One-Dimensional Kronig-Penney Model with Positional Disorder: Theory versus Experiment

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We study the effects of random positional disorder in the transmission of waves in a 1D Kronig-Penney model. For weak disorder we derive an analytical expression for the localization length and relate it to the transmission coefficient for finite samples. The obtained results describe very well the experimental frequency dependence of the transmission in a microwave realization of the model. Our results can be applied both to photonic crystals and semiconductor super lattices.

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I. INTRODUCTION

In recent years there is a high activity in the study of wave (electron) propagation through one-dimensional periodic structures (see, for example, Ref.[1] and references therein). Much is already known about band structures of perfectly propagating waves in strictly periodic and relatively simple devices, and one of the current interests is the influence of random imperfections that are commonly present in real experiments. These imperfections are originated, for example, from the variations of the medium parameters such as the dielectric constant, magnetic permeability, barrier widths or heights [2, 3, 4, 5, 6, 7, 8, 9].

The analysis of scattering properties of periodic-on-average (when periodic systems are slightly affected by a disorder) models with various kinds of disorder is mainly related to numerical methods. It is obvious that giving important results for specific models and parameters, the numerical approaches can not serve as a guide for the understanding of generic properties caused by disorder. In this paper we try to fill this gap in the theory by the derivation of the localization length for the 1D Kronig-Penney model, relating it to the properties of transmission through a finite number of disordered barriers.

Our analytical results are compared with the experimental data obtained for a single-mode microwave guide. We show that in spite of the standard restrictions of analytical results (restricted to infinite samples and weak disorder), comparison between theory and experiment is quite good. This fact is highly non-trivial since the experimental data are strongly influenced by absorption in the waveguide walls; effect that is also not taken into account analytically.

Our study is relevant to other types of 1D stratified media, for example, to electron transport through random superlattices [10] (disordered arrays of semiconductor quantum wells/barriers) or acoustic waves in random layered media [11]. Also, similar properties of the transmission are expected to occur in the 1D quantum Kronig-Penney model (with alternating rectangular wells and barriers).

In Sect. II the model is specified and the transfer matrix equations are derived. In Sect. III the experimental setup is briefly discussed and the numerical simulations for the transmission coefficient are compared with experimental results for the case of an array of 26 cells and different amounts of positional disorder. In Sect. IV we present the main experimental results and discuss some of the properties of transmission. In Sect. V we derive, for the regime of weak disorder, the analytical expression for the logarithm of the transmission in connection with the inverse localization length. We compare there too the numerical simulations and analytical results with the experimental data and show the effectiveness of our analytical approach. In Sect. VI, we summarize our results.

II. MODEL

We consider an array formed by two alternating dielectric slabs with refractive indices $n_a$ and $n_b$ placed in an electromagnetic metallic-wall waveguide of constant width $w$ and height $h$, see Fig. 1 For convenience, the layers with the refractive index $n_a$ ($n_b$) shall be referred as to the $a$-layers ($b$-layers). The lengths of the $n$th $a$- and $b$-layers are denoted, respectively, by $d_a(n)$ and $d_b(n)$. The positional disorder in our model consists in randomly varying lengths of only one type of layer, say...
the a-layer, such that
\[ d_a(n) = d_a + \sigma \eta(n), \quad \langle \eta(n) \rangle = d_a, \quad d_b(n) = d_b. \quad (2.1) \]

Here \( \sigma \) is r.m.s deviation of \( d_a(n) \) and \( \sigma^2 \) its variance. Hence, \( \eta(n) \) is a sequence with zero average and unit variance. In this work we assume that \( \eta(n) \) is random uncorrelated, i.e.
\[ \langle \eta(n) \eta(n') \rangle = \delta_{nn'}, \quad \langle \eta(n) \rangle = 0. \quad (2.2) \]

The angular brackets \( \langle \ldots \rangle \) stand for a statistical average over different realizations of randomly layered structure. Note that the random structure is periodic on average with the period \( d = d_a + d_b \).

In this work we shall treat the lowest TE mode of frequency \( \nu \) whose electric field \( \mathbf{E} \) is defined by
\[ E_y = \sin(\pi z/w) \Psi(x), \quad E_z = E_x = 0, \quad (2.3) \]
(see Fig. 1). Within every a- or b-layer, the function \( \Psi(x) \) obeys the 1D Helmholtz equation,
\[ \left( \frac{d^2}{dx^2} + k_{a,b}^2 \right) \Psi_{a,b}(x) = 0, \quad (2.4) \]
with the wave numbers
\[ k_{a,b} = \frac{2\pi}{c} \sqrt{n_{a,b}^2 \nu^2 - (c/2w)^2}. \quad (2.5) \]

In the incoming-outgoing wave representation, the solution of Eq. (2.4) for the \( n \)th elementary cell reads
\[ \Psi_{a_n}(x) = A^+_n \exp \{ i k_a [x - x_a(n)] \} + A^-_n \exp \{ -i k_a [x - x_a(n)] \}, \quad (2.6) \]
inside the \( a_n \)-layer, \( x_a(n) \leq x \leq x_b(n) \), and
\[ \Psi_{b_n}(x) = B^+_n \exp \{ i k_b [x - x_b(n)] \} + B^-_n \exp \{ -i k_b [x - x_b(n)] \} \quad (2.7) \]
inside the \( b_n \)-layer, \( x_b(n) \leq x \leq x_a(n+1) \). Here \( A^\pm_n \) and \( B^\pm_n \) are complex amplitudes of the forward/backward traveling wave, the coordinates \( x_a(n), x_b(n) \) denote the left-hand boundaries of the \( a_n \) and \( b_n \) layers, respectively.

With the use of the continuity conditions for the wave function \( \Psi_{a,b}(x) \) and its derivative at the boundaries \( x = x_a(n+1) \) one can obtain the transfer relation for the amplitudes \( A^\pm_{n+1} \) and \( A^\pm_n \) of two adjacent cells,
\[ \begin{pmatrix} A^+_n \\ A^-_n \end{pmatrix} = \begin{pmatrix} Q_{11}(n) & Q_{12}(n) \\ Q_{21}(n) & Q_{22}(n) \end{pmatrix} \begin{pmatrix} A^+_n \\ A^-_n \end{pmatrix}. \quad (2.8) \]

The transfer matrix \( \hat{Q}(n) \) has the following elements,
\[ \begin{aligned} Q_{11}(n) &= [\cos(k_b d_b) + i \alpha_+ \sin(k_b d_b)] \exp[i k_b d_a(n)] \\ &= Q_{22}(n); \quad (2.9a) \\ Q_{12}(n) &= i \alpha_- \sin(k_b d_b) \exp[i k_b d_a(n)] \\ &= Q_{21}(n). \quad (2.9b) \end{aligned} \]

Here the asterisk stands for the complex conjugation, and we introduced the parameters \( \alpha_\pm \),
\[ \alpha_\pm = \frac{1}{2} \left( \frac{k_a}{k_b} \pm \frac{k_b}{k_a} \right), \quad \alpha_+^2 - \alpha_-^2 = 1. \quad (2.10) \]

The determinant of \( \hat{Q}(n) \) is equal to unit, \( \det \hat{Q}(n) = 1 \). Note that the transfer matrix \( \hat{Q}(n) \) differs from cell to cell only in the phase factor \( \exp[i k_a d_a(n)] \).

The transfer matrix equation for the array of \( N \) cells with or without positional disorder is
\[ \begin{pmatrix} A^+_N \\ A^-_N \end{pmatrix} = \hat{Q}^N \begin{pmatrix} A^+_1 \\ A^-_1 \end{pmatrix}, \quad (2.11) \]
where,
\[ \hat{Q}^N = \hat{Q}(N) \hat{Q}(N-1) \ldots \hat{Q}(n) \ldots \hat{Q}(2) \hat{Q}(1). \quad (2.12) \]

All matrices \( \hat{Q}(n) \) \( (n = 1, 2, \ldots, N) \) have the same form (2.9), only differing in the values of \( d_a(n) \). In our following numerical simulations and experimental set-up we have \( A^\pm_{N+1} = 0 \). Thus the transmittance of \( N \) cells is given by
\[ T_N \equiv |A^+_N/A^+_1|^2 = |Q^N_{11}|^2 = |S^N_{12}|^2, \quad (2.13) \]
where \( S^N_{12} \) is the scattering matrix element in the relation
\[ \begin{pmatrix} A^+_N \\ A^-_{N+1} \end{pmatrix} = \begin{pmatrix} S^N_{11} & S^N_{12} \\ S^N_{21} & S^N_{22} \end{pmatrix} \begin{pmatrix} A^+_1 \\ A^-_{N+1} \end{pmatrix}. \quad (2.14) \]

In the case of no disorder, \( \eta(n) = 0 \), the length of \( a \)-layer does not depend on the cell number \( n \), \( d_a(n) = d_a \). Therefore, the unperturbed transfer matrix \( \hat{Q}^{(0)} \) is described by Eq. (2.9) with \( d_a(n) \) replaced by the constant length \( d_a \). As is known (see, e.g., Ref. 1), the transmission through \( N \) identical cells is expressed in closed form as
\[ T^{(0)}_N = \frac{1}{1 + |Q^{(0)}_{12} \sin(N \nu d)/\sin(\nu d)|^2} = |S^{(0)}_{12}|^2, \quad (2.15) \]
where \( \kappa \) is the Bloch wave number defined by the dispersion relation

\[
\cos(\kappa d) = \cos(k_a d_a) \cos(k_b d_b) - \alpha_+ \sin(k_a d_a) \sin(k_b d_b). \tag{2.16}
\]

Expression (2.15) indicates that the transmission is perfect \( (S_{12}^{(0)N} = 1) \) for all \( N \) where \( Q_{12}^{(0)} = 0 \) or when \( \sin(N \kappa d)/\sin(\kappa d) = 0 \). The former occurs at the Fabry-Perot resonances \( k_b d_b = n\pi \) in the \( b \)-layers. The latter produces \( N - 1 \) Fabry-Perot oscillations in each spectral band, associated with the total system length \( Nd \). We shall refer to the resonances \( k_b d_b = n\pi \) as “teflon resonances” since in the experiment the \( b \)-slabs are made of teflon, whereas the \( a \)-slabs are just air.

### III. EXPERIMENTAL SETUP

In Fig. 2 we show the experimental setup of a microwave ring guide of height \( h = 1 \) cm, width \( w = 2 \) cm and perimeter \( P = 234 \) cm. The waveguide consists of \( N = 26 \) cells, where the \( b \)-layers are pieces of teflon of length \( d_b = 4.078 \) cm and the refractive index \( n_b = \sqrt{2.08} \). Two electric dipole antennas connect to a network analyzer. Also shown are the two carbon pieces used to absorb the electric field at both ends of the waveguide and thus mimic two infinite leads connected to each side of the array of teflon pieces and air segments. The frequency range is 7.5 to 15 GHz, corresponding to wave lengths from 4 to 2 cm. This arrangement has been used to study the transport effects of single impurities in the photonic Kronig-Penny model [12]. Earlier, in an analogous model with metallic screws instead of teflon pieces, the microwave realization of the Hofstadter butterfly [13] was studied. The same configuration (metallic screws) was used to investigate transport properties of on-site correlated disorder [14, 15, 16].

Obviously this waveguide is not rectilinear; in the experiment the teflon pieces and air segments are not perfect parallelepipeds: one side is longer than the other by 5 percent. However since the perimeter (234 cm) is much larger than the wave length of the electric field even in the regime of the first mode, it is expected that the rectilinear model is a good approximation. In fact, as shown in detail in [12], a good quantitative agreement is found by defining an effective length of the teflon pieces and the air segments that are 1.95 percent larger than the smaller side of the teflon pieces and air segments. For example, the length of the smaller (inner) length of the teflon pieces used in the experiments reported here is 4 cm, so the effective width \( d_a \) we use in our calculations is 4.078 cm. We remark that this value is found by best fitting and is the only fitting, good for the whole frequency range of all our results presented here.

In Fig. 3 curve (a), we plot the experimentally measured value of \( |S_{12}^{(0)N}| \) (in what follows, the transmission spectrum) for the 26-cells periodic array. Curve (c) is the transmission spectrum \( |S_{12}^{(0)N}| \), calculated according to Eqs. (2.15), (2.16) and unperturbed Eq. (2.2). Note that the experimental transmission spectrum is about 1/5 of the theoretical one. This decrease of the signal is due to absorption by the metal walls of the waveguide. The value of the transmission spectrum is roughly constant over the frequency range, and can be taken into account phenomenologically by introducing an absorption factor. However, it is not the purpose of our work here to study the absorption effects. Our question is the global frequency dependence of the transmission spectrum on the frequency, giving us a possibility to reveal resonance effects and the role of disorder. Note that the band structure of the spectrum remains practically the same in spite of a strong absorption (see discussion in Ref. [13, 15, 16]).

Inspection of the transmission spectrum \( |S_{12}^{(0)N}| \) of the perfectly periodic array, Fig. 3 curve (c), demonstrates the effect of the teflon resonances on the transmission bands. We see two types of bands; namely, bands 1, 2, 4, and 6 show the \( N - 1 = 25 \) oscillations mentioned above, whereas bands 3, 5 and 7 are flat \( (|S_{12}^{(0)N}| \approx 1) \) around the resonance since \( |Q_{12}^{(0)}| \) is zero at the resonance, and very small in some neighborhood around it. For future reference we shall refer to the second type of bands as the “resonance bands”. Clearly, these bands disappear when \( d_a \rightarrow 0 \), turning into those with \( N - 1 \) oscillations occurring for delta-function potentials.

In accordance with Eq. (2.3), the first (lowest) mode in the air spacings \( (a \)-slabs) opens at the cut-off frequency \( \nu_a^{\text{cut}} = (c/2\omega n_a) = 7.5 \) GHz while in the teflon \( b \)-layers it opens at \( \nu_b^{\text{cut}} = (c/2\omega n_b) = 5.2 \) GHz, which is less than \( \nu_a^{\text{cut}} (n_a < n_b) \). On the other hand, Fig. 3 specifies the bottom of the first transmission band at \( \nu_1^{\text{cut}} = 7.387 \) GHz. So that the real cutoff \( \nu_1^{\text{cut}} \) per-
The experimental data. Inspection shows that indeed the assumed small error is enough to break the regularity of the band oscillations giving a better agreement with the experimental data of the supposedly regular array.

### IV. DISORDERED ARRAY

Let us now move to intentionally disordered arrays, with the lengths of all teflon bars constant, \( d_b(n) = d_b \) while the air layers have random lengths given by Eqs. (2.1), (2.2). In our experimental and numerical calculations, the sequence \( \eta(n) \) is an uncorrelated random function uniformly distributed in the interval \([-\sqrt{3}, \sqrt{3}]\), with unit variance.

**Apriori** it is not known how large a random deviation from the average value \( d_a \) should be to observe weak, medium or strong disorder effects in the transmission. We tentatively classify the amount of disorder by the value of the maximum deviation from the average length of the air spacing divided by the average length of the cells,

\[
\epsilon \equiv \frac{|d_a(n) - d_a|_{\text{max}}}{d} = \sigma \sqrt{3}/d. \tag{4.1}
\]

Table I shows the values of \( \epsilon \) we consider in this work, together with corresponding values of relative r.m.s. \( \sigma/d \) and \( (\sigma/d)^2 \). The latter quantity is needed to ease the comparison with analytical results obtained below. The case of \( \epsilon = 0.49 \times 10^{-2} \) was discussed above to simulate the errors in the experimental setup. We call it the case of extremely weak disorder. Similarly, \( \epsilon = 3.0 \times 10^{-2}, 12.3 \times 10^{-2} \) and \( 49.0 \times 10^{-2} \) cm, respectively, are called the weak, medium, and strong disordered cases.

Fig. 4 shows the transmission for the array of 26-cells with the positional disorder. Compared with the case of weak disorder (Fig. 3a), for medium disorder (Fig. 3b) only the first two gaps are clearly distinguishable; the third only partially. There is no trace of the \( N-1 \) oscillations in the transmission bands, and the second, fourth, and sixth transmission bands have decayed substantially. However, remnants of the resonance bands are still recognized, and so this can be considered the regime of medium disorder. For strong disorder (Fig. 3c) the first two transmission bands have disappeared. There is no longer any evidence of the band structure of the unperturbed array. But still the transmission spectrum is close to one in the vicinity of the teflon resonances.

**TABLE I:** Parameter values of random disorder. Here \( d = d_a + d_b = 2d_a \approx 8.16 \text{ cm} \)

| Case | \( \epsilon/10^{-2} \) | \( \sigma \) (\( \tau \)) \text{/cms} | \( (\sigma/d)^2 \) |
|------|------------------------|---------------------------------|------------------|
| very weak | 0.49 | 0.28 \times 10^{-2} | 8.0 \times 10^{-6} |
| weak | 3.00 | 1.77 \times 10^{-2} | 3.0 \times 10^{-4} |
| medium | 12.30 | 7.07 \times 10^{-2} | 5.0 \times 10^{-5} |
| strong | 49.00 | 28.30 \times 10^{-2} | 8.0 \times 10^{-2} |
The results so far discussed pertain to the array of $N = 26$ cells so the question arises about the effects on larger arrays. Our experimental setup does not allow for the implementation of much larger arrays. However, given that the numerical simulations are in good correspondence with the experimental data for $N = 26$ cells, we now consider only numerically, and later analytically, larger arrays for the same three cases: weak, medium, and strong disorder. In Figs. 5a – 5c we plot $|S_{12}^N|$ for an array with $N = 100$ cells, for weak (Fig. 5a), medium (Fig. 5b), and strong (Fig. 5c) disorder. Similarly, in Figs. 5d – 5f we plot $|S_{12}^N|$ for an array with $N = 400$ cells, again for weak (Fig 5d), medium (Fig. 5e), and strong (Fig. 5f) disorder. Comparing, for example, Figs. 5a and 5d, corresponding to weak disorder for $N = 100$ and $N = 400$ cells, respectively, we see that the effect of increasing the size of the array, keeping the same amount of disorder, is to further decrease the transmission, consistent with the localization theory. However, this decrease occurs only away from the teflon resonance frequencies. For medium and strong disorder, transmission has decayed below $10^{-3}$ for most of the frequencies except around the teflon resonances. Thus the localization is not homogeneous at all: the teflon Fabry-Perot resonances strongly suppress the localization.

V. LOCALIZATION LENGTH

In this section we derive an analytical expression for the localization length $L_{\text{loc}}$, and relate it with the experimental data. In the case of weak positional disorder,$$ (k_a \sigma)^2 \ll 1, \quad (5.1) $$ an analytical expression for this quantity can be obtained as follows. First, we expand the transfer matrix $Q(n)$ defined by Eq. (2.9), up to quadratic terms in the perturbation parameter $k_a \sigma n(n)$,

$$ \hat{Q}(n) \approx \left(1 - \frac{(k_a \sigma)^2 n^2(n)}{2} \right) \hat{Q}^{(0)} + k_a \sigma n(n) \hat{Q}^{(1)}. \quad (5.2) $$

The unperturbed $\hat{Q}^{(0)}$ and first-order $\hat{Q}^{(1)}$ matrices are suitable to be presented in the form

$$ \hat{Q}^{(0)} = \begin{pmatrix} u & v^* \\ v & u^* \end{pmatrix}, \quad \hat{Q}^{(1)} = \begin{pmatrix} iu & -iv^* \\ iv & -iu^* \end{pmatrix}; \quad (5.3a) $$
FIG. 5: (color online) Transmission spectrum $|S_{12}^N|$ (solid curves) and $\exp(-l_{\text{loc}})$ (dashed curves) for $N = 100$ and $N = 400$: (a) to (c) $N=100$ cells; a) weak, b) medium and c) strong disorder. d) to f) $N=400$ cells; d) weak, e) medium and f) strong disorder.

$$u = [\cos(k_0d_0) + i\alpha_+ \sin(k_0d_0)] \exp(i k_0 d_0),$$

$$v = i \alpha_- \sin(k_0d_0) \exp(i k_0 d_0),$$

$$\det \hat{Q}^{(0)} = \det \hat{Q}^{(1)} = |u|^2 - |v|^2 = 1.$$ 

Also, it is useful for further calculations to take into account that the real and imaginary parts, $u_r \equiv \text{Re} u$ and $u_i \equiv \text{Im} u$, of the matrix element $u$ can be expressed as

$$u_r = \cos(\kappa d), \quad u_r^2 = \sin^2(\kappa d) + |v|^2.$$ 

The first equality is identical to the dispersion relation (2.16), while the second one is a direct consequence of the matrix unimodularity (5.3d).

In order to extract the effects that are solely due to disorder, it is conventional to perform a canonical transformation to the Bloch normal-mode representation in the transfer relation (2.8),

$$\left( \begin{array}{c} A^+_{n+1} \\ A^-_{n+1} \end{array} \right) = \hat{P} \hat{Q} \hat{P}^{-1} \left( \begin{array}{c} A^+_{n} \\ A^-_{n} \end{array} \right),$$

$$\left( \begin{array}{c} A^+_{n} \\ A^-_{n} \end{array} \right) = \hat{P} \left( \begin{array}{c} A^+_{n} \\ A^-_{n} \end{array} \right).$$ 

The transformation matrix $\hat{P}$ is specified in such a manner to make the unperturbed matrix $\hat{Q}^{(0)}$ diagonal,

$$\hat{P} \hat{Q}^{(0)} \hat{P}^{-1} = \begin{pmatrix} \exp(+i\kappa d) & 0 \\ 0 & \exp(-i\kappa d) \end{pmatrix},$$ 

in complete accordance with the Floquet theorem [17], or the same, the Bloch condition [18]. The solution of the problem for the eigenvectors and eigenvalues of $\hat{Q}^{(0)}$ results in

$$\hat{P} = \begin{pmatrix} |v|/\beta_+ & -iv^*/\beta_- \\ iv/\beta_- & |v|/\beta_+ \end{pmatrix},$$

$$\beta_+^2 = 2\sqrt{1 - u_r^2 (u_i + \sqrt{1 - u_r^2})} = 2 \sin(\kappa d)[u_i \mp \sin(\kappa d)],$$

$$\beta_+^2 \beta_-^2 = 4|v|^2 \sin^2(\kappa d),$$

$$\det \hat{P} = \det \hat{P}^{-1} = |v|^2 (\beta_+^{-2} - \beta_-^{-2}) = 1.$$ 

After substituting Eqs. (5.2), (5.3a) and (5.7a) into the canonical transfer relation (5.5a), one can obtain the explicit perturbative recursion relations for the new com-
plex amplitudes,
\[
\tilde{A}_{n+1}^+ = \left[ 1 - \frac{k_a^2 \sigma^2 \eta^2(n)}{2} + \frac{ik_a \sigma \eta(n) u_i}{\sin(kd)} \right] \exp(i kd) \tilde{A}_{n}^+ - \frac{k_a \sigma \eta(n) v}{\sin(kd)} \exp(i kd) \tilde{A}_{n}^- \quad \text{for } n = 1, 2, \ldots.
\]
\[
\tilde{A}_{n+1}^- = \left[ 1 - \frac{k_a^2 \sigma^2 \eta^2(n)}{2} - \frac{ik_a \sigma \eta(n) u_i}{\sin(kd)} \right] \exp(-i kd) \tilde{A}_{n}^- - \frac{k_a \sigma \eta(n) v}{\sin(kd)} \exp(-i kd) \tilde{A}_{n}^+. \quad \text{(5.9)}
\]

Now one can see from these equations that one equation can be directly obtained from the other just by complex conjugation, if we suppose that \( \tilde{A}_n^+ = \tilde{A}_n^- \). In other words, it is convenient to seek the amplitudes \( \tilde{A}_n^\pm \) in terms of action-angle variables,
\[
\tilde{A}_n^\pm = R_n \exp(\pm i \theta_n). \quad \text{(5.10)}
\]

In order to derive the equation for the real amplitude \( R_n \), we multiply Eq. (5.8) by Eq. (5.9). Within the second order of approximation in the perturbation parameter \( k_a \sigma \eta(n) \), we have
\[
\frac{R_{n+1}^2}{R_n^2} = 1 + \frac{2k_a \sigma \eta(n)|v|}{\sin(kd)} \sin(2\theta_n + k_a d_n) - k_a^2 \sigma^2 \eta^2(n)
\]
\[
+ \frac{k_a^2 \sigma^2 \eta^2(n)}{\sin^2(kd)} \left[ u_i^2 + |v|^2 + 2u_i|v| \cos(2\theta_n + k_a d_n) \right]. \quad \text{(5.11)}
\]

The logarithm of Eq. (5.11), that determines the localization length, is also be expanded within the quadratic approximation,
\[
\ln \left( \frac{R_{n+1}^2}{R_n^2} \right) = \frac{2k_a \sigma \eta(n)|v|}{\sin(kd)} \sin(2\theta_n + k_a d_n)
\]
\[
+ \frac{2k_a^2 \sigma^2 \eta^2(n)|v|^2}{\sin^2(kd)} \left[ 1 - \sin^2(2\theta_n + k_a d_n) \right]
\]
\[
+ \frac{u_i}{|v|} \cos(2\theta_n + k_a d_n). \quad \text{(5.12)}
\]

Now we are in a position to write down the expression for the inverse localization length \( L_{\text{loc}}^{-1} \) that is known to be defined as follows [15],
\[
L_{\text{loc}}^{-1} = \frac{1}{2d} \ln \left( \frac{R_{n+1}^2}{R_n^2} \right). \quad \text{(5.13)}
\]

The average \( \langle \theta \rangle \) is performed over the disorder \( \eta(n) \) and the average \( \overline{\theta} \) is carried out over the rapid random phase \( \theta_n \). Within the accepted approximation and for uncorrelated disorder, see Eq. (5.22), we may regard the random quantities \( \eta(n) \) and \( \eta^2(n) \) to be uncorrelated with trigonometrical functions, containing the angle variable \( \theta_n \). Moreover, it can be shown (see, e.g., Ref. [19]) that the distribution of phase \( \theta_n \), within the first order of approximation in a weak disorder, is homogeneous (the corresponding distribution function is constant). Therefore, after averaging over \( \theta_n \) of Eq. (5.14), the term linear in \( \eta(n) \) and the last term in the brackets vanish and \( \sin^2(2\theta_n + k_a d_n) \) is replaced with \( 1/2 \). As a result, we get
\[
L_{\text{loc}}^{-1} = \frac{(k_a \sigma)^2 \alpha^2 \sin^2(k d_b)}{2d \sin^2(kd)} \quad \text{(5.14)}
\]

This expression is in complete correspondence with that obtained in Refs. [11,12] using a different approach, and reduces to Eq. (13) of Ref. [21] for the limiting case of delta-like barriers. The appearance of the term \( \sin^2(k d_b) \) in the numerator of Eq. (5.14) indicates that at frequencies obeying the teflon resonance condition \( k d_b = m \pi \), the localization length turns into infinity. That is, the random array becomes transparent and this is what is observed in the experimental and numerical transmission coefficient plotted in Figs. 3-5.

As is known, the localization length is directly related to the transmittance \( T_N \) for a finite array of the length \( L = N d \), according to the famous relation (\( \ln T_N = -2L/L_{\text{loc}} \)). In view of this relation and recalling that \( T_N = |S_{12}^N|^2 \), see Eq. (2.13), it is convenient to introduce the rescaled inverse localization length \( l_{\text{loc}}^{-1} \) as
\[
|\ln |S_{12}^N|| = -L/L_{\text{loc}} \equiv -l_{\text{loc}}^{-1}. \quad \text{(5.15)}
\]

According to Eq. (5.14), one can get
\[
l_{\text{loc}}^{-1} = (\sigma/d)^2 F(\nu) N. \quad \text{(5.16)}
\]

Here we introduced the form-factor
\[
F(\nu) = (k_a d)^2 \frac{\alpha^2 \sin^2(k d_b)}{2d \sin^2(kd)} \quad \text{(5.17)}
\]

that specifies the frequency profile of the inverse localization length and is determined only by the parameters of the underlying regular array. The rescaled inverse localization length \( l_{\text{loc}}^{-1} \), increases linearly with the number of cells \( N \) and quadratically with the amount of disorder \( \sigma/d \).

In Fig. 6a, we plot the form factor \( F(\nu) \) for the parameters of our system, \( d_b = 9b = 4.078 \) cm, \( n_a = 1, n_b = \sqrt{2.08} \). In Fig. 6b, we present the frequency dependence of \( l_{\text{loc}}^{-1} \) for an array of 400 cells for the cases of weak, medium, and strong disorder. These figures show that the attenuation of transmission is larger for the 6th order than for the 12th, see Eq. (2.12), it is convenient to introduce the rescaled inverse localization length \( l_{\text{loc}}^{-1} \), increases linearly with the number of cells \( N \) and quadratically with the amount of disorder \( \sigma/d \).

In order to compare our analytical results with the experimental and numerical ones, we define the theoretical value \( |S_{12}^N|_{\text{th}} \) of the transmission spectrum as follows,
\[
|S_{12}^N|_{\text{th}} \equiv \exp(-l_{\text{loc}}^{-1}) = \exp(-(\sigma/d)^2 F(\nu) N). \quad \text{(5.18)}
\]

This is what is plotted in green dashed lines in Fig. 4 for the cases of weak, medium and strong disorder in
fact. Indeed, in contrast with $N = 26$ (Fig. 4), for $N = 100$ and $N = 400$ (Fig. 5) we can see much better agreement between theory and experiments/numerics for longer chains. The quantity $\exp(-l_{loc}^{-1})$ gives a good fit to the numerical data even for the case of medium disorder and up to $\nu = 11.5$ GHz. Observe also that already for $N = 100$ the non-resonance bands have completely disappeared for the medium disorder case.

VI. SUMMARY

We studied the effects of the uncorrelated positional disorder in the 1D Kronig-Penney model paying main attention to the transmission for weak, medium, and strong disorder. We compare experimental data obtained in the microwave Kronig-Penney setup with the direct calculations based on the transfer matrix method. We show that in spite of a very strong absorption in the metallic walls of waveguide, the experimental frequency dependence of the transmission is similar to that obtained numerically by neglecting the absorption effect. In particular, the position of the $N - 1$ Fabry-Perot resonances, as well as the teflon resonances are in correspondence with numerical data. The structured similarity between experimental and numerical data can be explained by the different nature of the absorption in comparison with the resonance effects. Namely, the latter are due coherent effects, in contrast with the non-coherent nature of the absorption.

Another part of our study is related to the expression for the localization length obtained in the perturbative approach for an infinite chain of scattering barriers. Although in the experiment we have a relatively small number of 26 teflon barriers, we show that the expression for the localization length can be effectively used for the description of the transmission coefficient through the finite sample of barriers. To do this, we have used the relation between the transmission coefficient and localization length that involves the finite size of samples, and applied it to the experimental situation. As a result, we have found that the analytical expression for the transmission coefficient reproduces quite well the frequency dependence of experimentally obtained data. Note again, that the global correspondence between analytical and experimental results occurs in the presence of strong absorption that is non-avoidable experimentally. The observed correspondence between experimental and analytical results indicate that the expression for the localization length is a working quantity for finite samples, a fact that is not commonly used in applications.

The method we used to derive the expression for the localization length can be generalized to a more general case of the correlated disorder [14, 15, 20, 21, 22, 23, 24, 25, 26], for which anomalous effects in the transmission are predicted and experimentally observed. Our results can be applied to photonic crystals and electron superlattices, as well as to propagation of acoustic waves in disordered systems.
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