Local Pareto optimality conditions for vector quadratic fractional optimization problems

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Abstract

There are several concepts and definitions that characterize and give optimality conditions for solutions of a vector optimization problem. One of the most important is the first-order necessary optimality condition that generalizes the Karush-Kuhn-Tucker condition. This condition ensures the existence of an arbitrary neighborhood that contains an local optimal solution. The present work we introduce an alternative concept to identify the local optimal solution neighborhood in vector optimization problems. The main aspect of this contribution is the development of necessary and sufficient Pareto optimality conditions for the solutions of a particular vector optimization problem, where each objective function consists of a ratio quadratic functions and the feasible set is defined by linear inequalities. We show how to calculate the largest radius of the spherical

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region centered on a local Pareto solution in which this solution is optimal. In this process we may conclude that the solution is also globally optimal. These conditions might be useful to determine termination criteria in the development of algorithms, including more general problems in which quadratic approximations are used locally.

**Keywords:** Pareto optimality conditions ; efficient solutions ; vector optimization ; vector quadratic fractional optimization problems

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1. Introduction

In general the vector optimization problem (VOP) appears in the processes for decision-making and is represented as the following problem:

(VOP) Minimize \( f(x) = (f_1(x), \ldots, f_m(x)) \)

subject to \( h_j(x) \leq 0, \quad j \in J, \)

\( x \in \Omega \subseteq \mathbb{R}^n, \)

where \( f_i : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m, i \in I \equiv \{1, \ldots, m\}, \) are functions of \( n \) real variables and \( \Omega \) is a nonempty subset. The fields of real functions \( h_j \)'s contains \( \Omega \). We denote by \( S \) the feasible set that is the intersection of \( \Omega \) with the set of points \( x \) in which \( h_j(x) \leq 0, j \in J \equiv \{1, \ldots, p\} \). If \( x \in S \), we say that \( x \) is a feasible point. \( f_i(x) \) is the result of the \( i \)th objective function if the decision maker chooses the action \( x \in S \).

Let \( \mathbb{R}_+ \) denote the nonnegative real numbers and \( x^T \) denote the transpose of the vector \( x \in \mathbb{R}^n \). Furthermore, we will adopt the following conventions for inequalities among vectors. If \( x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m \) and \( y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m \), then

\[
\begin{align*}
x &= y \quad \text{if and only if} \quad x_i = y_i, \quad \text{for all} \ i \in I; \\
x &< y \quad \text{if and only if} \quad x_i < y_i, \quad \text{for all} \ i \in I; \\
x &\leq y \quad \text{if and only if} \quad x_i \leq y_i, \quad \text{for all} \ i \in I; \\
x &\leq y \quad \text{if and only if} \quad x \leq y \text{ and } x \not= y.
\end{align*}
\]

Similarly we consider the equivalent convention for inequalities \( >, \geq \) and \( \geq \).

We say that \( x \in S \) dominates \( z \in S \) in (VOP) if \( f(x) \leq f(z) \), that \( N(y) \subseteq S \) is a neighborhood of \( y \in S \), that \( B(x, r) \) is an open ball centered on \( x \), the radius \( r \)
and the boundary \( \partial B(x,r) \), and that \( \bar{B}(x,r) \) is a \textit{closed ball}, defined by Euclidean distance.

Different optimality notions for the problem (VOP) are referred to as a Pareto optimal solution \([32]\), two of which are defined as follows.

**Definition 1.** A feasible point \( x^* \) is said to be a \textit{(locally) Pareto-efficient optimal solution} of (VOP), if there does not exist another \( (x \in N(x^*)) x \in S \) such that \( f(x) \leq f(x^*) \).

**Definition 2.** A feasible point \( x^* \) is said to be a \textit{(locally) weakly Pareto-efficient optimal solution} of (VOP), if there does not exist another \( (x \in N(x^*)) x \in S \) such that \( f(x) < f(x^*) \).

Every Pareto-efficient optimal solution is a weakly Pareto-efficient optimal solution. The Pareto-efficient optimal solution set is denoted by \( \text{Eff}(VOP) \), and locally Pareto-efficient optimal solution by \( \text{Leff}(VOP) \). The Pareto-efficient optimal solutions are also known as globally Pareto-efficient optimal solutions. Let the set \( \text{Leff}(VOP) \subseteq \mathbb{R}^n \), we say that \( f(\text{Leff}(VOP)) \subseteq \mathbb{R}^m \) is the \textit{Pareto-optimal curve}.

There are many contributions, concepts, and definitions that characterize and give the Pareto-efficient optimality conditions for solutions of a vector optimization problem (see, for instances \([9, 17, 28, 30, 35]\)). One of the most important is the first-order necessary optimality condition that generalizes the Karush-Kuhn-Tucker (KKT) condition. However, to obtain the sufficient optimality conditions, it is necessary to impose additional assumptions (like convexity and its generalizations) in the objective functions and in the constraint set. Otherwise, is only possible to characterize the locally Pareto-efficient optimal solutions. In general the non-equivalence between locally and globally Pareto-efficient optimal solutions is a difficult problem to be faced. The following example shows a simple problem which illustrates this. We recall that \( \nabla f(x) \) denote the gradient of the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) at point \( x \).

Exemplo: Consider the following problem:

\[
\begin{align*}
\text{Minimize} & \quad f(x) = (4x_1^2 - x_2^2, -(x_1 - 2)^2 + 4(x_2 + 1)^2) \\
\text{subject to} & \quad x = (x_1, x_2) \in \mathbb{R}^2,
\end{align*}
\]

which their objective functions can be expressed as
\[ f_1(x) = 0.5 x^T A_1 x + b_1^T x + c_1 = 4x_1^2 - x_2^2, \]
\[ f_2(x) = 0.5 x^T A_2 x + b_2^T x + c_2 = -(x_1 - 2)^2 + 4(x_2 + 1)^2, \]
where \( c_1 = 0, c_2 = 0, b_1 = (0, 0)^T, b_2 = (4, 8)^T, \) and whose matrices \( A_1 \) and \( A_2 \) are
\[
A_1 = \begin{pmatrix} 8 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} -2 & 0 \\ 0 & 8 \end{pmatrix}.
\]

This problem is an unconstrained two objectives optimization problem, which the objectives are quadratic functions with indefinite Hessian matrices. If \( d \neq 0 \), it can not simultaneously occurs
\[
d^T A_1 d \leq 0 \quad \text{and} \quad d^T A_2 d \leq 0, \quad \text{for all} \quad d \in \partial B(0, 1). \tag{1}
\]
In fact, let \( d = (d_1, d_2)^T \neq (0, 0)^T \). If \( d^T A_1 d = 8d_1^2 - 2d_2^2 \leq 0 \) and \( d^T A_2 d = -2d_1^2 + 8d_2^2 \leq 0 \), then
\[
0 < 6d_1^2 + 6d_2^2 = (8d_1^2 - 2d_2^2) + (8d_2^2 - 2d_1^2) \leq 0.
\]
This is a contradiction, therefore inequalities (1) only occur if \( d \equiv 0 \). A necessary condition for a point \( x^* \) be a locally Pareto-efficient optimal solution \([2]\) is that there are real numbers \( \tau_1, \tau_2 \geq 0 \), not all zero, such that
\[
\tau_1 \nabla f_1(x^*) + \tau_2 \nabla f_2(x^*) = 0. \tag{2}
\]
If \( \tau_2 = 0 \) and \( \tau_1 > 0 \), then \( x^* = (0, 0)^T \). In the direction \( d = \left( \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right)^T \in \partial B(0, 1) \), for every point of the form \( x' = x^* + \lambda d \), for each real value \( \lambda \in (0, \frac{4}{\sqrt{5}}) \), simultaneously occur \( f_1(x') = f_1(x^*) \) and \( f_2(x') < f_2(x^*) \). Therefore, the point \( x^* = (0, 0)^T \) is not a locally Pareto-efficient optimal solution.

If \( \tau_1 = 0 \) and \( \tau_2 > 0 \), then \( x^* = (2, -1)^T \). In the direction \( d = \left( \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right)^T \in \partial B(0, 1) \), for every point of the form \( x' = x^* + \lambda d \), for each real value \( \lambda \in (0, 2\sqrt{5}) \), simultaneously occurs \( f_1(x') < f_1(x^*) \) and \( f_2(x') = f_2(x^*) \). Therefore, the point \( x^* = (2, -1)^T \) is not a locally Pareto-efficient optimal solution.

Now, if \( \tau_1, \tau_2 > 0 \), an equivalent way to write the condition (2) is
\[
\tau \nabla f_1(x^*) + \nabla f_2(x^*) = 0, \quad \text{where} \quad \tau = \frac{\tau_1}{\tau_2} > 0. \tag{3}
\]
But, this condition is also sufficient for the points \( x^* \), unlike \((0, 0)^T\) and \((2, -1)^T\), are locally Pareto-efficient optimal solution. In fact, suppose that \( x^* \) is not locally
Pareto-efficient optimal solution, then there exists a direction $d \in \partial B(0, 1)$ and a number real $\lambda > 0$ such that $f(x^* + \lambda d) \leq f(x^*)$ is valid. That is, there is a descent direction $d$ which locally occurs $\nabla f_1(x^*)^T d \leq 0$ and $\nabla f_2(x^*)^T d \leq 0$, with $\nabla f_1(x^*) \neq 0$ and $\nabla f_2(x^*) \neq 0$. However, for $x^*$ and $\tau > 0$ if the equation (3) is valid, by Stiemke’s alternative theorem [41], $\nabla f_1(x^*)^T d < 0$ and $\nabla f_2(x^*)^T d \leq 0$ has no solution, and $\nabla f_1(x^*)^T d \leq 0$ and $\nabla f_2(x^*)^T d < 0$ has no solution.

Thus, when $f(x^* + \lambda d) \leq f(x^*)$ and Stiemke’s alternative theorem are valid to must occur $\nabla f_1(x^*)^T d = 0$ and $\nabla f_2(x^*)^T d = 0$, and to must simultaneously occurs $d^T A_1 d \leq 0$ and $d^T A_2 d < 0$, or $d^T A_1 d < 0$ and $d^T A_2 d \leq 0$. What is impossible, because (1). Therefore, the locally Pareto-efficient optimal solutions are those that satisfy (3), that is, $8\tau x_1^* - 2(x_1^* - 2) = 0$ and $-2\tau x_2^* + 8(x_2^* + 1) = 0$, that is,

$$
\begin{align*}
    x_1^* &= \frac{2}{4\tau+1}, \quad 0 < \tau \neq \frac{1}{4},
    \\
    x_2^* &= \frac{4}{\tau-4}, \quad 0 < \tau \neq 4.
\end{align*}
$$

(4)

![Figure 1: Locally Pareto-efficient optimal solutions for the Example 1](image)

The points $(x_1^*, x_2^*)$ satisfying (4) are represented in Figure 1. The locally Pareto-efficient optimal solutions to this problem are grouped into three disconnected branches: $A$, $B$ and $C$. In addition, it is possible to verify that only the solutions set generated by inequality $\frac{1}{4} < \tau < 4$ are globally Pareto-efficient optimal solutions (branch $C$ plotted in the graph). We accept this by looking Figure 2.
It illustrates the graph in the plane representing the Pareto-optimal curve, that is, the branches images $A$, $B$ and $C$, by functions $f_1$ and $f_2$ plotted in the coordinates $(f_1(x_1^*, x_2^*), f_2(x_1^*, x_2^*))$. $f(A) = (f_1(A), f_2(A))$, $f(B) = (f_1(B), f_2(B))$ and $f(C) = (f_1(C), f_2(C))$ represent the branches images $A$, $B$ and $C$, respectively. We see in Figure 2 that the locally Pareto-efficient optimal solutions set belonging to the branch $C$ also are globally Pareto-efficient optimal solutions, that is, globally non-dominated.

![Figure 2: Pareto-optimal curve for the Example](image)

We have just seen a simple example which the globally Pareto-efficient optimal solutions set is strictly contained in the locally Pareto-efficient optimal solutions set, but investigate the locally or globally efficient optimal solutions are not a trivial task. In this paper, we present new ways that can facilitate this analysis.

This paper is organized as follows. We start by defining some notations and basic properties. In Section 3 the new concept of radius of efficiency and the relationships among associated problem are presented. In Section 4 using this concept some necessarily and sufficient Pareto optimality conditions are established. Finally, comments and concluding remarks are presented in Section 5.
2. Quadratic fractional problem and literature review

In this paper, we deal with a particular case of (VOP), which each objective function consists of a ratio of two quadratic functions. Without generalized convexity assumptions in the objective functions, we show how to calculate the largest radius of the spherical region centered on a local Pareto-efficient solution in which this solution is optimal. Let us consider the following vector quadratic fractional optimization problem:

\[
\text{(VQFP)} \quad \text{Minimize} \quad \frac{f(x)}{g(x)} = \left( \frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)} \right)
\]

subject to
\[
\begin{align*}
    h_j(x) &\leq 0, \quad j \in J, \\
    x &\in \Omega \subseteq \mathbb{R}^n,
\end{align*}
\]

where \( f_i \) and \( g_i, i \in I \), are quadratic functions of \( n \) real variables. In addition, we assume that \( g_i(x) > 0, i \in I \), for any \( x \) in nonempty \( \Omega \). The fields of real functions \( h_j \)'s contains \( \Omega \). We denote by \( S \) the feasible set that is the intersection of \( \Omega \) with the set of points \( x \) in which \( h_j(x) \leq 0, j \in J \). We choose the functions \( g_i \)'s that preserves the signal. We denote the unconstrained (VQFP) by (VQFP').

The fractional optimization problems arise frequently in the decision making applications, including science management, portfolio selection, cutting and stock, game theory, in the optimization of the ratio performance/cost, or profit/investment, or cost/time and so on.

There are many contributions dealing with the scalar (single-objective) fractional optimization problem (FP) and vector fractional optimization problem (VFP). In most of them, using convexity or their generalizations, optimality conditions in the KKT sense and the main duality theorems for optimal points are obtained. With a parametric approach, which transforms the original problem in a simpler associated problem, Dinkelbach [13], Jagannathan [16] and Antczak [1] establish optimality conditions, presents algorithms and apply their approaches in an example (FP) consisting of quadratic functions. For example, Dinkelbach [13] presents some theoretical results that relate the two problems, and proposes an algorithm that converges to the minimum of problem (FP) to perform a sequence of operations in associated parametric problem. Using some known generalized convexity, Antczak [1], Khan and Hanson [20], Reddy and Mukherjee [33], Jeyakumar [18], Liang et al. [25] establish optimality conditions and theorems that relate the pair primal-dual of problem (FP). In Craven [10, 11] and Weir [42], other results for the scalar optimization (FP) can be found.

Further, Liang et al. [26] extended their approach to the vector optimization case (VFP) considering the type duals of Mond-Weir [29], Schaible [37, 38] and
Bector [5]. Considering the parametric approach of Dinkelbach [13], Jagannathan [16], Bector et al. [6] and two classes of generalized convexity, Osuna-Gómez et al. [31] establish weak Pareto-efficient optimality conditions and the main duality theorems for the differentiable vector optimization case (VFP). Santos et al. [36] deepened these results to the more general non-differentiable case (VFP). Jeyakumar and Mond [19] use generalized convexity to study the problem (VFP) and with the parametric approach, Singh and Hanson [40] extended the results obtained by Geoffrion [14].

Few studies are found involving quadratic functions at both the numerator and denominator in the ratio objective function. Most of them involve the mixing of linear and quadratic functions. The closest approaches of the scalar quadratic fractional optimization problem (QFP) are considered by Crouzeix et al. [12], Schaible and Shi [39], Gotoh and Konno [15], Lo and MacKinlay [27] and Cambini et al. [8]. On the other hand, Benson [7] considered a pure (QFP) consisting of the convex function, develop some theoretical properties and optimality conditions, he presents an algorithm and its convergence properties.

The closest approaches of the vector optimization case (VQFP) are considered by Beato et al. [3, 4], Arévalo and Zapata [2], Konno and Inori [21], Rhode and Weber [34], Kornbluth and Steuer [24], Korhonen and Yu [22, 23]. Using an iterative computational test, Beato et al. [3, 4] characterize the Pareto-efficient optimal point for the problem (VQFP), consisting of a linear and quadratic functions, and using the function linearization approach of Bector et al. [6], some theoretical results are obtained. Arévalo and Zapata [2], Konno and Inori [21], Rhode and Weber [34] analyze the portfolio selection problem. Kornbluth and Steuer [24] use an adapted Simplex method in the problem (VFP) consisting of linear functions. Korhonen and Yu [22, 23] propose an iterative computational method for solving the problem (VQFP), consisting of the linear and quadratic functions, based on search directions and weighted sums.

The approach taken in this work is different from the previous ones. We believe that the approach presented here facilitates the resolution of the problem (VQFP). The main aspect of this contribution is the development of necessary and sufficient Pareto-efficient optimality conditions for a particular vector optimization problems based on the calculation of largest radius of the spherical region centered on a local Pareto-efficient solution in which this solution is optimal. In this process we may conclude that the solution is also globally Pareto-efficient optimal. These conditions might be useful to determine termination criteria in the development of algorithms, including more general problems in which quadratic approximations are used locally.
3. Radius of efficiency

We introduce the new concept radius of efficiency for (VOP) and present some Pareto-efficient optimality conditions for \((VQFP')\) from this alternative approach, and then we extend the results to the constrained case (VQFP).

3.1. Radius of efficiency in the (VOP)

From certain particular properties verified in some search directions of the objective functions, we want to detect when a local Pareto-efficient optimal solution is equivalent to a global Pareto-efficient optimal solution. With this approach is possible to identify a spherical region of feasible points where a local Pareto-efficient optimal solution is not dominated.

We say that a point \(x^* \in S\) is \(\lambda\)-efficient or has radius of efficiency \(\lambda\) in (VOP) if \(x^* \in \text{Leff}(VOP)\) and there does not exist another point \(x' \in B(x^*, \lambda) \cap S\) that dominates \(x^*\).

Note an important difference between the locally Pareto-efficient optimal solution definition and the radius of efficiency definition of a locally Pareto-efficient optimal solution. In the first case, from a theoretical point of view we know that always there is an arbitrary neighbourhood \(N(x^*) \subseteq S\) of \(x^*\), where \(x^*\) is not dominated. Naturally, this neighbourhood always can be regarded as a ball of arbitrary radius in \(\mathbb{R}^n\). In the second case, is possible to calculate the largest radius \(\lambda^* > 0\) of the spherical region \(B(x^*, \lambda^*)\) such that \(x^*\) is not dominated in \(B(x^*, \lambda^*) \cap S\), or is possible to conclude that \(x^*\) is not dominated everywhere the feasible set \(S\).

Some important reasons for the use of the radius of efficiency in the problems (VOP) can be indicate. We always can consider a local solution in a fixed well-know spherical region instead in an arbitrary spherical region. We can determine a subset compact where there does not exist another points that dominate a specific solution, what is useful in the resolution of (VOP), because if the decision maker knows the radius of efficiency, he can estimates the cost to try to find a new solution, and also choose a more suitable search procedure. Auxiliary problems induced by the concept of radius of efficiency can be used to conclude the global Pareto-efficiency.

Naturally, if \(x^*\) is \(\lambda\)-efficient, then it is \(\beta\)-efficient, \(\forall \ \beta < \lambda\). Similarly we say that \(x^*\) is \(\infty\)-efficient if it is efficient in \(S\).

3.2. Radius of efficiency in the (VQFP)

We calculate the radius of efficiency of a solution \(x^*\) and some results from the calculation are presented. By hypothesis, we assume that \(x^* \in \text{Leff}(VQFP)\). To
determine the radius of efficiency \( \lambda > 0 \) for this solution \( x^* \), we must ask: What is the smallest value of \( \lambda > 0 \) such that among every unitary direction \( d \) in which

\[
\frac{f(x^* + \lambda d)}{g(x^* + \lambda d)} \leq \frac{f(x^*)}{g(x^*)}
\]

is valid? The answer to this question provides the maximum radius of efficiency of \( x^* \).

The next theorems allow us to characterise when a locally Pareto-efficient optimal solution is equivalent to a Pareto-efficient optimal solution. We will use these theorems to identify the maximum radius of efficiency and analyse the dominance of a locally Pareto-efficient optimal solution in the feasible set.

Similarly to Dinkelbach [13] and Jagannathan [16], which transform the fractional optimization in a new problem, we consider the following problem associated with the (VQFP).

\[
(VQFP)_{x^*} \quad \text{Minimize} \quad f(x) - \frac{f(x^*)}{g(x^*)} g(x) = \left( f_1(x) - \frac{f_1(x^*)}{g_1(x^*)} g_1(x), \ldots, f_m(x) - \frac{f_m(x^*)}{g_m(x^*)} g_m(x) \right)
\]

subject to

\[
h_j(x) \leq 0 \quad j \in J, \\
x \in \Omega \subseteq \mathbb{R}^n,
\]

were \( x^* \in S \) and \( f_i, g_i, i \in I, h_j, j \in J \), are the same functions defined in (VQFP).

The \((VQFP)_{x^*}\) is similarly introduced by Osuna-Gómez et al., which derive optimality conditions and duality results for the weakly Pareto-efficient optimal solutions. The following theorem and its proof is an approach equivalent to Lemma 1.1 presented in [31], but we consider the Pareto-efficient optimal solutions.

**Theorem 1.** \( x^* \in \text{Leff}(VQFP) \) if and only if \( x^* \in \text{Leff}(VQFP)_{x^*} \). In addition, \( x^* \) is locally Pareto-efficient optimal solution for \((VQFP)\) in \( N(x^*) \) if and only if \( x^* \) is locally Pareto-efficient optimal solution for \((VQFP)_{x^*}\) in \( N(x^*) \).

**Proof** \((\Rightarrow)\) Let \( N(x^*) \subseteq S \) and \( x^* \) be locally Pareto-efficient optimal solution for \((VQFP)\) in \( N(x^*) \). Suppose that \( x^* \notin \text{Leff}(VQFP)_{x^*} \), then there exists another point \( x' \in N(x^*) \) satisfying

\[
f(x') - \frac{f(x^*)}{g(x^*)} g(x') \leq f(x^*) - \frac{f(x^*)}{g(x^*)} g(x^*) = 0 \quad \Rightarrow \quad \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)}.
\]

Which contradicts \( x^* \in \text{Leff}(VQFP) \) in \( N(x^*) \), and therefore \( x^* \in \text{Leff}(VQFP)_{x^*} \) in \( N(x^*) \).
4.1. Radius of efficiency in the unconstrained case

and then we extend the results for the feasible set defined by linear inequalities.

Similarly, let \( N(x^*) \subseteq S \) and \( x^* \) be locally Pareto-efficient optimal solution for \((VQFP)_{x^*}\) in \( N(x^*) \). Suppose that \( x^* \notin \text{Leff}(VQFP) \), then there exists another point \( x' \in N(x^*) \), satisfying

\[
\frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \implies f(x') - \frac{f(x^*)}{g(x^*)}g(x') \leq 0 = f(x^*) - \frac{f(x^*)}{g(x^*)}g(x^*). 
\]

Which contradicts \( x^* \in \text{Leff}(VQFP)_{x^*} \) in \( N(x^*) \), and therefore \( x^* \in \text{Leff}(VQFP) \) in \( N(x^*) \).

Note that on associated problems, for each \( x^* \in \text{Leff}(VQFP) \) we can only ensure that \( \text{Leff}(VQFP) \subset \text{Leff}(VQFP)_{x^*} \).

The same arbitrary neighborhood \( N(x^*) \) equivalent to a ball of arbitrary radius in \( \mathbb{R}^n \) centered at \( x^* \) appears in both problems \((VQFP)\) and \((VQFP)_{x^*}\), and we want to compute a neighborhood \( B(x^*, \lambda^*) \) of maximum radius \( \lambda^* \), such that \( x^* \in \text{Leff}(VQFP) \) in \( B(x^*, \lambda^*) \). Then, calculate the radius of efficiency \( \lambda^* \) of the solution \( x^* \) in the \((VQFP)\) is equivalent to calculate the radius of efficiency \( \lambda^* \) of the solution \( x^* \) in the \((VQFP)_{x^*}\), and so we can choose between the two problems one whose calculation is easier.

4. Optimality Conditions from Radius of efficiency

First the optimality conditions are established for the unconstrained problem and then we extend the results for the feasible set defined by linear inequalities.

4.1. Radius of efficiency in the unconstrained case

For each \( i \in I \) and all \( x \in \mathbb{R}^n \) we consider the objective functions defined as \( f_i(x) = x^T A_i x + a_i^T x + \tilde{a}_i \) and \( g_i(x) = x^T B_i x + b_i^T x + \tilde{b}_i \), where \( A_i, B_i \in \mathbb{R}^{n \times n} \), \( A_i \) symmetric, \( B_i \) symmetric and positive semidefinite, \( a_i, b_i \in \mathbb{R}^n \) and \( \tilde{a}_i, \tilde{b}_i \in \mathbb{R} \), with \( \tilde{b}_i > -(w^T B_i w + b_i^T w) \), where \( w \) is the solution of the system \( 2B_i x + b_i = 0 \), that is, \( w \) is the point where the function \( x^T B_i x + b_i^T x \) reaches its minimum and this ensures that \( g_i(x) > 0, \forall x \in \mathbb{R}^n \). We cannot consider cases where \( 2B_i x + b_i = 0 \) has no solution. Similarly, we denote by \((VQFP')_x\) the unconstrained \((VQFP)_{x^*}\). We recall that \( \nabla^2 f(x) \) denote the Hessian matrix of the function \( f : \mathbb{R}^n \to \mathbb{R} \) at point \( x \).

Further, we define some sets and parameters that are used throughout this work. Given \( x^* \in \mathbb{R}^n \), we define the following quadratic function

\[
p_i(x) = x^T \left( A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \right) x + \left( a_i^T - \frac{f_i(x^*)}{g_i(x^*)} b_i^T \right) x, \quad i \in I, \quad (5)
\]
and given an unitary direction $d$, we define

$$X_0 = \{ i \in I \mid [d^T \nabla^2 p_i(x^*)d, \nabla p_i(x^*)^T d]^T \geq [0,0]^T \},$$

$$X_1 = \{ i \in I \mid d^T \nabla^2 p_i(x^*)d > 0 \text{ and } \nabla p_i(x^*)^T d < 0 \},$$

$$X_2 = \{ i \in I \mid d^T \nabla^2 p_i(x^*)d < 0 \text{ and } \nabla p_i(x^*)^T d > 0 \},$$

$$\lambda^d_2 = \max_{i \in X_2} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*)d} \right\}, \quad \lambda^d_1 = \begin{cases} \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*)d} \right\}, & \text{if } X_1 \neq \emptyset \\ +\infty, & \text{if } X_1 = \emptyset \end{cases},$$

$$\Lambda^d_2 = [\lambda^d_2, \infty), \quad \Lambda^d_1 = \begin{cases} (0, \lambda^d_1], & \text{if } X_1 \neq \emptyset \\ (0, \lambda^d_2], & \text{if } X_1 = \emptyset \end{cases}, \quad \Lambda^d = \Lambda^d_2 \cap \Lambda^d_1.$$ 

The functions $p_i, i \in I$, defined in [5] are the same objective functions of the (VQFP)$_{\lambda^*}$ unless the constants term $\bar{a}_i - \frac{f_i(x^*)}{g_i(x^*)} \bar{b}_i$, and so $p_i(x^*) = -[\bar{a}_i - \frac{f_i(x^*)}{g_i(x^*)} \bar{b}_i]$. We denote by $|A|$ the number of elements in the set $A$.

For a better understanding of sets and parameters above, consider the Taylor expansion around zero of the function $r_i(\lambda) = p_i(x^* + \lambda d), i \in I, \lambda \in \mathbb{R}$. Since $r_i$ is a quadratic function, we obtain

$$r_i(\lambda) = p_i(x^*) + \lambda \nabla p_i(x^*)^T d + \frac{\lambda^2}{2} d^T \nabla^2 p_i(x^*)d.$$ 

In other words, $r_i$ is a real function of one variable, whose graph is a parable, and has constant term $p_i(x^*)$, linear term $\nabla p_i(x^*)^T d$ and quadratic term $\frac{\lambda^2}{2} d^T \nabla^2 p_i(x^*)d$. Thus, it becomes easier to interpret the sets $X_0, X_1$ and $X_2$. Given an unitary direction $d$, the set $X_1$ is formed by the indices $i \in I$ whose $r_i$ functions decrease to values close to $\lambda = 0$ and are convex, while the set $X_2$ is formed by the indices $i \in I$ whose $r_i$ functions grow to values close to $\lambda = 0$ and are concave.

Figure [3] shows typical elements of the sets $X_1$ and $X_2$, in which the graphs represent two functions $r_i$, and are plotted the points of function $r_i(\lambda) = p_i(x^* + \lambda d)$ in the coordinates $(\lambda, r_i(\lambda))$ for $\lambda > 0$, which are two parables: a convex indicating that the function index $r_i$ belongs to the set $X_1$ (where $d^T \nabla^2 p_i(x^*)d > 0$
and $\nabla p_i(x^*)^Td < 0$; and a concave indicating that the function index $r_i$ belongs to the set $X_2$ (where, $d^T \nabla^2 p_i(x^*)d < 0$ and $\nabla p_i(x^*)^Td > 0$).

The set $X_0$ is formed by the indices $i \in I$ where each quadratic function $r_i$ has non-negative linear term and positive quadratic term or has positive linear term and non-negative quadratic term.

Figure 4 shows typical elements of the set $X_0$ and $X_2$, in which the graphs represent three functions $r_i$. In relation the function index $r_i$ belongs to the set $X_0$, two examples are plotted: a convex parable (where, $d^T \nabla^2 p_i(x^*)d > 0$ and $\nabla p_i(x^*)^Td \geq 0$); and a growing straight line (where, $d^T \nabla^2 p_i(x^*)d = 0$ and $\nabla p_i(x^*)^Td > 0$).

The parameter $\lambda d_2$ is defined as the maximum of the positive roots of the equations $r_i(\lambda) - p_i(x^*) = 0$ in which indices $i$ belong to $X_2$. Similarly, if $X_1 \neq \emptyset$, the parameter $\lambda d_1$ is defined as the minimum of the positive roots of the equations $r_i(\lambda) - p_i(x^*) = 0$ in which indices $i$ belong to $X_1$. If $X_1 = \emptyset$, we define $\lambda d_1 \equiv +\infty$.

For each index $i$, another root of the equation $r_i(\lambda) - p_i(x^*) = 0$ is $\lambda = 0$, but we are interested in the values $\lambda > 0$. Figure 3 shows these parameters, the parameter $\lambda d_2$ and $\lambda d_1$ are the crossover point between the curve $r_i(\lambda)$ and the dotted straight line $p_i(x^*)$ for $i \in X_2$ and $i \in X_1$, respectively. Figure 4 shows a positive root of the equation $r_i(\lambda) - p_i(x^*)$ for $i \in X_2$.

Given an unitary direction $d$, the sets $\Lambda d_2$, $\Lambda d_1$ and $\Lambda d$ are real intervals contained in $\mathbb{R}_+ \setminus \{0\}$, where $\lambda d_2$ is the extreme left value of the interval $\Lambda d_2$ and $\lambda d_1$ is the extreme right value of the interval $\Lambda d_1$. When occur $\lambda d_2 < \lambda d_1$, they are the end points of the interval $\Lambda d$ and if occur $\lambda d_2 > \lambda d_1$, we obtain $\Lambda d = \emptyset$. However, in
some particular cases, for example when occur $|X_1| = |X_2| = 1$ and $\lambda^d_2 = \lambda^d_1$, we obtain $\Lambda^d = \{\lambda^d_2\}$, that is $\Lambda^d \neq \emptyset$, but the most suitable in this case is set $\Lambda^d = \emptyset$. This case and others like it are important and are explained in detail in due course.

Figure 3 illustrates examples of the intervals $\Lambda^d_i$, $\Lambda^d_1$ and $\Lambda^d$. In it, we represent over the dotted straight line $p_i(x^*)$ the interval $\Lambda^d_2 = [\lambda^d_2, \infty)$, over the $\lambda$-axis the interval $\Lambda^d_1 = (0, \lambda^d_1]$, and below the $\lambda$-axis the interval $\Lambda^d = \Lambda^d_2 \cap \Lambda^d_1$. When $X_1 = \emptyset$ we choose $\lambda^d_1 = \infty$, because when $X_2 \neq \emptyset$ we always obtain $\Lambda^d \neq \emptyset$. Note that in Figure 3 we have $p_i(x') \leq p_i(x^*)$ for all $\lambda \in \Lambda^d$ and $x' = x^* + \lambda d$. On the other hand, Figure 6 illustrates an example in which $\Lambda^d = \emptyset$.

Before the next result, we present some important details in order to understand the possibility of obtaining an unitary direction $d$ and a constant $\lambda > 0$, in which $f(x^* + \lambda d) \leq f(x^*)$ is valid for a solution $x^* \in \text{Leff}(VQFP')$, and what are their relations with the sets $X_0$, $X_1$ and $X_2$.

Consider the Taylor expansion around zero of each function $\bar{r}_i(\lambda) = f_i(x^* + \lambda d)$ and $\tilde{r}_i(\lambda) = g_i(x^* + \lambda d)$, $i \in I$, in solution $x^*$ and along the direction $d$,

\[
\bar{r}_i(\lambda) = f_i(x^* + \lambda d) = f_i(x^*) + \lambda \nabla f_i(x^*)^T d + \lambda^2 d^T A_i d,
\]

\[
\tilde{r}_i(\lambda) = g_i(x^* + \lambda d) = g_i(x^*) + \lambda \nabla g_i(x^*)^T d + \lambda^2 d^T B_i d.
\]

Performing some manipulations, we obtain

\[
\frac{f_i(x^* + \lambda d)}{g_i(x^* + \lambda d)} \leq \frac{f_i(x^*)}{g_i(x^*)} \iff \lambda d^T \left(\lambda A_i - \lambda f_i(x^*)^T g_i(x^*) - \lambda \nabla f_i(x^*)^T g_i(x^*) - \lambda \nabla f_i(x^*) + \lambda \nabla g_i(x^*) - \lambda \nabla f_i(x^*)\right) d \leq 0.
\]
From (5) consider the function \( p_i(x) \), then

\[
\lambda d^T \left( \lambda A_i - \lambda \frac{f_i(x^*)}{g_i(x^*)} B_i \right) d \leq \left( \lambda \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) - \lambda \nabla f_i(x^*) \right)^T d \iff \\
\iff \lambda \left( \nabla p_i(x^*)^T d + \frac{\lambda}{2} d^T \nabla^2 p_i(x^*) d \right) \leq 0, \quad (6)
\]

where \( \frac{1}{2} \nabla^2 p_i(x^*) = A_i - \frac{f_i(x^*)}{g_i(x^*)} B_i \) and \( \nabla p_i(x^*) = \nabla f_i(x^*) - \frac{f_i(x^*)}{g_i(x^*)} \nabla g_i(x^*) \). From (6), we obtain

\[
p_i(x^* + \lambda d) - p_i(x^*) = \lambda \left( \nabla p_i(x^*)^T d + \frac{\lambda}{2} d^T \nabla^2 p_i(x^*) d \right) \leq 0 \iff \\
\iff r_i(\lambda) = p_i(x^* + \lambda d) \leq p_i(x^*).
\]

The following situations about the existence of positive solutions to \( \lambda \) in inequalities (6) can occur for each direction \( d \) and for each \( i \in I \).

1. If \( d^T \nabla^2 p_i(x^*) d > 0 \),
   (a) and if \( \nabla p_i(x^*)^T d > 0 \), does not exist \( \lambda > 0 \) satisfying (6) and \( i \in X_0 \),
   (b) and if \( \nabla p_i(x^*)^T d = 0 \), does not exist \( \lambda > 0 \) satisfying (6) and \( i \in X_0 \),
   (c) and if \( \nabla p_i(x^*)^T d < 0 \), \( \lambda \in \left( 0, \frac{2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right] \) satisfying (6) and \( i \in X_1 \).

2. If \( d^T \nabla^2 p_i(x^*) d = 0 \),

\section*{Figure 5: Graphs of \( r_i \) in which index \( i \) belongs to the set \( X_1 \), and a root \( \lambda \) of the equation \( r_i(\lambda) - p_i(x^*) = 0 \).}
Figure 6: A search direction by $x'$ that dominates $x^* \in \text{Leff}(VQFP')$. $X_0 = \emptyset$, $X_1 = \{1,3\}$, $X_2 = \{2,4\}$ and $\Lambda^d = \emptyset$

(a) and if $\nabla p_i(x^*)^T d > 0$, does not exist $\lambda > 0$ satisfying (6) and $i \in X_0$,
(b) and if $\nabla p_i(x^*)^T d = 0$, $\lambda \in (0,\infty)$ satisfying (6),
(c) and if $\nabla p_i(x^*)^T d < 0$, $\lambda \in (0,\infty)$ satisfying (6).

3. If $d^T \nabla^2 p_i(x^*) d < 0$,
   (a) and if $\nabla p_i(x^*)^T d > 0$, $\lambda \in \left[ -\frac{2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}, \infty \right)$ satisfying (6) and $i \in X_2$,
   (b) and if $\nabla p_i(x^*)^T d = 0$, $\lambda \in (0,\infty)$ satisfying (6),
   (c) and if $\nabla p_i(x^*)^T d < 0$, $\lambda \in (0,\infty)$ satisfying (6).

The above items inform whenever exists $\lambda > 0$ satisfying the inequality (6) and in which interval it exists. They also inform the behavior of each function $r_i$ along the direction $d$, e.g. items 1.(a), 1.(b) and 2.(a) represent the functions $r_i$ that are crescent along the direction $d$ and, therefore, does not exist $\lambda > 0$ satisfying (6). On such cases the index of function $r_i$ belongs to the set $X_0$ and occurs $r_i(\lambda) > p_i(x^*)$. The behavior of function $r_i$ for items 1.(a), 1.(b) and 2.(a) is illustrated in Figure 4, where appears a convex parable and a growing straight line for $\lambda > 0$.

In Figure 5 is represented the event of item 1.(c) above, a function $r_i$ that is the graph of a convex parable. In item 1.(c), exists $\lambda > 0$ satisfying (6) and it gives the conditions for $r_i$ to be a convex quadratic function, in which the equation $r_i(\lambda) = p_i(x^*) = 0$ has a positive root $\lambda = -\frac{2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}$. Then, to occur $p_i(x^* + \lambda d) \leq p_i(x^*)$, we must have $\lambda \in \left(0, -\frac{2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right]$. We exclude zero in this interval because we are interested on points $x^* + \lambda d$ distinct to $x^*$. 

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verify the items above for $m$ along the unitary direction $x$ or when $r$ functions $r$ equation $r \in i$ is the graphic of a concave parabola. In this item, exists $\lambda$ that $d$ direction $x$ possible to occur two situations whenever we have $X$. Figure 7: A search direction by $i$ whose indices $\lambda$ is valid $p_i(x^*)$. Figure 6 is represented what occurs in item 3.(a). There, the function $r_i$ which are not crescent along the unitary direction $d$, therefore, all $\lambda \in (0, \infty)$ satisfy (6). The behavior of functions $r_i$ for such items are illustrated in Figure 5. Item 2.(b) imply that for one $i \in I$, the function $r_i$ is constant, because your linear and quadratic term are zero. The items 2.(c), 3.(b) and 3.(c) occur when $d^T \nabla p_i(x^*)d < 0$ and $\nabla p_i(x^*)^T d \leq 0$, or when $d^T \nabla^2 p_i(x^*)d \leq 0$ and $\nabla p_i(x^*)^T d < 0$, for one $i \in I$. In this case, it is possible to occur two situations whenever we have $X_0 = \emptyset$ and $X_2 \neq \emptyset$ on the direction $d$. In the first, if $X_1 = \emptyset$ then $\lambda_i^d = \infty$ and the inequality $r_i(\lambda) \leq p_i(x^*)$ is valid $\lambda > 0$ determined by the positive roots of equations $r_i(\lambda) = p_i(x^*) = 0$ whose indices $i \in X_2$. In the second, when $X_1 \neq \emptyset$, if exists one index $k \in I$ such that $d^T \nabla^2 p_k(x^*)d \leq 0$ and $\nabla p_k(x^*)^T d \leq 0$, the value of $\lambda_i^d$ is not influenced by the function $r_k$, because $\lambda_i^d$ is determined by the positive roots of equations $r_i(\lambda) = p_i(x^*) = 0$ whose indices $i \in X_1$.

In Figure 4 is represented what occurs in item 3.(a). There, the function $r_i$ is the graphic of a concave parabola. In this item, exists $\lambda > 0$ satisfying (6) and equation $r_i(\lambda) - p_i(x^*) = 0$ has a positive root $\lambda = \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*)d}$. Therefore, to occur $p_i(x^* + \lambda d) \leq p_i(x^*)$ we must have $\lambda \in \left[ \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*)d}, \infty \right)$.

Given an unitary direction $d$, we look for a nonempty interval $\Lambda^d$ such that $x' = x^* + \lambda d$, $p_i(x') \leq p_i(x^*)$, $\forall i \in I$, where $\lambda \in \Lambda^d$ and $X_0 = \emptyset$. Hence, we must verify the items above for $m$ functions. In Figures 6, 7, 8 and 9 are plotted the...
points of the parabolas that are the graphics of functions $r_i(\lambda) = p_i(x^* + \lambda d)$, $i \in \{1, 2, 3, 4\}$, at coordinates $(\lambda, r_i(\lambda))$. Figure 7 illustrates a case which $\Lambda^d \neq \emptyset$, $X_1 = \{2, 3\}$ and $X_2 = \{1, 4\}$, where we see that $\lambda_2^d \leq \lambda_1^d$. Note also that if $\lambda \in \Lambda^d$ and $x' = x^* + \lambda d$, we obtain $p_i(x') \leq p_i(x^*)$, $\forall i \in \{1, 2, 3, 4\}$.

Figure 8 illustrates a case which $X_0 \neq \emptyset$, where we have $X_0 = \{1\}$, $X_1 = \{3\}$, and $X_2 = \{2, 4\}$. In this case, $r_1(\lambda)$ always increases for $\lambda > 0$, then $x^*$ is globally Pareto-efficient solution in this direction. On the other hand, given a direction $d$, if $X_0 = \emptyset$ and $X_1 \neq \emptyset$, then not occur $X_2 = \emptyset$ in this direction, because the solution $x^*$ would not be locally Pareto-efficient solution, since $p(x^* + \lambda d) \leq p(x^*)$ would be valid for all $\lambda > 0$. This situation can be observed in Figures 6 and 7 needing only to assume $X_2 = \emptyset$ in both figures. In fact, suppose that $X_2 = \emptyset$ in figure 7 then when ignoring the graphics of functions $r_1$ and $r_4$, we obtain $r_1(\lambda) \leq p_1(x^*)$, $r_2(\lambda) < p_2(x^*)$, $r_3(\lambda) < p_3(x^*)$ and $r_4(\lambda) \leq p_4(x^*)$ for all $\lambda > 0$. That is, $p(x^* + \lambda d) \leq p(x^*)$ is valid for all $\lambda > 0$.

Therefore, whenever occur $X_0 = \emptyset$ in a determined direction $d$ and $x^*$ being locally Pareto-efficient optimal solution, two important situations to our analysis may happen: $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, or $X_1 = \emptyset$ and $X_2 \neq \emptyset$. Figure 9 illustrates the case $X_0 = \emptyset$, $X_1 = \emptyset$ and only $X_2 \neq \emptyset$, where $X_2 = \{1, 2, 3, 4\}$, $X_0 = \emptyset$ and $X_1 = \emptyset$, then we obtain $\lambda_1^d = \infty$, $\lambda_2^d < \lambda_1^d$ and $\Lambda^d \neq \emptyset$. Note in the unconstrained problems that, in the direction $d$, always is possible calculation the positive root of the equation.

Figure 8: A search direction by $x'$ that dominates $x^* \in \text{Leff}(VQFP)^*, X_0 = \{1\}, X_1 = \{3\}$ and $X_2 = \{2, 4\}$
such that $X$ explained. For example, we must see what happens when there exists a direction at least that one inequality be strict.

Therefore, to grant that $\lambda$ we must choose correctly $\lambda$ of $r$.

Next, some possibility which harden the demonstration of our results are ex-

We can conclude that if $X_0 = \emptyset$, $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$ in a direction $d$, the values of $\lambda > 0$, such that, $\frac{f(x^* + \lambda d)}{g(x^* + \lambda d)} \leq \frac{f(x^*)}{g(x^*)}$, $\forall i \in I$, must satisfy

$$\lambda \leq \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \text{ if } i \in X_1 \quad \text{and} \quad \lambda \geq \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \text{ if } i \in X_2. \quad (7)$$

Therefore, to grant that $\frac{f(x^* + \lambda d)}{g(x^* + \lambda d)} \leq \frac{f(x^*)}{g(x^*)}$ have solution in the unitary direction $d$, we must choose correctly $\lambda > 0$ such that $\lambda^d_2 \leq \lambda \leq \lambda^d_1$, and this choose must grant at least that one inequality be strict.

Next, some possibility which harden the demonstration of our results are explained. For example, we must see what happens when there exists a direction $d$ such that $X_0 = \emptyset$, $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and $\lambda^d_2 = \lambda^d_1 = \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}$, for all $i \in X_1 \cup X_2$, and also when there exists a direction $d$ such that $X_0 = \emptyset$, $X_1 = \emptyset$ and $X_2 = \emptyset$. First, suppose that we obtain $X_0 = \emptyset$, $X_1 \neq \emptyset$ and $X_2 \neq \emptyset$, then:

4. if $|X_1| \geq 2$,
   (a) $|X_2| \geq 1$ and $\lambda^d_2 = \lambda^d_1 = \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}$, for all $i \in X_1 \cup X_2$. As $\Lambda^d = (0, \lambda^d_1]$, we obtain $\Lambda^d = \Lambda^d_2 \cap \Lambda^d_1 = \{\lambda^d_1\}$, and then exists $\lambda \in \Lambda^d$. However, it

Figure 9: A search direction by $x'$ that dominates $x^* \in \text{Eff}(VQFP')_x$. $X_0 = \emptyset$, $X_1 = \emptyset$, $X_2 = \{1,2,3,4\}$ and $\Lambda^d \neq \emptyset$
is elementary to verify that \( p(x^* + \lambda d) \leq p(x^*) \) does not have solution for each \( \lambda > 0 \). We need to be careful in this case, because we obtain a nonempty interval \( \Lambda^d = \{\lambda^d_i\} \), however \( p(x^* + \lambda^d_i d) \leq p(x^*) \) does not have solution.

(b) \(|X_2| \geq 1\) and there exists \( j \in X_2 \) such that \(-\frac{2\nabla p_i(x^*)^T d}{\nabla^2 p_i(x^*)d} \leq \lambda^d_j \). As \( \Lambda^d = (0, \lambda^d_j] \), we obtain \( \Lambda^d = \Lambda^d_j \cap \Lambda^d_1 = \{\lambda^d_j\} \). Hence, there exists \( \lambda \in \Lambda^d \) and as \( j \in X_2 \) such that \(-\frac{2\nabla p_i(x^*)^T d}{\nabla^2 p_i(x^*)d} \leq \lambda^d_j \), it is elementary to verify that, unlike item 4.(a), \( p(x^* + \lambda^d_j d) \leq p(x^*) \) has solution.

(c) \(|X_2| \geq 1\) and, analogously item 4.(b), there exists \( j \in X_1 \) such that \( \lambda^d_j \leq \frac{2\nabla p_i(x^*)^T d}{\nabla^2 p_i(x^*)d} \). Hence, we obtain \( \Lambda^d = \Lambda^d_j \cap \Lambda^d_1 = \{\lambda^d_j\} \), there exists \( \lambda \in \Lambda^d \) and as \( j \in X_1 \) such that \( \lambda^d_j \leq \frac{2\nabla p_i(x^*)^T d}{\nabla^2 p_i(x^*)d} \), it is elementary to verify that, unlike item 4.(a), \( p(x^* + \lambda^d_j d) \leq p(x^*) \) has solution.

Figure 7 helps to understand the items discussed above. Now, suppose that given a direction \( d \) we obtain \( X_0 = 0 \), \( X_1 = 0 \) and \( X_2 = 0 \). This means that for the existence of positive solutions to inequality (6), only item 2.(b) occurs for all \( i \in I \). That is, simultaneously \( \nabla p_i(x^*)^T d = d^T \nabla^2 p_i(x^*)d = 0 \), for all \( i \in I \), and inequality (6) is valid for all \( \lambda \in (0, \infty) \). In this case, among other hypothesis, the following can occur.

5. \( d \) is orthogonal to set \( \{\nabla p_i(x^*)\}_{i \in I} \),
   - (a) and \( d \) is orthogonal to set \( \{\nabla^2 p_i(x^*)d\}_{i \in I} \),
   - (b) and there exists \( \bar{I} \subseteq I \) such that, \( d \) is orthogonal to set \( \{\nabla^2 p_i(x^*)d\}_{i \in \bar{I}} \) and \( d \in N_u(\nabla^2 p_i(x^*)) \), for all \( i \in I \setminus \bar{I} \), where \( N_u(\nabla^2 p_i(x^*)) \) is the kernel of the matrix \( \nabla^2 p_i(x^*) \).

6. There exists \( \bar{I} \subseteq I \), such that, \( d \) is orthogonal to set \( \{\nabla p_i(x^*)\}_{i \in \bar{I}} \) and \( \nabla p_i(x^*) = 0 \), for all \( i \in I \setminus \bar{I} \),
   - (a) and \( d \) is orthogonal to set \( \{\nabla^2 p_i(x^*)d\}_{i \in \bar{I}} \),
   - (b) and there exists \( \bar{I} \subseteq I \) such that, \( d \) is orthogonal to set \( \{\nabla^2 p_i(x^*)d\}_{i \in \bar{I}} \) and \( d \in N_u(\nabla^2 p_i(x^*)) \), for all \( i \in I \setminus \bar{I} \).

We believe that such events 4.(a)–(c), 5.(a)–(b) or 6.(a)–(b) are rare, because in general the objective functions of the vector optimization problems are conflicting. Aiming to facilitate the demonstrations of our results, we impose the following two conditions.
Condition 1. Let \( d \in \partial B(0,1) \) and suppose that \( X_0 = \emptyset, X_1 \neq \emptyset \) and \( X_2 \neq \emptyset \). If \( \lambda_2^d = \lambda_1^d = -\frac{2\nabla g(x^*_i)^Td}{\nabla^2 p_i(x^*_i)} \), for all \( i \in X_1 \cup X_2 \), then we define \( \Lambda^d := \emptyset \).

Condition 2. Let \( x^* \in \text{Leff}(VQFP') \) and \( d \in \partial B(0,1) \). Then does not simultaneously occurs \( \nabla p_i(x^*)^Td = d^T\nabla^2 p_i(x^*)d = 0 \), for all \( i \in I \).

Note that if we were dealing with weakly Pareto-efficient optimal solutions, items 4–6 and Conditions[1] and[2] would not be necessary.

Lemma 1. Let \( x^* \in \text{Leff}(VQFP') \). Then \( x' \) dominates \( x^* \) if and only if there exists \( d \in \partial B(0,1) \) such that \( X_0 = \emptyset, X_2 \neq \emptyset \) and \( \Lambda^d \neq \emptyset \). In this case, there exists \( \lambda^* \in \Lambda^d \) such that \( x' = x^* + \lambda^*d \).

Proof (\( \Rightarrow \)) If \( x^* \in \text{Leff}(VQFP') \), then from Theorem[1] \( x^* \in \text{Leff}(VQFP') \) and there exists a neighborhood \( N(x^*) \), such that, \( f(x) - \frac{f(x^*)}{g(x^*)}g(x) \leq 0 \) does not have solution for each \( x \in N(x^*) \), and by (5), \( p(x) \leq p(x^*) \) does not have solution for each \( x \in N(x^*) \). Consider \( N(x^*) \) as a ball of \( \mathbb{R}^n \) whith radius \( \bar{\lambda} \), that is, \( p(x) \leq p(x^*) \) does not have solution for each \( x \in N(x^*) \equiv B(x^*, \bar{\lambda}) \) and, if \( d \in \partial B(0,1) \), \( p(x) \leq p(x^*) \) does not have solution for each \( x = x^* + \lambda d \) such that \( \|x - x^*\| \leq \lambda \), for all \( \lambda \in (0, \bar{\lambda}) \), where \( \| \cdot \| \) is the Euclidean norm. Therefore, if \( x' \) dominates \( x^* \), \( x' \notin B(x^*, \bar{\lambda}) \) and there exists \( d \in \partial B(0,1) \) such that \( x' = x^* + \lambda^*d \) and \( \lambda^* \geq \bar{\lambda} \). We need to verify what happens to the sets \( X_0, X_1, X_2 \) and \( \Lambda^d \) in this direction \( d \). Figure[10] illustrates an unidimensional neighborhood \( B(0, \bar{\lambda}) \) at point \( \lambda = 0 \), which represents an \( n \)-dimensional neighborhood \( B(x^*, \bar{\lambda}) \) at point \( x^* \). We observe that in a fixed direction \( d \), used as example, there exists functions \( r_i \) whose graphics show that their indices \( i \) belong to \( X_0 \) (increasing curve in dashed line), or \( i \) belong to \( X_1 \) (convex parabola in continuous line), or \( i \) belong to \( X_2 \) (concave parabola in dashed line), and more, whenever \( \lambda \) satisfy items 3.(b) and 3.(c) for any index \( i \) (decreasing curve in continuous line). It is also possible to observe that if \( X_0 = \emptyset \) in this figure, \( r_i(\lambda) \leq p_i(x^*) \), for all \( i \in I \), only have solution if \( \lambda \in \Lambda^d \) (represented below of the \( \lambda \)-axis of the cartesian plane). That is, if \( X_0 = \emptyset \) in figure[10] \( p(x^* + \lambda d) \leq p(x^*) \) has solution if and only if \( \lambda_2^d \leq \lambda \leq \lambda_1^d \). The set \( \Lambda^d \) is represented above the \( \lambda \)-axis, while the set \( \Lambda^d \) is represented above the straight line \( p_i(x^*) \) in this figure.

Thus, given an arbitrary \( d \in \partial B(0,1) \), suppose that \( X_0 \neq \emptyset \) in this direction. Then there exists an index \( i \in I \), such that, the item 1.(a), 1.(b) or 2.(a) is satisfied, that is, \( d^T\nabla^2 p_i(x^*)d > 0 \) and \( \nabla p_i(x^*)^Td \geq 0 \) or \( d^T\nabla^2 p_i(x^*)d \geq 0 \) and
\[ \nabla p_i(x^*)^T d > 0. \] In this case, \( r_i(\lambda) \) grows indefinitely for \( \lambda > 0 \) and therefore given any neighborhood of \( x^* \) in direction \( d \) (e.g., \( B(0, \lambda) \) with \( \lambda \geq \tilde{\lambda} > 0 \), in Figure 10, \( p(x) \leq p(x^*) \) does not have solution for each \( x = x^* + \lambda d \) in this neighborhood, hence does not exist \( x' = x^* + \lambda d \) such that \( x' \) dominates \( x^* \) in this direction. Therefore, \( X_0 \) has to be empty.

![Figure 10: Neighborhood of point \( x^* \) and the behavior of the functions \( r_i \)](image)

Suppose now that \( X_0 = \emptyset \) and \( X_2 = \emptyset \) in direction \( d \). As \( x^* \in \text{Left}(VQFP') \) in \( B(x^*, \tilde{\lambda}) \), \( x^* \) is also locally Pareto-efficient optimal solution in \( B(0, \tilde{\lambda}) \) in direction \( d \). Then in this direction cannot occur \( X_1 \neq \emptyset \) and none of items 2.(c), 3.(b) or 3.(c) (decaying curve in continuous line shown in Figure 10) can be satisfied, that is, cannot occur \( d^T \nabla^2 p_i(x^*) d > 0 \) and \( \nabla p_i(x^*)^T d < 0 \), as well, cannot occur \( d^T \nabla^2 p_i(x^*) d \leq 0 \) and \( \nabla p_i(x^*)^T d < 0 \), or \( d^T \nabla^2 p_i(x^*) d < 0 \) and \( \nabla p_i(x^*)^T d \leq 0 \).

Hence, only item 2.(b) could be satisfied in this direction, that is, \( d^T \nabla^2 p_i(x^*) d = 0 \) and \( \nabla p_i(x^*)^T d = 0 \), \( \forall i \in I \). Which is impossible, due to Condition 2. Therefore, \( X_0 = \emptyset \) and \( X_2 \neq \emptyset \) in direction \( d \).

Suppose now that \( X_0 = \emptyset \) and \( X_2 \neq \emptyset \), but \( \Lambda^d = \emptyset \) in direction \( d \). With such hypothesis, \( X_1 = \emptyset \) cannot occur in direction \( d \), otherwise we would have \( \lambda^d_1 = \infty \) and \( \Lambda^d \neq \emptyset \). Then, if \( X_0 = \emptyset \), \( X_1 \neq \emptyset \), \( X_2 \neq \emptyset \) and \( \Lambda^d = \emptyset \) two cases may occur. Either \( \lambda^d_1 < \lambda^d_2 \) and \( p(x^* + \lambda d) \leq p(x^*) \) does not have solution for \( \lambda > 0 \), or by Condition 1 and item 4.(a), \( \lambda^d_2 = \lambda^d_1 = \frac{-2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \), for all \( i \in X_1 \cup X_2 \), and \( p(x^* + \lambda d) \leq p(x^*) \) does not have solution for \( \lambda > 0 \). Therefore, \( x' = x^* + \lambda d \) does not dominate \( x^* \) in this direction. We conclude that if \( x' \) dominates \( x^* \), then there exists \( d \in \partial B(0, 1) \) where \( X_0 = \emptyset \) and \( X_2 \neq \emptyset \), and more \( \lambda^* \in \Lambda^d \neq \emptyset \) such that \( x' = x^* + \lambda^* d \).
(⇐) Suppose that in the direction \(d\), we obtain \(X_0 = \emptyset\), \(X_2 \neq \emptyset\) and \(\Lambda^d \neq \emptyset\). If \(X_1 = \emptyset\), then there exists \(\lambda^* \in \Lambda^d\) (see Figure 9), such that, if \(x' = x^* + \lambda^* d\) then \(p(x') \leq p(x^*)\) has solution. If \(X_1 \neq \emptyset\), then there are items 4.(b)–(c) grant that exists \(\lambda^* \in \Lambda^d\), such that, if \(x' = x^* + \lambda^* d\) then \(p(x') \leq p(x^*)\) has solution, or \(\lambda_2^d < \lambda_1^d\) and there exists \(\lambda^* \in \Lambda^d = [\lambda_2^d, \lambda_1^d]\), such that, if \(x' = x^* + \lambda^* d\) then \(p(x') \leq p(x^*)\) has solution. Therefore, \(x'\) dominates \(x^*\).

**Theorem 2.** Let \(x^* \in \text{Eff}(\text{VQFP}')\). Then \(x^* \in \text{Eff}(\text{VQFP}')\) if and only if for all \(d \in \partial B(0, 1), X_0 \neq \emptyset\) or \(\Lambda^d = \emptyset\) whenever \(X_1 \neq \emptyset\).

*Proof* \((\Rightarrow)\) Suppose that exists \(d \in \partial B(0, 1)\) where \(X_1 \neq \emptyset\), \(X_0 = \emptyset\) and \(\Lambda^d \neq \emptyset\). As \(x^* \in \text{Eff}(\text{VQFP}')\), from Theorem 1 \(x^* \in \text{Eff}(\text{VQFP}')\) and \(X_2 \neq \emptyset\) in this direction. By Lemma 1 there exists \(\lambda^* \in \Lambda^d\) and \(x' = x^* + \lambda^* d\) such that \(p(x') \leq p(x^*)\) has solution. By (5), \(\frac{f(x)}{g(x')} \leq \frac{f(x^*)}{g(x^*)}\) has solution. What contradicts \(x^* \in \text{Eff}(\text{VQFP}')\).

(⇐) By Theorem 1 \(x^* \in \text{Eff}(\text{VQFP}')\). Let an arbitrary \(d \in \partial B(0, 1)\) and suppose that \(X_0 \neq \emptyset\) in direction \(d\). Then there exists an index \(i \in I\), such that, one of items 1.(a) and 1.(b) or 2.(a) is satisfied, that is, \(d^T \nabla^2 p_i(x^*) d > 0\) and \(\nabla p_i(x^*)^T d \geq 0\), or \(d^T \nabla^2 p_i(x^*) d \geq 0\) and \(\nabla p_i(x^*)^T d > 0\). Hence, \(r_i(\lambda)\) grows indefinitely for \(\lambda > 0\) and given any neighborhood of \(x^*\) in direction \(d\), as large as it is, \(p(x') \leq p(x^*)\) does not have solution for each \(x' = x^* + \lambda d\) in this neighborhood. Therefore, \(x' = x^* + \lambda d\) does not dominates \(x^*\) and \(x^*\) is Pareto-efficient optimal solution in this direction. Figure 8 illustrates this possibility, where the function \(r_1, 1 \in X_0\), grows indefinitely for \(\lambda > 0\). On the other hand, suppose that \(X_1 \neq \emptyset\) and \(X_0 = \emptyset\), but \(\Lambda^d = \emptyset\) in direction \(d\). As \(x^* \in \text{Eff}(\text{VQFP}')\), necessarily \(X_2 \neq \emptyset\) in this direction. We have then \(X_0 = \emptyset\), \(X_1 \neq \emptyset\), \(X_2 \neq \emptyset\), \(\Lambda^d = \emptyset\) and two cases may occur. Either \(\lambda_1^d < \lambda_2^d\) and \(p(x^* + \lambda d) \leq p(x^*)\) does not have solution for each \(\lambda > 0\), or by Condition 1 and item 4.(a), \(\lambda_2^d = \lambda_1^d = \frac{-2
abla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}\), for all \(i \in X_1 \cup X_2\), and \(p(x^* + \lambda d) \leq p(x^*)\) does not have solution for each \(\lambda > 0\). Therefore, \(x' = x^* + \lambda d\) does not dominates \(x^*\) and \(x^*\) is Pareto-efficient optimal solution in this direction. Figure 9 illustrates this possibility, where we observe \(X_0 = \emptyset\), \(X_1 = \{1, 3\}, X_2 = \{2, 4\}, \lambda_1^d < \lambda_2^d, \Lambda^d = \emptyset\) and \(p(x^* + \lambda d) \leq p(x^*)\) does not have solution for each \(\lambda > 0\). Therefore, given an arbitrary direction \(d \in \partial B_2(0, 1)\), if \(X_0 \neq \emptyset\) we obtain \(x^*\) non-dominated, or whenever \(X_1 \neq \emptyset\) and \(\Lambda^d = \emptyset\) we obtain again \(x^*\) non-dominated. Hence, does not exist another point \(x' = x^* + \lambda d\) that dominates \(x^*\), and \(x^* \in \text{Eff}(\text{VQFP}')\).
Given a direction \( d \), in the proof of Theorem 2 when \( X_0 = \emptyset \) and \( \Lambda^d = \emptyset \), we require that \( X_1 \) be nonempty, otherwise the Condition 2 grant us only \( X_2 \neq \emptyset \) (when \( X_0 = \emptyset \) and \( X_1 = \emptyset \)) and we obtain \( \lambda_1^d = \infty \) and \( \lambda_2^d < \lambda_1^d \). When this last case occurs in the unconstrained problem, we always have \( \Lambda^d \neq \emptyset \), that is, there exists \( \lambda \in \Lambda^d \) such that \( x^* + \lambda d \) dominates \( x^* \) in direction \( d \). Figure 9 illustrates this possibility, there we observe that \( X_0 = \emptyset \), \( X_1 = \emptyset \), \( X_2 = \{1, 2, 3, 4\} \) and the interval \( \Lambda^d \) is nonempty. But, this does not always happens in the constrained problem, because the feasible set can be limited in this direction.

Note, unlike the results obtained by Osuna-Gómez et al. [31], that on the objective functions of the auxiliary problem (VQFP) we do not require any kind of generalized convexity to obtain the Pareto optimality conditions of Theorem 2.

In the results that follows we define \( L = \{d \in \partial B(0, 1) \mid \Lambda^d \neq \emptyset\} \).

**Corollary 3.** Let \( x^* \in \text{Leff}(VQFP) \) and \( \beta = \inf_{d \in L} \{\lambda^d_2\} \). Then does not exist another point \( x' \in B(x^*, \beta) \) such that \( \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \).

**Proof** It follows immediately from Theorem 2. If \( d \in L \), then there exists \( \lambda > 0 \) in \( \Lambda^d \) such that another point \( x' \in \mathbb{R}^n \) dominates \( x^* \) in this direction. However, the first point \( x' \) that dominates \( x^* \) along the direction \( d \) is \( x' = x^* + \lambda_2^d d \). By checking all set \( L \), we conclude that the point \( x' \) which dominates \( x^* \) is \( x' = x^* + \beta d \). Therefore \( x^* \) is \( \beta \)-efficient and does not exist another point \( x' \in B(x^*, \beta) \) such that \( \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \). Figure 7 illustrates this possibility, there we observe that \( X_0 = \emptyset \), \( X_1 = \{2, 3\} \), \( X_2 = \{1, 4\} \), \( \Lambda^d \neq \emptyset \) and if \( x' = x^* + \lambda_2^d d \), \( p(x') \leq p(x^*) \) has solution.

Although complicate your proof, it is possible to rewrite Corollary 3 to replace the set \( L \) with the set \( L' = \{d \in \partial B_n(0, 1) \mid X_0 = \emptyset\} \) (see in the charts in Figures 6 and 7). Next, a very useful result which gives a lower bound for the radius of efficiency of the solution \( x^* \in \text{Leff}(VQFP) \) is presented.

**Corollary 4.** Let \( x^* \in \text{Leff}(VQFP) \) and \( F(d) = \max_{i \in I} \{2 \nabla p_i(x^*)^T d\} \). Suppose that exists \( \rho \in \mathbb{R} \), such that for all \( d \in \partial B(0, 1) \) we have \( F(d) \geq \rho \). Then does not exist another point \( x' \in B(x^*, \frac{\rho}{\gamma}) \) such that \( \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \), where \( \gamma < 0 \), \( \gamma = \min_{i \in I} \{\gamma_i\} \) and \( \gamma_i \) is the smallest negative eigenvalue of the matrix \( \nabla^2 p_i(x^*) \), \( i \in I \).
Proof By the proof of theorem 2 and the value of $\beta$ in Corollary 3, if we search a point $x' = x^* + \lambda d$ that dominates $x^*$, we have to find a value $\lambda > 0$ that satisfy

$$\lambda \geq \beta = \inf_{d \in L} \left\{ \max_{i \in X_2} \left\{ -\frac{2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} \right\}.$$ 

Given the symmetric and diagonalizable matrix $\nabla^2 p_i(x^*), i \in X_2$, we obtain $0 > d^T \nabla^2 p_i(x^*) d \geq \gamma_i$, where $\gamma_i$ is the smallest negative eigenvalue of the matrix $\nabla^2 p_i(x^*)$, therefore

$$\lambda \geq \beta = \inf_{d \in L} \left\{ \max_{i \in X_2} \left\{ -\frac{2 \nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} \right\} \geq \inf_{d \in L} \left\{ \max_{i \in X_2} \left\{ -\frac{2 \nabla p_i(x^*)^T d}{\gamma_i} \right\} \right\} = \inf_{d \in L} \frac{F(d)}{-\gamma},$$

where $\gamma = \min_{i \in I} \{\gamma_i\}$. Since $F(d) \geq \rho$, we have $\beta \geq \frac{\rho}{\gamma}$. If $\Lambda^d = \emptyset$ for all $d \in \partial B(0, 1)$, then $x^*$ is $\infty$-efficient, but if there exists $d$ such that $\Lambda^d \neq \emptyset$, then $x^*$ is $\beta$-efficient. As $\infty > \beta \geq \frac{\rho}{\gamma}$, the theorem holds.

Because $i \in X_2$, the passage from the first to the second inequality of the above demonstration is valid, and this also grants that $\rho > 0$. Therefore $\frac{\rho}{\gamma} > 0$.

In order to limit the search space, another very useful result which gives an upper bound for the radius of efficiency of the solution $x^* \in \text{Leff}(VQFP')$ is presented.

Theorem 5. Let $x^* \in \text{Leff}(VQFP')$ and $M = \min_{i \in X} \left\{ \|2 \nabla p_i(x^*)\|_\alpha \right\}$. Suppose that $d^T \nabla^2 p_i(x^*) d \geq \alpha > 0$, for some $d \in L$ and for all $i \in X_1 \neq \emptyset$. If there does not exist another point $x' \in B(x^*, M)$ such that $\frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)}$, then $x^* \in \text{Eff}(VQFP')$.

Proof We deduce from Lemma 1 and Corollary 3 that if there exists a point $x'$ that dominates $x^*$ in direction $d$, then $X_2 \neq \emptyset$ and $\lambda_d^2 \leq \|x' - x^*\| \leq \lambda_d^d$. If $P = \sup_{d \in L} \{\lambda_d^d\}$,
Figure 11: Some interesting neighborhoods of the solution $x^* \in \text{Eff}(VQFP')$

we have $\|x' - x^*\| \leq P$. By the hypothesis, we also obtain

\[
P = \sup_{d \in \mathcal{L}} \left\{ \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} \right\}
\leq \sup_{d \in \mathcal{L}} \left\{ \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{\alpha} \right\} \right\}
\leq \min_{i \in X_1} \left\{ \frac{\|\nabla p_i(x^*)\|}{\alpha} \right\} = M.
\]

Therefore, if there does not exist a point that dominates $x^*$ in $B(x^*, M)$, then does not exist another point in $\mathbb{R}^n$ with this property, and $x^* \in \text{Eff}(VQFP')$.

Because $i \in X_1$, the passage from the first to the second inequality of the above demonstration is valid. Theorem 5 only has usefulness if $M < \infty$.

The problem of determining whether a locally Pareto-efficient optimal solution is dominated can be reduced by comparing it within a limited subset of $\mathbb{R}^n$ using the above results. Note that Collorary 3 provides the maximum radius $\beta = \inf_{d \in \mathcal{L}} \left\{ \lambda_d \right\}$ of the neighborhood $B(x^*, \beta)$ where $x^*$ is not dominated. Collo-
Figure 12: Search space for $x'$ that dominates $x^* \in \text{Left}(VQFP')$

Notations

- $B(x^*, \rho - \gamma)$
- $B(x^*, \beta)$
- $B(x^*, P)$
- $B(x^*, M)$

Theorem 4 provides a lower bound for $\beta$, thus a new neighborhood $B(x^*, \frac{\rho}{\gamma})$ for $x^*$. On the other hand, Theorem 5 provides two neighborhoods for $x^*$, one with radius $P = \sup_{d \in L} \{\lambda_i^d\}$, and another with radius $M = \min_{i \in X_1} \{\frac{\|2\nabla p_i(x^*)\|}{\alpha}\}$ as a upper bound for $P$. Therefore, we obtain four neighborhoods satisfying $B(x^*, \frac{\rho}{\gamma}) \subseteq B(x^*, \beta) \subseteq B(x^*, P) \subseteq B(x^*, M)$.

The lower and upper bounds are more attractive computationally to calculate.

If we have a good computational search method to find a point that dominates the solution $x^*$, it is enough that this search is made in the subset $\bar{B}(x^*, P) \setminus B(x^*, \beta)$, or alternatively in the subset $\bar{B}(x^*, M) \setminus B(x^*, \frac{\rho}{\gamma})$.

In general, the former neighborhoods are spheres in $\mathbb{R}^n$, but a particular case when the domain of the functions is $\mathbb{R}^2$ is illustrated in Figure 11. It shows four spheres centered in solution $x^*$: in dashed lines and nearer to the center appears the radius of neighborhood $B(x^*, \frac{\rho}{\gamma})$; in continuous lines and going little far away from the center appears the radius of neighborhoods $B(x^*, \beta)$ and $B(x^*, P)$, respectively; in shading appears the closed subset $\bar{B}(x^*, P) \setminus B(x^*, \beta)$; finally, in dashed lines appears the radius of the biggest neighborhood $B(x^*, M)$. If there exists a point $x'$ that dominates $x^*$, it must belong to subset $\bar{B}(x^*, P) \setminus B(x^*, \beta)$, as shown in Figure 11.

Auxiliary computational methods can be designed to further reduce the size of
subset $\overline{B}(x^*, P) \setminus B(x^*, \beta)$. It is enough to observe that $x' \in \overline{B}(x^*, P) \setminus B(x^*, \beta)$ if and only if there exists an unitary direction $d$ such that $\lambda_2^d \leq \lambda \leq \lambda_1^d$ and $x' = x^* + \lambda d$. Therefore, all those directions participating in subset $\overline{B}(x^*, P) \setminus B(x^*, \beta)$, but with $\Lambda^d = 0$, can be excluded. Figure 12 illustrates this possibility, where there exists two shaded subsets in $\overline{B}(x^*, P) \setminus B(x^*, \beta)$, each containing a possible point $x'$. The shaded subsets represent only the directions $d$ which occurs $\lambda_2^d \leq \lambda \leq \lambda_1^d$.

4.2. Radius of efficiency in the constrained case

Now, we can extend the results achieved in Section 4.1 for the constrained problem (VQFP) defined by feasible set $S$. Let $\text{diam}(S) = \sup \{ \|x - y\| : x, y \in S\}$. The radius of efficiency $\text{diam}(S)$ is equivalent to the radius of efficiency $\infty$ in the (VQFP'). If $x^*$ is $\lambda$-efficient, then it is $\beta$-efficient, for all $\beta < \lambda \leq \text{diam}(S)$. We say that $x^*$ is $\text{diam}(S)$-efficient if it is globally Pareto-efficient optimal solution in $S$.

In (VOP), many constraint sets can be considered, however as a first contributed we consider this section the problem (VQFP) with the feasible set $S$ defined as $S = \{x \in \mathbb{R}^n | Cx \leq b, C \in \mathbb{R}^{p \times n}, b \in \mathbb{R}^p\}$.

This choice makes it easy to calculate the distance from a point $x^* \in S$ to a boundary $S$, which is indispensable in the calculus of radius of efficiency. However, we believe that the results can extended for the more general constraint sets than linear inequalities.

To apply the concept radius of efficiency, we begin again from a solution $x^* \in \text{Leff}(VQFP)$. The objective is to calculate its radius of efficiency to validate if $x^*$ is globally Pareto-efficient optimal solution and also obtain some information regarding the neighborhoods of $x^*$.

The next two results extend naturally the existing relation between the associated problems (VQFP) and (VQFP) shown in Section 4.1. Now we can admit the existence of a particular neighborhood of fixed radius for the solution $x^*$, where it is not dominated, and we can be calculate this radius using our methodology. The relationship is that the same fixed neighborhood can be used in both problems.

**Theorem 6.** Let $x^* \in \text{Leff}(VQFP)$. $x^*$ is $\lambda$-efficient for (VQFP) if and only if $x^*$ is $\lambda$-efficient for (VQFP)$_{x^*}$.

**Proof** To calculate the radius of efficiency of the solution $x^* \in \text{Leff}(PMFQ)_{x^*}$, we must find, if any exists, the values of $\lambda > 0$ and a feasible unitary direction $d$ such that $x^* + \lambda d \in S$ dominates $x^*$. In other words, following the ideas until now, we
must find a pair \((\tilde{\lambda}, \tilde{d})\) which solves the following inequalities

\[
\frac{f_i(x^* + \lambda d)}{g_i(x^* + \lambda d)} \leq \frac{f_i(x^*)}{g_i(x^*)} \leq \frac{f_i(x^*)}{g_i(x^*)} = 0, \text{ for all } i \in I.
\]

As defined in the Section 4.1, this implies the resolution of the inequalities (6). Hence, if there exists a solution \((\lambda, d)\), we must find a pair \((\tilde{\lambda}, \tilde{d})\) which solves the following inequalities

\[
\frac{f_i(x^* + \lambda d)}{g_i(x^* + \lambda d)} \leq \frac{f_i(x^*)}{g_i(x^*)} \iff \lambda \left(\nabla p_i(x^*)^T d + \frac{\lambda}{2} d^T \nabla^2 p_i(x^*)d \right) \leq 0, \text{ for all } i \in I.
\]

This implies again the resolution of the inequalities (6). Hence, if there exists a pair \((\tilde{\lambda}, \tilde{d})\) for the solution \(x^* \in \text{Eff}(\text{PMFQ})\), and a pair \((\bar{\lambda}, \bar{d})\) for the same solution \(x^* \in \text{Eff}(\text{PMFQ})\), they are obtained from the same set of inequalities (6). Therefore, \((\tilde{\lambda}, \tilde{d}) = (\bar{\lambda}, \bar{d})\) and \(x^*\) is \(\lambda\)-efficient for both problems, \(\lambda = \bar{\lambda} = \tilde{\lambda}\).

**Corollary 7.** \(x^* \in \text{Eff}(\text{PMFQ})\) if and only if \(x^* \in \text{Eff}(\text{PMFQ})_{x^*}\).

**Proof** It is sufficient to replace the pair \((\lambda, d)\) to the pair \((\infty, d)\) in Theorem 6.

We represent the active constraints set in \(x^* \in S\) as \(K(x^*) = \{ j \in J \mid C_j x^* = b_j \}\), where \(C_j\) is the \(j\)-th row of the matrix \(C\). We also represent the tangent cone to \(S\) in the point \(x^* \in S\) as \(T(x^*) = \{ y \in \mathbb{R}^n \mid C_j y \leq 0, \forall j \in K(x^*), \|y\| = 1 \}\). We say that \(T(x^*) = T(x^*) \cap \partial B(0,1)\) is the feasible directions set in the point \(x^*\).

Next we extracted some results as in the unconstrained case, however by their similarities, some demonstrations are omitted.

Given \(x^* \in S\), for each \(d \in T(x^*)\) such that \(x^* + \lambda^d d \in S\), we say that the real number \(\lambda^d_x \in (0, +\infty)\) is the limiting for the radius of efficiency of \(x^*\). And analogously to the Section 4.1 we define

\[
\lambda^d_x = \min_{j \in J, C_j d > 0} \left\{ \frac{b_j - C_j x^*}{C_j d} \right\},
\]

\[
\lambda^d_x = \max_{i \in X_2} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*)d} \right\}, \quad \lambda^d_x = \begin{cases} \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*)d} \right\}, \lambda^d_x, & \text{if } X_1 \neq \emptyset \\ \inf \{ \infty, \lambda^d_x \}, & \text{if } X_1 = \emptyset. \end{cases}
\]
\( \Lambda_2^d = [\lambda_2^d, \infty), \quad \Lambda_1^d = \begin{cases} (0, \lambda_1^d], & \text{if } X_1 \neq \emptyset \\ (0, \lambda_1^d), & \text{if } X_1 = \emptyset, \end{cases} \quad \Lambda^d = \Lambda_2^d \cap \Lambda_1^d, \)

where \( p_i, i \in I, X_0, X_1 \) and \( X_2 \) are the same functions and the same sets defined in Section 4.1.

Let \( x^* \) be locally Pareto-efficient optimal solution and \( d \in T(x^*) \). Recall that if \( X_0 = \emptyset \) and \( X_1 \neq \emptyset \), then \( X_2 = \emptyset \) cannot occurs, however, if \( X_0 \neq \emptyset \) then \( x^* \) is Pareto-efficient optimal solution in this direction. Also, if \( X_0 = \emptyset \), then either \( X_1 \neq \emptyset \) and \( X_2 \neq \emptyset \), or only \( X_2 \neq \emptyset \). Some of these possibilities are illustrated in Figures 13, 14, 15 and 16 including now \( \lambda_i^d \) as possible limiting for the radius of efficiency in direction \( d \). For each \( i \in \{1, 2, 3, 4\} \), the functions

\[
  r_i(\lambda) = p_i(x^*) + \lambda \nabla p_i(x^*)^T d + \frac{\lambda^2}{2} d^T \nabla^2 p_i(x^*) d,
\]

are plotted in the coordinates \((\lambda, r_i(\lambda))\), \( \lambda > 0 \), whose graphics are parabolas and we can verify examples of the sets \( X_0, X_1, X_2 \) and \( \Lambda^d \). Figure 13 illustrates an example of direction where \( X_1 = \{1, 3\} \) is formed by the indices of the functions \( r_1 \) and \( r_3 \), \( X_2 = \{2, 4\} \) is formed by the indices of the functions \( r_2 \) and \( r_4 \), and \( \Lambda^d = \emptyset \). In Figures 13, 14 and 15 are shown the cases in which \( X_0 = \emptyset \) and \( X_1 \neq \emptyset \),
Figure 14: A search direction by $x'$ that dominates $x^* \in \text{Leff}(VQFP)$, $X_0 = \emptyset$, $X_1 = \{2, 3\}$, $X_2 = \{1, 4\}$, $\lambda^d_4 < \lambda^d_2$, $\lambda^d_1 \leftarrow \lambda^d_2$, and $\Lambda^d = \emptyset$

therefore $X_2 \neq \emptyset$. However, Figure 16 illustrates the case in which $X_1 = \emptyset$ and only $X_2 \neq \emptyset$.

An important change is made in relation the parameter $\lambda^d_1$. Now, we have to include the possibility of $\lambda^d_1$ become $\lambda^d_2$. In the unconstrained problems we defined $\lambda^d_1 = \min_{i \in X_1} \left\{-\frac{2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}\right\}$ and, when $X_1 \neq \emptyset$, the interest was in the directions $d$ such that $\lambda^d_2 \leq \lambda^d_1$. We continue interested in those directions, however observe that if occur $\lambda^d_1 < \lambda^d_2$ does not make sense choose $\lambda$, $\lambda^d_1 < \lambda^d_2 \leq \lambda \leq \lambda^d_2$ such that $x' = x^* + \lambda d$ dominates $x^*$, because $x' \notin S$. Therefore, as was defined above, the correct is let the new $\lambda^d_1 := \min\{\lambda^d_1, \lambda^d_2\}$ when $X_1 \neq \emptyset$, and $\lambda^d_1 := \inf\{+\infty, \lambda^d_1\}$ when $X_1 = \emptyset$.

The role of the limiting $\lambda^d_1$ in the calculation of the radius of efficiency of $x^* \in \text{Leff}(VQFP)$ can be observed in the Figures 13, 14, 15 and 16. $\lambda^d_1$ is represented by a continuous vertical line that represents, in the position that it crosses the $x$-axis of the cartesian plane and in the direction $d$, the boundary of the set $S$. Figure 13 illustrates the case in which $\lambda^d_1 < \lambda^d_2 \leq \lambda^d_4$, and therefore we have $\Lambda^d = \emptyset$. Figure 15 illustrates the case in which $\lambda^d_2 \leq \lambda^d_4 < \lambda^d_1$, and therefore we have $\Lambda^d \neq \emptyset$.

On the other hand, Figure 14 illustrates the case in which $\lambda^d_1$ become $\lambda^d_2$, that is $\lambda^d_1 := \lambda^d_2 < \lambda^d_4$, and therefore $\Lambda^d = \emptyset$. Another possible case is illustrated in Figure 16, in which the sets $X_0$ and $X_1$ are empty and we have only $X_2 \neq \emptyset$. Therefore $\lambda^d_1$ become $\lambda^d_2$, that is $\lambda^d_1 := \lambda^d_2 < \lambda^d_2$, and we obtain $\Lambda^d = \emptyset$. 

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Similar situations are possible, however among them we can exemplify two in which there exists a feasible point $x'$ that dominates $x^*$ in the direction $d \in T(x^*)$. First is shown in Figure 15 where there exists $\lambda > 0, \lambda_d^2 \leq \lambda \leq \lambda_d^1 < \lambda_d^\ell$ such that $x^* + \lambda d$ dominates $x^*$, the interval $\Lambda^d$ is nonempty and $p(x') \leq p(x^*)$ has solution.

Second can be observed in Figure 16, if we move $\lambda_d^\ell$ towards the right of the $\lambda$-axis until that $\lambda_d^2 \leq \lambda_d^\ell$, then $\lambda_d^1$ become $\lambda_d^\ell$, that is we would have $\lambda_d^2 \leq \lambda_d^1 := \lambda_d^\ell$ and obtain $\Lambda^d \neq \emptyset$.

**Theorem 8.** Let $x^* \in \text{Leff}(VQFP)$. Then $x^* \in \text{Eff}(VQFP)$ if and only if for all $d \in T(x^*), X_0 \neq \emptyset$ or $\Lambda^d = \emptyset$.

**Proof (⇒)** Identical to the shown in Theorem 2 considering now the directions in $T(x^*)$ and the fact of $S$ be limited or not in those directions.

**Proof (⇐)** By Theorem 6, $x^* \in \text{Leff}(VQFP)$. Let an arbitrary $d \in T(x^*)$ and suppose that $X_0 \neq \emptyset$ in direction $d$. Then there exists an index $i \in I$ such that one of items 1.(a) and 1.(b) or 2.(a) is satisfied, that is, $d^T \nabla^2 p_i(x^*)d \geq 0$ and $\nabla p_i(x^*)^Td \geq 0$, or $d^T \nabla^2 p_i(x^*)d \geq 0$ and $\nabla p_i(x^*)^Td > 0$. Hence, $r_i(\lambda)$ grows indefinitely for $\lambda > 0$ and given any neighborhood of $x^*$ in direction $d$, as large as it is, $p(x) \leq p(x^*)$ does not have solution for each $x = x^* + \lambda d$ in this neighborhood. Therefore, $x' = x^* + \lambda d$ does not dominates $x^*$ and $x^*$ is Pareto-efficient optimal solution in this direction. Figure 16 illustrates this possibility, where the function $r_1, 1 \in X_0$, grows indefinitely for $\lambda > 0$. On the other hand, suppose that $X_0 = \emptyset$ and
For each $\lambda$, in the second case, Figure 14 shows $X_\lambda$ as defined in Theorem 2. In fact, either $\lambda^*_x$ exists or another feasible point $x^*$ does not exist. Suppose $x^*$ is Pareto-efficient optimal solution in this direction. Figures 13 and 14 illustrate this possibility, which can be divided in two cases. In the first case, Figure 13 shows $\lambda^*_d = \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} < \lambda^*_d$ and $\lambda^*_d < \lambda^*_2$, however $\Lambda^d = \emptyset$ means that $\lambda^*_d < \lambda^*_2$ and inequalities (7), defined in Section 4.1, are not satisfied for $\lambda > 0$. Hence, does not exist another feasible point $x' = x^* + \lambda d$ that dominates $x^*$ in the direction $d$. In the second case, Figure 14 shows $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and $\lambda^*_d \leq \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\}$.

$\Lambda^d = \emptyset$ in the direction $d$, then we can divide in two cases regarding the size of $X_1$. First, suppose also that $X_1 = \emptyset$, then the Condition 2 grants that only $X_2 \neq \emptyset$. Since $\Lambda^d = \emptyset$, $X_2 \neq \emptyset$, and, as $\lambda^*_d := \lambda^*_d < \lambda^*_2$, we return to the unconstrained case as in Theorem 2. In fact, either $\lambda^*_d < \lambda^*_2$ and $p(x^* + \lambda d) \leq p(x^*)$ does not have solution for each $\lambda > 0$, or by Condition 2 and item 4(a), $\lambda^*_d = \lambda^*_d = \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d}$, for all $i \in X_1 \cup X_2$, and $p(x^* + \lambda d) \leq p(x^*)$ does not have solution for each $\lambda > 0$. Therefore, does not exist another point $x' = x^* + \lambda d \in S$ that dominates $x^*$ and $x^*$ is Pareto-efficient optimal solution in this direction. Figures 13 and 14 illustrate this possibility, which can be divided in two cases. In the first case, Figure 13 shows $\lambda^*_d = \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\} < \lambda^*_d$ and $\lambda^*_d \leq \lambda^*_2$, however $\Lambda^d = \emptyset$ means that $\lambda^*_d < \lambda^*_2$ and inequalities (7), defined in Section 4.1, are not satisfied for $\lambda > 0$. Hence, does not exist another feasible point $x' = x^* + \lambda d$ that dominates $x^*$ in the direction $d$. In the second case, Figure 14 shows $X_1 \neq \emptyset$, $X_2 \neq \emptyset$ and $\lambda^*_d \leq \min_{i \in X_1} \left\{ \frac{-2\nabla p_i(x^*)^T d}{d^T \nabla^2 p_i(x^*) d} \right\}$,
However \( \Lambda^d = \emptyset \) means that there exists \( j \in X_2 \) such that \( \lambda^d_j \leq \frac{-2 \nabla p_j(x) T d}{\nabla^2 p_j(x) d} \) and \( \lambda^d_i < \lambda^d_j \). Hence, for all \( \lambda \) such that \( \lambda^d_2 \leq \lambda \leq \min_{i \in X_1} \left\{ \frac{-2 \nabla p_i(x) T d}{\nabla^2 p_i(x) d} \right\} \), any other point \( x' = x^* + \lambda d \notin S \), and does not exist another feasible point that dominates \( x^* \) in this direction.

Therefore, given an arbitrary direction \( d \in T(x^*) \), if \( X_0 \neq \emptyset \) we obtain \( x^* \) non-dominated, or if \( \Lambda^d = \emptyset \) we obtain again \( x^* \) non-dominated. Hence, does not exist another point \( x' = x^* + \lambda d \in S \) that dominates \( x^* \), and \( x^* \in \text{Eff}(\text{VQFP}) \).

It is important to observe that if \( X_0 = \emptyset, X_1 = \emptyset \) and only \( X_2 \neq \emptyset \) in a direction \( d \in T(x^*) \) it is possible to obtain \( \Lambda^d = \emptyset \). However, in the unconstrained case always \( \Lambda^d \neq \emptyset \) (see Figure 9). Figure 16 illustrates the constrained case, where we have \( \lambda^d_i < \lambda^d_i \), in which \( \lambda^d_1 \) become \( \lambda^d_i \), that is \( \lambda^d_1 = \lambda^d_i = \lambda^d_i \), and we obtain \( \Lambda^d = \emptyset \). As we define \( \lambda^d_i = \inf \{ +\infty, \lambda^d_i \} \), if for all \( j \in X_2 \), \( \frac{-2 \nabla p_i(x) T d}{\nabla^2 p_i(x) d} \leq \lambda^d_i \), we return to the unconstrained case, because we would have \( \lambda^d_1 \leq \lambda^d_i \) and \( \Lambda^d \neq \emptyset \). That is, there exists \( \lambda \in \Lambda^d \) such that \( x^* + \lambda d \in S \) dominates \( x^* \) (see Figure 16 and move \( \lambda^d_i \) towards the right of the \( \lambda \)-axis until \( \lambda^d_2 \leq \lambda^d_i \)).

The next results are extensions of the Corollaries 3, 4 and Theorem 5, so their proofs are equivalent and can be omitted. To simplify the presentation we define \( \hat{L} = \{ d \in T(x^*) \mid \Lambda^d \neq \emptyset \} \).

**Corollary 9.** Let \( x^* \in \text{Eff}(\text{VQFP}) \) and \( \beta = \inf_{d \in \hat{L}} \{ \lambda^d_2 \} \). Then does not exist another point \( x' \in B(x^*, \beta) \cap S \) such that \( \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \).

**Corollary 10.** Let \( x^* \in \text{Eff}(\text{VQFP}) \) and \( F(d) = \max_{i \in X_2} \{ 2 \nabla p_i(x) T d \} \). Suppose that exists \( \rho \in \mathbb{R} \) such that for all \( d \in T(x^*) \) we have \( F(d) \geq \rho \). Then does not exist another point \( x' \in B(x^*, \frac{\rho}{\gamma}) \cap S \) such that \( \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \), where \( \gamma < 0 \), \( \gamma = \min_{i \in I} \{ \gamma_i \} \) and \( \gamma_i \) is the smallest negative eigenvalue of the matrix \( \nabla^2 p_i(x^*) \), \( i \in I \).

**Theorem 11.** Let \( x^* \in \text{Eff}(\text{VQFP}) \) and \( M = \min \left\{ \min_{i \in X_1} \frac{\|2 \nabla p_i(x^*)\|}{\alpha}, \text{diam}(S) \right\} \). Suppose that \( d^T \nabla^2 p_i(x^*) d \geq \alpha > 0 \), for some \( d \in \hat{L} \) and for all \( i \in X_1 \neq \emptyset \). If there does not exist another point \( x' \in B(x^*, M) \cap S \) such that \( \frac{f(x')}{g(x')} \leq \frac{f(x^*)}{g(x^*)} \), then \( x^* \in \text{Eff}(\text{VQFP}) \).
Similarly to the shown in Section 4.1 we can identify in the Corollaries 9, 10 and Theorem 11 four important subsets related to solution \( x^* \in \text{Leff}(PMFQ) \) satisfying 
\[
(B(x^*, \rho - \gamma) \cap S) \subseteq (B(x^*, \beta) \cap S) \subseteq (B(x^*, P) \cap S) \subseteq (B(x^*, M) \cap S),
\]
where
\[
P = \sup_{d \in L} \{ \lambda_d \}.
\]
If we have a good computational search method to find a point \( x' \) that dominates the solution \( x^* \), it is enough that this search is made in the subset 
\[
(\bar{B}(x^*, P) \cap S) \setminus (B(x^*, \beta) \cap S),
\]
or alternatively in the subset 
\[
(\bar{B}(x^*, M) \cap S) \setminus (B(x^*, \rho - \gamma) \cap S).
\]
This way, we can present the following corollary.

**Corollary 12.** Let \( x^* \in \text{Leff}(PMFQ) \). Suppose that the hypothesis of Corollaries 9, 10 and Theorem 11 are satisfied, if there exists \( x' \) that dominates \( x^* \) in problem (VQFP), then \( x' \in (\bar{B}(x^*, P) \cap S) \setminus (B(x^*, \beta) \cap S) \).

Figure 17 explains the Corollary 12, it is shown a solution \( x^* \in \text{Leff}(PMFQ) \) in the boundary of the \( S \subseteq \mathbb{R}^2 \). In dashed lines, two subsets of interest are shown: 
\( B(x^*, \rho - \gamma) \cap S \) and \( B(x^*, M) \cap S \). And in continuous lines, another two are shown: 
\( B(x^*, \beta) \cap S \) and \( B(x^*, P) \cap S \). If there exists a point \( x' \) that dominates \( x^* \), it must belong to subset 
\( (\bar{B}(x^*, P) \cap S) \setminus (B(x^*, \beta) \cap S) \).
5. Conclusions

The main contribution of this work is the development of necessary and sufficient Pareto optimality conditions for the solutions of a particular vector optimization problem, where each objective function consists of a ratio of two quadratic functions and the feasible set is defined by linear inequalities. We introduce the new concept of radius of efficiency in order to identify the neighborhoods of a locally Pareto-efficient optimal solution in vector optimization problems. We show how to calculate the radius of the very useful two spherical regions centered in this solution, and if there exists another point that dominates the former solution, it belongs to the subtraction of those spherical regions. In this process we may conclude that the solution is also globally optimal. Theorems are established for it. These results might be useful to determine termination criteria in the development of algorithms, and new extensions can be established from these to more general vector optimization problems in which quadratic approximations are used locally. In future work we plan to develop algorithms using the concept presented here.

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