Radii of the $\beta$—uniformly convex of order $\alpha$ of Lommel and Struve functions

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Abstract

In this paper, we determine the radii of $\beta$—uniformly convex of order $\alpha$ for six kinds of normalized Lommel and Struve functions of the first kind. In the cases considered the normalized Lommel and Struve functions are $\beta$—uniformly convex functions of order $\alpha$ on the determined disks.

Keywords: Lommel functions, Struve functions, Univalent functions, $\beta$—uniformly convex functions of order $\alpha$, Zeros of Lommel functions of the first kind, Zeros of Struve functions of the first kind.

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1 Introduction and Preliminaries

It is well known that the concepts of convexity, starlikeness, close-to-convexity and uniform convexity including necessary and sufficient conditions, have a long history as a part of geometric function theory. It is known that special functions, like Bessel, Struve and Lommel functions of the first kind have some beautiful geometric properties. Recently, the above geometric properties of the Bessel functions were investigated by some earlier results (see [1,5,12]). On the other hand the radii of convexity and starlikeness of the Struve and Lommel functions were studied by Baricz and his coauthors [8,10]. Motivated by the above developments in this topic, in this paper our aim is to give some new results for the radius of $\beta$—uniformly convex functions of order $\alpha$ of the normalized Struve and Lommel functions of the first kind. In the special cases of the parameters $\alpha$ and $\beta$ we can obtain some earlier results. The key tools in their proofs were some Mittag-Leffler
expansions of Lommel and Struve functions of the first kind, special properties of the zeros of these functions and their derivatives.

Let \( U(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \} \) denote the disk of radius \( r \) and center \( z_0 \). We let \( U(r) = U(0, r) \) and \( U = U(0, 1) = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \((a_n)_{n \geq 2}\) be a sequence of complex numbers with

\[
d = \limsup_{n \to \infty} |a_n|^\frac{1}{n} \geq 0, \text{ and } r_f = \frac{1}{d}.
\]

If \( d = 0 \) then \( r_f = +\infty \). As usual, with \( \mathcal{A} \) we denote the class of analytic functions \( f : U(r_f) \to \mathbb{C} \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

(1.1)

We say that a function \( f \) of the form (1.1) is convex if \( f \) is univalent and \( f(U(r)) \) is a convex domain in \( \mathbb{C} \). An analytic description of this definition is

\[
f \in \mathcal{A} \text{ is convex if and only if } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in U(r).
\]

The radius of convexity of the function \( f \) is defined by

\[
r^c_f = \sup \left\{ r \in (0, r_f) : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in U(r) \right\}.
\]

In the following we deal with the class of the uniformly convex functions. Goodman in [13] introduced the concept of uniform convexity for functions of the form (1.1). A function \( f \) is said to be uniformly convex in \( U(r) \) if \( f \) is of the form (1.1), it is convex, and has the property that for every circular arc \( \gamma \) contained in \( U(r) \), with center \( \varsigma \), also in \( U(r) \), the arc \( f(\gamma) \) is convex. In 1993, Rønning [21] determined necessary and sufficient conditions of analytic functions to be uniformly convex in the open unit disk, while in 2002 Ravichandran [20] also presented simpler criteria for uniform convexity. Rønning in [21] give an analytic description of the uniformly convex functions in the following theorem.

**Theorem 1.1** Let \( f \) be a function of the form \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) in the disk \( U(r) \). The function \( f \) is uniformly convex in the disk \( U(r) \) if and only if

\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U(r).
\]

(1.2)

The class of uniformly convex function denote by \( UC \). The radius of uniform convexity
is defined by
\[ r_{f}^{\text{uc}} = \sup \left\{ r \in (0, r_f) : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \frac{zf''(z)}{f'(z)} \right\}, \]

In 1997, Bharti, Parvatham and Swaminathan defined \( \beta \)-uniformly convex functions of order \( \alpha \) which is subclass of uniformly convex functions. A function \( f \in \mathcal{A} \) is said to be in the class of \( \beta \)-uniformly convex functions of order \( \alpha \), denoted by \( \beta - UC(\alpha) \), if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta \left| \frac{zf''(z)}{f'(z)} \right| + \alpha, \quad (1.3)
\]
where \( \beta \geq 0, \alpha \in [0, 1) \) (see [11]).

This class generalize various other classes which are worthy to mention here. For example, the class \( \beta - UC(0) = \beta - UC \) is the class of \( \beta \)-uniformly convex functions [14] (also see [15] and [16]) and \( 1 - UC(0) = UC \) is the class of uniformly convex functions defined by Goodman [13] and Ronning [21], respectively.

**Geometric Interpretation.** It is known that \( f \in \beta - UC(\alpha) \) if and only if \( 1 + \frac{zf''(z)}{f'(z)} \), respectively, takes all the values in the conic domain \( \mathcal{R}_{\beta,\alpha} \) which is included in the right half plane given by
\[
\mathcal{R}_{\beta,\alpha} := \left\{ w = u + iv \in \mathbb{C} : u > \beta \sqrt{(u-1)^2 + v^2} + \alpha, \beta \geq 0 \text{ and } \alpha \in [0, 1) \right\}. \quad (1.4)
\]

Denote by \( \mathcal{P}(P_{\beta,\alpha}) \), \( (\beta \geq 0, \ 0 \leq \alpha < 1) \) the family of functions \( p \), such that \( p \in \mathcal{P} \), where \( \mathcal{P} \) denotes the well-known class of Caratheodory functions and \( p < P_{\beta,\alpha} \) in \( U \). The function \( P_{\beta,\alpha} \) maps the unit disk conformally onto the domain \( \mathcal{R}_{\beta,\alpha} \) such that \( 1 \in \mathcal{R}_{\beta,\alpha} \) and \( \partial \mathcal{R}_{\beta,\alpha} \) is a curve defined by the equality
\[
\partial \mathcal{R}_{\beta,\alpha} := \left\{ w = u + iv \in \mathbb{C} : u^2 = \left( \beta \sqrt{(u-1)^2 + v^2} + \alpha \right)^2, \beta \geq 0 \text{ and } \alpha \in [0, 1) \right\}. \quad (1.5)
\]

From elementary computations we see that \( (1.5) \) represents conic sections symmetric about the real axis. Thus \( \mathcal{R}_{\beta,\alpha} \) is an elliptic domain for \( \beta > 1 \), a parabolic domain for \( \beta = 1 \), a hyperbolic domain for \( 0 < \beta < 1 \) and the right half plane \( u > \alpha \), for \( \beta = 0 \).

The radius of \( \beta \)-uniform convexity of order \( \alpha \) is defined by
\[
r_{f}^{\beta - uc(\alpha)} = \sup \left\{ r \in (0, r_f) : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta \left| \frac{zf''(z)}{f'(z)} \right| + \alpha, \beta \geq 0, \alpha \in [0, 1), \ z \in U(r) \right\}. \]

Our main aim to determine the radii of \( \beta \)-uniform convexity of order \( \alpha \) of Lommel and Struve functions.

In order to prove the main results, we need the following lemma given in [12].
Lemma 1.1 i. If \( a > b > r \geq |z| \), and \( \lambda \in [0, 1] \), then
\[
\left| \frac{z}{b - z} - \lambda \frac{z}{a - z} \right| \leq \frac{r}{b - r} - \lambda \frac{r}{a - r}.
\] (1.6)

The followings are very simple consequences of this inequality
\[
\Re \left( \frac{z}{b - z} - \lambda \frac{z}{a - z} \right) \leq \frac{r}{b - r} - \lambda \frac{r}{a - r}.
\] (1.7)

and
\[
\Re \left( \frac{z}{b - z} \right) \leq \left| \frac{z}{b - z} \right| \leq \frac{r}{b - r}.
\] (1.8)

ii. If \( b > a > r \geq |z| \), then
\[
\left| \frac{1}{(a + z)(b - z)} \right| \leq \frac{1}{(a - r)(b + r)}.
\] (1.9)

2 Main results

In this paper our aim is to consider two classical special functions, the Lommel function of the first kind \( s_{\mu, \nu} \) and the Struve function of the first kind \( H_\nu \). They are explicitly defined in terms of the hypergeometric function \( _1 F_2 \) by
\[
s_{\mu, \nu}(z) = \frac{z^{\mu+1}}{(\mu - \nu + 1)(\mu + \nu + 1)} \ _1 F_2 \left( 1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4} \right),
\]
\[
\frac{1}{2} (-\mu \pm \nu - 3) \notin \mathbb{N}
\]
and
\[
H_\nu(z) = \frac{\left( \frac{z}{2} \right)^{\nu+1}}{\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right)} \ _1 F_2 \left( 1; \frac{3}{2}, \nu + \frac{3}{2}; -\frac{z^2}{4} \right), \quad -\nu - \frac{3}{2} \notin \mathbb{N}.
\]

Observe that
\[
s_{\mu, \nu}(z) = 2^{\nu - 1} \sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right) \ H_\nu(z).
\]

A common feature of these functions is that they are solutions of inhomogeneous Bessel differential equations [24]. Indeed, the Lommel functions of the first kind \( s_{\mu, \nu} \) is a solution of
\[
z^2 w''(z) + zw'(z) + (z^2 - \nu^2) w(z) = z^{\mu+1}
\]
while the Struve function \( H_\nu \) obeys
\[
z^2 w''(z) + zw'(z) + (z^2 - \nu^2) w(z) = \frac{4 \left( \frac{z}{2} \right)^{\nu+1}}{\sqrt{\pi} \Gamma \left( \nu + \frac{1}{2} \right)}.
\]
We refer to Watson’s treatise [24] for comprehensive information about these functions and recall here briefly some contributions. In 1970 Steinig [22] investigated the real zeros of the Struve function $H_\nu$, while in 1972 he [23] examined the sign of $s_{\mu,\nu}(z)$ for real $\mu, \nu$ and positive $z$. He showed, among other things, that for $\mu < \frac{1}{2}$ the function $s_{\mu,\nu}$ has infinitely many changes of sign on $(0, \infty)$. In 2012 Koumandos and Lamprecht [17] obtained sharp estimates for the location of the zeros of $s_{\mu^2,\nu}$ when $\mu \in (0, 1)$. Turán type inequalities for $s_{\mu^2,\nu}$ were established in [6] while those for Struve function were proved in [9]. Geometric properties of the Lommel function $s_{\mu^2,\nu}$ and of the Struve function $H_\nu$ were obtained in [7,8,10,19,25]. Motivated by those results, in this paper we are interested on the radii of $\beta$–uniformly convex of order $\alpha$ of certain analytic functions related to the classical special functions under discussion. Since neither $s_{\mu^2,\nu}$ nor $H_\nu$ belongs to the class analytic functions, first we perform some natural normalizations, as in [8]. We define three functions related to $s_{\mu^2,\nu}$:

$$f_\mu(z) = f_{\mu^2,\frac{1}{2}}(z) = \left( \mu + 1 \right) s_{\mu^2,\frac{1}{2}}(z)$$

$$g_\mu(z) = g_{\mu^2,\frac{1}{2}}(z) = \mu + 1 \frac{z}{s_{\mu^2,\frac{1}{2}}}(z),$$

and

$$h_\mu(z) = h_{\mu^2,\frac{1}{2}}(z) = \mu + 1 \frac{z^3}{s_{\mu^2,\frac{1}{2}}}(\sqrt{z}).$$

Similarly, we associate with $H_\nu$ the functions

$$u_\nu(z) = \left( \sqrt{\pi} 2^\nu \Gamma \left( \nu + \frac{3}{2} \right) H_\nu(z) \right) \frac{1}{\sqrt{z}}$$

$$v_\nu(z) = \sqrt{\pi} 2^\nu \frac{z^{-\nu}}{\Gamma \left( \nu + \frac{3}{2} \right)} H_\nu(z),$$

and

$$w_\nu(z) = \sqrt{\pi} 2^\nu z \frac{1-\nu}{\Gamma \left( \nu + \frac{3}{2} \right)} H_\nu(\sqrt{z}).$$

Clearly the functions $f_\mu$, $g_\mu$, $h_\mu$, $u_\nu$, $v_\nu$ and $w_\nu$ belong to the class of analytic functions $A$. The main results in the present paper concern some interlacing properties of the zeros of Lommel and Struve functions and derivatives, as well as the exact values of the radii of $\beta$–uniform convexity of order $\alpha$ for these six functions, for some ranges of the parameters.

In the following, we present some lemmas given by Baricz and Yağmur [10], on the zeros of derivatives of Lommel and Struve functions of first kind. These lemmas is one of the key tools in the proof of our main results.

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Lemma 2.1 The zeros of the Lommel function \( s_{\mu - \frac{1}{2}, \frac{1}{2}} \) and its derivative interlace when \( \mu \in (-1, 1), \mu \neq 0 \). Moreover, the zeros \( \xi'_{\mu, n} \) of the function \( s'_{\mu - \frac{1}{2}, \frac{1}{2}} \) are all real and simple when \( \mu \in (-1, 1), \mu \neq 0 \).

Lemma 2.2 The zeros of the function \( H_\nu \) and its derivative interlace when \( |\nu| \leq \frac{1}{2} \). Moreover, the zeros \( h'_{\nu, n} \) of the function \( H'_\nu \) are all real and simple when \( |\nu| \leq \frac{1}{2} \).

Now, the first main result of this section presents the radii of \( \beta \)-uniform convexity of order \( \alpha \) of functions \( f_\mu, g_\mu \) and \( h_\mu \).

Theorem 2.1 Let \( \mu \in (-1, 1), \mu \neq 0 \) and suppose that \( \mu \neq -\frac{1}{2} \). Then the radius of \( \beta \)-uniform convexity of order \( \alpha \) of the function \( f_\mu \) is the smallest positive root of the equation

\[
(1 - \alpha) + (1 + \beta) \left( \frac{r s''_{\mu - \frac{1}{2}, \frac{1}{2}}(r)}{s'_{\mu - \frac{1}{2}, \frac{1}{2}}(r)} + \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \frac{r s'_{\mu - \frac{1}{2}, \frac{1}{2}}(r)}{s_{\mu - \frac{1}{2}, \frac{1}{2}}(r)} \right) = 0.
\]

Moreover \( r^{-uc(\alpha)} f_\mu < r^{c}_f, \xi'_{\mu, 1} < \xi_{\mu, 1} \), where \( \xi_{\mu, 1} \) and \( \xi'_{\mu, 1} \) denote the first positive zeros of \( s_{\mu - \frac{1}{2}, \frac{1}{2}} \) and \( s'_{\mu - \frac{1}{2}, \frac{1}{2}} \), respectively and \( r^{c}_f \) is the radius of convexity of the function \( f_\mu \).

Proof. In [10], authors proved the Mittag-Leffler expansions of \( s_{\mu - \frac{1}{2}, \frac{1}{2}}(z) \) and \( s'_{\mu - \frac{1}{2}, \frac{1}{2}}(z) \) as follows:

\[
s_{\mu - \frac{1}{2}, \frac{1}{2}}(z) = \frac{z^{\mu + \frac{1}{2}}}{\mu(\mu + 1)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{\xi_{\mu, n}^2} \right) \tag{2.1}
\]

and

\[
s'_{\mu - \frac{1}{2}, \frac{1}{2}}(z) = \frac{(\mu + \frac{1}{2}) z^{\mu - \frac{1}{2}}}{\mu(\mu + 1)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{\xi_{\mu, n}^2} \right) \tag{2.2}
\]

where \( \xi_{\mu, n} \) and \( \xi'_{\mu, n} \) denote the \( n \)-th positive roots of \( s_{\mu - \frac{1}{2}, \frac{1}{2}} \) and \( s'_{\mu - \frac{1}{2}, \frac{1}{2}} \), respectively. Observe also that

\[
1 + \frac{z f''_\mu(z)}{f'_\mu(z)} = 1 + \frac{z s''_{\mu - \frac{1}{2}, \frac{1}{2}}(z)}{s'_{\mu - \frac{1}{2}, \frac{1}{2}}(z)} + \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \frac{z s'_{\mu - \frac{1}{2}, \frac{1}{2}}(z)}{s_{\mu - \frac{1}{2}, \frac{1}{2}}(z)}. \tag{2.3}
\]

Thus, from (2.1), (2.2) and (2.3), we have

\[
1 + \frac{z f''_\mu(z)}{f'_\mu(z)} = 1 - \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \frac{2 z^2}{\xi_{\mu, n}^2} \sum_{n \geq 1} \frac{2 z^2}{\xi_{\mu, n}^2} = \sum_{n \geq 1} \frac{2 z^2}{\xi_{\mu, n}^2} - \sum_{n \geq 1} \frac{2 z^2}{\xi_{\mu, n}^2}.
\]

Now, the proof will be presented in three cases by considering the intervals of \( \mu \).

Firstly, suppose that \( \mu \in \left( 0, \frac{1}{2} \right) \). Since \( \frac{1}{\mu + \frac{1}{2}} - 1 \geq 0 \), inequality (1.8) implies for \( |z| < r < \)
\( \xi_{\mu,1} < \xi_{\mu,1} \)

\[ \Re \left( 1 + \frac{zf''(z)}{f''(z)} \right) = 1 - \sum_{n \geq 1} \Re \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} - \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \Re \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \right) \]  \( (2.4) \)

\[ \geq 1 - \sum_{n \geq 1} \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \]

\[ = 1 + \frac{r f''(r)}{f''(r)}. \]

On the other hand, if in the second part of inequality \( (1.8) \) we replace \( z \) by \( z^2 \) and \( b \) by \( \xi_{\mu,n} \) and \( \xi_{\mu,n} \), respectively, then it follows that

\[ \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \leq \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \quad \text{and} \quad \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \leq \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \]

provided that \(|z| \leq r < \xi'_{\mu,1} < \xi_{\mu,1}\). These two inequalities and the conditions \( \frac{1}{\mu + \frac{1}{2}} - 1 \geq 0 \) and \( \beta \geq 0 \), imply that

\[ \beta \left| \frac{zf''(z)}{f''(z)} \right| = \beta \sum_{n \geq 1} \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} + \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \] \( (2.5) \)

\[ \leq \beta \sum_{n \geq 1} \left| \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right| + \beta \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \left| \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right| \]

\[ \leq \beta \sum_{n \geq 1} \left( \frac{2r^2}{\xi_{\mu,n}^2 - r^2} + \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \right) = -\beta \frac{r f''(r)}{f''(r)}. \]

From \( (2.4) \) and \( (2.5) \) we infer

\[ \Re \left( 1 + \frac{zf''(z)}{f''(z)} \right) - \beta \left| \frac{zf''(z)}{f''(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) \frac{r f''(r)}{f''(r)}, \] \( (2.6) \)

where \(|z| \leq r < \xi'_{\mu,1} \) and \( \alpha \in [0,1), \beta \geq 0 \).

In the second step we will prove that inequalities \( (2.4) \) and \( (2.5) \) hold in the case \( \mu \in \left( \frac{1}{2}, 1 \right) \), too. Indeed in the case \( \mu \in \left( \frac{1}{2}, 1 \right) \) the roots \( 0 < \xi'_{\mu,n} < \xi_{\mu,n} \) are real for every natural number \( n \). Moreover, inequality \( (1.8) \) implies that

\[ \Re \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \leq \frac{2r^2}{\xi_{\mu,n}^2 - r^2}, \quad |z| \leq r < \xi'_{\mu,1} < \xi_{\mu,1} \]

and

\[ \Re \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \leq \frac{2r^2}{\xi_{\mu,n}^2 - r^2}, \quad |z| \leq r < \xi'_{\mu,1} < \xi_{\mu,1}. \]
Putting $\lambda = 1 - \frac{1}{\mu + \frac{1}{2}}$ inequality (1.7) implies
\[ \Re \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \leq \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2r^2}{\xi_{\mu,n}^2 - r^2}, \]
for $|z| \leq r < \xi_{\mu,1}' < \xi_{\mu,1}$, and we get
\[ \Re \left( 1 + \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right) = 1 - \sum_{n \geq 1} \Re \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \geq 1 - \sum_{n \geq 1} \left( \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \right) = 1 + \frac{rf''_{\mu}(r)}{f'_{\mu}(r)}. \]
Now, if in the inequality (1.6) we replace $z$ by $z^2$ and $b$ by $\xi_{\mu,n}$ and $\xi_{\mu,n}$ we again put $\lambda = 1 - \frac{1}{\mu + \frac{1}{2}}$, it follows that
\[ \left| \frac{2z^2}{\xi_{\mu,n}^2 - z^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right| \leq \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2r^2}{\xi_{\mu,n}^2 - r^2}, \]
provided that $|z| \leq r < \xi_{\mu,1}' < \xi_{\mu,1}$. Thus, for $\beta \geq 0$ we obtain
\[ \beta \left| \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right| = \beta \sum_{n \geq 1} \left( \frac{2z^2}{\xi_{\mu,n}^2 - z^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2z^2}{\xi_{\mu,n}^2 - z^2} \right) \leq \beta \sum_{n \geq 1} \left( \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \right) \leq \beta \sum_{n \geq 1} \left( \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2r^2}{\xi_{\mu,n}^2 - r^2} \right) = -\beta \frac{rf''_{\mu}(r)}{f'_{\mu}(r)}. \]
Finally the following inequality be inferred from (2.7) and (2.8) for $\mu \in (\frac{1}{2}, 1)$
\[ \Re \left( 1 + \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right) - \beta \left| \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) \frac{rf''_{\mu}(r)}{f'_{\mu}(r)}, \]
where $|z| \leq r < \xi_{\mu,1}'$ and $\alpha \in [0, 1)$, $\beta \geq 0$.
Equality holds (2.7) if and only if $z = r$. Thus it follows that
\[ \inf_{|z| < r} \left[ \Re \left( 1 + \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right) - \beta \left| \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right| - \alpha \right] = 1 - \alpha + (1 + \beta) \frac{rf''_{\mu}(r)}{f'_{\mu}(r)}, \]
for $r \in (0, \xi_{\mu,1})$, $\alpha \in [0, 1)$, $\beta \geq 0$ and $\mu \in (0, 1)$. 
The mapping \( \psi_{\mu} : (0, \xi_{\mu,1}'') \to \mathbb{R} \) defined by

\[
\psi_{\mu}(r) = 1 - \alpha + (1 + \beta) \frac{r f''_{\mu}(r)}{f'_{\mu}(r)} = 1 - \alpha - (1 + \beta) \sum_{n \geq 1} \left( \frac{2r^2}{\xi_{\mu,n}^2 - r^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{2r^2}{\xi_{\mu,n}^2-r^2} \right)
\]

is strictly decreasing for all \( \mu \in (0,1) \) and \( \alpha \in [0, 1) \), \( \beta \geq 0 \). Namely, we obtain

\[
\psi'_{\mu}(r) = -(1 + \beta) \sum_{n \geq 1} \left( \frac{4r^2 \xi_{\mu,n}^2}{(\xi_{\mu,n}^2 - r^2)^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{4r^2 \xi_{\mu,n}^2}{(\xi_{\mu,n}^2 - r^2)^2} \right)
\]

\[
< (1 + \beta) \sum_{n \geq 1} \left( \frac{4r^2 \xi_{\mu,n}^2}{(\xi_{\mu,n}^2 - r^2)^2} - \frac{4r^2 \xi_{\mu,n}^2}{(\xi_{\mu,n}^2 - r^2)^2} \right) < 0
\]

for \( \mu \in \left( \frac{1}{2}, 1 \right), r \in (0, \xi_{\mu,1}'') \) and \( \beta \geq 0 \). Here we used again that the zeros \( \xi_{\mu,n} \) and \( \xi_{\mu,n}' \) interlace, and for all \( n \in \mathbb{N}, \mu \in (0,1) \) and \( r < \sqrt{\xi_{\mu,n} \xi_{\mu,n}'} \) we have that

\[
\xi_{\mu,n}^2 \left( \xi_{\mu,n}'' - r^2 \right)^2 < \xi_{\mu,n}^2 \left( \xi_{\mu,n}'' - r^2 \right)^2.
\]

Let now \( \mu \in (0, \frac{1}{2}] \) and \( r > 0 \). Thus the following inequality

\[
\psi'_{\mu}(r) = -(1 + \beta) \sum_{n \geq 1} \left( \frac{4r^2 \xi_{\mu,n}^2}{(\xi_{\mu,n}^2 - r^2)^2} - \left( 1 - \frac{1}{\mu + \frac{1}{2}} \right) \frac{4r^2 \xi_{\mu,n}^2}{(\xi_{\mu,n}^2 - r^2)^2} \right) < 0
\]

is satisfy and thus \( \psi_{\mu} \) is indeed strictly decreasing for all \( \mu \in (0, 1) \) and \( \beta \geq 0 \).

Now, since \( \lim_{r \to 0} \psi_{\mu}(r) = 1 \) and \( \lim_{r \to \xi_{\mu,1}''} \psi_{\mu}(r) = -\infty \), in wiew of the minimum principle for harmonic functions it follows that for \( \mu \in (0,1) \) and \( z \in U(r_{f_{\mu}}^{\beta-uc(\alpha)}) \) we have

\[
\Re \left( 1 + \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right) - \beta \left| \frac{zf''_{\mu}(z)}{f'_{\mu}(z)} \right| - \alpha > 0 \quad (2.10)
\]

if and only if \( r_{f_{\mu}}^{\beta-uc(\alpha)} \) is the unique root of

\[
1 + (1 + \beta) \frac{r f''_{\mu}(r)}{f'_{\mu}(r)} = \alpha, \quad \alpha \in [0,1) \text{ and } \beta \geq 0.
\]

In the final step we will proved that inequality (2.10) also holds when \( \mu \in (-1,0) \).

In order to this, suppose that \( \mu \in (0,1) \) and since (see [6]) the function \( z \to \varphi_0(z) = \mu(\mu+1)z^{-\mu-\frac{1}{2}}s_{\mu-\frac{1}{2},\frac{1}{2}}(z) \) belongs to the Laguerre-Pólya class of entire functions, it satisfies the Laguerre inequality

\[
\left( \varphi_0^{(n)}(z) \right)^2 - \left( \varphi_0^{(n-1)}(z) \right) \left( \varphi_0^{(n+1)}(z) \right) > 0,
\]
where \( \mu \in (0, 1) \) and \( z \in \mathbb{R} \).

Substituting \( \mu \) by \( \mu - 1 \), \( \varphi_0 \) by the function \( \varphi_1(z) = \mu(\mu - 1)z^{-\mu + \frac{1}{2}}s_{\mu - \frac{1}{2}}(z) \) and taking into account that the \( n \)th positive zeros of \( \varphi_1 \) and \( \varphi'_1 \), denoted by \( \zeta_{\mu,n} \) and \( \zeta'_{\mu,n} \), interlace, since \( \varphi_1 \) belongs also to the Laguerre-Pólya class of entire functions (see [6]). It is worth mentioning that

\[
\Re \left( 1 + \frac{zf''_{\mu-1}(z)}{f'_{\mu-1}(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{2r^2}{s_{\mu,n}^{-\frac{1}{2}} - r^2} - \left( \frac{1}{\mu - \frac{1}{2}} - 1 \right) \sum_{n \geq 1} \frac{2r^2}{s_{\mu,n}^{-\frac{1}{2}} - r^2} \tag{2.12}
\]

and

\[
\beta \left| \frac{zf''_{\mu-1}(z)}{f'_{\mu-1}(z)} \right| \leq \beta \sum_{n \geq 1} \left( \frac{2r^2}{s_{\mu,n}^{-\frac{1}{2}} - r^2} + \left( \frac{1}{\mu - \frac{1}{2}} - 1 \right) \frac{2r^2}{s_{\mu,n}^{-\frac{1}{2}} - r^2} \right) \tag{2.13}
\]

holds for \( \mu \in (0, 1) \), \( \beta \geq 0 \) and \( \mu \neq \frac{1}{2} \). In this case we use again minimum principle for harmonic functions to ensure that (2.10) is valid for \( \mu - 1 \) instead of \( \mu \). Consequently, replacing \( \mu \) by \( \mu + 1 \), the equation (2.11) is satisfy for \( \mu \in (-1, 0) \), \( \alpha \in [0, 1) \) and \( \beta \geq 0 \). Thus the proof is complete. \( \blacksquare \)

As a result of the Theorem 2.1, the following corollary is obtained by taking \( \alpha = 0 \) ve \( \beta = 1 \).

**Corollary 2.1** Let \( \mu \in (-1, 1) \), \( \mu \neq 0 \) and suppose that \( \mu \neq -\frac{1}{2} \). Then the radius of uniform convexity of the function \( f_\mu \) is the smallest positive root of the equation

\[
1 + 2 \left( \frac{rs'_{\mu-\frac{1}{2}}(r)}{s_{\mu-\frac{1}{2}}(r)} + \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \frac{rs'_{\mu-\frac{1}{2}}(r)}{s_{\mu-\frac{1}{2}}(r)} \right) = 0.
\]

Moreover \( r_{f_\mu}^{uc} < r_{f_\mu}^c < \xi_{\mu,1} < \xi_{\mu,1}', \) where \( \xi_{\mu,1} \) and \( \xi'_{\mu,1} \) denote the first positive zeros of \( s_{\mu-\frac{1}{2}} \) and \( s'_{\mu-\frac{1}{2}} \), respectively and \( r_{f_\mu}^c \) is the radius of convexity of the function \( f_\mu \).
The graph of the function $r \mapsto 1 + 2 \left( \frac{r s'_{\mu - \frac{1}{2}}(r)}{s'_{\mu - \frac{1}{2}}(r)} + \left( \frac{1}{\mu + \frac{1}{2}} - 1 \right) \frac{r s''_{\mu - \frac{1}{2}}(r)}{s'_{\mu - \frac{1}{2}}(r)} \right)$

for $\mu \in \{ -0.25, -0.2, 0.1, 0.3 \}$ on $[0, 0.9]$

**Theorem 2.2** Let $\mu \in (-1, 1)$, $\mu \neq 0$ and suppose that $\mu \neq -\frac{1}{2}$. Then the radius of $\beta$–uniform convexity of order $\alpha$ of the function $g_\mu$ is the smallest positive root of the equation

$$(1 - \alpha) - (1 + \beta) \left( \frac{1}{2} + \mu - r \frac{\frac{3}{2} - \mu) s'_{\mu - \frac{1}{2}}(r) + r s''_{\mu - \frac{1}{2}}(r)}{\mu - \frac{1}{2}} \right) = 0.$$  

Moreover $r_{g_{\mu}}^{\beta \text{–uc}(\alpha)} < r_{g_{\mu}}^{\alpha} < \gamma'_{\mu,1} < \xi_{\mu,1}$, where $\xi_{\mu,1}$ and $\gamma_{\mu,1}$ denote the first positive zeros of $s_{\mu - \frac{1}{2}}$ and $g'_{\mu}$, respectively.

**Proof.** Let $\xi_{\mu,n}$ and $\gamma_{\mu,n}$ denote the $n$-th positive root of $s_{\mu - \frac{1}{2}}$ and $g'_{\mu}$, respectively and the smallest positive root of $g'_{\mu}$ does not exceed the first positive root of $s_{\mu - \frac{1}{2}}$. In 

with the help of Hadamard’s Theorem [18, p.26], the following equality was proved:

$$1 + \frac{z g''_{\mu}(z)}{g'_{\mu}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{\gamma_{\mu,n}^2 - z^2}.$$  

By using inequality (1.8), for all $z \in U(\gamma_{\mu,1})$ we have the inequality

$$\Re \left( 1 + \frac{z g''_{\mu}(z)}{g'_{\mu}(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{2r^2}{\gamma_{\mu,n}^2 - r^2} \tag{2.14}$$

where $|z| = r$.

On the other hand, again by using inequality (1.8), for all $z \in U(\gamma_{\mu,1})$ and $\beta \geq 0$ we get the inequality

$$\beta \left| z \frac{g''_{\mu}(z)}{g'_{\mu}(z)} \right| = \beta \left| \sum_{n \geq 1} \frac{2z^2}{\gamma_{\mu,n}^2 - z^2} \right| \tag{2.15}$$

$$\leq \beta \sum_{n \geq 1} \left| \frac{2z^2}{\gamma_{\mu,n}^2 - z^2} \right|$$

$$\leq \beta \sum_{n \geq 1} \frac{2r^2}{\gamma_{\mu,n}^2 - r^2} = -\beta r g''_{\mu}(r).$$

Finally the following inequality be inferred from (2.14) and (2.15)

$$\Re \left( 1 + \frac{z g''_{\mu}(z)}{g'_{\mu}(z)} \right) - \beta \left| z \frac{g''_{\mu}(z)}{g'_{\mu}(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) \frac{r g''_{\mu}(r)}{g'_{\mu}(r)}, \quad \beta \geq 0, \alpha \in [0, 1), \tag{2.16}$$
where \( |z| = r \). Thus, for \( r \in (0, \gamma_{\mu,1}) \), \( \beta \geq 0 \) and \( \alpha \in [0, 1) \) we obtain

\[
\inf_{|z| < r} \left[ \Re \left( 1 + \frac{zg''_{\mu}(z)}{g'_{\mu}(z)} \right) - \beta \left| \frac{zg''_{\mu}(z)}{g'_{\mu}(z)} \right| - \alpha \right] = 1 - \alpha + (1 + \beta) \frac{rg''_{\mu}(r)}{g'_{\mu}(r)}.
\]

The mapping \( \Theta_{\mu} : (0, \gamma_{\mu,1}) \to \mathbb{R} \) defined by

\[
\Theta_{\mu}(r) = 1 - \alpha + (1 + \beta) \frac{rg''_{\mu}(r)}{g'_{\mu}(r)} = 1 - \alpha - (1 + \beta) \sum_{n \geq 1} \frac{2r^2}{\gamma_{\mu,n}^2 - r^2}
\]

is strictly decreasing since \( \lim_{r \searrow 0} \Theta_{\mu}(r) = 1 \) and \( \lim_{r \nearrow \gamma_{\mu,1}} \Theta_{\mu}(r) = -\infty \). As a result in view of the minimum principle for harmonic functions it follows that for \( \alpha \in [0, 1) \), \( \beta \geq 0 \) and \( z \in U(r_1) \) we have

\[
\Re \left( 1 + \frac{zg''_{\mu}(z)}{g'_{\mu}(z)} \right) - \beta \left| \frac{zg''_{\mu}(z)}{g'_{\mu}(z)} \right| - \alpha > 0
\]

if and only if \( r_1 \) is the unique root of

\[
1 + (1 + \beta) \frac{rg''_{\mu}(r)}{g'_{\mu}(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0
\]

situated in \((0, \gamma_{\mu,1})\). \( \blacksquare \)

As a result of the Theorem 2.2, the next corollary is obtained by taking \( \alpha = 0 \) and \( \beta = 1 \).

**Corollary 2.2** Let \( \mu \in (-1, 1) \), \( \mu \neq 0 \) and suppose that \( \mu \neq -\frac{1}{2} \). Then the radius of uniform convexity of the function \( g_{\mu} \) is the smallest positive root of the equation

\[
1 - 2 \left( \frac{3}{2} + \mu - r \frac{(\frac{3}{2} - \mu) s'_{\mu - \frac{1}{2}, \frac{1}{2}}(r) + rs''_{\mu - \frac{1}{2}, \frac{1}{2}}(r)}{(\frac{1}{2} - \mu) s'_{\mu - \frac{3}{2}, \frac{1}{2}}(r) + rs''_{\mu - \frac{3}{2}, \frac{1}{2}}(r)} \right) = 0.
\]

Moreover \( r_{uc}^{g_{\mu}} < r_{uc}^{g_{\mu}} < \gamma_{\mu,1}' < \xi_{\mu,1} \), where \( \xi_{\mu,1} \) and \( \gamma_{\mu,1} \) denote the first positive zeros of \( s_{\mu - \frac{1}{2}, \frac{1}{2}} \) and \( g'_{\mu} \), respectively.
The graph of the function $r \mapsto 1 - 2 \left( \frac{1}{2} + \mu - r \frac{(\frac{3}{2} - \mu)s_{\mu - \frac{1}{2}}^\prime((\sqrt{r})) + \sqrt{r}s_{\mu - \frac{1}{2}}''((\sqrt{r}))}{(\frac{3}{2} - \mu)s_{\mu - \frac{1}{2}}^\prime((\sqrt{r})) + \sqrt{r}s_{\mu - \frac{1}{2}}''((\sqrt{r}))} \right)$

for $\mu \in \{-0.25, -0.2, 0.1, 0.3\}$ on $[0, 0.9].$

**Theorem 2.3** Let $\mu \in (-1, 1)$, $\mu \neq 0$ and suppose that $\mu \neq -\frac{1}{2}$. Then the radius of $\beta$–uniform convexity of order $\alpha$ of the function $h_\mu$ is the smallest positive root of the equation

$$4(1 - \alpha) - (1 + \beta) \left( 1 + 2\mu - 2\sqrt{r} \frac{(\frac{3}{2} - \mu)s_{\mu - \frac{1}{2}}^\prime((\sqrt{r})) + \sqrt{r}s_{\mu - \frac{1}{2}}''((\sqrt{r}))}{(\frac{3}{2} - \mu)s_{\mu - \frac{1}{2}}^\prime((\sqrt{r})) + \sqrt{r}s_{\mu - \frac{1}{2}}''((\sqrt{r}))} \right) = 0.$$  

Moreover $\rho_{\beta-uc}^{c}(\alpha) < r^c_{\mu} < \delta_{\mu,1}^\prime < \xi_{\mu,1}$, where $\xi_{\mu,1}$ and $\delta_{\mu,1}$ denote the first positive zeros of $s_{\mu - \frac{1}{2}}'$ and $h_\mu'$, respectively.

**Proof.** Let $\xi_{\mu,n}$ and $\delta_{\mu,n}$ denote the $n$-th positive root of $s_{\mu - \frac{1}{2}}'$ and $h_\mu'$, respectively and the smallest positive root of $h_\mu'$ does not exceed the first positive root of $s_{\mu - \frac{1}{2}}'$. In [10] with the help of Hadamard’s Theorem [18, p.26], the following equality was proved:

$$1 + \frac{zh_\mu''(z)}{h_\mu'(z)} = 1 - \sum_{n \geq 1} \frac{z}{\delta_{\mu,n}^2 - z}.$$  

By using inequality (1.8), for all $z \in U(\delta_{\mu,1})$ we obtain the inequality

$$\Re \left( 1 + \frac{zh_\mu''(z)}{h_\mu'(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{\delta_{\mu,n}^2 - r}$$  \hspace{1cm} (2.17)

where $|z| = r$.

Moreover, again by using inequality (1.8), for all $z \in U(\delta_{\mu,1})$ and $\beta \geq 0$ we get the
inequality

\[
\begin{align*}
\beta \left| \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right| &= \beta \left| \sum_{n\geq 1} \frac{z}{\delta_{\mu,n}^2 - z} \right| \\
&\leq \beta \sum_{n\geq 1} \left| \frac{z}{\delta_{\mu,n}^2 - z} \right| \\
&\leq \beta \sum_{n\geq 1} \frac{r}{\delta_{\mu,n}^2 - r} = -\beta \frac{rh''_{\mu}(r)}{h'_{\mu}(r)}.
\end{align*}
\] (2.18)

As a result, from (2.17) and (2.18) we have

\[
\Re \left( 1 + \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right) - \beta \left| \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) \frac{rh''_{\mu}(r)}{h'_{\mu}(r)}
\] (2.19)

where \(|z| = r\). Thus, for \(r \in (0, \delta_{\mu,1})\), \(\beta \geq 0\) and \(\alpha \in [0, 1)\) we have

\[
\inf_{|z| < r} \left[ \Re \left( 1 + \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right) - \beta \left| \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right| - \alpha \right] = 1 - \alpha + (1 + \beta) \frac{rh''_{\mu}(r)}{h'_{\mu}(r)}.
\]

The mapping \(\Phi_\mu : (0, \delta_{\mu,1}) \rightarrow \mathbb{R}\) defined by

\[
\Phi_\mu(r) = 1 - \alpha + (1 + \beta) \frac{rh''_{\mu}(r)}{h'_{\mu}(r)} = 1 - \alpha - (1 + \beta) \sum_{n\geq 1} \frac{r}{\delta_{\mu,n}^2 - r}
\]

is strictly decreasing since \(\lim_{r \searrow 0} \Phi_\mu(r) = 1 > \alpha\) and \(\lim_{r \nearrow \delta_{\mu,1}} \Phi_\mu(r) = -\infty\). Consequently, in view of the minimum principle for harmonic functions it follows that for \(\alpha \in [0, 1), \beta \geq 0\) and \(z \in U(r_2)\) we have

\[
\Re \left( 1 + \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right) - \beta \left| \frac{zh''_{\mu}(z)}{h'_{\mu}(z)} \right| - \alpha > 0
\]

if and only if \(r_2\) is the unique root of

\[
1 + (1 + \beta) \frac{rh''_{\mu}(r)}{h'_{\mu}(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0.
\]

situated in \((0, \delta_{\mu,1})\). ■

As a result of the Theorem 2.3 the following corollary is obtained by taking \(\alpha = 0\) and \(\beta = 1\).

**Corollary 2.3** Let \(\mu \in (-1, 1), \mu \neq 0\) and suppose that \(\mu \neq -\frac{1}{2}\). Then the radius of
uniform convexity of the function $h_\mu$ is the smallest positive root of the equation

$$\frac{1}{2} - \mu + \sqrt{\frac{(\frac{5}{2} - \mu)s'_{\mu-\frac{1}{2}}(\sqrt{r}) + \sqrt{r}s''_{\mu-\frac{1}{2}}(\sqrt{r})}{(\frac{5}{2} - \mu)s_{\mu-\frac{1}{2}}(\sqrt{r}) + \sqrt{r}s'_{\mu-\frac{1}{2}}(\sqrt{r})}} = 0.$$  

Moreover $r_{hc} < r_{hc} < \delta'_{\mu,1} < \xi_{\mu,1}$, where $\xi_{\mu,1}$ and $\delta_{\mu,1}$ denote the first positive zeros of $s_{\mu-\frac{1}{2}}$ and $h'_{\mu}$, respectively.

The graph of the function $r \mapsto \frac{1}{2} - \mu + \sqrt{\frac{(\frac{5}{2} - \mu)s'_{\mu-\frac{1}{2}}(\sqrt{r}) + \sqrt{r}s''_{\mu-\frac{1}{2}}(\sqrt{r})}{(\frac{5}{2} - \mu)s_{\mu-\frac{1}{2}}(\sqrt{r}) + \sqrt{r}s'_{\mu-\frac{1}{2}}(\sqrt{r})}}$ for $\mu \in \{-0.25, -0.2, 0.1, 0.3\}$ on $[0, 2]$.

For $\mu = \frac{1}{3}$, Lommel functions defined in terms of the hypergeometric function $\, _1F_2$ as follows:

$$s_{\frac{1}{3}}(z) = \frac{100z^{4/5}}{39} \, _1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z^2}{4}\right).$$

Then, we have

$$f_{\frac{1}{m}}(z) = z \left[ \, _1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z^2}{4}\right) \right]^{5/4}, \quad g_{\frac{1}{m}}(z) = z \, _1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z^2}{4}\right)$$

and

$$h_{\frac{1}{m}}(z) = z \, _1F_2\left(1; \frac{23}{20}, \frac{33}{20}; -\frac{z}{4}\right).$$

We obtain the following results for the functions $f_{\frac{1}{m}}$, $g_{\frac{1}{m}}$, and $h_{\frac{1}{m}}$:

- $f_{\frac{1}{m}}(z) \in UC$ in the disk $U(r_1 = 0.6623)$,
- $g_{\frac{1}{m}}(z) \in UC$ in the disk $U(r_2 = 0.7376)$,
- $h_{\frac{1}{m}}(z) \in UC$ in the disk $U(r_3 = 1.4961)$,
where $r_1$, $r_2$ and $r_3$ is the smallest positive root of the equations given Corollary 2.1-2.3 for $\mu = \frac{1}{3}$.

Secondly, the other main result of this section presents the $\beta-$uniform convexity of order $\alpha$ of functions $u_\nu, v_\nu$ and $w_\nu$, related to Struve’s one. The first part of next theorem is an interesting of Lemma 2.2.

**Theorem 2.4** Let $|\nu| \leq \frac{1}{2}$ and $0 \leq \alpha < 1$. Then the radius of $\beta-$uniform convexity of order $\alpha$ of the function $u_\nu$ is the smallest positive root of the equation

$$(1 - \alpha) + (1 + \beta) \left( \frac{r H''_\nu(r)}{H'_\nu(r)} + \left( \frac{1}{\nu + 1} - 1 \right) \frac{r H'_\nu(r)}{H_\nu(r)} \right) = 0.$$

Moreover $r_{u_\nu}^{\beta-uc(\alpha)} < r_{u_\nu}^c < h_{\nu,1} < h_{\nu,1}$, where $h_{\nu,1}$ and $h_{\nu,1}'$ denote the first positive zeros of $H_\nu$ and $H'_\nu$, respectively.

**Proof.** We note that

$$1 + zu''_\nu(z) + u'_\nu(z) = 1 + z \frac{H''_\nu(z)}{H'_\nu(z)} + \left( \frac{1}{\nu + 1} - 1 \right) \frac{z H'_\nu(z)}{H_\nu(z)}$$

Using the Mittag-Leffler expansions of $H_\nu$ and $H'_\nu$ [10, Theorem 4] given by

$$H_\nu(z) = \frac{z^{\nu+1}}{\sqrt{\pi}\Gamma\left(\nu + \frac{3}{2}\right)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{h_{\nu,n}^2} \right)$$

and

$$H'_\nu(z) = \frac{(\nu + 1) z^\nu}{\sqrt{\pi} 2^{\nu-\frac{3}{2}} \Gamma\left(\nu + \frac{3}{2}\right)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{h'_{\nu,n}^2} \right)$$

where $h_{\nu,n}$ and $h'_{\nu,n}$ denote the $n$-th positive root of $H_\nu$ and $H'_\nu$, respectively. From (2.20) and (2.21), we obtain

$$\frac{z H''_\nu(z)}{H'_\nu(z)} = \nu + 1 - \sum_{n \geq 1} \frac{2z^2}{h_{\nu,n}^2 - z^2}, \quad \frac{z H'_\nu(z)}{H_\nu(z)} = \nu + 1 - \sum_{n \geq 1} \frac{2z^2}{h'_{\nu,n}^2 - z^2}.$$

Thus, we have

$$1 + \frac{z u''_\nu(z)}{u'_\nu(z)} = 1 - \left( \frac{1}{\nu + 1} - 1 \right) \sum_{n \geq 1} \frac{2z^2}{h_{\nu,n}^2 - z^2} - \sum_{n \geq 1} \frac{2z^2}{h'_{\nu,n}^2 - z^2}.$$

Now, the proof will be presented in two cases by considering the intervals of $\nu$. 

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Firstly, suppose that $\nu \in \left[ -\frac{1}{2}, 0 \right]$. Since $\frac{1}{\nu+1} - 1 \geq 0$, inequality (1.8) implies

$$\Re \left( 1 + \frac{z u_{\nu}''(z)}{u_{\nu}'(z)} \right) = 1 - \sum_{n \geq 1} \Re \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) - \left( \frac{1}{\nu + 1} - 1 \right) \sum_{n \geq 1} \Re \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \tag{2.22}$$

$$\geq 1 - \sum_{n \geq 1} \frac{2r^2}{h_{\nu,n}^2 - r^2} - \left( \frac{1}{\nu + 1} - 1 \right) \sum_{n \geq 1} \frac{2r^2}{h_{\nu,n}^2 - r^2}$$

$$= 1 + \frac{ru_{\nu}''(r)}{u_{\nu}'(r)}.$$  

On the other hand, if in the second part of inequality (1.8) we replace $z$ by $z^2$ and by $h'_{\nu,1}$ and $h_{\nu,1}$, respectively, then it follows that

$$\left| \frac{2z^2}{h_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{h_{\nu,n}^2 - r^2} \quad \text{and} \quad \left| \frac{2z^2}{h_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{h_{\nu,n}^2 - r^2}$$

provided that $|z| \leq r < h'_{\nu,1} < h_{\nu,1}$. These two inequalities and the conditions $\frac{1}{\nu+1} - 1 \geq 0$ and $\beta \geq 0$, imply that

$$\beta \left| \frac{z u_{\nu}''(z)}{u_{\nu}'(z)} \right| = \beta \sum_{n \geq 1} \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} + \left( \frac{1}{\nu + 1} - 1 \right) \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \leq \beta \sum_{n \geq 1} \frac{2z^2}{h_{\nu,n}^2 - z^2} + \beta \left( \frac{1}{\nu + 1} - 1 \right) \sum_{n \geq 1} \frac{2z^2}{h_{\nu,n}^2 - z^2} \leq \beta \sum_{n \geq 1} \left( \frac{2r^2}{h_{\nu,n}^2 - r^2} + \left( \frac{1}{\nu + 1} - 1 \right) \frac{2r^2}{h_{\nu,n}^2 - r^2} \right) = -\beta \frac{ru_{\nu}''(r)}{u_{\nu}'(r)}. \tag{2.23}$$

From (2.22) and (2.23) we get

$$\Re \left( 1 + \frac{z u_{\nu}''(z)}{u_{\nu}'(z)} \right) - \beta \left| \frac{z u_{\nu}''(z)}{u_{\nu}'(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) \frac{ru_{\nu}''(r)}{u_{\nu}'(r)}, \tag{2.24}$$

where $|z| \leq r < h'_{\nu,1}$ and $\beta \geq 0, \alpha \in [0, 1)$.

Secondly, in the case $\nu \in \left[ 0, \frac{1}{2} \right]$ the roots $0 < h'_{\nu,1} < h_{\nu,1}$, are real for every natural number $n$. Moreover, inequality (1.8) implies that

$$\Re \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \leq \left| \frac{2z^2}{h_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{h_{\nu,n}^2 - r^2}, \quad |z| \leq r < h'_{\nu,1} < h_{\nu,1}$$

and

$$\Re \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \leq \left| \frac{2z^2}{h_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{h_{\nu,n}^2 - r^2}, \quad |z| \leq r < h'_{\nu,1} < h_{\nu,1}.$$
Putting $\lambda = 1 - \frac{1}{\nu+1}$, inequality (1.7) implies

$$
\Re \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \leq \frac{2r^2}{h_{\nu,n}^2 - r^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2r^2}{h_{\nu,n}^2 - r^2},
$$

for $|z| \leq r < h_{\nu,1}^\prime < h_{\nu,1}$, and we get

$$
\Re \left( 1 + \frac{zu''_{\nu}(z)}{u'_{\nu}(z)} \right) = 1 - \sum_{n \geq 1} \Re \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \geq 1 - \sum_{n \geq 1} \left( \frac{2r^2}{h_{\nu,n}^2 - r^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2r^2}{h_{\nu,n}^2 - r^2} \right) = 1 + \frac{ru''_{\nu}(r)}{u'_{\nu}(r)}.
$$

Now, if in the inequality (1.6) we replace $z$ by $z^2$ and we again put $\lambda = 1 - \frac{1}{\nu+1}$, it follows that

$$
\left| \frac{2z^2}{h_{\nu,n}^2 - z^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2z^2}{h_{\nu,n}^2 - z^2} \right| \leq \frac{2r^2}{h_{\nu,n}^2 - r^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2r^2}{h_{\nu,n}^2 - r^2},
$$

provided that $|z| \leq r < h_{\nu,1}^\prime < h_{\nu,1}$. Thus, for $\beta \geq 0$ we have

$$
\beta \left| \frac{zu''_{\nu}(z)}{u'_{\nu}(z)} \right| = \beta \left| \sum_{n \geq 1} \left( \frac{2z^2}{h_{\nu,n}^2 - z^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2z^2}{h_{\nu,n}^2 - z^2} \right) \right| \leq \beta \left| \sum_{n \geq 1} \left( \frac{2r^2}{h_{\nu,n}^2 - r^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2r^2}{h_{\nu,n}^2 - r^2} \right) \right| = -\beta RU''_{\nu}(r) / U'_{\nu}(r).
$$

As a result, the following inequality be inferred from (2.25) and (2.26) such as (2.22) and (2.23)

$$
\Re \left( 1 + \frac{zu''_{\nu}(z)}{u'_{\nu}(z)} \right) - \beta \left| \frac{zu''_{\nu}(z)}{u'_{\nu}(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) RU''_{\nu}(r) / U'_{\nu}(r),
$$

where $|z| \leq r < h_{\nu,1}^\prime$ and $\beta \geq 0, \alpha \in [0, 1)$.

Equality holds (2.24) and (2.27) if and only if $z = r$. Thus it follows that

$$
\inf_{|z| < r} \left[ \Re \left( 1 + \frac{zu''_{\nu}(z)}{u'_{\nu}(z)} \right) - \beta \left| \frac{zu''_{\nu}(z)}{u'_{\nu}(z)} \right| - \alpha \right] = 1 - \alpha + (1 + \beta) RU''_{\nu}(r) / U'_{\nu}(r),
$$

where $r \in (0, h_{\nu,1}^\prime)$ and $\beta \geq 0, \alpha \in [0, 1)$. 
The mapping $\psi_\nu : (0, h_{\nu,1}') \to \mathbb{R}$ defined by

$$\psi_\nu(r) = 1 + (1 + \beta) \frac{ru''_\nu(r)}{u'_\nu(r)} = 1 - (1 + \beta) \sum_{n \geq 1} \left( \frac{2r^2}{h_{\nu,n}'^2 - r^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{2r^2}{h_{\nu,n}^2 - r^2} \right)$$

is strictly decreasing for all $|\nu| \leq \frac{1}{2}$ and $\beta \geq 0$. Namely, we obtain

$$\psi'_\nu(r) = -(1 + \beta) \sum_{n \geq 1} \left( \frac{4rh_{\nu,n}^2}{(h_{\nu,n}'^2 - r^2)^2} - \left( 1 - \frac{1}{\nu + 1} \right) \frac{4rh_{\nu,n}^2}{(h_{\nu,n}^2 - r^2)^2} \right) < 0$$

for $\nu \in [0, \frac{1}{2}]$, $r \in (0, h_{\nu,1}')$ and $\beta \geq 0$. Here we used again that the zeros $h_{\nu,n}$ and $h_{\nu,n}'$ interlace, and for all $n \in \mathbb{N}$, $|\nu| \leq \frac{1}{2}$ and $r < \sqrt{h_{\nu,n}h_{\nu,n}'}$ we have that

$$h_{\nu,n}'^2 (h_{\nu,n}'^2 - r^2)^2 < h_{\nu,n}^2 (h_{\nu,n}^2 - r^2)^2.$$ 

Observe that when $\nu \in [\frac{1}{2}, 0]$ and $r > 0$ we have also that $\psi'_\nu(r) < 0$, and thus $\psi_\nu$ is indeed strictly decreasing for all $|\nu| \leq \frac{1}{2}$ and $\beta \geq 0$.

Now, since $\lim_{r \searrow 0} \psi_\nu(r) = 1$ and $\lim_{r \nearrow h_{\nu,1}'} \psi_\nu(r) = -\infty$, in view of the minimum principle for harmonic functions it follows that for $|\nu| \leq \frac{1}{2}$ and $z \in U(r_3)$ we get

$$\Re \left( 1 + \frac{zu''_\nu(z)}{u'_\nu(z)} \right) - \beta \left| \frac{zu'_\nu(z)}{u'_\nu(z)} \right| > \alpha.$$  \hspace{1cm} (2.28)

if and only if $r_3$ is the unique root of

$$1 + (1 + \beta) \frac{ru''_\nu(r)}{u'_\nu(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0.$$

situates in $(0, h_{\nu,1}')$. 

As a result of the Theorem 2.4, the next corollary is obtained by taking $\alpha = 0$ ve $\beta = 1$.

**Corollary 2.4** Let $|\nu| \leq \frac{1}{2}$. Then the radius of uniform convexity of the function $u_\nu$ is the smallest positive root of the equation

$$1 + 2 \left( \frac{rH''_\nu(r)}{H'_\nu(r)} + \left( \frac{1}{\nu + 1} - 1 \right) \frac{rH'_\nu(r)}{H'_\nu(r)} \right) = 0.$$ 

Moreover $r_{u_\nu}^{wc} < r_{u_\nu}^{cc} < h_{\nu,1}' < h_{\nu,1}$, where $h_{\nu,1}$ and $h_{\nu,1}'$ denote the first positive zeros of $H_\nu$ and $H'_\nu$, respectively.
The graph of the function \( r \mapsto 1 + 2 \left( \frac{rH''(r)}{H'(r)} + \left( \frac{1}{\nu+1} - 1 \right) \frac{rH'(r)}{H''(r)} \right) \)

for \( \nu \in \{-0.3, -0.25, 0, 0.5\} \) on \([0, 1.2]\)

**Theorem 2.5** Let \( |\nu| \leq \frac{1}{2} \) and \( 0 \leq \alpha < 1 \). Then the radius of \( \beta \)-uniform convexity of order \( \alpha \) of the function \( v_\nu \) is the smallest positive root of the equation

\[
(1 - \alpha) - (1 + \beta) \left( 1 + \nu - r \frac{(1 - \nu)H'_\nu(r) + rH''_\nu(r)}{-\nu H_\nu(r) + rH'_\nu(r)} \right) = 0.
\]

Moreover \( r_{\nu,1}^\beta < r_{\nu}^c < h_{\nu,1} \), where \( h_{\nu,1} \) and \( s_{\nu,1} \) denote the first positive zeros of \( H_\nu \) and \( v'_\nu \), respectively.

**Proof.** Let \( h_{\nu,n} \) and \( s_{\nu,n} \) denote the \( n \)-th positive root of \( H_\nu \) and \( v'_\nu \), respectively and the smallest positive root of \( v'_\nu \) does not exceed the first positive root of \( H_\nu \). In [10], the following equality was proved:

\[
1 + \frac{zv''_\nu(z)}{v'_\nu(z)} = -\nu + z \frac{(1 - \nu)H'_\nu(z) + zH''_\nu(z)}{-\nu H_\nu(z) + zH'_\nu(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{s_{\nu,n}^2 - z^2}.
\]

By using inequality (1.8), for all \( z \in U(s_{\nu,1}) \) we have the inequality

\[
\Re \left( 1 + \frac{zv''_\nu(z)}{v'_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{2r^2}{s_{\nu,n}^2 - r^2} \tag{2.29}
\]

where \( |z| = r \).

On the other hand, again by using inequality (1.8), for all \( z \in U(s_{\nu,1}) \) and \( \beta \geq 0 \) we get
the inequality

\[
\beta \frac{z \nu''(z)}{v'_\nu(z)} = \beta \left| \sum_{n \geq 1} \frac{2z^2}{\zeta_{\nu,n}^2 - z^2} \right| \\
\leq \beta \sum_{n \geq 1} \left| \frac{2z^2}{\zeta_{\nu,n}^2 - z^2} \right| \\
\leq \beta \sum_{n \geq 1} \frac{2r^2}{\zeta_{\nu,n}^2 - r^2} = - \beta r v''_{\nu}(r) v'_{\nu}(r). 
\]

Finally the following inequality be infered from (2.29) and (2.30)

\[
\Re \left( 1 + \frac{z \nu''(z)}{v'_\nu(z)} \right) - \beta \left| \frac{z \nu''(z)}{v'_\nu(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) r v''_{\nu}(r) v'_{\nu}(r), \quad \beta \geq 0, 
\]

where \(|z| = r\). Thus, for \(r \in (0, \varsigma_{\nu,1})\), \(\beta \geq 0\) and \(\alpha \in [0, 1)\) we obtain

\[
\inf_{|z|<r} \left[ \Re \left( 1 + \frac{z \nu''(z)}{v'_\nu(z)} \right) - \beta \left| \frac{z \nu''(z)}{v'_\nu(z)} \right| - \alpha \right] = 1 - \alpha + (1 + \beta) r v''_{\nu}(r) v'_{\nu}(r).
\]

The mapping \(\Theta_{\nu} : (0, \varsigma_{\nu,1}) \to \mathbb{R}\) defined by

\[
\Theta_{\nu}(r) = 1 + (1 + \beta) \frac{r v''_{\nu}(r)}{v'_\nu(r)} = 1 - (1 + \beta) \sum_{n \geq 1} \frac{2r^2}{\zeta_{\nu,n}^2 - r^2}
\]

is strictly decreasing since \(\lim_{r \searrow 0} \Theta_{\nu}(r) = 1\) and \(\lim_{r \nearrow \varsigma_{\nu,1}} \Theta_{\nu}(r) = -\infty\). As a result in wiew of the minimum principle for harmonic functions it follows that for \(\alpha \in [0, 1)\), \(\beta \geq 0\) and \(z \in U(r_4)\) we have

\[
\Re \left( 1 + \frac{z \nu''(z)}{v'_\nu(z)} \right) - \beta \left| \frac{z \nu''(z)}{v'_\nu(z)} \right| > \alpha.
\]

if and only if \(r_4\) is the unique root of

\[
1 + (1 + \beta) \frac{r v''_{\nu}(r)}{v'_\nu(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0,
\]

situated in \((0, \varsigma_{\nu,1})\).

As a result of the Theorem 2.5, the following corollary is obtained by taking \(\alpha = 0\) ve \(\beta = 1\).

**Corollary 2.5** Let \(|\nu| \leq \frac{1}{2}\). Then the radius of uniform convexity of the function \(v_{\nu}\) is
the smallest positive root of the equation

$$1 - 2 \left( 1 + \nu - r \frac{(1 - \nu) H'_\nu(r) + r H''_\nu(r)}{-\nu H_\nu(r) + r H'_\nu(r)} \right) = 0.$$ 

Moreover \( r_{w_c} < r_{v_c} < \varsigma_{\nu,1} < h_{\nu,1}, \) where \( h_{\nu,1} \) and \( \varsigma_{\nu,1} \) denote the first positive zeros of \( H_\nu \) and \( v'_\nu, \) respectively.

The graph of the function \( r \mapsto 1 - 2 \left( 1 + \nu - r \frac{(1 - \nu) H'_\nu(r) + r H''_\nu(r)}{-\nu H_\nu(r) + r H'_\nu(r)} \right) \)

for \( \nu \in \{-0.3, -0.25, 0, 0.5\} \) on \([0, 1.2]\)

**Theorem 2.6** Let \( |\nu| \leq \frac{1}{2} \) and \( 0 \leq \alpha < 1. \) Then the radius of \( \beta- \)uniformly convex of order \( \alpha \) of the function \( w_\nu \) is the smallest positive root of the equation

$$2(1 - \alpha) - (1 + \beta) \left( 1 + \nu - \sqrt{r} \frac{(2 - \nu) H'_\nu(\sqrt{r}) + \sqrt{r} H''_\nu(\sqrt{r})}{(1 - \nu) H_\nu(\sqrt{r}) + \sqrt{r} H'_\nu(\sqrt{r})} \right) = 0.$$ 

Moreover \( r_{\beta-w_c} < r_{w_c} < \varsigma_{\nu,1} < h_{\nu,1}, \) where \( h_{\nu,1} \) and \( \varsigma_{\nu,1} \) denote the first positive zeros of \( H_\nu \) and \( w'_\nu, \) respectively.

**Proof.** Let \( h_{\nu,n} \) and \( \sigma_{\nu,n} \) denote the \( n \)-th positive root of \( H_\nu \) and \( w'_\nu, \) respectively and the smallest positive root of \( w'_\nu \) does not exceed the first positive root of \( H_\nu. \) In [10], the following equality was proved,

$$1 + \frac{z w''_\nu(z)}{w'_\nu(z)} = \frac{1}{2} \left[ 1 - \nu + \sqrt{r} \frac{(2 - \nu) H'_\nu(\sqrt{r}) + \sqrt{r} H''_\nu(\sqrt{r})}{(1 - \nu) H_\nu(\sqrt{r}) + \sqrt{r} H'_\nu(\sqrt{r})} \right] = 1 - \sum_{n \geq 1} \frac{z}{\sigma_{\nu,n}^2 - z}.$$ 

By using inequality (1.8), for all \( z \in U(\sigma_{\nu,1}) \) we obtain the inequality

$$\Re \left( 1 + \frac{z w''_\nu(z)}{w'_\nu(z)} \right) \geq 1 - \sum_{n \geq 1} \frac{r}{\sigma_{\nu,n}^2 - r} \quad (2.32)$$

where \( |z| = r. \)

Moreover, again by using inequality (1.8), for all \( z \in U(\sigma_{\nu,1}) \) and \( \beta \geq 0 \) we get the
inequality

\[
\beta \left| \frac{z w''(z)}{w'(z)} \right| = \beta \left| \sum_{n \geq 1} \frac{z}{\sigma^2_{\nu,n} - z} \right|
\leq \beta \sum_{n \geq 1} \left| \frac{z}{\sigma^2_{\nu,n} - z} \right|
\leq \beta \sum_{n \geq 1} \frac{r}{\sigma^2_{\nu,n} - r} = -\beta \frac{r w''(r)}{w'(r)}. \tag{2.33}
\]

As a result, the following inequality be infered from (2.32) and (2.33)

\[
\Re \left( 1 + \frac{z w''(z)}{w'(z)} \right) - \beta \left| \frac{z w''(z)}{w'(z)} \right| - \alpha \geq 1 - \alpha + (1 + \beta) \frac{r w''(r)}{w'(r)}, \quad \beta \geq 0 \text{ and } \alpha \in [0, 1). \tag{2.34}
\]

where \(|z| = r\). So, for \(r \in (0, \sigma_{\nu,1})\), \(\beta \geq 0\) and \(\alpha \in [0, 1)\) we have

\[
\inf_{|z|<r} \left[ \Re \left( 1 + \frac{z w''(z)}{w'(z)} \right) - \beta \left| \frac{z w''(z)}{w'(z)} \right| - \alpha \right] = 1 - \alpha + (1 + \beta) \frac{r w''(r)}{w'(r)}.
\]

The mapping \(\Phi_{\nu} : (0, \sigma_{\nu,1}) \rightarrow \mathbb{R}\) defined by

\[
\Phi_{\nu}(r) = 1 + (1 + \beta) \frac{r w''(r)}{w'(r)} = 1 - (1 + \beta) \sum_{n \geq 1} \frac{r}{\sigma^2_{\nu,n} - r}
\]

is strictly decreasing since \(\lim_{r \searrow 0} \Phi_{\nu}(r) = 1 > \alpha\) and \(\lim_{r \nearrow \sigma_{\nu,1}} \Phi_{\nu}(r) = -\infty\). Consequently, in view of the minimum principle for harmonic functions it follows that for \(\alpha \in [0, 1)\), \(\beta \geq 0\) and \(z \in U(r_5)\) we obtain

\[
\Re \left( 1 + \frac{z w''(z)}{w'(z)} \right) - \beta \left| \frac{z w''(z)}{w'(z)} \right| > \alpha.
\]

if and only if \(r_5\) is the unique root of

\[
1 + (1 + \beta) \frac{r w''(r)}{w'(r)} = \alpha, \quad \alpha \in [0, 1) \text{ and } \beta \geq 0.
\]

situated in \((0, \sigma_{\nu,1})\). ■

As a result of the Theorem [2.6], the next corollary is obtained by taking \(\alpha = 0\) ve \(\beta = 1\).

**Corollary 2.6** Let \(|\nu| \leq \frac{1}{2}\). Then the uniformly convexity of the function \(w_{\nu}\) is the
smallest positive root of the equation

\[-\nu + \sqrt{r}(2 - \nu)H'_\nu(\sqrt{r}) + \sqrt{r}H''_\nu(\sqrt{r}) = 0.\]

Moreover, \(r^{uc} < r^{cw} < \sigma_{\nu,1} < h_{\nu,1}\), where \(h_{\nu,1}\) and \(\sigma_{\nu,1}\) denote the first positive zeros of \(H_\nu\) and \(w'_\nu\), respectively.

The graph of the function \(r \mapsto -\nu + \sqrt{r}(2 - \nu)H'_\nu(\sqrt{r}) + \sqrt{r}H''_\nu(\sqrt{r})\)

for \(\nu \in \{-0.3, -0.25, 0, 0.5\}\) on \([0, 2.7]\)

Using the following representation of Struve functions of order 1/2 in terms of elementary trigonometric functions

\[H_{1/2}(z) = \sqrt{\frac{2}{\pi z}}(1 - \cos z)\]

we obtain

\[u_{1/2}(z) = 2 \left( \frac{1 - \cos z}{\sqrt{z}} \right)^2, \quad v_{1/2}(z) = 2 \left( \frac{1 - \cos z}{z} \right) \text{ and } w_{1/2}(z) = 2 \left( 1 - \cos \sqrt{z} \right).\]

We state the following results for the functions \(u_{1/2}, v_{1/2}\) and \(w_{1/2}\).

- \(u_{1/2}(z) \in UC\) in the disk \(U(r_1 = 1.1382)\),
- \(v_{1/2}(z) \in UC\) in the disk \(U(r_2 = 0.9349)\),
- \(w_{1/2}(z) \in UC\) in the disk \(U(r_3 = 2.4674)\),

where \(r_1, r_2\) and \(r_3\) is the smallest positive root of the equations

- \(5 + (-5 + 8z^2) \cos z - 2z(2z + \sin z) = 0,\)
- \(\cos z(2z^2 - 3) - z \sin z + 3 = 0,\)
\[
\sqrt{z} \cot \sqrt{z} = 0, \text{ respectively.}
\]

**Remark 2.7** For \( \beta = 0 \), Theorems 2.1, 2.2 and 2.3 reduce to [10, Thm. 3, (a),(b) and (c)], respectively, for the case \( \mu \in (-1, 1), \mu \neq 0 \) and \( \mu \neq -\frac{1}{2} \).

Moreover, Theorems 2.4, 2.5 and 2.6 reduce to [10, Thm. 4, (a),(b) and (c)], respectively, on putting \( \beta = 0 \), for the case \( |\nu| \leq \frac{1}{2} \).

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