RICCI FLOW ON SURFACES WITH CONICAL SINGULARITIES, II

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Abstract. In this paper, we study the (normalized) Ricci flow on surfaces with conical singularities. Long time existence is proved for cone angle smaller than $2\pi$. In this case, convergence results are obtained if the Euler number is nonpositive.

1. Introduction

This is a continuation of our previous work on Ricci flow on surfaces with conical singularities [16]. For smooth surfaces, in 1982, Hamilton [8] used his Ricci flow to prove the (normalized) Ricci flow converges to a metric of constant curvature in a given conformal class. Chow [5], Chen, Lu and Tian [3] removed various technical conditions to show that Ricci flow can be used as a tool to prove the Uniformization theorem.

Let $S$ be a smooth Riemann surface and $p \in S$. A metric $g$ on $S$ is said to have a conical singularity of order $\beta (\beta > -1)$, or of angle $\theta = (2\pi(\beta + 1))$ at $p$, if in a neighborhood $U$,

$$g = w(z) |z|^{2\beta} |dz|^2,$$

where $z$ is a conformal coordinate in $U$ with $z(p) = 0$ and $w$ is some continuous positive function. In general, consider a divisor $\beta = \sum_{i=1}^{n} \beta_i p_i$ and a conformal metric $g$ on $S$ which has a conical singularity of order $\beta_i$ at $p_i$. The set of all such metrics is a singular conformal class, which we denote by $(S, \beta)$.

We are interested in generalizing the theory about Ricci flow to surfaces with conical singularities. In particular, we hope Ricci flow could be a useful tool in the study of canonical metric in the conformal class $(S, \beta)$. There are already several papers devoted to this subject, for example, [10], [13], [14]. It suffices for us to note here that not all $(S, \beta)$ admits a metric of constant curvature and it is still an open question to find a 'canonical' metric in $(S, \beta)$ for general $S$ and $\beta$. We refer to [2], [9], [1] for related study.

In [16], the author showed that there is a reasonable notion of normalized Ricci flow despite that the metric is not complete at the cone tips. For a 'good' initial value, there is a family of metrics $g(t), t \in [0, T]$ in the conformal class $(S, \beta)$ satisfying

$$\frac{\partial g}{\partial t} = (r - R)g.$$ 

Here $R$ is the scalar curvature of $g(t)$ and $r$ is some constant. Moreover, the Gauss-Bonnet formula remains true for $g(t)$ and the volume of $g(t)$ does not depend on $t$. 

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Precisely, we have
\[ \int_S K_g dV_g = 2\pi \tilde{\chi} = \frac{r^2}{2} V(g_t), \]
where \( K_g \) is the Gauss curvature, \( V(g_t) \) is the total area of \( g_t \) and \( \tilde{\chi} = \chi(S) + \sum_{i=1}^{n} \beta_i \) is the Euler number of \((S, \beta)\).

Due to technical difficulties, in [16], the author was not able to show any long time existence result. It was not even clear whether the curvature is bounded along the flow. The main purpose of this paper is to solve these problems. In fact, we reprove the local existence result by iterating in a smaller function space. The solution obtained this way automatically has bounded Gauss curvature. We then discuss long time existence and convergence result. We follow closely the argument in [8]. With the presence of conical singularity, we must be careful in very application of maximum principle. It turns out that for cone singularities with \( \beta_i < 0 \), some arguments in [8] work.

To introduce our main result, we set up some notions. Throughout this paper, we choose a background cone metric \( \tilde{g} \) in the conformal class \((S, \beta)\) which is smooth away from the singularity and is the standard flat cone metric in small neighborhoods of the singularities. That is, if \( p \) is a singular point, then there exists some complex coordinate \( z \) compatible to the complex structure of \( S \) near \( p \), then
\[ \tilde{g} = |z|^{2\beta} |dz|^2. \]

We denote the Laplace of \( \tilde{g} \) by \( \tilde{\Delta} \), the Gauss curvature by \( \tilde{K} \), the gradient by \( \tilde{\nabla} \) and the volume form by \( d\tilde{V} \). \( g_0 = e^{2u_0} \tilde{g} \) will be the initial metric of the normalized Ricci flow and \( g(x, t) \) will be the solution. \( K_t \) and \( dV_t \) are the Gauss curvature and volume element of \( g_t \) for \( t \geq 0 \).

Our result is

**Theorem 1.1.** Assume both \( u_0 \) and \( K_0 \) are in \( C^{0,\alpha}(S, \beta) \) and have bounded Dirichlet energy and \( \Delta K_0 \) is bounded. Then there exists a local solution to (1.1) defined on \([0, T] \). If we choose \( r \) properly, the volume of \( g(t) \) will be a constant and Gauss-Bonnet theorem remains true for \( t > 0 \). Moreover,

1. if \( K_0 < 0 \), then the solution exists for all \( t \) and converges to a metric of constant negative curvature.
2. if \( \beta_i < 0 \) for all \( i \), then the solution exists for all \( t \).
3. if \( \beta_i < 0 \) for all \( i \) and \( \chi(S, \beta) \leq 0 \), then the solution converges to a metric of constant curvature.

**Remark 1.2.** \( C^{0,\alpha}(S, \beta) \) is a weighted Hölder space defined in [10] and will be recalled in the next section.

The proof of this theorem depends on further understanding of both the analysis involved in the cone singularity and the nature of the Ricci flow equation (on surface) in comparison with [16]. The central theme is that what regularity can be proved for the singular equation and whether this regularity enables us to reproduce the methods used in the smooth setting. The idea behind our previous paper on this subject [16] can be summarized as follows: (1) approximate the cone manifold by manifold with boundary; (2) obtain uniform \( C^0 \) estimate in the approximation and leave higher order estimates to interior estimates away from the singularity; (3) show the finiteness of Dirichlet energy, which justifies the maximum principle.
In this paper, several new observations and methods are introduced. For the convenience of the reader, we list them below.

(1) The approximation scheme we used in [16] is robust if we take $t-$derivative of the evolution equation. Precisely, we approximate $u$ (the solution on conical surface) by $u_k$ (the solution on the manifold with boundary), while at the same time $\partial_t u$ is approximated by $\partial_t u_k$. This enables us to prove linear estimate for $\partial_t u$ and the Dirichlet energy of $\partial_t u$ (see Lemma 2.5 and Lemma 2.6) in Section 2. Using this double-layer linear estimate, we do contractive mapping as before to obtain local solution to the normalized Ricci flow with bounded curvature in Section 4.

(2) In principle, we can do multi-layer linear estimate by keep taking $t-$derivatives. In that sense, we can obtain local solution with better and better regularity, at the expenses of assuming more and more of the initial data. A difficulty is that to obtain a local solution with bounded curvature $K$, we need to assume that $\Delta_0 K_0$ is bounded for the initial value (see Theorem 4.1). Therefore, even if we use the evolution equation of $K$ to show that $K$ is bound up to time $T$, Theorem 4.1 does not extend the solution beyond $T$.

This problem can be solved as follows. We note that the normalized Ricci flow equation is quasi-linear, the evolution equation for $\partial_t u$ (or equivalently, $\Delta u$, or $K$) is semi-linear and fortunately, the evolution equation for $\partial_t K$ (or equivalently $\Delta K$ if $u$ and $K$ are bounded) is linear (with $u$ and $K$ as coefficients). The point is the linear equation never blows up. Our proof of Lemma 5.2 follows this line.

(3) This last observation is not a necessary logical part in the proof. It is a point of view about the regularity of parabolic equations on conical surfaces. If $u$ is a solution to (1.1), what is the regularity near the cone point? Is the gradient of $u$ bounded? the Hessian? A 'correct' way of answering these questions is that $t-$derivatives of $u$ are bounded (as long as $u$ is) and one should consult the regularity of Poisson equation for the boundedness of $\nabla u$ and $\nabla^2 u$, which we know depends on the cone angle. We discuss the Poisson equation in Section 3. We need it to justify the application of maximum principle to $|\nabla f|^2$, where $f$ is the potential of curvature ($\Delta f = r - R$).

Other proofs in this paper are more or less known techniques in the theory of Ricci flow on surfaces.

It is natural to ask what happens for positive Euler number case. With the existence of conical singularities, there may not be a constant curvature metric. It is expected that when constant curvature metric exists, the flow converges to it and when no such metric exists, the flow approximates a Ricci soliton. A complete discussion involves a stability condition and is beyond the ability of the author. Therefore, we stop here and hope that we can come back to the problem later.

Remark 1.3. Recently, we notice the preprint [4] of Chen and Wang on Kähler Ricci flow on manifolds with edge singularities. The problem discussed in [16] and this paper is the complex dimension one case of their paper. In [4], the authors proved local existence of the flow and results on the long-time behavior of the flow are announced there.
2. Linear parabolic equation and maximum principle

In this section, we study the following linear parabolic equation

(2.1) \[ \frac{\partial u}{\partial t} = a(x,t)\tilde{\Delta}u + b(x,t)u + f(x,t) \]

with initial value \[ u|_{t=0} = u_0 \]
on S, where \( \tilde{\Delta} \) is the Laplacian of the fixed background metric \( \tilde{g} \).

2.1. basic estimates. Important to our discussion are some weighted Hölder spaces defined in [16]. The elliptic version is \( \mathcal{E}^{l,\alpha}(S,\beta) \) and the parabolic version is \( \mathcal{P}^{l,\alpha,T} \). In this paper, for simplicity, we denote them by \( E^{l,\alpha} \) and \( P^{l,\alpha,T} \) respectively. For the precise definition, we refer the readers to Section 2 of [16]. In spite of the tedious definition, it is not difficult to understand the meaning of these weighted Hölder space. Away from the singularity, they are just the normal Hölder space. Near a singularity, the \( \mathcal{E}^{l,\alpha} \) norm is the bound for up to \( l \)-th derivatives which one may obtain for a bounded harmonic function via applying interior estimate on a ball away from the singularity. A similar characterization is true for \( \mathcal{P}^{l,\alpha,T} \) if we replace the harmonic function by a solution to linear heat equation defined on \( S \times [0,T] \).

Remark 2.1. Weighted Hölder space is nothing new in the study of degenerate elliptic operators. In fact, \( \mathcal{E}^{l,\alpha} \) coincides with the edge Hölder space in [12] in the case of conical surfaces.

The following is almost Theorem 3.1 in [16]. The difference is that we have an extra \( b(x,t)u \) term in our equation, which does not affect the proof so that the proof in [16] works here with almost no modification.

Lemma 2.2. If \( a, b, f \in \mathcal{P}^{0,\alpha,T} \) and \( u_0 \in \mathcal{E}^{2,\alpha} \), then there exists a solution \( u \) to equation (2.1) with initial value \( u_0 \) such that

\[ \|u\|_{P^{2,\alpha,T}} \leq C \]

where \( C \) depends on \( T \) and the \( \mathcal{P}^{0,\alpha,T} \) norm of \( a, b, f \) and the \( \mathcal{E}^{2,\alpha} \) norm of \( u_0 \).

Remark 2.3. In fact, \( C \) depends on \( \max(T,1) \) instead of \( T \). It follows from the \( C^0 \) estimate in the proof. We shall need this fact later.

To illustrate the idea of the proof, assume that there is only one singularity. On a neighborhood \( U \) of the singularity, the background metric \( \tilde{g} \) is given by

\[ \tilde{g} = r^{2\beta}(dr^2 + r^2d\theta^2). \]

Set

\[ S_k = S \setminus \{(x, y) \in U | \rho(x, y) < 2^{-k}\} \]

where \( \rho = \frac{1}{\beta+1}r^{\beta+1} \). The following boundary value problem has a unique solution \( u_k \).

(2.2) \[ \begin{cases} \frac{\partial u_k}{\partial t} = a(x,t)\tilde{\Delta}u_k + b(x,t)u_k + f(x,t) & \text{on } S_k \\ \frac{\partial u_k}{\partial \nu} |_{S_k} = 0 \\ u_k(x,0) = u_0(x) & \text{on } S_k \end{cases} \]
Here $\nu$ is the normal vector to the boundary $\partial S_k$. We can obtain a uniform $C^0$ estimate for $u_k$ so that by sending $k$ to infinity, we get the solution $u$ in Lemma 2.2. The estimate of $u$ follows from the interior estimates.

The next lemma is an estimate for $\int_S |\nabla u|^2 \, d\tilde{V}$. It turns out that the boundedness of the Dirichlet energy is very important to us. For example, we need some maximum principle for parabolic equations. However, the presence of a conical singularity implies that ordinary maximum principle can’t be true without extra assumptions. Indeed, being bounded in the Dirichlet energy is part of the assumption. For more detail, see the next subsection on maximum principle.

**Lemma 2.4.** Let $u$ be the solution constructed in Lemma 2.2 and assume $u_0$, $b$ and $f(\cdot,t)$ have uniformly bounded Dirichlet energy. Then

\begin{equation}
\frac{d}{dt} \int_S |\nabla u|^2 \, d\tilde{V} \leq C_1 \int_S |\nabla u|^2 \, d\tilde{V} + C_2 \int_S |\nabla f|^2 + |\nabla b|^2 \, d\tilde{V}.
\end{equation}

Here $C_1$ depends on the $C^0$ norm of $b$ and $C_2$ depends on the $C^0$ norm of $u$. In particular, $u(\cdot,t)$ has bounded energy.

**Proof.** Recall that $u$ is the limit $u_k$. Moreover,

\begin{align*}
\frac{d}{dt} \int_{S_k} |\nabla u_k|^2 \, d\tilde{V} &= 2 \int_{S_k} \nabla u_k \cdot \nabla \frac{\partial u_k}{\partial t} \, d\tilde{V} \\
&= -2 \int_{S_k} a(x,t)(\Delta u_k)^2 d\tilde{V} + C \int_{S_k} \nabla u_k \cdot \nabla (bu_k) d\tilde{V} + 2 \int_{S_k} \nabla u_k \cdot \nabla f(x,t) d\tilde{V} \\
&\leq C_1 \int_{S_k} |\nabla u_k|^2 \, d\tilde{V} + C_2 \int_{S_k} |\nabla f|^2 + |\nabla b|^2 \, d\tilde{V}.
\end{align*}

Here $C_1$ depends on the $C^0$ norm of $b$ and $C_2$ depends on the $C^0$ norm of $u_k$.

By taking the limit of $k \to \infty$, we finish the proof of the lemma. \qed

In [16], we applied Lemma 2.2 and Lemma 2.4 to a linear equation to show the local existence of the normalized Ricci flow. The linear equation was of the form

\begin{equation}
\frac{\partial u}{\partial t} = a(x,t)\Delta u + f(x,t).
\end{equation}

We add an extra term $h(x,t)u$ in this paper because in addition to (2.4), we would also like to study the evolution equation of $\frac{\partial u}{\partial t}$, which is of the form (2.1). The new observation is that during the construction of $u$ in Lemma 2.2 we can obtain some estimate for $\frac{\partial u_k}{\partial t}$, which passes on to $\frac{\partial u}{\partial t}$. This extra estimate will enable us to prove the local existence of the normalized Ricci flow in a smaller space, which guarantees, among other things, the boundedness of the Gauss curvature.

Taking $t-$derivative of (2.2) with $b = 0$, we have

\[
\begin{aligned}
\partial_t(\partial_t u_k) &= a(x,t)\Delta (\partial_t u_k) + \frac{\partial a}{\partial t}(\partial_t u_k - f) + \partial_t f(x,t) \\
\frac{\partial(\partial_t u_k)}{\partial\nu}|_{S_k} &= 0 \\
(\partial_t u_k)(x,0) &= a(x,0)\Delta u_0(x) + f(x,0)
\end{aligned}
\]  

on $S_k$

The important observation is that $\partial_t u_k$ still satisfies the Neumann boundary condition so that a uniform maximum principle holds for all $k$. As in Proposition 3.2 of [16], we can now derive estimates of $\partial_t u_k$ as before and pass it to $\partial_t u$, which
satisfies
\[
\begin{align*}
\partial_t (\partial_t u) &= a(x,t) \tilde{\Delta} (\partial_t u) + \frac{\partial a}{\partial t}(\partial_t u - f) + \partial_t f(x,t) & \text{on } S \\
(\partial_t u)(x,0) &= a(x,0) \tilde{\Delta} u_0(x) + f(x,0) & \text{on } S
\end{align*}
\]

In summary, we have the following lemma,

**Lemma 2.5.** Let \( u \) be a solution to (2.4) given by Lemma 2.2. In addition to the assumption in Lemma 2.2 if \( \tilde{\Delta} u_0, a(x,0) \) and \( f(x,0) \) are in \( \mathcal{E}^{2,\alpha} \), \( \partial_t a \) and \( \partial_t f \) are in \( P^{0,\alpha,T} \) and

\[ a(x,t) > c > 0 \]

for some uniform constant \( c \), then

\[
\|\partial_t u\|_{P^{2,\alpha,T}} \leq C
\]

where \( C \) depends on \( T \), \( c \) and the all the norms mentioned above.

There is also a version of Lemma 2.4 for \( \partial u/\partial t \).

**Lemma 2.6.** In addition to all the assumptions of Lemma 2.5, if \( \tilde{\Delta} u_0, a(x,0), f(x,0), \partial_t a \) and \( \partial_t f \) have uniformly bounded Dirichlet energy, then

\[
\frac{d}{dt} \int_S |\tilde{\nabla} \partial_t u|^2 d\tilde{V} \leq C_1 \int_S |\tilde{\nabla} \partial_t u|^2 d\tilde{V} + C_2 \int_S (|\tilde{\nabla} a|^2 + |\tilde{\nabla} \partial_t a|^2 + |\tilde{\nabla} f|^2 + |\tilde{\nabla} \partial_t f|^2) d\tilde{V}.
\]

Here \( C_1 \) depends on \( C^0 \) norm of \( \partial_t a \) and \( c \) in Lemma 2.6 and \( C_2 \) depends on the \( C^0 \) norm of \( \partial_t u, c, \partial_t a \) and \( f \). In particular, \( \partial_t u \) has bounded energy.

### 2.2. maximum principle

It is obvious that a maximum principle is very important to us. As noted earlier, it can not be true without further assumptions. For our purpose, the assumption is being bounded and having finite Dirichlet energy.

It should be noted that a maximum principle was used in the proof of Lemma 2.2, see Proposition 3.2 and Corollary 3.5 in [16]. That one applies only to functions constructed from a specific process. The point here is that we use some regularity assumption instead of knowing where the functions come from.

**Remark 2.7.** The assumption being used here, i.e. being bounded and having finite Dirichlet energy, is absolutely not necessary for the validity of integration by parts and maximum principle, as can be seen from the proof below. One can take different approaches to deal with the parabolic equations on a conical surfaces. It turns out that our approach favors the assumption.

One may want to compare with Jeffres’ idea in [11] and its parabolic version, Lemma 11.4 in [4].

The following basic lemma shows that this assumption justifies the integration by parts, which is important to the proof of the maximum principle of this subsection.

**Lemma 2.8.** Consider a functions \( u \) defined on \( S \). If \( u \) is bounded and has bounded Dirichlet energy, then

\[
\int_S \triangle u = 0 \quad \text{and} \quad \int_S u \cdot \triangle u = -\int_S |\nabla u|^2
\]

**Proof.** Note that these two equalities are invariant under conformal change of the metric, hence we may assume the metric is \( \tilde{g} \). Since the proof is similar, we prove the second equality only.
For simplicity, assume that there is only one singularity. Near the singularity,
\[ \int_0^\delta \int_{\partial B(r)} |\tilde{\nabla} u|^2 d\sigma dr < \infty. \]
Here \( B(r) \) is the ball of radius \( r \) centered at the singularity measured with respect to the cone metric \( \tilde{g} \). For any \( \varepsilon > 0 \), there is a sequence \( r_i \) such that
\[ \int_{\partial B(r_i)} |\tilde{\nabla} u|^2 d\sigma \leq \frac{\varepsilon}{r_i}. \]
Since \( u \) is bounded, it suffices to show that
\[ \lim_{i \to \infty} \int_{\partial B(r_i)} |\tilde{\nabla} u|^2 d\sigma = 0, \]
because the boundary term in the integration by parts is \( \frac{\partial u}{\partial \nu} u \). The Hölder inequality implies
\[ \int_{\partial B(r_i)} |\tilde{\nabla} u|^2 d\sigma \leq C \left( \int_{\partial B(r_i)} |\tilde{\nabla} u|^2 d\sigma \right)^{1/2} r_i^{1/2}. \]
Since \( \varepsilon \) is arbitrary, we are done. \( \square \)

The following are two maximum principles. The proofs are almost the same, but they are prepared for different purposes.

**Lemma 2.9.** Let \( u(x, t) \) be a bounded function with bounded Dirichlet energy satisfying
\[ \partial_t u \leq a(x, t) \tilde{\Delta} u(x, t) + F(u), \]
where \( F \) is a smooth function. We assume that \( \partial_t a \), \( a \) and \( a^{-1} \) are bounded. If \( C_0 = \max_S u(\cdot, 0) \) and \( h(t) \) is the solution of ODE,
\[ \frac{dh}{dt} = F(h) \text{ and } h(0) = C_0, \]
then
\[ \max_S u(\cdot, t) \leq h(t). \]

**Proof.** For any function \( w \), we will denote by \( w^+ \) the positive part of \( w \) given by \( \max\{w, 0\} \).
\[
\frac{d}{dt} \int_S [(u(x, t) - h(t))^+]^2 a^{-1} d\tilde{V} \\
= \int_S 2(u - h)^+ \frac{d}{dt} (u - h) a^{-1} d\tilde{V} \\
+ \int_S [(u - h)^+]^2 \partial_t (a^{-1}) d\tilde{V} \\
\leq \int_S 2(u - h)^+ [\tilde{\Delta} (u - h) + a^{-1}(F(u) - F(h))] d\tilde{V} \\
+ C \int_S [(u - h)^+]^2 a^{-1} d\tilde{V} \\
= -2 \int_S |\tilde{\nabla} (u - h)^+|^2 d\tilde{V} + \int_S 2(u - h)^+ a^{-1} F'(\xi)(u - h) d\tilde{V} \\
\leq C \int_S [(u - h)^+]^2 a^{-1} d\tilde{V}. 
\]
The integration by parts is justified by Lemma 2.8. When \( t = 0 \), we know \( \int_S [(u - h)^+]^2 d\tilde{V} \) is zero. Hence, it is zero forever.

**Lemma 2.10.** Let \( u \) be a solution to (2.1) which is bounded and has bounded Dirichlet energy. Assume that \( \partial_t a, a \) and \( a^{-1} \) are bounded. If \( \tilde{\Delta} u_0 \) is bounded, then we have

\[
\| u(\cdot, t) - u_0 \|_{C^0} \leq e^{C_1 t} \int_0^t e^{-C_1 s} C_2 ds,
\]

where \( C_1 \) is the \( C^0 \) norm of \( b \) and \( C_2 \) is the \( C_0 \) norm of \( f + bu_0 + a\tilde{\Delta} u_0 \).

**Proof.** It suffices to apply the method of proof in Lemma 2.9 to the equation of \( u - u_0 \)

\[
\partial_t (u - u_0) = a\tilde{\Delta}(u - u_0) + b(u - u_0) + f + bu_0 + a\tilde{\Delta} u_0.
\]

Let \( h(t) \) be the solution of ODE

\[
\frac{dh}{dt} = C_1 h + C_2, \quad h(0) = 0.
\]

Here \( C_1 \) is the \( C^0 \) norm of \( b \) and \( C_2 \) is the \( C_0 \) norm of \( f + bu_0 + a\tilde{\Delta} u_0 \).

\[
\partial_t (u - u_0 - h) = a\tilde{\Delta}(u - u_0 - h) + b(u - u_0) - C_1 h + f + bu_0 + a\tilde{\Delta} u_0 - C_2 \\
\leq a\tilde{\Delta}(u - u_0 - h) + b(u - u_0) - bh,
\]

where in the last step we make use of the fact that \( h \geq 0 \).

The same proof as in Lemma 2.9 shows the \( u - u_0 - h \leq 0 \) for all \( t \). The other side of estimate is similar. \( \square \)

### 3. Poisson equation

In this section, we discuss Poisson equation

\[
\tilde{\Delta} v = f
\]

on surfaces with conical singularities.

There are two main results in this section. The first is about the existence of solution to (3.1) and the second is about the regularity of \( v \). It should be noted that for a general conformal metric \( g = e^{2u}\tilde{g} \) and the Poisson equation \( \tilde{\Delta} v = f \), we can always move the conformal factor to the right hand side to reduce it to (3.1). This method applies to both the existence problem and the regularity issue.

**Theorem 3.1.** For any \( f \) satisfying

\[
\int_S f^2 d\tilde{V} \leq C \quad \text{and} \quad \int_S f d\tilde{V} = 0,
\]

there is a solution \( v \) to (3.1). Moreover, \( v \) is bounded with bounded Dirichlet energy. Among all bounded functions with bounded Dirichlet energy, the solution is unique up to a constant.

**Proof.** Let \( g_0 \) be the metric compatible with the conformal structure of \( S \) such that \( \tilde{g} = wg_0 \) near the cone tip and

\[
w = r^{2\beta},
\]

where \( r \) is the distance to the singular point with respect to \( g_0 \).

We then consider the equation

\[
\triangle_{g_0} v = w f,
\]
which is equivalent to (3.1). By our assumption, we have
\[ \int_S w f dV_{g_0} = 0. \]
Therefore, there is a solution \( v \) to (3.1). Since \( \beta > -1 \), there exists some \( q > 2 \) such that
\[ w^{1/2} \in L^q(g_0). \]
Hence, we have \( w f \in L^{2q}(g_0) \) because \( w^{1/2} f \in L^2(g_0) \). The boundedness of \( v \) follows from Sobolev embedding, \( L^p \) estimate and the fact \( \frac{2q}{2+q} > 1 \). For the same reason,
\[ \int_S \left| \tilde{\nabla} v \right|^2 d\tilde{V} = \int_S \left| \nabla_{g_0} v \right|^2 dV_{g_0} \leq C. \]
For the uniqueness, it suffices to show the only bounded harmonic functions with bounded Dirichlet energy are constants. This is true by integration by parts. \( \square \)

The rest of this section is devoted to the regularity issue of the Poisson equation. We shall not try to present the sharpest result, since a complete regularity theory is not necessary. We prove only the following result which is useful when we tried to apply the maximum principle to \( |\nabla f|^2 \) for \( \triangle f = R - r \).

For simplicity, assume that there is only one conical singularity on \( S \). Recall that \( \tilde{g} \) is a fixed reference metric which is flat near the cone tip.

**Lemma 3.2.** Assume that \( \beta < 0 \). Let \( v \) be a solution to
\[ \tilde{\triangle} v = f. \]
If \( v \) is bounded with bounded Dirichlet energy and \( f \) is bounded, then \( |\nabla v|^2 \) is bounded.

**Proof.** In a neighborhood \( U \) of the singular point, we have local coordinates so that
\[ \tilde{g} = r^{2\beta} (dr^2 + r^2 d\theta^2) = r^{2\beta} (dx^2 + dy^2). \]
As above, we denote the flat metric \( dx^2 + dy^2 \) by \( g_0 \) and write \( W^{2,p}(g_0) \) for the Sobolev space with respect to \( g_0 \).

Assume first that \( 0 > \beta > -\frac{1}{2} \). In this case,
\[ \triangle_{g_0} v = r^{2\beta} f \in L^q(g_0) \]
for some \( q > 2 \).

We claim that \( v \in W^{2,p}(g_0) \). (Note that since \( v \) satisfies the above equation away from the origin, the claim does not follow directly from usual \( L^p \) estimate.) To see this, let \( \tilde{v} \) be the usual solution of
\[ \triangle_{g_0} \tilde{v} = r^{2\beta} f \]
with boundary value \( \tilde{v} = v \) on \( \{ r = 1 \} \). We know \( \tilde{v} \in W^{2,q}(g_0) \). On the other hand, the difference \( \tilde{v} - v \) is a harmonic function defined on \( \{ 0 < r < 1 \} \). Moreover, it is bounded and has bounded Dirichlet energy. Hence it is zero and \( v = \tilde{v} \).

By the Sobolev embedding, \( \partial_x v \) and \( \partial_y v \) is bounded. Hence, \( |\nabla v|^2 \) is bounded because
\[ |\nabla v|^2 = r^{-2\beta} \left( (\partial_x v)^2 + (\partial_y v)^2 \right). \]
Since the Dirichlet energy is conformal invariant, we may compute with respect to $g_0$:

$$\partial_x \left( |\nabla v|^2 \right) = (-2\beta) r^{-2\beta-1} x \left( (\partial_x v)^2 + (\partial_y v)^2 \right) + r^{-2\beta} \left( 2\partial^2_{xx} v \partial_x v + 2\partial^2_{xy} v \partial_y v \right).$$

This is square integrable because $\partial v$ is bounded and $\partial^2 v$ is in $L^q$ for $q > 2$. This finishes the proof when $0 > \beta > -1/2$.

If $-1 < \beta \leq -1/2$, we can find positive integer $m$ and $\beta_0 \in (-1/2, 0]$ such that $1 + \beta_0 = m(1 + \beta)$. Hence, we may consider a cone of order $\beta_0$, which is $m$–fold cover of the original one. Then the lemma with cone of order $\beta$ follows from the one with cone of order $\beta_0$.

Precisely, by setting $\rho = \frac{1}{1+\beta} r^{\beta+1}$, we have

$$\tilde{g} = d\rho^2 + (1 + \beta_0)^2 \rho^2 d\theta^2.$$ Consider another cone of order $\beta_0$, whose metric is given by

$$\hat{g} = d\rho^2 + (1 + \beta_0)^2 \rho^2 d\eta^2.$$ The map $\Psi$ from $(\rho, \eta)$ to $(\rho, m\eta \text{mod } 2\pi)$ is an $m$–fold isometric covering. By setting $\hat{v} = v \circ \Psi$ and $\hat{f} = f \circ \Psi$, we have

$$\Delta_{\hat{g}} \hat{v} = \hat{f}.$$ Since $\hat{f}$, $\hat{v}$ is bounded and $\hat{v}$ has bounded Dirichlet energy, we know $|\nabla \hat{v}|^2$ is bounded and has bounded Dirichlet energy. So is $v$ and the lemma is proved. □

**Corollary 3.3.** If $g = e^{2u} \tilde{g}$ for some $u$ bounded and with bounded Dirichlet energy, $v$ is a solution to

$$\Delta_g v = f$$

for bounded $f$. If $v$ is bounded with bounded Dirichlet energy, so is $|\nabla v|^2$.

**Proof.** Moving the conformal factor to the right, we have

$$\tilde{\Delta} v = e^{2u} f.$$ Lemma 3.2 implies that $|\nabla \tilde{v}|^2$ is bounded and has bounded Dirichlet energy. The claim follows from our assumption on $u$ and the fact that

$$|\nabla v|^2 = e^{-2u} |\nabla \tilde{v}|^2.$$ □

### 4. Local existence

The purpose of this section is to show a local solution to the normalized Ricci flow if the initial metric is good.

**Theorem 4.1.** If $u_0$ and $K_0$ are bounded in $E^{2,\alpha}$, $\tilde{\Delta} \tilde{K}_0$ is bounded and $u_0$ and $K_0$ have finite Dirichlet energy, then there exists some $T > 0$ depending on the $E^{2,\alpha}$ norms of $u_0$, $K_{u_0}$ and $C^0$ norm of $\tilde{\Delta} \tilde{K}_0$, such that we have a solution $u(x, t), t \in [0, T)$ with $u_0$ as initial value satisfying

$$\frac{\partial u}{\partial t} = e^{-2u} \tilde{\Delta} u + \frac{r}{2} - e^{-2u} \tilde{K}.$$
Proof. It follows immediately from the formula \( K = e^{-2u}(-\Delta u + \tilde{K}) \) that \( \Delta u_0 \) is also bounded in \( \mathcal{E}^{2,\alpha} \).

For the proof, we need to consider an iteration. Consider a linear equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = e^{-2v} \Delta u + \frac{r}{2} - e^{-2v} \tilde{K} \\
u(x, 0) = u_0,
\end{cases}
\]

(4.2)

To start the process, let \( v(x, t) = u_0(x) \) for \( x \in S \) and \( t \in [0, T] \). It is trivial that \( v(x, t) \) satisfies the following conditions:

(a) \[
\|v - u_0\|_{p_0, \alpha, T} + \|\partial_t v - \left(e^{-2u_0} \Delta u_0 + \frac{r}{2} - e^{-2u_0} \tilde{K}\right)\|_{p_0, \alpha, T} \leq C_0,
\]

where \( C_0 \) is larger than \( \mathcal{E}^{2,\alpha} \) norms of \( u_0 \) and \( \Delta u_0 \). It follows that \[
\|v\|_{p_0, \alpha, T} + \|\partial_t v\|_{p_0, \alpha, T} \leq C(C_0),
\]

(b) \[
v(x, 0) = u_0(x).
\]

Let \( u(x, t) \) be the solution to (1.2). We now argue that if \( T \) is small, then \( u \) also satisfies (a) and (b). Since (b) is just the initial condition of (4.2), it suffices to check (a), which is done in several steps.

(1) The \( p_0, \alpha, T \) norm of \( a = e^{-2v} \) and \( f = \frac{r}{2} - e^{-2v} \tilde{K} \) are bounded by \( p_0, \alpha, T \) norm of \( v; u_0 \in \mathcal{E}^{2,\alpha} \); Hence, Lemma 2.2 implies that

\[
\|u\|_{p_0, \alpha, T} \leq C(C_0).
\]

(2) \( a(x, 0) \) and \( f(x, 0) \) depends only on \( v(x, 0) \), hence their \( \mathcal{E}^{2,\alpha} \) norms are bounded; by assumption \( \mathcal{E}^{2,\alpha} \) norm of \( \Delta u_0 \) is also bounded; by (a),

\[
\|\partial_t a\|_{p_0, \alpha, T} + \|\partial_t f\|_{p_0, \alpha, T} \leq C(C_0).
\]

Then by Lemma 2.5 we have

\[
\|\partial_t u\|_{p_2, \alpha, T} \leq C(C_0).
\]

(3) By (1) and (2),

\[
\|u - u_0\|_{p_2, \alpha, T} + \|\partial_t u - \left(e^{-2u_0} \Delta u_0 + \frac{r}{2} - e^{-2u_0} \tilde{K}\right)\|_{p_2, \alpha, T} \leq C(C_0).
\]

(4) Applying Lemma 2.10 to the equations of both \( u \) and \( \partial_t u \), we have

\[
\|u - u_0\|_{C_0} + \|\partial_t u - \left(e^{-2u_0} \Delta u_0 + \frac{r}{2} - e^{-2u_0} \tilde{K}\right)\|_{C_0} \leq e^{C_1 T} \int_0^T e^{-C_1 s} C_2 ds.
\]

Here \( C_1 \) depends only on \( C_0 \) and \( C_2 \) depends on both \( C_0 \) and \( \Delta K_0 \) (see remark below). It implies that we can choose \( T \) small so that the above is smaller than anything we want.

\textbf{Remark 4.2.} \( e^{-2u_0} \Delta u_0 + \frac{r}{2} - e^{-2u_0} \tilde{K} \) is just \( \frac{r}{2} - K_0 \). Lemma 2.17 says the \( C^0 \) norm of \( \Delta (\frac{r}{2} - K_0) \) is involved in the determination of the above constants. In fact, this is the only place where we use the assumption on the bound of \( \Delta K_0 \).

(5) We then use an interpolation inequality as given in Lemma 4.3 of [16] to show (a) is true for \( u \).
By taking \( u \) as \( v \) and repeating the above argument, we have a sequence of \( u_i \) such that the \( P^{2,α,T} \) norms of \( u_i \) and \( \tilde{\Delta} u_i \) are bounded. To see the later, recall that
\[
\partial_t u_i = e^{-2u_i-1} \tilde{\Delta} u_i + \frac{r}{2} - e^{-2u_i-1} \tilde{K}.
\]
By taking a converging subsequence, we have a solution \( u \) to (4.1) for \( t \in [0, T] \). \( \square \)

The above theorem provides a local solution to (4.1).

**Corollary 4.3.** Let \( u(t) \) be a local solution to (4.1) given above. If
\[
r = \frac{4π\chi(S, β)}{V_0},
\]
then the volume of \( g(t) \) is a constant and
\[
2π\chi(S, β) = \int_S K_i dV_i.
\]

*Proof.* Since \( \tilde{g} \) is flat near the singularities, then by using a version of Gauss-Bonnet theorem involving the geodesic curvature on the boundary, it is easy to show
\[
2π\chi(S, β) = \int_S \tilde{K} d\tilde{V}.
\]
Since the Gauss curvature of \( e^{2u} \tilde{g} \) is given by \( e^{-2u}(-\tilde{\Delta} u + \tilde{K}) \), Gauss-Bonnet theorem remains true for \( e^{2u} \tilde{g} \) if and only if
\[
\int_S \tilde{\Delta} u d\tilde{V} = 0,
\]
which is true for our \( u(t) \) (this is a consequence of Lemma 2.8 and Proposition 4.4 below). Denote the volume of \( g(t) \) by \( V(t) \).
\[
\frac{d}{dt} V(t) = \int_S 2\tilde{\Delta} u + re^{2u} - 2\tilde{K} d\tilde{V} = rV(t) - rV_0.
\]
Since \( V(0) = V_0 \), \( V(t) \equiv V_0 \). \( \square \)

For the local solution constructed above, we know \( u \) and \( \partial_t u \) (hence \( K \)) are bounded. Later, to show the long time existence and convergence, we will apply maximum principles to the Gauss curvature \( K \) of \( e^{2u} \tilde{g} \). As we have shown before, for that purpose, we need to show the Dirichlet energy of \( K \) (or equivalently, \( \Delta u \)) is uniformly bounded for \( x \in S \) and \( t \in [0, T] \).

**Proposition 4.4.** For the solution \( u \) obtained above, we have
\[
\int_S |\tilde{\nabla} u|^2 + |\tilde{\nabla} \tilde{\Delta} u|^2 d\tilde{V} \leq C
\]
for \( t \in [0, T] \).

**Remark 4.5.** We note that the Proposition is still true if we replace \( \tilde{\nabla} \), \( \tilde{\Delta} \) and \( d\tilde{V} \), with \( \nabla \), \( \Delta \) and \( dV \) (with respect to some conformal metric \( g = e^{2v} \tilde{g} \), if \( v \) is bounded and has bounded Dirichlet energy). This follows from two facts: (1) the Dirichlet energy is conformal invariant; (2) \( \Delta u = e^{-2v} \tilde{\Delta} u \).
Proof: Let \( u_i \) be the sequence we used in the iteration process of the previous theorem to obtain \( u \). Since \( u \) is the limit of \( u_i \), it suffices to show that

\[
\int_S \left| \nabla u_i \right|^2 + \left| \nabla (\Delta u_i) \right|^2 \, dx \leq C
\]

uniformly for all \( i \) and \( t \in [0, T] \).

We apply the derivation of (2.3) to (4.3) to see that

\[
\frac{d}{dt} \int_S \left| \nabla u_i \right|^2 \, dV \leq C_1 \int_S \left| \nabla u_i \right|^2 \, dV + C_2 \int_S \left| \nabla u_{i-1} \right|^2 \, d\tilde{V}.
\]

Let \( f_i(t) \) be \( \int_S \left| \nabla u_i \right|^2 \, d\tilde{V} \). Then \( f_0(t) = \int_S \left| \nabla u_0 \right|^2 \, d\tilde{V} \) is a constant, which we denote by \( C_3 \). We have

\[
\frac{d}{dt} f_i(t) \leq C_1 f_i + C_2 f_{i-1}
\]

and

\[
f_i(0) = C_3.
\]

We claim that \( f_i(t) \leq C_3 e^{(C_1 + C_2) t} \) for \( t \in [0, T] \). To see this, set \( g_0(t) \equiv C_3 \) and

\[
\frac{d}{dt} g_i(t) = C_1 g_i + C_2 g_{i-1}
\]

and

\[
g_i(0) = C_3.
\]

It is easy to prove by induction that \( f_i(t) \leq g_i(t) \). On the other hand, \( g_1(t) \geq g_0(t) \).

Again, by induction, we can show that

\[
g_i(t) \geq g_{i-1}(t).
\]

Therefore,

\[
\frac{d}{dt} g_i(t) \leq (C_1 + C_2) g_i,
\]

from which our claim follows. Hence, we have proved

\[
\int_S \left| \nabla u \right|^2 \, d\tilde{V} \leq C
\]

for \( t \in [0, T] \).

For the energy of \( \tilde{\Delta} u \), due to the equation (4.1), it suffices to consider \( \partial_t u \). We take a \( t \)-derivative of (4.3) to get

\[
\partial_t (\partial_t u_i) = e^{-2u_{i-1}} \tilde{\Delta} (\partial_t u_i) - 2e^{-2u_{i-1}} (\partial_t u_{i-1}) (\Delta u_i) + 2e^{-2u_{i-1}} \tilde{K} (\partial_t u_{i-1})
\]

\[
= e^{-2u_{i-1}} \tilde{\Delta} (\partial_t u_i) - 2(\partial_t u_{i-1}) (\partial_t u_i - \frac{r}{2} + e^{-2u_{i-1}} \tilde{K}) + 2e^{-2u_{i-1}} \tilde{K} (\partial_t u_{i-1})
\]

\[
= a(x, t) \tilde{\Delta} (\partial_t u_i) + b(x, t) (\partial_t u_i) + f(x, t).
\]

Here

\[
b(x, t) = -2(\partial_t u_{i-1})
\]

and

\[
f(x, t) = -2(\partial_t u_{i-1}) (-\frac{r}{2} + e^{-2u_{i-1}} \tilde{K}) + 2e^{-2u_{i-1}} \tilde{K} (\partial_t u_{i-1}).
\]

(There is a cancellation there, which is not important at all.) We can argue as before to show that

\[
\int_S \left| \nabla (\partial_t u_i) \right|^2 \, d\tilde{V}
\]
is bounded. We can use (4.3) again with the fact that \( \int_S \left| \nabla u_{i-1} \right|^2 dV \) are bounded to conclude the second part of our claim.

To conclude this section, we will prove a uniqueness result to (4.1).

**Lemma 4.6.** If \( u_1 \) and \( u_2 \) are two solution to (4.1) with the same initial value and both of them are bounded with bounded Dirichlet energy and \( \tilde{\Delta} u_1 \) and \( \tilde{\Delta} u_2 \) are bounded, then they are the same.

**Proof.** Set \( w = u_1 - u_2 \). Then
\[
\partial_t w = e^{-2u_1} \tilde{\Delta} w + (e^{-2u_1} - e^{-2u_2})(\tilde{\Delta} u_2 - \tilde{K}),
\]
which is equivalent to
\[
\partial_t w = e^{-2u_1} \tilde{\Delta} w - 2 \left( \int_0^1 e^{-2u_2 - 2t(u_1 - u_2)} dt \right) (\tilde{\Delta} u_2 - \tilde{K})w.
\]
Since \( w \) is bounded with bounded Dirichlet energy and the coefficients in the above equation are bounded, then maximum principle (Lemma 2.10 with \( f \equiv 0 \) and \( u_0 \equiv 0 \)) shows that \( w \) is always zero.

\[ \square \]

5. Long time existence and convergence

The section consists of two subsections. In the first one, we define a notion of maximal solution and then show that the solution exists as long as its curvature remains bounded. In the second one, we discuss the long time existence and convergence of the normalized Ricci flow under various assumptions.

5.1. Maximal solution.

**Definition 5.1.** Let \( u \) be a solution to (4.1) defined on \([0, T)\) such that both \( u \) and \( K \) are in \( \mathcal{P}^{2, \alpha, T} \) and have finite Dirichlet energy for \( t \in [0, T) \). \( u \) is said to be maximal if there are no other solutions defined on \([0, T')\) with \( T' > T \) which satisfies the same restrictions and agrees with \( u \) on \([0, T)\).

**Lemma 5.2.** Assume that \( u \) is a maximal solution defined above. Then either \( T \) is infinity, or \( T \) is finite and
\[
\lim_{t \to T} \max_S |K(x, t)| = \infty.
\]

**Proof.** Assume without loss of generality that \( T \geq 1 \). It suffices to show that if
\[
\lim_{t \to T} \max_S |K(x, t)| \leq C_1,
\]
then the solution can be extended to \([0, T + \delta]\) for some \( \delta > 0 \).

(5.1) and the evolution equation for \( u \) imply that
\[
\lim_{t \to T} \max_S |u(x, t)| \leq C(C_1, T).
\]
By interior estimates, we know that the \( \mathcal{P}^{2, \alpha, T} \) norms of both \( u \) and \( K \) are bounded by \( C(C_1, T) \).

We claim that
\[
\lim_{t \to T} \max_S \left| \tilde{\Delta} K \right| \leq C(C_1, T).
\]

(5.2) To see this, let’s consider the evolution equation of \( R \)
\[
\partial_t R = e^{-2u} \tilde{\Delta} R + R(R - r).
\]
Before going into the detail, we outline the idea of the proof. First, notice that it is equivalent to have a bound for \( \partial_t R \) since \( R \) and \( u \) is already uniformly bounded. If we take an time derivative of (5.3), we will see that \( \partial_t R \) satisfies a linear parabolic equation. Since we have assume that \( \tilde{\Delta} R_0 \) is bounded, it is natural to expect that our claim is true. The technical issue is that we can’t apply maximum principle to \( \partial_t R \) directly, because we don’t know if it is bounded and has bounded Dirichlet energy. The idea is to go back to the construction of solutions to the linear equation, prove some estimate for approximation sequence and show that the limit is the same as \( \partial_t R_0 \).

Let \( S_k \) be as before. Consider the solution \( R_k \) to the following equation.

\[
\begin{align*}
\partial_t R_k &= e^{-2u} \tilde{\Delta} R_k + R_k (R_k - r) \\
\frac{\partial R_k}{\partial n}|_{\partial S_k} &= 0 \\
R_k(x,0) &= 2K_0.
\end{align*}
\]

Here \( K_0 \) is the initial value of Gauss curvature. We have assumed that \( \tilde{\Delta} K_0 \) is bounded.

A priori, we don’t know whether the solutions exist for \( t \in [0,T] \). However, we do have a uniform estimate independent of \( k \). For any \( k \), since \( 2K_0 \leq C \), we have an estimate for the \( C^0 \) norm of \( R_k \) given by maximum principle on some time interval \( [0,T'] \). Denote the limit of \( R_k \) by \( \hat{R} \). Let’s show some properties of this \( \hat{R} \).

\[
\frac{d}{dt} \int_{S_k} \left| \nabla R_k \right|^2 d\hat{V} = 2 \int_{S_k} \nabla (\partial_t R_k) \cdot \nabla R_k d\hat{V}
\]

\[
= -2 \int_{S_k} \partial_t R_k \tilde{\Delta} R_k d\hat{V}
\]

\[
= -2 \int_{S_k} e^{-2u} \tilde{\Delta} R_k \nabla R_k d\hat{V} + \int_{S_k} \tilde{\Delta} (R_k (R_k - r)) \nabla R_k d\hat{V}
\]

\[
\leq C \int_{S_k} \left| \nabla R_k \right|^2 d\hat{V}.
\]

Here \( C \) depends on the \( C^0 \) norm of \( R_k \), which we know is uniformly bounded on \( [0,T'] \).

By taking a limit, we know \( \hat{R} \) is bounded and has bounded Dirichlet energy for \( t \in [0,T] \). Since both \( R \) and \( \hat{R} \) satisfy the same equation and both of them are bounded with bounded Dirichlet energy, we can show that \( R \equiv \hat{R} \) for \( t \in [0,T'] \) by subtracting the two equations and applying maximum principle to \( R - \hat{R} \).

Taking the \( t \)-derivative of (5.3), we have

\[
\begin{align*}
\partial_t (\partial_t R_k) &= e^{-2u} \tilde{\Delta} (\partial_t R_k) + (\partial_t R_k)(2R_k - r - 2\partial_t u) + 2\partial_t u R_k (R_k - r) \\
\frac{\partial (\partial_t R_k)}{\partial n}|_{\partial S_k} &= 0 \\
(\partial_t R_k)|_{t=0} &= e^{-2u_0} \tilde{\Delta} R_0 + R_0(R_0 - r).
\end{align*}
\]

Maximum principle implies that

\[
|\partial_t R_k|(x,t) \leq C
\]

for \( t \in [0,T'] \). By passing the limit \( k \to \infty \), we obtain that \( \partial_t R \) is bounded for \( t \in [0,T] \). We then take \( R(x,t) \) as initial value in (5.3) and repeat the above argument to show

\[
|\partial_t R|(x,t) \leq C(T)
\]

for all \( t \in [0,T] \). This finishes the proof of (5.2).
In summary, we have proved that for any $0 \leq t < T$, $E^{2,\alpha}$ norm of $u$ and $K$, $C^0$ norm of $\Delta K$ are bounded by $C(T)$. Moreover, by Proposition 4.3, the Dirichlet energy of $u$ and $K$ are bounded. Hence, Theorem 4.4 implies that for any $t_1 < T$, we can solve (4.4) with $u(x, t_1)$ as the initial value for a solution defined on $[t_1, t_1 + \delta(T)]$. We may choose $t_1$ such that $t_1 + \delta(T) > T$. Due to Lemma 4.6, we know the solution on $[t_1, t_1 + \delta(T)]$ is an extension of the maximal solution, which is a contradiction. 

\[ \square \]

5.2. Long time existence and convergence. For the solution we have constructed (by iteration), $K$ is bounded and has finite energy so that the maximum principle applies.

Due to the evolution equation of $K$,

\[ \partial_t K = e^{-2u} \Delta K + K(2K - r), \]

we may conclude that

1. The Gauss curvature has a lower bound as long as the solution exists;
2. If the initial curvature is negative, then the Gauss curvature will converge to a negative number exponentially fast. Hence, we proved the first statement of Theorem 1.1.

If the initial curvature is not negative, we can only deal with the sharp cone case ($\beta_i < 0$), for a reason which will be clear soon.

At time $t = 0$, solve the Poisson equation

\[ \Delta u_0 f_0 = R_0 - r. \]

According to Theorem 3.1 (move the conformal factor to the right), we know $f_0$ is bounded with bounded Dirichlet energy. Solve linear PDE by Lemma 2.2,

(5.5)

\[ \frac{\partial f}{\partial t} = e^{-2u} \Delta f + rf. \]

Lemma 2.4 implies that $f$ has bounded Dirichlet energy. Thanks to Lemma 2.5 and Lemma 2.6, $\partial_t f$ is also bounded with bounded energy.

Now we claim that for $t > 0$

(5.6)

\[ e^{-2u} \Delta f = R - r. \]

The above is true for $t = 0$. Write $\Delta_t = e^{-2u} \Delta$ and compute

\[
\partial_t (\Delta_t f - R + r) = (R - r) \Delta_t f + \Delta_t (\partial_t f) - \partial_t R
= (R - r) R + (R - r)(\Delta_t f - R) + \Delta_t (\Delta_t f + rf) - \Delta_t R - R(R - r)
= \Delta_t (\Delta_t f - R) + r \Delta_t f + (R - r)(\Delta_t f - R)
= \Delta_t (\Delta_t f - R + r) + R(\Delta_t f - R + r).
\]

As discussed before $\Delta_t f - R + r$ is a bounded function with bounded Dirichlet energy (since $\partial_t f$ is), it satisfies the above equation and with zero initial value. Hence, it is zero as long as it exists.

Following Hamilton, set

\[ H = R - r + |\nabla f|^2. \]

Compute

\[ \frac{\partial}{\partial t} H = \Delta H - 2 |M|^2 + rH, \]
where \( M = \nabla \nabla f - \frac{1}{2} \Delta f \cdot g \). Therefore, \( H \) is a subsolution to
\[
(5.7) \quad \frac{\partial H}{\partial t} \leq \Delta H + rH.
\]
By (5.6) and Lemma 3.2, \( H \) is bounded and has bounded energy, then maximum principle can be applied and we know that

1. For any \( T > 0 \), we have an upper bound of \( R \) depending on \( T \). This together with Lemma 5.2 implies the second statement of Theorem 1.1.
2. If the Euler number is smaller than zero, then the curvature will become negative everywhere after some time and the normalized Ricci flow in this case will converge to constant curvature metric.

Hence to complete the proof of Theorem 1.1 it remains to consider the case \( \chi = 0 \). The proof is somewhat different from known ones, which can not be applied directly due to technical reasons. We will first establish some uniform estimates for all \( t > 0 \), which imply a sequentially convergence. We then show that the limit is a flat cone metric and the limit is unique. Hence, the flow converges to this limit as \( t \to \infty \).

In the remaining part of the paper, we are talking about a global solution \( u(x, t) \) to the flow (4.1) when \( \chi = 0 \). For simplicity, we omit this assumption from the statement of lemmas.

**Lemma 5.3.** There is some constant \( C \) independent of \( t \) such that
\[
|u| + |K| + \int_M |\nabla u|^2 dV \leq C.
\]

**Proof.** Applying the maximum principle to (5.7), we obtain that \( K \) is uniformly bounded. For \( u \), consider the evolution equation (5.5) of the potential function \( f \).

Since \( r = 0 \), we have uniform bound for \( |f| \). The flow equation can be written as
\[
\partial_t g = (-e^{2u} \tilde{\Delta} f) g = (-\partial_t f) g.
\]
Integrating over \( t \), we have
\[
g(t) = e^{f_0(x) - f(x, t)} g(0).
\]
This provides the uniform bound for \( |u| \). For the energy of \( u \), note that \( \Delta u \) is uniformly bounded since \( K = e^{-2u}(-\tilde{\Delta} u + \tilde{K}) \) and
\[
\int_M |\nabla u|^2 dV = -\int_M u \Delta u dV.
\]
It remains to justify the integration by parts, which follows from Proposition 4.4 and Lemma 2.8.

With this uniform estimate, we know that for any \( t_i \) going to \( +\infty \), there is a subsequence (still denoted by \( t_i \)) such that \( u(\cdot, t_i) \) converges to some \( u_\infty \). By interior estimates of (4.1), the convergence is smooth away from the singularity. Moreover, \( u_\infty \) is bounded and has bounded energy. Next, we prove the Gauss curvature of the limit is zero. It follows from Proposition 5.29 of [6].

**Lemma 5.4.** There is a constant depending on the initial value such that
\[
\sup_{x \in M} |\nabla f|^2 \leq \frac{C}{1 + t}.
\]
Proof. Exactly the same proof. It suffices to note that we can apply the maximum principle to \(t|\nabla f|^2 + f^2\) for reason discussed above.

Let \(\Omega\) be some domain of \(M\) away from the singularity. For any \(t_i\), set \(v_i(x, t) = u(x, t - t_i)\) and \(f_i(x, t) = f(x, t - t_i)\). Then \(v_i\) is a solution to (1.1) on \(\Omega \times [-1, 0]\) and \(f_i\) is still the potential function in the sense that if \(R_i\) is the scalar curvature of \(e^{2v_i}g_0\),

\[
e^{-2v_i} \tilde{\Delta} f_i = r - R_i
\]

and \(f_i\) solves

\[
\partial_t f_i = e^{-2v_i} \tilde{\Delta} f_i.
\]

Moreover, we have \(C^k\) uniform bound of \(v_i\) and \(f_i\) over \(\Omega \times [-1, 0]\). Passing to the limit, denote the limit by \(v_\infty\) and \(f_\infty\) respectively. Thanks to the above lemma, \(f_\infty\) is constant in space. The limit equation of \(f_\infty\) then implies that \(f_\infty\) is constant in \(\Omega' \times [-1/2, 0]\). (\(\Omega'\) is a smaller domain.) Hence, the limit metric \(e^{2v_\infty}g_0\) has constant curvature zero.

Finally, if we have two sequence of \(t_i\) going to infinity and obtain two limit metric \(e^{2u_1}g_0\) and \(e^{2u_2}g_0\), we need to show that they are the same. Note that both metrics are flat away from the singularity and both \(u_1\) and \(u_2\) are bounded with bounded energy. Hence,

\[
-K = -\tilde{\Delta} u_1 = -\tilde{\Delta} u_2.
\]

The difference is a constant and the constant must be zero, since the two limit metric have the same volume. The proof to Theorem 1.1 is done.

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