ON THE LOCAL ISOMORPHISM PROPERTY FOR FAMILIES OF K3 SURFACES

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1. Introduction

Let \( f: X \rightarrow S \) and \( f': X' \rightarrow S \) be two families of compact complex manifolds over a reduced complex space \( S \). Following Meersseman [3, p. 496] we say that these families are pointwise isomorphic when for all points \( s \in S \) the fibers \( X_s = f^{-1}(s) \) and \( X'_s = f'^{-1}(s) \) of the families are biholomorphic. When \( U \) is an open complex subspace of \( S \), we write \( f_U: X_U \rightarrow U \) for the induced holomorphic map where \( X_U \subseteq X \) denotes the inverse image of \( U \subseteq S \) under \( f \); we adopt the analogous notation for \( f' \). Given a point \( o \in S \) we say that the family \( f' \) is locally isomorphic at \( o \) when there exist an open complex subspace \( T \subseteq S \) and a biholomorphism \( g: X_T \rightarrow X'_T \) such that \( o \in T \) and \( f_T = f'_T \circ g \). We say that the family \( f \) has the local isomorphism property at \( o \) when all families that are pointwise isomorphic to \( f \) are locally isomorphic at \( o \) to \( f \).

In 1977, assuming \( S \) nonsingular, Wehler states [4, p. 77] that it is unclear whether \( f \) has the local isomorphism property at all points of \( S \) if the function
\[
s \mapsto h^0(X_s; \Theta_{X_s}) := \dim \mathbb{C}H^0(X_s; \Theta_{X_s}) = \dim \text{Aut}(X_s)
\]
is constant on \( S \). Meersseman [3, Theorem 3] asserts that Wehler’s question becomes a valid criterion. We contend that the opposite is true.

Theorem 1. There exist two families of K3 surfaces over a complex manifold \( S \) together with a point \( o \in S \) such that the families are pointwise isomorphic but not locally isomorphic at \( o \).

Recall [1, §1.3.3] that when for all \( s \in S \) the fiber \( X_s \) of \( f \) is a K3 surface, the assignment of eq. (1) defines the identically zero function.

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2. Construction of the families

We start with a K3 surface \( F \) and a \((-2)\)-class \( d \) on \( F \); that is, \( d \in H^{1,1}(F) \subseteq H^2(F; \mathbb{C}) \) is an integral cohomology class with \( \langle d, d \rangle = -2 \). The angle brackets denote the topological intersection form on \( H^2(F; \mathbb{C}) \), which is given by the cup product and the evaluation at the homology class that determines the orientation of \( F \). Explicitly \( F \) could be the Fermat quartic in \( \mathbb{P}^3 \) and \( d \) the class of a projective line that is contained in \( F \).

Observe that the group \( H^2(F; \Theta_F) \) is trivial [1, §6.2.3]. Hence by the theorem of Kodaira, Nirenberg, and Spencer [2, p. 452] there exist a family of compact complex

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manifolds \( f : X \to S \) over a complex manifold \( S \) and a point \( o \in S \) such that the Kodaira--Spencer map
\[
\rho_o : T_o S \to H^1(X_o ; \Theta_{X_o})
\]
is an isomorphism and \( F \cong X_o \). In fact the quoted theorem yields more—namely, that \( S \) is, setting \( m := h^1(F ; \Theta_F) \), an open ball in \( \mathbb{C}^m \) and that topologically \( f \) is nothing but the Cartesian projection \( \operatorname{pr}_1 : S \times F \to S \). In particular for all \( s \in S \) the identity map on \( F \) induces a vector space isomorphism
\[
\mu_s : H^2(X_s ; \mathbb{C}) \to H^2(X_o ; \mathbb{C}) =: V
\]
which restricts to an isomorphism between the integral cohomology groups. Therefore
\[
\mu_s[H^2,0(X_s)] = s, \quad \forall s \in S.
\]
By virtue of the canonical identification \( F \cong \{o\} \times F = X_o \) we regard \( d \) as an element of \( V \) and consider the reflection map
\[
\phi : V \to V, \quad \phi(v) = v + (v, d).
\]
The map \( \phi \) is an involutary linear isometry of \( V \) with respect to the intersection form. Hence we obtain an involutary biholomorphism \( h : Q \to Q \) satisfying \( h(s) = \phi[s] \) for all \( s \in Q \). Since
\[
o = \mu_o[H^2,0(X_o)] = H^2,0(X_o) \perp H^{1,1}(X_o) \ni d,
\]
we see that \( o \) is a fixed point of \( h \). Thus \( h^{-1}(S) \subseteq Q \) is an open subspace with \( o \in h^{-1}(S) \). Replacing \( S \) by \( S \cap h^{-1}(S) \), we may assume that the map \( h \) restricts to a biholomorphism \( i := h|_S : S \to S \). We define
\[
f' := i \circ f : X' := X \to S.
\]
It is evident that \( f' \) is then, just like \( f \), a family of K3 surfaces over the complex manifold \( S \).

3. Verification of the isomorphism properties

In the first place, we show that \( f \) and \( f' \) are pointwise isomorphic. For that matter let \( t \in S \) be an arbitrary point. The definition of \( f' \) implies that \( X'_t = X_{i^{-1}(t)} \).

Setting
\[
\psi := \mu^{-1}_t \circ \phi \circ \mu_{i^{-1}(t)} : H^2(X_{i^{-1}(t)} ; \mathbb{C}) \to H^2(X_t ; \mathbb{C})
\]
and applying eq. (2) we see that
\[
\psi[H^2,0(X_{i^{-1}(t)})] = (\mu^{-1}_t \circ \phi)[i^{-1}(t)] = \mu^{-1}_t[i] = H^2,0(X_t).
\]
In addition, \( \phi \) and whence \( \psi \) is an isometry with respect to the intersection forms and restricts to an isomorphism between the integral cohomology groups. Therefore \( X_t \) and \( X'_t \) are biholomorphic by virtue of the (weak form of the) global Torelli theorem for K3 surfaces [1, Theorem 7.5.3].

In the second place, we show that \( f' \) is not locally isomorphic at \( o \) to \( f \).
Lemma 1. Let \( \alpha: V \to V \) be a linear automorphism and \( U \subseteq Q \) be a nonempty open subset such that \( \alpha[t] = t \) for all \( t \in U \). Then \( \alpha = \lambda \text{id}_V \) for a nonzero complex number \( \lambda \).

Proof. We will use that the rank of the intersection form on \( V \) is \( \geq 3 \) so that \( Q \) is an irreducible quadric in \( \mathbb{P}(V) \). Indeed, of course, we know that \( n := \dim_{\mathbb{C}} V = 22 \) and that the intersection form is nondegenerate [1, §1.3.3].

Assume that \( W \subseteq V \) is a linear subspace and \( U \subseteq \mathbb{P}(W) \). Then, as \( Q \) is irreducible and \( U \subseteq Q \) is nonempty and open, the identity theorem for holomorphic functions entails that \( Q \subseteq \mathbb{P}(W) \). Since a quadric of rank \( \geq 2 \) is not contained in a linear hyperplane, we conclude that \( W = V \). This means that the affine cone over \( U \) spans the vector space \( V \). Hence there exists a basis \((v_l, \ldots, v_n)\) of \( V \) such that \( \mathbb{C}v_k \subseteq U \) for all \( k \in \{1, \ldots, n\} \). For every \( l \in \{1, \ldots, n\} \) define

\[
W_l := \text{span}(\{v_k \mid k \neq l\}) \subseteq V.
\]

Then \( U \setminus \mathbb{P}(W_l) \) is nonempty and open in \( Q \). Repeating this argument inductively, we conclude that \( U \setminus \bigcup_{l=1}^{n} \mathbb{P}(W_l) \) is nonempty and open in \( Q \). In particular there exists a vector \( v = \mu_1 v_1 + \cdots + \mu_n v_n \) in \( V \) such that \( \mathbb{C}v \subseteq U \) and \( \mu_l \neq 0 \) for all \( l \).

Since \( \alpha[t] = t \) for all \( t \in U \), there exist nonzero complex numbers \( \lambda_1, \ldots, \lambda_n \) and \( \lambda \) such that \( \alpha(v_k) = \lambda_k v_k \) for all \( k \) and \( \alpha(v) = \lambda v \). Therefore

\[
(\mu_1 \lambda_1)v_1 + \cdots + (\mu_n \lambda_n)v_n = \mu_1 \alpha(v_1) + \cdots + \mu_n \alpha(v_n)
= \alpha(v)
= \lambda v
= (\lambda_1 \mu_1)v_1 + \cdots + (\lambda_\mu_n)v_n.
\]

Comparing coefficients we see that \( \mu_k \lambda_k = \lambda \mu_k \), whence \( \lambda_k = \lambda \), for all \( k \). This proves that \( \alpha = \lambda \text{id}_V \). \( \square \)

Now assume \( f' \) is locally isomorphic at \( o \) to \( f \). By Section 2 we dispose of an open subspace \( T \subseteq S \) and a biholomorphism \( g: X_T \to X'_{T'} \) such that \( o \in T \) and \( f_T = f'_T \circ g \). Fix a point \( t \in T \). Then \( g \) induces a biholomorphism \( g_t: X_t \to X'_{t'} \). Observing that \( X'_{X_t} = X_{t'}(T) \), we obtain a commutative diagram of linear maps:

\[
\begin{array}{ccc}
H^2(X_{t'}(T); \mathbb{C}) & \xrightarrow{g^*} & H^2(X_T; \mathbb{C}) \\
\text{rest.} & & \text{rest.}
\end{array}
\]

\[
\begin{array}{ccc}
H^2(X_{t'}(t); \mathbb{C}) & \xrightarrow{g_t^*} & H^2(X_t; \mathbb{C})
\end{array}
\]

Without loss of generality we may assume that \( T \) is biholomorphic to a ball in \( \mathbb{C}^m \). This makes the vertical arrows in the diagram isomorphisms. Employing the commutativity of the diagram twice, once for \( t \) and once for \( t = o \), we deduce that

\[
g_o^* \circ \mu_{t^{-1}(t)} = \mu_t \circ g_t^*.
\]

Invoking eq. (2), we furthermore deduce that

\[
(g_o^* \circ \phi^{-1})[t] = g_o^*[\phi^{-1}(t)]
= (g_o^* \circ \mu_{t^{-1}(t)})[H^2(0)(X_{t^-1(t))}]
= (\mu_t \circ g_t^*)[H^2(0)(X_{t^{-1}(t))}]
= \mu_t[H^2(0)(X_t)] = t.
\]

Hence Lemma 1 implies that \( g_o^* \circ \phi^{-1} = \lambda \text{id}_V \) for a complex number \( \lambda \neq 0 \). Seeing that \( g_o^* \circ \phi^{-1} \) restricts to an automorphism of \( H^2(X_o; \mathbb{Z}) \), we infer that \( \lambda \) is either 1 or \(-1\). Both alternatives lead to a contradiction.
Consider the real cone
\[ C := \{ v \in H^{1,1}(X_0) \mid \pi = v, \langle v, v \rangle > 0 \} \subseteq H^{1,1}(X_0) \]
and notice that both \( g_\pi \) and \( \phi \) map \( C \) homeomorphically onto itself. Notice moreover that \( C \) has precisely two connected components; one of these components, say \( C_+ \), contains all Kähler classes of \( X_0 \) while the other component is just \(-C_+\) \cite{1}, §8.5.1. Since \( g_\pi \) takes Kähler classes to Kähler classes, \( g_\pi \) preserves the connected components of \( C \). For all \( w \in C \) we see that the line segment joining \( w \) and \( \phi(w) \) is contained in \( C \). Thus \( \phi \) preserves the connected components of \( C \), too, while \(-\phi\) swaps the components. So if \( g_\pi = -\phi \), we obtain a contradiction. If on the other hand \( g_\pi = \phi \), then \( g_\pi(d) = -d \). As mentioned at the outset of section 2, we can assume that \( d \) is the class of a smooth rational curve on \( X_0 \). Then, however, for every Kähler class \( c \) on \( X_0 \),
\[ 0 < \langle c, d \rangle = \langle g_\pi(c), g_\pi(d) \rangle = -\langle g_\pi(c), d \rangle < 0, \]
which completes our proof of \( \text{Theorem 1} \).

4. Closing remarks

In our discussions with Schwald we have realized that an alternative, yet effectively related, proof strategy for \( \text{Theorem 1} \) uses the existence of the flop of a \((-2)\)-curve in a complex threefold. Using this idea we can construct pointwise but not everywhere locally isomorphic families of K3 surfaces over a 1-dimensional complex manifold \( S \), whereas the complex manifold \( S \) in section 2 is of dimension 20. We refrain from going into the details.

As regards Meersseman’s work it seems worthwhile to investigate whether and to what extend a weakened form of his result \cite{3}, Theorem 3 remains true—for instance, assuming that eq. (1) defines a constant map, does the family of compact complex manifolds \( f \) have the local isomorphism property at the (very) general point of \( S \)?

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