G-CENTRAL LIMIT THEOREMS AND G-INVARIANCE PRINCIPLES FOR ASSOCIATED RANDOM VARIABLES

ALADJI BABACAR NIANG, AKYM ADEKPEDJOU, HAROUNA SANGARÉ, AND GANE SAMB LO

ABSTRACT. The investigation asymptotic limits on associated data mainly focused on limit theorems of summands of associated data and on the related invariance principles. In a series of papers, we are going to set the general frame of the theory by considering an arbitrary infinitely decomposable (divisible) limit law for summands and study the associated functional laws converging to Lévy processes. The asymptotic frame of Newman (1980) is still used as a main tool. Detailed results are given when $G$ is a Gaussian law (as confirmation of known results) and when $G$ is a Poisson law. In the later case, classical results for independent and identically distributed data are extended to stationary and non-stationary associated data.

† Aladji Babacar Niang
LERSTAD, Gaston Berger University, Saint-Louis, Sénégal.
Email: niang.aladji-babacar@ugb.edu.sn, aladji93@gmail.com

†† Akim Adekpedjou, PhD
Professor of Statistics
Department of Mathematics & Statistics
Missouri University of Science & Technology
Office: +1(573)-341-4649
E-mail: akima@mst.edu
URL: web.mst.edu/akima
ROLLa, MO 65409

††† Harouna Sangaré
Main Affiliation: DER MI, FST, Université des Sciences, des Techniques et des Technologies de Bamako (USTT-B), Mali.
Affiliation: LERSTAD, Université Gaston Berger (UGB), Saint-Louis, Sénégal.
Email: harounasangare@fst-usttb-edu.ml, harouna.sangare@mesrs.ml
sangare.harouna@ugb.edu.sn, harounasangareusttb@gmail.com

†††† Gane Samb Lo.
LERSTAD, Gaston Berger University, Saint-Louis, Sénégal (main affiliation).
LSTA, Pierre and Marie Curie University, Paris VI, France.
AUST - African University of Sciences and Technology, Abuja, Nigeria
gane-samb.lo@edu.ugb.sn, gslo@aust.edu.ng, ganesamblo@ganesamblo.net
Permanent address: 1178 Evanston Dr NW T3P0J9, Calgary, Alberta, Canada.

Keywords. associated data; Newman approximation; infinitely divisible law; Poisson and Gaussian laws; invariance principles; central limit theorems; summands of random variables; stationary and non-stationary sequences;

AMS 2010 Mathematics Subject Classification: 60GXX, 62GXX
1. Introduction

Recently, the investigation of fundamental aspects of probability theory on associated data has been very active. Since the pioneering work of Newman (1980), the type of dependence called association is highly present in the literature. Since independent sequences are associated, the research community tries to extend results of the independence frame to the association frame. Because this concept is not popular yet, we are going to give a brief account of it below, before we describe our aims and motivation with more precision.

1.1. A brief account on associated data.

The definition is given by Esary et al. (1967) as follows.

**Definition 1.** A finite sequence of rv’s $(X_1, \cdots, X_n)$ are associated when for any couple of real valued and coordinate-wise non-decreasing functions $h$ and $g$ defined on $\mathbb{R}^n$, we have

$$
\text{Cov}(h(X_1, \cdots, X_n), g(X_1, \cdots, X_n)) \geq 0.
$$

An infinite sequence of rv’s are associated whenever all its finite sub-sequences are associated.

There is a few number of interesting properties to be found in Prakasa Rao (2012):

**P1** A sequence of independent rv’s is associated. **P2** Partial sums of associated rv’s are associated. **P3** Order statistics of independent rv’s are associated. **P4** Non-decreasing functions and non-increasing functions of associated variables are associated. **P5** Suppose that $Z_1, Z_2, \cdots, Z_n$ is associated and $(a_i)_{1 \leq i \leq n}$ is sequence of non-negative numbers and $(b_i)_{1 \leq i \leq n}$ real numbers, then the rv’s $a_i(Z_i - b_i)$ are associated.

The concept of association also arises in Physics, and is quoted under the name of FKG property (Fortuin et al. (1968)), in percolation theory and even in Finance (see Joazhu Daley (2002)).
However, the concept took its place in pure probability theory only with papers by Newman and Wright (1981, 1982) which addressed the central limit theorem, the strong law of large numbers and the functional limiting laws. Since then, a great number of contributions appeared in the literature: [Newman (1980), Sangaré and Lo (2015), etc., for example for the strong law of large numbers], [Yu (1968), Louhichi (2000), Sangaré et al. (2020), etc, for the empirical processes; Cox and Grimmett (1984), Oliveira (2012), Lo et al. and Lo (2018), Sangaré and Lo (2018), Adekpejou et al. (2021) for the central limit theorem, Oliveira (2012) for the law of the iterated logarithm, Burton et al. (1986), Dabrowski and Dehling (1988), Newman and Wright (1982), Traoré et al. (2016) for the invariance principles, etc.]. There are three important manuscripts aiming at reviewing the current research [Bulinski and Shashkin (2007), Oliveira (2012), Prakasa Rao (2012)]. Sangaré and Lo (2015) provides a short but interesting review. Important results in mathematical statistics using associated data also exist and some of them are given in Prakasa Rao (2012) in particular.

From the review above, we see that the central limit theorem and the invariance principle topics are central in the asymptotic theory on associated data. However, it appeared that so far the investigations focused in the case where the weak limits are Gaussian random variables [for the CLT] or re-scaled Brownian motions [for the invariance principles]. This motivates us to have a general theory for CLT’s with arbitrary limiting law $G$ for an infinitely decomposable random variable corresponding to invariance principles to Lévy processes. This extension is really interesting since the scope of the study passes from one limiting law to an infinite number of limiting laws. From the general results we obtain, we particularize Poisson limits (for CLT’s) and compound Poisson processes (for invariance principles).

We organize the rest of the paper as follows. We will present the main notation in Subsection 1.2 where we recall the most important results on limits of summands for independent random variables with finite variances. Therein, we also describe the general method of using the Newman approximation method called the block methods and state intermediate results on the convergence of rates induced by Newman method. In Section 2, we use the general results of Subsection 1.5 of Section 1 to have a general theory of the CLT to associated data for both sequences and arrays, for both stationary and non-stationary. In Section 3, we specialized the
results in Subsection 1.5 for stationary associated sequences. In section 4, we consider two iconic examples of convergence to Poisson law, namely the summands of Bernoulli rv.’s and corrected geometric rv.’s in the by-row iid case, and extend the related results to non-stationary by-row independent arrays and next to the association frame for both stationary and non-stationary.

The obtained CLT’s should be extended to invariance principles. However in fear of presenting a too lengthy paper, we decided to devote a coming paper to those invariance principles.

1.2. Notation and $G$-CLT for summands of random variables random variables.

Let us consider the array

\[ X \equiv \{ X_{k,n}, 1 \leq k \leq k(n), n \geq 1 \}, \]

of square integrable random variables defined on the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\). We denote \(F_{k,n}\) as the cumulative distribution function (cdf) of \(X_{k,n}\). We also denote by \(a_{k,n} = \mathbb{E}(X_{k,n})\) and \(\sigma_{k,n}^2 = \text{Var}(X_{k,n})\), \(1 \leq k \leq k(n)\), if these expectations or variances exist. We also suppose that

\[ k(n) \to +\infty \text{ as } n \to +\infty. \]

The central limit theorem problem consists in finding, whenever possible, the weak limit law of the by-row sums of the array \(X\), i.e. the summands:

\[ S_n[X] = \sum_{k=1}^{k(n)} X_{k,n}, \quad n \geq 1. \]

Historically, the CLT was discovered with the convergence of a Binomial law (which has the same law as the sum of iid Bernoulli random variables) to the standard Gaussian law (due to Laplace, De Moivre, etc., around 1731, see Loève (1977) for a review). For a long period of time, the Gaussian limit was automatically meant in the CLT problem. Many authors, among them Lévy, Gnedenko, Kolmogorov, etc., characterized the class of possible limit laws under the Uniform Asymptotic Negligibility (UAN) condition, exactly as
the class of infinitely decomposable distributions (see below). The longtime association of CLT’s to Gaussian limit explains that some authors reserve the vocable CLT for the Gaussian limit and for other possible limits, they use the vocable Non-Central limit theorem. Here we use G-CLT to cover all possible limit laws G beyond the Gaussian law.

Here we suppose that the \( X_{k,n} \)’s are integrable with finite variances. For an array \( X \), we define some important hypotheses used in the formulation of the CLT problem.

(1) **The UAN condition**: for any \( \varepsilon > 0 \),

\[
U(n, \varepsilon, X) = \sup_{1 \leq k \leq k(n)} \mathbb{P}(|X_{k,n} - a_{k,n}| \geq \varepsilon) \to 0.
\]

(2) **The Bounded Variance Hypothesis (BVH)**: there exists a constant \( c > 0 \),

\[
\sup_{n \geq 1} MV(n, X) \leq c,
\]

where

\[
MV(n, X) = \text{Var}(S_n[X]), \ n \geq 1.
\]

(3) **The Variance Convergence Hypothesis (VCH)**:

\[
MV(n, X) \to c \in ]0, +\infty[.
\]

According to the state of the art in CLT’s theory for centered, square integrable and independent array of random variables, the summands weakly converges to a rv associated to the cdf \( G \) and to the cha.f \( \psi_G \) under the UAN condition and the BVH if and only if the sequence of distribution functions (df)

\[
K_n(x) = \sum_{k=1}^{k(n)} \int_{-\infty}^{x} y^2 dF_{k,n}(y), \ x \in \mathbb{R}, \ n \geq 1,
\]

pre-weakly converges to a df \( K \), denoted \( K_n \leadsto_{pre} K \), that is for any continuity point \( x \) of \( K \) denoted as \([x \in C(K)]\).
\[ K_n(x) \to K(x), \]

and the \( \text{cha.f} \ \psi_G(\circ) \) of \( G \) is given by \( \exp(\psi[K](\circ)) \) with

\begin{equation}
\forall u \in \mathbb{R}, \ \psi[K](u) = \int \frac{e^{iux} - 1 - iux}{x^2} \, dK(x).
\end{equation}

If we have the VCH, the convergence criterion is replaced by the weak convergence \( K_n \rightsquigarrow K \). Moreover, the limit law \( G \) is necessarily an infinitely decomposable law.

In the non centered case, with the same hypotheses above on the random variables of the array, the summands weakly converges to a probability law associated to the \( \text{cdf} \ G^* \) and to the \( \text{cha.f} \ \psi_{G^*} \) under the UAN condition and the BVH if and only if

\[ \sum_{k=1}^{k(n)} a_{k,n} \to a, \ a \in \mathbb{R} \]

and the sequence of distribution functions (\( df \))

\[ K_n^*(x) = \sum_{k=1}^{k(n)} \int_{-\infty}^{x} y^2 dF_{k,n}(y + a_{k,n}), \ x \in \mathbb{R}, \ n \geq 1, \]

pre-weakly converges to a \( df \ K^* \) and the \( \text{cha.f} \ \psi_{G^*}(\circ) \) of \( G^* \) is given by \( \exp(\psi[K^*](\circ)) \) with

\[ \forall u \in \mathbb{R}, \ \psi[K^*](u) = \int \frac{e^{iux} - 1 - iux}{x^2} \, dK^*(x). \]

If we have the VCH, the convergence criterion is replaced by the weak convergence \( K_n^* \rightsquigarrow K^* \). Moreover, the limit law \( G^* \) is of the form \( G^* = G + a \), with \( G \) is necessarily a centered and infinitely decomposable law.
1.3. The data-regrouping method.

In many situations with dependent data, there are efforts to transform the summands $S_n[X]$ into blocks with increasing lengths such that the dependence between the summands of blocks decreases as $n$ grows to infinity. We are going to apply that idea here.

We consider sequences of non-negative integer numbers $(m(n), \ell(n), r(n))$ such that:

$$\forall n \geq 1, \ k(n) = m(n)\ell(n) + r(n)$$

and

$$1 \leq m(n) \to +\infty, \ 0 \leq r(n) < \ell(n) \text{ and } \ell(n)/m(n) \to 0.$$

Accordingly, we define

$$Y_{j,n} = \sum_{h=1}^{\ell(n)} X_{(j-1)\ell(n)+h,n}, \ 1 \leq j \leq m(n); \ Y_n^* = \sum_{h=1}^{r(n)} X_{m(n)\ell(n)+h,n}$$

and

$$S_{m(n)\ell(n)} = \sum_{j=1}^{m(n)\ell(n)} X_{j,n} = \sum_{j=1}^{m(n)} Y_{j,n}.$$

So, we have

$$S_n = \sum_{j=1}^{m(n)\ell(n)} X_{j,n} + Y_n^* = S_{m(n)\ell(n)} + Y_n^*, \ n \geq 1.$$

We denote the new array as

$$Y \equiv \left\{ \{Y_{j,n}, \ 1 \leq j \leq m(n)\}, \ n \geq 1 \right\}.$$

We can denote
\[ S_{m(n)}[Y] = \sum_{j=1}^{m(n)} Y_{j,n}, \quad s_{j,n}^2 = \text{Var}(Y_{j,n}) \quad \text{and} \quad s_{*,n}^2 = \text{Var}(Y^*_n). \]

1.4. **A general method for handling the blocks.**

We need two steps:

**Step 1.** We try to transform the study of the summands \( S_n[X] \) by those of \( S_{m(n)}[Y] \) by getting rid of \( Y^*_n \). It is easy to check that for any \( u \in \mathbb{R} \),

\[(1.5) \quad \left| \psi_{S_n[X]}(u) - \psi_{S_{m(n)}[Y]}(u) \right| \leq \text{Var}(Y^*_n)^{1/2} \to 0.\]

**Step 2.** If step 1 is validated, we try to replace the summands \( S_{m(n)}[Y] \) by the summands \( S_{m(n)}[T] \) where the array

\[ T \equiv \left\{ \{T_{j,n}, 1 \leq j \leq m(n)\}, n \geq 1 \right\}, \]

has independent elements in each row and

\[ \forall n \geq 1, \forall 1 \leq j \leq m(n), \quad T_{j,n} \equiv Y_{j,n}. \]

We need to have for any \( u \in \mathbb{R} \),

\[(1.6) \quad \left| \psi_{S_{m(n)}[Y]}(u) - \psi_{S_{m(n)}[T]}(u) \right| \to 0,\]

that is

\[ \left| \psi_{S_{m(n)}[Y]}(u) - \prod_{j=1}^{m(n)} \psi_{Y_{j,n}}(u) \right| \to 0. \]

Now let us apply the methodology to associated data.
1.5. **How works the block method for associated data?**

We begin by requiring (1.5) as an hypothesis and we recall that it holds whenever each row, i.e. the \((X_{k,n})_{1 \leq k \leq k(n)}\), is a stationary sequence.

Let us address (1.6). Let us suppose that the elements of the \(k(n)\)-th row \((X_{k,n})_{1 \leq k \leq k(n)}\) are associated, so are the \((Y_{j,n})_{1 \leq j \leq m(n)}\). By a Newman’s Lemma (See Newman and Wright (1981, 1982)), we have for any \(u \in \mathbb{R}\),

\[
|\psi_{S_{m(n)}[Y]}(u) - \psi_{S_{m(n)}[T]}(u)| \leq R_{m(n),\ell(n)}(u),
\]

with

\[
R_{m(n),\ell(n)}(u) = \frac{u^2}{2} \sum_{1 \leq r \neq s \leq m(n)} \text{Cov}(Y_{r,n}, Y_{s,n}).
\]

So Hypothesis (1.6) is achieved through

\[
R_{m(n),\ell(n)}(u) \to 0.
\]

We get the following general theorem.

**Theorem 1.** (General Theorem for centered random variables) Let

\[
X \equiv \left\{ X_{k,n}, \ 1 \leq k \leq k_n = k(n), \ n \geq 1 \right\},
\]

be an array of centered, square integrable and associated random variables defined on the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\).

We denote \(F_{k,n}\) as the cumulative distribution function (cdf) of \(X_{k,n}\) and let

\[
S_n[X] = \sum_{k=1}^{k(n)} X_{k,n} \quad \text{and} \quad s_n^2[X] = \text{Var}(S_n[X]), \ n \geq 1.
\]

Let \((m(n), \ell(n), r(n))_{n \geq 1}\) be sequences of non-negative integer numbers such that \(k(n) = m(n)\ell(n) + r(n)\) for any \(n \geq 1\) and

\[
1 \leq m(n) \to +\infty, \quad 0 \leq r(n) < \ell(n) \quad \text{and} \quad \ell(n)/m(n) \to 0.
\]
We define

\[ Y_{j,n} = \sum_{h=1}^{\ell(n)} X_{(j-1)\ell(n)+h,n}, \quad 1 \leq j \leq m(n); \quad Y^*_n = \sum_{h=1}^{r(n)} X_{m(n)\ell(n)+h,n} \]

and an array

\[ T \equiv \left\{ \{T_{j,n}, \ 1 \leq j \leq m(n)\}, \ n \geq 1 \right\}, \]

which has independent elements in each row and

\[ \forall n \geq 1, \ \forall 1 \leq j \leq m(n), \ T_{j,n} \overset{d}{=} Y_{j,n}. \]

We set

\[ S_{m(n)}[T] = \sum_{j=1}^{m(n)} T_{j,n} \]

and denote \( G_{j,n} \) the cdf of \( T_{j,n} \). Then, given (1.2), (1.5), (1.6) and assuming that \( UAN[T], BVH[T] \) or \( CVH[T] \) hold, we have the following characterizations.

1. \( S_n[X] \) converges to a random variable with cdf \( G \) if and only if \( G \) is ide-

2. \( S_n[X] \) converges to a random variable with df \( G \) if and only if \( \psi_G(\phi) = \exp(\psi[K](\phi)) \), where

\[ \forall u \in \mathbb{R}, \ \psi[K](u) = \int \frac{e^{iux} - 1 - iux}{x^2} dK(x), \]

with \( K \) is df on \( \mathbb{R} \) such that

\[ K_n(\phi) = \sum_{j=1}^{m(n)} \int_{-\infty}^{0} y^2 dG_{j,n}(y) \sim_{pre} K, \]

under \( BVH[T] \) and
Next, we have the second theorem for not necessarily centered random variables.

**Theorem 2. (General Theorem for non-centered random variables)** Let us adopt the notation and assumptions of Theorem 1 except that the random variables are not necessarily centered. Let us denote

\[ a_{k,n} = \mathbb{E}(X_{k,n}), \quad 1 \leq k \leq k(n), \quad a_n = \sum_{k=1}^{k(n)} a_{k,n}, \]

\[ a_n^* = \sum_{j=1}^{m(n)\ell(n)} a_{j,n}, \quad n \geq 1 \]

and

\[ b_{j,n} = \sum_{h=1}^{\ell(n)} a_{(j-1)\ell(n)+h,n}, \quad 1 \leq j \leq m(n). \]

Let us denote \( G_{j,n}^*(\circ) = G_{j,n}(\circ + b_{j,n}) \). Then, given (1.2), (1.5) and (1.6) and assuming that UAN[T], BHV[T] (or CVH[T]) hold, we have the following results.

(A) If BHV[T] holds (respectively CVH[T] holds), if \( a_n \to b \) (equivalently \( a_n^* \to b \) if \( b_n^* = \sum_{h=1}^{r(n)} a_{m(n)\ell(n)+h,n} \to 0 \)) and

\[ K_{n,*}^*(\circ) = \sum_{j=1}^{m(n)} \int_{-\infty}^{\circ} y^2 \, dG_{j,n,*}^*(y) \overset{\text{pre}}{\Rightarrow} K^*, \]

(resp.)
\[ K^*_n(\circ) = \sum_{j=1}^{m(n)} \int_{-\infty}^{\circ} y^2 \, dG^*_{j,n}(y) \sim K^*, \]

\( J \), then \( S_n[X] \sim b + G \), where \( G \) is idecomp and \( \psi_G(\circ) = \exp(\psi[K^*](\circ)) \), with

\[ \forall u \in \mathbb{R}, \psi[K^*](u) = \int \frac{e^{iux} - 1 - iux}{x^2} \, dK^*(x). \]

(B) If \( S_n[X] \sim G^* \), where \( G^* \) is the cdf of an a.s. finite random variable, then the sequence \((a_n)_{n \geq 1}\) converges to a finite number \( b \) and \( G^* = b + G \), with \( G \) idecomp and (1.12) holds under the BVH[T] or (1.13) holds under the CVH[T] and in both case, (1.14) holds.

2. General results for \( G\text{-CLT} \) by-row-associated arrays

From this general description, we are going to state the overall general results for \( G\text{-CLT} \) of associated random variables. As we already mentioned, existing results up to our knowledge only cover the Gaussian CLT.

In the two above theorems, we need to check the conditions of validity of CLT for by-row independent arrays on the grouped data \( \{T_{j,n}, 1 \leq j \leq m(n)\} \). However, we may get sufficient conditions on the non-grouped data \( \{X_{k,n}, 1 \leq k \leq k(n)\} \), implying them. Here are some examples.

**Corollary 1.** Let us suppose that we have all the notation of theorems 1 and 2.

(I) Suppose that the conditions (1.2), (1.5) and (1.6) hold. If

(i) BVH[X] (resp. CVH[X]) holds

and

(ii) \( \ell(n) \, U(n, \epsilon/\ell(n), X) \rightarrow 0 \),
then $BV[H[T]$ (resp. $CV[H[T]$) and $UA[N[T]$ hold and the conclusions of (1) and (2) of theorem 1 are still validated.

(2) If the Gaussian-Lynderberg condition related to $X$ is such that:

$$\forall \epsilon > 0, \quad \ell(n)^2 L_n[X](\epsilon/\ell(n)) \to 0,$$

then the Gaussian-Lynderberg condition related to $T$ holds and we have

$$S_{m(n)}[T] \sim N(0, 1) \quad \text{and} \quad B_{m(n)}[T] \to 0.$$ 

Similarly, if $B_n[X]$ satisfies

$$\ell(n)^2 B_n[X] \to 0,$$

then

$$S_{m(n)}[T] \sim N(0, 1)$$

implies that for any $\epsilon > 0$,

$$L_{m(n)}[T](\epsilon) \to 0.$$

**General comment.** Using the conditions on grouped and non-grouped data is a matter of situations.

Dealing directly with the conditions on $T_{j,n}$’s is far better and more precise. If not, we may try to establish the conditions on the non-grouped data. However, we should be aware that they are only sufficient conditions.

If they failing to have them, does not mean that the conditions on the grouped data fail.

**Elements of Proof.** These both theorems are entirely proved by the whole description in section (1.5). For the proof of the Corollary 1, we only need to have the comparison between the condition on grouped and non-grouped data as below.

**Conditions on the regrouped and non-regrouping by-row data.**
To shorten the notation, we use the following notation below

\[ X'_{j,n,h} = X_{(j-1)\ell(n)+h,n}, \ 1 \leq j \leq m(n), \ 1 \leq h \leq \ell(n). \]

**1) The (CVH) condition.** Let us begin to see that, since \( \mathbb{V}ar(S_n[X]) \geq 0 \) for any \( n \geq 1 \), the sequence of variances \( (\mathbb{V}ar(S_n[X]))_{n \geq 1} \) converges to a some \( \sigma^2 > 0 \), otherwise, it diverges to infinity. By equation (3.4), we have

\[
1 = \frac{\mathbb{V}ar(S_{m(n)}[T])}{\mathbb{V}ar(S_n[X])} + \frac{\mathbb{V}ar(Y^*_n)}{\mathbb{V}ar(S_n[X])} + 2 \frac{\mathbb{C}ov(S_{m(n)}[T], Y^*_n)}{\mathbb{V}ar(S_n[X])}
\]

and so, since \( \mathbb{V}ar(Y^*_n) \to 0 \) (condition (1.5)), and next

\[
\frac{\mathbb{C}ov(S_{m(n)}[T], Y^*_n)}{\mathbb{V}ar(S_n[X])} \to 0
\]

by Cauchy Schwartz inequality, we get

\[
\frac{\mathbb{V}ar(S_{m(n)}[T])}{\mathbb{V}ar(S_n[X])} \to 1.
\]

Hence

\[
(2.1) \quad MV(n, X) = (1 + o(1)) \ MV(m(n), T),
\]

with

\[
MV(m(n), T) = \mathbb{V}ar(S_{m(n)}[T]).
\]

**2) The UAN condition.** We have, for any \( \varepsilon > 0 \), the (UAN) condition for the sequence \( \{T_{j,n}, \ 1 \leq j \leq m(n)\} \) is
\[
U(m(n), \varepsilon, T) = \sup_{1 \leq j \leq m(n)} \mathbb{P}(|T_{j,n}| \geq \varepsilon)
\]
\[
= \sup_{1 \leq j \leq m(n)} \mathbb{P}\left(\left|\sum_{h=1}^{\ell(n)} X'_{j,n,h}\right| \geq \varepsilon\right)
\]
\[
\leq \sup_{1 \leq j \leq m(n)} \mathbb{P}\left(\sum_{h=1}^{\ell(n)} (|X'_{j,n,h}| > \varepsilon/\ell(n))\right)
\]
\[
\leq \sup_{1 \leq j \leq m(n)} \sum_{h=1}^{\ell(n)} \mathbb{P}(|X'_{j,n,h}| > \varepsilon/\ell(n))
\]
\[
\leq \ell(n) \sup_{1 \leq j \leq m(n)} \sup_{1 \leq h \leq \ell(n)} \mathbb{P}(|X'_{j,n,h}| > \varepsilon/\ell(n))
\]
\[
\leq \ell(n) \sup_{1 \leq k \leq k(n)} \mathbb{P}(|X_{k,n}| > \varepsilon/\ell(n)).
\]

Hence we get for any \( \varepsilon > 0, \)

\[(2.2) \quad U(m(n), \varepsilon, T) \leq \ell(n) U(n, \varepsilon/\ell(n), X), \]

where we recall that

\[
U(n, \varepsilon, X) = \sup_{1 \leq k \leq k(n)} \mathbb{P}(|X_{k,n}| > \varepsilon).
\]

\( (3) \) A sufficient condition to obtain (UAN) condition. Since for any \( \varepsilon > 0, \)

\[
U(m(n), \varepsilon, T) \leq \varepsilon^{-2} \sup_{1 \leq j \leq m(n)} \mathbb{E}(T_{j,n}^2),
\]

then the UAN condition for the sequence \( \{T_{j,n}, 1 \leq j \leq m(n)\} \) is controlled as:

\[
B_{m(n)}[T] := \sup_{1 \leq j \leq m(n)} \mathbb{E}(T_{j,n}^2) \rightarrow 0
\]

and we have
\[
B_{m(n)}[T] = \sup_{1 \leq j \leq m(n)} \mathbb{E} \left( \sum_{h=1}^{\ell(n)} X'_{j,n,h} \right)^2 \\
= \sup_{1 \leq j \leq m(n)} \mathbb{E} \left| \ell(n) \left( \sum_{h=1}^{\ell(n)} \{X'_{j,n,h}/\ell(n)\} \right) \right|^2 \\
= \ell(n)^2 \sup_{1 \leq j \leq m(n)} \mathbb{E} \left| \sum_{h=1}^{\ell(n)} \{X'_{j,n,h}/\ell(n)\} \right|^2 \\
\leq \ell(n) \sup_{1 \leq j \leq m(n)} \sum_{h=1}^{\ell(n)} \mathbb{E} \left( X'^2_{n,j,h} \right) \quad (L3) \\
\leq \ell(n)^2 \sup_{1 \leq j \leq m(n)} \sup_{1 \leq h \leq \ell(n)} \mathbb{E} \left( X'^2_{n,j,h} \right) \\
\leq \ell(n)^2 \sup_{1 \leq k \leq k(n)} \mathbb{E} \left( X'^2_{k,n} \right),
\]

where we use in line (L3), the convexity of \( \mathbb{R}_+ \ni x \mapsto x^2 \). By denoting, for \( \varepsilon > 0 \),

\[
B_n[X] = \sup_{1 \leq k \leq k(n)} \mathbb{E} \left( X'^2_{k,n} \right),
\]

we have,

\[
(2.3) \quad B_{m(n)}[T] \leq \ell(n)^2 B_n[X].
\]

(4) - Lyapounov condition. For \( \delta > 0 \), the Lyapounov condition for the sequence \( \{T_{j,n}, 1 \leq j \leq m(n)\} \) is

\[
A_{m(n)}[T](\delta) = \sum_{j=1}^{m(n)} \mathbb{E} |T_{j,n}|^{2+\delta} \\
= \sum_{j=1}^{m(n)} \mathbb{E} \left| \sum_{h=1}^{\ell(n)} X'_{j,n,h} \right|^{2+\delta}.
\]
Now, by using the convexity of $\mathbb{R}_+ \ni x \mapsto x^{2+\delta}$, we have

$$A_{m(n)}[T](\delta) = \sum_{j=1}^{m(n)} \mathbb{E} \left[ \ell(n) \left( \sum_{h=1}^{\ell(n)} \{X'_{j,n,h} / \ell(n)\} \right)^{2+\delta} \right]$$

$$\leq \ell(n)^{1+\delta} \sum_{j=1}^{m(n)} \sum_{h=1}^{\ell(n)} \mathbb{E} \left| X'_{j,n,h} \right|^{2+\delta}$$

$$\leq \ell(n)^{1+\delta} \sum_{k=1}^{k(n)} \left| X_{k,n} \right|^{2+\delta}.$$

Hence, we get for any $\delta > 0$,

(2.4) $$A_{m(n)}[T](\delta) \leq \ell(n)^{1+\delta} A_n[X](\delta),$$

where we recall that

$$A_n[X](\delta) = \sum_{k=1}^{k(n)} \mathbb{E} \left| X_{k,n} \right|^{2+\delta}.$$

(5) - Lynderberg-Gaussian Condition. For $\varepsilon > 0$, the Lynderberg-Gaussian Condition for the sequence $\{T_{j,n}, 1 \leq j \leq m(n)\}$ is

$$L_{m(n)}[T](\varepsilon) = \sum_{j=1}^{m(n)} \int_{\{|T_{j,n}| \geq \varepsilon\}} |T_{j,n}|^2 \, d\mathbb{P}.$$

Let us set, for $n \geq 1$, $1 \leq j \leq m(n)$, $1 \leq h \leq \ell(n)$,

$$A'_{j,n,h} = \left( \left| X'_{j,n,h} \right| \geq \varepsilon / \ell(n) \right)$$

and

$$A'_{j,n} = \bigcup_{h=1}^{\ell(n)} A'_{j,n,h}.$$

We have for any $\varepsilon > 0$,
\[ L_{m(n)}[T](\varepsilon) \leq \sum_{j=1}^{m(n)} \int_{A'_{j,n}} |T_{j,n}|^2 \, d\mathbb{P}. \]

Let \( M_{j,n} = \max_{1 \leq r \leq \ell(n)} |X'_{j,n,r}|. \) Since we have
\[
\bigcup_{r=1}^{\ell(n)} \left( M_{j,n} = |X'_{j,n,r}| \right) = \Omega,
\]
we get
\[
L_{m(n)}[T](\varepsilon) \leq \sum_{j=1}^{m(n)} \sum_{r=1}^{\ell(n)} \int_{A'_{j,n} \cap \{ M_{j,n} = |X'_{j,n,r}| \}} |T_{j,n}|^2 \, d\mathbb{P}
\leq \sum_{j=1}^{m(n)} \sum_{r=1}^{\ell(n)} \int_{A'_{j,n} \cap \{ M_{j,n} = |X'_{j,n,r}| \}} \left( \ell(n)|X'_{j,n,r}| \right)^2 \, d\mathbb{P}
\leq \ell(n)^2 \sum_{j=1}^{m(n)} \sum_{r=1}^{\ell(n)} \int \left( |X'_{j,n,r}| \geq \varepsilon/\ell(n) \right) X'_{j,n,r}^2 \, d\mathbb{P}
\leq \ell(n)^2 \sum_{k=1}^{k(n)} \int \left( |X_{k,n}| \geq \varepsilon/\ell(n) \right) X_{k,n}^2 \, d\mathbb{P}.
\]

Let us put for \( \varepsilon > 0, \)
\[
L_n[X](\varepsilon) = \sum_{k=1}^{k(n)} \int \left( |X_{k,n}| \geq \varepsilon \right) X_{k,n}^2 \, d\mathbb{P}.
\]

Hence, we get for any \( \varepsilon > 0, \)
\[
(2.5) \quad L_{m(n)}[T](\varepsilon) \leq \ell(n)^2 L_n[X](\varepsilon/\ell(n)).
\]

We are going to specialize these results for stationary associated sequences.
3. $G$-CLT FOR STATIONARY ASSOCIATED SEQUENCES

We will rediscover here the initial Gaussian CLT of Newman without any other condition that $\sqrt{\text{Var}(S_n[X])}$ converges to $\sigma^2 > 0$ for an infinite stationary sequence. But this result will be hardly extended to arrays. This is interesting since non-Gaussian CLT’s usually arise for arrays as we will see it.

After the statement of the results, we will treat important examples of samples of Bernoulli and corrected geometric laws.

Let us begin by evaluating Condition (1.5).

3.1. (1.5) Automatically holds for stationary and associated data.

Let us suppose that each row $X_{k,n}$, $1 \leq k \leq k(n)$ in the array $X$, is centered, square integrable, stationary and associated.

So, because of the stationarity, we have

$$\sqrt{\text{Var}(S_n[X])} = k(n)\sigma_{1,n}^2 + 2\sum_{j=2}^{k(n)}(k(n) - j + 1)\text{Cov}(X_{1,n}, X_{j,n})$$

and next suppose that

$$k(n)\sigma_{1,n}^2 \rightarrow \lambda_1 > 0$$

and

$$2\sum_{j=2}^{k(n)}(k(n) - j + 1)\text{Cov}(X_{1,n}, X_{j,n}) \rightarrow \lambda_2 \geq 0.$$
\[ \text{Var}(Y^*_n) = \text{Var} \left( \sum_{h=1}^{r(n)} X_{m(n)\ell(n)+h,n} \right) \]

\[ = \sum_{h=1}^{r(n)} \text{Var} \left( X_{m(n)\ell(n)+h,n} \right) \]

\[ + \sum_{1 \leq i \neq j \leq r(n)} \text{Cov} \left( X_{m(n)\ell(n)+i,n}; X_{m(n)\ell(n)+j,n} \right) \]

\[ = r(n)\sigma^2_{1,n} + 2 \sum_{j=2}^{r(n)} (r(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}). \]

But

\[ r(n)\sigma^2_{1,n} = \frac{r(n)}{k(n)} \times (k(n)\sigma^2_{1,n}) \to 0 \times \lambda_1 = 0 \]

and

\[ 2 \sum_{j=2}^{r(n)} (r(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) = \frac{2r(n)}{k(n)} \sum_{j=2}^{r(n)} \left( k(n) - \frac{k(n)}{r(n)} (j-1) \right) \text{Cov}(X_{1,n}, X_{j,n}) \]

\[ \leq \frac{2r(n)}{k(n)} \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) \]

\[ \to 0 \times \lambda_2 = 0. \]

So

\[ \text{Var}(Y^*_n) \to 0 \]

and next (1.5) holds.
3.2. **Gaussian CLT for an infinite associated and stationary sequence.**

Let us take for each \( n \geq 1 \),

\[
X_{k,n} = \frac{X_k}{\sqrt{n}}, \quad 1 \leq k \leq n =: k(n),
\]

where the \( X_k \)'s are centered, square integrable, stationary, associated and defined on the same probability space \((\Omega, \mathcal{A}, \mathbb{P})\).

We already know that Condition (1.5) holds (see Subsection 3.1, page 19). We let \( \ell(n) \equiv \ell \) fix at the beginning. So

\[
R_{m(n), \ell(n)}(u) = \frac{u^2}{2} \sum_{1 \leq r \neq s \leq m(n)} \text{Cov}(Y_{r,n}, Y_{s,n})
= \frac{u^2}{2n} \sum_{1 \leq i \neq j \leq m(n) \ell(n)} \text{Cov}(X_i, X_j)
- \frac{u^2}{2n} \sum_{h=1}^{m(n)} \sum_{(r,s) \in I_h; r \neq s} \text{Cov}(X_s, X_r),
\]

with for \( h \in \{1, \cdots, m(n)\} \),

\[
I_h = \{(h - 1)\ell(n) + k, \quad 1 \leq k \leq \ell(n)\}.
\]

Hence by using the stationarity, we have
\[ R_{m(n),\ell(n)}(u) = \frac{u^2}{2n} \left( m(n)\ell(n)\text{Var}(X_1) + 2 \sum_{j=2}^{m(n)\ell(n)} (m(n)\ell(n) - j + 1) \text{Cov}(X_1, X_j) \right) \]

\[ - \frac{u^2}{2n} \sum_{h=1}^{m(n)} \left( \ell(n)\text{Var}(X_1) + 2 \sum_{j=2}^{\ell(n)} (\ell(n) - j + 1) \text{Cov}(X_1, X_j) \right) \]

\[ = \frac{u^2}{2} \frac{m(n)\ell(n)}{n} \left( \frac{2}{m(n)\ell(n)} \sum_{j=2}^{m(n)\ell(n)} (m(n)\ell(n) - j + 1) \text{Cov}(X_1, X_j) \right) \]

\[ - \frac{2}{\ell(n)} \sum_{j=2}^{\ell(n)} (\ell(n) - j + 1) \text{Cov}(X_1, X_j) \right). \]

(3.1)

We know that

\[ \text{Var}(S_n[X]) = \text{Var}(X_1) + \frac{2}{n} \sum_{j=2}^{n} (n - j + 1) \text{Cov}(X_1, X_j), \]

see (Sangharé and Lo (2016)).

Let us suppose that

\[ \text{Var}(S_n[X]) \to \sigma^2 = \text{Var}(X_1) + 2 \sum_{j \geq 2} \text{Cov}(X_1, X_j) =: \text{Var}(X_1) + \sigma_2^2 < +\infty. \]

[Recall that, by associativity, \( \text{Cov}(X_1, X_j) \geq 0 \) for any \( j \geq 2 \).]

Then, from (3.1), we have

\[ R_{m(n),\ell(n)}(u) \to \frac{u^2}{2} \left( \sigma_2^2 - \frac{2}{\ell} \sum_{j=2}^{\ell} (\ell - j + 1) \text{Cov}(X_1, X_j) \right) \]

\[ = \frac{u^2}{2} B(\ell). \]

But, by stationarity, we have
$S_{m(n)}[T] = d \sum_{j=1}^{m(n)} Y_j,n = d \sqrt{\frac{m(n)\ell(n)}{n}} \times \frac{1}{\sqrt{m(n)}} \sum_{j=1}^{m(n)} Z_{j,\ell},$

where the $Z_{j,\ell}$’s, $j = 1, \cdots, m(n)$ are iid and having the same law as

$$Z := \frac{X_1 + \cdots + X_\ell}{\sqrt{\ell}}.$$ 

So

$$S_{m(n)}[T] = d (1 + o(1)) S^*_m[T],$$

with

$$S^*_m[T] = \frac{1}{\sqrt{m(n)}} \sum_{j=1}^{m(n)} Z_{j,\ell},$$

where $Z_1, \cdots, Z_{m(n),\ell}$ are iid $\sim Z$, with

$$\mathbb{E}(Z) = 0 \text{ and } \mathbb{V}(Z) = \text{Var}(S_\ell) =: \sigma^2_\ell.$$ 

By the standard CLT for iid sequence,

$$S^*_m[T] \rightsquigarrow \mathcal{N}(0, \sigma^2_\ell)$$

and this implies that for any $u \in \mathbb{R}$,

$$\psi_{S^*_m[T]}(u) \to e^{-\sigma^2_\ell u^2/2} \text{ as } n \to +\infty.$$ 

Hence for any $\ell$ fixed, for any $u \in \mathbb{R}$,

$$\psi_{S_{m(n)}[Y]}(u) - e^{-\sigma^2 u^2/2} \leq \psi_{S^*_m[T]}(u) - e^{-\sigma^2_\ell u^2/2} + \psi_{S_{m(n)}[Y]}(u) - \psi_{S^*_m[T]}(u) + \psi_{S_{m(n)}[T]}(u) - e^{-u^2\sigma^2_\ell /2}.$$ 

So
\[
\limsup_{n \to +\infty} \left| \psi_{S_{m(n)}}(u) - e^{-\sigma^2 u^2/2} \right| \leq \left| e^{-\sigma^2 u^2/2} - e^{-\sigma^2 u^2/2} \right| + \frac{u^2}{2} B(\ell),
\]

for any \(\ell\). Hence when we let \(\ell \to +\infty\), we arrive at
\[
\limsup_{n \to +\infty} \left| \psi_{S_{m(n)}}(u) - e^{-\sigma^2 u^2/2} \right| = 0
\]

since \(\sigma^2_1 \to \sigma^2\) and so \(B(\ell) \to 0\). \(\square\)

### 3.3. CLT for an array with associated and stationary sequence of random variables by rows.

Here, we consider an array
\[
X \equiv \left\{ \{X_{k,n}, \ 1 \leq k \leq k(n)\}, \ n \geq 1 \right\}
\]
of random variables defined on the same probability space \((\Omega, A, P)\) such that each row \(\{X_{k,n}, \ 1 \leq k \leq k(n)\}\) is centered, square integrable, stationary and associated and so
\[
\text{Var}(S_n[X]) = k(n)\sigma^2_{1,n} + 2 \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}).
\]

We require that
\[
\text{Var}(S_n[X]) \to \lambda > 0.
\]

But this can be achieved for example if:

\[
(3.2) \quad k(n)\sigma^2_{1,n} \to \lambda_1
\]

and
\begin{equation}
2 \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) \to \lambda_2,
\end{equation}

with $\lambda_1 > 0$ and $\lambda_2 \geq 0$.

We already know that, with conditions (3.2) and (3.3), (1.5) holds (see Subsection 3.1, page 19), but we are not able to directly show that (1.6) holds as we proved it in the last subsection, because each $S_n[X]$ uses here a different sequence of random variables, for any $n \geq 1$, and we do not know how the lines are related, contrary to the situation of an infinite sequence of stationary random variables as in Subsection 3.1.

We need to check that $R_{m(n), \ell(n)}(u)$ goes to zero for $u$ fixed, with

$$R_{m(n), \ell(n)}(u) = \frac{u^2}{2} \left\{ 2 \sum_{j=2}^{\ell(n)} (m(n)\ell(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) - 2m(n) \sum_{j=2}^{\ell(n)} (\ell(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) \right\} =: \frac{u^2}{2} (A_n - B_n).$$

The general conditions we may require are:

$$A_n \to \lambda_2 \text{ and } B_n \to \lambda_2.$$

But, we have

$$S_n[X] = S_{m(n)\ell(n)} + Y_n^*$$

and so

$$\text{Var}(S_n[X]) = \text{Var}(S_{m(n)\ell(n)}) + \text{Var}(Y_n^*) + 2 \text{Cov}(S_{m(n)\ell(n)}, Y_n^*),$$

which implies that
\[
1 = \frac{\text{Var}(S_{m(n)\ell(n)})}{\text{Var}(S_n[X])} + \frac{\text{Var}(Y_n^{*})}{\text{Var}(S_n[X])} + 2 \frac{\text{Cov}(S_{m(n)\ell(n)}, Y_n^{*})}{\text{Var}(S_n[X])}.
\]

Moreover, by Cauchy Schwartz inequality, we have
\[
\text{Cov}(S_{m(n)\ell(n)}, Y_n^{*}) \leq \text{Var}(S_{m(n)\ell(n)})^{1/2} \text{Var}(Y_n^{*})^{1/2} \leq \text{Var}(S_n[X])^{1/2} \text{Var}(Y_n^{*})^{1/2},
\]
where we used the association to bound \(\text{Var}(S_{m(n)\ell(n)})\) by \(\text{Var}(S_n[X])\).

Hence, by using Condition (1.5) and the hypotheses (3.2) and (3.3), we get
\[
\frac{\text{Var}(S_{m(n)\ell(n)})}{\text{Var}(S_n[X])} \rightarrow 1.
\]

Now, we have
\[
\text{Var}(S_{m(n)\ell(n)}) = m(n)\ell(n)\sigma_{1,n}^2 + A_n
\]
and since
\[
\frac{m(n)\ell(n)}{k(n)} \rightarrow 1,
\]
we get
\[
A_n \rightarrow \lambda_2.
\]

Hence, the only condition we require is
\[
(\text{GC}) \quad 2m(n) \sum_{j=2}^{\ell(n)} (\ell(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) \rightarrow \lambda_2.
\]

Let us give two simple conditions under which (GC) holds.

\textbf{(PC1)} For \(\lambda_2 = 0\),
\[
m(n)\ell(n)^2 \sup_{2 \leq j \leq k(n)} \text{Cov}(X_{1,n}, X_{j,n}) \rightarrow 0.
\]
(PC2) For $\lambda_2 > 0$,
\[
\text{Cov}(X_{1,n}, X_{j,n}) = \frac{\lambda_2 + o(1)}{m(n)\ell(n)^2}, \text{ uniformly in } j \in \{2, \cdots, \ell(n)\}.
\]

Let us summarize this into the following theorem.

**Theorem 3.** Let
\[
X \equiv \left\{ \{X_{k,n} \mid 1 \leq k \leq k(n)\}, \ n \geq 1 \right\},
\]
be an array of random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that each row $\{X_{k,n} \mid 1 \leq k \leq k(n)\}$ is centered, square integrable, stationary and associated and that
\[
k(n)\sigma^2_{1,n} \to \lambda_1
\]
and
\[
2 \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) \to \lambda_2,
\]
with $\lambda_1 > 0$ and $\lambda_2 \geq 0$.

If one of the conditions (PC1) or (PC2) holds, then $S_n[X]$ and $S_{m(n)}[T]$ have the same weak limit law or do not.

4. **Two iconic examples**

Let us extend the two classical examples of Poisson-CLT in the independent case to associated case.

**Proposition 1.** Let $X_n \sim B(n, p_n), \ n \geq 1$, with
\[
p_n \to 0 \text{ and } np_n \to \lambda > 0.
\]

Then
\[
X_n \sim \mathcal{P}(\lambda), \text{ as } n \to +\infty.
\]
Proposition 2. Let $X_k \sim NB(k, p_k)$, $k \geq 1$, with

$$p_k \to 1 \quad \text{and} \quad k(1-p_k) \to \lambda > 0.$$ 

Then

$$X_k - k \sim \mathcal{P}(\lambda), \quad \text{as} \quad k \to +\infty.$$ 

Proofs of Proposition 1 and 2 can be found in Lo (2018), chapter 1.

Now, we are going to extend these results for arrays of associated data by row.

Here, we suppose that for each $n \geq 1$, the elements of the row $\{X_{k,n} \mid 1 \leq k \leq k(n)\}$ are associated and for any $k \in \{1, \ldots, k(n)\}$,

$$X_{k,n} \sim B(p_{k,n}), \quad \text{or} \quad X_{k,n} \sim G(p_{k,n}) - 1, \quad 0 < p_{k,n} < 1.$$ 

First, we focus on the stationary case.

(A) Stationary case with $p_{k,n} = p_n$, $1 \leq k \leq k(n)$.

Let us proceed for each case.

(A1) Case of sums of associated Bernoulli laws. We have the following result.

Theorem 4. Let $X$ be an array of by-row stationary associated random variables such that:

(i) $\forall 1 \leq k \leq k(n)$, $X_{k,n} \sim B(p_n)$;

(ii) $p_n \to 0$ and $k(n)p_n \to \lambda_1$, as $n \to +\infty$;

(iii) Condition (3.3) holds (which is justified here by (PC1)).

Then

$$S_n[X] \sim \mathcal{P}(\lambda_1).$$
That result can be extend almost word by word for an array of corrected Geometric laws.

**Case of sums of associated corrected Geometric laws.** We have the following result.

**Theorem 5.** Let \( X \) be an array of by-row stationary associated random variables such that:

(i) each \( X_{k,n}, 1 \leq k \leq k(n) \), follows the corrected Geometric law of parameter \( p_n \), that is \( X_{k,n} \) has the same law of \( Z_n - 1 \), where \( Z_n \) is a Geometric-law of parameter \( p_n \);

(ii) \( p_n \to 1 \) and \( k(n)(1 - p_n) \to \lambda_1 > 0 \), as \( n \to +\infty \);

(iii) Condition (3.3) holds.

Then

\[
S_n[X] \Rightarrow \mathcal{P}(\lambda_1).
\]

**Proof of Theorem 4.** The condition (3.3) implies (1.6) holds. By using the stationarity and Conditions (3.3) and

\[
k(n)p_n \to \lambda_1 > 0,
\]

we have for \( n \) large enough,

\[
\text{Var}(S_n[X]) = (1+o(1)) k(n)p_n + \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n}) \to \lambda = \lambda_1 + \lambda_2 > 0
\]

and hence Condition (1.5) automatically holds (see Subsection 3.1, page 19).

So, for any \( u \in \mathbb{R} \),

\[
\left| \psi_{S_n[X]}(u) - \psi_{S_{m(n)}[T]}(u) \right| \to 0
\]
and

\[ |\psi_{S_{m(n)}}[T](u) - \prod_{j=1}^{m(n)} \psi_{T_{j,n}}(u) - \prod_{j=1}^{m(n)} \psi_{T_{j,n}}(u)| \to 0. \]

Now, we may conclude in the following way: for any \( u \in \mathbb{R} \), for any \( n \geq 1 \),

\[ |\psi_{S_n}[X](u) - \exp(\lambda_1(e^{iu} - 1))| \leq |\psi_{S_n}[X](u) - \psi_{S_{m(n)}}[T](u)| + |\psi_{S_{m(n)}}[T](u) - \prod_{j=1}^{m(n)} \psi_{T_{j,n}}(u)| \]

\[ + \prod_{j=1}^{m(n)} \psi_{T_{j,n}}(u) - \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X'_{h,j,n}}(u) \]

\[ + \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X'_{h,j,n}}(u) - \exp(\lambda_1(e^{iu} - 1)) |\]

\[ = R_{1,n}(u) + R_{2,n}(u) + R_{3,n}(u) + R_{4,n}(u). \]

We already know that for any \( u \in \mathbb{R} \),

\[ R_{1,n}(u) \to 0 \text{ and } R_{2,n}(u) \to 0. \]

Next, we are going to use the simple following fact.

**Fact 1.** Let \( \{(x_i, y_i), i \in \{1, \cdots, n\}, n \geq 1\} \) be complex numbers such that

\[ \forall i \in \{1, \cdots, n\}, |x_i| \leq 1 \text{ and } |y_i| \leq 1. \]

Then

\[ \left| \prod_{i=1}^{n} x_i - \prod_{i=1}^{n} y_i \right| \leq \sum_{i=1}^{n} |x_i - y_i|. \]

Moreover, for any \( u \in \mathbb{R} \),
\( R_{3,n}(u) = \left| \prod_{j=1}^{m(n)} \psi_{T,j,n}(u) - \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X_{h,j,n}}(u) \right| \)

\[ \leq \sum_{j=1}^{m(n)} \left| \psi_{T,j,n}(u) - \prod_{h=1}^{\ell(n)} \psi_{X_{h,j,n}}(u) \right| \quad (L2) \]

\[ \leq \sum_{j=1}^{m(n)} \frac{u^2}{2} \sum_{1 \leq r \neq s \leq \ell(n)} \text{Cov}(X'_{r,j,n}, X'_{s,j,n}) \quad (L3) \]

\[ \leq u^2 m(n) \sup_{1 \leq j \leq m(n)} \sum_{1 \leq r \neq s \leq \ell(n)} \text{Cov}(X'_{r,j,n}, X'_{s,j,n}), \quad (4.1) \]

where we use Fact 1 in the line (L2) and Newman’s Lemma (See Newman and Wright (1981, 1982)) in the line (L3).

Hence, we use the stationarity in formula (4.1) to get, for any \( u \in \mathbb{R} \),

\[ R_{3,n}(u) \leq u^2 m(n) \sup_{1 \leq j \leq m(n)} \sum_{h=2}^{\ell(n)} (\ell(n) - h + 1) \text{Cov}(X_{1,n}, X_{h,n}) \]

\[ \leq u^2 m(n) \ell(n)^2 \sup_{2 \leq h \leq \ell(n)} \text{Cov}(X_{1,n}, X_{h,n}), \]

which converges to zero by (PC1).

Finally, for any \( u \in \mathbb{R} \),

\[ C_n(u) := \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X_{h,j,n}}(u) \]

\[ = \left( (1 - p_n) + p_n e^{iu} \right)^{m(n)\ell(n)} \]

\[ = \left( (1 - p_n) + p_n e^{iu} \right)^{m(n)\ell(n)/k(n)}. \]

Here, by Proposition 1, we have the asymptotic law of
\[ Z_n \sim \mathcal{B}(k(n), p_n), \]

with

\[ p_n \to 0 \text{ and } k(n)p_n \to \lambda_1 \]

to Poisson law \( \mathcal{P}(\lambda) \).

Then for any \( u \in \mathbb{R} \),

\[
((1 - p_n) + p_n e^{iu})^{k(n)} \to \exp(\lambda_1(e^{iu} - 1))
\]

and next

\[ C_n(u) \to \exp(\lambda_1(e^{iu} - 1)) \]

since

\[
\frac{m(n)\ell(n)}{k(n)} \to 1.
\]

The proof is over. \( \Box \)

**Proof of Theorem 5.** We are going to follow the lines of the proof of Theorem 4. Let us put \( q_n := 1 - p_n \). So, by using the stationarity and condition (ii), we have for \( n \) large enough,

\[
\text{Var}(S_n[X]) = k(n) \frac{q_n}{p_n^2} + \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n})
\]

\[
= (1 + o(1)) k(n)q_n + \sum_{j=2}^{k(n)} (k(n) - j + 1) \text{Cov}(X_{1,n}, X_{j,n})
\]

and next by Conditions (ii) and (iii),

\[
\text{Var}(S_n[X]) \to \lambda_1 + \lambda_2 =: \lambda > 0.
\]
Hence, Conditions (1.5) and (1.6) hold (see Subsection 3.1, page 19 and (3.3)) and next for any $u \in \mathbb{R}$,
\[
\left| \psi_{S_n[X]}(u) - \psi_{S_{m(n)}[T]}(u) \right| \to 0
\]
and
\[
\left| \psi_{S_{m(n)}[T]}(u) - \prod_{j=1}^{m(n)} \psi_{T_j,n}(u) \right| \to 0.
\]

Now, we are going to use the same method as the proof of Theorem 4 to show that: for any $u \in \mathbb{R}$, for any $n \geq 1$,
\[
\left| \psi_{S_n[X]}(u) - \exp \left( \lambda_1 (e^{iu} - 1) \right) \right| \leq R_{1,n}(u) + R_{2,n}(u) + R_{3,n}(u) + R_{4,n}(u),
\]
with the $R_{q,n}(u)$, $q \in \{1, 2, 3, 4\}$, are already defined in the proof of Theorem 4 and for any $u \in \mathbb{R}$,
\[
R_{q,n}(u) \to 0 \text{ for } q \in \{1, 2, 3\}.
\]

Finally, for any $u \in \mathbb{R}$,
\[
C_n(u) := \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X_{h,j,n}}(u)
= \left( \frac{p_n}{1 - q_ne^{iu}} \right)^{m(n)\ell(n)}
= \left( \frac{p_n}{1 - q_ne^{iu}} \right)^{k(n)\ell(n)/k(n)}
\]

Let us conclude as follows: let
\[
Z_n \sim NB(k(n), p_n),
\]
with here
\[
p_n \to 1 \text{ and } k(n)q_n \to \lambda_1.
\]
Then, by Proposition 2,

$$Z_n - n \sim \mathcal{P}(\lambda)$$

and this implies that, for any $u \in \mathbb{R}$,

$$\left(\frac{p_n}{1 - q_ne^{iu}}\right)^{k(n)} = \Phi_{Z_n - n}(u) \to \exp(\lambda(e^{iu} - 1))$$

and next

$$C_n(u) \to \exp(\lambda(e^{iu} - 1))$$

since

$$\frac{m(n)\ell(n)}{k(n)} \to 1.$$ 

The proof is over. □

(B) General case. We are going to generalize these results above for an array non necessarily stationary by row.

(B1) Case of sums of associated Bernoulli laws. We have the following result.

**Theorem 6.** Let $X$ be an array of by-row associated random variables such that:

1. $\forall 1 \leq k \leq k(n),$

   $$X_{k,n} \sim \mathcal{B}(p_{k,n});$$

2. as $n \to +\infty$,

   (a) $\overline{p}_n := \sup_{1 \leq k \leq k(n)} p_{k,n} \to 0$,

   $k(n)$

   (b) $\sum_{k=1}^{k(n)} p_{k,n} \to \lambda > 0$, 

   $k(n)$
and

\( (c) \ k(n) \ p_n^2 \to 0 \)

and

\( (d) \ k(n)^2 \sup_{1 \leq r \neq s \leq k(n)} \text{Cov}(X_{r,n}, X_{s,n}) \to 0; \)

(3) as \( n \to +\infty \),

\[ \sum_{h=1}^{r(n)} p_{m(n) \ell(n)+h,n} \to 0; \]

(4)

\( (a) \ \lim_{n \to +\infty} m(n) \sup_{1 \leq j \leq m(n)} \sum_{1 \leq r \neq s \leq \ell(n)} \text{Cov}(X'_{r,j,n}, X'_{s,j,n}) = 0 \)

and

\( (b) \ \lim_{n \to +\infty} \sum_{1 \leq r \neq s \leq r(n)} \text{Cov}(X_{m(n) \ell(n)+r,n}, X_{m(n) \ell(n)+s,n}) \to 0. \)

Then

\[ S_n[X] \rightsquigarrow \mathcal{P}(\lambda). \]

**Proof of Theorem 6.** Throughout this proof, the notation \( \ell_k,n = o_n(1) \), for \( k \) ranging over some set \( I_n \) means that the sequence \( \ell_k,n \) goes to zero as \( n \to +\infty \) uniformly in \( k \in I_n \). So Assumption (2a) means that

\[ p_{k,n} = \mathcal{O}_n(1) \quad \text{and} \quad q_{k,n} = 1 - p_{k,n} = 1 + \mathcal{O}_n(1). \]

We have the convergence of variances since by (2a),

\[ \text{Var}(S_n[X]) = (1 + \mathcal{O}_n(1)) \sum_{k=1}^{k(n)} p_{k,n} + \sum_{1 \leq r \neq s \leq k(n)} \text{Cov}(X_{r,n}, X_{s,n}), \]

which converges by (2b) and (2d).
Next, we can see that Condition (1.5) holds by (3) and (4b). Also, Condition (1.6) holds by (2d). Indeed, that last Condition is controlled by

\[ \sum_{1 \leq r \neq s \leq m(n)} \text{Cov}(T_{r,n}, T_{s,n}) \to 0 \]

and that

\[ \sum_{1 \leq r \neq s \leq m(n)} \text{Cov}(T_{r,n}, T_{s,n}) = \sum_{1 \leq r \neq s \leq m(n) \ell(n)} \text{Cov}(X_{r,n}, X_{s,n}) \]
\[ - \sum_{j=1}^{m(n)} \sum_{1 \leq r \neq s \leq \ell(n)} \text{Cov}(X'_{r,j,n}, X'_{s,j,n}) \]
\[ \leq \sum_{1 \leq r \neq s \leq k(n)} \text{Cov}(X_{r,n}, X_{s,n}) \quad (L2) \]
\[ \leq k(n)^2 \sup_{1 \leq r \neq s \leq k(n)} \text{Cov}(X_{r,n}, X_{s,n}), \]

where the line (L2) is justified by the non-negativity of covariances for associated data.

Hence, we are going to use the same method as in the Proof of Theorem 4 to conclude by showing

\[ \forall u \in \mathbb{R}, \; R_{q,n}(u) \to 0, \; \text{as} \; n \to +\infty \; \text{for} \; q \in \{1, 2, 3, 4\}. \]

But it is clear that for any \( u \in \mathbb{R} \),

\[ R_{q,n}(u) \to 0, \; \text{as} \; n \to +\infty \; \text{for} \; q \in \{1, 2\}. \]

Moreover, we easily see by (4.1) and the hypothesis (4a) that, for any \( u \in \mathbb{R} \),

(4.2) \[ R_{3,n}(u) \to 0, \; \text{as} \; n \to +\infty. \]

Hence, we should show that for any \( u \in \mathbb{R} \),

\[ \overline{\phi}_n(u) := \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X'_{h,j,n}}(u) \to \exp \left( \lambda (e^{iu} - 1) \right) \]
to conclude. But, for $u$ fixed, we have
\[
\phi_n(u) = \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \left(1 + p(j-1)\ell(n) + e^{iu} - 1\right)
\]
and hence
\[
\log \phi_n(u) = \sum_{j=1}^{m(n)} \sum_{h=1}^{\ell(n)} \left( p(j-1)\ell(n) + e^{iu} - 1 + O(1) \right)
\]
By (2b) and (3),
\[
\sum_{k=1}^{m(n)\ell(n)} p_{k,n} \to \lambda
\]
and hence, we conclude by (2c) that,
\[
\log \phi_n(u) \to \lambda(e^{iu} - 1),
\]
that is,
\[
\phi_n(u) \to \exp(\lambda(e^{iu} - 1)) = \phi_{\lambda}(u).
\]

**Remark 1.** The assumption (2c) is very reasonable, since for the stationary case, we have
\[
\tilde{p}_n = p_n
\]
and since
\[
k(n)p_n \to \lambda,
\]
we have
\[
k(n)p_n^2 = \frac{(k(n)p_n)^2}{k(n)} = \frac{\lambda^2(1 + o(1))}{k(n)} \to 0.
\]
(B2) Case of sums of associated corrected Geometric laws. We have the following result.

**Theorem 7.** Let \( X \) be an array of by-row associated random variables such that:

1. \( \forall 1 \leq k \leq k(n), \)
   \[ X_{k,n} = d Z_{k,n} - 1, \quad Z_{k,n} \sim \mathcal{G}(p_{k,n}); \]
2. for \( q_{k,n} = 1 - p_{k,n} \), as \( n \to +\infty, \)
   
   (a) \( \bar{q}_n := \sup_{1 \leq k \leq k(n)} q_{k,n} \to 0, \)

   and

   (b) \( k(n) \bar{q}_n^2 \to 0; \)
3. as \( n \to +\infty, \)
   
   (a) \( \sum_{k=1}^{k(n)} q_{k,n} \to \lambda, \)

   (b) \( \sum_{h=1}^{r(n)} q_{m(n)\ell(n)+h,n} \to 0 \)

   and

   (c) \( k(n)^2 \sup_{1 \leq r \neq s \leq k(n)} \text{Cov}(X_{r,n}, X_{s,n}) \to 0; \)
4. \( \lim_{n \to +\infty} m(n) \sup_{1 \leq j \leq m(n)} \sum_{1 \leq r \neq s \leq \ell(n)} \text{Cov}(X'_{r,j,n}, X'_{s,j,n}) = 0 \)

   and

   (b) \( \lim_{n \to +\infty} \sum_{1 \leq r \neq s \leq r(n)} \text{Cov}(X_{m(n)\ell(n)+r,n}, X_{m(n)\ell(n)+s,n}) \to 0. \)
Then

\[ S_n[X] \xrightarrow{\mathcal{D}} \mathcal{P}(\lambda). \]

**Remark 2.** The assumption (2b) is very reasonable, since for the stationary case, we have

\[ \overline{q}_n = q_n \]

and since

\[ k(n)q_n \to \lambda, \]

we have

\[ k(n)q_n^2 \frac{(k(n)q_n)^2}{k(n)} = \frac{\lambda^2(1 + o(1))}{k(n)} \to 0. \]

**Proof of Theorem 7.** We are going to follow the lines of the proof of Theorem 6. Under the hypotheses, we can easily show that Conditions (1.5) and (1.6) hold and next for any \( u \in \mathbb{R}, \)

\[ R_{q,n}(u) \to 0, \ \text{as} \ \ n \to +\infty \ \text{for} \ q \in \{1, 2, 3\}. \]

Hence, we should show that for any \( u \in \mathbb{R}, \)

\[ \overline{\phi}_n(u) := \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \psi_{X_{h,j,n}}(u) \to \exp \left( \lambda(e^{iu} - 1) \right) \]

to conclude. But, for \( u \) fixed, we have

\[ \overline{\phi}_n(u) = \prod_{j=1}^{m(n)} \prod_{h=1}^{\ell(n)} \left( \frac{p(j-1)\ell(n)+h,n}{1-q(j-1)\ell(n)+h,n e^{iu}} \right) \]

and hence
\[
\log \phi_n(u) = \sum_{j=1}^{m(n)} \sum_{h=1}^{\ell(n)} \left( \log(1 - q_{(j-1)\ell(n)+h,n} e^{iu}) - \log(1 - q_{(j-1)\ell(n)+h,n} e^{iu}) \right) \\
= \sum_{j=1}^{m(n)} \sum_{h=1}^{\ell(n)} \left( -q_{(j-1)\ell(n)+h,n} + q_{(j-1)\ell(n)+h,n} e^{iu} + O(r_n^2) \right) \\
= \left( \sum_{k=1}^{m(n)\ell(n)} q_{k,n} \right) (e^{iu} - 1) + O(k(n) r_n^2).
\]

By (3a) and (3b),
\[
\sum_{k=1}^{m(n)\ell(n)} q_{k,n} \to \lambda
\]
and hence, we conclude by (2b) that,
\[
\log \phi_n(u) \to \lambda(e^{iu} - 1),
\]
that is,
\[
\phi_n(u) \to \exp \left( \lambda(e^{iu} - 1) \right) = \phi_p(\lambda)(u).
\]

5. CONCLUSION

We have obtained a general CLT theory of stationary and non-stationary associated arrays in the frame of Newman approximation under specific conditions. This theory can be improved by trying to loosen the conditions. However, we think they might be optimal since we closely work on the borders of the Newman’s frame. The asymptotic theory on associated data outside that frame is still a major concern and until now, there is not a single result of this.

Now, it is natural to extend our results to functional limit laws (invariance principles) with a deterministic or a random horizon. This will be the main topic of a coming paper.
References

Adekpejou A., Traoré C.M.M., Lo G.S. and Niang A.B and Lo G.S. (2020). Central limit theorems for associated possibly moving partial sums and application to the non-stationary invariance principle. arXiv:2108.10906.

Bulinski A. and Shashkin A.(2007). Limit theorems for associated random fields and related systems. World Scientific Publishing, Singapore.

Burton, R.M., Dabrowski, A.R. and Dehling, H. (1986). An invariance principle for weakly associated random variables, *Stoch. Proc. Appl.*, 23, 301-306.

Cox, J.T. and Grimmett, G. (1984) Central limit theorems for associated random variables and the percolation model, *Ann. Probab.*, 12, 514-528.

Dabrowski, A.R. and Dehling, H. (1988). A Berry-Esseen theorem and a functional law of the iterated logarithm for weakly associated random variables, *Stochastic Process. Appl.*, 30, 247-289.

Esary, J., Proschan, F. and Walkup, D.(1967). Association of random variables with application. *Ann. Math Statist.*, 38.

Fortuin, C., Kastelyn, P. and Ginibre, J.(1971). Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22, 89-103.

Jiazhu, P.(2002). Tail dependence of random variables from ARCH and heavy-tailed bilinear models. *Sciences in China*, 45 (6), Ser. A, 749-760.

Lo, G.S.(2018). Weak Convergence (IA). Sequences of random vectors. SPAS Books Series.(2016). Doi : 10.16929/sbs/2016.0001.

Lo¨eve, M.(1977). *Probability Theory I*. Springer-Verlag, New-York.

Louhichi, S.(2000). Weak convergence for empirical processes of associated sequences. *Ann. Inst. Henri Poincaré, Probabilités et Statistiques* 36 (5), pp. 547-567.

Newman C.M. (1980) Normal fluctuations and the FKG inequalities. *Comm. Math. Phys.* 74, 119-128.

Newman, C.M and Wright, A.L.(1981). An invariance principle for certain dependent sequences. *Ann. probab.*, 9 (4), 671-675.

Newman, C.M and Wright, A.L.(1982). Associated random variables and martingale inequalities. Z. Wahrscheinlichkeitstheorie verw. Gebiete 59, 361-371.
Newman, C.M and Wright, A.L.(1981). An invariance principle for certain dependent sequences. *Ann. probab.*, 9 (4), 671-675.

Newman, C.M and Wright, A.L.(1982). Associated random variables and martingale inequalities. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 59, 361-371.

Oliveira, P.E.(2012). *Asymptotics for Associated Random Variables*. DOI 10.1007/978-3-642-25532-8, Springer-Verlag Berlin Heidelberg.

Prakasa Rao, B. L. S.(2012). *Associated sequences, Demimartingales and Nonparametric Inference. Probability and its applications*. Springer Basel Dordrecht, Heidelberg, London, New York.

A general strong law of large numbers and applications to associated sequences and to extreme value theory, Annales Mathematicae et Informaticae, 45 (2015) pp. 111–132, http://ami.ektf.hu.

Sangaré, H. and Lo, G. S.(2016) A Review on asymptotic normality of sums of associated random variables. *Afrika Statistika*, 11 (1), pp.855-867. Doi : 10.16929/as/2016.855.79. Arxiv 1405.4316.

Sangaré H. and Lo, G. S. (2018). General Central Limit Theorems for Associated Sequences. A Collection of Papers in Mathematics and Related Sciences. Spas Editions, Euclid Series Book, pp. 289-308. Doi: http://dx.doi.org/10.16929/sbs/2018.100-04-01, url: https://projecteuclid.org/euclid.spaseds/1569509478.

Sangaré H., Lo, G. S. and Traoré M.C.M. (2020). Arbitrary functional Glivenko-Cantelli classes and applications to different types of dependence. Far East Journal of Theoretical Statistics. Vol. 60 (1-2), 41-62. ISSN: 0972-0863. http://dx.doi.org/10.17654/TS060020041.

Traoré C.M.M., Lo G.S. and Niang A.B (2016) Invariance principles for random sums of non-stationary independent and associated random variables. arXiv:1610.02700

Yu, H.(1993). A Glivenko-Cantelli lemma and weak convergence for empirical processes of associated sequences. *Probab. Theory Related Fields* 95, 357-370.