QUANTIZATION OF QUASI-LIE BIALGEBRAS

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Abstract. We construct quantization functors of quasi-Lie bialgebras. We establish a bijection between this set of quantization functors, modulo equivalence and twist equivalence, and the set of quantization functors of Lie bialgebras, modulo equivalence. This is based on the acyclicity of the kernel of the natural morphism from the universal deformation complex of quasi-Lie bialgebras to that of Lie bialgebras. The proof of this acyclicity consists in several steps, ending up in the acyclicity of a complex related to free Lie algebras, namely, the universal version of the Lie algebra cohomology complex of a Lie algebra in its enveloping algebra, viewed as the left regular module. Using the same arguments, we also prove the compatibility of quantization functors of quasi-Lie bialgebras with twists, which allows us to recover our earlier results on compatibility of quantization functors with twists in the case of Lie bialgebras.

Let \( k \) be a field of characteristic 0. Unless specified otherwise, “Lie algebra”, “vector space”, etc., means “Lie algebra over \( k \)”, etc.

1. Introduction and main results

The main result of this paper is the construction of quantization functors for quasi-Lie bialgebras. This problem was posed in [Dr4, Section 5]; let us recall its formulation ([Dr2]). A quasi-bialgebra is a 6-uple \( (A, m, \Delta, \Phi, \epsilon, \eta) \), where \( (A, m, \eta) \) is an algebra with unit, \( \Delta : A \to A^\otimes 2 \) is an algebra morphism with counit \( \epsilon \), and \( \Phi \in A^\otimes 3 \) satisfies the identity \((\text{id} \otimes \Delta)(\Delta(a)) = \Phi(\Delta \otimes \text{id})(\Delta(a))\Phi^{-1}\) and the pentagon identity. Examples of quasi-bialgebras are \( A_0 = U(\mathfrak{a}) \), where \( \mathfrak{a} \) is a Lie algebra, equipped with its bialgebra structure and \( \Phi = 1 \). A deformation of \( A_0 \) in the category of topologically free \( k[[\hbar]] \)-modules (i.e., a QUE quasi-bialgebra) gives rise to a quasi-Lie bialgebra \( (\mathfrak{a}, \mu, \delta, \varphi) \), the classical limit of \( A \). A quantization functor of quasi-Lie bialgebras is a section of the classical limit functor \( \{\text{QUE quasi-bialgebras}\} \to \{\text{quasi-Lie bialgebras}\} \). The paper [Dr2] contains examples of quasi-Lie bialgebras, whose quantization is not explicitly known (e.g., the quasi-Lie bialgebras from p. 1437, in the non-split case).

As in the case of Lie bialgebras, the problem of construction of quantization functors of quasi-Lie bialgebras can be rephrased in the language of props ([McL]). In Section 2 we introduce a complete prop \( QLBA \) of quasi-Lie bialgebras and define the notion of a quantized symmetric quasi-bialgebra (QSQB) in \( QLBA \). A QSB in \( QLBA \) (which we will also call a universal quantization functor of quasi-Lie bialgebras) then gives rise to a quantization functor as above (it also allows to construct quantizations of quasi-Lie bialgebras in symmetric monoidal categories, more general than that of vector spaces, like super-vector spaces, d.g.vector spaces, etc.). We introduce the notions of equivalence and twist equivalence on the set of QSB’s in \( QLBA \). We similarly introduce the notion of a QSB (quantized symmetric bialgebra) in the prop \( LBA \) of Lie bialgebras, which is the same as the universal quantization functors from [EK2]; there is a notion of equivalence for these QSB’s. Our main result is (Theorem 2.1):

Theorem 1.1. The map \( \{\text{universal quantization functors of quasi-Lie bialgebras}\}/(\text{equivalence, twist equivalence}) \to \{\text{universal quantization functors of Lie bialgebras}\}/(\text{equivalence}) \) is a bijection.
Together with the results of [EK1, EK2], where is constructed a map \{associators over \( k \)} \( \rightarrow \) \{universal quantization functors of Lie bialgebras\}, and of [Dr3] on the existence of associators over \( k \), this result implies the existence of universal quantization functors for quasi-Lie bialgebras.

Let us explain the idea of the proof of Theorem [1.1]. According to deformation theory, there are complexes \( C_{QLBA} \) and \( C_{LBA} \), equipped with gradings, whose second and first cohomology groups respectively contain the obstruction to lifting a quantization from degree \( n \) to degree \( n + 1 \), and parametrize such liftings. This viewpoint is not used in the quantization of Lie bialgebras, since the groups \( H^i_{LBA} \) are not known. (In the same way, it is not known how to construct associators using deformation theory, see Remark 2, p. 854 in [Dr3].) However, this viewpoint can be used in our context. Namely, we will prove:

**Theorem 1.2.** The canonical map \( H^i_{QLBA} \rightarrow H^i_{LBA} \) is an isomorphism for any \( i \geq 0 \).

This immediately implies our main result (see Section 2.4).

Let us give some details of the proof of Theorem [1.2]. Our aim is to prove that the relative complex \( \text{Ker}(C_{QLBA} \rightarrow C_{LBA}) \) is acyclic. We introduce a filtration of the prop \( QLBA \) by the powers of an ideal \( \langle \phi \rangle \). Our first main result is Theorem [3.1] which gives an isomorphism \( \text{gr} \, QLBA \cong LBA_\alpha \) of the associated graded prop with an explicitly presented prop. For this, one constructs a morphism \( LBA_\alpha \rightarrow \text{gr} \, QLBA \), which is clearly surjective (here \( LBA_\alpha \) is an explicitly presented prop); to prove its injectivity, we use the existence of “many” quasi-Lie bialgebras, namely, the classical twists of Lie bialgebras of the form \( F(c) \) (where \( c \) is a Lie coalgebra) by an element \( r \in \wedge^2(c) \) (in the same way, the existence of the Lie bialgebras \( F(c) \) is the argument underlying the structure theorem for the prop \( LBA \), see [E3, Pos].)

The associated graded (for the filtration of \( QLBA \)) of the relative complex is then the positive degree part of the complex \( C_{LBA_\alpha} \). To prove that it is acyclic, we prove that its lines are.

These lines are of the form \( 0 \rightarrow LBA_\alpha(1, \wedge^q) \rightarrow LBA_\alpha(\text{id}, \wedge^q) \rightarrow LBA_\alpha(\wedge^2, \wedge^q) \rightarrow \ldots \), where \( LBA_\alpha(X,Y) = \text{Coker}(LBA(D \otimes X, Y) \rightarrow LBA(C \otimes X,Y)) \) and the map corresponds to \( \kappa \in \text{LCA}(C,D) \). Here \( C, D \) are sums of Schur functors of positive degree, and the differential is the universal version of the differential of Lie algebra cohomology.

The proof of the acyclicity of this complex (Theorem 2.2; proof in Section 3) involves several reductions. We first show that in this complex, the spaces of cochains may be modified as follows: \( LBA_\alpha(\wedge^p, \wedge^q) \) is replaced by \( LBA_2(\wedge^p, \wedge^q) = LBA(Z \otimes \wedge^p, \wedge^q) \), where \( Z \) is an irreducible Schur functor, and the space of cochains is reduced to the sum of its components, where the “intermediate Schur functor between \( Z \) and \( \wedge^\alpha \) is \( Z \) (this notion is based on the structure theorem of \( LBA \), see Proposition 3.1) we say that the intermediate Schur functor between \( X_i \) and \( Y_j \) in the summand appearing in the r.h.s. of \( (i) \) is \( Z_{ij} \). We next introduce a filtration on the complex, viewing \( \wedge^p \) as a subobject of \( \text{id} \otimes \wedge^q \) and counting the number of intermediate Schur functor between the \( p \) factors \( \text{id} \) and \( \wedge^q \) which equal \( \text{id} \). We identify the associated graded complex with a subcomplex of \( 0 \rightarrow LBA(Z \otimes 1 \otimes \text{id} \otimes \wedge^p, \wedge^q) \rightarrow \ldots \rightarrow LBA(Z \otimes \wedge^p \otimes \text{id} \otimes \wedge^p, \wedge^q) \rightarrow \ldots \), where the differential involves Lie brackets between the components of \( \text{id} \otimes \wedge^p \otimes \wedge^q \) and of these components with \( Z \), formed by the sums of components, where the intermediate Schur functor between a component \( \text{id} \otimes \wedge^q \) (resp., of \( \text{id} \otimes \wedge^p \)) and \( \wedge^q \) is \( \text{id} \) (resp., has degree \( > 1 \)) and antisymmetric w.r.t. \( \wedge^p \). This subcomplex decomposes according to the intermediate Schur functors between the factors of \( \text{id} \otimes \wedge^p \) and \( \wedge^q \), and these subcomplexes are obtained from the complexes \( C^2_{Z} = (0 \rightarrow LA(Z \otimes 1 \otimes (\otimes_{i=1}^p Z_{i}^q), \wedge^q) \rightarrow \ldots \rightarrow LA(Z \otimes \wedge^p \otimes (\otimes_{i=1}^q Z_{i}^p), \wedge^q) \rightarrow \ldots \) with the same differentials (the \( Z_{i}^p \) are irreducible Schur functors of degree \( > 1 \)) by taking tensor products with vector spaces \( \text{LCA}(\text{id}, Z_{i}^p) \) and taking antiinvariants under \( \wedge^p \). We therefore have to show the acyclicity of the complexes \( C^*_{Z} \).
For this, we show that \( \Lambda^g \) may be replaced by \( \text{id}^\otimes q \), \( Z \) by \( \text{id}^\otimes z \), and \( (\otimes_{i=1}^{p'} Z''_i) \) by \( \text{id}^\otimes N \), and express the corresponding complex as a sum of tensor products of complexes, which reduces the problem to a complex \( 0 \to LA(\text{id}^\otimes 1) \to \cdots \to LA(\text{id}^\otimes \Lambda^P \otimes \text{id}^\otimes N, \text{id}) \to \cdots \). The spaces of chains are now spaces of multilinear Lie polynomials. Using Dynkin’s correspondence between free Lie and free associative polynomials, we identify the complex with a complex subspaces of chains are now spaces of multilinear Lie polynomials. Using Dynkin’s correspondence between free Lie and free associative polynomials, we identify the complex with a complex of subcomplexes, indexed by permutations. We next identify each summand \( A^\bullet_{\epsilon,N,1} \) with a tensor product of “elementary” complexes. These complexes \( E_{\epsilon',\epsilon}^\bullet \) (\( \epsilon, \epsilon' \in \{0,1\} \)) are 1-dimensional in each degree, and are universal versions of the complexes computing the Lie algebra cohomology of a Lie algebra \( g \) in \( U(g) \), equipped with one of its trivial, adjoint, left or right \( g \)-module structures. We show that two of these complexes are acyclic, using the PBW filtration of free associative algebras (when \( g \) is a finite dimensional Lie algebra, the corresponding complexes have 1-dimensional cohomology, concentrated in degree \( \dim g \)). As \( E_{0,1}^\bullet \) necessarily enters the tensor product decomposition of each subcomplex \( A^\bullet_{\epsilon,N,1} \), each of the \( A^\bullet_{\epsilon,N,1} \) is acyclic, which implies that \( A^\bullet_{\epsilon,N,1} \) is acyclic.

In the final section of the paper, we apply Theorem 2.2 for proving that quantization functors of quasi-Lie bialgebras are compatible with twists (Proposition 6.1). This allows us to generalize our earlier results (EH) on compatibility of quantization functors of Lie bialgebras with twists, see Proposition 5.2 (in EH, this result was established for Etingof-Kazhdan quantization functors, while Proposition 5.2 applies to any quantization functor of Lie bialgebras).

2. Quantization of (quasi)Lie bialgebras

In this section, we recall the general formalism of props and its relation with the quantization problems of (quasi)Lie bialgebras. In particular, we show that this formalism allows to recover biquantization results of [KT]. We also formulate our main result (Theorem 2.1) and explain the strategy of its proof.

2.1. Props. Recall the definitions of the Schur categories \( \text{Sch} \) and \( \text{Sch} \) (EH). These are braided symmetric tensor categories, defined as follows. The objects of \( \text{Sch} \) are Schur functors, i.e., finitely supported families \( X = (X_{\rho})_{\rho} \) of finite dimensional vector spaces, where \( \rho \in \bigcup_{n \geq 0} \widehat{\mathfrak{S}}_n \) (\( \rho \) is therefore a pair \( (n, \pi_\rho) \), where \( n \geq 0 \) and \( \pi_\rho \) is an irreducible representation of \( \mathfrak{S}_n \); \( n \) is called the degree of \( \rho \) by convention, \( \mathfrak{S}_0 \) is the trivial group). The set of morphisms from \( X \) to \( Y \) is \( \text{Sch}(X,Y) := \bigoplus_{\rho} \text{Vect}(X_{\rho}, Y_{\rho}) \). The direct sum of objects is \( X \oplus Y = (X_{\rho} \oplus Y_{\rho})_{\rho \in \bigcup_{n \geq 0} \widehat{\mathfrak{S}}_n} \). Their tensor product is \( X \otimes Y = (\otimes_{\rho,\rho'} M_{\rho,\rho'}^{\rho'} \otimes X_{\rho'} \otimes Y_{\rho''})_{\rho} \), where for \( \rho \in \widehat{\mathfrak{S}}_n, \rho' \in \widehat{\mathfrak{S}}_{n'}, \rho'' \in \widehat{\mathfrak{S}}_{n''}, \) then \( M_{\rho,\rho'}^{\rho''} = [\pi_{\rho} : \text{Ind}_{\mathfrak{S}_{n'}}^{\mathfrak{S}_n} (\pi_{\rho'} \otimes \pi_{\rho''})] \) if \( n = n' + n'' \) and 0 otherwise. \( \text{Sch} \) is defined similarly, dropping the condition that \( X \) is finitely supported.

An object \( X \) of \( \text{Sch} \) or \( \text{Sch} \) is called homogeneous of degree \( n \) if \( X_{\rho} = 0 \) if the degree of \( \rho \) is \( \neq n \). If \( X \) is homogeneous, we denote by \( |X| \) its degree.

We have a bijection \( \text{Irr}(\text{Sch}) \simeq \bigcup_{n \geq 0} \widehat{\mathfrak{S}}_n \), where \( \text{Irr}(\text{Sch}) \) is the set of equivalence classes of irreducible objects in \( \text{Sch} \). The unit object of \( \text{Sch} \) is \( 1 \), which corresponds to the element of \( \mathfrak{S}_0 \). We also define \( \text{id}, S^p, \Lambda^p \) as the objects corresponding to: the element of \( \mathfrak{S}_1 \), the trivial, and the signature character of \( \mathfrak{S}_p \). We set \( T_p := \text{id}^\otimes p \) and \( S := \bigoplus_{p \geq 0} S^p \in \text{Ob}(\text{Sch}) \).

Recall that a prop (resp., a \( \text{Sch} \)-prop) is an additive symmetric monoidal category \( C \), equipped with a tensor functor \( \text{Sch} \to C \) (resp., \( \text{Sch} \to C \)), which is the identity on objects.

A prop morphism \( C \to D \) is a tensor functor, such that the functors \( \text{Sch} \to C \to D \) and \( \text{Sch} \to D \) coincide. An ideal \( I \) of the prop \( C \) is an assignment \( (X,Y) \mapsto I(X,Y) \), such that

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\(^{1}\)For \( U, V \) finite dimensional vector spaces, \( \text{Vect}(U,V) \) is the set of morphisms \( U \to V \).
(X, Y) → C/I(X, Y) is a monoidal category. C/I is then the quotient prop. If f_α ∈ C(X_α, Y_α) are morphisms in C, then the ideal (f_α) is the smallest ideal I in C such that f_α ∈ I(X_α, Y_α).

Props may be presented by generators and relations. If V = (V_{n,m})_{n,m≥0} is a collection of vector spaces, there is a unique prop Free(V), such that for any prop C, we have a bijection ∏_{n,m} Vect(V_{n,m}, C(T_n, T_m)) ≃ Prop(Free(V), C) (where Prop denotes the set of prop morphisms). If R = (R_{n,m})_{n,m≥0} is a collection of subspaces of the Free(V)(T_n, T_m), then the ideal generated by V with relations R is the quotient prop Free(V)/(R).

We say that C is a topological prop if for any X, Y ∈ Ob(Sch) we have a filtration C(X, Y) = C^{≥0}(X, Y) ⊃ C^{≥1}(X, Y) ⊃ ..., complete and separated, and compatible with the prop operations; and if C(X, Y) = C^{≥v(|X|, |Y|)}(X, Y) for any homogeneous X, Y, where v(x, y) → ∞ when x → ∞, y being fixed, or y → ∞, x being fixed. Such a C gives rise to a Sch-prop C, given by C(X, Y) = ⊕_i C(X_i, Y_j), where X = ⊕_i X_i, Y = ⊕_j Y_j are the homogeneous decompositions of X, Y ∈ Ob(Sch) (⊕ is the direct product); C is then equipped with a complete and separated filtration, compatible with the prop operations.

2.2. Quantization functors. If C is a topological prop, the associated graded prop is gr(C) with gr(C)(X, Y) := ⊕_{i≥0} C^{≥i}/C^{≥i+1}(X, Y) for X, Y ∈ Ob(Sch). We denote by gr(C) its completion.

A quantized symmetric bialgebra (QSB) in C is a bialgebra structure on S in Ĉ, i.e., morphisms m_C ∈ Ĉ(S^{⊙2}, S), Δ_C ∈ Ĉ(S, S^{⊙2}), ε_C ∈ Ĉ(1, S), satisfying the bialgebra relations, and whose reductions mod Ĉ^{≥1} coincide with the standard (commutative, cocommutative) bialgebra structure on S, induced by the morphism Sch → Ĉ.

A QSB in Ĉ gives rise to a Lie bialgebra structure on on id in gr(C), with morphisms of degree 1 (its classical limit), as follows: one shows that there exist unique morphisms μ_C ∈ gr^1(C)(S^2, id) and δ_C ∈ gr^1(C)(id, S^2), such that (m_C ◦ Alt_2 ◦ inj_1^{⊗2} mod Ĉ^{≥2}) = inj_1 ◦ μ_C and (Alt_2 ◦ Δ_C ◦ inj_1 mod Ĉ^{≥2}) = inj_1^{⊗2} ◦ δ_C (here Alt_2 : S^{⊗2} → S^{⊗2} is the antisymmetrization, without factor 1/2), we identify Ĉ( miệng, S^2) with a subspace of Ĉ(id^p, id^q) and inj_1 : id → S is the canonical morphism); (μ_C, δ_C) then obey the Lie bialgebra relations. We say that the QSB (m_C, Δ_C, ε_C, η_C) is a quantization of (μ_C, δ_C).

Let Ĉ(S, S) be the preimage of id_S under Ĉ(S, S) → Ĉ/Ĉ^{≥1}(S, S); this is a group under composition. This group acts on the set of QSB’s by i*C * (m_C, Δ_C, ε_C, η_C) := (i*C ◦ m_C ◦ (i*C)^{-1}, i*C ◦ Δ_C ◦ i*C^{-1}, i*C ◦ ε_C, i*C ◦ η_C). Two QSB’s related by this group action are called equivalent. Equivalent QSB’s have the same classical limit.

Recall that LBA is the prop with generators μ ∈ LBA(⟨^2, id⟩, δ ∈ LBA(id, ^2)) and relations μ ◦ (μ ⊗ id_{id}) ◦ Alt_3 = 0, Alt_3 ◦ (δ ⊗ id_{id}) ◦ δ = 0, δ ◦ μ = Alt_2 ◦ (μ ⊗ id_{id}) ◦ (id_{id} ⊗ δ) ◦ Alt_2 (recall that μ, δ are identified with morphisms in LBA(id^{⊗2}, id) and LBA(id, id^{⊗2})). Then LBA is a Lie bialgebra in LBA, and it is an initial object in the category of props equipped with a Lie bialgebra structure on id.

LBA is graded by N^2, with μ, δ of degrees (1, 0), (0, 1); we denote by (deg_μ, deg_δ) this grading; LBA is then N-graded by the total degree deg_μ + deg_δ. If x ∈ LBA(X, Y) and x, y are homogeneous, then deg_μ(x) − deg_δ(x) = |X| − |Y|, so LBA(X, Y) = LBA^{≥|X|−|Y|}(X, Y).

So the total degree completion of LBA is a topological prop, with associated graded LBA. We denote by LBA the corresponding Sch-prop.

\[2\] If X is a symmetric monoidal category and X is an object of X, recall that a Lie bialgebra (resp., algebra, Lie algebra, bialgebra...) structure on X is a pair of morphisms μ_X ∈ X^{⊗2}, X) and δ_X ∈ X^{⊗2} (resp., m_X ∈ X^{⊗2}, X; etc.), satisfying the axioms of Lie bialgebra (resp., algebra, etc.).
We then define a quantization functor of Lie bialgebras as a QSB in \( \mathbf{LBA} \) quantizing \((\mu, \delta)\). Two quantization functors are equivalent if they are equivalent as QSB’s. Quantization functors of Lie bialgebras were constructed in \([\text{EK1}, \text{EK2}]\).

If now \( \mathcal{C} \) is a topological prop and \((\mu_{\mathcal{C}}, \delta_{\mathcal{C}})\) is a Lie bialgebra structure on \( \text{id} \) in \( \mathcal{C} \), where the structure morphisms have positive valuation, then a quantization functor gives rise to a QSB in \( \mathcal{C} \), quantizing \((\mu_{\mathcal{C}} \text{ mod } \mathcal{C} \geq 2, \delta_{\mathcal{C}} \text{ mod } \mathcal{C} \geq 2)\); the structure morphisms of this QSB are the images of the morphisms of the QSB in \( \mathbf{LBA} \) under the prop morphism \( \mathbf{LBA} \rightarrow \mathcal{C} \) taking \( \mu, \delta \) to \( \mu_{\mathcal{C}}, \delta_{\mathcal{C}} \).

**Remark 1.** (QUE-QFSH equivalence) If \( V \) is a vector space and \( X, Y \in \text{Ob}(\text{Sch}) \), we set \( C_{0}^{V}(X, Y) := \text{Vect}(X(V), Y(V)) \); this defines a prop. By extension of scalars, it gives rise to props \( C_{0}^{V}[[h]] \), \( C_{0}^{V}(\langle h \rangle) \), with \( C_{0}^{V}(\langle h \rangle)(X, Y) := C_{0}^{V}(X, Y)(\langle h \rangle) \), etc. We define two gradings on \( C_{0}^{V}(\langle h \rangle) \): for \( X, Y \) homogeneous, \( \text{deg}_{\text{QUE}}(C_{0}^{V}(X, Y)h^{i}) = |X| - |Y| + i \) and \( \text{deg}_{\text{QFSH}}(C_{0}^{V}(X, Y)h^{i}) = |Y| - |X| + i \). We then define \( C_{0}^{V,\text{QUE}} \) by \( C_{0}^{V,\text{QUE}}(X, Y) := C_{0}^{V}(X, Y)(\langle h \rangle) \) with QUE-degree \( \geq ||X| - |Y|| \) and \( C_{0}^{V,\text{QFSH}}(X, Y) := C_{0}^{V}(X, Y)(\langle h \rangle) \) of QFSH-degree \( \geq ||X| - |Y|| \), for \( X, Y \) homogeneous; these are topological props. Explicitly, \( C_{0}^{V,\text{QUE}}(X, Y) = h^{\max(|Y| - |X|, 0)}C_{0}^{V}(X, Y)[[h]] \) and \( C_{0}^{V,\text{QFSH}}(X, Y) = h^{\max(|X| - |Y|, 0)}C_{0}^{V}(X, Y)[[h]] \). Then a QSB in \( C_{0}^{V,\text{QUE}} \) (resp., \( C_{0}^{V,\text{QFSH}} \)) gives rise to a QUE deforming \( U(V) \) (resp., a QFSH deforming \( S(V) \)), where in both cases \( V \) is equipped with a Lie bialgebra structure. The prop automorphism of \( C_{0}^{V}(\langle h \rangle) \) given by \( C_{0}^{V}(\langle X, Y \rangle)(\langle h \rangle) \rightarrow C_{0}^{V}(\langle X, Y \rangle)(\langle h \rangle) \), \( x \rightarrow h^{\max(|Y| - |X|)}x \) for \( X, Y \) homogeneous, restricts to a prop isomorphism \( C_{0}^{V,\text{QUE}} \simeq \simeq C_{0}^{V,\text{QFSH}} \), which is compatible with the correspondence between QUE and QFSH algebras \( \mathbf{Dr} \Rightarrow \mathbf{Gas} \); namely, we have a commuting diagram

\[
\begin{array}{ccc}
\{\text{QSB’s in } C_{0}^{V,\text{QUE}}\} & \xrightarrow{\simeq} & \{\text{QUE algebras}\} \\
\{\text{QSB’s in } C_{0}^{V,\text{QFSH}}\} & \xrightarrow{\simeq} & \{\text{QFSH algebras}\}
\end{array}
\]

**Remark 2.** (Biquantization) Let \( A_{0} := k[[uv]] \otimes k[u, v] \subset k[u, v] \); this is the subring of \( k[[u, v]] \) of series \( a(u, v) = \sum_{i, j \geq 0} a_{ij}u^{i}v^{j} \) such that for some \( N_{a} \), the support of \( (a_{ij}) \) is contained in \( \{(i, j) \mid |i - j| \leq N_{a}\} \). We have \( A_{0} \simeq k[[u]][[v]] \oplus r k[[u]][[v]] \) as vector spaces. We have ring morphisms \( A_{0} \rightarrow k[[u]] \) and \( A_{0} \rightarrow k[[v]] \) obtained by setting \( v = 1 \), resp., \( u = 1 \).

\( C_{0}^{V} \) gives rise to the prop \( C_{0}^{V}(u, v) \) where \( C_{0}^{V}(u, v)(X, Y) := C_{0}^{V}(X, Y)((u, v)) \) and to the subprop \( C_{0}^{V}(u, v) \) of \( C_{0}^{V}(X, Y)((u, v)) \) obtained by setting \( v = 1 \), resp., \( u = 1 \). Explicitly, we have \( C_{0}^{V,\text{QUE}}(X, Y) = u^{\max(|X| - |Y|)}C_{0}^{V}(X, Y)[[uv]] \) if \( |X| \geq |Y| \) and \( C_{0}^{V,\text{QFSH}}(X, Y) = v^{\max(|Y| - |X|)}C_{0}^{V}(X, Y)[[uv]] \) if \( |Y| > |X| \). \( C_{0}^{V}(u, v) \) is also graded by the total degree \( \text{deg}_{\text{QUE}} + \text{deg}_{\text{QFSH}} \); this induces a filtration on \( C_{0}^{V}(u, v) \), which is topological as \( C_{0}^{V}(u, v) = (C_{0}^{V})(u, v) \).

A QSB in the prop \( C_{0}^{V}(u, v) \) then gives rise to a biquantization of a Lie bialgebra structure on \( V \), in the sense of \( \text{[KT]} \).

Setting \( v = 1 \) (resp., \( u = 1 \)), we get a prop morphisms \( C_{0}^{V}(u, v) \rightarrow C_{0}^{V}(u) \) (resp., \( C_{0}^{V}(u, v) \rightarrow C_{0}^{V}(v) \)), which restrict to prop morphisms \( C_{V}^{\text{QUE}} \rightarrow C_{0}^{\text{QUE}} \) (resp., \( C_{V}^{\text{QFSH}} \rightarrow C_{0}^{\text{QFSH}} \)), where in the target props \( h \) is replaced by \( u \) (resp., by \( v \)). The diagram of props

\[
\begin{array}{ccc}
C_{V}^{\text{QUE}} & \xrightarrow{\simeq} & C_{0}^{\text{QUE}} \\
\downarrow \simeq & & \downarrow \simeq \\
C_{V}^{\text{QFSH}} & \rightarrow & C_{0}^{\text{QFSH}}
\end{array}
\]

commutes. It follows that the corresponding diagram between sets of QSB’s in all three props commutes, so a biquantization of \( V \) arising from a QSB in \( C_{0}^{V}(u, v) \) gives rise to QUE and QFSH
algebras, which correspond to each other under the category equivalence between QUE and QFSH algebras.

If now \((V, \mu_V, \delta_V)\) is a Lie bialgebra, \((u_0 \mu_V, v_0 \delta_V)\) defines a Lie bialgebra structure on \(id \in \text{gr}^1(C_{K,T}) = C_{K,T}^1\), where the morphisms have positive valuation (in fact, total degree 1). A quantization functor of Lie bialgebras then gives rise to a QSB in \(C_{K,T}\), i.e., a biquantization of \((V, \mu_V, \delta_V)\) in the sense of \([KT]\). We recover in this way Theorem 2.3 of \([KT]\) (where a biquantization was constructed using the Etingof-Kazhdan construction).

2.3. Quantization functors of quasi-Lie bialgebras. Let again \(C\) be a topological prop. A quantized symmetric quasi-bialgebra (QSQB) in \(C\) is a quasi-bialgebra structure on \(S\) in \(C\), whose reduction mod \(\hat{C}^1\) coincides with the standard bialgebra structure on \(S\), induced by the morphism \(\text{Sch} \to S\). Explicitly, this is the data of morphisms \((m_C, \Delta_C, \Phi_C, \epsilon_C, \eta_C)\) (we often drop \(\epsilon_C, \eta_C\) in the same spaces as above, with \(\Phi_C \in C(1, S^{0,2})\), satisfying the quasi-bialgebra axioms

\[
m_C \circ (m_C \otimes \text{id}_S) = m_C \circ (\text{id}_S \otimes m_C), \quad \Delta_C \circ m_C = m_C^{0,2} \circ \beta_2 \circ \Delta_C^{0,2},
\]

\[
[(\text{id}_C \otimes \Delta_C) \circ \Delta_C] \ast \Phi_C = \Phi_C \ast [(\Delta_C \otimes \text{id}_C) \circ \Delta_C],
\]

\[
(\eta_C \otimes \Phi_C) \ast [(\text{id}_S \otimes \Delta_C \otimes \text{id}_S) \circ \Phi_C] \ast (\Phi_C \otimes \eta_C) = [(\text{id}_S \otimes \Delta_C) \circ \Phi_C] \ast [(\Delta_C \otimes \text{id}_S^2) \circ \Phi_C],
\]

\[
(\text{id}_S \otimes \epsilon_C \otimes \text{id}_S) \circ \Phi_C = \eta_C^{0,2}, \quad (\epsilon_C \otimes \text{id}_S) \circ \Delta_C = (\text{id}_C \otimes \epsilon_C) \circ \Delta_C = m_C \circ (\eta_C \otimes \text{id}_S) = m_C \circ (\text{id}_S \otimes \eta_C) = \text{id}_S,
\]

(we use the associativity of \(m_C\) to define an associative operation \(*: \hat{C}(X, S^{0,k}) \otimes \hat{C}(Y, S^{0,k}) \to \hat{C}(X \otimes Y, S^{0,k})\) by \(x \otimes y = m_C^{0,k} \circ \beta_2 \circ (x \otimes y)\), where \(\beta_2\) is the categorical version of \(x_1 \otimes x_2 \otimes x_3 \otimes y_1 \otimes y_2 \to x_1 \otimes y_1 \otimes x_2 \otimes y_2 \otimes x_3 \otimes y_3\); the isomorphisms \(1 \otimes X \simeq X \otimes 1\) for \(X = 1, \text{id}\) are implied); and the same reduction conditions as above, together with \((\Phi_C \text{ mod } \hat{C}^1) = \eta_C^{0,2}\), where \(\eta_C^{0,2}: 1 \to S\) is the canonical morphism (note that this condition implies that \(\Phi_C\) is invertible for the product \(*\) on \(C(1, S^{0,3}))\).

Let \(\hat{C}(1, S^{0,2})_1 \subset \hat{C}(1, S^{0,2})\) be the set of \(F_C\) such that \((F_C \text{ mod } \hat{C}^1) = \eta_C^{0,2}\) and \((\epsilon_C \otimes \text{id}_S) \circ F_C = (\text{id}_S \otimes \epsilon_C) \circ F_C = \eta_C\). Let \((m_C, \ldots)\) be a QSQB in \(\hat{C}\); when equipped with the product \(*\) induced by \(m_C\), \(\hat{C}(1, S^{0,2})_1\) is a group. This group acts on the set of all QSQB’s in \(\hat{C}\) with fixed \(m_C\) by \(F_C \mapsto (m_C \circ \Delta_C \otimes \epsilon_C \otimes \text{id}_S) \circ F_C = (m_C \circ \Delta_C \ast F_C = \epsilon_C \otimes \text{id}_S) \circ F_C \ast \Phi_C \ast [(\text{id}_S \otimes \Delta_C) \circ F_C]^{-1}\). Two QSQB’s related in this way are called twist equivalent.

The group \(\hat{C}(S, 1)\) acts on the set of QSQB’s in \(\hat{C}\) by \(i_C \ast (m_C, \ldots, \Phi_C) := (i_C \circ m_C \otimes (i_C^{-1})^{0,2} \otimes \ldots \otimes \eta_C) \circ \Phi_C\). Two QSQB’s related by this group action are called equivalent.

As in \([DT2]\), one proves:

**Proposition 2.1.** 1) For any QSQB \((m_C, \Delta_C, \Phi_C)\) in \(\hat{C}\), there exists \((\mu_C, \delta_C, \varphi_C)\), where \(\mu_C \in \text{gr}^1(C, \text{id}, \text{id})\), \(\delta_C \in \text{gr}^1(C/\text{id}, \text{id})\), \(\varphi_C \in \text{gr}^1(C/1, \text{id}, \text{id})\), such that: \((m_C \circ \text{Alt}_2 \circ \text{inj}_0 \circ \text{mod } \hat{C}^2) = \text{inj}_0 \circ \mu_C\), \((\text{Alt}_2 \circ \Delta_C \circ \text{inj}_1 \circ \text{mod } \hat{C}^2) = \text{inj}_1 \circ \delta_C\), \((\text{Alt}_3 \circ \Phi_C \circ \text{mod } \hat{C}^2) = \text{inj}_0^{0,3} \circ \varphi_C\). The triple \((\mu_C, \delta_C, \varphi_C)\) equips the object \(i_C\) with a quasi-Lie bialgebra structure in \(\text{gr}(C)\), where the morphisms have degree 1. We call it the classical limit of \((m_C, \Delta_C, \Phi_C)\).

2) Equivalent and twist equivalent QSQB’s have the same classical limit.

**Proof.** Let \((m_C, \Delta_C, \Phi_C)\) in \(\hat{C}\). Let \(\tilde{\mu}_C := (m_C \circ \text{Alt}_2 \circ \text{inj}_0 \circ \text{mod } \hat{C}^2)\); we have \(\tilde{\mu}_C \in \text{gr}^1(C/\text{id}, \text{id}, S)\). Composing \(\Delta_C \circ (m_C \circ \text{Alt}_2) = [(m_C \circ \text{Alt}_2) \circ m_C + (m_C \circ \beta) \circ (m_C \circ \text{Alt}_2)] \circ \beta_2 \circ \Delta_C \circ m_C \circ \text{inj}_0 \circ \text{mod } \hat{C}^2\), we get \(\Delta_S \circ \tilde{\mu}_C = \tilde{\mu}_C \circ \text{inj}_0 + \text{inj}_0 \circ \tilde{\mu}_C\), which implies the existence of \(\mu_C\) (here \(\Delta_S : S \to S^{0,2}\) is the image in \(\text{gr}(C)\) of the coproduct morphism of \(S \in \text{Ob}(\text{Sch})\)).

We next prove that if \(x \in \hat{C}^{2,1}(X, S^{0,n})\) and \(\Psi \in \hat{C}(1, S^{0,2})\) is such that \((\Psi \text{ mod } \hat{C}^1) = \text{inj}_0^{0,2}\), then \(\Psi \ast x \ast \Psi^{-1} = x \text{ mod } \hat{C}^{2,1}\). Indeed, \(\Psi \ast x \ast \Psi^{-1} = [(\Psi - \eta_C^{0,2}) \ast x \ast (\Psi - \eta_C^{0,2})] \ast \Psi^{-1}\); the result then follows from \(\Psi - \eta_C^{0,2} \in \hat{C}^{2,1}(X, S^{0,n})\) and \(m_C - m_C \circ \beta \in \hat{C}^{2,1}(S^{0,2}, S)\).
It follows that \((\text{id}_S \circ \Delta_C) \circ \Delta_C = (\Delta_C \circ \text{id}_S) \circ \Delta_C \mod \hat{C}^{\geq 2}\). Let \(\beta_{321}\); \(S^\otimes 3 \to S^\otimes 3\) be the analogue of \(x_1 \otimes x_2 \otimes x_3 \to x_3 \otimes x_2 \otimes x_1\). As \((\Delta_C \circ \text{id}_S) \circ \Delta_C - \beta_{321}\) \((\text{id}_S \circ \Delta_C) \circ \Delta_C = \left(\left[\Delta_C - \beta \circ \Delta_C\right] \circ \text{id}_S\right) \circ \left(\left[\Delta_C - \beta \circ \Delta_C\right] \circ \text{id}_S\right) = \left[\left[\Delta_C - \beta \circ \Delta_C\right] \circ \text{id}_S\right] \circ \left[\left[\Delta_C - \beta \circ \Delta_C\right] \circ \text{id}_S\right]\), we get \(\left(\left[\Delta_C - \beta \circ \Delta_C\right] \circ \text{id}_S\right) \circ \left(\left[\Delta_C - \beta \circ \Delta_C\right] \circ \text{id}_S\right) \circ \Delta_C \mod \hat{C}^{\geq 2}\).

Let \(\delta_C := (\text{id}_S \circ \Delta_C \circ \text{id}_j_1 \circ \text{id}_{S^\otimes 2}) \mod \hat{C}^{\geq 2}\); we have \(\delta_C \in \text{gr}^1(\text{id}_S, S^\otimes 2)\). Composing this with \(\text{id}_j_1\), we get \(\delta_C \circ \text{id}_j_0 - \text{id}_j_0 \circ \delta_C + (\Delta_0 \circ \text{id}_S - \text{id}_S \circ \Delta_0) \circ \delta_C = 0\). The degree of the cohomology group of the co-Hochschild (or Cartier) complex \(\ldots \to \text{gr}^1(C)(X, S^\otimes n) \to \text{gr}^1(C)(X, S^\otimes n + 1) \to \ldots\) is \(\text{gr}^1(C)(X, \wedge^n)\), and the map taking a co-boundary to its cohomology class is \((1/n!)\text{Alt}_n\). As \(\delta_C\) is a coboundary and is antisymmetric, it coincides with its cohomology class, so there exists the announced \(\gamma_C\).

The pentagon equation implies that \((\Phi_C - \eta_{C}^3 \circ \text{id}_S) \circ \Delta_C \circ \text{id}_S = \text{id}_S \circ \Delta_C \circ \text{id}_S (\text{co-Hochschild})\); it follows that its image by \(\text{Alt}_3\) coincides with its cohomology class, which implies the existence of \(\gamma_C\).

Before showing that the classical limit \(\mu_C, \delta_C, \varphi_C\) of \((m_C, \Delta_C, \Phi_C)\) satisfies the quasi-Lie bialgebra relations, let us show that it is invariant under twist equivalence. Let \(F_C \in \text{C}(S, S)_1\), let \((m_C, \Delta_C, \Phi_C) := := F_C \ast (m_C, \Delta_C, \Phi_C) = (\mu_C, \delta_C, \varphi_C)\) be its classical limit. As \(m_C = m_C, \mu_C = \mu_C\). We have \(\Delta_C - \beta \circ \Delta_C = F_C \ast \Delta_C - \beta \circ F_C = F_C \ast \Delta_C - \beta \circ F_C - (\beta \circ \Delta_C) - (\beta \circ F_C) - (\beta \circ \Delta_C) - (\beta \circ F_C) - (\beta \circ \Delta_C) - (\beta \circ F_C)\); it follows that \(\Delta_C - \beta \circ \Delta_C = \Delta_C - \beta \circ \Delta_C \mod \hat{C}^{\geq 2}, \) so \(\delta_C = \delta_C\). Finally, \(\Phi_C - \eta_{C}^3 \mod \hat{C}^{\geq 2} = (\Phi_C - \eta_{C}^3 \circ \text{id}_S) \circ \Delta_C \circ \text{id}_S = \text{id}_S \circ \Delta_C \circ \text{id}_S (\text{co-Hochschild})\), where \(d\) is the co-Hochschild differential, which implies that \(\varphi_C = \varphi_C\).

We have \((m_C \circ \text{id}_S) \circ (m_C \circ \text{id}_S) \circ \Delta_C = 0\) (equality in \(\hat{C}^{\geq 2}\)); composing with \(\text{id}_j_1\) and taking the class modulo \(\hat{C}^{\geq 3}\), we get \(\mu_C \circ (\mu_C \circ \text{id}_S) \circ \Delta_C = 0\).

We have \(|m_C \circ \text{id}_S| \circ (m_C \circ \text{id}_S) \circ \Delta_C = 0\) (equality in \(\hat{C}^{\geq 2}\)); composing with \(\text{id}_j_1\) and taking the class modulo \(\hat{C}^{\geq 3}\), we get \(\mu_C \circ (\mu_C \circ \text{id}_S) \circ \Delta_C = 0\).

We have \((\text{id}_S \circ \Delta_C - \Delta_C \circ \text{id}_S) \circ \Delta_C = 0\) (equality in \(\hat{C}^{\geq 2}\)); composing with \(\text{id}_j_1\) and taking the class modulo \(\hat{C}^{\geq 3}\), we get \(\mu_C \circ (\mu_C \circ \text{id}_S) \circ \Delta_C = 0\).

Using the co-Hochschild complex, one may find a twist \(F_C\) with \(F_C \ast (m_C, \Delta_C, \Phi_C) = (m_C, \Delta_C, \Phi_C)\), where \((\Phi_C - \eta_{C}^3 \circ \text{id}_S) \circ \Delta_C \circ \text{id}_S = \mu_C \circ (\mu_C \circ \text{id}_S) \circ \Delta_C \circ \text{id}_S\). Set 0.5cm \(\varphi_C := \Phi_C - \eta_{C}^3 \circ \Phi_C + \varphi_C \circ \text{id}_S + \varphi_C \circ \text{id}_S \circ \varphi_C\). Then the pentagon equation yields \(\varphi_C = \varphi_C \circ \text{id}_S + \varphi_C \circ \text{id}_S \circ \varphi_C\). As \(\varphi_C \in \hat{C}^{\geq 2}\), we have \(\varphi_C = \varphi_C \circ \text{id}_S + \varphi_C \circ \text{id}_S \circ \varphi_C\). It follows that \(\varphi_C \circ \text{id}_S + \varphi_C \circ \text{id}_S \circ \varphi_C\) is symmetric in indices 3, 4 modulo \(\hat{C}^{\geq 3}\); the image in \(\text{gr}^2(C)\) of the composition of this term with \(\text{Alt}_3\) is then zero; in the same way, we get the image of the composition of the r.h.s. with \(\text{Alt}_4\). It follows that the image in \(\text{gr}^2(C)\) of \(\text{Alt}_4 \circ \varphi_C \circ \text{id}_S \circ \varphi_C \circ \text{id}_S \circ \varphi_C \circ \text{id}_S \circ \varphi_C = 0\). As \(\delta_C = \delta_C, \Delta_C \circ \text{id}_S \circ \text{id}_S \circ \varphi_C \circ \text{id}_S \circ \varphi_C = 0\). We then get \(\text{id}_S \circ (\delta_C \circ \text{id}_S) \circ \varphi_C = 0\).
If now \( i_C \in \hat{C}(S, S)_1 \), we have \( i_C \circ \text{id}_{j_1} = \text{id}_{j_1} \mod \hat{C}^{\geq 2} \), which implies that the classical limit is invariant under equivalence.

Let QLBA be the prop with generators \( \mu \in \text{QLBA}(\Lambda^2, \text{id}), \delta \in \text{QLBA}(\text{id}, \Lambda^2), \varphi \in \text{QLBA}(1, \Lambda^3) \) and relations

\[
\mu \circ (\mu \otimes \text{id}_{\text{id}}) \circ \text{Alt}_3 = 0, \quad \delta \circ \mu = \text{Alt}_2 \circ (\mu \otimes \text{id}_{\text{id}}) \circ (\text{id}_{\text{id}} \otimes \delta) \circ \text{Alt}_2,
\]

\[
\text{Alt}_3 \circ (\delta \otimes \text{id}_{\text{id}}) \circ \delta = \text{Alt}_3 \circ (\mu \otimes \text{id}_{\text{id}}^3) \circ (\text{id}_{\text{id}} \otimes \varphi), \quad \text{Alt}_4 \circ (\delta \otimes \text{id}_{\text{id}}^2) \circ \varphi = 0.
\]

This prop is graded by \( \{(u, v) \in \mathbb{Z}^2 | u, v \geq 0, 2u + v \geq 0 \} \), with \( \mu, \delta, \varphi \) of degrees \((1, 0), (0, 1), (-1, 2)\); we denote this degree \((\text{deg}_\mu, \text{deg}_\delta)\). QLBA is then \( \mathbb{N} \)-graded by the total degree \( \text{deg}_\mu + \text{deg}_\delta \); the generators have then degree 1. If \( x \in \text{QLBA}(X, Y) \) and \( x, X, Y \) are homogeneous, then \( \text{deg}_\mu(x) - \text{deg}_\delta(x) = |X| - |Y| \), so \( \text{QLBA}(X, Y) = \text{QLBA}^{\geq |X| - |Y|}(X, Y) \), which implies that the total degree completion of QLBA is a topological prop, with associated graded QLBA. We denote by QLBA the corresponding Sch-prop. In the prop QLBA, the object \( \text{id} \) is equipped with a quasi-Lie bialgebra structure, and QLBA is an initial object in the category of props equipped with a quasi-Lie bialgebra structure on \( \text{id} \). We have an obvious prop morphism QLBA \( \rightarrow \text{LBA} \), given by \( \mu, \delta, \varphi \mapsto \mu, \delta, 0 \).

A quantization functor of quasi-Lie bialgebras is then a QSQB in QLBA admitting \((\mu, \delta, \varphi)\) as its classical limit. Two quantization functors are (twist) equivalent if they are as QSQB’s.

We have a map \{quantization functors of quasi-Lie bialgebras\} \( \rightarrow \{\text{quantization functors of Lie bialgebras}\} \), induced by the above morphism QLBA \( \rightarrow \text{LBA} \); this map takes equivalent functors to equivalent functors, and it takes two twist equivalent functors to the same image.

The main result of our paper is:

**Theorem 2.1.** The map \{quantization functors of quasi-Lie bialgebras\}/(equivalence, twist equivalence) \( \rightarrow \{\text{quantization functors of Lie bialgebras}\}/(\text{equivalence}) \) is a bijection.

**Remark 3.** If \( V \) is a vector space, then a QSQB in \( C^{\text{QHQUE}}_{\text{V}, \text{E}} \) gives rise to a QHQUE algebra deforming \( U(V) \), in the sense of [Dr2]. A quasi-Lie bialgebra structure \((\mu_V, \delta_V, \varphi_V)\) on \( V \) then gives rise to a quasi-Lie bialgebra structure on \( \text{id} \) in \( C^{\text{QHQUE}}_{\text{V}, \text{E}} \), namely \((\mu_V, h\delta_V, h^2\varphi_V)\).

As above, if \( C \) is a topological prop and \((\mu_C, \delta_C, \varphi_C)\) is a quasi-Lie bialgebra structure on the object \( \text{id} \) in \( C \), with morphisms of valuation \( \geq 1 \), then a quantization functor for quasi-Lie bialgebras yield a QSQB in \( C \) quantizing \((\mu_C, \delta_C, \varphi_C)\) mod \( \hat{C}^{\geq 2} \).

In particular, for each quasi-Lie bialgebra structure \((\mu_V, \delta_V, \varphi_V)\) on a vector space \( V \), a quantization functor for quasi-Lie bialgebras gives rise to a QHQUE algebra quantizing \( V \), in the sense of [Dr2].

### 2.4. Deformation complexes

Let \( D \) be a prop. Set \( C^p_D := D(\Lambda^{p+1}, \Lambda^{p+1}) \) for \( p, q \in \mathbb{Z} \) and \( C^p_D := \bigoplus_{p, q \in \mathbb{Z}} C^p_D \); this is a \( \mathbb{Z}^2 \)-graded vector space (hence \( \mathbb{Z} \)-graded by the total degree). We define a \( \mathbb{Z}^2 \)-graded Lie superalgebra structure on \( C^p_D \) as follows. For \( a \in C^p_D, a' \in C^{p'}_D \), we set \( a' \circ a := \text{Alt}_{q+q'+1}(a' \otimes \text{id}_{\text{id}}^q \otimes a) \circ \text{Alt}_{p+p'+1}, \) and define their Schouten bracket \( [a, a'] := a \circ a' - \text{deg}(a') \text{id}_{\text{id}}^q \text{deg}(a) \). The condition that \((\mu_D, \delta_D)\) defines a Lie bialgebra structure on \( \text{id} \) in \( D \) is then \( [\mu_D \otimes \delta_D, \mu_D \otimes \delta_D] = 0 \); the condition that \((\mu_D, \delta_D, \varphi_D)\) defines a quasi-Lie bialgebra structure on \( \text{id} \) in \( D \) is \( [\mu_D \otimes \delta_D \otimes \varphi_D, \mu_D \otimes \delta_D \otimes \varphi_D] = 0 \). The bracket with \( \mu_D \otimes \delta_D \) (resp., \( \mu_D \otimes \delta_D \otimes \varphi_D \)) then defines a complex structure on \( C^p_D \) (graded by the total degree); we denote by \( H^2_D \) the corresponding cohomology groups.

If \( C \) is a topological prop, we define a quotient prop \( C_{\leq n} \) by \( C_{\leq n}(X, Y) = C/C^{n+1}(X, Y) \); this is also a topological prop; we have a projective system \( \ldots \rightarrow C_{\leq n} \rightarrow \ldots \rightarrow C_{\leq 0} \) and \( C(X, Y) = \lim_{\leftarrow} C_{\leq n}(X, Y) \).

If now \((m_C^{\leq n}, \Delta_C^{\leq n})\) is a QS in \( C_{\leq n} \) with classical limit \((\mu_C, \delta_C)\), then the obstruction to extend it to a QS in \( C_{\leq n+1} \) belongs to \( H^2_{\text{gr}(C)}[n + 2] \), and the set of such extensions modulo
equivalence is an affine space over $H^1_{\text{gr}(C)}[n + 1]$ (here $[n]$ means the degree $n$ part for the grading of gr$(C)$); see [Dr1, GS, ShSt].

In the same way, if $(m_1^{<n}, \Delta_{<n}, \Phi_{<n})$ is a QSQB in $C_{<n}$ with classical limit $(\mu_C, \delta_C, \varphi_C)$, then the obstruction to extend it to degree $n + 1$ belongs to $H^2_{\text{gr}(C)}[n + 2]$, and the set of extensions, modulo equivalence and twist equivalence, is an affine space over $H^1_{\text{gr}(C)}[n + 1]$ (see [ShSt]). In both cases, cohomologies are relative to the differentials defined by the classical limits.

Note that when $C = \text{LBA}$ or $\text{QLBA}$, $\text{gr}(C) = C$. The prop morphism $\text{QLBA} \to \text{LBA}$ induces morphisms of complexes $C_{\text{QLBA}} \to C_{\text{LBA}}$ and of the cohomologies $H^1_{\text{QLBA}} \to H^1_{\text{LBA}}$.

2.5. Proof of the main theorem. To show Theorem 2.1 we will prove:

**Theorem 2.2.** The maps $H^i_{\text{QLBA}} \to H^i_{\text{LBA}}$ are isomorphisms for any $i \geq 0$.

We will prove this in Section 3.

Let us explain why this implies Theorem 2.1. We will prove inductively over $n$ that the map $\text{red}_n : \{\text{QSB's in } \text{QLBA}_{\leq n}\text{ quantizing } (\mu, \delta, \varphi)\}/(\text{equivalence, twist equivalence}) \to \{\text{QSB's in } \text{LBA}_{\leq n}\text{ quantizing } (\mu, \delta)\}$ is bijective. We denote by $\text{pr}^C_{n+1,n}$ the reduction map $\{\text{QSB in } C_{\leq n+1}\} \to \{\text{QSB in } C_{\leq n}\}$. Then $\text{red}_n \circ \text{pr}^C_{n+1,n} = \text{pr}^{\text{LBA}}_{n+1,n} \circ \text{red}_{n+1}$.

Assume that $\text{red}_n$ is bijective and let us show that $\text{red}_{n+1}$ is bijective. We first show that it is injective. If $[(m, \Delta, \Phi)]$ and $[(m', \Delta', \Phi')]$ are two classes of QSB's in $\text{QLBA}_{\leq n+1}$ with the same image by $\text{red}_n$, then the injectivity of $\text{red}_n$ implies that their images by $\text{pr}^C_{n+1,n}$ coincide. So $[(m, \Delta, \Phi)]$ and $[(m', \Delta', \Phi')]$ differ by an element $\omega \in H^1_{\text{QLBA}}[n+1]$. Their images by $\text{red}_{n+1}$ are classes of QSB's in $\text{QLBA}_{\leq n+1}$, whose reductions in $\text{LBA}_{\leq n}$ are equivalent; these classes differ by the image of $\omega$ under $H^1_{\text{QLBA}}[n+1] \to H^1_{\text{LBA}}[n+1]$. As this map is injective, $\omega = 0$ so $[(m, \Delta, \Phi)] = [(m', \Delta', \Phi')]$, which proves the injectivity of $\text{red}_{n+1}$.

Let us now show that $\text{red}_{n+1}$ is surjective. Let $[(m_{\leq n+1}, \Delta_{\leq n+1})]$ be a class of QSB in $\text{LBA}_{\leq n+1}$. Set $[(m_{\leq n}, \Delta_{\leq n})] := \text{pr}^{\text{LBA}}_{n+1,n}([(m_{\leq n+1}, \Delta_{\leq n+1})])$: this is the class of a QSB in $\text{LBA}_{\leq n}$. Let $[(\tilde{m}_{\leq n}, \tilde{\Delta}_{\leq n}, \tilde{\Phi}_{\leq n})]$ be the preimage of $[(m_{\leq n}, \Delta_{\leq n})]$ by $\text{red}_n$. The obstruction to extending it to a QSB in $\text{QLBA}_{\leq n+1}$ is a cohomology class in $H^2_{\text{QLBA}}[n+2]$. The image of this class by $H^2_{\text{QLBA}}[n+2] \to H^2_{\text{LBA}}[n+2]$ is the obstruction to extending $[(m_{\leq n}, \Delta_{\leq n})]$ to a QSB in $\text{LBA}_{\leq n+1}$. The existence of $[(m_{\leq n+1}, \Delta_{\leq n+1})]$ implies that this class in $H^2_{\text{LBA}}[n+2]$ is zero. As the map $H^2_{\text{QLBA}}[n+2] \to H^2_{\text{LBA}}[n+2]$ is injective, the obstruction class in $H^2_{\text{QLBA}}[n+2]$ is zero, and $[(\tilde{m}_{\leq n}, \tilde{\Delta}_{\leq n}, \tilde{\Phi}_{\leq n})]$ may be extended to a QSB in $\text{QLBA}_{\leq n+1}$. Let $[(\tilde{m}_{\leq n+1}, \tilde{\Delta}_{\leq n+1}, \tilde{\Phi}_{\leq n+1})]$ be such an extension. The difference between $\text{red}_{n+1}([(\tilde{m}_{\leq n+1}, \tilde{\Delta}_{\leq n+1}, \tilde{\Phi}_{\leq n+1})])$ and $[(m_{\leq n+1}, \Delta_{\leq n+1})]$ is a cohomology class in $H^1_{\text{LBA}}[n+1]$. As the map $H^1_{\text{QLBA}}[n+1] \to H^1_{\text{LBA}}[n+1]$ is surjective, this is the image of a cohomology class in $H^1_{\text{QLBA}}[n+1]$. Subtracting this cohomology class from $[(\tilde{m}_{\leq n+1}, \tilde{\Delta}_{\leq n+1}, \tilde{\Phi}_{\leq n+1})]$, we obtain a preimage of $[(m_{\leq n+1}, \Delta_{\leq n+1})]$ by $\text{red}_{n+1}$.

3. Structure of the prop $\text{QLBA}$

In order to establish Theorem 2.2 we study the structure of $\text{QLBA}$.

3.1. Products of ideals in props. If $C$ is a prop and $I_1, ..., I_n$ are ideals of $C$, then the product $I_1 I_2 ... I_n$ is the smallest ideal containing the morphisms $f_1 \ast ... \ast f_n$, where $f_i$ is morphism in $I_i$ and $\ast$ is $*$ or $\otimes$. One defines in this way the powers $I^n$ of an ideal.

3.2. Structure of the prop $\text{LBA}$. Define LA (resp., LCA) as the prop generated by $\mu \in \text{LA}(\text{id} \otimes \text{id})$ subject to the antisymmetry and Jacobi relation (resp., $\delta \in \text{LCA}(\text{id}, \text{id} \otimes \text{id})$ subject to antisymmetry and the co-Jacobi relation). We have prop morphisms $\text{LA}, \text{LCA} \to \text{LBA}$. The structure of LBA is given by
3.3. A filtration on $QLBA$. Let $\langle \varphi \rangle$ be the prop ideal of $QLBA$ generated by $\varphi$ and by $\langle \varphi \rangle^n$ is $n$th power. For $X,Y \in Ob(Sch)$, we have a decreasing filtration $QLBA(X,Y) \supset (\varphi(X,Y) \supset (\varphi)^2(X,Y) \supset \ldots$. As $\varphi$ is homogeneous for the $\mathbb{Z}^2$-grading, so are the $\langle \varphi \rangle^n$, i.e., $\langle \varphi \rangle^n(X,Y) = \oplus_{n \in \mathbb{Z}^2} \langle \varphi \rangle^n(X,Y)[\alpha]$.

Lemma 3.1. This filtration is complete, i.e., $\cap_{n \geq 0} \langle \varphi \rangle^n(X,Y) = 0$.

Proof. Observe that $\langle \varphi \rangle^n(X,Y)$ is supported in $n(-1,2) + N(1,0) + N(0,1) + N(-1,2) \subset (2n + N)(0,1) + Z(1,0)$. Then $\cap_{n \geq 0} \langle \varphi \rangle^n(X,Y)$ is supported in $\cap_{n \geq 0} (2n + N)(0,1) + Z(1,0)$, which is empty. So this intersection is zero. □

The composition of $QLBA$ restricts to a map $\langle \varphi \rangle^m(G,H) \otimes \langle \varphi \rangle^n(F,G) \rightarrow \langle \varphi \rangle^{n+m}(F,H)$, and the tensor product restricts to $\langle \varphi \rangle^n(F,G) \otimes \langle \varphi \rangle^n(F',G') \rightarrow \langle \varphi \rangle^{n+n'}(F \otimes F',G \otimes G')$, so $QLBA \supset \langle \varphi \rangle \supset \ldots$ is a prop filtration. The associated graded prop is defined by $grQLBA(F,G) := \oplus_{n \geq 0} gr_n QLBA(F,G)$, where $gr_n QLBA(F,G) = \langle \varphi \rangle^n(F,G)/\langle \varphi \rangle^{n+1}(F,G)$.

3.4. The graded prop $LBA_\alpha$. Define $P$ to be the prop with the same generators $\tilde{\mu}, \tilde{\delta}, \tilde{\phi}$ as $QLBA$ and the same relations, except for the third which is replaced by $Alt_3 \circ (\tilde{\delta} \otimes id_{id}) \circ \tilde{\delta} = 0$.

We now construct a prop isomorphic to $P$. The following general construction goes back to [EH]. For $C \in Ob(Sch)$, we have a prop $LBA_C$ defined by $LBA_C(F,G) := \oplus_{n \geq 0} LBA(F \otimes S^n(C), G)$ (the composition is induced by the coproduct $S \rightarrow S^{\otimes 2}$). For $D \in Ob(Sch)$, we set $LBA_C,D(F,G) := \oplus_{n \geq 0} LBA(F \otimes S^n(C) \otimes D, G)$; for $\alpha \in LBA(C,D)$, we have a map $LBA(F \otimes S^n(C) \otimes D, G) \rightarrow LBA(F \otimes S^{n+1}(C), G)$, $x \mapsto x \circ [id_F \otimes S^n(C) \otimes \alpha] \circ [id_F \otimes \Delta_{n,1}]$, where $\Delta_{n,1} : S^{n+1}(C) \rightarrow S^n(C) \otimes C$ is the component $n+1 \rightarrow (n,1)$ of the coproduct $S(C) \rightarrow S(C)^{\otimes 2}$.

We then have commutative diagrams:

\[
\begin{array}{ccc}
LBA_C,D(F,G) \otimes LBA_C(G,H) & \rightarrow & LBA_C,D(F,H) \\
\oplus LBA_C(F,G) \otimes LBA_C,D(G,H) & \downarrow & \\
LBA_C(F,G) \otimes LBA_C(G,H) & \rightarrow & LBA_C(F,H)
\end{array}
\]

and

\[
\begin{array}{ccc}
LBA_C,D(F,G) \otimes LBA_C(F',G') & \rightarrow & LBA_C,D(F \otimes F', G \otimes G') \\
\oplus LBA_C(F,G) \otimes LBA_C,D(F',G') & \downarrow & \\
LBA_C(F,G) \otimes LBA_C(F',G') & \rightarrow & LBA_C(F \otimes F', G \otimes G')
\end{array}
\]

induced by the composition and tensor product, which implies that if $LBA_\alpha(F,G) := Coker[LBA_C,D(F,G) \rightarrow LBA_C(F,G)]$, then we have a prop morphism $LBA_C \rightarrow LBA_\alpha$.

In what follows, we will set $C := \wedge^3, D := \wedge^4, \alpha := pr_4 \circ Alt_4 \circ (\delta \otimes id_{id}) \circ in_{j_3} \in LBA(\wedge^3, \wedge^4)$, where $in_{j_3} : \wedge^3 \rightarrow id^{\otimes 3}$ and $pr_4 : id^{\otimes 4} \rightarrow \wedge^4$ are the canonical injection and projection.

Lemma 3.2. We have a prop isomorphism $LBA_\alpha \simeq P$. 

Proof. Let \( \hat{P} \) be the prop with generators \( \hat{\mu}, \hat{\delta}, \hat{\varphi} \) and only relations: Lie bialgebra relations between \( \hat{\mu}, \hat{\delta} \) and \( \hat{\varphi} = \frac{1}{\delta} \text{Alt}_3(\hat{\varphi}) \). We have a morphism \( \hat{P} \to LBA_{\lambda_3} \), defined by \( \hat{\mu} \mapsto \mu \in LBA(\id \otimes S^0(\wedge^1), \id) \subseteq LBA_{\lambda_3}(\id \otimes \id) \); \( \hat{\delta} \mapsto \delta \in LBA(\id \otimes S^0(\wedge^3), \id \otimes \id) \subseteq LBA_{\lambda_3}(\id, \id) \); \( \hat{\varphi} \mapsto \text{inj}_3 \in LBA(1 \otimes S^3(\wedge^3), \id \otimes \id) \subseteq LBA_{\lambda_3}(1, \id) \), as \( \text{inj}_3 = \frac{1}{\delta} \text{Alt}_3 \circ \text{inj}_3 \).

We also have a morphism \( LBA_{\lambda_3} \to \hat{P} \), defined by \( LBA_{\lambda_3}(F, G) \to LBA(F \otimes S^n(\wedge^3), G) \to f \mapsto \text{can}(f) \circ (\id_F \otimes S^n(\hat{\varphi})) \in \hat{P}(F, G) \), where \( \text{can} : LBA \to \hat{P} \) is the prop morphism defined by \( \mu, \delta \mapsto \hat{\mu}, \hat{\delta} \). One proves that these are inverse isomorphisms, which induce an isomorphism \( LBA_{\alpha} \simeq P \). \( \square \)

3.5. A graded prop morphism \( LBA_{\alpha} \to \text{gr} \text{QLBA} \).

Lemma 3.3. There is a unique prop morphism \( LBA_{\alpha} \simeq P \to \text{gr} \text{QLBA} \), defined by \( P(\id \otimes \id) \ni \hat{\mu} \mapsto \mu \in LBA(\id \otimes \id) \subseteq \text{gr} \text{QLBA}(\id \otimes \id) \), \( P(\id \otimes \id) \ni \hat{\delta} \mapsto \delta \in LBA(\id, \id) \subseteq \text{gr} \text{QLBA}(\id, \id) \), \( P(\id) \ni \hat{\varphi} \mapsto \varphi \in \text{gr} \text{QLBA}(\id) \).

Proof. The images in \( \text{gr}^0 \text{QLBA} \) of the Jacobi relation for \( \mu \), of the cocycle relation between \( \mu, \delta \), and of the quasi-co-Jacobi relation between \( \mu, \delta, \varphi \) (which hold in \( \langle \varphi \rangle^0 = \text{QLBA} \)) are respectively, the Jacobi relation for \( [\mu] \), the cocycle relation between \( [\mu], [\delta] \) and the co-Jacobi relation for \( [\delta] \). The images in \( \text{gr}^1 \text{QLBA} \) of the relations \( \varphi = \frac{1}{8} \text{Alt}_3(\varphi) \), \( \text{Alt}_4(\delta \otimes \id \otimes \id)(\varphi) = 0 \) (which hold in \( \langle \varphi \rangle \)) are the similar relations, with \( \delta, \varphi \) replaced by \( [\delta], [\varphi] \). It follows that we have a prop morphism \( P \to \text{gr} \text{QLBA} \), \( \mu, \delta, \varphi \mapsto [\mu], [\delta], [\varphi] \). \( \square \)

3.6. Theorem 3.1. The morphism \( LBA_{\alpha} \to \text{gr} \text{QLBA} \) is a prop isomorphism.

Proof. We say that a prop morphism \( C \to D \) is surjective (resp., injective) if the maps \( C(F, G) \to D(F, G) \) are.

As \( \text{QLBA} \) is generated by \( \mu, \delta, \varphi \), the prop \( \text{gr} \text{QLBA} \) is generated by their classes \( [\mu], [\delta], [\varphi] \), and since the generators of \( P \simeq LBA_{\alpha} \) map to these elements, the morphism \( LBA_{\alpha} \to \text{gr} \text{QLBA} \) is surjective.

We now prove the injectivity of \( LBA_{\alpha} \to \text{gr} \text{QLBA} \). For this, we construct a filtered prop morphism \( \text{QLBA} \to L(\text{LCA}_{\lambda_2}) \); composing the associated graded morphism \( \text{gr} \text{QLBA} \to L(\text{LCA}_{\lambda_2}) \) with \( LBA_{\alpha} \to \text{gr} \text{QLBA} \), we obtain a morphism \( LBA_{\alpha} \to L(\text{LCA}_{\lambda_2}) \). This morphism factors as \( LBA_{\alpha} \to L(\text{LCA}_{\alpha}) \to L(\text{LCA}_{\lambda_2}) \). The injectivity of \( LBA_{\alpha} \to L(\text{LCA}_{\alpha}) \) is a consequence of a general argument (already used in the proof of the structure result for the prop \( LBA \), see Appendix [A]), while the injectivity of the second morphism follows from that of a morphism \( LBA_{\alpha} \to \text{LCA}_{\lambda_2} \), which is a consequence of Lemma 3.6. This establishes the injectivity of \( LBA_{\alpha} \to L(\text{LCA}_{\lambda_2}) \) and therefore of \( LBA_{\alpha} \to \text{gr} \text{QLBA} \). Let us now proceed with the details of the proof.

We first define the auxiliary props mentioned above. \( \text{LCA}_{\lambda_2} \) is the prop with generators \( \delta_{\text{LCA}} : \id \to \id \otimes \id \), \( r : 1 \to \id \otimes \id \), and relations: antisymmetry and co-Jacobi for \( \delta_{\text{LCA}} \); and antisymmetry for \( r \). Similarly, \( \text{LCA}_{\alpha} \) is the prop with generators \( \hat{\delta} : \id \to \id \otimes \id \) and \( \hat{\varphi} : 1 \to \id \otimes \id \otimes \id \), and relations: antisymmetry and co-Jacobi for \( \hat{\delta} \); antisymmetry for \( \hat{\varphi} \); and \( \text{Alt}_4(\hat{\delta} \otimes \id \otimes \id) \circ \hat{\varphi} = 0 \). One checks that there are unique \( \text{Sch} \)-props \( \text{LCA}_{\lambda_2}, \text{LCA}_{\alpha} \) associated to these props (for example, \( \text{LCA}_{\lambda_2}(F, G) = \oplus_{i,j} \text{LCA}_{\lambda_2}(F_i, G_j) \) for \( F = \oplus F_i, G = \oplus G_j \)). We denote by \( L \in \text{Ob}(\text{Sch}) \) the “free Lie algebra” Schur functor, i.e., if \( V \) is a vector space, then \( L(V) \) is the free Lie algebra generated by \( V \); so \( L = L_1 \oplus L_2 \oplus \ldots \), where \( L_1 = \id, L_2 = \wedge^2, \text{etc.} \)

We now define the prop morphism \( \text{QLBA} \to L(\text{LCA}_{\lambda_2}) \). The universal version of the Lie algebra bracket on \( L(V) \) is an element \( \mu_{\text{free}} \in \text{Sch}(L \otimes L) \). The prop morphism \( \text{QLBA} \to \text{gr} \text{QLBA} \)
whose vertical cokernel is an isomorphism.

δ is the unique derivation extending a quasi-Lie bialgebra.

propic version of the following construction: if \((\wedge LCA)\) (here \(\tilde{\mu}\) is then given by \(\tilde{\mu}(2) LBA_{p,q} \oplus (3) LBA_{\alpha}\)), we define a filtration on the prop \(LCA_{\lambda^2}\); this is the unique derivation extending \(\delta_\epsilon\) (where \(\text{inj}_j : \epsilon \to L\) is the canonical injection). This morphism is the propic version of the following construction: if \((\epsilon, \delta_\epsilon)\) is a Lie coalgebra and \(r_\epsilon \in \Lambda^2(\epsilon)\), we consider the twist by \(\wedge^2(\text{inj}_j^1)(r_\epsilon)\) of the Lie bialgebra \((L(\epsilon), \delta_{L(\epsilon)})\), where \(\delta_{L(\epsilon)} : L(\epsilon) \to L(\epsilon)^{\otimes 2}\) is the unique derivation extending \(\delta_\epsilon\) (where \(\text{inj}_j^1 : \epsilon \to L(\epsilon)\) is the canonical injection); this is a quasi-Lie bialgebra.

The powers of the prop ideal \(r\) define a filtration on the prop \(LCA_{\lambda^2}\); the associated graded prop \(\text{gr} LCA_{\lambda^2}\) is canonically isomorphic to \(LCA_{\lambda^2}\). The prop morphism \(QLBA \to L(LCA_{\lambda^2})\) is compatible with the filtrations (as it takes \(\varphi\) to \((r)\)), and the associated graded morphism \(\text{gr} QLBA \to L(LCA_{\lambda^2})\) is given by \([\mu] \mapsto \mu_{\text{free}}, [\delta] \mapsto \delta_{\text{free}}\) and \([\varphi] \mapsto \frac{1}{2} \text{Alt}_3 \circ (\delta_{\text{free}} \otimes id_L) \circ \text{inj}_j^{\otimes 2} \circ r\).

We now define two prop morphisms \(BLA_\alpha \to L(LCA_{\lambda^2})\) and \(LCA_\alpha \to L(LCA_{\lambda^2})\), such that the above morphism \(BLA_\alpha \to L(LCA_{\lambda^2})\) coincides with \(BLA_\alpha \to L(LCA_{\lambda^2})\) and \(LCA_\alpha \to L(LCA_{\lambda^2})\).

First define \(LBA_\alpha \to L(LCA_{\lambda^2})\). There is a unique morphism \(LBA_\alpha \to L(LCA_{\lambda^2})\), taking \(\alpha\), \(\delta\) to \(\mu_{\text{free}}, \delta_{\text{free}}\) (see Appendix A); this is the propic version of the functor \{Lie coalgebras\} \(\to\) \{Lie bialgebras\}, \((\epsilon, \delta_\epsilon) \mapsto (L(\epsilon), \text{free Lie bracket, unique cobracket extending} \delta_\epsilon)\). We define \(LBA_\alpha \to L(LCA_{\lambda^2})\) by \(\tilde{\mu}, \tilde{\delta} \mapsto \mu_{\text{free}}, \delta_{\text{free}}\) (we identify \(\mu_{\text{free}}\) with their images in \(L(LCA_{\lambda^2})\)) and \(\tilde{\varphi} \mapsto \text{inj}_j^{\otimes 2} \circ \tilde{\varphi}\). This morphism is the propic version of \(\{(\epsilon, \delta_\epsilon, \varphi_\epsilon)\} \mapsto \{(\epsilon, \delta_\epsilon)\}\); it is a Lie coalgebra, \(\varphi_\epsilon \in \Lambda^3(\epsilon), \text{Alt}_4 \circ (\delta_\epsilon \otimes id^{\otimes 2}_\epsilon)(\varphi_\epsilon) = 0\) \(\to\) \(\{(a, \delta_\alpha, \varphi_a)\} \mapsto \langle a, \mu_a, \sigma_a \rangle\) is a Lie bialgebra, \(\varphi_a \in \Lambda^3(a), \text{Alt}_4 \circ (\delta_\alpha \otimes id^{\otimes 2}_\alpha) \circ \varphi_a = 0\), extending the above functor by \(\varphi_a := \Lambda^3(\text{inj}_j^1)(\varphi_a)\).

We then define the morphism \(LCA_\alpha \to L(LCA_{\lambda^2})\) by \(\tilde{\delta} \mapsto \delta_{\text{LCA}}, \tilde{\varphi} \mapsto \frac{1}{2} \text{Alt}_3(\delta_{\text{LCA}} \otimes id_D) \circ r\).

One checks that \((\tilde{\mu}, \tilde{\delta})\) coincides with \(LBA_\alpha \to L(LCA_{\lambda^2})\).

Let us prove that \(LBA_\alpha \to L(LCA_{\lambda^2})\) is injective. Using the symmetric group actions, this is equivalent to proving that for any \(p, q \geq 0\), the map

\[
LBA_\alpha(T_p, T_q) \to LCA_\alpha(L^{\otimes p}, L^{\otimes q})
\]

is injective.

**Lemma 3.4.** The map \(\bigoplus_{Z \in \text{Irr}(Sch)} LCA_\alpha(T_p, Z) \otimes LA(Z, T_q) \to LBA_\alpha(T_p, T_q)\), induced by the prop morphisms \(LCA_\alpha, LA \to LBA_\alpha (\tilde{\delta}, \tilde{\varphi} \mapsto \tilde{\delta}, \tilde{\varphi}, \mu \mapsto \tilde{\mu})\) and by composition, is an isomorphism of vector spaces.

**Proof of Lemma.** Recall that \(C = \Lambda^3, D = \Lambda^4, \alpha \in LBA(D, C)\). One may construct as above a prop \(LCA_{\alpha}\) by \(LCA_{\alpha}(F, G) := \bigoplus_{n \geq 0} LCA(F \otimes S^n(C), G)\), set \(LCA_{\alpha}(D, F, G) := \bigoplus_{n \geq 0} LCA(F \otimes S^n(D), G), \) then using the fact that \(\alpha \in LBA(D, C)\), one constructs a map \(LCA_{\alpha}(D, F, G) \to LCA_{\alpha}(F, G)\) and one then checks that \(LCA_{\alpha}(F, G) = \text{Coker}[LCA_{\alpha}(D, F, G) \to LCA_{\alpha}(F, G)]\). For \(F, G \in \text{Ob}(Sch)\), we have a commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{Z \in \text{Irr}(Sch)} LCA_{\alpha}(F, Z) \otimes LA(Z, G) & \xrightarrow{\sim} & LBA_{\alpha}(F, G) \\
\downarrow & & \downarrow \\
\bigoplus_{Z \in \text{Irr}(Sch)} LCA_{\alpha}(F, Z) \otimes LA(Z, G) & \xrightarrow{\sim} & LBA_{\alpha}(F, G)
\end{array}
\]

whose vertical cokernel is an isomorphism

\[
\bigoplus_{Z \in \text{Irr}(Sch)} LCA_{\alpha}(F, Z) \otimes LA(Z, G) \xrightarrow{\sim} LBA_{\alpha}(F, G);
\]
this isomorphism coincides with the map described in the statement of the lemma. □

We now consider the composite map
\[ (4) \]
\[ \bigoplus_{Z \in \text{Irr}(\text{Sch})} \text{LCA}_\alpha(T_p, Z) \otimes \text{LA}(Z, T_q) \to \text{LBA}_\alpha(T_p, T_q) \to \text{LCA}_\alpha(L^{\otimes p}, L^{\otimes q}) \to \text{LCA}_\alpha(T_p, L^{\otimes q}), \]
where the first map is described in Lemma 3.4, the middle map is (3), and the last map is induced by the injection \( T_p = \text{id}^{\otimes p} \to L^{\otimes p} \).

**Lemma 3.5.** The map (4) coincides with the composite map \( \bigoplus_Z \text{LCA}_\alpha(T_p, Z) \otimes \text{LA}(Z, T_q) \cong \bigoplus_Z \text{LCA}_\alpha(T_p, Z) \otimes \text{Sch}(Z, L^{\otimes q}) \to \text{LCA}_\alpha(T_p, L^{\otimes q}) \), where the first map is induced by the isomorphism \( \text{LA}(Z, T_q) \cong \text{Sch}(Z, L^{\otimes q}) \) and the second by composition.

**Proof.** Note that the isomorphism \( \text{LA}(Z, T_q) \cong \text{Sch}(Z, L^{\otimes q}) \) is proved in Appendix A. By using symmetric group actions, one shows that it suffices to prove the above statement with \( Z \) replaced by \( T_N \). We have composed prop morphisms \( \rho : \text{LCA}_\alpha \to \text{LBA}_\alpha \to \text{L(LCA}_\alpha) \) and \( \sigma : \text{LA} \to \text{LBA}_\alpha \to \text{L(LCA}_\alpha) \); actually \( \sigma \) factors through \( \text{LA} \to \text{L(Sch)} \). The map (4) (with \( Z \) replaced by \( T_N \)) then takes \( f \otimes g \) to \( \sigma(g) \circ \rho(f) \circ \text{inj}_1^{\otimes p} \), where \( \text{inj}_1 : \text{id} \to L \) is the canonical morphism, \( f \in \text{LCA}_\alpha(T_p, T_N), g \in \text{Sch}(T_N, T_q), \rho(f) \in \text{LCA}_\alpha(L^{\otimes p}, L^{\otimes N}), \sigma(g) \in \text{Sch}(L^{\otimes N}, L^{\otimes q}) \).

We have \( \rho(f) \circ \text{inj}_1^{\otimes p} = \& f, \) as this property can be checked for \( f = \delta_{\text{LCA}}, r \) and is preserved by composition and tensor products. Moreover, \( \sigma(g) \circ \text{inj}_1^{\otimes N} \in \text{LA}(T_N, L^{\otimes N}) \) is the image \( \tilde{g} \) of \( g \) under \( \text{LA}(Z, T_q) \cong \text{Sch}(Z, L^{\otimes q}) \). It follows that (4) coincides with \( f \otimes g \mapsto \tilde{g} \circ f, \) which was to be proved. □

According to Lemma A.1, the composite map \( \bigoplus_Z \text{LCA}_\alpha(T_p, Z) \otimes \text{LA}(Z, T_q) \cong \bigoplus_Z \text{LCA}_\alpha(T_p, Z) \otimes \text{Sch}(Z, L^{\otimes q}) \to \text{LCA}_\alpha(T_p, L^{\otimes q}) \) is an isomorphism, so Lemma 3.2 implies that the composite map (4) is an isomorphism. The first map in (4) is also an isomorphism by Lemma 3.4, so the map (3) is injective, which was to be proved.

Let us now prove that \( \text{LCA}_\alpha \to \text{LCA}_\Lambda^2 \) is injective. For this, we outline the structure of these props. We have
\[ \text{LCA}_\alpha(T_p, T_z) = \bigoplus_{k \geq 0} \left[ \bigoplus_{z', z_1, \ldots, z_k \se t_{z', z_1 + \ldots + z_k = z}} \text{Ind}\_{\text{Sch}_{z'}}^{\text{Sch}_{z_1} \otimes \cdots \otimes \text{Sch}_{z_k}} \left[ \text{LCA}(T_p, T_{z'}) \otimes \otimes_{i=1}^k \text{LCA}(\Lambda^2, T_{z_i}) \right] \right] \]
and
\[ \text{LCA}(T_p, T_z) = \bigoplus_{k \geq 0} \left[ \bigoplus_{z', z_1, \ldots, z_k \se t_{z', z_1 + \ldots + z_k = z}} \text{Ind}\_{\text{Sch}_{z'}}^{\text{Sch}_{z_1} \otimes \cdots \otimes \text{Sch}_{z_k}} \left[ \text{LCA}(T_p, T_{z'}) \otimes \otimes_{i=1}^k \text{Coker}\{ \text{LCA}(\Lambda^4, T_{z_i}) \to \text{LCA}(\Lambda^3, T_{z_i}) \} \right] \right] \]

The injectivity of \( \text{LCA}_\alpha \to \text{LCA}_\Lambda^2 \), is therefore equivalent to that of \( \text{Coker}\{ \text{LCA}(\Lambda^4, T_{z_1}) \to \text{LCA}(\Lambda^3, T_{z_1}) \}; \) in other terms, we have a sequence \( \text{LCA}(\Lambda^4, T_{z_1}) \to \text{LCA}(\Lambda^3, T_{z_1}) \to \text{Coker}\{ \text{LCA}(\Lambda^4, T_{z_1}) \to \text{LCA}(\Lambda^3, T_{z_1}) \} \to \text{LCA}(\Lambda^3, T_{z_1}) \sim \text{Coker}\{ \text{LCA}(\Lambda^4, T_{z_1}) \to \text{LCA}(\Lambda^3, T_{z_1}) \} \to 0 \), where the composite map is zero, and we should prove that the homology vanishes.

To prove this, we shall prove that the second homology of the complex
\[ \ldots \to \text{LA}(T_z, \Lambda^4) \xrightarrow{\text{Alt}_3 \circ (\phi \otimes \text{id}_{\Lambda^2})} \text{LA}(T_z, \Lambda^3) \xrightarrow{\text{Alt}_2 \circ (\phi \otimes \text{id}_{\Lambda^1})} \text{LA}(T_z, \Lambda^2) \xrightarrow{\text{Id}_{\Lambda^1}} \text{LA}(T_z, \Lambda) \xrightarrow{0} 0 \]
vanishes. We will prove more generally:

**Lemma 3.6.** If \( z \geq 2 \), the complex (5) is acyclic; if \( z = 1 \), its homology is 1-dimensional, concentrated in degree 0.
Proof. Let \( L_z \) (resp., \( A_z \)) be the free Lie (resp., associative) algebra with generators \( x_1, \ldots, x_z \). The spaces are both graded by \( \oplus_{i=1}^\infty \mathbb{N} \delta_i \), where \( |x_i| = \delta_i \). For \( V \) a vector space graded by \( \oplus_{i=1}^\infty \mathbb{N} \delta_i \), and \( I \subset [1, z] \), we denote by \( V_I \) the part of \( V \) of degree \( \sum_{i \in I} \delta_i \). We have \( \Lambda(T_z, \wedge^k) \simeq (\wedge^k(L_z))[1, z] \). This isomorphism takes the complex \( 5 \) to

\[
\text{Alt}_3 \circ (\mu_L \otimes \text{id}) (\wedge^3(L_z))[1, z] \xrightarrow{\text{Alt}_2 \circ (\mu_L \otimes \text{id})} (\wedge^2(L_z))[1, z] \xrightarrow{\mu_L} (L_z)[1, z] \to 0
\]

where \( \mu_L \) is the Lie bracket of \( L_z \).

Let \( \text{Part}_k(I) \) be the set of \( k \)-partitions of a set \( I \), i.e., of the k-tuples \( (I_1, \ldots, I_k) \) with \( \bigcup_{i=1}^k I_i = I \). The group \( \mathfrak{S}_k \) acts on \( \text{Part}_k([1, z]) \), and we have a decomposition

\[
(\wedge^k(L_z))[1, z] = \bigoplus_{(I_1, \ldots, I_k) \in \text{Part}_k([1, z])} (\wedge^k(L_z))(I_1, \ldots, I_k)
\]

where the summand in the r.h.s. is the space of antisymmetric tensors in \( \bigotimes_{\sigma \in \mathfrak{S}_k} (L_z)_{I_{\sigma(1)}} \otimes \ldots \otimes (L_z)_{I_{\sigma(k)}} \).

We have a bijection \( \{(I_1, ([I_2, \ldots, I_k])) | I'_1 \subset [2, z], ([I_2, \ldots, I_k]) \in \text{Part}_{k-1}([2, z] - I'_1) / \mathfrak{S}_{k-1} \} \to \text{Part}_k([1, z]) \), taking \( (I_1, ([I_2, \ldots, I_k])) \) to \( ([I'_1 \cup \{1\}, I_2, \ldots, I_k]) \). The inverse bijection takes \( ([I_1, I_k]) \) to \( (I_1 - \{1\}, [I_1, \ldots, I_{k-1}, I_{k+1}, \ldots, I_k]) \), where \( i \in [1, k] \) is the index such that \( 1 \in I_i \).

For \( (I'_1, ([I_2, \ldots, I_k])) \) in the first set, we have an isomorphism

\[
(\wedge^k(L_z))[I'_1 \cup \{1\}, I_2, \ldots, I_k] \simeq (L_z)_{I'_1 \cup \{1\}} \otimes (\wedge^{k-1}(L_z))[I_2, \ldots, I_k]
\]

(whose inverse is given by \( \text{Alt}_k \), or, up to a factor, by the sum of all cyclic permutations if \( k \) is odd, and their alternated sum if \( k \) is even), which gives rise to an isomorphism

\[
(\wedge^k(L_z))[1, z] \simeq \bigoplus_{(I'_1, ([I_2, \ldots, I_k]))} (L_z)_{I'_1 \cup \{1\}} \otimes (\wedge^{k-1}(L_z))[I_2, \ldots, I_k] \subset (L_z \otimes \wedge^{k-1}(L_z))[1, z].
\]

We have a complex

\[
\ldots \to (L_z \otimes \wedge^2(L_z))[1, z] \to (L_z \otimes L_z)[1, z] \to (L_z)[1, z] \to 0
\]

where the differential \( (L_z \otimes \wedge^k(L_z))[1, z] \to (L_z \otimes \wedge^{k-1}(L_z))[1, z] \) is induced by \( x_0 \otimes (x_1 \wedge \ldots \wedge x_k) \mapsto \sum_{i=1}^k (-1)^{i+1} [x_0, x_i] \otimes (x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_k) + \sum_{1 \leq i < j \leq k} (-1)^{j+1} x_0 \otimes (x_1 \wedge \ldots \wedge [x_i, x_j] \wedge \ldots \wedge x_k) \).

If \( I, J \subset [1, z] \) are disjoint, we have \( [L_z]_I \cap [L_z]_J \subset [L_z]_{I \cup J} \), which implies that if

\[
C_k := \bigoplus_{(I'_1, ([I_2, \ldots, I_k]))} (L_z)_{I'_1 \cup \{1\}} \otimes (\wedge^{k-1}(L_z))[I_2, \ldots, I_k],
\]

then

\[
\ldots \to C_2 \to C_1 \to 0
\]

is a subcomplex of \( 4 \), isomorphic to \( 6 \).

For \( I' \subset [2, z] \), we have an isomorphism \( (A_z)_{I'} \to (L_z)_{I' \cup \{1\}} \), given by \( x_{i_1} \ldots x_{i_k} \mapsto [[x_{i_1}, x_{i_2}], x_{i_3}, \ldots, x_{i_k}] \); the inverse isomorphism is the restriction of the map \( (A_z)_{I' \cup \{1\}} \to (A_z)_{I'} \) taking a monomial not starting with \( x_1 \) to 0, and a monomial starting with \( x_1 \) to the same monomial with the \( x_1 \) removed (see \( [B] \)).

The compatibility of these isomorphisms with the Lie bracket can be described as follows: for \( I', I \subset [2, z] \) disjoint, we have a commutative diagram

\[
\begin{array}{ccc}
(A_z)_{I'} \otimes (L_z)_I & \to & (A_z)_I \\
\downarrow & & \downarrow \\
(L_z)_{I' \cup \{1\}} \otimes (L_z)_I & \to & (L_z)_{I' \cup \{1\}}
\end{array}
\]

where the upper horizontal map is induced by the product in \( A_z \) (\( L_z \) being viewed as a subspace of \( A_z \)) and the lower horizontal map is induced by the Lie bracket of \( L_z \).

We have a complex

\[
\ldots \to (A_z \otimes \wedge^2(L_z))[2, z] \to (A_z \otimes L_z)[2, z] \to (A_z)[2, z] \to 0,
\]
where the map \((A_z \otimes \wedge^k(L_z))[2,z] \to (A_z \otimes \wedge^{k-1}(L_z))[2,z]\) is induced by \(x_0 \otimes (x_1 \wedge ... \wedge x_k) \mapsto \sum_{i=1}^{k}(-1)^{i+1}x_0x_i \otimes x_1 \wedge ... \wedge \hat{x}_i ... \wedge x_k + \sum_{1 \leq i < j \leq k}(-1)^{i+j}x_0 \otimes x_i \wedge ... \wedge \hat{x}_i ... \wedge \hat{x}_j ... \wedge x_k.\)

The isomorphisms \((L_z)_i \cup (1) \simeq (A_z)_i\) induce isomorphisms \(C_k \simeq \oplus (L_z)[(i_1,...,i_k)](A)_i \otimes (\wedge^{k-1}(L_z))[i_1,...,i_k]\), which are compatible with the differentials. Hence the complex \(\mathfrak{S}\) is isomorphic to ... \(\to C_2 \to C_1 \to C_0 \to 0.\)

The complex \(\mathfrak{S}\) is the degree \(\delta_2 + ... + \delta_k\) part of the complex

\[
... \to A_z \otimes \wedge^2(L_z) \to A_z \otimes L_z \to A_z \to 0,
\]

where the differentials are defined by the same formulas.

Define a complete increasing filtration on \(\mathfrak{S}\) by \(\text{Fil}^n[A_z \otimes \wedge^k(L_z)] = (A_z)_{\leq n-k} \otimes \wedge^k(L_z),\) where \((A_z)_{\leq n}\) is the part of degree \(\leq n\) of \(A_z\) \(\simeq U(L_z)\) (i.e., the span of products of \(\leq n\) elements of \(L_z\)). The associated graded complex is the sum over \(n \geq 0\) of complexes \(\wedge^n(L_z) \to \to S^{n-1}(L_z) \otimes L_z \to S^n(L_z) \to 0), which add up to the Koszul complex

\[
... \to S(L_z) \otimes \wedge^2(L_z) \to S(L_z) \otimes L_z \to S(L_z) \to 0,
\]

where the differential \(S(L_z) \otimes \wedge^k(L_z) \to S(L_z) \otimes \wedge^{k-1}(L_z)\) is \(f \otimes (x_1 \wedge ... \wedge x_k) \mapsto \sum_{i=1}^{k}(-1)^{i+1}f_i \otimes (x_1 \wedge ... \hat{x}_i ... \wedge x_k).

Now if \(V\) is a vector space, the Koszul complex

\[
C(V) := [... \to S(V) \otimes \wedge^2(V) \to S(V) \otimes V \to S(V) \to 0]
\]

is a sum of complexes, graded by \(\mathbb{N}\) (this degree corresponds to \(p + q\) in \(S^p(V) \otimes \wedge^q(V)\)). It is well-known that the homology of this complex is concentrated in homological degree 0 and in degree 0, where it is equal to \(k\). Recall a proof. One checks this directly when \(V\) is one-dimensional; we have isomorphisms \(C(V \oplus W) \simeq C(V) \otimes C(W)\) of \(\mathbb{N}\)-graded complexes, which implies the statement when \(V\) is finite dimensional. It follows that the Koszul complex in \(\text{Sch}\)

\[
... \to S \otimes \wedge^2 \to S \otimes \text{id} \to S \to 0\]

has its homology concentrated in homological degree 0 and degree 0, where it equals 1. This implies the statement in general.

It follows that the homology of \(\mathfrak{S}\) is concentrated in degree 0, where it is equal to \(k\); a non-trivial homology class is that of 1 \(\in A_z\). It follows that the degree \(\delta_2 + ... + \delta_k\) part of this complex is acyclic if \(z \geq 2\), i.e. \(\mathfrak{S}\) is acyclic if \(z \geq 2\). The computation of the homology of \(\mathfrak{S}\) is straightforward when \(z = 1\).

This ends the proof of Theorem 3.1. \(\square\)

4. Comparison of cohomology groups

We now prove Theorem 2.2. The morphism of complexes \(C_{QLBA} \to C_{LBA}\) is surjective, so we have an exact sequence \(0 \to \text{Ker}(C_{QLBA} \to C_{LBA}) \to C_{QLBA} \to C_{LBA} \to 0\), inducing a long exact sequence in cohomology. The isomorphisms \(H^i_{QLBA} \simeq H^i_{LBA}\) will then follow from the vanishing of the relative cohomology, i.e., the cohomology of the complex \(\text{Ker}(C_{QLBA} \to C_{LBA})\).

Note that the complex \(C_{QLBA}\) has a complete descending filtration \(F^i(C_{QLBA}) := \langle \varphi \rangle^i(\wedge^p, \wedge^q)\).

The associated graded complex is the Schouten complex \(C_{LBA\alpha}\) of \(LBA\), equipped with the differential \([\tilde{\mu} \oplus \tilde{\delta}, -]\); unlike \(C_{QLBA}\), this is the total complex of a bicomplex, as the differentials have now degrees \((1, 0)\) and \((0, 1)\). The relative complex \(\text{Ker}(C_{QLBA} \to C_{LBA})\) coincides with the first step \(F^1(C_{QLBA})\) of the complex \(C_{QLBA}\); its associated graded is the positive degree (in \(\tilde{\varphi}\)) part of the Schouten complex \(C_{LBA\alpha}\). To prove that the relative complex is acyclic, it then suffices to prove that the positive degree part in \(\tilde{\varphi}\) of the complex \(C_{LBA\alpha}\) is acyclic.

Explicitly, recall that \(C_{LBA\alpha} = \oplus_{p,q} LBA\alpha(\wedge^p, \wedge^q)\), and denoting by \(LBA\alpha^{(i)}(X,Y)\) the degree \(i\) (in \(\tilde{\varphi}\)) of \(LBA\alpha^{(i)}(X,Y)\), the bicomplex \((C_{LBA\alpha},[\tilde{\mu} \oplus \tilde{\delta}, -])\) splits up as \(\oplus_{i \geq 0}(C_{LBA\alpha}^{(i)},[\tilde{\mu} \oplus \tilde{\delta}, -])\),
where \((C^{(i)}_{\text{LBA}})_{0}^{\otimes q} = \text{LBA}_{\kappa}(\Lambda^{p}, \Lambda^{q})\), and we wish to prove that for \(i > 0\), the total cohomology of \((C^{(i)}_{\text{LBA}}), [\bar{\mu} \otimes \bar{\delta}, -] \) is zero. For this, we will prove that the lines of this complex are acyclic.

We will prove more generally:

**Theorem 4.1.** Let \(C, D\) be homogeneous Schur functors of positive degrees, let \(\kappa \in \text{LCA}(C, D)\). Let \(\text{LBA}_{\kappa}(X, Y) := \text{Coker}[\text{LBA}(D \otimes X, Y) \to \text{LBA}(C \otimes X, Y)]\). Then for any \(q \geq 0\), the complex \((\text{LBA}_{\kappa}(\Lambda^{p}, \Lambda^{q}), [\mu, -])_{P \geq 0}\) is acyclic.

**Proof.** Let us make this complex explicit. For \(Z \in \text{Irr}(\text{Sch})\), define \(\mu_{Z} \in \text{LA}(\text{id} \otimes Z, Z)\) and \(\tilde{\mu}_{Z} \in \text{LA}(Z \otimes \text{id}, Z)\) as follows: \(\mu_{P} \in \text{LA}(\text{id} \otimes T_{p}, T_{p})\) is the universal version of \(x \otimes x_{1} \otimes \ldots \otimes x_{p} \mapsto \sum_{i=1}^{p} x_{i} \otimes \ldots \otimes [x, x_{i}] \otimes \ldots \otimes x_{p}\); as it is \(\Theta_{p}\)-equivariant, it decomposes under \(\text{LA}(\text{id} \otimes T_{p}, T_{p}) \simeq \oplus_{W, [W]=W} \text{LA}(\text{id} \otimes Z, W) \otimes \text{Vect}(\pi_{Z}, \pi_{W})\) as \(\oplus_{Z} \mu_{Z} \otimes \text{id}_{\pi_{Z}}\). We then set \(\bar{\mu}_{Z} := -\mu_{Z} \circ \beta_{Z, \text{id}}\), where \(\beta_{Z, \text{id}} : Z \otimes \text{id} \to \text{id} \otimes Z\) is the braiding morphism.

Then \(\mu_{-} : \text{LBA}(\Lambda^{p}, \Lambda^{q}) \to \text{LBA}(\Lambda^{p+1}, \Lambda^{q})\) is the composed map \(\text{LBA}(\Lambda^{p}, \Lambda^{q}) \to \text{LBA}(C \otimes \Lambda^{p}, \Lambda^{q}) \to \text{LBA}(\Lambda^{p}, \Lambda^{q})\), where the first map is \(x \mapsto x \circ (\text{id}_{Z} \otimes \tilde{\mu}_{Z}) - \mu_{Z} \circ (x \otimes \text{id})\), and the second map is \(y \mapsto y \circ \text{Alt}_{p+1}\). We have a similar differential, with \(C\) replaced by \(D\), and \(\kappa\) induces a commutative diagram

\[
\begin{array}{c}
\text{LBA}(D \otimes \Lambda^{p}, \Lambda^{q}) \\
\downarrow \\
\text{LBA}(C \otimes \Lambda^{p}, \Lambda^{q})
\end{array} \rightarrow \begin{array}{c}
\text{LBA}(D \otimes \Lambda^{p} \otimes \text{id}, \Lambda^{q}) \\
\downarrow \\
\text{LBA}(C \otimes \Lambda^{p} \otimes \text{id}, \Lambda^{q})
\end{array} \rightarrow \begin{array}{c}
\text{LBA}(D \otimes \Lambda^{p+1}, \Lambda^{q}) \\
\downarrow \\
\text{LBA}(C \otimes \Lambda^{p+1}, \Lambda^{q})
\end{array}
\]

The cokernel of this diagram is \(\text{LBA}_{\kappa}(\Lambda^{p}, \Lambda^{q}) \to \text{LBA}_{\kappa}(\Lambda^{p} \otimes \text{id}, \Lambda^{q}) \to \text{LBA}_{\kappa}(\Lambda^{p+1}, \Lambda^{q})\) and the composed map is the differential of our complex.

Recall that for \(X_{i}, Y \in \text{Ob}(\text{Sch})\), \(i = 1, \ldots, n\), we have an isomorphism \(\text{LBA}(X_{1} \otimes \ldots \otimes X_{n}, Y) \simeq \oplus_{Z_{1}, \ldots, Z_{n} \in \text{Irr}(\text{Sch})} \text{LC}(X_{1}, Z_{1}) \otimes \ldots \otimes \text{LC}(X_{n}, Z_{n}) \otimes \text{LA}(Z_{1} \otimes \ldots \otimes Z_{n}, Y) = \oplus_{Z_{1}, \ldots, Z_{n}} \text{LBA}(X_{1} \otimes \ldots \otimes X_{n}, Y)_{Z_{1}, \ldots, Z_{n}}\). The inverse isomorphism is the direct sum of the maps \(c_{1} \otimes \ldots \otimes c_{n} \otimes a \mapsto a \circ (c_{1} \otimes \ldots \otimes c_{n})\). If \(X_{i}\) is homogeneous of positive degree, \(\text{LCA}(X_{i}, 1) = 0\), so the above sum may be restricted by the condition \(|Z_{i}| > 0\).

We now define a complex \(0 \to C^{0} \to C^{1} \to \ldots\) as follows. The analogue of the above complex \([\mu_{-}] : \text{LBA}(\Lambda^{p}, \Lambda^{q}) \to \text{LBA}(\Lambda^{p+1}, \Lambda^{q})\) (with \(C\) replaced by \(Z\)) admits a subcomplex, namely \(Z_{q}^{0}, := \oplus_{Z \in \text{Irr}(\text{Sch})} \text{LBA}(Z \otimes \Lambda^{p}, \Lambda^{q})_{Z, z, z'}\):

\[
d^{p+1}_{z, q} : Z_{q}^{0, p+1} \to Z_{q}^{0, p+1}
\]

is then the restriction of the differential \([\mu_{-}]\). We then have an isomorphism between the complexes \(\text{LBA}(\Lambda^{p}, \Lambda^{q}), [\mu_{-}])_{P \geq 0}\) and \(\oplus_{Z \in \text{Irr}(\text{Sch}), |Z| > 0} \text{LCA}(C, Z) \otimes (\text{id}^{0, p+1})_{p \geq 0}\). We have a similar isomorphism replacing \(C\) by \(D\), and these isomorphisms are compatible with the morphisms of complexes induced by \(\kappa\). Taking cokernels, we get an isomorphism of complexes

\[
(\text{LBA}_{\kappa}(\Lambda^{p}, \Lambda^{q}), [\mu_{-}])_{P \geq 0} \simeq \oplus_{Z \in \text{Irr}(\text{Sch}), |Z| > 0} \text{Coker}[\text{LCA}(D, Z) \to \text{LCA}(C, Z)] \otimes (C^{0, p+1})_{p \geq 0}.
\]

We now prove the acyclicity of \((C_{Z, q}^{0}, d^{p+1}_{Z, q})_{P \geq 0}\), for any \(q \geq 0\) and any \(Z \in \text{Irr}(\text{Sch}), |Z| > 0\). To lighten notation, we will denote it \((C_{q}^{p}, d^{p+1}_{q})_{P \geq 0}\). We reexpress this complex as follows. View \(C^{p}\) as the antisymmetric part (under the action of \(\Theta_{p}\)) of \(\tilde{C}^{p} := \oplus_{Z_{1}, \ldots, Z_{p} \in \text{Irr}(\text{Sch})} \text{LBA}(Z \otimes \text{id}^{0, p+1})_{Z_{1}, \ldots, Z_{p} < C^{0, p+1} \otimes \text{id}^{0, p+1}, \Lambda^{q}}\) (we may restrict this sum by the conditions \(|Z_{i}| > 0\)).

Define

\[
d^{p+1}_{q} : C^{p} \simeq \text{LBA}(Z \otimes \text{id}^{0, p+1}, \Lambda^{q}) \to \text{LBA}(Z \otimes \text{id}^{0, p+1}, \Lambda^{q})
\]

by \(\tilde{d}^{p+1}(x) := x \circ (\text{id}_{Z} \otimes \text{id}_{0} \otimes \text{id}_{0}) \circ (\sum_{1 \leq i \leq (p+1)(-1)^{i+1} \beta_{ij}) + \mu_{\Lambda} \circ (\text{id}_{Z} \otimes x) \circ (\sum_{1 \leq i \leq p+1} (-1)^{i+1} \beta_{ij})\), where \(\beta_{ij}\) is the automorphism of \(Z \otimes x_{1} \otimes \ldots \otimes x_{p+1} \mapsto \ldots\).
Lemma 4.1. \( \oplus (\Box (C) \subset C_p) + 1 \) \( \rightarrow \) 0 is the universal version of \( \oplus x_1 \otimes x_2 \otimes x_3 \otimes \ldots x_{p+1} \Rightarrow x_1 \otimes x_2 \otimes x_3 \otimes \ldots x_{p+1} \) \( \rightarrow \) 0. Then \( \Delta_{p+1} \rightarrow C_p \rightarrow C_{p+1} \).

We now introduce a filtration on \( C_p \). Let \( (\check{C}_p)^{\leq p'} \subset \check{C}_p \) be the sum of all terms where \( \text{card}(\{Z_i = id\}) \leq p' \). This subspace is invariant under the action of \( \mathcal{E}_p \), so its totally antisymmetric part is a subspace \( (C_p)^{\leq p'} \subset C_p \).

Lemma 4.4. \( \Delta_{p+1}((C_p)^{\leq p'}) \subset (C_{p+1})^{\leq p' + 1} \).

Proof. To prove this, we will show that \( \Delta_{p+1}((\check{C}_p)^{\leq p'}) \subset (\check{C}_{p+1})^{\leq p' + 1} \). If \( x \in LBA(Z \otimes id^{\otimes p}, \wedge^q) Z, Z_1, \ldots, Z_p \), then \( \mu_{1, p} (id \otimes id \otimes \beta_1) \) is clearly in \( LBA(Z \otimes id^{\otimes p} + 1, \wedge^q) Z, Z_1, \ldots, Z_{p-1}, \beta_1, \ldots, \beta_p \). Here \( \text{card}(\{Z_i = id\}) \) has been increased by 1. Moreover, for any \( W \in Irr(Sch) \), the image of \( LCA(id, W) \rightarrow LBA(id^{\otimes 2}, W) \), \( c \rightarrow c \otimes \mu \) lies in \( \oplus_{W_1 + W_2 \in Irr(Sch) \mid |W_1| > 0} LBA(id^{\otimes 2}, W_1 + W_2) \). So \( x \circ (id \otimes \mu \otimes id^{\otimes p-1}) \circ \beta_{ij} \) lies in

\[
(10) \oplus_{W_1 + W_2 \in Irr(Sch) \mid |W_1| = |W_2| = |Z_i| + 1} LBA(Z \otimes id^{\otimes p} + 1, \wedge^q) Z, Z_1, Z_2, \ldots, Z_p, Z_{i+1}, W_1, Z_{i+2}, \ldots, Z_{p-1}, W_2, Z_{p+1}, Z_p.
\]

When \( (W_1, W_2) \in Irr(Sch) \) are such that \( |W_i| > 0, |W_1| + |W_2| = |Z| + 1, \{i \mid |W_i| = 1\} \leq 1 \) if \( |Z_i| > 1 \) and \( 2 \) if \( |Z_i| = 1 \). So in the summands of \( (10) \), \( \text{card}(\{Z_i = id\}) \) is increased by \( 1 \).

It follows that the differential \( \Delta_{p+1} \) is compatible with the filtration \( (C_p)^{\leq 0} \subset (C_p)^{\leq 1} \subset \ldots \subset (C_p)^{\leq p} = C_p \). To prove that it is acyclic, we will prove that the associated graded complex is acyclic. For this, we first determine this associated graded complex.

For \( p' + p'' = p \), let \( \bar{C}_p : = \oplus_{Z_i^p \in Irr(Sch), |Z_i^p| \neq 0, 1} LBA(Z \otimes id^{\otimes p}, \wedge^q) Z, id, Z_i^p, \ldots, Z_i^p \).

Let \( C_{p'} \cdot p'' \) be the antisymmetric part of this space w.r.t. the action of \( \mathcal{E}_{p'} \times \mathcal{E}_{p''} \).

Lemma 4.5. \( (C_p)^{\leq p}' / (C_{p})^{\leq p - 1} = C_{p'} \cdot p'' \), where \( p'' = p - p' \).

Proof. \( (\check{C}_p)^{\leq p}' / (\check{C}_p)^{\leq p - 1} = \oplus_{Z_i \in Irr(Sch), |Z_i| > 0} LBA(Z \otimes id^{\otimes p}, \wedge^q) Z, Z_1, \ldots, Z_p \).

\( (C_p)^{\leq p}' / (C_{p})^{\leq p - 1} \) is the \( \mathcal{E}_p \)-antiinvariant part of this space, which identifies with the \( \mathcal{E}_{p'} \times \mathcal{E}_{p''} \)-antiinvariant part of \( \check{C}_{p'} \cdot p'' \), i.e., \( C_{p'} \cdot p'' \). The isomorphic isomorphism

\[
[\oplus_{\text{card}(\{Z_i = id\})} LBA(Z \otimes id^{\otimes p}, \wedge^q)] \otimes \mathcal{E}_{p'} \rightarrow [\oplus_{|Z_i^p| > 1} LBA(Z \otimes id^{\otimes p}, \wedge^q) Z, id, Z_i^p, \ldots, Z_i^p)] \otimes \mathcal{E}_{p'}
\]

is given by projection on the relevant components, and the inverse isomorphism is given by the action of \( (1/p!) \sum_{\sigma \in S_p} \epsilon(\sigma) \sigma \) (or \( (p')!/p! \sum_{\sigma \in S_{p'}, \sigma'} \epsilon(\sigma) \sigma \), where \( S_{p'} \cdot p'' \) is the set of \( p', p'' \)-shuffle permutations).

Define

\[
\Delta_{p+1} : LBA(Z \otimes id^{\otimes p} + 1, \wedge^q) \rightarrow LBA(Z \otimes id^{\otimes p + 1}, \wedge^q)
\]

by \( x \mapsto x \circ (id \otimes \mu \otimes id^{\otimes p - 1}) \circ \beta_{ij} \). \( \sum_{1 \leq i < j \leq p+1} (-1)^{i+j+1} \beta_{ij} \).

Lemma 4.3. The map \( \Delta_{p+1} \) restricts to maps \( \bar{C}_{p'} \cdot p'' \rightarrow \bar{C}_{p'+1} \cdot p'' \) and \( \bar{C}_{p'} \cdot p'' \rightarrow \bar{C}_{p'+1} \cdot p'' \), and the map \( (C_p)^{\leq p}' / (C_{p})^{\leq p - 1} \rightarrow (C_{p+1})^{\leq p+1} / (C_{p+1})^{\leq p} \) induced by \( \Delta_{p+1} \) coincides with \( \bar{C}_{p'} \cdot p'' \), where \( p'' = p - p' \).

For each \( p'' \), \( (C_{p'})^p, \bar{C}_{p'} \cdot p'' \) is therefore a complex (this can be checked directly); it is embedded in the similar complex, where the restrictions \( |Z''| \neq 0, 1 \) are dropped, which is

---

3For \( M \) a module over \( \prod_{i} \mathcal{E}_p \), we denote by \( M \otimes \mathcal{E}_p \rightarrow \) the component of \( M \) of type \( \otimes \mathcal{E}_p \), where \( \epsilon_i \) is the signature character of \( \mathcal{E}_p \).
the universal version of the complex computing $H^{p'}(a, Z(a)^*) \otimes \wedge^{p''}(a)^* \otimes \wedge^{q}(a)$, where $a$ is a Lie bialgebra.

Proof. If $x \in \text{LBA}(Z \otimes \text{id}^{\otimes p'} \otimes \text{id}^{\otimes p''}, \Lambda^q)$, then one checks that both $x \circ (|_{\text{id}^{\otimes p-1}} \beta_{ij})$ and $x \circ (|_{\text{id}^{\otimes p+1}} \beta_{ij})$ lie in $\text{LBA}(Z \otimes \text{id}^{\otimes p'} \otimes \text{id}^{\otimes p''}, \Lambda^q)$, which implies that $\tilde{d}^{p',p''}$ induces a map $\tilde{C}^{p',p''} \rightarrow \tilde{C}^{p'+1,p''}$. The map $\tilde{d}^{p',p''+1}$ maps the $\mathfrak{g}_p \times \mathfrak{g}_p$-antisymmetric part of $\text{LBA}(Z \otimes \text{id}^{\otimes p'} \otimes \text{id}^{\otimes p''}, \Lambda^q)$ to its analogue with $p'$ increased by 1, so it restricts to a map $d^{p',p''+1}$: $C^{p',p''} \rightarrow C^{p'+1,p''}$.

Let us now show that the map $C^{p',p''} \rightarrow C^{p'+1,p''}$ induced by $d^{p,p+1}$: $(C^{p'})_{\leq p'} \rightarrow (C^{p+1})_{\leq p'+1}$ is $d^{p',p''+1}$.

Let $Z_1, ..., Z_p \in \text{Irr}(\text{Sch})$ be such that $Z_i = \text{id}$ for $i \leq p'$ and $|Z_i| > 1$ if $i > p'$. Let $y \in \text{LBA}(Z \otimes \text{id}^{\otimes p}, \Lambda^q)_{\leq p'}$ be of the form $a \circ (|_{\text{id}^{\otimes 1}} \otimes ... \otimes c_p)$, where $c_i \in \text{LCA}(\text{id}, Z_1)$ and $a \in \text{LA}(Z \otimes Z_1 \otimes ... \otimes Z_{p'}).$ Let $x := y \circ (Z \otimes (\sum_{\sigma \in \mathfrak{g}_p} \epsilon(\sigma)\sigma))$, and $\tilde{x} := y \circ (Z \otimes (\sum_{\sigma \in \mathfrak{g}_p} \epsilon(\sigma)\sigma))$. Then $x \in C^{p',p''}$, $\tilde{x} \in (C^{p'})_{\leq p'}$, and $x$ corresponds to the class of $\tilde{x}$ under $C^{p',p''} \simeq (\tilde{C}^{p'})_{/C^{p'}}$. Let us compute $d^{p,p+1}(\tilde{x})$. We have

\begin{equation}
\phi \circ (|_{\text{id}^{\otimes 1}} \otimes c_1) + \mu \circ (c \otimes \text{id}_\mathfrak{g}) + \kappa(c),
\end{equation}

where $\phi \in \text{LBA}(\text{id}^{\otimes 2}, Z)$ is such that:

- $\phi(c) \in \oplus_{|W_1|,|W_2| > 1} \text{LBA}(\text{id}^{\otimes 2}, Z)_{|W_1|,|W_2|}$ if $|Z| > 1$,
- $\phi(c) = -c \circ \phi$ if $Z = \text{id}$.

(12) is proved as follows: it is obvious when $Z = \text{id}$; we first prove it when $Z = T_p$ ($p > 0$) and $c = (\delta \otimes \text{id}^{\otimes p-1}) \circ ... \circ \delta$ (iterating the use of the cocycle identity); as this element generates $\mathfrak{g}_p$-module $\text{LCA}(\text{id}, T_p)$, this implies the identity when $Z = T_p$. The case of $Z \in \text{Irr}(\text{Sch})$, $|Z| = p$ is derived from there by taking isotypic components under the action of $\mathfrak{g}_p$.

When $|Z_{\sigma(1)}| > 1$, the contribution of $\kappa(c_{\sigma(1)})$ to (11) belongs to $(\tilde{C}^{p'})_{/p'}$. The class of (11) in $(\tilde{C}^{p'})_{/p'} \otimes (\tilde{C}^{p'+1})_{/p'+1}$ is then the same as that of

\begin{equation}
\sum_{1 \leq i < j \leq p+1} (-1)^{i+j} a \circ (Z \otimes (c_{\sigma(1)} \otimes \text{id}^{\otimes 1})) \otimes (c_{\sigma(2)} \otimes ... \otimes c_{\sigma(p)}) \circ \beta_{ij},
\end{equation}

The first line may be rewritten as follows. Let $a_\tau \in \mathfrak{g}_p$ be the cycle $a_{j+1} = 2, ..., a_{j+1} = j - 1, a_{j} = j, ..., a_{j} = p$. In terms of $\tau := a \circ a_\tau$.

\begin{equation}
\sum_{j \in \{1,...,p+1\}} \sum_{i < j} \sum_{\tau \in \mathfrak{g}_p} (-1)^{i+j} \epsilon(\tau) a \circ (Z \otimes (c_{\tau(1)} \otimes ... \otimes c_{\tau(j-1)}) \otimes ... \otimes c_{\tau(p)}) \circ \gamma_{ij},
\end{equation}
where $\gamma_{ij} \in \text{Aut}(Z \otimes \text{id}^{\otimes p})$ is the categorical version of $z \otimes x_1 \otimes \ldots \otimes x_{p+1} \mapsto z \otimes x_1 \otimes \ldots \otimes x_{j-1} \otimes x_i \otimes x_{j+1} \otimes \ldots \otimes x_{p+1}$. In the same way, one shows that the second line has the same expression, with the condition $i < j$ replaced by $i > j$ and $\gamma_{ij}$ the categorical version of $z \otimes x_1 \otimes \ldots \otimes x_{p+1} \mapsto z \otimes x_1 \otimes \ldots \otimes x_{j-1} \otimes x_i \otimes x_{j+1} \otimes \ldots \otimes x_{p+1}$.

Adding up these lines, and using the identity

$$
\mu_W \circ (\text{id}_{id} \otimes a) = \sum_{a=1}^{k} a \circ (\text{id}_{W_1} \otimes \ldots \otimes \mu_W \otimes \ldots \otimes \text{id}_{W_k}) \circ \beta_\alpha,
$$

in $\text{LA}(\text{id} \otimes W_1 \otimes \ldots \otimes W_k, W)$, where $a \in \text{LA}(W_1 \otimes \ldots \otimes W_k, W)$ and $\beta_\alpha$ is the braiding $\text{id} \otimes W_1 \otimes \ldots \otimes W_k \mapsto W_1 \otimes \ldots \otimes W_k \otimes \text{id} \otimes W_k \otimes \ldots \otimes W_k$, we express the contribution of (11) as (last line of (13)) + $\mu_{\lambda^+} \circ (\text{id}_{id} \otimes \tilde{x}) \circ (\sum_{i=1}^{p+1} (-1)^i \beta_j) + \tilde{x} \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ (\sum_{i=1}^{p+1} (-1)^{i+1} \beta_j)$.

The class of $d^{p,p+1}(\tilde{x})$ in $(C^{p+1}_{\lambda^+}/(C^{p+1})^{\otimes p})$ is therefore the same as that of (last line of (13)) + $\tilde{x} \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ (\sum_{i=1}^{p+1} (-1)^{i+1} \beta_j)$. To evaluate its image in $C^{p'+1, p''}$, we apply the projection of $\otimes Z_{i+1}, \ldots, Z_{p+1}$, $\text{LBA}(Z \otimes \text{id}^{\otimes p+1}, \wedge^q)_{Z_{i+1}, \ldots, Z_{p+1}}$ on the sum of components with $Z_i = \ldots = Z_{p'=1} = \text{id}$, $Z_{p'+1} = \ldots, Z_{p+1} > 1$, and the antisymmetric part (w.r.t. the condition $i > j$).

We have $\tilde{x} \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ \beta_i = \sum_{\sigma \in S_p} \epsilon(\sigma) \circ (\text{id}_{id} \otimes \beta_i) \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ (\sum_{i=1}^{p+1} (-1)^{i+1} \beta_j)$, and the summand corresponding to $\sigma$ belongs to $\text{LBA}(Z \otimes \text{id}^{\otimes p}, \wedge^q)_{Z_{i+1}, \ldots, Z_{p+1}, \text{id}, Z_{i+1}, \ldots, Z_{p+1}}$, i.e., $\text{id} \equiv \text{id} \otimes \text{id}^{\otimes p+1}$, and zero on the other ones. So the projection of $\tilde{x} \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ (\sum_{i=1}^{p+1} (-1)^{i+1} \beta_j)$ is $x \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ (\sum_{i=1}^{p+1} (-1)^{i+1} \beta_j)$.

Let us compute the projection of the last line of (13). The term in this line corresponding to $i, j, \sigma$ belongs to $\text{LBA}(Z \otimes \text{id}^{\otimes p}, \wedge^q)_{Z_{i+1}, \ldots, Z_{p+1}, \text{id}, Z_{i+1}, \ldots, Z_{p+1}}$, i.e., $\text{id} \equiv \text{id} \otimes \text{id}^{\otimes p+1}$, and zero on the other ones. The projection of the last line of (13) is therefore $x \circ (\text{id}_{id} \otimes \mu_Z \otimes \text{id}_{id}^{\otimes p-1})$. The projection of $\tilde{x} \circ (\mu_Z \otimes \text{id}_{id}^{\otimes p}) \circ (\sum_{i=1}^{p+1} (-1)^{i+1} \beta_j)$ is the sum of projections, i.e., $d^{p', p+1}(\tilde{x})$.

The associated graded of the complex $(C^p, d^{p,p+1})_{p \geq 0}$ is therefore $\oplus_{p \geq 0} (C^{p', p''})_{p' \geq 0}$. We now prove that for each $p'' \geq 0$, the complex $(C^{p', p''}, d^{p', p''+1})_{p' \geq 0}$ is acyclic.

For $Z'' = (Z''_1, \ldots, Z''_p) \in \text{Irr}(\text{Sch})$, let

$$
\tilde{d}^{p', p''+1}_{Z''} : \text{LA}(Z \otimes \text{id}^{\otimes p'}, \wedge^q) \rightarrow \text{LA}(Z \otimes \text{id}^{\otimes p'+1} \otimes (\otimes_i Z''_i), \wedge^q)
$$

be defined by the same formula as $d^{p', p''+1}_{Z''}$, replacing $\text{id}^{\otimes p-1}_{id} \otimes \text{id}^{\otimes p'}_{id}$ by $\text{id}^{\otimes p-1}_{id} \otimes \text{id}^{\otimes i}_Z$, $\text{id}^{\otimes p'}_{id} \otimes \text{id}^{\otimes Z''}$. Let $C'_{Z''}$ be the antisymmetric part of $\text{LA}(Z \otimes \text{id}^{\otimes p'} \otimes (\otimes_i Z''_i), \wedge^q)$ (w.r.t. the action of $\text{S}_{p'}$). Then $d^{p', p''+1}_{Z''}$ restricts to a differential $d^{p', p''+1}_{Z''} : C'_{Z''} \rightarrow C^{p', p''+1}$, moreover, we have an isomorphism between $(C^{p', p''}, d^{p', p''+1}_{Z''})_{p' \geq 0}$ and the antisymmetric part (w.r.t. the action of $\text{S}_{p'}$) of

$$
\oplus_{Z''_i \in \text{Irr}(\text{Sch})} (Z''_i) \otimes (\otimes_i Z''_i) \rightarrow \text{LA}(\text{id}, (\otimes_i Z''_i) \otimes (\otimes_i Z''_i)) \otimes (\otimes_i Z''_i) \otimes (\otimes_i Z''_i) \otimes (\otimes_i Z''_i).
$$

Since the differential of this complex is $\text{S}_{p'}$-equivariant, it suffices to prove that each component $(C^{p', p''}_{Z''})_{p' \geq 0}$ is acyclic.

Let $z = |Z|, N := \sum_i |Z''_i|$, let

$$
\tilde{d}^{p', p''+1}_{Z''_{N,q}} : \text{LA}(\text{id}^{\otimes z} \otimes \text{id}^{\otimes p'} \otimes \text{id}^{\otimes N}, \text{id}^{\otimes q}) \rightarrow \text{LA}(\text{id}^{\otimes z} \otimes \text{id}^{\otimes p'+1} \otimes \text{id}^{\otimes N}, \text{id}^{\otimes q})
$$

be defined by the same formula as $d^{p', p''+1}_{Z''}$, replacing $\otimes_i Z''_i$ by $\text{id}^{\otimes N}$ and $\mu_Z$ by $\mu_{\text{id}^{\otimes N}}$. Let $C'_{N,q}$ be the antisymmetric part of $\text{LA}(\text{id}^{\otimes z} \otimes \text{id}^{\otimes p'} \otimes \text{id}^{\otimes N}, \text{id}^{\otimes q})$ (w.r.t. the action of $\text{S}_{p'}$). Then
$d_{z,N,q}^{p',p'+1}$ restricts to a differential $d_{z,N,q}^{p',p'+1}: C_{z,N,q}^{p'} \to C_{z,N,q}^{p'+1}$. The complex $(C_{z,N,q}^{p}, d_{z,N,q}^{p',p'+1})_{p \geq 0}$ is equipped with a natural action of $\mathfrak{S}_z \times \prod_{i} \mathfrak{S}_i[z_i^{p_i}] \times \mathfrak{S}_q$, and $(C_{z,N,q}^{p}, d_{z,N,q}^{p',p'+1})$ is an isotypic component of this action. It suffices therefore to prove that $(C_{z,N,q}^{p}, d_{z,N,q}^{p',p'+1})_{p \geq 0}$ is acyclic.

In what follows, we denote by $\mathcal{L}(u_1, \ldots, u_s)$ (resp., $\mathcal{A}(u_1, \ldots, u_s)$) the free Lie (resp., associative) algebra generated by $u_1, \ldots, u_p$. These spaces are graded by $\oplus_{i \in [1,p]} \mathbb{N} \delta_i$ and for $S \subset [1,p]$ we denote by $\mathcal{L}(u_1, \ldots, u_s)_S$, $\mathcal{A}(u_1, \ldots, u_s)_S$ the subspaces of degree of $\mathfrak{S}_i \in S \delta_i$. In the case of two sets of generating variables $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_t)$, the spaces are graded by $\oplus_{i \in [1,p]} \mathbb{N} \delta_i$ and we use the same notation for homogeneous subspaces.

**Lemma 4.4.** We have an isomorphism of complexes

\begin{equation}
C_{z,N,q}^\bullet \cong \bigoplus_{i \geq 1} \bigoplus_{a=1}^q C_{[1,i],[i],[a,1],[1,N]} \circ \alpha \cong C_{[1,i],[a,1],[1,N]}
\end{equation}

**Proof.** Identify $C_{z,N,q}^{p'}$ with $[\mathcal{L}(a_1, \ldots, a_{z+N}, x_1, \ldots, x_{p'}) \otimes |1,z+N| \otimes |1,p'|]$ which is the part of the $q$th tensor power of $\mathcal{L}(a_1, \ldots, x_{p'})$, multilinear in $a_1, \ldots, a_{z+N}, x_1, \ldots, x_{p'}$, and antisymmetric in $x_1, \ldots, x_{p'}$. Let $a_1, \ldots, a_z$ correspond to the $z$ factors of $\mathfrak{id}^\otimes z$, $a_{z+1}, \ldots, a_{z+N}$ to the $N$ factors of $\mathfrak{id}^\otimes N$, and $x_1, \ldots, x_{p'}$ to the $p'$ factors of $\mathfrak{id}^\otimes p'$. The differential $d_{z,N,q}^{p',p'+1}$ then expresses as

\begin{equation}
F(a_1, \ldots, a_{z+N}, x_1, \ldots, x_{p'}) \mapsto \sum_{1 \leq i < j \leq p'+1} (-1)^{i+j+1} F(a_1, \ldots, a_{z+N}, [x_i, x_j], x_1, \ldots, x_i \ldots x_j \ldots x_{p'+1}) + \sum_{1 \leq \ell \leq p'+1} \sum_{1 \leq i \ldots i' = 1} (-1)^{i_1 + \ldots + i_{p'+1}} F(a_1, \ldots, [x_{i_1}, \ldots, x_{i_{p'+1}}], a_{z+N}, \ldots, x_{p'+1}).
\end{equation}

On the other hand, we have an isomorphism $\mathcal{L}_A(\mathfrak{id}^\otimes N, \mathfrak{id}^\otimes q) \cong \bigoplus_{i \in [1,N] = [1,i]} \mathcal{L}_A(\mathfrak{id}^\otimes [1,i], \mathfrak{id})$, with inverse given by the sum of maps $a_1 \otimes \ldots \otimes a_q \mapsto (a_1 \otimes \ldots \otimes a_q) \circ \beta_{1, \ldots, 1, a_1, \ldots, a_q}$, where $\beta_{1, \ldots, 1, a_1, \ldots, a_q}: \mathfrak{id}^\otimes N \to \bigotimes \mathfrak{id}^\otimes [1,i]$ is the braiding induced by the maps $[1,i] \to \bigcup_{[1,1]} [1,1]$, taking $I_a$ to $[1,|I_a|]$ by preserving the order. Analyzing the action of $\mathfrak{S}_N$ on the set of $q$-compositions of $[1,N]$ we derive an isomorphism $\mathcal{L}_A(\Lambda^N, \mathfrak{id}^\otimes q) \cong \bigotimes_{i \in [1,N]} \mathcal{L}_A(\mathfrak{id}^\otimes [1,i], \mathfrak{id})$, with inverse given by the direct sum of the maps $a_1 \otimes \ldots \otimes a_q \mapsto (a_1 \otimes \ldots \otimes a_q) \circ \beta_{N, \ldots, N, a_1}$. The composite $\mathfrak{id}^\otimes N \to \bigotimes \mathfrak{id}^\otimes [1,i]$ is an isotypic representation of $\Lambda^N \to \bigotimes \mathfrak{id}^\otimes [1,i]$. The braiding induced by the maps $[1,i] \to \bigcup_{[1,1]} [1,1]$ is strictly preserved by $\mathfrak{id}^\otimes z \to \bigotimes \mathfrak{id}^\otimes [1,i]$.

One proves similarly that we have an isomorphism $\mathcal{L}_A(\mathfrak{id}^\otimes z \otimes \Lambda^p \otimes \mathfrak{id}^\otimes N, \mathfrak{id}^\otimes q) \cong \bigotimes_{i \in [1,z]} \mathcal{L}_A(\mathfrak{id}^\otimes [1,i], \mathfrak{id})$, with inverse given by the direct sum of the maps $a_1 \to (a_1) \circ \beta_{I_1, (J_1), (p_1)}$, where $\beta_{I_1, (J_1), (p_1)}: \mathfrak{id}^\otimes z \to \bigotimes \mathfrak{id}^\otimes [1,i] \otimes \Lambda^p \otimes \mathfrak{id}^\otimes [1,i]$. This construction is reversible from the Schur morphism. It follows that we have the isomorphism $\mathfrak{i}d$ between graded vector spaces. Let us show that it is compatible with differentials.

For $\bigcup I_1 = [1,z], \bigcup I_2 = z + [1,N]$, $\sum_{i \in [1,N]} p_1 = p'$, the map $\bigotimes a_1^q \mathcal{L}_A(\mathfrak{id}^\otimes [1,i], \mathfrak{id}) \to \mathcal{L}_A(\mathfrak{id}^\otimes z \otimes \Lambda^p \otimes \mathfrak{id}^\otimes N, \mathfrak{id}^\otimes q)$ identifies with the map

\begin{equation}
\begin{align*}
\bigotimes_{i \in [1,z]} \mathcal{L}_A(\mathfrak{id}^\otimes [1,i], \mathfrak{id}) & \to \mathcal{L}_A(\mathfrak{id}^\otimes z \otimes \Lambda^p \otimes \mathfrak{id}^\otimes N, \mathfrak{id}^\otimes q) \\
\sum_{\sigma} \sigma_{i}^\otimes \mathcal{L}_A(\mathfrak{id}^\otimes [1,i], \mathfrak{id}) & \to \mathcal{L}_A(\mathfrak{id}^\otimes z \otimes \Lambda^p \otimes \mathfrak{id}^\otimes N, \mathfrak{id}^\otimes q),
\end{align*}
\end{equation}

where $\mathfrak{S}_{p_1 \ldots p_\alpha}$ is the set of shuffle permutations of $\mathfrak{S}_{p_1 \ldots p_\alpha}$ (preserving the order of the elements of $[1,p_1], p_1 + [1,p_2], \ldots$) and $\ast$ is the permutation action on $x_1, \ldots, x_{p'}$. The projection $\mathcal{L}_A(\mathfrak{id}^\otimes z \otimes \Lambda^p \otimes \mathfrak{id}^\otimes N, \mathfrak{id}^\otimes q)$ to the component indexed by $(I_1, (J_1))$
along the other components can then be described as follows: the composite map
\[
\mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{[1, z+N]|1, p'} \simeq \bigoplus_{\alpha \in \mathbb{I}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{I_{\alpha}} \cup \bigcup_{\alpha \in \mathbb{I}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{I_{\alpha} \cup J_{\alpha} \cup P_{\alpha}} \\
\rightarrow \bigoplus_{\alpha \in \mathbb{I}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{I_{\alpha} \cup J_{\alpha} \cup (p_1 + \ldots + p_{\alpha-1} + 1, p_{\alpha})}
\]
(where the second map is the projection along all other components) restricts to
\[
\mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{[1, z+N]|1, p'} \rightarrow \bigoplus_{\alpha \in \mathbb{I}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{I_{\alpha} \cup J_{\alpha} \cup (p_1 + \ldots + p_{\alpha-1} + 1, p_{\alpha})},
\]
which identifies with the projection \(C_{z, N, q}^p \rightarrow \bigoplus_{\alpha \in \mathbb{I}} C_{I_{\alpha}, [1, z + N]}.\)

Extending formula (15) defining \(d_{z, N, q}^{p, p+1}\), we define a map
\[
d_{z, N, q}^{p, p+1} : \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{[1, z+N]|1, p'} \rightarrow \mathcal{L}(a_1, \ldots, x_{z+N}, x_1, \ldots, x_{p'+1})^{q'}_{[1, z+N]|1, p'+1}.
\]
It follows that the map \(d_{z, N, q}^{p, p+1} : C_{z, N, q}^p \rightarrow C_{z, N, q}^{p+1}\) may be identified with the composite map
\[
\bigoplus_{\sum_{\alpha \in \mathbb{I}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{I_{\alpha}} \cup J_{\alpha} \cup P_{\alpha}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{[1, z+N]|1, p'} \\
\rightarrow \bigoplus_{\sum_{\alpha \in \mathbb{I}} \mathcal{L}(a_1, \ldots, x_{p'}^{q'})_{I_{\alpha}} \cup J_{\alpha} \cup P_{\alpha}} \mathcal{L}(a_1, \ldots, x_{p'+1})^{q'}_{[1, z+N]|1, p'+1}
\]
We have a decomposition \(\tilde{d}^{p, p+1}_{z, N, q} = \sum_{1 \leq i < j \leq p'} \tilde{d}^{ij} + \sum_{i=1}^{p'+1} \sum_{z'=1}^{z} \tilde{d}^{z'} z').

Then:

\bullet \ \tilde{d}^{ij}\) takes the summand indexed by \((I_{\alpha})_{\alpha}, (J_{\alpha})_{\alpha}, (P_{\alpha})_{\alpha}\) to the summand indexed by
\((I_{\alpha})_{\alpha}, (J_{\alpha})_{\alpha}, (P_{\alpha})_{\alpha}\), where \((P_{\alpha})_{\alpha}\) is the partition of \([1, p'+1]\) given by \(P_{\alpha} = \{(P_{\alpha} \cap [i, j]) - 1\} \cup (P_{\alpha} \cap [i+1, j-1]) \cup (P_{\alpha} \cap [j, p]) + 1\) if \(i \notin P_{\alpha}\), and the union of the same set with \(\{i, j\}\)
if \(1 \in P_{\alpha}\) (all these unions are disjoint):

\bullet \ \tilde{d}^{z'}\) takes the summand indexed by \((I_{\alpha})_{\alpha}, (J_{\alpha})_{\alpha}, (P_{\alpha})_{\alpha}\) to the summand indexed by
\((I_{\alpha})_{\alpha}, (J_{\alpha})_{\alpha}, (P_{\alpha})_{\alpha}\), where \((P_{\alpha})_{\alpha}\) is the partition of \([1, p'+1]\) given by \(P_{\alpha} = \{(P_{\alpha} \cap [i, j-1]) \cup (P_{\alpha} \cap [i, p']) + 1\} if \(z' \notin I_{\alpha}\), and the union of the same set with \(\{i, j\}\)
if \(z' \in I_{\alpha}\) (all these unions are disjoint).

As the partitions \((I_{\alpha})_{\alpha}\) and \((J_{\alpha})_{\alpha}\) of \([1, z] + (z+1, N]\) are not modified, (14) is a decomposition of complexes. If \((P_{\alpha})_{\alpha}\) is one of the partitions \((P_{\alpha}^{ij})_{\alpha}\) or \((P_{\alpha}^{z'})_{\alpha}\), then the sequence \(|\tilde{P}_{\alpha}| = 1, \ldots, q\) has the form \((p_{\alpha}, \delta_{\alpha}, \alpha)\), where \(\beta \in [1, q]\) and \(p_{\alpha} = |P_{\alpha}|.

Fix \(\beta \in [1, q]\) and set \(p_{\alpha}^\beta := p_{\alpha} + \delta_{\alpha, \beta}\). The partition \((P_{\alpha})_{\alpha}\) coincides with \((p_{\alpha}^0 + + p_{\alpha-1} + [1, p_{\alpha}^0])_{\alpha}\) if:

(a) \(P_{\alpha} = p_1 + \ldots + p_{\alpha-1} + [1, p_{\alpha}]\) if \(\alpha < \beta, P_{\beta} = [1 + p_1 + \ldots + p_{\beta-1} + [1, p_{\beta}] - 1]\) \(\cup \{1\}, P_{\alpha} = p_1 + \ldots + p_{\alpha-1} + [1, p_{\alpha}]\) if \(\alpha > \beta\) and \(p_1 + \ldots + p_{\alpha-1} + 1 \leq i < j \leq p_1 + \ldots + p_{\beta} + 1\); in that case, \((P_{\alpha})_{\alpha}\) is given by \(P_{\alpha}^0 = p_1 + \ldots + p_{\alpha-1} + [1, p_{\alpha}]\) for \(\alpha < \beta, P_{\beta}^\alpha = p_1 + \ldots + p_{\beta} - 1 + [1, p_{\beta} + 1],\) and \(\tilde{P}_{\alpha}^\beta = 1 + p_1 + \ldots + p_{\alpha-1} + [1, p_{\alpha}]\) if \(\alpha > \beta\). In particular, \(i \in P_{\beta}^\alpha\) and \(z' \in I_{\beta}\).

(b) \(P_{\alpha} = p_1 + \ldots + p_{\alpha-1} + [1, p_{\alpha}]\) for any \(\alpha, p_1 + \ldots + p_{\alpha-1} + 1 \leq i \leq p_1 + \ldots + p_{\beta} + 1\) and \(z' \in I_{\beta}\); in that case, \((P_{\alpha}^{z'})_{\alpha}\) is given by \(\tilde{P}_{\alpha}^{z'} = p_1 + \ldots + p_{\alpha-1} + [1, p_{\alpha}]\) for \(\alpha < \beta, P_{\beta}^{z'} = p_1 + \ldots + p_{\beta} - 1 + [1, p_{\beta}]\) and \(\tilde{P}_{\alpha}^{z'} = 1 + p_1 + \ldots + p_{\beta-1} + [1, p_{\beta}]\) for \(\alpha > \beta\). In particular, \(i \in P_{\beta}^{z'}\) and \(z' \in I_{\beta}\).
Let now \( \otimes_\alpha F_\alpha(a_1, ..., x_{p'}) \) belong to \( \otimes_\alpha \mathcal{L}(a_1, ..., x_{p'})_{I_1 \cup J_1 \cup \{p_1 + \ldots + p_{n-1} + 1, p_n\}} \). The image of this element in \( \mathcal{L}(a_1, ..., x_{p'})_{I_1 \cup J_1 \cup \{p_1 + \ldots + p_{n-1} + 1, p_n\}} \) is \( (\sum_{\sigma \in \mathcal{S}_p} e(\sigma)\sigma) * (\otimes_\alpha F_\alpha) \). Let us apply \( d^{p', p'+1}_\beta \) to this element, and let us project the result to \( \otimes_\beta \mathcal{L}(a_1, ..., x_{p'+1})_{I_1 \cup J_1 \cup \{p_1 + \ldots + p_{n-1} + 1, p_n\}} \).

According to what we have seen, the nontrivial contributions to the summand indexed by \( \beta \) are:

- For \( i < j \) in \( p_1 + \ldots + p_{\beta-1} + 1, p_{\beta} + 1 \), the projection of \( d^{i-j}(e(\sigma)\sigma * (\otimes_\alpha F_\alpha)) \), where \( \sigma \) is the shuffle permutation taking the \( p_1 + \ldots + p_{\alpha-1} + 1, p_\alpha \) to \( p_\alpha \) described in (a) above;

- For \( i \in p_1 + \ldots + p_{\beta-1} + 1, p_{\beta} + 1 \) and \( z' \in I_\beta \), the projection of \( d^{p'}(\otimes_\alpha F_\alpha) \), where \( d^{p'} \) is the summand of \( d^{p', p'+1} \) corresponding to \( (i, z') \).

Let \( d^{p, p+1}_\beta : \mathcal{L}(a_1, ..., x_{p'})_{I_1 \cup J_1 \cup \{p_1 + \ldots + p_{n-1} + 1, p_n\}} \rightarrow \mathcal{L}(a_1, ..., x_{p'+1})_{I_1 \cup J_1 \cup \{p_1 + \ldots + p_{n-1} + 1, p_n\}} \) be the differential of the complex \( \mathcal{C}_{[I_1, |J_1|]} \) and let \( d^{i-j}_\beta, d^{p'}_\beta \) be its components. We have \( d^{i-j}(\sigma * (\otimes_\alpha F_\alpha)) = F_1 \otimes \ldots \otimes d^{i-j}_\beta(F_{i-j}) \otimes \ldots \otimes F_q \) (to prove this equality, note that the \( x_1 \) present in the \( \beta \)th factor of \( \sigma * (\otimes_\alpha F_\alpha) \) gets replaced by \( [x_i, x_j] \) in both sides; the signs coincide since the “usual” indices of variables \( x_i, x_j \) are shifts of \( i, j \) by the same quantity, and this does not alter \((1)^{i+j+1}\)), while \( e(\sigma) = (-1)^{p_1 + \ldots + p_{\beta-1} - 1} \); on the other hand, \( d^{p'}(\otimes_\alpha F_\alpha) = (-1)^{p_1 + \ldots + p_{\beta-1} - 1} F_1 \otimes \ldots \otimes d^{p'}_\beta(F_{\beta}) \otimes \ldots \otimes F_q \) (here the sign is due to the fact that the index of \( x_i \) is, in the usual ordering, \( i - (p_1 + \ldots + p_{\beta-1}) \)). It follows that the contribution to the summand indexed by \( \beta \) is \( (-1)^{p_1 + \ldots + p_{\beta-1} - 1} F_1 \otimes \ldots \otimes d^{p, p+1}_\beta(F_{\beta}) \otimes \ldots \otimes F_q \). So the projection of \( d^{p', p'+1}((\sum_{\sigma \in \mathcal{S}_p} e(\sigma)\sigma) * (\otimes_\alpha F_\alpha)) \) is

\[
\sum_{\beta} (-1)^{p_1 + \ldots + p_{\beta-1} - 1} \text{id} \otimes \ldots \otimes d^{p, p+1}_\beta \otimes \ldots \otimes \text{id} \otimes (\otimes_\alpha F_\alpha),
\]

as was to be proved.

As \( z \neq 0 \), for each partition \((I_1, ..., I_q)\) of \([1, z]\), there exists \( i \) such that \( |I_i| \neq 0 \). So renaming \(|I_1|, |J_1| \) by \( z, N \), it suffices to prove that if \( z \neq 0 \), then \( C^{p'}_{z, N, 1} \) is acyclic.

Recall that \( C^{p'}_{z, N, 1} \simeq \mathcal{L}(a_1, ..., x_{p'})_{[1, z+N]|I_{[1, p']}} \) and \( d^{p', p'+1}_{z, N, 1} \) is given by (15). On the other hand, the map \( a \mapsto \text{ad}(a)(a_1) \) gives rise to an isomorphism

\[
\mathcal{A}^{p'}_{z, N, 1} := \mathcal{A}(a_2, ..., x_{p'})_{[2, z+N]|I_{[1, p']}} \simeq C^{p'}_{z, N, 1},
\]

where \( \mathcal{A}(u_1, ..., u_n) \) is the free associative algebra generated by \( u_1, ..., u_n \) and \( \text{ad} : \mathcal{A}(u_1, ..., u_n) \rightarrow \text{End}(\mathcal{L}(u_1, ..., u_n)) \) is the algebra morphism derived from the adjoint action of \( \mathcal{L}(u_1, ..., u_n) \) on itself. The differential \( d^{p', p'+1}_{z, N, 1} : \mathcal{A}^{p'}_{z, N, 1} \rightarrow \mathcal{A}^{p'+1}_{z, N, 1} \) is given by

\[
Q(a_2, ..., a_{z+N}, x_1, ..., x_{p'}) \mapsto \sum_{1 \leq i < j \leq p'+1} (-1)^{i+j+1} Q(a_2, ..., a_{z+N}, [x_i, x_j], x_1, ..., \hat{x}_i, \hat{x}_j, ..., x_{p'+1})
\]

\[
+ \sum_{i=1}^{p'+1} (-1)^{i+1} \left( Q(a_2, ..., a_{z+N}, x_1, ..., \hat{x}_i, x_i, x_{p'+1}) + \sum_{z'=2}^z Q(a_2, ..., x_i, a_{z'}, ..., a_{z+N}, x_1, ..., \hat{x}_i, ..., x_{p'+1}) \right),
\]

as \( \text{ad}(a)([x, a_1]) = \text{ad}(ax)(a_1) \), for any \( x \in \mathcal{L}(a_1, ..., x_{p'}) \) and \( a \in \mathcal{A}(a_1, ..., x_{p'}) \).

We have an isomorphism

\[
\mathcal{A}^{p'}_{z, N, 1} \simeq \oplus_{\sigma \in \text{Perm}(\{2, ..., z+N\})} \mathcal{A}_\sigma^{p'},
\]

\[
\mathcal{A}^{p'}_{z, N, 1} \simeq \oplus_{\sigma \in \text{Perm}(\{2, ..., z+N\})} \mathcal{A}_\sigma^{p'},
\]
where $\mathcal{A}_z^\bullet := (\mathcal{A}(x_1, ..., x_{p'})^\otimes z+1 \mathcal{S}_{[1,p']}^\bullet)$, whose inverse is the direct sum of the maps induced by

$$\otimes_{\alpha=1}^{\mathcal{S}^\alpha_{[1,p']} Q_\alpha(x_1, ..., x_{p'}) \to Q_1(x_1, ..., x_{p'})a_{\sigma(2)}Q_2(x_1, ..., x_{p'})a_{\sigma(3)}...a_{\sigma(z+N)}Q_{z+N}(x_1, ..., x_{p'}) \text{.}$$

The explicit formula (10) shows that if $Q(a_1, ..., x_{p'})$ is a multilinear monomial, then the image of $Q$ by the extension of $d_{\mathcal{S}^\alpha_{[1,p']} \mathcal{S}_{p'+1}}$ given by the same formula is a linear combination of monomials, where the $a_i$ appear in the same order as in $Q(a_1, ..., x_{p'})$. It follows that for each $\sigma \in \text{Perm}(\{2, ..., z + N\})$, $\mathcal{A}_z^\bullet$ is a subcomplex of $\mathcal{A}_{z,N,1}^\bullet$, and that we have a direct sum decomposition of the complex $\mathcal{A}_{z,N,1}^\bullet$

$$\mathcal{A}_{z,N,1}^\bullet \simeq \oplus_{\sigma} \mathcal{A}_z^\bullet. \quad (17)$$

The acyclicity of $\mathcal{A}_{z,N,1}^\bullet$ is then a consequence of that of each subcomplex $\mathcal{A}_z^\bullet$, which we now prove. Let us fix $\sigma \in \text{Perm}(\{2, ..., z + N\})$. There is a unique linear map

$$\tilde{d}_{\mathcal{S}^\alpha_{[1,p']} \mathcal{S}_{p'+1}} : (\mathcal{A}(x_1, ..., x_{p'})^\otimes z+1 \mathcal{S}_{[1,p']}^\bullet) \to (\mathcal{A}(x_1, ..., x_{p'+1})^\otimes z+1 \mathcal{S}_{[1,p'+1]}^\bullet),$$

given by

$$\tilde{Q}(x_1, ..., x_{p'}) \mapsto \sum_{1 \leq i \leq j \leq p'+1} (-1)^{i+j+1} \tilde{Q}(x_i, x_j), x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{p'+1})$$

$$+ \sum_{i=1}^{p'+1} (-1)^{i+1} \left( \tilde{Q}(x_1, ..., \hat{x}_i, ..., x_{p'+1}) \left[ \sum_{\alpha \in \sigma^{-1}(\{2, z\} - 1)} x_i^{(\alpha)} - \sum_{\alpha \in \sigma^{-1}(\{2, z\})} x_i^{(\alpha)} \right] \tilde{Q}(x_1, ..., \hat{x}_i, ..., x_{p'+1}) \right),$$

where $f^{(\alpha)} = 1^{\otimes \alpha-1} \otimes f \otimes 1^{\otimes z+N-\alpha}$. If we set $\epsilon_1 = 0$, $\epsilon_{z+N+1} = 1$, and

$$\epsilon_\alpha = 1 \leftrightarrow \sigma(\alpha) \in \{2, z\}, \quad \epsilon_\alpha = 0 \leftrightarrow \sigma(\alpha) \in z + [1, N],$$

for $\alpha \in [2, z + N]$, then this map is

$$\tilde{Q}(x_1, ..., x_{p'}) \mapsto \sum_{1 \leq i \leq j \leq p'+1} (-1)^{i+j+1} \tilde{Q}(x_i, x_j), x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{p'+1})$$

$$+ \sum_{\alpha \in [1, z+N]} \sum_{i=1}^{p'+1} (-1)^{i+1} \left[ \epsilon_{\alpha+1} \tilde{Q}(x_1, ..., \hat{x}_i, ..., x_{p'+1}) x_i^{(\alpha)} - \epsilon_\alpha x_i^{(\alpha)} \tilde{Q}(x_1, ..., \hat{x}_i, ..., x_{p'+1}) \right].$$

The map $\tilde{d}_{\mathcal{S}^\alpha_{[1,p']} \mathcal{S}_{p'+1}}$ then restricts to a linear map between the subspaces of totally antisymmetric tensor (under the actions of $\mathcal{S}_{p'}$ on the left side and $\mathcal{S}_{p'+1}$ on the right side), which coincides with $d_{\mathcal{S}^\alpha_{[1,p']} \mathcal{S}_{p'+1}}$.

For $\epsilon, \epsilon' \in \{0, 1\}$, define the “elementary” complexes $\mathcal{E}_{\epsilon, \epsilon'}$ as follows. We set $\mathcal{E}_{\epsilon, \epsilon'} := \mathcal{A}(x_1, ..., x_{p'})^\otimes \mathcal{S}_{[1,p']}$, and define $d_{\epsilon, \epsilon'} : \mathcal{E}_{\epsilon, \epsilon'} \to \mathcal{E}_{\epsilon, \epsilon'}^{p'+1}$ by

$$(d_{\epsilon, \epsilon'}^{p'+1}) E(x_1, ..., x_{p'+1}) := \sum_{1 \leq i \leq j \leq p'+1} (-1)^{i+j+1} E(x_i, x_j), x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{p'+1})$$

$$+ \epsilon \sum_{i=1}^{p'+1} (-1)^{i} x_i E(x_1, ..., \hat{x}_i, ..., x_{p'+1}) + \epsilon' \sum_{i=1}^{p'+1} (-1)^{i+1} E(x_1, ..., \hat{x}_i, ..., x_{p'+1}) x_i.$$

**Lemma 4.5.** For $\epsilon, \epsilon' \in \{0, 1\}$, $\mathcal{E}_{\epsilon, \epsilon'} := (\mathcal{E}_{\epsilon, \epsilon'}, d_{\epsilon, \epsilon'}^{p'+1})_{p' \geq 0}$ is a complex.

**Proof.** Note first that for any $p' \geq 0$, $\mathcal{E}_{\epsilon, \epsilon'}^{p'}$ is 1-dimensional, spanned by $e_{p'}(x_1, ..., x_{p'}) := \sum_{\sigma \in \mathcal{S}_{p'}} \epsilon(\sigma)x_{\sigma(1)} ... x_{\sigma(p')}$. 
If $\mathfrak{g}$ is a Lie algebra, let $U(\mathfrak{g}, e, \epsilon)$ be the universal enveloping algebra of $\mathfrak{g}$, equipped with the trivial $\mathfrak{g}$-module structure if $\epsilon = (0, 0)$, the left (resp., right) regular $\mathfrak{g}$-module structure if $\epsilon = (1, 0)$ (resp., $(0, 1)$), and the adjoint $\mathfrak{g}$-module structure if $(\epsilon, e') = (1, 1)$. Let $(C_{e, e'}^{\mathfrak{g}}(\mathfrak{g}), d_{e, e'}^{\mathfrak{g}'}(p))$ be the cochain complex computing the cohomology of $\mathfrak{g}$ in these modules. We have $C_{e, e'}^{\mathfrak{g}}(\mathfrak{g}) = \text{Hom}(\wedge^{e'}(\mathfrak{g}), U(\mathfrak{g}))$. There is a unique linear map $\mathcal{E}_{e, e'}^{\mathfrak{g}'} \rightarrow C_{e, e'}^{\mathfrak{g}'}(\mathfrak{g})$, taking $e_{e'}$ to the composite map $\wedge^{e'}(\mathfrak{g}) \rightarrow \mathfrak{g} \otimes^{e'} \rightarrow U(\mathfrak{g})$, where the last map is the product map, and one checks that the diagram

\[
\begin{array}{ccc}
\mathcal{E}_{e, e'}^{\mathfrak{g}'} & \xrightarrow{d_{e, e'}^{\mathfrak{g}'} + 1} & \mathcal{E}_{e, e'}^{\mathfrak{g}'} \\
\downarrow & & \downarrow \\
C_{e, e'}^{\mathfrak{g}'} & \xrightarrow{d_{e, e'}^{\mathfrak{g}'} + 1} & C_{e, e'}^{\mathfrak{g}'}(\mathfrak{g})
\end{array}
\]

commutes. Since $C_{e, e'}^{\mathfrak{g}'}(\mathfrak{g})$ is a complex, and there exists a Lie algebra $\mathfrak{g}$ such that the morphisms $\mathcal{E}_{e, e'}^{\mathfrak{g}'} \rightarrow C_{e, e'}^{\mathfrak{g}'}(\mathfrak{g})$ are injective (for example, $\mathfrak{g}$ is a free Lie algebra with countably many generators), $\mathcal{E}_{e, e'}^{\mathfrak{g}'}$ is also a complex.

**Lemma 4.6.** We have an isomorphism of complexes $A_{e} \simeq \mathcal{E}_{e, e_2} \otimes \mathcal{E}_{e_2, e_3} \otimes \ldots \otimes \mathcal{E}_{e_{z+1}, 1}$, where $(e_2, ..., e_{z+1})$ is as in [12].

**Proof.** The proof is parallel to that of Lemma 4.3. Let us set

\[A^{e'} := A_{e}^{e'} := [A(x_1, ..., x_{p'})^{0 \otimes z + N}]^{S_{p'}}_{[1, p']}, \quad A^{e'} := [A(x_1, ..., x_{p'})^{0 \otimes z + N}]_{[1, p']}\]

if $p_1 + ... + p_{z+N} = p'$, set

\[A_{p_1, ..., p_{z+N}} := \otimes_{\alpha = 1}^{z+N} A(x_1, ..., x_{p'})^{S_{p_\alpha}}_{p_1 + ... + p_{\alpha-1} + 1, p_{\alpha}}\]

and if $\bigcup_{\alpha = 1}^{z+N} p_{\alpha} = [1, p']$, set

\[A_{p_1, ..., p_{z+N}} := \otimes_{\alpha = 1}^{z+N} A(x_1, ..., x_{p'})_{p_{\alpha}}.\]

We have a decomposition

\[A^{e'} = \bigoplus_{p_1 + ... + p_{z+N} = p'} A_{p_1, ..., p_{z+N}} \simeq A^{e'}.\]

We will define the support of an element $x$ of $A^{e'}$ as the set of partitions $(P_1, ..., P_{z+N})$ of $[1, p']$ such that the component $x_{(P_1, ..., P_{z+N})}$ is nonzero. We also have natural morphisms $A_{p_1, ..., p_{z+N}} \rightarrow A^{e'}$, given by $x \mapsto (\sum_{\sigma \in S_{p_1, ..., p_{z+N}}} \epsilon(\sigma) * x$, where $*$ is the permutation action of $S_{p'}$ on $x_{1, ..., x_{p'}}$. The direct sum of these morphisms gives rise to an isomorphism

\[\bigoplus_{p_1 + ... + p_{z+N} = p'} A_{p_1, ..., p_{z+N}} \simeq A^{e'}.\]

As the l.h.s. identifies with $\bigoplus_{p_1 + ... + p_{z+N} = p'} \mathcal{E}_{e_0, e_1} \bigotimes \ldots \mathcal{E}_{e_{z+1}, 1}$, we obtain the identification $A_{e} \simeq \mathcal{E}_{e_0, e_{z+1}}$ at the level of graded vector spaces. We now show that this identification is compatible with the differentials.

The composite map

\[\bigoplus_{p_1 + ... + p_{z+N} = p'} A_{p_1, ..., p_{z+N}} \simeq A^{e'} \xrightarrow{\text{can}} A^{e'} \xrightarrow{\pi} \bigoplus_{p_1 + ... + p_{z+N} = p'} A_{[1, p_1] + [1, p_2], ..., [1, p_{z+N}]} \]

where the last map is the projection along the components indexed by the other (non-consecutive) partitions, is the canonical inclusion map. It follows that the map $\bigoplus_{p_1 + ... + p_{z+N} = p'} \mathcal{E}_{e_0, e_{z+1}} \rightarrow \bigoplus_{p_1 + ... + p_{z+N} = p'} A_{[1, p_1] + [1, p_2], ..., [1, p_{z+N}]}$ may be identified with the composite map

\[\bigoplus_{p_1 + ... + p_{z+N} = p'} A_{p_1, ..., p_{z+N}} \simeq A^{e'} \xrightarrow{\text{can}} A^{e'} \xrightarrow{d_{e, e'}^{e'} + 1} A^{e'} \xrightarrow{\pi} \bigoplus_{p_1 + ... + p_{z+N} = p'} A_{[1, p_1] + [1, p_2], ..., [1, p_{z+N}]} \]
Let now $Q_\alpha \in \mathcal{E}_{p_{\alpha} + 1} \simeq \mathcal{A}(x_1, \ldots, x_p)\delta_{\sigma_1, \ldots, \delta_{p-1}, p_{\alpha}, [1, p_{\alpha}]}$ and $Q := \otimes_\alpha Q_\alpha \in \mathcal{A}_{[1, p_{\alpha} + 1]}$. The image of this element in $\mathcal{A}'$ is $(\sum_{\sigma} \delta_{\sigma_1, \ldots, \delta_{p-1}, p_{\alpha}, [1, p_{\alpha}]}) \epsilon(\sigma) \sigma \ast Q$. The summand $\epsilon(\sigma) \sigma \ast Q$ belongs to $\mathcal{A}_{[1, p_{\alpha} + 1]}$, where $p_{\alpha} := \sigma(p_1 + \ldots + p_{\alpha} - 1 + [1, p_{\alpha}])$.

Decompose $d_{\delta_{p'}} \ast Q$ as a sum $d_{\delta_{p'}} \ast Q = \sum_{1 \leq i < j \leq p'} d_{\delta_{p'}} + \sum_{1 \leq i = 1}^{\sum_{(1, z + N]} \delta_{\alpha}$. If $\cup_\gamma p_\alpha = [1, p']$, then $\hat{d}_{\delta_{p'}}(\mathcal{A}_{p_{\alpha} + 1}) \subset \mathcal{A}_{[1, p_{\alpha} + 1]}$ and $\hat{d}_{\delta_{p'}}(\mathcal{A}_{p_{\alpha} + 1}) \subset \mathcal{A}_{[1, p_{\alpha} + 1]}(\mathcal{A}_{p_{\alpha} + 1} \subset \mathcal{A}_{[1, p_{\alpha} + 1]}(\mathcal{A}_{p_{\alpha} + 1}

where

- $(P_{ij}, \ldots, P_{jN})$ is given by $P_{ij} := [(P_\alpha \cap [2, i]) - 1] \cup ((P_\alpha \cap [j, p])] + 1)$ if $1 \notin P_\alpha$, and the union of the same set with $(i, j)$ if $1 \in P_\alpha$;

- $(P_{i, \ldots, P_{N+1}})$ is given by $P_{i, \ldots, P_{N+1}} = (P_\gamma \cap [1, i - 1]) \cup ((P_\gamma \cap [j, p')] + 1)$ if $\gamma \neq \alpha$, and the union of the same set with $(i, j)$ if $\gamma = \alpha$.

Note that the sequences $(|P_{ij}|, \ldots, |P_{jN}|)$ and $(|P_{i, \ldots, P_{N+1}}|)$ are necessary of the form $(p_1, \ldots, p_{jN}) := (p_1 + \delta_{1, \beta}, p_{i, \ldots, p_{N+1}} + \delta_{i, \beta, \gamma})$, where $\beta \in [1, z + N]$ is the index such that $1 \in P_\alpha$ in the first case, and $\alpha$ in the second case. Then:

(a) for any $i, j (1 \leq i < j \leq p' + 1)$ and any $\beta \in [1, z + N]$, $(P_{ij}, \ldots, P_{jN})$ coincides with $(|P_1|, \ldots, p_{i}^\beta + \ldots + p_{jN}^\beta + 1)$ if $P_\alpha = P_{\gamma} \cap [1, i - 1] \cup (P_\gamma \cap [j, p'] + 1]$; and $P_\alpha = P_{\gamma} \cap [1, i - 1] \cup (P_\gamma \cap [j, p'] + 1]$ for $\beta > \alpha$, and $i, j \in P_{\gamma} \cap [1, j + 1]$;

(b) for $i \in [1, p' + 1]$ and $\alpha \in [1, z + N]$, $(P_{i, \ldots, P_{N+1}})$ coincides with $(|P_1|, \ldots, p_{i}^\beta + \ldots + p_{jN}^\beta + 1]$ if $P_\alpha = P_{\gamma} \cap [1, i - 1] \cup (P_\gamma \cap [j, p'] + 1]$ for any $\alpha$ and $i \in P_{\gamma} \cap [1, i - 1] \cup (P_\gamma \cap [j, p'] + 1]$.

If $i, j$ are such that $1 \leq i < j \leq p' + 1$, then the condition on $\alpha \in \mathcal{E}_{p_{\alpha} + 1}$. For the support $\hat{d}_{\delta_{p'}}(\epsilon(\sigma) \ast Q)$ to consist in a consecutive partition of $[1, p' + 1]$ is therefore: there exists $\beta \in [1, z + N]$ such that $i, j \in P_{\gamma} \cap [1, i - 1] \cup (P_\gamma \cap [j, p'] + 1]$, and $\sigma$ is the shuffle permutation taking $[1, p_{\alpha}]$ to the partition described in (a) above.

If $i \in [1, p' + 1]$ and $\alpha \in [1, z + N]$, then the condition on $\sigma \in \mathcal{E}_{p_{\alpha} + 1}$ for the support of $\hat{d}_{\delta_{p'}}(\epsilon(\sigma) \ast Q)$ to consist in a consecutive partition of $[1, p' + 1]$ is therefore: $\sigma = i$ and $i \in P_{\gamma} \cap [1, i - 1] \cup (P_\gamma \cap [j, p'] + 1]$. In the first case, we have $\epsilon(\sigma) = (-1)^{p_{\alpha} + \ldots + p_{\beta} - 1}$ and $\pi \circ \hat{d}_{\delta_{p'}}(Q) = Q_1 \otimes \ldots \otimes Q_{z + N}$; in the second case, $\pi \circ \hat{d}_{\delta_{p'}}(Q) = (-1)^{p_{\alpha} + \ldots + p_{\beta} - 1} Q_1 \otimes \ldots \otimes Q_{z + N}$. Here $\hat{d}_{\delta_{p'}}(x_{i' + \ldots + p_{\beta} - 1}, [1, p_{\alpha}]) \rightarrow \mathcal{A}(x_1, \ldots, x_{p'+1})_{p_{\alpha} + \ldots + p_{\beta} + 1}$ is decomposed as $\hat{d}_{\delta_{p'}}(x_{i' + \ldots + p_{\beta} + 1}) = \sum_{p_{i' + \ldots + p_{\beta} + 1}}^{p_{i' + \ldots + p_{\beta} + 1}} (Q_1 \otimes \ldots \otimes Q_{z + N})$.

Then $\pi \circ d_{\delta_{p'}}(Q) = \sum_{\beta = 1}^{z + N} (-1)^{p_{\alpha} + \ldots + p_{\beta} - 1} Q_1 \otimes \ldots \otimes Q_{z + N}$, which proves our claim.

\[ \square \]

**Proposition 4.1.** The complexes $E_{0,1}$ and $E_{*,0}$ are acyclic; moreover, for $\epsilon \in \{0, 1\}$, $H^{p'}(E_{*,\epsilon})$ is zero for any $p' \neq 0$ and $k$ for $p' = 0$.

**Proof.** If $u_1, \ldots, u_n$ are free variables, let $k = A_{i} \subset C_{i} \subset A_{i} \subset \mathcal{A}(u_1, \ldots, u_n)$ be the increasing PBW filtration of $\mathcal{A}(u_1, \ldots, u_n)$, induced by its identification with $U(L(u_1, \ldots, u_n))$. The symmetrization isomorphism $\mathcal{A}(u_1, \ldots, u_n) \simeq S(L(u_1, \ldots, u_n))$ identifies $A_{i} \subset \mathcal{A}(u_1, \ldots, u_n)$ with $\otimes_{i'=i}^j S'(L(u_1, \ldots, u_n))$. The graded space associated to this filtration is the free Poisson algebra $\mathcal{P}(u_1, \ldots, u_n) = S(L(u_1, \ldots, u_n))$; its degree $i$ part is $\mathcal{P}[i](u_1, \ldots, u_n) = S'(L(u_1, \ldots, u_n))$. 


Define a filtration on $E^*_e$ by $F_u(E^*_e) := A_≤(x_1, ..., x_{p'})(1, p')$ for $u ≥ 0$. If $E(x_1, ..., x_{p'}) ∈ A_≤(x_1, ..., x_{p'})(1, p')$, then: $E([x_i, x_j], x_1, ..., x_{i-1}, x_{i+1}, ..., x_{p'} + 1) ∈ A_≤(x_1, ..., x_{p'} + 1)(1, p' + 1)$

while $[x_i, E(x_1, ..., x_{p'} + 1)] ∈ A_≤(x_1, ..., x_{p'} + 1)(1, p' + 1)$. It follows that for $ε ∈ \{0, 1\}$, we have

$$d_{e,ε}^{p'}(F_u(E^*_e)) = F_u(E^*_e)^{p'}$$

while for $ε ≠ ε'$ in $\{0, 1\}$,

$$d_{e,ε}^{p', p'}(F_u(E^*_e')) = F_{u+1}(E^{p'}_{e, ε'})$$

The associated graded complex is $P_{e, ε'}$, where

$$(P_{e, ε'}) = \mathcal{P}(x_1, ..., x_{p'})(1, p') = \oplus_{u≥0} \mathcal{P}[u](x_1, ..., x_{p'})(1, p'),$$

with differential

$$\text{gr} d_{e, ε'}^{p', p'} : \mathcal{P}_{e, ε'}^{p'} → \mathcal{P}_{e, ε'}^{p', p'}$$

given by

$$\text{gr} d_{e, ε'}^{p', p'}(x_1, ..., x_{p'} + 1) := \sum_{1 ≤ i < j ≤ p'} (-1)^{i+j+1} P([x_i, x_j], x_1, ..., x_{i-1}, x_{i+1}, ..., x_{j-1}, x_{j+1}, ..., x_{p'} + 1)$$

for $ε ∈ \{0, 1\}$,

$$\text{gr} d_{0, 0}^{p', p'}(x_1, ..., x_{p'} + 1) := \sum_{i=1}^{p' + 1} (-1)^{i+1} x_i P(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{p'} + 1),$$

and $\text{gr} d_{0, 1}^{p', p'} = 1 - \text{gr} d_{1, 0}^{p, p'}$ (when $ε = ε'$, the commutators give rise to brackets in the associated graded differential, while if $ε ≠ ε'$, the only part of the differential with nontrivial contribution to the associated graded differential is the second line of (20)). The differentials $\text{gr} d_{e, ε'}^{p', p'}$ have degree 0, and the differentials $\text{gr} d_{e, ε'}^{p', p'}$ have degree 1 (if $ε ≠ ε'$) with respect to the N-grading on $P_{e, ε'}$, induced by (21). We therefore have direct sum decompositions

$$P_{e, ε'} = \oplus_{u∈\mathbb{Z}} P_{e, ε'}^*[u], \quad P_{e, ε'} = \oplus_{u∈\mathbb{N}} P_{e, ε'}^*[u] \quad (\text{if } ε ≠ ε'),$$

where for any $ε, ε'$, we set $P_{e, ε'}^*[u] := P[u](x_1, ..., x_{p'})(1, p')$ and $P_{e, ε'}^*[u] = P_{e, ε'}^*[u + p']$.

**Lemma 4.7.** For $n, u ≥ 0$, $P_{e, ε'}^*[u]$ have the following values:

- if $n = 2m$, $P_{e, ε'}^{2m}[m]$ is 1-dimensional, spanned by

$$P_{2m}(x_1, ..., x_{2m}) := \sum_{σ ∈ g_2, ..., 2} ε(σ)\{x_{σ(1)}, x_{σ(2)}\}^{σ_1}x_{σ(1), x_{σ(2)}}, x_{σ(2m-1), x_{σ(2m)}}$$

and $P_{e, ε'}^{2m}[u] = 0$ for $u ≠ m$;

- if $n = 2m + 1$, $P_{e, ε'}^{2m+1}[m + 1]$ is 1-dimensional, spanned by

$$P_{2m+1}(x_1, ..., x_{2m+1}) := \sum_{σ ∈ g_1, ..., 2} ε(σ)x_{σ(1)}\{x_{σ(2)}, x_{σ(3)}\}^{σ_1}x_{σ(2), x_{σ(3)}}, x_{σ(2m), x_{σ(2m+1)}}$$

and $P_{e, ε'}^{2m+1}[u] = 0$ for $u ≠ m + 1$. 
Proof of Lemma. As the category of $\mathfrak{g}_n$-modules is semisimple, the $\mathfrak{g}_n$-modules $A(x_1, ..., x_n)|_{[1,n]}$ and $P(x_1, ..., x_n)|_{[1,n]}$ are equivalent. It follows that $P(x_1, ..., x_n)|_{[1,n]}$ is 1-dimensional. Since this space is equal to $\oplus_{n \geq 0} P[u]|_{[1,n]}$, it follows that exactly one of these summands is 1-dimensional, and the others are zero. It then remains to prove that $p_n \in P_{e\epsilon'}[[n(n + 1)/2]]$ and $p_n \neq 0$, where $[x]$ is the integral part of $x$.

If $n = 2m$, we have $p_{2m}(x_1, ..., x_{2m}) = 2^{-m} \sum_{\sigma \in S_{2m}} \epsilon(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}\} ... \{x_{\sigma(2m-1)}, x_{\sigma(2m)}\}$, so $p_{2m}$ is $\mathfrak{g}_n$-antiinvariant; and if $\Gamma$ is the set of $\sigma \in S_{2m}$, such that $\sigma(1) < \sigma(3) < ... < \sigma(2m - 1)$ and $\sigma(2i + 1) < \sigma(2i + 2)$ for $i = 0, ..., m - 1$ (this identifies with the set of partitions of $[1, 2m]$ in subsets of cardinality 2, modulo permutation of the subsets), we have $p_{2m}(x_1, ..., x_{2m}) = m! \sum_{\sigma \in \Gamma} \epsilon(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}\} ... \{x_{\sigma(2m-1)}, x_{\sigma(2m)}\}$, and as the summands in this expression are linearly independent, $p_{2m} \neq 0$.

If $n = 2m + 1$, we have similarly

$p_{2m+1}(x_1, ..., x_{2m+1}) = 2^{-m} \sum_{\sigma \in S_{2m+1}} \epsilon(\sigma)x_{\sigma(1)}\{x_{\sigma(2)}, x_{\sigma(3)}\} ... \{x_{\sigma(2m)}, x_{\sigma(2m+1)}\}$

which implies that $p_{2m+1}$ is $\mathfrak{g}_n$-antiinvariant, and

$p_{2m+1}(x_1, ..., x_{2m+1}) = m! \sum_{\sigma \in \Gamma} \epsilon(\sigma)x_{\sigma(1)}\{x_{\sigma(2)}, x_{\sigma(3)}\} ... \{x_{\sigma(2m)}, x_{\sigma(2m+1)}\}$

where $\Gamma$ is the set of permutations $\sigma \in S_n$ such that $\sigma(2) < \sigma(4) < ... < \sigma(2m)$ and $\sigma(2i) < \sigma(2i + 1)$ for $i = 1, ..., m$, which implies that $p_{2m+1}$ is nonzero, as the summands in this expression are linearly independent.

End of proof of Proposition [4.7]. For $u \in \mathbb{Z}$, the complex $P_{0,1}^*[u]$ is $0 \to P_{0,1}^0[u] \to P_{0,1}^1[u + 1] \to ...$. For $u > 0$, the groups of this complex are all zero, so $P_{0,1}^*[u]$ is acyclic. For $u \leq 0$, this complex is $0 \to ... \to 0 \to P_{0,1}^0[u] \to P_{0,1}^1[u + 1] \to ...$, where $m = -u$. The nontrivial map in this complex is $p_{2m} \mapsto d_{2m+1}^0(p_{2m}) = p_{2m+1}$, which is an isomorphism, so $P_{0,1}^*[u]$ is acyclic. It follows that $P_{0,1}^*$ is acyclic. As the differential of $P_{1,0}^*$ is the negative of that of $P_{0,1}^*$, $P_{1,0}^*$ is acyclic as well.

Let $\epsilon \in \{0, 1\}$ and $u \in \mathbb{N}$. The complex $P_{\epsilon,\epsilon'}^*[u]$ is $0 \to P_{\epsilon,\epsilon'}^0[u] \to P_{\epsilon,\epsilon'}^1[u] \to ...$; if $i = u$, this complex is $0 \to k \to 0 \to ...$, whose cohomology is 1-dimensional, concentrated in degree 0; if $u > 0$, this complex is $0 \to ... \to 0 \to P_{\epsilon,\epsilon'}^{2u}[u] \to P_{\epsilon,\epsilon'}^{2u+1}[u] \to 0 \to ...$; the nontrivial map in this complex is $p_{2m-1} \mapsto d_{2u+1}^{m-1}(p_{2m-1}) = u_{2u}$, where $u_{2u} = 1$. As this is an isomorphism in both cases, $P_{\epsilon,\epsilon'}^*[u]$ is acyclic for $u > 0$. It follows that the cohomology of $P_{\epsilon,\epsilon'}^*$ is 1-dimensional, concentrated in degree 0.

This implies that $E_{\epsilon,\epsilon'}^*$ is acyclic for $\epsilon \neq \epsilon'$, and that the cohomology of $E_{\epsilon,\epsilon'}^*$ is concentrated in degree 0. As $d_{0,1}^{0,1} = 0$, we have in degree 0, $H^0(E_{\epsilon,\epsilon'}^*) = E_{\epsilon,\epsilon'}^0 = k$.

Remark 4. If $\mathfrak{g}$ is a Lie algebra, we have natural maps

$$H^*(E_{\epsilon,\epsilon'}^*) \to H^*(\mathfrak{g}, U(\mathfrak{g}); e, \epsilon')$$

When $(\epsilon, \epsilon') = (0, 1)$ and $\mathfrak{g}$ is finite dimensional, then $H^n(\mathfrak{g}, U(\mathfrak{g}); 0, 1) = k$ if $n = \dim \mathfrak{g}$, and = 0 otherwise. Indeed, if $C^n(\mathfrak{g}) := \bigwedge^n(\mathfrak{g}) \otimes U(\mathfrak{g})$, then the differential $d^{n+1}_n = C^n(\mathfrak{g}) \to C^{n+1}(\mathfrak{g})$ is given by $\omega \otimes x \mapsto \delta(\omega) \otimes x + \sum_{n=1}^{\dim \mathfrak{g}} (\omega \wedge e_a) \otimes (e_a)_{\alpha}$, where $(e_a)_{\alpha}, (e_a)_{\alpha}$ are dual bases of $\mathfrak{g}^*$ and $\mathfrak{g}$ and $\delta : \bigwedge^n(\mathfrak{g}^*) \to \bigwedge^{n+1}(\mathfrak{g}^*)$ is induced by the Lie coalgebra structure of $\mathfrak{g}^*$. For $i \in \mathbb{Z}$, set $F_i(C^n(\mathfrak{g})) := \bigwedge \cap U(\mathfrak{g}) \leq \mathfrak{g}$, (where the last term is the subspace of elements of degree $\leq n + i$ for the PBW filtration). Then $d_{1,0}^{n+1} F_i(C^n(\mathfrak{g})) \subset F_i(C^{n+1}(\mathfrak{g}))$, so $\cdots \subset F_i(C^n(\mathfrak{g})) \subset \cdots \subset C^n(\mathfrak{g})$ is a complete filtration of $C^n(\mathfrak{g})$. The associated graded complex is $\bar{C}^n(\mathfrak{g}) := \bigwedge^{n+1}(\mathfrak{g}) \otimes S(\mathfrak{g})$, with differential $d^{n+1}_n : \bar{C}^{n+1}(\mathfrak{g}) \to \bar{C}^{n+1}(\mathfrak{g})$, $\omega \otimes x \mapsto \sum_{n=1}^{\dim \mathfrak{g}} (\omega \wedge e_a) \otimes (e_a)_{\alpha}$. 


This complex only depends on the vector space structure of \( g \); if we denote it by \( \tilde{C}^*(g) \), then we have an isomorphism \( \tilde{C}^*(g_1 \oplus g_2) \simeq \tilde{C}^*(g_1) \otimes \tilde{C}^*(g_2) \), so \( C^*(g) \simeq C^*(k)^{\otimes \dim g} \). As the cohomology of \( C^*(k) \) is 1-dimensional, concentrated in degree 1, the cohomology of \( C^*(g) \) is 1-dimensional, concentrated in degree \( \dim g \). It follows that \( C^*(g) \) is acyclic in every degree \( \neq \dim g \), and its cohomology in degree \( \dim g \) has dimension \( \leq 1 \). If \( \omega \in \wedge^{\dim g} g \) is nonzero, then \( \omega \otimes 1 \in C^n(g) \) is a nontrivial cocycle, so the cohomology of \( C^*(g) \) coincides with that of \( \tilde{C}^*(g) \). As \( U(g)_{1,0} \simeq U(g)_{0,1} \) (using the antipode), we have \( \tilde{H}^*(g, U(g)) \simeq H^*(g, U(g)) \).

When \( \epsilon \neq \epsilon' \), \((\tilde{\epsilon})\) is the zero map. If \( \epsilon = \epsilon' \), then the map \( k = H^0(\tilde{E}^*_\epsilon) \to H^0(g, U(g)) \) takes 1 to the class of \( 1 \in U(g)_{\epsilon, \epsilon} \) (which is invariant, both under the trivial and the adjoint actions of \( g \) on \( U(g) \)). 

\[ \square \]

5. Compatibility of quantization functors with twists

We first prove the compatibility of quantization functors of quasi-Lie bialgebras with twists of quasi-Lie bialgebras; we derive from there the compatibility of quantization functors of Lie bialgebras with twists of Lie bialgebra (a result which was obtained in [BH] in the case of Etingof-Kazhdan quantization functors).

5.1. Twists of quasi-Lie bialgebras. Let \( \text{QLBA}_f \) be the prop with the same generators as \( \text{QLBA} \) with the additional \( f \in \text{QLBA}_f(\wedge^2, \text{id}) \) and the same relations. This prop is \( \mathbb{N} \)-graded, if we extend the degree \( \deg_{\mu} + \deg_{\phi} \) in \( \text{QLBA} \) by \( |f| = 1 \).

We then have \( \text{QLBA}_f(X, Y) = \text{QLBA}(S(\wedge^2) \otimes X, Y) \). The filtration of \( \text{QLBA}_f \) induced by the degree is such that

\[ \text{QLBA}^{\geq n}(X, Y) = \oplus_{k \geq 0} \text{QLBA}^{\geq n-k}(S^k(\wedge^2) \otimes X, Y). \]

It follows that \( \text{QLBA}_f(X, Y) \subset \text{QLBA}^{\geq v_f(|X|, |Y|)}(X, Y) \), where \( v_f(|X|, |Y|) = \inf \{ v(|X| + 2k, |Y|) + k, k \geq 0 \} \) and \( v(|X|, |Y|) = \frac{1}{2}|X| - |Y| \). As \( v_f(|X|, |Y|) \geq v(|X|, |Y|) \), \( \text{QLBA}_f \) gives rise to a topological prop \( \text{QLBA}_f \).

We have two prop morphisms \( \kappa_i : \text{QLBA} \to \text{QLBA}_f \), defined by

\[ \kappa_1 : \mu, \delta, \phi \mapsto \mu, \delta, \phi, \]

\[ \kappa_2 : \mu \mapsto \mu, \delta \mapsto \delta + \text{Alt}_2 \circ (\text{id}_\text{id} \otimes \mu) \circ (f \otimes \text{id}_\text{id}), \]

\[ \phi \mapsto \phi + \frac{1}{2} \text{Alt}_3 \circ (\delta \otimes \text{id}_\text{id}) \circ f + (\text{id}_\text{id} \otimes \mu \otimes \text{id}_\text{id}) \circ (f \otimes f) \];

this is the universal version of the operation of twisting of a quasi-Lie bialgebra structure. The prop morphisms \( \kappa_i \) extend to topological props.

Let \( (m, \Delta, \Phi, \epsilon, \eta) \) be a QSB in \( \text{QLBA} \) (i.e., a quantization functor of quasi-Lie bialgebras).

For \( i = 1, 2 \), set \( (m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i) := \kappa_i(m, \Delta, \Phi, \epsilon, \eta) \). Then \( (m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i) \) are QSB’s in \( \text{QLBA}_f \).

Proposition 5.1. The QSB’s \( (m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i) \) are related by equivalence and twist, i.e., there exists \( F \in \text{QLBA}_f(1, S^{\otimes 2}) \) and \( i \in \text{QLBA}_f(S, S) \) such that \( (\epsilon \otimes \text{id}_S) \circ F = (\text{id}_S \otimes \epsilon) \circ F = \eta \), \( F = \text{id}_{S^{\otimes 2}} \) and \( i \equiv \text{id}_S \mod \text{QLBA}_f^{\otimes 1} \), such that \( (m_2, \Delta_2, \Phi_2, \epsilon_2, \eta_2) = \epsilon \ast \{ F \ast (m_1, \Delta_1, \Phi_1, \epsilon_1, \eta_1) \} \).
This implies that the quantization functors of quasi-Lie bialgebras take a pair of quasi-Lie bialgebras related by a classical twist to a pair of quasi-Hopf QUE algebras related by a quantum twist.

**Proof.** As \((\mu, \delta, \varphi)\) defines a quasi-Lie bialgebra structure on \(Q\text{LBA}_f\), we have \([\mu \oplus \delta \oplus \varphi, \mu \oplus \delta \oplus \varphi] = 0\) (in \(Q\text{LBA}_f\)), so the bracket with \(\mu \oplus \delta \oplus \varphi\) defines a complex structure on \(Q\text{LBA}_f\). This gives rise to cohomology groups \(H^i_{Q\text{LBA}_f}\), which are graded by the grading of \(Q\text{LBA}_f\).

**Lemma 5.1.** The canonical maps \(H^i_{Q\text{LBA}_f} \to H^i_{\text{LBA}}\) are isomorphisms.

**Proof of Lemma.** As before, we will show that the relative complex is acyclic. This relative complex is filtered by the powers of the ideal \((\varphi)_f\) of \(Q\text{LBA}_f\). The associated graded prop of \(Q\text{LBA}_f\) w.r.t. this filtration is \(\text{LBA}_{\alpha,f}\) defined by \(\text{LBA}_{\alpha,f}(X,Y) = \text{Coker}[\text{LBA}(S(\wedge^2) \otimes S(\wedge^3) \otimes S(\wedge^4) \otimes X, Y) \to \text{LBA}(S(\wedge^2) \otimes S(\wedge^3) \otimes S(\wedge^4) \otimes X, Y)]\), induced by the morphism \(S(\wedge^2) \otimes S(\wedge^3) \to S(\wedge^2) \otimes S(\wedge^3) \otimes S(\wedge^4) \otimes X, Y)\). The associated graded complex of the relative complex is the positive degree part of \(C_{\text{LBA}_{\alpha,f}}\) w.r.t. the degree on \(\text{LBA}_{\alpha,f}\) induced by the filtration, i.e., for \(k \geq 0\), the degree \(k\) part of \(\text{LBA}_{\alpha,f}(X,Y)\) is \(\text{Coker}[\text{LBA}(S(\wedge^2) \otimes S^{k-1}(\wedge^3) \otimes \wedge^4) \otimes X, Y) \to \text{LBA}(S(\wedge^2) \otimes S^{k-1}(\wedge^3) \otimes \wedge^4) \otimes X, Y)]\), by convention \(S^{-1}(\wedge^3) = 0\).

For \(k > 0\), the degree \(k\) part of the associated graded complex has the form \(C^p_{Q\text{LBA}_f}(k) = \text{Coker}[\text{LBA}(S(\wedge^2) \otimes S^{k-1}(\wedge^3) \otimes \wedge^4) \otimes \wedge^{p+1}, \wedge^{q+1}) \to \text{LBA}(S(\wedge^2) \otimes S^{k}(\wedge^3) \otimes \wedge^{p+1}, \wedge^{q+1})]\) \((p, q \geq -1)\), equipped with the differential \([\mu \oplus \delta, -]\). For \(k > 0\), both \(S(\wedge^2) \otimes S^{k-1}(\wedge^3) \otimes \wedge^4\) and \(S(\wedge^2) \otimes S^{k}(\wedge^3) \otimes \wedge^4\) are sums of Schur functors of positive degrees, so Theorem 4.1 implies that the lines of \(C^p_{Q\text{LBA}_f}(k)\) are acyclic, so for each \(k > 0\), this complex is acyclic. So the relative complex \(\text{Ker}(C_{Q\text{LBA}_f} \to C_{\text{LBA}})\) is acyclic.

**End of proof of Proposition 5.1.** Recall that we have a prop morphism \(\pi : Q\text{LBA} \to \text{LBA}\) defined by \(\pi : \mu, \delta, \varphi \mapsto \mu, \delta, 0\). We also have a prop morphism \(\pi_f : Q\text{LBA}_f \to \text{LBA}_f\), defined by \(\pi_f : \mu, \delta, \varphi, f \mapsto \mu, \delta, 0, 0\). These morphisms extend to topological props and satisfy \(\pi_f \circ \kappa_i = \kappa\) for \(i = 1, 2\).

Then \(\pi_f(m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i) = \pi_f \circ \kappa_i(m, \Delta, \Phi, \epsilon, \eta) = \pi(m, \Delta, \Phi, \epsilon, \eta)\) for \(i = 1, 2\). The classical limits of \((m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i)\) are \((\mu, \delta, Z)\) for \(i = 1, 2\) (as the additional terms \(\text{Alt}_2 \circ ([\text{id}_{\wedge^4} \otimes \mu]) \circ (f \otimes \text{id}_{\wedge^4}) + \frac{1}{2} \text{Alt}_2 \circ ([\delta \otimes \text{id}_{\wedge^4}] \circ f + ([\text{id}_{\wedge^4} \otimes \mu \otimes \text{id}_{\wedge^4}] \circ (f \otimes f)]\) have degree > 1). As in the proof of Theorem 2.1 (Subsection 2.5), Lemma 5.1 implies that two QSB’s in \(Q\text{LBA}_f\) with the same image in \(\text{LBA}\) and the same classical limit are related by twist and equivalence. It follows that \((m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i)\) \((i = 1, 2)\) are related by twist and equivalence.

**5.2. Twists of Lie bialgebras.** In [EH], we introduced the prop \(\text{LBA}_f\) of Lie bialgebras with a twist: its generators are \(\mu, \delta, f \in \text{LBA}_f(1, \wedge^2)\) and its relations as the same as the relations of LBA, together with \(\text{Alt}_2 \circ ([\delta \otimes \text{id}_{\wedge^4}] \circ f + ([\text{id}_{\wedge^4} \otimes \mu \otimes \text{id}_{\wedge^4}] \circ (f \otimes f)]\) = 0. We defined prop morphisms \(\kappa^0_i : \text{LBA} \to \text{LBA}_f\) by \(\kappa_1 : \mu, \delta \mapsto \mu, \delta\) and \(\kappa_2 : \mu, \delta \mapsto \mu, \delta + \text{Alt}_2 \circ ([\text{id}_{\wedge^4} \otimes \mu] \circ (f \otimes \text{id}_{\wedge^4})\).

Let \((m_0, \Delta_0, \epsilon_0, \eta_0)\) be a QSB in \(\text{LBA}\), quantizing \((\mu, \delta)\) (i.e., a quantization functor of Lie bialgebras). Set \((m_0^0, \Delta_0^0, \epsilon_0^0, \eta_0^0) := \kappa^0_1(m_0, \Delta_0, \epsilon_0, \eta_0)\). These are QSB’s in \(\text{LBA}_f\).

**Proposition 5.2.** The QSB’s \((m_i^0, \Delta_i^0, \epsilon_i^0, \eta_i^0)\) are related by a and equivalence and a bialgebra twist. More explicitly, there exists \(F_0 \in \text{LBA}_f(1, S^2)\) and \(i_0 \in \text{LBA}_f(S, S)\), whose reduction
mod $\mathrm{LBA}_f^> \geq_1$ is $\mathrm{inv}_{>0}^\otimes$ and $\mathrm{id}_S$, such that $(\epsilon_0 \otimes \mathrm{id}_S) \circ F_0 = (\mathrm{id}_S \otimes \epsilon_0) \circ F_0 = \eta_0$, $[\varsigma_0 \otimes \eta_0, F_0] * ([\varsigma_0 \otimes \Delta_0] \circ F_0) = [F_0 \otimes \eta_0] * ([\Delta_0 \otimes \varsigma_0] \circ F_0)$, and $[\varsigma_0 \otimes \Delta_0, \epsilon_0] = [\varsigma_0 \otimes \Delta_0, F_0] = [\varsigma_0 \otimes \Delta_0, F_0] = [\varsigma_0 \otimes \Delta_0, \epsilon_0]$.}

**Proof.** There are uniquely defined prop morphisms $\tilde{\pi}_f : \mathrm{QLBA}_f \to \mathrm{LBA}_f$ and $\pi^0_f : \mathrm{LBA}_f \to \mathrm{LBA}$, such that $\tilde{\pi}_f : \mu, \delta, \varphi, f \mapsto \mu, \delta, 0, f$ and $\pi^0_f : \mu, \delta, f \mapsto \mu, \delta, 0$. We then have $\mathrm{id}_{\mathrm{LBA}} = \pi^0_f \circ \varsigma_0$ for $i = 1, 2$, $\kappa^0_i \circ \pi = \tilde{\pi}_f \circ \kappa_i$, $\pi_f = \pi^0_f \circ \tilde{\pi}_f$. All the information on compositions can be summarized in the commutative diagram $\begin{array}{ccc} \mathrm{QLBA}_f & \xrightarrow{\kappa_f} & \mathrm{LBA} \\ \downarrow \pi & & \downarrow \pi_f \\ \mathrm{LBA} & \xrightarrow{\kappa_f^0} & \mathrm{LBA}_f \end{array}$ together with the relations $\pi = \pi_f \circ \kappa_{1,2}$, $\mathrm{id}_{\mathrm{LBA}} = \pi^0_f \circ \kappa_{1,2}$. These morphisms induce morphisms between completed props.

According to Theorem 2.1 we may lift $(m_0, \Delta_0, \epsilon_0, \eta_0)$ to a QSQB $(m, \Delta, \Phi, \epsilon, \eta)$ in $\mathrm{QLBA}$ with classical limit $(\mu, \delta, \varphi)$; this means that $\pi (m, \Delta, \Phi, \epsilon, \eta) = (m_0, \Delta_0, \epsilon_0, \eta_0)$. Set $(m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i) := \kappa_i(m, \Delta, \Phi, \epsilon, \eta)$, then $\pi (m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i) = (m_0^i, \Delta_0^i, \epsilon_0^i, \eta_0^i)$. Let then $i, F$ be as in Proposition 3.1. Let $i_0 := \tilde{\pi}_f(i)$, $F_0 := \tilde{\pi}_f(F)$. As $\Phi$ is a twisting relation of $\Phi_1$ and $i^{-1} \Phi_2$, and as the images of $\Phi_1, i^{-1} \Phi_2$ under $\tilde{\pi}_f$ are $\eta_0^3$, $F_0$ is a twisting relation of $\tilde{\pi}_f$ with itself, i.e., it satisfies the announced cocycle relation. The image under $\tilde{\pi}_f$ of the statement that the $(m_i, \Delta_i, \Phi_i, \epsilon_i, \eta_i)$ are related by $(i, F)$ is then that the $(m_i^0, \Delta_0^i, \epsilon_i^0, \eta_0^i)$ are related by $(i_0, F_0)$. \hfill \Box

**APPENDIX A. STRUCTURE OF THE PROP LBA**

The following structure theorem of the prop LBA was proved in [3] [Pon]. We reformulate here this proof using the language of props. In [3], we derived Proposition 3.2 from Theorem A.1 below.

**Theorem A.1.** If $F, G \in \mathrm{Ob}(\mathcal{Sch})$, then the map $\oplus_{Z \in \mathrm{Irr}(\mathcal{Sch})} \mathrm{LCA}(F, Z) \otimes \mathrm{LA}(Z, G) \to \mathrm{LBA}(F, G)$ induced by composition and the prop morphisms $\mathrm{LCA} \to \mathrm{LA}$, $\mathrm{LBA} \to \mathrm{LA}$ is a linear isomorphism.

**Proof.** It suffices to prove this when $F, G \in \mathrm{Irr}(\mathcal{Sch})$, and then (using the action of $\mathfrak{S}_n, \mathfrak{S}_m$) for $F = T_n$, $G = T_m$. Using the cocycle relation, and the isomorphism of the l.h.s. with $\oplus_{z \geq 0} \mathrm{LCA}(T_n, T_z) \otimes \mathrm{LA}(T_z, T_m)\mathfrak{S}_z$, one proves that the morphism is surjective. We now prove that it is injective. We have $\mathrm{LBA}(T_n, T_m) = \oplus_{a,b \geq 0} \mathbb{Z} \geq \min(n,m) \mathrm{LBA}(T_n, T_m)[a,b] = \oplus_{a \geq \min(n, m)} \mathrm{LBA}(T_n, T_m)[z - m, z - n]$, and the morphism is the direct of over $z \geq \min(n, m)$ of the maps $\oplus_{Z \in \mathrm{Irr}(\mathcal{Sch})} |Z| = z \mathrm{LCA}(T_n, Z) \otimes \mathrm{LA}(Z, T_m) \to \mathrm{LBA}(T_n, T_m)[z - m, z - n]$. It remains to show that each of the maps is injective.

There is a unique morphism $\mathrm{LBA} \to \mathcal{L}(\mathrm{LCA})$ (the generators of $\mathrm{LBA}$ are $\mu, \delta, \phi$, the generator of $\mathrm{LCA}$ is $\delta_{\mathrm{LCA}}$, taking $\mu$ to $\mu_{\mathrm{free}} : L^{\otimes 2} \to L$ and $\delta$ to the unique $\delta_{\mathrm{free}} : L \to L^{\otimes 2}$, such that $\mathrm{id} \to L^{\otimes 2} \xrightarrow{\delta_{\mathrm{free}}} L^{\otimes 2}$ is $\delta_{\mathrm{LCA}} : \mathrm{id} \xrightarrow{\delta_{\mathrm{LCA}}} L^{\otimes 2}$ and $\delta_{\mathrm{free}} = ((\mu_{\mathrm{free}} \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \beta_{L,L}) + \mathrm{id}_L \otimes \mu_{\mathrm{free}}) \circ (\delta_{\mathrm{free}} \otimes \mathrm{id}_L) + (\mu_{\mathrm{free}} \otimes \mathrm{id}_L + (\mathrm{id}_L \otimes \mu_{\mathrm{free}}) \circ (\beta_{L,L} \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \delta_{\mathrm{free}})$. The prop $\mathrm{LCA}$ is $\mathbb{Z}$-graded, with $\mathrm{deg} \delta_{\mathrm{LCA}} = 1$, then the morphism $\mathrm{LBA} \to \mathcal{L}(\mathrm{LCA})$ is compatible with the morphism $Z \to Z$, $(1, 0) \mapsto 0, (0, 1) \mapsto 1$.

We then have maps $\mathrm{LBA}(T_n, T_m) \to \mathcal{L}(\mathrm{LCA})(T_n, T_m) = \mathcal{LCA}(L^{\otimes n}, L^{\otimes m}) \to \mathcal{LCA}(T_n, L^{\otimes m})$, where the last map is induced by $\mathrm{id} \to L$, which restricts to $\mathrm{LBA}(T_n, T_m)[z - m, z - n] \to \mathcal{LCA}(T_n, L^{\otimes m})[z - n] = \mathcal{LCA}(T_n, (L^{\otimes m})_z)$, where the index $z$ denotes the (Schur functor) degree $z$ part.
Lemma A.1. If $X$ is any prop and $F \in \text{Ob}(\text{Sch})$, we have an isomorphism $X(F, (L^\otimes m)_z) \simeq \bigoplus_{Z \in \text{Irr}(\text{Sch})} X(F, Z) \otimes L\text{A}(Z, T_m)$.

Proof of Lemma. We have isomorphisms $L\text{A}(T_z, \text{id}) \simeq \text{multilinear part of the free Lie algebra in } z$ ordered generators $\simeq \text{Sch}(T_z, L_z)$. So if $|Z| = z$, $L\text{A}(Z, \text{id}) \simeq \text{Sch}(Z, L_z)$, which may be expressed as $L_z = \bigoplus_{|Z| = z} L\text{A}(Z, \text{id}) \otimes Z$.

So

$$X(F, (L^\otimes m)_z) = \bigoplus_{|Z| = z} X(F, \otimes_{i=1}^m L_z_i)$$

$$= \bigoplus_{|Z| = z} X(F, \otimes_{i=1}^m Z_i) \otimes (\otimes_{i=1}^m L\text{A}(Z_i, \text{id}))$$

$$= \bigoplus_{|Z| = z} X(F, Z) \otimes \text{Sch}(Z, \otimes_{i=1}^m Z_i) \otimes (\otimes_{i=1}^m L\text{A}(Z_i, \text{id}))$$

where the last equality follows from $L\text{A}(Z, T_m) = \bigoplus_{Z_1, \ldots, Z_m \in \text{Irr}(\text{Sch})} \prod_i |Z_i| = z \text{Sch}(Z, \otimes Z_i) \otimes \otimes_i L\text{A}(Z_i, \text{id})$, for $Z \in \text{Ob}(\text{Sch})$ (see [EH])

End of proof of Theorem. We have constructed a map $L\text{BA}(T_n, T_m)[z - m, z - n] \to \bigoplus_{Z \in \text{Irr}(\text{Sch})} X(F, Z) \otimes L\text{A}(Z, T_m)$, and one proves that is a section of the morphism $\bigoplus_{Z \in \text{Irr}(\text{Sch})} |Z| = z L\text{BA}(T_n, T_m) \otimes L\text{A}(Z, T_m) \to L\text{BA}(T_n, T_m)[z - m, z - n]$, which is therefore injective.

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