LINEAR ALGEBRA AND UNIFICATION OF GEOMETRIES IN ALL SCALES

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Abstract. We present an idea of unifying small scale (topology, proximity spaces, uniform spaces) and large scale (coarse spaces, large scale spaces). It relies on an analog of multilinear forms from Linear Algebra. As an application we get simple proofs of results generalizing well-known theorems from coarse topology. A new result (at least to the author) is the following (see 13.7):

A coarse bornologous function $f : X \rightarrow Y$ of metrizable large scale spaces is a large scale equivalence if and only if it induces a homeomorphism of Higson coronas.

This paper is an extension of [5] and, at the same time, it overrides [5]. It is in a preliminary form but, since the author gives talks about it at many conferences, it makes sense to post it on arXiv now.

1. Introduction

A topology on a set $X$ is the same as a projection (i.e. an idempotent linear operator) $cl : 2^X \rightarrow 2^X$ satisfying $A \subset cl(A)$ for all $A \subset X$. That’s a good way to summarize Kuratowski’s closure operator.

Basic geometry on a set $X$ is a dot product $\cdot : 2^X \times 2^X \rightarrow \{0, \infty\}$. Its equivalent form is an orthogonality relation on subsets of $X$. The optimal case is when the orthogonality relation satisfies a variant of parallel-perpendicular decomposition from linear algebra. Dot products are a special case of forms which act on arbitrary vectors based on a given set $X$.

We show that this concept unifies small scale (topology, proximity spaces, uniform spaces) and large scale (coarse spaces, large scale spaces). Using forms we define large scale compactifications that generalize all well-known compactifications: Higson corona, Gromov boundary, Čech-Stone compactification, Samuel-Smirnov compactification, and Freudenthal compactification. This allows to generalize many results in coarse topology from proper metric spaces to arbitrary metric spaces or even to arbitrary large scale spaces.

Example 1.1. (see 19.4) Let $X$ be a metric space and let $G$ be a finite group acting isometrically on $X$. Then $X/G$ has the same asymptotic dimension as $X$.

In case of proper metric spaces $X$, Theorem 19.4 was proved by Daniel Kasprowski [13].

Example 1.2. (see 13.5) If $n \geq 1$ and $f : X \rightarrow Y$ is a coarsely $n$-to-1 bornologous map of large scale spaces, then $\text{asdim}(Y) \leq \text{asdim}(Y) + n - 1$.

Theorem 13.5 was proved by Austin-Virk in [1] for proper metric spaces $X$ and $Y$.

2. Multilinear forms on sets

Definition 2.1. The semi-group $\{0, \infty\}$ has the following binary operation:

1. $0 + 0 = 0$,
2. $0 + \infty = \infty + 0 = \infty + \infty = \infty$.

Recall that a bornology $B$ on a set $X$ is any family of subsets closed under finite unions so that $B \subset B' \in B$ implies $B \in B$.

Notice that bornologies $B$ on a set $X$ are identical with kernels of basic linear operators $\omega : 2^X \rightarrow \{0, \infty\}$, i.e. functions satisfying
1. \( \omega(\emptyset) = 0 \),
2. \( \omega(C \cup D) = \omega(C) + \omega(D) \) for all \( C, D \in 2^X \).

That observation leads to the following generalization:

**Definition 2.2.** A \( k \)-vector in \( X \) \((k \geq 1)\) is a \( k \)-tuple \((C_1, \ldots, C_k)\) of subsets in \( X \). The set of all vectors in \( X \) will be denoted by \( \text{Vect}(X) \).

The concatenation \( V_1 \ast V_2 \) of a \( k \)-vector \( V_1 = (C_1, \ldots, C_k) \) and an \( m \)-vector \( V_2 = (D_1, \ldots, D_m) \) is the \((k + m)\)-vector \((C_1, \ldots, C_k, D_1, \ldots, D_m)\).

A basic multilinear form \( \omega \) on \( X \) is a symmetric function on all \( k \)-vectors of \( X \) \((k \geq 1)\) to \( \{0, \infty\} \) satisfying the following properties:
1. \( \omega((C_1 \cup C_2) \ast V) = \omega(C_1 \ast V) + \omega(C_2 \ast V) \) for any \( k \)-vector \( V \) and any two 1-vectors \( C_1, C_2 \),
2. \( \omega(\emptyset) = 0 \),
3. \( \omega(C \ast V) = \omega(V) \) if one of the coordinates of \( V \) is contained in \( C \).

**Remark 2.3.** One can consider forms with values in any join-semilattice, i.e. a partially ordered set \((L, \leq)\) in which each two-element subset \( \{a, b\} \subseteq L \) has a join (i.e. least upper bound). However, it is not clear what new applications would arise from that concept.

Notice that \( \{0, \infty\} \) has a natural order: namely \( 0 < \infty \) (in fact, it is the smallest non-trivial lattice).

**Lemma 2.4.** If \( V \) and \( V' \) are two \( k \)-vectors such that \( V(i) \subseteq V'(i) \) for each \( 1 \leq i \leq k \), then \( \omega(V) \leq \omega(V') \).

**Proof.** Apply Axiom 1 repeatedly to see that \( \omega(V) \) is a summand of \( \omega(V') \). \(\square\)

**Corollary 2.5.** \( \omega(V) \leq \omega(C) \) if some coordinate of \( V \) is contained in \( C \).

**Proof.** If \( V \) is a \( k \)-vector, we may assume \( V(k) \subseteq C \). Now, \( \omega(V) \leq \omega(X \ast \ldots \ast X \ast C) \) by 2.4 and the latter equals \( \omega(C) \) by Axiom 3. \(\square\)

**Proposition 2.6.** \( \omega(V \ast V') \leq \omega(V) \) for all vectors \( V \) and \( V' \).

**Proof.** By 2.4 \( \omega(V \ast V') \leq \omega(V \ast X \ast \ldots \ast X) \) and the latter equals \( \omega(V) \) by Axiom 3. \(\square\)

**Definition 2.7.** A bornological space \((X, \mathcal{B})\) is a set \( X \) equipped with a bornology \( \mathcal{B} \).

A formed space \((X, \omega)\) is a set \( X \) equipped with a basic multilinear form \( \omega \).

**Proposition 2.8.** Each formed space \((X, \omega)\) induces a bornology on \( X \) via

\[
\mathcal{B}(\omega) := \{ C \subseteq X \mid \omega(C) = 0 \}.
\]

Conversely, given a bornology \( \mathcal{B} \) on \( X \),

\[
\omega(C_1, \ldots, C_k) = 0 \iff C_1 \cap \ldots \cap C_k \in \mathcal{B}
\]

defines a basic multilinear form on \( X \) that induces \( \mathcal{B} \).

**Proof.** Left to the reader. \(\square\)

**Observation 2.9.** Notice that, given a bornology \( \mathcal{B} \) on \( X \), the form defined above is the largest form that induces \( \mathcal{B} \). The smallest form inducing \( \mathcal{B} \) is given by the formula

\[
\omega(C_1, \ldots, C_k) = 0 \iff C_i \in \mathcal{B} \text{ for some } i.
\]
Example 2.10. Given a topological space \((X, T)\), the **basic topological form** \(\omega(X, T)\) is defined as follows:

\[
\omega(X, T)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} \text{cl}(C_i) = \emptyset.
\]

Example 2.11. Given a topological space \((X, T)\), the **basic functional form** \(\omega_f(X, T)\) is defined as follows:

\[
\omega_f(X, T)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} D_i = \emptyset \text{ for some zero-sets } D_i \supset C_i.
\]

Recall \(D\) is a **zero-set** in a topological space \(X\) if there is a continuous function \(f : X \to [0, 1]\) such that \(D = f^{-1}(0)\).

Example 2.12. Given a metric space \((X, d)\), the **basic small scale form** \(\omega_s(X, d)\) is defined as follows:

\[
\omega_s(X, d)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} B(C_i, r) = \emptyset
\]

for some \(r > 0\).

Example 2.13. Given a metric space \((X, d)\), the **basic large scale form** \(\omega_l(X, d)\) is defined as follows:

\[
\omega_l(X, d)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} B(C_i, r) \text{ is bounded}
\]

for each \(r > 0\).

Example 2.14. Given a metric space \((X, d)\), the \(C_0\)-form \(\omega_0(X, d)\) is defined as follows:

\[
\omega_0(X, d)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} B(C_i, r) \text{ is bounded}
\]

for some \(r > 0\).

Example 2.15. Given a uniform space \((X, U)\), the form \(\omega(X, U)\) is defined as follows:

\[
\omega(X, U)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} \text{st}(C_i, V) = \emptyset
\]

for some uniform cover \(V\) of \(X\).

Example 2.16. Given a large scale space \((X, \mathcal{L})\), the form \(\omega(X, \mathcal{L})\) is defined as follows:

\[
\omega(X, \mathcal{L})(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} \text{st}(C_i, V) \text{ is bounded}
\]

for each uniformly bounded cover \(V\) of \(X\).
Example 2.17. Given a topological space \((X, T)\) and a subset \(A\) of \(X\), the form \(\omega(X, A, T)\) on \(X \setminus A\) is defined as follows:

\[
\omega(X, A, T)(C_1, \ldots, C_k) = 0 \iff \bigcap_{i=1}^{k} \text{cl}(C_i) \subset X \setminus A.
\]

Here the closures are taken in \(X\) via \(T\).

3. Topology induced by forms

Proposition 3.1. Given a formed space \((X, \omega)\), the topology induced by \(\omega\) on \(X\) consists of sets \(U\) with the property that \(\omega(x, X \setminus U) = 0\) for all \(x \in U\).

Proof. If \(x \in U \cap W\) and \(\omega(x, X \setminus U) = 0 = \omega(x, X \setminus W)\), then \(\omega((X \setminus U) \cup (X \setminus W)) = 0\) and \((X \setminus U) \cup (X \setminus W) = X \setminus U \cap W\).

Here is a typical construction of an open set in the topology induced by \(\omega\):

Proposition 3.2. If \(\{C_n\}_{n=1}^{\infty}\) is an increasing sequence of subsets of a formed space \((X, \omega)\) such that \(\omega(C_k, X \setminus C_{k+1}) = 0\) for each \(k \geq 1\), then \(U := \bigcup_{i=1}^{\infty} C_i\) is an open set in the topology induced by \(\omega\).

Proof. Given \(x \in U\), there is \(k \geq 1\) such that \(x \in C_k\). Therefore \(\omega(x, X \setminus C_{k+1}) = 0\). Since \(X \setminus U \subset X \setminus C_{k+1}\), \(\omega(x, X \setminus U) = 0\) by 2.4. □

Proposition 3.3. The topology induced by \(\omega\) is \(T_1\) if and only if \(\omega(x, y) = 0\) whenever \(x \neq y\).

Proof. If \(\omega(x, y) = 0\) whenever \(x \neq y\), then \(X \setminus \{y\}\) is open. If \(X \setminus \{y\}\) is open and \(x \neq y\), then \(\omega(x, y) = 0\). □

Corollary 3.4. If the topology induced by the basic topological form \(\omega\) of \((X, T)\) is \(T_0\), then \((X, T)\) is \(T_1\).

Proof. Given \(x \neq y\) in \(X\), there is an open set \(U\) containing exactly one of the points \(x, y\). Say \(y \in U\) and \(x \notin U\). Therefore \(\omega(y, X \setminus U) = 0\) implies \(\omega(y, x) = 0\). □

Proposition 3.5. If \((X, T)\) is a \(T_1\) topological space, then its basic topological form induces \(T\).

Proof. Let \(\omega\) be the basic topological form of \(X\). \(\omega(x, X \setminus U) = 0\) means \(\text{cl}(x) \cap \text{cl}(X \setminus U) = \emptyset\). Consequently, if \(U \in T \in T_1\), then \(U\) is open in the topology induced by \(\omega\). Conversely, if \(C\) is closed in the topology induced by \(\omega\) and \(x \notin C\), then \(\omega(x, C) = 0\), hence \(x \notin \text{cl}(C)\). Thus \(C = \text{cl}(C)\) in \(T\). □

Of special interest are cases where the topology induced by a form is related to compactness.
3.1. Large scale topology.

**Definition 3.6.** A large scale topological space \((X, T, B)\) is a topological space \((X, T)\) in which a bornology \(B\) of open-closed subspaces is selected.

**Definition 3.7.** A large scale topological space \((X, T, B)\) is **large scale compact** if and only if, for any family \(\{U_x\}_{x \in S}\) of open subsets of \(X\), \(X = \bigcup \cup s \in S U_s\) implies existence of a finite subset \(F\) of \(S\) such that \(X \setminus \bigcup_{s \in F} U_s\) belongs to \(B\).

**Proposition 3.8.** 1) If \((X, T, B)\) is large scale compact and topologically Hausdorff, then it is topologically regular.

2) If \((X, T, B)\) is large scale compact and topologically regular, then it is topologically normal.

**Proof.** 1). Suppose \(A\) is a closed subset of \(X\) not containing \(x_0\). If \(x_0\) is open-closed, then \(A \subset X \setminus \{x_0\}\) is disjoint from \(x_0\) and we are done. Assume \(x_0\) is not open. For each point \(x \in A\) choose disjoint open sets \(U_x\) containing \(x\) and \(V_x\) containing \(x_0\). Notice \(X = (X \setminus A) \cup \bigcup_{x \in A} U_x\), so there is an open-closed set \(B \in \mathcal{B}\) and a finite subset \(F\) of \(A\) such that \(X = B \cup (X \setminus A) \cup \bigcup_{x \in A} U_x\). Notice \(B\) does not contain \(x_0\).

Therefore, \(A \subset B \cup \bigcup_{x \in A} U_x\) is disjoint from \(\bigcap_{x \in F} V_x \setminus B\) which contains \(x_0\).

The proof of 2) is similar or apply 1) to \(X/A\). \(\square\)

**Proposition 3.9.** A continuous map \(f : (X, T_X, \mathcal{B}_X) \rightarrow (Y, T_Y, \mathcal{B}_Y)\) of large scale topological spaces is closed if the following conditions are satisfied:

1. \((X, T_X, \mathcal{B}_X)\) is large scale compact.
2. \((Y, T_Y, \mathcal{B}_Y)\) is Hausdorff.
3. \(f(\mathcal{B}_X) \subset \mathcal{B}_Y\).

**Proof.** It suffices to show \(f(X)\) is closed and then apply it to \(f|A\), \(A\) any closed subset of \(X\).

Given \(y \in Y \setminus f(X)\) that is not open choose, for any \(z \in f(X)\), disjoint open neighborhoods \(U(z)\) of \(y\) and \(W(z)\) of \(z\). Since \(\{f^{-1}(W(z))\}_{z \in f(X)}\) covers \(X\), there exist points \(z_1, \ldots, z_k\) of \(f(X)\) such that \(X \setminus \bigcup_{i=1}^k f^{-1}(W(z_i)) \in \mathcal{B}_X\). Hence, \(D := f(X) \setminus \bigcup_{i=1}^k W(z_i) \in \mathcal{B}_Y\) is open-closed in \(Y\), it does not contain \(y\), and \(U := \bigcap_{i=1}^k U(z_i)\setminus D\) is a neighborhood of \(y\) missing \(f(X)\). \(\square\)

4. **Boundaries of large scale topological spaces**

**Definition 4.1.** The boundary \(\partial(X, T, B)\) of a large scale topological space \((X, T, B)\) is the collection of all maximal families \(\mathcal{P}\) in \(2^X\) consisting of zero-sets satisfying the following condition: intersection of any finite subfamily of \(\mathcal{P}\) is not in \(\mathcal{B}\).

If \(\{x\} \notin \mathcal{B}\), then we can identify \(x\) with the family consisting of all zero subsets of \(X\) containing \(x\). This way we can talk about \(X \cup \partial(X, T, B)\).

**Observation 4.2.** Technically speaking, points \(x_1, x_2 \in X\) such that there is no continuous function from \(X\) to \([0,1]\) separating them, are identified in \(\partial(X, T, B)\). Namely, they generate the same element of \(\partial(X, T, B)\) consisting of all zero subsets.
of $X$ containing $x_1$. Thus, to consider $X \cup \partial(X, T, B)$ in set-theoretic sense, we need $X$ to be functionally Hausdorff (i.e. every pair of different points of $X$ can be separated by a real-valued continuous function). We will continue to use notation $X \cup \partial(X, T, B)$ for the union of a quotient of $X$ and $\partial(X, T, B)$.

Remark 4.3. In case of a bornological space $(X, B)$ we can define its boundary as $\partial(X, T, B)$, where $T$ is the discrete topology on $X$.

Observation 4.4. Suppose $A_1$ and $A_2$ are zero-subsets of $(X, T, B)$. If $A_1 \cup A_2 \in P \in \partial(X, B)$, then $A_1 \in P$ or $A_2 \in P$.

Proof. Suppose $A_1 \cup A_2 \in P$ but $A_1 \notin P$ and $A_2 \notin P$. In that case there exist $C, D \in P$ such that $A_1 \cap C, A_2 \cap D \in B$. Consequently, $(A_1 \cup A_2) \cap C \cap D \in B$, a contradiction.

Corollary 4.5. If $A \in P$ and $B \in B$, then $A \setminus B \in P$.

Definition 4.6. $X \cup \partial(X, T, B)$ has a natural topology whose basis is formed (in view of 4.4) by sets

$$N(U) := U \cup \{Q \in \partial(X, T, B) \mid X \setminus U \notin Q\},$$

where $U$ is a co-zero-subset of $X$ (i.e. the complement of a zero-set in $X$).

Lemma 4.7. $N(U) \cap N(V) = N(U \cap V)$ and $N(U) \cup N(V) = N(U \cup V)$.

Proof. Reformulate 4.4 as follows: $A_1 \cup A_2 \notin P \in \partial(X, B)$ if and only if $A_1 \notin P$ and $A_2 \notin P$. The equality $N(U) \cap N(V) = N(U \cap V)$ is now obvious as well as $N(U) \cup N(V) = N(U \cup V)$.

Corollary 4.8. Each $B \in B$ is open-closed in $X \cup \partial(X, T, B)$.

Proof. By 4.3 $N(B) = B$ and $N(X \setminus B) = X \setminus B$. As $N(X) = N(B) \cup N(X \setminus B)$, both $N(B)$ and $N(X \setminus B)$ are closed.

Proposition 4.9. Suppose $C$ is a zero-subset of $X$. $P \in \partial(X, T, B)$ belongs to the closure of $C \subset X$ if and only if $C \in P$.

Proof. If $C \in P$, then every base element $N(U)$ containing $P$ intersects $C$ as $C \cap U$ cannot be empty. Indeed, $C \cap U = \emptyset$ means $C \subset X \setminus U$ resulting in $X \setminus U \in P$ which contradicts $P \in N(U)$.

Conversely, if $C \notin P$, then $P \in N(X \setminus C)$ and $C \cap N(X \setminus C) = \emptyset$. Indeed, any $Q \in C \cap N(X \setminus C)$ must be equal to all zero subsets containing some $x \in C$. Hence $C \in Q$ which contradicts $Q \in N(X \setminus C)$.

Corollary 4.10. The basis of $X \cup \partial(X, B)$ consists of open-closed sets.

Proof. In this case the topology on $X$ is discrete and the complement of $N(U)$ is $N(X \setminus U)$.

Theorem 4.11. $X \cup \partial(X, T, B)$ is large scale compact and topologically normal.

Proof. Suppose $\{N(U)\}_{U \in S}$ is a cover of $X \cup \partial(X, T, B)$ such that $X \cup \partial(X, T, B) \setminus \bigcup_{U \in F} N(U)$ does not belong to $B$ for all finite subfamilies $F$ of $S$. Consider the family $\{X \setminus \bigcup_{U \in F} U\}_{F \subset S}$, where $F$ runs over all finite subfamilies of $S$. That family has the property that it is closed under finite intersections and none of its elements belongs
to $B$. Therefore it extends to an element $P$ of $\partial(X, T, B)$. Hence, $P \in N(U)$ for some $U \in S$. However, $X \setminus U$ also belongs to $P$, a contradiction.

To prove $X \cup \partial(X, T, B)$ is topologically normal it suffices to show it is Hausdorff. Clearly, any two points $x_1 \neq x_2$ in $X$ have disjoint neighborhoods in $X \cup \partial(X, T, B)$ if $x_1 \in B$. Namely, it is $\{x_1\}$ and its complement. If $P \neq Q$ are points of $\partial(X, T, B)$, then there is $C \in P \setminus Q$. Hence, there is $D \in Q$ such that $C \cap D \in B$. Notice $E := D \setminus C \cap D \in \mathcal{Q}$ (see [11]) and there exist disjoint co-zero subsets $U$ and $W$ of $X$ containing $C$ and $E$, respectively. Observe $P \in N(U)$, $Q \in N(W)$, and $N(U) \cap N(W) = \emptyset$.

**Theorem 4.12.** If $f : (X, T_X, B_X) \to (Y, T_Y, B_Y)$ is a continuous function of large scale topological spaces such that $f^{-1}(B_Y) \subset B_X$, then there is a unique continuous extension

$$\tilde{f} : X \cup \partial(X, T_X, B_X) \to Y \cup \partial(Y, T_Y, B_Y)$$

of $f$. It is given by

$$\tilde{f}(P) \in \bigcap_{C \in P} cl(f(C))$$

for all $P \in \partial(X, T_X, B_X)$.

**Proof.** First of all, let us show that $\bigcap_{C \in P} cl(f(C)) \neq \emptyset$ for all $P$, i.e. $\tilde{f}$ exists. Suppose

$$\bigcap_{C \in P} cl(f(C)) = \emptyset$$

for some $P$. Since $Y \cup \partial(Y, T_Y, B_Y)$ is large scale compact, there exist $C_i \in P$, $1 \leq i \leq k$, such that $\bigcap_{i=1}^k cl(f(C_i)) \subset B_Y$. Let $C := \bigcap_{i=1}^k C_i$. Now, $C \in P$ and $f(C) \subset B_Y$. Therefore $C \in B_X$, a contradiction.

Our next step is to show continuity of $\tilde{f}$. Suppose $\tilde{f}(P) \in N(U)$ for some co-zero set $U$ in $Y$. By a similar argument as above, $D \subset U$ for some $D \in \tilde{f}(P)$, so we can choose a co-zero set $W$ in $Y$ and a zero-set $E$ in $Y$ satisfying $D \subset W \subset E \subset U$. Suppose $Q \in N(f^{-1}(W))$. There is $C \in Q$, $C \subset f^{-1}(W)$. Hence $\tilde{f}(Q) \in cl(f(C)) \subset cl(W) \subset cl(E)$ which results in $E \in \tilde{f}(Q)$. Now, $Y \setminus U \notin \tilde{f}(Q)$, so $\tilde{f}(Q) \in N(U)$ which completes the proof of continuity of $\tilde{f}$ provided $P \in N(f^{-1}(W))$. It is so as $G := X \setminus f^{-1}(W) \in P$ implies $\tilde{f}(P) \in cl(f(G)) \subset cl(Y \setminus W)$, i.e. $Y \setminus W \notin \tilde{f}(P)$ which contradicts $D \in \tilde{f}(P)$.

The uniqueness of $\tilde{f}$ follows from the fact the range is Hausdorff (see [11]) and $X$ is dense in $X \cup \partial(X, T_X, B_X)$.

**Observation 4.13.** [4, 12] is a generalization of the classical Čech-Stone compactification. Namely, it reduces to the basic property of Čech-Stone compactification in the case of bornologies being empty. See [3, 14] for more details.

**Proposition 4.14.** If $(X, T_X, B_X)$ is large scale compact, then for every $P \in \partial(X, T_X, B_X)$ there is $x \in X \setminus \bigcup B$ such that $P$ equals the family of all zero-sets in $X$ containing $x$. In particular, if $(X, T_X, B_X)$ is large scale compact and Hausdorff, then $\partial(X, T_X, B_X) = X \setminus \bigcup B$ and $id : (X, T_X, B_X) \to X \cup \partial(X, T_X, B_X)$ is a homeomorphism.

**Proof.** Suppose for each $x \in X \setminus \bigcup B$ there is $C_x \in P$ not containing $x$. For $x \in \bigcup B$ put $C_x := X \setminus \{x\}$. Since $X = \bigcup_{x \in X} (X \setminus C_x)$, there exists a finite subset $F$ of $X$ such that $\bigcap_{x \in F} C_x \in B_X$. However, $\bigcap_{x \in F} C_x \in P$, a contradiction. \qed
5. Boundaries of formed spaces

Definition 5.1. Given a set $X$ equipped with a basic multilinear form $\omega$, the **boundary** $\partial(X, \omega)$ is defined as the set of maximal families $P$ of subsets of $X$ so that

$$\omega(C_1, \ldots, C_k) = \infty$$

for all sets $C_1, \ldots, C_k \in P$. Such $P$ may be referred to as a **point at infinity** of $X$ despite the possibility of $X$ being identified with some point of $X$.

Definition 5.2. If $\{x\}$ is $\omega$-unbounded, then we can identify $x$ with the principal ultrafilter consisting of all subsets of $X$ containing $x$. This way we can talk about $X \cup \partial(X, \omega)$. To distinguish between $X$ and its boundary, for each $C \subset X$ we define its **non-boundary points** $C^o$ as $\{x \in C|\omega(x) = 0\} = C \setminus \partial(X, \omega)$.

Remark 5.3. There may be points $P$ of the boundary containing $x \in X$ but not equal to $x$. That is the case if $\omega(x) = \omega(X \setminus \{x\}) = \omega(x, X \setminus \{x\}) = \infty$, for example.

Observation 5.4. If $C \cup D \in Q \in \partial(X, \omega)$, then $C \in Q$ or $D \in Q$.

Proof. Suppose $C \cup D \in Q \in \partial(X, \omega)$ but $C \notin Q$ and $D \notin Q$. In that case there exist $V_1, V_2 \in Vect(Q)$ such that $\omega(C \ast V_1) = \omega(D\ast V_2) = 0$. Consequently, $\omega((C\cup D)\ast V_1 \ast V_2) = 0$ by 2.6, a contradiction. □

In order to extend the topology on $(X, \omega)$ induced by $\omega$ over $X \cup \partial(X, \omega)$ we need the following concept:

Definition 5.5. Suppose $(X, \omega)$ is a formed space and $C \subset X$. $o(C)$ is defined as

$$o(C) := C^o \cup \{Q \in \partial(X, \omega) \mid X \setminus C \notin Q\},$$

Observation 5.6. Notice $C = o(C) \cap X$. Indeed, if $x \in C \setminus C^o$, then $X \setminus C$ does not belong to the principal ultrafilter generated by $x$. Conversely, if $x \in X \cap o(C)$, then either $x \in C^o \subset C$ or $x$ represents the principal ultrafilter not containing $X \setminus C$ which implies $x \in C$.

Lemma 5.7. $o(C) \cap o(D) = o(C \cap D)$.

Proof. The equality follows from 5.4 and $(C \cap D)^o = C^o \cap D^o$. □

Definition 5.8. Suppose $(X, \omega)$ is a set $X$ equipped with a basic multilinear form $\omega$ and $B(\omega)$ is the bornology induced by $\omega$. $X \cup \partial(X, \omega)$ has a **natural topology** defined as follows: $U \subset X \cup \partial(X, \omega)$ is declared open if $X \cap U$ is open in $T(\omega)$ and for each $Q \in U \cap \partial(X, \omega)$ there is an open (in $T(\omega)$) $C \subset X$ such that $Q \in o(C) \subset U$.

Remark 5.9. In view of 5.4 the above topology is well-defined and sets $o(U)$, $U$ open in $X$, form its basis.

Lemma 5.10. 1. $Q \in \partial(X, \omega)$ belongs to $cl(C)$ if $C \in Q$.

2. If $C$ is closed in $X$ and $Q \in cl(C)$, then $C \in Q$.

Proof. 1. Suppose $C \in Q$ and $Q \notin cl(C)$. There is $o(D)$ containing $Q$ and disjoint with $C$. Hence $D \subset X \setminus C$ resulting in $C \subset X \setminus D$. Consequently, $X \setminus D \in Q$ contradicting $Q \in o(D)$.

2. If $C \notin Q$, then $Q \in o(X \setminus C)$. However, $C \cap o(X \setminus C) = \emptyset$ contradicting $Q \in cl(C)$. □

Corollary 5.11. $C \subset o(C) \subset cl(C)$ for all $C \subset X$. In particular, $cl(C) = cl(o(C))$. 


Proof. If \( Q \in o(C) \), then \( X \setminus C \notin Q \), hence \( C \in Q \). By \[5.10\] \( Q \in cl(C) \).

**Lemma 5.12.** 1. If \( \bigcap_{i=1}^{k} cl(C_i) \subset X \setminus \partial(X, \omega) \), then \( \omega(C_1, \ldots, C_k) = 0 \).

2. If \( C_i \), \( i \leq k \), are closed in \( X \), \( \bigcap_{i=1}^{k} C_i = \emptyset \), and \( \omega(C_1, \ldots, C_k) = 0 \), then
\[
\bigcap_{i=1}^{k} cl(o(C_i)) \subset X \setminus \partial(X, \omega).
\]

Proof. 1. Suppose \( \bigcap_{i=1}^{k} cl(C_i) \subset X \setminus \partial(X, \omega) \) but \( \omega(C_1, \ldots, C_k) = \infty \). Hence all \( C_i \) are \( \omega \)-unbounded and there is \( Q \in \partial(X, \omega) \) containing all \( C_i \), \( i \leq k \). By Lemma \[5.10\] \( Q \in \bigcap_{i=1}^{k} cl(C_i) \), a contradiction.

2. Suppose \( Q \in \bigcap_{i=1}^{k} cl(o(C_i)) \), \( \bigcap_{i=1}^{k} C_i = \emptyset \), and \( \omega(C_1, \ldots, C_k) = 0 \). Therefore \( Q \) cannot contain all \( C_i \)'s, so assume \( C_1 \notin Q \). Hence \( Q \in o(X \setminus C_1) \). However, \( o(C_1) \cap o(X \setminus C_1) = \emptyset \), contradicting \( Q \in cl(o(C_1)) \).

**Proposition 5.13.** Suppose \( D \subset X \) is closed. If \( \{C_i\}_{i=1}^{k} \) is a finite family of subsets of \( X \), then the following conditions are equivalent:
1. \( \bigcup_{i=1}^{k} o(C_i) \) contains \( cl(D) \cap \partial(X, \omega) \).
2. \( \omega(D, X \setminus C_1, \ldots, X \setminus C_k) = 0 \) and \( D \setminus \bigcup_{i=1}^{k} C_i \in B(\omega) \).
3. \( \omega(D, X \setminus C_1, \ldots, X \setminus C_k) = 0 \).

Proof. Notice \( Q \in cl(D) \cap \partial(X, \omega) \setminus \bigcup_{i=1}^{k} o(C_i) \) if and only if \( D, X \setminus C_1, \ldots, X \setminus C_k \in Q \) (see \[5.10\]), so \( cl(D) \cap \partial(X, \omega) \setminus \bigcup_{i=1}^{k} o(C_i) = \emptyset \) if and only if \( \omega(D, X \setminus C_1, \ldots, X \setminus C_k) = 0 \). Therefore 1) \( \iff \) 3).

2) \( \iff \) 3) follows from the fact that \( \omega(D, X \setminus C_1, \ldots, X \setminus C_k) = 0 \) implies \( \omega(D \setminus \bigcup_{i=1}^{k} C_i) = 0 \) by \[2.4\].

**Corollary 5.14.** Given a covering \( \{o(C)\}_{C \in S} \) of \( cl(D) \), \( D \subset X \) being closed, there is a finite subfamily \( F \) of \( S \) covering \( cl(D) \cap \partial(X, \omega) \) such that \( D \setminus \bigcup_{C \in F} C \) belongs to \( B(\omega) \).

Proof. Suppose \( \{o(C)\}_{C \in S} \) is a cover of \( cl(D) \) that has no finite subcover of \( cl(D) \cap \partial(X, \omega) \). By \[5.13\] it means \( \omega(D, X \setminus C_1, \ldots, X \setminus C_k) = \infty \) for all finite subfamilies \( \{C_i\}_{i=1}^{k} \) of \( S \). Therefore there is \( Q \in \partial(X, \omega) \) containing \( D \) and all sets \( X \setminus C, C \in S \). By \[5.10\] \( Q \in cl(D) \). Since \( Q \notin \bigcup_{C \in S} o(C) \), we arrive at a contradiction.

**Corollary 5.15.** \( X \cup \partial(X, \omega) \) is large scale compact with respect to \( B(\omega) \).

**Proposition 5.16.** Suppose \( \omega \) is a form on \( X \). The following conditions are equivalent:
a. $X \cup \partial(X, \omega)$ is large scale compact with respect to $B(\omega)$ and Hausdorff.

b. For every two different points $Q$ and $R$ of $\partial(X, \omega)$ there exist $\omega$-closed subsets $C$ and $D$ of $X$ such that $C \cup D = X$, $C \notin Q$, and $D \notin R$.

Proof. a) $\implies$ b). Choose disjoint sets $o(U)$ and $o(W)$, where $Q \in o(U)$, $R \in o(W)$, and $U, W$ are $\omega$-open. Therefore $U \cap W = \emptyset$ resulting in $C := X \setminus U$, $D := X \setminus W$ satisfying the required conditions.

b) $\implies$ a). Since every point of $X \setminus \partial(X, \omega)$ is open-closed, it suffices to show that every two distinct points $Q, R \in \partial(X, \omega)$ have disjoint neighborhoods. Choose $\omega$-closed subsets $C$ and $D$ of $X$ such that $C \cup D = X$, $C \notin Q$, and $D \notin R$. Notice that $Q \in o(X \setminus C)$, $R \in o(X \setminus D)$. Suppose $S \in o(X \setminus C) \cap o(X \setminus D)$. Therefore $C \notin S$ and $D \notin S$ contradicting $X = C \cup D \in S$. □

Proposition 5.17. If $(X, \omega)$ is a formed space, then $\omega(C_1, \ldots, C_k) = 0$ iff $\bigcap_{i=1}^{k} \text{cl}(C_i) \subset X \setminus \partial(X, \omega)$.

Proof. $\omega(C_1, \ldots, C_k) = \infty$ if and only if there is $Q \in \partial(X, \omega)$ containing all of $C_i$.

That is equivalent to $Q \in \bigcap_{i=1}^{k} \text{cl}(C_i)$. □

6. NORMAL FORMS

Definition 6.1. A formed space $(X, \omega)$ is normal if $\omega(C_1, \ldots, C_k) = 0$ implies existence of subsets $D_i$, $1 \leq i \leq k$, of $X$ such that $\bigcup_{i=1}^{k} D_i = X$ and $\omega(C_i, D_i) = 0$ for each $1 \leq i \leq k$.

Lemma 6.2. If $\{C_i\}_{i=1}^{k}$ is a family of zero-sets in a topological space $(X, T)$ whose intersection is empty, then there exists a family of zero-sets $\{D_i\}_{i=1}^{k}$ satisfying the following conditions:

a. $C_i \cap D_i = \emptyset$ for each $i \leq k$,

b. $\bigcup_{i=1}^{k} D_i = X$.

Proof. Choose continuous functions $f_i : X \rightarrow [0, 1]$ such that $C_i = f_i^{-1}(0)$ for each $i \leq k$. Notice that $g := \sum_{i=1}^{k} f_i$ is positive and let $g_i := f_i / g$. Define $D_i$ as $g_i^{-1}[1/k, 1]$ and one can easily see that $\{D_i\}_{i=1}^{k}$ satisfies the required conditions. □

Corollary 6.3. The basic functional form $\omega_f(X, T)$ of a topological space $(X, T)$ is normal.

Proof. Apply 6.2. □

Corollary 6.4. The basic topological form of a $T_1$ space $(X, T)$ is normal if and only if $(X, T)$ is normal.

Proof. If $(X, T)$ is normal, then its functional form equals the basic topological form, hence is normal by 6.3.

Suppose $\omega(T)$ is normal and $C, D$ are two disjoint closed subsets of $X$. Thus $\omega(T)(C, D) = 0$ and there exist subsets $C', D'$ of $X$ such that $C' \cup D' = X$, $\omega(T)(C, C') = 0$, and $\omega(T)(D, D') = 0$. That implies $C \subset U := X \setminus \text{cl}(C')$, $D \subset W := X \setminus \text{cl}(D')$ and $U \cap W = \emptyset$. Since $(X, T)$ is $T_1$, it is normal. □
Proposition 6.5. If a large scale space \((X, L)\) is metrizable, then the form \(\omega(L)\) induced by \(L\) is normal.

Proof. Suppose \(\omega(C_1, \ldots, C_k) = 0\) for some subsets of \(X\) and \(d\) is a metric on \(X\) inducing \(L\). Choose \(x_0 \in X\) and an increasing function \(f : \mathbb{N} \to \mathbb{N}\) such that for each \(n \geq 1\), \(\bigcap_{i=1}^{k} B(C_i, n) \subset B(x_0, f(n))\).

For each \(i \leq k\), define \(D_i\) as
\[
\{x \in X | \text{for all } n \geq 1, \ d(x, x_0) > f(n) \implies d(x, C_i) \geq n\}.
\]
If \(\omega(C_i, D_i) = \infty\) for some \(i \leq k\), then there is \(m \geq 1\) such that \(B(C_i, m) \cap B(D_i, m)\) is unbounded. In particular, there is \(y \in B(C_i, m) \cap B(D_i, m)\) satisfying \(d(y, x_0) > f(m)\). Thus, \(d(y, C_i) \geq m\) contradicting \(y \in B(C_i, m)\).

Suppose \(z \in X \setminus \bigcup_{i=1}^{k} D_i\). Therefore, for each \(i \leq k\), there is \(m_i\) such that \(d(z, x_0) > f(m_i)\) but \(d(z, C_i) < m_i\). Put \(M = \max\{m_i | i \leq k\}\). Notice \(z \in B(x_0, f(M))\) as \(z \in \bigcap_{i=1}^{k} B(C_i, M) \subset B(x_0, f(M))\) which contradicts \(d(z, x_0) > f(M)\).

\[\square\]

Lemma 6.6. Suppose \(\omega\) is a normal form on \(X\). If \(D \cap C = \emptyset\) and \(\omega(D, C) = 0\), then there is an \(\omega\)-open set \(U\) containing \(D\), disjoint with \(C\), and satisfying \(\omega(C, U) = 0\).

Proof. Claim: There exist disjoint sets \(C'\) and \(D'\) such that
1. \(C' \cup D' = X\),
2. \(C \subset C'\),
3. \(D \subset D'\),
4. \(\omega(C, D') = 0\) and \(\omega(D, C') = 0\).

Proof of Claim:
Choose sets \(E\) and \(F\) such that \(\omega(C, E) = 0 = \omega(D, F) = 0\) and \(E \cup F = X\). Notice \(C \cap E, D \cap F \in B(\omega)\). Put \(C' := C \cup (F \setminus F \cap D)\) and \(D' := D \cup (E \setminus E \cap C)\).

By induction construct an increasing sequence of sets \(D_n\) containing \(D\) and disjoint with \(C'\) so that \(\omega(D_n, X \setminus D_{n+1}) = 0\). By \(6.8\) \(U := \bigcup_{i=1}^{\infty} D_n\) is \(\omega\)-open, \(D \subset U\), and \(U \cap C' = \emptyset\). Therefore \(U \subset D'\) and \(\omega(C, U) = 0\).

\[\square\]

Corollary 6.7. If \(\omega\) is a normal form on \(X\), then \(x\) belongs to \(\omega\)-closure of \(C \subset X\)
if and only if \(x \in C\) or \(\omega(x, C) = \infty\).

Proof. Suppose \(x \notin C\) and \(\omega(x, C) = 0\). By \(6.6\) \(x \notin cl(C)\).

If \(x \notin cl(C)\), then \(x \in U := X \setminus cl(C)\) and \(\omega(x, X \setminus U) = 0\) as \(U\) is \(\omega\)-open. Since \(C \subset X \setminus U\), \(\omega(x, C) = 0\).

\[\square\]

Corollary 6.8. If \(\omega\) is a normal form on \(X\) and \(\omega(C_1, \ldots, C_k) = 0\), then
\[
\omega(cl(C_1), \ldots, cl(C_k)) = 0.
\]

Proof. Claim: If \(\omega(C, D) = 0\), then \(\omega(cl(C), cl(D)) = 0\).

Proof of Claim: It is sufficient to show \(\omega(cl(C), D) = 0\). Choose sets \(C'\) and \(D'\) such that \(C' \cup D' = X\), \(\omega(C, C') = 0\), and \(\omega(D, D') = 0\). If \(x \in D' \setminus C\), then \(x \notin cl(C)\) by \(6.7\). Thus, \(cl(C) \subset C \cup D'\). Since \(\omega(C \cup D', D) = 0\), the proof of Claim is completed.

Suppose \(\omega(cl(C_1), \ldots, cl(C_k)) = \infty\) and choose \(D_i \subset X, i \leq k\), such that \(\omega(C_i, D_i) = \infty\).
0 for all \( i \leq k \), and \( \bigcup_{i=1}^{k} D_i = X \). There is \( Q \in \partial(X, \omega) \) containing each \( cl(C_i), i \leq k \).

As \( \omega(cl(C_i), D_i) = 0 \) for each \( i \), none of \( D_i \)'s belong to \( Q \), a contradiction as \( \bigcup_{i=1}^{k} D_i = X \).

**Theorem 6.9.** \( X \cup \partial(X, \omega) \) is large scale compact with respect to \( \mathcal{B}(\omega) \) and Hausdorff if and only if \( \omega \) is normal and \( T_1 \).

**Proof.** Suppose \( \omega \) is normal. Given \( Q \not\in \mathcal{R} \) there is \( D \in \mathcal{Q} \) and \( C_1, \ldots, C_k \in \mathcal{R} \) such that \( \omega(D, C_1, \ldots, C_k) = 0 \). Choose \( D', D_1, \ldots, D_k \subset X \) such that \( \omega(D, D') = \omega(C_i, D_i) = 0 \) for \( i \leq k \) and \( D' \cup \bigcup_{i=1}^{k} D_i = X \). By 6.8 we may assume \( D' \) is \( \omega \)-closed and all \( D_i \)'s are \( \omega \)-closed. Notice \( D' \notin \mathcal{Q} \). Also, \( \bigcup_{i=1}^{k} D_i \notin \mathcal{R} \) (as otherwise \( D_j \in \mathcal{R} \) for some \( j \) contradicting \( \omega(C_j, D_j) = 0 \)). By 5.13 and 5.16, \( X \cup \partial(X, \omega) \) is Hausdorff and large scale compact with respect to \( \mathcal{B}(\omega) \).

Suppose \( X \cup \partial(X, \omega) \) is Hausdorff, large scale compact with respect to \( \mathcal{B}(\omega) \), and \( \omega(C(1), \ldots, C(k)) = 0 \) for some \( C(i) \subset X \). That means the intersection of closures of \( C(i) \) (in \( X \cup \partial(X, \omega) \)) is an element \( B \) of \( \mathcal{B}(\omega) \). Choose, for each \( \mathcal{Q} \), an index \( i(Q) \) such that \( Q \notin cl(C(i(Q))) \). Then choose an \( \omega \)-open \( D(Q) \) such that \( cl(o(D(Q))) \cap cl(C(i(Q))) = \emptyset \) and \( Q \notin o(D(Q)) \). Notice \( \omega(D(Q), cl(C(i(Q))) = 0 \) as otherwise there is \( Q \in cl(o(D(Q)) \cap cl(C(i(Q))) \).

There are finitely many points \( Q(j) \) so that \( B' := (X \cup \partial(X, \omega)) \setminus \bigcup_{j=1}^{n} o(D(Q(j))) \in B(\omega) \). For each index \( s \leq k \) define \( E_s \) as the union of all \( D(Q(j)) \) so that \( i(Q(j)) = s \). Notice \( \omega(B' \cup B \cup E_s, C_s) = 0 \) for each \( s \leq k \) and \( \bigcup_{s=1}^{k} (B' \cup B \cup E_s) = X \).

**Lemma 6.10.** Suppose \( \omega_1, \omega_2 \) are normal forms on \( X \). If \( \omega_1(C_1, C_2) = \omega_2(C_1, C_2) \) for all 2-vectors \( (C_1, C_2) \) in \( X \), then \( \omega_2 = \omega_1 \).

**Proof.** Notice \( \omega_1(C) = \omega_1(C, C) = \omega_2(C, C) = \omega_2(C) \) for all \( C \subset X \). Now, it suffices to show \( \omega_1(C_1, \ldots, C_k) = 0 \) implies \( \omega_2(C_1, \ldots, C_k) = 0 \) for all \( k \geq 3 \) and all \( k \)-vectors \( V = C_1, \ldots, C_k \) of \( X \). Choose subsets \( D_i, 1 \leq i \leq k \), of \( X \) such that \( \bigcup_{i=1}^{k} D_i = X \) and \( \omega_1(C_i, D_i) = 0 \) for each \( 1 \leq i \leq k \) which implies \( \omega_2(C_i, D_i) = 0 \) for each \( 1 \leq i \leq k \). Therefore \( \omega_2(V \ast D_i) = 0 \) for each \( 1 \leq i \leq k \). Finally, \( \omega_2(V) = \omega_2(V \ast X) = \sum_{i=1}^{k} \omega_2(V \ast D_i) = 0 \).

**Theorem 6.11.** Suppose \( \omega_1, \omega_2 \) are normal forms on \( X \) and \( (X, T(\omega_1), B(\omega_1)) \) is large scale compact and Hausdorff. If \( T(\omega_2) = T(\omega_1) \) and \( B(\omega_2) = B(\omega_1) \), then \( \omega_2 = \omega_1 \).

**Proof.** Notice \( B(\omega_2) = B(\omega_1) \) means exactly that \( \omega_1(C) = \omega_2(C) \) for all \( C \subset X \), so our next goal is to prove \( \omega_1(C_1, C_2) = \omega_2(C_1, C_2) \) for all 2-vectors \( (C_1, C_2) \) in \( X \). It suffices to show \( \omega_1(C_1, C_2) = 0 \) implies \( \omega_2(C_1, C_2) = 0 \) for disjoint subsets \( C_1, C_2 \) of \( X \) (remove their intersection if it is not empty). Enlarge \( C_2 \) to an open subset \( U \) of \( X \) so that \( \omega_2(C_2, U) = 0 \) (see 5.8 and put \( D := cl(C_1) \). Now, \( \omega_2(D, U) = 0 \) by 6.8. Given \( x \in D \setminus U \), there is a neighborhood \( W(x) \) of \( x \) satisfying \( \omega_2(W(x), U) = 0 \) as
Definition 7.1. An orthogonality relation on subsets of a set $X$ is a symmetric relation $\perp$ satisfying the following properties:
1. $\emptyset \perp X$,
2. $A \perp (C \cup C') \iff A \perp C$ and $A \perp C'$.

Observation 7.2. One can reduce the number of axioms by dropping symmetry and replacing Axiom 2 by
2'. $A \perp (C \cup C') \iff C \perp A$ and $C' \perp A$.

Example 7.3. For every bornology $B$ on a set $X$ the relation $A \perp C$ defined as $A \cap C \in B$ is an orthogonality relation.

7.1. Bounded sets.

Definition 7.4. Given an orthogonality relation $\perp$ on subsets of $X$, a bounded subset $B$ of $X$ is one that is orthogonal to the whole set:

$$B \perp X.$$ 

Definition 7.5. An orthogonality relation $\perp$ on subsets of $X$ is small scale if the empty set is the only subset of $X$ that is orthogonal to itself. In particular, the only bounded subset of $X$ is the empty set.

Definition 7.6. An orthogonality relation $\perp$ on subsets of $X$ is large scale if each point is a bounded subset of $X$.
7.2. Examples of small scale orthogonality.

Example 7.7. 1. Set-theoretic orthogonality: Disjointness,
2. Topological orthogonality: Disjointness of closures,
3. Metric orthogonality: Disjointness of \( r \)-balls for some \( r > 0 \),
4. Uniform orthogonality: Disjointness of \( \mathcal{U} \)-neighborhoods for some uniform cover \( \mathcal{U} \).

7.3. Examples of large scale orthogonality.

Example 7.8. 1. Set-theoretic large scale orthogonality: Finiteness of intersection,
2. Metric large scale orthogonality: Boundedness of intersection of \( r \)-balls for all \( r > 0 \),
3. Group large scale orthogonality: Finiteness of \( (A \cdot F) \cap (C \cdot F) \) for all finite subsets \( F \) of a group \( G \).
   Same as metric ls-orthogonality for word metrics if \( G \) is finitely generated.
4. Topological ls-orthogonality: Disjointness of coronas of closures in a fixed compactification \( \bar{X} \) of \( X \).

7.4. Hyperbolic orthogonality. Given a metric space \((X, d)\), the Gromov product of \( x \) and \( y \) with respect to \( a \in X \) is defined by
\[
\langle x, y \rangle_a = \frac{1}{2}(d(x,a) + d(y,a) - d(x,y)).
\]
Recall that metric space \((X, d)\) is (Gromov) \( \delta \)-hyperbolic if it satisfies the \( \delta/4 \)-inequality:
\[
\langle x, y \rangle_a \geq \min\{\langle x, z \rangle_a, \langle z, y \rangle_a\} - \delta/4, \quad \forall x, y, z, a \in X.
\]
\((X, d)\) is Gromov hyperbolic if it is \( \delta \)-hyperbolic for some \( \delta > 0 \).

Definition 7.9. Two subsets \( A \) and \( C \) of a hyperbolic space \( X \) are hyperbolically orthogonal if there is \( r > 0 \) such that
\[
\langle a, c \rangle_p < r
\]
for some fixed \( p \) and all \( (a, c) \in A \times C \).

7.5. Freundenthal orthogonality.

Definition 7.10. Suppose \( X \) is a locally compact and locally connect topological space. Two subsets \( A \) and \( C \) of \( X \) are Freundenthal orthogonal if there is a compact subset \( K \) of \( X \) such that the union of all components of \( X \setminus K \) intersecting \( A \) is disjoint from the union of all components of \( X \setminus K \) intersecting \( C \).

7.6. Normal orthogonality relations.

Definition 7.11. Given an orthogonality relation \( \perp \) on subsets of \( X \), two subsets \( C \) and \( D \) \( \perp \)-span \( X \) if the following conditions are satisfied:
1. \( C \perp D \),
2. \( X \) can be decomposed as \( X = C' \cup D' \), where \( C' \perp D \) and \( D' \perp C \).

Remark 7.12. Obviously, we may interpret the word "decompose" in the definition above as \( C' \cap D' = \emptyset \) since \( D' \) can be replaced by \( X \setminus C' \). The other extreme is when \( C \subset C' \) and \( D \subset D' \) which can be accomplished by replacing \( C' \) by \( C \cup C' \) and replacing \( D' \) by \( D \cup D' \). In that case we may think of \( C \) being parallel to
$C'$, $D$ being parallel to $D'$ and interpret Definition 7.11 as an analog of parallel-perpendicular decomposition in Linear Algebra.

**Proposition 7.13.** If $C$ and $D \perp$-span $X$, then $C \cap D$ is $\perp$-bounded.

**Proof.** $X = C' \cup D'$, where $C' \perp D$ and $D' \perp C$. Therefore $C' \perp (C \cap D)$ and $D' \perp (C \cap D)$ resulting in $(C \cap D) \perp X$. □

**Definition 7.14.** An orthogonality relation $\perp$ on subsets of $X$ is **normal** if $C$ and $D \perp$-span $X$ whenever $C \perp D$.

**Example 7.15.** The topological orthogonality relation on a topological space is normal if $X$ is topologically normal.

**Definition 7.16.** The functional orthogonality relation $\perp$ on a topological space $X$ is defined as follows: $C \perp D$ if there is a continuous function $f : X \to [0,1]$ such that $f(C) \subset \{0\}$ and $f(D) \subset \{1\}$.

**Proposition 7.17.** The functional orthogonality relation $\perp$ on a topological space $X$ is always normal.

**Proof.** For any continuous $f : X \to [0,1]$ satisfying $f(C) \subset \{0\}$ and $f(D) \subset \{1\}$, one puts $C' = f^{-1}[0.5,1]$, $D' = f^{-1}[0,0.5]$ and observe $X = C' \cup D'$, $C' \perp C$, and $D' \perp D$. □

### 7.7. Proximity spaces

There is a more general structure than uniform spaces, namely a proximity (see [15]). In this section we show that those structures correspond to normal small scale orthogonal relations.

**Definition 7.18.** A **proximity space** $(X, \delta)$ is a set $X$ with a relation $\delta$ between subsets of $X$ satisfying the following properties:

1. $A \delta B \implies B \delta A$
2. $A \delta B \implies A \neq \emptyset$
3. $A \cap B \neq \emptyset \implies A \delta B$
4. $A \delta (B \cup C) \iff (A \delta B \text{ or } A \delta C)$
5. $\forall E, A \delta E \text{ or } B \delta (X \setminus E) \implies A \delta B$

**Proposition 7.19.** Normal small scale orthogonality relations are in one-to-one correspondence with proximity relations.

**Proof.** Given a small scale orthogonal relation $\perp$ we define $A \delta C$ as $\neg (A \perp C)$.

Conversely, given a proximity relation $\delta$ we define $A \perp C$ as $\neg (A \delta C)$.

The proof amounts to negating implications, so let’s show only the implication $A \cap B \neq \emptyset \implies A \delta B$. If it fails, then we have two orthogonal sets $A$ and $B$ with non-empty intersection $A \cap B$. However, in this case $A \cap B$ is self-orthogonal, a contradiction. □

### 8. $\perp$-Continuous functions

**Definition 8.1.** Given two sets $X$ and $Y$ equipped with orthogonality relations $\perp_X$ and $\perp_Y$, a function $f : X \to Y$ is **$\perp$-continuous** if

$$f(A) \perp_Y f(C) \implies A \perp_X C$$

for all subsets $A, C$ of $X$. 
8.1. Small Scale Examples. In the small scale \( \perp \)-continuous functions are exactly neighborhood-continuous functions with respect to the induced neighborhood operator. Therefore both examples below follow from \([8]\) in view of \([9, 7]\).

**Example 8.2.** If both \( X \) and \( Y \) are normal spaces equipped with topological orthogonality relations, then \( \perp \)-continuity is ordinary topological continuity.

**Example 8.3.** If both \( X \) and \( Y \) are uniform spaces equipped with uniform orthogonality relations, then \( \perp \)-continuity is ordinary uniform continuity.

8.2. Large Scale Examples.

**Example 8.4.** If both \( X \) and \( Y \) are metric spaces equipped with metric \( l_s \)-orthogonality relations and \( f : X \to Y \) preserves bounded sets, then \( \perp \)-continuity is the same as \( f \) being coarse and bornologous.

**Proof.** Recall that \( f : X \to Y \) is bornologous if, for each \( r > 0 \), there is \( s > 0 \) such that \( \text{diam}(f(A)) < s \) if \( \text{diam}(A) < r \).

Notice that every \( \perp \)-continuous function co-preserves bounded sets, i.e. it is coarse. Suppose \( f \) is \( \perp \)-continuous but not bornologous. Hence, there is a sequence \( B_n \) of uniformly bounded subsets of \( X \) whose images \( f(B_n) \) have diameters diverging to infinity. We may reduce it to the case of each \( B_n \) consisting of exactly two points \( x_n \) and \( y_n \) so that both \( f(x_n) \) and \( f(y_n) \) diverge to infinity. Notice \( A := \{ f(x_n) \}_{n \geq 1} \) and \( C := \{ f(y_n) \}_{n \geq 1} \) are orthogonal in \( Y \) but their point-inverses are not orthogonal in \( X \), a contradiction.

Suppose \( f \) is coarse and bornologous but not \( \perp \)-continuous. Choose two orthogonal subsets \( A \) and \( C \) of \( Y \) whose point-inverses are not orthogonal. Therefore the intersection of \( B(f^{-1}(A), r) \) and \( B(f^{-1}(C), r) \) is unbounded for some \( r > 0 \) and the image of that intersection is unbounded. There is \( s > 0 \) satisfying \( f(B(Z, r)) \subset B(f(Z), s) \) for all subsets \( Z \) of \( X \). Therefore, the intersection of \( B(A, s) \) and \( B(C, s) \) is unbounded, a contradiction. \( \square \)

**Example 8.5.** If \( X \) is a metric space equipped with metric \( l_s \)-orthogonality relation and \( Y \) is a compact metric space equipped with small scale metric orthogonality, then \( \perp \)-continuity is the same as \( f \) being slowly oscillating.

**Proof.** Recall that \( f : X \to Y \) is slowly oscillating if, for every pair of sequences \( \{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \) in \( X \), \( \lim_{n \to \infty} d_Y(f(x_n), f(y_n)) = 0 \) if \( \{d_X(x_n, y_n)\}_{n \geq 1} \) is uniformly bounded and \( d(x_n, x_1) \to \infty \) as \( n \to \infty \).

Suppose \( f \) is \( \perp \)-continuous but not slowly oscillating. Hence, there is a pair of sequences \( \{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \) in \( X \), and \( \epsilon > 0 \) such that \( d_Y(f(x_n), f(y_n)) > \epsilon \) for each \( n \geq 1 \) and \( \{d_X(x_n, y_n)\}_{n \geq 1} \) is uniformly bounded. We may assume that the limit of \( f(x_n) \) is \( z_1 \), the limit of \( f(y_n) \) is \( z_2 \). In particular \( d_Y(z_1, z_2) \geq \epsilon \). The sets \( B(z_1, \epsilon/3) \) and \( B(z_2, \epsilon/3) \) are orthogonal in \( Y \) but their point-inverses in \( X \) are not, a contradiction.

Suppose \( f \) is slowly oscillating but not \( \perp \)-continuous. Choose two orthogonal subsets \( A \) and \( C \) of \( Y \) whose point-inverses are not orthogonal. Therefore the intersection of \( B(f^{-1}(A), r) \) and \( B(f^{-1}(C), r) \) is unbounded for some \( r > 0 \). Therefore there are two sequences diverging to infinity in \( X \): \( \{x_n\}_{n \geq 1} \) in \( f^{-1}(A) \) and \( \{y_n\}_{n \geq 1} \) in \( f^{-1}(C) \) such that \( d_X(x_n, y_n) < 2r \) for each \( n \). Consequently, \( \lim_{n \to \infty} d_Y(f(x_n), f(y_n)) = 0 \) contradicting orthogonality of \( A \) and \( C \). \( \square \)
8.3. Quotient structures. It is well-known that defining quotient maps in both the uniform category and in the coarse category is tricky. In contrast, in sets equipped with orthogonality relations it is quite easy.

**Definition 8.6.** Suppose \( \perp X \) is an orthogonality relation on a set \( X \). Given a surjective function \( f : X \to Y \) define \( C \perp Y D \) to mean \( f^{-1}(C) \perp X f^{-1}(D) \).

It is easy to check that \( \perp Y \) is an orthogonality relation on \( Y \), called the quotient orthogonality relation. Also, it is clear that the following holds:

**Proposition 8.7.** Suppose \( \perp X \) is an orthogonality relation on a set \( X \), \( f : X \to Y \) is a surjective function, and \( Y \) is equipped with the quotient orthogonality relation \( \perp Y \). Given any \( \perp - \)continuous \( h : X \to Z \) that is constant on fibers of \( f \), there is unique \( \perp - \)continuous \( g : Y \to Z \) such that \( h = g \circ f \).

9. Neighborhood operators

This section is devoted to exploring the relation between orthogonality relations and neighborhood operators.

**Definition 9.1.** A neighborhood operator \( \prec \) on a set \( X \) is a relation between its subsets satisfying the following conditions:

- (N0) \( A \prec X \) for all \( A \subseteq X \).
- (N1) If \( A \prec B \) then \( X \setminus B \prec X \setminus A \).
- (N2) If \( A \prec B \subseteq C \), then \( A \prec C \).
- (N3) If \( A \prec N \) and \( A' \prec N' \) then \( A \cap A' \prec N \cap N' \).

**Observation 9.2.** Note that (N0) is implied by (N1) and the condition \( X \prec X \). Also, it is easy to see that, together, axioms (N0) – (N3) imply:

- (N0') \( \emptyset \prec A \) for all \( A \subseteq X \).
- (N2') If \( A \subseteq B \prec C \) then \( A \prec C \).
- (N3') If \( A \prec N \) and \( A' \prec N' \) then \( A \cap A' \prec N \cap N' \).

**Definition 9.3.** A normal neighborhood operator \( \prec \) satisfies the following condition:

- (N4) for every pair of subsets \( A \prec C \), there is a subset \( B \) with \( A \prec B \prec C \).

**Proposition 9.4.** Each orthogonality relation \( \perp \) on \( X \) induces a neighborhood operator \( \prec \) defined as follows: \( A \prec U \) if \( A \perp X \setminus U \) and \( A \subseteq U \). It is normal if and only if \( \perp \) is normal.

**Proof.** Left to the reader. \(\square\)

**Proposition 9.5.** Each neighborhood operator \( \prec \) on \( X \) induces a small scale orthogonality relation \( \perp \) defined as follows: \( A \perp U \) if \( A \prec X \setminus U \). \( \perp \) is normal if and only if \( \prec \) is normal.

**Proof.** Left to the reader. \(\square\)

**Definition 9.6.** Let \( X \) be a set and \( \prec \) a neighborhood operator. If \( A \) is a subset of \( X \), then the induced neighborhood operator \( \prec_A \) on subsets of \( A \) is defined as follows: \( S \prec_A T \) precisely when there exists a subset \( T' \) of \( X \) such that \( S \prec T \) as subsets of \( X \) and \( T = T' \cap A \).
Proposition 9.7. Suppose $X$ is a set equipped with an orthogonality relation $\perp_X$ and $Y$ is a set equipped with a small scale orthogonality relation $\perp_Y$. A function $f : A \subset X \to Y$ is neighborhood continuous (with respect to the induced neighborhood operators) if and only if it is $\perp$-continuous.

Proof. Suppose $f : A \subset X \to Y$ is neighborhood continuous and $C \perp_Y D$. Therefore $C \prec_Y (Y \setminus D)$ and $f^{-1}(C) \prec_A f^{-1}(Y \setminus D)$. That means existence of $S \subset X$ such that $S \cap A = f^{-1}(Y \setminus D)$ and $f^{-1}(C) \prec_X S$. Consequently, $f^{-1}(C) \perp_X (X \setminus S)$. Since $f^{-1}(D) \subset X \setminus S$, $f^{-1}(D) \perp_X f^{-1}(C)$.

Suppose $f : A \subset X \to Y$ is $\perp$-continuous and $C \prec_Y D$. Hence $C \perp_Y (Y \setminus D)$ and $f^{-1}(C) \perp_X f^{-1}(Y \setminus D)$. That implies $f^{-1}(C) \prec_X S$, where $S := X \setminus f^{-1}(Y \setminus D)$. Since $S \cap A = f^{-1}(D)$, $f$ is neighborhood continuous. □

Corollary 9.8. Suppose $X$ is a set equipped with a normal orthogonality relation $\perp_X$ and $[a, b] \subset \mathbb{R}$ is equipped with the topological orthogonality relation $\perp$. If $f : A \subset X \to [a, b]$ is $\perp$-continuous, then it extends to a $\perp$-continuous $\bar{f} : X \to [a, b]$.

Proof. In view of [9,7] it suffices to switch to neighborhood continuity and that case is done in [8] (Theorem 8.5). □

Corollary 9.9. Suppose $X$ is a set equipped with a normal orthogonality relation $\perp_X$ and the set of complex numbers $\mathbb{C}$ is equipped with the topological orthogonality relation $\perp$. If $f : A \subset X \to \mathbb{C}$ is $\perp$-continuous with metrically bounded image, then it extends to a $\perp$-continuous $\bar{f} : X \to \mathbb{C}$ with metrically bounded image.

Proof. To apply [9,8] it suffices to show that $g, h : A \to [a, b]$ are $\perp$-continuous if and only $g \Delta h : A \to [a, b] \times [a, b], (g \Delta h)(x) := (g(x), h(x))$, is $\perp$-continuous.

In one direction it is obvious, so assume $C, D \subset [a, b] \times [a, b]$ are metrically separated. That means there is $\epsilon > 0$ such that $|z_1 - z_2| \geq \epsilon$ if $z_1 \in C$ and $z_2 \in D$. Cover $[a, b] \times [a, b]$ by finitely many sets of the form $B_1 \times B_2$, where $B_1$ and $B_2$ are intervals of length $\epsilon/4$. Notice $(g \Delta h)^{-1}((C \cap (B_1 \times B_2)) \perp ((g \Delta h)^{-1}(C) \setminus (B_1' \times B_2'))) = (g \Delta h)^{-1}((C \cap (B_1 \times B_2)) \perp (g \Delta h)^{-1}(D))$ for any choice of $B_1, B_2$. Therefore, $(g \Delta h)^{-1}(C \cap (B_1 \times B_2)) \perp ((g \Delta h)^{-1}(D))$ for any choice of $B_1, B_2$. Finally, $(g \Delta h)^{-1}(C) \perp ((g \Delta h)^{-1}(D))$. □

Observation 9.10. Observe that the proof of 9.9 can be used to prove that, given two functions $f, g : X \to [0, 1]$ from a set equipped with an orthogonality relation $\perp$, the function $h : X \to [0, 1] \times [0, 1]$ is $\perp$-continuous if and only if both $f$ and $g$ are $\perp$-continuous.

10. Orthogonality relations vs forms

The purpose of this section is to show that normal orthogonality relations on a set are in one-to-one correspondence with normal forms on $X$.

Proposition 10.1. Every form $\omega$ on $X$ induces a natural orthogonality relation $\perp_\omega$ on $X$ defined by $C \perp_\omega D$ if and only if $\omega(C, D) = 0$. If $\omega$ is normal, then $\perp_\omega$ is normal.

Proof. Left to the reader. □

Theorem 10.2. Every normal orthogonality relation $\perp$ on a set $X$ extends uniquely to a normal multilinear form $\omega(\perp)$. 

Proof. Define \( \omega(\bot) \) as follows: \( \omega(\bot)(C_1, \ldots, C_k) = 0 \) if and only if there exist sets \( D_i, i \leq k \) such that \( D_i \cap C_i \) for each \( 1 \leq i \leq k \) and \( \bigcup_{i=1}^{k} D_i = X \).

Suppose \( \omega(\bot)(E_0, C_1, \ldots, C_k) = 0 \) and \( \omega(\bot)(E_0', C_1', \ldots, C_k') = 0 \). Choose sets \( D_i, 0 \leq k \) such that \( D_i \cap C_i \) for each \( 1 \leq i \leq k \), \( D_0 \cap E_0 = \emptyset \), and \( \bigcup_{i=0}^{k} D_i = X \). Choose sets \( D'_i, 0 \leq k \) such that \( D'_i \cap C_i \) for each \( 1 \leq i \leq k \), \( D'_0 \cap E_0 = \emptyset \), and \( \bigcup_{i=0}^{k} D'_i = X \). Observe \( (D_i \cup D'_i) \cap C_i \) for each \( 1 \leq i \leq k \), \( (D_0 \cap D'_0) \cap (E_0 \cup E'_0) \), and \( (D_0 \cap D'_0) \cup \bigcup_{i=1}^{k} (D_i \cup D'_i) = X \). Thus, \( \omega(\bot)(E_0 \cup E'_0, C_1, \ldots, C_k) = 0 \).

Suppose a normal form \( \omega \) induces \( \bot \). That means \( \omega(C_1, C_2) = \omega(\bot)(C_1, C_2) \) for all subsets \( C_1, C_2 \) of \( X \). By [6.10] \( \omega = \omega(\bot) \).

Similarly to forms, every orthogonality relation \( \bot \) on \( X \) induces a topology on \( X \) as follows: \( U \) is open if and only if \( x \perp (X \setminus U) \) for all \( x \in U \).

11. Form-continuous functions

Definition 11.1. A function \( f : (X, \omega_X) \to (Y, \omega_Y) \) of formed spaces is form-
continuous if \( \omega_Y(f(C_1), \ldots, f(C_k)) = 0 \) implies \( \omega_X(C_1, \ldots, C_k) = 0 \) for all subsets \( C_i, i \leq k, \) of \( X \).

Proposition 11.2. If \( f : (X, \omega_X) \to (Y, \omega_Y) \) is a function of formed spaces such
that \( \omega_Y \) is normal, then \( f \) is form-continuous if and only if it is \( \bot \)-continuous with respect to orthogonality relations induced by \( \omega_X \) and \( \omega_Y \), respectively.

Proof. We need to show that \( f \) is form-continuous if and only if for all subsets \( C, D \) of \( X \), the equality \( \omega_Y(f(C), f(D)) = 0 \) implies \( \omega_X(C, D) = 0 \).

The implication in one direction is obvious, so assume \( \omega_Y(f(C_1), \ldots, f(C_k)) = 0 \) but \( \omega_X(C_1, \ldots, C_k) = \infty \) for some subsets \( C_i \) of \( X \). Choose \( Q \in \partial(X, \omega_X) \) containing all \( C_i \)'s and choose subsets \( D_i \) of \( Y \) such that \( \omega_Y(D_i, f(C_i)) = 0 \) for each \( 1 \leq i \leq k \) and \( \bigcup_{i=1}^{k} D_i = Y \). There is \( j \) so that \( Q \) contains \( E := f^{-1}(D_j) \). Now, \( \omega_Y(f(E), f(C_j)) = 0 \) as \( f(E) \subset D_j \) which implies \( \omega_X(E, C_j) = 0 \), a contradiction to \( E, C_j \in Q \).

Theorem 11.3. Suppose \( f : (X, \omega_X) \to (Y, \omega_Y) \) is a form-continuous function
of formed spaces. If \( \omega_Y \) is normal and \( T_1 \), then the unique continuous extension
\( \hat{f} : X \cup \partial(X, \omega_X) \to Y \cup \partial(Y, \omega_Y) \)
of \( f \) is given by the formula
\[ \hat{f}(Q) := \{ C \subset Y \mid \omega_Y(C, f(D)) = \infty \text{ for all } D \in Q \}. \]

In particular, the following statements hold:
1. \( \hat{f}(Q) \) is the unique element of \( \partial(Y, \omega_Y) \) containing all \( f(D), D \in Q \).
2. \( \hat{f}(Q) \) is the unique element of the intersection of closures of \( f(D) \) in \( Y \cup \partial(Y, \omega_Y) \), where \( D \in Q \).

Proof. By [6.9] \( Y \cup \partial(Y, \omega_Y) \) is normal, so all we have to show is \( \hat{f} \) is well-defined and continuous.
First of all, we have to make sure that \( \hat{f}(Q) \) exists. Observe that \( \omega_Y(f(E), f(D)) = \infty \) for all \( E, D \in Q \), so \( \hat{f}(Q) \) contains \( f(Q) \). In particular, it shows \( \hat{f}(\partial(X, \omega_X)) \subset \partial(Y, \omega_Y) \).

Suppose \( \omega(C_1, \ldots, C_k) = 0 \) for some \( C_i \in \hat{f}(Q) \). Choose \( D_i \subset Y \) such that \( \omega(C_i, D_i) = 0 \) for each \( i \) and \( \bigcup_{i=1}^{k} D_i = Y \). Therefore \( E = f^{-1}(D_j) \in Q \) for some \( j \) contradicting \( \omega_Y(C_j, f(E)) = \infty \).

Suppose \( \omega_Y(C, C_1, \ldots, C_k) = \infty \) for all \( C_i \in \hat{f}(Q) \) but \( \omega(C, f(E)) = 0 \) for some \( E \in Q \). Notice \( f(E) \in \hat{f}(Q) \), a contradiction. Thus, \( \hat{f}(Q) \in \partial(Y, \omega_Y) \).

Suppose \( \hat{f}(Q) \in o(C) \), \( C \) being \( \omega_Y \)-open. Choose \( \omega_Y \)-open \( D \subset Y \) such that \( \hat{f}(Q) \subset o(D) \subset cl(o(D)) \subset o(C) \). In particular, \( \omega_Y(D, Y \setminus C) = 0 \), as otherwise there is \( P \in \partial(Y, \omega_Y) \) containing both \( D \) and \( Y \setminus C \) which belongs to \( cl(D) \) by \( 5.10 \) but not to \( o(C) \).

Hence \( Q \in o(f^{-1}(D)) \) and for any \( R \in o(f^{-1}(D)) \) we have \( \hat{f}(R) \in o(C) \). Indeed, \( D \in \hat{f}(R) \), so \( Y \setminus C \notin \hat{f}(R) \) which means \( \hat{f}(R) \in o(C) \).

Since \( f(D) \in P \) implies \( P \in cl(f(D)) \) (see \( 5.10 \)), it suffices to show the validity of Statement 2. Suppose \( P \neq \hat{f}(Q) \) and pick \( C \in P \) that does not belong to \( \hat{f}(Q) \). Hence, \( \omega_Y(C, f(D)) = 0 \) for some \( D \in Q \). By \( 6.8 \), \( \omega_Y(C, cl(f(D))) = 0 \) and using \( 6.6 \) we can produce an open set \( W \) in \( X \) disjoint with \( E := cl_X(f(D)) \) and satisfying \( W \in P, \omega_Y(W, E) = 0 \). Since \( P \in o(W) \), there is \( R \in o(W) \cap cl(E) \). Now (see \( 5.10 \)), \( E \in R \) and \( W \in R \), a contradiction.

**Proposition 11.4.** Suppose If \( f : (X, \omega_X) \to (Y, \omega_Y) \) is a function of formed spaces such that \( \omega_Y(f(C)) = 0 \) implies \( \omega_X(C) = 0 \) for all subsets \( C \) of \( X \). If \( f \) has a continuous extension

\[
\tilde{f} : X \cup \partial(X, \omega_X) \to Y \cup \partial(Y, \omega_Y)
\]

then it is form-continuous.

**Proof.** Suppose \( \omega_Y(f(C_1), \ldots, f(C_k)) = 0 \) but \( \omega_X(C_1, \ldots, C_k) = \infty \) for some subsets \( C_i, i \leq k \), of \( X \). Choose \( Q \in \partial(X, \omega_X) \) containing all \( C_i, i \leq k \). There is \( j \) such that \( f(C_j) \notin \hat{f}(Q) \). Hence \( \hat{f}(Q) \notin cl(f(C_j)) \) contradicting \( Q \in cl(C_j) \).

### 12. Forms vs Large Scale Structures

This section is devoted to interaction between forms and large scale structures. Namely, every form \( \omega \) on \( X \) induces a large scale structure \( LS(\omega) \) and every large scale structure \( LS \) on \( X \) induces a form \( \omega(LS) \) on \( X \).

For basic facts related to the coarse category see \( 17 \).

Recall that a coarse structure \( C \) on \( X \) is a family of subsets \( E \) (called controlled sets) of \( X \times X \) satisfying the following properties:

1. The diagonal \( \Delta = \{(x, x)\}_{x \in X} \) belongs to \( C \).
2. \( E_1 \in C \) implies \( E_2 \in C \) for every \( E_2 \subset E_1 \).
3. \( E \in C \) implies \( E^{-1} \in C \), where \( E^{-1} = \{(y, x)\}_{(x, y) \in E} \).
4. \( E_1, E_2 \in C \) implies \( E_1 \cup E_2 \in C \).
5. \( E, F \in C \) implies \( E \circ F \in C \), where \( E \circ F \) consists of \( (x, y) \) such that there is \( z \in X \) so that \( (x, z) \in E \) and \( (z, y) \in F \).

Recall that the star \( st(B, \mathcal{U}) \) of a subset \( B \) of \( X \) with respect to a family \( \mathcal{U} \) of subsets of \( X \) is the union of those elements of \( \mathcal{U} \) that intersect \( B \). Given two
families \(B\) and \(U\) of subsets of \(X\), \(st(B,U)\) is the family \(\{st(B,U)\}\), \(B \in B\), of all stars of elements of \(B\) with respect to \(U\).

**Definition 12.1.** A large scale structure \(\mathcal{LSS}_X\) on a set \(X\) is a non-empty set of families \(B\) of subsets of \(X\) (called uniformly \(\mathcal{LSS}_X\)-bounded or uniformly bounded once \(\mathcal{LSS}_X\) is fixed) satisfying the following conditions:

1. \(B_1 \in \mathcal{LSS}_X\) implies \(B_2 \in \mathcal{LSS}_X\) if each element of \(B_2\) consisting of more than one point is contained in some element of \(B_1\).
2. \(B_1, B_2 \in \mathcal{LSS}_X\) implies \(st(B_1,B_2) \in \mathcal{LSS}_X\).

A subset \(B\) of \(X\) is bounded with respect to \(\mathcal{LSS}_X\) if the family \(\{B\}\) belongs to \(\mathcal{LSS}_X\). Thus, each large scale structure on \(X\) induces a bornology on \(X\) if every finite subset of \(X\) is bounded.

As described in [7], the transition between the two structures is as follows:

1. Given a uniformly bounded family \(U\) in \(X\), the set \(\bigcup_{B \in U} B \times B\) is a controlled set, \(U = \{B\} \cup \bigcup_{x \in X} \{x\}\), where \(E[x] := \{y \in X \mid (x,y) \in E\}\).

**Proposition 12.2.** Given a form \(\omega\) on \(X\), the family of all covers \(U\) of \(X\) with the property

\[\omega(C_1,\ldots,C_k) = \omega(st(C_1,U),\ldots,st(C_k,U))\]

for all vectors \((C_1,\ldots,C_k)\) in \(X\) induces a large scale structure \(LS(\omega)\) whose family of bounded subsets contains all \(\omega\)-bounded sets. If \(B\) is bounded with respect to \(LS(\omega)\) and \(\omega(B) = \infty\), then \(\omega(C) = \infty\) for all \(C\) intersecting \(B\).

**Proof.** Suppose \(B_1 \in LS(\omega)\) and each element of a cover \(B_2\) of \(X\) consisting of more than one point is contained in some element of \(B_1\). For all vectors \((C_1,\ldots,C_k)\) in \(X\), \(\omega(C_1,\ldots,C_k) \leq \omega(st(C_1,B_2),\ldots,st(C_k,B_2)) \leq \omega(st(C_1,B_1),\ldots,st(C_k,B_1)) = \omega(C_1,\ldots,C_k)\). \(B_1, B_2 \in LS(\omega)\) implies \(st(B_1,B_2) \in LS(\omega)\) follows from the fact \(st(C, st(B_1,B_2)) \subset st(st(C,B_1),B_2) \cup st(st(C,B_2),B_1)\). Thus, \(LS(\omega)\) is indeed a large scale structure.

If \(\omega(B) = 0\) and \(U = \{B\} \cup \bigcup_{x \in X} \{x\}\), then for all vectors \((C_1,\ldots,C_k)\) in \(X\),

\[\omega(C_1,\ldots,C_k) \leq \omega(st(C_1,U),\ldots,st(C_k,U)) \leq \omega(C_1 \cup B,\ldots,C_k \cup B) = \omega(C_1,\ldots,C_k)\].

That means \(B\) is a bounded subset of \(X\) with respect to \(LS(\omega)\). Suppose \(B \neq \emptyset\) is a bounded subset of \(X\) with respect to \(LS(\omega)\) and \(\omega(B) = \infty\). Let \(U := \{B\} \cup \bigcup_{x \in X} \{x\}\) and suppose \(C\) intersects \(B\). Now, \(\omega(C) = \omega(st(C,U)) = \omega(C \cup B) = \infty\).

**Proposition 12.3.** If \(\omega\) is a normal \(T_1\) form, then the following conditions are equivalent:

a. \(U \in LS(\omega)\),
b. For each neighborhood \(W\) of \(P \in \partial(X,\omega)\) in \(X \cup \partial(X,\omega)\) there is a neighborhood \(W'\) of \(P\) such that \(st(W',U) \subset W\),
c. The coronas of \(C\) and \(st(C,U)\) coincide for each \(C \subset X\).

**Proof.** a) \(\Rightarrow\) b). Since \(X \cup \partial(X,\omega)\) is compact Hausdorff, there is a neighborhood \(U\) of \(P\) in \(X \cup \partial(X,\omega)\) whose closure is contained in \(W\). In particular, \(\omega(U \cap X, W) = 0\). Now, \(\omega(st(U \cap X,U),st(X \cap W,U)) = 0\) and \(B := st(U \cap X,U)\cap st(X \setminus W,U)\) is \(\omega\)-bounded. Put \(W' := U \setminus st(B,U)\).

b) \(\Rightarrow\) c). The corona of \(D\) is defined as all \(P \in \partial(X,\omega)\) contained in \(cl(D)\). Equivalently, all \(P\) containing \(D\).
Suppose \( \mathcal{P} \) contains \( \text{st}(C, \mathcal{U}) \) but not \( C \). There exists a vector \((C_1, \ldots, C_k)\) consisting of elements of \( \mathcal{P} \) such that \( \omega(C, C_1, \ldots, C_k) = 0 \). Consequently, \( \omega(\text{st}(C, \mathcal{U}), \text{st}(C_1, \mathcal{U}), \ldots, \text{st}(C_k, \mathcal{U})) = 0 \), a contradiction as \( \mathcal{P} \) of these sets.

a) \( \implies \) \( \omega \). Suppose \( U \notin \text{LS}(\omega) \). There is a vector \((C_1, \ldots, C_k)\) satisfying \( \omega(C_1, \ldots, C_k) = 0 \) and \( \omega(\text{st}(C_1, \mathcal{U}), \ldots, \text{st}(C_k, \mathcal{U})) = \infty \). Choose sets \( D_1, \ldots, D_k \) such that \( \omega(C_i, D_i) = 0 \) for each \( i \leq k \) and \( \bigcup_{i=1}^{k} D_i = X \). Thus, coronas of \( D_i \) and \( C_i \) are disjoint for each \( i \) resulting in \( \omega(\text{st}(C_i, \mathcal{U}), \text{st}(D_i, \mathcal{U})) = 0 \) for each \( i \leq k \). That contradicts \( \omega(\text{st}(C_1, \mathcal{U}), \ldots, \text{st}(C_k, \mathcal{U})) = \infty \).

\[ \Box \]

**Proposition 12.4.** Given an infinite set \( X \) consider the maximal bounded geometry large scale structure \( \text{LS} \) on \( X \). It consists of all covers \( \mathcal{U} \) of \( X \) satisfying the following properties:

a. There is a natural number \( n \geq 1 \) such that each element \( U \) of \( \mathcal{U} \) has at most \( n \) points.

b. There is a natural number \( m \geq 1 \) such that each point \( x \) of \( X \) belongs to at most \( m \) elements \( U \) of \( \mathcal{U} \).

The form \( \omega(\text{LS}) \) induced by \( \text{LS} \) is characterized by the following property: \( \omega(\text{LS})(V) = 0 \) if and only if at least one coordinate of \( V \) is finite.

**Proof.** Suppose all \( C_i \) are infinite and \( \omega(\text{LS})(C_1, \ldots, C_k) = 0 \). We may assume \( k \) is the smallest number with that property. Reduce to \( C_i \) being mutually disjoint and countable. Pick bijections \( f_i : C_i \rightarrow C_i \) and consider the trivial extension \( \mathcal{U} \) of the family \( U := \{x, f_2(x), \ldots, f_k(x)\} \). Notice that \( C_1 \subset \text{st}(C_i, \mathcal{U}) \) for all \( i \leq k \), a contradiction.

\[ \Box \]

**Proposition 12.5.** Let \( \omega \) be defined as follows: \( \omega(\text{LS})(V) = 0 \) if and only if at least one coordinate of \( V \) is finite. The large scale structure \( \text{LS}(\omega) \) induced by \( \omega \) consists of all covers \( \mathcal{U} \) of \( X \) with the property \( \text{st}(F, \mathcal{U}) \) is finite for all finite subsets \( F \) of \( X \).

**Proof.** Suppose \( \mathcal{U} \in \text{LS}(\omega) \) and \( F \subset X \) is finite. Now, \( 0 = \omega(F) = \omega(\text{st}(F, \mathcal{U})) \), so \( \text{st}(F, \mathcal{U}) \) is finite.

The remainder of the proof is obvious.

\[ \Box \]

**Question 12.6.** Is \( \text{LS}(\omega) \) normal if \( \omega \) is the form defined in [12.3]?

**Proposition 12.7.** If \( \lambda \) is a normal form on \( X \), then \( \omega(\text{LS}(\lambda)) = \lambda \).

**Proof.** See [2.10] for the definition of the form induced by a large scale structure. It is easy to observe that \( \omega(\text{LS}(\lambda)) \leq \lambda \).

Indeed, if \( \lambda(C_1, \ldots, C_k) = 0 \), then for any \( \mathcal{U} \in \text{LS}(\lambda) \) one has \( \lambda(\text{st}(C_1, \mathcal{U}), \ldots, \text{st}(C_k, \mathcal{U})) = 0 \). In particular, \( \bigcap_{i=1}^{k} \text{st}(C_i, \mathcal{U}) \) must be \( \text{LS}(\lambda) \)-bounded (see [12.2]), hence \( \omega(\text{LS}(\lambda))(C_1, \ldots, C_k) = 0 \).

Conversely, suppose \( \omega(\text{LS}(\lambda))(C_1, \ldots, C_k) = 0 \) and \( \lambda(C_1, \ldots, C_k) = \infty \). xxx

\[ \Box \]

**Observation 12.8.** The large scale structure \( \text{LS} \) described in [12.4] has the property that \( \text{LS}(\omega(\text{LS})) \neq \text{LS} \).
\begin{conjecture}
\textbf{Conjecture 12.9.}\ If $LS(\omega)$ is normal iff for every $k$-vector $V$ such that $\omega(V) = 0$ there is a slowly oscillating $f : X \rightarrow \Delta^k$ and a bounded set $B$ of $X$ such that $C_i \setminus B \subset f^{-1}(st(v_i))$ for each $i \leq k$. \[xxx\]

\textbf{Question 12.10.}\ Is $LS(\omega)$ normal if $\omega$ is normal?

\textbf{Question 12.11.}\ Is $\omega(LS)$ normal if $LS$ is normal?

\section{Dimension of formed spaces}

The goal of this section is to give simple proofs of results that generalize the work of Austin-Virk [1].

\begin{definition}
\textbf{Definition 13.1.}\ A form-continuous function $f : (X, \omega_X) \rightarrow (Y, \omega_Y)$ of formed spaces is \textit{proper} if images of $\omega_X$-bounded sets are $\omega_Y$-bounded.
\end{definition}

\begin{theorem}
\textbf{Theorem 13.2.}\ Suppose $f : (X, \omega_X) \rightarrow (Y, \omega_Y)$ is a surjective and proper form-continuous function of normal $T_1$ formed spaces and $n \geq 1$. The induced map
\[\partial f : \partial(X, \omega_X) \rightarrow \partial(Y, \omega_Y)\]
is surjective and $n$-to-1 if and only if for each sequence $\{A_i\}_{i=1}^{n+1}$ of subsets of $X$, the condition $\omega_X(A_i, A_j) = 0$ for all $i \neq j$ implies $\omega_Y(f(A_1), \ldots, f(A_{n+1})) = 0$.
\end{theorem}

\begin{proof}
See [11.3] for a description of $\partial f$. Using [3.9] observe that the continuous extension $\tilde{f}$ of $f$ is closed, hence the extension $\tilde{f}$ of $f$ is surjective ($Y$ is dense in $Y \cup \partial(Y, \omega_Y)$). In particular $\partial f$ is surjective as no point in $X \setminus \partial(X, \omega_X)$ can be mapped to $\partial(Y, \omega_Y)$.

Suppose $\partial f$ is not $n$-to-1, i.e. there exists $Q \in \partial(Y, \omega_Y)$ and $Q_i \in \partial(X, \omega_X)$, $1 \leq i \leq n + 1$, such that $(\partial f)(Q) = Q$ and $Q_i \neq Q_j$ if $i \neq j$. Since $X \cup \partial(X, \omega_X)$ is Hausdorff, there exist mutually disjoint closed neighborhoods $D_i$ of $Q_i$ in $X \cup \partial(X, \omega_X)$. Put $A_i := D_i \cap X$. Notice $A_i \in Q_i$ for each $i \leq n + 1$. Also, $\omega_X(A_i, A_j) = 0$ for all $i \neq j$. However, $\omega_Y(f(A_1), \ldots, f(A_{n+1})) = \infty$ as $f(A_i) \in Q$ for each $i \leq n + 1$.

Suppose there exist sets $A_i, i \leq n + 1$, such that $\omega_X(A_i, A_j) = 0$ whenever $i \neq j$ but $\omega_Y(f(A_1), \ldots, f(A_{n+1})) = \infty$. There is $Q \in \partial(Y, \omega_Y)$ containing $f(A_i)$ for each $i$. In particular, $Q \in \partial(f(A_i), \omega_Y)$ for each $i$. Therefore, for each $i \leq n + 1$, there is $Q_i \in \partial(A_i, \omega_X)$ so that $(\partial f)(Q_i) = Q$. Since $Q_i \neq Q_j$ in $\partial(X, \omega_X)$, we conclude that $\partial f$ is not $n$-to-1.
\end{proof}

\begin{theorem}
\textbf{Theorem 13.3.}\ Suppose $(X, L_X)$ and $(Y, L_Y)$ are large scale spaces such that the induced forms $\omega_X, \omega_Y$ are normal and $T_1$. If $f : (X, L_X) \rightarrow (Y, L_Y)$ is coarsely $n$-to-1 for some $n \geq 1$, then the induced map $\partial(X, \omega_X) \rightarrow \partial(Y, \omega_Y)$ of form coronas is $n$-to-1.
\end{theorem}

\begin{proof}
Suppose the induced map $\partial(X, \omega_X) \rightarrow \partial(Y, \omega_Y)$ of form coronas is not $n$-to-1. By Theorem 13.2 there exist sets $A_i \subset X, i \leq n + 1$ such that $\omega_X(A_i, A_j) = 0$ if $i \neq j$ but $\omega_Y(f(A_1), \ldots, f(A_{n+1})) = \infty$. Therefore, there exists a uniformly bounded cover $U$ of $Y$ such that $\bigcap_{i=1}^{n+1} st(f(A_i), U)$ is $\omega_Y$-unbounded. Choose a uniformly bounded cover $W$ of $X$ such that each set $f^{-1}(U), U \in st(U, U)$, can be covered by at most $n$-elements of $W$. Define $C_i$ as $A_i \setminus B_i$, where $B_i := \bigcup_{j \neq i} st(A_i, W) \cap \partial(X, \omega_X) \rightarrow \partial(Y, \omega_Y)$ of form coronas is not $n$-to-1.
Remark. Proof. □

Theorem 13.4. Suppose $f : X \to Y$ is a coarse bornologous function of large scale spaces whose induced forms are normal and $T_1$. If $X$ is metrizable and $n \geq 1$, then $f$ is coarsely $n$-to-1 if and only if the induced map of their boundaries at infinity is $n$-to-1.

Proof. Suppose $f$ is not coarsely $n$-to-1 and the induced map of the boundaries at infinity is $n$-to-1. That means the existence of a uniformly bounded cover $U$ of $Y$ such that for each $k \geq 1$ there is $U_k \in U$ such that $f^{-1}(U_k)$ cannot be covered by at most $n$ sets of diameter at most $2k$. That implies existence, for each $k \geq 1$, of points $x_{i,1}, x_{i,2}, \ldots, x_{i,n+1}$ of $f^{-1}(U_k)$ such that $d(x_{i,k}, x_{j,k}) \geq k$ whenever $i \neq j$. Let $A_i := \{x_{i,k}^k \}_{k=1}^\infty$. Notice $\omega_X(A_i, A_j) = 0$ if $i \neq j$. By Theorem 13.2 $\omega_Y(f(A_1), \ldots, f(A_{n+1})) = 0$. Let $C_i = st(f(A_i), U)$ for $i \leq n + 1$. Therefore $\omega_Y(C_1, \ldots, C_{n+1}) = 0$. Since $W := \bigcup_{i=1}^\infty U_i \in C_i$ for each $i$, $\omega_Y(W) = 0$ and $W$ is bounded in $Y$. Hence $f^{-1}(W)$ is bounded in $X$, a contradiction as in that case, for $k > diam(f^{-1}(W))$, $f^{-1}(U_k)$ can be covered by one set of diameter at most $2k$.

Apply [13.8] to conclude the proof. □

Theorem 13.5. If $n \geq 1$ and $f : X \to Y$ is a coarsely $n$-to-1 bornologous map of large scale spaces, then $\text{asdim}(Y) \leq \text{asdim}(Y) + n - 1$.

Proof.

Remark 13.6. Theorem 13.5 was proved by Austin-Virk in [1] for proper metric spaces $X$ and $Y$.

Theorem 13.7. A coarse bornologous function $f : X \to Y$ of metrizable large scale spaces is a large scale equivalence if and only if it induces a homeomorphism of Higson coronas.

Proof. Suppose $f : X \to Y$ induces a homeomorphism of Higson coronas. Notice $Z := f(X)$ induces a large scale equivalence $i : Z \to Y$. Apply [13.3] to conclude $f : X \to Z$ is coarsely 1-to-1. That means precisely that the inverse of a uniformly bounded cover of $Z$ is uniformly bounded in $X$ from which it follows that $f$ is a large scale equivalence.

Corollary 13.8. A coarse bornologous function $f : X \to Y$ of metrizable large scale spaces is a large scale embedding if and only if it induces a topological embedding of Higson coronas.

Proof. $f : X \to f(X)$ induces a homeomorphism of Higson coronas. Apply [13.7] □

Theorem 13.9. Suppose $(X, \omega)$ is a normal $T_1$ formed space. $(X, \omega)$ is metrizable if and only if there is a formed continuous map $f : M \to X$ from a metric space inducing a homeomorphism of boundaries at infinity.
14. Metrizability of form compactifications

Theorem 14.1. Suppose $(X, \omega)$ is a normal $T_1$ formed space. The following conditions are equivalent:
1. $X \cup \partial(X, \omega)$ is metrizable,
2. There exists a countable family $S$ of subsets of $X$ such that for any subsets $C_1, \ldots, C_k$ of $X$ satisfying $\omega(C_1, \ldots, C_k) = 0$ there are $D_i \in S$, $i \leq k$, such that $C_i \subset D_i$ for each $i \leq k$ and $\omega(D_1, \ldots, D_k) = 0$,
3. There exist subsets $C_n$, $n \geq 1$, of $X$ such that for any two subsets $D, E$ of $X$ satisfying $\omega(D, E) = 0$ there is $C_k$ such that $D \subset C_k$ and $\omega(C_k, E) = 0$.

Proof. 1) $\implies$ 2). Given a metric $d$ on $X \cup \partial(X, \omega)$, notice that complements $B_n$ of all balls $B(\partial(X, \omega), \frac{1}{n})$, $n \geq 1$, are $\omega$-bounded and we start $S$ by enlisting all of them. As $(\partial(X, \omega), d)$ is compact metric, there is a countable family $\{D\}$ of subsets of $X$ such that sets $o(D)$ form a basis at points of $\partial(X, \omega)$. Add all finite unions of elements of $\{D\} \cup \{B_n\}_{n=1}^\infty$ to $S$. If subsets $C_1, \ldots, C_k$ of $X$ satisfy $\omega(C_1, \ldots, C_k) = 0$, then the intersection of sets $\cl{cl(C_i)} \cap \partial(X, \omega)$ is empty, so there is a finite union $E_i$ of elements of $\{D\}$ such that $C_i \setminus E_i$ is $\omega$-bounded and the intersection of sets $\cl{\cl{E_i}} \cap \partial(X, \omega)$ is empty. Each set $C_i \setminus E_i$ must be contained in some $B_n$, so all of them are contained in a single $B_m$. Define $D_i$ as $E_i \cup B_m$ and we are done.

3) $\implies$ 1). Notice that $\omega$-bounded elements of the family $\{C_n\}$ form a basis of $\omega$-bounded subsets of $X$ in the following sense: if $B$ is $\omega$-bounded, then there is $n$ such that $C_n$ is $\omega$-bounded and $B \subset C_n$. List those sets as $\{B_n\}$.

Claim: The family $\{\cl{\cl{C_n}} \setminus B_m\}$ forms a basis of points of $\partial(X, \omega)$ in $X \cup \partial(X, \omega)$.

Proof of Claim: Suppose $Q \in \partial(X, \omega)$ and $Q \in o(U) \subset cl(o(U)) \subset o(W)$. Now, $\omega(U, X \setminus W) = 0$, so there exists $D \in \{C_n\}$ containing $U$ such that $\omega(D, X \setminus W) = 0$. Choose $B_n$ containing $\cl{\cl{D}} \cap \cl{\cl{X \setminus W}}$ and notice $\cl{\cl{D \setminus B_n}} \cap \cl{\cl{X \setminus W}} = \emptyset$. Therefore $Q \in \cl{\cl{D \setminus B_n}} \subset o(W)$.

Given a countable basis $\{U_n\}$ of points of $\partial(X, \omega)$ in $X \cup \partial(X, \omega)$ choose, for every pair $(U_n, U_m)$ such that $\cl{\cl{U_n}} \subset U_m$, a continuous function $f_{m,n} : X \cup \partial(X, \omega) \rightarrow [0, 2^{-m-n}]$ such that $f(\cl{\cl{U_n}}) \subset \{0\}$ and $f(X \cup \partial(X, \omega) \setminus U_m) \subset \{2^{-m-n}\}$. Add all Dirac functions of points in $X$ that are $\omega$-bounded and the resulting family $\{f_s\}_{s \in S}$ has continuous sum $f := \sum_{s \in S} f_s$. Now, $\{f_s/\omega\}_{s \in S}$ is a partition of unity on $X \cup \partial(X, \omega)$ whose carriers form a basis of $X \cup \partial(X, \omega)$. By [6], $X \cup \partial(X, \omega)$ is metrizable.

15. Forms on large scale spaces

Proposition 15.1. If $A$ is a subset of a large scale space $(X, \mathcal{L})$, then the form induced by $\mathcal{L}|A$ coincides with $\omega(\mathcal{L})|A$.

Proof. Suppose $C_i \subset A$, $i \leq k$, and for each uniformly bounded cover $U$ of $X$, the set $\bigcap_{i=1}^k st(C_i, U)$ is bounded. Suppose $W$ is a uniformly bounded cover of $X$. To
show \( \omega(\mathcal{L})(C_1, \ldots, C_k) = 0 \) it suffices to prove inclusion
\[
\bigcap_{i=1}^{k} \text{st}(C_i, \mathcal{W}) \subset \text{st}\left( \bigcap_{i=1}^{k} \text{st}(C_i, \mathcal{W}|A), \mathcal{W} \right).
\]
If \( x \in \bigcap_{i=1}^{k} \text{st}(C_i, \mathcal{W}) \setminus A \), then for each \( i \) there is \( W_i \in \mathcal{W} \) such that \( C_i \cap W \neq \emptyset \) and \( x \in C_i \cup W_i \). Choose \( y_i \in C_i \cap W \) and notice \( y_j \in \text{st}(C_i, \mathcal{W}|A) \) for all \( i, j \leq k \). Therefore \( x \in \text{st}\left( \bigcap_{i=1}^{k} \text{st}(C_i, \mathcal{W}|A), \mathcal{W} \right) \).

**Proposition 15.2.** Suppose \( f, g : (X, \mathcal{L}_X) \to (Y, \mathcal{L}_Y) \) are coarse maps of large scale spaces.

a. If the induced form \( \omega_Y := \omega(\mathcal{L}_Y) \) is normal \( T_1 \) and \( f \) is close to \( g \), then \( \partial f = \partial g \).

b. If \( (Y, \mathcal{L}_Y) \) is metrizable and \( \partial f = \partial g \), then \( f \) is close to \( g \).

**Proof.**

a. \( \mathcal{U} := \{ f(x), g(x) \}_{x \in X} \) is a uniformly bounded family in \( Y \). Suppose \( \mathcal{Q} \in \partial(X, \omega(\mathcal{L}_X)) \) and \( (\partial f)(\mathcal{Q}) \neq (\partial g)(\mathcal{Q}) \). By \( 11.3 \) there is \( C \in \mathcal{Q} \) such that \( f(C) \notin (\partial g)(\mathcal{Q}) \). Using \( 11.3 \) again, we detect \( D \in \mathcal{Q} \) so that \( \omega_Y(f(C), g(D)) = 0 \). Hence \( \omega_Y(st(f(C)), \mathcal{U}), g(D)) = 0 \), a contradiction as \( g(C) \subset st(f(C)), \mathcal{U}) \).

b. If \( f \) is not close to \( g \), then for each \( k \geq 1 \) there exists \( x_k \in X \) such that \( d(f(x_k), g(x_k)) > k \). Put \( A := \{ x_k \}_{k=1}^{\infty} \) and notice it is unbounded. Notice \( \omega_Y(f(A), g(A)) = 0 \), so for any \( \mathcal{Q} \in \partial(X, \omega(\mathcal{L}_X)) \) containing \( A \) one has \( (\partial f)(\mathcal{Q}) \neq (\partial g)(\mathcal{Q}) \), a contradiction.

\[
16. \text{Asymptotic dimension 0}
\]

**Definition 16.1.** \( X \) is **coarsely totally disconnected** if \( \mathcal{U} \)-components of every uniformly bounded cover \( \mathcal{U} \) of \( X \) are bounded.

**Lemma 16.2.** If \( X \) is coarsely totally disconnected and \( \mathcal{W} \) is the family of all \( \mathcal{U} \)-components of \( X \) for some uniformly bounded cover \( \mathcal{U} \), then \( \text{st}(B, \mathcal{W}) \) is bounded for all bounded sets \( B \).

**Proof.** If \( \text{st}(B, \mathcal{W}) \) is unbounded, then the \( \mathcal{U} \)-component of \( X \) containing \( B \), \( \mathcal{U} := \mathcal{U} \cup \{ B \} \), contains \( \text{st}(B, \mathcal{W}) \) and is unbounded, a contradiction.

**Theorem 16.3.** Suppose \( \omega \) is a normal \( T_1 \) form on \( X \). If \( \text{dim}(X \cup \partial(X, \omega)) = 0 \) and \( \text{LS}(\omega) \) is coarsely totally disconnected, then the asymptotic dimension of \( \text{LS}(\omega) \) equals 0.

**Proof.** Suppose \( \mathcal{U} \) is an element of \( \text{LS}(\omega) \) such that \( \mathcal{W} \), the collection of \( \mathcal{U} \)-components of \( X \), is not uniformly bounded. That means existence of an \( \omega \)-unbounded \( C \subset X \) such that \( X_0 \cap \text{cl}(C) \) (\( X_0 := \partial(X, \omega) \)) is a proper subset of \( X_0 \cap \text{cl}(st(C, \mathcal{W})) \). Choose \( \mathcal{Q} \in X_0 \) belonging to \( \text{cl}(st(C, \mathcal{W})) \setminus \text{cl}(C) \). There is an open-closed subset \( Y \subset X \cup \partial(X, \omega) \) containing \( \text{cl}(C) \) and missing \( \mathcal{Q} \). Let \( Y_1 := X \cap Y \).

Choose a continuous function \( \alpha : X \cup \partial(X, \omega) \to S^0 \) such that \( \alpha(Y_1) \) and \( \alpha(X \setminus Y_1) \) are disjoint. Consider \( \mathcal{S} := \{ U \in \mathcal{U} | \alpha(U) = S^0 \} \). \( B := \bigcup \mathcal{S} \) is \( \omega \)-bounded as otherwise, for each \( U \in \mathcal{S} \), we can pick \( x_U, y_U \in U \) satisfying \( \alpha(x_U) = -1 \) and \( \alpha(y_U) = 1 \). In that case both \( D := \{ x_U \}_{U \in \mathcal{S}} \) and \( E := \{ y_U \}_{U \in \mathcal{S}} \) are \( \omega \)-unbounded with disjoint coronas contradicting \( E \subset \text{st}(D, \mathcal{U}) \).
Consider \( Y_1 \setminus st(B, W) \) and \( (X \setminus Y_1) \setminus st(B, W) \). There is no \( \mathcal{U} \)-chain joining these two sets, hence the stars of the two sets with respect to \( W \) are disjoint, contradicting \( Q \) belonging to \( cl(st(C, W)) \setminus cl(C) \). \( \square \)

**Example 16.4.** Given an infinite set \( X \) consider \( \omega \) defined as follows: \( \omega(V) = 0 \) if and only if at least one coordinate of \( V \) is finite. In that case \( LS(\omega) \) consists of all covers \( U \) of \( X \) by finite sets such that \( st(F, U) \) is finite for each finite subset \( F \) of \( X \). Notice \( LS(\omega) \) is not coarsely totally disconnected. Indeed, choose an infinite sequence \( \{x_n\}_{n=1}^\infty \) in \( X \) and notice the trivial extension \( U \) of \( \{x_n, x_{n+1}\}_{n=1}^\infty \) has infinite \( U \)-component.

**Conjecture 16.5.** If \( \text{dim}(X \cup \partial(X, \omega)) = 0 \) and the asymptotic dimension of \( LS(\omega) \) does not equal 0, then there is an unbounded subset \( C \) of \( X \) whose corona consists of exactly one point.

### 17. Parallelism

\( C \) is parallel to \( D \) if their coronas are equal.

Same as \( E \perp C \iff E \perp D \) for all \( E \subset X \).

### 18. Visual forms

**Definition 18.1.** Given a pointed geodesic space \((X, x_0)\) and \( C \subset X \), the \( r \)-projection \( P_r(C, x_0) \) of \( C \), where \( r > 0 \), is the closure of \( D \) consisting of intersections of geodesics from \( x \in C \) to \( x_0 \) with the \( r \)-sphere centered at \( x_0 \).

**Example 18.2.** The first visual form \( \omega_1 \) is defined as follows:

\[
\omega_1(C_1, \ldots, C_k) = 0
\]

iff there is \( M > 0 \) such that \( \bigcap_{i=1}^k P_r(C_i, x_0) = \emptyset \) for all \( r > M \).

**Example 18.3.** The second visual form \( \omega_2 \) is defined as follows:

\[
\omega_2(C_1, \ldots, C_k) = 0
\]

iff for each \( \epsilon > 0 \) there is \( M > 0 \) such that \( \bigcap_{i=1}^k B(P_r(C_i, x_0), \epsilon) = \emptyset \) for all \( r > M \).

**Example 18.4.** The third visual form \( \omega_3 \) is defined as follows:

\[
\omega_3(C_1, \ldots, C_k) = 0
\]

iff for each sequence \( \epsilon_n > 0 \) there is a sequence \( M_n > 0 \) such that \( \bigcap_{i=1}^k B(P_r(C_i, x_0), \epsilon_n) = \emptyset \) for all \( r > M_n \).

### 19. Group actions on formed spaces

**Definition 19.1.** Suppose a group \( G \) acts on a formed space \((X, \omega)\). \( \omega \) is \( G \)-invariant if

\[
\omega(V) = \omega(g \cdot V)
\]

for each \( g \in G \) and each vector \( V \) in \( X \).
The space of orbits $X/G$ consists of sets of the form $G \cdot x$, there is a natural projection $\pi : X \to X/G$ ($\pi(x) := G \cdot x$) and $X/G$ is equipped with the form $\omega_G$ defined as follows:

$$\omega_G(C_1, \ldots, C_k) = \omega(\pi^{-1}(C_1), \ldots, \pi^{-1}(C_k))$$

for all vectors $(C_1, \ldots, C_k)$ in $X/G$.

**Proposition 19.2.** Suppose $G$ is a finite group acting on $X$. If $\omega$ is a normal, $T_1$, and $G$-invariant form on $X$, then the induced form $\omega_G$ on $X/G$ is normal and $T_1$.

**Proof.** Given two different orbits $G \cdot x$ and $G \cdot y$, $\omega(G \cdot x, G \cdot y) = 0$ (use additivity of $\omega$). Thus $\omega_G$ is $T_1$.

Suppose $\omega(\pi^{-1}(C_1), \ldots, \pi^{-1}(C_k)) = 0$ for a vector $(C_1, \ldots, C_k)$ in $X/G$. Choose sets $D_i$ in $X$ so that $\omega(D_i, \pi^{-1}(C_i)) = 0$ for each $i \leq k$ and $\{D_i\}_{i=1}^k$ covers $X$. For each $g \in G$, $\omega(g \cdot D_i, \pi^{-1}(C_i)) = \omega(g \cdot D_i, g \cdot \pi^{-1}(C_i)) = \omega(D_i, \pi^{-1}(C_i)) = 0$. Therefore, $\omega(G \cdot D_i, \pi^{-1}(C_i)) = 0$ and sets $E_i := \pi(G \cdot D_i)$ form a cover of $X/G$ such that $\omega_G(E_i, C_i) = 0$ for each $i \leq k$.

To show $\tilde{\pi}$ is closed we apply [3,4] by observing $\pi(B)$ is $\omega_G$-bounded if $G$ is $\omega$-bounded. Indeed, $\omega_G(\pi(B), X/G) = \omega(G \cdot B, X) = 0$ as $G \cdot B$ is a finite union of $\omega$-bounded sets $g \cdot B$. \hfill $\Box$

**Corollary 19.3.** Suppose $G$ is a finite group acting on $X$. If $\omega$ is a normal, $T_1$, and $G$-invariant form on $X$, then $\dim(X \cup \partial(X, \omega)) = \dim(X/G \cup \partial(X/G, \omega_G))$.

**Proof.** In order to apply Proposition 9.2.16 from [10] we need to show that $\tilde{\pi} : X \cup \partial(X, \omega) \to X/G \cup \partial(X/G, \omega_G)$ is open and its fibers are finite. Here is the full statement of that proposition:

Let $X, Y$ be weakly paracompact, normal spaces. Let $f : X \to Y$ be a continuous, open surjection. If for every point $y \in Y$ the pre-image $f^{-1}(y)$ is finite, then $\dim(X) = \dim(Y)$.

Notice we can extend the action of $G$ over $X \cup \partial(X, \omega)$. Namely, $g \cdot \mathcal{P} := \{g \cdot C | C \in \mathcal{P}\}$. Observe $g \cdot o(U) = o(g \cdot U)$, so $G$ acts by homeomorphisms on $X \cup \partial(X, \omega)$. It suffices to show that fibers of $\tilde{\pi}$ coincide with the orbits of $G$, then $\tilde{\pi}$ is open as it is quotient (it is actually closed) and the inverse of $\tilde{\pi}(W)$ is $G \cdot W$ for each open subset $W$ of $X \cup \partial(X, \omega)$.

Suppose $\tilde{\pi}(\mathcal{Q}) = \tilde{\pi}(\mathcal{P})$ but $\mathcal{Q} \neq g \cdot \mathcal{P}$ for all $g \in G$. For each $g \in G$ there are open subsets $U_g, W_g$ of $X$ such that $\mathcal{Q} \in o(U_g), g \cdot \mathcal{P} \in o(W_g)$ and closures of $o(U_g)$ and $o(W_g)$ are disjoint. In particular, $\omega(U_g, W_g) = 0$. Put $U = \bigcap_{g \in G} U_g$ and $W = \bigcap_{g \in G} g^{-1} \cdot W_g, \mathcal{Q} \in o(U), \mathcal{P} \in o(W)$, and $\omega(G \cdot U, G \cdot W) = 0$, a contradiction as $G \cdot U \in \tilde{\pi}((\mathcal{Q})$ and $G \cdot W \in \tilde{\pi}(\mathcal{P})$. \hfill $\Box$

**Theorem 19.4.** Let $X$ be a metric space and let $G$ be a finite group acting isometrically on $X$. Then $X/G$ has the same asymptotic dimension as $X$.

**Proof.**

**Remark 19.5.** In case of proper metric spaces $X$, Theorem 19.4 was proved by Daniel Kasprzak [13].

**Theorem 19.6.** If $G$ acts coarsely on $X$ and $S^k \in \text{LSAE}(G), S^m \in \text{LSAE}(X/G)$, then $S^{k+m} \in \text{LSAE}(X)$. 
**Proof.** \( \nu(G) \to \nu(X) \) is the fiber of \( \nu(X) \to \nu(X/G) \).

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