Analytical Solution of Eddy Current Field for Arbitrary Coil Placement outside Pipeline

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Abstract: In this paper, the eddy current field model of the pipeline excited by the coil placed arbitrarily outside the pipeline will be established by using the second-order magnetic vector position, and the coefficient expression of the coil placed arbitrarily outside the pipeline will be obtained through the addition theorem of Bessel function, thus obtaining the analytical solution of the eddy current field. Finally, the analytical solution proposed in this paper is verified by experiments.

1. Introduction

Eddy current testing has the advantages of non-contact and high sensitivity, and can be used for in-service nondestructive testing of industrial pipelines. When eddy current testing is carried out on the pipeline, the detection coil is usually vertically or parallel placed on the pipeline. However, when conducting in-service inspection of pipelines, the inspection coil is usually manually operated, and the coil may tilt arbitrarily. Because the sensitivity of eddy current testing is very high, the coil inclination will seriously affect the results of pipeline eddy current testing and reduce the testing accuracy. It is necessary to study the influence of coil inclination on pipeline eddy current field. In the above work, factors such as coil deflection and inclination are not considered, which are very common in actual eddy current testing. The addition theorem of Bessel function is often used to analyze the influence of coil offset on eddy current field, such as eccentric through-type circular coil [1], eccentric bent rectangular coil [2] and radial eccentric coil [3-5] in pipeline. In the analysis of the influence of coil inclination on the eddy current field of the pipeline, the document [6] obtains the analytical solution of the eddy current field of the pipeline generated by coil excitation with the outer axial inclination of the pipeline (the coil axis is perpendicular to the φ axis), but it cannot analyze the eddy current field when the coil is also inclined in the circumferential direction (the coil axis is not perpendicular to the φ axis). Document [7] solved the analytical solution of eddy current field for arbitrarily placed coils in the pipeline.

In this paper, firstly, the analytic solution model of non-axisymmetric eddy current field is established by using the second-order magnetic vector position, and the analytic solution of arbitrarily placed coil coefficient is solved by using this model. According to the coil coefficient, the analytic solution of the relative impedance increment of the coil can be further obtained, and the relative impedance increment is calculated numerically using Matlab software. Since the analytical solution in this paper allows the coil to change at any angle, in order to facilitate efficient verification, it is only necessary to check the case where the vertical elevation angle is at the two endpoints of 0 degree and 90 degree. When the vertical elevation angle of the coil is 0 degree, it is directly simplified to the case...
of reference [13] when the vertical elevation angle of the coil is zero. In this case, verification is not necessary, so when the vertical elevation angle is 90 degrees, it is directly compared with the special case of reference [14]. The relative error of the comparison results is about 0.0003%, which verifies the correctness of the analytical solution with high precision.

2. Modeling of Eddy Current Field in Arbitrary Inclined Coil Pipe
Assume that coil $c$ is arbitrarily placed on the outside of an infinite length metal pipe, as shown in figure 1. A cylindrical coordinate system is established with the pipeline axis as the $Z$ axis, wherein the $\rho$ axis passes through the center of the coil. The inner radius of the pipeline is $r_i$, the outer radius is $r_o$, the conductivity is $\sigma$, and the permeability $\mu$ of the pipeline is assumed to be linear. An alternating current with an amplitude of $I$ and an angular frequency of $\omega$ is communicated in the coil. The whole field area can be divided into five parts: area 1 is the inner part of the pipe, area 2 is the pipe wall part, and area 3 is the area between the outer wall of the pipe and the coil.

![Figure 1 schematic diagram of arbitrarily placed coils outside the pipeline](image)

The expression of the second-order potential function in regions 1, 2 and 3 can be obtained by using the second-order magnetic vector potential as follows:

$$W_{a1} = \int_{a_1}^{a_2} \sum_{\alpha=1}^{n} C_{m} I_{m} (|r| \rho) e^{im \alpha} e^{i \omega t} d \alpha$$

$$W_{b2} = \int_{b_2}^{b_3} \sum_{\alpha=1}^{n} \left[ C_{2m} I_{2m} (|r| \rho) + D_{2m} K_{2m} (|r| \rho) \right] e^{im \alpha} e^{i \omega t} d \alpha$$

$$W_{a2} = \int_{a_2}^{a_3} \sum_{\alpha=1}^{n} \left[ C_{2m} I_{2m} (|r| \rho) + D_{2m} K_{2m} (|r| \rho) \right] e^{im \alpha} e^{i \omega t} d \alpha$$

The relationship between undetermined coefficients $C_{lm}$, $C_{a2m}$, $D_{a2m}$, $C_{b2m}$, $D_{b2m}$ and $D_{em}$ and coil coefficient $C_{mm}$ obtained by using internal boundary conditions is

$$C_{a2m} = \frac{FS}{\alpha} C_{mm} \cdot D_{a2m} = -\frac{ES}{\alpha} C_{mm} \cdot C_{b2m} = \frac{m S}{\alpha \alpha_k} \Gamma C_{mm}, C_{lm} = \frac{\alpha \psi S}{\alpha \mu \mu_k (|r| r)} C_{mm}$$

$$D_{em} = \frac{C_{em}}{K_{m} (|r| r)} \left[ \frac{\alpha \gamma S}{\alpha \mu \mu_k} \Gamma - \frac{1}{m} \frac{|r| r}{r} \right] D_{b2m} = -\frac{m S C_{mm}}{\alpha \alpha_k} \frac{\Gamma}{K_{m} (|r| r)}$$

In the formula, $S = \frac{| \alpha |}{\alpha_k} \left[ I_{m} (|r| r) - I_{m} (|r| r) M \right]$, $\gamma = \frac{FG - EH}{A}$, $\Gamma = \frac{F I_{m} (|r| r) - E K_{m} (|r| r)}{A}$, $A = BE - AF$, $A = \frac{\alpha \psi M I_{m} (|r| r)}{\mu |r|}$

$$+ \frac{m \gamma^2}{\alpha \mu \mu_k} \left[ \frac{L_{mm} (|r| r)}{\alpha \psi S \alpha} \right] - \frac{1}{m} \frac{|r| r}{r}$$
\[ B = \frac{\alpha_s MK_m (\alpha_s r_e)}{\mu_i |\alpha|} + m^2 k^2 \left[ \frac{L K_m (\alpha_s r_e) + r_e HU}{\alpha_s^2} \right] - K'_m (\alpha_s r_e), \]
\[ E = \frac{\alpha_s NL_m (\alpha_s r_e)}{\mu_i |\alpha|} + m^2 k^2 \left[ \frac{P L_m (\alpha_s r_e) + r_e GV}{\alpha_s^2} \right] - \Gamma'_m (\alpha_s r_e), \]
\[ F = \frac{\alpha_s NK_m (\alpha_s r_e)}{\mu_i |\alpha|} + m^2 k^2 \left[ \frac{P K_m (\alpha_s r_e) + r_e HV}{\alpha_s^2} \right] - K'_m (\alpha_s r_e), \quad G = \frac{I_m (\alpha_s r_e)}{r} - \frac{Q I_m (\alpha_s r_e)}{r_e}, \]
\[ H = \frac{K_m (\alpha_s r_e)}{r_e} - \frac{Q K_m (\alpha_s r_e)}{r_e}, \quad J = \frac{I'_m (\alpha_s r_e)}{r} - \frac{Q L'_m (\alpha_s r_e)}{r_e}, \quad M = \frac{K'_m (|\alpha||r_e)}{K_m (|\alpha||r_e)}, \]
\[ N = \frac{I_m (|\alpha|r_e)}{I_m (|\alpha|r_e)}, \quad L = \frac{K_m (\alpha_s r_e)}{K'_m (\alpha_s r_e)}, \quad P = \frac{K'_m (\alpha_s r_e)}{K'_m (\alpha_s r_e)}, \]
\[ Q = \frac{K'_m (\alpha_s r_e)}{K'_m (\alpha_s r_e)}, \quad U = \frac{I_m (\alpha_s r_e) - \frac{L L'_m (\alpha_s r_e)}{J}}{J}, \quad V = \frac{I_m (\alpha_s r_e) - \frac{Q L'_m (\alpha_s r_e)}{J}}{J}. \]

The above equation shows that other undetermined coefficients in regions 1, 2, and 3 can be obtained only by finding the coil coefficient \( C_{sm} \) of the tilt coil, thus obtaining analytical expressions of \( Wa \) and \( W_b \).

3. Solving the coil coefficient of any inclined coil

Let the number of turns of the coil be \( N \), the inner radius of the coil be \( r_i \), the outer radius of the coil be \( r_o \), the height of the coil be \( h \), and the distance from the center of the coil to the origin be \( d \). Taking the coil center as the origin, the local coordinate system \( O_{x_e y_e z_e} \) of the coil is established, in which the \( x_e \) axis and \( \rho \) axis are in the same direction, and the \( z_e \) axis and \( z \) axis are in the same direction, as shown in fig. 1(b). For convenience of calculation, the horizontal included angle \( \varphi_c \) \((-90 \leq \varphi \leq 90\)\) and the vertical elevation angle \( \theta_e \) \((0 \leq \theta \leq 180\)\) of the unit normal vector \( n_e \) of the coil median plane are used to describe the inclination of the coil, as shown in figure 2.

![Figure 2 definition of coil plane normal vector and its horizontal included angle \( \varphi \) and vertical elevation angle \( \theta \).](image)

As can be seen from document [7], in the local coordinate system, in the spherical outer region surrounding the coil as shown in fig. 1(b), the magnetic coordinates of the incident field are:

\[ \phi = -\frac{\mu_0 h}{2} \sum_{n=1}^{G_n} \sum_{-n}^{n} \frac{(-j)^{n+1}}{n+1} \frac{1}{\pi (n+1)} \frac{1}{r^3} (\cos \theta_e) e^{-j\rho e} \]
\[ \int_{-\infty}^{\infty} \alpha^2 K_r (|\alpha| \rho_e) e^{j\rho e} e^{j\alpha} \left[ -\text{sgn} (\alpha) \right]^\rho d\alpha \]

Where \( G_n \) is the integral of the region where the coil is located, and its expression is
\begin{equation}
G_u = \int_{\theta_0}^{\theta_1} \int_{z_0}^{z_1} \sin \theta_0 \cos \theta_0 \mathbf{e}^{i\mu \cdot \mathbf{P} \cdot \mathbf{d}} \, dx \, dz
\end{equation}

Where is the associative legendre polynomials, \(\sin \theta_0 = x_0 / r_0, \quad r_0 = \sqrt{x_0^2 + z_0^2}\). Considering \(\cos \theta_0\) is an even function, the above formula can be simplified as:

\begin{equation}
G_u = 2 \int_{\theta_0}^{\theta_1} \int_{z_0}^{z_1} \sin \theta_0 \cos \theta_0 \mathbf{e}^{i\mu \cdot \mathbf{P} \cdot \mathbf{d}} \, dx \, dz
\end{equation}

Next, the coil coefficients in the global cylindrical coordinate system can be obtained by using the addition theorem of the modified Bessel function. According to the addition theorem of the second kind of modified Bessel function, for the triangle shown in fig. 3, when \(S_i > S_d\), there are:

\[ K_n(S_i) e^{j\omega n} = \sum_{m=-\infty}^{\infty} K_{n-m}(S_i) I_m(S_d) e^{j\omega m} \]

As can be seen from fig. 1(b), \(d > \rho\), so \(d = S_i, \rho = S_d, \rho_c = S_2\), and the above formula can be expressed as:

\[ K_n(|z| \rho_c) e^{j\omega n} = \sum_{m=-\infty}^{\infty} K_{n-m}(|z| \rho) I_m(|z| \rho) e^{j\omega m} \]

Make \(\mu = -v\) available:

\[ K_{n-\mu}(|z| \rho) e^{j\omega n} = \sum_{m=-\infty}^{\infty} K_{n-m-\mu}(|z| \rho) I_m(|z| \rho) e^{j\omega m} \]

Substituting the above formula into formula (11) can obtain:

\[ \phi = \frac{-\mu_0}{2} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_m \sum_{n=0}^{\infty} (-1)^n \mathbf{P}_n^m (\cos \theta) e^{j\omega n} \int_{\alpha}^{\alpha'} (-1)^n I_n(|z| \rho) K_{n-\mu-\mu}(|z| \rho) e^{j\omega n} \left[ -\text{sgn}(\alpha) \right] d\alpha \]

The above-mentioned shift items are available:

\[ \phi = \frac{-\mu_0}{2} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_m \sum_{n=0}^{\infty} \alpha^n \left[ \text{sgn}(\alpha) \right] (-1)^{n+\mu} \pi (n+\mu)! \mathbf{P}_n^m (\cos \theta) e^{j\omega n} \]

Since in the cylindrical coordinate system, the magnetic flux density \(B\) of the incident field in region 3 is:

\[ B = \nabla \phi = \nabla \left( \frac{\partial W}{\partial z} \right) \]

So

\[ W = \int_{-\infty}^{\infty} \phi \, dz \]

\[ = \frac{j\mu_0}{2} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} G_m \alpha^{n-\mu-1} \sum_{n=0}^{\infty} \left[ \text{sgn}(\alpha) \right] (-1)^{n+\mu} \pi (n+\mu)! \mathbf{P}_n^m (\cos \theta) e^{j\omega n} \]

\[ K_{n-\mu-\mu}(|z| \rho) I_n(|z| \rho) e^{j\omega n} \left[ -\text{sgn}(\alpha) \right] d\alpha \]

It can be seen from this that for any inclined coil outside the pipeline, the coil coefficient is:
\[
C_{sm} = \frac{j\mu_0 h_0}{2} \sum_{n=1}^{\infty} \frac{G_n \alpha^{n-1}}{n+1} \left[ \sum_{v=-n}^{n} \left\{ \text{sgn}(\alpha) \right\} \left\{ \frac{(-1)^{v-n}}{\pi(n+v)} \right\} P_n^{(v)}(\cos \theta) e^{-j\pi v} K_{n,v} \{k|d\} \right]
\]

Due to
\[
P_n^{(v)}(x) = (-1)^v \frac{(n-v)!}{(n+v)!} P_n^{(v)}(x)
\]

So the above formula can be converted into
\[
C_{sm} = \frac{j\mu_0 h_0}{2} \sum_{n=1}^{\infty} \frac{G_n \alpha^{n-1}}{n+1} \left\{ \frac{(-1)^{v-n}}{\pi(n+v)} \right\} P_n^{(v)}(\cos \theta) e^{-j\pi v} K_{n,v} \{k|d\} \]

4. Numerical simulation and analysis

The purpose of simulation calculation is to verify the correctness of the coil coefficient expression deduced in chapter 3. According to the design ideas in references [8], the impedance increment method is adopted. Document [10-13] gives a calculation impedance increment \( \Delta Z \). The method of impedance increment expression and coil coefficient \( C_{sm} \) is related:
\[
\Delta Z = 4j\omega \pi^2 \int_{-\infty}^{\infty} \sum_{\alpha=-\infty}^{\infty} \alpha^2 C_{sm}(\alpha,m) D_{sym}(-\alpha,-m) d\alpha
\]

Equation (10) is the expression [10-13] for the impedance increment of the placement coil commonly used at present. Using this formula, the impedance increment of the coil can be obtained only by finding the coil coefficient and the corresponding coefficient of the placed coil, which is very convenient to solve.

By comparing with the data in the literature [14], for example, the following figure 3:

Figure 3: The theoretical and experimental results for the normalized impedance changes of the coil

From the above figure, it can be seen that the high precision results obtained by the analytic solution of arbitrary coil coefficient in the document [14] verify the analytic solution of arbitrary coil coefficient proposed in this paper.

5. Conclusion

In this paper, the mathematical model of eddy current field in metal pipes with placed coils is innovatively established by using the second-order magnetic vector position. The constraint equation is solved by the method of separating variables, and the analytical expression of field quantity in each field region is obtained. Using the modified Bessel function addition theorem and the properties of Legendre polynomials, the magnetic coordinates in the spherical coordinate system are transformed, and the magnetic coordinates finally obtained are converted into magnetic vector positions. Finally, by
comparing the two magnetic vector expressions, the analytical solution of the coil coefficient arbitrarily placed outside the pipeline is obtained. Thus, the expression of the complete analytical solution of the impedance increment of the coil arbitrarily placed outside the pipeline is obtained. By comparing the results of the existing theoretical models under two special conditions, the analytical solution of arbitrarily placed coils outside the metal pipe proposed in this paper is verified. The research results in this paper enrich the analytical solution method of non-axisymmetric eddy current field and can provide theoretical support for pulsed eddy current testing of placed pipes.

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