CLASSIFICATION OF FINITELY GENERATED MODULES FOR \( k[x, y]/(xy) \) AND GENERALIZATIONS

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Abstract. We relax the definition by Butler and Ringel of a ‘string algebra’ to also include infinite-dimensional algebras such as \( k[x, y]/(xy) \). We show that finitely generated modules (and more generally what we call ‘finitely controlled’ modules) for such algebras are direct sums of string and band modules. This subsumes the known classifications of finite-dimensional modules for string algebras and of finitely generated modules for \( k[x, y]/(xy) \). The string and band modules are parameterized by certain ‘words’ which, unlike in the finite-dimensional case, may be infinite. We show that the Krull-Remak-Schmidt property holds.

1. Introduction

Setup. Let \( k \) be a field, \( Q \) a quiver, not necessarily finite, and \( kQ \) the path algebra. Let \( \rho \) be a set of zero relations in \( kQ \), that is, paths of length \( \geq 2 \), and write \( (\rho) \) for the ideal generated by \( \rho \).

Definition 1.1. A string algebra is an algebra \( \Lambda = kQ/(\rho) \) satisfying:

(a) Any vertex of \( Q \) is the head of at most two arrows and the tail of at most two arrows, and

(b) Given any arrow \( y \) in \( Q \), there is at most one path \( xy \) of length 2 with \( xy \not\in \rho \) and at most one path \( yz \) of length 2 with \( yz \not\in \rho \).

This notion is based on the ‘special biserial’ algebras of Skowroński and Waschbüscher [14]. For simplicity we restrict to the case of zero relations, in which case Butler and Ringel [2] used the name ‘string algebra’. But both sets of authors included finiteness conditions which we omit.

For example the algebra \( k[x, y]/(xy) \) arises from the quiver with one vertex and loops \( x \) and \( y \) with \( \rho = \{xy, yx\} \). The relations \( \rho = \{x^2, y^2\} \) give the algebra \( k(x, y)/(x^2, y^2) \). As another example one can take \( \Gamma = kQ/(\rho) \) where \( Q \) is the quiver

\[
\cdots \xrightarrow{x_{-1}} \bullet \xrightarrow{x_0} \bullet \xrightarrow{x_1} \bullet \xrightarrow{x_2} \bullet \xrightarrow{x_3} \cdots
\]

and \( \rho = \{x_i y_{i-1} : i \in \mathbb{Z}\} \cup \{y_i x_{i-1} : i \in \mathbb{Z}\} \).

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Let \( \Lambda \) be a string algebra. We consider left \( \Lambda \)-modules \( M \) which are \textit{unital} in the sense that \( \Lambda M = M \). If \( Q \) is finite, then \( \Lambda \) has a one, and this corresponds to the usual notion. It is equivalent that \( M \) is the direct sum of its subspaces \( e_vM \), where \( v \) runs through the vertices in \( Q \) and \( e_v \) denotes the trivial path at vertex \( v \), considered as an idempotent element in \( \Lambda \). Clearly unital \( \Lambda \)-modules correspond to representations of \( Q \) satisfying the zero relations in \( \rho \), with the vector space at vertex \( v \) being \( e_vM \). In this way \( \Gamma \)-modules correspond to \( \mathbb{Z} \)-graded modules for \( k[x,y]/(xy) \), where \( x \) and \( y \) have degree 1.

As usual, a module \( M \) is \textit{finitely generated} if \( M = \Lambda m_1 + \cdots + \Lambda m_n \) for some elements \( m_1, \ldots, m_n \in M \). The following notion is slightly more general if \( Q \) has infinitely many vertices.

\textbf{Definition 1.2.} A module \( M \) is \textit{finitely controlled} if for every \( a \in \Lambda \) the set \( aM \) is contained in a finitely generated submodule of \( M \).

It is equivalent that \( e_vM \) is contained in a finitely generated submodule for every vertex \( v \). One sees easily that finitely controlled \( \Gamma \)-modules correspond to \( \mathbb{Z} \)-graded \( k[x,y]/(xy) \)-modules whose homogeneous components are finite-dimensional.

\textbf{Background.} In this paper we classify finitely controlled modules for string algebras. This is related to existing work as follows.

(i) Finite-dimensional modules for special biserial/string algebras have been classified in terms of string and band modules by several authors \([5, 12, 4, 15, 2]\). The method of proof is sometimes called the ‘functorial filtration’ method, as it relies on certain functorially-defined filtrations of modules. The original work of Gelfand and Ponomarev \([5]\) applied to \( k[x,y]/(xy) \), and Ringel \([12]\), in probably the best reference for the method, adapted it to \( k(x,y)/(x^2, y^2) \). For the most part we follow this method, making appropriate modifications for infinite-dimensional (but finitely controlled) modules.

(ii) Instead of a string algebra, one can consider its localization or completion with respect to the ideal generated by the arrows. Algebras of this type have occasionally been studied by matrix reductions. For example Burban and Drozd \([1]\) study the derived category for certain ‘nodal’ algebras,
including \( k\langle x, y \rangle / \langle x^2, y^2 \rangle \). The functorial filtration method should adapt to classify finitely generated modules for such localizations and completions. Note that Theorem 9.2 would no longer be necessary in this case, as there would be no primitive simples.

**Words.** Let \( \Lambda = kQ/\langle \rho \rangle \) be a string algebra. As for finite-dimensional modules, we use certain ‘words’ to classify indecomposables. Unlike the finite-dimensional case, we need to allow infinite words. Such words have been used before by Ringel [13].

By a letter \( \ell \) one means either an arrow \( x \) in \( Q \) (a direct letter) or its formal inverse \( x^{-1} \) (an inverse letter). We define the inverse of any letter by setting \((x^{-1})^{-1} = x\). The head and tail of an arrow \( x \) in \( Q \) are already defined, and we extend them to all letters \( \ell \) so that the head of \( \ell^{-1} \) is the tail of \( \ell \).

We choose a sign \( \epsilon = \pm 1 \) for each letter \( \ell \), such that if distinct letters \( \ell \) and \( \ell' \) have the same head and sign, then \( \{\ell, \ell'\} = \{x^{-1}, y\} \) for some zero relation \( xy \in \rho \). (This is equivalent to the use of \( \sigma \) and \( \epsilon \) in [2].)

If \( I \) is one of the sets \( \{0, 1, \ldots, n\} \) with \( n \geq 0 \), or \( N = \{0, 1, 2, \ldots\} \), or \(-N = \{0, -1, -2, \ldots\} \) or \( \mathbb{Z} \), we define an \( I \)-word \( C \) as follows. If \( I \neq \emptyset \), then \( C \) consists of a sequence of letters \( C_i \) for all \( i \in I \) with \( i - 1 \in I \), so

\[
C = \begin{cases}
C_1C_2\ldots C_n & (\text{if } I = \{0, 1, \ldots, n\}) \\
C_1C_2C_3\ldots & (\text{if } I = \mathbb{N}) \\
\ldots C_{-2}C_{-1}C_0 & (\text{if } I = -\mathbb{N}) \\
\ldots C_{-1}C_0|C_1C_2\ldots & (\text{if } I = \mathbb{Z})
\end{cases}
\]

(a bar shows the position of \( C_0 \) and \( C_1 \) if \( I = \mathbb{Z} \)) satisfying:

- (a) if \( C_i \) and \( C_{i+1} \) are consecutive letters, then the tail of \( C_i \) is equal to the head of \( C_{i+1} \);
- (b) if \( C_i \) and \( C_{i+1} \) are consecutive letters, then \( C_i^{-1} \neq C_{i+1} \); and
- (c) no zero relation \( x_1, \ldots, x_m \) in \( \rho \), nor its inverse \( x_m^{-1} \ldots x_1^{-1} \) occurs as a sequence of consecutive letters in \( C \).

It follows that if \( C_i \) and \( C_{i+1} \) are consecutive letters, then \( C_i^{-1} \) and \( C_{i+1} \) have opposite signs. In case \( I = \{0\} \) there are trivial \( I \)-words \( 1_{v, \epsilon} \) for each vertex \( v \) in \( Q \) and \( \epsilon \) \( = \pm 1 \). By a word, we mean an \( I \)-word for some \( I \); it is a finite word of length \( n \) if \( I = \{0, 1, \ldots, n\} \). If \( C \) is an \( I \)-word, then for each \( i \in I \) there is associated a vertex \( v_i(C) \), the tail of \( C_i \) or the head of \( C_{i+1} \), or \( v \) for \( 1_{v, \epsilon} \).

The inverse \( C^{-1} \) of a word \( C \) is defined by inverting its letters and reversing their order. For example the inverse of a \( N \)-word is a \((-N)\)-word, and vice versa. By convention \( (1_{v, \epsilon})^{-1} = 1_{v, -\epsilon} \), and the inverse of a \( \mathbb{Z} \)-word is indexed so that

\[
(\ldots C_0|C_1\ldots)^{-1} = \ldots C_{-1}^{-1}|C_0^{-1} \ldots
\]

If \( C \) is a \( \mathbb{Z} \)-word and \( n \in \mathbb{Z} \), the shift \( C[n] \) is the word \( \ldots C_n|C_{n+1} \ldots \). We say that a \( \mathbb{Z} \)-word \( C \) is periodic if \( C = C[n] \) for some \( n > 0 \). If so, the minimal such \( n \) is called the period. We extend the shift to \( I \)-words \( C \) with \( I \neq \mathbb{Z} \) by defining \( C[n] = C \). There is an equivalence relation \( \sim \) on the set
of all words defined by \( C \sim D \) if and only if \( D = C[n] \) or \( D = (C^{-1})[n] \) for some \( n \).

**Modules.** String and band modules already appeared in the classification of finite-dimensional modules for string algebras.

Given any \( I \)-word \( C \) the corresponding *string module* \( M(C) \) is defined to be the module with basis the symbols \( b_i \) \((i \in I)\), with the action of \( \Lambda \) given by

\[
e_v b_i = \begin{cases} 
  b_i & \text{if } v_i(C) = v \\
  0 & \text{otherwise}
\end{cases}
\]

for a trivial path \( e_v \) in \( \Lambda \) \((v \text{ a vertex in } Q)\), and

\[
x b_i = \begin{cases} 
  b_{i-1} & \text{if } i - 1 \in I \text{ and } C_i = x \\
  b_{i+1} & \text{if } i + 1 \in I \text{ and } C_{i+1} = x^{-1} \\
  0 & \text{otherwise}
\end{cases}
\]

for an arrow \( x \).

For example, the string module \( M(y^{-1}xxy^{-1}y^{-1}y^{-1}... \) for the algebra \( k[x, y]/(xy) \) may be depicted as

![Diagram of string module](image)

where the vertices in the diagram correspond to elements of \( I \), so to basis elements \( b_i \), and the arrows show the action of \( x \) and \( y \).

For any word \( C \) there is an isomorphism \( i_C : M(C) \to M(C^{-1}) \) given by reversing the basis, and for \( n \in \mathbb{Z} \) there is an isomorphism \( t_{C,n} : M(C) \to M(C[n]) \), given for a \( \mathbb{Z} \)-word by \( t_{C,n}(b_i) = b_{i-n} \). Thus string modules corresponding to equivalent words are isomorphic.

If \( C \) is a periodic \( \mathbb{Z} \)-word of period \( n \), then \( M(C) \) becomes a \( \Lambda \)-\( k[T, T^{-1}] \)-bimodule with \( T \) acting as \( t_{C,n} \), and a *band module* is one of the form

\[
M(C, V) = M(C) \otimes_{k[T, T^{-1}]} V
\]

with \( V \) a finite-dimensional indecomposable \( k[T, T^{-1}] \)-module. It is clear that \( M(C) \) is free over \( k[T, T^{-1}] \) of rank \( n \), so band modules are finite-dimensional.

**Results.** Recall that \( \Lambda = kQ/(\rho) \) is a string algebra. Our main result is as follows.

**Theorem 1.3.** Every finitely controlled \( \Lambda \)-module is isomorphic to a direct sum of string and band modules.

Note that not every string module, or direct sum, is finitely controlled. An \( I \)-word \( C \) is said to be *eventually inverse* if there are only finitely many \( i > 0 \) in \( I \) with \( C_i \) a direct letter, and *vertex-finite* if for each vertex \( v \) there are only finitely many \( i > 0 \) in \( I \) with \( v_i(C) = v \). Both hold if \( C \) is finite or a \((-\mathbb{N})\)-word.
Theorem 1.4. (i) String and band modules are indecomposable.

(ii) The only isomorphisms between string and band modules are those arising from the equivalence relation on words.

(iii) A string module $M(C)$ is finitely generated if and only if $C$ and $C^{-1}$ are eventually inverse; it is finitely controlled if and only if $C$ and $C^{-1}$ are eventually inverse or vertex-finite.

(iv) A direct sum of string and band modules is finitely generated if and only if the string modules are finitely generated and the sum is finite; it is finitely controlled if and only if the string modules are finitely controlled and, for every vertex $v$, only finitely many summands are supported at $v$.

For example the $k[x, y]/(xy)$-module $M(C)$, where

$$C = y^{-1}xxy^{-1}y^{-1}y^{-1}\ldots,$$

is finitely generated as $C$ and $C^{-1}$ are eventually inverse. The $\Gamma$-module $M(D)$, where

$$D = \ldots y_3 y_2 y_1 x_2 y_1 x_1 y_0 x_0^{-1} \mid x_1^{-1} x_2^{-1} x_3^{-1} \ldots$$

is finitely controlled, but not finitely generated, as $D$ is eventually inverse while $D^{-1}$ is vertex-finite but not eventually inverse.

Theorem 1.5 (Krull-Remak-Schmidt property). If a finitely controlled module is written as a direct sum of indecomposables in two different ways, then there is a bijection between the summands in such a way that corresponding summands are isomorphic.

2. Linear Relations

In this section we generalize known results about linear relations to the infinite-dimensional case. Let $V$ and $W$ be vector spaces. Recall that a linear relation from $V$ to $W$ is a subspace $C$ of $V \oplus W$, for example the graph of a linear map $f : V \to W$. If $C$ is a linear relation from $V$ to $W$, $v \in V$ and $H \subseteq V$ we define

$$Cv = \{w \in W : (v, w) \in C\} \quad \text{and} \quad CH = \bigcup_{v \in H} Cv,$$

and in this way we can think of $C$ as a mapping from elements of $V$ (or subsets of $V$) to subsets of $W$. If $D$ is a linear relation from $U$ to $V$ then $CD$ is the linear relation from $U$ to $W$ given by

$$CD = \{(u, w) : \exists v \in V \text{ with } w \in Cv \text{ and } v \in Du\}.$$ 

We write $C^{-1}$ for the linear relation from $W$ to $V$ given by

$$C^{-1} = \{(w, v) : (v, w) \in C\},$$

and hence we can define powers $C^n$ for all $n \in \mathbb{Z}$.

Definition 2.1. If $C$ is a linear relation on a vector space $V$ (that is, from $V$ to itself), we define subspaces $C' \subseteq C'' \subseteq V$ by

$$C'' = \{v \in V : \exists v_0, v_1, v_2, \ldots \text{ with } v = v_0 \text{ and } v_n \in Cv_{n+1} \forall n\},$$

and

$$C' = \bigcup_{n \geq 0} C''.$$
The first of these differs from the definition used previously, for example in [12], but that work only involved relations on finite-dimensional vector spaces, for which the two definitions agree:

**Lemma 2.2.** If $C$ is a linear relation on $V$ then

$$C'' \subseteq \bigcap_{n \geq 0} C'' V$$

with equality if $V$ is finite-dimensional.

**Proof.** The inclusion is clear. If $V$ is finite-dimensional, the chain of subspaces $V \supseteq C V \supseteq C^2 V \supseteq \ldots$ stabilizes, with $C^r V = C^{r+1} V = \ldots$ for some $r$. Then any $v \in C^r V$ belongs to $C''$ since for any $v_n \in C^r V$ we can choose $v_{n+1} \in C^r V$ with $v_n \in C v_{n+1}$. □

**Definition 2.3.** If $C$ is a linear relation on $V$ we define subspaces $C^b \subseteq C^2 \subseteq V$ by

$$C^b = C'' \cap (C^{-1})'' \quad \text{and} \quad C^b = C'' \cap (C^{-1})' + C' \cap (C^{-1})''.$$

**Lemma 2.4.** (i) $C^t \subseteq C C^t$, (ii) $C^b = C^t \cap C C^b$, (iii) $C^t \subseteq C^{-1} C^t$, and (iv) $C^t = C^b \cap C^{-1} C^b$.

**Proof.** (i) If $v \in C^b$ then there are $v_n (n \in \mathbb{Z})$ with $v = v_0 = v$, $v_n \in C v_{n+1}$ for all $n$. Now $v \in C v_1$ and clearly $v_1 \in C^b$, so $C^b \subseteq C C^t$.

(ii) Suppose $b \in C^b$. We write it as $b = b^+ + b^-$ with $b^+ \in C'' \cap (C^{-1})'$ and $b^- \in C' \cap (C^{-1})''$. Now there are $b^+_n (n \in \mathbb{Z})$ with $b^+_n = b^+_0$, $b^-_n \in C b^+_n$ for all $n$, $b^+_n = 0$ for $n \ll 0$ and $b^-_n \in C^b$ for $n \gg 0$. Clearly $b^+_1 + b^-_1 \in C^b$ and $b = b^+ + b^- \in C_1 (b^+_1 + b^-_1)$, so $C^b \subseteq C^t \cap C C^b$. Conversely, suppose that $v \in C^t \cap C b$. Then $b^+_1 \subseteq C b^+ - b^-$, so

$$v - b^+_1 - b^-_1 \in C^t \cap C (b - b^+ - b^-) = C^t \cap C 0 \subseteq C^2 \cap C' \subseteq C^b.$$

Clearly also $b^+_1 \subseteq C^b$, so $v \in C^b$.

(iii) and (iv) follow by symmetry between $C$ and $C^{-1}$. □

**Lemma 2.5.** A linear relation $C$ on $V$ induces an automorphism $\theta$ of $C^2 / C^b$ with $\theta(C^b + v) = C^b + w$ if and only if $w \in C^2 \cap (C^b + C v)$.

**Proof.** For $v \in C^2$ we define $\theta$ by $\theta(C^b + v) = C^b + w$ where $w$ is any element of $C^2 \cap (C^b + C v)$. There always is some $w$ by Lemma 2.4 (iii), and $\theta$ is well-defined since if $w' \in C^2 \cap (C^b + C v')$ and $v - v' \in C^b$, then

$$w - w' \in C^2 \cap (C^b + C (v - v')) \subseteq C^b + C^2 \cap C (v - v') \subseteq C^b + C^2 \cap C C^b = C^b$$

by Lemma 2.4 (ii). Clearly $\theta$ is a linear map, and by symmetry between $C$ and $C^{-1}$ it is an automorphism. □

**Lemma 2.6** (Splitting Lemma). If $C$ is a linear relation on $V$ and $C^2 / C^b$ is finite-dimensional, then there is a subspace $U$ of $V$ such that $C^2 = C^b \oplus U$ and the restriction of $C$ to $U$ is an automorphism.
By assumption it is finite-dimensional, so spanned by $C$. Thus

$$\theta(C^0 + v_j) = \sum_{i=1}^{k} a_{ij}(C^0 + v_i) = C^0 + \sum_{i=1}^{k} a_{ij}v_i$$

so there are $b_1, \ldots, b_k \in C^0$ with

$$b_j + \sum_{i=1}^{k} a_{ij}v_i \in Cv_j$$

for all $j$. We write $b_j = b^+_j + b^-_j$ with $b^+_j \in C'' \cap (C^{-1})'$ and $b^-_j \in C' \cap (C^{-1})''$. Now there are $b^+_j(n \in \mathbb{Z})$ with $b^+_j = b^+_j, 0$, $b^+_j, n \in Cb^+_j, n+1$ for all $n$, $b^+_j, n = 0$ for $n \ll 0$ and $b^-_j, n = 0$ for $n \gg 0$. Define matrices $M^\pm, n = (m^\pm, n)$ for $n \in \mathbb{Z}$ by

$$M^+, n = \begin{cases} 0 & (n > 0) \\ (A^{-1})^{1-n} & (n \leq 0) \end{cases} \quad \text{and} \quad M^-, n = \begin{cases} -A^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}.$$

and let

$$u_j = v_j + \sum_{n \in \mathbb{Z}} \sum_{i=1}^{k} m^+, n b^+_i, n + \sum_{n \in \mathbb{Z}} \sum_{i=1}^{k} m^-, n b_-, i, n.$$

These are finite sums since $M^+, n = 0$ for $n > 0$ and $b^+_i, n = 0$ for $n \ll 0$, and $M^-, n = 0$ for $n \leq 0$ and $b^-_i, n = 0$ for $n \gg 0$. Now

$$b_j + \sum_{i=1}^{k} a_{ij}v_i \in Cv_j$$

implies

$$b_j, 0 + \sum_{i=1}^{k} a_{ij}v_i + \sum_{n \in \mathbb{Z}} \sum_{i=1}^{k} m^+, n b^+_i, n-1 + \sum_{n \in \mathbb{Z}} \sum_{i=1}^{k} m^-, n b_-, i, n-1 \in Cu_j.$$

If $\delta_{pq}$ is the Kronecker delta function, we have

$$\delta_{n0} I + M^\pm, n+1 = AM^\pm, n$$

which enables this to be rewritten as

$$\sum_{i=1}^{k} a_{ij}u_i \in Cu_j$$

for all $j$. Then $C^2 = C^0 \oplus U$ where $U$ has basis $u_1, \ldots, u_k$, and $C$ induces on $U$ the automorphism with matrix $A$. \hfill \Box

**Lemma 2.7.** Suppose that $C$ is a linear relation on a vector space $V$ and that $C''/C'$ is finite-dimensional. Then (i) $C'' \cap (C^{-1})' \subseteq C'$, and (ii) $C'' \subseteq \bigcap_{n \geq 0} (C^0 + C^{-n}V)$.

**Proof.** (i) If $v \in C'' \cap (C^{-1})'$ then there are $v_n (n \in \mathbb{Z})$ with $v_0 = v$, $v_n \in Cv_{n+1}$ for all $n$ and $v_n = 0$ for $n \ll 0$. Choose $m \geq 0$ with $v_n = 0$ for $n \leq -m$. Let $U$ be the subspace of $C''/C'$ spanned by $C' + v_n$ for $n \in \mathbb{Z}$. By assumption it is finite-dimensional, so spanned by $C' + v_n$ for $n \leq r$, for
some \( r \geq 0 \). Let \( s = m + r \). Now \( C' + v_s \) is a linear combination of the \( C' + v_n \) with \( n \leq r \), so
\[
v_s = \sum_{n \leq r} \lambda_nv_n + b
\]
with \( b \in C' \). For \( n \leq r \) we have \( 0 = v_{n-s} \in C^s v_n \). Also \( v = v_0 \in C^s v_s \). It follows that \( v \in C^s b \subseteq C' \).

(ii) If \( v \in C'' \) then there are \( v_n (n \geq 0) \) with \( v_0 = v \) and \( v_n \in Cv_{n+1} \) for all \( n \). For \( n \geq 0 \), let \( U_n \) be the subspace of \( C''/C' \) spanned by \( C' + v_r \) for \( r \geq n \). The descending chain of subspaces \( U_0 \supseteq U_1 \supseteq \ldots \) of \( C''/C' \) must stabilize, say with \( U_s = U_{s+1} = \ldots \). For any \( n \geq 0 \) we have \( C' + v_s \in U_s = U_{s+n} \), so we can write \( v_s \) as a linear combination
\[
v_s = \sum_{r \geq s+n} \lambda_r v_r + b
\]
with \( b \in C' \). Now \( v_{r-s} \in C^s v_r \) for all \( r \), so
\[
v - \sum_{r \geq s+n} \lambda_r v_{r-s} \in C^s \left( v_s - \sum_{r \geq s+n} \lambda_r v_r \right) \subseteq C^s b \subseteq C'.
\]
Also, for \( r \geq s + n \) we have \( v_{r-s} \in C^{-n} v_{r-s-n} \subseteq C^{-n} V \), and hence \( v \in C' + C^{-n} V \), as required. \( \Box \)

3. More about words

We introduce more conventions about words, which were not needed in the introduction, but will be needed later.

The head of a finite word or \( \mathbb{N} \)-word \( C \) is defined to be \( v_0(C) \), so it is the head of \( C_1 \), or \( v \) for \( C = 1_{v, \epsilon} \). The sign of a finite word or \( \mathbb{N} \)-word \( C \) is defined to be that of \( C_1 \), or \( \epsilon \) for \( C = 1_{v, \epsilon} \). The tail is defined for a word \( C \) of length \( n \) to be \( v_n(C) \) and for \( C \) a \((-\mathbb{N})\)-word to be \( v_0(C) \).

The composition \( CD \) of a word \( C \) and a word \( D \) is obtained by concatenating the sequences of letters, provided that the tail of \( C \) is equal to the head of \( D \), the words \( C^{-1} \) and \( D \) have opposite signs, and the result is a word. By convention \( 1_{v, \epsilon} 1_{v, \epsilon} = 1_{v, \epsilon} \) and the composition of a \((-\mathbb{N})\)-word \( C \) and an \( \mathbb{N} \)-word \( D \) is indexed so that
\[
CD = \ldots C_{-1}C_0|D_1D_2\ldots.
\]
If \( C = C_1 C_2 \ldots C_n \) is a non-trivial finite word and all powers \( C^m \) are words, we write \( C^\infty \) and \( \infty C^\infty \) for the \( \mathbb{N} \)-word and periodic \( \mathbb{Z} \)-word
\[
C_1 \ldots C_n C_1 \ldots C_n C_1 \ldots \text{ and } \ldots C_1 \ldots C_n | C_1 \ldots C_n C_1 \ldots.
\]
If \( C \) is an \( I \)-word and \( i \in I \), there are words
\[
C_{>i} = C_{i+1} C_{i+2} \ldots \text{ and } \ C_{\leq i} = \ldots C_{i-1} C_i
\]
with appropriate conventions if \( i \) is maximal or minimal in \( I \), such that
\[
C = (C_{\leq i} C_{>i})[-i].
\]

We say that a word \( C \) is direct or inverse if every letter in \( C \) is direct or inverse respectively. We say that a word \( C \) is repeating if \( C = D^\infty \) for some non-trivial finite word \( D \). We say that a word \( C \) is eventually repeating if \( C_{>i} \) is repeating for some \( i \).
If $M$ is a $\Lambda$-module and $x$ is an arrow with head $v$ and tail $u$, then multiplication by $x$ defines a linear map $e_u M \to e_v M$, and hence a linear relation from $e_u M$ to $e_v M$. By composing such relations and their inverses, any finite word $C$ defines a linear relation from $e_u M$ to $e_v M$, where $u$ is the head of $C$ and $v$ is the tail of $C$. We denote this relation also by $C$. Thus, for any subspace $U$ of $e_u M$, one obtains a subspace $CU$ of $e_v M$. We write $C0$ for the case $U = \{0\}$ and $CM$ for the case $U = e_u M$.

**Lemma 3.1.** No word can be equal to a shift of its inverse.

**Proof.** If $C$ is finite of length $n$, then $C = C^{-1}$ implies $C_i^{-1} = C_{n+1-i}$ for all $i$. The same holds if $C$ is a $\mathbb{Z}$-word and $C = C^{-1}[-n]$. Now if $n$ is even, then $C_i^{-1} = C_{i+1}$ for $i = n/2$, which is impossible, and if $n$ is odd, then $C_i^{-1} = C_i$ for $i = (n + 1)/2$, which is also impossible. □

### 4. Primitive cycles

By a primitive cycle $P$ we mean a non-trivial finite direct word such that $\infty P \infty$ is a periodic $\mathbb{Z}$-word of period equal to the length of $P$. Equivalently, $P$ is not itself a power of another word, and every power of $P$ is a word. For example the primitive cycles for $k[x, y]/(xy)$ are $x$ and $y$; for $k[x, y]/(x^2, y^2)$ they are $xy$ and $yx$; the algebra $\Gamma$ in the introduction has no primitive cycles.

For any vertex $v$ we define $z_v \in e_v \Lambda e_v$ to be the sum of all primitive cycles with head $v$. A non-trivial finite direct word is uniquely determined by its first arrow and length, so there are at most two primitive cycles with any given head $v$. If $M$ is a $\Lambda$-module, we define an action of the polynomial ring $k[z]$ on $M$, with $z$ acting as multiplication by $z_v$ on $e_v M$.

If $P$ and $R$ are distinct primitive cycles with head $v$ then $PR = RP = 0$ in $\Lambda$. Thus for example $z^n_v = P^n + R^n$ and $z^n_v P = P^{n+1}$.

**Lemma 4.1.** (i) The actions of $k[z]$ and $\Lambda$ on $M$ commute.

(ii) $M$ is finitely controlled if and only if $e_v M$ is a finitely generated $k[z]$-module for all $v$.

**Proof.** (i) If $a$ is an arrow with head $v$ and tail $u$ then $z_v a = a z_u$, for $z_v a$ is either zero, or it is a word of the form $Pa$ where $P$ is a primitive cycle whose first letter is $a$. Then $Pa = a R$ where $R$ is a primitive cycle at $u$, so $aR = az_u$.

(ii) We show first that $e_v \Lambda e_u$ is a finitely generated $k[z]$-module. Consider non-trivial paths from $u$ to $v$ in $Q$ which are non-zero in $\Lambda$. They correspond to finite direct words $C$ with head $v$ and tail $u$. By the string algebra condition all such words with the same sign must be of the form $D, PD, P^2 D, \ldots$ for some non-trivial words $D$ and $P$. If there are infinitely many such words, then $P$ is a primitive cycle, and these words are equal in $e_v \Lambda e_u$ to $D, z_v D, z_v^2 D, \ldots$. Thus $e_v \Lambda e_u$ is a finitely generated $k[z]$-module.

Now suppose that $M$ is finitely controlled. Then $e_v M$ is contained in a finitely generated submodule $\sum_{i=1}^k \Lambda m_i$. We may assume that each $m_i$ belongs to $e_v M$ for some $v_i$. Then $e_v M$ is contained in a $k[z]$-submodule of $M$ which is isomorphic to a quotient of $\sum_{i=1}^k e_v \Lambda e_{v_i}$, so is finitely generated as a $k[z]$-module.

Conversely, suppose $e_v M$ is a finitely generated $k[z]$-module and let $m_i \in e_v M$ be generators. Then $e_v M \subseteq \sum_i \Lambda m_i$, so $M$ is finitely controlled. □
If $X$ is a $k[z]$-module, its torsion submodule decomposes as a direct sum $\tau(X) = \tau^0(X) \oplus \tau^1(X)$, where
\[
\tau^0(X) = \{ x \in X : z^n x = 0 \text{ for some } n \geq 0 \}, \quad \text{and}
\tau^1(X) = \{ x \in X : f(z)x = 0 \text{ for some } f(z) \in k[z] \text{ with } f(0) = 1 \}
\]
are the nilpotent torsion and primitive torsion submodules of $X$. They are finite-dimensional if $X$ is finitely generated over $k[z]$.

If $M$ is a $\Lambda$-module, we consider it as a $k[z]$-module, and hence define $\tau(M)$, $\tau^0(M)$ and $\tau^1(M)$. They are $\Lambda$-submodules of $M$. We say that $M$ is nilpotent torsion if $M = \tau^0(M)$ and primitive torsion if $M = \tau^1(M)$. If $P$ is a primitive cycle with head $v$ we can also consider $e_vM$ as a $k[P]$-module, and hence define
\[
\tau_P(M) = \tau^0_P(M) \oplus \tau^1_P(M)
\]
as above, with $z$ replaced by $P$, and call these the $P$-torsion, $P$-nilpotent torsion and $P$-primitive torsion subspaces of $e_vM$. They are $k[z]$-submodules of $e_vM$.

**Lemma 4.2.** We have
\[
\tau^0(e_vM) = \bigcap_P \tau^0_P(M) \quad \text{and} \quad \tau^1(e_vM) = \bigoplus_P \tau^1_P(M)
\]
where $P$ runs through the (up to two) primitive cycles with head $v$.

**Proof.** We only need to deal with the case when there are two primitive cycles $P, R$ with head $v$. If $m \in \tau^0(e_vM)$ then $z^n m = 0$ for some $n$. Thus $(P^n + R^n)m = 0$, so $P^{n+1}m = R^{n+1}m = 0$, and hence $m \in \tau^0_P(M) \cap \tau^0_R(M)$. By the observations above $\tau^1_P(M)$ is annihilated by $R$, so its intersection with $\tau^1_R(M)$ must be zero. Now suppose that $m \in e_vM$ and $f(z)m = 0$ with $f(z) = 1 + g(z)$ where $g(0) = 0$. Then $0 = f(P+R)m = m + g(P)m + g(R)m$. Thus $0 = g(P)(m + g(P)m + g(R)m) = g(P)m + g(P)^2m = f(P)g(P)m$, so $g(P)m \in \tau^1_P(M)$. Similarly $g(R)m \in \tau_R(M)$, so $m = g(P)m + g(R)m$ is in the direct sum. \hfill \Box

A primitive cycle $P$ with head $v$ defines a relation on $e_vM$, also denoted $P$, as discussed in Section 3. Trivially we have $P' = 0$, $(P^{-1})'' = e_vM$ and $(P^{-1})' = \tau^1_P(M)$.

**Lemma 4.3.** If $P$ be a primitive cycle and $M$ is a finitely controlled $\Lambda$-module, then
\begin{enumerate}[label=(\roman*)]
\item $P'' = \bigcap_{n \geq 0} P^n M = \tau^1_P(M)$, and
\item $\tau_P(M) = \bigcap_{n \geq 0} (\tau^0_P(M) + P^n M)$.
\end{enumerate}

**Proof.** (i) If $m \in \tau^1_P(M)$ then $f(P)m = 0$ for some $f(z)$ with $f(z) = 1$. Writing $f(z) = 1 - zg(z)$, we have $m = P g(P)m \in P \tau^1_P(M)$. Thus
\[
\tau^1_P(M) \subseteq P'' \subseteq \bigcap_{n \geq 0} P^n M.
\]
The equality of the two outside terms should be Krull’s Theorem [9, Theorem 8.9] applied to the $k[P]$-module $e_vM$. The only complication is that $e_vM$ need not be finitely generated over $k[P]$ if there is another primitive cycle $R$ with head $v$. In this case one can, for example, apply Krull’s Theorem to

\[...\]
\[ e_v M \text{ considered as a module for } k[P,R]/(PR), \text{ and the ideal generated by } P. \]

(ii) Clearly \( \tau_P(M) \subseteq \tau_P^0(M) + P^n M \) for all \( n \). Now \( L = e_v M/\tau_P^0(M) \) is a finitely generated \( k[P] \)-module, so by Krull’s Theorem

\[
\bigcap_{n \geq 0} P^n L = \bigcap_{n \geq 0} \left( \tau_P^0(M) + P^n M \right) / \tau_P^0(M)
\]

is a torsion \( k[P] \)-module. Thus \( \bigcap_{n \geq 0} \left( \tau_P^0(M) + P^n M \right) \) is torsion, so contained in \( \tau_P(M) \).

\[\square\]

5. Functorial filtration given by words

For \( v \) a vertex and \( \epsilon = \pm 1 \) we define \( W_{v,\epsilon} \) to be the set of all finite words and \( \mathbb{N} \)-words with head \( v \) and sign \( \epsilon \). There is a total order on \( W_{v,\epsilon} \) given by \( C < C' \) if

(a) \( C = ByD \) and \( C' = Bx^{-1}D' \) where \( B \) is a finite word, \( x,y \) are arrows, and \( D,D' \) are words, or

(b) \( C' \) is a finite word and \( C = C'yD \) where \( y \) is an arrow and \( D \) is a word, or

(c) \( C \) is a finite word and \( C' = Cx^{-1}D' \) where \( x \) is an arrow and \( D' \) is a word.

For any \( \Lambda \)-module \( M \) and \( C \in W_{v,\epsilon} \) we define subspaces

\[
C^-(M) \subseteq C^+(M) \subseteq e_v M
\]
as follows. First suppose that \( C \) is a finite word. Then \( C^+(M) = Cx^{-1}0 \) if there is an arrow \( x \) such that \( Cx^{-1} \) is a word, and otherwise \( C^+(M) = CM \). Similarly, \( C^-(M) = CyM \) if there is an arrow \( y \) such that \( Cy \) is a word, and otherwise \( C^-(M) = C0 \). Now suppose that \( C \) is an \( \mathbb{N} \)-word. Then \( C^+(M) \) is the set of \( m \in M \) such that there is a sequence \( m_n \ (n \geq 0) \) such that \( m_0 = m \) and \( m_{n-1} \in C_n m_n \) for all \( n \). One defines \( C^-(M) \) to be the set of \( m \in M \) such that there is a sequence \( m_n \) as above which is eventually zero. Equivalently \( C^-(M) = \bigcup_n C_{\leq n} \mathbb{N} \). Observe that if \( C \in W_{v,\epsilon} \) is repeating, say \( C = D^\infty \), then \( C^-(M) = D^r \) and \( C^+(M) = D^s \), where \( D \) is considered as a linear relation on \( e_v M \).

Clearly one has \( \theta(C^+(M)) \subseteq C^+(N) \) for a homomorphism \( \theta : M \to N \) of \( \Lambda \)-modules. Thus \( C^\pm \) define subfunctors of the forgetful functor from \( \Lambda \)-modules to vector spaces. The following is standard. For finite words, see the lemma on page 23 of [12].

Lemma 5.1. If \( C, D \in W_{v,\epsilon} \) and \( C < D \), then \( C^+(M) \subseteq D^-(M) \).

Lemma 5.2 (Covering property). Let \( M \) be a \( \Lambda \)-module, let \( v \) be a vertex and \( \epsilon = \pm 1 \). Suppose that \( S \) is a non-empty subset of \( e_v M \) with \( 0 \notin S \). Then there is a word \( C \in W_{v,\epsilon} \) such that either (a) \( C \) is finite and \( S \) meets \( C^+(M) \) but does not meet \( C^-(M) \), or (b) \( C \) is an \( \mathbb{N} \)-word and \( S \) meets \( C_{\leq n} M \) for all \( n \) but does not meet \( C^-(M) \).

Proof. Suppose there is no finite word \( C \in W_{v,\epsilon} \) such that \( S \) meets \( C^+(M) \) but not \( C^-(M) \). Starting with the trivial word \( 1_{v,\epsilon} \), we iteratively construct an \( \mathbb{N} \)-word \( C \in W_{v,\epsilon} \) such that \( S \) meets \( C_{\leq n} M \) but not \( C_{\leq n} 0 \). Suppose we have constructed \( D = C_{\leq n} \). If there is a letter \( y \) with \( Dy \) a word, and \( S \)
meets $DyM$, then we define $C_{n+1} = y$ and repeat. Otherwise $S$ does not meet $D^{-}(M)$. If there is a letter $x$ with $Dx^{-1}$ a word, and $S$ does not meet $Dx^{-1}0$ then we define $C_{n+1} = x^{-1}$ and repeat. Otherwise $S$ meets $D^{+}(M)$. By our assumption, one of these two possibilities must occur. □

**Lemma 5.3.** If $C \in \mathcal{W}_{v,\epsilon}$ and $D \in \mathcal{W}_{v,-\epsilon}$ are not inverse N-words, then $C^{+}(M) \cap D^{+}(M) \subseteq \tau^{0}(e_{v}M)$, so it is finite-dimensional if $M$ is finitely controlled.

**Proof.** If $P$ is a primitive cycle with head $v$, then $P^{-1}$ has the same sign as one of $C$ and $D$, say $C$, and since $C$ is not an inverse N-word, $C < (P^{-1})^{\infty}$. Thus $C^{+}(M) \subseteq ((P^{-1})^{\infty})^{-}(M)$. It follows that any element of $C^{+}(M)$ is annihilated by some power of $P$. Thus any element $m$ of the intersection is annihilated by some power of any primitive cycle with head $v$. Since distinct primitive cycles with head $v$ have composition zero in $\Lambda$, it follows that $m$ is annihilated by some power of $v_{0}$. □

**Lemma 5.4.** If $C \in \mathcal{W}_{v,\epsilon}$ is finite, not an inverse word and $D \in \mathcal{W}_{v,-\epsilon}$ is not an inverse N-word, then $CM \cap D^{+}(M) \subseteq \tau^{0}(e_{v}M)$, so it is finite-dimensional if $M$ is finitely controlled.

**Proof.** We can write $C = Bx_{E}$, for some arrow $x$ and words $B, E$. Then $CM \subseteq Bx_{M} = B^{-}(M) \subseteq B^{+}(M)$, so the assertion follows from Lemma 5.3. □

**Lemma 5.5.** Let $C \in \mathcal{W}_{v,\epsilon}$ and suppose that $C$ is not (direct and repeating). If $P$ is a primitive cycle with head $v$ and sign $\epsilon$, then $PC^{+}(M) \subseteq C^{-}(M)$.

**Proof.** If $PC$ is not a word, it must involve a zero relation, so $PC^{+}(M) = 0$. Thus we may suppose that $PC$ is a word. Since $PC^{+}(M) \subseteq (PC)^{+}(M)$ it suffices to show that $PC < C$. If not, then $C \leq PC$ since the two words must have the same head and sign. This forces $C$ to have the form $PD$ for some $D$. Then $PD \leq PPD$, so $D \leq PD$, which forces $D$ to have the form $PE$, and so on. Thus $C = P^{\infty}$, a contradiction. □

**Lemma 5.6.** If $C \in \mathcal{W}_{v,\epsilon}$ is not (inverse and repeating), and $M$ is finitely controlled, then $C^{+}(M)/C^{-}(M)$ is finite-dimensional.

**Proof.** We may suppose that $C$ is not (direct and repeating), for otherwise $C = P^{\infty}$, for some primitive cycle $P$, so $C^{+}(M) = \tau^{1}_{P}(M)$ by Lemma 4.3, which is finite-dimensional.

Let $P$ be a primitive cycle with head $v$. If $P$ has sign $\epsilon$ then $PC^{+}(M) \subseteq C^{-}(M)$ by Lemma 5.5. If $P$ has sign $-\epsilon$, then $C$ and $(P^{-1})^{\infty}$ both belong to $\mathcal{W}_{v,\epsilon}$. They are distinct since $C$ is not (inverse and repeating). Thus $C < (P^{-1})^{\infty}$, so $C^{+}(M)$ is contained in $((P^{-1})^{\infty})^{-}(M)$. Since $e_{v}M$ is a finitely generated $k[z]$-module, the ascending chain condition ensures that $C^{+}(M) \subseteq P^{-n}0$ for some $n$, so $P^{n}C^{+}(M) = 0$ for some $n$.

It follows that $z^{n}C^{+}(M) \subseteq C^{-}(M)$, so $C^{+}(M)/C^{-}(M)$ is a finitely generated torsion $k[z]$-module, hence finite-dimensional. □

**Lemma 5.7.** Let $C$ be an N-word which is not (direct and repeating). If $M$ is finitely controlled, then the descending chain

$$C_{\leq 1}M \supseteq C_{\leq 2}M \supseteq C_{\leq 3}M \supseteq \ldots$$

stabilizes.
Proof. First suppose that $C$ is direct. If $P$ is a primitive cycle with the same head as $C$ and length $r$, then the first letter $C_1$ cannot be the same as $P_1$, for that would force $C = P^n$, which is direct and repeating. Thus $P_1C_1 = 0$ in $\Lambda$, so $PC_1M = 0$. It follows that $C_1M$ is a torsion $k[z]$-module, so finite-dimensional. Thus the terms in the descending chain are finite-dimensional, so it must stabilize.

Also, if $C$ is eventually inverse the chain stabilizes at $C_{\leq n}M$ with $n$ chosen so that $C_{> n}$ is inverse.

Thus we may suppose $C$ is not direct and not eventually inverse. It follows that $C = D x^{-1} y B$ for some words $D, B$, and distinct arrows $x, y$, say with head $v$. We thus need the chain

$$D x^{-1} y B_{\leq 1} M \supseteq D x^{-1} y B_{\leq 2} M \supseteq \ldots$$

to stabilize. Let $U_n = \tau^0(e_vM) \cap y B_{\leq n} M$. Then

$$D x^{-1} y B_{\leq n} M = D x^{-1} U_n$$

for if $m \in D x^{-1} y B_{\leq n} M$, then $m \in D x^{-1} m'$ for some $m' \in y B_{\leq n} M$. But then

$$m' \in y B_{\leq n} M \cap x D^{-1} m \subseteq \tau^0(e_vM)$$

since any arrow with tail $v$ has zero composition with $x$ or $y$, so $m' \in U_n$.

Now since $\tau^0(e_vM)$ is finite-dimensional, the chain $U_1 \supseteq U_2 \supseteq \ldots$ stabilizes, and hence so does

$$D x^{-1} U_1 \supseteq D x^{-1} U_2 \supseteq \ldots,$$

giving the result.

\[ \square \]

Lemma 5.8 (Realization lemma). If $M$ is finitely controlled and $C$ is an $\mathbb{N}$-word, then $C^+(M) = \bigcap_{n \geq 0} C_{\leq n} M$.

Proof. It suffices to show that if $C$ factorizes as $C = \ell D$, with $\ell$ a letter, and if $m \in \bigcap_{n \geq 0} C_{\leq n} M$, then $m \in \ell m'$ for some $m' \in \bigcap_{n \geq 0} D_{\leq n} M$.

If $\ell$ is an inverse letter, say $x^{-1}$, then this is trivial, taking $m' = x m$. Thus suppose $\ell = x$ is a direct letter.

If $D$ is not (direct and repeating), then $\bigcap_{n \geq 0} D_{\leq n} M = D_{\leq k} M$ for some $k$ by Lemma 5.7, and then $m \in x m'$ for some $m' \in D_{\leq k} M$, giving the result.

Thus suppose that $D$ is direct and repeating. Since $\ell$ is a direct letter, it follows that $C$ is direct and repeating, so of the form $C = P^\infty$ for a primitive cycle $P$. Say $P = \ell B$. By Lemma 4.3 we get

$$C^+(M) = P'' = P(P'') = \ell B \left( \bigcap_{n \geq 0} P^n M \right) \subseteq \ell \left( \bigcap_{n \geq 0} B P^n M \right)$$

giving the assertion.

\[ \square \]

Using this we obtain another version of the covering property.

Lemma 5.9. Let $M$ be a finitely controlled $\mathbb{L}$-module. Let $\epsilon = \pm 1$ and let $v$ be a vertex. Suppose that $0 \neq m \in e_v M$. Then there is a word $C \in W_{v, \epsilon}$ such that $m \in C^+(M) \setminus C^-(M)$.

Proof. Take $S = \{m\}$ in Lemma 5.2 and use Lemma 5.8. \[ \square \]
6. Refined Functors

If $B \in \mathcal{W}_{v,\epsilon}$ and $D \in \mathcal{W}_{v,-\epsilon}$, for some vertex $v$ and $\epsilon = \pm 1$, and $M$ is a $\Lambda$-module, we define

$$F_{B,D}^+(M) = B^+(M) \cap D^+(M),$$
$$F_{B,D}^-(M) = (B^+(M) \cap D^-(M)) + (B^-(M) \cap D^+(M)),$$
$$F_{B,D}(M) = F_{B,D}^+(M)/F_{B,D}^-(M).$$

In general we consider $F_{B,D}$ as a functor from the category of $\Lambda$-modules to vector spaces. It does not depend on the order of $B$ and $D$.

If $B^{-1}D$ is a periodic $\mathbb{Z}$-word of period $n$, then it is equal to $\infty C \infty$ for some word $C$ of length $n$ and head $v$, say. If $M$ is a $\Lambda$-module, then $C$ induces a linear relation on $\epsilon_v M$, and $F_{B,D}^+(M) = C^2$ and $F_{B,D}^-(M) = C^2$ as in Section\[2\]. Thus $C$ induces an automorphism of $F_{B,D}(M)$, and $F_{B,D}$ defines a functor from $\Lambda$-modules to $k[T,T^{-1}]$-modules. In this case exchanging $B$ and $D$ has the effect of exchanging the actions of $T$ and $T^{-1}$.

**Lemma 6.1.** (i) If $B^{-1}D$ is not a word, then $F_{B,D} = 0$.

(ii) If $E$ is a fixed word, the functors $F_{B,D}$ with $B^{-1}D = E[n]$, for any $n$, are all isomorphic.

**Proof.** (i) $B^{-1}D$ must involve a zero relation, and exchanging $B$ and $D$ if necessary, we may assume that $B = x_n^{-1} \ldots x_1^{-1} C$ and $D = y_1 \ldots y_m E$ with $x_1 \ldots x_n y_1 \ldots y_m \in \rho$. If $m \in F_{B,D}^+(M)$ then $m = y_1 \ldots y_m m'$ with $m' \in E^+(M)$, so $m = x_n^{-1} \ldots x_1^{-1} 0 \subseteq B^-(M)$, so $m \in F_{B,D}^-(M)$.

(ii) This is the same as the corresponding lemma at the top of page 25 in [12]. The extension to functors to $k[T,T^{-1}]$-modules in case $E$ is a periodic $\mathbb{Z}$-word is straightforward. \qed

**Lemma 6.2.** $F_{B,D}(M)$ is finite-dimensional for $M$ finitely controlled.

**Proof.** Say $B$ and $D$ have head $v$. We may suppose that $B$ and $D$ are both inverse and repeating, for otherwise the result follows from Lemma 5.6. Say $B = (P^{-1})^\infty$ and $D = (R^{-1})^\infty$ for primitive cycles $P, R$. Then for $m \in \epsilon_v M$ we have

$$zm = Pm + Rm \in R^{-1} 0 + P^{-1} 0 \subseteq D^-(M) + B^-(M).$$

Thus $z$ acts as zero on $F_{B,D}(M) = \epsilon_v M/(D^-(M) + B^-(M))$. But this is a finitely generated $k[\mathbb{Z}]$-module, so it is finite-dimensional. \qed

Let $v$ be a vertex. If $(B,D) \in \mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$ and $M$ is a $\Lambda$-module, we define

$$G_{B,D}^+(M) = B^-(M) + D^\pm(M) \cap B^+(M) \subseteq \epsilon_v M.$$ 

Clearly $G_{B,D}^-(M) \subseteq G_{B,D}^+(M)$ and $G_{B,D}^+(M)/G_{B,D}^-(M) \cong F_{B,D}(M)$. We totally order $\mathcal{W}_{v,1} \times \mathcal{W}_{v,-1}$ lexicographically, so

$$(B,D) < (B',D') \iff \text{ if } B < B' \text{ or } (B = B' \text{ and } D < D').$$

We have $G_{B,D}^+(M) \subseteq G_{B',D'}^-(M)$ for $(B,D) < (B',D')$ by Lemma 5.1.
Lemma 6.3. If $M$ is a finitely controlled $A$-module and $0 \neq m \in e_v M$, then there is $(B, D) \in W_{e,1} \times W_{e,-1}$ such that $m$ is in $G^+_{B,D}(M)$ but not in $G^+_{B,D}(M)$, and hence $F_{B,D}(M) \neq 0$.

Proof. By Lemma 5.9 there is $B \in W_{e,1}$ such that $m \in B^+(M) \setminus B^-(M)$. Thus

$$S = (m + B^- (M)) \cap B^+(M)$$

is non-empty and does not contain 0. By Lemma 5.2 there is a word $D \in W_{e,-1}$ such that either (a) $D$ is finite and $S$ meets $D^+(M)$ but not $D^-(M)$, or (b) $D$ is an $N$-word and $S$ meets $D_{\leq n} M$ for all $n$ but not $D^-(M)$.

The conditions in the statement of the lemma are equivalent to the statement that $S$ meets $D^+(M)$ but not $D^-(M)$, so we already have the result in case (a). To prove the result in case (b), assume for a contradiction that $S$ meets $D_{\leq n} M$ for all $n$ but not $D^+(M)$.

Now $D$ is direct and repeating, for otherwise $D^+(M) = D_{\leq n} M$ for some $n$ by Lemma 5.7. Thus $D = P^\infty$ for some primitive cycle $P$.

Suppose $B \neq (P^{-1})^\infty$. This is the only inverse $N$-word with head $v$ and sign 1, so $B$ is not an inverse $N$-word. Then by Lemma 5.4 $PM \cap B^+(M)$ is contained in $r^0 (e_v M)$, so is finite-dimensional. Thus the descending chain of subspaces

$$PM \cap B^+(M) \supseteq P^2 M \cap B^+(M) \supseteq P^3 M \cap B^+(M) \supseteq \ldots$$

stabilizes, so $P^N M \cap B^+(M) = D^+(M) \cap B^+(M)$ for some $N$ by Lemma 4.3. This implies $S$ meets $D^+(M)$, a contradiction.

Thus $B = (P^{-1})^\infty$. For all $n$, we know that $S$ meets $D_{\leq n} M$. It follows that $m \in B^-(M) + D_{\leq n} M \cap B^+(M)$. Thus

$$m \in \bigcap_{i \geq 0} \left( (P^{-1})^i + P^i M \right) .$$

By Lemma 4.3(ii), it follows that

$$m \in (P^{-1})^i + P^i = B^-(M) + D^+(M) \cap B^+(M),$$

or equivalently $S$ meets $D^+(M)$, again a contradiction. \null\null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null \null
Proof. Let $M = M(C)$. Using the ordering on words and functors, it suffices to show that $b_i \in C(i, \epsilon)^+(M)$ and that if a linear combination $m$ of the basis elements $b_j$ belongs to $C(i, \epsilon)^-(M)$, then the coefficient of $b_i$ in $m$ is zero.

If $C(i, \epsilon)$ is finite, let $1_{u,n}$ be the trivial word with $C(i, \epsilon)1_{u,n}$ defined (and hence equal to $C(i, \epsilon)$). Define $d = d_i(C, \epsilon)$. For $n \geq 1$, and not greater than the length of $C(i, \epsilon)$, we have $b_{i+d(n-1)} \in C(i, \epsilon)_n b_{i+dn}$. Moreover, if $C(i, \epsilon)$ has length $n$ then $b_{i+dn} \in 1_{1^n, q}(M)$. It follows that $b_i \in C(i, \epsilon)^+(M)$.

By induction on $n$, the following is straightforward. Suppose $b_i$ is not greater than the length of $C(i, \epsilon)$. If $m$ is an element of $M$ whose coefficient of $b_i$ is $\lambda$, and $m \in C(i, \epsilon) \leq n m'$, then the coefficient of $b_{i+dn}$ in $m'$ is also $\lambda$. Clearly if $C(i, \epsilon)$ has length $n$, then no element of $1_{u,n}(M)$ has $b_{i+dn}$ occurring with non-zero coefficient. It follows that no element of $C(i, \epsilon)^-(M)$ can have $b_i$ occurring with non-zero coefficient.

$\square$

Lemma 7.2. Let $M = M(C)$ where $C$ is an I-word which is not a periodic Z-word.

(i) If $i \in I$, then $F^+_{C(i,1),C(i,-1)}(M) = F^-_{C(i,1),C(i,-1)}(M) \oplus kb_i$.

(ii) If $B^{-1}D = C$, then $F_{B,D}(M) \cong k$.

(iii) If $B^{-1}D$ is not equivalent to $C$, then $F_{B,D}(M) = 0$.

Proof. (i) By Lemma 7.1

$$F^+_{C(i,1),C(i,-1)}(M) = F^-_{C(i,1),C(i,-1)}(M) \oplus U$$

where $U$ is spanned the $b_j$ with $C(j,1) = C(i,1)$ and $C(j,-1) = C(i,-1)$. By Lemma 3.1 and since $C$ is not a periodic Z-word, this condition holds only for $j = i$.

(ii) We have $\{B, D\} = \{C(i,1), C(i,-1)\}$ for some $i$.

(iii) Exchanging $B$ and $D$ if necessary, and letting $v$ be the head of $B$ and $D$, we have $(B, D) \in W_{v,1} \times W_{v,-1}$. Lemma 7.1 implies that the spaces $G_{B,D}^\pm(M)$ are spanned by sets of basis elements $b_j$, so if $F_{B,D}(M) \neq 0$, then some $b_i$ belongs to $G_{B,D}^+(M)$ but not to $G_{B,D}^-(M)$. But by (i) we have

$$b_i \in G_{C(i,1),C(i,-1)}^+(M) \setminus G_{C(i,1),C(i,-1)}^-(M).$$

Then $(B, D) = (C(i,1), C(i,-1))$ by the total ordering of the $G_{B,D}^\pm$, so $B^{-1}D$ is equivalent to $C$. $\square$

Lemma 7.3. Suppose that $C$ is an I-word which is not a periodic Z-word. Suppose that $i \in I$, $B = C(i,1)$ and $D = C(i,-1)$. Let $M$ be a finitely controlled module and let $n = \dim F_{B,D}(M)$. Then there is a map $\theta_{B,D,M}: M(C)^n \to M$ such that $F_{B,D}(\theta_{B,D,M})$ is an isomorphism.

Proof. Given any $m \in F_{B,D}(M) = B^+(M) \cap D^+(M)$ there is a $\Lambda$-module map $M(C) \to M$ sending $b_i$ to $m$. Lifting a basis of $F_{B,D}(M)$ to $F_{B,D}(M)$ thus gives a suitable map $\theta_{B,D,M}$. $\square$

Band modules. Suppose that $C$ is a periodic Z-word of period $n$ and $V$ is a $k[T, T^{-1}]$-module. The module $M(C, V) = M(C) \otimes_{k[T, T^{-1}]} V$ can be written as

$$M(C, V) = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$$
where each $V_i = b_i \otimes V$ is identified with a copy of $V$. (It is a band module provided $V$ is finite-dimensional and indecomposable.)

**Lemma 7.4.** If $D \in \mathcal{W}_{e_0}$ then

1. $D^+(M) = \bigoplus_{i \in I^+} V_i$, $I^+ = \{0 \leq i < n : v_i(C) = v, C(i, \epsilon) \leq D\}$.
2. $D^-(M) = \bigoplus_{i \in I^-} V_i$, $I^- = \{0 \leq i < n : v_i(C) = v, C(i, \epsilon) < D\}$.

**Proof.** Similar to Lemma 7.1 \(\square\)

**Lemma 7.5.** Let $M = M(C, V)$.

1. If $0 \leq i < n$, then $F_{C[(i,1), C(i, -1)]}^+(M) = F_{C(i,1), C(i, -1)}^-(M) \oplus V_i$.
2. If $B^{-1}D = C$, then $F_{B,D}(M) \cong V$ as $k[T, T^{-1}]$-modules.
3. If $B^{-1}D$ is not equivalent to $C$ then $F_{B,D}(M(C, V)) = 0$.

**Proof.** Similar to Lemma 7.2 \(\square\)

**Lemma 7.6.** Suppose that $B, D$ are words and $B^{-1}D = C$ is a periodic $Z$-word. Suppose $M$ is finitely controlled. Then $V = F_{B,D}(M)$ is a finite-dimensional $k[T, T^{-1}]$-module, and there is a map $\theta_{B,D,M} : M(C, V) \to M$ such that $F_{B,D}(\theta_{B,D,M})$ is an isomorphism.

**Proof.** Let $D = E^\infty$ with $E$ of length $n$, equal to the period of $C$. Let $v$ be the head of $E$. Then $E$ defines a linear relation on $v_0 M$ and by definition $V = E^2/E^3$ and the action of $T$ is induced by $E$. Now $V$ is finite-dimensional since $B$ and $D$ cannot both be inverse words, so $B^+(M)/B^-(M)$ or $D^+(M)/D^-(M)$ is finite-dimensional. By Lemma 2.6 there is a subspace $U$ of $v_0 M$ with $E^2 = E^3 \oplus U$ and such that $E$ induces an automorphism $T$ on $U$, and of course $U \cong V$. As in \[12\] §5, Proposition, one gets a map $\theta_{B,D,M} : M(C, V) \to M$ such that $F_{B,D}(\theta_{B,D,M})$ is an isomorphism. Namely, there are elements $u_{r,i} \in M$ for $1 \leq r \leq s$ and $0 \leq i \leq n$ with $u_{1,0}, \ldots, u_{s,0}$ and $u_{1,n}, \ldots, u_{s,n}$ bases of $U$ connected by $u_{r,0} = Tu_{r,n}$, and $u_{r,i-1} \in E_i u_{r,i}$ for all $r, i$. Using these elements one defines $\theta_{B,D,M} : M(C, U) \to M$, sending $b_i \otimes v_{r,0} \in V_i$ for $0 \leq i < n$ to $u_{r,i}$. \(\square\)

8. **Applications of the refined functors**

Let $\Sigma$ be a set of representative of the equivalence classes of words. As discussed in the introduction, we have $M(C) \cong M(D)$ if $C$ and $D$ are equivalent. Also, if $C$ is a periodic $Z$-word we have $M(C) \cong M(C, k[T, T^{-1}])$ and

$$M(C, V) \cong M(C[n], V) \cong M((C^{-1})[n], V^{-1})$$

where $V^{-1}$ is the $k[T, T^{-1}]$ module with the same underlying vector space as $V$, but with the actions of $T$ and $T^{-1}$ exchanged. Thus when considering a direct sum of modules of type $M(C)$ and $M(C, V)$, we may assume that the summands are copies of $M(C)$ for $C \in \Sigma$ not a periodic $Z$-word and $M(C, V)$ for $C \in \Sigma$ a periodic $Z$-word.

**Theorem 8.1.** Suppose a module $M$ is written as a direct sum of copies of string modules $M(C)$ for $C \in \Sigma$ not a periodic $Z$-word and modules $M(C, V)$ for $C \in \Sigma$ a periodic $Z$-word.

1. If $C = B^{-1}D$ in $\Sigma$ is not a periodic $Z$-word, then the number of copies of $M(C)$ in the direct sum is equal to $\dim F_{B,D}(M)$.
(ii) If $C = B^{-1}D$ in $\Sigma$ is a periodic $\mathbb{Z}$-word, and the summands of type $M(C,V)$ are $M(C,V_t)$ with $t$ running through some indexing set, then $F_{B,D}(M) \cong \bigoplus_t V_t$ as a $k[T,T^{-1}]$-module.

Proof. Follows immediately from Lemmas 7.2 and 7.5. □

If $V$ is a finite-dimensional $k[T,T^{-1}]$-module, we can write it as a direct sum of indecomposables $V = V_1 \oplus \cdots \oplus V_n$, and hence we can write $M(C,V) \cong M(C,V_1) \oplus \cdots \oplus M(C,V_n)$, a direct sum of band modules. Then Lemmas 7.3 and 7.6 give the following.

**Theorem 8.2.** Given a finitely controlled module $M$, there is a homomorphism $\theta : N \to M$ such that $N$ is a direct sum of string and band modules and $F_{B,D}(\theta)$ is an isomorphism for all refined functors $F_{B,D}$.

Proof. Write each $C \in \Sigma$ in the form $C = B^{-1}D$ and let $N$ be the direct sum of powers of the string modules $M(C)$ and modules $M(C,V)$ as in Lemmas 7.3 and 7.6. The maps $\theta_{B,D,M}$ combine to give a map $\theta : N \to M$ and all refined functors $F_{B,D}(\theta)$ are isomorphisms. □

**Theorem 8.3.** Suppose $\theta : N \to M$ is a homomorphism, with $M$ finitely controlled and such that $F_{B,D}(\theta)$ is an isomorphism for all refined functors $F_{B,D}$. Then

1. $\theta$ is injective.
2. $\text{Im}(\theta)$ contains the primitive torsion submodule $\tau^1(M)$ of $M$.
3. The cokernel of $\theta$ is primitive torsion.
4. If $e_v M$ is finite-dimensional for all $v$, then $\theta$ is an isomorphism.

Proof. (1) Say $0 \neq n \in e_v N$ and $\theta(n) = 0$. By Lemma 6.3 there are $B,D$ with (i) $n \in B^-(N) + (D^+(N) \cap B^+(N))$ and (ii) $n \notin B^-(N) + (D^-(N) \cap B^+(N))$. By (i) we can write $n = n' + n''$ with $n' \in B^-(N)$ and $n'' \in D^+(N) \cap B^+(N) = F_{B,D}(N)$. Then it follows from (ii) that $n''$ induces a non-zero element in $F_{B,D}(N)$. Thus by assumption $\theta(n'')$ induces a non-zero element in $F_{B,D}(M)$. But $\theta(n'') \in B^+(M) \cap D^+ (M)$ and $\theta(n'') = -\theta(n'') \in B^+(M)$, so $\theta(n'') \in F_{B,D}(M)$, a contradiction.

(2) By Lemma 1.2 it suffices to show $\tau^1_p(M) \subseteq \text{Im}(\theta)$ for $P$ a primitive cycle. Let $m \in \tau^1_p(M)$. By Lemma 1.3

$$m \in P'' \cap (P^{-1})'' = F_{B,D}^+(M)$$

where $B = (P^{-1})^\infty$ and $D = P^\infty$. Thus by hypothesis $m = m' + \theta(n)$ for some $n \in N$ and

$$m' \in F_{B,D}^+(M) = (P' \cap (P^{-1})'') + (P'' \cap (P^{-1})').$$

Now $P' = 0$ since $P$ is direct, and $P'' \cap (P^{-1})' = \tau^1_p(M) \cap \tau^0_p(M) = 0$. Thus $m' = 0$, so $m = \theta(n) \in \text{Im}(\theta)$.

(3) Since $M/\text{Im}(\theta) = \bigoplus_v e_v (M/\text{Im}(\theta))$ is a direct sum of finitely generated $k[\mathbb{Z}]$-modules, if it is not primitive torsion, then there is a $k[\mathbb{Z}]$-submodule $U$ of codimension 1 in $M$ with $\text{Im}(\theta) \subseteq U$ and $z \mathbb{Z} \subseteq U$. Choose $m \in e_v M$ for some $v$, with $m \notin U$.

By Lemma 5.2 applied to $S = U + m$ there is $D \in W_{v,-1}$ such that either (a) $D$ is finite and $m$ is in $U + D^+(M)$ but not in $U + D^-(M)$, or (b) $D$ is...
an $N$-word and $m$ is in $U + D_{<n}M$ for all $n$ but not in $U + D^{-}(M)$. If $D$ is direct and repeating, then $D = P^\infty$ for a primitive cycle of length $p$, say. Then $m \in U + D_{<p}M$ and $D_{<p}M = P^2M = zPM \subseteq zM \subseteq U$, so $m \in U$, a contradiction. Thus $D$ is not (direct and repeating). By Lemma 5.7, if $D$ is a $N$-word, then $D^+(M) = D_{\leq n}M$ for some $n$. Thus $m \in U + D^+(M)$ and $m \notin U + D^-(M)$ in both cases (a) and (b). Since $U$ has codimension 1 in $M$, we have $U + D^+(M) = M$ and $U + D^-(M) = U$, so $D^-(M) \subseteq U$. Now we can write $m = u + m'$ with $u \in U$ and $m' \in D^+(M)$.

Applying Lemma 5.2 with the set $T = (U + m') \cap D^+(M)$, there is word $B \in W_{v,1}$ such that either (a) $B$ is finite and $T$ meets $B^+(M)$ but not $B^-(M)$, or (b) $B$ is a $N$-word and $T$ meets $B_{\leq n}(M)$ for all $n$ but not $B^-(M)$.

If $B$ is direct and repeating, then $B = P^\infty$ with $P$ a primitive cycle of length $p$, say. Then $T$ meets $B_{\leq p}M = P^2M = zPM \subseteq zM \subseteq U$, as before, which implies $m' \in U$, and then $m \in U$, a contradiction. Thus $B$ is not (direct and repeating), so $B^+(M) = B_{\leq n}M$ for some $n$. Thus $T$ meets $B^+(M)$ but not $B^-(M)$ in both cases (a) and (b). It follows that $m' \in U + (B^+(M) \cap D^+(M))$ and $m' \notin U + (B^-(M) \cap D^+(M))$.

Write $m' = u' + m''$ with $u' \in U$ and $m'' \in B^+(M) \cap D^+(M)$. Since $F_{B,D}(\theta)$ is an isomorphism, there is some $n \in B^+(N) \cap D^+(N)$ such that $m'' - \theta(n) \in (B^+(M) \cap D^-(M)) + (B^-(M) \cap D^+(M))$. Write this as $a + a'$. Then $m = (u + u' + \theta(n) + a) + a' \in U + B^-(M) \cap D^+(M)$, a contradiction.

(4) If all $e_vM$ are finite-dimensional, then $M$ is torsion. Now the cokernel of $\theta$ is nilpotent torsion by (2) and primitive torsion by (3), so it must be zero.

Combining Theorems 8.2 and 8.3 one obtains the following special case of Theorem 1.3

**Corollary 8.4.** If $M$ is a $\Lambda$-module with $e_vM$ finite-dimensional for all $v$, then $M$ is isomorphic to a direct sum of string and band modules.

9. Extensions by a primitive simple

We fix a primitive simple $S$ for $\Lambda$, that is, a simple, primitive torsion module. It is easy to see (for example using Corollary 8.3) that it is of the form $S = M(\infty P^\infty, V)$ where $P$ is a primitive cycle, say with head $v$, sign $\epsilon$ and length $p$, and $V$ is a simple $k[T, T^{-1}]$-module, so of the form $V = k[T, T^{-1}]/(f(T))$ where $f(T)$ is an irreducible polynomial in $k[T]$ with $f(0) = 1$. Since $P$ has sign $\epsilon$, it follows that $P^{-1}$ and $(P^{-1})^\infty$ have sign $-\epsilon$.

**Definition 9.1.** Let $C$ be an $I$-word. We say that $i \in I$ is $P$-deep for $C$ if $C(i, -\epsilon) = (P^{-1})^\infty$. Equivalently if the basis element $b_i$ in $M(C)$ is not killed by any power of $P$. We say that $i \in I$ is a $P$-peak for $C$ if it is $P$-deep for $C$ and $C(i, \epsilon)$ is not of the form $PD$ for some word $D$. Equivalently, it is $P$-deep for $C$ and $b_i$ is not in $PM(C)$.

Clearly only an infinite word can have a $P$-peak, and then it has at most two $P$-peaks (and if so it is a $Z$-word). Our aim in this section is to prove the following result.
Lemma 9.5. The middle term formed from the pushout of the projective resolution in Lemma 9.3 along the homomorphism $\Lambda e_z$ since to $M_k$ two actions of $\Lambda$. By using a projective resolution of $C$, which has a $P$-peak, such that $M = N'_\mu \oplus N''$ where

$$N'' = \bigoplus_{\lambda \in \Phi \setminus \{\mu\}} N_\lambda,$$

and $N'_\mu$ is a submodule of $M$ with $N'_\mu \cong N_\mu$.

The following is straightforward.

**Lemma 9.3.** There is a projective resolution

$$0 \to \Lambda e_v \to \Lambda e_v \to S \to 0$$

where the first map is right multiplication by $f(P)$.

For any $\Lambda$-module $M$, the resolution of $S$ gives an exact sequence

$$0 \to \operatorname{Hom}(S,M) \to e_vM \xrightarrow{f(P)} e_vM \xrightarrow{\alpha_M} \operatorname{Ext}^1(S,M) \to 0.$$ 

We denote the pullback of $\xi \in \operatorname{Ext}^1(S,M)$ along $a \in \operatorname{End}(S)$ by $\xi a$, and if $\theta : M \to N$ is a homomorphism, we denote the pushout map $\operatorname{Ext}^1(S,M) \to \operatorname{Ext}^1(S,N)$ by $\theta_\ast$.

**Lemma 9.4.** If $a \in \operatorname{End}(S)$ and $\xi \in \operatorname{Ext}^1(S,M)$, then $\xi a = \psi_\ast(\xi)$ for some $\psi$ in the centre of $\operatorname{End}(M)$.

**Proof.** For any $\Lambda$-module $M$, the action of $k[z]$ on $M$ defines a homomorphism $\gamma_M : k[z] \to \operatorname{End}(M)$. If $N$ is another $\Lambda$-module, the actions of $k[z]$ on $M$ and $N$ induce left and right actions of $k[z]$ on $\operatorname{Hom}(N,M)$, but these are the same, for if $\theta \in \operatorname{Hom}(N,M)$ and $m \in e_vM$ then

$$\quad (\theta z)(m) = \theta(zm) = \theta(z_vm) = z_v\theta(m) = z\theta(m) = (z\theta)(m)$$

since $z_v \in \Lambda$. By using a projective resolution of $N$, the same holds for the two actions of $k[z]$ on $\operatorname{Ext}^1(N,M)$. It is clear that $\gamma_S$ induces an isomorphism

$$k[z]/(f(z)) \cong \operatorname{End}(S).$$

Thus, writing $a = \gamma_S(h(z))$ for some $h(z) \in k[z]$, we can take $\psi = \gamma_M(h(z))$. It is central by the discussion above.

If $C$ is an $I$-word and $i$ is a $P$-peak for $C$, consider the exact sequence

$$\xi_{C,i} : 0 \to M(C) \to E_{C,i} \to S \to 0$$

formed from the pushout of the projective resolution in Lemma 9.3 along the homomorphism $\Lambda e_v \to M(C)$ sending $e_v$ to $b_i$. Thus

$$\xi_{C,i} = \alpha_{M(C)}(b_i) \in \operatorname{Ext}^1(S,M(C)).$$

**Lemma 9.5.** The middle term $E_{C,i}$ of the exact sequence $\xi_{C,i}$ is isomorphic to $M(C)$. 

Theorem 9.2. Suppose that $M$ is a finitely controlled $\Lambda$-module and $N$ is a submodule of $M$ with $\tau^1(M) \subseteq N$ and $M/N \cong S$. Suppose that $N$ is a direct sum of string and band modules

$$N = \bigoplus_{\lambda \in \Phi} N_\lambda,$$

indexed by some set $\Phi$. Then there is some $\mu \in \Phi$ with $N_\mu$ of the form $M(C)$ for some word $C$, which has a $P$-peak, such that $M = N'_\mu \oplus N''$ where

$$N'' = \bigoplus_{\lambda \in \Phi \setminus \{\mu\}} N_\lambda,$$

and $N'_\mu$ is a submodule of $M$ with $N'_\mu \cong N_\mu$. 

The following is straightforward.
Proof. We define \( \phi \in \text{End}(M(C)) \) as follows. If \( d(C, -\epsilon) = 1 \), so that \( C_{>i} = (P^{-1})^\infty \), let \( j \) be minimal with \( C_{>j} \) an inverse word. Since \( i \) is a \( P \)-peak for \( C \), we have \( i - p < j \leq i \), where \( p \) is the length of \( P \). We define \( \phi(b_k) = b_{k+p} \) for \( k \geq j \) and \( \phi(b_k) = 0 \) for \( k < j \). Dually, if \( d_i(C, -\epsilon) = -1 \), so that \( (C_{\leq i})^{-1} = (P^{-1})^\infty \), let \( j \in I \) be maximal such that \( (C_{\leq j})^{-1} \) is an inverse word, and define \( \phi(b_k) = b_{k-p} \) for \( k \leq j \) and \( \phi(b_k) = 0 \) for \( k > j \).

It is straightforward to see that \( f(\phi) \) is an injective endomorphism of \( M(C) \) with cokernel isomorphic to \( S \). We fix an isomorphism between \( S \) and the cokernel of \( f(\phi) \), and hence obtain an exact sequence

\[
\eta_{C,i} : 0 \to M(C) \xrightarrow{f(\phi)} M(C) \xrightarrow{g} S \to 0.
\]

Let \( M = M(C) \). The exact sequences above lead to a commutative diagram with exact rows and columns

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & \hom(S, M) & \to & \hom(S, M) & \to & \text{End}(S) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & e_rM & \xrightarrow{f(\phi)} & e_rM & \to & e_rS & \to & 0 \\
\downarrow{f(P)} & \downarrow{f(P)} & \downarrow{f(P)} & \downarrow & \downarrow & \\
0 & \to & e_rM & \xrightarrow{f(\phi)} & e_rM & \to & e_rS & \to & 0 \\
\downarrow{\alpha_M} & \downarrow{\alpha_M} & \downarrow{\alpha_S} & \downarrow & \downarrow & \\
\text{Ext}^1(S, M) & \to & \text{Ext}^1(S, M) & \to & \text{Ext}^1(S, S) & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0 & 0.
\end{array}
\]

The Snake Lemma gives a connecting map \( c : \text{End}(S) \to \text{Ext}^1(S, M) \) sending \( a \in \text{End}(S) \) to \( \eta_{C,i}a \). Now \( f(\phi)b_i = f(P)b_i \) so by the diagram chase defining the connecting map there is \( a \in \text{End}(S) \) with \( c(a) = \alpha_M(b_i) \). Moreover \( a \neq 0 \) since \( b_i \notin f(\phi)M \), so \( g(b_i) \neq 0 \). Then \( \eta_{C,i}a = \alpha_M(b_i) = \xi_{C,i} \), so there is map of exact sequences

\[
\begin{array}{cccc}
\xi_{C,i} : 0 & \to & M(C) & \to & E_{C,i} & \to & S & \to & 0 \\
\eta_{C,i} : 0 & \to & M(C) & \to & M(C) & \to & S & \to & 0 \\
\end{array}
\]

and since \( a \) is an isomorphism, \( E_{C,i} \cong M(C) \). \( \square \)

**Lemma 9.6.** If \( C \) is a word which is not equivalent to \( \infty P^\infty \), then the elements \( \xi_{C,i} \) with \( i \) a \( P \)-peak for \( C \), form an \( \text{End}(S) \)-basis for \( \text{Ext}^1(S, M(C)) \).

**Proof.** Observe that \( e_rM(C) \), as a \( k[P] \)-module, is the direct sum of free submodules \( k[P]b_i \) where \( i \) runs through the \( P \)-peaks, and a nilpotent torsion submodule spanned by the \( b_i \) with \( v_i(C) = v \) and \( i \) not \( P \)-deep. It follows
that the elements $\xi_{C,i} = \alpha_{M(C)}(b_i)$ form an $\text{End}(S)$ basis for $\text{Ext}^1(S, M(C))$. □

Let $\Sigma$ be a set of representative of the equivalence classes of words.

**Definition 9.7.** We define the set of $P$-classes to be the set pairs $(C, i)$ where $C \in \Sigma$ and $i$ is a $P$-peak for $C$, The set of $P$-classes is totally ordered by $(C, i) > (D, j)$ if $C(i, \epsilon) > D(j, \epsilon)$.

Henceforth, we write $b_C^\epsilon$ instead of $b_i$ for the basis elements of $M(C)$, so as to identify the word $C$.

**Lemma 9.8.** Suppose that $(C, i) > (D, j)$ are $P$-classes. Then there is a homomorphism $\theta_{ij} : M(C) \rightarrow M(D)$ such $(\theta_{ij})_*(\xi_{C,i}) = \xi_{D,j}$. Moreover, if $C = D$ then $\theta_{ij}^2 = 0$.

**Proof.** By assumption $C(i, \epsilon) > D(j, \epsilon)$. Let $r$ be maximal with $C(i, \epsilon)_{<r} = D(j, \epsilon)_{<r} = B$.

say. Then $C(i, \epsilon)_{r+1}$ is an inverse letter and $D(j, \epsilon)_{r+1}$ is a direct letter (or one of them is absent because the relevant word $C(i, \epsilon)$ or $D(j, \epsilon)$ has length $r$). Let $c = d_i(C, \epsilon)$ and $d = d_j(D, \epsilon)$. We define

$\theta_{ij}(b_C^\epsilon) = \begin{cases} b_D^j & (c(i - k) \geq -r) \\ 0 & (c(i - k) < -r). \end{cases}$

This is a homomorphism and it sends $b_C^\epsilon$ to $b_D^j$. Thus

$$(\theta_{ij})_*(\xi_{C,i}) = (\theta_{ij})_*(\alpha_{M(C)}(b_C^\epsilon)) = \alpha_{M(D)}(b_D^j) = \xi_{D,j}.$$ 

Now suppose that $C = D$. Then $C(i, \epsilon)$ is of the form $E(P^{-1})^\infty$ and $C(j, \epsilon)$ is of the form $E^{-1}(P^{-1})^\infty$, where $E$ has length $|i - j|$. Then $E > E^{-1}$ and $r$ is maximal with $E_{<r} = (E^{-1})_{<r}$. Then Lemma 8.1 implies that $E$ has length $> 2r$, and that $E = BFB^{-1}$ for some word $F$ of length $\geq 1$ whose first and last letters are inverse. But then the basis elements $b_C^\epsilon$ in the image of $\theta_{ij}$ are all sent to zero by $\theta_{ij}$.

**Lemma 9.9.** Let $M = M(D, U)$ be a band module.

(i) If $D$ is not equivalent to $\infty P^\infty$ then $\text{Ext}^1(S, M) = 0$.

(ii) If $D = \infty P^\infty$ then $\text{Ext}^1(S, M)$ has dimension $\leq 1$ as a vector space over $\text{End}(S)$.

(iii) If $\text{Ext}^1(S, M) \neq 0$ and $(C, i)$ is a $P$-class, then $\psi_*(\xi_{C,i}) \neq 0$ for some homomorphism $\psi : M(C) \rightarrow M$.

**Proof.** (i) The projective resolution of $S$ realizes $\text{Ext}^1(S, M)$ as the cokernel of the map $f(P)$ from $e_v M$ to $e_v M$. If $D$ is not equivalent to $\infty P^\infty$ then there are no $P$-deep basis elements for $D$. It follows that each element of $e_v M$ is killed by a power of $P$, so $f(P)$ acts invertibly on $e_v M$.

(ii) We have $M = U_0 \oplus U_1 \oplus \cdots \oplus U_{p-1}$ using the notation preceding Lemma 7.4 where $p$ is the length of $P$. Now as a $k[P]$-module, $e_v M$ is isomorphic to the direct sum of $U_0$, which is a copy of $U$ with $P$ acting as $T$, and a nilpotent torsion submodule, spanned by the other $U_i$ with $U_i = e_v U_i$.

Thus

$$\text{Ext}^1(S, M) \cong e_v M/f(P)M \cong U/f(T)U \cong \text{Ext}^1(V, U).$$
Since \( U \) is an indecomposable \( k[T, T^{-1}] \)-module and \( V \) is simple, this has dimension \( \leq 1 \) as a module for \( \text{End}(V) \cong \text{End}(S) \).

(iii) We may assume we are in case (ii). Then \( \text{Ext}^1(V, U) \neq 0 \), so we can identify \( U = k[T]/(f(T))^r \) for some \( r > 0 \). There is a homomorphism \( M(C) \to M(D) \) sending \( b_i^C \) to \( b_i^D \). It induces a homomorphism \( \psi : M(C) \to M \) sending \( b_i^C \) to \( m = b_i^D \otimes T \in e_v M \), and

\[
\psi_*(\xi_{C,i}) = \psi_*(\alpha_{M(C)}(b_i^C)) = \alpha_M(\psi(b_i^C)) = \alpha_M(m).
\]

This is non-zero since \( m \notin f(P)M \), which follows from the observation in (ii) about the \( k[P] \)-module structure of \( e_v M \), as we can identify \( m \) with the element \( T \in U_0 \).

Proof of Theorem 9.3. Letting \( i_\lambda \) denote the inclusion of \( N_\lambda \) in \( N \), we can write the class \( \zeta \in \text{Ext}^1(S, N) \) of the extension

\[
0 \to N \to M \to S \to 0
\]

as

\[
\zeta = \sum_{\lambda \in \Phi}(i_\lambda)_*(\xi_\lambda)
\]

for elements \( \xi_\lambda \in \text{Ext}^1(S, N_\lambda) \), all but finitely many zero.

Consider the summands \( N_\lambda \) which are string modules. Since equivalent words give isomorphic string modules, we may write them in the form \( M(C^\lambda) \) for words \( C^\lambda \in \Sigma \), our chosen set of representative of the equivalence classes of words. Moreover, \( M \), and hence \( N \), is finitely controlled, but the module \( M(\infty P^\infty) \) is not, so none of the \( C^\lambda \) can be equivalent to \( \infty P^\infty \). Thus by Lemma 9.6 we can write

\[
\zeta_\lambda = \sum_i \xi_{C^\lambda,i} a_{\lambda i}
\]

where \( i \) runs through the \( P \)-peaks for \( C^\lambda \) and \( a_{\lambda i} \in \text{End}(S) \).

There must be at least one string module \( N_\lambda \) with \( \zeta_\lambda \neq 0 \), for otherwise, by Lemma 9.9 \( S \) only extends band modules which are primitive torsion, so there is a primitive torsion submodule of \( M \) mapping onto \( S \), contradicting the assumption that \( \tau^1(M) \subset N \). Among all pairs \((\lambda, i)\) where \( N_\lambda \) is a string module \( M(C^\lambda) \), \( i \) is a \( P \)-peak for \( C^\lambda \) and \( a_{\lambda i} \neq 0 \), choose a pair \((\lambda, i) = (\mu, j)\) for which the \( P \)-class \((C^\lambda, i)\) is maximal.

Suppose that \( N_\lambda \) is a band module and \( \zeta_\lambda \neq 0 \). By Lemma 9.9 there is a map \( \theta_\lambda : N_\mu \to N_\lambda \) such that \((\theta_\lambda)_*(\xi_{C^\mu,j}) \neq 0 \). Then by Lemma 9.4 and Lemma 9.9(ii) there is \( \psi_\lambda \in \text{End}(N_\mu) \) such that \( \phi_\lambda = \psi_\lambda \theta_\lambda \) satisfies \((\phi_\lambda)_*(\xi_{C^\mu,j} a_{\mu j}) = \zeta_\lambda \).

Suppose that \( N_\lambda \) is a string module and \( \zeta_\lambda \neq 0 \). If \( i \) is a \( P \)-peak for \( C^\lambda \) with \((\lambda, i) \neq (\mu, j)\) and \( a_{\lambda i} \neq 0 \), then by the choice of \((\mu, j)\), by Lemma 9.8 (or trivially if \((C^\lambda, i) = (C^\mu, j)\)), there is a homomorphism \( \theta_{\lambda i} : N_\mu \to N_\lambda \) such that \((\theta_{\lambda i})_*(\xi_{C^\mu,j}) = \xi_{C^\lambda,i} \). By Lemma 9.4 there is \( \psi_{\lambda i} \) in the centre of \( \text{End}(N_\lambda) \) such that \((\psi_{\lambda i} \theta_{\lambda i})_*(\xi_{C^\mu,j} a_{\mu j}) = \xi_{C^\lambda,i} a_{\lambda i} \). We define \( \phi_\lambda : N_\mu \to N_\lambda \) by

\[
\phi_\lambda = \begin{cases} 
\sum_i \psi_{\lambda i} \theta_{\lambda i} & \text{if } \lambda \neq \mu \\
1 + \sum_i \psi_{\lambda i} \theta_{\lambda i} & \text{if } \lambda = \mu 
\end{cases}
\]
where \( i \) runs through the \( P \)-peaks for \( C^\lambda \) (with \( i \neq j \) in case \( \lambda = \mu \), so the second sum has at most one term). It follows that \((\phi_\lambda)_{\ast}(\xi_{Cw,j}a_{\mu j}) = \zeta_\lambda\). Observe that \( \phi_\mu \) is invertible since \( \psi_\mu \) is in the centre of \( \text{End}(N_\mu) \), so \((\psi_\mu \theta_\mu)^2 = \psi_\mu^2 \theta_\mu^2 = 0 \) by Lemma 9.8.

Now consider the pullback diagram

\[
\begin{array}{ccc}
\xi_{Cw,j}a_{\mu j} : & 0 & \rightarrow N_\mu \\
\downarrow & & \downarrow a_{\mu j} \\
\xi_{Cw,j} & : & 0 \rightarrow N_\mu \rightarrow N_\mu \rightarrow S \rightarrow 0.
\end{array}
\]

Since \( a_{\mu j} \) is an isomorphism, so is \( a_{\mu j} \). The map \( \phi = \sum_\lambda i_\lambda \phi_\lambda : N_\mu \rightarrow N \) satisfies \( \phi_{\ast}(\xi_{Cw,j}a_{\mu j}) = \zeta \), so there is a pushout diagram

\[
\begin{array}{ccc}
\xi_{Cw,j}a_{\mu j} : & 0 & \rightarrow N_\mu \\
\downarrow \phi & & \downarrow t \\
\xi_{Cw,j} & : & 0 \rightarrow N \rightarrow M \rightarrow S \rightarrow 0.
\end{array}
\]

Since \( \phi_\mu \) is invertible, \( \phi \) is a split monomorphism and \( N = N' \oplus \text{Im}(\phi) \). It follows that \( M = N' \oplus \text{Im}(t) \) and \( \text{Im}(t) \cong E \cong N_\mu. \) □

10. Proofs of the main results

**Proof of Theorem 1.3** We may suppose that \( Q \) is connected. By Theorem 8.3 there is a submodule \( N \) of \( M \), containing \( \tau^1(M) \), such that

\[
N = \bigoplus_{\lambda \in \Phi} N_\lambda,
\]

a direct sum of string and band modules, and with \( L = M/N \) primitive torsion.

Since \( Q \) is connected it has only countably many vertices, and since \( L \) is finitely controlled and primitive torsion, \( e_vL \) is finite-dimensional for all \( v \). It follows that we can write \( L \) as a union \( L = \bigcup L_j \) of a finite or infinite sequence of submodules

\[
0 = L_0 \subset L_1 \subset L_2 \subset \ldots
\]

with the quotients \( S_j = L_j/L_{j-1} \) being primitive simples. Let \( M_j \) be the inverse image of \( L_j \) in \( M \). Thus we have exact sequences

\[
0 \rightarrow M_{j-1} \rightarrow M_j \rightarrow S_j \rightarrow 0
\]

with \( M_0 = N \) and \( M = \bigcup M_\mu. \)

Let \( N_{\lambda,0} = N_\lambda \). By Theorem 9.2 we can write \( M_j = \bigoplus_{\lambda \in \Phi} N_{\lambda,j} \) for submodules \( N_{\lambda,j} \cong N_\lambda \) and such that \( N_{\lambda,j} = N_{\lambda,j-1} \) unless \( N_\lambda \) is isomorphic to a string module \( M(C) \) such that \( C \) has a \( P \)-peak for some primitive cycle \( P \) with \( S_j \) supported at the head of \( P \).

For any vertex \( v \), only finitely many of the simples \( S_j \) can be supported at \( v \). It follows that for each \( \lambda \) there is some \( j \) with

\[
N_{\lambda,j} = N_{\lambda,j+1} = N_{\lambda,j+2} = \ldots
\]

Defining \( N_{\lambda,\infty} = N_{\lambda,j} \), it follows easily that \( M = \bigoplus_{\lambda \in \Phi} N_{\lambda,\infty}. \) □
Proof of Theorem 1.4. (i) It is known that string modules are indecomposable. See Krause [6] for a special case and [3, §1.4] in general.

If a band module decomposes as a direct sum of two submodules, then by Corollary 8.4, each summand is a direct sum of string and band modules. Then Theorem 8.1 implies that one summand is zero.

(ii) More precisely, the claim is that the only isomorphisms between string and band modules are as follows. It follows from Theorem 8.1.

(a) \( M(C) \cong M(D) \) if and only if \( C \sim D \).

(b) \( M(C, V) \cong M(D, U) \) if and only if \( C \sim D \) and either \( U \cong V \) (if \( D = C[n] \)) or \( U \cong V^{-1} \) (if \( D = (C^{-1})[n] \)).

(c) There are no isomorphisms between string modules and band modules. (If \( C \) is a periodic \( \mathbb{Z} \)-word, then \( M(C) \cong M(C, k[T, T^{-1}]) \), but by convention a band module is one of the form \( M(C, V) \) with \( V \) finite-dimensional and indecomposable.)

(iii) Straightforward.

(iv) Follows from Lemma 4.1. □

Proof of Theorem 1.5. By Theorem 1.3 the indecomposable summands are string and band modules. The result thus follows from Theorem 8.1 and the Krull-Remak-Schmidt property for finite-dimensional \( k[T, T^{-1}] \)-modules. □

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