Isogeometric Parametrization Inspired by Large Elastic Deformation

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Abstract: Construction of volumetric parametrizations for computational domains is a key step in the pipeline of isogeometric analysis. Here we propose an alternative solution to this problem that uses the nonlinear elasticity analogy. The desired domain is modeled as a deformed configuration of an initial simple geometry. Assuming that the parametrization of the initial domain is bijective and that it is possible to find a locally invertible displacement field, the method yields a bijective parametrization of the target domain. We show that, in many cases, an unknown deformation can be computed using the Saint Venant-Kirchhoff model. Moreover, if sufficiently small incremental steps for the deformation are used, a linear elasticity solver is enough to produce high quality parametrizations for nontrivial domains, what makes the methodology less demanding on implementation and computational time. Additionally, a great control over the resulting parametrization can be exerted by varying the material properties used in the elasticity model. Experiments indicate that when the material is close to being incompressible and the initial domain has uniformly sized elements, the method yields parametrizations with uniformly sized elements as well. The method is not restricted to a single patch scenario but can be utilized to construct multi-patch parametrizations with naturally looking boundaries between neighboring patches. We illustrate the performance of the method and study the quality of the resulting computational meshes on a range of two-dimensional and three-dimensional examples.

Keywords: isogeometric analysis, parametrization, elastic deformation.

1 Introduction

In the world of Computer Aided Design one usually models only the surfaces of objects whereas their interior is of no interest for the design process. If, in order to analyze a given object, Finite Element Method or Isogeometric Analysis are applied, one has to provide a computational mesh for the interior as well. In case of IGA, this gives rise to a problem of generating a planar or volumetric parametrization from a given boundary parametrization. While being trivial for relatively simple domains like distorted quads or cubes, this problem poses a serious challenge as soon as a more challenging domain is encountered.
We would like to augment a wide variety of already existing solutions to this problem by an alternative approach which, in certain situations, can be superior in terms of quality of the resulting parametrization or computational efficiency. It is based on an analogy to structural mechanics following which a computational mesh is deformed as if it represented an elastic body. This approach is especially well-known in the context of fluid-structure interaction [3, 22, 9] but was also applied, considering relatively small deformations, to generate curvilinear meshes from piece-wise linear triangulations [19]. Here, we try to substitute a nontrivial problem of generating a bijective parametrization for a complicated domain by two conceptually simpler steps: first, choose a simple initial domain that can be readily parametrized and which is geometrically close to the target domain. Then, deform it such that its boundary coincides with the boundary of the target domain. The result is a deformed configuration of the initial domain that serves as a desired parametrization. Assuming that the parametrization of the initial domain is bijective and that it is possible to find a locally invertible displacement field, the method yields a bijective parametrization of the target domain as well. Throughout the paper, we show that valid deformation can be computed using the Saint Venant-Kirchhoff material model, a relatively simple nonlinear formulation of elasticity. Moreover, we demonstrate that using small incremental steps the deformation can be computed using as little as the linear elasticity model. A choice of the initial domain, of course, plays just as important role. We illustrate it while applying the proposed method to various geometries and try to draw a list of recommendations concerning this ingredient of the method. Additionally, we show that the resulting parametrization depends on material parameters used in the elasticity model. In particular, we show that by using values of Poisson’s ratio close to 0.5 one can achieve uniformly sized elements of the corresponding mesh if the initial domain possesses the same property.

Let us give a short overview of the state-of-the-art in the field. The first method one should try when generating a domain parametrization with tensor-product splines is a Coons patch [20]. Although nothing guarantees that the resulting parametrization is bijective, the method is explicit and its output can be used as an initial guess for more sophisticated parametrization techniques. A number of other explicit methods is described in [11]. This paper also presents a general constrained optimization framework where the domain parametrization is optimized with respect to some quality measure. Constrained optimization techniques are further discussed in papers from this probably not exhaustive list [23, 10, 24]. The harmonic energy often plays a role of the objective function being optimized, a choice motivated by beautiful geometric properties of harmonic mappings. Rather than considering optimization problems, many search for harmonic mappings between the parametric and the target domains by directly constructing it or by solving PDEs [17, 18, 12]. If a domain belongs to a certain class of geometries, specialized techniques can be developed to deal with
The examples are the swept volumes [1] and the star-shaped domains [7].

The rest of this paper is structured as follows: Section 2 briefly states the parametrization problem to be solved and then outlines the mathematical concepts behind the proposed solution in broad brush-strokes. Section 3 deals with the elasticity model used to compute deformation of the initial domain. In Section 4, a choice of the initial domain is illustrated on a range of 2D and 3D examples. Moreover, a behavior of the algorithm with respect to different parameters is studied and the comparison between the proposed method and the elliptic grid generation techniques is conducted. Finally, Section 5 draws a conclusion and outlines further research directions concerning possible extensions and enhancements of the proposed method.

2 Problem of domain parametrization

In what follows, we briefly formulate the problem to fix the notation and to describe our approach to its solution. For the sake of simplicity, we restrict the discussion to a single-patch case in two dimensions, however, the proposed technique is readily applicable to multi-patch and 3D problems, as we demonstrate in Section 4.

2.1 Problem setting

Assume that the boundary $\partial \Omega$ of the domain $\Omega$ is given as four compatible B-spline curves. By compatible we mean that the oppositely lying curves have the same set of B-spline basis functions and, therefore, the same number of control points. In this case, we can search for an unknown parametrization $G : [0, 1]^2 \rightarrow \Omega$ of the domain $\Omega$ in form of a tensor product B-spline function

$$G(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} M_i(\xi) N_j(\eta),$$  \hspace{1cm} (1)

with $\{M_i(\xi)\}_{i=1}^{n}$ and $\{N_j(\eta)\}_{j=1}^{m}$ being the bases of the boundary curves. In the following, we will often distinguish between the boundary and the interior control points. The former are given by the boundary parametrization, while the latter are unknown. We will denote the set of indices corresponding to the boundary control points by $\mathcal{B}$ and its counterpart, the set of indices of the interior control points, by $\mathcal{I}$. In fact, the whole parametrization problem boils down to allocation of the unknown interior control points $\{c_{ij}\}_{\mathcal{I}}$ such that the resulting parametrization $G$ is bijective, see Fig. 1.
Figure 1: Domain given by four compatible boundary curves. How to choose the interior control points to obtain a bijective planar parametrization?

We would like to mention here that when the boundary of the domain is given by four curves, the assumption on their compatibility is not as severe as it might seem. Often it can be satisfied by application of knot insertion and degree elevation algorithms or, in the worst case, by reparametrization of the curves. A true limitation is the use of tensor-product B-splines since it narrows the class of domains which can be parametrized down to those topologically equivalent to a unit square. To circumvent this issue in practice, a multi-patch approach can be utilized for complicated domains or domains with holes. Then the parametrization problem has to be solved on every single patch. Another way to break free from the topological constraints is the use of unstructured spline technologies, such as triangular splines \[14, 4\] or subdivision schemes like Catmull-Clark \[6, 16\].

### 2.2 Mesh deformation method

We propose a solution to the parametrization problem that mimics a deformation of a solid object to construct a bijective parametrization \(G\) of a given domain \(\Omega\). It involves choosing an initial domain \(\Omega_0\), which is straightforward to parametrize and is geometrically close to the given domain, and then deforming this initial domain such that its boundary coincides with the boundary of the given domain. The resulting deformed configuration of the initial domain defines the sought parametrization \(G\). In other words, the parametrization \(G : [0, 1]^2 \rightarrow \Omega\) is constructed as a superposition of a parametrization of the initial domain \(G_0 : [0, 1]^2 \rightarrow \Omega_0\) and a deformation \(D : \Omega_0 \rightarrow \Omega\):

\[
G = D \circ G_0.
\]  \(2\)
The process of choosing an initial domain $\Omega_0$ is illustrated on the examples in Section 4. For now, let us assume that its parametrization $G_0$ is given as a tensor product B-spline function

$$G_0(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij}^0 M_i(\xi) N_j(\eta),$$

where, importantly, the bases $\{M_i(\xi)\}_{i=1}^{n}$ and $\{N_j(\eta)\}_{j=1}^{m}$ are the bases given by the boundary $\partial \Omega$. The control points $\{c_{ij}^0\}$ are assumed to be known.

Next, let the deformation $D$ be represented as $D(x) = x + u(x)$ for any point $x \in \Omega_0$, where $u : \Omega_0 \to \mathbb{R}^2$ is the unknown displacement field. Following the isogeometric approach, we introduce a discretization $u_h : \Omega_0 \to \mathbb{R}^2$ and its pull-back counterpart $\tilde{u}_h : [0,1]^2 \to \mathbb{R}^2$ as

$$u_h(x) = \tilde{u}_h(G_0^{-1}(x)), \quad \tilde{u}_h(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} M_i(\xi) N_j(\eta)$$

using, once again, the same bases $\{M_i(\xi)\}_{i=1}^{n}$ and $\{N_j(\eta)\}_{j=1}^{m}$. Note that only the interior part of the coefficients $\{d_{ij}\}$ is unknown. The boundary part is given as a difference between the boundary control points of the target and the initial domains

$$\{d_{ij}\}_B = \{c_{ij} - c_{ij}^0\}_B$$

and will later serve as the Dirichlet boundary condition for the elasticity problem which has to be solved to find the displacement field.
Having $G_0$ and $u_h$ introduced, we can now define $G$ as

$$G(\xi, \eta) = D(G_0(\xi, \eta)) = G_0(\xi, \eta) + u_h(G_0(\xi, \eta)) = G_0(\xi, \eta) + \tilde{u}_h(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} (c^0_{ij} + d_{ij})M_i(\xi)N_j(\eta).$$

That means that the control points of $G$ are defined simply as a sum of the control points of $G_0$ and the coefficients of $\tilde{u}_h$. In other words, in order to deform the initial domain $\Omega_0$, one has to move the control points of its parametrization $G_0$ along the vectors defining the displacement field. The described method is demonstrated once again in Figure 3.

Figure 3: Left: undeformed configuration of the initial domain. Right: deformed configuration of the initial domain / parametrization of the target domain

## 3 Deformation technique

So far, we just reformulated the problem of domain parametrization as an elasticity problem involving large deformations. In order to deform the initial domain, a valid displacement field must be found. To that end, we solve a system of nonlinear elasticity equations. Among many formulations of nonlinear elasticity available, we are interested in the simplest one that is still able to model large deformations. The Saint Venant-Kirchhoff model [5] satisfies these conditions since it includes full geometric nonlinearities but uses only a linear material law. In Section 4, we show that this formulation of elasticity is enough to compute parametrizations for rather nontrivial domains.

### 3.1 Saint Venant-Kirchhoff material

The initial domain $\Omega_0$ serves as a reference configuration for the elasticity problem. We formulate all equation in Lagrangian coordinates, that is with respect
to the undeformed reference configuration. We search for the displacement field \( u : \Omega_0 \rightarrow \mathbb{R}^2 \) which has a prescribed value \( u_B \) on the boundary \( \partial \Omega_0 \) given by (6).

With the St. Venant-Kirchhoff material model, the displacement \( u \) satisfies the following system of equations

\[
-\nabla \cdot (F(u)(2\mu E(u) + \lambda \text{tr}(E(u))I)) = 0 \quad \text{in } \Omega_0, \\
u = u_B \quad \text{on } \partial \Omega_0,
\]

where the full strain tensor \( E \) and the deformation gradient \( F \) are defined as

\[
E(u) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u \nabla u^T), \\
F(u) = I + \nabla u.
\]

With the initial domain \( \Omega_0 \) and the Dirichlet condition \( u_B \) given, the solution of system (8-9) depends only on the chosen material parameters \( \lambda \) and \( \mu \). We are free to vary them in an attempt to achieve certain properties of the solution. If necessary, additional control can be exerted by choosing a nonzero right-hand side. We would also like to point out that system (8-9) does not necessarily yield a valid displacement field, i.e. one with the nonzero Jacobain determinant. In practice, however, if the prescribed boundary deformation \( u_B \) is not extreme, i.e. if the initial domain \( \Omega_0 \) is geometrically close to the target domain \( \Omega \), the solution \( u \) of system (8-9) is a valid displacement field. Consequently, the resulting parametrization of the target domain \( \Omega \) is bijective.

### 3.2 Incremental deformation with linear elasticity

Being nonlinear, system (8-9) is usually solved using Newton’s method. A natural choice for the initial guess is a zero deformation, however, the algorithm may not converge to a true solution. To overcome this problem, system (8-9) can be solved in an incremental fashion. Instead of being solved for the full boundary deformation \( u_B \), the system is solved for a fraction of it. The solution to this intermediate problem serves as an initial guess for the full problem. With \( N \) steps, \( N \) nonlinear systems have to solved granting a sequence of \( N \) deformed configurations:

\[
-\nabla \cdot (F(u_i)(2\mu E(u_i) + \lambda \text{tr}(E(u_i))I)) = 0 \quad \text{in } \Omega_0, \\
u_i = \frac{i}{N} u_B \quad \text{on } \partial \Omega_0,
\]

where \( i = 1, \ldots, N \). On every step, solution \( u_i \) serves as an initial guess for the next step. Schematically, this incremental procedure can be presented as

\[
\Omega_0 \overset{u_1}{\longrightarrow} \Omega_1 \\
\Omega_0 \overset{u_i}{\longrightarrow} \Omega_i \\
\Omega_0 \overset{u_N}{\longrightarrow} \Omega_N = \Omega,
\]
where $\Omega_i$ is a deformed configuration on every step. Note that each system (12-13) is formulated in the initial domain $\Omega_0$.

An obvious problem with this approach is that solving a sequence of nonlinear problems is computationally expensive. If $p$ is an average number of iterations necessary for Newton’s method to converge, then the total number of iterations is $O(pN)$. To mitigate it, one can observe that the difference between the solutions at two consecutive incremental steps is small if the number of steps $N$ is big enough. In that case, a deformation $u_l^i$ from $\Omega_{i-1}$ to $\Omega_i$ can be found using a linear elasticity model:

$$-\nabla \cdot (2\mu \varepsilon(u_l^i) + \lambda \text{tr}(\varepsilon(u_l^i))I) = 0 \text{ in } \Omega_{i-1},$$

(14)

$$u_l^i = \frac{u_B}{N} \text{ on } \partial \Omega_{i-1},$$

(15)

where $\varepsilon(u) = \frac{1}{2} (\nabla u + \nabla u^T)$ stands for the linear strain tensor. In contrast to (12-13), which is always solved in the initial domain $\Omega_0$, the linear system (14-15) is solved in the deformed configuration obtained from the previous step. Thus, the linearized procedure looks like this:

$$\Omega_0 \xrightarrow{u_l^i} \Omega_1 \xrightarrow{} \Omega_i \xrightarrow{u_l^i} \Omega_{i+1} \xrightarrow{} \Omega_{N-1} \xrightarrow{u_l^N} \Omega_N = \Omega.$$

The linearized procedure requires only a linear elasticity solver and is therefore simple to implement. Since solving a system of linear elasticity equations can be considered as one iteration of Newton’s method for a nonlinear elasticity system, the total number of iterations necessary to compute the deformation of the initial domain drops down to $N$. Numerical experiments indicate that this linearized incremental procedure converges to a solution of the original nonlinear system (8-9) as the number of incremental steps grows, i.e.

$$\sum_{i=1}^{N} u_l^i \xrightarrow{N \to \infty} u,$$

(16)

however, further research is necessary in order to rigorously prove, or invalidate, this statement.

4 Examples

In this section, we present a number of examples where the proposed method is applied. The examples include 2D and 3D geometries and illustrate the choice of the initial domain in different cases. We also use the first example, a puzzle piece,
to illustrate how the algorithm reacts to a change of parameters and to compare it
with the elliptic grid generation techniques. The elasticity simulations necessary
for the method were implemented within G+Smo, an open-source C++ library
for IGA [15].

4.1 2D puzzle piece

We start with the puzzle piece, a simple geometry defined by four boundary
curves. As an initial domain serves a quad spanned by the corner points, an intu-
itive choice for many simple geometries. The parametrization for the quad can be
easily generated by interpolation of the corner control points. The corresponding
initial mesh has uniformly sized well-shape elements, and one may hope that this
property will be, at least partially, inherited by the deformed mesh. The defor-
mation process, conducted in six incremental steps, is shown in Figure 4. The
resulting parametrization is bijective and has rather uniformly sized elements.

![Figure 4: Parametrization for the puzzle piece (bottom-right). A corner-spanned quad serves as an initial domain (top, left). Two intermediate configurations illustrate the deformation process.](image)

In the proposed method, the number of incremental steps defines how accu-
rately the incremental procedure (14-15) approximates the solution of the non-linear problem (12-13). Note that even with one incremental step, which corresponds to an attempt of deforming the mesh using the linear elasticity model, the boundary of the target domain is reproduced exactly if, of course, the Dirichlet boundary conditions are enforced strongly. However, the resulting parametrization may not be bijective, as shown in Figure 5 which gives a glimpse on the dependence of the algorithm on the number of incremental steps. We consider a number of incremental steps used for this example, six, as relatively low, especially in comparison to optimization techniques where, at least in our experience, a number of iterations necessary for the minimization algorithm to converge can reach twenty or thirty.

![Figure 5](image)

Figure 5: Behavior of the algorithm with respect to a number of incremental steps (1, 2 and 6 from left to right). With only one step the parametrization is not bijective.

Another way of controlling the output of the algorithm in via material parameters used in the elasticity model. Observe, for example, how the resulting parametrization reacts to different values of Poisson’s ratio, see Fig. 6. It appears that with Poisson’s ratio close to 0.5, that is with an almost incompressible material, the resulting parametrization has the most uniformly sized elements. This can be explained by the fact that the initial mesh possesses the same property which was preserved by local incompressibility. For lower values of Poisson’s ratio the effect is less prominent. Moreover, isoparametric lines are distributed denser next to concave regions of the domain boundary and sparser in the vicinity of convex regions. This behavior is well known for elliptic grid generation techniques.
Figure 6: Dependence of the resulting parametrization on Poisson’s ratio used in the underlying elasticity model (-0.9, 0 and 0.49 from left to right).

Figure 7: Comparison between two different approaches to parametrization: mesh deformation with Poisson’s ratio $\nu = 0.49$ (left) and elliptic grid generation (right).

Lastly, we want to compare the proposed mesh deformation (MD) method with the elliptic (EG) grid generation technique as another PDE-bases approach. Figure 7 depicts two parametrization generated using these two method. The MD-parametrization is an obvious winner if the uniformly sized elements is the criterion. On the other hand, the EG-parametrization sports almost orthogonal isoparametric lines and, in general, has a more structured look. As usual, it is hard to pick an overall better parametrization because the choice varies together with the quality measure. It is interesting to note, however, that the proposed
method can produce the results similar to the elliptic grid generation techniques. It may be possible that by varying other material parameters or by using another material law one can make the method mimic other mesh quality measures.

4.2 2D multi-patch L-shape domain

As mentioned in Subsection 2.1, it may not be worth trying to parametrize a given domain using only one patch. Often, it is easier to split the domain into several patches and to work with the resulting multi-patch structure. Study the wavy L-shape domain example in Figure 8. The initial domain is formed by three squares, each generated by interpolating the corner control points. The deformation is conducted in just two incremental steps with a value of 0.49 for Poisson’s ratio.

![Figure 8: Multi-patch parametrization for a wavy L-shape domain. Patch-to-patch interfaces assume a natural shape via deformation.](image)

One advantageous property of this algorithm when applied to a multi-patch structure is that all patches can be deformed simultaneously. This is opposed to patch-by-patch parametrization methods where each patch is parametrized individually. The result is that the patch-to-patch interfaces are free to deform and can assume a natural shape, improving the global parametrization quality. Moreover, since the initial multi-patch structure has isoparametric lines that connect smoothly on the patch-to-patch interfaces, the deformed structure also possess this property, at least far from the inward-looking corner. That is a consequence of the smoothness of the solution to the elasticity problem (12-13) everywhere but in the inward looking corner. Although the used isogeometric discretization is only $C^0$-continuous on the patch-to-patch interfaces, it approximates the smooth solution and thus produces an almost smooth patch-to-patch connection.
4.3 2D SB-parametrizations for rotor geometries

Multi-patch structures are known to be very case-dependent. A good alternative approach, especially for many rotor-type geometries, can be the so-called scaled boundary (SB) parametrizations [2]. Within this approach, the interior of the domain can be parametrized using only one patch as illustrated on an example of the screw machine’s female rotor [21] shown in Figure 9. As an initial domain serves a disk, another domain that is straightforward to parametrize by interpolation of the control points. For Poisson’s ratio, once again, a value of 0.49 is used. Three incremental steps are used for the deformation.

![Figure 9: Scaled-boundary parametrization of the screw machine’s female rotor generated as a deformed configuration of a disk.](image)

4.4 3D screw machine rotor

The proposed mesh deformation method is applicable to 3D domains as well. Figure 10 illustrates it on the example of the screw machine’s male rotor. One possible option for the initial domain is just a cube. Poisson’s ratio is 0.49 as before, and ten incremental steps are used. With such an initial domain, however, the method would fail if the screw pitch reached 360° since the intermediate configurations, specified by the scaled Dirichlet boundary condition [5], would have a degenerate or self-intersecting boundary. To combat this, a better initial domain can be chosen, for example, a pre-twisted cube, see Fig. 11. By investing more effort into a choice of the initial domain, we can make the computation of deformation easier. In this case, with a pre-twisted cube as the initial domain it suffices to use just two incremental steps to find a valid displacement field.
Figure 10: Volumetric parametrization for the screw machine’s male rotor generated as a deformed configuration of a cube.

Figure 11: A better initial domain for the screw geometry: a pre-twisted cube.

5 Conclusions

In this paper, we proposed an isogeometric parametrization technique that mimics the behavior of a deformable solid object. It starts with a simple domain that is readily parameterizable and deforms it such that its boundary coincides with
the boundary of the target domain. The deformed configuration of the initial
domain serves as a desired parametrization. As follows from the description,
the method relies on the solution of the elasticity problem and depends heavily
on the underlying elasticity model and used material parameters. We showed,
however, that already the Saint Venant-Kirchhoff model is enough to acquire
quite promising results for nontrivial geometries. Experiments with Poisson’s
ratio indicated that the method yields parametrizations with uniformly sized
elements when the material is close to being incompressible and the elements
of the initial domain are also uniformly sized. Further investigation is to be
made around the choice of the initial domain since it affects the reliability and
the output of the algorithm. Moreover, the proposed method may benefit from
the use of non-homogeneous distribution of the material parameters or a nonzero
right-hand side. Last but not least, it is planned to combine the proposed method
with elliptic grid generation techniques to benefit from the diametrically opposite
advantages that these two methods have to offer.

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