We consider two $\mathbb{Z}_2$-actions on the Podleś generic quantum spheres. They yield, as noncommutative quotient spaces, the Klimek-Leszniowski $q$-disc and the quantum real projective space, respectively. The $C^*$-algebras of all these quantum spaces are described as graph $C^*$-algebras. The $K$-groups of the thus presented $C^*$-algebras are then easily determined from the general theory of graph $C^*$-algebras. For the quantum real projective space, we also recall the classification of the classes of irreducible $*$-representations of its algebra and give a linear basis for this algebra.

**Keywords:** $K$-theory of $C^*$-algebras, Galois extensions of noncommutative rings, quantum-group homogeneous spaces
1. Introduction

In noncommutative geometry, one thinks of quantum spaces as objects dual to noncommutative algebras in the sense of the Gelfand-Neumark correspondence between spaces and function algebras. Analogously to classical topology, where taking quotients of proper group actions is a standard way of obtaining new topological spaces, one is then lead to think of Hopf-algebra coinvariant subalgebras as encoding noncommutative quotient spaces. For Hopf algebras that are function algebras on finite groups, their coactions on algebras can be easily understood as group actions on algebras. Then the coinvariant subalgebras are simply fixed-point subalgebras. In this note, we study two simple examples of such actions, notably actions of $\mathbb{Z}_2$ on the algebra of the Podleś quantum sphere $S^2_{q,1}$. For this noncommutative sphere, there is still an analogue of an equator formed by the classical points (one-dimensional representations). One can quotient $S^2_{q,1}$ by the reflection with respect to the equator plane, and verify that the quotient coincides with a well-known quantum disc of Klimcz and Lesniewski [KL93]. On the other hand, one can also define the antipodal action. The quotient under this action is the quantum real projective space [HPM96]. Here we mostly review results from [HMS] concerning the aforementioned noncommutative quotient spaces, recast in the more recent language of graph $C^*$-algebras [HS].

2. Graph $C^*$-algebras

For the convenience of the reader, we briefly recall the definition of a graph $C^*$-algebra (e.g., see [FLR00]). Let $G$ be a countable graph with the set of vertices $G^0$ and the set of directed edges $G^1$. (If $e \in G^1$ then $s(e)$ and $r(e)$ are the source and range of $e$, respectively.) For the sake of simplicity, we assume that every vertex in $G$ emits only finitely many edges. Then $C^*(G)$ is, by definition, the universal $C^*$-algebra generated by partial isometries $\{S_e | e \in G^1\}$ with mutually orthogonal ranges and by mutually orthogonal projections $\{P_v | v \in G^0\}$ such that

1. $S_e^*S_e = P_{r(e)}$ for any $e \in G^1$,
2. $P_v = \sum_{s(f)=v} S_f S_f^*$, for any $v \in G^0$ emitting at least one edge.

These so-called graph $C^*$-algebras generalize the classical Cuntz-Krieger algebras [CK80].

According to the general Cuntz formula valid for graph $C^*$-algebras [RS, Theorem 3.2], the $K$-theory of $C^*(G)$ can be calculated as follows. Let $G^0_+$ be the set of those vertices of a graph $G$ that emit at least one edge, and let $\mathbb{Z}G^0$ and $\mathbb{Z}G^0_+$ be the free abelian groups with generators $G^0$ and $G^0_+$, respectively. Let $A_G : \mathbb{Z}G^0_+ \to \mathbb{Z}G^0$ be the map defined by

$$A_G(v) = \left( \sum_{s(e)=v, e \in G^1} r(e) \right) - v. \quad (1)$$

Then

$$K_0(C^*(G)) \simeq \text{Coker}(A_G), \quad K_1(C^*(G)) \simeq \text{Ker}(A_G). \quad (2)$$
3. Quantum spheres

The quantum spheres $S_{q,c}^2$, $0 < |q| < 1$, $c \in [0,\infty]$, were discovered by Podleś as $SU_q(2)$-homogeneous spaces [P-P87]. A uniform description of all these spheres can be given in the following way. First, we change the Podleś parameter $c \in [0,\infty]$ into $s \in [0,1]$ via the formula $c = (s^{-1} - s)^{-2}$ (equivalently, $s = \frac{2\sqrt{s}}{1+\sqrt{1+4c}}$) [BM00]. Then we can rescale the Podleś generators $A$ and $B$:

$$K := \begin{cases} (1-s^2)A & \text{for } s \in [0,1] \text{ (i.e., } c \in [0,\infty]) \\ A & \text{for } s = 1 \text{ (i.e., } c = \infty) \end{cases}$$

$$L := \begin{cases} (1-s^2)B & \text{for } s \in [0,1] \text{ (i.e., } c \in [0,\infty]) \\ B & \text{for } s = 1 \text{ (i.e., } c = \infty) \end{cases}$$

This allows us to define the coordinate $*$-algebras of the family of Podleś quantum spheres $S_{q,s}^2$ as the $*$-algebras generated by $K$ and $L$ satisfying the relations:

$$K = K^*, \quad LK = q^2 KL, \quad L^* L + K^2 = (1-s^2)K + s^2, \quad LL^* + q^4 K^2 = (1-s^2)q^2 K + s^2.$$  \hfill (5)

The $C^*$-algebra $C(S_{0,s}^2)$ can be given as the norm closure of $O(S_{0,s}^2)$.

It seems interesting that the algebra $O(S_{0,1}^2)$ is isomorphic with $O(SU_{q^2}(2))/\langle b - b^* \rangle$, where $O(SU_{q^2}(2))$ is the polynomial algebra of $SU_{q^2}(2)$ and $b$ is the upper-off-diagonal generator in the fundamental representation $\begin{pmatrix} a & b \\ -q^{-2}b^* & a^* \end{pmatrix}$ of $SU_{q^2}(2)$. (Cf. the paragraph above Proposition 3.2 in [HS] and the relevant considerations in [HL].) Indeed, it follows immediately from the defining relations of $O(S_{0,1}^2)$ and $O(SU_{q^2}(2))$ (e.g., see [W-SL87] and put $\nu = q^2$, $\alpha = a$, $\gamma = q^{-2}b^*$) that the assignment $K \mapsto q^{-2}b$, $L \mapsto a$ defines an algebra epimorphism $F : O(S_{0,1}^2) \to O(SU_{q^2}(2))/\langle b - b^* \rangle$. The injectivity of $F$ follows from the representation theory of these algebras. The representations $\rho_\pm$ of $O(SU_{q^2}(2))$ [VSS88]

$$\rho_+(b)e_k = \pm q^{2(k+1)}e_k, \quad \rho_-(a)e_k = \sqrt{1-q^{4l}}e_{k-1}, \quad \langle e_k|e_l \rangle = \delta_{kl}, \quad k, l \in \mathbb{N},$$

annihilate the ideal $\langle b - b^* \rangle$ and composed with $F$ agree with Podleś representations [P-P87]: $\pi_\pm = \rho_\pm \circ F$. Since $\pi_+ \oplus \pi_-$ is faithful, one can conclude that $F$ is injective. Hence it is an isomorphism. The geometric meaning of this isomorphism is that $S_{0,1}^2$ is embedded as an equator in $SU_{q^2}(2)$ thought of as a quantum 3-sphere. The other extreme value of $s$, i.e., $s = 0$, also simplifies the relations. In this case, it turns out that $O(S_{0,0}^2)$ is isomorphic with the fixed-point subalgebra $O(SU_q(2))^{U(1)}$, so that we can interpret $S_{0,0}^2$ as the quotient sphere $SU_q(2)/U(1)$ in the spirit of the Hopf fibration. (Here the action of $U(1)$ is given by rescaling the generator $a$ by $e^{i\theta}$ and $b$ by $-e^{i\theta}$.) This way we can view the family of Podleś spheres $S_{q,s}^2$ as an approximation between the quotient sphere $S_{0,0}^2 \simeq SU_q(2)/U(1)$ and the embedded sphere $S_{0,1}^2 \subset SU_q(2)$. Although the desired $\mathbb{Z}_2$-actions can be defined on the $C^*$-level for any $s \in [0,1]$ (for $s > 0$, $C(S_{q,s}^2) \simeq C(S_{0,1}^2)$), in what follows, we restrict our attention to $S_{0,1}^2$ because this is where we can define these actions on the algebraic level.
It is shown in [HS, Proposition 3.1] that, for any $q \in ]0,1[$, there exists a $C^*$-algebra isomorphism $\phi_q$ from $C(S^2_{q,1})$ to the graph $C^*$-algebra $C^*(G_1)$ corresponding to the following graph:

![Graph Image]

(Note that $C^*(G_1)$ is generated by partial isometries $S_e$, $S_{f_1}$ and $S_{f_2}$, corresponding to the edges of $G_1$.) The matrix of the group homomorphism $A_{G_1}$ (see (1)) is

$$A_{G_1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z} \rightarrow \mathbb{Z}^2.$$  

(7)

Hence the formulas (2) yield $K_0(C(S^2_{q,1})) \simeq \mathbb{Z}^2$, $K_1(C(S^2_{q,1})) \simeq 0$, which agrees with the computation of [MNW91] and with the classical situation.

Note that the primitive ideal space $\text{Prim}(C(S^2_{q,1}))$ (coinciding with $\text{Prim}(C(S^2_{q,s}))$ for all $s \in ]0,1[$) can be described as follows: One has a circle $S^1$ with the usual topology and two extra points, which are separated from each other but can not be separated from the circle. This is easily derived from the irreducible representations [P-P87] or using general arguments of the theory of graph $C^*$-algebras.

4. Quantum disc

The $*$-algebra of the quantum disc $D_q$ is defined as follows:

$$\mathcal{O}(D_q) := \mathbb{C} \langle x, x^* \rangle / (x^*x - qx^*- (1-q)), \quad 0 < q < 1.$$  

(8)

This is a one-parameter subfamily of a two-parameter family of quantum discs introduced by Klimek and Lesniewski [KL93] as homogeneous spaces of $SU_q(1,1)$. It was shown in [KL93] that $\|\pi(x)\| = 1$ in every bounded $*$-representation of $\mathcal{O}(D_q)$. Thus the $C^*$-algebra $C(D_q)$ can be defined using the supremum over the norms in bounded $*$-representations. (Note here that there are unbounded representations – the disc algebra is easily transformed into the $q$-oscillator algebra.) As also shown in [KL93], $C(D_q)$ is isomorphic to the $C^*$-algebra of the one-sided shift (Toeplitz algebra). This isomorphism is provided by a faithful infinite dimensional representation $\pi$ [KL93, p.14]. The one-dimensional representations of $C(D_q)$ form a circle that can be considered the boundary of the quantum disc $D_q$. The infinite dimensional representation corresponds to the interior of that disc.
We define now a $\mathbb{Z}_2$-action by sending $1 \in \mathbb{Z}_2$ to $r_1 : C(S^2_{q,1}) \to C(S^2_{q,1})$ that “identifies upper and lower hemispheres”:

$$r_1(L) = L, \quad r_1(K) = -K.$$  \hfill (9)

We have shown in [HMS] the following results concerning this action:

- The fixed-point polynomial algebra $\mathcal{O}(S^2_{q,1}/\mathbb{Z}_2) := \{ a \in \mathcal{O}(S^2_{q,1}) \mid r_1(a) = a \}$ is the $\ast$-subalgebra generated by $L$, and can be identified with $\mathcal{O}(D^4_q)$ by sending the generator $x$ to the generator $L$. This extends to the polynomial level the fact that the $r_1$-fixed point subalgebra of $C(S^2_{q,1})$ coincides with the Toeplitz algebra.

- The $\mathbb{Z}_2$-extension $\mathcal{O}(D^4_q) \subset \mathcal{O}(S^2_{q,1})$ defined in this way is not Galois. (The $\mathbb{Z}_2$-action is not free.)

- All $\ast$-representations of $\mathcal{O}(D^4_q)$ are restrictions of $\ast$-representations of $\mathcal{O}(S^2_{q,1})$.

- The automorphism $r_1$ commutes with the $SU_q(2)$-induced $U(1)$-action on $S^2_{q,1}$.

Under the aforementioned isomorphism $\phi_q : C(S^2_{q,1}) \to C^*(G_1)$, the order two automorphism $r_1$ of $C(S^2_{q,1})$ is transformed into the automorphism of $C^*(G_1)$ determined by

$$S_e \mapsto S_e, \quad S_{f_1} \mapsto S_{f_{3-1}}.$$  \hfill (10)

This automorphism of $C^*(G_1)$ is induced from the automorphism of the graph $G_1$ that fixes the edge $e$ and interchanges $f_1$ with $f_2$. The fixed-point subalgebra for this $\mathbb{Z}_2$-action coincides with the Toeplitz algebra and corresponds to the following graph:

\[ \text{G}_2 \]

The $K$-theory of the Toeplitz algebra is well-known (e.g. see [W-NE93]). Here, we can directly determine it from (1) and (2):

$$A_{G_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{Z} \to \mathbb{Z}^2, \quad K_0(C(D_q)) \simeq \mathbb{Z}, \quad K_1(C(D_q)) \simeq 0.$$  \hfill (11)

5. Quantum two-dimensional real projective space

We define the antipodal $\mathbb{Z}_2$-action on $S^2_{q,1}$ by sending $1 \in \mathbb{Z}_2$ to the $\ast$-automorphism $r_2$ of $C(S^2_{q,1})$ defined by

$$r_2(K) = -K, \quad r_2(L) = -L.$$
The \( \mathcal{O}(\mathbb{RP}_2^2) := \{ a \in \mathcal{O}(S^2_{q,1}) | r_2(a) = a \} \) describes the quantum two-dimensional real projective space \( \mathbb{RP}_2^2 \). This algebra is generated by \( P = K^2, \ R = L^2, \ T = KL \). They fulfill the relations

\[
\begin{align*}
P &= P^*, \\
RP &= q^8 PR, \quad RT = q^4 TR, \quad PT = q^{-4} TP, \\
T^2 &= q^2 PR, \quad RT^* = q^2 T(-q^4 P + I), \quad R^* T = q^{-2} T^*(-P + I), \\
RR^* &= q^{12} P^2 - q^4 (1 + q^4) P + I, \quad R^* R = q^{-4} P^2 - (1 + q^{-4}) P + I, \\
TT^* &= -q^4 P^2 + P, \quad T^* T = q^{-4} (P - P^2). 
\end{align*}
\]

One can show that \( \mathcal{O}(\mathbb{RP}_2^2) \) is isomorphic to the free *-algebra generated by \( P, R, T \) divided by the ideal of relations (12)–(16), see [HMS]. Concerning representations, we have:

**Theorem 1.** [HMS] There are no unbounded *-representations of the *-algebra \( \mathcal{O}(\mathbb{RP}_2^2) \).

Up to unitary equivalence, all irreducible *-representations of this algebra are the following:

(i) A family of one-dimensional representations \( \rho_\theta : \mathcal{O}(\mathbb{RP}_2^2) \to \mathbb{C} \), parameterised by \( \theta \in [0, 2\pi) \), given by

\[
\rho_\theta(P) = \rho_\theta(T) = 0, \quad \rho_\theta(R) = e^{i\theta}. \tag{17}
\]

(ii) An infinite dimensional representation \( \rho \) (in a Hilbert space \( H \) with an orthonormal basis \( \{ e_k \}_{k \in \mathbb{N}} \) given by

\[
\begin{align*}
\rho(P)e_k &= q^{4k} e_k, \\
\rho(T)e_k &= q^{2(k-1)} \sqrt{1 - q^{4k}} e_{k-1}, \\
\rho(R)e_k &= \sqrt{(1 - q^{4k})(1 - q^{4(k-1)})} e_{k-2}. \tag{20}
\end{align*}
\]

**Proof (sketch):** The boundness of \( \rho(P) = \rho(P^*) \) comes from the relation \( T^* T = q^{-4}(P - P^2) \). Then it is an easy consequence of the relations (15) and (16) that \( \rho(R) \) and \( \rho(T) \) are also bounded. Next, the assumption \( \rho(P) = 0 \) immediately leads to \( \rho(T) = 0 \), and the only remaining relations are \( \rho(R^* R) = 1 = \rho(R R^*) \). This yields the one-dimensional representations. On the other hand, assuming \( \rho(P) \neq 0 \) implies, by the irreducibility of \( \rho \), that \( \text{Ker}\rho(P) = 0 \). Using the characterisation of the spectrum by approximate eigenvectors, and taking advantage of the relations, it is possible to identify the spectrum of \( \rho(P) \) with \( \{ 0 \} \cup \{ q^{4k} | k \in \mathbb{N} \} \). Now, one builds a Hilbert space out of the eigenvectors of \( \rho(P) \), and identifies \( \rho(T) \) and \( \rho(R) \) as weighted-shift operators.

Note that the irreducible *-representations of \( \mathcal{O}(\mathbb{RP}_2^2) \) are the restrictions of the irreducible *-representations of \( \mathcal{O}(S^2_{q,1}) \).

**Proposition 1**

(i) The infinite dimensional representation \( \rho \) of \( \mathcal{O}(\mathbb{RP}_2^2) \) is faithful.

(ii) The set \( \{ P^k R^l | k, l \in \mathbb{N} \} \cup \{ P^k R^{*-l} | k \in \mathbb{N}, l \in \mathbb{N} \setminus \{ 0 \} \} \cup \{ P^k R^l T | k, l \in \mathbb{N} \} \cup \{ P^k R^{*-l} T^* | k, l \in \mathbb{N} \} \) is a basis of the vector space \( \mathcal{O}(\mathbb{RP}_2^2) \).
We know from the proof of [HMS, Proposition 4.3] that the set (ii) generates $\O(\mathbb{R}P^2_q)$ as a vector space. Thus, a general element of $\O(\mathbb{R}P^2_q)$ is of the form

$$x = \sum_{k,l \geq 0} a_{kl} P^k R^l T + \sum_{k,l \geq 0} b_{kl} P^k R^l T^* + \sum_{k,l \geq 0} c_{kl} P^k R^l + \sum_{k \geq 0, l \geq 1} d_{kl} P^k R^l. \quad (21)$$

For the proof of both claims of the proposition it is sufficient to show that it follows from $\rho(x) = 0$ that all the coefficients $a_{kl}, b_{kl}, c_{kl}, d_{kl}$ vanish. Acting with $\rho(x)$ onto a basis vector $e_n$ of the representation space, making use of (18) – (20) and the formulas

$$\rho(T^*) e_n = q^{2n} \sqrt{1 - q^{4(n+1)}} e_{n+1},$$

$$\rho(R^*) e_n = \sqrt{(1 - q^{4(n+1)})(1 - q^{4(n+2)})} e_{n+2},$$

we obtain

$$\rho(x) e_n = \sum_{k \geq 0, l \leq \frac{n-1}{2}} a_{kl} q^{4k(n-1-2l)} \sqrt{(1 - q^{4(n-2l)})} \cdots (1 - q^{4(n-1)}) q^{2(n-1)} \sqrt{1 - q^{4n}} e_{n-1-2l}$$

$$+ \sum_{k,l \geq 0} b_{kl} q^{4k(n+1+2l)} \sqrt{(1 - q^{4(n+1+2l)})} \cdots (1 - q^{4(n+2)}) q^{2n} \sqrt{1 - q^{4(n+1)}} e_{n+1+2l}$$

$$+ \sum_{k \geq 0, l \leq \frac{n}{2}} c_{kl} q^{4k(n-2l)} \sqrt{(1 - q^{4(n-2l+1)})} \cdots (1 - q^{4n}) e_{n-2l}$$

$$+ \sum_{k \geq 0, l \geq 1} d_{kl} q^{4k(n+2l)} \sqrt{(1 - q^{4(n+2l)})} \cdots (1 - q^{4(n+1)}) e_{n+2l}$$

$$= 0.$$

Fixing $l$ we get the following four sets of equations:

$$\sum_{k \geq 0} a_{kl} q^{4k(n-1-2l)} = 0, \quad n \geq 1, \quad l \leq \frac{n-1}{2}, \quad (22)$$

$$\sum_{k \geq 0} b_{kl} q^{4k(n+1+2l)} = 0, \quad n \geq 0, \quad l \geq 0, \quad (23)$$

$$\sum_{k \geq 0} c_{kl} q^{4k(n-2l)} = 0, \quad n \geq 0, \quad l \leq \frac{n}{2}, \quad (24)$$

$$\sum_{k \geq 0} d_{kl} q^{4k(n+2l)} = 0, \quad n \geq 0, \quad l \geq 1. \quad (25)$$

Let us consider the equations (23) for fixed $l \geq 0$. Since $b_{kl}$ is different from 0 only for a finite number of indices $k$, there is a biggest $k = N$ with $b_{kl} \neq 0$. Then the first $N$ (i.e., $n = 0, \ldots, N$) of the equations (23) are a system of linear equations for $b_{0l}, \ldots, b_{Nl}$
whose matrix is a Vandermonde matrix
\[
\begin{pmatrix}
1 & x_0 & x_3^2 & \ldots & x_N^N \\
1 & x_1 & x_1^2 & \ldots & x_1^N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_N & x_N^2 & \ldots & x_N^N
\end{pmatrix}
\]
with \(x_i = q^{4(i+1/2)}\). Since \(x_i \neq x_j\) for \(i \neq j\), the determinant \(\prod_{0 \leq i < j \leq N}(x_j - x_i)\) of this matrix is nonzero. It follows that \(b_{kl} = 0\). One argues analogously for the remaining equations (22), (24) and (25). Note that for (22) and (24) one has to consider a set of \(x\) with \(x\) determined by \(C\) algebra.

The fixed-point subalgebra for this action coincides with \(C\) algebra of the following graph \([H-S]\):

\[
G_3
\]
The group homomorphism $A_G$ (see (1)) has the form

$$A_G = \begin{pmatrix} 0 \\ 2 \end{pmatrix} : \mathbb{Z} \rightarrow \mathbb{Z}^2.$$  

(26)

It follows now from (2) that the $K$-groups of $C(\mathbb{R}P^2_q)$ agree with their classical counterparts:

**Theorem 2.** [HMS] $K_0(C(\mathbb{R}P^2_q)) \simeq \mathbb{Z} \oplus \mathbb{Z}_2, \quad K_1(C(\mathbb{R}P^2_q)) \simeq 0.$

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