AdS$_2$ Models in an Embedding Superspace

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Abstract
An embedding superspace, whose Bosonic part is the flat 2 + 1 dimensional embedding space for AdS$_2$, is introduced. Superfields and several supersymmetric models are examined in the embedded AdS$_2$ superspace.
The $N = 1$ supersymmetry (SUSY) algebra in $AdS_2$ space can be written

$$[J_{\mu \nu}, J_{\lambda \sigma}] = \eta_{\mu \lambda} J_{\nu \sigma} - \eta_{\nu \lambda} J_{\mu \sigma} + \eta_{\nu \sigma} J_{\mu \lambda} - \eta_{\mu \sigma} J_{\nu \lambda}$$

(1a)

$$[J_{\mu \nu}, Q] = -\Sigma_{\mu \nu} Q$$

(1b)

$$\{Q, \tilde{Q}\} = 2\Sigma_{\mu \nu} J^{\mu \nu}.$$  

(1c)

$AdS_2$ corresponds to a surface of constant curvature embedded in flat 2 + 1 dimensional space time (2+1D) described by the constraint

$$x^\mu x_\mu = a^2.$$  

(2)

In this letter, we follow Dirac’s approach to $dS_4$ space [1]; we work in the embedding space, rather than directly in the two dimensional curved space itself, and we use the $SO(2,1)$ generators associated with the symmetries on this curved surface. A similar approach has been used by Adler [3] and Drummond and Shore [4] to analyze massless gauge theories on the surface $S_4$, using the $SO(5)$ generators associated with $S_4$.

In a previous paper [2] we constructed, using this approach, a component field supersymmetric model involving two real spin zero fields and one Majorana spinor field. These fields we have shown to carry full off shell implementation of the SUSY algebra (1) above.

In this letter we wish to report on some progress in developing a superspace and superfield method of addressing this problem. Our approach is modelled directly on Dirac’s approach. What we do in this paper is distinct from the approach employed in [5,6] where $AdS_2$ supersymmetry is implemented directly in 2|2 curved superspace [7,8,9] by specializing to the appropriate fixed geometry.

We consider a superspace with a 2 + 1 D Bosonic part, corresponding to the embedding space for $AdS_2$, with coordinates $x^\mu$ ($\mu = 1, 2, 3$), and a Fermionic part with Grassmann coordinates given by a Majorana spinor $\theta_i$ ($i = 1, 2$). The $AdS_2$ superspace is an embedded surface in this superspace given by the supersymmetric generalization of the above constraint.

1The metric is $\eta_{\mu \nu} = \text{diag}(+, - , +)$ and we take $\gamma^\mu \gamma^\nu = -\eta^{\mu \nu} + 2\Sigma^{\mu \nu}$ with $\gamma_1 = i\tau_1$, $\gamma_2 = \tau_2$, $\gamma_3 = i\tau_3$ so that $\Sigma^{\mu \nu} L_{\mu \nu} L^{\lambda \sigma} - \Sigma^{\mu \nu} L_{\mu \nu} = -\frac{1}{2} L^{\mu \nu} L_{\mu \nu}$. $Q$ is Majorana with $\bar{Q} = Q^1 \gamma_2$, $\tilde{Q} = Q^T \gamma_2$, $Q = Q_C = \gamma_2 \bar{Q}^T$. 

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In this embedding superspace we can realise the generators in the algebra (1) as follows

\[ J_{\mu\nu} = \frac{\partial}{\partial \theta} \Sigma_{\mu\nu} - (x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \equiv \frac{\partial}{\partial \theta} \Sigma_{\mu\nu} + L_{\mu\nu} \]  

(3)

\[ Q = \gamma^\mu \partial_{\mu} \theta + \gamma^\mu x_{\mu} \frac{\partial}{\partial \theta} \]  

(4)

\[ \bar{Q} = -\bar{\theta} \gamma^\mu \partial_{\mu} + \frac{\partial}{\partial \bar{\theta}} \gamma^\mu x_{\mu} . \]  

(5)

(We do not include the translation operator in the Bosonic sector of this algebra, as in [13].)

We can identify at once the supersymmetric generalization of two SO(2,1) invariants which are crucial to Dirac’s approach. First, the generalization of the SO(2,1) invariant \( x^\mu x_{\mu} \) is

\[ R^2 \equiv x^\mu x_{\mu} - \bar{\theta} \theta \]  

(6)

and second, the generalization of the SO(2,1) invariant \( x^\mu \partial_{\mu} \) is

\[ \Delta \equiv x^\mu \partial_{\mu} + \theta_i \frac{\partial}{\partial \theta_i} = x^\mu \partial_{\mu} + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i} . \]  

(7)

In other words, both \( R^2 \) and \( \Delta \) commute with \( J_{\mu\nu} \) and \( Q \). Dirac [1] used the constraint \( x^\mu x_{\mu} = a^2 \) to go from the embedding space to \( dS_4 \) space; we use its supersymmetric generalization to go from the embedding superspace to the \( AdS_2 \) superspace. This entails including a factor of \( \delta (R^2 - a^2) = \delta (x^2 - a^2) - \bar{\theta} \theta \delta ' (x - a^2) \) in the action. Also Dirac used \( x^\mu \partial_{\mu} \) to define fields, initially given on the \( dS_4 \) surface, away from that surface; likewise we use \( \Delta \) to define superfields, initially given on the \( AdS_2 \) supersurface, away from that surface.

A scalar superfield will be a scalar function defined on the embedding superspace, \( \Phi(x, \theta) \). In the Dirac approach, the superfield is considered defined on the surface \( x^2 = a^2 \) by including the supersymmetry invariant factor of \( \delta (R^2 - a^2) \) in the action. It is defined away from this surface by imposing the condition

\[ \Delta \Phi = w \Phi \]  

(8)

where \( w \) is a real eigenvalue. We note that this condition is preserved under a supersymmetry transformation. Expanding \( \Phi \) in powers of \( \theta \)

\[ \Phi(x, \theta) = \phi(x) + \bar{\lambda}(x)\theta + F(x)\bar{\theta} \theta . \]  

(9)
we can at once identify the component fields, namely, two scalar fields $\phi$ and $F$ and a Majorana spinor field $\lambda$. The condition (8) translates into the following homogeneity conditions for the component fields

$$(x^\mu \partial_\mu - w)\phi = (x^\mu \partial_\mu + 1 - w)\lambda = (x^\mu \partial_\mu + 2 - w)F = 0.$$  

(10)

The SUSY transformations for the component fields are induced from

$$\delta \Phi = i \left[ \tilde{\xi} Q, \Phi \right]$$  

(11)

by the realization (4); we find

$$\delta \phi = i \tilde{\xi} \gamma^\mu x_\mu \lambda$$  

(12a)

$$\delta \tilde{\lambda} = i \tilde{\xi} \gamma^\mu \left( \partial_\mu \phi + 2 x_\mu F \right)$$  

(12b)

$$\delta F = - \frac{i}{2} \tilde{\xi} \gamma^\cdot \partial \lambda.$$  

(12c)

It is now possible to specify an action in terms of the superfield $\Phi$ that is automatically invariant under the SUSY transformations associated with the $AdS_2$ algebra (1). To do this we introduce the operators $E(\alpha, \beta)$ defined by

$$E(\alpha, \beta) = \alpha \gamma^\cdot \partial \theta + \beta \gamma^\cdot x \frac{\partial}{\partial \theta}$$  

(13a)

$$\tilde{E}(\alpha, \beta) = - \alpha \tilde{\theta} \gamma^\cdot \partial + \beta \frac{\partial}{\partial \theta} \gamma^\cdot x.$$  

(13b)

If $\alpha = 1/\beta$ (and only in this case), $E(1/\beta, \beta) (\equiv Q)$ satisfies the algebra of eq. (1c). Furthermore,

$$[E(\alpha, \beta), \Delta] = 0$$  

(14a)

for all $\alpha, \beta$ while

$$[E(\alpha, \beta), R^2] \neq 0$$  

(14b)

unless $\alpha = \beta$. It is now possible to write down a family of free supersymmetric actions for the superfield $\Phi$ in the embedding $AdS_2$ superspace, namely

$$S_1(\alpha, \beta, \rho) = \int d^3 x d^2 \theta \delta \left( R^2 - a^2 \right) \Phi \left( \tilde{E} \Phi + \rho \right).$$  

(15)
The role of the $\delta$-function is to constrain the integration over the embedding superspace to just the $AdS_2$ superspace part of it. It implies that the Bosonic part of the superspace is constrained to lie on the $AdS_2$ surface embedded in $2 + 1$ dimensions as

$$\delta(R^2 - a^2) = \delta(x^2 - a^2) - \delta'(x^2 - a^2) \tilde{\theta} \theta .$$  \hfill (16)

Clearly $S_1$ is $SO(2,1)$ invariant; it is also supersymmetric. (The product of the three scalar superfields $\Phi$, $\tilde{E}E\Phi$ and $\delta (R^2 - a^2)$ is also a scalar superfield, and by (12c) the $\tilde{\theta}\theta$ component of any scalar superfield transforms as a divergence. Upon defining $\int d^2\theta \tilde{\theta} \theta = 1$, the $\theta$ integration in (15) picks out the $\tilde{\theta}\theta$ component of the superfield integrand, and thus it follows that (15) is supersymmetric for all values of $\alpha$, $\beta$, and $\rho$.)

The component field form of this action can be deduced using

$$\tilde{E}E = -\beta^2 x^2 \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tilde{\theta}} + \frac{\alpha^2}{2x^2} (L^{\mu\nu}L_{\mu\nu} + 2(x \cdot \partial)^2 + 2(x \cdot \partial)) \tilde{\theta} \theta \quad \hfill (17a)$$

$$-\alpha \beta \left[ 2(x \cdot \partial) + 2\tilde{\theta} (-x \cdot \partial + \Sigma^{\mu\nu}L_{\mu\nu}) \frac{\partial}{\partial \tilde{\theta}} - 3\tilde{\theta}ight].$$

We find that

$$S_1(\alpha, \beta, \rho) = \int d^3x \left\{ \delta(x^2 - a^2) \left[ +\tilde{\lambda} \left( \alpha\beta\Sigma^{\mu\nu}L_{\mu\nu} - \frac{\rho + 3\alpha\beta}{2} \right) \lambda + \frac{\alpha^2}{2x^2} \phi (L^{\mu\nu}L_{\mu\nu} + 2w(1 + w)) \phi \right. \right.$$

$$-2x^2\beta^2F^2 + 2(\rho + \alpha\beta)\phi F \left. \right] + \delta' \left( x^2 - a^2 \right) \left[ 2x^2\beta^2F - (\rho - 2\alpha\beta\omega)\phi^2 \right]\} .$$

Here the action is expressed as an integral over the embedding space of $AdS_2$. Integrating over $\sqrt{x^\mu x_\mu}$ using the $\delta$-function, we find the action integral defined on the $AdS_2$ surface

$$S_1(\alpha, \beta, \rho) = \int d^2A \left\{ \lambda \left[ \alpha\beta \left( \Sigma^{\mu\nu}L_{\mu\nu} - \frac{3}{2} \right) - \rho/2 \right] \lambda \right. \quad \hfill (18)$$

$^2$We use

$$\int d^3x \delta' (x^2 - a^2) f(x) = -\frac{1}{2a^2} \int d^3x \delta (x^2 - a^2) (x \cdot \partial + 1)f(x)$$

$$= -\frac{1}{2a^2} \int d^2A \frac{a}{2} (x \cdot \partial + 1)f(x) \bigg|_{(x^2=a^2)}$$
\[ + \frac{1}{2a^2} \phi \left[ \alpha^2 (L^\mu L_\mu + 2w(1 + w)) + (2w + 1)(\rho - 2\alpha \beta w) \right] \phi \\
- 2a^2 \beta^2 F^2 + \left[ -\beta^2(2w + 1) + 2(\rho + \alpha \beta) \right] \phi F \].

It is of interest to compare this supersymmetric action on AdS$_2$ derived by superfield methods with the supersymmetric action constructed previously [2] without the benefit of superfield techniques. The free part of that action was (with $a^2 = 1$),

\[ \overline{S} = \int d^2A \left[ -\tilde{\Psi} (\Sigma^{\mu\nu} L_{\mu\nu} + \chi + \lambda_1) \Psi + A \left( \frac{1}{2} L^{\mu\nu} L_{\mu\nu} \right. \right. \]
\[ \left. \left. + \chi(1 + \chi) + \lambda_1(1 + 2\chi) \right) A - B^2 + 2\lambda_1 AB \right], \tag{19} \]

where $A$ and $B$ are real scalar fields and $\Psi$ is a Majorana spinor. The SUSY transformations under which $\overline{S}$ is invariant are

\[ \delta A = \tilde{\xi} \Psi \] \hspace{1cm} (20a)
\[ \delta \Psi = - \left( [\Sigma^{\mu\nu} L_{\mu\nu} - (1 + \chi)] A - B \right] \xi \] \hspace{1cm} (20b)
\[ \delta B = - \tilde{\xi} \left[ \Sigma^{\mu\nu} L_{\mu\nu} + \chi \right] \Psi. \] \hspace{1cm} (20c)

These transformations satisfy the algebra of eq. (1c).

Let us compare the two actions (18) and (19) and the SUSY transformations under which they are invariant, (12) and (20) respectively. We note that the transformations (12) and (20) are identical if we identify $A = \phi$, $\Psi = i\gamma \cdot x \lambda$, $B = -2x^2 F$ and $\chi = -1 - w$. However, the action $\overline{S}$ in (19) does not yield $S_1$ in (18) if we make this identification; we find (using $\Sigma^{\mu\nu} L_{\mu\nu} \gamma \cdot x = \gamma \cdot x \Sigma^{\mu\nu} L_{\mu\nu} - 2\gamma \cdot x$)

\[ \overline{S} = \int d^2A \left\{ \tilde{\lambda} \left( \Sigma^{\mu\nu} L_{\mu\nu} - 2 + \chi + \lambda_1 \right) \lambda \right. \left. \right. \]
\[ \left. \left. + \phi \left( \frac{1}{2} L^{\mu\nu} L_{\mu\nu} + \chi(1 + \chi) + \lambda_1(1 + 2\chi) \right) \phi \right\} - 4F^2 - 4\lambda_1 \phi F \right]. \tag{21} \]

This form of $\overline{S}$ and the form of $S_1$ in (18) cannot be equated for any values of $\alpha$, $\beta$ and $\rho$. However, both are invariant under the same SUSY transformations, namely eqs. (12).
It is possible to write down another family of supersymmetric actions in superfield formalism,

\[ S_2(\alpha, \beta, \rho) = \int d^3x d^2\theta \delta \left( R^2 - a^2 \right) \left[ - \left( \tilde{E}\Phi \right) (E\Phi) + \rho \Phi^2 \right] . \]  

The actions \( S_2 \) are distinct from the actions of \( S_1 \) unless \([E, R^2] = 0\), which only occurs when \( \alpha = \beta \). \( S_2 \) is clearly supersymmetric; the product of the spinor superfields \( \tilde{E}\Phi \) and \( E\Phi \) is a scalar superfield and hence the integrand is a scalar superfield whose \( F \) component transform as a total divergence (12c).

The actions of eqs. (15) and (19) do not involve interactions. A suitable interaction in the superfield approach is given by

\[ S_N = g_N \int d^3x d^2\theta \delta \left( R^2 - a^2 \right) \Phi^N \quad (N \geq 3). \]  

It is also possible to supplement the action \( \mathbf{S} \) of eq. (19) with an interaction Lagrangian

\[ \mathbf{S}_N = \lambda_N \int d^3x \left[ (1 + 2\chi)A^2 - \tilde{\Psi}\Psi + 2AB \right]^N \]  

\((N \geq 2)\). For all \( N, \mathbf{S}_N \) in (24) is invariant under the transformations of eq. (20). The interactions \( S_N \) and \( \mathbf{S}_N \) are clearly distinct, as \( \mathbf{S}_N \) contains a contribution of order \((AB)^N\) while \( S_N \) is at most linear in \( F \).

The \( N = 2 \) SUSY algebra on \( AdS_2 \) involves replacing (1a) by

\[ \{ Q_i, \tilde{Q}_j \} = -i\epsilon_{ij} \alpha \pm 2\delta_{ij} \Sigma^{\mu\nu} J_{\mu\nu} \]  

(25a)

and introducing

\[ [\alpha, Q_i] = \pm i\epsilon_{ij} \alpha . \]  

(25b)

(Either sign is consistent with the Jacobi identities.) The occurrence of an ‘internal symmetry’ generator \( \alpha \) in (25) is reminiscent of the \( Z \) generator in the simplest SUSY algebras on the surface of a sphere \( S_2 \) embedded in three dimensions. On \( S_2 \) one can only have Dirac or symplectic Majorana (but not Majorana) spinors. These SUSY algebras on \( S_2 \) are

\[ \{ R, R^\dagger \} = Z \mp 2\sigma \cdot \vec{J} \quad [J_a, R] = -\frac{1}{2} \sigma_a R \]  

(26a, b)
\[ [Z, R] = iR \quad [J_a, J_b] = i\epsilon_{abc}J_c \quad (26c, d) \]

where \( R \) is a Dirac spinorial generator. A multiplet consisting of a Dirac spinor and two complex scalars has a supersymmetric action given in [2] which is similar in form to the sum of (19) and (24). It is interesting to speculate that a superfield formulation, which carries a representation of either of the algebras of (26), can be found for the case of \( S_2 \). Indeed, it is possible to realize the operators occurring in (26), namely \( R, R^\dagger, Z \) and \( J \), on a 4\(|4\) space

\[ R = (\bar{\sigma} \cdot \bar{x} + \beta) \frac{\partial}{\partial \theta^\dagger} \pm \left( \frac{\partial}{\partial \beta} - \bar{\sigma} \cdot \nabla \right) \theta \quad R^\dagger = \frac{\partial}{\partial \theta} (\bar{\sigma} \cdot \bar{x} + \beta) \mp \theta^\dagger \left( \frac{\partial}{\partial \beta} - \bar{\sigma} \cdot \nabla \right) \quad (27a, b) \]

\[ J_a = \frac{1}{2} \left[ \theta^\dagger \sigma_a \frac{\partial}{\partial \theta^\dagger} + \frac{\partial}{\partial \theta} \sigma_a \theta \right] + (-i\bar{x} \times \nabla)_a \quad Z = \pm \left[ \theta^\dagger \frac{\partial}{\partial \theta^\dagger} - \theta \frac{\partial}{\partial \theta} \right] \quad (27c, d) \]

To achieve this realization we have introduced an auxiliary variable \( \beta \) which does not seem to have anything to do with the sphere \( S_2 \). The \( SO(3) \) invariants \( x^2 \) and \( \bar{x} \cdot \nabla \) can be generalized to \( x^2 - \beta^2 - 2\theta^\dagger \theta \) and \( \bar{x} \cdot \nabla + \beta \frac{\partial}{\partial \beta} + \theta^\dagger \frac{\partial}{\partial \theta^\dagger} + \theta \frac{\partial}{\partial \theta} \), both of which commute with \( R, R^\dagger, \bar{J} \) and \( Z \). However, no such superfield formulation has yet been found. Indeed, it may be necessary to consider harmonic superspace, as has been done for \( N = 2 \) supersymmetry in four dimensional Minkowski space [10], to achieve this formulation on \( S_2 \).

In this paper and in our previous work [2], we have chosen to follow the approach of Dirac [1] and work in the flat embedding space for \( AdS_2 \). A more conventional approach has been used to construct supersymmetric models in \( AdS_2 \) [5,6]. In this approach one works directly in the two dimensional constant curvature \( AdS_2 \) space. Although a superfield formalism in 2\(|2\) dimensions using supergravity techniques is known [7-9], no superfield version of the \( AdS_2 \) models of [5,6] has been given explicitly.

It remains to be seen what precise relationship (if any) there is between the models which we have presented in this paper and the models of [5,6]. A somewhat similar problem for the spinor field has been considered some time ago [11]. In that case the relationship between Dirac’s \( dS_4 \) wave equation and the spinor wave equation in a four dimensional space of the appropriate constant curvature was examined. We establish below, and in the appendix, the mappings between the Bosonic and spinor free field equations of motion in the two formulations.
We can parametrize $AdS_2$ space using coordinates $t$ and $\rho$ where

$$
\begin{align*}
x^1 &= a \sin t \sec \rho, \\
x^2 &= a \tan \rho, \\
x^3 &= -a \cos t \sec \rho.
\end{align*}
$$

(28a)

This implies that in $AdS_2$ space

$$
\begin{align*}
g_{11} &= -g_{22} = \frac{a^2}{\cos^2 \rho}, \\
w_1^{12} &= -\tan \rho, \\
w_2^{12} &= 0
\end{align*}
$$

(29b, c, d)

while the $SO(2, 1)$ generators are

$$
\begin{align*}
L_{21} &= \cos t \sin \rho \partial_t + \sin t \cos \rho \partial_\rho, \\
L_{31} &= \partial_t \\
L_{32} &= -\sin \rho \sin t \partial_t + \cos t \cos \rho \partial_\rho.
\end{align*}
$$

(29e)

It follows that

$$
\frac{1}{2} L^{\mu\nu} L_{\mu\nu} = \cos^2 \rho \left( \partial_t^2 - \partial_\rho^2 \right) = \frac{1}{a^2} g^{ab} \partial_a \partial_b
$$

(30)

showing that the Bosonic field equations of motion are equivalent in the two formulations.

In the models of [5,6] the spinor equation of motion is, using our conventions,

$$
0 = (\gamma^a D_a - m) \chi = \left[ \cos \rho \left( \gamma^1 \partial_t + \gamma^2 \partial_\rho + \gamma^1 \Sigma_{12} \tan \rho \right) - m \right] \chi.
$$

(31a)

In the models we have considered, in this paper, the spinor equation is of the form

$$
0 = (\Sigma^{\mu\nu} L_{\mu\nu} - M) \lambda = \left\{ i \left[ \gamma^3 \left( \cos t \sin \rho \partial_t + \sin t \cos \rho \partial_\rho \right) + \gamma^2 \left( -\sin \rho \sin t \partial_t + \cos t \cos \rho \partial_\rho \right) \right] - M \right\} \lambda.
$$

(31b)

Although equations (31a) and (31b) appear to be different, they are actually equivalent. We can provide a linear mapping between their solutions $\chi$ and $\lambda$. The details of this mapping are provided in the appendix.

The whole question of quantization and the computation of radiative effects requires examination. Some preliminary work, in the context of the multiplet approach discussed above, has been reported in [2]. With the development of superfield versions of models on $AdS_2$, the possibility of quantization in superspace and of doing computations using superspace Feynman rules arises.
The models presented in this paper may well allow the construction of supersymmetric non-linear sigma models in which the world sheet is an $AdS_2$ space and the target space is a general $N$ dimensional space. [12]

Supersymmetric models on higher dimensional spaces of constant curvature merit consideration.

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**References**

[1] P.A.M. Dirac, Ann. of Math. 36 (1935) 657.

[2] D.G.C. McKeon and T.N. Sherry, Can. J. Phys. (in press).

[3] S. Adler, Phys. Rev. D6 (1973) 2400.

[4] I. Drummond and G. Shore, Ann. of Phys. 117 (1979) 89.

[5] W.A. Bardeen and D.Z. Freedman, Nucl. Phys. B253 (1985) 635.

[6] N. Sakai and Y. Tani, Nucl. Phys. B258 (1985) 661.

[7] P. Howe, J. Phys. A12 (1979) 393.

[8] K. Hagashijima, T. Uematsu and Y.Z. Yu, Phys. Lett. 139B (1984) 161.

[9] M. Brown and S.J. Gates, Ann. of Phys. 122 (1979) 443.
Appendix

There is a linear relationship $\chi = T \lambda$ between the wave functions $\chi$ and $\lambda$ appearing in (31a) and (31b) respectively provided

$$ (\gamma^a D_a - m) T \lambda = k T (\Sigma^{\mu\nu} L_{\mu\nu} - M) \lambda. \quad (A.1) $$

Here $k$ is a constant and

$$ T = \begin{pmatrix} A_- & B_- \\ A_+ & B_+ \end{pmatrix}. \quad (A.2) $$

With the Dirac matrix conventions of footnote (1), we find that (A.1) leads to the following consistency conditions upon identifying corresponding terms dependent on $\lambda$, $\partial_t \lambda$ and $\partial_\rho \lambda$:

$$ \left[ C_\rho (\partial_t \pm \partial_\rho) \pm \frac{1}{2} S_\rho \right] A_\pm = -i \Delta A_\mp \quad (A.3a) $$

$$ \left[ C_\rho (\partial_t \pm \partial_\rho) \pm \frac{1}{2} S_\rho \right] B_\pm = -i \Delta B_\mp \quad (A.3b) $$

$$ i C_\rho A_\pm = k \left[ (-C_t S_\rho) A_\mp + (1 + S_\rho S_t) B_\mp \right] \quad (A.4a) $$

$$ i C_\rho B_\pm = k \left[ (-1 + S_\rho S_t) A_\mp + (C_t S_\rho) B_\mp \right] \quad (A.4b) $$

$$ i A_\pm = \mp k [S_t A_\mp + C_t B_\mp] \quad (A.5a) $$

$$ i B_\pm = \mp k [C_t A_\mp - S_t B_\mp]. \quad (A.5b) $$

(Here, $C_\rho = \cos \rho$, $S_\rho = \sin \rho$ etc., $\Delta = m - kM$ and we have set $a = 1$.)

The algebraic eqs. (A.4 - A.5) are solved by

$$ B_- = \tan \left( \frac{\rho - t}{2} \right) A_- = \frac{S_\rho - S_t}{C_\rho - C_t} A_- \quad (A.6a) $$
\[ A_+ = \tan \left( \frac{\rho + t}{2} \right) B_+ \]  \hspace{1cm} (A.6b)

\[ B_+ = \frac{i}{k} \left( \frac{C_t + C_\rho}{S_{\rho - t}} \right) B_- \]  \hspace{1cm} (A.6c)

and

\[ k^2 = 1 \].  \hspace{1cm} (A.7)

The differential eqs. of (A.3) are consistent with (A.6) if \( \Delta = -k \), or in other words

\[ m = k(M - 1) \].  \hspace{1cm} (A.8)

Finally, by using (A.6) and (A.3b) we find that \( B_- \) satisfies the equations

\[ \partial_- B_- = \frac{1}{\cos(x_+ - x_-)} \left[ \frac{1}{2} \sin(x_+ - x_-) + \frac{\cos(x_+)}{\sin(x_-)} \right] B_- \]  \hspace{1cm} (A.9a)

and

\[ \cos (x_+ - x_-) \partial_+ \left( \frac{\cos(x_+)}{\sin(x_-)} B_- \right) = -\frac{1}{2} \left[ \frac{1}{2} \sin(x_+ - x_-) \left( \frac{\cos(x_+)}{\sin(x_-)} \right) + 1 \right] B_- \]  \hspace{1cm} (A.9b)

where \( x_\pm = \frac{1}{2}(t \pm \rho) \). Eq. (A.9) serves to fix the dependence of \( B_- \) on \( x_+ \) and \( x_- \); from eq. (A.6) we then find \( B_+ \), \( A_+ \) and \( A_- \). A linear relationship between \( \chi \) and \( \lambda \) has been established.