Homogenization of an advection equation with locally stationary random coefficients

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Abstract

In the paper we consider the solution of an advection equation with rapidly changing
coefficients \( \partial_t u_\varepsilon + \left(1/\varepsilon\right) V(t \varepsilon^{-2}, x/\varepsilon) \cdot \nabla_x u_\varepsilon = 0 \) for \( t < T \) and \( u_\varepsilon(T, x) = u_0(x), x \in \mathbb{R}^d \). Here \( \varepsilon > 0 \) is some small parameter and the drift term \( (V(t, x))_{(t,x) \in \mathbb{R}^{1+d}} \) is assumed to be a \( d \)-dimensional, vector valued random field with incompressible spatial realizations. We prove that when the field is Gaussian, locally stationary, quasi-periodic in the \( x \) variable and strongly mixing in time the solutions \( u_\varepsilon(t, x) \) converge in law, as \( \varepsilon \rightarrow 0 \), to \( u_0(x(T; t, x)) \), where \( (x(s; t, x))_{s \geq t} \) is a diffusion satisfying \( x(t; t, x) = x \). The averages of \( u_\varepsilon(T, x) \) converge then to the solution of the corresponding Kolmogorov backward equation.

1 Introduction

In the present paper, we consider solutions of linear advection equations with rapidly oscillating random coefficients of the form

\[
\partial_t u_\varepsilon(t, x) + \frac{1}{\varepsilon} V \left( \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right) \cdot \nabla_x u_\varepsilon(t, x) = 0,
\]
\[
u_\varepsilon(T, x) = u_0(x), \quad t < T, x \in \mathbb{R}^d.
\] (1.1)

Here, \((V(t, x))_{(t,x) \in \mathbb{R}^{1+d}}\) is a random, zero-mean, incompressible, Gaussian, vector-valued random field and \( \varepsilon > 0 \). We are interested in the diffusive scaling limit of the solutions, as the parameter \( \varepsilon \) tends to 0. Equation (1.1) appears e.g. in the passive scalar model that describes a concentration of particles drifting in a time-dependent, incompressible random flow and has

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applications in both turbulent diffusion and stochastic homogenization, see e.g. [25, 23, 32, 30] and the references therein. The model has been extensively studied, both in the mathematics and physics literature, under various assumptions on the advection term $V(t, x)$. A typical result states that, if the field is stationary and sufficiently strongly mixing, then the underlying random characteristics (that correspond to the trajectory realizations of the drifting particle) converge in law to a zero mean Brownian motion $(\beta_t)_{t \geq 0}$ whose covariance matrix $[a_{p,q}]_{p,q=1,\ldots,d}$ is determined by the statistics of $V(\cdot, \cdot)$, see e.g. [21, 6, 7, 8, 12, 20, 18]. In that case the laws of the solutions of (1.1) converge, as $\varepsilon \to 0$, to $u_0(x + \beta_{T-t})$. Its expectation $\bar{u}(t, x)$ satisfies

$$\partial_t \bar{u}(t, x) + \frac{1}{2} \sum_{p,q=1}^{d} a_{pq} \partial^2_{x_p x_q} \bar{u}(t, x) = 0, \quad t \leq T,$$

$$\bar{u}(T, x) = u_0(x).$$

Since the coefficients of equation (1.2) do not depend on the spatial variable, the limiting procedure is sometimes referred to as homogenization. Stationarity and ergodicity of the velocity field play a crucial role in substantiating the existence of the limit in homogenization, as the argument relies on an application of some form of an ergodic theorem.

The main purpose of the present article is to investigate the situation when the coefficients of the advection equation (1.1) are no longer stationary. We assume instead that the velocity can be written as $V(t, x, \varepsilon x)$, for some random vector field $V(t, x, y)$, where for a fixed $y$ the field is assumed to be stationary and ergodic in the variables $(t, x)$. The variable $y$ represents a 'slow' parameter i.e. when $\varepsilon \ll 1$ then the statistics of the field $V(t, x, \varepsilon x)$ suffer a significant change only when $|x| \sim 1/\varepsilon$. For technical reasons we shall also assume that $V(t, x, y)$ is quasi-periodic in the $x$ variable. A more precise description of the fields considered in the paper is given in Section 2.1. In our main result, see Theorem 3.1 below, we show that, as $\varepsilon \to 0$, the limit of $u_\varepsilon(t, x)$, in the law, is given by $u_0(x(T; t, x))$, where $(x(t; s, x))_{t \geq s}$ is the diffusion, starting at $s$ at position $x$ with the generator given by the differential operator defined in (3.3). Then $\bar{u}(t, x)$ - the expectation of $u_0(x(T; t, x))$ - is the solution of the respective Kolmogorov backward parabolic equation

$$\partial_t \bar{u}(t, x) + \sum_{p=1}^{d} B_p(x) \partial_{x_p} \bar{u}(t, x) + \frac{1}{2} \sum_{p,q=1}^{d} A_{pq}(x) \partial^2_{x_p x_q} \bar{u}(t, x) = 0, \quad t < T,$$

$$\bar{u}(T, x) = u_0(x)$$

with the respective coefficients appearing in the definition of the generator.

Homogenization of parabolic and elliptic equations with locally periodic coefficients has been considered in Chapter 6 of the book [4]. The generalization to the case of random parabolic equations in divergence form with locally stationary and ergodic coefficients, has been done in [29]. An analogous question in the case of difference equations in divergence
form in dimension one has been considered in [27]. The notion of local ergodicity used in ibid. differs from the one in [29] and is conceptually closer to the one considered in the present paper. Homogenization of linear parabolic equations in non-divergence form with non-stationary coefficients has been treated in [5]. Somewhat related problem of averaging with two scale (fast and slow) motion, but under a scaling different from ours, has been also considered in the literature, see e.g. [17, 16] and references therein.

Our proof is based on an analysis of the asymptotics of the random characteristics corresponding to the advection equation (1.1). We apply the corrector method to eliminate the large amplitude terms that arise in the description of the characteristics. This requires showing regularity of the correctors with respect to the parameter that corresponds to the slow variable of the velocity field. In Section 6 we prove several results concerning the regularity properties of the corrector, which seem to be of independent interest. They are obtained by a technique based on an application of the Malliavin calculus, which is related to the method used in [14] to establish asymptotic strong Feller property for the solutions of stochastic Navier-Stokes equations in two dimensions. It is essentially the only place in our argument that requires the hypothesis of quasi-periodicity of the flow. To show the existence of the limit (in law) of the processes corresponding to the random characteristics we apply an averaging lemma, see Lemma 5.2 below, which is a version of a suitable ergodic theorem.

The organization of the paper is as follows. In Section 2 we present more detailed description of the model, which we are going to study and formulate some of its basic properties. The main result, see Theorem 3.1 below, is formulated in Section 3. Its proof is contained in Section 5. For the notational convenience we conduct the argument only for dimension $d = 2$. Section 4 contains a detailed description of the two dimensional case. It is clear from our proof that it can be easily generalized to the case of an arbitrary dimension. However this can be done at the expense of a considerably heavier notation, see Section 5.5 for the discussion of the general dimension situation. Finally, Sections 6–8 are devoted to showing some technical results needed for the proof of our main theorem.

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2 Preliminaries

2.1 Quasi-periodic, locally stationary fields of coefficients

Given \( \varepsilon > 0 \) we let \( V_\varepsilon = (V_{1,\varepsilon}, \ldots, V_{d,\varepsilon} : \mathbb{R}^{1+d} \times \Omega \rightarrow \mathbb{R}^d \) be a random, vector field. Here \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space. We let \( \mathbb{E} \) be the expectation with respect to \( \mathbb{P} \). To ensure that the field has divergence free realizations we let

\[
V_{m,\varepsilon}(t, x) = \sum_{l=1}^{d} \partial_{x_l} H^\varepsilon_{l,m}(t, x), \quad m = 1, \ldots, d, \tag{2.1}
\]

where \( H^\varepsilon(t, x) := [H^\varepsilon_{l,m}(t, x)]_{l,m=1,\ldots,d} \) is a \( d \times d \) anti-symmetric matrix valued random, quasi-periodic field of the form \( H^\varepsilon_{l,m}(t, x) = H_{l,m}(t, x, \varepsilon x) \), with

\[
H_{l,m}(t, x, y) = \sum_{i=1}^{N} [a_j(t, y) \cos(k_i \cdot x) + b_j(t, y) \sin(k_i \cdot x)], \quad (t, x, y) \in \mathbb{R}^{1+2d},
\]

where \( N \) is fixed natural number and \( j \) denotes the multi-index \((i, l, m)\) made of three components \( i = 1, \ldots, N \) and \( l, m = 1, \ldots, d \). Here we let also \( k_i = (k_{i,1}, \ldots, k_{i,d}) \in \mathbb{R}^d \).

The random fields \((a_j(t, y))_{(t,y)\in\mathbb{R}^{1+d}}, (b_j(t, y))_{(t,y)\in\mathbb{R}^{1+d}}\) for

\[
j \in Z := \{(i, l, m) : i = 1, \ldots, N, 1 \leq l < m \leq d\} \tag{2.2}
\]

are of the form

\[
a_j(t; y) = \sqrt{2\alpha_j(y)} \sigma_j(y) \int_{-\infty}^{t} e^{-\alpha_j(y)(t-s)} dw_{j,a}(s),
\]

\[
b_j(t; y) = \sqrt{2\alpha_j(y)} \sigma_j(y) \int_{-\infty}^{t} e^{-\alpha_j(y)(t-s)} dw_{j,b}(s). \tag{2.3}
\]

Here \( w_{j,a}(t), w_{j,b}(t), j \in Z \) are independent, two-sided one dimensional standard Brownian motions. For the indices \( j = (i, l, m) \), with \( m \geq l \) we let

\[
a_{i,l,m}(t; y) = -a_{i,m,l}(t; y), \quad b_{i,l,m}(t; y) = -b_{i,m,l}(t; y), \quad i = 1, \ldots, N.
\]

The above implies in particular that

\[
a_{i,m,m}(t; y) = b_{i,m,m}(t; y) \equiv 0, \quad m = 1, \ldots, d, i = 1, \ldots, N.
\]

Functions \( \alpha_j(\cdot), \sigma_j(\cdot) \) are assumed to belong to \( C^2_b(\mathbb{R}^d) \) - the class of twice, continuously differentiable functions with bounded derivatives and satisfy

\[
1/\sigma_* \geq \sigma_j(y) \geq \sigma_*, \quad 1/\gamma_0 \geq \alpha_j(y) \geq \gamma_0 \quad \text{for } y \in \mathbb{R}^d \tag{2.4}
\]

and some \( \gamma_0, \sigma_* \in (0, 1) \). It is clear from (2.3) that for each \( y \) the processes \((a_j(t, y))_{t\in\mathbb{R}}, (b_j(t, y))_{(t,y)\in\mathbb{R}}\) are the stationary solutions of the Itô stochastic differential equations

\[
da_j(t; y) = -\alpha_j(y)a_j(t; y)dt + \sqrt{2\alpha_j(y)} \sigma_j(y)dw_{j,a}(t),
\]

\[
 db_j(t; y) = -\alpha_j(y)b_j(t; y)dt + \sqrt{2\alpha_j(y)} \sigma_j(y)dw_{j,b}(t). \tag{2.5}
\]
2.2 Markov property of the process

The generator \( L^y \) of the \( \mathbb{R}^{2S} \)-valued process \( \alpha(t, y) := (a_j(t; y), b_j(t; y))_{j \in \mathbb{Z}} \) equals \( (2.5) \)

\[
L^y F(a) = \sum_{j \in \mathbb{Z}} \left( L_{j,a_j}^y + L_{j,b_j}^y \right) F(a), \quad F \in C^2(\mathbb{R}^{2S}), \ a := (a_j, b_j)_{j \in \mathbb{Z}} \in \mathbb{R}^{2S}. \tag{2.6}
\]

Here \( S \) denotes the cardinality of \( \mathbb{Z} \). The one dimensional differential operators \( L_{j,a_j}^y, L_{j,b_j}^y \) act on the \( a_j \) and \( b_j \) variables, respectively, with

\[
L_{j,a_j}^y f(a) := \alpha_j(y) \left[ \sigma_j^2(y) f''(a) - a f'(a) \right], \quad f \in C^2(\mathbb{R}). \tag{2.7}
\]

The Gaussian product measure

\[
\nu^y_x(da) = \prod_{j \in \mathbb{Z}} \Phi_{\sigma_j(y)}(a_j) \Phi_{\sigma_j(y)}(b_j) da_j db_j, \tag{2.8}
\]

where for \( \sigma > 0 \)

\[
\Phi_\sigma(a) := \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{a^2}{2\sigma^2} \right\}, \quad a \in \mathbb{R},
\]

is invariant under the dynamics corresponding to the generator \( L^y \), i.e.

\[
\int_{\mathbb{R}^{2S}} L^y F d\nu^y_x = 0 \tag{2.9}
\]

for any \( F \in C^2(\mathbb{R}^{2S}) \) of at most polynomial growth.

The following result is a consequence of Propositions 12.4 and 12.14, part v) of \([19]\).

**Proposition 2.1** (Spectral gap property of the generator). Fix \( y \in \mathbb{R}^d \). The set \( \mathcal{P} \) of polynomials on \( \mathbb{R}^{2S} \) constitutes a core of \( L^y \). Assume that \( F \in \mathcal{P} \) satisfies

\[
\int_{\mathbb{R}^{2S}} F d\nu^y_x = 0. \tag{2.10}
\]

Then,

\[
-\langle L^y F, F \rangle_{\nu^y_x} \geq \gamma_0 ||F||^2_{L^2(\nu^x_y)}, \tag{2.11}
\]

where \( \gamma_0 \) was introduced in \([2.4]\).

2.3 Homogeneous fields

For \( x \in \mathbb{R}^d \) and \( \alpha = ((a_j, b_j))_{j \in \mathbb{Z}} \) we let \( \tau_x : \mathbb{R}^{2S} \to \mathbb{R}^{2S} \) by the formula \( \tau_x(\alpha) = (a'_j, b'_j)_{j \in \mathbb{Z}} \), where

\[
a'_j := a_j \cos(k_i \cdot x) + b_j \sin(k_i \cdot x), \quad b'_j := -a_j \sin(k_i \cdot x) + b_j \cos(k_i \cdot x), \quad j \in \mathbb{Z}. \tag{2.12}
\]
It is easy to check that \((\tau_x)_{x \in \mathbb{R}^d}\) forms a group of transformations with \(\tau_x \tau_y = \tau_{x+y}, x, y \in \mathbb{R}^d\). For the function \(G : \mathbb{R}^{2N} \times \mathbb{R}^d \to \mathbb{R}\) we denote by \(\tilde{G} : \mathbb{R} \times \mathbb{R}^{2d} \times \Omega \to \mathbb{R}\) the random field given by

\[
\tilde{G}(t, x, y) := G(\tau_x (a(t;y))),
\]
where \((a(t;y))\) is the process given by (2.3).

### 3 Statement of the main result

Our main result concerns the diffusive scaling limit for the random characteristics of (1.1). They are given by the trajectories of solutions of the ordinary differential equation with the random right hand side given by the field \(V_\epsilon(t,x)\), defined in Section 2.1. More precisely, suppose that

\[
\begin{aligned}
\frac{dX_{\epsilon}^{s,x_0}(t)}{dt} &= V_\epsilon(t, X_{\epsilon}^{s,x_0}(t)), \\
X_{\epsilon}^{s,x_0}(s) &= x_0,
\end{aligned}
\]
where \(\epsilon > 0\) and \(x_0 \in \mathbb{R}^d\) and \(s,t \in \mathbb{R}\). The diffusively scaled processes \(x_\epsilon(t;s,x_0) := \epsilon X_{\epsilon^{2},x_0/\epsilon^2}(t/\epsilon^2)\) satisfy

\[
\begin{aligned}
\frac{dx_\epsilon(t;s,x_0)}{dt} &= \frac{1}{\epsilon} W \left( \frac{t}{\epsilon^2}, \frac{x_\epsilon(t;s,x_0)}{\epsilon}, x_\epsilon(t;s,x_0) \right) + U \left( \frac{t}{\epsilon^2}, \frac{x_\epsilon(t;s,x_0)}{\epsilon}, x_\epsilon(t;s,x_0) \right), \\
x_\epsilon(s;s,x_0) &= x_0,
\end{aligned}
\]
where \(W = (W_1, \ldots, W_d)\), \(U = (U_1, \ldots, U_d)\) are given by

\[
W_m(t,x,y) = \sum_{l=1}^{d} \partial_{x_l} H_{l,m}(t,x,y), \quad U_m(t,x,y) = \sum_{l=1}^{d} \partial_{y_l} H_{l,m}(t,x,y).
\]

The main result of the present paper can be formulated as follows.

**Theorem 3.1.** For a given \((s,x_0) \in \mathbb{R}^{1+d}\) the processes \((x_\epsilon(t;s,x_0))_{t \geq s}\) converge in law over \(C([s,\infty);\mathbb{R}^d]\), when \(\epsilon \to 0\), to the diffusion \((x(t;s,x_0))_{t \geq s}\), which starts at time \(s\) at \(x_0\) and whose generator is given by

\[
\mathcal{L} f(y) = \sum_{l=1}^{d} B_l(y) \partial_{y_l} f(y) + \frac{1}{2} \sum_{l,l'=1}^{d} A_{l,l'}(y) \partial_{y_l}^2 f(y), \quad f \in C^2(\mathbb{R}^d), \quad y \in \mathbb{R}^d,
\]
where coefficients \(B_l(\cdot)\) and \(A_{l,l'}(\cdot)\), \(l,l' = 1, \ldots, d\) are defined by formulas (5.50) below.

Using the characteristics of (1.1) we can write the solution in the form

\[
u_\epsilon(t,x) = u_0(x_\epsilon(T;t,x)), \quad t \leq T, \quad x \in \mathbb{R}^d.
\]
As an immediate corollary of Theorem 3.1 we conclude the following.
Corollary 3.2. Suppose that $u_\varepsilon(t, x)$ is the solution of (1.1) with $u_0$ that is bounded and continuous. Then, the random variables $u_\varepsilon(t, x)$ converge in law, as $\varepsilon \to 0$, to $u_0(x(T; t, x))$. In particular, $\bar{u}(t, x) := \lim_{\varepsilon \to 0} \mathbb{E} u_\varepsilon(x(T; t, x))$ is the bounded solution of

\[ \partial_t \bar{u}(t, x) + \sum_{i=1}^d B_i(x) \partial_{x_i} \bar{u}(t, x) + \frac{1}{2} \sum_{i,i'=1}^d A_{i,i'}(x) \partial_{x_{i,i'}}^2 \bar{u}(t, x) = 0, \quad t < T, \ x \in \mathbb{R}^d, \]

\[ \bar{u}(T, x) = u_0(x), \]

where coefficients $B_i(\cdot)$ and $A_{i,i'}(\cdot)$, $i, i' = 1, \ldots, d$ are as in Theorem 3.1.

4 Prelude to the proof of Theorem 3.1

4.1 Two dimensional case

To simplify the notation we shall assume that $(s, x_0) = (0, 0)$. In that case we shall write $x_\varepsilon(t) := x_\varepsilon(t; 0, 0)$. To further lighten up the notation we shall present the argument for the case $d = 2$. Then, the antisymmetric matrix $H(t, x, y)$ can be described by a scalar and, as a result, this allows to reduce the multi-index $j$ to just a scalar index $i \in \{1, \ldots, N\}$. The case of an arbitrary dimension $d$ requires the same consideration, however the argument will be obscured by some heavy notation.

In this case velocity field $V_\varepsilon = (V_{\varepsilon,1}, V_{\varepsilon,2})$ is given by the formula $V_\varepsilon(t; x) = \nabla_x \perp H_\varepsilon(t; x)$, with $\nabla_x \perp := [\partial_{x_2}, -\partial_{x_1}]$ and $H_\varepsilon(t; x) := H(t, x, \varepsilon x)$, where

\[ H(t, x, y) = \sum_{i=1}^N [a_i(t; y) \cos(k_i \cdot x) + b_i(t; y) \sin(k_i \cdot x)]. \] (4.1)

The processes $a_i(t, y)$ and $b_i(t, y)$ are described by (2.3), with the multi-index $j$ replaced by $i$. The formulas (2.6) and (2.8) for the generator and the invariant measure are modified in an obvious fashion, with $S = N$.

We can write $V_\varepsilon(t, x) = V(t, x, \varepsilon x)$ with

\[ V(t, x, y) = W(t, x, y) + \varepsilon U(t, x, y), \] (4.2)

where

\[ W(t, x, y) = \sum_{i=1}^N k_i^\perp [-a_i(t; y) \sin(k_i \cdot x) + b_i(t; y) \cos(k_i \cdot x)], \] (4.3)

\[ U(t, x, y) = \sum_{i=1}^N a_{i,y}^\perp(t; y) \cos(k_i \cdot x) + b_{i,y}^\perp(t; y) \sin(k_i \cdot x). \]

We shall denote

\[ k_i^\perp := (k_{i,2}, -k_{i,1}) \] (4.4)

for $k_i = (k_{i,1}, k_{i,2})$ and $a_{i,y}^\perp(t; y) := \nabla_y a_i(t; y)$, and likewise for $b_{i,y}^\perp(t; y)$.
4.2 Auxiliary dynamics

For any $y \in \mathbb{R}^d$ we consider the auxiliary dynamics $z(t, y)$ given by

$$
\frac{dz(t, y)}{dt} = W(t, z(t, y), y),
$$

$$
z(0, y) = 0.
$$

(4.5)

Define $(\tilde{a}(t; y))_{t \geq 0}$ an $\mathbb{R}^{2N}$-valued process, given by $\tilde{a}(t; y) := \left(\tilde{a}_i(t; y), \tilde{b}_i(t; y)\right)_{i=1, \ldots, N}$, with

$$
\begin{align*}
\tilde{a}_i(t; y) &:= a_i(t; y) \cos(k_i \cdot z(t, y)) + b_i(t; y) \sin(k_i \cdot z(t, y)), \\
\tilde{b}_i(t; y) &:= -a_i(t; y) \sin(k_i \cdot z(t, y)) + b_i(t; y) \cos(k_i \cdot z(t, y)).
\end{align*}
$$

(4.6)

Equation (4.5) can be rewritten in the form

$$
\frac{dz(t, y)}{dt} = \sum_{i=1}^{N} \tilde{b}_i(t, y) k_i^\perp,
$$

$$
z(0, y) = 0,
$$

(4.7)

For any $k = (k_1, k_2), \ell = (\ell_1, \ell_2) \in \mathbb{R}^2$ we let

$$
\delta(k, \ell) = k \cdot \ell^\perp = k_1 \ell_2 - k_2 \ell_1.
$$

(4.8)

A simple application of Itô formula shows that the components of $\tilde{a}(t; y)$ satisfy the following Itô stochastic differential equation

$$
\begin{align*}
\frac{d\tilde{a}_i(t)}{dt} &= \left\{-\alpha_i \tilde{a}_i(t) + \sum_{j=1}^{N} \delta(k_i, k_j) \tilde{b}_j(t) \tilde{b}_i(t)\right\} dt + \sqrt{2\alpha_i \sigma_i} d\tilde{w}_{i,a}(t), \\
\frac{d\tilde{b}_i(t)}{dt} &= \left\{-\alpha_i \tilde{b}_i(t) - \sum_{j=1}^{N} \delta(k_i, k_j) \tilde{b}_j(t) \tilde{a}_i(t)\right\} dt + \sqrt{2\alpha_i \sigma_i} d\tilde{w}_{i,b}(t),
\end{align*}
$$

(4.9)

Here $\tilde{w}_{i,a}, \tilde{w}_{i,b}, i = 1, \ldots, N$ are i.i.d. standard, one dimensional Brownian motions. To shornen the notation we have omitted writing the argument $y$. Let $\tilde{a} \in \mathbb{R}^{2N}$. We denote by $\tilde{a}^a(t) = \left(\tilde{a}_i^a(t), \tilde{b}_i^a(t)\right)$ the solution of (4.9) satisfying $\tilde{a}^a(0) = a$.

The generator of the diffusion (4.9) is given by

$$
L_y F = \mathbb{L}^y F + \mathbf{w} \cdot DF, \quad F \in C^2(\mathbb{R}^{2N}),
$$

(4.10)

where

$$
DF := \sum_{i=1}^{N} k_i \mathcal{R}_i F,
$$

(4.11)
with the differential operator
\[ R_i F := (b_i \partial_{a_i} - a_i \partial_{b_i}) F, \quad F \in C^2(\mathbb{R}^{2N}). \]  
(4.12)
The mapping \( \mathbf{w} = (w_1, w_2) : \mathbb{R}^{2N} \to \mathbb{R}^2 \) is given by
\[ \mathbf{w}(\mathbf{a}) = \sum_{i=1}^{N} k_i^+ b_i, \quad \mathbf{a} = (a_i, b_i)_{i=1,\ldots,N}. \]  
(4.13)
Obviously the components of \( \mathbf{w} \) belong to \( L^p(\nu_*) \) for any \( p \in [1, +\infty) \). Define also
\[ D^\perp F := \sum_{i=1}^{N} k_i^+ R_i F. \]  
(4.14)
Let \( (P_t^y)_{t \geq 0} \) be the transition semigroup of the diffusion given by (4.9). For any \( p \in [1, +\infty) \) and a natural number \( m \) let \( W^{p,m}(\nu_*) \) be the Sobolev space made of those \( F \in L^p(\nu_*) \) whose partials of order \( m \) are \( L^p \) integrable with respect to \( \nu_* \) equipped with the norm
\[ \| F \|_{W^{p,m}(\nu_*)} := \left\{ \sum_{\ell=0}^{m} \| \nabla_\mathbf{a}^\ell F \|_{L^p(\nu_*)}^p \right\}^{1/p}. \]
Here \( \nabla_\mathbf{a}^\ell \) denotes the \( \ell \)-th order, derivative tensor with respect to the variable \( \mathbf{a} \). By \( W^{p,m}(\mathbb{R}^{2N}) \), \( W^{p,m}_{loc}(\mathbb{R}^{2N}) \) we denote correspondingly the standard Sobolev space with respect to the ”flat” Lebesgue measure and the space of functions that belong to the respective Sobolev space on any ball.

Define the set
\[ C_0 := \bigcap_{p \geq 1} W^{p,2}(\nu_*^y). \]  
(4.15)
A simple calculation shows that
\[ \int_{\mathbb{R}^{2N}} \mathbf{w} \cdot D F G d\nu_*^y = - \int_{\mathbb{R}^{2N}} F \mathbf{w} \cdot D G d\nu_*^y \]  
(4.16)
for any \( F, G \in C_0 \). As a result from (2.9) and (4.16) (with \( G \equiv 1 \)) we conclude that
\[ \int_{\mathbb{R}^{2N}} \mathcal{L}_y F d\nu_*^y = 0 \]  
(4.17)
and, thanks to (2.11),
\[ - \langle \mathcal{L}_y F(\cdot; y), F(\cdot; y) \rangle_{\nu_*^y} \geq \gamma_0 \| F(\cdot; y) \|_{L^2(\nu_*^y)}^2, \quad y \in \mathbb{R}^2, \]  
provided that \( \int_{\mathbb{R}^{2N}} F d\nu_*^y = 0 \)  
(4.18)
for any \( F \in C_0 \). In particular, from (4.17) we obtain that \( \nu_*^y \) is an invariant measure for the dynamics described by (4.9). In consequence,
\[ \int_{\mathbb{R}^{2N}} P_t^y F d\nu_*^y = \int_{\mathbb{R}^{2N}} F d\nu_*^y, \quad t \geq 0, y \in \mathbb{R}^2, \]  
for any bounded and measurable \( F \). The semigroup extends therefore to a Markovian contraction \( C_0 \)-semigroup on \( L^p(\nu_*^y) \) for any \( p \in [1, +\infty) \). We have the following.
Proposition 4.1. (see [19] Section 12.3) $C_0$ is a core of the generator $\mathcal{L}_y$ of the semigroup $(P^y_t)_{t \geq 0}$. On this set, the generator coincides with the differential operator $\mathcal{L}_y$ given by (4.10).

As a direct conclusion from (4.10) we obtain also.

Proposition 4.2. Fix $y \in \mathbb{R}^2$. Assume that a function $F : \mathbb{R}^{2N} \to \mathbb{R}$, is such that $F \in L^2(\nu^y_*)$ and
\[
\int_{\mathbb{R}^{2N}} F d\nu^y_* = 0. \quad (4.19)
\]
Then,
\[
||P^y_t F||_{L^2(\nu^y_*)} \leq e^{-\gamma_0 t} ||F||_{L^2(\nu^y_*)}, \quad t \geq 0. \quad (4.20)
\]
where $\gamma_0$ was introduced in (2.4).

Remark 4.3. Fix $y \in \mathbb{R}^2$. Suppose that $F$ satisfies (4.19). Applying an interpolation argument we conclude, from (4.20) and the fact that $P^y_t$ is contraction in both $L^1(\nu^y_*)$ and $L^\infty(\nu^y_*)$, that for any $p \in (1, +\infty)$ there exists $\gamma(p) \in (0, \gamma_0)$ such that
\[
||P^y_t F(\cdot; y)||_{L^p(\nu^y_*)} \leq e^{-\gamma(p) t} ||F(\cdot; y)||_{L^p(\nu^y_*)}, \quad t \geq 0. \quad (4.21)
\]

4.3 Corrector

Since
\[
\int_{\mathbb{R}^{2N}} w_q d\nu_*^y = 0, \quad q = 1, 2
\]
for any $y \in \mathbb{R}^2$, we can define the corrector in direction $e_q = (\delta_{1,q}, \delta_{2,q})$ by letting
\[
\chi_q(\cdot; y) := \int_0^{+\infty} P^y_t w_q dt, \quad q = 1, 2, \ y \in \mathbb{R}^2.
\]
Thanks to (4.21) it belongs to the $L^p(\nu^y_*)$-domain of the generator $\mathcal{L}_y$ and it is the unique solution of the problem
\[
-\mathcal{L}_y \chi_q(\cdot; y) = w_q \quad \text{and} \quad \int_{\mathbb{R}^{2N}} \chi_q(\cdot; y) d\nu_*^y = 0. \quad (4.22)
\]

From the standard regularity results for diffusions with smooth coefficients it follows that corrector $\chi_q(\cdot; y)$ belongs to the Sobolev space $W^{m,p}_{loc}(\mathbb{R}^{2N})$ for any $p \in [1, +\infty)$ and $m \geq 1$. Far less trivial is the regularity of the corrector in the $y$-variable. From Theorem 6.4 below we can conclude the following.

Proposition 4.4. We have $\chi_q \in W^{2,p}_{loc}(\mathbb{R}^{2N+2})$ for any $p \in [1, +\infty)$ and $q = 1, 2$.\]
5 Proof of Theorem 3.1

To abbreviate the notation for a given function $G : \mathbb{R}^{2N+2} \to \mathbb{R}$ we write

$$G^{(e)}(t) := \tilde{G}\left(\frac{t}{\varepsilon^2}, \bar{x}_\varepsilon(t), x_\varepsilon(t)\right), \quad (5.1)$$

where $\tilde{G}$ is given by (2.13), $x_\varepsilon(t)$ is the scaled trajectory as in (3.2), with $s = 0$, $x_0 = 0$, and $\bar{x}_\varepsilon(t) := \varepsilon^{-1}x_\varepsilon(t)$. Using the Itô-Krylov formula (see [24], Theorem 1, p. 122) and the above convention we obtain

$$d\left[\varepsilon \chi_q^{(e)}(t)\right] = \left\{\frac{1}{\varepsilon}(L\chi_q)^{(e)}(t) + U_\varepsilon(t) \cdot (D\chi_q)^{(e)}(t) + V_\varepsilon(t) \cdot \chi_{q,y}^{(e)}(t) \right. + \sum_{i=1}^{N} V_i^{(e)}(t) \left[ \chi_{q,a_i}^{(e)}(t) a_{i,y}^{(e)}(t) + \chi_{q,b_i}^{(e)}(t) b_{i,y}^{(e)}(t) \right] \right\} dt + \mathcal{M}_q^{(e)}(dt), \quad q = 1, 2, \quad (5.2)$$

Here the processes $\chi_{q,a_i}(t)$, $\chi_{q,b_i}(t)$, $\chi_{q,y}(t)$ are formed from the partials $\chi_{q,a_i}(a, y)$, $\chi_{q,b_i}(a, y)$ and $\chi_{q,y}(a, y)$ using the convention introduced in (2.13) and (5.1). The processes $(L\chi_q)^{(e)}(t)$, $(D\chi_q)^{(e)}(t)$ are obtained from the fields $L\chi_q(t, x, y)$ and $D\chi_q(t, x, y)$ (operator $D$ is defined in (4.11)). We let

$$a^{(e)}(t) = \left(a_i^{(e)}(t), b_i^{(e)}(t)\right)_{i=1, \ldots, N}, \quad (5.3)$$

where

$$a_i^{(e)}(t) := a_i(t, x_\varepsilon(t)) \cos(k_i \cdot \bar{x}_\varepsilon(t)) + b_i(t, x_\varepsilon(t)) \sin(k_i \cdot \bar{x}_\varepsilon(t)), \quad (5.4)$$

$$b_i^{(e)}(t) := -a_i(t, x_\varepsilon(t)) \sin(k_i \cdot \bar{x}_\varepsilon(t)) + b_i(t, x_\varepsilon(t)) \cos(k_i \cdot \bar{x}_\varepsilon(t)).$$

Similarly we define $a_y^{(e)}(t) = \left(a_{i,y}^{(e)}(t), b_{i,y}^{(e)}(t)\right)_{i=1, \ldots, N}$, using the processes $a_{i,y}(t, y)$ and $b_{i,y}(t, y)$ correspondingly.

The martingale term is given by

$$\mathcal{M}_q^{(e)}(dt) := \sum_{i=1}^{N} (\sigma_i \sqrt{2\alpha_i}) (x_\varepsilon(t)) \left[ \chi_{q,a_i}^{(e)}(t) dw_{i,a}(t) + \chi_{q,b_i}^{(e)}(t) dw_{i,b}(t) \right], \quad (5.5)$$

with $w_{i,a}(t), w_{i,b}(t), i = 1, \ldots, N$ independent standard Brownian motions.

Using the fact that $-L\chi_q(t, x, y) = W_q(t, x, y)$ we obtain

$$x_\varepsilon, q(t) = \varepsilon \chi_q^{(e)}(0) - \varepsilon \chi_q^{(e)}(t) + \sum_{i=1}^{2} \int_0^t W_q^{(e)}(s) ds + \int_0^t \mathcal{M}_q^{(e)}(ds), \quad q = 1, 2, \quad (5.6)$$

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where $W_q^{(e,s)}$ are formed (by means of (2.13)) from

$$
W_q^1(t, x, y) := U_q(t, x, y) + U(t, x, y) \cdot D\bar{x}_q(t, x, y) \\
+ \sum_{i=1}^N W(t, x, y) \cdot \left[ \bar{x}_{q,a_i}(t, x, y)\bar{a}_{i,y}(t, x, y) + \bar{x}_{q,b_i}(t, x, y)\bar{b}_{i,y}(t, x, y) \right],
$$

$$
W_q^2(t, x, y) := V(t, x, y) \cdot \bar{x}_{q,y}(t, x, y)
+ \varepsilon \sum_{i=1}^N U(t, x, y) \cdot \left[ \bar{x}_{q,a_i}(t, x, y)\bar{a}_{i,y}(t, x, y) + \bar{x}_{q,b_i}(t, x, y)\bar{b}_{i,y}(t, x, y) \right].
$$

Observe that (cf (4.3)) that the terms constituting $W_q^1$ contain the fields $\bar{a}_{i,y}(t, x, y)$ and $\bar{b}_{i,y}(t, x, y)$ and are of apparent order of magnitude $O(1)$. They are not of the form (5.1) and therefore require a separate treatment. We are going to deal with these terms in Section 5.1. Expressions included in $W_q^2$ are either the terms of order $O(1)$ that are of the form (5.1), or terms that are of apparent order of magnitude $O(\varepsilon)$. We shall deal with them in Section 5.2.

### 5.1 Term $W_q^1$

To avoid using multitude of constants appearing in our estimates, for any two expressions $f, g : A \to [0, +\infty)$, where $A$ is some set, we shall write $f \leq g$ iff there exists $C > 0$ such that $f(a) \leq Cg(a), a \in A$. We shall also write $f \approx g$ iff $f \leq g$ and $g \leq f$.

Comparing (4.3) with the first formula of (5.7) we conclude that

$$
W_q^{(1,e)}(t) = \sum_{i,j} a_{i,y}^{(e)}(t) F_{i,j}^{(e)}(t) + \sum_{i,j} b_{i,y}^{(e)}(t) G_{i,j}^{(e)}(t)
$$

where the summations extend over $i = 1, \ldots, N, j = 1, 2$, $a_{i,y}^{(e)}(t), b_{i,y}^{(e)}(t)$ are given by

$$
a_{i,y}^{(e)}(t) := a_{i,y}(t, x_\varepsilon(t)) \cos(k_i \cdot \bar{x}_\varepsilon(t)) + b_{i,y}(t, x_\varepsilon(t)) \sin(k_i \cdot \bar{x}_\varepsilon(t)),
$$

$$
b_{i,y}^{(e)}(t) := -a_{i,y}(t, x_\varepsilon(t)) \sin(k_i \cdot \bar{x}_\varepsilon(t)) + b_{i,y}(t, x_\varepsilon(t)) \cos(k_i \cdot \bar{x}_\varepsilon(t))\)$$

and $F_{i,j}^{(e)}(t), G_{i,j}^{(e)}(t)$ are the processes obtained (using (5.1)) from

$$
F_{i,j}(a, y) = e_{q,j}^{L} + D_{j}^{L} \chi_{q}(a, y) + w_{j}(a)\chi_{q,a_i}(a, y),
$$

$$
G_{i,j}(a, y) = w_{j}(a)\chi_{q,b_i}(a, y), \quad i = 1, \ldots, N, j = 1, 2,
$$

with $e_{q}^{L}$, $w$ and $D^{L}$ given by (4.4), (4.13) and (4.14) respectively. Using Theorem 6.4 and Proposition 7.1 one can conclude that these functions satisfy

$$
\sup_{y \in \mathbb{R}^2} \left( ||F_{i,j}(\cdot; y)||_{L^p(\mu^2)} + ||G_{i,j}(\cdot; y)||_{L^p(\nu^2)} \right) < +\infty, \quad j = 1, 2, i = 1, \ldots, N
$$

for any $p \in [1, +\infty)$.  

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To average expressions of the form \(5.8\) we represent them, using appropriately defined correctors, as functionals of the process \(a^{(e)}(t)\) (cf. \(5.3\)). This enables us to apply our averaging result, see Theorem \(5.2\) below, to identify their appropriate limits, as \(\varepsilon \to 0\).

We introduce the correctors \(\Theta_{i,j}^{(1)}, \Theta_{i,j}^{(2)} : \mathbb{R}^{2N+2} \to \mathbb{R}\), which are the solutions of the following system of equations

\[
\begin{align*}
[\mathcal{L}_y - \alpha_i(y)] \Theta_{i,j}^{(1)}(\cdot; y) - (k_i \cdot w)\Theta_{i,j}^{(2)}(\cdot; y) &= F_{i,j}(\cdot; y), \\
[\mathcal{L}_y - \alpha_i(y)] \Theta_{i,j}^{(2)}(\cdot; y) + (k_i \cdot w)\Theta_{i,j}^{(1)}(\cdot; y) &= G_{i,j}(\cdot; y),
\end{align*}
\]

such that

\[
\left\| \Theta_{i,j}^{(e)}(\cdot; y) \right\|_{L^p(E^\omega)} < +\infty, \quad i,j = 1, 2, \quad i = 1, \ldots, N, \quad p \in [1, +\infty), \quad y \in \mathbb{R}^2.
\]

To solve the above system, note that the function \(\Psi_{i,j} : \mathbb{R}^{2N+2} \to \mathbb{C}\)

\[
\Psi_{i,j} := \Theta_{i,j}^{(1)} + i\Theta_{i,j}^{(2)}
\]

satisfies

\[
[\mathcal{L}_y - \alpha_i(y)] \Psi_{i,j}(\cdot; y) + i(k_i \cdot w)\Psi_{i,j}(\cdot; y) = K_{i,j}(\cdot; y),
\]

with \(K_{i,j} := F_{i,j} + iG_{i,j}\). Here \(i = \sqrt{-1}\) denotes the imaginary unit and \(w\) is given by \(5.1\).

Using the Feynman-Kac formula we obtain that the solution \(5.13\) is given by

\[
\Psi_{i,j}(a; y) = \int_0^{+\infty} \mathbb{E} \left[ \exp \left\{ -\alpha_i(y)t + i \int_0^t (k_i \cdot w)(\tilde{a}^s(t; y))ds \right\} K_{i,j}(\tilde{a}^t(t; y); y) \right] dt,
\]

\(i = 1, \ldots, N, \quad j = 1, 2\). Here \(\tilde{a}^s(t) = (\tilde{a}^s(t), \tilde{b}^s(t))\) the solution of \(4.9\) satisfying \(\tilde{a}^0(0) = a\). The improper integral above converges absolutely, thanks to assumption \(2.4\).

The following lemma allows us to replace expressions containing processes \(a^{(e)}_{i,y_j}(t)\) and \(b^{(e)}_{i,y_j}(t)\) by functionals of the processes \(a^{(e)}_i(t)\) and \(b^{(e)}_i(t)\).

**Lemma 5.1.** Assume that \(F_{i,j}, \quad G_{i,j}, \quad i = 1, \ldots, N, \quad j = 1, 2\) satisfy \(5.10\) and \(\Theta_{i,j}^{(e)}\) are the respective solutions of \(5.11\). Suppose also that the processes \(\left(a^{(e)}_{i,y_j}(t)\right)_{t \geq 0}, \left(b^{(e)}_{i,y_j}(t)\right)_{t \geq 0}\) are given by \(5.9\) and \(\left(F^{(e)}_{i,j}(t)\right)_{t \geq 0}, \left(G^{(e)}_{i,j}(t)\right)_{t \geq 0}\) are obtained from \(F_{i,j}, \quad G_{i,j}\) using \(5.1\). Then,

\[
\begin{align*}
\int_0^t \left[ a^{(e)}_{i,y_j}(s)F^{(e)}_{i,j}(s) + b^{(e)}_{i,y_j}(s)G^{(e)}_{i,j}(s) \right] ds \\
= \int_0^t \left\{ \partial_y \alpha_i(x_c(s)) \left( \Theta_{i,j}^{(1,e)}(s)a^{(e)}_i(s) + \Theta_{i,j}^{(2,e)}(s)b^{(e)}_i(s) \right) \right\} ds + \varepsilon \tilde{N}^{(e)}(t),
\end{align*}
\]

where \(\varepsilon \tilde{N}^{(e)}(t)\) is a negligible semi-martingale i.e. for any \(T > 0\) we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \sup_{0 \leq t \leq T} \left| \tilde{N}^{(e)}(t) \right| = 0.
\]
Proof. From (2.3) we obtain

\[
a_{i,y}(t; y) = \sqrt{2\alpha_i(y)}\gamma_{i,j}(y) \int_{-\infty}^t e^{-\alpha_i(y)(t-s)}dw_{i,a}(s) \tag{5.17}
\]

\[-\sqrt{2\alpha_i(y)}\sigma_i(y) \int_{-\infty}^t (t-s)\alpha_{i,y}(y)e^{-\alpha_i(y)(t-s)}dw_{i,a}(s).
\]

with

\[
\gamma_{i,j}(y) := \sigma_{i,y}(y) + \frac{\alpha_{i,y}(y)}{2\alpha_i(y)}\sigma_i(y), \quad y \in \mathbb{R}^2.
\]

Its Itô stochastic differential equals

\[
da_{i,y}(t; y) = -\left[\alpha_{i,y}(y)a_i(t; y) + \alpha_i(y)a_{i,y}(t; y)\right] dt + \sqrt{2\alpha_i(y)}\gamma_{i,j}(y)dw_{i,a}(t). \tag{5.18}
\]

Similar formulas hold for \((b_{i,y}(t; y))\). Recall that (cf (2.13) and (4.3))

\[
\Theta^{(1,e)}_{i,j}(s) := \tilde{\Theta}^{(i)}_{i,j} \left( \frac{t}{\varepsilon}, \bar{x}(t), x(t) \right), \quad W^{(e)}(t) := W \left( \frac{t}{\varepsilon}, \bar{x}(t), x(t) \right)
\]

and likewise for \(U^{(e)}(t)\).

From the Leibnitz rule applied to the respective stochastic differentials and straightforward (but rather lengthy calculation) we obtain

\[
\int_0^t \left\{ \alpha_{i,y}(x_{\varepsilon}(s)) \left( \Theta^{(1,e)}_{i,j}(s)a_i^{(e)}(s) + \Theta^{(2,e)}_{i,j}(s)b_i^{(e)}(s) \right) \right\} ds + \varepsilon \tilde{N}^{(e)}(t)
\]

\[
= \int_0^t \left[ \mathcal{L} - \alpha_i(x_{\varepsilon}(s)) \right] \Theta^{(1,e)}_{i,j}(s) - (k_i \cdot W^{(e)}(s))\Theta^{(2,e)}_{i,j}(s) \right\} a_{i,y}(s)ds \tag{5.19}
\]

\[
+ \int_0^t \left[ \mathcal{L} - \alpha_i(x_{\varepsilon}(s)) \right] \Theta^{(2,e)}_{i,j}(s) + (k_i \cdot W^{(e)}(s))\Theta^{(1,e)}_{i,j}(s) \right\} b_{i,y}(s)ds.
\]

We have denoted by

\[
\tilde{N}^{(e)}(t) := \varepsilon \left[ \Theta^{(1,e)}_{i,j}(s)a_{i,y}(s) + \Theta^{(2,e)}_{i,j}(s)b_{i,y}(s) \right]_s^t - \int_0^t \Gamma^{(e)}(s)ds - M^{(e)}(t), \tag{5.20}
\]

where \([f(s)]_s^{s=0} := f(t) - f(0)\). We let \(M_0^{(e)} = 0\) and

\[
\Gamma^{(e)}(t) := (k_i \cdot U^{(e)}(t))\left[ \Theta^{(1,e)}_{i,j}(t) b_{i,y}(t) - \Theta^{(2,e)}_{i,j}(t) a_{i,y}(t) \right]
\]

\[
+ a_{i,y}(t) \left\{ U^{(e)}(t) \cdot \left( D\Theta^{(1,e)}_{i,j}(e) + V^{(e)}(t) \cdot \left\{ \sum_{i'} \left[ \Theta^{(1,e)}_{i,i',a}(t)a_{i,y}(t) + \Theta^{(1,e)}_{i,i',b}(t)b_{i,y}(t) \right] + \Theta^{(1,e)}_{i,y}(t) \right\} \right) \right\}
\]

\[
+ b_{i,y}(t) \left\{ U^{(e)}(t) \cdot \left( D\Theta^{(2,e)}_{i,j}(e) + V^{(e)}(t) \cdot \left\{ \sum_{i'} \left[ \Theta^{(2,e)}_{i,i',a}(t)a_{i,y}(t) + \Theta^{(2,e)}_{i,i',b}(t)b_{i,y}(t) \right] + \Theta^{(1,e)}_{i,y}(t) \right\} \right) \right\}
\]

\[
+ 2\varepsilon (\alpha_i \gamma_{i,j} \sigma_i)(x_{\varepsilon}(t)) \left\{ \cos(k_i \cdot \bar{x}(t))\Theta^{(1,e)}_{i,j,a}(t) - \sin(k_i \cdot \bar{x}(t))\Theta^{(2,e)}_{i,j,a}(t) \right\}
\]

\[+ \sin(k_i \cdot \bar{x}(t))\Theta^{(1,e)}_{i,j,b}(t) + \cos(k_i \cdot \bar{x}(t))\Theta^{(2,e)}_{i,j,b}(t) \right) + V^{(e)}(t) \cdot \left[ \Theta^{(1,e)}_{i,y}(t)a_{i,y}(t) + \Theta^{(2,e)}_{i,y}(t)b_{i,y}(t) \right], \tag{5.21}
\]
and
\[
dM^{(e)}(t) := \sum_{i'} a^{(e)}_{i',y_j}(t) (\sigma_{i'} \sqrt{2\alpha_{i'}})(x_{i'}(t)) \left[ \Theta^{(1,e)}_{i,j,a_{i'}(t)} dw_{i',a}(t) + \Theta^{(1,e)}_{i,j,b_{i'}(t)} dw_{i',b}(t) \right] \\
+ \sum_{i'} b^{(e)}_{i,y_j}(t) (\sigma_{i'} \sqrt{2\alpha_{i'}})(x_{i'}(t)) \left[ \Theta^{(2,e)}_{i,j,a_{i'}(t)} dw_{i',a}(t) + \Theta^{(2,e)}_{i,j,b_{i'}(t)} dw_{i',b}(t) \right] \\
+ \Theta^{(1,e)}_{i,j}(t) \left\{ (\gamma_{i,j} \sqrt{2\alpha_{i'}})(x_{i'}(t)) \left[ \cos(k_i \cdot \bar{x}_{i'}(t)) dw_{i,a}(t) + \sin(k_i \cdot \bar{x}_{i'}(t)) dw_{i,b}(t) \right] \right\} \\
+ \Theta^{(2,e)}_{i,j}(t) \left\{ (\gamma_{i,j} \sqrt{2\alpha_{i'}})(x_{i'}(t)) \left[ -\sin(k_i \cdot \bar{x}_{i'}(t)) dw_{i,a}(t) + \cos(k_i \cdot \bar{x}_{i'}(t)) dw_{i,b}(t) \right] \right\}. \tag{5.22}
\]

The processes \(a^{(e)}_{i',y_j}(t)\) and \(b^{(e)}_{i,y_j}(t)\) appearing in the above formulas are given by analogues of (5.9), where the first derivatives in \(y\) are replaced by the respective second derivatives.

Substituting from (5.13) we obtain equality (5.15). In order to prove (5.16) we consider the three terms that appear on the right hand side of (5.20). Concerning the boundary terms, by the Cauchy-Schwarz inequality, we can estimate as follows
\[
\varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Theta^{(1,e)}_{i,j}(t) a^{(e)}_{i,y_j}(t)| \right] \leq \varepsilon \left[ \mathbb{E} \sup_{0 \leq t \leq T} \left( a^{(e)}_{i,y_j}(t) \right)^2 \right]^{1/2} \left[ \mathbb{E} \sup_{0 \leq t \leq T} \left( \Theta^{(1,e)}_{i,j}(t) \right)^2 \right]^{1/2}. \tag{5.23}
\]
Furthermore
\[
\sup_{0 \leq t \leq T} |a^{(e)}_{i,y_j}(t)| \leq \sup_{0 \leq t \leq T, y \in \mathbb{R}^2} \left\{ |a_{i,y_j}(t/\varepsilon^2; y)| + |b_{i,y_j}(t/\varepsilon^2; y)| \right\}. \tag{5.24}
\]

The fields \((a_{i,y_j}(t; y))_{(t,y) \in \mathbb{R}^3}, (b_{i,y_j}(t; y))_{(t,y) \in \mathbb{R}^3}\) are Gaussian. Using the results of Section 7 we conclude that for any \(\gamma \in (0, 1)\) we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( a^{(e)}_{i,y_j}(t) \right)^2 \right] \leq \varepsilon^{-\gamma}, \quad \varepsilon \in (0, 1). \tag{5.25}
\]

From (5.25) we obtain that
\[
\varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Theta^{(1,e)}_{i,j}(t) a^{(e)}_{i,y_j}(t)| \right] \leq \varepsilon^{1-\gamma/2} \left\{ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left( \Theta^{(1,e)}_{i,j}(t) \right)^2 \right] \right\}^{1/2}. \tag{5.26}
\]

Applying Lemma 8.1 we conclude that the right hand side vanishes, as \(\varepsilon \to 0\). Therefore
\[
\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\Theta^{(1,e)}_{i,j}(t) a^{(e)}_{i,y_j}(t)| \right] = 0, \quad T > 0. \tag{5.27}
\]

Concerning the bounded variation term on the right hand side of (5.20) note that the last expression on the right hand side of (5.21) can be estimated as follows
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \int_0^t V^{(e)}(s) \cdot \left[ \Theta^{(1,e)}_{i,j}(s) a^{(e)}_{i,y_j}(s) + \Theta^{(2,e)}_{i,j}(s) b^{(e)}_{i,y_j}(s) \right] ds \right] \\
\leq T \mathbb{E} \left\{ \sup_{0 \leq t \leq T} \left\{ |V^{(e)}(t)| \left[ |\Theta^{(1,e)}_{i,j}(t) a^{(e)}_{i,y_j}(t)| + |\Theta^{(2,e)}_{i,j}(t) b^{(e)}_{i,y_j}(t)| \right] \right\} \right\}. \tag*{15}
From this point on the estimate can be conducted in a similar way as for the boundary terms and we conclude that

$$\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t V^{(\varepsilon)}(s) \cdot \left[ \Theta^{(1,\varepsilon)}_{i,j}(s) a^{(\varepsilon)}_{i,y,j}(s) + \Theta^{(2,\varepsilon)}_{i,j}(s) b^{(\varepsilon)}_{i,y,j}(s) \right] ds \right| \right] = 0.$$  

The remaining terms appearing in the bounded variation expression on the right hand side of (5.21) can be dealt with similarly and we obtain

$$\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \Gamma^{(\varepsilon)}(s) ds \right| \right] = 0.$$  

For the martingale on the right hand side of (5.20), its first term, given in the right hand side of (5.22), can be estimated first by Jensen’s and then by Doob’s inequality leading to

$$\varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left| \int_0^t \sum_{i'} \sum_{i'j'} \left( a^{(\varepsilon)}_{i',j'}(s) (\sigma_{i'} \sqrt{2\alpha_{i'}})(x_{\varepsilon}(t)) \left[ \Theta^{(1,\varepsilon)}_{i',j',a_{i'}}(s) dw_{i',a}(s) + \Theta^{(1,\varepsilon)}_{i',j',b_{i'}}(s) dw_{i',b}(s) \right] \right) \right| \right] \leq \varepsilon \left\{ \mathbb{E} \left[ \left( \int_0^T \sum_{i'} \sum_{i'j'} \left( a^{(\varepsilon)}_{i',j'}(s) (\sigma_{i'} \sqrt{2\alpha_{i'}})(x_{\varepsilon}(t)) \right)^2 \left[ \left( \Theta^{(1,\varepsilon)}_{i',j',a_{i'}}(s) \right)^2 + \left( \Theta^{(1,\varepsilon)}_{i',j',b_{i'}}(s) \right)^2 \right] ds \right] \right\}^{1/2}.$$ (5.28)  

This expression vanishes, as $\varepsilon \to 0$, thanks to the results of Section 7. The other terms forming the martingale $M^{(\varepsilon)}(t)$ can be estimated analogously. This ends the proof of (5.16).  

Coming back to (5.6), using (5.8) together with (5.15), we can write

$$x_{\varepsilon,q}(t) = \int_0^t \tilde{W}^{(\varepsilon)}_q(s) ds + \int_0^t M^{(\varepsilon)}_q(ds) + \varepsilon N^{(\varepsilon)}_q(t), \quad q = 1, 2,$$ (5.29)  

where $\tilde{W}^{(\varepsilon)}_q(t)$ corresponds (via (5.11)) to the field

$$\tilde{W}_q(t, x, y) := \sum_{i,j} \left\{ \alpha_{i,y}(y) \left( \tilde{\Theta}_{i,j}^{(1)}(t, x, y) a_i(t, x, y) + \tilde{\Theta}_{i,j}^{(2)}(t, x, y) b_i(t, x, y) \right) \right\}$$

$$+ W(t, x, y) \cdot \chi_{q,y}(t, x, y).$$

The martingale part is given by (5.5). The semi-martingale $\varepsilon N^{(\varepsilon)}_q(t)$, which turns out to be negligible (see (5.31) below), is given by

$$N^{(\varepsilon)}_q(t) := \chi^{(\varepsilon)}_q(0) - \chi^{(\varepsilon)}_q(t) + \tilde{N}^{(\varepsilon)}(t) + \int_0^t U_{\varepsilon}(s) \cdot \tilde{\chi}^{(\varepsilon)}_{q,y}(s) ds$$

$$+ \sum_{i,j} \int_0^t U_{\varepsilon}(s) \cdot \left[ \chi^{(\varepsilon)}_{q,a_i}(s) a^{(\varepsilon)}_{i,y}(s) + \chi^{(\varepsilon)}_{q,b_i}(s) b^{(\varepsilon)}_{i,y}(s) \right] ds,$$ (5.30)
where \( \tilde{N}^{(e)}(t) \) is the process given by (5.20). From (5.16), the results of Section 7 and the argument used in the proof of Lemma 5.1 we conclude that
\[
\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left[ \sup_{0 \leq t \leq T} |N^{(e)}(t)| \right] = 0,
\]
with \( N^{(e)}(t) := (N_1^{(e)}(t), N_2^{(e)}(t)) \).

### 5.2 Averaging lemma

In this section we present a result, which allows us to average out the "fast variables", i.e. \( t/\varepsilon, x_{\varepsilon}(t)/\varepsilon \), for processes of the form \( F^{(e)}(t) := \tilde{F}(t/\varepsilon^2, x_{\varepsilon}(t)/\varepsilon, x_{\varepsilon}(t)) \), that appear in the significant terms of the decomposition (5.29). The following result allows to replace such terms by their averaged out counterparts and therefore it shall be crucial in the limit identification argument for the trajectory process given by (5.29).

**Lemma 5.2.** Assume that \( F : \mathbb{R}^{2N} \times \mathbb{R}^2 \to \mathbb{R} \) is continuous in all variables and such that \( y \mapsto F(\cdot, y) \), \( y \in \mathbb{R}^2 \) is twice differentiable in the \( L^p(\nu_y^\ast) \)-sense. Then, the process \( F^{(e)}(t) \) obtained from \( F \) by formulas (5.1) and (2.13) satisfies
\[
\lim_{\varepsilon \to 0^+} \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \int_0^t F^{(e)}(s) \, ds - \int_0^t \tilde{F}(x_{\varepsilon}(s)) \, ds \right) \right] = 0, \quad T > 0,
\]
where
\[
\tilde{F}(y) := \int_{\mathbb{R}^{2N}} F(a; y) \nu_y^\ast(da).
\]

**Proof.** By considering \( F'(a, y) := F(a, y) - \tilde{F}(y) \) we can and shall assume that \( F(\cdot, y) \) is of zero \( \nu_y^\ast \)-mean. Suppose that \( \Theta(\cdot, y) \) is the \( \nu_y^\ast \)-mean zero solution of the cell problem
\[
-\mathcal{L}\Theta(\cdot, y) = F(\cdot, y).
\]

Thanks to the results of Section 3 we can apply the Itô-Krylov formula to the process \( \Theta^{(e)}(t) \), obtained from \( \Theta \) by an application of (5.1). Using (5.33) we get
\[
d \left[ \varepsilon^2 \Theta^{(e)}(t) \right] = \left\{ -F^{(e)}(t) + \varepsilon R_1^{(e)}(t) + \varepsilon^2 R_2^{(e)}(t) \right\} dt
\]
\[
+ \varepsilon \sum_i \left\{ (\sigma_i \sqrt{2\alpha_i})(x_{\varepsilon}(t)) \left[ \Theta^{(e)}_{a_i}(t) \, dw_{i, a}(t) + \Theta^{(e)}_{b_i}(t) \, dw_{i, b}(t) \right] \right\}
\]
for some standard, independent Brownian motions \( dw_{i, a}(t), dw_{i, b}(t) \) and
\[
R_1^{(e)}(t) := \sum_{i,j} \left( F_{i,j}^{(e)}(t) a_{i, y_j}^{(e)}(t) + G_{i,j}^{(e)}(t) b_{i, y_j}^{(e)}(t) \right) + V^{(e)}(t) \cdot \Theta_y^{(e)}(t),
\]
\[
R_2^{(e)}(t) := \sum_{i,j} U_{i,j}^{(e)}(t) \left[ \Theta_{a_{i,j}}^{(e)}(t) a_{i, y_j}^{(e)}(t) + \Theta_{b_{i,j}}^{(e)}(t) b_{i, y_j}^{(e)}(t) \right],
\]

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with
\[ F_{i,j} := D_j^i \Theta + w_j \Theta a_i, \quad G_{i,j} := w_j \Theta b_i, \quad i = 1, \ldots, N, \quad j = 1, 2. \] (5.35)

In addition, Lemma 8.1 implies that for any \( T, r > 0 \) we have
\[
\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left\{ \sup_{t \in [0, T]} \left[ \left| \Theta^{(\varepsilon)}(t) \right|^r + \sum_i \left( \left| \Theta^{(\varepsilon)}_{a_i}(t) \right|^r + \left| \Theta^{(\varepsilon)}_{b_i}(t) \right|^r \right) + \left| \Theta^{(\varepsilon)}_y(t) \right|^r \right] \right\} = 0.
\]

Estimating as in (5.23) – (5.27) we conclude that all terms appearing with factor \( \varepsilon \), or \( \varepsilon^2 \) in (5.34) vanish, as \( \varepsilon \to 0 \), hence (5.32) follows. \( \square \)

### 5.3 Tightness

Next step in the proof of convergence in law of the processes \((x^{(\varepsilon)}(t))_{t \geq 0}\), as \( \varepsilon \to 0 \), is establishing the tightness of their laws of over \( C([0, +\infty); \mathbb{R}^2) \).

From (5.29) we can write
\[
x^{(\varepsilon)}_{q}(t) = B^{(\varepsilon)}_{q}(t) + \int_{0}^{t} \mathcal{M}^{(\varepsilon)}_{q}(ds) + R^{(\varepsilon)}_{q}(t) + \varepsilon \mathcal{N}^{(\varepsilon)}_{q}(t), \quad q = 1, 2, t \geq 0, \] (5.36)

where
\[
B^{(\varepsilon)}_{q}(t) := \int_{0}^{t} B_{q}(x^{(\varepsilon)}(s))ds
\] (5.37)

and
\[
B_{q}(y) := \sum_{i,j} \left\{ \alpha_{i,j}(y) \int_{\mathbb{R}^{2N}} \left( \Theta^{(1)}_{i,j}(a, y)a_i + \Theta^{(2)}_{i,j}(a, y)b_i \right) \nu^y_a(da) \right. \right.$
\[
+ \left. \left. \int_{\mathbb{R}^{2N}} w(a) \cdot \chi_{a,y}(a, y) \nu^y_a(da). \right. \right.$
\] (5.38)

In addition,
\[
R^{(\varepsilon)}_{q}(t) := \int_{0}^{t} \tilde{\mathcal{W}}^{(\varepsilon)}_{q}(s)ds - B^{(\varepsilon)}_{q}(t). \]

The martingale term in (5.36) is given by (5.3). Its covariation process is given by
\[
\left\langle \int_{0}^{t} \mathcal{M}^{(\varepsilon)}_{q}(ds), \int_{0}^{t} \mathcal{M}^{(\varepsilon)}_{q'}(ds) \right\rangle_t = \int_{0}^{t} m^{(\varepsilon)}_{q,q'}(s)ds, \quad \text{where}
\]
\[
m^{(\varepsilon)}_{q,q'}(s) := 2 \sum_{i=1}^{N} \left( \alpha_i \sigma^2 \right) (x^{(\varepsilon)}(s)) \left[ \chi^{(\varepsilon)}_{a_i}(s) \chi^{(\varepsilon)}_{a_i}(s) + \chi^{(\varepsilon)}_{b_i}(s) \chi^{(\varepsilon)}_{b_i}(s) \right], \quad q, q' = 1, 2. \] (5.39)

An elementary application of Lemma 5.2 implies that
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{t \in [0, T]} \left| R^{(\varepsilon)}_{q}(t) \right| \right] = 0, \quad T > 0. \] (5.40)
Combining (5.40) with (5.31) we conclude the tightness of \((x_\varepsilon(t))_{t \geq 0}\) is equivalent with the tightness of the laws of
\[
y_{\varepsilon,q}(t) = \mathcal{B}^{(\varepsilon)}_{q}(t) + \int_0^t \mathcal{M}^{(\varepsilon)}_q(ds), \quad q = 1, 2, t \geq 0,
\]
over \(C([0, +\infty); \mathbb{R}^2)\). The latter follows, provided that we show tightness of the family of processes corresponding to the terms appearing on the right hand side of (5.41). Tightness of \(\left(\mathcal{B}^{(\varepsilon)}_{q}(t)\right)_{t \geq 0}\) follows easily from an application of Theorem VI.5.17 of [15] and the fact that coefficients \(B_q\) are bounded. Concerning tightness of the laws of the processes corresponding to the martingale part, according to Theorem VI.4.13 of ibid., it is a consequence of the respective tightness of
\[
\sum_{q=1}^2 \int_0^t m^{(\varepsilon)}_{q,q}(s) ds, \quad t \geq 0,
\]
as \(\varepsilon \to 0\). Using Lemma 5.2 we conclude that the latter is a consequence of tightness of the laws of processes
\[
\sum_{q=1}^2 \int_0^t A_{q,q}(x_\varepsilon(s)) ds, \quad t \geq 0,
\]
with \(A_{q,q'}(y)\) given by
\[
A_{q,q'}(y) := 2 \sum_{i=1}^N (\alpha_i \sigma_i^2)(y) \int_{\mathbb{R}^N} \left[ (x_{q,a}, x_{q',a_i}) (a, y) + (x_{q,b_i} x_{q',b_i}) (a, y) \right] \nu^y(da).
\]
The latter follows from yet another application of Theorem VI.5.17 of [15].

5.4 Identification of the limit

We have already mentioned that limiting laws of \((x_\varepsilon(t))_{t \geq 0}\) and \((y_\varepsilon(t))_{t \geq 0}\) coincide, as
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left[ \sup_{t \in [0,T]} |x_\varepsilon(t) - y_\varepsilon(t)| \right] = 0, \quad T > 0.
\]

Using the Itô formula we conclude from (5.41) that for any function \(f \in C^2(\mathbb{R}^2)\),
\[
M_\varepsilon(t; f) := f(y_\varepsilon(t)) - f(y_\varepsilon(0)) - \sum_{q=1}^2 \int_0^t \partial_{y_q} f(y_\varepsilon(s)) B_q(x_\varepsilon(s)) ds
\]
\[
- \frac{1}{2} \sum_{q,q'=1}^2 \int_0^t \partial_{y_q y_{q'}}^2 f(y_\varepsilon(s)) A_{q,q'}(x_\varepsilon(s)) ds, \quad t \geq 0
\]
is a martingale, where \(B_q(y), A_{q,q'}(y)\) are given by (5.38) and (5.43). Applying the results of Section 6 we conclude that the coefficients are continuous. Thanks to (5.44) we conclude that
any limiting law of \((x_\varepsilon(t))_{t \geq 0}\), as \(\varepsilon \to 0\) has to solve the martingale problem corresponding to the operator
\[
\mathcal{L} f(y) = \sum_{q=1}^{2} B_q(y) \partial_{y_q} f(y) + \frac{1}{2} \sum_{q,q'=1}^{2} A_{q,q'}(y) \partial_{y_q,y_{q'}}^{2} f(y),
\]
which by virtue of Theorem 7.2.1 of [31] is well posed. So the limiting law is uniquely determined. This ends the proof of Theorem 3.1.

\[\square\]

### 5.5 The case of an arbitrary dimension

Because of the notational convenience we have proved Theorem 3.1 only in the two dimensional case. The proof in an arbitrary dimension is virtually the same. Here we discuss briefly how to modify the respective formulas in order to obtain the expressions for the drift and diffusivity coefficients \(B_q, A_{q,q'}\), \(q,q' = 1, \ldots, d\) in the \(d\)-dimensional situation. Recall that then the modes of the velocity field given by (2.1) are indexed by the elements of the set \(Z\) defined in (2.2).

The correctors \(\chi_q(\cdot; y)\) corresponding to (4.22) are given by the zero \(\nu^y\)-mean solutions of equations
\[
- \mathcal{L}_y \chi_q(\cdot; y) = w_q,
\]
with \(w = (w_1, \ldots, w_d)\) given by
\[
w_q(a) := \sum_{i,l} k_{i,l} b_{i,l,q}, \quad q = 1, \ldots, d,
\]
where \(\mathcal{L}_y\) is the generator of the diffusion \(\tilde{a}(t; y) := \left(\tilde{a}_j(t; y), \tilde{b}_j(t; y)\right)_{j \in Z}\), with
\[
\tilde{a}_j(t; y) := a_j(t; y) \cos(k_i \cdot z(t; y)) + b_j(t; y) \sin(k_i \cdot z(t; y)),
\]
\[
\tilde{b}_j(t; y) := -a_j(t; y) \sin(k_i \cdot z(t; y)) + b_j(t; y) \cos(k_i \cdot z(t; y)).
\]

Here \(z(t; y)\) is the solution of (4.5). The generator takes the form (4.10) with
\[
D_q F := \sum_j k_{i,q} \mathcal{R}_j F, \quad q = 1, \ldots, d
\]
and
\[
\mathcal{R}_j F := (b_j \partial_{a_j} - a_j \partial_{b_j}) F, \quad j = (i, l, m) \in Z, \quad F \in C^1(\mathbb{R}^2S)
\]
(recall that \(S\) is the cardinality of \(Z\)). Finally, we solve the systems
\[
\begin{align*}
[\mathcal{L}_y - \alpha_j(y)] \Theta^{(1,q)}_{j,j}(\cdot; y) - (k_i \cdot w) \Theta^{(2,q)}_{j,j}(\cdot; y) &= F^{q}_{j,j}(\cdot; y), \\
[\mathcal{L}_y - \alpha_j(y)] \Theta^{(2,q)}_{j,j}(\cdot; y) + (k_i \cdot w) \Theta^{(1,q)}_{j,j}(\cdot; y) &= G^{q}_{j,j}(\cdot; y),
\end{align*}
\]
with
\[
F_{j,j}^q(a, y) := \delta_{i,j} (\delta_{m,q} + D_m \chi_q(a)) + w_j(a) \chi_{q,aj}(a, y),
\]
\[
G_{j,j}^q(a, y) := w_j(a) \chi_{q,bj}(a, y), \quad (i, l, m) \in Z, \quad j, q = 1, \ldots, d.
\]

Then, the formulas for the coefficients of the limiting diffusions are as follows
\[
B_q(y) := \int_{\mathbb{R}^2} w(a) \cdot \chi_{q,y}(a, y) \nu_y^y(da) + \sum_{j,j} \left\{ \alpha_{j,y}(y) \int_{\mathbb{R}^2} \left( \Theta_{j,j}^{(1,q)}(a, y) a_j + \Theta_{j,j}^{(2,q)}(a, y) b_j \right) \nu_y^{y'}(da) \right\},
\]
\[
A_{q,q'}(y) := 2 \sum_j (\alpha_j q_j^2)(y) \int_{\mathbb{R}^2} \left[ \chi_{q,aj}(a, y) \chi_{q',aj}(a, y) + \chi_{q,bj}(a, y) \chi_{q',bj}(a, y) \right] \nu_y^{y'}(da)
\]
\[
(5.50)
\]
for \(q, q' = 1, \ldots, d\).

6 Regularity of the corrector

6.1 Corrector problem

The present section is concerned with regularity of solutions \(\Xi : \mathbb{R}^{2N+2} \to \mathbb{C}\) of the equation
\[
\mathcal{L}_y \Xi(a, y) + c(a, y) \Xi(a, y) = f(a, y), \quad (a, y) \in \mathbb{R}^{2N+2}.
\]

A complex valued function \(f : \mathbb{R}^{2N+2} \to \mathbb{C}\) is assumed to satisfy
\[
\sum_{m=0}^{2} \sup_{y \in \mathbb{R}^2} \| \nabla^m_y f(\cdot, y) \|_{L^p(\nu_y^y)} < +\infty, \quad \text{for each } p \in [1, +\infty).
\]

Concerning the function \(c\) we will consider two cases: either 1) \(c \equiv 0\) and then we assume
\[
\int_{\mathbb{R}^2} f(a, y) \nu_y^y(da) = 0, \quad y \in \mathbb{R}^2,
\]
or 2) \(c(a, y) = -\alpha_i(y) + iq(a, y)\) (i - the imaginary unit), where \(i \in \{1, \ldots, N\}\) is fixed while \(q\) is a real valued polynomial of the second degree in the variable \(a\). The coefficients of the polynomial \(q(a; y)\) are assumed to be \(C^2_b(\mathbb{R}^2)\) regular functions of the variable \(y\). The operator \(\mathcal{L}_y\) was defined in (4.10).

According to the results of Section 4.3 and formula (5.14), under the above assumptions there exists a unique solution to (6.1), which in the case 1) satisfies
\[
\int_{\mathbb{R}^{2N}} \Xi(a, y) \nu_y^{y'}(da) = 0, \quad y \in \mathbb{R}^2.
\]
Using the Feynman-Kac formula we obtain that
\[
\Xi(a; y) = \int_0^{+\infty} \mathbb{E} \left[ \exp \left\{ \int_0^t c(\tilde{a}(s; y), y) ds \right\} f(\tilde{a}(t; y); y) \right] dt, \quad a \in \mathbb{R}^{2N}. \tag{6.3}
\]

Here \( \tilde{a}(t, y) = (\tilde{a}_i(t, y), \tilde{b}_i(t, y)) \) is the solution of \( \text{(4.19)} \) satisfying \( \tilde{a}(0, y) = a \). Thanks to \( \text{(4.21)} \) we conclude from \( \text{(6.3)} \) that for each \( p \in (1, +\infty) \) we have
\[
\|\Xi(\cdot; y)\|_{L^p(\nu_y^q)} \leq \|f(\cdot; y)\|_{L^p(\nu_y^q)}, \quad y \in \mathbb{R}^2. \tag{6.4}
\]

6.2 \( L^p \) regularity of the corrector in the \( a \)-variable

Our first result concerns the \( L^p \) regularity of the solutions of the equation
\[
-\mathcal{L}_y \Xi(a; y) = f(a; y), \quad (a, y) \in \mathbb{R}^{2N+2}, \tag{6.5}
\]
in the \( a \) variable. Here \( f: \mathbb{R}^{2N+2} \to \mathbb{C} \) is such that \( f(\cdot, y) \in L^q(\nu_y^q) \) for some \( q > 1 \). Thanks to \( \text{(4.17)} \) we conclude that
\[
\int_{\mathbb{R}^{2N}} f(a; y) \nu_y^q(\text{d}a) = 0, \quad y \in \mathbb{R}^2, \tag{6.6}
\]
is a necessary condition for its solvability. Using \( \text{(6.3)} \) we can write
\[
\Xi(a, y) = \int_0^{+\infty} P_t^y f(a; y) dt = \int_0^{+\infty} \mathbb{E} [f(\tilde{a}(t; y), y)] dt. \tag{6.7}
\]

**Theorem 6.1.** Assume that \( \Xi \) is given by \( \text{(6.7)} \) and \( q \in (1, +\infty) \). Then, for any \( p \in [1, q) \) there exists \( C > 0 \) (independent of \( y \)) such that
\[
\|\nabla \Xi(\cdot; y)\|_{L^p(\nu_y^q)} + \|\nabla^2 \Xi(\cdot; y)\|_{L^p(\nu_y^q)} \leq C \|f(\cdot; y)\|_{L^q(\nu_y^q)}, \quad y \in \mathbb{R}^2 \tag{6.8}
\]
for any \( f: \mathbb{R}^{2N+2} \to \mathbb{C} \) such that \( f(\cdot, y) \in L^q(\nu_y^q) \) for all \( y \in \mathbb{R}^2 \).

The proof of the above result is presented in Section \[6.3\] but first we apply it to conclude the following.

**Corollary 6.2.** Under the assumptions of Section \[6.1\] for any \( 1 \leq p < q < +\infty \) there exists \( C > 0 \) such that \( \text{(6.1)} \) satisfies
\[
\|\Xi(\cdot; y)\|_{W^{2,p}(\nu_y^q)} \leq C \|f(\cdot; y)\|_{L^q(\nu_y^q)} \quad \text{for all } y \in \mathbb{R}^2, \quad f \in L^q(\nu_y^q). \tag{6.9}
\]

**Proof.** We let
\[
f(a; y) = -c(a; y) \Xi(a; y) + f(a; y). \tag{6.10}
\]

Thanks to \( \text{(6.2)} \) and \( \text{(6.4)} \) we conclude that \( f(\cdot, y) \in L^q(\nu_y^q) \) for any \( q \in [1, +\infty) \) and \( y \in \mathbb{R}^2 \). Then \( \text{(6.9)} \) is a direct consequence of \( \text{(6.8)} \) and \( \text{(6.4)} \). \( \square \)
6.3 Proof of Theorem 6.1

We show first that for each \( p \in [1, q) \) there exists \( C > 0 \) such that

\[
\| \nabla_a \Xi(\cdot, y) \|_{L^p(\nu_y)} \leq C \| f(\cdot, y) \|_{L^q(\nu_y)}, \quad y \in \mathbb{R}^2. \tag{6.11}
\]

In the proof we focus only on estimating the \( L^p(\nu_y) \) norm of \( \Xi_{bi}(\cdot, y), i = 1, \ldots, N \). The argument for \( \Xi_{ai}(\cdot, y) \) is similar. In addition, we assume that \( f \) is differentiable in the \( a \) variable. The constant \( C > 0 \) in estimate (6.11) turns out not to depend on \( \nabla_a f \) so we can relax this assumption by approximation.

From (6.7) we obtain

\[
\Xi_{bj}(a, y) = \int_0^{+\infty} v_j(t) dt, \tag{6.12}
\]

where

\[
v_j(t) := \partial_{bj} P^y f(a) = \mathbb{E} \left[ (\nabla_a f)(\tilde{a}^a(t, y)) \cdot D_{bj} \tilde{a}^a(t, y) \right] \tag{6.13}
\]

and \( D_{bj} \tilde{a}^a(t, y) = (\tilde{\xi}_{i,j}(t), \tilde{\eta}_{i,j}(t))_{i=1,\ldots,N} \) is the Fréchet derivative of the stochastic flow \( a \mapsto \tilde{a}^a(t, y) \), with \( \tilde{\xi}_{i,j}(t) := \tilde{a}_{i,bj}^a(t, y) \) and \( \tilde{\eta}_{i,j}(t) := \tilde{b}_{i,bj}^a(t, y) \).

Differentiating (4.9) with respect to the initial condition we conclude that

\[
\frac{d\xi_{i,j}}{dt} = -\alpha_i \xi_{i,j} + \sum_{k'} \delta(k_i, k_{i'}) \left( \tilde{b}_{i,j}^a \eta_{i,j} + \tilde{b}_{i,j}^a \eta_{i,j} \right),
\]

\[
\frac{d\eta_{i,j}}{dt} = -\alpha_i \eta_{i,j} - \sum_{k'} \delta(k_i, k_{i'}) \left( \tilde{a}_{i,j} \eta_{i',j} + \tilde{b}_{i,j} \xi_{i,j} \right), \tag{6.14}
\]

\( \xi_{i,j}(0) := 0, \quad \eta_{i,j}(0) := \delta_{i,j}, \quad i, j = 1, \ldots, N, \)

where \( \delta_{i,j} \) is the Kronecker symbol, i.e. \( \delta_{i,i} = 1 \) and \( \delta_{i,j} = 0 \) if \( i \neq j \).

We shall prove that

\[
\| v_j(t) \|_{L^p(\nu_y)} \leq \begin{cases} 
    e^{-\gamma(q)t} \| f(\cdot, y) \|_{L^q(\nu_y)}, & t \geq 1, \\
    \frac{1}{t^{1/2}} \| f(\cdot, y) \|_{L^q(\nu_y)} & t \in (0, 1),
\end{cases} \tag{6.15}
\]

where \( \gamma(q) \) is the same as in (4.21). Estimate (6.11) then follows from the above bound and formula (6.12).

Consider first the case when \( t \geq 1 \). For given \( g(t) = (g_i^{(a)}(t), g_i^{(b)}(t))_{i=1,\ldots,N} \) with \( g_i^{(a)}, g_i^{(b)} \in
Let \( f \in L^2[0, +\infty) \), \( i = 1, \ldots, N \) we let

\[
\|g\|_{r,t} := \sup_{y \in \mathbb{R}^2} \left\{ \sum_i \left\{ \int_{\mathbb{R}^2} \nu^y_s(da) \mathbb{E} \left[ \frac{1}{t} \int_0^t |g_i(s, y, a)|^2 ds \right]^{r/2} \right\}\right\}. 
\]

Therefore (see (4.9)), from the chain rule for the Malliavin derivative, see Proposition 1.2.3, p. 28 of [26], we obtain

\[
v_j(t) = \tilde{v}_j(t) + \mathbb{E} \left[ D_g f(\tilde{a}^a(t, y), y) \right], \quad (6.20)
\]

\[
(6.16)
\]

\[
\|g\|_r := \|g\|_{r,1}. \quad \text{Let also } h(t) = (h_i^{(a)}(t), h_i^{(b)}(t))_{i=1,\ldots,N}, \text{ where}
\]

\[
h_i^{(a)}(t) := \int_0^t g_i^{(a)}(s) ds, \quad h_i^{(b)}(t) := \int_0^t g_i^{(b)}(s) ds, \quad t \geq 0. \quad (6.17)
\]

The difference of the Fréchet and Malliavin derivatives

\[
\Gamma(t, y) := D_b \tilde{a}^a(t, y) - D_b \tilde{a}^a(t, y) = (\Upsilon_{i,j}(t), \Theta_{i,j}(t))_{i=1,\ldots,N}
\]

solves the following system of equations

\[
\frac{d\Upsilon_{i,j}}{dt} = -\alpha_i \Upsilon_{i,j} + \sum_{k'} \delta(k_i, k_{i'}) \left( \tilde{b}_i^{\alpha} \Theta_{k',i} + \tilde{b}_i^{\alpha} \Theta_{k',j} \right) - \sqrt{2\alpha_i \sigma_i} g_i^{(a)} ,
\]

\[
\frac{d\Theta_{i,j}}{dt} = -\alpha_i \Theta_i - \sum_{k'} \delta(k_i, k_{i'}) \left( \tilde{a}_i^{\alpha} \Theta_{k',i} + \tilde{a}_i^{\alpha} \Upsilon_{k',j} \right) - \sqrt{2\alpha_i \sigma_i} g_i^{(b)} , \quad (6.19)
\]

\[
\Upsilon_{i,j}(0) := 0, \quad \Theta_{i,j}(0) := 0, \quad i = 1, \ldots, N.
\]

Therefore (see (6.13)), from the chain rule for the Malliavin derivative, see Proposition 1.2.3, p. 28 of [26], we obtain

\[
v_j(t) = \tilde{v}_j(t) + \mathbb{E} \left[ D_g f(\tilde{a}^a(t, y), y) \right], \quad (6.20)
\]

\[
(6.20)
\]
where
\[ \tilde{v}_j(t) := \sum_i \mathbb{E} \left[ \partial_{a_i} f(\bar{a}^a(t, y), y) \bar{\gamma}_{i,j}(t) + \partial_{b_i} f(\bar{a}^a(t, y), y) \Theta_{i,j}(t) \right]. \]

Integrating by parts the second term on the right hand side of (6.20) (see Lemma 1.2.1 p. 25, of [26]) we conclude that
\[ v_j(t) = \tilde{v}_j(t) + \sum_i \mathbb{E} \left[ f(\bar{a}^a(t, y), y) \left( \int_0^t g^{(a)}_i(s) dw_{i,a}(s) + \int_0^t g^{(b)}_i(s) dw_{i,b}(s) \right) \right] \quad (6.21) \]

We shall look for the control \( g(t, y, a) = \left( g^{(a)}_i(t, y, a), g^{(b)}_i(t, y, a) \right) \) for \( i = 1, \ldots, N \), which satisfies the following conditions:

i) it is adapted with respect to the natural filtration of \( (w(t))_{t \geq 0} \);

ii) the respective \( \Gamma(t, y) \equiv 0 \) and \( g(t, y, a) \equiv 0 \) for \( t \geq 1 \), \( (a, y) \in \mathbb{R}^{2N+2} \);

iii) we have (cf (6.16))
\[ \|g\|_r = \|g\|_{r,1} < +\infty \quad (6.22) \]

Then, thanks to ii), we conclude that \( \tilde{v}_j(t) \equiv 0 \). Using formula (6.21) and the Markov property of \( (\bar{a}^a(t, y))_{t \geq 0} \) we can write
\[ v_j(t) = \sum_i \mathbb{E} \left[ P_{t-1}^\nu f(\bar{a}^a(1, y)) \left( \int_0^t g^{(a)}_i(s, y) dw_{i,a}(s) + \int_0^t g^{(b)}_i(s, y) dw_{i,b}(s) \right) \right] \quad \text{for } t \geq 1. \]

Applying Hölder’s inequality with \( 1/q + 1/r = 1/p \) we obtain
\[ \|v_j(t)\|_{L^p(\nu^2)} \leq \|P_{t-1}^\nu f(\cdot, y)\|_{L^q(\nu^2)} \times \sum_i \left\{ \left( \int_{\mathbb{R}^{2N}} \nu^2_i(da) \mathbb{E} \left[ \int_0^t g^{(a)}_i(s, y, a) dw_{i,a}(s) \right]^r \right)^{1/r} + \left( \int_{\mathbb{R}^{2N}} \nu^2_i(da) \mathbb{E} \left[ \int_0^t g^{(b)}_i(s, y, a) dw_{i,b}(s) \right]^r \right)^{1/r} \right\}. \]

Thanks to (6.6) we can apply spectral gap estimate (4.20). This and the Burkholder-Davis-Gundy inequality imply the following bound
\[ \|v_j(t)\|_{L^p(\nu^2)} \leq e^{-\gamma(q)(t-1)} \|f(\cdot, y)\|_{L^q(\nu^2)} \|g\|_r, \quad t \geq 1, \ y \in \mathbb{R}^2 \quad (6.23) \]

for \( \gamma(q) > 0 \) as in (4.21).

When, on the other hand \( t \in (0, 1) \) we represent \( v_j(t) = \partial_{b_j} P_{t}^\nu f(a) \) using the Bismut-Elworthy-Li formula, see e.g. formula (3.3.24), p. 75 of [11],
\[ \partial_{b_j} P_{t}^\nu f(a) = \frac{1}{t} \mathbb{E} \left[ f(\bar{a}^a(t, y)) \int_0^t \Sigma^{-1} D_{b_j} \bar{a}^a(s, y) \cdot dw(s) \right] \quad (6.24) \]
The matrix $\Sigma$ is diagonal and given by formula

$$\Sigma := \text{diag}[\sqrt{2\alpha_1}\sigma_1, \ldots, \sqrt{2\alpha_N}\sigma_N, \sqrt{2\alpha_1}\sigma_1, \ldots, \sqrt{2\alpha_N}\sigma_N].$$

(6.25)

Hence, after using the Hölder inequality and lower bounds (2.4), we get

$$\|v_j(t)\|_{L^p(\nu^2)} \leq \frac{1}{\sigma_*\gamma_0t^2}\|f(\cdot, y)\|_{L^q(\nu^2)}$$

(6.26)

$$\times \sum_i \left\{ \left\{ \int_{\mathbb{R}^2N} \nu^i_*(d\alpha) \mathbb{E} \left[ \int_0^t \xi_{i,j}(s) dw_{i,a}(s) \right]^{1/r} \right\} + \left\{ \int_{\mathbb{R}^2N} \nu^i_*(d\alpha) \mathbb{E} \left[ \int_0^t \eta_{i,j}(s) dw_{i,b}(s) \right]^{1/r} \right\} \right\}.$$

Applying subsequently Burkholder-Davis-Gundy and Jensen inequalities, we obtain

$$\|v_j(t)\|_{L^p(\nu^2)} \leq \frac{1}{t^{1/2}}\|f(\cdot, y)\|_{L^q(\nu^2)} D_{b_{\cdot}, a_{\cdot}}(\cdot, y) \|_{\mathcal{M}, t} \leq \frac{1}{t^{1/2}}\|f(\cdot, y)\|_{L^q(\nu^2)}$$

(6.27)

$$\times \sum_i \left\{ \left\{ \frac{1}{t} \int_0^t \int_{\mathbb{R}^2N} \nu^i_*(d\alpha) \mathbb{E} |\xi_{i,j}(s)|^r \right\}^{1/r} ds \right\} + \left\{ \frac{1}{t} \int_0^t \int_{\mathbb{R}^2N} \nu^i_*(d\alpha) \mathbb{E} |\eta_{i,j}(s)|^r \right\}^{1/r} ds \right\}.$$

Thanks to (6.40) we conclude that

$$\sup_{t \in (0, 1), y \in \mathbb{R}^2} \sum_i \left\{ \left\{ \frac{1}{t} \int_0^t \int_{\mathbb{R}^2N} \nu^i_*(d\alpha) \mathbb{E} |\xi_{i,j}(s)|^r \right\}^{1/r} ds \right\} + \left\{ \frac{1}{t} \int_0^t \int_{\mathbb{R}^2N} \nu^i_*(d\alpha) \mathbb{E} |\eta_{i,j}(s)|^r \right\}^{1/r} ds \right\} < +\infty.$$  

(6.28)

Thus,

$$\|v_j(t)\|_{L^p(\nu^2)} \leq \frac{1}{t^{1/2}}\|f(\cdot, y)\|_{L^q(\nu^2)}, \quad t \in (0, 1), \quad y \in \mathbb{R}^2.$$  

(6.29)

From estimates (6.23) and (6.29) we conclude (6.15), which ends the proof of (6.11), provided we can find a control $g$ which satisfies conditions i) - iii) and show estimate (6.28). We shall deal with these issues in Section 6.4. The above argument can be conducted in the case of $a_j$ variables as well, so we conclude (6.11).

Using (6.11) we infer that for each $p \in [1, +\infty)$

$$\|\mathcal{L} \Xi(\cdot, y)\|_{L^p(\nu^2)} \leq \|f(\cdot, y)\|_{L^q(\nu^2)} + \|\mathbf{w} \cdot D \Xi(\cdot, y)\|_{L^p(\nu^2)}.$$  

(6.30)

From the definition of the operator $D$, see (1.11), and Hölder inequality, applied to the second term on the right hand side of (6.30), we obtain

$$\|\mathbf{w} \cdot D \Xi(\cdot, y)\|_{L^p(\nu^2)} \leq \|\Phi\|_{L^p(\nu^2)} \|\nabla \Xi(\cdot, y)\|_{L^p(\nu^2)}.$$  

(6.31)

where $\Phi$ is some second degree polynomial in $a$ with constant coefficients. We have assumed that $q' \in (p, q)$ and $r'$ are such that $1/q' + 1/r' = 1/p$. Using the already proved estimate (6.11) to bound the norm of the gradient on the right hand side of (6.31), we conclude that

$$\|\mathbf{w} \cdot D \Xi(\cdot, y)\|_{L^p(\nu^2)} \leq \|f(\cdot, y)\|_{L^q(\nu^2)}, \quad y \in \mathbb{R}^2.$$
Thus,
\[ \| \mathbb{L}^y \Xi(\cdot, y) \|_{L^p(\nu^y)} \leq \| f(\cdot, y) \|_{L^q(\nu^y)}, \quad y \in \mathbb{R}^2. \] (6.32)

Since, see Theorem 1.5.1 of [26], p. 72, for each \( p \in (1, +\infty) \) we have
\[ \| \nabla_0^2 a \mathbb{L}^y \Xi(\cdot, y) \|_{L^p(\nu^y)} \leq \| \mathbb{L}^y \Xi(\cdot, y) \|_{L^p(\nu^y)}, \quad y \in \mathbb{R}^2. \]

This estimate allows us to conclude the proof of Theorem 6.1.

### 6.4 Construction of a control \( g \) and proof of (6.28)

Denote by \( C(t, s) = [C_{i,i'}(t, s)]_{i,i'=1,...,2N} \) the fundamental matrix of the system (6.14). It is a \( 2N \times 2N \)-matrix, which is the solution of the equation
\[ \frac{d}{dt} C(t, s) = A(t)C(t, s), \quad C(s, s) = I_{2N}, \quad t, s \geq 0, \] (6.33)

where \( I_{2N} \) is the identity \( 2N \times 2N \)-matrix and \( A(t) = [A_{i,i'}(t)]_{i,i'=1,...,2N} \), where
\[
\begin{align*}
A_{i,i'}(t) &:= -\alpha_i \delta_{i,i'}, & A_{i+N,i'+N}(t) &:= -[\alpha_i \delta_{i,i'} + \delta(k_i, k_{i'}) \tilde{a}^a_i(t, y)], \\
A_{i,i'+N}(t) &:= \delta(k_i, k_{i'}) \left( \tilde{b}^a_i(t, y) + \delta_{i,i'} \sum_{i''} \delta(k_i, k_{i''}) \tilde{b}^a_{i''}(t, y) \right), \\
A_{i+N,i'}(t) &:= \delta_{i,i'} \sum_{i''} \delta(k_i, k_{i''}) \tilde{b}^a_{i''}(t, y), \quad 1 \leq i, i' \leq N.
\end{align*}
\]

We have
\[ C(u, t)C(t, s) = C(u, s), \quad u, t, s \in \mathbb{R}. \]

System (6.19) can be rewritten as follows
\[ \frac{d\Phi}{dt} = A(t)\Phi - \Sigma g(t, y), \quad \Phi(0) = E, \] (6.35)

where \( \Phi \) is the \( 2N \times N \)-dimensional matrix such that
\[
\begin{align*}
\Phi_{i,j} &:= \Upsilon_{i,j}, \quad 1 \leq i, j \leq N, & \Phi_{i,j} &:= \Theta_{i,j}, \quad N + 1 \leq i \leq 2N, 1 \leq j \leq N,
\end{align*}
\]

and \( E \) is a block vector, such that \( E^T := [0_N, I_N] \), where \( 0_N, I_N \) are the \( N \times N \) null matrix and identity matrix, respectively. Here the diagonal matrix \( \Sigma \) is defined in (6.25) and
\[ g := [g^{(a)}_1, \ldots, g^{(a)}_N, g^{(b)}_1, \ldots, g^{(b)}_N]^T. \] The solution of equation (6.35) can be expressed by the fundamental solution as follows
\[ \Phi(t) = -\int_0^t C(t, s)\Sigma g(s)ds + C(t, 0)E, \quad t \geq 0, \] (6.36)

which in turn implies that \( \Gamma(t, y) \equiv 0 \), so condition ii) is satisfied.
Let
\[ g(t, y, a) := \sum_{i=0}^{\infty} C(t, 0) E, \quad t \in [0, 1], \] (6.37)
and \( g(t, y, a) \equiv 0 \) for \( t \geq 1 \). The process is adapted with respect to the natural filtration of \((w_t)_{t \geq 0}\), satisfying therefore condition i). Additionally, we have
\[- \int_0^1 C(1, s) \sum_{i=0}^{\infty} g(s, y, a) ds + C(1, 0) E = - \int_0^1 C(1, 0) E ds + C(1, 0) E = 0.\]
Thus, \( \Phi(t) \equiv 0 \) for \( t \geq 1 \), which in turn implies that \( \Gamma(t, y) \equiv 0, t \geq 1 \). Condition ii) is therefore fulfilled.

It remains to be checked that \( (g(t, y, a))_{t \geq 0} \), constructed above, satisfies the estimate (6.22). From (6.33) we conclude that
\[ \| C(t, 0) \| \leq 1 + \int_0^t A(s) \| C(s, 0) \| ds, \quad t \geq 0, \]
where
\[ \| C(t, s) \| := \max_{i, i'} |C_{i,i'}(t, s)| \quad \text{and} \quad A(t) := \sum_{i, i'} |A_{i,i'}(t)| \]
Using first Gronwall and then the Jensen inequality for the integral in \( ds \), we obtain
\[ \sup_{t \in [0, 1]} \| C(t, 0) \| \leq \exp \left\{ \int_0^1 \mathfrak{A}(s) ds \right\} \leq \int_0^1 \exp \{ \mathfrak{A}(s) \} ds. \] (6.38)
It is clear from (6.34) that
\[ \mathfrak{A}(t) \leq 1 + \sum_i (|a_i^a(t, y)| + |b_i^a(t, y)|), \] (6.39)
where \( (a_i^a(t, y), b_i^a(t, y))_{t \geq 0} \) is the solution of (2.5), with \( (a_0^a(0, y), b_0^a(0, y))_{i=1, \ldots, N} = a \).

Recall that \( \min_i \{ \sigma_i \} \geq \sigma_* > 0 \) and \( \min_i \{ \alpha_i \} \geq \gamma_0 > 0 \). Therefore, from (6.37), (6.39) and gaussianity of \( (a_i^a(t, y), b_i^a(t, y))_{t \geq 0} \), we obtain that for each \( r \geq 1 \) we have
\[ \| \mathfrak{g} \| \leq \sup_{y \in \mathbb{R}^2} \left\{ \int_0^1 ds \int_{\mathbb{R}^N} \nu_y^\nu(d\alpha) E \exp \{ r \mathfrak{A}(s) \} \right\}^{1/r} < +\infty. \]

Since \( D_{b_j} \tilde{\alpha}^a(t, y) = C(t, 0) E_j \), where \( E_j \) is the \( j \)-th column vector of the matrix \( E \), from (6.38) we conclude that
\[ \sup_{t \in [0, 1], y \in \mathbb{R}^2} \int_{\mathbb{R}^N} \nu_y^\nu(d\alpha) E \| D_{b_j} \tilde{\alpha}^a(t, y) \|^r \leq \sup_{y \in \mathbb{R}^2} \int_0^1 ds \int_{\mathbb{R}^N} \nu_y^\nu(d\alpha) E \exp \{ r \mathfrak{A}(s) \} < +\infty. \] (6.40)
6.5 Regularity of the corrector in the $y$-variable

We start with the following simple lemma.

**Lemma 6.3.** For any $1 \leq p < q < +\infty$ there exist $C, r > 0$ such that

$$
\sup_{|y-y_0| \leq r} \|g\|_{L^p(\nu^y_0)} \leq C\|g\|_{L^q(\nu^y_0)}, \quad y_0 \in \mathbb{R}^2. \tag{6.41}
$$

**Proof.** Observe that

$$
\|g\|_{L^p(\nu^y_0)}^p = \int_{\mathbb{R}^2} g^p \rho_{\nu^y_0} \quad \|g\|_{L^q(\nu^y_0)}^q = \int_{\mathbb{R}^2} g^q (\rho - 1) + \int_{\mathbb{R}^2} g^p \rho_{\nu^y_0}, \tag{6.42}
$$

where

$$
\rho(y) := \prod_{i=1}^N \left\{ \frac{\sigma_i^2(y_0)}{\sigma_i(y)} \exp \left\{ -\frac{(a_i^2 + b_i^2)(\sigma_i^2(y) - \sigma_i^2(y_0))}{2\sigma_i^2(y_0)\sigma_i^2(y)} \right\} \right\}, \quad y \in \mathbb{R}^2.
$$

Using an elementary inequality $|e^x - 1| \leq e^{2|x|}$ we obtain that

$$
\int |g|^p \rho - 1 \quad \int \|g\|^p \exp \left\{ \sum_{i=1}^N \frac{(a_i^2 + b_i^2)(\sigma_i^2(y) - \sigma_i^2(y_0))}{\sigma_i^2(y_0)\sigma_i^2(y)} \right\} \rho_{\nu^y_0}. \tag{6.43}
$$

From uniform continuity of the function $\sigma_i^2$ (because $\sigma_i \in C^2_b(\mathbb{R}^d)$) for any $\mu > 0$ there exists $\delta > 0$ such that $|\sigma_i^2(y) - \sigma_i^2(y_0)| < \mu$, provided that $|y - y_0| < \delta$. From the Hölder inequality with $1/q + 1/q' = 1/p$ and the lower bound (2.4) on $\sigma$, we obtain that the expression (6.43) is less than or equal

$$
\|g\|_{L^q(\nu^y_0)}^p \left( \int_{\mathbb{R}^2} \exp \left\{ \frac{q'\mu}{\sigma^2_\nu} \sum_{i=1}^N (a_i^2 + b_i^2) \right\} \rho_{\nu^y_0} \right)^{1/2}.
$$

One can choose $q'\mu$ sufficiently small so that the second factor is finite, which ends the proof of the lemma. \hfill \Box

The main result of this section is the following.

**Theorem 6.4.** Suppose that $\Xi : \mathbb{R}^{2N+2} \to \mathbb{C}$ is the solution of (6.5). Then, under the assumptions made in Section 6.1 for any $p \in [1, +\infty)$ there exists $r > 0$ such that for any $y_0 \in \mathbb{R}^2$ we have $\Xi(\cdot, y) \in W^{2,p}(\nu^y_0)$, provided that $|y - y_0| < r$ and

$$
\lim_{y \to y_0} \|\Xi(\cdot, y) - \Xi(\cdot, y_0)\|_{W^{2,p}(\nu^y_0)} = 0. \tag{6.44}
$$

For each $p \in [1, +\infty)$ the derivatives $\nabla^m_y \Xi(\cdot, y)$, $m = 1, 2$ exist in the $W^{p,2}(\nu^y_0)$-sense for each $y \in \mathbb{R}^2$. In addition,

$$
\sum_{m=0}^2 \sup_{y \in \mathbb{R}^2} \|\nabla^m_y \Xi(\cdot, y)\|_{W^{p,2}(\nu^y_0)} < +\infty.
$$
The main purpose of this section is the proof of the following result. Given a differentiable function $f : \mathbb{R}^{2N+2} \to \mathbb{C}$ we let $\delta f(\cdot, y_0) := f(\cdot, y) - f(\cdot, y_0)$. From (6.1) we can write

$$L_{y_0} \delta \Xi (a; y) + c(a, y_0) \delta \Xi (a; y) = -\delta L_{y_0} \Xi (a; y) - \delta c(a, y_0) \Xi (a; y) + \delta f(a, y_0), \quad (6.45)$$

where $\delta L_{y_0}$ is the differential operator obtained from $L_{y_0}$ by taking the corresponding differences of the coefficients. Using Lemma 6.3 and Corollary 6.2 we conclude that for each $q \in [1, +\infty)$ the $L^q (\nu^{y_0})$-norm of the right hand side tends to 0, as $y \to y_0$. Equality (6.44) is then a consequence of (6.9).

To prove the existence of the derivative $\partial_y \Xi (a; y)$ denote by

$$D_h f(\cdot, y_0) := \frac{1}{h} [f(\cdot, y_0 + h) - f(\cdot, y_0)]$$

for a given function $f : \mathbb{R}^{2N+2} \to \mathbb{C}$ and $h \neq 0$. We show that

$$\lim_{h \to 0} \| R_h \Xi (\cdot, y_0) \|_{W^2 (\nu^{y_0})} = 0, \quad (6.46)$$

where

$$R_h \Xi (\cdot, y_0) := D_h \Xi (\cdot, y_0) - \partial_y \Xi (\cdot, y_0)$$

and $\nabla_y \Xi (\cdot, y_0)$ is the solution of

$$L_{y_0} \partial_y \Xi (a; y_0) + c(a, y_0) \partial_y \Xi (a; y_0) = -L'_{y_0} \Xi (a; y_0) - \partial_y c(a, y_0) \Xi (a; y_0) + \partial_y f(a, y_0).$$

Here $L'_{y_0}$ is the differential operator obtained from $L_{y_0}$ by differentiating in $y$ its coefficients. We have

$$L_{y_0} R_h \Xi (\cdot, y_0) + c(a, y_0) R_h \Xi (\cdot, y_0) = L'_{y_0} \Xi (a; y_0) - D_h L_{y_0} \Xi (a; y_0 + h) + \partial_y c(a, y_0) \Xi (a; y_0) - D_h \partial_y c(a, y_0) \Xi (a; y_0 + h) + \partial_y f(a, y_0) - D_h f(a, y_0).$$

Here $D_h L_{y_0}$ is the differential operator obtained from $L_{y_0}$ by taking the respective quotients of its coefficients. Using again estimate (6.9) we conclude that

$$\lim_{h \to 0} \| R_h \Xi (\cdot, y_0) \|_{W^2 (\nu^{y_0})} = 0 \quad \text{for any } y_0 \in \mathbb{R}.$$

The proof of the existence of the second derivative is analogous.

6.6 $L^\infty$ estimates of the corrector

Given a differentiable function $f : \mathbb{R}^{2N} \to \mathbb{C}$ and $R > 0$ we define the norm

$$\| f \|_{1, \infty}^{(R)} := \sup_{|a| \leq R} (|f(a)| + |\nabla a f(a)|).$$

The main purpose of this section is the proof of the following result.
Proposition 6.5. Under the assumptions made in Section 6.1 for any $C_* > 0$ there exists $C > 0$ such that
\[
\sup_{y \in \mathbb{R}^2} \sum_{m=0}^{2} \| \nabla_y^m \Xi(\cdot; y) \|^{(R)}_{1,\infty} \leq C e^{C_* R^2} \quad \text{for all } R > 0. \tag{6.47}
\]

Proof. Let $C_* > 0$ be arbitrary. We choose $p \in [1, +\infty)$ to be specified further later on. Note that (cf (2.4))
\[
\| \Xi(\cdot; y) \|_{W^{2,p}(B_R)} \leq \| \Xi(\cdot; y) \|_{W^{2,p}(\nu y)} e^{R^2/(p \sigma^2)}, \quad \text{for all } R \geq 0, y \in \mathbb{R}^2. \tag{6.48}
\]
Due to Sobolev embedding, see e.g. Theorem 7.10, p. 155 of [13], space $W^{2,p}(B_R)$ can be embedded into $C^1(B_R)$, provided that $p > d$. In consequence there exists $C > 0$ such that
\[
\| \Xi(\cdot; y) \|^{(R)}_{1,\infty} \leq C (R + 1)^{2-d/p} \| \Xi(\cdot; y) \|_{W^{2,p}(B_R)} \quad \text{for all } y \in \mathbb{R}^2, R > 0. \tag{6.49}
\]
From (6.49) and (6.48) we conclude that there exists $C > 0$ such that
\[
\| \Xi(\cdot; y) \|^{(R)}_{1,\infty} \leq C (R + 1)^{2-d/p} e^{R^2/(p \sigma^2)} \quad \text{for all } y \in \mathbb{R}^2, R > 0. \tag{6.50}
\]
Choosing $p > 2/(C_* \sigma^2)$, we conclude that for some $R_0$
\[
(R + 1)^{2-d/p} e^{R^2/(p \sigma^2)} \leq e^{C_* R^2} \quad \text{for all } R \geq R_0.
\]
Increasing suitably the constant $C > 0$, if necessary and recalling (6.4) we conclude that
\[
\sup_{y \in \mathbb{R}^2} \| \Xi(\cdot; y) \|^{(R)}_{1,\infty} \leq C e^{C_* R^2} \quad \text{for all } R.
\]
The proof of the bounds on the respective norms of $\nabla_y \Xi(\cdot; y)$ and $\nabla_y^2 \Xi(\cdot; y)$ can be done analogously, thus (6.47) follows. \qed

7 Bounds on the moments of suprema of some Gaussian processes

Let $I$ be the arbitrary set. We say that a field $(A(t, z))_{(t,z) \in [0, +\infty) \times I}$ is stationary in the $t$-variable if for any $h \geq 0$ the laws of the field and that of $(A(t + h, z))_{(t,z) \in [0, +\infty) \times I}$ are identical.

Proposition 7.1. Let $I$ be a compact metric space and $N$ some natural number. Assume that $(A(t, z))_{(t,z) \in [0, +\infty) \times I}$ is an $\mathbb{R}^N$-valued, Gaussian and stationary in the $t$-variable random field with continuous realizations. Then for any $\gamma \in (0, 1), T > 1$, there exist $C, C' > 0$ such that
\[
\mathbb{E} \left\{ \sup_{t \in [0,T], z \in I} \exp \left\{ C \left| A \left( \frac{t}{\varepsilon}, z \right) \right|^2 \right\} \right\} \leq C' \varepsilon^{-\gamma}, \quad \varepsilon \in (0, 1]. \tag{7.1}
\]
Proof. Consider the Banach space $E := C([0, 1] \times I)$ with the standard supremum norm $\| \cdot \|_E$. From the Borell-Fernique-Talagrand theorem, see e.g. Theorem 2.6, p. 37 of [9], there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ we have

$$\mathbb{E} \exp \{ \lambda \| A(\cdot) \|^2_E \} < +\infty.$$ (7.2)

Choose any $C \in (0, \gamma \lambda_0)$. Consider the random variables

$$S_i := \exp \left\{ C \sup_{(t, z) \in [i, i+1] \times I} |A(t, z)|^2 \right\}, \quad i = 0, \ldots, T(\varepsilon) := \left\lceil \frac{T}{\varepsilon} \right\rceil + 1,$$

where $[x]$ denotes the largest integer that is less than, or equal to $x \in \mathbb{R}$. Thanks to the stationarity in $t$ of the field $A$ the laws of all these random variables are identical. Let $\lambda := C/\gamma$. From (7.2) we have

$$\mathbb{E} S_0^{1/\gamma} \leq \mathbb{E} \exp \{ \lambda \| A(\cdot) \|^2_E \} < +\infty.$$ (7.3)

The left hand side of (7.1) can be estimated by from above by

$$\mathbb{E} \left[ \max_{i=0, \ldots, T(\varepsilon)} S_i \right] \leq \left( \mathbb{E} \sum_{i=0}^{T(\varepsilon)} S_i^{1/\gamma} \right)^{\gamma} \leq (2T)^{\gamma} \varepsilon^{-\gamma} \left( \mathbb{E} S_0^{1/\gamma} \right)^{\gamma}.$$ and (7.1) follows.

**Corollary 7.2.** Suppose that $(a(t, y))_{(t, y) \in \mathbb{R}^3}$ is the field defined in (2.3). For any $\gamma \in (0, 1)$ and $T > 0$ there exist $C, C' > 0$ such that

$$\sum_{m=0}^2 \mathbb{E} \left\{ \sup_{t \in [0, T]} \sup_{y \in \mathbb{R}^d} \exp \left\{ C \left| \nabla_y^m a \left( \frac{t}{\varepsilon^2}, y \right) \right|^2 \right\} \right\} \leq C' \varepsilon^{-\gamma}, \quad \varepsilon \in (0, 1].$$ (7.4)

**Proof.** Indeed let (cf (2.4) and (2.3))

$$A_i(t, z, z') = z' \int_{-\infty}^t e^{-z(t-s)} dw_{i,a}(s),$$

$$B_i(t, z, z') = z' \int_{-\infty}^t e^{-z(t-s)} dw_{i,b}(s), \quad (z, z') \in I,$$

where $I := [\gamma_0, \gamma_0^{-1}] \times \left[ \sqrt{2\gamma_0/\sigma_*}, \sqrt{2/\sqrt{\gamma_0 \sigma_*}} \right]$, the Brownian motions $w_{i,a}(s), w_{i,b}(s)$ are as in (2.5) for $i = 1, \ldots, N$. The conclusion of the corollary concerning the term of (7.4) corresponding to $m = 0$ follows from an application of Proposition 7.1 to the field $A(t, z, z') = (A_i(t, z, z'), B_i(t, z, z'))_{i=1, \ldots, N}$. Since the functions $\alpha_i, \sigma_i$ appearing in the definitions of the respective fields $a_i(t, y)$ and $b_i(t, y)$ are of $C^2_b(\mathbb{R}^2)$ class of regularity we can repeat the above argument for the terms corresponding to $m = 1, 2$ as well. \qed
8 Application to estimates of moments of suprema related to the corrector along the tracer path

Recall that \( \Xi^{(\epsilon)}(t) = \tilde{\Xi}(t, \bar{x}(t), x(t)) \), where \( \tilde{\Xi}(t, x, y) = \Xi(t, \mathbf{a}(t), y) \) and \( \bar{x}(t) := \epsilon^{-1} x(t) \).

The processes \( \nabla_a \Xi^{(\epsilon)}(t) \) and \( \nabla_y \Xi^{(\epsilon)}(t) \) are formed similarly, using \( \nabla_a \Xi \) and \( \nabla_y \Xi \) instead of \( \Xi \).

Lemma 8.1. Under the assumptions made in Section 6.1 for any \( T, r > 0 \) we have

\[
\lim_{\epsilon \to 0} \epsilon \mathbb{E} \left\{ \sup_{t \in [0, T]} \left[ |\Xi^{(\epsilon)}(t)|^r + |\nabla_a \Xi^{(\epsilon)}(t)|^r + |\nabla_y \Xi^{(\epsilon)}(t)|^r \right] \right\} = 0. \tag{8.1}
\]

Proof. We conduct the proof for the process \( \Xi^{(\epsilon)}(t) \). For the other processes appearing in (8.1) the argument is similar. From Proposition 6.5 we know that for any \( r > 0 \) and constant \( C_* \) there exists constant \( C > 0 \) such that for all \( \epsilon \in (0, 1), \ t \geq 0 \)

\[
|\Xi^{(\epsilon)}(t)|^r \leq C \exp \left\{ C_* \left\| \tau_{\bar{x}(t)} \mathbf{a} \left( \frac{t}{\epsilon^2}, x(\bar{x}(t)) \right) \right\|^2 \right\} \leq C \sup_{y \in \mathbb{R}^d} \left\{ C_* \left\| \mathbf{a} \left( \frac{t}{\epsilon^2}, y \right) \right\|^2 \right\}. \tag{8.2}
\]

Thanks to Corollary 7.2 we can choose \( C_* \) in such a way that for some \( \gamma \in (0, 1) \) and \( C' > 0 \) the supremum of the right hand side over \( T \in [0, T] \) of (8.2) can be estimated by \( C' \epsilon^{-\gamma} \). Thus,

\[
\lim_{\epsilon \to 0} \epsilon \mathbb{E} \left( \sup_{t \in [0, T]} \left| \Xi^{(\epsilon)}(t) \right|^r \right) = 0.
\]

Lemma 8.2. For any \( t, r \geq 0 \) we have

\[
\limsup_{\epsilon \to 0} \mathbb{E} \left\{ |\Xi^{(\epsilon)}(t)|^r + |\nabla_a \Xi^{(\epsilon)}(t)|^r + |\nabla_y \Xi^{(\epsilon)}(t)|^r \right\} < +\infty. \tag{8.3}
\]

Proof. Again, we present the argument only for \( \Xi^{(\epsilon)}(t) \). Using estimate (8.2) we can see that the conclusion of the lemma follows, provided we can show that for some \( C_* > 0 \)

\[
\limsup_{\epsilon \to 0} \mathbb{E} \left( \sup_y \exp \left\{ C_* \left\| \mathbf{a} \left( \frac{t}{\epsilon^2}, y \right) \right\|^2 \right\} \right) < +\infty. \tag{8.4}
\]

From time stationarity of the Ornstein-Uhlenbeck processes we conclude that

\[
\mathbb{E} \left( \sup_y \exp \left\{ C_* \left\| \mathbf{a} \left( \frac{t}{\epsilon^2}, y \right) \right\|^2 \right\} \right) = \mathbb{E} \left\{ \exp \left\{ C_* \sup_y |\mathbf{a}(0; y)|^2 \right\} \right\}.
\]

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Recall that from (2.3) we have
\[ a_i(0; y) = \sqrt{2\alpha_i(y)\sigma_i(y)} \int_{-\infty}^{0} e^{\alpha_i(y)s} dw_{i,a}(s), \quad i = 1, \ldots, N, \quad y \in \mathbb{R}^2. \]

Therefore (cf (2.4))
\[ \sup_y |a_i(0; y)| \leq \frac{1}{\sigma^*} \sup_{z \in [\gamma_0, \gamma_0^{-1}]} |A_i(z)|, \]
where
\[ A_i(z) := \sqrt{2z} \int_{-\infty}^{0} e^{zs} dw_{i,a}(s), \quad z \in [\gamma_0, \gamma_0^{-1}]. \]

Field \( A_i(\cdot) \) is Gaussian with covariance function
\[ R(z, z') = \frac{2\sqrt{zz'}}{z + z'}, \quad z, z' \in [\gamma_0, \gamma_0^{-1}]. \]

Using results of [2], (see Theorem 5.2, p. 120) we know that there exists constants \( C, K, \lambda_0 > 0 \) such that
\[ \mathbb{P}\left( \sup_{z \in [\gamma_0, 1/\gamma_0]} |A_i(z)| \geq \lambda \right) \leq C\lambda^K \exp\left\{ -\frac{\lambda^2}{2R_*} \right\}, \quad i = 1, \ldots, N, \quad \lambda > \lambda_0, \]
where \( R_* := \sup \{ R(z, z'), z, z' \in [\gamma_0, 1/\gamma_0] \} \). Therefore for a sufficiently small \( C_* > 0 \) formula (8.4) holds.

\[ \square \]

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