The fractional and nonlinear magneto-flexible rod

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Abstract. This paper deals with a system involving a flexible rod subjected to magnetic forces that can bend it while simultaneously subjected to external excitations produces complex and nonlinear dynamic behavior, which may present different types of solutions for its different movement-related responses. This fact motivated us to analyze such a mechanical system based on modeling and numerical simulation involving both, integer order calculus (IOC) and fractional order calculus (FOC) approaches. The time responses, pseudo phase portraits and Fourier spectra have been presented. The results obtained can be used as a source for conduct experiments in order to obtain more realistic and more accurate results about fractional-order models when compared to the integer-order models.

1. Introduction

The theory of fractional calculus dates back to the birth of the theory of differential calculus, but its inherent complexity delayed the application of its associated concepts. In fact, fractional calculus is a natural extension of classical mathematics. Since the inception of the theory of differential and integral calculus, mathematicians such as Euler and Liouville developed their ideas about the calculation of non-integer order derivatives and integrals. Perhaps the subject would be more aptly called “integration and differentiation of arbitrary order.”

The basic aspects of the theory of fractional calculus are outlined in [14]. Insofar as it concerns the application of its concepts, we can cite research in different areas such as viscoelastic damping [1], robotics and control [7-9], signal processing [15], electric circuits [10]. As for the adoption of this concept in other scientific areas, several researchers have been inspired to examine this new possibility. Some work has been carried out in the field of dynamical systems theory, but the proposed models and algorithms are still in the preliminary stage.

With these ideas in mind, this work introduces the fundamentals of fractional order calculus (FOC) in order to investigate, by means of numerical simulations, the modified motion equation of a flexible rod induced by two electromagnets and submitted to external periodic excitations.

2. Fundamentals of fractional calculus

Fractional order calculus can represent systems with high-order dynamics and complex nonlinear phenomena using few coefficients, since the arbitrary order of the derivatives provides an additional degree of freedom to fit a specific behavior.

Numerous mathematicians have contributed to the history of fractional calculus by attempting to solve a fundamental problem to the best of their understanding [14]. Each researcher sought a definition and therefore different approaches, which has led to various definitions of differentiation

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and antidifferentiation of non-integer orders that are provenly equivalent. Although all these definitions may be equivalent, from one specific standpoint, i.e., for a specific application, some definitions seem more attractive.

The three most commonly used definitions are the Caputo, Riemann-Liouville and Laurent definitions. In this paper, we emphasize our simulations in Riemann-Liouville definitions, i.e.,

\[ C D^\nu_x f(x) = \frac{d^m}{dx^m} \left[ \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f(t) dt \right] \]

where, for convenience, \( m \) is considered the smallest integer larger than \( \nu \) and \( 0 < \rho \leq 1 \).

3. Mechanical system-theoretical model

Many technical devices such as motors, generators, transformers, and fusion reactors are known to employ wide elastic plates in magnetic fields. A system involving a flexible rod subjected to magnetic forces that can bend it while simultaneously subjected to external excitations produces complex and nonlinear dynamic behavior [3-5], which may present different types of solutions for its different movement-related responses [2]. This fact motivated us to analyze such a mechanical system based on modeling and numerical simulation involving both, (IOC) and (FOC), approaches.

A continuum model based on linear elastic and nonlinear magnetic forces was developed, which can be reduced to an oscillator model with a single degree of freedom using the Lagrangian formalism [6] and the Galerkin’s method [4-6].

The mechanical system whose theoretical model is developed is shown in Figure 1. A flexible rod is clamped in a rigid base. The electromagnets pull the beam in opposite directions and intensity of the field is strong enough to deflect the rod from one side to another.

![Figure 1: Mechanical system](image)

The electromagnets generate a magnetic field that induces a magnetization \( \mathbf{M} \) per unit volume in the solid. The rod can be modeled as a soft magnetic material where \( \mathbf{M} \) is proportional to the local magnetic field in the solid [12]; i.e.,

\[ \bar{\mathbf{M}} = \left[ \frac{\chi}{(\chi + 1)} \right] \bar{\mathbf{B}} \]

\( \mu_0 \) is the magnetic permeability of a vacuum; \( \chi \) is the magnetic susceptibility

The field \( \mathbf{B} \) can be written in terms of the field \( \mathbf{B}^0 \) produced by external magnets, and a field produced by the magnetization itself, \( \mathbf{B}^1 \). If self-forces on the rod are neglected then the external magnets produce both a force and moment distribution on the beam given by [12,13]

\[ \bar{\mathbf{F}} = \bar{\mathbf{M}} \cdot \nabla \mathbf{B}^0 \]

\[ \bar{\mathbf{C}} = \bar{\mathbf{M}} \times \mathbf{B}^0 \]

These forces can be derived from a magnetic potential and therefore they are conservative. Thus,
\[ W = -\frac{1}{2} \int \boldsymbol{M} : \dot{\boldsymbol{B}}^0 \, dv \]  \hspace{1cm} (5)

The non-linearities included in the analysis reflect the inhomogeneous nature of the magnetic field \( \mathbf{B}^0 \) [13] and the magnetic force and couple. If the \( x \), \( y \) components of \( \mathbf{B}_0 \) are introduced, defined by \( \mathbf{B}_{0x} = B_0 \cos \alpha \) and \( \mathbf{B}_{0y} = B_0 \sin \alpha \) and the local slope of the rod with the \( x \) axis by \( \theta \), then the magnetic energy potential then takes the form
\[ W = -\frac{\chi}{4 \mu_0 \mu_r} \int_0^L \left( B_1 + B_2 \sin 2 \theta + B_3 \cos 2 \theta \right) ds \]  \hspace{1cm} (6)

with:
\[ B_1 = \left( \mu_r + 1 \right) \left( B_{0x}^2 + B_{0y}^2 \right) \]  \hspace{1cm} (7)
\[ B_2 = 2 \left( \mu_r - 1 \right) B_{0x} B_{0y} \]  \hspace{1cm} (8)
\[ B_3 = \left( \mu_r - 1 \right) \left( B_{0x}^2 - B_{0y}^2 \right) \]  \hspace{1cm} (9)

Here the integration is carried out over the original length of the rod and \( B_1 \), \( B_2 \) and \( B_3 \) are functions of the rod displacement. The nonlinear elastic forces are small even for the large displacements of the rod tip. Moreover, if a single mode approximation is made for the beam deformation, the elastic energy can be write in the form
\[ P = \frac{1}{2} K y^2 + \text{(higher order terms)} \]  \hspace{1cm} (10)

With the usage Lagrangian's, the Galerkin's method, and a typical choice for the \( \phi \), in vibration problems are the normal modes of the associated linear problem, which when substitutes in equation of motion allow us write it with the dimensionless form [2],
\[ \ddot{x} + \delta \dot{x} - x + x^3 = F \cos \omega t \]  \hspace{1cm} (11)

where:
\( \delta > 0 \) is the damping constant, \( F \) is the forcing strength and \( \omega \) is the forcing frequency.

In this article we investigate, by means of numerical simulations, the modified magneto-flexible rod dynamic equations (12), i.e.,
\[ x^{\lambda+1} + \delta x^\lambda - x + x^3 = F \cos \omega t \]  \hspace{1cm} (12)

We presented the time responses of \( x \) and \( x^\lambda \), pseudo phase portraits and Fourier spectra for some values of \( \lambda \), \( \omega \), \( F \) and for \( \delta = 0.2 \).

4. Some numerical simulation results

We would like to illustrate some simulations results for non-integer order equation (12) in comparison to integer order for \( F = 0.18 \text{N} \), \( \omega = 1 \text{rad/s} \) and \( \lambda = [0.8 \text{(red line)}; 1.0 \text{(blue line)}; 1.2 \text{(green line)}] \).
We have illustrated too some simulations results for non-integer order equation (12) in comparison to integer order for $F = 4N$, $\omega = 1.12 \text{ rad/s}$ and $\lambda = \{0.8\text{(red line)}; 1.0\text{(blue line)}; 1.2\text{(green line)}\}$. 

Figure 4: Phase plane ($x^2$ vs. $x$), for $\lambda = \{0.8, 1\}$.

Figure 5: Phase plane ($x^2$ vs. $x$), for $\lambda = \{1, 1.2\}$.

Figure 6: FFT for $\lambda = \{0.8, 1\}$.

Figure 7: FFT for $\lambda = \{1, 1.2\}$.

Figure 8: $x$ vs. $t$.

Figure 9: $x^4$ vs. $t$.

Figure 10: Phase plane, ($x^2$ vs. $x$), for $\lambda = \{0.8, 1\}$.

Figure 11: Phase plane, ($x^2$ vs. $x$), for $\lambda = \{1, 1.2\}$.
We also have illustrated some simulations results for non-integer order equation (12) in comparison to integer order for $F = 0.5N$, $\omega = 1.5$ rad/s and $\lambda = [0.8$ (red line); $1.0$ (blue line); $1.2$ (green line)].
5. Conclusions

In this paper we have proposed some versions of the modified magneto-elastic dynamic equation like presented by equation (12). Such modifications consisted in the introduction of a non-integer order time derivative in the standard equation (11). The results reveal that the non-integer order systems can exhibit different and curious behaviour from those obtained with the standard magneto-flexible rod dynamic system. The non-integer order can be useful for a better understanding and control of such nonlinear systems. Also, the results obtained can be used as a source for conduct experiments in order to obtain more realistic and more accurate results about fractional-order models when compared to the integer-order models.

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