A NEW FAMILY OF TRIANGULATIONS OF $\mathbb{R}P^d$

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Abstract. We construct a family of PL triangulations of the $d$-dimensional real projective space $\mathbb{R}P^d$ on $\Theta\left(\frac{(1+\sqrt{5})^d}{2}\right)$ vertices for every $d \geq 1$. This improves a construction due to Kühnel on $2^{d+1} - 1$ vertices.

1. Introduction and main results

Triangulations of topological spaces play an important role in many areas of mathematics, from the more theoretical to the applied ones. A classical problem in PL topology asks for the minimum number of vertices that a simplicial complex with a certain geometric realization can have. This invariant depends on the underlying topological space: triangulable spaces with complicated homology or homotopy groups tend to need more vertices in their vertex-minimal triangulations. However, to determine this number is in general very hard, even if we restrict our discussion to manifolds or PL manifolds. One of the few families for which this number is known is that of sphere bundles over the circle. Kühnel [Küh86] showed that the boundary complex of the $(2d+3)$-vertex stacked $(d+1)$-manifold whose facet-ridge graph is a cycle is a PL triangulation of $S^{d-1} \times S^1$ for even $d$, and a PL triangulation of the twisted bundle $S^{d-1} \times S^1$ for odd $d$. Furthermore, Kühnel’s triangulations are vertex-minimal. In the remaining cases, namely when $d$ is odd and the bundle is orientable, or when $d$ is even and the bundle is non-orientable, the minimum number of vertices is $2d+4$, and it is attained for every $d$ [BD08, CSS08]. Novik and Swartz [NS09] (in the orientable case) and later Murai [Mur15] proved that the number of vertices $f_0(\Delta)$ of a $k$-homology $d$-manifold $\Delta$ must satisfy

$$f_0(\Delta) \geq \binom{d+2}{2} + 1,$$

for $d \geq 3$ and any field $k$. Equality in this formula is attained precisely by members in the Walkup class which are moreover 2-neighborly, i.e., their graphs are complete. In low dimensions computational methods are a precious source. Using the program BISTELLAR [BL00] several vertex-minimal triangulations of 3- and 4-dimensional manifolds have been constructed [Lut99].

For a manifold whose homology (computed with coefficients in $\mathbb{Z}$) has a nontrivial torsion part, the Novik-Swartz-Murai lower bound is far from being tight. In this article we focus on the triangulations of the real projective space $\mathbb{R}P^d$. A lower bound on the number of vertices of a triangulation of $\mathbb{R}P^d$ was given in [AM91].

Theorem 1.1. [AM91] Let $\Delta$ be a triangulation of $\mathbb{R}P^d$, with $d \geq 3$. Then

$$f_0(\Delta) \geq \binom{d+2}{2} + 1.$$
It is well known that there is a unique vertex-minimal triangulation of $\mathbb{R}P^2$ on 6 vertices, and Walkup [Wal70] proved that the number of vertices needed to triangulate $\mathbb{R}P^3$ is at least 11, and constructed one such complex. More recently, an enumeration of manifolds on 11 vertices was obtained in [SL09], revealing 30 non-isomorphic such triangulations, exhibiting 5 different $f$-vectors (see [SL09, Table 10]). This also shows that Walkup’s construction is the unique $f$-vectorwise minimal triangulation of $\mathbb{R}P^3$. Again by computer search, a vertex-minimal triangulation of $\mathbb{R}P^4$ on 16 vertices was found. This highly symmetric simplicial complex was studied by Balagopalan [Bal17] who described three different ways to construct this complex. Even though the 3- and 4-dimensional cases suggest the formula in Theorem 1.1 is tight in $d = 3, 4$, for $d = 5$ the computer search could not find triangulations of $\mathbb{R}P^5$ on less than 24 vertices. It is now very tempting to conjecture a tight lower bound of $\binom{d+2}{2} + \lfloor \frac{d+1}{2} \rfloor$, which fits all the known cases and reflects the fact that the number of nontrivial integral homology groups of $\mathbb{R}P^d$ depends on the parity of $d$. Unfortunately, in higher dimensions we don’t know any triangulation of $\mathbb{R}P^d$ with $O(d^2)$ or even $O(d)$ vertices, for any $i$. The current record is due to Kühnel [Küh87]. He observed that the barycentric subdivision of the boundary of the $(d+1)$-simplex possess a free involution, and the quotient w.r.t. the involution is PL homeomorphic to the $d$-dimensional real projective space. This construction provides a PL triangulation of $\mathbb{R}P^d$ on $2d+1 - 1$ vertices. Indeed, a natural way to construct a triangulation of $\mathbb{R}P^d$ is to find a centrally symmetric triangulation of $S^d$, the double cover of $\mathbb{R}P^d$, with an additional combinatorial condition:

**Lemma 1.2** ([Wal70, Proposition (8.1)]). Let $\Delta$ be a cs PL $d$-sphere with free involution $\sigma$ and with no induced cs 4-cycle. Then $\Delta/\sigma$ is a PL triangulation of $\mathbb{R}P^d$.

In this article, we construct a family of PL $d$-spheres as in Lemma 1.2 for every $d \geq 0$. Let $F_i$ be the $i$-th Fibonacci number, i.e., $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every $n \geq 2$. Our main result is the following.

**Theorem 1.3.** There exists a cs PL $d$-sphere $S_d$ with no cs induced 4-cycles and $f_0(S_d) = 3F_{d+1} + 7F_d + 3F_{d-1} - 4$.

Via Lemma 1.2 we obtain PL triangulations of $\mathbb{R}P^d$, for every $d \geq 1$.

**Theorem 1.4.** There exists a PL triangulation $\Delta$ of $\mathbb{R}P^d$ with $f_0(\Delta) = \frac{3}{2}F_{d+1} + \frac{7}{2}F_d + \frac{3}{2}F_{d-1} - 2$.

Observe that $\frac{3}{2}F_{d+1} + \frac{7}{2}F_d + \frac{3}{2}F_{d-1} - 2 < 2^{d+1} - 1$ for every $d \geq 3$, hence improving Kühnel’s construction in any dimension. Moreover the improvement of the bound is asymptotically significant, since $\frac{3}{2}F_{d+1} \leq \frac{3}{2\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^{d+1} \sim \frac{3}{2\sqrt{5}} (1.61803\ldots)^{d+1}$.

2. **Definitions**

A simplicial complex $\Delta$ with vertex set $V = V(\Delta)$ is a collection of subsets of $V$ that is closed under inclusion. The elements of $\Delta$ are called faces. For brevity, we usually denote $\{v\}$ as $v$ and with $f_0(\Delta)$ the number $|V(\Delta)|$. The dimension of a face $F \in \Delta$ is dim $F := |F| - 1$. The dimension of $\Delta$, dim $\Delta$, is the maximum dimension of its faces.

If $F$ is a face of $\Delta$, then the star of $F$ and the link of $F$ in $\Delta$ are the simplicial complexes

$$
st_\Delta(F) := \{ \sigma \in \Delta : \sigma \cup F \in \Delta \} \quad \text{and} \quad \text{lk}_\Delta(F) := \{ \sigma \in \text{st}_\Delta(F) : \sigma \cap F = \emptyset \}.$$


If $\Delta$ and $\Gamma$ are simplicial complexes on disjoint vertex sets, then the join of $\Delta$ and $\Gamma$ is the simplicial complex $\Delta \ast \Gamma = \{ \sigma \cup \tau : \sigma \in \Delta \text{ and } \tau \in \Gamma \}$. In particular, if $\Delta_1 = \emptyset \cup \{ v \}$, then $\Delta_1 \ast \Delta_2$ is called the cone over $\Delta_2$ with apex $v$.

Let $\Delta$ be a simplicial complex and $W \subseteq V(\Delta)$. The induced subcomplex of $\Delta$ on $W$ is $\Delta_W = \{ F \in \Delta : F \subseteq W \}$. If $\Gamma$ is a subcomplex of $\Delta$, define

$$\Delta \setminus \Gamma = \Delta_{V(\Delta) \setminus V(\Gamma)} = \{ F \in \Delta : F \cap V(\Gamma) = \emptyset \}.$$  

The complement of $\Gamma$ in $\Delta$, denoted by $\Delta - \Gamma$, is the set of faces in $\Delta$ but not in $\Gamma$. The closure $\overline{\Delta - \Gamma}$ is the subcomplex of $\Delta$ generated by the facets of $\Delta - \Gamma$. If $f$ is an automorphism of $\Delta$, then the quotient of $\Delta$ w.r.t. $f$, $\Delta/f$, is the simplicial complex obtained identifying the vertices in the same orbit and all the faces with the same vertex set. Observe that in general $|\Delta/f| \neq |\Delta|/\overline{f}$, where $\overline{f} : |\Delta| \rightarrow |\Delta|$ is the continuous map induced by $f$. For instance, if $\Delta$ is a square and $f$ maps every vertex $v$ to the unique vertex not adjacent to $v$ then $\Delta/f$ is the 1-dimensional simplex, while $|\Delta/\overline{f}| \cong S^1$.

We say two simplicial complexes $\Delta_1$ and $\Delta_2$ are PL homeomorphic, denoted as $\Delta_1^{\mathrm{PL}} \cong \Delta_2$, if there exists subdivisions $\Delta'_1$ of $\Delta_1$ and $\Delta'_2$ of $\Delta_2$ that are simplicially isomorphic. A PL $d$-ball is a simplicial complex PL homeomorphic to a $d$-simplex. Similarly, a PL $d$-sphere is a simplicial complex PL homeomorphic to the boundary complex of a $(d + 1)$-simplex. A $d$-dimensional simplicial complex $\Delta$ is called a PL $d$-manifold if the link of every non-empty face $F$ of $\Delta$ is a $(d - |F|)$-dimensional PL ball or sphere; in the former case, we say $F$ is a boundary face while in the latter case $F$ is an interior face. The boundary complex $\partial \Delta$ is the subcomplex of $\Delta$ that consists of all boundary faces of $\Delta$. A PL manifold whose geometric realization is homeomorphic to a closed manifold $M$ is called a PL triangulation of $M$. In the literature PL manifolds as defined above are sometimes called combinatorial manifolds or combinatorial triangulations of manifolds.

**Remark 2.1.** For $d \geq 5$ the class of PL $d$-manifolds is strictly contained in that of triangulated $d$-manifolds. In particular, the double suspension of any homology 3-sphere with a non-trivial fundamental group is a non-PL simplicial 5-sphere (see e.g., [RS82]).

PL manifolds have the following nice properties, see [Ale30] and [Lic90]:

**Lemma 2.2.** Let $\Delta_1$ and $\Delta_2$ be PL $d_1$- and $d_2$-manifolds, respectively.

1. If $\Delta_1$ and $\Delta_2$ are PL balls, so is $\Delta_1 \ast \Delta_2$.
2. If $d_1 = d_2 = d$ and $\Gamma := \Delta_1 \cap \Delta_2 = \partial \Delta_1 \cap \partial \Delta_2$ is a PL $(d - 1)$-manifold, then $\Delta_1 \cup \Delta_2$ is a PL $d$-manifold. If furthermore $\Delta_2$ and $\Gamma$ are PL balls, then $\Delta_1 \cup \Delta_2^{\mathrm{PL}} \cong \Delta_1$.

**Lemma 2.3.** (Newman’s theorem) Let $\Delta$ be a PL $d$-sphere and $\Psi \subseteq \Delta$ be a PL $d$-ball. Then the closure of the complement of $\Psi$ in $\Delta$ is a PL $d$-ball.

Let $\Delta$ be a PL manifold without boundary. The prism over $\Delta$ is the pure polyhedral complex $\Delta \times [-1, 1]$, whose cells are of the form $F \times \{1\}$, $F \times \{-1\}$ or $F \times [-1, 1]$, for every $F \in \Delta$. Clearly $|\Delta \times [-1, 1]| \cong |\Delta| \times [-1, 1]$. The boundary of the prism consists of the cells $F \times \{1\}$ and $F \times \{-1\}$, where $F \in \Delta$.

A simplicial complex $\Delta$ is centrally symmetric or cs if its vertex set is endowed with a free involution $\sigma : V(\Delta) \rightarrow V(\Delta)$ that induces a free involution on the set of all non-empty
faces of $\Delta$. Let $\Delta$ be a centrally symmetric PL $d$-sphere and let $\sigma$ be the free involution on $\Delta$. An induced cs $4$-cycle in $\Delta$ is an induced subcomplex $C \subseteq \Delta$ isomorphic to a $4$-cycle, with $\sigma(C) = C$. In particular, the vertices of $C$ are $v, w, \sigma(v), \sigma(w)$, for some $v, w \in V(\Delta)$. The complex $\Delta$ has no induced cs $4$-cycle if and only if $\text{st}_\Delta(v) \cap \text{st}_\Delta(\sigma(v)) = \{\emptyset\}$ for every $v \in V(\Delta)$.

In what follows we define two properties in the structure of a cs PL $d$-sphere.

**Definition 2.4 (Property $P_d$).** Let $S$ be a cs PL $d$-sphere with free involution $\sigma$. Assume that there exists a PL $d$-ball $B \subset S$ satisfying the following conditions:

i. $S = B \cup \sigma(B)$, where $B \cap \sigma(B) = \partial B = \partial \sigma(B)$.

ii. $S \setminus \partial B = D \cup \sigma(D)$, with $D \subseteq S$ a PL $d$-ball.

iii. There exists $v \in V(D)$ such that $V(\text{st}_S(v)) \cup V(\text{st}_S(\sigma(v))) = V(S)$ and furthermore, $D \subseteq \text{st}_S(v) \subseteq B$.

Then we say that $S$ satisfies Property $P_d$ w.r.t. the triple $(B, D, v)$.

**Definition 2.5 (Property $Q_d$).** We say that a cs PL $d$-sphere $S$ satisfies Property $Q_d$ if there exists a sequence of subcomplexes $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_d = S$ such that each $S_i$ is a cs PL $i$-sphere that satisfies Property $P_i$ w.r.t. a triple $(B_i, D_i, v_i)$ with $\partial B_i = S_{i-1}$ for all $1 \leq i \leq d$.

**Example 2.6.** The cs $6$-cycle $S_1$ satisfies Property $Q_1$.

**Example 2.7.** The boundary complex of the icosahedron can be realized as a cs PL $2$-sphere $S_2$ on $12$ vertices which satisfies Property $Q_2$: indeed there is an induced cs $6$-cycle $S_1$ in $S_2$ that divides $S_2$ into two antipodal $2$-balls. In this case $S_2 \setminus S_1$ is the disjoint union of two triangles $F, \sigma(F)$ and any vertex $v \in F$ satisfies the third condition in Definition 2.4.

### 3. FROM A TRIANGULATION OF $S^{d-1}$ TO A TRIANGULATION OF $S^d$

Given a cs PL $(d - 1)$-sphere $S$ with $2n$ vertices but with no induced cs $4$-cycles, one may build a cs PL $d$-sphere with $4n + 2$ vertices as follows: first build the prism over $S$; then triangulate the prism in such a way that no additional induced cs $4$-cycles and interior vertices are created; finally cone over the boundaries of the prism (as two disjoint copies of $S$) with two new vertices. The PL $d$-sphere obtained in this way has no induced $4$-cycles. In this section, we will modify the above approach to reduce the number of vertices in the construction.

In what follows, assume that $S$ is a cs PL $(d - 1)$-sphere with involution $\sigma$ and it satisfies Property $Q_{d-1}$; in particular, we have $S$ satisfies Property $P_{d-1}$ under the triple $(B, D, p)$ and $S' := \partial B$ is a cs PL $(d - 2)$-sphere that satisfies Property $P_{d-2}$ under the triple $(B', D', p')$. We will first give a triangulation $\Sigma$ of $S \times [-1, 1]$ such that

- $\Sigma$ is centrally symmetric;
- there exists a cs subcomplex $\Gamma \subseteq \Sigma$ with $\Gamma \cong S$ and $D \times \{1\}, \sigma(D) \times \{-1\} \subseteq \Gamma$.

This will be done in two steps. First we construct a polyhedral complex that satisfies the above conditions, see Proposition 3.1. Then we triangulate it while preserving these properties, see Corollary 3.10. Finally, we build a cs PL $d$-sphere from $\Sigma$, see Corollary 3.15.
3.1. **The prism over** $S$. The main point of this section is Proposition 3.1. By *refinement* of a polyhedral complex $P$ we mean a polyhedral complex $\Delta$ with $|\Delta| \cong |P|$ obtained by subdividing $P$.

**Proposition 3.1.** Let $S$ be a cs PL $(d-1)$-sphere satisfying Property $Q_{d-1}$. There exists a refinement $\Sigma''$ of $S \times [-1,1]$ such that:

- $\Sigma''$ is centrally symmetric.
- There exists a cs subcomplex $\Gamma \subseteq \Sigma''$ with $\Gamma \cong S$ and $D \times \{1\}, \sigma(D) \times \{-1\} \subseteq \Gamma$.

In order to prove Proposition 3.1 we first refine $S \times [-1,1]$ to a three layered prism over $S$.

**Definition 3.2.** We define $\Sigma'$ to be the polyhedral complex

$$\Sigma' = (S \times [-1,0]) \cup (S \times [0,1]).$$

In other words, $\Sigma'$ is a prism over $S$ with three layers $S \times \{-1\}, S \times \{0\}$ and $S \times \{1\}$. Furthermore, $\Sigma'$ is centrally symmetric with the free involution $\sigma'$ induced by $(v,i) \mapsto (\sigma(v),-i)$, for $i = -1,0,1$. We define the following maps:

$$\psi_{-1} : V(S \times \{1\}) \to V(\Sigma')$$

$$(v,1) \mapsto \begin{cases} (v,1) & \text{if } v \in V(D) \cup V(st_\sigma(v')) \\ (v,0) & \text{if } v \in V(\sigma(D)) \cup V(\sigma(st_\sigma(v'))) \end{cases}.$$

(3.1)

$$\psi_{-1} : V(S \times \{-1\}) \to V(\Sigma')$$

$$(v,-1) \mapsto \begin{cases} (v,-1) & \text{if } v \in V(\sigma(D)) \cup V(\sigma(st_\sigma(v'))) \\ (v,0) & \text{if } v \in V(D) \cup V(st_\sigma(v')) \end{cases}.$$

Observe that the maps $\psi_{+1}$ and $\psi_{-1}$ induce maps from the polyhedral complex $\Sigma'$ to itself.

**Definition 3.3.** We define

$$\Sigma'' := \Sigma' / \sim,$$

where $v \sim w$ if and only if $w = \psi_{+1}(v)$ or $w = \psi_{-1}(v)$.

Informally speaking, we identify the subcomplexes of the middle layer of $\Sigma$ with the upper and lower layers. In this way we obtain a polyhedral complex containing an induced subcomplex isomorphic to $S$ (see Figure 1).

**Lemma 3.4.** The complex $\Sigma''$ in Definition 3.3 has the following properties:

i. $|\Sigma''| \cong S^{d-1} \times [0,1]$.

ii. $\partial \Sigma'' \cong S \cup S$.

iii. There exists an induced cs subcomplex $\Gamma \subseteq \Sigma''$ with $\Gamma \cong S$ and the images of $D \times \{1\}$ and $\sigma(D) \times \{-1\}$ in $\Sigma''$ under the maps $\psi_{+1}$ and $\psi_{-1}$ are subcomplexes of $\Gamma$.

iv. $\Sigma''$ is centrally symmetric under an involution $\sigma''$ induced from $\sigma'$.

**Proof:** Part i is clear, and for ii it suffices to observe that the image of $S \times \{1\}$ and $S \times \{-1\}$ in $\Sigma''$ are disjoint and isomorphic to $S$.

Let $\Gamma$ be the image of $S \times \{0\}$ in $\Sigma'$. Since the vertices of $S \times \{0\}$ are not subject to any identification, $\Gamma = S \times \{0\} \cong S$. Moreover $\psi_{+1}(D \times \{1\}) = D \times \{1\} \subseteq \Gamma$ and $\psi_{-1}(\sigma(D) \times \{-1\}) = \sigma(D) \times \{-1\} \subseteq \Gamma$.
The complexes $\Sigma'$ and $\Sigma''$, together with a cs triangulation $\Sigma$.  

$\sigma(D) \times \{-1\} \subseteq \Gamma$. This proves $iii$. 

Finally, we observe that if $F \sim G$ in $\Sigma''$ then $\sigma(F) \sim \sigma(G)$. Equivalently, the map that assigns the equivalence class of the vertex $(v,i)$ to the class of $\sigma'((v,i))$ is well defined. Therefore, it induces a free involution of $\sigma'' : \Sigma'' \rightarrow \Sigma''$. 

3.2. The cs triangulation $\Sigma$ of $\Sigma''$. In this subsection we construct a centrally symmetric simplicial complex $\Sigma$ which refines $\Sigma''$ such that 

- $V(\Sigma) = V(\Sigma'')$, i.e., no new vertex is introduced. 
- $\Sigma$ is centrally symmetric with a free involution $\tau$, such that $\tau|_{\Sigma''} = \sigma''$. 

Our construction is based on certain orientations of the graph of a simplicial complex called locally acyclic orientations. We refer to [JDLS10, Section 7.2] for a more detailed treatment of the subject. 

**Definition 3.5.** A locally acyclic orientation (l.a.o.) of $\Delta$ is an orientation of the edges of its graph such that none of the 2-simplices of $\Delta$ contains an oriented cycle. 

**Lemma 3.6 ([JDLS10, Lemma 7.2.9]).** Let $\Delta$ be a simplicial complex. The simplicial refinements of $\Delta \times [0,1]$ are in bijection with locally acyclic orientations of $\Delta$. 

Every simplicial complex has a locally acyclic orientation obtained from acyclic orientations of its graph. The bijection in Lemma 3.6 is easy to describe: if $\{i,j\}$ is an edge of $\Delta$ with $i \rightarrow j$, then $\{(i,0),(j,1)\}$ is an edge of $\Delta \times [0,1]$ and vice versa. This induces a triangulation of $F \times [-1,1]$ for every face $F \in \Delta$. Furthermore, the locally acyclicity guarantees that
the union of triangulations of individual cells can be coherently completed to a triangulation of $\Delta \times [-1,1]$. The following observation is straightforward.

**Lemma 3.7.** Let $\Delta$ be a centrally symmetric simplicial complex with involution $\sigma$ and consider a refinement of $\Delta \times [-1,1]$ which is centrally symmetric with involution induced by $\tilde{\sigma} : (v,-1) \mapsto (\sigma(v),1)$, $(v,1) \mapsto (\sigma(v),-1)$ for any vertex $v \in \Delta$. Then the corresponding locally acyclic orientation of $\Delta$ is order reversing w.r.t. the symmetry, i.e., $v \rightarrow w$ if and only if $\sigma(w) \rightarrow \sigma(v)$.

**Proof:** In any cs triangulation of $\Delta \times [-1,1]$ the set $\{(v,-1),(w,1)\}$ is an edge if and only if $\{\tilde{\sigma}((v,-1)),\tilde{\sigma}((w,1))\} = \{(\sigma(v),1), (\sigma(w),-1)\}$ is an edge, which implies that on the corresponding l.a.o. we have that $v \rightarrow w$ if and only if $\sigma(w) \rightarrow \sigma(v)$. □

Recall that $S'$ is a cs PL $(d-2)$-sphere in $S$ that satisfies $P_{d-2}$ under the triple $(B',D',p')$. Hence $V(S') = V(st_{S'}(p')) \cup V(st_{S'}(\sigma(p')))$. Then $V(S) = V(\sigma(D)) \cup V(st_{S'}(\sigma(p'))) \cup V(D) \cup V(st_{S'}(p'))$.

We define a subset $A$ of the edges of $S$ as follows:

$A = \{(v,w) : v \in V(D) \cup V(st_{S'}(p')) \text{ and } w \in V(\sigma(D)) \cup V(st_{S'}(\sigma(p')))\}$.

Observe that $\sigma(e) \in A$ for every $e \in A$.

**Lemma 3.8.** There exists a l.a.o. of $S$ such that:

- $v \rightarrow w$ for every edge $\{v,w\}$ in $A$.
- For every edge $\{v,w\} \in S$, $v \rightarrow w$ if and only if $\sigma(w) \rightarrow \sigma(v)$.

**Proof:** First we choose any l.a.o. of the induced subcomplex of $S$ on $V(D) \cup V(st_{S'}(p'))$. Following Lemma 3.7 we impose on the induced subcomplex on $V(\sigma(D)) \cup V(st_{S'}(\sigma(p'))) \cup V(D)$ a reverse orientation. By (3.2), the remaining edges of $S$ are those in $A$. We orient all edges $\{v,w\}$ of type $A$ as $v \rightarrow w$. This orientation is by definition acyclic on every 2-simplex not containing edges in $A$. It suffices to check that no edge in $A$ is contained in an oriented cycle. Since every 2-simplex containing an edge in $A$ also contains another edge in $A$, it contains a vertex $v$ or a vertex $w$ such that either $w_1 \leftarrow v \rightarrow w_2$ or $v_1 \rightarrow w \leftarrow v_2$. This proves the claim. □

The l.a.o. in the proof of Lemma 3.8 defines a centrally symmetric triangulation of $S^{d-1} \times [-1,1]$. The following lemma shows that this triangulation is a PL manifold with boundary without interior vertices. For any prism $\Delta \times [-1,1]$, we denote by $(\Delta \times [-1,1])^{\ell}$ the refinement of $\Delta \times [-1,1]$ induced by the l.a.o. $\ell$ on $\Delta$.

**Lemma 3.9.** Let $\Gamma$ be a PL $(d-1)$-sphere, $\Delta' = \Delta \times [-1,1]$ and let $\ell$ be any locally acyclic orientation on $\Gamma$. Then, the refinement $\Delta'^{\ell}$ is a PL $d$-manifold with boundary and without interior vertices.

**Proof:** Since $\Gamma$ is a PL $(d-1)$-sphere, there exists a subdivision of $\Gamma'$ for which every simplex can be linearly embedded in $\mathbb{R}^N$ for some $N$. Moreover, for every locally acyclic orientation $\ell$ of $\Gamma$, consider a linear ordering of the vertices of $\Gamma'$ such that $v > w$ for every $v \in V(\Gamma') \setminus V(\Gamma)$ and $w \in V(\Gamma)$. Orient all the edges $\{u,v\} \not\in \Gamma$ with $u \rightarrow v$ if $u > v$. This extends to a locally acyclic orientation $\ell'$ of $\Gamma'$ that agrees with $\ell$ when restricted to the
edges of $\Gamma$. The corresponding refinement of $\Gamma' \times [-1,1]$ is a subdivision of $\Gamma \times [-1,1]$ in which every simplex is linearly embedded in $\mathbb{R}^{N+1}$. Therefore, $\Delta^\ell$ is a PL manifold with boundary. Furthermore $V(\Delta^\ell) = V(\Gamma \times \{-1\}) \cup V(\Gamma \times \{1\}) = V(\partial \Delta^\ell)$ and hence $\Delta^\ell$ has no interior vertices.

Corollary 3.10. There exists a l.a.o. $\ell$ on the cs $(d-1)$-sphere $S$ such that

- $(S \times [-1,1])^\ell$ is a cs PL manifold with boundary that refines $\Sigma''$;
- $(S \times [-1,1])^\ell$ contains an induced cs subcomplex $\Gamma$ isomorphic to $S$ which contains $(D \times \{1\}) \cup \sigma(D) \times \{-1\}$;
- $V((S \times [-1,1])^\ell) = V(\Sigma'')$.

Proof: Choose any l.a.o. $\ell$ on $S$ that satisfies the conditions in Lemma 3.8. The orientation on any edge $\{v,w\} \in A$ generates a new edge $\{(v,1),(w,-1)\} \in \Sigma$. This coincides with those edges in $\Sigma''$ but not in $S \times [-1,1]$. Hence $(S \times [-1,1])^\ell$ is a refinement of $\Sigma''$. By Lemma 3.9, $(S \times [-1,1])^\ell$ is a PL manifold. The other two properties follow from Lemmas 3.4 and 3.9.

Remark 3.11. In what follows, we fix any l.a.o. on $S$ as in Lemma 3.8 and denote by $\Sigma$ the PL manifold with boundary obtained in Corollary 3.10. Although $\Sigma$ can be defined directly from the l.a.o., the second bullet point in Corollary 3.10 (which follows from subsection 3.1) will play a key role in our inductive construction in Section 4.

3.3. From $\Sigma$ to a PL triangulation of $S^d$. In this subsection we complete the cs triangulation of the prism over a PL $(d-1)$-sphere as in Corollary 3.10 to a PL $d$-sphere. We introduce two pairs of new vertices $v_+, w_+, v_-, w_-$ and define

$$\Phi' := \Sigma \cup (v_+ \ast K_+) \cup (v_- \ast K_-) \cup (w_+ \ast L+) \cup (w_- \ast L-),$$

where $K_\pm := B \times \{\pm 1\}$ and $L_\pm := \sigma(B) \times \{\pm 1\} \cup_{\partial \sigma(B) \times \{\pm 1\}} (v_\pm \ast \sigma(B) \times \{\pm 1\})$. The second complex in Figure 3 offers a visualization of $\Phi'$ in the 2-dimensional case.

Proposition 3.12. The simplicial complex $\Phi'$ is a cs PL triangulation of $S^d$.

Proof: By Corollary 3.10, $\Sigma$ is a PL manifold with boundary. By Property $Q_{d-1}$, $K_\pm$ is a PL $(d-1)$-ball and by Lemma 2.2, $L_\pm$ is a PL $(d-1)$-sphere. Again by Lemma 2.2, $\Phi'$ is a PL manifold. Finally it is clear that $\Phi'$ is centrally symmetric and $|\Phi'|$ is homeomorphic to $S^d$. 

![Figure 2. Two different locally acyclic orientations of $S_1$ satisfying Lemma 3.9 and the corresponding triangulations of the prism.](image)
We next contract certain edges of $\Phi'$ in order to reduce the number of vertices. A simplicial complex is obtained from $\Delta$ via an edge contraction of $e = \{u, v\} \in \Delta$ if it is the image of $\Delta$ w.r.t. the simplicial map identifying $u$ and $v$. The edges we will contract are those of the form $\{(v, 1), (v, -1)\}$, where $v \in D \cup \sigma(D)$. In other words, we identify $D \times \{1\}$ and $\sigma(D) \times \{-1\}$ with $D \times \{-1\}$ and $\sigma(D) \times \{1\}$. It is easy to see that in this case the procedure does not depend on the order in which contractions are applied. To prove that the resulting complex is still a PL sphere we use a result by Nevo [Nev07].

**Theorem 3.13 ([Nev07, Theorem 1.4]).** Let $\Delta$ be a PL manifold and $\Delta'$ be the contraction of $\Delta$ at the edge $\{u, v\}$. Then $\Delta'$ is PL homeomorphic to $\Delta$ if and only if $\text{lk}_\Delta(u) \cap \text{lk}_\Delta(v) = \text{lk}_\Delta(\{u, v\})$.

**Lemma 3.14.** The complex $\Sigma$ satisfies the following link condition:

$$\text{lk}_\Sigma((v, 1)) \cap \text{lk}_\Sigma((v, -1)) = \text{lk}_\Sigma(\{(v, 1), (v, -1)\}).$$

**Proof:** By the definition of $\Sigma$, a face $F = \{(w_1, t_1), \ldots (w_k, t_k)\} \in \Sigma$ belongs to $\text{lk}_\Sigma((v, \pm 1))$ if and only if

- $\{w_1, \ldots, w_k\} \in \text{lk}_F(v)$;
- $\{(w_i, t_i), (v, \pm 1)\}$ is an edge of $\Sigma$ for every $i = 1, \ldots, n$.

Similarly, a face is in $\text{lk}(\{(v, 1), (v, -1)\})$ if and only if the above two conditions hold. This proves the claim. 

**Corollary 3.15.** The simplicial complex $\Phi$ obtained contracting $\Phi'$ along every edge of the form $\{(v, 1), (v, -1)\}$ with $v \in V(D) \cup V(\sigma(D))$, is a cs PL $d$-sphere.

**Proof:** Let $V(D) = \{v_0, \ldots, v_k\}$, $e_i = \{(v_i, 1), (v_i, -1)\}$ for $i = 0, 1, \ldots, k$ and

$$\Phi' = \Phi_0 \xrightarrow{e_0, \sigma(e_0)} \Phi_1 \xrightarrow{e_1, \sigma(e_1)} \cdots \Phi_{k-1} \xrightarrow{e_{k-1}, \sigma(e_{k-1})} \Phi_k = \Phi,$$

the sequence of contracting antipodal edges $e_i, \sigma(e_i)$ from $\Phi'$ to $\Phi$. For every vertex $v \in S'$ we have that

$$\text{lk}_{\Phi'}((v, 1)) \cap \text{lk}_{\Phi'}((v, -1)) = \text{lk}_\Sigma((v, 1)) \cap \text{lk}_\Sigma((v, -1))$$

and

$$\text{lk}_{\Phi'}(\{(v, 1), (v, -1)\}) = \text{lk}_\Sigma(\{(v, 1), (v, -1)\}).$$

By Theorem 3.13 and Lemma 3.14 and the fact that the links of $\{(v, 1), (v, -1)\}$, and $\{(\sigma(v), 1), (\sigma(v), -1)\}$ in $\Sigma$ are disjoint, it follows that the simplicial complex $\Phi_1$ is a PL $d$-sphere. Observe that $\{(v_{i-1}, -1), (v_{i-1}, 1)\}$ belongs to at most one of the complexes $\text{lk}_{\Phi_{i-1}}((v_i, 1))$ and $\text{lk}_{\Phi_{i-1}}((v_i, -1))$. In particular, for every $i = 1, \ldots, k$,

$$\{(v_{i-1}, -1), (v_{i-1}, 1)\} \notin \text{lk}_{\Phi_{i-1}}((v_i, 1)) \cap \text{lk}_{\Phi_{i-1}}((v_i, -1))$$

and

$$\{(v_{i-1}, -1), (v_{i-1}, 1)\} \notin \text{lk}_{\Phi_{i-1}}(\{(v_i, 1), (v_i, -1)\}).$$

This fact, together with Lemma 3.14, implies that

$$\text{lk}_{\Phi_{i-1}}((v_i, 1)) \cap \text{lk}_{\Phi_{i-1}}((v_i, -1)) = \text{lk}_{\Phi'}((v_i, 1)) \cap \text{lk}_{\Phi'}((v_i, -1))$$

$$= \text{lk}_{\Phi'}(\{(v, 1), (v, -1)\})$$

$$= \text{lk}_{\Phi_{i-1}}(\{(v_i, 1), (v_i, -1)\}),$$

which shows that every $\Phi_i$ is a PL $d$-sphere.
Remark 3.16. In fact, we can contract more edges $\{(v,-1),(v,1)\}$ and their antipodes with $v \in V(\Gamma)$ and still obtain a cs PL $d$-sphere. However, contracting too many edges would create induced cs 4 cycles in the resulting complex.

Remark 3.17. By Corollary 3.10, the cs $d$-sphere $\Phi$ that we define is usually not unique. (Indeed, the number of combinatorial types of $\Phi$ is related to the number of l.a.o. on $S$ that satisfies the conditions in Lemma 3.8.) With a slight abuse of notation we will often write “a sphere $\Phi$” to indicate any PL $d$-sphere that could be constructed from $S$ as in this section.

4. The induction step

In this section, we show that the sphere $\Phi$ constructed in the previous section satisfies property $Q_d$ w.r.t. a certain flag of spheres $S_0 \subseteq S_2 \subseteq \cdots \subseteq S_d = \Phi$. Finally, we show that if $S_{d-1}$ does not have cs induced 4-cycles then the same holds for $S_d$. Using this fact, together with the initial cases being the 0-dimensional sphere $S_0$ and the cs 6-cycle $S_1$, we prove Theorem 1.3. Assume that inductively we’ve constructed a sequence of cs $i$-spheres $S_i$, $0 \leq i \leq d-1$, that satisfies Property $Q_i$. In particular, the triple $(B_i, D_i, v_i)$ is such that

- $B_i$ is a PL $i$-ball in $S_i$ with boundary $S_{i-1}$.
- $D_i \cong \{v_i, w_i\} \ast \text{st}_{S_{i-2}}(v_{i-2})$ for some $w_i \in S_i$.
- $V(\text{st}_{S_i}(v_i)) = V(D_i) \cup V(\text{st}_{S_{i-1}}(v_{i-1}))$.

We will now show that the PL $d$-sphere $\Phi$ constructed in the previous section by setting $\Gamma := S_{d-1}$ satisfies Property $Q_d$. With the notation introduced earlier we define:

- $D_d := \{v_+, w_+\} \ast \text{st}_{S_{d-2}}(v_{d-2})$.
- $B_d$ is the closure of one of the two connected components of $\Phi - \Gamma := \{F \in \Phi \mid F \cap \Gamma = \emptyset\}$.
- $v_d := v_+$.

Remark 4.1. By Jordan’s theorem, the geometric realization of $\Phi - \Gamma$ consists of two connected components. Since $S$ and $\Gamma$ are PL spheres, it is known that $B_d$ is a simplicial ball. However, it is an open problem in PL topology (known as PL Schoenflies problem) to decide whether $B_d$ is also a PL ball. The following lemma gives a positive answer in the special case of our construction.

Lemma 4.2. The triple $(B_d, D_d, v_d)$ satisfies the following properties:

![Figure 3. An illustration of Theorem 4.4 in the case $d = 2$](image)
• $B_d$ is a PL $d$-ball in $\Phi$ with boundary $\Gamma \simeq S_{d-1}$.
• $D_d$ is isomorphic to $\{v_d, w_+\} \ast \text{st}_{S_{d-2}}(v_{d-2})$.
• $D_d \subseteq \text{st}_\Phi(v_d) \subseteq B_d$ and $V(\text{st}_\Phi(v_d)) = V(D_d) \cup V(\text{st}_\Gamma(v_{d-1}))$.

In particular, $\Phi$ satisfies Property $P_d$ under the triple $(B_d, D_d, v_d)$.

Proof: We only need to verify the first and third bullet points. By the inductive hypothesis and the definition of $D_d$, we obtain

$$V(\text{st}_\Phi(v_d)) = V(B_{d-1} \times \{1\}) \cup \{v_d, w_+\}$$

$$= \left( V(D_{d-1} \times \{1\}) \cup V(\text{st}_{S_{d-2}}(\sigma(v_{d-2}) \times \{1\})) \right) \cup \left( V(\text{st}_{S_{d-2}}(v_{d-2}) \times \{1\}) \cup \{v_d, w_+\} \right)$$

$$= V(\text{st}_\Gamma(v_{d-1})) \cup V(D_d).$$

To see that $B_d$ is a PL $d$-ball, we consider the simplicial complex $\Phi^*$ obtained from $\Phi$ by contracting all the edges of the form $\{(v, -1), (v, 1)\}$ with $v \in V(\text{st}_\Phi(v_d))$. By Lemma 3.14 and Theorem 3.13, $\Phi^*$ is a PL $d$-sphere. The image of $\Sigma \cup (v_+ \ast K_+) \cup (w_+ \ast L_+)$ is a PL $d$-ball, since $K_+$ and $L_+$ are a PL $(d-1)$-ball and a $(d-2)$-sphere respectively. By Lemma 2.3 the (closure of) the complement of the PL $d$-ball $\Sigma \cup (v_+ \ast K_+) \cup (w_+ \ast L_+)$ w.r.t. a PL $d$-sphere $\Phi^*$, $B_d$, is a PL $d$-ball.

Finally, by the definition, we have that $B_d \setminus \Gamma = D_d$ and $D_d \subseteq \text{st}_\Phi(v_d) \subseteq B_d$. Furthermore, $V(\text{st}_\Phi(v_d)) \cup V(\text{st}_\Phi(\sigma(v_d))) = V(\Phi)$ follows from the inductive assumption that $V(\text{st}_\Gamma(v_{d-1})) \cup V(\text{st}_\Gamma(\sigma(v_{d-1}))) = V(\Gamma)$.

The property which motivates our construction is the content of the following lemma.

**Proposition 4.3.** The complex $\Phi$ as constructed above has no induced cs 4-cycle if $\Gamma \simeq S_{d-1}$ has no induced cs 4-cycle.

Proof: As we see from the construction, since $\text{lk}_\Phi(v_+) \cap \text{lk}_\Phi(v_-) = \text{lk}_\Phi(w_+) \cap \text{lk}_\Phi(w_-) = \emptyset$, it follows that none of the vertices $v_+, v_-, w_+, w_-$ belongs to any induced cs 4-cycle. Furthermore, any vertex $(a, 1) \in S_{d-2} \times \{1\}$ is only adjacent to either $v_+, w_+$, or its neighbors in $S_{d-2} \times \{1\}$, or some $(\sigma(b), -1) \in S_{d-2} \times \{-1\}$ where $\{\sigma(a), \sigma(b)\} \in S_{d-2} \times \{-1\}$. By the inductive hypothesis, $S_{d-2}$ and $S_{d-1}$ do not contain any induced cs 4-cycle. We conclude that $(a, 1)$ is also not in any induced cs 4-cycle in $S$. As any cs 4-cycle must include two vertices outside $D_d \cup \sigma(D_d) \in \Phi$, this proves our claim that $\Phi$ has no induced cs 4-cycle.

Finally, we conclude with the main result.

**Theorem 4.4.** There exists a family of cs PL $i$-spheres $S_0 \subseteq S_1 \subseteq S_2 \ldots$ such that each $S_i$ satisfies Property $P_d$ with respect to $S_{i-1}$ and each $S_i$ has no induced cs 4-cycles. Furthermore $f_0(S_{i+1}) = f_0(S_i) + f_0(S_{i-1}) + 4$ for $i \leq 1$.

Proof: The first statement follows directly from Corollary 3.15 and Proposition 4.3. The number of vertices in the prism over $S_{i-1}$ equals $2f_0(S_{i-1})$, and together with $v_+, v_-, w_+, w_-$ sums up to $2f_0(S_{i-1}) + 4$. Identifying vertices of $D_{i-1} \times \{\pm 1\}$ and $\sigma(D_{i-1}) \times \{\pm 1\}$ decreases the number of vertices by $2f_0(D_{i-1})$. Since $f_0(S_{i-1}) = 2f_0(D_{i-1}) + f_0(S_{i-2})$, the claim follows. □
Proof of Theorem 1.3: Via Theorem 4.4 we know there exists a PL $d$-sphere $S_d$ whose number of vertices $n_d$ is given by the sequence $n_0 = 2$, $n_1 = 6$, $n_{i+1} = n_i + n_{i-1} + 4$. Solving the recursion we obtain the desired formula. □

Proof of Theorem 1.4: The result follows from a direct application of Theorem 1.3 and Lemma 1.2. □

Remark 4.5. Our inductive method produces the minimal triangulation of $\mathbb{R}P^2$, the boundary of icosahedron, from the minimal triangulation of $\mathbb{R}P^1$, the 6-cycle; see Figure 3. Furthermore, we find PL triangulations of $\mathbb{R}P^3$ with the $f$-vector $(11, 52, 82, 41)$, starting from the boundary of the icosahedron. They are vertex-minimal but not $f$-vectorwise minimal triangulations. For $d = 4, 5$, our construction is not vertex-minimal.

5. Open problems

We conclude this article with a few questions. The first one is about the (asymptotic) tight lower bound on the number of vertices required for a vertex-minimal triangulation of $\mathbb{R}P^d$.

Question 5.1. Does there exist a PL triangulation of $\mathbb{R}P^d$ with $\binom{d+2}{2} + \left\lfloor \frac{d-1}{2} \right\rfloor$ vertices for every $d \geq 1$? Does at least a construction with a number of vertices that is polynomial in $d$ exist?

We do not know if the PL spheres constructed in Theorem 4.4 are polytopal, i.e., they can be realized as the boundary complex of a simplicial polytope. It is natural to ask the following question.

Question 5.2. What is the minimum number of vertices required for a cs $d$-polytope with no induced cs 4-cycles?

Frequently in the literature, additional combinatorial properties are imposed on a triangulation. We focus on two properties, namely flagness and balancedness. A simplicial complex is flag if all minimal subsets of the vertices which do not form a face are edges. A $d$-dimensional simplicial complex is balanced if there exists a simplicial projection (often called coloring) to the $d$-simplex which preserves the dimension of faces. Recently in [BOW+19] and [Ven19] local flips and transformations have been implemented to obtain flag and balanced triangulations of manifolds which are vertex-minimal w.r.t. these properties. In particular the authors obtained a flag and balanced vertex-minimal triangulation of $\mathbb{R}P^2$ on 11 and 9 vertices respectively, and a balanced vertex-minimal triangulation of $\mathbb{R}P^3$ on 16 vertices. For higher $d$, a flag and balanced triangulation of $\mathbb{R}P^d$ can be obtained by considering the barycentric subdivision of the boundary complex of the $(d + 1)$-dimensional cross-polytope, and identifying antipodal vertices. These simplicial complexes have $\frac{3d+1-1}{2}$ vertices.

Problem 5.3. Construct flag or balanced PL triangulations of $\mathbb{R}P^d$ for every $d$ with less than $\frac{3d+1-1}{2}$ vertices. Does a construction on a number of vertices that is polynomial in $d$ exist?

Acknowledgements

We would like to thank Basudeb Datta and Isabella Novik for helpful comments.
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