Extrinsic symplectic symmetric spaces

Michel Cahen*,1, Simone Gutt*,1,2, Nicolas Richard1 and Lorenz Schwachhöfer3

mcahen@ulb.ac.be, sgutt@ulb.ac.be

nrichard@ulb.ac.be, Lorenz.Schwachhoefer@math.uni-dortmund.de

* Académie Royale de Belgique
1 Université Libre de Bruxelles, Campus Plaine CP 218,
Bvd du Triomphe, B-1050 Brussels, Belgium
2 Université Paul Verlaine - Metz, LMAM
Ile du Saulcy, F-57045 Metz Cedex 01, France
3 Universität Dortmund, Fachbereich Mathematik, Lehrstuhl LSVII
Vogelpothsweg 87, 44221 Dortmund, Germany

Abstract

We define the notion of extrinsic symplectic symmetric spaces and exhibit some of their properties. We construct large families of examples and show how they fit in the perspective of a complete classification of these manifolds. We also build a natural ⋆-quantization on a class of examples.

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Introduction

From its very early days differential geometry has had an intrinsic and an extrinsic viewpoint. The theory of surfaces in Euclidean 3-space was originally devoted to the description of properties of some subsets of the Euclidean 3-space, before Gauss discovered that certain properties depended only on the metric induced by the ambient space on the surface. This opened the way to study abstract objects of dimension 2 or higher endowed with a metric or some other structures. The consideration of actions of symmetry groups led naturally to the notion of symmetric spaces which can be characterized by their automorphism group, independently of any embedding in some large Euclidean space.

Symmetric symplectic spaces have been studied from different viewpoints, but they have not been completely classified (contrary to Riemannian symmetric spaces or Lorentz type pseudo-Riemannian symmetric spaces) and such a classification seems remote. What is known is the classification of symplectic symmetric spaces with completely reducible holonomy [2], the classification of those spaces with Ricci-type curvature [4] and the classification of symmetric spaces of dimension 2 and 4 [2].

It seems worthwhile to introduce the extrinsic point of view and to try and determine those symmetric spaces which are nice symmetric submanifolds of a symplectic vector space.

In this paper, we define the extrinsic symmetric subspaces of a symplectic vector space and we give large families of examples; we also prove that this class of examples exhaust all possibilities in codimension 2. It also turns out that an easy construction of quantization is possible.

We stress the fact that extrinsic symmetric spaces have been defined by D. Ferus in a Riemannian context [5] and that the reading of his paper was certainly an incentive for our work. More recently, extrinsic symmetric spaces have been studied in a pseudo-Riemannian context [7] and in a CR-context [6].

In section 1 we give the basic definitions and elementary properties of extrinsic symmetric symplectic spaces. We also give an algebraic characterization of those spaces. In section 2 we construct a family of submanifolds of codimension 2p of \((\mathbb{R}^{2(n+p)}, \Omega)\) and prove that they are indeed extrinsic symmetric symplectic spaces. In section 3 we give a class of solutions of the algebraic equations and show how they correspond to the class of examples built in section 2. In section 4 we describe the extrinsic symmetric symplectic spaces of dimension 2n embedded in \(\mathbb{R}^{2n+2}\). Finally in section 5 we show how to build an explicit \(\star\)-quantization on a class of examples.

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1 Definitions and elementary properties

A symplectic manifold \((M, \omega)\) is called symmetric \([3]\) if it is endowed with a smooth map

\[ s : M \times M \to M : (x, y) \to s(x, y) =: s_x(y) \]

such that, for all \(x \in M\), \(s_x\) is an involutive symplectic diffeomorphism of \((M, \omega)\),
called the symmetry at \(x\), for which \(x\) is an isolated fixed point, and such that for all \(x, y \in M\) one has

\[ s_x s_y s_x = s_x s_y s_x. \]

On a symplectic symmetric space \((M, \omega, s)\), there is a unique symplectic connec-
tion \(\nabla\), called the symmetric connection, for which each symmetry \(s_x\) is an affine
transformation (a symplectic connection being a torsion free connection for which
the symplectic form is parallel). The curvature of this connection is parallel so
that \((M, \nabla)\) is an affine symmetric space.

An extrinsic symmetric space is a manifold whose symmetric structure comes
from an embedding in a larger manifold. Let us now make this precise in our
symplectic context.

Let \((M, \omega)\) and \((P, \nu)\) be two smooth symplectic manifolds of dimension \(2n\)
and \(2(n+p)\) and let \(j : M \to P\) be a smooth symplectic embedding (i.e. \(j^*\nu = \omega\)).
With a slight abuse of notation, if \(x \in M\), we write:

\[ T_x P = T_x M \oplus (T_x M)^\perp \]

where \(\perp\) means orthogonal with respect to \(\nu_x\). We denote by \(p_x\) (resp. \(q_x\)) the
projection \(T_x P \to T_x M\) (resp. \(T_x P \to (T_x M)^\perp\)) relative to this decomposition.
We denote by \(N^M\) the normal bundle to \(M\) with fiber \(N^M_x := (T_x M)^\perp\).

Let \(\nabla\) be any symplectic connection on \((P, \nu)\) (i.e. \(\nabla\) is torsion free and \(\nabla \nu = 0\)).
For \(X, Y \in \mathfrak{X}(M)\) (= set of smooth vector fields on \(M\)) and \(\xi, \eta\) smooth sections
of \(N^M\), let us define:

\[ (\nabla_{\xi} Y)(x) := p_x(\nabla_{\xi} Y)(x) \]
\[ (\nabla_{\xi} \eta)(x) := q_x(\nabla_{\xi} \eta)(x) \]
\[ \alpha_x(X, Y) := q_x(\nabla_{\xi} Y)(x) \]
\[ A_x X(x) := p_x(\nabla_{\xi} \eta)(x). \]

Then \(\nabla\) defines a symplectic connection on \((M, \omega)\) called the symplectic connection
induced through the embedding \(j\) by \(\nabla\), and \(\nabla\) is a connection on \(N^M\) preserving
the symplectic structure on the fibers, i.e.

\[ X\nu(\xi, \eta) = \nu(\tilde{\nabla}_X \xi, \eta) + \nu(\xi, \tilde{\nabla}_X \eta) \]

for any smooth sections \( \xi, \eta \) of \( \mathcal{N}M \). Finally \( \alpha \) is a symmetric bilinear form on \( M \) with values in \( \mathcal{N}M \), called the second fundamental form of \( M \) and \( \alpha \) is known as the shape operator. The sign has been chosen so that

\[ \Omega(\alpha(X, Y), \xi) = \Omega(A_{\xi} X, Y). \]

The symplectic curvature tensor of the symplectic connection \( \nabla \) of \( M \) reads:

\[ R^\nabla(X, Y; Z, T) = \omega(R^\nabla(X, Y)Z, T) = \omega((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} Z, T) \]

\[ = R^\nabla(X, Y; Z, T) + \nu(\alpha(Y, Z), \alpha(X, T)) - \nu(\alpha(X, Z), \alpha(Y, T)) \]

(1.2)

where \( R^\nabla \) is the symplectic curvature of \( \tilde{\nabla} \) on \( P \). From now on we consider only embedded symplectic submanifolds of a symplectic vector space, i.e. \( (P, \nu) = (\mathbb{R}^{2(n+p)}, \Omega) \) where \( \Omega \) is the standard symplectic form on \( \mathbb{R}^{2(n+p)} \) and \( \tilde{\nabla} \) is the standard flat symplectic connection on \( (\mathbb{R}^{2(n+p)}, \Omega) \). In this case formula (1.2) becomes:

\[ R^\nabla(X, Y, Z, T) = \nu(\alpha(Y, Z), \alpha(X, T)) - \nu(\alpha(X, Z), \alpha(Y, T)) \]  

(1.3)

Let \( x \in (\mathbb{R}^{2(n+p)}, \Omega) \) and let \( W \) be a 2\( n \)-dimensional symplectic vector subspace of \( \mathbb{R}^{2(n+p)} \). We define the symmetry \( S^W_x \) at \( x \), relative to \( W \), as the affine symplectic transformation of \( (\mathbb{R}^{2(n+p)}, \Omega) \) given by:

\[ S^W_x y = S^W_x (x + (y - x)) = x - p^W(y - x) + q^W(y - x) = y - 2p^W(y - x) \]  

(1.4)

where \( p^W \) (resp. \( q^W \)) defines the projection of \( \mathbb{R}^{2(n+p)} \) on \( W \) (resp. \( W^\perp \)) relative to the decomposition \( \mathbb{R}^{2(n+p)} = W \oplus W^\perp \), where \( \perp \) means orthogonal with respect to \( \Omega \). The symmetry \( S^W_x \) is involutive \( ((S^W_x)^2 = \text{Id}) \), fixes all points of the affine subspace \( x + W^\perp \) and induces the usual symmetry of the affine subspace \( x + W \) relative to \( x \).

**Definition 1.1.** An embedded 2\( n \)-dimensional symplectic submanifold \((M, \omega, j)\) of \( (\mathbb{R}^{2(n+p)}, \Omega) \), where \( j : M \rightarrow \mathbb{R}^{2(n+p)} \) is the embedding, will be called an *extrinsic symplectic symmetric space* if for all \( x \in M \), the symmetry \( S^M_{Tx} \) stabilizes \( M \) (i.e. \( S^M_{Tx} M \subset M \)).

We identify implicitly \( T_x M \) with the symplectic subspace \( j_\ast T_x M \) of \( (\mathbb{R}^{2(n+p)}, \Omega) \). The symmetry \( S^M_{Tx} \) will, from now on, simply be denoted by \( S_x \); its restriction to \( M \) will be written \( s_x \); it is smooth since \( M \) is an embedded submanifold.
Let $\nabla$ be the symplectic connection on $(M, \omega)$ induced through the embedding $j$ by the standard flat symplectic connection $\tilde{\nabla}$ on $(\mathbb{R}^{2(n+p)}, \Omega)$. Since $S_x$ is symplectic and maps $M$ into $M$ we have

$$S_{x*} T_y M = T_{S_x y} M \quad S_{x*} T_y M^\perp = T_{S_x y} M^\perp,$$

and since $S_x$ is an affine transformation of $\mathbb{R}^{2(n+p)}$:

$$S_{x*} \nabla_X Y = \nabla_{S_x X} S_{x*} Y \quad \forall X, Y \in \mathfrak{X}(\mathbb{R}^{2(n+p)}).$$

Projecting on the tangent space and on the normal space we have:

$$s_{x*} \nabla_X Y = \nabla_{s_{x*} X} s_{x*} Y \quad \forall X, Y \in \mathfrak{X}(M),
S_{x*} \omega_{y}(X, Y) = \alpha_{S_{x*} y}(S_{x*} X, S_{x*} Y).$$

Hence the proposition (which justifies the terminology):

**Proposition 1.2.** Let $(M, \omega, j)$ be an extrinsic symplectic symmetric space. Then for each point $x \in M$, $s_x$ coincides with the geodesic symmetry at $x$. It is a global involutive symplectic diffeomorphism of $M$, admitting $x$ as isolated fixed point. Hence $(M, \omega, s)$ is a symmetric symplectic space. The corresponding symmetric connection is the induced symplectic connection $\nabla$.

The symmetries $S_x$ also stabilize the connection $\tilde{\nabla}$ in the normal bundle and hence the second fundamental form is parallel

$$\tilde{\nabla}_X \alpha = 0.$$

To a symmetric symplectic space $(M, \omega, s)$ one associates an algebraic object called a symmetric triple. A symmetric triple is a triple $(\mathcal{G}, \sigma, \mu)$ where $\mathcal{G}$ is a real Lie algebra, $\sigma$ is an involutive automorphism of $\mathcal{G}$ and $\mu$ is a Chevalley 2-cocycle of $\mathcal{G}$ with values in $\mathbb{R}$ for the trivial representation of $\mathcal{G}$ on $\mathbb{R}$, with the following properties

(a) $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$, $\sigma|_{\mathcal{K}} = \text{Id}|_{\mathcal{K}}$, $\sigma|_{\mathcal{P}} = - \text{Id}|_{\mathcal{P}}$

(b) The representation $\text{ad}|_{\mathcal{P}} : \mathcal{K} \to \text{End} \mathcal{P}$ is faithful

(c) $[\mathcal{P}, \mathcal{P}] = \mathcal{K}$

(d) $\forall k \in \mathcal{K}, \mu(k, \cdot) = 0$ and $\mu|_{\mathcal{P} \times \mathcal{P}}$ is symplectic.
The symmetric triple associated to a symmetric symplectic space is built as follows: \( G \) is the Lie algebra of the transvection group \( G \) of \( M \) (i.e. the subgroup of the affine symplectic group of \((M, \omega, \nabla)\) generated by products of an even number of symmetries); \( \sigma \) is the differential at the identity of the involutive automorphism of \( G \) which is the conjugation by \( s_{m_o} \) (the symmetry at a base point \( m_o \) of \( M \)); if \( \pi : G \to M : g \mapsto g(m_o) \), the Chevalley 2-cocycle is the pullback by the differential \( \pi_{se} \) (where \( e \) is the neutral element of \( G \)) of the symplectic 2-form \( \omega_{m_o} \). Remark that \( K \) is the algebra of the stabilizer \( K \) of \( m_o \) in \( G \) and that the differential \( \pi_{se} \) induces a linear isomorphism of \( P \) with \( T_{m_o} M \).

In the situation we are interested in, the symmetric space \((M, \omega)\) is embedded in a symplectic vector space \((\mathbb{R}^{2(n+p)}, \Omega)\) and the symmetries \( s_x \) of \( M \) are induced by affine symplectic transformations \( S_x \) of \( \mathbb{R}^{2(n+p)} \). We want to encode this additional information. We assume that \( M \) is connected.

We define \( G_1 \) to be the subgroup of the affine symplectic group of \((\mathbb{R}^{2(n+p)}, \Omega)\), \( A(\mathbb{R}^{2(n+p)}, \Omega) := \text{Sp}(\mathbb{R}^{2(n+p)}, \Omega) \ltimes \mathbb{R}^{2(n+p)} \), generated by the products of an even number of \( S_x \) for \( x \in M \).

Let \( X \in T_{m_o} M \) and let \( \exp_{m_o} tX \) be the corresponding geodesic of \( M \). Consider

\[
\psi_t^X = S_{\exp_{m_o} tX} S_{m_o}. \tag{1.7}
\]

It is a 1-parametric subgroup of the affine symplectic group \( A(\mathbb{R}^{2(n+p)}, \Omega) \) which yields the parallel transport (both in the tangent and in the normal bundle to \( M \)) along the geodesic of \( M \) defined by \( s \mapsto \exp_{m_o} sX \) of a parametric distance \( t \). The restriction of \( \psi_t^X \) to \( M \) is a 1-parametric group of transvections of \( M \). Let \( \tilde{X} \) be the affine symplectic vector field on \( \mathbb{R}^{2(n+p)} \) associated to \( \psi_t^X \) (i.e. \( \tilde{X}_u = \frac{d}{dt}|_0 \psi_t^X(u) \)); remark that \( \tilde{X}_{m_o} = X \). Let \( G_1 \) be the subalgebra of the affine symplectic algebra \( A(\mathbb{R}^{2(n+p)}, \Omega) := \text{sp}(\mathbb{R}^{2(n+p)}, \Omega) \ltimes \mathbb{R}^{2(n+p)} \) of \((\mathbb{R}^{2(n+p)}, \Omega)\) generated by the \( \tilde{X} \)'s. Then \( G_1 \) is the connected Lie subgroup of the affine symplectic group with algebra \( G_1 \). The group \( G_1 \) stabilizes \( M \) and the homomorphism \( G_1 \to \text{Diff} M : g \mapsto g_{|M} \) has for image the transvection group \( G \) of \( M \).

We define \( \bar{\sigma}_1 \) to be the involutive automorphism of \( G_1 \) given by conjugation by the symmetry \( S_{m_o} \) and \( \sigma_1 \) to be the automorphism of \( G_1 \) given by its differential. Remark that \( S_{m_o} \sigma_1 S_{m_o} = \psi_{-t} \), hence \( \sigma_1(\tilde{X}) = -\tilde{X} \).

Consider \( \pi_1 : G_1 \to M : g_1 \mapsto g_1(m_o) \); let \( K_1 \) be the stabilizer of \( m_o \) in \( G_1 \) and let \( G_1^{\tilde{\sigma}_1} \) be the group of fixed points of \( \bar{\sigma}_1 \). Then

\[
g_1 \in G_1^{\tilde{\sigma}_1} \Rightarrow g_1(m_o) = S_{m_o} g_1 S_{m_o}(m_o) = S_{m_o} g_1(m_o)
\]

implies that \( g_1(m_o) \) is a fixed point of \( s_{m_o} \), hence \( G_1^{\tilde{\sigma}_1}(0) \), the connected component of \( G_1^{\tilde{\sigma}_1} \), is contained in \( K_1 \). On the other hand:

\[
g_1 \in K_1 \Rightarrow S_{m_o} g_1 S_{m_o}(m_o) = m_o \text{ and } (S_{m_o} g_1 S_{m_o})_{s_{m_o}} = g_1 s_{m_o}
\]
since \( g_{1|m_0} \) stabilizes \( T_{m_0}M \) and \( T_{m_0}M^\perp \). Hence \( K_1 \subset G^2_{1^+} \) because an affine transformation of \( \mathbb{R}^{2(n+p)} \) is determined by its 1-jet at any point.

Let \( K_1 \) be the Lie algebra of \( K_1 \); the above shows that \( K_1 = \{ X \in G_1 \mid \sigma_1X = X \} \).

Define \( P_1 = \{ X \in G_1 \mid \sigma_1X = -X \} \). Clearly

\[
G_1 = K_1 + P_1.
\]

The map \( \pi_{1+} \circ \text{Id} : G_1 \to T_{m_0}M \) induces a linear isomorphism of \( P_1 \) with \( T_{m_0}M \).

Hence \( P_1 = \{ X \in G_1 \mid X \in T_{m_0}M \} \). Remark that \( \pi_{1+} \circ \text{Id}(A,a) = Am_o + a \) for \( (A,a) \in G_1 \subset \text{sp}(\mathbb{R}^{2(n+p)}, \Omega) \ltimes \mathbb{R}^{2(n+p)} \). To summarize the above:

**Lemma 1.3.** To an extrinsic symplectic symmetric space \((M, \omega, j)\), one associates (having chosen a basepoint \( m_o \)) a subalgebra \( G_1 \) of the symplectic affine algebra \( A(\mathbb{R}^{2(n+p)}, \Omega) \), stable under \( \sigma_1 \), the conjugation by \( S_{m_o} = -\text{Id}_{T_{m_0}M} \oplus \text{Id}_{T_{m_0}M^\perp} \), so that, writing

\[
G_1 = K_1 \oplus P_1 \quad \text{with} \quad \sigma_1|_{K_1} = \text{Id}|_{K_1} \quad \text{and} \quad \sigma_1|_{P_1} = -\text{Id}|_{P_1}
\]

we have \([P_1, P_1] = K_1\) and there is a bijective linear map \( \lambda : T_{m_0}M \to P_1 \) so that, if \( \lambda(x) = (A(x), a(x)) \in \text{sp}(\mathbb{R}^{2(n+p)}, \Omega) \ltimes \mathbb{R}^{2(n+p)} \) then \( A(x)m_o + a(x) = x \quad \forall x \in T_{m_0}M \).

**Definition 1.4.** Two extrinsic symplectic symmetric spaces \((M, \omega, j)\), \((M', \omega', j')\) of \((\mathbb{R}^{2(n+p)}, \Omega)\) will be called isomorphic if there exists an element \( \Phi \) of the affine symplectic group \( A(\mathbb{R}^{2(n+p)}, \Omega) \) such that \( \Phi \circ j = j' \).

Without any loss of generality, since we can compose the embedding \( j : M \to \mathbb{R}^{2(n+p)} \) with any symplectic affine transformation of \( \mathbb{R}^{2(n+p)} \) to get a new isomorphic extrinsic symmetric space, we may assume that the embedded symmetric space \((M, \omega)\) contains the origin \( 0 \) of \( \mathbb{R}^{2(n+p)} \) and that the tangent space \( T_0M \) coincides with the symplectic subspace \( \mathbb{R}^{2n} \) of \( \mathbb{R}^{2(n+p)} \) spanned by the \( 2n \) first basis vectors. We shall choose \( m_o = 0 \) as basepoint for \( M \). Then \( S_0 \) identifies with \( S_{0o} \). We also choose a symplectic basis adapted to the decomposition \( \mathbb{R}^{2(n+p)} = \mathbb{R}^{2n} + \mathbb{R}^{2p} \). Then \( \pi_{1+} \circ \text{Id} : G_1 \subset A(\mathbb{R}^{2(n+p)}, \Omega) = \text{sp}(n+p) \ltimes \mathbb{R}^{2(n+p)} \to T_{m_0}M \) reads:

\[
\pi_{1+}(Y, y) = y.
\]  

The affine symplectic algebra \( A(\mathbb{R}^{2(n+p)}, \Omega) \) has Lie bracket given by

\[
[(Y, y), (Y', y')] = ([Y, Y'], Y'y' - Y'y).
\]  

An element \((Y, y) \in P_1\) has the property

\[
S_0(Y, y)S_0 = (S_0YS_0, S_0y) = -(Y, y)
\]  

(1.10)
hence $S_0YS_0 = -Y$ and $y \in \mathbb{R}^{2n}$. Thus, there exists a linear map $\Lambda: \mathbb{R}^{2n} \to \text{sp}(n + p)$ such that

$$\mathcal{P}_1 = \{ (\Lambda(x), x) \mid x \in \mathbb{R}^{2n}, S_0 \Lambda(x) S_0 = -\Lambda(x) \} \quad (1.11)$$

In our chosen basis $\{e_{\alpha}; \alpha \leq 2n\}$ of $\mathbb{R}^{2n}$ and $\{f_i; i \leq 2p\}$ of $\mathbb{R}^{2p}$, the matrix of $\Omega$ reads

$$\Omega = \begin{pmatrix} \omega_0 & 0 \\ 0 & \Omega^N_0 \end{pmatrix} \quad (1.12a)$$

and the matrix of $S_0$ is

$$S_0 = \begin{pmatrix} -\text{Id}_{2n} & 0 \\ 0 & \text{Id}_{2p} \end{pmatrix} \quad (1.12b)$$

so that the matrix of $\Lambda(x)$ is of the form

$$\Lambda(x) = \begin{pmatrix} 0 & \Lambda_1(x) \\ \Lambda_1'(x) & 0 \end{pmatrix} \quad (1.12c)$$

with

$$^t \Lambda_1(x) \omega_0 + \Omega^N_0 \Lambda_1'(x) = 0 \quad (1.12d)$$

The algebra $\mathcal{G}_1$ is given by $\mathcal{G}_1 = \mathcal{P}_1 \oplus [\mathcal{P}_1, \mathcal{P}_1]$. Since $[\mathcal{P}_1, \mathcal{P}_1] = \mathcal{K}_1$ and since, by equation (1.8), $\mathcal{K}_1 \subset (\text{sp}(n + p), 0)$ we have:

$$\Lambda(x)y - \Lambda(y)x = 0 \quad \forall x, y \in \mathbb{R}^{2n}. \quad (1.13)$$

Finally, $[\mathcal{K}_1, \mathcal{P}_1] \subset \mathcal{P}_1$ implies:

$$\Lambda([\Lambda(x), \Lambda(y)]z) = [[\Lambda(x), \Lambda(y)], \Lambda(z)] \quad \forall x, y, z \in \mathbb{R}^{2n} \quad (1.14)$$

Lemma 1.5. Let $(M, \omega, j)$ be a $2n$-dimensional extrinsic symplectic symmetric subspace of $(\mathbb{R}^{2(n+p)}, \Omega)$ such that $0 \in M$ and $T_0M = \mathbb{R}^{2n}$. Then one can associate to $(M, \omega, j)$ a linear map $\Lambda: \mathbb{R}^{2n} \to \text{sp}(n + p)$ such that

$$S_0 \Lambda(x) S_0 = -\Lambda(x) \quad \forall x \in \mathbb{R}^{2n} \quad (1.12)$$

$$\Lambda(x)y - \Lambda(y)x = 0 \quad \forall x, y \in \mathbb{R}^{2n} \quad (1.13)$$

$$\Lambda([\Lambda(x), \Lambda(y)]z) = [[\Lambda(x), \Lambda(y)], \Lambda(z)] \quad \forall x, y, z \in \mathbb{R}^{2n} \quad (1.14)$$

Given two isomorphic extrinsic symmetric spaces $(M, \omega, j)$, $(M', \omega', j')$ one can construct two other isomorphic spaces $(M, \omega, j_1)$, $(M', \omega', j'_1)$ having the additional property that they both contain the origin and that the tangent space at the origin is $\mathbb{R}^{2n}$. Finally the isomorphism $\Phi_1$ between these two spaces can be modified (by composing if necessary with an element of the group $G'_1$) in such a
way that \( \Phi_1(0) = 0 \). In the basis of \( \mathbb{R}^{2(n+p)} \) already described, the element \( \Phi_1 \) of \( \text{Sp}(n + p) \) has matrix of the form

\[
\Phi_1 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad A \in \text{Sp}(n), B \in \text{Sp}(p)
\]

As \( \Phi_1 \circ j = j' \) we have \( \exp_m' = \frac{1}{2} \Phi_1 X = \Phi_1 \exp_m = \frac{1}{2} X \) \( (X \in \mathbb{R}^{2n}) \) and the group \( G_1' = \Phi_1 G_1 \Phi_1^{-1} \). Hence

**Lemma 1.6.** Let \((M, \omega, j), (M', \omega', j')\) be isomorphic 2n-dimensional extrinsic symmetric spaces containing the origin 0 and having \( \mathbb{R}^{2n} \) as tangent space at 0. Then there exists an element \( \Phi_1 \in \text{Sp}(n) \times \text{Sp}(p) \) such that the associated linear maps \( \Lambda \) and \( \Lambda' : \mathbb{R}^{2n} \to \text{sp}(n + p) \) are related by

\[
\Lambda' (\Phi_1 (x)) = \Phi_1 \Lambda (x) \Phi_1^{-1} \quad x \in \mathbb{R}^{2n}
\]

Conversely given a linear map \( \Lambda : \mathbb{R}^{2n} \to \text{sp}(n + p) \) satisfying properties (1.12, 1.13) of lemma 1.5 one can reconstruct a subalgebra \( \mathcal{G}_1 \) of \( \mathcal{A}(\mathbb{R}^{2(n+p)}, \Omega) \) by \( \mathcal{G}_1 = \mathcal{P}_1 \oplus \mathcal{K}_1 \) \( \mathcal{P}_1 = \{ (\Lambda (x), x) \mid x \in \mathbb{R}^{2n} \} \) \( \mathcal{K}_1 = \{ ([\Lambda (x), \Lambda (y)] , 0) \mid x, y \in \mathbb{R}^{2n} \} \).

One considers the connected Lie subgroup \( G_1 \) of the affine symplectic group \( \mathcal{A}(\mathbb{R}^{2(n+p)}, \Omega) \) with Lie algebra \( \mathcal{G}_1 \) and the orbit \( M \) of 0 in \( \mathbb{R}^{2n+2p} \) under the action of \( G_1, M = G_1(0) \); the isotropy of 0 is a Lie subgroup \( K_1 \) with algebra \( \mathcal{K}_1 \). This orbit is a symmetric symplectic space and is an extrinsic symmetric space if the orbit is embedded.

**Lemma 1.7.** Given a linear map \( \Lambda : \mathbb{R}^{2n} \to \text{sp}(n + p) \) as in lemma 1.5 it determines a connected Lie subgroup \( G_1 \) of the affine symplectic group. The orbit of the origin under \( G_1 \) has a structure of symplectic symmetric space and is an extrinsic symmetric symmetric space if the orbit is an embedded submanifold.

To conclude this paragraph let us relate the linear map \( \Lambda \) to the geometrical properties of \((M, \omega)\). Let \( x \in \mathbb{R}^{2n} \) and let \( X = (\Lambda (x), x) \) be the corresponding element of \( \mathcal{P}_1 \). The associated fundamental vector field \( X^* \) on \( \mathbb{R}^{2(n+p)} \) is given by

\[
X^*_u = \frac{d}{dt}|_0 \exp (-t \Lambda (x), x) \ u
\]

\[
= \frac{d}{dt}|_0 \left( e^{-t \Lambda (x)} , \frac{e^{-t \Lambda (x)} - 1}{\Lambda (x)} x \right) \ u = \frac{d}{dt}|_0 \left( e^{-t \Lambda (x)} u + \frac{e^{-t \Lambda (x)} - 1}{\Lambda (x)} x \right)
\]

\[
= -\Lambda (x) u - x
\]

In particular \( X^*_0 = -x \). The second fundamental form at 0 now reads:

\[
\alpha_0 (x, x') = \alpha_0 (X^*_0, X^*_0) = q_0 \nabla (-\Lambda (0) - x) (-\Lambda (x') v - x') = q_0 \Lambda (x') x = \Lambda (x') x.
\]

(1.16)
The symplectic curvature tensor at 0 can be expressed using (1.3), (1.12a) and (1.16)

\[ R_0(x, y, z, t) = \Omega_N^0(\Lambda(y)z, \Lambda(x)t) - \Omega_N^0(\Lambda(x)z, \Lambda(y)t) = -\omega([\Lambda(x), \Lambda(y)]z, t) \]  

(1.17)

It is convenient to express in a slightly different way the map \( \Lambda \), the curvature and the second fundamental form. We use the basis \( \{ e_{\alpha}; \alpha \leq 2n; f_i; i \leq 2p \} \) of \( \mathbb{R}^{2(n+p)} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2p} \) and the associated dual basis \( \{ f^i; i \leq 2p \} \) of \( \mathbb{R}^{2p} \), so that \( \Omega(f^i, f_j) = \delta^i_j \). Condition (1.12) allows us to write

\[ \Lambda(x) = \sum_{i=1}^{2p} (C_i(x) \otimes f^i + f^i \otimes C_i(x)) \]  

(1.18)

where \( u := \Omega(u, \cdot) \) and \( C_i(x) := \Lambda(x)f_i \). Condition (1.13) reads

\[ \Omega(C_i, x, y) = \Omega(\Lambda(x)f_i, y) = -\Omega(f_i, \Lambda(x)y) = -\Omega(f_i, \Lambda(y)x) = \Omega(C_i, y, x) \]

which says that \( \forall i \leq 2p, C_i \in \text{sp}(n) \).

Remark that \( \Omega(C_i, x, y) = -\Omega(f_i, \Lambda(x)y) = \Omega(\alpha(x, y), f_i) \); this gives

\[ C_i = A_{f_i}(0) \]  

(1.19)

so that the \( C_i \) define the shape operator at the base point. We also have

\[ \Lambda(x) = \sum_{i,k=1}^{2p} \Omega_{N_0}^{ik}(C_i x \otimes f_k + f_k \otimes C_i x) \]  

(1.20)

where \( \sum_{r} \Omega_{N_0}^{ik} \Omega_{N_0}^{lr} = \delta^k_i \). Hence :

\[ \alpha(x, y) = \sum_{i,k=1}^{2p} \Omega_{N_0}^{ik} f_k \omega(C_i x, y), \]  

(1.21)

\[ R(x, y)z = \sum_{i,j=1}^{2p} \Omega_{N_0}^{ji} (\omega(C_j y, z)C_i x - \omega(C_i x, z)C_j y), \]  

(1.22)

\[ R(x, y) = \sum_{i,j=1}^{2p} \Omega_{N_0}^{ji} (C_j x \otimes C_i y - C_j y \otimes C_i x). \]  

(1.23)

Condition (1.14) becomes

\[ \sum_{i,j=1}^{2p} \Omega_{N_0}^{ji} [\omega(C_j y, z)C_i x - \omega(C_i x, z)C_i y] \]

\[ = \sum_{i,j=1}^{2p} \Omega_{N_0}^{ji} [\omega(C_j y, C_i z)C_i x - \omega(C_j x, C_i z)C_i y + \omega(C_j y, C_i x)C_i z - \omega(C_j x, C_i y)C_i z]. \]  

(1.24)
Lemma 1.8. Let \((V, \omega)\) be a symplectic vector space of dimension \(2n\); let \(\mathcal{R} \subset \wedge^2 V^* \otimes S^2 V^*\) be the space of algebraic symplectic curvature tensors on \((V, \omega)\):
\[
\mathcal{R} = \{ R \in \wedge^2 V^* \otimes S^2 V^* \mid \bigoplus_{X,Y,Z} R(X,Y,Z,T) = 0 \ \forall X,Y,Z,T \in V \}.
\]
Let \(\mathcal{L} = \text{sp}(V, \Omega) \simeq \text{sp}(n, \mathbb{R})\). Then the linear map \(\varphi : \Lambda^2 \mathcal{L} \to \mathcal{R}\) defined by
\[
\varphi(A \wedge B)(X,Y,Z,T) = \omega(BY,Z)\omega(AX,T) - \omega(AY,Z)\omega(BX,T)
- \omega(BX,Z)\omega(AY,T) + \omega(AX,Z)\omega(BY,T)
\]
is a \(\text{Sp}(n, \mathbb{R})\) equivariant linear isomorphism. Furthermore the map
\[
\psi : \Lambda^2 \mathcal{L} \to \mathcal{L} : A \wedge B \mapsto [A, B]
\]
is also \(\text{Sp}(n)-\text{equivariant}\). If \(n \geq 2\), the space \(\mathcal{R}\) is the sum of two irreducible \(\text{Sp}(n, \mathbb{R})\) modules, \(\mathcal{R} = \mathcal{E} \oplus \mathcal{W}\), where \(\mathcal{E}\) is the space of curvature tensors of Ricci type. Then \(\ker \psi = \varphi^{-1}(\mathcal{W})\).

Proof. The map \(\varphi\) is well defined; indeed if
\[
A' = aA + bB \quad B' = cA + dB \quad ad - bc = 1
\]
\(\varphi(A \wedge B) = \varphi(A' \wedge B')\); also \(\varphi(A \wedge B) = -\varphi(B \wedge A)\). One checks that \(\varphi(A \wedge B)\) belongs to \(\mathcal{R}\) and
\[
\varphi(SAS^{-1} \wedge SBS^{-1})(X,Y,Z,T) = \varphi(A \wedge B)(S^{-1}X, S^{-1}Y, S^{-1}Z, S^{-1}T)
= (S\varphi(A \wedge B))(X,Y,Z,T)
\]
Furthermore
\[
\text{ric}_{\varphi(A \wedge B)}(X,Y) = \text{Tr}[Z \mapsto \varphi(A \wedge B)(X,Z)Y] = -\omega([A, B]X, Y)
\]
This implies that \(\varphi(\Lambda^2 \mathcal{L}) \supset \mathcal{E}\); also if \(n \geq 2\) one constructs an example of \(A, B \in \text{sp}(n, \mathbb{R})\) such that \(\varphi(A \wedge B) \neq 0\) and \([A, B] = 0\). Hence \(\varphi(\Lambda^2 \mathcal{L}) \supset \mathcal{W}\) and \(\varphi\) is surjective. Equality of dimension implies that it is a linear isomorphism.

From (1.22) and (1.25) one sees that
\[
\varphi^{-1}R = \frac{1}{2} \Omega^{ij}C_i \wedge C_j
\]
The left hand side of (1.24) is \(-C_i R(x,y)z\); the two first terms of the right hand side are \(-R(x,y)C_i z\); the last two terms depend only on \(\varphi^{-1}R\). Hence we have

Lemma 1.9. The relation (1.14) of lemma 1.5 is a condition depending only on curvature.
2 A family of examples

We consider a symplectic submanifold of \((\mathbb{R}^{2(n+p)}, \Omega)\) which is the set of common zeros of \(2p\) polynomials of degree 2. More precisely, let \(\{X_i = (A_i, a_i); i \leq 2p\}\) be a \(2p\)-dimensional subalgebra of the affine symplectic algebra. If \(X_i^*\) are the associated fundamental vector fields on \(\mathbb{R}^{2p}\) (i.e. \(X_i^*(x) = \frac{d}{dt}\big|_0 \exp(-tX_i) \cdot x = -(A_i x + a_i)\)), the corresponding hamiltonians \(F_i\) (functions so that \(\iota(X_i^*)\Omega = dF_i\)) can be chosen to be:

\[ F_i(x) = \frac{1}{2} \Omega(x, A_i x) - \Omega(a_i, x). \tag{2.1} \]

The set \(\Sigma = \{x \mid F_i(x) = 0; i \leq 2p\}\) is a \(2n\)-dimensional embedded symplectic submanifold if \(\forall x \in \Sigma\), the subspace of \(T_x \mathbb{R}^{2(n+p)}\) spanned by the \(X_i^*(x)\) is a \(2p\)-dimensional symplectic subspace. Indeed, in this case the map

\[ \Phi : \mathbb{R}^{2(n+p)} \to \mathbb{R}^{2p} : x \mapsto (F_1(x), \ldots, F_{2p}(x)) \]

is a submersion at all \(x \in \Sigma\) because

\[ \Phi_{xx}(X_i^*) = \sum_{k=1}^{2p} \Omega(X_k^*(x), X_j^*(x)) \partial_k. \]

The condition that the matrix \(\Omega(X_k^*(x), X_j^*(x))\) has rank \(2p\) reads

\[ \Omega(X_j^*(x), X_k^*(x)) = \Omega(A_j x + a_j, A_k x + a_k) = \frac{1}{2} \Omega(x, [A_k, A_j] x) - \Omega(A_k a_j - A_j a_k, x) + \Omega(a_j, a_k) \]

and since the \(X_i\) span an algebra

\[ [X_i, X_j] = ([A_i, A_j], A_i a_j - A_j a_i) = \sum_k c^k_{ij} (A_k, a_k) \]

we have, when \(x \in \Sigma\),

\[ \Omega(X_j^*(x), X_k^*(x)) = \sum_l c^l_{kj} F_l(x) + \Omega(a_j, a_k) = \Omega(a_j, a_k). \tag{2.2} \]

The assumption is thus that the \(2p \times 2p\) matrix \(\Omega^{N_0}_{jk} := \Omega(a_j, a_k)\) has rank \(2p\). The normal space to \(\Sigma\) at \(x, N_x \Sigma\) is spanned by the \(X_i^*(x)\); in particular the normal space at the origin is spanned by \(\{a_i; i \leq 2p\}\). The connection induced on \(\Sigma\) by the flat connection \(\tilde{\nabla}\) on \(\mathbb{R}^{2(n+p)}\) is:

\[
\nabla_X Y = \tilde{\nabla}_X Y - \sum_{i,j} \Omega^{ij}_{X_0} \Omega(\tilde{\nabla}_X Y, X_i)^* X_j^*
\]

\[ = \tilde{\nabla}_X Y - \sum_{i,j} \Omega^{ij}_{X_0} \Omega(Y, A_i X)^* X_j^* \tag{2.3} \]
if $X, Y \in \mathfrak{X}(\Sigma)$ and $\Omega_{N_0}^{ik} \Omega_{N_0}^{kj} = \delta^i_j$. Its curvature at $x$ is given by:

$$R_x(X, Y) = \sum_{i,j} \Omega_{N_0}^{ij}(p_x A_j Y \otimes p_x A_i X + p_x A_i X \otimes p_x A_j Y)$$

(2.4)

where $p_x$ is the symplectic orthogonal projection on $T_x \Sigma$ and $\underline{u} = \Omega(u, \cdot)$. The expression of the covariant derivative of the curvature involves the covariant derivative of the following endomorphisms

$$\nabla_T(A_i(\cdot) - \sum_{j,k} \Omega_{N_0}^{kj} \Omega(A_i(\cdot), X_j^*) X_j^*)_x = \sum_{j,k} \Omega_{N_0}^{kj}(p_x A_j T \otimes p_x A_i X_k^* + p_x A_i X_k^* \otimes p_x A_j T).$$

It vanishes identically if $\forall i, k$, $(A_i X_k^*)(x)$ belongs to $\mathcal{N}_x \Sigma$. This is the case if there exist some constants $B_{ij}^k(x)$ such that:

$$A_i X_j^* = \sum_k B_{ij}^k X_k^*$$

(2.5)

which yield

$$A_i A_j = \sum_k B_{ij}^k A_k$$

(2.6a)

$$A_i a_j = \sum_k B_{ij}^k a_k.$$

(2.6b)

Remark that the $A_i \in \text{sp}(n + p)$ stabilize $\mathcal{N}_0 \Sigma$ which is the space spanned by $\{a_i; i \leq 2p\}$, and thus stabilize also $(\mathcal{N}_0 \Sigma)^\perp = T_0 \Sigma$. We denote $A_{i|\mathcal{N}_0 \Sigma} =: B_i$ and $A_{i|T_0 \Sigma} =: C_i$.

We now prove that the $2n$-dimensional submanifolds $\Sigma$ constructed above are extrinsic symmetric spaces by showing that for any $x, y \in \Sigma$, $S_x y$ belongs to $\Sigma$:

$$F_i(S_x y) = \frac{1}{2} \Omega(S_x y, A_i S_x y) - \Omega(a_i, S_x y)$$

$$= \frac{1}{2} \Omega(y - 2p_x(y - x), A_i(y - 2p_x(y - x))) - \Omega(a_i, y - 2p_x(y - x))$$

$$= \frac{1}{2} \Omega(y, A_i y) - \Omega(a_i, y) - \Omega(p_x(y - x), A_i y) - \Omega(y, A_i p_x(y - x))$$

$$+ 2 \Omega(p_x(y - x), A_i p_x(y - x)) + 2 \Omega(a_i, p_x(y - x))$$

$$= F_i(y) - 2 \Omega(p_x(y - x), A_i y) + 2 \Omega(p_x(y - x), A_i(p_x(y - x) + q_x(y - x)))$$

$$+ 2 \Omega(a_i, p_x(y - x)) - 2 \Omega(p_x(y - x), A_i q_x(y - x))$$

$$= F_i(y) - 2 \Omega(p_x(y - x), A_i y - A_i(y - x) + a_i))$$

$$= F_i(y) + 2 \Omega(p_x(y - x), -A_i x - a_i) = F_i(y).$$

since $A_i q_x(\cdot)$ is in $\mathcal{N}_x \Sigma$ so that $\Omega(p_x(\cdot), A_i q_x(\cdot)) = 0$. Hence:
Theorem 2.1. Let \( \{X_i = (A_i, a_i); i \leq 2p\} \) be \( 2p \) elements of the affine symplectic algebra \( \mathcal{A}(\mathbb{R}^{2(n+p)}, \Omega) \) such that

1. \( \{a_i; i \leq 2p\} \) span a \( 2p \)-dimensional symplectic subspace of \( \mathbb{R}^{2(n+p)} \)

2. there exist constants \( B_{ij}^k \) so that

\[
\forall i, j, \quad A_i A_j = \sum_k B_{ij}^k A_k \quad \text{and} \quad A_i a_j = \sum_k B_{ij}^k a_k.
\]

Then

\[
\Sigma := \{ x \in \mathbb{R}^{2(n+p)} \mid F_i(x) := \frac{1}{2} \Omega(x, A_i x) - \Omega(a_i, x) = 0 \quad \forall i \leq 2p \}
\]

is a \( 2n \)-dimensional extrinsic symmetric symplectic space in \( \mathbb{R}^{2(n+p)} \). This space has zero curvature if and only if the element \( \sum_{ij} \Omega^N_{ij} C_i \wedge C_j \) of \( \Lambda^2(\text{sp}(n)) \) vanishes, where \( \Omega^N_{ij} := \Omega(a_i, a_j) \), \( \sum_j \Omega^N_{ij} \Omega^N_{jk} = \delta^i_k \) and where \( C_i = A_i |_{\Sigma} \).

We shall now exhibit a few properties of the \( A_i \)'s. We have

\[
A_i A_j + A_j A_i = \sum_k (B_{ij}^k + B_{ji}^k) A_k.
\]

The left hand side is antisymplectic and the right hand side is symplectic, hence they both vanish

\[
A_i A_j + A_j A_i = 0 \quad \sum_k (B_{ij}^k + B_{ji}^k) A_k = 0. \tag{2.7}
\]

This shows in particular that

\[
A_i A_j A_k = 0 \quad \forall i, j, k. \tag{2.8}
\]

The \( B_i := A_i |_{N_0 \Sigma} \) satisfy \( B_i B_j = \sum_k B_{ij}^k B_k \) hence \( B(u)B(v) = B(B(u)v) \) if \( B(u) = \sum_i u^i B_i \) for \( u = \sum_i u^i a_i \). On the other hand, each \( B_i \) is in \( \text{sp}(N_0 \Sigma, \Omega^N_{ij}) \):

\[
\Omega(B_i a_j, a_k) = \Omega(A_i a_j, a_k) = -\Omega(a_j, A_i a_k) = -\Omega(a_j, B_i a_k). \tag{2.9}
\]

Hence there is an associative structure on \( \mathbb{R}^{2p} := N_0 \Sigma \) (which is the space spanned by \( \{a_1, \ldots, a_{2p}\} \)) defined by

\[
u \bullet v := B(u)v
\]

so that \( \Omega^N_{ij}(B(u)v, w) + \Omega^N_{ij}(v, B(u)w) = 0 \quad \forall u, v, w. \)

An example of such an associative structure is given as follows. If we choose a basis
\{ g_i \mid i \leq 2p \} of \mathcal{N}_0 \Sigma relative to which the matrix of \Omega^{\mathcal{N}_0} reads \begin{pmatrix} 0 & \text{Id}_p \\ -\text{Id}_p & 0 \end{pmatrix} we can define \( u \cdot v = B(u)v \) where

\[
B(g_k) = 0 \quad \text{and} \quad B(g_{p+k}) = \begin{pmatrix} 0 & D_k \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad D_k = t D_k \quad \forall 1 \leq k \leq p.
\]

**Lemma 2.2.** If \( A_i A_j = 0 \) \( \forall i, j \), the symmetric space depends only on the restrictions \( C_i := A_i|_{T_0\Sigma} \) where \( T_0\Sigma = (\mathcal{N}_0 \Sigma)^\perp \) and \( \mathcal{N}_0 \Sigma \) is the space spanned by \{ \( a_i \mid i \leq 2p \} \). More precisely, we have:

Let \( B : \mathbb{R}^{2p} \to \text{sp}(p) \) and \( C : \mathbb{R}^{2p} \to \text{sp}(n) \) be maps such that for all \( \xi, \eta \in \mathbb{R}^{2p} \)

\[
B(B(\xi)\eta) = B(\xi)B(\eta) \quad \text{and} \quad C(B(\xi)\eta) = C(\xi)C(\eta) = 0.
\]

Let \( \mathcal{M} \) and \( \mathcal{N} \) be the sets

\[
\mathcal{M} = \left\{ (x, u) \in \mathbb{R}^{2n} \times \mathbb{R}^{2p} \mid \frac{1}{2} \Omega(x, C(\xi)x) + \frac{1}{2} \Omega(u, B(\xi)u) - \Omega(u, \xi) = 0 \quad \forall \xi \in \mathbb{R}^{2p} \right\}
\]

\[
\mathcal{N} = \left\{ (x, u) \in \mathbb{R}^{2n} \times \mathbb{R}^{2p} \mid \frac{1}{2} \Omega(x, C(\xi)x) - \Omega(u, \xi) = 0 \quad \forall \xi \in \mathbb{R}^{2p} \right\}.
\]

Then \( \mathcal{M} = \mathcal{N} \).

**Proof.** We first observe, as in (2.8), that \( B(\xi)B(\eta)B(\zeta) = 0 \) for all \( \xi, \eta, \zeta \). Indeed, since

\[
B(\xi)B(\eta) + B(\eta)B(\xi) = B(B(\xi)\eta + B(\eta)\xi)
\]

the right-hand side is both symplectic and antisymplectic, and thus vanishes. This shows that the \( B \)'s anticommute. But then we have

\[
B(\xi)B(\eta)B(\zeta) = -B(\eta)B(\xi)B(\zeta) = B(\eta)B(\zeta)B(\xi)
\]

\[
= B(B(\eta)\zeta)B(\xi) = -B(\xi)B(B(\eta)\zeta) = -B(\xi)B(\eta)B(\zeta)
\]

hence the result.

Now if \( (x, u) \) is in \( \mathcal{M} \), then for all \( \xi, \eta \) we have

\[
\Omega(B(\xi)u, \eta) = -\Omega(u, B(\xi)\eta)
\]

\[
= -\frac{1}{2} \Omega(x, C(B(\xi)\eta)x) - \frac{1}{2} \Omega(u, B(B(\xi)\eta)u)
\]

\[
= -\frac{1}{2} \Omega(u, B(\xi)B(\eta)u)
\]

\[
= \frac{1}{2} \Omega(B(\xi)u, B(\eta)u).
\]
Replacing $\eta$ with $B(\eta)u$, the above calculation gives
\[
\Omega(B(\xi)u, B(\eta)u) = \frac{1}{2} \Omega(B(\xi)u, B(B(\eta)u)u)
\]
which is zero because $B(\xi)B(\eta)B(u)$ is zero. But this was the last term of the previous equation which holds for all $\eta$, hence $B(\xi)u = 0$. In particular this means that $(x, u)$ is in $N$.

Conversely, if $(x, u)$ is in $N$, we do the same calculation and get immediately
\[
\Omega(B(\xi)u, \eta) = -\Omega(u, B(\xi)\eta) = -\frac{1}{2} \Omega(x, C(B(\xi)\eta)x) = 0
\]
hence $B(\xi)u = 0$, thus $(x, u)$ is in $M$.

\section*{Lemma 2.3.} If the $A_i$'s are linearly independent, then necessarily $A_i A_j = 0 \quad \forall i, j$.

\begin{proof}
If the $A_i$'s are linearly independent, then we have
\[
B_{ij}^k + B_{ji}^k = 0
\]
On the other hand if we write:
\[
B_{ijk} = \sum_l B_{il}^j \Omega_{lk}^{N_0}
\] (2.10)
$B_{ijk}$ is symmetric in the last pair of indices (since $B_i \in \text{Sp}(\mathbb{R}^{2p}, \Omega_{N_0})$. Being antisymmetric in the first pair of indices and symmetric in the last pair, it is identically zero.

If $C_i C_j = 0 \quad \forall i, j$, then $Im := \bigoplus_{i=1}^{2p} \text{Im}(C_i)$ is included in $K := \cap_{i=1}^{2p} \text{Ker}(C_i)$ and $\Omega(Im, K) = 0$. There is thus a Lagrangian subspace containing $Im$ and contained in $K$. In an adapted basis, we have
\[
C_i = \begin{pmatrix} 0 & \tilde{C}_i \\ 0 & 0 \end{pmatrix}
\]
with $\tilde{C}_i = ^t\tilde{C}_i, \forall i \leq 2p$. (2.11)
This is realizable with linearly independent $C_i$'s if $2p \leq \frac{n(n+1)}{2}$.

When $A_i A_j = 0 \quad \forall i, j$, we may choose $\{a_i; i \leq 2p\}$ as a basis of $\mathbb{R}^{2p}$ and write an element $x \in \mathbb{R}^{2(n+p)} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2p}$, $x = y + u$. The functions defining $\Sigma$ can be chosen in view of Lemma 2.2 to have the special form:
\[
F_i(x = y + u) = \frac{1}{2} \omega(y, C_i y) - \Omega(a_i, u)
\] (2.12)
Thus $\Sigma$ is a graph of a function $\mathbb{R}^{2n} \to \mathbb{R}^{2p}$ and in particular is diffeomorphic to $\mathbb{R}^{2n}$.

Remark that there exist solutions where the $A_i A_j$'s and even the $C_i C_j$'s are not all zero. For instance, on $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$ with the symplectic structure

$$
\Omega = \begin{pmatrix}
0 & I_2 & 0 & 0 \\
-I_2 & 0 & 0 & 0 \\
0 & 0 & 0 & I_2 \\
0 & 0 & -I_2 & 0
\end{pmatrix}.
$$

one can define

$$A_1 = 0 \quad A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$A_4 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

then $A_3 A_4 = -A_4 A_3 = A_2$ and all other products vanish.

### 3 The algebraic equations on $\Lambda$

We have seen that one can associate to a $2n$-dimensional extrinsic symmetric space $(M, \omega)$ embedded symplectically in $(\mathbb{R}^{2(n+p)}, \Omega)$ in such a way that $0 \in M$ and $T_0 M = \mathbb{R}^{2n}$, a linear map $\Lambda : \mathbb{R}^{2n} \to \text{sp}(n + p)$ such that; $\forall x, y, z \in \mathbb{R}^{2n}$:

$$S_0 \Lambda(x)S_0 = -\Lambda(x) \quad (1.12) \quad \Lambda(x)y = \Lambda(y)x \quad (1.13) \quad \text{and} \quad \Lambda([\Lambda(x), \Lambda(y)]z) = [[\Lambda(x), \Lambda(y)], \Lambda(z)] \quad (1.14).$$
Equivalently, one can consider a linear map

\[ C : \mathbb{R}^{2p} \to \text{sp}(n) \quad f_i \mapsto C_i \]

where the \( f_i (i \leq 2p) \) form a basis of \( \mathbb{R}^{2p} = T_0 \mathbf{M}^4 \), so that :

\[
\begin{align*}
\sum_{i,j=1}^{2p} \Omega_{N_0}^{ij} \left[ \omega(C_j y, z) C_i C_i x - \omega(C_j x, z) C_i y \right] \\
= \sum_{i,j=1}^{2p} \Omega_{N_0}^{ij} \left[ \omega(C_j y, C_i z) C_i x - \omega(C_j x, C_i z) C_i y \right] \\
+ \omega(C_j y, C_i x) C_i z - \omega(C_j x, C_i y) C_i z
\end{align*}
\]

(1.24)

with \( \Omega_{ij}^{N_0} = \Omega(f_i, f_j) \) and \( \sum_j \Omega_{N_0}^{ij} \Omega_{N_0}^{jk} = \delta_i^k \).

Given \( \Lambda \), the map \( C \) is defined by \( C_i x = \Lambda(x) f_i \); given \( C \) the map \( \Lambda \) is given by

\[
\Lambda(x) = \sum_{ik} \Omega_{N_0}^{ik} (C_i x \otimes f_k + f_k \otimes C_i x). \tag{1.20}
\]

We have also seen that given such a \( \Lambda \) or \( C \), one can construct a Lie subgroup \( \mathbf{G}_1 \) of the affine symplectic group and that the orbit \( \mathbf{G}_1(0) \) is an extrinsic symmetric space if it is an embedded submanifold.

**Lemma 3.1.** The subspace of \( \mathbb{R}^{2p} \) spanned by \( \{ \alpha_0(x, y) \mid x, y \in \mathbb{R}^{2n} \} \) is isotropic if and only if the curvature \( R_0 \) vanishes.

**Proof.** If the subspace spanned by the \( \alpha(x, y) \) is isotropic \( R = 0 \) (using (1.3)).

Conversely if \( R = 0 \), using (1.3) one has

\[
\nu(\alpha(x, t), \alpha(y, z)) = \nu(\alpha(y, t), \alpha(x, z)) = \nu(\alpha(t, y), \alpha(z, x))
\]

\[
= \nu(\alpha(z, y), \alpha(t, x)) = \nu(\alpha(y, z), \alpha(x, t)) = 0
\]

\[
\Box
\]

**Lemma 3.2.** The only flat extrinsic symmetric symplectic spaces are the graph of quadratic polynomial functions \( \mathbb{R}^{2n} \to \mathbb{R}^{2p} \) whose image is isotropic.

**Proof.** The extrinsic symmetric space is the orbit of the origin under the action of the group generated by \( \{ e^{i(\Lambda(x), x)} \mid x \in \mathbb{R}^{2n} \} \). Since \( \alpha(x, \cdot) = -\sum_{i,k} \Omega_{N_0}^{ki} f_k \otimes C_i x \) and the image of \( \alpha \) is isotropic we have:

\[
\Lambda(x)^2 = \sum_{ikrs} \Omega_{N_0}^{ik} \Omega_{N_0}^{rs} \omega(C_i x, C_r x) f_k \otimes f_s
\]

\[
\Lambda(x)^3 = 0
\]
and hence:
\[ \frac{e^{t\Lambda(x)} - 1}{\Lambda(x)} x = tx - \frac{t^2}{2} \sum_{ki} \Omega_{N_0}^{ki} f_k \Omega(C_i x, x) \]
which proves the lemma.

**Theorem 3.3.** Let \( \{f_i; i \leq 2p\} \) be a basis of \( (\mathbb{R}^{2p}, \Omega^{N_0}) \) and let \( \{f^i; i \leq 2p\} \) be the dual basis \( (\Omega(f^i, f_j) = \delta^i_j) \). Let \( C_i, i \leq 2p \) be \( 2p \) elements of the symplectic algebra \( \text{sp}(n) \) so that
\[
C_i C_j = \sum_k B^k_{ij} C_k
\]
for some constants \( B^k_{ij} \) satisfying
\[
\sum_r B^r_{ij} \Omega_{N_0}^{N_0} + \sum_r B^r_{ik} \Omega_{N_0}^{N_0} = 0.
\]
Then the \( C_i \) are a solution of (1.24); the corresponding group \( G_1 \) is a nilpotent subgroup of the affine symplectic group \( A_{2(n+p)} \); the orbit of the origin \( 0 \in \mathbb{R}^{2(n+p)} \) under this group is the connected component of \( 0 \) of the extrinsic symmetric space \( \Sigma \) defined by:
\[
\begin{align*}
\Sigma &= \{ (x, u) \in \mathbb{R}^{2(n+p)} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2p} | \\
& \quad \frac{1}{2} \Omega(x, C_i x) + \frac{1}{2} \Omega(u, B_i u) - \Omega(f_i, u) = 0 \quad 1 \leq i \leq 2p \}. \tag{3.3}
\end{align*}
\]

where \( B_i : \mathbb{R}^{2p} \to \mathbb{R}^{2p} \) is the linear symplectic map such that \( B_i(f_j) = \sum_k B^k_{ij} f_k \).

**Proof.** The fact that such \( C_i \)'s give a solution of (1.24) is straightforward. Observe that, as before, all \( C_i C_j + C_j C_i \) vanish, so \( C_i C_j C_k = 0 \forall i, j, k \). One has
\[
\begin{align*}
\Lambda(v) &= \sum_i (f^i \otimes C_i v + C_i v \otimes f^i) \\
\Lambda(v) \Lambda(w) &= \sum_{ij} (\Omega(C_i v, C_j w) f^i \otimes f^j - \Omega_{N_0}^{ij} C_i v \otimes C_j w) \\
\Lambda(v) \Lambda(w) \Lambda(x) &= -\sum_{ijk} (\Omega(C_i v, C_j w) \Omega_{N_0}^{jk} f^i \otimes C_k x \quad + \Omega(C_j w, C_k x) \Omega_{N_0}^{ij} C_i v \otimes f^k) \\
\Lambda(v) \Lambda(w) \Lambda(x) \Lambda(y) &= \sum_{ijkr} (\Omega(C_j w, C_k x) \Omega_{N_0}^{ij} \Omega_{N_0}^{kr} C_i v \otimes C_r y) \\
\Lambda(v) \Lambda(w) \Lambda(x) \Lambda(y) \Lambda(z) &= \sum_{ijkrs} (\Omega(C_j w, C_k x) \Omega_{N_0}^{ij} \Omega_{N_0}^{kr} \Omega_{N_0}^{ls} \Omega_{N_0}^{rs} C_i v \otimes f^s) \\
&= 0
\end{align*}
\]
because \(\sum_j \Omega(C_i v, C_j w)\Omega^{jk}_{N_0} = -\sum_{jn} B_{ji}^n \Omega^{jk}_{N_0} \Omega(C_n v, w) = -\sum_{jn} B_{ji}^k \Omega^{jn}_{N_0} \Omega(C_n v, w)\)
and \(\sum_{js} B_{ji}s C_s = C_k C_i C_i = 0\) hence \(\sum_{jk} (\Omega(C_i v, C_j w)\Omega^{jk}_{N_0} \Omega(C_k x, C_r y) = 0\).

The algebra \(G_1 = P_1 + K_1 \subset \mathfrak{sp}(n + p) \times \mathbb{R}^{2(n+p)}\) is defined by

\[
P_1 = \{ (\Lambda(x), x) \mid x \in \mathbb{R}^{2n}\}
K_1 = [P_1, P_1] = \{ (\Lambda(x), \Lambda(x')), 0) \mid x, x' \in \mathbb{R}^{2n}\}.
\]

One sees that \(K_1\) is a 2-step nilpotent algebra and that \(G_1\) is nilpotent. Since \(\Lambda(x)^5 = 0, \forall x \in \mathbb{R}^{2n}\) the orbit of the origin under the action of the group \(\exp t(\Lambda(x), x)\) is the set of points \((\tilde{x}(t), \tilde{u}(t)) = \frac{e^{t\Lambda(x)} - 1}{\Lambda(x)} x \in \mathbb{R}^{2n} \oplus \mathbb{R}^{2p}\) with

\[
\tilde{x}(t) = tx + \frac{t^3}{6} (\Lambda(x))^2 x + \frac{t^5}{5!} (\Lambda(x))^4 x
= tx - \frac{t^3}{6} \sum_{ij} \Omega^{ij}_{N_0} C_i x \Omega(C_j x, x) + \frac{t^5}{5!} \sum_{ijkr} \Omega(C_j x, C_k x) \Omega^{ijkr}_{N_0} \Omega(C_i x) \Omega(C_r x, x))
\]

\[
\tilde{u}(t) = \frac{t^2}{2} \Lambda(x) x + \frac{t^4}{4!} (\Lambda(x))^3 x
= \sum_i f_i \frac{t^2}{2} \omega(C_i x, x) - \frac{t^4}{4!} \sum_{jk} (\Omega(C_i x, C_j x) \Omega^{jk}_{N_0} \Omega(C_k x, x))
\]

Thus \(C_k \tilde{x}(t) = t C_k x - \frac{t^3}{6} \sum_{ij} \Omega^{ij}_{N_0} C_k C_i x \Omega(C_j x, x)\) and

\[
\Omega(\tilde{x}(t), C_k \tilde{x}(t)) = t^2 \Omega(x, C_k x) - \frac{t^4}{6} \sum_{ij} \Omega^{ij}_{N_0} \Omega(x, C_k C_i x) \Omega(C_j x, x)
- \frac{t^4}{6} \sum_{js} \Omega^{js}_{N_0} \Omega(C_j x, C_k x) \Omega(C_s x, x).
\]

Also \(B_k \tilde{u}(t) = \frac{t^2}{2} \sum_{ijr} B_{kr}^i \Omega^{ij}_{N_0} \omega(C_i x, x) f_r\) and

\[
\Omega(\tilde{u}(t), B_k \tilde{u}(t)) = \frac{t^4}{4!} \sum_{sji} \omega(C_i x, x) B_{kr}^i \Omega^{ij}_{N_0} \omega(C_s x, x)
\]

whereas

\[
\Omega(\tilde{u}(t), f_k) = \frac{t^2}{2} \omega(C_k x, x) + \frac{t^4}{4!} \sum_{jkr} B_{kr}^i \Omega^{ij}_{N_0} \Omega(C_r x, x) \Omega(C_s x, x)
\]

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so that
\[ \frac{1}{2} \Omega(x(t), C_kx(t)) + \frac{1}{2} \Omega(u(t), B_ku(t)) = \Omega(f_k, u(t)). \]

Hence the orbit \( G_0 \) coincides with the connected component of 0 of the surface
defined by (3.3), since this surface is an extrinsic symmetric symplectic space. \( \blacksquare \)

4 The situation in codimension 2

We prove:

**Theorem 4.1.** Let \( \{f_1, f_2\} \) be a symplectic basis of \( \mathbb{R}^2 (= \mathbb{R}^2p \) for \( p = 1 \)) and let
\( \{f^1, f^2\} \) be the dual basis. Let \( \Lambda : \mathbb{R}^{2n} \rightarrow \text{sp}(n + 1) \) be defined by
\[ \Lambda(x) = \sum_{i=1}^{2} f_i \otimes C_i x + C_i x \otimes f^i \]
where \( C_1, C_2 \) are elements of \( \text{sp}(n) \). Assume the \( C_i \)'s obey the relations (1.24). Then either \( \Lambda \) corresponds to a flat extrinsic symmetric space, or \( C_i C_j = 0 \) for all \( i, j \), hence \( \Lambda \) corresponds to extrinsic symmetric spaces described in theorem 3.3 (for vanishing \( B \)'s).

**Proof** Let \( \hat{C} \) be the subspace of \( \text{sp}(n) \) spanned by the elements \( C_1 \) and \( C_2 \). If \( \dim \hat{C} \leq 1 \), the curvature of the space associated to \( \Lambda \) is zero by lemma 3.1.

So assume from now on that \( \dim \hat{C} = 2 \). The rest of the proof is divided into a sequence of lemmas.

**Lemma 4.2.** All elements \( C \in \hat{C} \) are nilpotent.

**Proof.** Let us complexify \( \mathbb{R}^{2n} \), and denote by \( V \) its complexification. We may assume that \( (V, \omega) \) is a complex symplectic vector space. Assume there exists \( C \in \hat{C} \) which is not nilpotent; hence \( C \) has non-zero eigenvalues. Let \( \lambda \neq 0 \) be an eigenvalue such that \( |\lambda| \) is maximal; let \( x \) be an eigenvector and let \( y \in V \) be such that \( \omega(x, y) = 1 \).

In the case \( p = 1 \) the equations (1.24) can be written
\[ C^2_1 x \circ C_2 y - C_1 C_2 x \circ C_1 y - C^2_2 y \circ C_2 x + C_1 C_2 y \circ C_1 x + \omega((C_1 C_2 + C_2 C_1)y, x)C_1 - 2\omega(C^2_1 y, x)C_2 = 0 \quad (a) \]
\[ C_2 C_1 x \circ C_2 y - C^2_2 x \circ C_1 y - C_2 C_1 y \circ C_2 x + C^2_2 y \circ C_1 x + 2\omega(C^2_2 y, x)C_1 - \omega((C_2 C_1 + C_1 C_2)y, x)C_2 = 0 \quad (b) \]
where 
\[ u \circ v \not= u \otimes v + v \otimes u \]

We can choose the basis \{f_1, f_2\} of \( \mathbb{R}^2 \) such that \( C = C_1 \). Now
\[ \omega(C_1 x, y) = \lambda \omega(x, y) = \lambda \not= 0 \]

Hence we adjust \( f_2 \) so that
\[ \omega(C_1 x, y) = 0 \]

Then (a) becomes
\[
\lambda^2 x \circ C_2 y - C_1 C_2 x \circ C_1 y - C_1^2 y \circ C_2 x + \lambda C_2 C_1 y \circ x + \omega(C_2 C_1 y, x) C_1 + 2\lambda^2 C_2 = 0
\]

Apply this endomorphism to \( x \)
\[
- \lambda C_1 C_2 x + \lambda C_1 y \omega(C_2 x, x) - C_1^2 y \omega(C_2 x, x) + \lambda^2 C_2 x + \omega(C_2 C_1 y, x) C_1 x + 2\lambda C_2 x = 0
\]

(d.i) \[
\lambda(3\lambda - C_1) C_2 x - \omega(C_2 x, x) C_1 (C_1 - \lambda) y + \lambda \omega(C_2 C_1 y, x) x = 0
\]

(d.ii)

Pairing (d.i) with \( x \) gives:

\[
4\lambda^2 \omega(C_2 x, x) + 2\lambda^2 \omega(C_2 x, x) = 6\lambda^2 \omega(C_2 x, x) = 0
\]

Hence
\[ \omega(C_2 x, x) = 0 \]

(e)

Substituting in (d.i)
\[
\lambda(3\lambda - C_1) C_2 x - \lambda \omega(C_1 C_2 x, y) x = 0
\]
\[
(3\lambda - C_1) C_2 x = \omega(C_1 C_2 x, y) x
\]

(f)

But \( 3\lambda \) is not an eigenvalue of \( C_1 \) by maximality; hence \( 3\lambda - C_1 \) is invertible; on the other hand
\[
2\lambda(3\lambda - C_1)^{-1} x \not= u \Rightarrow (3\lambda - C_1) u = 2\lambda x = (3\lambda - C_1) x
\]
\[
2\lambda(3\lambda - C_1)^{-1} x = x
\]

Hence
\[ 2\lambda C_2 x = \omega(C_1 C_2 x, y) x \]

(g)

Pairing with \( y \) gives
\[ 0 = \omega(C_1 C_2 x, y) \]
hence $C_2 x = 0$ by (g). Substituting in (c) gives

$$x \circ (\lambda + C_1) C_2 y + 2 \lambda C_2 = 0 \quad (h)$$

Apply this endomorphism to $y$

$$\omega((\lambda + C_1)C_2 y, y) x + (3 \lambda + C_1) C_2 y = 0 \quad (i)$$

Again $-3 \lambda$ is not an eigenvalue of $C_1$ by maximality; i.e $3 \lambda + C_1$ is invertible. As above we get

$$4 \lambda C_2 y + \omega((\lambda + C_1)C_2 y, y) x = 0$$

where the second line in (i) is obtained from the first by multiplication with $(\lambda + C_1)$. Pairing with $y$ gives:

$$6 \lambda \omega((\lambda + C_1)C_2 y, y) = 0$$

Hence by (i)

$$(3 \lambda + C_1) C_2 y = 0$$

Hence $C_2 y = 0$; using (i) we have $C_2 = 0$ which contradicts our assumption, $\dim \hat{C} = 2$.

**Lemma 4.3. Assume that all elements $C \in \hat{C}$ are nilpotent and that equations (1.24) are satisfied. Assume there exists $C_1 \in \hat{C}$ such that $C_1^2 \neq 0$. Then for all $C \in \hat{C}$, $\ker C_1 \subset \ker C$.**

**Proof.** Let $C_2$ be linearly independent of $C_1$ and $x \in \ker C_1$. Then (m) gives:

$$- C_1 C_2 x \circ C_1 y - C_2 y \circ C_2 x + \omega(C_2 C_1 y, x) C_1 = 0 \quad (k)$$

Apply this to $y$

$$-\omega(C_1 y, y) C_1 C_2 x - 2 \omega(C_1 C_2 x, y) C_1 y - \omega(C_2 x, y) C_1^2 y = 0$$

Pairing with $y$:

$$-3 \omega(C_1 C_2 x, y) \omega(C_1 y, y) = 0$$

As $C_1 \neq 0$, there exists a dense open set $U$ in $\mathbb{R}^{2n}$ such that $\forall y \in U$, $\omega(C_1 y, y) \neq 0$; thus $\forall y \in U$

$$\omega(C_1 C_2 x, y) = 0$$

Thus $\omega(C_1 C_2 x, y) = 0 \quad \forall y \in \mathbb{R}^{2n}$; hence:

$$C_1 C_2 x = 0$$
Substituting in \(k\) gives:
\[
C_1^2 y \circ C_2 x = 0
\]
As \(C_1^2 \neq 0\), we can choose \(y\) such that \(C_1^2 y \neq 0\). Thus
\[
C_2 x = 0
\]
which proves the lemma.

**Lemma 4.4.** Assume there exists \(C_1 \in \hat{C}\) such that \(C_1^2 \neq 0\) and that relations (1.24) are satisfied. Then there is a \(0 \neq C_2 \in \hat{C}\) such that \(C_2^2 = C_2 C_1 = C_1 C_2 = 0\).

**Proof.** Choose \(C_2\) linearly independent of \(C_1\) and \(x\) such that \(C_1^2 x = 0, C_1 x \neq 0\). Thus \(C_1 x \in \ker C_1\) and by the previous lemma \(C_2 C_1 x = 0\). Going back to (23):
\[
-C_1 C_2 x \circ C_1 y - C_1^2 y \circ C_2 x + C_1 C_2 y \circ C_1 x + \omega(C_2 C_1 y, x)C_1 = 0 \tag{l}
\]
Apply this to \(x\):
\[
-\omega(C_1 y, x)C_1 C_2 x - \omega(C_2 x, x)C_1^2 y + \omega(C_1 x, x)C_1 C_2 y + \omega(C_2 C_1 y, x)C_1 x = 0 \tag{m}
\]
Pairing with \(x\):
\[
\omega(C_2 C_1 y, x)\omega(C_1 x, x) = 0 \tag{n}
\]
Assume \(C_1 C_2 x \neq 0\); then (m) implies \(\omega(C_1 x, x) = 0\). Pairing (m) with \(y\) gives:
\[
-2\omega(C_1 y, x)\omega(C_1 C_2 x, y) = 0
\]
As \(C_1 x \neq 0\), there exists an open dense set of \(y\)'s such that \(\omega(C_1 y, x) \neq 0\); hence for this open dense set \(\omega(C_1 C_2 x, y) = 0\), which contradicts our assumption. Hence \(C_1 C_2 x = 0\).

We may thus rewrite (l) as:
\[
-C_1 C_2 x \circ C_1 y - C_1^2 y \circ C_2 x + C_1 C_2 y \circ C_1 x = 0 \tag{o}
\]
Assume \(C_1 x\) and \(C_2 x\) are linearly independent. Then one checks that there exists \(\mu\) such that
\[
C_1 C_2 y = \mu(y)C_2 x \quad C_1^2 y = \mu(y)C_1 x
\]
Hence:
\[
0 = \omega(C_1^2 y, y) = \mu(y)\omega(C_1 x, y)
\]
As \(C_1 x \neq 0\), \(\omega(C_1 y, x) \neq 0\) on an open dense set; hence \(\mu(y) = 0\) everywhere and \(C_1^2 y = 0, \forall y\); hence \(C_1^2 = 0\) a contradiction. Thus \(C_1 x\) and \(C_2 x\) are linearly dependent.

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We can thus change basis in $\mathbb{R}^2$ (replace $C_2$ by $C_2 + \nu C_1$) in such a way that $C_2 x = 0$ (by assumption $C_1 x \neq 0$). Going back to (a) we have:

$$C_1 C_2 y \circ C_1 x = 0$$

This implies $C_1 C_2 y = 0, \forall y$; hence $C_1 C_2 = 0$. But

$$0 = \omega(C_1 C_2 y, z) = \omega(y, C_2 C_1 z)$$

implies $C_2 C_1 = 0$. By $C_1 C_2 = 0$ we have $\text{Im} C_2 \subset \ker C_1$; by lemma 4.3 $\ker C_1 \subset \ker C_2$; hence $C_2 = 0$.

**Lemma 4.5.** Let $C_1, C_2$ be two linearly independent elements of $\hat{\mathcal{C}}$. Assume equation (1.24) is satisfied. Then for all $C \in \hat{\mathcal{C}}$, $C^2 = 0$.

**Proof.** Assume $C_1^2 \neq 0$; let $C_2$ be an element given by lemma 4.4 such that $C_2^2 = C_2 C_1 = C_1 C_2 = 0$. Then relation (a) reads:

$$C_1^2 x \circ C_2 y - C_1^2 y \circ C_2 x - 2 \omega(C_1^2 y, x) C_2 = 0$$

Apply this to $y$:

$$C_1^2 x \omega(C_2 y, y) + 3 C_2 y \omega(C_1^2 x, y) - C_1^2 y \omega(C_2 x, y) = 0$$

Pairing with $y$:

$$4 \omega(C_1^2 x, y) \omega(C_2 y, y) = 0$$

Choose $x$ such that $C_1^2 x \neq 0$. Then an open dense set $\omega(C_1^2 x, y) \neq 0$. Hence $\omega(C_2 y, y) = 0, \forall y$. Hence by polarizing $\omega(C_2 y, z) = 0 \forall y, z$ and $C_2 = 0$ a contradiction. ■

We can now complete the proof of the theorem. To do this we first prove that $\hat{\mathcal{C}}$ must contain an element of rank $\geq 2$.

Let $C_1, C_2$ be a basis of $\hat{\mathcal{C}}$ such that $\text{rk} C_1 = \text{rk} C_2 = 1$. Then (replacing if necessary $C_1$ and (or) $C_2$ by $-C_1$ (resp $-C_2$) we have

$$C_1 = x_1 \circ x_1 \quad C_2 = x_2 \circ x_2$$

and $x_1, x_2$ are linearly independent elements of $\mathbb{R}^{2n}$. But then:

$$C_1 + C_2 = x_1 \circ x_1 + x_2 \circ x_2$$

and clearly the image of this operator has dimension 2.

Let us thus assume $\text{rk} C_1 \geq 2$. By lemma 4.3 $\forall a, b \in \mathbb{R}$

$$(a C_1 + b C_2)^2 = 0$$
Hence $C_1^2 = C_2^2 = C_1C_2 + C_2C_1 = 0$. Thus equation (a) reads:

$$-C_1C_2x \circ C_1y + C_1C_2y \circ C_1x = 0$$

Choose $x, y$ such that $C_1x$ and $C_1y$ are linearly independent. This is possible as $\text{rk} C_1 \geq 2$. This implies the existence of $k \in \mathbb{R}$ such that:

$$C_1C_2x = kC_1x = -C_2C_1x$$

Hence $k$ is an eigenvalue of $C_2$; but $C_2$ is nilpotent; thus $k = 0$ and $C_2C_1x = 0$. This is valid for all $x$; indeed if $C_1x = 0$ it is true; and if $C_1x \neq 0$, one chooses $y$ as above and come to the conclusion $C_2C_1x = 0$. So $C_2C_1 = 0$.

5 Quantization of a class of examples

Consider the extrinsic symmetric space $\Sigma \subset (V = \mathbb{R}^{2(n+p)} = \mathbb{R}^{2n} \oplus \mathbb{R}^{2p}, \Omega)$ defined, as before by

$$\Sigma = \left\{ z = (x, u) \in V | F_i(z) = \frac{1}{2} \Omega(x, C_i x) - \Omega(a_i, u) = 0 \quad 1 \leq i \leq 2p \right\}$$

where $C_i$ for $1 \leq i \leq 2p$ are elements in $\text{sp}(\mathbb{R}^{2n}, \omega_0)$ such that $C_iC_j = 0 \quad \forall i, j$, and where $\{a_i \quad 1 \leq i \leq 2p\}$ is a basis of $\mathbb{R}^{2p}$.

We denote by $\Omega_{N_0}$ the matrix of the restriction of $\Omega$ to $\mathbb{R}^{2p}$ in the basis given by the elements $a_k$’s so that $\Omega_{N_0}^{ij} := \Omega(a_i, a_j)$ and $\Omega_{N_0}$ its inverse matrix.

The submanifold $\Sigma$ is a graph; writing $u = \sum_{i=1}^{2p} u_i a_i$ we have:

$$\Sigma = \{(x, u) \in \mathbb{R}^{2n} \oplus \mathbb{R}^{2p} | u^j = \frac{1}{2} \sum_{i} \Omega_{N_0}^{ij} \Omega(x, C_i x)\}$$

and we shall use $x \in \mathbb{R}^{2n}$ as global coordinates on $\Sigma$.

Observe that we have two symplectic transverse foliations on $V$. One defined by the involutive distribution $\mathcal{N}$ spanned by the vector fields $(X_{F_i})_{(x,u)} = (-C_i x, -a_i)$ (which commute) and the other defined by $\mathcal{N}_{(x,u)}^{\perp \Omega}$ for which the integral submanifolds are

$$\Sigma^c = \{ z \in V | F_i(z) = c_i \quad 1 \leq i \leq 2p \} \quad \text{for } c \in \mathbb{R}^{2p}.$$ We have a natural projection $\pi$ of $V$ on $\Sigma = \Sigma^0$ along the integral submanifolds of $\mathcal{N}$. Observe that the integral curve of $X_{F_i}$ with initial condition $z = (x, u)$ is
given by $z(t) = (x - tC_i(x), u - ta_i)$ and $F_j(z(t)) = F_j(z) - t\Omega_{ij}^{N_0}$. Hence $\pi$ is given by

$$\pi(x, u) = (x - \sum_{i=1}^{2p} (u_i - \frac{1}{2} \sum_{j=1}^{2p} \Omega_{N_0}^{ij} \Omega(x, C_j(x)) C_ix), \frac{1}{2} \sum_{j,k} \Omega_{N_0}^{kj} \Omega(x, C_j(x) a_k)$$

We identify $C^\infty(\Sigma)$ with the functions which are constant along the leaves of $\mathcal{N}$ i.e.

$$C^\infty(\Sigma) \simeq \pi^*C^\infty(\Sigma) = \{ f \in C^\infty(V) \mid X_{F_i}(f) = 0 \quad \forall 1 \leq i \leq 2p \}$$

The Poisson bracket of two functions on $V$ is given by

$$\{u, v\}_V = \sum_{ij} \Omega^{ij} \partial_i u \partial_j v$$

(5.1)

where $\Omega^{ij}$ are the components of the inverse matrix $\Omega^{-1}$.

Given any point $z = (x, u)$ in $V$, we have the splitting $V \simeq T_{(x,u)}V = \mathcal{N}_{(x,u)} \oplus \Omega T_{(x,u)}\Sigma^c$ for $c = F(z)$. The Lie derivative of $\Omega$ in the direction of $X_{F_j}$ vanishes: $L_{X_{F_i}} \Omega = 0$. Hence

$$\{\pi^*\tilde{f}, \pi^*\tilde{g}\}_V = \pi^*\{\tilde{f}, \tilde{g}\}_\Sigma \quad \forall \tilde{f}, \tilde{g} \in C^\infty(\Sigma).$$

(5.2)

We consider the Moyal $\star$ product on $(V = \mathbb{R}^{2(n+p)}, \Omega)$. It is explicitly given by

$$u \star v = \sum_{r=0}^{\infty} \frac{1}{r!} \left( \frac{\nu}{2} \right)^r C_r(u, v)$$

(5.3)

for $u, v \in C^\infty(V)[[\nu]]$, with

$$C_r(u, v) = \sum_{i'j's} \Omega^{i'j_1} \ldots \Omega^{i'j_r} \nabla_{i_1\ldots i_r} u \nabla_{j_1\ldots j_r} v.$$  

(5.4)

It is invariant under the affine symplectic group of $(V, \Omega)$. We have

$$X_{F_i}(u \star v) = (X_{F_i}u) \star v + u \star (X_{F_i}v) \quad \forall u, v \in C^\infty(V)[[\nu]].$$

Hence

**Proposition 5.1.** A deformation quantization is induced on $\Sigma$ from the Moyal deformation quantization on $V$. The star product of two functions on $\Sigma$ is the restriction to $\Sigma$ of the Moyal star product of the pullback to $V$ of those two functions under the projection $\pi$ along the integral submanifolds of $\mathcal{N}$:

$$\tilde{f} \star_\Sigma \tilde{g} = (\pi^*\tilde{f} \star \pi^*\tilde{g})_\Sigma \quad \forall \tilde{f}, \tilde{g} \in C^\infty(\Sigma)[[\nu]]$$

This star product on $\Sigma$ is invariant under the action of the group of transvections.
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