ON THE SINGULAR SCHEME OF CODIMENSION ONE
HOLOMORPHIC FOLLATIONS IN $\mathbb{P}^3$

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Abstract. In this work, we begin by showing that a holomorphic foliation with singularities is reduced if and only if its normal sheaf is torsion free. In addition, when the codimension of the singular locus is at least two, it is shown that being reduced is equivalent to the reflexivity of the tangent sheaf. Our main results state on one hand, that the tangent sheaf of a codimension one foliation in $\mathbb{P}^3$ is locally free if and only the singular scheme is a curve, and that it splits if and only if that curve is arithmetically Cohen-Macaulay. On the other hand, we discuss when a split foliation in $\mathbb{P}^3$ is determined by its singular scheme.

1. Introduction

The study of holomorphic foliations with singularities in complex manifolds has attracted a lot of interest over the last 40 years. There are very good sources to learn about the different aspects of the theory, and we just mention here the classical paper [1] where a general theory in terms of coherent sheaves was developed, and the recent book [19].

The tangent sheaf of the foliation, its normal sheaf and singular locus are the key elements for this study. Needless to say, a very important problem is to analyze if the tangent sheaf of the foliation is locally free as well as further properties (as the splitting) when it is. These properties have important consequences, as for example the ones recently noted in [3] and [6], where it has been proved that the fact that the tangent sheaf splits, along with some properties of the singular locus, give stability under deformations of the foliation and make it possible to characterize certain components of the space that parameterizes holomorphic foliations.

The first section of this paper is devoted to the study of the tangent and normal sheaves. We prove some key technical results for the main goals of our research that, even when mentioned in some references, were not explicitly found in the literature.
on singular foliations. We think that they could be useful for other people working in the field. Namely, we prove that a foliation is reduced if and only if its normal sheaf is torsion free and that, under the assumption that the singular locus is of codimension at least two, a foliation is reduced if and only if the tangent sheaf is reflexive.

As an immediate application of the reflexivity of the tangent sheaf, we give an affirmative answer to the question posed by Suwa in [19]: Is the tangent sheaf to a 1 dimensional reduced foliation locally free?

The main results contained in this paper establish a link between the above mentioned properties of the tangent sheaf and some algebro-geometric properties of the singular scheme. For codimension one foliations in $\mathbb{P}^3$, using some results by Roggero on reflexive sheaves (see [17]) we prove that the tangent sheaf is locally free if and only if the singular scheme is a curve, i.e. a Cohen-Macaulay scheme of pure dimension one (Theorem 3.2); moreover, this sheaf splits if and only if the singular scheme is an arithmetically Cohen-Macaulay curve (Theorem 3.3).

We also prove (Theorem 3.5) an algebraic characterization of the splitting of a locally free tangent sheaf. It is given by the fact that the homogeneous ideal generated by the coefficients of a form defining the foliation, that defines the singular scheme, is saturated.

An interesting problem is to decide when the singular scheme uniquely determines the foliation. That is, given a foliation defined by a 1-form $\omega$, and having $Z$ as singular scheme, is it true that if the 1-form $\omega'$ defines a foliation with $Z$ as its singular scheme then we have $\omega = \lambda \omega'$ for $\lambda \in \mathbb{C}^*$?

In dimension 2, the problem was completely solved recently by Campillo and Olivares in [4]: for reduced foliations of degree $d \neq 1$ there are no two different foliations with the same singular scheme.

With our approach, we get simpler proofs of their main results, and more importantly we prove some results in the $\mathbb{P}^3$ case. Namely, we prove unique determination by the singular scheme for codimension one degree $d$ foliations in $\mathbb{P}^3$ with split tangent sheaf, when the splitting type is $\mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$, with $a, b \leq -1$.

We also show that if there is a subbundle of the tangent bundle of the foliation of the form $\mathcal{O}_{\mathbb{P}^3}(1)$, then it is a linear pull-back of a foliation in a plane, and so is determined by its singular scheme if $d \neq 1$.

When there is a $\mathcal{O}_{\mathbb{P}^3}$ subbundle of $\mathcal{F}$, we just show some examples that illustrate the situation: there are foliations determined by the singular scheme (exceptional foliations, see [3]), and others that are not: the logarithmic foliations of type $\mathcal{L}(1,1,1,1)$. 

2. Basic facts concerning the tangent sheaf

2.1. Preliminaries. In this section we recall the basics of the theory. The following definition is taken from [1]:

**Definition 2.1.** Let $M$ be a complex $n$-dimensional manifold, and let $TM$ be its tangent sheaf.

- A codimension $k$ holomorphic foliation with singularities in $M$ is an injective morphism of sheaves $\varphi : F \rightarrow TM$, such that $\varphi(F)$ is an integrable coherent subsheaf of $TM$, of rank $n - k$. The integrability means that for each point $x \in M$, $\varphi(F)_x$ is closed under the bracket operation for vector fields.

Notation: we will denote by $F_\varphi$ the foliation given by $\varphi : F \rightarrow TM$.

- The sheaves $F$ and the $N_{F_\varphi} := TM/\varphi(F)$ are, respectively, the tangent and normal sheaves of the foliation.

- $\text{Sing}(F_\varphi) = \{ x \in M \mid (N_{F_\varphi})_x \text{ is not a free } \mathcal{O}_{M,x} \text{-module} \}$ is the singular set of the foliation $F_\varphi$.

The following facts can be found, for instance, in [16]:

1. The normal sheaf is coherent.
2. By definition, $\text{Sing}(F_\varphi)$ is the singular set of the sheaf $N_{F_\varphi}$. It is a closed analytic subvariety of $M$ of codimension at least one.
3. There is a rank $n - k$ holomorphic vector bundle $F$ on $M \setminus \text{Sing}(F_\varphi)$ whose sheaf of sections is $\varphi(F)|_{M \setminus \text{Sing}(F_\varphi)}$.

Note also that being a subsheaf of a locally free sheaf, $F$ is torsion free and therefore its singular set $S(F) = \{ x \in M \mid F_x \text{ is not a free } \mathcal{O}_{M,x} \text{-module} \}$ is of codimension $\geq 2$.

Moreover, it is also clear that $S(F) \subset \text{Sing}(F_\varphi)$. Indeed, if $x \notin \text{Sing}(F_\varphi)$ there is a neighbourhood $V$ of $x$ such that the sequence

$$0 \rightarrow F(V) \rightarrow TM(V) \rightarrow N_{F_\varphi}(V) \rightarrow 0$$

is an exact sequence of $\mathcal{O}_M(V)$ modules, and the second and third modules are free. Hence (see [8], Appendix 3) the sequence splits since $N_{F_\varphi}(V)$ is projective and $F(V)$ is a direct summand of a free module, and so it is projective. We conclude that, for every $p \in V$, $F_p$ is a free $\mathcal{O}_{M,p}$ module. In particular, $x \notin S(F)$.

In the study of singular holomorphic foliations, it is usual to deal with reduced ones. Let us recall their definition ([1]):

**Definition 2.2.** Let $F_\varphi$ be a foliation:

- $\varphi(F)$ is full if given any open set $U \subset M$, and $\gamma$ a holomorphic section of $TM|_U$ such that $\gamma(x) \in \varphi(F)_x$ for each $x \in U \cap (M \setminus \text{Sing}(F_\varphi))$, then at each point $p \in U \cap \text{Sing}(F_\varphi)$ the germ of the holomorphic vector field $\gamma$ is in $\varphi(F)_p$.
- $F_\varphi$ is reduced if $\varphi(F)$ is full.
2.2. Some remarks. For the sake of completeness and lack of reference, we now state and prove some properties of the tangent and normal sheaves of a holomorphic foliation. Recall that a coherent sheaf $F$ is reflexive if and only if it is isomorphic to its bidual $F^{**}$ ([10], [12]).

**Remark 2.3.** The foliation $F_\varphi$ is reduced if and only if $N_{F_\varphi}$ is torsion free. If the singular locus is of codimension at least 2, the foliation is reduced if and only if $F$ is a reflexive sheaf.

Indeed, assume that $F_\varphi$ is reduced, hence $\varphi(F)$ is full. Torsion freeness is a local property and locally free sheaves are torsion free, so it is enough to show that $(N_{F_\varphi})_x$ is a torsion free $O_{M,x}$-module for $x \in \text{Sing}(F_\varphi)$.

Let us suppose, on the contrary, that it is not. Then, there is a nonzero element $m \in (N_{F_\varphi})_x$ and a nonzero element $a \in O_{M,x}$ such that $am = 0$ in $(N_{F_\varphi})_x$. Let $U$ be a small enough neighborhood of $x$ in $M$, so that we can take $a \in O_M(U)$ and $\gamma \in TM(U)$ such that $\alpha_x = a$, $\pi_x(\gamma_x) = m$.

$N_{F_\varphi}$ is locally free in $V = U \cap (M \setminus \text{Sing}(F_\varphi))$, hence torsion free and we have that

$$\gamma(U \cap (M \setminus \text{Sing}(F_\varphi)) \in \varphi(F)(U \cap (M \setminus \text{Sing}(F_\varphi)).$$

As $\varphi(F)$ is full, we have that $\gamma(U) \in \varphi(F)(U)$, which gives $m = 0$ in $(N_{F_\varphi})_x$, a contradiction.

Conversely, suppose that $N_{F_\varphi}$ is torsion free. If $U \subset M$ is an open subset and $\gamma$ is a section of $TM(U)$ such that $\gamma_x \in (\varphi(F))_x$ for each $x \in U \cap (M \setminus \text{Sing}(F_\varphi))$, then for a nonzero element $f \in O_M(U)$ vanishing on the analytic set $\text{Sing}(F_\varphi)$, and for each $p \in U \cap \text{Sing}(F_\varphi)$, $(f \gamma)_p = f_p \gamma_p$ gives the zero element in $(N_{F_\varphi})_p$. Hence, $\gamma_p \in (\varphi(F))_p$. Thus, $\varphi(F)$ is full and the foliation is reduced.

The second part of the statement follows from the first, using that a torsion free sheaf $F$ is reflexive if and only if there is a locally free sheaf $E$ such that $F \subset E$ and $E/F$ is torsion free (see [15], p.61).

We can make the following further remarks:

- The singular locus of a reflexive sheaf is of codimension $\geq 3$ (see [12], [16]). Hence, the same is true for the singular set of the tangent sheaf of a reduced holomorphic foliation.

- The tangent sheaf of a foliation $F$ is torsion free and we have $(F^*) \subset F^{**}$. Hence, if the singular locus of the foliation is of codimension $\geq 2$ we can consider the reduced foliation given by the double dual of $F$.

Let us finally obtain a consequence of Remark 2.3. Suwa, in [19], made a distinction between 1-dimensional singular holomorphic foliations and “foliations by curves”, a concept that was defined in [9] where it is shown that they correspond
to holomorphic foliations with singularities with locally free rank 1 tangent sheaf. We now answer the question that he posed in [19], Remark 1.9, page 179:

The tangent sheaf of a dimension one reduced foliation on a complex $n$-dimensional manifold is a line bundle.

It is a consequence of our Theorem 2.3 and of the fact that a reflexive sheaf of rank one is a line bundle (see Lemma 1.1.15, page 154, in [16]).

We obtain an immediate proof of Proposition 1.7 in [19] (with no use of [14] or [18]):

Let $\mathcal{F}_\varphi$ be a holomorphic foliation with singularities.

1. If it is reduced, the codimension of its singular locus is $\geq 2$;
2. If its tangent sheaf $\mathcal{F}$ is locally free, and the codimension of the singular locus of $\mathcal{F}_\varphi$ is at least 2, then the foliation is reduced.

The first assertion follows from Proposition 2.3 and so does the second after noting that locally free sheaves are reflexive.

3. Tangent sheaf vs. singular scheme

From now on, all the foliations that we will consider will be defined in projective space, of codimension one, reduced, and with singular set of codimension $\geq 2$. In this section we characterize when the tangent sheaf is locally free or split (i.e., direct sum of line bundles), in terms of the geometry of the singular scheme of the foliation.

Recall that a degree $d$ codimension one holomorphic foliation with singularities is defined by a global section $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1(d + 2))$. The form $\omega$ satisfies the integrability condition $\omega \wedge d\omega = 0$. The degree is the number of tangencies (counted with multiplicities) of a generic line with the foliation. We can write $\omega = \sum F_j dz_j$, where the $F_j$ are homogeneous polynomials of the same degree $\deg F_j = d + 1$ satisfying $\sum_{j=0}^n z_j F_j = 0$.

The singular set of the foliation is given by $F_0 = \cdots = F_n = 0$, and it has a natural structure of closed subscheme of $\mathbb{P}^n$, given by the homogeneous ideal $(F_0, \ldots, F_n)$. Recall (11) that two homogeneous ideal $I, J$ define the same projective scheme $X$ if and only if they have the same saturation, i.e., $I^{sat} = J^{sat}$, where

$$I^{sat} = \bigcup_{l \geq 0} I : (z_0, \ldots, z_n)^l = \bigoplus_n H^0(\mathbb{P}^n, \mathcal{I}_X(n)),$$

$\mathcal{I}_X$ being the ideal sheaf of the subscheme $X$.

The singular scheme can be also obtained (see 7) as the closed subscheme of $\mathbb{P}^n$ whose ideal sheaf is the image of the co-section

$$\omega^*: (\Omega_{\mathbb{P}^n}(d + 2))^* \longrightarrow \mathcal{O}_{\mathbb{P}^n}$$
To simplify the notation, let us write \( Z := \text{Sing}(\mathcal{F}_x) \), and \( \mathcal{I}_Z = \text{Im}(\omega^*) \) to denote the ideal sheaf defining the singular subscheme in \( \mathbb{P}^n \). The induced map

\[
T_{\mathbb{P}^n} \to \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(d+2)
\]

is surjective, and moreover, it makes the following sequence exact:

\[
0 \to \mathcal{F} \xrightarrow{\varphi} T_{\mathbb{P}^n} \to \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(d+2) \to 0.
\]

Observe that \( \mathcal{N}_\mathcal{F} = \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^n}(d+2) \).

We will focus on the case \( n = 3 \). As we have pointed out, the tangent sheaf \( \mathcal{F} \) is reflexive. Its first Chern class can be computed from (1) and equals \( 2 - d \).

We will make use of a Theorem in [17], that we state in the particular case that will be of use for us. Let \( \epsilon = \frac{d}{2} - 1 \) when \( d \) is even, and \( \epsilon = \frac{d-1}{2} - 1 \) for \( d \) odd. Let \( h^i(\mathcal{F}(\ell)) := \dim_{\mathbb{C}} H^i(\mathbb{P}^3, \mathcal{F}(\ell)) \).

**Theorem 3.1 ([17]).** The tangent sheaf \( \mathcal{F} \) is locally free if and only if \( h^2(\mathcal{F}(p)) = 0 \), for some \( p \leq \epsilon - 2 \).

If \( h^2(\mathcal{F}(p)) = 0 \) for \( p = \epsilon - 3 \) when \( d \) is even, or for \( p \in \{ \epsilon - 4, \epsilon - 3, \epsilon - 2 \} \) when \( d \) is odd, then \( \mathcal{F} \) splits.

We now obtain some results relating algebro-geometric properties of the singular scheme to the fact that the tangent sheaf is locally free, and also to its being split. For us, a curve will be an equidimensional, locally Cohen-Macaulay, dimension one subscheme of \( \mathbb{P}^n \). Given a curve \( C \) in \( \mathbb{P}^3 \), the Hartshorne-Rao module is defined to be

\[
\sum_{k \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(k)).
\]

It is well known that this module is finite dimensional as a \( \mathbb{C} \)-vector space, and that it is trivial if and only if \( C \) is arithmetically Cohen-Macaulay.

First, we characterize those foliations having locally free tangent sheaf.

**Theorem 3.2.** The tangent sheaf \( \mathcal{F} \) is locally free if and only if the singular scheme \( Z \) is a curve.

**Proof.** Suppose that the tangent sheaf is locally free, then \( Z \) has no isolated points. The reason is that, in this case, the singular scheme is given by the vanishing of the \( 2 \times 2 \) minors of the \( 3 \times 2 \) matrix corresponding to the local expression of \( \varphi \). Hence, \( Z \) is locally determinantal and Cohen-Macaulay [8]. As the codimension of \( Z \) is at least two, we are done. Observe that the singular scheme has no embedded points and no isolated points.

To prove the converse, suppose that \( Z \) satisfies the properties in the statement. We will make use of Theorem 3.1 for...
By considering the short exact sequence (1), after tensoring with \( O_{P^3}(-q) \), and taking the long exact sequence of cohomology we get:

\[
\cdots \longrightarrow H^1(P^3, F(-q)) \longrightarrow H^1(P^3, T_{P^3}(-q)) \longrightarrow H^1(P^3, \mathcal{I}_Z(d + 2 - q)) \longrightarrow \\
\cdots \longrightarrow H^2(P^3, F(-q)) \longrightarrow H^2(P^3, T_{P^3}(-q)) \longrightarrow H^2(P^3, \mathcal{I}_Z(d + 2 - q)) \longrightarrow \cdots
\]

Observe that, from Bott’s formula (see [16]), if we take \( q > 4 \), then

\[
h^1(P^3, T_{P^3}(-q)) = h^2(P^3, T_{P^3}(-q)) = 0,
\]

so

\[
H^2(P^3, F(-q)) \simeq H^1(P^3, \mathcal{I}_Z(d + 2 - q)).
\]

Now, since the Harshorne-Rao module has finite dimension, we have that

\[
H^1(P^3, \mathcal{I}_Z(d + 2 - q)) = 0
\]

for \( q \) large enough, and hence for some \( -q < \epsilon - 2 \). Hence, \( F \) is locally free.

Now, we characterize foliations whose tangent sheaf splits.

**Theorem 3.3.** Suppose \( d > 1 \). \( F \) splits if and only if \( Z \) is an arithmetically Cohen-Macaulay curve.

**Proof.** Suppose \( F \) splits. In particular, it is locally free, and hence \( Z \) is a curve. Consider the exact sequence

\[
0 \longrightarrow F(p) \xrightarrow{\varphi} T_{P^3}(p) \longrightarrow \mathcal{I}_Z(d + 2 + p) \longrightarrow 0
\]

and the cohomology exact sequence:

\[
(2) \quad \cdots \longrightarrow H^1(P^3, T_{P^3}(p)) \longrightarrow H^1(P^3, \mathcal{I}_Z(d + 2 + p)) \longrightarrow H^2(P^3, F(p)) \longrightarrow \cdots
\]

Bott’s formula tells us that \( H^1(P^3, T_{P^3}(p)) = 0 \) for every \( p \). In addition, since \( F \) splits, \( H^2(P^3, F(p)) = 0 \) for every \( p \), too. Then the Hartshorne-Rao module is trivial.

Conversely, suppose \( Z \) is an arithmetically Cohen-Macaulay curve. We use Proposition 3.1. Take \( p = \epsilon - 3 \) if \( d \) is even, and \( p = \epsilon - 2 \) if \( d \) is odd. It will be sufficient to show that

\[
H^2(P^3, F(p)) = 0.
\]

Consider the following piece of the long exact cohomology sequence (2):

\[
\cdots \longrightarrow H^1(P^3, \mathcal{I}_Z(d + 2 + p)) \longrightarrow H^2(P^3, F(p)) \longrightarrow H^2(P^3, T_{P^3}(p)) \longrightarrow \cdots
\]

and note that since the Hartshorne-Rao module is trivial, we just need to check that

\[
H^2(P^3, T_{P^3}(p)) = 0.
\]
Suppose first that $d$ is even; then, $\varepsilon - 3 = \frac{d}{2} - 4$ and by Bott’s formula, $H^2(\mathbb{P}^3, T\mathbb{P}^3(\frac{d}{2} - 4)) = 0$ for $d > 0$. If $d$ is odd, $\varepsilon - 2 = \frac{d - 7}{2}$, and so, $H^2(\mathbb{P}^3, T\mathbb{P}^3(\frac{d - 7}{2})) = 0$ for $d > -1$.

It is well known that every arithmetically Cohen-Macaulay curve is connected, so we have:

**Corollary 3.4.** If the tangent sheaf $\mathcal{F}$ splits, then $Z$ is connected.

We now obtain another characterization for the splitting of the tangent sheaf of a foliation:

**Theorem 3.5.** $\mathcal{F}$ splits if and only if it is locally free and the ideal $I = (F_0, F_1, F_2, F_3)$ is saturated.

**Proof.** From the exact sequence (1) and Euler’s sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow T\mathbb{P}^3 \longrightarrow 0$$

we get the exact sequence:

$$0 \longrightarrow \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^3}(d + 2) \longrightarrow 0.$$  

(3)

If $\mathcal{F}$ splits, $H^1(\mathbb{P}^3, \mathcal{F}(q) \oplus \mathcal{O}_{\mathbb{P}^3}(q)) = 0$ for every $q$, and so we have a surjective map:

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1 + q)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^3}(d + 2 + q))$$

(4)

By simple inspection of the map inducing this surjection we conclude that

$$H^0(\mathbb{P}^3, \mathcal{I}_Z \otimes \mathcal{O}_{\mathbb{P}^3}(d + 2 + q)) = I_{d+2+q}$$

for every $q \in \mathbb{Z}$, and therefore $I$ is saturated.

Conversely, if the ideal $I$ is saturated the map (1) is surjective. Since

$$H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 0,$$

we get that $H^1(\mathbb{P}^3, \mathcal{F}(q) \oplus \mathcal{O}_{\mathbb{P}^3}(q)) = 0$, and hence $H^1(\mathbb{P}^3, \mathcal{F}(q)) = 0$, for every $q$.

In order to apply Horrocks’ criterion for the splitting of the vector bundle $\mathcal{F}$ (see [16], p. 39), we just need to prove also that $H^2(\mathbb{P}^3, \mathcal{F}(q)) = 0$ for every $q$. By Serre’s duality

$$H^2(\mathbb{P}^3, \mathcal{F}(q)) = H^1(\mathbb{P}^3, \mathcal{F}^*(-q - 4)) \text{ for every } q.$$

But (this can be found in Hirzebruch [13])

$$\mathcal{F}^* \simeq \det \mathcal{F} \otimes \mathcal{F},$$

and so the vanishing follows from the previous calculation. \qed
Let us observe that Theorem 3.5 translates into an algebraic setting the splitting of the tangent sheaf of the foliation. Thus, it could be an effective tool in trying to solve the following important open problem, posed in [3]: Given a foliation in \( \mathbb{P}^3 \) with locally free tangent sheaf, does it split?

The question above can be restated as follows: let \( \omega = \sum_{i=0}^{3} F_i dz_i \) define a degree \( d \) foliation in \( \mathbb{P}^3 \), with locally free tangent sheaf (i.e. the scheme defined by \( F_0 = \cdots = F_3 = 0 \) is a curve), is the ideal \( (F_0, F_1, F_2, F_3) \) saturated? Furthermore, is this ideal saturated for every reduced foliation?

We note that given a reduced foliation defined by \( \omega = \sum_{i=0}^{3} F_i dz_i \), using for example the algorithm \texttt{nsatiety( )} implemented in the \textsc{Singular} library “noether.lib” [10], one immediately checks whether the ideal \( (F_0, F_1, F_2, F_3) \) is saturated or not.

Let us point out that Theorem 3.3 in the paper of Campillo and Olivares ([4]) can be restated as follows: for any foliation in \( \mathbb{P}^2 \) defined by a 1-form

\[
\omega = \sum_{i=0}^{2} F_i dx_i
\]

with singular set of codimension at least two, the ideal \( (F_0, F_1, F_2) \) is saturated. Our approach gives a much simpler proof of that result, since the tangent sheaf of such a foliation is a line bundle.

4. ON THE DETERMINATION OF A SPLIT FOLIATION BY THE SINGULAR SCHEME

Now we face the problem of deciding when a codimension one foliation in \( \mathbb{P}^3 \) is determined by its singular scheme, as stated in the first Introduction. In \( \mathbb{P}^2 \), the problem was completely solved in [4], Theorem 3.5: a degree \( r = 0 \), or \( r \geq 2 \) foliation is uniquely determined by its singular subscheme; for degree \( r = 1 \) they construct a 1-dimensional family of distinct foliations with the same singular subscheme.

In \( \mathbb{P}^3 \), we study this problem just for foliations whose tangent sheaf splits (recall that this is trivially true for foliations in the plane). We develop a method to deal with the problem, and obtain affirmative results for certain splitting types (Theorem 4.3, Theorem 4.5). We also show examples for the remaining splitting types, showing that the answer depends on the particular singular scheme \( Z \).

We begin by noting the following

**Remark 4.1.** Two split foliations defined by \( F_0 dz_0 + F_1 dz_1 + F_2 dz_2 + F_3 dz_3 \) and \( G_0 dz_0 + G_1 dz_1 + G_2 dz_2 + G_3 dz_3 \), with the same singular scheme \( Z \) are of the same degree \( d \). Indeed, it is enough to observe that by Theorem 3.5 the homogenous ideal \( (F_0, \ldots, F_3) = (G_0, \ldots, G_3) \) is saturated.
We now prove a lemma that contains a basic idea for the problem. Recall (see, for example [6]) that a codimension one holomorphic distribution is defined just in the same way as a foliation, removing the integrability condition.

**Lemma 4.2.** Let $\mathcal{F}_\phi$ be a foliation defined by a projective integrable 1-form $\omega = \sum_{i=0}^{3} F_i dz_i$ with split tangent sheaf, and singular scheme $Z$. Any distribution $\mathcal{F}'_{\phi'}$, with singular scheme $Z'$, is split, and induces a linear syzygy

$$\ell_0 F_0 + \ell_1 F_1 + \ell_2 F_2 + \ell_3 F_3 = 0, \quad \ell_i \in \mathbb{C}[z_0, z_1, z_2, z_3], \quad \deg \ell_i = 1, \ i = 0, \ldots, 3.$$

**Proof.** First note that Theorem 3.3 applies for distributions, as the integrability does not enter in the proof; thus we conclude that $\mathcal{F}'$ splits. Then

$$\mathcal{F}' = \mathcal{O}_{\mathbb{P}^3}(a') \oplus \mathcal{O}_{\mathbb{P}^3}(b'), \quad \text{with} \quad a' + b' = 2 - d, \ and \ a' \leq b' \leq 1.$$

Note that $a' \leq b' \leq 1$ follows from the stability of $T \mathbb{P}^3$ (see [16]). Furthermore, $a = a'$ and $b = b'$, as can be deduced by considering the long cohomology sequences obtained from (1) and

$$0 \longrightarrow \mathcal{F}' \longrightarrow T \mathbb{P}^3 \longrightarrow I_Z \otimes \mathcal{O}_{\mathbb{P}^n}(d+2) \longrightarrow 0,$$

as we have that $h^0(\mathbb{P}^3, I_Z(\ell))$ equals

$$h^0(\mathbb{P}^3, T \mathbb{P}^3(\ell - d - 2)) - h^0(\mathbb{P}^3, \mathcal{F}(\ell - d - 2)),$$

and also

$$h^0(\mathbb{P}^3, T \mathbb{P}^3(\ell - d - 2)) - h^0(\mathbb{P}^3, \mathcal{F}'(\ell - d - 2))$$

for every $\ell \in \mathbb{Z}$.

The singular subscheme of the distribution $\mathcal{F}'_{\phi'}$ is $Z$, and it is defined by the ideal $(F'_0, F'_1, F'_2, F'_3)$, where the $F'_i$ are degree $d + 1$ homogeneous polynomials such that $\omega' = \sum_{i=0}^{3} F'_i dz_i$ defines $\mathcal{F}'_{\phi'}$.

Hence, there is a $4 \times 4$ matrix $M = (a_{ij}) \in \text{GL}(4, \mathbb{C})$ such that

$$M \cdot \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} F'_0 \\ F'_1 \\ F'_2 \\ F'_3 \end{pmatrix}.$$

Now, the Euler condition $\sum_{j=0}^{3} z_j F'_j = 0$, gives

$$(z_0 z_1 z_2 z_3) \cdot M \cdot \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} = 0,$$

which produces a linear syzygy:

$$\ell_0 F_0 + \ell_1 F_1 + \ell_2 F_2 + \ell_3 F_3 = 0.$$
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Now then, $H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3})$ is the vector space of linear syzygies because of (3).

So such a distribution as in the Lemma above gives a 1-dimensional linear subspace of the vector space of linear syzygies $H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3})$. Conversely, given a linear subspace of $H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3})$ of dimension one, we can take a generator and express it in the form

$$(z_0 \ z_1 \ z_2 \ z_3) \cdot M.$$ 

If $M \in GL(4, \mathbb{C})$, then equation (6) gives coefficients for a homogeneous 1-form defining a distribution with the same singular scheme. Therefore, we can assure that the family of distributions with the same singular scheme as the foliation $\mathcal{F}_\varphi$ is parameterized by a Zariski open subset $D_{\mathcal{F}_\varphi}$ of $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3})) \subset \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4}))$, obtained after removing the algebraic subset corresponding to non-invertible matrices.

Therefore, foliations sharing the singular scheme $Z$ correspond to an algebraic subset of $D_{\mathcal{F}_\varphi}$, defined by the equations expressing the integrability condition.

Let us just point out that if we find a basis of $H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3})$ it is easy to write explicit equations for the integrability condition.

Now we present our first result on the characterization of a foliation by its singular scheme:

**Theorem 4.3.** Let $\mathcal{F}_\varphi$ be a degree $d$ reduced foliation in $\mathbb{P}^3$, with $\mathcal{F}$ a rank two split vector bundle, defined by a projective integrable one form $\omega = \sum_{i=0}^3 F_i dz_i$. Let $Z$ be its singular subscheme. Suppose that

$$\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b), \quad \text{with } a \leq b \leq -1, \ a + b = 2 - d.$$ 

Then, if $\mathcal{F}'_{\varphi'}$ is another foliation in $\mathbb{P}^3$ with the same singular subscheme $Z$, defined by the form $\omega' = \sum_{i=0}^3 F'_i dz_i$, we have $\omega = \lambda \omega'$ for $\lambda \in \mathbb{C}^*$.

**Proof.** By Lemma 4.2, we get a linear syzygy

$$\ell_0 F_0 + \ell_1 F_1 + \ell_2 F_2 + \ell_3 F_3 = 0.$$ 

From the sequence \[3\), we conclude that this syzygy is a multiple of the Euler relation $\sum_{j=0}^3 z_j F_j = 0$, so there is a $\lambda \in \mathbb{C}^*$ with $M = \lambda Id$ and also $\omega' = \lambda \omega$. 

We analyze now the other possible splitting types. To start with we prove,

**Proposition 4.4.** Let $\mathcal{F}_\varphi$ be a degree $d$ reduced foliation in $\mathbb{P}^3$. $\mathcal{F}_\varphi$ is the linear pull-back of a degree $d$ reduced foliation in $\mathbb{P}^2$ if and only if $\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1 - d)$.

**Proof.** When $\mathcal{F}_\varphi$ is a linear pull-back it is known (see \[3\) that the tangent sheaf splits, and that the splitting type is as in the statement of the theorem. Indeed, note that $\mathcal{O}_{\mathbb{P}^3}(1)$ is the tangent sheaf of the foliation whose leaves are the fibers of the projection, and the splitting is given by the projection.
For the converse, suppose that the tangent sheaf of $\mathcal{F}_\varphi$ splits as

$$\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1 - d).$$

Then, by first tensoring with $\mathcal{O}_{\mathbb{P}^3}(-1)$ and considering the exact cohomology sequence, we get four complex numbers $a_0, \ldots, a_3$ (not all of them equal to zero) such that $a_0F_0 + a_1F_1 + a_2F_2 + a_3F_3 = 0$. Suppose $a_0 \neq 0$.

If we change coordinates:

$$\begin{align*}
  z_0 &= a_0 z'_0 \\
  z_1 &= z'_1 + a_1 z'_0 \\
  z_2 &= z'_2 + a_2 z'_0 \\
  z_3 &= z'_3 + a_3 z'_0
\end{align*}$$

we get a new expression for the form defining $\mathcal{F}_\varphi$:

$$\eta = G_1 dz'_1 + G_2 dz'_2 + G_3 dz'_3$$

where $G_1$, $G_2$ and $G_3$ are homogeneous in the new variables.

Let us prove that $G_i \in \mathbb{C}[z'_1, z'_2, z'_3]$. To do that, we go to the affine open $z'_3 = 1$, and get a 1-form

$$\eta_3 = G_1(z'_0, z'_1, z'_2, 1) dz'_1 + G_2(z'_0, z'_1, z'_2, 1) dz'_2$$

that defines the foliation in that affine chart. To simplify, we write:

$$f(z'_0, z'_1, z'_2) = G_1(z'_0, z'_1, z'_2, 1), \quad g(z'_0, z'_1, z'_2) = G_2(z'_0, z'_1, z'_2, 1).$$

As the singular set of the foliation is of codimension two, $f$ and $g$ have no common irreducible factor.

Note that $\eta_3$ is integrable (i.e. $\eta_3 \wedge d\eta_3 = 0$), and hence we have:

$$\frac{\partial f}{\partial z'_0} g = \frac{\partial g}{\partial z'_0} f.$$

From the equality above, and as $f$ and $g$ have no common irreducible factor, it follows that $f$ divides $\frac{\partial f}{\partial z'_0}$ and hence $\frac{\partial f}{\partial z'_0} = 0$. Analogously, $\frac{\partial g}{\partial z'_0} = 0$. Thus $f$ and $g$ do not depend on $z'_0$. Proceeding in a analogous way in another affine open set, we can deduce that $G_0, G_1$ and $G_2$ are in $\mathbb{C}[z'_1, z'_2, z'_3]$.

We can finally conclude that $\mathcal{F}_\varphi$ is a linear pull-back of the foliation given by

$$\eta = G_1 dz'_1 + G_2 dz'_2 + G_3 dz'_3$$

in the projective plane $z'_0 = 0$. \hfill $\square$

Now, we come back to the problem of deciding whether the singular scheme characterizes split foliations and prove:
Theorem 4.5. Let \( \mathcal{F}_\varphi \) be a degree \( d \neq 1 \) reduced foliation in \( \mathbb{P}^3 \), with
\[
\mathcal{F} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1 - d),
\]
and let \( Z \) be its singular scheme. There is no other foliation with singular scheme \( Z \).

Proof. Suppose there is another foliation \( \mathcal{G}_\psi \) with singular scheme \( Z \). As \( Z \) is an arithmetically Cohen-Macaulay curve, its tangent sheaf \( \mathcal{G} \) splits, with the same splitting type as that of \( \mathcal{F} \). Hence, from the previous Proposition 4.4 both \( \mathcal{F}_\varphi \) and \( \mathcal{G}_\psi \) are linear pull-backs of two foliations of a projective plane in \( \mathbb{P}^3 \).

As \( Z \) is a cone, taking any plane \( H \subset \mathbb{P}^3 \) not passing through the vertex \( p \) of \( Z \), we obtain both \( \mathcal{F}_\varphi \) and \( \mathcal{G}_\psi \) as linear pull-backs from \( p \) of two degree \( d \) foliations in \( H \) (see for example Section 2 in \([5]\)). Now, as \( d \neq 1 \), by Theorem 3.5 in \([4]\) these two foliations are the same, as their common singular subscheme is \( H \cap Z \).

Finally, after Theorems 4.3 and 4.5 there is just one splitting type for each degree \( d \) that remains to be considered:
\[
\mathcal{F} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2 - d).
\]

We can make the following remarks:

- If \( d = 1 \), a foliation with splitting type \( \mathcal{F} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \) is a linear pull-back (by Proposition 4.4). The intersection with a general plane \( H \) gives a degree 1 foliation in \( H \) whose singular scheme determines \( Z \). However (see Remark 3.6 in \([4]\)), there is a one dimensional family of different foliations in the plane \( H \) with the same singular scheme, giving different foliations in \( \mathbb{P}^3 \) by pull-back with \textit{the same singular scheme as} \( \mathcal{F}_\varphi \). In fact, for any degree one foliation in \( \mathbb{P}^2 \), using the immediate translation for dimension 2 of Lemma 4.2 we conclude that the family of plane foliations with the same singular scheme is of dimension one.

- If \( d = 2 \), for a foliation \( \mathcal{G}_\psi \) with split tangent sheaf \( \mathcal{G} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \) there is a pair of linear vector fields \( X \) and \( Y \) in \( \psi(\mathcal{G}) \) generating a Lie algebra. According to the classification, this can be abelian or isomorphic to the affine Lie algebra.

  In the abelian case, we can diagonalize simultaneously both vector fields, and the resulting foliation is logarithmic of type \( \mathcal{L}(1, 1, 1, 1) \), as can be seen in \([6]\). Note that the other possible type of degree 2 logarithmic foliations \( \mathcal{L}(1, 1, 2) \) have a tangent sheaf which is not locally free, as it contains isolated points (see \([7]\)).

  A foliation in \( \mathcal{L}(1, 1, 1, 1) \) is defined by a form (see \([2]\)):
\[
\omega = \ell_0 \ell_1 \ell_2 \ell_3 \sum_{i=0}^{3} \lambda_i \frac{d\ell_i}{\ell_i},
\]
where the \( \ell_i \) are linear forms in general position, and the scalars satisfy

\[
\lambda_i \in \mathbb{C}^*, \quad \sum_{i=0}^{3} \lambda_i = 0.
\]

Its singular scheme \( Z \) is given by six lines giving the edges of a tetrahedron obtaining by intersecting any two of the \( \ell_i \).

Now, the same linear forms \( \ell_i \) with different choices of scalars satisfying (7) provide distinct foliations with the same singular scheme, as these determine the holonomy of the foliation.

If the Lie algebra generated by the linear vector fields is isomorphic to the affine Lie algebra, after a linear change of coordinates we can choose generators \( X' \) and \( Y' \) with

\[ [X', Y'] = Y'. \]

Thus, we are in the setting of the exceptional component of degree 2 foliations in \( \mathbb{P}^3 \), introduced by Cerveau and Lins Neto (see [5]): a general member is given by a split foliation \((\lambda, 1)\), with tangent sheaf \( \mathcal{F} = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3} \), and they prove that this foliation \( \mathcal{F}_\varphi \) is rigid (i.e. an open dense subset of the component is described by the action of the group \( PGL(4, \mathbb{C}) \) on that foliation).

The singular scheme \( Z \) has three irreducible components: a line \( \ell \), a conic \( C \) tangent to \( \ell \) at a point \( p \), and a twisted cubic with the line \( \ell \) as an inflection line at \( p \).

In this setting, the reader can check that the equations expressing the integrability, obtained from the basis of \( H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3}) \) given by \( X', Y' \) (see their explicit expression in [5]) and the radial vector field \( R \) have a unique solution: the one corresponding to the foliation \( \mathcal{F}_\varphi \) itself.

\[ \bullet \] If \( d > 2 \), we have that \( \dim H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3}) = 2 \). In [3], it is proven that there is an exceptional component of the space of (codimension one) degree \( d \) foliations in \( \mathbb{P}^3 \), whose general member is a foliation \( \mathcal{F}_\varphi \) with split tangent sheaf of the form \( \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(2 - d) \).

\( \mathcal{F}_\varphi \) is associated to a representation of the affine Lie algebra, in such a way that in an affine open set, \( \mathcal{F}_\varphi \) is defined by a one form \( \omega = iS \ i_X (dVol) \), where \( S \) is a linear vector field and \( X \) is quasi-homogeneous:

\[
S = (1 + d + d^2)z_1 \frac{\partial}{\partial z_1} + (1 + d)z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3},
\]

\[
X = (1 + d + d^2)z_2 \frac{\partial}{\partial z_1} + (1 + d)z_3 \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}.
\]

Thus, we can take \( R \) and \( S \) as a basis of the vector space \( H^0(\mathbb{P}^3, \mathcal{F} \oplus \mathcal{O}_{\mathbb{P}^3}) \).

The vector field \( S \) corresponds to a linear syzygy that can be interpreted as a matrix \( M \in GL(4, \mathbb{C}) \) acting on the coefficients of the projective form \( \varphi = \).
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$n\sum_{i=0}^{3} F_i dz_i$, extending

$\omega = (1+d)(z_2-z_3^{d+1})dz_1-(1+d+d^2)(z_1-z_2^d z_3)dz_2-(1+2d+2d^2+d^3)(z_2^d+1-z_1 z_3^d)dz_3$

to projective space, and defining $F_\phi$. By direct computation, we note that the form $\omega_1$ with coefficients given by the equation (6) is not integrable.

Hence, any distribution with the same singular scheme $Z$ as the foliation $F_\phi$ (see [3] for its explicit geometric description) can be defined by a form $\alpha_0 \omega_0 + \alpha_1 \omega_1$, where $\omega_0 = \bar{\omega}$. The integrability condition is

$$\alpha_0 \alpha_1 \omega_0 \wedge d\omega_1 + \alpha_0 \alpha_1 \omega_1 \wedge d\omega_0 + \alpha_1^2 \omega_1 \wedge d\omega_1.$$ 

An explicit computation shows that the only solution is $\alpha_1 = 0$, and so the only foliation with singular scheme $Z$ is $F_\phi$.

In principle, we do not know whether there are other irreducible components of the space of foliations with the same split tangent sheaf. In fact, despite the recent important results in the literature, the knowledge of these spaces is still quite incomplete.

Remark 4.6. Observe that the proofs of the results in Sections 2 and 3 except those of Proposition 4.4 and Theorem 4.5 could be immediately adapted to deal with singular distributions, since no use is made of the integrability condition.

Note also that we can extend our approach to foliations in $\mathbb{P}^n$, and get immediate results: Theorem 3.5, Lemma 4.2 and Theorem 4.3 are valid in that setting.

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