Near-Horizon Conformal Structure and Entropy of Schwarzschild Black Holes

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Abstract

Near-horizon conformal structure of a massive Schwarzschild black hole of mass $M$ is analyzed using a scalar field as a simple probe of the background geometry. The near-horizon dynamics is governed by an operator which is related to the Virasoro algebra and admits a one-parameter family of self-adjoint extensions described by a real parameter $z$. When $z$ satisfies a suitable constraint, the corresponding wavefunctions exhibit scaling behaviour in a band-like region near the horizon of the black hole. This formalism is consistent with the Bekenstein-Hawking entropy formula and naturally produces the $-\frac{3}{2}\log M^2$ correction term to the black hole entropy with other subleading corrections exponentially suppressed. This precise form for the black hole entropy is expected on general grounds in any conformal field theoretic description of the problem. The presence of the Virasoro algebra and the scaling properties of the associated wavefunctions in the near-horizon region together with the appearance of the logarithmic correction to the Bekenstein-Hawking entropy provide strong evidence for the near-horizon conformal structure in this system.

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1. Introduction

The discovery of a conformal symmetry in the near-horizon region of a black hole has led to interesting developments in quantum gravity [1, 2, 3]. By imposing a suitable boundary condition at the black hole horizon, it has been shown that the algebra of surface deformations contains a Virasoro subalgebra [2, 3, 4]. This approach is based on extension of the idea of Brown and Henneaux [5] regarding asymptotic symmetries of three-dimensional anti-de-Sitter gravity and is essentially classical in nature. The knowledge of the Virasoro algebra in the near-horizon region provides a method to calculate the black hole entropy [1, 2] and is indicative of the holographic nature of the system [6, 7, 8, 9]. The boundary conditions satisfied by the metric components at the horizon encode the appropriate physical requirements for holography.

In this paper we provide an alternate approach for analyzing the near-horizon conformal symmetry of black holes. Our method uses a massless scalar field coupled to the black hole background as a probe of the near-horizon geometry. It is useful to choose the probe in such a way so that unnecessary complications do not obscure the essential physical features of the system. We shall initially restrict our attention to massive Schwarzschild black holes and generalization to other geometries would be discussed later. With this in mind we consider the time-independent, zero angular momentum modes of a massless scalar field as a simple probe of the near-horizon geometry. The Klein-Gordon operator $H$ governing the dynamics of the probe in the near-horizon region contains an inverse square potential term [10]. Systems with such a potential have been known to be associated with conformal mechanics [11, 12, 13, 14]. We shall show below that indeed the full Virasoro algebra arises naturally in this system. $H$ however is not an element of the Virasoro algebra but belongs to the corresponding enveloping algebra [15]. This result is obtained using a factorization of the operator $H$ and the inverse square term plays a crucial role in this algebraic analysis.

The quantum properties of the near-horizon KG operator $H$ yields further valuable information about the system. In the quantum theory, $H$ admits a one-parameter family of self-adjoint extensions labelled by $e^{iz}$ where $z$ is a real number [16, 17]. For a generic value of $z$, the system admits an infinite number of bound states. However, it is only when $z$ is positive and satisfies the consistency condition $z \sim 0$, that the bound states exhibit a scaling behaviour in the near-horizon region of the black hole. This property of the bound states reflects the existence of an underlying conformal structure in the near-horizon region. The self-adjoint parameter $z$ labels the domains of the Hamiltonian and is therefore directly related to the boundary conditions satisfied by $H$. The constraint on $z$ is thus conceptually analogous to the boundary conditions satisfied by the metric components in Refs. [2, 3] required for holography.

The spectrum of the operator $H$ is determined by boundary conditions which are encoded in the choice of the corresponding domain. At an operator level, the properties of the Hamiltonian are thus determined by the parameter $z$. On the other hand, the only physical input in the problem is the mass $M$ of the black hole which must also play a role in determining the spectrum. The parameters $z$ and $M$ thus play a conceptually similar role and they are likely to be related to each other. Establishing a relation between $z$ and $M$, which is consistent with the constraints of our model, is a crucial step in our approach.

The relationship between $z$ and $M$ naturally leads to the identification of black hole entropy within this formalism [18]. Our analysis is based on the identification of the bound states described above with the quantum excitations of the black hole resulting from the capture of the scalar field probe. These excitations will eventually decay through the emission of Hawking
radiation. The corresponding density of states of the black hole is usually a function of the mass $M$. On the other hand, the density of states in our algebraic formalism has a smooth expression in terms of the self-adjoint parameter $z$. In view of the relation between $z$ and $M$, the density of states written as a function of $z$ can be re-expressed in terms of the variable $M$. This process leads to the identification of the black hole entropy consistent with the constraints of the model. The entropy so obtained is given by the Bekenstein-Hawking term together with a leading logarithmic correction whose coefficient is $-\frac{3}{2}$. The other subleading corrections terms are found to be exponentially suppressed. Such a logarithmic correction term to the Bekenstein-Hawking entropy with a $-\frac{3}{2}$ coefficient was first found in the quantum geometry formalism [19] and has subsequently appeared in several other publications [20, 21, 22, 23, 24]. In particular, using an exact convergent expansion for the partition of a number due to Rademacher [25], rather than the asymptotic formula due to Hardy and Ramanujan [26], it was shown in Ref. [23] that within any conformal field theoretic description, the black hole entropy can be expressed in a series where the leading logarithmic correction to Bekenstein-Hawking term always appears with a universal coefficient of $-\frac{3}{2}$ with the other subleading terms exponentially suppressed. Here we show that the entropy of the Schwarzschild black hole contains correction terms precisely of this structure, which provides a strong evidence for the underlying near-horizon conformal structure of the system.

The plan of this paper is as follows. In Section 2 we provide an algebraic formulation of the near-horizon dynamics where in terms of operators appearing in the factorization of $H$. We also show how the Virasoro algebra appears in this context. Some aspects of the representation theory of Virasoro algebra is discussed in Section 3. The particular representation to which $H$ belongs is explicitly obtained there. Section 4 discusses the near-horizon quantization in terms of the self-adjoint extensions of $H$ and the scaling properties of the corresponding wavefunctions is discussed in Section 5. The appearance of the Bekenstein-Hawking entropy together with the universal $-\frac{3}{2}$ logarithmic correction term is discussed in Section 6. The paper is concluded in Section 7 with some remarks and an outlook.

2. Algebraic Formulation of the Near-Horizon Dynamics

In this section, we consider the case of a scalar field probing the near-horizon geometry of a massive Schwarzschild black hole of mass $M$. The metric for an asymptotically flat, spherically symmetric and static black hole in 3+1 dimensions is given by

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2d\Omega^2 = g_{ij}dx^i dx^j.$$  \hspace{1cm} (2.1)

For Schwarzschild black hole $F(r) = \left(1 - \frac{2M}{r}\right)$.

The action for a massless scalar field in the above background can be written as

$$S = -\frac{1}{2} \int \sqrt{|g|} g^{ij} \partial_i \phi \partial_j \phi.$$ \hspace{1cm} (2.2)

We shall restrict the analysis to the spherically symmetric and time-independent modes of the scalar field. The Klein-Gordon operator governing the near-horizon dynamics of these modes can then be written as [10]

$$H = -\frac{d^2}{dx^2} + \frac{a}{x^2},$$ \hspace{1cm} (2.3)
where $a$ is a real dimensionless constant, and $x = r - 2M$ is the near-horizon coordinate. For the Schwarzschild background, we have $a = -\frac{1}{4}$. For the moment, however, we can consider a general value of $a$.

The starting point of our formulation is the observation that the operator $H$ can be factorized as

$$H = A_+ A_-,$$

where

$$A_\pm = \pm \frac{d}{dx} + \frac{b}{x},$$

and

$$b = \frac{1}{2} \pm \sqrt{1 + 4a^2}. \quad (2.6)$$

We note that $a = -\frac{1}{4}$ is the minimum value of $a$ for which $b$ is real. For real values of $b$, $A_+$ and $A_-$ are formal adjoints of each other (with respect to the measure $dx$), and consequently $H$ is formally a positive quantity (there are some subtleties to this argument arising from the self-adjoint extensions of $H$ which will be discussed later). When $a < -\frac{1}{4}$, $b$ is no longer real and $A_+$ and $A_-$ are not even formal adjoints of each other. However, $H$ can still be factorized as in Eqn. (2.4), but it is no longer a positive definite quantity. It can still be made self-adjoint [27], but remains unbounded from below; this case has been analyzed in [28].

Let us now define the operators

$$L_n = -x^{n+1} \frac{d}{dx}, \quad n \in \mathbb{Z},$$

$$P_m = \frac{1}{x^m}, \quad m \in \mathbb{Z}. \quad (2.8)$$

In terms of these operators, $A_\pm$ and $H$ can be written as

$$A_\pm = \mp L_{-1} + bP_1,$$

$$H = (-L_{-1} + bP_1)(L_{-1} + bP_1). \quad (2.10)$$

Thus, $L_{-1}$ and $P_1$ are the basic operators appearing in the factorization of $H$. Taking all possible commutators of these operators between themselves and with $H$, we obtain the following relations

$$[P_m, P_n] = 0,$$  \hspace{1cm} (2.11)

$$[L_m, P_n] = nP_{n-m},$$  \hspace{1cm} (2.12)

$$[L_m, L_n] = (m-n)L_{m+n} + c \frac{m^3 - m}{12} \delta_{m+n,0},$$  \hspace{1cm} (2.13)

$$[P_m, H] = m(m+1)P_{m+2} + 2mL_{-m-2},$$  \hspace{1cm} (2.14)

$$[L_m, H] = 2b(b-1)P_{2-m} - (m+1)(L_{-1}L_{m-1} + L_{m-1}L_{-1}).$$  \hspace{1cm} (2.15)

Eqn. (2.13) describes a Virasoro algebra with central charge $c$. Note that the algebra of the generators defined in Eqn. (2.7) would lead to $[L_m, L_n] = (m-n)L_{m+n}$. However, this algebra is known to admit a non-trivial central extension. Moreover, for any irreducible unitary highest weight representation of this algebra, $c \neq 0$ [29]. For these reasons, we have included the central term explicitly in Eqn. (2.13).
Eqns. (2.11 - 2.13) describe the semidirect product of the Virasoro algebra with an abelian algebra satisfied by the shift operators \( \{ P_m \} \). Henceforth, we denote this semidirect product algebra by \( \mathcal{A} \). Note that \( L_{-1} \) and \( P_1 \) are the only generators that appear in \( H \). Starting with these two generators, and using Eqns. (2.14) and (2.15), we see that the only operators which appear are the Virasoro generators with negative index (except \( L_{-2} \)), and the shift generators with positive index. Thus, \( L_m \) with \( m \geq 0 \) and \( P_m \) with \( m \leq 0 \) do not appear in the above expressions. In the next section, we will discuss how these quantities are generated.

Although the algebra of Virasoro and shift generators has a semidirect product structure, the operator \( H \) however does not belong to this algebra. This is due to the fact that the right-hand side of Eqn. (2.15) contains products of the Virasoro generators. While such products are not elements of the algebra, they do belong to the corresponding enveloping algebra. The given system is thus seen to be described by the enveloping algebra of the Virasoro generators, together with the abelian algebra of the shift operators. This algebraic system has been extensively studied in the literature [30].

3. Representation

We shall now discuss the representation theory of the algebra \( \mathcal{A} \), and the implications for the quantum properties of the operator \( H \). The eigenvalue equation of interest is

\[ H|\psi\rangle = E|\psi\rangle, \tag{3.1} \]

with the boundary condition that \( \psi(0) = 0 \). We are especially interested in the bound state sector of \( H \). As we have seen, the operator \( H \) can be expressed in terms of certain operators that belong to the algebra \( \mathcal{A} \). This observation allows us to give a description of the states of \( H \) in terms of the representation spaces of \( \mathcal{A} \). We first recall the relevant aspects of the representation theory of \( \mathcal{A} \).

Following [30], we introduce the space \( V_{\alpha,\beta} \) of densities containing elements of the form \( P(x)x^\alpha(dx)^\beta \), where \( \alpha, \beta \) are complex numbers, in general. Here, \( P(x) \) is an arbitrary polynomial in \( x \) and \( x^{-1} \), where \( x \) is now treated as a complex variable. It may be noted that the algebra \( \mathcal{A} \) remains unchanged even when \( x \) is complex. It is known that \( V_{\alpha,\beta} \) carries a representation of the algebra \( \mathcal{A} \). The space \( V_{\alpha,\beta} \) is spanned by a set of vectors, \( \omega_m = x^{m+\alpha}(dx)^\beta \), where \( m \in \mathbb{Z} \). The Virasoro generators and the shift operators have the following action on the basis vectors \( \omega_m \),

\[ P_n(\omega_m) = \omega_{m-n}, \tag{3.2} \]
\[ L_n(\omega_m) = -(m+\alpha+\beta+n\beta)\omega_{m+n}. \tag{3.3} \]

The representation \( V_{\alpha,\beta} \) is reducible if \( \alpha \in \mathbb{Z} \) and if \( \beta = 0 \) or 1; otherwise it is irreducible.

The requirement of unitarity of the representation \( V_{\alpha,\beta} \) leads to several important consequences. In any unitary representation of \( \mathcal{A} \), the Virasoro generators must satisfy the condition \( L_{-m} = L_m \). In the previous section, we saw that \( L_{-2} \) and \( L_m \) for \( m \geq 0 \) did not appear in the algebraic structure generated by the basic operators appearing in the factorization of \( H \). However, the requirement of a unitary representation now leads to the inclusion of \( L_m \) for \( m > 0 \). The remaining generators now appear through appropriate commutators, thus completing the algebra \( \mathcal{A} \).
Unitarity also constrains the parameters $\alpha$ and $\beta$, which must now satisfy the conditions
\begin{align}
\beta + \bar{\beta} &= 1, \quad (3.4) \\
\alpha + \beta &= \bar{\alpha} + \bar{\beta}, \quad (3.5)
\end{align}
where $\bar{\alpha}$ denotes the complex conjugate of $\alpha$. Finally, the central charge $c$ in the representation $V_{\alpha,\beta}$ is given by
\begin{equation}
c(\beta) = -12\beta^2 + 12\beta - 2. \quad (3.6)
\end{equation}

The above representation of $\mathcal{A}$ can now be used to analyze the eigenvalue problem of Eqn. (3.1). We would like to have a series solution to the differential Eqn. (3.1), and consequently choose an ansatz for the wave function $|\psi\rangle$ given by
\begin{equation}
|\psi\rangle = \sum_{n=0}^{\infty} c_n \omega_n. \quad (3.7)
\end{equation}
Furthermore, the operator $H$, as written in Eqn. (2.4), has a well-defined action on $|\psi\rangle$. From Eqn. (3.3), it may be seen that
\begin{equation}
L_{-1}(\omega_n) = -(n + \alpha)\omega_{n-1}, \quad (3.8)
\end{equation}
which is independent of $\beta$. Therefore, it appears that an eigenfunction of $H$ may be constructed from elements of $V_{\alpha,\beta}$ for arbitrary $\beta$. However, the unitarity conditions of Eqns. (3.4-3.5) put severe restrictions on $\beta$, as we shall see below.

The indicial equation obtained by substituting Eqn. (3.7) in Eqn. (3.1) gives
\begin{equation}
\alpha = b, \text{ or } (1 - b). \quad (3.9)
\end{equation}
To proceed, we analyze the cases (i) $a \geq -\frac{1}{4}$, and (ii) $a < -\frac{1}{4}$ separately.

(i) $a \geq -\frac{1}{4}$.

This is the main case of interest as it includes the value of $a$ for the Schwarzschild background. It follows from Eqn. (2.4) and Eqn. (3.9) that $b$ and $\alpha$ are real. The unitarity condition of Eqn (3.4) now fixes the value of $\beta = \frac{1}{2}$, and the corresponding central charge is given by $c = 1$. It may be noted that relation of the central charge calculated here to that appearing in the calculation of black hole entropy depends on geometric properties of the black hole in question. We do not address this issue here. Thus, we see that for the Schwarzschild black hole, we have identified the relevant representation space as $V_{1/2,1/2}$.

(ii) $a < -\frac{1}{4}$.

In this case, we can write $a = -\frac{1}{4} - \mu^2$ where $\mu \in \mathbb{R}$. It follows from Eqn. (2.4), that $b = \frac{1}{2} \pm i\mu$. Eqn. (3.9) then gives $\alpha = \frac{1}{2} \pm i\mu$, or $-\frac{1}{2} \mp i\mu$. Let us take the case when $\alpha = \frac{1}{2} + i\mu$, the other cases being similar. From Eqns. (3.4) and (3.5), we find $\beta = \frac{1}{2} - i\mu$. The value of the corresponding central charge is given by $c = 1 + 12\mu^2$. The operator $H$ in this case can be made self-adjoint but its spectrum remains unbounded from below [27, 28]. The algebraic description, however, always leads to a well-defined representation.
4. Near-Horizon Quantization

We return now to the eigenvalue problem for the differential operator $H$, and focus attention on the Schwarzschild background, for which $a = -\frac{1}{4}$. As already mentioned, we are interested in the bound state sector of $H$. The corresponding Schrödinger’s equation can be written as

$$H \psi = \mathcal{E} \psi, \quad \psi(0) = 0$$

where $\psi \in L^2[R^+, dr]$. $H$ appearing in the above equation is an example of an unbounded linear operator on a Hilbert space. Below we shall first summarize some basic properties of these operators which would be useful for our analysis.

Let $T$ be an unbounded differential operator acting on a Hilbert space $\mathcal{H}$ and let $(\gamma, \delta)$ denote the inner product of the elements $\gamma, \delta \in \mathcal{H}$. By the Hellinger-Toeplitz theorem [17], $T$ has a well defined action only on a dense subset $D(T)$ of the Hilbert space $\mathcal{H}$. $D(T)$ is known as the domain of the operator $T$. Let $D(T^*)$ be the set of $\phi \in \mathcal{H}$ for which there is a unique $\eta \in \mathcal{H}$ with $(T\xi, \phi) = (\xi, \eta) \forall \xi \in D(T)$. For each such $\phi \in D(T^*)$ we define $T^* \phi = \eta$. $T^*$ is called the adjoint of the operator $T$ and $D(T^*)$ is the corresponding domain of the adjoint.

The operator $T$ is called symmetric or Hermitian if $T \subset T^*$, i.e. if $D(T) \subset D(T^*)$ and $T \phi = T^* \phi \forall \phi \in D(T)$. Equivalently, $T$ is symmetric iff $(T \phi, \eta) = (\phi, T \eta) \forall \phi, \eta \in D(T)$. The operator $T$ is called self-adjoint iff $T = T^*$ and $D(T) = D(T^*)$.

We now state the criterion to determine if a symmetric operator $T$ is self-adjoint. For this purpose let us define the deficiency subspaces $K_\pm \equiv \text{Ker}(i \mp T^*)$ and the deficiency indices $n_\pm(T) \equiv \text{dim}[K_\pm]$. $T$ is (essentially) self-adjoint iff $(n_+, n_-) = (0, 0)$. $T$ has self-adjoint extensions iff $n_+ = n_-$. There is a one-to-one correspondence between self-adjoint extensions of $T$ and unitary maps from $K_+$ into $K_-$. Finally if $n_+ \neq n_-$, then $T$ has no self-adjoint extensions.

We now return to the discussion of the operator $H$. On a domain $D(H) \equiv \{ \phi(0) = \phi'(0) = 0, \phi, \phi' \text{ absolutely continuous} \}$, $H$ is a symmetric operator with deficiency indices $(1,1)$. The corresponding deficiency subspaces $K_\pm$ are 1-dimensional and are spanned by

$$\phi_+(x) = x^{\frac{1}{2}} H_0^{(1)}(xe^{i\pi}), \quad (4.2)$$
$$\phi_-(x) = x^{\frac{1}{2}} H_0^{(2)}(xe^{-i\pi}), \quad (4.3)$$

respectively, where $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions. Thus, the operator $H$ is not self-adjoint on $D(H)$ but admits a one-parameter family of self-adjoint extensions, labelled by unitary maps from $K_+$ into $K_-$. The self-adjoint extensions of $H$ are thus labelled by $e^{iz}$ where $z \in \mathbb{R}$. The operator $H$ is self-adjoint in the domain $D_z(H)$ which contains all the elements of $D(H)$ together with elements of the form $\phi_+(r) + e^{iz}\phi_-$. Each value of the parameter $z$ defines a particular domain $D_z(H)$ on which $H$ is self-adjoint and thus corresponds to a particular choice of boundary condition.

For an arbitrary value of the self-adjoint parameter $z$, the normalized bound state solutions of Eqn. (5.1) are given by

$$\psi_n(x) = \sqrt{2E_n} K_0 \left( \sqrt{E_n} x \right) \quad (4.4)$$

where $\mathcal{E}_n = -E_n$. In order to find the energy, we use the fact that if $H$ has to be self-adjoint, the eigenfunction $\psi_n(x)$ must belong to the domain $D_z(H)$. This leads to the expression of
energy given by and
\[
\mathcal{E}_n = -E_n = -\exp \left[ \frac{\pi}{2} (1 - 8n) \cot \frac{z}{2} \right]
\]
respectively, where \( n \) is an integer and \( K_0 \) is the modified Bessel function \([10,10]\). Thus for each value of \( z \), the operator \( H \) admits an infinite number of negative energy solutions. In our formalism, these solutions are interpreted as bound state excitations of the black hole due to the capture of the scalar field. As is obvious from Eqns. (4.4) and (4.5), different choices of \( z \) leads to inequivalent quantization of the system. The physics of the system is thus encoded in the choice of the parameter \( z \).

5. Scaling Properties

As we have seen, the Virasoro algebra plays an important role in determining the spectrum of \( H \). Since this operator is associated with a probe of the near-horizon geometry, one might expect that the corresponding wave functions would exhibit certain scaling behaviour in this region.

Firstly, let us recall that the horizon in this picture is located at \( x = 0 \). However, the wave functions \( \psi_n \) vanish at \( x = 0 \), and therefore do not exhibit any non-trivial scaling. Nevertheless, it is of interest to examine the behaviour of the wave functions near the horizon. For \( x \sim 0 \), the wave functions have the form
\[
\psi_n = \sqrt{2E_n} \left( B - \ln \left( \sqrt{E_n}x \right) \right),
\]
where \( B = \ln 2 - \gamma \), and \( \gamma \) is Euler’s constant \([31]\). While the logarithmic term, in general, breaks the scaling property, one notices that it vanishes at the point \( x_0 \sim 1/\sqrt{E_n} \), where the wave functions exhibit a scaling behaviour. The entire analysis so far, including the existence of the Virasoro algebra, is valid only in the near-horizon region of the black hole. Therefore, consistency of the above scaling behaviour requires that \( x_0 \) belongs to the near-horizon region. The minimum value of \( x_0 \) is obtained when \( E_n \) is maximum. When the parameter \( z \) appearing in the self-adjoint extension of \( H \) is positive, the maximum value of \( E_n \) is given by
\[
E_0 = \exp \left[ \frac{\pi}{2} \cot \frac{z}{2} \right].
\]
However, when \( z \) is negative, the maximum value of \( E_n \) is obtained when \( n \to \infty \). In this case, \( x_0 \to 0 \) where, as we have seen before, the wave function vanishes and scaling becomes trivial. We therefore conclude that
\[
x_0 \sim \frac{1}{\sqrt{E_0}}, \quad z > 0
\]
is the minimum value of \( x_0 \). It remains to show that \( x_0 \) given by Eqn. (5.3) belongs to the near-horizon region. We first note that we are free to set \( z \) to an arbitrary positive value. Thus, we consider \( z > 0 \) such that \( \cot \frac{z}{2} \gg 1 \); this is achieved by choosing \( z \sim 0 \). For all such \( z \), we find that \( x_0 \) is small but nonzero, and thus belongs to the near-horizon region. In effect, we can use the freedom in the choice of \( z \) to restrict \( x_0 \) to the near-horizon region.

We now consider a band-like region \( \Delta = [x_0 - \delta/\sqrt{E_0}, x_0 + \delta/\sqrt{E_0}] \), where \( \delta \sim 0 \) is real and positive. The region \( \Delta \) thus belongs to the near-horizon region of the black hole. At a point \( x \)
in the region $\Delta$, the leading behaviour of $\psi_n$ is given by

$$\psi_n = \sqrt{2E_n x} \left( B + 2\pi n \cot \frac{z}{2} \right). \quad (5.4)$$

Thus, all the eigenfunctions of $H$ exhibit a scaling behaviour, i.e. $\psi_n \sim \sqrt{x}$, in the near-horizon region $\Delta$. It should be stressed that this analysis is made possible by utilizing the freedom in the choice of $z$. The parameter $z$, which labels the self-adjoint extensions of $H$, thus plays a crucial role in establishing the self-consistency of this analysis.

We conclude this section with the following remarks:

1. A particular choice of $z$ is equivalent to a choice of domain for the differential operator $H$. Physically, the domain of an operator is specified by boundary conditions. A specific value of $z$ is thus directly related to a specific choice of boundary conditions for $H$. Thus, we see that the system exhibits non-trivial scaling behaviour only for a certain class of boundary conditions. These boundary conditions play a conceptually similar role to the fall-off conditions as discussed in Ref. [2, 3].

2. The analysis above provides a qualitative argument which suggests that the scaling behaviour in the presence of a black hole should be observed within a region $\Delta$. Although $\Delta$ belongs to the near-horizon region of the black hole, it does not actually contain the event horizon. Our picture is thus similar in spirit to the stretched horizon scenario of Ref. [8].

6. Density of States and Entropy

In the analysis presented above, the information about the spectrum of the Hamiltonian in the Schwarzschild background is coded in the parameter $z$. The wavefunctions and the energies of the bound states depends smoothly on $z$. Thus, within the near-horizon region $\Delta$, we propose to identify

$$\tilde{\rho}(z) \equiv \sum_{n=0}^{\infty} |\psi_n(z)|^2 \quad (6.1)$$

as the density of states for this system written in terms of the variable $z$. $\tilde{\rho}(z)dz$ counts the number of states when the self-adjoint parameter lies between $z$ and $z+dz$. As mentioned before, within the region $\Delta$ $z$ is positive and satisfies the consistency condition $z \sim 0$. From Eqns. (4.5) and (5.4) we therefore see that the term with $n = 0$ provides the dominant contribution to the sum in Eqn. (6.1). The contribution of the terms with $n \neq 0$ to the sum in Eqn. (6.1) is exponentially small for large $\cot \frac{z}{2}$. Physically this implies that the capture of the minimal probe excites only the lowest energy state in the near-horizon region of the massive black hole. The density of energy states of the black hole in the region $\Delta$ can therefore be written as

$$\tilde{\rho}(z) \approx |\psi_0|^2 = 2B^2 e^{\frac{\pi \cot \frac{z}{2}}{2}}. \quad (6.2)$$

As mentioned before, within the region $\Delta$, $\cot \frac{z}{2}$ is a large and positive number. We thus find that the density of states of a massive black hole is very large in the near-horizon region.

In order to proceed, we shall first provide a physical interpretation of the self-adjoint parameter $z$ using the Bekenstein-Hawking entropy formula. To this end, recall that in our formalism, the capture of the scalar field probe gives rise to the excitations of the black hole which subsequently decay by emitting Hawking radiation. A method of deriving density of states and
entropy for a black hole in a similar physical setting using quantum mechanical scattering theory has been suggested by 't Hooft [9]. This simple and robust derivation uses the black hole mass and the Hawking temperature as the only physical inputs and is independent of the microscopic details of the system. The interaction of infalling matter with the black hole is assumed to be described by Schrodinger's equation and the relevant emission and absorption cross sections are calculated using Fermi's Golden Rule. Finally, time reversal invariance (which is equivalent to CPT invariance in this case) is used to relate the emission and absorption cross sections. The density of states for a massive black hole of mass $M$ obtained from this scattering calculation is given by

$$\rho(M) = e^{4\pi M^2 + C'} = e^{S},$$  \hspace{1cm} (6.3)$$

where $C'$ is a constant and $S$ is the black hole entropy. It may be noted that for the purpose of deriving Eqn. (6.3) the infalling matter was described as particles. However, the above derivation of the density of states is independent of the microscopic details and is valid for a general class of infalling matter.

We are now ready to provide a physical interpretation of the parameter $z$. First note that the density of states calculated in Eqns. (6.2) and (6.3) correspond to the same physical situation described in terms of different variables. In our picture, the near horizon dynamics of the scalar field probe contains information regarding the black hole background through the self-adjoint parameter $z$. The same information in the formalism of 't Hooft is contained in the black hole mass $M$. It is thus meaningful to relate the density of states in our framework (cf. Eqn. 6.2) to that given by Eqn. (6.3). If these expression describe the same physical situation, we are led to the identification

$$\frac{\pi}{4}\cot\frac{z}{2} = 4\pi M^2.$$  \hspace{1cm} (6.4)$$

Note that the analysis presented here is valid only for massive black holes. We have also seen that in the near-horizon region $z$ must be positive and obey a consistency condition such that that $\cot\frac{z}{2}$ is large positive number. Thus the relation between $z$ and $M$ given by Eqn. (6.4) is consistent with the constraints of our formalism. We therefore conclude that the self-adjoint parameter $z$ has a physical interpretation in terms of the mass of the black hole.

As stated above, the density of states of the black hole can be expressed either in terms of the variable $z$ or in terms of $M$. In view of the relation between $z$ and $M$, we can write

$$\tilde{\rho}(z)dz \sim |J|\rho(M)dM,$$  \hspace{1cm} (6.5)$$

where $J = \frac{dz}{dM}$ is the Jacobian of the transformation from the variable $z$ to $M$. When $z \sim 0$, using Eqns. (6.3) and (6.4) we get

$$\tilde{\rho}(z)dz \sim e^{4\pi M^2} \left(\frac{1}{M^3}dM \sim e^{4\pi M^2 - \frac{3}{2}\ln M^2}dM.\right.$$  \hspace{1cm} (6.6)$$

The presence of the logarithmic correction term in the above equation is thus due to the effect of the Jacobian.

Finally, the entropy for the Schwarzschild black hole obtained from Eqn. (6.6) can be written as

$$S = S(0) - \frac{3}{2}\ln S(0) + C,$$  \hspace{1cm} (6.7)$$

where $S(0) = 4\pi M^2$ is the Bekenstein-Hawking entropy and $C$ is a constant. Thus the leading correction to the Bekenstein-Hawking entropy is provided by the logarithmic term in Eqn. (6.7)
with a coefficient of \(-\frac{3}{2}\). The subleading corrections to the entropy coming from the \(n \neq 0\) terms of Eqn. (6.1) are exponentially suppressed. As stated before, this is precisely the structure associated with the expression of black hole entropy whenever the same is calculated within a conformal field theoretic formalism \[23\]. We are thus led to conclude that the expression of the black hole entropy obtained in our formalism provides a strong indication of the underlying near horizon-conformal structure present in the system.

7. Conclusion

In this paper, we have analyzed the near-horizon properties of the Schwarzschild black hole, using a scalar field as a simple probe of the system. We restricted attention to the time-independent modes of the scalar field, and this allowed us to obtain a number of interesting results regarding the near-horizon properties of the black hole. It is possible that more sophisticated probes of general field configurations may lead to additional information.

The factorization of \(H\), leading to the algebraic formulation of section 2, is a process which appears to be essentially classical. As discussed, the algebra appearing in Eqns. (2.11-2.15) does not at first contain all the Virasoro generators. The requirement of unitarity of the representation leads to the inclusion of all the generators. It is thus fair to say that the full Virasoro algebra appears in our framework only at the quantum level. The operator \(H\) does not belong to \(\mathcal{A}\) but is contained in the enveloping algebra of the Virasoro generators. The enveloping algebra is the natural tool that is used to obtain representations of \(\mathcal{A}\). Thus, even though \(H\) is not an element of \(\mathcal{A}\), it nevertheless has a well-defined action in any representation of \(\mathcal{A}\). It is this feature that makes the algebraic description useful.

In section 3, we summarized some results from the representation theory of \(\mathcal{A}\). The operator \(H\) is now treated at the quantum level, and the corresponding eigenvalue problem is studied using the representations of \(\mathcal{A}\). Unitarity again plays a role in restricting the space of allowed representations. It is interesting to note that for all values of the coupling \(a \geq -\frac{1}{4}\), the value of the central charge in the representation of \(\mathcal{A}\) is equal to \(1\). Other black holes which have \(a\) in this range would exhibit a universality in this regard. As mentioned in section 3, the relationship of \(c\) calculated here to that appearing in the entropy calculation of a particular black hole would depend on other factors which are likely to break the universality.

If a Virasoro algebra is associated with the near-horizon dynamics, then some reflection of it should appear in the spectrum of \(H\). In particular, we can expect that the wave functions of \(H\) in the near-horizon region should exhibit scaling behaviour. Such a property was indeed found in a band-like region near the horizon. It is interesting to note that this band excludes the actual horizon. This is similar in spirit to the stretched horizon scenario of black hole dynamics. The parameter \(z\) describing the self-adjoint extensions of \(H\) is restricted to a set of values in this process. This implies that the near-horizon wave functions exhibit scaling behaviour only for a certain class of boundary conditions. It is important to note that boundary conditions also played a crucial role in proving the existence of a Virasoro algebra in Ref. \[2, 3\]. This feature provides a common thread in these different approaches towards the problem.

In order to investigate this idea further, we first observe that the self-adjoint parameter \(z\) describes the domain of the Hamiltonian and directly determines the spectrum within this

\[3\] It may be noted that the representation space for the \(c = 1\) conformal field theory is spanned by tensor densities of weight \(\frac{1}{2}\), i.e. spin \(\frac{1}{2}\) \[30\].
formalism. On the other hand, the black hole mass $M$ must also play a role in determining the spectrum. These two parameters thus play a conceptually similar role and it is expected that they will be related.

The next step of our analysis was based on the identification of these bound states with the excitations of the black hole resulting from the capture of the scalar field probe. These excitation would eventually decay through the emission of Hawking radiation. This process is described by quantum mechanical scattering theory in terms of the density of states of the black hole which is a function of the variable $M$. On the other hand, the density of states following from our formalism is a smooth function of $z$. Identifying these two expressions of the density of states leads to a quantitative relation between $z$ and $M$ which is consistent with the constraints of the system. Such a relation also provides a physical interpretation of the self-adjoint parameter in terms of the mass of the black hole.

The relation between $z$ and $M$ naturally leads to the identification of black hole entropy within this formalism. The entropy thus obtained contains the usual Bekenstein-Hawking term together with a leading logarithmic correction which has $-\frac{3}{2}$ as the coefficient. Moreover, the subleading non-constant corrections are shown to be exponentially suppressed. It has been observed that the expression for the black hole entropy is expected to have precisely this structure whenever it is calculated within the conformal field theoretic formalism and possibly even for the non-unitary case. Thus the expression that we obtain for the black hole entropy provides strong support to the hypothesis of an underlying conformal structure in the near-horizon region of the Schwarzschild black hole.

It is known that the near-horizon dynamics of various black holes is described by an operator of the form $H$, for different values of $\alpha \geq -\frac{1}{4}$. Any such operator can be factorised as in Eqn. (2.2) and the above analysis will also apply to these black holes. It has been claimed in that a Virasoro algebra is associated with a large class of black holes in arbitrary dimensions. It seems plausible that the near-horizon dynamics of probes in the background of these black holes would be described by an operator of the form of $H$ and the method presented here can be used to analyze the entropy for such black holes as well.

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