CONTAINMENT-DIVISION RINGS AND NEW CHARACTERIZATIONS OF DEDEKIND DOMAINS

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Abstract. We introduce a new class of commutative rings with unity, namely, the Containment-Division Rings (CDR-s). We show that this notion has a very exceptional origin, since it was essentially co-discovered with the qualitative help of a computer program (i.e. The Heterogeneous Tool Set (HETS)). Besides, we show that in a Noetherian setting, the CDR-s are just another way of describing Dedekind domains. Simultaneously, we see that for CDR-s, the Noetherian condition can be replaced by a weaker Divisor Chain Condition.

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Introduction

During the last decades we have seen very rapid advances in the development of artificial devices which allow us to verify and complete some quite intricate mathematical proofs e.g. the four-color theorem [1], [2], [10]; the Feit-Thompson odd order theorem [11]; and the Kepler Conjecture [12], [13]; among others.

Moreover, in the former cases the essential concepts behind the proofs were developed by mathematicians and a sophisticated computational support was required in order to handle the huge number of cases to be verified.

On the other hand, the outstanding work of S. Colon on automated theory formation [7] and the early versions of his program HR [6], were able to rediscover the number-theoretical notion of refactorable number [5].

Now, in [3] and [9], it was shown how a new valuable and interesting mathematical notion was co-invented with the help of the Heterogeneous Tool Set (HETS) [14].

Effectively, during the process of expressing in HETS the mathematical concept of a prime ideal of a commutative ring with unity as a formal blending (i.e. colimit) of the two notions of a meta-prime number and an ideal of a commutative ring with unity; a new condition appears involving the containment and the divisibility relations among the ideals of a ring. So, inspired by
this condition the notion of Containment-Division Ring (CDR) was proposed (see next section) and subsequently it was discovered that such a class of rings are strongly related with Dedekind Domains.

More specifically, in [3] and [9] a formalization of concepts as theories in a many-sorted first-order logic with proper signatures was used. Besides, the notion of conceptual blending of two input concepts with commonalities codified through a generic space was computed as the colimit of the corresponding ‘V’-diagram.

So, the sentence defining the notion of containment-division ring emerged when we consider the axiom defining the upside-down divisibility relation:

\[(\forall a, b \in \mathbb{Z})(a \mid b \leftrightarrow (\exists c \in \mathbb{Z})(a = c \cdot b)),\]

which belongs to one of the input spaces, i.e., the notion of prime element in a quasi-monoid \((\mathbb{Z}, \ast, 1)\) (i.e., \(\ast\) is a binary operation with neutral element 1). Now, the former condition was reinterpreted in the colimit (blend) concept computed by HETS using as a second concept, the notion of ideals over commutative rings with unity, including a sort \(G\) for the collection of all ideals. Effectively, the ‘conceptual’ morphisms between the generic and the input spaces induced syntactic replacements ‘merging’ (i.e., identifying) the sort denoting the quasi-monoid \(\mathbb{Z}\) with the sort denoting the collection of all ideals of a commutative ring with unity \(R\); the upside-down divisibility relation with the containment relation between ideals; and the binary operation in \(\mathbb{Z}\) with the product of ideals. Thus, the resulting axiom in the blended concept was of the form

\[(\forall a, b \in G)\left[\left(a \subseteq b\right) \leftrightarrow (\exists c \in G)(a = c \cdot_\ast b)\right],\]

where \(G\) denotes the set (sort) of ideals of \(R\), and \(\cdot_\ast\) denotes the product of ideals.

Now, the former axiom was exactly the kind of ‘surprising condition’, which we baptized as the containment-division condition, allowing us to discover a new class of commutative rings with unity that we will study in the next section.

1. Containment-Division Rings

Definition 1.1. Let \(R\) denote a commutative ring with unity. We say that \(R\) is a Containment-Division Ring (CDR), if for any two ideals \(I\) and \(J\), it holds that \(I \subseteq J\) if and only if \(J\) divides \(I\), i.e., if there exists an ideal \(H\) such that \(I = HJ\).

Now, it is straightforward to see that principal ideal domains are, in fact, CDR-s. Besides, in the setting of integer domains and as it was stated in a more informal context in [9], we will prove formally in this section that the Noetherian CDR-s are very close related to the Dedekind domains, i.e.,
integral domains with the additional property that every proper ideal can be written as a finite product of ideals \([4, \text{Theorem 37.1 and 37.8}].\) At the same time, we will see that for Noetherian CDR-s the ascending chain condition can be re-written in the form of a Divisor Chain Condition (DiCC), i.e., a commutative ring with unity fulfills the DiCC if for any chain of ideals \(I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots\) such that \(I_{j+1}\) divides \(I_j\), the chain is stationary.

The formal statement is the following:

**Theorem 1.2.** The following two conditions are equivalent for an integral domain \(R\):

1. \(R\) is a Dedekind domain.
2. \(R\) is a Noetherian CDR.
3. \(R\) is a CDR fulfilling the Divisor Chain Condition.

**Proof.** First let us prove that (1) is equivalent to (2): On the one hand, it is a clear fact that Dedekind domains are Noetherian \([4, \text{Theorem 37.1}].\) Furthermore, it is a classical result that Dedekind Domains fulfill the Containment-Division Condition (\([15, \text{Fundamental Theorem of OAK-s}]\)). In fact, if we consider two ideals \(I\) and \(J\) of \(R\), if \(I \subseteq J\), then by factoring each ideal, and then localizing on the primes appearing on their factorizations one obtains the desired division condition \([4, \text{Theorem 37.11}].\)

On the other hand, let us consider a proper ideal \(I\) of \(R\). The case where \(I\) is a prime ideal is clear. Otherwise, let \(P_1\) be a prime ideal of \(R\) such that \(I \subseteq P_1\). Then, there exists a proper ideal \(I_1\) such that \(I = I_1P_1\), because \(R\) is a CDR. If \(I_1\) is a prime ideal, then we can clearly express \(I\) as a product of two ideals. Otherwise, let us choose again a prime ideal \(P_2\) containing \(I_1\).

Hence, there is another proper ideal \(I_2\) such that \(I_1 = I_2P_2\). In the case that \(I_2\) is prime, we can express \(I = I_2P_2P_1\) as a finite product of prime ideals. Otherwise, we could continue inductively in a similar manner. If for some \(r\), the ideal \(I_r\) is prime, we can write \(I\) as a finite product of prime ideals. If not, one obtains an ascending chain of ideals

\[I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots\]

Because \(R\) is Noetherian, this sequence stops at some point (i.e., there exists some \(m \in \mathbb{N}\) such that for all \(i \geq m\), \(I_i = I_m\)). Moreover, \(I_m = I_{m+1}P_{m+1} = I_mP_{m+1}\) and so \(I_m = I_mP_{m+1} \subseteq I_m\), for all \(i \in \mathbb{N}\).

Therefore, \(I\) and \(I_m\) are contained in the intersection of the powers of \(I_m\), \(\cap_{i \geq 1} I_m^i\). Now, by Krull’s Intersection Theorem \([8, \text{Corollary 5.4}].\) this intersection must be zero. So, \(I = (0)\), which is a prime ideal. In conclusion, \(R\) is a Dedekind domain.

Finally, we verify that conditions (2) and (3) are equivalent:

In fact, due to the Containment-Division condition, any ascending chain of ideals \(I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots\) is a divisor chain of ideals and vice versa. So, the ascending chain condition defining Noetherian rings, which in general
implies the DiCC, turns out to be equivalent to the last one in the setting of CDR-s.

Remark 1.3. The Divisor Chain Condition involved in the third numeral above has also emerged as a natural consequence of replacing the containment relation with the divisibility one within the Ascending Chain Condition defining a Noetherian ring. So, this intermediate notion between the classes of Noetherian and Non-Noetherian rings can be seen as a natural human-program discovery, since the seminal idea of identifying, in a common setting, the former relations between ideals (i.e., containment and divisibility) was first (co-) suggested by HETS in the context of [3] and [9] as mentioned before. In conclusion, the former theorem is one of the very exceptional instances of a mathematical result whose involved concepts were co-discovered with the qualitative help of a computer program, and which has, simultaneously, enough mathematical value on its own.

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