A MONOTONICITY FORMULA FOR MEAN CURVATURE FLOW WITH SURGERY

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ABSTRACT. We prove a monotonicity formula for mean curvature flow with surgery. This formula differs from Huisken’s monotonicity formula by an extra term involving the mean curvature.

1. INTRODUCTION

Our goal in this paper is to establish a monotonicity formula for mean curvature flow with surgery in dimension 3. For smooth solutions of the mean curvature flow in $\mathbb{R}^3$, there is a well-known monotonicity formula due to Huisken [9], which asserts that the Gaussian integral

$$\tau \mapsto \int_{M_t - \tau} \frac{1}{4\pi \tau} e^{-\frac{|x-p|^2}{4\tau}}$$

is monotone increasing in $\tau$. This monotonicity property can be extended to weak solutions of the mean curvature flow, and plays a crucial role in the analysis of singularities; see e.g. [3], [7], [14], [15], [16], [17], [18].

In a fundamental paper [10], Huisken and Sinestrari introduced a notion of mean curvature flow with surgery for two-convex hypersurfaces in $\mathbb{R}^{n+1}$, where $n \geq 3$. In a recent joint work with Gerhard Huisken [3], we extended this construction to the case $n = 2$, and defined a notion of mean curvature flow with surgery for mean convex surfaces in $\mathbb{R}^3$. The arguments in [3] rely, among other things, on a sharp estimate for the inscribed radius established earlier in [2]. We note that there are similar surgery constructions, due to Hamilton [6] and Perelman [12], [13] for the Ricci flow in Riemannian geometry.

In this paper, we establish a monotonicity formula which is valid in the presence of surgeries. Our formula is closed related to the one considered by Huisken; however, we have to include an extra term involving the mean curvature in order to guarantee that the monotonicity is unaffected by surgeries. Specifically, if $M_t$ is solution of mean curvature flow with surgery in $\mathbb{R}^3$, then the quantity

$$\tau \mapsto \int_{M_t - \tau} \frac{1}{4\pi \tau} e^{-\frac{|x-p|^2}{4\tau}} - \frac{H}{200 H_t}$$

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is monotone increasing for $\tau \geq \frac{5}{9} H_1^{-2}$. Here, $H_1$ is a positive constant which represents the so-called surgery scale; in other words, each neck on which we perform surgery has mean curvature between $H_1$ and $2H_1$.

In Section 2, we establish a number of auxiliary results. This results will be used in Section 3 to prove the main monotonicity formula for mean curvature flow with surgery in $\mathbb{R}^3$. In Sections 4 and 5, we extend the monotonicity formula to solutions of mean curvature flow with surgery in Riemannian three-manifolds. Finally, in Section 6 we apply our results to the analysis of the longtime behavior of mean curvature flow with surgery in Riemannian three-manifolds.

2. Behavior of a Gaussian integral under a single surgery

Lemma 2.1. There exists a real number $\beta > 0$ with the following significance. Suppose that $\Gamma$ is a curve in $\mathbb{R}^2$ which is $\beta$-close to the unit circle in the $C^1$-norm. Moreover, suppose that $\psi$ is a real-valued function defined on $\Gamma$ satisfying $\sup_{\Gamma} |\psi - 1| < \beta$. Then

$$\int_{\Gamma} \psi e^{-\frac{|x|^2}{4\tau} + \langle q, x \rangle} \left( \frac{10}{11} - \frac{|x|^2}{2\tau} + \langle q, x \rangle \right) \geq 0$$

for all $\tau \geq \frac{5}{9}$ and all $q \in \mathbb{R}^2$.

Proof. Let us fix a large constant $Q$ with the property that

$$e^{\frac{|q|^2}{2}} \int_{\Gamma \cap \{\langle q, x \rangle \geq \frac{|q|^2}{4}\}} \psi \geq 16 \int_{\Gamma \cap \{\langle q, x \rangle \leq 0\}} \psi$$

for all points $q \in \mathbb{R}^2$ satisfying $|q| \geq Q$. This implies

$$e^{\frac{|q|^2}{2}} \int_{\Gamma \cap \{\langle q, x \rangle \geq \frac{|q|^2}{4}\}} \psi e^{\frac{|x|^2}{4\tau} + \langle q, x \rangle} \geq 4 \int_{\Gamma \cap \{\langle q, x \rangle \leq 0\}} \psi e^{-\frac{|x|^2}{4\tau}}$$

for all $\tau \geq \frac{5}{9}$ and all $q \in \mathbb{R}^2$ satisfying $|q| \geq Q$. From this, we deduce that

$$\int_{\Gamma \cap \{\langle q, x \rangle \geq \frac{|q|^2}{4}\}} \psi e^{-\frac{|x|^2}{4\tau} + \langle q, x \rangle} \langle q, x \rangle \geq 2|q| \int_{\Gamma \cap \{\langle q, x \rangle \leq 0\}} \psi e^{-\frac{|x|^2}{4\tau}}$$

for all $\tau \geq \frac{5}{9}$ and all $q \in \mathbb{R}^2$ satisfying $|q| \geq Q$. Therefore, we obtain

$$\int_{\Gamma} \psi e^{-\frac{|x|^2}{4\tau} + \langle q, x \rangle} \langle q, x \rangle \geq 0,$$

hence

$$\int_{\Gamma} \psi e^{-\frac{|x|^2}{4\tau} + \langle q, x \rangle} \left( \frac{10}{11} - \frac{|x|^2}{2\tau} + \langle q, x \rangle \right) \geq 0.$$
for all $\tau \geq \frac{5}{9}$ and all $q \in \mathbb{R}^2$ satisfying $|q| \geq Q$. On the other hand, by choosing $\beta > 0$ sufficiently small, we can arrange that
$$
\int_\Gamma \psi e^{-\frac{|x|^2}{4\tau} + \langle q, x \rangle} \left( \frac{10}{11} - \frac{|x|^2}{2\tau} + \langle q, x \rangle \right) \geq 0
$$
for all $\tau \geq \frac{5}{9}$ and all $q \in \mathbb{R}^2$ satisfying $|q| \leq Q$. This proves the assertion.

**Lemma 2.2.** Let $\beta > 0$ be chosen as in Lemma 2.1. Suppose that $\Gamma$ is a curve in $\mathbb{R}^2$ which is $\beta$-close to the unit circle in the $C^1$-norm. Moreover, suppose that $\psi$ is a real-valued function defined on $\Gamma$ satisfying $\sup_{\Gamma} |\psi - 1| < \beta$. Then
$$
\rho^{10} \int_\Gamma \psi e^{-\frac{\rho^2}{4\tau} + \rho \langle q, x \rangle} \leq \int_\Gamma \psi e^{-\frac{|x|^2}{2\tau} + \rho \langle q, x \rangle}
$$
for all $\rho \in (0, 1)$, all $\tau \geq \frac{5}{9}$, and all $q \in \mathbb{R}^2$.

**Proof.** Let us fix a real number $\tau \geq \frac{5}{9}$ and a point $q \in \mathbb{R}^2$. Applying Lemma 2.1 to $\bar{\tau} = \frac{5}{9}$ and $\bar{q} = \rho q$ gives
$$
\int_\Gamma \psi e^{-\frac{\rho^2}{4\tau} + \rho \langle q, x \rangle} \left( \frac{10}{11} - \frac{\rho^2 |x|^2}{2\tau} + \rho \langle q, x \rangle \right) \geq 0
$$
for all $\rho \in (0, 1)$. Therefore, we obtain
$$
\rho \frac{d}{d\rho} \left( \int_\Gamma \psi e^{-\frac{\rho^2}{4\tau} + \rho \langle q, x \rangle} \right) = \int_\Gamma \psi e^{-\frac{\rho^2}{4\tau} + \rho \langle q, x \rangle} \left( - \frac{\rho^2 |x|^2}{2\tau} + \rho \langle q, x \rangle \right)
$$
$$
\geq - \frac{10}{11} \left( \int_\Gamma \psi e^{-\frac{\rho^2}{4\tau} + \rho \langle q, x \rangle} \right)
$$
for all $\rho \in (0, 1)$. Consequently, the function
$$
\rho \mapsto \rho^{10} \int_\Gamma \psi e^{-\frac{\rho^2}{4\tau} + \rho \langle q, x \rangle}
$$
is monotone increasing for $\rho \in (0, 1)$. From this, the assertion follows.

In the remainder of this section, we consider an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck $N$ of size 1 (see $\mathbb{K}$ for the definition). It is understood that $\varepsilon$ is much smaller than $\hat{\delta}$. By definition, we can find a simple closed, convex curve $\Gamma$ with the property that $\text{dist}_{C^2}(N, \Gamma \times [-L, L]) \leq \varepsilon$. Since $\text{dist}_{C^2}(N, \Gamma \times [-L, L]) \leq \varepsilon$, we can find a collection of curves $\Gamma_s$ such that
$$
\{(\gamma_s(t), s) : s \in [-L - 1, L - 1], t \in [0, 1]\} \subset N
$$
and
$$
\sum_{k+l \leq 20} \left| \frac{\partial^k}{\partial s^k} \frac{\partial^l}{\partial t^l} (\gamma_s(t) - \gamma(t)) \right| \leq O(\varepsilon).
$$
Here, we have used the notation $\Gamma = \{\gamma(t) : t \in [0, 1]\}$ and $\Gamma_s = \{\gamma_s(t) : t \in [0, 1]\}$. 

Lemma 2.3. \( x \in (0, \Delta^+ \tilde{x}) \). Hence, if we choose \( \Delta \) sufficiently large, then we have

\[
\begin{align*}
H & \geq H + \omega - \frac{3}{4} \frac{A^2}{4} \\
& \geq \frac{1}{1 + (\omega - \Delta) / \Delta}\frac{A^2}{4} \\
& \leq \frac{1}{1 - \omega / \Delta}\frac{A^2}{4}.
\end{align*}
\]

The curve \( \tilde{N} \cap \{z_2 = s\} \) is obtained by dilating the curve \( N \cap \{z_2 = s\} \) by the factor \( \rho(s) \). It is easy to see that for all \( s \in (0, \Delta^+ \tilde{x}) \), all \( r_0 \in [0, 1] \), \( r \in [0, 1] \), \( t \in [0, 1] \), and all \( \tau \in [0, 1] \), the curve \( \tilde{N} \cap \{z_2 = s\} \) is smooth. Moreover, \( \tilde{N} \) is axially symmetric for \( s > \Delta \).

Proof. For abbreviation, let \( \rho(s) = 1 - e^{-s} \tilde{x} \). The curve \( \tilde{N} \cap \{z_2 = s\} \) is obtained by dilating the curve \( N \cap \{z_2 = s\} \) by the factor \( \rho(s) \).

for \( s \in [0, \Delta^+] \), \( \forall r_0 \in [0, 1] \), \( r \in [0, 1] \), \( t \in [0, 1] \), and \( \tau \in [0, 1] \), the curve \( \tilde{N} \cap \{z_2 = s\} \) is smooth. Moreover, \( \tilde{N} \) is axially symmetric for \( s > \Delta \).

Finally, we choose a smooth cutoff function \( \chi : [0, 1] \to [0, 1] \) such that \( \chi = 1 \) on \( \{0, 1\} \) and \( \chi = 0 \) on \( [2, \infty) \). We then define a surface \( \tilde{F}(\psi) : [L, \Delta^+] \times [0, 1] \to \mathbb{R}^3 \).

As in \( \mathbb{R}^3 \), we may translate the neck \( N \) in space so that the center of mass of \( \tilde{F} \) is at the origin. Using the curve shortening flow, we can construct a homotopy \( \hat{\gamma}(r) \), \( (r, t) \in [0, 1] \times [0, 1] \), with the following properties:

- \( \gamma(0) = \gamma(t) \) for \( r \in [0, 1] \).
- For each \( r \in [0, 1] \), the curve \( \hat{\gamma}(r) \) is noncollapsed.
- We have \( \sup_{r \in [0, 1]} \gamma(r, t) = \tilde{\gamma}(r) \) for \( r \in [0, 1] \).

where \( \omega(\delta) \to 0 \) as \( \delta \to 0 \).

Finally, we choose a smooth cutoff function \( \chi : [0, 1] \to [0, 1] \) such that \( \chi = 1 \) on \( \{0, 1\} \) and \( \chi = 0 \) on \( [2, \infty] \). We then define a surface \( \tilde{F}(\psi) : [L, \Delta^+] \times [0, 1] \to \mathbb{R}^3 \).
for $x_3 \in (0, \Lambda^+/4]$. From this, we deduce that
\[
\int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2-r_0\bar{H}} \frac{1}{|\nabla N x_3|} \leq \rho(s) \int_{\bar{N}\cap\{x_3=s\}} e^{-\rho(s)^2\frac{(s^2+x_2^2)}{4\tau}+\rho(s)(q_1x_1+q_2x_2)} e^{-r_0H} \frac{1}{|\nabla N x_3|}
\]
for $s \in (0, \Lambda^+/4]$. On the other hand, applying Lemma 2.2 with $\psi = e^{r_0(1-H)} \frac{1}{|\nabla N x_3|}$, we obtain
\[
\rho(s) \int_{\bar{N}\cap\{x_3=s\}} \psi e^{-\frac{\rho(s)^2(s^2+x_2^2)}{4\tau}+\rho(s)(q_1x_1+q_2x_2)} e^{-r_0H} \frac{1}{|\nabla N x_3|} \leq \int_{\bar{N}\cap\{x_3=s\}} \psi e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2},
\]
and hence
\[
\rho(s) \int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{\rho(s)^2(s^2+x_2^2)}{4\tau}+\rho(s)(q_1x_1+q_2x_2)} e^{-r_0H} \frac{1}{|\nabla N x_3|} \leq \int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2-r_0H} \frac{1}{|\nabla N x_3|}
\]
for $s \in (0, \Lambda^+/4]$. Putting these facts together, the assertion follows.

**Lemma 2.4.** We can find positive real numbers $\delta_*$ and $\Lambda_*$, and a positive function $E_*(\Lambda)$ with the following property. If $\delta < \delta_*$, $\Lambda > \Lambda_*$, and $\varepsilon < E_*(\Lambda)$, then we have
\[
\int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2-r_0\bar{H}} \frac{1}{|\nabla N x_3|} \leq \int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2-r_0H} \frac{1}{|\nabla N x_3|}
\]
for all $s \in (\Lambda^+/4, 1/4]$, all $r_0 \in [\frac{1}{1000}, 1]$, all $\tau \geq \frac{\epsilon}{g}$, and all $(q_1, q_2) \in \mathbb{R}^2$.

**Proof.** We first observe that the claim is true if $\sqrt{q_1^2+q_2^2}$ is sufficiently large. More precisely, we can find positive real numbers $\delta_*$ and $\Lambda_*$, and positive functions $E_1(\Lambda)$ and $Q(\Lambda)$ with the following significance. If $\delta < \delta_*$, $\Lambda > \Lambda_*$, $\varepsilon < E_1(\Lambda)$, and $\sqrt{q_1^2+q_2^2} > Q(\Lambda)$, then we have
\[
\int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2-r_0\bar{H}} \frac{1}{|\nabla N x_3|} \leq \int_{\bar{N}\cap\{x_3=s\}} e^{-\frac{s^2+x_2^2}{4\tau}+q_1x_1+q_2x_2-r_0H} \frac{1}{|\nabla N x_3|}
\]
for all $s \in (\Lambda^+/4, 1/4]$, all $r_0 \in [\frac{1}{1000}, 1]$, and all $\tau \geq \frac{\epsilon}{g}$. 
Hence, it remains to consider the case \( \sqrt{q_1^2 + q_2^2} \leq Q(\Lambda) \). In this case, we have
\[
\int_{\tilde{N} \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2} \leq (1 + C(\Lambda) \varepsilon) \rho(s) \int_{N \cap \{ x_3 = s \}} e^{-\frac{\rho(s)^2 (x_1^2 + x_2^2)}{4\tau} + \rho(s) (q_1 x_1 + q_2 x_2)}
\]
for each \( s \in (\Lambda^{\frac{1}{4}}, \Lambda^\frac{1}{4}] \), where \( \rho(s) = 1 - e^{-\frac{4s}{x_3}} \). At each point on the surface \( N \cap \{ x_3 \in (\Lambda^{\frac{1}{4}}, \Lambda^\frac{1}{4}] \} \), we have
\[
\frac{1}{|\nabla N x_3|} \geq 1
\]
and
\[
H \leq 1 + C \varepsilon.
\]
Moreover, at each point on the surface \( \tilde{N} \cap \{ x_3 \in (\Lambda^{\frac{1}{4}}, \Lambda^\frac{1}{4}] \} \), we have
\[
\frac{1}{|\nabla N x_3|} \leq 1 + C \left( \frac{\Lambda}{x_3} e^{-\frac{4s}{x_3}} \right)^2 + C \varepsilon \leq 1 + C \Lambda^{-2} e^{-\frac{4s}{x_3}} + C \varepsilon
\]
and
\[
\tilde{H} \geq 1 - C \varepsilon.
\]
Therefore, we obtain
\[
\int_{\tilde{N} \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 \tilde{H}} \frac{1}{|\nabla N x_3|} \leq (1 + C \Lambda^{-2} e^{-\frac{4s}{x_3}} + C(\Lambda) \varepsilon) \rho(s) \int_{N \cap \{ x_3 = s \}} e^{-\frac{\rho(s)^2 (x_1^2 + x_2^2)}{4\tau} + \rho(s) (q_1 x_1 + q_2 x_2) - r_0 H} \frac{1}{|\nabla N x_3|}
\]
for \( s \in (\Lambda^{\frac{1}{4}}, \Lambda^\frac{1}{4}] \). On the other hand, using Lemma 2.2 with \( \psi = e^{r_0 (1 - H)} \frac{1}{|\nabla N x_3|} \), we obtain
\[
\rho(s) \frac{10}{11} \int_{N \cap \{ x_3 = s \}} \psi e^{-\frac{\rho(s)^2 (x_1^2 + x_2^2)}{4\tau} + \rho(s) (q_1 x_1 + q_2 x_2)} \leq \int_{N \cap \{ x_3 = s \}} \psi e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2},
\]
hence
\[
\rho(s) \frac{10}{11} \int_{N \cap \{ x_3 = s \}} e^{-\frac{\rho(s)^2 (x_1^2 + x_2^2)}{4\tau} + \rho(s) (q_1 x_1 + q_2 x_2) - r_0 H} \frac{1}{|\nabla N x_3|} \leq \int_{N \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 H} \frac{1}{|\nabla N x_3|}
\]
for $s \in (\Lambda^{1/2}, \frac{1}{4}]$. Thus, we can find a positive function $E_2(\Lambda)$ with the following property. If $\sqrt{q_1} + q_2^2 \leq Q(\Lambda)$ and $\varepsilon < E_2(\Lambda)$, then

$$\int_{\tilde{N} \cap \{x_3 = s\}} e^{-\frac{x_1^2 + x_2^2}{4s} + q_1 x_1 + q_2 x_2 - r_0 \tilde{H}} \frac{1}{|\nabla \tilde{x}_3|} \leq \int_{N \cap \{x_3 = s\}} e^{-\frac{x_1^2 + x_2^2}{4s} + q_1 x_1 + q_2 x_2 - r_0 H} \frac{1}{|\nabla x_3|}$$

for all $s \in (\Lambda^{1/2}, \frac{1}{4}]$, all $r_0 \in [\frac{1}{1000}, 1]$, all $\tau \geq \frac{5}{9}$. Hence, if we put $E_*(\Lambda) = \min\{E_1(\Lambda), E_2(\Lambda)\}$, then the assertion follows.

**Lemma 2.5.** There exist positive real numbers $\delta_*$ and $\Lambda_*$ with the following property. If $\delta < \delta_*$ and $\Lambda > \Lambda_*$, then we have

$$\int_{\tilde{N} \cap \{x_3 = s\}} e^{-\frac{x_1^2 + x_2^2}{4s} + q_1 x_1 + q_2 x_2 - r_0 \tilde{H}} \frac{1}{|\nabla \tilde{x}_3|} \leq \int_{N \cap \{x_3 = s\}} e^{-\frac{x_1^2 + x_2^2}{4s} + q_1 x_1 + q_2 x_2 - r_0 H} \frac{1}{|\nabla x_3|}$$

for all $s > \frac{4}{3}$, all $r_0 \in [\frac{1}{1000}, 1]$, all $\tau \geq \frac{5}{9}$, and all $(q_1, q_2) \in \mathbb{R}^2$.

**Proof.** It is not difficult to verify the assertion in the asymptotically cylindrical region when $s \in (\frac{4}{3}, \Lambda + \Lambda^{\frac{1}{2}}]$. We now consider the case $s \in (\Lambda + \Lambda^{\frac{1}{2}}, \Lambda + 2 \Lambda^{\frac{1}{2}})$. In this regime, our surgical cap is axially symmetric with radius

$$\rho(s) = a \sqrt{b - s}$$

where $a = 1 - e^{-4} + \frac{1}{3}(1 - e^{-4})^2 \Lambda^{-\frac{1}{4}} < 1$ and $b = \Lambda + 2 \Lambda^{\frac{1}{2}}$. Note that

$$\frac{1}{|\nabla \tilde{x}_3|} = \sqrt{1 + \rho'(s)^2}$$

and

$$\tilde{H} \geq \frac{1}{a}$$
on the set \( \tilde{N} \cap \{ x_3 = s \} \). Therefore,

\[
\int_{\tilde{N} \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 \tilde{H}} \frac{1}{|\nabla_x N_3|} \\
\leq \int_{\tilde{N} \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 \tilde{H}} \frac{1}{|\nabla_x N_3|} \\
= \rho(s) \sqrt{1 + \rho'(s)^2} e^{-\frac{\rho(s)^2}{4\tau} - \frac{r_0}{a}} \int_{S^1} e^{\rho(s)} (q_1 x_1 + q_2 x_2) \\
= a \sqrt{1 - \frac{a}{a + b - s} + \frac{a^4}{4(a + b - s)^4} e^{\frac{a^2 - a^2}{a + b - s}} (q_1 x_1 + q_2 x_2)} \\
\cdot \int_{S^1} e^a \sqrt{\frac{b - s}{a + b - s}} (q_1 x_1 + q_2 x_2)
\]

for all \( \tau \geq \frac{5}{9} \) and all \( s < b \). Here, \( S^1 \) denotes the unit circle of radius 1. It is elementary to check that

\[
1 - z + \frac{z^4}{4} \leq e^{-z},
\]

hence

\[
\sqrt{1 - z + \frac{z^4}{4}} e^{\frac{z^2}{2}} \leq 1
\]

for all \( z \in [0, 1] \). Consequently, we have

\[
\sqrt{1 - \frac{a}{a + b - s} + \frac{a^4}{4(a + b - s)^4} e^{\frac{a^2 - a^2}{a + b - s}} (q_1 x_1 + q_2 x_2)} \leq 1
\]

for all \( \tau \geq \frac{5}{9} \) and \( s < b \). From this, we deduce that

\[
\int_{\tilde{N} \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 \tilde{H}} \frac{1}{|\nabla_x N_3|} \\
\leq a e^{-\frac{\rho^2}{4\tau} - \frac{r_0}{a}} \int_{S^1} e^a \sqrt{\frac{b - s}{a + b - s}} (q_1 x_1 + q_2 x_2)
\]

for all \( \tau \geq \frac{5}{9} \) and \( s < b \).

On the other hand, if \( \tilde{\delta} \) is sufficiently small, then we have

\[
\int_{S^1} e^a \sqrt{\frac{b - s}{a + b - s}} (q_1 x_1 + q_2 x_2) \\
\leq \int_{\tilde{N} \cap \{ x_3 = s \}} e^{-\frac{x_1^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 + r_0 (\frac{1}{a} - \tilde{H})} \frac{1}{|\nabla_x N_3|}
\]
for all $s \in (\Lambda + \Lambda \frac{1}{2}, \Lambda + 2 \Lambda \frac{1}{2})$, all $r_0 \in \left[\frac{1}{1000}, 1\right]$, all $\tau \geq \frac{5}{9}$, and all $(q_1, q_2) \in \mathbb{R}^2$. This implies
\[
e^{-\frac{1}{4\tau} - \frac{r_0}{\alpha}} \int_{\mathbb{S}^1} e^{a \sqrt{\frac{b - s}{a + b - s}} (q_1 x_1 + q_2 x_2)} \leq \int_{\mathbb{N} \cap \{x_3 = s\}} e^{-\frac{s^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 H} \frac{1}{|\nabla N x_3|}
\]
for all $s \in (\Lambda + \Lambda \frac{1}{2}, \Lambda + 2 \Lambda \frac{1}{2})$, all $r_0 \in \left[\frac{1}{1000}, 1\right]$, all $\tau \geq \frac{5}{9}$, and all $(q_1, q_2) \in \mathbb{R}^2$. Putting these facts together and using the inequality $a e^{-\frac{a^2}{4\tau}} \leq e^{-\frac{1}{4\tau}}$, the assertion follows.

Combining these results, we can draw the following conclusion:

**Proposition 2.6.** There exist real numbers $\delta_*$ and $\Lambda_*$, and a function $E_*(\Lambda)$ with the following property. If $\hat{\delta} < \delta_*$, $\Lambda > \Lambda_*$, and $\varepsilon < E_*(\Lambda)$, then we have
\[
\int_{\mathbb{N} \cap \{0 \leq x_3 \leq \Lambda + 2 \Lambda \frac{1}{2}\}} e^{-\frac{|x-p|^2}{4\tau} - r_0 H} \leq \int_{\mathbb{N} \cap \{0 \leq x_3 \leq \Lambda + 2 \Lambda \frac{1}{2}\}} e^{-\frac{|x-p|^2}{4\tau} - r_0 H}
\]
for all $r_0 \in \left[\frac{1}{1000}, 1\right]$, all $\tau \geq \frac{5}{9}$, and all $p \in \mathbb{R}^3$.

**Proof.** In view of Lemma 2.5, Lemma 2.4, and Lemma 2.5 we can find positive real numbers $\delta_*$ and $\Lambda_*$, and a positive function $E_*(\Lambda)$ with the following property. If $\hat{\delta} < \delta_*$, $\Lambda > \Lambda_*$, and $\varepsilon < E_*(\Lambda)$, then we have
\[
\int_{\mathbb{N} \cap \{x_3 = s\}} e^{-\frac{s^2 + x_2^2}{4\tau} + q_1 x_1 + q_2 x_2 - r_0 H} \frac{1}{|\nabla N x_3|}
\]
for all $s \in \mathbb{R}$, all $r_0 \in \left[\frac{1}{1000}, 1\right]$, all $\tau \geq \frac{5}{9}$, and all $(q_1, q_2) \in \mathbb{R}^2$. This implies
\[
\int_{\mathbb{N} \cap \{x_3 = s\}} e^{-\frac{|x-p|^2}{4\tau} - r_0 H} \frac{1}{|\nabla N x_3|}
\]
for all $s \in \mathbb{R}$, all $r_0 \in \left[\frac{1}{1000}, 1\right]$, all $\tau \geq \frac{5}{9}$, and all $p \in \mathbb{R}^3$. If we integrate over $s$ and apply the co-area formula, the assertion follows.

### 3. A monotonicity formula for mean curvature flow with surgery in $\mathbb{R}^3$

**Proposition 3.1.** Let $M_t$ be a family of surfaces in $\mathbb{R}^3$ which evolve under smooth mean curvature flow. For each $r_0 > 0$, the function
\[
\tau \mapsto \int_{M_{t_0 - \tau}} \frac{1}{4\pi \tau} e^{-\frac{|x-p|^2}{4\tau} - r_0 H}
\]
is monotone increasing in $\tau$.

**Proof.** We compute
\[
\left(\frac{\partial}{\partial t} - \Delta\right)e^{-r_0H} = -r_0e^{-r_0H}\left(\frac{\partial}{\partial t} - \Delta\right)H - r_0^2 e^{-r_0H} |\nabla H|^2 \\
= -r_0 e^{-r_0H} |A|^2 H - r_0^2 e^{-r_0H} |\nabla H|^2 \leq 0.
\]

Hence, the assertion follows from Ecker’s weighted monotonicity formula (see Theorem 4.13 in [5]).

We next consider a solution of mean curvature flow with surgery in $\mathbb{R}^3$. We assume that each surgery procedure involves performing $\Lambda$-surgery on an $(\hat{\alpha}, \hat{\delta}, \varepsilon, L)$-neck of size $r \in \left[\frac{1}{2H_1}, \frac{1}{H_1}\right]$ (see [3] for definitions).

**Theorem 3.2.** Let $\delta_*, \Lambda_*$, and $E_*(\Lambda)$ be defined as in Proposition 2.6. Moreover, suppose that $M_t$ is a solution of mean curvature flow with surgery, and that the surgery parameters satisfy $\hat{\delta} < \delta_*$, $\Lambda > \Lambda_*$, and $\varepsilon < E_*(\Lambda)$. Then the function
\[
\tau \mapsto \int_{M_{t_0+\tau}} \frac{1}{4\pi\tau} e^{-\frac{|x-p|^2}{4\tau} - \frac{H}{2\tau H_1}}
\]
is monotone increasing for $\tau \geq \frac{5}{9} H_1^{-2}$.

**Proof.** Proposition 3.1 guarantees that the monotonicity is correct in between surgery times. Moreover, it follows from Proposition 2.6 that the monotonicity holds across surgery times.

Theorem 3.2 allows us to draw the following conclusion:

**Corollary 3.3.** Let $\delta_*, \Lambda_*$, and $E_*(\Lambda)$ be defined as in Proposition 2.6. Moreover, we can find a numerical constant $L_*$ with the following property. Suppose that $M_t$ is a solution of mean curvature flow with surgery, and that the surgery parameters satisfy $\hat{\delta} < \delta_*$, $\Lambda > \Lambda_*$, $\varepsilon < E_*(\Lambda)$, and $L > L_*$. Then we have
\[
\int_{M_{t_0+\frac{5}{9} H_1^{-2} - \tau}} \frac{1}{4\pi\tau} e^{-\frac{|x-p|^2}{4\tau}} \geq 1.01
\]
for all $\tau \geq \frac{5}{9} H_1^{-2}$. 
Proof. We first observe that
\[
\int_{S^1(r) \times \mathbb{R}} \frac{9 H^2}{20 \pi} e^{-\frac{9 H^2 |x|^2}{20} - \frac{\mu}{200 H_1}}
\]
\[
= \sqrt{\frac{9 \pi}{5}} H_1 r e^{-\frac{9 H^2 r^2}{20} - \frac{1}{200 H_1}}
\]
\[
\geq \sqrt{\frac{9 \pi}{5}} H_1 r e^{-\frac{9 H^2 r^2}{20} - \frac{1}{200}}
\]
for \( r \in \left[ \frac{1}{2 \hat{H}} \right) \). Since \( \sqrt{\frac{9 \pi}{5}} e^{-\frac{1}{3}} > 1.02 \), we conclude that
\[
\int_{M_{0-}} \frac{9 H^2}{20 \pi} e^{-\frac{9 H^2 |x-p|^2}{20} - \frac{\mu}{200 H_1}} > 1.01,
\]
provided that \( r \in \left[ \frac{1}{2 \hat{H}}, \frac{1}{H} \right) \) and \( L \) is sufficiently large. Using Theorem 3.2, we obtain
\[
\int_{M_{0-}} \frac{1}{4 \pi \tau} e^{-\frac{|x-p|^2}{4 \tau}}
\]
\[
\geq \int_{M_{0-}} \frac{1}{4 \pi \tau} e^{-\frac{|x-p|^2}{4 \tau} - \frac{\mu}{200 H_1}}
\]
\[
\geq \int_{M_{0-}} \frac{9 H^2}{20 \pi} e^{-\frac{9 H^2 |x-p|^2}{20} - \frac{\mu}{200 H_1}}
\]
\[
> 1.01
\]
for all \( \tau \geq \frac{5}{9} H_1^{-2} \). This completes the proof.

**Theorem 3.4.** Fix an open interval \( I \) and a compact interval \( J \subset I \). Moreover, suppose that \( \mathcal{M} \) is an embedded smooth solution of the mean curvature flow which is defined for \( t \in I \). Moreover, suppose that \( \mathcal{M}^{(j)} \) is a sequence of mean curvature flows with surgery, each of which is defined for \( t \in I \). We assume that each surgery of the flow \( \mathcal{M}^{(j)} \) involves performing \( \Lambda \)-surgery on an \( (\hat{\alpha}, \hat{\delta}, \varepsilon, L) \)-neck of size \( r \in \left[ \frac{1}{2 \hat{H}}, \frac{1}{H} \right] \), where \( H_1^{(j)} \to \infty \). We assume further that the surgery parameters satisfy \( \hat{\delta} < \delta_* \), \( \Lambda > \Lambda_* \), \( \varepsilon < E_*(\Lambda) \), and \( L > L_* \). Finally, we assume that the flows \( \mathcal{M}^{(j)} \) converge to \( \mathcal{M} \) in the sense of geometric measure theory. Then, if \( j \) is sufficiently large, the flow \( \mathcal{M}^{(j)} \) is free of surgeries for all \( t \in J \). Furthermore, the flows \( \mathcal{M}^{(j)} \) converge smoothly to \( \mathcal{M} \) as \( j \to \infty \).

**Proof.** We first show that, for \( j \) sufficiently large, the flow \( \mathcal{M}^{(j)} \) is free of surgeries for all \( t \in J \). Suppose this is false. After passing to a subsequence
if necessary, we may assume that each flow $M^{(j)}$ has at most one surgery time $t_j \in J$. Let us pick a point $p_j$ in $\mathbb{R}^3$ which lies at the center of a neck of size $r_j \in [\frac{1}{2H_1^{(j)}}, \frac{1}{H_1^{(j)}}]$ in $M_{t_j}$. Using Corollary 3.3 we obtain

$$\int_{M_{t_j + \frac{5}{9}H_1^{(j)} - 2 - \tau}} \frac{1}{4\pi \tau} e^{-\frac{|x - p_j|^2}{4\tau}} \geq 1.01$$

for all $\tau \geq \frac{5}{9}H_1^{-2}$. We now pass to the limit as $j \to \infty$. If we define $\bar{t} = \lim_{j \to \infty} t_j \in J$ and $\bar{p} = \lim_{j \to \infty} p_j$, then we obtain

$$\int_{\bar{M}_{\bar{t} - \tau}} \frac{1}{4\pi \tau} e^{-\frac{|x - \bar{p}|^2}{4\tau}} \geq 1.01$$

for $\tau \in (0, \inf J - \inf I)$. On the other hand, since the solution $\bar{M}$ is smooth, we have

$$\int_{\bar{M}_{\bar{t} - \tau}} \frac{1}{4\pi \tau} e^{-\frac{|x - \bar{p}|^2}{4\tau}} \to 1$$

as $\tau \to 0$. This is a contradiction. Therefore, the flow $M^{(j)}$ is free of surgeries for $t \in J$. Using standard local regularity theorems for mean curvature flow (cf. [1], [16]), we conclude that the flows $M^{(j)}$ converge smoothly to $\bar{M}$ as $j \to \infty$.

4. ADAPTING THE MONOTONICITY FORMULA TO FLOWS IN RIEMANNIAN THREE-MANIFOLDS

Let $M_t$ be a solution of mean curvature flow in a Riemannian three-manifold. Let us fix a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(s) = s^2$ for $s \leq \frac{\text{inj}}{\varphi}$ and $\varphi(s) = \text{inj}^2$ for $s \geq \frac{\text{inj}}{\varphi}$, where inj denotes the injectivity radius of the ambient three-manifold. We define $\Phi(x, t) = \varphi(d(p, x))$.

Proposition 4.1. We can find positive constants $K_0$, $K_1$, and $K_2$ such that

$$\frac{d}{dt} \left( \int_{M_t} \frac{1}{4\pi (t_0 - t)} e^{-\frac{\Phi}{4(t_0 - t)} + K_0 (t_0 - t)^{\frac{3}{2}} - r_0 e^{-K_2 (t_0 - t)} H} \right) \leq K_1 |M_t|$$

whenever $t_0 - t \in (0, 1]$. The constants $K_0$, $K_1$, and $K_2$ depend only on the ambient three-manifold.

Proof. The function

$$u(x, t) = \frac{1}{4\pi (t_0 - t)} e^{-\frac{\Phi}{4(t_0 - t)}}$$

...
satisfies
\[
\left( \frac{\partial}{\partial t} + \Delta - H^2 \right) u
= \left( \frac{1}{t_0 - t} - \frac{\Phi}{4(t_0 - t)^2} + \frac{1}{4(t_0 - t)} H \bar{D}_\nu \Phi \right.
- \frac{1}{4(t_0 - t)} \Delta \Phi + \frac{1}{16(t_0 - t)^2} |\nabla M_t \Phi|^2 - H^2 \left. \right) u
\]
\[
= \left( \frac{1}{t_0 - t} + \frac{1}{4(t_0 - t)} H \bar{D}_\nu \Phi \right.
- \frac{1}{4(t_0 - t)} \Delta \Phi - \frac{1}{16(t_0 - t)^2} (\bar{D}_\nu \Phi)^2 - H^2 \left. \right) u
\]
\[
= \left( \frac{1}{t_0 - t} - \frac{1}{4(t_0 - t)} \sum_{i=1}^{2} \bar{D}^2 \Phi(e_i, e_i) - \frac{1}{16(t_0 - t)^2} (\bar{D}_\nu \Phi)^2 - H^2 \right) u
\]
\[
\leq \left( \frac{1}{t_0 - t} - \frac{1}{4(t_0 - t)} \sum_{i=1}^{2} \bar{D}^2 \Phi(e_i, e_i) \right) u
\]
\[
\leq K_0 \frac{d(p, x)^2}{t_0 - t} u
\]
for \(d(p, x) \leq \frac{\text{inj}}{4}\). Here, \(K_0\) is a positive constant that depends only on the ambient three-manifold. Hence, if we define
\[
v(x, t) = \frac{1}{4\pi(t_0 - t)} e^{-\frac{\Phi}{4(t_0 - t)}} + K_0 (t_0 - t)^{\frac{1}{2}},
\]
then we obtain
\[
\left( \frac{\partial}{\partial t} + \Delta - H^2 \right) v \leq K_0 \left( \frac{d(p, x)^2}{t_0 - t} - \frac{1}{2} (t_0 - t)^{-\frac{1}{2}} \right) v
\]
for \(d(p, x) \leq \frac{\text{inj}}{4}\). In particular, we have
\[
\left( \frac{\partial}{\partial t} + \Delta - H^2 \right) v \leq 0
\]
whenever \(d(p, x) \leq \frac{1}{4} \min\{\text{inj}, (t_0 - t)^{\frac{1}{2}}\}\). Hence, if \(t_0 - t \in (0, 1]\), then we obtain
\[
\left( \frac{\partial}{\partial t} + \Delta - H^2 \right) v \leq K_1
\]
at each point on \(M_t\). Here, \(K_1\) is a positive constant that depends only on the ambient three-manifold.
We now fix a positive constant $K_2$ with the property that the Ricci curvature of the ambient three-manifold is bounded from below by $-K_2$. Then
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \left( e^{-K_2(t_0-t)} H \right) \geq e^{-K_2(t_0-t)} |A|^2 H \geq 0,
\]
hence
\[
\left( \frac{\partial}{\partial t} - \Delta \right) e^{-r_0 e^{-K_2(t_0-t)}} H =-r_0 e^{-r_0 e^{-K_2(t_0-t)}} H \left( \frac{\partial}{\partial t} - \Delta \right) \left( e^{-K_2(t_0-t)} H \right) - r_0^2 e^{-r_0 e^{-K_2(t_0-t)}} H |\nabla (e^{-K_2(t_0-t)} H)|^2 \leq 0.
\]
Putting these facts together, we obtain
\[
\frac{d}{dt} \left( \int_{M_t} v e^{-r_0 e^{-K_2(t_0-t)}} H \right) \leq K_1 \int_{M_t} e^{-r_0 e^{-K_2(t_0-t)}} H \leq K_1 |M_t|
\]
provided that $t_0 - t \in (0, 1]$. This completes the proof.

5. Behavior of the monotone quantity during a single surgery

We now analyze how the monotone quantity described in the previous section changes under a single surgery.

**Definition 5.1.** Let $M$ be a closed surface in a Riemannian three-manifold, and let $N$ be a region in $M$. We say that $N$ is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, \Lambda)$-neck of size $r$ if there exists a point $o \in N$ with the property that the surface $\text{exp}_{o^{-1}}(N)$ is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, \Lambda)$-neck in Euclidean space.

We now consider an $(\hat{\alpha}, \hat{\delta}, \varepsilon, \Lambda)$-neck $N$ of size $r$ in the ambient three-manifold. By definition, we can find a point $o \in N$ with the property that the surface $\text{exp}_{o^{-1}}(N)$ is an $(\hat{\alpha}, \hat{\delta}, \varepsilon, \Lambda)$-neck in Euclidean space. It will be convenient to work in geodesic normal coordinates around $o$. This allows us to identify a neighborhood of $o$ in the ambient manifold with a ball in $\mathbb{R}^3$ centered at the origin. With this identification, we can now identify $N$ with an $(\hat{\alpha}, \hat{\delta}, \varepsilon, \Lambda)$-neck in $\mathbb{R}^3$. Without loss of generality, we may assume that the axis of the neck $N$ is parallel to the $x_3$-axis. We note that the origin lies on $N$, so the axis of the neck does not pass through the origin. Finally, we denote by $\tilde{N}$ the surface obtained from $N$ by performing a $\Lambda$-surgery on $N$.

**Proposition 5.2.** Assume that the surgery parameters $\hat{\delta}$, $\Lambda$, and $\varepsilon$ satisfy $\hat{\delta} < \delta_*, \Lambda > \Lambda_*$, and $\varepsilon < E_*(\Lambda)$, where $\delta_*$, $\Lambda_*$, and $E_*$ are defined as in Proposition 2.6. Then
\[
\int_{\tilde{N} \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^+\}} e^{-\frac{\phi}{4\tau} - r_0 e^{-K_2\tau} H} d\mu 
\leq \int_{N \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^+\}} e^{-\frac{\phi}{4\tau} - r_0 e^{-K_2\tau} H} d\mu + C(\Lambda) \tau r^2
\]
provided that $\frac{1}{K_2} \geq \tau \geq \frac{5}{9} r^2$ and $r_0 \in \left[ \frac{1}{300} r, r \right]$.

**Proof.** It suffices to consider the case that $d(o,p) \leq \frac{1}{300} r$, and it follows from Proposition 2.10 that

$$\int_{N \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}} e^{-\frac{|x-p|^2}{4\tau}} - r_0 e^{-K_2 \tau} \tilde{H}_{\text{eucl}} \, d\mu_{\text{eucl}}$$

provided that $\tau \geq \frac{5}{9} r^2$ and $r_0 \in \left[ \frac{1}{300} r, r \right]$. Here, $|x-p|$ denotes the Euclidean distance of $x$ and $p$; $H_{\text{eucl}}$ and $\tilde{H}_{\text{eucl}}$ denote the mean curvatures of $N$ and $\tilde{N}$ with respect to the Euclidean metric; and $d\mu_{\text{eucl}}$ denotes the area form with respect to the Euclidean metric on $\mathbb{R}^3$.

Since the metric in geodesic normal coordinates agrees with the Euclidean metric up to quadratic terms, we have

$$d(p,x)^2 - |x-p|^2 = O(|x|^2 |p|^2 + |x|^3 |p| + |x|^4).$$

If $x \in (N \cup \tilde{N}) \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}$, then we have $|x| \leq C(\Lambda) r^2$, hence $|x|^2 \leq C(\Lambda) \tau$. This implies

$$\left| \frac{d(p,x)^2}{4\tau} - \frac{|x-p|^2}{4\tau} \right| \leq C(\Lambda) (|p|^2 + \tau)$$

for all points $x \in (N \cup \tilde{N}) \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}$. Consequently, we have

$$e^{-\frac{|x-p|^2}{4\tau}} \leq (1 + C(\Lambda) (|p|^2 + \tau)) e^{-\frac{d(p,x)^2}{4\tau}}$$

for all points $x \in N \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}$. Since $|H - H_{\text{eucl}}| \leq C(\Lambda) r$ for all points $x \in N \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}$, it follows that

$$e^{-\frac{|x-p|^2}{4\tau}} - r_0 e^{-K_2 \tau} H_{\text{eucl}} \leq (1 + C(\Lambda) (|p|^2 + \tau)) e^{-\frac{d(p,x)^2}{4\tau}} - r_0 e^{-K_2 \tau} H$$

for all points $x \in N \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}$. Thus,

$$\int_{N \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}} e^{-\frac{|x-p|^2}{4\tau}} - r_0 e^{-K_2 \tau} H_{\text{eucl}} \, d\mu_{\text{eucl}}$$

$$\leq (1 + C(\Lambda) (|p|^2 + \tau)) \int_{N \cap \{0 \leq x_3 \leq \Lambda + \frac{\Lambda}{4} \}} e^{-\frac{d(p,x)^2}{4\tau}} - r_0 e^{-K_2 \tau} H \, d\mu.$$
for all points $x \in \tilde{N} \cap \{0 < x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}$. This gives
\[
\int_{\tilde{N} \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}} e^{-\frac{(p,x)^2}{4\tau} - r_0 e^{-K_2^* \tilde{H}}} \, d\mu \
\leq (1 + C(\Lambda) (|p|^2 + \tau)) \int_{\tilde{N} \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}} e^{-\frac{|x-p|^2}{4\tau} - r_0 e^{-K_2^* \tilde{H}_{\text{eucl}}}} \, d\mu_{\text{eucl}}.
\]
Putting these facts together, we conclude that
\[
\int_{\tilde{N} \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}} e^{-\frac{(p,x)^2}{4\tau} - r_0 e^{-K_2^* \tilde{H}}} \, d\mu \
\leq (1 + C(\Lambda) (|p|^2 + \tau)) \int_{\tilde{N} \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}} e^{-\frac{(p,x)^2}{4\tau} - r_0 e^{-K_2^* \tilde{H}}} \, d\mu.
\]
Finally, we have the pointwise estimate $(|p|^2 + \tau) e^{-\frac{(p,x)^2}{4\tau}} \leq C(\Lambda) \tau$ at each point on $N \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}$. Therefore, we obtain
\[
\int_{\tilde{N} \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}} e^{-\frac{(p,x)^2}{4\tau} - r_0 e^{-K_2^* \tilde{H}}} \, d\mu \
\leq \int_{N \cap \{0 \leq x_3 \leq \Lambda + 2\Lambda^\frac{1}{4}\}} e^{-\frac{(p,x)^2}{4\tau} - r_0 e^{-K_2^* \tilde{H}}} \, d\mu + C(\Lambda) \tau \tau^2,
\]
as claimed.

Proposition 5.2 implies that, during each surgery, the integral
\[
\int_{M_{t_0-\tau}} \frac{1}{4\pi \tau} e^{-\frac{(p,x)^2}{4\tau} + K_0 \tau^\frac{3}{2} - r_0 e^{-K_2^* \tilde{H}}} \, d\mu
\]
increases by at most $C(\Lambda) \tau^2$, provided that $\frac{1}{K_2} \geq \tau \geq \frac{5}{9} H_1^{-2}$. On the other hand, each surgery procedure reduces the area by at least $\frac{1}{10} L \tau^2$. Thus, the total number of surgeries is bounded from above by $C L^{-1} \tau^{-2}$, and the sum of all the error terms is bounded from above by $C(\Lambda) L^{-1}$. Combining this argument with Proposition 4.1, we arrive at the following conclusion:

**Theorem 5.3.** Let $M_t$ be a solution of mean curvature flow with surgery in a Riemannian three-manifold. Suppose that the surgery parameters satisfy $\delta < \delta_\star$, $\Lambda > \Lambda_\star$, and $\varepsilon < E_\star(\Lambda)$, where $\delta_\star$, $\Lambda_\star$, and $E_\star(\Lambda)$ are defined as in Proposition 4.1. Then
\[
\int_{M_{t_0-\tau}} \frac{1}{4\pi \tau} e^{-\frac{(p,x)^2}{4\tau} + K_0 \tau^\frac{3}{2} - e^{-K_2^* \tilde{H}}} \, d\mu \leq \int_{M_{t_0-\tau}} \frac{1}{4\pi \tau} e^{-\frac{(p,x)^2}{4\tau} + K_0 \tau^\frac{3}{2} - e^{-K_2^* \tilde{H}}} \, d\mu \\
+ K_1 |M_{t_0-\tau}| \tau + C(\Lambda) L^{-1},
\]
provided that $\frac{1}{K_2} \geq \tau \geq \frac{5}{9} H_1^{-2}$.

As a consequence of Theorem 5.3, we obtain an analogue of Theorem 3.4 in the Riemannian setting.
**Theorem 5.4.** Fix an open interval $I$ and a compact interval $J \subset I$. Moreover, suppose that $\mathcal{M}$ is an embedded smooth solution of the mean curvature flow in a Riemannian three-manifold which is defined for $t \in I$. Moreover, suppose that $\mathcal{M}^{(j)}$ is a sequence of mean curvature flows with surgery in the same Riemannian three-manifold, each of which is defined for $t \in I$. We assume that each surgery of the flow $\mathcal{M}^{(j)}$ involves performing $\Lambda$-surgery on an $\left(\hat{\alpha}, \hat{\delta}, \varepsilon, L\right)$-neck of size $r \in \left[\frac{1}{2H_1^{(j)}}, \frac{1}{H_1^{(j)}}\right]$, where $H_1^{(j)} \to \infty$. We assume further that the surgery parameters satisfy $\hat{\delta} < \delta^*, \Lambda > \Lambda^*, \varepsilon < E_\varepsilon(\Lambda)$, and $L > L_\varepsilon(\Lambda)$. Finally, we assume that the flows $\mathcal{M}^{(j)}$ converge to $\mathcal{M}$ in the sense of geometric measure theory. Then, if $j$ is sufficiently large, the flow $\mathcal{M}^{(j)}$ is free of surgeries for all $t \in J$. Furthermore, the flows $\mathcal{M}^{(j)}$ converge smoothly to $\mathcal{M}$ as $j \to \infty$.

### 6. Longtime Behavior of Mean Curvature Flow with Surgery in a Riemannian Three-Manifold

In this section, we describe an application to the longtime behavior of mean curvature flow with surgery in dimension 3.

**Theorem 6.1** (S. Brendle, G. Huisken). Let $M_0$ be a closed, embedded surface in a Riemannian three-manifold. We assume that $M_0$ is the boundary of a domain in the ambient three-manifold, and has positive mean curvature. Then there exists a solution of mean curvature flow with surgery which is defined for all $t \in [0, \infty)$. Furthermore, the solution either becomes extinct in finite time, or else the flow is smooth for $t$ sufficiently large, and the surfaces converge smoothly to a union of finitely many embedded stable minimal surfaces.

We recall that, under mean curvature flow with surgery, certain components of the surface may be removed. Any component that is being removed consists of a tube-like surface which either closes up, or is capped off at the ends.

Theorem 6.1 follows by combining Theorem 5.4 with arguments of Lauer [11] and Head [8] and a theorem of White [14]. In the following, we will give an outline of the argument; the details of the surgery construction will appear elsewhere.

Let $\bar{\mathcal{M}}$ denote the level-set solution of mean curvature flow with initial surface $M_0$. If the level-set solution becomes extinct in finite time, then the solution of mean curvature flow with surgery also becomes extinct in finite time. Hence, it is enough to consider the case that the level-set flow does not become extinct in finite time. By a theorem of Brian White (cf. Theorem 11.1 in [14]), there exists a positive real number $T$ and a nonnegative integer $m$ with the following property: For $t \in (T - 2, \infty)$, the level-set solution $\bar{\mathcal{M}}_t$ is smooth and has exactly $m$ components which will be denoted by $\bar{M}_t^{(l)}$, $l \in \{1, \ldots, m\}$. Moreover, as $t \to \infty$, each component $\bar{M}_t^{(l)}$ converges in $C^\infty$ to an embedded stable minimal surface $S^{(l)}$. Note that White's
that the same minimal surface from opposite sides. The limiting surface \( S^{(l)} \) may have multiplicity 2 if it is one-sided; otherwise it must have multiplicity 1.

We next consider the mean curvature flow with surgery on the interval \([0, T]\). Let \( \mathcal{M}^{(j)} \) be a sequence of mean curvature flows with surgery starting from the initial surface \( M_0 \). We assume that each surgery of the flow \( \mathcal{M}^{(j)} \) involves performing \( \Lambda \)-surgery on an \((\hat{\alpha}, \delta, \varepsilon, L)\)-neck of size \( r \in \left[ \frac{1}{2H_1^{(j)}}, \frac{1}{H_1^{(j)}} \right] \), where \( H_1^{(j)} \to \infty \). We assume further that the surgery parameters satisfy \( \hat{\delta} < \delta_c, \Lambda > \Lambda_c, \varepsilon < E_*(\Lambda) \), and \( L > L_*(\Lambda) \). Note that the mean curvature flow with surgery can be defined on any given compact time interval; in particular, we can define the flow \( \mathcal{M}^{(j)} \) on the time interval \([0, T]\). It follows from work of Lauer [11] that the flows \( \mathcal{M}^{(j)} \) converge to the level-set flow \( \mathcal{M} \) in the Hausdorff sense. Using the outward-minimizing property, we conclude that the flows \( \mathcal{M}^{(j)} \) converge to \( \mathcal{M} \) in the sense of geometric measure theory (cf. [8]). Since the flow \( \mathcal{M} \) is smooth for all \( t \in (T - 2, \infty) \), Theorem 5.3 implies that the flow \( \mathcal{M}^{(j)} \) is free of surgeries for \( t \in (T - 1, T] \), provided that \( j \) is sufficiently large. Moreover, as \( j \to \infty \), the flows \( \mathcal{M}^{(j)} \) converge smoothly to \( \mathcal{M} \) for all \( t \in (T - 1, T] \).

For each \( j \), we consider the unique maximal solution of the smooth mean curvature flow with initial surface \( M_T^{(j)} \). Let us denote this solution by \( \{M_t^{(j)} : t \in [T, T_j]\} \). At this point, we allow the possibility that \( T_j \) is finite, though we will later show that \( T_j = \infty \) if \( j \) is sufficiently large. It follows from work of Lauer [11] that there exists a sequence of positive real numbers \( a_j \to 0 \) with the property that the surface \( M_t^{(j)} \) lies in between the surface \( \hat{M}_t \) and the surface \( M_{t + a_j} \) for all \( t \in [T, T_j] \). If \( j \) is sufficiently large, then the surfaces \( M_t^{(j)} \) have exactly \( m \) connected components for \( t \in [T, T_j] \). Let us denote these components by \( M_l^{(j)} \), where \( l \in \{1, \ldots, m\} \).

We next show that the flows \( \{M_t^{(j)} : t \in [T, T_j]\} \) have uniformly bounded curvature if \( j \) is sufficiently large.

**Proposition 6.2.** There exists a large integer \( j_0 \) such that

\[
\sup_{j \geq j_0} \sup_{t \in [T, T_j]} \sup_{A \in M_t^{(j)}} |A| < \infty.
\]

**Proof.** Suppose that the assertion is false. Then we can find an integer \( l \in \{1, \ldots, m\} \) and a sequence of times \( t_j \in [T, T_j] \) such that \( \sup_{M_{t_j}^{(j_l)}} |A| \to \infty \). Let us consider the flows \( \{M_{t_j+s}^{(j_l)} : s \in (-1, 0]\} \). For each \( j \), this is a smooth solution to the mean curvature flow. There are two possibilities:

**Case 1:** After passing to a subsequence, the sequence \( t_j \) converges to a finite number \( \tilde{t} \in [T, \infty) \). Since the surface \( M_{t_j+s}^{(j_l)} \) lies between \( \hat{M}_{t_j+s}^{(j_l)} \) and \( \tilde{M}_{t_j+s+a_j}^{(j_l)} \) and is outward-minimizing, we conclude that the flows \( \{M_{t_j+s}^{(j_l)} : s \in (-1, 0]\} \)
$s \in (-1, 0]$ converge in the measure-theoretic sense to the flow $\{\tilde{M}^{(l)}_{s+i} : s \in (-1, 0]\}$. Since the limit flow is smooth and has multiplicity 1, standard local regularity results (cf. [1], [16]) imply that the flows $\{M^{(l)}_{s+i} : s \in (-\frac{1}{2}, 0]\}$ have uniformly bounded curvature if $j$ is sufficiently large. This contradicts our choice of $t_j$.

Case 2: After passing to a subsequence, we have $\lim_{j \to \infty} t_j = \infty$. Since the surface $M^{(j,l)}_{s+i}$ lies between $\tilde{M}^{(l)}_{s+i}$ and $\tilde{M}^{(l)}_{s+i+a_j}$ and is outward-minimizing, we conclude that the flows $\{M^{(j,l)}_{s+i} : s \in (-1, 0]\}$ converge in the measure-theoretic sense to the static flow $\{S^{(l)} : s \in (-1, 0]\}$. If the limiting surface $S^{(l)}$ is two-sided, then the limit flow has multiplicity 1. Hence, the local regularity theorem (cf. [1], [16]) implies that the flows $\{M^{(j,l)}_{s+i} : s \in (-\frac{1}{2}, 0]\}$ have uniformly bounded curvature if $j$ is sufficiently large. This contradicts our choice of the sequence $t_j$. Finally, it remains to consider the case that $S^{(l)}$ is one-sided and the limit flow has multiplicity 2. In this case, we locally pass to a double cover, and then apply the local regularity theorem. This again implies that the flows $\{M^{(j,l)}_{s+i} : s \in (-\frac{1}{2}, 0]\}$ have uniformly bounded curvature if $j$ is sufficiently large. This again contradicts the choice of $t_j$. This completes the proof of Proposition 6.2.

Proposition 6.2 directly implies that $T_j = \infty$ if $j$ is sufficiently large. From this, the conclusion of Theorem 6.1 follows easily.

References

[1] K. Brakke, The motion of a surface by its mean curvature, Princeton University Press (1978)
[2] S. Brendle, A sharp bound for the inscribed radius under mean curvature flow, preprint (2013)
[3] S. Brendle and G. Huisken, Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$, preprint (2013)
[4] T. Colding and W. Minicozzi, Generic mean curvature flow I: generic singularities, Ann. of Math. 175, 755–833 (2012)
[5] K. Ecker, Regularity Theory for Mean Curvature Flow, Birkhäuser, Boston, 2004
[6] R. Hamilton, Four-manifolds with positive isotropic curvature, Comm. Anal. Geom. 5, 1–92 (1997)
[7] R. Haslhofer and B. Kleiner, Mean curvature flow of mean convex hypersurfaces, preprint (2013)
[8] J. Head, On the mean curvature evolution of two-convex hypersurfaces, J. Diff. Geom. 94, 241–266 (2013)
[9] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Diff. Geom. 31, 285-299 (1990)
[10] G. Huisken and C. Sinestrari, Mean curvature flow with surgeries of two-convex hypersurfaces, Invent. Math. 175, 137–221 (2009)
[11] J. Lauer, Convergence of mean curvature flows with surgery, Comm. Anal. Geom. 21, 355–363 (2013)
[12] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arxiv:0211159
[13] G. Perelman, *Ricci flow with surgery on three-manifolds*, arxiv:0303109
[14] B. White, *The size of the singular set in mean curvature flow of mean convex sets*, J. Amer. Math. Soc. 13, 665–695 (2000)
[15] B. White, *The nature of singularities in mean curvature flow of mean convex sets*, J. Amer. Math. Soc. 16, 123–138 (2003)
[16] B. White, *A local regularity theorem for mean curvature flow*, Ann. of Math. 161, 1487–1519 (2005)
[17] B. White, *Subsequent singularities in mean-convex mean curvature flow*, arXiv:1103.1469
[18] B. White, *Topological change in mean convex mean curvature flow*, Invent. Math. 191, 501–525 (2013)

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