Existence of peakons for a cubic generalization of the Camassa-Holm equation

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Abstract In this paper, we study the following generalized Camassa-Holm equation with both cubic and quadratic nonlinearities:

\[ m_t + k_1(3uu_xm + u^2m_x) + k_2(2mu_x + m_xu) = 0, \quad m = u - u_{xx}, \]

which is presented as a linear combination of the Novikov equation and the Camassa-Holm equation with constants \( k_1 \) and \( k_2 \). The model is a cubic generalization of the Camassa-Holm equation. It is shown that the equation admits single-peaked soliton and periodic peakons.

Keywords: Generalization of Camassa-Holm equation, Peakons, Periodic peakons
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1 Introduction

The well-known Camassa-Holm(CH) equation

\[ m_t + um_x + 2u_xm = 0, \quad m = u - u_{xx}, \quad (1.1) \]

which was proposed by Camassa and Holm as a nonlinear model for the unidirectional propagation of the shallow water waves over a flat bottom with \( u(x, t) \) representing the water’s free surface \([6, 20, 21]\). It has attracted much attention in the past decades. In addition, the CH equation \((1.1)\) has several nice geometrical structures due to, for example, its description about a geodesic flow on the diffeomorphism group on the circle \([22]\) and its derivation from a non-stretching invariant planar curve flow in the centro-equi-affine geometry \([18]\). Moreover, well-posedness theory and wave breaking phenomenon of the CH equation were studied extensively, and many interesting results have been deduced, see \([2, 7, 9, 10, 29, 32, 35]\). The stability and interaction of peakons were discussed in several references \([11, 12, 31]\). Among these properties, a remarkable one is that it admits the single peakons and periodic peakons in the following forms

\[ \varphi_c(x, t) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \quad (1.2) \]

and

\[ u_c(x, t) = \frac{c}{\text{sh}(1/2)} \text{ch}(\frac{1}{2} - (x - ct) + [x - ct]), \quad c \in \mathbb{R}, \quad (1.3) \]

where the notation \([x]\) denotes the largest integer part of the real number \( x \in \mathbb{R} \).

In addition to the CH equation being an integrable model with peakons, other integrable peakon models, which include the Degasperis-Procesi equation...
and the cubic nonlinear peakon equations \[3, 23\], have been found. Indeed, two integrable CH-type equations with cubic nonlinearity have been discovered recently. The first one is mCH equation:

\[ m_t + \left( (u^2 - u_x^2) m \right)_x = 0, \quad m = u - u_{xx}, \tag{1.4} \]

and the second one is the so-called Novikov equation:

\[ u_t - u_{txx} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \quad t > 0, \quad x \in \mathbb{R}. \tag{1.5} \]

The perturbative symmetry approach \[5\] yielded necessary conditions for PDEs to admit infinitely many symmetries. Using this approach, Novikov \[23\] was able to isolate Eq.(1.5) in a symmetry classification and also found its first few symmetries. He subsequently found a scalar Lax pair for it, and also proved that the equation is integrable. Hone and Wang \[3\] showed that equation (1.5) arised as a zero curvature equation \(F_t - G_x + [F, G] = 0\), which is the compatibility condition for the linear system

\[ \begin{cases} 
\Psi_t = F \Psi, \\
\Psi_x = G \Psi, 
\end{cases} \]

where \(m = u - u_{xx}\),

\[ F = \begin{pmatrix} 0 & m \lambda & 1 \\
0 & 0 & m \lambda \\
1 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{1}{3 \lambda} - uu_x & \frac{u_x}{\lambda} - u^2 m \lambda & \frac{-u^2}{\lambda} \\
\frac{u_x}{\lambda} & \frac{-u^2}{\lambda} & \frac{-u^2}{\lambda} + uu_x \\
-uu_x & \frac{u_x}{\lambda} & \frac{-u^2}{\lambda} \end{pmatrix}. \]

They also proved that it was related to a negative flow in the Sawada-Kotera hierarchy by a reciprocal transformation. By defining a new dependent variable \(m\), Eq.(1.5) can be written as

\[ m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx}. \tag{1.6} \]

Analogous to the Camassa-Holm equation, the Novikov equation has a bi-Hamiltonian structure and an infinite sequence of conserved quantities. In addition, the single peakon for the Novikov equation was obtained in \[3\], which takes the form

\[ u(t, x) = \pm \sqrt{c} e^{-|x-ct|}, \quad c > 0, \]

and the periodic peakons \[30\]

\[ u_c(x, t) = \sqrt{c} \frac{\text{ch}(\frac{1}{2} - (x - ct) + [x - ct])}{\text{ch}(1/2)}, \quad c > 0. \]

Then, Liu, Liu and Qu \[26\] proved the single peakons are orbital stable. Wang, Tian \[30\] also proved the existence and orbital stability of periodic peakons.

On the other hand, applying tri-Hamiltonian duality to the modified Korteweg-de Vries (mKdV) equation leads to the modified Camassa-Holm (mCH) equation with cubic nonlinearity. More generally, applying tri-Hamiltonian duality to the bi-Hamiltonian Gardner equation

\[ Gu_t + u_{xxx} + k_1 u^2 u_x + k_2 uu_x = 0, \tag{1.7} \]
the resulting dual system is the following generalized modified Camassa-Holm (gmCH) equation with both cubic and quadratic nonlinearities \[13\]:

\[ m_t + k_1 \left( (u^2 - u_x^2) m \right)_x + k_2 (2u_x m + um_x) = 0, \quad m = u - u_{xx}. \]  

(1.8)

Recently, it was found \[33\] that, for \( k_1 \neq 0 \), the gmCH equation (1.8) admits a single peackon of the form

\[ \varphi_c(t,x) = ae^{-|x-ct|}, \quad c \in \mathbb{R}, \]

with

\[ a = \frac{3}{4} \frac{-k_2 \pm \sqrt{k_2^2 + \frac{8}{3}k_1c}}{k_1}, \quad k_2^2 + \frac{8}{3}k_1c \geq 0, \]

and also found \[27\] that, for \( k_1 \neq 0 \), the gmCH equation (1.8) admits periodic peackons of the form

\[ u_c(t,x) = a \text{ch} \left( \frac{1}{2} - (x - ct) + [x - ct] \right), \]

where

\[ a = \frac{3}{4} \frac{-k_2 \text{ch}(1/2) \pm \sqrt{k_2^2 \text{ch}^2(1/2) + \frac{8}{3}k_1c(1 + 2 \text{ch}^2(1/2))}}{k_1(1 + 2 \text{ch}^2(1/2))} \]

and

\[ k_2^2 \text{ch}^2(1/2) + \frac{4}{3}k_1c(1 + 2 \text{ch}^2(1/2)) \geq 0. \]

The existence of (periodic) peackons is of interest for the nonlinear integrable equations since they are relatively new solitary waves. More importantly, in the theory of water waves many papers have investigated the Stokes waves of greatest height, traveling waves which are smooth everywhere except at the crest where the lateral tangents differ. There is no closed form available for these waves, and the peackons capture the essential features of the extreme waveforms of great amplitude that are exact solutions of the governing equations for irrotational water waves, see the discussion in \[1, 8\]. Inspired by \[27, 33\], we focus on the following generalized Camassa-Holm equation with both cubic and quadratic nonlinearities:

\[ m_t + k_1 (3u^2u_x m + u_x^2m_x) + k_2 (2ma_x + m_xu) = 0, \quad m = u - u_{xx}. \]  

(1.9)

where \( k_1 \) and \( k_2 \) are arbitrary constants. It is clear that equation (1.9) reduces to the CH equation for \( k_1 = 0, k_2 = 1 \) and the Novikov equation for \( k_1 = 1, k_2 = 0 \), respectively. Equation (1.9) is actually a linear combination of CH equation (1.1) and cubic nonlinear equation (1.3). Therefore, we may view equation (1.9) as a generalization of the CH equation, or simply call equation (1.9) a generalized CH equation. Like the Camassa-Holm and Novikov equations, the new equation admits peaked soliton solutions.
2 Preliminaries

In this paper, we are concerned with the Cauchy problem for the generalized CH equation on both line and the unit circle:

\[
\begin{align*}
\begin{cases}
m_t + k_1(3u u_x m + u^2 m_x) + k_2(2m u_x + m_x u) = 0, & \ t > 0, \ x \in \mathbb{R} \text{ or } S, \\
 m(t, x) = u(t, x) - u_{xx}(t, x), \\
u(0, x) = u_0(x), & \ x \in X.
\end{cases}
\end{align*}
\]  

(2.1)

First, we will require the notion of strong (or classical) solutions as follows:

**Definition 2.1.** If \( u \in C([0, T); H^s(X)) \cap C^1([0, T); H^{s-1}(X)) \) with \( s > \frac{5}{2} \) and some \( T > 0 \) satisfies (2.1), then \( u \) is a strong solution on \([0, T)\). If \( u \) is a strong solution on \([0, T)\) for every \( T > 0 \), then it is called a global strong solution.

The following local well-posedness result and properties for strong solutions on the line and unit circle can be established using the same approach as in [15].

**Proposition 2.1.** Let \( u_0 \in H^s(X) \) with \( s > \frac{5}{2} \). Then there exists a time \( T > 0 \) such that the initial value problem (2.1) has a unique strong solution \( u \in C([0, T); H^s(X)) \cap C^1([0, T); H^{s-1}(X)) \) and the map \( u_0 \to u \) is continuous from a neighborhood of \( u_0 \) in \( H^s(X) \) into \( u \in C([0, T); H^s(X)) \cap C^1([0, T); H^{s-1}(X)) \).

If \( m = u - u_{xx} \) is substituted in terms of \( u \) into the generalized CH equation (2.1), then the resulting fully nonlinear partial differential equation takes the following form:

\[
u_t + k_1 u^2 u_x + \frac{1}{2} k_1 (1 - \partial_x^2)^{-1} u_x^3 + k_1 (1 - \partial_x^2)^{-1} \partial_x (u^3 + \frac{3}{2} uu_x^2) + k_2 uu_x + k_2 \partial_x (1 - \partial_x^2)^{-1} (u^2 + \frac{1}{2} u_x^2) = 0.\]  

(2.2)

Taking the convolution with the Green’s function for the Helmholtz operator \((1 - \partial_x^2)\), equation (2.2) can be rewritten as

\[
u_t + k_1 u^2 u_x + \frac{1}{2} k_1 G(x) * u_x^3 + k_1 G(x) * \partial_x (u^3 + \frac{3}{2} uu_x^2) + k_2 uu_x + k_2 G(x) * \partial_x (u^2 + \frac{1}{2} u_x^2) = 0.\]  

(2.3)

Note that \( u \) can be formulated by the Green function \( G(x) \) as

\[
u = (1 - \partial_x^2)^{-1} m = G * m,\]  

(2.4)

where \( G(x) = \frac{1}{2} e^{-|x|} \) for the non-periodic case, \( G(x) = \frac{\text{ch}(1/2 - x + [x])}{2 \text{sh}(1/2)} \) for the periodic case, and \(*\) denotes the convolution product on \( X \), defined by

\[ (f * g)(x) = \int_X f(y) g(x - y) dy.\]

The above formulation (2.4) allows us to define the periodic weak solutions as follows.
Definition 2.2. Given initial data $u_0 \in W^{1,3}(S)$, the function $u \in L^\infty_{loc}([0,T), W^{1,3}(S))$ is called a periodic weak solution to the initial value problem (2.1) if it satisfies the following identity:

\[
\int_0^T \int_S \left[ u \partial_t \phi + k_1 \frac{u^3 + 3}{2} \partial_x \phi + k_1 G(x) \ast \left( u^3 + \frac{3}{2} uu_x^2 \right) \partial_x \phi - k_1 G(x) \ast \left( \frac{u_x^3}{2} \right) \phi \\
+ k_2 u^2 \partial_x \phi + k_2 G(x) \ast (u^2 + \frac{1}{2} u_x^2) \partial_x \phi \right] \, dx \, dt + \int_S u_0(x) \phi(0,x) \, dx = 0,
\]

for any smooth test function $\phi(t,x) \in C^\infty_c([0,T) \times S)$. If $u$ is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global periodic weak solution.

3 Peakon solutions

In this section, we derive the single-peaked solutions and periodic peakons of equation (1.9).

3.1 Single-peaked solutions

Firstly, we give the result on the existence of single-peaked solutions for the generalized CH equation (1.9).

Theorem 3.1. (Single peakons) For the wave speed $c$ satisfying $k_2^2 + 4k_1c \geq 0$, equation (1.9) with $k_1 \neq 0$ admits the single peakons of the form:

\[
u = Ae^{-|x-ct|},
\]

where $A = \frac{-k_2 \pm \sqrt{k_2^2 + 4k_1c}}{2k_1}$.

Proof. Firstly, let us suppose the single-peaked solutions of equation (1.9) in the form of

\[
u = Ae^{-|x-ct|},
\]

where $A$ is to be determined. The derivatives of expression (3.2) do not exist at $x = ct$, thus (3.2) cannot satisfy equation (1.9) in the classical sense. However, in the weak sense, we can write out the expressions of $u_x$ and $u_t$ under the help of distribution:

\[
u_x = -Asgn(x-ct)e^{-|x-ct|}, \quad \nu_t = cAsgn(x-ct)e^{-|x-ct|}.
\]

Next, we need consider two cases (i) $x > ct$ and (ii) $x < ct$.

For $x > ct$, we calculate from (3.2) and (3.3) that

\[
u_t + k_1 u^2 u_x + k_2 uu_x = Ae^{-|x-ct|} - k_1 A^2 e^{-3|x-ct|} - k_2 A^2 e^{-2|x-ct|},
\]

Note that the Green function $G(x) = \frac{1}{2} e^{-|x|}$ in the non-periodic case, it is thus deduced that

\[
\frac{1}{2} k_1 G(x) \ast u_x^3 + k_1 G(x) \ast \partial_x (u^3 + \frac{3}{2} uu_x^2)
\]

\[
= -4k_1 A^3 \left( \int_{-\infty}^{ct} + \int_{ct}^x + \int_x^{+\infty} \right) e^{-|x-y|-3|y-ct|} \, dy
\]

\[
= -k_1 A^3 e^{-|x-ct|} + k_1 A^3 e^{-3|x-ct|},
\]
and
\[ k_2G(x) \ast \partial_x (u^2 + \frac{1}{2} u_x^2) \]
\[ = -\frac{3}{2} k_2 A^2 \int_R sgn(y - ct) e^{-|x-y|-2|y-ct|} dy \]
\[ = -\frac{3}{2} k_2 A^2 \left( \int_{-\infty}^c + \int_c^x + \int_x^\infty \right) e^{-|x-y|-2|y-ct|} dy \]
\[ = -k_2 A^2 e^{-(x-ct)} + k_2 A^2 e^{-2(x-ct)}. \]

The case \( x < ct \) is similar to \( x > ct \), here we do not compute in detail.

Plugging (3.4), (3.5) and (3.6) into (2.3), we deduce that
\[ u_t + k_1 u^2 u_x + \frac{1}{2} k_1 G(x) \ast u_x^3 + k_1 G(x) \ast \partial_x (u^3 + \frac{3}{2} uu_x^2) \]
\[ + k_2 uu_x + k_2 G(x) \ast \partial_x (u^2 + \frac{1}{2} u_x^2) \]
\[ = (Ac - k_1 A^3 - k_2 A^2) e^{-(x-ct)} \]
\[ = 0. \]

Therefore, we are able to conclude from (3.7) that \( A \) should satisfy
\[ k_1 A^2 + k_2 A - c = 0. \]

In general, we obtain
\[ A = \frac{-k_2 \pm \sqrt{k_2^2 + 4k_1 c}}{2k_1} \]
with \( k_2^2 + 4k_1 c \geq 0 \) and \( k_1 \neq 0 \).

### 3.2 Periodic peakons

The following theorem shows the existence of periodic peakons for the generalized CH equation (1.9).

**Theorem 3.2.** (Periodic peakons) For the wave speed \( c \) satisfying \( k_2^2 \text{ch}^2(1/2) + 4k_1 c (1 + \text{sh}^2(1/2)) \geq 0 \), equation (1.9) with \( k_1 \neq 0 \) possesses the periodic peakons of the form:
\[ u_c(x,t) = a \text{ch}(\zeta), \quad \zeta = \frac{1}{2} - (x - ct) + [x - ct], \]
(3.10)
where
\[ a = \frac{-k_2 \text{ch}(1/2) \pm \sqrt{k_2^2 \text{ch}^2(1/2) + 4k_1 c (1 + \text{sh}^2(1/2))}}{2k_1 (1 + \text{sh}^2(1/2))} \]
(3.11)
as the global periodic weak solutions to (2.1) in the sense of Definition 2.2.

**Proof.** Firstly, we identify \( S = [0,1) \) and regard \( u_c(t,x) \) as spatial periodic function on \( S \) with period one. On one hand, it is noted that \( u_c \) is continuous on \( S \) with peak at \( x = 0 \). On the other hand, \( u_c \) is smooth on \( (0,1) \) and for all \( t \in \mathbb{R}^+ \),
\[ \partial_x u_c(t,x) = -a \text{sh}(\zeta) \in L^\infty(S). \]
(3.12)
Hence, we denote \( u_{c,0}(x) = u_c(0,x) \) with \( x \in S \), then it holds that
\[
\lim_{t \to 0^+} \| u_c(t, \cdot) - u_{c,0}(\cdot) \|_{W^{1,\infty}(S)} = 0.
\] (3.13)

As in (3.12), it is found that
\[
\partial_t u_c(x,t) = \frac{a}{c} \text{sh}(\zeta) \in L^\infty(S), \ t \geq 0.
\] (3.14)

Using (3.12)-(3.14) and integration by parts, it is thus deduced that, for every test function \( \phi(t, x) \in C^\infty_0([0, \infty) \times S) \),
\[
\int_0^\infty \int_S \left[ k_1 G(x) * \left( u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2 \right) \partial_x \phi - \frac{k_1}{2} G(x) * (\partial_x u_c)^3 \phi \right] dx dt
= k_1 a^3 \int_0^\infty \int_S \phi G(x) * \left( 3 \text{sh}(\zeta) + \frac{7}{2} \text{sh}^3(\zeta) \right) dx dt
- \frac{3}{2} k_1 a^3 \int_0^\infty \int_S \phi G_x(x) * (\text{ch}(\zeta) \text{sh}^2(\zeta)) dx dt
= k_1 a^3 \int_0^\infty \int_S \phi \left( \text{sh}^2(1/2) \cdot \text{sh}(\zeta) - \text{sh}^3(\zeta) \right) dx dt.
\] (3.15)

On the other hand, noticing from the explicit form of the Green function \( G(x) \) for the periodic case that
\[
G(x) = \frac{\text{ch}(1/2 - x + |x|)}{2 \text{sh}(1/2)} \quad \text{and} \quad G_x(x) = -\frac{\text{sh}(1/2 - x + |x|)}{2 \text{sh}(1/2)}, \ x \in \mathbb{R},
\]

it follows from (3.12), (3.14) and the proof of Theorem 4.1 in [14] that
\[
\int_0^\infty \int_S \left[ k_2 G(x) * \left( u_c^2 + \frac{1}{2} (\partial_x u_c)^2 \right) \partial_x \phi \right] dx dt
= k_2 a^2 \int_0^\infty \int_S \phi G(x) * \text{sh}(2\zeta) dx dt.
\] (3.16)

Next, we compute directly that
\[
\int_0^\infty \int_S k_2 G(x) * \left( u_c^2 + \frac{1}{2} (\partial_x u_c)^2 \right) \partial_x \phi dx dt
= \frac{3k_2}{2} a^2 \int_0^\infty \int_S \phi G(x) * \text{sh}(2\zeta) dx dt.
\] (3.17)

To prove, we consider two cases: (i) \( x > ct \) and (ii) \( x < ct \). When \( x > ct \), a
direct calculation gives rise to

\[ G(x) \ast \text{sh}(2\zeta)(t, x) \]

\[
= \frac{1}{2 \text{sh}(1/2)} \int_S \text{ch}(1/2 - (x - y) + |x - y|) \cdot \text{sh}(1 - 2(y - ct) + 2[y - ct]) dy \\
= \frac{1}{2 \text{sh}(1/2)} \left[ \int_0^{ct} \text{ch}(1/2 - x + y) \cdot \text{sh}(-1 - 2y + 2ct)dy \\
+ \int_t^x \text{ch}(1/2 - x + y) \cdot \text{sh}(1 - 2y + 2ct)dy \\
+ \int_x^{ct} \text{ch}(1/2 + x - y) \cdot \text{sh}(1 - 2y + 2ct)dy \right] \\
= \frac{2}{3} [\text{sh}(1/2) \text{sh}(1/2 - (x - ct)) - \text{sh}(1/2 - (x - ct)) \text{ch}(1/2 - (x - ct))].
\] (3.18)

In a similar manner, for \( x < ct \),

\[ G(x) \ast \text{sh}(2\zeta)(t, x) \]

\[
= \frac{2}{3} [-\text{ch}(1/2) \text{sh}(1/2 + (x - ct)) + \text{sh}(1/2 + (x - ct)) \text{ch}(1/2 + (x - ct))].
\] (3.19)

Plugging (3.18) and (3.19) into (3.17), it is deduced by a straightforward computation that

\[
\int_0^\infty \int_S k_2 G(x) \ast \left( u_c^2 + \frac{1}{2} (\partial_x u_c)^2 \right) \partial_x \phi dx dt \\
= k_2 a^2 \int_0^\infty \int_S \phi (\text{ch}(1/2) \text{sh}(\zeta) - \text{sh}(\zeta) \text{ch}(\zeta)) dx dt.
\] (3.20)

In view of (3.15), (3.16) and (3.20), we have

\[
\int_0^\infty \int_S [u_c \partial_t \phi + \frac{k_1}{3} u_c^3 \partial_x \phi + \frac{k_2}{2} u_c^2 \partial_x \phi \\
+ k_1 G(x) \ast \left( u_c^3 + \frac{3}{2} u_c (\partial_x u_c)^2 \right) \partial_x \phi - k_1 G(x) \ast \left( \frac{(\partial_x u_c)^3}{2} \right) \phi \\
+ k_2 G(x) \ast (u_c^2 + 1/2 (\partial_x u_c)^2) \partial_x \phi] dx dt + \int_S u_{c,0}(x) \phi(0, x) dx \\
= \int_0^\infty \int_S \phi a \left[ k_1 (1 + \text{sh}^2(1/2)) a^2 + k_2 \text{ch}(1/2) a - c \right] \text{sh}(\zeta) dx dt.
\] (3.21)

If \( a \) takes value as (3.11), then

\[
k_1 (1 + \text{sh}^2(1/2)) a^2 + k_2 \text{ch}(1/2) a - c = 0,
\]
which implies that

\[
\int_0^\infty \int_S [u_\epsilon \partial_t \phi + \frac{k_1}{3} u_\epsilon^3 \partial_x \phi + \frac{k_2}{2} u_\epsilon^2 \partial_x \phi \\
+ k_1 G(x) \ast \left( u_\epsilon^2 + \frac{3}{2} u_\epsilon (\partial_x u_\epsilon)^2 \right) \partial_x \phi - \frac{k_1}{2} G(x) \ast (\partial_x u_\epsilon)^3 \phi \\
+ k_2 G(x) \ast (u_\epsilon^2 + \frac{1}{2} (\partial_x u_\epsilon)^2) \partial_x \phi] \, dx \, dt + \int_S u_\epsilon,0(x) \phi(0, x) \, dx = 0,
\]

for any test function \( \phi(x, t) \in C^\infty_c([0, \infty) \times S) \). Thus the theorem is proved.

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