The Application of G-heat equation and Numerical Properties

Xiaolin Gong † Shuzhen Yang ‡

Abstract: We consider a nonlinear expectation G-expectation which was established by Peng. In order to compute the nonlinear probability under the G-expectation, we prove that a function (special point is the nonlinear probability) is the viscosity solution of the G-heat equation, and show that the fully implicit discretization convergence to the viscosity solution of the G-heat equation.

Keywords: nonlinear probability; nonlinear PDE; G-heat equation; Newton iteration; fully implicit; viscosity solution

1 Introduction

In the mathematical Finance, we focus on the compute of probability of default. Under the assumption of linear probability (expectation) space, we use log normal distribution to describe the return of stock, and we could easily calculus probability of default by normal distribution. For general case, there is not only one probability. We need introduce volatility uncertainty (including much more probabilities) in the market.

A nonlinear expectation (probability) G-expectation was established by Peng in recent years, which could be equivalent to a set of probabilities (see [2]). In the theory of G-expectation, the G-normal distribution and G-Brownian motion were introduced and the corresponding stochastic calculus of Ito’s type were established (see [5], [6], [7]). In Markovian case, the G-expectation is associated with fully nonlinear PDEs, and is applied among economic and financial models with volatility uncertainty (see [3]).

The next equation is used to compute the nonlinear probability ([5]):

\[
\partial_t u - \frac{1}{2}(\bar{\sigma}^2(D_{xx}^2 u)^+ - \bar{\sigma}^2(D_{xx}^2 u)^-) = 0, \quad u(0, x) = \varphi(x), x \in \mathbb{R},
\] (1.1)

*This work was supported by National Natural Science Foundation of China (No. 11171187, No. 10871118 and No. 10921101).
†Institute for Economics and Institute of Financial Studies, Shandong University, Jinan, Shandong 250100, PR China(agcaelyn@gmail.com).
‡School of mathematics, Shandong University, Jinan, Shandong 250100, PR China(yangsz@mail.sdu.edu.cn).
where $\varphi(x) = 1_{\{x < 0\}}, x \in R$.

We show that $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)], (t, x) \in [0, \infty) \times R^d$, is the viscosity solution of the equation (1.1), where $\hat{E}$ is the nonlinear expectation.

Following the work of [3], [8], [9], [10], we prove that the fully implicit discretization convergence to the viscosity solution of the G-heat equation.

Under the same maximum volatility, we compare the nonlinear probability $u(1, 0)$ and linear probability $\hat{u}(1, 0)$ of the next two equations:

\begin{align*}
\partial_t u - \frac{1}{2}(D^2_{xx} u)^+ - \frac{1}{4}(D^2_{xx} u)^- &= 0, \\
u(0, x) &= I_{x \leq 0}, \quad x \in [-10, 10], \\
u(t, -10) &= 1, \quad \nu(t, 10) = 0, \quad t \in [0, 1];
\end{align*}

and

\begin{align*}
\partial_t \hat{u} - \frac{1}{2}(D^2_{xx} \hat{u})^+ - (D^2_{xx} \hat{u})^- &= 0, \\
\hat{u}(0, x) &= I_{x \leq 0}, \quad x \in [-10, 10], \\
\hat{u}(t, -10) &= 1, \quad \hat{u}(t, 10) = 0, \quad t \in [0, 1].
\end{align*}

By calculation, we have

\[\hat{E}[I_{X \leq 0}] = u(1, 0) = 0.6680, \quad P(X \leq 0) = \hat{u}(1, 0) = 0.5010.\]

The paper is organized as follows: In section 2, the notations and results on G-expectation is presented. In section 3, we prove that a function is the viscosity solution of the G-heat equation. Then, we compare the value of nonlinear probability and linear probability in section 4. The fully implicit numerical convergence to the viscosity solution of the G-heat equation is established in section 5.

\section{Preliminaries}

Firstly, we give the basic theory of G-expectation.

Let $\mathcal{O}$ be a given set and $\mathcal{H}$ a vector lattice of real valued functions defined on $\mathcal{O}$, namely $c \in \mathcal{H}$ for each constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables.

\textbf{Definition 2.1} A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to R$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

\begin{enumerate}
\end{enumerate}
(a) Monotonicity: If $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$;
(b) Constant preservation: $\hat{E}[c] = c$;
(c) Sub-additivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;
(d) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for each $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ be the space of real continuous functions defined on $\mathbb{R}^n$ such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \text{ for all } x, y \in \mathbb{R}^n,$$

where $k$ and $C$ depend only on $\varphi$.

**Definition 2.3** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}[]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, y)]_{x=X}]$.

**Definition 2.4** ($G$-normal distribution) A $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-normal distributed if for each $a, b \geq 0$ we have

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of $X$, i.e., $\bar{X} \overset{d}{=} X$ and $\bar{X} \perp X$. Here the letter $G$ denotes the function

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : \mathbb{S}_d \to \mathbb{R},$$

where $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

Peng [6] showed that $X = (X_1, \ldots, X_d)$ is $G$-normally distributed if and only if for each $\varphi \in C_{l.Lip}(\mathbb{R}^d)$, $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the viscosity solution of the following $G$-heat equation:

$$\partial_t u - G(D^2_{xx} u) = 0, \quad u(0, x) = \varphi(x). \quad (2.1)$$

The function $G(\cdot) : \mathbb{S}_d \to \mathbb{R}$ is a monotonic, sublinear mapping on $\mathbb{S}_d$ and $G(A) = \frac{1}{2} \hat{E}[(AX, X)] \leq \frac{1}{2} |A| \hat{E}[|X|^2] =: \frac{1}{2} |A| \sigma^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset \mathbb{S}_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma A],$$

3
where $S^+_d$ denotes the collection of nonnegative elements in $S_d$.

Let $\{W_t\}$ be a classical $d$-dimensional Brownian motion on a probability space $(\Omega^0, \mathcal{F}^0, P^0)$ and let $F^0 = \{\mathcal{F}_t^0\}$ be the augmented filtration generated by $W$. Set

$$P_M := \{P_\theta : P_\theta = P^0 \circ (B_{\cdot}^0)^{-1}, B_{\cdot}^0 = \int_0^\cdot \theta_s dW_s, \theta \in L^2_{F^0}([0, T]; \Gamma)\},$$

where $L^2_{F^0}([0, T]; \Gamma)$ is the collection of $F^0$-adapted square integrable measurable processes with values in $\Gamma$. Set $\mathcal{P} = \overline{P_M}$ the closure of $P_M$ under the topology of weak convergence, then $\mathcal{P}$ is weakly compact. \cite{2} proved that $\mathcal{P}$ represents $\hat{E}$ on $L^1_G(\Omega_T)$.

Let $d = 1$, we consider the finite difference method to the next G-heat equation:

$$\partial_t u - \frac{1}{2}(\sigma^2(D^2_{xx}u)^+ - \sigma^2(D^2_{xx}u)^-) = 0, \quad x \in R, \quad t > 0,$$

$$u(0, x) = \varphi(x), \quad x \in R,$$

which

$$(D^2_{xx}u)^+ = \begin{cases} D^2_{xx}u, & D^2_{xx}u \geq 0, \\ 0, & D^2_{xx}u < 0, \end{cases}$$

and

$$(D^2_{xx}u)^- = \begin{cases} -D^2_{xx}u, & D^2_{xx}u \leq 0, \\ 0, & D^2_{xx}u > 0. \end{cases}$$

For generally, we focus on the case $\varphi(x) = 1_{\{x < y\}}, y \in R$. Set $\bar{\sigma} = \bar{\sigma} = \sigma_0$, then $u(1, 0) = P(X < y), X \sim N(0, \sigma_0^2)$, specially. Next, we consider the function: $\varphi(x) = 1_{\{x < 0\}}, x \in R$.

## 3 The viscosity solution of G-heat equation

We will show that $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)], (t, x) \in [0, \infty) \times R^d$, is the viscosity solution of the following G-heat equation:

$$\partial_t u - G(D^2_{xx}u) = 0, \quad u(0, x) = \varphi(x), x \in R,$$

where $\varphi(x) = 1_{\{x < 0\}}, x \in R$.

**Lemma 3.1** \(\lim_{n \to \infty} \hat{E}[\varphi_n(X)] = \hat{E}[\varphi(X)]\),

where

$$\varphi_n(x) = \begin{cases} 1, & x \leq 0 \\ 1 - nx, & 0 < x < \frac{1}{n} \\ 0, & \frac{1}{n} \leq x \end{cases}.$$
Proof. For $\varphi(x) = 1_{\{x < 0\}}$, $\hat{E}[\varphi_n(X)] - \hat{E}[\varphi(X)] \leq \hat{E}[1_{\{0 < X < \frac{1}{n}\}}]$, and $X$ is a $G$-normal distribution, we have

$$\hat{E}[1_{\{0 < X < \frac{1}{n}\}}] = \sup_{p_0 \in \mathcal{P}_M} P_0(0 < X < \frac{1}{n})$$

$$\leq \sup_{\theta \in L_{p_0}^2([0, T]; \Gamma)} P^0(0 < \int_0^1 \theta_s dW_s < \frac{1}{n})$$

$$\leq P^0(-\frac{\theta}{2} < \sigma W_1 < \frac{\theta}{2}) \cdot \frac{1}{n}.$$ 

So $\hat{E}[\varphi_n(X)] - \hat{E}[\varphi(X)] \to 0$, as $n \to \infty$.

Theorem 3.2 The function $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, is the viscosity solution of equation (3.1).

Proof. Firstly, we show that $u$ is continuous in $[0, \infty) \times \mathbb{R}$. For all $\delta > 0$, $u(t + \delta, x) - u(t, x) = \hat{E}[\varphi(x + \sqrt{t + \delta}X)] - \hat{E}[\varphi(x + \sqrt{t}X)]$, by Lemma 3.1 and the definition of $\mathcal{P}_M$:

$$\hat{E}[\varphi(x + \sqrt{t + \delta}X)] - \hat{E}[\varphi(x + \sqrt{t}X)]$$

$$= \lim_{n \to \infty}(\hat{E}[\varphi_n(x + \sqrt{t + \delta}X)] - \hat{E}[\varphi_n(x + \sqrt{t}X)])$$

$$= \lim_{n \to \infty}(\hat{E}[\varphi_n(x + \sqrt{\delta}X + \sqrt{t}X)] - \hat{E}[\varphi_n(x + \sqrt{t}X)])$$

$$= \hat{E}[\varphi(x + \sqrt{\delta}X + \sqrt{t}X)] - \hat{E}[\varphi(x + \sqrt{t}X)]$$

$$\leq \sup_{p_0 \in \mathcal{P}_M} P_0(\{x + \sqrt{t}X \leq -\sqrt{\delta}X\}/\{x + \sqrt{t}X \leq 0\})$$

$$\leq \sup_{\theta \in L_{p_0}^2([0, T]; \Gamma)} |P^0(0 < x + \sqrt{t} \int_0^1 \theta_s dW_s \leq -\sqrt{\delta} \int_0^1 \theta_s dW_s)$$

$$+ P^0(-\sqrt{\delta} \int_0^1 \theta_s dW_s < x + \sqrt{t} \int_0^1 \theta_s dW_s \leq 0)|,$$

where $W$ is independent identically distributed with $W$ in the linear expectation space $(\Omega^0, \mathcal{F}^0, P^0)$. By simple calculus, we have

$$\sup_{\theta \in L_{p_0}^2([0, T]; \Gamma)} P^0(0 < x + \sqrt{t} \int_0^1 \theta_s dW_s \leq -\sqrt{\delta} \int_0^1 \theta_s dW_s)$$

$$\leq P^0(\frac{\sqrt{\delta}}{2} \sqrt{\delta} \sigma W_1 < \frac{\sqrt{\delta}}{2} x + \frac{\sqrt{\delta}}{2} \sqrt{t} \sigma W_1 \leq -\frac{\sqrt{\delta}}{2} \sqrt{\delta} \sigma W_1),$$

then, by dominated convergence theorem,

$$\sup_{\theta \in L_{p_0}^2([0, T]; \Gamma)} P^0(0 < x + \sqrt{t} \int_0^1 \theta_s dW_s \leq -\sqrt{\delta} \int_0^1 \theta_s dW_s) \to 0, \text{ as } \delta \to 0.$$ 

So $u$ is continuous in $t$. Similarly we could prove that $u$ is continuous in $x$. 

5
Now, we prove that $u$ is a viscosity subsolution of equation (3.1).

For a fixed $(t, x) \in [0, \infty) \times R^d$, let $\psi \in C^{2,3}_b([0, \infty) \times R^d)$, such that $\psi \geq u$ and $\psi(t, x) = u(t, x)$.

By Taylor’s expansion, it follows that, for $\delta \in (0, t)$,

$$0 \leq \hat{E}[\psi(t - \delta, x + \sqrt{\delta} X) - \psi(t, x)]$$

$$\leq o(\delta) - \partial_t \psi(t, x) \delta + \hat{E}[\langle D_x \psi(t, x), X \rangle \sqrt{\delta} + \frac{1}{2} \langle D_{xx}^2 \psi(t, x) X, X \rangle \delta]$$

$$= o(\delta) - \partial_t \psi(t, x) \delta + \hat{E}[\frac{1}{2} \langle D_{xx}^2 \psi(t, x) X, X \rangle \delta]$$

$$= -\partial_t \psi(t, x) \delta + \delta G(D_{xx}^2 \psi)(t, x) + o(\delta),$$

so

$$\partial_t \psi(t, x) - G(D_{xx}^2 \psi)(t, x) \leq 0.$$ 

Thus $u$ is a viscosity subsolution of (3.1). Similarly, we have $u$ is a viscosity solution of (3.1).

This completes the proof.

4 Numerical Example

In this section, we give an example which is important for financial market. By Theorem 3.2, the nonlinear probability $u(t, x) := \hat{E}[I_{x + \sqrt{t} X \leq 0}], (t, x) \in [0, \infty) \times R$, is the viscosity solution of the following $G$-heat equation:

$$\partial_t u - G(D_{xx}^2 u) = 0, \ u(0, x) = I_{x \leq 0}. \quad (4.1)$$

i.e.,

$$\partial_t u - \frac{1}{2}(\bar{\sigma}^2(D_{xx}^2 u)^+ - \underline{\sigma}^2(D_{xx}^2 u)^-) = 0, \ x \in R, \ t > 0, \quad (4.2)$$

$$u(0, x) = I_{x \leq 0}, \ x \in R.$$  

Next, we consider a boundary problem of (4.2), i.e.,

$$\partial_t u - \frac{1}{2}(D_{xx}^2 u)^+ - \frac{1}{4}(D_{xx}^2 u)^-) = 0, \quad (4.3)$$

$$u(0, x) = I_{x \leq 0}, \ x \in [-10, 10],$$

$$u(t, -10) = 1, \ u(t, 10) = 0, \ t \in [0, 1].$$

For a given probability space $(\Omega, \mathcal{F}, P)$, the linear probability $u(t, x) := E[I_{x + \sqrt{t} X \leq 0}], (t, x) \in [0, \infty) \times R$, is the viscosity solution of the following heat equation:
\[
\partial_t u - D_{xx}^2 u = 0, \quad u(0, x) = I_{x \leq 0}.
\]

We also consider a boundary problem of (4.4), i.e.,
\[
\partial_t \hat{u} - \frac{1}{2}((D_{xx}^2 \hat{u})^+ - (D_{xx}^2 \hat{u})^-) = 0,
\]
\[
\hat{u}(0, x) = I_{x \leq 0}, \quad x \in [-10, 10],
\]
\[
\hat{u}(t, -10) = 1, \quad \hat{u}(t, 10) = 0, \quad t \in [0, 1].
\]

Comparing the value of \(u(1, x)\) and \(\hat{u}(1, x)\), \(x \in [-10, 10]\):

The red line is the value of function \(u(1, x), x \in [-10, 10]\), and the blue line is the value of function \(\hat{u}(1, x), x \in [-10, 10]\). By \(u(t, x) := \hat{E}[I_{x + \sqrt{t}X \leq 0}]\) and \(\hat{u}(1, x) = E[I_{x + \sqrt{t}X \leq 0}]\), we have
\[
\hat{E}[I_{X \leq 0}] = u(1, 0) = 0.6680, \quad P(X \leq 0) = \hat{u}(1, 0) = 0.5010.
\]

5 Numerical Analysis

In this section, we consider the bounded boundary problem of (2.2), i.e.,
\[
\partial_t u - \frac{1}{2}(\sigma^2(D_{xx}^2 u)^+ - \sigma^2(D_{xx}^2 u)^-) = 0,
\]
\[
u(0, x) = \varphi(x), \quad x \in [a, b],
\]
\[
u(t, a) = g(t), \quad u(t, b) = h(t) \quad t \in [0, T].
\]

where \(\varphi, g, h\) are bounded and measurable functions.
5.1 A Finite Difference Discretization

The equation (5.1) can be discretized by a standard finite difference method with variable timeweighting to give

\[ u_{i}^{n+1} - u_{i}^{n} = \theta \alpha_{i}^{n}[u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n}] + (1 - \theta)\alpha_{i}^{n+1}[u_{i+1}^{n+1} - 2u_{i}^{n+1} + u_{i-1}^{n+1}], \quad (5.2) \]

where

\[ \alpha_{i}^{n} := \frac{\sigma(\Gamma_{i})^2 \Delta t_i}{2(x_{i+1} - x_{i})(x_{i} - x_{i-1})}, \]

\[ \sigma(\Gamma_{i}) := \begin{cases} \bar{\sigma}, & \text{if } \Gamma_{i} \geq 0 \\ \sigma, & \text{if } \Gamma_{i} < 0 \end{cases}, \quad (5.3) \]

\[ \Gamma_{i} := \frac{u_{i}^{n+1} - 2u_{i}^{n} + u_{i-1}^{n}}{(x_{i+1} - x_{i})(x_{i} - x_{i-1})}. \]

In this paper, we consider the fully implicit schemes with \( \theta = 0 \), i.e.,

\[ u_{i}^{n+1} - u_{i}^{n} = \alpha_{i}^{n+1}[u_{i+1}^{n+1} - 2u_{i}^{n+1} + u_{i-1}^{n+1}], \quad (5.4) \]

The set of algebraic equation (5.1) is nonlinear for the formula of \( \sigma(\Gamma_{i}) \). So we consider the discrete equation at each node as

\[ \phi_{i}^{n} := u_{i}^{n} - u_{i}^{n+1} + \alpha_{i}^{n+1}[u_{i+1}^{n+1} - 2u_{i}^{n+1} + u_{i-1}^{n+1}]. \]

Following the work of D.M. Pooey [3] (more see, Pang and Qi [8]; Qi and Sun [9]; Sun and Han [10]), we must specify the element of the generalized Jacobian that will be used in the Newton iteration. We define the derivatives as

\[ \frac{\partial \sigma(\Gamma)^2 \Gamma}{\partial \Gamma} = \begin{cases} \bar{\sigma}^2, & \text{if } \Gamma \geq 0 \\ \sigma^2, & \text{if } \Gamma < 0 \end{cases}, \]

For further analysis the Newton iteration, we rewrite the discrete equation (5.2) in matrix form. Let

\[ U^{n+1} = [u_{0}^{n+1}, u_{1}^{n+1}, \ldots, u_{m}^{n+1}]', \quad U^{n} = [u_{0}^{n}, u_{1}^{n}, \ldots, u_{m}^{n}]', \]

\[ [M^{n}U^{n}]_{i} := -\alpha_{i}^{n}[u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n}]. \]

For convenience, we modify the first and last rows of \( M \) as needed to handle the bounded boundary conditions. By the discretization schemes in (5.4), the matrix \( M \) is a diagonally dominant
matric with positive diagonals and non-positive off-diagonals. Note that all the elements of the inverse of $M$ are non-negative. The discrete equation (5.4) can be rewritten as:

$$[I + M^{n+1}]U^{n+1} = U^n, \quad (5.5)$$

where $I$ is the identity matrix. Next we prove the convergence of the Newton iteration for full implicit schemes.

### 5.2 Convergence of the Newton Iteration Schemes

For the matrix $M$ is a diagonally dominant matrix, we can analysis the Newton iteration of equation (5.5). We adopt the Newton timestep as the following scheme:

(a) Let $(U^{n+1})^0 = U^n$;
(b) For $k = 0, 1, 2, \cdots$ Solve

$$[I + M((U^{n+1})^k)](U^{n+1})^{k+1} = U^n \quad (5.6)$$

where $(U^{n+1})^{k+1}$ is the $(k + 1)$th iteration, and $M((U^{n+1})^k)$ means $M$ be dependent on $(U^{n+1})^k$.
(c) For a given small number $\varepsilon$, if

$$\max_i |(u_i^{n+1})^{k+1} - (u_i^{n+1})^k| < \varepsilon \cdot \max_i |(u_i^{n+1})^{k+1}|,$$

(d) we end the scheme.

We show the convergence results about the above Newton iteration as follows:

**Theorem 5.1** The nonlinear iteration (5.6) convergence to the unique solution of (5.5), for given initial iterate $(U^{n+1})^0 = U^n$.

**Proof.** For notional convergence, we denote $\hat{M}^k = M((U^{n+1})^k)$ and $\hat{U}^k = (U^{n+1})^k$. So equation (5.6) can be rewritten as

$$[I + \hat{M}^k]\hat{U}^{k+1} = U^n. \quad (5.7)$$

Firstly, we show that the sequence $\{\hat{U}^k\}_{0 \leq k}$ is monotonically. The $k$ iteration of equation (5.7) gives

$$[I + \hat{M}^{k-1}]\hat{U}^k = U^n. \quad (5.8)$$

Subtracting equation (5.7) from equation (5.8), we have

$$[I + \hat{M}^k](\hat{U}^{k+1} - \hat{U}^k) = [\hat{M}^{k-1} - \hat{M}^k]\hat{U}^k. \quad (5.9)$$

We consider the right side of (5.9) for each $i$

9
\[
(\hat{M}^{k-1} - \hat{M}^k) \hat{U}^k = \frac{\Delta t_i (\sigma(\hat{\Gamma}^k) - \sigma(\hat{\Gamma}^{k-1})^2)}{2} \hat{\Gamma}^k,
\]

where

\[
\hat{\Gamma}^k = \frac{\hat{u}_{i+1}^k - 2\hat{u}_i^k + \hat{u}_{i-1}^k}{(x_{i+1} - x_i)(x_i - x_{i-1})},
\]

\[
\hat{U}^k := [\hat{u}_0^k, \hat{u}_1^k, \ldots, \hat{u}_m^k]',
\]

\[
\sigma(\hat{\Gamma}^k) := \begin{cases} 
\sigma, & \text{if } \hat{\Gamma}^k_i \geq 0 \\
\bar{\sigma}, & \text{if } \hat{\Gamma}^k_i < 0
\end{cases}.
\]

By the equation (5.3), if \( \hat{\Gamma}^k_i \leq 0 \), \( \sigma(\hat{\Gamma}^k_i) = \sigma \), then

\[
\frac{\Delta t_i (\sigma(\hat{\Gamma}^k) - \sigma(\hat{\Gamma}^{k-1})^2)}{2} \hat{\Gamma}^k \geq 0;
\]

Similarly, if \( \hat{\Gamma}^k_i \geq 0 \), \( \sigma(\hat{\Gamma}^k_i) = \bar{\sigma} \), then

\[
\frac{\Delta t_i (\sigma(\hat{\Gamma}^k) - \sigma(\hat{\Gamma}^{k-1})^2)}{2} \hat{\Gamma}^k \geq 0.
\]

For the matric \( I + \hat{M}^k \) is a diagonally dominant matric, the inverse of matric \( I + \hat{M}^k \) is nonnegative, we have

\[\hat{U}^{k+1} - \hat{U}^k \geq 0, \quad k \geq 1.\] (5.10)

Next, we need to prove the sequence \( \{\hat{U}^k\}_{0 \leq k} \) is bounded. Set \( C_{\max} = \max_i u^n_i \), \( C_{\min} = \min_i u^n_i \), \( \hat{U}_{\max} = \max_i \hat{u}_i^k \), \( \hat{U}_{\min} = \min_i \hat{u}_i^k \). By the equation (5.8), we have

\[
\hat{u}_i^k - \alpha_i^k \hat{u}_{i+1}^k - 2\hat{u}_i^k + \hat{u}_{i-1}^k = u_i^n,
\]

where

\[
\alpha_i^k := \frac{\sigma(\hat{\Gamma}^k_i)^2 \Delta t_i}{2(x_{i+1} - x_i)(x_i - x_{i-1})}.\] (5.12)

By the equation (5.11), and \( \alpha_i^k - 1 \geq 0 \), then

\[
(1 + 2\alpha_i^k - 1)\hat{u}_i^k \leq 2\alpha_i^k - 1 \hat{U}_{\max} + C_{\max},
\]

and

\[
(1 + 2\alpha_i^k - 1)\hat{u}_i^k \geq 2\alpha_i^k - 1 \hat{U}_{\min} + C_{\min}.
\]

So
\[ \hat{u}_i^k \leq \frac{2\hat{\alpha}_i^{k-1}}{1 + 2\hat{\alpha}_i^{k-1}} \hat{U}_{\text{max}} + C_{\text{max}}, \]

and

\[ \hat{u}_i^k \geq \frac{2\hat{\alpha}_i^{k-1}}{1 + 2\hat{\alpha}_i^{k-1}} \hat{U}_{\text{min}} + C_{\text{min}}. \]

Set \( \max_i \frac{2\hat{\alpha}_i^{k-1}}{1 + 2\hat{\alpha}_i^{k-1}} = b_1, \max_i \frac{2\hat{\alpha}_i^{k-1}}{1 + 2\hat{\alpha}_i^{k-1}} = b_2, \) and \( 0 \leq b_1, b_2 < 1, \) then we have

\[ \hat{U}_{\text{max}} \leq \frac{1}{1 - b_1} C_{\text{max}}, \quad \hat{U}_{\text{min}} \geq \frac{1}{1 - b_2} C_{\text{min}}. \]

Now, we prove the uniqueness. Suppose there are two solutions to equation (5.5), \( U_{1}^{n+1} \) and \( U_{2}^{n+1}, \) such that

\[ [I + M_1]U_{1}^{n+1} = U^n, \quad [I + M_2]U_{2}^{n+1} = U^n. \]

Similar the proof of the monotonicity sequence \( \{\hat{U}^k\}_{0 \leq k}, \) we have

\[ [I + M_2](U_{2}^{n+1} - U_{1}^{n+1}) = [M_1 - M_2]U_{1}^{n+1}, \]

and

\[ U_{2}^{n+1} - U_{1}^{n+1} \geq 0 \]

By the equality of \( U_{1}^{n+1} \) and \( U_{2}^{n+1}, \) we have \( U_{2}^{n+1} - U_{1}^{n+1} = 0. \)

Thus, we complete the proof.

### 5.3 The Convergence of Fully Implicit

In the above section, we have proved the convergence of the Newton iteration for the nonlinear equation (5.4). Next, we would to prove the full implicit schemes convergence to the viscosity solution of (5.1). By the work of Barles in [1], we know that a stable, consistent, and monotone discretization will convergence to the viscosity solution.

**Theorem 5.2** The fully implicit discretization (5.5) convergences to the solution of the equation (5.1), as \( \Delta t, \Delta x \to 0. \)

We first give some important lemmas for prove Theorem 5.2.

Review the discrete equation at each node as

\[ \varphi_i^n := u_i^n - u_i^{n+1} + \alpha_i^{n+1}[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] \quad (5.13) \]

then at each step

\[ 11 \]
Proof: For any given \( \Delta \) Lemma 5.4
The fully implicit discretization (5.13) is monotone, independent of any choice of \( \Delta t \).

\[
\varphi_i^n(u_{i+1}^{n+1}, u_{i-1}^{n+1}, u_i^n) = 0, \quad \forall i.
\]  

(5.14)

In the case of nondifferentiable \( \varphi^n_i \), we use the following definition of monotonicity:

**Definition 5.3** A discretization of the form (5.17) is monotone if either

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \geq \varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),
\]

\[
\forall \varepsilon^{n+1}_i, \varepsilon^{n+1}_i, \varepsilon^n_i \geq 0,
\]

or

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \leq \varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),
\]

\[
\forall \varepsilon^{n+1}_i, \varepsilon^{n+1}_i, \varepsilon^n_i \geq 0.
\]

Next, we prove the monotonicity of the fully implicit discretization.

**Lemma 5.4** The fully implicit discretization (5.13) is monotone, independent of any choice of \( \Delta t \) and \( \Delta x \).

**Proof:** For any given \( \varepsilon > 0 \), we just to check the next two equations:

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \geq \varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n),
\]

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) \leq \varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n).
\]

By the definition of \( \varphi^n_i \), we have

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) = u_i^n - u_{i+1}^{n+1} + \alpha_{i+1}^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] + \alpha_{i+1}^{n+1} \cdot \varepsilon
\]

\[
\geq u_i^n - u_{i+1}^{n+1} + \alpha_{i+1}^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}]
\]

\[
= \varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n)
\]

and

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n) = u_i^n - u_{i+1}^{n+1} + \alpha_{i+1}^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] - (2\alpha_{i+1}^{n+1} + 1) \cdot \varepsilon
\]

\[
\leq u_i^n - u_{i+1}^{n+1} + \alpha_{i+1}^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}]
\]

\[
\varphi^n_i(u_{i+1}^{n+1}, u_i^{n+1}, u_{i-1}^{n+1}, u_i^n).
\]
This completes the proof.

**Proof of Theorem 5.2**

By the results of Barles, we just to check that the fully implicit discretization is consistent, stable, monotone. Fristly, the formula (5.14) is a consistent discretization. Then Theorem 5.1 shows that the fully implicit discretization is monotone. So we need to prove the discretization is stable. Set

\[ U_{\text{max}}^n = \max \left( \max_i U_i^n, g^n, h^n \right), \quad U_{\text{min}}^n = \min \left( \min_i U_i^n, g^n, h^n \right). \]

where \( g^n, h^n \) is the boundary value of the \( n \)th times step. Using the same method as in Lemma 5.4, we have the more exact results:

\[ U_{\text{min}}^n \leq u^n_{i+1} \leq U_{\text{max}}^n. \]

Thus, we complete the proof.

### 5.4 The Superlinear Expectation

For reader convenience, we still use the same notions as in sublinear expectation (G-expectation), and show the main results of superlinear expectation.

**Definition 5.5** A superlinear expectation \( \hat{E} \) on \( \mathcal{H} \) is a functional \( \hat{E} : \mathcal{H} \rightarrow \mathbb{R} \) satisfying the following properties: for all \( X, Y \in \mathcal{H} \), we have

(a) Monotonicity: If \( X \geq Y \) then \( \hat{E}[X] \geq \hat{E}[Y] \);

(b) Constant preservation: \( \hat{E}[c] = c \);

(c) Sub-additivity: \( \hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y] \);

(d) Positive homogeneity: \( \hat{E}[\lambda X] = \lambda \hat{E}[X] \) for each \( \lambda \geq 0 \).

\( (\Omega, \mathcal{H}, \hat{E}) \) is called a sublinear expectation space.

The bounded boundary problem is

\[
\begin{align*}
\partial_t u - \frac{1}{2}(\sigma^2(D_{xx}^2 u)^+) - \bar{\sigma}^2(D_{xx}^2 u^-) &= 0, \\
u(0, x) &= \varphi(x), \quad x \in [a, b], \\
u(t, a) &= g(t), \quad u(t, b) = h(t) \quad t \in [0, T].
\end{align*}
\]

(5.15)

where \( \varphi, g, h \) are measurable functions.

The equation (5.15) can be discretized by a standard finite difference method with variable timeweighting to give
\[
    u_i^{n+1} - u_i^n = \theta \alpha_i^n [u_{i+1}^n - 2u_i^n + u_{i-1}^n] + (1 - \theta) \alpha_i^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}],
\]

where

\[
    \alpha_i^n := \frac{\sigma(\Gamma_i^n)^2 \Delta t_i}{2(x_i+1-x_i)(x_i-x_{i-1})},
\]
\[
    \sigma(\Gamma_i^n) := \begin{cases}
    \sigma, & \text{if } \Gamma_i^n \geq 0 \\
    \bar{\sigma}, & \text{if } \Gamma_i^n < 0
    \end{cases},
\]
\[
    \Gamma_i^n := \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(x_i+1-x_i)(x_i-x_{i-1})}.
\]

In this paper, we consider the fully implicit schemes with \( \theta = 0 \), i.e.,

\[
    u_i^{n+1} - u_i^n = \alpha_i^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}],
\]

The set of algebraic equation (5.15) is nonlinear for the formula of \( \sigma(\Gamma_i^n) \). So we consider the discrete equation at each node as

\[
    \phi_i^n := u_i^n - u_i^{n+1} + \alpha_i^{n+1} [u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}]
\]

For further analysis the Newton iteration, we rewrite the discrete equation (5.16) in matrix form. Let

\[
    U^{n+1} = [u_0^{n+1}, u_1^{n+1}, \cdots, u_m^{n+1}]', \quad U^n = [u_0^n, u_1^n, \cdots, u_m^n]',
\]
\[
    [M^n U^n]_i := -\alpha_i^n [u_{i+1}^n - 2u_i^n + u_{i-1}^n].
\]
\[
    [I + M^{n+1}] U^{n+1} = U^n,
\]

**Theorem 5.6** The fully implicit discretization (5.19) convergences to the solution of the equation (5.15), as \( \Delta t, \Delta x \to 0 \).

**References**

[1] G. Barles.(1997) Convergence of numerical schemes for degenerate parabolic equations arising in finance, In L. C. G. Rogers and D. Talay (Eds), Numerical Methods in Finance, 1-21.

[2] L. Denis, M. Hu and S. Peng.(2011) Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths, Potential Anal, 139-161.
[3] D.M. Pooey, P.A. Forsyth and K.R. Vetzal. (2003) \textit{Numerical convergence properties of option pricing PDEs with uncertain volatility}. IMA. J. of Numerical Analysis 23, 241-267.

[4] S. Peng. (2005) \textit{Nonlinear expectations and nonlinear Markov chains}, Chin. Ann. Math. 26B(2), 159–184.

[5] S. Peng. (2007) \textit{G-expectation, G-Brownian Motion and Related Stochastic Calculus of Itô type}, Stochastic analysis and applications, Abel Symp., 2, Springer, Berlin. 541-567.

[6] S. Peng. (2008) \textit{Multi-Dimensional G-Brownian Motion and Related Stochastic Calculus under G-Expectation}, Stochastic Processes and their Applications, 118(12), 2223-2253.

[7] S. Peng. (2009) \textit{Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations}, Science in China Series A: Mathematics, 52(7), 1391-1411.

[8] J.S. Pang, L. Qi. (1993) \textit{Nonsmooth equations: Motivation and algorithms}. SIAM Journal on Optimization 3, 443-465.

[9] L. Qi, J. Sun. (1993) \textit{A nonsmooth version of Newton's method}. Mathematical Programming 58, 353-367.

[10] D. Sun, J. Han. (1997) \textit{Newton and quasi-Newton methods for a class of nonsmooth equations and related problems}. SIAM Journal on Numerical Analysis 17, 33-38.