THE SPACE OF NON-DEGENERATE CLOSED CURVES IN A RIEMANNIAN MANIFOLD

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ABSTRACT. Let $\mathcal{L}M$ be the semigroup of non-degenerate based loops with a fixed initial/final frame in a Riemannian manifold $M$ of dimension at least three. We compare the topology of $\mathcal{L}M$ to that of the loop space $\Omega F T M$ on the bundle of frames in the tangent bundle of $M$. We show that $\Omega F T M$ is the group completion of $\mathcal{L}M$, and prove that it is obtained by localizing $\mathcal{L}M$ with respect to adding a “small twist”.

1. Introduction

A smooth curve in an $n$-dimensional Riemannian manifold $M$ is called free of order $k$ if its first $k$ covariant derivatives are linearly independent at each point. Free curves of order smaller than $n$ were studied by Smale [21], Feldman [5] and Gromov [6], who showed that they satisfy the $h$-principle. In particular, when $k < n$, the space of all based free curves of order $k$ is homotopy equivalent to the loop space on the bundle of $k$-dimensional frames in the tangent bundle of $M$.

When $k = n$, the $h$-principle fails to be true already for the simplest examples $M = S^2$, see [8], and $M = \mathbb{R}^{2m}$ with $m > 1$, see [20]. We show that a somewhat weaker statement holds: for any $n > 2$ the loop space on the bundle of $n$-frames in $TM$ is the group completion of the monoid of free curves of order $n$ in $M$. Moreover, the group completion is achieved by localizing with respect to the multiplication by any “small” curve.

The spaces of free curves of order $n$ in $n$-manifolds, also called non-degenerate curves (the terminology varies in the literature) are the subject of an extensive and ongoing study, with significant advances in special cases [14]. In particular, our result builds heavily on the work of M. Shapiro [20] and N. Saldanha and B. Shapiro [16]. The main technical tool that we borrow from [16] are the “telephone wires” which allow to approximate in an appropriate sense any curve by a non-degenerate one. This idea goes back to Gromov (convex integration, [6]), Eliashberg-Mishachev (holonomic approximation theorem, [3]) and Thurston (corrugations, [15]), and in some form was used by Rourke and Sanderson (compression theorem, [13]).

A strong motivation for much of the research in the area is the fact that non-degenerate curves appear in the study of linear ordinary differential equations [1, 2, 7, 18, 17]. For example, the space of linear ordinary differential equations of order $n + 1$ on $S^1$ that have $n + 1$ independent solutions is homotopy equivalent to the space of based non-degenerate curves in $S^n$.

2. Statement of the results

A $C^n$-differentiable curve $\gamma$ in a Riemannian manifold $M$ of dimension $n$ is said to be non-degenerate if at each moment of time $t$ the set of covariant derivatives

$$\{\gamma'(t), \ldots, \gamma^{(n-1)}(t), \gamma^{(n)}(t)\}$$

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spans the tangent space $T_{\gamma(t)}M$. Here $\gamma^{(k)}(t)$ for $k > 1$ is defined as $\nabla \gamma^{(k-1)}(t)$. If the curve $\gamma$ is non-degenerate, the ordered set of its first $n$ covariant derivatives is referred to as the frame of $\gamma$ at time $t$ and is denoted by $F_\gamma(t)$.

Assume that a basis $F_0$ is chosen in the tangent space to $M$ at a point $x_0 \in M$. Consider the topological space $\mathcal{L}M$ consisting of all non-degenerate curves $\gamma : [0, a] \to M$, such that $\gamma(0) = \gamma(a) = x_0$ and $F_\gamma(0) = F_\gamma(a) = F_0$, where $a$ is a positive number.

The space $\mathcal{L}M$ is a topological semigroup: the multiplication is given by concatenating the curves. Let $\mathcal{F}TM$ be the bundle of frames in the tangent bundle $TM$, with the basepoint being the frame $F_0$, and $\Omega \mathcal{F}TM$ be its Moore loop space. The frame map $\mathcal{L}M \to \Omega \mathcal{F}TM$ is defined as $\gamma \mapsto F_\gamma$. It is a homomorphism of topological semigroups.

Remark. Often, the frame of the derivatives of a curve is orthogonalized and normalized; the result is called the Frenet frame. Accordingly, instead of the frame map that we consider here it is customary to consider the Frenet frame map. If the space of all frames is identified with the Lie group $SL(n)$, the orthonormal frames correspond to $SO(n)$. The two groups are homotopy equivalent and, in the context of the present paper, it will make no difference which version is used to state the results.

Observe that if we add a disjoint basepoint to $\mathcal{L}M$, the resulting space $\mathcal{L}M_+$ is a topological monoid: the basepoint can be thought of as representing the “curve of zero length” and is the neutral element with respect to the concatenation. We shall prove the following result:

**Theorem 1.** For a Riemannian manifold $M$ of dimension at least three, the homotopy-theoretic group completion $\Omega B\mathcal{L}M_+$ of the monoid $\mathcal{L}M_+$ is homotopy equivalent to $\Omega \mathcal{F}TM$.

In fact, we shall describe the group completion of $\mathcal{L}M_+$ explicitly, in terms of the localization with respect to the left multiplication by one fixed element.

In the case when $M = \mathbb{R}^n$ we take the origin to be the basepoint and the standard basis to be the chosen frame. The space $\mathcal{F}\mathbb{R}^n$ is homeomorphic to $SL(n) \times \mathbb{R}^n$. For a general $M$, identifying $\mathbb{R}^n$ with $T_{x_0}M$ in the manner that preserves the chosen frames, we get the exponential map

$$\exp_{x_0} : B^n_R \to M$$

defined on an open ball $B_R \subset \mathbb{R}^n$ of radius $R$; we can take $R$ to be infinite if $M$ is complete. Take an arbitrary curve $\alpha$ in $\mathcal{L}\mathbb{R}^n$. We shall see that there exists a positive number $\lambda_0$ such that for all positive $\lambda \leq \lambda_0$ the curves $\lambda \alpha$ are in $B^n_R$ and their images $\exp_{x_0}(\lambda \alpha)$ in $M$ are all non-degenerate. Define $\omega \in \mathcal{L}M$ as

$$\omega = \exp_{x_0}(\lambda_0 \alpha).$$
Write $LM_\omega$ for the localization of $LM$ by the left multiplication by $\omega$, that is, the direct limit of the sequence of embeddings

$$LM \xrightarrow{\omega} LM \xrightarrow{\omega} LM \xrightarrow{\omega} \ldots$$

The frame map sends the left multiplication by $\omega$ in $LM$ to the left multiplication by $F_\omega$ in $\Omega FTM$. This latter map is a homotopy equivalence, and the direct limit of a sequence of multiplications by $F_\omega$ in $\Omega FTM$ is again homotopy equivalent to $\Omega FTM$. In particular, the frame map descends to a map

$$LM_\omega \to \Omega FTM.$$  

The main technical result of this note is the following statement:

**Theorem 2.** For a Riemannian manifold $M$ of dimension at least three, the space $LM_\omega$ is weakly homotopy equivalent to $\Omega FTM$ and the equivalence is given by the frame map.

Theorem 2 is proved in Section 3 and in Section 4 we show how it implies Theorem 1.

### 3. Proof of Theorem 2

#### 3.1. The proof for non-degenerate curves in $\mathbb{R}^n$

It is known [9, 20, 19] that the set $\pi_0L\mathbb{R}^n$ consists of two elements when $n$ is odd and of three elements when $n$ is even. The concatenation of curves gives $\pi_0L\mathbb{R}^n$ the structure of a semigroup and the frame map $\pi_0L\mathbb{R}^n \to \pi_0\Omega SL(n) = \mathbb{Z}/2$ is an isomorphism when $n$ is odd and a group completion when $n$ is even. Of the three components of $\pi_0L\mathbb{R}^n$ two are mapped to the generator of $\mathbb{Z}/2$; of these two one is sent to the other by the left multiplication by the class of any element in $L\mathbb{R}^{2k}$ defining a contractible loop in $SL(2k)$ (see [20]). In particular, the localization of $\pi_0L\mathbb{R}^n$ with respect to the left multiplication by any of its elements is the same thing as its group completion, and is isomorphic to $\mathbb{Z}/2$.

Therefore, we only need to show that each component of $L\mathbb{R}^n_\omega$ has the same homotopy groups as a component of $\Omega SL(n)$ and the isomorphism is induced by the frame map. This, essentially, was proved by Saldanha and Shapiro [16]. Their argument goes as follows.

Fix $\omega \in L\mathbb{R}^n$ parametrized by $[0, 1]$ and consider a curve

$$A : [0, a] \to SL(n)$$

with the property that $A(t) = Id$ when $t \in [0, \varepsilon] \cup [a - \varepsilon, a]$ for some small $\varepsilon > 0$. For a positive integer $N$ let

$$A[N] : [0, a] \to \mathbb{R}^n$$

be the curve defined as

$$A[N](t) = A(t)\omega(Nt/a).$$

In what follows, when referring to a compact family of curves $\gamma_r$, we shall mean that $r$ varies over a compact set.

**Lemma 3.** Given a compact family of curves $A_r \in \Omega SL(m)$, each $A_r$ parametrized by $[0, a_r]$ and constant in a neighbourhood of $\{0\} \cup \{a_r\}$, we can find $N$ such that

- each curve $A_r[N]$ is non-degenerate;
- the frame map sends the family $A_r[N]$ to a family of curves homotopic to $A_r$.

Moreover, if the curve $A_r$ is in the image of the frame map, that is, $A_r = F\gamma_r$, then $A_r[N]$ is homotopic to $\gamma_r \cdot \omega^N$ through non-degenerate curves.
Proof. Indeed, the non-degeneracy of $A^{[N]}_r$ follows from the fact that asymptotically, as $N \to \infty$, we have

$$\left( A(t) \omega(Nt/a) \right)^{(k)} \sim N^k \cdot A(t) \omega^{(k)}(Nt/a).$$

To prove the second claim, take $N$ to be even. Cut the curve $A$ into $N/2$ pieces

$$A_i = A|_{[2(i-1)a/N, 2ia/N]}$$

and write $f_i$ for the curve $A(2ia/N)F_{\omega^2}$. Consider the concatenation of curves

$$A_1 f_1 A_2 f_2 \ldots A_{N/2} f_{N/2}.$$

On one hand, this curve is homotopic to $A$. This is obvious since $F_{\omega^2}$ is null-homotopic. On the other hand, it is close, and homotopic, to $F_{A^{[N]}(t)}$ if $N$ is sufficiently big. See [16] for details.

If $A$ is in the image of the frame map, $A = F_\gamma$, the above cut and paste construction can be described in terms of non-degenerate curves. One cuts the curve $\gamma$ into $N/2$ segments of equal length and inserts a suitably translated copy of $\omega^2$ after each segment; then one should take the frame map. Now, if we shrink the lengths of all the segments of $\gamma$, apart from the first one, to zero, while preserving the total length, we obtain the curve $\gamma \cdot \omega^N$. Performing this shrinking uniformly on all the curves of a family $\gamma_r$, we get a homotopy between the family $\gamma_r \cdot \omega^N$ and a family which is close and, hence, homotopic, to $A^{[N]}_r$.

As a consequence of Lemma 3, all the elements of all the homotopy groups of $\Omega SL(n)$ can be represented by families of frames of non-degenerate curves. Moreover, if two classes in $\pi_k \Omega SL(n)$ have the same image in $\pi_k \Omega SL(n)$, they become equal in $\pi_k \Omega SL(n)$.

3.2. Piecewise $C^n$ curves in Riemannian manifolds. In order to extend the argument to arbitrary Riemannian manifolds, we need to enlarge the space of non-degenerate curves by the curves whose frame can experience a small jump in a finite number of points.

Let us say that two $n$-frames $L_1$ and $L_2$ in $\mathbb{R}^n$ are $\varepsilon$-close if $L_1 L_2^{-1}$ and $L_2 L_1^{-1}$, thought of as elements of $SL(n)$, are $\varepsilon$-close to the identity in the usual matrix norm. Define $\mathcal{LM}(\delta)$ as the space of $C^1$-curves which are piecewise $C^n$ and non-degenerate, with the initial and the final frame $\delta$-close to the chosen frame in $M$ at $x_0$, and such that the limits of the frames on the right and on the left at any point are $\delta$-close. The topology on $\mathcal{LM}(\delta)$ is given by the Sobolev $H^n$ metric.

Proposition 4. The inclusion map $\mathcal{LM} \to \mathcal{LM}(\delta)$ is a weak homotopy equivalence for sufficiently small $\delta$.

Proof. We have to show that any compact family of curves $\gamma_r$ in $\mathcal{LM}(\delta)$ can be deformed into $\mathcal{LM}$ by a homotopy that sends $\mathcal{LM}$ to itself. Choose $\varepsilon_1, \varepsilon_2 > 0$, a curve $\gamma_0 \in \mathcal{LM}$ parametrized by $[0, 1]$, and a family of functions $f_\varepsilon : [0, 1] \to [0, 1]$ as in the figure such that all the derivatives of $f_\varepsilon$ vanish at 0 and 1:

Write $a_r$ for the length of the interval that parametrizes $\gamma_r$. 
First, we perform a smoothing of each $\gamma_r$ with the help of the standard mollifier with the parameter $\tau \in [0, \varepsilon_1]$ (see [4]). If $\varepsilon_1$ is sufficiently small, the resulting curves $\gamma_{r, \tau}$ will be non-degenerate, but their initial and final frames might differ from the chosen frame $F_0$. In order to remedy this, we replace $\gamma_{r, \tau}$ with

$$\tilde{\gamma}_{r, \tau}(t) = \gamma_{r, \tau}(t)(1 - f_{\varepsilon_2}(a_r t)) + \gamma_0(a_r t)f_{\varepsilon_2}(a_r t).$$

One can choose $\varepsilon_1$ and $\varepsilon_2$ so small that $\tilde{\gamma}_{r, \tau}$ is non-degenerate for all $\tau \in [0, \varepsilon_1]$ and that $\tilde{\gamma}_{r, 0}$ is homotopic to $\gamma_r$. This gives the desired deformation. \qed

3.3. The proof for curves in arbitrary $M$. First, we describe the behaviour of non-degenerate curves under the exponential map.

**Lemma 5.** Let $\alpha : [0, a] \to T_x M$ be a non-degenerate curve in the tangent space to $x \in M$. There exists a positive $\lambda_0$ such that for all positive $\lambda \leq \lambda_0$ the curve $\exp_{\lambda}(\lambda \alpha)$ in $M$ is non-degenerate.

**Proof.** In the normal coordinates around $x_0$ the exponential map is, tautologically, the identity map. Therefore, all we need to prove is that the difference between the usual derivatives, suitably scaled, and the covariant derivatives of $\lambda \alpha$ along the vector field $\lambda \alpha'$ is of the higher order in $\lambda$ than the derivatives themselves.

This is clear for the first derivatives, since they simply coincide. For the second derivatives, we have

$$(\lambda \alpha)^{(2)} = \nabla_{\lambda \alpha'} \lambda \alpha' = \lambda^2 (\alpha'' + \Gamma^k_{ij}(\lambda \alpha) \alpha'_j \cdot e_k),$$

where $e_k$ are the basis vectors and $\alpha'_j$ is the $j$th component of $\alpha'$. Since for the metric connection the Christoffel symbols $\Gamma^k_{ij}(\lambda \alpha)$ tend to zero with $\lambda$, the case $n = 2$ is also settled.

In general, the differences between the derivatives will involve Christoffel symbols $\Gamma^k_{ij}(\lambda \alpha)$ and their derivatives, which tend to zero even faster with $\lambda$. Hence, for $\lambda$ small enough the first $n$ covariant derivatives of $\lambda \alpha$ will be linearly independent for each $t \in [0, a]$. \qed

In fact, Lemma 5 holds for compact families of curves. Namely, for a family of curves $\alpha_r \subset T_{x_r} M$, where $r$ ranges over a compact set, we can find $\lambda_0$ as in the statement of Lemma 5 the same for all the curves $\alpha_r$. This follows directly from continuity of the Christoffel symbols and their derivatives. As a consequence, we have:

**Corollary 6.** For all $k \geq 0$ there is a well-defined map $\pi_k \mathcal{L} \mathbb{R}^n \to \pi_k \mathcal{L} M$.

**Proof.** Indeed, since the Christoffel symbols vanish at the origin, for any given $\delta$ and for any compact family $\gamma_r \in \mathcal{L} \mathbb{R}^n$ we can find $\lambda_0$ small enough so that for all $\lambda < \lambda_0$ the curves $\exp(\lambda \gamma_r(t/\lambda))$ are well-defined and lie in $\mathcal{L} M(\delta)$. Applying Proposition 4, we get the result. \qed

Now, let

$$\Gamma : [0, a] \to FTM$$

be an arbitrary curve and

$$\gamma : [0, a] \to M$$

its projection to $M$. The pullback $\gamma^* TM$ of the tangent bundle of $M$ to $[0, a]$ has a trivialization given by $\Gamma$, and we can identify it with $[0, a] \times \mathbb{R}^n$. There exists a positive $\varepsilon$ such that the map

$$\exp_{\Gamma} : [0, a] \times B_{\varepsilon} \to M$$

given by

$$(t, v) \to \exp_{\gamma(t)}(v)$$

is well-defined. Clearly, the restriction of $\exp_{\Gamma}$ to $[0, a] \times 0$ is the curve $\gamma$. We stress that $\exp_{\Gamma}$ depends on the choice of a basis in each tangent space $T_{\gamma(t)} M$ and not just on the curve $\gamma$. 

Let us now prove that \( \gamma \) can be approximated by a non-degenerate curve, and, moreover, that this can be done in families.

Assume that \( \alpha : [0, 1] \to \mathbb{R}^n \) lies in \( \mathcal{L}\mathbb{R}^n \). It will be convenient to extend \( \alpha \) to a periodic function on \( \mathbb{R} \), with period 1, and keep the same notation for it. Let \( \gamma_r : [0, a_r] \to M \) be a compact family of curves and let \( \Gamma_r : [0, a_r] \to FTM \) be a lifting of \( \gamma_r \).

**Lemma 7.** There exist \( \lambda_0 > 0 \) and \( N_0 \in \mathbb{N} \) such that for all \( N > N_0 \) each curve

\[
\gamma_r^{[N]} = \exp_{\Gamma_r} \left( \frac{t}{\lambda_0 N}, \lambda_0 \alpha \left( \frac{t}{\lambda_0 a_r} \right) \right),
\]

defined on the interval \([0, a_r \cdot \lambda_0 N]\), is non-degenerate in \( M \). Moreover, for any \( \delta > 0 \) the number \( N_0 \) can be chosen so that the initial and the final frames of \( \gamma_r^{[N]} \) are \( \delta \)-close to the corresponding frames of \( \gamma_r \).

Essentially, \( \gamma_r^{[N]} \) is a “telephone wire” in the shape of \( \gamma_r \), as in [16].

**Proof.** For each \( r, t \) we have an identification of \( T_{\gamma_r(t)} M \) with \( \mathbb{R}^n \) and thus we can speak of a copy of \( \alpha \) in \( T_{\gamma_r(t)} M \). By Lemma 5 (or, rather, its version for families) there exists \( \lambda_0 \) such that the curve \( \exp_{\gamma_r(t)}(\lambda\alpha) \) is non-degenerate in \( M \) for all \( \lambda \leq \lambda_0 \). By rescaling \( \alpha \) and reparametrizing \( \gamma_r \) we can assume, without loss of generality, that \( \lambda_0 = 1 \) and \( a_r = 1 \) for all \( r \). In particular, we have that

\[
\gamma_r^{[N]} = \exp_{\Gamma_r}(t/N, \alpha(t))
\]

is defined on the interval \([0, N]\).

Now, for each \( \varepsilon > 0 \) we can find \( N_0 \) big enough so that each curve \( \exp_{\Gamma_r}(t/N, \alpha(t)) \), restricted to \( t \in [i, i + 1] \) has all its covariant derivatives \( \varepsilon \)-close to those of the curve \( \exp_{\Gamma_r}(i/N, \alpha(t)) \), for all \( N > N_0 \) and for \( i = 0, \ldots, N - 1 \). In particular, this means that we can choose \( N_0 \) so that \( \exp_{\Gamma_r}(t/N, \alpha(t)) \) for all \( N > N_0 \) is non-degenerate. Moreover, this also implies the second statement of the lemma. \( \square \)

In particular, if the family \( \gamma_r \) is in \( \mathcal{LM} \) to begin with and \( \Gamma_r \) is given by the frame map, we see that for any \( \delta > 0 \) the number \( N \) can be chosen so that \( \gamma_r^{[N]} \in \mathcal{LM}(\delta) \). In fact, the homotopy class of \( \gamma_r^{[N]} \) can be explicitly described.

**Lemma 8.** If \( \gamma_r \in \mathcal{LM} \) for all \( r \), \( \Gamma_r \) is given by the frame map, and \( N_0 \) is sufficiently big, then for all \( N > N_0 \) the family \( \gamma_r^{[N]} \) is homotopic in \( \mathcal{LM}(\delta) \) to the family \( \gamma_r \cdot \omega^N \), where \( \omega = \exp_{x_0} \alpha \).

**Proof.** Let us assume, as in the previous proof, that \( a_r = 1 \) and \( \lambda_0 = 1 \). Given \( N \in \mathbb{N} \) we write

\[
\omega_i := \exp_{\Gamma_r}(i/N, \alpha(t)),
\]

defined on \([i, i + 1]\), and

\[
\gamma_r^i := \gamma_r|_{[i/N, (i+1)/N]}.
\]

Choose \( \varepsilon > 0 \). As we have noted before, for \( N \) big enough \( \gamma_r^{[N]} \), restricted to \( t \in [i, i + 1] \) has its first \( n \) covariant derivatives \( \varepsilon \)-close to those of the curve \( \omega_i \). On the other hand, it is clear that, again, for \( N \) sufficiently big, \( \omega_i \) is \( \varepsilon \)-close to the concatenation \( \gamma_r^i \cdot \omega_i \), reparametrized by \([i, i + 1]\). Therefore, we see that the curve \( \gamma_r^{[N]} \) is \( 2\varepsilon \)-close, together with its derivatives, to the suitably parametrized concatenation

\[
\gamma_r^0 \cdot \omega_1 \cdot \gamma_r^1 \cdot \omega_2 \cdot \ldots \cdot \gamma_r^{N-1} \cdot \omega_N.
\]

(1)

This means that for \( \varepsilon \) small enough these curves are homotopic, since a linear interpolation between them would provide the required homotopy in \( \mathcal{LM}(\delta) \).
On the other hand, the curve $[1]$ is homotopic to $\gamma_r \cdot \omega^N$ in $ŁM(δ)$. Indeed, $\omega_1 \cdot \gamma_r^i$ can be carried to $\gamma_r^i \cdot \omega_1$, with the help of the homotopy given by

$$\gamma_r^i = \exp_\gamma(i/N \cdot \tau, \alpha) \cdot \gamma_r^i,$$

Here $\tau \in [0, 1]$ is the parameter of the homotopy. It remains to notice that $\gamma_r^0 \cdot \cdots \cdot \gamma_r^{N-1} = \gamma_r$ and $\omega_N = \omega$.

Remark. In the course of the last proof we have, actually, shown that the left and the right multiplications by $\omega$ in $ŁM$ are homotopic.

The curve $\gamma_r^{[N]}$ is homotopic to $\gamma_r$. Hence, a representative of any homotopy class $\nu \in \pi_1 ŁM$ can be approximated in this way by a class in the subspace consisting of non-degenerate curves. Moreover, representing $\nu$ by curves which are constant in a neighbourhood of their endpoints, we can achieve that the initial and the final Frenet frames of the corresponding non-degenerate curves coincide with the chosen frame. In particular, $\nu$ comes from a class in $\pi_k ŁM$.

Now, with the principal bundle

$$SL(n) \rightarrow FTM \rightarrow M$$

we can associate the fibration

$$ΩSL(n) \rightarrow ΩFTM \rightarrow ΩM.$$

Take some class $\eta \in \pi_k ΩFTM$ and denote by $\nu$ its projection to $\pi_k ΩM$. Approximate $\nu$ by a class $\nu' \in \pi_k ŁM$ and let $F_\nu(\nu')$ be the image of $\nu'$ under the Frenet frame map. The class $F_\nu(\nu')\eta^{-1}$ projects to zero in $\pi_k ΩM$, hence it is in the image of $\pi_k ΩSL(n)$. This, in turn, can be approximated by a class in $\pi_k ŁR^n$, well-defined up to localization by $\alpha$, which, by Corollary 6 gives rise to a class $c \in \pi_k ŁM$. Then

$$F_\nu(c^{-1} \nu') = F_\nu(c^{-1})F_\nu(\nu')\eta^{-1} \eta = \eta.$$

Now, take a homotopy class $c \in \pi_k ŁM$ that vanishes in $\pi_k FTM$. Approximating each curve in the null-homotopy of $c$ by non-degenerate curves using Lemma 7 what we get is a null-homotopy

4. ON THE GROUP COMPLETION OF $ŁM_+$

Here we shall show how Theorem 1 follows from Theorem 2. Let us first recall the group completion theorem [11 [12]. Let $M$ be a topological monoid and $π = π_0 M$ the monoid of its path components. The homology of $M$ is an algebra, with the product induced by that of $M$, and $π$ can be thought of as a multiplicative subset of $H_*(M)$. We say that the localization $H_*(M)[π^{-1}]$ can be calculated by right fractions if the following conditions hold:

1. for every $s_1, s_2 \in π$ there exist $t_1, t_2 \in π$ with $s_1 t_1 = s_2 t_2$;
2. given $s, s_1, s_2 \in π$ such that $ss_1 = ss_2$, there exists $t \in π$ with $s_1 t = s_2 t$;
3. given $r \in H_*(M)$ and $s \in π$ with $sr = 0$, there exists $t \in π$ with $rt = 0$;
4. given $r \in H_*(M)$ and $s \in π$ there exist $r' \in H_*(M)$ and $t \in π$ with $rt = sr'$.

The group completion theorem says that if the localization $H_*(M)[π^{-1}]$ can be calculated by right fractions, then

$$H_*(M)[π^{-1}] \approx H_*(ΩBM),$$

where the homology is taken with arbitrary (possibly, twisted) coefficients. Moreover, the localization with respect to $π$ is induced by the canonical map $M \rightarrow ΩBM$.

Lemma 9. The localization $H_*(ŁM_+)[\pi_0 ŁM_+^{-1}]$ can be calculated by right fractions.
Proof. Let $\omega \in \pi = \pi_0 LM$ be a class coming from $\pi_0 LR^n$. Observe that we have the following two facts:

(a) $\pi$ becomes a group if the class of $\omega$ is made invertible;

(b) the class of $\omega$ is in the centre of $\pi$; moreover, by the remark after Lemma [8], the left and the right multiplications by $\omega$ give the same map on $H_* (LM_\pm)$.

For $t \in \pi$ denote by $[t]$ its class in the group completion of $\pi$. By (a) there exist $u_1$ and $u_2$ such that $u_1 = [s_1]^{-1}$ and $u_2 = [s_2]^{-1}$. Therefore,

$$s_1 \cdot u_1 \omega^k = s_2 \cdot u_2 \omega^k$$

for some $k$; this establishes (1). In order to prove (2) notice that if $s s_1 = s s_2$, we have $[s_1] = [s_2]$. In particular, $s_1 \cdot \omega^k = s_2 \cdot \omega^k$ for some $k$.

Now, for $r \in H_* (LM_\pm)$ let $[r]$ be its image in $H_* (\Omega FTM)$. If $[r_1] = [r_2]$, there exists $k$ such that $r_1 \omega^k = r_2 \omega^k$. In order to establish (3), note that $sr = 0$ implies $[s][r] = 0$ and, hence, $[r] = 0$. This means that $r \cdot \omega^k = 0$ for some $k$. Finally, for $s \in \pi$ find $u$ with $[u] = [s]^{-1}$. Then $[s][u][r] = [r]$ in $H_* (\Omega FTM)$, which implies that $r \cdot \omega^k = s \cdot ur \omega^k$ for some $k$. This establishes (4).

In particular, by the group completion theorem, $H_* (\Omega BLM_\pm)$ is obtained from $H_* (LM_\pm)$ by localization with respect to $\pi_0 LM_\pm$.

On the other hand, the map $LM_\pm \to \Omega BLM_\pm$ factors through $LM_\omega$. Indeed, the multiplication by $\omega$ (or any other element) in $LM_\pm$ is sent to a multiplication by an invertible, up to homotopy, element in $\Omega BLM_\pm$; the localization with respect to an invertible element has no effect. Note that the homology of $LM_\omega$ is obtained from $H_* (LM_\pm)$ by localization with respect to the class of $\omega$. Localizing it any further with respect to other classes of $\pi_0 LM_\pm$ has no effect since $\pi_0 LM_\omega$ is already a group. In particular, this means that the map $LM_\omega \to \Omega BLM_\pm$ induces an isomorphism in homology.

The standard argument shows that the fundamental group of $LM_\omega$ is abelian. Indeed, since $\omega$ commutes, up to homotopy, with any compact subset of $LM$, the homotopy groups of the space $LM_\omega$ have the same properties as the homotopy groups of an $h$-space, and $h$-spaces have abelian fundamental groups. As a consequence, the homology equivalence $LM_\omega \to \Omega BLM_\pm$ is a weak homotopy equivalence. Now, applying Theorem [2] we see that $\Omega BLM_\pm$ is weakly homotopy equivalent to $\Omega FTM$. Since both spaces have homotopy types of CW-complexes (see [10]), they are homotopy equivalent.

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