Sign Pairity Difference Cordial Labeling of Digraphs

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ABSTRACT: Let \( D = (V,A) \) be a digraph. An injective function \( f:V(D) \rightarrow \{1,2,\ldots,n\} \) is said to be a sign pairity difference cordial labeling if the induced arc labeling \( f^*:A(D) \rightarrow \{0,1\} \) defined by \( f^*((u,v)) = \begin{cases} 1 & \text{if } f(u) - f(v) > 0 \\ 0 & \text{elsewhere} \end{cases} \) satisfies the condition that \( |f^*_e(0) - f^*_e(1)| \leq 1 \) where, \( e_f(0) \) is the number of arcs with label 0 and \( e_f(1) \) is the number of arcs with label 1. In this paper, we analyze the existence of sign pairity difference cordial labeling in the digraphs Directed Path(\( P_n^* \)), Alternating Path(\( A\overrightarrow{P}_n \)), Directed Cycle(\( \overrightarrow{C}_n \)), Outstar(\( o\overrightarrow{K}_{1,n} \)), Inwheel(\( i\overrightarrow{W}_{1,n} \)) and Downcomb(\( \overrightarrow{D}P_n,\overrightarrow{D}K_1 \)) defined by K.Palani et.al.

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1 INTRODUCTION

A directed graph or digraph \( D \) consists of a finite set \( V \) of vertices(points) and a collection of ordered pairs of distinct vertices. Any such pair \( (u,v) \) is called an arc or directed line and will usually be denoted by \( u \rightarrow v \). The arc \( u \rightarrow v \) goes from \( u \) to \( v \) and incident with \( u \) and \( v \), we also say \( u \) is adjacent to \( v \) and \( v \) is adjacent from \( u \). A digraph \( D \) with \( p \) vertices and \( q \) arcs is denoted by \( D(p,q) \). The indegree \( d^-(v) \) of a vertex \( v \) in a digraph \( D \) is the number of arcs having \( v \) as its terminal vertex. The outdegree \( d^+(v) \) of \( v \) is the number of arcs having \( v \) as its initial vertex[5]. A labeling of a graph \( G \) is an assignment of integers to either the vertices or the edges or both subject to certain conditions. Motivated by different labeling concepts, we introduce here sign pairity difference cordial labeling in digraphs and analyze the existence of the same in different digraphs defined by K.Palani et.al[7].

The following results[7] are used in the subsequent section.

1.1 Directed Path(\( \overrightarrow{P}_n \)) A path \( P_n \) in which all the edges are directed in the same direction is called a directed path and is denoted as \( \overrightarrow{P}_n \).

1.2 Alternating Path(\( A\overrightarrow{P}_n \)) A path \( P_n \) in which the edges are given alternative direction is called an alternating path and is denoted as \( A\overrightarrow{P}_n \).

1.3 Directed Cycle(\( \overrightarrow{C}_n \)) A cycle \( C_n \) in which all the edges are directed clockwise or anticlockwise is called a directed cycle and is denoted as \( \overrightarrow{C}_n \).
1.4 Outstar (o\(\overrightarrow{K_{1,n}}\)) A star graph \(K_{1,n}\) in which all the edges are directed away from the root vertex is called an outstar and is denoted as \(\overrightarrow{K_{1,n}}\).

1.5 Inwheel (\(i\overrightarrow{W_n}\)) A wheel graph \(W_n\) in which the edges of the outer cycle are directed clockwise or anticlockwise and the spoke edges are directed towards the central (or Hub) vertex is called an inwheel and is denoted as \(i\overrightarrow{W_n}\).

1.6 Downcomb (\(Down\overrightarrow{P_n}\overrightarrow{OK_1}\)) A comb graph \(P_n\overrightarrow{OK_1}\) in which the path edges are directed in the same direction and the pendent edges are oriented towards the end vertices is called a downcomb and is denoted as \(Down\overrightarrow{P_n}\overrightarrow{OK_1}\).

2 SIGN PAIRITY DIFFERENCE CORDIAL LABELING

2.1 Definition Let \(D = (V, A)\) be a digraph. An injective function \(f: V(D) \rightarrow \{1, 2, \ldots, \ell\}\) is said to be a sign parity difference cordial labeling if the induced arc labeling

\[
f^\ast: A(D) \rightarrow \{0, 1\}
\]

defined by

\[
f^\ast((u, v)) = \begin{cases} 1 & \text{if } f(u) - f(v) > 0 \\ 0 & \text{elsewhere} \end{cases}
\]

satisfies the condition that \(|e_f(0) - e_f(1)| \leq 1\) where, \(e_f(0)\) is the number of arcs with label 0 and \(e_f(1)\) is the number of arcs with label 1.

2.2 Theorem Directed Path \((\overrightarrow{P_n})\) admits sign parity difference cordial labeling.

**Proof:**

Let \(V(\overrightarrow{P_n}) = \{u_1, u_2, u_3, \ldots, u_n\}\)

Then, \(A(\overrightarrow{P_n}) = \{(u_i, u_{i+1})\}/1 \leq i \leq n - 1\}

Here, \(|V(\overrightarrow{P_n})| = n\) and \(|A(\overrightarrow{P_n})| = n - 1\)

**Case 1:** \(n \equiv 0(mod 2)\)

Define \(f: V(\overrightarrow{P_n}) \rightarrow \{1, 2, 3, \ldots, n\}\) by

\[f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n}{2}\]

Correspondingly, the induced arc labels are

\[f^\ast((u_i, u_{i+1})) = \begin{cases} < 0 & \text{if } i \text{ is odd} \\ > 0 & \text{if } i \text{ is even} \end{cases}
\]

**Case 2:** \(n \equiv 1(mod 2)\)

Define \(f: V(\overrightarrow{P_n}) \rightarrow \{1, 2, 3, \ldots, n\}\) by

\[f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+1}{2}\]

Correspondingly, the induced arc labels are

\[f^\ast((u_i, u_{i+1})) = \begin{cases} < 0 & \text{if } i \text{ is odd} \\ > 0 & \text{if } i \text{ is even} \end{cases}
\]

Correspondingly, \(|e_f(0) - e_f(1)| = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}\)

Therefore, \(f\) is a sign parity difference cordial labeling of \(\overrightarrow{P_n}\).

Hence, \(\overrightarrow{P_n}\) admits sign parity difference cordial labeling.
2.3 Theorem Alternating Path($\overrightarrow{A_{n}}$) admits sign pairity difference cordial labeling.

Proof:
Let $V(\overrightarrow{A_{n}}) = \{u_1, u_2, u_3 \ldots u_n\}$
Then, $A(\overrightarrow{A_{n}}) = \{(u_i, u_{i+1})/1 \leq i \leq n - 1, i \text{ is odd} \} \cup \{(u_{i+1}, u_i)/1 \leq i \leq n - 1, i \text{ is even} \}$
Here, $|V(\overrightarrow{A_{n}})| = n$ and $|A(\overrightarrow{A_{n}})| = n - 1$

Define $f: V(\overrightarrow{A_{n}}) \rightarrow \{1, 2, 3 \ldots \ldots \ldots n\}$ by
$f(u_i) = i, 1 \leq i \leq n$
Correspondingly, the induced arc labels are
For $1 \leq i \leq n - 1,
\begin{align*}
\hat{f}^*((u_i, u_{i+1})) &= i - i - 1 = -1 < 0 & \text{if } i \text{ is odd} \\
\hat{f}^*((u_{i+1}, u_i)) &= i + 1 - i = 1 > 0 & \text{if } i \text{ is even}
\end{align*}

Correspondingly, $|e_f(0) - e_f(1)| = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ and so $|e_f(0) - e_f(1)| \leq 1 \ \forall n$

Therefore, $f$ is a sign pairity difference cordial labeling of $\overrightarrow{A_{n}}$.
Hence, $\overrightarrow{A_{n}}$ admits sign pairity difference cordial labeling.

2.4 Theorem Directed Cycle($\overrightarrow{C_n}$) admits sign pairity difference cordial labeling.

Proof:
Let $V(\overrightarrow{C_n}) = \{u_1, u_2, u_3 \ldots u_n\}$
Then, $A(\overrightarrow{C_n}) = \{(u_i, u_{i+1})/1 \leq i \leq n - 1\} \cup \{(u_n, u_1)\}$
Here, $|V(\overrightarrow{C_n})| = n$ and $|A(\overrightarrow{C_n})| = n$

Case 1: $n \equiv 0 (\text{mod}2)$
Define $f: V(\overrightarrow{C_n}) \rightarrow \{1, 2, 3 \ldots \ldots \ldots n\}$ by
\begin{align*}
f(u_{2i-1}) &= i, \ 1 \leq i \leq \frac{n}{2} \text{ and} \\
f(u_{2i}) &= \frac{n}{2} + i, \ 1 \leq i \leq \frac{n}{2}
\end{align*}
Correspondingly, the induced arc labels are
For $1 \leq i \leq n - 1,
\begin{align*}
\hat{f}^*((u_i, u_{i+1})) &= \begin{cases} < 0 & \text{if } i \text{ is odd} \\ > 0 & \text{if } i \text{ is even} \end{cases} \\
\hat{f}^*((u_n, u_1)) &= n - 1 > 0
\end{align*}

Case 2: $n \equiv 1 (\text{mod}2)$
Define $f: V(\overrightarrow{C_n}) \rightarrow \{1, 2, 3 \ldots \ldots \ldots n\}$ by
\begin{align*}
f(u_{2i-1}) &= i, \ 1 \leq i \leq \frac{n+1}{2} \text{ and} \\
f(u_{2i}) &= \frac{n+1}{2} + i, \ 1 \leq i \leq \frac{n-1}{2}
\end{align*}
Correspondingly, the induced arc labels are
For $1 \leq i \leq n - 1,
\begin{align*}
\hat{f}^*((u_i, u_{i+1})) &= \begin{cases} < 0 & \text{if } i \text{ is odd} \\ > 0 & \text{if } i \text{ is even} \end{cases} \\
\hat{f}^*((u_n, u_1)) &= \frac{n+1}{2} - 1 = \frac{n-1}{2} > 0
\end{align*}

Correspondingly, $|e_f(0) - e_f(1)| = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$ and so $|e_f(0) - e_f(1)| \leq 1 \ \forall n$

Therefore, $f$ is a sign pairity difference cordial labeling of $\overrightarrow{C_n}$.
Hence, $\overrightarrow{C_n}$ admits sign pairity difference cordial labeling.
\section*{2.5 Theorem} Outstar($oK_{1,n}$) admits sign pairity difference cordial labeling.

\textbf{Proof}: 
Let $V(oK_{1,n}) = \{u_1, u_2, u_3, \ldots, u_n, v\}$ 
Then, $A(oK_{1,n}) = \{(u_i, u_j): 1 \leq i \leq n\}$ 
Here, $|V(oK_{1,n})| = n + 1$ and $|A(oK_{1,n})| = n$

\textbf{Case 1: $n \equiv 0 (mod 2)$}
Define $f: V(oK_{1,n}) \rightarrow \{1, 2, 3, \ldots, n + 1\}$ by 
\begin{align*}
f(v) &= \frac{n+1}{2}; \\
f(u_i) &= i, \quad 1 \leq i \leq \frac{n}{2} \text{ and} \\
f\left(u_{\frac{n-1}{2}+i}\right) &= n-i+2, \quad 1 \leq i \leq \frac{n+1}{2}.
\end{align*}
Correspondingly, the induced arc labels are 
For $1 \leq i \leq \frac{n-1}{2},$
\begin{align*}
f^*(v, u_i) &= \frac{n+1}{2} - i = \frac{n}{2} + 1 - i > 0 \\
f^*(v, u_{\frac{n-1}{2}+i}) &= \frac{n+1}{2} - (n-i+2) = -\frac{n}{2} + i - 1 < 0
\end{align*}
\textbf{Case 2: $n \equiv 1 (mod 2)$}
Define $f: V(oK_{1,n}) \rightarrow \{1, 2, 3, \ldots, n + 1\}$ by 
\begin{align*}
f(v) &= \frac{n+1}{2}; \\
f(u_i) &= i, \quad 1 \leq i \leq \frac{n-1}{2} \text{ and} \\
f\left(u_{\frac{n-1}{2}+i}\right) &= n-i+2, \quad 1 \leq i \leq \frac{n+1}{2}.
\end{align*}
Correspondingly, the induced arc labels are 
For $1 \leq i \leq \frac{n-1}{2},$
\begin{align*}
f^*(v, u_i) &= \frac{n+1}{2} - i = \frac{n+1-2i}{2} > 0 \\
f^*(v, u_{\frac{n-1}{2}+i}) &= \frac{n+1}{2} - (n-i+2) = \frac{n}{2} - \frac{3}{2} + i < 0
\end{align*}
Correspondingly, $|e_f(0) - e_f(1)| = \left\{ \begin{array}{ll} 1 & \text{if $n$ is odd} \\
0 & \text{if $n$ is even} \end{array} \right.$ and so $|e_f(0) - e_f(1)| \leq 1 \ \forall n$

Therefore, $f$ is a sign pairity difference cordial labeling of $oK_{1,n}$.
Hence, $oK_{1,n}$ admits sign pairity difference cordial labeling.

\section*{2.6 Theorem} Inwheel($iW_n$) admits sign pairity difference cordial labeling.

\textbf{Proof}: 
Let $V(iW_n) = \{u_1, u_2, u_3, \ldots, u_n, v\}$ 
Then, $A(iW_n) = \{((u_i, u_{i+1}))/1 \leq i \leq n - 1\} \cup \{((u_i, v))/1 \leq i \leq n\} \cup \{((u_n, u_1))\}$ 
Here, $|V(iW_n)| = n + 1$ and $|A(iW_n)| = 2n$

\textbf{Case 1: $n \equiv 0 (mod 2)$}
Define $f: V(iW_n) \rightarrow \{1, 2, 3, \ldots, n + 1\}$ by 
\begin{align*}
f(v) &= \frac{n}{2} + 1; \\
f(u_{2i-1}) &= i, \quad 1 \leq i \leq \frac{n}{2} \text{ and} \\
f(u_{2i}) &= \frac{n}{2} + i + 1, \quad 1 \leq i \leq \frac{n}{2}.
\end{align*}
Correspondingly, the induced arc labels are
For $1 \leq i \leq n - 1$,

$$f^*(u_i, u_{i+1}) = \begin{cases} < 0 & \text{if } i \text{ is odd} \\ > 0 & \text{if } i \text{ is even} \end{cases}$$

$$f^*(u_n, u_1) = n + 1 - 1 = n > 0$$

For $1 \leq i \leq \frac{n}{2}$

$$f^*(u_{2i-1}, v) = i - \left(\frac{n}{2} + 1\right) = -\frac{n}{2} - 1 + i < 0$$

$$f^*(u_{2i}, v) = \frac{n}{2} + i + \frac{n}{2} - 1 = i > 0$$

Case 2: $n \equiv 1 \pmod{2}$

Define $f: V(i\overline{W}'_n) \to \{1, 2, 3, \ldots, n + 1\}$ by

$$f(v) = \frac{n+3}{2};$$

$$f(u_{2i-1}) = i, \quad 1 \leq i \leq \frac{n+1}{2} \text{ and }$$

$$f(u_{2i}) = \frac{n+3}{2} + i, \quad 1 \leq i \leq \frac{n-1}{2}.$$

Correspondingly, the induced arc labels are

For $1 \leq i \leq n - 1$,

$$f^*(u_i, u_{i+1}) = \begin{cases} < 0 & \text{if } i \text{ is odd} \\ > 0 & \text{if } i \text{ is even} \end{cases}$$

$$f^*(u_n, u_1) = \frac{n+1}{2} - 1 = \frac{n-1}{2} > 0$$

For $1 \leq i \leq \frac{n+1}{2}$,

$$f^*(u_{2i-1}, v) = i - \left(\frac{n+3}{2}\right) = i - \frac{n}{2} - \frac{3}{2} < 0$$

For $1 \leq i \leq \frac{n-1}{2}$,

$$f^*(u_{2i}, v) = \frac{n+1}{2} + i + 1 - \frac{n}{2} = i > 0$$

Correspondingly, $|e_f(0) - e_f(1)| = 0 \forall n$ and so $|e_f(0) - e_f(1)| \leq 1 \forall n$

Therefore, $f$ is a sign pairity difference cordial labeling of $i\overline{W}'_n$.

Hence, $i\overline{W}'_n$ admits sign pairity difference cordial labeling.

**2.7 Theorem**

Downcomb($\overline{P\bigcap K'_1}$) admits sign pairity difference cordial labeling.

**Proof:**

Let $V(\overline{P\bigcap K'_1}) = \{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, \ldots, v_n\}$

Then, $\overline{A(\overline{P\bigcap K'_1})} = \{(u_i, u_{i+1})/1 \leq i \leq n - 1\} \cup \{(u_i, v_i)/1 \leq i \leq n\}$

Here, $|V(\overline{P\bigcap K'_1})| = 2n$ and $|\overline{A(\overline{P\bigcap K'_1})}| = 2n - 1$

Define $f: V(\overline{P\bigcap K'_1}) \to \{1, 2, 3, \ldots, 2n\}$ by

$$f(u_i) = n + i, \quad 1 \leq i \leq n$$

and

$$f(v_i) = i, \quad 1 \leq i \leq n$$

Correspondingly, the induced arc labels are

For $1 \leq i \leq n - 1$,

$$f^*(u_i, u_{i+1}) = n + i - (n + i + 1) = -1 < 0$$

For $1 \leq i \leq n$,

$$f^*(u_i, v_i) = n + i - i = n > 0$$

Correspondingly, $|e_f(0) - e_f(1)| = 1 \forall n$ and so $|e_f(0) - e_f(1)| \leq 1 \forall n$

Therefore, $f$ is a sign pairity difference cordial labeling of $\overline{P\bigcap K'_1}$.

Hence, $\overline{P\bigcap K'_1}$ admits sign pairity difference cordial labeling.
CONCLUSION
In this way, we tested the digraphs namely Directed Path, Alternating Path, Directed Cycle, Outstar, Inwheel and Downcomb for the existence of sign pairity difference cordial labeling.

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