Abstract. We define a moment-based estimator that maximizes the empirical saddlepoint (ESP) approximation of the distribution of solutions to empirical moment conditions. We call it the ESP estimator. We prove its existence, consistency and asymptotic normality, and we propose novel test statistics. We also show that the ESP estimator corresponds to the MM (method of moments) estimator shrunk toward parameter values with lower estimated variance, so it reduces the documented instability of existing moment-based estimators. In the case of just-identified moment conditions, which is the case we focus on, the ESP estimator is different from the MM estimator, unlike the recently proposed alternatives, such as the empirical-likelihood-type estimators.

Keywords: Empirical Saddlepoint Approximation; Method of Moments; Kullback-Leibler Divergence Criterion; Maximum-probability Estimator; Variance Penalization.

1. Introduction

The saddlepoint (SP) approximation has been developed to approximate distributions. Because of its accuracy it is regularly used in several fields, such as numerical analysis (e.g., Loader (2000)’s algorithm to approximate binomial distributions, and which is notably used in the statistical software R) and actuarial sciences (e.g., Esscher (1932)’s approximation for distributions tails). In statistics, the SP approximation and its empirical version —the empirical saddlepoint (ESP) approximation— have been used to approximate finite-sample distributions (e.g., Daniels 1954, Davison and Hinkley 1988).¹

In the present paper, we propose to use the ESP approximation to define a point estimator \( \hat{\theta}_T \). We call it the ESP estimator. It maximizes the Ronchetti and Welsh (1994)’s ESP approximation, i.e.,

\[
\hat{\theta}_T \in \arg\max_{\theta \in \Theta} \hat{f}_{\theta_T}^*(\theta)
\]

where \( \hat{f}_{\theta_T}^*(\cdot) \) is the ESP approximation of the distribution of solutions to empirical moment conditions

\[
\frac{1}{T} \sum_{t=1}^{T} \psi (X_t, \theta) = 0
\]

¹Standard monographs and introductions about the ESP and the SP approximation for statistics include Field and Ronchetti (1990), Kolassa (1994/2006), Jensen (1995), Goutis and Casella (1999) and Butler (2007).

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and where $\psi(.,.)$ denotes the moment function s.t. $E[\psi(X_1, \theta_0)] = 0_{m \times 1}$ an $m$-dimensional vector of zeros, $(X_t)_t=1^T$ i.i.d. data, $\theta_0 \in \Theta \subset \mathbb{R}^m$ the unknown parameter of interest, and $T$ the sample size. The exact formula for $\hat{f}_{\theta^*_T}(.)$ is reminded below in equation (3) on p. 3.

The ESP estimator is a moment-based estimator. Since Pearson (1894, 1902)'s method of moment (MM), moment-based estimators have been found useful in a variety of applications (e.g., covariance structure analysis in psychology, and asset pricing in economics). Their two main advantages are (i) they do not require a parametric family of probability distributions for the data so they are less prone to model misspecification, and (ii) they allow complex models for which the likelihood function is intractable.

Nevertheless, the increase use of the MM and its extensions has revealed that they can be unstable and perform poorly in finite samples (e.g., July 1996 special issue of JBES). The idea of the ESP estimator to improve on the MM estimator is the following. By definition, the MM estimate $\theta^*_T(\omega)$ solves a realization of the empirical moment conditions (2), but it typically does not solve the empirical moment condition for another realization $\omega'$ of the data. Thus, we might want an estimate that does only take into account the realized empirical moment conditions, but also their other potential realizations. More precisely, we want an estimate that accounts for all the potential realizations of the empirical moment conditions according to their probability weight of occurrence. This leads to the ESP estimate, which is a maximum-probability estimate. The ESP estimate maximizes the estimated probability weights of solving the empirical moment conditions. If the empirical moment conditions have a unique solution with a continuous distribution, the ESP estimator maximizes the ESP approximation of a probability density function of the solution $\theta^*_T$. We rely on the ESP approximation because simulation and theoretical evidence shows the ESP approximation can be very accurate in small sample (e.g., Davison and Hinkley 1988, Ronchetti and Welsh 1994).

Besides the maximum-probability motivation, we show that the ESP estimator corresponds to an MM estimator shrunk toward parameter values with lower estimated variance. More precisely, we decompose the logarithm of the ESP approximation as the sum of a term, which is maximized at the MM estimator, and a variance penalty, which discounts parameter values with high estimated variance. Under assumptions adapted from the entropy literature, we establish the ESP estimator has the same good asymptotic

2This is in contrast to the traditional motivation for ML estimators, which maximize the probability weights of obtaining a sample equal to the observed sample. In other words, the support of the ESP distribution is the parameter space, while the support of the distribution associated with a likelihood is the data space. Thus, if we are looking for relevant parameter values instead of data values —as it is typically the case—, a maximum-probability motivation appears more appealing than the traditional ML motivation.

3Another motivation for maximum probability estimators is decision theoretic. Maximum probability estimators follows from the minimization of the expectation of a loss “function” that equals zero when $\theta$ solves the empirical moment conditions and one otherwise by normalization. This motivation is similar to the decision-theoretic justification for the Bayesian maximum a posteriori estimator (e.g., Robert 2007 (1994) sec. 4.1.2). As in Bayesian analysis, the choice of other loss functions is possible. It is left for future research.
properties as the MM estimator, so the variance penalization is a finite-sample correction. We also derive the ESP counterparts of the Wald, Lagrange multiplier (LM), analogue likelihood-ratio (ALR) test statistics, as well as another test statistic. Then, we investigate the ESP estimator through Monte-Carlo simulations. We compare its performance with the exponential tilting (ET) estimator, which is equal to the MM estimator in the just-identified case (i.e., when the number of parameters is the same as the number of moment conditions). Results show that the variance penalization of the ESP estimator reduces the finite-sample instability of the ET estimator (or equivalently, of the MM estimator). An empirical application illustrates the gain from this greater stability in terms of inference.

The ESP estimator is not the first proposal to improve on the MM and its extensions. Alternative moment-based approaches have been proposed such as the empirical likelihood approach of Owen (Qin and Lawless 1994), the continuously updating approach (Hansen et al. 1996), the already-mentioned exponential tilting (ET) approach (Kitamura and Stutzer 1997, Imbens et al. 1998), and combinations of the aforementioned approaches (e.g., Schennach 2007). All these approaches yield an estimator closely related to the empirical likelihood estimator, so we call them empirical-likelihood-type estimators. In the just-identified case, when well-defined, all of these empirical-likelihood-type estimators are numerically equal to the original Pearson’s MM estimator \( \theta^*_T \). Because we focus on the just-identified case, it thus is sufficient for us to compare the ESP estimator with the MM estimator, or with one of any of these more recent estimators.

In addition to the already cited papers, the present paper, which supersedes the unpublished manuscript Sowell (2009), is related to many other ones. We clarify these relations in Section 5 (p. 10). To the best of our knowledge, none of the prior papers use the SP or the ESP to propose a novel moment-based point estimator. Overall, the present paper brings together the literature on the saddlepoint approximation and the literature on moment-based estimation.

2. Finite-sample analysis

In the present section, we remind the formula for the ESP approximation, and analyze its finite-sample structure. Then, we decompose the log-ESP into two terms and show that the ESP estimator is a MM estimator shrunk toward parameter values with lower estimated variance.

2.1. The ESP approximation. Formalizing and generalizing prior works (Davison and Hinkley 1988, Feuerverger 1989, Wang 1990, Young and Daniels 1990), Ronchetti and Welsh (1994) propose the following ESP approximation to estimate the distribution of a solution to the empirical moment conditions (2)

\[
\hat{f}_{\theta_T}(\theta) := \exp \left\{ T \ln \left( \frac{1}{T} \sum_{t=1}^{T} e^{T(\theta)\psi_t(\theta)} \right) \right\} \left( \frac{T}{2\pi} \right)^{m/2} |\Sigma_T(\theta)|^{-\frac{1}{2}}
\]
where $|\cdot|_{\text{det}}$ denotes the determinant function, $\theta^*_T$ a solution to $\psi_t(\cdot) := \psi(X_t, \cdot)$, and

$$
\Sigma_T(\theta) := \left[ \sum_{t=1}^T w_{t,\theta} (\psi_t(\theta) \psi_t'(\theta)') \right]^{-1} \left[ \sum_{t=1}^T w_{t,\theta} (\psi_t(\theta)') \psi_t(\theta) \right] \left[ \sum_{t=1}^T w_{t,\theta} (\psi_t'(\theta)) \right]^{-1}, \quad (4)
$$

$$
w_{t,\theta} := \frac{\exp[\tau_T(\theta)' \psi_t(\theta) \psi_t(\theta)]}{\sum_{i=1}^T \exp[\tau_T(\theta)' \psi_t(\theta)]}, \quad (5)
$$

$$
\tau_T(\theta) \text{ such that } \sum_{t=1}^T \psi_t(\theta) \frac{\exp[\tau_T(\theta)' \psi_t(\theta)]}{\sum_{i=1}^T \exp[\tau_T(\theta)' \psi_t(\theta)]} \times \frac{1}{T} = 0. \quad (6)
$$

The ESP approximation (3) is the empirical counterpart of the SP approximation of Field (1982). From a computational point of view, the ESP approximation (3) is not complicated. The only implicit quantity is $\tau_T(\theta)$, which solves the tilting equation (6), which, in turn, is just the FOC (first-order condition) of the unconstrained convex problem $\min_{\tau \in \mathbb{R}^m} \sum_{i=1}^T e^{\tau \psi_t(\theta)}$. A full understanding of the ESP approximation (3) arguably requires to work through higher-order asymptotic expansions along the lines of Field (1982). However, direct inspection of the ESP approximation (3) also provides insight for how it incorporates information from the data through two channels.

The first channel is the ET (exponential tilting) term $\exp \left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^T e^{\tau_T(\theta)' \psi_t(\theta)} \right] \right\}$. In equation (6), for any $\theta \in \Theta$, the terms $\sum_{i=1}^T \exp[\tau_T(\theta)' \psi_t(\theta)] \psi_t(\theta) / \sum_{i=1}^T \exp[\tau_T(\theta)' \psi_t(\theta)]$ REVT (i.e., reweight) the empirical weights $1/T$, so the finite-sample moment conditions (6) holds. This tilting determines, through equation (5), the multinomial distribution $(w_{t,\theta})_{t=1}^T$ that is the closest to the empirical distribution —in the sense of the Kullback-Leibler divergence criterion— s.t. the finite-sample moment conditions (6) holds: The tilting equation (6) is the FOC w.r.t. (with respect to) $\tau$ of the Lagrangian dual problem of the minimization problem

$$
\min_{(w_{1,\theta}, w_{2,\theta}, \ldots, w_{T,\theta}) \in [0,1]^T} \sum_{t=1}^T w_{t,\theta} \log \left( \frac{w_{t,\theta}}{1/T} \right)
$$

s.t. $\sum_{t=1}^T w_{t,\theta} \psi_t(\theta) = 0$ and $\sum_{t=1}^T w_{t,\theta} = 1, \quad (7)$

where $\sum_{t=1}^T w_{t,\theta} \log[w_{t,\theta}/(1/T)]$ is the Kullback-Leibler divergence criterion between the empirical distribution and the multinomial distribution $(w_{t,\theta})_{t=1}^T$ with the same support (e.g., Efron 1981, Kitamura and Stutzer 1997). Then, for the given $\theta \in \Theta$, in the ESP approximation (3), the ET term $\exp \left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^T e^{\tau_T(\theta)' \psi_t(\theta)} \right] \right\}$ indicates the extent of the tilting needed to set the finite-sample moment conditions (7) (or equivalently, equation (6)) to zero. The bigger is the tilting of the empirical distribution, the less compatible

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4We do not claim that the ESP estimator is as easy to compute as the MM estimator, but that its additional complexity is similar to the recently proposed empirical-likelihood-type estimators (e.g., ET estimator), and that it is worthwhile in several applications (e.g., Section 4). Moreover, it seems to make sense to develop novel estimation methods that take advantage of the increasingly available computational power.
are the data with \( \theta \) solving the empirical moment conditions, and the smaller should be the ET term \( \exp \left\{ \frac{T}{T} \sum_{t=1}^{T} e^{\tau_T(\theta)\psi_t(\theta)} \right\} \). It can be easily seen that \( \frac{T}{T} \sum_{t=1}^{T} e^{\tau_T(\theta)\psi_t(\theta)} \) reaches its maximum when \( \theta \) is a solution \( \theta^* \) of the empirical moment conditions \( \left(2\right) \), i.e., when \( \tau_T(\theta^*) = 0 \times 1 \) and no tilting is needed.\(^5\)

In the ESP approximation on equation \( \left(3\right) \), the second term \( \left(\frac{T}{2\pi}\right)^{m/2} \) comes from the multivariate Gaussian distribution that is the leading term of the Edgeworth’s asymptotic expansions underlying ESP approximations. However, because it is constant w.r.t. \( \theta \), it does not affect the maximization of the ESP approximation, so it is not an information channel for the ESP estimator. The remaining term \( \left| \Sigma_T(\theta) \right|^{-\frac{1}{2}} \), which we call the variance term, is the second channel through which the ESP approximation incorporates information from data. The variance term discounts the ET term according to the tilted estimated variance of the solution to the finite-sample moment conditions. Under standard assumptions, a consistent estimator of the asymptotic variance of \( \sqrt{T}(\theta^* - \theta_0) \) is

\[
\left[ \frac{1}{T} \sum_{t=1}^{T} \frac{\partial \psi_t(\theta^*)}{\partial \theta} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta^*)' \psi_t(\theta^*) \right] \left[ \frac{1}{T} \sum_{t=1}^{T} \frac{\partial \psi_t(\theta^*)}{\partial \theta} \right]^{-1}.
\]

The bigger the variance term is, the less plausible a solution takes exactly this value, and the smaller is \( \left| \Sigma_T(\theta) \right|^{-\frac{1}{2}} \) —note the negative power. Therefore, overall, for a given \( \theta \in \Theta \), the bigger the tilting or the estimated variance, the smaller the ESP approximation, i.e., the estimated probability weight that \( \theta \) solves the empirical moment conditions \( \left(2\right) \).

2.2. The ESP estimator as a shrinkage estimator. As explained in the introduction, the recently proposed moment-based estimators are numerically equal to the Pearson’s MM estimator in the just-identified case. Thus, it is sufficient to compare the ESP estimator with one of them in order to understand the difference between the former and the other proposed moment-based estimators. The ET estimator of Kitamura and Stutzer \( \left(1997\right) \) and Imbens et al. \( \left(1998\right) \) is particularly convenient for this purpose. Taking the logarithm of the ESP approximation \( \left(3\right) \), and removing the terms constant w.r.t. \( \theta \), it can be seen that, \( \mathbb{P}\text{-a.s. for } T \text{ big enough, the ESP estimator } \hat{\theta}_T \text{ maximizes the objective function}

\[
\ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta)\psi_t(\theta)} \right] - \frac{1}{2T} \ln \left| \Sigma_T(\theta) \right|_{\text{det}},
\]

where \( \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta)\psi_t(\theta)} \right] \) is an increasing transformation of the objective function of the ET estimator. Thus, the difference between the ESP estimator and the ET estimators comes only from the log-variance term \( -\frac{1}{2T} \ln \left| \Sigma_T(\theta) \right|_{\text{det}} \). The latter does not only incorporates additional information from data as explained in Section 2.1 but it also penalizes parameter values with higher estimated variance. Thus, the ESP estimator is an ET estimator —or equivalently, a MM estimator— shrunk toward parameter values with lower estimated variance. Now, as the factor \( \frac{1}{2T} \) suggests and the proofs of Section

\(^5\)For a complete proof, one can follow the same reasoning as in the proof of Lemma 10 in Holcblat and Sowell \( \left(2019\right) \) p. 32 with the empirical distribution in lieu of \( \mathbb{P} \).
3. Asymptotic properties

In the present section, we investigate the asymptotic properties of the ESP estimator. Good asymptotic properties can be regarded as a minimal requirement for the ESP estimator, which is based on a small-sample asymptotic approximation. All the proofs and assumptions are in the online Appendix Holcblat and Sowell (2019).

3.1. Existence, consistency and asymptotic normality. Under assumptions adapted from the entropy literature, the following theorem establishes the existence, the strong consistency, and the asymptotic normality of the ESP estimator \( \hat{\theta}_T \).

**Theorem 1** (Existence, consistency and asymptotic normality). Under Assumption \( \square \) \( \mathbb{P} \)-a.s. for \( T \) big enough, there exists \( \hat{\theta}_T \) s.t.

(i) \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \hat{\theta}_T \to \theta_0 \); and

(ii) under the additional Assumption \( \lozenge \) as \( T \to \infty \), \( \sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{D} \mathcal{N}(0, \Sigma(\theta_0)) \).

where \( \Sigma(\theta_0) := \left[ \mathbb{E} \frac{\partial \psi(X,\theta_0)}{\partial \theta} \right]^{-1} \mathbb{E} [\psi(X,\theta_0)\psi(X,\theta_0)'] \left[ \mathbb{E} \frac{\partial \psi(X,\theta_0)}{\partial \theta} \right]^{-1}, \) \( D \to \) denotes the convergence in distribution.

Theorem 1 shows that the ESP estimator has the same first-order asymptotic properties as the MM and hence the recently proposed moment-based estimators. Although the asymptotic properties of the ESP estimator are standard, the proof of Theorem 1 is quite involved. The crux of the proof is to show that the variance penalization \( -\frac{1}{2T} \ln |\Sigma_T(\theta)|_{\text{det}} \) vanishes sufficiently quickly asymptotically, so it does not distort the first-order asymptotic.

3.2. More on inference: The trinity+1. The ESP estimator provides different ways to test parameter restrictions

\[ H_0 : r(\theta_0) = 0_{q \times 1} \]  

where \( r : \Theta \to \mathbb{R}^q \) with \( q \in [1, \infty[ \). More precisely, within the ESP framework, there exist the usual trinity of Wald, LM and ALR tests statistics, plus another test statistic, which we call the exponential tilting (ET) test statistic. Our ET test has a structure similar to a test for over-identified moment conditions in Imbens et al. (1998).

Under a mild standard additional assumption, the following theorem shows that the Wald, LM ALR, and ET statistics asymptotically follow a chi-squared distribution with \( q \) degrees of freedom.
Theorem 2 (The trinity+1: Wald, LM, ALR and ET tests). Define \( R(\theta) := \frac{\partial r(\theta)}{\partial \theta} \), and the following Wald, LM, ALR and ET test statistics

\[
\text{Wald}_T := Tr(\hat{\theta}_T)'[R(\hat{\theta}_T)\Sigma(\theta_0)_T R(\hat{\theta}_T)^{-1} r(\hat{\theta}_T)]
\]

\[
\text{LM}_T := T\gamma_T'[R(\hat{\theta}_T)\Sigma(\theta_0)_T R(\hat{\theta}_T)^{-1}]\gamma_T = \frac{\partial \ln[\hat{f}_{\theta_T}(\hat{\theta}_T)]}{\partial \theta'} \Sigma(\theta_0)_T^{-1} \frac{\partial \ln[\hat{f}_{\theta_T}(\hat{\theta}_T)]}{\partial \theta}
\]

\[
\text{ALR}_T := 2\{\ln[\hat{f}_{\theta_T}(\hat{\theta}_T)] - \ln[\hat{f}_{\theta_T}(\hat{\theta}_T)]\}
\]

\[
\text{ET}_T := T\tau_T(\hat{\theta}_T)'\hat{V}_T \tau_T(\hat{\theta}_T)
\]

where \( \Sigma(\theta_0)_T \) and \( \hat{V}_T \) are symmetric matrices that converge in probability to \( \Sigma(\theta_0) \) and \( \mathbb{E}[(\psi(X_1, \theta_0) \psi(X_1, \theta_0)')'] \), respectively; and where \( \gamma_T \) and \( \hat{\theta}_T \) respectively denote the Lagrange multiplier and a solution to the maximization of \( \hat{f}_{\theta_T}(\theta) \) w.r.t. \( \theta \in \Theta \) under the constraint that \( r(\theta) = 0_{q \times 1} \). Under Assumptions \([1, 2, 3]\) if the test hypothesis \([9]\) holds, as \( T \to \infty \),

\[\text{Wald}_T, \text{LM}_T, \text{ALR}_T, \text{ET}_T \xrightarrow{D} \chi^2_q.\]

Theorem 2 can also be used to obtain valid confidence regions by the inversion of the test statistics with \( \hat{\theta}_T = \theta_0 \). Our Wald, LM and ALR test statistics share some similarity with the test statistics proposed in Kitamura and Stutzer \(1997\), Imbens et al. \(1998\) and Robinson et al. \(2003\). The main difference is that the latter are built around (possibly constrained) maximizers of the ET term, while our tests statistics are based on the (possibly constrained) ESP estimator, which maximizes the whole ESP approximation including the variance term.

4. Examples

In the present section, we further investigate and illustrate the finite-sample properties of the ESP estimator.\(^6\) We focus on the comparison with the ET estimator, as previously noted, (i) in the just-identified case, which is the case addressed in the present paper, the MM estimator and the recently proposed moment-based estimators are equal to the ET estimator so there is no loss of generality in terms of point estimation, and (ii) the ESP objective function nests the ET objective function, so that the source of the difference

\(^6\)In mathematical terms, \( \hat{\theta}_T \in \arg\max_{\theta \in \hat{\Theta}} \hat{f}_{\theta_T}(\theta) \) where \( \hat{\Theta} := \{\theta \in \Theta : r(\theta) = 0_{q \times 1}\} \) and \( \hat{\gamma}_T \) is the Lagrangian multiplier s.t. \( \frac{1}{T} \frac{\partial \ln[\hat{f}_{\theta_T}(\hat{\theta}_T)]}{\partial \theta'} + \frac{\partial r(\hat{\theta}_T)}{\partial \theta'} \hat{\gamma}_T = 0_{n \times 1}. \)

\(^7\)In addition to our finite-sample analysis of the ESP objective function (Section 1), our derivation of the first-order asymptotic properties (Section 2), our Monte-Carlo simulations and empirical application (present section), another way to shed light on the finite-sample properties of the ESP estimator would be to derive its higher-order asymptotic properties such as its second-order bias (e.g., Rilstone et al. \(1996\)). In the present paper, we do not follow this way because it would add several dozens of pages of proofs without much insight: Our preliminary derivations yield a long and complicated structure for the second-order bias, from which we struggle to gain insight. The length and the complexity of the second-order bias mainly comes from (i) the derivatives of the variance \( \Sigma_T(\theta) \)^{-1/2}; and (ii) the reliance on the exact FOCs instead of approximate FOCs. A mild preview of this complexity can be seen in Holcblat and Sowell \(2019\) Appendix B.2.)
between the two is easily understood —it necessarily comes from the variance term (see Section 2.2). For brevity, we present the main results for a numerical and an empirical example that are known to be challenging for moment-based estimation.

4.1. **Numerical example: Monte-Carlo simulations.** We simulate the just-identified version of the Hall and Horowitz (1996) model, which has become a standard benchmark to compare the performance of moment-based estimators in statistics (e.g., Schennach 2007, Lő and Ronchetti 2012) and econometrics (e.g., Imbens et al. 1998, Kitamura 2001). This model can be interpreted as a simplified consumption-based asset pricing model where $\beta$ is the relative risk aversion (RRA) parameter (Gregory et al. 2002). In the simulations, we estimate the two parameters $(\mu, \beta)$ with the moment function

$$
\psi_t(\beta, \mu) = \left[ \frac{\exp \{\mu - \beta (X_t + Y_t) + 3Y_t\} - 1}{Y_t \left( \exp \{\mu - \beta (X_t + Y_t) + 3Y_t\} - 1 \right)} \right]
$$

where $\mu_0 = -0.72$, $\beta_0 = 3$, and $X_t$ and $Y_t$ are jointly i.i.d. random variables with distribution $\mathcal{N}(0, \sigma^2)$.

### Table 1. ESP vs. ET estimator for the just-identified Hall and Horowitz model.

| $T$ | $\beta$ | $\mu$ |
|-----|---------|-------|
|     | ET      | ESP   | ET   | ESP   |
| MSE | 3.6228  | 0.7065| 1.5391| 0.2319|
| 25  | 0.4782  | -0.0048| -0.1855| 0.1089|
| Var. | 3.3941  | 0.7065| 1.5047| 0.2200|
| MSE | 1.7024  | 0.3342| 0.9959| 0.1292|
| 50  | 0.2670  | -0.0160| -0.1330| 0.0619|
| Var. | 1.6311  | 0.3342| 0.9782| 0.1254|
| MSE | 0.6812  | 0.1742| 0.4780| 0.0645|
| 100 | 0.1429  | -0.0119| -0.0735| 0.0388|
| Var. | 0.6608  | 0.1741| 0.4726| 0.0630|
| MSE | 0.2162  | 0.0830| 0.1457| 0.0324|
| 200 | 0.0684  | -0.0113| -0.0340| 0.0223|
| Var. | 0.2115  | 0.0829| 0.1445| 0.0319|

Note: The reported statistics are based on 10,000 simulated samples of sample size equal to the indicated $T$. For ET, the parameter space is restricted to $\beta < 15$ in order to limit the erratic behaviour of the estimator at sample sizes $T = 25$ and 50. No such parameter restriction is imposed for ESP.

Table I reports the mean-squared error (MSE), bias and variance of the ESP and ET estimators for different sample sizes. The MSE, the variance and the bias of the ESP estimator are always smaller than for the ET estimator, and the differences are notable, especially for small sample sizes. In fact, Table I understates the improvement delivered by the variance penalization of the ESP objective function. We help the ET
estimator (or equivalently, the MM estimator) by restricting its parameter space to $\beta < 15$. Without this parameter restriction, the behaviour of the ET estimator is very unstable. An analysis of the typical shape of the objective functions for small sample size explains this phenomenon. The typical ET objective function has a ridge that follows from around the population parameter values ($\beta_0 = 3, \mu_0 = -0.72$) towards $(1000, -600)$. The ridgeline is not totally flat, and it often has a gentle downward slope as we move away from the area near the population parameter values. However, regularly, for some simulated samples, the very top of the ridge is extremely far from the population parameter values, so that ET estimates are very far from the population parameter values. This does not happen for the ESP estimator. The variance term of the ESP objective function ensures that the ridge drops sufficiently as we move away from the maximum that is near the population parameter value. Thus, in line with our finite-sample analysis of the ESP objective (Section 2.2), the ESP estimator is much more stable.

4.2. Empirical example. In this section, we present an empirical example from asset pricing. Since Hansen and Singleton (1982), moment-based estimation is standard in consumption-based asset pricing. For brevity, we focus on the key features of the example. See Holcblat and Sowell (2019, Appendix F on p. 90) for additional information and comparisons.

We estimate the relative risk aversion (RRA) $\theta$ of a representative agent of the US economy. Previous studies have shown that existing moment-based estimation approaches often produce unstable RRA parameter estimates. We rely on the following moment condition

$$E \left[ \left( \frac{C_t}{C_{t-1}} \right)^{-\theta} (R_{m,t} - R_{f,t}) \right] = 0,$$

where $\frac{C_t}{C_{t-1}}$ is the growth consumption and $(R_{m,t} - R_{f,t})$ the market return in excess of the risk-free rate. The moment condition, which is common to many consumption-based asset pricing models, and the data are similar to Julliard and Ghosh (2012), corresponding to standard US data at yearly frequency from Shiller’s website spanning from 1890 to 2009. We report ET and ESP estimates as well as confidence regions based on the inversion of the ALR test statistics of Theorem 2 (p. 6) with $\hat{\theta}_T = \theta_0$. The latter have the advantage to take into account the whole shape of the objective function unlike $t$-statistics-based confidence regions, which only account for the shape of the objective function in a neighborhood of the estimate through its standard errors.

In Table 2, Figures (A) and (B) respectively display the ET term and the ESP approximation. For ease of comparison, the scale is the same, and we normalize both of them so they integrate to one. The normalized ET term is much flatter around its maximum than the normalized ESP approximation. Flatness of the objective function around the

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\footnote{We numerically check that they deliver the same estimates even for the small sample sizes $T = 25$ and $50$.}
Table 2. ET vs. ESP inference (1890–2009)

Empirical moment condition: \( \frac{1}{2009-1889} \sum_{t=1890}^{2009} \left[ \left( \frac{C_t}{C_{t-1}} \right)^{-\theta} (R_{m,t} - R_{f,t}) \right] = 0 \), where 
- \( R_{m,t} := \) gross market return, 
- \( R_{f,t} := \) risk-free asset gross return, 
- \( C_t := \) consumption, and 
- \( \theta := \) relative risk aversion.

Normalized ET := \( \exp \left\{ T \ln \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta) \psi_t(\theta)} \right) \right\} \)/\( \int_{\Theta} \exp \left\{ T \ln \left( \sum_{t=1}^{T} e^{\tau_T(\theta) \psi_t(\theta)} \right) \right\} d\theta \); 
Normalized ESP := \( \hat{f}_{\theta_{ET}}(\cdot)/\int_{\Theta} \hat{f}_{\theta_{ET}}(\theta) d\theta \); 
\( \hat{\theta}_{ET,T} = \hat{\theta}_{MM,T} = 50.3 \) (bullet) and \( \hat{\theta}_{ESP,T} = 32.21 \) (bullet); 
ET and ESP support = \([-218.2, 289.0]\); 95% ET ALR conf. region=\([18.3, 289.0]\) (stripe); 95% ESP ALR conf. region=\([15.0, 112.7]\) (stripe).

(A) ET est. and ALR conf. region. (B) ESP est. and ALR conf. region.

estimate has been documented for other existing moment-based estimators, and it has often been regarded as one of the main sources of the instability of the RRA estimates (e.g., Stock and Wright 2000, Neely et al. 2001). Figure (B) shows that the normalized ESP is sharp around the ESP estimator. The relative sharpness of the ESP yields sharper confidence regions: The ESP confidence region is less than half its ET counterpart. In light of the variance penalization term in the ESP objective function (Section 2.2 on p. 5) and the shrinkage-like behavior of the ESP estimator in the Monte-Carlo simulations (Section 4.1), the relative sharpness of the ESP inference is not surprising. In Holcblat and Sowell (2019), additional empirical evidences corroborate the increased stability and precision of the ESP estimator w.r.t. the ET estimator (or equivalently, MM estimator).

5. Connection to the literature and further research directions

The present paper demonstrates a previously unknown connection between the SP approximation and moment-based estimation, and hence it is related to many papers in addition to the ones already cited. Following Daniels (1954), the literature in statistics (e.g., Easton and Ronchetti 1986, Spady 1991, Jensen 1992, Vecchia et al. 2012, Broda and Kan 2015, Fasiolo et al. 2018) and econometrics (e.g., Phillips 1978, Holly and Phillips 1979, Phillips 1982, Lieberman 1994, Aït-Sahalia and Yu 2006) has used the SP (saddlepoint)
and ESP approximations to obtain accurate approximations of distributions, especially in
the tails. The strand of the SP literature that is closest to our paper derives SP approxima-
tions to the distribution of statistics that correspond to solutions of nonlinear estimating
equations. The latter strand of literature started with Field (1982) and continued with
Skovgaard (1990), Monti and Ronchetti (1993), Imbens (1997), Jensen and Wood (1998),
Almudevar et al. (2000), Robinson et al. (2003), and Ronchetti and Trojani (2003), among
others. More recently, Czellar and Ronchetti (2010), Ma and Ronchetti (2011), and Lô
and Ronchetti (2012), Kundhi and Rilstone (2013, 2015) propose more accurate tests for
indirect inference, functional measurement error models, moment condition models, non-
linear estimators and GEL (generalized empirical likelihood) estimators, respectively. To
the best of our knowledge, unlike the present paper, none of the prior papers use the SP
or the ESP to develop an estimation method that yields a novel moment-based estimator.
In ongoing work, we generalize the ESP approximation to the over-identified case, and
establish further good mathematical properties.

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This appendix mainly consists of the proofs of Theorem 1, existence and consistency and asymptotic normality of the ESP estimator, and Theorem 2, asymptotic distributions of the Trinity+1 test statistics. The proof of Theorem 1 builds on the traditional uniform convergence proof technique of Wald (1949). The proof of Theorem 2 adapts the usual way of deriving the trinity tests. The length of the proofs is mainly due to the variance term $|\Sigma_T(\theta)|^{-\frac{1}{2}}\text{det}$ and the high-level of details. The latter should make the proofs more transparent, and should ease the use of the intermediary results in further research.

In addition to the proofs, this appendix contains a table of contents, some formal definitions, the precise assumptions of the paper, a discussion thereof, and additional information regarding the examples.
C.1. Discussion
C.2. Implications of Assumption 1(h)
Appendix D. Remaining technical results
Appendix E. More on the numerical example
Appendix F. More on the empirical example
F.1. Additional empirical evidence
F.2. Data description

APPENDIX A. DEFINITIONS AND ASSUMPTIONS

Definition 1 (ESP approximation; Ronchetti and Welsh 1994). The ESP approximation of the distribution of the solution to the empirical moment conditions (2) is

$$f_{\hat{\theta}_T}(\theta) := \exp \left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t(\theta)^{\prime} \psi_t(\theta)} \right] \right\} \left( \frac{T}{2\pi} \right)^{m/2} |\Sigma_T(\theta)|^{-\frac{1}{2}} \right.$$ (11)

where $|.|_{\text{det}}$ denotes the determinant function, $\theta^{*}_T$ a solution to the empirical moment conditions (2), $\psi_t(.) := \psi(X_t,.)$, and

$$\Sigma_T(\theta) := \left[ \sum_{t=1}^{T} \hat{w}_{t,\theta} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]^{-1} = \left[ \sum_{t=1}^{T} \hat{w}_{t,\theta} \psi_t(\theta) \psi_t(\theta)' \right] \left[ \sum_{t=1}^{T} \hat{w}_{t,\theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \right]^{-1},$$ (12)

$$\hat{w}_{t,\theta} := \frac{\exp [\tau_t(\theta)^{\prime} \psi_t(\theta)]}{\sum_{i=1}^{T} \exp [\tau_t(\theta)^{\prime} \psi_t(\theta)]},$$ (13)

$$\tau_T(\theta) \text{ such that } (s.t.) \sum_{t=1}^{T} \psi_t(\theta) \exp [\tau_T(\theta)^{\prime} \psi_t(\theta)] = 0_{m \times 1}. \quad (14)$$

Definition 2 (ESP estimator). The ESP estimator $\hat{\theta}_T$ is a maximizer of the ESP approximation (11), i.e.,

$$\hat{\theta}_T \in \arg \max_{\theta \in \Theta} f_{\hat{\theta}_T}(\theta). \quad (15)$$

We require the following assumption to prove the existence and the consistency of the ESP estimator.

Assumption 1. (a) The data $(X_t)_{t=1}^{\infty}$ are a sequence of i.i.d. random vectors of dimension $p$ on the complete probability sample space $(\Omega,\mathcal{F},\mathbb{P})$. (b) Let the moment function $\psi : \mathbb{R}^p \times \Theta^c \rightarrow \mathbb{R}^m$ be s.t. $\theta \mapsto \psi(X_1, \theta)$ is continuously differentiable $\mathbb{P}$-a.s., and $\forall \theta \in \Theta^c$, $x \mapsto \psi(x, \theta)$ is $\mathcal{B}(\mathbb{R}^p)/\mathcal{B}(\mathbb{R}^m)$-measurable, where, for $\epsilon > 0$, $\Theta^\epsilon$ denotes the $\epsilon$-neighborhood of $\Theta$. (c) In the parameter space $\Theta$, there exists a unique $\theta \in \text{int}(\Theta)$ s.t. $\mathbb{E}[\psi(X_1, \theta_0)] = 0_{m \times 1}$ where $\mathbb{E}$ denotes the expectation under $\mathbb{P}$. (d) Let the parameter space $\Theta \subset \mathbb{R}^m$ be a compact set, s.t., for all $\theta \in \Theta$, there exists $\tau(\theta) \in \mathbb{R}^m$ that solves the equation $\mathbb{E} \left[ e^{\tau(\psi(X_1, \theta) \psi(X_1, \theta)} \right] = 0$ for $\tau$. (e) $\mathbb{E} \left[ \sup_{(\theta, \tau) \in \mathcal{S}} e^{2\tau(\psi(X_1, \theta)} \right] < \infty$ where $\mathcal{S} := \{ (\theta, \tau) : \theta \in \Theta \& \tau \in \mathcal{T}(\theta) \}$ and $\mathcal{T}(\theta) := \overline{B}_{\epsilon_T}(\tau(\theta))$ with $\overline{B}_{\epsilon_T}(\tau(\theta))$ the closed ball of radius $\epsilon_T > 0$ and center $\tau(\theta)$. (f) $\mathbb{E} \left[ \sup_{\theta \in \Theta} |\frac{\partial \psi(X_1, \theta)}{\partial \theta}|^2 \right] < \infty$, where $|.|$ denotes the Euclidean norm. (g) $\mathbb{E} \left[ \sup_{\theta \in \Theta} |\psi(X_1, \theta)\psi(X_1, \theta)|^2 \right] < \infty$. (h) For all
We prove that the joint parameter space for \( \theta \in \Theta \), the matrices
\[
\left[ \mathbb{E} e^{\tau(\theta)\psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right] \quad \text{and} \quad \mathbb{E} \left[ e^{\tau(\theta)\psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)^T \right]
\]
are invertible, so \( \Sigma(\theta) := \left[ \mathbb{E} e^{\tau(\theta)\psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right]^{-1} \mathbb{E} \left[ e^{\tau(\theta)\psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)^T \right] \left[ \mathbb{E} e^{\tau(\theta)\psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)^T}{\partial \theta} \right]^{-1} \) is also invertible.

Assumption 2. (a) The function \( \theta \mapsto \psi(X_1, \theta) \) is three times continuously differentiable in a neighborhood \( N \) of \( \theta_0 \) in \( \Theta \) \( \mathbb{P} \)-a.s. (b) There exists a \( \mathcal{B}(\mathbb{R}^p)/\mathcal{B}(\mathbb{R}) \)-measurable function \( b(.) \) satisfying
\[
\mathbb{E} \left[ \sup_{\theta \in N} \sup_{\tau \in T(\theta)} e^{k_1 \tau' \psi(X_1, \theta)} b(X_1) \right] < \infty \quad \text{for } k_1 \in [1, 2] \text{ and } k_2 \in [1, 4] \text{ s.t., for all } j \in [0, 3], \sup_{\theta \in N} |\nabla^j \psi(X_1, \theta)| < b(X_1) \text{ where } \nabla^j \psi(X_1, \theta) \text{ denotes a vector of all partial derivatives of } \theta \mapsto \psi(X_1, \theta) \text{ of order } j.
\]

Assumptions 1 and 2 are stronger than the usual assumptions in the MM literature, but are similar to assumptions used in the entropy literature and related literatures. Assumptions 1 and 2 are essentially adapted from Haberman (1984), Kitamura and Stutzer (1997), and Schennach (2007, Assumption 3). See also Chib et al. (2018) for similar assumptions. The Appendix C.1 (p. 84) contains a detailed discussion of Assumptions 1 and 2.

In addition to Assumptions 1 and 2, we require the following standard and mild assumption to establish the asymptotic distribution of the Wald, LM, ALR, and ET statistics.

Assumption 3 (For the trinity+1). (a) The function \( r : \Theta \rightarrow \mathbb{R}^q \) in the null hypothesis \( \theta_0 \) is continuously differentiable. (b) The derivative \( R(\theta) := \frac{\partial r(\theta)}{\partial \theta} \) is full rank at \( \theta_0 \).

**Appendix B. Proofs**

**B.1. Proof of Theorem 1(i): Existence and consistency.** The proof of Theorem 1(i) (i.e., consistency) adapts the Wald’s approach to consistency (Wald 1949) along the lines of Kitamura and Stutzer (1997), Schennach (2007), Chib et al. (2018) and others. More precisely, standardizing the logarithm of the ESP approximation, we show that, \( \mathbb{P} \)-a.s. for \( T \) big enough, the ESP estimator maximizes the LogESP function (8) on p. 5, where
\[
\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r(\theta)\psi_t(\theta)} - \ln \mathbb{E} [e^{r(\theta)\psi(X_1, \theta)}] \right| = o(1)
\]
and
\[
\sup_{\theta \in \Theta} \left| \frac{1}{2T} \ln |\Sigma_T(\theta)|_{\text{det}} \right| = O(T^{-1}).
\]

The two main differences between our proof of Theorem 1(i) and the proofs available in the entropy literature are the following. Firstly, we need to ensure that, for \( T \) big enough, for all \( \theta \in \Theta \), \( |\Sigma_T(\theta)|_{\text{det}} \) is bounded away from zero, so that the LogESP function (8) on p. 5 does not diverge to \( \infty \) on parts of the parameter space. Secondly, we prove that the joint parameter space for \( \theta \) and \( \tau \) (i.e., \( S \)) is a compact set.

**Core of the proof of Theorem 1.** Under Assumption 1(a)(b) and (d)-(h), by Lemma 1 (p. 20), \( \mathbb{P} \)-a.s. for \( T \) big enough, the ESP approximation and the ESP estimator exist. Moreover, under Assumption 1(a)-(h) and (d)-(h), by Lemma 6 (p. 25), \( \mathbb{P} \)-a.s. for \( T \) big enough, \( |\Sigma_T(\theta)|_{\text{det}} > 0 \), for all \( \theta \in \Theta \). Thus, we can apply the strictly increasing transformation \( x \mapsto \frac{1}{T} \ln(x) - \frac{m}{2} \ln(\frac{T}{2}) \).
to the ESP approximation in equation (11) on p. 18 so that, \( \mathbb{P} \)-a.s. for \( T \) big enough,

\[
\hat{\theta}_T \in \arg \max_{\theta \in \Theta} \hat{f}_{\theta_T}(\theta)
\]

\[
\Leftrightarrow \hat{\theta}_T \in \arg \max_{\theta \in \Theta} \left\{ \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t(\theta)\psi_t(\theta)} \right] - \frac{1}{2T} \ln |\Sigma_T(\theta)|_{\text{det}} \right\}.
\]

(16)

Now, by the triangle inequality,

\[
\sup_{\theta \in \Theta} \left\{ \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t(\theta)\psi_t(\theta)} \right] - \frac{1}{2T} \ln |\Sigma_T(\theta)|_{\text{det}} - \ln \mathbb{E}[e^{\tau(\theta)\psi(X_t,\theta)}] \right\} \\
\leq \sup_{\theta \in \Theta} \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t(\theta)\psi_t(\theta)} \right] - \ln \mathbb{E}[e^{\tau(\theta)\psi(X_t,\theta)}] + \sup_{\theta \in \Theta} \frac{1}{2T} \ln |\Sigma_T(\theta)|_{\text{det}}
\]

(17)

where the last equality follow from Lemma 2v (p. 21) and Lemma 6v (p. 25) under Assumption 1(a)-(b) and (d)-(h). Thus, regarding \( \hat{\theta}_T \), it is now sufficient to check the assumptions of the standard consistency theorem (e.g. Newey and McFadden 1994 pp. 2121-2122 Theorem 2.1, which is also valid in an almost-sure sense). Firstly, under Assumption 1(a)-(e) and (g)-(h), by Lemma 10v (p. 32), \( \theta \mapsto \ln \mathbb{E}[e^{\tau(\theta)\psi(X_t,\theta)}] \) is uniquely maximized at \( \theta_0 \), i.e., for all \( \theta \in \Theta \setminus \{\theta_0\} \), \( \ln \mathbb{E}[e^{\tau(\theta_0)\psi(X_t,\theta_0)}] < \ln \mathbb{E}[e^{\tau(\theta)\psi(X_t,\theta)}] = 0 \). Secondly, under Assumptions 1(a)(b)(d)(e)(g) and (h), by Lemma 3 (p. 23), \( \theta \mapsto \ln \mathbb{E}[e^{\tau(\theta)\psi(X_t,\theta)}] \) is continuous in \( \Theta \). Finally, by Assumption 1(d), the parameter space \( \Theta \) is compact. \( \square \)

**Lemma 1** (Existence of the ESP approximation and estimator). Under Assumption 1(a)(b) and (d)-(h), \( \mathbb{P} \)-a.s. for \( T \) big enough,

(i) the ESP approximation \( \tilde{f}_{\theta_T}(\cdot) \) exists;

(ii) \( \theta \mapsto \tau_T(\theta) \) is unique and continuously differentiable in \( \Theta \), so that the ESP approximation \( \theta \mapsto \tilde{f}_{\theta_T}(\theta) \) is also unique and continuous in \( \Theta \);

(iii) for all \( \theta \in \Theta \), the ESP approximation \( \omega \mapsto \tilde{f}_{\theta_T}(\theta) \) is \( \mathcal{E}/\mathcal{B}(\mathbb{R}) \)-measurable; and

(iv) there exists an ESP estimator \( \hat{\theta}_T \in \arg \max_{\theta \in \Theta} \tilde{f}_{\theta_T}(\theta) \) that is \( \mathcal{E}/\mathcal{B}(\mathbb{R}^m) \)-measurable.

**Proof.** The result follows from Lemmas 2 (p. 21), 3 (p. 23) and 6 (p. 25) and standard arguments. For completeness, a detailed proof is provided.

(i) Under Assumption 1(a)(b), (d)-(e)(g) and (h), by Lemma 2i (p. 21), \( \mathbb{P} \)-a.s. there exists a \( \mathcal{B}(\Theta) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R}^m) \)-measurable function \( \tau_T(\cdot) \) s.t., for \( T \) big enough, for all \( \theta \in \Theta \),

\[
\frac{1}{T} \sum_{t=1}^{T} e^{\tau_t(\theta)\psi_t(\theta)} = 0 \quad \text{and} \quad \tau_T(\theta) \in \text{int}(T(\theta)).
\]

Moreover, under Assumption 1(a)(b)(d)(e)(g) and (h), by Lemma 3 (p. 23) with \( P = \frac{1}{T} \sum_{t=1}^{T} \delta_{X_t} \), for all \( T \in [1, \infty] \), for all \( (\theta, \tau) \in \mathcal{S} \),

\[
0 < \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t(\theta)\psi_t(\theta)} \quad \text{so that, for all \( \theta \in \Theta \),} \quad 0 < \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)\psi_t(\theta)}.
\]

Thus, the ET term exists. Now, under Assumption 1(a)-(b) and (d)-(h), by Lemma 6v (p. 25), \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \inf_{\theta \in \Theta} |\Sigma_T(\theta)|_{\text{det}} > 0 \), so that the variance term of the ESP approximation exists. Thus, the ESP approximation exists.

(ii) By Assumption 1(b), \( \theta \mapsto \psi(X_1,\theta) \) is continuously differentiable in \( \Theta^c \) \( \mathbb{P} \)-a.s., so that it is sufficient to show that \( \tau_T(\cdot) \) is unique and continuous, which we prove at once with the standard implicit function theorem. Check its assumptions. Firstly, under Assumption 1(a)(b), (d)-(e)(g) and (h), by Lemma 2ii (p. 21), \( \mathbb{P} \)-a.s. there exists a function \( \tau_T(\cdot) \) s.t., for \( T \) big
enough, for all \( \theta \in \Theta \), \( \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) \psi_t(\theta) = 0_{m \times 1} \) and \( \tau_T(\theta) \in \text{int} [T(\theta)] \). Secondly, for all \( \hat{\theta} \in \Theta \), \( \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\hat{\theta})'} \psi_t(\hat{\theta}) \psi_t(\hat{\theta}) \) is full rank \( \mathbb{P} \)-a.s. for \( T \) big enough for all \( \theta \in \Theta \), because under Assumption 1(a)-(b) and (d)-(h), by Lemma 3(iii) \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \text{inf}_{\theta \in \Theta} |\Sigma_T(\theta)|_{\text{det}} > 0 \). Finally, by Assumption 1(b), \( (\theta, \tau) \mapsto \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) \psi_t(\theta) \) is continuously differentiable in \( S^c \).

(iii) By Assumption 1(b), for all \( \theta \in \Theta \), \( x \mapsto \psi(x, \theta) \) is \( \mathcal{B}(\mathbb{R}^p)/\mathcal{B}(\mathbb{R}^m) \)-measurable. Moreover, under Assumption 1(a)-(b)(d)(e)(g) and (h), by Lemma 3(iii) \( \mathbb{P} \)-a.s. \( \tau_T(\cdot) \) is a \( \mathcal{B}(\Theta) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R}^m) \)-measurable function. Thus, the result follows.

(iv) By Assumption 1(d), \( \Theta \) is compact, so that, by the statements (i)-(iii) of the present lemma, the result follows from the Schmetterer-Jennrich lemma (Schmetterer 1966 Chap. 5 Lemma 3.3; Jennrich 1969 Lemma 2). \( \square \)

**Lemma 2** (Asymptotic limit of the ET term). Under Assumption 1(a)(b), (d)-(e)(g) and (h),

\( (i) \) \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \), which implies that \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \);

\( (ii) \) \( \mathbb{P} \)-a.s. there exists a \( \mathcal{B}(\Theta) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R}^m) \)-measurable function \( \tau_T(\cdot) \) s.t., for \( T \) big enough, for all \( \theta \in \Theta \), \( \tau_T(\theta) \in \text{arg min}_{\tau \in \mathbb{R}^m} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \), which implies that \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \tau_T(\theta) - \tau(\theta) \right| = o(1) \);

\( (iii) \) \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{\theta \in \Theta} |\tau_T(\theta)| = o(1) \);

\( (iv) \) \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \), which implies that \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \).

**Proof.** (i) Under Assumptions 1(a)-(b)(d)(e)(g) and (h), by Lemma 3(iii) \( \mathbb{P} \)-a.s. \( \sup_{(\theta, \tau) \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \), which implies that \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \);

(ii) By Lemma 3(iii) \( \mathbb{P} \)-a.s. there exists a \( \mathcal{B}(\Theta) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R}^m) \)-measurable function \( \tau_T(\cdot) \) s.t., for \( T \) big enough, for all \( \theta \in \Theta \), \( \tau_T(\theta) \in \text{arg min}_{\tau \in \mathbb{R}^m} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r_T(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{r_T(X_1, \theta)}] \right| = o(1) \), which implies that \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{(\theta, \tau) \in \Theta} \left| \tau_T(\theta) - \tau(\theta) \right| = o(1) \);

(iii) By Assumption 1(d), \( \Theta \) is compact, so that, by the statements (i)-(iii) of the present lemma, the result follows from the Schmetterer-Jennrich lemma (Schmetterer 1966 Chap. 5 Lemma 3.3; Jennrich 1969 Lemma 2). \( \square \)

Note that, unlike what has been sometimes suggested in the entropy literature, if \( T(\theta) \) is an unspecified compact set, \( \{ (\theta, \tau) : \theta \in \Theta \land \tau \in T(\theta) \} \) does not need to be a compact set, \( \{ (\theta, \tau) : \theta \in \Theta \land \tau \in T(\theta) \} \) is not a Cartesian product, but the graph of a correspondence. See Lemma 3(i) \( \mathbb{P} \)-a.s. for more details.
$(\theta, \omega) \mapsto \frac{1}{T} \sum_{t=1}^{T} e^{r_t} \psi_t(\theta)$ is continuous w.r.t. $\theta$ and $\mathcal{E}/\mathcal{B}(\mathbb{R})$-measurable w.r.t. to $\omega$, so that it is $\mathcal{B}(\Theta) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R})$-measurable (e.g., Aliprantis and Border 2006/1999, Lemma 4.51). Moreover, under Assumptions 1(a)-(b)(d)(e)(g) and (h), by Lemma 4(i) (p. 24), $\theta \mapsto T(\theta)$ is a nonempty compact valued measurable correspondence. Then, by a generalization of the Schmetterer-Jennrich lemma (e.g., Aliprantis and Border 2006/1999, Theorem 18.19), we can define a $\mathcal{B}(\Theta) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R})$-measurable function $\tilde{\tau}_T(\theta)$ s.t., for all $\theta \in \Theta$, $\tilde{\tau}_T(\theta) \in \text{arg min}_{\tau \in \mathcal{T}(\theta)} \frac{1}{T} \sum_{t=1}^{T} e^{r_t} \psi_t(\theta)$. For the present proof, put $\varepsilon := \inf_{\theta \in \Theta} \inf_{\tau \in \mathcal{T}(\theta)} e^{r_t} \psi_t(\theta) \leq \mathbb{E}[e^{r_t} \psi_t(\theta)]$, which is strictly positive by Lemma 5 (p. 24) under Assumptions 1(a)(b)(d)(e) and (h)\(^{10}\). Then, by the definition of $\varepsilon$, whenever $\sup_{\theta \in \Theta} [\mathbb{E}[e^{r_t} \psi_t(\theta)] - \mathbb{E}[e^{r_t} \psi_t(\theta)]] < \varepsilon$, then $\sup_{\theta \in \Theta} |\tilde{\tau}_T(\theta) - \tau(\theta)| \leq \eta$. We now show that it is happening $\mathbb{P}$-a.s. as $T \to \infty$. Under Assumptions 1(a)(b)(d)(e)(g) and (h), by Lemma 10 (p. 32), $\tau(\theta) = \text{arg min}_{\tau \in \mathcal{R}} \mathbb{E}[e^{r_t} \psi_t(\theta)]$, so that

\[
\begin{align*}
\sup_{\theta \in \Theta} \left| \mathbb{E}[e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)] - \mathbb{E}[e^{\tau(\theta)'} \psi_t(\theta)] \right| \\
= \sup_{\theta \in \Theta} \left\{ \mathbb{E}[e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)] - \mathbb{E}[e^{\tau(\theta)'} \psi_t(\theta)] \right\} \\
\overset{(a)}{=} \sup_{\theta \in \Theta} \left\{ \mathbb{E}[e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)] - \frac{1}{T} \sum_{t=1}^{T} e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta) + \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta) - \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta) \right\} \\
+ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{\tau(\theta)'} \psi_t(\theta)] \\
\overset{(b)}{\leq} \sup_{\theta \in \Theta} \left\{ \mathbb{E}[e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)] - \frac{1}{T} \sum_{t=1}^{T} e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta) + \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{\tau(\theta)'} \psi_t(\theta)] \right\} \\
\overset{(c)}{\leq} \sup_{\theta \in \Theta} \left\{ \mathbb{E}[e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)] - \frac{1}{T} \sum_{t=1}^{T} e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta) + \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta) - \mathbb{E}[e^{\tau(\theta)'} \psi_t(\theta)] \right| \right\} \\
\overset{(d)}{=} o(1) \text{ $\mathbb{P}$-a.s. as } T \to \infty.
\end{align*}
\]

(a) Add and subtract $\frac{1}{T} \sum_{t=1}^{T} e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)$ and $\frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta)$. (b) Note that, under Assumption $(\theta, \omega) \mapsto \frac{1}{T} \sum_{t=1}^{T} e^{r_t} \psi_t(\theta)$ and $(\psi_t(\theta), \omega) \mapsto \mathbb{E}[e^{\psi_t(\theta)}]$, for all $\theta \in \Theta$, $T(\theta) \in \mathcal{T}(\theta)$ so that $\frac{1}{T} \sum_{t=1}^{T} e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta) - \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi_t(\theta) \leq 0$. (c) Triangle inequality w.r.t. the uniform norm. (d) Under Assumption 1(d), by definition, for all $\theta \in \Theta$, $\tau(\theta) \in \mathcal{T}(\theta)$ and $\tilde{\tau}_T(\theta) \in \mathcal{T}(\theta)$ so that the conclusion follows from statement (i).

Inequality (18) implies that $\sup_{\theta \in \Theta} |\tilde{\tau}_T(\theta) - \tau(\theta)| = o(1) \text{ $\mathbb{P}$-a.s. as } T \to \infty$. Moreover, by Assumption 1(e), for all $\theta \in \Theta$, $T(\theta) = \mathcal{B}_c(\tau(\theta))$ where $\mathcal{B}_c > 0$. Thus, $\mathbb{P}$-a.s., for $T$ big enough, for all $\theta \in \Theta$, $\tilde{\tau}_T(\theta) \in \text{int} [\mathcal{T}(\theta)]$. Now, for all $\theta \in \Theta$, $\tau \mapsto \frac{1}{T} \sum_{t=1}^{T} e^{r_t} \psi_t(\theta)$ is a convex function (Lemma 29 on p. 87) with $\mathbb{P} = \frac{1}{T} \sum_{t=1}^{T} \delta_{X_t}$ ensures that $\frac{\partial^2}{\partial r_t \partial r_s} \frac{1}{T} \sum_{t=1}^{T} e^{r_t} \psi_t(\theta) = \frac{1}{T} \sum_{t=1}^{T} e^{\psi_t(\theta)} \psi'_t(\theta) \psi_t(\theta)^T \geq 0$, and the local minimum of a convex function is a global minimum (e.g., Hiriart-Urruty and Lemaréchal 1993/1996, p. 253). Therefore, $\mathbb{P}$-a.s. for $T$ big enough, for

\footnote{The argument requires $\varepsilon > 0$. If $\varepsilon = 0$, then the upcoming inequality (18) is not sufficient to show that $\sup_{\theta \in \Theta} \mathbb{E}[e^{\tilde{\tau}_T(\theta)'} \psi_t(\theta)] - \mathbb{E}[e^{\tau(\theta)'} \psi_t(\theta)] < \varepsilon$.}

\footnote{Strict convexity of $\tau \mapsto \mathbb{E}[e^{r_t} \psi_t(\theta)]$ and compactness of $\Theta$ are not sufficient to ensure that $\varepsilon > 0$: We also need the continuity of the value function of the first infimum, which we obtain through Berge’s maximum theorem. See Lemma 5 (p. 24).}
all $\theta \in \Theta$, $\tilde{\tau}_T(\theta)$ minimizes $\frac{1}{T} \sum_{t=1}^T e^{r^T \psi_t(\theta)}$ not only over $T(\theta)$, but also over $\mathbb{R}^m$, which means that we can put $\tilde{\tau}_T(\theta) = \tau_T(\theta)$.

(iv) Addition and subtraction of $\mathbb{E}[e^{r^T (\theta) \psi(X_1, \theta)}]$, and the triangle inequality yield $\mathbb{P}$-a.s. for $T$ big enough

$$\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^T e^{r^T (\theta) \psi_t(\theta)} - \mathbb{E}[e^{r^T (\theta) \psi(X_1, \theta)}] \right| \leq \sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^T e^{r^T (\theta) \psi_t(\theta)} - \mathbb{E}[e^{r^T (\theta) \psi(X_1, \theta)}] \right| + \sup_{\theta \in \Theta} \left| \mathbb{E}[e^{r^T (\theta) \psi(X_1, \theta)}] - \mathbb{E}[e^{r^T (\theta) \psi(X_1, \theta)}] \right|$$

$$= o(1)$$, as $T \to \infty$,

where the explanations for the last equality are as follows. By the statement (i) of the present lemma, $\mathbb{P}$-a.s. as $T \to \infty$, $\sup_{(\theta, \tau) \in S} \left| \frac{1}{T} \sum_{t=1}^T e^{r^T \psi_t(\theta)} - \mathbb{E}[e^{r^T \psi(X_1, \theta)}] \right| = o(1)$. Moreover, by the statement (ii) of the present lemma, $\mathbb{P}$-a.s. for $T$ big enough, $\tau_T(\theta) \in \text{int}(T(\theta))$, so that, for all $\theta \in \Theta$, $(\theta, \tau_T(\theta)) \in S$. Thus, the first supremum is $o(1)$ as $T \to \infty$. Regarding the second supremum, under Assumption 1 (a)(b) and (h), by Lemma 3 (p. 23), $(\theta, \tau) \mapsto \mathbb{E}[e^{r^T \psi(X_1, \theta)}]$ is continuous in $\Theta$. Now, under Assumptions 1 (a)(b)(d)(e)(g) and (h), by Lemma 4iii (p. 24), $S$ is compact, so that $(\theta, \tau) \mapsto \mathbb{E}[e^{r^T \psi(X_1, \theta)}]$ is also uniformly continuous in $\Theta$—continuous functions on compact sets are uniformly continuous (e.g., Rudin 1953 Theorem 4.19). Thus, under Assumption 4(a)(b), (d)-(e), (g) and (h), by the statement (iii) of the present lemma, which states that $\mathbb{P}_{\tau} \sup_{\theta \in \Theta} |\tau_T(\theta) - \tau(\theta)| = o(1) \mathbb{P}$-a.s. as $T \to \infty$, the second supremum is also $o(1) \mathbb{P}$-a.s. as $T \to \infty$.

The second part of the result follows from the first part as in the proof of the statement (i) of the present lemma.

\[ \square \]

**Lemma 3.** Let $P$ be any probability measure, and $\mathbb{E}_P$ denote the expectation under $P$. Under Assumption 1 (a) and (b), if $\mathbb{E}_P[\sup_{(\theta, \tau) \in S} e^{r^T \psi(X_1, \theta)}] < \infty$, then

$$0 < \inf_{(\theta, \tau) \in S} \mathbb{E}_P[e^{r^T \psi(X_1, \theta)}]$$

so that $0 < \inf_{\theta \in \Theta} \mathbb{E}_P[e^{r^T (\theta) \psi(X_1, \theta)}]$. Moreover, $(\theta, \tau) \mapsto \mathbb{E}_P[e^{r^T (\theta) \psi(X_1, \theta)}]$ and $\theta \mapsto \mathbb{E}_P[e^{r^T (\theta) \psi(X_1, \theta)}]$ are continuous in $\Theta$ and $\Theta$, respectively. All of these results hold for $P = \mathbb{P}$ under the aforementioned assumptions.

**Proof.** Under Assumption 1 (a) and (b), the Lebesgue dominated convergence theorem and the lemma’s assumption $\mathbb{E}_P[\sup_{(\theta, \tau) \in S} e^{r^T \psi(X_1, \theta)}] < \infty$ imply that $(\theta, \tau) \mapsto \mathbb{E}_P[e^{r^T \psi(X_1, \theta)}]$ is continuous. Moreover, under Assumptions 1 (a)(b)(d)(e)(g) and (h), by Lemma 1 (p. 24), $S$ is compact, and continuous functions over compact sets reach a minimum (e.g., Rudin 1953 Theorem 4.16). Now, if there exist $(\hat{\tau}, \hat{\theta}) \in S$ s.t. $0 = \mathbb{E}_P[e^{r^T \psi(X_1, \hat{\theta})}]$, then $e^{r^T \psi(X_1, \theta)} = 0$ $P$-a.s. (e.g., Kallemberg 2002 (1997) Lemma 1.24), which is impossible by definition of the exponential function. Thus, $0 < \inf_{(\theta, \tau) \in S} \mathbb{E}_P[e^{r^T \psi(X_1, \theta)}]$, so that $0 < \inf_{\theta \in \Theta} \mathbb{E}_P[e^{r^T (\theta) \psi(X_1, \theta)}]$ because by the definition of $S$ in Assumption 1(e), for all $\theta \in \Theta$, $(\theta, \tau(\theta)) \in S$. Regarding the second part of the result, it immediately follows from the Lebesgue dominated convergence theorem, the lemma’s assumption that $\mathbb{E}_P[\sup_{(\theta, \tau) \in S} e^{r^T (\theta) \psi(X_1, \theta)}] < \infty$, and the continuity of $\tau : \Theta \to \mathbb{R}^m$ by Lemma 10iii (p. 32) under Assumptions 1 (a)(b)(d)(e)(g) and (h). Regarding the third part of the result, it is sufficient to note that, under Assumption 1 (a)(b), by the Cauchy-Schwarz
inequality, \( E[\sup_{(\theta, \tau) \in S} e^{\tau \psi(X_1, \theta)}] \leq E[\sup_{(\theta, \tau) \in S} e^{2\tau \psi(X_1, \theta)}]^{1/2} < \infty \), where the last inequality follows from Assumption (f).

**Lemma 4 (Compactness of \( S \)).** Under Assumptions (a)(b)(d)(e)(g) and (h),

(i) The closure of the \( \epsilon_T \)-neighborhood of \( \tau(\Theta) \) (i.e., \( \tau(\Theta)^{\epsilon_T} \)) is compact

(ii) For all \( \theta \in \Theta \), the correspondence \( \theta \to T(\theta) \) is nonempty compact-valued and uhc (upper hemi-continuous), and thus measurable;

(iii) The set \( S := \{ (\theta, \tau) : \theta \in \Theta \land \tau \in T(\theta) \} \) is compact.

**Proof.** (i) Under Assumptions (a)(b)(d)(e)(g) and (h), by Lemma 10(ii) (p. 32), \( \tau : \Theta \to R^m \) is continuous. Moreover, by Assumption (d), \( \Theta \) is compact. Thus, \( \tau(\Theta) \) is bounded — continuous mappings preserve compactness (e.g., Rudin 1953, Theorem 4.14). Consequently, \( \tau(\Theta)^{\epsilon_T} = \{ \tau \in R^m : \inf_{\tau \in \tau(\Theta)} |\tau - \bar{\tau}| < \epsilon_T \} \) is bounded, which means that its closure \( \tau(\Theta)^{\epsilon_T} \) is closed and bounded, i.e., compact.

(ii) Proof that \( T \) is nonempty and compact valued. By Assumption (d), for all \( \theta \in \Theta \), there exists \( \tau(\theta) \) s.t. \( E[e^{\tau(\theta) \psi(X_1, \theta)} \psi(X_1, \theta)] = 0 \). Thus, for all \( \theta \in \Theta \), \( T(\theta) = B_{\epsilon_T}(\tau(\theta)) \) is nonempty. Moreover, by construction, \( B_{\epsilon_T}(\tau(\theta)) \) is compact, so that it is nonempty compact valued.

Proof that \( T \) is uhc. Because \( T \) is compact valued, we can use the sequential characterization of upper hemicontinuity (e.g., Aliprantis and Border 2006/1999, Theorem 17.20). Let \( (\theta_n, \tau_n)_{n \in N} \in (S)^N \) be a sequence s.t., for all \( n \in N \), \( \tau_n \in T(\theta_n) \) and \( \theta_n \to \bar{\theta} \in \Theta \) as \( n \to \infty \). By construction, for all \( n \in N \), \( \tau_n \in B_{\epsilon_T}(\tau(\theta_n)) \subseteq \tau(\Theta)^{\epsilon_T} \). Moreover, by statement (i), \( \tau(\Theta)^{\epsilon_T} \) is compact, so that there exists a subsequence \( (\tau_{\alpha(n)})_{n \in N} \) s.t. \( \tau_{\alpha(n)} \to \bar{\tau} \in \tau(\Theta)^{\epsilon_T} \), as \( n \to \infty \). Again, by construction, for all \( n \in N \), \( (\theta_n, \tau_n) \in S \), so that \( |\tau_{\alpha(n)} - \tau(\theta_{\alpha(n)})| \leq \epsilon_T \). Now, under Assumptions (a)(b)(d)(e)(g) and (h), by Lemma 10, \( \tau : \Theta \to R^m \) is continuous. Thus, \( |\tau_{\alpha(n)} - \tau(\theta_{\alpha(n)})| \to |\bar{\tau} - \tau(\bar{\theta})| \) as \( n \to \infty \). Thus, \( |\bar{\tau} - \tau(\bar{\theta})| \leq \epsilon_T \), which means that \( \bar{\tau} \in B_{\epsilon_T}(\tau(\bar{\theta})) = T(\bar{\theta}) \).

Proof that \( T \) is measurable. Let \( F \) be a closed subset of \( R^m \). Then, its complement \( F^c \) is an open subset of \( R^m \). Now, a correspondence is uhc iff the upper inverse image of an open set is an open set (e.g., Aliprantis and Border 2006/1999, Lemma 17.4). Thus, by the previous paragraph, \( T^u(F^c) \subseteq B(\Theta) \), where \( T^u \) denotes the upper inverse of \( T \). Now, denote the lower inverse of \( T \) with \( T^l \), notice that \( T^u(F^c) = [T^l(F)]^c \) (e.g., Aliprantis and Border 2006/1999, p. 557), so that \( [T^l(F)]^c \subseteq B(\Theta) \), which, in turn implies that \( T^l(F) \subseteq B(\Theta) \) because of the stability of \( \sigma \)-algebras under complementation.

(iii) Note that the compactness of \( \Theta \) and \( T(\theta) \) are not sufficient to ensure the compactness of \( S \) because \( S \) is not a Cartesian product. By the statement (ii) of the present lemma, \( T \) is uhc and closed valued, so that it has a closed graph (e.g., Aliprantis and Border 2006/1999, Theorem 17.10), i.e., \( S \) is closed. Now, by construction, \( S \) is a subset \( \tau(\Theta)^{\epsilon_T} \times \Theta \), which is compact by statement (i) and Assumption (d). Thus, \( S \) is also compact — in metric spaces, closed subsets of compact sets are compact (e.g., Rudin 1953, Theorem 2.35).

**Lemma 5.** Under Assumptions (a)(b)(d)(e) and (h),

(i) for any constant \( \eta \in [0, \epsilon_T] \), there exists a continuous value function \( v : \Theta \to R_+ \) s.t., for all \( \theta \in \Theta \), \( v(\theta) = \inf_{\tau \in T(\theta) : |\tau - \theta| > \eta} \left[ E[e^{\tau \psi(X_1, \theta)}] - E[e^{\tau \psi(X_1, \theta)}] \right] \);

(ii) for any constant \( \eta \in [0, \epsilon_T] \), \( 0 < \inf_{\theta \in \Theta} \inf_{\tau \in T(\theta) : |\tau - \theta| > \eta} \left[ E[e^{\tau \psi(X_1, \theta)}] - E[e^{\tau \psi(X_1, \theta)}] \right] \).
Proof. (i) It is a consequence of Berge’s maximum theorem (e.g., Aliprantis and Border 2006/1999 Theorem 17.31). Thus, it remains to check its assumptions. For the present proof, define the correspondence \( \varphi : \Theta \to \mathbb{R}^m \) s.t. \( \varphi(\theta) = \{ \tau \in T(\theta) : |\tau - \tau(\theta)| \geq \eta \} \), and the function \( f : S \to \mathbb{R}_+ \) s.t. \( f(\theta, \tau) = |\mathbb{E}[e^{\tau\psi(X_1, \theta)}] - \mathbb{E}[e^{\tau(\theta)\psi(X_1, \theta)}]| \).

Proof of the continuity of \( f \). Under Assumption [1](a)(b)(d)(e)(g) and (h), by Lemma [3](p. 23), \( (\theta, \tau) \mapsto \mathbb{E}[e^{\tau\psi(X_1, \theta)}] \) and \( \theta \mapsto \mathbb{E}[e^{\tau(\theta)\psi(X_1, \theta)}] \) are continuous in \( S \) and \( \Theta \), respectively, so that the continuity of \( f \) follows immediately.

Proof that \( \varphi \) is nonempty compact valued. By the definition of \( T \) in Assumption [1](e), for all \( \theta \in \Theta \), \( T(\theta) = B_{\mathcal{N}}(\tau(\theta)) \), so that, for any \( \eta \in ]0, \epsilon_T[ \), \( \varphi(\theta) = \overline{B}_{\mathcal{N}}(\tau(\theta)) \cap \{ \tau \in \mathbb{R}^m : \eta \leq |\tau - \tau(\theta)| \} \neq \emptyset \), i.e., \( \varphi \) is nonempty valued. Moreover, for all \( \theta \in \Theta \), \( \overline{B}_{\mathcal{N}}(\tau(\theta)) \) is a compact set and \( \{ \tau \in \mathbb{R}^m : \eta \leq |\tau - \tau(\theta)| \} \) is a closed set, so that \( \varphi(\theta) \), which is their intersection, is compact (e.g., Rudin 1953, Theorem 2.35 and the following Corollary).

Proof of the upper hemicontinuity of \( \varphi \). Use the sequential characterization of the upper hemicontinuity (e.g., Aliprantis and Border 2006/1999 Theorem 17.20). Let \( ((\theta_n, \tau_n))_{n \in \mathbb{N}} \) be a sequence s.t., for all \( n \in \mathbb{N} \), \( \tau_n \in \varphi(\theta_n) \) and \( \theta_n \to \theta \in \Theta \) as \( n \to \infty \). Now, under Assumptions [1](a)(b)(d)(e)(g) and (h), Lemma [4](p. 24), \( S \) is a compact set, so that there exists a subsequence \( ((\theta_{\alpha(n)}, \tau_{\alpha(n)}))_{n \in \mathbb{N}} \) s.t. \( (\theta_{\alpha(n)}, \tau_{\alpha(n)}) \to (\theta, \bar{\tau}) \in S \), as \( n \to \infty \). The definition of \( S \) implies that \( \bar{\tau} \in T(\hat{\theta}) \), thus, it remains to show that \( \eta \leq |\bar{\tau} - \tau(\theta)| \) in order to conclude that \( \bar{\tau} \in \varphi(\theta) \). By construction, for all \( n \in \mathbb{N} \), \( \eta \leq |\tau_{\alpha(n)} - \tau(\theta_{\alpha(n)})| \). Moreover, under Assumptions [1](a)(b)(d)(e)(g) and (h), by Lemma [10](p. 32), \( \tau : \Theta \to \mathbb{R}^m \) is continuous, so that \( |\tau_{\alpha(n)} - \tau(\theta_{\alpha(n)})| \to |\bar{\tau} - \tau(\theta)| \) as \( n \to \infty \), which means that \( \eta \leq |\bar{\tau} - \tau(\theta)| \).

Proof of the lower hemicontinuity of \( \varphi \). Use the sequential characterization of the lower hemicontinuity (e.g., Aliprantis and Border 2006/1999 Theorem 17.21). Let \( (\theta_n)_{n \in \mathbb{N}} \in \Theta^\mathbb{N} \) be a sequence s.t. \( \theta_n \to \theta \in \Theta \) and \( \bar{\tau} \in \varphi(\theta) \). Define the sequence \( (\tau_n)_{n \in \mathbb{N}} \) s.t., for all \( n \in \mathbb{N} \), \( \tau_n = \tau(\theta_n) + \bar{\tau} - \tau(\theta) \). By definition of the correspondence \( \varphi \), for all \( n \in \mathbb{N} \), \( |\tau_n - \tau(\theta_n)| = |\bar{\tau} - \tau(\theta)| \in ]\eta, \epsilon_T[ \), which implies that \( \tau_n \in \varphi(\theta_n) \). Moreover, under Assumptions [1](a)(b)(d)(e)(g) and (h), by Lemma [10](p. 32), \( \tau : \Theta \to \mathbb{R}^m \) is continuous, so that \( \lim_{n \to \infty} \tau_n = \lim_{n \to \infty} \tau(\theta_n) + \bar{\tau} - \tau(\theta) = \tau(\hat{\theta}) + \bar{\tau} - \tau(\hat{\theta}) = \bar{\tau} \).

(ii) Under Assumptions [1](a)(b)(d)(e)(g) and (h), by Lemma [10](p. 32), for all \( \theta \in \Theta \), \( \tau(\theta) \) is the unique minimum of the strictly convex minimization problem \( \inf_{\tau \in \mathbb{R}^m} \mathbb{E}[e^{\tau\psi(X_1, \theta)}] \). Thus, for all \( \theta \in \Theta \), \( v(\theta) > 0 \). Moreover, by Assumption [1](d), \( \Theta \) is compact, and by statement (i) of the present lemma, \( v(.) \) is continuous. Thus, there exists \( \varepsilon_v > 0 \) s.t. \( \min_{\theta \in \Theta} v(\theta) > \varepsilon_v \) because a continuous function over a compact set reaches a minimum (e.g., Rudin 1953, Theorem 4.16). \( \square \)

Lemma 6 (Asymptotic limit of the variance term). Under Assumption [1](a)-(b) and (d)-(h),

(i) \( \mathbb{P} \)-a.s. for \( T \) big enough, \( 0 < \inf_{\theta \in \Theta} \left| \left[ \sum_{t=1}^{T} \tilde{w}_t, \theta \frac{\partial \psi(t, \theta)}{\partial \theta} \right] \right|_{\det} \);

(ii) \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{\theta \in \Theta} \left| \sum_{t=1}^{T} (\theta) - \mathbb{E}[e^{\tau(\theta)\psi(X_1, \theta)}] \mathbb{E}T \right| = o(1) \);

(iii) \( \theta \to \Sigma(\theta) \) and \( \theta \to \mathbb{E}[e^{\tau(\theta)\psi(X_1, \theta)}] \mathbb{E} \) are continuous in \( \Theta \);

(iv) \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \inf_{\theta \in \Theta} \left| \sum_{t=1}^{T} (\theta) \right|_{\det} > 0 \);

(v) \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{\theta \in \Theta} \left| \log \left| \sum_{t=1}^{T} (\theta) \right|_{\det} - \log \left| \mathbb{E}[e^{\tau(\theta)\psi(X_1, \theta)}] \mathbb{E} \right|_{\det} \right| = o(1) \), so that, for all \( \eta > 0 \), \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \sup_{\theta \in \Theta} \left| \frac{1}{2\eta T} \log \left| \sum_{t=1}^{T} (\theta) \right|_{\det} \right| = o(1) \).
Proof. (i) Under Assumption $\Pi(a)$-$(b)$ and $(d)$-$(h)$, by Lemma $\Pi$ (p. 27), $\mathbb{P}$-a.s. as $T \to \infty$, sup$_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} \hat{w}_{t,\theta} \frac{\partial \psi(X_{t},\theta)}{\partial \theta} \right] = o(1)$, so that it is sufficient to check the invertibility of 
\[ \frac{1}{E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]} \] for all $\theta \in \Theta$ and the continuity of $\theta \mapsto \frac{1}{E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]}$ (Lemma 30 on p. 88). Firstly, by Assumption $\Pi(h)$, for all $\theta \in \Theta$, $\Sigma(\theta) := E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]^{-1}$ is a positive-definite symmetric matrix, and thus $E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]$ is invertible. Moreover, under Assumption $\Pi(a)$-$(b)$-$(d)$-$(e)$-$(g)$ and $(h)$, by Lemma 9 (p. 23), and Assumption $\Pi(e)$, for all $\theta \in \Theta$, $0 < E[e^{\tau(\theta)\psi(X_{1},\theta)}] < \infty$, so that $\frac{1}{E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]}$ is invertible for all $\theta \in \Theta$. Secondly, under Assumption $\Pi(a)$-$(b)$, $(e)$-$(f)$, by Lemma 8 (p. 29), $E \left[ \sup_{(\theta,\tau) \in S} \left| e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta} \right| \right] < \infty$, so that the Lebesgue dominated convergence theorem and Assumption $\Pi(b)$ imply the continuity of $\theta \mapsto E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]$ in $S$. Moreover, by definition in Assumption $\Pi(e)$, for all $\theta \in \Theta$, $(\tau(\theta),\theta) \in S$, and under Assumption $\Pi(a)$-$(b)$-$(d)$-$(e)$-$(g)$ and $(h)$, by Lemma 10 (p. 32), $\tau : \Theta \rightarrow \mathbb{R}^m$ is continuous. Thus, $\theta \mapsto \frac{1}{E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]}$ is continuous. Then, the continuity of $\theta \mapsto \frac{1}{E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]}$ follows from Lemma 3 (p. 23) under Assumption $\Pi(a)$-$(b)$-$(d)$-$(e)$-$(g)$ and $(h)$.

(ii) On one hand, by definition, $\Sigma(\theta) := E[e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta}]^{-1}$, which is symmetric positive definite by Assumption $\Pi(h)$, and $\Sigma_T(\theta) := \left[ \sum_{t=1}^{T} \hat{w}_{t,\theta} \frac{\partial \psi(X_{t},\theta)}{\partial \theta} \right]^{-1} \left[ \sum_{t=1}^{T} \hat{w}_{t,\theta} \psi_t(\theta)(\theta) \right] \left[ \sum_{t=1}^{T} \hat{w}_{t,\theta} \frac{\partial \psi(X_{t},\theta)}{\partial \theta} \right]^{-1}$, which is well defined $\mathbb{P}$-a.s. for $T$ big enough by the statement (i) of the present lemma. On the other hand, under Assumption $\Pi(a)$-$(b)$ and $(d)$-$(h)$, by Lemma 7 (p. 27), $\mathbb{P}$-a.s. as $T \to \infty$, sup$_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} \hat{w}_{t,\theta} \frac{\partial \psi(X_{t},\theta)}{\partial \theta} \right] = o(1)$, and, under Assumptions $\Pi(a)$-$(b)$, $(d)$-$(e)$ and $(g)$-$(h)$, by Lemma 8 (p. 29), $\mathbb{P}$-a.s. as $T \to \infty$, sup$_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)\psi_t(\theta)\psi_t(\theta)} \right] = o(1)$. Thus, the claim follows from the continuity of the inverse transformation (e.g., Rudin 1953 Theorem 9.8) and the limiting functions, and the compactness of $\Theta$.

(iii) Under Assumption $\Pi(a)$-$(b)$, $(e)$-$(g)$, by Lemma 7 (p. 27) and 8 (p. 29), $\mathbb{E} \left[ \sup_{(\theta,\tau) \in S} \left| e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta} \right| \right] < \infty$ and $\mathbb{E} \left[ \sup_{(\theta,\tau) \in S} \left| e^{\tau(\theta)\psi(X_{1},\theta)} \psi(X_{1},\theta) \right| \right] < \infty$, so that, by the Lebesgue dominated convergence theorem and Assumption $\Pi(b)$, $(\theta,\tau) \mapsto \mathbb{E} \left[ e^{\tau(\theta)\psi(X_{1},\theta)} \frac{\partial \psi(X_{1},\theta)}{\partial \theta} \right]$ and $(\theta,\tau) \mapsto \mathbb{E} \left[ e^{\tau(\theta)\psi(X_{1},\theta)} \psi(X_{1},\theta) \right]$ are continuous in $S$. Moreover, by definition in Assumption $\Pi(e)$, for all $\theta \in \Theta$, $(\theta,\tau(\theta)) \in S$, and under Assumption $\Pi(a)$-$(b)$-$(d)$-$(e)$-$(g)$ and $(h)$, by Lemma 10 (p. 32), $\tau : \Theta \rightarrow \mathbb{R}^m$ is continuous. Thus $\theta \mapsto \Sigma(\theta)$ is continuous, which is the first result. Under Assumption $\Pi(a)$-$(b)$-$(d)$-$(e)$-$(g)$ and $(h)$, the second result follows from Lemma 3 (p. 23), which states that $(\theta,\tau \mapsto \mathbb{E} \left[ e^{\tau(\theta)\psi(X_{1},\theta)} \Sigma(\theta) \right]$ for all $\theta \in \Theta$ (Lemma 30 on p. 88). By Assumption $\Pi(h)$, for all $\theta \in \Theta$,
\[ \Sigma(\theta) := \left[ \mathbb{E} e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right]^{-1} \mathbb{E} e^{\tau(\theta)'} \psi(X_1, \theta) \psi(X_1, \theta)' \left[ \mathbb{E} e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right]^{-1} \]

is a positive-definite symmetric matrix, and thus a fortiori invertible. Moreover, under Assumption 1 (a)(b)(d)(e)(g) and (h), by Lemma 3 (p. 23), and Assumption 1 (e), for all \( \theta \in \Theta \), 0 \(<\mathbb{E}[e^{\tau(\theta)'} \psi(X_1, \theta)] < \infty \), so that it is also invertible.

(v) Under Assumption 1 (a)(b)(d)(e)(g) and (h), by Lemma 3 (p. 23) with \( P = \sum_{t=1}^T \delta_{X_t} \), and by the statement (iv) of the present lemma, \( \mathbb{P}\text{-a.s. for } T \to \infty \),

\[
\frac{1}{T^0} \sup_{\theta \in \Theta} \left| \ln |\Sigma_T(\theta)|_{det} \right| = o(1) \]

where the explanations of the last equality are as follows. Under Assumption 1(a)-(b) and (d)-(h), by the statement (iii) of the present lemma \( \theta \mapsto \mathbb{E}[e^{\tau(\theta)'} \psi(X_1, \theta)] |\Sigma(\theta)|_{det} \) is continuous in \( \Theta \), which is a compact set by Assumption 1(d). Now, continuous functions over compact sets are bounded (e.g., Rudin 1953, Theorem 4.16), so that \( \sup_{\theta \in \Theta} \left| \ln |\mathbb{E}[e^{\tau(\theta)'} \psi(X_1, \theta)] |\Sigma(\theta)|_{det} \right| = o(1) \), as \( T \to \infty \). Now the last equality follows from the statement (iv) of the present lemma. \( \square \)

**Lemma 7.** Under Assumptions 1(a)-(b) and (e)-(f),

1. \( \mathbb{E}\left[ \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right| \right] < \infty; \)
2. under additional Assumption 1(d)(g) and (h), \( \mathbb{P}\text{-a.s. as } T \to \infty \),

\[
\sup_{(\theta, \tau) \in S} \left[ \frac{1}{T} \sum_{t=1}^T e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] - \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] = o(1), \text{ so that} \]

\[
\sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^T e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] - \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] = o(1); \text{ and} \]

3. under additional Assumption 1(d)(g) and (h), \( \mathbb{P}\text{-a.s. as } T \to \infty \),

\[
\sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^T u_{t, \theta} \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] - \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] = o(1) \]

**Proof.** (i) The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus,

\[
\mathbb{E}\left[ \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right| \right] \leq \mathbb{E}\left[ \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \right| \sup_{(\theta, \tau) \in S} \left| \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right| \right] \]

\[
= \mathbb{E}\left[ \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \right| \right]^{1/2} \mathbb{E}\left[ \sup_{\theta \in \Theta} \left| \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right| \right]^{1/2} \]

\[
< \infty \quad (19) \]
(a) Firstly, note that the expression in the second supremum does not depend on \( \tau \), so that
\[
\sup_{(\theta, \tau) \in S} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| = \sup_{\theta \in \Theta} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right|.
\]
Secondly apply the Cauchy-Schwarz inequality. Finally, note that
\[
\sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \right|^2 = \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \right|^2 = \sup_{\theta \in \Theta} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right|^2
\]
because \( x \mapsto x^2 \) is increasing on \( \mathbb{R}_+ \). (b) Note that
\[
e^{2\tau(\theta)'} \psi(X_1, \theta)
\]
and then apply Assumption 1(e) to the first term. Then, application of Assumption 1(f) to the second term yields the result.

(ii) By the triangle inequality, as \( T \to \infty \) \( \mathbb{P} \)-a.s.,
\[
\sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] - \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right]
\]
\[
\leq \sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] - \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right]
\]
\[
+ \sup_{\theta \in \Theta} \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] - \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right]
\]
\[
= o(1)
\]
where the explanations for the last equality are as follows. Regarding the first supremum, under Assumptions 1(a)-(b)(d)(e)(g) and (h), by Lemma 2ii (p. 21), \( S := \{(\theta, \tau) : \theta \in \Theta \land \tau \in \mathbb{T}(\theta)\} \) is a compact set, so that Assumptions 1(a)-(b)(d)(e)(g) and (h), by Lemma 2ii (p. 21), \( \sup_{\theta \in \Theta} |\tau_T(\theta) - \tau(\theta)| = o(1) \) \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \( \theta \in \Theta \), \( \tau_T(\theta) \in \mathbb{T}(\theta) \). Moreover, under Assumptions 1(a)(b), (d)-(e), (g) and (h), by Lemma 2iii (p. 21),
\[
\sup_{\theta \in \Theta} |\tau_T(\theta) - \tau(\theta)| = o(1) \) \( \mathbb{P} \)-a.s. as \( T \to \infty \). Thus, the first supremum is \( o(1) \), i.e.,
\[
\sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] = o(1)
\]
\( \mathbb{P} \)-a.s. Regarding the second supremum, by Assumption 1(b), \( (\theta, \tau) \mapsto e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \) is continuous. Moreover under Assumptions 1(a)-(b), and (e)-(f), by the statement (i) of the present lemma,
\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right| \right] < \infty.
\]
Thus, by the Lebesgue dominated convergence theorem and Assumption 1(b), \( (\theta, \tau) \mapsto \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] \) is also continuous in \( S \). Now, under Assumptions 1(a)(b)(d)(e)(g) and (h), by Lemma 2iii (p. 24), \( S \) is compact, so that \( (\theta, \tau) \mapsto \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)'}{\partial \theta} \right] \) is uniformly continuous in \( S \) — continuous functions on compact sets are uniformly continuous (e.g., Rudin 1953, Theorem 4.19). Thus, under Assumption 1(a)(b), (d)-(e), (g) and (h), by Lemma 2iii (p. 21), which states that \( \sup_{\theta \in \Theta} |\tau_T(\theta) - \tau(\theta)| = o(1) \) \( \mathbb{P} \)-a.s. as \( T \to \infty \), the second supremum is also \( o(1) \) \( \mathbb{P} \)-a.s. as \( T \to \infty \).

(iii) Under Assumptions 1(a)(b)(d)(e)(g) and (h), Lemma 3 (p. 23) yields
\[
0 < \inf_{(\theta, \tau) \in S} \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'} \psi(X_1, \theta) \psi(X_1, \theta)' \psi(X_1, \theta)' \mathbb{E} \left[ e^{\tau(\theta)'} \psi(X_1, \theta) \right] \text{ with } P = \mathbb{P}
\]
Consequently, under Assumption 1(a)(b), (d)-(f), (g) and (h), by Lemma 2ii and iv (p. 21) and
Under Assumptions 1(a)-(b), (e) and (g), the statement (ii) of the present lemma, as

\[
\sum_{t=1}^{T} \tilde{w}_{t, \theta} \frac{\partial \psi_{t}(\theta)'}{\partial \theta} = \frac{1}{T} \sum_{i=1}^{T} e^{\tau_{T}(\theta)'} \psi_{t}(\theta) \frac{\partial \psi_{t}(\theta)'}{\partial \theta}
\]

\[
\rightarrow \frac{1}{E[e^{\tau(\theta)'}(X_{1}, \theta)]]} E \left[ e^{\tau(\theta)'}(X_{1}, \theta) \frac{\partial \psi_{t}(X_{1}, \theta)}{\partial \theta} \right].
\]

\[\square\]

Lemma 8. Under Assumptions 7(a)-(b), (e) and (g),

(i) \[E \left[ \sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau'(X_{1}, \theta)}\psi(X_{1}, \theta)\psi(X_{1}, \theta)'| \right] < \infty\]

(ii) under additional Assumptions 7(d) and (h), \(\mathbb{P}\)-a.s. as \(T \to \infty\),

\[\sup_{(\theta, \tau) \in \mathbf{S}} \left| \frac{1}{T} \sum_{t=1}^{T} e^{\tau'(\theta)} \psi_{t}(\theta) \psi_{t}(\theta)' \right| = o(1), \text{ so that} \]

\[\sup_{\theta \in \Theta} \left| \frac{1}{T} \sum_{t=1}^{T} e^{\tau'(\theta)'} \psi_{t}(\theta) \psi_{t}(\theta)' \right| = o(1)\]

(iii) under additional Assumptions 7(d)(f) and (h), \(\mathbb{P}\)-a.s. as \(T \to \infty\),

\[\sup_{\theta \in \Theta} \left| \sum_{t=1}^{T} \tilde{w}_{t, \theta} \psi_{t}(\theta) \psi_{t}(\theta)' \right| = o(1)\]

Proof. The proof is the same as for Lemma 7 with \(\psi(X_{1}, \theta)\psi(X_{1}, \theta)'\) and \(\psi_{t}(\theta)\psi_{t}(\theta)'\) in lieu of \(\frac{\partial \psi_{t}(X_{1}, \theta)'}{\partial \theta}\) and \(\frac{\partial \psi_{t}(\theta)'}{\partial \theta}\), respectively. For completeness, we provide a proof.

(i) The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus,

\[
E \left[ \sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau'(X_{1}, \theta)}\psi(X_{1}, \theta)\psi(X_{1}, \theta)'| \right]
\]

\[
\leq E \left[ \sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau'(X_{1}, \theta)}| \sup_{(\theta, \tau) \in \mathbf{S}} |\psi(X_{1}, \theta)\psi(X_{1}, \theta)'| \right]
\]

\[\leq E \left[ \sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau'(X_{1}, \theta)}| \right]^{1/2} \left[ \sup_{\theta \in \Theta} |\psi(X_{1}, \theta)\psi(X_{1}, \theta)'|^{2} \right]^{1/2}
\]

\[\leq \infty.
\]

(a) Firstly, for any \((\theta, \tau) \in \mathbf{S}', \theta \in \Theta'\) because, for all \((\tilde{\tau}, \tilde{\theta}) \in \mathbf{S}, |\theta - \tilde{\theta}| = \sqrt{\sum_{k=1}^{m} (\theta_{k} - \tilde{\theta}_{k})^{2}} \leq \sqrt{\sum_{k=1}^{m} (\theta_{k} - \tilde{\theta}_{k})^{2} + \sum_{k=1}^{m} (\tau_{k} - \tilde{\tau}_{k})^{2}} = ||(\theta, \tau) - (\tilde{\theta}, \tilde{\tau})|| < \epsilon. \) Thus, as the second supremum does not depend on \(\tau, \sup_{(\theta, \tau) \in \mathbf{S}} |\psi(X_{1}, \theta)\psi(X_{1}, \theta)'| \leq \sup_{\theta \in \Theta} |\psi(X_{1}, \theta)\psi(X_{1}, \theta)'|. \) Secondly apply the Cauchy-Schwarz inequality. Finally, \(\sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau'(\theta)'}(X_{1}, \theta)|^{2} = \sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau'(\theta)'}(X_{1}, \theta)|^{2}\) and \(\sup_{\theta \in \Theta} |\psi(X_{1}, \theta)\psi(X_{1}, \theta)'|^{2} = \sup_{\theta \in \Theta} |\psi(X_{1}, \theta)\psi(X_{1}, \theta)'|^{2}\) because \(x \mapsto x^{2}\) is increasing on \(\mathbb{R}_{+}. \) (b) Note that \(|e^{\tau'(\theta)'}(X_{1}, \theta)|^{2} = e^{2\tau'(\theta)'}(X_{1}, \theta)'\), and then apply Assumption 7(e) to the first term. Then, application of Assumption 7(g) to the second term yields the result.
(ii) By the triangle inequality, \( P \)-a.s. as \( T \to \infty \),
\[
\begin{align*}
\sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'}\psi_t(\theta)\psi_t(\theta)' \right] - & \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \right] \\
\leq & \sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'}\psi_t(\theta)\psi_t(\theta)' \right] - \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \right] \\
+ & \sup_{\theta \in \Theta} \left| \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \right] - \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \right] \right| = o(1)
\end{align*}
\]
where the explanations for the last equality are as follows. Regarding the first supremum, under Assumptions \( 1(a)-(b)(d)(e)(g) \) and \( h \), by Lemma 2iii (p. 21), \( S := \{ (\theta, \tau) : \theta \in \Theta \land \tau \in \mathbb{T}(\theta) \} \) is a compact set, so that Assumption \( 1(a)-(b) \), the statement (i) of the present lemma and the ULLN (uniform law of large numbers) à la Wald yields that (e.g., Ghosh and Ramamoorti 2003, pp. 24-25, Theorem 1.3.3), \( P \)-a.s. as \( T \to \infty \),
\[
\begin{align*}
\sup_{(\theta, \tau) \in S} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'}\psi_t(\theta)\psi_t(\theta)' \right] - & \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \right] = o(1).
\end{align*}
\]
Now, by Assumption \( 1(e) \), for all \( \theta \in \Theta \), \( \tau(\theta) \in \mathbb{T}(\theta) \), and under Assumption \( 1(a)(b), (d)-(e), (g) \) and \( h \), by Lemma 2i (p. 21), \( P \)-a.s. for \( T \) big enough, for all \( \theta \in \Theta \), \( \tau(\theta) \in \mathbb{T}(\theta) \). Moreover, under Assumption \( 1(a)(b), (d)-(e), (g) \) and \( h \), by Lemma 2iii (p. 21), \( \sup_{\theta \in \Theta} | \tau(\theta) - \tau(\theta) | = o(1) \) \( P \)-a.s. as \( T \to \infty \). Thus, the first supremum is \( o(1) \), i.e., \( \sup_{\theta \in \Theta} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'}\psi_t(\theta)\psi_t(\theta)' \right] - \mathbb{E} e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' = o(1) \), as \( T \to \infty \) \( P \)-a.s. Regarding the second supremum, by Assumption \( 1(b) \), \( (\theta, \tau) \mapsto e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \) is continuous in \( S \). Moreover under Assumptions \( 1(a)-(b), (e) \) and \( (g) \), by the statement (i) of the present lemma, \( E \left[ \sup_{(\theta, \tau) \in S} \left| e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \right| \right] < \infty \). Thus, by the Lebesgue dominated convergence theorem and Assumption \( 1(b) \), \( (\theta, \tau) \mapsto \mathbb{E} e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \) is also continuous. Now, under Assumptions \( 1(a)(b), (d)(e)(g) \) and \( (h) \), by Lemma 4ii (p. 24), \( S \) is compact, so that \( (\theta, \tau) \mapsto \mathbb{E} e^{\tau(\theta)'}\psi(X_1,\theta)\psi(X_1,\theta)' \) is uniformly continuous —continuous functions on compact sets are uniformly continuous (e.g., Rudin 1953, Theorem 4.19). Thus, under Assumption \( 1(a)(b), (d)-(e), (g) \) and \( (h) \), by Lemma 2ii (p. 21), which states that \( \sup_{\theta \in \Theta} | \tau(\theta) - \tau(\theta) | = o(1) \) \( P \)-a.s. as \( T \to \infty \), the second supremum is also \( o(1) \) \( P \)-a.s. as \( T \to \infty \).  

(iii) Under Assumptions \( 1(a)(b)(d)(e)(g) \) and \( h \), Lemma 3 (p. 23) yields
\[
0 < \inf_{(\theta, \tau) \in S} \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'}\psi_t(\theta) \quad \text{with} \quad P = \frac{1}{T} \sum_{t=1}^{T} \delta_{X_t}, \quad \text{and} \quad 0 < \inf_{(\theta, \tau) \in S} \mathbb{E} | e^{\tau(\theta)'} | \quad \text{with} \quad P = \mathbb{P}.
\]
Consequently, under Assumption \( 1(a)(b), (d)-(e), (g) \) and \( (h) \), by Lemma 2ii and iv (p. 21) and the statement (ii) of the present lemma, as \( T \to \infty \), \( P \)-a.s., uniformly w.r.t. \( \theta \),
\[
\begin{align*}
\sum_{t=1}^{T} \psi_t(\theta)\psi_t(\theta)' & = \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta)'}\psi_t(\theta)\psi_t(\theta)' \\
& \to \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta) \right] \mathbb{E} \left[ e^{\tau(\theta)'}\psi(X_1,\theta) \psi(X_1,\theta)' \right].
\end{align*}
\]
Lemma 9. Under Assumptions \([a,b](g)\),

(i) \(E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^4 \right] < \infty\), so that \(E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right] < \infty\); and

(ii) under additional Assumption \([c]\), \(E \left[ \sup_{(\theta, \tau) \in S} |e^{r\psi(X_1, \theta)}\psi(X_1, \theta)| \right] < \infty\).

Proof. (i) Put \(\psi(X_1, \theta) =: (\psi_1(X_1, \theta) \psi_2(X_1, \theta) \cdots \psi_m(X_1, \theta))^t\). Note that \(\sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^4 = [\sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2]^2\) because \(x \mapsto x^2\) is an increasing function. Thus, by the Cauchy-Schwarz inequality,

\[
E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right] \leq \sqrt{E \left[ \left( \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right)^2 \right]} = \sqrt{E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^4 \right]},
\]

so that it remains to show the first part of the statement. On one hand, by the definition of the Euclidean norm,

\[
\sqrt{E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^4 \right]} = \sqrt{E \left[ \sup_{\theta \in \Theta^*} \left( \sum_{k=1}^m \psi_k(X_1, \theta)^2 \right)^2 \right]} \leq m \sqrt{E \left[ \sup_{\theta \in \Theta^*} \left( \sum_{k=1}^m \psi_k(X_1, \theta)^4 \right) \right]} \tag{20}
\]

where the explanation for the last inequality is as follows. By the Jensen’s inequality,

\[
\left( \frac{1}{m} \sum_{k=1}^m a_k \right)^2 \leq \frac{1}{m} \sum_{k=1}^m a_k^2,
\]

so that \((\sum_{k=1}^m a_k)^2 \leq m \sum_{k=1}^m a_k^2\). Apply the later inequality with \(\psi_k(X_1, \theta)^2 = a_k\).

On the other hand,

\[
E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right] = E \left[ \left( \sum_{i,j} |\psi_i(X_1, \theta)\psi_j(X_1, \theta)|^2 \right) \right] \leq m \sqrt{E \left[ \sup_{\theta \in \Theta^*} \left( \sum_{k=1}^m \psi_k(X_1, \theta)^4 \right) \right]},
\]

Therefore, \(\sum_{k=1}^m \psi_k(X_1, \theta)^4 \leq \sum_{k=1}^m \psi_k(X_1, \theta)^4 + \sum_{(i,j) \in [1, m]^2, i \neq j} |\psi_i(X_1, \theta)\psi_j(X_1, \theta)|^2\), the later equality and inequality \([20]\) yield

\[
E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right] \leq \sqrt{m E \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)\psi(X_1, \theta)|^2 \right]} \leq \infty
\]

where the last inequality follows from Assumption \([l]g\).
(ii) The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus,

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in S^*} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta)| \right] \\
\leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in S^*} |e^{\tau \psi(X_1, \theta)}| \sup_{(\theta, \tau) \in S^*} |\psi(X_1, \theta)| \right] \\
\leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in S^*} |e^{\tau \psi(X_1, \theta)}|^2 \right]^{1/2} \mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right]^{1/2} \\
\leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in S^*} |e^{\tau \psi(X_1, \theta)}|^2 \right]^{1/2} \mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right]^{1/2} \\
\leq \infty
\]

(a) Firstly, for any \((\theta, \tau) \in S^c, \theta \in \Theta^c\) because, for all \((\tilde{\tau}, \tilde{\theta}) \in S, |\theta - \tilde{\theta}| = \sqrt{\sum_{k=1}^{m} (\theta_k - \tilde{\theta}_k)^2} \leq \sqrt{\sum_{k=1}^{m} (\tau_k - \tilde{\tau}_k)^2} = |(\theta, \tau) - (\tilde{\tau}, \tilde{\theta})| < \epsilon. Thus, as the expression in the second suprema does not depend on \tau, \sup_{(\theta, \tau) \in S^c} |\psi(X_1, \theta)\psi(X_1, \theta)'| \leq \sup_{\theta \in \Theta^c} |\psi(X_1, \theta)\psi(X_1, \theta)'|.

Secondly apply the Cauchy-Schwarz inequality. Finally, note that \(|\sup_{(\theta, \tau) \in S} |e^{\tau \psi(X_1, \theta)}|^2 = \sup_{(\theta, \tau) \in S} |\psi(X_1, \theta)\psi(X_1, \theta)'|^2, \] and \(|\sup_{\theta \in \Theta} |\psi(X_1, \theta)|^2 = \sup_{\theta \in \Theta} |\psi(X_1, \theta)|^2| because \(x \mapsto x^2\) is increasing on \(\mathbb{R}_+\). (b) Note that \(|e^{\tau \psi(X_1, \theta)}|^2 = e^{2\tau \psi(X_1, \theta)}, \) and then apply Assumption 1(c) to the first term. Then, application of the statement (i) of the present lemma to the second term yields the result. 

\[\square\]

Remark 1. The first step of the proof shows that even the fourth moment is uniformly bounded. \(\diamondsuit\)

Lemma 10 (Implicit function \(\tau(\cdot)\)). Under Assumption 1 \((a)(b)(e)(g)\) and \(h)\

(i) for all \(\theta \in \Theta, \tau \mapsto \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\right] \) is a strictly convex function s.t. \(\frac{\partial \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\right]}{\partial \tau} = \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\psi(X_1, \theta)\right];\)

(ii) under additional Assumption 1 \((d)\), for all \(\theta \in \Theta, \) there exists a unique \(\tau(\theta)\) such that \(\mathbb{E}\left[e^{\tau(\theta) \psi(X_1, \theta)}\psi(X_1, \theta)\right] = 0; \) and

(iii) under additional Assumption 1 \((d)\), \(\tau : \Theta \to \mathbb{R}^m\) is continuous; and

(iv) under additional Assumption 1 \((c)\) and \((d), \) for all \(\theta \in \Theta \setminus \{\theta_0\}, \mathbb{E}\left[e^{\tau(\theta) \psi(X_1, \theta)}\right] < 1 \) where \(\tau(\theta_0) = 0_{m \times 1}\).

Proof. (i) Under Assumption 1 \((a)\) and \((b), \) by the Cauchy-Schwarz inequality,

\[
\mathbb{E}\left[\sup_{(\theta, \tau) \in S^c} e^{\tau \psi(X_1, \theta)}\right] \leq \mathbb{E}\left[\sup_{(\theta, \tau) \in S^c} e^{2\tau \psi(X_1, \theta)}\right]^{1/2}, \] which is finite by Assumption 1 \((e). \)

Now, by Assumption 1 \((e), \) for all \(\tilde{\theta} \in \Theta, \) \(\tau(\tilde{\theta}) \in \text{int}[\textbf{T}(\tilde{\theta})]. \) Then, by a standard result on Laplace’s transform (e.g., Monfort (1980, Theorems 3 on p. 183), \(\tau \mapsto \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\right] \) is \(C^\infty\) in a neighborhood of \(\tau(\tilde{\theta}), \) and \(\tau \mapsto \frac{\partial \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\right]}{\partial \tau} = \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\psi(X_1, \theta)\right] \) and \(\tau \mapsto \frac{\partial^2 \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\right]}{\partial \tau^2} = \mathbb{E}\left[e^{\tau \psi(X_1, \theta)}\psi(X_1, \theta)'\right]. \) Moreover, under Assumptions 1 \((a)-(b), (e) and (g), Assumption 1 \((h)\) implies that,
\[ \mathbb{E}\left[ e^{r \psi(X_1, \hat{\theta})} \psi(X_1, \hat{\theta}) \psi(X_1, \hat{\theta})^T \right] \] is a symmetric positive-definite matrix because a well-defined covariance matrix is invertible iff it is invertible under an equivalent probability measure (Lemma 29 and Corollary 1 on p. 87).

(ii) Assumption (d) ensures existence, while the statement (i) of the present lemma ensures that \( \tau(\theta) \) is the solution of a strictly convex problem, so that it is unique.

(iii) Note that, under our assumptions, an application of the standard implicit function theorem (e.g., Rudin 1953 Theorem 9.28) is not directly possible as it requires \( \tau \) to be continuously differentiable in \( \mathcal{S}^c \), which, in turn, typically requires to uniformly bound the derivative of the latter in \( \mathcal{S}^c \) (e.g., Davidson 1994 Theorem 9.31). Thus, we apply the sufficiency part of Kumagai’s implicit function theorem (Kumagai 1980). Check its assumptions. Firstly, under Assumptions 1(a)(b)(e) and (g), by Lemma 9(iii) (p. 30) and the Lebesgue dominated convergence theorem, \( \tau \) is continuous in \( \mathcal{S}^c \), i.e., in an open neighborhood of every \( \theta, \tau \in \mathcal{S} \). Secondly, by the inverse function theorem applied to \( \tau \mapsto \mathbb{E}\left[ e^{r \psi(X_1, \theta)} \psi(X_1, \theta) \right] \) (e.g., Rudin 1953 Theorem 9.24), for all \( \theta \in \Theta^c \), \( \tau \mapsto \mathbb{E}\left[ e^{r \psi(X_1, \theta)} \psi(X_1, \theta) \right] \) is locally one-to-one. As explained in the proof of (i), under Assumption 1(a)(b)(e) and (h), \( \tau \mapsto \mathbb{E}\left[ e^{r \psi(X_1, \theta)} \psi(X_1, \theta) \right] \) is continuously differentiable and, under Assumption 1(a)(b)(e)(g) and (h), for all \( \theta \in \Theta \),

\[
\frac{\partial}{\partial \theta} \mathbb{E}\left[ e^{r \psi(X_1, \theta)} \psi(X_1, \theta) \right] = \mathbb{E}\left[ e^{r \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)^T \right]
\]

is invertible, so that the assumptions of the inverse function theorem are valid.

(iv) By the statements (i) and (ii) of the present lemma, for all \( \theta \in \Theta \), for all \( \tau \neq \tau(\theta) \), \( \mathbb{E}[e^{r(\lambda)\psi(X_1, \theta)}] < \mathbb{E}[e^{r\psi(X_1, \theta)}] \). Now, for all \( \theta \in \Theta \setminus \{\theta_0\} \), \( \tau(\theta) \neq 0 \) s.t. \( \tau(\hat{\theta}) = 0 \) if there existed \( \hat{\theta} \in \Theta \setminus \{\theta_0\} \). Thus, for all \( \theta \in \Theta \setminus \{\theta_0\} \), \( \mathbb{E}[e^{r(\tau(\theta))\psi(X_1, \theta)}] < \mathbb{E}[e^{0_{1 \times m}\psi(X_1, \theta)}] = 1 \). Then, the result follows by the statement (ii) of the present lemma because \( 0_{1 \times m}\psi(X_1, \theta_0) = \mathbb{E}[e^{0_{1 \times m}\psi(X_1, \theta_0)}] \).

\[ \blacksquare \]

B.2. Decomposition and derivatives of the log-ESP \( L_T(.,.) \). In this section, we simplify \( L_T(\theta, \tau) \) and study its derivatives. Such results are needed for the proof of Theorem 11 and other results afterwards.

Lemma 11. Under Assumption 1(a)-(e) and (g)(h), by Lemma 10 (p. 82), define \( \tau(\theta_0) = \tau_0 = 0_{m \times 1} \). Under Assumption 1(a)-(b), (c) and (h),

(i) under additional Assumption 1(d) and (g), there exist \( (\overline{M}_e, \underline{M}_e) \in \mathbb{R}_+ \setminus \{0\} \) s.t. \( \mathbb{P}\text{-a.s. for } T \text{ big enough, } \underline{M}_e < \inf_{(\theta, \tau) \in \mathcal{S}} \frac{1}{T} \sum_{t=1}^T e^{r \psi(\theta)} < \sup_{(\theta, \tau) \in \mathcal{S}} \frac{1}{T} \sum_{t=1}^T e^{r \psi(\theta)} < \overline{M}_e; \)

(ii) under additional Assumption 1(c)(d) and (g), there exists an open ball \( B_r(\theta, \tau_0) \) centered at \( (\theta_0, \tau_0) \) of radius \( r > 0 \), which is a subset of \( \mathcal{S} \);

(iii) under additional Assumption 1(c)(d)(f) and (g), for all \( (\theta, \tau) \) in a closed ball \( \overline{B}_{r_0}(\theta, \tau_0) \subset \mathcal{S} \) centered at \( (\theta_0, \tau_0) \) with radius \( r_0 > 0 \), \( \mathbb{E}[e^{r \psi(X_1, \theta)} \psi(X_1, \theta)] \) \( \mathbb{E}[e^{0_{1 \times m}\psi(X_1, \theta)}] \) is strictly positive, so that, \( \mathbb{P}\text{-a.s. for } T \text{ big enough, } \frac{1}{r_0^2} \sum_{t=1}^T e^{r \psi(\theta)} < \overline{M}_e; \)

(iv) under additional Assumption 1(g), \( \mathbb{P}\text{-a.s. for } T \text{ big enough, }
Under Assumption 1(a)(b)(d)(e)(g) and (h), by Lemma 2i (p. 21), which states that, 
\[ \text{Proof. (i)} \]
Firstly, under Assumptions 1(a)-(b), (e), (g) and (h), by Corollary 1 (p. 87), for all \( (\theta, \tau) \in B \) implies that 
\[ \sqrt{\sum \tau \psi_t^2} \leq \sqrt{\sum \tau t} \psi_t^2 \leq r < \frac{T}{2} \] by definition of \( r \). Secondly, and similarly, \( |\dot{\theta} - \theta_0| < \sqrt{\sum \tau \dot{k} - \theta_0, k}^2 \leq \sqrt{\sum \tau \dot{k} - \theta_0, k}^2 < r < \frac{T}{2} \) because \( B_{r_0}(\theta_0) \subset \tau^{-1}[B_{r_0}(\tau, 0)] \) and \( B_{r_0}(\theta_0) \subset \Theta \). Now, for this proof, put \( r = \min\{r_0, \epsilon_T/2\} \). Then, it remains to show that \( B_r(\theta_0, \tau_0) \subset S \), i.e., for all \( (\dot{\theta}, \dot{\tau}) \in B_r(\theta_0, \tau_0) \), \( |\dot{\tau} - \tau| < \frac{T}{2} \) because \( B_{r_0}(\theta_0) \subset \tau^{-1}[B_{r_0}(\tau, 0)] \).

(iii) Under Assumption 1(a)-(b) and (e)-(f), by Lemma 7i (p. 27), Assumption 1(b) and the Lebesgue dominated convergence theorem, \((\theta, \tau) \mapsto E[e^{r\psi(X_1, \theta)} \partial \psi(X_1, \theta)]\) is continuous in \( S \), and thus in a neighborhood of \((\theta_0, \tau_0)\) in \( S \) by Assumption 1(c) and (e). Then, \((\theta, \tau) \mapsto |E[e^{r\psi(X_1, \theta)} \partial \psi(X_1, \theta)]]_\text{det}^2 > 0 \), so that, under Assumption 1(a)-(e) and (g)-(h), by the statement (ii) of the present lemma, there exists a closed ball \( B_{r_0}(\theta_0, \tau_0) \subset S \) centered at \((\theta_0, \tau_0)\) with radius \( r_0 > 0 \), s.t., for all \((\theta, \tau) \in B_{r_0}(\theta_0, \tau_0) \), \( 0 < |E[e^{r\psi(X_1, \theta)} \partial \psi(X_1, \theta)]]_\text{det}^2 \), which is the first part of the result. By Lemma 30 (p. 88), the second part of the result follows from the continuity of \((\theta, \tau) \mapsto E[e^{r\psi(X_1, \theta)} \partial \psi(X_1, \theta)]]\), the invertibility of \( E[e^{r\psi(X_1, \theta)} \partial \psi(X_1, \theta)]] \) for all \((\theta, \tau) \in B_{r_0}(\theta_0, \tau_0) \), and Lemma 7i (p. 27), which, under Assumption 1(a)-(b) and (e)-(f), implies that 
\[ \text{sup}_{(\theta, \tau) \in B_{r_0}(\theta_0, \tau_0)} \left| \frac{1}{T} \sum_{t=1}^{T} e^{r\psi_t^2} \psi_t^2 \right| = o(1), \text{ P-a.s. as } T \to \infty. \]

(iv) It follows from Lemma 30 (p. 88), so that it is sufficient to check its assumptions. Firstly, under Assumptions 1(a)-(b), (e), (g) and (h), by Corollary 1 (p. 87), for all \((\theta, \tau) \in S\), \( E[e^{r\psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)]\) is a positive definite symmetric matrix, and thus it is invertible. Secondly, under Assumption 1(a)-(b), (e) and (g), by Lemma 3 (p. 23), \( E[e^{r\psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)] < \infty \), so that by the Lebesgue dominated convergence theorem and Assumption 1(b), \((\theta, \tau) \mapsto E[e^{r\psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)]\) is continuous in \( S \). Finally, under Assumptions 1(a)-(b), (d), (e), (g) and (h), P-a.s. as \( T \to \infty, \)

\[ \inf_{\theta, \tau} \left| \sum_{t=1}^{T} e^{r\psi_t^2} \psi_t^2 \right| > 0. \]
In order to simplify the analysis of the asymptotic properties of the ESP estimator, we decompose the LogESP into three terms.

**Lemma 12** (LogESP decomposition). Under Assumption I, \( \mathbb{P} \)-a.s. for \( T \) big enough, define, for all \( (\theta, \tau) \in B_{\tau_0}(\theta_0, \tau_0) \),

\[
L_T(\theta, \tau) := \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \psi_t(\theta)'} - \mathbb{E}[e^{\tau' \psi(X_1, \theta) \psi(X_1, \theta)'}] \right] = o(1).
\]

Proof. First of all, note that, under Assumption I by Lemma [11] (p. 33), \( L_T(\cdot) \) is well-defined \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \( (\theta, \tau) \in B_{\tau_0}(\theta_0, \tau_0) \). Thus, under Assumption I \( \mathbb{P} \)-a.s. for \( T \) big
enough, for all \((\theta, \tau) \in B_{\tau_0}(\theta_0, \tau_0)\).

\[
L_T(\theta, \tau) = \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right]
- \frac{1}{2T} \ln \left( \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \right) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right) - 1 \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \psi_t(\theta) \psi_t(\theta)' \right) \right]^{-1}
\]

\[
(a) \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right] + \frac{1}{2T} \ln \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right)^2 \right]
- \frac{1}{2T} \ln \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \psi_t(\theta) \psi_t(\theta)' \right) \right] \right]
\]

\[
(b) \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right] + \frac{1}{2T} \ln \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right)^2 \right]
- \frac{1}{2T} \ln \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \psi_t(\theta) \psi_t(\theta)' \right) \right] \right]
\]

\[
(c) \left( 1 - \frac{m}{2T} \right) \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right] + \frac{1}{2T} \ln \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right)^2 \right]
- \frac{1}{2T} \ln \left( \sum_{t=1}^{T} \frac{e^{\tau' \psi_t(\theta)}}{\sum_{i=1}^{T} e^{\tau' \psi_i(\theta)}} \psi_t(\theta) \psi_t(\theta)' \right) \right] \right]. \tag{23}
\]

(a) Firstly, use that the determinant of the product is the product of the determinants (e.g. Rudin 1953, Theorem 9.35). Secondly, the determinant of an inverse is the inverse of the determinant (e.g. Rudin 1953, p. 233). Finally, use basic properties of the logarithm, and note that we keep the square in the second logarithm in order to ensure the positivity of the argument (then the strict positivity is ensured by Lemma 11 on p. 33). (b) Use multilinearity of determinant. (c) Note that \(1 + \frac{2m}{2T} - \frac{m}{2T} = 1 - \frac{m}{2T}\). \(\square\)

B.2.1. Derivatives of \(M_{1,T}(\theta, \tau) := (1 - \frac{m}{2T}) \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right]\). First derivative \(\frac{\partial M_{1,T}(\theta, \tau)}{\partial \theta_j}\). By Assumption 1(b), \(\theta \mapsto \psi(X_1, \theta)\) is differentiable in \(\Theta\ \mathbb{P}\)-a.s. Thus, for all \((\theta, \tau) \in \mathcal{S}\), for all \(j \in [1, m]\),

\[
\frac{\partial M_{1,T}(\theta, \tau)}{\partial \theta_j} = \left( 1 - \frac{m}{2T} \right) \frac{\frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta_j}}{\frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)}}. \tag{24}
\]
Second derivative $\frac{\partial^2 M_{1,T}(\theta, \tau)}{\partial \theta_i \partial \theta_j}$. By Assumption 1(a), $\theta \mapsto \psi(X_1, \theta)$ are three times continuously differentiable in a neighborhood of $\theta_0$ a.s. Thus, by equation (24) on p. 36 under Assumptions 1(a)-(e), (g)-(h) and 2(a), by Lemma 11ii (p. 33), P-a.s., for all $(\theta, \tau)$ in a neighborhood of $(\theta_0, \tau_0)$, for all $(\ell, j) \in [1, m]^2$,

$$\frac{\partial^2 M_{1,T}(\theta, \tau)}{\partial \theta_i \partial \theta_j} = \left(1 - \frac{m}{2T}\right) \frac{1}{\left[\frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)}\right]^2} \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \left[ \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_i} \right] \left[ \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_j} \right] + e^{\tau \psi_t(\theta)} \left[ \tau' \frac{\partial^2 \psi_t(\theta)}{\partial \theta_i \partial \theta_j} \right] \right\} \times \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \left[ \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_i} \right] \right\}$$

$$- \frac{1 - m}{2T} \left[\frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)}\right]^2 \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_j} \right\} \times \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_i} \right\}. \quad (25)$$

Second derivative $\frac{\partial^2 M_{1,T}(\theta, \tau)}{\partial \tau_k \partial \theta_j}$. Under Assumption 1(a)-(b), by equation (24) on p. 36 P-a.s., for all $(\theta, \tau) \in S$, for all $(k, j) \in [1, m]^2$,

$$\frac{\partial^2 M_{1,T}(\theta, \tau)}{\partial \tau_k \partial \theta_j} = \left(1 - \frac{m}{2T}\right) \frac{1}{\left[\frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)}\right]^2} \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \frac{1}{T} \sum_{t=1}^{T} \left\{ e^{\tau \psi_t(\theta)} \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_j} \psi_{t,k}(\theta) + e^{\tau \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \psi_{t,k}(\theta) \right\} \right\}$$

$$- \left[\frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_j} \right] \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \psi_{1,k}(\theta) \right\}. \quad (26)$$

First derivative $\frac{\partial M_{1,T}(\theta, \tau)}{\partial \tau_k}$. By definition of $M_{1,T}(\theta, \tau)$ in Lemma 12 (p. 35), for all $(\theta, \tau) \in S$, for all $k \in [1, m]$,

$$\frac{\partial M_{1,T}(\theta, \tau)}{\partial \tau_k} = \left(1 - \frac{m}{2T}\right) \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \psi_{1,k}(\theta). \quad (27)$$
Second derivative \( \frac{\partial^2 M_{1,T}(\theta, \tau)}{\partial \theta_j \partial \tau_k} \). By the above equation (27), for all \((\theta, \tau) \in S\), for all \((h, k) \in [1, m]^2\),

\[
\frac{\partial^2 M_{1,T}(\theta, \tau)}{\partial \theta_j \partial \tau_k} = \left( 1 - \frac{m}{2T} \right) \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right]^2 \times \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_{t,k}(\theta) \psi_{t,k}(\theta) \right] - \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \psi_{t,h}(\theta)} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \psi_{t,i}(\theta)} \right] \right\}.
\]

(28)

B.2.2. Derivatives of \( M_{2,T}(\theta, \tau) := \frac{1}{2T} \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta_j}} \right]^2 \). First derivative \( \frac{\partial M_{2,T}(\theta, \tau)}{\partial \theta_j} \).

If \( F(.) \) is a differentiable matrix function s.t. \( |F(x)|_{det} \neq 0 \), then \( D \ln|F(x)|_{det}^2 = 2tr[F(x)^{-1} DF(x)] \) (Lemma 32) on p. 39 where \( DF(x) \) denotes the derivative of \( F(.) \) at \( x \). Now, under Assumption 1 by Lemma 11 ii (p. 33), \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \((\theta, \tau) \) in a neighborhood of \((\theta_0, \tau_0)\), \( \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta_j}} \) is invertible. In addition, under Assumption 1(a), by Assumption 2(a), \( \theta \mapsto \psi(X_1, \theta) \) is twice differentiable in a neighborhood of \((\theta_0, \tau_0)\), \( \mathbb{P} \)-a.s. that, under Assumption 1(a)-(e) and (g)-(h), by Lemma 11 ii (p. 33), \( (\theta, \tau) \mapsto \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta_j}} \) is also differentiable in a neighborhood of \((\theta_0, \tau_0)\) \( \mathbb{P} \)-a.s. Thus, under Assumptions 1 and 2(a), \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \((\theta, \tau) \) in a neighborhood of \((\theta_0, \tau_0)\), for all \( j \in [1, m] \),

\[
\frac{\partial M_{2,T}(\theta, \tau)}{\partial \theta_j} = \frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta_j}} \right]^{-1} \right\}
\]

(29)

Second derivative \( \frac{\partial^2 M_{2,T}(\theta, \tau)}{\partial \theta_j \partial \tau_k} \). The trace of a derivative is the derivative of the trace because both the trace and derivative operators are linear (e.g., Magnus and Neudecker 1999/1988 chap. 9 sec. 9). Moreover, if \( F(.) \) is a differentiable matrix function s.t., for all \( x \) in a neighborhood of \( \hat{x} \), \( |F(x)|_{det} \neq 0 \), then \( D \left[ F(\hat{x})^{-1} \right] = -F(\hat{x})^{-1}[DF(\hat{x})]F(\hat{x})^{-1} \) (e.g., Magnus and Neudecker 1999/1988 chap. 8 sec. 4). Now, as explained for the first derivative, under Assumption 1 by Lemma 11 ii (p. 33), \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \((\theta, \tau) \) in a neighborhood of \((\theta_0, \tau_0)\), \( \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta_j}} \) is invertible. In addition, by Assumption 2(a), \( \mathbb{P} \)-a.s. \( \theta \mapsto \psi(X_1, \theta) \) is three times continuously differentiable in a neighborhood of \( \theta_0 \), so that, under Assumption 1 and 2(a), \( \theta \mapsto \frac{\partial M_{2,T}(\theta, \tau)}{\partial \theta_j} \) is differentiable in a neighborhood of \((\theta_0, \tau_0)\). Thus, under Assumptions 1 and 2(a), by the above equation (29), \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \((\theta, \tau) \) in a neighborhood of
\[ \frac{\partial^2 M_{2,T}(\theta, \tau)}{\partial \theta_k \partial \theta_j} = \frac{1}{T} \text{tr} \left\{ - \left[ \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]^{-1} \cdot \left[ \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \frac{\partial^2 \psi_t(\theta)}{\partial \theta_k \partial \theta_j} + \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \left( r^t \frac{\partial \psi_t(\theta)}{\partial \theta_k} \right) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right\} \]

Second derivative \( \frac{\partial^2 M_{2,T}(\theta, \tau)}{\partial \tau_k \partial \theta_j} \). By a reasoning similar to the one for the derivative \( \frac{\partial^2 M_{2,T}(\theta, \tau)}{\partial \theta_k \partial \theta_j} \), under Assumptions 1 and 2(a), by the above equation (29), \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \( (\theta, \tau) \) in a neighborhood of \((\theta_0, \tau_0)\), for all \((k, j) \in [1, m]^2\),

\[ \frac{\partial^2 M_{2,T}(\theta, \tau)}{\partial \tau_k \partial \theta_j} = \frac{1}{T} \text{tr} \left\{ - \left[ \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]^{-1} \cdot \left[ \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \frac{\partial^2 \psi_t(\theta)}{\partial \tau_k \partial \theta_j} + \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \left( r^t \frac{\partial \psi_t(\theta)}{\partial \tau_k} \right) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \cdot \left[ \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \frac{\partial^2 \psi_t(\theta)}{\partial \theta_k \partial \theta_j} + \frac{1}{T} \sum_{t=1}^{T} e^{r^t \psi_t(\theta)} \left( r^t \frac{\partial \psi_t(\theta)}{\partial \theta_k} \right) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right\}. \]
First derivative $\frac{\partial M_2, T(\theta, \tau)}{\partial \tau_k}$. If $F(.)$ is a differentiable matrix function s.t. $|F(x)|_{det} \neq 0$, then $D \ln ||F(x)||_{det}^2 = 2tr[F(x)^{-1}DF(x)]$ (Lemma 32 on p. 39) where $DF(x)$ denotes the derivative of $F(.)$ at $x$. Now, under Assumption 1, by Lemma 11iii (p. 33), $P$-a.s. for $T$ big enough, for all $(\theta, \tau)$ in a neighborhood of $(\theta_0, \tau_0)$, $\frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta))$ is invertible. Thus, under Assumption 1 by definition of $M_2, T(\theta, \tau)$ in Lemma 12 (p. 35), $P$-a.s. for $T$ big enough, for all $(\theta, \tau)$ in a neighborhood of $(\theta_0, \tau_0)$, for all $k \in [1, m]$,

$$\frac{\partial M_2, T(\theta, \tau)}{\partial \tau_k} = \frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right\}.$$  (33)

Second derivative $\frac{\partial^2 M_2, T(\theta, \tau)}{\partial \tau_n \partial \tau_k}$. The trace of a derivative is the derivative of the trace because both the trace and derivative operators are linear (e.g., Magnus and Neudecker 1999/1988, chap. 9 sec. 9). Moreover, if $F(.)$ is a differentiable matrix function s.t., for all $x$ in a neighborhood of $\dot{x}$, $|F(x)|_{det} \neq 0$, then $D \left[ \frac{F(x)}{\dot{x}} \right] = -F(x)^{-1} \left[ DF(x) \right] F(x)^{-1}$ (e.g., Magnus and Neudecker 1999/1988, chap. 8 sec. 4). Now, as explained for the first derivative, under Assumption 1 by Lemma 11ii (p. 33), $P$-a.s. for $T$ big enough, for all $(\theta, \tau)$ in a neighborhood of $(\theta_0, \tau_0)$, $\frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta))$ is invertible. Thus, under Assumption 1 by the above equation (33), $P$-a.s. for $T$ big enough, for all $(\theta, \tau)$ in a neighborhood of $(\theta_0, \tau_0)$, for all $(h, k) \in [1, m]^2$,

$$\frac{\partial^2 M_2, T(\theta, \tau)}{\partial \tau_n \partial \tau_k} = -\frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right\} - 1 \times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right\} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right\}.$$

B.2.3. Derivatives of $M_3, T(\theta, \tau) = -\frac{1}{2T} \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \psi_t(\theta) \psi_t(\theta) \right]$. First derivative $\frac{\partial M_3, T(\theta, \tau)}{\partial \psi_j}$. If $F(.)$ is a differentiable matrix function s.t. $|F(x)|_{det} > 0$, then $D \ln ||F(x)||_{det} = \text{tr}[F(x)^{-1}DF(x)]$ (e.g., Magnus and Neudecker 1999/1988, chap. 8 sec. 3). Now, under Assumption 1(a)-(b), (e) and (g)(h), by Lemma 11v (p. 33), $P$-a.s. for $T$ big enough, for all $(\theta, \tau) \in S$, $\frac{1}{T} \sum_{t=1}^{T} e^{\tau'}(\psi_t(\theta)) \psi_t(\theta) \psi_t(\theta) |_{det} > 0$. In addition, by Assumption 1(b), $P$-a.s. $\theta \mapsto \psi(X_1, \theta)$ is continuously differentiable in $\Theta$, so that, $P$-a.s. for $T$ big enough, $\theta \mapsto M_3, T(\theta, \tau)$ is differentiable in $\Theta$, for all $(\theta, \tau) \in S$. Thus, under Assumption 1(a)-(b) and (e)(g)(h), $P$-a.s.
for $T$ big enough, for all $(\theta, \tau) \in S$, for all $j \in [1, m]$

$$\frac{\partial M_{3,T}(\theta, \tau)}{\partial \theta_j} = -\frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{t' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)' \right]^{-1} \right.$$ 

$$\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{t' \psi_t(\theta)} \left\{ \frac{\partial \psi_t(\theta)}{\partial \theta_j} \psi_t(\theta)' + \psi_t(\theta) \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right\} \right.$$ 

$$\left. + \frac{1}{T} \sum_{t=1}^{T} e^{t' \psi_t(\theta)} \left( \tau' \frac{\partial \psi_t(\theta)}{\partial \theta_j} \right) \psi_t(\theta) \psi_t(\theta)' \right\} \right\} (35)$$

Second derivative $\frac{\partial^2 M_{3,T}(\theta, \tau)}{\partial \theta_j \partial \theta_l}$. The trace of a derivative is the derivative of the trace because both the trace and differentiation operators are linear (e.g., Magnus and Neudecker 1999/1988, chap. 9 sec. 9). Moreover, if $F(\cdot)$ is a differentiable matrix function s.t., for all $x$ in a neighborhood of $\dot{x}$, $|F(x)|_{\text{det}} \neq 0$, then $D \left[ F(\dot{x})^{-1} \right] = -F(\dot{x})^{-1} [DF(\dot{x})] F(\dot{x})^{-1}$ (e.g., Magnus and Neudecker 1999/1988, chap. 8 sec. 4). Now, under Assumption 1(a)-(b), (e) and (g)(h), by Lemma 11iv (p. 33), $\mathbb{P}$-a.s. for $T$ big enough, for all $(\theta, \tau) \in S$, $\frac{1}{T} \sum_{t=1}^{T} e^{t' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'$ is invertible. In addition, by Assumption 2(a), $\mathbb{P}$-a.s. $\theta \mapsto \psi(X_1, \theta)$ is three times continuously differentiable in a neighborhood of $\theta_0$, so that, under Assumption 1(a)-(e) and (g)(h), by Lemma 11ii (p. 33), $\theta \mapsto \frac{\partial M_{3,T}(\theta, \tau)}{\partial \theta_j}$ is differentiable in a neighborhood of $(\theta_0, \tau_0)$. Thus, under Assumptions 1(a)(b), (e) and (g)(h), and 2(a), by the above equation (35), $\mathbb{P}$-a.s. for $T$ big enough, for


all \((\theta, \tau)\) in a neighborhood of \((\theta_0, \tau_0)\), for all \((\ell, j) \in [1, m]^2\),

\[
\frac{\partial^2 M_{3,T}(\theta, \tau)}{\partial \theta_{\ell} \partial \theta_j} = -\frac{1}{2T} \text{tr} \left\{ -\frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)' \right\}^{-1}
\]

\[
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \left( \frac{\partial^2 \psi_t(\theta)}{\partial \theta_{\ell} \partial \theta_j} \psi_t(\theta)' + \frac{\partial \psi_t(\theta)}{\partial \theta_{\ell}} \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right) + \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \left( \frac{\partial^2 \psi_t(\theta)}{\partial \theta_{\ell} \partial \theta_j} \psi_t(\theta)' + \frac{\partial \psi_t(\theta)}{\partial \theta_{\ell}} \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right) \right. \\
\left. + \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \left( \frac{\partial \psi_t(\theta)}{\partial \theta_{\ell}} \psi_t(\theta)' + \psi_t(\theta) \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right) \right]^{-1}
\]

\[
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \left( \frac{\partial \psi_t(\theta)}{\partial \theta_{\ell}} \psi_t(\theta)' + \psi_t(\theta) \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right) \right]^{-1}
\]

\[
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \left( \frac{\partial^2 \psi_t(\theta)}{\partial \theta_{\ell} \partial \theta_j} \psi_t(\theta)' + \frac{\partial \psi_t(\theta)}{\partial \theta_{\ell}} \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right) \right. \\
\left. + \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \left( \frac{\partial \psi_t(\theta)}{\partial \theta_{\ell}} \psi_t(\theta)' + \psi_t(\theta) \frac{\partial \psi_t(\theta)'}{\partial \theta_j} \right) \right]^{-1}
\]

Second derivative \(\frac{\partial^2 M_{3,T}(\theta, \tau)}{\partial \theta_{\ell} \partial \theta_j}\). Follow a reasoning similar to the one for the derivative \(\frac{\partial M_{2,T}(\theta, \tau)}{\partial \theta_{\ell} \partial \theta_j}\) . The trace of a derivative is the derivative of the trace because both the trace and differentiation operators are linear (e.g., Magnus and Neudecker [1999/1988] chap. 9 sec. 9). Moreover, if \(F(.)\) is a differentiable matrix function s.t., for all \(x\) in a neighborhood of \(\hat{x}\), \(|F(x)|_{\text{det}} \neq 0\), then \(D \left[ F(\hat{x})^{-1} \right] = -F(\hat{x})^{-1} [DF(\hat{x})] F(\hat{x})^{-1}\) (e.g., Magnus and Neudecker [1999/1988] chap. 8 sec. 4). Now, under Assumption \([\text{a)}\)-(b), (e) and (g)-(h)]\) by Lemma \([\text{a)}\]) v (p. 33), \(\mathbb{P}\)-a.s. for \(T\) big enough, for all \((\theta, \tau) \in \mathbb{S}\), \(\frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'\) is invertible. Thus, under Assumptions \([\text{a)}\)-(b), (e), (g)(h), by the above equation \([35]\), \(\mathbb{P}\)-a.s. for \(T\) big enough, for all \((\theta, \tau) \in \mathbb{S}\), for
Thus, under Assumptions 1(a)-(b)(e)(g)(h), by the above equation (38),

\[
D_M \approx 0. Thus, under Assumption 1(a)-(b)(e)(g)-(h), by Lemma 12 (p. 35), \( P \)-a.s. for \( T \) big enough, for all \( \theta, \tau \in \mathbf{S} \), \( 1 T \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)' \) \( \det \) > 0. Thus, under Assumption 1(a)-(b)(e)(g)(h), by definition of \( M_{3,T}(\theta, \tau) \) in Lemma 12 (p. 35), \( P \)-a.s. for \( T \) big enough, for all \( \theta, \tau \in \mathbf{S} \), for all \( k \in [1, m] \),

\[
\frac{\partial M_{3,T}(\theta, \tau)}{\partial \tau_k} = - \frac{1}{2T} \frac{1}{\frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'} \det \times \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'
\]

\[
\times \det \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'ight\}^{-1} \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'ight\}^{-1} \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'ight\}
\]

(37)

**First derivative** \( \frac{\partial M_{3,T}(\theta, \tau)}{\partial \tau_k} \). If \( F(\cdot) \) is a differentiable matrix function s.t. \( |F(x)|_\det > 0 \), then \( D \ln |F(x)|_\det = \text{tr}[F(x)^{-1} DF(x)] \) (e.g., Magnus and Neudecker [1999/1988], chap. 8 sec. 3). Now, under Assumption \( \Pi \)(a)-(b)(e)(g)(h), by Lemma 11v (p. 33), \( P \)-a.s. for \( T \) big enough, for all \( \theta, \tau \in \mathbf{S} \), \( 1 T \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)' \) \( \det \) > 0. Thus, under Assumption \( \Pi \)(a)-(b)(e)(g)(h), by definition of \( M_{3,T}(\theta, \tau) \) in Lemma 12 (p. 35), \( P \)-a.s. for \( T \) big enough, for all \( \theta, \tau \in \mathbf{S} \), for all \( k \in [1, m] \),

\[
\frac{\partial M_{3,T}(\theta, \tau)}{\partial \tau_k} = - \frac{1}{2T} \frac{1}{\frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'} \det \times \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'
\]

\[
\times \det \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'ight\}^{-1} \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'ight\}^{-1} \left\{ \frac{1}{T} \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)'ight\}
\]

(38)

**Second derivative** \( \frac{\partial^2 M_{3,T}(\theta, \tau)}{\partial \tau_k \partial \tau_j} \). The trace of a derivative is the derivative of the trace because both the trace and differentiation operators are linear (e.g., Magnus and Neudecker [1999/1988], chap. 9 sec. 9). Moreover, if \( F(\cdot) \) is a differentiable matrix function s.t., for all \( x \) in a neighborhood of \( \hat{x} \), \( |F(x)|_\det \neq 0 \), then \( D \left[ F(\hat{x})^{-1} \right] = -F(\hat{x})^{-1} [DF(\hat{x})] F(\hat{x})^{-1} \) (e.g., Magnus and Neudecker [1999/1988], chap. 8 sec. 4). Now, under Assumption \( \Pi \)(a)-(b)(e)(g)(h), by Lemma 11v (p. 33), \( P \)-a.s. for \( T \) big enough, for all \( \theta, \tau \in \mathbf{S} \), \( 1 T \sum_{t=1}^{T} e^{\tau' \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)' \) is invertible. Thus, under Assumptions \( \Pi \)(a)-(b)(e)(g)(h), by the above equation (38), \( P \)-a.s. for \( T \) big enough,
for all \((\theta, \tau)\) in a neighborhood of \((\theta_0, \tau_0)\), for all \((h, k) \in [1, m]^2\),

\[
\frac{\partial^2 M_{3,T}(\theta, \tau)}{\partial \tau_h \partial \tau_k}
= \frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right] \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right]^{-1} \right\}
\times \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right) \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right)^{-1}
\]

\[
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right] \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right)^{-1}
\]

\[
\times \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right) \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right)^{-1}
\]

(39)

\[\text{B.2.4. Derivatives of } \theta \mapsto L_T(\theta, \tau). \text{ First derivative. Under Assumption 1(a)-(e) and (g)-(h) and 2(a), by Lemma 11i (p. 33), } S \text{ contains an open neighborhood of } (\theta_0, \tau_0), \text{ so that the derivatives derived in } S \text{ also hold in a neighborhood of } (\theta_0, \tau_0). \text{ Thus, by equations (24), (29) and (35) on pp. 36-41. Therefore, under Assumptions 1 and 2(a), } P\text{-a.s. for } T \text{ big enough, for all } (\theta, \tau) \text{ in a neighborhood of } (\theta_0, \tau_0),\]

\[
\frac{\partial L_T(\theta, \tau)}{\partial \theta_j}
= \left( 1 - \frac{m}{2T} \right) \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta_j}
\]

\[+ \frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]^{-1} \right\}
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \frac{\partial^2 \psi_t(\theta)}{\partial \theta_j \partial \theta'} + \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta'} \right]
\]

\[- \frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right] \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \psi_t(\theta) \right]^{-1} \right\}
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta)} \left( \frac{\partial \psi_t(\theta)}{\partial \theta_j} + \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \right) \right]
\]

\[- \frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0) \psi_t(\theta_0)' \right] \left[ \frac{1}{T} \sum_{t=1}^{T} \left\{ \frac{\partial \psi_t(\theta_0)}{\partial \theta_j} + \psi_t(\theta_0) \frac{\partial \psi_t(\theta_0)}{\partial \theta_j} \right\} \right] \right\}
\]

(40)
because $\tau(\theta_0) = 0_{m \times 1}$ by Lemma 10.v (p. 32).

### B.3. Proof of Theorem 1(ii): Asymptotic normality.

The proof of Theorem 1(ii) (i.e., asymptotic normality) adapts the traditional approach of expanding the FOCs (first order conditions). The two main differences w.r.t. the proofs in the entropy literature are the following. Firstly, instead of expanding the FOC $\frac{\partial L_T(\theta,T)}{\partial \theta}$, we expand the approximate FOC $\frac{\partial L_T(\theta,T)}{\partial \theta}|_{(\theta,T)=(\hat{\theta},T(\hat{\theta}))}$ combined with the FOC (14) for $\tau$ on p. 18. Secondly, we need to control the asymptotic behaviour of the derivatives that come from $\ln |\Sigma_T(\theta)|_{det}$.

**Core of the proof of Theorem 1(ii).** We prove asymptotic normality adapting the traditional approach of expanding the FOCs (first order conditions). Note that our approximate FOCs are written as a function of the $2m$ variables $\theta$ and $\tau$. In other words, instead of using the implicit function $\tau_T(\theta)$, $\tau$ is an estimated parameter and hence the ET equation (14) on p. 18 is also included in the expansion.

Under Assumptions 1 and 2, by Proposition 1 (p. 45), $\mathbb{P}$-a.s. as $T \to \infty$,

$$
\sqrt{T} \left[ \begin{bmatrix} \hat{\theta} - \theta_0 \\ \tau_T(\hat{\theta}) \end{bmatrix} \right] = - \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right]^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_P(1)
$$

$$
= \frac{D}{(a)} \mathcal{N}\left( 0, \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right]^{-1} \mathbb{E} \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right] \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)' }{\partial \theta} \right] \right]^{-1} \right)
$$

$$
= \frac{D}{(b)} \mathcal{N}\left( 0, \left[ \Sigma(\theta_0) \right]_{0 \times m, 0 \times m} \right)
$$

where $\Sigma(\theta_0) = \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right]^{-1} \mathbb{E} \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right] \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)' }{\partial \theta} \right] \right]^{-1}$.

(a) Under Assumption 1(a)-(c) and (g), by the Lindeberg-Lévy CLT theorem,

$$
\sqrt{T} \sum_{t=1}^{T} \psi_t(\theta_0) \xrightarrow{D} \mathcal{N}(0, \mathbb{E} [\psi(X_1, \theta_0) \psi(X_1, \theta_0)']),
$$

as $T \to \infty$. (b) Firstly, the minus sign can be discarded because of the symmetry of the Gaussian distribution. Secondly, if $X$ is a random vector and $F$ is a (deterministic) matrix, then $\nabla F(X) = F'V(X)F'$.

\[\square\]

**Proposition 1** (Asymptotic expansion of $\sqrt{T}(\hat{\theta}_T - \theta_0)$). Under Assumptions 1 and 2, $\mathbb{P}$-a.s. as $T \to \infty$,

$$
\sqrt{T} \left[ \begin{bmatrix} \hat{\theta} - \theta_0 \\ \tau_T(\hat{\theta}) \end{bmatrix} \right] = - \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right]^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_P(1)
$$

**Proof.** The function $L_T(\theta,T)$ is well-defined and twice continuously differentiable in a neighborhood of $(\theta_0', \tau(\theta_0)')$ $\mathbb{P}$-a.s. for $T$ big enough by subsection B.2 (p. 33). under Assumptions 1 and 2(a). Similarly, let $S_T(\theta, \tau) := \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)}$, which is continuously differentiable in a neighborhood of $(\theta_0', \tau(\theta_0)')$ by Assumption 1(a)(b). Now, under Assumption 1 by Theorem 1 (p. 6), Lemma 2 ii (p. 21) and Lemma 10.v (p. 32), $\mathbb{P}$-a.s., $\hat{\theta}_T \to \theta_0$ and
\( \tau_T(\hat{\theta}_T) \rightarrow \tau(\theta_0) \), where \( \tau(\theta_0) = 0_{m \times 1} \), so that \( \mathbb{P}\)-a.s. for \( T \) big enough, \( (\hat{\theta}'_T, \tau_T(\hat{\theta}_T)') \) is in any arbitrary small neighborhood of \( (\theta'_0, \tau(\theta_0)') \). Therefore, under Assumptions 1 and 2(a), a stochastic first-order Taylor-Lagrange expansion (Jennrich, 1969, Lemma 3) around \( (\theta_0, \tau(\theta_0)) \) evaluated at \( (\hat{\theta}_T, \tau_T(\hat{\theta}_T)) \), yields, \( \mathbb{P}\)-a.s. for \( T \) big enough,

\[
\begin{bmatrix}
\frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'} \\
\frac{S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta'} \\
\frac{S_T(\theta_0, \tau(\theta_0))}
\end{bmatrix} + \begin{bmatrix}
\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2} \\
\frac{\partial S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2}
\end{bmatrix} \begin{bmatrix}
\hat{\theta}_T - \theta_0 \\
\tau_T(\hat{\theta}_T)
\end{bmatrix}
\]

(42)

where \( \hat{\theta}_T \) and \( \tau_T \) are between \( \hat{\theta}_T \) and \( \theta_0 \), and between \( \tau_T(\hat{\theta}_T) \) and \( \tau(\theta_0) \), respectively. Under Assumptions 1 and 2, by Lemma 20 (p. 61) and by definition of \( \tau_T(\cdot) \) (equation 14 on p. 18), \( \frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'} = O(T^{-1}) \) and \( S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T)) = 0 \), respectively. Moreover, under Assumptions 1 and 2, by Theorem 1, Lemma 2(iii) (p. 21) and Lemma 13 ii (p. 47), \( \mathbb{P}\)-a.s. for \( T \) big enough, \( \begin{bmatrix}
\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2} \\
\frac{\partial S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2}
\end{bmatrix} \) is invertible. Thus, under Assumptions 1 and 2, \( \mathbb{P}\)-a.s. for \( T \) big enough,

\[
\sqrt{T} \begin{bmatrix}
(\hat{\theta}_T - \theta_0) \\
\tau_T(\hat{\theta}_T)
\end{bmatrix} = - \begin{bmatrix}
\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2} \\
\frac{\partial S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2}
\end{bmatrix}^{-1} \sqrt{T} \begin{bmatrix}
\frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta'} \\
S_T(\theta_0, \tau(\theta_0))
\end{bmatrix} + O(T^{-1})
\]

(\( a \))

\[
\approx - \begin{bmatrix}
\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2} \\
\frac{\partial S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2}
\end{bmatrix}^{-1} \begin{bmatrix}
O(T^{-\frac{1}{2}}) \\
\sqrt{T} \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0)
\end{bmatrix}
\]

(\( b \))

\[
= - \begin{bmatrix}
\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2} \\
\frac{\partial S_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'^2}
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{\partial \psi(X_1, \theta_0)}{\partial \theta'}^{-1} \frac{\Sigma(\theta_0)}{0_{m \times m}}^{-1} \\
\sqrt{T} \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0)
\end{bmatrix} + \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta'}^{-1} \right]^{-1} O(T^{-\frac{1}{2}})
\]

(\( c \))

where \( \Sigma(\theta_0) = \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta'}^{-1} \right]^{-1} \mathbb{E} [\psi(X_1, \theta_0) \psi(X_1, \theta_0)'] \left[ \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta'} \right]^{-1} \right]^{-1} \). (\( a \)) Firstly, under Assumptions 1 and 2, by Lemma 14 (p. 48), \( \mathbb{P}\)-a.s. as \( T \rightarrow \infty \), \( \frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta'} = O(T^{-1}) \), so that \( \sqrt{T} \begin{bmatrix}
\frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta'} \\
S_T(\theta_0, \tau(\theta_0))
\end{bmatrix} + O(T^{-1}) = O(T^{-\frac{1}{2}}) \). Secondly, note that \( S_T(\theta_0, \tau(\theta_0)) = \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0) \). (\( b \))

Add and subtract the matrix

\[
\begin{bmatrix}
\frac{\partial \psi(X_1, \theta_0)}{\partial \theta'}^{-1} \frac{\Sigma(\theta_0)}{0_{m \times m}}^{-1} \\
\sqrt{T} \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0)
\end{bmatrix}
\]

(\( c \)) Firstly, the first column of the first square matrix cancels out because the first element of the vector is zero. Secondly, under Assumptions 1 and 2, by Lemma 13 ii (p. 47) and Theorem 1 (p. 6), \( \mathbb{P}\)-a.s. as \( T \rightarrow \infty \), the curly bracket is \( o(1) \), and, under Assumption 7(a)-(c) and (g), by the Lindeberg-Lvy CLT, \( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) = o_p(1) \), as \( T \rightarrow \infty \).
Remark 2 (Alternative approximate FOC). In the proof of Theorem 1ii, it is possible to use the approximate FOC \( \frac{\partial M_1,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = O(T^{-1}) \) instead of the approximate FOC \( \frac{\partial L_T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = O(T^{-1}) \). Under Assumption 1 and 2 (with \( k_2 \in [1,3] \) and \( j \in [0,2] \) in its part b), by Lemma 12 (p. 35) and 18v-vii-xi-xiv (p. 51) and the ULLN à la Wald, \( \frac{\partial L_T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = \frac{\partial M_1,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} + \frac{\partial M_2,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} + \frac{\partial M_3,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = O(T^{-1}) \). The approximate FOC \( \frac{\partial M_1,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = O(T^{-1}) \) would lead to replace expansion (42) on p. 46 with the following expansion

\[
\begin{bmatrix}
\frac{\partial M_1,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} \\
\frac{S_T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial M_1,T(\theta_0,\tau(\theta_0))}{\partial \theta} \\
\frac{S_T(\theta_0,\tau(\theta_0))}{\partial \theta}
\end{bmatrix} + \begin{bmatrix}
\frac{\partial^2 M_1,T(\theta_0,\tau(\theta_0))}{\partial \theta^2} & \frac{\partial^2 M_1,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta \partial \tau} \\
\frac{\partial S_T(\theta_0,\tau(\theta_0))}{\partial \theta} & \frac{\partial S_T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta}
\end{bmatrix} \begin{bmatrix}
\hat{\theta}_T - \theta_0 \\
\tau_T(\hat{\theta}_T)
\end{bmatrix}
\]

where \( \frac{\partial^2 M_1,T(\theta_0,\tau(\theta_0))}{\partial \theta^2} \) and \( \frac{\partial^2 M_1,T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta \partial \tau} \) can easily be controlled by Lemma 18iv (p. 51), Lemma 19v (p. 57), Lemma 23iii (p. 65) and ULLN à la Wald under Assumptions 1 and 2 (with \( k_2 \in [1,3] \) and \( j \in [0,2] \) in its part b). The approximate FOC \( \frac{\partial L_T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = O(T^{-1}) \) requires less assumptions than the approximate FOC \( \frac{\partial L_T(\hat{\theta}_T,\tau_T(\hat{\theta}_T))}{\partial \theta} = O(T^{-1}) \) because it does not require to control the 2nd derivatives of \( M_2,T(\theta,\tau) \) and \( M_3,T(\theta,\tau) \). However, it would not save space and it would require to add one more block of assumptions because our proof of Theorem 2 requires the full Assumption 2.

\[ \diamond \]

Lemma 13. Under Assumptions 1 and 2

(i) for any sequence \( (\theta_T,\tau_T) \in \mathbb{N} \) converging to \( (\theta_0,\tau(\theta_0)) \), \( \mathbb{P} \)-a.s. as \( T \to \infty \),

\[
\begin{bmatrix}
\frac{\partial^2 L_T(\theta_T,\tau_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T,\tau_T)}{\partial \theta \partial \tau} \\
\frac{\partial S_T(\theta_T,\tau_T)}{\partial \theta \partial \tau} & \frac{\partial S_T(\theta_T,\tau_T)}{\partial \theta}
\end{bmatrix} \rightarrow \begin{bmatrix}
0_{m \times m} & \mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]'
\end{bmatrix};
\]

(ii) \( \begin{bmatrix}
0_{m \times m} \\
\mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]'
\end{bmatrix} \) is invertible, so that, for any sequence \( (\theta_T,\tau_T) \in \mathbb{N} \) converging to \( (\theta_0,\tau(\theta_0)) \), \( \mathbb{P} \)-a.s., for \( T \) big enough, the matrix

\[
\begin{bmatrix}
\frac{\partial^2 L_T(\theta_T,\tau_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T,\tau_T)}{\partial \theta \partial \tau} \\
\frac{\partial S_T(\theta_T,\tau_T)}{\partial \theta \partial \tau} & \frac{\partial S_T(\theta_T,\tau_T)}{\partial \theta}
\end{bmatrix}
\]

is invertible; and

(iii) for any sequence \( (\theta_T,\tau_T) \in \mathbb{N} \) converging to \( (\theta_0,\tau(\theta_0)) \), \( \mathbb{P} \)-a.s. as \( T \to \infty \),

\[
\begin{bmatrix}
\frac{\partial^2 L_T(\theta_T,\tau_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T,\tau_T)}{\partial \theta \partial \tau} \\
\frac{\partial S_T(\theta_T,\tau_T)}{\partial \theta \partial \tau} & \frac{\partial S_T(\theta_T,\tau_T)}{\partial \theta}
\end{bmatrix}^{-1} \rightarrow \begin{bmatrix}
-\Sigma(\theta_0) & \mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]'
\end{bmatrix}^{-1}, \text{ where}
\]

\[
\begin{bmatrix}
-\Sigma(\theta_0) \\
\mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]'
\end{bmatrix}^{-1} \mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]^{-1} = \begin{bmatrix}
0_{m \times m} & \mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]'
\end{bmatrix}^{-1} \mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right]^{-1}.
\]

Proof. (i) Under Assumptions 1 and 2 it follows from Lemma 14; and iii (p. 50), given that \( \tau(\theta_0) = 0_{m \times 1} \) by Lemma 10ii (p. 32) and Assumption 1(c), under Assumption 1(a)(b)(d)(e)(g) and (h).

(ii) Assumption 1(h) implies the invertibility of

\[
\mathbb{E} \left[ e^{\tau(\theta_0)} \psi(X_1,\theta_0) \psi(X_1,\theta_0)' \right] = \mathbb{E} [\psi(X_1,\theta_0) \psi(X_1,\theta_0)'] \quad \text{and} \quad \mathbb{E} \left[ e^{\tau(\theta_0)} \psi(X_1,\theta_0) \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right] = \mathbb{E} \left[ \frac{\partial \psi(X_1,\theta_0)}{\partial \theta} \right] \quad \text{because} \ \tau(\theta_0) = 0_{m \times 1} \text{ by Lemma 10v (p. 32) under Assumption 1(a)-(e)(g)-(h).}
Thus, $E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]' E \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right]^{-1} E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]'$ is also invertible, so that the first part of the statement (ii) follows from Lemma 33i (p. 89) with $A = 0_{m \times m}$, $B = E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]'$.

C = $E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]$ and $D = E \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right]$. Then, the second part of the statement follows from a trivial case of the Lemma 30 (p. 88).

(iii) Under Assumption 1(a)(b)(c)(d)(e)(g)(h), by the statement (ii) of the present lemma, the limiting matrix is invertible. Thus, by the inverse formula for partitioned matrices (e.g., Magnus and Neudecker 1999/1988 Chap. 1 Sec. 11),

$$
\begin{bmatrix}
0_{m \times m} & E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]'
\end{bmatrix}
\begin{bmatrix}
E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] E \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right]^{-1} E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]' - \Sigma(\theta_0) - \Sigma(\theta_0)'^{-1} 0_{m \times 1}
\end{bmatrix}
$$

because $-M'V^{-1}M^{-1} = -M^{-1}V(M')^{-1} := -\Sigma(\theta_0)$. Then, the result follows from the continuity of the inverse transformation (e.g., Rudin 1953 Theorem 9.8).

\[ \square \]

**Lemma 14.** Under Assumptions 1 and 2,

(i) $\mathbb{P}$-a.s. as $T \to \infty$,

\[
T \frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j} \to \text{tr} \left\{ \left[ E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right]^{-1} E \left[ \frac{\partial^2 \psi_t(\theta_0)}{\partial \theta_j \partial \theta'} \right] \right\} - \frac{1}{2} \text{tr} \left\{ \left[ E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right]^{-1} \right\}, \tag{40}
\]

so that $\frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j} = O(T^{-1})$;

(ii) for any sequence $(\theta_T, \tau_T)_{T \in \mathbb{N}}$ converging to $(\theta_0, \tau(\theta_0))$, $\left[ \frac{\partial^2 L_T(\theta_T, \tau_T)}{\partial \theta_j \partial \theta'} \right] = o(1), \mathbb{P}$-a.s. as $T \to \infty$;

(iii) for any sequence $(\theta_T, \tau_T)_{T \in \mathbb{N}}$ converging to $(\theta_0, \tau(\theta_0))$, $\left[ \frac{\partial^2 L_T(\theta_T, \tau_T)}{\partial \theta' \partial \tau} \right] - E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta'} \right] = o(1), \mathbb{P}$-a.s. as $T \to \infty$.

**Proof.** (i) By equation (40) on p. 44, under Assumptions 1 and 2(a), for all $j \in [1, m]$, $\mathbb{P}$-a.s. for $T$ big enough, evaluating $\frac{\partial L_T(\theta_T, \tau_T)}{\partial \theta_j}$ at $(\theta_0, \tau(\theta_0))$ yields

\[
\frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j} = \frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} \frac{\partial \psi_t(\theta_0)}{\partial \theta'} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 \psi_t(\theta_0)}{\partial \theta_j \partial \theta'} \right] \right\} - \frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0) \psi_t(\theta_0)' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\partial \psi_t(\theta_0)}{\partial \theta_j} \psi_t(\theta_0)' + \psi_t(\theta_0) \frac{\partial \psi_t(\theta_0)}{\partial \theta_j}' \right) \right] \right\}. \tag{43}
\]

Now, under Assumption 1(a)(b),

- under additional Assumption 1(h), by the LLN and Lemma 33 (p. 88), $\mathbb{P}$-a.s. for $T$ big enough, $\frac{1}{T} \sum_{t=1}^{T} \frac{\partial \psi_t(\theta_0)}{\partial \theta'}$ is invertible, so that $\left[ \frac{1}{T} \sum_{t=1}^{T} \frac{\partial \psi_t(\theta_0)}{\partial \theta'} \right]^{-1} \to E \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta'} \right]^{-1}$;

- under additional Assumption 2(b), by the LLN, $\frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 \psi_t(\theta_0)}{\partial \theta_j \partial \theta'} \to E \left[ \frac{\partial^2 \psi_t(\theta_0)}{\partial \theta_j \partial \theta'} \right]$. 


• under additional Assumption $\text{(f)}(g)$, by the Cauchy-Schwarz inequality and the monotonicity of integration,

$$
\mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \psi(X_1, \theta_0) \right] \leq \sqrt{\mathbb{E}\left[ \sup_{\theta \in \Theta} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right|^2 \right] \mathbb{E}\left[ \sup_{\theta \in \Theta} \left| \psi(X_1, \theta) \right|^2 \right]} < \infty,
$$

so that, by the LLN,

$$
\frac{1}{T} \sum_{t=1}^{T} \left\{ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \psi(X_1, \theta_0) + \psi(X_1, \theta_0) \right\} \rightarrow \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \psi(X_1, \theta_0) \right] + \mathbb{E}\left[ \psi(X_1, \theta_0) \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right]
$$

$\mathbb{P}$-a.s. as $T \rightarrow \infty$.

Thus, under Assumptions $1$ and $2$ for all $j \in [1, m]$, $\mathbb{P}$-a.s. as $T \rightarrow \infty$, $\frac{1}{T} \frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j} \rightarrow 

\text{tr} \left\{ \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right]^{-1} \mathbb{E}\left[ \frac{\partial^2 \psi(X_1, \theta_0)}{\partial \theta_j \partial \theta_j} \right] \right\} - \frac{1}{T} \text{tr} \left\{ \mathbb{E}\left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0) \right]^{-1} \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \left( \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right) \psi(X_1, \theta_0) \right] \right\}

+ \mathbb{E}\left[ \psi(X_1, \theta_0) \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right] \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right],

where, so that $\frac{1}{T} \frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j} \rightarrow$

$$
\frac{1}{\mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right]} \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right] \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \left( \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right) \psi(X_1, \theta_0) \right] + \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right],
$$

and (h), by Lemma 19i (p. 32) and Assumption $\text{(f)}(g)$, put $\tau(\theta_0) = 0_{m \times 1}$, so that the result follows.

(iii) Under Assumptions $1$ and $2$ by Lemma 16 (p. 50) and Lemma 12 (p. 35), $\mathbb{P}$-a.s. as $T \rightarrow \infty$, uniformly over a closed ball around $(\theta_0, \tau(\theta_0))$ with strictly positive radius, $\frac{\partial^2 L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j \partial \theta_l} \rightarrow 

\frac{1}{\mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right]} \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right] \frac{\partial^2 \psi(X_1, \theta_0) \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \psi(X_1, \theta_0)}{\partial \theta_j} + \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right],

and (h), by Lemma 19i (p. 32) and Assumption $\text{(f)}(g)$, put $\tau(\theta_0) = 0_{m \times 1}$, so that $\mathbb{P}$-a.s. as $T \rightarrow \infty$, $\frac{\partial^2 L_T(\theta_0, \tau(\theta_0))}{\partial \theta_j \partial \theta_l} \rightarrow$

$$
\left\{ \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right] \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \left( \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right) \psi(X_1, \theta_0) \right] \right\} \times \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right],
$$

where the components together in order to obtain the result.

---

Lemma 15 (Uniform limit of $\frac{\partial^2 L_T(\theta, \tau)}{\partial \theta_j \partial \theta_l}$ in a neighborhood of $(\theta_0, \tau(\theta_0))$). Under Assumptions $1$ and $2$ for all $(j, l) \in [1, m] \times [1, m]$, $\mathbb{P}$-a.s. as $T \rightarrow \infty$, uniformly over a closed ball around $(\theta_0, \tau(\theta_0))$ with strictly positive radius,

(i) $\frac{\partial^2 L_T(\theta, \tau)}{\partial \theta_j \partial \theta_l} \rightarrow \frac{1}{\mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right]} \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right] \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \left( \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right) \psi(X_1, \theta_0) \right] + \mathbb{E}\left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta_j} \right],

(ii) $\frac{\partial^2 L_T(\theta, \tau)}{\partial \theta_j \partial \theta_l} \rightarrow 0$;

(iii) $\frac{\partial^2 L_T(\theta, \tau)}{\partial \theta_j \partial \theta_l} \rightarrow 0$.

Proof. (i) Under Assumptions $1$ and $2$ by Lemma 15 (p. 53), Assumption $1a$ and (b), all the averages in $\frac{\partial^2 M_{1, T}(\theta, \tau)}{\partial \theta_j \partial \theta_l}$ (equation 25 on p. 37) satisfy the assumptions of the ULLN à la Wald. Moreover, under Assumption $1a$-(b) (d)(e)(g) and (h), by Lemma 11 (p. 33) the averages in the denominators are bounded away from zero. Thus, the result follows from the ULLN à la Wald. Note that the coefficient $\frac{m}{T}$ vanishes as it goes to zero, as $T \rightarrow \infty$. 

---
(ii) Under Assumptions 1 and 2 by Lemma 18v-xii (p. 51), Assumption 1(a) and (b), all the averages in \( \frac{\partial^2 M_2}{\partial \theta \partial \tau} \) (equation (30) on p. 39) satisfy the assumptions of the ULLN à la Wald. Moreover, under Assumption 1 by Lemma 11iii (p. 33) the averages in the inverted matrices are invertible in a neighborhood of \((\theta_0, \tau(\theta_0))\) \(\mathbb{P}\)-a.s. for \(T\) big enough. Thus, the result follows from the ULLN à la Wald, the linearity of the trace operator and the scaling by \( \frac{1}{T} \).

(iii) Under Assumptions 1 and 2 by Lemma 18ii-xix (p. 51), Assumption 1(a) and (b), all the averages in \( \frac{\partial^2 M_3}{\partial \theta^2} \) (equation (36) on p. 42) satisfy the assumptions of the ULLN à la Wald. Moreover, under Assumption 1(a)(b)(e)(g) and (h), by Lemma 11v (p. 33) the averages in the inverted matrices are invertible in a neighborhood of \((\theta_0, \tau(\theta_0))\), \(\mathbb{P}\)-a.s. for \(T\) big enough. Thus, the result follows from the ULLN à la Wald, the linearity of the trace operator and the scaling by \( \frac{1}{T} \).

\[\square\]

**Lemma 16** (Uniform limit of \( \frac{\partial^2 L_T(\theta, \tau)}{\partial \theta \partial \tau} \) in a neighborhood of \((\theta_0, \tau(\theta_0))\)). Under Assumptions 1 and 2 for all \((k, \ell) \in [1, m]^2\), \(\mathbb{P}\)-a.s. as \(T \to \infty\), uniformly over a closed ball around \((\theta_0, \tau(\theta_0))\) with strictly positive radius,

\[
\begin{align*}
(i) \quad \frac{\partial^2 M_1}{\partial \theta \partial \tau} & \to \frac{1}{e^{\theta^* \psi(X_1, \theta)}} \left\{ \mathbb{E} \left[ e^{\theta^* \psi(X_1, \theta)} \right] \mathbb{E} \left[ e^{\theta^* \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta^2} \psi_k(X_1, \theta) + e^{\theta^* \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right] \right\}; \\
(ii) \quad \frac{\partial^2 M_2}{\partial \theta \partial \tau} & \to 0; \\
(iii) \quad \frac{\partial^2 M_3}{\partial \theta \partial \tau} & \to 0.
\end{align*}
\]

**Proof.** The proof is similar to the one of Lemma 15 (p. 49). (i) Under Assumptions 1 and 2 by Lemma 19-v (p. 57), Assumption 1(a) and (b), all the averages in \( \frac{\partial^2 M_1}{\partial \theta \partial \tau} \) (equation (26) on p. 37) satisfy the assumptions of the ULLN à la Wald. Moreover, under Assumption 1(a)(b)(d)(e)(g) and (h), by Lemma 11 (p. 33) the averages in the denominators are bounded away from zero. Thus, the result follows from the ULLN à la Wald. Note that the coefficient \( \frac{m}{2T} \) vanishes as it goes to zero, as \(T \to \infty\).

(ii) Under Assumptions 1 and 2 by Lemma 19vi-xii (p. 57), Assumption 1(a) and (b), all the averages in \( \frac{\partial^2 M_2}{\partial \theta \partial \tau} \) (equation (32) on p. 39) satisfy the assumptions of the ULLN à la Wald. Moreover, under Assumption 1 by Lemma 11iii (p. 33) the averages in the inverted matrices are invertible in a neighborhood of \((\theta_0, \tau(\theta_0))\) \(\mathbb{P}\)-a.s. for \(T\) big enough. Thus, the result follows from the ULLN à la Wald, the linearity of the trace operator and the scaling by \( \frac{1}{T} \).

(iii) Under Assumptions 1 and 2 by Lemma 19ii-xix (p. 57), Assumption 1(a) and (b), all the averages in \( \frac{\partial^2 M_3}{\partial \theta \partial \tau} \) (equation (34) on p. 41) satisfy the assumptions of the ULLN à la Wald. Moreover, under Assumption 1(a)(b)(e)(g) and (h), by Lemma 11v (p. 33) the averages in the inverted matrices are invertible in a neighborhood of \((\theta_0, \tau(\theta_0))\), \(\mathbb{P}\)-a.s. for \(T\) big enough. Thus, the result follows from the ULLN à la Wald, the linearity of the trace operator and the scaling by \( \frac{1}{T} \).

\[\square\]

**Lemma 17.** Put \( S_T(\theta, \tau) := \frac{1}{T} \sum_{t=1}^T e^{\theta^* \psi(t)} \psi(t) \). Under Assumptions 1 and 2, there exists a closed ball centered at \((\theta_0', \tau(\theta_0)')\) with strictly positive radius s.t., \(\mathbb{P}\)-a.s. as \(T \to \infty\),

\[
\begin{align*}
(i) \quad \sup_{(\theta, \tau) \in B_{r_{\ell}}((\theta_0, \tau_0))} \frac{\partial S_T}{\partial \theta} - \mathbb{E} \left[ e^{\theta^* \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta^2} \right] & = o(1); \\
(ii) \quad \sup_{(\theta, \tau) \in B_{r_{\ell}}((\theta_0, \tau_0))} \frac{\partial S_T}{\partial \tau} - \mathbb{E} \left[ e^{\theta^* \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta^2} \right] & = o(1).
\end{align*}
\]
Proof. (i) By definition of $S_T(\theta, \tau)$,
\[
\frac{\partial S_T(\theta, \tau)}{\partial \theta'} = \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \psi_t(\theta) \tau' \frac{\partial \psi_t(\theta)}{\partial \theta'} + \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \frac{\partial \psi_t(\theta)}{\partial \theta'}.
\]
Thus, by the triangle inequality,
\[
\sup_{(\theta, \tau) \in \overline{B}_L((\theta_0, \tau_0))} \left| \frac{\partial S_T(\theta, \tau)}{\partial \theta'} - \mathbb{E} \left[ e^{\tau \psi_t(\theta)} \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] - \mathbb{E} \left[ e^{\tau \psi(\theta)} \psi(\theta) \frac{\partial \psi(\theta)}{\partial \theta'} \right] \right| 
\leq \sup_{(\theta, \tau) \in \overline{B}_L((\theta_0, \tau_0))} \left| \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta'} - \mathbb{E} \left[ e^{\tau \psi_t(\theta)} \psi_t(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta'} \right] \right| 
+ \sup_{(\theta, \tau) \in \overline{B}_L((\theta_0, \tau_0))} \left| \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi(\theta)} \frac{\partial \psi(\theta)}{\partial \theta'} - \mathbb{E} \left[ e^{\tau \psi(\theta)} \frac{\partial \psi(\theta)}{\partial \theta'} \right] \right| 
= o(1) \ 	ext{P-a.s. as } T \to \infty.
\]
where the last equality follows from the ULLN à la Wald by Assumption \[I(a)(b)\] and Lemma \[I\] \[v\] (p. 51); under Assumptions \[I\] \[1\] and \[I\] \[2\].
(ii) By definition of $S_T(\theta, \tau)$, \[ \frac{\partial S_T(\theta, \tau)}{\partial \theta'} = \frac{1}{T} \sum_{t=1}^{T} e^{\tau \psi_t(\theta)} \psi_t(\theta) \psi_t(\theta)' \]. Now, under Assumption \[I(a)-(b)(e)\] and \[I\] \[g\], by Lemma \[I\] \[8\] (p. 29) and Assumption \[I\] \[1\](a)(b), the assumptions of the ULLN à la Wald are satisfied, so that the result follows from the latter. \hfill \Box

Lemma 18 (Finiteness of the expectations of the supremum of the terms from \[ \frac{\partial^2 L_T(\theta, \tau)}{\partial \theta \partial \theta'} \]. Under Assumptions \[I\] \[1\] and \[I\] \[2\], there exists a closed ball \[ \overline{B}_L \subset S \] centered at \( (\theta_0, \tau(\theta_0)) \) with strictly positive radius s.t., for all \( (j, k) \in [1, m]^2 \),

(i) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} e^{\tau \psi_t(\theta)} X_{1, \theta} \right) < \infty; \]
(ii) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \right| \right) < \infty; \]
(iii) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \right| \right) < \infty; \]
(iv) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(v) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \right| \right) < \infty; \]
(vi) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(vii) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(viii) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(ix) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(x) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(xi) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \frac{\partial \psi_t(\theta)}{\partial \theta} \frac{\partial \psi_t(\theta)}{\partial \theta_j} \frac{\partial \psi_t(\theta)}{\partial \theta_k} \frac{\partial \psi_t(\theta)}{\partial \theta_l} \right| \right) < \infty; \]
(xii) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} X_{1, \theta} \psi(X_1, \theta)' \psi(X_1, \theta) \right| \right) < \infty; \]
(xiii) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \right| \right) < \infty; \]
(xiv) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \psi(X_1, \theta) \right| \right) < \infty; \]
(xv) \[ \mathbb{E} \left( \sup_{(\theta, \tau) \in \overline{B}_L} \left| e^{\tau \psi_t(\theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \psi(X_1, \theta) \right| \right) < \infty; \]
(xvi) \[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left( \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right) \right| \right] < \infty; \]

(xvii) \[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left( \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta \partial \tau} \right) \right| \right] < \infty; \]

(xviii) \[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left( \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta \partial \tau} \right) \right| \right] < \infty; \quad \text{and} \]

(xix) \[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left( \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta \partial \tau} \right) \right| \right] < \infty. \]

**Proof.** (i) Under Assumption (a)-(e) and (g)-(h), by Lemma 11.1 (p. 333), \( S \) contains an open ball centered at \((\theta_0, \tau(\theta_0))\), so that the Cauchy-Schwarz inequality, for \( B_L \) of sufficiently small radius, \[ E \left[ \sup_{(\theta, \tau) \in B_L} e^{\tau \psi(X_1, \theta)} \right] \leq \sqrt{E \left[ (\sup_{(\theta, \tau) \in B_L} e^{\tau \psi(X_1, \theta)})^2 \right]} = \sqrt{E \left[ \sup_{(\theta, \tau) \in S} e^{2\tau \psi(X_1, \theta)} \right]} < \infty \]

where the equality follows from the fact that supremum of the square of a positive function is the square of the supremum of the function, and the last inequality from Assumption (e).

(ii) The norm of a product of matrices is smaller than the product of the norms (e.g., Rudin 1953, Theorem 9.7 and note that all norms are equivalent on finite dimensional spaces). Thus, for \( B_L \) of sufficiently small radius, for all \((\ell, j) \in [1, m]^2\),

\[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta} \tau \frac{\partial \psi(X_1, \theta)}{\partial \tau} \right| \right] \]

\[ \leq \left( \sup_{(\theta, \tau) \in B_L} |\tau|^2 \right) E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] \]

\[ \leq (a) \left( \sup_{(\theta, \tau) \in B_L} |\tau|^2 \right) E \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty. \]

(a) Firstly, under Assumption (a)-(e) and (g)-(h), by Lemma 11.1 (p. 333), \( S \) contains an open ball centered at \((\theta_0, \tau(\theta_0))\). Thus, under Assumption (a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, by definition of \( S \), \( B_L \subset \{ (\theta, \tau) : \theta \in \Theta \land \tau \in T(\theta) \} \subset S \), because \( \Theta \subset \Theta \) by Assumption 2(a). Secondly, by Assumption 2(b), \( \sup_{\theta \in \Theta} |\frac{\partial \psi(X_1, \theta)}{\partial \theta}| \leq b(X) \) and \( \sup_{\theta \in \Theta} |\frac{\partial \psi(X_1, \theta)}{\partial \tau}| \leq b(X). \) (b) Firstly, \( \sup_{(\theta, \tau) \in B_L} |\tau|^2 \leq \infty \) because \( B_L \) is bounded. Secondly, by Assumption 2(b), \( E \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty. \)

(iii) Similarly to the proof of statement (ii), under Assumption (a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \((\ell, j) \in [1, m]^2\), \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{2\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta} \tau \frac{\partial \psi(X_1, \theta)}{\partial \tau} \right| \right] \leq (sup_{(\theta, \tau) \in B_L} |\tau|) \]

\[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta} \tau \frac{\partial \psi(X_1, \theta)}{\partial \tau} \right| \right] \]

\[ \leq (sup_{(\theta, \tau) \in B_L} |\tau|) E \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty, \]

where the two last inequalities follow from Assumption 2(b) and the boundedness of \( B_L \).

(iv) Similarly to the proof of statement (ii), under Assumption (a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( \ell \in [1, m] \), \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] \leq (sup_{(\theta, \tau) \in B_L} |\tau|) \]

\[ E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] \]

\[ \leq (sup_{(\theta, \tau) \in B_L} |\tau|) E \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1) \right] < \infty, \]

where the two last inequalities follow from Assumption 2(b) and the boundedness of \( B_L \).

(v) Similarly to the proof of statement (ii), under Assumption (a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \right| \right] \leq E \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1) \right] < \infty, \]

where the last inequality follows from Assumption 2(b).

(vi) Similarly to the proof of statement (ii), under Assumption (a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( \ell \in [1, m] \), \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \right| \right] \)
\[ \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{r \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1) \right] < \infty, \text{ where the last inequality follows from Assumption 2(b).} \]

(vii) Similarly to the proof of statement (ii), under Assumption 1(a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( \ell \in [1, m] \), \( \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} r^{\frac{\partial \psi(X_1, \theta)}{\partial \theta}} \right| \right] < (\sup_{(\theta, r) \in \overline{B_L}} |\tau|) \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] \leq (\sup_{(\theta, r) \in \overline{B_L}} |\tau|)^2 \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{r \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty \text{ where the two last inequalities follow from Assumption 2(b) and the boundedness of } \overline{B_L}. \]

(viii) Similarly to the proof of statement (ii), under Assumption 1(a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( (\ell, j) \in [1, m]^2 \), \( \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta \partial \phi} \right| \right] < (\sup_{(\theta, r) \in \overline{B_L}} |\tau|)^2 \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{r \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty \text{ where the two last inequalities follow from Assumption 2(b) and the boundedness of } \overline{B_L}. \]

(ix) Similarly to the proof of statement (ii), under Assumption 1(a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( (\ell, j) \in [1, m]^2 \), \( \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta \partial \phi} \frac{\partial \psi(X_1, \theta)}{\partial \phi} \right| \right] < (\sup_{(\theta, r) \in \overline{B_L}} |\tau|) \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{r \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty \text{ where the two last inequalities follow from Assumption 2(b) and the boundedness of } \overline{B_L}. \]

(x) Similarly to the proof of statement (ii), under Assumption 1(a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( (\ell, j) \in [1, m]^2 \), \( \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial^3 \psi(X_1, \theta)}{\partial \theta \partial \phi^2} \frac{\partial \psi(X_1, \theta)}{\partial \phi} \right| \right] < (\sup_{(\theta, r) \in \overline{B_L}} |\tau|)^2 \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{r \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty \text{ where the two last inequalities follow from Assumption 2(b) and the boundedness of } \overline{B_L}. \]

(xi) Similarly to the proof of statement (ii), under Assumption 1(a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \( (\ell, j) \in [1, m]^2 \), \( \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial^3 \psi(X_1, \theta)}{\partial \theta \partial \phi^2} \frac{\partial \psi(X_1, \theta)}{\partial \phi} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] < (\sup_{(\theta, r) \in \overline{B_L}} |\tau|)^2 \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{r \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} b(X_1)^2 \right] < \infty \text{ where the two last inequalities follow from Assumption 2(b) and the boundedness of } \overline{B_L}. \]

(xii) Under Assumption 1(a)-(e) and (g)-(h), by Lemma 11 (p. 33), \( S \) contains an open ball centered at \((\theta_0, \tau(\theta_0))\), so that, for \( B_L \) of sufficiently small radius, \( \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \right| \right] \leq \mathbb{E} \left[ \sup_{(\theta, r) \in \overline{B_L}} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \right| \right] < \infty \text{ where the last inequality follows from Lemma 5 (p. 29) under Assumption 1(a)-(b)(e)(g).} \]

(xiii) The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under Assumption 1(a)(b), for \( B_L \) of sufficiently small
radius, for all $\ell \in [1, m]$,\]
\[
E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{r^\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_{\ell}} \psi(X_1, \theta)^{\prime} \right| \right] \leq E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{r^\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_{\ell}} \right| \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \right] \leq \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \right]^2} \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^2 \right]} \leq \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \right]^2} \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^2 \right]} \leq (c) \infty.
\]

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption 1(a)-(e) and (g)-(h), by Lemma 11i (p. 33), $S$ contains an open ball centered at $(\theta_0, \tau(\theta_0))$, so that, for $B_L$ of sufficiently small radius, $B_L \subset \{ (\theta, \tau) : \theta \in N \land \tau \in T(\theta) \} \subset S \subset S^\epsilon$ because $N \subset \Theta$ and $S = \{ (\theta, \tau) : \theta \in \Theta \land \tau \in T(\theta) \}$. Secondly, as the second supremum does not depend on $\tau$, $\sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \leq \sup_{\theta \in \Theta^c} |\psi(X_1, \theta)|^{\prime}$ because $B_L \subset S$ for $B_L$ of radius small enough. (c) By Assumption 2(b), the first expectation is bounded. Under Assumption 1(a)(b)(g), by Lemma 9 (p. 30), the second expectation is also bounded.

(xiv) Proof similar to the one of statement (xiii). The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under Assumption 1(a)(b), for $B_L$ of sufficiently small radius, for all $\ell \in [1, m]$,\]
\[
E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{r^\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_{\ell}} \psi(X_1, \theta)^{\prime} \right| \right] \leq (\sup_{(\theta, \tau) \in B_L} |\tau|) \left( \sup_{(\theta, \tau) \in B_L} \left| e^{r^\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_{\ell}} \right| \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \right) \leq (\sup_{(\theta, \tau) \in B_L} |\tau|) \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{r^\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_{\ell}} \right|^2 \right]} \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \right]^2} \leq (\sup_{(\theta, \tau) \in B_L} |\tau|) \sqrt{E \left[ \sup_{\theta \in \Theta^c} \left| e^{r^\tau \psi(X_1, \theta)} \psi(X_1, \theta)^{\prime} \right| \right]^2} \sqrt{E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{r^\tau \psi(X_1, \theta)} \psi(X_1, \theta)^{\prime} \right| \right]^2} \leq (c) \infty.
\]

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption 1(a)-(e) and (g)-(h), by Lemma 11i (p. 33), $S$ contains an open ball centered at $(\theta_0, \tau(\theta_0))$, so that, for $B_L$ of sufficiently small radius, $B_L \subset \{ (\theta, \tau) : \theta \in N \land \tau \in T(\theta) \} \subset S \subset S^\epsilon$. Secondly, as the second supremum does not depend on $\tau$, $\sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^{\prime} \leq \sup_{\theta \in \Theta^c} |\psi(X_1, \theta)|^{\prime}$ because $B_L \subset S$ for $B_L$ of radius small enough. (c) Firstly, because $B_L$ is bounded, $(\sup_{(\theta, \tau) \in B_L} |\tau|) < \infty$. Secondly, by Assumption 2(b), the first expectation is bounded. Thirdly, by Assumption 1(g), the second expectation is also bounded.
The proof is the same as for statement (xiii) with \( \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_i \partial \theta_j} \) instead of \( \frac{\partial \psi(X_1, \theta)}{\partial \theta_i} \). The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under Assumption 1(a)(b), for \( B_L \) of sufficiently small radius, for all \((\ell, j) \in [1, m]^2\),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau' \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_i \partial \theta_j} \psi(X_1, \theta)'| \right] 
\leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau' \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_i \partial \theta_j} | \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)'| \right] 
\leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)'|^2 \right] \leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)'|^2 \right] \leq \infty.
\]

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption 1(a)-(e) and (g)-(h), by Lemma 11i (p. 33), \( \mathbf{S} \) contains an open ball centered at \((\theta_0, \tau(\theta_0))\), so that, for \( B_L \) of sufficiently small radius, \( B_L \subset \{(\theta, \tau) : \theta \in \mathcal{N} \cap \tau \in \mathbf{T}(\theta)\} \subset \mathbf{S} \subset \mathbf{S'} \). Secondly, as the second supremum does not depend on \( \tau \), \( \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)'|^2 \leq \sup_{\theta \in \mathbf{S'}} |\psi(X_1, \theta)'|^2 \) because \( B_L \subset \mathbf{S} \), for \( B_L \) of radius small enough. (c) By Assumption 2(b), the first expectation is bounded. Under Assumption 1(a)(b)(g), by Lemma 9i (p. 30), the second expectation is also bounded.

(xvi) Similarly to the proof of statement (ii), under Assumption 1(a)-(e) and (g)-(h), for \( B_L \) of sufficiently small radius, for all \((\ell, j) \in [1, m]^2\),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_i} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} | \right] < \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_i} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} | \right] \leq \mathbb{E} \left[ \sup_{\theta \in \mathbf{N}} \sup_{\tau \in \mathbf{T}(\theta)} |e^{\tau' \psi(X_1, \theta)} | \mathbb{E} b(X_1)^2 \right] < \infty \]

where the last inequality follows from Assumption 2(b).

(xvii) Proof similar to the one of statement (xiii). The norm of a product of matrices is smaller than the product of the norms (e.g., Rudin 1953, Theorem 9.7 and note that all norms are equivalent on finite dimensional spaces). Moreover, the supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under
Assumption \((a)(b)\), for \(B_L\) of sufficiently small radius, for all \((\ell, j) \in [1, m]^2\),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left( \tau \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_1 \partial \theta_j} \right) \psi(X_1, \theta) \psi(X_1, \theta) \right| \right] < \infty
\]

\[
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_1} \right| \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|
\]

\[
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau| \right) \left\{ \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} e^{2\tau \psi(X_1, \theta)} |b(X_1)|^4 \right] \right\} \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^2 \right].
\]

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption \((a)-(e)\) and \((g)-(h)\), by Lemma 11 (p. 33), \(S\) contains an open ball centered at \((\theta, \tau(\theta_0))\), so that, for \(B_L\) of sufficiently small radius, \(\overline{B_L} \subset \{(\theta, \tau) : \theta \in \mathcal{N} \land \tau \in T(\theta)\} \subset S \subset S'\). Secondly, as the second supremum does not depend on \(\tau\), \(\sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^2 \leq \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2\) because \(\overline{B_L} \subset S\), for \(\overline{B_L}\) of radius small enough. (c) Firstly, because \(\overline{B_L}\) is bounded, \(\sup_{(\theta, \tau) \in \overline{B_L}} |\tau| < \infty\). Secondly, by Assumption 2 (b), the first expectation is bounded. Thirdly, under Assumption \((a)(b)(g)\), by Lemma 9 (p. 30), the second expectation is also bounded.

(xviii) Proof similar to the one of statement (xiii). The supremum of the absolute value of the product is smaller than the product of the supremum of the absolute values. Thus, under Assumption \((a)(b)\), for \(B_L\) of sufficiently small radius, for all \((j, \ell) \in [1, m]^2\),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left( \tau \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_1 \partial \theta_j} \right) \psi(X_1, \theta) \psi(X_1, \theta) \right| \right]
\]

\[
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_1} \right| \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|
\]

\[
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau| \right) \left\{ \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} e^{2\tau \psi(X_1, \theta)} |b(X_1)|^2 \right] \right\} \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^2 \right]
\]

\[
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau| \right) \left\{ \mathbb{E} \left[ \sup_{\theta \in \mathcal{N} \land \tau \in T(\theta)} e^{2\tau \psi(X_1, \theta)} |b(X_1)|^2 \right] \right\} \mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right] < \infty.
\]

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption \((a)-(e)\) and \((g)-(h)\), by Lemma 11 (p. 33), \(S\) contains an open ball centered at \((\theta, \tau(\theta_0))\), so that, for \(B_L\) of sufficiently small radius, \(\overline{B_L} \subset \{(\theta, \tau) : \theta \in \mathcal{N} \land \tau \in T(\theta)\} \subset S \subset S'\). Secondly, as the second supremum does not depend on \(\tau\), \(\sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta)|^2 \leq \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2\) because \(\overline{B_L} \subset S\), for \(\overline{B_L}\) of radius small enough. (c) Firstly, because \(\overline{B_L}\) is bounded, \(\sup_{(\theta, \tau) \in \overline{B_L}} |\tau| < \infty\). Secondly, by Assumption 2 (b), the first expectation is bounded. Thirdly, by Assumption \((g)\), the second expectation is also bounded.
(xix) Proof similar to the one of statement (xiii). The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under Assumption 1(a)(b), for $B_L$ of sufficiently small radius, for all $(j, \ell) \in \lbrack 1, m \rbrack^2$,

$$
\begin{align*}
E \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau_l \psi(X_1, \theta)} \left( \tau - \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell} \right)^{\frac{1}{2}} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta) \psi(X_1, \theta) \right] \\
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau|^2 \right) E \left[ \sup_{(\theta, \tau) \in B_L} \left( e^{\tau_l \psi(X_1, \theta)} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell} \right| \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right) \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta) \psi(X_1, \theta) | \right] \\
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau|^2 \right) E \left[ \sup_{(\theta, \tau) \in B_L} \left( e^{\tau_l \psi(X_1, \theta)} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell} \right| \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right)^2 \right] E \left[ \sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta) \psi(X_1, \theta) | ^2 \right] \\
\leq \left( \sup_{(\theta, \tau) \in B_L} |\tau|^2 \right) \sqrt{E \left[ \sup_{\theta \in \mathcal{N}} \sup_{\tau \in T(\theta)} e^{2\tau_l \psi(X_1, \theta)} b(X_1)^4 \right]} \sqrt{E \left[ \sup_{\theta \in \Theta} |\psi(X_1, \theta) \psi(X_1, \theta) | ^2 \right]} < \infty.
\end{align*}
$$

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption 1(a)-(c) and (g)-(h), by Lemma 11i (p. 33), $S$ contains an open ball centered at $(\theta_0, \tau(\theta_0))$, so that, for $B_L$ of sufficiently small radius, $B_L \subset \{ (\theta, \tau) : \theta \in \mathcal{N} \wedge \tau \in T(\theta) \} \subset S \subset S'$. Secondly, as the second supremum does not depend on $\tau$, $sup_{(\theta, \tau) \in B_L} |\psi(X_1, \theta) \psi(X_1, \theta) | ^2 \leq sup_{\theta \in \Theta} |\psi(X_1, \theta) \psi(X_1, \theta) | ^2$ because $B_L \subset S$, for $B_L$ of radius small enough. (c) Firstly, because $B_L$ is bounded, $(sup_{(\theta, \tau) \in B_L} |\tau|^2) < \infty$. Secondly, by Assumption 1(b), the first expectation is bounded. Thirdly, by Assumption 1(g), the second expectation is also bounded. $\Box$

Lemma 19 (Finiteness of the expectations of the supremum of the terms from $\frac{\partial^2 L(\theta, \tau)}{\partial \theta_\ell \partial \theta_j}$). Under Assumptions 1 and 3 there exists a closed ball $B_L$ centered at $(\theta_0, \tau(\theta_0))$ with strictly positive radius s.t. for all $(k, j) \in \lbrack 1, m \rbrack^2$,

\begin{enumerate}
\item[(i)] $E \left[ \sup_{(\theta, \tau) \in B_L} e^{\tau_l \psi(X_1, \theta)} \right] < \infty$;
\item[(ii)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell} \psi_k(X_1, \theta) \right| \right] < \infty$;
\item[(iii)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \frac{\partial \psi_k(X_1, \theta)}{\partial \theta_\ell} \right| \right] < \infty$;
\item[(iv)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] < \infty$;
\item[(v)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \psi_k(X_1, \theta) \right| \right] < \infty$;
\item[(vi)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] < \infty$;
\item[(vii)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] < \infty$;
\item[(viii)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] < \infty$;
\item[(ix)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell \partial \theta_j} \right| \right] < \infty$;
\item[(x)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell \partial \theta_j} \right| \right] < \infty$;
\item[(xi)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_\ell \partial \theta_j} \right| \right] < \infty$;
\item[(xii)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_\ell} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \right| \right] < \infty$;
\item[(xiii)] $E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau_l \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta) \right| \right] < \infty$;
\end{enumerate}
\[(xiv) \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi(X_1, \theta)'| \right] < \infty; \]
\[(xv) \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta) \psi(X_1, \theta)'| \right] < \infty; \]
\[(xvi) \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)'| \right] < \infty; \]
\[(xvii) \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta) \psi(X_1, \theta)'| \right] < \infty; \]
\[(xviii) \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)'| \right] < \infty; \text{ and} \]
\[(xix) \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)'| \right] < \infty. \]

**Proof.** The proofs are similar to the ones of Lemma [18](#) (p. 51). We only use more often the inequality that states that the norm of a component of a vector is smaller than the norm of the vector (e.g., $|\psi_k(X_1, \theta)| \leq \sqrt{\sum_{l=1}^m |\psi(X_1, \theta)|^2} = |\psi(X_1, \theta)|$). Thus, we only provide proof sketches.

(i) See Lemma [18](#) p. 51.

(ii) For $\overline{B}_L$ of sufficiently small radius, for all $(k, j) \in [1, m]^2$,
\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \tau \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi_k(X_1, \theta) | \right]
\leq ( \sup_{(\theta, \tau) \in \overline{B}_L} |\tau| ) \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)'| \sup_{(\theta, \tau) \in \overline{B}_L} |\psi_k(X_1, \theta)'| \right]
\leq ( \sup_{(\theta, \tau) \in \overline{B}_L} |\tau| ) \sqrt{\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} |^2 \right]} \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |\psi_k(X_1, \theta)'|^2 \right] < \infty,
\]
where the last inequality follows from Assumption [2](#)b, and Lemma [9](#) (p. 30), under Assumption [1](#)a(b)(g).

(iii) For $\overline{B}_L$ of sufficiently small radius, for all $(k, j) \in [1, m]^2$,
\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)' \right] \leq \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{\tau \in \mathcal{T}(\theta)} e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta)' \right] < \infty,
\]
where the last inequality follows from Assumption [2](#)b.

(iv) See Lemma [18](#) p. 51.

(v) For $\overline{B}_L$ of sufficiently small radius, for all $k \in [1, m]$,
\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta)'| \right] \leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta)'| \right] \leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta)'| \right] < \infty
\]
where the last inequality follows from Lemma [9](#)i (p. 30), under Assumption [1](#)a(b)(e)(g).

(vi) See Lemma [18](#) p. 51.
(vii) For $\overline{B_L}$ of sufficiently small radius, for all $k \in [1, m]$,
\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left| e^{\tau' \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \right| \right] \\
\leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left| e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \right| \sup_{(\theta, \tau) \in \overline{B_L}} |\psi_k(X_1, \theta)| \right] \\
\leq \sqrt{\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left( e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \right)^2 \right]} \sqrt{\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} |\psi_k(X_1, \theta)|^2 \right]} < \infty,
\]
where the last inequality follows from Assumption 2(b) and Lemma 9i (p. 30) under Assumption 1(a)(b)(g).

(viii) For all $j \in [1, m]$, \[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left| e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \psi_j(X_1, \theta) \right| \right] < \sup_{(\theta, \tau) \in \overline{B_L}} |\tau| \mathbb{E} \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{\tau' \psi(X_1, \theta)} b(X_1)^2 \right] < \infty
\]
where the two last inequalities follow from Assumption 2(b) and the boundedness of $\overline{B_L}$.

(ix) See Lemma 18 vi p. 51.

(x) Under Assumption 1(a)(b), for $\overline{B_L}$ of sufficiently small radius, for all $(j, k) \in [1, m]^2$, \[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left| e^{\tau' \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \right| \right] \leq \left( \sup_{(\theta, \tau) \in \overline{B_L}} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left( e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \right)^2 \right] \sup_{(\theta, \tau) \in \overline{B_L}} |\psi_k(X_1, \theta)| \\\n\leq \left( \sup_{(\theta, \tau) \in \overline{B_L}} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left( e^{\tau' \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'} \right)^2 \right] \sqrt{\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} |\psi_k(X_1, \theta)|^2 \right]} \\\n\leq \left( \sup_{(\theta, \tau) \in \overline{B_L}} |\tau| \right) \mathbb{E} \left[ \sup_{\theta \in \Theta} \sup_{\tau \in T(\theta)} e^{2\tau' \psi(X_1, \theta)} b(X_1)^4 \right] \mathbb{E} \left[ \sup_{\theta \in \Theta} |\psi(X_1, \theta)|^2 \right] < \infty,
\]
where the last inequality follows from the boundedness of $\overline{B_L}$, Assumption 2(b) and Lemma 9i (p. 30) under Assumption 1(a)(b)(g) and (e).
(xi) Under Assumption 1(a)(b), for $B_L$ of sufficiently small radius, for all $(j, k) \in [1, m]^2$, 

$$
\mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^j \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_j \partial \theta'} | \right] 
\leq \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^j \psi(X_1, \theta)} \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta_j \partial \theta'} | \sup_{(\theta, r) \in B_L} |\psi_k(X_1, \theta)'| \right] 
\leq \mathbb{E} \left[ \sup_{\theta \in \Theta} \sup_{r \in \mathcal{T}(\theta)} e^{2r^j \psi(X_1, \theta)} b(X_1)^2 \right] \sqrt{\mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)'|^2 \right]} < \infty, 
$$

where the last inequality follows from Assumption 2(b) and Lemma 9 (p. 30), under Assumption 1(a)(b)(g) and (e).

(xii) Under Assumption 1(a)-(e) and (g)-(h), for $B_L$ of sufficiently small radius, for all $(j, k) \in [1, m]^2$, 

$$
\mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^j \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} | \sup_{(\theta, r) \in B_L} |\frac{\partial \psi(X_1, \theta)}{\partial \theta_j} | \right] < \infty 
$$

$$
\mathbb{E} \left[ \sup_{\theta \in \Theta^*} \sup_{r \in \mathcal{T}(\theta)} e^{r^j \psi(X_1, \theta)} b(X_1)^2 \right] < \infty 
$$

where the last two inequalities follow from Assumption 2(b).

(xiii) See Lemma 18kii p. 51.

(xiv) Under Assumption 1(a)-(e) and (g)-(h), for $B_L$ of sufficiently small radius, for all $k \in [1, m]$, 

$$
\mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^j \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi(X_1, \theta)' | \right] 
\leq \left( \sup_{(\theta, r) \in B_L} |r|^2 \right) \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} e^{r^j \psi(X_1, \theta)} |\psi_k(X_1, \theta)| \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta) \psi(X_1, \theta)' | \right] 
\leq \left( \sup_{(\theta, r) \in B_L} |r|^2 \right) \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} e^{r^j \psi(X_1, \theta)} |\psi_k(X_1, \theta)| \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta) \psi(X_1, \theta)' | \right] 
\leq \left( \sup_{(\theta, r) \in B_L} |r|^2 \right) \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} e^{2r^j \psi(X_1, \theta)} b(X_1)^2 \right] \sqrt{\mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta) \psi(X_1, \theta)' | \right]} < \infty, 
$$

where the last inequality follows from the boundedness of $B_L$, Assumptions 1(g) and 2(b).

(xv) See Lemma 18kiv p. 51.

(xvi) See Lemma 18kii p. 51.
(xvii) Under Assumption [1(a)(b)], for $B_L$ of sufficiently small radius, for all $(k, j) \in [1, m]^2$,

$$\mathbb{E} \left[ \sup_{(\theta, r) \in B_L} e^{r^*\psi(X_1, \theta)} \psi_k(X_1, \theta) \tau' \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta) \psi(X_1, \theta)' \right]$$

$$\leq \left( \sup_{(\theta, r) \in B_L} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} \left( e^{r^*\psi(X_1, \theta)} \psi_k(X_1, \theta) \right) |\frac{\partial \psi(X_1, \theta)}{\partial \theta_j}| \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta)\psi(X_1, \theta)'| \right]$$

$$\leq \left( \sup_{(\theta, r) \in B_L} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} \left( e^{r^*\psi(X_1, \theta)} \psi_k(X_1, \theta) \right) |\frac{\partial \psi(X_1, \theta)}{\partial \theta_j}|^2 \right] \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta)\psi(X_1, \theta)'|^2 \right]$$

$$\leq \left( \sup_{(\theta, r) \in B_L} |\tau| \right) \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{\tau \in \mathcal{T}(\theta)} e^{2r^*\psi(X_1, \theta)} b(X_1)^4 \right] \mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)|^2 \right] < \infty,$$

where the last inequality follows from the boundedness of $B_L$, Assumption [2(b)] and Assumption [1(g)].

(xviii) Under Assumption [1(a)(b)], for $B_L$ of sufficiently small radius, for all $(k, j) \in [1, m]^2$,

$$\mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^*\psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)'| \right]$$

$$\leq \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^*\psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j}| \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta)'| \right]$$

$$\leq \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} \left( e^{r^*\psi(X_1, \theta)} \psi_k(X_1, \theta) \right) \frac{\partial \psi(X_1, \theta)}{\partial \theta_j} |^2 \right] \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta)'| \right]$$

$$\leq \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{\tau \in \mathcal{T}(\theta)} e^{2r^*\psi(X_1, \theta)} b(X_1)^4 \right] \mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)'| \right] < \infty,$$

where the last inequality follows from Assumption [2(b)], and Lemma 9 (p. 30), under Assumption [1(a)(b)(g)].

(xix) Under Assumption [1(a)(b)], for $B_L$ of sufficiently small radius, for all $(k, j) \in [1, m]^2$,

$$\mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^*\psi(X_1, \theta)} \frac{\partial \psi_k(X_1, \theta)}{\partial \theta_j} \psi(X_1, \theta)\psi(X_1, \theta)'| \right]$$

$$\leq \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |e^{r^*\psi(X_1, \theta)} \frac{\partial \psi_k(X_1, \theta)}{\partial \theta_j}| \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta)\psi(X_1, \theta)'| \right]$$

$$\leq \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} \left( e^{r^*\psi(X_1, \theta)} \frac{\partial \psi_k(X_1, \theta)}{\partial \theta_j} \right) |^2 \right] \mathbb{E} \left[ \sup_{(\theta, r) \in B_L} |\psi(X_1, \theta)\psi(X_1, \theta)'| \right]$$

$$\leq \mathbb{E} \left[ \sup_{\theta \in \mathcal{N}} \sup_{\tau \in \mathcal{T}(\theta)} e^{2r^*\psi(X_1, \theta)} b(X_1)^2 \right] \mathbb{E} \left[ \sup_{\theta \in \Theta^*} |\psi(X_1, \theta)\psi(X_1, \theta)'| \right] < \infty,$$

where the last inequality follows from Assumption [2(b)] and Assumption [1(g)].
**Lemma 20.** Under Assumptions 1 and 2, P-a.s. as $T \to \infty$, $\frac{\partial L_T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = O(T^{-1})$.

*Proof.* Unlike in most of the rest of the paper, for clarity, in this proof we do not use the potentially ambiguous notation that denotes $\frac{\partial L_T(\theta, \tau)}{\partial \theta}(\theta_T, \tau_T(\theta_T))$ with $\frac{\partial L_T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T}$.

Under Assumptions 1 and 2(a), by subsection B.2 (p. 33), the function $L_T(\theta, \tau)$ is well-defined and twice continuously differentiable in a neighborhood of $(\theta_0', \tau(\theta_0'))$ P-a.s. for $T$ big enough. Moreover, under Assumptions 1(a)(b) and (d)-(h), by Lemma 21(i) (p. 62), $\tau(\cdot)$ is continuously differentiable in $\Theta$. Now, under Assumption 1 by Theorem 1 (p. 6) and Lemma 21(ii) (p. 62), P-a.s., $\theta_T \to \theta_0$ and $\tau_T(\theta_T) \to \tau(\theta_0)$, so that P-a.s. for $T$ big enough, $(\theta_T', \tau_T(\theta_T))$ is in any arbitrary small neighborhood of $(\theta_0', \tau(\theta_0'))$. Therefore, under Assumption 1 and 2(a), by the chain rule theorem (e.g., Magnus and Neudecker 1999/1988 Chap. 5 sec. 11), P-a.s. for $T$ big enough, $\theta \mapsto L_T(\theta, \tau_T(\theta))$ is continuously differentiable in a neighborhood of $\bar{\theta}_T$, and, for all $j \in \{1, m\}$,

$$0 = \frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta_j} \bigg|_{\theta = \bar{\theta}_T}$$

$$\Leftrightarrow 0 = \frac{\partial L_T(\theta, \tau)}{\partial \theta_j} \bigg|_{(\theta, \tau) = (\bar{\theta}_T, \tau_T(\bar{\theta}_T))} + \frac{\partial L_T(\theta, \tau)}{\partial \tau'} \bigg|_{(\theta, \tau) = (\bar{\theta}_T, \tau_T(\bar{\theta}_T))} \frac{\partial \tau(\theta)}{\partial \theta_j} \bigg|_{\theta = \bar{\theta}_T}$$

$$\Leftrightarrow \frac{\partial L_T(\theta, \tau)}{\partial \theta_j} \bigg|_{(\theta, \tau) = (\bar{\theta}_T, \tau_T(\bar{\theta}_T))} = O(T^{-1})O(1) = O(T^{-1}).$$

(a) It is an immediate and standard implication of the chain rule (e.g., Magnus and Neudecker 1999/1988 chap. 5, sec. 12, exercise 3). (b) Firstly, under Assumptions 1 and 2, by Lemma 22v (p. 64), P-a.s. as $T \to \infty$, $\frac{\partial L_T(\theta, \tau)}{\partial \theta_j} \bigg|_{(\theta, \tau) = (\bar{\theta}_T, \tau_T(\bar{\theta}_T))} = O(T^{-1})$ because $(\bar{\theta}_T, \tau_T(\bar{\theta}_T)) \to (\theta_0, \tau(\theta_0))$, P-a.s. as $T \to \infty$, by Theorem 1 (p. 6) and Lemma 22i (p. 62). Secondly, under Assumptions 1 and 2, by Theorem 1 (p. 6) and Lemma 21ii (p. 62), P-a.s. as $T \to \infty$, $\frac{\partial \tau(\theta)}{\partial \theta_j} \bigg|_{(\theta, \tau) = (\bar{\theta}_T, \tau_T(\bar{\theta}_T))} = O(1)$.

**Lemma 21** (First Derivative of the implicit function $\tau_T(\cdot)$). Under Assumption 1(a)(b) and (d)-(h),

(i) P-a.s. for $T$ big enough, the function $\tau_T : \Theta \to \mathbb{R}^m$ is continuously differentiable in $\Theta$ and its first derivative is

$$\frac{\partial \tau_T(\theta)}{\partial \theta'} = \left[ \frac{1}{T} \sum_{t=1}^T \hat{e}^{\tau_T(\theta)'} \psi_1(\theta) \psi_1(\theta)' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^T \hat{e}^{\tau_T(\theta)'} \psi_1(\theta) \left( \frac{\partial \psi_1(\theta)}{\partial \theta'} + \psi_1(\theta) \tau_T(\theta) \frac{\partial \psi_1(\theta)}{\partial \theta'} \right) \right];$$

(ii) for any sequence $(\theta_T)_{T \in \mathbb{N}}$ converging to $\theta_0$, P-a.s. for $T$ big enough, there exists $\theta_T$ between $\theta_T$ and $\theta_0$ s.t. $\sqrt{T}[\tau_T(\theta_T) - \tau_T(\theta_0)] = \frac{\partial \tau_T(\theta_T)}{\partial \theta_T} \sqrt{T}(\theta_T - \theta_0);$
(iii) under additional Assumptions 2(c) and 2(b), for any sequence \((\theta_T)_{T \in \mathbb{N}} \in \Theta^N\) converging to \(\theta_0\), \(\mathbb{P}\)-a.s. as \(T \to \infty\), \(\partial_T(\theta_T) = -\mathbb{E}[\psi(X_1, \theta_0)\psi(X_1, \theta_0)']^{-1}\mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] ; \) and
(iv) under additional Assumptions 2(c) and 2(b), for any sequence \((\theta_T)_{T \in \mathbb{N}} \in \Theta^N\) converging to \(\theta_0\) s.t., as \(T \to \infty\), \(\sqrt{T}(\theta_T - \theta_0) = O_{\mathbb{P}}(1) \) \(\mathbb{P}\)-a.s. as \(T \to \infty\), \(\tau_T(\theta_T) - \tau_T(\theta_0) = -V^{-1}M(\theta_T - \theta_0) + o_{\mathbb{P}}(T^{-1/2})\), where \(V := \mathbb{E}[\psi(X_1, \theta_0)\psi(X_1, \theta_0)']\) and \(M := \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] .

Proof. (i) Under Assumption 1(a)(b) and (d)-(h), by Lemma 1i (p. 20) and its proof, \(\mathbb{P}\)-a.s. for \(T\) big enough, the assumptions of the standard implicit function theorem hold and \(\tau_T(.)\) is continuously differentiable. Thus, under Assumption 1(a)(b) and (d)-(h), \(\mathbb{P}\)-a.s. for \(T\) big enough, application of the implicit function theorem yields
\[
\frac{\partial \tau_T(\theta)}{\partial \theta} = -\left[ \frac{\partial}{\partial \theta} \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta) \psi_t(\theta)} \psi_t(\theta) \right) \right]^{-1} \left[ \frac{\partial}{\partial \theta} \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta) \psi_t(\theta)} \left( \frac{\partial \psi_t(\theta)}{\partial \theta} + \psi_t(\theta) \tau_T(\theta) \frac{\partial \psi_t(\theta)}{\partial \theta} \right) \right) \right] \bigg|_{\tau=\tau_T(\theta)}
\]

(ii) Again, under Assumption 1(a)(b) and (d)-(h), by Lemma 1i (p. 20), \(\mathbb{P}\)-a.s. for \(T\) big enough, \(\tau_T(.)\) is continuously differentiable, so that the result follows from a first-order stochastic Taylor-Lagrange expansion (Jennrich 1969, Lemma 3).

(iii) Firstly, under Assumption 1(a)(b)(d)(e)(g)(h), by Lemma 2iii (p. 21), \(\mathbb{P}\)-a.s. as \(T \to \infty\), \(\sup_{\theta \in \Theta} |\tau_T(\theta) - \tau(\theta)| = o(1)\), so that \(\tau_T(\theta_T) \to \tau(\theta_0)\). Secondly, under Assumptions 1 and 2 by Lemma 23v, vii and x (p. 65), for \(\bar{B}_L\) a ball around \((\theta_0, \tau(\theta_0))\) of sufficiently small radius, \(\mathbb{E} \left[ \sup_{(\theta, \tau) \in \bar{B}_L} |e^{\tau(\theta)}\psi(X_1, \theta)\tau(\theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta} |^{\frac{1}{2}} \right] < \infty\), and \(\mathbb{E} \left[ \sup_{(\theta, \tau) \in \bar{B}_L} |e^{\tau(\theta)}\psi(X_1, \theta)\tau(\theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta} |^{\frac{1}{2}} \right] < \infty\). Thus, by Assumptions 1(a)(b) and (d), the ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthy 2003, pp. 24-25, Theorem 1.3.3), implies that, for all \(k \in \{1, m\}\), \(\mathbb{P}\)-a.s. as \(T \to \infty\),
\[
\frac{\partial \tau_T(\theta_T)}{\partial \theta} \to -\mathbb{E}[e^{\tau(\theta_0)}\psi(X_1, \theta_0)\psi(X_1, \theta_0)']^{-1} \left\{ \mathbb{E} \left[ e^{\tau(\theta_0)}\psi(X_1, \theta_0) \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \right\}
\]

because \(\tau(\theta_0) = 0_{m \times 1}\) by Lemma 10v (p. 32) under Assumption 3(a)-(e) and (g)-(h).
(iv) Under Assumption 1(b), by the statement (ii) of the present lemma, \( \mathbb{P} \)-a.s. as \( T \to \infty \), there exists \( \bar{\theta}_T \) between \( \theta_T \) and \( \theta_0 \) s.t.

\[
\tau_T(\theta_T) - \tau_T(\theta_0) = \frac{\partial \tau_T(\theta_T)}{\partial \theta_T}(\theta_T - \theta_0)
\]

\[
\overset{(a)}{=} -V^{-1}M(\theta_T - \theta_0) + \left[ \frac{\partial \tau_T(\theta_T)}{\partial \theta_T} + V^{-1}M \right] (\theta_T - \theta_0)
\]

\[
\overset{(b)}{=} -V^{-1}M(\theta_T - \theta_0) + o_p(T^{-1/2})
\]

(a) Add and subtract \( V^{-1}M(\theta_T - \theta_0) \). (b) Under Assumption 1(b), by the statement (ii) of the present lemma, \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \frac{\partial \tau_T(\theta_T)}{\partial \theta_T} + V^{-1}M = o(1) \). Moreover, by assumption, as \( T \to \infty \), \( \theta_T - \theta_0 = O_\mathbb{P}(T^{-1/2}) \), so that \( \left[ \frac{\partial \tau_T(\theta_T)}{\partial \theta_T} + V^{-1}M \right] (\theta_T - \theta_0) = o_\mathbb{P}(T^{-1/2}) \). \( \square \)

**Remark 3.** As notation indicates, \( \frac{\partial \tau_T(\theta_T)}{\partial \theta_T} \) corresponds to a partial derivative as \( \tau_T(\theta_T) \) is also a function of the data. \( \diamond \)

**Lemma 22** (Asymptotic limit of \( \frac{\partial L_T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} \)). Under Assumptions 1 and 2, for any sequence \( (\theta_T)_{T \in \mathbb{N}} \in \Theta^\mathbb{N} \) converging to \( \theta_0 \), for all \( k \in [1, m] \), \( \mathbb{P} \)-a.s. as \( T \to \infty \),

(i) \( \frac{\partial M_1.T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = 0 \);

(ii) \( \frac{\partial M_2.T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = O(T^{-1}) \);

(iii) \( \frac{\partial M_3.T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = O(T^{-1}) \); and

(iv) \( \frac{\partial L_T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = O(T^{-1}) \).

**Proof.** (i) Under Assumption 1(a)(b) and (d)-(h), by Lemma 21 (p. 21), \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \tau_T(\theta_T) \) exists, so that, by equation (27) on p. 37 \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \( k \in [1, m] \),

\[
\frac{\partial M_1.T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = \left( 1 - \frac{m}{2T} \right) \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta_T) \psi_t(\theta_T)} \psi_{t,k}(\theta_T)
\]

\[
= 0
\]

because, by definition of \( \tau_T(\theta) \) in equation (14) on p. 18, \( \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta_T) \psi_t(\theta_T)} \psi_{t,k}(\theta_T) = 0 \).

(ii) Similarly, under Assumption 1 by equation (33) on p. 40 \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \( k \in [1, m] \),

\[
\frac{\partial M_2.T(\theta_T, \tau_T(\theta_T))}{\partial \theta_T} = \frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta_T) \psi_t(\theta_T)} \psi_{t,k}(\theta_T) \right]^{-1} \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T(\theta_T) \psi_t(\theta_T)} \psi_{t,k}(\theta_T) \frac{\partial \psi_{t,k}(\theta_T)}{\partial \theta_T} \right\}
\]

where \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \theta_T \to \theta_0 \) by the lemma’s assumption and Lemma 2ii (p. 21). Now, under Assumptions 1 and 2, by Lemma 23v and v (p. 65), for \( B_L \) a ball around \( \theta_0, \tau(\theta_0) \) of sufficiently small radius, \( \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau(\theta, \theta) \psi_k(X_1, \theta)} \frac{\partial \psi_k(X_1, \theta)}{\partial \theta} \right| \right] < \infty \), and, for all \( k \in [1, m] \), \( \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau(\theta, \theta) \psi_k(X_1, \theta)} \frac{\partial \psi_k(X_1, \theta)}{\partial \theta} \right| \right] < \infty \). Thus, by Assumptions 1(a)(b) and (d), the ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthi...
2003 pp. 24-25, Theorem 1.3.3), implies that, for all \( k \in [1, m] \), \( \mathbb{P} \)-a.s. as \( T \to \infty \),

\[
T \frac{\partial M_{2,T}(\theta_T, \tau_T(\theta_T))}{\partial \tau_k}
\]

\[
= \frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta_0)'X_t} \psi_t(\theta_T) \psi_t(\theta_T) \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau'X_t} \psi_t(\theta_T) \psi_t(\theta_T) \right] \right\}
\]

because \( \tau(\theta_0) = 0_{m \times 1} \) by Lemma \( 10 \) v (p. 32) under Assumption \( 1 \)-a)-(e) and (g)-(h). Therefore, \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \frac{\partial M_{2,T}(\theta_T, \tau_T(\theta_T))}{\partial \tau_k} = O(T^{-1}) \).

(iii) Under Assumption \( 1 \) by equation (38) (p. 43), for all \( k \in [1, m] \),

\[
\frac{\partial M_{3,T}(\theta_T, \tau_T(\theta_T))}{\partial \tau_k}
\]

\[
= -\frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta_0)'X_t} \psi_t(\theta_T) \psi_t(\theta_T) \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau(\theta_0)'X_t} \psi_t(\theta_T) \psi_t(\theta_T) \right] \right\}
\]

where \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( (\theta_T')' \to \theta_T' \) \( \tau(\theta_T)' \to \tau(\theta)' \) by Theorem \( 1 \) (p. 6). Now, under Assumptions \( 1 \) and \( 2 \) by Lemma 23 vii and viii (p. 65), there exists a closed ball \( B_L \subset S \) centered at \( (\theta_0, \tau(\theta_0)) \) with strictly positive radius \( s \), for all \( k \in [1, m] \),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left| e^{\tau(\theta_0)'X_t} \psi_t(\theta) \psi(\theta)' \right| \right] < \infty
\]

and

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B_L}} \left| e^{\tau(\theta_0)'X_t} \psi_k(\theta) \psi(\theta)' \right| \right] < \infty.
\]

Thus, under Assumptions \( 1 \) and \( 2 \) by ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthi 2003, pp. 24-25, Theorem 1.3.3), for all \( k \in [1, m] \), \( \mathbb{P} \)-a.s. as \( T \to \infty \),

\[
T \frac{\partial M_{3,T}(\theta_T, \tau_T(\theta_T))}{\partial \tau_k}
\]

\[
= -\frac{1}{2T} \text{tr} \left\{ \mathbb{E} \left[ e^{\tau(\theta_0)'X_t} \psi_t(\theta) \psi(\theta)' \right]^{-1} \mathbb{E} \left[ e^{\tau(\theta_0)'X_t} \psi_t(\theta) \psi(\theta)' \right] \right\}
\]

because \( \tau(\theta_0) = 0_{m \times 1} \) by Lemma \( 10 \) v (p. 32) under Assumption \( 1 \)-a)-(e) and (g)-(h). Therefore, \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \frac{\partial M_{3,T}(\theta_T, \tau_T(\theta_T))}{\partial \tau_k} = O(T^{-1}) \).

(iv) Under Assumption \( 1 \)-a)-(b) and (d)-(h), by Lemma \( 12 \) (p. 35), \( L_T(\theta, \tau) = M_{1,T}(\theta, \tau) + M_{2,T}(\theta, \tau) + M_{3,T}(\theta, \tau) \), so that the result follows from the statement (i)-(iii) of the present lemma.

\[ \diamond \]

Remark 4. In the case in which \( \theta_T = \hat{\theta}_T \), there exist at least one other way to prove Lemma \( 22 \) that do not require Assumption \( 2 \). This way follows an approach à la Newey and Smith (2004), which relies on ULLN with \( \mathbb{T}_T(\theta) = \{ \tau \in \mathbb{R}^m : |\tau| \leq T^{-\zeta} \} \) and \( \zeta > 0 \). We do not follow this ways because (i) Other parts of the proof of Theorem \( 1 \) (p. 6) require the asymptotic normality of \( \hat{\theta}_T \) and thus Assumption \( 2 \); (ii) It would lengthen the proofs and complicate their logic; (iii) We later use Lemma \( 22 \) with \( \theta_T = \hat{\theta}_T \), where \( \hat{\theta}_T \) is a constrained estimator.

\[ \diamond \]
Lemma 23 (Finiteness of the expectations of supremum of the terms from \( \frac{\partial L_T(\theta, \tau)}{\partial \tau} \) and \( \frac{\partial^2 L_T(\theta, \tau)}{\partial \tau^2} \)). Under Assumptions 1 and 2, there exists a closed ball \( B_L \subset S \) centered at \((\theta_0, \tau(\theta_0))\) with strictly positive radius s.t., for all \((h, k) \in [1, m]^2\),

(i) \( E \left[ \sup_{(\theta, \tau) \in B_L} e^{\tau \psi(X_1, \theta)} \right] < \infty; \)
(ii) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_h(X_1, \theta) \right| \right] < \infty; \)
(iii) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \right| \right] < \infty; \)
(iv) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] < \infty; \)
(v) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] < \infty; \)
(vi) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] < \infty; \)
(vii) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \right| \right] < \infty; \)
(viii) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)' \psi(X_1, \theta)' \right| \right] < \infty; \)
(ix) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_h(X_1, \theta) \psi(X_1, \theta) \psi(X_1, \theta)' \right| \right] < \infty; \) and
(x) \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \tau' \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \right] < \infty. \)

Proof. (i) Apply Lemma 18 (p. 51) under Assumptions 1 and 2. Note that it does not immediately follow from Assumption 1(e) and the Cauchy-Schwarz inequality because we need additional assumptions to ensure that there exists \( B_L \subset S \): See Lemma 11(ii) on p. 33.

(ii) For all \((h, k) \in [1, m]^2\), for all \((\theta, \tau) \in B_L\), \( e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_h(X_1, \theta) = e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_h(X_1, \theta) \sqrt{\psi_k(X_1, \theta) \psi_h(X_1, \theta)}^2 \leq \psi_k(X_1, \theta) \psi_h(X_1, \theta) \sqrt{\sum_{i,j \in [1, m]}^2 \psi_i(X_1, \theta) \psi_j(X_1, \theta)}^2 = e^{\tau \psi(X_1, \theta)} \psi(X_1, \theta) \psi(X_1, \theta)', \)

so that \( E \left[ \sup_{(\theta, \tau) \in B_L} \left| e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_h(X_1, \theta) \right| \right] < \infty, \) where the last inequality follows from Lemma 18kii (p. 51) under Assumptions 1 and 2.

(iii) Apply Lemma 19v (p. 57) under Assumptions 1 and 2.
(iv) Apply Lemma 18v (p. 51) under Assumptions 1 and 2.
(v) Apply Lemma 19vii (p. 57) under Assumptions 1 and 2.
(vi) Proof similar to the one of Lemma 18kiii (p. 51). The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under
Assumption (a)(b), for $\overline{B}_L$ of sufficiently small radius, for all $(h, k) \in [1, m]^2$,

$$E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_l(X_1, \theta) \frac{\partial \psi(X_1, \theta)}{\partial \theta'}| \right]$$

$$\leq E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'}| \sup_{(\theta, \tau) \in \overline{B}_L} |\psi_k(X_1, \theta) \psi_l(X_1, \theta)| \right]$$

$$\leq (a) E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \frac{\partial \psi(X_1, \theta)}{\partial \theta'}| \sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta) \psi(X_1, \theta)'| \right]$$

$$\leq (b) \sqrt{E \left[ \sup_{\theta \in \Theta, \tau \in T(\theta)} |e^{2\tau \psi(X_1, \theta)} b(X_1)|^2 \right]} \sqrt{E \left[ \sup_{\theta \in \Theta} |\psi(X_1, \theta) \psi(X_1, \theta)'|^2 \right]} \leq \infty.$$

(a) As in the proof of statement (ii), for all $(h, k) \in [1, m]^2$, for all $(\theta, \tau) \in \overline{B}_L$, $|\psi_k(X_1, \theta) \psi_l(X_1, \theta)| \leq |\psi(X_1, \theta) \psi(X_1, \theta)'|$. (b) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (c) Firstly, under Assumption (a)-(c) and (g)-(h), by Lemma 11 (p. 33), $S$ contains an open ball centered at $(\theta_0, \tau(\theta_0))$, so that, for $\overline{B}_L$ of sufficiently small radius, $\overline{B}_L \subset \{ (\theta, \tau) : \theta \in \mathcal{N} \land \tau \in T(\theta) \} \subset S \subset \mathcal{S}$. Secondly, as the second supremum does not depend on $\tau$, $\sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta) \psi(X_1, \theta)'|^2 \leq \sup_{\theta \in \Theta} |\psi(X_1, \theta) \psi(X_1, \theta)'|^2$ because $\overline{B}_L \subset S$, for $\overline{B}_L$ of radius small enough. (d) Firstly, by Assumption 2(b), the first expectation is bounded. Secondly, by Assumption 2(g), the second expectation is also bounded.

(vii) Apply Lemma 18xii (p. 51) under Assumptions 1 and 2.

(viii) Proof similar to the one of Lemma 18xii (p. 51) and to the statement (vi) of the present lemma. The supremum of the absolute value of the product is smaller than the product of the suprema of the absolute values. Thus, under Assumption (a)(b), for $\overline{B}_L$ of sufficiently small radius, for all $(h, k) \in [1, m]^2$,

$$E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta) \psi_l(X_1, \theta) \psi(X_1, \theta)| \right]$$

$$\leq E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta)| \sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta) \psi(X_1, \theta)'| \right]$$

$$\leq (a) \sqrt{E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{\tau \psi(X_1, \theta)} \psi_k(X_1, \theta)'|^2 \right]} \sqrt{E \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta) \psi(X_1, \theta)'|^2 \right]} \leq \infty.$$

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption (a)-(e) and (g)-(h), by Lemma 11ii (p. 33), $S$ contains an open ball centered at $(\theta_0, \tau(\theta_0))$.
Moreover, for all radius, for all \((\theta, \tau)\), the first expectation is bounded. Secondly, as the second supremum does not depend on \(\tau\), \(\sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta)\psi(X_1, \theta)'| \leq \sup_{\theta \in \Theta} |\psi(X_1, \theta)\psi(X_1, \theta)'|^2\) because \(\overline{B}_L \subset S\), for \(\overline{B}_L\) of radius small enough. (c) Firstly, by Assumption (2)(b), the first expectation is bounded. Secondly, by Assumption (1)(g), the second expectation is also bounded.

(ix) Proof similar to the one of Lemma 18iii (p. 51) and to the statement (vi) of the present lemma. The supremum of the absolute value of the product is smaller than the product of the supremum of the absolute values. Thus, under Assumption (1)(a)(b), for \(\overline{B}_L\) of sufficiently small radius, for all \((h, k) \in [1, n]^2\),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{r^2} \psi(X_1, \theta) \psi^h(X_1, \theta) \psi(X_1, \theta)| \right] \leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{r^2} \psi^h(X_1, \theta) \psi(X_1, \theta)| \sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta)\psi(X_1, \theta)'| \right] \leq \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{r^2} \psi^h(X_1, \theta) \psi(X_1, \theta)| \right] \sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta)\psi(X_1, \theta)'| \leq \mathbb{E} \left[ \sup_{\theta \in \Theta} |\psi(X_1, \theta)\psi(X_1, \theta)'|^2 \right] < \infty.
\]

(a) Apply the Cauchy-Schwarz inequality, and note that the supremum of the square of a positive function is the square of the supremum of the function. (b) Firstly, under Assumption (1)(a)-(e) and (g)-(h), by Lemma 11ii (p. 33), \(S\) contains an open ball centered at \((\theta_0, \tau(\theta_0))\), so that, for \(\overline{B}_L\) of sufficiently small radius, \(\overline{B}_L \subset \{(\theta, \tau) : \theta \in \mathcal{N} \wedge \tau \in T(\theta)\} \subset S \subset S^c\). Moreover, for all \(k \in [1, n]\), for all \(\theta \in \Theta\), \(|\psi^k(X_1, \theta)| \leq |\psi(X_1, \theta)| \leq b(X)\) where the last inequality follows from Assumption (2)(b). Secondly, as the second supremum does not depend on \(\tau\), \(\sup_{(\theta, \tau) \in \overline{B}_L} |\psi(X_1, \theta)\psi(X_1, \theta)'|^2 \leq \sup_{\theta \in \Theta} |\psi(X_1, \theta)\psi(X_1, \theta)'|^2\) because \(\overline{B}_L \subset S\), for \(\overline{B}_L\) of radius small enough. (c) Firstly, by Assumption (2)(b), the first expectation is bounded. Secondly, by Assumption (1)(g), the second expectation is also bounded.

(iii) The norm of a product of matrices is smaller than the product of the norms (e.g., Rudin 1953, Theorem 9.7 and note that all norms are equivalent on finite dimensional spaces). Thus, for \(\overline{B}_L\) of sufficiently small radius, for all \((\ell, j) \in [1, n]^2\),

\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} |e^{r^2} \psi(X_1, \theta) \psi^\ell(X_1, \theta) \psi(X_1, \theta)\frac{\partial \psi(X_1, \theta)}{\partial \theta^\ell}| \right] \leq \left( \sup_{(\theta, \tau) \in \overline{B}_L} |\tau| \right) \mathbb{E} \left[ \sup_{(\theta, \tau) \in \overline{B}_L} e^{r^2} \psi(X_1, \theta)|\psi(X_1, \theta)||\frac{\partial \psi(X_1, \theta)}{\partial \theta^\ell}| \right] \leq \left( \sup_{(\theta, \tau) \in \overline{B}_L} |\tau| \right) \mathbb{E} \left[ \sup_{\theta \in \mathcal{N} \tau \in T(\theta)} e^{r^2} \psi(X_1, \theta) b(X_1)^2 \right] < \infty.
\]
In addition, by Assumption 1(h) and 3(b), \( \Sigma(\gamma) \) on p. 6 holds, as follows from the Cochran’s theorem.

\[ \begin{align*}
\text{(a) Firstly, under Assumption } & \text{1(a)-(e) and } \text{2(a), } \text{by Lemma } 11 \text{ii (p. 33), } \text{S contains an open ball centered at } \theta_0, \text{ thus, under Assumption } 1\text{(a)-(e) and } \text{2(a), for } \overline{D}_L \text{ of sufficiently small radius, by definition of } S, \overline{D}_L \subset \{ (\theta, \tau) : \theta \in \mathcal{N} \wedge \tau \in T(\theta) \} \subset S, \text{ because } \mathcal{N} \subset \Theta \text{ by Assumption 2(a). Secondly, by Assumption 2(b), } \sup_{\theta \in \mathcal{N}} |\psi(X_1, \theta)| \leq b(X) \text{ and } \sup_{\theta \in \mathcal{N}} \left| \frac{\partial \psi(X_1, \theta)}{\partial \theta} \right| \leq b(X). \text{ (b) Firstly, } \sup_{(\theta, \tau) \in \overline{D}_L} |\tau|^2 < \infty \text{ because } \overline{D}_L \text{ is bounded. Secondly, by Assumption 2(b),} \end{align*} \]

\[ \text{by Assumption 3(a). Moreover, by the theorem’s assumption, as } T \to \infty, \text{ } \hat{\theta}_T \to \theta_0. \text{ Thus, under Assumptions 1, 2 and 3, by continuity of } R(\cdot), \mathbb{P}\text{-a.s. as } T \to \infty, \text{ } R(\hat{\theta}_T) \to R(\theta_0), \text{ so that the result follows by the Slutsky’s theorem.} \]

Now, under Assumptions 1 and 2, by Lemma 28 (p. 83), \( \mathbb{P}\text{-a.s. as } T \to \infty, \sqrt{T}(\hat{\theta}_T - \theta_0) \overset{D}{\to} N(0, \Sigma(\theta_0)) \), which also implies that \( \hat{\theta}_T \to \theta_0 \). Thus, under Assumptions 1 and 3, by continuity of \( R(\cdot) \), \( \mathbb{P}\text{-a.s. as } T \to \infty, \text{ } R(\hat{\theta}_T) \to R(\theta_0) \), so that the result follows by the Slutzky’s theorem.

Asymptotic distribution of Wald. By Assumption 3(a), \( r : \Theta \to \mathbb{R}^2 \) is continuously differentiable. Thus, under Assumptions 1 and 2, if the test hypothesis \( \theta_0 \) on p. 6 holds, a first-order Taylor-Lagrange expansion at \( \theta_0 \), \( \omega \) by \( \omega \), yields, \( \mathbb{P}\text{-a.s. as } T \to \infty, \)

\[ r(\hat{\theta}_T) = r(\theta_0) + R(\theta_0)(\hat{\theta}_T - \theta_0), \text{ where } \hat{\theta}_T \text{ is between } \hat{\theta}_T \text{ and } \theta_0; \]

\[ \begin{align*}
(\text{a) By definition, if the test hypothesis } & \theta_0 \text{ on p. 6 holds, } r(\theta_0) = 0_{(k \times 1)}. \text{ (b) Under Assumptions } 1 \text{ and } 2 \text{ by Theorem } 1 \text{ii (p. 33), } \mathbb{P}\text{-a.s. as } T \to \infty, \sqrt{T}(\hat{\theta}_T - \theta_0) \overset{D}{\to} N(0, \Sigma(\theta_0)), \text{ which also implies that } \hat{\theta}_T \to \theta_0. \text{ Thus, under Assumptions } 1 \text{ and } 3, \text{ by continuity of } R(\cdot), \mathbb{P}\text{-a.s. as } T \to \infty, \text{ } R(\hat{\theta}_T) \to R(\theta_0), \text{ so that the result follows by the Slutzky’s theorem.} \end{align*} \]

Asymptotic distribution of LM. Under Assumptions 1 and 3, by Proposition 2ii (p. 76), if the test hypothesis \( \theta_0 \) on p. 6 holds, as \( T \to \infty, \gamma_T \overset{D}{\to} N(0, (R(\theta_0)\Sigma(\theta_0)R(\theta_0)' - 1)^{-1} \Sigma(\theta_0)). \) Now, under Assumptions 1 and 3, by Lemma 28 (p. 83), \( \mathbb{P}\text{-a.s. as } T \to \infty, \hat{\theta}_T \to \theta_0, \) so that \( R(\hat{\theta}_T) \to R(\theta_0) \) by Assumption 3(a). Moreover, by the theorem’s assumption, as \( T \to \infty, \) \( \Sigma(\theta_0) \) and \( R(\theta_0) \) are full rank, so that \( \Sigma(\theta_0) \) and \( R(\theta_0) \) are full rank w.p.a.1 as \( T \to \infty \) (Lemma 30 p. 88). Then, the result follows from the Cochran’s theorem.

Finally, under Assumptions 1 and 3, by Lemma 28ii (p. 83), \( R(\hat{\theta}_T)'\gamma_T = - \frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta} \bigg|_{\theta=\hat{\theta}_T} ; \) so \( T(\hat{\theta}_T)'\gamma_T = T(\hat{\theta}_T)'R(\hat{\theta}_T)\Sigma(\theta_0)R(\hat{\theta}_T)'^{-1} \gamma_T = T(\hat{\theta}_T)'R(\hat{\theta}_T)\Sigma(\theta_0)R(\hat{\theta}_T)'^{-1} \gamma_T = T \left( \frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta} \right)' \bigg|_{\theta=\hat{\theta}_T} \Sigma(\theta_0) = \frac{\partial \ln |\Sigma(\theta_0)|}{\partial \theta} \bigg|_{\theta=\hat{\theta}_T} \Sigma(\theta_0) = \frac{\partial \ln |\Sigma(\theta_0)|}{\partial \theta} \bigg|_{\theta=\hat{\theta}_T} \Sigma(\theta_0) = \frac{\partial \ln |\Sigma(\theta_0)|}{\partial \theta} \bigg|_{\theta=\hat{\theta}_T} \Sigma(\theta_0) = \frac{\partial \ln |\Sigma(\theta_0)|}{\partial \theta} \bigg|_{\theta=\hat{\theta}_T} \Sigma(\theta_0) \]
Asymptotic distribution of $\text{ALR}_T$. Under Assumptions \ref{assumption:1} and \ref{assumption:2} if the test hypothesis \ref{hyp:9} on p. \ref{page:6} holds, by Lemma \ref{lemma:24} (p. \ref{page:71}), $\mathbb{P}$-a.s. as $T \to \infty$,

$$2\{\ln[\hat{f}_T(\theta_T)] - \ln[\hat{f}_T(\bar{\theta}_T)]\} = -\left[\sqrt{T}(\hat{\theta}_T - \bar{\theta}_T)\right] \Sigma^{-1} \left[\sqrt{T}(\hat{\theta}_T - \bar{\theta}_T)\right] + o_T(1)$$

\begin{align*}
        &\stackrel{(a)}{=} -\left[\Sigma R'(R\Sigma R')^{-1}RM^{-1}\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)\right] \Sigma^{-1} \left[\Sigma R'(R\Sigma R')^{-1}RM^{-1}\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)\right] + o_T(1) \\
        &\stackrel{(b)}{=} -\left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0)\right] \left(M' - R'(R\Sigma R')^{-1}R\Sigma^{-1}\Sigma R'(R\Sigma R')^{-1}RM^{-1}\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)\right] \\
        &\stackrel{(c)}{=} -\left[V^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0)\right] V^{-1/2} \left(M' - R'(R\Sigma R')^{-1}RM^{-1}V^{-1/2} = V^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)\right] \\
        &\stackrel{(d)}{=} -\left[V^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0)\right] P_{\Sigma_{1/2}R'} \left[V^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0)\right] + o_T(1) \\
        &\stackrel{(f)}{=} \chi^2_q
\end{align*}

(a) Under Assumptions \ref{assumption:1} and \ref{assumption:2} if the test hypothesis \ref{hyp:9} on p. \ref{page:6} holds, by Lemma \ref{lemma:24} (p. \ref{page:71}), $\mathbb{P}$-a.s. as $T \to \infty$, $\sqrt{T}(\hat{\theta}_T - \bar{\theta}_T) = \Sigma R'(R\Sigma R')^{-1}RM^{-1}\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)$. (b) Transpose the content of the first square bracket, and then note that $\Sigma = \Sigma'$ by symmetry. (c) Note that $R\Sigma^{-1} \Sigma R'(R\Sigma R')^{-1} = I$. (d) Use that $V^{1/2}V^{-1/2} = I$. (e) Note that $P_{\Sigma_{1/2}R'} = V^{-1/2} (M' - R'(R\Sigma R')^{-1}R'M^{-1} V^{-1/2}$. (f) Under Assumption \ref{assumption:1}(a)-(c) and (g), by the Lindeberg-Lévy CLT theorem, as $T \to \infty$, $\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \xrightarrow{D} \mathcal{N}(0, V)$ where $V := \mathbb{E}[\psi'(X_1, \theta_0)\psi'(X_1, \theta_0)]$. Moreover, the orthogonal projection matrix $P_{\Sigma_{1/2}R'}$ has rank $q$ because $R$ is of rank $q$ and $\Sigma$ has full rank by Assumptions \ref{assumption:3}(b) and \ref{assumption:4}(h), respectively. Thus, the result follows from the Cochran’s theorem.

Asymptotic distribution of $ET_T$. Under Assumptions \ref{assumption:1} and \ref{assumption:2} if the test hypothesis \ref{hyp:9} on p. \ref{page:6} holds, by Proposition \ref{proposition:29} (p. \ref{page:76}), $\mathbb{P}$-a.s. as $T \to \infty$,

$$\sqrt{T}(\hat{\theta}_T) = (M' - \Sigma^{-1/2} P_{\Sigma_{1/2}R'} \Sigma^{-1/2} M^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)$$

\begin{align*}
        &\stackrel{(a)}{=} (M' - [V^{-1/2}(M')^{-1}]^{-1} P_{\Sigma_{1/2}R'} [M^{-1} V^{-1/2}]^{-1} M^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1) \\
        &\stackrel{(b)}{=} (M' - M' V^{-1/2} P_{\Sigma_{1/2}R'} V^{-1/2} M M^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1) \\
        &\stackrel{(c)}{=} V^{-1/2} P_{\Sigma_{1/2}R'} V^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_T(1)
\end{align*}
(a) By definition, $M^{-1}V(M')^{-1} = \Sigma = \Sigma^{1/2}\Sigma^{1/2}$, so that $\Sigma^{-1/2} := (\Sigma^{1/2})^{-1} = [V^{1/2}(M')^{-1}]^{-1}$ and $\Sigma^{-1/2'} := (\Sigma^{1/2'})^{-1} = [V^{-1/2'}]^{-1}$. (b) By standard property of inverses, $[V^{1/2}(M')^{-1}]^{-1}$ and $[M^{-1}V^{-1/2'}]^{-1} = V^{-1/2'}M$.

Thus, under Assumptions 1(a) and 3 if the test hypothesis 9 on p. 8 holds, by Proposition 2 P-a.s. as $T \to \infty$,

$T^2(\theta_T) \sim \psi_T(\theta_T)$

$= \left[ V^{-1/2}P_{\Sigma^{1/2}R}V^{-1/2'} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_p(1) \right]^T \psi_T \left[ V^{-1/2}P_{\Sigma^{1/2}R}V^{-1/2'} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_p(1) \right]$

$\sim (a) \left[ V^{-1/2}P_{\Sigma^{1/2}R}V^{-1/2'} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \right]^T \psi_T \left[ V^{-1/2}P_{\Sigma^{1/2}R}V^{-1/2'} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \right]$

$= 2 \left[ V^{-1/2}P_{\Sigma^{1/2}R}V^{-1/2'} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \right]^T \psi_T \left[ V^{-1/2}P_{\Sigma^{1/2}R}V^{-1/2'} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \right] + o_p(1)$

$\sim (b) \left[ \sum_{t=1}^{T} \psi_t(\theta_0) \right]^T \psi_T \left[ \sum_{t=1}^{T} \psi_t(\theta_0) \right]$

$\sim (c) \left[ \sum_{t=1}^{T} \psi_t(\theta_0) \right]^T \left[ \sum_{t=1}^{T} \psi_t(\theta_0) \right]$ \left[ \left( V^{-1/2} \psi_T V^{-1/2} \right) - I \right] \left[ V^{-1/2} \psi_T V^{-1/2} \right] + o_p(1)$

$\sim (d) \left[ \sum_{t=1}^{T} \psi_t(\theta_0) \right]^T \left[ \sum_{t=1}^{T} \psi_t(\theta_0) \right]$ \left[ \left( V^{-1/2} \psi_T V^{-1/2} \right) \right] + o_p(1)$

Denoting the convergence in probability with \( \psi_T \), by the present theorem assumption, as $T \to \infty$, \( \psi_T \rightarrow V \), where $V$ is a positive definite symmetric matrix by Assumption 1(h). Thus, by Lemma 13 (p. 88), w.p.a.1 as $T \to \infty$, \( \psi_T \) is p-d.m. so that it has a square root s.t. \( \psi_T = \psi_T^{1/2} \psi_T^{1/2} \), where \( \psi_T^{1/2} \rightarrow V^{1/2} \).

(d) Under Assumption 1(a)-(c) and (g), by the Lindeberg-Lévy CLT theorem, as $T \to \infty$, \( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \rightarrow \mathcal{N}(0, V) \) where $V := \mathbb{E}[\psi(X_1, \theta_0)^T \psi(X_1, \theta_0)]$. Moreover, the orthogonal projection matrix $P_{\Sigma^{1/2}R}$ has rank $q$ because $R$ is of rank $q$ and $\Sigma$ has full rank by Assumptions 3(b) and 1(h), respectively. Thus, the result follows from the Cochran’s theorem.

\[ \square \]

**Lemma 24** (Asymptotic expansions for ALR$_T$). Under Assumptions 1(a) and 3 if the test hypothesis 9 on p. 8 holds, P-a.s. as $T \to \infty$,
(i) $2\{\ln[f_{\theta_T}^*(\hat{\theta}_T)] - \ln[f_{\theta_T}^*(\hat{\theta}_T)]\} = T(\hat{\theta}_T - \hat{\theta}_T)'\Sigma^{-1}(\hat{\theta}_T - \hat{\theta}_T) + o_P(1)$;
(ii) $\sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) = \Sigma R'(R\Sigma R')^{-1}R M^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_P(1)$

where $\Sigma := \Sigma(\theta_0) := M^{-1}V(M')^{-1}$, $M := \mathbb{E}\left[\frac{\partial \psi(X_1, \theta_0)}{\partial \theta'}\right]$, $V := \mathbb{E}[\psi(X_1, \theta_0)\psi(X_1, \theta_0)]$, and $R := \frac{\partial \psi(\theta_0)}{\partial \theta'}$.

Proof. (i) Under Assumption 1 by Lemma 12 (p. 35), $\mathbb{P}$-a.s. for $T$ big enough, for all $(\theta, \tau)$ in a neighborhood of $(\theta_0, \tau(\theta_0))$, $L_T(\theta, \tau)$ exists. Moreover, under Assumptions 1, 2 and 3, if the test hypothesis (9) on p. 6 holds, by Theorem 1i (p. 6), Lemma 28i (p. 83) and Lemma 2iii (p. 62), $\hat{\theta}_T \rightarrow \theta_0$, $\hat{\tau}_T \rightarrow \tau_0$, $\tau_T(\hat{\theta}_T) \rightarrow \tau(\theta_0)$ and $\tau_T(\hat{\tau}_T) \rightarrow \tau(\theta_0)$, $\mathbb{P}$-a.s. as $T \rightarrow \infty$. Thus, noting that $\ln[f_{\theta_T}^*(\theta)] = L_T(\theta, \tau_T(\theta))$, under Assumptions 1, 2 and 3, if the test hypothesis (9) on p. 6 holds, $\mathbb{P}$-a.s. for $T$ big enough,

$$2\{\ln[f_{\theta_T}^*(\hat{\theta}_T)] - \ln[f_{\theta_T}^*(\hat{\theta}_T)]\} = -2T[L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T)) - L_T(\hat{T}_T, \tau_T(\hat{T}_T))]$$

Now, under Assumptions 1 and 2, by subsection 3.2 (p. 33), $L_T(\theta, \tau)$ is twice continuously differentiable in a neighborhood of $(\theta_0, \tau(\theta_0))$, $\mathbb{P}$-a.s. for $T$ big enough, so that a stochastic second-order Taylor-Lagrange expansion (e.g., Aliprantis and Border 2006/1999, Theorem 18.18) around $(\hat{\theta}_T, \tau_T(\hat{\theta}_T))$ and evaluated $(\hat{\theta}_T, \tau_T(\hat{\theta}_T))$ yields, $\mathbb{P}$-a.s. for $T$ big enough,

$$L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T)) = L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T)) + \left[\frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'} \frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \tau'}\right] \left[\begin{array}{c} \hat{\theta}_T - \hat{\theta}_T \\ \tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T) \end{array}\right]$$

$$+ \frac{1}{2} \left((\hat{\theta}_T - \hat{\theta}_T)'(\tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T))'\right) \left[\begin{array}{cc} \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta' \partial \theta} & \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta' \partial \tau} \\ \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta \partial \tau} & \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \tau' \partial \tau} \end{array}\right] \left[\begin{array}{c} \hat{\theta}_T - \hat{\theta}_T \\ \tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T) \end{array}\right]$$

where $(\hat{\theta}_T, \tau_T)$ is between $(\hat{\theta}_T, \tau_T(\hat{\theta}_T))$ and $(\hat{\theta}_T, \tau_T(\hat{\theta}_T))$;

$$\Rightarrow L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T)) - L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))$$

$$= \frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'}(\hat{\theta}_T - \hat{\theta}_T) + \frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \tau'}(\tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T))$$

$$+ \frac{1}{2}(\hat{\theta}_T - \hat{\theta}_T)'\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta' \partial \theta}(\hat{\theta}_T - \hat{\theta}_T) + \frac{1}{2}(\tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T))\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta' \partial \tau}(\tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T))$$

$$+ (\hat{\theta}_T - \hat{\theta}_T)'\frac{\partial^2 L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta \partial \tau}(\tau_T(\hat{\theta}_T) - \tau_T(\hat{\theta}_T)),$$

where

- Under Assumptions 1, 2 and 3 if the test hypothesis (9) on p. 6 holds, by Theorem 1i (p. 6) and Proposition 2ii (p. 76), $\mathbb{P}$-a.s. as $T \rightarrow \infty$, $\hat{\theta}_T - \hat{\theta}_T = (\hat{\theta}_T - \theta_0) - (\hat{\theta}_T - \theta_0) = O_P(T^{-\frac{1}{2}}) + O_P(T^{-\frac{1}{2}}) = O_P(T^{-\frac{1}{2}})$;
- Under Assumptions 1 and 2 by Lemma 20 (p. 61), $\mathbb{P}$-a.s. as $T \rightarrow \infty$, $\frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'} = O(T^{-1})$, so that $\frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta'}(\hat{\theta}_T - \hat{\theta}_T) = O_P(T^{-\frac{1}{2}})$ by the first bullet point;
- Under Assumptions 1 and 2 if the test hypothesis (9) on p. 6 holds, by Lemma 21v (p. 62) and Theorem 1i (p. 6) and Lemma 28 (p. 83), $\mathbb{P}$-a.s. as $T \rightarrow \infty$, there exists $\theta_T$ between $\hat{\theta}_T$ and $\theta_0$ such that $\sqrt{T}[\tau_T(\theta_T) - \tau_T(\hat{\theta}_T)] = \sqrt{T}[\tau_T(\theta_T) - \tau_T(\theta_T)] = \sqrt{T}[\tau_T(\theta_T) - \tau_T(\theta_T)] = \sqrt{T}[\tau_T(\theta_T) - \tau_T(\theta_T)] = -V^{-1}M(\hat{\theta}_T - \theta_0) + o_P(1) - \left[-V^{-1}M(\hat{\theta}_T - \theta_0) + o_P(T^{-1/2})\right] = -V^{-1}M(\hat{\theta}_T - \theta_0) + V^{-1}M(\hat{\theta}_T - \theta_0) + o_P(1) = -V^{-1}M(\hat{\theta}_T - \hat{\theta}_T) + o_P(1);$
• Under Assumptions 1 and 2 by Lemma 22iv (p. 64), \( P \)-a.s. as \( T \to \infty \), \( \frac{\partial L_T(\hat{\theta}, \tau_T(\hat{\theta}))}{\partial \tau} = O(T^{-1}) \), so that, by the first and third bullet point, \( \frac{\partial L_T(\hat{\theta}, \tau_T(\hat{\theta}))}{\partial \tau} (\hat{\theta}_T - \tau_T(\hat{\theta})) = O_p(T^{-3/2}) \), under Assumptions 1, 2 and 3, if the test hypothesis (9) on p. 6 holds;

• Under Assumptions 1, 2 and 3, if the test hypothesis (9) on p. 6 holds, by Lemma 21iv (p. 62) and Theorem 3 (p. 6) and Lemma 28 (p. 83), \( P \)-a.s. as \( T \to \infty \), there exists \( \hat{\theta}_T \) between \( \hat{\theta}_T \) and \( \theta_0 \) s.t. \( \sqrt{T}[\tau_T(\hat{\theta}_T) - \tau_T(\theta_0)] - \sqrt{T}[\tau_T(\hat{\theta}_T) - \tau_T(\theta_0)] = -V^{-1}M(\hat{\theta}_T - \theta_0) + o_p(1) \) - \( -V^{-1}M(\hat{\theta}_T - \theta_0) + o_p(1) \) - \( -V^{-1}M(\hat{\theta}_T - \theta_0) + V^{-1}M(\hat{\theta}_T - \theta_0) + o_p(1) = -V^{-1}M(\hat{\theta}_T - \theta_0) + o_p(1) \); and

• under Assumptions 1 and 2, by Lemma 14ii (p. 48), \( P \)-a.s. as \( T \to \infty \) \( \left| \frac{\partial^2 L_T(\theta, \tau_T)}{\partial \theta \partial \tau} \right| = o(1) \), so that, by Theorem 1ii (p. 6), \( (\hat{\theta}_T - \hat{\theta}_T) \frac{\partial^2 L_T(\hat{\theta}, \tau_T)}{\partial \theta \partial \tau} (\hat{\theta}_T - \hat{\theta}_T) = o_p(T^{-1}) \) by the first bullet point.

Therefore, Assumptions 1, 2 and 3, if the test hypothesis (9) on p. 6 holds,

\[
2[\ln[\hat{\theta} \tau_T(\hat{\theta})] - \ln[\hat{\theta} \tau_T(\hat{\theta})]] = -2T \left[ L_T(\hat{\theta}, \tau_T(\hat{\theta})) - L_T(\hat{\theta}, \tau_T(\hat{\theta})) \right]
\]

\[
= \left[ -V^{-1}M - \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) \right] \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T)}{\partial \theta \partial \tau} \left[ -V^{-1}M \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) \right] + 2\sqrt{T}(\theta_0 - \hat{\theta}_T) \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T)}{\partial \theta \partial \theta} \left[ -V^{-1}M \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) \right] + o_p(1)
\]

\[
= \left[ -V^{-1}M - \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) \right] \frac{M'V^{-1} \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T)}{\partial \theta \partial \tau} - 2 \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T)}{\partial \theta \partial \tau} \left[ V^{-1}M \right] \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) + o_p(1)
\]

\[
= \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) \frac{M'V^{-1} \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T)}{\partial \theta \partial \tau} - 2 \frac{\partial^2 L_T(\hat{\theta}_T, \tau_T)}{\partial \theta \partial \tau} \left[ V^{-1}M \right] \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) + o_p(1)
\]

where the explanations for the convergence are as follow. Firstly, under Assumptions 1 and 2 by Lemma 14ii (p. 48), for any sequence \((\theta_T, \tau_T)_{T \in \mathbb{N}}\) converging to \((\theta_0, \tau(\theta_0))\), \( P \)-a.s. as \( T \to \infty \), \( \frac{\partial^2 L_T(\hat{\theta}, \tau_T)}{\partial \theta \partial \tau} \to E \left[ \frac{\partial^2 \psi(X_1, \theta)}{\partial \theta \partial \tau} \right] = M \). Secondly, under Assumptions 1 and 2 by Lemma 25iv (p. 74), \( P \)-a.s. as \( T \to \infty \), \( \frac{\partial L_T(\hat{\theta}, \tau_T)}{\partial \theta} \to E[\psi(X_1, \theta_0)\psi(X_1, \theta_0)] =: V \). Therefore, \( P \)-a.s. as \( T \to \infty \),

\[
M'V^{-1} - \frac{\partial^2 L_T(\hat{\theta}, \tau_T)}{\partial \theta \partial \tau} V^{-1}M - 2 \frac{\partial^2 L_T(\hat{\theta}, \tau_T)}{\partial \theta \partial \tau} V^{-1}M
\]

\[
\to M'V^{-1} - 2M'V^{-1}M = -M'V^{-1}M = -\Sigma(\theta_0)^{-1}
\]

(ii) Under Assumptions 1, 2 and 3, if the test hypothesis (9) on p. 6 holds, \( P \)-a.s. as \( T \to \infty \), addition and subtraction of \( \sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) \) yield

\[
\sqrt{T}(\hat{\theta}_T - \hat{\theta}_T) = \sqrt{T}(\hat{\theta}_T - \theta_0) - \sqrt{T}(\hat{\theta}_T - \theta_0)
\]

\[
= -M^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_p(1) - \left[ M^{-1} - \Sigma R'(R\Sigma R')^{-1} RM^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_p(1) \right]
\]

\[
= \Sigma R'(R\Sigma R')^{-1} RM^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_p(1)
\]

where the explanations for the second equality are the following. Firstly, under Assumptions 1 and 2 by Proposition 1 (p. 45), \( P \)-a.s. as \( T \to \infty \), \( \sqrt{T}(\hat{\theta}_T - \theta_0) = -M^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_p(1) \).
Secondly, under Assumptions 1 and 3 by Proposition 2 on p. 76, if the test hypothesis (9) on p. 74 holds, $\mathbb{P}$-a.s. as $T \to \infty$, $\sqrt{T}(\theta_T - \theta_0) = M^{-1} - \Sigma_R(R \Sigma R')^{-1} RM^{-1} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) + o_P(1)$. □

Lemma 25 (Asymptotic limit of $\frac{\partial^2 L_T(\theta_T, \tau_T)}{\partial \theta_T \partial \tau_T}$). Under Assumptions 1 and 2 for any sequence $(\theta_T, \tau_T)_{T \in \mathbb{N}}$ converging to $(\theta_0, \tau(\theta_0))$, $\mathbb{P}$-a.s. as $T \to \infty$, for all $(h, k) \in [1, m]^2$, $\mathbb{P}$-a.s. as $T \to \infty$,

(i) $\frac{\partial^2 M_{1,T}(\theta_T, \tau_T)}{\partial \tau_T \partial \tau_k} \to \mathbb{E}[\psi_h(X_1, \theta_0) \psi_h(X_1, \theta_0)];$

(ii) $\frac{\partial^2 M_{1,T}(\theta_T, \tau_T)}{\partial \theta_T \partial \tau_k} = O(T^{-1});$

(iii) $\frac{\partial^2 M_{1,T}(\theta_T, \tau_T)}{\partial \theta_T \partial \tau_T} = O(T^{-1});$ and

(iv) $\frac{\partial^2 L_T(\theta_T, \tau_T)}{\partial \theta_T \partial \tau_T} \to \mathbb{E}[\psi(X_1, \theta_0) \psi(X_1, \theta_0)^\prime].$

Proof. (i) By equation (28) on p. 38 for all $(h, k) \in [1, m]^2$,

$$\frac{\partial^2 M_{1,T}(\theta_T, \tau_T)}{\partial \tau_T \partial \tau_k} = \left(1 - \frac{m}{2T}\right) \left[\frac{1}{T} \sum_{i=1}^{T} e^{\tau_i \psi_i(\theta_T)}\right] \left\{ \left[\frac{1}{T} \sum_{i=1}^{T} e^{\tau_i \psi_i(\theta_T) \psi_{t,h}(\theta_T) \psi_{t,k}(\theta_T)}\right] \right. \right.$$  

$$- \left[\frac{1}{T} \sum_{i=1}^{T} e^{\tau_i \psi_i(\theta_T) \psi_{t,h}(\theta_T)}\right] \left[\frac{1}{T} \sum_{i=1}^{T} e^{\tau_i \psi_i(\theta_T) \psi_{t,k}(\theta_T)}\right]\right\}.\right.$$  

where, as $T \to \infty$, $(\theta_T, \tau_T) \to (\theta_0, \tau(\theta_0))$ by assumption. Now, under Assumptions 1 and 2 by Lemma 23-iii (p. 65), for $BL$ a ball around $(\theta_0, \tau(\theta_0))$ of sufficiently small radius, $\mathbb{E} \left[\sup_{(\theta, \tau) \in BL} e^{\tau \psi(X_1, \theta)}\right] < \infty$, $\mathbb{E} \left[\sup_{(\theta, \tau) \in BL} |e^{\tau \psi(X_1, \theta) \psi(X_1, \theta)}\right] < \infty$, and $\mathbb{E} \left[\sup_{(\theta, \tau) \in BL} |e^{\tau \psi(X_1, \theta) \psi(X_1, \theta)}\right]$. Thus, by Assumption 1(a)(b) and (d), the ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthi 2003, pp. 24-25, Theorem 1.3.3), implies that, for all $(h, k) \in [1, m]^2$, $\mathbb{P}$-a.s. as $T \to \infty$,

$$\frac{\partial^2 M_{1,T}(\theta_T, \tau_T)}{\partial \theta_T \partial \tau_h} \to \mathbb{E} \left[e^{\tau(\theta_0) \psi(X_1, \theta_0)}\right] \mathbb{E} \left[e^{\tau(\theta_0) \psi(X_1, \theta_0) \psi(X_1, \theta_0)}\right]$$

because $\mathbb{E}[\psi(X_1, \theta_0)] = 0_{m \times 1}$ by Assumption 1(c), and $\tau(\theta_0) = 0_{m \times 1}$ by Lemma 10(v (p. 32) under Assumption 1(a)-(e) and (g)-(h).
(ii) Under Assumptions \([1]\) by equation \([34]\) on p. \([40]\) \(\mathbb{P}\)-a.s. for \(T\) big enough, for all \((h,k) \in [1,m]^2\),
\[
\frac{\partial^2 M_{2,T}(\theta_T, \tau_T)}{\partial \tau_h \partial \tau_k}
= -\frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T \psi_t(\theta_T)} \frac{\partial \psi_t(\theta_T)}{\partial \theta'} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T \psi_t(\theta_T)} \psi_{t,k}(\theta_T) \frac{\partial \psi_t(\theta_T)}{\partial \theta'} \right] \right\}
\times \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T \psi_t(\theta_T)} \frac{\partial \psi_t(\theta_T)}{\partial \theta'} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T \psi_t(\theta_T)} \psi_{t,h}(\theta_T) \frac{\partial \psi_t(\theta_T)}{\partial \theta'} \right] \right\}
+ \frac{1}{T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T \psi_t(\theta_T)} \frac{\partial \psi_t(\theta_T)}{\partial \theta'} \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_T \psi_t(\theta_T)} \psi_{t,k}(\theta_T) \psi_{t,h}(\theta_T) \frac{\partial \psi_t(\theta_T)}{\partial \theta'} \right] \right\}.
\]
where, as \(T \to \infty\), \((\theta_T, \tau_T) \to (\theta_0, \tau(\theta_0))\) by assumption. Now, under Assumptions \([1]\) and \([2]\) by Lemma \([23]\) v-i (p. \([65]\), for \(B_L\) a ball around \((\theta_0, \tau(\theta_0))\) of sufficiently small radius,
\[
\mathbb{E} \left[ \sup_{(\theta,\tau) \in B_L} \left| e^{\tau \psi(X_1,\theta)} \frac{\partial \psi(X_1,\theta)}{\partial \theta'} \right|^1 \right], \mathbb{E} \left[ \sup_{(\theta,\tau) \in B_L} \left| e^{\tau \psi(X_1,\theta)} \psi_k(X_1,\theta) \frac{\partial \psi(X_1,\theta)}{\partial \theta'} \right|^1 \right],
\]
and
\[
\mathbb{E} \left[ \sup_{(\theta,\tau) \in B_L} \left| e^{\tau \psi(X_1,\theta)} \psi_h(X_1,\theta) \frac{\partial \psi(X_1,\theta)}{\partial \theta'} \right|^1 \right].
\]
Thus, by Assumptions \([1]\)(a)(b) and (d), the ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthi \([2003]\) pp. 24-25, Theorem 1.3.3), implies that, for all \((h,k) \in [1,m]^2\), \(\mathbb{P}\)-a.s. as \(T \to \infty\),
\[
T \frac{\partial^2 M_{2,T}(\theta_T, \tau_T)}{\partial \tau_h \partial \tau_k}
\to \text{tr} \left\{ \mathbb{E} \left[ e^{\tau(\theta_0)' \psi(X_1,\theta_0)} \frac{\partial \psi(X_1,\theta_0)}{\partial \theta'} \right]^{-1} \mathbb{E} \left[ e^{\tau(\theta_0)' \psi(X_1,\theta_0)} \psi_k(X_1,\theta_0) \frac{\partial \psi(X_1,\theta_0)}{\partial \theta'} \right] \right\}
\times \mathbb{E} \left[ e^{\tau(\theta_0)' \psi(X_1,\theta_0)} \frac{\partial \psi(X_1,\theta_0)}{\partial \theta'} \right]^{-1} \mathbb{E} \left[ e^{\tau(\theta_0)' \psi(X_1,\theta_0)} \psi_h(X_1,\theta_0) \frac{\partial \psi(X_1,\theta_0)}{\partial \theta'} \right] \right\}
+ \text{tr} \left\{ \mathbb{E} \left[ e^{\tau(\theta_0)' \psi(X_1,\theta_0)} \frac{\partial \psi(X_1,\theta_0)}{\partial \theta'} \right]^{-1} \mathbb{E} \left[ e^{\tau(\theta_0)' \psi(X_1,\theta_0)} \psi_k(X_1,\theta_0) \psi_h(X_1,\theta_0) \frac{\partial \psi(X_1,\theta_0)}{\partial \theta'} \right] \right\}.
\]
because \(\tau(\theta_0) = 0_m \times 1\) by Lemma \([10]\) v (p. \([32]\) under Assumption \([1]\)(a)-(c) and (g)-(h). Therefore, \(\mathbb{P}\)-a.s. as \(T \to \infty\),
\[
\frac{\partial^2 M_{2,T}(\theta_T, \tau_T)}{\partial \tau_h \partial \tau_k} = O(T^{-1}).
\]
(iii) Under Assumptions \([1](a)(b)(e)(g)(h)\), by equation (39) (p. 44), for all \((h, k) \in [1, m]^2\),
\[
\frac{\partial^2 M_{3,T}(\theta_T, \tau_T)}{\partial \tau_h \partial \tau_k} = \frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta_T)} \psi_t(\theta_T) \psi_t(\theta_T)' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta_T)} \psi_{t,h} (\theta_T) \psi_t(\theta_T)' \right] \right\}
\times \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta_T)} \psi_t(\theta_T) \psi_t(\theta_T)' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta_T)} \psi_{t,h} (\theta_T) \psi_t(\theta_T)' \right] \right\}^{-1}
\frac{1}{2T} \text{tr} \left\{ \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta_T)} \psi_t(\theta_T) \psi_t(\theta_T)' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau_t \psi_t(\theta_T)} \psi_{t,k} (\theta_T) \psi_t(\theta_T)' \right] \right\}
\]}

where, as \(T \to \infty\), \((\theta_T, \tau_T) \to (\theta_0, \tau(\theta_0))\) by assumption. Now, under Assumptions \([1]\) and \([2]\) by Lemma \([23]\) vii-ix (p. 65), there exists a closed ball \(B_L \subset S\) centered at \((\theta_0, \tau(\theta_0))\) with strictly positive radius s.t., for all \(k \in [1, m]\),
\[
\mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau(x, \theta)} \psi_k(x, \theta) \psi(x, \theta)\psi(x, \theta)'| \right] < \infty, \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau(x, \theta)} \psi_k(x, \theta) \psi(x, \theta)\psi(x, \theta)'| \right] < \infty, \quad \mathbb{E} \left[ \sup_{(\theta, \tau) \in B_L} |e^{\tau(x, \theta)} \psi_k(x, \theta) \psi(x, \theta)\psi(x, \theta)'| \right] < \infty.
\]

Thus, by Assumptions \([1](a)(b)\) and \((d)\), the ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthi 2003 pp. 24-25, Theorem 1.3.3), implies that, for all \((h, k) \in [1, m]^2\), \(\mathbb{P}\)-a.s. as \(T \to \infty\),
\[
T \frac{\partial^2 M_{3,T}(\theta_T, \tau_T)}{\partial \tau_h \partial \tau_k} \to \frac{1}{2} \text{tr} \left\{ \mathbb{E} \left[ e^{\tau(x, \theta)} \psi_k(x, \theta) \psi(x, \theta)' \right] \right\}^{-1} \mathbb{E} \left[ e^{\tau(x, \theta)} \psi_k(x, \theta) \psi(x, \theta)' \right] \mathbb{E} \left[ e^{\tau(x, \theta)} \psi_k(x, \theta) \psi(x, \theta)' \right] \]}

because \(\tau(\theta_0) = 0_{m \times 1}\) by Lemma \([10]\) (p. 32) under Assumption \([1](a)-(e)\) and \((g)-(h)\). Therefore, \(\mathbb{P}\)-a.s. as \(T \to \infty\),
\[
\frac{\partial^2 M_{3,T}(\theta_T, \tau_T)}{\partial \tau_h \partial \tau_k} = O(T^{-1}).
\]

(iv) Under Assumptions \([1](a)-(b)\) and \((d)-(h)\), by Lemma \([12]\) (p. 35), for all \((\theta, \tau)\) in a neighborhood of \((\theta_0, \tau(\theta_0))\), \(L_T(\theta, \tau) = M_{1,T}(\theta, \tau) + M_{2,T}(\theta, \tau) + M_{3,T}(\theta, \tau)\), so that the result follows from the statement (i)-(iii) of the present lemma.

\[\square\]

**Proposition 2** (Asymptotic normality of \(\hat{\theta}_T, \hat{\tau}_T(\hat{\theta}_T)\) and \(\hat{\gamma}_T\)). Under Assumptions \([1]\), \([2]\) and \([3]\) if the test hypothesis \([6]\) on p. 6 holds, \(\mathbb{P}\)-a.s. as \(T \to \infty\),
where \(\Sigma := \Sigma(\theta_0) := M^{-1}V(M')^{-1}, M := E\left[\frac{\partial \nu(X_1, \theta_0)}{\partial \theta}\right], V := E[\psi(X_1, \theta_0)\psi(X_1, \theta_0)^t], \) and \(R := \frac{\partial r(\theta_0)}{\partial \theta} \).

Proof. (i)-(ii) The function \(L_T(\theta, \tau)\) is well-defined and twice continuously differentiable in a neighborhood of \((\theta_0', \tau(\theta_0'))\) P-a.s. for \(T\) big enough by subsection B.2 (p. 336), under Assumptions L and 2a). Similarly, the function \(S_T(\theta, \tau) := \frac{1}{T} \sum_{t=1}^T e^{r(\theta)}\psi\theta(\theta)\) and \(\theta \rightarrow r(\theta)\) are continuously differentiable in a neighborhood of \((\theta_0', \tau(\theta_0'))\) by Assumption 1(a)(b) and 3(a).

Now, under Assumptions 1, 2 and 3(a), by Lemma 28 (p. 83), Lemma 2 iii (p. 211), P-a.s., \(\tilde{\theta}_T \rightarrow \theta_0\) and \(\tau_T(\tilde{\theta}_T) \rightarrow \tau(\theta_0)\), so that P-a.s. for \(T\) big enough, \((\theta_T', \tau_T(\theta_T'))\) is in an arbitrary small neighborhood of \((\theta_0', \tau(\theta_0'))\). Therefore, under Assumptions 1, 2 and 3(a), stochastic first-order Taylor-Lagrange expansions (Jenrich 1969, Lemma 3) around \((\theta_0, \tau(\theta_0))\) evaluated at \((\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))\) yield, P-a.s. for \(T\) big enough:

\[
\frac{\partial L_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta} = \frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta} + \frac{\partial^2 L_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta \partial \theta} (\tilde{\theta}_T - \theta_0) + \frac{\partial^2 L_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta' \partial \theta} \tau_T(\tilde{\theta}_T)
\]

\[
S_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T)) = S_T(\theta_0, \tau(\theta_0)) + \frac{\partial S_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta} (\tilde{\theta}_T - \theta_0) + \frac{\partial S_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta'} \tau_T(\tilde{\theta}_T)
\]

\[
r(\tilde{\theta}_T) = r(\theta_0) + \frac{\partial r(\tilde{\theta}_T)}{\partial \theta} (\tilde{\theta}_T - \theta_0)
\]

because \(\tau(\theta_0) = 0_{m \times 1}\) by Lemma 10v (p. 327), and where \(\tilde{\theta}_T\) and \(\tau_T\) are between \(\tilde{\theta}_T\) and \(\theta_0\), and between \(\tau_T(\tilde{\theta}_T)\) and \(\tau(\theta_0)\), respectively. Now, under Assumptions 1 and 2 by definition of \(\tilde{\theta}_T\) and by definition of \(\tau_T()\) (equation 14 on p. 151), \(r(\tilde{\theta}_T) = 0_{q \times 1}\) and \(S_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T)) = 0_{q \times 1}\), respectively. Moreover, under Assumptions 1, 2 and 3(a), by Lemma 28v (p. 83), P-a.s. as \(T \rightarrow \infty\),

\[
\frac{\partial L_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta} = -\frac{\partial \nu(X_1, \theta_0)}{\partial \theta} \gamma_T + O(T^{-1}).
\]

Therefore, under Assumptions 1, 2 and 3(a), P-a.s. as \(T \rightarrow \infty\),

\[
\begin{pmatrix}
O(T^{-1}) \\
0_{m \times 1}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial L_T(\theta_0, \tau(\theta_0))}{\partial \theta} \\
S_T(\theta_0, \tau(\theta_0)) \\
r(\theta_0)
\end{pmatrix} + \begin{pmatrix}
\frac{\partial^2 L_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta \partial \theta} \\
\frac{\partial S_T(\tilde{\theta}_T, \tau_T(\tilde{\theta}_T))}{\partial \theta} \\
\frac{\partial r(\tilde{\theta}_T)}{\partial \theta}
\end{pmatrix} \begin{pmatrix}
\tilde{\theta}_T - \theta_0 \\
\tau_T(\tilde{\theta}_T)
\end{pmatrix}.
\]
Now, under Assumptions 1 and 2 by Lemma 14 (p. 38), \( \mathbb{P} \)-a.s. as \( T \to \infty \), the matrix
\[
\begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}
\]
is invertible. Then, under Assumptions 1, 2 and 3, solving for the parameters and multiplying by \( \sqrt{T} \) yield, \( \mathbb{P} \)-a.s. as \( T \to \infty \),
\[
\sqrt{T} \begin{bmatrix} \theta_T - \theta_0 \\ \tau_T(\theta_T) \end{bmatrix} = - \begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}^{-1} \sqrt{T} \begin{bmatrix} \frac{\partial L_T(\theta_T, \gamma_T)}{\partial \theta} + O(T^{-1}) \\ S_T(\theta_T, \gamma_T) - r(\theta_T) \end{bmatrix}
\]
\[
= - \begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}^{-1} \sqrt{T} \begin{bmatrix} O(T^{-\frac{1}{2}}) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \end{bmatrix}
\]
\[
(\text{a}) + \begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}^{-1} \begin{bmatrix} O(T^{-\frac{1}{2}}) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \end{bmatrix}
\]
\[
(\text{b}) + \begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}^{-1} \begin{bmatrix} O(T^{-\frac{1}{2}}) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \end{bmatrix}
\]
\[
(\text{c}) + \begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}^{-1} \begin{bmatrix} O(T^{-\frac{1}{2}}) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \end{bmatrix} + o_P(1)
\]
\[
(\text{d}) + \begin{bmatrix}
\frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta^2} & \frac{\partial^2 L_T(\theta_T, \gamma_T)}{\partial \theta \partial \gamma} \\
\frac{\partial S_T(\theta_T, \gamma_T)}{\partial \theta} & \frac{\partial S_T(\theta_T, \gamma_T)}{\partial \gamma}
\end{bmatrix}^{-1} \begin{bmatrix} O(T^{-\frac{1}{2}}) \\ \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) \end{bmatrix} + o_P(1)
\]

(a) Firstly, under Assumptions 1 and 2 by Lemma 14 (p. 38), \( \mathbb{P} \)-a.s. as \( T \to \infty \), \( \frac{\partial L_T(\theta_T, \gamma_T)}{\partial \theta} = O(T^{-1}) \), so that \( \sqrt{T} \left[ \frac{\partial L_T(\theta_T, \gamma_T)}{\partial \theta} + O(T^{-1}) \right] = O(T^{-\frac{1}{2}}) \). Secondly, note that \( S_T(\theta_T, \gamma_T) - r(\theta_T) = \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta_0) \) because \( \tau(\theta_0) = 0_{m \times 1} \) by Lemma 10 (p. 32) under Assumption 1(a)-(e) and (g)-(h). Finally, if the test hypothesis (9) on p. 6 holds, then \( r(\theta_0) = 0_{q \times 1} \). (b) Add and subtract the
matrix \[
\begin{pmatrix}
-\Sigma + \Sigma R'(R\Sigma R')^{-1}R\Sigma & M^{-1}-\Sigma R'(R\Sigma R')^{-1}RM^{-1} & \Sigma R'(R\Sigma R')^{-1} \\
(M')^{-1}-(M')^{-1}R'(R\Sigma R')^{-1}R\Sigma & (M')^{-1}R'(R\Sigma R')^{-1}RM^{-1} & -(M')^{-1}R'(R\Sigma R')^{-1} \\
(R\Sigma R')^{-1}R\Sigma & -(R\Sigma R')^{-1}RM^{-1} & (R\Sigma R')^{-1}
\end{pmatrix}.
\]

(c) Firstly, the first and third column of the first square matrix cancel out because the first element and third element of the vector are zeros. Secondly, under Assumptions 2 and 3 by Lemma 26iii (p. 80) and Theorem 1 (p. 6), P.a.s. as \( T \to \infty \), the curly bracket is \( o(1) \), and, under Assumption 1a)-(c) and (g), by the Lindeberg-Lévy CLT, \( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \psi_t(\theta_0) = O_P(1) \), as \( T \to \infty \). (d) By definition \( \Sigma = \Sigma^{1/2}\Sigma^{1/2} \) and \( \Sigma^{-1/2} = [\Sigma^{1/2}]^{-1} \). Thus,

- \( M^{-1} - \Sigma R'(R\Sigma R')^{-1}RM^{-1} = \Sigma^{1/2}P_{\Sigma^{1/2}R'}^{-1}\Sigma^{-1/2}M^{-1} \) where \( P_{\Sigma^{1/2}R'}^{-1} \) denotes the orthogonal projection on the orthogonal of the space spanned by the columns of \( \Sigma^{1/2}R' \).
- \( (M')^{-1}R'(R\Sigma R')^{-1}RM^{-1} = (M')^{-1}\Sigma^{-1/2}P_{\Sigma^{1/2}R'}^{-1}R\Sigma^{-1/2}M^{-1} \)
- \( = (M')^{-1}\Sigma^{-1/2}P_{\Sigma^{1/2}R'}^{-1}P_{\Sigma^{1/2}R'}\Sigma^{-1/2}M^{-1} = (M')^{-1}\Sigma^{-1/2}P_{\Sigma^{1/2}R'}^{-1}\Sigma^{-1/2}M^{-1} \)
- \( = (M')^{-1}M^{-1}V \)

(iii) Under Assumptions 1a and 3 by the statement (ii) of the present proposition, P.a.s. as \( T \to \infty \),
\[
\sqrt{T}\begin{bmatrix}
\hat{\theta}_T - \theta_0 \\
\gamma_T
\end{bmatrix} \rightarrow \mathcal{N}(0,V)
\]

\( \frac{\sqrt{T}}{\tau T} \sum_{t=1}^{T} \psi_t(\theta_0) + o(1) \)

\( \frac{\sqrt{T}}{\tau T} \sum_{t=1}^{T} \psi_t(\theta_0) \)

\( \mathcal{N}(0,V) \)

\( \mathcal{N}(0,V) \)

\( \mathcal{N}(0,V) \)
be discarded because of the symmetry of the Gaussian distribution. Secondly, if \( X \) is a random vector and \( F \) is a matrix, then \( V(FX) = FV(X)F' \). (c) Denote the final asymptotic variance matrix with \( \Gamma \), and its \((i,j)\) block components with \( \Gamma_{i,j} \). Then,

\[
\begin{align*}
\Gamma_{1,1} &= \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} V(M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{1/2} \\
&= \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} \Sigma^{-1/2} \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{1/2} = (\Sigma^{1/2})' P_{\Sigma_{1/2}^2 R} \Sigma^{1/2} \text{ because } M^{-1} V(M')^{-1} =: \Sigma = \Sigma^{1/2} \Sigma^{1/2} = (\Sigma^{1/2})^{-1}, \text{ and } P_{\Sigma_{1/2}^2 R} = P_{\Sigma_{1/2}^2 R} \\
&= \Sigma^{1/2} / \Sigma^{1/2}, \Sigma^{-1/2} := (\Sigma^{1/2})^{-1}, \text{ by idempotence of projections on linear spaces;}
\end{align*}
\]

\[
\begin{align*}
\Gamma_{2,2} &= (M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} V(M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} = \\
&= (M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} = (M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} \\
&= (M')^{-1} [V^{1/2}(M')^{-1}]' P_{\Sigma_{1/2}^2 R} [M^{-1} V^{1/2}]^{-1} M^{-1} \\
&= (V^{1/2})^{-1} P_{\Sigma_{1/2}^2 R} [V^{1/2}]^{-1} \text{ because } M^{-1} V(M')^{-1} =: \Sigma = \Sigma^{1/2} \Sigma^{1/2}, \Sigma^{-1/2} := (\Sigma^{1/2})^{-1} = [V^{1/2}(M')^{-1}]^{-1} = M' V^{-1/2}, \Sigma^{-1/2} := (\Sigma^{1/2})^{-1} = [M^{-1} V^{1/2}]^{-1} = V^{-1/2} M, \text{ and } P_{\Sigma_{1/2}^2 R} = P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} \text{ by idempotence;}
\end{align*}
\]

\[
\begin{align*}
\Gamma_{3,3} &= (R \Sigma R')^{-1} R M^{-1} V(M')^{-1} R' (R \Sigma R')^{-1} = (R \Sigma R')^{-1} R \Sigma R' (R \Sigma R')^{-1} = (R \Sigma R')^{-1} \\
&= \Sigma - M^{-1} V(M')^{-1} \text{ because } M^{-1} V(M')^{-1} =: \Sigma; \\
\Gamma_{1,2} &= \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} V(M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} = \\
&= \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} = \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} = 0 \text{ because } M^{-1} V(M')^{-1} =: \Sigma = \Sigma^{1/2} \Sigma^{1/2}, \Sigma^{-1/2} := (\Sigma^{1/2})^{-1}, \Sigma^{-1/2} := (\Sigma^{1/2})^{-1}, \text{ and } P_{\Sigma_{1/2}^2 R} = P_{\Sigma_{1/2}^2 R} = 0_{m \times n}; \\
\Gamma_{1,3} &= -\Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} V(M')^{-1} R' (R \Sigma R')^{-1} = \\
&= -\Sigma^{1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} (R \Sigma R')^{-1} \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} (R \Sigma R')^{-1} = 0 \text{ because } M^{-1} V(M')^{-1} =: \Sigma = \Sigma^{1/2} \Sigma^{1/2}, \Sigma^{-1/2} := (\Sigma^{1/2})^{-1}, \text{ and } P_{\Sigma_{1/2}^2 R} = P_{\Sigma_{1/2}^2 R} = 0_{m \times q}; \\
\Gamma_{2,3} &= -(M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} M^{-1} V(M')^{-1} R' (R \Sigma R')^{-1} = \\
&= -(M')^{-1} \Sigma^{-1/2} P_{\Sigma_{1/2}^2 R} \Sigma^{-1/2} R' (R \Sigma R')^{-1} \Sigma^{1/2} P_{\Sigma_{1/2}^2 R} (R \Sigma R')^{-1} = \\
&= -(M')^{-1} R' (R \Sigma R')^{-1} R' (R \Sigma R')^{-1} = -(M')^{-1} R' (R \Sigma R')^{-1} \\
&= -(M')^{-1} R' (R \Sigma R')^{-1} \Sigma R' (R \Sigma R')^{-1} = -(M')^{-1} R' (R \Sigma R')^{-1} \\
&= -M^{-1} V(M')^{-1} =: \Sigma \text{ and } P_{\Sigma_{1/2}^2 R} = [\Sigma^{1/2} R' (R \Sigma R')^{-1} \Sigma^{1/2}]'.
\end{align*}
\]

\(\square\)

**Lemma 26.** Using the notation of Proposition 3 (p. 76), under Assumptions 1, 2 and 3,

(i) for any sequence \((\theta_T, \tau_T)\) \(T \in \mathbb{N}\) converging to \((\theta_0, \tau(\theta_0))\), \(\mathbb{P}\)-a.s. as \(T \to \infty\),

\[
\begin{bmatrix}
\frac{\partial^2 T(\theta_T, \tau_T)}{\partial \theta \partial \theta} & \frac{\partial^2 T(\theta_T, \tau_T)}{\partial \theta \partial \tau} & \frac{\partial T(\theta_T, \tau_T)}{\partial \theta} \\
\frac{\partial T(\theta_T, \tau_T)}{\partial \theta} & \frac{\partial T(\theta_T, \tau_T)}{\partial \tau} & 0 \\
\frac{\partial T(\theta_T, \tau_T)}{\partial \tau} & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0_{m \times m} & \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]' & \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \\
\mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]' & \mathbb{E} \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right] & 0 \\
\mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] & \mathbb{E} \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right] & 0
\end{bmatrix};
\]

(ii) \(\mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right]' \mathbb{E} \left[ \psi(X_1, \theta_0) \psi(X_1, \theta_0)' \right] 0 \) is invertible, so that, for any sequence \((\theta_T, \tau_T)\) \(T \in \mathbb{N}\) converging to \((\theta_0, \tau(\theta_0))\), \(\mathbb{P}\)-a.s., for \(T\) big enough, the matrix
under Assumptions 1(a)(b)(c)(d)(e)(g)(h), by the statement (ii) of the present lemma, the follow-

Proof. (i) Under Assumptions 1, 2 and 3(a), it follows from the continuity of \( \frac{\partial \psi}{\partial \theta} \), which is implied by Assumption 3(a), and Lemma 14(i) and iii (p. 48) and Lemma 17 (p. 50), given that \( \tau(\theta_0) = 0_{m \times 1} \) by Lemma 10(i) (p. 32) and Assumption 1(c), under Assumption 1(a)(b)(d)(e)(g) and (h).

(ii) It is sufficient to check the assumptions of Corollary 2i (p. 89) with \( A = \begin{bmatrix} 0_{m \times m} & M' \\ M & V \end{bmatrix} \) and \( B = \begin{bmatrix} R' \\ 0_{m \times q} \end{bmatrix} \) in order to establish the first part of the statement. Firstly, under Assumptions 1 and 2, by Lemma 13(ii) (p. 47), \( A = \begin{bmatrix} 0_{m \times m} & M' \\ M & V \end{bmatrix} \) is invertible. Secondly, by Assumptions 1(h) and 3(b), \( (B'A^{-1}B) = -(R\Sigma R') \) is also invertible. Then, the second part of the statement follows from a trivial case of the Lemma 30 (p. 88).

(iii) Under Assumption 1(a)(b)(c)(d)(e)(g)(h), by the statement (ii) of the present lemma, the limiting matrix is invertible. Thus, using the notation of Proposition 2 (p. 76),

\[
\begin{pmatrix}
\frac{\partial^2 L_T(\hat{\theta}, \tau_T)}{\partial \theta \partial \theta} & \frac{\partial^2 L_T(\hat{\theta}, \tau_T)}{\partial \theta \partial \tau} \\
\frac{\partial S_T(\hat{\theta}, \tau_T)}{\partial \theta} & \frac{\partial S_T(\hat{\theta}, \tau_T)}{\partial \tau} \\
\frac{\partial \psi}{\partial \theta} & \frac{\partial \psi}{\partial \tau}
\end{pmatrix}
\]

is invertible; and

\[
\begin{vmatrix}
-\Sigma + \Sigma R'(R\Sigma R')^{-1}R\Sigma & M^{-1} - \Sigma R'(R\Sigma R')^{-1}RM^{-1} & \Sigma R'(R\Sigma R')^{-1} \\
(M')^{-1} - R'(R\Sigma R')^{-1}R\Sigma & (M')^{-1} R'(R\Sigma R')^{-1}RM^{-1} & -(M')^{-1} R'(R\Sigma R')^{-1} \\
(R\Sigma R')^{-1}R\Sigma & -(R\Sigma R')^{-1}RM^{-1} & (R\Sigma R')^{-1}
\end{vmatrix}
\]
where the explanation for the last equality is as follows. Apply Corollary 2i (p. 89) with
\[ A = \begin{bmatrix} 0_{m \times m} & M' \\ M & V \end{bmatrix} \] and \[ B = \begin{bmatrix} R' \\ 0_{m \times q} \end{bmatrix} , \]
and note that, by Lemma 27iii, iv and vi (p. 82),

\[ A^{-1} - A^{-1}B(B'A^{-1}B)B'A^{-1} = \begin{bmatrix} -\Sigma + \Sigma R'(R\Sigma R')^{-1}R\Sigma & M^{-1} - \Sigma R'(R\Sigma R')^{-1}RM^{-1} \\ (M')^{-1} - (M')^{-1}R'(R\Sigma R')^{-1}R\Sigma & (M')^{-1}R'(R\Sigma R')^{-1}RM^{-1} \end{bmatrix} \]

\[ A^{-1}B(B'A^{-1}B)^{-1} = \begin{bmatrix} \Sigma R'(R\Sigma R')^{-1} \\ -(M')^{-1}R'(R\Sigma R')^{-1} \end{bmatrix} \]

\[ (B'A^{-1})^{-1} = -(R\Sigma R')^{-1} . \]

Then, the result follows from the continuity of the inverse transformation (e.g., Rudin 1953, Theorem 9.8).

**Lemma 27.** Let \[ A = \begin{bmatrix} 0_{m \times m} & M' \\ M & V \end{bmatrix} \] and \[ B = \begin{bmatrix} R' \\ 0_{m \times q} \end{bmatrix} \] where \( \Sigma := \Sigma(\theta_0) := M^{-1}V(M')^{-1} \), \( M := \mathbb{E} \left[ \frac{\partial \psi(X_1, \theta_0)}{\partial \theta} \right] \), \( V := \mathbb{E}[\psi(X_1, \theta_0)\psi(X_1, \theta_0)'] \), and \( R = \frac{\partial \psi(\theta_0)}{\partial \theta} \). Then, under Assumption 1(a)(b)(h) and 3(b), the following equalities hold

(i) \( A^{-1} = \begin{bmatrix} -\Sigma & M^{-1} \\ (M')^{-1} & 0_{m \times m} \end{bmatrix} \);

(ii) \( A^{-1}B = \begin{bmatrix} -\Sigma R' \\ (M')^{-1}R' \end{bmatrix} , \) so that \( B'A^{-1} = \begin{bmatrix} -R\Sigma & RM^{-1} \end{bmatrix} ; \)

(iii) \( (B'A^{-1})^{-1} = -(R\Sigma R')^{-1} ; \)

(iv) \( A^{-1}B(B'A^{-1}B)^{-1} = \begin{bmatrix} \Sigma R'(R\Sigma R')^{-1} \\ -(M')^{-1}R'(R\Sigma R')^{-1} \end{bmatrix} ; \)

(v) \( A^{-1}B(B'A^{-1}B)B'A^{-1} = \begin{bmatrix} -\Sigma R'(R\Sigma R')^{-1}R\Sigma & \Sigma R'(R\Sigma R')^{-1}RM^{-1} \\ (M')^{-1}R'(R\Sigma R')^{-1}R\Sigma & -(M')^{-1}R'(R\Sigma R')^{-1}RM^{-1} \end{bmatrix} ; \) and

(vi) \( A^{-1} - A^{-1}B(B'A^{-1}B)B'A^{-1} = \begin{bmatrix} -\Sigma + \Sigma R'(R\Sigma R')^{-1}R\Sigma & M^{-1} - \Sigma R'(R\Sigma R')^{-1}RM^{-1} \\ (M')^{-1} - (M')^{-1}R'(R\Sigma R')^{-1}R\Sigma & (M')^{-1}R'(R\Sigma R')^{-1}RM^{-1} \end{bmatrix} . \)

**Proof.** (i) It corresponds to a part of Lemma 13iii (p. 47) under Assumptions 1 and 2

(ii) \[ \begin{bmatrix} -\Sigma \\ (M')^{-1} \end{bmatrix} \begin{bmatrix} 0_{m \times m} & M' \\ M & V \end{bmatrix} \begin{bmatrix} R' \\ 0_{m \times q} \end{bmatrix} = \begin{bmatrix} -\Sigma R' \\ (M')^{-1}R' \end{bmatrix} = A^{-1}B \]

(iii) \[ \begin{bmatrix} R \\ 0_{q \times m} \end{bmatrix} \begin{bmatrix} -\Sigma R' \\ (M')^{-1}R' \end{bmatrix} = -R\Sigma R' = B'A^{-1}B \]
Lemma 28 (Constrained estimator and its Lagrangian). Under Assumptions 1, 2 and 3(a), if the test hypothesis (b) on p. 20 holds, \( P \)-a.s. for \( T \) big enough,

(i) the constrained estimator \( \hat{\theta}_T \) exists, and \( \hat{\theta}_T \to \theta_0 \), as \( T \to \infty \);

(ii) \( \theta \to L_T(\theta, \tau_T(\theta)) \) is continuously differentiable in a neighborhood of \( \hat{\theta}_T \);

(iii) under additional Assumption 3(b), there exists a unique vector, \( \gamma_T \), called the Lagrangian multiplier, s.t. \( \frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta} + \frac{\partial r(\theta, \tau_T(\theta))}{\partial \theta} \gamma_T = 0_{m \times 1} \);

(iv) under additional Assumption 3(b), \( \frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta} \) is continuous and, for all \( (\theta, \tau, \theta) \),

\[
\frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta} \bigg|_{(\theta, \tau, \theta)} = 0, \text{ as } T \to \infty,
\]

where \( \frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta} \) is continuously differentiable in a neighborhood of \( \hat{\theta}_T \).

Proof. (i) The constrained set \( \tilde{\Theta} := \{ \theta \in \Theta : r(\theta) = 0 \} \) is bounded as a subset of the compact (and thus bounded) set \( \Theta \). The constrained set \( \tilde{\Theta} \) is also closed: For all \( (\theta_n)_{n \in \mathbb{N}} \in \tilde{\Theta} \), \( \tilde{\Theta} \subseteq \tilde{\Theta} \) s.t. \( \lim_{n \to \infty} \theta_n = \tilde{\theta} \), \( \tilde{\theta} \in \tilde{\Theta} \) because (i) by compactness of \( \Theta \), \( \tilde{\theta} \in \Theta \); and (ii) by the continuity of \( r : \Theta \to \mathbb{R}^q \) (i.e., Assumption 3(a)), \( r(\tilde{\theta}) = \lim_{n \to \infty} r(\theta_n) = \lim_{n \to \infty} 0 = 0 \). Therefore, the constrained set \( \tilde{\Theta} \) is itself compact. Moreover, under Assumption 1(a)(b) and (d)-(h), by Lemma 1i-iii (p. 20), \( P \)-a.s. for \( T \) big enough, \( \theta \to \hat{\theta}_T(\theta) \) is continuous and, for all \( \theta \in \Theta \), \( \omega \to \hat{\theta}_T(\theta) \) is measurable. Thus, the existence and measurability of the constrained estimator \( \hat{\theta}_T \) follows from the Schmetterer-Jennrich lemma (Schmetterer 1966 Chap. 5 Lemma 3.3; Jennrich 1969 Lemma 2).

In order to establish the consistency of \( \hat{\theta}_T \), it remains to check the other assumptions of the standard consistency theorem (e.g. Newey and McFadden 1994, pp. 2121-2122 Theorem 2.1, which is also valid in an almost-sure sense), where the constrained set \( \tilde{\Theta} := \{ \theta \in \Theta : r(\theta) = 0 \} \) is the parameter space. Because \( \tilde{\Theta} \subseteq \Theta \), \( P \)-a.s. as \( T \to \infty \),

\[
\sup_{\theta \in \Theta} \left| \ln \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau T(\theta) \psi(\theta)} \right) - \frac{1}{2T} \ln |\Sigma_T(\theta)|_{\det} - \ln \mathbb{E}[e^{\tau T(\theta) \psi(X_1, \theta)}] \right| \leq \sup_{\theta \in \Theta} \left| \ln \left( \frac{1}{T} \sum_{t=1}^{T} e^{\tau T(\theta) \psi(\theta)} \right) - \frac{1}{2T} \ln |\Sigma_T(\theta)|_{\det} - \ln \mathbb{E}[e^{\tau T(\theta) \psi(X_1, \theta)}] \right| \to 0
\]

where the convergence to zero follows from equation (17) on p. 20 under Assumption 1. In addition, under Assumption 1(a)-(c) and (g)-(h), by Lemma 10v (p. 32), \( \theta \to \ln \mathbb{E}[e^{\tau T(\theta) \psi(X_1, \theta)}] \) is uniquely maximized at \( \theta_0 \), i.e., for all \( \theta \in \Theta \setminus \{ \theta_0 \} \), \( \ln \mathbb{E}[e^{\tau T(\theta) \psi(X_1, \theta)}] < \ln \mathbb{E}[e^{\tau T(\theta_0) \psi(X_1, \theta_0)}] = 0 \),
and, under Assumptions 1(a)(b)(d)(e)(g) and (h), by Lemma 3(p. 23), \( \theta \mapsto \ln \mathbb{E}[e^{r'(\theta)\psi(X_1,\theta)}] \) is continuous in \( \Theta \in T \subset \Theta \).

(ii) Under Assumptions 1 and 2(a), by subsection B.2 (p. 33), the function \( L_T(\theta, \tau) \) is well-defined and twice continuously differentiable in a neighborhood of \( (\theta_0, \tau(\theta_0))' \) \( \mathbb{P} \)-a.s. for \( T \) big enough. Moreover, under Assumption 1(a)(b) and (d)-(h), by Lemma 21i (p. 62), \( \tau(\cdot) \) is continuously differentiable in \( \Theta \). Now, under Assumption 1 by the statement (i) of the present lemma and Lemma 21ii (p. 21), \( \mathbb{P} \)-a.s., \( \hat{\theta}_T \rightarrow \theta_0 \) and \( \tau_T(\hat{\theta}_T) \rightarrow \tau(\theta_0) \), so that \( \mathbb{P} \)-a.s. for \( T \) big enough, \( (\hat{\theta}_T, \tau_T(\hat{\theta}_T))' \) is in any arbitrary small neighborhood of \( (\theta_0, \tau(\theta_0))' \). Therefore, under Assumption 1 and 2(a), by the chain rule theorem (e.g., Magnus and Neudecker 1999/1988 Chap. 5 sec. 11), \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \theta \mapsto L_T(\theta, \tau_T(\theta)) \) is continuously differentiable at \( \hat{\theta}_T \).

(iii) It is a consequence of the Lagrange theorem (e.g., Magnus and Neudecker 1999/1988 Chap. 7 sec. 12). Check its assumptions. Firstly, under Assumptions 1 and 2(a), by the statement (i) of the present lemma, \( \mathbb{P} \)-a.s. for \( T \) big enough, the constrained estimator \( \hat{\theta}_T \) exists and that it is in the interior of \( \Theta \) by consistency and Assumption 1(c). Then, we should check the other assumptions of the Lagrange theorem \( \omega \) by \( \omega \) on the subset of \( \Omega \) where \( \hat{\theta}_T \) exists. Firstly, by Assumption 3(a), \( r : \Theta \rightarrow \mathbb{R}^q \) is continuously differentiable. Secondly, under Assumptions 12 and 3(a), if the test hypothesis (9) on p. 6 holds, \( \mathbb{P} \)-a.s. as \( T \rightarrow \infty, \hat{\theta}_T \rightarrow \theta_0 \), and, by Assumption 3(b), \( \frac{\partial r(\theta)}{\partial \theta} \) is full rank, Thus, \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \frac{\partial r(\theta)}{\partial \theta} \) is full rank by continuity of the determinant function. Finally, by the statement (iv) of the present lemma \( \theta \mapsto L_T(\theta, \tau_T(\theta)) \) is differentiable at \( \hat{\theta}_T \).

(iv) First of all, note that it does not immediately follow from the statement (iii) because \( \frac{\partial L_T(\hat{\theta}_T, \tau_T(\hat{\theta}_T))}{\partial \theta} \) denotes \( \frac{\partial L_T(\theta, \tau)}{\partial \theta} \big|_{(\theta, \tau) = (\hat{\theta}_T, \tau_T(\hat{\theta}_T))} \) instead of \( \frac{\partial L_T(\theta, \tau)}{\partial \theta} \big|_{\theta = \hat{\theta}_T} \) (see footnote 14 on p. 62). Under Assumption 12 and 3(a), by Lemma 21 (p. 62), \( \tau_T(\cdot) \) is continuously differentiable in \( \Theta \). Moreover, under Assumptions 12 and 3(a), if by the statement (ii) of the present lemma, \( \mathbb{P} \)-a.s. for \( T \) big enough, \( \theta \mapsto L_T(\theta, \tau_T(\theta)) \) is continuously differentiable in a neighborhood of \( \hat{\theta}_T \). Thus, by an immediate and standard implication of the chain rule (e.g., Magnus and Neudecker 1999/1988 chap. 5, sec. 12, exercise 3), \( \mathbb{P} \)-a.s. for \( T \) big enough, for all \( j \in \{1, m\} \),

\[
\frac{\partial L_T(\theta, \tau_T(\theta))}{\partial \theta_j} \bigg|_{\theta = \hat{\theta}_T} = \frac{\partial L_T(\theta, \tau)}{\partial \theta_j} \bigg|_{(\theta, \tau) = (\hat{\theta}_T, \tau_T(\hat{\theta}_T))} + \frac{\partial L_T(\theta, \tau)}{\partial \tau} \bigg|_{(\theta, \tau) = (\hat{\theta}_T, \tau_T(\hat{\theta}_T))} \frac{\partial \tau(\theta)}{\partial \theta_j} \bigg|_{\theta = \hat{\theta}_T} + O(T^{-1})
\]

where the explanations for the last equality are as follow. Firstly, under Assumptions 12 and 3(a), by Lemma 22v (p. 64), \( \mathbb{P} \)-a.s. as \( T \rightarrow \infty, \frac{\partial L_T(\theta, \tau)}{\partial \tau} \bigg|_{(\theta, \tau) = (\hat{\theta}_T, \tau_T(\hat{\theta}_T))} = O(T^{-1}) \) because \( \theta_T \rightarrow \theta_0 \), \( \mathbb{P} \)-a.s. as \( T \rightarrow \infty \), by the second part of the statement (i) of the present lemma. Secondly, under Assumptions 12 and 3(a), by the second part of the statement (i) of the present lemma and Lemma 21 (p. 62), \( \mathbb{P} \)-a.s. as \( T \rightarrow \infty, \frac{\partial \tau(\theta)}{\partial \theta_j} \bigg|_{\theta = \hat{\theta}_T} = O(1) \).

Now the results follows by plugging the above equation (44) into the Lagrangian FOC of the statement (iii) of the present lemma.

\[ \square \]

APPENDIX C. ON THE ASSUMPTIONS
C.1. Discussion. Assumptions 1 and 2 are mainly adapted from the entropy literature. Assumption 1(a) ensures the basic requirement for inference, that is, data contain different pieces of information (independence) about the same phenomenon (identically distributed). The conditions “independence and identically distributed” are much stronger than needed, and can be relaxed to allow for time dependence along the lines of Kitamura and Stutzer (1997). We restrain ourselves to the i.i.d. case for brevity and clarity. Assumption 1(a) also requires completeness of the probability space so that we can define functions only a probability-one subset of \( \Omega \) without generating potential measurability complications. The completeness of the probability space is without significant loss of generality (e.g., Kallenberg 2002 (1997, p. 13), and it is often implicitly or explicitly required in the literature.

Assumption 1(b) mainly requires standard regularity conditions for the moment function \( \psi(\cdot, \cdot) \). As usual in nonlinear econometrics, the existence of the estimator relies on such regularity conditions. An alternative would be to rely on empirical process theory, but it seems here inappropriate as the implicit nature of the definition of the ESP approximation requires smooth functions. We require Assumption 1(b), as well as some of the following assumptions, to hold in an \( \epsilon \)-neighborhood of the parameter space \( \Theta \), so that we can deal with its boundary \( \partial \Theta \) in the same way as with its interior. In particular, it ensures that \( \Sigma(\theta) \) is invertible for \( \theta \in \partial \Theta \) under probability measures equivalent to \( \mathbb{P} \) (Corollary 1ii on p. 87), and it allows to apply an implicit function theorem to \( \tau(\theta) \), also for \( \theta \in \partial \Theta \) (Lemma 10 on p. 32). For the latter reason, the entropy literature often appears to also (implicitly) assume that assumptions hold in an \( \epsilon \)-neighborhood of the parameter space. In applications, this is often innocuous as the boundary of the parameter space is often loosely specified. However, in some specific situations, which we rule out, this may be problematic (e.g., Andrews 1999, and references therein).

Assumption 1(c) requires global identification, which is a necessary condition to prove the consistency of an estimator. If we were interested in the ESP approximation instead of its maximizer (i.e., the ESP estimator), global identification could be relaxed as Holcblat (2012) and a companion paper show. Assumption 1(c) also requires equality between the dimension of the parameter space and the number of moment conditions, i.e., just-identified moment conditions. We impose the latter for mainly three reasons. Firstly, it appears reasonable to investigate the ESP estimator in the just-identified case before moving to the over-identified case, which requires to generalize the ESP approximation. Secondly, the just-identified case makes clear the difference between the ESP estimator and the existing alternatives, which are all equal in this case (see section 2.2). Thirdly, this is a standard assumption in the saddlepoint literature. However, note that (i) this assumption is less restrictive than it seems at first sight because, in the linear case, over-identified moment conditions correspond to just-identified moment conditions through the FOCs, and, in the nonlinear case, we can transform over-identified estimating equations into just-identified estimating equations through an extension of the parameter space (e.g., Newey and McFadden 1994, p. 2232); (ii) ongoing work show how to generalize the ESP approximation to over-identified moment conditions.

Assumption 1(d) requires the compactness of the parameter space \( \Theta \), and the existence of a solution \( \tau(\theta) \in \mathbb{R}^m \) that solves the equation \( \mathbb{E}\left[ e^{\tau(\theta)\psi(X_1, \theta)}\psi(X_1, \theta) \right] = 0 \), for all \( \theta \in \Theta \). Schennach (2005) also makes this assumption. Compactness of the parameter space is a convenient standard mathematical assumption that is often relevant in practice. A computer can only handle a
bounded parameter space—finite memory of a computer. Regarding the existence of \( \tau(\theta) \), it is necessary to ensure the asymptotic existence of the ESP approximation. From a theoretical point of view, the existence of \( \tau(\theta) \) looks like a reasonable assumption: If, for some \( \theta \in \Theta \), \( 0_{m \times 1} \) is outside the convex hull of the support of \( \psi(X_1, \theta) \), there is not such a solution \( \tau(\theta) \), which also means that \( \theta \) cannot be \( \theta_0 \), so that it should be excluded from the parameter space. However, the existence of \( \tau(\theta) \) might be difficult to check in practice. A way to get around this assumption is to (i) assume the existence of \( \tau(\theta) \) only in a neighborhood of \( \theta_0 \); and (ii) to set the ESP approximation to zero for the \( \theta \) values that do not have a solution to the finite-sample moment conditions \( (14) \). Holcblat (2012) follows such an approach. We do not follow such an approach because it significantly complicates the proofs and the presentation.

Assumptions \([1(e), 2(b)]\) rule out fat-tailed distributions. More precisely, they require the existence of exponential moments. They are necessary to apply the the ULLN (uniform law of large numbers) à la Wald (e.g., Ghosh and Ramamoorthi 2003 pp. 24-25, Theorem 1.3.3) to components of the ESP approximation. Assumptions \([1(e), 2(b)]\) are stronger than the moment existence assumption in Hansen (1982), but they are a common type of assumptions in the entropy literature (e.g., Hansen 1982, Haberman 1984, Kitamura and Stutzer 1997, Schennach 2007), the saddlepoint literature (e.g., Amadeo et al. 2000) and the literature on exponential models (e.g., Berk 1972). In particular, Assumptions \([1(d), 2(b)]\) are a convenient variant of Assumptions 3.4 and 3.5 in Schennach (2007). Both in Schennach (2007) and in the present paper, the successful estimation of the Hall and Horowitz model, which does not satisfy Assumptions \([1(e), 2(b)]\), suggests that the latter can be relaxed. In practice, Assumptions \([1(e), 2(b)]\) are not as strong as it may appear because observable quantities have finite support (finite memory of computers), which, in turn, implies that they have all finite moments. Moreover, in the case in which unboundedness is a concern (e.g., moment conditions derived from a likelihood), Ronchetti and Trojani (2001) provide a way to bound moment functions.

Assumptions \([1(f), g)]\) play the same role as Assumptions \([1(e), 2(b)]\), although they are less stringent. Assumption \([1(h)]\) requires the invertibility of the asymptotic variance of standard estimators (scaled by \( \sqrt{T} \)) of any solution to the tilted moment condition. In the present paper, this assumption has two main roles. Firstly, it ensures that the determinant term \( \det \Sigma_T^2(\theta) \) in the ESP approximation \( (11) \) does not explode, asymptotically. Secondly, it ensures the positive definiteness of the symmetric matrix \( \mathbb{E} \left[ e^{\tau(\psi(X_1, \theta))} \psi(X_1, \theta) \psi(X_1, \theta)' \right] \) for all \( (\theta, \tau) \in S \), so that the \( \min_{\tau \in \mathbb{R}^m} \mathbb{E} \left[ e^{\tau(\psi(X_1, \theta))} \right] \) is a strictly convex problem, which, in turn, implies the unicity of its solution \( \tau(\theta) \). In the setup of the present paper, Assumption \([1(g)]\) is equivalent to the invertibility of \( \mathbb{E} \left[ e^{\tau(\theta) \psi(X_1, \theta)} \partial\psi(X_1, \theta) \right] \) and \( \mathbb{E} \left[ \psi(X_1, \theta) \psi(X_1, \theta)' \right] \), for all \( \theta \in \Theta \) (Lemma 29 on p. 87 with \( P = \mathbb{P} \) and \( \frac{dQ}{dP} = \frac{1}{e^{\tau(\theta) \psi(X_1, \theta)}} \)). In this way, it is stronger than the Assumption 4 in Kitamura and Stutzer (1997), but it is close to Stock and Wright (2000 Assumption C). Note that Schennach (2007) also implicitly assumes that \( \mathbb{E} \left[ e^{\tau(\psi(X_1, \theta))} \psi(X_1, \theta) \psi(X_1, \theta)' \right] \) is full rank for all \( (\theta, \tau) \in S \), because Schennach (2007 p. 649) regards \( \tau(\theta) \) as a solution to a strictly convex problem (e.g., Hiriart-Urruty and Lemaréchal 1993/1996 chap. 4, Theorem 4.3.1). Assumption \([1(g)]\) should
often be reasonable because the set of singular matrices has zero Lebesgue measure in the space of square matrices.\footnote{The set of singular matrices corresponds to the set of zeros of the determinant, which is nonzero polynomial in several variables. Moreover, by induction over the number of variables with the fundamental theorem of algebra for the base step, a nonzero polynomials has a finite number of zeros.}

C.2. Implications of Assumption 1(h).

Lemma 29. Let \((\Omega_A, \mathcal{A})\) be a measurable space, \(Z : \Omega \to \mathbb{R}^k\) be a \(k\)-dimensional random vectors with \(k \in [1, \infty]\) and \(P\) and \(Q\) two probability measures on \((\Omega_A, \mathcal{A})\). Denote the expectation and the variance under \(P\) with \(\mathbb{E}_P\) and \(\mathbb{V}_P\), respectively.

(i) For all \(\tau \in \mathbb{R}^k\), \(\mathbb{E}_P\left(e^{\tau'ZZ'}\right) > 0\), it is a positive semi-definite symmetric matrix.

(ii) If \(P \sim Q\) (i.e., they are equivalent), \(\mathbb{E}_P(|ZZ'|) < \infty\) and \(\mathbb{E}_Q(|ZZ'|) < \infty\), then

\[
\mathbb{E}_P(ZZ') \text{ invertible } \iff \mathbb{E}_Q(ZZ') \text{ invertible}
\]

Proof. (i) Symmetry follows from the invariance under transposition of \(\mathbb{E}_P \left(ZZ'e^{\tau'Z}\right)\). It remains to show positive semi-definiteness. For all \(y \in \mathbb{R}^k\),

\[
\forall \omega \in \Omega,\ y'e^{\tau'Z}ZZ'y = e^{\tau'Z}[y'Z]^2 \geq 0
\]

\[
\Rightarrow y'\mathbb{E}_P \left[e^{\tau'Z}ZZ'\right] y = \mathbb{E}_P \left[y'e^{\tau'Z}ZZ'y\right] \geq 0.
\]

where the implication follows from the monotonicity of the Lebesgue integral (e.g., Monfort 1997, p. 47).

(ii) By contraposition, it is equivalent to prove that \(\mathbb{E}_P(ZZ')\) noninvertible iff \(\mathbb{E}_Q(ZZ')\) noninvertible. By statement (i),

\[
\mathbb{E}_P(ZZ') \text{ noninvertible}
\]

\[
\iff \exists y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : y'\mathbb{E}_P(ZZ')y = 0
\]

\[
\forall y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : \mathbb{E}_P([y'Z]^2) = 0
\]

\[
\text{(a)} \iff \exists y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : \mathbb{E}_P([y'Z]^2) = 0 \text{ P-a.s.}
\]

\[
\text{(b)} \iff \exists y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : (y'Z)^2 = 0 \text{ Q-a.s.}
\]

\[
\text{(c)} \iff \exists y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : (y'Z)^2 = 0 \text{ Q-a.s.}
\]

\[
\text{(d)} \iff \exists y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : \mathbb{E}_Q([y'Z]^2) = 0
\]

\[
\text{(e)} \iff \exists y \in \mathbb{R}^k \setminus \{0_{k \times 1}\} : y'\mathbb{E}_Q(ZZ')y = 0
\]

\[
\Rightarrow \mathbb{E}_Q(ZZ') \text{ noninvertible}
\]

(a) \(y'\mathbb{E}_P(ZZ')y = \mathbb{E}_P[y'Z(y'Z)'] = \mathbb{E}_P([y'Z]^2)\) (b) The integral of a positive function w.r.t a measure is null iff the function is null almost-surely (e.g., Kallenberg 2002 (1997) Lemma 1.24).

(c) By assumption, \(P \sim Q\). (d) Same as (b). (a) Same as (a) with \(Q\) instead of \(P\).

□

Corollary 1 (Implication of Assumption 1(h)). Under Assumptions 1(a)-(b), (e) and (g), Assumption 1(h) implies that, for all \((\theta, \tau) \in \mathcal{S}\),

\[
\mathbb{E} \left[e^{\tau'\psi(X_1, \theta)\psi(X_1, \theta)'}\right] \text{ is a positive definite symmetric matrix.}
\]
Proof. By Lemma 31 (p. 87) with $Z = \psi(X_1, \theta)$, it is a positive semi-definite matrix. Thus, it remains to show that it is invertible, i.e., definite instead of only semi-definite.

Under Assumption 1 (a)(b)(d)(e)(g) and (h), by Lemma 3 (p. 23) and Assumption 1(d)(e), for all $\theta \in \Theta$, $0 < E[e^{\tau(\theta)^2}\psi(X_1, \theta)] < \infty$. Moreover, by Assumption 1(h), for all $\theta \in \Theta$, $E[e^{\tau(\theta)^2}\psi(X_1, \theta)\psi(X_1, \theta)^T]$ is invertible, so that $\frac{1}{E[e^{\tau(\theta)^2}\psi(X_1, \theta)]} E[e^{\tau(\theta)^2}\psi(X_1, \theta)\psi(X_1, \theta)^T]$ is also invertible. For every $(\theta, \tau) \in \mathbf{S}$, the check of the assumptions of Lemma 31 (p. 87) with $Z = \psi(X_1, \theta)$, $\frac{dQ_{\theta, \tau}}{d\theta} = \frac{e^{\tau(\theta)^2}\psi(X_1, \theta)}{E[e^{\tau(\theta)^2}\psi(X_1, \theta)]}$ and $\frac{dQ_{\theta, \tau}}{d\tau} = \frac{1}{E[e^{\tau(\theta)^2}\psi(X_1, \theta)]} E[e^{\tau(\theta)^2}\psi(X_1, \theta)\psi(X_1, \theta)^T]$, so that $\frac{dQ_{\theta, \tau}}{d\theta}$.

Firstly, for all $(\omega, \tau) \in \Omega \times T \times \Theta$, $0 < \frac{dQ_{\theta, \tau}}{d\theta}$ and $0 < \frac{dQ_{\theta, \tau}}{d\tau}$, so that $Q_{(\theta, \tau)} \sim P_{\theta} \sim P$. Secondly, by monotonicity of integration and the Cauchy-Schwarz inequality, for all $\theta \in \Theta$, $E[|\psi(X_1, \theta')\psi(X_1, \theta)|] \leq E[\sup_{\theta \in \Theta} |\psi(X_1, \theta)\psi(X_1, \theta')|] < \sqrt{E[\sup_{\theta \in \Theta} |\psi(X_1, \theta)|]} |\psi(X_1, \theta')| < \infty$, where the last inequality follows from Assumption 1(g). Thirdly, under Assumption 1(a)(b)(d)(e)(g) and (h), by Lemma 3 (p. 23) and Assumption 1(d)(e), for all $(\theta, \tau) \in \mathbf{S}$, $0 < E[e^{\tau(\theta)^2}\psi(X_1, \theta)] < \infty$. Moreover, under Assumptions 1(a)(b), (e) and (g), by Lemma 3 (p. 29), $E[\sup_{(\theta, \tau) \in \mathbf{S}} |e^{\tau(\theta)^2}\psi(X_1, \theta)\psi(X_1, \theta')|] < \infty$, so that, for all $(\theta, \tau) \in \mathbf{S}$,

\[ E\left[\frac{e^{\tau(\theta)^2}\psi(X_1, \theta)}{E[e^{\tau(\theta)^2}\psi(X_1, \theta)]} \psi(X_1, \theta)\psi(X_1, \theta')\right] < \infty. \]

Thus, for each $(\theta, \tau) \in \mathbf{S}$, apply Lemma 29i (p. 87) to show the result. □

**APPENDIX D. REMAINING TECHNICAL RESULTS**

**Lemma 30 (Asymptotic invertibility of sequence of matrix functions).** Let $A(\gamma)$ be a family of invertible matrices indexed by $\gamma \in \Gamma$ s.t. $\gamma \mapsto A(\gamma)$ is continuous, and where $\Gamma$ is a compact subset of a Euclidean space. Let $(A_T(\gamma))_{T \in [1, \infty]}$ be a sequence of square matrices. If, as $T \to \infty$, $\sup_{\gamma \in \Gamma} |A_T(\gamma) - A(\gamma)| \to 0$, then there exist a constant $\varepsilon_A > 0$ and $T_A \in \mathbb{N}$ s.t. for all $T \in [T_A, \infty]$, for all $\gamma \in \Gamma$, $||A_T(\gamma)||_{\text{det}} \geq \varepsilon_A$.

**Proof.** The function $A \mapsto ||A||_{\text{det}}$ is a continuous function. Moreover, by assumption, for all $\gamma \in \Gamma$, $||A(\gamma)||_{\text{det}} > 0$. Thus, by continuity of $\gamma \mapsto A(\gamma)$ and compactness of $\Gamma$, there exists $\varepsilon_A$ s.t. $\min_{\gamma \in \Gamma} ||A(\gamma)||_{\text{det}} > 2\varepsilon_A$. Now continuity of $A \mapsto ||A||_{\text{det}}$ on the compact set $\Gamma$ implies uniform continuity (e.g., Rudin [1953] Theorem 4.19), so that there exists $T_{\varepsilon_A} \in \mathbb{N}$ s.t., for all $T \in [T_{\varepsilon_A}, \infty]$, $\sup_{\gamma \in \Gamma} ||A_T(\gamma)||_{\text{det}} - ||A(\gamma)||_{\text{det}} \leq \varepsilon_A$. Then, for all $\gamma \in \Gamma$, the triangle inequality $||A(\gamma)||_{\text{det}} \leq ||A(\gamma)||_{\text{det}} - ||A_T(\gamma)||_{\text{det}} + ||A_T(\gamma)||_{\text{det}}$ implies that $\varepsilon_A = 2\varepsilon_A - \varepsilon_A \leq ||A(\gamma)||_{\text{det}} - ||A(\gamma)||_{\text{det}} - ||A_T(\gamma)||_{\text{det}} \leq ||A_T(\gamma)||_{\text{det}}$. □

**Lemma 31 (Asymptotic positivity and definiteness of matrices).** Let $(A_T)_{T \geq 1}$ a sequence of square matrices converging to $A$ as $T \to \infty$. Then, if $(A_T)_{T \geq 1}$ is a sequence of symmetric matrices and $A$ is a positive-definite matrix (p.d.m.), then there exists $\hat{T} \in \mathbb{N}$ such that $T \geq \hat{T}$ implies $A_T$ is p.d.m.

**Proof.** On one hand, $A_T$ is a p.d.m. if and only if all its eigenvalues are strictly positive (e.g., Magnus and Neudecker [1999/1988], Ch. 1 Sec. 13 Theorem 8). On the other hand, $\min \text{sp}A_T = \min_{z ||z|| = 1} z^T A_T z$, where $\text{sp}A_T$ denotes the set of eigenvalues of $A$ (e.g., Magnus and Neudecker [1999/1988], Ch. 11 Sec. 5). Thus, it is sufficient to prove that $\lim_{T \to \infty} \min_{z ||z|| = 1} z^T A_T z = 16Note that we do not need to specify the norm as all norms are equivalent in finite-dimensional spaces.
min_{z:||z||=1}|z'A\overline{z} - z'Az|, which in turn implies that it is sufficient to prove that sup_{z:||z||=1}|z'A_Tz - z'Az| \to 0, as \( T \to \infty \). Prove this last result by contradiction.

Assume that sup_{z:||z||=1}|z'A_Tz - z'Az| does not converge to 0 as \( T \to \infty \). Then, there exists \( \varepsilon > 0 \) and an increasing function \( \alpha_1: \mathbb{N} \to \mathbb{N} \) defining a subsequence of vectors of norm 1, \((z_{\alpha_1}(T))_{T \geq 1}\), and a subsequence of matrices, \((A_{\alpha_1}(T))_{T \geq 1}\), such that

\[
\varepsilon < \left| z'_{\alpha_1}(T)A_{\alpha_1}(T)z_{\alpha_1}(T) - z'_{\alpha_1}(T)Az_{\alpha_1}(T) \right|
\]

\[
= \left| z'_{\alpha_1}(T)(A_{\alpha_1}(T) - A)z_{\alpha_1}(T) \right| \leq \sum_{(k,l)\in[1,m]^2} \left| a_{\alpha_1}(k,l) - a(k,l) \right| z_{\alpha_1}(T)z_{\alpha_1}(T)
\]

\[
\leq m^2 \max_{(k,l)\in[1,m]^2} \left| a_{\alpha_1}(k,l) - a(k,l) \right| \to 0 \quad \text{as} \quad T \to \infty.
\]

Thus, there is a contradiction. \( \square \)

**Lemma 32** (Differential of a log of a squared determinant). Let \( G \) be an open set of \( \mathbb{R}^q \) with \( q \in [1, \infty[ \), and \( F: G \to \mathbb{R}^{m \times m} \) a differentiable function on \( G \). Then \( |F|_{\det} : G \to \mathbb{R} \) is also differentiable on \( G \). Moreover, if \( |F(x)|_{\det} \neq 0 \) where \( x \in G \), then

\( i \) \( \quad D|F(x)|_{\det} = |F(x)|_{\det} \text{tr}[F(x)^{-1}DF(x)]; \)

\( ii \) \( \quad D \log |F(x)|_{\det}^2 = 2 \text{tr}[F(x)^{-1}DF(x)]. \)

**Proof.** (i) It is a consequence of the so-called Jacobi’s formula (e.g., Magnus and Neudecker 1999/1988, chap. 8 sec. 3).

(ii) First of all, note that the logarithm is well-defined as its argument is strictly positive by assumption. Then, by the statement (i) of the present lemma and the chain rule,

\[
D \log |F(x)|_{\det}^2 = \frac{1}{|F(x)|_{\det}^2} 2|F(x)|_{\det} |F(x)|_{\det} \text{tr}[F(x)^{-1}DF(x)].
\]

\( \square \)

**Lemma 33** (Inverse of a \( 2 \times 2 \) partitioned matrix). Let \( F \) be a square matrix s.t.

\[
F = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

where \( A \) and \( D \) are square matrices. Then, the following statements hold.

(i) If \( A \) is invertible, then \( F \) invertible \( \Leftrightarrow \) \((D - CA^{-1}B) \) invertible. Moreover,

\[
F^{-1} = \begin{bmatrix}
A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1}
\end{bmatrix}.
\]

(ii) If \( D \) is invertible, then \( F \) invertible \( \Leftrightarrow \) \((A - BD^{-1}C)^{-1} \) invertible. Moreover,

\[
F^{-1} = \begin{bmatrix}
(A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\
-D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1}
\end{bmatrix}.
\]

**Proof.** This is a standard result (e.g., Magnus and Neudecker 1999/1988, Chap. 1 sec. 11). \( \square \)
Corollary 2 (Inverse of a $2 \times 2$ partitioned matrix in a special case). Let $E$ be a square matrix s.t.

$$E = \begin{bmatrix} A & B \\ B' & 0 \end{bmatrix}.$$ 

Then,

(i) If $A$ and $B' A^{-1} B$ are invertible, then $E$ is invertible; and

(ii) \[\begin{bmatrix} A & B \\ B' & 0 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} - A^{-1} B (B' A^{-1} B)^{-1} B' A^{-1} & A^{-1} B (B' A^{-1} B)^{-1} \\ (B' A^{-1} B)^{-1} B' A^{-1} & -(B' A^{-1} B)^{-1} \end{bmatrix}.\]

Proof. Apply the above Lemma 33 with $F = E$, $C = B'$ and $D = 0$. 

Appendix E. More on the numerical example

The simulations were performed in R. Each model parameterization is simulated 10,000 times. The robustness of the simulation results was checked with different optimization algorithms, starting values and tolerance parameter values. The estimation for a single sample is typically performed in less than a few seconds. The calculations were done on a 24 CPU cores of a Dell server with 4 AMD Opteron 8425 HE processors running at 2.1 GHz. We numerically checked that the reported statistics have a converging behaviour as we increase the number of simulated samples to 10,000.

Appendix F. More on the empirical example

In empirical consumption-based asset pricing, the literature has found little common ground about the value of the relative risk aversion (RRA) of the representative agent: In most studies, point estimates from economically similar moment conditions are generally outside of each other’s confidence intervals. Section 4.2 (p. 9) and the present appendix revisit the estimation of the RRA. The popularity of moment-based estimation in consumption-based asset pricing, and more generally in economics is due to the fact that moment-based estimation does not necessarily require the specification of a family of distributions for the data (e.g., Hansen 2013, sec. 3). Typically, an economic model does not imply such family of distributions, except for tractability reasons. Imposing a family of distributions makes it difficult to disentangle the part of the inference results due to the empirical relevance of the economic model from the part due to these additional restrictions. Under regularity conditions, assuming a distribution corresponds to imposing an infinite number of extra moment restrictions (e.g., Feller 1971/1966, chap. VII, sec. 3).

In Section 4.2 (p. 9) and the present appendix, we rely on the moment condition (10) on p. 9. This moment condition has several advantages. Firstly, it is as consistent with Lucas (1978) as with more recent consumption-based asset-pricing models, such as Barro (2006) or Gabaix (2012). In other words, despite its simplicity it also correspond to sophisticated models, and it allows us to obtain estimates that are robust to different variations of consumption-based asset pricing theory. Secondly, without loss of generality, it does not require to estimate the time discount rate, about which there is little debate: The time discount rate of the representative agent is consistently found to be between .9 and 1. Note also that it has been common to use
moment conditions with a separate parameter for the so-called intertemporal elasticity of substitution, i.e., use Epstein-Zin-Weil preferences (e.g., Epstein and Zin 1991). However, Bommier et al. (2017) show that such a specification makes the economic interpretation of the parameters difficult. In particular, they show that an increase of the so-called RRA (relative risk-aversion) parameter does not yield a behaviour that would be considered more risk averse. E.g., All other things being equal, savings can be a decreasing function of the so-called RRA parameter for an agent with Epstein-Zin-Weil preferences (e.g., Bommier et al. 2017, sec. 6). This difficulty of interpretation comes from a violation of the monotonicity axiom according to which an agent does not choose an action if another available action is preferable in every state of the world.

F.1. Additional empirical evidence.

Table 3 (p. 93) is the same as Table 2 (p. 10) with the additional Table 3 Figures (A). The latter clearly shows that the normalized ESP is relatively sharp around the ESP estimator. Table 4 (p. 94) is the counterpart of Table 3 (p. 93) for the 1930-2009 data set. The 95% ET ALR confidence region is based on the inversion of the ALR ET statistic 2T \[ \ln \left( \frac{1}{T} \sum_{t=1}^{T} \psi_t(\hat{\theta}_{MM,T}) \psi_t(\hat{\theta}_{MM,T})' \right) = 0 \] because, in the just-identified case, \[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\hat{\theta}_{T}) = 0 \] so that \[ \tau_T(\hat{\theta}_T) = 0 \times 1 \]. The ET and ESP support correspond to the parameter values \( \theta \in \Theta \) for which there exists a solution \( \tau_T(\theta) \) to the equation (14) on p. 18. Table 4 confirms the findings of Table 3 (p. 93) in Section 4.2: The ESP is sharper than the ET around its maximum, so that the ESP confidence region is also shorter. Note also that the ESP estimate is almost the same as for the data set 1890-2009. These results are in line with the ESP shrinkage-like behaviour documented in the Monte-Carlo simulations of the section 4.1.

Tables 5 (p. 95) and 7 (p. 96) report the MM estimates and the confidence regions based on the inversion of the MM ALR test statistic T \[ Q_{MM,T}(\theta) - Q_{MM,T}(\hat{\theta}_{MM,T}) \] as \( T \to \infty \), (e.g., Newey and McFadden 1994, Theorem 9.2), where \( Q_{MM,T}(\theta) := \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\hat{\theta}_{MM,T}) \psi_t(\hat{\theta}_{MM,T})' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta) \right] \) and \( Q_{MM,T}(\hat{\theta}_{MM,T}) = 0 \) because \[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\hat{\theta}_{T}) = 0 \] in the just-identified case. The MM objective function is sharper around its minimum for the 1930-2009 data set than for the 1890-2009. However, the former sharpness appears misleading as it yields a confidence region that does not include the MM estimate of the 1890-2009 data set.

Tables 6 (p. 95) and 8 (p. 96) report the CU (continuously updating) MM estimates and the confidence regions based on the inversion of the CU ALR test statistic T \[ Q_{CU,T}(\theta) - Q_{CU,T}(\hat{\theta}_{MM,T}) \] as \( T \to \infty \), where \( Q_{CU,T}(\theta) := \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta) \right] \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta) \psi_t(\theta)' \right]^{-1} \left[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\theta) \right] \) and \( Q_{CU,T}(\hat{\theta}_{CU,T}) = 0 \) because \[ \frac{1}{T} \sum_{t=1}^{T} \psi_t(\hat{\theta}_{T}) = 0 \] in the just-identified case. In the just-identified case, which is the case addressed in the present paper, such confidence regions correspond to the S-sets, which
were proposed by Stock and Wright (2000)—following Hansen et al. (1996)—as a solution to the flatness of GMM objective functions. As previously documented in the literature (e.g., Hansen et al. 1996), CU GMM objective functions tend to be flat and low in the tails. Thus, the CU ALR confidence regions (and $S$-sets in the just-identified case) are huge, and hardly informative.

F.2. **Data description.** As in Julliard and Ghosh (2012), our data are standard. For the 1890-2009 data set, our source is the Robert Shiller’s web site. The prime commercial paper and the S&P stock price index play the role of proxies for the risk-less asset and the market return. For the 1930-2009 data set, the proxies for the risk-less asset and the market return are the one month Treasury-bill and the Center for Research in Security Prices (CRSP) value-weighted index of all stocks on the NYSE, AMEX, and NASDAQ. The computation of the growth consumption is based per capita real personal consumption expenditures on nondurable goods from the National Income and Product Accounts (NIPA). Quantities are deflated from the inflation.

Tables 9 and 10 indicate that there is no significant autocorrelation for the excess returns, and only a mild clustering effect (Figures (E) and (F) in Table 10 on p. 97). Thus, the i.i.d. assumption (Assumption 1(a)) appears to be a good approximation for the excess returns for both data set. For the growth consumption, the i.i.d. assumption may appear less appropriate. Table 11 indicates a mild autocorrelation for the growth consumption, and, more strikingly, a change of variance at the end of WWII. However, in the moment function, the growth consumption is multiplied by the excess returns, whose variance is several orders of magnitude higher (Table 9 on p. 97), so that the change of variance is dampened.
Table 3. ET vs. ESP inference (1890–2009)

Empirical moment condition: \[
\frac{1}{2009-1889} \sum_{t=1890}^{2009} \left( \frac{C_t}{C_{t-1}} \right)^{-\theta} (R_{m,t} - R_{f,t}) = 0,
\]
where \( R_{m,t} := \text{gross market return} \), \( R_{f,t} := \text{risk-free asset gross return} \), \( C_t := \text{consumption} \), and \( \theta := \text{relative risk aversion} \);

Normalized ET:=\[\exp\left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau T(\cdot)'\psi(\cdot)} \right] \right\} / \int_{\Omega} \exp\left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau T(\cdot)'\psi(\cdot)} \right] \right\} d\theta;\]

Normalized ESP:=\[\hat{f}_{\hat{\theta}_T}(\cdot)/\int_{\Theta} \hat{f}_{\hat{\theta}_T}(\theta)d\theta;\]

\( \hat{\theta}_{\text{ET},T} = \hat{\theta}_{\text{MM},T} = 50.3 \) (bullet) and \( \hat{\theta}_{\text{ESP},T} = 32.21 \) (bullet);

ET and ESP support = \([-218.2, 289.0]\); 95% ET ALR conf. region=\([18.3, 289.0]\) (stripe); 95% ESP ALR conf. region=\([15.0, 112.7]\) (stripe).
Table 4. ET vs. ESP inference (1930–2009)

Empirical moment condition: \[
\frac{1}{2009-1889} \sum_{t=1890}^{2009} \left( \frac{C_t}{C_{t-1}} \right)^{-\theta} (R_{m,t} - R_{f,t}) = 0,
\]
where \( R_{m,t} := \) gross market return, \( R_{f,t} := \) risk-free asset gross return, \( C_t := \) consumption, and \( \theta := \) relative risk aversion;

Normalized ET: \( = \exp \left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau t(\cdot)\psi_t(\cdot)} \right] \right\} / \int_{\Theta} \exp \left\{ T \ln \left[ \frac{1}{T} \sum_{t=1}^{T} e^{\tau t(\cdot)\psi_t(\cdot)} \right] \right\} d\theta; \)

Normalized ESP: \( = \hat{f}_{\theta_T}(\cdot) / \int_{\Theta} \hat{f}_{\theta_T}(\cdot) d\theta; \)

\( \hat{\theta}_{ET,T} = 35.0 \) (bullet) and \( \hat{\theta}_{ESP,T} = 32.5 \) (bullet); ET and ESP support = \([-202.8, 813.3]\) 95% ET ALR conf. region = \([-202.8, -76.0] \cup \[17.7, 197.8]\) (stripe);

95% ESP ALR conf. region = \([17.7, 58.7]\) (stripe).

(A) Normalized ET (light green) vs. normalized ESP (dark blue).

(B) ET est. and ALR conf. region.

(C) ESP est. and ALR conf. region.
Table 5. MM inference (1890–2009)

Empirical moment condition: $\frac{1}{2009-1889} \sum_{t=1890}^{2009} \left( \left( \frac{C_t}{C_{t-1}} \right)^{\theta} (R_{m,t} - R_{f,t}) \right) = 0$, where $R_{m,t}$ := gross market return, $R_{f,t}$ := risk-free asset gross return, $C_t$ := consumption, and $\theta$ := relative risk aversion.

$\hat{\theta}_{\text{GMM},T} = 50.3$ (bullet); 95% ALR confidence region $=[-41.7, 71.5]$ (stripe).

Table 6. Continuously updated (CU) GMM inference (1890–2009)

Empirical moment condition: $\frac{1}{1990-1889} \sum_{t=1890}^{1990} \left( \left( \frac{C_t}{C_{t-1}} \right)^{\theta} (R_{m,t} - R_{f,t}) \right) = 0$, where $R_{m,t}$ := gross market return, $R_{f,t}$ := risk-free asset gross return, $C_t$ := consumption, and $\theta$ := relative risk aversion.

$\hat{\theta}_{\text{CU},T} = 50.3$ (bullet); 95% ALR confidence region (and $S$-set) $= [\ldots, -59.1] \cup [18.2, \ldots]$ (stripe).

Rk: We constrain the numerical search for point estimate to discard large values of $\theta$.

(A) MM objective function and point estimate. (A zoom) MM obj. function ALR conf. region.

(A) Objective function and point estimate. (B) Truncated ALR conf. region (and $S$-set).
### Table 7. MM inference (1930-2009)

Empirical moment condition: \[
\frac{1}{2009-1930} \sum_{t=1930}^{2009} \left[ \left( \frac{C_t}{C_{t-1}} \right)^{-\theta} (R_{m,t} - R_{f,t}) \right] = 0, \]
where
- \( R_{m,t} := \) gross market return,
- \( R_{f,t} := \) risk-free asset gross return,
- \( C_t := \) consumption,
- \( \theta := \) relative risk aversion.

\( \hat{\theta}_{\text{MM},T} = 35.0 \) (bullet), ALR confidence region = \([-10.4, 46.5]\) (stripe)

(A) MM objective function and point estimate. (A zoom) Objective function and point estimate.

### Table 8. Continuously updated (CU) GMM inference (1930–2009)

Empirical moment condition: \[
\frac{1}{2009-1890} \sum_{t=1890}^{2009} \left[ \left( \frac{C_t}{C_{t-1}} \right)^{-\theta} (R_{m,t} - R_{f,t}) \right] = 0, \]
where
- \( R_{m,t} := \) gross market return,
- \( R_{f,t} := \) risk-free asset gross return,
- \( C_t := \) consumption,
- \( \theta := \) relative risk aversion.

\( \hat{\theta}_{\text{CU},T} = 50.3 \) (bullet); ALR confidence region (and S-set) = \([-\ldots, -35.8] \cup [17.9, \ldots]\) (stripe).

Rk: We constrain the numerical search for point estimate to discard large values of \( \theta \).

(A) Objective function and point estimate. (B) Truncated ALR conf. region (and S-set).
Table 9. **Descriptive statistics.**

| Variable | Mean (Variance) 1890-2009 | Mean (Variance) 1930-2009 |
|----------|---------------------------|---------------------------|
| $C_t/C_{t-1}$ | 1.0182 (.0009) | 1.014 (.0007) |
| $R_{m,t} - R_{f,t}$ | .0630 (.0367) | .074 (.0424) |

Table 10. **Excess returns: $R_{m,t} - R_{f,t}$**

(A) Time series 1890-2009  (B) Time series 1930-2009

(C) Autocorr. function of $R_{m,t} - R_{f,t}$  (D) Autocorr. function of $R_{m,t} - R_{f,t}$

(E) Autocorr. function of $(R_{m,t} - R_{f,t})^2$  (F) Autocorr. function of $(R_{m,t} - R_{f,t})^2$
Table 11. Growth consumption: $C_t/C_{t-1}$.

| Year Range | 1890-2009 | 1930-2009 |
|------------|-----------|-----------|
| (A) Time series | ![Time series](image) | ![Time series](image) |
| (B) Time series | ![Time series](image) | ![Time series](image) |
| (C) Autocorr. function of $C_t/C_{t-1}$ | ![Autocorr. function](image) | ![Autocorr. function](image) |
| (D) Autocorr. function of $C_t/C_{t-1}$ | ![Autocorr. function](image) | ![Autocorr. function](image) |