DIFFERENTIAL INCLUSIONS INVOLVING OSCILLATORY TERMS

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ABSTRACT. Motivated by mechanical problems where external forces are non-smooth, we consider the differential inclusion problem

\[
\begin{align*}
-\Delta u(x) &\in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in } \Omega; \\
u &\geq 0, & \text{in } \Omega; \\
u & = 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded open domain, and \( \partial F \) and \( \partial G \) stand for the generalized gradients of the locally Lipschitz functions \( F \) and \( G \). In this paper we provide a quite complete picture on the number of solutions of \((D_\lambda)\) whenever \( \partial F \) oscillates near the origin/infinity and \( \partial G \) is a generic perturbation of order \( p > 0 \) at the origin/infinity, respectively. Our results extend in several aspects those of Kristály and Moroşanu [J. Math. Pures Appl., 2010].

1. INTRODUCTION

We consider the model Dirichlet problem

\[
\begin{align*}
-\Delta u(x) & = f(u(x)) & \text{in } \Omega; \\
u &\geq 0, & \text{in } \Omega; \\
u & = 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Delta \) is the usual Laplace operator, \( \Omega \subset \mathbb{R}^n \) is a bounded open domain \( (n \geq 2) \), and \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function verifying certain growth conditions at the origin and infinity. Usually, such a problem is studied on the Sobolev space \( H_0^1(\Omega) \) and weak solutions of \((P_0)\) become classical/strong solutions whenever \( f \) has further regularity. There are several approaches to treat problem \((P_0)\), mainly depending on the behavior of the function \( f \). When \( f \) is superlinear and subcritical at infinity (and superlinear at the origin), the seminal paper of Ambrosetti and Rabinowitz [2] guarantees the existence of at least a nontrivial solution of \((P_0)\) by using variational methods. An important extension of \((P_0)\) is its perturbation, i.e.,

\[
\begin{align*}
-\Delta u(x) & = f(u(x)) + \lambda g(u(x)) & \text{in } \Omega; \\
u &\geq 0, & \text{in } \Omega; \\
u & = 0, & \text{on } \partial \Omega,
\end{align*}
\]

where \( g : \mathbb{R} \to \mathbb{R} \) is another continuous function which is going to compete with the original function \( f \). When both functions \( f \) and \( g \) are of polynomial type of sub- and super-unit degree, – the right hand side being called as a concave-convex nonlinearity – the existence of at least one or two nontrivial solutions of \((P_\lambda)\) is guaranteed, depending on the range of \( \lambda > 0 \), see e.g., Ambrosetti, Brezis and Cerami [1], Autuori and Pucci [4], de Figueiredo, Gossez and Ubilla [8]. In these papers variational arguments, sub- and super-solution methods as well as fixed point arguments are employed.

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Dedicated to Professor Gheorghe Moroşanu on the occasion of his 70th birthday.
Another important class of problems of the type \((P_{\lambda})\) is studied whenever \(f\) has a certain\oscillation (near the origin or at infinity) and \(g\) is a \perturbation. Although oscillatory functions seemingly call forth the existence of infinitely many solutions, it turns out that 'too classical' oscillatory functions do not have such a feature. Indeed, when \(f(s) = c \sin s\) and \(g = 0\), with \(c > 0\) small enough, a simple use of the Poincaré inequality implies that problem \((P_{\lambda})\) has only the zero solution. However, when \(f\) strongly oscillates, problem \((P_{0})\) has indeed infinitely many different solutions; see e.g. Omari and Zanolin [19], Saint Raymond [21]. Furthermore, if \(g(s) = s^p\) \((s > 0)\), a novel competition phenomena has been described for \((P_{\lambda})\) by Kristály and Moroşanu [12]. We notice that several extensions of [12] can be found in the literature, see e.g. Ambrosio, D’Onofrio and Molica Bisci [3] and Molica Bisci and Pizzimenti [16] for nonlocal fractional Laplacians; Molica Bisci, Rădulescu and Servadei [17] for general operators in divergence form; Mălin and Rădulescu [15] for difference equations. We emphasize that in the aforementioned papers the perturbations are either zero or have a (smooth) polynomial form.

In mechanical applications, however, the perturbation may occur in a \discontinuous manner as a non-regular external force, see e.g. the gluing force in von Kármán laminated plates, cf. Bocea, Panagiotopoulos and Rădulescu [5], Motreanu and Panagiotopoulos [18] and Panagiotopoulos [20]. In order to give a reasonable reformulation of problem \((P_{\lambda})\) in such a non-regular setting, the idea is to 'fill the gaps' of the discontinuities, considering instead of the discontinuous nonlinearity a \set-valued map appearing as the generalized gradient of a locally Lipschitz function. In this way, we deal with an \elliptic differential inclusion problem rather than an elliptic differential equation, see e.g. Chang [6], Gazzolla and Rădulescu [9] and Kristály [10]; this problem can be formulated generically as

\[
\begin{cases}
-\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in } \Omega; \\
u \geq 0, & \text{in } \Omega; \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \(F\) and \(G\) are both nonsmooth, locally Lipschitz functions having various growths, while \(\partial F\) and \(\partial G\) stand for the generalized gradients of \(F\) and \(G\), respectively.

The main purpose of the present paper is to extend the main results of Kristály and Moroşanu [12] in two directions:

(a) to allow the presence of nonsmooth nonlinear terms – reformulated into the inclusion \((D_{\lambda})\) – which are more suitable from mechanical point of view (mostly due to the perturbation term \(G\), although we allow non-smoothness for the oscillatory term \(F\) as well);

(b) to consider a generic \(p\)-order perturbation \(\partial G\) at the origin/infinity, not necessarily of polynomial growth as in [12], \(p > 0\).

In the present paper we study the inclusion \((D_{\lambda})\) in two different settings, i.e., we analyze the number of distinct solutions of \((D_{\lambda})\) whenever \(\partial F\) oscillates near the origin/infinity and \(\partial G\) is of order \(p > 0\) near the origin/infinity. Roughly speaking, when \(\partial F\) oscillates near the origin and \(\partial G\) is of order \(p > 0\) at the origin, we prove that the number of distinct, nontrivial solutions of \((D_{\lambda})\) is

- infinitely many whenever \(p > 1\) \((\lambda \geq 0\) is arbitrary\) or \(p = 1\) and \(\lambda\) is small enough (see Theorem 2.1);
- at least \((a\ prescribed\ number)\) \(k \in \mathbb{N}\) whenever \(0 < p < 1\) and \(\lambda\) is small enough (see Theorem 2.2).
As we can observe, in the first case, the term $\partial G(s) \sim s^p$ as $s \to 0^+$ with $p > 1$ has no effect on the number of solutions (i.e., the oscillatory term is the leading one), while in the second case, the situation changes dramatically, i.e., $\partial G$ has a 'truth' competition with respect to the oscillatory term $\partial F$.

We can state a very similar result as above whenever $\partial F$ oscillates at infinity and $\partial G$ is of order $p > 0$ at infinity by proving that the number of distinct, nontrivial solutions of the differential inclusion $(D_\lambda)$ is

- infinitely many whenever $p < 1$ ($\lambda \geq 0$ is arbitrary) or $p = 1$ and $\lambda$ is small enough (see Theorem 2.3);
- at least (a prescribed number) $k \in \mathbb{N}$ whenever $p > 1$ and $\lambda$ is small enough (see Theorem 2.4).

Contrary to the competition at the origin, in the first case the term $\partial G(s) \sim s^p$ as $s \to \infty$ with $p < 1$ has no effect on the number of solutions (i.e., the oscillatory term is the leading one), while in the second case, the perturbation term $\partial G$ competes with the oscillator function $\partial F$.

We admit that the line of the proofs is conceptually similar to that of Kristály and Moroşanu [12]; however, the presence of the nonsmooth terms $\partial F$ and $\partial G$ requires a deep argumentation by fully exploring the nonsmooth calculus of locally Lipschitz functions in the sense of Clarke [7]. In addition, the presence of the generic $p$-order perturbation $\partial G$ needs a special attention with respect to [12]; in particular, the $p$-order growth of $\partial G$ is new even in smooth settings.

The organization of the present paper is the following. In Section 2 we state our main assumptions and results, providing also some examples of functions fulfilling the assumptions. Section 3 contains a generic localization theorem for differential inclusions, while Sections 4 and 5 are devoted to the proof of our main results. In Section 6 we formulate some concluding remarks, while in the Appendix (Section 7) we collect those notions and results on locally Lipschitz functions that are used throughout our arguments.

2. Main theorems

Let $F, G : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz functions and as usual, let us denote by $\partial F$ and $\partial G$ their generalized gradients in the sense of Clarke (see the Appendix). Hereafter, $\mathbb{R}_+ = [0, \infty)$. Let $p > 0$, $\lambda \geq 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open domain, and consider the elliptic differential inclusion problem

$$
\begin{cases}
-\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in } \Omega; \\
u \geq 0 & \text{in } \Omega; \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\quad (D_\lambda)
$$

We distinguish the cases when $\partial F$ oscillates near the origin or at infinity.

2.1. Oscillation near the origin. We assume:

- $(F_0^0)$ $F(0) = 0$;
- $(F_1^0)$ $-\infty < \liminf_{s \to 0^+} \frac{F(s)}{s^p}, \limsup_{s \to 0^+} \frac{F(s)}{s^p} = +\infty$;
- $(F_2^0)$ $l_0 := \liminf_{s \to 0^+} \frac{\max\{\xi : \xi \in \partial F(s)\}}{s} < 0$.
- $(G_0^0)$ $G(0) = 0$;
- $(G_1^0)$ There exist $p > 0$ and $\xi, \bar{\xi} \in \mathbb{R}$ such that

$$
\xi = \liminf_{s \to 0^+} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \leq \limsup_{s \to 0^+} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \bar{\xi}.
$$
Remark 2.1. Hypotheses \((F_0^1)\) and \((F_0^2)\) imply a strong oscillatory behavior of \(\partial F\) near the origin. Moreover, it turns out that \(0 \in \partial F(0)\); indeed, if we assume the contrary, by the upper semicontinuity of \(\partial F\) we also have that \(0 \notin \partial F(s)\) for every small \(s > 0\). Thus, by \((F_0^2)\) we have that \(\partial F(s) \subset (-\infty, 0]\) for these values of \(s > 0\). By using \((F_0^0)\) and Lebourg’s mean value theorem (see Proposition 7.3 in the Appendix), it follows that \(F(0) = F(s) - F(0) = \xi s \leq 0\) for some \(\xi \in \partial F(\theta s) \subset (-\infty, 0]\) with \(\theta \in (0, 1)\). The latter inequality contradicts the second assumption from \((F_0^0)\). Similarly, one obtains that \(0 \in \partial G(0)\) by exploring \((G_0^0)\) and \((G_0^1)\), respectively.

In conclusion, since \(0 \in \partial F(0)\) and \(0 \in \partial G(0)\), it turns out that \(0 \in H^1_0(\Omega)\) is a solution of the differential inclusion \((D_\lambda)\). Clearly, we are interested in nonzero solutions of \((D_\lambda)\).

Example 2.1. Let us consider \(F_0(s) = \int_0^s f_0(t), s \geq 0, \) where \(f_0(t) = \sqrt{t}(\frac{1}{2} + \sin t^{-1}), t > 0\) and \(f_0(0) = 0\), or some of its jumping variants. One can prove that \(\partial F_0 = f_0\) verifies the assumptions \((F_0^0) - (F_0^2)\). For a fixed \(p > 0\), let \(G_0(s) = \ln(1 + s^p + 2\max\{0, \cos s^{-1}\}, s > 0\) and \(G_0(0) = 0\). It is clear that \(G_0\) is not of class \(C^1\) and verifies \((G_0^0)\) with \(\bar{c} = -1\) and \(\bar{c} = 1\), respectively; see Figure 1 representing both \(f_0\) and \(G_0\) (for \(p = 2\)).

In the sequel, we provide a quite complete picture about the competition concerning the terms \(s \mapsto \partial F(s)\) and \(s \mapsto \partial G(s)\), respectively. First, we are going to show that when \(p \geq 1\) then the ‘leading’ term is the oscillatory function \(\partial F\); roughly speaking, one can say that the effect of \(s \mapsto \partial G(s)\) is negligible in this competition. More precisely, we prove the following result.

Theorem 2.1. (Case \(p \geq 1\)) Assume that \(p \geq 1\) and the locally Lipschitz functions \(F,G : \mathbb{R}_+ \rightarrow \mathbb{R}\) satisfy \((F_0^0) - (F_0^2)\) and \((G_0^0) - (G_0^1)\). If

(i) either \(p = 1\) and \(\lambda \bar{c} < -l_0\) (with \(\lambda \geq 0\)),

(ii) or \(p > 1\) and \(\lambda \geq 0\) is arbitrary,

then the differential inclusion problem \((D_\lambda)\) admits a sequence \(\{u_i\}_i \subset H^1_0(\Omega)\) of distinct weak solutions such that

\[
\lim_{i \rightarrow \infty} \|u_i\|_{H^1_0} = \lim_{i \rightarrow \infty} \|u_i\|_{L^\infty} = 0. \tag{2.1}
\]
In the case when $p < 1$, the perturbation term $\partial G$ may compete with the oscillatory function $\partial F$; namely, we have:

**Theorem 2.2.** (Case $0 < p < 1$) Assume $0 < p < 1$ and that the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(F_0^0) - (F_2^0)$ and $(G_0^0) - (G_0^1)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k > 0$ such that the differential inclusion $(D_\lambda)$ has at least $k$ distinct weak solutions \( \{u_{1,\lambda}, \ldots, u_{k,\lambda}\} \subset H_0^1(\Omega) \) whenever $\lambda \in [0, \lambda_k]$. Moreover,

\[
\|u_{i,\lambda}\|_{H_0^1} < i^{-1} \quad \text{and} \quad \|u_{i,\lambda}\|_{L^\infty} < i^{-1} \quad \text{for any} \ i = 1, \ldots, k; \ \lambda \in [0, \lambda_k].
\] (2.2)

2.2. Oscillation at infinity. Let assume:

\( (F_0^\infty) \) $F(0) = 0$;
\( (F_1^\infty) \) $-\infty < \liminf_{s \to \infty} \frac{F(s)}{s^p} \leq \limsup_{s \to \infty} \frac{F(s)}{s^p} = +\infty$;
\( (F_2^\infty) \) \( l_\infty := \liminf_{s \to \infty} \frac{\max\{\xi : \xi \in \partial F(s)\}}{s} < 0 \);
\( (G_0^\infty) \) $G(0) = 0$;
\( (G_1^\infty) \) There exist $p > 0$ and $\varrho, \varpi \in \mathbb{R}$ such that

\[
\varrho = \liminf_{s \to \infty} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \leq \limsup_{s \to \infty} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \varpi.
\]

**Remark 2.2.** Hypotheses $(F_1^\infty)$ and $(F_2^\infty)$ imply a strong oscillatory behavior of the set-valued map $\partial F$ at infinity.

**Example 2.2.** We consider $F_\infty(s) = \int_0^s f_\infty(t), \ s \geq 0$, where $f_\infty(t) = \sqrt{t} (\frac{1}{2} + \sin t), \ t \geq 0$, or some of its jumping variants; one has that $F_\infty$ verifies the assumptions $(F_0^\infty) - (F_2^\infty)$. For a fixed $p > 0$, let $G_\infty(s) = s^p \max\{0, \sin s\}, \ s \geq 0$; it is clear that $G_\infty$ is a typically locally Lipschitz function on $[0, \infty)$ (not being of class $C^1$) and verifies $(G_1^\infty)$ with $\varrho = -1$ and $\varpi = 1$; see Figure 2 representing both $f_\infty$ and $G_\infty$ (for $p = 2$), respectively.

![Figure 2. Graphs of $f_\infty$ and $G_\infty$ at infinity, respectively.](image-url)
In the sequel, we investigate the competition at infinity concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we show that when $p \leq 1$ then the 'leading' term is the oscillatory function $F$, i.e., the effect of $s \mapsto \partial G(s)$ is negligible. More precisely, we prove the following result:

**Theorem 2.3.** (Case $p \leq 1$) Assume that $p \leq 1$ and the locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ satisfy $(F_0^\infty) - (F_2^\infty)$ and $(G_0^\infty) - (G_1^\infty)$. If

(i) either $p = 1$ and $\lambda \xi \leq -l_0$ (with $\lambda \geq 0$),

(ii) or $p < 1$ and $\lambda \geq 0$ is arbitrary,

then the differential inclusion $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i \to \infty} \|u_i^\infty\|_{L^\infty} = \infty. \quad (2.3)$$

**Remark 2.3.** Let $2^*$ be the usual critical Sobolev exponent. In addition to (2.3), we also have $\lim_{i \to \infty} \|u_i^\infty\|_{H_0^1} = \infty$ whenever

$$\sup_{s \in [0, \infty)} \frac{\max\{|\xi| : \xi \in \partial F(s)\}}{1 + s^{2^*-1}} < \infty. \quad (2.4)$$

In the case when $p > 1$, it turns out that the perturbation term $\partial G$ may compete with the oscillatory function $\partial F$; more precisely, we have:

**Theorem 2.4.** (Case $p > 1$) Assume that $p > 1$ and the locally Lipschitz functions $F, G : \mathbb{R} \to \mathbb{R}$ satisfy $(F_0^\infty) - (F_2^\infty)$ and $(G_0^\infty) - (G_1^\infty)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^\infty > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least $k$ distinct weak solutions $\{u_{1,\lambda}, \ldots, u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k^\infty]$. Moreover,

$$\|u_{i,\lambda}\|_{L^\infty} > i - 1 \text{ for any } i = 1, k; \lambda \in [0, \lambda_k^\infty]. \quad (2.5)$$

**Remark 2.4.** If (2.4) holds and $p \leq 2^* - 1$ in Theorem 2.4, then we have in addition that

$$\|u_{i,\lambda}\|_{H_0^1} > i - 1 \text{ for any } i = 1, k; \lambda \in [0, \lambda_k^\infty].$$

3. **Localization: a generic result**

We consider the following differential inclusion problem

$$\begin{cases}
-\Delta u(x) + ku(x) \in \partial A(u(x)), & u(x) \geq 0 \quad x \in \Omega, \\
u(x) = 0 & x \in \partial \Omega,
\end{cases} \quad (\mathcal{D}_\lambda^k)$$

where $k > 0$ and

$(H_\lambda^k)$: $A : [0, \infty) \to \mathbb{R}$ is a locally Lipschitz function with $A(0) = 0$, and there is $M_A > 0$ such that

$$\max\{|\partial A(s)|\} := \max\{|\xi| : \xi \in \partial A(s)\} \leq M_A$$

for every $s \geq 0$;

$(H_\delta^\lambda)$: there are $0 < \delta < \eta$ such that $\max\{|\xi| : \xi \in \partial A(s)\} \leq 0$ for every $s \in [\delta, \eta]$.

For simplicity, we extend the function $A$ by $A(s) = 0$ for $s \leq 0$: the extended function is locally Lipschitz on the whole $\mathbb{R}$. The natural energy functional $\mathcal{T} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem $(\mathcal{D}_\lambda^k)$ is defined by

$$\mathcal{T}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 + k \int_{\Delta} u^2 dx - \int_{\partial \Omega} A(u(x)) dx.$$
The energy functional $\mathcal{T}$ is well defined and locally Lipschitz on $H_0^1(\Omega)$, while its critical points in the sense of Chang (see Definition 7.3 in the Appendix) are precisely the weak solutions of the differential inclusion problem

$$\begin{cases}
-\Delta u(x) + ku(x) \in \partial A(u(x)), & x \in \Omega, \\
u(x) = 0 & x \in \partial \Omega;
\end{cases} \quad (D^k_A)$$

note that at this stage we have no information on the sign of $u$.

Indeed, if $0 \in \partial \mathcal{T}(u)$, then for every $v \in H_0^1(\Omega)$ we have

$$\int_{\Omega} \nabla u(x) \nabla v(x) \, dx - k \int_{\Omega} u(x) v(x) \, dx - \int_{\Omega} \xi_x(x)v(x) \, dx = 0,$$

where $\xi_x \in \partial A(u(x))$ a.e. $x \in \Omega$, see e.g. Motreanu and Panagiotopoulos [18]. By using the divergence theorem for the first term at the left hand side (and exploring the Dirichlet boundary condition), we obtain that

$$\int_{\Omega} \nabla u(x) \nabla v(x) \, dx = - \int_{\Omega} \text{div}(\nabla u(x))v(x) \, dx = - \int_{\Omega} \Delta u(x)v(x) \, dx.$$

Accordingly, we have that

$$- \int_{\Omega} \Delta u(x)v(x) \, dx + k \int_{\Omega} u(x)v(x) = \int_{\Omega} \xi_x v(x) \, dx$$

for every test function $v \in H_0^1(\Omega)$ which means that $-\Delta u(x) + ku(x) \in \partial A(u(x))$ in the weak sense in $\Omega$, as claimed before.

Let us consider the number $\eta \in \mathbb{R}$ from $(H^2_A)$ and the set

$$W^\eta = \{ u \in H_0^1(\Omega) : \|u\|_{L^\infty} \leq \eta \}.$$

Our localization result reads as follows (see [12, Theorem 2.1] for its smooth form):

**Theorem 3.1.** Let $k > 0$ and assume that hypotheses $(H^1_A)$ and $(H^2_A)$ hold. Then

(i) the energy functional $\mathcal{T}$ is bounded from below on $W^\eta$ and its infimum is attained at some $\tilde{u} \in W^\eta$;

(ii) $\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$;

(iii) $\tilde{u}$ is a weak solution of the differential inclusion $(D^k_A)$.

**Proof.** The proof is similar to that of Kristály and Moroşanu [12]; for completeness, we provide its main steps.

(i) Due to $(H^1_A)$, it is clear that the energy functional $\mathcal{T}$ is bounded from below on $H_0^1(\Omega)$. Moreover, due to the compactness of the embedding $H_0^1(\Omega) \subset L^q(\Omega)$, $q \in [2, 2^*)$, it turns out that $\mathcal{T}$ is sequentially weak semi-continuous on $H_0^1(\Omega)$. In addition, the set $W^\eta$ is weakly closed, being convex and closed in $H_0^1(\Omega)$. Thus, there is $\tilde{u} \in W^\eta$ which is a minimum point of $\mathcal{T}$ on the set $W^\eta$, cf. Zeidler [24].

(ii) We introduce the set $L = \{ x \in \Omega : \tilde{u}(x) \notin [0, \delta] \}$ and suppose indirectly that $m(L) > 0$. Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, \delta)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ \tilde{u}$. It is clear that $\gamma$ is a Lipschitz function and $\gamma(0) = 0$. Accordingly, based on the superposition theorem of Marcus and Mizel [14], one has that $w \in H_0^1(\Omega)$. Moreover, $0 \leq w(x) \leq \delta$ for a.e. $\Omega$. Consequently, $w \in W^\eta$. 7
Let us introduce the sets
\[
L_1 = \{ x \in L : \tilde{u}(x) < 0 \} \quad \text{and} \quad L_2 = \{ x \in L : \tilde{u}(x) > \delta \}.
\]
In particular, \( L = L_1 \cup L_2 \), and by definition, it follows that \( w(x) = \tilde{u}(x) \) for all \( x \in \Omega \setminus L \), \( w(x) = 0 \) for all \( x \in L_1 \), and \( w(x) = \delta \) for all \( x \in L_2 \). In addition, one has
\[
T(w) - T(\tilde{u}) = \frac{1}{2} \left[ \|w\|_{H^0}^2 - \|\tilde{u}\|_{H^0}^2 \right] + \frac{k}{2} \int_\Omega [w^2 - \tilde{u}^2] - \int_\Omega [A(w(x)) - A(\tilde{u}(x))].
\]
On account of \( k > 0 \), we have
\[
k \int_L [w^2 - \tilde{u}^2] = -k \int_{L_1} \tilde{u}^2 + k \int_{L_2} [\delta^2 - \tilde{u}^2] \leq 0.
\]
Since \( A(s) = 0 \) for all \( s \leq 0 \), we have
\[
\int_{L_1} [A(w(x)) - A(\tilde{u}(x))] = 0.
\]
By means of the Lebourg’s mean value theorem, for a.e. \( x \in L_2 \), there exists \( \theta(x) \in [\delta, \tilde{u}(x)] \subseteq [\delta, \eta] \) such that
\[
A(w(x)) - A(\tilde{u}(x)) = A(\delta) - A(\tilde{u}(x)) = a(\theta(x))(\delta - \tilde{u}(x)),
\]
where \( a(\theta(x)) \in \partial A(\theta(x)) \). Due to \( (H^2_\lambda) \), it turns out that
\[
\int_{L_2} [A(w(x)) - A(\tilde{u}(x))] \geq 0.
\]
Therefore, we obtain that \( T(w) - T(\tilde{u}) \leq 0 \). On the other hand, since \( w \in W^\eta \), then \( T(w) \geq T(\tilde{u}) = \inf_{W^\eta} T \), thus every term in the difference \( T(w) - T(\tilde{u}) \) should be zero; in particular,
\[
\int_{L_1} \tilde{u}^2 = \int_{L_2} [\tilde{u}^2 - \delta^2] = 0.
\]
The latter relation implies in particular that \( m(L) = 0 \), which is a contradiction, completing the proof of (ii).

(iii) Since \( \tilde{u}(x) \in [0, \delta] \) for a.e. \( x \in \Omega \), an arbitrarily small perturbation \( \tilde{u} + \epsilon v \) of \( \tilde{u} \) with \( 0 < \epsilon \ll 1 \) and \( v \in C_0^\infty(\Omega) \) still implies that \( T(\tilde{u} + \epsilon v) \geq T(\tilde{u}) \); accordingly, \( \tilde{u} \) is a minimum point for \( T \) in the strong topology of \( H^1_0(\Omega) \), thus \( 0 \in \partial T(\tilde{u}) \), cf. Remark 7.1 in the Appendix. Consequently, it follows that \( \tilde{u} \) is a weak solution of the differential inclusion \( (D^\lambda) \).

In the sequel, we need a truncation function of \( H^1_0(\Omega) \), see also [12]. To construct this function, let \( B(x_0, r) \subseteq \Omega \) be the \( n \)-dimensional ball with radius \( r > 0 \) and center \( x_0 \in \Omega \). For \( s > 0 \), define
\[
w_s(x) = \begin{cases} 
0, & \text{if } x \in \Omega \setminus B(x_0, r); \\
s, & \text{if } x \in B(x_0, r/2); \\
\frac{2s}{r}(r - |x - x_0|), & \text{if } x \in B(x_0, r) \setminus B(x_0, r/2).
\end{cases}
\]  
(3.1)
Note that \( w_s \in H^1_0(\Omega) \), \( \|w_s\|_{L^\infty} = s \) and
\[
\|w_s\|_{H^1_0}^2 = \int_\Omega |\nabla w_s|^2 = 4r^{n-2}(1 - 2^{-n})\omega_n s^2 \equiv C(r, n)s^2 > 0;
\]  
(3.2)
hereafter \( \omega_n \) stands for the volume of \( B(0, 1) \subseteq \mathbb{R}^n \).
4. PROOF OF THEOREMS 2.1 AND 2.2

Before giving the proof of Theorems 2.1 and 2.2, in the first part of this section we study the differential inclusion problem

\[
\begin{aligned}
-\triangle u(x) + ku(x) & \in \partial A(u(x)), & x & \in \Omega, \\
u(x) & = 0, & x & \in \partial \Omega,
\end{aligned}
\]

where \( k > 0 \) and the locally Lipschitz function \( A : \mathbb{R}_+ \to \mathbb{R} \) verifies

- \((H^0_0)\): \( A(0) = 0 \);
- \((H^0_1)\): \( -\infty < \liminf_{s \to 0^+} \frac{A(s)}{s} \) and \( \limsup_{s \to 0^+} \frac{A(s)}{s} = +\infty \);
- \((H^0_2)\): there are two sequences \( \{\delta_i\}, \{\eta_i\} \) with \( 0 < \eta_{i+1} < \delta_i < \eta_i \), \( \lim_{i \to \infty} \eta_i = 0 \), and

\[
\max \{ \partial A(s) \} := \max \{ \xi \in \partial A(s) \} \leq 0
\]

for every \( s \in [\delta_i, \eta_i], i \in \mathbb{N} \).

**Theorem 4.1.** Let \( k > 0 \) and assume hypotheses \((H^0_0)\), \((H^0_1)\) and \((H^0_2)\) hold. Then there exists a sequence \( \{u^i_0\}_i \subset H^1_0(\Omega) \) of distinct weak solutions of the differential inclusion problem \((D^k_A)\) such that

\[
\lim_{i \to \infty} \|u^i_0\|_{H^1_0} = \lim_{i \to \infty} \|u^i_0\|_{L^\infty} = 0.
\] (4.1)

**Proof.** We may assume that \( \{\delta_i\}, \{\eta_i\} \subset (0, 1) \). For any fixed number \( i \in \mathbb{N} \), we define the locally Lipschitz function \( A_i : \mathbb{R} \to \mathbb{R} \) by

\[
A_i(s) = A(\tau_{\eta_i}(s)),
\] (4.2)

where \( A(s) = 0 \) for \( s \leq 0 \) and \( \tau_{\eta} : \mathbb{R} \to \mathbb{R} \) denotes the truncation function \( \tau_{\eta}(s) = \min(s, \eta) \), \( \eta > 0 \). For further use, we introduce the energy functional \( \mathcal{T}_i : H^1_0(\Omega) \to \mathbb{R} \) associated with problem \((D^k_{A_i})\).

We notice that for \( s \geq 0 \), the chain rule (see Proposition 7.4 in the Appendix) gives

\[
\partial A_i(s) = \begin{cases} 
\partial A(s) & \text{if } s < \eta_i, \\
\square(0, \partial A(\eta_i)) & \text{if } s = \eta_i, \\
\{0\} & \text{if } s > \eta_i.
\end{cases}
\]

It turns out that on the compact set \([0, \eta_i]\), the upper semicontinuous set-valued map \( s \mapsto \partial A_i(s) \) attains its supremum (see Proposition 7.1 in the Appendix); therefore, there exists \( M_{A_i} > 0 \) such that

\[
\max | \partial A_i(s) | := \max \{ |\xi| : \xi \in \partial A_i(s) \} \leq M_{A_i}
\]

for every \( s \geq 0 \), i.e., \((H^1_{A_i})\) holds. The same is true for \((H^2_{A_i})\) by using \((H^0_2)\) on \([\delta_i, \eta_i], i \in \mathbb{N}\).

Accordingly, the assumptions of Theorem 3.1 are verified for every \( i \in \mathbb{N} \) with \([\delta_i, \eta_i]\), thus there exists \( u^0_i \in W^{1, \infty} \) such that

\[
u^0_i \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^{1, \infty},
\] (4.3)

\[
u^0_i(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega,
\] (4.4)

\[
u^0_i \text{ is a solution of } (D^k_{A_i}).
\] (4.5)

On account of relations (4.2), (4.4) and (4.5), \( u^0_i \) is a weak solution also for the differential inclusion problem \((D^k_A)\).
We are going to prove that there are infinitely many distinct elements in the sequence \( \{ u_i^0 \} \). To conclude it, we first prove that

\[
\mathcal{T}_i(u_i^0) < 0 \quad \text{for all } i \in \mathbb{N}; \quad \text{and} \quad \lim_{i \to \infty} \mathcal{T}_i(u_i^0) = 0. 
\tag{4.6}
\]

The left part of (H\(_0^1\)) implies the existence of some \( l_0 > 0 \) and \( \zeta \in (0, \eta) \) such that

\[
A(s) \geq -l_0 s^2 \quad \text{for all } s \in (0, \zeta). 
\tag{4.8}
\]

One can choose \( L_0 > 0 \) such that

\[
\frac{1}{2} C(r, n) + \left( \frac{k}{2} + l_0 \right) m(\Omega) < L_0(r/2)^n \omega_n, 
\tag{4.9}
\]

where \( r > 0 \) and \( C(r, n) > 0 \) come from (3.2). Based on the right part of (H\(_1^0\)), one can find a sequence \( \{ \tilde{s}_i \} \subset (0, \zeta) \) such that \( \tilde{s}_i \leq \delta_i \) and

\[
A(\tilde{s}_i) > L_0 \tilde{s}_i^2 \quad \text{for all } i \in \mathbb{N}. 
\tag{4.10}
\]

Let \( i \in \mathbb{N} \) be a fixed number and let \( w_{\tilde{s}_i} \in H^1_0(\Omega) \) be the function from (3.1) corresponding to the value \( \tilde{s}_i > 0 \). Then \( w_{\tilde{s}_i} \in \mathcal{W}^n \), and due to (4.8), (4.10) and (3.2) one has

\[
\mathcal{T}_i(w_{\tilde{s}_i}) = \frac{1}{2} \| w_{\tilde{s}_i} \|^2_{H^1_0} + \frac{k}{2} \int_\Omega w_{\tilde{s}_i}^2 - \int_\Omega A_i(w_{\tilde{s}_i}(x)) \ dx \\
= \frac{1}{2} C(r, n) \tilde{s}_i^2 + \frac{k}{2} \int_\Omega w_{\tilde{s}_i}^2 - \int_{B(x_0, r/2)} A(\tilde{s}_i) \ dx - \int_{B(x_0, r/2) \setminus B(x_0, r/2)} A(w_{\tilde{s}_i}(x)) \ dx \\
\leq \left[ \frac{1}{2} C(r, n) + \frac{k}{2} m(\Omega) - L_0(r/2)^n \omega_n + l_0 m(\Omega) \right] \tilde{s}_i^2.
\]

Accordingly, with (4.3) and (4.9), we conclude that

\[
\mathcal{T}_i(u_i^0) = \min_{\mathcal{W}^n} \mathcal{T}_i \leq \mathcal{T}_i(w_{\tilde{s}_i}) < 0 
\tag{4.11}
\]

which completes the proof of (4.6).

Now, we prove (4.7). For every \( i \in \mathbb{N} \), by using the Lebourg’s mean value theorem, relations (4.2) and (4.4) and (H\(_0^1\)), we have

\[
\mathcal{T}_i(u_i^0) \geq -\int_\Omega A_i(u_i^0(x)) \ dx = -\int_\Omega A_1(u_i^0(x)) \ dx \geq -M_{A_1} m(\Omega) \delta_i.
\]

Since \( \lim_{i \to \infty} \delta_i = 0 \), the latter estimate and (4.11) provides relation (4.7).

Based on (4.2) and (4.4), we have that \( \mathcal{T}_i(u_i^0) = \mathcal{T}_i(u_i^0) \) for all \( i \in \mathbb{N} \). This relation with (4.6) and (4.7) means that the sequence \( \{ u_i^0 \} \) contains infinitely many distinct elements.

We now prove (4.1). One can prove the former limit by (4.4), i.e. \( \| u_i^0 \|_{L^\infty} \leq \delta_i \) for all \( i \in \mathbb{N} \), combined with \( \lim_{i \to \infty} \delta_i = 0 \). For the latter limit, we use \( k > 0 \), (4.11), (4.2) and (4.4) to get for all \( i \in \mathbb{N} \) that

\[
\frac{1}{2} \| u_i^0 \|_{H^1_0}^2 \leq \frac{1}{2} \| u_i^0 \|_{H^1_0}^2 + \frac{k}{2} \int_\Omega (u_i^0)^2 < \int_\Omega A_i(u_i^0(x)) = \int_\Omega A_1(u_i^0(x)) \leq M_{A_1} m(\Omega) \delta_i,
\]

which completes the proof. \( \square \)
Proof of Theorem 2.1. We split the proof into two parts.

(i) Case $p = 1$. Let $\lambda \geq 0$ with $\lambda c < -l_0$ and fix $\lambda_0 \in \mathbb{R}$ such that $\lambda c < \lambda_0 < -l_0$. With these choices we define

$$k := \lambda_0 - \lambda c > 0$$

and $A(s) := F(s) + \frac{\lambda_0}{2}s^2 + \lambda \left(G(s) - \frac{c}{2}s^2\right)$ for every $s \in [0, \infty)$. (4.12)

It is clear that $A(0) = 0$, i.e., $(H_0^0)$ is verified. Since $p = 1$, by $(G_0^0)$ one has

$$\zeta = \liminf_{s \to 0^+} \frac{\min \{\partial G(s)\}}{s} \leq \limsup_{s \to 0^+} \frac{\max \{\partial G(s)\}}{s} = \tau.$$ 

In particular, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max \{\partial G(s)\} - \tau s < \epsilon s, \ \forall s \in [0, \gamma],$$

and

$$\min \{\partial G(s)\} - \epsilon s > -\epsilon s, \ \forall s \in [0, \gamma].$$

For $s \in [0, \gamma]$, Lebourg’s mean value theorem and $G(0) = 0$ implies that there exists $\xi_s \in \partial G(\theta_s s)$ for some $\theta_s \in [0, 1]$ such that $G(s) - G(0) = \xi_s s$. Accordingly, for every $s \in [0, \gamma]$ we have that

$$(\zeta - \epsilon)s^2 \leq G(s) \leq (\tau + \epsilon)s^2.$$ 

By (4.13) and $(F_0^0)$ we have that

$$\liminf_{s \to 0^+} \frac{A(s)}{s^2} \geq \liminf_{s \to 0^+} \frac{F(s)}{s^2} + \frac{\lambda_0 - \lambda c}{2} + \lambda \liminf_{s \to 0^+} \frac{G(s)}{s^2} \geq \liminf_{s \to 0^+} \frac{F(s)}{s^2} + \frac{\lambda_0 - \lambda c}{2} + \lambda \gamma > -\infty$$

and

$$\limsup_{s \to 0^+} \frac{A(s)}{s^2} \geq \limsup_{s \to 0^+} \frac{F(s)}{s^2} + \frac{\lambda_0 - \lambda c}{2} + \lambda \liminf_{s \to 0^+} \frac{G(s)}{s^2} = +\infty,$$

i.e., $(H_0^0)$ is verified.

Since

$$\partial A(s) \subseteq \partial F(s) + \lambda_0 s + \lambda (\partial G(s) - \tau s),$$

and $\lambda \geq 0$, we have that

$$\max \{\partial A(s)\} \leq \max \{\partial F(s) + \lambda_0 s\} + \lambda \max \{\partial G(s) - \tau s\}.\quad (4.14)$$

Since

$$\limsup_{s \to 0^+} \frac{\max \{\partial G(s)\}}{s} = \tau,$$

cf. $(G_1^0)$, and

$$\liminf_{s \to 0^+} \frac{\max \{\partial F(s)\}}{s} = l_0 < 0,$$

cf. $(F_1^0)$, it turns out by (4.15) that

$$\liminf_{s \to 0^+} \frac{\max \{\partial A(s)\}}{s} \leq \liminf_{s \to 0^+} \frac{\max \{\partial F(s)\}}{s} + \lambda_0 - \lambda c + \lambda \limsup_{s \to 0^+} \frac{\max \{\partial G(s)\}}{s} \leq l_0 + \lambda_0 < 0.$$ 

Therefore, one has a sequence $\{s_i\}_i \subset (0, 1)$ converging to 0 such that $\frac{\max \{\partial A(s_i)\}}{s_i} < 0$ i.e., $\max \{\partial A(s_i)\} < 0$ for all $i \in \mathbb{N}$. By using the upper semicontinuity of $s \mapsto \partial A(s)$, we may choose two numbers $\delta_i, \eta_i \in (0, 1)$ with $\delta_i < s_i < \eta_i$ such that $\partial A(s) \subset \partial A(s_i) + [-\epsilon_i, \epsilon_i]$ for every $s \in [\delta_i, \eta_i]$, where $\epsilon_i := -\max \{\partial A(s_i)\}/2 > 0$. In particular, $\max \{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$. Thus, one may fix two sequences $\{\delta_i\}_i, \{\eta_i\}_i \subset (0, 1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i \to \infty} \eta_i = 0$, and $\max \{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$. Accordingly, $(H_2^0)$ is verified as...
well. Let us apply Theorem 4.1 with the choice (4.12), i.e., there exists a sequence \( \{u_i\}_i \subset H^1_0(\Omega) \) of different elements such that

\[
\begin{cases}
-\triangle u_i(x) + (\lambda_0 - \lambda^2) u_i(x) \in \partial F(u_i(x)) + \lambda_0 u_i(x) + \lambda (\partial G(u_i(x)) - \tau u_i(x)) & x \in \Omega, \\
u_i(x) \geq 0 & x \in \Omega, \\
u_i(x) = 0 & x \in \partial \Omega,
\end{cases}
\]

where we used the inclusion (4.14). In particular, \( u_i \) solves problem \( (D_\lambda) \), \( i \in \mathbb{N} \), which completes the proof of (i).

(ii) Case \( p > 1 \). Let \( \lambda \geq 0 \) be arbitrary fixed and choose a number \( \lambda_0 \in (0, -l_0) \). Let

\[
k := \lambda_0 > 0 \quad \text{and} \quad A(s) := F(s) + \lambda G(s) + \lambda_0 s^2 \quad \text{for every} \quad s \in [0, \infty).
\]

Since \( F(0) = G(0) = 0 \), hypothesis \( (H^0_0) \) clearly holds. By \( (G^0_1) \) one has

\[
\mathcal{C} = \liminf_{s \to 0^+} \min \left\{ \frac{\partial G(s)}{s^p} \right\} \leq \limsup_{s \to 0^+} \max \left\{ \frac{\partial G(s)}{s^p} \right\} = \tau.
\]

In particular, since \( p > 1 \), then

\[
\lim_{s \to 0^+} \min \left\{ \frac{\partial G(s)}{s} \right\} = \lim_{s \to 0^+} \max \left\{ \frac{\partial G(s)}{s} \right\} = 0
\]

and for sufficiently small \( \epsilon > 0 \) there exists \( \gamma = \gamma(\epsilon) > 0 \) such that

\[
\max \left\{ \frac{\partial G(s)}{s} \right\} - \gamma s^p < \epsilon s^p, \quad \forall s \in [0, \gamma]
\]

and

\[
\min \left\{ \frac{\partial G(s)}{s} \right\} - \epsilon s^p > -\epsilon s^p, \quad \forall s \in [0, \gamma].
\]

For a fixed \( s \in [0, \gamma] \), by Lebourg’s mean value theorem and \( G(0) = 0 \) we conclude again that \( G(s) - G(0) = \xi_s s \). Accordingly, for sufficiently small \( \epsilon > 0 \) there exists \( \gamma = \gamma(\epsilon) > 0 \) such that \( (\mathcal{C} - \epsilon) s^p + 1 \leq G(s) \leq (\mathcal{C} + \epsilon) s^p + 1 \) for every \( s \in [0, \gamma] \). Thus, since \( p > 1 \),

\[
\lim_{s \to 0^+} \frac{G(s)}{s^2} = \lim_{s \to 0^+} \frac{G(s)}{s^{p+1}} s^{p-1} = 0.
\]

Therefore, by using (4.16) and \( (F^0_1) \), we conclude that

\[
\liminf_{s \to 0^+} \frac{A(s)}{s^2} = \liminf_{s \to 0^+} \frac{F(s)}{s^2} + \lambda \lim_{s \to 0^+} \frac{G(s)}{s^2} + \frac{\lambda_0}{2} > -\infty,
\]

and

\[
\limsup_{s \to 0^+} \frac{A(s)}{s^2} = \infty,
\]

i.e., \( (H^1_0) \) holds. Since

\[
\partial A(s) \subseteq \partial F(s) + \lambda \partial G(s) + \lambda_0 s, \quad \lambda \geq 0,
\]

and \( \lambda \geq 0 \), we have that

\[
\max \{\partial A(s)\} \leq \max \{\partial F(s)\} + \max \{\lambda \partial G(s) + \lambda_0 s\}.
\]

Since

\[
\limsup_{s \to 0^+} \frac{\max \{\partial G(s)\}}{s^p} = \tau,
\]

cf. \( (G^1_1) \), and

\[
\liminf_{s \to 0^+} \frac{\max \{\partial F(s)\}}{s} = l_0,
\]
cf. \((F^0_2)\), by relation \((4.17)\) it turns out that
\[
\liminf_{s \to 0^+} \frac{\max\{\partial A(s)\}}{s} = \liminf_{s \to 0^+} \frac{\max\{\partial F(s)\}}{s} + \lambda \lim_{s \to 0^+} \frac{\max\{\partial G(s)\}}{s} + \lambda_0 = l_0 + \lambda_0 < 0,
\]
and the upper semicontinuity of \(\partial A\) implies the existence of two sequences \(\{\delta_i\}_i\) and \(\{\eta_i\}_i \subset (0, 1)\) such that \(0 < \eta_{i+1} < \delta_i < s_i < \eta_i, \lim_{i \to \infty} \eta_i = 0\), and \(\max\{\partial A(s)\} \leq 0\) for all \(s \in [\delta_i, \eta_i]\) and \(i \in \mathbb{N}\). Therefore, hypothesis \((\Pi^0_2)\) holds. Now, we can apply Theorem 4.1, i.e., there is a sequence \(\{u_i\}_i \subset H^1_0(\Omega)\) of different elements such that
\[
\begin{cases}
-\Delta u_i(x) + \lambda_0 u_i(x) \in \partial A(u_i(x)) \subseteq \partial F(u_i(x)) + \lambda \partial G(u_i(x)) + \lambda_0 u_i(x) & x \in \Omega,
\qquad \liminf_{s \to 0^+} \frac{\max\{\partial A(s)\}}{s} \\
u_i(x) \geq 0 & x \in \Omega,
\qquad \lim_{s \to 0^+} \frac{\max\{\partial G(s)\}}{s} \\
u_i(x) = 0 & x \in \partial \Omega.
\end{cases}
\]
which means that \(u_i\) solves problem \((D^k)\), \(i \in \mathbb{N}\). This completes the proof of Theorem 2.1. \(\square\)

**Proof of Theorem 2.2.** The proof is done in two steps:

(i) Let \(\lambda_0 \in (0, -l_0)\), \(\lambda \geq 0\) and define
\[
k := \lambda_0 > 0 \quad \text{and} \quad A^\lambda(s) := F(s) + \lambda G(s) + \frac{\lambda_0 s^2}{2} \quad \text{for every} \quad s \in [0, \infty).
\]
One can observe that \(\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_0 s + \lambda \partial G(s)\) for every \(s \geq 0\). On account of \((F^0_2)\), there is a sequence \(\{s_i\}_i \subset (0, 1)\) converging to 0 such that
\[
\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_0 s_i < 0.
\]
Thus, due to the upper semicontinuity of \((s, \lambda) \mapsto \partial A^\lambda(s)\), we can choose three sequences \(\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0, 1)\) such that \(0 < \eta_{i+1} < \delta_i < s_i < \eta_i, \lim_{i \to \infty} \eta_i = 0\), and
\[
\max\{\partial A^\lambda(s)\} \leq 0 \quad \text{for all} \quad \lambda \in [0, \lambda_i], s \in [\delta_i, \eta_i], \quad i \in \mathbb{N}.
\]
Without any loss of generality, we may choose
\[
\delta_i \leq \min\{i^{-1}, 2^{-1}i^{-2}[1 + m(\Omega)(\max_{s \in [0, 1]} |\partial F(s)| + \max_{s \in [0, 1]} |\partial G(s)|)^{-1}\}. \quad (4.19)
\]
For every \(i \in \mathbb{N}\) and \(\lambda \in [0, \lambda_i]\), let \(A^\lambda_i : [0, \infty) \to \mathbb{R}\) be defined as
\[
A^\lambda_i(s) = A^\lambda(\tau_{\eta_i}(s)), \quad (4.20)
\]
and the energy functional \(T_{i, \lambda} : H^1_0(\Omega) \to \mathbb{R}\) associated with the differential inclusion problem \((D^k)\) is given by
\[
T_{i, \lambda}(u) = \frac{1}{2} \|u\|_{H^1_0}^2 + k \int_\Omega u^2 \, dx - \int_\Omega A^\lambda_i(u(x)) \, dx.
\]
One can easily check that for every \(i \in \mathbb{N}\) and \(\lambda \in [0, \lambda_i]\), the function \(A^\lambda_i\) verifies the hypotheses of Theorem 3.1. Accordingly, for every \(i \in \mathbb{N}\) and \(\lambda \in [0, \lambda_i]\):
\[
\begin{align*}
T_{i, \lambda} & \text{attains its infimum on} \ W^m_h \text{at some} \ u_{i, \lambda}^0 \in W^m_h \quad (4.21) \\
u_{i, \lambda}^0(x) & \in [0, \delta_i] \text{ for a.e.} \ x \in \Omega; \quad (4.22) \\
u_{i, \lambda} & \text{is a weak solution of} \ (D^k_{A^\lambda_i}); \quad (4.23)
\end{align*}
\]
By the choice of the function \(A^\lambda\) and \(k > 0\), \(u_{i, \lambda}^0\) is also a solution to the differential inclusion problem \((D^k_{A^\lambda})\), so \((D^\lambda)\).
Based on these inequalities, it turns out that the elements \( u_\lambda \) are trivial whenever

\[ \mathcal{T}_i(u_\lambda^0) = \min_{W^m} \mathcal{T}_i(w_{\lambda^0}) < 0 \]  

for all \( i \in \mathbb{N} \). (4.24)

Similarly to Kristály and Moroșanu [12], let \( \{\theta_i\}_i \) be a sequence with negative terms such that \( \lim_{i \to \infty} \theta_i = 0 \). Due to (4.24) we may assume that

\[ \theta_i < \mathcal{T}_i(u_\lambda^0) \leq \mathcal{T}_i(w_{\lambda^0}) < \theta_{i+1}. \]  

(4.25)

Let us choose

\[ \lambda_i' = \frac{\theta_{i+1} - \mathcal{T}_i(w_{\lambda^0})}{m(\Omega) \max_{s \in [0,1]} |G(s)| + 1} \]  

and

\[ \lambda_i'' = \frac{\mathcal{T}_i(u_\lambda^0) - \theta_i}{m(\Omega) \max_{s \in [0,1]} |G(s)| + 1}, \]  

\( i \in \mathbb{N} \). (4.26)

and for a fixed \( k \in \mathbb{N} \), set

\[ \lambda_k^i = \min(1, \lambda_1, ..., \lambda_k, \lambda_i', ..., \lambda_k', \lambda_i'', ..., \lambda_k'') > 0. \]  

(4.27)

Having in mind these choices, for every \( i \in \{1, ..., k\} \) and \( \lambda \in [0, \lambda_k^0] \) one has

\[ \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) \leq \mathcal{T}_{i,\lambda}(w_{\lambda^0}) = \frac{1}{2} \|w_{\lambda^0}\|_{H^1}^2 - \int_{\Omega} F(w_{\lambda^0}(x)) dx - \lambda \int_{\Omega} G(w_{\lambda^0}(x)) dx \]

\[ = \mathcal{T}_i(w_{\lambda^0}) - \lambda \int_{\Omega} G(w_{\lambda^0}(x)) dx \]

\[ < \theta_{i+1}. \]  

(4.28)

and due to \( u_{i,\lambda}^0 \in W^m \) and to the fact that \( u_i^0 \) is the minimum point of \( \mathcal{T}_i \) on the set \( W^m \), by (4.25) we also have

\[ \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{T}_i(u_{i,\lambda}^0) - \lambda \int_{\Omega} G(u_{i,\lambda}^0(x)) dx \geq \mathcal{T}_i(u_\lambda^0) - \lambda \int_{\Omega} G(u_{i,\lambda}^0(x)) dx > \theta_i. \]  

(4.29)

Therefore, by (4.28) and (4.29), for every \( i \in \{1, ..., k\} \) and \( \lambda \in [0, \lambda_k^0] \), one has

\[ \theta_i < \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1}, \]  

thus

\[ \mathcal{T}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{T}_{k,\lambda}(u_{k,\lambda}^0) < 0. \]

We notice that \( u_i^0 \in W^m \) for every \( i \in \{1, ..., k\} \), so \( \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) \) because of (4.20). Therefore, we conclude that for every \( \lambda \in [0, \lambda_k^0] \),

\[ \mathcal{T}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{T}_{k,\lambda}(u_{k,\lambda}^0) < 0 = \mathcal{T}_{1,\lambda}(0). \]

Based on these inequalities, it turns out that the elements \( u_{1,\lambda}^0, ..., u_{k,\lambda}^0 \) are distinct and non-trivial whenever \( \lambda \in [0, \lambda_k^0] \).

Now, we are going to prove the estimate (2.2). We have for every \( i \in \{1, ..., k\} \) and \( \lambda \in [0, \lambda_k^0] \):

\[ \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{T}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1} < 0. \]
By Lebourg’s mean value theorem and (4.19), we have for every \( i \in \{1, ..., k\} \) and \( \lambda \in [0, \lambda^0_i] \) that
\[
\frac{1}{2} \| u_i^0, \lambda \|_{H^1_0}^2 < \int_{\Omega} F(u_i^0, \lambda)(x) \, dx + \lambda \int_{\Omega} G(u_i^0, \lambda)(x) \, dx \leq m(\Omega) \delta_i \max_{s \in [0,1]} |\partial F(s)| + \max_{s \in [0,1]} |\partial G(s)| \leq \frac{1}{2\delta_i}.
\]
This completes the proof of Theorem 2.2. \( \square \)

5. Proof of Theorems 2.3 and 2.4

We consider again the differential inclusion problem
\[
\begin{aligned}
&-\Delta u(x) + ku(x) \in \partial A(u(x)), \quad u(x) \geq 0 \quad x \in \Omega, \\
u(x) = 0 \quad x \in \partial \Omega,
\end{aligned}
\]
where \( k > 0 \) and the locally Lipschitz function \( A : \mathbb{R}_+ \to \mathbb{R} \) verifies
- (H\(^{2^*_\infty}\)): \( A(0) = 0 \);
- (H\(^{1^*_\infty}\)): \( -\infty < \liminf_{s \to \infty} A(s) \frac{\lambda}{s^2} \) and \( \limsup_{s \to \infty} A(s) \frac{\lambda}{s^2} = +\infty \);
- (H\(^{1^*_\infty}\)): there are two sequences \( \{\delta_i\}, \{\eta_i\} \) with \( 0 < \delta_i < \eta_i < \delta_{i+1}, \lim_{i \to \infty} \delta_i = \infty \), and
\[
\max \{\partial A(s)\} := \max \{\xi : \xi \in \partial A(s)\} \leq 0
\]
for every \( s \in [\delta_i, \eta_i], i \in \mathbb{N} \).

The counterpart of Theorem 4.1 reads as follows.

**Theorem 5.1.** Let \( k > 0 \) and assume the hypotheses (H\(^{2^*_\infty}\)), (H\(^{1^*_\infty}\)) and (H\(^{1^*_\infty}\)) hold. Then the differential inclusion problem (D\(^k_\lambda\)) admits a sequence \( \{u_i^\infty\} \subset H^1_0(\Omega) \) of distinct weak solutions such that
\[
\lim_{i \to \infty} \| u_i^\infty \|_{L^\infty} = \infty.
\]

**Proof.** The proof is similar to the one performed in Theorem 4.1; we shall show the differences only. We associate the energy functional \( \mathcal{T}_i : H^1_0(\Omega) \to \mathbb{R} \) with problem (D\(^k_\lambda\)), where \( A_i : \mathbb{R} \to \mathbb{R} \) is given by
\[
A_i(s) = A(\tau_{\eta_i}(s)),
\]
with \( A(s) = 0 \) for \( s \leq 0 \). One can show that there exists \( M_{A_i} > 0 \) such that
\[
\max |\partial A_i(s)| := \max \{|\xi| : \xi \in \partial A_i(s)\} \leq M_{A_i}
\]
for all \( s \geq 0 \), i.e., hypothesis (H\(^{1^*_\lambda_i}\)) holds. Moreover, (H\(^{2^*_\lambda_i}\)) follows by (H\(^{2^*_\infty}\)). Thus Theorem 4.1 can be applied for all \( i \in \mathbb{N} \), i.e., we have an element \( u_i^\infty \in W^m \) such that
\[
u_i^\infty \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^m,
\]
\[
u_i^\infty(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega,
\]
\[
u_i^\infty \text{ is a weak solution of (D}^k_{\lambda_i} \text{).}
\]
By (5.2), \( u_i^\infty \) turns to be a weak solution also for differential inclusion problem (D\(^k_{\lambda_i}\)).

We shall prove that there are infinitely many distinct elements in the sequence \( \{u_i^\infty\} \) by showing that
\[
\lim_{i \to \infty} \mathcal{T}_i(u_i^\infty) = -\infty.
\]
By the left part of (H^∞_1) we can find l^A_∞ > 0 and ζ > 0 such that
\[ A(s) \geq -l^A_∞ \text{ for all } s > ζ. \] (5.7)

Let us choose L^A_∞ > 0 large enough such that
\[ \frac{1}{2}C(r, n) + \left( \frac{k}{2} + l^A_∞ \right) m(Ω) < L^A_∞ (r/2)^n ω_n. \] (5.8)

On account of the right part of (H^∞_1), one can fix a sequence \{s_ι\}_ι ⊂ (0, ∞) such that \lim_{ι→∞} s_ι = ∞ and
\[ A(s_ι) > L^A_∞ s_ι^2 \text{ for every } ι \in \mathbb{N}. \] (5.9)

We know from (H^∞_2) that \lim_{ι→∞} δ_ι = ∞, therefore one has a subsequence \{δ_{m_ι}\}_ι of \{δ_ι\}_ι such that \( s_ι \leq δ_{m_ι} \) for all \ι \in \mathbb{N}. Let ι \in \mathbb{N}, and recall \( w_{s_ι} ∈ H^1_0(Ω) \) from (3.1) with \( s_ι := s_ι > 0. \) Then \( w_{s_ι} ∈ W^{m_ι} \) and according to (3.2), (5.7) and (5.9) we have
\[
T_{m_ι}(w_{s_ι}) = \frac{1}{2} ||w_{s_ι}||^2_{H^1_0} + \frac{k}{2} \int_Ω (w_{s_ι})^2 - \int_Ω A_{m_ι}(w_{s_ι}(x))dx
\]
\[
= \frac{1}{2} C(r, n) s_ι^2 + \frac{k}{2} \int_Ω (w_{s_ι})^2 - \int_{B(x_0,r/2)} A(s_ι)dx
\]
\[ - \int_{(B(x_0,r)\setminus B(x_0,r/2)) \cap \{w_{s_ι} > ζ\}} A(w_{s_ι}(x))dx
\]
\[ - \int_{(B(x_0,r)\setminus B(x_0,r/2)) \cap \{w_{s_ι} ≤ ζ\}} A(w_{s_ι}(x))dx
\]
\[
\leq \left[ \frac{1}{2} C(r, n) + \frac{k}{2} m(Ω) - L^A_∞ (r/2)^n ω_n + l^A_∞ m(Ω) \right] s_ι^2 + \tilde{M}_A m(Ω)\zeta,
\]
where \( \tilde{M}_A = \max\{|A(s)| : s ∈ [0, ζ]\} \) does not depend on \ι \in \mathbb{N}. This estimate combined by (5.8) and \lim_{ι→∞} s_ι = ∞ yields that
\[ \lim_{ι→∞} T_{m_ι}(w_{s_ι}) = -∞. \] (5.10)

By equation (5.3), one has
\[ T_{m_ι}(u^∞_{m_ι}) = \min_{W^{m_ι}} T_{m_ι} ≤ T_{m_ι}(w_{s_ι}). \] (5.11)

It follows by (5.10) that \lim_{ι→∞} T_{m_ι}(u^∞_{m_ι}) = -∞.

We notice that the sequence \{T_i(u^∞_{m_ι})\}_ι is non-increasing. Indeed, let \ι < k; due to (5.2) one has that
\[ T_i(u^∞_{m_ι}) = \min_{W^{m_i}} T_i = \min_{W^{m_k}} T_k ≥ \min_{W^{m_k}} T_k = T_k(u^∞_{m_k}), \] (5.12)
which completes the proof of (5.6).

The proof of (5.1) goes in a similar way as in [12].

Proof of Theorem 2.3. We split the proof into two parts.

(i) Case p = 1. Let λ ≥ 0 with λc < -l_∞ and fix \( \tilde{λ}_∞ ∈ \mathbb{R} \) such that λc < \( \tilde{λ}_∞ < -l_∞. \) With these choices, we define
\[ k := \tilde{λ}_∞ - λc > 0 \text{ and } A(s) := F(s) + \frac{\tilde{λ}_∞}{2} s^2 + λ \left( G(s) - \frac{c}{2} s^2 \right) \text{ for every } s ∈ [0, ∞). \] (5.13)
It is clear that \(A(0) = 0\), i.e., \((H^\infty_0)\) is verified. A similar argument for the \(p\)-order perturbation \(\partial G\) as before shows that
\[
\liminf_{s \to \infty} \frac{A(s)}{s^2} \geq \liminf_{s \to \infty} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda \tilde{\tau}}{2} + \lambda \liminf_{s \to \infty} \frac{G(s)}{s^2} \geq \liminf_{s \to \infty} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda \tilde{\tau}}{2} + \lambda \epsilon > -\infty,
\]
and
\[
\limsup_{s \to \infty} \frac{A(s)}{s^2} \geq \limsup_{s \to \infty} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda \tilde{\tau}}{2} + \lambda \limsup_{s \to \infty} \frac{G(s)}{s^2} = +\infty,
\]
i.e., \((H^\infty_1)\) is verified.

Since
\[
\frac{\partial A(s)}{s^2} \leq \frac{\partial F(s)}{s^2} + \frac{\tilde{\lambda}_\infty + \lambda \tilde{\tau}}{2} + \lambda \limsup_{s \to \infty} \frac{\partial G(s)}{s^2} = l_\infty + \tilde{\lambda}_\infty < 0,
\]

it turns out that
\[
\liminf_{s \to \infty} \frac{\max\{\partial A(s)\}}{s} \leq \liminf_{s \to \infty} \frac{\max\{\partial F(s)\}}{s} + \frac{\tilde{\lambda}_\infty - \lambda \tilde{\tau} + \lambda \limsup_{s \to \infty} \frac{\max\{\partial G(s)\}}{s}}{s} = l_\infty + \tilde{\lambda}_\infty < 0.
\]

By using the upper semicontinuity of \(s \mapsto \partial A(s)\), one may fix two sequences \(\{\delta_i\}_i, \{\eta_i\}_i \subset (0, \infty)\) such that \(0 < \delta_i < s_i < \eta_i < \delta_{i+1}\), \(\lim_{i \to \infty} \delta_i = \infty\), and \(\max\{\partial A(s)\} \leq 0\) for all \(s \in [\delta_i, \eta_i]\) and \(i \in \mathbb{N}\). Thus, \((H^\infty_2)\) is verified as well. By applying the inclusion (5.14) and Theorem 4.1 with the choice \((5.13)\), there exists a sequence \(\{u_i\}_i \subset H^1_0(\Omega)\) of different elements such that
\[
\begin{align*}
\left\{ -\Delta u_i(x) + (\tilde{\lambda}_\infty - \lambda \tilde{\tau})u_i(x) \in \partial F(u_i(x)) + \tilde{\lambda}_\infty u_i(x) + \lambda(\partial G(u_i(x)) - \tilde{\tau}u_i(x)) & \quad x \in \Omega, \\
u_i(x) \geq 0 & \quad x \in \Omega, \\
u_i(x) = 0 & \quad x \in \partial \Omega,
\end{align*}
\]
i.e., \(u_i\) solves problem \((D)\), \(i \in \mathbb{N}\).

(ii) Case \(p < 1\). Let \(\lambda \geq 0\) be arbitrary fixed and choose a number \(\lambda_\infty \in (0, -l_\infty)\). Let
\[
k := \lambda_\infty > 0 \quad \text{and} \quad A(s) := F(s) + \lambda G(s) + \lambda_\infty \frac{s^2}{2} \quad \text{for every} \quad s \in [0, \infty).
\]

Since \(F(0) = G(0) = 0\), hypothesis \((H^\infty_0)\) clearly holds. Moreover, by \((G^\infty)\), for sufficiently small \(\epsilon > 0\) there exists \(s_0 > 0\), such that \((\tilde{\tau} - \epsilon)s^{p+1} \leq G(s) \leq (\tilde{\tau} + \epsilon)s^{p+1}\) for every \(s > s_0\). Thus, since \(p < 1\),
\[
\lim_{s \to \infty} \frac{G(s)}{s^p} = \lim_{s \to \infty} \frac{G(s)}{s^{p+1}}s^{p-1} = 0.
\]
Accordingly, by using (5.15) we obtain that hypothesis \((H^\infty_1)\) holds. A similar argument as above implies that
\[
\liminf_{s \to \infty} \frac{\max\{\partial A(s)\}}{s} \leq l_0 + \lambda_\infty < 0,
\]
and the upper semicontinuity of \(\partial A\) implies the existence of two sequences \(\{\delta_i\}_i\) and \(\{\eta_i\}_i \subset (0, 1)\) such that \(0 < \delta_i < s_i < \eta_i < \delta_{i+1}\), \(\lim_{i \to \infty} \delta_i = \infty\), and \(\max\{\partial A(s)\} \leq 0\) for all \(s \in [\delta_i, \eta_i]\) and \(i \in \mathbb{N}\). Therefore, hypothesis \((H^\infty_2)\) holds. Now, we can apply Theorem 4.1, i.e., there is a sequence \(\{u_i\}_i \subset H^1_0(\Omega)\) of different elements such that
\[
\begin{align*}
\left\{ -\Delta u_i(x) + \lambda_\infty u_i(x) \in \partial A(u_i(x)) \subseteq \partial F(u_i(x)) + \lambda \partial G(u_i(x)) + \lambda_\infty u_i(x) & \quad x \in \Omega, \\
u_i(x) \geq 0 & \quad x \in \Omega, \\
u_i(x) = 0 & \quad x \in \partial \Omega,
\end{align*}
\]
which means that \(u_i\) solves problem \((D)\), \(i \in \mathbb{N}\), which completes the proof. \(\Box\)
Proof of Theorem 2.4. The proof is done in two steps:

(i) Let $\lambda_\infty \in (0, -l_\infty)$, $\lambda \geq 0$ and define

$$k := \lambda_\infty > 0 \quad \text{and} \quad A^\lambda(s) := F(s) + \lambda G(s) + \lambda_\infty \frac{s^2}{2} \quad \text{for every} \quad s \in [0, \infty). \quad (5.16)$$

One has clearly that $\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_\infty s + \lambda \partial G(s)$ for every $s \in \mathbb{R}$. On account of $(F_2^\infty)$, there is a sequence $\{s_i\}_i \subset (0, \infty)$ converging to $\infty$ such that

$$\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_\infty s_i < 0.$$ 

By the upper semicontinuity of $(s, \lambda) \mapsto \partial A^\lambda(s)$, we can choose the sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0, \infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i \to \infty} \delta_i = \infty$, and

$$\max\{\partial A^\lambda(s)\} \leq 0$$

for all $\lambda \in [0, \lambda_i], s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$.

For every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, let $A^\lambda_i : [0, \infty) \to \mathbb{R}$ be defined by

$$A^\lambda_i(s) = A^\lambda(\tau_{\eta_i}(s)), \quad (5.17)$$

and accordingly, the energy functional $\mathcal{T}_{i,\lambda} : H^1_0(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem $(D^k_{A^\lambda_i})$ is

$$\mathcal{T}_{i,\lambda}(u) = \frac{1}{2} \|u\|^2_{H^1_0} + \frac{k}{2} \int_{\Omega} u^2 \, dx - \int_{\Omega} A^\lambda_i(u(x)) \, dx.$$ 

Then for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, the function $A^\lambda_i$ clearly verifies the hypotheses of Theorem 3.1. Accordingly, for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$ there exists

$$\tilde{u}_{i,\lambda}^\infty \in W^{\eta_i}$$

(5.18)

$$\tilde{u}_{i,\lambda}^\infty \in [0, \delta_i] \quad \text{for a.e.} \ x \in \Omega; \quad (5.19)$$

$$\tilde{u}_{i,\lambda}^\infty(x) \quad \text{is a weak solution of} \ (D^k_{A^\lambda_i}). \quad (5.20)$$

Due to (5.17), $\tilde{u}_{i,\lambda}^\infty$ is not only a solution to $(D^k_{A^\lambda_i})$ but also to the differential inclusion problem $(D_{A^\lambda_i})$, so $(D_\lambda)$.

(ii) For $\lambda = 0$, the function $\partial A^\lambda = \partial A^0$ verifies the hypotheses of Theorem 4.1. Moreover, $\mathcal{T}_i := \mathcal{T}_{i,0}$ is the energy functional associated with problem $(D^k_{A^\lambda_i})$. Consequently, the elements $u_i^\infty := u_i^\infty_{i,0}$ verify not only (5.18)-(5.20) but also

$$\mathcal{T}_{m_i}(u_{m_i}^\infty) = \min_{W^{m_i}}(\mathcal{T}_{m_i}) \leq \mathcal{T}_{m_i}(w_{s_i}) \quad \text{for all} \ i \in \mathbb{N}, \quad (5.21)$$

where the subsequence $\{u_{m_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ and $w_{s_i} \in W$ appear in the proof of Theorem 5.1.

Similarly to Kristály and Moroşanu [12], let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i \to \infty} \theta_i = -\infty$. On account of (5.21) we may assume that

$$\theta_i < \mathcal{T}_{m_i}(u_{m_i}^\infty) \leq \mathcal{T}_{m_i}(w_{s_i}) < \theta_{i+1}. \quad (5.22)$$

Let

$$\lambda_i' = \frac{\theta_i - \mathcal{T}_{m_i}(w_{s_i})}{m(\Omega) \max_{s \in [0, 1]} |G(s)| + 1} \quad \text{and} \lambda_i'' = \frac{\mathcal{T}_{m_i}(u_{m_i}^\infty) - \theta_{i+1}}{m(\Omega) \max_{s \in [0, 1]} |G(s)| + 1}, \quad i \in \mathbb{N}, \quad (5.23)$$

and for a fixed $k \in \mathbb{N}$, we set

$$\lambda_k = \min(1, \lambda_1, \ldots, \lambda_k, \lambda_1', \ldots, \lambda_k'), \lambda_1'', \ldots, \lambda_k'') > 0. \quad (5.24)$$
Then, for every $i \in \{1, \ldots, k\}$ and $\lambda \in [0, \lambda^*_k]$, due to (5.22) we have that
\[
\mathcal{T}_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda}) \leq \mathcal{T}_{m_i, \lambda}(w_{s_i}) = \frac{1}{2}\|w_{s_i}\|_{H^1}^2 - \int_{\Omega} F(w_{s_i}(x))dx - \lambda \int_{\Omega} G(w_{s_i}(x))dx = T_{m_i}(w_{s_i}) - \lambda \int_{\Omega} G(w_{s_i}(x))dx < \theta_i.
\] (5.25)
Similarly, since $\tilde{u}^{\infty}_{m_i, \lambda} \in W^{m_i}$ and $w_{s_i}$ is the minimum point of $\mathcal{T}_i$ on the set $W^{m_i}$, on account of (5.22) we have
\[
\mathcal{T}_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda}) = T_{m_i}(\tilde{u}^{\infty}_{m_i, \lambda}) - \lambda \int_{\Omega} G(\tilde{u}^{\infty}_{m_i, \lambda})dx \geq T_{m_i}(\tilde{u}^{\infty}_{m_i, \lambda}) - \lambda \int_{\Omega} G(\tilde{u}^{\infty}_{m_i, \lambda})dx > \theta_{i+1}.
\] (5.26)
Therefore, for every $i \in \{1, \ldots, k\}$ and $\lambda \in [0, \lambda^*_k]$,  
\[
\theta_{i+1} < \mathcal{T}_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda}) < \theta_i < 0,
\] (5.27)
thus
\[
\mathcal{T}_{m_k, \lambda}(\tilde{u}^{\infty}_{m_k, \lambda}) < \ldots < \mathcal{T}_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda}) < 0.
\] (5.28)
Because of (5.17), we notice that $\tilde{u}^{\infty}_{m_i, \lambda} \in W^{m_i}$ for every $i \in \{1, \ldots, k\}$, thus $\mathcal{T}_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda}) = T_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda})$. Therefore, for every $\lambda \in [0, \lambda^*_k]$, 
\[
\mathcal{T}_{m_k, \lambda}(\tilde{u}^{\infty}_{m_k, \lambda}) < \ldots < \mathcal{T}_{m_i, \lambda}(\tilde{u}^{\infty}_{m_i, \lambda}) < 0 = \mathcal{T}_{m_k, \lambda}(0),
\]
i.e., the elements $\tilde{u}^{\infty}_{m_1, \lambda}, \ldots, \tilde{u}^{\infty}_{m_k, \lambda}$ are distinct and non-trivial whenever $\lambda \in [0, \lambda^*_k]$. The estimate (2.5) follows in a similar manner as in [12].

6. Concluding remarks

1. Suitable modification of our arguments provide multiplicity results for the differential inclusion problem
\[
\begin{cases}
-\Delta u(x) + u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)) & \text{in } \mathbb{R}^n; \\
u \geq 0, & \text{in } \mathbb{R}^n,
\end{cases}
\] (\tilde{D}_\lambda)
where $\partial F$ and $\partial G$ behave in a similar manner as before. The main difficulty in the investigation of (\tilde{D}_\lambda) is the lack of compact embedding of the Sobolev space $H^1(\mathbb{R}^n)$ into the Lebesgue spaces $L^p(\mathbb{R}^n)$, $n \geq 2$, $q \in [2, 2^*)$. However, by using Strauss-type estimates and Lions-type embedding results for radially symmetric functions of $H^1(\mathbb{R}^n)$ (see e.g. Willem [23]), the principle of symmetric criticality for non-smooth functionals (see Kobayashi and Otani [13] and Squassina [22]) provides the expected results. A related result in the smooth setting can be found in Kristály [11].

2. Assume that $\partial F$ oscillates at a point $l \in [0, +\infty]$ and $\partial G$ has a $p$-order growth at $l$. We are wondering if our results, valid for $l = 0$ and $l = +\infty$, can be extended to any $l \in (0, \infty)$, even in the smooth framework.
Proposition 7.2. (see [6])

Remark 7.1. (see [7]) Let \( x \) be a critical point of \( f \) in the sense of Chang [6]. Let \((X, \parallel \cdot \parallel)\) be a real Banach space and \( U \subset X \) be an open set; we denote by \((\cdot, \cdot)\) the duality mapping between \( X^* \) and \( X \).

Definition 7.1. (see [7]) A function \( f : X \to \mathbb{R} \) is locally Lipschitz if, for every \( x \in X \), there exist a neighborhood \( U \) of \( x \) and a constant \( L > 0 \) such that

\[
|f(x_1) - f(x_2)| \leq L\|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in U.
\]

Definition 7.2. (see [7]) Let \( f \) be a locally Lipschitz function near the point \( x \) and let \( v \) be any arbitrary vector in \( X \). The generalized directional derivative in the sense of Clarke of \( f \) at the point \( x \in X \) in the direction \( v \in X \) is

\[
f^\circ(x; v) = \limsup_{z \to x, \tau \to 0^+} \frac{f(z + \tau v) - f(z)}{\tau}.
\]

The generalized gradient of \( f \) at \( x \in X \) is the set

\[
\partial f(x) = \{ x^* \in X^* : \langle x^*, v \rangle \leq f^\circ(x; v) \quad \text{for all } v \in X \}.
\]

For all \( x \in X \), the functional \( f^\circ(x, \cdot) \) is finite and positively homogeneous. Moreover, we have the following properties.

Proposition 7.1. (see [7]) Let \( X \) be a real Banach space, \( U \subset X \) an open subset and \( f, g : U \to \mathbb{R} \) be locally Lipschitz functions. The following properties hold:

(a) For every \( x \in U \), \( \partial f(x) \) is a nonempty, convex and weakly* compact subset of \( X^* \) which is bounded by the Lipschitz constant \( L > 0 \) of \( f \) near \( x \);

(b) \( f^\circ(x; v) = \max\{\langle \xi, v \rangle : \xi \in \partial f(x)\} \) for all \( v \in X \);

(c) \( (f + g)^\circ(x; v) \leq f^\circ(x; v) + g^\circ(x; v) \) for all \( x \in U, v \in X \);

(d) \( \partial(f + g)(u) \subset \partial f(u) + \partial g(u) \) for all \( u \in U \);

(e) \( (-f)^\circ(x; v) = f^\circ(x; -v) \) for all \( x \in U \);

(f) The function \( (x, v) \mapsto f^\circ(x; v) \) is upper semicontinuous;

(g) The set-valued map \( \partial f : U \to 2^{X^*} \) is weakly* closed, that is, if \( \{x_i\} \subset U \) and \( \{w_i\} \subset X^* \) are sequences such that \( x_i \to x \) strongly in \( X \) and \( w_i \in \partial f(x_i) \) with \( w_i \rightharpoonup w \) weakly* in \( X^* \), then \( z \in \partial f(x) \). In particular, if \( X \) is finite dimensional, then \( \partial f \) is upper semicontinuous, i.e., for every \( \epsilon > 0 \) there exists \( \gamma > 0 \) such that \( \partial f(x') \subseteq \partial f(x) + B_{X^*}(0, \epsilon) \), \( \forall x' \in B_X(x, \gamma) \);

Proposition 7.2. (see [6]) The number \( \lambda_f(u) = \inf_{w \in \partial f(u)} \|w\|_{X^*} \) is well defined and

\[
\liminf_{u \to u_0} \lambda_f(u) \geq \lambda_f(u_0).
\]

Definition 7.3. (see [6]) Let \( f : X \to \mathbb{R} \) be a locally Lipschitz function. We say that \( u \in X \) is a critical point (in the sense of Chang) of \( f \), if \( \lambda_f(u) = 0 \), i.e., \( 0 \in \partial f(u) \).

Remark 7.1. (see [7]) (a) \( u \in X \) is a critical point of \( f \) if \( f^\circ(u; v) \geq 0 \) for all \( v \in X \).

(b) If \( x \in U \) is a local minimum or maximum of the locally Lipschitz function \( f : X \to \mathbb{R} \) on an open set of a Banach space, then \( x \) is a critical point of \( f \).
Proposition 7.3. (see [7]) (Lebourg’s mean value theorem) Let $X$ be a Banach space, $x, y \in X$ and $f : X \to \mathbb{R}$ be Lipschitz on an open set containing the line segment $[x, y]$. Then there is a point $a \in (x, y)$ such that
\[ f(y) - f(x) \in \langle \partial f(a), y - x \rangle. \]

Proposition 7.4. (see [7]) (Chain Rule) Let $X$ be Banach space, let us consider the composite function $f = g \circ h$ where $h : X \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. Let denote $h_i$, $i \in \{1, ..., n\}$ be the component functions of $h$. We assume $h_i$ is locally Lipschitz near $x$ and $g$ is too near $h(x)$. Then $f$ is locally Lipschitz near $x$ as well. Let us denote by $\alpha_i$ the elements of $\partial g$, and let $\alpha = (\alpha_1, ..., \alpha_n)$; then
\[ \partial f(x) \subset \overline{\text{co}}\{ \sum \alpha_i \xi_i : \xi_i \in \partial h_i(x), \alpha \in \partial g(h(x)) \}, \]
where $\overline{\text{co}}$ denotes the weak-closed convex hull.

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