Pricing Average Price Advertisement Options When Underlying Spot Market Prices Are Discontinuous

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Abstract

Advertisement (ad) options have been recently studied as a novel guaranteed delivery (GD) system in online advertising. In essence, an ad option is a contract that gives an advertiser a right but not obligation to enter into transactions to purchase ad inventories such as page views or link clicks from a specific slot at one or multiple pre-specified prices in a specific future period. Compared to guaranteed contracts, the advertiser pays a lower upfront fee but can have greater flexibility and more control in advertising. So far ad option studies have been restricted to the situations where the option payoff is determined by the underlying auction payment price at a specific time point and the price evolution over time is assumed to be continuous. The former leads to a biased option payoff calculation and the latter is invalid empirically for many ad slots. This paper discusses a new option pricing framework which can be applied to a general situation. The option payoff is calculated based on the average price over a specific future period. As we use the general mean, our framework contains different payoff functions as special cases. Further, we use jump-diffusion stochastic models to describe the auction payment price movement, which have Markov and price discontinuity properties, and those properties are validated by our statistical investigation of ad auctions from different datasets. In the paper, we propose a general option pricing solution based on Monte Carlo simulation and also give an explicit pricing formula for a special case. The latter is also a generalisation of the option pricing models in some other recent developments.

1 Introduction

Online advertising has become a significant source of revenue for web-based businesses, where ad inventories like page views (also called impressions) or link clicks are usually sold by publishers or search engines to advertisers in the non-guaranteed way via ad auctions, such as the Second-Price (SP) auction, the Generalised Second-Price (GSP) auction and the Vickrey-Clarke-Groves (VCG) auction. In theory, ad auctions have many desirable economic properties. For example, the GSP auction has a locally envy-free equilibrium [15, 40], and the VCG auction is efficient and incentive compatible [28]. However, ad auctions have several limitations. Firstly, the uncertainty in ad auctions makes: (1) advertisers difficult to predict campaign costs; (2) the revenue of publishers and search engines be volatile on some ad slots. Secondly, the “pay-as-you-go” nature of auction mechanisms does not encourage advertisers’ engagement because an advertiser can switch from one ad platform or marketplace to another in the next bidding at near-zero cost. To alleviate these problems, GD systems have been recently studied and contributions include [11, 5, 31, 2, 39, 16, 10, 19, 7].

Ad options are a special kind of GD systems, which allow an advertiser to pay a small upfront fee to exchange for a priority buying right of ad inventories in the future and the per-inventory payment in
In this paper, we discuss a robust option pricing framework that can be used for a general situation. We make the following contributions. Firstly, the option payoff function is based on a variable that measures the mean of the underlying prices rather than a time point. It is less biased and gives a better overall measurement on the payment price movement in the future period, particularly, if there is any price jumps and spikes. Secondly, we use the general mean to calculate the average value, whose special cases and limiting cases offer several different option payoff structures. Therefore, the proposed average price ad option becomes a generalised version of many relevant options studied in both advertising and financial markets. Thirdly, we study jump-diffusion stochastic models to describe the situations when auction payment prices are discontinuous over time. The discussed models are validated by our statistical investigation of ad auctions with real data. Finally, we propose a general solution for ad option pricing via Monte Carlo simulation as well as obtain an explicit solution for a special case. The latter is also a generalisation of some relevant ad option pricing models.

The rest of this paper is organised as follows: Section 2 reviews the related work; Section 3 introduces the proposed average price ad option and discusses its pricing; Section 4 presents our experimental results; and Section 5 concludes the paper.

## 2 Related Work

Options have been widely used in many fields for speculating profits, hedging risk and project management. Option pricing can be traced back to [1], in which the Brownian motion (also called the Wiener process) was proposed as the underlying process to price a call option written on a stock. It is a continuous-path stochastic process \( \{ W(t), t \geq 0 \} \) such that \( W(0) = 0 \), the increment \( W(t + dt) - W(t) \) is distributed as normal \( N(0, dt) \), and is independent of filtration \( \mathcal{F}_t \) – the history of what the process did up to time \( t \). In essence, it is simultaneously a Markov process and a martingale [32]. These two processes are important tools. The former describes a random system that changes states according to a transition rule that only depends on the current state, e.g., \( W(t + dt) - W(t) \) is independent of \( \mathcal{F}_t \). The latter is the mathematical representation of a player’s fortune in a fair game. Simply, the expected fortune at some later time is equal to the current fortune, e.g., \( \mathbb{E}_t[W(t + dt)] = W(t) \), where \( \mathbb{E}_t[\cdot] \) represents the conditional expectation given \( \mathcal{F}_t \). Since a Brownian motion allows negative values, it was then replaced with a geometric form in [32], called the geometric Brownian motion (GBM). It satisfies a stochastic differential equation (SDE) and has

\[
\text{d} S_t = \mu S_t \text{d} t + \sigma S_t \text{d} W_t
\]

where \( S_t \) is the underlying process, \( \mu \) and \( \sigma \) are drift and volatility, respectively, and \( W_t \) is the standard Brownian motion.

The future is pre-specified according to the targeted ad format. For example, it can be a fixed cost-per-mille (CPM) in display advertising or cost-per-click (CPC) in sponsored search. The upfront fee is called the option price and the future per-inventory payment is called the exercise price. Therefore, the future payments will be based on the number of future deliveries through option exercising. The advantages of ad options are obvious. Compared to ad auctions, the advertiser can guarantee her targeted deliveries in the future under the given budget. The prepaid option price functions as an “insurance” to cap the cost of advertising. Compared to guaranteed contracts, ad options give the advertiser greater flexibility and more control in advertising. She can request her targeted inventories in advance as well as decide when and whether to exercise the option. Ad options can be seamlessly integrated with the existing ad auctions because the advertiser’s cost in ad option is just her pre-paid option price and she can join ad auctions if she doesn’t want to exercise the purchased option in the future. On the sell side, selling ad options gives publishers and search engines some upfront incomes. More importantly, they are able to establish a contractual relationship with advertisers, which has great potential to increase the long-term revenue.

Option pricing refers to the calculation of option price for the given specifications. It contains several building blocks: the modelling of underlying price movement; the formulation of option payoff; and the pricing objective. Previous studies on ad options have been restricted to two situations. Firstly, the ad options are path-independent and their payoffs are calculated based on the value of the underlying price at a specific time point. Since ad options allow advertisers to purchase but not sell, the optimal time of option exercising for advertisers are the option expiration date [41, 9]. This leads to the biased calculation of option payoff towards the terminal value. The second limitation is that previous research assumes that the underlying auction payment price follows a continuous stochastic process. This assumption is not valid for many ad slots [9, 8]. As shown in Fig. 1 and Table 4, price discontinuity such as spikes and jumps is an important empirical property of auction payment prices which has been rarely discussed.

In this paper, we discuss a robust option pricing framework that can be used for a general situation.
an explicit solution by checking Itô’s stochastic calculus [42]. Based on the GBM, [5] constructed a replicating portfolio for an option and proposed a risk-neutral option pricing method. In the same year, [24] also discussed a similar idea to price an option. Their seminal contributions (called the Black–Scholes–Merton (BSM) model) revolutionized the financial industry and spurred the research in option pricing. We simply classify option pricing into four subfields [36, 18]: complex underlying stochastic processes; valuation of exotic options; numerical pricing approaches; and transaction cost models. Our research in this paper is based on the developments in the first three subfields, which are briefly reviewed in the following discussion.

This paper discusses an exotic option tailored to the unique environment of online advertising. Exotic options have been traded for many years in financial markets since the 1980s. In finance, the average price options are one popular type of exotic options, also called Asian options [43], whose payoff is determined by the average value of prices over a pre-specified period of future time. Therefore, average price options are path-dependent. This is different to the path-independent options such as European options and American options [42], where the option payoff is calculated for the price at exercise (i.e., on or prior to the option expiration date). The average price options can be divided into two subgroups: fixed exercise price and floating exercise price. Our proposed average price ad options are in the former group, where the exercise price is fixed and the random variable is the average underlying price. Several studies of average price options are worth mentioning here. A pricing method for financial options whose payoffs are based on a geometric mean was discussed in [14]; an option pricing model for the arithmetic mean case was explored in [30]; and the general mean option payoff was then discussed in [43]. These studies offer solid analytical fundamentals for our research. However, they all assume the underlying price movement follows a GBM so that the price needs to be continuous and there are no spikes and jumps.

The concept of ad option was initially proposed by [26], where an advertiser is allowed to make a choice of payment after winning a campaign at either CPM or CPC in the future. This option is similar to the option paying the worst and cash [43] but the option price in [26] is determined by a Nash bargaining game between an advertiser and a publisher. The first ad option that allows an advertiser to secure her targeted inventories was discussed in [41]. It is a simple European ad option that considers buying and non-buying the future impressions and whose price is calculated based on a single-period binomial lattice for a risk-averse publisher. This research was then further developed into a multi-period case in [8]. This work also discussed a stochastic volatility (SV) underlying model for the situations when the GBM assumption is not valid empirically. [9] discussed an ad option for sponsored search. The ad option allows an advertiser to target a set of candidate keywords for a certain number of total clicks in the future. Therefore, it can specify multiple underlying variables and their movements follow a multivariate GBM. Each candidate keyword can also be specified with a unique fixed CPC, and the advertiser can exercise the option multiple times at any time prior to or on the option expiration date. This design is a generalisation of the dual-strike call option [43] and the multi-exercise option [22]. Due to the contingent nature of ad options, they are able to provide greater flexibility to advertisers. This paper is one of the very first studies that discusses contingent payment in online advertising, and the two major limitations of the previous studies are addressed by employing a different payoff function and a different stochastic underlying framework.

3 Average Price Ad Options

The following example introduces the structure of the proposed ad option. Suppose that a university’s computer science department (i.e., advertiser) creates a new master degree programme and is interested in displaying the programme’s banner ad online for three months. In the current display advertising market, the advertiser will join RTB to purchase impressions. However, it is difficult for her to guarantee that her ad can be displayed for the needed number of times because the cost of future campaigns is uncertain. To secure adequate ad exposure under the given budget, the advertiser can purchase a guaranteed delivery contract or an ad option. The former needs the advertiser to pay the full amount upfront while the latter only charges her a small amount – the option price – because the ad option gives her a right but not obligation in the future to purchase the targeted impressions at a fixed payment. Consider today the advertiser submits her buy request of an ad option to a publisher, which includes: (1) the number of targeted impressions; (2) the exercise price; and (3) the future period that the option can be exercised for purchasing impressions. The publisher then calculates how much to charge this guarantee service upfront. This process is called option pricing. The publisher
then returns the advertiser with the calculated option price. If she pays the option price, she will have a right to make the decision whether or not to purchase her targeted impressions via option in the future. If the option is exercised in the future, the publisher will reserve the specified impressions for the advertiser until the needed number of impressions is fulfilled or the option expires. For each impression, the advertiser will pay the pre-specified exercise price. If the option is not exercised, the publisher will not reserve any inventories for the advertiser so the advertiser's cost is just the option price and she can use the remaining budget to join RTB. Below we discuss the building blocks of ad option pricing.

3.1 Jump-Diffusion Stochastic Models

Let \( X(t) \) be the payment price of an inventory from a specific ad slot at time \( t \). The inventory can be an impression in display advertising or a click in sponsored search. Hence, \( X(t) \) can be either CPM or CPC. The evolution of \( X(t) \) can be described by a stochastic process \( \{X(t) : t \geq 0\} \), which is defined under a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{F}, \mathbb{P})\), where \( \Omega \) is the sample space, \( \mathcal{F} \) is a collection of subsets of \( \Omega \), \( \mathbb{P} \) specifies the probability of each event in \( \mathcal{F} \), and \((\mathcal{F}_t)_{t \geq 0}\) is a filtration satisfying \( \mathcal{F}_s \subset \mathcal{F}_t \) for any \( 0 \leq s < t \). Therefore, \( X(t) \) is \( \mathcal{F}_t \)-measurable \([42]\). Given \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})\), \( X(t) \) can be modelled by the following jump-diffusion process:

\[
\frac{dX(t)}{X(t^-)} = \mu dt + \sigma dW(t) + d \left( \sum_{i=1}^{N(t)} (Y_i - 1) \right),
\]

(1)

where \( \mu \) and \( \sigma \) are the constant drift and volatility terms, \( W(t) \) is a standard Brownian motion, \( X(t^-) \) stands for the value of \( X \) just before a jump at time \( t \) (if there is one), \( N(t) \) is a homogeneous Poisson process with intensity \( \lambda \) so that

\[
\begin{align*}
&\mathbb{P}(\text{Price jumps once in } dt) = \lambda dt + O(dt), \\
&\mathbb{P}(\text{Price jumps more than once in } dt) = O(dt), \\
&\mathbb{P}(\text{Price does not jump in } dt) = 1 - \lambda dt + O(dt),
\end{align*}
\]

where \( O(dt) \) is the asymptotic order symbol which does not depend on \( t \), and \( \{Y_i, i = 1, 2, \cdots\} \) is a sequence of i.i.d. non-negative variables representing the jump sizes. In the model, all sources of randomness (i.e., \( N(t), W(t) \), and \( Y_i \)) are assumed to be independent. We simply explain the term \( Y_i = X(t_i)/X(t_i^-) \geq 0 \) because \( X(t_i) \geq 0 \). The proposed framework is a general version of the GBM model, which captures jumps and spikes in the auction payment price and is valid in most of ad instances (see Section 4.2). By checking the Itô Lemma \([42]\), the solution to Eq. (1) can be obtained (see Appendix A):

\[
X(t) = X(0) \exp \left\{ (\mu - \frac{1}{2} \sigma^2) t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} Y_i,
\]

(2)

where \( \prod_{k=1}^{N(t)} Y_i = 1 \). This is also called the exponential Lévy model \([13]\) and the jump sizes can have different distributions. Below, we discuss three popular choices of the jump size distribution.

1. In \([25]\), the logarithm of jump size \( Y_i = \ln \{Y_i \} \) is assumed to follow a normal distribution \( \mathcal{N}(\alpha, \beta^2) \) so that \( \mathbb{E}[Y_i] = e^{\alpha + \frac{1}{2} \beta^2} \), \( \text{var}[Y_i] = e^{2\alpha + \beta^2}(e^{\beta^2} - 1) \), and

\[
Y_i - 1 \sim \mathcal{N}(e^{\alpha + \frac{1}{2} \beta^2} - 1, e^{2\alpha + \beta^2}(e^{\beta^2} - 1)).
\]

2. In \([20]\), \( V_i \) is considered to follow an asymmetric double exponential distribution, denoted by \( \text{ADE}(\eta_1, \eta_2, p_1, p_2) \), whose density is

\[
f_V(v; \eta_1, \eta_2, p_1, p_2) = p_1 \eta_1 e^{-\eta_1 v}1_{v \geq 0} + p_2 \eta_2 e^{\eta_2 v}1_{v < 0},
\]

(3)

where \( 1 \) is an indicator function, \( p_1 \) and \( p_2 \) represent the probabilities of upward and downward jumps, \( \eta_1 \) > 1 and \( \eta_2 \) > 0 are model parameters. The condition \( \eta_1 > 1 \) ensures \( \mathbb{E}(Y_i) < \infty, \mathbb{E}(X(t)) < \infty \). \( V_i \) can be rewritten as the combination of exponential variables:

\[
V_i = \begin{cases}
\omega_1, & \text{with probability } p_1, \\
-\omega_2, & \text{with probability } p_2,
\end{cases}
\]

(4)
\[
\begin{array}{|c|c|}
\hline
\text{Model} & \text{Distribution of } V_t & \zeta = \mathbb{E}[e^{V_t}] - 1 \\
\hline
\text{Log-normal} & N(\alpha, \beta^2) & e^{\alpha + \frac{1}{2} \beta^2} - 1 \\
\text{Log-ADE} & \text{ADE}(\eta_1, \eta_2, p_1, p_2) & p_1 \frac{\eta_1}{1} + p_2 \frac{\eta_2}{1} - 1 \\
\text{Log-laplacian} & \text{LAP}(\varphi, \eta) & e^{\frac{\eta}{\varphi}} - 1 \\
\hline
\end{array}
\]

Table 1: Calculation of \( \zeta \).

where \( \varpi_1 \sim \text{EXP}(\eta_1) \) and \( \varpi_2 \sim \text{EXP}(\eta_2) \). Hence, \( \mathbb{E}[V_t] = p_1 \mathbb{E}[\varpi_1] - p_2 \mathbb{E}[\varpi_2] = \frac{p_1}{\eta_1} \frac{p_2}{\eta_2} \), \( \text{var}[V_t] = \left( \frac{p_1}{\eta_1^2} + \frac{p_2}{\eta_2^2} \right) + p_1 p_2 \left( \frac{1}{\eta_1^2} + \frac{1}{\eta_2^2} \right) \), and \( \mathbb{E}[Y_t] = p_1 \frac{\eta_1}{1} + p_2 \frac{\eta_2}{1} + \frac{\eta_1}{1} + \frac{\eta_2}{1} \). Here we have also shown why the condition \( \eta_1 > 1 \) is needed.

3. There is a special case of the ADE distribution: \( p_1 = p_2 = \frac{1}{2}, \eta_1 = \eta_2 = \frac{1}{\eta} \) and \( V \) has the mean \( \varphi \). It is called the Laplacian distribution, denoted by \( \text{LAP}(\varphi, \eta) \), and its density is

\[
f_V(v; \varphi, \eta) = \frac{1}{2\eta} e^{-|v - \varphi|/\eta},
\]

so \( \mathbb{E}[V_t] = \varphi \), \( \text{var}[V_t] = 2\eta^2 \), and \( \mathbb{E}[Y_t] = e^{\frac{\varphi}{\eta}} \). For finite samples, the Laplacian distribution is similar to the Student-t distribution but the latter gives a higher probability concentration such as higher peak around its mean \([38]\).

3.2 Risk-Neutral Probability

We make several assumptions about an advertiser. Firstly, she can act as an intermediate agent—she can buy and sell ad inventories in advertising markets. Secondly, the advertiser’s objective is to make money and she is risk neutral—she is indifferent between a sure thing and a risky alternative with the same expectation. A sure thing is usually described by a risk-less interest rate, denoted by \( r \). In online advertising, the auction payment price is risky because the supply and demand of inventories are uncertain in ad auctions. As auction payment prices can be contributed by different advertisers over time and each advertiser has her own CTR. To simplify the discussion and without loss of generality, \( \frac{c_M}{c} X(t) \) can be considered as the advertiser’s payment based on her ad quality, where \( c_M \) is the average value of all winning advertisers’ CTRs for inventories from the same ad slot in previous auctions and \( c \) is the average value of the advertiser’s CTR. Here \( c_M \) and \( c \) are considered as constants.

At time 0, an advertiser can receive \( \frac{c_M}{c} X(0) \) if she agrees to sell an impression or click to another advertiser with the same ad quality in future time \( t \). She can now invest the income into a bank account and can then receive \( \frac{c_M}{c} X(0) e^{rt} \) at time \( t \). To avoid arbitrage (“free lunch”) \([42]\), the expectation of the future cost of advertising should be equal to this amount. This is estimated under the so-called risk-neutral probability measure \([42]\), denoted by \( \tilde{Q} \). Hence, \( \mathbb{E}_{\tilde{Q}}[X(t)/X(0)] = e^{rt} \). Let \( Z(t) = \ln\{X(t)/x(0)\} \), the moment generating function \( \phi_{Z(t)}(1) = e^{rt} e^{\lambda (\mathbb{E}[e^{V_t}] - 1)} \). Comparing it with \( e^{rt} \) then gives the solution under \( \tilde{Q} \):

\[
X(t) = X(0) \exp\left\{ \left( r - \lambda\zeta - \frac{1}{2} \sigma^2 \right) t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} Y_i,
\]

where \( \zeta := \mathbb{E}[e^{V_t}] - 1 \) and Table [1] provides the detailed calculations.

3.3 General Mean Option Payoff

To measure an advertiser’s relative gain or loss in the future, the option payoff function is defined based on the general mean:

\[
\Phi(X) = \theta \left( \frac{c_M}{c} \left( \frac{1}{m} \sum_{i=m+1}^{m+m} X_i^\gamma \right)^{1/\gamma} - K \right)^+, \tag{7}
\]

where \((\cdot)^+ := \max\{\cdot, 0\}, \theta \) is the advertiser’s requested number of impressions or clicks, \( K \) is the exercise price, and \( \left( \frac{1}{m} \sum_{i=m+1}^{m+m} X_i^\gamma \right)^{1/\gamma} \) is the general mean measure for the spot market prices in
| $\gamma$ | $\psi(\gamma \mid X)$ | Description |
|------|----------------|-------------|
| $-\infty$ | $\min\{X_{\tilde{m}+1}, \ldots, X_{\tilde{m}+m}\}$ | Minimum value |
| $-1$ | $m/(\sum_{i=\tilde{m}+1}^{\tilde{m}+m} X_i)$ | Harmonic mean |
| $0$ | $(\prod_{i=\tilde{m}+1}^{\tilde{m}+m} X_i)^{\frac{1}{m}}$ | Geometric mean |
| $1$ | $1/\sum_{i=\tilde{m}+1}^{\tilde{m}+m} X_i$ | Arithmetic mean |
| $2$ | $\sqrt{\sum_{i=\tilde{m}+1}^{\tilde{m}+m} X_i^2}$ | Quadratic mean |
| $\infty$ | $\max\{X_{\tilde{m}+1}, \ldots, X_{\tilde{m}+m}\}$ | Maximum value |

Table 2: Special cases of the general mean.

**Input:** $X(0), r, \sigma, S, T, m, K, c, c_M, z, \theta, \gamma, \Upsilon$

where $\Upsilon = \{\alpha, \beta\}$ or $\{\eta_1, \eta_2, p_1, p_2\}$ or $\{\varrho, \eta\}$

**Output:** $\pi_0$

1: $\Delta t \leftarrow \frac{1}{z}(T - S)$; $\tilde{m} \leftarrow \left\lceil \frac{S}{\Delta t} \right\rceil$; $\zeta \leftarrow \text{Table 1}$ $z \leftarrow \text{Number of simulations}$
2: for $j \leftarrow 1$ to $z$
3: $X_0^{(j)} \leftarrow X(0)$.
4: for $i \leftarrow 1$ to $\tilde{m} + m$ do
5: $a_t \leftarrow N(r - \frac{3}{2} \sigma^2) \Delta t, \sigma^2 \Delta t)$
6: $\xi_t \leftarrow \text{BER}(a_t)$
7: $v_t \leftarrow N(0, \beta^2)$ or $\text{ADE}(\eta_1, \eta_2, p_1, p_2)$ or $\text{LAP}(\varrho, \eta)$
8: $\ln\{X_t^{(j)}\} \leftarrow \ln\{X_{t-1}^{(j)}\} + a_t + \xi_t v_t$
9: end for
10: $\Phi^{(j)} \leftarrow \text{Eq. (7)}$.
11: end for
12: $\pi_0 \leftarrow e^{-rT} \left\{ \frac{1}{z} \sum_{j=1}^{z} \Phi^{(j)} \right\}$.

Algorithm 1: General solution.

The future period $[S, T]$. There are $m$ future spot market prices in $[S, T]$, indexed from $\tilde{m} + 1$ to $\tilde{m} + m$. As described earlier, the term $C_M/c$ adds quality effects on the general mean and converts the spot market prices in $[S, T]$ into the advertiser’s own cost. We can simply consider it as the average winning payment for the advertiser if she participates in auctions in the same period. The general mean can also be considered as a function of $\gamma$, denoted by $\psi(\gamma \mid X)$. As listed in Table 2, it includes the arithmetic mean, the harmonic mean, the quadratic mean and the geometric mean as special cases, and the maximum and minimum observations as limiting cases. It is not difficult to prove that $\psi(\gamma_1 \mid X) \leq \psi(\gamma_2 \mid X)$ for $\gamma_1 \leq \gamma_2$. The geometric mean (i.e., $\gamma = 0$) is log-normally distributed whereas other means do not. This is an important property for option pricing. More discussion will be given in Theorem 1.

### 3.4 Option Pricing Methods

The concept of net present value (NPV) is used to value an ad option, in which the incoming and outgoing cash flows can be described as the option benefit and cost, respectively. The benefit of an ad option can be considered as the final value of the investment in the option, and the cost is the cost associated with the option. Therefore, NPV (Option) = PV(Option Benefit) − PV(Option Cost). We assume that an ad option adds no monetary value to both buyer and seller so that NPV = 0. Then, the option price

$$\pi_0 = e^{-rT} \mathbb{E}^Q_0[\Phi(X)],$$

where $\mathbb{E}^Q_0[\cdot]$ is the expectation conditioned on $\mathcal{F}_0$ under $Q$.

Algorithm 1 gives a general solution to Eq. (8) via Monte Carlo simulations. The time interval $[S, T]$ has been divided into $m$ equal sub-periods $\Delta t$ for a sufficiently large averaging observation. The steps in $[0, S]$ are then $\tilde{m} = \lceil \frac{S}{\Delta t} \rceil$. Hence, the total number of steps in $[0, T]$ is $\tilde{m} + m$, and we denote $t_i = \frac{i}{\tilde{m} + m}, i = 1, \ldots, \tilde{m} + m$. Do Monte Carlo replications for $z$ times. For each $j = 1, \ldots, z$, run the sub-procedures for $\tilde{m} + m$ steps. For $i = 1, \ldots, (\tilde{m} + m)$, the step $\xi_t$ follows a Bernoulli distribution because in a very small time period, no more than one jump can occur almost surely. The confidence interval for $\pi_0$ is between $\pi_0 - e^{-rT} 1.96\sigma(\Phi)/\sqrt{z}$ and $\pi_0 + e^{-rT} 1.96\sigma(\Phi)/\sqrt{z}$, where $\sigma(\Phi) = \text{std}\{\Phi^{(j)}\}_{j=1}^{z}$. The bounds can be further reduced by either increasing the number of simulations.
Table 3: Summary of datasets.

| Dataset      | SSP  | Google (UK) | Google (US) |
|--------------|------|-------------|-------------|
| Ad format    | Display | Search     | Search      |
| Auction model| SP    | GSP         | GSP         |
| Ad position  | NA    | 1st         | 1st         |
| Bid quote    | GBP/CPM | GBP/CPC    | GBP/CPC    |
| Market       | UK    | UK          | US          |
| From         | 08/01/2013 | 26/11/2011 | 26/11/2011 |
| To           | 14/02/2013 | 14/01/2013 | 14/01/2013 |
| Number of total ads | 31    | 106         | 141         |
| Record frequency | By auction | By day     | By day      |
| Number of total auctions | 6,646,643 | NA         | NA          |
| Number of total bids | 33,043,127 | NA        | NA          |

replications or reducing the variance of option payoffs. Variance reduction techniques can be used for the latter while we do not further discuss them here. Below Theorem 1 further discusses an explicit solution for the case when $\gamma = 0$ and $V_t \sim \mathcal{N}(\alpha, \beta^2)$ (see Appendix B for its proof). It has also two special cases: (1) the averaging period is small, the average price can be replaced by the terminal spot market price and the option price can be calculated by using Merton’s option pricing model [25]; (2) the jumps are not significant, e.g., the estimated $\lambda$ or $\alpha$ is very small. Then the discontinuous jump component can be removed and the underlying model becomes a GBM. Hence, the pricing framework is similar to the European geometric Asian call option [43].

**Theorem 1** If $\gamma = 0$ and $V_t \sim \mathcal{N}(\alpha, \beta^2)$, the option price can then be obtained by the formula

$$
\pi_0 = \theta e^{-rT} \sum_{k=0}^{\infty} \frac{(\lambda T)^k}{k!} \left( \frac{c_M}{c} X_0 \mathcal{N}(\xi_1) - K \mathcal{N}(\xi_2) \right),
$$

(9)

where $\mathcal{N}(\cdot)$ is the cumulative standard normal distribution function, and

$$
A = \frac{1}{2}(r - \lambda \zeta - \frac{1}{2} \sigma^2)(T + S) + k\alpha, \quad B^2 = \frac{1}{3} \sigma^2 T + \frac{2}{3} \sigma^2 S + k\beta^2,
$$

$$
\Omega = e^{\frac{A}{2}(B^2 + 2A)}, \quad \xi_1 = B - \frac{\phi}{B} + \frac{A}{B}, \quad \xi_2 = \frac{A}{B} - \frac{\phi}{B}.
$$

4 Experiments

This section describes our datasets and experimental settings, investigates the statistical properties of payment prices in ad auctions, discusses the estimation of model parameters, and presents the results of option pricing and revenue analysis.

4.1 Data and Experimental Settings

Our empirical findings and evaluation are based on three datasets: an RTB dataset from a medium-sized supply-side platform (SSP) in the UK; and two sponsored search datasets from Google AdWords. Table 3 summarises these datasets. It should be noted that different auction models are used. In display advertising, multiple ad slots on a webpage are sold separately in RTB through the SP auction model [27]. Therefore, each auction in the SSP dataset is for a specific slot. In sponsored search, Google uses the GSP auction model to sell a list of ad slots on its search result page for a specific keyword. Here we only look at the ads located in the first position in the mainline paid listing. Table 4 describes our experimental settings. The SSP dataset has all bids for each auction, so different time scale $\Delta t$ can be used to extract time series of the payment prices. However, there are only daily payment prices in Google’s datasets. In the experiments, we randomly select the time period which has consecutively reported data so that the evolution of payment prices can be analysed. The period $[S, T]$ is used to calculate the general mean of the ad option payoff. If $S = T$, there is only one time point and the option becomes the European call option.
| Dataset | SSP | Google (UK) | Google (US) |
|---------|-----|-------------|-------------|
| Time scale | 1 hour | 4 hours | 6 hours | 12 hours | 1 day | 1 day | 1 day |
| $\Delta t$ | $1.1416e-04$ | $4.5662e-04$ | $6.8493e-04$ | $0.0014$ | $0.0027$ | $0.0027$ | $0.0027$ |
| Data size on each ad: | | | | | | | |
| Training set | 60 | 40 | 30 | 20 | 14 | 60 | 60 |
| Development and test set | 60 | 20 | 15 | 5 | 1 | 60 | 60 |
| Number of total ads | 31 | 23 | 22 | 20 | 12 | 106 | 141 |
| $[S, T]$ | $[0.0034, 0.0068]$ | $[0.0046, 0.0091]$ | $[0.0034, 0.0103]$ | $[0.0041, 0.0068]$ | $[0, 0.0027]$ | $[0.0822, 0.1644]$ | $[0.0822, 0.1644]$ |
| Jumps and spikes | 100.00% (9.07) | 100.00% (5.80) | 100.00% (4.22) | 100.00% (2.42) | 100.00% (1.56) | 100.00% (8.33) | 100.00% (9.08) |
| Normality | | | | | | | |
| Kolmogorov-Smirnov test | 19.35% | 82.61% | 86.36% | 55.00% | 75.00% | 0.00% | 0.00% |
| Shapiro-Wilk test | 9.68% | 78.26% | 31.82% | 50.00% | 83.33% | 0.94% | 0.71% |
| Heavy tails: kurtosis | 100.00% (6.78) | 26.09% (3.48) | 40.91% (3.25) | 5.00% (1.83) | 100% (4.21) | 100% (23.17) | 100% (23.42) |
| Absence of autocorrelations | | | | | | | |
| Ljung-Box-Q test at lags 5 | 77.42% | 30.43% | 9.09% | 10.00% | 100% | 80.19% | 90.07% |
| Ljung-Box-Q test at lags 10 | 87.10% | 8.70% | 9.09% | 20.00% | 100.00% | 86.79% | 92.20 |
| Ljung-Box-Q test at lags 15 | 90.32% | 13.04% | 9.09% | 20.00% | 100.00% | 95.28% | 92.91 |
| Volatility clustering: | | | | | | | |
| Ljung-Box-Q test at lags 5 (abs) | 19.35% | 21.74% | 86.36% | 10.00% | 0.00% | 16.04% | 10.64% |
| Ljung-Box-Q test at lags 10 (abs) | 16.13% | 30.43% | 90.91% | 15.00% | 0.00% | 5.66% | 7.80% |
| Ljung-Box-Q test at lags 15 (abs) | 16.13% | 34.78% | 90.91% | 15.00% | 0.00% | 4.72% | 5.67% |
| Ljung-Box-Q test at lags 5 (square) | 19.35% | 13.04% | 72.72% | 5.00% | 0.00% | 16.04% | 9.93% |
| Ljung-Box-Q test at lags 10 (square) | 19.35% | 13.04% | 72.72% | 15.00% | 0.00% | 1.89% | 3.55% |
| Ljung-Box-Q test at lags 15 (square) | 3.23% | 13.04% | 72.72% | 15.00% | 0.00% | 0.94% | 2.84% |
| Ads with absence of both autocorrelations and volatility clustering | 58.06% (18) | 8.70% (2) | 0.00% (0) | 10.00% (2) | 100.00% (12) | 72.64% (77) | 75.18% (106) |

The number in brackets represents: (1) the average number of jumps; (2) the average kurtosis value; (3) the number of ads.

Table 4: Experimental settings and statistical investigation of stylised facts from the training data.
4.2 Stylised Facts

We now examine the so-called stylised facts of payment prices in ad auctions. Stylised facts refer to empirical findings or properties that are so consistent across markets [12, 34]. Since $X(t) \geq 0$, for mathematical convenience, its logarithm is usually analysed. Given a time scale $\Delta t$, which can range from a few seconds to a day, the log change rate of $X(t)$ at scale $\Delta t$ is defined as $R(t, \Delta t) = \ln \{ X(t + \Delta t) \} - \ln \{ X(t) \}$. It is also called the log return or continuously compounded return in time series analysis [38]. One important reason that the log change rate is used is because it has more tractable statistical properties than the simple rate, e.g., $R(t, k\Delta t) = \sum_{i=0}^{k-1} R(t + i, \Delta t)$ for $k$ periods. A set of stylised facts which are common to a wide set of online ads is summarised below:

**Jumps and Spikes** As shown in Fig. 1, the auction payment price from a display ad slot exhibits sudden jumps and spikes. From a modelling point of view, the price process exhibits a non-Markovian behaviour in short time intervals and prices increase or decrease significantly in a continuous way. In our datasets, all slots show price jumps and spikes while they have different jump frequencies. Several jump detection techniques will be discussed in Section 4.3. This stylised fact has also been discussed in electricity prices and the typical explanation is a non-linear supply-demand curve in combination with the electricity’s non-storability [4]. This explanation can possibly be applied to online ads, as show in Fig. 2, the advertising supply is triggered by ad-hoc web surf or search made by online users and the demand is the number of advertisers who join ad auctions. For a specific ad slot, the competition level in ad auctions affects the payment price nonlinearly.

**Non-Normality and Heavy Tails** The non-normal character of the unconditional distribution of log change rates has been observed in many ad slots. Normality can be graphically checked by a histogram or Q-Q plot, and can be statistically verified by hypothesis testing such as the one-sample Kolmogorov-Smirnov (KS) test [23] and the Shapiro-Wilk (SW) test [35]. Fig. 1 exhibits that the the distribution of log change rates has two tails heavier than those of the normal distribution. This is also called leptokurtic. One way to quantify the deviation from the normal distribution is by checking the kurtosis statistic [38], which describes the tail behaviour of a series. Since the kurtosis of a standard normal distribution is 3, then the empirical distribution will have a higher peak and two heavy tails if the kurtosis is larger than 3.

**Absence of Autocorrelations** Consider whether the future log change rates can be predicted from the current values, we can formulate this question by asking whether they remain stable and whether they are correlated over time [12]. The process is assumed to be weakly stationary so that the first moment and autocovariance do not vary with respect to time. The (linear) autocorrelation functions (ACFs) of log change rates are insignificant in Fig. 1. Therefore, the process can be constructed by using the Markov property. In fact, as discussed in Section 4.3, most of the classical models in economics and finance assume that asset prices follow a GBM and the latter is based on independent asset returns [33]. In experiments, we also use the Ljung-Box Q-test to check the autocorrelation for a fixed number of lags. Table 4 shows that most of hourly and daily log change rates (overall more than 80%) do not have autocorrelations. However, most of the 4-hour, 6-hour, and 12-hour rates in the SSP dataset exhibit autocorrelations.

**Volatility Clustering** A stochastic process can have uncorrelated but not independent increments. The magnitude of price fluctuations is measured by volatility, and volatility clustering is referred to the property that large price variations are more likely to be followed by large price variations [12]. To detect volatility clustering, two commonly used methods are: (1) the ACF of absolute log change rates; and (2) the ACF of squared log change rates. Volatility clustering has been observed in Fig. 1. However, Table 4 shows that it is still not the property for the majority of ad slots, particularly, for hourly and daily rates.

In this paper, the discussed jump-diffusion stochastic models incorporate the first three properties but not volatility clustering. This property can be incorporated by adding another dynamic for volatility such as the SV model discussed in [8]. In the following experiments, only hourly and daily data will be used for developing stochatic models and then option pricing.
4.3 Estimation of Model Parameters

One of the widely accepted interpretations of price jumps considers them as time-dependent outliers. Simply, a price jump is an observation that lies in an abnormal distance from other values. In [17], an extensive simulation study was conducted to compare the relative performance of several detection methods for price jumps, including the global centiles (GC), the price-jump index (PJI), the centiles over block-windows (COBW), and various bipower variation methods. The comparison results showed that: (1) the GC and the COBW outperformed others in the case of false positive probability; (2) the bipower variation method proposed by Lee and Mykland [21] (abbreviated as BV-LM) performed best in the case of false negative probability. In our experiments, we implement all these methods, and the hamper filter [29] because it has been widely used for outlier detection in signal processing. The identified jumps for the training sets are presented in Table 4.

Fig. 3 gives an illustration of detecting price jumps on an ad slot from the SSP dataset. There are large differences in terms of performance among methods. If there is no jump, our discussed stochastic underlying framework in Eq. (1) then becomes a GBM. Therefore, after removing the detected jumps,
the kurtosis of the training log change rates should approach 3 and we use this criterion to select the 
best jump detection method. An illustration of our model selection is described in Fig. 4, where the 
COBW performances best for the SSP dataset based on hourly time scale as it has the highest number 
of slots which lie in the range $[2, 4]$. Fig. 5 further summarises the overall results of model selection 
in our training sets. It should be noted that the BV-LM is not used for the Google datasets because 
there are no intraday campaign records. In summary, after removing the identified jumps, the COBW 
performs best as it has 40.50% of slots that the kurtosis of log change rates lies in the range $[2, 4]$, 
followed by the hamper filter with 39.80%. In the following experiments, jumps are identified by the 
COBW.

We follow [6] and estimate other model parameters as follows (Appendix C):

$$\arg\max_{\sigma \geq 0, \mu_V, \sigma_V \geq 0} \ln \left\{ \prod_{j=1}^{\tilde{n}} \left( (1 - \lambda \Delta t) f_1(\tilde{z}_j) + \lambda \Delta t f_2(\tilde{z}_j) \right) \right\},$$

(10)

where $f_1(\tilde{z}_j)$ is the density of $N((\mu - \frac{1}{2} \sigma^2) \Delta t, \sigma^2 \Delta t)$, $f_2(\tilde{z}_j)$ is the density of $N((\mu - \frac{1}{2} \sigma^2) \Delta t + \mu_V, \sigma^2 \Delta t + \sigma_V^2)$.

An alternative way to estimate the values of parameters is to minimise the squared L2-norm of the difference between the option prices from market and from model, i.e.,

$$\arg\min_{\mu^*, \sigma^*, \mu_V, \sigma_V} \left\| \pi_{0, \text{Model}} - \pi_{0, \text{Market}} \right\|^2_2.$$

This is also called model calibration in financial engineering [37]. However, as there is no such ad option market at the moment, this method cannot be used in our research.

### 4.4 Option Pricing and Revenue Analysis

Fig. 6 presents empirical examples of pricing an ad option written on the keyword ‘panasonic dmc’ 
from the Google (UK) dataset. The training and test prices of the first (position) slot from this 
keyword search result page exhibit frequent jumps. The histogram of observed log change rates 
tends to have a higher peak than normal distribution. By estimating the parameters of different
jump-diffusion stochastic models, we then simulate the paths of the underlying auction payment price for the future period. Each path is generated by lines 4-9 in Algorithm 1. The differences of path patterns from three models are obvious. The generated paths are used to calculate the option payoffs and then be used to calculate the option price. The advertiser only pays a small upfront fee and she can then have the right to purchase the targeted inventory at the pre-specified exercise price. In this example, the exercise price is set less than the current auction payment price.

The overall results of option pricing for ad slots in our datasets and the effects of ad options on the seller’s revenue have been presented in Table 5. Here we further consider the market performance and pricing specifications. The market performance can be simply divided into bull market and bear market. The former describes the situation that the average auction payment price in the future (i.e., the test set) is equal to or higher than its current value (i.e., \( X_0 \)) while the latter means the market is going down. Each ad option has also been priced under three different specifications [42]: in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM). For an ITM ad option, the exercise price is below the current auction payment price and in experiments we set \( K = 0.75X_0 \). For an ATM ad option, the exercise price is as same as its current underlying auction payment price, i.e., \( K = X_0 \). An option is OTM if the exercise price is above where the current payment price and in experiments \( K = 1.25X_0 \). In a bull market, as the auction payment is high, advertisers will exercise their purchased options to hedge price risk. This also means that a seller’s revenue will decrease. Table 5 shows only a small number of slots can still generate more revenues in the bull market. This may because the calculated option prices are high and also the future average payment is just slightly higher than its current payment. Also, as an ITM option has a lower exercise price compared to ATM and OTM options, its option price should be more expensive. However, in the bull market, it also saves an advertiser more. On the other hand, in a bear market, advertisers will not exercise the purchased options. They will join ad auctions instead and the seller can benefit by obtaining upfront income. In fact, how a publisher or search engine strategically sells ad options to advertisers can be further discussed in depth.
| Dataset | Bull market | Bear market |
|---------|-------------|-------------|
|         | ITM ($K = 0.75X_0$) | ATM ($K = X_0$) | OTM ($K = 1.25X_0$) | ITM ($K = 0.75X_0$) | ATM ($K = X_0$) | OTM ($K = 1.25X_0$) |
| Log-normal jumps (explicit solution) |     |     |     |     |     |     |
| SSP (1 hour) | < 66.67%, 33.33% > | 0.00% (-83.89%) | 0.00% (-79.83%) | 8.33% (-75.73%) | 100.00% (214.46%) | 100.00% (261.64%) | 100.00% (309.17%) |
| Google (UK, 1 day) | < 12.00%, 88.00% > | 22.22% (-30.63%) | 22.22% (-17.54%) | 33.33% (-4.41%) | 96.97% (272.85%) | 100.00% (370.00%) | 100.00% (467.29%) |
| Google (US, 1 day) | < 75.00%, 25.00% > | 3.84% (-36.77%) | 26.92% (-22.06%) | 53.85% (-7.12%) | 73.07% (98.23%) | 100.00% (146.81%) | 100.00% (195.76%) |
| Log-normal jumps (Monte Carlo simulation) |     |     |     |     |     |     |
| SSP (1 hour) | < 66.67%, 33.33% > | 0.00% (-77.74%) | 8.33% (-74.07%) | 8.33% (-70.29%) | 100.00% (75.66%) | 100.00% (122.06%) | 100.00% (168.92%) |
| Google (UK, 1 day) | < 12.00%, 88.00% > | 0.00% (-48.46%) | 22.22% (-36.25%) | 33.33% (-23.78%) | 96.97% (282.04%) | 100.00% (374.75%) | 100.00% (468.55%) |
| Google (US, 1 day) | < 75.00%, 25.00% > | 21.79% (-26.51%) | 39.74% (-14.15%) | 57.69% (-0.86%) | 76.92% (107.63%) | 100.00% (152.23%) | 100.00% (198.43%) |
| Log-ADE jumps (Monte Carlo simulation) |     |     |     |     |     |     |
| SSP (1 hour) | < 66.67%, 33.33% > | 0.00% (-84.50%) | 0.00% (-80.64%) | 0.00% (-76.63%) | 100.00% (101.71%) | 100.00% (143.82%) | 100.00% (187.67%) |
| Google (UK, 1 day) | < 12.00%, 88.00% > | 33.33% (-26.01%) | 33.33% (-17.46%) | 33.33% (-9.65%) | 83.33% (255.44%) | 84.84% (353.42%) | 84.84% (453.39%) |
| Google (US, 1 day) | < 75.00%, 25.00% > | 16.67% (-30.34%) | 24.36% (-20.26%) | 44.87% (-8.24%) | 61.53% (71.50%) | 100.00% (114.23%) | 100.00% (163.57%) |
| Log-laplacian jumps (Monte Carlo simulation) |     |     |     |     |     |     |
| SSP (1 hour) | < 66.67%, 33.33% > | 0.00% (-82.22%) | 8.33% (-78.45%) | 8.33% (-74.56%) | 100.00% (74.80%) | 100.00% (121.34%) | 100.00% (168.39%) |
| Google (UK, 1 day) | < 12.00%, 88.00% > | 0.00% (-54.12%) | 11.11% (-41.62%) | 33.33% (-28.93%) | 81.82% (240.84%) | 81.82% (334.64%) | 81.82% (429.33%) |
| Google (US, 1 day) | < 75.00%, 25.00% > | 2.56% (-40.95%) | 17.95% (-27.79%) | 41.02% (-13.82%) | 65.38% (69.07%) | 100.00% (115.04%) | 100.00% (162.38%) |

The number outside the round brackets represents the percentage of ad slots that have revenue growth and the number in the round brackets represents average change rates of slots under that group. The numbers inside the angle brackets represent the percentages of slots in bull market and bear market, respectively.

Table 5: Comparison of revenues from selling ad options to ad auctions.
Figure 6: Empirical example of ad option pricing for keyword ‘equity line of credit’ from Google (UK) dataset.

5 Conclusion

This paper has discussed a robust ad option pricing framework. The option payoff is based on the general mean of underlying auction payment prices in the future period, and the underlying price is modelled by jump-diffusion stochastic models, which allow discontinuities in price evolution and can capture most of the stylised empirical properties. We have provided a general option pricing solution based on Monte Carlo simulations and have also discussed an explicit pricing formula for a special case. The latter is also a generalised solution of some related studies. Future research is likely to proceed in two directions. Firstly, the estimation of jump-diffusion models can be further discussed in depth. This contains the identification of jumps and exploring the boundaries that making the estimated model parameters satisfying the theoretical assumptions or conditions. Secondly, capacity issue can be considered in ad option pricing. In this paper, we consider a single inventory issue and assume that a publisher or search engine has a good estimation of future inventories and rationally sells them in advance via options. Discussing capacity will include a game-theoretical analysis on the combined strategies of both buy-side and sell-side markets.
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Appendix

A Solution to Eq. (2)

Let $t_1 < t_2 < \ldots$ be the jump times of $N(t)$. For $0 = t_0 \leq t < t_1$, Eq. (1) becomes a GBM, by checking the Itô Lemma [22], we have $d(\ln(X(t))) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW(t)$, then $X(t) = X(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$. For $t_1 \leq t < t_2$, the solution is similar and we just need to multiply the price with $Y_1$. By following the same procedure, the solution to Eq. (1) can be obtained

$$X(t) = \begin{cases} 
X(0) \exp \left\{ (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \right\}, & 0 \leq t < t_1, \\
X(0) \exp \left\{ (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \right\} Y_1, & t_1 \leq t < t_2, \\
X(0) \exp \left\{ (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \right\} Y_1 Y_2, & t_2 \leq t < t_3, \\
\cdots, \\
X(0) \exp \left\{ (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t) \right\} \prod_{i=1}^{N(t)} Y_i. 
\end{cases}$$

B Proof of Theorem 1

$\psi(0 | X)$ can be rewritten as $e^{-\frac{1}{2} \int_{S}^{T} \ln(X(t)) dt}$, then

$$Z(T) |_{N(T)=k} \sim N \left( (r - \lambda \zeta - \frac{1}{2}\sigma^2)T + k \alpha, \sigma^2 T + k \beta^2 \right).$$

Below we show $\psi(0 | X)$ is log-normally distributed.

$$\psi(0 | X) = X_0 \left( \prod_{i=\tilde{m}+1}^{m} \frac{X_i}{X_{\tilde{m}+1}} \right)^{1/m}$$

$$= X_0 \exp \left\{ \frac{1}{m} \ln \left\{ \left( \frac{X_{\tilde{m}}}{X_0} \right)^{m} \left( \frac{X_{\tilde{m}+1}}{X_{\tilde{m}}} \right)^{m} \left( \frac{X_{\tilde{m}+2}}{X_{\tilde{m}+1}} \right)^{m-1} \cdots \left( \frac{X_{m+m}}{X_{m+m-1}} \right) \right\} \right\}$$

Since $\Delta t = \frac{T-S}{m}$, so $\tilde{m} = \frac{S}{\Delta t} = \frac{S}{m}$, and then

$$\ln \left\{ \frac{X_{\tilde{m}}}{X_0} \right\} \bigg|_{N(T)=k} \sim N \left( (r - \lambda \zeta - \frac{1}{2}\sigma^2)S + k \alpha, \sigma^2 S + k \beta^2 \right),$$

$$\ln \left( \frac{X_{\tilde{m}+i}}{X_{\tilde{m}+i}} \right) \bigg|_{N(T)=k} \sim N \left( (r - \lambda \zeta - \frac{1}{2}\sigma^2)\Delta t, \sigma^2 \Delta t \right), \forall i = 0, \ldots, (m - 1).$$

Let $\Theta = \frac{1}{T-S} \int_{S}^{T} Z(t) dt$, then $\Theta |_{N(T)=k} \sim N (\tilde{A}, \tilde{B}^2)$, where

$$\tilde{A} = (r - \lambda \zeta - \frac{1}{2}\sigma^2)(\frac{(m+1)(T-S)}{m} + S) + k \alpha,$$

$$\tilde{B}^2 = \frac{(m+1)(2m+1)}{6m^2} \sigma^2 (T-S) + \sigma^2 S + k \beta^2.$$

If $m \to \infty$, $\Theta |_{N(T)=k} \sim N (A, B^2)$, where

$$A = \frac{1}{2} (r - \lambda \zeta - \frac{1}{2}\sigma^2)(T + S) + k \alpha,$$

$$B^2 = \frac{1}{3} \sigma^2 T + \frac{2}{3} \sigma^2 S + k \beta^2.$$
Hence, the option price can be obtained as

\[
\pi_0 = \theta e^{-rT} \mathbb{E}_0^Q \left[ \mathbb{E}_0^Q \left( \left( \frac{CM}{c} X_0 e^{\Theta} - K \right)^+ \right) \Bigg| N(T) = k \right] \\
= \theta e^{-rT} \sum_{k=0}^{\infty} \frac{(\lambda T)^k}{k!} e^{-\lambda T} \mathbb{E}_0^Q \left[ \left( \frac{CM}{c} X_0 e^{\Theta} - K \right)^+ \right] \\
= \theta e^{-rT} \sum_{k=0}^{\infty} \frac{(\lambda T)^k}{k!} e^{-\lambda T} \int_{\phi}^{\infty} \left( \frac{CM}{c} X_0 e^{\Theta} - K \right) f(\Theta) d\Theta,
\]

solving the integral terms then completes the proof.

C Discussion on Eq. (10)

The discretisation of Eq. (2) is

\[
\frac{X(t)}{X(t - \Delta t)} = \exp \left\{ (\mu - \frac{1}{2} \sigma^2) \Delta t + \sigma \sqrt{\Delta t} \varepsilon_t \right\} \prod_{i=1}^{n_t} Y_i,
\]

where \( \varepsilon_t \sim \mathcal{N}(0, 1) \), and \( n_t = N(t) - N(t - \Delta t) \) representing the number of price jumps between \( t - \Delta t \) and \( t \). Let \( \tilde{Z}(t) = \ln \{ X(t)/X(t - \Delta t) \} \), \( \mu_V = \mathbb{E}[V_i] \), \( \sigma_V^2 = \text{Var}[V_i] \), where \( \mu^* = \mu - \frac{1}{2} \sigma^2 + \lambda \mu_V \) and \( \Delta J^*_t = \sum_{i=1}^{n_t} V_i - \lambda \mu_V \Delta t \). Then \( \mathbb{E}[\Delta J^*_t] = \mathbb{E}[n_t] \mu_V - \lambda \Delta t \mu_V = 0 \), \( \mathbb{E}[\Delta J^*_t | n_t] = n_t \mu_V - \lambda \Delta t \mu_V \) and \( \text{Var}[\Delta J^*_t | n_t] = n_t^2 \sigma_V^2 \). Hence, \( \mathbb{E}[\tilde{Z}(t) | n_t] = (\mu - \frac{1}{2} \sigma^2) \Delta t + n_t \mu_V \), \( \text{Var}[\tilde{Z}(t) | n_t] = \sigma^2 \Delta t + n_t^2 \sigma_V^2 \). For simplicity, \( \tilde{Z}(t) | n_t \) is considered to be normal, then

\[
\arg \max_{\mu^*, \sigma^2 \geq 0, \mu_V, \sigma_V \geq 0} \ln \left\{ L(\mu^*, \sigma, \mu_V, \sigma_V) \right\} \\
= \ln \left\{ \prod_{j=1}^{\tilde{n}} f(\tilde{z}_j; \mu^*, \sigma, \mu_V, \sigma_V) \right\} \\
= \ln \left\{ \prod_{j=1}^{\tilde{n}} \sum_{k=0}^{\infty} \mathbb{P}(n_t = k) f(\tilde{z}_j | n_t) \right\},
\]

where \( \tilde{n} \) is the number of observations, the density \( f(\tilde{z}_j) \) is the sum of the conditional probabilities density \( f(\tilde{z}_j | n_t) \) weighted by the probability of the number of jumps \( \mathbb{P}(n_t) \). This is an infinite mixture of normal variables, and there is usually one price jump if \( \Delta t \) is small. Therefore, the estimation becomes Eq. (10).