RESOLUTIONS AND CHARACTERS OF IRREDUCIBLE REPRESENTATIONS
OF THE $N=2$ SUPERCONFORMAL ALGEBRA

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Abstract. We evaluate characters of irreducible representations of the $N=2$ supersymmetric extension of the Virasoro algebra. We do so by deriving the BGG-resolution of the admissible $N=2$ representations and also a new “3, 5, 7, . . .”-resolution in terms of twisted massive Verma modules. We analyse how the characters behave under the automorphisms of the algebra, whose most significant part is the spectral flow transformations. The possibility to express the characters in terms of theta functions is determined by their behaviour under the spectral flow. We also derive the identity expressing every $\hat{sl}(2)$ character as a linear combination of $N=2$ characters; this identity involves a finite number of $N=2$ characters in the case of unitary representations. Conversely, we find an integral representation for the admissible $N=2$ characters as contour integrals of admissible $\hat{sl}(2)$ characters.

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1. Introduction

We consider representations of the $N=2$ supersymmetric extension of the Virasoro algebra [1] and of the affine Lie algebra $\hat{sl}(2)$ with the emphasis on the relation between the two representation theories [2]. We construct the resolutions that allow us to evaluate characters of the admissible $N=2$ representations and then show how these characters are related to the admissible $\hat{sl}(2)$ characters [3].
Expressions for some (unitary) $N=2$ characters were first proposed in [4, 5, 7] (in a closed form—in terms of theta-functions—in [5, 6]), with the modular properties of the unitary representation characters given in [4, 8]. It was noticed later on [9] that the embedding structure of $N=2$ Verma modules is not as had previously been assumed. This might have invalidated the derivation of characters, since the structure of the resolution used to derive the characters is obviously sensitive to the embedding structure.

The structure of submodules in $N=2$ Verma modules was described in [10] (or, in the equivalent language of relaxed $\hat{sl}(2)$ modules, in [3]), which has allowed the complete classification of the $N=2$ embedding diagrams [11]. There are two different types of Verma-like modules, called the massive and the topological (≡chiral) ones, of which the latter may appear as submodules of the former, but not vice versa; more precisely, it is the twisted topological Verma modules that appear as submodules. The embedding structure [11] of massive $N=2$ Verma modules is indeed more complicated than, e.g., in the well-known $\hat{sl}(2)$ Verma-module case [12, 13].

However, the admissible $N=2$ representations that we mostly concentrate on in this paper are the quotients of massive Verma modules such that at least one charged singular vector [23] is necessarily quotiented away. A crucial fact is then that the quotient of a massive Verma module over the submodule generated from a charged singular vector is a twisted topological Verma module [13, 10]. Therefore, the admissible $N=2$ representations are also the quotients of twisted topological Verma modules. Now, the embedding structure of topological Verma modules is equivalent to the embedding structure of $\hat{sl}(2)$ Verma modules; therefore, the resolutions are isomorphic and, thus, the evaluation of the $N=2$ characters is rather straightforward; in particular, it is not affected by the superfluous subsingular vectors, which for the $N=2$ algebra are an artifact of a restricted choice of vectors from which to generate the submodules (see [10] and references therein).

This equivalence between the embedding structures of $\hat{sl}(2)$ Verma modules and topological $N=2$ Verma modules follows from the fact [3] that the categories of $\hat{sl}(2)$ and $N=2$ representations are equivalent modulo the spectral flows. Categories (roughly, collections of objects, some of which are connected by arrows—morphisms) can be related by functors (“mappings” that send objects into objects and morphisms, into morphisms). Two categories are called equivalent if there is a functor $F$ between them and the inverse functor $F^{-1}$ (such that $F^{-1}F$ takes every object into an isomorphic object); a well-known example is that of (finite-dimensional) Lie algebras and connected simply-connected Lie groups. The $\hat{sl}(2)$ and $N=2$ representation categories are equivalent modulo the spectral flow; this can be formalised by introducing chains of spectral-flow-transformed modules (on which the equivalence is strict, see [3]) or by considering two representations “isomorphic” if they are related by the spectral flow (which is not an isomorphism in the usual sense!).

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1In this paper, twist and twisted refer to the spectral-flow transform, to be defined below.
2Among the recent findings related to the $N=2$ superconformal algebra, note also the calculation of semi-infinite $N=2$ cohomology, i.e., the cohomology of the critical $N=2$ string [14].
The \( \widehat{sl}(2) \) and \( N = 2 \) spectral flows are known from \([16, 4]\) (see also \([17]\)) and \([18]\), respectively. More generally, given the basic symmetry algebra (of, e.g., a conformal model) and its automorphism group, the full space of states of the theory can be taken to include the modules subjected to the automorphisms (unless these give rise to equivalent representations). For the algebras dealt with in this paper, the most significant part of the automorphisms is given by the spectral flow transformations (for short, twist, particularly when applied to representations). In the familiar case of a positive integer \( \widehat{sl}(2) \) level \( k \) (see, e.g., \([19]\) for applications to the WZW model), adding the twisted representations does not give anything new because the unitary (integrable) representations are invariant under the spectral flow with even transformation parameters, while the spectral flow with an odd parameter maps one unitary representation into another. The situation is different for rational \( k \): in that case, the theory is extended by adding the twisted representations for all \( \theta \in \mathbb{Z} \). On the \( N = 2 \) side, the significance of the spectral flow is already seen for the unitary representations, since it is needed for enumerating all the \((k + 1)(k + 2)/2\) unitary representations; also, submodules of the topological Verma modules are necessarily the twisted topological Verma modules.

The characters must carry a representation of the automorphism group. As regards the spectral flow transform, this requirement (Eqs. (2.9) and (2.22) for the \( \widehat{sl}(2) \) and \( N = 2 \) characters, respectively) is very useful in deriving certain properties of characters (in this respect, it is similar to the modular ‘covariance’ condition; note that the modular and the spectral flow transformations are not completely independent, they are combined by taking a semidirect product). On the \( \widehat{sl}(2) \) side, the general transformation properties are illustrated by the known characters, while on the \( N = 2 \) side, the behaviour under the spectral flow is crucial as regards rewriting the characters in terms of theta-functions.

There is a subtlety, partly of a terminological nature. When dealing with characters, one usually defines them as formal Laurent series; in some cases, such a series determines a holomorphic or meromorphic function, or a distribution. In the cases we are interested in (particularly for the \( \widehat{sl}(2) \) representations), the series may converge once the variables belong to some complex domain, in which case the character would determine a meromorphic function. This function may, however, have different series expansions in different domains, which in general correspond to characters of different modules (such as those obtained by the action of the automorphism group). We, thus, have sometimes to distinguish between the character defined as a series and the function to which this series converges inside a certain domain (but which can be continued outside of it). To stress this point, we will sometimes refer to the meromorphic function representing the character as the character function.

Some character functions may be invariant under the spectral flow. This is the case with the unitary \( N = 2 \) and \( \widehat{sl}(2) \) representations, which are invariant under the spectral flow transformations with the parameter \( \theta = k + 2 \) and \( \theta = 2 \), respectively. This shows up in the quasiperiodicity of the theta functions through which these character functions are expressed.\(^3\) On the unitary representations, the orbits of

\(^3\)On the other hand, the admissible \( \widehat{sl}(2) \) characters are invariant under the spectral flow with \( \theta = 2p' \) (where \( k + 2 = \frac{2p'}{p} \)), whereas the representations are not, which is in accordance with the fact that a chosen character function gives characters of different modules (in the present case, of spectral-flow-transformed modules) when expanded in different domains.
the $N=2$ spectral flow have length $k + 2$, with inequivalent unitary representation filling a half of the boxes in the table on p. [23] (into which the Kač table is included as the first column). At the same time, the $\widehat{sl}(2)$ spectral flow orbits on unitary representations are of length 2; in accordance with the equivalence theorem, however, both algebras have the same number $\left\lfloor \frac{k+2}{2} \right\rfloor$ of the equivalence classes of unitary representations modulo the spectral flow.

Including the twisted (spectral-flow-transformed) representations in both the $\widehat{sl}(2)$ and $N=2$ theories is crucial in order to derive the essential equivalence of the corresponding representation categories [2]. An immediate consequence of the equivalence, as we see in Sec. [5.2], is that the $\widehat{sl}(2)$ characters are given by infinite sums of the $N=2$ characters transformed by the spectral flow with all $\theta \in \mathbb{Z}$:

$$
\chi_{j,t}(z, q) \vartheta_{1,0}(z, q) = \sum_{\theta \in \mathbb{Z}} \omega_{-\frac{2\theta}{z}} \left( y^{1-q^{-\theta}}, q \right) y^{\theta} z^{-\theta} q^{\frac{2\theta}{z} + \frac{\theta^{2}}{2} + \frac{\theta^{2} - 2j\theta}{2} + \frac{1}{2}},
$$

which can also be viewed as a “branching” of $\widehat{sl}(2)$ characters into $N=2$ ones (or, a “sumrule” for the $N=2$ characters). This applies to any $N=2$ representation; in particular, in the simple case where these are taken as the topological Verma modules, this formula becomes the “truly remarkable identity” from [3] (where it was derived using different representation-theory considerations).

More identities are obtained by taking other representations. For the unitary ones, the above-mentioned invariance under the spectral flow with $\theta = k + 2$ on the $N=2$ side results in that the identity contains only a finite number (precisely $k + 2$) of different $N=2$ characters, their linear combination giving a unitary $\widehat{sl}(2)$ character (see [5.23]). For the admissible representations, on the other hand, the identity contains an infinite number of $N=2$ characters. The reason is that neither the admissible $N=2$ representations themselves nor their character functions are periodic under the spectral flow. This also implies that the admissible $N=2$ characters cannot be algebraically expressed in terms of the theta functions. However, there is an integral representation for the admissible $N=2$ character functions $\omega_{r,s,p,p'}^3$ through those of the corresponding (hence, admissible) $\widehat{sl}(2)$ representations,

$$
\omega_{r,s,p,p'}^3(y, q) y^\delta_p (r-1+1-s) = \frac{1}{2\pi i} \int \frac{dz}{z} \vartheta_{1,0}(zy, q) q^{-\frac{z}{p} pp'} \times
$$

$$
\times \left( \vartheta_{1,0}(zp q^{-pp'+rp'-(s-1)p}, q^{2pp'}) - z^{-r} q^{r(s-1)} \vartheta_{1,0}(zp q^{-pp'-rp'-(s-1)p}, q^{2pp'}) \right),
$$

see (5.21) for more details; in fact, a similar representation exists for any $N=2$ character, see (5.14).

The means to derive characters of irreducible representations is provided by resolutions of these representations in terms of Verma modules. As has already been said, the $N=2$ BGG-resolution in terms of only (twisted) topological Verma modules is of the same structure as the well-known $\widehat{sl}(2)$ resolution [28]. However, using the massive $N=2$ Verma modules opens up other possibilities.

To arrive at a resolution in terms of massive Verma modules, we start with the $N=2$ embedding diagrams [11]: these contain, generically, modules of both types, the twisted topological and the massive ones. We then rewrite the embedding diagram as an exact sequence (in particular, the mappings between modules acquire kernels, thereby no longer being embeddings); this ‘re-interpretation’ is not
completely automatic for sufficiently complicated embedding diagrams (as, e.g., the $\text{III}^6_2(2, -+)$ one, which we explicitly consider in what follows). Next, we replace every topological Verma module with its resolution in terms of massive Verma modules. Examining the resulting mappings, we can read off the resolution of a new type, with the number of modules growing from term to term as $3, 5, 7, \ldots$, see (3.9).

All these structures exist also in the $\hat{\mathfrak{sl}}(2)$ guise: as explained in [2, 11], the $\hat{\mathfrak{sl}}(2) \leftrightarrow N=2$ equivalence implies the following “dictionary” between the objects of the two representation theories:

| $\mathfrak{sl}(2)$ Verma module | $N=2$ |
|---------------------------------|-------|
| (twisted) Verma module          | twisted topological (equivariant) Verma module |
| singular vector in Verma module | topological singular vector |
| (twisted) relaxed Verma module | (twisted) massive Verma module |
| unitary representations $(k+1$ inequivalent representations for level $k \in \mathbb{N}$) | unitary representations $(k+1)(k+2)/2$ inequivalent representations for central charge $c = 3(1 - 2/(k+2))$ |
| periodicity 2 under the spectral flow for the unitary representations | periodicity $k+2$ under the spectral flow for the unitary representations |

In what follows, we add to the list the admissible representations of each algebra. However, we do not consider the relaxed $\hat{\mathfrak{sl}}(2)$ Verma modules here, although their $N=2$ counterpart will play a role in our constructions.

In Sec. 2, we introduce the $\hat{\mathfrak{sl}}(2)$ and $N=2$ algebras and the respective spectral flows. In Sec. 3, we recall the BGG resolution of the $\hat{\mathfrak{sl}}(2)$ admissible representations and then derive the BGG resolution of the $N=2$ admissible representations and a new resolution consisting entirely of massive Verma modules. In Sec. 4, we evaluate the admissible representation characters and find how they behave under the spectral flow (for each of the two algebras). We also consider in some detail the unitary representations, which are interesting to us here primarily because they are periodic under the spectral flow, which has a number of consequences. In Sec. 5, we relate the $N=2$ and $\hat{\mathfrak{sl}}(2)$ characters using the equivalence of categories, which implies “sum rules” and an integral representation for the $N=2$ characters. Appendix A summarizes our theta-function conventions.

**Notation.** We use the following notations for various modules:

- (twisted) topological $N=2$ Verma modules, $\mathfrak{V}$, $\mathfrak{V}$,
- $\mathfrak{sl}(2)$ Verma modules, $\mathfrak{V}$,
- (twisted) massive $N=2$ Verma modules, $\mathfrak{U}$,
- $\hat{\mathfrak{sl}}(2)$ admissible representations, $\mathfrak{J}$,
- $\hat{\mathfrak{sl}}(2)$ unitary representations, $\mathfrak{J}$.

We also write $\omega$ and $\chi$ for the $N=2$ and $\hat{\mathfrak{sl}}(2)$ characters, respectively.

2. **The algebras, their Verma modules, and spectral flows**

In this section, we introduce the $\hat{\mathfrak{sl}}(2)$ and $N=2$ algebras and show how the respective spectral flows act between representations and, thus, act also on characters.
2.1. The \( \hat{\mathfrak{sl}}(2) \) side.

2.1.1. The algebra and the spectral flow. The affine \( \mathfrak{sl}(2) \) algebra is defined by the commutation relations

\[
[j^0_m, j^+_n] = \pm j^+_m j^0_{m+n}, \quad [j^0_m, j^-_n] = \frac{K}{\pi} m \delta_{m+n,0},
\]

\[(2.1)\]

\[
[j^-_m, j^+_n] = Km \delta_{m+n,0} + 2j^0_{m+n},
\]

\[(2.2)\]

with \( K \) being the central element, whose eigenvalue will be denoted by \( t - 2 \) with \( t \in \mathbb{C} \setminus \{0\} \).

For \( \theta \in \mathbb{Z} \), we have the automorphisms given by the spectral flow transformations \( \hat{U} \) (a general definition can be found in [16])

\[
\hat{U}_\theta : \quad j^+_n \mapsto j^+_{n+\theta}, \quad j^-_n \mapsto j^-_{n-\theta}, \quad j^0_n \mapsto j^0_n + \frac{\theta}{2} \delta_{n,0}.
\]

\[(2.3)\]

We also have the involutive automorphism

\[
T : \quad j^+_n \mapsto j^-_n, \quad j^-_n \mapsto j^+_n, \quad j^0_n \mapsto -j^0_n.
\]

\[(2.4)\]

These satisfy

\[
U_\theta U_{\theta'} = U_{\theta + \theta'}, \quad U_0 = T^2 = 1, \quad U_\theta T = T U_{-\theta}.
\]

\[(2.5)\]

The automorphism group is thus the semidirect product \( \text{Aut} = \mathbb{Z} \ltimes \mathbb{Z}_2 \). It contains the affine Weyl group \( (2\mathbb{Z}) \ltimes \mathbb{Z}_2 \) as the inner automorphisms. We also have the involutive anti-automorphism

\[
T' : \quad j^+_n \mapsto j^-_n, \quad j^-_n \mapsto j^+_n, \quad j^0_n \mapsto j^0_n
\]

\[(2.6)\]

that commutes with the group \( \text{Aut} \).

The character \( \chi^\mathfrak{c}_t \equiv \chi^\mathfrak{c} \) of a module \( \mathfrak{c}_t \) with a definite value of the level \( K = t - 2 \) is defined by

\[
\chi^\mathfrak{c}_t(z, q) = \text{Tr}_{\mathfrak{c}_t}(q^{L_0} z^{J^0_0}),
\]

\[(2.7)\]

where \( L_0 \) is the zero mode of the Sugawara energy-momentum tensor. We assume here a sesquilinear (Shapovalov) form such that for any \( A \in \hat{\mathfrak{sl}}(2) \) we have \( A^\dagger = T'A \). As to \( q \) and \( z \), these are complex variables and we assume \( |q| < 1 \) in what follows. The series implied in \( (2.7) \) would in general converge only in some domain depending on the module under consideration. One often divides \( \chi^\mathfrak{c}_t(z, q) \) by \( z^j q^\Delta \), where \( j \) and \( \Delta \) are the charge and the Sugawara dimension of the highest-weight vector, but we do not do it here.

Elements of the automorphism group \( \text{Aut} \) can be applied to any \( \hat{\mathfrak{sl}}(2) \) module \( \mathfrak{c} \); we thus obtain the modules \( U_\theta \mathfrak{c} = \mathfrak{c}_\theta \) and \( T \mathfrak{c} \) (the former are called the 'twisted' modules). Hence the automorphism group also acts on character functions of \( \hat{\mathfrak{sl}}(2) \) modules:

\[
U_\theta \chi^\mathfrak{c}(z, q) = \chi^\mathfrak{c}_\theta(z, q), \quad T \chi^\mathfrak{c}(z, q) = \chi^{T\mathfrak{c}}(z, q).
\]

\[(2.8)\]

For a module with a definite level \( K = t - 2 \), we write \( \chi^\mathfrak{c}_{t,\theta}(z, q) \) for \( U_\theta \chi^\mathfrak{c}_t(z, q) \).

**Lemma 2.1.** Let \( \mathfrak{c} \equiv \mathfrak{c}_t \) be an \( \hat{\mathfrak{sl}}(2) \) module with the level \( K = t - 2 \). Then its character function transforms under the spectral flow transform and the involution as

\[
\chi^\mathfrak{c}_{t,\theta}(z, q) = q^{-\frac{1-\theta}{2} \Delta} z^{\frac{1-\theta}{2} \Delta} \chi^\mathfrak{c}_t(z q^{-\theta}, q), \quad T \chi^\mathfrak{c}_t(z, q) = \chi^\mathfrak{c}_t(z^{-1}, q).
\]

\[(2.9)\]
If, further, the series for \( \chi_{t,\theta}(z, q) \) converges in an annulus \( \mathcal{A} \), the transformed character converges in the annulus

\[
U_\theta \mathcal{A} = \mathcal{A}_\theta = \{ z \mid z q^{-\theta} \in \mathcal{A} \}, \quad T \mathcal{A} = \mathcal{A}^{-1} = \{ z \mid z^{-1} \in \mathcal{A} \}.
\]

2.1.2. Verma modules and twisted Verma modules. The action of the spectral flow on Verma modules gives twisted Verma modules \( \mathfrak{M}_{j,t;\theta} \), which are described as follows. For \( \theta \in \mathbb{Z} \), the twisted Verma module \( \mathfrak{M}_{j,t;\theta} \) is freely generated by \( J^+ \leq \theta - 1 \), \( J^- \leq -\theta \), and \( J^0 \leq -1 \) from a twisted highest-weight vector \( |j, t; \theta \rangle_{s\ell(2)} \) defined by

\[
\begin{align*}
J^+_\geq \theta |j, t; \theta \rangle_{s\ell(2)} &= J^0_{\geq 1} |j, t; \theta \rangle_{s\ell(2)} = J^-_{\geq -\theta + 1} |j, t; \theta \rangle_{s\ell(2)} = 0, \\
\left(J^0_{\theta} + \frac{t-2\theta}{2}\right) |j, t; \theta \rangle_{s\ell(2)} &= j |j, t; \theta \rangle_{s\ell(2)} ,
\end{align*}
\]

The highest-weight vector \( |j, t; \theta \rangle_{s\ell(2)} \) of the twisted Verma module has the Sugawara dimension

\[
\Delta(j, t; \theta) = \frac{j^2+2j}{t} - \theta j + \frac{t-2\theta^2}{4}.
\]

Setting \( \theta = 0 \) in the above formula gives the usual (‘untwisted’) Verma modules. We define \( |j, t \rangle_{s\ell(2)} = |j, t; 0 \rangle_{s\ell(2)} \), and similarly, denote \( \mathfrak{M}_{j,t} = \mathfrak{M}_{j,t;0} \).

The character of a Verma module \( \mathfrak{M}_{j,t} \) converges for \( |q| < 1 \) and \( z \in \mathcal{A} \),

\[
\mathcal{A} = \{ 1 < |z| < |q|^{-1} \},
\]

where it can be summed to the character function

\[
\chi_{s\ell(2)}^{\mathfrak{m}}(z, q) = \frac{q^{j^2+2j/4} z^j}{\eta_{1,1}(z, q)}
\]

(see the Appendix for our conventions on theta functions). To obtain occupation numbers in each grade, one should expand (2.13) assuming that \( z \in \mathcal{A} \). As to the behaviour under the spectral flow, the character of a twisted Verma module converges in

\[
\mathcal{A}_\theta = \{ z \mid |q|^\theta < |z| < |q|^{\theta-1} \},
\]

where Eq. (2.9) allows us to find

\[
\chi_{s\ell(2)}^{\mathfrak{m}}(z, q) = (-1)^\theta q^{\frac{j^2+1}{4} - \theta (j+\frac{1}{4})} z^{-\frac{j^2}{4}} \chi_{s\ell(2)}^{\mathfrak{m}}(z, q).
\]

To obtain occupation numbers in each grade, this should be expanded assuming that \( z \) is inside the annulus \( \mathcal{A}_\theta \). We also assume \( |q| < 1 \) everywhere in what follows.

2.2. The \( N = 2 \) side. We now introduce the \( N = 2 \) superconformal algebra and the corresponding analogues of the structures considered above for \( \hat{s\ell(2)} \). Note that the proper \( N = 2 \) counterpart of the \( \hat{s\ell(2)} \) Verma modules are the topological Verma modules considered in Sec. 2.2.2, while the massive Verma modules introduced in Sec. 2.2.3 are somewhat different objects, whose \( s\ell(2) \) analogues are not considered here.
2.2.1. The algebra and its automorphisms. The $N=2$ superconformal algebra is taken in the basis where

\[ [\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad [\mathcal{H}_m, \mathcal{H}_n] = \frac{c}{3} m \delta_{m+n,0}, \]
\[ [\mathcal{L}_m, \mathcal{G}_n] = (m-n)\mathcal{G}_{m+n}, \quad [\mathcal{H}_m, \mathcal{G}_n] = \mathcal{G}_{m+n}, \]
\[ (2.15) \]
\[ [\mathcal{L}_m, \mathcal{Q}_n] = -n\mathcal{Q}_{m+n}, \quad [\mathcal{H}_m, \mathcal{Q}_n] = -\mathcal{Q}_{m+n}, \]
\[ [\mathcal{L}_m, \mathcal{H}_n] = -n\mathcal{H}_{m+n} + \frac{c}{6}(m^2 + m)\delta_{m+n,0}, \]
\[ \mathcal{G}_{m+n} = 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2 + m)\delta_{m+n,0}, \]

and $m, n \in \mathbb{Z}$. In what follows, we do not distinguish between the central element $C$ and its eigenvalue $c$, which we assume to be $c \neq 3$ and parametrise as $c = 3(1 - \frac{\delta}{2})$ with $t \in \mathbb{C} \setminus \{0\}$.

The automorphisms of the $N=2$ algebra consist of the spectral flow \[18\] and the involutive automorphism. In the basis chosen in \[(2.15)\], the spectral flow acts as

\[ \tilde{U}_\theta : \]
\[ \begin{align*}
\mathcal{L}_n &\mapsto \mathcal{L}_n + \theta \mathcal{H}_n + \frac{c}{6}(\theta^2 + \theta)\delta_{n,0}, \\
\mathcal{H}_n &\mapsto \mathcal{H}_n + \frac{c}{3}\theta\delta_{n,0}, \\
\mathcal{G}_n &\mapsto \mathcal{G}_n + \theta, \\
\mathcal{Q}_n &\mapsto \mathcal{Q}_{n-\theta}, \\
\mathcal{Q}_n &\mapsto \mathcal{Q}_{n+\theta},
\end{align*} \tag{2.16} \]

where $\theta \in \mathbb{Z}$. The involutive automorphism is given by

\[ \bar{T} : \]
\[ \begin{align*}
\mathcal{G}_n &\mapsto \mathcal{Q}_n, \\
\mathcal{Q}_n &\mapsto \mathcal{G}_n, \\
\mathcal{H}_n &\mapsto -\mathcal{H}_n - \frac{c}{3}\delta_{n,0}, \\
\mathcal{L}_n &\mapsto \mathcal{L}_n - n\mathcal{H}_n.
\end{align*} \tag{2.17} \]

As in the $\hat{sl}(2)$ case, the operations $\tilde{U}_\theta$ and $\bar{T}$ satisfy \[(2.5)\] and thus constitute the group of automorphisms $\text{Aut} = \mathbb{Z} \times \mathbb{Z}_2$. We also have an involutive \text{anti-}automorphism

\[ \bar{T}' : \]
\[ \begin{align*}
\mathcal{G}_n &\mapsto \mathcal{Q}_{-n}, \\
\mathcal{Q}_n &\mapsto \mathcal{G}_{-n}, \\
\mathcal{H}_n &\mapsto \mathcal{H}_{-n}, \\
\mathcal{L}_n &\mapsto \mathcal{L}_{-n} + n\mathcal{H}_{-n},
\end{align*} \tag{2.18} \]

which commutes with the group $\text{Aut}$.

We introduce the characters as

\[ \omega^\mathcal{D}(z, q) = \text{Tr}_\mathcal{D}(z^\mathcal{H}_0 q^C) \tag{2.19} \]

and write $\omega^\mathcal{D}_\theta(z, q) \equiv \bar{T}' \omega^\mathcal{D}(z, q)$ for a module $\mathcal{D}_t$ with a definite central charge $c = 3(1 - \frac{\delta}{2})$. We assume here a sesquilinear form \[23\] such that for any element $A$ from the $N=2$ algebra we have $A^\dagger = \bar{T}' A$ or equivalently,

\[ \begin{align*}
(|y\rangle, \mathcal{G}_n|z\rangle) &= (\mathcal{G}_{-n}|y\rangle, |z\rangle), \\
(|y\rangle, \mathcal{Q}_n|z\rangle) &= (\mathcal{Q}_{-n}|y\rangle, |z\rangle), \\
(|y\rangle, \mathcal{L}_n|z\rangle) &= ((\mathcal{L}_{-n} + n\mathcal{H}_{-n})|y\rangle, |z\rangle), \\
(|y\rangle, \mathcal{H}_n|z\rangle) &= (\mathcal{H}_{-n}|y\rangle, |z\rangle),
\end{align*} \tag{2.20} \]

for $|y\rangle$ and $|z\rangle$ from the module.

The modules are mapped under automorphisms, which we denote as $\mathcal{D}_{t,\theta} \equiv \tilde{U}_\theta \mathcal{D}_t$ in the case of the spectral flow. Therefore, the automorphisms act on characters, for which we write $\omega^\mathcal{D}_{t,\theta} \equiv \tilde{U}_\theta \omega^\mathcal{D}_t$.

**Lemma 2.2.** Let $\mathcal{D} \equiv \mathcal{D}_t$ be an $N=2$ module with a definite central charge $c = 3(1 - \frac{\delta}{2})$. Then the character function of the spectral-flow transformed module $\mathcal{D}_{t,\theta}$ is

\[ \omega^\mathcal{D}_{t,\theta}(z, q) = z^{-\frac{\delta}{2} \theta q^c(\theta^2 - \theta)} \omega^\mathcal{D}_t(z q^{-\theta}, q). \tag{2.22} \]
For the involutive automorphism, similarly,
\begin{equation}
\bar{T} \omega^D_t(z, q) = z^{-\frac{c}{3}} \omega^D_t(z^{-1}, q).
\end{equation}

Remark 2.3. For a given \( \omega^D_t(z, q) \), Eq. (2.22) also allows one to determine the characters transformed by the spectral flow with half-integral \( \theta \), in particular \( \theta = \frac{1}{2} \), thereby recovering the "NS sector" of the algebra. We will not repeat this point and keep on working with the characters that directly pertain to the algebra written in the basis (2.13).

Note also that in the \( N=2 \) context, it is not necessary to distinguish between characters (defined via formal series) and character functions, since the series converge for \( z \in \mathbb{C} \) and, thus, the characters are holomorphic functions.

2.2.2. Topological Verma modules. We now consider the class of \( N=2 \) Verma modules that we call topological Verma modules following \( 2 \). We give the definitions applicable to the twisted case as well. First, for a fixed \( \theta \in \mathbb{Z} \), we define the twisted topological highest-weight vector \( |h, t; \theta\rangle_{\text{top}} \) to satisfy the annihilation conditions
\begin{equation}
Q_{-\theta+m} |h, t; \theta\rangle_{\text{top}} = G_{\theta+m} |h, t; \theta\rangle_{\text{top}} = L_{m+1} |h, t; \theta\rangle_{\text{top}} = H_{m+1} |h, t; \theta\rangle_{\text{top}} = 0, \quad m \in \mathbb{N}_0,
\end{equation}
with the following eigenvalues of the Cartan generators (where the second equation follows from the annihilation conditions):
\begin{align}
(\mathcal{H}_0 + \frac{c}{3} \theta) |h, t; \theta\rangle_{\text{top}} &= h |h, t; \theta\rangle_{\text{top}}, \\
(\mathcal{L}_0 + \theta \mathcal{H}_0 + \frac{c}{6} (\theta^2 + \theta)) |h, t; \theta\rangle_{\text{top}} &= 0.
\end{align}

Definition 2.4. The twisted topological Verma module \( V_{h,t;\theta} \) is the module freely generated from the topological highest-weight vector \( |h, t; \theta\rangle_{\text{top}} \) by \( Q_{\leq -1-\theta}, G_{\leq -1+\theta}, L_{\leq -1}, \) and \( H_{\leq -1} \).

The twisted topological highest-weight vectors \( |h, t; \theta\rangle_{\text{top}} \) are defined in accordance with the action of \( \bar{U} \), so that \( \bar{U} \theta |h, t; \theta\rangle_{\text{top}} = |h, t; \theta + \theta'\rangle_{\text{top}} \). We write \( |h, t\rangle_{\text{top}} \equiv |h, t; 0\rangle_{\text{top}} \) in the ‘untwisted’ case of \( \theta = 0 \) and also denote by \( V_{h,t} \equiv V_{h,t;0} \) the untwisted module.

The character of the topological Verma module \( V_{h,t} \) converges for \( |q| < 1 \) and \( z \in \mathbb{C} \) and can then be summed to
\begin{equation}
\omega^V_{h,t}(z, q) = z^h \frac{\theta_{1,0}(z, q)}{(1 + z^{-1}) \eta(q)^3}.
\end{equation}

The character function of a twisted topological Verma module, thus, reads as
\begin{equation}
\omega^V_{h,t;\theta}(z, q) = z^{h + \frac{2\theta}{3}} q^{-h\theta - \frac{\theta^2 - \theta}{3}} \frac{\theta_{1,0}(z, q)}{(1 + z^{-1} q^{\theta}) \eta(q)^3}.
\end{equation}

\(^4\)chiral, in a different set of conventions, see, e.g., \( 22 \).

\(^5\)The untwisted modules are in a certain sense more “rare” in the \( N=2 \) context than in the \( \widehat{\mathfrak{s}\ell}(2) \) one, because submodules of a topological Verma module are always the twisted topological Verma modules \( 10 \).
2.2.3. **Massive Verma modules over the \( N = 2 \) algebra.** A different class of Verma-like \( N = 2 \) modules are defined as follows \([10]\).

**Definition 2.5.** A twisted massive Verma module \( \mathcal{U}_{h,\ell,t;\theta} \) is freely generated from a twisted massive highest-weight vector \( |h,\ell,t;\theta\rangle \) by the generators

\[
\mathcal{L}_m, \ m \in \mathbb{N}, \quad \mathcal{H}_m, \ m \in \mathbb{N}, \quad \mathcal{Q}_m, \ m \in \mathbb{N}_0, \quad \mathcal{G}_m, \ m \in \mathbb{N}.
\]

The massive highest-weight vector \( |h,\ell,t;\theta\rangle \) satisfies the following conditions:

\[
(2.30) \quad \mathcal{Q}_{m+1-\theta} |h,\ell,t;\theta\rangle = \mathcal{G}_{m+\theta} |h,\ell,t;\theta\rangle = \mathcal{L}_{m+1} |h,\ell,t;\theta\rangle = \mathcal{H}_{m+1} |h,\ell,t;\theta\rangle = 0, \ m \in \mathbb{N}_0,
\]

\[
\begin{align*}
& (\mathcal{H}_0 + \xi(t) |h,\ell,t;\theta\rangle = h |h,\ell,t;\theta\rangle, \\
& (\mathcal{L}_0 + \theta \mathcal{H}_0 + \xi(t^2 + \theta)) |h,\ell,t;\theta\rangle = \ell |h,\ell,t;\theta\rangle.
\end{align*}
\]

We also write \( |h,\ell,t\rangle = |h,\ell,t;0\rangle \) and \( \mathcal{U}_{h,\ell,t} = \mathcal{U}_{h,\ell,t;0} \).

The character of the twisted massive Verma module is given by

\[
\omega_{h,\ell,t;\theta}(z) = z^{h+\frac{2\theta}{t}} q^{-\theta h - \frac{\theta^2}{t}} \frac{\eta(zq)}{\eta(z)}.
\]

A charged singular vector occurs in \( \mathcal{U}_{h,\ell,t} \) whenever \( \ell = \ell_{ch}(n,h,t) \equiv -n(h - \frac{n+1}{t}), \ n \in \mathbb{Z} \), and reads as \([15, 10]\)

\[
(2.33) \quad |E(n,h,t)\rangle_{ch} = \begin{cases} \\
\mathcal{Q}_{-n} \ldots \mathcal{Q}_0 |h,\ell_{ch}(n,h,t),t\rangle, & n \geq 0, \\
\mathcal{G}_n \ldots \mathcal{G}_{-1} |h,\ell_{ch}(n,h,t),t\rangle, & n \leq -1.
\end{cases}
\]

This state on the extremal diagram \([10]\) satisfies the twisted topological highest-weight conditions with the twist \( \theta = n \), the submodule generated from \( |E(n,h,t)\rangle_{ch} \) being the twisted topological Verma module \( \mathcal{U}_{h,\ell,t;\theta} \) if \( n \geq 0 \) and \( \mathcal{U}_{h,\ell,t;\theta} \) if \( n \leq -1 \).

3. **Some \( \hat{\mathfrak{sl}}(2) \) and \( N = 2 \) resolutions**

For the rest of the paper, we fix two coprime positive integers \( p \) and \( p' \) and parametrise \( t = \frac{p}{p'} \).

For positive rational \( t \), we consider (twisted) topological Verma modules with dominant highest-weights. Such a module is not embedded into another (twisted) topological Verma module and admits infinitely many singular vectors. We define the admissible \( N = 2 \) representations to be the (irreducible) quotients of these Verma modules over the maximal submodules.

The functors from \([2]\) (which we review in Sec. \([5.4]\)) relate the \( N = 2 \) and \( \hat{\mathfrak{sl}}(2) \) admissible representations to each other (note that we extend “admissible representations” to mean also all the integral twists of the “standard” admissible \( \hat{\mathfrak{sl}}(2) \) representations \([2]\)). Therefore, the embedding diagrams of the \( N = 2 \) topological Verma modules are isomorphic to the embedding diagrams of the \( \hat{\mathfrak{sl}}(2) \) Verma modules and, thus, the corresponding resolutions are isomorphic as well (as objects in the category of exact sequences). In addition to the thus obtained BGG-resolution, the admissible \( N = 2 \) representations possess “massive” resolutions constructed in the category of twisted massive Verma modules.

3.1. **The standard BGG resolution**
3.1.1. The $\tilde{\mathfrak{sl}}(2)$ side. As is well-known [14], the admissible representation $\mathcal{J}_{r,s,p,p'}$ is the quotient over the maximal submodule of the Verma module $M_{j(r,s,p,p'),p'}$, where

\begin{equation}
2j(r,s,\frac{p'}{p}) + 1 = r - (s-1)\frac{p'}{p}, \quad 1 \leq r \leq p - 1, \quad 1 \leq s \leq p'.
\end{equation}

The BGG-resolution for the admissible $\tilde{\mathfrak{sl}}(2)$ representations is the exact sequence

\begin{equation}
\cdots \to \mathcal{N}_-(m) \oplus \mathcal{N}_+(m) \to \cdots \to \mathcal{N}_-(2) \oplus \mathcal{N}_+(2) \to \mathcal{N}_-(1) \oplus \mathcal{N}_+(1) \to \mathcal{N} \to \mathcal{I} \to 0,
\end{equation}

where $\mathcal{I} \approx \mathcal{J}_{r,s,p,p'}$ is the irreducible representation in question, $\mathcal{N} \approx M_{\frac{r-1}{2} - \frac{p'}{p}(s-1),\frac{p'}{p}}$ is the Verma module whose quotient is $\mathcal{J}_{r,s,p,p'}$, and

\begin{equation}
\begin{align*}
\mathcal{N}_+(2m) &\approx M_{\frac{r-1}{2} - \frac{p'}{p}(s-1) - pm,\frac{p'}{p}}, & \mathcal{N}_-(2m) &\approx M_{\frac{r-1}{2} - \frac{p'}{p}(s-1) + pm,\frac{p'}{p}}, \\
\mathcal{N}_+(2m-1) &\approx M_{\frac{r-1}{2} - \frac{p'}{p}(s-1) - p(m-1),\frac{p'}{p}}, & \mathcal{N}_-(2m-1) &\approx M_{\frac{r-1}{2} - \frac{p'}{p}(s-1) + p(m-1),\frac{p'}{p}},
\end{align*}
\end{equation}

are the Verma modules embedded into $\mathcal{N}$.

In order to enumerate all the admissible representations (including, in accordance with our conventions, the spectral-flow-transformed ones), we should take all the twists $\mathcal{J}_{r,s,p,p';\theta}$, $\theta \in \mathbb{Z}$, of $\mathcal{J}_{r,s,p,p'}$. This then results in that every module in $\mathcal{N}_+(\mathfrak{sl}_2)$ acquires the additional twist $\theta$.

3.1.2. The $N = 2$ side. All the $N = 2$ admissible representations (which we denote as $\mathcal{J}_{r,s,p,p';\theta}$) can be obtained as the quotients of twisted topological Verma modules $\mathfrak{U}_{h(r,s,\theta)}$, where $\theta \in \mathbb{Z}$ and

\begin{equation}
h(r,s,t) = \frac{-r-1}{t} + s - 1, \quad t = \frac{p'}{p}, \quad 1 \leq r \leq p - 1, \quad 1 \leq s \leq p'.
\end{equation}

**Theorem 3.1.** There is an exact sequence of $N = 2$ representations given by $\mathfrak{U}_h$, where $\mathcal{I} \approx \mathcal{J}_{r,s,p,p';\theta}$ is an admissible representation, $\mathcal{N} \approx \mathfrak{U}_{-\frac{r'}{p}(r-1)+s-1,\frac{p'}{p},\theta}$ is a twisted topological Verma module, and, for $m \in \mathbb{N}$,

\begin{align*}
\mathcal{N}_+(2m) &\approx \mathfrak{U}_{-\frac{r'}{p}(r-1)+s-1+2pm,\frac{p'}{p},\theta-mp}, \\
\mathcal{N}_+(2m-1) &\approx \mathfrak{U}_{-\frac{r'}{p}(r-1)+s-1+2p(m-1),\frac{p'}{p},\theta-(m-1)p}, \\
\mathcal{N}_-(2m) &\approx \mathfrak{U}_{-\frac{r'}{p}(r-1)+s-1-2pm,\frac{p'}{p},\theta+mp}, \\
\mathcal{N}_-(2m-1) &\approx \mathfrak{U}_{-\frac{r'}{p}(r-1)+s-1-2p(m-1),\frac{p'}{p},\theta-r-(m-1)p},
\end{align*}

are twisted topological Verma modules.

**Sketch of the Proof.** The only difference from the $\tilde{\mathfrak{sl}}(2)$ case is that the resolution consists of twisted topological Verma modules with different twists. Although the submodules in a twisted topological Verma module $\mathfrak{U}$ appear simultaneously with submodules in the corresponding $\tilde{\mathfrak{sl}}(2)$ Verma module $M$, ....
yet the submodules in $\mathcal{V}$ are necessarily twisted (all with different twists, in general):

\[\begin{align*}
N = 2 & \quad \text{and} \\
\hat{\mathfrak{s}}\ell(2) &
\end{align*}\]

The equivalence statement does not fix the twists, and these have to be found separately. They are determined by the properties of topological singular vectors. To a singular vector $|MFF(r, s, t)\rangle_{\pm}$, $r, s \in \mathbb{N}$ (see [2] for the definition in the current notations) in the $\hat{\mathfrak{s}}\ell(2)$ Verma module $\mathfrak{M}_{j,t}$, there corresponds the topological singular vector $|E(r, s, t)\rangle_{\pm}$, $r, s \in \mathbb{N}$ (see [2] for the definition in the current notations) in the $\hat{\mathfrak{s}}\ell(2)$ Verma module $\mathfrak{M}_{j,t}$, that satisfies the $\theta = \mp r$-twisted topological highest-weight conditions:

\[Q_{\geq r}|E(r, s, t)\rangle_{\pm} = G_{\geq r}|E(r, s, t)\rangle_{\pm} = L_{\geq 1}|E(r, s, t)\rangle_{\pm} = H_{\geq 1}|E(r, s, t)\rangle_{\pm} = 0\]

(3.6)

(3.7)

over the maximal submodule. The embedding diagrams of the Verma modules with parameters (3.7) are different for $s = 1$ and $2 \leq s \leq p'$. We now concentrate on the case of $s = 1$ (where the embedding
diagram is more complicated). Then the massive Verma module is included into the following diagram of morphisms of twisted massive Verma modules:

\[
\cdots \longrightarrow \bigoplus_{b=-a}^{a} \mathcal{U}(a, b) \longrightarrow \cdots \longrightarrow \mathcal{U}(1, 1) \oplus \mathcal{U}(1, 0) \oplus \mathcal{U}(1, -1) \longrightarrow \mathcal{U}(0, 0) \longrightarrow 3_{r, 1, p, p', 0} \longrightarrow 0,
\]

where

\[
\bigoplus_{i=-2n}^{2n} \mathcal{U}(2n, i) = \bigoplus_{j=1}^{n} \mathcal{U}_{i, j}^{\prime} \mathcal{U}(r+1) + 2p' j + 2(n-j), 1-\mathcal{U}_{i, j}^{\prime}(r+1) + 2p' j + 2(n-j); 1-pj
\]

\[
\bigoplus_{j=0}^{n-1} \mathcal{U}_{i, j}^{\prime}(r+1) - 2p' j - 2(n-j), 0; pj + \bigoplus_{j=0}^{n-1} \mathcal{U}_{i, j}^{\prime}(1-r) - 2p' n, 0; np
\]

and

\[
\bigoplus_{i=-2n-1}^{2n+1} \mathcal{U}(2n+1, i) = \bigoplus_{j=0}^{n} \mathcal{U}_{i, j}^{\prime} \mathcal{U}(r+1) + 2p' j + 2(n-j), 1-\mathcal{U}_{i, j}^{\prime}(r+1) + 2p' j + 2(n-j); 1-pj
\]

\[
\bigoplus_{j=1}^{n} \mathcal{U}_{i, j}^{\prime}(r+1) - 2p' j + 2(n-j), 0; (n+1)p-r + \bigoplus_{j=0}^{n} \mathcal{U}_{i, j}^{\prime} (1-r) - 2p' j - 2(n-j), 0; pj
\]

\[
\bigoplus_{j=1}^{n} \mathcal{U}_{i, j}^{\prime} (1+r) - 2p' j - 2(1+n-j), 0; pj - r
\]

We label the module at the intersection of the \(a\)-th row and the \(b\)-th column by \(\mathcal{U}(a, b)\) and set \(\theta = 0\), with the nonzero \(\theta\) to be restored in the end. We also temporarily omit \(t = \frac{p}{p'}\) from the notations for modules, writing \(\mathcal{U}_{i, j, \ell, \theta}\) for \(\mathcal{U}_{i, j, \ell, t, \theta}\) (and, accordingly, \(\mathcal{U}_{i, j, \ell}\) for \(\mathcal{U}_{i, j, \ell, t}\)).

**Theorem 3.2.** For \(s = 1\), there is the exact sequence

\[
\cdots \longrightarrow \bigoplus_{b=-a}^{a} \mathcal{U}(a, b) \longrightarrow \cdots \longrightarrow \mathcal{U}(1, 1) \oplus \mathcal{U}(1, 0) \oplus \mathcal{U}(1, -1) \longrightarrow \mathcal{U}(0, 0) \longrightarrow 3_{r, 1, p, p', 0} \longrightarrow 0,
\]

The rest of this subsection is devoted to proving this.

The starting point is the relevant embedding diagram from [11], which is given by case III\(\text{a}_2\)(2, +). To describe it, we also borrow some notations from [11]. The arrows in the embedding diagrams are drawn from the ‘parent’ module to its submodule. Horizontal arrows denote embeddings onto dense \(Q\)-or \(G\)-descendants (see [11, 11]). The (twisted) massive Verma submodules are denoted by \(\bullet\), while \(\bigcirc\) and
\(\circ\) denote twisted topological Verma submodules. Now the \(\Pi^0_{+}(2,--)\) embedding diagram reads as

\[
\begin{array}{c}
\bullet \quad \circ \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \circ \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \circ \\
\end{array}
\]

\[
\begin{array}{c}
\bullet \quad \circ \\
\end{array}
\]

The frame around the pairs of \(\bullet\) and \(\circ\) modules represents the direct sum of these modules. Accordingly, an arrow drawn from that frame (symbolized by a \(*\)) to the corresponding twisted massive Verma module indicates that the latter is embedded into the direct sum. On the other hand, the twisted massive Verma module has two submodules associated with charged singular vectors, as shown by the arrows drawn from the massive Verma module.

At the top level in (3.12), we have the massive Verma module \(\mathcal{U}\) with its two twisted topological Verma submodules \(\mathcal{V}_-\) and \(\mathcal{V}_+\) generated from the charged singular vectors. Each of these submodules has the standard lattice of submodules represented by a “double-braid” embedding diagram; the corresponding submodules are, again, twisted topological Verma modules. We denote by \(\mathcal{V}_-\) the \(\bullet\)-submodule generated from \(|E(n,h,t)\rangle_{ch}\) with \(n \leq -1\), and by \(\mathcal{V}_+\), the \(\circ\)-module generated from \(|E(m,h,t)\rangle_{ch}\) with \(m \geq 0\). Then the submodules of the \(\bullet\) type are those embedded into \(\mathcal{V}_-\) (with \(\mathcal{V}_-\) itself being the top one), while all those of the \(\circ\) type are embedded into (and including) \(\mathcal{V}_+\).

We now need to introduce notations for the remaining modules in (3.12). At the next level, we have a (twisted) massive Verma module \(\mathcal{U}^1\) with its two submodules embedded via charged singular vectors, \(\mathcal{V}^1_-\) and \(\mathcal{V}^1_+\) (such that, \(\mathcal{V}^1_- \subset \mathcal{V}_+\), in fact \(\mathcal{V}^1_- = \mathcal{U}^1 \cap \mathcal{V}_+\)). One level down, we have a (twisted) massive Verma module \(\mathcal{U}^2\) and four twisted topological Verma modules. The \(\bullet\) ones are denoted by \(\mathcal{V}^2_-\) and \(\mathcal{V}^2_+\), so that \(\mathcal{V}^2_- \subset \mathcal{V}^2_+\), and the \(\circ\) ones are \(\tilde{\mathcal{V}}^2_- \supset \mathcal{V}^2_+\). We have [10, 11] \(\mathcal{U}^2 \cap \mathcal{V}^2_- = \mathcal{V}^2_-\) and \(\mathcal{U}^2 \cap \mathcal{V}^2_+ = \mathcal{V}^2_+\). This pattern then repeats at every embedding level: we have the \(\bullet\)-modules \(\mathcal{V}^i_- \subset \mathcal{V}^i\) and the \(\circ\)-ones

\[\text{The difference between the } \bullet \text{ and } \circ \text{ modules is that the former are generated from a twisted topological highest-weight state } |e'\rangle \text{ satisfying twisted topological highest-weight conditions with the twist parameter } \theta' \text{ and } H_0|e'\rangle = (h_0 - \theta')|e'\rangle, \text{ with } h_0 \text{ being the eigenvalue of } H_0 \text{ on the highest-weight vector of } \mathcal{U}, \text{ while the latter are generated from } |e'\rangle \text{ satisfying twisted topological highest-weight conditions with the twist parameter } \theta' \text{ and } H_0|e'\rangle = (h_0 - \theta' - 1)|e'\rangle.\]
\( \mathfrak{V}_i^+ \supset \mathfrak{W}_+^i \), such that
\[
(3.13) \quad \mathfrak{U}^i \cap \mathfrak{V}_-^i = \mathfrak{W}_-^i, \quad \mathfrak{U}^i \cap \mathfrak{V}_+^i = \mathfrak{W}_+^i, \quad i \geq 2.
\]

A crucial step leading from the embedding diagram \((3.12)\) to the resolution is the fact that the massive Verma module \(\mathfrak{U}^i\) is embedded \([11]\) into the direct sum of the corresponding \(\mathfrak{V}_-^i\) and \(\mathfrak{V}_+^i\),
\[
(3.14) \quad \mathfrak{U}^i \hookrightarrow \mathfrak{V}_-^i \oplus \mathfrak{V}_+^i, \quad i \geq 2,
\]
with \(\mathfrak{U}^i \cap \mathfrak{V}_-^i = \mathfrak{V}_-^{i-1}\) and \(\mathfrak{U}^i \cap \mathfrak{V}_+^i = \mathfrak{V}_+^{i-1}\). This leads to the following Lemma. We first note that the irreducible representation \(\mathfrak{J} \equiv \mathfrak{J}_{r,1,p,p'}\) is given by \(0 \to \mathfrak{W}_- + \mathfrak{U}^1 + \mathfrak{W}_+ \to \mathfrak{U} \to \mathfrak{J} \to 0\) (where the sum is not direct) with the embedding given by, obviously,
\[
(3.15) \quad \mathfrak{W}_- + \mathfrak{U}^1 + \mathfrak{W}_+ \to \mathfrak{U}, \quad (x_-, u^1, x_+) \mapsto x_- + u^1 + x_+.
\]

**Lemma 3.3.** There is the exact sequence
\[
(3.16) \quad \ldots \to \mathfrak{W}_-^2 \oplus \mathfrak{U}^3 \oplus \mathfrak{W}_+^2 \to \mathfrak{W}_-^1 \oplus \mathfrak{U}^2 \oplus \mathfrak{W}_+^1 \to \mathfrak{W}_- \oplus \mathfrak{U}^1 \oplus \mathfrak{W}_+ \to \mathfrak{U} \to \mathfrak{J} \to 0.
\]

**Sketch of the Proof.** To see what these mappings are, consider, e.g., \(\mathfrak{W}_-^1 \oplus \mathfrak{U}^2 \oplus \mathfrak{W}_+^1 \to \mathfrak{W}_- \oplus \mathfrak{U}^1 \oplus \mathfrak{W}_+\). Using the embedding \(\mathfrak{U}^2 \hookrightarrow \mathfrak{V}_-^2 \oplus \mathfrak{V}_+^2\), we represent an element \(u^2 \in \mathfrak{U}^2\) as \(\bar{x}_2^- + \bar{x}_2^+\), then the mapping \(\mathfrak{W}_-^1 \oplus \mathfrak{U}^2 \oplus \mathfrak{W}_+^1 \to \mathfrak{W}_- \oplus \mathfrak{U}^1 \oplus \mathfrak{W}_+\) given by
\[
(3.17) \quad (x_-^1, \bar{x}_2^-, \bar{x}_2^+, x_+^1) \mapsto (x_-^1 - \bar{x}_2^-, \bar{x}_2^+ - x_+^1 - x_-^1, x_+^1 - \bar{x}_2^+)
\]
spans the kernel of \(\mathfrak{W}_-^1 \oplus \mathfrak{U}^2 \oplus \mathfrak{W}_+^1 \to \mathfrak{W}_- \oplus \mathfrak{U}^1 \oplus \mathfrak{W}_+\). It is also instructive to consider the next mapping, \(\mathfrak{W}_-^2 \oplus \mathfrak{U}^3 \oplus \mathfrak{W}_+^2 \to \mathfrak{W}_-^1 \oplus \mathfrak{U}^2 \oplus \mathfrak{W}_+^1\). As before, we use the fact that \(\mathfrak{U}^3 \hookrightarrow \mathfrak{V}_-^3 \oplus \mathfrak{V}_+^3\), then
\[
(3.18) \quad (x_-^2, \bar{x}_3^-, \bar{x}_3^+, x_+^2) \mapsto (x_-^2 + \bar{x}_3^-, \bar{x}_3^+ + \bar{x}_3^- + x_+^2 + x_-^2, x_+^2 + \bar{x}_3^+).
\]

Because of the embeddings \(\mathfrak{V}_+^3 \hookrightarrow \mathfrak{V}_+^2\) (which by themselves are parts of the embedding diagrams of \(\mathfrak{W}_-\) and \(\mathfrak{W}_+\), respectively), the element \(\bar{x}_3^- + \bar{x}_3^+ \in \mathfrak{U}^2\) can naturally be viewed as an element of \(\mathfrak{V}_-^2 \oplus \mathfrak{V}_+^2\), whence we see that the composition of \((3.18)\) and \((3.17)\) vanishes.

Continuing in the same way, we complete the proof by taking into account that the quotient with respect to either \(\mathfrak{W}_-\) or \(\mathfrak{W}_+\) gives a twisted topological Verma module. For example, taking the quotient \(\mathfrak{W} = \mathfrak{U}/\mathfrak{W}_+\), we obtain the same irreducible representation simply as \(0 \to \mathfrak{W}_- + \mathfrak{U}^1 \to \mathfrak{W} \to \mathfrak{J} \to 0\), where \(\mathfrak{W}^1 = \mathfrak{U}^1/\mathfrak{W}_+^1\). This then has the BGG-resolution
\[
(3.19) \quad \ldots \to \mathfrak{W}_-^2 \oplus \mathfrak{W}_-^3 \to \mathfrak{W}_-^1 \oplus \mathfrak{W}_-^2 \to \mathfrak{W}_- \oplus \mathfrak{W}^1 \to \mathfrak{W} \to \mathfrak{J} \to 0
\]
(note the occurrence of the \(\mathfrak{W}_-^i\) modules, see \((3.13)\) and the text before that formula). A crucial point is that
\[
(\mathfrak{W}_-^i + \mathfrak{W}_+^i)/((\mathfrak{W}_-^i + \mathfrak{W}_+^i) \cap \mathfrak{W}_+) = (\mathfrak{W}_-^i + \mathfrak{W}_+^i)/\mathfrak{W}_+^i = \mathfrak{W}_-^i, \quad i \geq 2.
\]
The resolutions are therefore related as shown in the following exact commutative diagram:

\[ \begin{array}{cccccccc}
\ldots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\ldots & \rightarrow & V_2^1 \oplus V_3^1 & \rightarrow & V_2^1 \oplus V_3^1 & \rightarrow & V_2^1 \oplus V_3^1 & \rightarrow & 0 \\
\ldots & \rightarrow & V_2^2 \oplus V_3^2 & \rightarrow & V_2^1 \oplus V_3^1 & \rightarrow & V_2^1 \oplus V_3^1 & \rightarrow & 0 \\
\ldots & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array} \]

Applying here the Lemma on Five Homomorphisms shows (3.16) (the overall spectral transform by \( \theta \) goes through the above argument by simply adding \( \theta \) to the twist of every module).

A straightforward analysis shows that the exact sequence (3.16) actually reads as (omitting \( t = \frac{p}{p'} \) from the notations for the modules, as in (3.10)–(3.11))

\[ \begin{array}{cccccccc}
\ldots & \rightarrow & V_{p'(r+1)+2p'm;-r-m;p} \oplus U_{p'(r+1)-2p'(m+1),0}(m+1)p-r \oplus V_{p'(r+1)-1-2p'm;} & \rightarrow & \\
\ldots & \rightarrow & V_{p'(r+1)+2p'm;-r-m;p} \oplus U_{p'(r+1)-2p'm,0;m;p} \oplus V_{p'(r+1)-1-2p'm;} & \rightarrow & \\
\ldots & \rightarrow & V_{p'(r+1)-2p';p} \oplus U_{p'(r+1)-1-2p';p} & \rightarrow & 0. \\
\end{array} \]

Finally, every twisted topological Verma module is the quotient of a massive Verma module over its submodule which is a twisted topological Verma module. The following Lemma, which can be shown in a straightforward way, gives resolutions of twisted topological Verma modules in terms of twisted massive Verma modules (from now on, we restore \( t \) in the notation for the modules).

**Lemma 3.4.** Let \( \mathcal{V}_{h,t,\theta} \) be a twisted topological Verma module. There are the exact sequences

\[ \begin{array}{cccccccc}
\ldots & \rightarrow & U_{h+3,\frac{p}{p'}}h+3,\frac{p}{p'},t;\theta+1 & \rightarrow & U_{h+2,\frac{p}{p'}}h+2,\frac{p}{p'},t;\theta+1 & \rightarrow & U_{h+1,\frac{p}{p'}}h+1,\frac{p}{p'},t;\theta+1 & \rightarrow & \mathcal{V}_{h,t,\theta} & \rightarrow & 0, \\
0 & \leftarrow & \mathcal{V}_{h,t,\theta} & \leftarrow & \mathcal{V}_{h,0,t,\theta} & \leftarrow & \mathcal{V}_{h-1,0,t,\theta} & \leftarrow & \mathcal{V}_{h-2,0,t,\theta} & \leftarrow & \ldots. \\
\end{array} \]

Using these in (3.22) for the \( \bullet \) and \( \circ \) modules, respectively, we arrive at the resolution (3.9).

### 3.3. The massive-admissible representations.

Along with the admissible \( N = 2 \) representations defined above, one can consider a more ‘exotic’ case where an irreducible \( N = 2 \) representation is obtained by taking the quotient of a massive Verma module with respect to only massive Verma submodules (that is, one starts with a massive Verma module without charged singular vectors). A characteristic feature of this situation is that all submodules can be considered to have zero twist (because none of them have
charged singular vectors either). The embedding diagram of the massive Verma module in question is case \( \Pi_+ (0) \) of \( \Pi \). Then, similarly to the standard admissible case, we can single out the “massive-admissible” representations as the quotients of those massive Verma modules that are not embedded into another massive Verma module. These are the modules \( \mathcal{U}_{\hat{r},l(r,s,h,p')}, \hat{p'} \), where \( h \in \mathbb{C} \) and

\[
(l(r,s,h,\hat{p'})) = \frac{p'}{4p}((h\hat{p'} - 1)^2 - (r - \frac{p'}{p'}(s - 1))^2).
\]

The resolution reads as

\[
\ldots \rightarrow \mathcal{N}_+(m) \oplus \mathcal{N}_-(m) \rightarrow \ldots \rightarrow \mathcal{N}_+(2) \oplus \mathcal{N}_-(2) \rightarrow \mathcal{N}_+(1) \oplus \mathcal{N}_-(1) \rightarrow \mathcal{N}_+(1) \oplus \mathcal{N}_-(1) \rightarrow \mathcal{U}_{\hat{r},l(r,s,h,\hat{p'}), \hat{p'}} \rightarrow \mathcal{U}_{\hat{r},s,h,p'} \rightarrow 0,
\]

where \( \mathcal{N}_\pm (m) = \mathcal{U}_{h, l(r,s,h,\hat{p'}, \hat{p'})}((h\hat{p'} - 1)^2 - \Delta_\pm (m)^2), \hat{p'} \) are the massive Verma modules with

\[
\Delta_+ (2m) = r - \frac{p'}{p'}(s - 1) - 2pm,
\Delta_- (2m) = r - \frac{p'}{p'}(s - 1) + 2pm,
\Delta_+ (2m - 1) = -r - \frac{p'}{p'}(s - 1) - 2p(m - 1),
\Delta_- (2m - 1) = -r - \frac{p'}{p'}(s - 1) + 2pm,
\]

where \( m \in \mathbb{N} \). Since, as we have noted, all submodules in \( \mathcal{U}_{\hat{r},l(r,s,h,\hat{p'}), \hat{p'}} \) are untwisted massive Verma modules, the value of \( h \) does not change along the resolution (the entire resolution being thus, in a sense, a “Virasoro effect”).

4. Characters of irreducible representations

In this section, we use the resolution constructed above to derive characters of the admissible \( N = 2 \) representations and to study their properties.

4.1. The \( \hat{\mathfrak{sl}}(2) \) side. We start with rederiving the known results for the \( \hat{\mathfrak{sl}}(2) \) characters in order to show the similarities (in fact, uniformity) with the \( N = 2 \) ones and to find the behaviour of characters under the spectral flow.

4.1.1. Admissible representations. We consider the admissible representations as defined in Sec. 3.4. The Sugawara dimension of the highest-weight state \( |j(r,s,\hat{p'}, \hat{p'})_{\mathfrak{sl}(2)} \rangle \) then equals

\[
\frac{j(j+1)}{l} = \frac{(r^2 - 1)p'}{4p} - \frac{r}{2}(s - 1) + \frac{(s-1)^2 p}{4p}.
\]

The character can be found from the BGG-resolution in a standard manner. It converges in the annulus \( 1 < |z| < |q|^{-1} \), where it can be summed to the character function

\[
\chi^{\mathfrak{g}}_{\hat{r},s,p',p}(z, q) = \frac{(r^2 - 1)p - r(s - 1) + (s-1)^2 p + \frac{r}{2} - \frac{1}{4} p p'}{\vartheta_{1,1}(z, q)} z^{\frac{r-1}{2} - \frac{s-1}{2}} \times
\]

\[
\vartheta_{1,0}(z^p q^{-pp' + r p' - (s-1)p}, q^{2pp'}) - z^{-r} q^{r(s-1)} \vartheta_{1,0}(z^p q^{-pp' + r p' - (s-1)p}, q^{2pp'})
\]

\[
\times \left( \vartheta_{1,0}(z^p q^{-pp' + r p' - (s-1)p}, q^{2pp'}) - z^{-r} q^{r(s-1)} \vartheta_{1,0}(z^p q^{-pp' + r p' - (s-1)p}, q^{2pp'}) \right).
\]

Remark 4.1. Whenever \( p = 2r \), we use (3.7) to rewrite the above formula as

\[
\chi^{\mathfrak{g}}_{\hat{r},s,2r,p'}(z, q) = q^{-\frac{p'}{2} + \frac{(s-1)^2}{2} + \frac{r}{2} - \frac{s}{2}(s-1)} z^{\frac{r-1}{2} - \frac{s-1}{2}} \frac{\vartheta_{1,1}(z^r q^{-r(s-1)}, q^{rr'})}{\vartheta_{1,1}(z, q)}.
\]
A direct calculation (using the formulae of Appendix A) shows that the admissible \( \hat{\mathfrak{sl}}(2) \) character functions are invariant under the spectral flow with \( \theta = 2p' \):

\[
\chi_{r,s,p,p'}^3(z, q) = \chi_{r,s,p,p'}^3(z, q) , \quad z \in \mathbb{C}.
\]

Thus, explicitly,

\[
\chi_{r,s,p,p'}^3(z q^{-2p'}, q) = z^{p-2p'} q^{2p^2-pp'} \chi_{r,s,p,p'}^3(z, q).
\]

In addition, whenever \( p = 2r \), we have \( \chi_{r,s,2r,p'}^3(z, q) = (-1)^{p'-1} \chi_{r,s,2r,p'}^3(z, q) \).

In view of this quasiperiodicity, the distinct character functions are found by applying the spectral flow transform with \( 0 \leq \theta \leq 2p' - 1 \) whenever \( p \neq 2r \), and with \( 0 \leq \theta \leq p' - 1 \) when \( p = 2r \). For such \( \theta \), we find the spectral-flow transformed character functions as

\[
\chi_{r,s,p,p',\theta}^3(z, q) = (-1)^\theta q^{(z^2-1)p' + \frac{2p'}{3}(s+2\theta-1) - \frac{1}{4}2(s+\theta-1)+ \frac{1}{2}p' - \frac{1}{2} \theta^2} \cdot \frac{\theta_{1,0}(z, q)}{\theta_{1,1}(z, q)} \times (\theta_{1,0}(zp q^{-pp'} - \frac{1}{2} \theta^2 - p + \theta - 1, z^{pp'+pp').}
\]

To summarise, while generically the spectral flow transform action on twisted characters is

\[
U_{\theta'} \chi_{r,s,p,p',\theta}^3(z, q) = \chi_{r,s,p,p',\theta+\theta'}^3(z, q),
\]

we have the special cases described as follows.

**Theorem 4.2.** Character functions (4.6) of the admissible \( \hat{\mathfrak{sl}}(2) \) representations carry a representation of the group \( \text{Aut} \) given by (1.7) and

\[
U_{2p'} \chi_{r,s,p,p',\theta}^3(z, q) = \chi_{r,s,p,p',\theta}^3(z, q),
\]

\[
U_{p'} \chi_{r,s,p,p',\theta}^3(z, q) = (-1)^{p'-1} \chi_{r,s,p,p',\theta}^3(z, q),
\]

\[
T \chi_{r,s,p,p',\theta}^3(z, q) = (-1)^{p'-1} \chi_{r,s,p,s+1,p',p',\theta-1}^3(z, q).
\]

**Remark 4.3.** Equations (4.8) and (4.9) may seem puzzling if we recall that the twisted admissible representations certainly contain states in the bigradings\(^7\) in which, e.g., the untwisted representation contains no states. However, in order to derive from a given character function \( \chi(z, q) \) the occupation number \( \chi_{m,n} \) in grade \((m, n)\), we have to expand it as \( \chi(z, q) = \sum_{m,n} \chi_{m,n} \xi^m q^n \) in the annulus determined by the chosen representation. The expansion would depend on the annulus as soon as the character function has poles in \( z \).

As regards the poles of \( \chi_{r,s,p,p'}^3(z) \), we have

**Lemma 4.4.** As a function of \( z \in \mathbb{C} \), the character function of the admissible \( \hat{\mathfrak{sl}}(2) \) representation \( \mathcal{J}_{r,s,p,p'} \) has poles at the points \( z = q^n, n \in \mathbb{Z} \setminus (p'\mathbb{Z} + s - 1) \).

\(^7\)with respect to (level, charge), i.e., the eigenvalues of (the zero mode of) the Sugawara energy-momentum tensor and \( J_0^+ \), respectively.
In particular, in the case where
\begin{equation}
U \chi_{(4.14)}(z, q)
\end{equation}
flow with even transformation parameters (not surprisingly, since 
\begin{equation}
(2n) as \\text{n} as \text{mod}\ p as \text{n} = p' \ell + s - 1 + a with \ell \in \mathbb{Z}, a = 0, 1, \ldots, p' - 1, and evaluate the bracket in (4.12) at \ z = q^n:
\begin{equation}
(4.11) \quad \left( \sum_{m \in \mathbb{Z}} q^{pp'm^2 - pp' \ell m - rpm + rpm} \right)
\end{equation}
Replacing \( m \mapsto \ell - m \) in the second sum, we rewrite this as
\begin{equation}
(4.12) \quad \sum_{m \in \mathbb{Z}} q^{pp'm^2 - pp' \ell m - rpm + rpm} \left( q^{\frac{\ell}{2} - pam} - q^{-\frac{\ell}{2} + pam - \ell pa} \right),
\end{equation}
which vanishes whenever \( a = 0 \). \hfill \Box

4.1.2. Unitary representations. A module \( \Sigma \) is called unitary if the Shapovalov form \( (\cdot, \cdot) \) is positive definite. We now consider those admissible representations that are unitary, namely the ones with \( s = p' = 1 \); these are the integrable representations (thus, our usage of the term “unitary” is limited to the unitary admissible representations). The characters of these representation are therefore given by
\begin{equation}
(4.13) \quad \chi_{r,p}^\Sigma(z, q) = \frac{q^\frac{z^2 - 1}{4p} + \frac{z - \mathbf{q}}{2} \mathbf{z}}{\partial_{1,1}(z, q)} \left( \partial_{1,0}(z^p q^{p+1}, q^{2p}) - z^{-r} \partial_{1,0}(z^p q^{-p-r}, q^{2p}) \right).
\end{equation}

Lemma 4.5. The character function of the unitary \( \hat{sl}(2) \) representation is holomorphic in \( z \in \mathbb{C} \).

The modular transformation properties of these characters are well-known, and we do not repeat them here. Instead, we point out how the characters behave under the spectral flow and the involution:

Lemma 4.6. The character functions of the unitary \( \hat{sl}(2) \) modules carry the following representation of the group Aut:
\begin{align}
(4.14) & \quad \mathbf{U}_1 \chi_{r,p}^\Sigma(z, q) = \chi_{p-r,p}^\Sigma(z, q), \\
(4.15) & \quad \mathbf{T} \chi_{r,p}^\Sigma(z, q) = \chi_{r,p}^\Sigma(z, q).
\end{align}

In particular, in the case where \( p = 2r \), we have \( \mathbf{U}_1 \chi_{r,p}^\Sigma(z, q) = \chi_{r,p}^\Sigma(z, q) \).

We thus see that \( \mathbf{U}_2 \chi_{r,p}^\Sigma(z, q) = \chi_{r,p}^\Sigma(z, q) \), i.e., all the unitary characters are invariant under the spectral flow with even transformation parameters (not surprisingly, since \( (2\mathbb{Z}) \ltimes \mathbb{Z}_2 \subset \text{Aut} \) is the affine Weyl group).

Remark 4.7. Unlike the case with admissible representations, it is not only the characters but actually the unitary representations themselves that are mapped into each other by the group Aut (this agrees with the fact that the unitary characters are holomorphic, cf. [4.3]). It is easy to see how the unitary representations are mapped by the spectral flow at the level of their extremal diagrams [4]. Such representations are the quotients of Verma modules with singular vectors of the form \( (J_0^-)^r \) and \( (J_{-1}^+)^{p-r} \) (which are understood to act on the highest-weight vector). In the generic case where \( r \neq 1 \) or \( p - 1 \),
the extremal diagram of the unitary representation $\mathfrak{U}_{r,p}$ looks like (in the conventions of [2])

$$
(J_0)^{r-1} \quad (J_i)^{p-r-1} \quad (J_{i-1})^{p-r-1} \quad (J_{i-2})^{r-1} \quad (J_i)^{r-1} \quad (J_{i+1})^{p-r-1} \quad (J_i^+)^{r-1} \quad (J_i^+)^{p-r-1} \quad (J_i^+)^{r-1} \quad (J_i^+)^{p-r-1}
$$

(4.16)

The sections between the cusps are such that their horizontal projections (onto the charge axis) have lengths $\ldots, p-r-1, r-1, p-r-1, r-1, \ldots$. Thus, marking each section with the action of a $J_n^-$ operator, the extremal diagram of $\mathfrak{U}_{r,p}$ can be described as

$$
\ldots (J_{i-3})^{p-r-1}(J_{i-2})^{r-1}(J_{i-1})^{p-r-1}(J_i^+)^{r-1}(J_{i+1})^{p-r-1} \ldots \longleftrightarrow \mathfrak{U}_{r,p}.
$$

(4.17)

After the spectral flow transformation $U_1$, this takes the form

$$
\ldots (J_{i-3})^{r-1}(J_{i-2})^{p-r-1}(J_{i-1})^{r-1}(J_i^+)^{p-r-1}(J_{i+1})^{r-1} \ldots \longleftrightarrow \mathfrak{U}_{p-r,p},
$$

in agreement with $U_1 \mathfrak{U}_{r,p} = \mathfrak{U}_{p-r,p}$.

4.2. **The $N=2$ side.** We now derive the $N=2$ characters using the BGG resolution constructed in Sec. 3.1 and also the massive resolutions from Sec. 7.2.

4.2.1. **Admissible representations.** Consider the admissible $N=2$ representations as defined in Sec. 3. We can start with the untwisted topological Verma modules and then take the twist. Let, thus, the admissible representation be $\mathfrak{U}_{r,s,p,p'} = \mathfrak{U}_{h(r,s,p)} \frac{x}{p}$/(maximal submodule).

The alternating sum of characters over the resolution from Theorem 3.1 reads as

$$
\omega_{r,s,p,p'}^{\mathfrak{U}}(z,q) = \sum_{m=-\infty}^{\infty} \omega_{(r-1)+s-2p',m,p,r,m,p} \mathfrak{U}_{r,s,p,p'}(z,q) - \sum_{m=-\infty}^{\infty} \omega_{(r+1)+s-2p',m,p,r,m,p} \mathfrak{U}_{r,s,p,p'}(z,q).
$$

(4.19)

As we have seen in Sec. 7.1.2, the twists $\omega_{r,s,p,p'}^{\mathfrak{U}}(z,q)$ and $\omega_{r,s,p,p'}^{\mathfrak{U}}(z,q)$ follow from the fact that the corresponding submodules are generated from twisted topological highest-weight states. Explicitly substituting the topological Verma module characters from [2.27] leads to the following Theorem.

**Theorem 4.8.** The admissible $N=2$ characters read as

$$
\omega_{r,s,p,p'}^{\mathfrak{U}}(z,q) = \varphi_{r,s,p,p'}(z,q) \omega_{(r-1)+s-1,0}^{\mathfrak{U}}(z,q),
$$

where

$$
\omega_{(r-1)+s-1,0}^{\mathfrak{U}}(z,q) = z^{-\eta \phi_{(r-1)+s-1,0}^{\mathfrak{U}}(z,q)} / \eta(q)^3.
$$

(4.20)

(4.21)

\[\]
and

$$\varphi_{r,s,p,p'}(z, q) = \sum_{m \in \mathbb{Z}} q^{mp^r m^2 - mp(s-1)} \left( \frac{q^{mp^r}}{1 + z^{-1} q^{mp}} - q^{r(s-1)} \frac{q^{-mp^r}}{1 + z^{-1} q^{mp-r}} \right).$$

(We have separated into \[4.21\] a factor equal to the massive Verma module character). The following statement is obvious:

**Lemma 4.9.** The admissible $N=2$ character functions are holomorphic in $z \in \mathbb{C}$.

Next, we study the behaviour of $\omega^{3}_{r,s,p,p'}$ under the spectral flow. The transformation law \[2.22\] induces a transformation of $\varphi_{r,s,p,p'}$, which we now evaluate. As follows by a direct calculation, the following identities describe the behaviour of $\varphi_{r,s,p,p'}(z, q)$ under the spectral flow transform: for $n \geq 1$, we have

$$\varphi_{r,s,p,p'}(z, q) = \sum_{a=0}^{2p'n-1} (-1)^a z^{a+1} q^{-mp'/4} \left( \vartheta_{1,0}(q^{-pp'+(s+a)-p'r}, q^{2mp'}) - q^{r(s+a)} \vartheta_{1,0}(q^{-pp'+p(s+a)+p'r}, q^{2mp'}) \right).$$

and for $n \leq -1$, similarly,

$$\varphi_{r,s,p,p'}(z, q) = \sum_{a=2p'n}^{1} (-1)^a z^{a+1} q^{-mp'/4} \left( \vartheta_{1,0}(q^{-pp'+(s+a)-p'r}, q^{2mp'}) - q^{r(s+a)} \vartheta_{1,0}(q^{-pp'+p(s+a)+p'r}, q^{2mp'}) \right).$$

In the same way as in the proof of Lemma 4.4, we can see that those terms on the right-hand side of \[1.23\] where $a = \ell p' - s$, $1 \leq \ell \leq 2n$, actually vanish. In general, $\varphi_{r,s,p,p'}(z, q)$ is not quasiperiodic in $z$ and, thus, cannot be algebraically expressed through theta functions. An exception is provided by the unitary representations considered below.

**Remark 4.10.** It is amusing to observe that the inhomogeneous terms in the transformation law of $\varphi_{r,s,p,p'}(z, q)$ under the spectral flow are the residues of the admissible $\hat{sl}(2)$ characters; while presently this comes as an “experimental fact”, we see in the next section why it must be so.

**Remark 4.11.** When $|q|^{-r+pm} < |z| < |q|^{-p+pm}$, Eq. \[1.22\] can also be written as

$$\varphi_{r,s,p,p'}(z, q) = \sum_{m \geq n+1} - \sum_{m \leq n-1} \sum_{k \leq -1} q^{mp^r m^2 - mp(s-1)+mpk} (1)_{k} z^{-k} \left( q^{mp^r} - q^{r(s-1)-mp^r-rk} \right).$$

Similarly, when $|q|^{pm} < |z| < |q|^{-r+pm}$, we can expand as

$$\varphi_{r,s,p,p'}(z, q) = \sum_{m \geq n+1} - \sum_{m \leq n-1} \sum_{k \leq -1} q^{mp^r m^2 - mp(s-1)+mpk} (1)_{k} z^{-k} \left( q^{mp^r} - q^{r(s-1)-mp^r-rk} \right) + q^{-pp'n^2 - np(s-1)} \sum_{k \geq 0} (-1)^k z^{-k} q^{npk} + q^{r(s-1)-np^r} \sum_{k \leq -1} (-1)^k z^{-k} q^{npk-rk}.$$
Using this in (4.20), we recall the factor given by the character of a massive \( N=2 \) Verma module, which suggests that the character of an admissible representation is given by an infinite sum of massive Verma module characters. This is not accidental in view of the following result (even though it is by itself limited to the case of \( s=1 \)).

**Lemma 4.12.** The character formula for the admissible \( N=2 \) representation \( \mathcal{J}_{r,1,p,p'} \) following from the massive resolution (3.3) reads as (4.20) (with \( s=1 \)), where

(4.27)

\[
\varphi_{r,1,p,p'}(z,q) = \sum_{m \geq 0} \left( \sum_{j=1}^{m} z^{1+2(m-j)} q^{j}[pp'j+p+2p(m-j)-p'r] + \sum_{j=0}^{m} z^{2(m-j)} q^{(pj+r)[p'j+2(m-j)]} \right)
+ \sum_{j=1}^{m} z^{1-2(m-j)} q^{(pj-r)[p'j+1+2(m-j)]} + \sum_{j=0}^{m-1} z^{2-(m-j)} q^{j}[pp'j+p'+r+2p(m-j)]
\]

This converges for \( 1 < |z| < |q|^{-r} \) and is then equal to (4.26).

**Proof.** The character formula is straightforward to derive by inserting the twisted massive Verma module characters (2.32) into the massive resolution (3.3)–(3.11). The organization of (4.27) is in accordance with the 3, 5, 7, \ldots structure of the resolution. For \( 1 < |z| < |q|^{-r} \), the sum in (4.27) can be brought to the form (4.26) by changing the order of summations in each term and then redefining the summation indices. \( \square \)

4.2.2. A series of unitary representations. A module \( \mathfrak{K} \) is called unitary if the form (2.20) is positive definite. Taking the untwisted admissible representations with \( s = p' = 1 \) [23], we obtain \( p-1 \) from the total of \( \frac{1}{2}p(p-1) \) unitary representations (all of which are considered in Sec. 4.2.3). Therefore, the characters of the unitary representations \( \mathfrak{K}_{r,p} \equiv \mathcal{J}_{r,1,p,1} \) are given by

(4.28)

\[
\omega_{r,p}^{\mathfrak{K}}(z,q) = \frac{\eta(\frac{1}{2},q)}{(q\eta(q))^{3}} z^{-\frac{r-1}{p}} \sum_{m \in \mathbb{Z}} q^{m^2} \left( \frac{q^{mr}}{1+z^{-1}q^{mp}} - \frac{q^{-mr}}{1+z^{-1}q^{-mp-r}} \right), \quad 1 \leq r \leq p-1.
\]

Obviously, the character functions of unitary \( N=2 \) representations are holomorphic in \( z \in \mathbb{C} \). Moreover,

**Lemma 4.13.** The characters of unitary \( N=2 \) representations are invariant under the spectral flow transform with \( \theta = p \):

(4.29)

\[
\omega_{r,pcp}^{\mathfrak{K}}(z,q) = \omega_{r,p}^{\mathfrak{K}}(z,q).
\]
In addition, whenever \( p = 2r \), the character is invariant under the spectral flow with \( \theta = r \):

\[
\omega^R_{r,2r,r}(z,q) = \omega^R_{r,2r}(z,q).
\]

Proof. We recall that the right-hand side of (4.32) (where now \( n = 1 \)) vanishes when \( a = p' - s \) and \( a = 2p' - s \). In the unitary case, where \( p' = s = 1 \), this means that these terms vanish altogether, and therefore,

\[
z^2 q^{r-p} \varphi_{r,1,p,1}(z q^{-p},q) = \varphi_{r,1,p,1}(z,q),
\]

which shows (4.32); the second assertion is proved similarly.

Explicitly,

\[
\omega^R_{r,p}(z q^{-pn},q) = z^{n(p-2)} q^{-\frac{1}{2}(p-2)n(pn-1)} \omega^R_{r,p}(z,q).
\]

Thus, \( \omega^R_{r,p}(z,q) \) are quasiperiodic holomorphic functions of \( z \in \mathbb{C} \), and therefore, they can be expressed through the theta functions. This allows us to arrive at the following statement [5, 8]:

**Theorem 4.14.** The characters of the unitary \( N=2 \) representations \( J_{r,p} \equiv J_{r,1,p,1} \) are given by

\[
\omega^R_{r,p}(z,q) = -z^{-\frac{r+1}{2}} \eta(q^p)^3 \frac{\vartheta_{1,0}(z,q) \vartheta_{1,1}(q^r,q^p)}{\eta(q)^3 \vartheta_{1,0}(z,q^p) \vartheta_{1,0}(z q^r,q^p)}.
\]

**Remark 4.15.** In the special case where \( p = 2r \), the last formula rewrites as

\[
\omega^R_{r,2r}(z,q) = z^{-\frac{r+1}{2}} \frac{\vartheta_{1,0}(z,q) \vartheta_{1,1}(q^r,q^p)}{\eta(q)^3 \vartheta_{1,0}(z,q^p) \vartheta_{1,0}(z q^r,q^p)}.
\]

Proof. We see from (4.33) that

\[
\varphi_{r,1,p,1}(z,q) = \frac{f_{r,p}(q)}{z \vartheta_{1,0}(z,q^p) \vartheta_{1,0}(z q^r,q^p)}.
\]

Indeed, the right-hand side of (4.33) considered as a function of \( z \) has all the poles of (4.22) and, for \( s = p' = 1 \), possesses the same periodicity properties. To find \( f_{r,p}(q) \), we evaluate the residues of both sides of (4.35) at \( z = -q^{pn}, n \in \mathbb{Z} \) using (A,6):

\[
\text{res}_{z=-q^{pn}} \sum_{m \in \mathbb{Z}} q^{pm^2} \left( \frac{q^{mr}}{1 + z^{-1} q^{mp-r}} - \frac{q^{mr}}{1 + z^{-1} q^{-mp+r}} \right) = -q^{pn^2+pn+n r},
\]

\[
\text{res}_{z=-q^{pn}} \frac{f_{r,p}(q)}{z \vartheta_{1,0}(z,q^p) \vartheta_{1,0}(z q^r,q^p)} = f_{r,p}(q) \frac{q^{pn^2+pn+n r}}{\eta(q^p)^3 \vartheta_{1,1}(q^r,q^p)}.
\]

Therefore, \( f_{r,p}(q) = -\eta(q^p)^3 \vartheta_{1,1}(q^r,q^p) \). 

For the modular properties of these characters, see [5, 8].
4.2.3. All unitary representations. We have seen that the unitary $N=2$ representations can be obtained by taking quotients of twisted topological Verma modules $\mathcal{M}_{h,p,\theta}$ with $h = \frac{1-r}{p}$, where $1 \leq r \leq p-1$, $p-2 \in \mathbb{N}$, over two topological singular vectors. The unitary modules can be also obtained by taking the quotients of the massive Verma modules $\mathcal{M}_{h,\ell,p}$ with $h = \frac{1+n-m}{p}$ and $\ell = \frac{m}{p}$, where $1 \leq n + m \leq p - 1$, $1 \leq m$, and $0 \leq n$, over two charged (see (2.33)) and one massive singular vectors. The $r$ and $\theta$ parameters are then recovered as $r = n + m$ and $\theta = n$. The highest-weight vector of the quotient module lies on the extremal diagram of the massive Verma module and is a twisted topological highest-weight vector. Different twists give rise to generically non-isomorphic representations, however there are the isomorphisms

$$\mathcal{R}_{r,p,\theta+r} \approx \mathcal{R}_{r,p,\theta} \quad \text{and} \quad \mathcal{R}_{r,p,\theta+p} \approx \mathcal{R}_{r-p,p,\theta} .$$

The first of these isomorphisms means periodicity with respect to the spectral flow with period $p$ (therefore, we can take $\theta \in \mathbb{Z}_p$), while the second one shows that in the table consisting of $p(p-1)$ boxes that correspond to $0 \leq \theta \leq p - 1$ and $1 \leq r \leq p - 1$, there are $\frac{1}{2}p(p-1)$ non-isomorphic unitary representations. Some of the isomorphisms described by (4.36) are shown in the diagram (the boxes with the same values of $[\theta r]$). Whenever $p$ is even, the row labelled by $r = p/2$ is obtained by repeating twice the first $p/2$ representations, i.e., it reads as $([0, \frac{p}{2}], \ldots, [\frac{p}{2} - 1, \frac{p}{2}], [0, \frac{p}{2}], \ldots, [\frac{p}{2} - 1, \frac{p}{2}])$.

If the $r$ and $\theta$ parameters are expressed through $n$ and $m$ as above, it follows that $0 \leq \theta \leq r - 1$, thereby ensuring that each unitary representation is counted precisely once.

We now outline an alternative description of the unitary modules $\mathcal{M}$ (which is a counterpart of the $\widehat{sl}(2)$ unitary module construction $[23]$). The operator $S(z) = \partial^{p-2}G(z) \ldots \partial G(z)\partial(z)$ is a null field (because $S(0)|0\rangle$ is a singular vector in the vacuum representation) and is therefore quotiented away in any unitary representation. This gives the relations

$$\sum_{i_0 < \ldots < i_{p-2}} \left( \prod_{m<n} (i_m - i_n) \right) G_{i_0} \ldots G_{i_{p-2}} = 0 , \quad a \in \mathbb{Z}$$

that hold in every unitary module. These allow us to describe the structure of the extremal diagram of unitary modules. Every product

$$G_{a_1} \ldots G_{a_N} , \quad a_1 < \ldots < a_N$$

(where the ordering can always be achieved at the expense of an overall sign) can be encoded by saying which of the numbers from $\{a_1, a_1 + 1, \ldots, a_N - 1, a_N\}$ are occupied, i.e., are such that the corresponding mode of $G$ is present in (4.38). Now, when constructing extremal states $\ldots G_{\theta-2}G_{\theta-1}$ (which is assumed to act on state satisfying the $\theta$-twisted topological highest-weight conditions), every
such product vanishes as soon as there are $p - 1$ consecutive numbers that are occupied in $\{a_1, a_1 + 1, \ldots, a_N - 1, a_N\}$.

The unitary extremal diagram can be described by periodic sequences with period $p$ consisting of $\times$ or $\circ$ (occupied/unoccupied positions) and exactly $p - 2$ occupied numbers (the crosses) per period:

$$\ldots \overset{r-1}{\cdots} \times \overset{p-r-1}{\cdots} \overset{}{\circ} \underset{\wedge}{\cdots} \times \overset{}{\circ} \underset{\wedge}{\cdots} \times \overset{}{\circ} \ldots$$

(4.39)

One of the positions is marked as the basepoint. Two such periodic infinite sequences are identified whenever they coincide (including their basepoints) after a translation by $ip$, $i \in \mathbb{Z}$.

With the modes $Q_{-n}$ on the $Q$-side of the extremal diagram formally replaced with $G_n$, such sequences are in a 1 : 1 correspondence with extremal diagrams of the unitary modules; the unoccupied positions correspond to cusps in the extremal diagram, and the basepoint is placed to grade 0 with respect to $\mathcal{H}_0$:

$$\text{The outer parabola is the extremal diagram of a massive Verma module, which also can be represented as a dense fill}$$

$$\ldots \times \times \times \ldots$$

(4.40)

Moreover, the basis for all states in the unitary module is given by a semi-infinite construction where a finite number of length-$p$ fragments contain not more than $p - 2$ occupied positions each, while the infinite tail $G_{a_1} \ldots G_{a_{i-1}} G_{a_i} \ldots$ with $a_1 < \ldots < a_{i-1} < a_i < \ldots$, is such that the associated sequence of the occupied positions is periodic with period $p$ and has exactly $p - 2$ occupied positions per period.

An infinite $p$-periodic sequence (with a basepoint) consisting of $\times$ and $\circ$ with $p - 2$ $\times$’s per period can be characterised by its length-$p$ section starting at the basepoint,

$$\overset{a}{\times} \overset{}{\cdots} \overset{b}{\times} \overset{}{\circ} \overset{}{\cdots} \overset{c}{\times} \overset{}{\circ} \overset{}{\cdots} \overset{}{\times}$$

(4.41)

where $0 \leq a \leq p - 2$, $0 \leq c \leq p - 2$, $0 \leq a + c \leq p - 2$ (and $a + b + c = p - 2$). This corresponds to the unitary representation $\mathcal{R}_{r,p,\theta}$ with $r = a + c + 1$ and $\theta = a$. We note again that this implies $0 \leq \theta \leq r - 1$ with $1 \leq r \leq p - 1$, and, therefore, each unitary representation is labelled once.
For example, in the case of $p = 4$ the possible extremal diagrams are

\[
\begin{array}{ccc}
\ldots \circ \circ \times \ldots & \ldots \circ \circ \circ \ldots & \ldots \circ \circ \circ \circ \ldots \\
\wedge & \wedge & \wedge \\
\ldots \times \circ \circ \circ \ldots & \ldots \circ \circ \circ \ldots & \ldots \times \circ \circ \circ \ldots \\
\wedge & \wedge & \wedge 
\end{array}
\]

A unitary module is characterised by its extremal diagram.

**Lemma 4.16.** There exist $\frac{1}{2}p(p - 1)$ different $p$-periodic infinite sequences $(\lambda_i)_{i \in \mathbb{Z}}$, $\lambda_i \in \{\times, \circ\}$ with a basepoint and with $p - 2$ crosses $\times$ per period. Thus, there exist $\frac{1}{2}p(p - 1)$ non-isomorphic unitary $N = 2$ representations with the central charge $c = 3(1 - \frac{2}{p})$, $p \in \mathbb{N}$, $p \geq 2$.

The characters of all the $p(p - 1)/2$ inequivalent unitary representations are found by taking the twists of the above characters (4.33). This is done in accordance with formula (2.22) for the spectral flow transform of the $N = 2$ characters. While generically one has

\[
\tilde{U}_{\theta'} \omega^R_{r, p, \theta}(z, q) = \omega^R_{r, p, \theta + \theta'}(z, q),
\]

we see that there are the following special cases of the spectral flow transform, in accordance with (4.36):

\[
\tilde{U}_p \omega^R_{r, p, \theta}(z, q) = \omega^R_{r, p, \theta}(z, q),
\]

\[
\tilde{U}_r \omega^R_{r, p, \theta}(z, q) = \omega^R_{p-r, p, \theta}(z, q).
\]

When, in particular, $p = 2r$, we have

\[
\tilde{U}_r \omega^R_{r, 2r, \theta}(z, q) = \omega^R_{r, 2r, \theta}(z, q).
\]

**Remark 4.17.** Note that if we identify the representations that differ by the spectral flow, we will be left with only $[p/2]$ classes of unitary $N = 2$ representations, which is the same as the number of classes of unitary $\tilde{\ell}(2)$ representations modulo the spectral flow [3].

4.2.4. More on the admissible $N = 2$ characters. Unlike the unitary representations, the characters of general admissible representations, which are not quasi-periodic, cannot be expressed algebraically in terms of a finite number of theta functions. In Sec. 2.4.3, we consider an integral representation for the admissible $N = 2$ characters, while now we briefly comment on how one can arrive at equations for these characters.

Observe first of all that although setting $p' = 1$ implies that $s = 1$ in the case of admissible representations, we can formally define $\varphi_{r, s, p, 1}(z, q)$ by the same formula (4.22) even for $s > 1$. Since $\varphi_{r, s, p, 1}(z, q)$ is still quasi-periodic under $z \mapsto z q^p$, we can proceed similarly to the above to derive

\[
\varphi_{r, s, p, 1}(z, q) = -\frac{\eta(q^p)^3}{z^s} \partial_{1, 0}(z, q^p) \partial_{1, 0}(z q^r, q^p).
\]

Now, let $e_{p'} = e^{2\pi i/p'}$ be the primitive $p'$th root of unity. We then see that

\[
\sum_{a=0}^{p'-1} \varphi_{r, s, p, p'}(-e_{p'}^{-a} z, q) = p' \varphi_{r, 1 + \frac{z}{p'}, p, 1}(-z, q).
\]
Thus, we arrive at the following equation for the sought function \( \tilde{\varphi}_{r,s,p,p'}(z,q) \):

\[
\sum_{a=0}^{p'-1} \varphi_{r,s,p,p'}(e_{p'}^a z, q) = -\frac{p' - 1}{q^{p' - s}} \frac{\eta(q^{p'})^3}{\vartheta_{1,1}(( -z)^{p'}, q^{p'}) \vartheta_{1,1}(( -z)^{p'} q^{p'}, q^{p'})}.
\]

This is to be solved under the condition that the poles of \( \varphi_{r,s,p,p'}(z,q) \) are at \( z = -q^{np}, n \in \mathbb{N} \), and at \( z = -q^{np-r}, n \in \mathbb{N} \). Note that \( z = -c_{p'}^0 q^{np} \) and \( z = -c_{p'}^0 q^{np-r} \), with \( a = 0, \ldots, p' - 1 \) and \( n \in \mathbb{N} \), are then the complete set of poles of the function of the right-hand side of (4.45).

4.3. **Massive admissible characters.** Characters of the massive admissible representation \( \mathfrak{P}_{r,s,h,p,p'} \) from Sec. 3.3 are read off from the resolution (3.23). We thus find

\[
\omega_{r,s,h,p,p'}(z,q) = z^h \frac{\vartheta_{1,0}(z,q)}{\eta(q)^3} q^{\frac{r}{2}[(r-p)(s-1)^2-\frac{r}{2}(s-p)(s-1)(s-2)]-\frac{rp'}{2}} \times \\
\times \left( \vartheta_{1,0}(q^{-p'-(s-1)p+r}, q^{2p'}) - q^{r(s-1)} \vartheta_{1,0}(q^{2p'}-(s-1)p-r, q^{2p'}) \right),
\]

which, as we see, has the structure of a massive Verma module character times a Virasoro character (more precisely, the ratio of the irreducible Virasoro character to the Virasoro Verma module character).

The relation between the embedding case III(0) and the Virasoro embedding structure has been pointed out in [1].

5. **The \( \hat{\mathfrak{sl}}(2) \leftrightarrow \mathbb{N}=2 \) correspondence at the level of characters**

The construction [20] of the \( \mathbb{N}=2 \) algebra, which in fact determines the simplest Kazama–Suzuki coset \( \hat{\mathfrak{sl}}(2) \oplus u(1)/u(1) \), allows one to completely analyse the representations involved in this construction. This was done in [2] where the \( \hat{\mathfrak{sl}}(2) \) and \( \mathbb{N}=2 \) representation categories were proved to be equivalent up to the respective spectral flows. In this section, we recall the equivalence and then derive the consequences for characters.

5.1. **Reminder on the equivalence of categories.** For simplicity, we avoid introducing chains of modules and speak instead of representations *modulo the spectral flow*; exact definitions can be found in [2]. We will only need representations with a highest-weight vector, so all the general statements below are limited to this case.

**Theorem 5.1 ([2]).** There exists a functor \( F \) that assigns to every \( \hat{\mathfrak{sl}}(2) \) representation with a highest-weight vector \( |j,t\rangle_{\hat{\mathfrak{sl}}(2)} \) an \( \mathbb{N}=2 \) representation with the highest-weight vector \( |-2j/t, t\rangle_{\mathbb{N}=2} \) up to the spectral flow,

\[
F(\mathcal{C}_{j,t}) = \mathcal{D}_{-\frac{2j}{t}, t}. \]

There exists the inverse functor modulo the spectral flow. In particular, a homomorphism \( \mathcal{C}_{j',t} \rightarrow \mathcal{C}_{j,t} \) of \( \hat{\mathfrak{sl}}(2) \) representations exists if and only if there is a homomorphism of \( \mathbb{N}=2 \) representations \( \mathcal{D}_{j',t} \rightarrow \mathcal{D}_{-\frac{2j}{t}, t} \) with some \( \theta \) and

\[
F(\mathcal{C}_{j',t}) = \mathcal{D}_{j',t}. \]
Underlying this Theorem is the construction of the $N=2$ algebra in terms of $\hat{s}\ell(2)$ and the ghosts \cite{20,21} and the analysis of representations initiated in \cite{20}; introducing the fermionic first-order $BC$ system, one can construct the $N=2$ generators as

\begin{align}
(5.2) & \quad Q = CJ^+, \quad G = \frac{1}{2}BJ^-, \\
(5.3) & \quad \mathcal{H} = \frac{1}{4}BC - \frac{1}{2}J^0, \quad \mathcal{T} = \frac{1}{4}(J^+J^-) - \frac{1}{4}B\partial C - \frac{1}{4}BCJ^0,
\end{align}

where $G = \sum_{n \in \mathbb{Z}} G_n z^{-n-2}$, $T = \sum_{n \in \mathbb{Z}} T_n z^{-n-2}$, $Q = \sum_{n \in \mathbb{Z}} Q_n z^{-n-1}$, $\mathcal{H} = \sum_{n \in \mathbb{Z}} \mathcal{H}_n z^{-n-1}$, $J^{\pm,0} = \sum_{n \in \mathbb{Z}} J_n^{\pm,0} z^{-n-1}$, and $B(z)C(w) = \frac{1}{z-w}$. There also exists an extra current

\begin{align}
(5.4) & \quad I^+ = \sqrt{\frac{1}{4}}(BC + J^0)
\end{align}

that commutes with the $N=2$ generators \eqref{5.2}-\eqref{5.3}.

Now, let $\Omega$ be the module generated from the vacuum $|0\rangle_{gh}$ defined by the highest-weight conditions $C_{\geq 1} |0\rangle_{gh} = B_{\geq 0} |0\rangle_{gh} = 0$ and let $\mathcal{F}^+_\rho$ be the Fock space generated from the highest-weight vector determined by $I_{\geq 1} |\rho\rangle^+ = 0$ and $I_0 |\rho\rangle^+ = \rho |\rho\rangle^+$.

Given an $\hat{s}\ell(2)$ representation $\mathcal{E}_{j,t}$ (e.g., from category $\mathcal{O}$), the tensor product $\mathcal{E}_{j,t} \otimes \Omega$ carries the representation \eqref{5.2}-\eqref{5.3} of the $N=2$ algebra. This representation is highly reducible; decomposing it, we arrive at the representations $\mathcal{D}_{\frac{1}{2}j,t,\theta} = F(\mathcal{E}_{j,t})$:

\begin{align}
(5.5) & \quad \mathcal{E}_{j,t,\theta} \otimes \Omega \approx \bigoplus_{m \in \mathbb{Z}} \mathcal{D}_{\frac{1}{2}j,t,m-\theta} \otimes \mathcal{F}^+_\sqrt{\frac{1}{4}(j-\omega^2 m)}.
\end{align}

This formula, proved in \cite{2}, is essentially equivalent to the statement of the theorem.

5.2. Identities relating $N = 2$ and $\hat{s}\ell(2)$ characters. Theorem 5.1 has a direct implication for characters. Equations \eqref{5.2}-\eqref{5.4} imply the identities

\begin{align}
(5.6) & \quad T_{\text{Sug}} + T_{\text{gh}} = \mathcal{T} + T^+ \\
(5.7) & \quad J^0 - BC = -2\mathcal{H} + \frac{1}{4\sqrt{2}t} I^+,
\end{align}

where $T_{\text{Sug}}$ is the Sugawara energy-momentum tensor for the affine $\hat{s}\ell(2)$, $T_{\text{gh}} = -B\partial C$ is the ghost energy-momentum tensor, and

\begin{align}
(5.8) & \quad T^+ = \frac{1}{2} (I^+)^2 - \frac{1}{\sqrt{2}t} \partial I^+.
\end{align}

We thus have a formal equality

\begin{align}
(5.9) & \quad q_0^{T_{\text{Sug}} + T_{\text{gh}}} z^j_0 (zy^{-1})_0^{(BC)_0} = q_{C_{\geq 0} + L^+_0 y^{-\mathcal{H}_0 - \sqrt{\frac{1}{2}}} I^+} z^{\sqrt{\frac{1}{2}}} l^+_0.
\end{align}

We now take the traces of both sides over the spaces on the respective sides of Eq. \eqref{5.5}. Then, for any two modules related by the functor, $\mathcal{E}_{j,t}$ on the $\hat{s}\ell(2)$ side and $\mathcal{D}_{\frac{1}{2}j,t}$ on the $N=2$ side, we obtain the following relation between their characters:

\begin{align}
(5.10) & \quad \chi_{\mathcal{E},j,t}(z,q) \chi^{\text{gh}}(zy^{-1},q) = \sum_{\theta \in \mathbb{Z}} \omega_{\frac{1}{2}j,t,\theta} (y^{-1},q) \cdot \left(\frac{y^{-\sqrt{\frac{1}{2}}} z^{\sqrt{\frac{1}{2}}} \sqrt{\frac{1}{2}(j-\omega^2 \theta)^2 + j-\omega^2 \theta} q^{i_{j-\theta}^2 + j-\omega^2 \theta}}{1-q^m}\right) \prod_{m \geq 1} (1-q^m),
\end{align}
where $1 < |z| < |q|^{-1}$. Recall now that the ghost character is

$$\chi_{gh}^{\ell}(z, q) = \text{Tr}_{\ell}(q^L_{\ell} z^{-2}(BC)_0) = \prod_{m=0}^{\infty} (1 + q^m z) \prod_{m=1}^{\infty} (1 + q^m z^{-1}) = q^{-\frac{1}{12}} z^{\theta_{1,0}(z, q)} \eta(q).$$

Thus, we have arrived at

**Theorem 5.2.** Let $\mathcal{C}_{j,t}$ be an $\hat{\mathfrak{sl}}(2)$ representation from category $\mathcal{O}$ with the highest-weight vector $|j, t\rangle_{\hat{\mathfrak{sl}}(2)}$ and let $F(\mathcal{C}_{j,t}) = \mathcal{D}_{\frac{j}{2}, t, \theta}$. Let $\chi_{\mathcal{C}_{j,t}}(z, q)$ be the character of $\mathcal{C}_{j,t}$ and $\omega_{\mathcal{D}_{\frac{j}{2}, t, \theta}}(z, q)$ be the character of one of the representations $\mathcal{D}_{\frac{j}{2}, t, \theta}$. Then (assuming $1 < |z| < |q|^{-1}$),

$$\chi_{\mathcal{C}_{j,t}}(z, q) = \sum_{\theta \in \mathbb{Z}} \omega_{\mathcal{D}_{\frac{j}{2}, t, \theta}}(y^{-1}, q) y^{-\frac{j}{4}(j-\theta)} y^{-j-\theta} q^{(j-\theta)^2 + j + \theta + \frac{1}{8}}$$

$$= \sum_{\theta \in \mathbb{Z}} \omega_{\mathcal{D}_{\frac{j}{2}, t, \theta}}(y^{-1} q^{-\theta}, q) y^{\theta - \frac{j}{4} j - j - \theta} q^{\theta^2 + j + \theta + \frac{1}{8}},$$

**Remark 5.3.** Taking a twisted module on the $\hat{\mathfrak{sl}}(2)$ side, we can proceed in a similar way and arrive at

$$\chi_{\mathcal{C}_{j,t, \mu}}(z, q) = \sum_{\theta \in \mathbb{Z}} \omega_{\mathcal{D}_{\frac{j}{2}, t, \theta}}(y^{-1} q^{-\theta}, q) y^{\theta - \frac{j}{4} j - j - \theta} q^{\theta^2 + j + \theta + \frac{1}{8}},$$

where $|q|^\mu < |z| < |q|^{-1}$.

Equation (5.12) can be viewed as a “generalised branching rule” (or, if read from right to left, a “sum rule”), termed “generalised” because it contains an infinite number of terms on the right-hand side. The most interesting cases are when the characters on the right-hand side are quasi-periodic and, thus, the summation contains only a finite number of terms. This will be considered in Sec. 5.4 where we obtain true branching relations. However, even with the nonperiodic Verma-module characters, Eq. (5.12) becomes the $\hat{\mathfrak{sl}}(2)$ denominator identity, see (5.13).

It should be kept in mind that $F$ has the inverse functor, and therefore, every $N=2$ representation admits a “sum rule” yielding the character of the corresponding $\hat{\mathfrak{sl}}(2)$ representation after taking the trace over the spectral flow transforms, as in (5.12).

**5.3. An integral representation for $N=2$ characters.** Relation (5.12) implies an integral representation for the $N=2$ characters:

$$\omega_{\mathcal{D}_{\frac{j}{2}, t}}(y, q) = y^h q^{-\frac{j}{4} (h(t-2) - h^2)} \frac{1}{2 \pi i} \oint dz z^{-\frac{h}{4} - 1} \chi_{\mathcal{D}_{\frac{j}{2}, t}}(z, q) \vartheta_{1,0}(z y, q).$$

As before, $\mathcal{D}_{\frac{j}{2}, t}$ and $\mathcal{C}_{j,t, \bullet}$ are some $N=2$ and $\hat{\mathfrak{sl}}(2)$ representations, respectively, related by the functor. The integration contour encompasses the origin counterclockwise and lies inside the annulus determined by the twist of the chosen $\mathcal{C}$ representation.

**5.4. Examples.** Let us see how the general formula of Theorem 5.2 rewrites for particular representations.
5.4.1. **Trivial representations.** For the trivial representations \((t = 2, j = 0, \chi_{0,2,H}(z,q) = 1,\) and \(\omega_{0,2}(z,q) = 1), Eq. (5.12) becomes simply the definition of \(\vartheta_{1,0}(z,q)\).

5.4.2. **Topological Verma modules.** Taking the \(N=2\) representations to be the twisted topological Verma modules, and the \(\hat{sl}(2)\) representation, accordingly, the Verma module, we see that Eq. (5.12) becomes the “truly remarkable identity” from [8, Example 4.1]:

\[
\frac{\vartheta_{1,0}(z y^{-1}, q) \eta(q)^3}{\vartheta_{1,0}(y^{-1}, q) \vartheta_{1,1}(z, q)} = \sum_{\theta \in \mathbb{Z}} \frac{z^{-\theta}}{1 + y q^\theta}, \quad 1 < |z| < |q|^{-1},
\]

\[
(\sum_{m \geq 0} \sum_{k \leq -1} z_m (-1)^k y^k q^{mk}, \quad 1 < |z| < |q|^{-1}, \quad |q|^{-n+1} < |y| < |q|^{-n}.
\]

Taking the \(\hat{sl}(2)\) Verma module with an arbitrary twist, we have, similarly,

\[
(-1)^\mu y^{-\mu} \frac{\vartheta_{1,0}(z y^{-1}, q) \eta(q)^3}{\vartheta_{1,0}(y^{-1}, q) \vartheta_{1,1}(z, q)} = \sum_{m \in \mathbb{Z}} \frac{z^{-m} q^{\mu m}}{1 + y q^m}, \quad |q|^{\mu} < |z| < |q|^{\mu-1}.
\]

As a particular case, we recover the identity that was written out already in [8]:

\[
\frac{\vartheta_{1,0}(z, q)}{\vartheta_{1,1}(z, q)} = \frac{\vartheta_{1,0}(z^2, q)}{\eta(q)^3} \sum_{\theta \in \mathbb{Z}} \frac{z^{-\theta+1}}{1 + z^2 q^\theta}, \quad 1 < |z| < |q|^{-1}.
\]

5.4.3. **Admissible representations: resummation and the integral formula.** For the admissible representation characters given by (4.20) and (4.2), Eq. (5.12) becomes

\[
\sum_{\theta \in \mathbb{Z}} z^{-\theta} \varphi_{r,s,p,p'}(y^{-1} q^{-\theta}, q) = q^{-\frac{pp'}{4}} \frac{\vartheta_{1,0}(z y^{-1}, q) \eta(q)^3}{\vartheta_{1,0}(y^{-1}, q)} \left( \vartheta_{1,0}(z^p q^{-pp'-p(s-1)+p' r}, q^{2pp'}) - q^{r(s-1)} z^{-r} \vartheta_{1,0}(z^p q^{-pp'-p(s-1)-p' r}, q^{2pp'}) \right),
\]

where \(1 < |z| < |q|^{-1}\). This can also be obtained from the previous identities by first deriving

\[
\sum_{\theta \in \mathbb{Z}} z^{-\theta} \varphi_{r,s,p,p'}(y^{-1} q^{-\theta}, q) = q^{-\frac{pp'}{4}} \sum_{\theta \in \mathbb{Z}} \frac{z^{-\theta}}{1 + y q^\theta} \left( \vartheta_{1,0}(z^p q^{-pp'-p(s-1)+p' r}, q^{2pp'}) - q^{r(s-1)} z^{-r} \vartheta_{1,0}(z^p q^{-pp'-p(s-1)-p' r}, q^{2pp'}) \right)
\]

and then combining this with (5.15).

Recall that the function \(\varphi_{r,s,p,p'}(z,q)\) defined in (4.22) cannot be algebraically expressed through theta functions (unless \(p' = 1\)), since it is not quasi-periodic. On the other hand, there is an integral representation for \(\varphi_{r,s,p,p'}(z,q)\) with the integrand given by a combination of theta functions. The integral formula for the admissible \(N=2\) characters takes the form of the following representation for \(\varphi_{r,s,p,p'}\):

\[
\frac{\vartheta_{1,0}(y, q)}{\eta(q)^3} \varphi_{r,s,p,p'}(y, q) = \frac{1}{2\pi i} \oint_{C_0} \frac{dz}{z} \frac{z^{-pp'}}{\vartheta_{1,0}(z, q)} \left( \vartheta_{1,0}(z^p q^{-pp'+p'(s-1)p}, q^{2pp'}) - z^{-r} q^{r(s-1)} \vartheta_{1,0}(z^p q^{-pp'-p(s-1)-p' r}, q^{2pp'}) \right),
\]
where we use the notation $C_p(n)$ for the integration contour that surrounds the origin counterclockwise and lies in the annulus

$$|q|^{-2p'n} < |z| < |q|^{-2p'n-1}.$$ 

In the present case of $n = 0$, the integral is given by the sum over the poles inside the unit circle. Now, under the spectral flow transform with $\theta = pn$, the integration contour changes as $C_p(0) \rightarrow C_p(n)$, which is described by adding or subtracting the corresponding poles described in Lemma 4.4. We parametrise these as $z = q^n$, $n = -2p' \ell - a - 1$, $\ell \in \mathbb{Z}$. Then

\begin{equation}
(5.24) \quad \text{res}_{z=q^{-2p'\ell-a-1}} \left( z^{-1} \frac{\partial_{1,0}(zy^{-1}, q)}{\vartheta_{1,1}(z, q)} \times \left( \vartheta_{1,0}(zp q^{-pp'+rp'}-(s-1)p, q^{2pp'}) - z^{-r} q^{r(s-1)} \vartheta_{1,0}(zp q^{-pp'-rp'}-(s-1)p, q^{2pp'}) \right) \right) = \frac{\vartheta_{1,0}(y^{-1}, q)}{\eta(q)^3} (-1)^{a+1} y^{-2p'\ell-a-1} q^{-pp'/4} q^{-pp'\ell^2 - tp\ell a + ta + tp\ell r} \times \left( \vartheta_{1,0}(q^{-pa-ps-pp'+rp'}, q^{2pp'}) - q^{r(a+s)} \vartheta_{1,0}(q^{-pa-ps-pp'-rp'}, q^{2pp'}) \right),
\end{equation}

which allows us to rederive Eqs. (4.23)–(4.24) that were obtained directly from (4.22), and, as a by-product, explains why the terms violating quasiperiodicity of the admissible $N = 2$ characters are precisely the residues of the $\hat{sl}(2)$ admissible characters.

### 5.4.4. Unitary representations: branching, or sum rule, relations.

This case is of some special interest from our present point of view because the $N = 2$ characters are quasiperiodic, see (1.32). This allows us to rewrite (5.22) in the form with only a finite number of terms on the right-hand side. The summation goes over the orbit of the $N = 2$ spectral flow on the unitary representations:

\begin{equation}
(5.23) \quad \chi_{r,p}(z, q) \vartheta_{1,0}(z^{-1}, q) = q^{\frac{r+1}{4p}} \cdot \frac{\eta(q^p)^3}{\eta(q)^3} \sum_{a=0}^{p-1} \omega_{r,p}^{\frac{r+1}{4p}} q^{r-a} q^a y^{-1} \vartheta_{1,0}(z^{p-q^{-r}}, q^{2p}) \vartheta_{1,0}(z^{p q^{-r-2a}}, q^{2p}).
\end{equation}

Explicitly substituting the $\hat{sl}(2)$ and $N = 2$ characters given by (1.13) and (4.33), respectively, we obtain

\begin{equation}
(5.24) \quad \frac{\vartheta_{1,0}(zp q^{-p+r}, q^{2p}) - z^{-r} \vartheta_{1,0}(zp q^{-p-r}, q^{2p})}{\vartheta_{1,1}(z, q)} \vartheta_{1,0}(zy, q) = - \frac{\eta(q^p)^3}{\eta(q)^3} \vartheta_{1,1}(q^r, q^p) \vartheta_{1,0}(y, q) \sum_{a=0}^{p-1} z^{-a} q^{a} q^{-1} \vartheta_{1,0}(z^{p} q^{2r-p-2a}, q^{2p}) \vartheta_{1,0}(y q^{-r-a}, q^p).
\end{equation}

In the special case where $p = 2r$, the theta-function identity contains only $r$ terms on the right-hand side,

\begin{equation}
(5.25) \quad \frac{\vartheta_{1,1}(z^r, q^r)}{\vartheta_{1,1}(z, q)} \vartheta_{1,0}(zy, q) = \frac{\eta(q^r)^3}{\eta(q)^3} \vartheta_{1,0}(y, q) \sum_{a=0}^{r-1} q^{-r} \vartheta_{1,0}(z^{r} y q^{-a}, q^r) z^{-a},
\end{equation}

where we have used Eqs. (33) and (4.34) for the $\hat{sl}(2)$ and $N = 2$ characters, respectively.
6. Concluding remarks

In this paper, we have evaluated the characters of admissible $N = 2$ representations, making the derivation parallel to the affine-$\mathfrak{sl}(2)$ case and explicitly deriving the exact relations between the $N = 2$ and $\hat{\mathfrak{sl}}(2)$ characters. An important tool in our analysis of characters has been the spectral flow transform of the $\hat{\mathfrak{sl}}(2)$ and $N = 2$ algebras.

We have constructed different resolutions of the admissible $N = 2$ representations. Although the embedding structure \cite{11} of massive $N = 2$ Verma modules is more complicated than in the familiar case of $\mathfrak{sl}(2)$ Verma modules, it is still not very difficult to rewrite the embedding diagrams as the resolutions consisting of massive Verma modules (we have considered in detail the most involved case $\text{III}_0^0(2, -+)$ from \cite{11}, others can be dealt with along the same lines). In the massive resolutions, the number of modules grows from term to term.

There also exists a simpler resolution in terms of twisted topological Verma modules, whose structure is in fact equivalent to the well-known BGG resolution for irreducible $\mathfrak{sl}(2)$ representations (the character formulae following from the massive resolution have been shown to agree with those derived from the BGG resolution). This has a deep reason that rests in the equivalence of $\mathfrak{sl}(2)$ and $N = 2$ representation categories up to the spectral flows \cite{2}. The equivalence can also be viewed as the “reason” why the earlier proposal \cite{3} for the unitary $N = 2$ characters is correct despite a number of subtleties that have not been explicitly taken into account in \cite{5}. A detailed analysis of the structure of the $N = 2$ Verma modules \cite{10, 11} has allowed us to find the characters in a considerably more general case of the admissible $N = 2$ representations. Not being quasiperiodic under the spectral flow, these characters cannot be algebraically expressed through the theta-functions. On the other hand, we have found an integral representation for these characters through the respective (hence, admissible) $\mathfrak{sl}(2)$ characters (which are expressed in terms of the theta-functions).

We have shown that a certain sum of the $N = 2$ characters over the orbit of the $N = 2$ spectral flow provides a kind of “harmonic decomposition” of an $\hat{\mathfrak{sl}}(2)$ character. This identity relating the $N = 2$ and $\hat{\mathfrak{sl}}(2)$ characters can be considered as the exact “branching rule” that holds for any representation, not necessarily the unitary ones, although it is only for the unitary representations that the identity involves a finite number of terms on the $N = 2$-side. The resulting theta-function identities are, therefore, also a consequence of the equivalence of categories.

The equivalence of categories, which we have used for the topological Verma modules and the irreducible representations of the corresponding highest-weight type, can also be formulated for the massive $N = 2$ Verma modules (relating them to the relaxed Verma modules over $\hat{\mathfrak{sl}}(2)$ \cite{4}) and for the massive-admissible representations (3.25). The corresponding character identities can easily be derived, however they are not very interesting.

The analysis of this paper can be viewed from the broader perspective of coset models; a part of what we have done was to investigate the $N = 2$ algebra pretending to know only that this is $\hat{\mathfrak{sl}}(2) \oplus u(1)/u(1)$. As regards more general cosets, very suggestive pieces of the picture that we have completely described
in the $N = 2$ case can be found in [23] in a more complicated case of $N = 4$ algebras (in fact, one can ask about different “sumrules” for characters whether they follow from some relations between representations). Note also that both the $N = 2$ and $\hat{sl}(2)$ algebras are closely related to the affine Lie superalgebra $\hat{sl}(2|1)$, see [31, 33] and references therein. Thus, it would be interesting to apply the present approach to the results of [32], especially because the denominator $\hat{sl}(2|1)$ identity is already the particular case of the formula expressing the $N = 2 \leftrightarrow \hat{sl}(2)$ equivalence obtained by taking the Verma modules in the general identity (5.12).

**Acknowledgements.** We are grateful to V. Ya. Fainberg, A. V. Odessky, and M. A. Soloviev for useful discussions. AMS and IYT were encouraged by very stimulating discussions with F. Malikov. IYT wishes to thank A. A. Kirillov and I. Paramonova for a discussion. AMS is grateful to P. H. Damgaard for kind hospitality at the Niels Bohr Institute, where a part of this paper was written, and to I. Shchepochkina for a discussion. VAS is thankful to A. V. Gurevich for his kind attention to her work. This work was supported in part by the RFBR Grant 98-01-01155.

**Appendix A. Theta-function conventions**

Since we deal with Verma-module characters along with the characters of various irreducible representations, we prefer using Jacobi theta-functions (rather than the higher-level theta functions) for all of the characters that can be expressed through theta-functions. We thus introduce the Jacobi theta functions

(A.1) \[ \vartheta_{1,1}(z, q) = q^{1/8} \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{1}{2}(m^2 - m)} z^{-m} = q^{1/8} \prod_{m \geq 0} (1 - z^{-1} q^m) \prod_{m \geq 1} (1 - z q^m) \prod_{m \geq 1} (1 - q^m), \]

(A.2) \[ \vartheta_{1,0}(z, q) = q^{1/8} \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}(m^2 - m)} z^{-m} = q^{1/8} \prod_{m \geq 0} (1 + z^{-1} q^m) \prod_{m \geq 1} (1 + z q^m) \prod_{m \geq 1} (1 - q^m). \]

Then, for $\theta \in \mathbb{Z}$,

(A.3) \[ \vartheta_{1,1}(z \theta^q, q) = (-1)^\theta q^{-\frac{1}{2} \theta^2} \vartheta_{1,1}(z, q), \]

(A.4) \[ \vartheta_{1,0}(z \theta^q, q) = q^{-\frac{1}{2} \theta} \vartheta_{1,0}(z, q). \]

We also use the Dedekind eta-function

(A.5) \[ \eta(q) = q^{1/24} \sum_{m = 0}^{\infty} (-1)^m q^{\frac{1}{2}(3m^2 + m)} = q^{1/24} \prod_{m = 1}^{\infty} (1 - q^m). \]

The following useful identities are elementary to prove (the prime means $\partial/\partial z$):

(A.6) \[ \vartheta_{1,0}(z, q) = \frac{q^{z^2} \eta(q)}{\eta(q^2)} \vartheta_{1,0}(z, q^2) \vartheta_{1,0}(zq, q^2); \]

(A.7) \[ \vartheta_{1,1}(z, q) = q^{-\frac{1}{8}} (\vartheta_{1,0}(z^2 q^{-1}, q^4) - z^{-1} \vartheta_{1,0}(z^2 q^{-3}, q^4)), \]

(A.8) \[ \vartheta_{1,0}(z, q)|_{z = -q^n} = (-1)^{n+1} \eta(q)^3 q^{-\frac{n^2}{2} - \frac{3n}{2}}, \quad n \in \mathbb{Z}, \]

(A.9) \[ \vartheta_{1,1}(z, q)|_{z = q^n} = (-1)^n \eta(q)^3 q^{-\frac{n^2}{2} - \frac{3n}{2}}, \quad n \in \mathbb{Z}, \]
\[ \vartheta_{1,1}(q, q^2) = -q^{-\frac{3}{4}} \frac{\eta(q)^2}{\eta(q^2)}, \quad \vartheta_{1,0}(1, q) = 2 \frac{\eta(q^2)^2}{\eta(q)}. \]

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