ARITHMETIC OF CHARACTER VARIETIES OF FREE GROUPS

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ABSTRACT. Via counting over finite fields, we derive explicit formulas for the \( E \)-polynomials and Euler characteristics of \( \text{GL}_d \) and \( \text{PGL}_d \)-character varieties of free groups. We prove a positivity property for these polynomials and relate them to the number of subgroups of finite index.

1. Introduction and statement of the results

Given a finitely generated group \( \Gamma \) and a complex reductive algebraic group \( G \), one can consider the character variety

\[
X_\Gamma(G) = \text{Hom}(\Gamma, G)/G = \text{Spec}(\mathbb{C}[\text{Hom}(\Gamma, G)])^{\text{G}}
\]

the spectrum of the (finitely generated) ring of \( G \)-invariant functions on the space of representations of \( \Gamma \) in \( G \). The study of these varieties, in particular for \( \Gamma \) the fundamental group of a manifold, is a common theme in geometry and topology. Arithmetic aspects of character varieties have been studied very fruitfully e.g. in [7].

From now on let \( \Gamma = F_m \) be the free group in \( m \geq 0 \) generators (i.e. the fundamental group of an \((m+1)\)-punctured sphere). The character varieties \( X_\Gamma(G) \) for the groups \( G = \text{GL}_d(\mathbb{C}) \), \( G = \text{PGL}_d(\mathbb{C}) \), and \( G = \text{SL}_d(\mathbb{C}) \) were studied e.g. in [4, 5], and from an arithmetic point of view in [3]. We summarize some basic geometric properties of these representation varieties. For \( m \geq 2 \), the variety \( X_\Gamma(\text{GL}_d(\mathbb{C})) \) is an irreducible affine variety of dimension \((m-1)d^2 + 1\). It is usually a singular variety; its smooth locus typically reduces to \( X_\Gamma^{\text{irr}}(\text{GL}_d(\mathbb{C})) \), the subset corresponding to irreducible representations.

Basic to the arithmetic study of character varieties is the following approach: there exists a \( \mathbb{Z} \)-model \( X_d \) of \( X_\Gamma(\text{GL}_d(\mathbb{C})) \) using Seshadri’s GIT over base rings \( \mathbb{Z} \). Namely, consider the scheme

\[
X_d = \text{Spec}(\mathbb{Z}[\text{Hom}(\Gamma, \text{GL}_d(\mathbb{Z}))]^{\text{GL}_d(\mathbb{Z})})
\]

over \text{Spec}(\mathbb{Z}). Then

\[
\text{Spec}(\mathbb{C}) \times_{\text{Spec}(\mathbb{Z})} X_d \simeq X_\Gamma(\text{GL}_d(\mathbb{C})).
\]

We find a similar open subscheme \( X_d^{\text{irr}} \subset X_d \) of irreducible representations (over some open subscheme of \text{Spec}(\mathbb{Z})).

We can thus reduce to finite fields and consider the counting functions

\[
A_d(q) = |X_d(\mathbb{F}_q)|, \quad A_d^{\text{irr}}(q) = |X_d^{\text{irr}}(\mathbb{F}_q)|
\]

defined on prime powers \( q \). By definition, \( A_d(q) \) (resp. \( A_d^{\text{irr}}(q) \)) equals the number of isomorphism classes of completely reducible representations (resp. of absolutely irreducible representations, that is, representations that stay irreducible under any...
finite field extension) of the group $F_m$ of dimension $d$ over $F_q$. Our first main result is the following:

**Theorem 1.1.** The varieties $X_\Gamma(\text{GL}_d(\mathbb{C}))$ and $X_\Gamma^{\text{irr}}(\text{GL}_d(\mathbb{C}))$ are polynomial count \[\text{[7, Appendix]}, that is, the functions $A_d(q)$ and $A_d^{\text{irr}}(q)$ are polynomials with integer coefficients in $q$. Consequently, the $E$-polynomials of these varieties are

$$E(X_\Gamma(\text{GL}_d(\mathbb{C})); u, v) = A_d(uv), \quad E(X_\Gamma^{\text{irr}}(\text{GL}_d(\mathbb{C})); u, v) = A_d^{\text{irr}}(uv).$$

Similar formulas for the group $\text{PGL}_d(\mathbb{C})$ are proved in Corollary \[\text{[2, 4]}. The above theorem immediately follows from explicit formulas for $A_d(q)$ and $A_d^{\text{irr}}$ which we now formulate. Consider the formal power series ring $\mathbb{Q}(q)[[t]]$ with the maximal ideal $m$. We define the $\mathbb{Q}(q)$-linear shift operator $T$ on $\mathbb{Q}(q)[[t]]$ by

$$T(t^d) = q^{(1-m)(\frac{d}{2})}t^d.$$  

We define the plethystic exponential $\text{Exp} : m \to 1 + m$ by

$$\text{Exp}(q^d t^d) = (1 - q^d t^d)^{-1} \quad \text{and} \quad \text{Exp}(f + g) = \text{Exp}(f) \text{Exp}(g).$$

It admits an inverse $\text{Log} : 1 + m \to m$. Define the power operator $\text{Pow}$ on $\mathbb{Q}(q)[[t]]$ by $\text{Pow} : (1 + m) \times \mathbb{Q}(q)[[t]] \to 1 + m$, $\text{Pow}(f, g) = \text{Exp}(g \text{Log}(f))$.

Finally, define the series

$$F(t) = \sum_{d \geq 0} ((q - 1) \ldots (q^d - 1))^{m-1} t^d \in \mathbb{Q}(q)[[t]].$$

**Theorem 1.2.** In $\mathbb{Q}(q)[[t]]$, we have

$$\sum_{d \geq 0} A_d(q)t^d = \text{Pow}(T^{-1}F(t)^{-1}, 1 - q),$$

$$\sum_{d \geq 1} A_d^{\text{irr}}(q)t^d = (1 - q) \text{Log}(T^{-1}F(t)^{-1}).$$

Using combinatorial methods, we prove a quite surprising positivity property of the counting polynomials $A_d(q)$:

**Theorem 1.3.** We have $A_d(q) \in \mathbb{N}[q - 1]$.

Contrarily, $A_d^{\text{irr}}(q)$ does not fulfill this property. It is tempting to interpret this result geometrically, for example as a prediction for a paving of $X_\Gamma(\text{GL}_d(\mathbb{C}))$ by tori (or at least quotients by finite groups of tori); but such a paving can not be expected to be very natural since it cannot be compatible with $X_\Gamma^{\text{irr}}(\text{GL}_d(\mathbb{C}))$. It would be interesting to test this prediction for the rather explicitly known $\text{GL}_2(\mathbb{C})$-character varieties. We can determine explicitly the lowest order Taylor coefficient of $A_d(q)$ around $q = 1$ to derive:

**Theorem 1.4.** For $m \geq 2$, the topological Euler characteristic of the varieties $X_\Gamma(\text{PGL}_d(\mathbb{C}))$ and $X_\Gamma^{\text{irr}}(\text{PGL}_d(\mathbb{C}))$ are given by

$$\chi(X_\Gamma(\text{PGL}_d(\mathbb{C}))) = \varphi(d)d^{m-2},$$

$$\chi(X_\Gamma^{\text{irr}}(\text{PGL}_d(\mathbb{C}))) = \mu(d)d^{m-2},$$

where $\varphi$ and $\mu$ denote the Euler totient function, resp. the M"obius function.
Again, it would be interesting to have a geometrical explanation for these Euler characteristics. We can also count absolutely indecomposable representations (that is, representations that stay indecomposable under any finite field extension) of $F_m$ over finite fields:

**Theorem 1.5.** For any $d \geq 1$, there exists a polynomial $A^\text{ind}_d(q) \in \mathbb{Z}[q]$ that counts isomorphism classes of absolutely indecomposable representations of $F_m$ of dimension $d$ over $\mathbb{F}_q$ for every prime power $q$. These polynomials satisfy

$$
\sum_{d \geq 1} A^\text{ind}_d(q)t^d = (q - 1) \log \left( \sum_{\lambda \in \mathcal{P}} r^m_{\lambda} t^{\lambda} \right),
$$

where the sum ranges over all partitions and, for any partition $\lambda$,

$$
r_{\lambda} = \prod_{n \geq 1} q^{\lambda_n^2} (q^{-1})^{\lambda_n - \lambda_{n+1}}, \quad (q)_n = (1 - q) \cdots (1 - q^n).
$$

In contrast to [3], our approach to the arithmetic of the character varieties is non-geometric and purely formal. It is based on the Hall algebra approach to the arithmetic of moduli spaces of representations of quivers developed in [14]. In fact, we can directly adopt the methods there to derive the explicit formula for $A_d(q)$. The other results follow by a detailed study of the right hand side of this formula and explicit combinatorics. One can expect this approach to even work in a motivic Hall algebra [1], leading to a formula similar to Theorem 1.2 for the motives of character varieties.

The paper is organized as follows: in Section 2 we recall the Hall algebra methods of [14] to prove Theorems 1.1 and 1.2. In Section 3, we derive Theorem 1.4 using an elementary argument. Combinatorial notions are introduced in Section 4 to derive Theorem 1.3. In Section 5, we derive a formula for the number of absolutely indecomposable representations of $F_m$ over finite fields. In Section 6, we explain how the numbers of subgroups of fixed index in free groups [6] can be reconstructed from the counting polynomials, providing a potential link of the present study with the topic of subgroup growth [10].

2. Computation of the counting polynomial

Let $\Gamma = F_m$ be the free group in $m$ generators and let $k$ be a field. A representation of the group algebra $k\Gamma$ can be identified with a representation of the free associative algebra $A$ with $m$ generators such that all generators act bijectively on the representation. Therefore the category of $k\Gamma$-representations can be identified with an exact subcategory of the abelian category of $A$-representations. Since the algebra $A$ is a path algebra of the quiver with one vertex and $m$ loops, the methods of [14] for the explicit calculation of the number of isomorphism classes of absolutely irreducible representations of quivers continue to work for $\Gamma$-representations. We recall the main steps of this calculation and refer to [14] for the proofs which hold without any changes.

In the following, let $k$ be a finite field with $q$ elements. We first define the Hall algebra $H(k\Gamma)$ of the group algebra $k\Gamma$. As a (complete, $\mathbb{Z}_{\geq 0}$-graded) $\mathbb{Q}$-vector space, we define

$$
H(k\Gamma) = \prod_{[V]} \mathbb{Q} \cdot [V],
$$

where $[V]$ denotes the conjugacy class of the subgroup $V$.
where the direct product ranges over all isomorphism classes of (finite-dimensional) representations \( V \) of \( k\Gamma \). We have a natural grading by the dimension \( \dim V \). We define a product on \( H((k\Gamma)) \) by

\[
[V] \cdot [W] = \sum_{[X]} g^X_{V,W}[X],
\]

where \( g^X_{V,W} \) equals the number of subrepresentations \( U \subset X \) such that \( U \) is isomorphic to \( W \) and \( X/U \) is isomorphic to \( V \). Then we have \(^{14} 3.3\):

**Lemma 2.1.** This product endows \( H((k\Gamma)) \) with a structure of a \( \mathbb{Z}_{\geq 0} \)-graded complete local associative unital \( \mathbb{Q} \)-algebra. In particular, every element with constant term 1 (the class of the zero-dimensional representation) is invertible in \( H((k\Gamma)) \).

We define an evaluation map

\[
I : H((k\Gamma)) \to \mathbb{Q}[t], \quad [V] \mapsto \frac{1}{|\text{Aut}(V)|} t^{\dim V}.
\]

By \(^{14} \text{Lemma 3.4}\), we have:

**Lemma 2.2.** The composition \( T \circ I : H((k\Gamma)) \to \mathbb{Q}[t] \) is a \( \mathbb{Q} \)-algebra homomorphism, where the operator \( T \) is defined by \(^1\).

We consider the (invertible) element

\[
e = \sum_{[V]} [V] \in H((k\Gamma)).
\]

Using \( \text{Hom}(\Gamma, \text{GL}_{d}(k)) \simeq \text{GL}_{d}(k)^{m} \) and

\[
|\text{GL}_{d}(k)| = \prod_{i=0}^{d-1} (q^{d} - q^{i}) = q^{\binom{d}{2}} \prod_{i=1}^{d} (q^{i} - 1),
\]

the following is easily verified:

**Lemma 2.3.** We have

\[
I(e) = T^{-1} F(t), \quad I(e^{-1}) = T^{-1} F(t)^{-1},
\]

where \( F(t) \) is defined by \(^3\).

**Proof.** By the definitions, we have

\[
I(e) = \sum_{d \geq 0} \frac{|\text{GL}_{d}(k)^{m}|}{|\text{GL}_{d}(k)|} t^{d} = \sum_{d \geq 0} \left( q^{\binom{d}{2}} \prod_{i=1}^{d} (q^{i} - 1) \right)^{m-1} t^{d} = T^{-1} F(t).
\]

This implies \( TI(e) = F(t) \) and therefore \( TI(e^{-1}) = F(t)^{-1} \) by Lemma \(^2\). Therefore \( I(e^{-1}) = T^{-1} F(t)^{-1} \).

The key lemma towards counting absolutely irreducible representations is \(^{14} \text{Lemma 3.5}\):

**Lemma 2.4.** Writing

\[
e^{-1} = \sum_{[V]} \gamma_{V} [V]
\]

in \( H((k\Gamma)) \), we have the following description of the coefficients \( \gamma_{V} \)

1. If \( V \) is not completely reducible, then \( \gamma_{V} = 0 \).
(2) If $V = \bigoplus_S S^{m_S}$ is completely reducible (the direct sum ranging over all isomorphism classes of irreducible representations $S$ of $k\Gamma$), then

$$\gamma_V = \prod_{|S|} (-1)^{m_S} \text{End}(S)^{\binom{m_S}{2}}.$$ 

Based on this lemma, one can prove (see [14, Theorem 4.2]) that:

**Theorem 2.5.** The function $A_{d}^{\text{irr}}(q)$ is a polynomial in $q$ and we have in $\mathbb{Q}[t]$ 

$$I(e^{-1}) = \text{Exp} \left( \frac{1}{1-q} \sum_{d \geq 1} A_{d}^{\text{irr}}(q)t^d \right).$$

**Proof of Theorems 2.2 and 2.3.** We obtain from Theorem 2.5 and Lemma 2.3 that $A_{d}^{\text{irr}}(q)$ is a polynomial in $q$ satisfying in $\mathbb{Q}[q][t]$ 

$$\sum_{d \geq 1} A_{d}^{\text{irr}}(q)t^d = (1-q) \text{Log} \left( T^{-1}F(t)^{-1} \right).$$

This implies that $A_{d}^{\text{irr}}(q)$ has integer coefficients. Using [12, Lemma 5], one can prove that $A_{d}(q)$ is a polynomial in $q$ satisfying in $\mathbb{Q}[q][t]$ 

$$\sum_{d \geq 0} A_{d}(q)t^d = \text{Exp} \left( \sum_{d \geq 1} A_{d}^{\text{irr}}(q)t^d \right).$$

Therefore 

$$\sum_{d \geq 0} A_{d}(q)t^d = \text{Exp} \left( (1-q) \text{Log} \left( T^{-1}F(t)^{-1} \right) \right) = \text{Pow} \left( T^{-1}F(t)^{-1}, 1-q \right)$$

and $A_{d}(q)$ has integer coefficients. This finishes the proof of Theorem 1.2. To prove Theorem 1.2 we note that by [7, Appendix], the $E$-polynomials are given by the counting polynomials evaluated at $q = uv$. \qed

Now we pass to the $\text{PGL}_d(\mathbb{C})$-character varieties. There is a free action of the torus $(\mathbb{C}^*)^m$ on $X_\Gamma(\text{GL}_d(\mathbb{C}))$. The quotient fibration $X_\Gamma(\text{GL}_d(\mathbb{C})) \to X_\Gamma(\text{PGL}_d(\mathbb{C}))$ has fibres isomorphic to $(\mathbb{C}^*)^m$ and is Zariski locally trivial by Hilbert 90. Similarly for the open subsets corresponding to irreducible representations.

Consequently, we see that the $E$-polynomials of the $\text{PGL}_d(\mathbb{C})$-character varieties are known:

**Corollary 2.6.** We have 

$$E(X_\Gamma(\text{PGL}_d(\mathbb{C})); u, v) = \frac{1}{(uv-1)^m} A_{d}(uv),$$

$$E(X_\Gamma(\text{PGL}_d(\mathbb{C}))^{\text{irr}}; u, v) = \frac{1}{(uv-1)^m} A_{d}^{\text{irr}}(uv).$$

We conclude this section with an example. First, we have 

$$A_{1}^{\text{irr}}(q) = A_{1}(q) = (q-1)^m.$$ 

As the first nontrivial special case of Theorem 1.2 we have $A_{2}^{\text{irr}}(q) = (q-1)^m \left( q^{m-1}(q-1)^{m-1}((q+1)^{m-1} - 1) - \frac{1}{2}(q+1)^{m-1} + \frac{1}{2}(q-1)^{m-1} \right)$.
This implies

\[ A_2(q) = A_2^{\text{rt}}(q) + \frac{1}{2} \left( A_1^{\text{rt}}(q^2) + A_1^{\text{rt}}(q)^2 \right) = \]

\[ (q-1)^m \left( q^{m-1}(q-1)^{m-1} \left( (q+1)^{m-1} - 1 \right) + \frac{1}{2} q \left( (q+1)^{m-1} + (q-1)^{m-1} \right) \right) \]

and thus

\[ E(X_1(\text{PGL}_2(\mathbb{C})); u, v) = \frac{1}{(uv-1)^m} A_2(uv) \]

\[ = (uv)^{m-1}(uv - 1)^{m-1} \left( (uv + 1)^{m-1} - 1 \right) + \frac{1}{2} uv \left( (uv + 1)^{m-1} + (uv - 1)^{m-1} \right), \]

which should be compared with [3, Theorem B].

3. Euler characteristic

In this section, we prove Theorem [124]. We first need a lemma on the behaviour of specialization at \( q = 1 \) for special plethystic exponentials and logarithms. Let \( A \) be the subring of \( \mathbb{Q}(q) \) consisting of all rational functions without pole at \( q = 1 \).

**Lemma 3.1.** Assume that

\[ \log \left( 1 + (q-1)^m \sum_{n \geq 1} a_n(q) t^n \right) = (q-1)^m \sum_{n \geq 1} b_n(q) t^n \in \mathbb{Q}(q)[[t]]. \]

Then \( a_n(q) \in A \) for all \( n \geq 1 \) if and only if \( b_n(q) \in A \) for all \( n \geq 1 \). Moreover,

\[ b_n(1) = \sum_{d \mid n} a_{n/d}(1) \mu(d) d^{m-1}, \quad a_n(1) = \sum_{d \mid n} b_{n/d}(1) d^{m-1}. \]

**Proof.** We recall a more direct definition of the operators \( \text{Exp} \) and \( \text{Log} \) on formal power series: for \( n \geq 1 \), define the \( \mathbb{Q} \)-linear Adams operator \( \psi_n \) on \( R = \mathbb{Q}(q)[[t]] \) by \( \psi_n(q^t t^j) = q^{nj} t^n j! \), and define \( \Psi = \sum_{n \geq 1} \frac{1}{n} \psi_n \) with inverse \( \Psi^{-1} = \sum_{n \geq 1} \frac{\mu(n)}{n} \psi_n \).

Then \( \text{Exp} = \exp \circ \Psi \) and \( \text{Log} = \Psi^{-1} \circ \log \).

Using this description, we immediately derive the following formula:

\[ \log \left( 1 + \sum_{n \geq 1} r_n(q) t^n \right) = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\mu(i)}{i} \sum_{j=c_1+\cdots+c_l} \frac{(-1)^{l-1}}{l} \prod_{k=1}^{l} r_{c_k}(q^j t^n). \]

Now if \( r_n(q) = (q-1)^m a_n(q) \), then

\[ \sum_{n \geq 1} b_n(q) t^n = (q-1)^{-m} \log \left( 1 + (q-1)^m \sum_{n \geq 1} a_n(q) t^n \right) \]

\[ = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{\mu(i)}{i} \sum_{j=c_1+\cdots+c_l} \frac{(-1)^{l-1}}{l} \frac{(q^i - 1)^{ml}}{(q-1)^m} \prod_{k=1}^{l} a_{c_k}(q^j t^n). \]

We see that the summand corresponding to the decomposition \( j = c_1 + \cdots + c_l \) has a zero of order at least \( m(l-1) \) at \( q = 1 \). Specializing this formula at \( q = 1 \),
we see that only terms with \( l = 1 \) (and thus \( c_1 = j \)) contribute. Thus the formula simplifies to

\[
\sum_{ij=n} a_j(1) \mu(i) t^m = \sum_{ij=n} a_j(1) \mu(i) t^m - 1,
\]

and the first claim follows.

Similarly, we prove the second claim. Using the above description of \( \text{Exp} \) we derive the following formula:

\[
\text{Exp} \left( \sum_{n \geq 1} r_n(q) t^n \right) = 1 + \sum_{n=1}^{\infty} \sum_{n=c_1 + \ldots + c_l} \frac{1}{l!} \sum_{k=1}^{l} \prod_{i=1}^{k} \frac{1}{i_k} r_{ij}^{(q_{i_k})} (q_{i_k} - 1)^{m-1} (q-1)^m t^n.
\]

Now if \( r_n(q) = (q-1)^m b_n(q) \), this formula yields

\[
\sum_{n \geq 1} a_n(q) t^n = \frac{1}{(q-1)^m} \left( \text{Exp} \left( (q-1)^m \sum_{n \geq 1} b_n(q) t^n \right) - 1 \right) = \sum_{n=1}^{\infty} \sum_{n=c_1 + \ldots + c_l} \frac{1}{l!} \sum_{k=1}^{l} \prod_{i=1}^{k} \frac{1}{i_k} r_{ij}^{(q_{i_k})} (q_{i_k} - 1)^{m-1} (q-1)^m t^n.
\]

As soon as \( l \geq 2 \) in a summand on the right hand side, this summand specializes to zero at \( q = 1 \). Thus, after this specialization, we only have to consider summands with \( l = 1 \), and thus \( c_1 = n \), which reads

\[
a_n(1) = \sum_{ij=n} \frac{b_j(1)}{i} \frac{(q^j - 1)^m}{(q-1)^m} \bigg|_{q=1} = \sum_{ij=n} b_j(1) i^{m-1},
\]

as claimed. \( \square \)

We can now prove Theorem \[1.3\].

**Proof.** We can write the series \( T^{-1} F(t)^{-1} \) of the previous sections in the form

\[
T^{-1} F(t)^{-1} = 1 + \sum_{d \geq 1} (q-1)^d (m-1) a_d(q) t^d
\]

for some \( a_d(q) \in \mathbb{Q}[q] \) with \( a_1(q) = -1 \). By Theorem \[1.2\] we have

\[
\frac{1}{(q-1)^m} \sum_{d \geq 1} A_d^{irr}(q) t^d = - \frac{1}{(q-1)^{m-1}} \log \left( 1 + \sum_{d \geq 1} (q-1)^d (m-1) a_d(q) t^d \right)
\]

and applying Lemma \[3.4\] we obtain

\[
\frac{1}{(q-1)^m} A_d^{irr}(q) \bigg|_{q=1} = -a_1(1) \mu(d) d^{m-2} = \mu(d) d^{m-2},
\]

which, together with Corollary \[2.3\] proves the first part of the theorem. Setting \( b_d(q) = (q-1)^{-m} A_d^{irr}(q) \) we can write, by Theorem \[1.2\]

\[
\sum_{d \geq 1} A_d(q) t^d = \text{Exp} \left( (q-1)^m \sum_{d \geq 1} b_d(q) t^d \right)
\]
and applying Lemma 3.1 again, we derive

\[
\frac{1}{(q - 1)^m} A_d(q) \bigg|_{q=1} = \sum_{ij=d} b_j(1)i^{m-1} = \sum_{ij=d} \mu(j)j^{m-2}i^{m-1} = \sum_{ij=d} \mu(j)i \cdot d^{m-2} = \varphi(d)d^{m-2},
\]

proving the second part of the theorem, again by Corollary 2.6.

4. Positivity

The goal of this section is to prove Theorem 1.3, that is, that the polynomials \( A_d(q) \) determined in the previous sections satisfy

\[
A_d \in \mathbb{N}[s], \quad s = q - 1.
\]

By Theorem 1.2 we have

\[
\sum_{d \geq 0} A_d t^d = \text{Pow}\left(T^{-1} F(t)^{-1}, 1 - q\right),
\]

where the series \( F(t) \), defined in (3), can be written in the form

\[
F(t) = \sum_{d \geq 0} [d]_q^l(m-1)((q - 1)^{m-1} t)^d,
\]

with \([d]_q^l = \prod_{i=1}^d \frac{q^i - 1}{q - 1}\). We will prove positivity in several steps.

4.1. Positivity of the inverse. Let

\[
F(t)^{-1} = 1 - \sum_{n \geq 1} a_n(q)t^n.
\]

We claim that \( a_n \in \mathbb{N}[s] \), where \( s = q - 1 \). We will prove actually a stronger result.

Let \( S_n \) be the group of permutations of \([n] = \{1, \ldots, n\}\). For any \( \sigma \in S_n \), let \( l(\sigma) \) denote its length. It can be described as a number of inversions

\[
l(\sigma) = |\{i < j \mid \sigma(i) > \sigma(j)\}|.
\]

Let \( G_n = S_n^{m-1} \) and, for any \( \sigma = (\sigma_1, \ldots, \sigma_{m-1}) \in G_n \), let

\[
l(\sigma) = l(\sigma_1) + \cdots + l(\sigma_{m-1}).
\]

For any \( n \geq 1 \), let \( P_n \subset G_n \) be the set of connected elements, that is, elements \( \sigma = (\sigma_1, \ldots, \sigma_{m-1}) \) such that there is no subinterval \([k]\) for \( k < n \), fixed by all \( \sigma_i \).

Theorem 4.1. Let

\[
\left(\sum_{n \geq 0} [n]_q^l(m-1)t^n\right)^{-1} = 1 - \sum_{n \geq 1} a_n t^n.
\]

Then

\[
a_n = \sum_{\sigma \in P_n} q^{l(\sigma)} \in \mathbb{N}[q].
\]
Proof. It is known [11, Ch. III, Eq. 1.3.viii] that
\[ \sum_{\sigma \in S_n} q^{l(\sigma)} = [n]_q! \]
This implies that
\[ \sum_{n \geq 0} [n]_q! t^n = \sum_{n \geq 0} \sum_{\sigma \in G_n} q^{l(\sigma)} t^n. \]
The theorem will be proved if we will show that
\[ \left( \sum_{n \geq 0} \sum_{\sigma \in G_n} q^{l(\sigma)} t^n \right) \left( 1 - \sum_{n \geq 1} \sum_{\sigma \in P_n} q^{l(\sigma)} t^n \right) = 1 \]
or equivalently
\[ \sum_{\sigma \in G_n} q^{l(\sigma)} = \sum_{k+l=n, l \geq 1} \sum_{\sigma \in G_k} \sum_{\tau \in P_l} q^{l(\sigma) + l(\tau)}. \]
Any element in \( G_n \) can be uniquely written in the form \((\sigma, \tau) \in G_k \times P_l\) where \( k+l = n, k \geq 0, \) and \( l \geq 1. \) It is clear that the length of \((\sigma, \tau)\) is equal to \( l(\sigma) + l(\tau)\) and the theorem follows. \( \square \)

Applying this theorem to \([10]\), we see that the polynomials \( a_n \) determined by \([17]\) are contained in \( \mathbb{N}[s] \). The same is then true if we substitute \( F(t)^{-1} \) by \( T^{-1}F(t)^{-1} \).

4.2. Positivity of the power. The goal of this section is to prove the following result

**Theorem 4.2.** Let \( f = 1 - \sum_{n \geq 1} a_n(q)t^n \in \mathbb{Q}[q][t], \) where \( a_n \in \mathbb{Q}_{\geq 0}[q-1]. \) Then
\[ \text{Pow}(f, 1 - q) \in \mathbb{Q}_{\geq 0}[q-1][t]. \]

Applying this result to \( f = T^{-1}F(t)^{-1} \), we prove \([11]\) and therefore prove Theorem \([13]\). In order to prove the above theorem we will use the formula for \( \text{Pow}(f, 1 - q) \) proved in \([12]\) Lemma 22]. Let \( \Phi_d(q) \) be the number of irreducible monic polynomials of degree \( d \) over \( \mathbb{F}_q \) with a nonzero constant coefficient. Then
\[ \psi_n(q - 1) = q^n - 1 = \sum_{d|n} d\Phi_d(q), \]
\[ \Phi_n(q) = \frac{1}{n} \sum_{d|n} \mu(n/d)(q^d - 1) = \frac{1}{n} \sum_{d|n} \mu(n/d)q^d - \delta_{n1}. \]
It is proved in \([12]\) Lemma 22] that
\[ (8) \quad \text{Pow}(f, 1 - q) = \prod_{d \geq 1} \psi_d(f)^{-\Phi_d}, \]
where on the right we use \( f^g = \text{pow}(f, g) = \exp(g \log(f)) \). We will show that for \( f \) as in the theorem, each multiple on the right is in \( \mathbb{Q}_{\geq 0}[s][t] \), where \( s = q - 1, \) and therefore \( \text{Pow}(f, 1 - q) \in \mathbb{Q}_{\geq 0}[s][t] \) as required.

**Remark 4.3.** Note that \( \psi_n(s) = (s + 1)^n - 1 \in \mathbb{N}[s] \). Therefore, if \( f \in \mathbb{Q}_{\geq 0}[s], \) then \( \psi_n(f) \in \mathbb{Q}_{\geq 0}[s] \).
Remark 4.4. Computer tests show that if \( f \in \mathbb{Q}_{\geq 0}[s][t] \), then \( \text{Pow}(f, q - 1) \in \mathbb{Q}_{\geq 0}[s][t] \). This is slightly different from our statement. Our strategy of the proof will not work in this case as the multiples on the right of the form \( \text{pow}(f, s^k) \) are not in \( \mathbb{Q}_{\geq 0}[s][t] \) in general.

Lemma 4.5. For any \( n \geq 1 \), we have
\[
n\Phi_n(q) \in \mathbb{N}[q - 1].
\]
Proof. We will use the idea from [2]. For \( n \geq 2 \), we have
\[
n\Phi_n(s + 1) = \sum_{d \mid n} \mu(d)(s + 1)^{n/d} = \sum_{k \geq 0} \sum_{d \mid n} \mu(d) \binom{n/d}{k} s^k.
\]
For any \( d \mid n \), we have
\[
\binom{n/d}{k} = |\{1 \leq a_1 < \cdots < a_k \leq n/d\}|
\]
\[
= |\{1 \leq a_1 < \cdots < a_k \leq n \mid d \mid \gcd(a_1, \ldots, a_k)\}|.
\]
Therefore
\[
\sum_{d \mid n} \mu(d) \binom{n/d}{k} = \sum_{d \mid n} \mu(d) \cdot |\{1 \leq a_1 < \cdots < a_k \leq n \mid d \mid \gcd(a_1, \ldots, a_k, n)\}|
\]
\[
= |\{1 \leq a_1 < \cdots < a_k \leq n \mid \gcd(a_1, \ldots, a_k, n) = 1\}|.
\]
Indeed, for any tuple \( 1 \leq a_1 < \cdots < a_k \leq n \) with \( \gcd(a_1, \ldots, a_k) = m \), its contribution to the second sum is
\[
\sum_{d \mid \gcd(m, n)} \mu(d) = \delta_{\gcd(m, n), 1}.
\]
\[\square\]

Lemma 4.6. Let
\[
f = 1 - \sum_{n \geq 1} a_n(q)t^n \in \mathbb{Q}(q)[t],
\]
where \( a_n \in \mathbb{Q}_{\geq 0}[s] \) and let \( g \in \mathbb{Q}_{\geq 0}[s] \). Then
\[
\text{pow}(f, -g) \in \mathbb{Q}_{\geq 0}[s][t].
\]
Proof. It is enough to show that \( \text{pow}(1 - t, -g) \in \mathbb{Q}_{\geq 0}[s][t] \). But
\[
\text{pow}(1 - t, -g) = \exp(-g \log(1 - t)) = \exp \left( g \sum_{n \geq 1} \frac{t^n}{n} \right) \in \mathbb{Q}_{\geq 0}[s][t].
\]
\[\square\]

Applying this lemma to \( \psi_d(f)^{-\Phi_d} \) for \( d \geq 1 \) and using Formula (8), we prove Theorem 4.2.
5. Counting indecomposable representations

Let $k$ be a finite field. For any $d \geq 1$, let $G_d = \text{GL}_d(k)$ and $R_d = \text{Hom}(\Gamma_m, G_d)$. Then $G_d$ acts on $R_d$ by conjugation. Using the Kac-Stanley-Hua approach \cite{8, 12}, we will prove a formula for the number of $G_d$-orbits in $R_d$.

**Theorem 5.1.** We have

$$
\sum_{d \geq 0} |R_d/G_d| t^d = \text{Pow} \left( \sum_\lambda r_\lambda(q^m - 1) \right),
$$

where the sum ranges over all partitions and, for any partition $\lambda$,

$$r_\lambda(q) = \prod_{n \geq 1} q^{\lambda_n}(q^{-1})^{\lambda_n - \lambda_{n+1}}, \quad (q)_n = (1 - q) \cdots (1 - q^n).
$$

**Proof.** Using the Burnside formula, we can write

$$
|R_d/G_d| = \sum_{|g| \in G_d/\sim} |R_d^g|/|G_d^g| = \sum_{g \in G_d/\sim} |G_d^g|^{m-1},
$$

where the sum ranges over the conjugacy classes, $R_d^g$ is the set of $g$-invariant elements in $R_d$, and $G_d^g$ is the centralizer of $g$.

Let us describe the set of conjugacy classes of $G_d$. Let $\Phi$ be the set of all monic irreducible polynomials in a variable $t$ over $k$ with a nonzero constant coefficient. The set of invertible matrices are parametrized by pairs $(n, f)$, where $n \geq 1$ and $f \in \Phi$. The Jordan block $J(n, f)$ corresponds to the action of $x$ on $k[x]/(f^n)$ and has size $n \cdot \deg f$. Conjugacy classes in $G_d$ are parametrized by maps $\varphi : \mathbb{N}_{>0} \times \Phi \to \mathbb{N}$ such that $\sum_{f \in \Phi} \deg f \sum_{n \geq 1} n \varphi(n, f) = d$. Equivalently, we can consider $\varphi$ as a map $\Phi \to \mathcal{P}$, where $\mathcal{P} = \text{Map}_0(\mathbb{N}_{>0}, \mathbb{N})$ is the set of maps with finite support. The set $\mathcal{P}$ can be identified with the set of all partitions, where to any $m \in \mathcal{P}$ we associate the usual partition $\lambda$ with $\lambda_n = \sum_{i \geq n} m_i$. Note that $|\lambda| = \sum_{n \geq 1} \lambda_n = \sum_{n \geq 1} n m_n$.

The conjugacy class corresponding to $\varphi : \Phi \to \mathcal{P}$ is given by the action of $x$ on $V_\varphi = \bigoplus_{f \in \Phi, n \geq 1} (k[x]/(f^n))^{\varphi(n, f)}$.

Its centralizer has order \cite[Theorem 2.1]{13}.

$$|\text{Aut}(V_\varphi)| = |\text{End}(V_\varphi)| \prod_{f \in \Phi} \prod_{n \geq 1} (q^{-\deg f})^{\varphi(n, f)},$$

where $q = |k|$.

$$\dim \text{End}(V_\varphi) = \sum_{f \in \Phi} \deg f \sum_{k, l \geq 1} \min\{k, l\} \varphi(k, f) \varphi(l, f) = \sum_{f \in \Phi} \deg f \sum_{n \geq 1} \lambda(f)_n^2$$

and $\lambda(f)$ is the partition corresponding to $\varphi(-, f) \in \mathcal{P}$. This implies

$$|\text{Aut}(V_\varphi)| = \prod_{f \in \Phi} \prod_{n \geq 1} q^{\deg f \lambda(f)_n^2} (q^{-\deg f})^{\varphi(n, f)}$$

$$= \prod_{f \in \Phi} \psi_{\deg f} \left( \prod_{n \geq 1} q^{\lambda(f)_n^2} (q^{-1})^{\lambda(f)_n - \lambda(f)_{n+1}} \right) = \prod_{f \in \Phi} \psi_{\deg f} \left( r_\lambda(f)(q) \right),$$
where $\psi_n$ is the Adams operation. Therefore, applying \([2]\), we obtain

$$|R_d/G_d| = \sum_{\lambda: \Phi \to \mathcal{P}} \prod_{f \in \Phi} \psi_{\deg f} \left( r_{\lambda(f)}(q)^{m-1} \right).$$

This implies

$$\sum_{d \geq 1} |R_d/G_d| t^d = \sum_{\lambda: \Phi \to \mathcal{P}} \left( \prod_{f \in \Phi} \psi_{\deg f} \left( r_{\lambda(f)}(q)^{m-1} \right) \right) t^{\sum_{f \in \Phi} \deg f |\lambda(f)|}$$

$$= \sum_{\lambda: \Phi \to \mathcal{P}} \prod_{f \in \Phi} \psi_{\deg f} \left( r_{\lambda(f)}(q)^{m-1} t^{|\lambda(f)|} \right) = \prod_{\lambda \in \mathcal{P}} \psi_{\deg f} \left( \sum_{\lambda \in \mathcal{P}} r_{\lambda}(q)^{m-1} t^{|\lambda|} \right)$$

$$= \prod_{d \geq 1} \psi_d \left( \sum_{\lambda \in \mathcal{P}} r_{\lambda}(q)^{m-1} t^{|\lambda|} \right)^{\Phi_d(q)} = \text{Pow} \left( \sum_{\lambda \in \mathcal{P}} r_{\lambda}(q)^{m-1} t^{|\lambda|}, q-1 \right).$$

For any $d \geq 1$ let $R^\text{ind}_d \subset R_d$ denote the set of absolutely indecomposable representations. This subset is invariant under the action of $G_d$ and the quotient can be identified with the set of isomorphism classes of $d$-dimensional absolutely indecomposable representations.

**Corollary 5.2.** We have

$$\sum_{d \geq 0} |R^\text{ind}_d/G_d| t^d = (q-1) \Log \left( \sum_{\lambda \in \mathcal{P}} r_{\lambda}(q)^{m-1} t^{|\lambda|} \right)_{q=|k|},$$

where the sum ranges over all partitions.

**Proof.** For any finite field $\mathbb{F}_q$, let

$$M_d(q) = |R_d(\mathbb{F}_q)/G_d(\mathbb{F}_q)|, \quad A_d^\text{ind}(q) = \left| R^\text{ind}_d(\mathbb{F}_q)/G_d(\mathbb{F}_q) \right|.$$ 

We know from the previous theorem that $M_d(q)$ is a polynomial in $q$. One can prove \([12] \text{ Lemma 5} \) that $A_d^\text{ind}(q)$ is also a polynomial in $q$ and

$$\sum_{d \geq 0} M_d(q) t^d = \text{Exp} \left( \sum_{d \geq 1} A_d^\text{ind}(q) t^d \right).$$

Applying the previous theorem, we obtain

$$\sum_{d \geq 1} A_d^\text{ind}(q) t^d =$$

$$= \Log \text{Pow} \left( \sum_{\lambda \in \mathcal{P}} r_{\lambda}(q)^{m-1} t^{|\lambda|}, q-1 \right) = (q-1) \Log \left( \sum_{\lambda \in \mathcal{P}} r_{\lambda}(q)^{m-1} t^{|\lambda|} \right).$$

$\square$
6. Counting subgroups

The aim of this section is to relate the counting polynomials $A^d_{\text{irr}}(q)$ with the number of index $d$ subgroups of $\Gamma = F_m$. Such a relation is motivated by ideas of $\mathbb{F}_1$-geometry [9]: viewing the symmetric group $S_n$ as $\text{GL}_n$ over a hypothetical field with one element, one can expect counts of permutation representations of a group, or, equivalently, (conjugacy classes of) subgroups of finite index, to arise from the counts of representations over fields with $q$ elements by an appropriate limit process $q \to 1$. This is accomplished in Lemma 6.4.

6.1. Subgroups and permutation representations. Let $G$ be a finitely generated group. A (permutation) representation of $G$ of order $n$ is an action of $G$ on $[n] = \{1, \ldots, n\}$. It is called irreducible if the action of $G$ is transitive. One can define the notion of an isomorphism between two representations in a natural way. Any irreducible order $n$ representation can be written in the form $G/H$, where $H$ is an index $n$ subgroup of $G$. Two irreducible representations $G/H$ and $G/H'$ are isomorphic if and only if $H$ and $H'$ are conjugate. Moreover
\[
\text{Aut}(G/H) \simeq N_G H/H.
\]

Let $J_n$ denote the set of index $n$ subgroups and $I_n$ denote the set of conjugacy classes of index $n$ subgroups (or isoclasses of irreducible order $n$ representations). Given a representation $M = G/H$ in $I_n$, the number of elements in the conjugacy class of $H$ is equal to
\[
\frac{|G|}{|N_G H/H|} = \frac{n}{|\text{Aut } M|}.
\]

Therefore
\[
(10) \quad \frac{|J_n|}{n} = \sum_{M \in I_n} \frac{1}{|\text{Aut } M|}.
\]

Remark 6.1. We should call an irreducible representation $G/H$ absolutely irreducible if $|\text{Aut}(G/H)| = 1$, that is, $N_G H = H$. This means that $|J_n|/n$ counts all irreducible representations and not just the absolutely irreducible.

6.2. Two formulas. The set $R_n = \text{Hom}(G, S_n)$ is equipped with an action of $S_n$ by conjugation. Let $R^{\text{irr}}_n \subset R_n$ be the set of irreducible representations. Then
\[
I_n = R^{\text{irr}}_n / S_n.
\]

According to [16, Ex. 5.13]
\[
(11) \quad \sum_{n \geq 0} \sum_{M \in R_n/S_n} \frac{t^n}{|\text{Aut } M|} = \sum_{n \geq 0} \frac{|R_n|}{|S_n|} t^n = \exp \left( \sum_{n \geq 1} \sum_{M \in I_n} \frac{t^n}{|\text{Aut } M|} \right),
\]

\[
(12) \quad \sum_{n \geq 0} |R_n/S_n| t^n = \exp \left( \sum_{n \geq 1} |I_n| t^n \right).
\]

The second formula is a standard relation between all representations and indecomposable representations.
Remark 6.2. The second formula is equivalent to the statement of [16], Ex. 5.13.c, because
\[ \text{Hom}(G \times \mathbb{Z}, S_n) = \{(M, g) \in R_n \times S_n \mid g \in \text{Aut } M\} \]
and therefore
\[ \frac{|\text{Hom}(G \times \mathbb{Z}, S_n)|}{|S_n|} = \sum_{M \in R_n} \frac{|\text{Aut } M|}{|S_n|} = |R_n/S_n|. \]

Remark 6.3. One can define the "formal" Hall algebra for permutation representations as follows. Let \( J = \bigcup_{n \geq 1} I_n \) be the set of isoclasses of all irreducible representations. Each \( S \in I_n \) has two integer parameters \( \dim S = n \) and \( |\text{Aut } S| \).

All representations are parametrized by maps \( m: J \to \mathbb{N} \) with finite support. Note that
\[ |\text{Aut } m| = \prod_{S \in J} m_S! \cdot |\text{Aut } S|^{m_S}, \]
while in the usual semisimple category \( |\text{Aut } m| = \prod_{S \in J} |\text{GL}_{m_S}(\text{End } S)| \).

The "formal" Hall algebra \( H \) has these maps as a basis and has multiplication given by
\[ m \circ n = \prod_{S \in J} \left( m_S + n_S \right) \cdot |m \oplus n|. \]

There is an integration map \( I: H \to \mathbb{Q}[t] \) (the latter algebra has the usual multiplication) given by
\[ m \mapsto \frac{t^{\dim m}}{|\text{Aut } m|} = \prod_{S \in J} \frac{t^{m_S \dim S}}{m_S! |\text{Aut } S|^{m_S}}. \]

Formula (11) can be obtained by analyzing this integration map.

6.3. The case \( G = F_m \). The polynomials \( A_{d \mid}(q) \) count absolutely irreducible representations, while \( |J_n|/n \) counts all irreducible permutation representations. There is a straightforward relation between these counts:

Lemma 6.4. We have
\[ \Psi \left( \sum_{n \geq 1} \frac{A_{n \mid}(q)}{q - 1} t^n \right) = \sum_{n \geq 1} \frac{|J_n|}{n} x^n. \]

Proof. This follows from
\[ \Psi \left( \sum_{n \geq 1} \frac{A_{n \mid}(q)}{1 - q} t^n \right) = \log \left( T^{-1} \left( \sum_{n \geq 0} [n]_q^{(m-1)}((q - 1)^{m-1} t)^n \right)^{-1} \right), \]
\[ \sum_{n \geq 1} \frac{|J_n|}{n} x^n = \log \left( \sum_{n \geq 0} n!^{m-1} x^n \right). \]

An explanation for this relation could be that in order to pass from absolutely irreducible representations to all irreducible representations we have to apply Adams operations as in [14], Lemma 3.2.
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