A dielectric response model for FEM solutions of ICRF wave fields

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Abstract. Modelling of fast wave ICRF heating in large machines with high density such as DEMO is challenging because of the short wave lengths. Therefore, fast, efficient global wave solvers are necessary. A major difficulty with calculating the wave field in a spatial dispersive medium is that the dielectric tensor becomes a function of the local wave vector, which in its turn depends on the solution. Furthermore, the solution may consist of several waves co-existing at the same location subjected to separate response functions. In order to model upshift of the parallel wave vector, higher order FLR-effects on the cyclotron absorption and TTMP damping for the electron absorption methods based on iteration, suitable for FEM codes, are proposed.

1. Introduction

Modelling fast wave ICRF (ion cyclotron range of frequencies) heating in large machines with high density such as DEMO is a challenging task because of the spatial dispersive effects in a hot plasma and the short wave lengths. A major difficulty when solving the wave equation for spatial dispersive media is that the dielectric tensor depends on the wave vector, which in its turn depends on the solution. The predominant effect of spatial dispersion of the fast magnetosonic wave comes from the parallel wave number. Other important spatial dispersive effects are higher order FLR (finite Larmor radius) terms, TTMP (transit time magnetic pumping) damping and linear mode conversion. Geometrical optics within the applicability of the theory describes how the wave vector changes in a transparent way, but misses out on interference. The interference can be taken into account by using global wave solvers based on local discretisation by finite differences and finite elements or spectral methods. However, the co-existence of several waves complicates the problem because the dielectric tensors for the different waves differ, which requires that the waves are separated into different components having their own response. This is important for waves being upshifted after a reflection so that they are not damped by upshift before the reflection has taken place. The straightforward way to tackle this problem is to solve an integro-differential wave equation [1]. For global wave solvers the problem is handled by Fourier decomposition the wave field into planar waves, which has the advantage that the wave numbers are known for each component. Such an approach allows the response to be constructed consistently with that for planar waves in quasi-homogenous plasma, but requires that the coupling between the modes is taken into account. For
routine calculations fast efficient wave solvers being able to resolve the wave field are required. Thus, there is a need for approximating the response function allowing simpler methods to be used. When solving the wave field with a FEM code like LION \[2, 3\] the field is correctly calculated for an approximate response function. If the wave field consists of a single wave, the response can be corrected by iterations taking into account the wave field from the previous iteration when calculating the response. Here methods based on iteration are proposed, suitable for FEM codes, to model upshift of the parallel wave vector, higher order FLR-effects on the cyclotron absorption and TTMP damping for the electron absorption.

2. Upshift
The up or down shift of the parallel wave number, \( k_\parallel = n_\phi B / R + m_\theta B / \rho \), appears because the poloidal mode number, \( m_\theta \), is not an invariant in a torus. The common approximation of the parallel wave number by \( k_\parallel = n_\phi / R \) considerably simplifies the calculations, and is appropriate when the relative changes of \( k_\parallel \) due to the poloidal mode number is small. The change in the poloidal mode number is taken into account in the geometrical optics in a transparent way. In global wave codes it can be taken into account by Fourier decomposing the wave field, which gives a spectrum of coupled modes. Note that the wave field corresponding to a narrow beam has a wide spectrum of poloidal Fourier components that is required to describe the cancellation of the wave field outside the beam. The wave field within the beam may well be described by a slowly varying single wave vector as in geometrical optics. Here we assume the damping to be strong and the wave field to consist of one wave with a local well defined wave vector, which is the typical case for most ICRH scenarios for DEMO.

Wave-particle interactions in inhomogeneous plasma, like a torus, differ fundamentally from that of a homogeneous plasma. In homogeneous plasma only a small fraction of the particles are resonant. In an inhomogeneous plasma the cyclotron frequency varies and, in general, all particles passing the cyclotron resonance are resonant, but at different locations, and the resonance occurs only at a short time, but still over many wave periods. Here we assume the wave-particle interactions to be fully decorrelated, and do not consider bounce resonances \[4\]. The fully decorrelated interactions are localised around the resonance, \( \omega - \omega_c - k_\parallel v_\parallel = 0 \), and only the change of the poloidal mode number around the resonance needs to be taken into account, where \( \omega_c \) denotes the cyclotron resonance and \( v_\parallel \) de parallel velocity of the ion. Fourier decomposition of the field in poloidal direction requires that all the poloidal modes are included with proper correlation between the interactions in order to cancel out the wave-particle interactions in regions where the wave field vanishes. Here we calculate the upshift consistent with the stationary phase method. The change in energy \( \Delta W \) due to wave-particle interactions is obtained by integrating the local change in energy of the particle along the drift orbit

\[
\Delta W = Ze \int_0^\tau v \cdot E dt
\]  

(1a)

Expanding around the resonance, where \( \omega_c = \omega_{c0} \) yields

\[
\Delta W = Ze \int_0^\tau \left( v_\perp E_\perp \exp \left[ i (\omega_{c0} - \omega) t + i \int_0^t dt' v_\parallel \left[ \frac{\partial \omega_c}{\partial l} t' + \frac{\partial}{\partial l} \left[ \text{Im} (\ln E_\perp(t')) \right] \right] \right] \right) dt
\]  

(1b)

For simplicity we assume the waves interacting with passing particles or trapped far away from the turning point so that \( v_\parallel \) and \( k_\parallel \) can be regarded as constants. For a wave \( E = E_0 \exp(k_\perp x - \omega t) \) the integral in Eq. (1) oscillates rapidly as the particle moves away from the resonance, \( \omega - \omega_c - k_\parallel v_\parallel = 0 \), because of the variation of \( \omega_c \). To calculate the width of the resonance region we assume \( E_\perp = E_0 \exp(k_\perp x - \omega t) \), which gives \( k_\parallel = \text{Im} (\partial l (\ln E_\parallel)) / \partial l \) and integrate the phase up to \( \pi / 2 \) from the resonance, which defines one half of the resonance region

\[
\frac{\pi}{2} = \int_0^\tau \left( \omega - \omega_{c0} - \int_0^t v_\parallel \left[ \frac{\partial \omega_c}{\partial l} dt' - k_\parallel v_\parallel \right] \right) dt
\]  

(2)
The width of the resonance region along the magnetic field becomes \( \Delta l_r = 2t_r \), or in the poloidal direction \( \Delta \theta_r = 2t_r \partial \theta / \partial l \). Here we have used \( \omega_0 = \sqrt{2qR^2/v} \), and assumed \( B(r, \theta) = B_0 (1 + r \cos \theta / R) \). The width of the resonance depends on the parallel velocity of the particle. For trapped particles having their turning points near the resonance, for which case \( v || \to 0 \) and \( t_r \to \infty \), the variation of \( v || \) has to be taken into account [5]. For tangential resonance, for which \( \partial \omega_0 / \partial l \to 0 \), the change in \( \omega_0 \) has to be expanded up to second order. For a single wave the local wave vector can be obtained by taking the averaged value of the imaginary part of the logarithmic derivative of the electric field over the resonant region

\[
k_r = \left\{ \Im \partial \ln E_r / \partial l \right\}_{r}
\]

or

\[
k_\theta = \left\{ \Im \partial \ln E_r / \partial \theta \right\}_{r},
\]

where \( \langle \ldots \rangle_r = \frac{1}{2\Delta \theta_r} \int_{\theta_r - \Delta \theta_r}^{\theta_r + \Delta \theta_r} \langle \ldots \rangle d\theta \). The region, over which one has to average \( k_r \), depends on the parallel velocity of the resonant ions. Further away from the resonance only particles with higher energy can satisfy the resonance condition and the region, over which one has to average increases.

For strong single pass damping when the wave field can be regarded as arising from one wave and the finite \( k_\theta \) only makes a perturbation of \( k_r \), the wave field can be obtained by means of iterations replacing \( k_r = n \phi / R \) in the susceptibility tensor with \( k_r = n \phi / R + k_\theta B \phi / B + k_\theta B \phi / B \) when \( k_\theta \) is given by Eq. (3b). As a verification of that the solution consists of one propagating wave rather than a standing wave, one can compare \( k_\theta \) calculated from \( k_\theta = \left\{ \Im \partial \ln E_r / \partial \theta \right\}_{r} \) and from

\[
k_\theta^2 = -\langle E_r^{-1} \partial^2 E_r / \partial \theta^2 \rangle_r.
\]

3. Higher order FLR terms
The dielectric response for hot homogeneous plasma can be found in e.g. Ref. [6]. The spatial dispersive effects caused by FLR depend in leading order only on \( k_r \). The higher order terms depend also on the direction of \( k_r \), and are important for describing how the absorption by the \( E_+ \) and the \( E_- \) components add up. The perpendicular components of the susceptibility tensor in an arbitrary coordinate system can be written as

\[
\chi_{\perp} = \begin{pmatrix}
T + \sigma \cos 2\theta & -i \chi_{xy} - \sigma \sin 2\theta \\
T - \sigma \sin 2\theta & i \chi_{xy} - \sigma \cos 2\theta
\end{pmatrix},
\]

where \( T = (\chi_{xx} + \chi_{yy})/2, \sigma = (\chi_{xx} - \chi_{yy})/2 \). The angle \( \phi \) depends on the direction of \( k_r \), and \( \sigma \) is a constant for which \( k_y = 0 \). In the limit of small Larmor radius \( \sigma \to 0 \) and \( T \to \chi_{xx} \), the absorbtion depends only on \( |E_+| \). For finite Larmor radius the \( E_r \) component also contributes to the absorption.

When defining \( E_+ = 0.5 (E_n + i E_0) \) and \( E_- = 0.5 (E_n - i E_0) \), where \( E_n \) and \( E_0 \) are the normal and bi-normal components of the electric wave field, respectively, the absorbed power can be written as

\[
p = -2|E_+|^2 \Im(T + \chi_{xy}) - 2|E_-|^2 \Im(T - \chi_{xy}) - 4 \Im \sigma \left( \Re(E_+^* E_-) \cos 2\theta - \Im(E_+^* E_-) \sin 2\theta \right)
\]

The contributions to the absorption from the \( |E_+|^2 \) and \( |E_-|^2 \) terms do not depend on how the
coordinate system is rotated, but the contributions proportional to \( E_+ \) do. In a local coordinate system \((x, y, z)\) with \( k_y = 0 \) in which we denote the wave field by \( \vec{E} \) the absorbed power becomes

\[
p = -2 \left| \vec{E}_y \right|^2 \text{Im} \left( T + \chi_{xy} \right) - 2 \left| \vec{E}_x \right|^2 \text{Im} \left( T - \chi_{xy} \right) - 4 \text{Re} \left( \vec{E}_x \vec{E}_y \right) \text{Im} \sigma ,
\]

where \( \vec{E}_y = 0.5 \left( \vec{E}_x + i \vec{E}_y \right) \) and \( \vec{E}_x = 0.5 \left( \vec{E}_x - i \vec{E}_y \right) \). Since the absorbed power is independent of the coordinate system it follows that

\[
\text{Re} \left( \vec{E}_x \vec{E}_y \right) \cos 2\vartheta + \text{Im} \left( \vec{E}_x \vec{E}_y \right) \sin 2\vartheta = \text{Re} \left( \vec{E}_x \vec{E}_y \right).
\]

Let \( \alpha_+ \) and \( \alpha_- \) be defined by \( e^{i\alpha_+} |E_+| = 0.5 (E_n + i E_b) \) and \( e^{i\alpha_-} |E_-| = 0.5 (E_n - i E_b) \), the angle \( \vartheta \) then becomes \( \vartheta = (\alpha_+ - \alpha_-) / 2 \). Self-consistent calculations of the wave field and the susceptibility tensor can be obtained by means of iteration using a local susceptibility tensor on the form of Eq. (4) calculated from the wave field components \( E_n \) and \( E_b \) obtained from the previous iteration, starting the initial calculation with \( \vartheta = 0 \).

4. Electron damping
The electron damping by the fast magnetosonic wave arises from TTMP and ELD (electron Landau damping). The TTMP damping arises from the acceleration of the electrons by the compression of the magnetic field by the wave due to the finite magnetic moment of the electrons. Here we denote the corresponding force as the mirror force. The force is proportional to the parallel gradient of the parallel component of the compressional magnetic wave field, \( \vec{B}_|| \). The power density caused by TTMP damping is given by

\[
P_{\text{TTMP}} = 2 \left( \frac{\omega}{k_{||}} \right)^2 \frac{\sqrt{\pi}}{m_\perp} e^{\left( \frac{\omega}{k_{||}} \right)^2} \left| n_e \right|^2 \left\{ m_e w_{\perp}^2 \left| k_{||} \vec{B}_|| \right| \right\}^2 ,
\]

by ELD

\[
P_{\text{ELD}} = \left( \frac{\omega}{k_{||}} \right)^2 \frac{\sqrt{\pi}}{m_{\perp}} e^{\left( \frac{\omega}{k_{||}} \right)^2} \left| n_e \right|^2 \left| E_{||} \right|^2 ,
\]

and the cross term by

\[
P_m = \left( \frac{\omega}{k_{||}} \right)^2 \frac{\sqrt{\pi}}{m_\perp} e^{\left( \frac{\omega}{k_{||}} \right)^2} \left| n_e \right|^2 \frac{m_e w_{\perp}^2 \left| k_{||} \vec{B}_|| \right| e E_{||}}{2B} + cc ,
\]

where \( w_{\perp}^2 = 2kT_{\perp} / m_e \) and \( w_{\perp}^2 = 2kT_{\parallel} / m_e \). The TTMP appears as a FLR effect because the damping by the mirror force depends on the magnetic moment. For planar propagating waves in a homogeneous plasma in a coordinate system with \( k_y = 0 \) and the magnetic field is in the \( z \)-direction, the TTMP damping appears in the electron susceptibility tensor in the \( \chi_{yz}^{(e)} \) component acting on \( E_y \). The electron damping from ELD appears in \( \chi_{zz}^{(e)} \) component. The coupling between \( E_y \) and \( E_z \) is caused by the \( \chi_{yz}^{(e)} \) and the \( \chi_{zy}^{(e)} \) components. Coupling between the parallel electric wave field, \( E_\parallel \), and the \( E_z \) component occurs through the \( k_k k_z \) term, when neglecting this coupling one obtains

\[
E_\parallel \approx - \frac{m_e w_{\perp}^2}{2eB} \vec{\nabla}_\parallel \vec{B}_||.
\]
The parallel electric wave field component, caused by acceleration of the electrons along the magnetic field lines by the mirror force, counteracts the acceleration by the mirror force, and reduce the total damping to about half of that for TTMP alone [6]. To include the correction of the electron damping by the finite parallel electric field, when it is not included in the wave solution as in the LION code, only half of the electron susceptibility is then included.

\[
\text{Im} \chi^{(eTTMP/ELD)}_{yy} = \frac{k_z^2 w_z^2 T_z}{2 \omega_e^2 T_\parallel} \sqrt{\frac{\pi}{\omega_\parallel k_{\parallel}}} e^{-\left(\frac{\omega_\parallel}{\omega_\parallel k_{\parallel}}\right)^2} 
\]

(12)

The above procedure is justified because the Hermitian part of the dielectric tensor component \( \chi_{yy} \) is dominated by the ion contributions, and the parallel electric field has very little effect on the dispersion relation of the magnetosonic wave.

For incompressible waves, as the ideal Alfvén wave, the parallel electric field arises due to the coupling by \( k_x k_z \)-term, which we are not treated here.

Since the TTMP is a magnetic response and \( \vec{B}_\parallel \) is related to the electric wave field by \( \partial \vec{B}_\parallel / \partial t = -(\nabla \times \vec{E})_\parallel \), the response should be added to the derivatives of the electric wave field rather than the electric field itself. If one approximates \( |\nabla \times \vec{E}|^2 \approx k_\perp^2 |E_\perp|^2 \) the electron absorption can first be calculated by solving the wave equation with the anti-Hermitian part of the electron susceptibility tensor element \( \chi^{(e)}_{bb} \), given by Eq. (12), where \( b \) denote the bi normal direction and then correct it by re-evaluating \( \vec{B}_\parallel \) after solving the wave equation by multiplying the electron absorption with \( |\nabla \times \vec{E}|^2 / k_\perp^2 |E_\perp|^2 \) using \( i \omega \vec{B}_\parallel = (\nabla \times \vec{E})_\parallel \) with \( \nabla \times \vec{E} \) given by the solution. This correction factor can be used in the tensor in the next iteration allowing the wave field and the dielectric tensor to be solved consistently.

The upshift for the TTMP damping should be calculated in the same way as in section 2. The parallel wave number, \( k_\parallel \), for the transit time magnetic pumping should be evaluated from

\[
\Delta W = \mu \int_0^{\tau_\parallel} \vec{v}_\parallel \cdot \vec{\nabla} \vec{B}_\parallel dt \, ,
\]

(13)

which gives

\[
k_\parallel = \frac{B_s n_s}{B R} + \frac{B_s}{r B} \text{Im} \left[ \left( \frac{\partial \vec{B}_\parallel}{\partial l} \right)^{-1} \frac{\partial \vec{\nabla} \vec{B}_\parallel}{\partial \theta} \frac{\partial \vec{\nabla} \vec{B}_\parallel}{\partial l} \right] \, .
\]

(14)

5. Conclusions and Discussions

The proposed methods of calculating effects of spatial dispersion for the fast magnetosonic wave are well suited for FEM codes, which only describe one single wave, and allow calculations of the upshift of the parallel wave number, the cyclotron absorption including higher order FLR terms, and TTMP damping. Higher order FLR terms are important for modelling harmonic damping. The width of the resonance region depends on the parallel wave number whereas the total harmonic damping depends only weakly on it. The FLR correction should therefore be a good approximation also for weakly damped modes as long as the magnitudes of their perpendicular wave vectors do not differ too much. The FLR model in section 3 gives a positive definite absorption for distribution function decreasing with energy, and differs from the frequently used so called “reduced FLR wave equation” [7, 8] by including the absorption due to the \( E \) component. It takes into account the decrease of the absorption at the fundamental cyclotron resonance due to the FLR terms and avoids the problem of creating negative absorption. The TTMP damping is important for fast wave current drive. The correction term is found to be most important near the magnetic axis reducing the
electron damping. This correction is important for current drive, since in this region current drive has the highest efficiency because of the low fraction of trapped particles. The proposed method for calculating the upshift cannot handle weak single pass damping where the wave field consists of different waves (reflected wave). In this case one can only handle waves with large magnitudes of $n_p$ for which the upshift is negligible.

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