ATIYAH-SEGAL THEOREM FOR DELIGNE-MUMFORD STACKS
AND APPLICATIONS

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Abstract. We prove an Atiyah-Segal isomorphism for the higher $K$-theory of coherent sheaves on quotient Deligne-Mumford stacks over $\mathbb{C}$. As an application, we prove the Grothendieck-Riemann-Roch theorem for such stacks. This theorem establishes an isomorphism between the higher $K$-theory of coherent sheaves on a Deligne-Mumford stack and the higher Chow groups of its inertia stack. Furthermore, this isomorphism is covariant for proper maps between Deligne-Mumford stacks.

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1. Introduction

The Grothendieck-Riemann-Roch theorem provides an isomorphism between the rational Grothendieck group of coherent sheaves on a regular scheme and its rational Chow groups. It also shows that this isomorphism is covariant for proper morphisms between schemes. This result of Grothendieck was extended to the case of singular schemes by Baum, Fulton and MacPherson [3]. It was later generalized to the level of higher $K$-theory and higher Chow groups of all quasi-projective (possibly singular) schemes by Bloch [5].

The Riemann-Roch theorem for higher $K$-theory of quasi-projective schemes is arguably one of the most important and deep results in algebraic geometry. The famous result of Riemann and Roch that computes the dimension of the space of sections of a line bundle on a Riemann surface in terms of its topological invariants, is a special case of this. The celebrated index theorem of Atiyah and Singer that computes the index of elliptic operators on a compact manifold, is a differential geometric avatar of the Riemann-Roch theorem.

One reason for the outstanding nature of the Grothendieck-Riemann-Roch theorem is that it identifies two seemingly very different theories in algebraic geometry in a functorial manner. The Atiyah-Segal theorem for higher $K$-theory of Deligne-Mumford stacks further enriches this identification by showing that the higher $K$-theory of coherent sheaves on such stacks is isomorphic to the higher Chow groups of its inertia stack. This isomorphism is covariant for proper maps between Deligne-Mumford stacks.

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way. One of these ($K$-theory) is abstractly defined and hence is much less accessible, and hence, much harder to compute while the other (higher Chow groups) is described in terms of explicit generators and relations and hence, in principle, more computable.

Many of the recent works in moduli theory seek a version of the Riemann-Roch theorem for algebraic stacks. For instance, in Gromov-Witten theory, the virtual cycles, whose degrees give rise to the Gromov-Witten invariants, are the Chern classes of vector bundles on a moduli space (see [4]). These vector bundles can be lifted to virtual vector bundles on the moduli stack, and the Riemann-Roch theorem can then be used to compute the degrees of the virtual cycles. Note that these stacks can be highly singular in general. For example, Kontsevich’s moduli stack $M_{g,n}(X, \beta)$ of stable maps from $n$-pointed stable curves of genus $g$ to a projective variety $X$ is known to be singular even if $X$ is smooth.

In string theory, physicists are often interested in computing some cohomological invariants such as Euler characteristics of orbifolds (see [9], [10]). They want to compute these invariants for an orbifold and its resolution of singularities (the orbifold cohomology). Since the $K$-theory and Chow groups of these orbifolds are quotients of the corresponding groups for the associated quotient stacks, the Riemann-Roch for quotient stacks are the natural tools to analyze these Chow cohomology and Euler characteristics.

The higher $K$-theory of coherent sheaves and vector bundles of algebraic stacks is defined exactly like that of schemes. But there is less clarity on the correct definition of higher Chow groups. The rational Chow groups of Deligne-Mumford stacks were defined independently by Gillet [20] and Vistoli [46]. The integral Chow groups of algebraic stacks with affine stabilizers were defined by Kresch [27]. For stacks which occur as quotients of quasi-projective schemes by actions of linear algebraic groups, the theory of (Borel style) higher Chow groups was defined by Edidin and Graham [12]. This construction is based on an earlier construction of the Chow groups of classifying stacks by Totaro [43]. These Chow groups satisfy all the expected properties of a Borel-Moore homology theory in the category of quotient stacks and representable maps (see [29] for details).

When Vistoli constructed intersection theory on Deligne-Mumford stacks [46], he asked if his Chow groups are related to the $K$-theory of the stack, and secondly, if there exists a Riemann-Roch theorem for proper maps between Deligne-Mumford stacks. Note that for quotient stacks, the Chow groups of Gillet, Vistoli, Kresch and Edidin-Graham all agree with rational coefficients.

When one tries to generalize the Grothendieck-Riemann-Roch theorem to quotient stacks, one runs into two serious obstacles. The first problem is that the direct generalization of the Riemann-Roch theorem for schemes actually fails for quotient stacks. Already for the classifying stack $\mathcal{X} = [\text{Spec}(\mathbb{C})/G]$, where $G = \mathbb{Z}/n$, one knows that $G_0(\mathcal{X})_{\mathbb{C}} \simeq \mathbb{C}[t]/(t^n - 1)$ while $\text{CH}_*(\mathcal{X})_{\mathbb{C}} \simeq \mathbb{C}$. Hence, there can be no Riemann-Roch map $G_0(\mathcal{X})_{\mathbb{C}} \to \text{CH}_*(\mathcal{X})_{\mathbb{C}}$ which is an isomorphism! This shows that the Borel style higher Chow groups of a stack $\mathcal{X}$ are not the right objects which can describe its $K$-theory. A version of Grothendieck-Riemann-Roch theorem using the higher Chow groups of a stack was established in [13] for $G_0$ and in [30] for higher $K$-theory. But these Riemann-Roch maps fail to describe $K$-theory completely.

The second main problem in Riemann-Roch for stacks is that unlike for schemes, a morphism between stacks may not be representable. This does not allow the techniques of Riemann-Roch for schemes to directly generalize to stacks. This creates a very serious obstacle in proving the covariance of any possible Riemann-Roch map.

It was observed by Edidin and Graham in [14] that the inertia stack should be the right object for the Chow groups while studying the Riemann-Roch transformation for
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Using the Chow groups of inertia stack, they proved a version of the Riemann-Roch theorem for the Grothendieck group of stacks when we assume the stack to be smooth Deligne-Mumford and the morphism to be the coarse moduli space map [14].

Other than this, the Riemann-Roch problem, identifying the higher $K$-theory and higher Chow groups of stacks in a functorial way, is completely open. The purpose of this work is to fill this gap in the study of the cohomology theories of separated quotient stacks. In particular, we verify the expectation of Edidin and Graham that the higher $K$-theory (with complex coefficients) of a separated quotient stack $X$ can be completely described in terms of the Borel style higher Chow groups of the inertia stack of $X$.

1.1. Main results. Given a stack $X$ which is of finite type over $\mathbb{C}$, let $I_X$ denote its inertia stack. Let $G_i(X)$ denote the homotopy groups of the Quillen $K$-theory spectrum of the category of coherent sheaves on $X$. Let $\text{CH}_*(X, \bullet)$ denote its higher Chow groups, as defined in [12]. Given a map of stacks $f : X \to Y$, let $f^I : I_X \to I_Y$ denote the induced map on the inertia stacks. In Theorems 1.1, 1.2 and 1.3 below, all $K$-groups and higher Chow groups of stacks are taken with complex coefficients. The main result of this article is the following extension of the Grothendieck-Riemann-Roch theorem to separated quotient stacks.

Theorem 1.1. Let $f : X \to Y$ be a proper morphism of separated quotient stacks of finite type over $\mathbb{C}$ with quasi-projective coarse moduli spaces. Then for any integer $i \geq 0$, there exists a commutative diagram

\[
\begin{array}{ccc}
G_i(X) \xrightarrow{\tau_X} & \text{CH}_*(I_X, i) \\
\downarrow f_* & & \downarrow \bar{f}_* \\
G_i(Y) \xrightarrow{\tau_Y} & \text{CH}_*(I_Y, i)
\end{array}
\]

such that the horizontal arrows are isomorphisms.

The assumption in Theorem 1.1 that the coarse moduli spaces are quasi-projective is made because the theory of higher Chow groups of general schemes is not well behaved and many fundamental results for this theory are not known in the general context. If we ignore this assumption, we can prove the following.

Theorem 1.2. Let $f : X \to Y$ be a proper morphism of separated quotient stacks of finite type over $\mathbb{C}$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
G_0(X) \xrightarrow{\tau_X} & \text{CH}_*(I_X) \\
\downarrow f_* & & \downarrow \bar{f}_* \\
G_0(Y) \xrightarrow{\tau_Y} & \text{CH}_*(I_Y)
\end{array}
\]

such that the horizontal arrows are isomorphisms.

As we have already mentioned, a special case of Theorem 1.2 where $X$ is smooth and $Y$ is its coarse moduli space, is a direct consequence of the main results of [14].

We deduce the Riemann-Roch theorem for stacks as a consequence of the following more general Atiyah-Segal isomorphism which describes the higher $K$-theory of coherent sheaves on a stack in terms of the geometric part of the higher $K$-theory of the inertia stack. For a stack $X$, the higher $K$-theory of coherent sheaves $G_i(X)$ is a module over $K_0(X)$. Let $m_X \subset K_0(X)_\mathbb{C}$ denote the ideal of virtual vector bundles on $X$ whose rank is zero on all connected components of $X$. 
Theorem 1.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of separated quotient stacks of finite type over $\mathbb{C}$. Let $i \geq 0$ be an integer. Then there is a commutative diagram

$$
\begin{array}{ccc}
G_i(\mathcal{X}) & \xrightarrow{\vartheta_{\mathcal{X}}} & G_i(I_{X})_{m_{I_{X}}} \\
f_* & \downarrow & f'_* \\
G_i(\mathcal{Y}) & \xrightarrow{\vartheta_{\mathcal{Y}}} & G_i(I_{Y})_{m_{I_{Y}}} 
\end{array}
$$

such that the horizontal arrows are isomorphisms.

Note that unlike Theorem 1.1 the Atiyah-Segal isomorphism makes no assumption on the coarse moduli spaces.

Another point one needs to note here is that the map $f'_I$ in (1.3) is very subtle. The proper map $f^I : I_X \to I_Y$ induces a push-forward map $f'_I : G_i(I_X) \to G_i(I_Y)$. However, there is no guarantee that $f'_I$ is continuous with respect to the $m_{I_{X}}$-adic topology on $G_i(I_X)$ and the $m_{I_{Y}}$-adic topology on $G_i(I_Y)$. In particular, $f'_I$ does not automatically induce the map on the localizations.

A very crucial part of the proof of Theorem 1.3 is to show that the map $f'_I$ on the right side of (1.3) is defined. Existence of this map is in fact part of an old conjecture of Köck [25]. As part of the proof of Theorem 1.3 therefore, we also prove a version of Köck’s conjecture for separated quotient stacks. We refer to Theorem 6.2 for a precise statement of our result. For a representable map of more general smooth quotient stacks, this was proven by Edidin and Graham [13].

When $G$ is a finite group acting on a compact oriented differentiable manifold, a version of the isomorphism $\vartheta_{\mathcal{X}}$ was proven long ago by Segal. When the finite group $G$ acts on a scheme $X$, a finer version of the isomorphism $\vartheta_{\mathcal{X}}$ was obtained for $\mathcal{X} = [X/G]$ by Vistoli [47]. If we are working with $K$-theory with complex coefficients, Theorem 1.3 thus provides a complete generalization of Vistoli’s theorem to all separated quotient stacks. Additionally, it also proves the covariance of this isomorphism.

The equivariant higher Chow groups of schemes with group action are defined as the ordinary higher Chow groups (see [5]) of the Borel spaces. In particular, these equivariant Chow groups are actually a non-equivariant cohomology theory and much more explicitly defined. On the other hand, the equivariant $K$-theory is a very abstract object and there is no explicit way to compute them. Novelty of Theorems 1.1 and 1.2 is that they provide a formula for the equivariant $K$-theory in terms of the ordinary Chow groups of Borel spaces. Apart from this, they also provide formula to compute the Euler characteristics of coherent sheaves and vector bundles on separated quotient stacks. Note that such a formula was known before only for smooth quotient stacks (see [11] and [14], see also [2] in the topological case).

The results obtained above motivate two important questions for future investigation. The first concerns one objective of the paper, namely, to describe $K$-theory of stacks in terms of more explicit objects like higher Chow groups. Going beyond the Riemann-Roch theorem, it was proven by Bloch-Lichtenbaum [6] and Friedlander-Suslin [19] that the integral $K$-theory of quasi-projective schemes could be described by higher Chow groups in terms of an Atiyah-Hirzebruch spectral sequence. Our results provide a strong indication that there should be a similar spectral sequence consisting of the Borel style higher Chow groups of the inertia stack that should converge to the $K$-theory of the underlying stack. This is a very important but a challenging problem in the study of cohomology theory of stacks. The second question concerns going from Deligne-Mumford to Artin stacks. Here, one could ask if there is an analogue of the Atiyah-Segal isomorphism or the Riemann-Roch isomorphism for Artin quotient stacks. We hope to come back to these questions in future projects.
We end the description of our main results with few comments on their comparison with the Riemann-Roch theorems for stacks that are available in the literature. In [22] and [23], Joshua proved a version of Grothendieck-Riemann-Roch theorem. In his results, the target of the Riemann-Roch map is a version of abstractly defined Bousfield localized topological $K$-theory of the underlying stack. In particular, his Riemann-Roch map is not an isomorphism. In [42], Toen proved a version of the Grothendieck Riemann-Roch theorem for quotient Deligne-Mumford stacks with quasi-projective coarse moduli space. In Toen’s Riemann-Roch theorem, the target of the Riemann-Roch map is a generalized cohomology theory with coefficients in the sheaves of representations such as, the étale and de Rham hypercohomology. In particular, the Riemann-Roch map is not an isomorphism in this case too. To authors’ knowledge, a Riemann-Roch theorem connecting the $K$-theory of stacks with Chow groups first appeared in the work of Edidin and Graham [14]. Our results provide the most general version of the Riemann-Roch theorem of Edidin and Graham.

1.2. Outline of the proofs. Since the proof of Theorem 1.1 is significantly involved, we try here to give a brief sketch of its main steps. Our hope is that this will help the reader keep his/her focus on the final proof and not get intimidated by the several intermediate steps, which are mostly of implementational nature.

The proofs of our Riemann-Roch theorems are deduced from Theorem 1.3. So the most of this paper is devoted to the proof of this theorem. To prove Theorem 1.3 using the fact that we are dealing with quotient stacks, we first break the map $f : \mathcal{X} \rightarrow \mathcal{Y}$ into the product $f = f_2 \circ f_1$ of two maps of the following kinds.

1. We find a separated quotient Deligne-Mumford stack $\mathcal{X}'$ with action by an algebraic group $F$ such that the coarse moduli space map $p_1 : \mathcal{X}' \rightarrow \mathcal{X}'$ is $F$-equivariant. The map $f_1 : \mathcal{X} = [\mathcal{X}'/F] \rightarrow [\mathcal{X}'/F]$ is then the ‘stacky’ coarse moduli space map.

2. There is a $F$-equivariant map $p_1 : \mathcal{X}' \rightarrow \mathcal{Y}'$ of quasi-projective schemes such that $f_2 : [\mathcal{X}'/F] \rightarrow [\mathcal{Y}'/F] = \mathcal{Y}$ is the induced maps between the quotient stacks. In particular, $f_2$ is representable.

The Atiyah-Segal isomorphism for case (2) is proven in §7. Apart from the usage of twisting operators and Morita isomorphisms, this case crucially relies on the Riemann-Roch theorem of [30]. This shows that a part of Atiyah-Segal isomorphism actually uses a weaker version of the Riemann-Roch theorem (see Lemma 8.4).

In order to prove (1), we write $\mathcal{X} = [X/(H \times F)]$ so that $\mathcal{X}' = [X/H]$ and $\mathcal{X}' = X/H$. We then note that the canonical map $\mathcal{X}' \rightarrow \mathcal{X}'$ reduces to the identity map on taking the coarse moduli spaces. This allows us to reduce to the case when $F$ is trivial so that $f_1$ is just the coarse moduli space map. When $\mathcal{X} = [X/G]$ and $f$ is the coarse moduli space map, we first prove the case of the free action and then reduce to this case by finding an equivariant finite surjective cover of $X$ where the group action is free. The full implementation of these steps is the most subtle part of the proof of Theorem 1.3 and is done in §8.

A brief description of the other sections of this paper is as follows. In §2 we recollect some known facts about Deligne-Mumford stacks and prove some geometric properties about them. In §3 we recall some results about the equivariant $K$-theory for proper action. The results here are mostly recollected from [14]. In §4 we define a twisting operator on the equivariant $K$-theory which plays a role in the construction of the Atiyah-Segal transformation. We prove some results associated to the Morita isomorphism in equivariant $K$-theory in §5 and §6 is devoted to the construction of the Atiyah-Segal map. We prove the Atiyah-Segal correspondence for separated quotient stacks in §9. The final section is devoted to deducing the Riemann-Roch theorem from Theorem 1.3.
2. Group actions and quotient stacks

In this section, we set up our assumptions and notations. We also very briefly recall some basic definitions related to quotient Deligne-Mumford stacks. We prove some basic properties of these stacks and group actions on algebraic spaces. These properties will be used throughout the rest of this text.

2.1. Assumptions and Notations. In this text, we work over the base field $k = \mathbb{C}$. By a scheme, we shall mean a reduced quasi-projective scheme over $\mathbb{C}$ and denote this category by $\text{Sch}_{\mathbb{C}}$. We let $\text{Sm}_{\mathbb{C}}$ be the full subcategory of $\text{Sch}_{\mathbb{C}}$ consisting of smooth schemes. All products in the category $\text{Sch}_{\mathbb{C}}$ will be taken over $\text{Spec}(\mathbb{C})$ unless we specifically decorate them. In this text, all stacks and algebraic spaces will be separated and of finite type over $\mathbb{C}$.

A linear algebraic group $G$ is a smooth affine group scheme over $\mathbb{C}$. We assume the action of $G$ on a scheme $X$ is always linear in the sense that there is a $G$-equivariant ample line bundle on $X$. We let $\text{Sch}_{G}^{\mathbb{C}}$ denote the category of quasi-projective $\mathbb{C}$-schemes with linear $G$-action. We let $\text{Sm}_{G}^{\mathbb{C}}$ denote the full subcategory of $\text{Sch}_{G}^{\mathbb{C}}$ consisting smooth $\mathbb{C}$-schemes. We let $\text{Qproj}_{G}^{\mathbb{C}}$ denote the category of $\mathbb{C}$-schemes of finite type $X$ on which $G$ acts properly such that the quotient $X/G$ is quasi-projective. Note that an object of $\text{Qproj}_{G}^{\mathbb{C}}$ is quasi-projective over $\mathbb{C}$ with linear $G$-action (see [26, Remark 4.3]) so that $\text{Qproj}_{G}^{\mathbb{C}}$ is a full subcategory of $\text{Sch}_{G}^{\mathbb{C}}$.

Recall that the Riemann-Roch transformation for schemes is meaningful only when we consider the algebraic $K$-theory and higher Chow groups with rational coefficients. Since our constructions rely on [14] and [15], we shall actually consider these groups with complex coefficients and not just rational. We shall therefore assume throughout this text that all abelian groups are replaced by their base change by $\mathbb{C}$. In particular, all equivariant $K$-theory groups $G_{i}(G, X)$ and all equivariant higher Chow groups $\text{CH}_{i}^{G}(X, i)$ will actually mean $G_{i}(G, X)_{\mathbb{C}}$ and $\text{CH}_{i}^{G}(X, i)_{\mathbb{C}}$, respectively.

2.2. Deligne-Mumford stacks. Recall that a quotient stack is a stack $\mathcal{X}$ equivalent to the stack quotient $[X/G]$ (see [32, Example 2.4.2]), where $X$ is an algebraic space and $G$ is a linear algebraic group acting on $X$. The stack $[\text{Spec}(\mathbb{C})/G]$ is called the classifying stack of $G$, which we shall denote by $BG$.

Recall that a stack $\mathcal{X}$ of finite type over $\mathbb{C}$ is called a Deligne-Mumford stack if the diagonal $\Delta_{\mathcal{X}}: \mathcal{X} \to \mathcal{X} \times_{\mathbb{C}} \mathcal{X}$ is representable, quasi-compact and separated, and there is a scheme $U$ with an étale surjective morphism $U \to \mathcal{X}$. The scheme $U$ is called an atlas of $\mathcal{X}$. The stack $\mathcal{X}$ is called separated if $\Delta_{\mathcal{X}}$ is proper. It is well known that a separated quotient stack over $\mathbb{C}$ is a Deligne-Mumford stack.

Recall from [24] that a Deligne-Mumford stack $\mathcal{X}$ has a coarse moduli space $p: \mathcal{X} \to X$ such that $X$ is an algebraic space. For a Deligne-Mumford stack $\mathcal{X}$, we let $\text{Et}_{\mathcal{X}}$ denote the small étale site of $\mathcal{X}$. An object of $\text{Et}_{\mathcal{X}}$ is a finite type $\mathbb{C}$-scheme $U$ with an étale map $u: U \to \mathcal{X}$. This map is representable (by schemes) if $\mathcal{X}$ is separated. In this text, we shall be dealing with Deligne-Mumford stacks which arise as follows.

**Theorem 2.1** (Deligne-Mumford [8]). Let $X \in \text{Sch}_{\mathbb{C}}$ and let $G$ be a smooth affine group scheme over $\mathbb{C}$ acting on $X$ such that the stabilizers of geometric points are finite. Then the stack quotient $[X/G]$ (see, for example, [46, 7.17]) is a Deligne-Mumford stack. If the stabilizers are trivial, then $[X/G]$ is an algebraic space. Furthermore, the stack is separated if and only if the action is proper.

Recall that it is possible in general that a Deligne-Mumford stack may not be a quotient stack. However, one does not know any example of a separated Deligne-Mumford stack over $\mathbb{C}$ which is not a quotient stack. Moreover, there are definite results (for
Lemma 2.3. We give a very brief sketch of the proofs. The last assertion is [26, Remark 4.3].

Proof. The item (1) is [26, Theorem 4.4] while (2) follows from [16, Theorem 2.14].

Let $H \subset G$ be a closed normal subgroup of a linear algebraic group $G$ and let $F = G/H$. Let $f : X \to Y$ be a $G$-equivariant morphism which is an $H$-torsor. Then $f$ induces an isomorphism of quotient stacks $f : [X/G] \to [Y/F]$.

Proof. It is enough to show the equivalence of the two functors of groupoids on $\text{Sch}_C$. Given $B \in \text{Sch}_C$, a $B$-valued point of $[X/G]$ is the datum $B \xra{q} P \xra{\iota} X$, where $P \in \text{Sch}_C^G$ and $f$ is $G$-equivariant and $p$ is a $G$-torsor. Taking the quotients of $P$ and $X$ by $H \subset G$, we get a datum $B \xra{q} P/H \xra{\iota} Y$ which is a $B$-valued point of $[Y/F]$.

Conversely, given a $B$-valued point of $[Y/F]$, given by the datum $B \xra{g} Q \xra{\iota} Y$, we let $P = X \times_Y Q$ and easily check that $B \xra{q} P \xra{\iota} X$ gives a $B$-valued point of $[X/G]$, where $P \to B$ is the composite $P \to Q \to B$. It is left for the reader to check that this provides inverse natural transformations between the stacks $[X/G]$ and $[Y/F]$. 

To apply Lemma 2.3, consider the following situation. Let $G$ be a linear algebraic group and let $H \subset G$ be a closed subgroup which acts properly on an algebraic space $X$. We have the $H$-equivariant closed embedding $H \times X \hookrightarrow G \times X$, where $h(g, x) = (gh^{-1}, hx)$. This gives rise to the closed embedding of the quotients $\iota : X \xrightarrow{\sim} H \times X \hookrightarrow G \times X$, where the first isomorphism is induced by $x \mapsto (x, e)$. Let $G$ act on $Y := G \times X$ by $g(g', x) = (gg', x)$. One checks that $\iota : X \hookrightarrow Y$ is $H$-equivariant and hence it induces the map between the quotient stacks $\iota : [X/H] \to [Y/H]$. Let $\iota^{GH}_X$ denote the composite map $[X/H] \to [Y/H] \to [Y/G]$.

Lemma 2.4. With the above notations, the following hold.

(1) $Y \in \text{Sch}_C^G$ if $X \in \text{Sch}_C^H$ and $Y \in \text{Sm}^G_C$ if $X \in \text{Sm}^H_C$.

(2) The map $\iota^{GH}_X : [X/H] \to [Y/G]$ is an isomorphism.

(3) If $f : X' \to X$ is an $H$-equivariant map of algebraic spaces with proper $H$-actions and $Y' = G \times H Y$, then $\iota^{GH}_X \circ f = f \circ \iota^{GH}_{X'}$.

Proof. Since $G/H$ is a smooth quasi-projective scheme by [7, Theorem 6.8], part (1) follows from [13, Proposition 23] and [29, Lemma 2.1].

To prove (2), let us define an $H \times G$ action on $G \times X$ by $(h, g) \ast (g', x) = (gg'h^{-1}, hx)$. Similarly let us define an $H \times G$ action on $X$ by $(h, g) \bullet x = hx$. Then the projection map $p : G \times X \to X$ is $(H \times G)$-equivariant and is an $(1_H \times G)$-torsor. Similarly, the map $q : G \times X \to G \times H X$ is $(H \times G)$-equivariant and an $(H \times 1_G)$-torsor.
Letting $H = H \times 1_G \subset H \times G$, we get the maps of stacks

\[
\begin{array}{c}
\xymatrix{[X/H] \ar[r]^\phi & [(G \times X)/(H \times G)] \ar[r]^{\bar{q}} & [Y/G]}
\end{array}
\]

such that $\iota_{X}^{GH} = \bar{q} \circ \iota'$ and $\bar{p} \circ \iota'$ is identity. Since the maps $\bar{p}$ and $\bar{q}$ are isomorphisms by Lemma 2.3, it follows that $\iota'$ and $\iota_{X}^{GH}$ are isomorphisms too. The identity $\iota_{X}^{GH} \circ \bar{f} = \bar{f} \circ \iota_{X}$ follows directly from the proof of Lemma 2.3.

\section{Inertia space and inertia stack}

The Atiyah-Segal and Riemann-Roch transformations for stacks involve the inertia stacks. We recall their definitions and prove some basic properties. Given a group action $\mu: G \times X \to X$, the inertia space $I_X$ is the algebraic space defined by the Cartesian square

\[
\begin{array}{c}
\xymatrix{I_X \ar[r]^{\phi} \ar[d] & X \ar[d]^{\Delta_X} \\
G \times X \ar[r]_{(Id,\mu)} & X \times X}
\end{array}
\]

where $\Delta_X : X \to X \times X$ is the diagonal morphism. It is clear that $I_X \in \text{Sch}^G_G$ if $X \in \text{Sch}^G_G$, where the $G$-action on $I_X$ is induced from the above square. More specifically, the conjugation action of $G$ on itself and its diagonal action on $G \times X$ keeps $I_X$ a $G$-invariant closed subspace of $G \times X$. On geometric points, this inclusion is given by

\[
I_X = \{(g, x) \in G \times X | gx = x\} \quad \text{and} \quad g' \cdot (g, x) = (g'gg'^{-1}, g'x).
\]

One checks that $\phi : I_X \to X$ is a $G$-equivariant morphism which is finite if $G$-action on $X$ is proper (which means $(Id, \mu)$ is a proper map). Furthermore, $\phi$ makes $I_X$ an affine group scheme over $X$ whose geometric fibers are the stabilizers of points on $X$ under its $G$-action. The composite map $\text{fix}_X : I_X \to G \times X \to G$ is $G$-equivariant with respect to the conjugation action of $G$ on itself and the fiber of this map over a point $g \in G$ is the fix point locus $X^g$. We let $I_X^\psi \subset I_X$ denote $\text{fix}_X^{-1}(\psi)$ for a semi-simple conjugacy class $\psi \subset G$ (which is closed). It follows from [13, Remark 4.2] that there is a decomposition $I_X = \coprod_{\psi \in \Sigma^G_G} I_X^\psi$ (see Definition B.7).

If we write $X = [X/G]$, then the quotient stack $[I_X/G]$ is often denoted by $I_X$ and is called the Inertia stack of $X$. It is easy to check that for a finite group $G$, one has $I_BG \simeq [G/G]$, where $G$ acts on itself by conjugation.

\begin{lemma}
Let $G = H \times F$ be a linear algebraic group acting properly on an algebraic space $Z$ such that $f: Z \to X$ is an $F$-torsor of algebraic spaces. Then the diagram

\[
\begin{array}{c}
\xymatrix{I_Z \ar[r] & Z \\
I_X \ar[r] \ar[u] & X \ar[u]}
\end{array}
\]

is Cartesian. In particular, $I_Z \to I_X$ is an $F$-torsor.

\begin{proof}
We consider the commutative diagram

\[
\begin{array}{c}
\xymatrix{I_Z \ar[r]^{p_X} \ar[d] & I_X \ar[r] \ar[d] & I_X \ar[d] \\
Z \ar[r] & X \ar[r] & X}
\end{array}
\]

\end{proof}

\section{Inertia space and inertia stack}

The Atiyah-Segal and Riemann-Roch transformations for stacks involve the inertia stacks. We recall their definitions and prove some basic properties. Given a group action $\mu: G \times X \to X$, the inertia space $I_X$ is the algebraic space defined by the Cartesian square

\[
\begin{array}{c}
\xymatrix{I_X \ar[r]^{\phi} \ar[d] & X \ar[d]^{\Delta_X} \\
G \times X \ar[r]_{(Id,\mu)} & X \times X}
\end{array}
\]

where $\Delta_X : X \to X \times X$ is the diagonal morphism. It is clear that $I_X \in \text{Sch}^G_G$ if $X \in \text{Sch}^G_G$, where the $G$-action on $I_X$ is induced from the above square. More specifically, the conjugation action of $G$ on itself and its diagonal action on $G \times X$ keeps $I_X$ a $G$-invariant closed subspace of $G \times X$. On geometric points, this inclusion is given by

\[
I_X = \{(g, x) \in G \times X | gx = x\} \quad \text{and} \quad g' \cdot (g, x) = (g'gg'^{-1}, g'x).
\]

One checks that $\phi : I_X \to X$ is a $G$-equivariant morphism which is finite if $G$-action on $X$ is proper (which means $(Id, \mu)$ is a proper map). Furthermore, $\phi$ makes $I_X$ an affine group scheme over $X$ whose geometric fibers are the stabilizers of points on $X$ under its $G$-action. The composite map $\text{fix}_X : I_X \to G \times X \to G$ is $G$-equivariant with respect to the conjugation action of $G$ on itself and the fiber of this map over a point $g \in G$ is the fix point locus $X^g$. We let $I_X^\psi \subset I_X$ denote $\text{fix}_X^{-1}(\psi)$ for a semi-simple conjugacy class $\psi \subset G$ (which is closed). It follows from [13, Remark 4.2] that there is a decomposition $I_X = \coprod_{\psi \in \Sigma^G_G} I_X^\psi$ (see Definition B.7).

If we write $X = [X/G]$, then the quotient stack $[I_X/G]$ is often denoted by $I_X$ and is called the Inertia stack of $X$. It is easy to check that for a finite group $G$, one has $I_BG \simeq [G/G]$, where $G$ acts on itself by conjugation.

\begin{lemma}
Let $G = H \times F$ be a linear algebraic group acting properly on an algebraic space $Z$ such that $f: Z \to X$ is an $F$-torsor of algebraic spaces. Then the diagram

\[
\begin{array}{c}
\xymatrix{I_Z \ar[r] & Z \\
I_X \ar[r] \ar[u] & X \ar[u]}
\end{array}
\]

is Cartesian. In particular, $I_Z \to I_X$ is an $F$-torsor.

\begin{proof}
We consider the commutative diagram

\[
\begin{array}{c}
\xymatrix{I_Z \ar[r]^{p_X} \ar[d] & I_X \ar[r] \ar[d] & I_X \ar[d] \\
Z \ar[r] & X \ar[r] & X}
\end{array}
\]

\end{proof}
where $\mathcal{X} = [X/H] \simeq [Z/G]$. Under this identification, the big outer square and the right square are both Cartesian, essentially by definition (see [36 Tag 050P]). It follows at once that the left square is Cartesian (for example, see [17, p. 11]).

**Lemma 2.6.** Let $G$ be a linear algebraic group acting properly on an algebraic space $X$ and let $\psi \subset G$ be a semi-simple conjugacy class. Then the following hold.

1. For any $g \in \psi$, the $G$-invariant subspace $GX^g$ is closed in $X$.
2. The map $G \times X^g \rightarrow I_X^\psi$, given by $(g', x) \mapsto (g'gg'^{-1}, g'x)$, is a $Z_g$-torsor. In particular, there is a $G$-equivariant isomorphism $\psi : G \times Z_g X^g \rightarrow I_X^\psi$ with respect to the $G$-actions given in Lemma 2.4 and (2.3).
3. The map $\mu^\psi : I_X^\psi \rightarrow X^\psi := GX^g$ induced by $\phi$ in (2.2) is finite and surjective $G$-equivariant map.

**Proof.** We first show that $GX^g$ is closed in $X$. Let $x' = g'x$, where $g' \in G$ and $x \in X^g$. We then have $(g'gg'^{-1}, g'x) \in I_X^\psi$ and $g'gg'^{-1} \in \psi$. It follows that $GX^g \subset \phi(I_X^\psi)$. Conversely, any element of $I_X^\psi$ can be written as $(g', x)$ such that $g'x = x$ and $g' \in \psi$ and therefore $\phi(g', x) = x$. As $g' \in \psi$, we can write $g' = hgh^{-1}$ for some $h \in G$. Since $x = g'x = hgh^{-1}x$, we get $h^{-1}x = h^{-1}x$ so that $h^{-1}x \in X^g$. Equivalently, we get $\phi(g', x) = x \in hX^g \subset GX^g$. We have thus shown that $\phi : I_X^\psi \rightarrow X$ is a finite morphism whose image is $GX^g$. We conclude that $GX^g \subset X$ is closed.

The part (2) of the lemma is the restatement of [13, Lemma 4.3] and (3) follows from the proof of (1). \hfill \Box

2.4. Some properties of morphisms between stacks. We now prove some properties of the morphisms between Deligne-Mumford stacks that we shall frequently use.

**Lemma 2.7.** Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. Let $H \subset G$ be a normal subgroup with quotient $F$. Let $Z = X/G$ and $W = X/H$ denote the coarse moduli spaces. Then the following hold.

1. The $F$-action on $W$ is proper.
2. If $X/G$ is quasi-projective, then $W$ is quasi-projective with a linear $F$-action.

**Proof.** As the $G$-action on $X$ is proper, it implies that the $H$-action is also proper. Therefore, $[X/H]$ is a separated Deligne-Mumford stack with $W$ its coarse moduli space. By the Keel-Mori theorem [24], it follows that $W$ is a separated algebraic space over $\mathbb{C}$. As $G$ acts properly on $X$, we can find a finite and surjective $G$-equivariant map $f : X' \rightarrow X$ such that $G$-acts freely on $X'$ with $G$-torsor $X' \rightarrow X'/G$ and the induced maps on the quotients $f : X'/G \rightarrow X/G$ is also finite and surjective (see [12, Proposition 10]).

We first show that $u : T = X'/H \rightarrow W$ is an $F$-equivariant, finite and surjective map. It is clear that the map of stacks $[X'/H] \rightarrow [X/H]$ is finite and surjective. Since $[X/H]$ is a separated quotient Deligne-Mumford stack, the coarse moduli space map $[X/H] \rightarrow W$ is proper, surjective and quasi-finite. Therefore, the composite map $T = [X'/H] \rightarrow [X/H] \rightarrow W$ is finite and surjective. The $F$-equivariance of this map is clear.

To show (1), we need to show that the $F$-action map $\Phi_W : F \times W \rightarrow W \times W$ is proper. For this, we consider the diagram

\[
\begin{array}{ccc}
F \times T & \xrightarrow{\Phi_T} & T \times T \\
\phi \downarrow & & \downarrow u \times u \\
F \times W & \xleftarrow{\Phi_W} & W \times W,
\end{array}
\]

where $\phi = (Id_F, u)$. From what we have shown above, it follows that this diagram commutes and the vertical arrows are finite and surjective. As $F$ acts freely on $T$, the
map $\Phi_T$ is a closed immersion. In particular, the composite $\Phi_W \circ \phi = (u \times u) \circ \Phi_T$ is finite. Since $\phi$ is finite and surjective, it easily follows that $\Phi_W$ is in fact finite.

Suppose now that $X/G$ is quasi-projective. It follows from (1) and Theorem [2.1] that $[W/F] = [(X/H)/F]$ is a separated quotient Deligne-Mumford stack with quasi-projective coarse moduli space $X/G$. Since $W \to W/F = X/G$ is affine (for example, see [18] Proposition 0.7), it follows from [26] Remark 4.3 that $W$ must be quasi-projective with linear $F$-action.

Let $\mathcal{X} = [X/H]$ and $\mathcal{Y} = [Y/F]$ be two separated quotient stacks and let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism. We can then construct a Cartesian diagram

$$
\begin{array}{c}
Z \xrightarrow{s} \mathcal{X}' \xrightarrow{f'} \mathcal{Y} \\
\downarrow t \quad \quad \quad \downarrow p \\
X \xrightarrow{q} \mathcal{X} \xrightarrow{f} \mathcal{Y}.
\end{array}
$$

(2.7)

It follows that $t$ is an $F$-torsor and $s$ is an $H$-torsor. In particular, $Z$ is an algebraic space with $H$ and $F$-actions. Note that if $X$ and $Y$ are schemes, then $Z$ is also a scheme. We let $\gamma = f \circ q$ and $\beta = f' \circ s$. The $H$-action of $X$ induces by pull-back an $H$-action $\nu : H \times Z \to Z$ and the $F$-action of $Y$ induces an $F$-action $\mu : F \times Z \to Z$.

**Lemma 2.8.** With the above notations, the following hold.

1. $F$ and $H$ act on $Z$ such that their actions commute. In particular, $G = H \times F$ acts on $Z$.
2. The action of $G$ on $Z$ is proper.
3. The maps $t$ and $\beta$ are $G$-equivariant.
4. There is a factorization

$$
f : \mathcal{X} \cong [Z/G] \to [(Z/H)/F] \to [Y/F] = \mathcal{Y}
$$

such that the two maps are proper.

5. If $X \in \text{Sch}^H$ and $Y \in \text{Sch}^F$, then $Z \in \text{Sch}^G$.
6. If $X \in \text{Qproj}^H$ and $Y \in \text{Qproj}^F$, then $Z \in \text{Qproj}^G$.

**Proof.** The first assertion is standard and known to the experts. We give a sketch of the construction. We have a commutative diagram

$$
\begin{array}{c}
F \times H \times Z \xrightarrow{\overline{\nu}} F \times Z \xrightarrow{\mu} Z \\
\downarrow \nu^H \quad \quad \quad \downarrow t \\
H \times Z \xrightarrow{\nu^Z} Z \xrightarrow{t} X \\
\downarrow s \\
Z \xrightarrow{s} \mathcal{X}' \xrightarrow{f'} \mathcal{X}.
\end{array}
$$

(2.9)

In the above diagram, the vertical arrows on the left are projections. The top square on the right and the bottom square on the left are Cartesian by definition of $Z$. If we let $\overline{\nu} = (id_F, \nu)$, then the top left square commutes. In particular, we get maps $F \times H \times Z \to (H \times Z) \times_Z (F \times Z) \to H \times Z$ both of which are $F$-equivariant (with respect to the trivial $F$-action on $H$ and $Z$) and the second map and the composite map are both $F$-torsors. It follows that the top left square is Cartesian. It can now be checked that $\theta_1 = \mu \circ \overline{\nu} : F \times H \times Z \to Z$ defines a $G$-action on $Z$ (with $G = H \times F$) which makes the above diagram commute. As the composite vertical arrow on the left is just the projection and $p' \circ s = q \circ t$, it follows that $[Z/G] = \mathcal{X}$ and $Z \to \mathcal{X}$ is the stack quotient map for the $G$-action.
We next consider the diagram

\[
\begin{array}{ccc}
H \times F \times Z & \xrightarrow{\overline{\mu}} & H \times Z \\
\downarrow & & \downarrow e \\
F \times Z & \xrightarrow{\mu} & Z \\
\downarrow t & & \downarrow s \\
Z & \xrightarrow{t} & X' \\
\end{array}
\]

(2.10)

where the left vertical arrows are the projection maps. By a similar argument, one checks that \( \theta_2 = v \circ \overline{\pi} : H \times F \times Z \to Z \) defines a group action making (2.10) commute. Furthermore, (2.9) and (2.10) together show that the stack quotients \([Z/(H \times F)]\) for actions via \( \theta_1 \) and \( \theta_2 \) both coincide with \( \mathcal{X} \). It follows that \( \theta_1 \) and \( \theta_2 \) define the same action on \( Z \) and this implies that

\[
v(h, \mu(f, z)) = v \circ \overline{\pi}(h, f, z) = \mu \circ \overline{\pi}(f, h, z) = \mu(f, v(h, z))
\]

and this shows that the actions of \( H \) and \( F \) on \( Z \) commute. Since \( \mathcal{X} \) is separated, it follows from Theorem 2.1 that the \( G \)-action on \( Z \) is proper. It is also clear that the maps \( t \) and \( \beta = f' \circ s \) are \( G \)-equivariant. This proves (1) and (2) and (3).

By the Keel-Mori theorem, the coarse moduli space map \([Z/H] \to Z/H\) is proper. If we let \( W = Z/H \) and \( W = [W/F] \), it follows that the map \( \mathcal{X} = [Z/G] \to [W/F] \) is also proper. Since \( Y \) is an algebraic space, we have a factorization \([Z/H] = \mathcal{X}' \to W \to Y\).

Hence \( \beta' : W \to Y \) is the map induced on the coarse moduli spaces by the proper map of stacks \( \mathcal{X}' \to Y \), it follows that \( \beta' \) is proper. This proves (4).

We now prove (5) and (6). It follows from (2.7) that there is a Cartesian square

\[
\begin{array}{ccc}
Z & \xrightarrow{(t, \beta)} & Y \\
\downarrow \delta_Y & & \downarrow \delta_Y \\
X \times Y & \xrightarrow{\gamma \times \rho} & Y \times Y.
\end{array}
\]

(2.11)

Since \( Y \) is a separated Deligne-Mumford stack, its diagonal \( \delta_Y \) is representable, proper and quasi-finite. In particular, \( (t, \beta) \) is a representable proper and quasi-finite map of algebraic spaces. Hence, it is finite.

Suppose now that \( X \) and \( Y \) are \( H \)-quasi-projective and \( F \)-quasi-projective, respectively. Then \( X \times Y \) is \( G \)-quasi-projective with respect to the coordinate-wise action. Let \( \mathcal{L} \) be a \( G \)-equivariant ample line bundle on \( X \times Y \). Since \( Z \xrightarrow{(t, \beta)} X \times Y \) is finite, we get \( Z \in \text{Sch}_C \). Since this map is furthermore \( G \)-equivariant, we conclude from [21] Exc. III.5.7] that the pull-back of \( \mathcal{L} \) on \( Z \) is a \( G \)-equivariant ample line bundle. This proves (5). If \( X \in \text{Qproj}_H^G \) and \( Y \in \text{Qproj}_F^C \), then \( Z/G = X/H \) is quasi-projective and hence \( Z \in \text{Qproj}_C^G \). This proves (6). \( \square \)

**Lemma 2.9.** Let \( f : \mathcal{X} = [X/H] \to \mathcal{Y} = [Y/F] \) be a proper map of separated quotient stacks with coarse moduli spaces \( M \) and \( N \), respectively. Then the induced map \( \tilde{f} : M \to N \) is proper. If \( f \) is finite, then so is \( \tilde{f} \).

**Proof.** The lemma is a consequence of the following basic property of separated algebraic spaces of finite type over \( \mathbb{C} \). Let \( W_1 \xrightarrow{f_1} W_2 \xrightarrow{f_2} W_3 \) be morphisms of separated algebraic spaces of finite type over \( \mathbb{C} \) such that \( W_1 \) is a scheme, \( f_1 \) is finite and surjective and \( f_2 \circ f_1 \) is proper. Then \( f_2 \) is proper.

We now prove the lemma using the above fact as follows. Let \( p_X : \mathcal{X} \to M \) and \( p_Y : \mathcal{Y} \to N \) denote the coarse moduli space maps. As in Lemma [2.7] we let \( g : Z \to
Let \([X/H]\) be a finite and surjective map where \(Z\) is a finite type separated \(\mathbb{C}\)-scheme. Let us consider the commutative diagram of stacks

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & [X/H] \\
\downarrow{\bar{g}} & & \downarrow{\bar{f}} \\
M & \xrightarrow{\bar{f}} & [Y/F]
\end{array}
\]

(2.12)

Since \(g\) is finite and surjective, it follows that \(\bar{g}\) is a finite and surjective morphism of algebraic spaces. Since \(f\) and \(p_Y\) are proper, it follows that \(\bar{f} \circ \bar{g}\) is proper. Since \(\bar{g}\) is finite and surjective, \(\bar{f}\) must be proper. If \(f\) is finite, then so is \(\bar{f} \circ g\). In particular, the composite map \(\bar{f} \circ \bar{g} = p_Y \circ f \circ g\) is a finite morphism of finite type separated algebraic spaces with \(Z\) a scheme. Since \(\bar{g}\) is finite and surjective, \(\bar{f}\) must be quasi-finite and proper. Hence, it is finite by \([30\text{ Tag } 05W7]\).

2.5. **Cohomological dimension of morphisms of stacks.** For a stack \(\mathcal{X}\), let \(\text{Coh}_{\mathcal{X}}\) denote the abelian category of coherent sheaves on \(\mathcal{X}\) and let \(D(\mathcal{X})\) denote the unbounded derived category of quasi-coherent sheaves on \(\mathcal{X}\). Given a linear algebraic group \(G\) and an algebraic space \(X\) with \(G\)-action, we let \(\text{Coh}^G_X\) denote the abelian category of \(G\)-equivariant coherent sheaves on \(X\) and let \(D^G(X)\) denote the unbounded derived category of \(G\)-equivariant coherent sheaves on \(X\). It is well known that \(\text{Coh}^G_X \simeq \text{Coh}_{[X/G]}\).

Recall that a proper map \(f : \mathcal{X} \to \mathcal{Y}\) of Deligne-Mumford stacks is said to have finite cohomological dimension if the higher direct image functor \(Rf_* : D(\mathcal{X}) \to D(\mathcal{Y})\) has finite cohomological dimension.

**Lemma 2.10.** Let \(f : \mathcal{X} \to \mathcal{Y}\) be a proper morphism of separated quotient stacks. Then \(f\) has finite cohomological dimension.

**Proof.** We write \(\mathcal{X} = [X/H]\) and \(\mathcal{Y} = [Y/F]\) and follow the notations of (2.7) and Lemma 2.8 in this proof. Let \(\phi : \mathcal{X}' \to W\) be the coarse moduli space map and let \(\beta' : W \to \mathcal{Y}\) be the induced map on the coarse moduli spaces such that \(f' = \beta' \circ \phi\). We have the factorization of \(f\) (see (2.8)):

\[
\mathcal{X} = [Z/G] \xrightarrow{\bar{\phi}} W \xrightarrow{\beta'} [Y/F] = \mathcal{Y}.
\]

(2.13)

Since \(\phi : \mathcal{X}' \to W\) is the coarse moduli space map, it is proper. It follows from the fppf-descent that \(\bar{\phi}\) is also proper. Since the functor \(\phi_* : \text{Coh}_{\mathcal{X}'} \to \text{Coh}_W\) is exact by [11, Lemma 2.3.4], it follows again from the fppf-descent that \(\bar{\phi}_* : \text{Coh}_X \to \text{Coh}_W\) is exact. Since \(\mathcal{X}\) and \(W\) have finite diagonals, this is same as saying that \(\bar{\phi}_* = R\bar{\phi}_*\) at the level of the derived categories of quasi-coherent sheaves.

Next, \(\beta' : W \to \mathcal{Y}\) is a finite type morphism of algebraic spaces of finite type over \(\mathbb{C}\). Hence, it is of finite cohomological dimension. Since \(W = Y \times_Y W\), it follows from the fppf-descent again that \(\beta' : W \to \mathcal{Y}\) is of finite cohomological dimension. We conclude using the Grothendieck spectral sequence that \(f = \beta' \circ \bar{\phi}\) has finite cohomological dimension.

3. **Localization in equivariant \(K\)-theory**

Given a linear algebraic group \(G\) and an algebraic space \(X\) with \(G\)-action, we let \(G(G, X)\) denote the \(K\)-theory spectrum of the category of \(G\)-equivariant coherent sheaves on \(X\). We let \(G_i(G, X)\) denote the homotopy groups of \(G(G, X)\) for \(i \geq 0\) and let \(G_*(G, X) = \oplus_{i \geq 0} G_i(G, X)\). The equivalence of abelian categories \(\text{Coh}^G_X \simeq \text{Coh}_{[X/G]}\) induces a homotopy equivalence of their \(K\)-theory spectra. This equivalence will be used throughout this text.
3.1. Representation ring. Let $R(G)$ denote the representation ring of $G$ which in our notation is same as $K_0(G, \text{Spec } \mathbb{C})$. One knows that $R(G)$ is a finitely generated $\mathbb{C}$-algebra (see [34 Corollary 3.3]). In particular, it is Noetherian. The ideal $m_G \subset R(G)$, defined as the kernel of the rank map $R(G) \to \mathbb{C}$, is called the augmentation ideal of $G$. We refer to [30 Appendix A] for a list of basic properties of equivariant $K$-theory that we shall mostly use in this text.

For a linear algebraic group $G$ over $\mathbb{C}$ and any $g \in G(\mathbb{C})$, we let $C_G(g)$ denote the conjugacy class of $g$ and $Z_g := Z_G(g)$ the centralizer of $g$. A point $g \in G$ will always denote a closed point, unless we particularly specify it. A conjugacy class $\psi = C_G(g)$ is called a semi-simple conjugacy class if $g \in G$ is a semi-simple element of $G$. We recall below some well known facts about the structure of $R(G)$ for which the reader is referred to [14 § 2.1].

For an affine variety $X$, let $C[X]$ denote its coordinate ring. Let $G$ be a reductive group and let $C[G]^G$ denote the ring of regular functions on $G$ which are invariant under the conjugation action. There is a natural ring homomorphism $\chi : R(G) \to C[G]^G$ defined by $\chi([V]) = \chi_V$, where $\chi_V(g) = \text{trace}(g|_V)$.

**Proposition 3.1.** ([14 Propositions 2.2, 2.4]) For a reductive group $G$, the map $\chi : R(G) \to C[G]^G$, taking a virtual representation to its character, is an isomorphism. Furthermore, $m \subset R(G)$ is a maximal ideal if and only if there is a unique semi-simple conjugacy class $\psi$ in $G$ such that $m = \{ v \in R(G) | \chi_v(g) = 0 \text{ for some } g \in \psi \}$.

Let $G$ be any linear algebraic group which is not necessarily reductive. There is a Levi decomposition $G = U \times L$, where $U$ is the unipotent radical of $G$ and $L$ is reductive. If $X \in \text{Sch}_G^c$, the Morita isomorphism (see § 5) shows that there is a weak equivalence of spectra $G(G, U \times X) \simeq G(L, X)$ and the homotopy invariance of the equivariant $G$-theory tells us that $G(G, X) \xrightarrow{\sim} G(G, U \times X)$. It follows that the restriction map $G(G, X) \to G(L, X)$ is a weak equivalence. The same holds for the equivariant higher Chow groups too (see [30 Proposition B.6]). This shows that we can assume our group to be reductive in order to study the equivariant $K$-theory and equivariant higher Chow groups. The special case $R(G) \xrightarrow{\sim} R(L)$ will be frequently used in this text without any further explanation.

For a linear algebraic group $G$ as above and $g \in G$, there is a ring homomorphism $\chi^g : R(G) \to \mathbb{C}$ given by $\chi^g([V]) = \chi^g_{|V}(g)$. As $\chi^g(1) = 1$, this map is surjective. Hence $m_g := \ker(\chi^g)$ is a maximal ideal of $R(G)$. Note that $\chi^g$ depends only on $\psi = C_G(g)$ and hence there is no ambiguity in writing $m_g$ and $m_\psi$ interchangeably. If $G$ is not necessarily reductive, even though there is no identification between $R(G)$ and $C[G]^G$, we can still say the following.

**Proposition 3.2.** ([14 Proposition 2.5]) For a linear algebraic group $G$, the assignment $\psi \mapsto m_\psi$ yields a bijection between the semi-simple conjugacy classes in $G$ and the maximal ideals in $R(G)$.

For a surjective morphism of algebraic groups, the following is the correspondence between the conjugacy classes. We refer to [35 Theorem 2.4.8] for a proof of the first part of the lemma and the remaining part follow easily by direct verification.

**Proposition 3.3.** Let $f : G \to F$ be a surjective morphism of algebraic groups. Then $f$ takes a semi-simple (unipotent) conjugacy class to a semi-simple (unipotent) conjugacy class. If $f_* : R(F) \to R(G)$ is the restriction map, then $f_*^{-1}(m_\psi) = m_\phi$ if and only if $f(\psi) = \phi$. If $\text{Ker}(f)$ is finite, then the inverse image of a semi-simple conjugacy class in $F$ is a finite disjoint union of semi-simple conjugacy classes in $G$. 

For a closed immersion of algebraic groups, we have the following correspondence between the conjugacy classes of the two groups. We refer to [14, Propositions 2.3, 2.6, Remark 2.7] for a proof.

**Lemma 3.4.** Let $H \hookrightarrow G$ be a closed embedding of linear algebraic groups. Then the following hold.

1. The ring homomorphism $R(G) \to R(H)$ is finite.
2. For a semi-simple conjugacy class $\psi \subset G$, $\psi \cap H = \psi_1 \amalg \cdots \amalg \psi_n$ is a disjoint union of semi-simple conjugacy classes in $H$.
3. Given a semi-simple conjugacy class $\psi \subset G$ and $\phi \subset H$, $m_\psi R(H) \subset m_\phi$ if and only if $\phi \subset \psi \cap H$.
4. For a semi-simple class $\psi \subset G$, $R(H)_{m_\psi}$ is a semi-local ring with maximal ideals $m_{\psi_1}, \ldots, m_{\psi_n}$.

### 3.2. Fixed point loci for action of semi-simple elements.

If a linear algebraic group $G$ acts on an algebraic space $X$ and if $g \in G$, let $X^g \subset X$ denote the maximal closed subspace of $X$ which is fixed by $g$. In general, $X^g$ is a $\mathbb{Z}_g$-invariant closed subspace of $X$. The following results (see [15, Theorem 5.4]) show that there are only finitely many semi-simple conjugacy classes in $G$ for which the invariant subspaces $X^g$ are non-empty, provided the action is proper.

**Lemma 3.5.** Let $G$ be a diagonalizable torus acting properly on an algebraic space $X$. Then there are only finitely many elements $g \in G$ such that $X^g \neq \emptyset$.

**Proof.** Since $X$ is a separated algebraic space, there is a largest open dense subspace which is the scheme locus of $X$. We denote it by $V$. It follows from Thomason’s generic slice theorem [10, Proposition 4.10] that there is a $G$-invariant non-empty open subset $U \subset V \subset X$ and a subgroup $H \subset G$ such that $H$ acts trivially on $U$ and $F = G/H$ acts freely on $U$ with quotient algebraic space $U/F = U/G$. The freeness of $F$-action on $U$ implies that if there is $g \in G$ such that $U^g \neq \emptyset$, then $g$ must lie in $H$. Furthermore, the properness of $G$-action implies that the stabilizer of any $x \in X$ is finite. It follows that $H$ is finite. The result thus follows for $U$.

Since $Z = X \setminus U$ is a proper $G$-invariant closed subscheme of $X$, the Noetherian induction implies that there are only finitely many $g \in G$ such that $Z^g \neq \emptyset$. As $X^g \neq \emptyset$ if and only if either $U^g \neq \emptyset$ or $Z^g \neq \emptyset$, the lemma follows.

**Proposition 3.6.** Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. Then there exists a finite set of semi-simple conjugacy classes $\Sigma_X^G = \{\psi_1, \ldots, \psi_n\}$ in $G$ such that $X^g \neq \emptyset$ if and only if $g \in \psi_i$ for some $1 \leq i \leq n$.

**Proof.** If $G$ is a diagonalizable torus, then the proposition follows from Lemma 3.5. Let us assume that $G$ is connected but not necessarily diagonalizable. Let $T$ be a fixed maximal torus of $G$. Then the $T$-action on $X$ is also proper. Therefore, it follows from Lemma 3.5 that there are finitely many $g_1, \ldots, g_n \in T$ such that $X^{g_i} \neq \emptyset$. As each $g_i \in T$, it is semi-simple and hence $\psi_i := C_G(g_i)$ is a semi-simple conjugacy class in $G$. We let $\psi_i = C_G(g_i)$ and $\Sigma_X^G = \{\psi_1, \ldots, \psi_n\}$. Note that $\Sigma_X^G$ is never empty because it always contains the conjugacy class of the identity element.

Suppose now that $g \in G$ is such that $X^g \neq \emptyset$. The properness of the $G$-action implies that $g$ must be an element of finite order. In particular, $g$ must be a semi-simple element of $G$ and hence must belong to a maximal torus of $G$. Since all maximal tori of $G$ are conjugate (for example, see [25, Theorem 6.4.1]), there is $h \in T$ such that $g \in C_G(h)$. It is then clear that $X^h \neq \emptyset$ and hence $h \in \psi_i$ for some $1 \leq i \leq n$. But this implies that $g \in \psi_i$. This proves the proposition when $G$ is connected.

If $G$ is not necessarily connected, then any element $g \in G$ such that $X^g \neq \emptyset$ may not lie in a maximal torus and so the above proof breaks. In this case, we reduce to the previous
case as follows. We choose a closed embedding \( G \subseteq GL_n \) and let \( Y = X \times^G GL_n \), where \( G \) acts on \( X \times GL_n \) by \( g(x, g') = (gx, g'g^{-1}) \) and \( GL_n \) acts on \( Y \) which is induced by the action \( h(x, g') = (x, hg') \). It follows from Lemma 2.3 that \([X/G] \simeq [Y/GL_n]\). Since \( G \) acts properly on \( X \), it follows from Theorem 2.4 that \( GL_n \) acts properly on \( Y \).

We prove the proposition by contradiction. Suppose that there are infinitely many semi-simple conjugacy classes \( S = \{\psi_i\} \) in \( G \) such that \( X^g \neq \emptyset \) for \( g \in \psi_i \). Let \( g_i \in \psi_i \) be representatives of the conjugacy classes and set \( \psi_i = C_{GL_n}(g_i^{-1}) \). It follows from Lemma 3.4 that \( \psi' = \{\psi_i | \psi_i \in S\} \) is an infinite set. The proposition will now follow from the case of connected groups if we can show that \( Y^{g^{-1}} \neq \emptyset \) whenever \( g \in \psi_i \) and \( \psi_i \in \psi' \).

So fix any \( \psi_i \in S \) and \( g \in \psi_i \). Since \( X^g \neq \emptyset \), we choose a closed point \( x \in X^g \) and let \( y \) denote the image of \((x, e)\) under the quotient map \( \pi : X \times GL_n \rightarrow Y \). We then have \( g^{-1}y = \pi((hx, g^{-1}h^{-1}) | h \in G) = \pi((hgx, g^{-1}h^{-1}) | h \in G) = \pi((h, h^{-1}) | h \in G) = y \), where the second equality follows from the fact that \( x \in X^g \). It follows that \( Y^{g^{-1}} \neq \emptyset \) and we are done.

### 3.3. Support of equivariant K-theory for proper action.

**Definition 3.7.** Given an algebraic space \( X \) with proper action by an algebraic group \( G \), we let \( \Sigma^G_X \) denote the set of semi-simple conjugacy classes in \( G \) such that \( X^g \neq \emptyset \) if and only if \( g \in \psi \) for some \( \psi \in \Sigma^G_X \). It follows from Proposition 3.6 that \( \Sigma^G_X \) is a non-empty finite set. We let \( S_G \) denote the set of semi-simple conjugacy classes in \( G \).

**Lemma 3.8.** (13) Remark 5.1) Let \( G \) be a diagonalizable torus. Given any \( G \)-space \( \text{X} \) with proper action, there is an ideal \( J \subset R(G) \) such that \( R(G)/J \) has finite support containing \( \Sigma^G_X \) and \( JG_i(G, X) = 0 \) for every \( i \geq 0 \).

**Proof.** By Thomason’s generic slice theorem, there exists a non-empty \( G \)-invariant smooth open dense subspace \( U \subset X \) which is an affine scheme, and there is a finite subgroup \( H \subset G \) such that \( F = G/H \) acts freely on \( U \) with quotient algebraic space \( U/G \simeq U/F \). As \( F \) acts freely on \( U \), we have \( G_i(F, U) \simeq G_i(U/F) \). Since the augmentation ideal \( \ker(K_0(U/F) \rightarrow \mathbb{C}) \) of \( K_0(U/F) \) is nilpotent and since each \( G_i(U/F) \) is an \( K_0(U/F) \)-module, it follows that there exists an integer \( n \gg 0 \) (depending only on \( K_0(U/F) \)) such that \( m^i_{K_0} G_i(U/F) = 0 \) for all \( i \geq 0 \).

We have a short exact sequence of character groups

\[
0 \rightarrow T(F) \rightarrow T(G) \rightarrow T(H) \rightarrow 0
\]

and since the representation ring of a diagonalizable group over \( \mathbb{C} \) is the group ring over its character group, it follows that the map \( R(G)/m^i_{K_0} R(G) \rightarrow R(H) \) is an isomorphism. In particular, \( J_1 := m^i_{K_0} R(G) \) is an ideal in \( R(G) \simeq \mathbb{C} [t_1^{\pm 1}, \cdots, t_r^{\pm 1}] \) (where \( r = \text{rank}(G) \)) whose support is \( H \).

Since the maps \( R(G) \otimes_{R(F)} G_i(F, U) \overset{\cong}{\rightarrow} R(G) \otimes_{R(F)} G_i(U/F) \overset{\cong}{\rightarrow} G_i(U) \) are isomorphisms by 3.6, we conclude that \( J_1 G_i(G, U) = 0 \) for all \( i \geq 0 \). Note that as \( F \) acts freely on \( U \), we must have \( \Sigma^G_X \subset H \). So the lemma is proven for \( U \).

We now finish the proof of the lemma using the Noetherian induction. This induction shows that there is an ideal \( J_2 \subset R(G) \) whose support is a finite subset of \( G \) containing \( \Sigma^G_X \) such that \( J_2 G_i(G, X \setminus U) = 0 \) for all \( i \geq 0 \). It follows that \( J = J_1 J_2 \) annihilates \( G_i(G, X \setminus U) \) and \( G_i(G, U) \) for all \( i \geq 0 \).

The support of \( J \) is a finite set containing \( \Sigma^G_X \) as it is the union of \( \Sigma^G_X \setminus U \) and \( \Sigma^G \). All we are left to show is \( JG_i(G, X) = 0 \) for all \( i \geq 0 \). But this follows immediately from the localization exact sequence \( G_i(G, X \setminus U) \rightarrow G_i(G, X) \rightarrow G_i(G, U) \) (see 3.6 Proposition A.1).
Lemma 3.9. ([14] Remark 5.1) Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. Then there is an ideal $J \subset R(G)$ such that $R(G)/J$ has finite support containing $\Sigma^G_X$ and $J G_i(G, X) = 0$ for every $i \geq 0$.

Proof. If $G$ is a torus it follows from Lemma 3.8. We now assume $G = GL_n$ and let $T$ be a maximal torus of $G$. It follows from Lemma 3.8 that there is an ideal $I \subset R(T)$ which satisfies the claim of the lemma for the $T$-action on $X$.

Since $R(G) \to R(T)$ is a finite map, it follows that $J = I \cap R(G)$ is an ideal $R(G)$ with finite support. As shown in Proposition 3.6, the image of $\Sigma^G_T$ is $\Sigma^G_X$ under the map $\text{Spec}(R(T)) \to \text{Spec}(R(G))$. The split monomorphism of $R(G)$ modules $G_i(G, X) \hookrightarrow G_i(T, X)$, induced by the inclusion $T \subset G$ (see [14] (1.9)), now shows that $J G_i(G, X) = 0$ for all $i \geq 0$. This proves the lemma for $GL_n$.

To prove the general case, we embed $G \hookrightarrow GL_n$ and set $Y = X \times^G GL_n$, where the $G$ and $GL_n$ actions on $X \times GL_n$ are described in the proof of Proposition 3.6. Let $p : X \times GL_n \to X, q : GL_n \to pt, \pi_G : GL_n/G \to pt, \pi_X : X \to pt, r : X \times GL_n \to GL_n$ be the obvious projection maps. Then we see that for any $G$-representation $V$, one has $q^*(V) = V^G \times GL_n$. In particular, for any $GL_n$-representation $W$, we get

$$q^* \circ \text{res}(W) = W \times GL_n = W \times (GL_n/G) = \pi^*_G(W).$$

In particular, the left triangle in the diagram

$$\begin{array}{ccc}
R(GL_n) & \xrightarrow{\text{res}} & R(G) \\
\pi^*_G & \downarrow & \pi^*_X \\
K_0(GL_n, GL_n/G) & \xrightarrow{r_*} & K_0(GL_n, Y)
\end{array}$$

commutes. Since the right square commutes by the functoriality of the Morita isomorphism (see 3), we see that the $R(GL_n)$-module structure on $G^GL_n(Y)$ is same as the restriction of its $R(G)$-module structure acquired via the Morita isomorphism $p^* : G_*(GL_n, Y) \cong G_*(G, X)$. Using this compatibility and the proof of the lemma for $GL_n$, we conclude that there is an ideal $I \subset R(GL_n)$ with finite support containing $\Sigma^G_Y$ such that $IG_*(GL_n, Y) = JG_*(G, X) = 0$, where $J := IR(G) \subset R(G)$.

Since $R(GL_n) \to R(G)$ is finite by Lemma 3.4, we see that $J$ has finite support containing the inverse image of $\Sigma^G_Y$. Moreover, we have shown in the proof of Proposition 3.6 that $\Sigma^G_X$ is contained in the inverse image of $\Sigma^G_Y$. This finishes the proof. □

Let $G$ act properly on an algebraic space $X$. For any semi-simple conjugacy class $\psi \subset G$, let $j^\psi : X_\psi \to X$ denote the image of the map $\phi : I^\psi_X \to X$ (see (2.2)). We know from Lemma 2.6 that $X_\psi \subset X$ is a closed $G$-invariant closed subspace. We shall repeatedly use the following localization theorem in the rest of the text.

Theorem 3.10. ([14] Theorem 3.3]) Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. Then for any semi-simple conjugacy class $\psi \subset G$ and $i \geq 0$, the push-forward map $j^\psi_* : G_i(G, X_\psi) \to G_i(G, X)$ induces an isomorphism of $R(G)$-modules

$$G_i(G, X_\psi)_m \to G_i(G, X)_m.$$
Proposition 3.11. Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. Then for every $i \geq 0$, the natural map
\[
G_i(G, X) \to \oplus_{\psi \in S_G} G_i(G, X)_{m_\psi} = \oplus_{\psi \in \Sigma_X^G} G_i(G, X)_{m_\psi}
\]
is an isomorphism of $R(G)$-modules.

Proposition 3.12. Let $G$ act properly on an algebraic space $X$. The following hold.

1. Given a closed subgroup $H \subset G$ and a semi-simple conjugacy class $\psi \subset G$ with $\psi \cap H = \{ \psi_1, \cdots, \psi_r \}$, we have
\[
G_*(H, X)_{m_\psi} \cong \bigoplus_{i=1}^r G_*(H, X)_{m_{\psi_i}},
\]
where the localization on the left and the right sides are with respect to the $R(G)$-module (via the map $R(G) \to R(H)$) and $R(H)$-module structures on $G_*(H, X)$, respectively. The map in (3.4) is product of various localizations.

2. Given an epimorphism $u : G \twoheadrightarrow F$, the map $R(F) \to R(G)$ induces an isomorphism of $R(F)$-modules
\[
u_* : G_*(G, X) \cong \bigoplus_{\phi \in S_F} G_*(G, X)_{m_\phi}.
\]

Proof. The first part follows directly from Proposition 3.11 in combination with Lemma 3.4 which says that $R(H)_{m_\psi}$ is the semi-local ring with maximal ideals $\{ m_{\psi_1}, \cdots, m_{\psi_r} \}$.

To prove (2), we fix a $\phi \in S_F$ and consider the commutative diagram of $R(F)$-modules
\[
\begin{array}{ccc}
G_*(G, X) & \longrightarrow & \bigoplus_{\psi \in S_G} G_*(G, X)_{m_\psi} \\
\downarrow & & \downarrow \\
G_*(G, X)_{m_\phi} & \longrightarrow & \bigoplus_{\psi \in S_G} G_*(G, X)_{m_\phi},
\end{array}
\]
where the bottom row is the localization of the top row at the maximal ideal $m_\phi$ of $R(F)$.

We now show that the bottom row in (3.6) induces an isomorphism
\[
G_*(G, X)_{m_\phi} \cong \bigoplus_{u(\psi) = \phi} G_*(G, X)_{m_\psi}.
\]

Suppose first $\psi \in S_G$ is such that $u(\psi) = \phi$. Then $m_\phi = u_*^{-1}(m_\psi)$ by Proposition 3.3 and therefore the map $G_*(G, X) \to G_*(G, X)_{m_\phi}$ factors through $G_*(G, X)_{m_\psi}$. This implies in particular that $(G_*(G, X)_{m_\psi})_{m_\phi} = G_*(G, X)_{m_\phi}$.

Suppose next that $u(\psi) \neq \phi$. To show that $(G_*(G, X)_{m_\psi})_{m_\phi} = 0$, we can assume $\psi \in \Sigma_X^G$. Since $u(\psi) \neq \phi$, Proposition 3.3 says that there exists $a \in m_\psi \setminus m_\phi$, which must act invertibly on $G_*(G, X)_{m_\psi}$ and hence on $(G_*(G, X)_{m_\psi})_{m_\phi} = (G_*(G, X)_{m_\phi})_{m_\phi}$. On the other hand, Lemma 3.3 says that there is an ideal $J \subset m_\psi \subset R(G)$ such that $JG_*(G, X) = 0$ and $J$ has finite support. Hence, $m_\psi$ acts nilpotently on $G_*(G, X)_{m_\psi}$ and hence on $(G_*(G, X)_{m_\psi})_{m_\phi}$. But this implies that this module must be zero. $\square$

3.4. The functor of invariants. Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. Let $p : X \to X/G$ be the quotient map. For a $G$-equivariant coherent sheaf $F$ on $X$, one knows that $p_*(F)$ is a coherent sheaf on $X/G$ with $G$-action. Moreover, the subsheaf of $G$-invariant sections $(p_*(F))^G \subset p_*(F)$ is a coherent sheaf on $X/G$. It is shown in [34, Lemma 6.2] that this is an exact functor. Our goal here is to prove a generalization of this construction in the setting of higher $K$-theory. Our result is the following.
Theorem 3.13. Let $H \subset G$ be a normal subgroup with quotient $F$. Let $G$ act properly on an algebraic space $X$ and let $W = X/H$ be the quotient for the $H$-action. Then, there is a functor of `$H$-invariants' which induces an $R(F)$-linear map

\[(3.8) \quad \text{Inv}_X^H : G_*(G, X) \to G_*(F, W)\]

Given a $G$-equivariant proper map $u : X' \to X$ with $W' = X'/H$, there is a commutative diagram

\[(3.9) \quad \begin{array}{ccc}
G_*(G, X') & \xrightarrow{u_*} & G_*(G, X) \\
\downarrow \text{Inv}_X^H & & \downarrow \text{Inv}_X^H \\
G_*(F, W') & \xrightarrow{u_*} & G_*(F, W).
\end{array}\]

Proof. Using the Keel-Mori theorem, we first observe that $W$ and $W'$ are separated algebraic spaces with $F$-actions and there are proper and quasi-finite maps $f^{X,H} : [X/H] \to W$ and $f^{X',H} : [X'/H] \to W'$ which are $F$-equivariant.

We first consider the case when $H = G$ as in [14 § 6.1]. Since $[X/G]$ is a separated Deligne-Mumford stack over $\mathbb{C}$, it follows from [11 Lemma 2.3.4] that $f_*^{X,G}$ is an exact functor that takes coherent sheaves on $[X/G]$ to coherent sheaves on $W$. This induces a proper push-forward map $f_*^{X,G} : G_*(G, X) \simeq G_*([X/G]) \to G_*(W)$. We denote this map by $\text{Inv}_X^G$.

In the general case, we have the proper maps of separated Deligne-Mumford stacks

\[(3.10) \quad [X/G] \xrightarrow{g'} [W/F] \xrightarrow{g} W/F = X/G.\]

Using Lemma 2.10 the Thomason-Trobaugh (see [11 § 3.16.1]) construction of the push-forward map on $K$-theory gives us the maps $G([X/G]) \xrightarrow{g_*} G([W/F]) \xrightarrow{g} G(X/G)$, where the first map is $K_0([W/F])$-linear and the second map is $K_0(W/F)$-linear. In particular, $g_*$ is $R(F)$-linear. We let $g_*$ be denoted by $\text{Inv}_X^H$. The second part of the theorem follows from the first part and the commutative diagram of proper maps

\[(3.11) \quad \begin{array}{ccc}
[X'/G] & \xrightarrow{u_*} & [X/G] \\
\downarrow g' & & \downarrow g \\
[W'/F] & \xrightarrow{u_*} & [W/F]
\end{array}\]

and the covariant functoriality of the push-forward maps $(g \circ u)_* = g_* \circ u_*$ at the level of $K$-theory spectra. \hfill \square

4. Twisting in equivariant $K$-theory

This section is the starting point of our construction of the Atiyah-Segal correspondence. Here, we define the twisting action on higher equivariant $K$-theory. This action was introduced for $G_0$ in [14 § 6.2]. We shall generalize it to higher equivariant $K$-theory by constructing twisting type functors at the level of the exact categories of sheaves. This will play a crucial role in the construction of the Atiyah-Segal map in § 6.

Let $Q$ be a linear algebraic group acting properly on a separated algebraic space $T$. Let $P \subset Q$ be a finite central subgroup which acts trivially on $T$. Note that $P$ is then a finite abelian group. In particular, $P$ is semi-simple. For any $p \in P$, we want to define an automorphism $t_\rho : G_*(Q, T) \to G_*(Q, T)$. Notice that our notations here for the group and the algebraic space deviate from the ones used in the previous sections. The reason for this deviation can be seen in the following situation where we are going to apply the twisting action.
Let $G$ act properly on an algebraic space $X$. Let $g \in G$ be a closed point of finite order. Then $X^g$ is not a $G$-invariant closed subspace but it naturally is $Z_g$-invariant. In this situation, one would like to define a twisting action on $G_s(Z_g, X^g)$. In the above notation, it would translate as $Q := Z_g$, $P = \langle g \rangle$, the finite closed subgroup generated by $g$. As $g$ is a semi-simple element of finite order, $P = \langle g \rangle \subset Z_g$ is a finite closed central subgroup of $Z_g$ which acts trivially on $X^g$.

We now return to the notations of the first paragraph. Since $p \in P$ is a semi-simple element of $Q$, Proposition 3.2 says that there is a unique maximal ideal in $R(Q)$ corresponding to $p$ which we denote by $m_p$. We are interested in defining a twisting action $t_p$ on $G_s(Q, T)$ such that this is an isomorphism and takes the summand $G_s(Q, T)_{m_p}$ to $G_s(Q, T)_{m_1}$.

4.1. **Decomposing the category of $Q$-equivariant coherent sheaves.** As $P$ acts trivially on $T$, the action of $P$ on a $Q$-equivariant coherent sheaf $F$ is fiber-wise over the points of $T$. Equivalently, this action is given in terms of a group homomorphism $P \to \text{Aut}_T(F)$. Further note that every étale open subset of $T$ can be considered as a $P$-invariant open subset with the trivial action of $P$. We let $\hat{P}$ denote the dual of the finite abelian group $P$ (the group of characters of $P$). For each $\chi \in \hat{P}$, we let $C^\chi_T$ be the full subcategory of $Q$-equivariant coherent sheaves defined by

\begin{equation}
\text{Obj}(C^\chi_T) = \{ F \in \text{Coh}^Q_T | h.f = \chi(h)f \ \forall \ U \in \text{Et}_T, f \in F(U), h \in P \}.
\end{equation}

Given $F \in \text{Coh}^Q_T$ and $\chi \in \hat{P}$, we let $F^\chi_T$ be the subsheaf of $F$ defined by

\begin{equation}
F^\chi_T(U) := \{ f \in F(U) | h.f = \chi(h)f \ \forall \ h \in P \}.
\end{equation}

**Lemma 4.1.** With the above notations, the following hold.

1. For each $F \in \text{Coh}^Q_T$ and $\chi \in \hat{P}$, the subsheaf $F^\chi_T$ is coherent and $Q$-equivariant.
2. The natural map $\bigoplus_{\chi \in \hat{P}} F^\chi_T \to F$ is an isomorphism in $\text{Coh}^Q_T$.
3. $C^\chi_T$ is a full abelian subcategory of $\text{Coh}^Q_T$.
4. The natural inclusions $C^\chi_T \hookrightarrow \text{Coh}^Q_T$ induce an equivalence of categories

\[ \Pi \chi \in \hat{P} C^\chi_T \cong \text{Coh}^Q_T. \]

In particular, the natural map of spectra $\Pi_{\chi \in \hat{P}} K(C^\chi_T) \to G(Q, T)$ is a homotopy equivalence, so that $G_s(Q, T) \simeq \bigoplus_{\chi \in \hat{P}} K(C^\chi_T)$.

**Proof.** Since $P$ acts trivially on $T$, we can assume that $T$ is affine in order to prove that $F^\chi_T$ is coherent. If we let $T = \text{Spec}(A)$, it suffices to show that $F^\chi_T$ is an $A$-submodule of $F$ (since $A$ is Noetherian). But this is immediate from (4.2). To prove that $F^\chi_T$ is $Q$-equivariant, it suffices to show that for every open $U \subset T$ and every $q \in Q$, the isomorphism $q^* : F(U) \to F(qU)$ (induced by the $Q$-action on $F$) preserves $F^\chi_T$. That is, we need to show that for every $h \in P$, $\chi \in \hat{P}$ and $f \in F^\chi_T(U)$, the equality $h \cdot q^*(f) = \chi(h)q^*(f)$ holds. But

\[
\begin{align*}
\chi(h)q^*(f) &= q^*(\chi(h)f) = q^*(h^*(f)) = (gh)^*(f) = (hq)^*(f) = h^*(q^*(f)) = h \cdot q^*(f),
\end{align*}
\]

where $=^1$ holds because $q^*$ is $\mathbb{C}$-linear, $=^2$ holds because $f \in F^\chi_T(U)$ and $=^3$ holds because $P$ is central in $Q$. We have thus proven (1).
We now prove (2). Since each $\mathcal{F}_\chi \subset \mathcal{F}$ is $Q$-equivariant, it suffices to show the decomposition as $P$-modules. But this is a direct consequence of the diagonalizability of $P$ and triviality of its action on $T$ (see [14 § 6.2]). We briefly outline its proof. Since $P$ acts trivially on $T$, it suffices to show that the map $\oplus_{\chi \in \hat{P}} \mathcal{F}_\chi \to \mathcal{F}$ is an isomorphism on affine open subsets of $T$. We can thus assume $T$ is affine. Since $P$ is finite abelian, it is diagonalizable. The isomorphism then follows from [38 Lemma 5.6].

Since each $C_T^\chi$ is a full subcategory of $\text{Coh}_T^Q$ by definition, we need to show only that it is closed under taking kernels and cokernels in order to prove (3). For this, we need to show that the kernel and cokernel of a map $F \to G$ in $C_T^\chi$ also lie in $C_T^\chi$. We can prove this also by considering $\mathcal{F}$ and $\mathcal{G}$ as $P$-equivariant coherent sheaves. In this case, it is enough to check this at each affine open in $T$, where one can check directly.

We now prove the first part of (4). As each $C_T^\chi$ is a full abelian subcategory, all we have to show is that the inclusion functors induce the desired equivalence of categories. From (2), it follows that the functor $\oplus_{\chi \in \hat{P}} i_\chi$ is essentially surjective, where $i_\chi : C_T^\chi \to \text{Coh}_T^Q$ is the inclusion. To show it is fully faithful, we can work in the category of $P$-equivariant coherent sheaves. But $P$ is diagonalizable and acts trivially on $T$. We can thus restrict to affine open subsets of $T$. In this case, the assertion follows from [38 Lemma 5.6].

The second part of (4) follows from its first part and [38 § 1, (4), § 2, (8)] since $P$ is finite.

4.2. The twisting map and its properties. Let $P \subset Q$ and $T$ be as above. We let $G^\chi(P, T)$ denote the spectrum $K(C_T^\chi)$ and let $G^\chi_*(P, T)$ denote the sum of its homotopy groups. We define the twisting action of $p \in P$ on $G^\chi_*(P, T)$ by

$$(4.3) \quad t_p(\alpha) = \chi(p^{-1})\alpha \text{ for } \alpha \in G^\chi_*(P, T).$$

Note that this makes sense since we consider $K$-groups with complex coefficients. We extend this action to all of $G_*(Q, T)$ using Lemma 4.1. Since $P$ is finite, every $\chi$ is a multiplicative homomorphism $\chi : P \to \mathbb{C}^\times$ and this implies that $4.3$ defines a $\mathbb{C}$-linear action of $P$ on $G_*(Q, T)$ which keeps each $G^\chi_*(P, T)$ invariant. When we restrict to the case $i = 0$, this action is given by

$$t_p([F]) = t_p(\Sigma_{\chi \in \hat{P}} [F\chi]) = \Sigma_{\chi \in \hat{P}} \chi(p^{-1})[F\chi].$$

This is same as the one considered in [14 § 6.2]. Note that if $\mathcal{F}$ is a $Q$-equivariant vector bundle, then each $\mathcal{F}_\chi$ is also a $Q$-equivariant vector bundle by Lemma 4.4. In particular, the twisting map $t_p$ in (4.4) is defined for vector bundles as well and there is an automorphism $t_p : K_0(Q, T) \to K_0(Q, T)$.

Now, for any $\mathcal{F} \in C_T^\chi$ and $\mathcal{G} \in C_T^{\chi'}$, we have $\mathcal{F} \otimes_{C_T^Q} \mathcal{G} \in C_T^{\chi \chi'}$. Since $G_i(Q, T)$ has a structure of a $K_0(Q, T)$-module via the tensor product of $O_T$-modules, it follows that

$$(4.5) \quad t_p(\alpha \cdot \beta) = t_p(\alpha) \cdot t_p(\beta)$$

for $\alpha \in K_0(Q, T)$ and $\beta \in G_i(Q, T)$. In particular, $t_p : K_0(Q, T) \to K_0(Q, T)$ is a $\mathbb{C}$-algebra automorphism. The following are some more properties of the twisting maps.

Proposition 4.2. Let $Q$ be a linear algebraic group acting properly on algebraic spaces $T$ and $T'$. Let $P \subset Q$ be a finite central subgroup of $Q$ which acts trivially on $T$ and $T'$. Let $f : T' \to T$ be a $Q$-equivariant map. Then the following hold.

1. The map $f^* : K_0(Q, T) \to K_0(Q, T')$ commutes with the twisting action.
2. If $f$ is flat, the map $f^* : G_*(Q, T) \to G_*(Q, T')$ commutes with the twisting action.
3. For $p \in P$, we have $t_{p^{-1}}(G_*(Q, T)_m) = G_*(Q, X)_m$ under the decomposition of $G_*(Q, T)$ given in Proposition [3.7].
Proof. The first and the second properties are immediate from (1.3) since $t_p$ is just a scalar multiplication on $\mathbb{C}$-vector spaces $G_*(Q, -)$. On $R(Q)$, it is an easy verification from (4.4) that $t_p^{-1}(m_p) = m_1$ (for example, see [33 § 6.2]). Using this, the last property follows directly from (4.5) and (1), which together say that $t_p(\alpha \cdot \beta) = t_p(\alpha) \cdot t_p(\beta)$ for $\alpha \in R(Q)$ and $\beta \in G_*(Q, T)$. \qed

Proposition 4.3. Let $f : T' \to T$ be a $Q$-equivariant proper morphism of quasi-projective $\mathbb{C}$-schemes with proper $Q$-actions. Let $P \subset Q$ be a finite central subgroup of $Q$. Then $f$ induces a push-forward map $f_*^\chi : G^\chi_* (P, T') \to G^\chi_* (P, T)$ and a commutative diagram

\[
\begin{align*}
\oplus_{\chi \in P} G^\chi_* (P, T') & \xrightarrow{f^\chi_*} G^\chi_*(Q, T') \\
\oplus_{\chi \in P} G^\chi_* (P, T) & \xrightarrow{f_*} G^\chi_*(Q, T).
\end{align*}
\]

If $T$ and $T'$ are only algebraic spaces, the same holds for $G_0$.

Proof. We shall mimic Quillen’s construction [33] of the push-forward map in $K$-theory, which was adapted to the equivariant set up in [38, § 6.2]. We first observe as an immediate consequence the definition of $C^\chi_*$ and $f_*$ that for any coherent sheaf $\mathcal{F} \in C^\chi_*$, one has $f_*(\mathcal{F}) \in C^\chi_*$.

Let us first assume that $T'$ and $T$ are quasi-projective $\mathbb{C}$-schemes. In this case, $f$ must be projective. Since $G$ acts linearly on $T'$ (and also on $T$), we can find a $Q$-equivariant line bundle $\mathcal{O}(1)$ on $T'$ which is very ample. In particular, this is very ample relative to the $Q$-equivariant map $f : T' \to T$. In this case, there is a $Q$-equivariant factorization $T' \xrightarrow{P} \mathbb{P}_T(\mathcal{E}) \xrightarrow{P} T$ of $f$, where the first map is a closed embedding and $\mathcal{E}$ is a $Q$-equivariant vector bundle on $T$. This gives a $Q$-equivariant surjection $f^* (\mathcal{E}) \to \mathcal{O}(1)$, and hence for every integer $n$, a surjection $f^*(\mathcal{E})^\otimes n \to \mathcal{O}(n)$.

Dualizing this surjection and twisting by $\mathcal{O}(n)$, we get a $Q$-equivariant short exact sequence of vector bundles

\[
0 \to \mathcal{O}_{T'} \to \mathcal{E}'(n) \to \mathcal{E}'' \to 0,
\]

where we take $\mathcal{E}' = (f^* (\mathcal{E})^\otimes n)'$. Tensoring this with any given $\mathcal{F} \in Coh^Q_{T'}$, we get an inclusion $\mathcal{F} \hookrightarrow (\mathcal{F} \otimes \mathcal{E}')^*(n)$ for every integer $n$. Now, we know that there exists $n \geq 0$ such that $R^i f_* ((\mathcal{F} \otimes \mathcal{E}')^*(n)) = 0$ for all $i \geq 1$. We let $\mathcal{G} = (\mathcal{F} \otimes \mathcal{E}')^*(n)$. Since the direct summand $\mathcal{F}_\chi \hookrightarrow \mathcal{F}$ is functorial, we also get an inclusion $\mathcal{F}_\chi \hookrightarrow \mathcal{G}_\chi$ and $R^i f_* (\mathcal{G}_\chi) = 0$ for $i \geq 1$. In particular, if $\mathcal{F} \in C^\chi_*$, so that $\mathcal{F} = \mathcal{F}_\chi$, we get an inclusion $\mathcal{F} \hookrightarrow \mathcal{G}$ with $\mathcal{G} \in C^\chi_*$, such that $R^i f_* (\mathcal{G}) = 0$ for $i \geq 1$.

If we let $Coh^Q_{T'}(f)$ (resp. $C^\chi_{T'}(f)$) denote the subcategory of $Coh^Q_{T'}$ (resp. $C^\chi_{T'}$) which are $f$-acyclic, then we have shown that every $\mathcal{F} \in Coh^Q_{T'}$ (resp. $C^\chi_{T'}$) admits a $Q$-equivariant injection into an object of $Coh^Q_{T'}(f)$ (resp. $C^\chi_{T'}(f)$). We conclude from [33 § 3, Corollary 3] that

\[
K(Coh^Q_{T'}(f)) \simeq K((Coh^Q_{T'}(f))^{op}) \simeq K((Coh^Q_{T'}(f)))^{op} \simeq K(Coh^Q_{T'}(f)).
\]

$K(C^\chi_{T'}(f)) \simeq K((C^\chi_{T'}(f))^{op}) \simeq K((C^\chi_{T'}(f)))^{op} \simeq K(C^\chi_{T'}(f)).$

In particular, the inclusions $Coh^Q_{T'}(f) \subset Coh^Q_{T'}$ and $C^\chi_{T'}(f) \subset C^\chi_{T'}$ induce a commutative diagram of $K$-theory spectra.
Let $f$ be a finite central subgroup of $G$.

As shown in [38, 1.0. 1.11], the restriction of the functor $f_*$ on $G_0(Q,T')$ is given by $f_*(\mathcal{F}) = \sum_i (-1)^i R^i f_* \mathcal{F}$ (see (1.4)). Since each $f_*^\chi$ is simply the restriction of $f_*$ on $G_0^\chi(P,T')$, the same expression holds for $f_*^\chi$ too.

Let us now assume that $T$ and $T'$ are algebraic spaces which are not necessarily quasi-projective. In this case, we first observe that the definition (1.2) makes sense for quasi-coherent sheaves as well. If we denote this category by $QC^\chi_T$, then the proof of Lemma 4.11(3) also shows that $QC^\chi_T$ is abelian. In particular, it has enough injectives. It follows that the functors $R^i f_*$ from the category of $Q$-equivariant quasi-coherent sheaves on $T'$ to that on $T$ restrict to $QC^\chi_T$. Moreover, $R^i f_*$ has finite cohomological dimension by Lemma 2.10. We conclude that the map $f_* : G_0(Q,T') \rightarrow G_0(Q,T)$, given by $f_*(\mathcal{F}) = \sum_i (-1)^i R^i f_* \mathcal{F}$, is well defined and it restricts to a similar push-forward map $f_*^\chi : G_0^\chi(P,T') \rightarrow G_0^\chi(P,T)$. This completes the proof of the proposition.

As an immediate consequence of Proposition 4.3 and (4.3), we get

**Corollary 4.4.** Let $Q$ be a linear algebraic group acting properly on quasi-projective $\mathbb{C}$-schemes $T$ and $T'$. Let $P$ be a finite central subgroup of $Q$ such that $P$ acts trivially on $T$ and $T'$. Let $f : T' \rightarrow T$ be a $Q$-equivariant proper map. Then $f_* : G_0(Q,T') \rightarrow G_0(Q,T)$ is equivariant for the twisting by $P$-action. That is, for each $p \in P$, the diagram

\[
\begin{array}{ccc}
G_i(Q,T') & \xrightarrow{f_\ast} & G_i(Q,T) \\
\downarrow f_* & & \downarrow f_* \\
G_i(Q,T) & \xrightarrow{f_\ast} & G_i(Q,T)
\end{array}
\]

commutes. If $T$ and $T'$ are only algebraic spaces, the same holds for $G_0$.

**Proposition 4.5.** Let $Q$ be a linear algebraic group acting properly on an algebraic space $T$. Let $P$ be a central subgroup of $Q$ of finite order which acts trivially on $T$. Then for any $p \in P$ and $\alpha \in G_*(Q,T)$, we have $\text{Inv}_Q^\alpha \circ t_p(\alpha) = \text{Inv}_T^\alpha(\alpha)$.

**Proof.** For $G_0(Q,T)$, this is shown in [38, Lemma 6.6], and we follow a similar argument. By Lemma 4.11, we can replace $G_*(Q,T)$ by $G^\chi(P,T)$. Now, it follows from (4.11) that no sheaf $\mathcal{F} \in C^\chi_T$ can be $P$-invariant unless $\chi = 1$. Hence, it can not be $Q$-invariant.

If $p : T \rightarrow T/Q$ is the quotient map, it follows that $(p_*(\mathcal{F}))^Q = 0$ unless $\chi = 1$. In
particular, $p_* : G_r^X(P,T) \to G_*(T/Q)$ is the zero map unless $\chi = 1$. When $\chi = 1$, the twisting map $t_\phi : C^T_1 \to C^T_2$ is identity by (13) so the assertion is obvious. □

5. Morita isomorphisms

In this section, we define some Morita isomorphisms for $K$-theory and prove their functorial properties that will be needed for the Atiyah-Segal map in §6.

5.1. The map $\mu_\psi$. Let $G$ be an linear algebraic group acting properly on an algebraic space $X$. Let $\psi$ be a semi-simple conjugacy class in $G$. If $g,h \in \psi$, then there exists $k \in G$ such that $h = k g k^{-1}$. There exists an isomorphism between the centralizers $Z_g$ and $Z_h$ given by

$$\phi : Z_g \xrightarrow{\sim} Z_h; \quad \phi(z) = k z k^{-1}. \tag{5.1}$$

There is an isomorphism $u : X^g \to X^h$ between the fixed point loci, given by $u(x) = k x$. If we let $Z_g$ act on $X^h$ via the canonical action of $Z_h$ composed with $\phi$, then $u$ is $Z_g$-equivariant. We call this the $\star$-action of $Z_g$ on a $Z_h$-space. This allows us to define the isomorphisms at the level of $K$-theory

$$u_* : G_*(Z_g, X^g) \xrightarrow{\sim} G_*(Z_h, X^h); \quad \phi_* : G_*(Z_g, X^h) \xrightarrow{\sim} G_*(Z_h, X^h). \tag{5.2}$$

We let

$$\theta_X = \phi_* \circ u_* : G_*(X_g, X^g) \xrightarrow{\sim} G_*(X_h, X^h). \tag{5.3}$$

We let $G \times Z_g$ act on $G \times X$ by $(g,z) \cdot (h,x) = (ghz^{-1},zx)$. Then $1 \times Z_g$ acts on $G \times X$ freely with quotient $G \times Z^g X$. Note that this action of $1 \times Z_g$ coincides with the $Z_g$-action on $G \times X$ given in Lemma [2, 3]. Let $p_{X^g} : G \times X^g \to X^g$ and $p_{X^h} : G \times X^h \to X^h$ be the projections.

Definition 5.1. The Morita equivalence $\mu_\psi$ is the composition of the functors:

$$\mu_\psi : \text{Coh}_{X_g} \xrightarrow{p_{X^g}} \text{Coh}_{G \times X} \xrightarrow{\text{Inv}^1_{G \times X}} \text{Coh}_{G \times Z_g X^g} \xrightarrow{p_{X^g}} \text{Coh}_{I^G_{X}}. \tag{5.4}$$

In the notations of (2.1), $p_{X^g}$ is same as $\bar{p}_s$. But we have seen in the proof of Lemma [2, 3] that $t'$ and $\bar{p}$ are inverses to each other as maps of stacks. It follows that $p_{X^g} = \bar{p} = t'$. Since $\text{Inv}^1_{G \times X}$ is same as $\bar{q}_s$ (see Theorem 5.13), we see that $\mu_\psi$ is the push-forward map

$$\mu_\psi = (t_{X^g})_* = \bar{q}_s \circ t'_* : \text{Coh}_{[X^g/Z_g]} \xrightarrow{\sim} \text{Coh}_{[I^G_{X}/G]} \tag{5.5}.$$ 

The functor $\mu_\psi$ induces weak-equivalence of the $K$-theory spectra and hence an isomorphism $G_*(Z_g, X^g) \xrightarrow{\sim} G_*(G, I^G_{X})$. This is an $R(G)$-linear isomorphism, where $R(G)$ acts on $G_*(Z_g, X^g)$ via the restriction map $R(G) \to R(Z_g)$ (see [14] Remark 3.2). We shall denote the induced composite map on the $K$-theory also by $\mu_\psi$. Our goal is to show that this map is compatible with respect to the choice of the representatives of $\psi$. So we fix $h = k g k^{-1} \in \psi$.

Lemma 5.2. For an algebraic space $X$ with a proper $G$-action, we have $\mu_h \circ \theta_X = \mu_g$.

Proof. We let $Z_g$ act on $X^h$ and $G \times Z_g$ act on $G \times X^h$ via $\phi$ as

$$g_2 \star z = \phi(g_2) \cdot z \quad \text{and} \quad (g_2, z) \star (g', x) := (g_2 g' \phi(z)^{-1}, \phi(z)x) = (g_2 g' (k z k^{-1})^{-1}, k z k^{-1} x).$$

Since $Z_h$ acts on $X^h$ and $G \times Z_h$ acts on $G \times X^h$ by $z \cdot x = x z$ and $(g_2, z) \cdot (g', x) = (g_2 g' z^{-1}, z x)$, respectively, it follows that these two actions correspond to the above $\star$-actions of $Z_g$ and $G \times Z_g$ on $X^h$ and $G \times X^h$ via the isomorphism $\phi$, respectively. We
thus have a commutative diagram of equivalences

\[
\begin{array}{ccc}
\text{Coh}^{|G|} X_h^h & \xrightarrow{p^h} & \text{Coh}^{|G\times Z_g|} G_{X^h} \text{Inv}^{|G\times X^h|} \text{Coh}^{|G\times Z_g X^h|} p_h \text{Coh}^{|G\times I_X^\psi|} \\
\phi_* & \downarrow (Id \times \phi)_* & \downarrow (Id \times \phi)_* \\
\text{Coh}^{|G|} X_h^h & \xrightarrow{p^h} & \text{Coh}^{|G\times Z_g X^h|} \text{Inv}^{|G\times X^h|} \text{Coh}^{|G\times Z_g X^h|} p_h \text{Coh}^{|G\times I_X^\psi|},
\end{array}
\]

where the top horizontal arrow is same as the bottom one and is induced by the map \( p_h : G \times X^h \to I_X^\psi \), given by \( p_h(g', x) = (g'h^{-1}, g'x) \). This is a \( Z_g \)-torsor with respect to the \( \star \)-action.

We now let \( \delta : G \times X^g \to G \times X^h \) be the map \( \delta(g', x) = (g'k^{-1}, kx) \). With respect to the canonical action of \( G \times Z_g \) on \( G \times X^g \) and \( \star \)-action on \( G \times X^h \), we have

\[
\delta ((g_1, z_1) (g', x)) = \delta ((g_1 g' z_1^{-1}, z_1 x)) = (g_1 g' k^{-1} z_1^{-1} k z_1^{-1} k^{-1}, k x) = (g_1 z_1) \ast (g' k^{-1}, k x)
\]

and this shows that \( \delta \) is \((G \times Z_g)\)-equivariant. Furthermore, one verifies immediately that the diagram

\[
\begin{array}{ccc}
G \times X^g & \xrightarrow{p_X^g} & X^g \\
\delta & \downarrow & \downarrow \mu \\
G \times X^h & \xrightarrow{p_X^h} & X^h
\end{array}
\]

commutes in which all maps are \((G \times Z_g)\)-equivariant with respect to the \( \star \)-action on the bottom row (where we give \( X^g \) and \( X^h \) the trivial \( G \)-action).

We next claim that the map \( p_g : G \times X^g \to I_X^\psi \) is same as the map \( p_h \circ \delta \). But this can be directly checked as follows.

\[
\begin{align*}
p_h \circ \delta (g_1, x) &= p_h (g_1 k^{-1}, k x) = ((g_1 k^{-1}) h (g_1 k^{-1})^{-1}, g_1 x) \\
&= (g_1 k^{-1} k g^{-1} k_1^{-1} g_1 x) = (g_1 g_1^{-1}, g_1 x) \\
&= p_g (g_1, x).
\end{align*}
\]

Combining what we have shown above, we get a commutative diagram

\[
\begin{array}{ccc}
G_*(Z_g, X^g) & \xrightarrow{p_X^h} & G_*(G \times Z_g, G \times X^g) \xrightarrow{\text{Inv}^{|G\times X^g|}} G_* (G \times Z_g X^g) \xrightarrow{p_g} G_* (G, I_X^\psi) \\
\phi_* & \downarrow \delta_* & \downarrow \delta_* & \downarrow \text{Id} \\
G_*(Z_h, X^h) & \xrightarrow{p_X^h} & G_* (G \times Z_h, G \times X^h) \xrightarrow{\text{Inv}^{|G\times X^h|}} G_* (G \times Z_h X^h) \xrightarrow{p_h} G_* (G, I_X^\psi) \\
(1d \times \phi)_* & \downarrow (1d \times \phi)_* & \downarrow \text{Id}
\end{array}
\]

where the top squares commute by \((5.7), (5.8), (5.9)\), and the bottom squares commute by \((5.6)\). Since the composite arrows on the top and the bottom are \( \mu_g \) and \( \mu_h \), respectively, and the composite vertical arrow on the left is \( \theta_X \), the lemma follows.

**Lemma 5.3.** Let \( f : X \to Y \) be a \( G \)-equivariant proper morphism of algebraic spaces with proper \( G \)-actions and let \( f^I : I_X \to I_Y \) be the induced map on the inertia spaces.
Let $g \in G$ be a semi-simple element of $G$ with $\psi$ its conjugacy class. Let $\mu_g^X$ and $\mu_g^Y$ be the Morita isomorphisms for $X$ and $Y$. Then there is a commutative diagram

\[
\begin{array}{ccc}
G_*(Z_g, X^g) & \xrightarrow{\mu_g^X} & G_*(G, I_X^\psi) \\
\downarrow f & & \downarrow f' \\
G_*(Z_g, Y^g) & \xrightarrow{\mu_g^Y} & G_*(G, I_Y^\psi).
\end{array}
\]

**Proof.** It is well known and an elementary exercise that $f^I$ is proper if $f$ is proper. We have the commutative diagram

\[
\begin{array}{ccc}
[X^g/Z_g] & \xrightarrow{i^*} & [I_X^\psi/G] \\
\downarrow & & \downarrow \\
[Y^g/Z_g] & \xrightarrow{i^*} & [I_Y^\psi/G],
\end{array}
\]

where the all maps are proper. The lemma now follows from the functoriality of proper push-forward on $K$-theory of coherent sheaves because $\mu_g^X$ and $\mu_g^Y$ are just the push-forward on $K$-theory induced by $i_{X_g}^G$ and $i_{Y_g}^G$, respectively (see (5.5)). \qed

**Lemma 5.4.** Let $i : X \hookrightarrow Y$ be a $G$-equivariant closed immersion of algebraic spaces with proper $G$-actions with open complement $U$. Let $g \in G$ be a semi-simple element of $G$ with $\psi$ its conjugacy class. Then there is a commutative diagram of $R(Z_g)$-modules

\[
\begin{array}{c}
G_{i+1}(Z_g, U^g) \xrightarrow{i_*} G_i(Z_g, X^g) \xrightarrow{\mu_g} G_i(Z_g, Y^g) \xrightarrow{i_*} G_i(Z_g, U^g) \xrightarrow{\mu_g} G_{i-1}(Z_g, X^g) \\
G_{i+1}(G, I_X^\psi) \xrightarrow{\mu_g} G_i(G, I_X^\psi) \xrightarrow{i_*} G_i(G, I_Y^\psi) \xrightarrow{\mu_g} G_{i-1}(G, I_X^\psi)
\end{array}
\]

commutes with rows being exact.

**Proof.** It is clear that $I_X \hookrightarrow I_Y$ is a closed immersion with complement $I_U$. We have the following diagram of abelian categories

\[
\begin{array}{c}
\text{Coh}^G_{X_g} \xrightarrow{i_*} \text{Coh}^G_{Y_g} \\
\downarrow p'_X \downarrow \downarrow p'_Y \\
\text{Coh}^G_{G \times X_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Y_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g X_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g Y_g} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{Coh}^G_{G \times Z_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g X_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g Y_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g Y_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g Y_g} \xrightarrow{i_*} \text{Coh}^G_{G \times Z_g Y_g} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{Coh}^{I_X^\psi} \xrightarrow{i_*} \text{Coh}^{I_Y^\psi}.
\end{array}
\]

It is shown in Lemma 5.3 that all the squares in (5.14) are commutative. Moreover, in each square, every horizontal functor is an inclusion of a localizing Serre subcategory whose localization is canonically equivalent to the analogous abelian category for $U$ via restriction. In particular, every square gives rise to a commutative diagram of the localization sequences of $K$-theory spectra. This implies that each of the maps $p'_i(\_), \text{Inv}_{G \times (-)}^1 Z_g$ and $p'_g(\_)$ commutes with the localization sequences. It follows that the composite $\mu_g = p_g(-) \circ \text{Inv}_{G \times (-)}^1 Z_g \circ p'_i(\_)$ commutes with the localization sequence. Equivalently, (5.13) commutes. \qed
5.2. The map $\omega_\psi$. We continue with the above notations. Let $\psi \subset G$ be a semi-simple conjugacy class and let $g \in \psi$ with centralizer $Z_g$. We have seen in (5.4) that the Morita map $\mu_g : G_*(Z_g, X^g) \rightarrow G_*(G, I_X^\psi)$ is an $R(G)$-linear isomorphism where $R(G)$ acts on $G_*(Z_g, X^g)$ via the restriction $R(G) \rightarrow R(Z_g)$. This map induces a similar isomorphism of the localizations $\mu_g : G_*(Z_g, X^g)_{m_\psi} \xrightarrow{\sim} G_*(G, I_X^\psi)_{m_\psi}$ at the maximal ideals of $R(G)$ corresponding the semi-simple conjugacy classes $\Phi \in S_G$.

1. If we let $\Phi = \psi$, we have $\psi \cap Z_g = \{\{g\}, \psi_1, \cdots, \psi_r\}$. Letting $G_*(Z_g, X^h)_{nc_\psi} = \bigoplus_{i=1}^r G_*(Z_g, X^h)_{nc_\psi}$, we get

$$G_*(Z_g, X^h)_{m_\psi} \oplus G_*(Z_g, X^h)_{nc_\psi} \xrightarrow{\sim} G_*(G, I_X^\psi)_{m_\psi}. \tag{5.15}$$

2. If we let $\Phi = \{e\}$, we get $\Phi \cap Z_g = \{e\}$ and hence an isomorphism

$$G_*(Z_g, X^h)_{m_1} \xrightarrow{\sim} G_*(G, I_X^\psi)_{m_1}. \tag{5.16}$$

We shall denote this isomorphism by $\mu_g^1$. One should keep in mind that the localization on the left side of $\mu_g^1$ is at the augmentation ideal of $R(Z_g)$ while on the right, it is at the augmentation ideal of $R(G)$.

We shall let $\mu_g^\psi$ denote the composite map

$$\mu_g^\psi : G_*(Z_g, X^h)_{m_g} \rightarrow G_*(Z_g, X^h)_{m_\psi} \xrightarrow{\sim} G_*(G, I_X^\psi)_{m_\psi}. \tag{5.17}$$

Note that the inclusion map in (5.17) is $R(Z_g)_{m_\psi}$-linear while the isomorphism is $R(G)_{m_\psi}$-linear. In particular, $\mu_g^\psi$ is $R(G)_{m_\psi}$-linear.

Definition 5.5. We let $\omega_g$ be the composite map

$$\omega_g = \mu_g^\psi \circ \mu_g^1 : G_*(Z_g, X^h)_{m_g} \rightarrow G_*(G, I_X^\psi)_{m_\psi} \rightarrow G_*(G, X^\psi)_{m_\psi}. \tag{5.18}$$

We now show that $\omega_g$ is an isomorphism of $R(G)_{m_\psi}$-modules and is compatible with the choice of $g \in \psi$. Recall that the conjugation action of $G$ on itself fixes the identity element of $G$ and hence the tangent space of $G$ at its identity element (which is same as its Lie algebra $\mathfrak{g}$) is naturally a representation of $G$. This the called the adjoint representation of $G$. If $\psi$ is a semi-simple conjugacy class in $G$ and $g \in \psi$, we see that $\mathfrak{g}/\mathfrak{z}_g$ is representation of $Z_g$. We let $\lambda_{-1}((\mathfrak{g}/\mathfrak{z}_g)^*) \subset R(Z_g)$ be the top Chern class of the dual representation $(\mathfrak{g}/\mathfrak{z}_g)^*$, given by $\lambda_{-1}((\mathfrak{g}/\mathfrak{z}_g)^*) = \sum_i (-1)^i [\wedge^i ((\mathfrak{g}/\mathfrak{z}_g)^*)]$. We shall denote $\lambda_{-1}((\mathfrak{g}/\mathfrak{z}_g)^*)$ by $l_g$ in the rest of this text.

If we let $h = kgk^{-1}$, the conjugation automorphism $\phi : G \rightarrow G$ (taking $g'$ to $kgk^{-1}$) takes $Z_g$ to $Z_h$. In particular, $\phi_* : R(G) \xrightarrow{\sim} R(G)$ restricts to an isomorphism $R(Z_g) \xrightarrow{\sim} R(Z_h)$ such that

$$\phi_*(\mathfrak{g}) = \mathfrak{g}, \quad \phi_*(\mathfrak{z}_g) = \mathfrak{z}_h \quad \text{and} \quad \phi_*(m_g) = m_h. \tag{5.19}$$

To show that $\omega_g$ is an isomorphism when $X$ is smooth, we need the following representation theoretic fact.

Lemma 5.6. The element $l_g$ is invertible in $R(Z_g)_{m_g}$.

Proof. Using Proposition 3.22, it suffices to show that the character of $l_g$ does not vanish at $g$. Since $g$ is semi-simple, its action on $V = \mathfrak{g}/\mathfrak{z}_g$ is diagonalizable. In particular, we have $l_g = \prod_{i=1}^d (1 - [V_i])$, where each $V_i$ is an one-dimensional representation of $\langle g \rangle$, on which the action is by a character. It is therefore enough to show that $V^*$ has no non-zero vector which is left invariant by $g$. Since every representation of $\langle g \rangle$ is completely reducible, it is equivalent to show that $V$ has no non-zero $g$-invariant vector.
Now, it follows from \([7 \S \text{9.1 (⋆)}]\) that \(3_g = \{ v \in g | \text{Ad}_g(v) = v \}\), where \(\text{Ad}_g\) is the action of \(g\) on \(g\) under the adjoint representation. So we can write \(g = V \oplus 3_g\) as a representation of \(\langle g \rangle\) and it is clear that \(V^g = \{0\}\).

**Lemma 5.7.** The map

\[
\omega_g : G_s(Z_g, X^g)_{m_y} \to G_s(G, X_\psi)_{m_Y}
\]

is an isomorphism. Moreover, we have \(\omega_g = \omega_h \circ \theta_X\) if we let \(h = kgk^{-1} \in \psi\).

**Proof.** We first show that \(w_g\) is an isomorphism. Suppose that \(X\) is smooth. In this case, it follows from \([15, \text{Theorems 5.3, 5.9}]\) that the map \(j^\psi_\ast \circ \mu^\psi_\ast \circ l_g : G_s(Z_g, X^g)_{m_y} \to G_s(G, X)_{m_y}\) is an isomorphism, where \(j^\psi : X_\psi \hookrightarrow X\) is the inclusion and \(l_g : G_s(Z_g, X^g)_{m_y} \to G_s(Z_g, X^g)_{m_y}\) is multiplication by \(l_g \in R(Z_g)_{m_y}\). We conclude from Theorem \([5.10]\) and Lemma \([5.7]\) that \(\omega_g = \mu^\psi_\ast \circ \mu^\psi_\ast\) is an isomorphism. We prove the general case by the Noetherian induction.

If \(X\) is a reduced \(G\)-space, there exists a \(G\)-invariant dense open subspace \(U \subset X\) which is a smooth scheme. Letting \(i : Y = X \setminus U \hookrightarrow X\), we get a diagram

\[
\begin{array}{cccccccc}
G_{i+1}(Z_g, U^g)_{m_y} & \longrightarrow & G_i(Z_g, Y^g)_{m_y} & \longrightarrow & G_i(Z_g, X^g)_{m_y} & \longrightarrow & G_i(Z_g, U^g)_{m_y} & \longrightarrow & G_{i-1}(Z_g, U^g)_{m_y} \\
\omega_g & & \downarrow \omega_g & & \downarrow \omega_g & & \downarrow \omega_g & & \downarrow \omega_g \\
G_{i+1}(G, U_\psi)_{m_y} & \longrightarrow & G_i(G, Y_\psi)_{m_y} & \longrightarrow & G_i(G, X_\psi)_{m_y} & \longrightarrow & G_i(G, U_\psi)_{m_y} & \longrightarrow & G_{i-1}(G, U_\psi)_{m_y},
\end{array}
\]

where the rows are the long exact localization sequences of \(R(G)_{m_y}\)-modules (see \([34, \text{Theorem 2.7}]\)). The first and the fourth vertical arrows (from left) are isomorphisms since \(U\) is smooth. The second and the fifth vertical arrows are isomorphisms by the Noetherian induction. The lemma would now follow if know that all the squares in this diagram commute. All we are therefore left with showing is the commutativity of the diagram

\[
\begin{array}{cccccccc}
G_{i+1}(Z_g, U^g)_{m_y} & \longrightarrow & G_i(Z_g, Y^g)_{m_y} & \longrightarrow & G_i(Z_g, X^g)_{m_y} & \longrightarrow & G_i(Z_g, U^g)_{m_y} \\
\mu^\psi_\ast & & \downarrow \mu^\psi_\ast & & \downarrow \mu^\psi_\ast & & \downarrow \mu^\psi_\ast \\
G_{i+1}(G, I_U)_Y & \longrightarrow & G_i(G, I_Y)_Y & \longrightarrow & G_i(G, I_X)_Y & \longrightarrow & G_i(G, I_U)_Y \\
\mu^\psi_\ast & & \downarrow \mu^\psi_\ast & & \downarrow \mu^\psi_\ast & & \downarrow \mu^\psi_\ast \\
G_{i+1}(G, U_\psi)_{m_y} & \longrightarrow & G_i(G, Y_\psi)_{m_y} & \longrightarrow & G_i(G, X_\psi)_{m_y} & \longrightarrow & G_i(G, U_\psi)_{m_y},
\end{array}
\]

Now, the bottom squares commute because all the vertical arrows are induced by the finite \(G\)-equivariant map \(\mu^\psi : I^\psi_X \to X_\psi\) (see Lemma \([2.6]\) and its restrictions to \(Y\) and \(U\). In particular, the diagram of abelian categories

\[
\begin{array}{cccc}
\text{Coh}^G_{I_X^\psi} & \longrightarrow & \text{Coh}^G_{I_X^\psi} \\
\mu^\psi_\ast & & \downarrow \mu^\psi_\ast \\
\text{Coh}^G_{I_U^\psi} & \longrightarrow & \text{Coh}^G_{I_U^\psi}
\end{array}
\]

commutes. The top horizontal arrow is the inclusion of a localizing Serre subcategory whose localization is canonically equivalent to \(\text{Coh}^G_{I_X^\psi}\) via restriction. Similarly, \(\text{Coh}^G_{I_U^\psi}\) is the localization of the bottom horizontal arrow. The top squares commute by Lemma \([5.4]\). This proves that \(\omega_g\) is an isomorphism.
The second part follows directly from Lemma 5.2 and (5.19) which imply for any $\alpha \in G_i(Z_g; X^g)_{m_g}$ that $\omega_h \circ \theta_X(\alpha) = \mu_h \circ \theta_X(\alpha) = \mu_g \circ \theta_X(\alpha)$. \hfill \Box

6. The Atiyah-Segal map

Let a compact Lie group $G$ act on a compact Hausdorff space $X$. Let $I_G$ denote the augmentation ideal of $R(G)$. Recall from [2] that the Atiyah-Segal theorem in topology provides an isomorphism between the $I_G$-adic completion of the $G$-equivariant topological $K$-theory of $X$ and the usual topological $K$-theory of the Borel space $E G \times^G X$. The algebraic version of Atiyah-Segal theorem was studied in [31]. The $K$-theory of $E G \times^G X$ is often called the geometric part of the equivariant $K$-theory $K(G, X)$. As the goal of this text is to functorially describe the full equivariant $K$-theory of $G$-space in terms of the geometric part of the $K$-theory of the inertia space, we call such a connection the Atiyah-Segal correspondence. In this section, we define the Atiyah-Segal map which will describe this correspondence.

6.1. Two completions of $K$-theory and Köck’s conjecture. Let $X$ be a quotient (not necessarily separated) Deligne-Mumford stack of finite type over $\mathbb{C}$. Let $p : X \to M$ be the coarse moduli space map and let $\{X_1, \cdots, X_r\}$ denote the set of inverse images of the connected components of $M$ under this map. We shall say that $X$ is connected if $r = 1$. Let $K_0(X)$ denote the Grothendieck group of vector bundles on $X$. For each $1 \leq i \leq r$, let $m_i \subset K_0(X)$ denote the set of virtual vector bundles on $X$ whose restrictions to $X_i$ have rank zero. It is easy to see that $m_i$ is a maximal ideal of $K_0(X)$.

We let $m_X = \bigcap_{i=1}^r m_i$ and call this the augmentation ideal of $K_0(X)$.

Let $X$ be an algebraic space with an action of a linear algebraic group $G$ such that $X = [X/G]$ and let $p_X : X \to X$ be the quotient map. The pull-back map $p_X^* : K_0(G, X) \to K_0(X)$ is an isomorphism of rings and $p_X^* (m_X)$ is the set of $G$-equivariant virtual vector bundles on $X$ whose restrictions to each $G$-invariant closed subspace $X_i := (p_X)^{-1}(X_i)$ have rank zero. We shall denote $p_X^* (m_X)$ by $m_X$.

The $G$-equivariant maps $X_i \hookrightarrow X \to \text{Spec}(\mathbb{C})$ induce the ring homomorphisms $R(G) \to K_0(G, X) \to K_0(G, X_i)$ and it follows from this that $\pi^*(m_1) \subset p_X^* (m_X)$, where we use $m_1$ as a shorthand for the augmentation ideal $m_1 \subset R(G)$. In other words, we have the inclusion of ideals $m_1 K_0(X) \subset m_X$. In particular, given any $K_0(X)$-module $N$, there is a natural map of the completions $\tilde{N}_{m_1} \to \tilde{N}_{m_X}$. Since each $G_i(X)$ is a $K_0(X)$-module, we get a natural map $\tilde{G_i(X)}_{m_1} \to \tilde{G_i(X)}_{m_X}$.

Let $G_i(X)_{m_1}$ and $G_i(X)_{m_X}$ denote the localizations of $G_i(X)$ at $m_1$ and $m_X$, respectively. Since $m_1$ is a maximal ideal of $R(G)$ and $m_1 K_0(X) \subset m_X$, it follows that $m_1 = R(G) \cap m_X$. In particular, the localization map $G_i(X) \to G_i(X)_{m_X}$ factors through $G_i(X)_{m_1}$. To compare various localizations and completions of the $K$-theory of stacks, we shall use the following elementary result from commutative algebra.

Lemma 6.1. Let $A$ be a commutative ring (not necessarily Noetherian) and let $m$ be an ideal of $A$ which is intersection of maximal ideals $\{m_1, \cdots, m_r\}$. Then for any $A$-module $M$ and any integer $n \geq 0$, the map $M/m^n M \to M/m^n m$ is an isomorphism.

Proof. We first recall for the reader that $M_m$ is, by definition, the localization of $M$ obtained by inverting all elements in $A \setminus \bigcup_{i=1}^r m_i$. We next observe that $A_m$ must be a semi-local ring with maximal ideals $\{m_1 A_m, \cdots, m_r A_m\}$. In particular, any element of $A_m$ which is not in any of these maximal ideals must be a unit. This immediately implies the lemma when $M = A$. The general case follows from this because we can now write $M/m^n M \simeq M \otimes_A m^n \simeq (M \otimes_A A_m) \otimes_{A_m} A_m/m^n A_m \simeq M_m/m^n M_m$. \hfill \Box
Our main result on the two completions of the $K$-theory is the following.

**Theorem 6.2.** Let $\mathcal{X}$ be a quotient Deligne-Mumford stack of finite type over $\mathbb{C}$. Given any presentation $\mathcal{X} = [X/G]$ and integer $i \geq 0$, the following hold. 

1. $m^n_{\mathcal{X}}G_i(\mathcal{X})_{m_{\mathcal{X}}} = 0$ for all $n \gg 0$.
2. There is a commutative diagram

\[
\begin{array}{ccc}
G_i(\mathcal{X})_{m_1} & \longrightarrow & \hat{G}_i(\mathcal{X})_{m_1} \\
\downarrow & & \downarrow \\
G_i(\mathcal{X})_{m_{\mathcal{X}}} & \longrightarrow & \hat{G}_i(\mathcal{X})_{m_{\mathcal{X}}},
\end{array}
\]

in which all arrows are isomorphisms.

**Proof.** We first observe that

\[
\hat{G}_i(\mathcal{X})_{m_{\mathcal{X}}} = \lim_{n} G_i(\mathcal{X})/m^n_{\mathcal{X}}G_i(\mathcal{X}) \simeq \lim_{n} G_i(\mathcal{X})_{m_{\mathcal{X}}}/m^n_{\mathcal{X}}G_i(\mathcal{X})_{m_{\mathcal{X}}},
\]

where the second isomorphism follows from Lemma 6.1. Note here that $K_0(\mathcal{X})$ may not be a Noetherian ring. We similarly have

\[
\hat{G}_i(\mathcal{X})_{m_1} = \lim_{n} G_i(\mathcal{X})/m^n_{\mathcal{X}}G_i(\mathcal{X}) \simeq \lim_{n} G_i(\mathcal{X})_{m_1}/m^n_{\mathcal{X}}G_i(\mathcal{X})_{m_1}.
\]

We shall prove the theorem by the Noetherian induction on $X$. We first assume that $X$ is smooth. In this case, we have $K(\mathcal{X}) \overset{\sim}{\longrightarrow} G(\mathcal{X})$. We know from [30, Lemma 7.3] that $m^n_{\mathcal{X}}K_i(\mathcal{X})_{m_1} = 0$ for $n \gg 0$. Since $K_i(\mathcal{X})_{m_{\mathcal{X}}}$ is a localization of $K_i(\mathcal{X})_{m_1}$, we get $m^n_{\mathcal{X}}K_i(\mathcal{X})_{m_{\mathcal{X}}} = 0$ for $n \gg 0$. On the other hand, since each $K_i(\mathcal{X})$ is a $K_0(\mathcal{X})$-module, it follows from [13, Theorem 6.1] that for every $n \geq 0$, one has

\[
m^n_{\mathcal{X}}K_i(\mathcal{X}) \subseteq m^n_{\mathcal{X}}K_i(\mathcal{X}) \subseteq m^n_{\mathcal{X}}K_i(\mathcal{X})\text{ for } n' \gg 0.
\]

We conclude from this that $m^n_{\mathcal{X}}K_i(\mathcal{X})_{m_{\mathcal{X}}} = 0$ for all $n \gg 0$. It follows now from (6.2) and (6.3) that the bottom and the top horizontal arrows in (6.1) are isomorphisms. It follows from (6.2) that the $m_1$-adic and $m_{\mathcal{X}}$-adic topologies on $K_i(\mathcal{X})$ coincide. Hence, the right vertical arrow in (6.1) is an isomorphism. We conclude that the left vertical arrow is also an isomorphism. The theorem is thus proven when $X$ is smooth.

In general, there is a non-empty $G$-invariant open subspace $U \subset X$ which is smooth so that the open substack $\mathcal{U} = [U/G] \subset \mathcal{X}$ is smooth. We let $Z = X \setminus U$ and $Z = [Z/G]$.

Let $Z \overset{\sim}{\rightarrow} X \overset{\sim}{\rightarrow} \mathcal{U}$ denote the inclusion maps.

It is easy to check that $m_{\mathcal{X}}K_0(Z) \subseteq m_{\mathcal{X}}K_0(Z)$. It is also immediate from the definition of $m_{\mathcal{X}}$ that $m_{\mathcal{X}} \cap K_0(\mathcal{X}) = m_{\mathcal{X}} = m_{\mathcal{X}} \cap K_0(\mathcal{X})$. It follows that we have natural maps of localizations $G_i(Z)_{m_1} \rightarrow G_i(Z)_{m_{\mathcal{X}}} \rightarrow G_i(Z)_{m_{\mathcal{X}}}$. It follows by the Noetherian induction that the composite map is an isomorphism. In particular, the second map is surjective. We claim that this map is also injective, and hence an isomorphism. To prove this, we note that showing injectivity of this map is equivalent to showing that given any $a \in G_i(Z)$ and $s \in K_0(Z) \setminus m_{\mathcal{X}}$ such that $sa = 0$, there is $t \in K_0(\mathcal{X}) \setminus m_{\mathcal{X}}$ such that $ta = 0$. However, the injectivity of the map $G_i(Z)_{m_1} \rightarrow G_i(Z)_{m_{\mathcal{X}}}$ implies that there is $u \in R(G) \setminus m_1$ such that $ua = 0$. As $m_{\mathcal{X}} \cap R(G) = m_1$, if we let $t$ be the image of $u$ under the map $R(G) \rightarrow K_0(X)$, we get $t \not\in m_{\mathcal{X}}$ and $ta = 0$.

We similarly have the natural maps of localizations $G_i(\mathcal{U})_{m_1} \rightarrow G_i(\mathcal{U})_{m_{\mathcal{X}}} \rightarrow G_i(\mathcal{U})_{m_{\mathcal{X}}}$ and the composite map is an isomorphism since $\mathcal{U}$ is smooth. An identical reason as before now shows that these maps are isomorphisms.
We have the exact sequence of $K_0(\mathcal{X})$-modules
\[ G_i(\mathcal{Z}) \to G_i(\mathcal{X}) \to G_i(\mathcal{U}) \]
and hence this remains exact after localization at $m_\mathcal{X}$. For reader unfamiliar with the fact that the above is a sequence of $K_0(\mathcal{X})$-modules, recall that $K(\mathcal{X})$ is a ring spectrum and $G(\mathcal{Z}) \to G(\mathcal{X}) \to G(\mathcal{U})$ is a fiber sequence in the stable homotopy category of module spectra over the ring spectrum $K(\mathcal{X})$. In particular, the associated long exact sequence of homotopy groups is a sequence of $\pi_0(K(\mathcal{X}))$-modules. But $\pi_0(K(\mathcal{X}))$ is $K_0(\mathcal{X})$ by definition.

Now, we know that $m_\mathcal{U} G_i(\mathcal{U})_{m_\mathcal{U}} = 0$ as $\mathcal{U}$ is smooth. Since $G_i(\mathcal{U})_{m_\mathcal{X}} \simeq G_i(\mathcal{U})_{m_\mathcal{U}}$ as shown above, and since $m_\mathcal{Y} G_i(\mathcal{U}) \subset m_\mathcal{Y} G_i(\mathcal{U})$, it follows that $m_\mathcal{Y} G_i(\mathcal{U})_{m_\mathcal{X}} = 0$ for all $n \gg 0$. Similarly, we have $m_\mathcal{Y} G_i(\mathcal{Z})_{m_\mathcal{X}} = 0$ for $n \gg 0$. It easily follows from this and (6.5) that $m_\mathcal{X} G_i(\mathcal{X})_{m_\mathcal{X}} = 0$ for $n \gg 0$. This proves (1).

We now prove (2). It follows from (1) and (6.2) that the bottom horizontal arrow in (6.1) is an isomorphism. It follows from [30, Lemma 7.3] that $m_1 G_i(\mathcal{X})_{m_1} = 0$ for $n \gg 0$. Combining this with (6.3), it follows that the top horizontal arrow in (6.1) is an isomorphism. It is now enough to show that the left vertical arrow is an isomorphism.

To prove this, we consider the commutative diagram
\[ \begin{array}{cccc}
G_{i+1}(\mathcal{U})_{m_1} & \to & G_i(\mathcal{Z})_{m_1} & \to & G_i(\mathcal{X})_{m_1} & \to & G_i(\mathcal{U})_{m_1} & \to & G_{i-1}(\mathcal{Z})_{m_1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
G_{i+1}(\mathcal{U})_{m_\mathcal{X}} & \to & G_i(\mathcal{Z})_{m_\mathcal{X}} & \to & G_i(\mathcal{X})_{m_\mathcal{X}} & \to & G_i(\mathcal{U})_{m_\mathcal{X}} & \to & G_{i-1}(\mathcal{Z})_{m_\mathcal{X}}.
\end{array} \]

Since the localization exact sequence for the $K$-theory (of coherent sheaves) is a sequence of $K_0(\mathcal{X})$-modules, it follows that this sequence remains exact after localization at $m_\mathcal{X}$. Similarly, it remains exact after localization at $m_1 \subset R(G)$. We conclude that the top and the bottom rows of (6.6) are exact. We have shown that all vertical arrows in this diagram (except possibly the middle one) are isomorphisms. It follows that the middle one must also be an isomorphism. \[\square\]

6.1.1. K"ock’s conjecture. A conjecture of K"ock [25 Conjecture 5.6] in equivariant $K$-theory asserts that if $f : X \to Y$ is a $G$-equivariant proper map of schemes such that $f$ is a local complete intersection, then it induces a push-forward map of completions $f_* : K_1(G, X)_{m_{[X/C]}} \to K_1(G, Y)_{m_{[Y/C]}}$. The main result of [25] is based on the validity of this conjecture. If $X$ and $Y$ are smooth, this conjecture was settled by Edidin and Graham [13].

The following consequence of Theorem 6.2 proves K"ock’s conjecture when $Y$ is smooth but $X$ is possibly singular and the $G$-actions are with finite stabilizers. In fact, the result below is stronger than the one predicted by K"ock’s conjecture because it holds for any proper map of stacks which may not be representable. Recall that a stack $\mathcal{X}$ satisfies the resolution property if every coherent sheaf on $\mathcal{X}$ is a quotient of a vector bundle. If $\mathcal{X}$ is a quotient Deligne-Mumford stack with quasi-projective coarse moduli space, then it is known that it satisfies the resolution property (see [26 Proposition 5.1]).

**Corollary 6.3.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of quotient Deligne-Mumford stacks of finite type over $\mathbb{C}$. Let $i \geq 0$ be an integer. Then there is a push-forward map $f_* : G_i(\mathcal{X})_{m_\mathcal{X}} \to G_i(\mathcal{Y})_{m_\mathcal{Y}}$. If $\mathcal{Y}$ is smooth and satisfies the resolution property, then there is a push-forward map $f_* : K_i(\mathcal{X})_{m_\mathcal{X}} \to K_i(\mathcal{Y})_{m_\mathcal{Y}}$.

**Proof.** By (2.8), we can write $f$ as the product $\mathcal{X} \simeq [Z/G] \xrightarrow{p} [W/F] \xrightarrow{q} [Y/F] = \mathcal{Y}$, where $G = H \times F$ and $[Z/H] \to W$ is the coarse moduli space and $W \to Y$ is $F$-equivariant.
Proof. We need to show that the diagram

\[
\begin{array}{c}
\mu_t: G_i(G, Z)_{m_t} \rightarrow G_i(G, Z)_{m_{t_{g^{-1}}}G_i(Z_g, X^g)_{m_{g^{-1}t^{-1}}}G_i(Z_g, X^g)_{m_1}G_i(G, I_X^t)_{m_1}.
\end{array}
\]

(6.7)

Note that all arrows in (6.7) are isomorphisms.

Lemma 6.5. The map \( \theta_X^\psi \) is independent of the choice of \( g \) in \( \psi \). That is, the composite maps \( \mu^1_g \circ t_{g^{-1}} \circ \omega_g \) and \( \mu^1_h \circ t_{h^{-1}} \circ \omega_h \) coincide if \( g, h \in \psi \).

Proof. We need to show that the diagram

\[
\begin{array}{ccc}
G_i(G, X_m) & \xrightarrow{\omega_{g^{-1}}} & G_i(Z_g, X^g)_{m_g} & \xrightarrow{t_{g^{-1}}} & G_i(Z_g, X^g)_{m_1} & \xrightarrow{\mu^1_h} & G_i(G, I_X^t)_{m_1}
\end{array}
\]

commutes. Lemma 5.2 says that the right square commutes. To show that the left square commutes, we can replace the horizontal arrows in this square by their inverses. In this case, the desired commutativity follows from Lemma 5.7. We are now left with showing that the middle square commutes.

Recall from 5.2 that \( u_s: G_i(Z_g, X^g) \rightarrow G_i(Z_g, X^h) \) is an isomorphism, where \( X^h \) is given the \( \ast \)-action of \( Z_g \) via \( \phi \). As \( u_s \) is a \( Z_g \)-equivariant isomorphism, it follows from Proposition 4.2 that \( u_s \circ t_{g^{-1}} = t_{g^{-1}} \circ u_s \). Since \( \theta_X = \phi_X \circ u_s \), it suffices to show that \( t_{g^{-1}} \) commutes with \( \phi_X \). First, one checks directly from the definitions of \( t_{g^{-1}} \) and \( \phi_X \) that the square

\[
\begin{array}{ccc}
R(Z_g)_{m_g} & \xrightarrow{t_{g^{-1}}} & R(Z_g)_{m_1}
\end{array}
\]

commutes. This can also be deduced from [14 Lemma 6.5]. Indeed, an element of \( R(Z_g) \) is a virtual representation of \( Z_g \), which we can assume to be an actual representation where \( (g) \) acts by a fixed character \( \chi \) (see Lemma 4.1). If \( V \) is such a representation, then \( \phi_X([V]) \) is just \( V \) itself, considered as a representation of \( Z_h \) via the isomorphism \( \phi^{-1}: Z_h \xrightarrow{\cong} Z_g \). We thus have \( t_{h^{-1}} \circ \phi_X([V]) = \chi(\phi^{-1}(h)) \cdot [V] \). But this is clearly same as \( \chi(g) \cdot [V] = \phi_X \circ t_{g^{-1}}([V]) \).

In general, for \( a \in G^+_1((g), X^g) \), we have

\[
\phi_X \circ t_{g^{-1}}(a) = \phi_X(\chi(g)a) = \phi_X(t_{g^{-1}}([1_{Z_g}])\phi_X(a)) = \dot{t}_{h^{-1}}([1_{Z_g}])\phi_X(a) = t_{h^{-1}} \circ \phi_X(a),
\]
where $1_{Z_g}$ is the rank one trivial representation of $Z_g$ and $=_{^1}$ holds by case of $\phi_{\ast} : R(Z_g) \xrightarrow{\sim} R(Z_h)$ shown above. Since $G_\ast(Z_g, X^g) = \oplus_{\chi \in (g)} G_\chi^X((g), X^g)$ by Lemma 4.1, we have shown that the middle square in (6.8) commutes. This finishes the proof. □

Recall from §2.3 that there is an $G$-equivariant decomposition $I_X = \Pi_{\psi \in \Sigma_X^G} I_X^\psi$ and hence we have $\oplus_{\psi \in \Sigma_X^G} G_\ast(G, I_X^\psi) \xrightarrow{\sim} G_\ast(G, I_X)$. Using Theorem 3.10 and Proposition 3.11, we get the following.

Definition 6.6. Let $G$ act properly on an algebraic space $X$. For any integer $i \geq 0$, the Atiyah-Segal map $\vartheta_X^G : G_\ast(G, X) \to G_\ast(G, I_X^\psi)_{m_1}$ is defined to be the composite (6.10)

\[
\vartheta_X^G : G_\ast(G, X) \xrightarrow{\sim} \oplus_{\psi \in \Sigma_X^G} G_\ast(G, X^\psi)_{m_\psi} \xrightarrow{\oplus \vartheta_X^\psi} \oplus_{\psi \in \Sigma_X^G} G_\ast(G, I_X^\psi)_{m_1} \xrightarrow{\sim} G_\ast(G, I_X)_{m_1}.
\]

Since each $\vartheta_X^\psi$ is an isomorphism, it follows that $\vartheta_X^G$ is an isomorphism.

7. Atiyah-Segal Correspondence for the Coarse Moduli Space Map

The construction of the Atiyah-Segal correspondence for the $G$-theory of stacks involves two parts. The first is to show that the Atiyah-Segal map is well defined and is an isomorphism. The second part is to show that this map is covariant with respect to proper maps of separated quotient stacks. We have shown the first part in the previous section. The second part will be shown in several steps. In this section, we prove it for the representable maps and the coarse moduli space maps.

Proposition 7.1. For any $G$-equivariant proper morphism $f : X \to Y$ of algebraic spaces with proper $G$-action and for any integer $i \geq 0$, there is a commutative diagram

\[
\begin{array}{ccc}
G_\ast(G, X) & \xrightarrow{\vartheta_X^G} & G_\ast(G, I_X)_{m_1} \\
\downarrow f_{\ast} & & \downarrow f_{\ast} \\
G_\ast(G, Y) & \xrightarrow{\vartheta_Y^G} & G_\ast(G, I_Y)_{m_1}.
\end{array}
\]

Proof. Using (2.2) and (2.3), it is easy to check that $f_{\ast} : I_X \to I_Y$ is a $G$-equivariant proper map which takes $I_X^\psi$ to $I_Y^\psi$. Since $f_{\ast} : G_\ast(G, X) \to G_\ast(G, Y)$ is $R(G)$-linear, it takes the summand $G_\ast(G, X)_{m_\psi} \to G_\ast(G, Y)_{m_\psi}$.

If $\psi$ is a semi-simple conjugacy class of $G$ that does not belong to $\Sigma_X^G$ or $\Sigma_Y^G$, then both $X^\psi$ and $Y^\psi$ are empty hence there is nothing to prove. If $\psi$ is such that it belongs to only one of $\Sigma_X^G$ or $\Sigma_Y^G$, then either $G_\ast(G, X)_{m_\psi} = 0$ and $I_X^\psi = 0$ or $G_\ast(G, Y)_{m_\psi} = 0$ and $I_Y^\psi = 0$. So $f_{\ast}$ and $f_{\ast}^I$ are both zero. We can thus assume that $\psi \in \Sigma_X^G \cap \Sigma_Y^G$. This reduces us to showing that $\vartheta_Y^\psi \circ f_{\ast} = f_{\ast}^I \circ \vartheta_X^\psi : G_\ast(G, X)_{m_\psi} \to G_\ast(G, I_Y^\psi)_{m_1}$.

Now, going back to (6.7), we note that the map $t_{g^{-1}}$ commutes with $f_{\ast}$ by Corollary 4.4. The maps $\mu_{g}^\psi$ and $\mu_{g}^1$ commute with $f_{\ast}$ and $f_{\ast}^I$ by Lemma 5.3. Since

\[
\begin{array}{c}
I_X^\psi \\
\downarrow f_{\ast} \\
X
\end{array}
\xrightarrow{\mu_{g}}
\begin{array}{c}
I_Y^\psi \\
\downarrow f_{\ast}^I \\
Y
\end{array}
\]

is a commutative square of $G$-equivariant proper maps, it follows that $\mu_{g}^\psi$ and $f_{\ast}$ commute. Since $\omega_{g} = \mu_{g}^\psi \circ \mu_{g}^1$, it follows that $f_{\ast} \circ \omega_{g} = \omega_{g} \circ f_{\ast}$. Since $\omega_{g}$ is invertible, we get $f_{\ast} \circ \omega_{g}^{-1} = \omega_{g}^{-1} \circ f_{\ast}$. We thus get $\vartheta_Y^\psi \circ f_{\ast} = f_{\ast}^I \circ \vartheta_X^\psi$ and this finishes the proof. □
7.1. The case of coarse moduli space map. Our goal now is to prove that the Atiyah-Segal map is covariant with respect to the coarse moduli space map $[X/G] \to X/G$.

Let $X$ be an algebraic space with proper $G$-action and let $Y = X/G$ be the quotient. Let $\psi \subset G$ be a semi-simple conjugacy class. We have the $G$-equivariant closed immersion $j_X^\psi : X_\psi \subset X$ and a $G$-equivariant finite and surjective map $\mu^\psi : I_X^\psi \to X_\psi$. Let $\nu^\psi : I_X^\psi / G \to X_\psi / G := Y_\psi$ denote the induced map on the quotients. This is clearly finite and surjective. We fix an integer $i \geq 0$.

**Lemma 7.2.** There is a commutative diagram

\[
\begin{array}{ccc}
G_i(G, X_\psi)_{m_\psi} & \xrightarrow{\frac{\psi}{G}} & G_i(G, I_X^\psi)_{m_1} \\
\downarrow \text{Inv}^G_{X_\psi} & & \downarrow \nu^\psi \circ \text{Inv}^G_{I_X^\psi} \\
G_i(Y_\psi) & \xrightarrow{\text{Id}} & G_i(Y_\psi).
\end{array}
\]

**Proof.** We consider the diagram

\[
\begin{array}{ccc}
G_i(G, X_\psi)_{m_\psi} & \xrightarrow{\omega^{-1}_i} & G_i(Z_g, X^g)_{m_g} & \xrightarrow{\iota^{-1}_g} & G_i(Z_g, X^g)_{m_1} & \xrightarrow{\mu^1_0} & G_i(G, I_X^\psi)_{m_1} \\
\downarrow \text{Inv}^G_{X_\psi} & & \downarrow \text{Inv}^G_{Z_g} & & \downarrow \text{Inv}^G_{Z_g} & & \downarrow \text{Inv}^G_{I_X^\psi} \\
G_i(X^g/Z^g) & \xrightarrow{\text{Id}} & G_i(X^g/Z^g) & \xrightarrow{\gamma^*} & G_i(I_X^\psi / G) \\
\downarrow \nu^\psi \circ \gamma^* & & \downarrow \nu^\psi \\
G_i(Y_\psi) & \xrightarrow{\text{Id}} & G_i(Y_\psi).
\end{array}
\]

The lemma is equivalent to the commutativity of the big outer square. Using the $G$-equivariant isomorphism $p_g : G \times_{Z_g} X^g \xrightarrow{\sim} I_X^\psi$, we note from (5.5) that $\mu^1_0$ is the push-forward isomorphism induced on $K$-theory via the isomorphism of quotient stacks $t_{X^g}^G : [X^g/Z_g] \xrightarrow{\sim} [I_X^\psi / G]$ given by Lemma 2.4. We have a commutative square of separated quotient stacks and coarse moduli spaces

\[
\begin{array}{ccc}
[X^g/Z_g] & \xrightarrow{t_{X^g}^G} & [I_X^\psi / G] \\
\downarrow \pi^g & & \downarrow \pi^G \\
X^g/Z^g & \xrightarrow{\gamma^*} & I_X^\psi / G,
\end{array}
\]

where the vertical arrows are proper. Taking the corresponding maps on the $K$-theory, we see that the top square from the extreme right in (7.4) is commutative.

The middle square on the top in (7.4) commutes by Proposition 4.5. We are now left with showing that the trapezium on the left side of (7.4) commutes. For any $a \in G_i(G, X)_{m_\psi}$, we have

\[
\text{Inv}^G_{X_\psi}(a) = \text{Inv}^G_{X_\psi} \circ \omega_g \circ \omega^{-1}_g(a) = \text{Inv}^G_{X_\psi} \circ \mu^\psi \circ \mu^1_g \circ \omega^{-1}_g(a) = 1^1 \nu^\psi \circ \text{Inv}^G_{I_X^\psi} \circ \mu^1_g \circ \omega^{-1}_g(a) = 2^2 \nu^\psi \circ \gamma^*_g \circ \text{Inv}^G_{X^g} \circ \omega^{-1}_g(a),
\]

where $=^1$ follows from Theorem 3.13 and $=^2$ follows from (7.5). This proves the desired commutativity and finishes the proof of the lemma. \qed

**Theorem 7.3.** Let $X$ be an algebraic space with proper $G$-action and let $Y = X/G$ be the quotient. For a semi-simple conjugacy class $\psi \subset G$, let $j_X^\psi : Y_\psi \hookrightarrow Y$ denote the
closed immersion, where \( Y_\psi = X_\psi / G \). Then there is a commutative diagram

\[
\begin{array}{ccc}
G_i(G, X) & \longrightarrow & \oplus_{\psi \in S_G G_i(G, I_X^\psi)_{m_1}} G_i(Y_\psi) \\
\downarrow \quad \text{Inv}_{X_\psi}^G & & \downarrow \quad \text{Inv}_{Y_\psi}^G \\
G_i(Y) & \longrightarrow & G_i(Y)
\end{array}
\]

Proof. It suffices to show that the diagram commutes when restricted to each \( G_i(G, X)_{m_\psi} \) such that \( \psi \in \Sigma_X^G \). Continuing with the notations before and after Lemma 7.2 let \( a \in G_i(G, X)_{m_\psi} \). We then have

\[
\text{Inv}_{X_\psi}^G(a) = \text{Inv}_{Y_\psi}^G \circ j_{X_\psi}^\psi(a) = j_{Y_\psi}^\psi \circ \text{Inv}_{X_\psi}^G \circ \vartheta_X^\psi(a) = j_{Y_\psi}^\psi \circ \mu_{X_\psi}^\psi \circ \vartheta_X^\psi(a).
\]

The equality \( =^1 \) follows from Theorem 3.10 and \( =^2 \) follows from Lemma 7.2. The equality \( =^3 \) follows from the functoriality of the proper push-forward map in \( K \)-theory of coherent sheaves, applied to the commutative square of proper maps of stacks

\[
\begin{array}{ccc}
[I_X^\psi / G] & \stackrel{\mu^\psi}{\longrightarrow} & [X_\psi / G] \\
\downarrow & & \downarrow \\
I_X^\psi / G & \stackrel{\nu^\psi}{\longrightarrow} & Y_\psi
\end{array}
\]

and noting that \( \text{Inv}_{(-)}^G \) is the push-forward map on \( K \)-theory induced by the coarse moduli space map which is proper (see Theorem 3.13). This proves the theorem. \( \square \)

8. Atiyah-Segal correspondence for a partial quotient map

In this section, we shall prove a version of the Atiyah-Segal correspondence for a map which is the quotient by one of the factors of a product of groups acting properly on an algebraic space. More specifically, we shall work with the following set up.

Let \( G = H \times F \) be the product of two linear algebraic groups which acts properly on an algebraic space \( X \) with quotients \( Y = X / H \) and \( Z = X / G \). Let \( g = (g_1, g_2) \in H \times F = G \) be a semi-simple element. Let \( Z_{g_1} = Z_H(g_1) \) and \( Z_{g_2} = Z_F(g_2) \) be the centralizers of \( g_1 \) in \( H \) and \( g_2 \) in \( F \), respectively so that \( Z_g = Z_G(g) = Z_{g_1} \times Z_{g_2} \). Let \( \psi \) denote the conjugacy class of \( g \) in \( G \) and let \( \phi \) denote its image in \( F \). As \( g \) is a semi-simple element in \( G \), so is \( g_2 \) and hence \( \phi \) is a semi-simple conjugacy class in \( F \).

We let \( W = X^g / Z_{g_1} \) and let \( k^g : W \to Y_{g_2} \) denote the map induced by \( X \to Y \). Let \( \nu^\psi : I_X^\psi / H \to I_X^\psi \) and \( \nu : I_X^\psi / H \to I_Y \) denote the maps on the inertia spaces.

Let \( j_X^\psi : X_\psi \to X \) and \( j_Y^\psi : Y_\psi \to Y \) be the inclusion maps. Let \( s_X^\psi : I_X^\psi \to I_X \) and \( s_Y^\psi : I_Y^\psi \to I_Y \) be the inclusion maps. Let \( a^g : X_\psi / H \to Y_\psi \) denote the canonical map induced by quotient map \( X \to Y \). This is clearly \( F \)-equivariant.
We have the two commutative diagrams

\[
\begin{array}{ccc}
I_X^\psi & \xrightarrow{\mu^\psi} & X \\
\pi^\psi & & \\
I_X^\psi/H & \xrightarrow{\nu^\psi} & I_Y^\phi \\
\downarrow & & \\
I_X^\psi/G & \xrightarrow{\delta^\psi} & I_Y^\phi/F \\
\end{array}
\]

\[
\begin{array}{ccc}
X^g & \xrightarrow{G \times X^g} & G \times Z_g \times X^g \\
\downarrow & & \\
W & \xrightarrow{F \times W} & F \times Z_{g_2} W. \\
\end{array}
\]

Note that in the diagram on the right, all arrows are just the quotient maps. We shall use this diagram later (see, for instance, Lemma 8.2).

**Lemma 8.1.** With respect to the above set up, we have the following.

1. All horizontal arrows in the diagram on the left in (8.1) are finite. All maps are $G$-equivariant and all maps in the bottom squares are $F$-equivariant.
2. The map $u^\psi: X_\psi/H \to Y_\phi$ is finite.
3. There is an $F$-equivariant isomorphism $p^W_{g_2}: F \times Z_{g_2} W \xrightarrow{\simeq} I_X^\psi/H$.
4. There is a canonical isomorphism of quotient stacks $[W/Z_{g_2}] \simeq [(I_X^\psi/H)/F]$.
5. The map $k^g: W \to Y^g$ is $Z_{g_2}$-equivariant and finite.

**Proof.** The $G$-equivariance and $F$-equivariance of the maps in the left diagram in (8.1) are clear. Except possibly $\nu^\psi$ and $\delta^\psi$, all the horizontal arrows in (8.1) are finite by Lemmas 2.6 and 2.9. In particular, $\mu^\phi \circ \nu^\psi$ and $\gamma^\phi \circ \delta^\psi$ are finite. It follows that $\nu^\psi$ and $\delta^\psi$ are also finite. This proves (1).

We next note that there are maps $I^\psi/H \xrightarrow{u^\psi/H} X_\psi/H \xrightarrow{u^\psi} Y_\phi$, such that $u^\phi \circ (\mu^\psi/H) = \mu^\phi \circ \nu^\psi$. The first map is finite and surjective by Lemmas 2.6 and 2.9. We just showed above that the map $u^\phi \circ (\mu^\psi/H)$ is finite. It follows that $u^\psi$ is finite (see the proof of Lemma 2.9). This proves (2). We now consider the commutative diagrams

\[
\begin{array}{ccc}
X^g & \xrightarrow{G \times X^g} & G/\times Z_g \\
\downarrow & & \\
W & \xrightarrow{F \times Z_{g_2} W} & F/\times Z_{g_2} \\
\downarrow & & \\
Y^g & \xrightarrow{F \times Z_{g_2} W} & F/\times Z_{g_2} \\
\end{array}
\]

With respect to the above set up, we have the following.

We first explain the maps in the diagram on the left. The horizontal arrows on the left are all of the type $z \mapsto (e, z)$, where $e$ is the identity element of $G$ or $F$. The horizontal arrows on the right are the ones induced on the quotients by the projection maps to the first factors. It is then clear that the left column is the closed fiber of these maps over the identity cosets. In particular, horizontal arrows on the left are all closed immersions.

We now explain the maps in the diagram on the right. The maps $p^g$ and $p^\psi_{g_2}$ are given in Lemma 2.6. Since $p^g$ is $G$-equivariant with respect to the action $(g_1, g_2)(h_1, h_2, x) = (g_1'h_1, g_2'h_2, x)$, it follows that this descends to an $F$-equivariant map $(G \times Z_g X^g)/H \to$...
$I_X^\psi/H$. On the other hand, using the way various actions are defined, we get

$$
(G \times Z_g X^g)/H \xlongrightarrows (H \times F) \times (Z_g \times Z_{g^2}) \times X^g \xlongrightarrows (H \times F) \times X^g/H \times Z_{g^2} \times Z_{g^2} \xlongrightarrows F \times (X^g/Z_{g^2}) \xlongrightarrows F \times Z_{g^2} W.
$$

(8.3)

The second isomorphism holds because $H \simeq (H \times 1_F)$ and $(Z_{g_1} \times Z_{g_2})$-actions on $(H \times F) \times X^g$ commute. The third isomorphism holds because the $(H \times F)$-action on $(H \times F) \times X^g$ is free and this action on $X^g$-factor is trivial. The fourth isomorphism holds because $Z_{g_1}$ acts trivially on $F$. One checks easily that all the maps in (8.3) are $F$-equivariant. It follows that $p_g$ descends to an $F$-equivariant isomorphism $p^W_{Z_{g^2}}$ and the two squares on the right side of (8.2) commute. This proves (3) and (4) follows from this because of the isomorphism of stacks $[W/Z_{g^2}] \simeq [(F \times Z_{g^2} W)/F]$ by Lemma 2.4.

We now prove (5). Since the map $V^\psi$ is finite (see (8.1)), it follows that the map $\bar{d}_F \times k^g : F \times Z_{g^2} W \to F \times Z_{g^2} Y_{g^2}$ is finite $F$-equivariant. Since $W \to F \times Z_{g^2} W$ and $Y_{g^2} \to F \times Z_{g^2} Y_{g^2}$ are closed immersions, it follows by looking at the diagram on the left in (8.2) that $k^g$ is an $Z_{g^2}$-equivariant finite morphism. $\square$

8.1. Equivariant $K$-theory for action of $H \times F$. We shall let

$$
\mu_g : G(Z_g, X^g) \xlongrightarrows G(G, I_X^\psi), \ p_g : G(Z_{g^2}, W) \xlongrightarrows G(F, I_X^\psi/H) \text{ and }
$$

$$
\mu_{g_2} : G(Z_{g^2}, Y_{g^2}) \xlongrightarrows G(G, I_Y^\psi)
$$

(8.4)

denote the Morita isomorphisms on the $K$-theory induced by Lemma 8.1. Continuing with the above set up, we now prove the following statements.

**Lemma 8.2.** There is a commutative diagram

$$
\begin{array}{ccc}
G_s(Z_g, X^g) & \xlongrightarrows & G_s(Z_{g^2}, W) \\
\xlongrightarrows \mu_g & & \xlongrightarrows \mu_{g_2} \\
G_s(G, I_X^\psi) & \xlongrightarrows & G_s(F, I_Y^\psi)
\end{array}
$$

(8.5)

**Proof.** The commutativity of the right square follows by applying Lemma 2.4 to the $Z_{g^2}$-equivariant morphism $k^g : W \to Y_{g^2}$. The left square commutes because, by Theorem 8.1 and (8.5), it is induced by the commutative diagram of quotient stacks

$$
\begin{array}{ccc}
[X^g/Z_g] & \xlongrightarrows & [W/Z_{g^2}] \\
\xlongrightarrows G(Z_g, X^g) & & \xlongrightarrows F(Z_{g^2}) \\
[I_X^\psi/G] & \xlongrightarrows & [(I_Y^\psi/H)/F],
\end{array}
$$

(8.6)

where the horizontal arrows are proper and the vertical arrows are isomorphisms. $\square$
Lemma 8.3. There is a commutative diagram

\[
\begin{array}{cccc}
G_*(Z_g, X^g)_{m_g} & \xrightarrow{\mu_g^\psi} & G_*(G, I^\psi_X)_{m_\psi} & \xrightarrow{\mu_\psi^\phi} & G_*(G, X_\psi)_{m_\psi} \\
\text{Inv}^Z_{X^g} & & \text{Inv}^H_{I^\psi_X} & & \text{Inv}^H_{X_\psi} \\
G_*(Z_g, W)_{m_g} & \xrightarrow{\mu_g^\phi} & G_*(F, I^\psi_X/H)_{m_\phi} & \xrightarrow{\mu_\phi^\psi} & G_*(F, X_\psi/H)_{m_\phi} \\
& \xrightarrow{k^2_g} & \xrightarrow{u^\psi_g} & \xrightarrow{u^\phi_g} & \\
G_*(Z_g, Y^g)_{m_g} & \xrightarrow{\mu_g^\psi} & G_*(F, I^\psi_Y)_{m_\phi} & \xrightarrow{\mu_\phi^\psi} & G_*(F, Y_\phi)_{m_\phi}.
\end{array}
\]

Proof. Using the fact that all maps on the top squares are \(R(G)\)-linear and all maps on the bottom squares are \(R(F)\)-linear, the commutativity of the two squares on the left follows directly from Lemma 8.2. The commutativity of the two squares on the right follows from Theorem 3.13. \(\Box\)

Lemma 8.4. Let \(G\) be a linear algebraic group acting properly on an algebraic space \(X\) and let \(p : X \to X/G\) be the quotient. Then the map \(\text{Inv}^G_X : G_*(G, X)_{m_1} \to G_*(X/G)\) is an isomorphism.

Proof. If \(X/G\) is quasi-projective, then so is \(X\) and the result follows from Corollary 10.3. In general, it follows from [39, Proposition 2.4] that there is a dense open subspace \(U \subset X/G\) such that \(U\) and \(p^{-1}(U)\) are schemes of finite type over \(\mathbb{C}\). Since the map \(p^{-1}(U) \to U\) is affine, we can find an affine open subscheme \(W' \subset X/G\) so that \(W = p^{-1}(W') \subset p^{-1}(U)\) is \(G\)-invariant affine and \(W' = W/G\). It follows therefore from the quasi-projective case that the map \(\text{Inv}^G_W : G_*(G, W)_{m_1} \to G_*(W/G)\) is an isomorphism.

We now let \(Z = X \setminus W\) and consider the commutative diagram

\[
\begin{array}{cccc}
G_{i+1}(G, W)_{m_1} & \to & G_i(G, Z)_{m_1} & \to & G_i(G, X)_{m_1} & \to & G_i(G, W)_{m_1} & \to & G_{i-1}(G, Z)_{m_1} \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & G_{i+1}(W/G) & \to & G_i(Z/G) & \to & G_i(X/G) & \to & G_i(W/G) & \to & G_{i-1}(Z/G)
\end{array}
\]

of localization exact sequences. We showed above that the first and the fourth vertical arrows from the left are isomorphisms. The second and the fifth vertical arrows are isomorphisms by the Noetherian induction on \(X\). It follows that the middle vertical arrow in an isomorphism. \(\Box\)

Lemma 8.5. There is a commutative diagram

\[
\begin{array}{cccc}
G_*(Z_g, X^g)_{m_g} & \xrightarrow{\text{Inv}^Z_{X^g}} & G_*(Z_g, W)_{m_g} & \xrightarrow{\text{Inv}^Z_{W}} & G_*(X^g/Z_g) \\
& \xrightarrow{t_{g-1}} & & \xrightarrow{t_{g-1}} & \\
G_*(Z_g, X^g)_{m_1} & \xrightarrow{\text{Inv}^Z_{X^g}} & G_*(Z_g, W)_{m_1} & \xrightarrow{\text{Inv}^Z_{W}} & G_*(X^g/Z_g).
\end{array}
\]

Proof. We consider the diagram

\[
\begin{array}{cccc}
G_*(Z_g, X^g)_{m_g} & \xrightarrow{\text{Inv}^Z_{X^g}} & G_*(Z_g, W)_{m_g} & \xrightarrow{\text{Inv}^Z_{W}} & G_*(X^g/Z_g) \\
\xrightarrow{t_{g-1}} & & \xrightarrow{t_{g-1}} & & \xrightarrow{t_{g-1}} \\
G_*(Z_g, X^g)_{m_1} & \xrightarrow{\text{Inv}^Z_{X^g}} & G_*(Z_g, W)_{m_1} & \xrightarrow{\text{Inv}^Z_{W}} & G_*(X^g/Z_g).
\end{array}
\]

The right square and the outer rectangle commute by Proposition 4.5 and the map \(\text{Inv}^Z_{W}\) is an isomorphism by Lemma 8.4. It follows that the left square commutes. \(\Box\)
8.2. Atiyah-Segal for partial quotient map. We continue with our set up of proper action of an algebraic group \( G = H \times F \) on an algebraic space \( X \). We shall now prove the covariance of the Atiyah-Segal map with respect to the partial quotient map \( X \to X/H = Y \). Let \( i \geq 0 \) be a fixed integer. Recall from Theorem 3.13 that the map \( \text{Inv}^H_{X, \psi} : G_i(G, X_\psi) \to G_i(F, X_\psi/H) \) is \( R(F) \)-linear. On the localizations, we get the maps of \( R(F) \)-modules

\[
(8.11) \quad G_i(G, X_\psi)_{m_\psi} \to \oplus_{g \in F} G_i(G, X_\psi)_{m_g} \xrightarrow{\text{Inv}^H_{X, \psi}} G_i(G, X_\psi)_{m_\psi} \to G_i(F, X_\psi/H)_{m_\psi},
\]

where the first map is the canonical inclusion as a direct summand and the second map is an \( R(\Sigma G, X) \to \). We prove the covariance of the Atiyah-Segal map with respect to the partial quotient map.

There is a commutative diagram

\[
\text{Proposition 8.6. There is a commutative square}
\]

\[
G_i(G, X_\psi)_{m_\psi} \xrightarrow{g^\psi} G_i(G, I^\psi_X)_m \xrightarrow{u^\psi \circ \text{Inv}^H_{X, \psi}} G_i(F, I^\phi_Y)_m.
\]

Proof. We consider the diagram

\[
G_i(G, X_\psi)_{m_\psi} \xrightarrow{\text{Inv}^H_{X, \psi}} G_i(G, X_{\psi, H})_{m_{\psi, H}} \xrightarrow{t_{1, g}^{-1}} G_i(G, X_{g, H})_{m_{g, H}} \xrightarrow{\mu^{1, g}_\psi} G_i(G, I^\psi_X)_m
\]

\[
G_i(F, X_\psi/H)_{m_{\psi, H}} \xrightarrow{t_{1, g}^{-1}} G_i(F, X_{g, H})_{m_{g, H}} \xrightarrow{\mu^{1, g}_\psi} G_i(F, I^\phi_Y)_m
\]

\[
G_i(Z_{g, X}, Y_{g, H})_{m_{g, H}} \xrightarrow{t_{1, g}^{-1}} G_i(Z_{g, X}, Y_{g, H})_{m_{g, H}} \xrightarrow{\mu^{1, g}_\psi} G_i(F, I^\phi_Y)_m
\]

In this diagram, the top and the bottom squares on the right commute by Lemma 8.2. The middle square on the top commutes by Lemma 8.5 and the one on the bottom commutes by Corollary 4.4. To prove that the big rectangle on the left commutes, we can replace the horizontal arrows by their inverses. In this case, the rectangle commutes by Lemma 8.3. We have thus shown that (8.12) commutes. 

The final consequence of all the above results of this section is the following.

**Theorem 8.7.** There is a commutative square

\[
G_i(G, X) \xrightarrow{\varphi^G_X} G_i(G, I^\psi_X)_m \xrightarrow{v^\psi \circ \text{Inv}^H_{X, \psi}} G_i(F, I^\phi_Y)_m
\]

in which the horizontal arrows are isomorphisms.

Proof. We first observe that the \( G \)-equivariant decomposition \( I^G_X = \Pi_{\psi \in \Sigma G} I^\psi_X \) yields an \( F \)-equivariant decomposition \( I^G_X/H = \Pi_{\psi \in \Sigma G} I^\psi_X/H \). There is a similar decomposition of
\( I_Y \) in terms of \( I_Y^\phi \). We now look at the diagram

\[
\begin{array}{ccc}
\oplus_{\psi \in S_G} G_i(G, X^\psi)_{m_\psi} & \oplus_{\phi \in S_F} G_i(F, Y^\phi)_{m_\phi} & \oplus_{\psi \in S_G} G_i(G, I_Y^\psi)_{m_1} \\
\oplus \phi \in S_F G_i(F, Y^\phi)_{m_\phi} & \oplus \psi \in S_G G_i(G, I_Y^\psi)_{m_1} & \oplus s_{\psi, X} \\
\oplus \psi \in S_G G_i(G, X^\psi)_{m_\psi} & \oplus \phi \in S_F G_i(F, Y^\phi)_{m_\phi} & \oplus s_{\phi, Y} \\
\end{array}
\]

The back face of this cube commutes by Proposition 8.6. The top and the bottom faces commute by the definition of \( \vartheta_X^{\psi} \) and \( \vartheta_Y^{\phi} \) (see Definition 6.6). The left and the right faces commute by the functoriality of the proper push-forward maps in equivariant \( K \)-theory (of coherent sheaves). The maps \( \oplus j_X^\psi \) and \( \oplus j_Y^\phi \) are isomorphisms by Theorem 3.10 and Proposition 3.11. The maps \( \oplus s_X^{\psi} \) and \( \oplus s_Y^{\phi} \) are isomorphisms by the above decompositions of the inertia spaces. It follows that the front face commutes. Since each of \( \vartheta_X^{\psi} \) and \( \vartheta_Y^{\phi} \) is an isomorphism (see Definition 6.4), we conclude that \( \vartheta_X^G \) and \( \vartheta_Y^F \) are isomorphisms too. The proof of the theorem is complete. \( \square \)

9. Atiyah-Segal correspondence for quotient stacks

In this section, we shall prove our final results on the Atiyah-Segal correspondence for separated quotient stacks. In order to define the Atiyah-Segal map for such stacks, we need to prove the following result about these maps for the group action.

Let \( H \) and \( F \) be two linear algebraic groups. Let \( H \) and \( F \) act properly on algebraic spaces \( X \) and \( Y \), respectively. We let \( X = [X/H] \) and \( Y = [Y/F] \) denote the stack quotients. Let \( f : \mathcal{X} \to \mathcal{Y} \) be a proper morphism. If we let \( G = H \times F \), then this gives rise to the algebraic space \( Z \) with \( G \)-action and the Cartesian diagram (2.7), which we reproduce here for reader’s convenience.

\[
\begin{array}{ccc}
Z & \xrightarrow{s} & \mathcal{X}' \\
\downarrow t & & \downarrow f' \\
X & \xrightarrow{q} & \mathcal{X} \\
\end{array}
\]

Let us now assume that \( \mathcal{X} \) is a separated quotient stack with the coarse moduli space \( M \). Let \( \mathcal{X} = [X/H] = [Y/F] \) be two presentations of \( \mathcal{X} \). Letting \( Z = X \times \mathcal{X} Y \), we get a Cartesian diagram like (9.1), where \( f \) is the identity map. In particular, Lemma 2.8 applies. Let \( I_X \) denote the inertia stack of \( \mathcal{X} \). Let \( \epsilon : I_Z/F \to I_Y \) denote the canonical map induced by \( s \). We fix an integer \( i \geq 0 \).
Lemma 9.1. There exists a commutative diagram

\[
\begin{array}{c}
G_i(X) \xrightarrow{q^*} G_i(H, X) \xrightarrow{\vartheta_X} G_i(H, I_X)_{m_1} \xrightarrow{q^{*-1} \text{Inv}_Y} G_i(I_X)_{m_1} \\
\downarrow Id \quad \downarrow \text{Inv}_Y \quad \downarrow \vartheta_{I_X} \quad \downarrow Id \\
G_i(X)^{(qot)^*} \xrightarrow{\vartheta_Z} G_i(G, Z) \xrightarrow{\varphi_G} G_i(G, I_Z)_{m_1} \xrightarrow{\varphi^{*-1} \text{Inv}_Y} G_i(I_Z)_{m_1} \\
\downarrow Id \quad \downarrow \varphi_{I_Z} \quad \downarrow \text{Inv}_Y \quad \downarrow Id \\
G_i(X) \xrightarrow{p^*} G_i(F, Y) \xrightarrow{\varphi_Y} G_i(F, I_Y)_{m_1} \xrightarrow{p^{*-1} \text{Inv}_Y} G_i(I_Y)_{m_1}.
\end{array}
\]

Proof. We show that all the top squares commute as the argument for the commutativity of the bottom squares is identical. We first observe that \( F \) acts freely on \( I_Z \) and the map \( v : I_Z/F \to I_X \) is an isomorphism by Lemma 2.5. In particular, the map \( v_* \circ \text{Inv}_{I_Z} \) is same as the map \( \text{Inv}_{I_Z} : G_i(G, I_Z) \to G_i(H, I_X) \). Since the maps \( Z \to X \) and \( I_Z \to I_X \) are \( F \)-torsors, the induced maps \( \text{Inv}_{I_Z}^F : G_i(G, Z) \to G_i(H, X) \) and \( \text{Inv}_{I_Z}^F : G_i(G, I_Z) \to G_i(H, I_X) \) are isomorphisms which are inverses to the pull-back maps \( t^* : G_i(H, X) \to G_i(G, Z) \) and \( \pi^* : G_i(H, I_X) \to G_i(G, I_Z) \), respectively. Here, \( \pi : I_Z \to I_Z/F \xrightarrow{\cong} I_X \) is the quotient map (see (8.1)). The commutativity of the top squares in (9.2) is therefore equivalent to the commutativity of the diagram

\[
\begin{array}{c}
G_i(X) \xrightarrow{q^*} G_i(H, X) \xrightarrow{\vartheta_X} G_i(H, I_X)_{m_1} \xrightarrow{q^{*-1} \text{Inv}_Y} G_i(I_X)_{m_1} \\
\downarrow Id \quad \downarrow t^* \quad \downarrow \pi^* \quad \downarrow Id \\
G_i(X)^{(qot)^*} \xrightarrow{\vartheta_Z} G_i(G, Z) \xrightarrow{\varphi_G} G_i(G, I_Z)_{m_1} \xrightarrow{\varphi^{*-1} \text{Inv}_Y} G_i(I_Z)_{m_1} \\
\downarrow Id \quad \downarrow \varphi_{I_Z} \quad \downarrow \text{Inv}_Y \quad \downarrow Id \\
G_i(X) \xrightarrow{p^*} G_i(F, Y) \xrightarrow{\varphi_Y} G_i(F, I_Y)_{m_1} \xrightarrow{p^{*-1} \text{Inv}_Y} G_i(I_Y)_{m_1}.
\end{array}
\]

Now, the left square commutes by definition and the middle square commutes by Theorem 8.7. The right square commutes again by definition since \( t^* \circ v_X = v_Z \circ t^* : G_i(H, I_X)_{m_1} \to G_i(G, I_Z)_{m_1} \). This proves the lemma. \( \square \)

Definition 9.2. Let \( X = [X/G] \) be a separated quotient stack. We define the Atiyah-Segal map \( \vartheta_X \) to be the composite

\[
\vartheta_X = v_X \circ \vartheta_X \circ p_X^* : G_i(X) \to G_i(I_X)_{m_1}.
\]

Here, \( p_X : X \to X \) is the projection map. It follows from Lemma 9.1 that \( \vartheta_X \) is independent of the choice of the presentation of the stack. Furthermore, it follows from Theorem 8.7 that \( \vartheta_X \) is an isomorphism.

Our main result on the Atiyah-Segal correspondence for separated quotient stacks is the following.

Theorem 9.3. Let \( f : X \to Y \) be a proper morphism of separated quotient stacks of finite type over \( \mathbb{C} \). Let \( i \geq 0 \) be an integer. Then there is a commutative diagram

\[
\begin{array}{c}
G_i(X) \xrightarrow{\vartheta_X} G_i(I_X)_{m_1} \\
\downarrow f_* \quad \downarrow f'_* \\
G_i(Y) \xrightarrow{\vartheta_Y} G_i(I_Y)_{m_1}.
\end{array}
\]

such that the horizontal arrows are isomorphisms.

Proof. We have already seen in Definition 9.2 that the horizontal arrows are isomorphisms. We only need to show the commutativity. We let \( X = [X/H] \) and \( Y = [Y/F] \).
Let $Z$ be as in (9.1). It follows from Lemma 2.8 that $G = H \times F$ acts on $Z$ such that $t$ is $G$-equivariant and $F$-torsor. By Lemma 2.8 we have proper maps $\mathcal{X} = [Z/G] \xrightarrow{p} W = [W/F] \xrightarrow{q} \mathcal{Y} = [Y/F]$, where $W = Z/H$.

We consider the diagram

\[
\begin{array}{cccccc}
G_{i}(\mathcal{X}) & \xrightarrow{p_{*}} & G_{i}(G, Z) & \xrightarrow{\varphi_{G}^{Z}} & G_{i}(G, I_{Z})_{m_{1G}} & \xrightarrow{v_{Z}} & G_{i}(I_{Z})_{m_{1Z}} \\
p_{*} & & \text{Inv}_{Z}^{H} & & \text{Inv}_{Z}^{H} & & \text{Inv}_{Z}^{H} \\
G_{i}(W) & \xrightarrow{p_{*}^{W}} & G_{i}(F, W) & \xrightarrow{\varphi_{W}^{F}} & G_{i}(F, I_{W})_{m_{1F}} & \xrightarrow{v_{W}} & G_{i}(I_{W})_{m_{1W}} \\
q_{*} & & q_{*} & & q_{*} & & q_{*} \\
G_{i}(\mathcal{Y}) & \xrightarrow{p_{*}^{Y}} & G_{i}(F, Y) & \xrightarrow{\varphi_{Y}^{F}} & G_{i}(I_{Y})_{m_{1F}} & \xrightarrow{v_{Y}} & G_{i}(I_{Y})_{m_{1Y}} \\
\end{array}
\]

The top and the bottom squares on the left commute by the commutativity of proper push-forward and flat pull-back maps on $K$-theory of stacks, noting that Inv$^{-1}_{(-)}$ is just the proper push-forward map (see Theorem 3.13). The top and the bottom squares on the right commute by definition of the various maps. The bottom square in the middle commutes by Proposition 7.1 and the top square in the middle commutes by Theorem 8.7. It follows that (9.5) commutes.

\[\square\]

10. The Riemann-Roch theorem for quotient stacks

In this section, we shall apply the Atiyah-Segal isomorphism and the equivariant Riemann-Roch theorem of [30] to prove the Grothendieck Riemann-Roch theorem for separated quotient stacks. In order to do so, we need to recall the higher Chow groups of such stacks and prove some properties of these groups.

10.1. Equivariant higher Chow groups. Let $G$ be a linear algebraic group. Since most of the basic properties of equivariant higher Chow groups require us to assume that the underlying algebraic space with $G$-action is $G$-quasi-projective, we shall have to mostly work under this assumption in this section. Of course, all group actions on such schemes will be linear. However, one can check that all the properties that we need or prove below, hold for algebraic spaces too if we are only interested in the equivariant (not higher) Chow groups $\text{CH}_{i}^{G}(X)$ and Riemann-Roch for these.

The equivariant higher Chow groups of quasi-projective schemes were defined in [12] and their fundamental properties were described in [20]. We refer to these references without recalling the definition of equivariant higher Chow groups. We only recall that they share most of the properties of equivariant $K$-theory. In particular, they satisfy localization, homotopy invariance, flat pull-back, proper push-forward and Morita isomorphisms. Furthermore, they coincide with Bloch’s higher Chow groups of quotients under a free action. These properties will be used frequently in this section. We shall let $\text{CH}_{i}^{G}(X, i) = \oplus_{j \in \mathbb{Z}} \text{CH}_{i}^{G}(X, i)$ for $i \geq 0$ and $\text{CH}_{0}^{G}(X) = \text{CH}_{0}^{G}(X, 0)$. Recall also that these are all $\mathbb{C}$-vector spaces under our convention.

In this text, we shall use the following convention for differentiating the maps on $K$-theory and Chow groups, induced by a morphism of spaces $f: X \to Y$. We shall denote the maps on $K$-theory by $f^{-}$ (resp. $f_{+}$) and on Chow groups by $f^{*}$ (resp. $f_{*}$).

Let $H \subset G$ be a normal subgroup of an algebraic group $G$ with quotient $F$. Let $G$ act properly on quasi-projective $\mathbb{C}$-schemes $X$ and $X'$ whose quotients $X/G$ and $X'/G$ are quasi-projective. Let $f: X' \to X$ be a $G$-equivariant proper map. It follows from Lemma 2.7 that under this hypothesis, the quotients $Y = X/H$ and $Y' = X'/H$ are quasi-projective with proper and linear $F$-action.
We recall the following result from [12, § 4] and [14, § 6.4] on the functor of invariants for equivariant higher Chow groups.

**Theorem 10.1.** There is a natural map

\[
\Inv^H_X : \CH^G_*(X, i) \to \CH^F_*(Y, i)
\]

and a commutative diagram

\[
\begin{array}{ccc}
\CH^G_*(X', i) & \xrightarrow{\bar{f}^*} & \CH^G_*(X, i) \\
\Inv^H_{X'} & \downarrow & \Inv^H_X \\
\CH^F_*(Y', i) & \xrightarrow{f^*} & \CH^F_*(Y, i)
\end{array}
\]

such that \(\Inv^G_X = \Inv^F_Y \circ \Inv^H_X\).

If \(X\) and \(X'\) are only algebraic spaces, then the above hold for \(\CH^G_*(-)\).

**Proof.** Let \(X \xrightarrow{\pi_H} Y \xrightarrow{\pi_F} Y/F = X/G\) denote the quotient maps and let \(\pi_G = \pi_F \circ \pi_H\). Then the map \(\pi_G^* : \CH_*(X/G, i) \to \CH^G_*(X, i)\) is defined in the proof of [12, Theorem 3] and this is an isomorphism. The map \(\Inv^G_X\) is defined in [14, § 6.4] for Chow groups \(\CH^G_*(X)\). However, under the quasi-projective assumption, this extends to higher Chow groups as well which we now explain.

We first define \(\Inv^H_X\) as the inverse of \(\pi_G^*\). In particular, \(\Inv^F_Y\) is the inverse of \(\pi_F^*\). We let \(\Inv^H_X : \CH^G_*(X, i) \to \CH^F_*(Y, i)\) be the unique homomorphism such that the diagram

\[
\begin{array}{ccc}
\CH^G_*(X, i) & \xrightarrow{\Inv^H_X} & \CH^F_*(Y, i) \\
\Inv^F_Y & \downarrow & \Inv^F_Y \\
\CH_*(X/G, i) & \xrightarrow{\Inv^G_X} & \CH^G_*(X, i)
\end{array}
\]

commutes. To show that \(10.2\) commutes, we consider the diagram

\[
\begin{array}{ccc}
\CH^G_*(X', i) & \xrightarrow{\bar{f}^*} & \CH^F_*(Y', i) \\
\Inv^H_{X'} & \downarrow & \Inv^F_{Y'} \\
\CH^G_*(X, i) & \xrightarrow{f^*} & \CH^G_*(X, i)
\end{array}
\]

It follows from [12, Proposition 11(a)] that the right square and the big outer rectangle commute. Since \(\Inv^F_Y\) is an isomorphism, it follows that the left square also commutes. For \(i = 0\), an identical construction and proof work for algebraic spaces as well. \(\square\)

**Definition 10.2.** Let \(\mathcal{X}\) be a separated quotient stack with coarse moduli space \(M\). We define the Chow group of \(\mathcal{X}\) as \(\CH_*(\mathcal{X}) := \CH_*(M)\). If \(M\) is quasi-projective, then we define the higher Chow groups of \(\mathcal{X}\) as \(\CH_*(\mathcal{X}, i) := \CH_*(M, i)\).

Let \(f : \mathcal{X} = [X/G] \to \mathcal{Y} = [Y/F]\) be a proper map of separated quotient stacks with coarse moduli spaces \(M\) and \(N\), respectively. Let \(\bar{f}\) be the map induced on the coarse moduli spaces. It follows from Lemma [2.9] that \(\bar{f}\) is proper.

We define \(f_* : \CH_*(\mathcal{X}) \to \CH_*(\mathcal{Y})\) to be the proper push-forward map \(\bar{f}_* : \CH_*(M) \to \CH_*(N)\). If \(M\) and \(N\) are quasi-projective, then we define the push-forward map \(\bar{f}_* : \CH_*(\mathcal{X}, i) \to \CH_*(\mathcal{Y}, i)\) to be the corresponding proper push-forward map \(\bar{f}_* : \CH_*(M, i) \to \CH_*(N, i)\).
10.2. Some consequences of the equivariant Riemann-Roch theorem. Let $G$ be a linear algebraic group acting properly on an algebraic space $X$. The Riemann-Roch map $G_0(G, X) \to \text{CH}^G_*(X)$ was defined in [13]. If $X$ is quasi-projective, the Riemann-Roch map $G_i(G, X) \to \text{CH}^G_*(X, i)$ was defined in [30]. We recall the following.

**Theorem 10.3.** ([13 Theorem 3.1], [30 Theorem 1.4]) Let $G$ act properly on a quasi-projective scheme $X$. Then for every $i \geq 0$, there is a Riemann-Roch map

$$t^G_X : G_i(G, X) \to \text{CH}^G_*(X, i)$$

which is covariant for proper $G$-equivariant maps and the induced map $G_i(G, X)_{m_1} \overset{t^G_X}{\to} \text{CH}^G_*(X, i)$ is an isomorphism. If $X$ is only an algebraic space, the same holds for $i = 0$.

When $G$ is trivial, then the Riemann-Roch map $t^G_X$ of Theorem 10.3 coincides with the Riemann-Roch map in the non-equivariant $K$-theory, established by Bloch [5]. We shall denote this map by $\tau_X$ in the sequel.

**Lemma 10.4.** Let $G$ act properly on an algebraic space $X$ with quotient $Y$ and let $i \geq 0$ be an integer. If $Y$ is quasi-projective, there is a commutative diagram

$$
\begin{array}{ccc}
G_i(G, X)_{m_1} & \overset{t^G_X}{\longrightarrow} & \text{CH}^G_*(X, i) \\
\downarrow \text{Inv}^G_X & & \downarrow \text{Inv}^G_X \\
G_i(Y) & \overset{\tau_Y}{\longrightarrow} & \text{CH}^*_{\text{inv}}(Y, i). \\
\end{array}
$$

If $Y$ is not necessarily quasi-projective, this diagram is commutative for $i = 0$.

**Proof.** If $G$ freely acts on $X$, then $\Sigma_Y^G = \{1\}$ and hence $G_i(G, X) \simeq G_i(G, X)_{m_1}$. The lemma then follows from [30 Theorem 4.6] when $Y$ is quasi-projective and from [13 § 3.2] when $Y$ is just an algebraic space.

We now prove the general case when $Y$ is quasi-projective. Note in this case that $X$ is also quasi-projective. By [28 Theorem 1], we can find a finite, flat and surjective $G$-equivariant map $f : X' \to X$ such that $G$ acts freely on $X'$ with $G$-torsor $X' \to Y' := X'/G$ and the induced map $u : Y' \to Y$ is finite and surjective. We consider the diagram

$$
\begin{array}{ccc}
G_i(G, X')_{m_1} & \overset{\tau^G_{X'}}{\longrightarrow} & \text{CH}^G_*(X', i) \\
\downarrow \text{Inv}^G_{X'} & & \downarrow \text{Inv}^G_{X'} \\
G_i(Y') & \overset{\tau_Y'}{\longrightarrow} & \text{CH}^*_{\text{inv}}(Y', i) \\
\downarrow f_* & & \downarrow f_* \\
G_i(G, X)_{m_1} & \overset{\tau^G_X}{\longrightarrow} & \text{CH}^G_*(X, i) \\
\downarrow \text{Inv}^G_X & & \downarrow \text{Inv}^G_X \\
G_i(Y) & \overset{\tau_Y}{\longrightarrow} & \text{CH}^*_{\text{inv}}(Y, i). \\
\end{array}
$$

The back and the front faces of this cube commute by By Theorem 10.3 and its non-equivariant version. The left face commutes by the functoriality of the push-forward maps in $K$-theory of stacks because $\text{Inv}^G_X$ and $\text{Inv}^G_{X'}$ are just the push-forward maps induced by the proper maps $[X'/G] \to Y'$ and $[X/G] \to Y$. The right face commutes by Theorem 10.3 and the top face commutes because of the free action. Since $f$ is flat, the map $\bar{f}_*$ is surjective by [30 Lemma 11.1]. It follows from Theorem 10.3 that $f_*$ is also surjective. But this easily implies that the bottom face of the cube commutes. This finishes the proof when $Y$ is quasi-projective.
If \( Y \) is only an algebraic space, then we can use \([12, \text{Proposition 10}]\) to find a finite and surjective \( G \)-equivariant map \( f : X' \to X \) such that \( G \) acts freely on \( X' \) with \( G \)-torsor \( X' \to Y' := X'/G \) and the induced map \( u : Y' \to Y \) is finite and surjective. It is then easy to check that the map \( \bar{f}_*: \text{CH}^G(X') \to \text{CH}^G(X) \) is surjective (we are working with \( \mathbb{C} \)-coefficients). Now, the above argument works verbatim to prove that (10.6) commutes for \( i = 0 \).

**Corollary 10.5.** With notations of Lemma [10.4] the map \( \text{Inv}^{G}_X : G_i(G, X)_{m_1} \to G_i(Y) \) is an isomorphism for every \( i \geq 0 \) if \( Y \) is quasi-projective.

**Proof.** The corollary is an immediate consequence of Lemma [10.4] and Theorem [10.3] because the map \( \text{Inv}^{G}_X \) is an isomorphism (see Theorem [10.1]).

**Lemma 10.6.** Let \( G = H \times F \) act properly on an algebraic space \( X \) and let \( Y = X/H \). Let \( i \geq 0 \) be any integer. If \( X/G \) is quasi-projective, there is a commutative diagram

\[
\begin{array}{ccc}
G_i(G, X)_{m_1} & \xrightarrow{t^G_X} & \text{CH}^G(X, i) \\
\downarrow \text{Inv}^H_X & & \downarrow \text{Inv}^H_X \\
G_i(F, Y)_{m_1} & \xrightarrow{t^F_Y} & \text{CH}^F(Y, i).
\end{array}
\]

If \( i = 0 \), this diagram is commutative without any assumption on \( X/G \).

**Proof.** We let \( Z = X/G = Y/F \) and consider the diagram

\[
\begin{array}{ccc}
G_i(G, X)_{m_1} & \xrightarrow{t^G_X} & G_i(F, Y)_{m_1} \xrightarrow{t^F_Y} G_i(Z) \\
\downarrow \text{Inv}^H_X & & \downarrow \text{Inv}^H_X \\
\text{CH}^G(X, i) & \xrightarrow{t^G_X} & \text{CH}^F(Y, i) \xrightarrow{\tau^G_Y} \text{CH}^F(Z, i).
\end{array}
\]

It follows from Theorem [8.13] that \( \text{Inv}^F_Y \circ \text{Inv}^H_X = \text{Inv}^G_X \) and it follows from Theorem [10.1] that \( \text{Inv}^G_X = \text{Inv}^F_Y \circ \text{Inv}^H_X \). The the outer rectangle and the right square both commute by Lemma [10.4]. Since the map \( \text{Inv}^F_Y \) is an isomorphism (see the proof of Theorem [10.1]), it follows that the left square commutes. For \( i = 0 \), exactly the same proof works for algebraic spaces.

**10.3. Riemann-Roch for quotient stacks.** We now define the Riemann-Roch transformation for separated quotient stacks and prove the Riemann-Roch theorem for them.

**Definition 10.7.** Let \( \mathcal{X} = [X/G] \) be a separated quotient stack of finite type over \( \mathbb{C} \) and let \( M \) denote its coarse moduli space. Let \( i \geq 0 \) be an integer.

1. If \( M \) is quasi-projective, we let the Riemann-Roch maps be the composite arrows

\[
\tau^G_{\mathcal{X}} : G_i(G, X) \xrightarrow{\theta^G_X} G_i(G, I_X)_{m_1} \xrightarrow{t^G_X} \text{CH}^G(I_X, i).
\]

2. If \( M \) is only an algebraic space and \( i = 0 \), we let \( \tau^G_{\mathcal{X}} \) and \( \tau_{\mathcal{X}} \) be the maps as defined in (10.10) and (10.11), respectively.

It follows immediately from Lemma [9.1] and Theorem [10.1] that \( \tau_{\mathcal{X}} \) is independent of the presentation of \( \mathcal{X} \) as a quotient stack.
Lemma 2.8 that $G$ is a diagonalizable group, then the semi-simple conjugacy classes of $G$ are easily handled, since $G$ is an isomorphism. It follows from the above constructions that $\tau^G_X$ in this case is simply the direct sum of the Riemann-Roch maps $t^G_X : G_i(G, X^h)_{m_1} \cong CH^G_i(X^h, i)$. 

Example 10.8. If $G$ is a diagonalizable group, then the semi-simple conjugacy classes of $G$ are easily handled, since $G$ is an isomorphism. It follows from the above constructions that $\tau^G_X$ in this case is simply the direct sum of the Riemann-Roch maps $t^G_X : G_i(G, X^h)_{m_1} \cong CH^G_i(X^h, i)$.

Remark 10.9. If $X$ is smooth and $i = 0$, then an equivariant Riemann-Roch map was constructed by Edidin and Graham [14] Theorem 6.7. Using the non-abelian localization theorem of [15], we can check that the map $\tau^G_X$ of (10.10) coincides with that of [14].

It is enough to check this at every semi-simple conjugacy class $\psi$. Let $p_\phi$ denote the projection $G_i(Z, X_\phi)_{m_\phi} \to G_i(Z, X_\phi)_{m_\phi}$ and let $l_f = \lambda_1(N_f^{-1})$ (see [15] § 5.4]). To check the agreement, all we need to show is that $l_f \circ p_\phi \circ \mu^{-1} \circ f^* = \omega^{-1} \circ (j_\psi)^{-1}$, where $j_\psi : X_\psi \to X$ is the inclusion, $j_\phi : G_j(G, X_\phi)_{m_\phi} \to G_j(G, X)_{m_\psi}$ is the isomorphism of Theorem 3.10 and $f = j_\psi \circ \mu : I^G_X \to X$ is a map of smooth algebraic spaces. Equivalently, we need to show that $j_\psi^* \omega \circ (l_f \circ p_\phi \circ \mu^{-1} \circ f^*)$ is identity. Using our definition of $\omega_\phi$ and $j_\phi^* \mu_\phi = f_\phi$, this is equivalent to showing that $f_\psi \circ \mu_\phi \circ (l_f \circ p_\phi \circ \mu^{-1} \circ f^*)$ is identity. In the notations of [15] § 5.4], this is equivalent to saying that $f_\psi((l_f \cap f^*(\alpha))_{c_\phi}) = \alpha$ for $\alpha \in G_j(G, X)_{m_\psi}$. But this is precisely the identity (13) of [15].

The Grothendieck-Riemann-Roch theorems that we wish to prove in this text for separated quotient stacks are as follows.

Theorem 10.10. Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of separated quotient stacks of finite type over $\mathbb{C}$ with quasi-projective coarse moduli spaces. Let $i \geq 0$ be an integer. Then there is a commutative diagram

\begin{equation}
G_i(\mathcal{X}) \xrightarrow{\tau_X} CH_*(I\mathcal{X}, i) \\
\downarrow f_* \quad \quad \quad \downarrow f_!
\end{equation}

such that the horizontal arrows are isomorphisms.

Proof. We let $X = [X/H]$ and $Y = [Y/F]$. Let $Z$ be as in (9.1). It follows from Lemma 2.8 that $G = H \times F$ acts on $Z$ such that $t$ is $G$-equivariant and $F$-torsors. By Lemma 2.8, we have proper maps $\mathcal{X} = [Z/G] \xrightarrow{p_Z} \mathcal{W} = [W/F] \xrightarrow{p_W} \mathcal{Y} = [Y/F]$, where $W = Z/H$. Let $v : I_Z/H \to I_W$ denote the canonical map induced on the inertia schemes. We consider the diagram

\begin{equation}
G_i(\mathcal{X}) \xrightarrow{p_Z^*} G_i(G, Z) \xrightarrow{\varphi^G_Z} G_i(G, I_Z)_{m_1} \xrightarrow{\tau^G_X} CH^G_*(I_Z, i) \xrightarrow{Inv_{I_Z}^G} CH^G_*(I_X, i) \\
\end{equation}

\begin{equation}
G_i(\mathcal{W}) \xrightarrow{p_W^*} G_i(F, W) \xrightarrow{\varphi^F_W} G_i(F, I_W)_{m_1} \xrightarrow{\tau^F_W} CH^F_*(I_W, i) \xrightarrow{Inv_{I_W}^F} CH^F_*(I_Y, i) \\
\end{equation}

\begin{equation}
G_i(\mathcal{Y}) \xrightarrow{p_Y^*} G_i(F, Y) \xrightarrow{\varphi^F_Y} G_i(F, I_Y)_{m_1} \xrightarrow{\tau^F_Y} CH^F_*(I_Y, i) \xrightarrow{Inv_{I_Y}^F} CH^F_*(I_Y, i). \\
\end{equation}

The top and the bottom squares on the left commute by the commutativity of flat pull-back and proper push-forward maps in $K$-theory of stacks, noting that $Inv_{I_Z}^G$ is just the proper push-forward map (see Theorem 5.3.13). The top and the bottom squares on the right commute by the functoriality of the proper push-forward maps on the higher Chow groups of stacks (see §10.1). The second square from the left on the top commutes
by Theorem 8.7 and the corresponding bottom square commutes by Proposition 7.1. The second square from the right on the top commutes using Lemma 10.6 and Theorem 10.3. The corresponding bottom square commutes by Theorem 10.3. It follows that (10.12) commutes. We have already shown before that all horizontal arrows in (10.13) are isomorphisms. This finishes the proof.

□

An identical proof also gives us the following Riemann-Roch theorem without any assumption on the coarse moduli spaces. In the special case where we assume \( \mathcal{X} \) to be smooth and \( f : \mathcal{X} \rightarrow \mathcal{Y} = M \) to be the coarse moduli space map for \( \mathcal{X} \), this result is due to Edidin and Graham (see [14, Theorem 6.8] and [11, Theorem 5.1]).

**Theorem 10.11.** Let \( f : \mathcal{X} \rightarrow \mathcal{Y} \) be a proper morphism of separated quotient stacks. Then there is a commutative diagram

\[
\begin{array}{ccc}
G_0(\mathcal{X}) & \xrightarrow{\tau_\mathcal{X}} & CH_*(I_\mathcal{X}) \\
\downarrow f_* & & \downarrow f_* \\
G_0(\mathcal{Y}) & \xrightarrow{\tau_\mathcal{Y}} & CH_*(I_\mathcal{Y})
\end{array}
\]

in which the horizontal arrows are isomorphisms.

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