Homological mirror symmetry for Brieskorn-Pham singularities

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Abstract
We prove that the derived Fukaya category of the Lefschetz fibration defined by a Brieskorn-Pham polynomial is equivalent to the triangulated category of singularities associated with the same polynomial together with a grading by an abelian group of rank one. Symplectic Picard-Lefschetz theory developed by Seidel is an essential ingredient of the proof.

1 Introduction
A polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n] \) is said to be a Brieskorn-Pham polynomial if

\[
f = f_p = x_1^{p_1} + \cdots + x_n^{p_n}
\]

for a sequence \( p = (p_1, \ldots, p_n) \) of positive integers. A hypersurface singularity defined by a Brieskorn-Pham polynomial is called a Brieskorn-Pham singularity. This class of singularities includes a part of simple singularities, simple elliptic singularities, Arnold’s exceptional unimodal singularities, and many more.

An important invariant of a hypersurface singularity is the Milnor lattice, which is the homology group of the Milnor fiber equipped with the intersection form. More recently, Seidel [21] introduced the Fukaya category of a Lefschetz fibration, which is a categorification of the Milnor lattice in the sense that the Grothendieck group equipped with the symmetrized Euler form is naturally isomorphic to the Milnor lattice.

Although the Milnor lattice of a singularity is difficult to compute in general, the Milnor lattice of a Brieskorn-Pham singularity allows the following description: The Milnor lattice of an \( A_{p-1} \)-singularity, defined by \( f_p = x^p \) for an integer \( p \) greater than one, is a free Abelian group generated by \( C_i \) for \( i = 1, \ldots, p-1 \) with the intersection form given by

\[
(C_i, C_j) = \begin{cases} 
2 & i = j, \\
-1 & |i - j| = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that this is isomorphic to the root lattice of type \( A_{p-1} \). Let

\[
I_p = \{(i_1, \ldots, i_n) \in \mathbb{N}^n \mid 1 \leq i_k \leq p_k - 1, \ k = 1, \ldots, n\},
\]
be a finite set equipped with the lexicographic order

\[(i_1, \ldots, i_n) < (j_1, \ldots, j_n) \text{ if } i_k = j_k \text{ for } k < \ell \text{ and } i_\ell < j_\ell \text{ for some } \ell \in \{1, \ldots, n\}.\]

Then it follows from a theorem of Sebastiani and Thom [17] that the Milnor lattice of

\[f_p = x_1^{p_1} + \cdots + x_n^{p_n}\]

is the tensor product of the Milnor lattices for \(f_{p_k}\), so that there is a distinguished basis \((C_i)_{i \in I_p}\) of vanishing cycles satisfying

\[\langle C_i, C_j \rangle = \prod_{k=1}^n (C_{i_k}, C_{j_k}) \text{ if } i_k \leq j_k \text{ for } k = 1, \ldots, n,\]

\[0 \text{ otherwise}\]

for \(i < j\). The intersection form is determined by (1.1) together with the anti-symmetry

\[\langle C_i, C_j \rangle = -\langle C_j, C_i \rangle\]

if \(n\) is even, and the symmetry

\[\langle C_i, C_j \rangle = \langle C_j, C_i \rangle\]

and \(\langle C_i, C_i \rangle = 2\) if \(n\) is odd.

The main result in this paper is a categorification of the above description of the Milnor lattice. For an integer \(p\) greater than one, let \(\mathfrak{A}_p\) be the differential graded category whose set of objects is

\[\mathcal{O}b(\mathfrak{A}_p) = (C_1, \ldots, C_p),\]

and whose spaces of morphisms are

\[\text{hom}(C_i, C_j) = \begin{cases} \mathbb{C} \cdot \text{id}_{C_i} & \text{if } i = j, \\ \mathbb{C}[-1] & \text{if } i = j - 1, \\ 0 & \text{otherwise,} \end{cases}\]

with the trivial differential. The tensor product of differential graded categories \(\mathcal{A}\) and \(\mathcal{B}\) is defined by

\[\mathcal{O}b(\mathcal{A} \otimes \mathcal{B}) = \mathcal{O}b(\mathcal{A}) \times \mathcal{O}b(\mathcal{B})\]

and

\[\text{Hom}_{\mathcal{A} \otimes \mathcal{B}}(A_1 \times B_1, A_2 \times B_2) = \text{Hom}_\mathcal{A}(A_1, A_2) \otimes \text{Hom}_\mathcal{B}(B_1, B_2)\]

together with the differential determined by the Leibniz rule. Now consider the polynomial

\[W_p(x_1, \ldots, x_n) = f_p(x_1, \ldots, x_n) + \text{(lower order terms)}\]

obtained by Morsifying \(f_p\) and let \(\mathfrak{F}uk W_p\) be the Fukaya category of \(W_p : \mathbb{C}^n \to \mathbb{C}\) considered as an exact symplectic Lefschetz fibration with respect to the Euclidean Kähler structure on \(\mathbb{C}^n\).

**Theorem 1.1.** For any sequence \(p = (p_1, \ldots, p_n)\) of positive integers, one has a quasi-equivalence

\[\mathfrak{F}uk W_p \cong \mathfrak{A}_{p_1-1} \otimes \cdots \otimes \mathfrak{A}_{p_n-1}\]

of \(A_\infty\)-categories.
An essential ingredient of the proof is the symplectic Picard-Lefschetz theory developed by Seidel [21], which provides an inductive tool to compute the Fukaya category in a combinatorial way.

Another important category that one can associate with a singularity is the **stabilized derived category**, introduced by Buchweitz [1] as the quotient category

\[
D_{\text{Sg}}(A) = D^b(\text{mod } A)/D^\text{perf}(\text{mod } A)
\]

of the bounded derived category \(D^b(\text{mod } A)\) of finitely-generated \(A\)-modules by its full subcategory \(D^\text{perf}(\text{mod } A)\) consisting of perfect complexes. Here \(A\) is the coordinate ring of the singularity, and a complex of \(A\)-module is said to be **perfect** if it is quasi-isomorphic to a bounded complex of projective modules. The motivation for this category comes from **matrix factorizations**, introduced by Eisenbud [4] to study maximal Cohen-Macaulay modules on a hypersurface. The same category is studied by Orlov [13] under the name ‘triangulated category of singularities’.

If the ring \(A\) is graded by an abelian group, then there is a graded version of the stabilized derived category, defined as the quotient category

\[
D_{\text{Sg}}^\text{gr}(A) = D^b(\text{gr } A)/D^\text{perf}(\text{gr } A)
\]

of the bounded derived category \(D^b(\text{gr } A)\) of finitely-generated graded \(A\)-modules by its full subcategory \(D^\text{perf}(\text{gr } A)\) consisting of bounded complexes of projectives.

In the case of a Brieskorn-Pham singularity, we equip the coordinate ring

\[
A_p = \mathbb{C}[x_1, \ldots, x_n]/(f_p),
\]

with the grading given by the abelian group \(L(p)\) of rank one generated by \(n+1\) elements \(\bar{x}_1, \ldots, \bar{x}_n, \bar{c}\) with relations

\[
p_1 \bar{x}_1 = p_2 \bar{x}_2 = \cdots = p_n \bar{x}_n = \bar{c}.
\]

**Theorem 1.2.** For any sequence \(p = (p_1, \ldots, p_n)\) of positive integers, one has an equivalence

\[
D_{\text{Sg}}^\text{gr}(A_p) \cong D^b(\mathfrak{A}_{p_1-1} \otimes \cdots \otimes \mathfrak{A}_{p_n-1})
\]

of triangulated categories.

By combining Theorem 1.1 and Theorem 1.2 one obtains homological mirror symmetry for Brieskorn-Pham singularities:

**Theorem 1.3.** For any sequence \(p = (p_1, \ldots, p_n)\) of positive integers, one has an equivalence

\[
D^b \text{Fuk}_W p \cong D_{\text{Sg}}^\text{gr}(A_p)
\]

of triangulated categories.

The special case of \(n = 2\) in Theorem 1.3 is proved in [25]. Homological mirror symmetry is proposed by Kontsevich [10] for Calabi-Yau manifolds, and later generalized to more general classes of manifolds [11, 8, 18]. Ebeling and Takahashi [3, 23, 22] discuss the relation between mirror symmetry for singularities and Saito’s duality for regular
systems of weights [16]. Stabilized derived categories of singularities associated with regular systems of weights whose smallest exponents are \( \pm 1 \) are studied by Kajiura, Saito and Takahashi [24, 6, 7]. Okada [12] also discusses homological mirror symmetry for Brieskorn-Pham singularities.

If \( p \) satisfies a condition described below, then one can relate the stabilized derived category \( D^\text{gr}_{Sg}(A_p) \) with the derived category \( D^b \text{coh} Y_p \) of coherent sheaves on a stack \( Y_p \) defined as follows: Let \( \ell \) be the least common multiple of \( (p_1, \ldots, p_n) \) and equip \( A(p) \) with a \( \mathbb{Z} \)-grading given by

\[
de g x_i = a_i = \frac{\ell}{p_i}, \quad i = 1, \ldots, n.
\]

Then \( X_p = \text{Proj} A_p \) is a hypersurface of degree \( \ell \) in the weighted projective space \( \mathbb{P}(a_1, \ldots, a_n) \). Put

\[
K = \{ (\alpha_1, \ldots, \alpha_n) \in (\mathbb{C}^\times)^n \mid \alpha_1^{p_1} = \cdots = \alpha_n^{p_n} \}
\]

and define a homomorphism

\[
\phi : \mathbb{C}^\times \rightarrow K
\]

by

\[
\phi(\alpha) = (\alpha_1^a, \ldots, \alpha_n^a).
\]

Then the cokernel

\[
G_p = \text{coker} \phi
\]

of \( \phi \) is a finite abelian group acting on \( X_p \), and let

\[
Y_p = [X_p/G_p]
\]

be the quotient stack with respect to this action. The adaptation of the Calabi-Yau/Landau-Ginzburg correspondence proved by Orlov [14, Theorem 2.5] to the \( L(p) \)-graded situation gives the following:

**Theorem 1.4.** If a sequence \( p = (p_1, \ldots, p_n) \) satisfies

\[
\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1,
\]

then one has an equivalence

\[
D^\text{gr}_{Sg}(A_p) \cong D^b \text{coh} Y_p
\]

of triangulated categories.

By combining Theorem 1.3 and Theorem 1.4, one obtains an equivalence

\[
D^b \text{Fuk} W_p \cong D^b \text{coh}^G Y
\]

between the derived Fukaya category of the Lefschetz fibration \( W_p \) and the derived category of coherent sheaves on the stack \( Y_p \).

The organization of this paper is as follows: In Section 2, we give a description of the Fukaya category of \( f(x) + u^k \) in terms of the Fukaya category of \( f^{-1}(0) \). This is based on
symplectic Picard-Lefschetz theory developed by Seidel \cite{Seidel}, and the case when \( k = 2 \) is discussed in \cite{Audin}. In Section 3, we prove Theorem 1.1 by induction on \( n \). The proof of Theorem 1.2 is given in Section 4.

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2 The Fukaya category of \( f(x) + u^k \)

Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a polynomial in \( n \) variables. Assume that

- \( f \) is \emph{tame}, in the sense that the gradient \( ||\nabla f|| \) is bounded from below outside of a compact set by a positive number, and

- \( f \) has non-degenerate critical points with distinct critical values.

Then \( f \) gives an \emph{exact symplectic Lefschetz fibration} \cite{Seidel}, Section (15d)] with respect to the standard Euclidean Kähler structure on \( \mathbb{C}^n \). Assume for simplicity that the set of critical values of \( f \) is the set of \( m \)-th roots of unity, and let \( (\gamma_i)_{i=1}^m \) be the distinguished set of vanishing paths chosen as straight line segments from the origin. Let \( \mathcal{B} \) denote the Fukaya category of \( f^{-1}(0) \) whose objects are vanishing cycles \( C_i \subset f^{-1}(0) \) along \( \gamma_i \) for \( i = 1, \ldots, m \). It is an \( \infty \)-category whose spaces of morphisms are given by Lagrangian intersection Floer complexes, and \( \infty \)-operations are defined by counting virtual numbers of stable maps of genus zero with Lagrangian boundary conditions \cite{Seidel}. The directed subcategory of \( \mathcal{B} \) with respect to the order \( (C_1, \ldots, C_m) \) will be denoted by \( \mathcal{A} \). Although \( \mathcal{A} \) depends on the choice of a distinguished basis of vanishing cycles, the derived category \( D^b \mathcal{A} \) is independent of this choice \cite{Stern, Seidel} and gives an invariant of the Lefschetz fibration.

Let

\[
g(u) = u^k - \epsilon u
\]

be a perturbation of \( u^k \) and consider the \emph{Lefschetz bifibration} \cite{Seidel}, Section (15e)]

\[
\Psi = \psi \circ \varpi
\]

where

\[
\varpi(x, u) = (f(x) + g(u), u)
\]

and

\[
\psi(y_1, y_2) = y_1.
\]

We write the critical points of \( f \) and \( g \) as

\[
\text{Crit} f = \{ x_i \}_{i=1}^m
\]

and

\[
\text{Crit} g = \{ u \in \mathbb{C} \mid ku^{k-1} = \epsilon \} = \{ u_j \}_{j=1}^{k-1}.
\]
so that the set of critical points of $\Psi$ is given by

$$\text{Crit } \Psi = \text{Crit } f \times \text{Crit } g = \{p_{ij} = (x_i, u_j)\}_{i,j}$$

with critical values

$$\Psi(p_{ij}) = f(x_i) + g(u_j).$$

Assume that $\epsilon$ is sufficiently small and order the critical values of $f$ clockwise and those of $g$ counterclockwise as shown in Figure 2.1. The critical values of $\Psi$ are shown in Figure 2.2 and we choose a distinguished set of vanishing paths $\gamma_{ij}$ from the origin to $p_{ij}$ as in Figure 2.3. These figures are for the case $m = k = 4$, and the general case is similar.

For general $t \in \mathbb{C}$, the map

$$\mathcal{E}_t \xrightarrow{\omega_t} \mathcal{S}_t$$

from $\mathcal{E}_t = \Psi^{-1}(t)$ to $\mathcal{S}_t = \psi^{-1}(t)$ is a Lefschetz fibration. The vanishing cycle $C_{ij}$ of $\Psi$ along the path $\gamma_{ij}$ comes from a matching path $\mu_{ij}$, which is obtained as the trajectory of critical values of $\omega_t$ along the path $\gamma_{ij}$.
Figure 2.5: Matching paths on the $u$-plane

The fiber $\mathcal{S}_t$ of $\psi$ can be identified with the $u$-plane, and the image of the matching path by $g$ can be described as follows: Consider the map

$$
\pi_t : \Psi^{-1}(t) \ni (x, u) \mapsto s = t - f(x) = g(u).$

The fiber of $\pi_t$ over $s \in \mathbb{C}$ is given by

$$
\pi_t^{-1}(s) = f^{-1}(t - s) \times g^{-1}(s).
$$

The first factor becomes singular if $s$ is $t$ minus a critical value of $f$, and the second factor becomes singular if $s$ is a critical value of $g$. As one varies $t$ along vanishing paths in Figure 2.3 the critical values of minus $f$ move on the $s$-plane until one of them hits one of the critical values of $g$. The trajectory of $t$ minus a critical value of $f$ along $\gamma_{ij}$, starting from the $i$-th point outside and ending at the $j$-th point inside, is the image by $g$ of the matching path $\mu_{ij}$ on the $u$-plane corresponding to the vanishing path $\gamma_{ij}$. Figure 2.4 shows these trajectories, and Figure 2.5 shows the matching paths obtained as the inverse images of these trajectories by $g$.

For each critical point $x_i$ of $f$, there are $k$ critical values $g^{-1} \circ f(x_i) = \{x_{i1}, \ldots, x_{ik}\}$ of $\omega_0$, and the matching path $\mu_{ij}$ connects $x_{i,j+1}$ with $x_{ij}$. We write the straight line segment on the $u$-plane from the origin to $x_{ij}$ as $\delta_{ij}$. The fiber $\omega_0^{-1}(0)$ can naturally be identified with $f^{-1}(0)$, so that the vanishing cycle $\Delta_{i,j}$ of $\omega_0$ along $\delta_{ij}$ corresponds to $C_i$. This shows that the Fukaya category $\mathcal{B}_k$ of $\omega_0^{-1}(0)$ consisting of $\Delta_{i,j}$ is given by

$$
\text{hom}_{\mathcal{B}_k}(\Delta_{i,j}, \Delta_{i',j'}) = \text{hom}_{\mathcal{B}}(C_i, C_{i'})
$$

with the natural $A_\infty$-structure inherited from $\mathcal{B}$. 

7
Let $A_k$ be the directed subcategory of $B_k$ with respect to the order

$$(i, j) < (i', j') \quad \text{if} \quad j > j' \quad \text{or} \quad j = j' \quad \text{and} \quad i < i'$$

on the index set. Note that the $A_\infty$-category $A_k$ depends not only on $A$ but also on $B$.

Let further

$$S_{ij} = \text{Cone}(\Delta_{i,j+1} \to \Delta_{i,j}), \quad i = 1, \ldots, m, \quad j = 1, \ldots, k - 1$$

be the object in $D^b A_k$ which is the cone over the morphism $e_{i,j} : \Delta_{i,j+1} \to \Delta_{i,j}$ corresponding to $\text{id}_{C_i}$ under the isomorphism

$$\text{hom}_{A_k}(\Delta_{i,j+1}, \Delta_{i,j}) \cong \text{hom}_B(C_i, C_i).$$

The following theorem gives a description of the Fukaya category of $\Psi^{-1}(0)$ in terms of the Fukaya category of $f^{-1}(0)$:

**Theorem 2.1** (Seidel [21, Proposition 18.21]). If $n$ is greater than one, then the Fukaya category of $\Psi^{-1}(0)$ consisting of vanishing cycles $C_{ij}$ is quasi-equivalent to the full subcategory of $D^b A_k$ consisting of $S_{ij}$.

### 3 Inductive description of the Fukaya category

Let $p = (p_1, \ldots, p_n)$ be a sequence of natural numbers and $\tilde{p} = (p_1, \ldots, p_n, k)$ be another sequence obtained by appending $p_{n+1} = k$ to $p$. Let further $W_p$ be a perturbation of the Brieskorn-Pham polynomial of degree $p$ and $g(u)$ be a perturbation of $u^k$ as in Section 2, so that $W_{\tilde{p}} = W_p + g$ is a perturbation of the Brieskorn-Pham polynomial of degree $\tilde{p}$. The directed Fukaya category $\mathfrak{F}uk W_p$ consisting of a distinguished basis of vanishing cycles in $W_p^{-1}(0)$ will be denoted by $\mathcal{A}$.

Assume that Theorem 1.1 holds for $W_p$ so that one has a quasi-equivalence

$$\mathcal{A} = \bigotimes_{p=1}^{n} \mathcal{A}_{p-1}$$

of $A_\infty$-categories. Theorem 2.1 shows that $\mathfrak{F}uk W_{\tilde{p}}$ is quasi-equivalent to the directed subcategory of $D^b A_k$ consisting of

$$C_{ij} = \text{Cone}(\Delta_{i,j+1} \to \Delta_{i,j}), \quad (i, j) \in I_{\tilde{p}}$$

with respect to the order

$$(i, j) < (i', j') \quad \text{if} \quad i < i' \quad \text{or} \quad i = i' \quad \text{and} \quad j < j'.$$

Note that one has

$$\text{hom}_{D^b A_k}(C_{ij}, C_{i', j'}) = \left\{ \begin{array}{c}
\text{hom}_{A_k}(\Delta_{i,j}, \Delta_{i', j'+1}) \\
\text{hom}_{A_k}(\Delta_{i,j+1}, \Delta_{i', j'+1})
\end{array} \right\},$$

where

$$\text{hom}_{A_k}(\Delta_{i,j}, \Delta_{i', j'}) = \left\{ \begin{array}{c}
\text{hom}_{A_k}(\Delta_{i,j}, \Delta_{i', j'}) \\
\text{hom}_{A_k}(\Delta_{i,j+1}, \Delta_{i', j'})
\end{array} \right\}.$$
where the right hand side denotes the total complex of the double complex. If $j < j' - 1$, then the right hand side is trivial;

$$\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array} \simeq 0.$$

If $j = j' - 1$, then the right hand side is given by

$$\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{hom}_A(C_i, C_{i'})
\end{array} \simeq \text{hom}_A(C_i, C_{i'})[-1].$$

Natural representatives of a basis of the cohomology group of this complex are given by

$$\Delta_{i,j+1} \longrightarrow \Delta_{i,j}
\begin{array}{ccc}
x_{i,i'} & & \\
\downarrow & & \downarrow \\
\Delta_{i',j+2} & \longrightarrow & \Delta_{i',j+1}
\end{array}$$

where $x_{i,i'}$ runs over a basis of $\text{hom}_A(C_i, C_{i'})$. If $j = j'$ and $i < i'$, then the right hand side is given by

$$\begin{array}{ccc}
0 & \longrightarrow & \text{hom}_A(C_i, C_{i'}) \\
\downarrow & \text{id} & \downarrow \\
\text{hom}_A(C_i, C_{i'}) & \longrightarrow & \text{hom}_A(C_i, C_{i'})
\end{array} \simeq \text{hom}_A(C_i, C_{i'}).$$

whose cohomologies are spanned by

$$\Delta_{i,j+1} \longrightarrow \Delta_{i,j}
\begin{array}{ccc}
x_{i,i'} & & \\
\downarrow & & \downarrow \\
\Delta_{i',j+1} & \longrightarrow & \Delta_{i',j}.
\end{array}$$

If $j > j'$ and $i < i'$, then the right hand side is acyclic;

$$\begin{array}{ccc}
\text{hom}_A(C_i, C_{i'}) & \text{id} & \text{hom}_A(C_i, C_{i'}) \\
\text{id} & \text{id} & \\
\text{hom}_A(C_i, C_{i'}) & \text{id} & \text{hom}_A(C_i, C_{i'})
\end{array} \simeq 0.$$

It is straightforward to compute the compositions among the above basis to show that the cohomology category of $\mathfrak{Fuk} W_{\tilde{p}}$ is equivalent to $\mathfrak{A}_{p_1 - 1} \otimes \cdots \otimes \mathfrak{A}_{p_n - 1} \otimes \mathfrak{A}_{k-1}$ as a graded category. Higher $A_\infty$-operations on $\mathfrak{Fuk} W_{\tilde{p}}$ vanish for degree reasons, and one obtains a quasi-equivalence

$$\mathfrak{Fuk} W_{\tilde{p}} \simeq \mathfrak{A}_{p_1 - 1} \otimes \cdots \otimes \mathfrak{A}_{p_n - 1} \otimes \mathfrak{A}_{k-1},$$

of $A_\infty$-categories. It is easy to check the $n = 2$ case using Figure 2.5 since only triangles contribute because of the directedness, and Theorem 1.1 is proved.
4 The triangulated category of singularities

Let $V$ be a vector space and $W$ be an element of the polynomial ring $\mathbb{C}[V]$. Assume that $W$ has an isolated critical point at the origin and there is a one-form $\gamma$ such that

$$W = \gamma(\eta),$$

where $\eta = \sum_i x_i \partial_i$ is the Euler vector field. The following is a variation of [18, Lemma 12.3]:

**Lemma 4.1.** Let $P$ be the structure sheaf of the origin as an $O_X$-module, where $X$ is the zero locus of $W$. Then the chain complex $C = (C^i, \delta^i)$ given by

$$C^i = \begin{cases} \bigoplus_{j=0}^{[-i/2]} \Omega^{-i-2j}_V|_X & i \leq 0, \\ 0 & i > 0, \end{cases}$$

$$\delta^i = \iota_\eta + \gamma \wedge \cdot,$$

is an $O_X$-free resolution of $P$.

**Proof.** Let $Q$ be the structure sheaf of the origin as an $O_V$-module. Then the derived pull-back of $Q$ by the inclusion $i : X \hookrightarrow V$ is the direct sum of $P$ and $P[1]$:

$$i^* Q \cong P \oplus P[1].$$

Since the Koszul complex

$$K = \left\{ 0 \to \Omega^n_V \xrightarrow{i_\eta} \Omega^{n-1}_V \xrightarrow{i_\eta} \cdots \xrightarrow{i_\eta} \Omega^0_V \to 0 \right\}$$

is a $O_V$-free resolution of $Q$, this shows that its restriction

$$K|_X = \left\{ 0 \to \Omega^n_V|_X \xrightarrow{i_\eta} \Omega^{n-1}_V|_X \xrightarrow{i_\eta} \cdots \xrightarrow{i_\eta} \Omega^0_V|_X \to 0 \right\}$$

to $X$ is also isomorphic to the direct sum of $P$ and $P[1]$. Now consider the chain map

$$0 \to \Omega^n_V|_X \to \Omega^{n-1}_V|_X \to \cdots \to \Omega^1_V|_X \to \Omega^0_V|_X \to 0 \to 0$$

$$0 \to 0 \to \Omega^n_V|_X \to \cdots \to \Omega^1_V|_X \to \Omega^0_V|_X \to \Omega^0_V|_X \to 0$$

from $K|_X[1]$ to $K|_X$ where vertical arrows are given by $\gamma \wedge \cdot$. Since this induces the identity map on the $(-1)$-th cohomology group, which is $P$ for both $K|_X[1]$ and $K|_X$, the mapping cone for this map is isomorphic to $P \oplus P[2];$

$$\left\{ K|_X[1] \xrightarrow{\gamma \wedge} K|_X \right\} \cong P \oplus P[2].$$

By iterating this process, one obtains

$$\left\{ K|_X[i] \xrightarrow{\gamma \wedge} K|_X[i-1] \xrightarrow{\gamma \wedge} \cdots \xrightarrow{\gamma \wedge} K|_X \right\} \cong P \oplus P[i + 1].$$

Now the lemma follows by taking $i$ to infinity. $\Box$
Now we prove Theorem 1.2 along the lines of [25, Theorem 5]. Fix any weight \( p = (p_1, \ldots, p_n) \) and put \( A = A(p) \) and \( L = L(p) \). We will find a full triangulated subcategory \( \mathcal{T} \) of \( D^b(\text{gr } A) \) equivalent to \( D_{S_\mathbf{g}}^\text{gr}(A) \) such that \( (k(\vec{m}))_{\vec{m} \in I} \) is a full exceptional collection in \( \mathcal{T} \), where \( k = A/(x_1, \ldots, x_n) \) and

\[
I = \{a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n \in L \mid -p_1 + 2 \leq a_1 \leq 0, \ldots, -p_n + 2 \leq a_n \leq 0\}.
\]

Let \( L_+ \) be the subset of \( L \) defined by

\[
L_+ = \left\{- (n - 1) \vec{c} + a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n \mid a_i \geq 1, \quad i = 1, \ldots, n\right\},
\]

and \( L_- \) be the complement \( L \setminus L_+ \). Let further \( \mathcal{S}_- \) and \( \mathcal{P}_+ \) be the full triangulated subcategories of \( D^b(\text{gr } A) \) generated by \( k(\vec{n}) \) for \( \vec{n} \in L_+ \) and \( A(\vec{m}) \) for \( \vec{m} \in L_- \) respectively. Then \( \mathcal{S}_- \) and \( \mathcal{P}_+ \) are left admissible in \( D^b(\text{gr } A) \), and since \( \mathcal{P}_+ \subset \text{ad } \mathcal{S}_- \), one has a weak semiorthogonal decomposition

\[
D^b(\text{gr } A) = \langle \mathcal{S}_-, \mathcal{P}_+, \mathcal{T} \rangle
\]

such that \( \mathcal{T} \cong D_{S_\mathbf{g}}^\text{gr}(A) \). One can see that \( k(\vec{m}) \) for \( \vec{m} \in I \) belongs to \( \mathcal{T} \), since

\[
\mathbb{R} \text{ Hom}(k(\vec{m}), k(\vec{n})) = 0
\]

if \( \vec{m} \neq \vec{n} + \vec{x}_1 + \cdots + \vec{x}_n + \vec{n} \). The \( \mathbb{R} \text{ Hom's} \) between them can be calculated by the free resolution obtained in Lemma 4.1 to be

\[
\mathbb{R} \text{ Hom}(k(a_1 \vec{x}_1 + \cdots + a_n \vec{x}_n), k(b_1 \vec{x}_1 + \cdots + b_n \vec{x}_n)) = \text{hom}_{A_{p_1-1}}(C_{-a_1+1}, C_{-b_1+1}) \otimes \cdots \otimes \text{hom}_{A_{p_n-1}}(C_{-a_n+1}, C_{-b_n+1})
\]

Hence \( (k(\vec{n}))_{\vec{n} \in I} \) is an exceptional collection. It is straightforward to read off the structure of the Yoneda products from the above resolution to show that the the full subcategory of \( \mathcal{T} \) consisting of \( (k(\vec{n}))_{\vec{n} \in I} \) is isomorphic as a graded category to \( \mathcal{A}_{p_1-1} \otimes \cdots \otimes \mathcal{A}_{p_n-1} \). Moreover, \( \mathcal{T} \) has a differential graded enhancement induced from that of \( D^b(\text{gr } A) \), which is formal for degree reasons. This shows that \( \mathcal{A}_{p_1-1} \otimes \cdots \otimes \mathcal{A}_{p_n-1} \) is equivalent to the full triangulated subcategory of \( D_{S_\mathbf{g}}^\text{gr}(A) \) generated by the above exceptional collection.

To prove that the image of \( (k(\vec{m}))_{\vec{m} \in I} \) in \( D_{S_\mathbf{g}}^\text{gr}(A) \) is full, we use the following:

**Lemma 4.2.** The module \( k(\vec{m}) \) for any \( \vec{m} \in L \) can be obtained from \( (k(\vec{n}))_{\vec{n} \in I} \) by taking cones up to perfect complexes.

**Proof.** First note that the exact sequences

\[
0 \to k(-\vec{x}_1) \to k[x_1]/(x_1^2) \to k \to 0,
\]

\[
0 \to k(-2 \vec{x}_1) \to k[x_1]/(x_1^2) \to k[x_1]/(x_1^2) \to 0,
\]

\[
\vdots
\]

\[
0 \to k(-(p_1 - 1) \vec{x}_1) \to k[x_1]/(x_1^{p_1}) \to k[x_1]/(x_1^{p_1-1}) \to 0
\]
of $A$-modules show that $k(- (p_1 - 1)\vec{x}_1)$ can be obtained from $k, k(- \vec{x}_1), \ldots, k(- (p_1 - 2)\vec{x}_1)$ by taking cones up to the perfect module $k[x_1]/(x_1^{p_1}) \cong A/(x_2, \ldots, x_n)$. Then by shifting the degrees, one can see that for any $\vec{n} \in L$, $k(\vec{n})$ can be obtained from either $k(\vec{n} - \vec{x}_1), k(\vec{n} - 2\vec{x}_1), \ldots, k(\vec{n} - (p_1 - 1)\vec{x}_1)$ or $k(\vec{n} + \vec{x}_1), k(\vec{n} + 2\vec{x}_1), \ldots, k(\vec{n} + (p_1 - 1)\vec{x}_1)$ by taking cones up to perfect complexes. The same is true for $\vec{x}_i$ for $i = 2, \ldots, n$, and the lemma follows.

Now one can use [2, Corollary 4.3], [7, Theorem 4.5], [9, Proposition A.2], [15, Proposition 2.7] or [25, Section 4] to conclude that $(k(\vec{n}))_{\vec{n} \in I}$ is full.

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