ENERGY-CRITICAL NLS WITH POTENTIALS OF QUADRATIC GROWTH

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Abstract. We consider the global wellposedness problem for the nonlinear Schrödinger equation

\[ i \partial_t u = \left(-\frac{1}{2} \Delta + V(x)\right)u \pm |u|^{4/(d-2)}u, \quad u(0) \in \Sigma(\mathbb{R}^d), \]

where \( \Sigma \) is the weighted Sobolev space \( \dot{H}^1 \cap |x|^{-1}L^2 \). The case \( V(x) = \frac{1}{2}|x|^2 \) was recently treated by the author. This note generalizes the results to a class of "approximately quadratic" potentials.

We closely follow the previous concentration compactness arguments for the harmonic oscillator. A key technical difference is that in the absence of a concrete formula for the linear propagator, we apply more general tools from microlocal analysis, including a Fourier integral parametrix of Fujiwara.

1. Introduction. We consider the nonlinear Schrödinger equation

\[
\begin{cases}
  i \partial_t u = (-\frac{1}{2} \Delta + V)u + \mu|u|^{\frac{4}{d-2}}u, & \mu = \pm 1, \\
  u(0) = u_0 \in \Sigma(\mathbb{R}^d),
\end{cases}
\]

where \( V = V(x) \) is a real-valued potential The equation is defocusing or focusing if \( \mu = 1 \) or \( \mu = -1 \), respectively. In a recent work [14], we studied large-data global wellposedness of the Cauchy problem with the harmonic oscillator potential \( V(x) = \frac{1}{2}|x|^2 \), for which \( \Sigma := \dot{H}^1 \cap |x|^{-1}L^2 \), the weighted Sobolev space with norm \( \|f\|_{\Sigma}^2 := \|\nabla f\|_{L^2}^2 + \|xf\|_{L^2}^2 < \infty \), is precisely the function space associated with the conserved energy

\[
E(u(t)) = \int_{\mathbb{R}^d} \frac{1}{2} \vert \nabla u(t,x) \vert^2 + V(x) |u(t,x)|^2 + \mu(1 - \frac{2}{d}) |u(t,x)|^{\frac{4}{d-2}} \, dx = E(u(0)).
\]

This note extends the previous results to a wider class of potentials that grow approximately quadratically. More precisely, we assume that \( V \) is smooth and satisfies

\[
\partial_x^\alpha V \in L^\infty \quad \text{for all } |\alpha| \geq 2, \tag{2}
\]

\[
V(x) \geq \delta |x|^2 \quad \text{for some } \delta > 0. \tag{3}
\]

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These hypotheses ensure that $\delta |x|^2 \leq V(x) \leq \delta^{-1}(1+|x|^2)$ for some constant $\delta > 0$. Therefore, by Sobolev embedding $\Sigma$ is still the energy space and is also the form domain $Q(H) = D(H^{1/2})$ for the positive operator $H = -\frac{1}{2} \Delta + V$. It will be convenient at times to use the equivalent norm
\[
\|f\|_{Q(H)}^2 := \|H^{1/2}f\|_{L^2}^2 = \|\nabla f\|_{L^2}^2 + \|V^{1/2}f\|_{L^2}^2,
\]
which is exactly preserved by the propagator $e^{-itH}$.

This equation is closely linked to the energy-critical NLS
\[
(i\partial_t + \frac{1}{2} \Delta)u = \mu |u|^{\frac{d}{2}-2} u, \quad u(0) \in \dot{H}^1(\mathbb{R}^d)
\]
\[
E_{\Delta}(u) = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u|^2 + \mu(1 - \frac{2}{d}) |u|^{\frac{d}{2}} \, dx,
\]
which is invariant under the scaling $u \mapsto u^\lambda(t, x) = \lambda^{-\frac{d-2}{2}} u(\lambda^{-2}t, \lambda^{-1}x)$. Roughly speaking, if a solution $u$ to (1) is initially highly concentrated at some point $x_0$, it sees the potential $V$ as approximately a constant $V(x_0)$, and for short times the behavior of $u$ will be modelled, up to a temporal phase, by equation (4).

As with the harmonic oscillator [14], it will be essential to formulate this approximation precisely and understand the behavior of solutions to the limiting scale-invariant equation. Fortunately, the latter problem has received considerable attention in the past twenty years. We summarize the state of the art in the following conjecture and theorem, which we employ as a black box in our analysis:

**Conjecture 1.** When $\mu = 1$, solutions to (4) exist globally and scatter. That is, for any $u_0 \in \dot{H}^1(\mathbb{R}^d)$, there exists a unique global solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ to (4) with $u(0) = u_0$, and this solution satisfies a spacetime bound
\[
S_R(u) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u(t, x)| \frac{(d+2)}{2} \, dx \, dt \leq C(E_{\Delta}(u_0)) < \infty.
\]
Moreover, there exist functions $u_\pm \in \dot{H}^1(\mathbb{R}^d)$ such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{\pm \frac{i\mu \Delta}{2}} u_\pm\|_{\dot{H}^1} = 0,
\]
and the correspondences $u_0 \mapsto u_\pm(u_0)$ are homeomorphisms of $\dot{H}^1$.

When $\mu = -1$, one also has global wellposedness and scattering provided that
\[
E_{\Delta}(u_0) < E_{\Delta}(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2},
\]
where the ground state
\[
W(x) = \left(1 + \frac{2|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}} \in \dot{H}^1(\mathbb{R}^d)
\]
solves the elliptic equation $\frac{1}{2} \Delta + |W|^{\frac{d}{d-2}} W = 0$.

**Theorem 1.1.** Conjecture 1 holds for the defocusing equation. For the focusing equation, the conjecture holds for radial initial data when $d \geq 3$, and for all initial data when $d \geq 5$.

**Proof.** See [2, 5, 25, 27] for the defocusing case and [16, 20] for the focusing case. □

As $H = -\Delta + V$ has purely discrete spectrum, global-in-time spacetime bounds of the form (5) are not available even for the linear equation $i\partial_t u = (-\frac{1}{2} \Delta + V)u$. Therefore the natural setting is on a bounded time interval, and we consider
Conjecture 2. When μ = 1, equation (1) is globally wellposed. That is, for each
u₀ ∈ Q(H) there is a unique global solution u : R × Rᵈ → C with u(0) = u₀. This
solution obeys the spacetime bound

\[ S_I(u) := \int_I \int_{R^d} |u(t,x)|^{2(d+2)/(d-2)} \, dx \, dt \leq C(\|I\|, \|u_0\|_\Sigma) \tag{6} \]

for any compact interval I ⊂ R.

If μ = -1, then the same is true provided also that

\[ E(u_0) < E(\Delta(W)) \quad \text{and} \quad \|\nabla u_0\|_{L^2} \leq \|\nabla W\|_{L^2}. \]

The restriction on kinetic energy \(\|\nabla u\|_{L^2}\) in the focusing case is necessary, for as
with the harmonic oscillator, we have:

Theorem 1.2. If μ = -1, \(E(u_0) < E(\Delta(W))\), and \(\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}\), then the
solution to (1) blows up in finite time.

To prove this one need only make notational changes to the discussion in [14, Section 7], and we refer
the reader to there for details.

We state our main result in a conditional fashion to emphasize the pivotal role of
the exactly scale-invariant problem; by Theorem 1.1, however, the result is unconditionally valid except in the focusing case for nonradial data in dimensions \(d = 3\) and 4.

Theorem 1.3. Assume Conjecture 1. Then Conjecture 2 holds.

NLS with external potentials have both significant physical relevance (see for example [28]) and mathematical interest as a dispersive model with broken symmetries. Besides earlier work on the energy-critical harmonic oscillator [14, 23], we also mention the papers of Carles [4], who considered a large class of subquadratic potentials for the energy-subcritical problem

\[ i∂ₜ u = (-\frac{1}{2} \Delta + V) u + \mu |u|^p u, \quad p < \frac{4}{d-2}. \]

Taking initial data in \(Σ\), he established global wellposedness in the defocusing case
when \(4/d < p < 4/(d-2)\) and in the focusing case when \(0 < p < 4/d\). Carles did not require that \(V\) be bounded from below, and also allowed \(V = V(t,x)\) to depend on time. Oh [24] had previously proved large data global existence in the focusing case when \(p < 4/d\) and the potential is time-independent and subquadratic.

We consider a more restricted class of potentials but focus on the subtleties connected to the energy-critical exponent \(p = 4/(d-2)\). When \(V = 0\), perturbative arguments and conservation laws only yield local-in-time solutions whose lifespan depend on the shape of the initial data, not just on the energy. Thus unlike when \(p < 4/(d-2)\), conservation of energy alone is not sufficient to preclude finite time blowup.

Although our equation does not actually have scaling symmetry, it nonetheless contains the same essential difficulties as the scale-invariant problem. For if we consider initial data of the form \(u₀^{λ} = λ^{-(d-2)/2} φ(λ^{-1})\) for a fixed Schwartz function \(φ\), and take \(λ \rightarrow 0\), the energy \(E(u₀^{λ})\) barely depends on \(λ\). In Section 5, we shall see that if \(u^{λ}\) is the solution to (1) with \(u^{λ}(0) = u₀^{λ}\) and we restrict to a time window \(|t| \leq λ²\), then \(u^{λ}\) can be approximated in critical spacetime norms by \(v^{λ}\), where \(v^{λ}(t,x) = λ^{-(d-2)/2} v(λ^{-2} t, λ^{-1} x)\) solves the scale-invariant equation (4) with \(v(0) = φ\). Therefore, just as in the scale-invariant case, solutions to (1) with
bounded energies can accumulate nontrivial spacetime norm in arbitrarily short timeframes.

To prove Theorem (1) we apply the concentration compactness and rigidity method, which had been adapted previously to different critical equations [16, 18, 19, 13, 12, 11, 22]. The reader should also consult the references following Theorem 1.1 for the pioneering instances of this method in scale-invariant problems. We recall its main ingredients:

- **Stability theory.** If \( \tilde{u} \) approximately solves equation (1) with error sufficiently small in Strichartz norms, then there is an exact solution \( u \) to (1) with the same initial data as \( \tilde{u} \), and which is close to \( \tilde{u} \) in critical spacetime norms.

- **Linear and nonlinear profile decompositions.** Given a bounded sequence \( \{f_n\} \subset Q(H) \), there is a decomposition \( f_n = \sum_j \phi_{j,n}^l \), and corresponding decompositions of the linear and nonlinear solutions, where the profiles are asymptotically pairwise independent and reflect the “symmetries” of the problem.

- **Analysis of scaling limits.** A typical profile in the profile decomposition looks schematically like \( \phi_n = N_n^{(d-2)/2} \phi(N_n) \) where either \( N_n \equiv 1 \) or \( \lim_n N_n = \infty \). We will show that in the latter case, for \( n \) large enough the solution \( u_n \) to (1) with \( u_n(0) = \phi_n \) behaves so similarly to a solution to the globally wellposed equation (4) that, by stability theory, \( u_n \) itself must have finite spacetime norm on a length-1 time interval. This essentially rules out blowup for equation (1) when the initial data is highly concentrated at a point.

- **Induction on energy.** Introduced originally by Bourgain [2] and subsequently refined substantially [5, 17, 16], the idea is to assume that global wellposedness of (1) fails for some initial data, and consider the smallest energy \( E_c \) such that solutions \( u \) with \( E(u) \geq E_c \) fail to exist globally. This energy threshold is positive by the small data theory. Using the profile decomposition, the induction hypothesis that solutions with energy smaller than \( E_c \) do exist globally, and the scaling limit analysis, one proves the existence of a blowup solution \( u_c \) with energy \( E(u_c) = E_c \), and which must simultaneously obey an impossibly strong compactness property.

In view of the broken translation and scaling symmetry, constructing the required profile decompositions is rather involved and constituted a major component of our previous work on the harmonic oscillator. We concentrate in this note on the additional ingredients needed in the present, more general context. When \( H = -\frac{1}{2} \Delta + \frac{1}{2} |x|^2 \), we exploited at several junctures the classical Mehler formula for the linear fundamental solution (see for example [6]):

\[
e^{-itH}(x, y) = \frac{1}{(2\pi \sin t)^{d/2}} e^{\frac{\pi i}{4}t (\frac{x^2+y^2}{2} \cos t - xy)}.
\]

No such explicit formula is available for the general potentials considered in this paper. Instead, we appeal to more robust microlocal techniques, in particular the oscillatory integral parametrices of Fujiwara [8, 9].

For the more standard arguments, we provide the main steps and refer the reader to [14] for detailed proofs.

**Outline of paper.** In Section 2 we set our notation and collect some basic estimates regarding equation (1), including Fujiwara’s Fourier integral parametrix. Section 3 states some standard (but vital) local theory. The core of this note, section 4, discusses the profile decomposition mentioned above. The scaling limit
analysis of Section 5 and the compactness arguments of Section 6 parallel the ones given in [14]. As will be the case throughout the paper, we describe mainly the required adjustments and refer to [14] for a comprehensive presentation.

2. Preliminaries.

2.1. Notation and basic estimates. We write $X \lesssim Y$ to mean $X \leq CY$ for some constant $C$. Similarly $X \sim Y$ means $X \lesssim Y$ and $Y \lesssim X$. Denote by $L^p(\mathbb{R}^d)$ the usual Lebesgue spaces, whose norm we sometimes denote using the compact notation $\|f\|_p$. If $I \subset \mathbb{R}^d$ is an interval, the mixed Lebesgue norms on $I \times \mathbb{R}^d$ are defined by

$$\|f\|_{L^q_{I}L^r_{\mathbb{R}^d}(I \times \mathbb{R}^d)} = \left( \int_I \left( \int_{\mathbb{R}^d} |f(t,x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}}.$$

We use the following function space notation due to Schwartz:

$$B_k(\mathbb{R}^d) = \{ f \in C^\infty(\mathbb{R}^d) : D^\ell f \in L^\infty \text{ for all } \ell \geq k \},$$

$$B(\mathbb{R}^d) = B_0(\mathbb{R}^d).$$

We recall Fujiwara’s construction of the fundamental solution for $H$. Recall that the symbol $H(\xi,x) = \frac{1}{2} |\xi|^2 + V(x)$ defines the Hamiltonian flow

$$\begin{cases}
\dot{x} = \partial_\xi H, & x(0) = y \\
\dot{\xi} = -\partial_x H, & \xi(0) = \eta.
\end{cases}
$$

Suppose that $V$ is subquadratic in the sense that

$$|V(0)| + |\nabla^k V(x)| \leq C_k \text{ for all } k \geq 2. \quad (9)$$

Then the vector field $(-\partial_x H, \partial_\xi H)$ is globally Lipschitz, and we may regard $x$ and $\xi$ as functions of $(t,y,\eta) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$.

**Proposition 1** ([8, Proposition 1.7]). Suppose $V$ satisfies (9) and put $H(\xi,x) = \frac{1}{2} |\xi|^2 + V(x)$. Then the map $(y,\eta) \mapsto (x,y)$ obeys the derivative estimates

$$\frac{\partial x}{\partial y} = I + t^2 a(t,y,\eta), \quad \frac{\partial \xi}{\partial \eta} = t(I + t^2 b(t,y,\eta))$$

for some matrix-valued $a,b \in B(\mathbb{R}^d \times \mathbb{R}^d) \times B(\mathbb{R}^d \times \mathbb{R}^d)$.

Further, there exists $\delta_0$ such that whenever $0 \neq |t| \leq \delta_0$, for pairs $x,y \in \mathbb{R}^d$ there is a unique trajectory $(x(\tau),\xi(\tau))$ such that $x(0) = y$ and $x(t) = x$.

**Remark 1.** To get the second statement from the first, one invokes the Hadamard global inverse function theorem to see that $(y,\eta) \mapsto (x,y)$ is a diffeomorphism for $0 \neq t$ sufficiently small.

Consequently, when $0 < |t| \leq \delta_0$ we can define the action

$$S(t,x,y) = \int_0^t \frac{1}{2} |\xi(\tau)|^2 - V(x(\tau)) \, d\tau, \quad (10)$$

where $(x(\tau),\xi(\tau))$ is the unique trajectory with $x(0) = y$ and $x(t) = x$.

**Theorem 2.1** (Unitary propagator [8, 9]). Let $V$ be subquadratic as in the previous proposition. Then there exists $\delta_0 > 0$ such that:
• The action $S(t,x,y)$ is well-defined by (10) for all $0 < |t| < \delta_0$ and satisfies
\[
S(t,x,y) = \frac{1}{2t} |x - y|^2 + tw(t,x,y),
\]
where the term $\omega(t,\cdot,\cdot)$ belongs to $B_2$ uniformly for $|t| \leq \delta_0$. That is, there exist constants $C_k$ such that
\[
|\nabla^k_{x,y}\omega(t,x,y)| \leq C_k (1 + |x| + |y|)^{\max(2-k,0)}
\]
for all $k$.

• For all $0 < |t| < \delta_0$ and all $f \in C_c^\infty(\mathbb{R}^d)$ we have
\[
e^{-itH}f(x) = \frac{1}{(2\pi it)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{iS(t,x,y)}a(t,x,y)f(y)\,dy,
\]
where
\[
\|\nabla^k_{x,y}[a(t,\cdot,\cdot) - 1]\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} = O_k(t^2) \quad \text{for all } k \geq 0.
\]

The above integral representation immediately yields a dispersive estimate:

**Corollary 1** (Dispersive estimate). For $|t| \leq \delta_0$, we have
\[
\|e^{-itH}f\|_{\infty} \lesssim |t|^{-\frac{d}{2}}\|f\|_1.
\]

A pair of exponents is $(q,r)$ admissible if $2 \leq q \leq \infty$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. For an interval $I$, define the Strichartz spaces
\[
S(I) = L^\infty_t L^2_x L^2_{x,y}^{\frac{2d}{d+2}}(I \times \mathbb{R}^d), \quad N(I) = L^1_t L^2_x + L^2_t L^\frac{2d}{d+2} (I \times \mathbb{R}^d).
\]

By interpolation, the $S$ norm controls $\|u\|_{L^1_t L^r_x}$ for all admissible pairs $(q,r)$, while the $N$ norm is controlled by the dual $(q',r')$ of any admissible exponents.

**Lemma 2.2** (Strichartz [15]). Let $I$ be a compact time interval containing $t_0$, and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to the inhomogeneous Schrödinger equation
\[
(i\partial_t - H)u = F.
\]

Then there is a constant $C$, depending only on the length of the interval $I$, such that
\[
\|u\|_{S(I)} \leq C (\|u_0\|_{L^2} + \|F\|_{N(I)}).
\]

**Proof.** This follows from the abstract Keel-Tao theorem [15] as a consequence of the dispersive estimate of the previous corollary, and the unitarity of $e^{-itH}$ on $L^2(\mathbb{R}^d)$.

As $V$ is nonnegative, we have access to the following spectral multiplier theorem of Hebisch [10]:

**Theorem 2.3.** If $F : (0,\infty) \to \mathbb{C}$ is a bounded function which obeys the derivative estimates
\[
|\partial^k F(\lambda)| \lesssim_k |\lambda|^{-k} \quad \text{for all } 0 \leq k \leq \frac{d}{2} + 1,
\]
then the operator $F(H)$, defined initially on $L^2$ by the Borel functional calculus, is bounded from $L^p$ to $L^p$ for all $1 < p < \infty$.

The following norm equivalence was first proven for the quadratic potential by Killip-Visan-Zhang [23, Lemma 2.7]. Using the coercivity hypothesis 3, we adapt their result to the potentials considered here.
Proposition 2 (Equivalence of norms). For any $1 < p < \infty$ and $s \in [0, 1]$, we have
\[ \|H^s f\|_p \sim_{p,s} \|(-\Delta)^s f\|_p + \|V^s f\|_p \]
for all Schwartz functions $f$.

To prove this we shall need the following fact, which is classical when $V$ is exactly quadratic; we give a proof for the sake of completeness.

Lemma 2.4. Let $H = -\frac{1}{2}\Delta + V$ where $V \geq 0$ is smooth and satisfies the hypotheses 2, 3. Then the space of smooth vectors for $H$ is precisely Schwartz class:
\[ D(H^\infty) := \bigcap_{n \geq 0} D(H^n) = S(\mathbb{R}^d). \]

Proof of Proposition 2. We show first that
\[ \|(-\Delta)^s f\|_p + \|V^s f\|_p \lesssim_p \|H^s f\|_p \quad \text{for all} \quad f \in S(\mathbb{R}^d). \]  
(11)

As $f = H^{-s} H^s f$ and $H^s f \in S(\mathbb{R}^d)$ by Lemma 2.4, it suffices to prove
\[ \|(-\Delta)^s H^{-s} f\|_p + \|V^s H^{-s} f\|_p \lesssim_p \|f\|_p \quad \text{for all} \quad f \in S(\mathbb{R}^d). \]  
(12)

By hypothesis, there is some $\delta > 0$ such that $V(x) \geq \delta |x|^2$. Killip-Visan-Zhang [23] proved that
\[ \|V^s H^{-s} f\|_p \lesssim_p \|f\|_p, \]
where $H_\delta = -\frac{1}{2} \Delta + \delta |x|^2$. On the other hand, the parabolic maximum principle implies
\[ 0 \leq e^{-tH}(x, y) \leq e^{-tH_\delta}(x, y) \]
Combining this with the identity
\[ H^{-s}(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-tH}(x, y) t^{s-1} dt, \]
we obtain the kernel inequality
\[ 0 \leq H^{-s}(x, y) \leq H^{-s}_\delta(x, y) \]
In particular, $V^s H^{-s}$ and $V^s H^{-s}_\delta$ have nonnegative integral kernels. We may therefore bound
\[ \|V^s H^{-s} f\|_p \leq \|V^s H^{-s}_\delta f\|_p \leq \|V^s H^{-s}_\delta f\|_p \lesssim_p \|f\|_p. \]
This yields half of (12). Specializing to the case $s = 1$ and writing $-\Delta = 2(H - V)$, we obtain
\[ \|(-\Delta)^{-1} f\|_p \lesssim_p \|f\|_p. \]  
(13)

The rest of the argument is imported directly from [23], and is included to make the discussion self-contained. To show that $\|(-\Delta)^s H^{-s} f\|_p \lesssim_p \|f\|_p$ for all $s \in [0, 1]$, we use analytic interpolation. It suffices to verify that
\[ \|(-\Delta)^{s'} H^{-s} f, g\|_{L^2} \leq C\|f\|_p \|g\|_{p'} \]
for all Schwartz $f$ and $g$. By homogeneity, we may assume $\|f\|_p = \|g\|_{p'} = 1$. Put
\[ F(z) = \langle (-\Delta)^s H^{-s} f, g \rangle_{L^2}. \]
By the spectral theorem, $F(z)$ is bounded and continuous on the closed strip $\{0 \leq \Re(z) \leq 1\}$ and analytic on its interior. By the special case (13) and Theorem 2.3,
\[ |F(it)| \leq \|((-\Delta)^s H^{-s} f, g)_{L^2} \| \leq C_0 \]
\[ |F(1 + it)| \leq \|((-\Delta)^s H^{-s} f, g)_{L^2} \| \leq C_1. \]
Hadamard’s three-lines lemma implies that $|F(z)| \leq C$ on the whole strip. Thus (11) and (12) hold for all $p \in (1, \infty)$ and $s \in [0, 1]$.

Dualizing those estimates yields

$$||H^{-s}(-\Delta)^s f||_p + ||H^{-s}V^s f||_p \lesssim_{p,s} ||f||_p$$  for all $p \in (1, \infty)$, $s \in [0, 1]$. Writing $H^s f = H^{s-1} H f = \frac{1}{2} H^s(-\Delta)^{1-s}(-\Delta)^s f + H^{s-1} V^s V^s f$, we have

$$||H^s f||_p \lesssim_{p,s} ||(-\Delta)^s f||_p + ||V^s f||_p$$  for all $p \in (1, \infty)$, $s \in [0, 1]$. This completes the proof of the proposition modulo the lemma. □

**Proof of Lemma 2.4.** The inclusion $S(\mathbb{R}^d) \subset D(H^\infty)$ is clear. To prove the opposite inclusion, we show by an induction argument the equivalent assertion that

$$D(H^\infty) \subset \bigcap_{k \geq 0} \{u : x^\alpha \partial^\beta u \in L^2 \text{ for all } |\alpha| + |\beta| \leq k\}. \quad (14)$$

We have the following identities:

$$H \partial_j u = \partial_j Hu - (\partial_j V)u$$

$$H mu = m Hu - \frac{1}{2}(\Delta m) u - \nabla m \cdot \nabla u \quad (15)$$

Define for each $n \geq 1$ the following statements:

$$P_1(n) = "m : D(H^{n-1}) \to D(H^{n-1})\text{ for all } m \in B^n"$$

$$P_2(n) = "\partial_j : D(H^n) \to D(H^{n-1})"$$

$$P_3(n) = "\partial_j V : D(H^n) \to D(H^{n-1})".$$ As $D(H) \subset D(H^{1/2}) = \{u : \|\nabla u\|_{L^2} + \|x u\|_{L^2} < \infty\}$, these hold for $n = 1$.

Assume that they hold for some $n$. For $u \in D(H^n)$ and $m \in B$, use (15) and the statements $P_1(n)$, $P_2(n)$ to see that $H(mu) \in D(H^{n-1})$, so $mu \in D(H^n)$ and $P_1(n + 1)$ holds since $m$ was chosen arbitrarily in $B$. Similar reasoning shows that $P_2(n)$ and $P_3(n)$ imply $P_2(n + 1)$, and that $P_1(n)$, $P_2(n)$, $P_3(n)$ yield $P_3(n + 1)$. Hence, by induction these statements hold for all $n \geq 1$.

Next, apply (12) in the special case $s = 1$, $p = 2$ to see that

$$V : D(H) \to D(H^0) = L^2.$$ Suppose $u \in D(H^n)$ and $n \geq 2$. We have

$$H(V u) = VH u - \frac{1}{2}(\Delta V) u - \nabla V \cdot \nabla u.$$ By induction, $V H u \in D(H^{n-2})$, while $P_1(n)$, $P_2(n)$, and $P_3(n-1)$ imply that the second and third terms also belong to $D(H^{n-2})$. Thus $V u \in D(H^{n-1})$

Summing up, we find that

$$V : D(H^n) \to D(H^{n-1})$$  for all $n \geq 1$. These mapping properties, together with the coercivity hypothesis 3, immediately yield the claim (14). □

Thanks to this norm equivalence, $H^\gamma$ inherits many properties of the fractional derivative $(-\Delta)^\gamma$, including Sobolev embedding:

**Lemma 2.5** ([23, Lemma 2.8]). Suppose $\gamma \in [0, 1]$ and $1 < p < \frac{d}{2\gamma}$, and define $p^*$ by $\frac{1}{p^*} = \frac{1}{p} - \frac{2\gamma}{d}$. Then

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \lesssim \|H^\gamma f\|_{L^p(\mathbb{R}^d)}.$$ Similarly, the fractional chain and product rules carry over to the current setting:
Corollary 2 ([23, Proposition 2.10]). Let \( F(z) = \lvert z \rvert ^{\frac{1}{\gamma}} z \). For any \( 0 \leq \gamma \leq \frac{1}{p} \) and \( 1 < p < \infty \),
\[
\| H^\gamma F(u) \|_{L^p(\mathbb{R}^d)} \lesssim \| F'(u) \|_{L^{p_0}(\mathbb{R}^d)} \| H^\gamma f \|_{L^{p_1}(\mathbb{R}^d)}
\]
for all \( p_0, p_1 \in (1, \infty) \) with \( p^{-1} = p_0^{-1} + p_1^{-1} \).

Using Proposition 2 and the Christ-Weinstein fractional product rule for \((-\Delta)^\gamma\) (e.g. [26]), we obtain

Corollary 3. For \( \gamma \in (0,1], \ r, p_i, q_i \in (1, \infty) \) with \( r^{-1} = p_i^{-1} + q_i^{-1}, \ i = 1, 2, \) we have
\[
\| H^\gamma (fg) \|_r \lesssim \| H^\gamma f \|_{p_1} \| g \|_{q_1} + \| f \|_{p_2} \| H^\gamma g \|_{q_2}.
\]

2.2. FIO technology. We review some properties of Fourier integral operators tailored to the Schrödinger equation, which were developed by Fujiwara [7] and Asada-Fujiwara [1].

Definition 2.6. A phase function is a smooth \( \phi(x,y) \in \mathcal{B}_2(\mathbb{R}_x^d \times \mathbb{R}_y^d) \) which satisfies the nondegeneracy condition
\[
\inf_{x,y} \lvert \det \nabla^2_{xy} \phi(x,y) \rvert > 0. \tag{16}
\]

Given a phase \( \phi(x,y) \) and an amplitude \( a(x,y) \in \mathcal{B}(\mathbb{R}_x^d \times \mathbb{R}_y^d) \), define for each \( \lambda \neq 0 \) the integral operator
\[
A(\lambda) f(x) = \left( \frac{\lambda}{2\pi i} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\lambda \phi(x,y)} a(x,y) f(y) dy. \tag{17}
\]

Note that in this notation, which we have carried over from [1], \( \lambda \) plays the role of frequency or equivalently the inverse of the semiclassical parameter.

Remark 2. Asada and Fujiwara studied more general oscillatory integral operators of the form
\[
f \mapsto \left( \frac{\lambda}{2\pi i} \right)^{\frac{d+n}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i\phi(x,\theta,y)} a(x,\theta,y) f(y) dy d\theta
\]
where \( a(x,\theta,y) \in \mathcal{B}(\mathbb{R}_x^d \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n) \) and the phase \( \phi \) satisfies a nondegeneracy condition
\[
\left\lvert \det \begin{pmatrix} \nabla^2_{xy} \phi & \nabla^2_{x\theta} \phi \\ \nabla^2_{\theta y} \phi & \nabla^2_{\theta \theta} \phi \end{pmatrix} \right\rvert \geq \delta
\]

Theorem 2.7 (Fujiwara [7]). \( \| A(\lambda) \|_{L^2 \to L^2} \leq C \| a \|_{C^{d+1}} \)

Let \( \phi \) be a phase function. By the global inverse function theorem, the maps
\[
\chi_1(x,y) = (y,-\partial_y \phi) \quad \text{and} \quad \chi_2(x,y) = (x,\partial_x \phi)
\]
are diffeomorphisms of \( \mathbb{R}^d \times \mathbb{R}^d \). It follows that the relation
\[
(y,-\partial_y \phi) \mapsto (x,y) \mapsto (x,\partial_x \phi)
\]
defines a diffeomorphism
\[
\chi = \chi_2 \circ \chi_1^{-1} : \mathbb{R}_y^d \times \mathbb{R}_\eta^d \to \mathbb{R}_x^d \times \mathbb{R}_\xi^d,
\]
which preserves the standard symplectic form \( d\xi \wedge dx \). The map
\[
\chi(y,\eta) = (x(y,\eta),\xi(y,\eta))
\]
is the canonical transformation generated by the phase function \( \phi(x,y) \).
Proposition 4 (Stability). Let \( \phi \) be an approximate solution to \( (1) \) in the sense that
\[
i\partial_t \tilde{u} = H u \pm \tilde{u}^{\frac{d}{2}} \tilde{u} + e
\]
for some function \( e \). Assume that
\[
\|\tilde{u}\|_{L^{\frac{d}{d+2}}_{t,x}} \leq L, \quad \|H^\frac{1}{2} u\|_{L^\infty_t L^2_x} \leq E,
\]
(18)
and that for some \(0 < \varepsilon < \varepsilon_0(E, L)\) one has
\[
\|H^{1/2}(\tilde{u}(t_0) - u_0)\|_{L^2} + \|\dot{H}^{1/2}e\|_{N(I)} \leq \varepsilon. \tag{19}
\]
Then there exists a unique solution \(u : I \times \mathbb{R}^d \to \mathbb{C}\) to (1) with \(u(t_0) = u_0\) and which further satisfies the estimates
\[
\|\tilde{u} - u\|_{\frac{2(4+2)}{L_1, x}} + \|\dot{H}^{1/2}(\tilde{u} - u)\|_{S(I)} \lesssim C(E, L)e^c \tag{20}
\]
where \(0 < c = c(d) < 1\) and \(C(E, L)\) is a function which is nondecreasing in each variable.

4. Concentration compactness. Let \(0 \in I\) be a compact interval so that \(|I| \leq \delta_0\), where \(\delta_0\) is the constant in Theorem 2.1. As a basic building block in our analysis, we need suitable profile decompositions for the linear and nonlinear equations. The discussion here focuses on the linear case which already contains most of the subtleties. In view of the perturbative theory in Section 3, we seek to characterize initial data with nontrivial linear evolutions, i.e. which come close to saturating the Strichartz inequality
\[
\|e^{-itH}f\|_{\frac{2(4+2)}{L_1, x}}(I \times \mathbb{R}^d) \lesssim \|H^{1/2}f\|_{L^2}.
\]
A substantial part of our previous work on the harmonic oscillator was devoted to constructing profile decompositions for \(H = -\frac{1}{2}\Delta + \frac{1}{2}|x|^2\). We closely follow that exposition but highlight a key technical difference in the present setting. As alluded to in the introduction, we must compare the linear evolutions of a highly concentrated initial state under the propagators \(e^{-itH}\) and \(e^{it\Delta}\) with and without a potential, respectively (see Proposition 7 below). For the harmonic oscillator we relied on the Mehler formula to write
\[
e^{-itH} = m_t(x)e^{i\sin(t)\Delta} - m_t(x)
\]
where \(m_t(x) = \exp(i\frac{\sin(t)\Delta}{2\sin(t)})x^2\), which clearly manifests the relation between the two propagators. Here, we shall instead appeal to the general parametrix in Theorem (2.1) and apply the estimates from Section 2.2.

**Definition 4.1.** A frame is a sequence \((t_n, x_n, N_n) \in I \times \mathbb{R}^d \times 2^\mathbb{N}\) conforming to one of the following scenarios:

1. \(N_n \equiv 1, t_n \equiv 0, x_n \equiv 0\).
2. \(N_n \to \infty\) and \(N_n^{-1}V(x_n)^{1/2} \to r_\infty \in [0, \infty)\).

**Remark 3.** The quantity \(N_n^{-1}V(x_n)^{1/2}\) is the analog of the ratio \(N_n^{-1}|x_n|\) that was considered in [14].

These parameters will specify the temporal center, spatial center, and (inverse) length scale of a function. The hypothesis that \(V\) grows essentially quadratically ensures that \(|x_n| \lesssim N_n\), which reflects the fact that we only consider functions obeying some uniform bound in \(Q(H)\), and such functions cannot be centered arbitrarily far from the origin. We need to augment the frame \(\{(t_n, x_n, N_n)\}\) with an auxiliary parameter \(N_n^\prime\), which corresponds to a sequence of spatial cutoffs adapted to the frame.

**Definition 4.2.** An augmented frame is a sequence \((t_n, x_n, N_n, N_n^\prime) \in I \times \mathbb{R}^d \times 2^\mathbb{N} \times \mathbb{R}\) belonging to one of the following types:

1. \(N_n \equiv 1, t_n \equiv 0, x_n \equiv 0, N_n^\prime \equiv 1\).
2. \( N_n \to \infty, \ N_n^{-1} V(x_n)^{1/2} \to r_\infty \in [0, \infty), \) and either
   
   (a) \( N_n' \equiv 1 \) if \( r_\infty > 0, \) or
   
   (b) \( N_n'^{1/2} \leq N_n' \leq N_n, \ N_n^{-1} V(x_n)^{1/2} (N_n/N_n') \to 0, \) and \( N_n/N_n' \to \infty \) if \( r_\infty = 0. \)

Given an augmented frame \( (t_n, x_n, N_n, N_n') \), we define scaling and translation operators on functions of space and of spacetime by

\[
(G_n \phi)(x) = N_n^{-\frac{d-2}{2}} \phi(N_n(x - x_n))
\]

\[
(\tilde{G}_n f)(t, x) = N_n^{-\frac{d-2}{2}} f(N_n(t - t_n), N_n(x - x_n)).
\]

We also define spatial cutoff operators \( S_n \) by

\[
S_n \phi = \begin{cases} 
\phi, & \text{for frames of type 1 (i.e. } N_n \equiv 1), \\
\chi(N_n) \phi, & \text{for frames of type 2 (i.e. } N_n \to \infty),
\end{cases}
\]

where \( \chi \) is a smooth compactly supported function equal to 1 on the ball \( \{|x| \leq 1\} \). The following mapping properties of these operators are elementary:

\[
\lim_{n \to \infty} S_n = I \text{ strongly in } \dot{H}^1 \text{ and in } Q(H), \\
\limsup_{n \to \infty} \|G_n\|_{Q(H) \to Q(H)} < \infty.
\]

The next technical lemma is the counterpart of [14, Lemma 4.2] and is proved in the same manner (in particular we use the equivalence of norms furnished by Proposition 2).

**Lemma 4.3 (Approximation).** Let \((q, r)\) be an admissible pair of exponents with \( 2 \leq r < d \), and let \( \mathcal{F} = \{(t_n, x_n, N_n, N_n')\} \) be an augmented frame of type 2.

1. Suppose \( \mathcal{F} \) is of type 2a in Definition 4.2. Then for \( \{f_n\} \subseteq L^q_1 H^r_x(R \times R^d) \), we have

\[
\limsup_n \|H^{1/2} \tilde{G}_n S_n f_n\|_{L^q_1 L^r_x} \lesssim \limsup_n \|f_n\|_{L^q_1 H^r_x}.
\]

2. Suppose \( \mathcal{F} \) is of type 2b and \( f_n \in L^q_1 \dot{H}^{1, r}_x(R \times R^d) \). Then

\[
\limsup_n \|H^{1/2} \tilde{G}_n S_n f_n\|_{L^q_1 L^r_x} \lesssim \limsup_n \|f_n\|_{L^q_1 \dot{H}^{1, r}_x}.
\]

Here \( H^r_x(R^d) \) and \( \dot{H}^{1, r}_x(R^d) \) denote the inhomogeneous and homogeneous Sobolev spaces, respectively, equipped with the norms

\[
\|f\|_{H^r_x} = \|\langle \nabla \rangle f\|_{L^r_x(R^d)}, \quad \|f\|_{\dot{H}^{1, r}_x} = \|\langle \nabla \rangle f\|_{L^r_x(R^d)}.
\]

We come to the main results of this section.

**Proposition 5 (Inverse Strichartz).** Let \( I \) be a compact interval containing 0 of length at most \( \delta_0 \) and suppose \( f_n \) is a sequence of functions in \( Q(H) \) satisfying

\[
0 < \varepsilon \leq \|e^{-itH} f_n\|_{L^{2(q+2n)}_{t,x} (R \times R^d)} \lesssim \|H^{1/2}f_n\|_{L^2} \leq A < \infty.
\]

Then, after passing to a subsequence, there exists an augmented frame

\[
\mathcal{F} = \{(t_n, x_n, N_n, N_n')\}
\]

and a sequence of functions \( \phi_n \in Q(H) \) such that one of the following holds:

1. \( \mathcal{F} \) is of type 1 (i.e. \( N_n \equiv 1 \)) and \( \phi_n = \phi \) where \( \phi \in Q(H) \) is a weak limit of \( f_n \) in \( Q(H) \).
The functions $\phi_n$ have the following properties:

\begin{align}
\liminf_{n \to \infty} \|H^{1/2} \phi_n\|_{L^2} & \geq A \left( \frac{x}{N} \right)^{\frac{d(d+2)}{8}} \\
\lim_{n \to \infty} \|f_n\|_{L^{\frac{2d}{d+2}}} & - \|f_n - \phi_n\|_{L^{\frac{2d}{d+2}}} - \|\phi_n\|_{L^{\frac{2d}{d+2}}} = 0. \\
\lim_{n \to \infty} \|H^{1/2} f_n\|_{L^2}^2 & - \|H^{1/2}(f_n - \phi_n)\|_{L^2}^2 - \|H^{1/2} \phi_n\|_{L^2}^2 = 0
\end{align}

We recall that the proof of the analogous result in [14, Section 4.1] used the following ingredients:

- Littlewood-Paley theory adapted to the operator $H = -\frac{1}{2} \Delta + \frac{1}{2} |x|^2$, which depended on a spectral multiplier theorem (Theorem 2.3).
- A refined Strichartz inequality, proved using the Littlewood-Paley theory.
- Convergence properties of equivalent and orthogonal frames, in particular, the comparison of the linear flows generated by the Hamiltonians for the free particle and the harmonic oscillator, when acting on concentrated initial data.

It was here that we invoked the Mehler formula (7).

Once suitable analogues for these components are obtained, the rest of the proof carries over without difficulty. Adapting the first two to our situation requires little more than replacing all instances of $\frac{1}{2} |x|^2$ in the proofs with $V$. Consequently, we shall describe the main steps and refer to [14] for the details of various lemmas. The third requires elaboration, however, and will be the subject of the next section.

**Proof.** As in the case of the harmonic oscillator, one first develops a Littlewood-Paley theory based on the spectral theorem. We briefly sketch the (fairly standard) construction as follows.

Choose $\psi \in C_0^\infty((0, \infty))$ so that $1 = \sum_N \text{dyadic } \psi(\lambda/N)$, and define the spectral multipliers

$$P_N := \psi(\sqrt{H}/N), \quad P_{\leq N} := \sum_{M \leq N} P_N$$

$$\tilde{P}_N := e^{-H/2N^2} - e^{-2H/N^2}, \quad \tilde{P}_{\leq N} := \sum_{M \leq N} \tilde{P}_N.$$ 

The multipliers based on the heat kernel are more convenient at some junctures since the parabolic maximum principle implies the pointwise bound $0 \leq e^{-tH}(x, y) \leq e^{\frac{t^2}{4}}(x, y)$. Theorem 2.3 implies that $\|P_{\leq N} f\|_{L^p} \lesssim \|\tilde{P}_{\leq N} f\|_{L^p}$ for all $1 < p < \infty$. As in the usual Littlewood-Paley theory, one has Bernstein and square function estimates

$$\|\tilde{P}_{\leq N}\|_{L^p \to L^q} \lesssim N^\frac{q}{2} - \frac{d}{p} \quad \text{for all } 1 \leq p \leq q \leq \infty$$

$$\|f\|_{L^p} \sim \|P_N f\|_{L^p_{x,y}} \quad \text{for all } 1 < p < \infty.$$

Using the square function and Bernstein estimates, one deduces as in [14, Proposition 4.1] a preliminary refined Strichartz estimate

$$\|e^{-itH} f\|_{L^p_{x,y}} \lesssim \|f\|_{L^p_{x,y}} \sup_N \|e^{-itH} P_N f\|_{L^2_{x,y}}$$

2. $\mathcal{F}$ is of type 2, either $t_n \equiv 0$ or $N_2^n t_n \to \pm \infty$, and $\phi_n = e^{it_n H} G_n S_n \phi$ where $\phi \in H^1(\mathbb{R}^d)$ is a weak limit of $G_n^{-1} e^{-it_n H} f_n$ in $H^1$. Moreover, if $\mathcal{F}$ is of type 2a, then $\phi$ also belongs to $L^2(\mathbb{R}^d)$. 

The multipliers based on the heat kernel are more convenient at some junctures since the parabolic maximum principle implies the pointwise bound $0 \leq e^{-tH}(x, y) \leq e^{\frac{t^2}{4}}(x, y)$.

As in the usual Littlewood-Paley theory, one has Bernstein and square function estimates, which depend on a spectral multiplier theorem (Theorem 2.3). Adapting the first two to our situation requires little more than replacing all instances of $\frac{1}{2} |x|^2$ in the proofs with $V$. Consequently, we shall describe the main steps and refer to [14] for the details of various lemmas. The third requires elaboration, however, and will be the subject of the next section.
Under the hypotheses of the Proposition, we find for each \(n\) some frequency \(N_n\) such that
\[
\left\| \hat{P}_{N_n} e^{-itH} f_n \right\|_{L_{t,x}^{2(d+2)}} \gtrsim \left\| P_{N_n} e^{-itH} f_n \right\|_{L_{t,x}^{2(d+2)}} \gtrsim \varepsilon^{\frac{d+2}{2}} A^{-\frac{d+2}{8}}.
\]
By interpolating \(L^{2(d+2)}\) between \(L^{2(d+2)}\) and \(L^\infty\), and using the Strichartz and Bernstein estimates for the former, we find a sequence of times \(t_n \in I\) and positions \(x_n \in \mathbb{R}^d\) such that
\[
|e^{-it_n H} \hat{P}_{N_n} f_n(x_n)| \gtrsim N_n^{\frac{d-2}{2}} A \left( \frac{\varepsilon}{A} \right)^{\frac{d(d+2)}{8}}.
\]
(27)
On the other hand, the spatial weight in the space \(\Sigma\), together with pointwise heat kernel bounds, imply that
\[
|x_n| \leq C_{\Lambda, \varepsilon} N_n;
\]
see Lemma 4.4 in [14].

Passing to a subsequence, we may assume that either \(N_n \equiv N_\infty \in [1, \infty)\) or \(N_n \to \infty\). The profiles \(\phi_n \in \Sigma\) are constructed according to this dichotomy.

**Case 1.** \(N_n \equiv N_\infty\). In view of the bound (28), after passing to a further subsequence we may assume that \(x_n \to x_\infty \in \mathbb{R}^d\), \(t_n \to t_\infty \in I\), and that \(f_n\) converges weakly in \(\Sigma\) to some \(\phi\).

As in [14], we set \(\phi_n \equiv \phi \in \Sigma\). The energy lower bound (24) is obtained by pairing \(\phi\) with \(e^{it_\infty H} \hat{P}_{N_\infty} \delta_{x_\infty}\). On one hand,
\[
\langle \phi, e^{it_\infty H} \hat{P}_{N_\infty} \delta_{x_\infty} \rangle = \lim_{n \to \infty} \langle f_n, e^{it_\infty H} \hat{P}_{N_\infty} \delta_{x_\infty} \rangle
\]
\[
= \lim_{n \to \infty} e^{-it_\infty H} \hat{P}_{N_n} f_n(x_n) + o_{n \to \infty}(1)
\]
is bounded below by (27), while on the other hand, Hölder and the heat kernel bounds imply that
\[
|\langle \phi, e^{it_\infty H} \hat{P}_{N_\infty} \delta_{x_\infty} \rangle| \leq \left\| e^{-it_\infty H} \phi \right\|_{L_{t,x}^{2(d+2)}} \left\| \hat{P}_{N_\infty} \delta_{x_\infty} \right\|_{L_{t,x}^{2(d+2)}} \lesssim \left\| \phi \right\|_{\Sigma} N_\infty^{\frac{d-2}{2}}.
\]
The decoupling (26) follows from standard Hilbert space arguments. Finally, the decoupling in the \(L^{\frac{d(d+2)}{8}}\) norm can be deduced from the compactness of the embedding \(\Sigma \subset L^2\) and the Brezis-Lieb refined Fatou lemma:

**Lemma 4.4** (Refined Fatou [3]). Fix \(1 \leq p \leq \infty\), and suppose \(f_n\) is a sequence of functions in \(L^p(\mathbb{R}^d)\) such that \(\sup_n \|f_n\|_p < \infty\) and \(f_n \to f\) pointwise. Then
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \left| |f_n|^p - |f_n - f|^p - |f|^p \right| \, dx = 0.
\]

The parameters in this case \((t_n, x_n, N_n) \equiv (0, 0, 1)\) constitute a “type 1” frame.

**Case 2.** \(N_n \to \infty\). After passing to a subsequence, we define \(\phi \in H^1(\mathbb{R}^d)\) as the weak limit,
\[
N_n^{-\frac{d+2}{2}} (e^{-it_n H} f_n)(x_n + N_n^{-1}) \rightharpoonup \phi \quad \text{in} \quad H^1(\mathbb{R}^d),
\]
and in view of the bound (28) the quantities
\[
r_\infty := \lim_{n \to \infty} N_n^{-1} V(x_n)^{1/2} < \infty \quad \text{and} \quad t_\infty := \lim_{n \to \infty} N_n^2 t_n \in [-\infty, \infty]
\]
are well-defined. As in the case of the exact harmonic oscillator, the reason we distinguish between the cases \(r_\infty = 0\) ("type 2a") and \(r_\infty > 0\) ("type 2b") lies in the following
Lemma 4.5 ([14, Lemma 4.5]). If \( r_\infty > 0 \), then the weak limit \( \phi \) also belongs to \( L^2 \).

With the cutoff parameter \( N'_n \) defined in 2b, we set
\[
\phi_n := e^{it_nH}G_nS_n\phi = e^{it_nH}[N_n^{\pm2} \phi(N_n(-x_n))\chi(N'_n(-x_n))].
\]

The properties (24), (26) are verified much as in the case of the harmonic oscillator (see [14, Lemma 4.7]). Similarly, one obtains the decoupling (25) of \( L^{2d/\alpha} \) norms in the case \( t_\infty = \pm \infty \) by using a density argument and the \( L^{2d/\alpha} \to L^{2d/\alpha} \) dispersive estimate for \( e^{it_nH} \) to show that \( \lim_{n \to \infty} \| \phi_n \|_{L^{2d/\alpha}} = 0 \).

It remains to prove
\[
\lim_{n \to \infty} \| f_n \|^2_{L^{2d/\alpha}} - \| f_n - e^{it_nH}G_nS_n\phi \|^2_{L^{2d/\alpha}} - \| e^{it_nH}G_nS_n\phi \|^2_{L^{2d/\alpha}} = 0 \tag{30}
\]
when \( t_\infty \in (-\infty, \infty) \). We demonstrate here how it follows from the lemmas of the next section, which constitute the main technical improvement of this note compared to our earlier work on the harmonic oscillator.

Applying Corollary 5 below with the equivalent frames \( (t_n, x_n, N_n), (0, x_n, N_n) \), we have
\[
\| e^{it_nH}G_nS_n\phi - G_nS_nU_\infty\phi \|_{\Sigma} \to 0,
\]
where \( U_\infty = e^{-it_\infty(\cdot)^2}e^{it_\infty\Delta} \). By Sobolev embedding, the left side of (30) therefore simplifies to
\[
\| f_n \|^2_{L^{2d/\alpha}} - \| f_n - G_nS_nU_\infty\phi \|^2_{L^{2d/\alpha}} - \| G_nS_nU_\infty\phi \|^2_{L^{2d/\alpha}} = \| f_n \|^2_{L^{2d/\alpha}} - \| G_n^{-1}f_n - U_\infty\phi \|^2_{L^{2d/\alpha}} - \| U_\infty\phi \|^2_{L^{2d/\alpha}} + o(1).
\]

On the other hand, for any test function \( \psi \) one has, in view of Corollary 5 and Lemma 4.9,
\[
\langle G_n^{-1}f_n - U_\infty\phi, U_\infty\psi \rangle_{\hat{H}^1} = \langle f_n - G_nS_nU_\infty\phi, e^{it_nH}G_nS_n\psi \rangle_{\hat{H}^1} + o(1) = \langle f_n - e^{it_nH}G_nS_n\phi, e^{it_nH}G_nS_n\psi \rangle_{\hat{H}^1} + o(1) = \langle G_n^{-1}e^{-it_nH}f_n - \phi, \psi \rangle_{\hat{H}^1} + o(1) = o(1).
\]

Therefore \( G_n^{-1}f_n \) converges weakly in \( \hat{H}^1 \) to \( U_\infty\phi \). As \( \hat{H}^1 \) embeds compactly in \( L^2(K) \) for any compact \( K \subset \mathbb{R}^d \), after passing to a suitable subsequence we see that \( G_n^{-1}f_n \to \phi \) pointwise. The claim (30) then follows from the refined Fatou lemma.

\[\square\]

Proposition 6 (Linear profile decomposition). Let \( 0 \in I \) be an interval with \( |I| \leq \delta_0 \), and let \( f_n \) be a bounded sequence in \( Q(\hat{H}) \). After passing to a subsequence, there exists \( J^* \in \{0,1,\ldots\} \cup \{\infty\} \) such that for each finite \( 1 \leq j \leq J^* \), there exist an augmented frame \( \mathcal{F} = \{(t_n^{(j)}, x_n^{(j)}, N_n^{(j)}, N'_n^{(j)})\} \) and a function \( \hat{\phi}^j \) with the following properties.

- Either \( t_n^{(j)}_n \equiv 0 \) or \( (N_n^{(j)})^2(t_n^{(j)}) \to \pm \infty \) as \( n \to \infty \).
- \( \hat{\phi}^j \) belongs to \( Q(\hat{H}), \hat{H}^1, \) or \( \hat{H}^1 \) depending on whether \( \mathcal{F} \) is of type 1, 2a, or 2b, respectively.
For each finite \( J \leq J^* \), we have a decomposition

\[
f_n = \sum_{j=1}^{J} e^{it_n^j H} G_n^j S_n^j \phi^j + r_n^J,
\]

where \( G_n^j \), \( S_n^j \) are the \( \dot{H}^1 \)-isometry and spatial cutoff operators associated to \( F^j \). Writing \( \phi^j_n \) for \( e^{it_n^j H} G_n^j S_n^j \phi^j \), this decomposition has the following properties:

\[
(G_n^j)^{-1} e^{-it_n^j H} r_n^J \to 0 \quad \text{for all } J \leq J^*,
\]

\[
\sup \lim_{J \to \infty} \left\| H^{1/2} f_n \right\|_{L^2}^2 - \sum_{j=1}^{J} \left\| H^{1/2} \phi_n^j \right\|_{L^2}^2 - \left\| H^{1/2} r_n^J \right\|_{L^2}^2 = 0,
\]

\[
\sup \lim_{J \to \infty} \left\| f_n \right\|_{L^\infty}^{2(d+2)} - \sum_{j=1}^{J} \left\| \phi_n^j \right\|_{L^\infty}^{2(d+2)} - \left\| r_n^J \right\|_{L^\infty}^{2(d+2)} = 0.
\]

Whenever \( j \neq k \), the frames \( \{(t_n^j, x_n^j, N_n^j)\} \) and \( \{(t_n^k, x_n^k, N_n^k)\} \) are orthogonal:

\[
\lim_{n \to \infty} N_n^j x_n^j + N_n^j x_n^j - t_n^j + \sqrt{N_n^j N_n^k} |x_n^j - x_n^k| = \infty.
\]

Finally, we have

\[
\lim_{J \to J^*} \limsup_{n \to \infty} \left\| e^{-it_n^J H} r_n^J \right\|_{L^\infty}^{2(d+2)} = 0,
\]

Proof sketch. The argument is completely analogous to the one in [14, Proposition 4.14]. One inductively applies inverse Strichartz to extract the frames \( F^j \) and profiles \( \phi^j \), and deduces the orthogonality of frames (35) via the lemmas discussed in the next section.

4.1. Convergence of linear propagators. This section’s main result is Proposition 7, which compares the linear propagators \( e^{it_n^j H} \phi \) and \( e^{it_n^j (\frac{1}{2}-V)} \phi \) for highly concentrated \( \phi \). While the proposition is simply a translation of [14, Lemma 4.8], its proof is more involved and requires a closer study of the underlying classical dynamics.

**Definition 4.6.** We say two frames \( F^1 = \{(t_n^1, x_n^1, N_n^1)\} \) and \( F^2 = \{(t_n^2, x_n^2, N_n^2)\} \) (where the superscripts are indices, not exponents) are equivalent if

\[
\frac{N_n^1}{N_n^2} \to R_{\infty} \in (0, \infty), \quad N_n^1(x_n^1 - x_n^2) \to x_{\infty} \in \mathbb{R}^d, \quad (N_n^1)^2 (t_n^1 - t_n^2) \to t_{\infty} \in \mathbb{R}.
\]

If any of the above statements fail, we say that \( F_1 \) and \( F_2 \) are orthogonal. Note that replacing the \( N_n^1 \) in the second and third expressions above by \( N_n^2 \) yields an equivalent definition of orthogonality.

Two augmented frames \( (t_n, x_n, N_n, N_n') \) and \( (\tilde{t}_n, \tilde{x}_n, \tilde{N}_n, \tilde{N}_n') \) are said to be equivalent if their underlying frames \( (t_n, x_n, N_n) \) and \( (\tilde{t}_n, \tilde{x}_n, \tilde{N}_n) \) are equivalent.

**Proposition 7** (Strong convergence). Suppose \( F^M = (t_n^M, x_n, M_n) \) and \( F^N = (t_n^N, y_n, N_n) \) are equivalent frames. Define

\[
R_{\infty} = \lim_{n \to \infty} \frac{M_n^1}{N_n^1}, \quad t_{\infty} = \lim_{n \to \infty} M_n^2 (t_n^M - t_n^N),
\]

\[
x_{\infty} = \lim_{n \to \infty} M_n (y_n - x_n), \quad r_{\infty} = \lim_{n \to \infty} M_n^{-1} V(x_n)^{1/2}.
\]

(The last limit exists by the definition of a frame.) Let \( G_n^M, G_n^N \) be the scaling and translation operators associated with the frames \( F^M \) and \( F^N \) respectively. Then
the sequence \( (e^{-it_n^N H}G_n^N)^{-1}e^{-it_n^M H}G_n^M \) converges in the strong operator topology on \( B(\Sigma, \Sigma) \) to the operator \( U_\infty \) defined by

\[
U_\infty \phi = e^{-it_\infty (r_\infty)^2} R_\infty^{-\frac d2} e^{it_\infty \Delta} \phi (R_\infty \cdot + x_\infty).
\]

**Proof.** Write \( (e^{-it_n^N H}G_n^N)^{-1}e^{-it_n^M H}G_n^M = (G_n^N)^{-1}G_n^M (G_n^M)^{-1}G_n^M e^{-it_n^H G_n^M} \) where \( t_n = t_n^M - t_n^N \). As \( (G_n^N)^{-1}G_n^M \) converges strongly to the operator \( f \mapsto R_\infty^{-\frac d2} f(R_\infty \cdot + x_\infty) \), it suffices to show that

\[
(G_n^M)^{-1}e^{-it_n^H G_n^M} \to e^{-it_\infty (r_\infty)^2} e^{it_\infty \Delta}.
\]  

(37)

Recall from Theorem 2.1 that the phase in the Fourier integral formula for \( e^{-itH} \) is the classical action and has the form

\[
S(t, x, y) = \frac{|x-y|^2}{2t} - \int_0^t V(y + (x-y)\tau) d\tau + O(t^3(1 + |x|^2 + |y|^2)).
\]

**Proof.** The system (8) may be written equivalently as

\[
\begin{align*}
\xi(t) &= \eta - \int_0^t \partial_x V(x(\theta)) d\theta, \\
x(t) &= y + \int_0^t \xi(\tau) d\tau = y + t\eta - \int_0^t (t-\theta)\partial_x V(x(\theta)) d\theta.
\end{align*}
\]

(38)

As \( \partial_x V \) grows at most linearly, Gronwall’s inequality implies that for all initial data \( y, \eta \) we have

\[
|x(t)| \leq C(1 + |y| + |t\eta|).
\]

Fix a time \( t > 0 \) and positions \( x, y \in \mathbb{R}^d \). By Proposition 1, there is a unique initial velocity \( \eta = \eta(t, x, y) \) such that the solution \( (x(\tau), \xi(\tau)) \) to (38) satisfies \( x(0) = y \) and \( x(t) = x \).

Refraining to the definition (10) of the action, we estimate the error incurred by replacing the true trajectory by the straight line path from \( y \) to \( x \). Rearranging the above expression for \( x(t) \), we have

\[
\eta = \frac{x-y}{t} - \frac 1t \int_0^t (t-\theta)\partial_x V(x(\theta)) d\theta.
\]

(39)

For \( \tau \) between 0 and \( t \),

\[
|x(\tau)| \leq |y| + \frac{x-y}{t} \tau + C \int_0^t |t-\theta|(1 + |x(\theta)|) d\theta,
\]
hence $|x(\tau)| \leq C(1 + |x| + |y|)$. The preceding computations reveal that
\[
|x(\tau) - y - \tau \frac{x - y}{t}| \leq \frac{\tau}{2} \int_0^t |t - \theta| \partial_x V(x(\theta)) |d\theta + \int_0^T |\tau - \theta| \partial_x V(x(\theta))| d\theta \leq C(\tau t + \tau^2)(1 + |x| + |y|).
\]
By the fundamental theorem of calculus,
\[
\int_0^t |V(x(\tau)) - V(y + \tau \frac{x - y}{t})| d\tau \leq C \int_0^t (\tau t + \tau^2)(1 + |x| + |y|)^2 d\tau \leq C t^3 (1 + |x| + |y|)^2. \tag{40}
\]
Next, by combining the first line of (38) with (39), we find that
\[
\xi(\tau) = \frac{x - y}{t} + \frac{1}{t} \int_0^t (t - \theta) \partial_x V(x(\theta)) d\theta - \int_0^\tau \partial_x V(x(\theta)) d\theta.
\]
It is easy to see that second and third terms are bounded by $O(t(1 + |x| + |y|))$. Therefore,
\[
\int_0^t \frac{1}{2} |\xi(\tau)|^2 d\tau = \frac{|x - y|^2}{2t} + \frac{x - y}{t} \int_0^t (t - \theta) \partial_x V(x(\theta)) d\theta - \frac{|x - y|^2}{2t} + O(t^3 (1 + |x| + |y|)^2)
\]
Combining this with (40) establishes the lemma. \qed

By Theorem 2.1 and a change of variable,
\[
(G_n^M)^{-1} e^{-it\alpha H} G_n^M f(x) = \left(\frac{\lambda_n}{2\pi i}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\lambda_n \phi_n(x, y)} a_n(x, y) f(y) dy, \tag{41}
\]
where
\[
\lambda_n = (M_n^2 t_n)^{-1},
\]
\[
a_n(x, y) = a(t_n, x_n + M_n^{-1} x, x_n + M_n^{-1} y)
\]
\[
\phi_n(x, y) = \frac{1}{2} |x - y|^2 + \lambda_n^{-1} t_n \omega_n(t_n, x_n + M_n^{-1} x, x_n + M_n^{-1} y)
\]
\[
= \phi_0(x, y) + \lambda_n^{-1} t_n \omega_n(x, y).
\]

Theorem 2.1 and Lemma 4.7 imply that these quantities obey the following estimates:
\[
t_n \omega_n(x, y) = - \int_0^{t_n} V(x_n + M_n^{-1} y + \frac{x - y}{M_n^2} \tau) d\tau + O(t_n^3 (|x_n|^2 + M_n^{-2} |x|^2 + M_n^{-2} |y|^2))
\]
\[
= -t_n V_n(x_n) + O(M_n^{-2} (1 + |x|^2 + |y|^2)), \tag{42}
\]
\[
|\nabla_{x,y} \omega_n(x, y)| \lesssim \left\{ \begin{array}{ll}
M_n^{-1} (1 + |x_n + M_n^{-1} x| + |x_n + M_n^{-1} y|), & k = 1 \\
M_n^{-k}, & k \geq 2
\end{array} \right.
\]
\[
|\nabla_{x,y}^k [a_n(x, y) - 1]| \lesssim_k M_n^{-2-k} \text{ for all } k \geq 0.
\]
We need the following adaptation of [8, Proposition 4.15].

**Lemma 4.8.** The operators $(G_n^M)^{-1} e^{-it\alpha H} G_n^M$ are uniformly bounded on $\Sigma$. 
Proof. Let \( \chi_n : (y, -\partial_y \phi_n) \mapsto (x, \partial_x \phi_n) \) be the canonical transformation generated by the phase function \( \phi_n \). In terms of the variables \((y, \eta)\), we have
\[
\chi_n(y, \eta) = (y + \eta, \eta) + \lambda_n^{-1} t_n (\partial_y \omega_n, \partial_x \omega_n + \partial_y \omega_n)(x(t_n, y, \eta), y)
= (y + \eta, \eta) + (r_{1,n}(y, \eta), r_{2,n}(y, \eta)).
\]

First we show that
\[
\| \nabla (G_n^M)^{-1} e^{-it_n H} G_n^M f \|_{L^2} \lesssim \| f \|_{\Sigma}. \tag{43}
\]
Put \( p(x, \theta, y) = \theta \) and \( q_n(x, \theta, y) = \theta + r_{2,n}(y, \theta) \). By construction,
\[
p(x, \xi, x)|_{(x, \xi) = \chi_n(y, \eta)} = q_n(y, \eta, y).
\]
By the representation (41) and Theorem 2.8,
\[
D(G_n^M)^{-1} e^{-it_n H} G_n^M = \text{Op}(\lambda_n p, \lambda_n)(G_n^M)^{-1} e^{-it_n H} G_n^M
= (G_n^M)^{-1} e^{-it_n H} G_n^M \text{Op}(\lambda_n q_n, \lambda_n) + R_n(\lambda_n), \tag{44}
\]
where we write \( D = \frac{1}{2} \nabla \). In light of the estimates (42) and Theorem 2.7, it suffices to obtain a uniform bound
\[
\| \text{Op}(\lambda_n q_n, \lambda_n) \|_{\Sigma \mapsto L^2} \lesssim 1
\]
By definition
\[
\text{Op}(\lambda_n q_n, \lambda_n) f(x) = Df + \text{Op}(\lambda_n r_{2,n}, \lambda_n).
\]
Using (42) and Proposition 1, we see that
\[
\lambda_n r_{2,y}(y, \eta) = t_n (\partial_x \omega_n + \partial_y \omega_n)(t_n, x(t_n, 0, 0), 0)
+ t_n y \int_0^1 (\partial_{xy}^2 \omega_n)(t_n, x(t_n, sy, \eta), sy) \frac{\partial^2 x}{\partial y} + (\partial_y^2 \omega_n)(t_n, x(t_n, sy, \eta), sy) ds
+ t_n \eta \int_0^1 (\partial_{xy}^2 \omega_n)(t_n, x(t_n, sy, \eta), sy) \frac{\partial^2 x}{\partial y} ds
= c_n + y r_{2,n}^{(1)}(y, \eta) + \eta r_{2,n}^{(2)}(y, \eta),
\]
where \( |c_n| \lesssim M_n^{-2} \) and \( \| D^{k} r_{2,n} \|_{L^\infty} \lesssim M_n^{-4}, \| D^{k} r_{2,n} \|_{L^\infty} \lesssim M_n^{-6} \) for all \( k \). Thus
\[
\text{Op}(\lambda_n r_{2,n}, \lambda_n) = c_n I + \text{Op}(y r_{2,n}^{(1)}(y, \eta), \lambda_n) + \text{Op}(\eta r_{2,n}^{(2)}(y, \eta), \lambda_n)
= c_n I + \text{Op}(r_{2,n}^{(1)}(y, \lambda_n) X + \text{Op}(\lambda^{-1} r_{2,n}^{(2)}(y, \lambda_n) D + \text{Op}(\lambda^{-1} (D_{y} r_{2,n}^{(2)}(y, \lambda_n).
\]
The Calderón-Vaillancourt theorem now implies
\[
\| \text{Op}(\lambda_n r_{2,n}, \lambda_n) f \|_{L^2} \lesssim M_n^{-2} \| f \|_{L^2} + M_n^{-4} \| xf \|_{L^2} + M_n^{-6} \| Df \|_{L^2} \lesssim M_n^{-2} \| f \|_{\Sigma}. \tag{43}
\]
Altogether we obtain (43).
By setting \( p(x, \theta, y) = x \), \( q(x, \theta, y) = y + \theta + r_{1,n}(y, \eta) \) and making a similar analysis as above, we obtain
\[
\| x(G_n^M)^{-1} e^{-it_n H} G_n^M f \|_{L^2} \lesssim \| f \|_{\Sigma}.
\]
This concludes the proof of the lemma.
\[\square\]

We now verify the limit (37). As \( e^{i\frac{M_2^2 t_n \Delta}{2}} \to e^{i\frac{\wtilde{w}_n \Delta}{2}} \) strongly, it suffices to show that
\[
(G_n^M)^{-1} e^{-it_n H} G_n^M f - e^{-i\wtilde{w}_n (r_{w})^2} e^{i\frac{M_2^2 t_n \Delta}{2}} f
\]
converges to 0 for all $f \in \Sigma$. By Lemma 4.8 we may assume $f \in C_c^\infty$. The above difference may be written as
\[
\left(\frac{\lambda_n}{2\pi}\right)^{\frac{d}{2}} \int e^{\lambda_n \phi_n} [a_n - 1] f(y) \, dy + \left(\frac{\lambda_n}{2\pi}\right)^{\frac{d}{2}} \int [e^{\lambda_n \phi_n} - e^{-i t \infty (r_n)^2} e^{i \lambda_n \phi_n}] f(y) \, dy = A_n f + B_n f.
\]

Using Theorem 2.7 and the estimates (42), one argues as in the proof of Lemma 4.8 to see that $\|A_n f\|_\Sigma \lesssim M_n^{-2} \|f\|_\Sigma$.

It remains to bound $B_n f$. By hypothesis $f$ is supported in some ball $B(0, R)$, and the estimates (42) show that the integral kernel of $B_n$ converges to 0 in $C^\infty$. It follows that $|x B_n f|$ and $|\nabla B_n f|$ converge to 0 locally uniformly. On the other hand, integration by parts reveals that for all $n$ sufficiently large,
\[
|x B_n f| + |\nabla B_n f| \lesssim N |x|^{-N}
\]
for any $N > 0$ and for all $|x| \geq 4R$. Hence $\|B_n f\|_\Sigma \to 0$ by dominated convergence. This completes the proof of the proposition.

In the remainder of this section we collect other lemmata regarding equivalent and orthogonal frames. They can be proved in much the same manner as their counterparts in [14, Section 4.2].

**Corollary 5.** Let $\{\{t_n^M, x_n, M_n, M'_n\}\}$ and $\{\{t_n^N, y_n, N_n, N'_n\}\}$ be equivalent augmented frames. Let $S_n^M$, $S_n^N$ be the associated spatial cutoff operators. Then
\[
\lim_{n \to \infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma = 0 \tag{45}
\]
and
\[
\lim_{n \to \infty} \|e^{-it_n^M H} G_n^M S_n^M \phi - e^{-it_n^N H} G_n^N S_n^N U_\infty \phi\|_\Sigma = 0 \tag{46}
\]
whenever $\phi \in H^1$ if the frames conform to case 2a and $\phi \in H^1$ if they conform to case 2b in Definition 4.2.

**Proof.** Run an approximation argument using Lemma 4.3 in the manner of [14, Corollary 4.9].

The following “approximate adjoint” identity is the analogue of [14, Lemma 4.10].

**Lemma 4.9.** Suppose the frames $\{\{t_n^M, x_n, M_n\}\}$ and $\{\{t_n^N, y_n, N_n\}\}$ are equivalent. Put $t_n = t_n^M - t_n^N$. Then for $f, g \in \Sigma$ we have
\[
\langle (G_n^N)^{-1} e^{-it_n H} G_n^M f, g \rangle_{H^1} = \langle f, (G_n^M)^{-1} e^{it_n H} G_n^N g \rangle_{H^1} + R_n(f, g),
\]
where $|R_n(f, g)| \leq C|t_n| \|G_n^M f\|_\Sigma \|G_n^N g\|_\Sigma$.

**Proof.** The proof of Lemma 4.8 yields the following commutator estimate:
\[
\|[D, e^{-it H}]\|_{\Sigma \to L^2} = O(t).
\]
We have
\[
\langle D(G_n^N)^{-1} e^{-it_n H} G_n^M f, Dg \rangle_{L^2} = \langle Df, D(G_n^M)^{-1} e^{it_n H} G_n^N g \rangle_{L^2} + R_n(f, g)
\]
where $R_n(f, g) = \langle [D, e^{-it_n H}] G_n^M f, DG_n^N g \rangle_{L^2} - \langle DG_n^M f, [D, e^{it_n H}] G_n^N g \rangle_{L^2}$. The claim then follows from Cauchy-Schwarz and the above estimate.

The next lemma is a converse to Proposition 7.
Lemma 4.10 (Weak convergence). Assume the frames $\mathcal{F}^M = \{(t_n^M, x_n, M_n)\}$ and $\mathcal{F}^N = \{(t_n^N, y_n, N_n)\}$ are orthogonal. Then, for any $f \in \Sigma$, 
\[
(e^{-it_n^M H} G_n^M)^{-1} e^{-it_n^N H} G_n^N f \to 0 \quad \text{weakly in } H^1.
\]
Proof. Put $t_n = t_n^M - t_n^N$, and suppose that $|M_n^2 t_n| \to \infty$. Then 
\[
\|(G_n^N)^{-1} e^{-it_n^N H} G_n^M f\|_{L^2} \to 0
\]
for $f \in C_c^\infty$ by a change of variables and the dispersive estimate, thus for general $f \in \Sigma$ by a density argument. Therefore $(G_n^N)^{-1} e^{-it_n^N H} G_n^M f$ converges weakly in $H^1$ to 0. We consider next the case where $M_n^2 t_n \to t_\infty \in \mathbb{R}$. The orthogonality of $\mathcal{F}^M$ and $\mathcal{F}^N$ implies that either $N_n^{-2} M_n$ converges to 0 or $\infty$, or $M_n |x_n - y_n|$ diverges as $n \to \infty$. In either case, one verifies easily that the operators $(G_n^N)^{-1} G_n^M$ converge to zero in the weak operator topology on $B(H^1, H^1)$. Applying Proposition 7, we see that $(G_n^N)^{-1} e^{-it_n^N H} G_n^M f = (G_n^M)^{-1} G_n^M (G_n^N)^{-1} e^{-it_n^N H} G_n^M f$ converges to zero weakly in $H^1$. \hfill \Box

Corollary 6. Let $\{(t_n^M, x_n, M_n, M'_n)\}$ and $\{(t_n^N, y_n, N_n, N'_n)\}$ be augmented frames such that $\{(t_n^M, x_n, M_n)\}$ and $\{(t_n^N, y_n, N_n)\}$ are orthogonal. Let $G_n^M$, $S_n^M$ and $G_n^N$, $S_n^N$ be the associated operators. Then 
\[
(e^{-it_n^M H} G_n^M)^{-1} e^{-it_n^N H} G_n^M S_n^M \phi \to 0 \quad \text{in } H^1
\]
whenever $\phi \in H^1$ if $\mathcal{F}^M$ is of type 2a and $\phi \in \dot{H}^1$ if $\mathcal{F}^M$ is of type 2b.

Proof. If $\phi \in C_c^\infty$, then $S_n^M \phi = \phi$ for all large $n$, and the claim follows from Lemma 4.10. The case of general $\phi$ in $H^1$ or $\dot{H}^1$ then follows from an approximation argument similar to the one used in the proof of Corollary 5. \hfill \Box

5. The case of concentrated initial data. With the main complications out of the way, we sketch the rest of the wellposedness argument in the remaining two sections. The next step is to rule out blowup for equation (1) when the initial data is highly concentrated in space.

Proposition 8. Let $I = [-\delta_0/2, \delta_0/2]$, where $\delta_0$ is the constant in Theorem 2.1. Assume that Conjecture 1.1 holds. Let 
\[
\mathcal{F} = \{(t_n, x_n, N_n, N'_n)\}
\]
be an augmented frame with $t_n \in I$ and $N_n \to \infty$, such that either $t_n \equiv 0$ or $N_n^2 t_n \to \pm \infty$; that is, $\mathcal{F}$ is type 2a or 2b in Definition 4.2. Let $G_n$, $\tilde{G}_n$, and $S_n$ be the associated operators as defined in (21) and (22). Suppose $\phi$ belongs to $H^1$ or $\dot{H}^1$ depending on whether $\mathcal{F}$ is type 2a or 2b respectively. Then, for $n$ sufficiently large, there is a unique solution $u_n : I \times \mathbb{R}^d \to C$ to the defocusing equation (1), $\mu = 1$, with initial data 
\[
u u_n(0) = e^{it_n H} G_n S_n \phi.
\]
This solution satisfies a spacetime bound 
\[
\limsup_{n \to \infty} S_I(u_n) \leq C(E(u_n)).
\]
Suppose in addition that $\{(q_k, r_k)\}$ is any finite collection of admissible pairs with $2 < r_k < d$. Then for each $\varepsilon > 0$ there exists $\psi^\varepsilon \in C_c^\infty(\mathbb{R} \times \mathbb{R}^d)$ such that 
\[
\limsup_{n \to \infty} \sum_k \|H^{1/2}(u_n - \tilde{G}_n [e^{-it_n N_n^{-2} V(x_n) \psi}]^\varepsilon)\|_{L^2 T} \leq \varepsilon.
\]
Assuming also that \( \|\nabla \phi\|_{L^2} < \|\nabla W\|_{L^2} \) and \( E_\Delta(\phi) < E_\Delta(W) \), we have the same conclusion as above for the focusing equation (1), \( \mu = -1 \).

**Proof sketch.** We only give a rough idea as one can proceed just as in Proposition 5 of [14] and replace every instance of \( \frac{1}{2} |x_n|^2 \) with \( V(x_n) \). The idea is to show that for \( n \) large enough, one can fashion a sufficiently accurate approximate solution \( \tilde{u}_n \) on the interval \( I \) in the sense of Proposition 4, such that \( S_I(\tilde{u}_n) \) are bounded. This bound will then be transferred to the exact solution \( u_n \) by the stability theory.

While \( u_n \) remains highly concentrated (over time scales on the order of \( N_n^{-2} \)), it will be approximated by a modified solution to the scale-invariant equation (4) (whose solutions admit global spacetime bounds). By the time this approximation breaks down, the solution \( u_n \) will have dispersed to such an extent that the evolution of \( u_n \) is essentially linear.

If \( t_n \equiv 0 \), let \( v \) be the global solution to (4) furnished by Conjecture 1 with \( v(0) = \phi \). If \( N_n^2 t_n \to \pm \infty \), let \( v \) be the (unique) solution to (4) which scatters in \( H^1 \) to \( e^{it \Delta} \phi \) as \( t \to \mp \infty \). Note the reversal of signs.

The approximate solution is defined as follows. Let \( \tilde{G}_n \) and \( S_n \) be the operators defined in (21) and (22), and define for each \( n \) a Littlewood-Paley cutoff

\[
P_{\leq \tilde{N}_n} = \varphi(-\Delta/(\tilde{N}_n)^2), \quad \tilde{N}_n = \left( \frac{N_n}{N_n^2} \right)^{\frac{1}{2}},
\]

where \( \varphi : \mathbb{R} \to \mathbb{R} \) denotes a smooth function equal to 1 on the ball \( B(0,1) \) and supported in \( B(0,1.1) \). Fix a large \( T > 0 \), and define

\[
\tilde{v}_n^T(t) = \begin{cases} 
    e^{-itV(x_n)} \tilde{G}_n[S_n P_{\leq \tilde{N}_n} v](t + t_n) & |t| \leq TN_n^{-2} \\
    e^{-i(t - TN_n^{-2})^2} \tilde{v}_n^T(TN_n^{-2}), & TN_n^{-2} \leq t \leq \delta_0 \\
    e^{-i(t + TN_n^{-2})^2} \tilde{v}_n^T(-TN_n^{-2}), & -\delta_0 \leq t \leq -TN_n^{-2}
\end{cases}, \quad (48)
\]

Inside the “window of concentration”, \( \tilde{v}_n^T \) is essentially a modulated solution to (4) with cutoffs applied in both space, to place the solution in \( C_t \Sigma_x \), and frequency, to enable taking an extra derivative in the error analysis for the stability theory. The time translation by \( t_n \) is needed to undo the time translation built into the operator \( \tilde{G}_n \); see (21).

Essentially the same computations as in [14] yield the estimate

\[
\limsup_n \|H^{1/2} \tilde{v}_n^T \|_{L^\infty L^2([-\delta_0, \delta_0])} + \| \tilde{v}_n^T \|_{L_t^{2(4+2)} \left([-\delta_0, \delta_0] \times \mathbb{R}^d \right)} \lesssim C(\|\phi\|_{H^1}),
\]

uniformly in \( T \); one also sees that

\[
\lim_{T \to \infty} \limsup_n \|H^{1/2}[(i \partial_t - H)(\tilde{v}_n^T) - F(\tilde{v}_n^T)]\|_{N([-\delta_0, \delta_0])} = 0,
\]

where \( F(z) = \mu |z|^{\frac{4}{d-2}} z \) is the nonlinearity.

\[
\lim_{T \to \infty} \limsup_n \|H^{1/2}[\tilde{v}_n^T(t_n) - u_n(0)]\|_{L^2} = 0.
\]

Thus, for some fixed large \( T \) and all large \( n \), \( \tilde{u}_n(t, x) := \tilde{v}_n^T(t - t_n, x) \) is an approximate solution on the time interval \([-\delta_0/2, \delta_0/2]\) in the sense of Proposition 4. Thus one obtains the first part of Proposition 8. The last claim regarding approximation by smooth functions is proven by applying Lemma 4.3 to the functions \( \tilde{v}_n^T \) in the manner of [14, Lemma 5.6].
6. A compactness property for blowup sequences. In this section we give a Palais-Smale condition on blowup sequences of solutions to (1). This will quickly lead to the proof of Theorem 1.3.

For a maximal solution $u$ to (1), define
\[
S_u(u) = \sup \{S_I(u) : I \text{ is an open interval with } |I| \leq 1 \},
\]
where we set $S_I(u) = \infty$ if $u$ is not defined on $I$. Set
\[
\Lambda_d(E) = \sup \{S_u(u) : u \text{ solves } (1), \mu = +1, \ E(u) = E \}
\]
\[
\Lambda_f(E) = \sup \{S_u(u) : u \text{ solves } (1), \mu = -1, \ E(u) = E, \ |
\]
\[
\|\nabla u(0)\|_{L^2} < \|\nabla W\|_{L^2} \}. \]

Finally, define
\[
\mathcal{E}_d = \{E : \Lambda_d(E) < \infty \}, \quad \mathcal{E}_f = \{E : \Lambda_f(E) < \infty \},
\]

By the local theory, Theorem 1.3 is equivalent to the assertions
\[
\mathcal{E}_d = [0, \infty), \quad \mathcal{E}_f = [0, E_{\Delta}(W)).
\]

Suppose Theorem 1.3 failed. By the small data theory, $\mathcal{E}_d$, $\mathcal{E}_f$ are nonempty and open, and the failure of Theorem 1.3 implies the existence of a critical energy $E_c > 0$, with $E_c < E_{\Delta}(W)$ in the focusing case such that $\Lambda_d(E_c) = \Lambda_f(E_c) = \infty$ for $E > E_c$ and $\Lambda_d(E_c) = \Lambda_f(E_c) < \infty$ for all $E < E_c$. We have the following compactness property.

**Proposition 9** (Palais-Smale). Assume Conjecture 1.1 holds. Suppose that $u_n : (t_n - \delta_0, t_n + \delta_0) \times \mathbb{R}^d \to \mathbb{C}$ is a sequence of solutions with
\[
\lim_{n \to \infty} E(u_n) = E_c, \quad \lim_{n \to \infty} S_{(t_n - \delta_0, t_n]}(u_n) = \lim_{n \to \infty} S_{(t_n, t_n + \delta_0]}(u_n) = \infty,
\]
where $\delta_0$ is the constant in Theorem 2.1. In the focusing case, assume also that $E_c < E_{\Delta}(W)$ and $\|\nabla u_n(t_n)\|_{L^2} < \|\nabla W\|_{L^2}$. Then there exists a subsequence such that $(u_n(t_n))$ converges in $Q(H)$.

**Proof.** We refer to the presentation following Proposition 6.1 in [14]. The proof uses a local smoothing estimate for the propagator $e^{-i t H}$, which can be obtained via a multiplier argument just as in Corollary 2.10 of [14]. In the focusing case, one also uses energy trapping arguments (see Section 7 of [14]) to see that the hypotheses are in fact equivalent to $\|H^{1/2} u_n(t_n)\|_{L^2} < \|\nabla W\|_{L^2}$.

**Proof of Theorem 1.3.** Suppose the theorem failed, and let $E_c$ be as above. Then, after applying suitable time translations, there is a sequence of solutions $u_n$ with $E(u_n) \to E_c$ and $S_{(-\delta_0/4, \delta_0/4]}(u_n) \to \infty$. Choose $t_n$ such that $S_{(-\delta_0/4, t_n]}(u_n) = \frac{1}{2} S_{(-\delta_0/4, \delta_0/4]}(u_n)$. By Proposition 9, after passing to a subsequence we have $\|u(t_n) - \phi\|_{\Sigma} \to 0$ for some $\phi \in \Sigma$. Then $E(\phi) = \lim_n E(u_n(t_n)) = E_c$.

Let $v : (-T_{\min}, T_{\max}) \to \mathbb{C}$ be the maximum-lifespan solution to (1) with $v(0) = \phi$. By comparing $v(t, x)$ with the solutions $u_n(t + t_n, x)$ and applying Proposition 4, we see that $S_{(0, \delta_0/2]}(v) = S_{(-\delta_0/2, 0]}(v) = \infty$. Thus $-\delta_0/2 \leq -T_{\min} < T_{\max} \leq \delta_0/2$. But the orbit $\{v(t) \}_{t \in (-T_{\min}, T_{\max})}$ is a precompact subset of $\Sigma$, by Proposition 9, so there is some sequence of times $t_n$ increasing to $T_{\max}$ such that $v(t_n)$ converges in $\Sigma$ to some $\psi$. By considering a local solution with initial data $\psi$ and invoking stability theory, we see that $v$ can actually be extended to some larger interval $(-T_{\min}, T_{\max} + \eta)$, in contradiction to the maximality of $v$. \qed
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REFERENCES

[1] K. Asada and D. Fujiwara, On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$, *Japan. J. Math. (N.S.)*, 4 (1978), 299–361.
[2] J. Bourgain, Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case, *J. Amer. Math. Soc.*, 12 (1999), 145–171.
[3] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, 88 (1983), 486–490.
[4] R. Carles, Nonlinear schrödinger equation with time-dependent potential, *Commun. Math. Sci.*, 9 (2011), 937–964.
[5] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, *Ann. of Math. (2)*, 167 (2008), 767–865.
[6] G. B. Folland, *Harmonic Analysis in Phase Space*, vol. 122 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1989.
[7] D. Fujiwara, On the boundedness of integral transformations with highly oscillatory kernels, *Proc. Japan Acad.*, 51 (1975), 96–99.
[8] D. Fujiwara, A construction of the fundamental solution for the Schrödinger equation, *J. Analyse Math.*, 35 (1979), 41–96.
[9] D. Fujiwara, Remarks on convergence of the Feynman path integrals, *Duke Math. J.*, 47 (1980), 559–600.
[10] W. Hebisch, A multiplier theorem for Schrödinger operators, *Colloq. Math.*, 60/61 (1990), 659–664.
[11] A. D. Ionescu and B. Pausader, The energy-critical defocusing NLS on $\mathbb{T}^3$, *Duke Math. J.*, 161 (2012), 1581–1612.
[12] A. D. Ionescu and B. Pausader, Global well-posedness of the energy-critical defocusing NLS on $\mathbb{R} \times \mathbb{T}^3$, *Comm. Math. Phys.*, 312 (2012), 781–831.
[13] A. D. Ionescu, B. Pausader and G. Staffilani, On the global well-posedness of energy-critical Schrödinger equations in curved spaces, *Anal. PDE*, 5 (2012), 705–746.
[14] C. Jao, The energy-critical quantum harmonic oscillator, *Comm. Partial Differential Equations*, 41 (2016), 79–133.
[15] M. Keel and T. Tao, Endpoint Strichartz estimates, *Amer. J. Math.*, 120 (1998), 955–980.
[16] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, *Invent. Math.*, 166 (2006), 645–675.
[17] S. Keraani, On the blow up phenomenon of the critical nonlinear Schrödinger equation, *J. Funct. Anal.*, 235 (2006), 171–192.
[18] R. Killip, S. Kwon, S. Shao and M. Visan, On the mass-critical generalized KdV equation, *Discrete Contin. Dyn. Syst.*, 32 (2012), 191–221.
[19] R. Killip, B. Stovall and M. Visan, Scattering for the cubic Klein-Gordon equation in two space dimensions, *Trans. Amer. Math. Soc.*, 364 (2012), 1571–1631.
[20] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, *Amer. J. Math.*, 132 (2010), 361–424.
[21] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity, in *Evolution equations*, vol. 17 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2013, 325–437.
[22] R. Killip, M. Visan and X. Zhang, Quintic NLS in the exterior of a strictly convex obstacle, *Amer. J. Math.*, 138 (2016), 1193–1346.
[23] R. Killip, M. Visan and X. Zhang, Energy-critical NLS with quadratic potentials, *Comm. Partial Differential Equations*, 34 (2009), 1531–1565.
[24] Y.-G. Oh, Cauchy problem and Ehrenfest’s law of nonlinear Schrödinger equations with potentials, *J. Differential Equations*, 81 (1989), 255–274.
[25] E. Ryckman and M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$, *Amer. J. Math.*, 129 (2007), 1–60.
[26] M. E. Taylor, *Tools for PDE*, vol. 81 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2000, Pseudodifferential operators, paradifferential operators, and layer potentials.

[27] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions, *Duke Math. J.*, 138 (2007), 281–374.

[28] J. Zhang, Stability of attractive Bose-Einstein condensates, *J. Statist. Phys.*, 101 (2000), 731–746.

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