On static spherically symmetric solutions of
the Bach-Einstein gravitational field equations

Hans - Jürgen Schmidt

Universität Potsdam, Institut für Mathematik, Am Neuen Palais 10
D-14469 Potsdam, Germany, E-mail: hjschmi@rz.uni-potsdam.de

Abstract

For field equations of 4th order, following from a Lagrangian “Ricci
calar plus Weyl scalar”, it is shown (using methods of non-standard
alysis) that in a neighbourhood of Minkowski space there do not
exist regular static spherically symmetric solutions. With that (be-
sides the known local expansions about $r = 0$ and $r = \infty$ resp.) for
the first time a global statement on the existence of such solutions is
given. Finally, this result will be discussed in connection with Ein-
stein’s particle programme.

Für die Feldgleichungen 4. Ordnung, die aus einem Lagrangeaus-
druck “Ricci-Skalar plus Weyl-Skalar” folgen, wird unter Zuhilfenahme
von Methoden der Nicht-Standard-Analysis gezeigt, daß in einer Umge-
bung des Minkowskiraumes keine statischen kugelsymmetrischen Lösun-
gen existieren. Damit wird erstmals neben den bekannten lokalen
Entwicklungen um $r = 0$ und $r = \infty$ eine globale Aussage über die
Existenz solcher Lösungen getroffen. Anschließend wird dies Ergebnis
im Zusammenhang mit Einstein’s Teilchenprogramm diskutiert.
1 Introduction

General Relativity Theory starts from the Einstein-Hilbert Lagrangian $L_{EH} = R/2\kappa$ with the Ricci scalar $R$ which leads to Einstein’s vacuum field equation $\nabla^k R_k^{\ i} = 0$ being of second order in the metrical tensor $g_{ik}$. Its validity is proven with high accuracy in space-time regions where the curvature is small only. Therefore, the additional presence of a term being quadratic in the curvature is not excluded by the standard weak field experiments.

Lagrangians with squared curvature have already been discussed by Weyl (1919), Bach (1921), and Einstein (1921). They were guided by ideas about conformal invariance, and Einstein (1921) proposed to look seriously to such alternatives. In Bach (1921) $L_W = C/2\kappa$ with the Weyl scalar $2C = C_{ijkl}C^{ijkl}$ was of special interest. Then it became quiet of them for a couple of decades because of the brilliant results of GRT but not at least because of mathematical difficulties.

Recently, the interest in such equations has anewed with arguments coming from quantum gravity, cf. e.g. Treder (1975), Borzeszkowski, Treder, Yourgrau (1978), Stelle (1978), Fiedler, Schimming (1983) and ref. cited there. In the large set of possible quadratic modifications a linear combination $L = L_{EH} + l^2 L_W$ enjoyed a special interest, cf. e.g. Treder (1977). The coupling constant $l$ has to be a length for dimensional reasons and it has to be a small one to avoid conflicts with observations: One often takes Heisenberg length (= Compton wave length of the proton) $= 1.3 \cdot 10^{-18}$ cm or Planck length $1.6 \cdot 10^{-33}$ cm.

In the present paper we consider this Lagrangian $L$ in connection with Einstein’s particle programme, see Einstein, Pauli (1943): One asks for spherically symmetric singularity-free asymptotically flat solutions of the vacuum field equations which shall be interpreted as particles, but this cannot be fulfilled within GRT itself; it is still a hope (cf. Borzeszkowski (1981)) to realize it in such 4th order field equations. Two partial answers have al-
ready been given: Stelle (1978) showed the gravitational potential of the linearized equations to be

\[ \Phi(r) = -\frac{m}{r} + \exp(-r/l) c/r \quad \text{for} \quad r \ll l \quad ; \quad (1) \]

he also gave an expansion series about \( r = 0 \), and Fiedler, Schimming (1983) proved its convergence and smoothness in a certain neighbourhood of \( r = 0 \), but said nothing about the convergence radius of this expansion.

Now we want to join these two local expansions. To this end we write down the field equations in different but equivalent versions (sct. 2), calculate some linearizations (sct. 3) and prove the statement of the summary in section 4 which will be followed by a discussion of Einstein’s particle programme in section 5.

## 2 Notations and field equations

We start from a space-time metric with signature \((+---)\), the Riemann and Ricci tensor being defined by \( R^i_{jkl} = \Gamma^i_{j,l,k} - \ldots \), \( R_{ik} = R^i_{ijk} \) resp. The Weyl tensor \( C^i_{jkl} \) is the traceless part of the Riemann tensor and the Weyl scalar is given by \( 2C = C_{ijkl} C^{ijkl} \). Light velocity is taken to be 1 and \( G = \kappa / 8\pi \) is Newton’s constant; \( g = \det g_{ij} \).

Then we consider the Lagrangian

\[ \mathcal{L} = \sqrt{-g} L = \sqrt{-g} (R + l^2 C) / 2\kappa + \mathcal{L}_{\text{mat}} , \quad (2) \]

where \( \mathcal{L}_{\text{mat}} \) is the matter Lagrangian. For writing down the corresponding field equation it is convenient to introduce the Bach tensor \( B_{ij} \), cf. Bach (1921) and Wünsch (1976), beforehand.

\[ \frac{1}{2} B_{ij} = \frac{\kappa \delta (\sqrt{-g} L_W)}{\sqrt{-g} \delta g^{ij}} = C^a_{ij} b_{ba} + \frac{1}{2} C^a_{ij} b R_{ba} . \quad (3) \]

It holds \( B^i_i = 0 \), \( B^i_{i;j} = 0 \), \( B_{ij} = B_{ji} \), and \( B_{ij} \) conformally invariant of weight -1.
Variation of (2) with respect to $g_{ij}$ leads to the field equation

$$R_{ij} - \frac{1}{2} g_{ij} R + l^2 B_{ij} = \kappa T_{ij}. \quad (4)$$

Now let us consider the vacuum case $T_{ij} = 0$. The trace of eq. (4) then simply reads $R = 0$, and eq. (4) becomes equivalent to the simpler one

$$R_{ij} + l^2 B_{ij} = 0. \quad (5)$$

Writing $k = l^{-2}$ and $\Box = g^{ij} (\cdot)_{;ij}$ one gets also the equivalent system

$$R = 0; \quad kR_{ij} + \Box R_{ij} = 2R_{iabj} R^{ab} + \frac{1}{2} g_{ij} R^{ab} R_{ab}. \quad (6)$$

The static spherically symmetric line element in Schwarzschild coordinates reads

$$ds^2 = (1 + \beta) e^{-2\lambda} dt^2 - (1 + \beta)^{-1} dr^2 - r^2 d\Omega^2, \quad (7)$$

where $\beta$ and $\lambda$ depend on $r$ only. The dot means differentiation with respect to $r$, and defining

$$\alpha = 2\beta - 2r\dot{\beta} + 2r\lambda (1 + \beta) + 2r^2 (\dot{\lambda}^2 - \ddot{\lambda}) (1 + \beta) + r^2 (\ddot{\beta} - 3\dot{\beta} \dot{\lambda}),$$

$$\zeta = r\dot{\alpha}, \quad \eta_1 = \alpha r^{-3}, \quad \eta_2 = 3\beta r^{-1}, \quad \eta_3 = \zeta r^{-3},$$

the field equations (6) are just eqs. (2.15.a-c) and (2.16) of the paper of Fiedler, Schimming (1983). To avoid the products $r\dot{\eta}_i$ we define a new independent variable $x$ by $r = le^x$. Then $r\dot{\eta}_i = \eta'_i = d\eta_i / dx$ and we obtain

$$0 = \eta'_1 + 3\eta_1 - \eta_3, \quad (8)$$

$$0 = (k + r\eta_3/6)\eta'_2 + \eta_1 + \eta_3 + r^2 k\eta_1 + r^3 \eta_1 \eta_3/6 + r\eta_2 \eta_3/2 + r^3 \eta_1^2/4, \quad (9)$$

$$0 = (1 + r\eta_2/3)\eta'_3 - 2\eta_1 + 2k\eta_2 + 2\eta_3 - r^2 k\eta_1 - r^3 \eta_1^2/2 - r^3 \eta_1 \eta_3/6 + 2r\eta_2 \eta_3/3. \quad (10)$$

Conversely, if the system eqs. (8-10) is solved, the metric can be obtained by

$$\beta = r\eta_2/3, \quad \lambda = (2r\dot{\eta}_2 + 2\eta_2 + r^2 \eta_1)/(6 + 2r\eta_2), \quad \lambda(0) = 0. \quad (11)$$
3 Linearizations

Now we make some approximations to obtain the rough behaviour of the solutions. First, for $r \to \infty$, Steelle (1978) has shown the following: Linearization about Minkowski space leads (using our notations (1) and (7)) to $\lambda = 0$ and $\beta = 2\Phi$, where powers of $\Phi$ are neglected and the term $\exp(r/l)/r$ in $\Phi$ must be suppressed because of asymptotical flatness. Further, a finite $\Phi(0)$ requires $c = m$ in (1). Then $\Phi$ is just the Bopp-Podolsky potential (stemming from 4th order electrodynamics), and it were Pechlaner, Sexl (1966) who proposed this form as representing a gravitational potential.

Now we have $\Phi(r) = m(\exp(-r/l) - 1)/r = -m/l + rm/(2l^2) + \ldots$, but also this finite potential gives rise to a singularity: the invariant $R_{ij}R^{ij} \approx m^2/r^4$ as $r \to 0$, that means, the linearization makes no sense for this region. From this one can already expect that in a neighbourhood of Minkowski space no regular solutions exist.

Second, for $r \to 0$, Fiedler, Schimming (1983) proved that there exists a one-parameter family of solutions being singularity-free and analytical in a neighbourhood of $r = 0$. The parameter will be called $\epsilon$ and can be defined as follows: neglecting the terms with $r$ in eqs. (8-10) one obtains a linear system with constant coefficients possessing just a one-parameter family of solutions being regular at $r = 0$; it reads

$$\eta_1 = \epsilon k^2 r, \quad \eta_2 = -5\epsilon kr, \quad \eta_3 = 4\epsilon k^2 r.$$ (12)

(The factors $k^n$ are chosen such that $\epsilon$ becomes dimensionless.) Now one can take (12) as the first term of a power series $\eta_i = \Sigma_n a_i^{(n)} r^n$, and inserting this into eqs. (8-10) one iteratively obtains the coefficients $a_i^{(n)}$. For $n$ even, $a_i^{(n)}$ vanishes: The $r^3$-terms are $k^3(\epsilon/14 + 10\epsilon^2/21)$, $-k^2(\epsilon/2 - 10\epsilon^2/3)$ and

\footnote{That means, loosely speaking, the right hand side of eq. (6) will be neglected. Connected with this one may doubt the relevance of the term $cr^{-1}e^{-r/l}$ in eq. (1) for $m \neq 0$ and $r \to \infty$, because it is small compared with the neglected terms.}
Furthermore, the $r^{2n-1}$-term is always a suitable power of $k$ times a polynomial in $\epsilon$ of the order $\leq n$.

Up to the $r^2$-terms the corresponding metric (7) reads

$$ds^2 = (1 + 5k\epsilon r^2/3)dt^2 - (1 + 5k\epsilon r^2/3)dr^2 - r^2d\Omega^2. \tag{13}$$

Third, we look for a linearization which holds uniformly for $0 \leq r < \infty$. A glance at (11) and (7) shows that $\eta_i = 0$ gives Minkowski space, and therefore we neglect terms containing products of $\eta_i$ in (8-10). In other words, instead of the linearization used before we additionally retain the term $r^2\eta_1$. Then we proceed as follows: from eqs. (8) and (10) we obtain

$$\eta_3 = \eta_1' + 3\eta_1 \quad \text{and} \quad \eta_2 = \frac{-\eta_1' - 5\eta_1 + (e^{2x} - 4)\eta_1 + 3e^x\eta_1(\eta_1 + \eta_1'/6)}{2k + le^x(\eta_1' + 5\eta_1 + 6\eta_1)/3}. \tag{15}$$

Inserting eqs. (14/15) into eq. (9) one obtains a third order equation for $\eta_1$ only whose linearization reads

$$0 = \eta_1''' + 5\eta_1'' + \eta_1'(2 - e^{2x}) - \eta_1(8 + 4e^{2x}). \tag{16}$$

The solution of eq. (16) which is bounded for $x \to -\infty$ reads

$$\eta_1 = \gamma \cdot \left[(3e^{-4x} + e^{-2x}) \sinh e^x - 3e^{-3x} \cosh e^x\right]. \tag{17}$$

A comparison with (12) gives $\gamma = 15\epsilon l^{-3}$.

---

2In MÜLLER, SCHMIDT (1983) the same vacuum field equations were discussed for axially symmetric Bianchi type I models. They possess a four-parameter group of isometries, too, and the essential field equation is also a third order equation for one function. The difference is that in the present case spherical symmetry implies an explicit coordinate dependence.

3A second solution is $\eta_1 = -12mr^{-4}$ leading to the Schwarzschild solution (this solution solves both the full and the linearized equations in accordance with the fact that it makes zero both sides of eq. (6)), and the third one can be obtained from them up to quadrature by usual methods.
Now we insert this $\eta_1$ into eq. (15/11) and neglect again all powers of $\eta_1$; then the metric reads

\[
\beta = 5\epsilon \left[ l r^{-1} \sinh(r l^{-1}) - \cosh(r l^{-1}) \right],
\]

\[
\lambda = \frac{5\epsilon}{2l} \int_0^r \left[ (l^2 z^{-2} - 1) \sinh(z l^{-1}) - lz^{-1} \cosh(z l^{-1}) \right] dz. \tag{18}
\]

This linearization has (in the contrary to (1) and (12)) the following property: to each $r_0 > 0$ and $\Delta > 0$ there exists an $\delta > 0$ such that for $-\delta < \epsilon < \delta$ the relative error of the linearized solution (18) does not exceed $\Delta$ uniformly on the interval $0 \leq r \leq r_0$.

\section*{4 Non-standard analysis}

In this section we will prove that in a neighbourhood of Minkowski space there do not exist any solutions. “Neighbourhood of Minkowski space” is in general a concept requiring additional explanations because of the large number of different topologies discussed in literature. To make the above statement sufficiently strong we apply a quite weak topology here: given a $\delta > 0$ then all space-times being diffeomorphic to Minkowski space and fulfilling $|R_{ij} R^{ij}| < \delta k^2$ form a neighbourhood about Minkowski space.

In (13) we have at $r = 0$ (independently of higher order terms in $r$)

\[
R_{ij} R^{ij} = 100k^2 \epsilon^2 / 3 \quad \text{and} \quad R_{00} = 5k \epsilon. \tag{19}
\]

Therefore it holds: the one-parameter family of solutions being regular at $r = 0$ is invariantly characterized by the real parameter $\epsilon$, and a necessary condition for it to be within a neighbourhood of Minkowski space is that $\epsilon$ lies in a neighbourhood of 0.

Now we suppose $\epsilon$ to be an infinitesimal number, i.e., a positive number which is smaller than any positive real number. The mathematical theory dealing with such infinitesimals is called non-standard analysis, cf. ROBINSON (1966). The clue is that one can handle non-standard numbers like real
numbers. Further we need the so-called *Permanence principle* (= Robinson lemma): let $A(\epsilon)$ be an internal statement holding for all infinitesimals $\epsilon$. Then there exists a positive standard real $\delta > 0$ such that $A(\epsilon)$ holds for all $\epsilon$ with $0 < \epsilon < \delta$. The presumption “internal” says, roughly speaking, that in the formulation of $A(.)$ the words “standard” and “infinitesimal” do not appear. For a more detailed explanation cf. the literature.

This permanence principle shall be applied as follows: $A(\epsilon)$ is the statement: “Take (12) as initial condition for eqs. (8-10) and calculate the corresponding metric (7/11). Then there exists an $r_0 \leq l$ such that $R_{ij} R^{ij} \geq k^2$ at $r = r_0$.”

**Remark.** At this point it is not essential whether $r_0$ is a standard or an (infinitely large) non-standard number.

**Proof** of $A(\epsilon)$ for infinitesimals $\epsilon$ with $\epsilon \neq 0$: the difference of (18) in relation to the exact solution is of the order $\epsilon^2$, i.e. the relative error is infinitesimally small. For increasing $r$, $|\beta|$ becomes arbitrary large, i.e. $(1 + \beta)^{-1}$ becomes small and $R_{ij} R^{ij}$ increases to arbitrarily large values, cf. eq. (18). Now take $r_0$ such that with metric (7/18) $R_{ij} R^{ij} \geq 2k^2$ holds. Then, for the exact solution, $R_{ij} R^{ij} \geq k^2$ holds at $r = r_0$, because their difference was shown to be infinitesimally small.

Now the permanence principle tells us that there exists a positive standard real $\delta > 0$ such that for all $\epsilon$ with $0 < |\epsilon| < \delta$ the corresponding exact solution has a point $r_0$ such that at $r = r_0$ $R_{ij} R^{ij} \geq k^2$ holds. But “$R_{ij} R^{ij} < k^2$” is another necessary condition for a solution to lie within a neighbourhood of Minkowski space.

**Remark.** Supposed this $r_0$ is an infinitely large non-standard number, then by continuity arguments also a standard (finite) number with the same property exists.

Now the statement is proved, but we have learned nothing about the actual value of the number $\delta$. Here, numerical calculations may help. They were
performed as follows: the power series for the functions \( \eta \) were calculated up to the \( r^6 \)-term, then these functions taken at \( x = -4 \) (i.e. \( r = 0.018 \) \( l \)) were used as initial conditions for a Runge–Kutta integration of eqs. (8-10). We got the following result: firstly, for \( \epsilon = \pm 10^{-5} \) and \( r \leq 10 \) \( l \), the relative difference between the linearized solution (17) and the numerical one is less than 2 per thousand. Secondly, for \( 0 < |\epsilon| \leq 1 \) the behaviour \( \beta \to -\infty \cdot sgn \epsilon \) is confirmed. That means, the statement made above keeps valid at least for \( \delta = 1 \).

Remark. For large values \( \epsilon \) the power series for the \( \eta \) converge very slowly, and therefore other methods would be necessary to decide about asymptotical flatness.

5 Discussion – Einstein’s particle programme

Fourth order gravitational field equations could be taken as a field theoretical model of ordinary matter, the energy–momentum tensor of which is defined by

\[
\kappa T^*_{ij} = R_{ij} - \frac{1}{2} g_{ij} R .
\]  

(20)

For our case one obtains at \( r = 0 \) \( T^*_{ij} \) to represent an ideal fluid with the equation of state \( p^* = \mu^*/3 \) and (cf. eq. (19))

\[
\mu^*(0) = 5k\epsilon \kappa^{-1} = 5\epsilon/8\pi G l^2 .
\]  

(21)

Inserting \( |\epsilon| \geq 1 \) and \( l \leq 1.3 \cdot 10^{-13} \) cm into eq. (21), we obtain

\[
|\mu^*(0)| \geq 1.5 \cdot 10^{53} \text{ g cm}^{-3} .
\]  

(22)

Therefore it holds: if there exists a non-trivial static spherically symmetric asymptotically flat singularity-free solution of eq. (5) at all, then the corresponding particle would be a very massive one: its phenomenological energy density exceeds that of a neutron star by at least 40 orders of magnitude.
The resulting statement can be understood as follows: For a small curvature the 4th order corrections to Einstein’s equations are small, too, and the situation should not be very different from that one we know from Einstein’s theory.

Now we want to refer to a problem concerning Schwarzschild coordinates: the transition from a general static spherically symmetric line element to Schwarzschild coordinates is possible only in the case that the function “invariant surface of the sphere \( r = \text{const.} \) in dependence on its invariant radius” has not any stationary point. Here two standpoints are possible: either one takes this as a natural condition for a reasonable particle model or one allows coordinate singularities in (7) like \( \beta \geq -1, \) and \( \beta = -1 \) at single points\(^4\) which require a special care. (The discussion made above is not influenced by this.)

The statement on the existence of solutions can be strengthened as follows: FIEDLER, SCHIMMING (1983) proved that the solutions are analytical in a neighbourhood of \( r = 0. \) Further, the differential equation is an analytical one and, therefore, in the subspace of singularity-free solutions they remain so in the limit \( r \to \infty. \)

Then, there exists only a finite or countably infinite set of values \( \epsilon_n \) such that the corresponding solution becomes asymptotically flat (the question, whether this set is empty or not, shall be subject of further investigation.); furthermore, the \( \epsilon_n \) have no finite accumulation point. That means, there exists at most a discrete spectrum of solutions.

With respect to this fact, we remark the following: as one knows, Einstein’s theory is a covariant one. But besides this symmetry, it is homothetically invariant, too. That means, if \( ds^2 \) is changed to \( e^{2\chi}ds^2 \) with constant \( \chi, \) then the tensor \( R_{ij} - \frac{1}{2}g_{ij}R \) remains unchanged, whereas the scalars \( R \) and

\(^4\)For \( \beta < -1 \) one would obtain a cosmological model of Kantowski-Sachs type. Eqs. (2-11) remain unchanged for this case.
$C$ will be divided by $e^{2\chi}$ and $e^{4\chi}$ resp. From this it follows: with one solution of Einstein’s vacuum equation one obtains by homothetical invariance just a one-parameter class of solutions. On the other hand, the sum $R + l^2 C$ has not such a symmetry and, therefore, one should not expect that a one-parameter family of solutions globally exists, and this is just in the scope of the particle programme where a definite particle’s mass is wanted.

Acknowledgement. Discussions with R. JOHN, U. KASPER and R. SCHIMMING on 4th order field equations and with J. REICHERT and H. TUSCHIK on non-standard analysis are gratefully acknowledged. Further I want to thank W. MAI and M. SCHULZ for supporting the numerical calculations.

References

BACH, R.: 1921, Math. Zeitschr. 9, 110.
BORZESZKOWSKI, H. v.: 1981, Ann. Phys. (Leipz.) 38, 239.
BORZESZKOWSKI, H. v., TREDER, H., YOURGRAU, W.: 1978, Ann. Phys. (Leipz.) 35, 471.
EINSTEIN, A.: 1921, Sitzungsber. AdW, Berlin, 1, 261.
EINSTEIN, A., PAULI, W.: 1943, Ann. Math. 44, 131.
FIEDLER, B., SCHIMMING, R.: 1983, Astron. Nachr. 304, 221.
MÜLLER, V., SCHMIDT, H.-J.: 1983, submitted to Gen. Rel. Grav. 5
PECHLANER, E., SEXL, R.: 1966, Comm. Math. Phys. 2, 165.
ROBINSON, A.: 1966, Non-standard analysis, North Holland, Amsterdam.
STELLE, K. S.: 1978, Gen. Rel. Grav. 9, 353.
TREDER, H.: 1975, Ann. Phys. (Leipz.) 32, 383.
TREDER, H.: 1977, p. 279 in: 75 Jahre Quantentheorie, ed. BRAUER, W.,

5The correct source is: H.-J. Schmidt, V. Müller: On Bianchi type I vacuum solutions in $R + R^2$ theories of gravitation II. The axially symmetric anisotropic case, Gen. Rel. Grav. 17 (1985) 971-980. (This footnote is not in the original.)
Akad.-Verlag Berlin.

WEYL, H.: 1919, Ann. Phys. (Leipz.) 59, 101.

WÜNSCH, V.: 1976, Math. Nachr. 73, 37.

(Received 1984 March 20)

In this reprint (done with the kind permission of the copyright owner) we removed only obvious misprints of the original, which was published in Astronomische Nachrichten under the title “On static spherical symmetric solutions of the Bach-Einstein gravitational field equations”, Astron. Nachr. 306 (1985) Nr. 2, pages 67 - 70; Author’s address that time: Zentralinstitut für Astrophysik der AdW der DDR, 1502 Potsdam–Babelsberg, R.-Luxemburg-Str. 17a.