Bifurcations to turbulence in transitional channel flow

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Abstract

In wall-bounded parallel flows, sustained turbulence can occur even while laminar flow is still stable. Channel flow is one of such flows and displays spatio-temporal fluctuating patterns of localized turbulence along its way from/to featureless turbulence. By direct numerical simulation, we study the observed inconsistency between turbulence decay according to a two-dimensional directed-percolation (2D-DP) scenario and the presence of sustained oblique localized turbulent bands (LTBs) below the DP critical point. Above Reynolds number $\text{Re}_g \sim 700$ sustained LTBs are observed; most LTBs have the same orientation so that the spanwise symmetry of the LTB pattern is broken below $\text{Re}_s \sim 1000$. The frequency of transversal splitting, by which an LTB generates another one with opposite obliqueness, so that turbulence spreading becomes intrinsically two dimensional, increases in the range $\text{Re}_g < \text{Re} < \text{Re}_2$. It reaches a critical rate at $\text{Re}_2$ beyond which symmetry is restored. 2D-DP behavior is retrieved only above $\text{Re}_2$. A mean-field model is proposed which qualitatively accounts for the above symmetry-restoring bifurcation by considering interactions between space-averaged densities of LTBs propagating in either direction.

I. INTRODUCTION

The ways shear flows become turbulent markedly depend on the shape of their laminar velocity profiles and the occurrence of instability mechanisms of Kelvin–Helmholtz type affecting them. Inflectional profiles like in free shear layer, wakes, or jets, become unstable at rather low values of the Reynolds number against such modes of inertial origin and then turn turbulent through a continuous cascade of successive instabilities [1], each step being in principle captured by weakly nonlinear perturbation theory, resulting in a globally supercritical transition to turbulence. By contrast, unidirectional flows controlled by the shear at solid walls do not display inflection points and are not prone to such instability modes. They may become unstable but only against more intricate and counter-intuitive mechanisms involving viscous dissipation [1, 2]. The so-produced Tollmien–Schlichting waves develop only at high Reynolds number, beyond a certain linear threshold $\text{Re}_c$ (later denoted $\text{Re}_{\text{TS}}$ for convenience). Pipe flow in tubes of circular and square sections or plane Couette

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flow between flat counter-translating plates remain linearly stable for all Re, Reₐ → ∞, whereas for channel flow (plane Poiseuille flow) Reₐ < ∞ [3]. For such flows the transition is *globally subcritical*. Physically infinitesimal perturbations are however not mathematically infinitesimal but only finite-but-small, and the possibility of a direct transition to turbulence, called *bypass*, is in practice observed at moderate values of Re due to the nonlinearity of the Navier–Stokes equations: At increasing Re the flow can in principle be maintained steady and laminar up to Reₐ ≤ ∞, but the introduction of finite perturbations with particular shapes and amplitudes may force it to become unsteady. There is therefore a full range of Reynolds numbers, called *transitional*, where the trivial laminar solution coexists with other nontrivial solutions. These solutions are usually chaotic and cannot be straightforwardly reached by standard perturbation analysis. A direct transition to turbulence is observed with strong hysteresis upon continuous up-and-down variation of Re, the flow jumping from the laminar solution branch to the nontrivial turbulent branch or back. Once in a state belonging to the nontrivial branch, the flow can be maintained turbulent down to a global stability threshold Reₔ below which the only possible stable regime is laminar.

Sub-criticality is often perceived as a problem to be studied within the theory of dynamical systems. Solutions to the Navier–Stokes equations are then searched under minimal flow unit (MFU) conditions [4] with periodic boundary conditions at short distances, focusing on coherent structures analyzed in the corresponding *phase space* perspective [5, 6]. This approach was progressively generalized to deal with extended systems in one or two in-plane directions, and mostly concentrated on the existence and properties of isolated localized solutions at the edge of chaos, see e.g. [7] for channel flow. On the other hand, when appreciated in *physical space*, provided that the system’s geometry is wide enough, states along the nontrivial branch in general display a coexistence of domains alternately laminar and turbulent separated by fluctuating interfaces.

In a number of cases, beyond Reₔ this laminar-turbulent coexistence happens in the form of a regular pattern. The iconic case is that of the turbulent spiral obtained in moderate-gap cylindrical Couette flow by Coles [8], and re-obtained later by Andereck *et al.* [9] who introduced the term *featureless* to qualify the regime obtained at higher shearing rates when turbulence has apparently returned to statistical uniformity. In small-gap experiments, several helical branches were obtained [10]. Close to the featureless regime, laminar-turbulent patterning involved superpositions of such helices with opposite pitch while, plane Couette
flow being understood as the zero-curvature limit of cylindrical Couette flow, helices were straightened into oblique bands found in a whole range of Re down to \( \text{Re}_t \) [10]. As Re increased, the amplitude of the turbulence modulation associated with these bands/helices was seen to decrease, and a threshold usually denoted as \( \text{Re}_t \) could be determined above which turbulence was featureless. The transition at \( \text{Re}_t \) seemed continuous with a major role played by the intense noise arising from the chaotic dynamics in the turbulent regime [10, 11]. Regular laminar-turbulent patterning with similar characteristics has been observed in a few other cases including stratified Ekman boundary layer [12], Couette–Poiseuille flow [13], or stratified plane Couette flow [14]. Experimental evidence for laminar-turbulent oblique patterning in channel flow was provided by Tsukahara’s group [15] as a follow-up of their simulations in wide domains [16]. Another numerical approach using the oblique-elongated-but-narrow domain assumption [17], was developed in [18] to yield similar patterns at a much more limited numerical cost.

The location of the lower threshold \( \text{Re}_t \) is another problem that can be studied either by increasing Re from a germ or by decreasing Re from a pattern. In the first case the study mostly relies on the capability to produce localized states and study their persistence. The first solid results were obtained in pipe flow (consult [19] for a general perspective and references back to Reynolds’ seminal work) for which the relevant localized states are puff states and the threshold \( \text{Re}_t \) identified the value of Re when the probability of turbulence extinction due to their decay is overcome by the probability of their proliferation due to repeated splittings [20]. In planar flows, such as in planar boundary layers, plane Couette flow [21], Couette–Poiseuille flow [13], or channel flow [22], equivalent localized solutions appear as turbulent spots (see [23] for a recent study). The specific effect of spanwise extension on localization has been studied in small aspect-ratio rectangular duct flow [24] and in annular Poiseuille flow [25]. In all cases the location of \( \text{Re}_t \) is derived from statistical studies, with the difficulty that a wide range of initial conditions needs to be considered by varying their shapes, structures and strengths. A value \( \text{Re}_t \sim 1000 \) was mentioned in [22] for channel flow, and was also reported in other systems provided that the Reynolds number is based on an appropriate physically-based estimate of the shear [26, Appendix]. The second approach, i.e., by decreasing Re, is also statistical but rather appeals to an early conjecture by Pomeau [27] who put forward the analogy between turbulence onset/decay in extended systems and directed percolation (DP). This conjecture was based on the recognition that DP
is a spatiotemporally intermittent process involving active (here turbulent) and absorbing (here laminar) states defined locally in physical space, and the probability of contamination of the latter by the former is the key factor. DP is the representative of a class of non-equilibrium phase transition with specific universal properties depending on the effective dimension $D$ of the physical space in which the process develops [28]. Conceptually, the corresponding critical point $Re_{DP}$ should have to be identified with $Re_g$ and the long turbulent transients observed for $Re < Re_g$ as the decay of a percolation cluster below threshold.

Flows evolve physically in three-dimensional domains but confinement in the wall-normal direction reduces the dimension to an effective value $D = 1$ in tubes and flow configurations similarly constrained by lateral boundary conditions, or $D = 2$ when the lateral boundaries are far enough when compared to the typical wall-normal distance. This is due to the fact that the transitional range takes place at moderate Re so that viscosity is able to impose a strong coherence except along the unbounded directions, one in tubes or two along plates. To our knowledge, 1D-DP universality has not been checked directly in tubes up to now, but a numerical model suggests it might apply [29]. On the other hand, universality has been shown to hold quantitatively in a numerical model of plane shear flow defined in a narrow oblique domain [30, 31] (a geometry introduced in [17]) and in a cylindrical Couette configuration with a small axial aspect ratio ensuring quasi-1D confinement [31]. Similar findings have also been obtained in a model of two-dimensional shear flow without walls called Kolmogorov flow in an elongated domain rendering the dynamics effectively 1D [32]. As to 2D-DP universality, the corresponding critical behavior has been obtained with great precision in a numerical experiment on a low order Galerkin approximation to Waleffe flow, which is a three-dimensional shear flow mimicking plane Couette flow but with stress-free boundary conditions [33]. Preliminary results on plane Couette flow proper, but in an under-resolved numerical context [34], suggest a similar conclusion. At last, agreement with 2D-DP universality has been obtained by Sano & Tamai [35] in a laboratory experiment on channel flow through a duct with a rectangular section and large spanwise aspect ratio. This finding has however to be re-interpreted in view of the discovery of sustained nontrivial solutions below $Re_{DP}$ that could therefore not be the expected global stability threshold.

These solutions were first obtained in the form of a localized turbulent band (LTB) by triggering [36] and later upon looking for the global stability threshold by slowly decreasing Re and following the mutations in the laminar-turbulent patterning [37, 38], by contrast
with experiments in \([35]\) where the decay from homogeneous turbulence was studied in a channel of limited length. They were later studied in more detail in \([39]\) where sustainment mechanisms were scrutinized and in \([40]\) where their decay was considered in relation with the size of the numerical domain. Localized solutions with an analogous structure were also obtained in laboratory experiments \([41, 42]\).

Below we present our simulation results for channel flow in wide domains. An overview of the observed flow regimes is given in \(\S\) I\(I\) from LTBs around \(\text{Re}_g\) to the oblique pattern regime described in \([15, 16]\) and to featureless turbulence above \(\text{Re}_t\). We next show in two steps how the contradiction between the observation of a 2D-DP scenario as reported in \([35]\) can be reconciled with that of LTBs at lower Re. In \(\S\) I\(I\) we enter the details of how LTBs grow and split as Re increases, and how spanwise symmetry is restored from the low-Re regime where LTBs move essentially in one direction \([36, 42]\). A simple phenomenological model is developed to show that this bifurcation is controlled by the increasing rate of transversal splitting, thereby supporting the existence of a well-defined threshold. Above that threshold the invasion of turbulence is genuinely 2D in the plane of the flow, in the form of a spatiotemporally intermittent network, where a DP-like behavior becomes relevant. The quantitative analysis of the turbulent fraction in \(\S\) IV will give evidence of a growth compatible with 2D-DP universality. Our findings are recapitulated in \(\S\) V where we discuss in particular how processes specific to particular flow configurations may come and breach the appealing concept of universality expected on general grounds. Details on our simulations and the filtering-thresholding methodology used to measure the turbulent fraction are presented in Appendices A and B respectively.

II. OVERVIEW OF THE TRANSITIONAL RANGE

Channel flow here is driven by a constant body force \(f\). The Reynolds number is defined as \(\text{Re} = fh^3/2\nu^2\), where \(h\) is the half-distance between two parallel walls and \(\nu\) the kinematic viscosity. All quantities below are written in units of the center-plane velocity of the corresponding laminar flow \(U = fh^2/2\nu\) and \(h\). For comparison with previous works, we also define \(\text{Re}_b = \frac{3}{2}\langle U_m \rangle \text{Re}\), where \(\langle U_m \rangle\) is the time-average of the dimensionless bulk velocity \(U_m\) (‘b’ stands for ‘bulk’ \([18]\)). Most of the time, the computational domain size is \(500 \times 250\) (streamwise \(\times\) spanwise) but we also consider \(250 \times 125\) and \(1000 \times 500\) to check for size
effects. Simulations are performed for durations sufficient to obtain statistically significant results, typically up to $1.5 \times 10^5$ time units. See Appendix A for a detailed description of the flow system and the numerical procedures.

Snapshots of flow patterns for typical Reynolds numbers are displayed in Fig. 1. For $Re = 850$ and 1050, Fig. 1(a,b), several localized turbulent bands (LTBs) are observed, propagating at an angle about $45^\circ$ with the streamwise direction, each driven by a downstream active head (DAH) [36–40, 44] located at its downstream extremity. DAHs entraining LTBs drift at a speed about 0.8 in the streamwise direction and about 0.1 in the spanwise direction. In agreement with previous studies [39, 40], these LTBs decay below $Re_c \approx 700$. At $Re = 850$ (a), all LTBs go in the same direction, therefore breaking the symmetry with respect to the spanwise direction. By contrast, LTBs go in both directions at $Re = 1050$ (b). These states are respectively called one-sided and two-sided. As Re increases, LTBs joint to form a loose continuous network of oblique bands and for $Re = 1200$ (c) DAHs practically cease to be seen. The pattern is strongly intermittent with turbulence intensity far from being

![Fig. 1](image)

**FIG. 1.** (a) $Re = 850$ ($Re_b \approx 789$): one-sided LTB regime. (b) $Re = 1050$ ($Re_b \approx 947$): two-sided LTB regime. (c) $Re = 1200$ ($Re_b \approx 1012$): strongly intermittent loose continuous network of LTBs. (d) $Re = 1800$ ($Re_b \approx 1237$): weakly intermittent loose banded pattern. (e) $Re = 3000$ ($Re_b \approx 1604$): tight banded pattern. (f) $Re = 4000$ ($Re_b \approx 1843$): nearly featureless state. Flow direction is from the left to right. The wall-normal velocity field on the center plane ($y = 0$) is displayed. Domain size is $500 \times 250$. See Supplemental Videos [43].
FIG. 2. Bifurcation diagram for plane channel flow as obtained in our simulations. (Corresponding values of Re_b between parentheses.) Laminar flow is always recovered in the long-time limit for Re < Re_g ≈ 700 (Re ≡ Re_b up to Re_g). Event A corresponds to the onset of transversal splitting at Re ∼ 800 (Re_b ∼ 795). The extrapolated 2D-DP threshold is found at Re_{DP} ≃ 984 (Re_b ∼ 905). The transition from one-sided to two-sided propagation of LTBs takes place at Re_2 ≃ 1011 (Re_b ≃ 924). Event B marks the opening of laminar gaps in the turbulent branches, sufficiently long-lived to allow the development of DAHs. It takes place at Re ≃ 1200 (Re_b ≃ 1012). Finally, “featureless” turbulence is present for Re > Re_t ≃ 3900 (Re_b ≃ 1820). The Tollmien–Schlichting instability threshold is at Re_{TS} = 5772 (analysis refers to laminar base flow, hence identical Re_b).

uniform along the bands. At larger values of Re, the network narrows, Re = 1800 (d), and wide laminar voids disappear while regular patterns form, which can be understood as crisscrossed more acute (∼ 25°) oblique turbulence modulations, Re = 3000 (e), similar to those obtained in circular Couette flow [10]. The amplitude of this modulation then decreases and the featureless regime eventually prevails for Re ≥ 4000 (f). Properties of the flow deeper inside the uniformly turbulent, developed regime achieved at even larger values of Re are briefly reviewed in [45]. Figure 2 is a sketch of the bifurcation diagram of channel flow indicating the different regimes illustrated above and anticipating the output of the quantitative study of phenomena developing for 800 ≤ Re ≤ 1200 to be presented below.

Information from the statistics over the time-series of typical global quantities is displayed as functions of Re or Re_b in Fig. 3 and 4. Transverse turbulent energies, \( E_y(t) = V^{-1} \int_V u_y^2 dV \) and \( E_z(t) = V^{-1} \int_V u_z^2 dV \), in Fig. 3(a) directly monitor the distance to the laminar base flow. Irregularities noted for Re < 1200 can be interpreted with the help of Fig. 1(a–c). The rapid growth of \( E_y \) and \( E_z \) for Re ≤ 850 is related to the increasing number of DAHs in the one-sided LTB regime. When Re > 850, the increasing fraction of LTBs with different orientations leads to a strong decrease in \( E_y \) and \( E_z \) until Re ≈ 1000. Next, as Re increases, \( E_y \) grows again owing to an increasing turbulent fraction while \( E_z \) slightly
FIG. 3. Means (a) and relative standard deviations (b) of the wall-normal energy $E_y(t)$ and spanwise energy $E_z(t)$.

increases up to $Re \approx 1200$ before decreasing in the band-network regime where the global flow around LTBs is inhibited. The variations of the standard deviations of fluctuations, once normalized by their respective means, are remarkably correlated, as seen in Fig. 3(b). Their rapid growth as $Re$ approaches $Re \approx 1000$ from above is reminiscent of the divergence of fluctuations observed for a phase transition at a threshold $Re_2$ precisely located later. Peaks at $Re = 725$ and $Re = 800$ mark the onset of longitudinal and transversal splittings to be examined below.

Further general observations about the transitional range as a whole can be extracted from our numerical results. Using a constant body force (mean applied pressure gradient), our numerical implementation of Navier–Stokes equations produces streamwise and spanwise net flux components $U_m$ and $W_m$ as time-fluctuating observables governed by (A5). In particular, the bulk Reynolds number $Re_b := \frac{3}{2} \langle U_m \rangle Re$ introduced earlier is obtained from the time average of $U_m$. Figure 4(a) displays $\langle U_m \rangle$ as a function of $Re$. Laminar flow corresponds to $U_{m}^{\text{lam}} = 2/3$. As soon as some turbulence is present we get $\langle U_m \rangle < U_{m}^{\text{lam}}$ and, for $Re$ above the one-sided regime, the observed decrease nicely fits an inverse square root over a large part of the transitional range. Using results for $Re \geq 1050$ we obtain $U_{m}^{\text{fit}} = w/\sqrt{Re}$ with $w \approx 19.5$, hence $Re_b \approx \frac{3}{2} w \sqrt{Re}$. Turning to wall units, the friction velocity $U_\tau$ and the friction Reynolds numbers $Re_\tau$ are obtained in our formulation as $U_\tau^2 = \tau_w = 2/Re$ and $Re_\tau = \sqrt{2Re}$, relation (A11). This means that $Re_\tau$ and $Re_b$ are roughly proportional.
as long as the flow remains textured, before entering the developed regime where Re_τ then 
grows as a function of Re_b at a slightly smaller rate (exponent \( \approx 0.88 \)) \[43\). This behavior
is directly reflected in the variation of the skin friction coefficient C_f with Re_b shown in
Fig. 4(b). Here we have C_f := \( \tau_w/\frac{1}{2} \langle U_m \rangle^2 = 4/(\text{Re} \langle U_m \rangle^2) \) (see §A). As soon as some
turbulence is present, C_f deviates from its laminar expression 9/Re_b, remains close to it
in the one-sided regime, next change to a near plateau dependence as soon as it enters in
the two-sided regime, as expected from the variation of \( \langle U_m \rangle^2 \) as Re^{-1} in the corresponding
Re-range. In our simulations, this plateau extends up to the transition to fully developed
turbulent channel flow marked by a dotted line in Fig. 4(b), theoretically expressed as
Re_b = \( \frac{3}{\sqrt{2}C_f} \exp\left(0.41(\sqrt{2/C_f} - 2.4)\right) \) \[45, 46\]. This plateau is better defined than in earlier
experiments \[46, 47\] and simulations \[16, 37\], also shown in the picture. It indicates that,
above Re_2, the transition to turbulence develops at a nearly constant dissipation rate \( \langle \epsilon_m \rangle \),
as stems from relation (A12). In the upper transitional range Re \( \gtrsim 4000 \), i.e. Re_b \( \gtrsim 1850 \),
the trend suggested by our data points seems to overestimate C_f as predicted for the fully
turbulent regime \[45\] (dotted line). This is presumably due to a lack of resolution close
to the wall at such high values of the Reynolds number. A detailed quantitative study of
the transition to the “featureless” regime around Re_t, beyond the preliminary result in \[48\]
indicated in Figure 2 is left to future work.

III. SYMMETRY-RESTORING BIFURCATION

Laminar–turbulent patterns below Re = 1200 were examined to better understand the
symmetry-restoring bifurcation observed at increasing Re. Processes involved in the dynamics
are illustrated in Fig. 5. The local spread and decay of turbulence respectively stem from
splittings and collisions of LTBs with either identical or opposite orientations. Figure 5(a)
does the nucleation of a new band by longitudinal splitting of an LTB at its tail. The active
region is downstream and the splitting takes place upstream where the turbulence level is
always weaker than near the DAH, contrary to what happens for puffs in pipe flow \[20, 49\].
When two parallel LTBs collide (longitudinal collision), following the large scale flow around
them \[37, 38, 40\], the upstream faster LTB catches the downstream LTB that disappears as
in Fig. 5(b). At larger Re, another splitting process here called transversal can take place
along an LTB: A turbulent ‘bud’ appears on the side of an LTB and forms an off-aligned
FIG. 4. (a) Average of streamwise bulk velocity \( \langle U_m \rangle \) as a function of Re; the solid line corresponds to the fit and the dashed line to laminar flow, \( U_{lam}^m = 2/3 \). Open symbols in the inset are for the different domain sizes. (b) Skin friction coefficient \( C_f \) as a function Re\(_b\). Filled and open symbols are from DNSs (Tsukahara et al. 2005 \[16\], Xiong et al. 2015 \[37\]) and experiments (Patel and Head 1969 \[47\], Dean 1979 \[46\]), respectively. The dashed line is for laminar flow and the dotted line for the fully turbulent regime.

This one-sided/two-sided spanwise-symmetry restoring bifurcation can be understood using a simple prey-predator model for the densities \( X_\pm \) of two species of LTBs, left-propagating and right-propagating:

\[
\frac{dX_+}{dt} = aX_+ - bX_+^2 + cX_- - dX_+X_-, \tag{1}
\]

\[
\frac{dX_-}{dt} = aX_- - bX_-^2 + cX_+ - dX_-X_+. \tag{2}
\]

By construction, these equations incorporate the built-in spanwise symmetry of the system and each term corresponds to a process in Fig. 5. Coefficient \( a \) represents the longitudinal splitting rate (Fig. 5(a)). The transversal splitting rate \( c \) (Fig. 5(c)), the natural control...
parameter, is assumed to increase with Re according to the observations \( c > 0 \) for creation of \( X_\pm \) out of \( X_\mp \neq 0 \), hence \( c \geq 0 \) for \( Re \gtrsim 800 \), event A in Fig. 2. Coefficients \( b \) and \( d \) account for the turbulence-level decrease by the collision between LTBs of either the same (Fig. 5(b)) or different (Fig. 5(d)) orientations. For collisions between differently oriented LTBs, the term \(-dX_+X_-\) models the decay rate of one of the species \( X_\pm \) taken as proportional to the cross-section of LTBs of the opposite kind \( X_\mp \). Coefficient \( d \), which is weakly dependent on Re and a function of the speed of colliding LTBs, is assumed constant, as well as \( b \) parameterizing a logistic self-interaction for predation among LTBs with the same orientation.
orientation, i.e. \(-bX^2_\pm\). A reduced cross-section and a very small relative velocity between LTBs of the same kind suggest \(b \ll d\). By contrast with works elaborating on reaction-diffusion models devised to account for local interactions in transitional flows \[51, 52\], our approach is Landau-like and deals with global observables minimally coupled by purely phenomenological coefficients.

The analysis of the model is straightforward when considering the total amount of turbulence \(S = X_+ + X_−\) and the degree of asymmetry \(A = X_+ - X_−\) as working variables. The equations for these variables become:

\[
\frac{dS}{dt} = (a + c)S - \frac{1}{2}(b + d)S^2 - \frac{1}{2}(b - d)A^2, \quad (3)
\]

\[
\frac{dA}{dt} = (a - c)A - bSA. \quad (4)
\]

The two-sided regime labeled “**” corresponds to \(A = 0\), while \(A \neq 0\) implies the dominance of one propagation direction. “\(A_{**} = 0\)” solves (4) in all circumstances. Using (3), the symmetrical fixed point is then given by:

\[
S_{**} = \frac{2(a + c)}{b + d}, \quad A_{**} = 0. \quad (5)
\]

This fixed point has eigenvalues \((s_S, s_A)\) with \(s_S = -(a + c) < 0\) and \(s_A = [a(d - b) - c(d + 3b)]/(b + d)\). The symmetric solution is then stable as long as \(s_A < 0\); hence, \(c > c_c = a(d - b)/(d + 3b)\) when \(b < d\) as assumed from the observations. The two-sided regime is then stable for large \(\text{Re}\) and becomes unstable below a threshold corresponding to \(c_c\).

The one-sided regime labeled “*” corresponds to \(A \neq 0\) at steady state (fixed point); hence, from (4) and next from (3):

\[
S_* = \frac{a - c}{b}, \quad A_*^2 = \frac{(a - c)(d + 3b)(c_c - c)}{b^2(d - b)}. \quad (6)
\]

This analysis shows that the system experiences a standard super-critical pitchfork bifurcation toward asymmetry by decreasing \(c\). At \(c = c_c\) as previously defined, \((S_{**, A_{**}})\) becomes unstable and is replaced by \((S_*, \pm A_*)\), which is stable for \(c < c_c\) as expected. Figure 6(a) displays the bifurcation diagram of model (3,4) with splitting rate \(c\) taken as the control parameter, and \(a = 10^{-4}, d = 10, b = d/10\). The thick line represents the total turbulence amount \(S\) given by \(S_* = (a - c)/b\) below \(c_c\) and \(S_{**} = 2(a + c)/(b + d)\) above. Thin lines correspond to \(X_\pm\). Dashed lines correspond to the unstable solutions (the dashed branch
for \( c > a \) is furthermore irrelevant to the present problem since it leads to negative values of \( X_- \).

Experimental support to the model is displayed in the boxed region in Fig. 6(b). The total transverse perturbation energy \( E_y + E_z = E_{2D} \) is taken as a proxy for \( S \). The mean spanwise velocity component \( W_m = \mathcal{V}^{-1} \int \mathcal{V} u_z d\mathcal{V} \) is interpreted as an instantaneous measure of the degree of asymmetry \( A \) because it cancels out statistically when symmetry is restored at high \( \text{Re} \), i.e. \( \langle W_m \rangle = 0 \), where \( \langle \cdot \rangle \) represents time average. Notice that the standard deviation of \( E_{2D} \) is multiplied by 6 and both \( \langle W_m \rangle \) and its standard deviation are divided by 4 so that variations of the observables can be more easily compared. As long as transversal splitting is negligible, \( \langle E_{2D} \rangle \) and \( \langle W_m \rangle \) increase with the mean length of LTBs. The two orientations correspond to \( \langle W_m \rangle \) with opposite signs. Periodic spanwise boundary conditions leave open the possibility of having \( W_m \neq 0 \) as a result of symmetry breaking. The case of an experimental system with solid lateral boundaries forbidding such a net transverse flux is considered later in \( \S \) V. In Figure 6(b) we only display \(|\langle W_m \rangle|\) since at steady state, depending on the orientation of the LTBs, \( \pm |\langle W_m \rangle| \) are equally possible. Both \( \langle E_{2D} \rangle \) and \(|\langle W_m \rangle|\) reach their maximum values for \( \text{Re} \approx 850 \), which can be understood as when transversal splitting –not observed below \( \text{Re} = 800 \)– becomes significant. For \( \text{Re} > 850 \), our two observables follow the trend suggested by the model: \( \langle E_{2D} \rangle \) decreases roughly linearly as \( \text{Re} \) increases up to \( \text{Re} \approx 1000 \) and slowly grows beyond, in agreement with equations \( (5, 6) \). Likewise, \(|\langle W_m \rangle|\) decreases rapidly to zero, similar to \( A^* \), as indicated by the fact that the standard deviation becomes larger than the mean, at \( \text{Re} \approx 1000 \). For larger \( \text{Re} \) outside the box, the system enters a developed two-sided regime where the model, designed to account for the one-sided/two-sided bifurcation, becomes insufficient. Inside the box, this oversimplified formulation well captures the phenomenology of the transition at a qualitative level.

The transition is then understood by assuming that parameter \( c \) measuring the transversal splitting rate increases with \( \text{Re} \). The analysis above shows that \( c_c \approx a \) assuming that \( b \ll d \), as stems from our observations. Furthermore, in the range \( 850 \leq \text{Re} \leq 1200 \), the variations of \( S \) and \( A \) around \( c_c \) are consistent with those of the turbulent energy and the spanwise mean velocity, respectively. In fact, the deviation of \( \langle U_m \rangle \) away from its behavior in the two-sided regime fitted as \( U_m^{\text{fit}} = \frac{w}{\sqrt{\text{Re}}} \) in Fig. 4(a) is a proxy for the change of total amount of turbulence at the one-sided/two-sided bifurcation. As displayed in Fig. 7, this variation is linear close to the bifurcation point \( \text{Re}_2 \approx 1011 \), in agreement with the predictions of the
FIG. 6. Transition from one-sided to two-sided flow. (a) Model. (b) Numerical simulation.

FIG. 7. Departure of observed streamwise mean flow $\langle U_m \rangle$ from the value predicted by the law $U_m^{\text{fit}} = 19.47/\sqrt{Re}$, solid line in Fig. 4(a). Fitting the data against a straight line for $850 \leq Re \leq 1000$ yields $Re_2 \simeq 1011$. Triangles are for the larger domain $1000 \times 2 \times 500$. As can be seen in the inset of Fig. 4(a) and in Fig. 7 the current domain size is appropriate to obtain this threshold.

IV. BEHAVIOR ABOVE $Re_2$

Beyond $Re_2$ and up to entrance in the tight-banded pattern regime, Fig. 1(e), channel flow exhibits a spatiotemporal intermittent behavior, Fig. 1(c,d), strongly reminiscent of DP
above the threshold. The work of Sano and Tamai [35] focusing on the critical properties of turbulence decay in a DP context [28] naturally suggests one to study this Reynolds number range in terms of turbulent fraction $F_t$. This in turn relies on the identification of appropriate laminar and turbulent local states, based on observables that vary sufficiently sharply in space to define the respective domains properly, allowing a precise measurement of their relative occupancy fractions. Figure 8 displays four possible candidates, the three velocity components in the mid-gap plane $y = 0$, and the mean transverse perturbation energy $E_{2D}(x, z; t) = \frac{1}{2} \int_{-1}^{1} (u_y^2 + u_z^2) \, dy$ around an LTB at $Re = 700$. In-plane components $u_x$ and $u_z$ display a large-scale, slowly decaying, structure around the LTB. By contrast, $u_y$ sharply discriminates non-laminar flow regions. $E_{2D}$ is slightly less close-fitting due to the limited contribution of the component $u_z$. Accordingly, the absolute value of the wall-normal velocity on the mid-plane $|u_y(x, 0, z)|$ will be used to evaluate $F_t$. However, inside LTBs, $u_y$ displays small-scale oscillations associated with the presence of streamwise vortices, which produces narrow regions with $|u_y(x, 0, z)| \approx 0$ that must not be counted as laminar. Accordingly, the field $|u_y(x, 0, z)|$ has to be smoothed beforehand, here by simple
box-averaging over cells of size $w \times w$, and next thresholded using the “moment-preserving” procedure [53], as explained in Appendix B.

The variation of $F_t$ as a function of Re is displayed in Fig. 9(a) for $700 \leq Re \leq 3000$ and for $w$ varying between 0 (no filtering) and 40$\delta$. For all values of $w$, similar variations are observed and one readily identifies the different stages illustrated in Fig. 1: one-sided growth for $Re \leq Re_2 \approx 1011$, with a clear change of regime for $Re \sim 850$ as transversal splitting sets in, before a rapid increase akin to a power-law growth as Re increases in the symmetry-restored regimes (Fig. 1(b)-(d)) up to the tight banded pattern regime (Fig. 1(e)).

The consistent square-root-like behavior is reminiscent of the growth of the order parameter of DP beyond threshold. In line with the idea that the critical properties of the stochastic DP process are relevant for channel flow, as put forward by Sano & Tamai, we first test the plausibility of exponent $\beta_{DP} \approx 0.583$ to describe the variation of the turbulent fraction with the relative distance to threshold. Figure 9(b) therefore displays $F_t^{1/\beta_{DP}}$ as a function of 1/Re. The linear behavior of the plots for different values of $w$, systematically extrapolating to zero for around $Re \approx 1000$ ($1000/Re \approx 1$) therefore strongly supports the expected behavior of $F_t$ as if it were produced by a 2D-DP process. From the least-square fits, the straight lines in Fig. 9(b), it is seen that some filtering is needed to eliminate spurious small scale oscillations of $u_y(x, 0, z)$ since raw data ($w = 0$, red data points) does not behave so satisfactorily. On the other hand, strong filtering ($w \geq 20$) leads to a reduction
of the range where a good linear fit is obtained \((1050 \leq \text{Re} \leq 2400 \text{ for } w \leq 12\delta, \text{ and} \ 1050 \leq \text{Re} \leq 2000 \text{ for } w = 20\delta \text{ and } 40\delta)\). To further develop the quantitative analysis we choose the best compromise which seems to be \(w = 12\delta\), i.e. the largest filtering possible in the widest Re interval with the expected property.

In these conditions, a direct fitting over the interval \(1050 \leq \text{Re} \leq 2400\) has been attempted against the function \(F_t = B \left(1 - \frac{\text{Re}_{DP}}{\text{Re}}\right)^\beta\) with \(B\), \(\text{Re}_{DP}\) and \(\beta\) as fitting parameters. We proceeded to a least-square minimization of the error \(\text{Err}^2 = \frac{1}{N} \sum_{n=1}^{N} \left(F_{tn} - B \left(1 - \frac{\text{Re}_{DP}}{\text{Re}_n}\right)^\beta\right)^2\) where \(N\) is the number of values \(\text{Re}_n\) of Re entering the fit, and \(F_{tn}\) the corresponding measured mean turbulent fraction. Figure 10(a) displays the minimum error as a function of \(\beta\), pointing to \(\beta \approx 0.58\), while panels (b) and (c) display similar results for \(B \approx 0.45\) and \(\text{Re}_{DP} \approx 984\) as best fitting values, respectively. Following this approach, the estimate for \(\beta\) turns out to be close to the theoretical value \(\beta_{DP} = 0.583\) that served as initial guess in Fig. 9(b). The error curves in each case however indicate that the optima are not sharply defined. Unfortunately, these estimates cannot be improved mainly because the critical point cannot be approached sufficiently closely. This also justifies that we did not try to adjust the other critical parameters related to space-time correlations since they are discriminating solely in an arbitrarily close neighborhood of \(\text{Re}_{DP}\).

Our results, used to draw the fitting line in Fig. 10(d), therefore point toward a plausible universal DP behavior for large enough turbulent fraction, a behavior inevitably truncated before criticality by a crossover to a different decay regime as \(\text{Re}\) decreases. As a matter of fact, the turbulent fraction becomes so small that laminar gaps open along oblique arms of the fluctuating laminar-turbulent network, which produces the development of DAHs –original to channel flow– so that the DP regime is superseded by the LTB-dominated regime and, as \(\text{Re}\) decreases even more, to the symmetry-breaking bifurcation below which turbulence decay takes up a novel turn.

V. DISCUSSION AND CONCLUDING REMARKS

Apart from a few illustrations of the laminar-turbulent patterning in the upper transitional range of channel flow in Fig. 1(e,f), our results all relate to its lower part featuring the decay to laminar flow where a DP scenario with universal properties expected for two-dimensional systems has been put forward [35], while sustained LTBs have been observed
FIG. 10. (a–c) Minimum of Err as a function of the different fit parameters for $F_t$ with $w = 12 \delta$ and control parameter $1/\text{Re}$. (a) Exponent $\beta$. (b) Amplitude $B$. (c) Threshold $\text{Re}_{DP}$. In these panels, the error is displayed as a function of one fitting parameter, the two other being supposed fixed at their optimal value. (d) Turbulent fraction $F_t$ as a function of $\text{Re}$ and its fit against the theoretical expression given in §IV.

at values of $\text{Re}$ lower than the measured DP critical point \[36\ 42\]. The first result comes in support of a theoretical conjecture by Pomeau \[27\] based on the recognition that the flow can be locally in one of the two possible states, turbulent or laminar. Furthermore, in the transitional range, developing turbulence presents itself as a contamination of the laminar state, locally linearly stable and therefore absorbing, by a highly fluctuating, unstable, chaotic active state. This framework is directly derived from a statistical-physics approach where the key concept is the thermodynamic limit of infinitely wide systems at statistically steady state (here non-equilibrium steady state). Though experiments and simulations are
performed in domains and for durations somewhat far from the thermodynamic-limit conditions (with \[33\] as a possible exception), the question of what we may infer on general grounds from our results is relevant.

Let us first notice that comparisons with laboratory experiments and computer simulations are made difficult owing to the several possibilities chosen to drive the flow, either generated by a fixed pressure gradient (or given average friction at the walls) or a constant mean streamwise flow rate. Though it is generally admitted that the results should be statistically identical at the thermodynamic limit, quantitative correspondence between different works is not easy to establish, and first of all with respect to how Re is defined owing to what is controlled and what is measured. Here we fix the mean pressure gradient (fixed bulk force \( f \)) and use the computed centerline velocity of the corresponding laminar flow \( U \) as speed unit, while the half-distance \( h \) between the plates is taken as length unit and \( h/U \) as time unit. The definition of \( \text{Re} = Uh/\nu \) follows. If wall units are preferred, e.g. in \[16\], one gets \( \text{Re}_r = \sqrt{2\text{Re}} \) as derived in §A. In the laminar regime the mean flow rate is \( U_m = 2/3U \) and the Reynolds number made out of it: \( \text{Re}_m = U_m h/\nu = 2/3\text{Re} \) is frequently used. However, when the flow is not fully laminar, the measured mean flow rate \( \langle U_m \rangle \) decreases below its nominal value \( 2/3 \) (Fig. 7). Accordingly, the Reynolds number constructed using \( \langle U_m \rangle \) is not a control parameter but an observable that has to be determined by averaging \( u_x \) over space and time. This lead us to define \( \text{Re}_b = \frac{3}{2}\langle \text{Re}_m \rangle = \frac{3}{2}\langle U_m \rangle h/\nu \). With this definition, when the flow is fully laminar, \( \text{Re}_b \equiv \text{Re} \) and, accordingly, the linear instability threshold is \( \text{Re}_{TS} = 5772 \) \[3\]. Willing to keep \( \langle U_m \rangle \) strictly constant, as done in some numerical experiments and only approximately achieved in the laboratory, is therefore a different experiment in which the applied pressure gradient is unknown and fluctuating. All in all, our results in Fig. \( \f \) are consistent with those found in the literature when expressed in terms of \( \text{Re}_b \), especially those in \[16, 40\], or even in \[18, 54\].

Our main contention is that the previously mentioned conflicting experimental observations about DP universality and the existence of LTBs can be reconciled provided that it is recognized that the DP scenario cannot be followed down to its critical point. We have shown that, when the turbulent fraction is small enough, processes develop which are not present in other systems where DP-universality is observed: As Re is slowly decreased, the flow evolves from the weakly intermittent loose banded pattern regime, Fig. \( \f \)(d), to the strongly intermittent continuous network regime, Fig. \( \f \)(c). Further decreasing Re induces
the opening of laminar gaps along the turbulent branches, here called *splittings*, and the formation of LTBs terminated by DAHs constitutive of the two-sided LTB regime, Fig. 1(b). At first the intermittent expansion/recession of turbulence is two-dimensional in the plane of the flow owing to comparable rates for transversal and longitudinal splittings, but the rate of transversal splittings decreases with Re faster than that of longitudinal splittings, which implies a change to one-sided LTB propagation, i.e. a symmetry breaking due to a deficit of regeneration of one of the two LTB species at a local scale. In the long term, the rate difference indeed brings about a global dominance of one orientation over the other and a mostly one-dimensional expansion/recession of turbulence. In this respect the cases of Waleffe flow \[33\] and plane Couette flow \[34\] appear different. In these two cases, laminar gaps also open through splittings, longitudinal and transversal, along the turbulent segments forming the fluctuating laminar-turbulent network but the upstream-downstream distinction does not make sense. Close to \(\text{Re}_g\), the turbulent patches accordingly take the form of fluctuating short oblique straight segments, v-shaped, or x-shaped spots \[33, \text{Fig. } 2\] with dynamics profoundly different from that of LTBs equipped with DAHs. As a result the turbulent fraction can then decrease indefinitely according to 2D-DP universality, down to \(\text{Re}_{\text{DP}}\) that is reached before any symmetry-breaking bifurcation has a chance to take place, presumably because the longitudinal and transversal splitting rates remain comparable as Re decreases. By contrast, specificities of channel flow prevent the observation of a complete 2D-DP scenario and the origin of this *imperfection* is fully characterized as the result of an unavoidable symmetry-breaking bifurcation when the turbulent fraction becomes small enough, before the expected \(\text{Re}_{\text{DP}}\) is reached.

We now focus the discussion on two main points: the variation of the turbulent fraction in the range \(1050 \leq \text{Re} \leq 2400\) in connection with the DP scenario above the symmetry-breaking bifurcation, and the non-zero mean transverse flow in the symmetry broken regime, \(\text{Re} \leq \text{Re}_2\), its observability at the thermodynamic limit or, at the opposite, in laterally bounded finite-width channels.

For the conceptual reasons already evoked, DP is relevant *as a process* in the lowest transitional range. In its simplest formulation used to typify the universality class, a single contamination probability is needed to control the transition but the nature of physical mechanisms involved and their dependence on the Reynolds number are not known. The expectation for universality rests on the Janssen–Grassberger conjecture \[28\] stipulating in
particular that the absorbing state is unique, the interactions are local, and that there are no weird parasitic effects such as quenched disorder, which is implicitly assumed if Pomeau’s educated guess is valid. Strictly speaking, the question whether the decay of turbulence in channel flow follows 2D-DP universality is void since the critical regime – the immediate neighborhood of the threshold – cannot be entered, but results presented in §IV suggest a positive answer if we accept to loosen this restriction (Fig. 10).

This argument however hides a difficulty since, as already mentioned, the relation between the chosen external control parameter and the internal control parameter, the effective contamination probability generated by stochastic processes at the scale of a few MFUs, is not known. On general grounds, the critical behavior is not sensitive to the choice of the external control parameter in an asymptotically close vicinity of the critical point while departures from universality become conspicuous at variable distances from the threshold depending on that choice. In the previously reported cases, the variation of the turbulent fraction as a function of the Reynolds number deviates from the expected power law dependence beyond a relative distance $\epsilon$ to the critical point $\sim 10^{-2}$ for quasi-1D plane Couette flow [31, Fig. 2], or $\sim 10^{-3}$ for quasi-2D Waleffe flow [33, Fig. 4]. For channel flow, the universal behavior of the turbulent fraction is reported for $\epsilon \lesssim 0.1$ in [35, Fig. 3], but the experimental conditions do not permit access to the LTB regime. By contrast, taking advantage of a streamwise periodically continued domain, our approach leaves sufficient time for turbulent patches to rearrange into LTBs, which truncates the DP behavior above threshold [55]. Our most striking finding is that, characterizing the flow by its natural control parameter $Re$, the turbulent fraction then varies in accordance with universality over a surprisingly wide interval from below $Re \approx 2400$ nearly down to $Re_2$ where the dynamics becomes controlled by LTBs. When expressed in terms of $\epsilon = 1 - Re_{DP}/Re$ with $Re_{DP} = 984$ as the extrapolated threshold, this range extends over $0.07 \lesssim \epsilon \lesssim 0.6$ (Figs. 9(b) and 10(d)). At this point, it is fair to add that other possibilities for the external control, namely using $Re_\tau$ or $Re_b$ in principle equivalent in the neighborhood of the threshold, perform badly at some distance above threshold, not even permitting to detect a neat power-law variation over such a wide interval. The physical meaning of $Re_\tau$ or $Re_b$ implies an averaging over the flow in a strongly inhomogeneous spatiotemporally intermittent state, which might explain that the probabilities issued from the dynamics at the MFU scale are less straightforwardly parameterized using them than using $Re$ (or $1/Re$).
Let us now turn to the symmetry breaking observed at decreasing Re and studied in §III. Elementary events involved in the dynamics of LTBs were described in detail (Fig. 5). A satisfactory simple phenomenological model was developed then in the spirit of Landau theory to account for the bifurcation at a qualitative and semi-quantitative level. Extrapolating our results in a finite size domain with periodic boundary conditions to a laterally unbounded system (thermodynamic limit), when transversal splitting is negligible (Re ≲ 800) one-sided propagation is expected because for Re low enough, any transversal collision destroys one of the colliders (Fig. 5(d)) and, as seen in the movies at low Re [43], the transient evolution of sparse turbulence follows a majority rule. At short times, a patchwork of domains with one or the other dominant orientation forms, with transversal collisions concentrated along the domain boundaries. At longer times, a ripening process develop during which domains compete with each other, locally applying the majority rule so that, in the very long time limit at very low LTB density, one can figure out an asymptotic steady state which is mostly uniformly oriented. Residual LTBs are then moving nearly at the same speed, along the same direction, and interact only through longitudinal splittings/collisions (Fig. 5(a,b)) maintaining a state free of transversal collisions. Such a regime is expected to persist down to Re_2 below which it decays, much like in pipe flow [20], due to the predominance of longitudinal destructive collisions over expanding turbulence through splittings to be discussed in a future publication [56].

The difficulty with this picture is that, on general grounds, such a symmetry breaking is associated to a large scale flow with a transverse component corresponding to a net, fluctuating, spanwise mean flux with non-zero average \( \langle W_m \rangle \). In systems with periodic spanwise boundary conditions, symmetries do not forbid the existence of such a flow. Close to the bifurcation point, for Re < Re_2 when transversal splitting plays a significant role, the patchwork alluded to above can reach a statistically steady state with \( \langle W_m \rangle \neq 0 \) but reduced by compensations between patches of different orientations, whereas for Re > Re_2 one gets \( \langle W_m \rangle \approx 0 \) with substantial fluctuations, see Fig. 6(b). These are the characteristics of an order parameter apt to quantify the symmetry-breaking bifurcation. From the observations reported in the previous paragraph, these properties are expected to hold also as t goes to infinity in a spanwise unbounded system (thermodynamic limit).

In finite-width channels with impervious lateral walls, any net spanwise mean flow \( W_m \) is not permitted. In laboratory experiments where this condition applies, the one-sided
regime has been observed \[41, 42\] but, in contrast with our simulations where LTBs drift obliquely, after an initial transient stage during which the flow equilibrates, they are advected strictly along the streamwise direction. This means that an additional spanwise pressure wave accompanies the passing of an LTB, producing a spanwise mean flow component able to deviate it, thus compensating for the \( W_m \) that it would naturally generate. Rigid lateral walls are obviously able to withstand such pressure fluctuations as LTBs pass by. Only simulations specially designed to implement the corresponding no-slip lateral boundary conditions, e.g. with spanwise Chebyshev polynomials \[24\], would permit a detailed account of the LTB propagation in the one-sided regime. However, while keeping spanwise periodic boundary conditions, one can think of correcting our governing equations for an additional spanwise fluctuating bulk force generating a mean transverse flow sufficient to compensate the drift of LTBs and statistically maintain a strictly streamwise propagation. Such a work remains to be done but could give hints on the one-sided regime in realistic experimental conditions, owing to the obvious robustness of LTBs.

Two complementary studies are in progress, above and below the range of Reynolds numbers considered in this work: one is dedicated to the decay of the one-sided regime in a domain \( 1000 \times 2 \times 500 \) or larger \[56\] and the other is focussed on the onset of the laminar-turbulent patterning \( via \) standard Fourier analysis rather than turbulent fraction determination \[48, see Fig. 4 for preliminary results\].

By way of conclusion, the introduction of concepts and methods of statistical physics, and notably directed percolation, have put stress on universal features of the transition from/to turbulence in wall-bounded shear flows. All over the spatiotemporally intermittent regimes along the transitional range of channel flow, laminar-turbulent coexistence with a high level of stochasticity legitimates Pomeau’s views \[27\] about the decay of turbulence at \( Re_g \) as a process in the 2\( D\)-DP universality class. We have shown that this claim is however only partly fulfilled because a specific phenomenon comes and renders the full scenario imperfect: spanwise symmetry is broken due to a sensitive balance between local processes (longitudinal \( vs. \) transversal splittings) with rates depending on \( Re \) in different ways, ending with recession/expansion of turbulence becoming mostly one-dimensional before 2\( D\)-DP criticality has a chance to be observed. In turn this symmetry breaking is an event that could be well understood within the standard framework of dynamical systems and bifurcation theory, bringing an original perspective to the debated issue of universality \( vs. \) specificity in
the transition to turbulence of wall-bounded flows.

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Appendix A: System and numerical procedures

We consider the flow between two parallel walls driven by a time-independent body force, usually called channel flow or plane Poiseuille flow. The equations governing the velocity field \( \mathbf{u} \) are as follows:

\[
\begin{align*}
    (a) \quad \nabla \cdot \mathbf{u} &= 0, \\
    (b) \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} + \mathbf{f} \hat{e}_x,
\end{align*}
\]

where \( \rho = \text{Const.} \) is the density, \( \nu \) the kinematic viscosity, and \( \mathbf{f} \) the body force specific density. The unit vector in the streamwise direction \( x \) is denoted as \( \hat{e}_x \). The \( y \)- and \( z \)-axes are along the wall-normal and spanwise directions, respectively. All fields are assumed in-plane periodic and the velocity fulfills the usual no-slip boundary conditions at the walls.

The center-plane velocity of the corresponding laminar flow is \( U = f h^2 / 2 \nu \), where \( 2h \) is the distance between the walls. The Reynolds number is defined as \( \text{Re} = Uh/\nu = fh^3 / 2 \nu^2 \). Using \( h \) and \( U \) as distance and velocity units, and \( h/U \) as the time unit, all variables below are assumed dimensionless without a notational change; the equations governing the velocity field \( \mathbf{u}(x, y, z, t) \) are as follows:

\[
\begin{align*}
    (a) \quad \nabla \cdot \mathbf{u} &= 0, \\
    (b) \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \frac{1}{\text{Re}} \Delta \mathbf{u} + \frac{2}{\text{Re}} \hat{e}_x.
\end{align*}
\]

In practice, these equations are rewritten for the wall-normal velocity component \( u_y(x, y, z, t) \) and the wall-normal vorticity \( \omega_y(x, y, z, t) = (\nabla \times \mathbf{u})_y \) obtained by applying
\( \nabla \times \) and \( \nabla \times \nabla \times \) to (A2b) and keeping its \( y \) component \( 2 \):

(a) \( \frac{\partial \omega_y}{\partial t} = (\nabla \times \mathbf{N})_y + \frac{1}{\text{Re}} \Delta \omega_y, \)

(b) \( \frac{\partial \Delta u_y}{\partial t} = -(\nabla \times \nabla \times \mathbf{N})_y + \frac{1}{\text{Re}} \Delta \Delta u_y, \)

(A3)

where \( \mathbf{N} = \mathbf{u} \times \mathbf{\omega} \).

The full solution requires additional equations for in-plane averaged velocity fields. We define the auxiliary fields \( \phi_x(y,t) \) and \( \phi_z(y,t) \) as \( \partial_y \phi_x = \langle u_x \rangle_{xz} \) and \( \partial_y \phi_z = \langle u_z \rangle_{xz} \), where \( \langle g \rangle_{xz} \) denotes in-plane averaging of field \( g \), namely (\( L_x L_z \)) \( \int_0^{L_x} \int_0^{L_z} g \, dx \, dz \). By averaging the \( z \) and \( x \) components of the vorticity equation \( \nabla \times (\nabla \times \mathbf{N}) \) we obtain:

(a) \( \frac{\partial}{\partial t} \frac{\partial^2 \phi_x}{\partial y^2} = \frac{\partial \langle N_x \rangle_{xz}}{\partial y} + \frac{1}{\text{Re}} \frac{\partial^4 \phi_x}{\partial y^4}, \)

(b) \( \frac{\partial}{\partial t} \frac{\partial^2 \phi_z}{\partial y^2} = \frac{\partial \langle N_z \rangle_{xz}}{\partial y} + \frac{1}{\text{Re}} \frac{\partial^4 \phi_z}{\partial y^4}. \)

(A4)

Fields \( \phi_x \) and \( \phi_z \) are defined up to arbitrary functions of time that can be fixed as follows:

The streamwise bulk velocity is given by the difference between the boundary values of \( \phi_x \) at \( y = \pm 1 \), \( U_m = \frac{1}{2} \int_{-1}^{+1} \langle u_x \rangle_{xz} \, dy = \frac{1}{2} [\phi_x(y,t)]_{y=+1}^{y=-1} \). The arbitrariness in the definition of \( \phi_x \) can then be lifted by choosing \( \phi_x(+1,t) = U_m(t) \); hence, \( \phi_x(-1,t) = -U_m(t) \). Similar conditions apply to \( \phi_z \) and \( W_m \).

Equations for \( U_m(t) \) and \( W_m(t) \) are obtained by averaging (A2b) with respect to full space \( (x,y,z) \):

(a) \( \frac{dU_m}{dt} = \frac{1}{2\text{Re}} \left[ \frac{\partial^2 \phi_x}{\partial y^2} \right]_{y=-1}^{y=+1} + \frac{2}{\text{Re}}, \)

(b) \( \frac{dW_m}{dt} = \frac{1}{2\text{Re}} \left[ \frac{\partial^2 \phi_z}{\partial y^2} \right]_{y=-1}^{y=+1}, \)

(A5)

where the term \( 2/\text{Re} \) in (A5a) accounts for the constant streamwise driving.

In addition to periodic boundary conditions applied at distances \( L_x \) and \( L_z \) to \( \omega_y \) and \( u_y \), the no-slip boundary conditions at the walls and the boundary conditions relative to \( \phi_x \) and \( \phi_z \) are:

\( u_y \big|_{y=\pm 1} = \frac{\partial u_y}{\partial y} \bigg|_{y=\pm 1} = \omega_y \big|_{y=\pm 1} = \frac{\partial \phi_x}{\partial y} \bigg|_{y=\pm 1} = \frac{\partial \phi_z}{\partial y} \bigg|_{y=\pm 1} = 0, \phi_x \big|_{y=\pm 1} = \pm U_m, \phi_z \big|_{y=\pm 1} = \pm W_m. \)

(A6)

The set of equations (A3)-(A5) with boundary conditions (A6) are numerically integrated as follows:

- Spatial treatment of (A3) uses Fourier series in streamwise and spanwise directions, \( x \) and \( z \), respectively. In view of numerical accuracy and efficiency, the wall-normal dependence of (A3) is dealt with combinations of Chebyshev polynomials satisfying the homogeneous
boundary conditions at \( y = \pm 1 \). The equations for \( \omega_y \) and \( u_y \) are solved using the following expansions:

\[
\omega_y = \sum_{l=0}^{L-2} \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{\omega}^{lmn} (T_{i+2}(y) - T_i(y)) \exp \left[ i \frac{2\pi}{L_x} mx + i \frac{2\pi}{L_z} nx \right], \tag{A7}
\]

\[
u_y = \sum_{l=0}^{L-4} \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{\nu}^{lmn} \left( T_{i+4}(y) - \frac{2l + 4}{l + 3} T_{i+2}(y) + \frac{l + 1}{l + 3} T_i(y) \right) \exp \left[ i \frac{2\pi}{L_x} mx + i \frac{2\pi}{L_z} nx \right], \tag{A8}
\]

where \( T_l \) is the Chebyshev polynomial of degree \( l \).

– Similarly, the auxiliary fields \( \phi_x \) and \( \phi_z \) in \( \text{(A4)} \) are expanded as

\[
\phi_x = \sum_{l=0}^{L-4} \hat{\phi}_x^l \left( T_{i+4}(y) - \frac{2l + 4}{l + 3} T_{i+2}(y) + \frac{l + 1}{l + 3} T_i(y) \right) \frac{U_m}{2} y (3 - y^2), \tag{A9}
\]

\[
\phi_z = \sum_{l=0}^{L-4} \hat{\phi}_z^l \left( T_{i+4}(y) - \frac{2l + 4}{l + 3} T_{i+2}(y) + \frac{l + 1}{l + 3} T_i(y) \right) \frac{W_m}{2} y (3 - y^2), \tag{A10}
\]

where the last terms in \( \text{(A9, A10)} \) follow from the boundary-condition homogenization technique \[58\]; these terms are the polynomials of lowest degree satisfying all boundary conditions for \( \phi_x \) and \( \phi_z \), respectively.

– A conventional Galerkin method is developed by taking the inner product of the basis functions and the evolution equations, yielding ordinary differential equations for each coefficient in the expansions. Periodic boundary conditions are imposed in the streamwise and spanwise directions at distances \( L_x = 500 \) and \( L_z = 250 \). Maximum Fourier wavenumbers are \( M = N = 767 \) and the maximum degree for Chebyshev polynomials in the \( y \) direction is \( L = 31 \). Aliasing errors involved in the evaluation of the quadratic nonlinear terms are removed in all directions using the 2/3 rule \[58\] in the \( x \) and \( z \) directions, and computing all coefficients of Chebyshev polynomials up to degree \( 2L \) in direction \( y \). The evaluation of nonlinear terms then involves \( (N_x, N_y, N_z) = (2304, 64, 2304) \) modes. For larger and smaller domain sizes \( (L_x, L_z) = (1000, 500) \) and \( (250, 125) \) used to check size effects, mode numbers are taken in the same proportions. This spatial resolution has been found appropriate from the comparison with other numerical simulations and a parallel study of plane Couette flow driven by counter-translating plates rather than by a constant body force. The equations are numerically time-integrated in a standard way using a second-order method, Crank–Nicolson for the viscous terms and Heun method for the other terms, with time-increment
$\delta t = 0.04$.

Other works may solve the flow using different definitions or scalings:
- In the literature, reference is often made to the mean flow rate. Here the mean streamwise bulk velocity is a measured quantity that can characterize the flow regime upon time averaging, see Fig. 3(a) displaying $\langle U_m \rangle$ as a function of Re. From it we can define a Reynolds number $Re_b$ of practical use, though not a control parameter, as $Re_b = 1.5 \Re \langle U_m \rangle$. Once our scaling is adopted for the velocity, $Re_b$ is then understood as the Reynolds number built using the center-plane velocity of a parabolic flow profile with the same mean velocity.
- Another popular choice is by using so-called wall units: The friction velocity is defined by $U_\tau^2 = \tau_w = \nu (|\partial_y u_x|) \text{wall}$. It is obtained by averaging (A2b) as $U_\tau^2 = 2/\Re$ in our unit system; then,

$$Re_\tau = Re U_\tau = \sqrt{2Re}. \tag{A11}$$

The friction coefficient is traditionally defined as $C_f = \tau_w / \frac{1}{2} \langle U_m \rangle^2$. In our approach, its expression directly comes from $\tau_w = U_\tau^2 = 2/\Re$ to give $C_f = 4/(\Re \langle U_m \rangle^2)$. Taking the inner product between $u$ and (2b) and averaging the equation then relates $C_f$ to the mean dissipation rate $\langle \epsilon_m \rangle$. Expressed using $\langle U_m \rangle$ as the velocity unit, the volume-averaged dissipation rate $\epsilon_m(t) = 1.5(Re_b \langle U_m \rangle^2)^{-1} \left[ \mathcal{V}^{-1} \int_\mathcal{V} (\nabla u \cdot \nabla u) \, d\mathcal{V} \right] \text{ is computed to give:} \tag{A12}$$

$$\langle \epsilon_m \rangle = C_f/2.$$

**Appendix B: Measurement of turbulent fraction**

Local laminar and turbulent states are discriminated according to a traditional method of computer image treatment called “moment-preserving thresholding” [53]. The principle is to separate pixels in an image into several classes, using thresholds defined in a systematic way from the statistics of the pixels’ values, typically gray levels, rather than from some empirically-defined rule. On general grounds, the thresholds are chosen so that the first few moments of the histograms of pixel values are preserved. In the present case, two classes distinguished through a single threshold, are needed: “turbulent $\equiv$ above” and “laminar $\equiv$ below”. Normalized histograms are obtained from the distribution of the observable, here simply called $u$, at all points in the domain, say $p(u)$. Its moments $m_k = \sum_u u^k p(u)$, $k = 1, 2, \ldots$ are next computed. Unknown reduced variables $\tilde{u}_\ell$ with probability $p_\ell$ for
laminar local states and \( \tilde{u}_t \) with probability \( p_t = 1 - p_\ell \) for turbulent ones are then determined using three equations for the three lowest order moments [53]:

\[
\tilde{u}_t^k p_\ell + \tilde{u}_t^k p_t = m_k, \quad k = \{1, 2, 3\}.
\]

(Multi-level thresholding would require more moments and, if the set of pixels were to be divided into \( M \) classes; it is easily be seen that \( 2M - 1 \) moments would then be required.) The reduced representation \( (\tilde{u}_\ell, \tilde{u}_t) \) is thus the best two-level representation of the original field, while probability \( p_t \) corresponds to our evaluation of the turbulent fraction \( F_t \).

The observable \( u \) of interest is here the box-filtered value of \( |u_y(x, 0, z)| \) and a single parameter remains, the width \( w \) of the squares \( w \times w \) over which \( |u_y(x, 0, z)| \) is averaged. In order to visualize the result of the procedure, the laminar/turbulent cut-off \( u_* \) has to be determined. This is done by expressing the condition defining \( p_t \) explicitly, that is:

\[
p_t = \frac{\#(u_i > u_*)}{N_x N_z},
\]

where \( \#(u_i > u_*) \) is the number of lattice nodes considered as turbulent. The most faithful reproduction of the contour of turbulent domains requires that \( w \) should be large enough to damp out irrelevant small-scale modulations seen in Fig. 11 “raw”. The other panels in Fig. 11 illustrate the output of the procedure for different values of \( w \). In the figure, \( \delta \) stands for the spanwise grid spacing \( \delta = L_z / N_z \simeq 0.109 \). As the filter size becomes larger, spurious laminar regions disappear. A thin line corresponding to the cut-off condition \( u = u_* \) delineates the turbulent region obtained with our thresholding method. Figure 12 displays the two snapshots in Fig. 1 belonging to the Re range studied in § IV using \( w = 12 \delta \).
FIG. 11. Color-level illustration of the DAH of the LTB shown in Fig. using $|u_y(x,0,z)|$. The size of the domain displayed is $50 \times 50 \simeq 460\delta \times 460\delta$. Flow direction is from left to right. Raw data (left) and after box-filtering with the width $w = 12\delta$ (routinely used here) and $w = 40\delta$. Light-blue lines mark the boundaries between turbulent and laminar regions as determined by the moment-preserving thresholding method.

FIG. 12. Moment-preserving thresholding of snapshots at 1200 and 1800 in Fig. 1(c,d) under box-filtering with $w = 12\delta$. Flow direction is from left to right. Domain size is $500 \times 250$.

Appendix C: Supplemental video

Re=725

One-sided LTB regime at Re = 725 [Fig. 4(a) and (b)]. Splittings and collisions between LTBs propagating in the same direction are observed but without any trace of a transversal splitting. Wall-normal velocity field on the mid-plane $u_y(x,0,z,t)$ is displayed in a co-moving reference frame with velocity 0.8 in stream-wise direction (to the right) for better visibility.
**Re=850**

One-sided LTB regime with rare transversal splittings at $\text{Re} = 850$ [Fig. 1(a)]. The displayed quantity and velocity of the reference frame are the same as in Supplemental Video Re=725.

**Re=900**

One-sided LTB regime with frequent transversal splittings at $\text{Re} = 900$ [Fig. 4(c) and (d)]. The displayed quantity and velocity of the reference frame are the same as in Supplemental Video Re=725.

**Re=1050**

Two-sided LTB regime at $\text{Re} = 1050$, slightly above the onset at $\text{Re}_2 \approx 1011$[Fig. 1(b)]. The displayed quantity is the same as in Supplemental Video Re=725. The velocity of the reference frame is 0.7.

**Re=1200**

Strongly intermittent loose continuous network of LTBs at $\text{Re} = 1200$ [Fig. 1(c)]. The displayed quantity is the same as in Supplemental Video Re=725. The velocity of the reference frame is 0.6.

**Re=1800**

Weakly intermittent loose banded pattern at $\text{Re} = 1800$ [Fig. 1(d)]. The displayed quantity is the same as in Supplemental Video Re=725. The velocity of the reference frame is 0.45.
Re=3000

Tight banded pattern at Re = 3000 [Fig. 1(e)]. The displayed quantity is the same as in Supplemental Video Re=725. The velocity of the reference frame is 0.35.

**One-sided regime in a larger domain at Re=900**

Everything is same as in Supplemental Video Re=900 except for the horizontal domain size, which is 1000 and 500 in streamwise and spanwise directions respectively. The initial transient state is included to show the transition from randomly distributed LTBs of both directions to the final one-sided state.

**Two-sided regime in a larger domain at Re=1050**

Everything is same as in Supplemental Video Re=1050 except for the horizontal domain size, which is 1000 and 500 in streamwise and spanwise directions respectively.

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[54] The agreement is qualitative because the Barkley–Tuckerman oblique geometry [17] is only appropriate for bands generated by periodic continuation at a short distance along their own direction, which (i) reinforce coherence along that direction, (ii) forbids the opening of laminar gaps and the account of LTBs around Re_g, (iii) kills orientation fluctuations and/or superpositions near Re_t, all together leading to an upward shift of the effective Reynolds number w.r. to its nominal value.

[55] Finite-size effects linked to this methodological difference presumably roots a systematic discrepancy between our definition of the mean-flow based Reynolds number and the one in [35], leaving us unable to match results about DP thresholds quantitatively.

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