MEASURING COMODULES AND ENRICHMENT

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Abstract. We study the existence of universal measuring comodules \( Q(M,N) \) for a pair of modules \( M, N \) in a braided monoidal closed category, and the associated enrichment of the global category of modules over the monoidal global category of comodules. In the process, we use results for general fibred adjunctions encompassing the fibred structure of modules over monoids and the opfibred structure of comodules over comonoids. We also explore applications to the theory of Hopf modules.

1. Introduction

The generalized Sweedler dual endofunctor \((-)^{\circ} = P(-,I)\), left adjoint to the classic dual algebra functor, has been studied in the past few years in different contexts; first introduced as the finite or Sweedler dual in vector spaces [21], its generalized version acts on more general monoidal categories, usually modules over certain classes of commutative rings. Related works are discussed in [20], and fall under the latter’s establishment of the functor as an adjoint to \( \text{Hom}_R \) for any commutative ring \( R \), with the purpose of identifying conditions under which the Hopf structure is preserved.

Independently of these results, [11] (and previous related work in [22, 23]) investigates a generalization of the Sweedler dual construction in a broader direction. The universal measuring coalgebra \( P(A,B) \) for arbitrary \( k \)-algebras is already constructed in [21], defined by the property that algebra maps \( A \to \text{Hom}_k(C,B) \) are in natural bijection with coalgebra maps \( C \to P(A,B) \); the fact that this induces an enrichment of algebras in coalgebras was part of the mathematical folklore. In [11], we show the existence of the universal measuring comonoid in any braided monoidal category \( \mathcal{V} \) under some mild assumptions, covered by the general case of a locally presentable and monoidal closed category. Moreover, we examine preservation properties for bimonomial and Hopf monoid structures, and we establish an enrichment of the category of monoids \( \text{Mon}(\mathcal{V}) \) in comonoids \( \text{Comon}(\mathcal{V}) \) via an internal hom-like, opmonoidal action of the latter to the former.

The present work constitutes an extension of such a development to the setting of modules and comodules. Defined by an analogous property as \( P(A,B) \), the notion of a universal measuring comodule \( Q(M,N) \) for modules \( M, N \) over \( k \)-algebras was introduced in [3] for vector spaces: module morphisms \( M \to \text{Hom}_k(X,N) \) bijectively correspond to comodule morphisms \( X \to Q(M,N) \). These objects have already been employed in applications relatively to connections on bundles, loop algebras and representations.

Our aim is to once again generalize the existence of this object in any braided monoidal category. To this effect, it is natural to consider the fibrational structure of the global category of modules over algebras, where the fibre over a monoid \( A \) is the category of its modules \( \text{Mod}_V(A) \), as well as the opfibration of comodules over coalgebras (see also [8]). Then, the existence of \( Q(M,N) \) follows from a far more general result regarding adjunctions between (op)fibulations over arbitrary bases. Not only does this perspective provide with a better understanding of how these categories interrelate, but also gives more information about the structure of the universal measuring comodule than a plain adjoint functor theorem would. Moreover, the same theory of actions and enrichment that resulted in the establishment of the enrichment of monoids in comonoids in [11] shall be used to show that modules are enriched in comodules. Finally, we are interested in connections of universal measuring comodules with the theory of Hopf modules.

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The outline of this article is as follows: in Section 2 we gather some known facts about (co)monoids and (co)modules, local presentability properties as well as the theory of (opmonoidal) actions inducing (monoidal) enrichment, and the construction of the universal measuring comonoid. In Section 3, we explore conditions under which fibrred 1-cells between fibrations with arbitrary base have adjoints. In Section 4 we describe the global categories of modules and comodules, establish the existence of the universal measuring comodule and as a result the desired enrichment of modules in comodules. In Section 5, we investigate how the universal measuring comodule functor lifts to a functor between Hopf modules.

2. Background

In this section, we recall some of the main concepts and constructions needed for the development of the current work. In particular, we will summarize some of the key results from [11] pertinent to this paper. We assume familiarity with the basics of the theory of monoidal categories, found for example in [14].

2.1. (Co)monoids and (co)modules. Suppose $(\mathcal{V}, \otimes, I)$ is a monoidal category. A monoid is an object $A$ equipped with a multiplication $m: A \otimes A \to A$ and unit $\eta: I \to A$ that satisfy usual associativity and unit laws; along with monoid morphisms, they form a category $\text{Mon}(\mathcal{V})$. Dually, we have comonoids $(\mathcal{C}, \Delta: \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}, \varepsilon: \mathcal{C} \to I)$ whose category is denoted by $\text{Comon}(\mathcal{V})$. Both these categories are monoidal only if $\mathcal{V}$ is braided monoidal; if $\mathcal{V}$ is moreover symmetric, they inherit the symmetry.

If $F: \mathcal{V} \to \mathcal{W}$ is a lax monoidal functor, with structure maps $\phi_{A,B}: F(A \otimes B) \to F(A) \otimes B$ and $\psi_0: I \to F(I)$, it induces a map between their categories of monoids $\text{Mon}^F: \text{Mon}(\mathcal{V}) \to \text{Mon}(\mathcal{W})$ by $(A, m, \eta) \mapsto (FA, Fm \circ \phi_{\mathcal{A},A}, F\eta \circ \psi_0)$. Dually, oplax functors induce maps between the categories of comonoids.

Standard doctrinal adjunction arguments imply that oplax monoidal structures on left adjoints correspond bijectively to lax monoidal structures on right adjoints between monoidal categories; this generalizes to parametrized adjunctions, as found in [23, 3.2.3] or for higher dimension in [5, Prop. 2]. Therefore, if $\mathcal{V}$ is braided monoidal closed, the internal hom functor $[-,-]: \mathcal{V}^{\text{op}} \times \mathcal{V} \to \mathcal{V}$ obtains a lax monoidal structure as the parametrized adjoint of the strong monoidal tensor product functor $(- \otimes -)$. The induced functor between the monoids is denoted by

$$\text{Mon}[-,-]: \text{Comon}(\mathcal{V})^{\text{op}} \times \text{Mon}(\mathcal{V}) \to \text{Mon}(\mathcal{V});$$

for $C$ a comonoid and $A$ a monoid, $[C, A]$ has the convolution monoid structure.

If $A \in \text{Mon}(\mathcal{V})$, a (left) $A$-module is an object $M$ of $\mathcal{V}$ equipped with an arrow $\mu: A \otimes M \to M$ called action, such that the diagrams

$$\begin{array}{ccc}
A \otimes A \otimes M & \xrightarrow{m \otimes 1} & A \otimes M \\
\downarrow{1 \otimes \mu} & & \downarrow{\mu} \\
A \otimes M & \xrightarrow{\mu} & M
\end{array} \quad \text{and} \quad \begin{array}{ccc}
A \otimes M & \xrightarrow{\eta \otimes 1} & I \otimes M \\
\downarrow{1} & & \downarrow{l_M} \\
M & \xrightarrow{\mu} & M
\end{array}$$

commute. An $A$-module morphism $(M, \mu) \to (M', \mu')$ is an arrow $f: M \to M'$ in $\mathcal{V}$ such that $\mu' \circ (1 \otimes f) = f \circ \mu$. For any monoid $A$ in $\mathcal{V}$, there is a category $\text{Mod}_\mathcal{V}(A)$ of left $A$-modules and $A$-module morphisms. Dually, we have a category of (right) $C$-comodules $\text{Comod}_\mathcal{V}(C)$ for every $C \in \text{Comon}(\mathcal{V})$.

In a very similar way, we can define categories of right $A$-modules and left $C$-comodules. If $\mathcal{V}$ is symmetric, there is an obvious isomorphism between categories of left and right $A$-modules and left and right $C$-comodules, so usually there is no distinction in the notation.

A lax monoidal functor between monoidal categories $F: \mathcal{V} \to \mathcal{W}$, on top of inducing $\text{Mon}^F$ between their categories of monoids, it also induces functors

$$\text{Mod}^F: \text{Mod}_\mathcal{V}(A) \to \text{Mod}_\mathcal{W}(FA)$$

where the $FA$-action on $FM$ is $FA \otimes FM \xrightarrow{\phi_{A,M}} F(A \otimes M) \xrightarrow{F\mu} FM$. In particular, the lax monoidal $[-,-]: \mathcal{V}^{\text{op}} \times \mathcal{V} \to \mathcal{V}$ in a braided monoidal closed category gives $\text{Mon}[-,-]$. 


of (1), and the induced modules functor is, for any comonoid \( C \) and monoid \( A \),
\[
\text{Mod}_{CA}[-, -] : \text{Comod}_V(C) \otimes \text{Mod}_V(A) \to \text{Mod}_V([C, A])
\] (3)
mapping a \( C \)-comodule \( X \) and an \( A \)-module \( M \) to \([X, A]\) with a \([C, A]\)-action.

Each monoid morphism \( f : A \to B \) determines a restriction of scalars functor
\[
f^* : \text{Mod}_V(B) \to \text{Mod}_V(A)
\]
which makes every \( B \)-module \((N, \mu)\) into an \( A \)-module \( f^*N \) via the action \( A \otimes N \xrightarrow{f \otimes 1} B \otimes N \xrightarrow{\mu} N \). We have a commutative triangle of categories and functors

\[
\begin{tikzcd}
\text{Mod}_V(B) \ar{dr}{f^*} \ar{r}{\cdot} & \text{Mod}_V(A) \\
& \mathcal{V}
\end{tikzcd}
\]

Dually, we have the corestriction of scalars \( g_! : \text{Comod}_V(C) \to \text{Comod}_V(D) \) which commutes with the comonadic forgetful to \( \mathcal{V} \). Notice how \( f^* \) preserves all limits and \( g_! \) all colimits that exist in \( \mathcal{V} \).

2.2. Local presentability. In this section we collect some known facts about locally presentable categories.

Recall that a category is \( \kappa \)-presentable, for a regular cardinal \( \kappa \), if each subcategory with less than \( \kappa \) arrows is the base of a co-cone. A \( \kappa \)-filtered colimit is a colimit of a functor whose domain is a \( \kappa \)-filtered category.

An accessible category \( \mathcal{C} \) is a category, with a small set of \( \kappa \)-presentable objects (i.e. objects \( C \) such that \( \mathcal{C}(C, -) \) preserves \( \kappa \)-filtered colimits) such that every object in \( \mathcal{C} \) is the \( \kappa \)-filtered colimit of presentable objects, for some regular cardinal \( \kappa \). A locally presentable category is an accessible category that is cocomplete. A functor between accessible categories is accessible if it preserves \( \kappa \)-filtered colimits, for some regular cardinal \( \kappa \). We refer the reader to [17, 1] for more on the theory of locally presentable categories.

If a monoidal category \( V \) is locally presentable, and moreover the tensor product is accessible in each variables (as is the case when \( V \) is closed), the categories \( \text{Mon}(V) \) and \( \text{Comon}(V) \) are both locally presentable. This result can be found in [19, § 2], and in fact it follows from the much more general ‘Limit Theorem’ [17, 5.1.6] since both categories can be written as 2-categorical limit of accessible functors.

Local presentability of \( \text{Mon}(V) \) can also be deduced from the following well-known result, where the first part can be found e.g. in [6, Satz 10.3] and the second e.g. in [23, 3.4.3].

**Theorem 2.1.** Suppose that \( \mathcal{C} \) is a locally presentable category.

- If \((T, m, \eta)\) is an accessible monad on \( \mathcal{C} \), the category of Eilenberg-Moore algebras \( \mathcal{C}^T \) is locally presentable.
- If \((S, \Delta, \epsilon)\) is an accessible comonad on \( \mathcal{C} \), the category of Eilenberg-Moore coalgebras \( \mathcal{C}^S \) is locally presentable.

An important fact which will be used repeatedly is that any cocontinuous functor with domain a locally presentable category has a right adjoint; this can be obtained as a corollary to the following adjoint functor theorem, since the set of presentable objects form a small dense subcategory of any locally presentable category.

**Theorem 2.2.** [15, 5.33] If the cocomplete \( \mathcal{C} \) has a small dense subcategory, every cocontinuous \( S : \mathcal{C} \to \mathcal{B} \) has a right adjoint.

As an application, we can deduce the following, proved in [11, § 2.11].

**Proposition 2.3.** For \( V \) a locally presentable braided monoidal closed category, \( \text{Comon}(V) \) is comonadic over \( \mathcal{V} \), and also monoidal closed; denote its internal hom by
\[
\text{Hom} : \text{Comon}(V)^{\text{op}} \times \text{Comon}(V) \to \text{Comon}(V).
\]
It is well-known that $A$-modules $\text{Mod}_V(A)$ and $C$-comodules $\text{Comod}_V(C)$ are respectively monadic and comonadic over $V$, via the monad $(A \otimes -, \eta \otimes -, m \otimes -)$ and the comonad $(- \otimes C, - \otimes \epsilon, - \otimes \Delta)$ on $V$. Due to that, the categories of modules and comodules often inherit the locally presentable structure from $V$; this follows from 2.1, and in particular it generalizes the results of [18] for $V = \text{Mod}_R$.

**Proposition 2.4.** Suppose $V$ is a locally presentable monoidal category, such that $\otimes$ is accessible in each variable. Then $\text{Mod}_V(A)$ for any monoid $A$ and $\text{Comod}_V(C)$ for any comonoid $C$ are locally presentable categories.

2.3. Actions and enrichment. Briefly recall [13] that an action of a monoidal category on an ordinary one is given by a functor $*: V \times D \to D$ expressing that $D$ is a pseudomodule for the pseudomonoid $V$ in the monoidal 2-category $(\text{Cat}, \times, 1)$; the strict version was (2).

An opmonoidal action of a braided monoidal category on a monoidal category is one inside $(\text{MonCat}^{op} \times \text{MonCat}^{op})$, the monoidal 2-category of monoidal categories, lax monoidal functors and monoidal natural transformations. A braided opmonoidal action on a braided monoidal category is one where the opmonoidal $*$ is braided.

As a central example, we have the action of the opposite monoidal category on itself via the internal hom, see [11, 3.7k5.1].

**Lemma 2.5.** Suppose $V$ is a braided monoidal closed category. The internal hom $[\_, \_] : V^{op} \times V \to V$ is a (lax) monoidal action of $V^{op}$ on $V$. It induces an action $\text{Mon}[\_, \_] : \text{Comon}(V)^{op} \times \text{Mon}(V) \to \text{Mon}(V)$. If $V$ is symmetric, the monoidal action $[\_, \_]$ is braided, and $\text{Mon}[\_, \_]$ is lax monoidal.

Taking opposites in the lemma, $[\_, \_]^{op}$ is an opmonoidal action of $V$ on $V^{op}$, and in the symmetric case, $\text{Mon}[\_, \_]^{op}$ is an opmonoidal action of $\text{Comon}(V)$ on $\text{Mon}(V)^{op}$.

The following two theorems give conditions under which an action induces an enrichment, and an opmonoidal action induces a monoidal enrichment. Recall that a $V$-enriched category $A$ is a monoidal $V$-category if it is equipped with a $V$-functor $\Box : A \otimes A \to A$, an object $J \in A$ and $V$-natural isomorphisms $(X \Box Y) \Box Z \cong X \Box (Y \Box Z)$, $(J \Box A) \cong A \cong (A \Box J)$ such that the underlying functor $\Box_0$ renders $(A_0, \Box_0, J)$ into a monoidal category.

**Theorem 2.6.** Suppose that $V$ is a monoidal category which acts on a category $D$ via a functor $* : V \times D \to D$, such that $-_*D$ has a right adjoint $F(D, -)$ for every $D \in D$ with a natural isomorphism

$$D(X * D, E) \cong V(X, F(D, E)).$$

Then we can enrich $D$ in $V$, in the sense that there is a $V$-category $\mathcal{D}$ with hom-objects $\mathcal{D}(A, B) = F(A, B)$ and underlying category $D$. Moreover, if $V$ is monoidal closed, the $\mathcal{D}$ is tensored, with $X * D$ the tensor of $X \in V$ and $D \in D$.

If $V$ is furthermore symmetric, the enrichment is cotensored if $X * -$ has a right adjoint; finally, we can also enrich $D^{op}$ in $V$.

The above follows from a much stronger result of [7] regarding categories enriched in bicategories; details can be found in [13] and [23, § 4.3].

**Theorem 2.7.** [11, Thm. 3.6] Suppose that $V$ is a braided monoidal category with an opmonoidal action on the monoidal category $D$. Then, the induced enriched category $\mathcal{D}$ is monoidally $V$-enriched, with underlying monoidal category $D$.

2.4. Universal measuring comonoid. One of the basic goals of [11] was to establish an enrichment of the category of monoids in the category of comonoids, under certain assumptions on $V$. Below we summarize the basic results; details can be found in Sections 4 and 5 therein.

**Theorem 2.8.** [11, Thm. 4.1] If $V$ is locally presentable braided monoidal closed category, the functor $\text{Mon}[\_, B]^{op} : \text{Comon}(V) \to \text{Mon}(V)^{op}$ has a right adjoint $P(-, B)$, i.e. there is a natural isomorphism

$$\text{Mon}(V)(A, [C, B]) \cong \text{Comon}(V)(C, P(A, B)).$$
The parametrized adjoint $P : \text{Mon}(\mathcal{V})^{\text{op}} \times \text{Mon}(\mathcal{V}) \to \text{Comon}(\mathcal{V})$ of $\text{Mon}[-,-]$ is called the Sweedler hom, and $P(A,B)$ is called the universal measuring comonoid. In particular, $P(A,I)$ is called the finite dual of the monoid $A$. When $\mathcal{V}$ is the category of vector spaces over a field, and $A$ is a $k$-algebra, $P(A,k)$ is the well-known Sweedler or finite dual $A^\circ$ of $A$; see [21].

The action $\text{Mon}[C,-]^{\text{op}}$ has a right adjoint $(C \triangleright -)^{\text{op}}$, and the functor of two variables $\triangleright : \text{Comon}(\mathcal{V}) \times \text{Mon}(\mathcal{V}) \to \text{Mon}(\mathcal{V})$ is called the Sweedler product in [2].

By applying Theorems 2.6 and 2.7 for $\mathcal{V} = \text{Comon}(\mathcal{V})$ and $\mathcal{D} = \text{Mon}(\mathcal{V})^{\text{op}}$, we obtain the desired enrichment.

**Theorem 2.9.** Suppose $\mathcal{V}$ is a locally presentable symmetric monoidal closed category.

1. The category $\text{Mon}(\mathcal{V})^{\text{op}}$ is a monoidal tensored and cotensored $\text{Comon}(\mathcal{V})$-category, with hom-objects $\text{Mon}(\mathcal{V})^{\text{op}}(A,B) = P(B,A)$.
2. The category $\text{Mon}(\mathcal{V})$ is a monoidal $\text{Comon}(\mathcal{V})$-category, tensored and cotensored, with $\text{Mon}(\mathcal{V})(A,B) = P(A,B)$, cotensor $[C,B]$ and tensor $C \triangleright B$ for any comonoid $C$ and monoid $B$.

3. **Existence of General Fibred Adjoints**

In this section, we review some basic definitions and constructions regarding adjunctions between fibred categories. Following the terminology and results of [9, 12], the goal is to extend the existing theory by examining under which assumptions a fibred 1-cell between fibrations over different bases has a (fibred) adjoint. A detailed treatment of relevant issues can also be found in [23, §5].

**3.1. Basic definitions.** Briefly really that $P : \mathcal{A} \to \mathcal{X}$ is a (cloven) fibration if and only if for all $f : X \to Y$ in $\mathcal{X}$ and $B \in \mathcal{A}_Y$, there is a canonical cartesian lifting of $B$ along $f$ denoted by $\text{Cart}(f,B) : f^*(B) \to B$, and dually for an opfibration. $\mathcal{A}$ is the total category, $\mathcal{X}$ is the base and $\mathcal{A}_X$ of objects above $X$ and morphisms above $\text{id}_X$ is the fibre category. Any arrow in the total category of an (op)fibration factorizes uniquely into a vertical morphism followed by a (co)cartesian one:

\[
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\downarrow{\gamma} & & \downarrow{\delta} \\
C & \xleftarrow{\alpha} & D
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
\text{Cart}(f,B) & \xrightarrow{f} & Y \\
\downarrow{\gamma} & & \downarrow{\delta} \\
\text{Cocart}(g,C) & \xleftarrow{g} & X
\end{array}
\end{array}
\]

(Co)cartesian liftings are unique up to vertical isomorphism.

For every morphism $f : X \to Y$ in the base $\mathcal{X}$, we have the so-called reindexing functor

\[
f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X
\]

which maps each object to the domain of the cartesian lifting along $f$. It can be verified that $1_{\mathcal{A}_X} \cong (1_{\mathcal{A}})^*$ and that for composable morphism in the base category, $(g \circ f)^* \cong g^* \circ f^*$. If these isomorphisms are equalities, we have the notion of a split fibration.

We now turn to the appropriate notions of 1-cells and 2-cells for fibrations. A morphism of fibrations $(S,F) : P \to Q$ between $P : \mathcal{A} \to \mathcal{X}$ and $Q : \mathcal{B} \to \mathcal{Y}$ is given by a commutative square of functors and categories

\[
\begin{array}{c}
\begin{array}{ccc}
A & \xrightarrow{S} & B \\
\downarrow{P} & & \downarrow{Q} \\
\mathcal{X} & \xleftarrow{F} & \mathcal{Y}
\end{array}
\end{array}
\]

where $S$ preserves cartesian arrows, meaning that if $\phi$ is $P$-cartesian, then $S\phi$ is $Q$-cartesian. The pair $(S,F)$ is called a fibred 1-cell. In particular, when $P$ and $Q$ are fibrations over the same base category $\mathcal{X}$, we may consider fibred 1-cells of the form $(S,1_{\mathcal{X}})$ when $S$ is called a
fibred functor. Dually, we have the notion of an \emph{opfibred 1-cell} \((K, F)\) and \emph{opfibred functor} \((K, 1_X)\).

Any fibred or opfibred 1-cell determines a collection of functors \(\{S_X : A_X \to B_{FX}\}\) between the fibre categories for all \(X \in \text{ob} X\):

\[
\begin{array}{ccc}
S_X : A_X & \xrightarrow{S|X} & B_{FX} \\
\downarrow^{SA} & & \downarrow^{SF} \\
A' & \xrightarrow{Sf} & A''
\end{array}
\]

This is well-defined, since \(Q(SA) = F(PA) = FX\) by commmutativity of \((5)\), and \(Q(Sf) = F(Pf) = F(1_X) = 1_{FX}\) so \(Sf\) is vertical.

Now given two fibred 1-cells \((S, F)\) and \((T, G)\) between fibrations \(P : A \to X\) and \(Q : B \to Y\), a \emph{fibred 2-cell} from \((S, F)\) to \((T, G)\) is a pair of natural transformations \((\alpha : S \Rightarrow T, \beta : F \Rightarrow G)\) with \(\alpha\) above \(\beta\), i.e. \(Q(\alpha_A) = \beta_{PA}\) for all \(A \in A\). We can display a fibred 2-cell \((\alpha, \beta)\) between two fibred 1-cells as

\[
\begin{array}{ccc}
A & \xrightarrow{S} & B \\
\downarrow^{\alpha} & & \downarrow^{\beta} \\
X & \xrightarrow{T} & Y
\end{array}
\]

In particular, when \(P\) and \(Q\) are fibrations over the same base category \(X\), we may consider fibred 2-cells of the form \((\alpha, 1_{1_X}) : (S, 1_X) \Rightarrow (T, 1_X)\) displayed as

\[
\begin{array}{ccc}
A & \xrightarrow{S} & B \\
\downarrow^{\alpha} & & \downarrow^{\beta} \\
X & \xrightarrow{T} & Y
\end{array}
\]

which are in fact natural transformations whose components are vertical arrows, \(Q(\alpha_A) = 1_{PA}\). A 2-cell like this is called a \emph{fibred natural transformation}. Dually, we have the notion of an \emph{opfibred 2-cell} and \emph{opfibred natural transformation} between opfibred 1-cells and functors respectively.

In this way, we obtain a 2-category \(\text{Fib}\) of fibrations over arbitrary base categories, fibred 1-cells and fibred 2-cells, with the evident compositions coming from \(\text{Cat}\). In particular, there is a 2-category \(\text{Fib}(X)\) of fibrations over a fixed base category \(X\), fibred functors and fibred natural transformations. Dually, we have the 2-categories \(\text{OpFib}\) and \(\text{OpFib}(X)\).

The fundamental Grothendieck construction (see e.g. \([4]\)) establishes a 2-equivalence

\[
\text{Fib}(X) \simeq [X^{op}, \text{Cat}]_{ps} = \text{ICat}(X) : \mathcal{G}
\]

between the 2-category of fibrations over \(X\) and the 2-category of \(X\)-indexed categories i.e. pseudofunctors \(X^{op} \to \text{Cat}\), pseudonatural transformations and modifications. If \(\mathcal{M}\) is such a pseudofunctor, then the fibres \((\mathcal{G}\mathcal{M})_X\) are \(\mathcal{M}X\), and the reindexing functors \(f^*\) are \(\mathcal{M}f\); we can then view an object in any fibred category as a pair \((A, X) \in A_X \times X\), and a morphism \((A, X) \to (B, Y)\) as a pair \((\phi : A \to f^*B, f : X \to Y)\) by definition of the corresponding Grothendieck category.

Moreover, there is also a 2-equivalence \(\text{ICat} \simeq \text{Fib}\) between fibrations over arbitrary bases and an appropriately defined 2-category of indexed categories with arbitrary domain; for more details, see \([10]\). Along with the dual versions for opfibrations, these equivalences allow us to freely change our perspective from (op)fibrations to indexed categories. For example, we can realize the following lemma either using fibred theory or pseudofunctors.
**Lemma 3.1.** Suppose we have two fibrations $P : A \to X$, $Q : B \to Y$ and a fibred 1-cell $(S, F)$ as in (5). Then the reindexing functors commute up to isomorphism with the induced functors between the fibres: there is a natural isomorphism

\[
\begin{array}{ccc}
A_Y & \xrightarrow{S_Y} & B_{FY} \\
\downarrow{f^*} & \cong & \downarrow{(Ff)^*} \\
A_X & \xrightarrow{S_X} & B_{FX}
\end{array}
\]

(7)

for $f : X \to Y$ in $\mathcal{X}$. In particular, for $S$ a fibred functor, $\tau f : f^* \circ S_Y \cong S_X \circ f^*$.

### 3.2. Fibred adjunctions

The notions of fibred and opfibred adjunction arise from the general definition of an adjunction in a 2-category, applied to $(\text{OpFib})$ and $(\text{OpFib}(X))$.

**Definition 3.2.** Given fibrations $P : A \to X$ and $Q : B \to Y$, a **general fibred adjunction** is given by a pair of fibred 1-cells $(L, F) : P \to Q$ and $(R, G) : Q \to P$ together with fibred 2-cells $(\zeta, \eta) : (1_A, 1_X) \Rightarrow (RL, GF)$ and $(\xi, \varepsilon) : (LR, FG) \Rightarrow (1_B, 1_Y)$ such that $L \dashv R$ via $\zeta, \xi$ and $F \dashv G$ via $\eta, \varepsilon$. This is displayed as

\[
\begin{array}{ccc}
A & \xleftarrow{L} & B \\
\downarrow{P} & \cong & \downarrow{Q} \\
X & \xleftarrow{F} & Y
\end{array}
\]

and we write $(L, F) \dashv (R, G) : Q \to P$. In particular, a **fibred adjunction** is

\[
\begin{array}{ccc}
A & \xleftarrow{L} & B \\
\downarrow{P} & \cong & \downarrow{Q} \\
X & \xleftarrow{F} & Y
\end{array}
\]

(8)

Notice that by definition, $\zeta$ is above $\eta$ and $\xi$ is above $\varepsilon$, hence $(P, Q)$ is in particular a map between adjunctions in the ordinary sense. Dually, we have the notions of **general opfibred adjunction** and **opfibred adjunction** for adjunctions in $\text{OpFib}$ and $\text{OpFib}(X)$.

It is clear that a fibred adjunction as in (8) induces fibrewise adjunctions

\[
\begin{array}{ccc}
A_X & \xleftarrow{L_X} & B_X \\
\downarrow{R_X} & \cong & \downarrow{} \\
\end{array}
\]

between the fibre categories. In the converse direction, we have the following result, see for example [4, 8.4.2] or [12, 1.8.9].

**Proposition 3.3.** Suppose $S : Q \to P$ is a fibred functor. Then $S$ has a fibred left adjoint $L$ if and only if for each $X \in \mathcal{X}$ we have $L_X \dashv S_X$, and the components $\chi_A : (L_X \circ f^*)A \to (f^* \circ L_Y)A$ of the mate of the canonical $\tau f : f^* \circ S_Y \cong S_X \circ f^*$ (7) are isomorphisms. Similarly, $S$ has a fibred right adjoint $R$ iff $S_X \dashv R_X$ and the canonical $(f^* \circ R_Y)B \to (R_X \circ f^*)B$ are isomorphisms.

Notice that in order to define an ordinary left adjoint $L : A \to B$ of $S$, the fibrewise adjunctions alone are sufficient. That $\chi$ is an isomorphism ensures that this adjoint is also cartesian, therefore constitutes a fibred adjoint of $K$. On the other hand, for the existence of a right adjoint of $S$, the mate of $\tau f$ being an isomorphism is required for the very construction of the functor $R$; this fact depicts a certain asymmetry between the existence of left and right adjoint functors between fibrations.
There are dual results concerning fibrewise adjunctions between opfibrations over a fixed base. These give rise to questions concerning adjunctions between fibrations over two different bases; in this direction, Theorem 3.6 below generalizes the dual of the above proposition. We primarily consider opfibrations because of the applications that follow.

**Lemma 3.4.** Suppose \((K, F) : U \to V\) is an opfibred 1-cell

\[
\begin{array}{c}
\mathcal{C} \\
\downarrow K \\
\uparrow U \\
\mathcal{D} \\
\downarrow V \\
\downarrow F \\
\mathcal{Y}
\end{array}
\]

and there is an adjunction \(F \dashv G\) between the base categories with counit \(\varepsilon\). If, for each \(Y \in \mathcal{Y}\), the composite functor between the fibres

\[
C_{GY} \xrightarrow{K_{GY}} D_{FGY} \xrightarrow{(\varepsilon_Y)_!} D_Y
\]

has a right adjoint \(R_Y\), then \(K : \mathcal{C} \to \mathcal{D}\) between the total categories has a right adjoint, with \(R(-)\) its mapping on objects.

**Proof.** We will show that there exists a bijection

\[
D(KC, D) \cong C(C, R_Y D)
\]

natural in \(C \in \mathcal{C}_X\), hence the assignment \(D \mapsto R_Y D\) canonically extends to a right adjoint functor \(R : \mathcal{D} \to \mathcal{C}\) of \(K\).

An element of the left hand side of (10) is an arrow \(m : KC \to D\) in \(\mathcal{D}\), i.e.

\[
\begin{cases}
f!(KC) \xrightarrow{k} D & \text{in } \mathcal{D}_Y \\
FX \xrightarrow{l} Y & \text{in } \mathcal{Y}
\end{cases}
\]

with \(k\) the unique vertical arrow of the factorization \(m = k \circ \text{Cocart}(f, KC)\) (4).

An element of the right hand side of (10) is \(n : C \to R_Y D\) in \(\mathcal{C}\), i.e.

\[
\begin{cases}
g!C \xrightarrow{l} R_Y D & \text{in } \mathcal{C}_{GY} \\
X \xrightarrow{\tilde{g}} GY & \text{in } \mathcal{X}
\end{cases}
\]

where \(n = l \circ \text{Cocart}(g, C)\). By hypothesis, this corresponds bijectively to a pair

\[
\begin{cases}
((\varepsilon_Y)_! K_{GY})(g!C) \xrightarrow{\tilde{l}} D & \text{in } \mathcal{D}_Y \\
FX \xrightarrow{\tilde{g}} Y & \text{in } \mathcal{Y}
\end{cases}
\]

where \(\tilde{l}\) is the adjunct of \(l\) under \((\varepsilon_Y)_! K_{GY} \dashv R_Y\) and \(\tilde{g}\) is the adjunct of \(g\) under \(F \dashv G\). In order for this pair to be as in (11), it is enough to show that \((\varepsilon_Y)_! K_{GY})(g!C) \cong \tilde{g}(KC)\) in the fibre \(\mathcal{D}_Y\). For that, observe that the diagram

\[
\begin{array}{ccc}
\mathcal{C}_X & \xrightarrow{g!} & \mathcal{C}_{GY} \\
\downarrow K_X \downarrow K_{GY} \downarrow \mathcal{D}_{FGY} \\
\mathcal{D}_{FX} & \xrightarrow{g!} & \mathcal{D}_Y \\
\downarrow \tilde{g}_! \downarrow (g!)_! \downarrow \mathcal{D}_Y \\
\mathcal{C}_Y & \xrightarrow{(\varepsilon_Y)_!} & \mathcal{D}_Y
\end{array}
\]

commutes up to isomorphism: the left part by the dual of (7) for the cocartesian \(K\), and the right part is \(\tilde{g}_! = (Fg \circ \varepsilon_Y)_! \cong (Fg)_! \circ (\varepsilon_Y)_!\). Naturality in \(C\) can be verified, so we obtain a right adjoint \(R\) of \(K\) between the total categories. \(\square\)

**Corollary 3.5.** Under the above assumptions, \((K, F) \dashv (R, G)\) is an adjunction in \(\text{Cat}^2\).
Proof. If $\sigma : (Ff)_! K_X \cong K_{G_Y} f_!$ by cocartesianness of $K$, and $\nu : (\varepsilon_W)_! (FGh)_! \cong (\varepsilon_W \circ F Gh)_! = (h \circ \varepsilon_Y)_! \cong h_! (\varepsilon_Y)_!$ by naturality of $\varepsilon$, we can form an invertible 2-cell

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{C}_{G_Y} & \xrightarrow{\eta} & \mathcal{D}_{F G Y} \cong \mathcal{D}_{Y} \\
(Gh)_! & \cong & \varepsilon_Gh \cong h_! \\
\mathcal{C}_{G_W} & \xrightarrow{\omega} & \mathcal{D}_{F G W} \cong \mathcal{D}_{W} \\
(Gh)_! & \cong & \varepsilon_Gh \cong h_!
\end{array}
\end{array} \quad (12)$$

Its mate $\omega$ under the adjunctions $(\varepsilon_Y)_! K_{G_Y} \dashv R_Y$ and $(\varepsilon_W)_! K_{G_W} \dashv R_W$ has components,

$$\begin{array}{c}
\begin{array}{ccc}
(Gh)_! R_Y D & \xrightarrow{\eta^W} & (R_W((\varepsilon_W)_! K_{G_W}))(Gh)_! R_Y D \xrightarrow{R_W \sigma \omega} R_W (h_! (\varepsilon_Y)_! K_{G_Y}) R_Y D \\
\omega_D & \xrightarrow{\omega_D} & (R_W h_! \varepsilon_Y)
\end{array}
\end{array} \quad (13)$$

where $\eta$ and $\varepsilon$ are the unit and counit of the adjunctions $\varepsilon_{(-)} K_{G(-)} \dashv R_{(-)}$. These are required for the constructed mapping of $R$ on morphisms:

$$\begin{array}{c}
\begin{array}{ccc}
D & \xrightarrow{k} & E \\
\xrightarrow{\psi} & \xrightarrow{} & \xrightarrow{R_Y D} \\
\xrightarrow{\text{Cocart}(h, D)} & \xrightarrow{h_! D} & \xrightarrow{R_W E} \\
\xrightarrow{\text{Cocart}(Gh, R_W D)} & \xrightarrow{R_W (h_! D)} & \xrightarrow{(Gh)_! R_Y D} \\
\xrightarrow{\omega_D} & \xrightarrow{(Gh)_! R_W D} & \xrightarrow{\omega_D} \\
\xrightarrow{Y} & \xrightarrow{h} & \xrightarrow{W} \\
\xrightarrow{\text{in } \mathcal{D}} & \xrightarrow{\text{in } \mathcal{Y}} & \xrightarrow{\text{in } \mathcal{X}}
\end{array}
\end{array} \quad (14)$$

It is now not hard to verify that the adjoints commute with the opfibrations, $U \circ R = G \circ V$, and moreover, if $(\zeta, \xi)$ is the unit and counit of $K \dashv R$, the pairs $(\zeta, \eta)$ and $(\xi, \varepsilon)$ are above each other.

The following generalizes dual Proposition 3.3, for general fibred adjunctions.

**Theorem 3.6.** Suppose $(K, F) : U \to V$ is an opfibrd 1-cell and $F \dashv G$ is an adjunction between the bases of the fibrations, as in

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\xrightarrow{U} & \xrightarrow{V} & \xrightarrow{G} \mathcal{Y}.
\end{array}$$

If the composite (9) has a right adjoint for each $Y \in \mathcal{Y}$, then $K$ has a right adjoint $R$ between the total categories, with $(K, F) \dashv (R, G)$ in $\text{Cat}^2$. If the mate

$$\begin{array}{c}
\begin{array}{ccc}
\mathcal{D}_Y & \xrightarrow{R_Y} & \mathcal{C}_{G_Y} \\
\xrightarrow{h_!} & \xrightarrow{\not\exists \omega \, \text{in } \mathcal{Y}} & \xrightarrow{(Gh)_!} \\
\mathcal{D}_W & \xrightarrow{R_W} & \mathcal{C}_{G_W}
\end{array}
\end{array}$$

of the composite invertible 2-cell (12) is moreover an isomorphism for any $h$, then $R$ is cocartesian and so $(K, F) \dashv (R, G)$ is a general opfibrd adjunction.

Conversely, if $(K, F) \dashv (R, G)$ in $\text{OpFib}$, then evidently $F \dashv G$, $K \dashv R$, $R$ is cocartesian, and moreover for every $Y \in \mathcal{Y}$ there is an adjunction $(\varepsilon_Y)_! K_{G_Y} \dashv R_Y$. 

Proof. The first part is established by Lemma 3.4, Corollary 3.5 and noticing that if we request that the $\omega_D$’s as in (13) are isomorphisms, inserting $k = \text{Cocart}(g, D)$ in the composite (14) exhibits $R$-cocartesianness.

For the converse, start with some $f : C \to R_Y D$ in $\mathcal{C}_{GY}$. There is a bijective correspondence

\[
\begin{array}{c}
(C, GY) \\ \downarrow \text{(f,1,GY)}
\end{array} \quad \begin{array}{c}
(R_Y D, GY) \equiv R(D, Y)
\end{array} \quad \text{in } \mathcal{C}
\]

\[
K(C, GY) \equiv (K_G Y C, FGY) \quad \begin{array}{c}
\downarrow \text{(f,ex)}
\end{array} \quad \begin{array}{c}
(D, Y)
\end{array} \quad \text{in } \mathcal{D}
\]

since $K \dashv R$, where the latter morphism is uniquely determined by a vertical $(\varepsilon_Y) : K_{GY} C \to D$ in $\mathcal{D}_Y$; hence the required fibrewise adjunction is established. □

Dually, we get the following version about adjunctions between fibrations.

**Theorem 3.7.** Suppose $(S, G) : Q \to P$ is a fibred 1-cell between two fibrations and $F \dashv G$ is an adjunction between the bases, as shown in the diagram

\[
\begin{array}{c}
A \\ S \\ B \\ P \\
\downarrow \text{f} \\
X \\ \downarrow \text{G} \\
Q \\ \downarrow \text{Y}
\end{array}
\]

If, for each $X \in \mathfrak{X}$, the composite functor $B_{FX} \overset{S_{FX}}{\to} A_{GFX} \overset{\eta_X^*}{\to} A_X$ has a left adjoint $L_X$, then $S$ has a left adjoint $L$ between the total categories, with $(L, F) \dashv (S, G)$ in $\text{Cat}^2$. Furthermore, if the mate

\[
\begin{array}{c}
A_Z \\ \downarrow \text{(f,ex)}
\end{array} \quad \begin{array}{c}
L_X \quad B_{FX} \\
\downarrow \text{LX} \\
A_X \\
\downarrow \text{FX}
\end{array}
\]

of the composite isomorphism

\[
\begin{array}{c}
B_{FX} \\ \downarrow \text{FX}
\end{array} \quad \begin{array}{c}
A_{GFX} \\ \downarrow \text{(\eta_X)^*}
\end{array} \quad \begin{array}{c}
A_Z \\
\downarrow \text{FX}
\end{array} \quad \begin{array}{c}
\text{(\eta_X)^*}
\end{array}
\]

is invertible for any $f : X \to Z$ in $\mathfrak{X}$, then $(L, F) \dashv (S, G)$ is a general fibred adjunction. Conversely, if $(L, F) \dashv (S, G)$ is an adjunction in $\text{Fib}$, we have adjunctions $L_X \dashv \eta_X^* S_{FX}$ for all $X \in \mathfrak{X}$.

In the above composite 2-cell, the 2-isomorphism $\tau^{\text{FJ}}$ comes from the cartesian functor $S$ as in (7) and $\kappa$ from naturality of $\eta$, the unit of the base adjunction.

4. Enrichment of modules in comodules

In this section, we will describe the total categories of modules and comodules over monoids and comonoids in a monoidal category $V$. The main result from the previous section, Theorem 3.6, is essential in order to establish an enrichment relation between them. This enrichment is directly connected to the enrichment of monoids in comonoids of $\mathbb{11}$, as sketched in Section 2.4. In what follows, emphasis is given to comodules because they will serve as the enrichment basis.
4.1. Global categories of modules and comodules. Suppose $\mathcal{V}$ is a monoidal category.

**Definition 4.1.** The **global category of modules** $\text{Mod}$ is the category of all $\mathcal{C}$-comodules $X$ for any comonoid $\mathcal{C}$, denoted by $\mathcal{X}_C$. A morphism $k_{\mathcal{C}} : X_{\mathcal{C}} \rightarrow Y_{\mathcal{D}}$ for $X$ a $\mathcal{C}$-comodule and $Y$ a $\mathcal{D}$-comodule consists of a comonoid morphism $g : \mathcal{C} \rightarrow \mathcal{D}$ and an arrow $k : X \rightarrow Y$ in $\mathcal{V}$ which makes the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\delta} & X \otimes C \\
& k \downarrow & \downarrow 1 \otimes g \\
Y & \xrightarrow{\delta} & Y \otimes D
\end{array}
\]

commute. Dually, the **global category of modules** $\text{Mod}$ has as objects all $\mathcal{A}$-modules $M$ for any monoid $\mathcal{A}$, and morphisms are $p_f : M_\mathcal{A} \rightarrow N_{\mathcal{B}}$ where $f : A \rightarrow B$ is a monoid morphism and $p : M \rightarrow N$ makes the dual diagram commute.

Conventionally, the modules considered will be left and the comodules will be right. There are obvious forgetful functors

\[
G : \text{Mod} \rightarrow \text{Mon}(\mathcal{V}) \quad \text{and} \quad V : \text{Comod} \rightarrow \text{Comon}(\mathcal{V})
\]

which simply map any module $M_{\mathcal{A}}/\text{comodule } X_{\mathcal{C}}$ to its monoid $\mathcal{A}/\text{comonoid } \mathcal{C}$. In fact, $G$ is a split fibration and $V$ is a split opfibration: the descriptions of the global categories agree with the Grothendieck categories for the functors

\[
\begin{array}{ccc}
\text{Mon}(\mathcal{V})^{\text{op}} & \xrightarrow{\text{Mod}_\mathcal{V}} & \text{Cat} \\
\mathcal{A} & \xrightarrow{f} & \text{Mod}_\mathcal{V}(A) \\
\downarrow & & \downarrow f^* \\
\mathcal{B} & \xrightarrow{g} & \text{Mod}_\mathcal{V}(B)
\end{array} \quad \quad \begin{array}{ccc}
\text{Comon}(\mathcal{V}) & \xrightarrow{\text{Comod}_\mathcal{V}} & \text{Cat} \\
\mathcal{C} & \xrightarrow{g} & \text{Comod}_\mathcal{V}(C) \\
\downarrow & & \downarrow g \\
\mathcal{D} & \xrightarrow{f} & \text{Comod}_\mathcal{V}(D)
\end{array}
\]

where $f^*$ and $g^*$ are the restriction and corestriction of scalars, see Section 2.1. Indeed, under (6) we can view objects in $\text{Comod} = \mathcal{S}(\text{Comod}_\mathcal{V})$ as pairs $(X, C) \in \text{Comon}_\mathcal{V}(C) \times \text{Comon}(\mathcal{V})$ and morphisms as

\[
\begin{cases}
g : X \xrightarrow{k} Y & \text{in } \text{Comod}_\mathcal{V}(D) \\
C \xrightarrow{g} D & \text{in } \text{Comon}(\mathcal{V})
\end{cases}
\]

and dually for $\text{Mod} = \mathcal{S}(\text{Mod}_\mathcal{V})$. The fibre over a comonoid $\mathcal{C}$ is clearly the category of $\mathcal{C}$-comodules, $\text{Comod}_\mathcal{V}(\mathcal{C})$, and for a monoid $\mathcal{A}$ it is $\text{Mod}_\mathcal{V}(\mathcal{A})$. The chosen cartesian and cocartesian liftings are

\[
\text{Cart}(f, N) : f^* N \xrightarrow{(1_f, N, f)} N \text{ in } \text{Mod},
\]

\[
\text{Cocart}(g, X) : X \xrightarrow{(g, X, g)} g_! X \text{ in } \text{Comod}.
\]

**Remark 4.2.** Another way of viewing the global categories is due to Steve Lack, based on the observation that to give a lax functor of bicategories $\mathcal{M} \mathcal{I} \rightarrow \mathcal{M} \mathcal{V}$ which is identity on objects is to give an object in $\text{Mod}$.

Specifically, these bicategories arise from the canonical actions of the monoidal categories $\mathcal{I}, \mathcal{V}$ on themselves via tensor product. They both have two objects $\{0, 1\}$, and hom-categories $\mathcal{M} \mathcal{I}(0, 0) = \mathcal{M} \mathcal{I}(0, 1) = \mathcal{M} \mathcal{I}(1, 1) = 1$ and $\mathcal{M} \mathcal{I}(1, 0) = \emptyset$, as well as $\mathcal{M} \mathcal{V}(0, 1) = \mathcal{M} \mathcal{V}(1, 1) = \emptyset$, $\mathcal{M} \mathcal{V}(1, 0) = \emptyset$ and $\mathcal{M}(0, 0) = 1$. An identity-on-objects lax functor $\mathcal{F}$ would in particular consist of functors

$\mathcal{F}_{0,1}, \mathcal{F}_{1,1} : 1 \Rightarrow \mathcal{V}$

which pick up two objects $M$ and $A$ in $\mathcal{V}$. The components of lax functoriality and unitality give arrows $\mu : A \otimes M \rightarrow A, m : A \otimes A \rightarrow A$ in $\mathcal{V}$ and $\eta : I \rightarrow A$ satisfying the appropriate axioms. Then, morphisms in $\text{Mod}$ are *icons*, see [16]: if $M_A, N_B$ are two identity-on-objects
lax functors between \( \mathcal{ML} \) and \( \mathcal{MV} \), an icon between them consists in particular of natural transformations

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (A) at (0,0) {$A$};
    \node (B) at (0,-1) {$B$};
    \node (C) at (1,0) {$Y$};
    \node (D) at (1,-1) {$V$};

    \draw[->] (A) to node {$f$} (B);
    \draw[->] (C) to node [swap] {$g$} (D);
\end{tikzpicture}
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\begin{tikzpicture}
    \node (E) at (0,0) {$M$};
    \node (F) at (0,-1) {$N$};
    \node (G) at (1,0) {$V$};

    \draw[->] (E) to node {$p$} (F);
\end{tikzpicture}
\end{array}
\]

which are two arrows \( f : A \to B \) and \( p : M \to N \) in \( V \), subject to conditions which coincide with those of Definition 4.1. Dually, colax natural transformations \( \mathcal{ML} \to \mathcal{MV} \) correspond to comodules over comonoids, and icons then turn out to be comodule morphisms. Therefore we have \( \text{Mod} = \text{Bicat}_2(\mathcal{ML}, \mathcal{MV})_l \) and \( \text{Comod} = \text{Bicat}_2(\mathcal{ML}, \mathcal{MV})_c \), where \( \text{Bicat}_2 \) is the 2-category of bicategories, lax/cotransformers and icons.

The following result is also mentioned in [24, Thm. 45] for the particular case of \( V = \text{Mod}_R \) for a commutative ring \( R \).

**Proposition 4.3.** The functor \( F : \text{Comod} \to \mathcal{V} \times \text{Comon}(\mathcal{V}) \) which maps an object \( X_C \) to the pair \((X, C)\) is comonadic.

**Proof.** Define a functor

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (A) at (0,0) {\text{Mod}};
    \node (B) at (2,0) {\text{Comod}};

    \draw[->] (A) to node [swap] {$R$} (B);

    \node (C) at (0,-1) {(A, D)};
    \node (D) at (2,-1) {(B, E)};

    \node (E) at (1,-2) {(A \otimes D)_D};
    \node (F) at (3,-2) {(B \otimes E)_E};

    \draw[->] (A) to node {$\iota \circ \varphi$} (C);
    \draw[->] (B) to node {$\iota \circ \varphi$} (D);
    \draw[->] (C) to node {$1 \otimes \Delta$} (E);
    \draw[->] (D) to node {$1 \otimes \Delta$} (F);
\end{tikzpicture}
\end{array}
\]

where the \( D \)-action on the object \( A \otimes D \) is given by \( A \otimes D \xrightarrow{1 \otimes \Delta} A \otimes D \otimes D \). This establishes an adjunction \( F \dashv R \); the induced comonad \( FR \) on \( \mathcal{V} \times \text{Comon}(\mathcal{V}) \), given by \( (A, D) \mapsto (A \otimes D, D) \), has \( \text{Comod} \) as its category of coalgebras.

This in particular implies that if \( V \) and \( \text{Comon}(\mathcal{V}) \) are cocomplete categories, then \( \text{Comod} \) is also cocomplete. Dually, \( \text{Mod} \) is monadic over the category \( \mathcal{V} \times \text{Mon}(\mathcal{V}) \). These facts are used to show the following.

**Proposition 4.4.** If \( \mathcal{V} \) is a locally presentable monoidal category such that \( - \otimes - \) is finitary on both entries, \( \text{Mod} \) and \( \text{Comod} \) are locally presentable.

**Proof.** The monad and comonad whose Eilenberg-Moore algebras and coalgebras are \( \text{Mod} \) and \( \text{Comod} \) are both finitary. The latter follows from comonadicity of \( \text{Comon}(\mathcal{V}) \to \mathcal{V} \) (Proposition 2.3): \( (\lambda_j \otimes \tau_j, \tau_j) : (X_j \otimes C_j, C_j) \to (X \otimes C, C) \) is filtered colimiting when \( \lambda_j \) and \( \tau_j \) are. The former holds because the monadic \( \text{Mon}(\mathcal{V}) \to \mathcal{V} \) creates all colimits that the finitary monad preserves. Since \( \mathcal{V} \), \( \text{Mon}(\mathcal{V}) \) and \( \text{Comon}(\mathcal{V}) \) are all locally presentable categories under the above assumptions, we can apply Theorem 2.1 and the result follows.

When \( \mathcal{V} \) is symmetric (even braided), \( \text{Comod} \) and \( \text{Mod} \) are symmetric monoidal categories as well: if \( s \) is the symmetry, the object \( X_C \otimes Y_D \) is a comodule over the comonoid \( C \otimes D \) via the coaction

\[
X \otimes Y \xrightarrow{s \otimes s \Delta} X \otimes C \otimes Y \otimes D \xrightarrow{1 \otimes \Delta \otimes 1} X \otimes Y \otimes C \otimes D.
\]

and similarly for \( M_A \otimes N_B \in \text{Mod} \); symmetry is inherited from \( \mathcal{V} \). Moreover, the functors \( V \) and \( G \) of (15) are strict braided monoidal.

As a first application of the general fibred adjunctions theory of the previous section, we can deduce monoidal closedness of \( \text{Comod} \) when \( \mathcal{V} \) is locally presentable and closed.

**Proposition 4.5.** If \( \mathcal{V} \) is a locally presentable symmetric monoidal closed category, the symmetric monoidal \( \text{Comod} \) is closed.
Proof. First, we observe that there exists an opfibred 1-cell

\[
\begin{array}{c}
\text{Comod} \xrightarrow{(- \otimes X_C)} \text{Comod} \\
V \downarrow \quad \downarrow V \\
\text{Comon}(V) \xrightarrow{(- \otimes C)} \text{Comon}(V).
\end{array}
\tag{17}
\]

Indeed, the top functor \(( - \otimes X_C )\) maps a cocartesian lifting to

\[
\begin{array}{ccc}
Y \xrightarrow{\text{Cocart}(f,Y)} fY & \xrightarrow{fY \otimes X} & fY \otimes X \\
D \xrightarrow{f} E & \xrightarrow{f \otimes 1} & E \otimes C
\end{array}
\]

where \(( f \otimes 1 )_! ( Y \otimes X )\) are both \( Y \otimes X \) as objects in \( V \), and the \(( E \otimes C )\)-coactions coincide. By the canonical liftings \((16)\) for \( V : \text{Comod} \to \text{Comon}(V) \), since \( 1 ( f \otimes 1 )_! ( Y \otimes X ) = 1_f Y \otimes 1_X \), the functor \(( - \otimes X_C )\) is deduced to be cocartesian.

By Proposition 2.3, there is an adjunction \(( - \otimes C ) \dashv \text{Hom}(C, -)\) between the bases of \((17)\). Also, if \( \varepsilon \) is its counit, the composite

\[
\text{Comod}_V(\text{Hom}(C, D)) \xrightarrow{(- \otimes X_C)} \text{Comod}_V(\text{Hom}(C, D) \otimes C) \xrightarrow{(\varepsilon D)_!} \text{Comod}_V(D)
\]

has a right adjoint \( \text{Hom}_D(X_C, -)\) by Theorem 2.2. Indeed \( \text{Comod}_V(\text{Hom}(C, D))\) is locally presentable by 2.4, reindexing functors preserve all colimits, and the commutative diagram

\[
\begin{array}{c}
\text{Comod} \xrightarrow{(- \otimes X_C)} \text{Comod} \\
\downarrow V \quad \downarrow F \\
V \times \text{Comon}(V) \xrightarrow{(- \otimes X) \times (- \otimes C)} V \times \text{Comon}(V)
\end{array}
\tag{18}
\]

implies that \(( - \otimes X_C )\) preserves all colimits, since the bottom arrow does by monoidal closedness of \( V \) and \( \text{Comon}(V) \), and \( F \) is comonadic.

By Theorem 3.6, the composite has a right adjoint \( \text{Hom}_D(X_C, -)\) between the fibres which produce a total adjoint \( \text{Hom}(X_C, -) : \text{Comod} \to \text{Comod} \) such that

\[
\begin{array}{c}
\text{Comod} \xrightarrow{- \otimes X_C} \text{Comod} \\
\downarrow \text{Hom}(X_C, -) \quad \downarrow \text{Hom}(C, -) \\
\text{Comon}(V) \xrightarrow{- \otimes C} \text{Comon}(V)
\end{array}
\]

is an adjunction in \( \text{Cat}^2 \). The uniquely defined parametrized adjoint

\[\text{HOM} : \text{Comod}^{op} \times \text{Comod} \longrightarrow \text{Comod} \]

\[\text{Hom}(X_C, Y_D) \longrightarrow \text{HOM}(X, Y)_{\text{Hom}(C, D)} \]

of \(( - \otimes - )\) is the internal hom of the global category of comodules \( \text{Comod} \).

We are now interested in a internal-hom flavored functor between the two total categories. If \( V \) is a symmetric monoidal closed category, the induced functors \( \text{Mod}_{C,A}[-, -] : \text{Comod}_V(C)^{op} \times \text{Mod}_V(A) \longrightarrow \text{Mod}_V([C, A])\) as in \((3)\) ‘glue’ together into a functor

\[
\text{Hom} : \text{Comod}^{op} \times \text{Mod} \longrightarrow \text{Mod} \]

\[\text{Hom}(X_C, M_A) \longrightarrow [X, M]_{[C, A]} \tag{19}\]
such that $\text{Mod}_{CA}^{[-, -]}$ are the functors induced between the fibres. If $(k_g, l_f) : (X_C, M_A) \to (Y_D, N_B)$ is a morphism in the cartesian product, the fact that $k$ and $l$ commute with the corestricted and restricted actions accordingly forces the arrow $[k, l] : [X, M] \to [Y, N]$ in $V$ to satisfy the appropriate property. In fact, the pair $\langle \text{Hom}, \text{Mon}[-, -] \rangle$, see 1, is a fibred 1-cell

$$
\begin{array}{ccc}
\text{Comod}^{\text{op}} \times \text{Mod} & \xrightarrow{\text{Hom}} & \text{Mod} \\
\downarrow V^{\text{op}} \times G & & \downarrow G \\
\text{Comon}(V)^{\text{op}} \times \text{Mon}(V) & \xrightarrow{\text{Mon}[-, -]} & \text{Mon}(V),
\end{array}
$$

Indeed, $G([X, N]|_{C,B}) = [VX_C, GN_B] = [C, B]$ and $\text{Hom}$ is a cartesian functor: it maps a pair of a cocartesian lifting in $\text{Comod}$ and a cartesian lifting in $\text{Mod}$ to

$$
[C,A] \xrightarrow{[g,f]} [D,B] \quad \text{in } \text{Mon}(V).
$$

By the canonical liftings (16), that arrow is $([1_g Y, 1_{f^* N}], [g, f]) = (1_{[g Y, f^* N]}, [g, f])$ – and the latter is identical to $1_{[g Y, f^* N]}$ as module maps. The commutativity below establishes that $\text{Hom}(-, N_B)^{\text{op}}$ is cocontinuous:

$$
\begin{array}{ccc}
\text{Comod} & \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} & \text{Mod}^{\text{op}} \\
\downarrow & & \downarrow \\
V \times \text{Comon}(V) & \xrightarrow{[-, N_B]^{\text{op}} \times [-, B]^{\text{op}}} & V^{\text{op}} \times \text{Mon}(V)^{\text{op}}
\end{array}
$$

The comonadic functors at the left and right create all colimits and both functors at the bottom have right adjoints, see Theorem 2.8. Moreover, since the fibres of the total categories $\text{Comod}$ and $\text{Mod}^{\text{op}}$ are closed under colimits, the fibrewise functor $\text{Hom}(-, N_B)^{\text{op}}_{P(A,B)}$ is cocontinuous too.

4.2. Universal measuring comodule. The notion of a universal measuring comodule in the category of vector spaces $\text{Vect}_k$ was first introduced by Batchelor in [3]. For its generalization, consider a symmetric monoidal closed category $V$. For $M_A, N_B \in \text{Mod}$, we define an object $Q(M, N)_{P(A,B)}$ in $\text{Comod}$, the universal measuring comodule, by a natural isomorphism

$$
\text{Comod}(X, Q(M, N)) \cong \text{Mod}(M, \text{Hom}(X, N))
$$

for any $X = X_C$ and $\text{Hom}(X, N) = [X, N]|_{C,B}$ as in 19. Hence for $Q(M, N)$ to exist, the functor $\text{Hom}(-, N_B)^{\text{op}} : \text{Comod} \to \text{Mod}^{\text{op}}$ for a fixed $B$-module $N$ has to have a right adjoint. The following result is an application of Theorem 3.6.

**Proposition 4.6.** Let $V$ be a locally presentable symmetric monoidal closed category. There is an adjunction

$$
\begin{array}{ccc}
\text{Comod} & \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} & \text{Mod}^{\text{op}} \\
\downarrow \text{Q}(-, N_B) & & \\
\text{Comod} & \xleftarrow{Q(-, N_B)} & \text{Mod}^{\text{op}}
\end{array}
$$

between the global categories of modules and comodules; moreover, $Q(M, N)$ is a comodule over the universal measuring comonoid $P(A, B)$.

**Proof.** The pair $\langle \text{Hom}, \text{Mon}[-, -] \rangle$ depicted as (20) constitutes a fibred 1-cell between the fibrations $V^{\text{op}} \times G$ and $G$. This implies that $\langle \text{Hom}(-, N_B), [-, B] \rangle$ for a fixed monoid $B$ and
a $B$-module $N$ is also a fibred 1-cell and so

$$\begin{array}{ccc}
\text{Comod} & \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} & \text{Mod}^{\text{op}} \\
V & \downarrow & G^{\text{op}} \\
\text{Comon}(V) & \xrightarrow{[-, B]^{\text{op}}} & \text{Mon}(V)^{\text{op}}
\end{array}$$

is an opfibred 1-cell between the opfibrations $V$ and $G^{\text{op}}$. Proposition 2.8 gives

$$\begin{array}{ccc}
\text{Comon}(V) & \xrightarrow{[-, B]^{\text{op}}} & \text{Mon}(V)^{\text{op}} \\
\text{Comod}(P(A, B)) & \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} & \text{Mod}^{\text{op}}(P(A, B)) \xrightarrow{\text{(ε, A)}} \text{Mod}^{\text{op}}(A)
\end{array}$$

where $\varepsilon_A^B : H(P(A, B), B) \to A$ in $\text{Mon}(V)^{\text{op}}$ is the counit of $\text{Mon}[-, -]^{\text{op}} \dashv P$, has a right adjoint by Theorem 2.2. $\text{Comod}_V(C)$ is a locally presentable category by Proposition 2.4, the reindexing functors are always cocontinuous as seen in Section 2.1, and the fibrewise $\text{Hom}(-, N_B)^{\text{op}}$ too, as remarked in the previous section. Its right adjoint is

$$Q_A(-, N_B) : \text{Mod}_V(A)^{\text{op}} \to \text{Comod}_V(P(A, B))$$

which lifts to a functor between the total categories such that

$$\begin{array}{ccc}
\text{Comod} & \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} & \text{Mod}^{\text{op}} \\
V & \downarrow & G^{\text{op}} \\
\text{Comon}(V) & \xrightarrow{[-, B]^{\text{op}}} & \text{Mon}(V)^{\text{op}}
\end{array}$$

is an adjunction in $\text{Cat}^2$. By construction of $Q$, the object $Q(M_A, N_B)$ has the structure of a $P(A, B)$-comodule.

By naturality, we have an induced functor of two variables

$$Q(-, -) : \text{Mod}^{\text{op}} \times \text{Mod} \to \text{Comod}$$

called the universal measuring comodule functor, which is the parametrized adjoint of the bifunctor $\text{Hom}^{\text{op}}$.

**Lemma 4.7.** Suppose $A$ and $B$ are monoids in $\mathcal{V}$ regarded as regular modules over themselves. Then there are natural isomorphisms of $P(A, B)$-comodules

$$[V, N] \otimes P(A, B) \cong Q(A \otimes V, N)$$

for any object $V$ in $\mathcal{V}$ and $B$-module $N$. In particular, $A^\circ A^\circ \cong Q(A, I)_{A^\circ}$, where $A^\circ = P(A, I)$ is the finite dual comonoid.

**Proof.** The diagram of the left adjoints below commutes by (22).

$$\begin{array}{ccc}
\text{Comod} & \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} & \text{Mod}^{\text{op}} \\
\xrightarrow{F} R & \xrightarrow{Q(-, N_B)} & \text{Mod}_{\text{op}} \\
\xrightarrow{\text{V} \times \text{Comon}(V)} \xrightarrow{\text{Hom}(-, N_B)^{\text{op}}} \xrightarrow{L_{\text{op}}} K^{\text{op}} \\
\xrightarrow{\text{V} \times \text{Comon}(V)} \xrightarrow{[-, N]^{\text{op}} \times H(-, B)^{\text{op}}} \xrightarrow{\text{V}^{\text{op}} \times \text{Mon}(V)^{\text{op}}}
\end{array}$$

Therefore the corresponding square of right adjoints commutes up to isomorphism. Given a monoid $A$ and an object $V$, we have the natural isomorphism (23), and for $V = N = B = I$ we get the particular case of the finite dual. □
Remark 4.8. An alternative approach for the existence of the functors \( \overline{\text{Hom}} \) (the internal hom in \( \text{Comod} \)) and \( Q \) would be to directly show that the functors

\[
- \otimes X_C : \text{Comod} \to \text{Comod}
\]

\[
\text{Hom}^{\text{op}}(-, N_B) : \text{Comod} \to \text{Mod}^{\text{op}}
\]

have right adjoints via Theorem 2.2. Indeed both functors are cocontinuous by diagrams (18) and (21) respectively, and the domain \( \text{Comod} \) is locally presentable by Theorem 4.4. However, the general fibred adjunctions method provides with a better understanding of the structures involved. For example, we obtain the extra information that \( \text{Hom}(X_C, Y_D) \) is a \( \text{Hom}(C, D) \)-comodule and \( Q(M_A, N_B) \) is a \( P(A, B) \)-comodule.

4.3. Enrichment of modules in comodules. Similarly to how the enrichment of monoids in comonoids, Theorem 2.9, was established in [11, §5], we will now combine the defining isomorphism for the universal measuring comodule with the theory of actions of monoidal categories, in order to show how we can enrich the global category of modules in the global category of comodules.

Proposition 4.9. If \( V \) is a symmetric monoidal closed category, the monoidal category \( \text{Comod} \) acts on \( \text{Mod}^{\text{op}} \) via the functor

\[
\text{Hom}^{\text{op}} : \text{Comod} \times \text{Mod}^{\text{op}} \to \text{Mod}^{\text{op}}
\]

Proof. For the opposite of (19) to be an action, we need to check that there exist natural isomorphisms in \( \text{Mod} \)

\[
\begin{align*}
[X \otimes Y, M]|_{C \otimes D, A} &\xrightarrow{\sim} [X, [Y, M]]|_{C, [D, A]} \\
[I, M]|_{I, A} &\xrightarrow{\sim} M_A
\end{align*}
\]

for any comonoids \( C, D \), monoids \( A \), comodules \( X_C, Y_D \) and modules \( M_A \), that satisfy the axioms of an action. This follows from \([-, -] : V^{\text{op}} \times V \to V \) and \( \text{Mon}[-, -] : \text{Comod}(V)^{\text{op}} \times \text{Mon}(V) \to \text{Mon}(V) \) being actions by Lemma 2.5, since the monadic \( \text{Mod} \to V \times \text{Mon}(V) \)

reflects isomorphisms.

Theorem 2.6 now applies to give the following result.

Theorem 4.10. Let \( V \) be a locally presentable symmetric monoidal closed category.

1. \( \text{Mod}^{\text{op}} \) is a tensored and cotensored \( \text{Comod} \)-enriched category, with hom-objects

\[
\text{Mod}^{\text{op}}(M_A, N_B) = Q(N, M)|_{P(B, A)}.
\]

2. \( \text{Mod} \) is a tensored and cotensored \( \text{Comod} \)-enriched category, with hom-objects

\[
\text{Mod}(M_A, N_B) = Q(M, N)|_{P(A, B)} \text{ and cotensor products } [X, N]|_{C, B}.
\]

Proof. The enrichments, as well as the (co)tensor product given by the action of the monoidal closed base \( \text{Comod} \), follow in a straightforward way from the action-enrichment theorem. The only part left to show is that the fixed-argument action \( \text{Hom}(X_C, -)^{\text{op}} : \text{Mod}^{\text{op}} \to \text{Mod}^{\text{op}} \) also has a right adjoint for every comodule \( X_C \), to obtain the tensor for \( \text{Mod} \).

Consider the commutative square

\[
\begin{array}{ccc}
\text{Mod} & \xrightarrow{\text{Hom}(X_C, -)} & \text{Mod} \\
\downarrow & & \downarrow \\
V \times \text{Mon}(V) & \xrightarrow{[X, -] \times \text{Mon}[C, -]} & V \times \text{Mon}(V)
\end{array}
\]

where the vertical functors are monadic, \( \text{Mod} \) is locally presentable by Theorem 4.4, \( [X, -] \vdash (- \otimes X) \) in \( V \) and \( \text{Mon}[C, -] \vdash C \vdash - \) as in Section 2.4. By Dubuc’s Adjoint Triangle Theorem, the top functor has a left adjoint \( X_C \square \) for all \( X_C \)’s, inducing

\[
\square : \text{Comod} \times \text{Mod} \to \text{Mod}
\]

which gives the tensor products of the \( \text{Comod} \)-enriched category \( \text{Mod} \).

In fact, since \( V \) is symmetric, both \( \text{Comod} \) and \( \text{Mod} \) are symmetric and we can deduce the same for the action.
**Proposition 4.11.** If $\mathcal{V}$ is symmetric, so $\text{Comod}$ and $\text{Mod}$ are symmetric too, then the action of Proposition 4.9 has an opmonoidal structure, which, furthermore, it is symmetric.

*Proof.* We know that the action of $\mathcal{V}$ on $\mathcal{V}^{op}$ is braided opmonoidal, by Lemma 2.5. It suffices, then, to prove that the natural transformations that give the opmonoidal structure and the symmetry lift from $\mathcal{V}$ to $\text{Mod}$. The opmonoidal structure has components

$$\chi: [X, M] \otimes [Y, N] \rightarrow [X \otimes Y, M \otimes N] \quad [I, I] \cong I,$$

(24)

for $X_C$, $Y_D$ in $\text{Comod}$ and $M_A$, $N_B$ in $\text{Mod}$. The condition that makes of the first of these morphisms a morphism in $\text{Mod}$ over $\chi: [C, A] \otimes [D, B] \rightarrow [C \otimes D, A \otimes B]$ is described by the commutativity of the outer diagram on the left below. Since its bottom inner square always commutes, we only need to show that the upper rectangle does. This can easily be seen to be a consequence of the associativity axiom of $\chi$ and the fact that the opmonoidal action of Lemma 2.5 is braided with respect to the symmetry c, i.e. that the diagram on the right commutes. The proof that the second morphism in (24) lifts to $\text{Mod}$ is trivial and left to the reader.

$$\begin{array}{ccc}
[C, A][D, B][X, M][Y, N] & \xrightarrow{\chi} & [CD, AB][XY, MN] \\
[C, A][Y, N] & \xrightarrow{\chi} & [CD, AB][XY, MN] \\
[CA][X, M][D, B][Y, N] & \xrightarrow{\chi} & [CDXY, ABMN] \\
[CA][X, M][D, B][Y, N] & \xrightarrow{\chi} & [CDXY, ABMN] \\
[CA][X, M][D, B][Y, N] & \xrightarrow{\chi} & [CDXY, ABMN] \\
\end{array}$$

It remains to show that the opmonoidal action is braided with respect to the symmetries of $\text{Comod}$ and $\text{Mod}$. This, again, follows from Lemma 2.5 because the forgetful functor $\text{Mod} \rightarrow \mathcal{V}$ is braided strict monoidal and faithful. \(\square\)

**Proposition 4.12.** The universal measuring comodule functor $\text{Mod}^{op} \times \text{Mod} \xrightarrow{Q} \text{Comod}$ is a lax monoidal functor.

*Proof.* By definition, $Q$ is a parametrised right adjoint of $\text{Mon}[-, -]^{op}$, which is an opmonoidal functor. So $Q$ carries a (lax) monoidal structure, by the comments at the beginning of Section 2.1. \(\square\)

5. **Measuring comodules and Hopf modules**

5.1. **Module comonoids.** In this section we describe an application of the measuring comodule construction to Hopf modules. Below we drop the notation $\text{Mon}(\mathcal{V})$ and $\text{Comon}(\mathcal{V})$ for the concise $\text{Mon}$ and $\text{Comon}$.

Consider the fibration $\text{Mod} \rightarrow \text{Mon}$, which we know to be a strong monoidal functor. We can then consider the induced functor between the respective categories of comonoids, or between the categories of monoids. Let us start with the description of the result of taking comonoids, which is a strong monoidal functor

$$\text{Comon}(\text{Mod}) \rightarrow \text{Comon}(\text{Mon}) \cong \text{Bimon}$$

(25)

The codomain is the category of bimonoids in $\mathcal{V}$. An object of the domain is a module $M$ over a bimonoid $H$ together with a comultiplication $\delta_M: M \rightarrow M \otimes M$ and a counit $\varepsilon_M: M \rightarrow k$ in $\text{Mod}$, satisfying the comonoid axioms; the former must be a morphism over the comultiplication $\delta_H: H \rightarrow H$ of $H$ and the latter a morphism over the counit $\varepsilon_H: H \rightarrow k$. These last conditions mean, by the definition of the fibration of modules, that $\delta_M$ is a morphism $M \rightarrow \delta_H(M \otimes M)$ and that $\varepsilon_M$ is a morphism $M \rightarrow \varepsilon_H(k)$, both in the category of $H$-modules. Expanding this description to commutative diagrams, an object of $\text{Comon}(\text{Mod})$ is a module $\nu: H \otimes M \rightarrow M$ over a bimonoid $H$, equipped with a
comonoid structure \((M, \delta_M, \varepsilon_M)\) that moreover satisfies the commutativity of the diagrams below (where we omit the tensor product symbol, to save space).

\[
\begin{array}{c}
\begin{array}{c}
HM \\
\downarrow \nu
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
HMM \\
\downarrow \nu
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
HHMM \\
\downarrow \nu \circ \mu
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
HM \varepsilon_M \varepsilon_M \\
\downarrow \delta_M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow \varepsilon_M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow \varepsilon_M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow \varepsilon_M
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
k \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
it \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
it \\
\downarrow
\end{array}
\end{array}\end{array}
\]

We call comonoids in \textbf{Mod} module comonoids, mirroring the standard nomenclature in Hopf algebra theory, where the term \(H\)-module coalgebra for a bialgebra \(H\) is widely used. A morphism of comonoids in \textbf{Mod} from \(M_H\) to \(M'_H\) is a morphism of comonoids \(f: H \rightarrow H'\) in \(\mathcal{V}\) with a morphism \(h: M \rightarrow M'\) in \(\mathcal{V}\) that is a morphism of monoids over \(f\) and a morphism of comonoids in \(\mathcal{V}\). The fibre of (25) over a bimonoid \(H\) is called the category of \(H\)-module comonoids.

5.2. Comodule monoids. We could also consider the category of comodules on \textbf{Mon},

\[
\text{Comod}(\text{Mon}) \rightarrow \text{Comon}(\text{Mon}) \cong \text{Bimon}. \tag{26}
\]

An object of \textbf{Comod}(\textbf{Mon}) over a bimonoid \(H\) is a monoid \(S\) with a \(H\)-comodule structure \(\chi_S: S \rightarrow H \otimes S\) that is a morphism of monoids. We call this structure a \(H\)-comodule monoid. In terms of commutative diagrams, the compatibility between comodule and monoid structure is expressed as follows.

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
SS \chi_S \otimes \chi_S \\
\downarrow \mu_S
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
HSHS \chi_S \\
\downarrow \mu_H \mu_S
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
HHSS \\
\downarrow \chi_S \otimes \chi_S
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
I \\
\downarrow \chi_S \otimes \chi_S
\end{array}
\end{array}
\end{array}
\end{array}
\]

Observe that (26) can be obtained as the result of taking monoids \textbf{Comod} \rightarrow \textbf{Comon}:

\[
\begin{array}{c}
\text{Mon(Comod)} \rightarrow \text{Mon(Comon)} \cong \text{Bimon}
\end{array}
\]

5.3. Hopf modules. Instead of taking comonoids on \textbf{Mod} \rightarrow \textbf{Mon} we could take comodules and get functors

\[
\begin{array}{c}
\text{Comod(Mod)} \rightarrow \text{Comod(Mon)} \\
\downarrow \\
\text{Comon(Comod)} \rightarrow \text{Bimon}
\end{array}
\]

Following the top and left arrows of the diagram, we see that an object of \textbf{Comod(Mod)} has an underlying \(H\)-comodule monoid \(S\) and an underlying \(H\)-module comonoid \(M\), for a bimonoid \(H\). An object of \textbf{Comod(Mod)} that is mapped to \(S\) and \(M\) is an \(S\)-module \(N\), with action \(\nu_N: S \otimes N \rightarrow N\), with an \(M\)-comodule structure \(\chi_N: N \rightarrow M \otimes N\), that are compatible, in the sense that \(\chi_M\) must be a morphism of modules over \(\chi_S: S \rightarrow H \otimes S\). This last condition amounts to the commutativity of the diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
SN \chi_S \otimes \chi_N \\
\downarrow \nu_N
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
HSMN \chi_S \otimes \chi_N \\
\downarrow \nu_M \nu_N
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
HMSN \\
\downarrow \chi_N \otimes \chi_N
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
M \chi_N \\
\downarrow \chi_N
\end{array}
\end{array}
\end{array}
\]

\textbf{Definition 5.1.} Given a \(H\)-comodule monoid \(S\) and a \(H\)-module comonoid \(M\) as in the previous paragraph, the \(N \in \text{Comod(Mod)}\) over \(S\) and \(M\) may be called the \(S\)-\(M\)-\textit{Hopf modules}. The category of such will be denoted by \textbf{HopfMod}\(^H_S\). Observe this category can also be obtained as \textbf{Mod(Comod)}. 


5.4. Module monoids. Next we describe the monoidal functor
\[ \text{Mon} \left( \text{Mod} \right) \to \text{Mon} \left( \text{Mon} \right) \cong \text{CMon} \]Its codomain is the category \( \text{CommMon} \) of commutative monoids in \( V \). An object of its domain over a commutative monoid \( A \) is an \( A \)-module \( \nu: A \otimes N \to N \) equipped with a monoid structure, with multiplication \( \mu_N: N \otimes N \to N \) and unit \( \eta_N: k \to N \), such that \( \mu_N \) should be a morphism of \( (A \otimes A) \)-modules \( N \otimes N \to \mu_A^*(N) \), and \( \eta_N \) should be a morphism of \( k \)-modules, but this latter condition is void. Unpacking this condition, we have that the diagram below should commute.

\[
\begin{array}{ccc}
AAN & \xrightarrow{\mu_A^*} & ANA \\
\downarrow{\mu_A \mu_N} & & \downarrow{\mu_N} \\
AN & \xrightarrow{\nu} & N \\
\end{array}
\]

We call these structures \( A \)-module monoids.

5.5. Measuring Hopf comodules. Recall that the universal measuring comodule functor \( Q: \text{Mod}^{\text{op}} \times \text{Mod} \to \text{Comod} \) is lax monoidal by Proposition 4.12. By taking monoids, we obtain functors
\[
\begin{array}{ccc}
\text{Comod} \left( \text{Mod} \right)^{\text{op}} \times \text{Mod} & \xrightarrow{\text{Mod}^Q} & \text{Mod} \left( \text{Comod} \right) \\
\downarrow & & \downarrow \\
\text{Comon} \left( \text{Mod} \right)^{\text{op}} \times \text{Mon} \left( \text{Mod} \right) & \xrightarrow{\text{Mon}^Q} & \text{Mon} \left( \text{Comod} \right)
\end{array}
\]

This implies:

**Proposition 5.2.** Given a bimonoid \( H \) and a commutative monoid \( A \) in \( V \), a \( H \)-module comonoid \( M \) and a commutative \( A \)-module \( N \), then \( Q(M, N) \) carries a canonical structure of a \( P(H, A) \)-comodule monoid.

**Proposition 5.3.** Let \( A \) be a commutative monoid in \( V \). The universal measuring comodule functor \( Q \) lifts to a functor
\[ \text{HopfMod}(\text{Mod})^{\text{op}} \to \text{HopfMod}(\text{Q}(K, A)) \]
for \( M \) an \( H \)-module comonoid, \( H \) a bimonoid and \( A \) a commutative monoid. In particular, it lifts to a functor between categories of Hopf modules
\[ \text{HopfMod}(H)^{\text{op}} \to \text{HopfMod}(H^o) \]
where \( H^o \) is the finite dual bimonoid.

Proof. If \( A \) be a commutative monoid in \( V \) and regard it as a module over itself (that we denote \( A_A \)). This makes \( A \) into an object of \( \text{Mon}(\text{Mod}) \). Furthermore, \( A \) is an object of \( \text{Mod}(\text{Mod}) \), sitting over \( A \in \text{Mon}(\text{Mod}) \), since the relevant instance of (27) commutes. Then (28) restricts to
\[
\begin{array}{ccc}
\text{Comod} \left( \text{Mod} \right)^{\text{op}} & \xrightarrow{Q(-, A_A)} & \text{Mod} \left( \text{Comod} \right) \\
\downarrow & & \downarrow \\
\text{Comon} \left( \text{Mod} \right)^{\text{op}} & \xrightarrow{Q(-, A_A)} & \text{Mon} \left( \text{Comod} \right)
\end{array}
\]
and we may restrict the top functor to the full subcategory \( \text{HopfMod}_M^H \) of \( \text{Comod}(\text{Mod}) \) (the category of objects \( X \) of \( V \) endowed with an comodule structure over a \( H \)-module comonoid \( M_H \) and over a \( H \)-comodule module \( S_H \), plus compatibility between the two structures; see Definition 5.1). If \( X \in \text{HopfMod}_M^H \), then \( Q(X, A) \) is a \( Q(M, A) \)-module. Here \( Q(M, A) \) The induced functor between the categories of modules is
\[ Q(-, A_A): \text{Comod}_{\text{Mod}}(M_H)^{\text{op}} \to \text{Mod}_{\text{Comod}}(Q(M, A)_{P(H, A)}) \]
for a bimonoid \( H \).
In particular, for $M_H = H^H$ as the regular $H$-monoid comonoid and $A = I$ as a commutative monoid, this functor becomes
\[
Q(-, I) : \text{Comod}_{\text{Mod}}(H^H)^{op} \to \text{Mod}_{\text{Comod}}(Q(H, I)p_{(H, I)}).
\] (29)

By Lemma 4.7 the $P(H, I)$-comodule $Q(H, I)$ is isomorphic to the cofree comodule $P(H, I)$. Thus, the codomain of (29) is the category of objects with compatible $P(H, I)$-module and $P(H, I)$-module structures, i.e., the category $\text{HopfMod}_{H^H}$, where $H^H = P(H, I)$. Therefore, (29) is of the form required in the statement, completing the proof. □

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