A CONSTRUCTION OF INFINITE SETS OF INTERTWINES
FOR PAIRS OF MATROIDS

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ABSTRACT. An intertwine of a pair of matroids is a matroid such that it, but none of its
proper minors, has minors that are isomorphic to each matroid in the pair. For pairs for
which neither matroid can be obtained, up to isomorphism, from the other by taking free
extensions, free coextensions, and minors, we construct a family of rank-$k$ intertwines for
each sufficiently large integer $k$. We also treat some properties of these intertwines.

1. INTRODUCTION

If the classes $C_1$ and $C_2$ of matroids are minor-closed, then so is $C_1 \cup C_2$. If $M$ is an
excluded minor for $C_1 \cup C_2$, then some minor of $M$ is an excluded minor for $C_1$ and another
is an excluded minor for $C_2$; furthermore, no proper minor of $M$ has this property. These
remarks motivate the following definition. A matroid $M$ is an intertwine of matroids $M_1$
and $M_2$ if $M$ but none of its proper minors has both an $M_1$-minor and an $M_2$-minor. Thus, each excluded minor for $C_1 \cup C_2$ is an intertwine of some excluded minor for $C_1$ and some excluded minor for $C_2$.

Many important results and problems in matroid theory involve the question of whether
the set of excluded minors for a given minor-closed class of matroids is finite; this leads
to the question of whether some pairs of matroids have infinitely many intertwines. This
question was raised by Tom Brylawski [2]; see also [5, Problem 14.4.6], where it is also
attributed to Neil Robertson and, in a different form, to Dominic Welsh. The question was
settled affirmatively by Dirk Vertigan in the mid 1990’s in unpublished work; we sketch his
construction in Section 5. Jim Geelen gave another construction [3, Section 5]: for each
pair of spikes, neither being a minor of the other and all elements of which are in dependent
transversals, he constructed infinitely many intertwines that are also spikes. (That the class
of spikes contains such infinite sets of intertwines follows from Vertigan’s construction
along with his embedding of the minor ordering on all matroids into that on the class
of spikes (for this intriguing embedding, see [2, Section 3]); Geelen’s construction is an
attractive realization of this phenomenon.) In this paper, we take weaker hypotheses than
the earlier constructions used; we assume only that neither $M_1$ nor $M_2$ can be obtained,
up to isomorphism, from the other via free extensions, free coextensions, and minors; for
such a pair $(M_1, M_2)$, we show that particular amalgams of certain free coextensions of
$M_1$ and $M_2$ are intertwines. This yields many intertwines of each sufficiently large rank;
indeed, for some pairs, a variation on our basic construction produces intertwines whose
number grows at least exponentially as a function of the rank.

We assume readers know basic matroid theory, an excellent account of which is in [5].
Key background topics are collected in Section 2 and the construction and a variation are

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given in Section [3]. In Section [4] we treat properties of these intertwines; for instance, we show that for large ranks, the intertwines we construct have large connectivity and uniform minors of large rank and corank; we show that if both matroids have no free elements, no cofree elements, no isthmuses, and no loops, then, for a fixed integer \( k \), the intertwines we construct cover the full range of possible sizes for the ground sets of rank-\( k \) intertwines of the pair; we also show that the construction preserves certain properties, such as being transversal and being a gammoid. In Section [5] we explain the relation between our construction and Dirk Vertigan’s.

2. BACKGROUND

The intertwines we construct are defined via cyclic flats and their ranks. A cyclic set in a matroid \( M \) is a (possibly empty) union of circuits. The cyclic flats of \( M \), ordered by inclusion, form a lattice; indeed, \( F \lor G = \text{cl}_M(F \cup G) \) and \( F \land G \) is the union of the circuits in \( F \cap G \). We let \( \mathcal{Z}(M) \) denote both the set and the lattice of cyclic flats of \( M \).

The following well-known results are easy to prove [5, Problem 2.1.13]:

1. \( \mathcal{Z}(M^*') = \{ S - F : F \in \mathcal{Z}(M) \} \), where \( S \) is the ground set of \( M \).
2. \( \mathcal{Z}(M_1 \oplus M_2) = \{ F_1 \cup F_2 : F_1 \in \mathcal{Z}(M_1) \text{ and } F_2 \in \mathcal{Z}(M_2) \} \), and
3. a matroid is determined by its cyclic flats and their ranks.

There are many ways to prove property (3); for instance, one can show how to get the circuits or the independent sets, or show that the rank of an arbitrary set \( Y \) in \( M \) is given by the formula

\[
(2.1) \quad r(Y) = \min \{ r(F) + |Y - F| : F \in \mathcal{Z}(M) \}.
\]

The following result from [7, 11] carries property (3) further.

**Proposition 2.1.** Let \( \mathcal{Z} \) be a collection of subsets of a set \( S \) and let \( r \) be an integer-valued function on \( \mathcal{Z} \). There is a matroid for which \( \mathcal{Z} \) is the collection of cyclic flats and \( r \) is the rank function restricted to the sets in \( \mathcal{Z} \) if and only if

1. \( \mathcal{Z} \) is a lattice under inclusion,
2. \( r(0_\mathcal{Z}) = 0 \), where \( 0_\mathcal{Z} \) is the least element of \( \mathcal{Z} \),
3. \( 0 < r(Y) - r(X) < |Y - X| \) for all sets \( X, Y \) in \( \mathcal{Z} \) with \( X \subset Y \), and
4. for all pairs of incomparable sets \( X, Y \) in \( \mathcal{Z} \),

\[
r(X) + r(Y) \geq r(X \lor Y) + r(X \land Y) + |(X \land Y) - (X \lor Y)|.
\]

Recall that the free extension \( M + x \) of the matroid \( M \) on \( S \) by the element \( x \notin S \) is the matroid on \( S \cup x \) whose circuits are those of \( M \) along with the sets \( B \cup x \) as \( B \) runs over the bases of \( M \). We extend this notation to sets: \( M + x \) is the result of applying free extension iteratively to add all elements of \( X \) to \( M \). From the perspective of Proposition [2.1] \( M + X \), for \( X \neq \emptyset \), is the matroid on \( S \cup X \) whose cyclic flats and ranks are (i) the proper cyclic flats \( F \) of \( M \), with rank \( r_M(F) \), and (ii) \( S \cup X \), of rank \( r(M) \). Dually, the cyclic flats and ranks of the free coextension \( M \times X = (M^* + X)^* \) are (i) the sets \( F \cup X \), of rank \( r_M(F) + |X| \), for \( F \in \mathcal{Z}(M) \) with \( F \neq \emptyset \), and (ii) the empty set, of rank 0.

We use only the simplest type of lift and truncation. The \( i \)-fold lift \( L^i(M) \) of \( M \) is \( (M \times X)^i \) where \( |X| = i \); dually, \( (M + X)^i \) is the \( i \)-fold truncation, \( T^i(M) \).

The nullity of a set \( Y \) is \( \eta(Y) = |Y| - r(Y) \). Let \( \mathcal{Z}'(M) \) be the set of nonempty proper cyclic flats of \( M \) and let \( \eta(\mathcal{Z}'(M)) \) be the sum of the nullities of these flats. The following lemma is easy to prove.
Lemma 2.2. If $F \in \mathcal{Z}(M \setminus x)$, then $\text{cl}_M(F) \in \mathcal{Z}(M)$. If $\text{cl}_M(F) = F$, then $\eta_{M \setminus x}(F)$ is $\eta_M(F)$; if $\text{cl}_M(F) = F \cup x$, then $\eta_{M \setminus x}(F) = \eta_M(F \cup x) - 1$.

Dually, if $F \in \mathcal{Z}(M/y)$, then exactly one of $F$ and $F \cup y$ is in $\mathcal{Z}(M)$. The nullities of $F$ in $M/y$ and the corresponding cyclic flat of $M$ agree unless $y$ is a loop of $M$, in which case $\eta_{M/y}(F) = \eta_M(F \cup y) - 1$.

Thus, if $N$ is a minor of $M$, then $\eta(Z'(N)) \leq \eta(Z'(M))$.

While a cyclic flat of a matroid may give rise to cyclic flats in its restrictions, the next lemma identifies a situation in which this does not happen.

Lemma 2.3. Let $Z$ be a cyclic flat of $M$. If a subset $U$ of $Z$ with $|U| \geq \eta(Z)$ is contained in all nonempty cyclic flats that are contained in $Z$, then $Z - U$ is independent.

Proof. Assume, to the contrary, that $Z - U$ contains some circuit $C$. The nonempty cyclic flat $\text{cl}(C)$ is contained in $Z$, so $U \subseteq \text{cl}(C)$. Thus $U \subseteq \text{cl}(Z - U) = Z$. Now $\eta(Z - U) > 0$ and $U \subseteq \text{cl}(Z - U)$ give $\eta(Z) > |U|$, contrary to the assumed inequality. \hfill \Box

For matroids $M_1$ on $S_1$ and $M_2$ on $S_2$, a matroid $M$ on $S_1 \cup S_2$ with $M|S_1 = M_1$ and $M|S_2 = M_2$ is called an amalgam of $M_1$ and $M_2$.

An element $x$ in a matroid $M$ is free in $M$ if $M = (M \setminus x) + x$. Dually, an element $y$ is cofree in $M$ if $M = (M/y) \times y$. Let $FI(M)$ be the set of all elements of $M$ that are in no proper cyclic flat of $M$; thus, $FI(M)$ consists of the free elements and isthmuses of $M$. Note that $FI(M^*)$ is the intersection of all nonempty cyclic flats of $M$; it consists of the cofree elements and loops of $M$.

3. INTERTWINES

We now construct the matroids of interest. The notation established in this paragraph is used in the rest of the paper. Assume the matroids $M_1$ and $M_2$ have positive rank and are defined on disjoint ground sets, $S_1$ and $S_2$, respectively. Let $r_1$ and $r_2$ be their rank functions, and let $\eta_1$ and $\eta_2$ be their nullity functions. Fix subsets $S_1'$ of $S_1$ and $S_2'$ of $S_2$, an integer $k$ with

$$k \geq r(M_1) + \eta_1(S_1') + r(M_2) + \eta_2(S_2'),$$

and sets $T_1$ and $T_2$ with

$$|T_1| = k - r(M_1) - |S_2'| \quad \text{and} \quad |T_2| = k - r(M_2) - |S_1'|$$

where $T_1, T_2$, and $S_1 \cup S_2$ are mutually disjoint. Let

$$Z = Z'((M_1 \times (T_1 \cup S_2')) \cup Z'((M_2 \times (T_2 \cup S_1'))) \cup \{\emptyset, S_1 \cup S_2 \cup T_1 \cup T_2\}.$$ (Note that inequality \((3.1)\) gives $|T_1| + |S_2'| \geq \eta_1(S_1') + r(M_2) + \eta_2(S_2')$, which is positive; therefore $M_1 \times (T_1 \cup S_2')$ is a proper coextension of $M_1$ and so has no loops. Likewise $M_2 \times (T_2 \cup S_1')$ has no loops. Thus, the least cyclic flat of these matroids is $\emptyset$.) Define $r : Z \to \mathbb{Z}$ by

1. $r(F \cup T_1 \cup S_2') = r_1(F) + |T_1| + |S_2'|$ for $F \in Z'(M_1)$,
2. $r(F \cup T_2 \cup S_1') = r_2(F) + |T_2| + |S_1'|$ for $F \in Z'(M_2)$,
3. $r(\emptyset) = 0$, and
4. $r(S_1 \cup S_2 \cup T_1 \cup T_2) = k$.

Theorem 3.1. The pair $(Z, r)$ satisfies properties (Z0)-(Z3) of Proposition 2.7. The rank-$k$ matroid $M$ on $S_1 \cup S_2 \cup T_1 \cup T_2$ thus defined has the following properties:

(i) $M$ is an amalgam of $M_1 \times (T_1 \cup S_2')$ and $M_2 \times (T_2 \cup S_1')$, and
(ii) \( \eta_1(F) = \eta_M(F \cup T_1 \cup S'_2) \) for \( F \in Z'(M_1) \) and \( \eta_2(F) = \eta_M(F \cup T_2 \cup S'_1) \) for \( F \in Z'(M_2) \).

**Proof.** Property (Z0) holds since any pair of sets in \( Z \) that does not have a join in one of \( Z'(M_1 \times (T_1 \cup S'_2)) \) and \( Z'(M_2 \times (T_2 \cup S'_1)) \) has \( S_1 \cup S_2 \cup T_1 \cup T_2 \) as the join. Property (Z1) is item (3) above. Property (Z2) follows from this property in \( Z(M_1 \times (T_1 \cup S'_2)) \) and \( Z(M_2 \times (T_2 \cup S'_1)) \), as do all instances of property (Z3) except in the case \( X = F_1 \cup T_1 \cup S'_2 \) with \( F_1 \in Z(M_1) \) and \( Y = F_2 \cup T_2 \cup S'_1 \) with \( F_2 \in Z(M_2) \). In this case, the required inequality is

\[
|F_1| + |T_1| + |S'_2| + |F_2| + |T_2| + |S'_1| \geq k + |F_1 \cap S'_1| + |F_2 \cap S'_2|
\]

which follows from inequality (3.1) and equations (3.2).

By symmetry, assertion (i) follows if we show that for any \( F \in Z'(M_2) \), the difference \( (F \cup T_2 \cup S'_2) - (T_2 \cup (S_2 \cup S'_1)) \) is independent in \( M \cup S_1 \cup T_1 \cup S'_2 \). All such differences are contained in \( S'_1 \cup S'_2 \), so it suffices to show that \( S'_1 \cup S'_2 \) is independent in \( M \). To show this, by equation (2.1) it suffices to prove

\[
r_M(Z) + |(S'_1 \cup S'_2) - Z| \geq |S'_1| + |S'_2|
\]

for all \( Z \in Z(M) \); again by symmetry, it suffices to show this for \( Z = F_1 \cup T_1 \cup S'_2 \) with \( F_1 \in Z'(M_1) \). For such \( Z \), the required inequality is

\[
r_1(F_1) + |T_1| + |S'_2| + |S'_1 - F_1| \geq |S'_1| + |S'_2|,
\]

or, using equations (3.2) and manipulating,

\[
k \geq r(M_1) + |S'_1 \cap F_1| - r_1(F_1) + |S'_2|.
\]

This inequality follows from inequality (3.1) since \( |S'_2| \leq r(M_2) + \eta_2(S'_2) \) and

\[
|S'_1 \cap F_1| - r_1(F_1) \leq |S'_1 \cap F_1| - r_1(S'_1 \cap F_1)
\]

\[
= \eta_1(S'_1 \cap F_1)
\]

\[
\leq \eta_1(S'_1).
\]

Assertion (ii) is evident. \( \square \)

The matroid so constructed depends on \( M_1, M_2, k, S'_1, S'_2, T_1, \) and \( T_2 \). If (as in the next result) listing all parameters aids clarity, we use \( M_k(M_1, S'_1, T_1; M_2, S'_2, T_2) \) to denote this matroid; otherwise we simply write \( M \).

The next result, which follows by comparing the cyclic flats and their ranks, shows that combining the construction with duality yields other instances of the same construction.

**Theorem 3.2.** With \( j = |S_1| + |S_2| + |T_1| + |T_2| - k \), we have

\[
(M_k(M_1, S'_1, T_1; M_2, S'_2, T_2))^* = M_j(M_1^*, S_1 - S'_1, T_2; M_2^*, S_2 - S'_2, T_1).
\]

Also, \( j \geq r(M_1^*) + \eta_{M_1^*}(S_1 - S'_1) + r(M_2^*) + \eta_{M_2^*}(S_2 - S'_2) \) if and only if \( k \) satisfies inequality (3.1).

We now treat the main result. A similar but somewhat longer argument would modestly increase the range for \( k \); we opt for the shorter proof since the main interest is in having infinitely many intertwines. Recall that \( FI(M) \) is the set of free elements and isthmuses of \( M \), so \( FI(M^*) \) is the set of cofree elements and loops of \( M \).
Theorem 3.3. Assume that the ground sets $S_1$ and $S_2$ of $M_1$ and $M_2$ are disjoint and that no matroid isomorphic to $M_1$ (resp., $M_2$) can be obtained from $M_2$ (resp., $M_1$) by any combination of minors, free extensions, and free coextensions. For $i \in \{1, 2\}$, fix a set $S'_i$ with $FI(M_i) \subseteq S'_i \subseteq S_i - FI(M_i^*)$. If $k \geq 4 \max\{|S_1|, |S_2|\}$, then the matroid $M$ defined above is an intertwine of $M_1$ and $M_2$.

Proof. Theorem 3.1 part (i) shows that $M_1$ and $M_2$ are minors of $M$. By symmetry, to prove that $M$ is an intertwine, it suffices to show that for $a \in S_1 \cup T_1$, neither $M \setminus a$ nor $M/a$ has a 1-minor and an $M_2$-minor; furthermore, by Theorem 3.2 and the observation that the hypotheses are invariant under duality, it suffices to treat $M \setminus a$. Now $|T_1| \geq r(M_i^*) + r(M_2) + |S_i| + 1$ since $k \geq 4 \max\{|S_1|, |S_2|\}$. If $M\setminus a \setminus X/Y \cong M_i$ with $i \in \{1, 2\}$, then $M\setminus a \setminus X/Y$ has $|S_i|$ elements, at least $|T_1| - |X \cap T_1| - |Y \cap T_1| - 1$ of which are in $T_1$, so $|X \cap T_1| + |Y \cap T_1| \geq |T_1| - |S_i| - 1$, and therefore

$$|X \cap T_1| + |Y \cap T_1| \geq r(M_i^*) + r(M_2).$$

Thus, either (i) $|X \cap T_1| \geq r(M_i^*)$ or (i*) $|Y \cap T_1| \geq r(M_2)$.

We claim that the three conclusions below follow when inequality (i) holds:

1. $M \setminus (X \cap T_1) = (M_2 \times (T_2 \cup S'_1)) + ((S_1 - S'_1) \cup (T_1 - X))$,
2. $M \setminus (X \cap T_1)$ has no $M_1$-minor, and
3. $\eta(F) = \eta_{M \setminus (X \cap T_1)}(F \cup T_2 \cup S_1)$ for $F \in Z'(M_1)$.

Item (1) holds since, using Lemma 2.2, we get that the proper cyclic flats and their ranks in the two matroids agree. Item (1) and the hypotheses give item (2). Item (3) is immediate.

Inequalities (i) and (i*) are related by duality, so Theorem 3.2 and the results in the last paragraph give the following conclusion if inequality (i*) holds:

1. $M \setminus (Y \cap T_1) = (M_1 \times ((T_1 - Y) \cup S'_2)) + ((S_2 - S'_2) \cup T_2)$,
2. $M \setminus (Y \cap T_1)$ has no $M_2$-minor, and
3. $\eta(F) = \eta_{M \setminus (Y \cap T_1)}(F \cup (T_1 - Y) \cup S'_2)$ for $F \in Z'(M_1)$.

For $a \in (S_1 - S'_1) \cup T_1$, assume $M \setminus a$ has an $M_1$-minor, say $M\setminus a \setminus X/Y$ By item (2), inequality (i*) holds. Since $a$ is in at least one set in $Z'(M \setminus (Y \cap T_1))$, item (3*) gives $\eta(Z'(M \setminus (Y \cap T_1) \cap a)) < \eta(Z'(M_1))$; the contradiction $M\setminus a \setminus X/Y \not\cong M_1$ now follows from Lemma 2.2.

Lastly, for $a \in S'_1$, assume $M \setminus a$ has an $M_2$-minor, say $M\setminus a \setminus X/Y$. Inequality (i) holds by item (2*). Now $a \in S'_1$ gives $\eta(Z'(M \setminus (X \cap T_1) \cap a)) < \eta(Z'(M_2))$, which, with Lemma 2.2, gives the contradiction $M\setminus a \setminus X/Y \not\cong M_2$. □

Assume $FI(M_i) = \emptyset = FI(M_i^*)$ for $i \in \{1, 2\}$. Reflecting on the proof above shows that $a \in S_1$ if and only if neither $M \setminus a$ nor $M/a$ has an $M_1$-minor, and likewise for $S_2$ and $M_2$. These conclusions and the structure of the cyclic flats of $M$ show that the counterparts of the sets $S_1, S_2, T_1, T_2, S'_1$, and $S'_2$ can be determined from any matroid that is isomorphic to $M$. This gives the following result.

Corollary 3.4. Assume $FI(M_i) = \emptyset = FI(M_i^*)$ for $i \in \{1, 2\}$. The construction gives at least $((|S_1| + 1)(|S_2| + 1))$ nonisomorphic rank-$k$ intertwines of $M_1$ and $M_2$ for each integer $k \geq 4 \max\{|S_1|, |S_2|\}$. If, in addition, both $M_1$ and $M_2$ have trivial automorphism groups, then the construction yields $2^{|S_1| + |S_2|}$ nonisomorphic rank-$k$ intertwines.
be a collection of \(k\)-subsets of \(T_1 \cup T_2\) with \(|H \cap H'| \leq k - 2\) whenever \(H\) and \(H'\) are distinct sets in \(\mathcal{H}\). In the construction, replace \(Z\) by \(Z \cup H\) and extend \(r\) to \(Z \cup H\) by setting \(r(H) = k - 1\) for all \(H \in \mathcal{H}\). Properties (Z0)–(Z3) of Proposition 2.1 are easily verified. Let \(M'\) be the matroid thus constructed. The sets in \(\mathcal{H}\) are the circuit-hyperplanes of \(M'\). By comparing the cyclic flats and their ranks, it follows that if \(M' \setminus X/Y\) has no circuit-hyperplanes, then \(M' \setminus X/Y = M \setminus X/Y\). Since neither \(M_1\) nor \(M_2\) has circuit-hyperplanes, it follows that if some single-element deletion or contraction of \(M'\) had both an \(M_1\)-minor and an \(M_2\)-minor, then the same would be true of the corresponding single-element deletion or contraction of \(M\), contrary to Theorem 3.3. Thus, \(M'\) is an intertwine of \(M_1\) and \(M_2\). Thus, we have the following result.

**Theorem 3.5.** Assume \(M_1\) and \(M_2\) satisfy the hypotheses of Theorem 3.3 and neither has circuit-hyperplanes. For each integer \(n\), there is an integer \(k_0\) so that if \(k \geq k_0\), then \(M_1\) and \(M_2\) have at least \(n\) intertwines of rank \(k\).

To take these ideas a step further, we give a simple proof that, as \(k\) grows, the number of nonisomorphic intertwines arising from the variation on the construction grows at least exponentially. To simplify the discussion slightly, assume both \(|T_1 \cup T_2|\) and \(k\) are even. Let \(\mathcal{H}'\) be the set of all sets \(\mathcal{H}\) of \(k\)-subsets of \(T_1 \cup T_2\) such that \(|H \cap H'| \leq k - 2\) whenever \(H\) and \(H'\) are distinct sets in \(\mathcal{H}\). One way to get a set \(\mathcal{H}\) in \(\mathcal{H}'\) is to pair off the elements in \(T_1 \cup T_2\) and, to get each set in \(\mathcal{H}\), choose \(k/2\) pairs. Even among sets \(\mathcal{H}\) formed in this limited way, their maximal size grows exponentially as a function of \(k\) (much as \(\binom{2n}{n}\) grows exponentially as a function of \(n\)). Among all sets in \(\mathcal{H}'\), let \(\mathcal{H}\) be one of maximal size. Subsets of \(\mathcal{H}\) of different sizes give rise to nonisomorphic intertwines (their numbers of circuit-hyperplanes differ), so these intertwines demonstrate our claim.

This discussion and the last two results suggest several problems. Let \(i(k; M_1, M_2)\) denote the number of rank-\(k\) intertwines of \(M_1\) and \(M_2\) up to isomorphism. What can be said about \(i(k; M_1, M_2)\) if \(M_1\) and \(M_2\) satisfy the hypotheses of Theorem 3.3? Is \(i(k; M_1, M_2)\) increasing as a function of \(k\), at least for sufficiently large \(k\)? If so, under what conditions on \(M_1\) and \(M_2\) is the difference \(i(k+1; M_1, M_2) - i(k; M_1, M_2)\) bounded above by a constant or by a polynomial? Under what conditions does \(i(k; M_1, M_2)\) grow exponentially or super-exponentially?

A matroid \(M\) is a labelled intertwine of \(M_1\) and \(M_2\) if \(M\) but none of its proper minors has minors equal to \(M_1\) and \(M_2\). We end this section by showing that weaker hypotheses than those in Theorem 3.4 suffice for our construction to yield labelled intertwines.

**Theorem 3.6.** Assume \(S_1\) and \(S_2\) are disjoint. If inequality (3.1) holds, neither \(M_1\) nor \(M_2\) is uniform, \(Z'(M_1) \neq \{S'_1\}\), and \(Z'(M_2) \neq \{S'_2\}\), then the matroid \(M\) constructed above is a labelled intertwine of \(M_1\) and \(M_2\).

**Proof.** By symmetry, to prove that no proper minor of \(M\) has both \(M_1\) and \(M_2\) as minors, it suffices to show that if \(M' \setminus X/Y = M_1\), then \(X = (S_2 - S'_2) \cup T_2\) and \(Y = T_1 \cup S'_2\). Thus, assume \(M' \setminus X/Y = M_1\). Fix \(F \in Z'(M_1) - \{S'_1\}\). By Lemma 2.2, the cyclic flat \(F\) of \(M' \setminus X/Y\) must arise from the cyclic flat \(F \cup T_1 \cup S'_2\) of \(M\); from Theorem 3.1 part (ii), it follows that for \(M' \setminus X/Y\) to yield the same nullity on \(F\) as in \(M_1\), each element of \(T_1 \cup S'_2\) must be contracted; dually, each element of \((S_2 - S'_2) \cup T_2\) must be deleted. Thus, \(X = (S_2 - S'_2) \cup T_2\) and \(Y = T_1 \cup S'_2\). \(\Box\)

4. **Further Results**

Among the pairs of matroids that Theorem 3.3 applies to are any two spikes of rank at least 4, neither of which is a minor of the other, provided that the one of smaller rank (if the
ranks differ) is not a free spike. (We use the definition of spikes in [3], which some sources call tip-less spikes. Free spikes are the only spikes that can be obtained from a spike by minors along with at least one lift or truncation.) Thus, the assumption in the construction in [3] that each element is in a dependent transversal is not needed here. However, unlike the construction in [3], the intertwine we get when \( M_1 \) and \( M_2 \) are spikes is not a spike.

The construction here and that in [3] give intertwines with contrasting properties and so show that some properties that hold for one construction need not hold for intertwines in general. For instance, the intertwines constructed here have neither small circuits nor small cocircuits, but those constructed in [3] have each element in a many 4-circuits and in many 4-cocircuits. Also, in our construction the number of cyclic flats does not depend on the rank, but in the construction in [3] the number of cyclic flats grows with the rank (as is true for the variation we discussed before Theorem 3.5).

4.1. Sizes of intertwines. We show that the intertwines constructed above can exhibit the full range of possible sizes for each rank.

**Theorem 4.1.** If \( S \) is the ground set of a rank-\( k \) intertwine of \( M_1 \) and \( M_2 \), then

\[
2k - r(M_1) - r(M_2) \leq |S| \leq 2k + r(M_1^*) + r(M_2^*).
\]

If \( FI(M_i) = \emptyset = FI(M_i^*) \) for \( i \in \{1, 2\} \), then the construction in Section 3 gives intertwines of each cardinality in this range.

**Proof.** Let \( M \), on the set \( S \), be a rank-\( k \) intertwine of \( M_1 \) and \( M_2 \), so \( M \setminus X/Y \simeq M_1 \) and \( M \setminus X'/Y' \simeq M_2 \) for some subsets \( X, Y, X', Y' \) of \( S \). Standard arguments about minors show that we may assume that \( Y \) and \( Y' \) are independent sets with \( |Y| = k - r(M_1) \) and \( |Y'| = k - r(M_2) \). No proper contraction of \( M \) has both \( M_1 \)- and \( M_2 \)-minors, so \( Y \cap Y' = \emptyset \); thus, \( |Y| + |Y'| \leq |S| \), so \( k - r(M_1) + k - r(M_2) \leq |S| \). This lower bound on \(|S|\) is achieved in the construction when \( S'_1 = S_1 \) and \( S'_2 = S_2 \). No proper deletion of \( M \) has both \( M_1 \)- and \( M_2 \)-minors, so \( X \cap X' = \emptyset \); thus, \( |X| \leq |S_2| + |Y'| \). This inequality, the equation \( |S| = |S_1| + |X| + |Y| \), and values for \(|Y|\) and \(|Y'|\) give the upper bound. This bound in attained when \( S'_1 = \emptyset = S'_2 \). By varying \(|S'_1|\) and \(|S'_2|\), all cardinalities between these bounds can be realized. \( \square \)

4.2. Representable matroids. All spikes are contained in \( \mathcal{E}(U_{2,6}, U_{4,6}) \), the class of matroids that have neither \( U_{2,6} \)-nor \( U_{4,6} \)-minors. The results in this subsection and the next are akin to a corollary that Vertigan got from his work on intertwines and spikes: some pairs of matroids in \( \mathcal{E}(U_{2,6}, U_{4,6}) \) have infinitely many intertwines in \( \mathcal{E}(U_{2,6}, U_{4,6}) \).

The result below uses the following equivalent formulations of two special cases of our construction. (The first assertion follows by comparing the cyclic flats and their ranks; the second is the dual of the first. Recall that \( T^k \) and \( L^j \) denote truncations and lifts.) If \( k \geq r(M_1) + r(M_2) \), then

\[
M_k(M_1, \emptyset, T_1; M_2, \emptyset, T_2) = T^k((M_1 \times T_1) \oplus (M_2 \times T_2)).
\]

If \( k \geq r(M_1) + |S_1| + r(M_2) + |S_2| \) and \( j = k - r(M_1) - r(M_2) \), then

\[
M_k(M_1, S_1, T_1; M_2, S_2, T_2) = L^j((M_1 + T_2) \oplus (M_2 + T_1)).
\]

**Corollary 4.2.** Assume a class \( C \) of matroids is closed under direct sum, free extension, free coextension, truncation, and lift. If \( M_1, M_2 \in C \) satisfy the hypotheses of Theorem 3.5 and if either \( FI(M_1) = \emptyset = FI(M_2) \) or \( FI(M_1^*) = \emptyset = FI(M_2^*) \), then \( C \) contains infinitely many intertwines of \( M_1 \) and \( M_2 \).
Such classes $C$ include the class of matroids that are representable over a given infinite field and the class of matroids that are representable over a given characteristic.

4.3. **Transversal matroids and gammoids.** We next show that the intertwine $M$ that we constructed is transversal if and only if $M_1$ and $M_2$ are; we also treat the corresponding statements for several related types of matroids. We will use the characterization of transversal matroids in Lemma 4.3, which is due to Ingleton [4] and refines a result of Mason. For a collection $\mathcal{F}$ of sets, let $\bigcap \mathcal{F}$ be $\bigcap_{X \in \mathcal{F}} X$ and $\bigcup \mathcal{F}$ be $\bigcup_{X \in \mathcal{F}} X$.

**Lemma 4.3.** A matroid $M$ is transversal if and only if for all $A \subseteq Z(M)$ with $A \neq \emptyset$,

$$
\sum_{\mathcal{F} \subseteq A} (-1)^{|\mathcal{F}|+1} r(\bigcup \mathcal{F}) \geq r(\bigcap A).
$$

In this result, it suffices to consider only antichains $A$ of cyclic flats since if $X, Y \in A$ with $X \subset Y$, then using $A - \{Y\}$ in place of $A$ does not change either side of inequality (4.1); with $A$, the terms on the left side that include $Y$ cancel via the involution that adjoins $X$ to, or omits $X$ from, $\mathcal{F}$. Also, it suffices to focus on inequality (4.1) for $|A| > 2$ since equality holds when $|A| = 1$ and the case of $|A| = 2$ is the semimodular inequality.

**Corollary 4.4.** A matroid $M$ is transversal if and only if $M \times x$ is.

Cotransversal matroids are duals of transversal matroids. Bitransversal matroids are both transversal and cotransversal. Gammoids are minors of transversal matroids. Restrictions of transversal matroids are transversal, so any gammoid is a contraction of some transversal matroid; it follows that any gammoid is a nullity-preserving contraction of some transversal matroids. The class of gammoids is closed under duality, so any gammoid has a rank-preserving extension to a cotransversal matroid.

**Theorem 4.5.** Assume inequality (4.1) holds. The matroids $M_1$ and $M_2$ are transversal if and only if $M = M_k(A_1, S'_1, T; M_2, S'_2, T_2)$ is. The corresponding statements hold for cotransversal matroids, bitransversal matroids, and gammoids.

**Proof.** Since $M_1 \times (T_1 \cup S'_2)$ and $M_2 \times (T_2 \cup S'_1)$ are restrictions of $M$, from Corollary 4.4 it follows that if $M$ is transversal, then so are $M_1$ and $M_2$. Now assume $M_1$ and $M_2$ are transversal. Let $A$ be an antichain in $Z(M)$ with $|A| > 2$. Set

$$
A_1 = A \cap Z'(M_1 \times (T_1 \cup S'_2)) \quad \text{and} \quad A_2 = A \cap Z'(M_2 \times (T_2 \cup S'_1)).
$$

Thus, $A$ is the disjoint union of $A_1$ and $A_2$. By Corollary 4.4 and Lemma 4.3, inequality (4.1) holds for $A_1$ if it is nonempty, and likewise for $A_2$; thus, this inequality holds for $A$ if one of $A_1$ and $A_2$ is empty. Assume neither is empty. For $F_1 \in A_1$ and $F_2 \in A_2$, we have $r_M(F_1 \cup F_2) = r(M)$, so

$$
\sum_{\mathcal{F} \subseteq A} (-1)^{|\mathcal{F}|+1} r(\bigcup \mathcal{F}) = \sum_{\mathcal{F}_{1,2} \subseteq A_1} (-1)^{|\mathcal{F}_{1,2}|+1} r(\bigcup \mathcal{F}_{1,2}) + \sum_{\mathcal{F}_{1,2} \subseteq A_2} (-1)^{|\mathcal{F}_{1,2}|+1} r(\bigcup \mathcal{F}_{1,2})
$$

$$
+ \sum_{\mathcal{F}_{1,2} \subseteq A_1, \mathcal{F}_{1,2} \neq \emptyset} (-1)^{|\mathcal{F}_{1,2}|+1} r(M)
$$

$$
\geq r(\bigcap A_1) + r(\bigcap A_2) - r(M)
$$

$$
\geq r(\bigcap A),
$$

where the last line follows from semimodularity along with the inclusions $T_1 \cup S'_2 \subseteq \bigcap A_1$ and $T_2 \cup S'_1 \subseteq \bigcap A_2$, and the fact that $T_1 \cup T_2 \cup S'_1 \cup S'_2$ spans $M$ (a consequence of equation (2.1) and inequality (3.1)). Thus, inequality (4.1) holds, so $M$ is transversal.
The assertions about cotransversal and bitransversal matroids follow by Theorem 3.2.

If \( M \) is a gammoid, then so are its minors \( M_1 \) and \( M_2 \). Now assume \( M_1 \) and \( M_2 \) are gammoids. Let \( M'_1 \) and \( M'_2 \) be rank-preserving cotransversal extensions of \( M_1 \) and \( M_2 \). Thus, \( M_2(M'_1, S'_1, T_1; M'_2, S'_2, T_2) \) is cotransversal since inequality (3.1) holds with \( M'_1 \) and \( M'_2 \) in place of \( M_1 \) and \( M_2 \). Comparing the cyclic flats and their ranks shows that \( M \) is a restriction of \( M_2(M'_1, S'_1, T_1; M'_2, S'_2, T_2) \), so \( M \) is a gammoid.

\[ \square \]

**Corollary 4.6.** If \( M_1 \) and \( M_2 \) satisfy the hypotheses of Theorem 3.3 and are transversal, then infinitely many intertwineds of \( M_1 \) and \( M_2 \) are transversal. The corresponding statements hold for cotransversal matroids, bitransversal matroids, and gammoids.

### 4.4. Uniform minors

We claim that \( M \mid B_1 \cup B_2 \cup T_1 \cup T_2 \), where \( B_i \) is a basis of \( M_i \), is the uniform matroid \( U_{k,2k-|S'_1| - |S'_2|} \). To see this, note that if \( C \) were a circuit in this restriction with \( r(C) < k \), then \( \text{cl}_M(C) \in Z'(M) \); however, it follows from the construction that flats in \( Z'(M) \) intersect \( B_1 \cup B_2 \cup T_1 \cup T_2 \) in independent sets.

**Corollary 4.7.** If \( M_1 \) and \( M_2 \) satisfy the hypotheses of Theorem 3.3, then for any integer \( n \), some intertwine of \( M_1 \) and \( M_2 \) has a \( U_{n,2n} \)-minor.

### 4.5. Connectivity

Recall that for any non-uniform matroid, \( \lambda(M) \leq \kappa(M) \) where \( \lambda(M) \) is the (Tutte) connectivity of \( M \) and \( \kappa(M) \) is its vertical connectivity. Thus, showing that the connectivity of intertwineds can be arbitrarily large gives the counterpart for vertical connectivity.

Qin [6] proved \( \lambda((M + p) \times q) - \lambda(M) \in \{1, 2\} \) for any matroid \( M \). For the matroid \( M \) constructed above, fix a subset \( T'_2 \) of \( T_2 \) with \( |T'_2| = \eta(M_2) \). Comparing the cyclic flats and their ranks, with the help of Lemma 2.3 gives

\[
M \setminus T'_2 = (M_1 \times (T_1 \cup S'_2)) + ((T_2 - T'_2) \cup (S_2 - S'_2)).
\]

After some number of free coextensions or free extensions of \( M_1 \) according to the difference between \( |T_1 \cup S'_2| \) and \( |(T_2 - T'_2) \cup (S_2 - S'_2)| \) (which does not change as \( k \) increases), the deletion \( M \setminus T'_2 \) can be seen as resulting from free extension/free coextension pairs, so the connectivity of such deletions \( M \setminus T'_2 \) grows with \( k \). Since extending as needed by the elements in \( T'_2 \) to obtain \( M \) preserves the rank and introduces no circuits of size \( |T_2| + |S'_1| \) or smaller, \( \lambda(M) \) also grows with \( k \).

**Corollary 4.8.** If \( M_1 \) and \( M_2 \) satisfy the hypotheses of Theorem 3.5, then for any integer \( n \), some intertwine of \( M_1 \) and \( M_2 \) is \( n \)-connected.

With the truncation that cuts the rank of the direct sum in half, it follows that the intertwine \( T^k((M_1 \times T_1) \oplus (M_2 \times T_2)) \) ( ARISING FROM \( S'_1 = 0 = S'_2 \) ) is rounded, that is, the ground set is not the union of two proper flats, or, equivalently, each cocircuit spans. (This notion, also called non-splitting, is equivalent to having \( \kappa(M) = r(M) \).)

Note that in a rank-\( n \) spike \( M \) with \( n \geq 4 \), if \( H \) is a hyperplane spanned by \( n - 2 \) legs (using the terminology of [3]), then \( (H, E(M) - H) \) is a vertical 3-separation of \( M \). Thus, the construction in this paper and that in [3] yield intertwineds with contrasting connectivity and vertical connectivity properties.

### 5. The Relation to Vertigan’s Construction

As mentioned in the introduction, the first construction of infinite sets of intertwineds for pairs of matroids was given by Dirk Vertigan. In this section we briefly outline his construction and show that, although the approaches differ, some instances of the two

\[
\]
constructions coincide; furthermore, both approaches can be extended to yield the same collections of intertwines. Vertigan’s theorem is as follows.

**Theorem 5.1.** Assume neither $M_1$ nor $M_2$ can be obtained, up to isomorphism, from the other by any combination of minors, free extensions, and free coextensions. If $FI(M_i) = \emptyset = FI(M_i')$ for $i \in \{1, 2\}$, then $M_1$ and $M_2$ have infinitely many intertwines.

The intertwines he constructed to prove this result are defined as follows. Let $S_1$ and $S_2$ be the ground sets of $M_1$ and $M_2$, which, in contrast to Theorem 3.3 need not be disjoint. Let $X$ and $Y$ be disjoint $k$-element sets, where $k \geq 10 \max\{|S_1|, |S_2|\}$, such that (i) $S_1 \cup S_2 \subseteq X \cup Y$, (ii) $X \cap S_1$ has $r(M_1)$ elements and is dependent in $M_1$, and (iii) $Y \cap S_2$ has $r(M_2)$ elements and is dependent in $M_2$. Set

$$M_1' = (M_1 + (Y - S_1)) \times (X - S_1) \quad \text{and} \quad M_2' = (M_2 + (X - S_2)) \times (Y - S_2).$$

Thus, $r(M_1') = k = r(M_2')$. He argues that the intersection of the collections of bases of $M_1'$ and $M_2'$ is the collection of bases of a matroid on $X \cup Y$, and that this matroid is an intertwine of $M_1$ and $M_2$. Thus, this intertwine has rank $k$ and has $2k$ elements. Vertigan observed that, as in Corollary 4.7, these intertwines have uniform minors of large rank and corank.

To relate this construction to ours, we first show that the bases of the intertwines we constructed can be described in a similar manner. Using the notation in Section 3 set

$$M_1'' = (M_1 \times (T_1 \cup S_2')) + (T_2 \cup (S_2 - S_2'))$$

and

$$M_2'' = (M_2 \times (T_2 \cup S_1')) + (T_1 \cup (S_1 - S_1')).$$

Both $M_1''$ and $M_2''$ have rank $k$. Observe that $Z(M) = Z(M_1'') \cup Z(M_2'')$. Using equation (3.3), it follows that a subset of $S_1 \cup S_2 \cup T_1 \cup T_2$ is a basis of $M$ if and only if it is a basis of both $M_1''$ and $M_2''$. In particular, the constructions coincide when applied under the same set up, and the basis approach can be extended to cover the results in this paper. In the other direction, it is easy to check that if we replace inequality (3.1) with a slightly stronger inequality, then Theorem 5.1 applies even when $S_1$ and $S_2$ are not disjoint; of course, then we need $S_1' \subseteq S_1 - S_2$ and $S_2' \subseteq S_2 - S_1$. Likewise, Theorem 3.2 can be adapted (for instance, instead of $S_1 - S_1'$ on the right, we need $S_1 - (S_2 \cup S_1')$). Consistent with the hypotheses in Theorem 5.1, Theorem 5.3 also applies provided that $S_1 \cap S_2$ is disjoint from $FI(M_i)$ and $FI(M_i')$ for $i \in \{1, 2\}$, Thus, an advantage of dealing with disjoint ground sets is that it eliminates the need for assumptions about $FI(M_i)$ and $FI(M_i')$.

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