An Efficient Topology Optimization Strategy for Structural Crashworthiness Using Model and ESLs Reduction Method

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\textbf{Abstract}. This paper presents an efficient structural topology optimization strategy for crashworthiness. Although equivalent static loads (ESLs) provides a relatively effective way to solve nonlinear dynamic response topology optimization problems, efficiency is still required to improve. To overcome above disadvantage, the proposed method combines the ESLs with model reduction (MR) method. The ESLs method transforms crashworthiness topology optimization problems into a series of linear static topology problems. The model reduction scheme is applicable to reduce the linear matrix equation and ESLs to interface degrees of freedom (DOFs), by selecting interface DOFs between design and non-design space as master DOFs. Therefore, the linear static topology optimization is performed in a reduction manner and the efficiency is improved. The topology optimization results are filtered into a black-white design for the crash simulation to avoid the element distortion problem. The process is repeated until the convergence criteria is satisfied. A numerical example is used to verify the reliability of the proposed method. The results are compared with the dynamic response optimization method using ESLs (ESLSO). The results show that, the proposed method can efficiently solve structural topology optimization problems for crashworthiness.

\textbf{Introduction}

Vehicle safety and environmental protection have become two challenges of rapid development of automobiles. [1]. As a capability to protect occupants during the crash, crashworthiness is an important attribute to be considered in the initial design stage [2]. Automotive lightweighting is an effective manner of developing vehicles for energy conservation and environmental protection [3]. The structural topology optimization considering crashworthiness, referred to as crashworthiness topology optimization, is a conceptual design method of structural lightweight under the crash loads [4]. Many researches have been developed for topology optimization with dynamic and nonlinear characteristics [5, 6, 7]. However, as a typical nonlinear dynamic response topology optimization problem, it is extremely difficulty to consider nonlinearities and dynamic responses simultaneously in topology optimization [8]. Structural optimization methods for nonlinear dynamic responses are generally classified into the gradient-based optimization, non-gradient-based optimization and ESLs-based optimization. Due to its easily implemented into tailored finite element codes, the ESLs-based method has been the popular and relatively feasible approach to simplify this process, which transform nonlinear dynamic response optimization problems into a series of linear static optimization problems [9]. Choi and Park first proposed the ESLs method [10, 11, 12], which originally developed for size optimization under dynamic loads, then it was further applied for topology optimization of linear dynamic structures. Recently, representative nonlinear topology optimization, considering geometric and material nonlinearities, have been realized based on ESLs method under static loads [13, 14]. In 2014, topology optimization has been introduced into structural crashworthiness topology design by incorporating the ESLs and a discrete-material topology optimizer [15]. To solve the mesh distortion issue, transformation variables were
employed to incorporate topology results into crash dynamic analysis [8]. Although, the ESLs
method provides an effective way to solve linear dynamic response topology problem, it still has
limited efficiency when applied to the nonlinear dynamic response topology optimization especially
for large-scale problems. Moreover, previous researches on dynamic response optimization using
the ESLs on all DOFs, which is not desirable in crashworthiness topology optimization process.
Since some elements can have a value of the lower bound during linear topology optimization,
directly applying the ESLs mostly beyond the linear range and lead to numerical defects such as
high compliance of each element, which significantly increases the number of iterations and
deteriorates the efficiency. Especially for crashworthiness topology optimization, the exaggerated
ESLs may be obtained due to large deformations. Model reduction (MR) methods have been widely
used to improve the efficiency of large-system analysis. A number of approaches have been
reported [16, 17, 18]. In previous studies, when combined with MR method, the linear dynamic
analysis of a large-scale system can be performed in an efficient and reliable manner. However,
there has been little discussion on the formulation of an efficient nonlinear dynamic response
topology optimization method that employs ESLs on selected DOFs. This paper proposes a
powerful crashworthiness topology optimization algorithm that combines ESLs with a model
reduction method. The ESLs method transforms crash dynamic problems into linear static
problems. The mode reduction scheme is applicable to the linear static topology optimization under
the ESLs with multiple loading conditions. The linear matrix equation and the ESLs are reduced to
interface DOFs, therefore the efficiency of each iteration enhanced. The effectiveness of the
proposed method is demonstrated by investing a numerical example and comparing with the results
of the ESLSO method. The results show that, the proposed method can provide a physically
appropriate topology layout and has high efficiency.

Statement of the Problem

Formulation of Crashworthiness Topology Optimization

The formulation of crashworthiness topology optimization is [19]:

\[
\text{find: } b \in \mathbb{R}^n \quad (1)
\]

\[
\text{to min: } F(b, z_N) \quad (2)
\]

\[
\text{subject to: } M(b)\dddot{z}_N(t) + C(b)\dot{z}_N(t) + K_N(b, z_N(t))z_N(t) = f(t), \quad t = 1, \ldots, l \quad (3)
\]

\[
g_i(b, z_N(t)) \leq 0, \quad i = 1, \ldots, m \quad (4)
\]

\[
b_{L,j} \leq b_j \leq b_{U,j}, \quad j = 1, \ldots, n \quad (5)
\]

where \(b\) is the vector of the design variables; \(F(b, z_N)\) is the objective function, representing sum of
compliance; \(M\) is the mass matrix; \(C\) is the damping matrix; \(K\) is the stiffness matrix; \(f(t)\) is the
external load vector at the \(t\)th time step; and \(\dddot{z}_N(t)\) and \(\ddot{z}_N(t)\) are the nodal velocity and acceleration
vectors, respectively. In addition, the subscript \(N\) indicates that the response has nonlinear dynamic
properties. \(b_{L,j}\) and \(b_{U,j}\) are the lower and upper bounds of the \(j\)th design variable \(b_j\), respectively. Eq.
(3) is the governing equation of nonlinear crash analysis, and the constants \(l\) and \(n\) are the number of
the constraints and design variables, respectively. \(g_i\) is the \(i\)th constrain.
The Theory of ESLSO

The idea of using the equivalent static loads method for linear dynamic response structural optimization (ESLSO) was initially propose by Choi and Park and used in linear dynamic size and shape optimization [20]. The ESLSO for nonlinear dynamic response topology optimization is formulated as a sequence of cycles with four modules: nonlinear dynamic crash analysis, equivalent static load construction, topology optimization, and crash model reconfiguration [21]. With this treatment as illustrated in Fig.1, topology optimization can be performed in the linear static framework, and mesh distortion problem can be avoided in the nonlinear dynamic analysis using the crash geometry updating. The details of each module can be briefly described as follows.

![Figure 1. Schematic flowchart of the ESLSO method.](image)

![Figure 2. Schematic process of the M&ER method.](image)

**Crash Dynamic Analysis**

A general equilibrium equation of a structure with nonlinear dynamic behavior can be formulated as Eq.(3). By solving it using commercial software program such as LS-Dyna, the response at each time step could be obtained.

**Calculation of ESLs**

The ESLs can be built by multiplying the linear stiffness matrix $K_L(b)$ to the displacement vector $z_N(t)$

$$F_{eq} = K_L(b) z_N(t), t = t_1, t_2, \ldots, t_q$$  \hspace{1cm} (6)

where $t$ is the time steps of the dynamic process, and $q$ is the total number of time steps during the crash analysis process. $K_L(b)$ is the linear stiffness matrix calculated by the linear elastic material parameters.

**Linear Static Topology Optimization**

Under the ESLs, a crashworthiness topology optimization can be convert to the sequence of linear static topology optimization problems. For stiffness design, the time step $t_c$ with maximal internal energy are normally selected as critical time steps. The special optimization problem at the time step can $t_c$ be formulated as.

$$\text{find } : b \in R^n$$ \hspace{1cm} (7)

$$\text{to min } : \sum_{c=1}^{q} \omega_c \left( F_{eq}(s_c) Z(s_c) \right)$$ \hspace{1cm} (8)

$$\text{s.t. } : K_L Z(s_c) = F^{eq}(s_c), \ c = 1, \ldots, q$$ \hspace{1cm} (9)

$$v^T b \leq v_j V$$ \hspace{1cm} (10)
where \( w_i \) is the \( i \)th weighting factor, \( Z(s_c) \) and \( F_{eq}(s_c) \) are corresponding displacement vector and equivalent static load vector at the \( t_c \) time step respectively. The static condition \( s_c \) is strictly corresponding with the crucial time step \( t_c \). \( V \) is the total volume of the designable space, \( v_j \) is the volume fraction and \( b_{min} \) is the minimal element density vector to avoid the numerical singularities.

**Crash Model Reconfiguration**

Direct employing the results of topology optimization for crash dynamic analysis will lead to mesh distortion phenomenon. To avoid this difficulty, the transformation variables were introduced. Transform the values of the design variables into the values of the transformation variables as follows:

\[
\alpha_i = \begin{cases} 
0 & \text{where } b_i \leq \varepsilon_i \\
1 & \text{where } b_i > \varepsilon_i \end{cases} : i = 1, \ldots, N_c
\]

where \( b_i \) is the \( i \)th optimum design variable. \( \alpha_i \) is the transformation variable for \( b_i \). The separation parameter \( \varepsilon_i \) is a certain value between \( b_{min} \) and 1, which is defined by the user. If the value of a design variable is smaller than the separation parameter, the corresponding transformation variable has value of 0. Otherwise, the corresponding transformation variable is regard as 1. Through this operation, all the filtered zero density elements will be deleted and not incorporated into the crash dynamic analysis.

**Existing Issues**

ESLSO method provides a relatively feasible way to solve crashworthiness topology optimization problems. However, in structural crashworthiness topology optimization, the standard ESLs are not suitable to be incorporated into the linear static topology optimization directly, since some elements can have a value of the lower bound during linear topology optimization, directly applying the ESLs mostly beyond the linear range and lead to numerical defects such as high compliance of each element, which significantly increases the number of iterations and deteriorates the efficiency. In addition, for crashworthiness topology optimization, the exaggerated ESLs may be obtained due to large deformations. Furthermore, the full element integral method must be used in the nonlinear analysis to ensure the stability of the calculation, which make the optimization more expensive.

**Proposed Model and ESLs Reduction (M&ER) Method**

To overcome above-mentioned issues, an efficient topology optimization strategy for structural crashworthiness using model and ESLs reduction method. As shown in Fig. 2, the whole optimization process is divided into the analysis, MR, optimization domains and crash model reconfiguration, which is quite different from the traditional ESLSO method (Fig. 1). The details of the proposed method are introduced as follows. A crash analysis is performed in the analysis domain to obtain the nonlinear response such as displacement. The standard ESLs are generated by multiplying the linear stiffness matrix by the displacement vector. By select the DOFs on interface between the design and non-design space as master DOFs, the model reduction reduces the linear matrix equation to interface DOFs through algebraic substitution in the MR domain. In addition, the ESLs are reduced to interface DOFs. As the inner iteration of the optimization process, the linear static topology optimization of the reduction model under multiple loading condition defined by the reduced ESLs is performed in the design domain, which enhance the iteration efficiency effectively. Then the optimized design variables are filtered and the crash model is reconfigured. Finally, the reconfigured crash model is incorporated into the analysis domain. The cycle among the analysis, MR, design domains and crash model reconfiguration is the outer cycle of the optimization process. This procedure is defined as a “cycle” and differs from “iteration” in the linear static topology optimization.
**Theoretical Derivation**

**Calculation of Reduced Model and ESLs**

The model reduction (MR) method can remarkably improve the analysis efficiency of structures and reduce optimization calculation cost by eliminating unwanted degrees of freedom [22, 23, 24], so it was extensively used in large-scale structural analysis [25, 26, 27]. The static reduction method is extensively used in the degree of freedom reduction of the finite element model and has been popularized in large-scale finite element analysis software. According to MR theory, all DOFs are categorized into master DOFs and slave DOFs which are simply referred as masters and slaves respectively in this paper. The DOFs on interface between design and non-design space in topology optimization are selected as masters. The displacement vectors of the masters and slaves are expressed as $Z_m$ and $Z_s$ respectively. Then, the static displacement vector can be expressed as [28]:

$$ Z = \begin{bmatrix} Z_m \\ Z_s \end{bmatrix} $$

(13)

The static equations of equilibrium, that is

$$ \begin{bmatrix} K_{mm} & K_{ms} \\ K_{sm}^T & K_{ss} \end{bmatrix} \cdot \begin{bmatrix} Z_m \\ Z_s \end{bmatrix} = \begin{bmatrix} f_m \\ f_s \end{bmatrix} $$

(14)

where $K$ is the stiffness matrix and $f$ is the force vector with the corresponding subscripts depicting the partitioning on the basis of the masters and slaves. Then, the static equilibrium equation of the reduced model can be expressed as:

$$ K_{red} \cdot Z_m = f_{red} $$

(15)

The reduced stiffness matrix can be expressed as:

$$ K_{red} = K_{mm} - K_{sm}^T K_{ss}^{-1} K_{sm} $$

(16)

The reduced load vector can be expressed as:

$$ f_{red} = f_m - K_{sm}^T K_{ss}^{-1} f_s $$

(17)

At time point $t_c$, the reduced ESLs acting on masters can be obtained using Eq. (17).

$$ f_{red,eq}(s_c) = f_{m,eq}(s_c) - K_{sm}^T K_{ss}^{-1} f_{s,eq}(s_c) $$

(18)

where, $f_{m,eq}(s_c)$ and $f_{s,eq}(s_c)$ are ESLs vectors acting on masters and slaves of full model, respectively, at time step $t_c$.

**Formulation of Optimization Model**

At time point $t_c$, the structural crashworthiness topology optimization problem can be equivalent to a linear static topology optimization problem for the reduced model under the reduced ESLs $f_{red,eq}(s_c)$. The specific formulation can be expressed as

$$ \text{find} : b \in R^n $$

$$ \text{to min} : f(b) = \sum_{i=1}^{q} Z_m(s_c) f_{red,eq}(s_c), \quad c = 1, \ldots, q $$

(19)

(20)

subject to: $$ \begin{bmatrix} K_{mm} - K_{sm}^T K_{ss}^{-1} K_{sm} \end{bmatrix} Z_m(s_c) = f_{red,eq}(s_c), \quad c = 1, \ldots, q $$

(21)
\[ v^T b \leq v_f^T V \]  
(22)

\[ 0.0 < b_{\min} \leq b_i \leq 1.0, \ i = 1, \ldots, n \]  
(23)

where the objective function is the sum of the linear compliance around critical time step \( t_c \). In this work, \( q \) is set to 3, and the time steps are selected as \( t_{c-1}, t_c, t_{c+1} \) and corresponding weighting factor are 0.1, 0.8, 0.1 respectively.

**Optimization Process**

The process of the proposed method is shown in Fig. 3. The specific optimization process is as follows:

1. **Step 1.** Prepare the crash simulation model for topology optimization design;
2. Define the topology optimization problem, and initialize the pre-defined parameters such as volume fraction for the designable region.
3. **Step 2.** Nonlinear dynamic analysis is performed;
4. Structural crash dynamics analysis is performed, and the displacement response vector of all the DOFs is obtained. Then, the time step with the maximal internal energy is selected as the critical time step \( t_c \), and the corresponding displacement vector \( z_h(t_c) \) is output.
5. **Step 3.** The ESLs for all the nodes are calculated;
6. Calculate the ESLs at the critical time steps and serve as a condition of the linear static analysis.
7. **Step 4.** The masters of the structure are selected;
8. DOFs on interface between the design space and non-design space in topology optimization are selected as masters for model reduction.
9. **Step 5.** The model and ESLs are reduced;
10. The linear matrix equation and ESLs are reduced to interface DOFs simultaneously;
11. **Step 6.** The topology optimization is performed in a reduced manner.
12. The linear static response topology optimization is performed in a reduced manner by using Eqs. (19–23), and the optimization design variables are obtained.
13. **Step 7.** Filter the topology optimization results and reconfigure the new crash simulation model;
14. The FE model is reconfigured for crash dynamics analysis by using the filtered design variables in Eq. (12).
15. **Step 8.** Check if the optimization is converged, and repeat Steps 2-8 until converged;
16. If converged condition is satisfied, the optimization process is terminated; otherwise, \( k = k + 1 \) and Step 2 is implemented.

![Flowchart of the proposed M&ER method.](image)
Numerical Examples

The proposed method is applied to a simplified vehicle structure under the conditions of crash to demonstrate its effectiveness. The optimization of the front-end structure is converted to 2D stiffness topology optimization under crash condition. The crash simulation is performed using LS_Dyna, and the ESLs calculation is implemented in MATLAB. The MR and the topology optimization are performed in Optistruct, and the entire optimization platform is established in the MATLAB environment.

Model Introduction

Fig. 4 (a) shows a vehicle crash problem that may be considered as a simplified idealization of side crash with a moving barrier that has an initial velocity. A cylinder with a diameter of 80 mm is simulated to impact the vehicle’s side at 2.5 m/s. The model consists of 10622 quadrilateral shell elements with a thickness of 2 mm, and the total number of model nodes is 11767. The entire model is divided into three parts. The region in red represents the design space for the front structure. The yellow region represents the exterior area of the design domain. The green remaining area represents the other simplistic idealization structure of the vehicle. The yellow and green areas are non-design spaces and cannot be modified by the optimizer. The total mass of the initial model is 105.6 kg. The entire design process uses each element density as the design variable and the minimum sum of compliance as the objective function. The volume fraction of less than 30% is regarded as the constraint that can improve lightweighting under the premise of crash safety. The time with maximal internal energy during the crash simulation is selected as the critical time step. Then the crashworthiness topology optimization problem is converted to a linear static topology optimization problem under the reduced ESLs.

(a) Topology optimization model
Figure 4. Topology optimization of side crash with rigid impactor.

Result Analysis

Figs. 4(b) and 4(c) show the topology optimization results of the M&ER and ESLSO methods. By comparison, the optimization result of the M&ER method proposed in this work is more explicit and the density distribution is more reasonable than the ESLSO method. Therefore, the proposed method effectively solve the numerical defects. The optimization iteration history of the objective functions and the topology evolution of the structure in two different methods are compared in Figs. 5(a) and 5(b). Evidently, the M&ER method proposed in this work has less fluctuation of the objective function and less optimization iterations during the entire iteration process than ESLSO. Thus, the proposed method has higher efficiency because the convergence is achieved after 24 optimization iterations with minimum objective function to satisfy the constraint conditions. Table 1 shows the optimization efficiency and the convergence result of the two methods. It can be seen that the stiffness of the proposed M&ER method increases by 15.7%, while the CPU time reduces by 30%.
Conclusions and Future Work

In this study, a novel approach for crashworthiness topology optimization is presented. The ESLs method is used to convert the structural crashworthiness topology optimization into the linear static topology optimization under the ESLs acting on all DOFs. The MR method is used to improve topology optimization efficiency and solve numerical defect problem by reducing linear matrix equation and ESLs are reduced to interface DOFs between design and non-design space. Thus, the efficiency of the crashworthiness topology optimization can be improved effectively. There are two major conclusions of the present work. First, directly applying those ESLs in linear static topology optimization mostly beyond the linear range and lead to numerical defects such as high compliance of each element, since some elements have a value of the lower bound during topology optimization. Second, the proposed strategy combining ESLs with model reduction method can effectively improve topology optimization efficiency for crashworthiness. We use numerical example to verify the two contributions and the efficiency. In addition, further research is needed to combine the ESLs with the model reduction for large-scale structural crashworthiness topology optimization.

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