ABSTRACT:
The concept of “Point Cloud” has played an increasingly important rôle in many areas of Engineering, Science, and Mathematics. Examples are: LIDAR, 3D-Printing, Data Analysis, Computer Graphics, Machine Learning, Mathematical Visualization, Numerical Analysis, and Monte Carlo Methods. Entering “point cloud” into Google returns nearly 3.5 million results! A point cloud for a finite volume manifold $M$ is a finite subset or a sequence in $M$, with the essential feature that it is a “representative” sample of $M$. The definition of a point cloud varies with its use, particularly what constitutes being representative. Point clouds arise in many different ways: in LIDAR they are just 3D data captured by a scanning device, while in Monte Carlo applications they are constructed using highly complex algorithms developed over many years. In this article we outline a rigorous mathematical theory of point clouds, based on the classic Cauchy-Crofton formula of Integral Geometry and its generalizations. We begin with point clouds on surfaces in $\mathbb{R}^3$, which simplifies the exposition and makes our constructions easily visualizable. We proceed to hypersurfaces and then submanifolds of arbitrary codimension in $\mathbb{R}^n$, and finally, using an elegant result of Jürgen Moser, to arbitrary smooth manifolds with a volume element.

1. Introduction

While the mathematical ideas to be discussed in this article have a theoretical flavor, they in fact evolved from our need to overcome vexing practical problems that arose during our development of mathematical visualization software.\(^1\) Those problems centered on difficulties we encountered while trying to devise appropriate rendering methods for displaying surfaces that are implicitly defined. Important examples of such surfaces are often of large genus, being defined as level sets of polynomials of high degree, and as a result cannot be explicitly parameterized. In addition they may be immersed rather than embedded and have singularities and self-intersections that both add to their interest but also make them difficult to visualize. A major problem contributing to the difficulty of working algorithmically with implicitly defined surfaces is that it is much harder to “choose” a point on an implicit surface than on one given parametrically, a consideration that we will discuss in more detail later. These inter-related factors make the customary approach to rendering implicit surfaces, based on “raytracing”, much slower and less flexible than the various methods available for rendering parametric surfaces, and also make it more difficult to display their important structural features. Not surprisingly then, we have kept searching for improved methods for visualizing implicit surfaces.

In raytracing a surface, one carries out a complex process for each pixel on the screen, even for those that turn out not to show a point of the surface. Moreover, even as graphics hardware has improved in speed, screen resolutions have increased.
in step, so that raytracing remains a time-consuming process. About twelve years
ago it occurred to us to try a “Pointilist” approach—that is to represent our implicit
surfaces by “point clouds”, i.e., relatively dense sets of points distributed evenly over
the surface. We guessed that if we could create such clouds of points and render
them in anaglyph stereo (the kind requiring red/green glasses), then not only could
we get away with processing far fewer pixels—for a major speedup—but as a side-
effect we would automatically achieve the transparency required to show all parts
of a surface simultaneously, and in this way reveal otherwise hidden structural
details that we often wished to display. Early experiments were highly encouraging,
confirming these hopes and expectations. In particular, when rotating a surface, a
point cloud, unlike a raytrace, only needs to be computed once, so even when using
slower hardware it gave an important speed-up. But a major problem remained
before these experiments could mature into a fully successful rendering technique;
namely we had to develop effective and efficient algorithms for generating point
clouds that had the required uniform density.

In the remainder of this article we will explain the mathematical ideas behind our
solution to that problem and its generalizations and some interesting explorations
we were led to while working on it. Using few (and metaphorical) words, what
eventually worked was throwing random darts and seeing where they hit the surface,
or in less picturesque language, seeing where randomly selected lines intersected
the surface. This approach was both motivated and justified by the famous Cauchy-
Crofton Formula from Integral Geometry, which says that the area of a piece of
surface is proportional to the average number of times lines intersect it.

But a lot is hidden in that word “random”! Our use of the term has little relation
to the randomness in physical processes like radioactivity, or the way repeated
measurements with a physical device are distributed about their mean. Rather, as
our experiments continued, we quickly discovered that to make our point clouds
look “right”, it was essential for the points of the cloud to be “equidistributed with
respect to area”, meaning roughly that parts of the surface with approximately
equal areas should contain approximately equal numbers of cloud points. Finding
out how to make rigorous mathematical sense of that, and how to implement it
algorithmically was a key step in developing our point cloud rendering techniques.
It led us on a learning adventure into an old and fascinating area of mathematics;
namely how to use a Random Number Generators (RNG) to distribute points uni-
formly in all manner of interesting spaces, and we hope that the results presented
below will count as a useful contribution to that circle of ideas.

2. Review of Required Background

1. Some Fixed Notation, Definitions, and Terminology.

We will denote by \( B^n \) and \( S^{n-1} \) respectively the open unit ball and the unit
sphere in \( \mathbb{R}^n \) (centered at the origin), and for \( v \in S^{n-1} \) we denote by \( v^\perp \) the \((n-1)\)-
dimensional subspace of \( \mathbb{R}^n \) orthogonal to \( v \). If \( D \) is a bounded open subset of \( v^\perp \) and
\( f \) is a smooth real-valued function defined on \( D \), then we call the subset \( \{(x, f(x)v) \mid x \in D \} \)
a smooth graph hypersurface in \( \mathbb{R}^n \). If \( D \) has a (piecewise) smooth boundary,
and if \( f \) extends to be smooth on the boundary of \( D \), then we call \( \{(x, f(x)v) \mid x \in \bar{D} \} \)
a smooth graph hypersurface with (piecewise) smooth boundary. A subset \( \Sigma \) of \( \mathbb{R}^n \)
will be called a hypersurface in \( \mathbb{R}^n \) if it can be represented as a finite union of smooth
graph hypersurfaces in \( \mathbb{R}^n \), some possibly with (piecewise) smooth boundary. If
\( F : \mathbb{R}^n \rightarrow \mathbb{R} \) is a smooth function, then \( p \in \mathbb{R}^n \) is a critical point of \( F \) if \( DF_p = 0 \),
and \( r \in \mathbb{R} \) is a regular value of \( F \) if the the level set \( \Sigma := \{ x \in \mathbb{R}^n \mid F(x) = r \} \)
contains no critical points of \( F \); it then follows from the Implicit Function Theorem
that any bounded part of such a \( \Sigma \) is a regular hypersurface, and such a hypersurface
is what we will mean by an implicit surface.
2. Sequences.

Sequences arise frequently in what follows and it will be convenient to establish notation to help deal efficiently with our uses of them. An infinite sequence of points, \( x_1, x_2, \ldots \), will be denoted by a symbol such as \( \{x_i\} \), in which case, for a positive integer \( N \), \( \{x_{N+i}\} \) denotes the infinite subsequence \( x_{N+1}, x_{N+2}, \ldots \), while \( \{x_i\}_N \) will denote the complementary finite sequence \( x_1, x_2, \ldots, x_N \). If \( E \) is a set, \( \#(\{x_i\}_N, E) \) will denote the number of indices \( i \) with \( 1 \leq i \leq N \) such that \( x_i \in E \).

Then a natural interpretation of the quotient, \( \frac{\#(\{x_i\}_N, E)}{N} \) (and one appropriate for our purposes) is the probability that if \( 1 \leq i \leq N \) then \( x_i \in E \). This leads us to the following:

**Definition.** Let \( X \) be a set and \( \{x_i\} \) an infinite sequence in \( X \). For each subset \( E \subseteq X \), if \( \lim_{N \to \infty} \frac{\#(\{x_i\}_N, E)}{N} \) exists and equals \( L \), then we say that \( \text{Prob}(\{x_i\} \in E) \) is defined and is equal to \( L \), otherwise we say that \( \text{Prob}(\{x_i\} \in E) \) is undefined.

3. Densities and Volume Forms.

We will assume familiarity with the theory of volume forms on manifolds and the associated measure theory. However these play such an important role in what follows that we provide next a quick review to establish consistent notation and terminology. For details see Chapter 3 of [Sp-1] and Chapter 7 of [Sp-2].

In what follows, \( M \) denotes a smooth (i.e., \( C^\infty \)) \( n \)-dimensional manifold, possibly with a piecewise smooth boundary, \( \partial M \). (Much of the time, \( M \) will be assumed compact.) \( \text{Diff}(M) \) will denote the group of diffeomorphisms of \( M \) with the \( C^\infty \) topology. A density for \( M \), \( d\mu \), is the absolute value \( |\omega| \) of a smooth \( n \)-form \( \omega \), and if \( \omega \) does not vanish anywhere we call \( d\mu \) a volume element. Since the space \( \Lambda^n(T^*M) \) of \( n \)-forms at a point \( p \) of \( M \) is one-dimensional, if \( d\mu \) is a volume element then any density \( d\gamma \) for \( M \) can be written uniquely in the form \( d\gamma = \rho \, d\mu \) where \( \rho : M \to \mathbb{R}^+ \) is a nonnegative smooth function.

In particular, suppose that \( (x_1, \ldots, x_n) \) is a local coordinate system for an open set \( O \subseteq M \), given by a diffeomorphism \( x : O \to x(O) \subseteq \mathbb{R}^n \). Then in \( O \) we have the coordinate representations: \( \omega = f(x_1, \ldots, x_n) \, dx_1 \wedge \ldots \wedge dx_n \) and \( d\mu = |f(x_1, \ldots, x_n)| \, dx_1 \wedge \ldots \wedge dx_n \) for some smooth function \( f : x(O) \to \mathbb{R} \).

If \( (\tilde{x}_1, \ldots, \tilde{x}_n) \) is a second coordinate system in \( M \), then the Change of Variable Formula of multi-variable calculus says that the corresponding coordinate representations of \( \omega \) and \( d\mu \) with respect to these new coordinates is given by replacing the function \( f(x_1, \ldots, x_n) \) by the function \( \tilde{f}(\tilde{x}_1, \ldots, \tilde{x}_n) = \text{Jac}(\tilde{x}, x) \, f(x_1(\tilde{x}), \ldots, x_n(\tilde{x})) \), where \( \text{Jac}(\tilde{x}, x) := \text{det} \left( \frac{\partial \tilde{x}_i}{\partial x_j} \right) \) is the Jacobian determinant of the transformation of coordinates \( \tilde{x}_i = \tilde{x}_i(x) \) from \( x(O) \) to \( \tilde{x}(O) \) and \( x_i = x_i(\tilde{x}) \) is the inverse transformation from \( \tilde{x}(O) \) to \( x(O) \). We will call the coordinate system \( (x_1, \ldots, x_n) \) canonical coordinates with respect to \( d\mu \) if \( f \) is identical one in \( O \), i.e., if \( \omega = dx_1 \wedge \ldots \wedge dx_n \) in \( O \) (so that \( d\mu = |dx_1 \wedge \ldots \wedge dx_n| \) in \( O \)). For such coordinates, the measure \( \mu(K) \) of a compact subset \( K \subseteq O \) is just the Lebesgue measure of the image \( x(K) \subseteq \mathbb{R}^n \) under the coordinate map \( x : O \to \mathbb{R}^n \). It is easy to see that at every point \( p \) of \( M \) we can choose canonical coordinates.

For if \( (y_1, \ldots, y_n) \) is any coordinate system centered at \( p \) and \( \omega = f(y_1, \ldots, y_n) \, dy_1 \wedge \ldots \wedge dy_n \), then we can define new coordinates \( (x_1, \ldots, x_n) \) by the transformation law \( x_1(y_1, \ldots, y_n) := \int_{y_1}^1 f(t, y_2, \ldots, y_n) \, dt \) and \( x_i := y_i \) for \( i > 1 \). Then \( dx_1 = f(y) \, dy_1 + \sum_{j=2}^n a_j(y) \, dy_j \) so \( dx_1 \wedge (dy_2 \wedge \ldots \wedge dy_n) = \omega \), proving that \( (x_1, \ldots, x_n) \) are canonical coordinates.

The density \( d\mu \) determines a Radon measure \( \mu \) on \( M \). If \( g \) is a continuous real-valued function on \( M \) with support a compact subset of the coordinate domain \( O \) above, then \( \int_M g(p) \, d\mu(p) = \int_O g(x_1, \ldots, x_n) \, f(x_1, \ldots, x_n) \, dx_1 \ldots dx_n \). The integral for a more general \( g \) can then be computed using a partition of unity subordinate to a covering by coordinate neighborhoods.

If \( F : M \to N \) is a diffeomorphism between two compact \( n \)-dimensional manifolds
and $d\nu$ is a density for $N$, then its pull-back, $d\mu := DF^*(d\nu)$, is a density for $M$, and if $d\nu$ is a volume element then so is $d\mu$. The corresponding measures, $\nu$ and $\mu$ on $N$ and $M$, are clearly related by $\nu = \mu \circ F^{-1}$, so in particular $\nu(N) = \mu \circ F^{-1}(N) = \mu(M)$, i.e., they have the same total volume, and it follows in particular that the natural action of $\text{Diff}(M)$ on the space of volume forms on $M$, given by pull-back, preserves total volumes.

4. Measured Manifolds.

By a measured manifold we mean a smooth manifold $M$ together with a choice of volume element, $d\mu_M$, and by canonical coordinates for $M$ we will mean canonical coordinates with respect to $d\mu_M$. Any smooth density $d\gamma$ on $M$ can be written uniquely as $d\gamma = \alpha \, d\mu_M$ where $\alpha$ is a non-negative smooth function on $M$ that we will refer to as the Radon-Nikodym derivative of $d\gamma$.

Various kinds of structured manifolds have natural choices of volume elements. For example if $M$ is a symplectic manifold of dimension $n = 2k$ and $\lambda$ is its symplectic form, then $d\mu_M := |\lambda^k|$ defines $M$ as a measured manifold, and similarly, for a Riemannian manifold $M$ the standard choice is $d\mu_M := |\omega|$, where for any point $p$ of $M$, $\omega_p = \theta_1 \wedge \ldots \wedge \theta_n$, with $\theta_1, \ldots, \theta_n$ any orthonormal basis for $T^*M_p$. Equivalently, in local coordinates $(x_1, \ldots, x_n)$, $d\mu_M = \sqrt{g} \, dx_1 \, dx_2 \ldots dx_n$, where $g := \det(g_{ij})$ is the determinant of the metric tensor. In particular we will consider any smooth submanifold of $\mathbb{R}^n$ to be a Riemannian, and hence measured, manifold, with its induced Riemannian metric. More particularly, a linear subspace, $V$, of $\mathbb{R}^n$ is a measured manifold with the associated measure being Lebesgue measure, and in this case we will as usual write integrals as $\int_V g(x) \, dx$.

**Definition.** If $M$ is a measured manifold, then an embedding $F : M \to \mathbb{R}^n$ of $M$ into a Euclidean space will be called measure preserving if it pulls back the Riemannian volume form on $F(M)$ to the volume form $d\mu_M$ on $M$.

Let $F : M \to N$ be a smooth map between two measured manifolds of the same dimension. The pull-back $DF^*(d\mu_N)$ is a smooth density on $M$ and so, as above, it can be written in the form $\alpha \, d\mu_M$ for a unique $\alpha : M \to \mathbb{R}^+$. For the case $M = N = \mathbb{R}^n$, the Change of Variable Theorem says that $\alpha = \left|\det \left( \frac{\partial F}{\partial x} \right) \right|$, the absolute value of the Jacobian determinant. If we choose canonical coordinates for $M$ at $p$ and for $N$ at $F(p)$, then using these coordinates we can consider $F$ as a map of $\mathbb{R}^n$ to $\mathbb{R}^n$, and again $\alpha$ will be the absolute value of the Jacobian determinant. For this reason we refer to $\alpha$ as the generalized Jacobian of $F$ and write it as $|\text{Jac}(F)|$. Clearly $|\text{Jac}(F)|$ is everywhere non-negative and vanishes precisely on the set $\Sigma$ of critical points of $F$, i.e., the points where $DF_p$ is not invertible, or equivalently the points where $F$ is not a local diffeomorphism. The subset $F(\Sigma)$ of $N$ is called the set of critical values of $F$ and it is an important fact, called The Morse-Sard Theorem ([H] page 69 or [SP-1] p. 72) that it has measure zero. With this formalism, the change of variable theorem takes the following form.

**Proposition.** (Change of Variable Formula for Measured Manifolds). Let $F : M \to N$ be a smooth injective map between two measured manifolds of the same dimension, and let $g$ be an integrable function on $N$. Then $g \circ F$ is integrable on $M$ and

$$\int_N g(y) \, d\mu_N(y) = \int_M g(F(x)) \, |\text{Jac}(F)|(x) \, d\mu_M(x).$$

In particular $\mu(N) = \int_M |\text{Jac}(F)| \, d\mu_M$.

5. A Theorem of Jürgen Moser.

As was noted above, for a compact, smooth manifold $M$, the natural action (given by pull-back) of $\text{Diff}(M)$ on the space of volume forms on $M$ preserves total volumes. In [M], Moser gave a straightforward and elegant proof of the following "converse" result.
Mosser’s Theorem. If $M$ is a compact, smooth manifold then $\text{Diff}(M)$ acts transitively on the space of volume forms on $M$ having any fixed total volume. In particular, if $M$ is a compact measured manifold and $d\nu$ is any volume element for $M$ with the same total volume as $d\mu_M$, then there is an $\phi \in \text{Diff}(M)$ such that $d\mu_M = D\phi^*(d\nu)$.

Corollary (Mosser-Whitney Theorem). A compact, smooth, measured manifold $M$ of dimension $n$ admits a volume preserving embedding in $\mathbb{R}^{2n}$.

Proof. By a theorem of Hassler Whitney [Wh], there exists a smooth embedding $F$ of $M$ onto a submanifold $M'$ of $\mathbb{R}^{2n}$. Scaling $\mathbb{R}^{2n}$ by a suitable positive constant $c$ will map $M'$ diffeomorphically onto a submanifold $M''$ of $\mathbb{R}^{2n}$ with the same volume as $M$, hence, replacing $F$ by $cF$, we can assume that the total volume of $M$ and of $M'$ are equal. Let $\nu$ denote the (Riemannian) volume element of $M'$. Then, since $F$ is a diffeomorphism, $d\mu_M$ and $DF^*(\nu)$ are volume elements for $M$ with the same total volume and, by Moser’s Theorem, there exists a diffeomorphism $\phi \in \text{Diff}(M)$ such that $d\mu_M = D\phi^*(DF^*(\nu))$, so that $F \circ \phi : M \to \mathbb{R}^{2n}$ gives the desired embedding.

6. Maps Between Measured Manifolds and The Co-area Formula.

We will need the following important generalization of the Change of Variable Formula that handles the case that $F$ is not assumed to be injective. We write $\#(X)$ for the number of points in a set $X$.

Theorem (Co-area Formula for measured manifolds of the same dimension).

If $F$ is a smooth map from a compact measured manifold $M$ to another smooth manifold $N$ of the same dimension then

$$\int_N \#(F^{-1}(p)) \, d\mu_N(p) = \int_M |\text{Jac}(F)| \, d\mu_M.$$  

Proof. For any open subset $O$ of $M$ we denote by $F_O : O \to N$ the restriction of $F$ to $O$ and we will prove that the formula \( \int_N \#(F_O^{-1}(p)) \, d\mu_N(p) = \int_O |\text{Jac}(F)| \, d\mu_M \) holds for all $O$. Of course the Co-Area Formula is just the special case where we take $O = M$, but using the more general statement will give us extra flexibility that we will make use of in the proof.

We first consider the case that $F_O$ is injective. Then $\#(F_O^{-1}(p))$ is identically one on $F(O)$ and zero on its complement, i.e., it is the characteristic function $\chi(p)$ of $F(O)$, and the claimed formula reduces to $\mu_N(F(O)) = \int_O |\text{Jac}(F)| \, d\mu_M$. So this first case follows from the preceding proposition by taking $F$ to be $F_O$ and $g$ to be $\chi$.

The set $\Xi$ of critical points of $F$ is clearly closed in $M$, and since $\text{Jac}(F)$ vanishes on $\Xi$, and $F(\Xi)$ has measure zero, we can replace the open set $O$ by $O \setminus \Xi$ without changing either side of the claimed formula. So, without loss of generality, we can assume that $F_O$ has no critical points and therefore, by the implicit function theorem, that it is a local diffeomorphism. Thus, for each $p \in O$ we can choose an open neighborhood $U$ of $p$ for which $F_U$ is a diffeomorphism onto $F(U)$, and we can choose $U$ so small that its closure $\bar{U}$ is compact and its boundary, $\partial U := \bar{U} \setminus U$ has measure zero. Since these $U$ form an open covering of $O$, by Lindelöf’s Theorem we can find a sequence $\{U_i\}$ of these $U$ that already cover $O$. By choice of the $U_i$, each $F_{U_i}$ is a diffeomorphism onto its image, so by the first case we have $\int_N \#(F_{U_i}^{-1}(p)) \, d\mu_N(p) = \int_{U_i} |\text{Jac}(F)| \, d\mu_M$, and moreover the same holds if we replace $U_i$ by any open set $O_i \subseteq U_i$.

We next observe that for each $p$ in $N$, the function $V \mapsto \#(F_V^{-1}(p))$ is a countably additive function on the subsets of $N$; in fact it is the measure given by the sum of
delta measures one at each point of $F^{-1}(p)$. To use this observation we “disjointify” the sequence $\{U_i\}$, that is we replace it by the sequence $O_i$ of disjoint open sets given by $O_i := U_1$ and $O_{i+1} := U_i \setminus \bigcup_{j=1}^{i} U_j$. While these $O_i$ no longer cover $O$, their union $O' := \bigcup_{i=1}^\infty O_i$ differs from $O$ only by the set of measure zero $\bigcup_{i=1}^\infty \partial U_i$. And since $O_i \subseteq U_i$ the first case gives $\int_N \#(F_{O_i}^{-1}(p)) \, d\mu_N(p) = \int_{O_i} |\text{Jac}(F)| \, d\mu_M$.

If we define $\Omega_i := \bigcup_{j=1}^i O_i$, then by the above observation and the disjointness of the $O_i$ we have:

$$\int_N \#(F_{\Omega_i}^{-1}(p)) \, d\mu_N(p) = \sum_{j=1}^i \int_N \#(F_{O_j}^{-1}(p)) \, d\mu_N(p)$$

$$= \sum_{j=1}^i \int_{O_j} |\text{Jac}(F)| \, d\mu_M$$

$$= \int_{\Omega_i} |\text{Jac}(F)| \, d\mu_M.$$

Now on the left, the sequence of integrands $\{\#(F_{\Omega_i}^{-1}(p))\}$ converges monotonically to $\#(F_{O'}^{-1}(p))$, so by the Monotone Convergence (aka Beppo-Levi) Theorem the limit of the sequence of integrals on the left is $\int_N \#(F_{O'}^{-1}(p)) \, d\mu_N(p)$, which equals $\int_N \#(F_{O}^{-1}(p)) \, d\mu_N(p)$ since the two integrands agree almost everywhere. And since $O'$ is the increasing union of the $\Omega_i$, the integrals on the right converge to $\int_{O'} |\text{Jac}(F)| \, d\mu_M = \int_{O} |\text{Jac}(F)| \, d\mu_M$, which completes the proof that $\int_N \#(F_{O}^{-1}(p)) \, d\mu_N(p) = \int_{O} |\text{Jac}(F)| \, d\mu_M$. \hfill \qed

3. WHAT DOES IT MEAN TO DISTRIBUTE POINTS “UNIFORMLY”

Let $\Sigma$ denote a compact, smoothly immersed hypersurface in $\mathbb{R}^n$, possibly with smooth boundary. To get started we need to make precise what it means to distribute points “uniformly” on $\Sigma$. Then, later, we will derive efficient algorithms for carrying this out. Roughly speaking, to say that an algorithm selects points uniformly on a surface means that the probability of selecting a point from a given subset should be proportional to the area of the subset. But giving a rigorous definition of this concept turns out to be considerably more involved than one might at first expect, and is of considerable independent interest, so we begin with a short discussion of the concept of equidistribution in a more general context, with references for readers who would like to see more detailed accounts. We will frame this discussion in measure theoretic terms, however the reader who is unfamiliar with (or has forgotten) measure theory can, without serious loss of generality, interpret “measure” to mean the usual volume measure on a smooth submanifold of $\mathbb{R}^n$.

1. Equidistributed Sequences.

Notation. In all that follows, $\mu$ will denote a finite, non-trivial Radon measure on a locally compact metric space $X$ and $\tilde{\mu}$ will denote the corresponding probability measure $\frac{1}{\mu(X)} \mu$. We will denote by $\{x_n\}$ a sequence of points in $X$. For a subset $E$ of $X$ and positive integer $N$ we recall from section 2.2 that we denote by $\#(\{x_n\}_N, E)$ the number of integers $n$ with $1 \leq n \leq N$ such that $x_n \in E$.

A seemingly intuitive definition of the sequence $\{x_n\}$ being “uniformly distributed” in $X$ would be to demand that, for every measurable $E$, the ratio $\frac{\#(\{x_n\}_N, E)}{N}$ should approach $\tilde{\mu}(E)$ as $N$ tends to infinity, or in the terminology of section 2.2, that $\text{Prob}(\{x_n\}_N \subseteq E)$ is defined and equals $\tilde{\mu}(E)$. But that condition could never be met by any sequence $\{x_n\}$, as is easily seen by taking for $E$ the set of all the $x_n$. That condition cannot hold even for all open sets $E$, since we could take for $E$ the
union of small open balls $B_n$ centered at the $x_n$ with $\mu(B_n) < \epsilon/2^n$ for arbitrarily small $\epsilon$. It turns out that we must restrict the subsets $E$ to those whose topological boundary\(^2\), $\partial E$, has measure zero.

**Definition 1.** If $\mu$ is a finite and non-trivial Radon measure on a locally compact metric space $X$, then a sequence of points $\{x_n\}$ in $X$ is said to be 1-equidistributed with respect to $\mu$ if it satisfies either and hence both of the following two equivalent conditions:

1) For every measurable set $E$ whose topological boundary has measure zero,

$$\text{Prob}(\{x_n\} \in E) \text{ is defined and equals } \frac{\mu(E)}{\mu(X)}.$$ 

2) \[ \frac{1}{\mu(X)} \int_X f(x) \, d\mu(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) \]

holds for every bounded continuous function $f : X \to \mathbb{R}$.

**Remark 1.** If we take for the function $f$ the characteristic function of the set $E$ then, while of course such an $f$ will not in general be continuous, it is worth noting that in this case 1) and 2) reduce to the same thing.

**Remark 2.** Whenever the equality 2) holds for a function $f$, we will say that the integral of the function $f$ can be evaluated by the Monte Carlo method, using the sequence $\{x_n\}$.

**Remark 3.** Recall that a sequence $\{\mu_n\}$ of finite Borel measures on $X$ is said to converge weakly to the Borel measure $\mu$ if for every bounded continuous function $f : X \to \mathbb{R}$, the sequence of integrals $\int f(x) \, d\mu_n(x)$ converges to $\int f(x) \, d\mu(x)$. If $\delta_x$ denotes the Dirac delta measure at $x$ then 2) just says that the sequence of averages $\frac{1}{N}(\delta_{x_1} + \ldots + \delta_{x_N})$ converges weakly to $\mu$.

The equivalence of the two conditions of Definition 1 is a consequence of a more general result of probability theory referred to as the Portmanteau Theorem. We shall not give a proof here but refer instead to Section 2, Chapter 1 of [B], where Billingsley proves the Portmanteau Theorem as his Theorem 2.1. Other good sources for a careful discussion of this equivalence and related questions are [K1] and [K-N]. In our case of primary interest, where $X$ is a smooth hypersurface embedded in some Euclidean space and $\mu$ is the area measure, we note that if the sequence $\{x_n\}$ is 1-equidistributed then in particular $\#(\{x_n\} \cap O)/N$ converges to $\mu(O)/\mu(X)$ for all open subsets $O$ of $X$ with smooth boundary, and this is a good intuitive way to think about the meaning of 1-equidistributed in this setting.

It turns out that being 1-equidistributed is a fairly weak restriction on the sequence $\{x_n\}$, and in particular it has little to do with its being random. We can however use it to define a sequence of increasingly more stringent conditions—called being $k$-equidistributed, and if $\{x_n\}$ is $k$-equidistributed for all positive integers $k$ then we will say that it is completely equidistributed. And being completely equidistributed does turn out to imply many of the conditions one would impose on a sequence for it to be called pseudo-randomly distributed with respect to $\mu$. (For the case $X = [0, 1]$ with $\mu$ Lebesgue measure, see [F].) First let’s look at the definition of 2-equidistributed. Given an infinite sequence $\{x_1, x_2, \ldots\}$ in our measure space $(X, \mu)$ as above, we get an associated sequence $\{(x_1, x_2), (x_2, x_3), \ldots\}$ in the Cartesian product $X^2 := X \times X$, and we say that the former sequence is 2-equidistributed in $(X, \mu)$ if the latter sequence is 1-equidistributed in $(X^2, \mu^2)$, where $\mu^2 := \mu \times \mu$ denotes the usual product measure. More generally:

\(^2\)I.e., the intersection of the closure of $E$ and the closure of its complement.
Definition 2. If \( X, \mu, \) and \( \{x_n\} \) are as in Definition 1 and \( k \geq 2 \) is a positive integer, then \( \{x_n\} \) is said to be \( k \)-equidistributed with respect to \( \mu \) if the sequence in \( X^k \) whose \( n \)-th element is \( (x_n, x_{n+1} \ldots, x_{n+k-1}) \) is 1-equidistributed with respect to \( \mu^k \), i.e., if for each bounded, continuous real-valued function \( f : X^k \to \mathbb{R} \)

\[
\frac{1}{\mu(X)^k} \int_X f(x_1, \ldots, x_k) \, d\mu(x_1) \ldots d\mu(x_k) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n, x_{n+1} \ldots, x_{n+k-1}).
\]

If \( \{x_n\} \) in \( X \) is \( k \)-equidistributed for all positive integers \( k \) then we say that the sequence \( \{x_n\} \) is completely equidistributed with respect to \( \mu \).

So \( \{x_n\} \) is \( k \)-equidistributed if and only if the Monte Carlo approach for evaluating \( k \)-fold integrals using the sequence \( \{x_n\} \) works for all bounded continuous real-valued functions of \( k \) variables on \( X \), and the Portmanteau Theorem says that this is equivalent to being able to estimate the product measure of a subset \( E \) of \( X^k \) as the limit of the average number of points of the sequence \( \{(x_n, x_{n+1} \ldots, x_{n+k-1})\} \) that belong to \( E \), provided the boundary of \( E \) has measure zero.

Since a function on \( X^k \) can be regarded as a function on \( X^{k+l} \) that is independent of its last \( l \) arguments, it follows that a \( k \)-equidistributed sequence \( \{x_n\} \) is also \( k' \)-equidistributed for all \( k' \) with \( 1 \leq k' < k \). However, a sequence that is \( k \)-equidistributed need not be \( k' \)-equidistributed for \( k' > k \); the following is an example of a 1-equidistributed sequence that is not 2-equidistributed.

2. The van der Corput Sequences.

It is usually not easy to write down a simple and explicit equidistributed sequence. However there is an old (1935) and elegant family of examples for the unit interval \((0, 1)\) that is due to J. G. van der Corput (for details, see [J] p.216). Namely for a fixed positive integer greater than 1 (the Radix) write the sequence of positive integers in that Radix, ending each with .0, and then “reflect across the Radix point”. For example, if Radix = 2, the sequence of integers in binary notation is:

\[
1.0, 10.0, 11.0, 100.0, 101.0, 110.0, 1000.0, 1001.0, 1010.0, 1011.0, 1100.0, 1101.0, 1110.0, 1111.0, \ldots
\]

and reflecting each element of this sequence in its binary point gives:

\[
0.1, 0.01, 0.11, 0.001, 0.101, 0.011, 0.111, 0.0001, 0.1001, 0.0101, 0.1101, 0.0011, 0.1011, 0.0111, 0.1111, \ldots
\]

or using fractional notation:

\[
\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{1}{8}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \ldots
\]

and this is, by definition, the Radix = 2 or binary van der Corput sequence. This recipe for forming the van der Corput sequences looks a bit mysterious, and, naively, it might appear more natural to rearrange the above sequence in the following order:

\[
\frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{1}{8}, \frac{5}{16}, \frac{13}{16}, \frac{3}{16}, \frac{11}{16}, \frac{7}{16}, \frac{15}{16}, \ldots
\]

i.e., arrange each successive sub-series of proper fractions with denominator \( 2^n \) and odd numerators in increasing order. However, while the original binary van der Corput sequence is particularly well equidistributed (technically, it is what is called a “low-discrepancy sequence”) the rearranged sequence turns out not to be equidistributed at all!
4. Random Number Generators (RNGs)

The further theory of equidistributed sequences is based on so-called Random Number Generators (or RNGs), which we shall always understand here to mean highly equidistributed sequences \( \{\xi_j\} \) in the interval \([0,1)\) of \( \mathbb{R} \) or, more precisely, algorithms for constructing such sequences. From its inception, this subject has drawn the attention of many famous mathematicians, including Herman Weyl [W], Stanislaw Ulam [U], John von Neumann [VN], and Donald Knuth [K1]. There is an extensive literature devoted to the subject, and many excellent articles and books that provide careful and detailed coverage of the theory, including: [B], [B-C], [E], [K1], [K-N]. Here we will only describe the basic facts concerning RNGs, concentrating on those we will need later in this article, and for proofs and further details we will refer to the above sources.

An obvious first question to consider is whether completely equidistributed RNGs even exist and if so, how to go about creating them. If one is satisfied with the pure mathematician’s usual non-constructive approach to the real numbers the answers to these questions seem encouraging. It was shown already in [F] that for almost all real \( \theta \) larger than one, the fractional part of \( \theta^n \) defines a completely equidistributed RNG, and while no explicit such \( \theta \) has been identified, in [K1], Knuth does describe an explicit completely equidistributed RNG.

But RNGs play an essential role in practical computer science, making it important to have RNG algorithms that can be coded efficiently on modern computers. Here the situation is considerably more complex. As is well-known, naive attempts at a rigorous machine-oriented theory of RNGs quickly founder on apparent paradoxes. Since even a 1-equidistributed sequence is dense, it must a fortiori be infinite, and in particular it cannot be periodic. But if one tries to design an algorithm to produce a RNG on a computer using a fixed precision to represent real numbers, there are only a finite set of reals that can be represented. And even if one uses “arbitrary precision arithmetic” (i.e., allows the number of computer words used to represent a real number to grow as more precision is required) then although there are potentially infinitely many real numbers available, the sequence will still be periodic. This is because any real machine has only finitely many states, and once the algorithm has produced enough elements of the sequence, it must eventually repeat a state of the machine and at this point the sequence will become periodic. There is no panacea for solving this problem, but a careful discussion of how to approach it will be found in the first section of L’Ecuyer’s review article [E]. Suffice it to say that for any realistic application it is possible to find a very highly equidistributed RNG that works “for all practical purposes”. To give some idea of what is available in this direction, there is the remarkable RNG, called The Mersenne Twister, due to Matsumoto and Nishimura [M-N], that has a period of \( 2^{19937} - 1 \) and, to 32-bit accuracy, is 623-equidistributed!

5. Methods for Constructing New Equidistributed Sequences from Old

For the remainder of this article we assume that we have available a \( k \)-equidistributed RNG for some large \( k \). We will denote it by \( \{\xi_n\} = \{\xi_1, \xi_2, \ldots\} \), and use it to help devise methods for constructing highly equidistributed sequences in various spaces. Here are some basic examples.

**Method 1. Products of Intervals.** If \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) are two \( n \)-tuples of real numbers with \( a_i < b_i \), then we can construct a \((k-n)\)-equidistributed sequence in the product \([a_1, b_1) \times \ldots \times [a_n, b_n)\) by taking the \( j \)-th element to be \( (a_1 + (b_1 - a_1)\xi_j, \ldots, a_n + (b_n - a_n)\xi_{j+n-1}) \). This follows trivially from the definition of \( k \)-equidistributed.

**Method 2. Subsets; The Method of Rejection.** Suppose that in our measure space \((X, \mu)\) we have a \( k \)-distributed sequence \( \{x_i\} \) and let \( Y \) be a measurable subset
of $X$ having positive measure and whose boundary $\partial Y$ has measure zero. Define a sequence $\{y_i\}$ in $Y$ by taking $y_j$ to be the $j$-th element of the sequence $\{x_i\}$ that belongs to $Y$; i.e., simply “reject” all elements of the original sequence that do not lie in $Y$. Then it is immediate from Definition 1 that the sequence $\{y_i\}$ is $k$-equidistributed in $Y$.

Method 3. Unions. We can also go the other way. That is, suppose we have a $k$-equidistributed sequence $\{x_j^i\}$ for each of $N$ disjoint measurable subsets $E_i$ of $(X, \mu)$. We can construct a $k$-equidistributed sequence $\{x_j\}$ for their union $E$ as follows. Define $M_n := \sum_{i=1}^N \mu(E_i)$. To obtain $x_j$, first choose one of the $E_i$ with a probability equal to its relative measure by seeing which subinterval $[M_{j+1}, M_{j+1}]$ of $[0, M_N)$ contains $M_n x_j$, and let $x_j$ be the first element of the sequence $\{x_j^i\}$ that has not already been chosen.

Method 4. Measure Preserving Maps. Assume that we have a $k$-equidistributed sequence $\{x_n\}$ for $(X, \mu)$, a second measure space $(Y, \nu)$, and a continuous map $f$ of $X$ onto $Y$ such that $\mu(f^{-1}(E))$ is proportional to $\nu(E)$. Then the sequence $\{f(x_n)\}$ is clearly $k$-equidistributed in $Y$.

Method 5. Equidistributed Sequences for the ball, $B^n$ and Sphere $S^{n-1}$. We next show how to combine the above methods to create ($k-n$)-equidistributed sequence $\{b_j\}$ for the $n$-dimensional ball $B^n$ and $\{s_j\}$ for the $(n-1)$-sphere $S^{n-1}$. We start by using Method 1 to define a ($k-n$)-equidistributed sequence $\{x_j\}$ for the cube $C_n := [-1, 1]^n$; namely $x_j := (2\xi_{j-1}, \ldots, 2\xi_{j+n-2})$. The (punctured) unit ball $B^n := \{x \in \mathbb{R}^n \mid 0 < ||x|| < 1\}$ is a subset of $C_n$ with smooth boundary, so we can apply rejection (Method 2) to the sequence $\{x_j\}$ to get a ($k-n$)-equidistributed sequence $\{b_j\}$ for $B^n$. The normalizing map $f : B^n \to S^{n-1}$, defined by $f(x) := \frac{x}{||x||}$, satisfies the condition of Method 4, that is, if $E$ is a measurable subset of $S^{n-1}$, its surface measure is proportional to the volume measure of $f^{-1}(E)$, the cone over $E$, hence $s_j := \frac{b_j}{||b_j||}$ is a ($k-n$)-equidistributed sequence for $S^{n-1}$.

A Small Problem! A careful analysis exposes a serious difficulty with the simple and elegant Method 5 when $n$ becomes large. Note that the probability of the point $x_j \in C_n$ being “accepted” (i.e., not rejected ) as an element of $B^n$ is the ratio $\rho_n := \kappa_n/2^n$ of the volume $\kappa_n$ of the $n$-ball to the volume $2^n$ of $C_n$. When $n = 3$ this ratio is $\rho_3 = (4\pi/3)/8 = \pi/6$, so slightly more than half of the $x_j$ will be accepted. But this is deceptive as a clue to what happens for $n$ large. There is a well-known formula for $\kappa_n$, namely $\kappa_n = \pi^{n/2}/\Gamma(\frac{n}{2} + 1)$. Restricting for simplicity to the case of even $n = 2k$, we have $\kappa_{2k} = \pi^k/k!$ so $\rho_{2k} = (\pi/4)^k/k!$, and we see that, already for $n = 10$, only about one in four hundred of the $x_j$ will be accepted! Moreover the situation gets rapidly worse as $n$ increases, making Method 5 impossibly slow, so we need another, faster way to select random points of $B^n$ and $S^{n-1}$ for large $n$. We will consider only the case of $S^{n-1}$, since it is easy to see that if $x$ is a random point of $S^{n-1}$ and we choose a random $u \in [0, 1)$ with our RNG, then $x \cdot u^{1/n}$ is a random point of $B^n$.

We recall that the Normal (or Gaussian) Distribution with mean $\mu$ and variance $\sigma^2$ is the continuous probability distribution (or density) on $\mathbb{R}$ given by

$$f(x) := \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

and we will denote by $\mathcal{N}(\mu, \sigma)$ the associated measured manifold. In particular, $\mathcal{N}(0, 1)$ is often referred to as the Standard Normal Deviate. There are numerous methods for starting from a highly equidistributed RNG and using it to construct a sequence of Standard Normal Deviates, that is a sequence $\{x_i\}$ of positive real numbers that are independent and are distributed according to the distribution
\(N(0, 1)\); see for example Algorithm M of [K1] page 110. Then it is easy to check that the sequence \(y_k := (x_k, x_{k+1}, \ldots, x_{k+n})\) of points of \(\mathbb{R}^n\) has a distribution that is invariant under rotation, and it follows that the normalized sequence \(\{y_k/\|y_k\|\}\) is highly equidistributed in \(S^{n-1}\). For details, see [K1] page 116.

6. The Space of Lines and its Kinematic Measure

1. The space \(L^n\) of Oriented Lines in \(\mathbb{R}^n\). We will denote by \(L^n\) the space of oriented lines in \(\mathbb{R}^n\). A parametric equation for some \(\ell \in L^n\) has the form \(x(t) = tv + p\), where \(p\) is any point of \(\ell\) and \(v\) is a non-zero vector giving its direction. Replacing \(p\) by any other point on \(\ell\) or multiplying \(v\) by a positive scalar gives another parametric equation for the same oriented line, so the mapping from \(L^n\) to pairs \((v, p)\) is highly non-unique, and since we wish to identify each \(\ell\) with a specific pair \((v, p)\) we next specify how to choose \(v\) and \(p\) canonically. We make \(v\) unique by normalizing it, i.e., we replace \(v\) by \(v/\|v\| \in S^{n-1}\). This is equivalent to making the parameter \(t\) the signed arc-length measured from \(p\). And to make \(p\) unique we choose it to be the point of \(\ell\) nearest to the origin, which is equivalent to requiring \(p\) to be orthogonal to \(v\). In this way we identify \(L^n\) with the set of all pairs \((v, p)\) with \(v \in S^{n-1}\) and \(p \in v^\perp\), the \(n-1\) dimensional subspace of \(\mathbb{R}^n\) orthogonal to \(v\). To each such \((v, p)\) we associate the oriented parametric line \(\sigma^{(v,p)}(t) := tv + p\), and we note that \(p\) is the orthogonal projection of each point of this line onto \(v^\perp\).

We will denote by \(L^n_\alpha\) the subset of \(L^n\) consisting of lines that meet the ball \(B_r\) of radius \(r\) centered at the origin. Clearly \(\ell = (v, p)\) belongs to \(L^n_\alpha\) if and only if \(\|p\| < r\).

2. The Action of Euclidean Motions on \(L^n\). The group of Euclidean motions of \(\mathbb{R}^n\) (transformations that can be written as a rotation followed by a translation) acts in an obvious way on oriented lines, and this action is transitive, since clearly any oriented line can be mapped onto any other by a rotation followed by a translation. A rotation \(g\) maps \(\sigma^{(v,p)}\) to \(\sigma^{(gv, gp)}\), so for \((v, p) \in L^n\), \(g(v, p) = (gv, gp)\). But how does a translation \(\tau_a : x \mapsto x + a\) act on the line \((v, p)\)? Clearly translation does not affect the direction component, \(v\), and translation by a multiple of \(v\) does not affect \(p\) either, so if we write \(a_{v^\perp}\) for the component of \(a\) orthogonal to \(v\), then \(\tau_a(v, p) = (v, p + a_{v^\perp})\). In particular, on \(\{v\} \times v^\perp\) (the space of all lines with direction \(v\)) \(\tau_a\) is just translation by \(a_{v^\perp}\) on the second component, and so in particular it is Lebesgue measure preserving.

3. The Kinematic Measure for \(L^n\). We next note that there is a natural measure on \(L^n\) that we denote by \(\mu\) and call *kinematic measure*. To specify it, we can use the identification of Radon measures with continuous linear functions on the Banach space of bounded, continuous functions with compact support. That is, given any bounded continuous function with compact support \(f(v, p)\) on \(L^n\), we must define its integral, \(\int f(v, p) \, d\mu\). For each \(v \in S^{n-1}\), first integrate \(f\) over \(v^\perp\) with respect to Lebesgue measure, getting in this way a function \(F\) on \(S^{n-1}\); \(F(v) := \int_{v^\perp} f(v, p) \, dp\), and then integrate \(F(v)\) over \(S^{n-1}\) with respect to its usual area measure \(d\sigma\), so the final result is

\[
\int_{L^n} f(v, p) \, d\mu := \int_{S^{n-1}} d\sigma(v) \int_{v^\perp} f(v, p) \, dp.
\]

We claim that both translations and rotations, and so the whole group of Euclidean motions, preserves kinematic measure. In fact, we saw above that translations act on \(L^n\) by a translation in each \(v^\perp\), and so they clearly preserve kinematic measure. And we saw that rotations act by \(g(v, p) = (gv, gp)\), from which it follows they also preserve the kinematic measure, since \(g\) maps \(v^\perp\) isometrically onto \((gv)^\perp\) and is

\[3\]Note that \(v^\perp\) is just \(T_xS^{n-1}\), the tangent space to \(S^{n-1}\) at \(v\), so we can identify \(L^n\) with \(TS^{n-1}\), the tangent bundle of \(S^{n-1}\), however we shall not have occasion to make use of this identification.
also an isometry of $S^{n-1}$. It is well-known (and elementary) that if a group acts smoothly and transitively on a smooth manifold $M$, then a non-trivial invariant volume form for $M$, if one exists, is unique up multiplication by a positive scalar. So in particular kinematic measure is characterized up to a positive scalar multiple by the fact that it is invariant under Euclidean motions.

4. Constructing A Highly Equidistributed Sequences of Lines.

Starting from our $k$-equidistributed RNG, $\{\xi_n\}$, we can now construct a sequence $\{\ell_j = (v_j, p_j)\}$ in $L^n_\ell$ that is $(k - 2n)$-equidistributed with respect to kinematic measure. In fact, it is clear from the definition of kinematic measure that we will get such a sequence if we use Method 5 of Section 5 to first select $v_j \in S^{n-1}$ and then to also select $p_j$ in the ball of radius $r$ of $v_j$.

Below we will explain the final steps that converts the sequence $\{\ell_j\}$ into a $(k - 2n)$-equidistributed sequence of points $\{x_j\}$ on any hypersurface $\Sigma \subset B^n_\ell$, essentially by intersecting the lines $\ell_j$ with $\Sigma$. But for a rigorous justification of this approach we must first recall some basic facts about transversality and The Cauchy-Crofton Formula.

7. Transversality

As just mentioned, a crucial step in our technique for constructing an equidistributed sequence of points on a bounded hypersurface $\Sigma \subset R^n$ involves producing finite subsets of $\Sigma$ by intersecting it with a line $\ell$. Intuitively we expect that a line should intersect a hypersurface in a discrete subset, and so a finite one if the hypersurface is bounded. But we must be careful; $\ell \cap \Sigma$ can be infinite, for example if $\Sigma$ is part of a cone or cylinder $C$ and $\ell$ is a generator of $C$. To make our technique rigorous we make a short excursion into transversality theory, where we will find that $\ell \cap \Sigma$ is indeed finite except possibly for a set of lines $\ell$ having kinematic measure zero. First we recall:

**Definition.** Let $M$, $N$ be smooth manifolds and $\Sigma$ a smooth submanifold of $N$. We call a smooth map $f : M \to N$ transverse to $\Sigma$ if for all $p \in f^{-1}(\Sigma)$, $\text{Im}(Df_p)$ together with $T\Sigma_{f(p)}$ span $TN_{f(p)}$.

**Remark.** We note that if $f$ is a submersion, that is if for each $p \in M$, $Df_p$ maps $TM_p$ onto $TN_{f(p)}$, then $f$ is automatically transverse to any submanifold of $N$.

**Theorem.** If a smooth map $f : M \to N$ is transverse to a submanifold $\Sigma$ of $N$ then $f^{-1}(\Sigma)$ is a smooth submanifold of $M$ whose codimension in $M$ is the same as the codimension of $\Sigma$ in $N$.

**Proof.** This is an elementary result; see Theorem 3.3 on page 22 of [H].

**Example.** If $\ell = (v, p) \in L^n$ and the map $\sigma^{(v, p)} : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto tv + p$ is transverse to a bounded hypersurface $\Sigma$ of $\mathbb{R}^n$, then the theorem says that the set of $t \in \mathbb{R}$ such that $\sigma^{(v, p)}(t)$ belongs to $\Sigma$ is a zero dimensional submanifold of $\mathbb{R}$, i.e., a discrete subset of $\mathbb{R}$, hence its image, $\ell \cap \Sigma$, is a discrete and hence finite subset of $\Sigma$.

We recall that a subset of a smooth manifold is called a null set if it meets each chart neighborhood in a subset that has Lebesgue measure zero in that chart, or equivalently if it has measure zero with respect to a measure defined by a non-vanishing differential form of top degree. A property is said to hold for almost all points if it holds outside of a null set.

**Parametric Transversality Theorem.** Let $M$, $N$ and $P$ be smooth manifolds, $F : M \times P \to N$ a smooth map and $\Sigma$ a smooth submanifold of $N$. Assume that $F$ is a submersion, or more generally that it is transverse to $\Sigma$. If for $p \in P$ we denote by $F_p : M \to N$ the map $m \mapsto F(m, p)$, then $F_p$ is transversal to $\Sigma$ for almost all $p$ in $P$.
Proof. This is Theorem 2.7 page 79 of [H].

We next apply this with \( M = \mathbb{R}, N = \mathbb{R}^n, P = \mathbb{L}^n \) as the parameter space, and defining \( F : \mathbb{R} \times \mathbb{L}^n \to \mathbb{R}^n \) by \( F(t, (v, p)) := \sigma_t^{(v, p)}(t) = tv + p \). If \( q = (t, (v, p)) \in \mathbb{R} \times \mathbb{L}^n \), recall that \( v \in \mathbb{S}^{n-1} \) and \( p \in v^\perp \), so then \( T(\mathbb{R} \times \mathbb{L}^n)_q = T(\mathbb{R})_t \oplus T(\mathbb{S}^{n-1})_v \oplus v^\perp \), and \( DF_q(\mathbb{L}^n) = v \) while for \( w \in v^\perp \), \( DF_q(w) = w \), so the image of \( DF_q \) includes all of \( \mathbb{R}v \oplus v^\perp = \mathbb{R}^n \), proving that \( F \) is a submersion. Hence the above example together with the Parametric Transversality Theorem imply that:

**Corollary.** If \( \Sigma \) is a smooth hypersurface in \( \mathbb{R}^n \), then all \( \ell \in \mathbb{L}^n \) except for a set of kinematic measure zero intersect \( \Sigma \) in a discrete set, and hence in a finite set if \( \Sigma \) is bounded.

8. **Classic Cauchy-Crofton**

Many interesting geometric quantities associated to submanifolds of \( \mathbb{R}^n \) can be expressed as an integral of some related function over \( \mathbb{L}^n \) with respect to kinematic measure. The study of such relations is known as Integral Geometry, and one of its earliest results is a formula of Crofton giving the length of a plane curve in terms of the integral over \( \mathbb{L}^2 \) of the number of times a line \( \ell \) intersects it. Both Crofton’s result and its proof generalize to a formula for the area of a compact hypersurface of \( \mathbb{R}^n \)—and this generalization, which we will call The Classic Cauchy-Crofton Formula, plays an important role in our algorithm for constructing point clouds on hypersurfaces.

In what follows \( \Sigma \) denotes a compact, smooth hypersurface in \( \mathbb{R}^n \). For \( \ell \in \mathbb{L}^n \) we define \( \#(\ell \cap \Sigma) \) to be zero if \( \ell \) does not meet \( \Sigma \) or if it meets it non-transversally, and otherwise it is the number of points in \( \ell \cap \Sigma \). We recall that we write the volume \((n - 1)\)-form of a hypersurface \( \Sigma \) as \( d\omega_\Sigma \) and its integral, \( A(\Sigma) \), over \( \Sigma \) is called the area of \( \Sigma \). Finally, we will denote by \( \kappa_m \) the volume of the \( m \)-ball \( \mathbb{B}^m \).

**The Classic Cauchy-Crofton Formula.** If \( \Sigma \) is a compact, smooth hypersurface in \( \mathbb{R}^n \), then

\[
A(\Sigma) = \frac{1}{2\kappa_{n-1}} \int_{\mathbb{L}^n} \#(\ell \cap \Sigma) \, d\mu(\ell).
\]

A standard reference for the proof of Crofton’s original formula, for a curve in \( \mathbb{R}^2 \), is pages 12–13 of Santaló’s classic *Introduction to Integral Geometry* [S1].

We begin our proof of Classic Cauchy-Crofton with a geometrically intuitive observation: If a cylindrical flashlight beam meets a flat surface at an angle \( \theta \), then the area \( A(S) \) of the region \( S \) of the surface illuminated by the beam is related to the area \( A(\tilde{S}) \) of the beam cross-section \( \tilde{S} \) by \( A(\tilde{S}) = A(S)|\cos(\theta)| \). The following is a generalization of this fact to higher dimensions, using more precise language.

**Proposition 1.** Let \( \nu, v \in \mathbb{S}^{n-1} \) and let \( \Pi_v \) denote orthogonal projection of \( \mathbb{R}^n \) onto \( v^\perp \), so that \( \Pi_v(x) := x - \langle x, v \rangle v \). If \( \pi_v : \nu^\perp \to v^\perp \) denotes the restriction of \( \Pi_v \) to \( \nu^\perp \), then the Jacobian of \( \pi_v \) has the constant value \( |\langle v, \nu \rangle| = |\cos(\theta_v)| \), where \( \theta_v \) is the angle between \( v \) and \( \nu \). (Equivalently, if \( S \) is a bounded region of \( \nu^\perp \), then the area \( A(S) \) of \( S \) and the area \( A(P_v(S)) \) of its projection onto \( v^\perp \) are related by: \( A(P_v(S)) = A(S)|\langle v, \nu \rangle| = A(S)|\cos(\theta_v)| \).

**Proof.** The case \( v = \nu \) is trivial, so we assume \( v \neq \nu \), identify \( \mathbb{R}^2 \) with the linear span of \( v \) and \( \nu \) and \( \mathbb{R}^{n-2} \) with its orthogonal complement, \( v^\perp \cap \nu^\perp \). Then \( \mathbb{R}^n \) is the

\[4\text{We recall the well-known formula, } \kappa_m = \frac{\sqrt{\pi}}{\Gamma(\frac{m}{2} + 1)}.
\[5\text{Most integral geometry sources omit the 2 in the denominator; this is because they use the space of unoriented lines rather than the space of oriented lines as we do here.}
orthogonal direct sum \( \mathbb{R}^2 \oplus \mathbb{R}^{n-2} \) and \( \Pi_v \) is the identity on \( \mathbb{R}^{n-2} \). This effectively reduces the Proposition to the case \( n = 2 \) where it is obvious.

If we regard \( \nu \) as the north pole, then the equatorial plane \( \nu^\perp \) splits the unit sphere \( S^{n-1} \) into two hemispheres \( S^\pm_{n-1} \), and each of \( \phi_{\pm} := P_v|_{S^\pm_{n-1}} \) is a diffeomorphism of \( S^\pm_{n-1} \) onto the unit ball \( B^{n-1} \) in that plane. Since the tangent space to \( S^{n-1} \) at a point \( v \) is \( v^\perp \), it follows from the calculation above that the absolute value of the Jacobian determinant of \( \phi_{\pm} \) at any point \( v \) is just \( |\langle v, \nu \rangle| \). By the change of variable formula of multi-variable calculus, this proves the following:

**Proposition 2.** For all \( \nu \in S^{n-1} \)

\[
\int_{S^{n-1}} |\langle v, \nu \rangle| \, d\sigma(v) = 2\kappa_{n-1},
\]
where as in Section 3, \( d\sigma \) denotes the Riemannian volume element of \( S^{n-1} \).

We are now ready to complete the proof of the Classic Cauchy-Crofton Formula. As above, for \( v \in S^{n-1} \), we denote orthogonal projection of \( \mathbb{R}^n \) onto \( v^\perp \) by \( \Pi_v \), and we will denote by \( P_v : \Sigma \to v^\perp \) its restriction to \( \Sigma \). For \( x \in \Sigma \) we denote by \( \nu_x \in S^{n-1} \) the unit normal to \( \Sigma \) at \( x \), so \( \nu^\perp_x \) is the tangent space to \( \Sigma \) at \( x \).

**Lemma.** \( |\operatorname{Jac}(P_v)(x)| = |\langle v, \nu_x \rangle| \).

**Proof.** Since \( \Pi_v \) is linear, its restriction \( \pi : \nu^\perp_x \to \nu^\perp \) to the tangent space \( \nu^\perp_x \) of \( \Sigma \) at \( x \) is the differential \( DP_x \) of the restriction \( P \) of \( \Pi_v \) to \( \Sigma \), so the Lemma follows from Proposition 1.

Then, recalling from Section 6 the definition of the kinematic measure on the space \( L^n \) of lines, and noting that if \( \ell = (v, p) \in L^n \) then \( (\ell \cap \Sigma) = P_v^{-1}(p) \),

\[
\int_{L^n} \#(\ell \cap \Sigma) \, d\mu(\ell) = \int_{S^{n-1}} d\sigma(v) \int_{v^\perp} \#(P_v^{-1}(p)) \, dp
\]

\[
= \int_{S^{n-1}} d\sigma(v) \int_{\Sigma} |\operatorname{Jac}(P_v)(x)| \, d\Sigma(x) \quad \text{by the Co-Area Formula of Section 2}
\]

\[
= \int_{S^{n-1}} d\sigma(v) \int_{\Sigma} |\langle v, \nu_x \rangle| \, d\Sigma(x) \quad \text{by the Lemma}
\]

\[
= \int_{\Sigma} d\Sigma(x) \int_{S^{n-1}} |\langle v, \nu_x \rangle| \, d\sigma(v) \quad \text{by Fubini}
\]

\[
= A(\Sigma) \times (2\kappa_{n-1}) \quad \text{by Proposition 2}
\]

which completes the proof of the Classic Cauchy-Crofton Formula.

**9. The Cauchy-Crofton Principle**

We will need a generalization of The Classic Cauchy-Crofton Formula that we will call The Cauchy-Crofton Principle. It says, roughly, that to integrate a function over a hypersurface \( \Sigma \) of \( \mathbb{R}^n \) you can first sum its values at the points where a line \( \ell \) meets \( \Sigma \), and then integrate that sum over the space \( L^n \) of all the lines in \( \mathbb{R}^n \).

When \( f \equiv 1 \) this of course is just a restatement of Classic Cauchy-Crofton and, as we will see in the next section, conversely The Cauchy-Crofton Principle follows from Classic Cauchy-Crofton.

**Definition.** If \( f : \Sigma \to \mathbb{R} \) is a real-valued function on a compact hypersurface \( \Sigma \) in \( \mathbb{R}^n \) we define \( f^\Sigma : L^n \to \mathbb{R} \) by \( f^\Sigma(\ell) := 0 \) if \( \ell \) is not transverse to \( \Sigma \), and otherwise
\( f^\Sigma(\ell) := \sum_{p \in \ell \cap \Sigma} f(p) \). More generally, if \( f : \Sigma^k \to \mathbb{R} \), we define \( f^\Sigma : (L^n)^k \to \mathbb{R} \) by \( f^\Sigma(\ell_1, \ldots, \ell_k) := 0 \) if some \( \ell_i \) is not transverse to \( \Sigma \), and otherwise

\[
f^\Sigma(\ell_1, \ldots, \ell_k) := \sum f(x_{j_1}^1, \ldots, x_{j_k}^k),
\]

where the sum is over all \( k \)-tuples \( (x_{j_1}^1, \ldots, x_{j_k}^k) \) such that \( x_{j_i}^i \in \ell_i \cap \Sigma \).

**The Cauchy-Crofton Principle.** If \( \Sigma \) is a compact, smooth hypersurface in \( \mathbb{R}^n \), and \( f : \Sigma \to \mathbb{R} \) is a continuous real-valued function on \( \Sigma \), then

\[
\int_\Sigma f(x) \, d\omega_\Sigma(x) = \frac{1}{2^{n-1}} \int_{L^n} f^\Sigma(\ell) \, d\mu(\ell).
\]

**Corollary: The Multivariable Cauchy-Crofton Principle.** If \( \Sigma \) is a compact, smooth hypersurface in \( \mathbb{R}^n \) and if \( f : \Sigma^k \to \mathbb{R} \) is a continuous real-valued function on \( \Sigma^k \), then

\[
\int_{\Sigma^k} f(x_1, \ldots, x_k) \, d\omega_\Sigma(x_1) \ldots d\omega_\Sigma(x_k) = \left( \frac{1}{2^{n-1}} \right)^k \int_{(L^n)^k} f^\Sigma(\ell_1, \ldots, \ell_k) \, d\mu(\ell_1) \ldots d\mu(\ell_k).
\]

**Proof of the Corollary.** We give the details for the case \( k = 2 \); the general case is only notationally more complicated. If for \( x \in \Sigma \) we define \( f_x : \Sigma \to \mathbb{R} \) by \( f_x(x_2) := f(x, x_2) \), then applying The Cauchy-Crofton Principle with \( f = f_{x_1} \),

\[
\int_{\Sigma^2} f(x_1, x_2) \, d\omega_\Sigma(x_1) \, d\omega_\Sigma(x_2) = \int_{\Sigma} d\omega_\Sigma(x_1) \int_{\Sigma} f_{x_1}(x_2) \, d\omega_\Sigma \, d\omega_\Sigma(x_2)
\]

\[
= \frac{1}{2^{n-1}} \int_{\Sigma} dA(x_1) \int_{L^n} f_{x_1}^\Sigma(\ell_2) \, d\mu(\ell_2),
\]

or by Fubini’s Theorem,

\[
\int_{\Sigma^2} f(x_1, x_2) \, dA(x_1) \, dA(x_2) = \frac{1}{2^{n-1}} \int_{L^n} d\mu(\ell_2) \int_{\Sigma} f_{x_1}^\Sigma(\ell_2) \, dA(x_1),
\]

and applying The Cauchy-Crofton Principle again, this time with \( f = g_{\ell_2} \) where \( g_{\ell_2}(x_1) := f_{x_1}^\Sigma(\ell_2) \),

\[
\int_{\Sigma^2} f(x_1, x_2) \, dA(x_1) \, dA(x_2) = \left( \frac{1}{2^{n-1}} \right)^2 \int_{L^n} d\mu(\ell_2) \int_{L^n} g_{\ell_2}^\Sigma(\ell_1) \, d\mu(\ell_1),
\]

and appealing to Fubini’s Theorem again we only have to check that \( g_{\ell_2}^\Sigma(\ell_1) = f_{\ell_2}^\Sigma(\ell_1, \ell_2) \). But

\[
g_{\ell_2}(x) = f_{x}^\Sigma(\ell_2) = \sum_{q \in (\ell_2 \cap \Sigma)} f_x(q) = \sum_{p \in (\ell_1 \cap \Sigma)} f(p, q) = f^\Sigma(\ell_1, \ell_2).
\]

\[
g_{\ell_2}^\Sigma(\ell_1) = \sum_{p \in (\ell_1 \cap \Sigma)} g_{\ell_2}(p) = \sum_{p \in (\ell_1 \cap \Sigma)} \sum_{q \in (\ell_2 \cap \Sigma)} f(p, q) = f^\Sigma(\ell_1, \ell_2).
\]
10. Demonstration of The Cauchy-Crofton Principle

Our proof of The Cauchy-Crofton Principle from The Classic Cauchy-Crofton Formula will proceed by a series of reductions. We note first that since the function $f : \Sigma \to \mathbb{R}$ in its statement clearly depends linearly on the function $f : \Sigma \to \mathbb{R}$ it follows that:

**Linearity Lemma.** If The Cauchy-Crofton Principle holds for functions $f_\alpha : \Sigma \to \mathbb{R}$ then it also holds for any finite linear combinations of the $f_\alpha$.

**Proposition 1.** If a compact, smooth hypersurface $\Sigma$ of $\mathbb{R}^n$ is a finite union of the interiors of hypersurfaces $\Sigma_i$ and if each $\Sigma_i$ satisfies The Cauchy-Crofton Principle, then so does $\Sigma$.

**Proof.** Let $\phi_i : \Sigma \to \mathbb{R}$ be a smooth partition of unity for $\Sigma$ with the support of $\phi_i$ a compact subset of the interior of $\Sigma_i$. Given a continuous $f : \Sigma \to \mathbb{R}$, the functions $f \phi_i$ satisfy The Cauchy-Crofton Principle and by the Linearity Lemma so does their sum $f$.

This shows that the Cauchy-Crofton Principle is local in nature; i.e., a compact hypersurface $\Sigma$ satisfies The Cauchy-Crofton Principle provided each point of $\Sigma$ has a neighborhood that does. It then follows from Section 2 that:

**Proposition 2.** To prove The Cauchy-Crofton Principle for a general compact, smooth hypersurface $\Sigma$ in $\mathbb{R}^n$ it suffices to treat the case of a smooth graph hypersurface.

**Continuity Lemma.** If $\Sigma$ is a compact, smooth hypersurface of $\mathbb{R}^n$ then $f \mapsto \int_{\Sigma} f^\Sigma(\ell) \, d\mu(\ell)$ is a well-defined continuous linear functional on the Banach space $L^\infty(\Sigma)$ of bounded, integrable, real-valued functions on $\Sigma$, i.e., it is continuous w.r.t. the sup norm, $\| \cdot \|_\infty$.

**Proof.** Recall that for $\ell \in \mathbb{L}^n$, $\#(\ell \cap \Sigma)$ is defined to be zero if $\ell$ does not meet $\Sigma$ or if $\ell$ meets it non-transversally, and otherwise it is the number of points in $\ell \cap \Sigma$. By Classic Cauchy-Crofton $A(\Sigma) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{L}^n} \#(\ell \cap \Sigma) \, d\mu(\ell)$. For non-negative integers $k$ define $\mathbb{L}_k^n := \{ \ell \in \mathbb{L}^n \mid \#(\ell \cap \Sigma) = k \}$, so $\mathbb{L}^n$ is the disjoint union of the $\mathbb{L}_k^n$ and the set of measure zero where $\ell$ does not meet $\Sigma$ transversally, and hence $\sum_{k=1}^\infty k \mu(\mathbb{L}_k^n) = \kappa_{n-1} A(\Sigma)$. If $f \in L^k(\Sigma)$, then by definition of $f^\Sigma$, $|f^\Sigma(\ell)| \leq k \|f\|_\infty$ for $\ell \in \mathbb{L}_k^n$. Hence:

$$\left| \int_{\mathbb{L}^n} f^\Sigma(\ell) \, d\mu(\ell) \right| = \sum_{k=1}^\infty \int_{\mathbb{L}_k^n} f^\Sigma(\ell) \, d\mu(\ell) \leq \sum_{k=1}^\infty \int_{\mathbb{L}_k^n} f^\Sigma(\ell) \, d\mu(\ell) \leq \sum_{k=1}^\infty \int_{\mathbb{L}_k^n} |f^\Sigma(\ell)| \, d\mu(\ell) \leq \sum_{k=1}^\infty k \|f\|_\infty \mu(\mathbb{L}_k^n) = \kappa_{n-1} A(\Sigma) \|f\|_\infty,$$

showing that $f \mapsto \int_{\mathbb{L}^n} f^\Sigma(\ell) \, d\mu(\ell)$ is bounded w.r.t. $\| \cdot \|_\infty$.

Completing the proof of The Cauchy-Crofton Principle is now easy. By the usual argument it is enough to prove it for a surface $\Sigma$ that is the graph of a continuous function $g : [0,1]^{n-1} \to \mathbb{R}$. If a function $f : \Sigma \to \mathbb{R}$ is the characteristic function of the image under $g$ of a sub-cube of $[0,1]^{n-1}$ then The Cauchy-Crofton Principle for $f$ follows from Classic Cauchy-Crofton, and so by the Linearity Lemma, The Cauchy-Crofton Principle also holds for “step functions”, i.e., for linear combinations of such characteristic functions. But since any continuous $f$ is clearly a uniform limit
of step functions, it follows from the Continuity Lemma that The Cauchy-Crofton Principle holds for all continuous $f : \Sigma \to \mathbb{R}$. ■

11. Constructing a Sequence $\{x_i\}$ Highly Equidistributed in a Hypersurface $\Sigma$

In this section $\{\ell_j\} = \{(v_j, p_j)\}$ denotes a fixed $k$-equidistributed sequence in $\mathbb{L}_n^k$, and $\Sigma$ a bounded hypersurface included in $\mathbb{B}_n^k$. We will show how to use the sequence of lines $\ell_j$ to construct a $k$-equidistributed sequence $\{x_i\}$ in $\Sigma$—essentially by just intersecting these lines with $\Sigma$.

We define a sequence $\sigma_j$ of non-negative integers by setting $\sigma_j$ to zero if $\ell_j$ does not meet $\Sigma$ or meets it non-transversely, and otherwise $\sigma_j$ is the positive integer denoting the cardinality of $\ell_j \cap \Sigma$; and we define $N_k := \sigma_1 + \ldots + \sigma_k$. (Thus, $N_k$ is the total number of points in which the first $k$ lines $\ell_j$ that are transversal to $\Sigma$ intersect $\Sigma$)

To construct the sequence $\{x_i\}$, we start with the empty sequence and consider each line $\ell_j$ in order and use it to append $\sigma_j$ further points to the sequence $\{x_i\}$ as follows:

i) If $\ell_j$ is disjoint from $\Sigma$ or if it meets $\Sigma$ non-transversely (so that $\sigma_j = 0$) then we append no new points to the sequence $\{x_i\}$.

ii) However, if $\ell_j$ does meet $\Sigma$, and meets it transversely, then it intersects $\Sigma$ in the $\sigma_j$ points, $\xi_j := \sigma_j^{(\ell_j)}(t_j)$, where $t_1 < t_2 < \ldots < t_{\sigma_j}$, and in this case we append the $\sigma_j$ new points $\xi_1, \ldots, \xi_{\sigma_j}$ of $\Sigma$ to the sequence $\{x_i\}$ in that order.

Clearly, after processing the first $m$ lines in this way the sequence $\{x_i\}$ will have length $N_m$, and if $f : \Sigma \to \mathbb{R}$ is a bounded continuous function it is immediate from i) and ii) and the definition of $f^2$ in section 8, that

$$
\frac{1}{N_m} \sum_{j=1}^{N_m} f(x_j) = \left( \frac{m}{N_m} \right) \frac{1}{m} \sum_{i=1}^{m} f^2(\ell_i).
$$

Lemma.

$$
\lim_{m \to \infty} \left( \frac{m}{N_m} \right) = \frac{1}{A(\Sigma) \kappa_{n-1}}.
$$

Proof. If we take $f \equiv 1$ in the Cauchy-Crofton Principle then, since $f^2(\ell)$ vanishes outside $\mathbb{L}_n^k$, $A(\Sigma) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{L}_n^k} f^2(\ell) \, d\mu(\ell)$. Since the sequence $\{\ell_j\}$ is by assumption $k$-equidistributed (and a fortiori 1-equidistributed) in $\mathbb{L}_n^k$, $A(\Sigma) = \frac{1}{\kappa_{n-1}} \lim_{m \to \infty} \frac{1}{m} (f^2(\ell_1) + \cdots + f^2(\ell_m))$. But clearly $f^2(\ell_j) = \sigma_j$ so

$$
A(\Sigma) = \frac{1}{\kappa_{n-1}} \lim_{m \to \infty} \frac{N_m}{m}.
$$

Applying the lemma and using again that the sequence $\{\ell_j\}$ is 1-equidistributed it now follows that

$$
\lim_{m \to \infty} \frac{1}{N_m} \sum_{j=1}^{N_m} f(x_j) = \lim_{m \to \infty} \left( \frac{m}{N_m} \right) \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} f^2(\ell_i) = \frac{1}{\kappa_{n-1} A(\Sigma)} \int_{\mathbb{L}_n^k} f^2(l) \, d\mu(l),
$$

so by the Cauchy-Crofton Principle again, applied now to $f$,

$$
\lim_{m \to \infty} \frac{1}{N_m} \sum_{j=1}^{N_m} f(x_j) = \frac{1}{A(\Sigma)} \int_{\Sigma} f(x) \, dx.
$$

This shows that:

---

6 for example, the one constructed in Subsection 4 of Section 6.
12. Principal Surface Types

Although we could continue developing the theory of point clouds living on hypersurfaces in $\mathbb{R}^n$, we will now specialize to the case $n = 3$. Not only does this make the discussion more intuitive and visualizable, but it also makes available some simplifications.

We will start by describing the three principal types of surfaces that arise in mathematics and computer graphics, implicit surfaces, parametric surfaces, and triangulated surfaces, discuss some of the important differences between them, and give a short intuitive description of the algorithms we use for generating point clouds for each type.

a) Implicit surfaces.

An implicitly defined surface is defined as the level set of a function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. In more detail, a smooth implicit surface, $\Sigma$, is the set of solutions of an equation of the form $f(x_1, x_2, x_3) = 0$, where $f$ is a smooth real-valued function of three real variables. For example, the theorem of Pythagoras says that the sphere of radius $r$ centered at the origin consists of the points $(x_1, x_2, x_3)$ satisfying $x_1^2 + x_2^2 + x_3^2 = r^2$, and so it is an implicit surface with the defining function $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - r^2$. For such surfaces it is easy to test if a point $(x_1, x_2, x_3)$ lies in the surface, since one only has to evaluate the function $f$ at the point—which we assume is easy—but it is often difficult to find points $(x_1, x_2, x_3)$ on the surface, since for that one has to solve the equation, and that can be a "hard" problem, even when $f$ is a polynomial.

We have already discussed in section 11 our algorithm, based on the Cauchy-Crofton Formula, for generating a point cloud on an implicit surface. Intuitively speaking, the idea is to find the points where a set of lines (with highly equidistributed directions and closest points to the origin) intersect the implicit surface. We accomplish this by evaluating the function defining the surface along those lines and use a standard numerical technique to see where it vanishes (see Section 13 for details).

A problem that must be addressed is that while some implicit surfaces, such as the sphere above, are bounded and have finite area, others (such as the plane, defined by $x_3 - x_1 - x_2 = 0$ or the paraboloid defined by $x_3 - x_1^2 - x_2^2 = 0$) are unbounded and have infinite area. For such surfaces, if the points in a point cloud are uniformly distributed in proportion to area, then the probability of a point being in a region of finite area would be zero! To avoid this problem we will, when constructing a point cloud, consider only a bounded piece of an implicit surface $\Sigma$; namely we replace $\Sigma$ by $\Sigma_r$, its intersection with the disk $D_r^3$ of radius $r$ in $\mathbb{R}^3$, and construct point clouds lying entirely in $\Sigma_r$.

b) Parametric surfaces.

A smooth parametric surface $\Sigma$ is defined by giving a smooth function:

$$\Phi(u, v) := (X_1(u, v), X_2(u, v), X_3(u, v))$$

from the $(u, v)$-plane into $(x_1, x_2, x_3)$-space. A point $(x_1, x_2, x_3)$ is on $\Sigma$ if and only if there are values of the parameters $u$ and $v$ such that $x_i = X_i(u, v)$ for $i = 1, 2, 3$. The three real-valued functions $X_i$ that define the mapping $\Phi$ from $(u, v)$-space to $(x_1, x_2, x_3)$-space are called the parametric functions for the parametric surface.

Theorem. The sequence $\{x_j\}$ defined above satisfies 2) of Definition 1 of Section 3, and so is 1-equidistributed in $\Sigma$.

To show that it is $k$-equidistributed, repeat the above argument but now for a bounded continuous function $f : \Sigma^k \rightarrow \mathbb{R}$ and instead of the Cauchy-Crofton Principle apply the Multivariable Cauchy-Crofton Principle. We omit these details.
Usually we restrict the parameters \((u, v)\) to lie in a restricted region of their plane (called the parameter domain) and often this is a rectangle with sides parallel to the axes. For example, the sphere of radius \(r\) centered at the origin, considered above as an implicit surface, can also be considered as a parametric surface, with \(u\) and \(v\) respectively the latitude and longitude (measured in radians). The parameter domain is \(-\pi/4 \leq u \leq \pi/4\) and \(-\pi/2 \leq v \leq \pi/2\) and \((X_1(u, v), X_2(u, v), X_3(u, v))\) are just the three spatial coordinates of the point of the sphere having latitude \(u\) and longitude \(v\). (See Remark 2 below for the explicit formulas.) For parametric surfaces it is easy to find many points on the surface, just substitute many pairs \((u, v)\) into \(X_1, X_2, X_3\), but it is usually hard to decide if a given point \((x_1, x_2, x_3)\) lies on the surface, since this requires solving the three simultaneous equations \(X_i(u, v) = x_i\) for \(u\) and \(v\). Our method for generating a point cloud for a parametric surface will be to define an associated triangulation of the surface and then use the method sketched below for triangulated surfaces.

c) Triangulated surfaces.

Three distinct points, \(v_1, v_2, v_3\), in \(\mathbb{R}^3\) define a planar surface called the triangle with vertices \(v_1, v_2, v_3\), and edges the line segments \([v_1, v_2], [v_2, v_3], [v_3, v_1]\) joining pairs of vertices. The points of this triangle are just all points that lie on some line joining a vertex to a point on the opposite edge. Points of the triangle not on any edge are called interior points of the triangle. A triangulated surface \(\Sigma\) is defined by giving a finite list of triangles (called the triangulation of the surface) and \(\Sigma\) is then just the union of these triangles. Each vertex or edge of any triangle of the triangulation is called a vertex or edge of \(\Sigma\), and if an edge of \(\Sigma\) belongs to exactly one triangle of the triangulation it is called a boundary edge, and its points are called boundary points of \(\Sigma\).

Our method for generating a point cloud for a triangulated surface is based on Method 3 of Section 5. We first compute the areas of the triangles to define a cumulative area distribution function. Each triangle is thereby associated with a subinterval of the unit interval whose length is proportional to the area it contributes to the total area of the surface. Then, given a random point in the unit interval, we can easily find which of these subintervals it belongs to, and then place a random point in the associated triangle using the following algorithm.

The standard 2-simplex is given by \(\Delta_2 := \{(u, v) \in [0, 1]^2 \mid u + v \leq 1\}\), and we can define a pseudo-random sequence \(\text{Rnd}_{\Delta_2}(n)\) from \(\text{Rnd}_{[0,1]}\) using rejection; i.e., we define a sequence \(\{(uk_i, vk_i)\}\) in \([0,1]^2\) by \(uk_i := \text{Rnd}_{[0,1]}(2k-1)\) and \(vk_i := \text{Rnd}_{[0,1]}(2k)\) and let \(\text{Rnd}_{\Delta_2}(n)\) to be the \(n\)-th element of this sequence such that \(u_n + v_n \leq 1\).

Then if \(\Delta\) is a triangle in \(\mathbb{R}^3\) having vertices \(v_1, v_2, v_3\), we can define a pseudo-random sequence for \(\Delta\) by \(\text{Rnd}_{\Delta}(n) := \Psi \circ \text{Rnd}_{\Delta_2}(n)\), where \(\Psi : \Delta_2 \to \mathbb{R}^3\) is the barycentric parametric representation \((u, v) \mapsto uv_1 + uv_2 + (1-u-v)v_3\) of \(\Delta\) (see Remark 3 above). This works because \(\Psi\) is clearly an affine map and so multiplies all areas by a fixed scale factor (its constant Jacobian determinant).

**Remark 1.** For each of the above surfaces types, one needs extra conditions to insure that what results is really what mathematicians call a surface (with boundary).

a) For an implicit surface, \(\Sigma\), given by \(f(x_1, x_2, x_3) = 0\), one needs to assume that the gradient of \(f\) does not vanish anywhere on the surface, i.e., at any point \((x_1, x_2, x_3)\) of \(\Sigma\), at least one of the three partial derivatives of \(f\) is non-zero. This insures that \(\Sigma\) has a two-dimensional tangent space at the point. In fact it then follows (from the Implicit Function Theorem) that near each of its points \(\Sigma\) can be represented locally as the graph of a smooth function.

b) For a parametric surface, \(\Sigma\), given by \(\Phi(u, v)\), one needs to assume that if \(u = \xi(t), \ v = \eta(t)\) are the equations of a straight line in the parameter domain, then...
the image curve on the surface (given by \( \Phi(\xi(t), \eta(t)) \)) has a non-zero tangent vector for all \( t \). This again insures that \( \Sigma \) has a two-dimensional tangent space at every point.

c) To make the definition of a triangulated surface \( \Sigma \) rigorous, one needs the following three conditions:

i) A point of \( \Sigma \) that is an interior point of some triangle of the triangulation does not belong to any other triangle of the triangulation.

ii) Any non-boundary edge of \( \Sigma \) should be an edge of exactly two triangles of the triangulation.

iii) If a vertex \( v \) of \( \Sigma \) lies on an edge \( e \), then \( v \) is one of the two endpoints of \( e \), and if \( e \) is a boundary edge then \( v \) is on exactly one other boundary edge.

(The first two condition insure that \( \Sigma \) looks right (i.e., in math jargon, “has the right local topology”) at interior points and edges and the third condition insures that it looks right at all vertices.) However, the conditions a), b), c) do not enter into the algorithms for defining point clouds and so will not play a significant role in what follows.

Remark 2. Sometimes an implicit surface can also be given parametrically, by explicitly solving its defining equation \( f(x_1, x_2, x_3) = 0 \). For example a plane is given implicitly by an equation of the form \( a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 = 0 \), or (assuming \( a_3 \neq 0 \)) parametrically by \( x_1 = u, x_2 = v, x_3 = (a_4 - a_1 u - a_2 v)/a_3 \), and similarly the implicitly defined ellipsoid \( (x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 - 1 = 0 \) can be given parametrically by \( x_1 = a_1 \cos(u) \sin(v), x_2 = a_2 \sin(u) \sin(v), x_3 = a_3 \cos(v) \). But these are exceptional cases, and in general no such explicit solution of an implicit equation in terms of elementary functions is possible.

Remark 3. A triangle \( \Delta \) with vertices \( v_1, v_2, v_3 \) can be considered a parametric surface with parameter domain \( \Delta_2 \), the standard 2-simplex, consisting of all \((u, v) \in [0, 1]^2 \) with \( u + v \leq 1 \), and defining \( \Psi(u, v) := uv_1 + vv_2 + (1-u-v)v_3 \). We call \( \Psi(u, v) \) the point of \( \Delta \) with barycentric coordinates \( u, v \), and \( 1-u-v \). Thus a triangulated surface can be considered as a finite union of parametric surfaces. In the reverse direction, it is easy to approximate a parametric surface \( \Sigma \) by a triangulated surface \( \Sigma' \). If we first triangulate the parameter domain of \( \Sigma \) into small triangles, and then take the images of the vertices of these triangles as the vertices of triangles of a triangulation, we get a triangulated surface \( \Sigma' \) that approximates \( \Sigma \).

Remark 4. While the surfaces that occur in mathematical contexts are usually of the implicit or parametric type, the surfaces of interest in computer graphics are frequently boundaries of some real world solid object, such as a human face or a teapot. Such surfaces are in general too irregular to have convenient representations as parametric or implicit surfaces, so it has been customary to represent them by ad hoc triangulations. However, recently there has been rapid progress in a process called LIDAR or laser scanning, a method for the rapid construction of a “point cloud”\(^7\) from a solid object. It works by scanning a rapidly pulsed laser beam over the surface of the object and collecting the reflected pulses in several optical devices. The 3D location of a point on the surface from which a pulse is reflected is calculated either by triangulation or by time-of-flight methods, and the totality of these calculated 3D locations gives the point cloud. There is associated software that can take the resulting LIDAR point cloud and construct from it a so-called Voronoi decomposition of the surface, and then from that in turn construct a triangulation. So it could be argued that the point clouds created by laser scanning are merely steppingstones to triangulations. But in fact, a major reason for creating the triangulation is that so much software has been developed over the years to work with

\(^7\) The points of these laser generated point clouds are not uniformly distributed or independent.
triangulated surfaces, while point clouds are relatively new. Since there is a loss of information in going from point cloud to triangulation, it seems likely that the point cloud data structure will gradually become another primary means for storing and working with surfaces—and perhaps eventually even the predominant one. This will make it important to also have good algorithms for creating high quality, uniform point clouds from implicit, parametric, and triangulated surfaces, in order to be able to work with all the various types of surfaces in a uniform way. Indeed, in virtual reality applications, one already sees combinations of real-world objects and mathematically defined surfaces in the same scene. It will also be important to be able to take a non-uniform LIDAR generated point cloud and “upgrade” it to one that is uniform. One way to do that is to construct a triangulated surface from the point cloud as outlined above and apply our algorithm for creating a uniform point cloud from a triangulation.

Remark 5. In addition to the somewhat conjectural reason just alluded to, there are many current applications requiring a point clouds approach. One is Monte Carlo based numerical methods, and we put off discussion of this until later. Another is visualization: when mathematicians would like to examine the geometry of a surface, either for research or teaching purposes, they can now use mathematical visualization software to create and rotate an image of the surface on a computer screen. As discussed earlier, the classic tool for displaying implicit surfaces is raytracing, and while this method produces realistic renderings, it does have drawbacks. In a raytrace of a surface with many layers, the layer nearest the viewpoint obscures those further away, and since raytracing is a comparatively slow process, it is difficult to rotate a raytraced image in real time to see the surface from different perspectives. Point cloud rendering is free of these drawbacks.

13. Point Clouds for Implicit Surfaces

1. Intersecting a line with an implicit surface.
   As remarked earlier, it may be difficult to find solutions of \( f(x_1, x_2, x_3) = 0 \) in a rectangular region \( m_i \leq x_i \leq M_i \). However, for the analogous single variable problem, namely that of solving the equation \( g(t) = 0 \) on an interval \( m \leq t \leq M \), there is an effective numerical approach to finding solutions. Recall that the Intermediate Value Theorem tells us that a continuous real-valued function of a real variable that has opposite signs at the endpoints of some interval vanishes for some point in that interval. So a good numerical method for finding zeros of \( g(t) \) on \( m \leq x \leq M \) is to divide \([m, M]\) into small subintervals, check on which subintervals \( g \) changes sign, and then use either bisection or Regula Falsi to locate quickly a zero on each of these subintervals. If \( x(t) = tv + p \) is the equation of a straight line \( L \) in \( \mathbb{R}^3 \) and we define \( g(t) := f(x(t)) \), we can use this method to find the solutions of \( g(t) = 0 \), i.e., to find the intersections of the line \( L \) with the implicit surface \( \Sigma \) defined by the equation \( f = 0 \), and this will be our point generating method for creating a point cloud on \( \Sigma \) (or rather on \( \Sigma_r \), the part of \( \Sigma \) that lies inside the ball of radius \( r \)).

But how should we choose the lines \( L \) to insure that the points obtained in this way are uniformly distributed? We consider this next.

2. Point clouds for Implicit Surfaces.
   We need an algorithm \( \text{Rnd}_{L^2} \) for selecting lines \((v, p)\) in \( L^2 \) uniformly with respect to kinematic measure. It is clear from the definition of kinematic measure that we get such a selection algorithm by first selecting \( v \in S^2 \) using \( \text{Rnd}_{S^2} \), and then selecting \( p \in v^\perp \) using the following algorithm \( \text{Rnd}_{D^2(v^\perp)} \) for selecting a pseudo-random sequence of points in \( D^2(v^\perp) \), the disk of radius \( r \) in \( v^\perp \).
Let $J_{2,3}(x_1, x_2) := (x_1, x_2, 0)$ be the inclusion of $\mathbb{R}^2$ into $\mathbb{R}^3$ and $T_v$ the rotation of $\mathbb{R}^3$ that is the identity on the line $P^\perp$ orthogonal to the plane $P$ spanned by $v$ and $e_3 := (0, 0, 1)$ and on $P$ rotates $e_3$ to $v$. Since $T_v$ maps $e_3$ to $v$, it maps $e_3^2$, the image of $J_{2,3}$, onto $v^{-1}$. Then we define $\text{Rnd}_{2,3}^l(v)(n) := T_v \circ J_{2,3} \circ \text{Rnd}_{2,3}^l(n)$, i.e., we use $J_{23}$ to inject $\text{Rnd}_{2,3}^l$ into $e_3^2$, and then rotate it with $T_v$ into $v^{-1}$. $T_v$ is a transvection, i.e., if $b$ is the midpoint of the great circle joining $e_3$ to $v$, then $T_v$ is the composition of reflections in two one-dimensional subspaces, namely reflection $\rho_b$ in the line spanned by $b$ followed by reflection $\rho_e$ in the line spanned by $v$, so finally our algorithm reads: $\text{Rnd}_{2,3}^l(v)(n) := \rho_e \circ \rho_b \circ J_{2,3} \circ \text{Rnd}_{2,3}^l(n)$.

It now follows from the corollary of the Generalized Cauchy-Crofton Formula that if $\Sigma$ is an implicit surface, then the expected number of intersections such a pseudo-randomly selected line will have with any region of $\Sigma$, is proportional to the area of that region. Thus, selecting a sequence of lines with $\text{Rnd}_{2,3}$ and then finding the intersections of these line with $\Sigma_r$, using the method just described above, provides our sought after algorithm for constructing a uniformly distributed sequence for an implicit surface $\Sigma$.

14. Point Clouds for Triangulated Surfaces

Let $\Sigma$ be a triangulated surface with triangulation given by a list of triangles $\Delta_i$, $i = 1, 2, \ldots, N$, having vertices $v_i^1, v_i^2, v_i^3$. We recall that the area of $\Delta_i$ is given by $A(\Delta_i) := \frac{1}{2}|| (v_i^2 - v_i^1) \times (v_i^3 - v_i^2)||$. We define $K_0 := 0$ and $K_i := K_{i-1} + A(\Delta_i)$, $i = 1, \ldots, N$; so $K_i$ is the sum of the areas of triangles $\Delta_1$ through $\Delta_i$ and in particular $K_N = A(\Sigma)$, the total area of $\Sigma$. Thus if we define $\delta_i := \frac{K_i}{K_N}$, then the sequence $0 = \delta_0 < \delta_1 < \ldots < \delta_N = 1$ defines a partition of $[0, 1]$ into $N$ subintervals $I_j := [\delta_{j-1}, \delta_j]$ with the length of the $j$-th subinterval being the fraction of the total area of $\Sigma$ that is in $\Delta_j$. This provides just what we need to define an algorithm, $\text{Rnd}_{\Sigma}$, for choosing uniformly distributed points of $\Sigma$. Namely, select a pseudo-random number $x = \text{Rnd}_{[0,1]}$ in $[0, 1]$, determine the subinterval $I_j$ that $x$ belongs to, and let $\text{Rnd}_{\Sigma} := \text{Rnd}_{\Delta_j}$, where $\text{Rnd}_{\Delta_j}$ denotes the pseudo-random selection function for $\Delta_j$ defined in 12 c).

A straightforward way to find the index $j$ such that $x$ belongs to $I_j$ is to test the inequality $\delta_i \leq x$ for $i = 1, 2, \ldots$ and let $j$ be the first index for which it is true. However this takes on the order of $N$ steps, and since in practice $N$ will be on the order of $10^4$ to $10^5$, it is much better to use a bisection search, which will produce $j$ in an order of $\log_2 N$ steps.

15. Point Clouds for Parametric Surfaces

Let $\Phi : [u\text{Min}, u\text{Max}] \times [v\text{Min}, v\text{Max}] \rightarrow \mathbb{R}^3$ define a parametric surface $\Sigma$. We will refer to its domain as the parameter rectangle. A standard technique in computer graphics is to divide the parameter rectangle into a grid of sub-rectangles by dividing each of the two intervals $[u\text{Min}, u\text{Max}]$ and $[v\text{Min}, v\text{Max}]$ intervals into equal length sub-intervals. By dividing each sub-rectangle by a diagonal into two triangles and mapping these triangles with $\Phi$, we get a triangulation of a piecewise linear approximation $\Sigma'$ of $\Sigma$. Thus the point cloud algorithm for $\Sigma'$ in the preceding section will give an approximate point cloud for $\Sigma$, and by a slight modification (see below) we can also get a point cloud for $\Sigma$ itself.

In more detail, let $u\text{Res} > 1$ and $v\text{Res} > 1$ be integers. We divide $[u\text{Min}, u\text{Max}]$ into the $(u\text{Res} - 1)$ subintervals $[u_i, u_{i+1}]$, $i = 1, \ldots, u\text{Res} - 1$, where we define $u\text{Step} := (u\text{Max} - u\text{Min})/(u\text{Res} - 1)$ and $u_i := u\text{Min} + (i - 1) \ast u\text{Step}$, and in a similar manner we divide $[v\text{Min}, v\text{Max}]$ into the $(v\text{Res} - 1)$ subintervals $[v_i, v_{i+1}]$, $i = 1, \ldots, v\text{Res} - 1$. The point $g_{i,j} := (u_i, v_j)$ of the parameter rectangle are called the parameter grid-points and their images $\gamma_{i,j} := \Phi(g_{i,j})$ are called the surface grid points. The rectangles $r_{i,j}$ with vertices $(g_{i,j}, g_{i+1,j}, g_{i+1,j+1}, g_{i,j+1})$ are called
the parameter sub-rectangles and we get a triangulation of the parameter rectangle by dividing each \( r_{i,j} \) into the two triangles \( \delta^+_{i,j} := (g_{i,j}, g_{i+1,j}, g_{i,j+1}) \) and \( \delta^-_{i,j} := (g_{i,j}, g_{i+1,j}, g_{i+1,j+1}) \). Finally we get a triangulated surface \( \Sigma' \) by taking their images \( \Delta^+_{i,j} := \Phi(\delta^+_{i,j}) = (\gamma_{i,j}, \gamma_{i+1,j}, \gamma_{i,j+1}) \) and \( \Delta^-_{i,j} := \Phi(\delta^-_{i,j}) = (\gamma_{i,j}, \gamma_{i+1,j}, \gamma_{i+1,j+1}) \).

Note that the surface grid points \( \gamma_{i,j} \) belong to the surface \( \Sigma \) by their definition, so the vertices of the triangles \( \Delta^+_{i,j} \) and \( \Delta^-_{i,j} \) lie in \( \Sigma \). Thus the smooth surfaces \( \Sigma \) and the piecewise linear surface \( \Sigma' \) coincide at the surface grid points, and it follows from the differentiability of \( \Phi \) that, as \( uRes \) and \( vRes \) approach infinity, the surface \( \Sigma' \) converges uniformly to \( \Sigma \). Hence, if we construct a point cloud for the triangulated surface \( \Sigma' \), using the algorithm of the preceding section, then for large \( uRes \) and \( vRes \) this will be a good approximation of a point cloud for \( \Sigma \), although the points will only lie near \( \Sigma \) on it. However it is easy to modify the algorithm for the triangulation point cloud slightly so that the points do lie on \( \Sigma \). Namely in the definition of the selection function \( \text{Rnd}_\Delta \), replace the barycentric parameterization \( \Psi \) by \( \Phi \).

16. Point Cloud Surface Normals

In many applications of point clouds associated to a surface \( \Sigma \), it is important to know the unit vector \( \nu_p \) normal to \( \Sigma \) at points \( p \) of the cloud. One important application of \( \nu_p \) arises when we try to visualize the surface by rendering the point cloud on a computer screen. After projecting each point to the view plane representing the screen we could simply render these points black on a white background. However, for a more realistic looking version of the surface, we can “paint” each point of the cloud the color it would get in a raytrace of the surface or, even better, use that color to paint a small disk tangent to the surface at the point. In fact, if the point cloud is sufficiently dense, such a rendering provides a quite good approximate raytrace that is much faster than a full one. Now to calculate this raytrace color at a cloud point \( p \) we have to know how a light ray from each of several colored light sources reflects off the surface when striking it at \( p \), and by the Law of Reflection for specular surfaces (“Angle of Incidence Equals Angle of Reflection”), this depends in a crucial way on \( \nu_p \). And of course the tangent plane at \( p \), \( T\Sigma_p = \nu_p^\perp \), is also determined by \( \nu_p \).

Other uses of \( \nu_p \) follow from the fact that it can be used to approximate \( \Sigma \) near \( p \). Indeed, by choosing two vectors \( f_1 \) and \( f_2 \) that are linearly independent of \( \nu_p \), and applying the Gram-Schmidt algorithm to \( \nu_p, f_1, f_2 \), we get an orthonormal basis \( e_1, e_2 \) for \( T\Sigma_p \), so that \( \Phi(u, v) := ue_1 + ve_2 \) gives a parameterization of \( T\Sigma_p \) that by Taylor’s Theorem will be a good first order approximation to \( \Sigma \) near \( p \).

For the point cloud algorithms for the three types of surfaces \( \Sigma \) discussed above, it is easy to construct the normal at a cloud point \( p \) using standard methods of multivariable Calculus. For example, if \( \Sigma \) is given by the implicit equation \( f(x_1, x_2, x_3) = 0 \), we can first calculate \( \nabla f_p \) by computing the three partial derivatives of \( f \), and then normalize it to get \( \nu_p \).

But what if we are given only the point cloud itself and do not know the surface \( \Sigma \) that it came from—for example, suppose the point cloud comes from a LIDAR scan of a solid object. Is there some way we can nevertheless construct at least a good approximation to the normal \( \nu_p \) to \( \Sigma \) at a cloud point \( p \)? If the point cloud is dense enough (relative to the maximal curvature of \( \Sigma \)) then there is a fairly obvious approach: choose two other points \( q \) and \( r \) of the cloud that are close to \( p \) and normalize the cross-product \((q - p) \times (r - p)\). And, to get a more accurate approximation to the normal, one can average the cross-products \((q_i - p) \times (r_i - p)\), for a number of different pairs of close neighbors \((q_i, r_i)\) of \( p \). It might look like there is a catch in this; finding those good close neighbors of \( p \) seems to require considerable computation, involving sorting the points of the cloud in various ways. But fortunately it turns out to be possible to carry it out surprisingly fast.
17. Applications: What are Point Clouds Good for?

1. Mathematical Visualization.

In the mid-1990s, while developing our mathematical visualization program (3D-XplorMath), we became dissatisfied with raytrace based methods for visualizing implicit surfaces, and our interest in point clouds grew out of attempts to improve on the tools available for this purpose. CPUs and graphic chips have improved considerably in speed since that time, and today a basic raytrace takes only a fraction of a minute on a modest laptop computer, but back then that process seemed glacially slow, even on a high-end desktop machine. In addition, for an immersed implicit surface that has many overlapping layers, the layer closest to the viewing camera can obscure interesting features on parts of the surface behind it, and even today the time required to redraw each frame is too slow to make mouse rotation of the surface an acceptable cure for that problem.

Point cloud rendering of implicit surfaces provided our sought for solution to these problems, particularly when coupled with the anaglyph (i.e., red/blue glasses based) stereo viewing technique. It is quite striking to see simultaneously the many layers of a complex surface floating in front of you, almost as if the surface were constructed out of vanishingly thin glass—and it becomes even more remarkable when this image is set smoothly rotating. Indeed, point cloud rendering performed so well for seeing implicit surfaces that we ended up including point cloud based techniques for viewing parametric and triangulated surfaces as a supplement to the more classic rendering methods already available for those surfaces.

In our original approach to generating implicit surface point clouds we intersected the surface with randomly chosen lines, each of which was parallel to one of the three coordinate axes. While such point clouds are reasonably uniform—enough so for most visualization purposes, a computation shows that their density can vary by a factor \( \sqrt{3} \) from one point to another on a surface, and this renders them unsuitable for use with Monte Carlo methods (see below) or other numerical applications whose accuracy depends on having highly uniform distributions of points. After considerable reflection and experimentation we realized that the Cauchy-Crofton Formula implied that if we selected the lines used to intersect the surface to be uniformly distributed with respect to kinematic measure, the resulting point clouds would have the uniformity properties required to support such computational applications. There is also a subtle visualization benefit from point cloud uniformity; namely as one looks towards a point on a surface “contour”, i.e., a point where the line of sight is tangent to the surface, the points of the cloud appear increasingly dense, and this not only helps make the contours visible but also gives the surface a more accurate three dimensional appearance.

2. Monte Carlo Methods.

It has been said that Monte Carlo methods “Lift the Curse of Dimensionality.” Let’s see what that means. Suppose we have a \( C^1 \) function defined on the \( n \)-cube, \( f : I^n \to \mathbb{R} \), and we want to estimate its integral \( J = \int f(x) \, dx \). A natural measure of the complexity of a numerical algorithm for estimating \( J \) is the number \( N(\epsilon) \) of evaluations of \( f \) required by the algorithm in order that the expected error in the estimate \( J' \) it generates will differ from the correct value by less than \( \epsilon \). We will estimate this is for both a simple Riemann integral approach and using a Monte Carlo method. For the Riemann integral inspired calculation we divide each side of \( I^n \) into \( k \) equal sub-intervals of length \( \frac{1}{k} \), thereby partitioning the cube into \( N_k := k^n \) sub-cubes \( \kappa_i \) of diameter \( \delta_k = \sqrt[3]{ \frac{1}{k} } \), and volume \( v_k = \frac{1}{k^n} \). To estimate \( J \), we select arbitrary points \( x_i \), one in each \( \kappa_i \), and compute the Riemann

\(^8\)This problem can be at least partially overcome by using “transparency”, however doing so slows down a raytrace considerably.
sum $J_k := \sum_{i=1}^{N_k} f(x_i)v_k$, and we note that this requires $N_k = k^n$ evaluations of $f$. On the other hand, if $x_i$ in $\kappa_i$ is a point where $f$ assumes its mean value in $\kappa_i$, (i.e., such that $f(x_i) = \frac{1}{|\kappa_i|} \int_{\kappa_i} f(x) \, dx$) then $J = \sum_{i=1}^{N_k} f(x_i)v_k$, and hence $|J - J_k| \leq \sum_{i=1}^{N_k} |f(x_i) - f(x_i)|v_k$. Since $f$ is $C^1$, $K := \sup \| \nabla f \|$ is a Lipschitz constant for $f$, hence if $|x_i - x_j| < \frac{1}{K}$ for all $i$ then $|f(x_i) - f(x_j)| < \epsilon$ for all $i$ and hence $|J - J_k| \leq \epsilon$. Since $x_i$ and $\bar{x}_i$ are arbitrary elements of $\kappa_i$, to insure $|\bar{x}_i - x_i| < \frac{1}{K}$ we need the diameter $\delta_k$ of the $\kappa_i$ less than $\frac{1}{K}$, i.e., $\frac{\sqrt{n}}{\epsilon^2} < \frac{1}{K}$ and hence $k \geq \frac{\sqrt{n}}{\epsilon^2}$. This means that for this algorithm the measure of complexity $N(\epsilon) = N_k = k^n \geq \left(\frac{K\epsilon^2}{\sqrt{n}}\right)^n$ and we see that, using standard “Big O” notation, for this algorithm $N(\epsilon) = O(\epsilon^{-n})$. While this estimate may at first glance appear innocuous, on further examination it leads us to typical examples of the “Curse of Dimensionality”. For there are many examples in theoretical physics where the answer to an important question requires computing a definite integral over $I^n$ where $n > 100$. Suppose we are satisfied with an accuracy of only $0.01$ in our calculation. Then the above method for evaluating the integral is clearly hopeless, since it requires on the order of $100^{100}$ evaluations of $f!$. Another way to look at the curse is to note that it says if one needs $N$ evaluations to get the expected error less than $\epsilon$, then it requires $2^N N$ evaluations to get it less than $\frac{1}{\epsilon}$. While it is true that similar but more sophisticated numerical methods, such as Simpson’s Rule, can cut down somewhat on the order of growth of $N(\epsilon)$ as a function of the dimension $n$, this improvement is too small to solve the problem—the curse remains. What we need is a method for which the order $N(\epsilon)$ of the error estimate is independent of $n$, and the remarkable fact is that the Monte Carlo approach provides just that. We will not prove it here (for details, see [J] p. 291) but it follows from the Central Limit Theorem that if we use an $n$-distributed RNG to estimate the integral using the Monte Carlo formula of Definition 2 in Section 3, then the expected error with $N$ evaluations is $O\left(\frac{1}{\sqrt{N}}\right)$ so $N(\epsilon) = O(\epsilon^{-2})$, and we only require four times as many function evaluations to double the expected accuracy.

18. Generalizations

While the theory developed above is restricted to hypersurfaces, it generalizes to provide a method for constructing equidistributed point clouds on submanifolds $\Sigma$ of $\mathbb{R}^n$ of higher codimension, say $k$. Although the details get somewhat more complex, the main constructions and proofs go over in a straightforward manner. The first modification is that the space $L^2$ of affine lines in $\mathbb{R}^n$ gets replaced by the space $\text{Aff}^n_k$ of $k$-dimensional affine subspaces of $\mathbb{R}^n$. (Of course, $\text{Aff}^1_k = L^2$.) The group of Euclidean motions acts transitively on $\text{Aff}^n_k$ and again there is a natural kinematic measure $d\mu$ that is invariant under this action. Except for a set of $V \in \text{Aff}^n_k$ having kinematic measure zero, the number $\#(V \cap \Sigma)$ of points where $V$ intersects $\Sigma$ is finite, and the obvious generalization of Cauchy-Crofton is valid, namely $\int_{\text{Aff}^n_k} \#(V \cap \Sigma) \, d\mu(V)$, the average number of intersection points, is a universal constant times the volume of $\Sigma$. When it comes to actually implementing this generalization numerically, there appears to be a problem; the technique described in the first subsection of Section 13 for calculating where a line meets an implicit hypersurface was based on the intermediate value theorem, i.e., the principle that if a continuous real-valued function has different signs at the endpoints of an interval then it vanishes somewhere in the interval, and this requires $k = 1$. Now even when $k = 1$, once we have a good approximation to the intersection point, it is more efficient to use Newton’s Method rather than bisection to find it with precision, and it is a higher dimensional version of this Newton’s Method approach that saves the day. The fact that Newton’s Method generalizes to an effective algorithm for solving our problem goes back to an old paper of Kantorovitch [Ka] and it is
usually known as the Newton-Kantorovich Theorem. We will not go into further detail here and instead refer the interested reader to a nice exposition by J. M. Ortega [O]. What makes this generalization to higher codimensional submanifolds of significant interest is that, coupled with the Moser-Whitney Theorem (section 1.5), it extends our methods for constructing highly equidistributed point clouds to arbitrary compact measured manifolds, and thereby provides an approach to implementing Monte Carlo methods in that degree of generality. We have not tried to estimate the complexity of the combination of algorithms that are required for such an implementation and consider that to be a worthwhile project.

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