NON-SEMISIMPLE TQFT’S AND BPS $q$-SERIES

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Abstract. We propose and in some cases prove a precise relation between 3-manifold invariants associated with quantum groups at roots of unity and at generic $q$. Both types of invariants are labeled by extra data which plays an important role in the proposed relation. Bridging the two sides — which until recently were developed independently, using very different methods — opens many new avenues. In one direction, it allows to study (and perhaps even to formulate) $q$-series invariants labeled by spin$^c$ structures in terms of non-semisimple invariants. In the opposite direction, it offers new insights and perspectives on various elements of non-semisimple TQFT’s, bringing the latter into one unifying framework with other invariants of knots and 3-manifolds that recently found realization in quantum field theory and in string theory.

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1. Introduction and Summary

As part of a larger quest for new quantum invariants of 3-manifolds, we wish to establish a precise relation between the CGP invariants $N_r(M, \omega)$ and the GPPV invariants $\hat{Z}_s(M,q)$, introduced in [17] and [33, 32] respectively.

Here we summarize some of the essential features of these invariants, compare side-by-side what they depend on, and present a motivation for why one should expect a connection between these two rather different sets of invariants. First, perhaps the most obvious part of input data that both sets of invariants need is a choice of 3-manifold $M$. In either case, however, it needs to be equipped with additional structure; in the case of CGP invariants $N_r(M, \omega)$ it involves a choice of $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z})$, whereas in the case of GPPV invariants $\hat{Z}_s(M,q)$ it depends on $s \in \text{Spin}^c(M)/\mathbb{Z}_2$. Although these structures are clearly different, they both are related\textsuperscript{1} to $H_1(M)$ and should be regarded as mutual counterparts in identifying the two sets of invariants, as we will see below.

Similarly, the dependence of $N_r(M, \omega)$ on a positive integer $r \neq 0$ mod 4 should be compared to the $q$-dependence of $\hat{Z}_s(M,q)$. Indeed, both sets of invariants are quantum group invariants of 3-manifolds, with $\xi = e^{\pi i r}$ and $q$ respectively playing the role of the quantum parameters. The definition of $N_r(M, \omega)$ is based [17] on the representation theory of the unrolled restricted quantum group $\widehat{U}^H_\xi (\mathfrak{sl}_2)$, whereas $\hat{Z}_s(M,q)$ should be thought of as a quantum group invariant associated with $U_q(\mathfrak{sl}_2)$ at generic $|q| < 1$.

\textsuperscript{1}Recall that non-canonically $\text{Spin}^c(M) \cong H_1(M; \mathbb{Z})$. 

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Indeed, $\hat{Z}_q(M,q)$ basically gives a non-perturbative definition of “$SL(2, \mathbb{C})$ Chern-Simons theory” that behaves well under surgery and has line operators labeled by Verma modules of arbitrary complex weight [28, 48].

\begin{align*}
\text{(1.1)} & \quad \begin{array}{c|c|c}
\text{quantum parameter} & \xi = e^{\pi i/r} \text{ root of unity} & \hat{Z}_q(M,q) \\
\text{quantum group} & U^H(\mathfrak{sl}_2) & U_q(\mathfrak{sl}_2) \\
\text{additional structure} & \omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z}) & s \in \text{Spin}^c(M)/\mathbb{Z}_2
\end{array}
\end{align*}

Since the relation between $U_q(\mathfrak{sl}_2)$ and $U^H_q(\mathfrak{sl}_2)$ involves, among other things, specializing $q$ to be a root of unity, one might expect $N_r(M,\omega)$ to be related to limiting values of $\hat{Z}_q(M,q)$ at roots of unity. This relation between quantum groups was a large part of the motivation in [29], where a relation of this form was established for knot complements. Namely, it was proposed that $F_K(x,q) := \hat{Z}(S^3 \setminus K)$ at roots of unity give polynomial knot invariants $\text{ADO}_r(x;K)$ introduced by Akutsu-Deguchi-Ohtsuki [1]:

**Conjecture 1.**

\begin{equation}
\text{(1.2)} \quad F_K(x,q)|_{q = \xi^2} = \frac{\text{ADO}_r(x/\xi^2;K)}{\Delta_K(x^r)} \cdot (x^{1/2} - x^{-1/2})
\end{equation}

and the problem of establishing a similar relation for more general 3-manifolds was posed. Addressing this problem requires 3-manifold invariants which specialize to ADO invariants for knot complements, and CGP invariants $N_r(M,\omega)$ perfectly fit the bill:

\begin{equation}
\text{(1.3)} \quad \text{ADO}_r(x^2/\xi^2) = \frac{x^r - x^{-r}}{x - x^{-1}} N_r(S^3, K_\alpha), \quad \text{where} \quad x = e^{\pi i/\alpha}, \xi = e^{\pi i/r}.
\end{equation}

The composition of (1.2) and (1.3) not only gives us the first instance of the sought-after relation between CGP and GPPV invariants in a certain class of 3-manifolds, but also provides a clue for the relation between parameters $r$ and $q$ for more general $M$:

\begin{equation}
\text{(1.4)} \quad N_r(M,\omega) \quad \longleftrightarrow \quad \hat{Z}(M,q)|_{q = \xi^2 = e^{2\pi i/r}}.
\end{equation}

Another useful clue that follows from (1.2)–(1.3) is that the relation between $N_r(M,\omega)$ and $\hat{Z}(M,q)$ should be linear,

\begin{equation}
\text{(1.5)} \quad N_r(M,\omega) = \sum_s c^\text{CGP}_s \hat{Z}_s(M,q)|_{q = e^{2\pi i/r}}
\end{equation}

much like analogous relations between $\hat{Z}(M,q)$ and other invariants of $M$, such as the inverse Turaev torsion [54, 52], Witten-Reshetikhin-Turaev (WRT) invariants [56, 51], and Rokhlin invariants.
Figure 1. One of our main goals is to explore possible relations between $N_r(M, \omega)$ and the limiting behavior of $\tilde{Z}(M, q)$ at roots of unity.

Establishing the relation (1.5) — and, in particular, determining the precise form of the coefficients $c_{CGP(\omega, s)}$ that relate additional structures on $M$ that enter the two sets of invariants — is one of our main goals. We find

Conjecture 2. Let $M$ be a rational homology sphere. Then, the following relation holds\(^2\):

\begin{equation}
N_r(M, \omega) = \sum_{s \in Spin^c(M)} c_{CGP(\omega, s)} \tilde{Z}_s(M, q) \Big|_{q \to e^{2\pi i r}}
\end{equation}

with

\begin{equation}
c_{CGP(\omega, s)} = \frac{T(M, [\omega])}{|H_1(M; \mathbb{Z})|} \times \begin{cases}
-e^{-\frac{\mu(M,s)}{2}} \sum_{a,f} e^{2\pi i \left(\frac{r-1}{4} \ell_k(a,a) + \ell_k(a,f-b) - \frac{1}{2} \omega(a) + \ell_k(f,f)\right)} & , \text{if } r \equiv 1 \mod 4, \\
\sqrt{|H_1(M; \mathbb{Z})|} \sum_{a} e^{-\frac{\mu(M,s)}{2} \ell_k(a,a) - 2\pi i \ell_k(a,b) - \pi i \omega(a)} & , \text{if } r \equiv 2 \mod 4, \\
-e^\frac{\pi i}{2} \mu(M,s) \sum_{a,f} e^{2\pi i \left(\frac{r+1}{4} \ell_k(a,a) - \ell_k(a,f+b) - \frac{1}{2} \omega(a) - \ell_k(f,f)\right)} & , \text{if } r \equiv 3 \mod 4
\end{cases}
\end{equation}

where:

- $T(M, [\omega])$ is a suitable version of the Reidemeister torsion (see Appendix A);
- $\ell_k(\cdot, \cdot)$ is the linking form on $H_1(M; \mathbb{Z})$, and in the sum $a, f \in H_1(M; \mathbb{Z})$;
- $\sigma$ is the canonical map

\begin{equation}\sigma : Spin(M) \times H_1(M; \mathbb{Z}) \longrightarrow Spin^c(M)\end{equation}

\(^2\text{Note that } \tilde{Z}_s(M, q), \text{ due to the overall factor } q^{\Delta_s}, \Delta_s \in \mathbb{Q}, \text{ are multivalued functions of } q (\text{the values on different branches differ by an overall phase}). \text{Throughout the paper, when writing the limit in the form } \sim q \to e^{i\phi}, \text{ it is assumed that the branch is such that } q^{\Delta_s} \to e^{i\phi \Delta_s}. \text{Equivalently, one can consider } \tilde{Z}_s(M, q) \text{ as single values functions of } \tau, \text{ with } q = e^{2\pi i \tau}. \text{Then } \sim q \to e^{i\phi} \text{ should be understood as } \sim \tau \to \phi/(2\pi) \text{ (taken from the upper half plane).}
producing a spin\textsuperscript{c} structure \( s = \sigma(b, s) \) on \( M \) from a spin structure\textsuperscript{3} \( s \) and \( b \in H_1(M; \mathbb{Z}) \).

- \( \mu(M, s) \) is the Rokhlin invariant of \( M \) for spin structure \( s \).

Note, the invariants \( N_r(M, \omega) \) are not defined for \( r = 0 \mod 4 \), which is why this case is omitted in (1.7). (For more details about this case, see Subsection 4.3.)

We prove the above conjecture under some technical hypotheses specified in Theorem 4.17, which can be stated in a simpler form as follows:

**Theorem 1.1** (Simplified form of Theorem 4.18). Let \( M \) be a rational homology sphere obtained by integral surgery on a framed link \( L \subset S^3 \) for which Conjecture 1 holds true. Then, under the technical hypotheses specified in Theorem 4.18, the Conjecture 2 is true for any pair \( (M, \omega) \) with \( \omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z}) \).

In the statement of the theorem though, we point out that the status of the invariant \( \hat{Z} \) is the following: assuming that Conjecture 1 holds for \( L \) (so that in particular a series \( F_L \) is defined), there is an explicit formula providing \( \hat{Z} \) for \( M \) but the full proof of the convergence of the formula and its invariance is not yet available.

An infinite family of cases to which Theorem 4.18 applies is provided in Example 4.20. The following is a list of cases for which the numerical evidence supports this result:

1. \( L \) is a plumbing link (for an infinite list of cases of these plumbing links the hypotheses of Theorem 4.18 can all be verified: see Example 4.20);
2. \( L \) is a trefoil knot.

We also provide another kind of numerical evidence for the case of surgeries on the figure eight knot by cross checking the above conjectures with similar conjectures relating \( \hat{Z} \) and the WRT invariants (see Section 6.2).

**Remark 1.2.** Conjecture 2 together with the conjectural relation between \( \hat{Z}_s \) and mod-2-cohomology-refined WRT invariant (see Appendix B) can be used to calculate the limiting values of \( \hat{Z}_s \) at \( q \to e^{\frac{2\pi i}{r}}, r \neq 0 \mod 4 \). Namely, assuming that \( T(M, [\omega]) \neq 0 \) for \( \omega \notin H^1(M; \mathbb{Z}/2\mathbb{Z}) \), one can invert the linear transform 1.6 formally substituting \( N_r(M, \omega)/T(M, [\omega]) \sim (-1)^r \text{WRT}_r(M, \omega)/(i\sqrt{8r}) \) when \( \omega \in H^1(M; \mathbb{Z}/2\mathbb{Z}) \). For \( r = 0 \mod 4 \) one can similarly use the relation between \( \hat{Z}_s \) and spin-version of CGP invariants (see Section 4.3) together with the spin-refined WRT invariant (see [31]). This can be especially useful since for many 3-manifolds \( \hat{Z}_s \) can be computed as \( q \)-series, but their modular properties, needed to determine the limiting values at roots of unity, are not known. Therefore, the relation to CGP and mod-2-cohomology-refined WRT invariants can serve as a substitute for modularity properties of \( \hat{Z}_s \) in problems that involve limiting values at roots of unity.

\textsuperscript{3}We use the standard typeface to denote spin structures (e.g. “\( s \)”) and Fraktur typeface to denote spin\textsuperscript{c} structures (e.g. “\( s \)”).

\textsuperscript{4}Note that conjecturally \( \frac{1}{2} \mu(M, s) - \ell k(b, b) = \Delta_{\sigma(b, s)} \mod 1 \), where \( \Delta_s \) is the overall rational shift in the powers of \( q \) in \( \hat{Z}_s \) for a given \( s \in \text{Spin}^c(M) \) [31].
When \( T(M, [\omega]) = 0 \) but \( N_r(M, \omega) \neq 0 \) for some \( \omega \notin H^1(M; \mathbb{Z}/2\mathbb{Z}) \) this indicates that \( \lim_{q \to e^{2\pi i}} \widehat{Z}_s = \infty \) at least for some \( s \). (The linear combinations of \( \widehat{Z}_s \) producing refined WRT invariants can still have finite limits.)

The study of the case of a manifold \( M \) with positive \( b_1 \) brought us to notice that the invariant \( \widehat{Z} \) for such manifolds requires an additional structure on \( M \), namely the choice of a splitting \( H_1(M; \mathbb{Z}) \) into its torsion part and its free part: \( \text{Tor}(H_1(M; \mathbb{Z})) \oplus \text{Free}(H_1(M; \mathbb{Z})) \), which we will denote from now on \( H_1(M; \mathbb{Z}) = T \oplus F \); we will also denote \( b' \) (resp. \( b'' \)) the projection of \( b \in H_1(M; \mathbb{Z}) \) in \( T \) (resp. \( F \)). For a fixed spin\(^c \) structure on \( M \), changing such a choice affects \( \widehat{Z}_s \) by multiplying it by a power of \( q \) (see (2.108) for the precise formulation) which we call the splitting anomaly.

A version of Conjecture 2 for the case of non-rational-homology-spheres is then the following:

**Conjecture 3.** Let \( M \) be any closed oriented 3-manifold. Choose a splitting \( H_1(M; \mathbb{Z}) = T \oplus F \) where \( T \) is the torsion part and \( F \) the free part. Then, the following holds:

\[
N_r(M, \omega) = \left. \sum_{b=b'+r'm, b' \in T, m \in F} c^\text{CGP}_{\omega, r'M(b, s)} \widehat{Z}_{\sigma(b, s)}(M, q) \right|_{q \to e^{2\pi i}}
\]

with

\[
c^\text{CGP}_{\omega, r'M(b, s)} = \frac{r'^1_T(M, [\omega])}{|\text{Tor} H_1(M; \mathbb{Z})|} \times \begin{cases} 
\sum_{a', f' \in T} e^{2\pi i \left( \frac{1}{4} \ell k(a', a') + \ell k(a', f') - b' - \frac{1}{2} \omega(a') + \ell k(f', f') + \omega'(m) - \frac{1}{4} \mu(M, s) + \frac{1}{2} \right)}, & \text{if } r = 1 \text{ mod } 4, \\
\sqrt{|\text{Tor} H_1(M; \mathbb{Z})|} \sum_{a' \in T} e^{-\frac{\pi i}{8} q_{\omega}(a') - 2\pi i \ell k(a', b') - \pi i \omega(a') + \pi i \omega'(m)}, & \text{if } r = 2 \text{ mod } 4, \\
\sum_{a', f' \in T} e^{2\pi i \left( \frac{1}{4} \ell k(a', a') + \ell k(a', f') - \frac{1}{2} \omega(a') + \ell k(f', f') + \omega'(m) + \frac{1}{4} \mu(M, s) + \frac{1}{2} \right)}, & \text{if } r = 3 \text{ mod } 4.
\end{cases}
\]

where \( r' = r \) if \( r \) is odd and \( \frac{r}{2} \) else and \( \omega' \) (resp. \( \omega'' \)) is the restriction of \( \omega \) on \( T \) (resp. on \( F \)).

**Remark 1.3.** In the above conjecture the choice of the splitting is auxiliary: as explained above, a different choice provides different values for \( \widehat{Z} \) but these choices are compensated by the change in the coefficients \( c^\text{CGP} \), so that the left-hand side is indeed independent on the splitting. Equivalently, the behavior of \( \widehat{Z} \) under the choice of such splitting is controlled by the behavior of the coefficients \( c^\text{CGP} \). Clearly, if \( M \) is a rational homology sphere the splitting is irrelevant and the above conjecture reduces to Conjecture 2.

Again, we can prove the above conjecture under suitable technical hypotheses which are omitted in the following statement:

**Theorem 1.4.** Under the technical hypotheses of Theorem 4.26, Conjecture 3 holds true for any pair \((M, \omega)\) with \( \omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z}) \).
(1) $M = \Sigma \times S^1$ for some closed oriented surface $\Sigma$;
(2) $M$ obtained by integral surgery on a framed link $L \subset S^3$ for which Conjecture 1 holds true.

Moreover, an infinite family of cases to which Theorem 4.26 applies is provided in Example 4.27.

In [7] the invariants $N_r$ where extended to a TQFT defined on a suitable category of cobordisms decorated with (relative) cohomology classes. Conjecturally, it should be possible to do the same for the invariant $\widehat{Z}$. In Section 5.2 we discuss this possibility and define (or sketch) two “operations on TQFTs” which, if applied to the conjectural TQFT for $\widehat{Z}$ would produce the TQFT build in [7], thus extending (unfortunately only partially) Conjecture 2 to the case of cobordisms.

The rest of the paper is organized as follows. In general we tried to be as self-contained as possible and we ascribed the physical motivations to dedicated sections so that the paper should be accessible for both mathematicians and physicists. Section 2 gives a self-contained review of all the relevant invariants that we wish to relate and introduces a number of technical tools that are used throughout the paper. After presenting a few families of concrete examples in Section 3, we then proceed to a more general and systematic discussion of the relation between $N_r$ and $\widehat{Z}$ invariants in Section 4. (A reader more interested in a general argument may prefer to read Section 4 first, before going through the examples in Section 3.) In Section 5, $\widehat{Z}$ and $N_r$ are considered as decorated TQFT’s and we propose how a relation between this richer structure can extend the relation between numerical 3-manifold invariants in the previous sections. Since both $\widehat{Z}$ and $N_r$ are related to Witten-Reshetikhin-Turaev (WRT) invariants, it is natural to ask whether our proposed relation between $\widehat{Z}$ and $N_r$ is compatible with the previously known relations. This question is answered in the affirmative in Section 6.

Finally, many useful facts about various refined invariants, gradings in CGP TQFT, and details related to the order of limits are collected in appendices.

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2. Preliminaries

2.1. ADO invariants of links in a three-sphere. One of the oldest non-semisimple invariants is the collection — labeled by integer $r \geq 2$ — of polynomial link invariants
introduced by Akutsu-Deguchi-Ohtsuki [1], or ADO invariants for short. Namely, let $L \subset S^3$ be an oriented framed link with $V$ components and $\xi = \exp \frac{2\pi i}{2r}$ for some $r \geq 2$. A coloring of $L$ is an assignment to each component $L_i$ of $L$ of a complex number $\alpha_i$. Then, the ADO invariant of $L$ colored by \{\alpha_i\} is an element of $\mathbb{Z}[\xi^\pm, \ldots, \xi^\pm]$ if $V \geq 2$ and an element of $\frac{1}{2\pi i} \mathbb{Z}[\xi^\pm]$ otherwise.

On the one hand, these polynomial link invariants are close cousins of the Alexander polynomial, and include the Alexander polynomial as a special case (corresponding to $r = 2$), cf. [43]. On the other hand, the ADO polynomials can be viewed as close cousins of the quantum group invariants that play a role in the Reshetikhin-Turaev construction. Indeed, the ADO polynomials were later re-formulated by Murakami [44] in terms of the R-matrix and quantum groups of at roots of unity. Specifically, the R-matrix (and its inverse) used in [44] is a map $V_\lambda \otimes V_\mu \to V_\mu \otimes V_\lambda$:

$$R_{kl}^{ij} = q^{\frac{1}{2}(\lambda - 2i - 2n)(\mu - 2j + 2n) + n(n-1)/2} \left\{ \frac{i + n; n}{n} \right\}_{\mu - j + n; n},$$

$$\left( R^{-1} \right)_{kl}^{ij} = (-1)^n q^{-\frac{1}{2}(\lambda - 2i)(\mu - 2j) - n(n-1)/2} \left\{ \frac{j + n; n}{n} \right\}_{\lambda - i + n; n},$$

where $n = l - i = j - k$, \{a\} = $q^a - q^{-a}$, \{x; n\} = $\prod_{i=0}^{n-1} (x - i)$ and $H_v^\lambda = (\lambda - 2i)v_i^\lambda$. We will return to this expression in Section 2.5 and compare it to the R-matrix that one encounters in the study of $\mathbb{Z}$-invariants.

Also, in Section 2.4 below we review how the ADO invariants were used in [17] to define invariants of 3-manifolds $M$ endowed with a cohomology class $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z})$. Namely, if $L$ is a link such that $M$ is obtained by integral surgery on $L$, then $\omega$ induces a coloring on $L$ by setting $\omega = \omega(m_I)$ where $m_I$ is the oriented meridian of $L_I$.

The study of these invariants remained detached from physics for almost 30 years. This is especially surprising given a large number of close ties that WRT invariants of knots and 3-manifolds have with quantum field theory and string theory. The situation started to change about a year ago [29] and we hope that the present paper can be another step toward bridging this gap, see also [15, 13, 19, 11, 12, 26, 50, 20, 25] for closely related work.

2.2. Combinatorics of spin and spin\(^c\)-structures. Let $M$ be obtained as an integral surgery on a framed oriented link $L$ in $S^3$ and let $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z})$. Let us index the components of $L$ by a set Vert and denote its $V \times V$ linking matrix by $B_{IJ}$, $I, J \in \text{Vert}$, $V := |\text{Vert}|$. Then we have the following identifications

$$H_1(M; \mathbb{Z}) \cong \mathbb{Z}^{\text{Vert}} / B\mathbb{Z}^{\text{Vert}},$$

$$H^1(M; \mathbb{Z}_2) \cong \{ c \in \mathbb{Z}_2^{\text{Vert}} \mid \sum_{I, J \in \text{Vert}} B_{IJ}c_I = 0 \pmod{2} \},$$

$$H^2(M; \mathbb{Z}) \cong \{ h \in \mathbb{Z}^{\text{Vert}} / B\mathbb{Z}^{\text{Vert}} \}.$$
Spin(M) \cong \{ s \in \mathbb{Z}_2^{\text{Vert}} \mid \sum_{I,J \in \text{Vert}} B_{IJ} s_{IJ} = B_{II} \mod 2 \}, \tag{2.6}

\text{Spin}^c(M) \cong \text{Char}(B)/2B\mathbb{Z}^{\text{Vert}} = \{ K \in \mathbb{Z}^{\text{Vert}}/2B\mathbb{Z}^{\text{Vert}} \mid K_I = B_{II} \mod 2 \}, \tag{2.7}

where \text{Char}(B) := \{ K \in \mathbb{Z}^{\text{Vert}} \mid K^T n = n^T B n \mod 2, \forall n \in \mathbb{Z}^{\text{Vert}} \} is the space of characteristic vectors of the lattice dual to the lattice \mathbb{Z}^{\text{Vert}} with the quadratic form \( B \). Here and in what follows we use \((\ldots)^T\) to denote vector transposition. The one-to-one correspondence between the elements \( a \) and \( K \) in (2.18) is given by \( a = K - B \varepsilon \), where

\( \varepsilon := (1, 1, 1, \ldots, 1) \in \mathbb{Z}^{\text{Vert}}. \)

The Bockstein homomorphism \( \beta : H^1(M; \mathbb{Z}_2) \to H^2(M; \mathbb{Z}) \) writes in the above notation as follows; for \( c \in \mathbb{Z}_2^{\text{Vert}} \) let \( \tilde{c} \in \mathbb{Z}^{\text{Vert}} \) be such that \( \tilde{c} \equiv c \mod 2 \). Then

\[ \beta(c) = \frac{1}{2} B \tilde{c} \in \mathbb{Z}^{\text{Vert}}/B\mathbb{Z}^{\text{Vert}} \]

where we observe that since \( B \tilde{c} \) is even, division by two is possible and its result in \( \mathbb{Z}^{\text{Vert}}/B\mathbb{Z}^{\text{Vert}} \) is independent of the choice of \( \tilde{c} \). We observe that \( \text{Spin}(M) \) is affine over \( H^1(M; \mathbb{Z}_2) \) via component-wise addition and \( \text{Spin}^c(M) \) is affine over \( H^2(M; \mathbb{Z}) \) by defining \( (K + [h])_I := K_I + 2h_I, \forall I \in \text{Vert} \). Observe also that there is a canonical map \( \iota : \text{Spin}(M) \to \text{Spin}^c(M) \) which in the above notation writes:

\[ \iota(s) = \sum_{I,J \in \text{Vert}} B_{IJ} \tilde{s}_{IJ} \]

where \( \tilde{s} \) is any lift of \( s \in \mathbb{Z}_2^{\text{Vert}} \) to \( \tilde{s} \in \mathbb{Z}^{\text{Vert}} \) so that \( s \equiv \tilde{s} \mod 2 \). Furthermore it is clear that \( \iota \) is affine over the Bockstein homomorphism:

\[ \iota(s + c) = \iota(s) + \beta(c) \forall s \in \text{Spin}(M), c \in H^1(M; \mathbb{Z}_2). \]

We will also need the linking pairing

\[ \ell k : \text{Tor} H_1(M; \mathbb{Z}) \otimes \text{Tor} H_1(M; \mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \]

and its quadratic refinement [36]:

\[ q_s : \text{Tor} H_1(M; \mathbb{Z}) \to \mathbb{Q}/2\mathbb{Z}, \quad q_s(a + b) - q_s(a) - q_s(b) = 2 \ell k(a, b). \]

depending on a spin structure \( s \in \text{Spin}(M) \) as follows:

\[ q_{s+c}(a) - q_s(a) = c(a), \ c \in H^1(M; \mathbb{Z}_2). \]

In terms of the identifications (2.3)-(2.18) we have

\[ \ell k(a, b) = a^T B^{-1} b \mod 1, \]

\[ q_s(a) = a^T B^{-1} a + s^T a \mod 2. \]

We will also use the following expression for the \( \mod 4 \) reduction of Rokhlin invariant (see e.g. [37]):

\[ \mu(M, s) = \sigma - s^T B s \mod 4 \]

\[ (2.17) \]
where $\sigma$ is the signature of the linking matrix $B$.

A special case is when $M$ is a plumbed manifold, i.e. all the components of $L$ are unknots which are linked to each other as Hopf links according to the combinatorial structure of a contractible graph $\Gamma$, the vertices of which are indexed by $\text{Vert}$. In this case, letting $\deg(I)$ be the degree of $I \in \text{Vert}$ (i.e. the number of edges containing $v$), we also have the following identification for spin$^c$-structures:

\[(2.18) \quad \text{Spin}^c(M) \cong \{ a \in \mathbb{Z}^{\text{Vert}} / 2B \mathbb{Z}^{\text{Vert}} \mid a_I = \deg(I) \mod 2 \}.\]

### 2.3. WRT invariants

Let $\xi = \exp \frac{\pi i}{\ell}$, $\mathcal{U}_\xi^H(\mathfrak{sl}_2)$ be the so-called “unrolled version” of the quantum $\mathfrak{sl}_2$ algebra as defined in [17], given by generators $E, F, H, K, K^{-1}$ and relations:

\[(2.19) \quad KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = \xi^2 E, \quad KFK^{-1} = \xi^{-2} F, \quad [E, F] = \frac{K - K^{-1}}{\xi - \xi^{-1}}, \]

\[(2.20) \quad HK = KH, \quad [H, E] = 2E, \quad [H, F] = -2F.\]

The algebra $\mathcal{U}_\xi^H(\mathfrak{sl}_2)$ is a Hopf algebra where the coproduct, counit and antipode are defined as follows

\[(2.21) \quad \Delta(E) = 1 \otimes E + E \otimes K, \quad \varepsilon(E) = 0, \quad S(E) = -EK^{-1}, \]

\[(2.22) \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \varepsilon(F) = 0, \quad S(F) = -KF, \]

\[(2.23) \quad \Delta(K) = K \otimes K, \quad \varepsilon(K) = 1, \quad S(K) = K^{-1}, \]

\[(2.24) \quad \Delta(H) = H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H.\]

Let $\mathcal{C}$ be the category of finite dimensional weight modules (i.e. modules on which $H$ acts diagonally) and on which $K$ acts as $q^H$ and such that $E^r$ and $F^r$ act as 0. We will now recall that the category $\mathcal{C}$ is a ribbon category. Let $V$ and $W$ be objects of $\mathcal{C}$. Let \{\(v_i\)\} be a basis of $V$ and \{\(v_i^*\)\} be a dual basis of $V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C})$. Then

\[(2.25) \quad \overrightarrow{\text{coev}}_V: \mathbb{C} \to V \otimes V^*, \text{ given by } 1 \mapsto \sum v_i \otimes v_i^*, \]

\[(2.26) \quad \overrightarrow{\text{ev}}_V: V^* \otimes V \to \mathbb{C}, \text{ given by } f \otimes w \mapsto f(w)\]

are duality morphisms of $\mathcal{C}$. In [46] Ohtsuki truncates the usual formula of the $h$-adic quantum $\mathfrak{sl}_2$ $R$-matrix to define an operator on $V \otimes W$ by

\[(2.27) \quad R = \xi^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} \xi^n n((n) / 2) E^n \otimes F^n \]

where $\xi^{H \otimes H/2}$ is the operator given by

\[(2.28) \quad \xi^{H \otimes H/2}(v \otimes v') = \xi^{\lambda \lambda'/2} v \otimes v'\]

for weight vectors $v$ and $v'$ of weights of $\lambda$ and $\lambda'$. The $R$-matrix is not an element in $\mathcal{U}_\xi^H(\mathfrak{sl}_2) \otimes \mathcal{U}_\xi^H(\mathfrak{sl}_2)$, however the action of $R$ on the tensor product of two objects of $\mathcal{C}$ is a well defined linear map on such a tensor product. So, $R$ gives rise to a braiding.
c_{V,W} : V \otimes W \rightarrow W \otimes V \text{ on } C \text{ defined by } v \otimes w \mapsto \tau(R(v \otimes w)) \text{ where } \tau \text{ is the permutation } x \otimes y \mapsto y \otimes x. \text{ Also, let } \theta \text{ be the operator given by}

\begin{equation}
\theta = K^{-r} \sum_{n=0}^{r-1} \frac{1}{n!} \frac{\xi^n(n-1)/2 S(F^n)}{\xi^{-H/2}} E^n \tag{2.29}
\end{equation}

where \( \xi^{-H/2} \) is an operator defined on a weight vector \( v_\lambda \) by \( \xi^{-H/2}.v_\lambda = \xi^{-2/3}v_\lambda \). Ohtsuki shows that the family of maps \( \theta_V : V \mapsto V \text{ in } C \text{ defined by } v \mapsto \theta^{-1}v \) is a twist (see [46]).

Now the ribbon structure on \( C \) yields right duality morphisms

\begin{equation}
\overleftarrow{ev}_V = ev_V \cdot (\theta_V \otimes \text{Id}_V) \quad \text{and} \quad \overrightarrow{coev}_V = (\text{Id}_V \otimes \theta_V) c_{V,V} \cdot \overrightarrow{coev}_V \tag{2.30}
\end{equation}

which are compatible with the left duality morphisms \{\overrightarrow{coev}_V\}_V and \{ev_V\}_V. These duality morphisms are given by

\begin{equation}
\overrightarrow{coev}_V : \mathbb{C} \rightarrow V^* \otimes V, \quad \text{where} \quad 1 \mapsto \sum K^{-r} v_i \otimes v_i^*, \quad \tag{2.31}
\end{equation}

\begin{equation}
\overleftarrow{ev}_V : V \otimes V^* \rightarrow \mathbb{C}, \quad \text{where} \quad v \otimes f \mapsto f(K^{-r}v). \tag{2.32}
\end{equation}

The quantum dimension \( \text{qdim}(V) \) of an object \( V \) in \( C \) is the \( \text{qdim}(V) = \overrightarrow{ev}_V \circ \overrightarrow{coev}_V = \sum v_i^*(K^{-r}v_i) \).

For each \( n \in \{0, \ldots, r-1\} \) let \( S_n \) be the usual \( (n+1) \)-dimensional simple highest weight \( U_{\xi}(sl_2) \)-module with highest weight \( n \). The module \( S_n \) is a highest weight module with a highest weight vector \( s_0 \) such that \( E.s_0 = 0 \) and \( H.s_0 = ns_0 \). Then \( \{s_0, s_1, \ldots, s_n\} \) is a basis of \( S_n \) where \( F.s_i = s_{i+1}, H.s_i = (n-2i)s_i, E.s_0 = 0 = F^{n+1}.s_0 \) and \( E.s_i = \frac{(i)[n+1-i]}{[n]!} s_{i-1} \). The quantum dimension of \( S_n \) is \( \text{qdim}(S_n) = (-1)^n \frac{[n+1]}{[1]} \). Next we consider a larger class of finite dimensional highest weight modules: for each \( \alpha \in \mathbb{C} \) we let \( V_\alpha \) be the \( r \)-dimensional highest weight \( U_{\xi}(sl_2) \)-module of highest weight \( \alpha + r - 1 \). The modules \( V_\alpha \) has a basis \( \{v_0, \ldots, v_{r-1}\} \) action on which is given by

\begin{equation}
H.v_i = (\alpha + r - 1 - 2i)v_i, \quad E.v_i = \frac{(i)[i-\alpha]}{[1]^2} v_{i-1}, \quad F.v_i = v_{i+1}. \tag{2.33}
\end{equation}

For all \( \alpha \in \mathbb{C} \), the quantum dimension of \( V_\alpha \) is zero:

\begin{equation}
\text{qdim}(V_\alpha) = \sum_{i=0}^{r-1} v_i^* (K^{-r}v_i) = \sum_{i=0}^{r-1} \xi^{(r-1)(\alpha+r-1-2i)} = \frac{\xi^{(r-1)(\alpha+r-1)} - \xi^{2r}}{1 - q^2} = 0. \tag{2.34}
\end{equation}

As shown in [18], if \( L \) is a framed oriented link in \( S^3 \) colored by modules \( S_{ci} \), where \( I \) runs over the components of the link, then

\begin{equation}
F(L) = J_c(L) \tag{2.35}
\end{equation}

where \( F \) is the Reshetikhin-Turaev functor and \( J_c \) is the (unnormalized) skein theoretical colored Jones polynomial of \( L \), so that in particular the value for the \( S_1 \)-colored unknot is \((-1)^{|i+1|} (n) \) (as usual \( [n] := \{n\}/\{1\} \)). Another well known version of the colored Jones polynomial, which we shall call “representation theoretical”, is obtained by considering
the full subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) generated by the modules \( S_n \) with the same braiding and ribbon structure as above, with the only difference in the following morphisms:

\[
\begin{align*}
\coev V : & \mathcal{C} \to V^* \otimes V, \quad \text{where} \quad 1 \mapsto \sum K^{-1} v_i \otimes v_i^*, \\
\ev_V : & V \otimes V^* \to \mathbb{C}, \quad \text{where} \quad v \otimes f \mapsto f(Kv).
\end{align*}
\]

Then the image of \( L \) via the Reshetikhin-Turaev functor associated to the category \( \mathcal{C}' \) is the version of the colored Jones considered for instance in [37], let us denote it \( V_c(L) \).

**Remark 2.1.** The definition of representation theoretical Jones polynomial can be extended to generic values of \( q \), as opposed to \( \xi = \exp(\frac{i\pi}{r}) \).

**Lemma 2.2.** It holds \( V_c(L) = (-1)^{\sum_i(B_{II}+1)c_i} J_c(L) \).

**Proof.** It is sufficient to compare when \( L \) is the closure of a braid of \( B_m \) such that \( B_{II} = 0 \) for all \( I \). Then the only difference between the two Reshetikhin-Turaev functors associated to \( \mathcal{C}' \) and \( \mathcal{C} \) is coming from the overall difference is \( (-1)^{\sum c_i} \) where the sum is taken over all the \( m \) strands of the braid. Indeed observe that \( B_{II} \) is not changed (modulo 2) if one switches the crossings of the braid so that we can prove the claim when the link is actually a disjoint union of unknots. Now for each such unknotted colored by odd \( n_I \) we observe that each Reidemeister 1 move changes by one both the number of maxima and \( B_{II} \), while the other Reidemeister moves do not change these values. Finally for the standard diagram of the unknot the statement is true.

**Lemma 2.3** (Symmetry principle for skein theoretical Jones polynomials). Let \( \xi = \exp(\frac{i\pi}{r}) \) and \( L \) be a framed oriented link colored by \( S_{c_1} \cdots S_{c_V} \) with \( 0 \leq c_i \leq r-2 \). For each \( a_I \in \{0, 1\} \) and \( c_I \in \{0, \ldots, r-2\} \) let \( a_I \star c_I := a_I(r-2-c_I) + (1-a_I)c_I \) and for \( a \in \{0, 1\}^V \) let \( a \star c := (a_1 \star c_1, \ldots, a_V \star c_V) \). Then it holds:

\[
(2.36) \quad J_{\text{asc}}(L) = i^{(r-2)\sum_{I,J} B_{IJ} a_I a_J} (-1)^{\sum_{I,J} B_{IJ} a_I c_J} (-1)^{\sum_i (B_{II}+1)(r-2)c_i} J_c(L).
\]

**Proof.** The symmetry principle for the representation theoretical Jones polynomial stated in [37], formula (4.20), is:

\[
(2.37) \quad V_{\text{asc}}(L) = i^{(r-2)\sum_{I,J} B_{IJ} a_I a_J} (-1)^{\sum_{I,J} B_{IJ} a_I c_J} V_c(L).
\]

By Lemma 2.3, for each component where \( a_I \neq 0 \) we acquire an additional factor \((-1)^{(B_{II}+1)(c_I-(r-2-c_I))}\).
Definition 2.4. Let $D = \sqrt{\frac{1}{2} (\sin(\frac{\pi}{r}))^{-1}}$ and $M$ be the three-manifold obtained by surgery on a framed oriented link $L$.

\[
WRT_r(M) = D^{-b_0(M) - b_1(M)} U_+^{-b_1} U_-^{-b_0} \sum_c d(c) J_c(L)
\]

where $U_\pm = \sum_{c=0}^{r-2} q\dim(c) J_c(u_\pm)$, $u_\pm$ is the unknot with framing $\pm$, $\mathcal{C}$ runs over all the colorings of $L$ with colors in $\{0, 1, \ldots, r - 2\}$ and $q\dim(c) = \prod_{i=1}^r q\dim(c_i)$.

We also recall that there exists a cohomological refinement of $WRT_r(M)$ defined for cohomology classes $\omega \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ (see [18]). Given such a class and an oriented link $L \subset S^3$ such that $M$ is obtained by surgery over $L$, then $\omega$ induces a parity on the components of $L$ via $\omega(L_I) := \omega(m_I) \in \mathbb{Z}/2\mathbb{Z}$ where $m_I$ is homology class of the meridian of $L_I$. Let

\[
\Delta_{SO(3)}^\pm := \begin{cases} 
\frac{\Delta_{-}^\pm}{(-1)^{r-1}(\xi - \xi^{-1})}, & r \equiv 0 \pmod{4} \\
(1-i)\frac{\Delta_{-}^\pm}{(-1)^{r-1}(\xi - \xi^{-1})}, & r \equiv 1 \pmod{4} \\
0, & r \equiv 2 \pmod{4} \\
-\frac{\Delta_{-}^\pm}{(-1)^{r-1}(\xi - \xi^{-1})}, & r \equiv 3 \pmod{4}
\end{cases}
\]

where $\Delta_{\pm}$ are defined in (2.44) and (2.45). Then one defines (see also Appendix B):

Definition 2.5.

\[
WRT_r(M, \omega) = D^{-b_0(M) - b_1(M)} (\Delta_{SO(3)}^+)^{b_1} (\Delta_{SO(3)}^-)^{b_0} \sum_{c=\omega \mod 2} d(c) J_c(L)
\]

where $c$ runs over all the colorings of $L$ with values in $\{0, 1, \ldots, r - 2\}$ which are congruent mod 2 to $\omega(L_I)$ for each component of $L$.

2.4. $N_r$ invariants. With the above notation, to calculate $N_r(M, \omega)$ in this case it is enough to introduce the following definitions:\

\[
d(\alpha) := \frac{\sin \frac{\pi \alpha}{r}}{\sin \frac{\pi \alpha}{r}} = \frac{\xi^\alpha - \xi^{-\alpha}}{\xi^{r\alpha} - \xi^{-r\alpha}},
\]

\[
S(\alpha, \beta) := \xi^{\alpha \beta},
\]

\[
T(\alpha) := \frac{\xi^{\alpha^2 - (r-1)^2}}{2}.
\]

\[
\Delta_{-} := \begin{cases} 
0, & r \equiv 0 \pmod{4} \\
i\xi^\frac{3}{2} r^\frac{1}{2}, & r \equiv 1 \pmod{4} \\
(1-i)\xi^\frac{3}{2} r^\frac{1}{2}, & r \equiv 2 \pmod{4} \\
-\xi^\frac{3}{2} r^\frac{1}{2}, & r \equiv 3 \pmod{4}
\end{cases}
\]

\[
\Delta_+ = \overline{\Delta_-}.
\]

\[\text{In [17] } q \text{ was used instead of } \xi := e^{2\pi i}, \text{ and, mainly, the definition of } d(\alpha) \text{ differs by a factor } (-1)^{r-1} r \text{ : this only affects } N_r \text{ by an overall scalar.}\]
As before, let us denote by \( m_I \in H_1(M; \mathbb{Z}) \) the homology class of the oriented meridian of the component of \( L \) indexed by \( I \in \text{Vert} \). Then define
\[
(2.46) \quad \mu_I := \omega(m_I) \in \mathbb{C}/2\mathbb{Z}.
\]
We will assume from now on that \( \mu_I \notin \mathbb{Z}/2\mathbb{Z}, \; \forall I \) (it was proved in [17] that one can always find \( L \) presenting \( M \) such that this condition is satisfied if \( \omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z}) \)). Note that
\[
(2.47) \quad \sum_J B_{IJ} \mu_J = 0 \mod 2.
\]

2.4.1. \( N_r \) for links, ADO and Conway-Alexander polynomials. As proved in [1] (but in what follows we use the notation used in [17]), there is an invariant \( N_r(S^3, L_\mu) \) associated to each framed oriented link with a \( \mathbb{C} \)-coloring of its components, which, up to a factor depending on the linking form, is valued in \( \mathbb{C}[\xi^{\pm \mu_1}, \ldots, \xi^{\pm \mu_V}] \) if \( V > 1 \) and in \( \xi^{\mu_1} \mathbb{C} \) in the knot case.

This invariant is computed by first cutting open \( L \) on one of its components (say the \( I^{th} \)) to get a \((1, 1)\)-tangle \( T \), then computing the Reshetikhin-Turaev functor \( F \) applied to \( T \) by considering a color \( \mu_J \) as the \( r \)-dimensional simple projective module \( V_{\mu_J} \) over \( \mathcal{U}^r_s(\mathfrak{sl}_2) \) thus getting
\[
(2.48) \quad F(T) = < T > \text{Id}_{V_{\mu_I}}
\]
for some scalar \( < T > \); then defining \( N_r(S^3, L_\mu) = f < T > d(\mu_I) \) where \( f \) is a certain factor related to the linking form (see later). It can be proved that this value does not depend on the way \( L \) was cut to obtain \( T \) and is then an invariant of the colored oriented framed link \( L_\mu \).

Comparing with the original definition of the ADO polynomial of links we remark the following: for each framed oriented link \( L \subset S^3 \) with \( V \geq 2 \) components we have
\[
(2.49) \quad \text{ADO}_r(L)(x_1 = \xi^{-2}\xi^{2\mu_1}, \ldots, x_V = \xi^{-2}\xi^{2\mu_V}) = \text{N}_r(S^3, L)\xi^{-\frac{\mu TB_\mu}{2} + \frac{(r-1)^2}{2} TB_\mu}.
\]

For knots, we remark that \( \text{N}_r \) is not a polynomial and the relation with ADO polynomial is as follows:
\[
(2.50) \quad \text{ADO}_r(K)(x = \xi^{-2}\xi^{2\mu}) = \frac{\xi^\mu - \xi^{-\mu}}{\xi^\mu - \xi^{-\mu}} N_r(S^3, K) \xi^{-\frac{\rho^2}{2} + \frac{r(r-1)}{2}}
\]
where \( f \) is the framing of \( K \).

A case of special interest is when \( r = 2 \) where, as proved in [7] \( N_r \) is equivalent to the Alexander-Conway function (see Corollary 6.19, taking into account the difference in the definition of the modified quantum dimension used here). The precise statement is the following, letting \( \xi = i \):
\[
(2.51) \quad N_2(L) = i \nabla_L(i\xi^{-\mu_1}, i\xi^{-\mu_1}, \ldots, i\xi^{-\mu_V})\xi^{-\frac{T_{B_\mu}}{2} - \frac{T_{B_\mu}}{2}} = (-1)^n i \nabla_L(-i\xi^{\mu_1}, -i\xi^{\mu_2}, \ldots, -i\xi^{\mu_V})\xi^{-\frac{T_{B_\mu}}{2} - \frac{T_{B_\mu}}{2}}.
\]
Example 2.6. Let $L$ be a plumbing link the components of which are colored by $\mu_I$, $I \in \text{Vert}$, and let $B$ be the linking matrix. Then:

\begin{equation}
N_r(S^3, L) = \xi^{-\text{Tr}_B(r-1)^2} \xi^{\frac{1}{2}(\mu^TB\mu)} \prod_{I \in \text{Vert}} d(\mu_I)^{1-\text{deg}(I)}.
\end{equation}

In the case $r = 2$, so that $\xi = i$ we get:

\begin{equation}
N_2(S^3, L) = \xi^{-\text{Tr}_B} \xi^{\frac{1}{2}(\mu^TB\mu)} \prod_{I \in \text{Vert}} (\xi^{\mu_I} + \xi^{-\mu_I})^{\text{deg}(I)-1}
\end{equation}

and

\begin{equation}
\nabla_L(x_1, \ldots, x_V) = \prod_I (x_I - x_I^{-1})^{\text{deg}(I)-1}.
\end{equation}

2.4.2. $N_r$ for manifolds. The definition of $N_r$ for a manifold $M$ represented by integral surgery on $L$ is as follows:

\begin{equation}
N_r(M, \omega) = \frac{1}{\Delta^b_+ \Delta^b_-} \sum_{k \in H^V_{\text{Vert}}} \prod_{I \in \text{Vert}} d(\alpha_{k_I}) T(\alpha_{k_I})^{B_{II}} N_r(S^3, L_{\alpha})
\end{equation}

where

\begin{equation}
\alpha_{k_I} := \mu_I + k_I, \quad H_r = \{-(r-1), -(r-3), \ldots, (r-3), r-1\}
\end{equation}

and $b_\pm$ are the number of positive/negative eigenvalues of $B$.

Remark 2.7. The invariant $N_r$ defined above differs by a constant with respect to the invariant $N'_r$ defined in [17] and its renormalisation $Z_r$ introduced in [6] because of the different definition of $d(\alpha)$ and consequently of $\Delta_\pm$ and of the link invariant. The general relation for a closed connected manifold is then:

\begin{equation}
N_r(M, \omega) = \frac{N'_r(M, \omega)}{((-1)^{r-1})^{b_1(M)+b_0(M)}} = \frac{r \sqrt{r^2} \sum_{k \in H^V_{\text{Vert}}} \prod_{I \in \text{Vert}} d(\alpha_{k_I})^{2-\text{deg}(I)} T(\alpha_{k_I})^{B_{II}} \prod_{(I, J) \in \text{Edges}} S(\alpha_{k_I}, \alpha_{k_J})}{((-1)^{r-1})^{b_1(M)+b_0(M)}}
\end{equation}

where $r' = r$ if $r$ is odd and $\frac{r}{2}$ else.

In the special case of a plumbed $M$ then it reads

\begin{equation}
N_r(M, \omega) = \frac{1}{\Delta^b_+ \Delta^b_-} \sum_{k \in H^V_{\text{Vert}}} \prod_{I \in \text{Vert}} d(\alpha_{k_I})^{2-\text{deg}(I)} T(\alpha_{k_I})^{B_{II}} \prod_{(I, J) \in \text{Edges}} S(\alpha_{k_I}, \alpha_{k_J}).
\end{equation}

Example 2.8 ($\Sigma \times S^1$). Let $M = \Sigma_g \times S^1$ for a closed oriented surface $\Sigma_g$ of genus $g$ and let $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z})$. If $\beta = \omega(pt) \times S^1$ then as shown in [6] (Thm 5.9) it holds:

\begin{equation}
N_r(M, \omega) = r^{2g} \sum_{k \in H_r} \left(\frac{r\beta}{\beta + k}\right)^{2g-2}.
\end{equation}
2.4.3. A relation between $N_r$ and $\text{WRT}_r$. The following result outlines a partial direct relation between the invariants $N_r$ and $\text{WRT}_r$ (cf. [3]):

**Theorem 2.9.** Let $r \geq 2$ be an integer non divisible by 4 and $K \subset S^3$ be an oriented zero framed knot and $M$ be the surgery on it. Let $\alpha \in \mathbb{C}$ be a color on $K$ and $\omega_\alpha \in H^1(M; \mathbb{C}/2\mathbb{Z})$ be the unique cohomology class the value of which on the positive meridian of $K$ is $\alpha \mod 2\mathbb{Z}$. Then the following holds:

\begin{equation}
(2.60) \quad \text{if } r \text{ is odd: } \lim_{\alpha \to 0} [r\alpha]^2N_r(M, \omega_\alpha) = D^2 \text{WRT}_r(M) \tag{2.60}
\end{equation}

\begin{equation}
(2.61) \quad \text{else: } \lim_{\alpha \to 0} [r\alpha]^2N_r(M, \omega_\alpha) = 2D^2 \text{WRT}_r(M, \omega_0) \tag{2.61}
\end{equation}

\begin{equation}
(2.62) \quad \text{and } \lim_{\alpha \to 1} [r\alpha]^2N_r(M, \omega_\alpha) = 2D^2 \text{WRT}_r(M, \omega_1) \tag{2.62}
\end{equation}

where $D = \sqrt{\frac{T}{2}}(\sin(\frac{\pi}{2}))^{-1}$, $\text{WRT}_r(M)$ is the standard WRT invariant of $M$, $\text{WRT}_r(M, \omega_i)$ are the cohomology refined invariants of $M$ and $\omega_i \in H^1(M; \mathbb{Z}/2\mathbb{Z})$ is the cohomology class on $M$ the value of which on the meridian of $K$ is $i \mod 2\mathbb{Z}$.

**Proof.** Present $K$ as the closure of a $(1,1)$-tangle $T$ and for each absolutely simple module $V$ (i.e. such that $\text{End}(V) = \mathbb{C}$) over $U_q^H(\mathfrak{sl}_2)$ let $T(V) \in \mathbb{C}$ be the scalar such that $F(T) = T(V)\text{Id}_V$ where $F$ is the Reshetikhin-Turaev functor. In particular we shall use the $n + 1$-dimensional highest weight simple module $S_n$ and the absolutely simple $r$-dimensional module $V_\alpha$ with highest weight $\alpha + r - 1$.

**r odd case.** By definition, for $r$ odd we have:

\begin{equation}
(2.63) \quad [r\alpha]^2N_r(M, \omega) = \sum_{k=-(r-1)}^{r-1} \frac{(\alpha + k)^2}{\{1\}^2} T(V_{\alpha+k}), \quad \text{by } 2
\end{equation}

\begin{equation}
(2.64) \quad \text{WRT}_r(M) = D^{-2} \sum_{j=0}^{r-2} \frac{\{j+1\}^2}{\{1\}^2} T(S_j).
\end{equation}

By the symmetry principle for the colored Jones polynomials of knots (Lemma 2.3), see [37] formula 4.20) and by the equality $\{r-1-j\} = \{j+1\}$ we have also:

\begin{equation}
(2.65) \quad \text{WRT}_r(M) = 2D^{-2} \sum_{j=0}^{r-3} \frac{\{j+1\}^2}{\{1\}^2} T(S_j).
\end{equation}

As shown in Proposition 4 of [18], it holds

\begin{equation}
(2.66) \quad T(S_{r-1-k}) = T(V_k) = T(V_{-k}) \quad \forall k \in \{0, \ldots, r-1\},
\end{equation}
so we have:

\[
\lim_{\alpha \to 0} [r\alpha]^2 N_r(M, \omega) = \sum_{k=-(r-1)}^{r-1} \frac{\{k\}^2}{\{1\}^2} T(V_k) = 2 \sum_{k=2}^{r-1} \frac{\{k\}^2}{\{1\}^2} T(V_k) = 2 \sum_{j=0}^{r-3} \frac{\{r-j\}^2}{\{1\}^2} T(S_j) = 2 \sum_{j=0}^{r-3} \frac{\{j+1\}^2}{\{1\}^2} T(S_j).
\]

\textbf{r even case.} By definition, for \( r \) even we have:

\[
[r\alpha]^2 N_r(M, \omega) = \sum_{k=-(r-1)}^{r-1} \frac{\{\alpha + k\}^2}{\{1\}^2} T(V_{\alpha+k}).
\]

\[
\text{WRT}_r(M, \omega_i) = D^{-2} \sum_{j=1}^{r-2+i} \frac{\{j+1\}^2}{\{1\}^2} T(S_j).
\]

So

\[
\lim_{\alpha \to 0} [r\alpha]^2 N_r(M, \omega) = \sum_{k=1}^{r-1} 2 \frac{\{k\}^2}{\{1\}^2} T(V_{\alpha+k}) = 2 \sum_{k=1}^{r-1} \frac{\{k\}^2}{\{1\}^2} T(S_r-1-k) = \sum_{j=0}^{r-2} 2 \frac{\{r-j\}^2}{\{1\}^2} T(S_j) = 2D^2 \text{WRT}_r(M, \omega_0).
\]

Similarly:

\[
\lim_{\alpha \to 1} [r\alpha]^2 N_r(M, \omega) = \sum_{k=-(r-1)}^{r-1} 2 \frac{\{k+1\}^2}{\{1\}^2} T(V_{k+1}) = \sum_{s=2}^{r-2} 2 \frac{\{s\}^2}{\{1\}^2} T(V_\alpha) = \sum_{j=1}^{r-3} 2 \frac{\{r-j\}^2}{\{1\}^2} T(S_j) = 2D^2 \text{WRT}_r(M, \omega_1)
\]

where in the second equality we used that \( \{r\} = \{0\} = 0 \).

2.5. \( \hat{Z} \) invariants. In this subsection we recall some general facts about the invariants \( \hat{Z}_b \) of 3-manifolds and their version \( F_L(q, x) := \hat{Z}(S^3 \setminus L) \) for links. We formulate some conjectures about the link invariant and outline how the 3-manifold invariant is built out of the invariant for links. We postpone examples to a later section.

2.5.1. \( \hat{Z} \) for links. A physical construction of new \( q \)-series invariants of 3-manifolds was proposed in [33, 32]. Much like invariants \( N_r(M) \) reviewed above, these \( q \)-series invariants are labeled by extra data, which originally was interpreted as the choice of
abelian flat connection or, equivalently, an element of $H_1(M)$. Soon, it was realized [30, 29] that the extra data $s$ which labels $\hat{Z}_s(M)$ should be understood as a spin$^c$ structure on $M$. Recall, that as a set $H_1(M)$ is isomorphic to Spin$^c(M)$, but the isomorphism is non-canonical. The difference between the two becomes apparent in cutting and gluing operations, which in part explains why it was not noticed until the invariants were extended to 3-manifolds with toral boundaries and link complements, where $s$ is a relative spin$^c$-structure. In particular, if $M$ is the complement of a knot $K$ in rational homology sphere, the invariant is expected to be read off from a single two-variable series:

$$F_K(x, q) \in 2^{-c} q^{\Delta} \mathbb{Z}[[q^{-1}, q]][[x^{\pm 1}]]$$

for some $c \in \mathbb{Z}_+$ and $\Delta \in \mathbb{Q}$. Apart from the original physics formulation, several approaches toward rigorous constructions of this invariant have been developed. For example, the approach based on recursion and resurgence, although in principle can be applied to any knot (or link) is rather laborious and was used in [30] to produce $F_K(q, x) := \hat{Z}(S^3 \setminus K)$ only for torus knots and a single hyperbolic knot, the figure-eight knot $K = 4_1$.

Another, much more efficient diagrammatic approach based on the R-matrix for Verma modules and quantum groups at generic $q$ was proposed by Park [48, 49]. Using this approach, one can quickly compute $F_K(x, q)$ for many hyperbolic knots up to roughly 10 crossings and also for many infinite families. Specifically, the R-matrix used by Park [48] is the universal R-matrix [38] applied to the lowest weight Verma modules with complex weights:

$$(2.73) \quad R_{ij}^{kl} = \left[ j \right] \prod_{t=1}^{n} (1 - y^{-1} q^{i+t}) x^{-(n-1)t} q^{-\frac{n}{2} x^{-n} + \frac{n}{2} x^{n} + \frac{n}{2} x^{n+1}}.$$ 

Here, and in relating it to the R-matrix (2.2) we are a little cavalier with the overall powers of $q$. Indeed, the fact that (2.2) and (2.73) are related by sending $q$ to a root of unity is the first indication for the relation (1.2) between the corresponding knot invariants. This also clarifies the quantum group origin of the $\hat{Z}$-invariants. Since they basically provide a non-perturbative formulation$^7$ of the $SL(2, \mathbb{C})$ Chern-Simons TQFT — where $q$ is a continuous complex variable and so are the highest weights of representations by which Wilson lines are colored — it was expected for a long time that $SL(2, \mathbb{C})$ Chern-Simons theory should be described by $U_q(g)$ with generic $q$.

In order to relate the R-matrices (2.2) and (2.73), we first need to replace $q \to q^{-2}$ and then take the root of unity limit. As in the generalized volume conjecture [28], all combinations of the form $q^{\text{weight}}$ must be treated as independent variables and kept

$^6$In this paper we only consider $SU(2)$ version of these invariants. The higher-rank version is also available, see e.g. [16, 47].

$^7$The all-order perturbative formulation was known for quite some time, see e.g. [23] and references therein.
fixed. In particular, we need to use the identifications \( x = q^\mu \) and \( y = q^\lambda \) and treat them as independent complex parameters. Then, using identities of the form

\[
(2.74) \quad \binom{n}{j} = \frac{[j]!}{n! [j - n]!} = \frac{\{1\} \cdots \{j\}}{\{1\} \cdots \{j\} \{1\} \cdots \{j - n\}} = \frac{\{j - n + 1\} \cdots \{j\}}{\{1\} \cdots \{n\}} = \frac{j}{n; n}
\]

we arrive at the desired relation between (2.73) and (2.2). Based on this relation between the R-matrices, it is not surprising to see the relation between the corresponding knot invariants [29] (see also [12]):

\[
(2.75) \quad \lim_{q \to \exp(2\pi i/r)} F_K(x_1, q) \Delta_K(x_1^r) = ADO_r(K)(\frac{x_1}{q}) |_{q = \exp \frac{2\pi i}{r}}
\]

where \( \Delta_K \) is the Alexander polynomial of \( K \).

To properly deal with the case of links, we consider the set

\[
(2.76) \quad \mathbb{C}[q^{-1}, q][[x_1^\pm 1, \ldots, x_V^\pm 1]]
\]

of formal power series in \( \{x_I\} \) with coefficients in Laurent series in \( q \). Such a set is not a ring but is a module over the ring:

\[
(2.77) \quad \mathbb{C}[q^{-1}, q][[x_1^\pm 1, \ldots, x_V^\pm 1]]
\]

of Laurent polynomials in \( \{x_I\} \) with coefficients in Laurent series in \( q \).

Taking into account the different normalisation between \( \text{ADO}_r(L) \) and \( N_r(S^3, L) \) we propose the following conjecture for the link case:

**Conjecture 4.** For each framed link \( L \) in \( S^3 \) there exists a non-zero formal power series \( F_L \in \mathbb{Z}[q^{-1}, q][[x_1, \ldots, x_V]] \) such that the following holds for every \( r \geq 2 \) and for every \( \bar{\alpha} \in \mathbb{C}^n \):

\[
(2.78) \quad \lim_{t \to 1} \lim_{q \to \exp(\frac{2\pi i}{r})} \frac{F^t_L(q^{\alpha_1}, \ldots, q^{\alpha_V}, q)}{F^t_L(q^{\bar{\alpha}_1}, \ldots, q^{\bar{\alpha}_V}, q^t)} = N_r(S^3, L_\alpha) q^{-\frac{1}{4}(\lambda^T B \alpha - (r-1)^2 \pi \nu) B}
\]

where \( B \) is the linking matrix of \( L \) and \( F^t_L \) is the formal power series in \( t \) obtained by replacing each \( x_I^n, n > 0 \) by \( tq^{\alpha_i} \) and each \( x_I^{-n} \) by \( tq^{-\alpha_i} \). In general the radius of convergence of \( F^t_L \) might be less than one but we conjecture that \( \lim_{q \to \exp(\frac{2\pi i}{r})} F^t_L \) is actually a rational function of \( t \) so that the limit \( t \to 1 \) makes sense via analytic continuation.

The relation of Conjecture 4 with the previous ones is clarified by the following:

**Conjecture 5.** If \( L \subset S^3 \) is a framed oriented link multivariable Alexander polynomial \( \nabla_L \) of which is non-zero then the following holds:

\[
(2.79) \quad \lim_{t \to 1} \lim_{q \to 1} F^t_L(q^{\alpha_1}, \ldots, q^{\alpha_V}, q) \nabla_L(q^{\alpha_1}, \ldots, q^{\alpha_V}) = 1.
\]

Indeed in the case of a knot \( K \) endowed with zero framing and colored by \( \alpha \), Conjecture 4 becomes:

\[
(2.80) \quad \lim_{t \to 1} \lim_{q \to \exp(\frac{2\pi i}{r})} \frac{F_K^t(q^{\alpha}, q)}{F_K^t(q^{\bar{\alpha}}, q^t)} = N_r(S^3, K_\alpha)
\]
so that using Conjecture 5 we get:

\[
(2.81) \quad \lim_{t \to 1} \lim_{q \to \exp(\frac{2\pi i}{r})} F^t_K(q^\alpha, q) \nabla (q^\rho) = N_r(S^3, K_\alpha).
\]

There are, however, links with vanishing Alexander polynomial, e.g.

\[9_{27}, 10_{32}, 10_{36}, 10_{107}, 11_{244}, 11_{247}, 11_{334}, 11_{381}, 11_{396}, 11_{404}, 11_{406}, \ldots\]

We then set the following definition

**Definition 2.10 (Good links).** A link \(L\) is very good if Conjectures 4 and 5 are verified for \(L\) and is good if only Conjecture 4 is satisfied for it.

We remark that the links with zero multivariable Alexander polynomial cannot be very good. Also, Conjecture 4 defines \(F_L\) only up to a scalar, which can be reduced to an indeterminacy of \(\pm 1\) by supposing that the greatest common divisor of the coefficients of \(F_L\) is \(\pm 1\).

When \(L\) is a knot, a version of Conjecture 4 was proven by S. Willets ([55]) in which \(F_L\) belongs to a suitable ring obtained as completion of the Laurent polynomials in two variables.

### 2.5.2. Examples of good links

Let \(L \subset S^3\) be a plumbing link, with the notation introduced in Example 2.6. In [31] the following formula was provided for \(F_L\)

\[
(2.82) \quad F_L(\{x^2_I\}, q) = \prod_i (x_I - x_I^{-1})^{1 - \deg(I)}.
\]

If we use this formula for \(F_L\) then it is clear that

\[
(2.83) \quad \lim_{q \to \exp(2\pi i/r)} \frac{F_L(\{x^2_I\}, q)}{F_L(\{q_i^2\}, q^r)} = \prod_i (x_I - x_I^{-1})^{1 - \deg(I)} = \xi^{-\frac{e^{TB_\alpha}}{2} + \frac{(r-1)^2}{2} \text{Tr}(B)} N_r(S^3, L)
\]

where we used Example 2.6 and we set \(x_I = \xi^{\alpha_I}\), so that Conjecture 4 holds for these links.

In general, \(F_L\) is to be considered as a formal power series and in the above case we do it as follows. Let \(t\) be a regularisation parameter and let \(F'_t\) be the power series development in \(t\) of

\[
(2.84) \quad F'_t_L(\{x^2_I\}, q) = \prod_{I: \deg(I) > 1} \frac{1}{2} ((x_I/t - t x_I^{-1})^{1 - \deg(I)} + (t x_I - x_I^{-1}/t)^{1 - \deg(I)}) \times \prod_{I: \deg(I) = 0} (t x_I - x_I^{-1}/t).
\]

Since \(F'_t\) is a rational function in \(t\) the limit \(F_t = \lim_{t \to 1} F'_t\) exists even though the power series \(F'_t\) might have radius of convergence in \(t\) less than 1. So that Conjecture 4 holds.
Similarly the Alexander-Conway polynomial of $L$ can be computed directly in these cases as $\nabla_L = \frac{1}{F}$ (see Example 2.6) so that also Conjecture 5 holds. The plumbing links are then very good links.

2.5.3. Whitehead link. Starting with the expression for $F_L(x_1, x_2, q)$ obtained by the R-matrix technique [48, 49]:

$$F_L(x_1, x_2, q) = \sum_{n \geq 0} (-1)^n q^{-\frac{n(n+1)}{2}} (q^{n+1}) (x_1^{1/2} - x_1^{-1/2})(x_2^{1/2} - x_2^{-1/2})$$

we get

$$F_L(x_1, x_2, q) \big|_{q \to -1} = \frac{1}{(x_1^{1/2} - x_1^{-1/2})(x_2^{1/2} - x_2^{-1/2})}.$$

According to the identification of the parameters in [29], we should compare this with $\text{ADO}_p$ with $p = 2$, evaluated at $xe^{-2\pi i/p}$. In the case of knots, $\text{ADO}_p$ is also multiplied by $\frac{1}{\Delta(x^p)}$.

Murakami works in conventions such that $q = e^{\pi i/p}$ is the $2p$-th root of unity and his $\text{ADO}_p$ with $p = 2$ gives the Alexander polynomial:

$$\text{ADO}_1 = 1,$$

$$\text{ADO}_2 = (z_1 - z_1^{-1})(z_2 - z_2^{-1}),$$

where we used $z_1 = q^x$ and $z_2 = q^y$ in Murakami’s notations. Comparing this with the multivariable Alexander polynomial

$$\Delta = (x_1^{1/2} - x_1^{-1/2})(x_2^{1/2} - x_2^{-1/2})$$

we see that Murakami’s $z_i = x_i^{1/2}$ and the relation between $F_L(\vec{x}, q)$ and $\text{ADO}_p(\vec{x})$ for links must be more complicated; it should convert powers of $x_i^{1/2} - x_i^{-1/2}$ in the denominator to the powers of $x_i^{1/2} - x_i^{-1/2}$ in the numerator.

Note, if as in case of knots we evaluate the ADO invariant at $xe^{-2\pi i/p}$, we would have $(x_1^{1/2} + x_1^{-1/2})(x_2^{1/2} + x_2^{-1/2})$ in the numerator. Then, dividing it by $\Delta(x_i^2)$ we would get $(x_1^{1/2} - x_1^{-1/2})(x_2^{1/2} - x_2^{-1/2})$ in the denominator. For $p = 3$, we get

$$F_L(x_1, x_2, q) \big|_{q \to e^{2\pi i/3}} = \frac{x_1^2 x_2 + x_1 x_2^2 + x_1^3 + x_1 x_2 + x_1 + x_2 + x_2 + 1}{(x_1^{1/2} - x_1^{-1/2})(x_2^{1/2} - x_2^{-1/2})(1 + x_1 + x_2^2)(1 + x_2 + x_2^2)}.$$

Apart from the familiar $\Delta(x_i)$, in the denominator we have a factor of

$$(x_1 + 1 + x_1^{-1})(x_2 + 1 + x_2^{-1}) = \frac{\Delta(x_i^p)}{\prod_i (x_i^{1/2} - x_i^{-1/2})},$$

so that the entire denominator is basically $\Delta(x_i^p)$. 

On the other hand, the numerator of (2.90) is precisely Murakami’s ADO3 evaluated at \( x_i e^{-2\pi i/p} \), just as in the case of knots. Therefore, we can write this relation as

\[
F_L(x_1, x_2, q) \bigg|_{q \to x_i e^{2\pi i/p}} = \frac{\text{ADO}_p(x_i e^{-2\pi i/p})}{\Delta(x_i^p)}. \tag{2.92}
\]

For reasons mentioned above, this also checks out in the \( p = 2 \) case. For \( p = 4 \) we get

\[
F_L(x_1, x_2, q) \bigg|_{q \to e^{2\pi i/4}} = \frac{x_1^{-3/2}x_2^{-3/2}(x_1 + 1)(x_2 + 1)(x_1^2(x_2^2 + 1) + 2ix_1x_2 + x_2^2 + 1)}{\Delta(x_i^4)}, \tag{2.93}
\]

again, in perfect agreement with (2.92).

As a side remark, we also note that, in general, ADO polynomials have many coefficients that are algebraic numbers; the coefficients of \( \text{ADO}_p(x_i e^{-2\pi i/p}) \) are also algebraic integers, but typically much simpler.

2.6. \( \hat{Z} \) for 3-manifolds. Let \( M = S^3(L) \) where \( L \) is a link with linking matrix \( B \) and the set of components \( \text{Vert} \). Let

\[
F(x, q) := F_L(x^2, q) \prod_{I \in \text{Vert}} (x_I - x_I^{-1}) = \sum_{\ell \in \mathbb{Z}_{\text{Vert}}} F_\ell \prod_{I \in \text{Vert}} x_I^{\ell_I}
\]

be a somewhat differently normalized \( \hat{Z} \)-invariant of the link \( L \) or, more precisely, the link complement \( S^3 \setminus L \). For example, \( F(x, q) = F_K(x^2, q)(x - x^{-1}) \) for a knot (complement), and \( F(x, q) = \prod_{I \in \text{Vert}} (x_I - x_I^{-1})^{2 - \deg(I)} \) for a plumbing link (complement).

Note, the two notations, \( F(x, q) \) and \( F_L(x, q) \), are very similar and we hope it will not cause a confusion. In fact, \( F_L(x, q) \) is used in most of the paper, and in a few places where \( F(x, q) \) is used we try to remind the reader about the relation between the two normalizations.

By an \( SL(V, \mathbb{Z}) \) transform we can bring the integral quadratic form \( B \) to \( B’ \oplus 0_{b_1} \) where \( 0_{b_1} \) is the trivial quadratic form on \( \mathbb{Z}^{b_1} \subset \mathbb{Z}^V \) [40]. Namely, there exists \( U \in SL(V, \mathbb{Z}) \) such that

\[
UBU^T = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix} \tag{2.95}
\]

where the right-hand side shows the block decomposition corresponding to the partition \( V = (V - b_1) + b_1 \). With \( \varepsilon = (1, 1, \ldots, 1) \) and \( s \) being mod 2 vectors defined as before \((\varepsilon = (1, 1, \ldots, 1), \sum_I B_{IJ} s_J = s_I \mod 2)\)

\[
\begin{pmatrix} s' \\ s'' \end{pmatrix} := (U^T)^{-1}s, \quad \begin{pmatrix} \varepsilon' \\ \varepsilon'' \end{pmatrix} := (U^T)^{-1}\varepsilon. \tag{2.96}
\]

Then

\[
\hat{Z}_{\sigma(b’ \oplus b'', s)} := (-1)^{b_1}q^{\frac{3s - \varepsilon x B}{4} + \sum_{I \in \text{Vert}} |\varepsilon''_I||b'_{I'}|} \sum_{\ell' = 2b' + B'(s' - \varepsilon')} \sum_{\ell'' = 2b'' \mod 2b'\mathbb{Z}^V - b_1} F_{U^{-1}} \left( \begin{pmatrix} \ell' \\ 2b'' \end{pmatrix} \right) q^{\ell'B''_1 \ell'.} \tag{2.97}
\]
where $b' \in \text{Coker}B' \cong \text{Tor} H_1(M; \mathbb{Z})$ and $b'' \in \mathbb{Z}^{b_1}$. Here we assume that the sum over $\ell'$ in the right-hand side is convergent in the space of formal power series $2^{-c}q^{\Delta}Z[q^{-1}, q]$. This means that any given power of $q$ gets contributions only from a finite number of terms.

If $b_1 = 0$, then $B = B'$ and the above formula simplifies to:

$$\hat{Z}_{\sigma(b,s)}(M) = q^{\frac{3a - \text{Tr} B}{4}} \sum_{\ell = 2b + B(s - \epsilon) \mod 2BZ^V} F_{\ell'} q^{-\ell'B^{-1} \ell'}.$$  

If $b_1 > 0$ one can show that the result is independent of the choice of $U$ preserving $b'$ and $b''$ and the Kirby moves only up to the following equivalence relation:

$$1 \sim q^{\text{LCM}(2, \text{GCD}(b''))}$$

where $\text{GCD}(b'') := \text{GCD}\{b''_i\}_{i=1}^{b_1}$ (which is invariant under $SL(b_1, \mathbb{Z})$ transformations). In other words, the invariants should be considered as:

$$\hat{Z}_{\sigma(b' \oplus b'', s)} \in 2^{-c}q^{\Delta}Z[[q]]/(1 - q^{\text{LCM}(2, \text{GCD}(b''))})Z[q]$$

It is easy to see that $\hat{Z}_{\sigma(\cdot, s)}$ transform covariantly (as functions on $H_1(M; \mathbb{Z}) \cong \text{Tor} H_1(M; \mathbb{Z}) \oplus \mathbb{Z}^{b_1}$) under the automorphisms preserving the splitting. They correspond to changes of the matrix $U$ of the following form:

$$U \sim \begin{pmatrix} \bar{\gamma} & 0 \\ 0 & \nu \end{pmatrix} U$$

where $\bar{\gamma} \in SL(V - b_1, \mathbb{Z})$ and $\nu \in SL(b_1, \mathbb{Z})$. Then from the Definition (2.97) it follows that

$$\hat{Z}_{\sigma(b' \oplus b'', s)} \sim \hat{Z}_{\sigma(\gamma^{-1}b' \oplus \nu^{-1}(b''), s)}$$

where $\gamma$ is the automorphism of $\text{Tor} H_1(M; \mathbb{Z})$ represented by $\bar{\gamma}$. Namely

$$\gamma(a') := \bar{\gamma}(a') \mod B'Z^{V - b_1}, \quad a' \in Z^{V - b_1}.$$ 

However there is an anomaly under the automorphisms changing the splitting. Modulo the automorphisms preserving the splitting, they are of the form

$$\text{Tor} H_1(M; \mathbb{Z}) \oplus \mathbb{Z}^{b_1} \xrightarrow{\begin{pmatrix} \text{id}_{\text{Tor} H_1} & \mu \\ 0 & \text{id}_{\mathbb{Z}^{b_1}} \end{pmatrix}} \text{Tor} H_1(M; \mathbb{Z}) \oplus \mathbb{Z}^{b_1}$$

where $\mu$ is a non-trivial homomorphism

$$\mu : \mathbb{Z}^{b_1} \rightarrow \text{Tor} H_1(M; \mathbb{Z}).$$

This automorphism is realized by the replacement

$$U \sim \begin{pmatrix} 1 \mu & \tilde{\mu} \\ 0 & 1 \end{pmatrix} U$$
where $\tilde{\mu}$ is a $(V - b_1) \times b_1$ matrix. Under the identification $\text{Tor} H_1(M; \mathbb{Z}) = \text{Coker} B'$, we have

$$\mu(a'') := \tilde{\mu}(a'') \mod B' \mathbb{Z}^{V - b_1}, \quad a'' \in \mathbb{Z}^{b_1}.$$  

The corresponding change of $\hat{Z}$ depends only on $\mu$, and not the representative $\tilde{\mu}$, if one takes into account the equivalence relation (2.99).

However it is not true that simply $\hat{Z}_\sigma(b' \oplus b'', s) \leadsto \hat{Z}_\sigma((b' - \mu(b'')) \oplus b'', s)$. The covariant transformation is corrected by an anomalous factor:

$$\hat{Z}_\sigma(b' \oplus b'', s) \leadsto q^{E(b', b'')} \hat{Z}_\sigma((b' - \mu(b'')) \oplus b'', s)$$

where

$$E(b', b'') = \text{GCD}(b'').$$

and

$$\hat{b}'' := \frac{b''}{\text{GCD}(b'')} \in \mathbb{Z}^{b_1}.$$  

The factor $q^{E(b', b'')}$ is well defined modulo the equivalence relation (2.99).

### 2.7. Physics of non-torsion fluxes.

For a given spin$^c$ structure $b$, let $\mathcal{B}_b$ be the corresponding boundary condition of the 6d fivebrane theory on $M \times D^2 \times_q S^1$. This defines the boundary condition in 4d gauge theory on $M \times \mathbb{R}_+$ (obtained by projecting $D^2 \times_q S^1 \to \mathbb{R}_+$) as well as the boundary condition in 3d theory $T[M]$ on $D^2 \times_q S^1$ (obtained by reducing on $M$):

$$\text{6d (0, 2) theory on } M \times D^2 \times_q S^1 \quad \text{on } M$$

$$\text{4d super-Yang-Mills on } M \times \mathbb{R}_+ \quad \text{3d theory } T[M] \text{ on } D^2 \times_q S^1.$$

In each of these descriptions, including the original 6d system viewed from the enumerative perspective of Calabi-Yau 3-fold and M2-branes, the ambiguity is naturally associated with the partition function on $M \times T^2 \times I$, where both $M \times T^2$ boundaries are colored by $\mathcal{B}_b$.

The effect of $\mathcal{B}_b$ is two-fold: i) first, it effectively abelianizes the theory, and ii) it also puts it in a non-trivial background, so that even abelian fluxes with one “leg” along $M$ and another “leg” along $I$ now carry a non-trivial $q$-degree. Note, this latter effect is
absent when $b$ is torsion. The sum over such fluxes gives a $q$-series unbounded in both directions:

$$
\sum_{m \in H^1(M)} q^{m \cdot b''}
$$

that we would like to remove or factor out, in order to make the partition function on $M \times D^2 \times q S^1$ well-defined. It would be interesting to explore various ways to do this. Relegating a more systematic study of this question for future work, here we merely sidestep the issue by imposing the identification $q^{\text{GCD}(b'')} \sim 1$ that leads to (2.99)-(2.100).

Let us make a few comments on this interesting phenomenon by examining it from various perspectives. Reduction to 4d gauge theory (2.111) yields a system of Kapustin-Witten PDE’s on $M \times \mathbb{R}_+$, with a Nahm pole boundary condition at $y = 0$, where $y$ is a coordinate along $\mathbb{R}_+$ [57]. The boundary condition at $y = \infty$ breaks the gauge group $G$ to a Levi subgroup $L \subseteq G$, which in applications to $\hat{Z}$-invariants is a maximal torus of $G$. In particular, $L = U(1)$ for $G = SU(2)$. Other choices of $L$ are also interesting, and lead to a generalization of $\hat{Z}$-invariants labeled by complex coadjoint orbits of $G_C$ or, equivalently, by $\rho : \mathfrak{sl}(2) \to \mathfrak{g}$ [34].

When $L = \mathbb{T}$, there are infinitely many different topological sectors labeled by monopole numbers $b''_i \in \Lambda_{\text{cochar}} = \text{Hom}(U(1), \mathbb{T})$ or, more precisely, by spin$^c$ structures. In order to keep track of dependence on $b$, one can introduce a topological term $\exp(2\pi i \eta b)$ in the action of 4d gauge theory. Then, using Pontryagin duality,

$$
\begin{align*}
\hat{x} = e^{2\pi i \eta} \in \text{Hom}(H_1(M), \mathbb{T})
\end{align*}
$$

can be identified with the variable by the same name in $F_K(x, q)$ and in the integrand of $\hat{Z}$-invariants. To summarize, on a closed 3-manifold with $b_1 > 0$ the boundary condition $B_0$ breaks the gauge group $G$ to $L = \mathbb{T}$ and creates a “flux” $b$. When this flux is non-torsion, the solutions to Kapustin-Witten PDE’s on $M \times \mathbb{R}_+$ can not approach a constant field configuration at $y = +\infty$. At best one can require solutions to approach a field configuration periodic in the $y$-direction, which leads us to conclude that the anomaly in question is controlled by the moduli space of solutions on $M \times S^1$ with gauge group $L$. Below we give another interpretation of this claim from the perspective of 3d-3d correspondence.

From the point of view of 3d $\mathcal{N} = 2$ theory $T[M]$, the boundary condition $B_0$ labels the background momentum / charge sectors of the 2d boundary theory, cf. [21]. In the partition function on $D^2 \times q S^1$, the parameter $q$ keeps track of the spin with respect to the rotation symmetry of $D^2$. It is defined up to spins of BPS states in 2d boundary theory or, equivalently, 3d theory $T[M, \mathbb{T}]$ on a slab $T^2 \times I$ with boundary conditions $B_0$ on both sides. This theory is a close cousin of $T[M \times S^1, \mathbb{T}]$ in the background of spin$^c$ structure $b$ on $M$. Indeed, both theories exhibit a qualitative change in behavior depending on whether $b_1 = 0$ or $b_1 > 0$. In the latter case the BPS states come in towers infinite in both direction, with spins in each tower being multiples of $b''$, which leads again to the ambiguity (2.99).
Similarly, from the curve counting perspective on $T^*M$, when $b_1 > 0$ in addition to open BPS states one also has a non-trivial “closed sector.”

3. Families of examples

In this section we illustrate the proposed relation with a number of instructive examples.

3.1. Plumbed 3-manifolds. Here we consider the case when $M$ is a rational homology sphere given by a weakly negative definite plumbing graph $\Gamma$ [32, 30]. We can then assume that

\begin{equation}
\omega \in H^1(M; \mathbb{Q}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z}).
\end{equation}

With the notation of Section 2.2, Formula (2.58) becomes:

\begin{equation}
N_r(M, \omega) = \frac{1}{\Delta^+_b \Delta^-_b} \sum_{k \in H^1_{\text{Vert}}} \prod_{I \in \text{Vert}} d(\alpha_{kI})^{2-\deg(I)} T(\alpha_{kI})^{B_{II}} \prod_{(I,J) \in \text{Edges}} S(\alpha_{kI}, \alpha_{kJ})
\end{equation}

where

\begin{equation}
\alpha_{kI} := \mu_I + k_I
\end{equation}

and $b_\pm$ are the number of positive/negative eigenvalues of $B$. In this section we will be somewhat cavalier with taking the limits and about convergence of infinite series. Such technical details will be properly addressed in Section 4.1 and Appendix D.

It is instructive to separate 3 factors:

\begin{equation}
N_r(M, \omega) = A \cdot B \cdot C,
\end{equation}

\begin{equation}
A = r^{-V/2} \xi^{\frac{3\sigma - T \cdot B}{4}}, \quad \begin{cases} 
\exp\left(\frac{\pi i (\sigma + T \cdot B)}{4}\right), & r = 1 \mod 4, \\
2^{-\sigma/2} e^{-\frac{\pi i \sigma}{4}}, & r = 2 \mod 4, \\
e^{-\frac{\pi i \sigma}{4} T \cdot B} (-1)^{\sigma}, & r = 3 \mod 4,
\end{cases}
\end{equation}

where $\sigma$ is the signature of $B$,

\begin{equation}
B = F(\{\exp(\pi i \mu_I)\}_{I \in \text{Vert}})^{-1},
\end{equation}

\begin{equation}
C = \sum_{k \in H^1_{\text{Vert}}} F(\{\xi^{\mu_I + k_I}\}_{I \in \text{Vert}}) \cdot \xi^{\frac{1}{2}(\mu + k)^T B(\mu + k)},
\end{equation}

and

\begin{equation}
F(x) := \prod_{I \in \text{Vert}} (x_I - 1/x_I)^{2-\deg(I)} = \sum_{\ell \in \mathbb{Z}_{\text{Vert}}} F_{\ell} \prod_{I \in \text{Vert}} x_{I}^{\ell_I}
\end{equation}

is a slightly different normalization of the invariant $F_L(x, q)$ for the case of the plumbing link $L$. Consider a contribution of a monomial $\prod_{I} x_{I}^{\ell_I}$ from $F(x)$ into (3.7):

\begin{equation}
C_{\ell} := \sum_{k \in H^1_{\text{Vert}}} \xi^{T(\mu + k)} \cdot \xi^{\frac{1}{2}(\mu + k)^T B(\mu + k)} = \sum_{n \in \mathbb{Z}_{\text{Vert}}/r \mathbb{Z}_{\text{Vert}}} \exp\left(\frac{\pi i}{4}(\tilde{\mu} + 2n)^T B(\tilde{\mu} + 2n) + \frac{\pi i}{2} \ell^T (\tilde{\mu} + 2n)\right)
\end{equation}
where

$$\tilde{\mu} := \mu + (r - 1) \varepsilon.$$  

We can now use the following version of Gauss reciprocity formula \[22, 35\]:

$$\sum_{n \in \mathbb{Z}/r\mathbb{Z}} \exp \left( \frac{2\pi i}{r} n^T Bn + \frac{2\pi i}{r} p^T n \right) = \frac{e^{\pi i \frac{r}{2}} (r/2)^{V/2}}{|\det B|^{1/2}} \sum_{\tilde{\alpha} \in \mathbb{Z}/2r\mathbb{Z}} \exp \left( -\frac{\pi ir}{2} (\tilde{\alpha} + \frac{p}{r})^T B^{-1}(\tilde{\alpha} + \frac{p}{r}) \right).$$

Applying it to (3.9) we have

$$\mathcal{C}_\ell = \xi^{-\frac{T B^{-1} \ell}{2}} e^{\pi i \frac{r}{2} (r/2)^{V/2}} \sum_{\tilde{\alpha} \in \mathbb{Z}/2r\mathbb{Z}} \exp \left\{ -\frac{\pi ir}{2} \tilde{\alpha}^T B^{-1} \tilde{\alpha} - \pi i \tilde{\alpha}^T B^{-1}(\ell + B \tilde{\mu}) \right\}.$$

Let us make the change of variables \(\tilde{a} = BA + a, A \in \mathbb{Z}/2\mathbb{Z}, a \in \mathbb{Z}/r\mathbb{Z}\):

$$\mathcal{C}'_\ell = \sum_{a \in \mathbb{Z}/2\mathbb{Z}} \sum_{A \in \mathbb{Z}/2\mathbb{Z}} e^{-\frac{\pi ir}{2} a^T B^{-1} a - \pi i a^T B^{-1}(s + B \tilde{\mu})} \cdot$$

where in the last line we used the fact that (3.8) only contains powers \(\prod I x_I^\ell_I\) satisfying \(\ell_I = \deg(I) \mod 2\), and therefore one can introduce \(b \in \mathbb{Z}/r\mathbb{Z}\) and \(s \in \mathbb{Z}/2\mathbb{Z}\), \(\sum J B_{IJ} s_J = B_{II} \mod 2\), such that

$$\ell = 2b + B(s - \varepsilon) \mod 2\mathbb{Z}.$$  

We also used the property (2.47). At this point we will need to consider the cases with different \(r \mod 4\) values separately.

3.2. Level \(r = 2 \mod 4\). Using the fact that \(r\) is even, while \(r/2\) is an odd integer, and condition on \(s\) the sum (3.13) simplifies to:

$$\mathcal{C}'_\ell = 2^V \sum_{a \in \mathbb{Z}/2\mathbb{Z}} \exp \left\{ -\frac{\pi ir}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T (s + \mu) \right\}.$$  

Combining everything together we then have:
\[(3.16) \quad N_r(M, \omega) = \frac{F(\{e^{\pi i I}\}_{I \in \text{Vert}})^{-1}}{|\det B|^{1/2}} \xi^{3\sigma - \frac{\chi B}{2}} \times \sum_{\ell \in \mathbb{Z}^{\text{Vert}}} \sum_{a \in \mathbb{Z}^{\text{Vert}}/BZ^{\text{Vert}}} F\ell^\xi - \frac{\ell^TB^{-1}\ell}{2} e^{-\frac{\pi i}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T (s + \mu)}. \]

Using the following expressions for \(\hat{Z}\) and Reidemeister torsion\(^8\) \(T\) for plumbed manifolds (see formula (A.7) in Appendix A)

\[(3.17) \quad \hat{Z}_a(M) = (-1)^{b_+} q^{\frac{3\sigma - \chi B}{4}} \sum_{\ell = a \mod 2BZ^{\text{Vert}}} F\ell^q - \frac{\ell^TB^{-1}\ell}{4}, \quad a_I = \deg(I) \mod 2, \]

\[(3.18) \quad T(M, [\omega]) = (-1)^{b_+} \prod_{I \in \text{Vert}} (e^{\pi i \mu I} - e^{-\pi i \mu I})^{\deg(I) - 2}, \quad [\omega] := \omega \mod H^1(M; \mathbb{Z}/2\mathbb{Z}), \]

and the identifications (2.3)-(2.18), (2.15)-(2.16) we can conjecture the following general relation for a rational homology sphere \(M\) and \(r = 2 \mod 4\):

\[(3.19) \quad N_r(M, \omega) = \frac{T(M, [\omega])}{\sqrt{|H_1(M; \mathbb{Z})|}} \sum_{a, b \in H_1(M; \mathbb{Z})} e^{-\frac{\pi i}{2} q_s(a) - 2\pi i \ell k(a, b) - \pi i \omega(a)} \hat{Z}_{\sigma(b, s)} \bigg|_{q \to 2^\pi} \]

where \(\sigma\) is the canonical map

\[(3.20) \quad \sigma : \text{Spin}(M) \times H_1(M; \mathbb{Z}) \rightarrow \text{Spin}^c(M) \]

producing a spin\(^c\) structure on \(M\) from a spin structure \(c\) and \(\tilde{b} \in H_1(M; \mathbb{Z})\). It is induced by the map \(B\text{Spin} \times BU(1) \rightarrow B\text{Spin}^c\) between the corresponding classifying spaces, combined with the isomorphisms \(BU(1) \cong B^2\mathbb{Z}, H_1(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z})\). In (3.19) we have introduced an auxiliary spin structure \(s \in \text{Spin}(M)\) (see also 3.14). The result is independent of it due to (2.14), so that the simultaneous change of \(b \in H_1(M; \mathbb{Z})\) and \(s \in \text{Spin}(M)\) leaving \(\sigma(b, s)\) invariant also leaves invariant the exponent in the sum in (3.19).

---

\(^8\)In principle the torsion \(T(M, \alpha)\), defined for \(\alpha \in H^1(M; \mathbb{Q}/\mathbb{Z}) \cong H_1(M; \mathbb{Z})\), has sign ambiguity, if no additional structures on \(M\) are introduced. One can fix the sign for example by introducing a spin structure \(s \in \text{Spin}(M)\) on \(M\), cf. [42]. The change of the spin structure \(s \rightarrow s + c, c \in H^2(M; \mathbb{Z}/2\mathbb{Z})\) then changes the sign by \((-1)^{\ell(k)}\) where \(\tilde{\alpha} \in H_1(M; \mathbb{Z})\) is dual to \(\alpha\). However, since in our case \(\alpha = \omega \mod 1, \tilde{\alpha}\) is even, and the dependence on spin structure drops out.
3.3. Level $r = 1 \mod 4$. The sum (3.13) reads

$$C'_\ell = \sum_{a \in \mathbb{Z} \text{Vert}/B \mathbb{Z} \text{Vert}} \sum_{A \in \mathbb{Z} \text{Vert}/2 \mathbb{Z} \text{Vert}} \exp \left\{ -\frac{\pi i r}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu - \pi i a^T (s - \varepsilon) - \frac{\pi i}{2} A^T B A + \pi i A^T (a + B(s - \varepsilon)) \right\}. $$

Applying a version of the Gauss reciprocity formula to the sum over $A$ we can rewrite it as follows:

$$C'_\ell = e^{-\pi i \sigma/4} \frac{V}{\det B}^{1/2} \sum_{a, f \in \mathbb{Z} \text{Vert}/B \mathbb{Z} \text{Vert}} \exp \left\{ -\frac{\pi i (r - 1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu + 2\pi i f^T B^{-1} f + 2\pi i f^T B^{-1} a + \frac{\pi i}{2} (s - \varepsilon)^T B(s - \varepsilon) \right\}. $$

Taking into account that

$$\frac{1}{4} \varepsilon^T B \varepsilon = \frac{V - 1}{2} + \frac{1}{4} \text{Tr} B \mod 1,$$

$$\frac{1}{2} \varepsilon^T B s = \frac{1}{2} \text{Tr} B \mod 1,$$

and combining everything together we then have:

$$N_r(M, \omega) = \frac{F\left(\{ e^{\pi i \mu}\}_{\ell \in \text{Vert}}\right)^{-1}}{|\det B|} e^{\pi i (2 - \sigma + s^T B s)} \xi^{3\sigma - \text{Tr} B} \times \sum_{\ell \in \mathbb{Z} \text{Vert}} \sum_{a, f \in \mathbb{Z} \text{Vert}/B \mathbb{Z} \text{Vert}} F_{\ell} \xi^{-\frac{\ell^T B^{-1} a}{2}} e^{-\frac{\pi i (r - 1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu + 2\pi i f^T B^{-1} f + 2\pi i f^T B^{-1} a}. $$

As in the case $r = 2 \mod 4$, we can then conjecture the following general relation for a rational homology sphere $M$ and $r = 1 \mod 4$:

$$N_r(M, \omega) = \frac{-e^{-\frac{\pi i}{4} \mu(M, s)}}{|H_1(M; \mathbb{Z})|} \times \sum_{a, b, f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r - 1}{4} \ell k(a, a) + \ell k(a, f - b) - \frac{1}{2} \omega(a) + \ell k(f, f) \right)} \tilde{Z}_{\sigma(b, s)}(q \to e^{2\pi i})$$

where we have used the surgery formula (2.17) for the mod 4 reduction of the Rokhlin invariant $\mu(M, s)$.

It is interesting to remark that unlike in the case $r = 2 \mod 4$ here the relation between $N_r$ and $\tilde{Z}$ is based on a triple summation (instead of double).
3.4. Level $r = 3 \mod 4$. This case is analogous to the case $r = 1 \mod 4$ considered above. When $r = 3 \mod 4$ the sum \((3.13)\) reads

\[
C'_\ell = \sum_{a \in \mathbb{Z}_{\text{Vert}}/B \mathbb{Z}_{\text{Vert}}} \sum_{A \in \mathbb{Z}_{\text{Vert}}/2 \mathbb{Z}_{\text{Vert}}} \exp \left\{ -\frac{\pi i r}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu 
- \pi i a^T (s - \varepsilon) + \frac{\pi i}{2} A^T B A + \pi i A^T (a + B (s - \varepsilon)) \right\}.
\]

Applying again the Gauss reciprocity formula to the sum over $A$ we have:

\[
C'_\ell = \frac{e^{\frac{\pi i}{8} 2^{V/2}}}{|\det B|^{1/2}} \sum_{a, f \in \mathbb{Z}_{\text{Vert}}/B \mathbb{Z}_{\text{Vert}}} \exp \left\{ -\frac{\pi i (r + 1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu 
- 2\pi i f^T B^{-1} f - 2\pi i f^T B^{-1} a - \frac{\pi i}{2} (s - \varepsilon)^T B (s - \varepsilon) \right\}.
\]

Combining everything together we then have:

\[
N_r(M, \omega) = \frac{F(\{e^{\pi i \mu}\}_{I \in \text{Vert}})^{-1}}{|\det B|} e^{\frac{\pi i}{8} (2 + s^{-T} B^{-1} s)} \xi^{2s - r \frac{B}{2}} \times
\sum_{\ell \in \mathbb{Z}_{\text{Vert}}} \sum_{a, f \in \mathbb{Z}_{\text{Vert}}/B \mathbb{Z}_{\text{Vert}}} F_\ell \xi^{-\frac{r^T B^{-1} f}{2}} e^{\frac{\pi i}{8} \frac{r^T (B^{-1} a)}{2}} e^{\frac{\pi i}{8} (s^{-T} B^{-1} s) a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu + 2\pi i f^T B^{-1} f + 2\pi i f^T B^{-1} a}.
\]

We can then conjecture the following general relation for a rational homology sphere $M$ and $r = 3 \mod 4$:

\begin{align}
N_r(M, \omega) &= e^{\frac{\pi i}{8} (\mu(M, s))} \frac{T(M, [\omega])}{|H_1(M; \mathbb{Z})|} \times \sum_{a, b, f \in H_1(M; \mathbb{Z})} \sum_{q \in e^{2\pi i/r}} e^{2\pi i \left( -\frac{r}{2} \ell(k(a, a) - k(a, f + b) - \frac{1}{2} \omega(a) - \ell(k(f, f)) \right)} \mathcal{Z}_{\sigma(b, s)}|_{q \rightarrow e^{2\pi i/r}}.
\end{align}

3.5. Generalization to $b_1 > 0$. Let $M$ be, as before, obtained by a surgery on a link with the linking matrix $B$. However now we will allow $B$ to be degenerate. By an $SL(V, \mathbb{Z})$ transform we can bring the quadratic form $B$ to $B' \oplus 0_{b_1}$ where $0_{b_1}$ is the trivial quadratic form on $\mathbb{Z}^{b_1} \subset \mathbb{Z}^V$ \cite{40}. The expression \((3.12)\) then will be modified to:

\[
C_\ell = \xi^{-\frac{r^T B^{-1} \ell}{2}} e^{\frac{\pi i}{8} \frac{r}{2}} e^{\frac{\pi i}{8} \frac{r}{2}} \sum_{\tilde{a} \in \mathbb{Z}^{V - b_1}/2B' \mathbb{Z}^{V - b_1}} e^{-\frac{\pi i}{2} \tilde{a}^T (B')^{-1} \tilde{a} - \pi i \tilde{a}^T (B')^{-1} (\ell' + B' \tilde{\mu})} \times \sum_{r^{b_1}} e^{\frac{\pi i}{8} \frac{r}{2}} \delta_{\ell'' \equiv 0} \mod r
\]

where $\ell = \ell' \oplus \ell''$ and $\tilde{\mu} = \tilde{\mu}' \oplus \tilde{\mu}''$ according to the splitting of $B$ above.
Consider first the case of \( r = 2 \mod 4 \). We have
\[
A = \xi_{\frac{3\sigma - 4}{4}}^{3\sigma - 4} r^{- V/2 + b_1/2} 2^{b_1/2} 2^{- V/2} e^{- \frac{\pi i a}{4}}.
\]
The relation (3.19) generalizes to
\[
N_r(M, \omega) = \frac{r^{b_1} \mathcal{T}(M, [\omega])}{\sqrt{|\text{Tor } H_1(M; \mathbb{Z})|}} \times \\
\sum_{a', b' \in \text{Tor } H_1(M; \mathbb{Z})} e^{- \frac{\pi i r}{2} q_{a'(\omega')} - 2\pi i \ell k(a', b') - \pi i \omega(\alpha') + \pi i \omega''(\alpha')} \widetilde{Z}_{\sigma(b \oplus b', 2, s)} |_{q \to e^{\frac{2\pi i}{q}}} = \\
\int_{H^1(M; \mathbb{Z} / 2 \mathbb{Z})} \mu(\alpha) \sum_{b \in H_1(M; \mathbb{Z})} e^{- \frac{\pi i r}{2} q_{a(\alpha')} - 2\pi i \ell k(a', b') - \pi i \omega(\alpha') + 2\pi i \omega''(b')} \times \\
\delta(\omega'' - \omega' / 2) \widetilde{Z}_{\sigma(b, s)} |_{q \to e^{\frac{2\pi i}{q}}}
\]
where we chose explicit splittings \( \omega = \omega' \oplus \omega'' \in H^1(M; \mathbb{C} / 2 \mathbb{Z}) \cong \text{Tor } H_1(M; \mathbb{Z}) \oplus (\mathbb{C} / 2 \mathbb{Z})^{b_1} \) and \( b = b' \oplus b'' \in H_1(M; \mathbb{Z}) \cong \text{Tor } H_1(M; \mathbb{Z}) \oplus \mathbb{Z}^{b_1} \) according to the splitting of the linking matrix \( B \) above. It is straightforward to see that the right-hand side is independent of the choice of representative of the equivalence class (2.99). The coefficients in the relation (3.33) are not invariant under the automorphisms (2.104). However this compensated by the non-covariance of \( \widetilde{Z} \). Taking into account (2.108), one can show that the total sum in (3.33) transforms covariantly (i.e. as a function on \( H^1(M; \mathbb{C} / 2 \mathbb{Z}) \cong \text{Hom}(\text{Tor } H_1(M; \mathbb{Z}) \oplus \mathbb{Z}^{b_1}, \mathbb{C} / 2 \mathbb{Z}) \)). Namely, considering that
\[
q^E(b', b'') |_{b'' = \frac{m}{r}} = e^{\frac{2\pi i m}{r} q_{a(\mu m) - 2\pi i \ell k(\mu m, b')}}
\]
and shifting the summation variables \( b' \to b' + \frac{a'}{r} \mu m, a' \to a' - \mu m \), we have indeed
\[
N_r(M, \omega' \oplus \omega'' \sim N_r(M, \omega' \oplus (\omega'' + 2\mu^* \omega))
\]
where
\[
\omega' \oplus \omega'' \in \text{Tor } H_1(M; \mathbb{Z}) \oplus (\mathbb{C} / 2 \mathbb{Z})^{b_1}
\]
and
\[
\mu^* : \text{Tor } H_1(M; \mathbb{Z}) \to (\mathbb{C} / \mathbb{Z})^{b_1}
\]
is the map dual to \( \mu \) in (2.105).

For \( r = 1 \mod 4 \) we have
\[
A = r^{- V/2 + b_1/2} \xi_{\frac{3\sigma - 4}{4}}^{3\sigma - 4} e^{\frac{\pi i}{2} (r b_1 + \sigma)}
\]
and

$$N_r(M, \omega) = \frac{r^{b_1} \mathcal{T}(M, [\omega])}{\text{Tor} \ H_1(M; \mathbb{Z})} \times \sum_{a', b', f' \in \text{Tor} \ H_1(M, \mathbb{Z})} e^{2\pi i \left( -\frac{\tau}{2} \ell k(a', a') + \ell k(a', f') - \frac{1}{2} \omega(a') + \ell k(f', f') - \Delta_{e(v, s)} - \ell k(v', b') + \omega''(m) \right)} \times \hat{Z}_s(v' \oplus r_m, s) \bigg|_{q \to e^{2\pi i}}.$$  

**Example 3.1.** An interesting example is $M = \Sigma_g \times S^1$. In this case $\ell k = 0$ and $q_s = 0$. Furthermore we have the identity:

$$\mathcal{T}(M, [\omega]) = (-2)^{b_1 + 1} \left( \frac{i}{4} \right)^{b_1} \frac{i}{2} N_2(M, 2\omega).$$

Thus replacing in the above formula we get

$$\frac{N_r(\Sigma_g \times S^1, \omega)}{N_2(\Sigma_g \times S^1, 2\omega)} = r^{b_1} (-2)^{1+b_1} \left( \frac{i}{4} \right)^{b_1} \frac{i}{2} \sum_{m \in \mathbb{Z}^{2g+1}} e^{\pi i \omega(m)} \hat{Z}_s(r_m, s) \bigg|_{q \to e^{2\pi i}}.$$  

If we now recall that

$$N_r(\Sigma_g \times S^1, \omega) = r^{2g} \sum_{k \in H_1} \left( \frac{r \beta}{\beta + k} \right)^{2g-2}$$

(so in particular $N_2(\Sigma_g \times S^1, 2\omega) = 2^{2g+1} \left( \frac{1}{4} \right)^{2g} (i^2 \beta - i^{-2\beta})^{2g-2}$) where $\beta = \omega(S^1)$ (note the formula does not depend on the orientation of $S^1$), then we get:

$$\frac{1}{r} \sum_{k \in H_1} \left( \frac{r \beta}{\beta + k} \right)^{2g-2} \left( i^2 \beta - i^{-2\beta} \right)^{2g-2} = \sum_{m \in \mathbb{Z}^{2g+1}} e^{\pi i \omega(m)} \hat{Z}_s(r_m, s) \bigg|_{q \to e^{2\pi i}}.$$  

In order to compute $\hat{Z}_s$ we observe that the formula (3.17) providing the value of $\hat{Z}$ for surgeries over plumbing links can be generalised to manifolds which are the boundaries of a plumbing of surfaces in a tree-like fashion. In particular, for $M = \Sigma_g \times S^1$,

$$\left( x - x^{-1} \right)^{2-2g} = \sum_{\ell \in \mathbb{Z}} F_{\ell} x^\ell$$

Since Tor $(H_1(M)) = 0$ the first Chern class provides a bijection between the set of spin$^c$ structures on $M$ and $H^2(M) = H_1(M)$, but $\hat{Z}_s$ is zero for all spin$^c$ structures $s$ such that $c_1(s) \neq PD(\ell \{pt\} \times S^1)$ for some $\ell \in \mathbb{Z}$. So letting $\ell$ be the spin$^c$ structure the first Chern class of which is Poincaré dual to $\ell \{pt\} \times S^1$ and using (2.97) with $U = \text{Id}$, $B = 0$, $\sigma = 0 = b_+$, we have: $\hat{Z}_\ell = q^\ell F_{\ell}$.

Therefore equation (3.43) becomes:

$$\sum_{k \in H_1} \frac{1}{e^{\frac{\pi i}{2} (\beta + k)} - e^{-\frac{\pi i}{2} (\beta + k)}} \left( \frac{r \beta}{\beta + k} \right)^{2g-2} = \sum_{m \in \mathbb{Z}} e^{\pi i \beta} \hat{Z}_s(r_m, s) \bigg|_{q \to e^{2\pi i}}.$$  

(3.44)
Then we have that the left hand side of (3.45) equals
\[ \sum_{\ell \in \mathbb{Z}} F_\ell \sum_{k \in H_r} e^{\pi i (\beta + k)} = r \sum_{\ell \in \mathbb{Z}} F_\ell e^{\pi i \beta} (-1)^{(r-1)l} \]
and the r.h.s. equals:
\[ r \sum_{m \in \mathbb{Z}} e^{\pi i m \beta} F_r m e^{\pi i m}. \]
So if \( r \) is even then (3.45) is verified directly and if \( r \) is odd, then the equality is true because \( F_\ell = 0 \) for odd \( \ell \).

3.6. Surgeries on knots. Consider \( M = S^3_p(K) \) and assume that \( -p \in \mathbb{Z}_+ \) for concreteness. Then
\[ \hat{Z}_a[S^3_p(K)] = q^{-\frac{3+p}{2}} \sum_{\ell = a \text{ mod } 2p} F_\ell q^{\frac{\ell^2}{4p}} \]
where the coefficients \( F_\ell \) appear in the expansion of
\[ F(x, q) := F_K(x^2, q)(x - x^{-1}) = \sum_\ell F_\ell x^\ell \]
with \( F_K \) introduced in [30]. We will use the facts that \( F_K(x^2, q) = -F_K(x^2, q) \) and that \( F_K(x^2, q) \in x\mathbb{Z}[[x^\pm 2, q]] \). In particular it follows that \( \hat{Z}_a \equiv 0 \) for \( a = 1 \mod 2 \).

As argued in [29], the series \( F_K(x, q) \) gives ADO polynomials at roots of unity (which, in turn, are related to the CGP invariants for knot complements). Note, this already establishes a relation between \( F_K(x, q) := \hat{Z}(S^3 \setminus K) \) and \( N_r(S^3, K) \) for knots, and tells us the relation between the parameters: in order to obtain \( N_r \) on the CGP side this large class of examples shows that on the GPPV side we need to take \( q = e^{2\pi i / r} \) (not \( q = e^{\pi i / r} \)). The normalizations are such that
\[ F_K(x, q)|_{q = \xi^2} = \frac{\text{ADO}_r(x/\xi^2, K)}{\Delta_K(x^r)} \cdot (x^{1/2} - x^{-1/2}), \quad \xi := e^{\pi i / r}, \]
where \( \Delta_K(t) \) denotes the Alexander polynomial of \( K \). We wish to compose this with the relation between ADO polynomials and CGP invariants for knot complements:
\[ \text{ADO}_r(x^2/\xi^2) = \frac{x^r - x^{-r}}{x - x^{-1}} N_r(S^3, K_\alpha), \quad \text{where } x = e^{\frac{3\pi i}{r}}. \]

Eliminating the ADO polynomial from the above two relations we get a more direct relation between GPPV and CGP invariants for knot complements
\[ N_r(S^3, K_\alpha) = \frac{F_K(e^{\frac{2\pi i \alpha}{r}}, e^{\frac{2\pi i}{r}}) \Delta_K(e^{2\pi i \alpha})}{(e^{\pi i \alpha} - e^{-\pi i \alpha})}. \]
On the other hand, we have
\[ N_r(S^3_p(K), \omega) = \frac{1}{\Delta} \sum_{k \in H_r} d(\alpha_k) N_r(S^3, K_{\alpha_k}) T(\alpha_k)^p \]
where $N_r(S^3, K_\alpha)$ denotes CGP invariant of a (with zero framing) knot in $S^3$ colored by $\alpha$ (e.g. for unknot $U$ with zero framing $N_r(S^3, U_\alpha) = d(\alpha)$). The other notations are the same as in Section 2.4. In particular, $\alpha_k := \alpha + \mu$, where $\mu = \omega(m) \in \mathbb{Z}/p\mathbb{Z}$ and $m \in H_1(S^3_p(K); \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ is the generator represented by the meridian of the knot $K$.

We want to check that the surgery formulas for homological blocks $\hat{\mathcal{Z}}$ and CGP invariant $N_r$ are consistent with the conjectural relations between these invariants for knot complements and closed manifolds. In other words, we want to check commutativity of the following schematic diagram:

$$
\begin{array}{ccc}
\hat{\mathcal{Z}}_b[S^3_p(K)](q) & q \to e^{2\pi i q} & N_r(S^3_p(K), \omega) \\
& \text{Laplace transform} & \\
& \text{Kirby color} & \\
F_K(x, q) & q \to e^{2\pi i q} & N_r(S^3, K) \\
\end{array}
$$

where, more concretely, the left vertical arrow is given by the equations (3.48)-(3.49), the right vertical arrow is given by (3.53), the top horizontal arrow is given by (3.26),(3.19),(3.30), and the bottom horizontal arrow is given by (3.52).

This check can be done by essentially repeating the analysis done for plumbings in case of a single vertex (with $F$ in (3.8) replaced by (3.49)) and using the following well known relation between the Reidemeister torsion of $M = S^3_p(K)$ and the Alexander polynomial of the knot $K$:

$$
\mathcal{T}(S^3_p(K), t) = \frac{t\Delta_K(t)}{(1 - t)^2} \bigg|_{t \in \mathbb{Z}/p \subset U(1)},
$$

where $t$ is the holonomy of $U(1)$ flat connection along the the meridian $m$ of the knot $K$. But let us write it explicitly anyway.

As for plumbings, after plugging (2.78) into (3.53), it is instructive to separate the result into three factors:

$$
N_r(S^3_p(K), \omega) = A \cdot B \cdot C,
$$

where

$$
A = r^{-1/2} \xi^{-2\pi i / 2} \cdot \left\{ \begin{array}{l}
e^{\pi i / 2} e^{-\pi i / 2}, \ r = 1 \ mod \ 4, \\
2^{-1/2} e^{\pi i / 2}, \ r = 2 \ mod \ 4, \\
-e^{-\pi i / 2}, \ r = 3 \ mod \ 4, 
\end{array} \right.
$$

$$
B = \frac{\Delta(e^{2\pi i \mu})}{(e^{\pi i \mu} - e^{-\pi i \mu})^2},
$$

$$
C = \sum_{k \in H_r} F(\xi^{\mu+k}, \xi^2) \cdot \xi^{\frac{2}{p} (\mu+k)^2}
$$

and $\Delta$ is the Alexander polynomial of the knot $K$. The other notations are the same as in Section 2.4.
where, as before, \( \xi := e^{\frac{\pi i}{r}} \). Consider a contribution of a monomial \( x^\ell \) from \( F_K(x) \) into (3.59):

\[
C_\ell := \sum_{k \in H_r} \xi^{\ell (\mu + k)} \cdot \xi^{\frac{\mu (\mu + k)^2}{2}} = \sum_{n \in \mathbb{Z}/r \mathbb{Z}} e^{\frac{\pi i}{2r}(\mu + 2n)^2 + \frac{\pi i}{r} \ell (\mu + 2n)}
\]

where

\[
\tilde{\mu} := \mu + (r - 1).
\]

We can now use the following one-dimensional Gauss reciprocity formula:

\[
\sum_{n \in \mathbb{Z}/r \mathbb{Z}} \exp \left( \frac{2\pi i}{r} (pn^2 + \ell n) \right) = e^{\frac{\pi i \text{sign}(p)}{4}} \sqrt{\frac{r}{2\lfloor p \rfloor}} \sum_{\tilde{\mu} \in \mathbb{Z}/2p \mathbb{Z}} \exp \left( -\frac{\pi ir}{2p} (\tilde{\mu} + \frac{\ell}{r})^2 \right)
\]

Applying it to (3.60) we have

\[
C_\ell = \xi^{-\frac{\pi i}{2p}} e^{-\frac{\pi i}{4}} \sqrt{\frac{r}{2\lfloor p \rfloor}} \sum_{\tilde{\mu} \in \mathbb{Z}/2p \mathbb{Z}} e^{-\frac{\pi ir}{2p} \tilde{\mu}^2 - \pi i \tilde{\mu} (\ell/p + \tilde{\mu})}.
\]

Therefore, taking into account (3.48) we can write

\[
\mathcal{C} = \sum_{\ell} C_\ell F_\ell = e^{-\frac{\pi i}{4}} \xi^{\frac{3+\mu}{2}} \sqrt{\frac{r}{2\lfloor p \rfloor}} \sum_{b \in \mathbb{Z}/p \mathbb{Z}} \sum_{\tilde{a} \in \mathbb{Z}/2p \mathbb{Z}} e^{-\frac{\pi i r}{2p} \tilde{a}^2 - \frac{2\pi i \bar{a} b}{p} - \pi i \mu \tilde{a} \bar{a} \hat{2b}[S^3_p(K)]} \bigg|_{q \rightarrow e^{\frac{2\pi i}{r}}}
\]

Combining this together with (3.57) and (3.58) we get

\[
N_r(S^3_p(K), \omega) = \frac{1}{\sqrt{|p|}} \Delta(e^{2\pi i \mu}) \left\{ \begin{array}{ll}
2^{-1}, & r = 2 \mod 4 \\
2^{-1/2} e^{\pm \frac{\pi i}{2}} e^{\mp \frac{3\pi i}{4}}, & r = \pm 1 \mod 4
\end{array} \right\} \times
\]

\[
\sum_{b \in \mathbb{Z}/p \mathbb{Z}} e^{-\frac{\pi i r}{2p} \tilde{a}^2 - \pi i (r-1)\tilde{a} - \frac{2\pi i a b}{p} - \pi i a \mu \hat{2b}[S^3_p(K)]} \bigg|_{q \rightarrow e^{\frac{2\pi i}{r}}}
\]

Using the formula (3.55) with \( t = e^{2\pi i \mu} \) we indeed arrive at the conjectural formula (3.19),(3.26) or (3.30), depending on the value \( r \mod 4 \), in the case of \( M = S^3_p(K) \). For example, when \( r = 2 \mod 4 \) the sum over \( \tilde{a} \) reduces to the sum over \( a = \tilde{a} \mod p \):

\[
N_r(S^3_p(K), \omega) = \frac{1}{\sqrt{|p|}} \Delta(e^{2\pi i \mu}) \times
\]

\[
\sum_{b \in \mathbb{Z}/p \mathbb{Z}} e^{-\frac{\pi i}{2p} a^2 - \pi i a - \frac{2\pi i a b}{p} - \pi i a \mu \hat{2b}[S^3_p(K)]} \bigg|_{q \rightarrow e^{\frac{2\pi i}{r}}}
\]
Similarly, for $r = 1 \text{ mod } 4$:

\begin{equation}
(3.67) \quad c_{a,b}^{CGP} = \frac{1}{|p|} \frac{\Delta(e^{4\pi ia/p})}{(e^{2\pi ia/p} - e^{-2\pi ia/p})^2} e^{2\pi i \frac{3 \text{sign}(p) - p}{4}} \sum_{c,f=0}^{p-1} e^{-\frac{2\pi i}{p} \left(-f^2 + (b-f)c + (r-1)c^2\right)}
\end{equation}

and for $r = 3 \text{ mod } 4$:

\begin{equation}
(3.68) \quad c_{a,b}^{CGP} = \frac{1}{|p|} \frac{\Delta(e^{4\pi ia/p})}{(e^{2\pi ia/p} - e^{-2\pi ia/p})^2} e^{2\pi i \frac{3 \text{sign}(p) - p}{4}} \sum_{c,f=0}^{p-1} e^{-\frac{2\pi i}{p} \left(f^2 + (b+f)c + (r+1)c^2\right)}.
\end{equation}

**Example 3.2.** As a concrete example, consider $p = -3$ surgery on the right-handed trefoil $K = 3_1$. In this case, $H_1(S^3_p(K)) = \mathbb{Z}_3$ and, therefore, there are two independent $\hat{Z}$-invariants, which can be expressed in terms of the false theta-functions (cf. [13, 30]):

\begin{equation}
(3.69) \quad \hat{Z}_0 = q^{\frac{71}{48}} \left(\Psi_{18}^{(1)} + \Psi_{18}^{(17)}\right) = q + q^5 - q^6 - q^{18} + q^{20} + \ldots,
\end{equation}

\begin{equation}
(3.70) \quad \hat{Z}_{\pm 1} = -\frac{1}{2} q^{\frac{71}{48}} \left(\Psi_{18}^{(5)} + \Psi_{18}^{(13)}\right) = -\frac{1}{2} q^{4/3} \left(1 + q^2 - q^7 - q^{13} + q^{23} + \ldots\right),
\end{equation}

where the factor $\frac{1}{2}$ in the latter expression and "±" in its label appear precisely because we write these expressions in the unfolded form. Evaluating the right-hand side of (3.66) for various values of $r$, we find

\begin{align}
\begin{array}{lcc}
\text{r} & \text{Value} \\
5 & 4.85591 - 6.4514i \\
6 & 5.30731 - 4.45336i \\
7 & 1.89035 + 3.49675i \\
9 & -6.77162 - 0.394402i \\
10 & -24.2779 + 7.76375i \\
11 & -6.01733 + 3.60533i \\
13 & -37.2754 - 1.28057i \\
14 & -11.4885 + 28.4093i \\
15 & -15.3891 + 13.8158i \\
17 & -11.2632 + 37.6555i \\
18 & 17.4965 + 18.5452i \\
19 & 59.3259 + 18.3538i \\
\end{array}
\end{align}

(3.71)

which match the corresponding values of $N_r(S^3_{-3}(3_1))$.

### 3.7. 0-surgeries on knots.

In general, the invariant of a knot complement is usually written as

\begin{equation}
(3.72) \quad F_K(x, q) = \frac{1}{2} \sum_{\substack{m \geq 1 \text{ odd}}} f_m(q) \cdot \left(x^{\frac{m}{2}} - x^{-\frac{m}{2}}\right)
\end{equation}
which after multiplying by \((x^{1/2} - x^{-1/2})\) in the surgery formula gives

\[
(3.73) \quad (x^{1/2} - x^{-1/2})F_K(x, q) = \frac{1}{2} \sum_{m \geq 1} \sum_{\text{odd}} f_m(q) \cdot (x^{m/2 + 1/2} - x^{m/2 - 1/2} - x^{-m/2 + 1/2} + x^{-m/2 - 1/2})
\]

which means (with \(n \in \mathbb{Z}\)):

\[
(3.74) \quad \text{Coeff}_x \left[ (x^{1/2} - x^{-1/2})F_K(x, q) \right] = \begin{cases} 
2f_{2n-1} - f_{2n+1}, & \text{if } n \geq 1, \\
-2f_1, & \text{if } n = 0, \\
f_{2|n|-1} - f_{2|n|+1}, & \text{if } n \leq -1.
\end{cases}
\]

For the unknot we have \(f_m(q) = \delta_{m,1}\), so that \(\tilde{Z}_n = \{\ldots, 0, 0, 1, -2, 1, 0, 0, \ldots\}\).

**Example 3.3.** **Trefoil knot:**

\[
(3.75) \quad N_r(S^3_0(3_1)) = \sum_{k \in H_r} d(\alpha + k) \frac{r^2 \xi^g(r-1)^2/4}{(2r\alpha + 2rk)} \sum_{n \in H_r} \xi^{3n\alpha + 3nk + 3n^2} \{2\alpha + 2k + n\} = (-1)^{r-1} \frac{r^2 \xi^g(r-1)^2/4}{(2r\alpha)} \sum_{k \in H_r} \sum_{n \in H_r} (-1)^k \{\alpha + k\} \{2\alpha + 2k + n\} \xi^{3n\alpha + 3nk + 3n^2}.
\]

For the trefoil knot \(f_m = \epsilon_m q^{m^2 + 23}/24\) and so

\[
(3.76) \quad \tilde{Z}_n = \begin{cases} 
\epsilon_{2n-1} q^{(2n-1)^2 + 23}/24 - \epsilon_{2n+1} q^{(2n+1)^2 + 23}/24, & \text{if } n \geq 1, \\
-2\epsilon_1 q^{2n}, & \text{if } n = 0, \\
\epsilon_{2|n|-1} q^{(2|n|-1)^2 + 23}/24 - \epsilon_{2|n|+1} q^{(2|n|+1)^2 + 23}/24, & \text{if } n \leq -1.
\end{cases}
\]

Again, we verified numerically and analytically that the proposed relations hold.

3.8. **Surgery on a link.** Let \(L\) be the link \(L_{027}\) in the Thistlethwaite table of links (see also Knot Atlas). Let \(M\) be obtained by \(a, b, c \in \mathbb{N}\) integral surgery on the three components of \(L\), where \(a\) corresponds to the blue component, \(b\) to the purple and \(c\) to the green one in Figure 2. The linking matrix of \(L\) is diagonal with entries \(a, b, c\) so that a cohomology class \(\omega \in H^1(M; \mathbb{C}/2\mathbb{Z})\) is described by a three-uple \((\alpha, \beta, \gamma)\) with \(\alpha \in \{\frac{2k}{a}, k = 0, \ldots a-1\}\), \(\beta \in \{\frac{2k}{b}, k = 0, \ldots b-1\}\) and \(\gamma \in \{\frac{2k}{c}, k = 0, \ldots c-1\}\).

\(L\) is one of the first links with the peculiar property that its multicolored Alexander polynomial is zero. Thus we have : \(N_2(M, \omega) = 0\) for any \(\omega \in H^1(M; \mathbb{C}/2\mathbb{Z})\).

On the other hand, a direct computer-based calculation gives:

\[
(3.77) \quad ADO_3(\alpha, \beta, \gamma) = q^{2\gamma} + (1 - i\sqrt{3}) + q^{-2\gamma}
\]
Figure 2. The $L_{9}27$ link (Knotscape image).

so that in particular it does not depend on $\alpha, \beta$ and we will denote it $P(\gamma)$. As a
consequence we have:

$$(3.78) \quad N_3(M, \omega) = \Delta_+^{-3} \left( \sum_{k_1 \in \{-2,0,2\}} q^\frac{\alpha}{2} (\alpha + k_1)^2 - (r-1)^2 d(\alpha + k_1) \right) \times$$

$$\left( \sum_{k_2 \in \{-2,0,2\}} q^\frac{\beta}{2} (\beta + k_2)^2 - (r-1)^2 d(\beta + k_2) \right) \times$$

$$\left( \sum_{k_3 \in \{-2,0,2\}} q^{\frac{c-1}{2}} (\gamma + k_3)^2 - (r-1)^2 d(\gamma + k_3) P(\gamma + k_3) \right)$$

(in the last factor of the above expression we used $c - 1$ as the self linking of the green
component is 1 in the given diagram). Choosing for instance $\alpha = \beta = \gamma = \frac{1}{2}$ one can compute directly that $N_3(M, \omega) \neq 0$. Since the torsion is 0
as the multivariable Alexander polynomial is, then in this case the invariants $\hat{Z}$ should
have infinite limits when $q \to \exp(2i\pi r)$.

4. The relation between $N_r$ and $\hat{Z}$

In this section we present a more general and systematic discussion of the proposed
relation.

4.1. Gauss sum vs Laplace transform. Let $M$ be the rational homology sphere
obtained by a surgery on a framed link $L$ in $S^3$ with a linking matrix $B$. We will
use the shorthand notations $x \equiv \{ x_I \}_{I \in \text{Vert}}$, $x^\pm \equiv \{ x^{\pm 1} I \}_{I \in \text{Vert}}$. In particular $K[[x^\pm]]$
denotes the space of formal Laurent series in $x^\pm$ with coefficients in $K$, considered as a
module over the ring of Laurent polynomials $K[x^\pm]$.

As before, $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z})$ and $\mu_I = \omega(m_I)$, which we assume to be fixed. We also
use notation

$$(4.1) \quad \xi \equiv e^{\frac{\alpha I}{2}}$$
in what follows.

Let $K$ be a field extension over $\mathbb{C}$ (some of the relevant cases are $K = \mathbb{C}$, $K = \mathbb{C}((q^{1/p}))$, $K = \mathbb{C}((q^{1/p}))(x)$). By $K(t)$ we will denote the localization of $K[t]$ by the subset of polynomials in $t$ non-vanishing at $t = 1$. This is a subset of the field $K(t)$ of rational functions in $t$ closed under multiplication. One can then consider the following ring homomorphism:

**Definition 4.1.**

\[
\lim_{t \to 1} : K(t) \to K,
\]

\[
P(t) \quad Q(t) \mapsto P(1) \quad Q(1).
\]

Similarly, by $K(x)$ we will denote the localization of $\mathbb{C}[x]$ by the subset of polynomials non-vanishing at $x = \xi^\mu + k$ for any $k \in H^\text{Vert}_r$. This is a subset of the field $K(x)$ of rational functions in $x$ closed under multiplication. We can then define the following $K$-linear “$\omega$-twisted Gauss sum” operation:

**Definition 4.2 (\(\omega\)-twisted Gauss sum).**

\[
G_\omega : K(x) \to K,
\]

\[
P(x) \quad Q(x) \mapsto \sum_{\ell \in H^\text{Vert}_r} P(\xi^{\mu + k}) \frac{Q(x)}{Q(x+1)} \xi^{\frac{\ell}{2}(\mu + k)^2} B(\mu + k).
\]

Denote $p := 4|\det B|$ and let $\mathbb{C}((q^{1/p})) \equiv \mathbb{C}[[q^{1/p}, q^{-1/p}]]$ be the field of fractions of $\mathbb{C}[[q^{1/p}]]$. We then define the $\mathbb{C}((q^{1/p}))$-linear “$\omega$-twisted Laplace transform” on a subspace of $\mathbb{C}((q^{1/p}))[x^\pm]$ by the following formula:

**Definition 4.3 (\(\omega\)-twisted Laplace transform).**

\[
L_\omega : \mathbb{C}((q^{1/p}))[x^\pm] \to \mathbb{C}((q^{1/p})),
\]

\[
\sum_{\ell \in \mathbb{Z}^\text{Vert}_r} A_{\ell,m} q^{\frac{m}{2}} x^\ell \mapsto \sum_{\ell \in \mathbb{Z}^\text{Vert}_r} q^{\frac{m}{2}} \sum_{m \in \mathbb{Z}} A_{\ell,m} \cdot C^\omega_\ell,
\]

where

\[
C^\omega_\ell = \frac{e^{\pi i \rho(\ell/2)^2/2}}{|\det B|^{1/2}} \sum_{\tilde{a} \in \mathbb{Z}^\text{Vert}_r/2B\mathbb{Z}^\text{Vert}} e^{-\pi i \tilde{a}^T B^{-1} \tilde{a} - \pi i \tilde{a}^T B^{-1} (\ell + B(\mu + (r-1)\epsilon))}.
\]

We say that $L_\omega$ is well defined if the interior sum in the right hand side of (4.4) has a finite number of non-zero terms.

**Remark 4.4.** $L_\omega$ restricted on $\mathbb{C}((q^{1/p}))[x^\pm]$ is well-defined.

**Remark 4.5.** Apart from $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z})$ (equivalently $\mu \in (\mathbb{C}/2\mathbb{Z})^\text{Vert}$ s.t. $B \mu = 0 \mod 2\mathbb{Z}^\text{Vert}$), the operations $L_\omega$ and $G_\omega$ also (implicitly) depend on $B$ and $r$.

Next we introduce the following limit operation (a morphism of $\mathbb{C}$-algebras):
Definition 4.6.  
\[
\lim_{q \to e^{2\pi i r}} : \mathbb{C}((q^{1/p})) \longrightarrow \mathbb{C},
\]
(4.6)  
\[
\sum_m A_m q^m \longmapsto \lim_{q \to e^{2\pi i r}} \sum_m A_m q^m.
\]
We say that the operation is well defined if the power series in \(q^{1/p}\) are convergent for \(0 < |q^{1/p}| < 1\) and the limit, taken to the root of unity along the radial direction, exists and is finite.

Remark 4.7. This operation can be extended to the polynomials/series with coefficients in \(\mathbb{C}((q^{1/p}))\) by applying it coefficient-wise.

Proposition 4.8. Let \(A(q, x) \in \mathbb{C}((q^{1/p}))[x]^{\pm}\) such that \(\lim_{q \to e^{2\pi i r}} A(q, x) \in \mathbb{C}[x]^{\pm}\) exists. Then  
(4.7)  
\[
\mathfrak{G}_\omega \lim_{q \to e^{2\pi i r}} A(q, x) = \lim_{q \to e^{2\pi i r}} \mathfrak{G}_\omega A(q, x) = \lim_{q \to e^{2\pi i r}} \mathcal{L}_\omega A(q, x).
\]
Proof. The first equality follows from the definition of \(\mathfrak{G}_\omega\), which involves taking a finite linear combination of the evaluations at certain values of \(x\). The second equality is shown by applying the Gauss reciprocity formula (3.11) to individual monomials in \(x\), the number of which is finite. \(\Box\)

Proposition 4.9. Let \(A(x) \in K(x)^{'}\), \(B(x^r) \in K(x)^{'} \cap K(x^r)\). Then  
(4.8)  
\[
\mathfrak{G}_\omega(B(x^r)A(x)) = B((-1)^{r-1} e^{\pi i \mu}) \mathfrak{G}_\omega(A(x)).
\]
Proof. Follows from the definition of \(\mathfrak{G}_\omega\). \(\Box\)

Remark 4.10. The statement of the Prop. 4.9 is directly extended to the series in \(K'(x)((t))\) by applying it coefficient-wise.

We also define the following limiting operation on a subalgebra of \(\mathbb{C}((q^{1/p}))(t)\):

Definition 4.11.  
\[
\lim_{t \to 1} : \mathbb{C}((q^{1/p}))(t) \longrightarrow \mathbb{C}((q^{1/p})),
\]
(4.9)  
\[
\sum_n (\sum_m A_{n,m} q^m) t^n \longmapsto \sum_m (\sum_n A_{n,m}) q^m.
\]
We say that the operation is well defined if there is a finite number of non-vanishing coefficients \(A_{n,m}\) for any fixed \(m\).

Remark 4.12. When restricted on the subspace \(C((q^{1/p}))(t)^{'} \subset C((q^{1/p}))(t)\) and is well-defined, the result of the operation in Def. 4.11 coincides with the result of the operation in Def. 4.1.
With such definitions, consider the following diagram of (partially defined) algebra homomorphisms, where the dotted arrow means that we make the hypothesis that in the cases of interest the image of \( \lim_{q \to e^{2\pi i/r}} \) map is contained in the subspace \( \mathbb{C}(t)' \):

\[
\begin{array}{ccc}
\mathbb{C}((q^{1/p}))(t) & \xrightarrow{\lim_{t \to 1}} & \mathbb{C}(t) \\
& \xrightarrow{\lim_{q \to e^{2\pi i/r}}} & \\
& \mathbb{C}((q^{1/p})) & \xrightarrow{\lim_{t \to \frac{1}{q}} \lim_{q \to e^{2\pi i/r}}} & \mathbb{C}
\end{array}
\]

(4.10)

**Proposition 4.13.** When \( r \neq 0 \mod 4 \) and

\[
\ell = 2b + B(s - \varepsilon) \mod 2B\mathbb{Z}^\text{Vert},
\]

where \( b \) and \( s \) represent elements of \( H_1(M; \mathbb{Z}) \) and \( \text{Spin}(M) \) respectively, the coefficients \( C^\omega_\ell \) in the Def. 4.3 admit the following expression:

\[
(4.12) \quad C^\omega_\ell = \frac{e^{\frac{\pi i}{4} r V/2}}{|H_1(M; \mathbb{Z})|^2} \times
\begin{cases}
\sum_{a, f \in H_1(M; \mathbb{Z})} e^{2\pi i (\frac{\sigma}{4} + \frac{1}{2} \ell k(a, a) + \frac{1}{2} \ell k(a, f) + \frac{1}{2} \mu(M, s) + \frac{1}{2})}, & r = 1 \mod 4, \\
2^{V/2}|H_1(M; \mathbb{Z})|^{1/2} \sum_{a \in H_1(M; \mathbb{Z})} e^{-\frac{\pi i}{4} q_0(a) - \frac{1}{2} \ell k(a, b) - \pi i \omega(a)} , & r = 2 \mod 4, \\
\sum_{a, f \in H_1(M; \mathbb{Z})} e^{2\pi i (\frac{\sigma}{4} - \frac{1}{2} \ell k(a, a) - \ell k(a, f) - \frac{1}{2} \omega(a) - \ell k(f, f) + \frac{1}{2} \mu(M, s) + \frac{1}{2})}, & r = 3 \mod 4.
\end{cases}
\]

**Proof.** Is contained in Section 3.1. \( \square \)

Assuming the identification \( \xi^{\alpha I} = x_I \) one can consider

\[
(4.13) \quad \xi^{-\frac{1}{2} \alpha^T B \alpha} N_r(S^3, L_\alpha) \in \mathbb{C}(x)' \subset \mathbb{C}(x).
\]

**Proposition 4.14.**

\[
(4.14) \quad \frac{1}{\Delta_{b^+} b^-} \Theta^\omega_\omega \left( \xi^{-\frac{1}{2} \alpha^T B \alpha} N_r(S^3, L_\alpha) \cdot \prod_{f \in \text{Vert}} \frac{(x_I - x_I^{-1})}{(x_I^r - x_I^{-r})} \right) = N_r(M, \omega).
\]

**Proof.** Follows from the definitions of \( \Theta^\omega_\omega \) and \( N_r(M, \omega) \). \( \square \)
Proposition 4.15.

\[
\mathcal{L}_\omega \left( F_L(x^2, q) \cdot \prod_{I \in \text{Vert}} (x_I - x_I^{-1}) \right) = \frac{e^{\frac{\pi ip}{4} q^{1/2}}}{|H_1(M; \mathbb{Z})|} \times \sum_{b \in H_1(M; \mathbb{Z})} \left\{ \begin{array}{ll}
eg e^{\frac{2\pi i}{4}(5\sigma - 2Tr B)} \sum_{a, f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r+1}{4} \ell k(a, a) + \ell k(a, f + b) - \frac{1}{2} \omega(a) - \ell k(f, f) + \frac{1}{2} \mu(M, s) + \frac{1}{2} \right)}, & r = 1 \mod 4, \\
2^{V/2}|H_1(M; \mathbb{Z})|^{1/2} \sum_{a \in H_1(M; \mathbb{Z})} e^{-\frac{\pi i}{8} q_s(a) - 2\pi i \ell k(a, b) - \pi i \omega(a)}, & r = 2 \mod 4, \\
e^{\frac{2\pi i}{4}(-3\sigma + 2Tr B)} \sum_{a, f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r+1}{4} \ell k(a, a) - \ell k(a, f + b) + \frac{1}{2} \omega(a) - \ell k(f, f) + \frac{1}{2} \mu(M, s) + \frac{1}{2} \right)}, & r = 3 \mod 4,
\end{array} \right.
\]

\[\times (-1)^{b_+} q^{\frac{Tr B - 3\sigma}{4}} \hat{Z}_{\sigma(b, s)},\]

when the left hand side is well defined.

Proof. Follows from the definition of \(\mathcal{L}_\omega\), Prop. 4.13 and the conditional definition of \(\hat{Z}_{\sigma(b, s)}\) through \(F_L\) (e.g. Eq. (2.97)). \(\Box\)

Note that \(t\)-regularization can be understood as the following \(\mathbb{C}\)-linear map:

Definition 4.16.

\[
(\cdot)^t : K[[x^\pm]] \rightarrow K[x^\pm][[t]]
\]

\[
f(x) = \sum_{\ell \in \mathbb{Z}^\text{Vert}} f_\ell x^\ell \mapsto f^t(x) = \sum_{m \geq 0} \left( \sum_{\ell \mid \ell || = m} f_\ell x^\ell \right) t^m
\]

where \(\| \cdot \|\) is the \(L^1\) norm.

The target space of this operation is an integral domain, and its ring of fractions is a subfield of \(K(x)((t))\). In particular,

\[
\frac{F^t_L(x^2, q)}{F^t_L(x^{2r}, q^r)} \in \mathbb{C}((q^\frac{1}{2}))(x)((t)).
\]

The Conjecture 4 then states that \(\lim_{q \to e^{2\pi i r}}\) takes it to an element in \(\mathbb{C}(x(t))' \subset \mathbb{C}(x)(t) \subset \mathbb{C}(x)((t))\), and \(\lim_{t \to 1}\) then takes it further to

\[
\xi^{1/2} \alpha^{TB} e^{-\frac{(r-1)^2 Tr B}{2}} N_r(S^3, L_\alpha) \in \mathbb{C}(x)' \subset \mathbb{C}(x).
\]

Theorem 4.17. Assume:

(i) \(\lim_{t \to 1} \lim_{q \to e^{2\pi i r}} = \lim_{q \to e^{2\pi i r}} \lim_{t \to 1}\) when applied to the ratio

\[
(4.19) \quad \mathcal{L}_\omega F^t_L(x^2, q) \prod_{I}(x_I - x_I^{-1}) = \left( \mathcal{L}_\omega |_{r=1} F^t_L(x^2, q) \prod_{I}(x_I - x_I^{-1}) \right) |_q \to q^r,
\]

that is the maps in the diagram (4.10) commute when restricted to this element in the top left corner.
\( (ii) \) \( \hat{Z} \) is defined via \( F_L \) (Eq. (2.97) where the right-hand side is assumed to converge in the space of series in \( q \)).

\( (iii) \) Conjecture 4 holds with the additional assumption that:

\( (iv) \) \( \exists \, \alpha(q, t) \in \mathbb{C}((q^{1/p}))((t)) \) such that \( \lim_{q \to e^{2\pi i}} \alpha(q, t)F_L^t(x^2, q) \in \mathbb{C}[x^\pm][(t)] \) exists and is non-zero.

Then

\[
N_r(M, \omega) = \lim_{q \to e^{2\pi i}} \frac{\sum_{b \in H_1(M; \mathbb{Z})} C_{\omega, b} \hat{Z}_{\sigma(b, s)}(q)}{\sum_{b \in H_1(M; \mathbb{Z})} e^{2\pi i \omega(b)} \hat{Z}_{\sigma(b, s)}(q^r)}
\]

where

\[
C_{\omega, b} := \frac{1}{|H_1(M; \mathbb{Z})|} \times \begin{cases} 
\sum_{a, f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r+1}{2} \ell k(a, a) + \ell k(a, f - b) - \frac{1}{2} \omega(a) + \ell k(f, f) - \frac{1}{2} \mu(M, s) + \frac{1}{2} \right)} & r \equiv 1 \mod 4, \\
|H_1(M; \mathbb{Z})|^{1/2} \sum_{a \in H_1(M; \mathbb{Z})} e^{-\pi i \omega(a) - 2\pi i \ell k(a, b) - \pi i \omega(a)} & r \equiv 2 \mod 4, \\
\sum_{a, f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r+1}{2} \ell k(a, a) - \ell k(a, f + b) - \frac{1}{2} \omega(a) - \ell k(f, f) + \frac{1}{2} \mu(M, s) + \frac{1}{2} \right)} & r \equiv 3 \mod 4.
\end{cases}
\]

**Proof.** The Conjecture 4 states that

\[
\xi^{\frac{1}{2} \alpha^T B a - \frac{(r-1)^2 \nu B}{2}} N_r(S^3, L_\alpha) = \lim_{t \to 1} \lim_{q \to e^{2\pi i}} \frac{F_L^t(x^2, q)}{F_L^t(x^{2r}; q^r)}.
\]

Multiplying both sides by \( \prod_I (x_I - x_I^{-1})/(x_I^r - x_I^{-r}) \) and applying \( \Phi_\omega \) we have

\[
\Phi_\omega \left( \xi^{\frac{1}{2} \alpha^T B a - \frac{(r-1)^2 \nu B}{2}} N_r(S^3, L_\alpha) \prod_I \frac{(x_I - x_I^{-1})}{(x_I^r - x_I^{-r})} \right) = \lim_{t \to 1} \Phi_\omega \lim_{q \to e^{2\pi i}} \frac{F_L^t(x^2, q)}{F_L^t(x^{2r}; q^r)} \prod_I (x_I^r - x_I^{-r})
\]

where we could bring \( \Phi_\omega \) inside the limit since its definition involves taking a finite sum of evaluations of the rational functions in \( x \) that appear in the coefficients of the series. By Proposition 4.14 the left hand side of the equation (4.23) gives the left hand side of the equation in the statement of the theorem, up to a simple factor. In the right hand
side, inside the limit \( \lim_{t \to 1} \) we have

\begin{equation}
(4.24) \quad \mathcal{G}_\omega \lim_{q \to e^{2\pi i}} \frac{F_L^t(x^2, q) \prod_I (x_I - x_I^{-1})}{F_L^t(x^{2r}, q^r) \prod_I (x_r - x_r^{-1})} =
\mathcal{G}_\omega \lim_{q \to e^{2\pi i}} \frac{F_L^t(x^2, q) \alpha(q^r, t) \prod_I (x_I - x_I^{-1})}{F_L^t(x^{2r}, q^r) \alpha(q^r, t) \prod_I (x_r - x_r^{-1})} =
\frac{\lim_{q \to e^{2\pi i}} \mathcal{G}_\omega F_L^t(x^2, q) \alpha(q^r, t) \prod_I (x_I - x_I^{-1})}{\lim_{q \to e^{2\pi i}} F_L^t(e^{2\pi i} q, t) \prod_I (e^{\pi i} - e^{-\pi i})}
\end{equation}

where in the first equality we used the assumption (iv) of the theorem. In the second equality we used the results of the Propositions 4.9 and 4.8.

Using Prop. 4.8 for \( r = 1 \) we have:

\begin{equation}
(4.25) \quad \xi^{\frac{1}{2}}_{\mu B} \lim_{q \to e^{2\pi i}} F_L^t(e^{2\pi i} q, \alpha(q^r, t) \prod_I (e^{\pi i} - e^{-\pi i})) =
\lim_{q \to e^{2\pi i}} \mathcal{G}_\omega |_{r=1} F_L^t(x^2, q) \alpha(q^r, t) \prod_I (x_I - x_I^{-1}) =
\lim_{q \to e^{2\pi i}} \mathcal{G}_\omega |_{r=1} F_L^t(x^2, q) \alpha(q^r, t) \prod_I (x_I - x_I^{-1}).
\end{equation}

It follows that the right hand side of (4.23) equals to

\begin{equation}
(4.26) \quad \xi^{\frac{1}{2}}_{\mu B} \lim_{t \to 1} \lim_{q \to e^{2\pi i}} \left( \frac{\mathcal{G}_\omega F_L^t(x^2, q) \prod_I (x_I - x_I^{-1})}{(\mathcal{G}_\omega |_{r=1} F_L^t(x^2, q) \prod_I (x_I - x_I^{-1})) |_{q \to q'}} \right) =
\xi^{\frac{1}{2}}_{\mu B} \lim_{q \to e^{2\pi i}} \lim_{t \to 1} \left( \frac{\mathcal{G}_\omega F_L^t(x^2, q) \prod_I (x_I - x_I^{-1})}{(\mathcal{G}_\omega |_{r=1} F_L^t(x^2, q) \prod_I (x_I - x_I^{-1})) |_{q \to q'}} \right) =
\xi^{\frac{1}{2}}_{\mu B} \lim_{q \to e^{2\pi i}} \left( \frac{\mathcal{G}_\omega F_L^t(x^2, q) \prod_I (x_I - x_I^{-1})}{(\mathcal{G}_\omega |_{r=1} F_L(x^2, q) \prod_I (x_I - x_I^{-1})) |_{q \to q'}} \right)
\end{equation}

where in the first equality we used the assumption (i) and in the second equality we used the fact that \( \mathcal{G}_\omega F_L(x^2; q) \prod_I (x_I - x_I^{-1}) \) is well defined (because \( \hat{Z} \) are assumed to be well-defined via \( F_L \) by assumption (ii)).
From Prop. 4.15 with \( r = 1 \) we also have

\[
L_\omega |_{r=1} F_L(x^2, q) \prod_I (x_I - x_I^{-1}) =
\]

\[
(4.27) \quad \frac{(-1)^{b_+} e^{\frac{\pi i}{2} (3\sigma - \text{Tr} B)}}{|H_1(M; \mathbb{Z})|} \sum_{a,b,f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( \ell k(a,b) - \frac{1}{2} \omega(a) + \ell k(f,f) - \frac{1}{4} \mu(M,s) + \frac{1}{2} \right)} \hat{Z}_{\sigma(b,s)} =
\]

\[
(4.28) \quad \frac{3\sigma - \text{Tr} B}{4} - \frac{(s - \varepsilon)^T B (s - \varepsilon)}{4} - b^T B^{-1} b = \frac{1}{2} + \frac{\mu(M,s)}{4} - \ell k(b,b) \mod 1.
\]

Using (4.27) in the right hand side of (4.26) and also using the Prop. 4.15 for general \( r \) we conclude that the right hand side of (4.23) gives the right hand side of the equation in the statement of the theorem, up to a simple phase factor. Taking care of the phase factors on both sides of (4.23) concludes the proof. \( \square \)

**Theorem 4.18.** Assume:

(i) \( \lim_{t \to 1} \lim_{q \to e^{2\pi i r}} F_L(x^2, q) = \lim_{q \to e^{2\pi i r}} \lim_{t \to 1} L_\omega F_L(x^2, q) \prod_I (x_I - x_I^{-1}) \)

that is the maps in the diagram (4.10) commute when restricted to this element in the top left corner.

(ii) \( \hat{Z} \) is defined via \( F_L \) (via eq. (2.97), where the right-hand side is assumed to converge in the space of series in \( q \)).

(iii) Conjectures 4 and 5 hold.

Then

\[
(4.30) \quad N_r(M, \omega) = T(M, [\omega]) \lim_{q \to e^{2\pi i r}} \sum_{b \in H_1(M; \mathbb{Z})} C_{\omega,b}^r \hat{Z}_{\sigma(b,s)}
\]

where \( C_{\omega,b}^r \) are the same as in Theorem 4.17.

**Proof.** One can follow the proof of the Theorem 4.17 since, from the Conjecture 5, the assumption (iv) of that theorem automatically holds with \( \alpha(q,t) = 1 \). Moreover, the Conjecture 5 provides a relation between \( \lim_{t \to 1} \lim_{q \to 1} F_L^t(x^2, q) \) which appears in the denominator of 4.24 and the Alexander-Conway function. Using the surgery formula for the torsion (see Appendix A) we arrive at the statement of the theorem. \( \square \)
Remark 4.19. In the case of plumbing surgeries, the assumption (ii) holds if and only if the plumbing is weakly negative definite, meaning the inverse of the linking matrix, $B^{-1}$, restricted on the vertices of degree $> 2$ is negative definite (cf. [30]).

Example 4.20. Let $M$ be a rational homology sphere obtained by surgery over a plumbing link corresponding to a “Y-shaped” plumbing graph i.e. one formed by a single trivalent vertex corresponding to an unknot with strictly negative framing, three 1-valent vertices and some 2-valent vertices. Then as shown in Appendix D the first hypothesis of Theorem 4.18 is satisfied. Hypothesis (iii) is satisfied as shown in Subsection 2.5.2 and $\hat{Z}$ is defined via $F_L$. So this provides an infinite family of examples in which Theorem 4.18 holds.

4.2. Generalization to $b_1 \geq 0$. In this section we briefly list modifications one needs to do in Section 4.1 in order to generalize the results to the case of general $b_1 \geq 0$. We will follow the conventions of Section 2.6. In particular we fix $U \in SL(V, \mathbb{Z})$ such that

$$(4.31) \quad UBU^T = \begin{pmatrix} B' & 0 \\ 0 & 0 \end{pmatrix}$$

where $\det B' \neq 0$ and use the following notations:

$$(4.32) \quad \begin{pmatrix} \ell' \\ \ell'' \end{pmatrix} := U\ell, \quad \begin{pmatrix} \mu' \\ \mu'' \end{pmatrix} := (U^T)^{-1}\mu, \quad \begin{pmatrix} s' \\ s'' \end{pmatrix} := (U^T)^{-1}s, \quad \begin{pmatrix} \varepsilon' \\ \varepsilon'' \end{pmatrix} := (U^T)^{-1}\varepsilon.$$

We redefine $p := 4|\det B'|$. Definitions 4.2, 4.1, 4.6, 4.11, 4.16 and Propositions 4.9, 4.14 do not need to be modified. Definition 4.3 is generalized to

Definition 4.21. Define the “$\omega$-twisted Laplace transform” on a subspace of $\mathbb{C}((q^{1/p})[[x^\pm]])$ by the following formula:

$$(4.33) \quad \mathcal{L}_\omega : \quad \mathbb{C}((q^{1/p})[[x^\pm]]) \longrightarrow \mathbb{C}((q^{1/p})),$$

$$\quad \sum_{\ell, m \in \mathbb{Z}} A_{\ell, m} q^{m/2} x^\ell \longrightarrow \sum_{n \in \mathbb{Z}} q^{n/2} \sum_{\ell, m \in \mathbb{Z}} A_{\ell, m} \cdot \mathcal{C}_\ell,$$

where

$$(4.34) \quad \mathcal{C}_\ell = \frac{e^{\pi i \nu/(2r)} r^{b_1} b_1}{|\det B'|^{1/2}} e^{\pi i (\ell' r + \nu')} \delta_{\ell'=0} \mod r \times \sum_{\tilde{a} \in \mathbb{Z}^{V-b_1}/2B'\mathbb{Z}^{V-b_1}} e^{\pi i (-\frac{r}{2} \tilde{a}^T (B')^{-1} \tilde{a} - \tilde{a}^T (B')^{-1} (\ell' + B' r + (r-1)\varepsilon'))}.$$

We say that $\mathcal{L}_\omega$ is well defined if the interior sum in the right hand side of (4.4) has finite number of non-zero terms.

With such modified definitions Proposition 4.8, then still holds by the similar argument. Propositions 4.13 and 4.15 are modified respectively to the following two:
Proposition 4.22. When \( r \neq 0 \mod 4 \) and
\[
\ell' = 2b' + B'(s' - \varepsilon') \mod 2B'\mathbb{Z}^{V-b_1},
\]
\[
\ell'' = \text{LCM}(r, 2) m \in \mathbb{Z}^{b_1},
\]
where \( b' \) and \( s' \) represent elements of \( \text{Tor} \, H_1(M; \mathbb{Z}) \) and \( \text{Spin}(M) \) respectively, the coefficients \( C^\omega_\ell \) in the Def. 4.21 admit the following expression:

\[
C^\omega_\ell = \frac{e^{\frac{x_\sigma}{4} r^{\frac{V-b_1}{2}}}}{|\text{Tor} \, H_1(M; \mathbb{Z})|} \frac{e^{2\pi \omega''(m)}}{\text{GCD}(r, 2)} \times \left\{ \begin{array}{ll}
e^{-\frac{\pi i}{2}(5\sigma -2\text{Tr} \, B)} \sum_{a, f \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{2\pi i(-\frac{r-1}{4}\ell(a,a) + \ell(k(a,f-b)) + \frac{1}{2}\omega(a) + \ell(k(f,f) - \frac{1}{2}\mu(M,s) + \frac{1}{2})}, & r = 1 \mod 4, \\
2^{\frac{V-b_1}{2}} |\text{Tor} \, H_1(M; \mathbb{Z})|^{1/2} \sum_{a \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{-\frac{\pi i}{2} q_s(a) - \pi i k(a,b) - \pi i \omega(a)}, & r = 2 \mod 4, \\
e^{\frac{\pi i}{4}(-5\sigma +2\text{Tr} \, B)} \sum_{a, f \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{2\pi i(-\frac{r-1}{4}\ell(a,a) - \ell(k(a,f+b)) + \frac{1}{2}\omega(a) - \ell(k(f,f) + \frac{1}{2}\mu(M,s) + \frac{1}{2})}, & r = 3 \mod 4. \\
\end{array} \right.
\]

Proof. Is contained in Section 3.1. \( \square \)

Proposition 4.23.

\[
L_\omega \left( F_L(x^2, q) \cdot \prod_{l \in \text{Vert}} (x_I - x_I^{-1}) \right) = \frac{e^{\frac{x_\sigma}{4} r^{\frac{V-b_1}{2}}}}{|\text{Tor} \, H_1(M; \mathbb{Z})|} \times \sum_{b \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{2\pi \omega''(m)} \times \left\{ \begin{array}{ll}
e^{-\frac{\pi i}{2}(5\sigma -2\text{Tr} \, B)} \sum_{a, f \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{2\pi i(-\frac{r-1}{4}\ell(a,a) + \ell(k(a,f-b)) + \frac{1}{2}\omega(a) + \ell(k(f,f) - \frac{1}{2}\mu(M,s) + \frac{1}{2})}, & r = 1 \mod 4, \\
2^{\frac{V-b_1}{2}} |\text{Tor} \, H_1(M; \mathbb{Z})|^{1/2} \sum_{a \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{-\frac{\pi i}{2} q_s(a) - \pi i k(a,b) - \pi i \omega(a)}, & r = 2 \mod 4, \\
e^{\frac{\pi i}{4}(-5\sigma +2\text{Tr} \, B)} \sum_{a, f \in \text{Tor} \, H_1(M; \mathbb{Z})} e^{2\pi i(-\frac{r-1}{4}\ell(a,a) - \ell(k(a,f+b)) + \frac{1}{2}\omega(a) - \ell(k(f,f) + \frac{1}{2}\mu(M,s) + \frac{1}{2})}, & r = 3 \mod 4, \\
\end{array} \right.
\]

\( \times (-1)^{b_s} q^{\frac{2\text{Tr} \, B - \ell}{4} - \frac{i}{4}} \hat{Z}(b; \eta_{\text{GCD}(r,2),s}) \)

when the left hand side is well defined.

Proof. Follows from the definition of \( L_\omega \), Prop. 4.22 and the conditional definition of \( \hat{Z}(b; a) \) through \( F_L \) (Eq. (2.97)). \( \square \)

Remark 4.24. In Proposition 4.23 and the theorems below, \( \hat{Z}(b; a) \) is understood as a particular representative in \( 2^{-c} q^{A} \mathbb{Z}[q] \subset \mathbb{C}(q^{\frac{1}{2}}) \), rather than an element of the quotient over the subspace \( 1 - q^{\text{LCM}(2, \text{GCD}(b'))} \mathbb{C}(q^{\frac{1}{2}}) \). This representative is fixed by the choice of the surgery link \( L \) in the definition of \( \hat{Z} \) via \( F_L \) by the formula (2.97).

The Theorems 4.17 and 4.18 are then modified respectively to the following two, with proofs following similar arguments:
Theorem 4.25. Assume:

(i) \( \lim_{t \to 1} \lim_{q \to e^{2\pi i}} = \lim_{q \to e^{2\pi i}} \lim_{t \to 1} \) when applied to the ratio

\[
(4.38) \quad \frac{\mathcal{L}_{\omega} F_{L}^{t}(x^2, q) \prod_{I}(x_{I} - x_{I}^{-1})}{(\mathcal{L}_{\omega}|_{r=1} F_{L}^{t}(x^2, q) \prod_{I}(x_{I} - x_{I}^{-1})) |_{q \to q^{r}}},
\]

that is the maps in the diagram (4.10) commute when restricted to this element in the top left corner.

(ii) \( \hat{Z} \) is defined via \( F_{L} \) (Eq. (2.97)).

(iii) Conjecture 4 holds with the additional assumption that:

(iv) \( \exists \alpha(q, t) \in \mathbb{C}((q^{1/p})((t))) \) such that \( \lim_{q \to e^{2\pi i}} \alpha(q, t) F_{L}^{t}(x^2, q) \in \mathbb{C}[x^{\pm}]((t)) \) exists.

Then

\[
(4.39) \quad N_{r}(M, \omega) = \lim_{q \to e^{2\pi i}} \frac{\sum_{b \in \text{Tor} H_{1}(M; \mathbb{Z})} C_{\omega, b, m}^{r} \hat{Z}_{\sigma(b \oplus \frac{mr}{\text{gcd}(r, 4)}, s)}}{\sum_{b \in \text{Tor} H_{1}(M; \mathbb{Z})} e^{2\pi i (b, q^{r})} \hat{Z}_{\sigma(b, s)}(q^{r})},
\]

where

\[
(4.40) \quad C_{\omega, b, m}^{r} := \frac{p_{b_{1}}}{|\text{Tor} H_{1}(M; \mathbb{Z})|} \times \begin{cases} \sum_{a, f \in \text{Tor} H_{1}(M; \mathbb{Z})} e^{2\pi i \left( -\frac{\pi}{4} \ell(k(a, a) + \ell(k(a, f-b)) - \frac{1}{2}(a) + \ell(k(f, f) - \frac{1}{2}(M, s)) + \frac{1}{2} \right)}, & r = 1 \mod 4, \\ \sum_{a \in \text{Tor} H_{1}(M; \mathbb{Z})} e^{-\frac{\pi}{4} \ell k(a, b) - \frac{\pi}{4} \omega(a)}, & r = 2 \mod 4, \\ \sum_{a, f \in \text{Tor} H_{1}(M; \mathbb{Z})} e^{2\pi i \left( -\frac{\pi}{4} \ell(k(a, a) - \ell(k(a, f+b)) - \frac{1}{2}(a) + \ell(k(f, f) + \frac{1}{2}(M, s)) + \frac{1}{2} \right)}, & r = 3 \mod 4. \end{cases}
\]

Theorem 4.26. Assume:

(i) \( \lim_{t \to 1} \lim_{q \to e^{2\pi i}} = \lim_{q \to e^{2\pi i}} \lim_{t \to 1} \) when applied to

\[
(4.41) \quad \mathcal{L}_{\omega} F_{L}^{t}(x^2, q) \prod_{I}(x_{I} - x_{I}^{-1})
\]

that is the maps in the diagram (4.10) commute when restricted to this element in the top left corner.

(ii) \( \hat{Z} \) is defined via \( F_{L} \) (Eq. (2.97)).

(iii) Conjectures 4 and 5 hold.

Then

\[
(4.42) \quad N_{r}(M, \omega) = T(M, [\omega]) \lim_{q \to e^{2\pi i}} \sum_{b \in \text{Tor} H_{1}(M; \mathbb{Z})} C_{\omega, b, m}^{r} \hat{Z}_{\sigma(b \oplus \frac{mr}{\text{gcd}(r, 4)}, s)}
\]
where $C_{\omega,b,m}^r$ are the same as in Theorem 4.25.

**Example 4.27.** Let $M$ be obtained by surgery over a plumbing link corresponding to a "Y-shaped" plumbing graph i.e. one formed by a single trivalent vertex corresponding to an unknot with strictly negative framing, three 1-valent vertices and some 2-valent vertices. Then, as shown in Appendix D, under a certain open condition on the linking matrix, the first hypothesis of Theorem 4.26 is satisfied. Hypothesis (iii) is satisfied as shown in Subsection 2.5.2 and $\hat{Z}$ is defined via $F_L$. So this provides an infinite family of examples in which Theorem 4.26 holds.

### 4.3. Generalization to the spin case.

In this section we briefly mention the relation between $\hat{Z}$ and the spin version of the $N_r$ invariant for $r = 0 \mod 4$ which was defined in [6]. We will omit any details and intermediate calculations, as they are analogous to the non-spin case. The spin version of the invariant depends on the choice of $\mathbb{C}/2\mathbb{Z}$-spin structure, which can be understood as a homotopy class of lifts

\[
\begin{array}{ccc}
B\text{Spin}(3,\mathbb{C}/2\mathbb{Z}) & \rightarrow & B\text{SO}(3) \\
\downarrow & & \\
M & \rightarrow & B\text{SO}(3)
\end{array}
\]

where $\text{Spin}(3,\mathbb{C}/2\mathbb{Z})$ is an extension of $\text{SO}(3)$ by $\mathbb{C}/2\mathbb{Z}$ equipped with discrete topology:

\[
(\mathbb{C}/2\mathbb{Z})_{\text{discrete}} \rightarrow \text{Spin}(3,\mathbb{C}/2\mathbb{Z}) \equiv \text{Spin}(3) \times_{\mathbb{Z}_2} (\mathbb{C}/2\mathbb{Z})_{\text{discrete}} \rightarrow \text{SO}(3),
\]

the vertical map in (4.43) corresponds to the projection, and the horizontal map is the classifying map of the bundle of orthonormal frames in the tangent bundle, which are non-canonically parametrized by $H^1(M;\mathbb{C}/2\mathbb{Z})$. To formulate the relation, it will be useful to consider the canonical map

\[
\tilde{\sigma} : H^1(M;\mathbb{C}/2\mathbb{Z}) \times \text{Spin}(M) \rightarrow \text{Spin}(M,\mathbb{C}/2\mathbb{Z})
\]

similar to the map $\sigma$. It is induced by the $\mathbb{Z}_2$ quotient map of the product $\text{Spin}(3) \times (\mathbb{C}/2\mathbb{Z})_{\text{discrete}}$, taking into account that $B(\mathbb{C}/2\mathbb{Z})_{\text{discrete}} = K(\mathbb{C}/2\mathbb{Z},1)$. Then, for general $b_1$ the relation for $r = 0 \mod 4$ reads

\[
N_r^{\text{spin}}(M,\tilde{\sigma}(\omega,s)) = \frac{(-1)^\mu(M,s)}{\sqrt{\text{Tor} H_1(M;\mathbb{Z})}} \times \lim_{q \to e^{2\pi i}} \sum_{a,b \in \text{Tor} H_1(M;\mathbb{Z})} e^{-\frac{\pi i}{2} \ell k(a,a) - \pi i \ell k(a,b) - \pi i \omega(a) + \pi i \omega'(m)} \hat{Z}_{\sigma(b \oplus \frac{m}{2},s)}(M)
\]

where $\mu(M, s)$ is the Rokhlin invariant, for which we used the surgery formula (2.17). Note that modulo two it is actually independent of spin structure $s$. For rational
homology spheres the relation simplifies to

\[ N^{\text{Spin}}_r(M, \tilde{\sigma}(\omega, s)) = \frac{(-1)^{\mu(M,s)}}{\sqrt{|H_1(M;\mathbb{Z})|}} \times \lim_{q \to e^{2\pi i/r}} \sum_{a,b \in H_1(M;\mathbb{Z})} e^{-\frac{\pi i}{2} \ell k(a,a) - 2\pi i \ell k(a,b) - \pi i \omega(a)} \tilde{Z}_{\sigma(b,s)}(M). \]

5. \( \tilde{Z} \) and \( N_r \) as decorated TQFTs

Both \( \tilde{Z} \) and \( N_r \) are examples of topological invariants that, on the one hand, behave well under cutting and gluing (admit surgery formulae) and, on the other hand, depend on additional data (decoration). In this section we describe how this additional data behaves under cutting and gluing as well as its transformation under the operation of taking the limit \( q \to e^{2\pi i/r} \) that relates \( \tilde{Z} \) and \( N_r \) invariants.

5.1. Hilbert space on a torus. In Table (1.1) we omitted the comparison of Hilbert spaces \( \mathcal{H}_{\text{BCGP}}(T^2) \) and \( \mathcal{H}_{\text{GPPV}}(T^2) \) in the two theories on a 2-torus.\(^9\) These spaces control surgery operations and deserve a separate section. Note, in a semisimple TQFT with a finite-dimensional Hilbert space, we have

\[ \mathcal{H}(T^2) = K^0(\text{MTC}(S^1)). \]

Neither of the two theories we are trying to compare fits into this standard paradigm of the Reshetikhin-Turaev construction, which nevertheless can be used as a good motivation. In particular, it is important that both BCGP and GPPV theories are “decorated” TQFTs, with additional structure \( \omega \) or \( b \in \text{Spin}^c(M)/\mathbb{Z}_2 \) originating from the equivariance under \( T_C \subset G_C \), the maximal torus of \( G_C = SL(2,\mathbb{C}) \).

Taking into account these extra structures, the first order approximation to \( \mathcal{H}(T^2) \) in our theories is as follows. From cutting and gluing (surgery) rules, we infer that the space \( \mathcal{H}_{\text{GPPV}}(T^2) \) has basis \(|n,x\rangle_a \) where \( n \in \mathbb{Z}, x \in \mathbb{C}^* \) and \( a \) is a relative spin\(^c \) structure. It is often convenient to replace \( x \) by a dual variable \( m \in \mathbb{Z} \), so that the Weyl group of \( G_C = SL(2,\mathbb{C}) \) maps \(|n,m\rangle_a \mapsto |-n,-m\rangle_a \). Therefore, for each given spin\(^c \) structure (= choice of background), we have

\[ \mathcal{H}_{\text{GPPV}}^{(a)}(T^2) \cong \mathbb{C} \left[ \frac{\Lambda \times \Lambda^\vee}{W} \right] \]

where, as usual, by \( \mathbb{C}[S] \) we denote the space of complex valued functions\(^10\) on set \( S \). Note that, compared to the Hilbert space \( \mathcal{H}_{\text{WRT}}(T^2) = \mathbb{C} [\Lambda/(W \times k\Lambda)] \), the space of states (5.2) has two copies of the lattice, i.e. corresponds to a toroidal algebra, and has no cut-off due to the level \([30, 14]\). The latter property is, of course, anticipated \textit{a priori}

\(^9\)BCGP stands for the TQFT build out of the CGP invariants by Blanchet, Costantino, Geer and Patureau [7].

\(^{10}\)In principle one has to impose a certain asymptotic behavior condition on the functions on the lattice. This condition should be coherent with a certain continuity condition on the allowed functions on the dual space, related by Fourier transform. In this work we do not specify such conditions.
since the invariants $\hat{Z}_b(M, q)$ depend on $q$, generic with $|q| < 1$. From the Kazhdan-Lusztig correspondence and the theory of $W$-algebras, it is also natural to attribute this “doubling” to a larger symmetry associated with the action of two (quantum) groups, such that (5.2) is basically a product of root lattices for $G$ and its Langlands dual $^L G$.

At a similar level of approximation, for a fixed choice of the “decoration” / equivariant parameter $\alpha$, we have

\begin{equation}
H^{(\alpha)}_{\text{BCGP}}(T^2) \cong \mathbb{C}[H_r], \quad H_r := \{1 - r, 3 - r, \ldots, r - 1\},
\end{equation}

which is a direct consequence of the corresponding statement for categories $\mathcal{C} = \bigoplus_{\pi, \tau} \mathcal{C}_{\pi, \tau}$ [7]. It is very well known that many affine algebras and VOAs at “critical level” have large center. The same is true about quantum groups at roots of unity. Very often, the center has a nice geometric meaning as the space of functions on some variety (moduli space) $M$, and in the case at hand $M = T_C$. Below we refine these descriptions of $H_{\text{BCGP}}(T^2)$ and $H_{\text{GPPV}}(T^2)$ by looking at these spaces from various angles.

Before continuing with $\Sigma = T^2$, it is instructive to pause for a moment and consider simpler cases of $\Sigma = S^2$ or $S^2$ with two points removed. In a 3d TQFT associated to $\mathcal{C}$, objects in $\mathcal{C}$ correspond to line operators whereas their non-trivial extensions are represented by non-trivial junctions, i.e. local operators where two line operators meet, illustrated in Figure 3. The space of such local operators is the space of states on $\Sigma = S^2 \setminus \{p_1, p_2\}$: in particular, it is non-trivial for BCGP theory indicating the non-semisimple nature of the TQFT. As a special case, the space of local operators at the junction of two trivial lines is simply $H(S^2)$, and for GPPV theory its structure is conveniently encoded in the unreduced version of the $\hat{Z}$-invariant for $S^1 \times S^2$:

\begin{equation}
\frac{1}{2} \qbinom{(x; q)_{\infty}}{(qt; q)_{\infty}} \frac{1}{(q; q)_{\infty}} \frac{1}{(x^{-1}; q)_{\infty}} \frac{1}{(qt^{-1}; q)_{\infty}}
\end{equation}

where the refinement variable $t$ corresponds to homological-type grading in the context of categorification. In general, in a Rozansky-Witten theory the space $H(S^2) = \mathbb{C}[X]$ encodes the geometry of the target space $X$, and the partition function on $S^1 \times S^2$ can be identified with the Hilbert series of $X$. In the case of $\hat{Z}$ theory, there is a manifest symmetry between the numerator and the denominator of the above expression. It is typical for a partition function of the cotangent bundles, $X = T^* M$, where the numerator and denominator correspond to the fiber and base, respectively. Indeed, “half” of the expression (5.4) is precisely the Hilbert series of the affine Grassmannian, whereas the complete expression (5.4) is the Hilbert series of a model for $T^* \text{Gr}_{G}$ that was used in [29].

\[11\] This is the version that includes the contribution of the Cartan component of the adjoint chiral, e.g. $\hat{Z}^{(\text{unred})}(S^3) = \frac{1}{(q^2; q^2)_{\infty}}$ for $G = SU(2)$. The reduced version is obtained by removing this contribution, i.e. via multiplying by $(qt; q)_{\infty}$. Moreover, there is also a factor of $(1; q)_{\infty}$ in the numerator which requires regularization. Sometimes it is simply removed. And, sometimes only the zero mode is removed so that $(1; q)_{\infty}$ is replaced by $(q; q)_{\infty}$, as in (5.4).
Recall, that the affine Grassmannian for $G = SU(2)$ has two connected components, with a similar Morse cell complex. In each component, the transverse slices are labeled by a pair of non-negative integers, $m$ and $n$. In particular, the transverse slices with $n = 0$ form a family of hyper-Kähler manifolds with quaternionic dimension $m^2$ that provide a finite-dimensional approximation to $Gr_G$. Their Hilbert series is

$$
(q^{m+1};q)_m/(qx;q)_m(q;q)_m(qx^{-1};q)_m.
$$

In the limit $m \to \infty$ we recover $Gr_G$ itself (or, more precisely, one of its connected components) and the denominator of the formula (5.4) at $t = 1$.

Now, returning to $\Sigma = T^2$, let $D^bCoh(X)$ be the (bounded) derived category of coherent sheaves on $X$. Then,

$$
\mathcal{H}_{GPPV}(T^2) \cong K^0(D^b_C(T^*Gr_G))
$$

where $Gr_G$ is the affine Grassmannian of $G$ or, rather, the affine Grassmannian of $G_C$ to be more precise. Again, one needs to use a suitable model for its cotangent bundle, which is a singular infinite-dimensional space, and the “ravioli” space of [9, 10, 45] provides the right candidate. Let $St_G$ be the affine Grassmannian analogue of the Steinberg variety. Then,

$$
K^{G(O)}(St_G) \cong \mathbb{C}[\mathbb{T}_C \times L\mathbb{T}_C]^W
$$

and

$$
\text{Spec}K(D^bCoh_{St_G}^{G(O)}(T^*Gr_G)) \cong \text{Spec}K^{G(O)}(Gr_G) \cong \frac{\mathbb{T}_C \times L\mathbb{T}_C}{W}
$$

where $O = \mathbb{C}[[t]]$, $F = \mathbb{C}((t))$, and $Gr_G = G(F)/G(O)$. Enhancing the equivariant $G(O)$-action to $G(O) \times U(1)$ action, where $U(1)$ acts by loop rotation, corresponds to a non-commutative deformation (quantization) of this space. Up to a two-fold cover, this is precisely the quantization of the space $\mathcal{M}_{\text{flat}}(G_C, T^2) \cong \frac{\mathbb{T}_C \times L\mathbb{T}_C}{W}$.

5.1.1. Quantization. The Hilbert space of Chern-Simons theory and its close cousins on a surface $\Sigma$ can be obtained by quantizing a suitable “phase space” $M$, that in many interesting examples can be realized as a submanifold in the moduli space $\mathcal{M}_H(G, \Sigma) \cong \mathcal{M}_{\text{flat}}(G_C, \Sigma)$. For example, a “real slice” $M = \mathcal{M}_{\text{flat}}(G_\mathbb{R}, \Sigma)$ that corresponds to a
real form $G_\mathbb{R}$ of $G_\mathbb{C}$ gives the Hilbert space of “$G_\mathbb{R}$ Chern-Simons theory,” whereas $M = \mathcal{M}_H(G, \Sigma)$ gives the Hilbert space of $\hat{Z}$ theory (which provides a non-perturbative definition to what one might call a “$G_\mathbb{C}$ Chern-Simons theory”). In all of these cases, we can represent $\mathcal{H}(\Sigma)$ as a Hom-space (= space of open strings) in the category of branes on $\mathcal{M}_H(G, \Sigma)$:

\begin{equation}
\mathcal{H}(\Sigma) = \text{Hom}(\mathcal{B}', \mathcal{B}_{cc})
\end{equation}

where $\mathcal{B}_{cc}$ is a rank-1 brane that carries a line bundle of curvature $c_1(\mathcal{L}) = \omega$ and $\mathcal{B}'$ is supported on $M$.

In the case of BCGP invariants, we have

\begin{equation}
\dim \mathcal{H}_{BCGP}(\Sigma) = \begin{cases} r', & \text{if } g = 1, \\ r^{3g-3}, & \text{if } g > 1 \text{ and } r \text{ is odd}, \\ \frac{r^{3g-3}}{2^{2g-1}}, & \text{if } g > 1 \text{ and } r \text{ is even}. \end{cases}
\end{equation}

Recall that $r'$ is $r$ (resp. $\frac{r}{2}$) when $r$ is odd (resp. even). The choice of a background (“decoration”) that appears in $N_r$ and $\hat{Z}$ invariants is of the same type as in $\mathbb{T}_C$-crossed modules, thus allowing to describe both TQFTs also in the language of $\mathbb{T}_C$-crossed modules.

### 5.1.2. A prototypical example.

Consider bosonic Chern-Simons theory with gauge group $U(1)$ and level $r \in \mathbb{Z}_+$. It has a topological $\mathbb{C}/\mathbb{Z}$ (0-form) global symmetry. On the level of the path integral, on a closed 3-manifold $M$, it can be coupled to a flat\footnote{Restriction of connections being flat can be interpreted as considering discrete topology on $\mathbb{C}/\mathbb{Z}$ symmetry group.} background $\mathbb{C}/\mathbb{Z}$ connection

\begin{equation}
\omega \in H^1(M; \mathbb{C}/\mathbb{Z}) \cong \text{Hom}(H_1(M; \mathbb{Z}), \mathbb{C}/\mathbb{Z}) \cong \text{Hom}(H^2(M; \mathbb{Z}), \mathbb{C}/\mathbb{Z})
\end{equation}

as follows:

\begin{equation}
Z_{U(1)}(M, \omega) = \int DA \exp \left\{ \frac{i r}{2\pi} \int_M \text{AdA} + 2\pi i \omega(c_1) \right\}
\end{equation}

where $c_1 \in H^2(M; \mathbb{Z})$ is the first Chern class of the $U(1)$ gauge connection (locally represented by 1-form $A$). On the level of charge/charged operators this 0-form global symmetry can be understood locally as follows (i.e. as in the general setting of [27]). \textit{Charged} operators are 0-dimensional. An operator with charge $m \in \mathbb{Z}$ can be understood as the puncture in the spacetime with magnetic flux $m = (2\pi)^{-1} \int_{S^2} F$ over a small 2-sphere $S^2$ surrounding the puncture (i.e. monopole). A \textit{charge} operator $O_g(\Sigma)$ corresponding to a group element $g \in \mathbb{C}/\mathbb{Z}$ and supported on a 2-dimensional surface $\Sigma$ is simply

\begin{equation}
O_g(\Sigma) = \exp(i g \int_{\Sigma} F).
\end{equation}
Globally, turning on a non-trivial \( \omega \) in (5.12) can be realized by the insertion of the charge operator supported on a 2-chain representing the Poincaré dual of \( \omega \), with charges on simplices given by the corresponding coefficients in \( \mathbb{C}/\mathbb{Z} \). The \( U(1) \) Chern-Simons TQFT, “enriched” by this \( \mathbb{C}/\mathbb{Z} \) global symmetry, then can be described in terms of a \( G \)-crossed MTC \( \mathcal{C} \), for \( G = \mathbb{C}/\mathbb{Z} \), using the general formalism of [2] (cf. also [4]) as follows. In the decomposition

\[
\mathcal{C} = \bigoplus_{g \in \mathbb{C}/\mathbb{Z}} \mathcal{C}_g
\]

the component \( \mathcal{C}_g \) has simple objects that correspond to the line operators

\[
W_e(\gamma) = \exp(ie \int_\gamma A)
\]

with a complex charge \( e \) with the fixed value \( e \mod 1 = g \in \mathbb{C}/\mathbb{Z} \). For any fixed \( g \) there are exactly \( r \) distinct simple objects, as \( W_r(\gamma) \) is known to be a trivial operator on the quantum level. That is, an equivalence class of the operator is determined by the value \( e \mod r \in \mathbb{C}/r\mathbb{Z} \). The fusion of the line operators is obviously consistent with the grading on the category.

Note that for \( g \neq 0 \), such line operators are not the ordinary line operators. They are not well defined by themselves, but are only allowed to live on a boundary of a surface \( \Sigma \) (i.e. locally \( \gamma = \partial \Sigma \)) where a charge operator \( O_g(\Sigma) \) is supported.

5.2. Decorated TQFTs, gradings, and Fourier transform. Here we consider some basic operations on decorated TQFTs. We propose that BCGP and GPPV TQFTs are related by the combination of such operations (with slight modification related to the simplifications we will impose below). Namely (for \( r = 2 \mod 4 \)), BCGP TQFT can be obtained by applying “Fourier transform” followed by “\( r \)-wrapping” to GPPV TQFT. Note that in this relation, at all intermediate steps \( r \) and \( q \) should be considered as independent parameters. Only at the final step one has to take the radial limit \( q \to e^{2\pi ir} \) (which, may lead to some divergences in certain cases). For the purpose of a more transparent exposition, instead of spin\(^c\)-TQFTs we consider \( H^2(\cdot; \mathbb{Z}) \)-decorated spin-TQFTs, and instead of \( H^1(\cdot; \mathbb{C}/2\mathbb{Z}) \)-decorated TQFTs we consider \( H^1(\cdot; \mathbb{R}/\mathbb{Z}) \)-decorated TQFTs.

5.2.1. Decorated TQFTs and grading of Hilbert spaces. “Decorated” TQFTs (i.e. TQFTs defined on bordisms with additional structure) in general have induced grading on the vector spaces \( V(\Sigma) \) associated to codimension-1 closed manifolds \( \Sigma \) (i.e. the objects of the bordism category). The general rule is that the choice of the decoration on \( \Sigma \times S^1 \) decomposes into a choice of decoration on \( \Sigma \) and a choice of the parameter dual to the grading on \( V(\Sigma) \). In particular if the decoration on \( \Sigma \times S^1 \) is just a pullback of the decoration on \( \Sigma \) (with respect to the projection \( \Sigma \times S^1 \to \Sigma \)), then \( Z(\Sigma \times S^1) = \dim V(\Sigma) \), where the right hand side is the total dimension (over all gradings).
3d $H^1(\cdot; \mathbb{R}/\mathbb{Z})$-decorated TQFTs. The choice of the decoration on $\Sigma \times S^1$ is an element of

\begin{equation}
H^1(\Sigma \times S^1; \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(H_1(\Sigma; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \oplus \text{Hom}(H_0(\Sigma; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong H^1(\Sigma; \mathbb{R}/\mathbb{Z}) \oplus \text{Hom}(H_0(\Sigma; \mathbb{Z}), \mathbb{R}/\mathbb{Z}).
\end{equation}

So $V(\Sigma)$ is naturally graded by $H_0(\Sigma) \cong H^2(\Sigma)$. The graded dimensions are given by the following relation:

\begin{equation}
\sum_{\omega \in H_0(\Sigma; \mathbb{Z})} \dim V_\omega(\Sigma, \omega) e^{2\pi i \alpha(n)} = Z(\Sigma \times S^1, \omega \oplus \alpha)
\end{equation}

where $\omega \in H^1(\Sigma)$ and $\alpha \in \text{Hom}(H_0(\Sigma; \mathbb{Z}), \mathbb{R}/\mathbb{Z})$.

Physically, $\mathbb{R}/\mathbb{Z}$ is a 0-form symmetry. There are 0-dimensional charged operators with charges in $\mathbb{Z}$ and 2-dimensional charge operators labelled by $\mathbb{R}/\mathbb{Z}$, as in the example in Section 5.1.2. Turning on $\alpha \in \text{Hom}(H_0(\Sigma; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong H_2(\Sigma; \mathbb{R}/\mathbb{Z})$ above corresponds to insertion of a charge operator along the spatial slice $\Sigma$.

3d $H^2(\cdot; \mathbb{Z})$-decorated TQFTs. The choice of the decoration on $\Sigma \times S^1$ is an element of

\begin{equation}
H^2(\Sigma \times S^1; \mathbb{Z}) \cong H^2(\Sigma; \mathbb{Z}) \oplus H^1(\Sigma; \mathbb{Z}) \cong H^2(\Sigma; \mathbb{Z}) \oplus H_1(\Sigma; \mathbb{Z}).
\end{equation}

So $V(\Sigma)$ is naturally graded by $\text{Hom}(H_1(\Sigma; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \cong H^1(\Sigma; \mathbb{R}/\mathbb{Z})$. The graded dimensions are given by the following relation:

\begin{equation}
\sum_{\omega \in H_1(\Sigma; \mathbb{R}/\mathbb{Z})} \dim V_\omega(\Sigma, \omega) e^{2\pi i \omega(\gamma)} = Z(\Sigma \times S^1, n + \gamma)
\end{equation}

where $n \in H^2(\Sigma; \mathbb{Z})$ and $\gamma \in H_1(\Sigma; \mathbb{Z})$. Since $H^1(\Sigma; \mathbb{R}/\mathbb{Z})$ is in general not discrete, one has to specify what is meant by $\sum_\omega$. Consider the case of connected $\Sigma$ (the generalization to the disconnected case is straightforward). Let $g$ be the genus of $\Sigma$. Then $H^1(\Sigma; \mathbb{R}/\mathbb{Z}) \cong (\mathbb{R}/\mathbb{Z})^{2g} = (S^1)^{2g}$, but non-canonically. There is a unique homogeneous form $\mu(\omega) \in \Omega^{2g}(H^1(\Sigma; \mathbb{R}/\mathbb{Z}))$ normalized such that $\int \mu(\omega) = 1$. Then

\begin{equation}
\sum_{\omega \in H^1(\Sigma; \mathbb{R}/\mathbb{Z})} \ldots := \int_{H^1(\Sigma; \mathbb{R}/\mathbb{Z})} \mu(\omega) \ldots
\end{equation}

It also satisfies

\begin{equation}
\int_{H^1(\Sigma; \mathbb{R}/\mathbb{Z})} \mu(\omega) e^{2\pi i \omega(\gamma)} = \delta_\gamma := \begin{cases} 1, & \gamma = 0, \\ 0, & \gamma \neq 0, \end{cases}
\end{equation}

where $\gamma \in H_1(\Sigma; \mathbb{Z})$.

Physically, $\mathbb{Z}$ is a 1-form symmetry. There are 1-dimensional charged operators with charges in $\mathbb{R}/\mathbb{Z}$ and 1-dimensional charge operators labelled by $\mathbb{Z}$. Turning on $\gamma \in H_1(\Sigma; \mathbb{Z})$ above corresponds to insertion of a charge operator supported on a 1-dimensional curve in the spatial slice $\Sigma$. 
5.2.2. **Fourier transform of TQFTs.** Let \( Z \) and \( Z' \) denote respectively \( H^2(\cdot; \mathbb{Z}) \) and \( H^1(\cdot; \mathbb{R}/\mathbb{Z}) \) decorated TQFTs for which the partition functions on a closed 3-manifold \( Y \) are related by a Fourier transform:

\[
Z'(Y, \omega) = \sum_{b \in H^2(Y; \mathbb{Z})} e^{2\pi i \int_Y \omega \cup b} Z(Y, b)
\]

where \( \omega \in H^1(Y; \mathbb{C}/\mathbb{Z}) \). Its inverse is

\[
Z(Y, b) = \frac{1}{|\text{Tor} H_1(Y; \mathbb{Z})|} \int_{H_1(Y; \mathbb{R}/\mathbb{Z})} \mu(\omega) e^{2\pi i \int_Y \omega \cup b'} Z'(Y, \omega)
\]

where \( \mu(\omega) \) is the homogeneous top degree form on \( H^1(Y; \mathbb{R}/\mathbb{Z}) \cong \text{Tor} H_1(Y; \mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})^{\text{ht}} \) uniquely fixed by the condition that \( \int_M \mu(\omega) = 1 \) for each connected component \( M \subset H^1(Y; \mathbb{R}/\mathbb{Z}) \). The relation between the values of TQFT on a closed 2-manifold \( \Sigma \) is given by the swap of grading with decoration:

\[
V'_n(\Sigma, \omega) = V_\omega(\Sigma, n)
\]

where \( n \in H^2(\Sigma; \mathbb{Z}) \cong H_0(\Sigma; \mathbb{Z}) \) and \( \omega \in H^1(\Sigma; \mathbb{R}/\mathbb{Z}) \). It is easy to see that (5.24) is consistent with (5.17), (5.19) combined with (5.22), (5.23).

In order to extend the relation between the TQFTs to cobordisms let us define the category of \( H^2 \)-decorated cobordisms as follows:

- The objects are pairs \((\Sigma, b)\) with \( b \in H^2(\Sigma, \Sigma \setminus \{\ast\}; \mathbb{Z})\) where \( \{\ast\} \) is the datum of one base point per connected component of \( \Sigma \).
- A morphism from \((\Sigma_-, b_-)\) to \((\Sigma_+, b_+)\) is a pair \((M, b)\) with \( b \in H^2(M, \partial M \setminus \{\ast\}; \mathbb{Z})\) where \( \partial M = \Sigma_+ \cup \Sigma_- \), \( \{\ast\} = \{\ast_-\} \cup \{\ast_+\} \) is the set of one basepoint per component of \( \Sigma_\pm \) and \( b_\pm \) is the restriction of \( b \) to \( \Sigma_\pm \).

The composition of \( M_1 : (\Sigma_-, b_-) \to (\Sigma_0, b_0) \) and \( M_2 : (\Sigma_0, b_0) \to (\Sigma_+, b_+) \) is obtained by gluing \( M_1 \) and \( M_2 \) along \( \Sigma_0 \) and defining \( b \) on \( M \) as \( r(\phi^{-1}(b_- + b_+)) \) where \( \{\ast\} = \{\ast_+, \ast_-, \ast_0\} \).

\[
\phi : H^2(M, \partial M \cup \Sigma_0 \setminus \{\ast\}) \to H^2(M_-, \partial M_- \setminus \{\ast_-, \ast_0\}) \oplus H^2(M_+, \partial M_+ \setminus \{\ast_+, \ast_0\})
\]
is coming from the Mayer-Vietoris sequence and
\begin{equation}
(5.26) \quad r : H^2(M, \partial M \cup \Sigma_0 \setminus \{*_+, *0_+, *-_\}) \to H^2(M, \partial M \setminus \{*_+, *-_\})
\end{equation}
is the restriction map induced by the long exact sequence of the triple \((M, (\partial M \cup \Sigma_0) \setminus \{*_0, *+, *-_\}, \partial M \setminus \{*_+, *-_\})\).

We shall now show that \(V(\Sigma, b)\) is endowed by a \(H^1(\Sigma; \mathbb{R}/\mathbb{Z})\)-grading as follows. First of all remark that
\begin{equation}
(5.27) \quad H^2(\Sigma \times [0, 1], (\Sigma \setminus \{\ast\}) \times \{0, 1\}; \mathbb{Z}) \cong H^1(\Sigma, \{\ast\}; \mathbb{Z}) \oplus H^2(\Sigma, \Sigma \setminus \{\ast\}; \mathbb{Z})
\end{equation}
where the injection \(\delta : H^1(\Sigma; \mathbb{Z}) \cong H^1(\Sigma, \{\ast\}; \mathbb{Z}) \to H^2(\Sigma \times [0, 1], (\Sigma \setminus \{\ast\}) \times \{0, 1\}; \mathbb{Z})\) is induced by the exact sequence of the triple \((\Sigma \times [0, 1], (\Sigma \setminus \{\ast\}) \times \{0, 1\}, (\Sigma \setminus \{\ast\}) \times \{0\})\); and the generators \(H^2(\Sigma, \Sigma \setminus \{\ast\}; \mathbb{Z}) = \mathbb{Z}^{\pi_0(\Sigma)}\) are the Poincaré duals of the arcs \(\{p\} \times [0, 1]\) for \(p \in \{\ast\}\).

Given \(\omega \in H^1(\Sigma; \mathbb{R}/\mathbb{Z})\) the \(\omega\)-homogeneous subspace of \(Z(\Sigma, b)\) is defined as follows:
\begin{equation}
(5.28) \quad V(\Sigma, b)_\omega := \{x \in V(\Sigma, b) | \forall c \in H^1(\Sigma; \mathbb{Z}) \quad Z(\Sigma \times [0, 1], b + \delta(c))(x) = \exp(-2\pi i \omega(c))x\}
\end{equation}
(remark that the restriction of \(b + \delta(c)\) to \(\Sigma \times \{0\}\) is \(b\)).

Now observe that there is a well defined map
\begin{equation}
(5.29) \quad \int_M : H^1(\Sigma, \{\ast\}; \mathbb{R}/\mathbb{Z}) \otimes H^2(M, \partial M \setminus \{\ast\}; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}
\end{equation}
defined equivalently as \(\int_M \omega \otimes b := \langle \omega \cup b, [M] \rangle = \omega(PD(b))\) where \([M] \in H^3(M, \partial M; \mathbb{Z})\) is the fundamental class and \(PD\) is Poincaré duality.

Considering all the morphisms \(M : (\Sigma_-, b_-) \to (\Sigma_+, b_+\) for which the underlying manifold is \(M\) we define their Fourier transform for any \(\omega \in H^1(\Sigma, \{\ast\}; \mathbb{R}/\mathbb{Z})\) as:
\begin{equation}
(5.30) \quad Z'(\Sigma, \omega)_b := \sum_{b \in H^2(\Sigma, \partial M \setminus \{\ast\}; \mathbb{Z})} \exp \left(2\pi i \int_M \omega \cup b\right) Z_\omega(M, b)
\end{equation}
where
\begin{equation}
(1) \quad Z_\omega(M, b) : V_{\omega_-}(\Sigma_-, b_-) \to V_{\omega_+}(\Sigma_+, b_+)
\end{equation}
is the restriction of \(Z(M, b)\) to the degree \(\omega_+ = \omega|_{\Sigma_+}\) vector subspaces and we use the identification \(V_{\omega_-}(\Sigma_-, b_-) = V_{\omega_+}'(\Sigma_+, \omega_+)\) to interpret it as a map \(V_{\omega_-}'(\Sigma_-, \omega_-) \to V_{\omega_+}'(\Sigma_+, \omega_+);\)
\begin{equation}
(2) \quad \delta : H^1(\Sigma_-; \mathbb{Z}) \to H^2(M, \partial M \setminus \{\ast\}; \mathbb{Z})
\end{equation}
is induced as above by the exact sequence of the triple \((M, \partial M \setminus \{\ast\}, \Sigma_- \setminus \{\ast_-\});\)
(3) the sum is over all the representatives of classes $b$ restricting to $b_-$ on $\Sigma_-$ and the choice of a representative is irrelevant because we have:

\[
\exp \left( 2\pi i \int_M \omega \cup (b + \delta(c)) \right) Z_\omega(M, b + \delta(c)) = \\
\exp \left( 2\pi i \int_M \omega \cup b + \omega \cup \delta(c) \right) Z_\omega(M, b) \circ Z_\omega(\Sigma_+ \times [0, 1], b_+ + \delta(c)) = \\
\exp \left( 2\pi i \int_M \omega \cup b \right) \exp (2\pi i \omega(c)) Z_\omega(M, b) \exp(-2\pi i \omega(c)) = \\
\exp \left( 2\pi i \int_M \omega \cup b \right) Z_\omega(M, b)
\]

where in the second equality we used the definition of the grading on $V(\Sigma, b)$.

Of course in the above formula we assume that the sum is convergent. So given a cobordism $M : \Sigma_- \to \Sigma_+$ and $\omega \in H^1(M, \{\ast\}; \mathbb{R}/\mathbb{Z})$ we will from now on say that $Z'(M, \omega)$ exists if it exists for all $b_- \in H^2(\Sigma_-, \Sigma_- \setminus \{\ast_\ast\}; \mathbb{Z})$.

**Proposition 5.1.** Suppose that $M_- : \Sigma_- \to \Sigma_0$ and $M_+ : \Sigma_0 \to \Sigma_+$ are two cobordisms and let $M = M_+ \circ M_-$. Let $\omega_\pm \in H^1(M_\pm, \{\ast_\ast, \ast\}; \mathbb{R}/\mathbb{Z})$ and $\omega \in H^1(M, \{\ast_+, \ast_\ast\}; \mathbb{R}/\mathbb{Z})$ be defined as $res(\omega')$ where $\omega' \in H^1(M, \{\ast_+, \ast_\ast, \ast\}; \mathbb{R}/\mathbb{Z})$ restricts to both $\omega_\pm$, and $res : H^1(M, \{\ast_+, \ast_\ast, \ast\}; \mathbb{R}/\mathbb{Z}) \to H^1(M, \{\ast_+, \ast_\ast\}; \mathbb{R}/\mathbb{Z})$ is the restriction map.

If $Z'(M_\pm, \omega_\pm)$ exist then also $Z'(M, \omega)$ exists and it holds:

\[
Z'(M, \omega) = Z'(M_+, \omega_+) \circ Z'(M_-, \omega_-).
\]

**Proof.** Let $b \in H^2(M, \Sigma_- \setminus \{\ast_\ast\} \cup \Sigma_+ \setminus \{\ast_+\}; \mathbb{Z})$. By the exact sequence of the triple

\[
(M, \Sigma_- \cup \Sigma_0 \cup \Sigma_+ \setminus \{\ast_-, \ast_0, \ast_+\}, \Sigma_- \cup \Sigma_+ \setminus \{\ast_-, \ast_+\})
\]

we have a surjective map

\[
\pi : H^2(M, \Sigma_- \cup \Sigma_0 \cup \Sigma_+ \setminus \{\ast_-, \ast_0, \ast_+\}; \mathbb{Z}) \to H^2(M, \Sigma_- \cup \Sigma_+ \setminus \{\ast_-, \ast_+\}; \mathbb{Z})
\]

so that $\pi^{-1}(b)$ is well defined up to elements of the form $\delta(c)$ for some $c \in H^1(\Sigma_0 \setminus \{\ast_0\}; \mathbb{R}/\mathbb{Z})$ where $\delta$ is induced by the same exact sequence. So if $b' \in \pi^{-1}(b)$ then there are well-defined restrictions $res_\pm(b') \in H^2(M_\pm, \Sigma_\pm \setminus \{\ast_\pm\} \cup \Sigma_0 \setminus \{\ast_0\}; \mathbb{Z})$. Furthermore one can check that the restrictions to $H^2(\Sigma_0, \Sigma_0 \setminus \{\ast_0\}; \mathbb{Z})$ of $res_+(b')$ and of $res_-(b')$ coincide and depend only on $b$ (not on the choice of $b'$). Let us then denote this common
restriction \( b_0 \in H^2(\Sigma_0, \Sigma_0 \setminus \{ \ast_0 \}; \mathbb{Z}) \). We have then

\[
(5.37) \quad Z'(M, \omega)_{b_-} = \sum_{b \in H^2(M, \Sigma_0 \setminus \{ \ast_0 \}; \mathbb{Z})} \exp \left( 2i\pi \int_M \omega \cup b \right) Z(M, b) = \\
= \sum_{b' \in H^2(M, \Sigma_0 \setminus \{ \ast_0 \}; \mathbb{Z})} \exp \left( 2i\pi \int_M \omega \cup \pi(b') \right) Z(M, \pi(b')) = \\
= \sum \exp \left( 2i\pi \int_{M_-} \omega_- \cup b'_- + \int_{M_+} \omega_+ \cup b'_+ \right) Z(M, b'_+)Z(M, b'_-)
\]

where the last sum ranges over

\[
(5.38) \quad b'_- \in H^2(M_-, \Sigma_0 \setminus \{ \ast_0 \}; \mathbb{Z})/\delta(H^1(\Sigma_0 \setminus \{ \ast_0 \}; \mathbb{Z}))
\]

such that \( res_-(\pi(b'_-)) = b_- \) and over

\[
(5.39) \quad b'_+ \in H^2(M_+, \Sigma_+ \setminus \{ \ast_+ \}; \mathbb{Z})/\delta(H^1(\Sigma_0 \setminus \{ \ast_0 \}; \mathbb{Z}))
\]

such that \( res_0(\pi(b'_+)) = res_0(\pi(b'_-)) \) (and where we let \( b'_\pm \) be the restriction of \( b' \) to \( (M_\pm, \Sigma_\pm \setminus \Sigma_0 \setminus \{ \ast_\pm, \ast_0 \}) \). The last equality uses the Mayer-Vietoris sequence for the pairs \( (M, \Sigma_0 \setminus \{ \ast_0 \}) \) and \( (M, \Sigma_\pm \setminus \{ \ast_\pm, \ast_0 \}) \) and the fact that \( Z \) is functorial. \( \square \)

5.2.3. "r-wrapping" of \( H^1(\cdot; \mathbb{R}/\mathbb{Z}) \)-TQFTs. The other operation which we will need to upgrade the relation (3.33) to TQFTs is the operation that takes an \( H^1(\cdot; \mathbb{R}/\mathbb{Z}) \)-TQFT \( Z \) and produces another \( H^1(\cdot; \mathbb{R}/\mathbb{Z}) \)-TQFT \( Z' \). For closed connected manifolds we have (assume \( r = 2 \mod 4 \))

\[
(5.40) \quad Z'(M, \omega) = \frac{r^{-(b_0+b_1)/2}}{\sqrt{|\text{Tor} H_1(M; \mathbb{Z})|}} \int_{H^1(M; \mathbb{R}/\mathbb{Z})} \mu(\alpha) e^{-\frac{2\pi i}{r}} q_\alpha(\alpha') - 2\pi i \ell k(\alpha', \omega') \frac{r^n}{r} \delta(r\alpha''-\omega'') Z(M, \alpha)
\]

where we choose an explicit splitting

\[
(5.41) \quad \omega = \omega' \oplus \omega'', \quad \alpha = \alpha' \oplus \alpha'' \in H^1(M; \mathbb{R}/\mathbb{Z}) \cong \text{Tor} H_1(M; \mathbb{R}/\mathbb{Z}) \oplus (\mathbb{R}/\mathbb{Z})^{b_1}
\]
as in Section 3.5.

Applying the formula (5.40) to \( M = \Sigma \times S^1 \), where \( \Sigma \) is a closed connected oriented genus \( g \) surface, we get the following relation between the dimensions of the corresponding graded vector spaces:

\[
(5.42) \quad \dim V_n'(\Sigma, \omega) = r^{-g} \sum_{m \in (\mathbb{Z}/r\mathbb{Z})^{2g}} \dim V_{rn}(\Sigma, \frac{\omega + m}{r})
\]

where \( n \in \mathbb{Z}, \omega \in H^1(\Sigma; \mathbb{R}/\mathbb{Z}) \cong (\mathbb{R}/\mathbb{Z})^{2g} \).

This formula suggests the following generalisation to TQFTs. Assume that after a choice of a symplectic basis in \( H_1(\Sigma; \mathbb{Z}) \) (that splits the generators into \( g \) A-cycles and
B-cycles) and the corresponding splitting $\omega = \omega_A \oplus \omega_B$ ($\omega_{A,B} \in (\mathbb{R}/\mathbb{Z})^g$), one can identify $V_n(\Sigma, \omega_A \oplus \omega_B)$ for a fixed $\omega_A$ and all possible $\omega_B$. That is one can explicitly drop the dependence on $\omega_B$: $V_n(\Sigma, \omega_A) := V_n(\Sigma, \omega_A \oplus \omega_B)$. Then the same is true for $V'_n(\Sigma, \omega_A \oplus \omega_B)$, and we have

\begin{equation}
V'_n(\Sigma, \omega_A) = \bigoplus_{s \in (\mathbb{Z}/r\mathbb{Z})^g} V_{rn}(\Sigma, \omega_A + \frac{s}{r})
\end{equation}

where $(\mathbb{Z}/r\mathbb{Z})^g \subset (\mathbb{R}/\mathbb{Z})^g$ is identified with the subgroup of holonomies of order $r$ along the A-cycles.

This is consistent with the conjectural relation (2.81) between the CGP and $\hat{Z}$ invariants of knots (which are valued in the corresponding vector spaces above for $\Sigma = T^2$).

The extension of the relation (5.40) to general bordisms turns out to be subtle and technically complicated. One of the reasons is that one requires to choose a splitting of $H_1(M; \mathbb{R}/\mathbb{Z})$ in (5.41), which is not a natural structure on manifolds and there are various ways one can extend it to the manifolds with boundary. We will not address this issue in this work.

6. Relation to WRT invariants

In the previous sections we focused on the relations between $\hat{Z}$ and $N_r$. But there are also relations between and $\hat{Z}$ and WRT invariants and between $N_r$ and WRT invariants (possibly in their cohomology-refined versions). In this section we compare this triangle of relations and show that they are compatible with each other.

6.1. Compatibility of the relations between $N^0_r$, WRT$_r$, and $\hat{Z}$. Let $M$ be a rational homology sphere. Consider

\begin{equation}
Z_a^{SO(3)}(M) := \sum_{s \in \text{Spin}^c(M)} S_{a,s}^{SO(3)} \hat{Z}_s(M), \quad a \in H_1(M; \mathbb{Z})
\end{equation}

where

\begin{equation}
S_{a,s}^{SO(3)} := \frac{1}{|H_1(M; \mathbb{Z})|} \times \\
\times \left\{ \begin{array}{ll}
\sum_f e^{2\pi i \left( -\frac{r}{4}\ell k(a,a) + \ell k(a,f-b) + \ell k(f,f) - \frac{1}{2}\mu(M,s) \right)} , & \text{if } r = 1 \text{ mod } 4, \\
\sqrt{|H_1(M; \mathbb{Z})|} e^{-\frac{\pi ir}{2} q_s(a) - 2\pi i \ell k(a,b)} , & \text{if } r = 2 \text{ mod } 4, \\
\sum_f e^{2\pi i \left( -\frac{r}{4}\ell k(a,a) - \ell k(a,f+b) - \ell k(f,f) - \frac{1}{2}\mu(M,s) \right)} , & \text{if } r = 3 \text{ mod } 4.
\end{array} \right.
\end{equation}

Then, conjecturally, we have (for any $r$)

\begin{equation}
N_r(M, \omega) = (-1)^r T(M, [\omega]) \sum_{a \in H_1(M; \mathbb{Z})} e^{-\pi i \omega(a)} Z_a^{SO(3)}(M) \bigg|_{q = e^{2\pi i r}}
\end{equation}
for \( \omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z}) \). Similarly, for the \( H^1(M; \mathbb{Z}/2\mathbb{Z}) \)-refined WRT invariant of [18] it is conjectured (see Appendix B for details):

\[
\text{WRT}_r(M, \omega) = \frac{1}{i\sqrt{8r}} \sum_{a \in H_1(M; \mathbb{Z})} e^{-\pi i \omega(a)} Z_{a}^{SO(3)}(M) \bigg|_{q=e^{2\pi i}}
\]

for \( \omega \in H^1(M; \mathbb{Z}/2\mathbb{Z}) \).

Given \( \omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \), as shown in [17] a normalized invariant \( N^0_r(M, \omega) \) can be defined by:

\[
N^0_r(M, \omega) := \frac{N_r(M \# M', \omega + \omega')}{N_r(M', \omega')}
\]

for any \( M' \) and \( \omega' \in H^1(M'; \mathbb{Z}/2\mathbb{Z}) \) such that both denominator and numerator in the right-hand side are well defined. For \( \omega \notin H^1(M; \mathbb{Z}/2\mathbb{Z}) \) we have

\[
N^0_r(M, \omega) = 0.
\]

And for \( \omega \in H^1(M; \mathbb{Z}/2\mathbb{Z}) \) the conjecture of [18] in this paper’s normalisation (theorem for knot surgeries) is that

\[
N^0_r(M, \omega) = D \cdot |H_1(M; \mathbb{Z})| \text{ WRT}_r(M, \omega).
\]

Let us check that (6.7) is consistent with our conjectures (6.3) and (6.4). Plugging (3.53) into the definition (6.5) we have

\[
N^0_r(M, \omega) = \frac{\mathcal{T}(M \# M', [\omega] \oplus [\omega'])}{\mathcal{T}(M', [\omega'])} \cdot \frac{\sum_{a \in H_1(M; \mathbb{Z})} e^{-\pi i \omega(a) - \pi i \omega'(a')} Z_{a}^{SO(3)}(M \# M')}{\sum_{a' \in H_1(M'; \mathbb{Z})} e^{-\pi i \omega'(a')} Z_{a'}^{SO(3)}(M')} \bigg|_{q=e^{2\pi i}}.
\]

Assume that \( b_1(M') > 0 \) and \( \omega' \) is a generic element of \( H^1(M; \mathbb{C}/2\mathbb{Z}) \). Then (see e.g. [53])

\[
\frac{\mathcal{T}(M \# M', [\omega] \oplus [\omega'])}{\mathcal{T}(M', [\omega'])} = |H_1(M; \mathbb{Z})|.
\]

Taking a limit (possible since \( b_1(M') > 0 \)) where \( \omega' \) tends to an element of \( H_1(M; \mathbb{Z}/2\mathbb{Z}) \) (e.g. zero) and using (6.4) we then have

\[
N^0_r(M, \omega) = |H_1(M; \mathbb{Z})| \frac{\text{WRT}_r(M \# M', \omega \oplus \omega')}{\text{WRT}_r(M', \omega')}.
\]

Using \( \text{WRT}_r(M \# M', \omega \oplus \omega') = D \cdot \text{WRT}_r(M, \omega) \text{ WRT}_r(M', \omega') \) we then indeed arrive at (6.7).

\[^{13}\text{For } b_1 > 0 \text{ it is assumed that } |H_1(M; \mathbb{Z})| = 0.\]
6.2. 0-surgeries on twist knots. 0-surgeries on knots, \( M = S^3_0(K) \), all have \( H_1(M) = \mathbb{Z} \). In such cases, the relation to the standard (not refined) WRT invariants is expected to be especially simple [14]:

\[
\text{WRT}_r(S^3_0(K)) = -\frac{1}{2D^2} \left[ \tilde{Z}_0^{(+)} + \tilde{Z}_0^{(-)} \right] \bigg|_{q \to e^{2\pi i/r}}.
\]

Here,

\[
\tilde{Z}_0^{(+)}(S^3_0(K)) = \text{Res}_{x=0} \frac{x^{1/2} - x^{-1/2}}{F_K(x, q)} = \frac{1}{2} f_1^r(q)
\]

is \( \tilde{Z} \)-invariant in the trivial spin\(^c \) structure. For twist knots we also have

\[
\tilde{Z}_0^{(-)}(S^3_0(K)) = \text{Res}_{x=x_0} \frac{x^{1/2} - x^{-1/2}}{F_K(x, q)}
\]

where \( x_0 \) is a suitable root of the Alexander polynomial (recall, that for a twist knot the Alexander polynomial has degree 2 and the residues differ by a sign.) Therefore, we can write

\[
\text{WRT}_r(S^3_0(K)) = -\frac{1}{2D^2} \lim_{q \to e^{2\pi i/k}} \oint_C \frac{dx}{2\pi i x} \left[ x^{1/2} - x^{-1/2} \right] F_K(x, q)
\]

where the contour \( C \) goes around \( x = 0 \) and \( x_0 \). It has been checked in [14] that this procedure indeed recovers the correct WRT invariants of \( M = S^3_0(K) \) for many twist knots \( K \).

This way of recovering \( \text{WRT}_r(S^3_0(K)) \) is a priori different from the strategy used earlier in this paper, where the CGP and WRT invariants of \( S^3_0(K) \) are obtained from \( F_K(x, q) \) by first specializing \( F_K(x, q) \) to a root of unity \( q = e^{2\pi i/r} \) and then summing over colors / decorations \( x \) as in a typical surgery formula. Roughly speaking, this approach — based on the relation (3.50) to \( \text{ADO}_r \) invariants — exchanges the order of operations, so that the limit \( q \to e^{2\pi i/r} \) comes first. It is instructive to verify that these two methods are compatible and, therefore, form a consistent network of proposed relations. Namely, repeating the arguments of Appendix B for the 0-surgeries on knots, we expect the following relation:

\[
\text{WRT}_r(S^3_0(K)) = -\frac{1}{2D^2} \lim_{q \to e^{2\pi i/k}} \sum_{n \in \mathbb{Z}} \tilde{Z}_{nk}
\]

or, according to (3.50),

\[
\text{WRT}_r(S^3_0(K)) = -\frac{1}{4D^2} \sum_{n=0}^{r-1} (\xi^n - \xi^{-n})^2 \text{ADO}_r(\xi^{2n-2}; K)
\]

where we used \( \Delta_K(1) = 1 \). Comparing the right-hand sides of the above two formulae we can eliminate \( \text{WRT}_r(S^3_0(K)) \) from these relations. Similar considerations apply to
the invariants $N_r(S_0^3(K), \omega)$; the only difference is that there is still $x$-dependence in all of the expressions, and, as explained in Section 3.7, we obtain

$$\lim_{q \to e^{2\pi i / r}} \sum_{n \in \mathbb{Z}} \hat{Z}_{nr} x^{nr} = \sum_{n=0}^{r-1} \frac{\text{ADO}_r(x \xi^{2n-2}; K)}{2 \Delta_K(x^r)} \cdot (\xi^n x^{1/2} - \xi^{-n} x^{-1/2})^2.$$  

Since $\text{ADO}_r(x; K)$ is a polynomial in $x$, the right-hand side is expected to be a rational function in $x$. Let us illustrate how this works for the figure-8 knot. In the conventions (3.72), for the figure-8 knot we have $f_1 = 1, f_2 = 2, f_5 = q^{-1} + 3 + q$, and so on. From (3.74) we get $\hat{Z}_0 = -2, \hat{Z}_1 = -1, \hat{Z}_2 = -q^{-1} - 1 - q, \text{ etc.}$ For example, for $r = 2$ and any knot $K$, with our conventions we have $\text{ADO}_2(x; K) = \Delta_K(x)$ and the right-hand side of the above relation becomes

$$\frac{(x - 2 + x^{-1}) \cdot \Delta_K(-x) + (-x - 2 - x^{-1}) \cdot \Delta_K(x)}{2 \Delta_K(x^2)}.$$  

For example, for the figure-8 knot, $\Delta_{4_1}(x) = -x^{-1} + 3 - x$ and we get

$$-x^2 - 4 + x^{-2} - x^2 - 3 + x^{-2} = -1 + x^2 + 3x^4 + 8x^6 + 21x^8 + 55x^{10} + 144x^{12} + 377x^{14} + 987x^{16} + \ldots$$

The coefficients of this expansion perfectly match $\hat{Z}_n(x)|_{q \to -1}$ for even values of $n$.\(^{14}\) Note, that (6.17) can be viewed as a close cousin of the Conjecture 1, obtained from it by multiplying with $x^{1/2} - x^{-1/2}$, replacing $x$ by $\xi^{2j} x$, and then summing over $j = 0, \ldots, r - 1$. This again verifies the consistency of various proposed relations.

In order to perform a similar computation for other values of $r$, it may be convenient, building on [24, 39], to express $\hat{Z}_n(q)$ in the quiver form,

$$\hat{Z}_n(q) = 2 \hat{f}_{n-1}(q) - \hat{f}_{n-2}(q) - \hat{f}_n(q), \quad (n > 1)$$

where, for the figure-8 knot,

$$\hat{f}_n(q) = \sum_{n_1 + \ldots + n_6 = n} (-q^{1/2})^{n_4 + n_5 + n_6} q^{(n_2 + n_5)(n_6 - n_1) + \frac{1}{2} (n_2^2 + n_5^2 + n_6^2)} \prod_{i=1}^6 \frac{1}{(q; q)_{n_i}}.$$  

Curiously, much like $f_m(q)$, the coefficients $\hat{f}_n(q)$ are all Laurent polynomials in $q$ with integer coefficients.

Now, once we managed to write the right-hand side of (6.15) with the regularization parameter $x$ as a rational function, it is straightforward to set $x = 1$ and obtain the WRT invariant. For example, for any knot $K$ we have

$$\text{WRT}_2(S_0^3(K)) = 1.$$  

This is indeed what we expect from the surgery formula for $\text{WRT}_r(S_0^3(K)) = D^{-2} \sum_{n=1}^{r-1} (-1)^{n+1} [n] J_{n-1}(q)|_{q = e^{2\pi i / r}}$.  

\(^{14}\)Turning this around, we can say that $(x^{1/2} - x^{-1/2}) \tilde{F}_K(x, q)$ restricted to even powers of $x$ is a $q$-deformation of the rational function (6.18) determined by the Alexander polynomial.
6.3. From CGP to WRT via $\hat{Z}$ for 0-surgeries on knots. Consider the general surgery formula (2.97) for $\hat{Z}$ in the case of 0-surgery on a knot $K$. We have $V = b_1 = 1$, $\varepsilon'' = 1$, $\sigma = 0$, $B = 0$ and

$$(6.23) \quad \hat{Z}_{\sigma(b'', s)} = q^{b''} F_{2b''},$$

where $b'' \in \mathbb{Z} \cong H_1(S^3_0(K); \mathbb{Z})$ runs over integers and $F_{2b''}$ are the coefficients of the following formal power series:

$$(6.24) \quad F(x, q) := F_K(x^2, q)(x - x^{-1}) = \sum_\ell F_\ell x^\ell.$$

For knots only even powers actually appear in the series $F(x, q)$ (not to be confused with $F_K(x, q)$).

6.3.1. Odd level $r$. As before, denote $\mu := \omega(m) \in \mathbb{C}/2\mathbb{Z}$, where $m$ is the class of the meridian of the know in $H_1(S^3_0(K); \mathbb{Z})$. The formula (3.39) in the case of a 0-surgery on $K$ then reads

$$(6.25) \quad N_r(S^3_0(K), \omega) = r \frac{\Delta(e^{2\pi i \mu})}{(e^{\pi i \mu} - e^{-\pi i \mu})^2} \sum_{m \in \mathbb{Z}} e^{2\pi i m} \hat{Z}_{\sigma(m, s)} \bigg|_{q \to e^{2\pi i}}.$$

Plugging in (6.23) and using $\Delta(1) = 1$, we then have

$$(6.26) \quad \lim_{\mu \to 0}[r\mu]^2 N_r(S^3_0(K), \omega) = \lim_{\mu \to 1}[r\mu]^2 N_r(S^3_0(K), \omega) = -\frac{D^2}{2} \sum_{m \in \mathbb{Z}} F_{2mr} |_{q \to e^{2\pi i}}.$$

On the other hand, for WRT invariant we have

$$(6.27) \quad \text{WRT}_r(S^3_0(K)) = D^{-2} \sum_{n=1}^{r-1} (-1)^{n+1} J_{n-1}(q)[n] = -\frac{1}{2r} \sum_{n \in \mathbb{Z}_r} F(q^{n/2}, q) |_{q \to e^{2\pi i}} - \frac{1}{2r} \sum_{n \in \mathbb{Z}_r} e^{2\pi i n/2} F_{2n} |_{q \to e^{2\pi i}} - \frac{1}{2} \sum_{m \in \mathbb{Z}} F_{2mr} |_{q \to e^{2\pi i}}.$$

Combining (6.26) and (6.27) we get

$$(6.28) \quad \lim_{\mu \to 0}[r\mu]^2 N_r(S^3_0(K), \omega) = \lim_{\mu \to 1}[r\mu]^2 N_r(S^3_0(K), \omega) = D^2 \text{WRT}_r(S^3_0(K))$$

which is in agreement with Theorem 2.9.

6.3.2. Even level $r$. The formula (3.33) in the case of a 0-surgery on $K$ then reads

$$(6.29) \quad N_r(M, \omega) = r \frac{\Delta_K(e^{2\pi i \mu})}{(e^{\pi i \mu} - e^{-\pi i \mu})^2} \sum_m e^{\pi i m} \hat{Z}_{\sigma(m, s)} \bigg|_{q \to e^{2\pi i}}.$$

Plugging in (6.23) we then have

$$(6.30) \quad \lim_{\mu \to 0}[r\mu]^2 N_r(S^3_0(K), \omega) = -\frac{D^2}{2} \sum_{m \in \mathbb{Z}} (-1)^m F_{mr} |_{q \to e^{2\pi i}},$$
and

\[(6.31) \lim_{\mu \to c} [r\mu]^2 N_r(S^3_0(K), \omega) = -\frac{D^2}{2} \sum_{m \in \mathbb{Z}} F_{m\nu}|_{q \to e^{2\pi i}}.\]

On the other hand, for \(r = 2 \mod 4\) one can consider WRT invariant refined by an element \(\gamma \in H^1(S^3_0(K); \mathbb{Z}_2) \cong \mathbb{Z}_2 [37]\). Let \(c = \gamma(m) \in \mathbb{Z}_2\). We have

\[(6.32) \text{WRT}_r(S^3_0(K), \gamma) = D^{-2} \sum_{n=c+1 \mod 2}^{r-1} (-1)^{n+1} J_{n-1}(q)[n] = D^{-2} \sum_{n \in 2\mathbb{Z}_{r/2} + c+1} F(q^{n/2})|_{q \to e^{2\pi i}} = -\frac{1}{2r} \sum_{\alpha \in 2\mathbb{Z}_{r/2}} e^{2\pi i (2a+c+1)x} F_{2\nu}|_{q \to e^{2\pi i}} = -\frac{1}{4} \sum_{m \in \mathbb{Z}} (-1)^{m(c+1)} F_{m}\nu|_{q \to e^{2\pi i}}.\]

Combining (6.30)-(6.31) and (6.32) we get

\[(6.33) \lim_{\mu \to c} [r\mu]^2 N_r(S^3_0(K), \omega) = 2D^2 \text{WRT}_r(S^3_0(K), \gamma), \quad c = 0, 1.\]

which is again consistent with Theorem 2.9.

**Appendix A. Spin and spin\(^c\) Sign-refined Torsion**

The Reidemeister torsion has an intrinsic sign ambiguity. As was shown by Turaev, it is possible to fix it by choosing an Euler structure [52]. In the case of 3-manifolds such choice is equivalent to a choice of a spin\(^c\) structure [52]. Consider a 3-manifold \(M = S^3(\mathcal{L})\) obtained by a surgery on a framed link \(\mathcal{L} \in S^3\). As before, let Vert be the set of components of the link and \(B_{IJ}, I, J \in \text{Vert}\) its linking matrix. Denote by \(\sigma_K(M) \in \text{Spin}^c\) a spin\(^c\) structure that corresponds to a characteristic vector \(K \in \mathbb{Z}^L/2B\mathbb{Z}^L\), \(K_I = B_{II} \mod 2\), as in (2.18). Let \(a \in H^1(M; \mathbb{C}/\mathbb{Z})\) and \(\alpha_I := a(m_I) \in \mathbb{C}/\mathbb{Z}\). They satisfy the condition \(\sum_I B_{IJ}\alpha_J = 0 \mod 1\). As before, denote \(\varepsilon = (1, 1, \ldots, 1) \in \mathbb{Z}^{\text{Vert}}\). The Turaev’s sign refined torsion of \(M = S^3(\mathcal{L})\) is then given by the following formula [53] (see also [7]):

\[(A.1) \quad \mathcal{T}(M, a, \sigma_K) = (-1)^{b_1} \prod_I \frac{1}{e^{\pi i} - e^{-\pi i}} \nabla(\mathcal{L})(\{e^{\pi i\alpha_I}\}_I) e^{\pi i(-\varepsilon^T K + \varepsilon^T B\alpha)}\]

where \(\nabla(\mathcal{L})\) is the Alexander-Conway function of the link \(\mathcal{L}\).

The above torsion relates to the invariant \(N_2(M, \omega)\) where \(\omega \in H^1(M; \mathbb{C}/2\mathbb{Z}) \setminus H^1(M; \mathbb{Z}/2\mathbb{Z})\). Indeed in [17] (Theorem 6.23, taking into account the different normalisation used in the present paper) the following was proved:

\[(A.2) \quad \mathcal{T}(M, \frac{\omega}{2}, \sigma_K) = (-2)^{1+b_1(M)} \left(\frac{i}{4}\right)^{b_1(M)} \frac{iN_2(M, \omega)}{2} e^{-\frac{\mu^T B\mu - K^T \mu}{2}}\]
where we used that $\frac{\omega}{2} \in H^1(M; \mathbb{C}/\mathbb{Z})$ is the well defined cohomology class the value of which on the meridian $m_I$ is $\frac{\omega(m_I)}{2} = \frac{\mu_I}{2}$ and we encoded the spin$^c$ structure $\sigma_K$ via $K$ as above.

In particular using the canonical map $i : \text{Spin}(M) \rightarrow \text{Spin}^c(M)$ we can define an invariant which depends only on a spin structure $s$ as

$$(A.3) \quad \mathcal{T}_s(M, \omega) := \mathcal{T}(M, \frac{\omega}{2}, i(s)).$$

More explicitly, if we now let $\tilde{s} \in \mathbb{Z}^\text{Vert}$ be such that $\tilde{s} = s \mod 2$ and $K = \sum_{I,J} B_{IJ} \tilde{s}_J$ then it holds:

$$(A.4) \quad \mathcal{T}_s(M, \omega) := (-2)^{1+b_1} \frac{i^{b_1(M)} iN_2(M, \omega)}{2} i - \frac{\omega_{\mu}}{2} - \tilde{s}_B.$$

It is possible to define the version of the torsion depending only on spin structure (cf. [42]). Using the map (3.20), consider

$$(A.5) \quad \mathcal{T}_s(M, a) := \mathcal{T}(M, a, \sigma(b, s)) e^{2\pi i a(b)}.$$

From the surgery formula, it is easy to see that the right hand side depends only on spin structure $s \in \text{Spin}(M)$, but not $b \in H_1(M; \mathbb{Z})$. Namely, using the correspondence (2.6) we have $K = 2b + B s$ for $\sigma_K = \sigma(b, s)$, where $b \in \mathbb{Z}^\text{Vert}/B\mathbb{Z}^\text{Vert}$, $s \in \mathbb{Z}^\text{Vert}/2\mathbb{Z}^\text{Vert}$, $\sum_J B_{IJ} s_J = B_{11} \mod 2$. Therefore, for $M = S^3(L)$ we have

$$(A.6) \quad \mathcal{T}_s(M, a) = (-1)^{b_+} \prod_I \frac{1}{e^{\pi i a_I} - e^{-\pi i a_I}} \nabla(L)(\{e^{\pi i a_I}\}_I) e^{\pi i(\epsilon - s)^T B a}.$$

In particular for a plumbed $M$ we have

$$(A.7) \quad \mathcal{T}_s(M, a) = (-1)^{b_+} \prod_I (e^{\pi i a_I} - e^{-\pi i a_I})^{\text{deg}(I)-2} e^{\pi i(\epsilon - s)^T B a}.$$

Note that if $a = \omega \mod H^1(M; \mathbb{Z}/2\mathbb{Z})$, where $\omega \in H^1(M; \mathbb{C}/2\mathbb{Z})$ the dependence on the spin structure disappears, as $B a = B \omega \in 2\mathbb{Z}^\text{Vert}$.

**Appendix B. $H^1(M; \mathbb{Z}/2\mathbb{Z})$-refined WRT invariant and $\hat{Z}$**

In [18] building on [37, 5] the authors define a refined Witten-Reshetikhin-Turaev invariant $\text{WRT}_r(M, \omega)$ for $r \neq 0 \mod 4$ that depends on a choice of $\omega \in H^1(M; \mathbb{Z}/2\mathbb{Z})$, see Definition 2.5. In this appendix we relate this invariant to $\hat{Z}$. Let $M = S^3(L)$ be the 3-manifold obtained by a surgery on a framed link $L$. We will use the same conventions as before. Let $J_{n-\epsilon}(L) \in \mathbb{Z}[q^{\pm 1/4}]$ be the Jones polynomial of $L$ colored by
\(\mathfrak{sl}_2\) representations of dimensions \(n \in \{1, 2, 3, \ldots\}\)^{Vert}. Then

\[
\text{WRT}_r(S^3(L), \omega) = \frac{D^{-b_0-b_1}}{(\Delta^{|SO(3)}|_{b_+})^b_0(\Delta^{|SO(3)}|_{b_-})} \times \sum_{n \in \{1, 2, \ldots, r-1\}^{\text{Vert}}} J_n [L] \prod_{I \in \text{Vert}} (-1)^{n_I+1} \frac{q^{n_I/2} - q^{-n_I/2}}{q^{1/2} - q^{-1/2}} \biggr|_{q = e^{2\pi i r}}
\]

where as before \(\mu_I := \omega_I (m_I)\). For \(r = 2 \mod 4\) this invariant is a slight modification of the invariant of [36] (see also [5]). For odd \(r\) the invariant satisfies

\[
\text{WRT}_r(M, 0) = \text{WRT}_r^{SO(3)}(M)
\]

where \(\text{WRT}_r^{SO(3)}(M)\) is the \(SO(3)\) version of the WRT invariant introduced in [36].

To conjecture a relationship between \(\text{WRT}_r(M, \omega)\) and \(\hat{Z}_b(M)\) consider again the case of plumbing surgery. In this case

\[
J[L]_{n-\epsilon} = \frac{q^{\sum_I B_{II}(n_I^2 - 1)}}{q^{1/2} - q^{-1/2}} \times \prod_{I \in \text{Vert}} (-1)^{(n_I+1)(B_{II}+1)} (q^{n_I/2} - q^{-n_I/2})^{1-\deg(I)} \prod_{(I,J) \in \text{Edges}} (q^{n_I n_J/2} - q^{-n_I n_J/2}).
\]

We proceed similarly to Section 3.1:

\[
\text{WRT}_r(M, \omega) = \tilde{A} \cdot \tilde{C},
\]

\[
\tilde{A} = \left( (-1)^{b-r-V/2} \xi^\frac{3r-\pi B}{2} \right) \frac{i \sqrt{8r}}{\sqrt{8r}} \begin{cases} e^{\frac{\pi i \rho}{2}}, & r = 1 \mod 4, \\ 2^{-V/2} e^{\frac{3\pi i}{4}}, & r = 2 \mod 4, \\ e^{\pi i \sigma}, & r = 3 \mod 4, \end{cases}
\]

\[
\tilde{C} = \sum_{\ell} \tilde{C}_\ell F_\ell,
\]

\[
\tilde{C}_\ell = \sum_{\tilde{n} \in (\mathbb{Z}/2r\mathbb{Z})^{\text{Vert}}} \xi^{\ell' \tilde{n} + \frac{1}{2} \tilde{n}' \tilde{B} \tilde{n}}
\]

We have used the following fact:

\[
(-1)^{\sum_I (n_I+1)B_{II}} \big|_{n = \mu + \epsilon \mod 2} = (-1)^{\sum_I \mu_I B_{II}} = (1)^{\mu^T B \mu}. 
\]
since $B\mu = 0 \mod 2$. where $F_\ell$ are the same as in Section 3.1. Applying Gauss reciprocity we have

\[
\tilde{C}_\ell = \xi^{\frac{r}{2}} (\frac{r}{2})^{V/2} \sum_{\tilde{a} \in \mathbb{Z}^V / \mathbb{Z}^V} e^{-\pi i \frac{r}{2} \tilde{a}^T B^{-1} \tilde{a}} e^{-\pi i \frac{r}{2} \tilde{a}^T B^{-1} (\ell + B(\mu + \varepsilon))},
\]

\[
= \tilde{C}'_\ell
\]

(B.9)

(B.10)

\[
\tilde{C}'_\ell = \sum_{\tilde{a} \in \mathbb{Z}^V / \mathbb{Z}^V} \sum_{A \in \mathbb{Z}^V / \mathbb{Z}^V} \exp \left\{ -\frac{\pi i r}{2} a^T B^{-1} a - \pi i r A^T a - \frac{\pi i r}{2} A^T B A - 2\pi i a^T B^{-1} b - \pi i a^T (s + \mu) - \pi i A^T B s \right\}.
\]

B.1. Level $r = 2 \mod 4$. The sum (B.10) simplifies to

\[
\tilde{C}'_\ell = 2^V \sum_{\tilde{a} \in \mathbb{Z}^V / \mathbb{Z}^V} \exp \left\{ -\frac{\pi i r}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T (s + \mu) \right\}.
\]

(B.11)

Combining everything together we then have:

\[
\text{WRT}_r(M, \omega) = \frac{(-1)^b_e}{i \sqrt{8r | H_1(M; \mathbb{Z})|}} \xi^{\frac{3s - T_B}{2}} \sum_{\ell \in \mathbb{Z}^V / \mathbb{Z}^V} \sum_{\tilde{a}, f \in \mathbb{Z}^V / \mathbb{Z}^V} F_\ell \xi^{\frac{r - 1}{2}} e^{-\pi i \frac{r}{2} \tilde{a}^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T (s + \mu)}.
\]

(B.12)

We can then conjecture the following general relation for a rational homology $M$ and $r = 2 \mod 4$:

\[
\text{WRT}_r(M, \omega) = \frac{1}{i \sqrt{8r | H_1(M; \mathbb{Z})|}} \sum_{a, b \in H_1(M; \mathbb{Z})} e^{-\pi i q_a(a) - 2\pi i \ell k(a, b) - \pi i \omega(a)} Z_{\sigma(h, a)} | q \rightarrow e^{2\pi i}.
\]

(B.13)

B.2. Level $r = 1 \mod 4$. Applying a version of the Gauss reciprocity formula to the sum over $A$ in (B.10) we can rewrite it as follows:

\[
\tilde{C}'_\ell = e^{-\pi i \frac{r}{2} 2^{V/2}} \sum_{a, f, \tilde{a} \in \mathbb{Z}^V / \mathbb{Z}^V} \exp \left\{ -\frac{\pi i (r - 1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu + 2\pi i f^T B^{-1} f + 2\pi i f^T B^{-1} a + \frac{\pi i}{2} s^T B s \right\}.
\]

(B.14)
Combining everything together we have:

\[(B.15) \quad \text{WRT}_r(M, \omega) = \frac{(-1)^{b_+}}{i \sqrt{8r} |\det B|} e^{\frac{\pi i}{T}(s^TB - \sigma)} \xi^{3a - \frac{3}{2} B} \times \]
\[
\sum_{\ell \in \mathbb{Z}^\text{Vert}_{a,f}} \sum_{a,b,f \in \mathbb{Z}^\text{Vert}_{/BZ^\text{Vert}}} F_{\ell} \xi^{\frac{-T B - 1}{2}} e^{\frac{-\pi i(r-1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu + 2\pi i f^T B^{-1} f + 2\pi i f^T B^{-1} a}.
\]

Taking into account (2.17), we can then conjecture the following general relation for a rational homology \(M\) and \(r = 1 \mod 4\):

\[(B.16) \quad \text{WRT}_r(M, \omega) = \frac{e^{-\frac{\pi i}{2} \mu(M,s)}}{i \sqrt{8r} |H_1(M; \mathbb{Z})|} \times \]
\[
\sum_{a,b,f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r-1}{2} k(a,a) - \frac{1}{2} k(a,f - b) - \frac{1}{2} \omega(a) - \frac{1}{2} k(f,f) \right)} \xi^{3a - \frac{3}{2} B} \sigma(b,s).\]

B.3. Level \(r = 3 \mod 4\). This case is analogous to the case \(r = 1 \mod 4\) considered above. Applying a version of the Gauss reciprocity formula to the sum over \(A\) in (B.10) we can rewrite it as follows:

\[(B.17) \quad \tilde{C}_t' = \frac{e^{\frac{\pi i}{2} 2V/2}}{|\det B|^{1/2}} \sum_{a,b,f \in \mathbb{Z}^\text{Vert}_{/BZ^\text{Vert}}} \exp \left\{ -\frac{\pi i(r+1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu - 2\pi i f^T B^{-1} f - 2\pi i f^T B^{-1} a - \frac{\pi i}{2} s^T B s \right\}.
\]

Combining everything together we then have:

\[(B.18) \quad \text{WRT}_r(M, \omega) = \frac{(-1)^{b_+}}{i \sqrt{8r} |\det B|} e^{\frac{\pi i}{T}(s^TB - \sigma)} \xi^{3a - \frac{3}{2} B} \times \]
\[
\sum_{\ell \in \mathbb{Z}^\text{Vert}_{a,f}} \sum_{a,b,f \in \mathbb{Z}^\text{Vert}_{/BZ^\text{Vert}}} F_{\ell} \xi^{\frac{-T B - 1}{2}} e^{\frac{-\pi i(r-1)}{2} a^T B^{-1} a - 2\pi i a^T B^{-1} b - \pi i a^T \mu + 2\pi i f^T B^{-1} f + 2\pi i f^T B^{-1} a}.
\]

We can then conjecture the following general relation for a rational homology \(M\) and \(r = 3 \mod 4\):

\[(B.19) \quad \text{WRT}_r(M, \omega) = \frac{e^{-\frac{\pi i}{2} \mu(M,s)}}{i \sqrt{8r} |H_1(M; \mathbb{Z})|} \times \]
\[
\sum_{a,b,f \in H_1(M; \mathbb{Z})} e^{2\pi i \left( -\frac{r+1}{2} k(a,a) - \frac{1}{2} k(a,f + b) - \frac{1}{2} \omega(a) - \frac{1}{2} k(f,f) \right)} \xi^{3a - \frac{3}{2} B} \sigma(b,s).\]
Appendix C. Graded bases of the CGP TQFT

In [7] a TQFT $\mathcal{V}$ was build by applying the universal construction to the invariants $N_r$. It turns out that the functor one gets is a symmetric monoidal one from a suitable category of decorated cobordisms into that of graded vector spaces, endowed with the symmetry which is the flip if $r$ is odd and is the supersymmetric exchange if $r$ is even.

The cobordisms considered for this construction are 3-manifolds $M$ with boundary endowed with cohomology classes $\omega \in H^1(M; \{\ast\}; \mathbb{C}/2\mathbb{Z})$ where $\{\ast\}$ is the choice of a base point per each connected component of the boundary (besides other standard decorations). Furthermore by definition

$$\Sigma = \text{instance when}$$

Example C.2.

Remark C.1. The value of the Verlinde formula coincides with the value of the invariant $Z_r$ which is NOT equal to $N_r$: $Z_r = (\frac{-1}{\sqrt{r}})^{b_0}(\frac{-1}{r})^{\sqrt{r}}b_1N_r$.

Example C.2. If $g = 3$ the graded dimensions of $\dim_{t^{-2r}} \mathcal{V}(\Sigma)$ are:

$$r = 2 : \dim_{t^{-2r}} \mathcal{V}(\Sigma) = t^4 - 4t^2 + 6 - 4t^{-2} + t^{-4},$$

$$r = 3 : \dim_{t^{-2r}} \mathcal{V}(\Sigma) = 108t^6 + 513 + 108t^{-6},$$

$$r = 4 : \dim_{t^{-2r}} \mathcal{V}(\Sigma) = -8t^{12} + 80t^8 - 248t^4 + 352 - 248t^{-4} + 80t^8 - 8t^{-12}.$$

Example C.3. The above Verlinde formula applies only when $(\Sigma, \omega)$ is admissible. For instance when $\Sigma = S^2$ this is not the case and indeed the invariant is not a Laurent polynomial:

$$N_r(S^2 \times S^1, \beta) = \sum_{k \in H_r} \frac{(q^\beta+k - q^{-\beta-k})^2}{(q^{r\beta} - q^{-r\beta})^2} = \frac{2r}{(q^{r\beta} - q^{-r\beta})^2}$$

where $\beta \in H^1(S^2 \times S^1; \mathbb{C}/2\mathbb{Z})$ is a cohomology class the value of which on $\{\text{pt}\} \times S^1$ is $\beta \in \mathbb{C}/2\mathbb{Z}$.
Example C.4. A special case is when \( r = 2 \). In this case, if \((\Sigma, \omega)\) is admissible one gets:

\[(C.8) \quad \dim_{t^{-2}} \mathbb{V}(\Sigma) = (t - t^{-1})^{2g-2} q^{2-2g}.\]

Then applying Proposition 6.22 in [7], one recovers the Conway polynomial of the link in Figure 4:

\[(C.9) \quad \nabla(L) = (q^{\alpha_0} - q^{-\alpha_0})^{2g-1} \prod_{i=1}^{2g} (q^{\alpha_i} - q^{-\alpha_i})\]

where \( \alpha_0 \) is the color of the main strand and \( \alpha_i \) are the colors of the remaining \( 2g \) ones.

The definition of the \( \mathbb{Z} \)-grading of \( \mathbb{V}(\Sigma, \omega) \) can be given in a more intrinsic way as follows. Let \( Y \) be a three-manifold obtained by taking the complement of an open ball in a three-dimensional handlebody, so that \( \partial Y = \Sigma \sqcup S^2 \). As detailed in [7] \( \mathbb{V}(\Sigma, \omega) \) is \( \mathbb{Z} \)-graded with the grading being induced by the action of \( H^0(\Sigma) \) as follows: if \( \phi \in H^0(\Sigma; \mathbb{C}/2\mathbb{Z}) \) we can map it to a cohomology class in \( H^0(\{*\}; \mathbb{C}/2\mathbb{Z}) \) (where \( \{*\} \) is set formed by base point on \( \Sigma \) and one on \( S^2 \) ) by extending it to 0 on \( S^2 \); then let \( \delta(\phi) \in H^1(Y, \{*\}; \mathbb{C}/2\mathbb{Z}) \) be the cohomology class induced by the exact sequence of the pair \( (Y, \{*\}) \). Observe that its restriction to \( (\partial Y \setminus \{*\}) \) is the zero cohomology class so that if \( W \in H^1(Y, \{*\}; \mathbb{C}/2\mathbb{Z}) \) is a class the restriction of which to \( H^1(\Sigma, \{*\}; \mathbb{C}/2\mathbb{Z}) = \omega \) then also \( W + \delta(\phi) \) is. We say that a vector \([Y, W] \in \mathbb{V}(\Sigma, \omega)\) is of degree \( k \) if for each \( \phi \in H^0(\Sigma; \mathbb{C}/2\mathbb{Z}) \) we have \([Y, \omega + \delta \phi] = [Y, \omega] q^{2r/k} \phi \). It turns out that only some integer values of \( k \) are possible.

For generic \( \omega \), a basis of \( \mathbb{V}_0(\Sigma_g, \omega) \) is obtained as follows. Let \( \Gamma \) be an oriented trivalent graph the thickening of which is a handlebody \( H_g \) of genus \( g \) the boundary of which is identified with \( \Sigma \). An edge \( e \) of \( \Gamma \) is colored by \( \overline{\sigma}(e) := \omega(m_e) \in \mathbb{C}/2\mathbb{Z} \) where \( m_e \) is the oriented meridian of the edge. Then consider a lift \( \alpha : Edges(\Gamma) \to \mathbb{C} \) of \( \overline{\sigma} \) such that \((\partial \alpha)(v) \in H_r \) for every vertex \( v \in \Gamma \). Furthermore restrict to those \( \alpha \) such that the real part of \( \alpha(e) \) is between \([0, 2r] \) for a fixed arbitrary edge \( e \). Such a set is a basis of \( \mathbb{V}_0(\Sigma_g, \omega) \).
Appendix D. Commutativity of limits

In this section we show that the assumption (i) of the Theorem 4.18 is satisfied for a certain subclass of plumbing links. We will need the following proposition, which is slight generalization of a corollary to a proposition in \[15\] [41]:

**Proposition D.1.** Let $C: \mathbb{Z} \to \mathbb{C}$ be a function with a period $M \in \mathbb{Z}_+$ and mean value 0. Then

(i)
\[
\lim_{\epsilon \to 0^+} \sum_{n \geq 1} C(n) e^{-\epsilon(n+\gamma)} = -\frac{M}{M} \sum_{n=1}^{\infty} n C(n),
\]

(ii)
\[
\lim_{\epsilon \to 0^+} \sum_{n \geq 1} C(n) e^{-\epsilon(n^2+2\alpha n+\beta)} = -\frac{M}{M} \sum_{n=1}^{\infty} n C(n),
\]

for any $\alpha, \beta, \gamma$.

**Proof.** Can be given for example using Euler-Maclaren asymptotic summation formula:

\[
\sum_{n \geq 0} f(n) = \int_0^\infty f(x) dx - \sum_{r \geq 1} B_r r! f^{(r-1)}(0).
\]

Only the integral part and the term $r = 1$ in the sum will contribute to constant and possibly singular terms of the expansion in $\epsilon$. The singular terms and the constant terms depending on $\alpha, \beta, \gamma$ will cancel out due to the zero mean value condition.

Consider a plumbing tree with vertex $I = 0$ of valency 3, three vertices of valency one ($I = 1, 2, 3$) and possibly other vertices of valency two. We then have

\[
F_L(x^2; q) \prod_I (x_I - x_I^{-1}) = (x_1 - x_1^{-1})(x_2 - x_2^{-1})(x_3 - x_3^{-1}) \sum_{n \geq 1} (x_0^{2n-1} - x_0^{-2n+1}).
\]

In the case $\det B \neq 0$ (i.e. $b_1 = 0$), after $t$-regularization and the Laplace transform the sum over $n$ above will take the following form (up to a finite number of terms, which do not affect the issue of commutativity of the limits):

\[
\mathfrak{L}_\omega F_L(x^2; q) \prod_I (x_I - x_I^{-1}) = \sum_{n \geq 1} \sum_{\alpha, \beta, \gamma} C_{\alpha, \beta, \gamma}(n) q^{-B_1^2/4} (n^2 + 2\alpha n + \beta) t^{2n+\gamma}
\]

where the sum over $\alpha, \beta, \gamma$ is over a finite set of rational numbers and $C_{\alpha, \beta, \gamma}(n)$ are periodic in $n$. For the sum to give a well-defined element in $\mathbb{C}((q^{1/p}))$ we require

\[\text{\footnote{The proposition in [41] considers a slightly less general regularization.}}\]
$B_{00}^{-1} < 0$. The zero mean value condition is satisfied because of the alternating signs in the expansion of (D.4). We then have

\[
\lim_{q \to e^{\frac{2\pi i}{r}}} \lim_{t \to 1} \mathcal{L}_\omega F_L(x^2; q) \prod_I (x - x_I^{-1}) = \lim_{\epsilon \to 0^+} \sum_{n \geq 1} \sum_{\alpha, \beta, \gamma} \tilde{C}_{\alpha, \beta, \gamma}(n) e^{-\epsilon(n^2 + 2\alpha n + \beta)}
\]

and

\[
\lim_{t \to 1} \lim_{q \to e^{\frac{2\pi i}{r}}} \mathcal{L}_\omega F_L(x^2; q) \prod_I (x - x_I^{-1}) = \lim_{\epsilon \to 0^+} \sum_{n \geq 1} \sum_{\alpha, \beta, \gamma} \tilde{C}_{\alpha, \beta, \gamma}(n) e^{-\epsilon(n + \gamma)}
\]

where

\[
\tilde{C}_{\alpha, \beta, \gamma}(n) := C_{\alpha, \beta, \gamma}(n) e^{-\frac{2\pi i B_{00}^{-1}}{r}(n^2 + 2\alpha n + \beta)}
\]

are also periodic (generally with a larger period). Using the Prop. D.1 we can then check commutativity of the limits.

The analysis can be extended to the case $b_1 > 0$. This in particular covers the case of 0-surgeries on torus knots, which can be related to the plumbings of this type by Kirby moves. The main modification is that in (D.5) one has to replace $B_{00}^{-1}$ with $\sum_{i,j=1}^{V-b_1} U_{i0} U_{j0} (B'_i)^{-1}_{ij}$ (see Section 2.6 for the notation), which is again required to be negative in order for $\mathcal{L}_\omega$ operation to be well-defined.

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