LEVEL ALGEBRAS AND \(s\)-LECTURE HALL POLYTOPES

FLORIAN KOHL AND MCCABE OLSEN

Abstract. Given a family of lattice polytopes, a common question in Ehrhart theory is classifying which polytopes in the family are Gorenstein. A less common question is classifying which polytopes in the family admit level semigroup algebras, a generalization of the Gorenstein property. In this article, we consider these questions for \(s\)-lecture hall polytopes, which are a family of simplices arising from lecture hall partitions. We provide a characterization of the Gorenstein property for a large subfamily of lecture hall polytopes. Additionally, we also provide a complete characterization for the level property in terms of \(s\)-inversion sequences and demonstrate some consequences of the classification result. We conclude briefly with some potential future work and extensions of these results.

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1. Introduction

Let \(P \subset \mathbb{R}^n\) be a convex lattice polytope. It is a common question in Ehrhart theory to determine if \(P\) is a Gorenstein polytope, that is the associated semigroup algebra of \(P\) is a Gorenstein algebra. Such polytopes are of interest within geometric combinatorics as \(P\) has some integer dilate \(cP\) which is a reflexive polytope [DNH97] and \(P\) has a palindromic
Ehrhart $h^*$-polynomial [Sta78]. Gorenstein polytopes are also of interest in algebraic geometry for a variety of reasons, including connections to mirror symmetry (see e.g. [Bat94] and [CLS11, Section 8.3]). Roughly speaking, a pair of reflexive lattice polytopes gives rise to a mirror pair of Calabi–Yau manifolds. We recommend [Cox15] for an excellent survey article about reflexive polytopes and their connection to mirror symmetry. Subsequently, classifications of the Gorenstein property have been extensively studied and are known for many families including order polytopes [Sta86, Hib87], twinned poset polytopes [HM16], and $r$-stable $(n, k)$-hypersimplices [HS16].

We say that $P$ is a level polytope if its associated semigroup algebra is a level algebra, a generalization of the notion of Gorenstein. This question has not been studied nearly to the degree as detecting the Gorenstein property (see e.g. [EHHSM15, HY18]). However, in addition to the independent interest in level algebras, if $P$ is level, we obtain nontrivial inequalities on the coefficients of the Ehrhart $h^*$-polynomial, which are not satisfied for general lattice polytopes.

One family of well-studied polytopes are the $s$-lecture hall polytopes. For a given $s \in \mathbb{Z}_{\geq 1}$, the $s$-lecture hall polytope is the simplex defined by

$$P_n(s) := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\}.$$  

These polytopes arise from the extensively investigated $s$-lecture hall partitions, introduced by Bousquet-Mélou and Eriksson [BME97a, BME97b]. To quote Savage and Schuster from [SS12]: “Since their discovery, lecture hall partitions and their generalizations have emerged as fundamental tools for interpreting classical partition identities and for discovering new ones.” Though many algebraic and geometric properties of $s$-lecture hall polytopes are known (see e.g. [Sav16]), the known Gorenstein results are very limited and there are no known levelness results.

Our focus is to work towards a classification of the Gorenstein and level properties in $s$-lecture hall polytopes. In particular, we provide a characterization for the Gorenstein property in the case that $s$ has at least one index $i$, $1 \leq i < n$, such that $\gcd(s_i, s_{i+1}) = 1$. We also provide a characterization for levelness which applies to all $s$-sequences in terms of $s$-inversion sequences. These main results are as follows:

**Theorem 1.1.** Let $s = (s_1, s_2, \ldots, s_n)$ be a sequence such that there exists some $1 < i \leq n$ such that $\gcd(s_{i-1}, s_i) = 1$ and define $\overline{s} = (\overline{s_1}, \ldots, \overline{s_n}) := (s_n, s_{n-1}, \ldots, s_1)$. Then $P_n(s)$ is Gorenstein if and only if there exists $c, d \in \mathbb{Z}^n$ satisfying

$$c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)$$

and

$$d_j \overline{s}_{j-1} = d_{j-1} \overline{s}_j + \gcd(\overline{s}_{j-1}, \overline{s}_j)$$

for $j > 1$ with $c_1 = d_1 = 1$.

**Theorem 1.2.** Let $s = (s_1, s_2, \ldots, s_n)$ and let $r = \max\{\text{asc}(e) : e \in I_n^{(s)}\}$. Then $P_n(s)$ is level if and only if for any $e \in I_n^{(s)}$ with $1 \leq k < r$ there exists some $e' \in I_n^{(s)}$ such that $(e + e') \in I_{n,k+1}^{(s)}$.

The structure of this note is as follows. In Section 2, we provide all necessary background, definitions, notation, and terminology for lattice polytopes and Ehrhart theory, Gorenstein algebras and level algebras, and polyhedral geometry related to lecture hall partitions. The
focus of Section 3 is proving the Gorenstein classification. In Section 4, we prove the characterization of the level property and use the characterization to arrive at several consequences of interest. We conclude in Section 5 with some potential ways to improve and extend these results and other future directions.

2. Background

In this section, we provide the necessary terminology and background literature necessary for our results. Specifically, we provide a review of lattice polytopes and Ehrhart theory, Gorenstein algebras and level algebras, and the polyhedral geometry of lecture hall partitions. Subsequently, some or all of the subsections may be safely skipped by the experts.

2.1. Lattice polytopes and Ehrhart theory. Recall that a lattice polytope $P \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Z}^n$. So we write $P = \text{conv}\{v_1, \ldots, v_r\}$, $v_i \in \mathbb{Z}^n$.

The Ehrhart polynomial of $P$ is the function $i(P, t) := \#(tP \cap \mathbb{Z}^n)$ which is a polynomial in the variable $t$ of degree $d = \dim(P)$ by a result of Ehrhart [Ehr62].

The Ehrhart series of $P$ is the rational generating function $1 + \sum_{t \geq 1} i(P, t) z^t = \frac{h^*(P, z)}{(1-z)^{d+1}}$, where the numerator is the polynomial $h^*(P, z) = \sum_{j=0}^{\dim(P)} h_j^*(P) z^j$,

which we call the Ehrhart $h^*$-polynomial of $P$. The coefficient vector $h^* = (h_0, h_1, \ldots, h_n)$ is known as the $h^*$-vector. By a result of Stanley [Sta80], we know that $h_j^*(P) \in \mathbb{Z}_{\geq 0}$ for all $j$. Many additional properties are known about Ehrhart $h^*$-polynomials (see e.g [BR07, Hib92]). Classifying the set of $h^*$-vectors is one of the most important open problems in Ehrhart theory. Therefore, inequalities for the coefficients are of special interest, see [Sta09, Sta16, Hib90, Sta91]. Hofscheier, Katthän, and Nill proved a structural result about $h^*$-vectors, see [HKN16, Thm. 3.1], where they showed that if the integer points of a lattice polytope span the integer lattice, then $h^*$ cannot have internal zeros. There are even some universal inequalities for $h^*$-vectors, see [BH17].

Given a lattice polytope $P$ with vertex set $V(P)$, define the cone over $P$ to be $\text{cone}(P) := \text{span}_{\mathbb{R}_{\geq 0}}\{(v, 1) : v \in V(P)\} \subset \mathbb{R}^n \times \mathbb{R}$.

Let $k$ be an algebraically closed field of characteristic zero. We define the affine semigroup algebra of $P$ to be $k[P] := k[\text{cone}(P) \cap \mathbb{Z}^{n+1}] = k[x^p \cdot y^m : (p, m) \in \text{cone}(P) \cap \mathbb{Z}^{n+1}] \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y]$.

This algebra is known to be a finitely generated, local $k$-algebra with a natural $\mathbb{Z}$-grading arising from the $y$-degree (see e.g. [MS05]). Moreover, we have that $k[P]$ is Cohen-Macaulay [Hoc72]. Given the observation that the lattice points $(p, m) \in \text{cone}(P) \cap \mathbb{Z}^{n+1}$ in the cone
are in clear bijection with elements in \( mP \cap \mathbb{Z}^n \), the Ehrhart polynomial is the Hilbert function for the algebra \( k[P] \).

We say that \( P \) satisfies the integer-decomposition property (or IDP for short) if for any \( x \in tP \cap \mathbb{Z}^n \), there exists \( t \) lattice points \( \{p_1, p_2, \ldots, p_t\} \in P \cap \mathbb{Z}^n \) such that \( p_1 + p_2 + \cdots + p_t = x \). In this case, we say that \( P \) has the IDP. This is equivalent to saying the semigroup algebra \( k[P] \) is generated entirely in degree 1.

Suppose that \( P \) is a simplex and has vertex set \( \{v_0, \cdots, v_d\} \). The (half-open) fundamental parallelepiped of \( P \) is the bounded region of \( \text{cone}(P) \) defined as follows

\[
\Pi_P := \left\{ \sum_{i=0}^{d} \eta_i(v_i, 1) : 0 \leq \eta_i < 1 \right\} \subset \text{cone}(P).
\]

For simplicies, we can use the fundamental parallelepiped to compute the Ehrhart \( h^* \)-polynomial. In particular, the coefficients are given by

\[
h^*_i(P) = \# \left( \{(q, i) \in \Pi_P \cap \mathbb{Z}^{n+1} \} \right),
\]

that is the number of lattice points at height \( i \) in \( \Pi_P \). For more details and exposition, the reader should consult [BR07].

2.2. Gorenstein algebras and level algebras. We now provide a brief review of Gorenstein and level algebras in general, as well as in the special case of semigroup algebras of polytopes. For additional details and expositions, the reader should consult [BH93, Sta96] as references.

Let \( k \) be an algebraically closed field of characteristic zero. Let \( \mathcal{R} = \bigoplus_{i \in \mathbb{Z}} \mathcal{R}_i \) be a finitely generated \( \mathbb{Z} \)-graded \( k \)-algebra of Krull dimension \( d \). Suppose that \( \mathcal{R} \) is local and Cohen-Macaulay. The canonical module of \( \mathcal{R} \), \( \omega_\mathcal{R} \), is the unique module (up to isomorphism) such that \( \text{Ext}^d_\mathcal{R}(k, \omega_\mathcal{R}) = k \) and \( \text{Ext}^i_\mathcal{R}(k, \omega_\mathcal{R}) = 0 \) when \( i \neq d \). We say that \( \mathcal{R} \) is Gorenstein if \( \omega_\mathcal{R} \cong \mathcal{R} \) as an \( \mathcal{R} \)-module, or equivalently is \( \omega_\mathcal{R} \) is generated by a single element.

One generalization of Gorenstein of which is also of interest is level. We say that \( \mathcal{R} \) is level if \( \omega_\mathcal{R} \) is generated by elements of the same degree, that is \( \omega_\mathcal{R} \) has minimal generating set \( \{\sigma_1, \ldots, \sigma_j\} \) such that \( \text{deg}(\sigma_1) = \text{deg}(\sigma_2) = \cdots = \text{deg}(\sigma_j) \). An equivalent formulation of the level property is often more fruitful for computational purposes. Recall for any \( \mathcal{R} \)-module \( M \), the socle of \( M \) is \( \text{soc}(M) := \{u \in M : R_+ u = 0\} \) where \( R_+ \) is the unique maximal ideal of \( \mathcal{R} \). It is equivalent to say that \( \mathcal{R} \) is level if for any homogeneous system of parameters \( \theta_1, \ldots, \theta_d \) of \( \mathcal{R} \), all the elements of the graded vector space \( \text{soc}(\mathcal{R}/(\theta_1, \ldots, \theta_d)) \) are of the same degree (see [Sta96, Chapter III, Proposition 3.2]).

In the case of \( k[P] \) for some lattice polytope \( P \), Stanley [Sta78] explicitly describes the canonical module as

\[
\omega_{k[P]} = k[\text{cone}(P)^\circ \cap \mathbb{Z}^{n+1}]
\]

where \( \text{cone}(P)^\circ \) denotes the relative interior of the cone. Subsequently, it is equivalent to say that \( k[P] \) is Gorenstein if there exists \( c \in \mathbb{Z}^{n+1} \) such that

\[
c + (\text{cone}(P) \cap \mathbb{Z}^{n+1}) = \text{cone}(P)^\circ \cap \mathbb{Z}^{n+1},
\]

and \( c \) is called the Gorenstein point of \( \text{cone}(P) \). Moreover, note that \( P \) is Gorenstein if and only if \( h^*(P, z) \) is a palindromic polynomial, see [Sta78, Thm. 4.4]. We can also provide a more concrete description of the level property. We say \( k[P] \) is level if there exists some
finite collection $c_1, \ldots, c_m \in \mathbb{Z}^{n+1}$ where

$$\sum_{i=1}^{m} (c_i + (\text{cone}(P) \cap \mathbb{Z}^{n+1})) = \text{cone}(P)^{\circ} \cap \mathbb{Z}^{n+1},$$

and the additional restriction that $c_{1_{n+1}} = c_{2_{n+1}} = \cdots = c_{m_{n+1}}$.

2.3. Polyhedral geometry of lecture hall partitions. In this subsection, we briefly recall relevant properties and results on lecture hall cones and lecture hall polytopes. For a more in-depth overview of some of these results and many other, the reader should consult the excellent survey of Savage [Sav16].

Let $s = (s_1, s_2, \ldots, s_n) \in \mathbb{Z}_{\geq 1}^n$ be a sequence. Given any $s$-sequence, define the $s$-lecture hall partitions to be the set

$$L_n^{(s)} := \left\{ \lambda \in \mathbb{Z}^n : 0 \leq \lambda_1 \leq s_1 \lambda_2 \leq s_2 \leq \cdots \leq \lambda_n \leq s_n \right\}.$$

We can associate to the set of $s$-lecture hall partitions several discrete geometric objects. In particular, the $s$-lecture hall polytope and the $s$-lecture hall cone. For a given $s$, the $s$-lecture hall polytope is defined as

$$P_n^{(s)} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \leq 1 \right\} = \text{conv}\{(0, \ldots, 0), (0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n) \text{ for } 1 \leq i \leq n\}.$$

See Figure 1 for two three-dimensional examples of lecture hall polytopes.

The Ehrhart $h^*$-polynomials of $P_n^{(s)}$ have been completely classified. Given $s$, the set of $s$-inversion sequences is defined as $I_n^{(s)} := \{e \in \mathbb{Z}_{\geq 0}^n : 0 \leq e_i < s_i \}$. For a given $e \in I_n^{(s)}$, the ascent set of $e$ is

$$\text{Asc}(e) := \left\{ i \in \{0, 1, \ldots, n-1\} : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

with the convention $s_0 = 1$ and $e_0 = 0$, and $\text{asc}(e) := \# \text{Asc}(e)$. With these definitions, we can give the explicit formulation for the Ehrhart $h^*$-polynomials.
Theorem 2.1 ([SS12, Theorem 8]). For a given $s \in \mathbb{Z}_{\geq 1}^n$,
\[ h^*(P_n^{(s)}, z) = \sum_{e \in I^*_n} z^{\text{asc}(e)}. \]

The polynomials $h^*(P_n^{(s)}, z)$ generalize Eulerian polynomials and are known as the $s$-Eulerian polynomials, as the sequence $s = (1, 2, \ldots, n)$ gives rise to
\[ \sum_{e \in I^*_n} z^{\text{asc}(e)} = \sum_{\pi \in S_n} z^{\text{des}(\pi)} = A_n(z) \]
which is the usual Eulerian polynomial. These polynomials are known to be real-rooted and, hence, unimodal [SV15].

These polytopes have been the subject of much additional study (see e.g. [HOT16, HOT17, PS13, PS13, SV12]). Of particular interest are algebraic and geometric structural results such as Gorenstein and IDP results. The second author along with Hibi and Tsuchiya in [HOT16] provide some Gorenstein results in particular circumstances. Additionally, the following theorem for IDP holds.

Theorem 2.2 ([BS]). $P_n^{(s)}$ has the IDP.

A proof for the case of monotonic $s$-sequences was given by the second author with Hibi and Tsuchiya in [HOT16] which Brändén and Solus [BS] show can be extended to any $s$ when they prove that all lecture hall order polytopes have the IDP. Moreover, in [BL16, Conj 5.4] it is conjectured that for any $s$, $P_n^{(s)}$ possesses a regular, unimodular triangulation.

For a given $s$, the $s$-lecture hall cone is defined to be
\[ C_n^{(s)} := \left\{ \lambda \in \mathbb{R}^n : 0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \cdots \leq \frac{\lambda_n}{s_n} \right\}. \]

These objects are related to $s$-lecture hall polytopes in that $C_n^{(s)}$ arises as the vertex cone of $P_n^{(s)}$ at the origin $(0, \ldots, 0)$. It is important to realize that $C_n^{(s)}$ is not the same object as cone($P_n^{(s)}$). In fact, $C_n^{(s)}$ arises as a nontrivial quotient of cone($P_n^{(s)}$). The $s$-lecture hall cones have been studied extensively (see e.g. [BBK+15, BBK+16, Ols17]) and the following Gorenstein results for the lecture hall cones are particularly of interest for our purposes.

Theorem 2.3 ([BBK+15, Corollary 2.6], [BME97b, Proposition 5.4]). For a positive integer sequence $s$, the $s$-lecture hall cone $C_n^{(s)}$ is Gorenstein if and only if there exists some $c \in \mathbb{Z}^n$ satisfying
\[ c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j) \]
for $j > 1$, with $c_1 = 1$.

Moreover, in the case of $s$-sequences where $\gcd(s_{i-1}, s_i) = 1$ holds for all $i$, we have a refinement to this theorem. We say that $s$ is $u$-generated by a sequence $u$ of positive integers if $s_2 = u_1 s_1 - 1$ and $s_{i+1} = u_i s_i - s_{i-1}$ for $i > 1$.

Theorem 2.4 ([BBK+15, Theorem 2.8], [BME97b, Proposition 5.5]). Let $s = (s_1, \cdots, s_n)$ be a sequence of positive integers such that $\gcd(s_{i-1}, s_i) = 1$ for $1 \leq i < n$. Then $C_n^{(s)}$ is Gorenstein if and only if $s$ is $u$-generated by some sequence $u = (u_1, u_2, \cdots, u_{n-1})$ of positive integers. When such a sequence exists, the Gorenstein point $c$ for $C_n^{(s)}$ is defined by $c_1 = 1$, $c_2 = u_1$, and for $2 \leq i < n$, $c_{i+1} = u_i c_i - c_{i-1}$.
3. Gorenstein lecture hall polytopes

In this section, we will give a characterization of Gorenstein s-lecture hall polytopes with the restriction that there exists some index \( i \) such that \( 1 \leq i < n \) satisfying \( \gcd(s_i, s_{i+1}) = 1 \). To give such a classification, we will analyze the structure of \( \text{cone}(P_n^{(s)}) \). The following lemma gives a half-space inequality description of this cone:

**Lemma 3.1.** With the notation from above, we have

\[
\text{cone}(P_n^{(s)}) = \{ \lambda \in \mathbb{R}^{n+1} : A\lambda \geq 0 \},
\]

where

\[
A = \begin{pmatrix}
\frac{1}{s_1} & 0 & 0 & \ldots & 0 \\
\frac{1}{s_1} & \frac{1}{s_2} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \frac{1}{s_{n-1}} & \frac{1}{s_n} & 0 \\
0 & \ldots & 0 & \frac{1}{s_n} & 1
\end{pmatrix}.
\]

Moreover, this cone is simplicial.

**Proof.** This directly follows from the half space description of \( P_n^{(s)} \). Assume that \( P_n^{(s)} = \{ \lambda : M\lambda \geq b \} \), where \( b = (0, 0, \ldots, 0, 1)^t \). Then on height \( \lambda_{n+1} \), we have \( M\lambda \geq \lambda_{n+1}b \). The statement now follows. \( \square \)

We now recall a technical lemma which we will use in our characterization.

**Lemma 3.2** ([BBK+15, Lemma 2.5]). Let \( \mathcal{C} = \{ \lambda \in \mathbb{R}^n : A\lambda \geq 0 \} \) be a full dimensional simplicial polyhedral cone where \( A \) is a rational matrix and denote the rows of \( A \) as linear functionals \( \alpha^1, \ldots, \alpha^n \) on \( \mathbb{R}^n \). For \( j = 1, \ldots, n \), let the projected lattice \( \alpha^j(\mathbb{Z}^n) \subset \mathbb{R} \) be generated by the number \( q_j \in \mathbb{Q}_{>0} \), so \( \alpha^j(\mathbb{Z}^n) = q_j\mathbb{Z} \).

1. The \( \mathcal{C} \) is Gorenstein if and only if there exists \( c \in \mathbb{Z}^n \) such that \( \alpha^j(c) = q_j \) for all \( j = 1, \ldots, n \).
2. Define a point \( \tilde{c} \in \mathcal{C} \cap \mathbb{Q}^n \) by \( \alpha^j(\tilde{c}) = q_j \) for all \( j = 1, \ldots, n \). Then \( \mathcal{C} \) is Gorenstein if and only if \( \tilde{c} \in \mathbb{Z}^n \).

We can now prove our main result.

**Proof of Theorem 1.1.** Let \( P_n^{(s)} \) be Gorenstein with Gorenstein point \( c \in \text{cone}(P_n^{(s)}) \). Then \( c \) has lattice distance 1 to all facets of \( \text{cone}(P_n^{(s)}) \). For a vertex \( v \), the vertex cone \( T_v(P_n^{(s)}) \) of \( P_n^{(s)} \) at \( v \) is obtained from \( \text{cone}(P_n^{(s)}) \) by quotienting \( \text{cone}(P_n^{(s)}) \) by \( (v, 1) \). The image of \( c \) under this quotient map can be seen to be the Gorenstein point of \( T_v(P_n^{(s)}) \), since this point has lattice distance 1 to all facets showing that all vertex cones are Gorenstein as well.

In particular, the vertex cone at the vertex \((0, 0, \ldots, 0)\) is of the form

\[
0 \leq \frac{\lambda_1}{s_1} \leq \cdots \leq \frac{\lambda_n}{s_n}
\]

and it is known by Theorem 2.3 that this cone is Gorenstein if and only if there exists a \( c \in \mathbb{Z}^n \) satisfying the recurrence above. Likewise, it straightforward to see that \( T_s(P_n^{(s)}) \cong T_0(P_n^{(s)}) \), where \( \cong \) means equivalence after an affine, unimodular transformation, and where \( T_0(P_n^{(s)}) \) is of the form

\[
0 \leq \frac{\lambda_1}{s_1} \leq \cdots \leq \frac{\lambda_n}{s_n}
\]
which is Gorenstein if and only if there exists a $d \in \mathbb{Z}^n$ satisfying the recurrence above. Therefore, this is certainly a necessary condition.

To show the other direction we employ Lemma 3.2. Since the characterization given in Lemma 3.2 essentially requires finding integer solutions to linear equations, we first deduce some divisibility conditions that will later prove useful.

Assume that we have $c, d \in \mathbb{Z}^n$ as described above and suppose that $\gcd(s_i, s_{i+1}) = 1$. Note that this gives us the following
\[ c_n s_{n-1} = c_{n-1} s_n + \gcd(s_{n-1}, s_n) \]
and
\[ d_2 s_1 = d_1 s_2 + \gcd(s_2, s_1) \quad \text{and} \quad d_2 s_n = d_1 s_{n-1} + \gcd(s_n, s_{n-1}) \]
where $d_1 = 1$. Subtracting both equalities, we get
\[ (d_2 + c_{n-1}) s_n = (1 + c_n) s_{n-1}. \]
Repeating the above process, we also have
\[ (d_3 + c_{n-2}) s_{n-1} = (d_2 + c_{n-1}) s_{n-2} \]
and in general for some $k$, we have
\[ (d_{k+1} + c_{n-k}) s_{n-k+1} = (d_k + c_{n-k+1}) s_{n-k}. \]
If we know that $i = n - k$, then $\gcd(s_{n-k}, s_{n-k+1}) = 1$ and we can deduce the division requirement $s_{n-k+1}(d_k + c_{n-k+1})$.

By Lemma 3.2, we get that a cone of the form $A \lambda \geq 0$ is Gorenstein if and only if there is a point $c$ such that $\alpha^i(c) = q_i$ for all $i$, where $\alpha^i$ is the $i$th row of $A$ and $q_i$ is defined as in Lemma 3.2. Lemma 3.1 explicitly describes the rows. From this it follows that
\[ q_1 = \frac{1}{s_1}, q_2 = \frac{1}{\text{lcm}(s_1, s_2)}, \ldots, q_n = \frac{1}{\text{lcm}(s_{n-1}, s_{n-2})}, q_{n+1} = \frac{1}{s_n}. \]
Now we need to find a point $c \in \mathbb{Z}$ such that $\alpha^i(c) = q_i$. This directly implies that $c_1 = 1$ and that
\[ c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j) \]
for $2 \leq j \leq n$. These conditions are all satisfied by assumption. However, we also need to satisfy the condition
\[ \frac{-c_n}{s_n} + 1 c_{n+1} = \frac{1}{s_n} \iff c_{n+1} s_n = 1 + c_n. \]
Now, we note that from Equation (1) it follows that
\[ s_n = \frac{(1 + c_n)}{(d_2 + c_{n-1})} s_{n-1}, \]
so we can rewrite
\[ c_{n+1} s_{n-1} = d_2 + c_{n-1}. \]
We can iterate these substitutions repeatedly to arrive at the equality
\[ c_{n+1} s_{n-k+1} = d_k + c_{n-k+1} \]
However, since $s_{n-k+1}(d_k + c_{n-k+1})$, $c_{n+1}$ is an integer. Here we are implicitly using that $c, d \in \mathbb{Z}_{\geq 1}$, which follows from the above condition. So we are done. This also shows that $s_{n-j+1}(d_j + c_{n-j+1})$ for all $j$. \qed
Remark 3.3. We mentioned before that this theorem applies to a large subfamily of \( s \)-lecture hall polytopes. This remark will make this statement more precise. Given two positive integers \( a \) and \( b \), the probability that \( \gcd(a, b) = 1 \) converges to \( \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \), where \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) is the Riemann \( \zeta \)-function. Heuristically, assuming that these events are independent (which they are not), we then get that roughly \( (1 - \frac{6}{\pi^2})^{n-1} \)-percent of sequences fall within the range of our theorem. Computer simulations suggest that this estimate is fairly precise for large dimensions.

Remark 3.4. In [BBK+15, Cor 2.7], the authors remark that if one truncates a sequences \( s \) with corresponding point \( c = (c_1, c_2, \ldots, c_n) \), the truncated sequence \( (s_1, s_2, \ldots, s_t) \) also has a corresponding point \( (c_1, c_2, \ldots, c_n) \). However, the direct analogue of this statement is not true in our case. The sequence \( (8, 6, 10, 10, 5, 2, 4) \) gives rise to a Gorenstein lecture hall polytope, whereas \( (8, 6, 10, 10, 5) \) does not give rise to a Gorenstein lecture hall polytope, since it has 39 interior lattice points. However, we can make the following statement: If \( (s_1, s_2, \ldots, s_n) \) has corresponding integer points \( c, d \), then the truncated sequence \( (s_1, s_2, \ldots, s_t) \) has corresponding integer points \( \tilde{c} = (c_1, c_2, \ldots, c_t), \tilde{d} = (d_{n-i+1}, \ldots, d_n) \) provided \( d_{n-i+1} = 1 \).

This theorem along with Theorem 2.4 implies the following more specialized characterization.

Corollary 3.5. Let \( s = (s_1, s_2, \ldots, s_n) \) be a sequence of positive integers satisfying \( \gcd(s_i, s_{i+1}) = 1 \) for all \( 1 \leq i < n \). Then \( P_n^{(s)} \) is Gorenstein if and only if \( s \) and \( \tilde{s} \) are \( u \)-generated sequences.

We have the following corollary on the level of \( s \)-Eulerian polynomials.

Corollary 3.6. Let \( s = (s_1, s_2, \ldots, s_n) \) be a sequence of positive integers satisfying \( \gcd(s_i, s_i) = 1 \) for some \( 1 < i \leq n \). The \( s \)-Eulerian polynomial is palindromic if and only if there exists \( c, d \in \mathbb{Z}^n \) satisfying

\[
c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1}, s_j)
\]

and

\[
d_j \tilde{s}_{j-1} = d_{j-1} \tilde{s}_j + \gcd(\tilde{s}_{j-1}, \tilde{s}_j)
\]

for \( j > 1 \) with \( c_1 = d_1 = 1 \).

Table 1 contains some examples of palindromic \( s \)-Eulerian polynomials.

4. Characterization of level lecture hall polytopes

We now give a characterization of which \( s \)-sequences admit level \( P_n^{(s)} \), which is given in terms of the structure of \( s \)-inversion sequences. It will be useful to define some new notation. Let \( I_{n,k}^{(s)} := \{ e \in I_n^{(s)} : \asc(e) = k \} \). Moreover, we define addition of inversion sequences as follows. Given \( e, e' \in I_n^{(s)} \) with \( e = (e_1, e_2, \ldots, e_n) \) and \( e' = (e_1', e_2', \ldots, e_n') \), then \( e + e' = (e_1 + e_1' \mod s_1, e_2 + e_2' \mod s_2, \ldots, e_n + e_n' \mod s_n) \).

4.1. Proof of characterization. Proving this characterization relies on understanding the link between the arithmetic structure of inversion sequences and the semigroup structure of lattice points in \( \Pi_{P_n^{(s)}} \). To fully understand and exploit this connection, we will need several lemmas.
|   | sequence \( s \) | corresponding \( c \) | corresponding \( d \) | \( s \)-Eulerian polynomial |
|---|-----------------|-----------------|-----------------|-----------------|
| (i) | \((2, 1, 3, 2, 1)\) | \((1, 1, 4, 3, 2)\) | \((1, 3, 5, 2, 5)\) | \(1 + 5z + 5z^2 + z^3\) |
| (ii) | \((3, 2, 3, 1, 2)\) | \((1, 1, 2, 1, 3)\) | \((1, 1, 4, 3, 5)\) | \(1 + 9z + 16z^2 + 9z^3 + z^4\) |
| (iii) | \((1, 4, 3, 2, 3)\) | \((1, 5, 4, 3, 5)\) | \((1, 1, 2, 3, 1)\) | \(1 + 16z + 38z^2 + 16z^3 + z^4\) |
| (iv) | \((3, 5, 2, 3, 1)\) | \((1, 2, 1, 2, 1)\) | \((1, 4, 3, 8, 5)\) | \(1 + 2v + 48z^2 + 20z^3 + z^4\) |
| (v) | \((1, 2, 3, 4, 5)\) | \((1, 3, 5, 7, 9)\) | \((1, 1, 1, 1, 1)\) | \(1 + 26v + 66z^2 + 26z^3 + z^4\) |
| (vi) | \((1, 2, 5, 8, 3)\) | \((1, 3, 8, 13, 5)\) | \((1, 3, 2, 1, 1)\) | \(1 + 50v + 138z^2 + 50z^3 + z^4\) |
| (vii) | \((4, 3, 2, 5, 3)\) | \((1, 1, 1, 3, 2)\) | \((1, 2, 1, 2, 3)\) | \(1 + 30v + 149z^2 + 149z^3 + 30z^4 + z^5\) |
| (viii) | \((4, 7, 3, 2, 3)\) | \((1, 2, 1, 1, 2)\) | \((1, 1, 2, 5, 3)\) | \(1 + 43v + 208z^2 + 208z^3 + 43z^4 + z^5\) |
| (ix) | \((5, 9, 4, 3, 2)\) | \((1, 2, 1, 1, 1)\) | \((1, 2, 3, 7, 6)\) | \(1 + 82v + 457z^2 + 457z^3 + 82z^4 + z^5\) |
| (x) | \((3, 5, 12, 7, 2)\) | \((1, 2, 5, 3, 1)\) | \((1, 4, 7, 3, 2)\) | \(1 + 175v + 1084z^2 + 1084z^3 + 175z^4 + z^5\) |
| (xi) | \((3, 11, 8, 5, 2)\) | \((1, 4, 3, 2, 1)\) | \((1, 3, 5, 7, 2)\) | \(1 + 180v + 1139z^2 + 1139z^3 + 180z^4 + z^5\) |
| (xii) | \((2, 7, 5, 10, 4)\) | \((1, 4, 3, 7, 3)\) | \((1, 3, 2, 3, 1)\) | \(1 + 181v + 1218z^2 + 1218z^3 + 181z^4 + z^5\) |
| (xiii) | \((3, 8, 13, 5, 2)\) | \((1, 3, 5, 2, 1)\) | \((1, 3, 8, 5, 2)\) | \(1 + 213v + 1346z^2 + 1346z^3 + 213z^4 + z^5\) |

**Table 1. Palindromic \( s \)-Eulerian Polynomials**

**Lemma 4.1.** Let \( V(P_n^{(s)}) = \{v_0, \ldots, v_n\} \) denote the set of vertices of \( P_n^{(s)} \). Let \( \mathcal{D}_n^{(s)} := (P_n^{(s)} \cap \mathbb{Z}^n) - V(P_n^{(s)}) \) There is an explicit bijection

\[
\varphi : \mathcal{D}_n^{(s)} \rightarrow \Gamma_{n,1}^{(s)}
\]

where \( \varphi(\lambda_1, \ldots, \lambda_n) \mapsto (e_1, \ldots, e_n) \) by \( e_i = s_i - \lambda_i \pmod{s_i} \).
Proof. Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}^{(s)}_n \). We have that

\[
0 \leq \frac{\lambda_1}{s_1} \leq \frac{\lambda_2}{s_2} \leq \ldots \leq \frac{\lambda_n}{s_n} \leq 1
\]

Note that this means that \( \lambda_i \leq s_i \) for all \( i \) and if \( \lambda_i = s_i \) then \( \lambda_j = s_j \) for all \( i \leq j \leq n \).

Additionally, note that the vertices of \( P^{(s)}_n \) are precisely the lattice points of the form

\[
(0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n)
\]

So, then \( \lambda \) can be expressed as the following:

\[
\lambda = (0, \ldots, 0, a_i, a_{i+1}, \ldots, a_j, s_{j+1}, \ldots, s_n)
\]

where each \( 0 < a_k < s_k \).

If we apply our map \( \lambda_i \mapsto s_i - \lambda_i \mod s_i \), we get the inversion sequence

\[
e = (0, 0, \ldots, 0, s_i - a_i, s_{i+1} - a_{i+1}, \ldots, s_j - a_j, 0, \ldots, 0)
\]

It is left to verify that \( e \in I^{(s)}_{n,1} \). Since we had that

\[
0 < a_i s_i \leq a_{i+1} s_{i+1} \leq \ldots \leq a_j s_j < 1
\]

which holds if and only if

\[
1 > \frac{s_i - a_i}{s_i} \geq \frac{s_{i+1} - a_{i+1}}{s_{i+1}} \geq \ldots \geq \frac{s_j - a_j}{s_j} > 0
\]

which means that \( e \) contains exactly one ascent at position \( i - 1 \).

This process is certainly reversible, so we have a bijection. \( \square \)

Note that \( \mathcal{P}^{(s)}_n \) is precisely the elements at height 1 in \( \Pi_{P^{(s)}_n} \). We can extend this bijection to apply to elements of \( I^{(s)}_{n,k} \) and all elements of \( P^{(s)}_n \) is the following manner.

**Lemma 4.2.** Let \( e = (e_1, e_2, \ldots, e_n) \in I^{(s)}_{n,k} \) and suppose that the \( k \) ascents at positions \( i_1, i_2, \ldots, i_k \). There is a bijective correspondence between \( e \) and lattice points \( \lambda = (\lambda_1, \ldots, \lambda_n, k) \in \Pi_{P^{(s)}_n} \cap \mathbb{Z}^{n+1} \). Suppose that \( i_\ell < j \leq i_{\ell+1} \), then we map \( e_j \mapsto \lambda_j \) by

\[
\lambda_j = \ell \cdot s_j - e_j
\]

and \( \lambda_j = 0 \) if \( 1 \leq j \leq i_1 \). Moreover, addition in the semigroup corresponds to entry-wise addition of the inversion sequences modulo \( s_i \) in the \( i \)th position. That is, any decomposition of \( \lambda \) as a sum of elements of height one in \( \Pi_{P^{(s)}_n} \) is consistent with the sum of inversion sequences.

**Remark 4.3.** By entry-wise addition of the inversion sequences modulo \( s_i \) in the \( i \)th position, we mean that we pick the unique representative of this equivalence class in \( \{0, 1, \ldots, s_i - 1\} \).

**Proof of Lemma 4.2.** It is clear that this map is injective. We must verify the following:

(A) The image of \( \lambda \) under this map is an element of \( \Pi_{P^{(s)}_n} \).

(B) Entry-wise addition of inversion sequences is consistent with addition in the semigroup.
To show (A), note that it is clear that $\lambda$ is at height $k$ in $\mathbb{R}^{n+1}$. Moreover, it is even clear that $(\lambda_1, \ldots, \lambda_n) \in k \cdot P_n^{(s)} \cap \mathbb{Z}^n$, as if $i_t < t < i_{t+1}$ then $\frac{e_t}{s_t} \geq \frac{e_{t+1}}{s_{t+1}}$ implies that

$$\frac{\ell \cdot s_t - e_t}{k \cdot s_t} \leq \frac{\ell \cdot s_{t+1} - e_{t+1}}{k \cdot s_{t+1}}$$

if $t = i_{t+1}$ we have

$$\frac{\ell \cdot s_t - e_t}{k \cdot s_t} \leq \frac{(\ell + 1) \cdot s_{t+1} - e_{t+1}}{k \cdot s_{t+1}}$$

immediate from $e_{t+1} < s_{t+1}$.

To verify that $\lambda$ is in fact in $\Pi_{P_n^{(s)}}$, we must show that neither of the following hold:

(i) $(\lambda_1, \ldots, \lambda_n) \in (k - 1) \cdot P_n^{(s)} \cap \mathbb{Z}^n$

(ii) $\lambda = \lambda' + \nu$ where $(\lambda_1', \ldots, \lambda_n') \in (k - 1) \cdot P_n^{(s)} \cap \mathbb{Z}^n$ and $\nu$ is a vertex of $P_n^{(s)}$.

Note that (i) is impossible as we have $\lambda_n = k \cdot s_n - e_n > (k - 1) s_n$ as $e_n < s_n$. For (ii), suppose that we write $\lambda = \lambda' + \nu$, where $\nu = (0, 0, \ldots, 0, s_{j+1}, \ldots, s_n)$ with $0 \leq j < n$. There are two possible cases: $j \in \text{Asc}(e)$ or $j \not\in \text{Asc}(e)$. If $j \in \text{Asc}(e)$, then we have $\frac{e_j}{s_j} < \frac{e_{j+1}}{s_{j+1}}$. Consider $\lambda'$ and suppose that

$$(\lambda_1', \ldots, \lambda_n') = (\lambda_1, \ldots, \lambda_j, \lambda_{j+1} - s_{j+1}, \ldots, \lambda_n - s_n) \in (k - 1) \cdot P_n^{(s)} \cap \mathbb{Z}^n$$

Given that $\lambda_j = (p - 1) \cdot s_j - e_j$ and $\lambda_{j+1} = p \cdot s_{j+1} - e_{j+1}$ where $j$ is the $p$th ascent, we have that the following inequality must hold:

$$\frac{(p - 1) \cdot s_j - e_j}{(k - 1) s_j} \leq \frac{p \cdot s_{j+1} - e_{j+1} - s_{j+1}}{(k - 1) s_{j+1}} = \frac{(p - 1) \cdot s_{j+1} - e_{j+1}}{(k - 1) s_{j+1}}.$$

However, this is equivalent to $\frac{e_j}{s_j} \geq \frac{e_{j+1}}{s_{j+1}}$ so this cannot occur. If $j \not\in \text{Asc}(e)$, say that $j > i_p$, the location of the $p$th ascent, so $\lambda_j = p \cdot s_j - e_j$ and $\lambda_{j+1} = p \cdot s_{j+1} - e_{j+1}$. For $(\lambda_1', \ldots, \lambda_n') \in (k - 1) \cdot P_n^{(s)} \cap \mathbb{Z}^n$, the following inequality must hold

$$\frac{p \cdot s_j - e_j}{(k - 1) s_j} \leq \frac{p \cdot s_{j+1} - e_{j+1} - s_{j+1}}{(k - 1) s_{j+1}} = \frac{(p - 1) \cdot s_{j+1} - e_{j+1}}{(k - 1) s_{j+1}}.$$

This inequality is equivalent to $\frac{e_j}{s_j} \geq \frac{e_{j+1}}{s_{j+1}} + 1$ which is a contradiction to $e_j < s_j$.

Therefore, we have shown (A). Note that this is sufficient for showing the bijection, as the map is clearly injective and the sets are of the same cardinality by previous work of Savage and Schuster [SS12]. That said, the bijection can also be realized through the fact that this map can clearly be reversed. In particular, suppose that $\lambda = (\lambda_1, \ldots, \lambda_n, k) \in \Pi_{P_n^{(s)}}$, we get our inversion sequence $e$ by

$$e_i = -\lambda_i \mod s_i.$$

Note that this inversion sequence will have precisely $k$ ascents and moreover the $p$th ascent in the sequence will occur at $i$ precisely when $(p - 1) \cdot s_i \leq \lambda_i < p \cdot s_i$ and $p \cdot s_{i+1} \leq \lambda_{i+1} < (p + 1) \cdot s_{i+1}$ for some $1 \leq p \leq k - 1$. This is the exact reversal of the constructive map form inversion sequences to lattice points is $\Pi_{P_n^{(s)}}$. 

To show (B), suppose that we have \( f \in \Pi_{n,k-1}^{(s)} \) and \( g \in \Pi_{n,1}^{(s)} \) such that \( f + g = e \in \Pi_{n,k}^{(s)} \). So we have
\[
f = (f_1, \ldots, f_{j-1}, f_j, \ldots, f_h, f_{h+1}, \ldots, f_n)
\]
and
\[
g = (0, \ldots, 0, g_j, \ldots, g_h, 0, \ldots, 0)
\]
and
\[
e = (f_1, \ldots, f_{j-1}, (f_j + g_j) \mod s_j, \ldots, (f_h + g_h) \mod s_h, f_{h+1}, \ldots, f_n).
\]
Consider the corresponding lattice points for \( f \) and \( g \) in \( \Pi_{n}^{(s)} \):
\[
\lambda_f = (\lambda_{f_1}, \ldots, \lambda_{f_{j-1}}, \lambda_{f_j} + \lambda_{g_j}, \ldots, \lambda_{f_h} + \lambda_{g_h}, \lambda_{f_{h+1}} + s_{h+1}, \ldots, \lambda_{f_n} + s_n, k - 1)
\]
and
\[
\lambda_g = (0, \ldots, 0, \alpha_{g_j}, \lambda_{g_{h+1}}, \ldots, \lambda_{g_n}, s_{h+1}, \ldots, s_n, 1).
\]
Adding these lattice points in the semigroup yields
\[
\lambda_f + \lambda_g = (\lambda_{f_1}, \ldots, \lambda_{f_{j-1}}, \lambda_{f_j} + \lambda_{g_j}, \ldots, \lambda_{f_h} + \lambda_{g_h}, \lambda_{f_{h+1}} + s_{h+1}, \ldots, \lambda_{f_n} + s_n, k).
\]
We have two possible cases: either \( \lambda_f + \lambda_g \in \Pi_{n}^{(s)} \) or \( \lambda_f + \lambda_g \not\in \Pi_{n}^{(s)} \).

If \( \lambda_f + \lambda_g \in \Pi_{n}^{(s)} \), we consider the reverse map which will give the inversion sequence
\[
(\ldots, \lambda_{f_{j-1}} \mod s_{j-1}, -\lambda_{f_j} \mod s_j, \ldots, -(\lambda_{f_h} + \lambda_{g_h}) \mod s_h, -\lambda_{f_{h+1}} + s_{h+1} \mod s_{h+1}, \ldots)
\]
and this inversion sequence is precisely \( e = f + g \), which is as desired.

Now suppose that \( \lambda_f + \lambda_g \not\in \Pi_{n}^{(s)} \). Note that we can express \( \lambda_f + \lambda_g = \lambda' + \sum_{i=1}^{n} \alpha_i v_i \) where \( \lambda' \in \Pi_{n}^{(s)} \), there is at least one \( \alpha_i \neq 0 \), \( \alpha_i \in \mathbb{Z}_{\geq 1} \), and \( \lambda' \) is at height \( r < k \). Additionally, given that \( v_i = (0, \ldots, 0, s_i, s_{i+1}, \ldots, s_n) \), it is clear that \( \lambda_f + \lambda_g \) maps to the same inversion sequence as \( \lambda' \) by definition of the inverse map. This implies that \( e \) maps to \( \lambda' \) and thus \( e \in \Pi_{n,r}^{(s)} \) for \( r < k \), which contradicts our initial assumption. \( \square \)

Remark 4.4. We should note that in the proof that addition is compatible, we only consider inversion sequences \( f \in \Pi_{n,k-1}^{(s)} \) and \( g \in \Pi_{n,1}^{(s)} \) such that \( f + g \in \Pi_{n,k}^{(s)} \), as this is the requirement of staying inside the fundamental parallelepiped. However, this need not always be the case. If \( f + g \in \Pi_{n,\ell}^{(s)} \) for some \( \ell \leq k - 1 \), the addition of the sequences is still consistent with addition in the semigroup, but this occurrence is precisely when \( \lambda_f + \lambda_g \not\in \Pi_{n}^{(s)} \). In particular, \( \lambda_f + \lambda_g = \lambda_{f+g} = (0, \ldots, 0, k - \ell) \), which lies in the equivalence class \( \lambda_{f+g} \), but is not the representative in \( \Pi_{n}^{(s)} \).

Remark 4.5. One could rephrase the statement to say that addition of inversion sequences is compatible with addition of lattice points in the semigroup modulo the equivalence class given by the fundamental parallelepiped.

With this understanding of the arithmetic structure of \( \Pi_{n,k}^{(s)} \), we can now give a proof of the characterization.

Proof of Theorem 1.2. First recall that if \( \mathcal{R} \) is a graded, local, Cohen-Macaulay algebra with \( \dim(\mathcal{R}) = d \), then \( \mathcal{R} \) is level if for some homogeneous system of parameters \( \theta_1, \ldots, \theta_d \) of \( \mathcal{R} \), all the elements of the graded vector space \( \text{soc}(\mathcal{R}/(\theta_1, \ldots, \theta_d)) \) are of the same degree. Consider the semigroup algebra \( k[\Pi_{n}^{(s)}] := k[\text{cone}(\Pi_{n}^{(s)}) \cap \mathbb{Z}^{n+1}] \). Notice that \( \Pi_{n}^{(s)} \) is a simplex and let \( \Pi_{n}^{(s)} \) denote the (half-open) fundamental parallelepiped. Note that \( \dim(k[\Pi_{n}^{(s)}]) = n + 1 \).
and \( k[\mathbf{P}_n^{(s)}] \) has a natural homogeneous system of parameters, namely the monomials corresponding to the vertices, which we denote by \( \theta_0, \theta_1, \ldots, \theta_n \). The quotient \( k[\mathbf{P}_n^{(s)}]/(\theta_0, \ldots, \theta_n) \) is precisely the equivalence classes of lattice point each in \( \Pi_{\mathbf{P}_n^{(s)}} \). Let \( m_1, \cdots, m_\alpha \in \Pi_{\mathbf{P}_n^{(s)}} \) be the elements at height 1. The socle \( \text{soc}(k[\mathbf{P}_n^{(s)}]/(\theta_0, \ldots, \theta_n)) \) are precisely the lattice points in \( \lambda \in \Pi_{\mathbf{P}_n^{(s)}} \) such that \( \lambda + m_i \not\in \Pi_{\mathbf{P}_n^{(s)}} \) for all \( m_i \) by Lemma 4.1 and Theorem 2.2. By Lemma 4.2, we know that semigroup addition corresponds to entry-wise addition on inversion sequences. Subsequently, this condition on inversion sequences is precisely the condition that only elements of highest degree in \( \Pi_{\mathbf{P}_n^{(s)}} \) are in \( \text{soc}(k[\mathbf{P}_n^{(s)}]/(\theta_0, \ldots, \theta_n)) \), which then must contain elements which are all the same degree.

\[ \square \]

4.2. Consequences of the characterization. First consider the following result giving the inequalities for the coefficients of \( s \)-Eulerian polynomials.

**Corollary 4.6.** Let \( s = (s_1, s_2, \ldots, s_n) \) be a sequence such that \( \mathbf{P}_n^{(s)} \) is level. Then the coefficients of the \( s \)-Eulerian polynomial \( h^*(\mathbf{P}_n^{(s)}, z) = 1 + h_1^*z + \cdots + h_r^*z^r \) satisfies the the inequalities \( h_i^* \leq h_j^*h_{i+j}^* \) for all pairs \( i \) and \( j \) such that \( h_{i+j}^* > 0 \).

These inequalities follow from [Sta96, Chapter III, Proposition 3.3] and provide additional information of the behavior of \( s \)-Eulerian polynomials to complement the known log-concave inequalities from [SV15]. It is worth noting that these inequalities need not be satisfied for arbitrary \( s \). For example, the sequence \( s = (2, 3, 5, 9) \) does not give rise to a level polytope as there exists no element \( f \in \mathbf{I}_{1,4}^{(2,3,5,9)} \) such that \( f + e \in \mathbf{I}_{4,4}^{(2,3,5,9)} \) for the inversion sequence \( e = (1, 1, 2, 4) \in \mathbf{I}_{1,3}^{(2,3,5,9)} \). Moreover, we have

\[ h^*(\mathbf{P}_4^{(2,3,5,9)}, z) = 1 + 48z + 154z^2 + 66z^3 + z^4 \]

and we notice that \( h_3^* > h_1^*h_4^* \).

In addition to the characterization of Gorenstein given in Section 3, we can also arrive at a different characterization by considering the following restriction of Theorem 1.2.

**Corollary 4.7.** Let \( s = (s_1, s_2, \cdots, s_n) \) and let \( r = \max\{\text{asc}(e) : e \in \mathbf{I}_{n}^{(s)}\} \). Then \( \mathbf{P}_n^{(s)} \) is Gorenstein if and only if for any \( e \in \mathbf{I}_{n}^{(s)} \) with \( 1 \leq k < r \) there exists some \( e' \in \mathbf{I}_{n,1}^{(s)} \) such that \( e + e' \in \mathbf{I}_{n,k+1}^{(s)} \) and \( |\mathbf{I}_{n,r}| = 1 \).

**Proof.** \( \mathbf{P}_n^{(s)} \) is Gorenstein if and only if \( \mathbf{P}_n^{(s)} \) is level with exactly one canonical module generator. The canonical module of \( k[\mathbf{P}_n^{(s)}] \) for \( \mathbf{P}_n^{(s)} \) level has \( |\mathbf{I}_{n,1}^{(s)}| \) generators, as this is the leading coefficient of the \( h^* \) polynomial of \( \mathbf{P}_n^{(s)} \). \( \square \)

We should note that in general Corollary 4.7 is less computationally useful than Theorem 1.1. However, it is unexpected, and indeed striking, that these conditions are equivalent when there exists an index \( i \) such that \( \gcd(s_{i-1}, s_i) = 1 \). Moreover, Corollary 4.7 has the added benefit of providing a characterization with no restrictions on \( s \).

In the case of \( s \in \mathbb{Z}_2^2 \), the conditions of Theorem 1.2 must always be satisfied. Therefore, we have the following result.

**Corollary 4.8.** The lecture hall polytope \( \mathbf{P}_2^{(s_1, s_2)} \) is level for any \( s = (s_1, s_2) \).
Proof. Without loss of generality, suppose that \( s_1 \leq s_2 \). First note that if \( I_{2,2}^{(s_1,s_2)} = \emptyset \), then levelness is trivial. This trivial case consists of sequences \( s = (1, s_2) \) and \( s = (2, 2) \), which can be concretely shown to not have any inversion sequence with 2 ascents. So, if \( I_{2,2}^{(s_1,s_2)} \neq \emptyset \), we have two cases either (i) \( 2 \leq s_1 < s_2 \), or (ii) \( 3 \leq s_1 = s_2 \). In both cases, given \( e \in I_{2,2}^{(s_1,s_2)} \) we will explicitly construct \( f \in I_{2,2}^{(s_1,s_2)} \) such that \( e + f \in I_{2,2}^{(s_1,s_2)} \).

In case (i), take \( e = (e_1, e_2) \in I_{2,1}^{(s_1,s_2)} \). We have three possible subcases:

- Suppose that \( e_1 \geq 1 \) and \( e_2 = 0 \). Let \( f = (f_1, f_2) \) where \( f_1 = 0 \) and \( f_2 = s_2 - 1 \). Then \( e + f = (e_1, s_2 - 1) \in I_{2,2}^{(s_1,s_2)} \) because

\[
\frac{e_1}{s_1} \leq \frac{s_1 - 1}{s_1} < \frac{s_2 - 1}{s_2}.
\]

- Suppose that \( e_1 \geq 1 \) and \( e_2 \geq 1 \). Note that \( e \in I_{2,1}^{(s_1,s_2)} \) implies that \( e_2 < s_2 - 1 \) because we have

\[
0 < \frac{e_1}{s_1} < \frac{s_1 - 1}{s_1} < \frac{s_2 - 1}{s_2}.
\]

So, if \( e_2 = s_2 - 1 \), then we will have that \( \text{Asc}((e_1, e_2)) = \{0, 1\} \) which contradicts that \( e \in I_{2,1}^{(s_1,s_2)} \). Let \( f = (0, s_2 - e_2 - 1) \). Then we have \( e + f = (e_1, s_2 - 1) \in I_{2,2}^{(s_1,s_2)} \) as shown above.

- Suppose that \( e_1 = 0 \) and \( e_2 \geq 1 \). Let \( f = (1, \min\{\lfloor \frac{s_2}{s_1} \rfloor, s_2 - e_2 - 1\}) \in I_{2,1}^{(s_1,s_2)} \). Then, we have that

\[
e + f = \begin{cases} (1, s_2 - 1) & \text{if } s_2 - e_2 - 1 \leq \lfloor \frac{s_2}{s_1} \rfloor \\ (1, e_2 + \lfloor \frac{s_2}{s_1} \rfloor) & \text{if } \lfloor \frac{s_2}{s_1} \rfloor < s_2 - e_2 - 1 \end{cases}
\]

If the first case is true, then clearly \( e + f \in I_{2,2}^{(s_1,s_2)} \) by previous arguments. In the second case, notice that

\[
\frac{1}{s_1} \leq \frac{\lfloor \frac{s_2}{s_1} \rfloor + 1}{s_2} \leq \frac{\lfloor \frac{s_2}{s_1} \rfloor + e_2}{s_2}
\]

and hence \( e + f \in I_{2,2}^{(s_1,s_2)} \).

Now for case (ii), take \( e = (e_1, e_2) \in I_{2,1}^{(s_1,s_2)} \). We have several possible subcases:

- Suppose that \( e_1 = 0 \) and \( e_2 \geq 1 \). If \( e_2 > 1 \), let \( f = (1, 0) \in I_{2,1}^{(s_1,s_2)} \) and we have \( e + f = (1, e_2) \in I_{2,2}^{(s_1,s_2)} \). If \( e_2 = 1 \), then let \( f = (1, 1) \in I_{2,1}^{(s_1,s_2)} \) and we have \( e + f = (1, 2) \in I_{2,2}^{(s_1,s_2)} \).

- Suppose that \( e_1 \geq 1 \) and \( e_2 = 0 \). If \( e_1 < s_1 - 1 \), let \( f = (0, s_2 - 1) \in I_{2,1}^{(s_1,s_2)} \) and we have that \( e + f = (e_1, s_2 - 1) \in I_{2,2}^{(s_1,s_2)} \). If \( e_1 = s_1 - 1 \), then let \( f = (s_1 - 1, s_2 - 1) \in I_{2,1}^{(s_1,s_2)} \) and we get that \( e + f = (s_1 - 2, s_2 - 1) \in I_{2,2}^{(s_1,s_2)} \).

- Suppose that \( e_1 \geq 1 \) and \( e_2 \geq 1 \). Note that \( e_1 \geq e_2 \). If \( e_1 < s_1 - 1 \), then let \( f = (0, s_2 - e_2 - 1) \in I_{2,1}^{(s_1,s_2)} \) and we have that \( e + f = (e_1, s_2 - 1) \in I_{2,2}^{(s_1,s_2)} \). If \( e_1 = s_1 - 1 \), the let \( f = (s_1 - 1, s_2 - e_2 - 1) \in I_{2,1}^{(s_1,s_2)} \) and we get that \( e + f = (s_1 - 2, s_2 - 1) \in I_{2,2}^{(s_1,s_2)} \).

Therefore, we satisfy the conditions of Theorem 1.2 in all cases. \( \Box \)
The characterization allows for the construction of new level $s$-lecture hall polytopes through the following corollaries.

**Corollary 4.9.** The lecture hall polytope $P_n^{(s)}$ is level if and only if the lecture hall polytope $P_{n+1}^{(1,s)}$ is level.

*Proof.* We can express any inversion sequence $e \in I_{n+1}^{(1,s)}$ as
\[ e = (0, e') \]
where $e' \in I_n^{(s)}$. Thus, $e$ satisfies the conditions of Theorem 1.2 exactly when $e'$ satisfies the conditions. \hfill $\square$

**Remark 4.10.** One also has $P_n^{(s)}$ level if and only if $P_{n+1}^{(s,1)}$ level by applying an analogous argument.

**Corollary 4.11.** If both $P_n^{(s)}$ and $P_m^{(t)}$ are level, then $P_{n+m+1}^{(s,1,t)}$ is level.

*Proof.* Any inversion sequence $e \in I_{n+m+1}^{(s,1,t)}$ can expressed as
\[ e = (e_1, 0, e_2) \]
where $e_1 \in I_n^{(s)}$ and $e_2 \in I_m^{(t)}$. Subsequently, $e$ satisfies the conditions of Theorem 1.2 when $e_1$ and $e_2$ both satisfy the conditions of Theorem 1.2. \hfill $\square$

**Remark 4.12.** It is worth noting that by combining Corollary 4.8 and Corollary 4.11 we can create an infinite family of level $s$-lecture hall polytopes of arbitrary dimension. In particular, $P_n^{(s)}$ is level when $s$ is any sequence satisfying $s_i = 1$ when $i = 0 \mod 3$.

5. **Concluding remarks and future directions**

There are two immediate avenues to continue this work, namely classifying the Gorenstein property in the case $\gcd(s_{i-1}, s_i) \geq 2$ for all $i$ and using the levelness characterization to produce more tractable results in special cases. With regards to the Gorenstein characterization, extensive computational evidence — using the Normaliz software [BIR+] — suggests that the $\gcd(s_i, s_{i+1}) = 1$ may not be necessary. We have the following conjecture:

**Conjecture 5.1.** Let $s = (s_1, s_2, \ldots, s_n)$ be any sequence. Then $P_n^{(s)}$ is Gorenstein if and only if there exists $c, d \in \mathbb{Z}^n$ satisfying
\[ c_j s_{j-1} = c_{j-1} s_j + \gcd(s_j, s_{j+1}) \]
and
\[ d_j \frac{s_{j-1}}{s_j} = d_{j-1} \frac{s_j}{s_{j+1}} + \gcd\left(\frac{s_j}{s_{j+1}}, \frac{s_{j+1}}{s_{j+2}}\right) \]
for $j > 1$ with $c_1 = d_1 = 1$.

Unfortunately, the condition $\gcd(s_{i-1}, s_i) = 1$ for some $i$ is necessary under our current method of proof. It is worth noting that examples of Gorenstein $P_n^{(s)}$ with the property that $\gcd(s_{i-1}, s_i) \geq 2$ seem to be very rare. In fact, most examples are well-structured so that reductions can be made to utilize the existing theorem. For example, the sequence $s = (2, 4, \ldots, 2n)$ produces a Gorenstein property and satisfies the condition of Conjecture 5.1. However, we can also realize $P_n^{(2,4,\ldots,2n)} = 2 \cdot P_n^{(1,2,\ldots,n)}$, $P_n^{(1,2,\ldots,n)}$ is Gorenstein by the classification and it is easy to see that $h_n^*(P_n^{(1,2,\ldots,n)}) \neq 0$ and $h_n^*(P_n^{(1,2,\ldots,n)}) = 0$. These
conditions together with results in [DNH97] all imply that \( \mathbf{P}^{(2,4,\ldots,2n)}_n \) must be a Gorenstein polytope as well. In fact, we have not found an example of a Gorenstein \( \mathbf{P}_n^{(s)} \) with \( \gcd(s_{i-1},s_i) \geq 2 \) which cannot alternatively be shown to be Gorenstein in a similar way.

Using the levelness characterization to produce more tractable results in special cases may prove fruitful. Based on experimental evidence, we have the following conjecture for levelness in a large family of lecture hall polytopes:

**Conjecture 5.2.** Let \( s \in \mathbb{Z}^n_{\geq 1} \) be a sequence such that there exists some \( c \in \mathbb{Z}^n \) satisfying
\[
c_j s_{j-1} = c_{j-1} s_j + \gcd(s_{j-1},s_j)
\]
for \( j > 1 \) with \( c_1 = 1 \). Then \( \mathbf{P}_n^{(s)} \) is level.

This conjecture, if true, implies that \( C_n^{(s)} \) a Gorenstein cone is sufficient for \( \mathbf{P}_n^{(s)} \) to be level. However, it should be noted that the characterization, though more efficient than explicitly computing the generators of the canonical module, can often be unwieldy for complicated computations. It may, in fact, be more effective to produce an alternative representation of the level property, perhaps in terms of local cohomology.

An additional future direction would be to consider levelness in \( s \)-lecture hall cones. There is no canonical choice of grading for the \( s \)-lecture hall cones as there is in the polytopes and the different gradings have different computational advantages (see [BBK+15, Ols17]). One must choose a grading before approaching this problem. Preliminary computations with respect to certain gradings suggests that (non-Gorenstein) level \( s \)-lecture hall cones are quite rare.

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FACHBEREICH MATHEMATIK UND INFORMATIK, FREIE UNIVERSITÄT BERLIN, GERMANY
E-mail address: fkohl@math.fu-berlin.de

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506–0027
E-mail address: mccabe.olsen@uky.edu