Nonlinear electrodynamics and black holes

N. Bretón and R. García-Salcedo.
Departamento de Física, Cinvestav-IPN, Mexico City.

January 31, 2022

Abstract

It is addressed the issue of black holes with nonlinear electromagnetic field, focussing mainly in the Born-Infeld case. The main features of these systems are described, for instance, geodesics, energy conditions, thermodynamics. Also it is revised some black hole solutions of alternative nonlinear electrodynamics and its inconveniences.

1 Introduction

Since the beginning of the past century, proposals were made of nonlinear electrodynamics with the aim to cure the singularity of Maxwell’s solution to the field of a point charge at the charge’s position. Among the most successful is the formulation of Born and Infeld (BI). The proposal by Born and Infeld [1] in 1934 has several desirable properties in a physical theory, for instance finiteness of the electric field and energy at the charge’s position, the freedom of duality rotations, propagation of discontinuities of the electromagnetic field on single characteristic surfaces, among others.

Born and Infeld inspired in a finiteness principle chose a particular “nonlinear” action with a maximum field strength \( b \); they solved the resulting field equations for the static spherically symmetric solution corresponding to a point charge. The resulting field was very different from a Coulombian field in the neighborhood of the point charge where the fields are most intense; they found, in fact, that the field was finite everywhere, including the location of the point charge.

Solutions to the Einstein equations coupled with Born-Infeld nonlinear electromagnetic theory were found as soon as 1937 by Hoffman and Infeld [2]. Nonlinear electrodynamics coupled to general relativity was also explored by Peres [3], finding some static spherically symmetric solutions as well as wavelike solutions.

Pellicer and Torrence [4] faced BI theory coupled with gravitational field and obtained a class of nonsingular static spherically symmetric (SSS) solutions, corresponding to a point charge source, that asymptotically behaves as a Reissner-Nordstrom (RN) solution. RN is the SSS solution of Einstein-Maxwell equations.
During the eighties, Plebański [5] and collaborators made an extensive study of nonlinear electrodynamics (NLED) in general relativity, in particular they addressed Einstein equations coupled to Born-Infeld field in metrics of type D in Petrov classification [6], [7]; among these solutions there is the nonlinear generalization of the Reissner-Nordström solution. They also investigated the causal propagation of signals and gave a classification of the characteristic surfaces (along which discontinuities of the field propagate) for NLED [8].

Demianski [9] found in 1986 a SSS solution of Einstein-Born-Infeld equations that is regular at the origin, the so called $EBlon$. This solution has a particle like structure with a regular center.

Critics are the arbitrariness in choosing the NLED Lagrangian, since there is a lot of Lagrangians that fulfill the physical requirements like the linear weak field limit. In this sense, BI Lagrangian is exceptional since it can be considered as an effective Lagrangian for quantum electrodynamics (QED) in the one-loop approximation, as was shown by Heisenberg and Euler [10] and later on by Schwinger [11] (see also the review by Delphenich [12]). If another justification was needed it came from string theory, where solutions of the BI equations represent states of D-branes [13].

In this contribution we review aspects of NLED black holes according to the following plan: In Sec. 2 NLED formalism is introduced; Sec. 3 deals with Einstein-NLED solutions that can be interpreted as black holes; we focus in the static spherically symmetric (SSS) solutions. Sec. 4 addresses the NLED black hole thermodynamics. In Sec. 5 the relationship between the ADM and horizon masses of NLED solitons and black holes is presented. Sec. 6 contains stability aspects of NLED SSS solutions and comments on recent related research are at the end.

2 NLED formalism

NLED Lagrangian depends in nonlinear way of the electromagnetic invariants. We assume that the nonlinear electromagnetic field can be described by a vector potential $A_\mu$,

$$F_{\mu\nu} = 2A_{[\mu,\nu]},$$

(1)

$F_{\mu\nu}$ possesses just one independent invariant and one independent pseudo-invariant:

$$F = \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}, \quad \tilde{G} = \frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta},$$

(2)

where $\tilde{F}^{\alpha\beta}$ denotes the dual of $F^{\alpha\beta}$ defined by $\tilde{F}^{\alpha\beta} = (i/2\sqrt{-g}) \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$. If NLED Lagrangian is to be invariant under Lorentz group including reflections, then it must depend on $F$ and $\tilde{G}^2$. The condition that for weak fields the nonlinear theory should approximate the linear one must be imposed as well.
2.1 \((F,\tilde{G})\) and \((P,\tilde{Q})\) frameworks

It is convenient to introduce the canonical formalism for the system, related to the Lagrangian by a Legendre transformation. Defining

\[
P^{\alpha\beta} = 2 \frac{\partial L}{\partial F_{\alpha\beta}} = \frac{\partial L}{\partial F} F^{\alpha\beta} + \frac{\partial L}{\partial \tilde{G}} \tilde{G}^{\alpha\beta},
\]

we can work with \(H(F,\tilde{G}^2)\) or with \(H = H(P,\tilde{Q}^2)\) the later depending on the invariants associated to \(P^{\alpha\beta}\),

\[
P = \frac{1}{4} P_{\alpha\beta} P^{\alpha\beta}, \quad \tilde{Q} = \frac{1}{4} P_{\alpha\beta} \tilde{P}^{\alpha\beta}.
\]

The material or constitutive equations express \(P_{\alpha\beta}\) through \(F_{\alpha\beta}\), \(F\) and \(\tilde{G}\); we shall assume that they can be inverted to express \(F_{\alpha\beta}\) through \(P^{\alpha\beta}\) and \(P,\tilde{Q}\). These equations (also called Hamilton’s equations) are

\[
F^{\alpha\beta} = 2 \frac{\partial H}{\partial P^{\alpha\beta}} = \frac{\partial H}{\partial P} P^{\alpha\beta} + \frac{\partial H}{\partial \tilde{Q}} \tilde{P}^{\alpha\beta}.
\]

The \((P,\tilde{Q})\) framework is an alternative form of NLED obtained from the original one, the \((F,\tilde{G})\) framework, by a Legendre transformation. Physically reasonable conditions must be imposed on \(H(P,\tilde{Q})\): \(H\) is real; for weak fields nonlinear effects are negligible and the theory must have the limit of linear electrodynamics, i.e. \(H(P,\tilde{Q}) = P + O(P^2,\tilde{Q}^2)\); if parity is to be conserved, then under transformation of \(\tilde{Q}\) into \(-\tilde{Q}\), \(H\) must stay invariant, therefore, \(H\) depends on \(P\) and \(\tilde{Q}^2\). Often it is also required that strong or dominant energy conditions hold.

In this formalism, \(F^{\alpha\beta}\) is the physically significant electromagnetic field tensor, while \(P^{\alpha\beta}\) is a tensor with no direct physical meaning. While the various nonlinear field theories succeed in making \(F^{\alpha\beta}\) well behaved, the tensor \(P^{\alpha\beta}\) is, in general, divergent at the location of point sources. The reason is that \(\delta\)-function sources continue to play a role in these theories and \(P^{\alpha\beta}\) absorbs the singularities of the sources allowing the field tensor \(F^{\alpha\beta}\) be well behaved.

Note that by passing from \((P,\tilde{Q})\) framework to \((F,\tilde{G})\), we are changing from one NLED theory, characterized by some Lagrangian \(L(P,\tilde{Q})\), to another, in general different, corresponding to the Lagrangian \(L(F,\tilde{G})\); in the case of Maxwell electrodynamics, both theories coincide, \(L = F = H = P\).

The coupled gravitational and NLED equations are derived from the action

\[
S = \int d^4x \sqrt{-g} \{ R(16\pi)^{-1} - L \},
\]

where \(R\) denotes the scalar curvature, \(g := \det|g_{\mu\nu}|\) and \(L\), the electromagnetic part, is assumed to depend in nonlinear way on the invariants of \(P_{\mu\nu}\), in the
\((P, \tilde{Q})\) framework, or \(L\) depending on the invariants of \(F_{\mu\nu}\) in \((F, \tilde{G})\) scheme. Let us refer here to the former one:

\[
L = \frac{1}{2} P^{\mu\nu} F_{\mu\nu} - H(P, \tilde{Q}),
\]

(8)

The energy-momentum tensor and the scalar of curvature are given, respectively, by

\[
4\pi T_{\mu\nu} = H, P P_{\mu\alpha} P^{\alpha}_{\nu} - g_{\mu\nu}(2PH, P + \tilde{Q}H, \tilde{\tilde{Q}} - H),
\]

\[
R = 8(PH, P + \tilde{Q}H, \tilde{\tilde{Q}} - H),
\]

(9)

where \(\partial H/\partial P = H, P\). Note that the curvature scalar, \(R\), and consequently the trace of \(T_{\mu\nu}\), may differ from zero.

The Born-Infeld nonlinear electrodynamics is given by the structural function \(H(P, \tilde{Q})\),

\[
H = b^2 \left( 1 - \sqrt{1 - 2P/b^2 + \tilde{\tilde{Q}}^2/b^4} \right),
\]

(10)

where \(b\) is the maximum field strength and the relevant parameter of the BI theory.

### 2.2 NLED energy conditions

Using a timelike vector, \(V^\alpha, V_{\alpha}V^\alpha < 1\), imposing local energy density being non-negative amounts to \(T_{\mu\nu}V^\mu V^\nu \geq 0\); while that the local energy flow vector be nonspacelike requires that \(T_{\alpha\beta}V^\alpha V^\beta \leq 0\); these are, respectively, the weak energy condition (WEC) and the dominant energy condition (DEC); both conditions hold provided

\[
H, P > 0, \quad (PH, P + \tilde{Q}H, \tilde{\tilde{Q}} - H) \geq 0.
\]

(11)

The strong energy condition (SEC) \(R_{\mu\nu}V^\mu V^\nu \geq 0\), using the Einstein equation can be settled as,

\[
R_{\mu\nu}V^\mu V^\nu = 8\pi(T_{\mu\nu}V^\mu V^\nu + \frac{T}{2}) \geq 0.
\]

(12)

Note that NLED matter can violate SEC if the trace of the energy-momentum tensor is negative enough; for instance, BI energy-momentum tensor can violate SEC. In the case of Maxwell ED \((T = 0)\) the fulfilment of WEC implies SEC.

### 3 NLED Black holes

Hoffmann and Infeld (HI) \([2]\) solved the Einstein-Born Infeld coupled equations (EBI) for the spherically symmetric case, imposing the condition of regularity on the electromagnetic tensor \(F_{kl}\) and its first derivative and the same condition.
to the metric tensor $g_{kl}$. The requirement that there be no infinities in $g_{kl}$ forces the identification of gravitational with electromagnetic mass. For the SSS line element

$$ds^2 = -\psi dt^2 + \psi^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

HI find the solution

$$\psi = 1 - \frac{8\pi}{r} \int_0^r (\sqrt{r^4 + 1} - r^3) dr. \quad (14)$$

However, this form leads to a conical singularity. In the same paper HI found the regular solution given by the fields $D_r = 1/r$, $E_r = r^2/(r^4 + 1)$ and the metric function

$$\psi_{HI} = 1 - \frac{k}{r} + \frac{8\pi \gamma}{r} \int_0^r r^2 \ln \left( \frac{r^4}{1 + r^4} \right) dr, \quad (15)$$

where $k$ is a constant of integration corresponding to $(-2m)$ in Schwarzschild solution. To have regularity at $r = 0$ it must be taken $k = 0$. Therefore we must consider as the gravitational mass the quantity $4\pi \int_0^r r^4 T_{tt} dr$, that is the total electromagnetic mass within a sphere having its center at $r = 0$. Thus the regularity condition shows that electromagnetic and gravitational mass are the same.

Pellicer and Torrence (PT) [4] following the lines of HI [2] searched for a EBI SSS solution well behaved at the origin; they imposed the continuity of $F_{\mu\nu}$ and the Lagrangian in the neighborhood of the charge; continuity on $F_{\mu\nu}$ and $L(F)$ leads to conditions on $H(F)$. They considered $\tilde{G} = 0$, but were not restricted to BI.

For the SSS line element [13] PT obtained

$$\psi_{PT} = 1 + \frac{d}{r} + \frac{8\pi}{r} \int_0^r H(x)x^2 dx, \quad (16)$$

the comparison with Schwarzschild solution gives $d = 2m$, the constant $d$ is related to the mass parameter $m$. If one considers the mass as arising from electromagnetic properties, one can put $m = d = 0$. The electromagnetic field for a charge $e$ is given by

$$F_{\mu\nu} = -\frac{e}{r^2} \frac{\partial H(P,0)}{\partial P} \delta^0_\mu \delta^c_\nu, \quad (17)$$

also $P = -e^2/4r^4$. Regularity takes place if the following conditions hold: the integral in (16) exists and is finite; the field (17) must be finite as $r \to 0$; besides, asymptotically, for large $r$, $H(P,0) \approx P$. These conditions guarantee the solution be well behaved. Note that there is still some freedom in chosing the function $H(P)$ so it can be selected ad hoc and numerous examples may be built.

Related to this point we shall address some cases of regular black hole solutions obtained with ad hoc NLEDs.
The existence of two alternative ways, $(P, \tilde{Q})$ and $(F, \tilde{G})$, of setting NLED, can generate some confusion. Let us refer to those regular electric black hole solutions that have been obtained recently [14]. In deriving the solutions, in the $(P, \tilde{Q} = 0)$ framework, an ad hoc NLED $H(P)$ was chosen to obtain a regular metric that describes black holes with regular center and RN asymptotics. The appropriate $H(P)$ was found imposing regularity conditions on the metric, in a similar way to the sketched by PT. The so constructed solutions [14] were using:

\begin{align}
H(P) &= P \frac{1 - 3\Pi}{(1 + \Pi)^3} + \frac{6}{q^2s} \left(\frac{\Pi}{1 + \Pi}\right)^{5/2}, \\
H(P) &= P / \cosh^2(s\sqrt{\Pi}), \\
H(P) &= P \exp\left(-s\sqrt{\Pi}\right) \left(1 + \frac{3\Pi}{s + \Pi}\right),
\end{align}

where $\Pi = \sqrt{-q^2P/2}$ and $s = |q|/(2m)$; $q = q_e$ and $m$ are, respectively, the parameters identified as charge and mass. The above presented functions $H(P)$ behave like $P$ at small $P$ and tend to finite limits as $P \to -\infty$. The solutions are called regular due to the finiteness of three invariants: $\mathcal{R}, \mathcal{R}_{\alpha\beta}, \mathcal{R}_{\alpha\beta\gamma\delta}$.

However, there is a clear contradiction with no go theorems that forbid the existence of SSS solutions with a regular center for whatever $L(F)$ chosen if it is required that for weak fields, $L \sim F$, i.e. if for weak fields it is demanded a Maxwellian behavior [15], [16].

The explanation of this apparent contradiction is the following: one can obtain regular solutions choosing ad hoc $H(P, \tilde{Q})$ as was explained above. These solutions correspond to certain lagrangian $L(F, \tilde{G})$ that is the one from which one derives the dynamical equations. That this Lagrangian be well behaved is not guaranteed by an adequate selection of the corresponding $H(P, \tilde{Q})$. This is the case, for instance in [14]: those solutions correspond to Lagrangians $L(F)$ that suffer branching. So, in spite that solutions are well behaved in the $P$ framework ($\tilde{G} = 0$), they correspond to different Lagrangians in different regions of space. The branching in $L(F)$ is due to extrema in the electromagnetic field; associated to extrema there are singularities that are seen only by photons. These singularities can either be hidden behind a horizon or naked depending on the value of one parameter. Additional features of those “regular” solutions include, for instance, that the energy density of electromagnetic field may be negative for some interval of the radial coordinate; this and more were investigated by Novello et al [17], [18].

Note, however, that regular solutions with only magnetic charge may exist [16]. Another way to avoid the prohibition of SSS electrically charged regular structures is to resign of having a regular center, like the solution by Dymnikova in [19] that has a de Sitter center. It is not excluded neither the possibility of regular solutions that correspond to Lagrangians depending on both invariants of the electromagnetic field, $L(F, \tilde{G})$. 
3.1 Type-D solutions with EBI

BI theory has the property that its equations have an exact $SO(2)$ electromagnetic duality invariance. In [6] Type-D solutions were constructed for BI coupled with Einstein equations allowing for the freedom of duality rotations. Under the assumption that the natural tetrads of type-D metrics coincide with the eigenvectors of the algebraically general nonlinear electromagnetic field and assuming that the two principal null directions are geodesic and shear-free, all D-type solutions that are compatible with the scheme of NLED endowed with duality rotations were found. It turned out that the presence of the acceleration or rotation parameters prohibits the existence of NLED type-D solutions with duality rotations. Moreover, by adding axion and dilaton fields, this invariance may be extended to $SL(2,\mathbb{R})$ S-duality, relevant to string theory, which implies a strong-weak coupling duality of such theories; for dualities in the context of more general nonlinear electrodynamics see [20], [21], [22].

All type-D solutions of the coupled Einstein and Born-Infeld equations were determined in [7]. There are two classes of them: static and stationary. The static solutions are exhausted by the BI generalizations of (i) the Bertotti-Robinson solution, (ii) Reissner-Nordstrom (RN) and (iii) anti-Reissner-Nordstrom. The stationary solutions belong to two subfamilies: the BI generalization of the NUT $\tilde{B}(\pm)$ metric which contains as a limit the BI generalization of RN, and the BI generalization of the anti-NUT $\tilde{B}(-)$ solution that contains as a limit the static case (iii). The NUT solution includes the NUT parameter, $m$, while anti-NUT changes from $m$ to $(-n)$. These EBI solutions were derived including a cosmological constant $\Lambda$ in their energy-momentum tensor. Since BI generalization of RN solution is of most interest, we analyze it in detail in what follows.

3.2 Born-Infeld black hole and EBIon

The EBI solution for a SSS spacetime [13] is given by the metric function $\psi_{BI}(r)$

$$\psi_{BI}(r) = 1 - \frac{2m}{r} + \frac{2}{3} b^2 (r^2 - \sqrt{r^4 + a^4}) + \frac{4g^2}{3r} G(r),$$

$$G'(r) = -(r^4 + a^4)^{-\frac{3}{2}},$$

where $G'(r)$ denotes the derivative of $G(r)$ with respect to the radial variable, $m$ is the mass parameter, $g$ is the magnetic (or electric) charge (both in length units), $a^4 = g^2/b^2$ and $b$ is the Born-Infeld parameter given in units of [length]$^{-1}$. The nonvanishing components of the electromagnetic field are

$$F_{rt} = g(r^4 + a^4)^{-\frac{3}{2}}, \quad P_{rt} = \frac{g}{r^2}. $$

The black hole solution given by García-Salazar-Plebański [7] corresponds to
\[ G(r) = \int_{r}^{\infty} \frac{ds}{\sqrt{s^4 + a^4}} = \frac{1}{2a} F \left[ \arccos \left( \frac{r^2 - a^2}{r^2 + a^2} \right), \frac{1}{\sqrt{2}} \right], \quad (24) \]

where \( F \) is the elliptic integral of the first kind. On the other side, the particle-like solution given by Demianski \[9\] is

\[ G(r) = \int_{0}^{r} -\frac{ds}{\sqrt{s^4 + a^4}} = -\frac{1}{2a} F \left[ \arccos \left( \frac{a^2 - r^2}{a^2 + r^2} \right), \frac{1}{\sqrt{2}} \right]. \quad (25) \]

The selection of \( G(r) \) as in Eq. \(24\) or Eq. \(25\) has as a consequence a different behavior of the solution at \( r = 0 \). The metric function \( \psi_{BI}(r) \) with \( G(r) \) given by Eq. \(24\) diverges at \( r \to 0 \) (even when \( m = 0 \)), corresponding to the black hole solution. The other one, meaning \( \psi_{BI}(r) \) with \( G(r) \) given by Eq. \(25\), is the so called EBIon solution that is finite at the origin (for \( m = 0 \)).

The integrals of Eqs. \(24\) and \(25\) are related by

\[ \int_{r}^{\infty} \frac{ds}{\sqrt{s^4 + a^4}} + \int_{0}^{r} \frac{ds}{\sqrt{s^4 + a^4}} = \frac{1}{a} K \left[ \frac{1}{2} \right], \quad (26) \]

where \( K[\frac{1}{2}] \) is the complete elliptic integral of the first kind. In the limit of large distances, \( r \to \infty \), asymptotically the solution approaches Reissner-Nordstrom (RN) solution, the SSS solution to Einstein-Maxwell equations. Also when the BI parameter goes to infinity, \( b \to \infty \), we recover the linear electromagnetic (Einstein-Maxwell) RN solution. In the uncharged limit, \( b = 0 \) (or \( g = 0 \)), it is recovered the Schwarzschild black hole. Note that due to the duality rotation both charges, electric \( e \) and magnetic \( g \), can be included in the solution by substituting \( g \to \sqrt{e^2 + g^2} \).

Since this solution is of type D, there is only one nonvanishing Weyl scalar, \( \Psi_2 \). For the black hole solution it is

\[ \Psi_2 = \frac{m}{r^3} - \frac{g^2 r^3}{6} \partial_{rr} \left( \frac{1}{r^2} \int_{r}^{\infty} \frac{ds}{s^2 + \sqrt{s^4 + a^4}} \right), \quad (27) \]

The invariants depend on \( \Psi_2 \), then at \( r = 0 \) there is a singularity of order \( 1/r^6 \), coming from the gravitational mass term, alike the Schwarzschild and RN singularities; furthermore, the second term also diverges at \( r = 0 \). There are also zeros in the metric function \( \psi_{BI}(r) \) that are coordinate singularities which can be removed using analytical extensions \[23\]. The zeros of \( \psi_{BI}(r) \) can be localized numerically and might be one, two or none, depending on the relative values of the parameters \( g, m, b \); it is illustrated in Fig.\[1\]. These parameters determine the position and size of the horizon as well \[24\]. Note that distinct \( b \) corresponds to different NLED theories.

### 3.3 Trajectories of test particles in BI black hole

Since the SSS spacetime possesses two Killing vectors, \( \partial_t \) and \( \partial_\phi \), the test particle conserves two motion quantities: its energy \( E \) and its angular momentum \( l \). Moreover, if we restrict ourselves to the equatorial plane \( (\theta = \pi/2) \), the timelike
Figure 1: It is shown the behavior of the BI metric function $\psi$ in terms of $u = r/m$; the label on each curve corresponds to the value of $bm$. Note that the position of the horizon, $\psi(r_{\Delta}) = 0$, depends on this parameter: the horizon shrinks as $bm$ is greater.

and null geodesics for the EBI spacetimes can be reduced to the problem of ordinary one-dimensional motion in an effective potential $U_{\text{eff}}$, alike in the Schwarzschild and RN cases.

3.3.1 Massive particles

Trajectories of massive particles are determined by the Lorentz equation. For a test particle of charge $\epsilon$ and mass $\mu$ it is

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = - \frac{\epsilon}{\mu} F^\nu_\sigma dx^\sigma d\tau,$$  \hspace{1cm} (28)

where $\tau$ is the affine parameter along the trajectory. Using the two conserved motion quantities: energy $E$ and angular momentum $l$, the geodesic for $t$ can be integrated once, obtaining the first derivative with respect to the proper time $\tau$,

$$\dot{t} \psi_{BI} = E + \frac{\epsilon g}{\mu} \sqrt{\frac{b}{4g}} \left[ \arccos \left( \frac{r^2 - g/b}{r^2 + g/b} \right), \frac{1}{\sqrt{2}} \right].$$  \hspace{1cm} (29)

From the line element for timelike geodesics we have,

$$1 = \psi \dot{t}^2 - \psi^{-1} \dot{r}^2 - \dot{\phi}^2$$  \hspace{1cm} (30)

substituting $l = g_{\phi\phi} \dot{\phi}$ it is obtained

$$r^2 + \psi \left( \frac{l^2}{r^2} + 1 \right) - \left[ E + \frac{\epsilon g}{\mu} \sqrt{\frac{b}{4g}} \left[ \arccos \left( \frac{r^2 - g/b}{r^2 + g/b} \right), \frac{1}{\sqrt{2}} \right] \right]^2 = 0.\hspace{1cm} (31)$$
Comparison with $\frac{1}{2} \dot{r}^2 + U_{\text{eff}}(E,l,r) = 0$, gives the effective potential that a charged test particle feels. The shape of the effective potential shows attractive regions with stable equilibrium positions. The equilibrium positions are of lower energy for greater angular momentum of the particles. If test particles reach the singularity or not depends on the value of the maximum field strength $b$.

### 3.3.2 Light trajectories in NLED

Discontinuities of the fields propagate obeying the equation of the characteristic surfaces, that in ordinary optics is the so-called eikonal equation. Locally these surfaces are normal to the light rays trajectories. In General Relativity when a linear Maxwellian field is present, its characteristics do coincide with the Einsteinian cone; but NLED photons do not propagate along null geodesics of the background geometry. Instead, they propagate along null geodesics of an effective geometry which depends on the nonlinear electromagnetic field. In a curved spacetime the characteristic surfaces equation is [8], [25]

$$g_{\mu\nu}S_{\nu}^{\mu} = 0. \tag{32}$$

However, if in the spacetime it is present a nonlinear electromagnetic field, the corresponding equation is

$$(g^{\mu\nu} + \frac{4\pi}{b^2} T^{\mu\nu}_{\text{NLED}})S_{\nu}^{\mu} = \gamma^{\mu\nu}S_{\mu}S_{\nu} = 0, \tag{33}$$

where $T^{\mu\nu}_{\text{NLED}}$ is the electromagnetic energy momentum density; remind that the linear limit is obtained when $b \rightarrow \infty$, hence in linear limit light trajectory, that is normal to characteristic surfaces, occurs on null geodesics. Hence, as far as $b$ is finite, there is a distinction between the propagation of gravitational discontinuities (gravitons, abusing of language) along Eq. 32 and the propa-
ation of electromagnetic discontinuities (photons) ruled by Eq. (33). For BI black holes, actually both trajectories converge at the horizon.

Using the two constants of motion, \( E = -\dot{\psi}t, \ l = \dot{\phi}r^2 \) and the line element for null geodesics, we obtain the derivative of \( r \) respect to an affine parameter,

\[
\dot{r} = \sqrt{E^2 - \frac{\psi_B l^2}{r^2}}. \tag{34}
\]

The trajectories for light rays, however, are given by an effective geometry, considering the same constants of motion \( E \) and \( l \):

\[
\dot{r}_{ph} = \sqrt{E^2 - \frac{\psi_B l^2}{r^2} \left(1 + \frac{a^4}{r^4}\right)^{-1}}. \tag{35}
\]

There is a correction factor in the last formula, \((1 + a^4/r^4)^{-1}\), due to the effective geometry; it vanishes if \( a = 0 \ (b \to \infty) \) or for photons with \( l = 0 \). Remind that usually it is considered \( \hbar E \) and \( \hbar l \) as the total energy and angular momentum of photon, respectively. The corresponding effective potentials are shown in Fig. 2.

From the above expressions it is easy to check that \((dr/dt)_{photon} < (dr/dt)_{grav}\), i. e. light travels slower that gravitational waves due to nonlinear effects, such as photon-photon interaction, for instance. This effect can also be described as if photons ruled by Maxwell ED were propagating inside a dielectric medium, with dielectric “constant” depending in nonlinear way of the fields. The possibility is not excluded of superluminal signals for other NLED [17].

4 NLED black hole thermodynamics

In a general context the zeroth and first laws of black hole mechanics (BHM) refer to equilibrium situations and small departures therefrom. First law of BHM is an identity relating the changes in mass, angular momentum and horizon area of a stationary black hole when it is perturbed. The variation applies for perturbations from one stationary axisymmetric solution of Einstein equations to another; moreover, it has been shown that the validity of this law depends only on very general properties of the field equations [26]. For the horizon mass \( M_\Delta \) the first law, when static spherically symmetric solutions are considered, is

\[
\delta M_\Delta = \frac{\kappa}{8\pi} \delta a_\Delta + \Phi_\Delta \delta Q_\Delta, \tag{36}
\]

where \( \kappa \) is the surface gravity at the horizon, \( a \) is the area of the horizon, \( Q \) is the electric charge and \( \Phi \) is the electric potential; the subindex \( \Delta \) indicates that the quantity is evaluated at the horizon of the black hole.

On the other side, the total mass is given by the Smarr’s formula

\[
M_\Delta = \frac{\kappa a_\Delta}{4\pi} + \Phi_\Delta Q_\Delta. \tag{37}
\]
In the case of Einstein-Maxwell theory, it is possible to deduce one, Eq. (36), directly from the other, Eq. (37), using the homogeneity of the mass as a function of $\sqrt{a}$ and $Q$. In the work by Ashtekar, Corichi and Sudarsky [27] the first law of BHM, for quantities defined only at the horizon, arises naturally as part of the requirements for a consistent Hamiltonian formulation.

Work on NLED black hole thermodynamics includes the derivation of the first law of black hole physics for some nonlinear matter models [28]. Rasheed [29] studied the zeroth and first laws of black hole mechanics in the context of non-linear electrodynamics coupled to gravity. In this case, the zeroth law, which states that the surface gravity of a stationary black hole is constant over the event horizon, is shown to hold even if the Dominant Energy Condition is violated. In NLED one no longer has homogeneity of the mass function and a priori one has no reason to expect that neither the first law or Smarr formula hold. Rasheed found that the usual first law (the general mass variation formula) holds true for the case of non-linear electrodynamics but the formula for the total mass (Smarr’s formula) does not.

However, we can propose the form that must have a Smarr-type formula for the horizon mass in order to be consistent with the variations expressed by the first law of BHM that indeed holds,

$$M_\Delta = \frac{k a_\Delta}{4\pi} + \Phi_\Delta Q_\Delta + V(a_\Delta, Q_\Delta, P_\Delta),$$

where $V$ is a so far undetermined potential that depends on the horizon parameters, $a_\Delta, Q_\Delta, P_\Delta$ and also on the coupling constants of the theory. In the variational principle this term plays no role, however in the Hamiltonian description it becomes essential.

Note that in the first law, Eq. (36), only variations of the electric charge are involved, and not variations of the magnetic charge, $P_\Delta$. On the other hand, the horizon mass, Eq. (38) might depend on $P_\Delta$ through $V$.

The equations to determine the potential $V(a_\Delta, Q_\Delta, P_\Delta)$ arise from the condition that the first law holds and demanding consistency between Eq. (36) and the variations of Eq. (38), these are [30],

$$a_\Delta \frac{\partial \psi}{\partial a_\Delta} + 8\pi r_\Delta Q_\Delta \frac{\partial \Phi}{\partial a_\Delta} + 8\pi r_\Delta \frac{\partial V}{\partial a_\Delta} = 0,$$

$$\frac{r_\Delta}{2} \frac{\partial \psi}{\partial Q_\Delta} + Q_\Delta \frac{\partial \Phi}{\partial Q_\Delta} + \frac{\partial V}{\partial Q_\Delta} = 0,$$

where $\psi$ is the metric function in a SSS line element (13), $\psi = 1 - 2m'(r)$, $a_\Delta = 4\pi r_\Delta^2$; $r_\Delta$ is the radius of the horizon.

The condition of consistency determines the set of parameters that can vary independently; in this case, the magnetic charge becomes a function of the area and electric charge, $P_\Delta = P_\Delta(r_\Delta, Q_\Delta)$. To illustrate the point, in what follows we shall determine the horizon mass from a Smarr type formula in agreement with the first law of BHM for the Bardeen black hole.
4.1 Smarr’s formula for Bardeen black hole

The Bardeen model was proposed some years ago as a regular black hole, however, only recently it has been shown [31] that it is an exact solution of the Einstein equations coupled to a kind of nonlinear electrodynamics characterized by the Lagrangian

\[
\mathcal{L}(F) = \frac{2}{2g^2} \left( \frac{2g^2 F}{1 + \sqrt{2g^2 F}} \right)^{5/2},
\]

where \(g\) and \(F\) are the magnetic charge and electromagnetic invariant, respectively and \(s = g/m\). The corresponding energy momentum tensor fulfills the weak energy condition and is regular everywhere. For a SSS space [13], the corresponding metric function is given by

\[
\psi_B = 1 - \frac{2m(r)}{r} = 1 - \frac{2mr^2}{(r^2 + g^2)^{3/2}},
\]

This solution is a self-gravitating magnetic monopole with charge \(g\). The solution is regular everywhere, although the invariants of the electromagnetic field exhibit the usual singular behaviour of magnetic monopoles, \(F = g^2/2r^4\). In the asymptotic behaviour of the solution the constant \(g\) vanishes as \(1/r^3\), and not as a Coulombian term \((1/r^2)\), that allows to interpret the constant \(g\) as a magnetic charge. The Bardeen solution does not involve electric charge, then the horizon mass depends only on the area of the horizon,

\[
M_\Delta = \frac{1}{8\pi} \int \kappa da = \int \left( 1 - m' \right) dr,
\]

the condition that the horizon mass be positive, from Eq. (42), gives that \(m(r) \leq r\); it also guarantees that \(\psi_B \geq 0\). Using the expression for \(\psi_B\) it amounts to \((r^2 + g^2)^{3} \geq 4m^2r^4\). In this case when \(g^2 = \frac{16}{27}m^2\) the two horizons that could be present shrink into a single one, being this value of \(g\) the corresponding to the extreme black hole; for \(g^2 < \frac{16}{27}m^2\) there exist both inner and event horizon. The potential \(V\) for the Smarr-type formula, Eq. (38), for the Bardeen black hole turns out to be, undetermined until an integration constant which we have put zero,

\[
V = mr^3 \frac{2g^2 - r^2}{(g^2 + r^2)^{3/2}},
\]

Substituting \(V\) in the Smarr-type formula one obtains the horizon mass

\[
M_\Delta = \frac{r}{2} - \frac{mr^3}{(r^2 + g^2)^{3/2}}
\]

This value for the horizon mass coincides with the one determined by integrating the first law, Eq. (12). Note that the horizon mass of the Bardeen black hole involved dependence only on the horizon area, since the magnetic charge
is not considered as a varying parameter of the horizon. In this case there is a full agreement between the horizon mass calculated with the first law of BHM and when it is determined by adding the appropriate potential to a Smarr-type formula; it is shown in Fig. 3.

The horizon mass, calculated with a Smarr type formula that is consistent with the first law of BHM, applies only to the magnetic sector of NLED solutions. If the variation of electric charge is taken into account in the potential \( V \) of the Smarr formula, the mentioned consistency does not longer hold. The potential \( V \) determined in agreement with the first law of BHM can not give the appropriate dependence for the terms corresponding to electric charge; Eqs. (39) do not describe confidently the potential \( V \) in general situations where nonlinear electromagnetic fields are present. Therefore, when the dependence of \( V \) on the electromagnetic field is of nonlinear nature Eqs. (39) are useless to determine \( V \).

5 Isolated horizon framework and mass relation

Remarkable properties of nonlinear electrodynamics black holes arise in the context of the isolated horizon formalism, recently put forward by Ashtekar et al [32]. In this approach it is pointed out the unsatisfactory (uncomplete) description of a black hole given by concepts such as ADM mass and event horizon, for instance, specially if one is dealing with hairy black holes. To remedy this uncompleteness, Ashtekar et al have proposed alternatively the isolated horizon formalism, that furnishes a more complete description of what happens in the neighborhood of the horizon of a hairy black hole.

In the isolated horizon formalism one considers spacetimes with an interior boundary, which satisfy quasi-local boundary conditions that insure that the horizon remains isolated. The boundary conditions imply that quasi-local charges can be defined at the horizon, which remain constant in time. In par-
ticular one can define a horizon mass, a horizon electric charge and a horizon magnetic charge.

Moreover, Ashtekar-Corichi-Sudarsky (ACS) conjecture about the relationship between the colored black holes and their solitonic analogs [27]: the Arnowitt-Deser-Misner (ADM) mass contains two contributions, one attributed to the black hole horizon and the other to the outside hair, captured by the solitonic residue. In this model, the hairy black hole can be regarded as a bound state of an ordinary black hole and a soliton. The proposed formula relating the horizon mass and the ADM mass of the colored black hole solution with the ADM mass of the soliton solution of the corresponding theory is

\[ M_{\text{sol}}^{(n)} = M_{\text{ADM}}^{(n)} - M_{\Delta}^{(n)}, \]  

where the superscript \( n \) indicates the colored version of the hole; in the papers of Ashtekar et al this \( n \) refers to the Yang-Mills hair, labeled by this parameter, corresponding to \( n = 0 \) the Schwarzschild limit (absence of YM charge). This relation has been proved numerically to work for the Einstein-Yang-Mills (EYM) black hole.

For the EBI black hole the location and size of the horizon depends on the parameter \( bq \), so \( b \) and \( q \) are not independent parameters; however, at infinity it is undistinguishable from a RN black hole characterized only by its charge \( q \) and mass \( m \). Provided that in the EBI theory there exist both exact solutions: the black hole and the soliton like solution, in spite that the EBI black hole is not a coloured one, we shall probe it with ACS model, considering \( b \) as a free parameter, for a fixed charge. Then for the case studied here the \( n \) version shall correspond to the distinct black holes labeled by distinct (continuous) BI parameter, \( b \).

It turns out that the EBI black hole and the corresponding EBIon solution fulfill the relation between the masses as well as most of the properties of the model as for the colored black hole [33]. For the EBI solution the horizon and ADM masses as functions of the horizon radius \( r_{\Delta} \) are given, respectively, by

\[ M_{\Delta}^{(b)}(r_{\Delta}) = \frac{r_{\Delta}}{2} + \frac{b^2 r_{\Delta}}{3} (r_{\Delta}^2 - \sqrt{r_{\Delta}^4 + a^4}) - \frac{2g^2}{3} \int_{0}^{r_{\Delta}} \frac{ds}{\sqrt{a^4 + s^4}}, \]  

\[ M_{\text{ADM}}^{(b)}(r_{\Delta}) = \frac{r_{\Delta}}{2} + \frac{b^2 r_{\Delta}}{3} (r_{\Delta}^2 - \sqrt{r_{\Delta}^4 + a^4}) + \frac{2g^2}{3} \int_{r_{\Delta}}^{\infty} \frac{ds}{\sqrt{a^4 + s^4}}. \]  

In Figs. 4 and 5 are displayed \( M_{\text{ADM}}^{(BI)} \) and \( M_{\Delta}^{(BI)} \) in comparison with the Schwarzschild and Reissner-Nordsstrom cases. The mass of the soliton can be obtained by letting \( r_{\Delta} \to 0 \) in the ADM mass, Eq. (17), obtaining \( M_{\text{sol}}^{(b)} = 2g\sqrt{b} K[\frac{4}{3}] \). From these expressions one can trivially check that they satisfy Eq. (45).

Other predictions of ACS model that are fulfilled by BI black holes and its soliton counterpart are: BI horizon mass is less than Schwarzschild horizon mass.
Figure 4: It is shown the ADM mass as function of the horizon radius $r_\Delta$, for the Schwarzschild (Schw), Reissner-Nordstrom (RN) and for BI black holes with BI parameters $b = 0.7$ and $b = 0.08$.

(see Fig. 5); horizon masses satisfy the inequality $M_{\Delta}^{RN} > M_{\Delta}^{Schw} > M_{\Delta}^{BI}$; for all $b$ and all $r_\Delta$, the surface gravity of the BI black hole is less than the one for Schwarzschild; $M_{\Delta}^{BI}$ and $\kappa_{BI}$ as functions of $r_\Delta$ are monotonically decreasing functions of $b$. However, EBI solutions do not fulfill that $M_{\Delta}^{BI}$ as function of $r_\Delta$ increases monotonically for all values of $b$ (see Fig. 5 for $b = 1$).

Since most of the ACS features are fulfilled, we can say that the static sector of the EBI theory is described by the heuristic model for the colored black holes proposed by Ashtekar et al when the BI parameter $b$ varies keeping the charge fixed.

6 Stability of NLED black holes

Stability properties in self-gravitating nonlinear electrodynamics were studied by Moreno and Sarbach [34]. They derived sufficient conditions for linear stability with respect to arbitrary linear fluctuations in the metric and in the gauge potential, $\delta g_{\mu\nu}$ and $\delta A_{\mu}$, respectively; the conditions were obtained in the form of inequalities to be fulfilled by the nonlinear electromagnetic Lagrangian $L(F)$ and its derivatives. The application of this criterion is restricted to static, spherically symmetric solutions of NLED coupled to gravity, that are purely electric or purely magnetic ($\tilde{G} = 0$). For these systems a gauge invariant perturbation formalism was used obtaining that linear fluctuations around a SSS purely electric (or purely magnetic) solution are governed by a wavelike equation with symmetric potential. The stability conditions are translated into some requirements on the Lagrangian and its derivatives, $L(F), L_F, L_{FF}$; in terms of the variable $y = \sqrt{2g^2F}$ these conditions are

$$L(y) > 0, \quad L(y)_y > 0, \quad L(y)_{yy} > 0.$$  \hspace{1cm} (48)
Figure 5: The horizon masses are shown for the Schwarzschild (Schw), Reissner-Nordstrom (RN) and the BI black holes with BI parameters \( b = 0.5 \) and \( b = 1 \). Note that \( M_{\Delta}^{RN} > M_{\Delta}^{Schw} > M_{\Delta}^{BI} \).

Besides, there are more inequalities to be fulfilled, that arise from the pulsation equations in the even-parity sector

\[ f(y) \equiv \frac{y L_{yy}}{L_y} > 0, \quad f(y)N(y) < 3. \]  \hfill (49)

where \( N(y) \) is the metric function in the SSS line element [13].

We shall apply this criterion to test the stability of the purely magnetic or purely electric EBI particle-like and black hole solutions [35]. In the former case the boundary point is the origin, \( r = 0 \), while for the black hole case the conditions must be held in the domain of outer communication (DOC), i.e. positions outside the horizon, \( r > r_{\Delta} \), \( r_{\Delta} \) being the radius of the horizon of the black hole.

The BI Lagrangian fulfills the stability conditions; in terms of the variable \( y \), the BI Lagrangian, with \( \tilde{G} = 0 \), is given by

\[ L(y) = b^2 \left[ \sqrt{1 + \frac{y^2}{b^2 g^2}} - 1 \right] > 0, \]  \hfill (50)

and the rest of the inequalities (49) read as:

\[ L_{,y} = \frac{y}{g^2} \left( 1 + \frac{y^2}{b^2 g^2} \right)^{-\frac{3}{2}} > 0, \]
\[ L_{,yy} = \frac{1}{g^2} \left( 1 + \frac{y^2}{b^2 g^2} \right)^{-\frac{5}{2}} > 0, \]
\[ f(y) = \frac{L_{yy}}{L_{,y}} = \left( 1 + \frac{y^2}{b^2 g^2} \right)^{-1} > 0. \]  \hfill (51)

Conditions (51) are fulfilled in all the range of \( y \). Moreover, \( f(y) \) is monotonically decreasing with \( f(y = 0) = 1, 0 < f(y) \leq 1 \); then the last stability
condition $f(y)N(y) < 3$ reduces to prove that $N(y) = \psi_{BI}(y) < 3$; for the black hole it must be fulfilled in DOC ($r > r_\Delta$), while for the particle-like solution the domain to be considered is $0 \leq r < \infty$.

In the black hole case, the metric function $\psi_{BI}(r)$ has a minimum in the extreme case ($g = m$) for $bm = 0.5224$ at $r_\Delta = 0.346m$; DOC is considered for distances larger than the radius of the horizon, $r > r_\Delta = 0.346m$. In terms of $y$, considering that $F = g^2/2r^4$ then $y = g^2/r^2$, the metric function $N(y) = \psi_{BI}(y)$ is

$$\psi_{BI}(y) = 1 - \frac{2m\sqrt{y}}{g} + \frac{2b^2g^2}{3y}[1 - \sqrt{1 + \frac{y^2}{b^2g^2}}] + \frac{2\sqrt{gb}y}{3y} \left[ \arccos\left(\frac{gb}{g + y}\right), \frac{1}{\sqrt{2}} \right],$$

\begin{equation}
(52)
\end{equation}

In the range $0 < y < y_\Delta = 8.35$ it turns out that $0 < \psi_{BI}(y) \leq 1$ with $\psi_{BI}(0) = 1$ therefore, $0 < N(y) = \psi_{BI}(y) < 1 < 3$, fulfilling the last inequality required as sufficient conditions for linear stability of the EBI black hole.

For the particle-like solution of the EBI equations, the metric function $\psi_{BI}(r)$ in terms of $y$ is

$$\psi_{BI}(y) = 1 - \frac{2m\sqrt{y}}{g} + \frac{2b^2g^2}{3y}[1 - \sqrt{1 + \frac{y^2}{b^2g^2}}] - \frac{2\sqrt{gb}y}{3y} \left[ \arccos\left(\frac{y - gb}{gb + y}\right), \frac{1}{\sqrt{2}} \right],$$

\begin{equation}
(53)
\end{equation}

The last stability condition $N = \psi < 3$ in fact occurs since $N(y = 0) = 1$ and the function is monotonically decreasing, having $N(y) \leq 1$, for $bg \neq 0$; the finiteness in the origin of $N(r)$ is valid when $m = 0$. Therefore, as far as this analysis proves, the EBI solutions, both black hole and particle-like one, are stable.

On the other side, stability of self-gravitating structures can also be approached from the isolated horizon framework: the ACS heuristic model explains the instability of the coloured black holes in terms of the instability of the solitons. The bound state of a bare black hole and a soliton is going to be unstable if the total energy in the initial bound state $E_{initial} = M^{(n)}(r_{initial}^\bullet) + M^{(n)}_{sol}$ exceeds the energy in the final black hole, $E_{final} = M^{(0)}(r_{final}^\bullet)$.

The available energy can be expressed (for a fixed $r_{initial}^\bullet$) as

$$E_{avail}^{(n)} = M^{(n)}_{ADM} - M^{(0)}_{ADM},$$

\begin{equation}
(54)
\end{equation}

where the superscript $n$ indicates the colored version of the hole and $M^{(0)}_{ADM}$ is the ADM mass of the bare (Schwarzschild) black hole. The positivity of the difference between the ADM mass and the horizon mass, $M_{ADM} - M_\Delta = E > 0$, indicates that there exists an energy $E$ available to be radiated. For static black holes this result can be interpreted as a potential instability, i.e. a slightly perturbation in the initial data will lead the solution to decay to a Schwarzschild black hole.
The difference between the ADM mass of the BI black hole and the bare black hole using (47), turns out to be

\[ M_{\text{ADM}}(r_\Delta) - \frac{r_\Delta}{2} = \frac{b^2}{3} \left( r_\Delta^3 \left(1 - \sqrt{1 + \frac{a^4}{r_\Delta^4}}\right) + a^3 \arccos \left( \frac{r_\Delta^2 - a^2}{r_\Delta^2 + a^2} \right) \left( \frac{1}{\sqrt{2}} \right) \right), \tag{55} \]

the difference is greater than zero (except if \( a = 0 \) that reduces to Schwarzschild black hole). On this basis, one might conjecture that NLED black holes are unstable. However, above was shown that SSS BI solutions are stable under linear perturbations. Hence for BI black hole there is no relation between the positivity of the masses difference and stability under perturbations of the metric and the electromagnetic potential.

ACS stability conjecture has been tested for other SSS NLED solutions (in the magnetic sector), proving to be true; we conclude then that ACS unstability conjecture does not apply generically for NLED solutions.

In relation to Born-Infeld black hole stability recently was presented in [36] the analysis of quasinormal modes for the gravitational perturbations, deriving a one dimensional Schrodinger type wave equation for the axial perturbations. From the behavior of the potentials it was concluded in [36] that the EBI black holes are classically stable, this in agreement with the analysis based on Lagrangian inequalities.

Current trends in Born-Infeld black holes include the coupling of EBI fields with dilaton and axion field as well as non Abelian Born Infeld structures [37]. Another aspect that has been explored is BI black holes as gravitational lenses: deflections depend on the BI parameter \( b \) [38], [39]. Lately also has been arised interest in BI black hole thermodynamics in extra dimensions [40].

References

[1] Born, M., Infeld, L.: *Foundations of the New Field Theory*, Proc. R. Soc. (London) A144, 425-451 (1934).
[2] Hoffmann, B., Infeld, L.: *On the Choice of the Action Function in the New Field Theory*, Phys. Rev. 51 765-773, (1937).
[3] Peres, A.: *Nonlinear Electrodynamics in General Relativity* Phys. Rev. 122 273-274, (1961).
[4] Pellicer, R., Torrence, R. J.: *Nonlinear Electrodynamics and General Relativity*, J. Math. Phys. 10, 1718-1723 (1969).
[5] Plebański, J. F.: *Lectures on Non-linear Electrodynamics*, (Copenhagen, NORDITA, 1970).
[6] Salazar, H., García, A., Plebański, J.F.: *Duality rotations and type D solutions to Einstein equations with nonlinear electromagnetic sources*, J. Math. Phys. 28, 2171-2181, (1987).
[7] García, A., Salazar, H., Plebański, J. F.: Type-D solutions of the Einstein and Born-Infeld Nonlinear Electrodynamics Equations, Nuovo Cim. 84, 65-90 (1984).

[8] Dudley, A., Alarcón, S., Plebański, J. F.: Signals and discontinuities in general relativistic nonlinear electrodynamics, J. Math. Phys. 22, 2835-2848 (1981).

[9] Demianski, M.: Static Electromagnetic Geon, Found. of Phys. 16, 187-190 (1986).

[10] Heisenberg, W., Euler, H.: Folgerungen aus der Diracschen Theorie des Positrons, (1936) Zeit. f. Phys. 98 714-732. Weisskopf, W.S. On the self-energy and the electromagnetic field of the electron, Phys. Rev. 56 72 (1939).

[11] Schwinger, J.: On Gauge Invariance and Vacuum Polarization, Phys. Rev. 82 664-679 (1951).

[12] Delphenich, D. H.: Nonlinear Electrodynamics and QED, arXiv: hep-th/0309108.

[13] Gibbons, G. W.: Born-Infeld particles and Dirichlet p-branes, Nucl. Phys. B 514 603-639 (1998).

[14] Ayón-Beato, E., García, A.: Regular black hole in general relativity coupled to nonlinear electrodynamics, Phys. Rev. Lett., 80, 5056-5059 (1998); Ayón-Beato, E., García, A.: Nonsingular charged black hole solution for nonlinear source, Gen. Rel. Gravit., 31, 629-633 (1999); Ayón-Beato, E., García, A.: New regular black hole solution from nonlinear electrodynamics, Phys. Lett. B 464, 25-28, (1999).

[15] Bronnikov, K. A.: Comment on “Regular black hole in general relativity coupled to nonlinear electrodynamics”, Phys. Rev. Lett. 85, 4641 (2000).

[16] Bronnikov, K. A.: Regular magnetic black holes and monopoles from nonlinear electrodynamics, Phys. Rev. D 63, 044005 (2001).

[17] Novello, M., De Lorenzi, V. A., Salim, J. M., Klippert, R.: Geometrical Aspects of Light Propagation in Nonlinear Electrodynamics Phys. Rev. D 61 (2000) 045001.

[18] Novello, M., Perez Bergliaffa, S.E., Salim, J. M.: Singularities in general relativity coupled to nonlinear electrodynamics Class. Quant. Grav. 17, 3821-3831, (2000).

[19] Dynnikova, I.: Regular electrically charged vacuum structures with de Sitter center in Nonlinear Electrodynamics coupled to General Relativity, Class. Quant. Grav. 21, 4417-4429, (2004).
[20] Gibbons, G. W., Rasheed, D. A.: Electric-magnetic duality rotations in non-linear electrodynamics, Nucl. Phys. B 454 185 (1995).

[21] Plebański, J. F., Przanowski, M.: Duality Transformations in Electrodynamics, Int. J. Theor. Phys. 33 1535-1551, (1994).

[22] Gibbons, G. W., Hashimoto, K.: Non-linear Electrodynamics in Curved Backgrounds, JHEP 0009 013 (2000).

[23] Graves, J. C., Brill D. R.: Oscillatory Character of Reissner-Nordström Metric for an Ideal Charged Wormhole, Phys. Rev. 120 1507-1513 (1960)

[24] Bretón, N.: Geodesic structure of the Born-Infeld black hole, Class. Quant. Grav. 19, 601-612, (2002).

[25] Boillat, G.: Nonlinear Electrodynamics: Lagrangians and Equations of Motion, J. Math. Phys. 11 941-951 (1970).

[26] Wald, R.: The First Law of Black Hole Mechanics, In ‘College Park 1993, Directions in general relativity, vol. 1’ 358-366. arXiv: gr-qc/9305022

[27] Ashtekar, A., Corichi, A., Sudarsky, D.: Hairy black holes, horizon mass and solitons, Class. Quant. Grav. 18, 919-940, (2001).

[28] Heusler, M., Straumann, N.: The First law of black hole physics for a class of nonlinear matter models, Class. Quant. Grav. 10, 1299-1322, (1993).

[29] Rasheed, D. A.: Non-Linear Electrodynamics: Zeroth and First Laws of Black Hole Mechanics, arXiv: hep-th/9702087

[30] Corichi, A., Nucamendi, U., Sudarsky, D.: Einstein-Yang-Mills isolated horizons: phase space, mechanics, hair and conjectures, Phys. Rev. D 62, 044046 (2000).

[31] Ayón-Beato, E., García, A.: The Bardeen model as a nonlinear magnetic monopole, Phys. Lett. B 493, 149-152, (2000).

[32] Ashtekar, A., Fairhurst, S., Krishnan, B.: Isolated horizons: Hamiltonian evolution and the first law, Phys. Rev. D 62, 104025 (2000).

[33] Bretón, N.: Born-Infeld black hole in the isolated horizon framework, Phys. Rev. D 67, 124004 (2003).

[34] Moreno, C., Sarbach, O. Stability properties of black holes in selfgravitating nonlinear electrodynamics, Phys. Rev. D 67, 024028 (2003).

[35] Bretón, N.: Stability of nonlinear magnetic black holes, Phys. Rev. D 72, 044015 (2005).

[36] Fernando, S.: Gravitational perturbations and quasi-normal modes of charged black holes in Einstein-Born-Infeld gravity, Gen. Rel. Grav. 37 585-604 (2005), hep-th/0407062
[37] Wirschins, M., Sood, A., Kunz, J.: Non-Abelian Einstein-Born-Infeld black holes, Phys. Rev. D 63, 084002 (2001).

[38] Eiroa, E. F.: Gravitational lensing by Einstein Born Infeld Black holes, Phys. Rev. D 73, 043002 (2006).

[39] Mosquera-Cuesta, H. J., de Freitas Pacheco, J. A., Salim, J. A.: Einstein’s gravitational lensing and nonlinear electrodynamics, Int. J. Mod. Phys. A 21, 43-55 (2006).

[40] Sheykhi, A., Riazi, N.: thermodynamics of black holes in (n+1) dimensional Einstein-Born-Infeld dilaton gravity arXiv: hep-th/0610085