TOPOLOGICAL MATTER IN TWO DIMENSIONS

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ABSTRACT

Topological quantum field theories containing matter fields are constructed by twisting $N=2$ supersymmetric quantum field theories. It is shown that $N=2$ chiral (antichiral) multiplets lead to topological sigma models while $N=2$ twisted chiral (twisted antichiral) multiplets lead to Landau-Ginzburg type topological quantum field theories. In addition, topological gravity in two dimensions is formulated using a gauge principle applied to the topological algebra which results after the twisting of $N=2$ supersymmetry.
1. INTRODUCTION

Recently, there has been a rapid development of topological gravity [1-6] as well as conformal topological matter in two dimensions [7] coupled to topological gravity [8,9,10]. Since the relation between topological gravity and one-matrix models [11] was pointed out by Witten [12], many efforts have been carried out to analyze the scope of this type of relation [13,14]. Similarly, it has been shown that topological gravity coupled to some kind of conformal topological matter is related to multimatrix models [8,15]. Though the study of topological matter in two dimensions is important in its own right, this connection has stimulated the study of this type of topological quantum field theories. Different aspects of conformal topological matter have been considered in recent works [16-25]. In this paper we will study topological matter in two dimensions from a general point of view, i.e., without restricting ourselves to the conformal case.

So far, all matter coupled to topological gravity has concerned conformal topological matter resulting after twisting $N = 2$ minimal models [26,7]. These theories have $c < 3$ and therefore, after the twisting, they correspond to topological gravity coupled to topological matter with $d < 1$. In this sense, the cases considered are below the $d = 1$ barrier. Presumably, beyond $d = 1$ the topological and the ordinary phases of non-critical strings are very different, contrary to what happens for $d < 1$. However, it would be interesting to understand the behavior of topological quantum field theories for $d > 1$ because, possibly, one could gain some insight on the structure of non-critical strings in more realistic dimensions. It is easy to find topological matter which leads us beyond the $d = 1$ barrier. For example, some interesting models are the ones corresponding to topological Wess-Zumino-Witten models.

In this paper we will set up the basis of the analysis of topological matter from the point of view of twisting $N = 2$ supersymmetric theories [27,28]. We will concentrate on theories resulting from the twisting of $N = 2$ chiral (antichiral) multiplets and $N = 2$ twisted chiral (twisted antichiral) multiplets as the basic
building multiplets of $N = 2$ supersymmetry [28]. We will consider theories which contain only one type of these multiplets. In future work we will enlarge our analysis to the case in which one has a mixture of these multiplets. This will lead to some types of topological Wess-Zumino-Witten models. Of course, there are other $N = 2$ multiplets [29] which should be also analyzed in this framework and that presumably lead to a rich class of topological matter.

Three types of topological matter in two dimensions are known. The first type, the topological sigma models, were found by Witten [30]. He showed that they correspond to some kind of twisted $N = 2$ matter. In this work we will show that this type of topological matter is the result of twisting $N = 2$ chiral multiplets. All topological matter constructed out of twisting $N = 2$ chiral multiplets will be referred as type A topological matter. The second type of models, conformal topological field theories in two dimensions, were first described from the point of view of their operator product algebra by Eguchi and Yang [7]. Finally, the third type of models, also called topological Landau-Ginzburg models, were proposed by Vafa [31]. In this paper we will show that this kind of topological matter is the result of twisting twisted chiral multiplets. This topological matter will be referred as type B topological matter. Our aim will be to construct the most general class of models corresponding to each of the non-conformal types. For type A it has been shown recently [32] that the topological sigma models constructed by Witten [30] can be enlarged by adding potential terms for the case in which the target manifold has isometries. As we will observe in this paper, these potential terms are not of the usual type in $N = 2$ supersymmetric theories (F-terms). Twisting $N = 2$ supersymmetric theories may lead to actions which are not gauge invariant. This is indeed what occurs for the case of type A matter. F-terms are not allowed and the only possibility to add potential terms to a topological sigma model is to make use of the formulation of $N = 2$ supersymmetric theories given by L. Alvarez-Gaumé and D. Z. Freedman [27]. For type B topological matter, however, F-terms are allowed and we will obtain the most general form of the action corresponding to this type of topological matter. The simplest of our models, the corresponding
to a flat target space, will be identified with the one constructed by Vafa [31].

Though it is known that twisting $N = 2$ supersymmetry one obtains topological sigma models or topological Landau-Ginzburg models, it is not clear which aspect (the type of twisting or the choice of $N = 2$ supermultiplet) is responsible for their difference. In this work we will show that once the twisting procedure is fixed, different multiplets lead to different topological models. Our analysis will provide the most general action for either kind of topological matter. These results will set up the framework for mixed multiplets which must be related to a topological version of Wess-Zumino-Witten models.

Another important aspect of our work is the appearance of the topological algebra. This algebra is a twisted version of $N = 2$ supersymmetry which besides the Poincare algebra contains two basic odd operators $Q$ and $G_\mu$. Indeed, the twisting of the spin 1/2 supersymmetric charges leads to two operators of spin 0 (one of them is $Q$) and one operator of spin 1 (with two components). In the conformal case this algebra becomes the one first obtained in [7]. The topological algebra provides a gauge principle to formulate topological gravity. Let us describe how this works. Starting with an $N = 2$ supersymmetric theory in a flat two-dimensional space, one finds, after twisting, a theory whose action is certainly invariant under transformations generated by the generators of the algebra. However, if now one places the resulting theory on an arbitrary two-dimensional manifold by introducing a metric $g_{\mu\nu}$, there is no reason to expect the action to be invariant under the whole algebra. A similar situation with the $N = 2$ supersymmetric case would lead to $N = 2$ supergravity. It turns out that the resulting action is invariant under $Q$ for an arbitrary two-dimensional manifold while, in general, it is not invariant under $G_\mu$. A simple way to understand the reason behind these facts is that $Q$ is a scalar operator and therefore one does not expect complication when analyzed on a curved space. However, $G_\mu$ is a vector operator and, as in ordinary gravity, general covariance implies the introduction of additional fields. A twisted theory by itself turns out to be a topological quantum field theory. On one hand its action is $Q$-invariant. On the other hand, the topological algebra almost guarantees that
the energy-momentum tensor is $Q$-exact*. We will introduce topological gravity by requiring the resulting action to be invariant under the vector-like symmetry. This is obtained by gauging the symmetry generated by $G_\mu$, i.e., by introducing an odd gauge field, $\psi_{\mu\nu}$, which is a $Q$-partner of $g_{\mu\nu}$.

The way we obtain topological gravity by using a gauge principle applied to the topological algebra is very reminiscent of the way gravity is induced in string theory. In this way we could think of topological gravity as induced topological gravity. Certainly, in this construction all couplings to matter are automatically generated. What is left is any additional term which involves just the gravity topological multiplet. It seems that in two dimensions there is not a possible non-trivial term of this kind to be added to the action. However, similar procedures carried out in higher dimensions may lead to additional structures. In principle, any term which is invariant under the transformations obtained after the gauging is allowed. Notice that from this point of view one can think of pure topological gravity as the case in which all matter is set to zero, similarly to the case of string theory.

The paper is organized as follows. In sect. 2 we will discuss the general form of the topological algebra which results after twisting $N = 2$ supersymmetry. In sect. 3 we will carry out the twist corresponding to chiral (antichiral) multiplets deriving the form of topological sigma models. This type of topological matter will be referred as type A. In sect. 4 twisted chiral (twisted antichiral) multiplets are analyzed and Landau-Ginzburg type of models are constructed. The resulting type of topological matter will be referred as type B. In sect. 5 we will construct topological gravity using a gauge principle. Finally, in sect. 6 we state our conclusions and we discuss the different lines of investigation opened in the framework of this paper. The appendix summarizes our conventions.

* As it shown in sect. 4, for type B topological matter this holds only on-shell.
2. The Topological Algebra

In this section we will derive the general form of the algebra corresponding to a topological quantum field theory (TQFT). We will derive it by twisting $N=2$ supersymmetry but it applies to a more general set of theories than the ones which can be obtained as the result of a twist.

The algebra of $N=2$ supersymmetry in two space-time dimensions is constituted by the following relations:

\[ \begin{align*}
\{ Q_{\alpha+}, Q_{\beta-} \} &= \gamma^\mu_{\alpha\beta} P_\mu, \\
\{ Q_{\alpha+}, Q_{\beta+} \} &= \{ Q_{\alpha-}, Q_{\beta-} \} = 0, \\
[Q_{\alpha a}, P_\mu] &= [P_\mu, P_\nu] = 0, \\
[Q_{\pm a}, Q_{\pm a}] &= \pm \frac{1}{2} Q_{\pm a}, \\
[Q_{\alpha\pm}, Q_{\beta\pm}] &= \{ Q_{\alpha}, Q_{\beta} \} = 0, \\
\{ Q_{\alpha\pm}, Q_{\beta}\} &= \{ Q_{\alpha-}, Q_{\beta+} \} = 0, \\
\{ Q_{\alpha\pm}, Q_{\beta}\} &= \{ Q_{\alpha+}, Q_{\beta-} \} = 0, \\
[Q_{\alpha}, P_\mu] &= -\epsilon^\mu_{\nu} P_\nu, \\
[Q_{\alpha}, P_\mu] &= 0, \\
[Q_{\alpha}, Q_{\beta}] &= 0, \\
[J, P_\mu] &= 0, \\
[R, P_\mu] &= 0,
\end{align*} \]

where $J$ is the generator of Lorentz $SO(2)$ transformations and $R$ the generator of the internal $SO(2)$ symmetry. In (2.1) greek indices from the beginning of the alphabet denote Lorentz $SO(2)$ spinor representations, and latin indices spinor representations of the internal $SO(2)$ symmetry. Greek indices from the middle of the alphabet denote Lorentz vector representations. The epsilon symbol in this expression is taken in such a way that $\epsilon^{12} = -\epsilon^{21} = 1$, and $\gamma_\mu$ are Dirac gamma matrices. Throughout this paper we use two-dimensional Euclidean space. Our conventions are summarized in the appendix. There are $N=2$ models where $R$-symmetry is broken by potential terms. Of course, all $N=2$ superconformal models are $R$-invariant. On the other hand, as it is described below, there are $N=2$ models which are not superconformal invariant and are $R$-invariant. From (2.1) is clear that if one is able to twist in such a way that some of the supersymmetric charges becomes scalar and the other ones components of a vector one possesses a momentum operator which is $Q$-exact. Furthermore, it seems plausible that in addition the new scalar charge squares to zero.

To carry out the twist we will perform a change in the spin of the supersymmetric charges. To do this we have to redefine the Lorentz generator in such a way
that under the new one some of the supersymmetric charges behave as scalars.
There are two obvious but equivalent possibilities to carry this out. Certainly, by
adding or subtracting the generators $R$ and $J$ one finds that some of the operators
$Q_{\alpha a}$ become scalars respect to the resulting angular momentum generator. Let us
define, for example,

\[ \tilde{J} = J + R. \]  

(2.2)

Respect to the new Lorentz generator $\tilde{J}$ one finds that $Q_{+,\text{--}}$ and $Q_{-,\text{+}}$ behave as scalars while the pair $Q_{+,\text{+}}$, $Q_{-,\text{--}}$ behave as a vector. Let us make the following
definitions to make manifest the new Lorentz structure of each of the generators:

\[ Q_L = Q_{+,\text{--}}, \]
\[ Q_R = Q_{-,\text{+}}, \]
\[ Q_{+,\text{+}} = \gamma^{\mu}_{++} G_{\mu}, \]
\[ Q_{-,\text{--}} = \gamma^{\mu}_{--} G_{\mu}. \]  

(2.3)

Notice that Lorentz $SO(2)$ and internal $SO(2)$ indices are separated by a comma
when specified explicitly. Clearly, from (2.1) follows that

\[ Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0. \]  

(2.4)

Let us rewrite the algebra (2.1) in terms of the new Lorentz generator $\tilde{J}$ and the
following operators defined from (2.3),

\[ Q = Q_L + Q_R, \quad M = Q_L - Q_R. \]  

(2.5)
It turns out that the resulting algebra takes the following form,

\[ Q^2 = M^2 = \{Q, M\} = [Q, P_\mu] = [M, P_\mu] = 0, \]
\[ \{Q, G_\mu\} = P_\mu, \]
\[ [Q, \tilde{J}] = [M, \tilde{J}] = 0, \]
\[ \{M, G_\mu\} = -i\epsilon_\mu^\nu P_\nu, \]
\[ [\tilde{J}, P_\mu] = -i\epsilon_\mu^\nu P_\nu, \]
\[ [\tilde{J}, G_\mu] = -i\epsilon_\mu^\nu G_\nu, \]
\[ [P_\mu, P_\nu] = \{G_\mu, G_\nu\} = [\tilde{J}, \tilde{J}] = 0. \tag{2.6} \]

This algebra contains the ingredients of a TQFT. It certainly possesses the generators of the ordinary Poincaré group plus a nilpotent operator, \(Q\). Furthermore, the momentum operator is \(Q\)-exact. In addition, the generators \(G_\mu\) and \(M\) can be thought as a kind of odd version of the Poincaré group. Notice however that the operator \(M\) which could play the role of odd Lorentz generator rotates \(G_\mu\) into \(P_\mu\).

This is on the other hand rather natural since it would be inconsistent with the odd nature of \(G_\mu\) and \(M\) to rotate it into \(G_\mu\).

It is worth to mention that in addition one has the \(R\) generator. Its action on the operators entering in the algebra (2.6) is,

\[ [R, Q] = -\frac{1}{2} M, \]
\[ [R, M] = -\frac{1}{2} Q, \]
\[ [R, G_\mu] = -\frac{i}{2} \epsilon_\mu^\nu G_\nu, \]
\[ [R, \tilde{J}] = [R, \tilde{J}] = [R, P_\mu] = 0. \tag{2.7} \]

The algebra (2.6) together with the relations (2.7) constitute what we will call topological algebra. In the coming section we will find out that the \(R\) symmetry can be redefined so that it can be regarded as “ghost number”. Actually, it is possible to introduce this ghost number symmetry in a more general framework
using the fermion-number symmetry of the $N = 2$ supersymmetric theory. If $F$ is the generator of this symmetry, its action on the operators entering the $N = 2$ supersymmetric algebra (2.1) is,

$$
\begin{align*}
\{F, Q_{+, -}\} &= Q_{+, -}, & \{F, Q_{+, +}\} &= -Q_{+, +}, \\
\{F, Q_{-, +}\} &= Q_{-, +}, & \{F, Q_{-, -}\} &= -Q_{-, -},
\end{align*}
$$

(2.8)

while it commutes with all other operators in (2.1). Notice that the first relation in (2.1) is consistent with (2.8) since $\gamma_\mu^\mu P_\mu = 0$ (see the appendix). After the twisting one finds that,

$$
\begin{align*}
\{F, Q\} &= Q, & \{F, M\} &= M, & \{F, G_\mu\} &= -G_\mu,
\end{align*}
$$

(2.9)

while its action on all other generators in (2.6) is trivial. Clearly, the natural interpretation of these transformations in the twisted theory is the corresponding to “ghost number”. The reason being that one would think that the generator $G_\mu$ lowers the ghost number by one unit while the $Q$-generator increases it by the same amount. This is consistent with the second relation in (2.6). Notice that $F$ treats both components of $G_\mu$ in the same footing while, according to (2.7), $R$ treats differently right and left components.

Before constructing the models of topological matter let us review the general features of a topological quantum field theory from the perspective of the topological algebra (2.6). First, we will recall what is understood by a topological quantum field theory. Let us consider a quantum field theory defined on a manifold $M$ endowed with a metric $g_{\mu \nu}$. This quantum field theory is topological if there exist some correlation functions involving some of the fields of the theory, $\langle \phi_{i_1} \phi_{i_2} \ldots \phi_{i_n} \rangle$ such that:

$$
\delta \left[ \frac{\delta}{\delta g_{\mu \nu}} \langle \phi_{i_1} \phi_{i_2} \ldots \phi_{i_n} \rangle \right] = 0.
$$

(2.10)

Here the indices of the field denote certain quantum numbers as well as, possibly, space-time points, curves or surfaces. If one considers only fields $\phi_i$ which are independent of the metric $g_{\mu \nu}$, one way to ensure a property like (2.10) in a quantum
field theory is the following. Let us assume that the theory possesses a symmetry and that the fields $\phi_i$ entering into the correlation functions above are invariant under such a symmetry,

$$\delta \phi_i = 0.$$  \hfill (2.11)

Property (2.10) follows if in addition there exist a tensor $G_{\mu\nu}$ such that the energy-momentum tensor of the theory, $T_{\mu\nu}$, can be written as

$$T_{\mu\nu} = \delta G_{\mu\nu}.$$  \hfill (2.12)

To verify this notice that,

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \phi_{i_1}\phi_{i_2}\ldots\phi_{i_n} \rangle = \langle \phi_{i_1}\phi_{i_2}\ldots\phi_{i_n} T_{\mu\nu} \rangle$$
$$= \langle \phi_{i_1}\phi_{i_2}\ldots\phi_{i_n} \delta G_{\mu\nu} \rangle = \langle \delta(\phi_{i_1}\phi_{i_2}\ldots\phi_{i_n} G_{\mu\nu}) \rangle = 0,$$  \hfill (2.13)

where in the last step we have used the fact the theory is invariant under the symmetry. For example, if one possesses a lagrangian formulation, this last statement means that the action and the functional integral measure are invariant under such a symmetry. Notice that in this analysis there is no need for $\delta$ to be a nilpotent transformation as is typically the case in topological quantum field theories. In fact, we will find in sect. 3 some kind of topological matter which possesses an operator $Q$ which is not nilpotent. Its origin from the point of view of twisting $N = 2$ supersymmetry corresponds to the fact that the $N = 2$ supersymmetric model possesses central charges. In general, twisting $N = 2$ theories with central charges will provide realizations of topological quantum field theories with $Q^2 \neq 0$.

We will consider the symmetry (2.11) as the one generated by $Q$ in the topological algebra (2.6). Notice that this is consistent with (2.12). Although (2.12) is stronger than $\{Q, G_{\mu}\} = P_{\mu}$, we will find that in the models obtained after twisting $N = 2$ supersymmetry (2.12) holds at least on-shell. Condition (2.11) implies that
the physical states of the theory, i.e., the ones associated to topological invariants must satisfy,

\[ Q|\Psi\rangle = 0. \quad (2.14) \]

Two states which differ by a \( Q \)-exact state must be identified since from (2.6) one has \( Q^2 = 0 \). In other words, physical states correspond to cohomology classes of \( Q \). Once we have a state satisfying (2.14) we may use the operator \( G_\mu \) to create its partners. The simplest partner consists of

\[ \int_{\gamma_1} G_\mu |\Psi\rangle \quad (2.15) \]

where \( \gamma_1 \) is a 1-cycle. One can easily verify using (2.6) that this new state satisfies (2.14):

\[ Q \int_{\gamma_1} G_\mu |\Psi\rangle = \int_{\gamma_1} \{ Q, G_\mu \} |\Psi\rangle = \int_{\gamma_1} P_\mu |\Psi\rangle = 0. \quad (2.16) \]

Similarly, one may construct other invariants tensoring \( n \) operators \( G_\mu \) and integrating over \( n \)-cycles \( \gamma_n \):

\[ \int_{\gamma_n} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_n} |\Psi\rangle. \quad (2.17) \]

Notice that since the operator \( G_\mu \) is odd the integrand in this expression is an \( n \)-form. It is straightforward to prove that these states also satisfy condition (2.14). Therefore, starting with a state \( |\Psi\rangle \in \text{Ker}Q \) we have built a set of partners or descendants constructing a topological multiplet. The members of a multiplet have well defined “ghost” number. If one assigns ghost number \(-1\) to the operator \( G_\mu \) the state in (2.17) has ghost number \(-n\) plus the ghost number of \( |\Psi\rangle \). Of course, \( n \) is bounded by the dimension of the manifold \( \mathcal{M} \). Among the states constructed in this way there may be many which are related via another state which is \( Q \)-exact, i.e., which can be written as \( Q \) acting on some other state. Let us try to single out representatives at each level of ghost number in a given topological multiplet.
Consider an \((n - 1)\)-cycle which is the boundary of an \(n\)-dimensional surface, \(\gamma_{n-1} = \partial S_n\). If one tried to build a state taking such a cycle one would have \((P_\mu = \partial_\mu)\),

\[
\int_{\gamma_{n-1}} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_{n-1}} | \Psi \rangle = \int_{S_n} P_{[\mu_1} G_{\mu_2} G_{\mu_3} \ldots G_{\mu_n]} | \Psi \rangle = Q \int_{S_n} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_n} | \Psi \rangle,
\]

i.e., it is \(Q\)-exact. The symbols \([\ ]\) in (2.18) indicate that all indices between them must be antisymmetrized. In (2.18) use has been made of (2.6). This result tells us that the representatives we are looking for are built out of the homology cycles of the manifold \(\mathcal{M}\). Given a manifold \(\mathcal{M}\), the homology cycles are equivalence classes among cycles, the equivalence relation being that two \(n\)-cycles are equivalent if they differ by a cycle which is the boundary of an \(n+1\) surface. Thus, knowledge on the homology of the manifold on which the TQFT is defined allows us to classify the representatives among the operators (2.17). Let us assume that \(\mathcal{M}\) has dimension \(d\) and that its homology cycles are \(\gamma_{i_n}, \ i_n = 1, \ldots, d_n, \ n = 1, \ldots, d\), being \(d_n\) the dimension of the \(n\)-homology group. Then, the non-trivial partners or descendants of a given \(|\Psi\rangle\) “highest-ghost-number” state are labeled in the following way:

\[
\int_{\gamma_{i_n}} G_{\mu_1} G_{\mu_2} \ldots G_{\mu_n} | \Psi \rangle, \quad i_n = 1, \ldots, d_n, \quad n = 1, \ldots, d.
\]

(2.19)

A similar construction to the one just described can be made for fields. Starting with a field \(\phi(x)\) which satisfies,

\[
[Q, \phi(x)] = 0
\]

(2.20)

one can construct other fields using the operators \(G_\mu\). These fields, which we will call partners or descendants are antisymmetric tensors defined as,

\[
\phi^{(n)}_{\mu_1 \mu_2 \ldots \mu_n} (x) = \frac{1}{n!} [G_{\mu_1}, [G_{\mu_2}, [G_{\mu_3}, \ldots [G_{\mu_n}, \phi(x)] \ldots]]], \quad n = 1, \ldots, d.
\]

(2.21)

Using (2.6) and (2.20) one finds that these fields satisfy the so-called “topological
descent equations":

\[ d\phi^{(n)} = [Q, \phi^{(n+1)}] \]  

(2.22)

where the subindices of the forms has been suppressed for simplicity, and the highest-ghost-number field \( \phi(x) \) has been denoted as \( \phi^{(0)}(x) \). These equations enclose all the relevant properties of the observables which are constructed out of them. As we will see in subsequent sections these equations are very useful to build the observables of the theory. Let us consider an \( n \)-cycle and the following quantity:

\[ W_{\phi}^{(\gamma_n)} = \int_{\gamma_n} \phi^{(n)}. \]  

(2.23)

The subindex of this quantity denotes the highest-ghost-number state out of which the form \( \phi^{(n)} \) is generated. The superindex denotes the order of such a form as well as the cycle which is utilized in the integration. Using the topological descent equations (2.22) it is immediate to prove that \( W_{\phi}^{(\gamma_n)} \) is indeed an observable

\[ [Q, W_{\phi}^{(\gamma_n)}] = \int_{\gamma_n} [Q, \phi^{(n)}] = \int_{\gamma_n} d\phi^{(n-1)} = 0. \]  

(2.24)

Furthermore, if \( \gamma_n \) is a trivial homology cycle, \( \gamma_n = \partial S_{n+1} \), one obtains that \( W_{\phi}^{(\gamma_n)} \) is \( Q \)-exact,

\[ W_{\phi}^{(\gamma_n)} = \int_{\gamma_n} \phi^{(n)} = \int_{S_{n+1}} d\phi^{(n)} = \int_{S_{n+1}} [Q, \phi^{(n+1)}] = [Q, \int_{S_{n+1}} \phi^{(n+1)}], \]  

(2.25)

and therefore its vacuum expectation value vanishes. Thus, similarly to the previous analysis leading to (2.19) the observables of the theory are operators of the form (2.23):

\[ W_{\phi}^{(\gamma_n, i_n)} \quad i_n = 1, \ldots, d_n, \quad n = 1, \ldots, d, \]  

(2.26)

where, as before, \( d_n \) denote the dimension of the \( n \)-homology group. Of course, these observables are a basis of observables but one can consider arbitrary products of them leading to new ones.
The main goal of this paper is to apply the twisting procedure just described to some $N = 2$ supersymmetric matter. We will analyze the two basic $N = 2$ supersymmetric multiplets, the chiral (antichiral) multiplet, and the twisted (not to be confused with the twisting just explained regarding the construction of TQFT out of $N = 2$ supersymmetric theories) chiral (twisted antichiral) multiplet. These multiplets are better described if one starts with its definition in $N = 2$ superspace. Let us therefore consider $N = 2$ superspace on which one has superspace covariant derivatives $D_{\alpha a}$ satisfying the following algebra:

\[
\{D_{+,+}, D_{+,+}\} = 2\partial_z, \\
\{D_{-,+}, D_{-,+}\} = 2\partial_{\bar{z}},
\]  

while all other anticommutators among the $D_{\alpha a}$ vanish.

The two basic $N = 2$ multiplets are described by a scalar $N = 2$ superfield $\Phi$ satisfying the following relations:

\[
D_{+,+}\Phi = D_{+,+}\Phi = 0, \quad \text{chiral}, \\
D_{+,+}\Phi = D_{+,+}\Phi = 0, \quad \text{twisted chiral}.
\]  

(2.28)

Of course, there exist also the antichiral and the twisted antichiral versions of these multiplets,

\[
D_{+,+}\hat{\Phi} = D_{+,+}\hat{\Phi} = 0, \quad \text{antichiral}, \\
D_{+,+}\hat{\Phi} = D_{+,+}\hat{\Phi} = 0, \quad \text{twisted antichiral}.
\]  

(2.29)

The two kinds of multiplets that we are going to treat lead, after twisting, to two different types of TQFT. They have different content and they allow different potential terms. We will refer to them as type A and type B topological matter. They will be described in the following sections.
3. Type A Topological Matter

In this section we will carry out the construction of theories involving topological matter of type A. We will describe this construction in full detail for this type of topological matter. The procedure concerning other types of topological matter is similar. In the next section we will concentrate mainly on the results concerning type B topological matter.

Let us consider a collection of chiral superfields $X^I$ and the corresponding set of antichiral superfields $\bar{X}^{\bar{I}}$, $(I, \bar{I} = 1, \ldots, d)$,

$$D_{\alpha,-}X^I = 0, \quad D_{\alpha,+}X^I = 0. \quad (3.1)$$

We will consider actions in $N = 2$ superspace that involve these sets of fields containing a non-chiral kinetic term (sometimes called D-term) and a chiral superpotential term (sometimes called F-term). The action takes the form,

$$S = \int d^2 z d^4 \theta K(X^I, X^{\bar{I}}) + \int d^2 \bar{z} (d^2 \theta W(X^I) + \hat{d}^2 \bar{\theta} \bar{W}(X^{\bar{I}})), \quad (3.2)$$

The quantities $X^I, \bar{X}^{\bar{I}}$ can be thought of as coordinates of a $2d$-dimensional Kahler manifold $M$ where $K$ is the Kahler potential. In this sense, the superpotential $W(X^I)$ ($\bar{W}(X^{\bar{I}})$) can be thought of as a holomorphic (antiholomorphic) scalar function on such a manifold.

The odd parts of the measure in both terms of the action are such that when projecting into components,

$$d^4 \theta \to D_{-,+}D_{+,+}D_{-,+}D_{+,-},$$
$$d^2 \theta \to D_{+,+}D_{-,+},$$
$$\hat{d}^2 \bar{\theta} \to D_{+,-}D_{-,+}. \quad (3.3)$$

This indicates that when twisting to obtain the corresponding TQFT we cannot allow the superpotential term since the measure is not Lorentz invariant respect
to the new Lorentz generator $\tilde{J}$. Recall from (2.2) that $\tilde{J}$ is such that while $D_{-,+}$ behaves as a scalar, $D_{++}$ behaves as the component of a vector. Therefore, we may have only the D-term for the case of type A topological matter. As we will discuss in sect. 4, for the case of type B topological matter this kind of term is allowed and one is able to build a TQFT containing potential terms. This is not however the end of the story regarding potential terms for type A topological matter. It is well known that there are some $N = 2$ supersymmetric matter models which contain potential terms and are such that they can not be derived from an $N = 2$ superspace formulation [27]. One may wonder if starting from those models one can construct additional TQFT. In particular, if those models are able to provide potential terms for type A topological matter. In [32] we have answered this question. It turns out that indeed, those models provide potential terms for type A topological matter but do not add any additional structure for the case of type B. Notice also that the F-term in (3.2) breaks $R$-symmetry. However, since we are not allowed to have that term in the topological model, $R$-invariance will be present. As we will discuss below, both type A and type B topological matter possess $R$-symmetry. $R$-invariance must be a feature of all TQFT which are constructed by twisting a Lorentz invariant supersymmetric theory since $R = \tilde{J} - J$ and in that case both Lorentz symmetries are preserved. At the end of this section we will make a brief summary of the results obtained in [32] for potential terms in type A topological matter. For the moment, however, we will restrict ourselves to consider the model (3.2) without F-term.

Let us define component fields in the following way,

$$
X^I| = x^I, \quad X^I| = x^I,
$$

$$
D_{+,+}X^I| = \psi^I_{+,+}, \quad D_{+,+}X^I| = \psi^I_{+,+},
$$

$$
D_{-,+}X^I| = \psi^I_{-,+}, \quad D_{-,+}X^I| = \psi^I_{-,+},
$$

$$
D_{-,+}D_{++,}X^I| = F^I_{++,}, \quad D_{-,+}D_{++,}X^I| = F^I_{++,},
$$

$$
D_{-,+}D_{++,}X^I| = F^I_{++,}, \quad D_{+,+}D_{-,-}X^I| = F^I_{++,}.
$$

(3.4)

Before proceeding with the twisting to obtain the corresponding TQFT let us work out the supersymmetry transformations of the component fields. This will
be very useful because it will allow us to determine the transformations under the symmetries of the topological algebra (2.6) and (2.7). The supersymmetry transformations are easily obtained taking into account that for an $N = 2$ superfield such a transformation takes the form,

$$\delta \Phi = \eta^{\alpha\alpha} Q_{\alpha\alpha} \Phi,$$

where $\eta^{\alpha\alpha}$ is a constant $N = 2$ supersymmetry parameter. Taking components in this transformation law and using the definitions (3.4) one finds,

$$\begin{align*}
\delta x^I &= \eta^{+,+} \psi^I_{+,+} + \eta^{-,+} \psi^I_{-,+}, \\
\delta \psi^I_{+,+} &= \eta^{-,+} F^I_{-,+,+} + 2 \eta^{-,-} \partial_z x^I, \\
\delta \psi^I_{-,+} &= - \eta^{+,+} F^I_{+,+,+} + 2 \eta^{-,-} \partial_z x^I, \\
\delta F^I_{-,+,+} &= 2 \eta^{-,-} \partial_z \psi^I_{+,-} - 2 \eta^{+,+} \partial_z \psi^I_{-,+}, \\
\delta \psi^I_{+,-} &= \eta^{+,+} \psi^I_{+,-} + \eta^{+,+} \psi^I_{-,-}, \\
\delta \psi^I_{+,-} &= - \eta^{-,-} F^I_{+,+,+} + 2 \eta^{+,-} \partial_z x^I, \\
\delta F^I_{+,+,+} &= 2 \eta^{-,-} \partial_z \psi^I_{+,+} - 2 \eta^{+,+} \partial_z \psi^I_{+,+}.
\end{align*}$$

The transformations under the $R$-symmetry in (2.1) are obvious from the $SO(2)$ indices carried out by the fields.

After this detailed description of the $N = 2$ supersymmetric model we are ready to carry out the twist. Under the new Lorentz generator $\tilde{J}$ in (2.2) the Lorentz structure of each of the fields is simple to find out since one just have to think of the indices as all belonging to the same $SO(2)$. To make manifest the new Lorentz structure we will make the following definitions:

$$\begin{align*}
\chi^I &= \psi^I_{-,+}, & \rho^I_z &= \psi^I_{+,+}, & F^I_z &= F^I_{-,+,+}, \\
\chi^I &= \psi^I_{+,+}, & \rho^I_z &= \psi^I_{-,+}, & F^I_z &= F^I_{+,+,+}.
\end{align*}$$

After these definitions the $R$-transformations of the fields are not manifest anymore.
Let us summarize them here:

\[
\begin{align*}
[R, x^I] &= 0, & [R, x^{ar{I}}] &= 0, \\
[R, \chi^I] &= \frac{1}{2} \chi^I, & [R, \chi^{ar{I}}] &= -\frac{1}{2} \chi^{ar{I}}, \\
[R, \rho^I_z] &= \frac{1}{2} \rho^I_z, & [R, \rho^{ar{I}}_z] &= -\frac{1}{2} \rho^{ar{I}}_z, \\
[R, F^I_z] &= F^I_z, & [R, F^{ar{I}}_z] &= -F^{ar{I}}_z.
\end{align*}
\]  

(3.8)

Ghost numbers for the fields (3.7) are easily obtained from the projections (3.4) and the fact that the ghost numbers of the superspace covariant derivatives are the same as the ones of the corresponding supersymmetry generators (2.8). Assuming that the superfields \(X^I, X^{ar{I}}\) have ghost number 0, one finds,

\[
\begin{align*}
[F, x^I] &= 0, & [F, x^{ar{I}}] &= 0, \\
[F, \chi^I] &= \chi^I, & [F, \chi^{ar{I}}] &= \chi^{ar{I}}, \\
[F, \rho^I_z] &= -\rho^I_z, & [F, \rho^{ar{I}}_z] &= -\rho^{ar{I}}_z, \\
[F, F^I_z] &= 0, & [F, F^{ar{I}}_z] &= 0.
\end{align*}
\]  

(3.9)

The transformations of the fields under the generators \(Q, M\) and \(G_{\mu}\) of the topological algebra (2.6) follow from (3.6):

\[
\begin{align*}
[Q, x^I] &= \chi^I, & [Q, x^{ar{I}}] &= \chi^{ar{I}}, \\
[M, x^I] &= -\chi^I, & [M, x^{ar{I}}] &= -\chi^{ar{I}}, \\
[G_z, x^I] &= \frac{1}{2} \rho^I_z, & [G_z, x^{ar{I}}] &= 0, \\
[G_{\bar{z}}, x^I] &= 0, & [G_{\bar{z}}, x^{ar{I}}] &= \frac{1}{2} \rho^{ar{I}}_z.
\end{align*}
\]  

(3.10)

\[
\begin{align*}
\{Q, \chi^I\} &= 0, & \{Q, \chi^{ar{I}}\} &= 0, \\
\{M, \chi^I\} &= 0, & \{M, \chi^{ar{I}}\} &= 0, \\
\{G_z, \chi^I\} &= -\frac{1}{2} F^I_z, & \{G_z, \chi^{ar{I}}\} &= \partial_z x^{ar{I}}, \\
\{G_{\bar{z}}, \chi^I\} &= \partial_{\bar{z}} x^I, & \{G_{\bar{z}}, \chi^{ar{I}}\} &= \frac{1}{2} F^{ar{I}}_z.
\end{align*}
\]  

(3.11)
\{ Q, \rho^I \} = 2\partial_z x^I + F^I, \quad \{ Q, \bar{\rho}^I \} = 2\partial_{\bar{z}} x^I + F^I, \\
\{ M, \rho^I \} = 2\partial_z x^I - F^I, \quad \{ M, \bar{\rho}^I \} = -2\partial_{\bar{z}} x^I + F^I, \\
\{ G_z, \rho^I \} = 0, \quad \{ G_z, \bar{\rho}^I \} = 0, \\
\{ G_{\bar{z}}, \rho^I \} = 0, \quad \{ G_{\bar{z}}, \bar{\rho}^I \} = 0, \\
\{ Q, F^I \} = -2\partial_z x^I, \quad \{ Q, F^I \} = -2\partial_{\bar{z}} x^I, \\
\{ M, F^I \} = -2\partial_z x^I, \quad \{ M, F^I \} = 2\partial_{\bar{z}} x^I, \\
\{ G_z, F^I \} = 0, \quad \{ G_{\bar{z}}, F^I \} = \partial_z \bar{\rho}^I, \\
\{ G_{\bar{z}}, F^I \} = \partial_{\bar{z}} \rho^I, \quad \{ G_{\bar{z}}, F^I \} = 0. 
\tag{3.12}

Let us write the action corresponding to type A topological matter in terms of the fields defined in (3.7),

\[ S = \int d^2z \left[ G_{IJ} \left( -F^I F^J - 2\rho^I D_z \chi^J - 2\bar{\rho}^I D_{\bar{z}} \chi^J + 4\partial_z x^I \partial_{\bar{z}} x^J \right) \right. \\
\left. + \partial_K \partial_{KL} G_{IJ} \rho^K \rho^I \chi^J - \partial_K G_{IJ} \chi^I F^J_{\bar{z}} \rho^K - \partial_K G_{IJ} \chi^I \bar{F}^J_{\bar{z}} \rho^K \right], \tag{3.14}
\]

where \( G_{IJ} \) is the metric of the Kahler manifold \( M \),

\[ G_{IJ} = \frac{\partial^2 K}{\partial x^I \partial x^J}, \tag{3.15} \]

and \( D_\mu \) represents a covariant derivative on sections of the pull-back of the tangent bundle,

\[ D_\mu \chi^I = \partial_\mu \chi^I + (\partial_\mu x^J) \Gamma^I_{JK} \chi^K, \tag{3.16} \]

being \( \Gamma^I_{JK} \) the Christoffel connection defined in (A22). Of course, the fields \( \rho^I_\mu, \rho^I_\bar{\mu}, \]
\( F^I_\mu \) and \( F^I_\bar{\mu} \) entering (3.14) satisfy the selfduality and anti-selfduality conditions,

\[ \rho^I_\mu = -i\epsilon^\mu_\nu \rho^I_\nu, \quad \rho^I_\bar{\mu} = i\epsilon^\mu_\nu \rho^I_\nu, \]
\[ F^I_\mu = -i\epsilon^\mu_\nu F^I_\nu, \quad F^I_\bar{\mu} = i\epsilon^\mu_\nu F^I_\nu, \tag{3.17} \]

where \( \epsilon^\mu_\nu \) is such that \( \epsilon^z_z = -\epsilon^{\bar{z}}_{\bar{z}} = i. \)
The action (3.14) possesses non-covariant looking terms which can be reorganized into covariant looking ones performing a redefinition of the auxiliary fields $F^I_\mu$ and $\tilde{F}^I_\mu$ in such a way that all dependence on them becomes gaussian. This is indeed the first step to carry out when integrating out those fields. Let us define:

\[
\begin{align*}
F^I_z &= F^I_z + \Gamma^I_{JK} \chi^J \rho^K_z, \\
\tilde{F}^I_z &= \tilde{F}^I_z + \Gamma^I_{JK} \chi^J \rho^K_z.
\end{align*}
\] (3.18)

The action (3.14) becomes,

\[
S = \int d^2z \left[ G_{IJ} \left( 4\partial_z x^I \partial_{\bar{z}} x^J - 2\rho_z^I D_z \chi^J - 2\rho_z^J D_z \chi^I - \tilde{F}_z^I \tilde{F}_z^J \right) + R_{IJKL} \rho_z^I \rho_z^J \chi^K \chi^L \right].
\] (3.19)

Notice the presence of a quartic term involving the curvature of the Kahler manifold. The redefinition of auxiliary fields modifies the symmetry transformations since they must be written in terms of the new fields. Let us write down, for example, the form of the $Q$-transformations,

\[
\begin{align*}
[Q, x^I] &= \chi^I, \\
\{Q, \chi^I\} &= 0, \\
\{Q, \rho^I_z\} &= \tilde{F}_z^I + 2\partial_z x^I - \Gamma^I_{JK} \chi^J \rho^K_z, \\
[Q, \tilde{F}_z^I] &= -2D_z \chi^I - \Gamma^I_{JK} \chi^J \tilde{F}_z^K - R^I_{JKL} \chi^K \chi^L \chi^J \rho^K_z, \\
\{Q, \chi^I\} &= 0, \\
\{Q, \rho^I_{\bar{z}}\} &= \tilde{F}^I_{\bar{z}} + 2\partial_{\bar{z}} x^I - \Gamma^I_{JK} \chi^J \rho^K_{\bar{z}}, \\
[Q, \tilde{F}^I_{\bar{z}}] &= -2D_{\bar{z}} \chi^I - \Gamma^I_{JK} \chi^J \tilde{F}^K_{\bar{z}} + R^I_{JKL} \chi^K \chi^J \rho^K_{\bar{z}}.
\end{align*}
\] (3.20)

Certainly, by construction, these transformations are such that $Q^2 = 0$. Notice that according to (3.8) and the definition (3.18) the $R$-transformations of the new auxiliary fields take the same form as the old ones.
So far we have considered the theory on a flat two-dimensional space. To analyze its topological character we must now place the theory on an arbitrary curved two-dimensional manifold $\Sigma$ and verify that it is still $Q$-invariant and that its energy-momentum tensor is $Q$-exact. On an arbitrary two-dimensional manifold endowed with a metric $g_{\mu\nu}$ the action (3.19) takes the form:

$$S = \int_\Sigma d^2z \sqrt{g} \left[ G_{IJ} \left( g^{\mu\nu} \partial_\mu x^I \partial_\nu x^J + \frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \partial_\mu x^I \partial_\nu x^J - g^{\mu\nu} \rho_\mu^I D_\nu \chi^J \right) 
- g^{\mu\nu} \rho_\mu^I D_\nu \chi^I - \frac{1}{2} g^{\mu\nu} \tilde{F}_\mu^I \tilde{F}_\nu^J + \frac{1}{2} g^{\mu\nu} R_{IKL} \rho_\mu^I \rho_\nu^J \chi^K \chi^L \right]. \tag{3.21}$$

One can indeed verify that this action is invariant under the covariantized form of the $Q$-transformations (3.20),

$$[Q, x^I] = \chi^I,$n
$$\{Q, \chi^I\} = 0,$n
$$\{Q, \rho_\mu^I\} = \tilde{F}_\mu^I + (\delta^\nu_\mu - \frac{i\epsilon^{\mu\nu}}{\sqrt{g}}) \partial_\nu x^I - \Gamma^K_{JK} \chi^K \rho_\mu^J,$n
$$[Q, \tilde{F}_\mu^I] = -\left( \delta^\nu_\mu - \frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \right) D_\nu \chi^I - \Gamma^K_{JK} \chi^K \tilde{F}_\mu^J - R^I_{IKL} \rho_\mu^I \chi^L,$n
$$[Q, \chi^I] = \chi^I,$n
$$\{Q, \chi^I\} = 0,$n
$$\{Q, \rho_\mu^I\} = \tilde{F}_\mu^I + (\delta^\nu_\mu + \frac{i\epsilon^{\mu\nu}}{\sqrt{g}}) \partial_\nu x^I - \Gamma^\mu_{JK} \chi^J \rho_\mu^K,$n
$$[Q, \tilde{F}_\mu^I] = -\left( \delta^\nu_\mu + \frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \right) D_\nu \chi^I - \Gamma^\mu_{JK} \chi^J \tilde{F}_\mu^K + R^I_{IKL} \rho_\mu^I \chi^L \rho^K_\mu.$n

Notice that the selfduality and anti-selfduality conditions (3.17) now take the form,

$$\rho_\mu^I = -\frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \rho_\nu^I, \quad \rho^K_\mu = \frac{i\epsilon^{\mu\nu}}{\sqrt{g}} \rho^K_\nu,$n
$$F_\mu^I = -\frac{i\epsilon^{\mu\nu}}{\sqrt{g}} F_\nu^I, \quad F^K_\mu = \frac{i\epsilon^{\mu\nu}}{\sqrt{g}} F^K_\nu. \tag{3.23}$$

The crucial test for the model that we have constructed being to polynomial is the verification of the $Q$-invariance of the action. It turns out that (3.21) is indeed
invariant under the symmetry transformations generated by $Q$ and $M$ but it is not invariant under the one generated by $G_\mu$. The invariance under $Q$ plus the fact that the energy-momentum tensor can be written as a $Q$-transformation, guarantees that the model just constructed is a TQFT. Certainly, besides reparametrizations, the other symmetries of the theory when considered on curved space are the $R$ and $M$ symmetries. The $R$-transformations are the ones given in (3.8). The transformations under $M$ follow directly from the relation in (2.7), $M = -2[R, Q]$. Notice that $Q \pm M$ generate independent nilpotent symmetries for holomorphic and antiholomorphic modes. Recall that according to (2.5) they correspond to $Q_L$ and $Q_R$, i.e., $Q_L = \frac{1}{2}(Q + M)$ and $Q_R = \frac{1}{2}(Q - M)$.

The construction that we have carried out guarantees that the action of the theory is $Q$-exact. In order to have a TQFT we must check if a stronger condition holds, namely, we must verify that the energy-momentum tensor is $Q$-exact. As we will prove now we have a much stronger result for these models of type A topological matter. It turns out that the action itself is $Q$-exact. It is simple to demonstrate that,

$$S = \left\{ Q, \frac{1}{2} \int d^2 z \sqrt{g} g^{\mu \nu} G_{IJ}\left[ \frac{1}{2} \rho^I_{\mu} \bar{F}^J_{\nu} + \frac{1}{2} \rho^J_{\mu} \bar{F}^I_{\nu} - (\rho^I_{\mu} \partial_{\nu} x^J + \rho^J_{\mu} \partial_{\nu} x^I) \right] \right\}, \quad (3.24)$$

and therefore the theory is certainly topological since this implies that the energy-momentum is also $Q$-exact.

Once we have succeeded in the formulation of this topological quantum field theory by twisting $N = 2$ supersymmetry we may ask if it can be generalized. All along our discussion the target space manifold $M$ was required to be Kahler. Indeed, it is well known that $N = 2$ supersymmetry requires a Kahler manifold. However, after the twisting, one may ask if this condition can be relaxed. Certainly, a field theory realization of the part of the topological algebra (2.6) which does not involve $G_\mu$ does not impose very restrictive conditions. The existence of a nilpotent operator $Q$ is much weaker than the realization of the supersymmetry algebra in
which the anticommutator of two supersymmetric charges must correspond to a vector operator. Thus in what regards to the $Q$ symmetry one could expect a more general framework. The realization of the rest of the algebra (2.6) is only important when coupling this type of matter to topological gravity. We will discuss that issue in sect. 5.

Topological sigma models as formulated by Witten [30] do indeed exist for almost Hermitian manifolds. Let us build its formulation from the one we have obtained for Kahler manifolds. Since we are going to relax the Kahler condition on $M$ we should first rewrite the theory changing the holomorphic (antiholomorphic) notation in target space indices by introducing a complex structure $J^{ij}, i,j = 1,...,2d$. For the case of a Kahler manifold this complex structure can be written locally as: $J^I_J = -i\delta^I_j, I,J = 1,...,d; J^I_J = i\delta^I_j, I,J = d+1,...,2d; J^I_I = J^{I_I} = 0, I = d+1,...,2d, J = 1,...,d$. Notice that $J^{i_k}J^{k_j} = -\delta^i_j$. Thus, the action (3.21) takes the form,

$$S = \int d^2z \sqrt{g} \left[ \frac{1}{2} G^{i\mu} g^{\mu\nu} \partial_\mu x^i \partial_\nu x^j + \frac{1}{2} \varepsilon^{i\mu\nu} J_{ij}^\rho \partial_\mu \chi^\rho \right] - \frac{1}{4} G^{i\mu} g^{\mu\nu} \tilde{F}_\mu^i \tilde{F}_\nu^j + \frac{1}{8} g^{\mu\nu} R_{ijklm} \rho^i_\mu \rho^j_\nu \chi^k \chi^m \right],$$  

(3.25)

and the selfduality and anti-selfduality conditions (3.23) become,

$$\rho^i_\mu = \varepsilon^{i\nu} J^j_\nu \rho^j_\mu, \quad \tilde{F}^i_\mu = \varepsilon^{i\nu} J^j_\nu \tilde{F}^j_\nu.$$

(3.26)

In (3.25) and (3.26) we have introduced the tensor,

$$\varepsilon^{i\mu\nu} = \frac{\epsilon^{i\mu\nu}}{\sqrt{g}}.$$

(3.27)

So far we have only rewritten the action of type A topological matter in compact notation with the help of a covariantly constant structure. Let us now release
this condition. Assume that the only requirements satisfied by \( J^i_j \) are

\[
J^j_k J^k_j = -\delta^j_j, \quad J^j_k J^j_m G_{ij} = G_{km},
\]

(3.28)

i.e., \( J^i_j \) is an almost-Hermitian structure. Notice that in a Kahler manifold in addition to (3.28) the condition \( D_k J^i_j = 0 \) is satisfied. Witten showed that the \( Q \) transformations of the theory can be generalized from the ones in (3.22) in such a way that the action (3.25) is \( Q \)-invariant. Furthermore, this generalization is such that the nilpotency of \( Q \) holds. Actually, it is rather simple to obtain the form of these \( Q \)-transformation from the ones given in (3.22). First, let us rewrite (3.22) in a compact form,

\[
\begin{align*}
\{Q, x^i\} &= \chi^i, \\
\{Q, \chi^i\} &= 0, \\
\{Q, \rho^i_\mu\} &= \bar{F}^i_\mu + (\delta^i_\mu \delta^j_j + \varepsilon^i_\mu \nu J^i_j) \partial_\nu x^j - \Gamma^i_{jk} \chi^j \rho^k_\mu, \\
\{Q, \bar{F}^i_\mu\} &= - (\delta^i_\mu \delta^j_j + \varepsilon^i_\mu \nu J^i_j) D_\nu \chi^j - \Gamma^i_{jk} \chi^j \bar{F}^k_\mu + \frac{1}{2} R^i_{kjm} \chi^k \chi^j \rho^m_\mu.
\end{align*}
\]

(3.29)

Certainly, the action (3.25) is invariant under these transformations when the manifold is Kahler. In addition, for such a case, the transformations are nilpotent. However, if the condition \( D_k J^i_j = 0 \) is released none of the facts holds. One must modify the transformations to achieve invariance and nilpotency. The procedure to carry this out is simple. First, we will redefine the transformation in such a way that \( Q \) is nilpotent. Then, since the action was \( Q \)-exact for the Kahler case, one may just take the compact version of (3.24) with the new form of \( Q \). To redefine the transformations (3.29), notice that the first two does not have to be modified. The \( Q \)-transformation of \( \rho^i_\mu \), however, has to be modified in such a way that the selfduality condition (2.11) is maintained under the \( Q \)-transformation. It turns out that this is simple to achieve by just adding a term of the form \( \varepsilon^i_\mu \nu \chi^k \rho^j_\sigma D_k J^j_j \). Once this is obtained one fixes the transformation of \( \bar{F}^i_\mu \) in such a way that \( Q \) on
\( \rho^i_\mu \) is nilpotent. This leads to the following new set of transformations:

\[
[Q, x^i] = \chi^i, \\
\{Q, \chi^i\} = 0, \\
\{Q, \rho^i_\mu\} = \tilde{F}^i_\mu + \partial_\mu x^i + \varepsilon_\mu^\nu J^i_j \partial_\nu x^j - \Gamma^i_{jk} \chi^j \rho^k_\mu + \frac{1}{2} \varepsilon_\mu^\nu \chi^k \rho^j_\nu D_k J^i_j, \\
[Q, \tilde{F}^i_\mu] = -D_\mu \chi^i - \varepsilon_\mu^\nu J^i_j D_\nu \chi^j - \Gamma^i_{jk} \chi^j \tilde{F}^k_\mu + \frac{1}{2} R_{mj}^i k \chi^m \chi^j \rho^k_\mu \\
- \frac{1}{2} \varepsilon_\mu^\nu \chi^m k \rho^j_\nu D_k \tilde{F}^i_j j - \frac{1}{2} \varepsilon_\mu^\nu (D_k J^i_j) \chi^k (\partial_\nu x^j - \varepsilon_\nu^\gamma J^j_m \partial_\gamma x^m) \\
- \frac{1}{4} \chi^k \chi^m \rho^i_m (D_k J^i_j) (D_m J^j_n) + \frac{1}{2} \varepsilon_\mu^\nu \chi^k \tilde{F}^i_\mu D_k J^i_j. \\
\tag{3.30}
\]

The crucial test for the validity of the construction resides on the fact that \( Q \) is also nilpotent on \( \tilde{F}^i_\mu \). It is not obvious from (3.30) that this is going to hold but an explicit computation shows that, remarkably, it is true. This fact was discovered by Witten in [30].

We have shown that the form of type A topological matter obtained after twisting \( N = 2 \) chiral multiplets can be generalized to the case of almost Hermitian manifolds. The analysis of observables of this theory was carried out in [30]. We will not discuss it here since our aim was to find out which kind of supersymmetric matter leads to Witten’s topological sigma models. This analysis will provide the adequate framework to carry out the coupling of this type of matter to topological gravity. This will be discussed in sect. 5.

So far we have not been able to introduce potential terms for type A topological matter. As we discussed at the beginning of this section, F-terms are not allowed for the chiral multiplet since they lead to actions which are not Lorentz invariant. There exists, however, some \( N = 2 \) supersymmetric models which can not be written in the form (3.2) and contain potential terms [27]. The twisting of these models has been recently carried out [32] and it turns out that they provide potential terms for type A topological matter for the case in which the target manifold \( M \) possesses some isometries. In particular, the potential terms involve
the Killing vectors associated to those isometries. Let us briefly review the results obtained in [32]. The analysis presented in this paper clarifies the origin of eq. (2) in [32], which might have seemed somehow obscure in that context. The action presented there is just the twisted version of the general $N = 2$ supersymmetric action obtained in [27] with F-terms set to zero. It turns out that regarding the $N = 2$ supersymmetric multiplet as a chiral multiplet, F-terms lead to non-Lorentz invariant expressions while terms involving Killing vectors are permitted. The opposite occurs if one regards the $N = 2$ supersymmetric multiplet as twisted chiral. In components fields, as it is the case of the construction given in [27], one may have one or the other multiplet depending on the $R$-charges which are assigned to each of the members of the multiplet. In this construction, nothing guarantees that the action that one obtains after the twisting can be generalized to the almost Hermitian case since the construction in [27] assumes a Kahler manifold with some isometries. As shown in [32] it turns out that, indeed, it can be generalized to that case proceeding in a similar way as in the case just described without potential terms. Actually, in the process one finds a small surprise. The $N = 2$ model under consideration in [27] possesses central charges. This implies that $Q$ is not nilpotent any more. As shown in the previous section, there is no need for $Q$ to be nilpotent to have a topological quantum field theory. The only requirement is a $Q$ invariant action and an energy-momentum tensor which is $Q$-exact. The model presented in [32] is such that $Q^2$ is not zero but just a Lie derivative respect to some Killing vector fields. We will summarize here the results presented in [32].

Let us consider an almost Hermitian manifold which possess two Killing vector fields $V_i$ and $U_i$ which satisfy

\[
D_i V_j + D_j V_i = 0, \quad V^k \partial_k J^i_n + J^i_k \partial_n V^k - J^k_n \partial_k V^i = 0, \quad U^k \partial_k J^i_n + J^i_k \partial_n U^k - J^k_n \partial_k U^i = 0, \quad (3.31)
\]

These conditions are rather natural. They represent the requirement that the metric and complex structure remain invariant under a variation along the Killing
vector fields. The most general action for type A topological matter takes the form:

\[
S = \left\{ Q, \int \sqrt{g} \left[ \frac{1}{2} g^{\mu \nu} G_{ij} \rho_\mu^i (\partial_\nu x^j - \frac{1}{2} \mathcal{F}_\nu^j) + \lambda^2 G_{ij} (V^i + U^i) \chi^j \right] \right\}
\]

\[
= \int \sqrt{g} \left\{ \frac{1}{2} G_{ij} g^{\mu \nu} \partial_\mu x^i \partial_\nu x^j + \frac{1}{2} \varepsilon^{\mu \nu} J_{ij} \partial_\mu x^i \partial_\nu x^j 
- g^{\mu \nu} G_{ij} \rho_\mu^i (D_\nu \chi^j + \frac{1}{2} (D_k J^j_i) \chi^k \varepsilon_\nu^\sigma \partial_\sigma x^i)
- \frac{1}{4} g^{\mu \nu} (G_{ij} \tilde{F}_\mu^{ij} - \frac{1}{2} R_{ijk\rho} \rho_\mu^i \chi^j \chi^m + \frac{1}{4} (D_k J^j_i) (D_m J^j_j) \rho_\mu^i \rho_\nu^j \chi^k \chi^m)
+ \lambda^2 G_{ij} (V^i V^j - U^i U^j) + \lambda^2 \chi^i \chi^j D_i (V_j + U_j) - \frac{1}{4} g^{\mu \nu} \rho_\mu^i \rho_\nu^j D_i (V_j - U_j) \right\}.
\]

(3.32)

The \(Q\)-transformations which leave this action invariant are the following,

\[
[Q, x^i] = \chi^i,
\]

\[
\{Q, \chi^i\} = (V^i - U^i),
\]

\[
\{Q, \rho_\mu^i\} = \tilde{F}_\mu^i + \partial_\mu x^i + \varepsilon_\mu^\nu J^j_i \partial_\nu x^j - \Gamma^i_{jk} \chi^j \rho_\mu^k + \frac{1}{2} \varepsilon_\mu^\nu \chi^k \rho_\nu^j D_k J^j_i,
\]

\[
[Q, \tilde{F}_\mu^i] = - D_\mu \chi^i - \varepsilon_\mu^\nu J^j_i D_\nu \chi^j - \Gamma^i_{jk} \chi^j \tilde{F}_\mu^k + \frac{1}{2} R_{mj} \chi^m \chi^j \rho_\mu^k
- \frac{1}{2} \varepsilon_\mu^\nu \chi^m \rho_\nu^j D_m D_k J^j_i - \frac{1}{2} \varepsilon_\mu^\nu (D_k J^j_i) \chi^k (\partial_\nu x^j - \varepsilon_\nu^\sigma J^j_m \partial_\sigma x^m)
- \frac{1}{4} \varepsilon_\mu^\nu \chi^m \rho_\nu^n (D_k J^j_i) (D_m J^j_n) + \frac{1}{2} \varepsilon_\mu^\nu \chi^k \tilde{F}_\nu^j D_k J^j_i
+ D_k (V^i - U^i) \rho_\mu^k - \frac{1}{2} \varepsilon_\mu^\nu (V^k - U^k) \rho_\nu^j D_k J^j_i.
\]

(3.33)

Form these transformation one can easily verify that, indeed, \(Q^2\) does not vanish.

It has the following action on the fields of the theory:

\[
[Q^2, x^i] = V^i - U^i,
\]

\[
[Q^2, \chi^i] = \partial_j (V^i - U^i) \chi^j,
\]

\[
[Q^2, \rho_\mu^i] = \partial_j (V^i - U^i) \rho_\mu^j,
\]

\[
[Q^2, \tilde{F}_\mu^i] = \partial_j (V^i - U^i) \tilde{F}_\mu^j,
\]

(3.34)

which just amounts to a Lie derivative respect to the Killing vector field \(V^i - U^i\).
The analysis of the observables of this theory was carried out in [32]. They turn out to be the same as in the standard topological sigma models [30] with the additional condition that the forms on $M$ involved are orthogonal to the difference of the Killing vector fields $V^i - U^i$. Explicit computations of some observables were presented in [32]. For example, it was shown there that the partition function for the case in which the two-dimensional manifold $\Sigma$ corresponds to the torus is just the Euler number of the target manifold $M$. This topological invariant was obtained as the number of singular points of the Killing vector field.
4. Type B Topological Matter

The construction carried out in the previous section can be followed to build theories involving topological matter of type B. We will briefly describe here the steps involved in this construction. Our starting point is a collection of twisted chiral superfields \( X^I \) and twisted antichiral superfields \( \bar{X}^{\bar{I}} \), \((I, \bar{I} = 1, \ldots, d)\),

\[
\begin{align*}
D_{+,-}X^I &= 0, & D_{-,+}X^I &= 0, \\
D_{+,+}X^I &= 0, & D_{-,-}X^I &= 0.
\end{align*}
\]  

As in the previous case, our starting superspace action is (3.2). However, due to conditions (4.1), the odd part of the measure corresponding to the F-term now takes the following form,

\[
\begin{align*}
d^2\theta &\to D_{+,+}D_{-,-}, \\
\tilde{d}^2\theta &\to D_{-,-}D_{+,+}.
\end{align*}
\]  

These measures are certainly invariant under the new Lorentz generator \( \tilde{J} \). This means that when twisting to obtain the corresponding TQFT we will have the superpotential term. This is a remarkable difference respect to the case of type A topological matter, where such terms were not allowed. Note on the other hand that, as announced, \( R \)-symmetry is preserved.

The next step in the construction is to define component fields and redefine them as in (3.7) to make manifest their Lorentz structure respect to the new Lorentz generator, \( \tilde{J} \), resulting after the twist. We define,

\[
\begin{align*}
X^I| &= x^I, & X^{\bar{I}}| &= \bar{x}^{\bar{I}}, \\
D_{+,+}X^I| &= \rho_z^I, & D_{+,+}X^{\bar{I}}| &= \chi^{\bar{I}}, \\
D_{-,-}X^I| &= \rho_{\bar{z}}^I, & D_{-,+}X^{\bar{I}}| &= \bar{\chi}^{\bar{I}}, \\
D_{+,+}D_{-,-}X^I| &= F^I, & D_{-,+}D_{-,+}X^{\bar{I}}| &= F^{\bar{I}},
\end{align*}
\]  

Notice that, contrary to the case of type A topological matter, the field \( \rho_z^I \) is not selfdual while the auxiliary fields \( F^I \) and \( F^{\bar{I}} \) are scalars. The \( R \)-transformations
of the fields appearing on the right hand side of (4.3) are not manifest any more.
Let us collect them here for later convenience,

\[
[R, x^I] = 0, \quad [R, F^I] = 0, \quad [R, x^\bar{I}] = 0, \quad [R, F^{\bar{I}}] = 0.
\]

\[
[R, \rho_z^I] = \frac{1}{2} \rho_z^I, \quad [R, \rho_{\bar{z}}^I] = -\frac{1}{2} \rho_{\bar{z}}^I, \quad [R, \rho_{\bar{z}}^\bar{I}] = \frac{1}{2} \rho_{\bar{z}}^\bar{I}, \quad [R, \rho_z^\bar{I}] = -\frac{1}{2} \rho_z^\bar{I}.
\]

The transformations of the fields under the generators \(Q, M\) and \(G_\mu\) of the topological algebra are easily obtained from (4.3) and the form of the \(N = 2\) supersymmetric transformations (3.5). They turn out to be,

\[
[Q, x^I] = 0, \quad [Q, x^\bar{I}] = \chi^I + \bar{\chi}^\bar{I},
\]

\[
[M, x^I] = 0, \quad [M, x^\bar{I}] = \chi^I - \bar{\chi}^\bar{I},
\]

\[
[G_z, x^I] = \frac{1}{2} \rho_z^I, \quad [G_z, x^\bar{I}] = 0, \quad [G_{\bar{z}}, x^I] = \frac{1}{2} \rho_{\bar{z}}^I, \quad [G_{\bar{z}}, x^\bar{I}] = 0.
\]

\[
\{Q, \rho_z^I\} = 2\partial_z x^I, \quad \{Q, \rho_{\bar{z}}^\bar{I}\} = 2\partial_{\bar{z}} x^\bar{I}, \quad \{Q, \rho_z^\bar{I}\} = 2\partial_z x^\bar{I}, \quad \{Q, \rho_{\bar{z}}^I\} = 2\partial_{\bar{z}} x^I,
\]

\[
\{M, \rho_z^I\} = 2\partial_z x^I, \quad \{M, \rho_{\bar{z}}^\bar{I}\} = -2\partial_{\bar{z}} x^\bar{I}, \quad \{M, \rho_z^\bar{I}\} = -2\partial_z x^\bar{I}, \quad \{M, \rho_{\bar{z}}^I\} = -2\partial_{\bar{z}} x^I,
\]

\[
\{G_z, \rho_z^I\} = 0, \quad \{G_z, \rho_{\bar{z}}^\bar{I}\} = \frac{1}{2} F^I, \quad \{G_{\bar{z}}, \rho_z^I\} = \frac{1}{2} F^\bar{I}, \quad \{G_{\bar{z}}, \rho_{\bar{z}}^\bar{I}\} = 0,
\]

\[
\{G_z, \rho_z^\bar{I}\} = -\frac{1}{2} F^\bar{I}, \quad \{G_z, \rho_{\bar{z}}^I\} = 0, \quad \{G_{\bar{z}}, \rho_z^\bar{I}\} = 0.
\]

\[
\{Q, \chi^I\} = \bar{F}^\bar{I}, \quad \{Q, \bar{\chi}^\bar{I}\} = -F^I,
\]

\[
\{M, \chi^I\} = -\bar{F}^\bar{I}, \quad \{M, \bar{\chi}^\bar{I}\} = -F^I,
\]

\[
\{G_z, \chi^I\} = \partial_z x^I, \quad \{G_z, \bar{\chi}^\bar{I}\} = 0, \quad \{G_{\bar{z}}, \chi^I\} = 0, \quad \{G_{\bar{z}}, \bar{\chi}^\bar{I}\} = \partial_{\bar{z}} x^\bar{I},
\]

\[
\{G_{\bar{z}}, \chi^\bar{I}\} = 0, \quad \{G_{\bar{z}}, \bar{\chi}^I\} = \partial_{\bar{z}} x^I.
\]
\[ [Q, F^I] = 2\partial_z \rho_z^I - 2\partial_{\bar{z}} \rho_z^I, \quad [Q, F^\bar{I}] = 0, \]
\[ [M, F^I] = 2\partial_z \rho_z^I + 2\partial_{\bar{z}} \rho_z^I, \quad [M, F^\bar{I}] = 0, \]
\[ [G_z, F^I] = 0, \quad [G_\bar{z}, F^I] = -\partial_z \chi^I, \]
\[ [G_z, F^\bar{I}] = 0, \quad [G_\bar{z}, F^\bar{I}] = \partial_{\bar{z}} \chi^\bar{I}. \] (4.8)

The action resulting from (3.2) after using the definitions (4.3) takes the following form,
\[
S' = \int d^2z \left[ G_{IJ}(F^IF^J + 2\rho_z^I D_z \chi^J + 2\rho_{\bar{z}}^I D_{\bar{z}} \chi^\bar{J} - 4\partial_z x^I \partial_{\bar{z}} x^\bar{J}) + \partial_K \partial_I G_{IJ} \bar{\rho}_z^K \rho_z^I \chi^J + \partial_K G_{IJ} \rho_{\bar{z}}^I \chi^J + \partial_K G_{IJ} \bar{\chi}^K \bar{F}^I \bar{F}^J + \partial_J \partial_I W) \rho_z^J \rho_{\bar{z}}^I + (\partial_{\bar{I}} W) F^I - (\partial_J \partial_{\bar{I}} \bar{W}) \chi^J \chi^\bar{I} + (\partial_{\bar{I}} \bar{W}) F^{\bar{I}} \right].
\] (4.9)

Notice the presence of potential terms in this action. For the moment we have denoted this action by \(S'\) since it will be convenient to redefine it by a global factor later. We will reserve the symbol \(S\) for its final form. The auxiliary fields \(F^I\) and \(F^{\bar{I}}\) can be integrated out in the usual way. Furthermore, as in the previous case, this procedure makes the action manifestly reparametrization invariant from the point of view of Kahler geometry. We will reduce the dependence of the action on \(F^I\) and \(F^{\bar{I}}\) to a simple quadratic term plus other terms involving the potentials \(W\) and \(\bar{W}\) carrying out the following definition,
\[
\bar{F}^I = F^I + \Gamma_{IK}^I \rho_z^K \rho_{\bar{z}}^I, \quad \bar{F}^{\bar{I}} = F^{\bar{I}} + \Gamma_{JK}^{\bar{I}} \chi^J \chi^K,
\] (4.10)

where \(\Gamma_{IK}^I\) and \(\Gamma_{JK}^{\bar{I}}\) are the Christoffel connections defined in (A22). The action (4.9) becomes,
\[
S' = \int d^2z \left[ G_{IJ}(- 4\partial_z x^I \partial_{\bar{z}} x^\bar{J} + 2\rho_z^I D_z \chi^J + 2\rho_{\bar{z}}^I D_{\bar{z}} \chi^\bar{J} + \bar{F}^I \bar{F}^J) + R_{ILJK} \rho_z^I \chi^L \rho_{\bar{z}}^K \chi^J + (D_I \partial_J \bar{W}) \rho_z^I \rho_{\bar{z}}^J + (\partial_{\bar{I}} W) \bar{F}^I - (D_I \partial_{\bar{I}} \bar{W}) \chi^J \chi^\bar{I} + (\partial_{\bar{I}} \bar{W}) \bar{F}^{\bar{I}} \right].
\] (4.11)

So far we have carried out the twisting procedure. Now we are in the position.
of verifying if the theory is topological. To this end we must first place it on an
arbitrary two dimensional manifold, and verify that the action is $Q$-invariant and
the energy-momentum tensor is $Q$-exact. Let us therefore consider a Riemann
surface $\Sigma$ endowed with a metric $g_{\mu\nu}$. The covariantization of the action (4.11)
takes the form,

\[
S' = \int_\Sigma d^2\sigma \sqrt{g} \left[ G_{IJ} \left( -g^{\mu\nu} \partial_\mu x^I \partial_\nu x^J + i\varepsilon^{\mu\nu} \partial_\mu x^I \partial_\nu x^J + \frac{1}{2} g^{\mu\nu} \rho^I_\mu D_\nu (\chi^J + \bar{\chi}^J) \right) 
+ \frac{i}{2} \varepsilon^{\mu\nu} \rho^I_\mu D_\nu (\chi^J - \bar{\chi}^J) + \tilde{F}^I \tilde{F}^J \right] + \frac{i}{4} \varepsilon^{\mu\nu} R_{IJKLM} \rho^I_\mu \chi^L \rho^K_\nu \chi^J 
+ (\partial_I W) \tilde{F}^I - (D_I \partial_J W) \chi^I \chi^J + (\partial_I \bar{W}) \tilde{F}^I + \frac{i}{4} \varepsilon^{\mu\nu} (D_I \partial_J W) \rho^I_\mu \rho^K_\nu \right], 
\]

(4.12)

where we have used (3.27). Notice that this action has a minus sign in the term
$-g^{\mu\nu} \partial_\mu x^I \partial_\nu x^J$ as compared to (3.21). This is reminiscent of a well known fact in
$N = 2$ supersymmetry, where chiral and twisted chiral multiplets provide kinetic
terms with opposite signs [28]. If one identifies $G_{IJ}$ with a positive definite metric
one should have to change the global sign of the whole action to have a bosonic
part leading to a convergent functional integral. Notice that although the bosonic
part of the action is not real, its imaginary part is a topological invariant. From
now on we will assume that $G_{IJ}$ is positive definite and we will introduce a global
negative sign to the action (4.12). The resulting action turns out to be,

\[
S = \int_\Sigma d^2\sigma \sqrt{g} \left[ G_{IJ} \left( g^{\mu\nu} \partial_\mu x^I \partial_\nu x^J - i\varepsilon^{\mu\nu} \partial_\mu x^I \partial_\nu x^J - \frac{1}{2} g^{\mu\nu} \rho^I_\mu D_\nu (\chi^J + \bar{\chi}^J) \right) 
- \frac{i}{2} \varepsilon^{\mu\nu} \rho^I_\mu D_\nu (\chi^J - \bar{\chi}^J) - \tilde{F}^I \tilde{F}^J \right] + \frac{i}{4} \varepsilon^{\mu\nu} R_{IJKLM} \rho^I_\mu \chi^L \rho^K_\nu \chi^J 
- (\partial_I W) \tilde{F}^I + (D_I \partial_J W) \chi^I \chi^J - (\partial_I \bar{W}) \tilde{F}^I - \frac{i}{4} \varepsilon^{\mu\nu} (D_I \partial_J W) \rho^I_\mu \rho^K_\nu \right]. 
\]

(4.13)

Taking into account the redefinitions (4.10), one easily derives the covariantized
form of the $Q$-transformations of the fields,
\[
[Q, x^I] = 0,
\]
\[
[Q, x^\bar{I}] = \chi^\bar{I} + \bar{\chi}^I,
\]
\[
\{Q, \rho^I_\mu\} = 2\partial_\mu x^I,
\]
\[
\{Q, \chi^I\} = \tilde{F}^\bar{I} - \Gamma^\bar{I}_{JK}\bar{\chi}^J\chi^K,
\]
\[
\{Q, \bar{\chi}^I\} = -\tilde{F}^I + \Gamma^I_{JK}\chi^J\bar{\chi}^K,
\]
\[
\{Q, \tilde{F}^I\} = i\varepsilon^{\mu\nu}[D_\mu \rho^I_\nu + \frac{1}{4} R^I_{JJK}(\chi^\bar{J} + \bar{\chi}^\bar{J})\rho^J_\mu \rho^K_\nu],
\]
\[
\{Q, \tilde{F}^\bar{I}\} = \Gamma^\bar{I}_{JK}\tilde{F}^J(\bar{\chi}^\bar{K} + \chi^K).
\]

(4.14)

It is simple to verify that, indeed, the action is invariant under this set of transformations. Furthermore, similarly to the case of type A topological matter, one can verify that the action is also invariant under $M$ and $R$ transformations. However, it is not invariant under transformations generated by $G_\mu$. Before carrying out the analysis of the energy-momentum tensor one could ask if the action (4.13) is $Q$-exact as in the previous case. The answer to this question is negative. Only when the potential terms are not present the action is $Q$-exact. Let us set $W = 0$ in (4.13). It is simple to verify that the remaining action can be written as,

\[
S|_{W=0} = \left\{Q, \int_{\Sigma} d^2\sigma \sqrt{g} \left[ G_{IJ}\left(\frac{1}{2}g^{\mu\nu}\rho^I_\mu \partial_\nu x^J - \frac{i}{2}\varepsilon^{\mu\nu}\rho^I_\mu \partial_\nu x^J - \tilde{F}^I \chi^J\right)\right]\right\}. \quad (4.15)
\]

When potential terms are present, the action (4.13) is not $Q$-exact. A short analysis shows that the transformations (4.14) can not generate a term like $(\partial_I W)\tilde{F}^I$ as the one present in the action (4.13). However, the requirement for the theory being topological is just that the energy-momentum tensor be $Q$-exact. The energy-momentum can be easily computed from the variation of the action (4.13) respect to the two-dimensional metric. One finds:

\[
T_{\mu\nu} = \frac{1}{2}(\delta^\sigma_\mu \delta^\tau_\nu + \delta^\sigma_\nu \delta^\tau_\mu)G_{IJ}(\partial_\sigma x^I \partial_\tau x^J - \frac{1}{2} \rho^I_\sigma D_\tau (\chi^J + \bar{\chi}^\bar{J}))
\]

\[
- \frac{1}{2}g_{\mu\nu} [G_{IJ}\left( g^{\sigma\tau}(\partial_\sigma x^I \partial_\tau x^J - \frac{1}{2} \rho^I_\sigma D_\tau (\chi^J + \bar{\chi}^\bar{J})) - \tilde{F}^I \tilde{F}^J\right)
\]

\[
- \partial_I W \tilde{F}^I - \partial_I \bar{W} \bar{F}^I + (D_I \partial_J \bar{W})\chi^I \chi^J]. \quad (4.16)
\]
Notice that, again, one has the same problem as before since the term \((\partial_IW)\tilde{F}^I\) is present. Since the field \(\tilde{F}^I\) is the source of the problem and, on the other hand, it is auxiliary, we will integrate it out. The price to pay in doing this is that the algebra will not close off-shell, \(i.e.,\) we will have \(Q^2 = 0\) modulo field equations. However, if we succeed in proving that after integrating out the auxiliary field the energy-momentum tensor of the theory is \(Q\)-exact we will have shown that the theory is topological. The integration of the auxiliary field will give some dependence on the two dimensional metric but this can be factorized and the rest must correspond to a topological invariant. In other words, here is no need of an off-shell realization for the theory being topological since the possible metric dependence originated from the auxiliary fields can be factorized. Let us define,

\[
\hat{F}^I = \tilde{F}^I + G^{IJ}\partial_J\tilde{W} \quad \hat{F}^I = \tilde{F}^{I} + G^{IJ}\partial_J\tilde{W}. \quad (4.17)
\]

The dependence of the action (4.13) on the auxiliary fields \(\hat{F}^I\) and \(\hat{F}^I\) becomes gaussian. The integration of the auxiliary fields \(\hat{F}^I\) and \(\hat{F}^I\) gives a dependence on the two dimensional metric which can be factorized from the functional integral. For example, the integration of the constant auxiliary modes gives a factor which is proportional to the volume of the two dimensional manifold to some power. The resulting action turns out to be,

\[
S = \int d^2\sigma \sqrt{g} \left[ G_{IJ}(g^{\mu\nu}\partial_\mu x^I\partial_\nu x^J - i\varepsilon^{\mu\nu}\partial_\mu x^I\partial_\nu x^J - \frac{1}{2}g^{\mu\nu}\rho^I_\mu D_\nu(\chi^J + \bar{\chi}^\bar{J})
\right.
\]

\[
- \frac{i}{2}\varepsilon^{\mu\nu}\rho^I_\mu D_\nu(\chi^J - \bar{\chi}^\bar{J})) + \frac{i}{4}\varepsilon^{\mu\nu}R_{IJK}\rho^I_\mu \bar{\chi}^\bar{J}\rho^K_\nu \chi^J + G^{IJ}(\partial_IW)\partial_J\tilde{W} + (D_I\partial_J\tilde{W})\chi^I + \bar{\chi}^\bar{J} - \frac{i}{4}\varepsilon^{\mu\nu}(D_I\partial_J\tilde{W})\rho^I_\mu \rho^J_\nu \right]. \quad (4.18)
\]

Now we have to check if the theory possess an energy-momentum tensor which is \(Q\)-exact. After using (4.17) and setting \(F^I = \tilde{F}^I = 0\) (since they have been
integrated out) the transformations (4.14) become,

\[
\begin{align*}
[Q, x^I] &= 0, \\
[Q, \bar{x}^\bar{I}] &= \chi^{\bar{I}} + \bar{\chi}^{\bar{I}}, \\
\{Q, \rho^I_\mu \} &= 2 \partial_\mu x^I, \\
\{Q, \chi^{\bar{I}} \} &= -G^{\bar{I}J} \partial J W - \Gamma^{\bar{I}JK} \chi^J \chi^K, \\
\{Q, \bar{\chi}^{\bar{I}} \} &= G^{\bar{I}J} \partial J W + \Gamma^{\bar{I}JK} \bar{\chi}^J \chi^K.
\end{align*}
\]

(4.19)

Using these transformations one finds that the energy-momentum tensor corresponding to the action (4.18) is \(Q\)-exact,

\[
T_{\mu\nu} = \left\{ Q, \frac{1}{2} g_{\mu\nu} \left( \eta^{\tau\sigma} G_{IJ} \rho^I_\sigma \partial_\tau x^J - (\chi^{\bar{I}} - \bar{\chi}^{\bar{I}}) \partial_\bar{I} W \right) \right. \\
- \left. \frac{1}{2} G_{\bar{I}J} (\rho^I_\mu \partial_\nu x^J + \rho^I_\nu \partial_\mu x^J) \right\}
\]

(4.20)

For the case in which the target space is flat, the theory we have constructed was first studied in [31].

The appearance of a topological quantum field theory where the \(Q\)-symmetry is only realized on-shell is not new. A classical example where this also occurs is topological Yang-Mills theory in four dimensions [33]. However, the theory we have constructed possess a feature which is not present in topological Yang-Mills theory in four dimensions. Namely, the action (4.18) is not \(Q\)-exact. In this sense the theory we have constructed has also features of other classes of topological quantum field theories as Chern-Simons theory in three dimensions [34].

Let us construct the observables corresponding to type B topological matter. First we will build the ones corresponding to zero forms and then, we will use (2.21) to obtain their descendants. This construction will show the usefulness of the operator \(G_\mu\) in dealing with observables. From the \(Q\)-transformations of the fields (4.19) follows trivially that any function which depends only on \(x^I\) and not on \(x^{\bar{I}}\), \textit{i.e.}, holomorphic from the point of view of the target space, is \(Q\)-invariant.
and therefore an observable. Then, in the notation used in (2.21) and (2.22), we have
\[ \phi^{(0)} = A(x^I). \]  

Using the operator \( G_\mu \), whose action on the fields is given in (4.5), (4.6), (4.7) and (4.8), one finds,
\[ \phi^{(1)} = [G_\mu, \phi^{(0)}] = \frac{1}{2} \partial_I A_\rho^I, \]
\[ \phi^{(2)} = \frac{1}{2} \{ G_\mu, \phi^{(1)} \} = \frac{1}{8} D_J \partial_I A_\rho^J A^I - i \frac{5}{4} \epsilon_{\mu \nu} \partial_I A (\hat{F}^I - \partial^I \hat{W}). \]  
Notice that we have restored all the dependence on the auxiliary fields in computing (4.22). The reason for this is that only off-shell the topological algebra holds. Otherwise it holds modulo field equations and then the analysis of observables is more complicated. The topological algebra (2.6) guarantees that the \( Q \)-transformations of the fields in (4.22) leads to a total derivative and therefore their integration over closed 1-cycles and 2-cycles respectively leads to observables. One can check this explicitly using the transformations listed in (4.5), (4.6), (4.7) and (4.8). Furthermore, as shown in (2.25), to obtain a possible non-vanishing vacuum expectation value one must consider closed 1-cycles which are homologically non-trivial. The only 2-cycle to be consider is the two-dimensional manifolds itself. The two-form operator (4.22) possesses a feature which is not standard in topological quantum field theories, namely, it depends on the auxiliary field \( \hat{F}^I \). Since, as discussed above, the presence of the auxiliary field leads to an energy-momentum tensor which is not \( Q \)-exact, we have to reanalyze the type of invariants obtained from operators as \( \phi^{(2)}_{\mu \nu} \) in (4.22). Before carrying out such analysis we will construct the second type of \( Q \)-invariant operators of the theory.

Let us consider a closed form of type \((0, p)\), i.e., a closed form with \( p \) antiholomorphic indices, \( A_{\tilde{I}_1 \tilde{I}_2 \ldots \tilde{I}_p} \). Certainly, the operators
\[ \tilde{\phi}^{(0)} = A_{\tilde{I}_1 \tilde{I}_2 \ldots \tilde{I}_p} (\chi^{\tilde{I}_1} + \bar{\chi}^{\tilde{I}_1}) (\chi^{\tilde{I}_2} + \bar{\chi}^{\tilde{I}_2}) \ldots (\chi^{\tilde{I}_p} + \bar{\chi}^{\tilde{I}_p}) \]  
are \( Q \)-invariant. The construction of their descendants is carried out using the
operator $G_\mu$ as before. One finds,

\begin{align*}
\tilde{\varphi}^{(1)}_{\mu} &= \frac{1}{2} \partial_J A_{I_1 I_2 \ldots I_p} \rho^J_\mu (x^{I_1} + \bar{x}^{I_1})(x^{I_2} + \bar{x}^{I_2}) \ldots (x^{I_p} + \bar{x}^{I_p}) \\
&\quad + A_{I_1 I_2 \ldots I_p} \sum_{s=1}^p (-1)^{s+1} (x^{I_1} + \bar{x}^{I_1}) \ldots \partial_\mu x^{I_s} \ldots (x^{I_p} + \bar{x}^{I_p}), \\
\tilde{\varphi}^{(2)}_{\mu \nu} &= \frac{1}{8} D_J \partial_K A_{I_1 I_2 \ldots I_p} \rho^J_\mu \rho^K_\nu (x^{I_1} + \bar{x}^{I_1})(x^{I_2} + \bar{x}^{I_2}) \ldots (x^{I_p} + \bar{x}^{I_p}) \\
&\quad - \frac{i}{4} \varepsilon_{\mu \nu} \partial_J A_{I_1 I_2 \ldots I_p} (\tilde{F}^J - \partial^J \tilde{W})(x^{I_1} + \bar{x}^{I_1})(x^{I_2} + \bar{x}^{I_2}) \ldots (x^{I_p} + \bar{x}^{I_p}) \\
&\quad + \frac{1}{4} \partial_J A_{I_1 I_2 \ldots I_p} \rho^J_\mu \sum_{s=1}^p (-1)^{s+1} (x^{I_1} + \bar{x}^{I_1}) \ldots \partial_\mu x^{I_s} \ldots (x^{I_p} + \bar{x}^{I_p}) \\
&\quad + \frac{1}{2} A_{I_1 I_2 \ldots I_p} \sum_{s, t=1 \atop s \neq t}^p (-1)^{t+s} (x^{I_1} + \bar{x}^{I_1}) \ldots \partial_\mu x^{I_s} \ldots \partial_\nu x^{I_t} \ldots (x^{I_p} + \bar{x}^{I_p}),
\end{align*}

where the squared brackets on indices denote antisymmetrization with no factor. Again, as in the previous operators, we observe the same feature regarding the dependence on the auxiliary field. Let us analyze the consequences of having a linear dependence on $F^I$ in (4.22) and (4.24).

We will denote by $\varphi^{(2)}_{\mu \nu}$ ($\tilde{\varphi}^{(2)}_{\mu \nu}$) the part of $\varphi^{(2)}_{\mu \nu}$ in (4.22) ($\tilde{\varphi}^{(2)}_{\mu \nu}$ in (4.24)) which does not contain $\tilde{F}^I$. Ones has,

\begin{align*}
\varphi^{(2)}_{\mu \nu} &= \varphi^{(2)}_{\mu \nu} + \frac{i}{4} \varepsilon_{\mu \nu} F^I \xi_I, \\
\tilde{\varphi}^{(2)}_{\mu \nu} &= \tilde{\varphi}^{(2)}_{\mu \nu} + \frac{i}{4} \varepsilon_{\mu \nu} F^I \tilde{\xi}_I,
\end{align*}

where,

\begin{align*}
\xi_I &= - \partial_I A, \\
\tilde{\xi}_I &= - \partial_I A_{I_1 I_2 \ldots I_p} (x^{I_1} + \bar{x}^{I_1})(x^{I_2} + \bar{x}^{I_2}) \ldots (x^{I_p} + \bar{x}^{I_p}).
\end{align*}

We will show that the vacuum expectation values of the integrated form of the operators $\varphi^{(2)}_{\mu \nu}$ and $\tilde{\varphi}^{(2)}_{\mu \nu}$ computed with the action (4.18) are topological invariants.
Namely, we will prove,
\[ \frac{\delta}{\delta g^{\mu\nu}} \langle \int_{\Sigma} \varphi^{(2)}_{\mu\nu} \rangle = 0, \]
\[ \frac{\delta}{\delta g^{\mu\nu}} \langle \int_{\Sigma} \tilde{\varphi}^{(2)}_{\mu\nu} \rangle = 0. \]
(4.27)

To prove this we will place back the auxiliary fields in the action of the theory and we will compute the vacuum expectation value of the integrated form of the operators \( \varphi^{(2)}_{\mu\nu} \) and \( \tilde{\varphi}^{(2)}_{\mu\nu} \). In other words we will consider the theory off-shell. We will carry out the analysis explicitly for \( \varphi^{(2)}_{\mu\nu} \) but, similarly, it follows for \( \tilde{\varphi}^{(2)}_{\mu\nu} \) since the only fact that we need to use is that the dependence on \( F^I \) is linear and this is a common feature to both operators. Let us therefore consider
\[ \langle \int_{\Sigma} \varphi^{(2)}_{\mu\nu} \rangle_{\text{off}} = \int [dX][dF] \int_{\Sigma} \varphi^{(2)}_{\mu\nu} + \frac{i}{4} \varepsilon_{\mu\nu} \hat{F}^I \xi_I e^{-S(X) + \int_{\Sigma} \sqrt{g} G_{IJ} \hat{F}^I \hat{F}^J}, \]
(4.28)
where \([dX]\) denotes the measure corresponding to the fields \( x^I, x^\bar{I}, \chi^I, \chi^\bar{I}, \rho_I \), and \([dF]\) the one corresponding to \( \hat{F}^I \) and \( \hat{F}^\bar{I} \). The subindex in \( \langle \rangle_{\text{off}} \) denotes that the vacuum expectation value is taken off-shell. Since the dependence of \( \varphi^{(2)}_{\mu\nu} \) on \( F^I \) is linear one has,
\[ \langle \int_{\Sigma} \varphi^{(2)}_{\mu\nu} \rangle_{\text{off}} = \int [dX][dF] \int_{\Sigma} \varphi^{(2)}_{\mu\nu} e^{-S(X) + \int_{\Sigma} \sqrt{g} G_{IJ} \hat{F}^I \hat{F}^J} \]
\[ = \int [dF] e^{\int_{\Sigma} \sqrt{g} G_{IJ} \hat{F}^I \hat{F}^J} \langle \int_{\Sigma} \varphi^{(2)}_{\mu\nu} \rangle. \]
(4.29)

This result shows that the vacuum expectation value (4.28) factorizes in a part which contains the integration on \( F^I \) and it is not topological invariant, times the vacuum expectation value \( \langle \int_{\Sigma} \varphi^{(2)}_{\mu\nu} \rangle \), where the functional integration is carried out without the fields \( F^I \) and \( F^\bar{I} \). Our aim is to show that this last factor is topological invariant. To carry this out we will take the vacuum expectation value (4.28) and
we will study its dependence on $g_{\mu\nu}$. We have,

$$
\frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\sigma\tau}} \langle \int_{\Sigma} \phi^{(2)}_{\mu\nu} \rangle_{\text{off}}
= \int [dX] [dF] \int_{\Sigma} \phi^{(2)}_{\mu\nu} (T_{\sigma\tau} + \frac{1}{2} g_{\sigma\tau} G_{I\bar{J}} \hat{F}^I \hat{F}^J) e^{-S(X)} + \int_{\Sigma} \sqrt{g} G_{I \bar{J}} \hat{F}^I \hat{F}^J
= \int [dX] [dF] \{Q, \int_{\Sigma} \phi^{(2)}_{\mu\nu} G_{\sigma\tau} \} e^{-S(X)} + \int_{\Sigma} \sqrt{g} G_{I \bar{J}} \hat{F}^I \hat{F}^J
+ \frac{1}{2} g_{\sigma\tau} \int [dX] [dF] \phi^{(2)}_{\mu\nu} G_{I \bar{J}} \hat{F}^I \hat{F}^J e^{-S(X)} + \int_{\Sigma} \sqrt{g} G_{I \bar{J}} \hat{F}^I \hat{F}^J,
$$

(4.30)

where we have used that $[Q, \phi^{(2)}_{\mu\nu}]$ is a total derivative and we have denoted by $G_{\sigma\tau}$ the quantity such that $T_{\sigma\tau} = \{Q, G_{\sigma\tau} \}$ in (4.20) (recall eq. (2.12)). The first term in (4.30) vanishes because, certainly, the off-shell action which appears in the exponent is $Q$-invariant. We are left with the second term which can be written as,

$$
\frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\sigma\tau}} \langle \int_{\Sigma} \phi^{(2)}_{\mu\nu} \rangle_{\text{off}} = - \frac{1}{2} g_{\sigma\tau} \int [dF] G_{I \bar{J}} \hat{F}^I \hat{F}^J e^{-S(X)} + \int_{\Sigma} \sqrt{g} G_{I \bar{J}} \hat{F}^I \hat{F}^J \langle \int_{\Sigma} \phi^{(2)}_{\mu\nu} \rangle,
$$

(4.31)

since the linear dependence on $F^I$ in $\phi^{(2)}_{\mu\nu}$ gives a vanishing contribution. Comparing (4.31) with (4.29) one concludes that, indeed,

$$
\frac{1}{\sqrt{g}} \frac{\delta}{\delta g_{\sigma\tau}} \langle \int_{\Sigma} \phi^{(2)}_{\mu\nu} \rangle = 0.
$$

(4.32)

This result proves that the on-shell theory leads to topological invariants. Notice that it has been essential in the proof that the dependence on $F^I$ of the observables is at most linear in the auxiliary fields.
5. Coupling to Topological Gravity

In this section we will describe a gauging procedure to build the coupling of the theories involving topological matter constructed in the two previous sections to topological gravity. The two types of topological quantum field theories which we have constructed possess the $Q$-symmetries generated by the transformations (3.33) and (4.19). If one considers a flat two-dimensional space it is clear from the construction that the actions (3.32) and (4.18) of the two types of theories are invariant under $G_\mu$-transformations. When considering the theories on a curved two-dimensional space, however, these actions are not $G_\mu$-invariant. The approach that we will describe in this section to couple topological matter to topological gravity will consists of a modification of these theories in such a way that the resulting theories are invariant under a local $G_\mu$-symmetry. This will be done by introducing a gauge field corresponding to this symmetry which we will denote by $\psi_{\mu\nu}$. Certainly, since $G_\mu$ is odd, this new gauge field is also odd. From the relation $\{Q, G_\mu\} = P_\mu$ in (2.6) follows that the gauge field associated to $G_\mu$ must be the $Q$-partner of the metric $g_{\mu\nu}$, which is the gauge field associated to $P_\mu$. We will find out in the construction that, indeed, invariance of the gauged action under $Q$ implies that $g_{\mu\nu}$ and $\psi_{\mu\nu}$ are $Q$-partners, i.e., $[Q, g_{\mu\nu}] = \psi_{\mu\nu}$. Notice that in this construction $P_\mu$ and $G_\mu$ are generators of local symmetries while $Q$ is a global one which relates both.

In this paper we will consider the coupling of a simple type B model. A treatment in full generality will be presented elsewhere. Let us consider type B topological matter corresponding to a flat $2d$-dimensional target space with no potential terms. The corresponding action is easily obtained from (4.18),

$$S = \int d^2\sigma \sqrt{g}(g^{\mu\nu}\partial_\mu x^I\partial_\nu x^\bar{I} - i\varepsilon^{\mu\nu}\partial_\mu x^I\partial_\nu x^\bar{I})$$

$$- \frac{1}{2}g_{\mu\nu}\rho^I_\mu\partial_\nu(x^I + \bar{x}^I) - \frac{i}{2}\varepsilon^{\mu\nu}\rho^I_\mu\partial_\nu(x^I - \bar{x}^I)).$$

(5.1)
This action is invariant under the $Q$-transformations (4.19) that now take the form,

\begin{align*}
[Q, x^I] &= 0, \\
[Q, \bar{x}^\bar{I}] &= \chi^\bar{I} + \bar{\chi}^\bar{I}, \\
\{Q, \chi^\bar{I}\} &= 0, \\
\{Q, \bar{\chi}^\bar{I}\} &= 0, \\
\{Q, \rho^I\} &= 2\partial_\mu x^I.
\end{align*}

However, (5.1) is not invariant under the $G_\mu$-transformation which can be easily obtained from (4.5), (4.6) and (4.7),

\begin{align*}
[G_\mu, x^I] &= \frac{1}{2}\rho^I, \\
[G_\mu, \bar{x}^\bar{I}] &= 0, \\
\{G_\mu, \chi^\bar{I}\} &= \frac{1}{2}(\partial_\mu x^I - i\varepsilon_{\mu\nu}\partial_\nu x^I), \\
\{G_\mu, \bar{\chi}^\bar{I}\} &= \frac{1}{2}(\partial_\mu x^I + i\varepsilon_{\mu\nu}\partial_\nu x^I), \\
\{G_\mu, \rho^I\} &= 0.
\end{align*}

The action (5.1) is reparametrization invariant. On the other hand, the $G_\mu$-transformations (5.3) are covariant. Since the action (5.1) is $G_\mu$-invariant for a flat two-dimensional space, it must be also invariant if the parameter associated to a $G_\mu$-transformation is covariantly constant. Let us introduce an odd local parameter $\eta^\mu$ as the one corresponding to $G_\mu$-transformations. From (5.3) these transformations can be written as,

\begin{align*}
\delta x^I &= \frac{1}{2}\eta^\mu \rho^I, \\
\delta \bar{x}^\bar{I} &= 0, \\
\delta \chi^\bar{I} &= \frac{1}{2}\eta^\mu (\partial_\mu x^I - i\varepsilon_{\mu\nu}\partial_\nu x^I), \\
\delta \bar{\chi}^\bar{I} &= \frac{1}{2}\eta^\mu (\partial_\mu x^I + i\varepsilon_{\mu\nu}\partial_\nu x^I), \\
\delta \rho^I &= 0.
\end{align*}

The variation of the action (5.1) under these transformations takes the form,

\begin{equation}
\delta S = \int_\Sigma d^2\sigma \sqrt{g}(\nabla^\mu \eta^\nu) P_{\mu\nu} \sigma^\tau \rho^I_{\sigma} \partial_\tau x^I, \tag{5.5}
\end{equation}
where $P_{\mu\nu}^{\sigma\tau}$ is a projector into the traceless symmetric part for tensors of rank two,

$$P_{\mu\nu}^{\sigma\tau} = \frac{1}{2} (\delta_{\sigma}^{\mu} \delta_{\tau}^{\nu} + \delta_{\nu}^{\mu} \delta_{\tau}^{\sigma} - g_{\mu\nu} g^{\sigma\tau})$$  \hfill (5.6)

and $\nabla_\mu$ is a two-dimensional covariant derivative. Notice that, as expected, the variation (5.5) vanishes for a covariantly constant parameter $\eta^\mu$.

Our next step is the introduction of a new odd field $\psi_{\mu\nu}$ which will play the role of gauge field for the transformations (5.4). From the variation (5.5) follows that this field must be symmetric and traceless,

$$\psi_{\mu\nu} = \psi_{\nu\mu}, \quad \psi_{\mu}^\mu = 0$$  \hfill (5.7)

and must transform as,

$$\delta \psi_{\mu\nu} = 2 P_{\mu\nu}^{\sigma\tau} \nabla_\sigma \eta_\tau$$  \hfill (5.8)

The term to be added to the action (5.1) must be such that the action is invariant under the transformations (5.4) and (5.8). This term is simple to guess. The gauged action turns out to be,

$$S_g = \int_\Sigma d^2\sigma \sqrt{g} \left( g^{\mu\nu} \partial_\mu x^I \partial_\nu x^I - i \varepsilon^{\mu\nu} \partial_\mu x^I \partial_\nu x^I ight.$$

$$- \frac{1}{2} g^{\mu\nu} \rho_{\mu} \partial_\nu (\chi^I + \bar{\chi}^I) - \frac{i}{2} \varepsilon^{\mu\nu} \rho_{\mu} \partial_\nu (\chi^I - \bar{\chi}^I) - \frac{1}{2} \psi^{\mu\nu} \rho_{\mu} \partial_\nu x^I )$$  \hfill (5.9)

Notice that the metric tensor $g_{\mu\nu}$ does not transform under $G_\mu$ transformations. The $Q$-variation of the action (5.9) is not defined since we have not specified the $Q$-transformation of $\psi_{\mu\nu}$. Furthermore, so far we have consider $g_{\mu\nu}$ as a $Q$-invariant quantity. It turns out that the only way to make (5.9) $Q$-invariant is defining the $Q$-transformations of $g_{\mu\nu}$ and $\psi_{\mu\nu}$ as,

$$[Q, g_{\mu\nu}] = \psi_{\mu\nu}, \quad \{Q, \psi_{\mu\nu}\} = 0$$  \hfill (5.10)

which are consistent with the nilpotency of $Q$. The gauge action is invariant under ordinary reparametrizations and the local symmetry listed in (5.4) and (5.8). To
quantize the gauged action (5.9) one needs to fix these local gauge symmetries. These gauge fixings leads to the introduction of ghost fields which build the standard content of topological gravity in two dimensions [1,2,6]. We will not describe the quantization procedure in this work. It follows the lines described in [1,6].

In this section we have coupled topological matter to topological gravity taking a simple model for type B matter. The procedure should be extended to the general case. The steps needed in the gauging procedure are rather standard and one does not expect unsurmountable complications. We expect to report on this in the future.
6. Concluding remarks

In this work we have constructed two types of theories containing topological matter after twisting $N = 2$ supersymmetry. In addition we have given a gauging procedure to couple matter to topological gravity. Type A topological matter is a topological quantum field theory whose action is $Q$-invariant. However, this feature is not shared by type B topological matter. It should be desirable to understand the nature of the observables associated to type B topological matter and, in particular, the role played by the potential which, certainly, is going to be non-trivial. One should also study the relation between these theories and conformal topological field theories. For example, it should be interesting to analyze if a relation as the one between supersymmetric Landau-Ginzburg models and $N=2$ superconformal models [35,36,37] holds. It is not clear in such a picture which one is the correspondence between the observables constructed in sect. 4 for type B matter and the ones in topological conformal field theory [7,8].

It is interesting to remark that associated to the two types of topological matter that we have studied there also exist their conjugate counterparts. The choice of new Lorentz generator (2.11), $\tilde{J} = J + R$ to carry out the twist could have been chosen differently, namely, $\tilde{J} = J - R$. It is clear from the construction that the resulting theories would have the same features as the ones constructed in sect. 3 and 4. In type A theories one would obtain a change of selfduality by anti-selfduality conditions. In type B theories the prominent role played by the anti-holomorphic coordinates will be played by the holomorphic ones.

Theories containing a mixture of type A and type B topological matter should be constructed. It would very interesting to analyze its geometric interpretation as well as the possible potential terms which they allow. The resulting models must be related to the supersymmetric ones constructed in [28] and therefore they will provide a topological version of Wess-Zumino-Witten models. Finally, the full coupling to topological gravity of all these types of topological matter should be carried out.
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APPENDIX

In this appendix we will give a summary of our conventions and we will recall a few facts about complex manifolds. The quantum field theories considered in this paper are defined on a two-dimensional manifold $\Sigma$ which is locally endowed with an Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$. This metric is invariant under $SO(2)$ tangent space rotations which generate the corresponding “Lorentz” group. Greek indices from the beginning of the alphabet label spinor representations of the Lorentz group while greek indices of the middle of the alphabet label vector representations. A real system of coordinates on $\Sigma$ is denoted by $x^1, x^2$ which combine to give holomorphic coordinates,

$$ z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \quad (A.1) $$

in such a way that a vector $V^\mu$ with real components $(V^1, V^2)$ has holomorphic components given by,

$$ V_z = \frac{1}{2}(V_1 - iV_2), \quad V_{\bar{z}} = \frac{1}{2}(V_1 + iV_2). \quad (A.2) $$

The components of the locally flat Euclidean metric are:

$$ g_{zz} = g_{\bar{z}\bar{z}} = \frac{1}{2}, \quad g_{\bar{z}z} = g_{z\bar{z}} = 2, \quad (A.3) $$

The epsilon symbol is chosen in such a way that,

$$ \epsilon^{12} = -\epsilon^{21} = 1, \quad (A.4) $$

or, in holomorphic coordinates,

$$ \epsilon^{\bar{z}z} = -\epsilon^{z\bar{z}} = 2i. \quad (A.5) $$

On a curved space, the epsilon symbol behaves as a tensor density in such a way
that $\varepsilon^{\mu\nu}$ defined by,

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{g}} \varepsilon^{\mu\nu}$$  \hspace{1cm} (A.6)

behaves as a tensor.

Our choice of Euclidean Dirac matrices $\gamma^\mu$ is,

$$(\gamma^1)_\alpha^\beta = \sigma^1, \quad (\gamma^2)_\alpha^\beta = \sigma^2,$$  \hspace{1cm} (A.7)

where $\sigma^1, \sigma^2$ are Pauli matrices. Lorentz spinor indices are lowered and raised by the matrix $C_{\alpha\beta} = \sigma^1$,

$$(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)^{\alpha\tau} C_{\tau\beta}.$$  \hspace{1cm} (A.8)

The algebra of the generators of $N = 2$ supersymmetry takes the form,

$$\{Q^a_\alpha, Q_{\beta b}\} = \delta_b^a \gamma^\mu_{\alpha\beta} P_\mu.$$  \hspace{1cm} (A.9)

where Latin indices label the spinor representation of the internal $SO(2)$ symmetry. When both, Lorentz $SO(2)$ and internal $SO(2)$ indices are written explicitly, they are separated by a comma. The algebra (A.9) reads:

$$\{Q_{+,+}, Q_{+, -}\} = 2P_z = 2\partial_z,$$

$$\{Q_{-,+}, Q_{-, -}\} = 2P_\bar{z} = 2\partial_{\bar{z}},$$  \hspace{1cm} (A.10)

while all others anticommutators vanish.

Let us recall a few facts about complex geometry which are useful for the comprehension of the paper. Let us consider even-dimensional manifold $M$. An almost complex structure $J_j^i$ on $M$ is a $(1, 1)$ tensor satisfying,

$$J_i^k J_k^j = -\delta_i^j.$$  \hspace{1cm} (A.11)

A manifold $M$ is called almost complex if it admits an almost complex structure $J_i^j$. This manifold $M$ is called a complex manifold if it admits an atlas with a
complex coordinate system in such a way that transition functions between neighbor charts are holomorphic. In other words, if the complex coordinates of two patches $U$ and $V$ ($U \cap V \neq \emptyset$) are $X^I$, $X^I$, and $Y^I$, $Y^I$, respectively, then the transition functions in $U \cap V$ are such that,

\begin{align*}
X^I &= X^I(Y^J), \quad \partial_{Y^I} X^I = 0, \\
X^J &= X^J(Y^J), \quad \partial_{Y^I} X^J = 0.
\end{align*}

(A.12)

In this holomorphic complex coordinate system the complex structure $J^i_j$ can be defined with constant entries,

\begin{align*}
J^i_j = -i \delta^i_j, \quad J^i_j = i \delta^i_j, \quad J^i_j = J^i_j = 0.
\end{align*}

(A.13)

A complex manifold with Riemannian metric $G_{ij}$ is called almost Hermitian if

\begin{align*}
G_{ij} = J^p_i J^l_j G_{pl}.
\end{align*}

(A.14)

Using property (A.11) this statement is equivalent to,

\begin{align*}
J_{ij} \equiv J^k_i G_{kj} = -J_{ji}.
\end{align*}

(A.15)

Hermiticity is a restriction on the metric, and not on the manifold. If a complex manifold admits a metric $H_{ij}$, then it also admits the metric $G_{ij}$ defined as,

\begin{align*}
G_{ij} = \frac{1}{2}(H_{ij} + J^p_i J^l_j H_{pl}),
\end{align*}

(A.16)

which is Hermitian. Moreover, in holomorphic coordinates, after using (A.13),

\begin{align*}
G_{I\bar{J}} = G_{I\bar{J}} = 0,
\end{align*}

(A.17)

being the nonvanishing components of the metric of the type $G_{I\bar{J}} = G_{\bar{J}I}$. 
From (A.15) follows that on an almost Hermitian manifold one can naturally define the two-form $J,\nabla\vspace{0.5em}
\begin{align}
J = \frac{1}{2} J_{ij} dX^i \wedge dX^j. \tag{A.18}
\end{align}
\nAn almost Hermitian manifold is called Kahler if this two-form is closed,
\n\begin{align}
dJ = 0, \tag{A.19}
\end{align}

which, in holomorphic components becomes,
\n\begin{align}
\partial K G_{IJ} = \partial I G_{KJ}, \quad \partial K G_{IJ} = \partial J G_{IK}. \tag{A.20}
\end{align}

This implies the existence of a Kahler potential $K(X^I, X^\bar{I})$ such that,
\n\begin{align}
G_{IJ} = \partial I \partial J K. \tag{A.21}
\end{align}

Christoffel connections can be computed straightforwardly from previous expressions. The nonvanishing components are:
\n\begin{align}
\Gamma^I_{JK} = G^I_{\bar{L} \bar{K}} \partial J G_{\bar{K} \bar{L}}, \quad \Gamma^\bar{I}_{\bar{J} \bar{K}} = G^{\bar{I} \bar{L}} \partial \bar{J} G_{\bar{L} \bar{K}}. \tag{A.22}
\end{align}

This fact implies that the only non-trivial components of the Riemann tensor are,
\n\begin{align}
R_{IJKL} = G_{IM} \partial_K \Gamma^M_{JL}, \tag{A.23}
\end{align}

plus all others which are obtained using the symmetry properties of the Riemann tensor.
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