-1}\right) \Omega_\tau$ in two different ways: by using formula (8.58) and by commuting $\left(\tilde{S}_\tau^{-1} \otimes \tilde{S}_\tau^{-1}\right)$ through the vertex operators of $\Omega_\tau$.

In the first case, up to a factor independent of $\lambda$ we get $\Omega_\tau([S_\tau q']_+, [S_\tau q'']_+)$. In the second one, using Lemmas 8.1 and 8.2 we get, up to factors independent of $\lambda$, the 1-form $\Omega_\infty(q', q'')$. □

**Proof of Theorem 1.5.** Let $r \in \mathbb{Z}$ be arbitrary and assume that $q', q'' \in \mathcal{H}_+$ are such that $\hat{\omega}'_\infty - \hat{\omega}''_\infty = r$, i.e.,

$$re = \Omega(w'_\infty, q') - \Omega(w''_\infty, q'').$$

Since $S_\tau$ is a symplectic transformation, the RHS equals

$$\Omega(w'_\tau, S_\tau q') - \Omega(w''_\tau, S_\tau q'') = \Omega(w'_\tau, [S_\tau q']_+) - \Omega(w''_\tau, [S_\tau q'']_+),$$

where the truncation operation $[\ ]_+$ does not change the value of the symplectic form because $w_\tau \in \mathcal{H}_-$. Theorem 1.5 follows from Lemma 8.3 and Theorem 1.3. □

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