Quantum States and Phases in Driven Open Quantum Systems with Cold Atoms

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An open quantum system, whose time evolution is governed by a master equation, can be driven into a given pure quantum state by an appropriate design of the system-reservoir coupling. This points out a route towards preparing many body states and non-equilibrium quantum phases by quantum reservoir engineering. Here we discuss in detail the example of a driven dissipative Bose Einstein Condensate of bosons and of paired fermions, where atoms in an optical lattice are coupled to a bath of Bogoliubov excitations via the atomic current representing local dissipation. In the absence of interactions the lattice gas is driven into a pure state with long range order. Weak interactions lead to a weakly mixed state, which in 3D can be understood as a depletion of the condensate, and in 1D and 2D exhibits properties reminiscent of a Luttinger liquid or a Kosterlitz-Thouless critical phase at finite temperature, with the role of the “finite temperature” played by the interactions.

I. INTRODUCTION

In condensed matter physics quantum phases and associated strongly correlated many body states are typically prepared by cooling the system to low temperatures, where its properties are dominated by the ground state of a Hamiltonian, \( H \ket{G} = E_G \ket{G} \), i.e. by considering a thermodynamic equilibrium situation, where \( \rho \sim e^{-H/k_B T} \rightarrow \ket{G} \bra{G} \) for temperature \( T \rightarrow 0 \). In particular, in the context of ultracold atomic quantum gases, much of the interest of the last few years has focused on engineering specific Hamiltonians based on control of microscopic system parameters via external fields, opening the door to a quantum simulation of strongly correlated ground states of bosons and of paired fermions, where atoms in an optical lattice are coupled to a bath of Bogoliubov excitations via the atomic current representing local dissipation. In the absence of interactions the lattice gas is driven into a pure state with long range order. Weak interactions lead to a weakly mixed state, which in 3D can be understood as a depletion of the condensate, and in 1D and 2D exhibits properties reminiscent of a Luttinger liquid or a Kosterlitz-Thouless critical phase at finite temperature, with the role of the “finite temperature” played by the interactions.

In contrast, quantum optics typically considers driven open quantum system, where a system of interest is driven by an external field and coupled to an environment inducing a non-equilibrium dynamics, with time evolution described by a master equation for the reduced system density operator (see e.g. 14),

\[
\dot{\rho} = -i[H, \rho] + \mathcal{L}\rho \\
\equiv -i[H, \rho] + \sum_\ell \kappa_\ell \left( 2c_\ell \rho c_\ell^\dagger - c_\ell^\dagger c_\ell \rho - \rho c_\ell^\dagger c_\ell \right). \tag{1}
\]

Here \( H \) is the Hamiltonian of the driven system, while the Liouvillean \( \mathcal{L} \) in Lindblad form represents the dissipative terms. The quantum jump operators \( c_\ell \) are system operators as they appear in the interaction Hamiltonian for the coupling to the bath of harmonic oscillators, and describe the time evolution (quantum jump) of the system associated with the emission of a quantum into the harmonic oscillator bath with rate \( \kappa_\ell \) in channel \( \ell \). The validity of the master equation is based on the Born-Markov approximation with system-bath coupling in rotating wave approximation, which in quantum optics is an excellent approximation because the (optical) system frequencies are much larger than the decay rates. For long times, the system described by (1) will approach a dynamical steady state, \( \rho(t) \rightarrow \rho_{\text{ss}} \), which in general will be a mixed state. However, under special circumstances the steady state can be a pure state, \( \rho_{\text{ss}} = |D\rangle \langle D| \), where in the language of quantum optics \( |D\rangle \) is called a dark state. Sufficient conditions for the existence of a unique dark state are (i) \( \forall \ell \; c_\ell |D\rangle = 0 \), i.e. the dark state is an eigenstate of the set of quantum jump operators with zero eigenvalue, which will be compatible with the dynamics induced by the Hamiltonian if (ii) \( H |D\rangle = E |D\rangle \). Uniqueness of the dark state is guaranteed if there is no other subspace of the system Hilbert space which is invariant under the action of the operators \( c_\ell \). In fact, it can be shown that for any given pure state there will be a master equation so that this state becomes the unique steady state.

Here, we are interested in this novel possibility of quantum state engineering by designing jump operators such that one drives the system into a desired many-body quantum state. This non-equilibrium approach is in strong contrast to conventional Hamiltonian engineering methods, as standard thermodynamics concepts are not valid in this driven system, and the dynamics governed by the master equation (1) is the only remaining principle determining the final state. While in quantum optics we know several examples of preparing single particle pure states dissipation, including dark state laser cooling to subrecoil temperatures, it is of interest to extend these ideas to many body systems, dissipatively driving the system into entangled states of interest, or preparing non-equilibrium quantum phases in condensed matter systems. Furthermore, for the example of a dissipative driven Bose Einstein condensate (BEC) discussed below, but also for stabilizer states in a system of spins-1/2 or qubits living on a lattice, the dissipation can be chosen to be quasi-local, i.e. the jump operators act non-trivially only on a small neighborhood of particles.

As an illustration of a many particle dark state, we discuss below a dissipatively driven BEC, where for non-
interacting atoms a pure state exhibiting long range order is generated as the steady state by quasi-local coupling to an environment with finite correlation length. Applying standard linearization schemes in the weakly interacting situations, allows us to determine the solution of the master equation \( \Pi \), and we find that the steady state exhibits similar properties as bosons in thermal contact to a heat bath: in 3D the effect of the interaction can be understood in terms of a depletion of the condensate, while in 1D and 2D the system exhibits properties reminiscent of a Luttinger liquid or a Kosterlitz-Thouless critical phase at finite temperature. In particular, we give a physical realization in terms of cold bosonic atoms in an optical lattice by immersion in a superfluid bath \( [19] \). As a second example we discuss a master equation whose steady state corresponds to an \( \eta \)-condensate \( [20, 21] \), i.e. a state of long range order of paired fermions, which remarkably corresponds to an exact excited eigenstate of a Hubbard Hamiltonian with repulsive or attractive interactions in \( d \) dimensions.

II. LONG RANGE ORDER BY LOCAL DISSIPATION

A. Dissipative Driven Condensate

Let us consider the dynamics of \( N \) bosonic atoms on \( d \)-dimensional lattice with \( M^d \) lattice sites, lattice vectors \( \mathbf{e}_\lambda \) and spacing \( a \) (see Fig. 1a). We assume that the coherent motion of the atoms can be described a single band Hubbard model with Hamiltonian

\[
H = H_0 + V = -J \sum_{\langle i,j \rangle} a_i^\dagger a_j + \frac{1}{2} U \sum_i a_i^2 a_i^\dagger^2,
\]

where \( H_0 \) represents the kinetic energy of bosons hopping between adjacent lattice sites with amplitude \( J \), and \( V \) is the onsite interaction with strength \( U \), and \( a_i \) (\( a_i^\dagger \)) are bosonic interactions (creation) operators for atoms at site \( i \). A physical realization of this situation is achieved by loading cold bosonic atoms into an optical lattice. By cooling to temperatures \( T \rightarrow 0 \) the system is prepared in the groundstate, which for noninteracting atoms \( |\text{BEC}\rangle = a_{i=0}^N |\text{vac}\rangle / \sqrt{N!} \) which corresponds to a state with macroscopic occupation of the quasimomentum \( \mathbf{q} = 0 \). Here \( a_\mathbf{q} = \sum_j a_j e^{i\mathbf{q}\mathbf{e}_\lambda} / \sqrt{M^d} \) is the destruction operator for quasimomentum \( \mathbf{q} \) in the Bloch band. For weak interactions we have in 3D a BEC, and a quasi-condensate in 1D and 2D, while with increasing interactions the system can undergo a quantum phase transition to a Mott phase \( [1] [6] [22] \).

In contrast, we are interested here in a dissipative Hubbard dynamics modelled by a master equation \( \Pi \). In atomic physics a bath will typically couple to the atoms via the atomic density, \( n_i = a_i^\dagger a_i \), as in the case of decoherence due to spontaneous emission in an optical lattice, or for collisional interactions. This will tend to dephase the condensate, and can heat the system. In contrast, our goal is to couple the system to a bath so that the system is driven to a purely many body state by quasi-local dissipation. This is achieved, for example, by choosing jump operators

\[
c_\ell \equiv c_{ij} = \left( a_i^\dagger + a_j^\dagger \right) \left( a_i - a_j \right)
\]

acting between a pair of adjacent lattice sites \( \ell \equiv \langle i, j \rangle \) with a dissipative rate \( \kappa_{ij} \equiv \kappa \). It is then easy to see that the state \( |\text{BEC}\rangle \) satisfies (i) \( \forall \langle i, j \rangle \) \( (a_i - a_j)|\text{BEC}\rangle = 0 \), and (ii) is an eigenstate of the kinetic energy \( H_0 \) \( |\text{BEC}\rangle = N c_{\mathbf{q}=0} |\text{BEC}\rangle \), where \( c_{\mathbf{q}} = 2J \sum_\lambda \sin^2 \mathbf{q} \mathbf{e}_\lambda / 2 \) is the single particle Bloch energy for quasimomentum \( \mathbf{q} \). In other words, \( |\text{BEC}\rangle \) is a many body dark state corresponding to a state of long range order for non-interacting bosons on the lattice. In fact (see Appendix A), this state is the unique steady state of this master equation, and any initial mixed state will evolve for long times into \( |\text{BEC}\rangle \).

The key in obtaining a state of long range order as the steady state is to couple to the bath involving the atomic current operator between two adjacent lattice sites; the concept that dissipative coupling to the current operator stabilizes superconductivity is well known in condensed matter \( [23] \). The jump operator \( c_{ij} \) describes a pumping process where the second factor \( a_i - a_j \) annihilates the anti-symmetric (out-phase) superposition on the pair of sites \( \langle i, j \rangle \), while \( a_i^\dagger + a_j^\dagger \) recycles the atoms into the symmetric (in-phase) state. Loosely speaking, we can interpret this process as a dissipative locking of the atomic phases of two adjacent lattice sites, which in turn results in a global phase locking, i.e. a condensate. Note that in view of \( (a_i - a_j)|\text{BEC}\rangle = 0 \) it is the destruction part of the jump operator which makes \( |\text{BEC}\rangle \) the dark state, while any linear combination \( a_i^\dagger \) and \( a_j^\dagger \) of recycling operators will do, except for a hermitian \( c_{ij} \) which would lead to a pure dephasing but no pumping into the dark state.

The above discussion becomes particularly clear in a momentum representation suggested by the translation invariance. The dissipative part of the master equation takes on the form \( \Pi \) with jump operators

\[
c_\ell \equiv c_{\mathbf{q},\lambda} = \frac{1}{\sqrt{M^d}} \sum_\mathbf{k} (1 + e^{i(k-\mathbf{q})\mathbf{e}_\lambda})(1 - e^{-i\mathbf{k}\mathbf{e}_\lambda}) a_{\mathbf{k}-\mathbf{q}} a_{\mathbf{k}}
\]

and \( \kappa_\ell \equiv \kappa \), which makes the appearance of the dark state \( \mathbf{q} = 0 \) decoupled from dissipation particularly apparent.

B. Implementation

The above master equation can be realized by immersing atoms \( a \) moving in an optical lattice in a large BEC of atoms \( b \) \( [12] \). The condensate interacts in the form of a contact potential with interspecies scattering length \( a_{ab} \) with the atoms \( a \), and acts as a bath of Bogoliubov excitations. This coupling provides an efficient mechanism for
decay of atoms \( \alpha \) from an excited to lower Bloch bands by emission of a Bogoliubov quasiparticle. Atoms \( \alpha \) moving in the lowest Bloch band can be driven by laser induced Raman processes to the first excited Bloch band, and can decay back to the ground band. This situation is reminiscent of optical pumping in quantum optics, or laser cooling [14, 17, 18]. There a laser excites electronic states of an atom, which return to the ground state by spontaneous emission of a photon. This formal analogy suggests and justifies a description of a driven dissipative Hubbard model. 

We consider a lattice as illustrated in Fig. 1a, with an additional auxiliary lattice site on each of the links. The optical lattice configuration corresponding to a single link is shown in Fig. 1b. It has the form of a Λ-system with the two Wannier functions of lattice sites 1 and 2 representing two ground states, and the auxiliary state in the middle representing an excited state. We drive this three-level system by Raman transitions from the two ground to the excited states with Rabi frequencies \( \Omega \) and \( -\Omega \), respectively. The excited atom can decay back to 1 or 2 by emission of a Bogoliubov quasiparticle. As is well-known from quantum optics, such a Λ-configuration supports a symmetric (out-of-phase) state \((|\alpha\rangle - |\beta\rangle)\) (crosses) of neighboring lattice sites \( i \) and \( j \) in the form of a dissipative current. (b) Schematic realization of the effective dissipative process in an optical super-lattice, which provides for excited states gapped by \( \varepsilon \) and localized on the links of neighboring lattice sites \((i,j)\): A Raman laser couples the ground and excited bands with effective Rabi-frequency \( \Omega \) and detuning \( \Delta = \omega - \varepsilon \) from the inter-band transition. The spatial modulation of the Raman-laser yields in a position dependent coupling, which excites only the anti-symmetric component of atoms sitting on neighboring lattice sites \( i \) and \( j \) into the upper band. The inter-band decay with a rate \( \Gamma \) back to their lower band is obtained via the emission of phonons into the surrounding BEC.

FIG. 1: Driven Dissipative Condensate: (a) A lattice gas \( a \) is immersed in a condensate \( b \), which acts on the links \((i,j)\) (crosses) of neighboring lattice sites \( i \) and \( j \) in the form a dissipative current. (b) Schematic realization of the effective dissipative process in an optical super-lattice, which provides for excited states gapped by \( \varepsilon \) and localized on the links of neighboring lattice sites \((i,j)\): A Raman laser couples the ground and excited bands with effective Rabi-frequency \( \Omega \) and detuning \( \Delta = \omega - \varepsilon \) from the inter-band transition. The spatial modulation of the Raman-laser yields in a position dependent coupling, which excites only the anti-symmetric component of atoms sitting on neighboring lattice sites \( i \) and \( j \) into the upper band. The inter-band decay with a rate \( \Gamma \) back to their lower band is obtained via the emission of phonons into the surrounding BEC.

III. COMPETITION OF HAMILTONIAN AND LIOUVILLIAN DYNAMICS

For a realistic system with a finite interaction \( V \), see Eq. (2), the \( |\text{BEC}\rangle \) is no longer a dark state of the master equation: the interactions tend to localize the particles, while the dissipative terms tend to enforce a pure condensate. In general, the competition between these two incompatible dynamics results in a mixed state. Below we present linearized theories, which allow us to solve for the density matrix \( \rho \), as well as to study the correlations functions under the time evolution.

A. Mean field theory

For weak interactions, one can expect that the pure \( |\text{BEC}\rangle \) state is only weakly perturbed with the zero momentum mode \( a_{q=0} \) still macroscopically occupied. We follow therefore the standard Bogoliubov prescription and replace in the master equation the zero momentum mode by its mean value, i.e., \( a_0 = \sqrt{n_0 M^2} \) with \( n_0 \) the condensate density \( (n - n_0 \ll n) \). In leading order, the jump operators in the master equation (3) reduce to

\[
c_\ell = a_\ell, \quad c_\ell^\dagger = a_\ell^\dagger,
\]

and the coupling rates \( \kappa_\ell = 16\pi n_0 \sum_\lambda \sin^2 (q\ell \lambda / 2) \). Here, the index \( \ell = \{q, \sigma\} \) with \( \sigma = \pm 1 \) characterizes the bosonic operators \( a_\ell \equiv a_{q,\sigma} = i^{1-\sigma} (a_q + \sigma a_{-q}) / \sqrt{2} \). In addition, the Hamiltonian in Eq. (2) simplifies to

\[
H = \sum_\ell \left\{ (\epsilon_\ell + Un) a_\ell^\dagger a_\ell + \frac{Un}{2} \left[ a_\ell^2 + (a_\ell^\dagger)^2 \right] \right\},
\]

with \( \epsilon_\ell \equiv \epsilon_q \). Note that \( a_{q,\sigma} = \sigma a_{-q,\sigma} \) and the summation over the index \( \ell \) avoids these double countings. It
follows that the master equation decouples for each mode $a_{f}$ and is quadratic in the bosonic operators. The analogous master equation in quantum optics is well known from parametric amplification. The general solution is given by a mixed Gaussian state, which, in steady state, takes the form $\rho = Z^{-1} \Pi_{f} \rho_{f}$ where

$$
\rho_{f} = \exp \left( -\beta_{f} b_{f}^{\dagger} b_{f} \right)
$$

(7)

with $\beta_{f}$ describing a finite occupation of the squeezed operators $b_{f} = e^{-i\phi_{f}} \cosh(\theta_{f}) a_{f} + e^{i\phi_{f}} \sinh(\theta_{f}) a_{f}^{\dagger}$. The squeezing parameter $\theta_{f}$ and $\beta_{f}$ are given by the relation

$$
\cosh^{2}(2\theta_{f}) = \coth^{2}(\beta_{f}/2) = \frac{\kappa_{f}^{2} + (\epsilon_{f} + U)n_{f}^{2}}{\kappa_{f}^{2} + E_{f}^{2}},
$$

(8)

with $E_{f} = \sqrt{\epsilon_{f}^{2} + 2Un_{f}} \equiv E_{q}$ the Bogoliubov energy. The phase $\cot \phi_{f} = (\epsilon_{f} + n_{f})/\kappa_{f}$ plays a minor role. From these relations, we recover the pure (BEC) in the limit of vanishing interactions $Un_{f}/\kappa_{f} \to 0$. On the other hand, for fixed interaction and dissipation $\kappa_{f}$, the modes $b_{f}$ essentially reduce to the well-known Bogoliubov modes for small momentum $|q|e_{f} \ll \sqrt{UJ/\kappa_{f}}$, while the parameter $\kappa_{f} \approx E_{f}/T_{eff}$ with an effective temperature

$$
T_{eff} = Un_{f}/2.
$$

(9)

The density matrix for the low momentum modes is therefore indistinguishable from the thermal state of a weakly interacting Bose gas with the role of a "finite temperature" played by the interactions $Un_{f}$. This similarity of the driven system with a thermal Bogoliubov state naturally rises the question on the validity of the mean field approximation: the ansatz assumes a small depletion $n_{D}$ of the zero momentum mode. Using the above solution for the steady state, the condensate depletion takes the form

$$
n_{D} = n - n_{0} = \frac{1}{2} \int \frac{dq}{v_{f}} \frac{(Un_{f})^{2}}{\kappa_{f}^{2} + E_{q}^{2}}
$$

(10)

with $v_{f}$ the volume of the Brillouin zone. This expression strongly depends on the dimension of the system: in 3D the depletion remains finite and small for weak interactions $Un_{f}/J \ll 1$. On the other hand, in one- and two-dimensional an infrared divergence appears indicating the absence of a macroscopic occupation and the breakdown of mean-field theory. Consequently, we find that a dissipatively driven system exhibits the same behavior as a bosonic system in thermal contact with a heat bath, where the appearance of long-range order at finite temperature is only possible above the lower critical dimension $d=2$.

The solution to the master equation also allows us to study the relaxation into the steady state, see Appendix C. The build up of the macroscopic occupation $n_{0}(t)$ obeys the long time behavior

$$
n_{0} - n_{0}(t) \sim \sqrt{Un_{f}} \frac{1}{8J 2\kappa_{f} n_{f}}
$$

(11)

The condensate $n_{0}(t)$ approaches the steady state according to a power law. In the strict absence of interactions the behavior is modified to $n_{0} - n_{0}(t) \sim t^{-3/2}$. The slower approach to equilibrium for the interacting system results from the scrambling of the particles via the interaction.

### B. Lower dimensions $d = 1,2$

In lower dimensions, the interaction drives strong phase fluctuations, which destroy the macroscopic occupation of the condensate. However, these fluctuations are only relevant on distances larger than the coherence length $\sqrt{J/Un_{f}}$ (a denotes the lattice spacing). Consequently, the formation of a local condensate still takes place on shorter distances, while only the phase between these local condensates is destroyed by the fluctuations. The influence of these phase fluctuations can be studied in a long wave length description by introducing a smoothly varying phase field $\phi_{i}$ and density field $\rho_{i}$ with $[\phi_{i}, \rho_{i}] = i\delta_{ij} \frac{23}{24}$. This behavior of the dissipative driven system is in close analogy to the thermodynamics of interacting bosons giving rise to Kosterlitz-Thouless critical phases in 2D, and Luttinger liquids in 1D. The jump operators simplify to

$$
c_{ij} = (n_{i} - n_{j}) - 2\sin(\phi_{i} - \phi_{j}),
$$

(12)

while the Hamiltonian (2) reduces to the harmonic model

$$
H = Jn \sum_{\langle ij \rangle} (\phi_{i} - \phi_{j})^{2} + \frac{U}{2} \sum_{i} n_{i}^{2}.
$$

(13)

Consequently, the master equation becomes again quadratic and introducing new bosonic operators $d_{f}$ with $\ell = \{q, \sigma\}$ allows us to decouple the master equation for each mode $\ell$: we define the bosonic operator $d_{\ell} = i^{1-\sigma}(d_{q} + \sigma d_{-q})$ with $d_{q} = (n_{q}/\sqrt{n} - i\sqrt{2n_{q}}\phi_{q})/\sqrt{2}$ . Surprisingly, with this definition the jump operator reduces to $c_{ij} \equiv d_{ij}$ and the master equation (1) in the long wave length limit $|q|a < \sqrt{UJ/\kappa_{f}}$ is mapped to the same form as the master equation in the previous section, with the operator $a_{f}$ replaced by the new operator $d_{f}$ and the chemical potential $Un_{f}$ replaced by $Un_{f} - \epsilon_{f}/2$. Consequently, we find again the exact solution to the master equation, which allows us to characterize the state via its correlation functions, see Appendix C; such a characterization of a states in cold gases has attracted a lot of interest recently $\frac{25}{26} \frac{27}{28}$.

First, we analyze the steady state. The interesting correlation function in lower dimensions is the Green’s function

$$
G_{\omega}(x,t) = \langle a_{i}(t_{1})a_{j}^{\dagger}(t_{0}) \rangle \sim \langle \exp \left( i (\phi_{i}(t_{1}) - \phi_{j}(t_{0})) \right) \rangle
$$

with $x$ the distance between the lattice sites $i$ and $j$ and $t = |t_{1} - t_{0}|$. In the long wave length limit with $x, tc \gg \sqrt{J/Un_{f}}$, the smooth part of the correlation function recovers rotational and translation invariance; here $c$ =
\[ G_t(x, 0) \sim \begin{cases} \frac{x}{ct}, & x \gg ct \\ \frac{t}{ct}, & x \ll ct \end{cases} \quad (14) \]

with the Kosterlitz-Thouless temperature \( T_{KT} = \pi J_n \gg T_{at} = Un/2 \), and the short distances scales \( x_0 \sim c/\kappa n \) and \( t_0 \sim (\kappa na/c)^2 a/c \). On the other hand, in one-dimension we find an exponential decay both in space and time with coherence length \( \xi_{1D} = 4cK/\pi T_{at} \), with the parameter \( K = \pi \sqrt{2Jn/\nu} \) playing the role of the Luttinger parameter. The concept of the effective temperature is particularly efficient in the low dimensional systems, where the correlations are dominated by the low energy phase modes, as implied by the exponential (1D) and algebraic decay (2D). Noticeably, the functional dependence of the correlation function is determined by the ratio \( U/J \) alone, while the dissipative coupling strength \( \kappa \) only gives rise to non-universal prefactors. This behavior is a consequence of the low momentum behavior of the dissipative damping \( \kappa_q \sim q^2 \) versus the linear sound spectrum of the Hamiltonian \( \{1\} \).

Finally, we study the time evolution of the spatial correlation function in two-dimensions i.e., \( G_t(x, 0) = \langle a_i \frac{\partial}{\partial x} a_j \rangle_t \) with the average defined by the density matrix \( \rho(t) \). The system is initially prepared in a disordered state characterized by a small correlation length \( \xi \), giving rise to an exponential decay of the correlations. The exact time evolution then allows us to study the appearance of the quasi-long range order Eq. \( \{1\} \) from this initially uncorrelated state. For \( t \gg x^2/(16\kappa n) \), we obtain in leading order

\[ G_t(x, 0) \sim \left( \frac{x}{x_0} \right)^\frac{\kappa n}{\pi T_{KT}} e^{-x^2/x_0^2}. \quad (15) \]

The length scale within which the system exhibits quasi-long range order is \( x_t = 2(\pi^2 \kappa n a t)^{1/4} \) and increases in time with the universal power \( x_t \sim t^{1/4} \), see Fig. \( \{2\} \).

**IV. CONDENSATE OF INTERACTING FERMIONIC DOUBLONS**

As a second example, we consider the dissipative preparation of an \( \eta \)-state, which is an exact excited eigenstate of the \( d \)-dimensional two-species fermionic Hubbard-Hamiltonian \( \{20\} \)

\[ H_{FH} = -J \sum_{\langle i, j \rangle, \sigma} f_{i\sigma}^\dagger f_{j\sigma} + U \sum_i f_{i\uparrow}^\dagger f_{i\downarrow} f_{i\downarrow} f_{i\uparrow}. \quad (16) \]

This state is created by the \( \eta \)-operators, \( \eta_i^\dagger = \sum_\sigma \phi_\sigma \eta_i^\dagger/\sqrt{M} \) with the local doublon operators \( \eta_i^\dagger = f_{i\uparrow}^\dagger f_{i\downarrow} \) and \( \phi_\sigma = \pm 1 \) denoting a sign alternating between sites in a checkerboard pattern. The \( N\eta \)-state is created by \( N \)-fold application of \( \eta_i^\dagger \), yielding an excited eigenstate of \( H_{FH} \), \( (\eta_i^\dagger) \langle \eta \rangle^{N-1} |0\rangle \), with energy \( N\nu U \). This excited state is a condensate of doublons into the quasimomentum-state \( (\pi, \ldots, \pi) \) in the corner of the \( d \)-dimensional Brillouin-zone. It exhibits superfluidity, with non-decaying off-diagonal long-range order in any spatial dimension. Together with the operator \( \sum_\sigma f_{i\sigma}^\dagger f_{i\sigma} \), the properties of \( \eta_i^\dagger \) allow the construction of symmetry generators of \( H_{FH} \), which has been used to investigate high-\( T_c \) superconductivity (c.f. \( \{23\} \)).

Following the lines presented above, we construct quasi-local operators at each pair of lattice sites having the \( N\eta \) state as a dark-state. As is easily verified, the quasilocal operators \( c_{ij} \equiv c_{ij}^{(k)} \) \( (k = 1, 2) \) given by

\[ c_{ij}^{(1)} = (\eta_i^\dagger - \eta_j^\dagger)(\eta_i + \eta_j), \quad (17) \]
\[ c_{ij}^{(2)} = n_{i\uparrow} f_{j\downarrow}^\dagger f_{i\downarrow} + n_{j\downarrow} f_{i\uparrow}^\dagger f_{i\downarrow}, \quad (18) \]

fulfill these requirements. Small-scale numerical simulations for open-boundary systems then show that a Liouvillian \( \{1\} \) constructed from these quantum jump-operators and the Hamiltonian \( H_{FH} \) drive any initial \( \rho(0) \) into the \( N\eta \)-state, assuming that \( N_t = N_i \) at all times. The result may be interpreted in the quantum jump picture: \( H_{FH} \) generates configurations with spin-up and spin-down particles on adjacent sites from any initial configuration. These configurations are then captured by \( c_{ij}^{(2)} \), associating them into doublons. Subsequent action of \( c_{ij}^{(1)} \) then generates the desired \( \eta \)-state by phase-locking, just as in the case of the jump-operators for the BEC.
V. CONCLUSIONS AND OUTLOOK

We have discussed a scenario where many body quantum states and entangled states are prepared as dark states in a non-equilibrium driven dissipative dynamics with quasi-local dissipation. While the present work has focused on condensed matter aspects of realizing non-equilibrium (quasi-)condensates of interacting bosons and paired fermions in optical lattices, the present ideas are readily extended to spin systems, and promise a new avenue towards preparing interesting entangled states of qubits for quantum information [16]. On the atomic physics side, control via external fields offers interesting new possibilities of engineering a broad class of quantum jump operators [3], one example being phase imprinting $a_i \rightarrow a_i e^{i\phi}$ with a laser.

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APPENDIX A: CRITERION FOR THE UNIQUENESS OF THE STATIONARY STATE

(SEE SEC. [II A] [16])

The BEC is a dark state for the set of jump operators $c_\ell$. For the uniqueness of this solution we further need it to be the only dark state. For a fixed particle number $N$, the first factor in $c_\ell$ (see Sec. [II A]) has no eigenvalues and in particular, no zero eigenvalues. Thus, in order to identify dark states $|D\rangle$ with zero eigenvalue, we may restrict to the equation $(a_i - a_j)|D\rangle = 0 \forall (i,j)$. Taking the Fourier transform, this translates to $(1 - e^{i\alpha q_i\alpha_j})a_q|D\rangle = 0 \forall q$. Thus the BEC state with $q = 0$ is the only dark state.

For the dark states to be the only stationary solution of $\dot{\rho} = L\rho$, we need to show now that if there exists another stationary state, then there must exist a subspace of the Hilbert space $\mathcal{H}$, which is invariant under any application of the operators $c_\ell$. Equivalently, in [16] we have the following

Theorem 1. Let $D$ be the space of dark states, i.e. $c_\ell D = 0$. If there exists no subspace $S \subseteq \mathcal{H}$ with $S \perp D$ such that $c_\ell S \subseteq S \forall c_\ell$, then the only stationary states are the dark states.

In order to show that the jump operators Eq. (??) indeed lead to the unique BEC steady state solution we construct a polynomial operator $O(c_\ell)$ with $\langle \text{BEC}(O\Phi) \neq 0 \forall |\Phi\rangle \in \mathcal{S} = \mathcal{H} \setminus \{\text{BEC}\}$. For this purpose we work in the momentum space representation such that

$$c_{\mathbf{q},\lambda} = \frac{1}{\sqrt{M^d}} \sum_{\mathbf{k}} (1 + e^{ik\cdot\mathbf{q}} e^{\lambda_\alpha}) (1 - e^{-ik\cdot\mathbf{q}} e^{\lambda_\alpha}) a_{\mathbf{k} + \mathbf{q}} a_{\mathbf{k}}.$$

with $e_\lambda$ characterizing the different lattice vectors connecting nearest neighbour sites. A basis in the Hilbert space can be written as $\{|n_{\mathbf{q}}\rangle\} = \prod_q (a_{\mathbf{q}})^{n_{\mathbf{q}}} |0\rangle$. Therefore the polynomial operator $O = \prod_q c_{\mathbf{q},\lambda}$ provides a finite overlap with the BEC. A general state can be written as $|\Phi\rangle = \sum f(n_{\mathbf{q}}) |n_{\mathbf{q}}\rangle$. Choose one basis state with nonvanishing coefficient with highest occupation in the zero mode. Then the polynomial operator $O$ for this basis state provides also a finite overlap of $|\Phi\rangle$ with the dark state.

APPENDIX B: IMPLEMENTATION OF THE BEC LIouvillian

Coherently driven two-band Hubbard model – We outline the mechanism for engineering Liouvillians driving into a BEC. We focus on a one-dimensional system, where the system atoms $a$ are moving in an optical superlattice $V_{\text{opt}} = \sum_{n=1}^{N} V_a \sin^2(n\pi z/a)/n^2$ with lattice spacing $a$ along the direction $z$, while being tightly confined by a harmonic potential with oscillator frequency $\omega_1$ in the transverse directions, $x$ and $y$. The lattice depths $V_2 > V_1 > 0$ of the superlattice are chosen such that the vibrational spacings $h \omega_{n=0,1} \approx 2 \sqrt{E_1(V_2 \pm V_1)}$ about the individual wells at positions $z_{n,j} = ja$ and $z_{1,j} = (j + 1/2)a$ are much larger than their gap $\varepsilon \approx V_1 + h(\omega_1 - \omega_0)/2$, cf. Fig.1(b). Here $E_1 = \hbar^2 \pi^2/2m_a a^2$ denotes the lattice recoil and $m_a$ the mass of the atoms $a$. The low-energy dynamics of the atoms $a$ is then given by a two-band Bose-Hubbard model

$$H_a = \sum_{n,q} \bar{\varepsilon}_{n,q} a_{n,q} a_{n,q} + \frac{1}{2} \sum_{\{\alpha_i\}} U_{\alpha_i} a_{\alpha_i} a_{\alpha_i} a_{\alpha_i} a_{\alpha_i},$$

where $a_{n,q} = \sum_{j=1}^{M} e^{i\alpha_j a_{n,j}^\dagger} / \sqrt{M}$ creates a Bloch-wave with quasi-momentum $0 < q \leq \pi/a$ and in the lower and upper band, $n = 0, 1$, respectively, and $a_{n,j}^\dagger$ creates an atom $a$ in the site at position $z_{n,j}$. We denote the dispersion relations for the two bands by $\bar{\varepsilon}_{n,q} = -2 J_n \cos(qa) + nc$ with $J_n$ the nearest neighbor inter-band hopping rates in band $n$ and $\varepsilon$ the band-separation. The second term in $H_a$ with $\alpha_i \equiv \{n_i,j_i\}$ account for intra- and inter-band interaction of two atoms $a$ in the superlattice in terms of the respective on- and off-site shift $U_{\alpha_i}$. The dominant intra- and inter-band interactions are given by density-density interactions $\sim \sum_{n,j} \bar{\varepsilon}_{n,q} a_{n,j}^\dagger a_{n,j} a_{n,j}$ and $\sim V_{0,1} \sum_{\{i,j\}} a_{0,j}^\dagger a_{1,j} a_{0,j} a_{1,j}$, respectively. Here, $U_{\alpha}$ denotes state-dependent on-site shifts, and $V_{0,1}$ the inter-band off-site shift for nearest neighbors $(i,j)$, c.f. $i - j = 0, 1$. A key ingredient for the BEC Liouvillian is a selective coherent drive between the two bands of the Hubbard model, which couples the antisymmetric superposition on a pair of lower sites to the upper level on the link in between. This is achieved via a Raman laser setup in the form of a set of standing waves locked at the position of one of its minima $z_{1,j}$. This allows to realize an effective dynamical coupling $V_{\text{las}}(t) = h\Omega \cos(\pi z/a) \sin(\omega t)$ with the same periodicity as the optical lattice. Here $\Omega$ is the
two-photon Rabi frequencies, and we denote the Raman-detuning from the upper band by $\Delta = \omega - \varepsilon / \hbar$. For a weak driving field, $\Omega \ll \varepsilon / \hbar$, the Raman-drive results in effective inter-band coupling

$$H_{\text{int}}(t) = \hbar \Omega e^{-i e \omega t} \sum_j (-1)^j a_{1,j}^\dagger (a_{0,j} - a_{0,j+1}) + \text{H.c.}$$

$$= \hbar \Omega e^{-i e \omega t} \sum_q (1 - e^{iqa}) a_{1,q+\pi/a}^\dagger a_{0,q} + \text{H.c.}$$

where we made use of the rotating wave approximation and dropped AC-Stark shifts, since they only renormalize the gap and dropped AC-Stark shifts, since they only renormalize the BEC and momentum $k$ while the upper-band displays the opposite behavior.

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differs in the interband-coupling $H_{\text{int}}$ for atoms sitting on neighboring sites in the lower band $n = 0$ originates from the modulation of the Raman-coupling. This results in a excitation of asymmetric (and in particular asymmetric) superposition of atoms sitting on adjacent sites from the lower band to the upper band, while their symmetric superposition is dark with respect to $H_a$, leading to the BEC dark state. From momentum representation of $H_{\text{int}}(t)$ we see that the two counter-propagating components $e^{\pm \pi i z / \alpha}$ provide a inter-band coupling of modes differing by $\pi / a$, and their interference results in the Bloch-waves to be selectively excited from the lower to the upper band based on their quasi-momentum $q$. In particular for the lower-band $n = 0$ the two components destructively (constructively) interfere for $q = 0$, ($q = \pi / a$), while the upper-band displays the opposite behavior.

**Coupling to the phonon reservoir** - The dissipative step is implemented by immersing the system into a large homogeneous 3D condensate of a distinguishable species of atoms $b$, which acts as a reservoir of phonon modes [19].

The corresponding phonon Hamiltonian is given in the Bogoliubov approximation by $H_b = \sum_{k \neq 0} E_k b_k^\dagger b_k$, where $b_k$ creates a Bogoliubov excitation with momentum $k$ and energy $E_k = \sqrt{[(\hbar c_k)^2 + (\hbar^2 k^2 / 2 m_b)^2] / 2 m_b}$ the mass of the atoms $b$ and $c_b$ their speed in sound in the condensate. The atoms $a$ and $b$ interact via a contact-interaction with a coupling constant $g_{ab} = 2 \pi \hbar^2 a_{ab} / \mu_{ab}$ given in terms of their intra-species scattering length $a_{ab}$ and their reduced mass $\mu_{ab} = m_a m_b / (m_a + m_b)$. Expanding these density-density interaction in terms of the fluctuations about the condensate wave-function we obtain first order (apart from an overall state-independent mean-field shift $g_{ab} N_a \rho_b$ proportional to the condensate density $\rho_b$) an effective coupling of the atoms $a$ to the Bogoliubov excitations in the form

$$H_{ab} = g_{ab} \sum_{k \neq 0} \sqrt{\rho_b S_k} A_k^\dagger b_k + \text{H.c.}$$

Here $S_k = \hbar^2 k^2 / 2 m_b E_k$ is the static structure factor of the BEC and $A_k^\dagger$ is the displacement-operator with momentum $k$ for the atoms $a$ associated with the recoil from the emission of a Bogoliubov excitation into the condensate.

In the following we are interested in the effective dynamics of the system atoms $a$ and consider the BEC as a reservoir of Bogoliubov excitation at essentially zero temperature, since under typical experimental conditions one can achieve temperatures $T_b \ll \varepsilon / \hbar$. We integrate out the bath dynamics in the Born-Markov approximation, and obtain a Master-equation for the density operator $\rho(t)$ of the atoms $a$ (within the rotating wave and independent rate approximation with respect to the laser excitation),

$$\frac{d \rho}{dt} = \frac{1}{i \hbar} [H_a(t), \rho] + \frac{1}{\sqrt{\pi}} \sum_k \left| \frac{g_k}{\hbar} \right|^2 \left[ [A_k, \rho A_k^\dagger(E_k)] + \text{H.c.} \right],$$

$$A_k^\dagger(E) = \int_0^\infty d \tau e^{|(H_a(t) - \varepsilon \tau) / \hbar} A_k^\dagger e^{i H_a \tau / \hbar}$$

with $H_a(t) = H_a + H_{\text{int}}(t)$ and where $A_k(E)$ denotes the Fourier-component of $A_k$ with frequency $\varepsilon / \hbar$ with the frequency $E_k / \hbar$ given by the dispersion relation of the Bogoliubov excitations. Given that intraband dissipative processes are suppressed by momentum conservation [19], we focus on the inter-band decay, and within the rotating frame approximation write the Master-equation with $A_k(E) = \sum_q C_{k,q}^\dagger \pi_{k,q} + (\varepsilon / \hbar) A_k^\dagger$ and their Fourier-components given by $A_k^\dagger(E) = A_k^\dagger \pi \delta(E - \varepsilon / \hbar) + i \pi [E / \hbar - \varepsilon / \hbar])$, where we neglected corrections $J_g, U_a \ll \varepsilon$ in the spectra and in the following will drop the Lamb-shifts, as they amount to (small) second-order shifts.

We take the continuum limit of $k$ and exploiting the radial symmetry of the Bogoliubov spectrum $E_k$, perform the integration over $k$ left with a integral over its azimuthal angle, which we rewrite in terms of $k_z / k$ as

$$\mathcal{L} \rho = \sum_{q,q'} \frac{\pi k_z^2}{v_0} \int d^2 n G_{k_0,q}^{(1,0)} G_{k_0,q'}^{(1,0)} \left[ \tilde{a}_{1,q}^\dagger \tilde{a}_{1,q} + \rho a_{1,q}^\dagger a_{1,q} \right] + \text{H.c.},$$

assuming a tight transverse confinement, $\varepsilon \ll \hbar \omega_\perp = \hbar^2 / 2 m_a a_\perp^2$, we approximate the Bloch-function by a Gaussian of width $a_\perp$ in transverse direction times periodic component along $z$, $\Phi_{s,q}(r) \approx \phi_{s,q}(z) e^{-r^2 / 2 a_\perp^2} / (\pi a_\perp^2)^{1/4}$ and perform the polar integration in $G^{(1,0)}$ as

$$G_{k,q}^{(1,0)} \approx g_k e^{-k_z^2 / 2 a_\perp^2} \phi_{1,q}^\dagger / \hbar,$n

$$f_{k,z}^{(1,0)} = \int dz \phi_{1,q+k_z}(z) e^{ik_z z} \phi_{0,q}(z)$$

**Long wavelength (Super/Sub-radiant) limit** - In general the recoil along $z$ for the bloch-wavefunction has to be computed numerically. However in the deeply bound limit a rough estimate may obtained by taking (orthogonalized) gaussians of width $a_z$ for the wanner functions

$$\int dz \phi_{1,q+k_z}(z) e^{ik_z z} \phi_{0,q}(z)$$

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$$\int dz \phi_{1,q+k_z}(z) e^{ik_z z} \phi_{0,q}(z)$$
and focus on nearest neighbor contributions, which yields
\[
f_{k,q}^{(1,0)} \approx \sum_{l=0,1} e^{-iqa} \int dz e^{ikz} w_1(z) w_0(z - (2l - 1)a/2)
\]
\[
\approx 2 \exp[-(ka_z/2)^2] \sin^2(ka_8/8) e^{-qqa^2/2} \sinh[(a/4a_z)^2]/\cos(ka/4 - qa/2).
\]

Thus the master-equation in momentum space reads
\[
\mathcal{L} \rho = \kappa_0 \int_{-k_0}^{+k_0} \frac{dk}{2\pi} e^{-k^2a_z^2} \sin^2(ka_8/8) \times
\sum_{q, q'} e^{iq(q' - a/2)} \cos \left( \frac{k^2a_z^2}{2} \right) \times \left[ a_{0,q}^\dagger a_{1,q+k}, \rho a_{1,q+k}^\dagger a_{1,q}^\dagger \right] + \text{H.c.},
\]
with \( \kappa_0 = 2\pi k_0 a_z^2 e^{-k^2a_z^2/2} / h^2 v_0 \sin^2[(a/4a_z)^2] \).

Short wavelength limit – For \( k_0 > \pi/2 \), the sum over \( k \) spans over \( n \) Brillouin zones, resulting in construct and destructive interference, which one associates to the finite correlation length of the Bogoliubov modes in the bath. In particular, for \( k_1 \gg k_0 = \pi/\alpha \) we notice that the summation in the master-equation runs over several Brillouin zones, which effectively suppresses the non-local / long-range decay. Thus it is convenient to rewrite the master-equation in position space, which yields the decay of the general form
\[
\mathcal{L} \rho = \sum_{j,j'} \sum_{\ell, \ell'} \Gamma_{j,j',\ell, \ell'} \left[ a_{0,j}^\dagger a_{1,\ell, \ell'}^\dagger \rho a_{1,\ell'}^\dagger a_{0,j'} \right] + \text{H.c.}
\]
where the summations \( j, j' \) and \( \ell, \ell' \) are over the lattice sites and links, respectively, and in the limit \( k \to \infty \) the decay rates take the simple form
\[
\Gamma_{j,j',\ell, \ell'} = \Gamma_0 \int dz w_{0,j} w_{1,\ell} w_{1,\ell'} w_{0,j'},
\]
where \( \Gamma_0 = g_{ab} S_{k_1}/1 - S_{k_1}^2 \) in terms of the being the static structure factor \( S_{k_1} \) at momentum \( k_1 \) and \( w_{n,i} \) denote the Wannier function for the lower \( (n = 0) \) and upper \( (n = 1) \) band, respectively. We notice that they are translationally invariant and rapidly fall off with increasing separation of \((j, j', \ell, \ell')\). Thus we can restrict ourselves to the largest quasi-local contributions, an in particular the ones for neighboring lattice sites yield
\[
\mathcal{L} \rho = \sum_j \sum_{p=0,1} \left[ \gamma_0 \left[ a_{0,j}^\dagger a_{1,j+p}^\dagger \rho a_{1,j+p}^\dagger a_{0,j} \right] + \gamma_1 \left[ a_{0,j+p}^\dagger a_{1,j+p}^\dagger \rho a_{1,j+p}^\dagger a_{0,j} \right] \right] + \text{H.c.}
\]
The first term describes the uncorrelated decay from the link \( j \) to the left site \( j \) and to the right site \( j + 1 \) with
\[
\gamma = \Gamma_{j,j+1} = \Gamma_{j+1,j+1}^\dagger.
\]

The second one accounts for the finite correlation between the two processes \( (\gamma \sim J_0) \) and the third is an analogous one that accounts for correlations in the decay between different links to the same site, i.e. \( \text{links } j-1 \) and \( j \) to site \( j \) \( (\gamma \sim J_1) \). It is convenient to rewrite the three-terms in terms of symmetric / antisymmetric combinations of particle localized on adjacent sites in the two bands, \( (a_{n,j+1}^\dagger \pm a_{n,j+1+1}^\dagger)/\sqrt{2} \).

This results in a master-equation in the form of a “chain” of concatenated \( \Lambda \)-systems and inverted \( \Lambda \) systems,
\[
\mathcal{L} \rho = \sum_{j, \pm} \left[ (\gamma_0 \pm \gamma_1) \left[ a_{0,j}^\dagger a_{0,j+1}^\dagger \right] \pm \left[ a_{0,j+1} \right] \right] + \text{H.c.}
\]
corresponding to a master-equation with jump operators \( C_{\pm}^{\text{L}}(ij) \) on links (\( \Lambda \)-systems) and vertices (\( V \)-systems) respectively, as
\[
C_{\pm}^{\text{L}}(ij) = \frac{\gamma_0 \pm \gamma_1}{2} a_{0,j}^\dagger a_{0,j+1}^\dagger
\]
\[
C_{V}^{\text{L}}(ij) = \frac{\gamma_0 \pm \gamma_1}{2} a_{0,j+1}^\dagger a_{0,j}^\dagger
\]
In the independent rate approximation we thus obtain the master-equation for the set of laser-driven excitations and quasi-local decay (that is in the regime where the wavelength of the Bogoliubov excitations is much smaller than the lattice-spacing, \( k_1 \gg \pi/\alpha \)) by
\[
d\rho/dt = \frac{1}{i\hbar} \left[ H_a + H_{\text{lin}}(t), \rho(t) \right] + \mathcal{L}_\Lambda \rho + \mathcal{L}_V \rho,
\]
\[
\mathcal{L}_\alpha \rho = \sum_{(ij)_k} \left[ 2C_{\pm}^{\text{L}}(ij) \rho(C_{\pm}^{\text{L}}(ij)) - ((C_{\pm}^{\text{L}}(ij)) \rho(C_{\pm}^{\text{L}}(ij)), \rho) \right],
\]
where \( \alpha_{ij} \) denote the summation over links and (connected) vertices for \( \alpha = \Lambda \) and \( \alpha = V \), respectively.

From the master-equation we see that (i) all excited states decay back to the ground-state and (ii) for \( U = 0 \) the phase-locked ground state (BEC) is a dark-state, i.e. an eigenstate of the hamiltonian that is dark with respect to the excitation and the decay.

Since all excited states are decaying, we adiabatically eliminate the excited band. In the limit that one has a weak far detuned laser, \( \Delta \gg \Omega, U_n, J_n, \gamma^\Lambda, \gamma^V \), results to lowest order \( a_{1,j} \sim \Omega(a_{0,j} - a_{0,j+1})/\sqrt{2} \Delta \) and thus the jump-operators transform to (up to global phases)
\[
C_{\lambda(j,+)} = \sqrt{\gamma_L^\Lambda (a_{0,j}^\dagger + a_{1,j+1}^\dagger)}(a_{0,j} - a_{0,j+1})/2,
\]
\[
C_{\varphi(j,+)} = \sqrt{\gamma_L^\Lambda (a_{0,j}^\dagger - (a_{0,j} + a_{0,j+1})/2),
\]
where we dropped the band-index \( n = 0 \) and we introduced the rates \( \gamma_L^\Lambda = \gamma_0^\Lambda (\Omega/\Delta)^2 \), and remark that the latter correspond to the diagonalized form of Eq. [4].
APPENDIX C: MOMENTUM SPACE CORRELATIONS

In the linearized theory all information is encoded in the first and second moments. Our construction implies the vanishing of the first moments, \( \langle a^\dagger_\ell a_\ell \rangle = 0 \). We may therefore concentrate on the time evolution equations of the second moments, whose solution then allows to reconstruct the Gaussian density operator and gives access to the correlation functions. In principle the relevant single particle sector of the density matrix is mapped out by the correlations \( \langle a^\dagger_\ell a_\ell' \rangle, \langle a_\ell a_\ell' \rangle \), but for our purposes it is sufficient to focus on the diagonal entries. The corresponding equations are obtained from the linearized version of Eq. (1) (cf. Sect. III) using the commutation relation \([a_\ell, a^\dagger_\ell'] = \delta_{\sigma,\sigma'}(\delta_{\mathbf{q},\mathbf{q}'} + \sigma\delta_{\mathbf{q},-\mathbf{q}'})\), \([a_\ell, a_\ell'] = [a^\dagger_\ell, a^\dagger_\ell'] = 0 \) (\( \ell = (\mathbf{q},\sigma) \)), with the result

\[
\partial_t \langle a^\dagger_\ell a_\ell \rangle = 2iUn(\langle a^\dagger_\ell a_\ell \rangle - \langle a^\dagger_\ell a^\dagger_\ell' \rangle) - 4\kappa_{\ell} \langle a^\dagger_\ell a_\ell \rangle, \quad (C1)
\]

\[
\partial_t \langle a_\ell a_\ell' \rangle = -4i[U(\langle a^\dagger_\ell a^\dagger_\ell' \rangle + \frac{i}{\hbar}) + (\epsilon_{\ell} + Un)\langle a^\dagger_\ell a_\ell \rangle] - 4\kappa_{\ell} \langle a_\ell a_\ell' \rangle
\]

where we omit the equation for \( \langle a^\dagger_\ell a^\dagger_\ell' \rangle \) trivially obtained from the second line. The equations of motion may be solved via the Laplace transform and form the basis for the computation of the density operator and the spatial and temporal correlation functions.

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