Footprints of Geodesics in Persistent Homology

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Abstract. Given a metric space $X$ and a subspace $A \subset X$, we prove that $A$ can generate various algebraic elements in persistent homology of $X$. We call such elements (algebraic) footprints of $A$. Our results imply that footprints typically appear in dimensions above $\dim(A)$. Higher dimensional persistent homology thus encodes lower dimensional geometric features of $X$. We pay special attention to a specific type of geodesics in a geodesic surface $X$ called geodesic circles. We explain how they may generate non-trivial odd-dimensional and two-dimensional footprints. In particular, we can detect even some contractible geodesics using two- and three-dimensional persistent homology. This provides a link between persistent homology and the length spectrum in Riemannian geometry.

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1. Introduction

Algebraic topology is based on the notions of homotopy and homology groups. These invariants have proved invaluable in most parts of mathematics. They were designed to detect holes of certain dimension, either through maps from spheres or via (co)chains. By construction, the $k$-dimensional homology $H_k$ of an $n$-dimensional complex is trivial for all $k > n$. The same was assumed to be true for homotopy groups, until the famous discovery [22] of the Hopf map, which demonstrated that $\pi_3(S^2)$ is non-trivial. This result completely transformed the perception of the homotopy groups, had profound consequences in mathematics, and launched an extensive research effort to understand the underlying machinery.

At the turn of the century, persistent homology appeared in the setting of computational topology, motivated in part by problems of data analysis of geometric shapes. The idea was to obtain a theoretical and computational...
framework, which would detect and measure sizes of holes. The resulting persistent homology has since had a profound impact on the development of theory and on applications in other sciences. At the heart of this approach is a filtration of simplicial complexes, often arising from Čech or Rips filtrations via increasing scale parameters. As decades before in algebraic topology, the intuition is that features of persistent homology in dimension $k$ correspond to $k$-dimensional holes in the shape. This intuition is well justified by the Nerve Theorem if our underlying space lies in the Euclidean space and we use the Euclidean distance to build the Čech filtration. However, it is known [9] that in the setting of Rips complexes, such expectation is overly optimistic. Furthermore, the ambient Euclidean metric is usually chosen for reasons of computational simplicity. In many cases, it might be more natural to consider a geodesic metric on the underlying space, which is the basic structure of Riemannian geometry, or perhaps some other statistically motivated distances [16] (for example the relative entropy, which is not even a metric; or the Fisher metric, which is in fact a geodesic metric).

In this paper, we will focus on persistent homology via Čech and Rips filtrations of geodesic spaces. Our starting point is a surprising deep Theorem 2.1, recently proved in [1]. The theorem states that the mentioned open filtrations of geodesic $S^1$ first attain the homotopy type of $S^1$, then of $S^3$, $S^5$, etc. The theorem thus plays a role of the Hopf map, demonstrating that 1-dimensional space $S^1$ generates non-trivial footprints in persistent higher dimensional homology. Consequently, it carries the same lesson for the perception of persistent homology as the original Hopf map had for the algebraic topology: non-trivial elements in persistent homology in dimension $k$ may be generated by features of dimension less than $k$.

In this paper, we intend to make use of this Hopf effect to extract new geometric information about the underlying geodesic space $X$ from persistent homology:

- First, we introduce a general framework for footprint detection (Theorems 4.5 and 5.3), i.e., a result stating that for suitable subspaces $A \subset X$, a part of the persistent homology of $A$ is contained in the persistent homology of $X$.
- We then prove Theorem 7.1, stating that for a loop $A$ on a surface $X$, a footprint of $A$ may sometimes contain a 2-dimensional interval not contained in the persistent homology of $A$.

In particular, we show how certain loops (appropriately isometrically embedded geodesic circles of positive circumference, regular polygons, and some ellipses) can be detected using persistent homology, even if they are contractible. For a graphical example of these results, see Fig. 1. An application of these results is demonstrated in Sect. 9.

Our eventual far-reaching goal, which is probably unattainable in its full generality, is understanding the correspondence between geometric features of a spaces and algebraic elements of persistent homology. We would like to know which geometric features generate footprints in persistent homology and how to recreate them. This paper establishes such a correspondence for nicely
embedded geodesic circles. There are now three different types of footprints that might be induced by a geodesic circle on a geodesic surface:

- A 1-dimensional **topological footprint** appearing if the loop is a member of a lexicographically shortest homology basis [27].
- A 3-dimensional (and higher dimensional) **combinatorial footprint** arising from the internal combinatorics of the Rips complex of a circle as described by Theorem 4.5.
- A 2-dimensional **geometric footprint** appearing under certain local geometric conditions at the loop in the absence of a topological footprint as described by Theorem 7.1.

Further footprint detection results could be obtained if we could compute the persistent homology of simple spaces, such as spheres, etc.

The results of this paper have further theoretical and practical consequences, some of which we plan to explore in the future work:

1. By results of [27], the critical values of the persistent fundamental group \( \{\pi_1(\text{Rips}(X, \bullet), r)\}_{r>0} \), which are in general incomputable due to the word problem in groups, correspond to geodesic circles. Results of this paper indicate that candidates for these critical values can in some cases be extracted from higher dimensional homology.

2. The collection of lengths of closed geodesics features prominently in differential geometry under the name of the length spectrum (for a modern treatment, see [23, Section 7.2] or [19]). It is closely related to the Laplacian spectrum and in some cases, even to the volume of the manifold. The results of this paper detect a part of this spectrum arising from geodesic circles. By extending our method, we hope establish a result describing how much of the length spectrum is encoded in the persistent homology.

3. As was already mentioned in [27], the setting of geodesic spaces provides a convenient venue for topological data analysis via persistent homology for a number of reasons: filtrations are smaller, they seem to be more stable [28] and seem to contain less noise (see example of Sect. 9), there seems to be an inherent structure to the corresponding persistence diagram, etc.

**Related work:** One of the first appearances of filtrations goes back to the introduction of Čech (co)homology. The approach later evolved into the Shape Theory [15], which studies the limiting behaviour as \( r \to 0 \). In the analogous setting, such approximations with \( r \to \infty \) have been employed in asymptotic topology [8,14]. Reconstructing the homotopy type of a manifold for small \( r \) was considered in [21]. Similar reconstructions in the geodesic setting were considered in [24] and are in general a subject of study in computational topology [6,26]. The study of filtrations for all values of \( r \) in the geodesic setting started with the case of \( S^1 \) in [1]. The 1-dimensional persistence was first considered in [18] for metric graphs, and completely developed in [27,28]. Further results that can be used in conjunction with our footprint detection procedure contain ellipses [2] and regular polygons [3]. A connection between persistent homology and some geometric notions in simplicial
Figure 1. An example of a footprint detection. The upper left side represents the homotopy type of the Vietoris–Rips filtration of a circle (i.e., odd-dimensional spheres) equipped with a geodesic metric by Theorem 2.1. The black shape on the right is a two-dimensional torus, and below, it is an excerpt from its barcode as retrieved from our results. Geodesic circle $a$ is a member of the shortest homology base and hence generates odd-dimensional spheres and bars by Theorem 4.5. Geodesic circle $c$ is not a member of such a base; hence, it generates higher odd-dimensional bars along with a 2-dimensional bar by Theorem 7.1.

complexes has recently been treated in [5]. A connection to the filling radius has been established in [25].

The structure of the paper is the following. In Sect. 2, we provide preliminaries. Section 3 introduces deformation contractions as our tool. Sections 4 and 5 provide a footprint detection framework in a specific and general setting, respectively. Section 6 describes the combinatorics of nullhomologies of loops. Section 7 is the most technical section, describing an emergence of a two-dimensional footprint. Section 8 extends the results to Čech complexes and closed filtrations. Section 9 provides an example of the interpretation using the results of this paper.
2. Preliminaries

Let \((X,d)\) be a metric space and \(r > 0\). For \(x \in X\), let \(B(x,r)\) and \(\overline{B}(x,r)\) denote the open and closed balls centred at \(x\) of radius \(r\). For \(A \subset X\), notation \(N_X(A,r)\) (or \(N(A,r)\), when it is clear what the ambient space is) represents the open neighborhood, and \(\overline{N}(A,r)\) represents the closed neighborhood around \(A\) of radius \(r\) in \(X\).

A space \(X\) is \textbf{geodesic}, if for each distinct \(x,y \in X\), there exists an isometric embedding \(g: [0,d(x,y)] \to X\) with \(g(0) = x\) and \(g(d(x,y)) = y\), i.e., if \(x\) and \(y\) are connected by a path of length \(|g| = d(x,y)\), which is called a \textbf{geodesic} (in the literature, the notion of a geodesic sometimes refers to what we would call a local geodesic, which differs from our notion of geodesic).

A \textbf{geodesic surface} is a surface equipped with a geodesic metric. A subset \(A\) of a geodesic space \(X\) is \textbf{geodesically convex}, if for each \(x,y \in A\), every geodesic between \(x\) and \(y\) in \(X\) is contained in \(A\). A \textbf{geodesic circle} in \(X\) is an isometrically embedded circle \((S,d_S) \hookrightarrow (X,d_X)\), where \(d_S\) is a geodesic metric. If \(G\) is an Abelian group and \(\alpha\) is a loop in \(X\), then \([\alpha]|_G \in H_1(X,G)\) is the homology element represented by \(\alpha\).

We next define the height of homotopy, which is motivated by the combinatorial version of the height in [10]. Given a homotopy \(H: S^1 \times I \to X\) between loops \(H_{S^1 \times \{0\}}\) and \(H_{S^1 \times \{1\}}\), its \textbf{height} is the length of the longest intermediate curve \(H_{S^1 \times \{t\}}\). A \textbf{homotopy height} between homotopic loops is the infimum of the heights of all homotopies between the loops. Similarly, a nullhomotopy height of a contractible loop is the infimum of the heights of all nullhomotopies.

Given \(r > 0\), we define various simplicial complexes with the vertex set \(X\). For a longer discussion on the subject, see [29]. The condition next to the name determines when a finite subset \(\sigma \subset X\) belongs to the complex.

1. \textbf{(Open) Rips} (or Vietoris-Rips) complex \(\text{Rips}(X,r)\): \(\text{Diam}(\sigma) < r\).
2. \textbf{Closed Rips complex} \(\overline{\text{Rips}}(X,r)\): \(\text{Diam}(\sigma) \leq r\).
3. \textbf{(Open) Čech complex} \(\text{Čech}(X,r)\): \(\cap_{z \in \sigma} B(z,r) \neq \emptyset\).
4. \textbf{Closed Čech complex} \(\overline{\text{Čech}}(X,r)\): \(\cap_{z \in \sigma} \overline{B}(z,r) \neq \emptyset\).

When considering Čech complexes of subsets \(A \subset X\), we need to specify where we look for an intersection. If the context is not clear, then \(\text{Čech}_X(A,r)\) consists of finite subsets \(\sigma \subset A\) for which \(\cap_{z \in \sigma} B_X(z,r) \neq \emptyset\) in \(X\); similarly, \(\text{Čech}_A(A,r)\) consists of finite subsets \(\sigma \subset A\) for which \(\cap_{z \in \sigma} B_A(z,r) \neq \emptyset\) in \(A\). The same goes for the closed Čech complexes. Note that if \(\alpha\) is a geodesic circle in \(X\), then \(\text{Čech}_X(\alpha,r) = \overline{\text{Čech}}_\alpha(\alpha,r)\) for all \(r > 0\).

We will refer to 1-dimensional simplices as edges and to 2-dimensional simplices as triangles. For a simplex \(\sigma\) in a Čech complex, we refer to any point \(w\) of \(\cap_{z \in \sigma} B(z,r)\) as a witness of \(\sigma\), or say that \(w\) witnesses \(\sigma\).

Given any complex \(\mathcal{C}\) mentioned above, we construct a corresponding \textbf{filtration} \(\{\mathcal{C}(X,r)\}_{r>0}\) as a collection of complexes \(\mathcal{C}(X,r)\) for all positive parameters, bound together by the bonding inclusions \(i_{p,q}: \mathcal{C}(X,p) \hookrightarrow \mathcal{C}(X,q)\), which are identities on the vertices for all \(p < q\). Two filtrations
that it is of circumference 1.

Theorem 2.1. Suppose $S$ is a circle, equipped with a geodesic metric, so that it is of circumference 1. Then, for all $l = 0, 1, 2, \ldots$

$\text{Rips}(S, r) \simeq S^{2l+1}$, for $\frac{l}{2l+1} < r \leq \frac{l+1}{2l+3}$,

$\text{Cech}(S, r) \simeq S^{2l+1}$, for $\frac{l}{2(l+1)} < r \leq \frac{l+1}{2(l+2)}$,

$\text{Rips}(S, r) \simeq \left\{ \begin{array}{ll}
S^{2l+1}, & \text{for } \frac{l}{2l+1} < r < \frac{l+1}{2l+3}, \\
\sqrt{\mathbb{R}} S^{2l}, & \text{for } r = \frac{l}{2l+1},
\end{array} \right.$

$\text{Cech}(S, r) \simeq \left\{ \begin{array}{ll}
S^{2l+1}, & \text{for } \frac{l}{2(l+1)} < r < \frac{l+1}{2(l+2)}, \\
\sqrt{\mathbb{R}} S^{2l}, & \text{for } r = \frac{l}{2(l+1)}.
\end{array} \right.$

Furthermore, the bonding maps on filtrations are homotopy equivalences whenever possible.

For $r \geq 1/2$, all the mentioned complexes are contractible.

For a loop (or a path) $\alpha: I \to X$ and $r > 0$, an $r$-sample of $\alpha$ is a sequence $\alpha(t_0), \alpha(t_1), \alpha(t_2), \ldots, \alpha(t_k)$, where $t_0 = 0 < t_1 < \ldots < t_k = 1$ and for each $i$, $\text{Diam}(\alpha|_{[t_i, t_{i+1}]}) < r$ holds. See [27, Section 3] for details and properties of $r$-samples. We will often identify an $r$-loop with the simplicial cycle in $\text{Rips}(X, r)$ consisting of edges $[t_i, t_{i+1}]$. We will often utilise a transition from the continuous setting of $X$ to the discrete setting of Rips and Čech complexes as introduced in [27] via $r$-samples of loops.

By persistence, we mean any object, obtained by applying any homology group functor to any filtration. For example, a $H_1(\_, \mathbb{Z})$ persistence via the closed Čech filtration of $X$ is a collection $\{H_1(\text{Čech}(X, r), \mathbb{Z})\}_{r > 0}$ along with the induced bonding maps. In the paper, we will sometimes consider restrictions of parameter $r$. We will often be using relations (such as isomorphisms) and operations (such as direct sums) on such persistences: such operations will always consist of level-wise maps, which are consistent (i.e., commutative) with the bonding maps. Where there is no ambiguity about the coefficients, such as in the proofs, we will omit them from the notation of homology. For an Abelian group $G$, the maps induced by the bonding maps $i_{p,q}$ on homology with coefficients in $G$ are denoted by $i^G_{p,q}$. Given $\epsilon > 0$, an $\epsilon$-interleaving between two filtrations $\{A_r\}_{r > 0}$ and $\{B_r\}_{r > 0}$ consists of collections of maps $\{f_r: A_r \to B_{r+\epsilon}\}_{r > 0}$ and $\{g_r: B_r \to A_{r+\epsilon}\}_{r > 0}$ that commute with the bonding maps.

For $b \leq d$, the notation $\langle b, d \rangle$ represents an interval. We use this notation when we do not want to commit to a specific type of endpoints of an interval. In particular, $\langle b, d \rangle$ can be either $(b, d)$ or $[b, d)$, etc.
For the rest of this section, let us assume $G$ is a field. Given an interval $\langle b, d \rangle \subseteq (0, \infty)$, $G_{\langle b, d \rangle}$, is a collection of vector spaces $\{U_r\}_{r>0}$ defined by

$$U_r = \begin{cases} G, & r \in \langle b, d \rangle \\ 0, & r \notin \langle b, d \rangle \end{cases}$$

(2.1)

and by setting all bonding maps $U_r \rightarrow U_s$, with $r, s \in \langle b, d \rangle$, to be isomorphisms. These are called (elementary) interval modules, intervals, or just bars. For compact spaces $X$, the $k$-dimensional $H_k(\cdot, G)$ persistence decomposes as a direct sum of such bars, which together constitute a barcode. A decomposition also exists for any persistence obtained through open Čech or Rips filtration (see q-tameness condition in Proposition 5.1 of [12] and the property of being radical in [11] for details). A persistence diagram PD is an alternative description of a barcode. It consists of a collection of points in a plane, one point corresponding to each such bar, with the coordinates of a point being the left endpoint (birth) and the right endpoint (death) of the corresponding bar.

Given a compact geodesic locally contractible space, there exists (see Definition 8.7 and Proposition 8.9 of [27] for details) a lexicographically minimal basis of $H_1(X, G)$ consisting of a finite collection of geodesic circles $a_1, a_2, \ldots, a_k$ with $|a_1| \leq |a_2| \leq \cdots \leq |a_k|$, such that:

1. for each $i$ homology class $[a_i]$ is not an element of the subgroup of $H_1(X, G)$ generated by $[a_1], [a_2], \ldots, [a_{i-1}]$, and
2. homology classes $[a_1], [a_2], \ldots, [a_k]$ generate $H_1(X, G)$.

The lexicographical minimality refers to the fact that if $b_1, b_2, \ldots, b_m$ with $|b_1| \leq |b_2| \leq \cdots \leq |b_m|$ is another collection of loops in $X$ satisfying (1) and (2) above and with $[b_1], [b_2], \ldots, [b_m]$ being lexicographically smaller than $([a_1], [a_2], \ldots, [a_k])$, then $m = k$ and $|b_i| = |a_i|, \forall i$.

**Theorem 2.2.** [27]. Given a compact geodesic locally contractible space, a field $G$, and a lexicographically minimal basis of $H_1(X, G)$ consisting of a finite collection of geodesic circles $a_1, a_2, \ldots, a_k$ with $|a_1| \leq |a_2| \leq \cdots \leq |a_k|$, persistence $\{H_1(\text{Rips}(X, r), G)\}_{r>0}$ is isomorphic to the direct sum of interval modules $G_{\langle 0, |a_1|/3 \rangle}$, $G_{\langle 0, |a_2|/3 \rangle}$, $\ldots$, $G_{\langle 0, |a_k|/3 \rangle}$.

### 3. Deformation Contractions

Crushings were first defined in [21] as a type of maps inducing homotopy equivalences on the corresponding Rips complexes. In this paper, we will refer to them as deformation contractions and prove that they also induce homotopy equivalences on the corresponding Čech complexes. In our context, deformation contractions are the crucial tool connecting metric properties of a space to the homotopy properties of the corresponding Rips or Čech complexes.

**Definition 3.1.** [21]. Let $X$ be a metric space and $A \subset X$. A continuous map $F : X \times [0, 1] \rightarrow X$ is called a deformation contraction (we will abbreviate it as DC and write $X \xrightarrow{\text{DC}} A$) if:
(1) $F(x, 0) = x, F(x, 1) \in A, F(a, t) = a, \forall x \in X, a \in A, t \in [0, 1]$, and
(2) $d(F(x, t'), F(y, t')) \leq d(F(x, t), F(y, t)), \forall x, y \in X, t' > t$.

If additionally $d(F(x, t'), F(y, t')) < d(F(x, t), F(y, t))$ holds for all pairs $(x, y) \in (X \setminus A) \times X$ with $x \neq y$ and for all $t' > t$, then $F$ is called a strict deformation contraction (SDC or $X \xrightarrow{SDC} A$).

It is easy to see that if $X$ is geodesic and $X \xrightarrow{SDC} A$, then $A$ is geodesically convex in $X$. Furthermore, if $X$ is geodesic and $X \xrightarrow{DC} A$, then $A$ equipped with the subspace metric is a geodesic space. If $N(A, r) \xrightarrow{DC} A$, where $A \subset X$ is a subspace, then $\text{Čech}_A(A, r) = \text{Čech}_X(A, r)$. We will be using this last fact generously throughout the paper whenever Čech complexes will be involved.

Proposition 3.2 for Rips complexes was first proved in [21]. Here, we present an adaptation of that proof to the case of Čech complexes.

**Proposition 3.2.** Suppose $X \xrightarrow{DC} A$. Then, the inclusions $\text{Rips}(A, r) \hookrightarrow \text{Rips}(X, r)$ and $\text{Čech}(A, r) \hookrightarrow \text{Čech}(X, r)$ are homotopy equivalences for each $r > 0$.

**Proof.** As mentioned above, we will only prove the Čech case by adjusting Hausmann’s proof.

We will prove the following claim: for each pair of finite simplicial complexes $K_0 \leq K$, each simplicial map

$$g : (K, K_0) \rightarrow (\text{Čech}(X, r), \text{Čech}(A, r))$$

is homotopic (rel $K_0$) to a simplicial map $f : (K, K_0) \rightarrow (\text{Čech}(X, r), \text{Čech}(A, r))$ with $f(K) \subset \text{Čech}(A, r)$. The claim implies that the induced maps on the homotopy groups are isomorphisms and the conclusion of the proposition follows from the Whitehead theorem as in [21] (see [20] for the necessary background in algebraic topology).

Define landmarks $L = g(K^{(0)}) \subset X$, and for each simplex $\sigma \in K$, choose a witness $w_\sigma \in \cap_{x \in \sigma} B(z, r) \subset X$. Choose also some $\varepsilon > 0$ with $\varepsilon < r - \max_{\sigma \in K} \max_{x \in \sigma} d(z, w_\sigma)$. At last, choose $p \in \mathbb{N}$, so that for all $k \in \{0, 1, \ldots, p - 1\}$ and for all $x \in L$

$$d \left( F \left( x, \frac{k}{p} \right), F \left( x, \frac{k + 1}{p} \right) \right) < \varepsilon,$$

where $F$ is the deformation contraction given in the hypotheses of the proposition. For each $k \in \{0, 1, \ldots, p\}$, a rule $z \mapsto F(g(z), \frac{k}{p})$ mapping $K^{(0)} \rightarrow X$ induces a simplicial map $f^k : K \rightarrow \text{Čech}(X, r)$. Note that for each simplex $\sigma \in K$, simplex $f^k(\sigma)$ is witnessed by $F(w_\sigma, \frac{k}{p})$ by the property of DC. Furthermore, each $f^k$ is constant on $K_0$.

Choose $\sigma \in K$. Note that for each $k \in \{0, 1, \ldots, p - 1\}$ and $z \in \sigma$, we have

$$d(f^{k+1}(z), F(w_\sigma, k/p)) \leq d(f^{k+1}(z), f^k(z)) + d(f^k(z), F(w_\sigma, k/p))$$

$$< \varepsilon + (r - \varepsilon) = r;$$
Figure 2. A demonstration that an SDC may not induce homotopy equivalence on closed filtrations. The underlying planar space $X$ (in bold) consists of two points (bullets $x_1$ and $x_2$) at distance 1, and two corresponding arcs connecting bullets to a point $*$ far away. The intersection of $\overline{B}(x_1, 1)$ and the opposite arc is precisely $x_2$, and vice versa. This implies that $[x_1, x_2]$ is a maximal simplex in $\overline{\text{Rips}}(X, 1)$, and thus, $\pi_1(\overline{\text{Rips}}(X, 1), *) \neq 1$. However, if the $y$-coordinates of arcs are changing monotonically, then $X \xrightarrow{\text{SDC}} *$ and $\overline{\text{Rips}}(*, 1)$ is just a point hence, $F(w_\sigma, p/k)$ witnesses a simplex in $\text{Čech}(X, r)$ containing $f^k(\sigma)$ and $f^{k+1}(\sigma)$. Consequently, $f^k$ and $f^{k+1}$ are contiguous rel $K_0$; hence, homotopic rel $K_0$. Inductively, we conclude that $g = f^0$ and $f = f^p$ are homotopic rel $K_0$, which proves our claim, since $f(K^{(0)}) \subset A$. □

**Corollary 3.3.** Suppose $X \xrightarrow{\text{DC}} A$. Then, open Rips filtrations of $X$ and $A$ are homotopy equivalent, and open Čech filtrations of $X$ and $A$ are homotopy equivalent.

**Proof.** The inclusions used in Proposition 3.2 and its counterpart in [21] obviously commute with the bonding inclusions of the filtrations. □

**Remark 3.4.** Proposition 3.2 does not hold for closed filtrations. Consider $X = [0, 1] \times \{0, 1\} \subset \mathbb{R}^2$ in the Euclidean metric. It is easy to see that its deformation contracts to $A = \{0\} \times \{0, 1\} \subset \mathbb{R}^2$. However, $\overline{\text{Rips}}(X, 1)$ has uncountable fundamental group, while $\overline{\text{Rips}}(A, 1)$ is an edge. Proposition 3.2 does not hold for closed filtrations and SDC either; see Fig. 2.

When considering deformation contraction of subspaces, we can use Proposition 3.2 to obtain induced maps on the Rips complexes. However, we do have to be a bit more careful when considering Čech complexes.

**Proposition 3.5.** Suppose $A \subset B \subset X$, $r > 0$, and $N_X(B, r) \xrightarrow{\text{DC}} A$. Then, $\text{Čech}_X(A, r) \simeq \text{Čech}_X(B, r)$.

**Proof.** The proof of Proposition 3.2 applies as witnesses of simplices of $\text{Čech}_X(B, r)$ are contained in $N_X(B, r)$. □

4. Basic Example of Footprint Detection

In this section, we present the prototype of a footprint detection. We use deformation contractions and the Mayer–Vietoris sequence on a surface to locally extract a footprint of a geodesic circle. More general conditions of this technique are provided in the subsequent section.
In this section, we assume that $X$ is a geodesic surface and a geodesic circle $\alpha$ has some orientable subsurface as a neighborhood.

**Definition 4.1.** Suppose $0 < D_1 \leq D_2$. A loop $\alpha \subset X$ is $DC(D_1, D_2)$ isolated (deformation contraction isolated) if there exist two closed nested neighborhoods $N_1 \subset N_2$ of $\alpha$, so that:

1. $N_1$ and $N_2$ are homeomorphic to $S^1 \times [0, 1]$;
2. $N_2 \supset N(N_1, D_2)$;
3. $\partial N_1$ consists of loops $\alpha_1$ and $\alpha_2$, which are at least $D_1$ apart from each other;
4. $\text{Rips}(\alpha_i, r) \simeq S^1, \forall i, \forall r < D_1$;
5. $N_2 \setminus \text{Int}(N_1) \xrightarrow{DC} \partial N_1$ and $N_1 \xrightarrow{DC} \alpha$.

See Fig. 3 for a sketch.

Loop $\alpha$ is $DC(D)$ isolated, if it is $DC(D, D)$ isolated. Loop $\alpha$ is $SDC(D)$ isolated or $SDC(D_1, D_2)$ isolated, if, furthermore, all deformation contractions involved are strict deformation contractions.

**Remark 4.2.** The conditions of Definition 4.1 stipulate that $\alpha$ has a sufficiently tame neighborhood, in which it is the shortest homotopy representative of its class.

Condition (5) implies $N_2 \xrightarrow{DC} \alpha$. Sufficient conditions implying condition (4), i.e., conditions for $r > 0$ and a topological circle $S$ in a metric space to have $\text{Rips}(S, r) \simeq S^1$, are provided in [29]. For example, (4) holds if for each $i$, the cover of $\alpha_i$ by maximal open sets of diameter $r$ is a good cover. If the $\alpha_i$ are isometric to planar loops, [4] also provides sufficient conditions.

**Lemma 4.3.** Using the notation of Definition 4.1, we have $N_2 \xrightarrow{DC} N_1$.

**Proof.** Let $F : N_2 \setminus \text{Int}(N_1) \xrightarrow{DC} \partial N_1$. It suffices to show that for each pair of points $x \in N_2 \setminus N_1, y \in N_1$, we have $d(F(x, t'), F(y, t')) \leq d(F(x, t), F(y, t))$.
for all \( t' > t \). Choose a geodesic \( g \) from \( x \) to \( y \) and let \( y^* \) be the intersection of \( g \) and \( \partial N_1 \). By the assumption, \( F \) is mapping \( x \) ever closer to \( y^* \) and thus also to \( y \). □

**Proposition 4.4.** Suppose \( D > 0 \) and the loop \( \alpha \) is \( DC(D) \) isolated. Then, for each \( r < D \):

1. \( \text{Rips}(N_2, r) \simeq \text{Rips}(N_1, r) \simeq \text{Rips}(\alpha, r) \),
2. \( \text{Rips}(N_2 \setminus \text{Int}(N_1)) \simeq \text{Rips}(\partial N_1) \), and
3. \( \text{Rips}(\partial N_1) = \text{Rips}(\alpha_1, r) \sqcup \text{Rips}(\alpha_2, r) \).

**Proof.** (1) and (2) follow by definitions, Lemma 4.3, and Proposition 3.2. (3) follows by Definition 4.1(3). □

We next provide our basic theorem about the footprint detection. Since the one-dimensional persistence was completely classified in [27], we focus on higher dimensional persistence.

**Theorem 4.5.** (*Footprint detection for loops on surfaces*) Suppose \( X \) is a geodesic surface, \( \alpha \) is a \( DC(D) \) isolated geodesic circle for some \( D > 0 \), and \( G \) is a group. Then, \( \{H_k(\text{Rips}(\alpha, r), G)\}_{r \leq D} \) is a direct summand of \( \{H_k(\text{Rips}(X, r), G)\}_{r \leq D} \) via the inclusion-induced map for all \( k \geq 2 \).

**Proof.** We set the Mayer–Vietoris long exact sequence using the notation of Definition 4.1 and Proposition 4.4. For a fixed \( r \leq D \), define

\[
A = \text{Rips}(N_2, r) \simeq \text{Rips}(\alpha, r), \quad B = \text{Rips}(X \setminus \text{Int}(N_1), r), \quad A \cap B = \text{Rips}(N_2 \setminus \text{Int}(N_1), r) \simeq \text{Rips}(\alpha_1, r) \sqcup \text{Rips}(\alpha_2, r) \simeq S^1 \sqcup S^1. 
\]

Since \( H_k(A \cap B) = H_{k-1}(A \cap B) = 0 \, \forall k \geq 3 \), we extract the following exact sequences:

\[
0 = H_k(A \cap B) \rightarrow H_k(A) \oplus H_k(B) \overset{f_k}{\rightarrow} H_k(\text{Rips}(X, r)) \rightarrow H_{k-1}(A \cap B) = 0.
\]

This proves that \( H_k(A) \cong H_k(\text{Rips}(\alpha, r)) \) is a direct summand in \( H_k(\text{Rips}(X, r)) \), \( \forall k \geq 3 \). By Theorem 2.1, \( H_2(\text{Rips}(\alpha, r)) \) is trivial; hence, the conclusion holds for all \( k \geq 2 \) and fixed \( r \).

When considering a range \( r \in (0, D) \), note that the bonding maps on persistences are induced by the inclusions, just as maps \( f_k \) in the Mayer–Vietoris sequence. Hence, all maps in question commute, implying that \( \{H_k(\text{Rips}(\alpha, r), G)\}_{r \leq D} \) is a direct summand of \( \{H_k(\text{Rips}(X, r), G)\}_{r \leq D} \) for all \( k \geq 2 \), which completes the proof. □

Since \( \alpha \) is a geodesic circle, its Rips filtration is known by Theorem 2.1, and so are the direct summands mentioned in Theorem 4.5. Corollary 4.6 summarizes such a situation in terms of bars of persistence diagrams.

**Corollary 4.6.** Suppose \( X \) is a totally bounded geodesic surface, \( \alpha \) is a \( DC(D) \) isolated geodesic circle for some \( D > 0 \), and \( G \) is a field.
(1) If \( \frac{l}{2l+1} |\alpha| < D \leq \frac{l+1}{2l+3} |\alpha| \) for some \( l \in \mathbb{N} \), then the following conclusion holds: the PD of \( X \) contains for each \( n \in \{1, 2, \ldots, l - 1\} \) a \((2n + 1)\)-dimensional bar \( (\frac{n}{2n+1} |\alpha|, \frac{n+1}{2n+3} |\alpha|) \) and a \((2l + 1)\)-dimensional bar \( (\frac{l}{2l+1} |\alpha|, w) \), for some \( w \in [D, \frac{l+1}{2l+3} |\alpha|] \), all generated by the included Rips complex of \( \alpha \).

(2) If \( \frac{l}{2l+1} |\alpha| < D \leq \frac{l+1}{2l+3} |\alpha| \) and \( \alpha \) is a member of some lexicographically shortest homology basis, then the conclusion of (1) holds for all \( n \in \{0, 1, \ldots, l - 1\} \).

(3) If \( D \geq |\alpha|/2 \), then the conclusion holds for all \( n \in \{1, 2, \ldots\} \).

**Proof.** Since \( X \) is totally bounded, the induced persistent homology is q-tame; hence, the PD exists in each dimension. The validity of (1) follows from Theorems 4.5 and 2.1. Statement (2) follows from [27], and (3) follows from Theorems 4.5 and 2.1. □

Theorem 4.5 considers a case of a single loop \( \alpha \). Using an inductive argument, Corollary 4.7 demonstrates that sufficiently disjoint geodesic circles generate separate footprints. In particular, for each loop of such a collection of geodesic circles, we get distinct bars mentioned in Corollary 4.6.

**Corollary 4.7.** Suppose \( X \) is a geodesic surface, \( m \in \mathbb{N} \), \( \alpha_1, \ldots, \alpha_m \) are DC\((D)\) isolated geodesic circles for some \( D > 0 \) with the corresponding \( N_2 \) neighborhoods \( N_{2,1}, \ldots, N_{2,m} \) from Definition 4.1 being disjoint, and \( G \) is a group. Then, \( \oplus_{i=1}^{m} \{H_k(\text{Rips}(\alpha_i, r), G)\}_{r \leq D} \) is a direct summand of \( \{H_k(\text{Rips}(X, r), G)\}_{r \leq D} \) via the inclusion-induced map for all \( k \geq 2 \).

**Proof.** Assume \( N_{1,1}, \ldots, N_{1,m} \) are neighborhoods of \( \alpha_1, \ldots, \alpha_m \) corresponding to neighborhood \( N_1 \) from Definition 4.1. Inductively apply the proof of Theorem 4.5 to \( X \), \( X \setminus \text{Int}(N_{1,1}) \), \( X \setminus (\text{Int}(N_{1,1}) \cup \text{Int}(N_{1,2})) \),... □

Theorem 4.5 is a prototype for footprint detection of subspaces onto which there are nice deformation contractions, and can be adapted to other subspaces. It is essentially a Hopf-type effect in persistence: one-dimensional geometric features generate higher dimensional algebraic objects. It explains the existence of all blue bars and the red 3-dimensional bar in Fig.1.

5. A General Framework for Footprint Detection via Deformation Contractions

Using the idea of the previous section, we provide a general framework for footprint detection in metric spaces via deformation contractions. In this case, we are detecting parts of the persistent homology of a subset \( Z \subset X \) in the persistent homology of \( X \).

Suppose \( X \) is a (not necessarily geodesic) metric space and let \( Z \subset X \) be a subspace. When considering a subset of \( X \) as a metric space, we always assume it is equipped with the restriction of the metric on \( X \).

**Definition 5.1.** Suppose \( 0 < a \leq b \), \( G \) is a group and \( k \in \mathbb{N} \). A subspace \( Z \) of a metric space \( X \) is DC\((\langle a, b \rangle; k, G)\) isolated if there exist two closed nested neighborhoods \( N_1 \subset N_2 \) of \( Z \), so that:
(1) \( N_2 \supset N(N_1, r), \forall r \in (a, b); \)
(2) for each \( r \in (a, b), \) the condition \( H_k(\text{Rips}(Z, r), G) \neq 0 \) implies that the following maps are trivial:
   (a) the inclusion-induced maps \( H_k(\text{Rips}(\partial N_1, r), G) \rightarrow H_k(\text{Rips}(N_1, r), G) \) and \( H_k(\text{Rips}(\partial N_1, r), G) \rightarrow H_k(\text{Rips}(X \setminus \text{Int}(N_1), r), G) \);
   (b) the boundary map \( H_k(\text{Rips}(X, r), G) \rightarrow H_{k-1}(\text{Rips}(N_2 \setminus \text{Int}(N_1), r)) \)
   arising from the Mayer–Vietoris long exact sequence for a decomposition of \( \text{Rips}(X, r) \) into \( A = \text{Rips}(N_2, r) \) and \( B = \text{Rips}(X \setminus \text{Int}(N_1), r) \);
(3) \( N_2 \setminus \text{Int}(N_1) \xrightarrow{DC} \partial N_1, N_2 \xrightarrow{DC} N_1, \) and \( N_1 \xrightarrow{DC} Z. \)

Notation \( SDC((a, b); k, G) \) denotes the strict version of the defined property, i.e., a version in which all deformation contractions of (3) are strict deformation contractions.

Proposition 5.2 is an adaptation of Proposition 4.4 and can be proved in the same way.

**Proposition 5.2.** Suppose \( 0 < a \leq b, X \) is a metric space, and \( Z \subset X \) is a subspace for which (1) and (3) of Definition 5.1 hold. Then, for each \( r \in (a, b): \)
(1) \( \text{Rips}(N_2, r) \simeq \text{Rips}(N_1, r) \simeq \text{Rips}(Z, r), \) and
(2) \( \text{Rips}(N_2 \setminus \text{Int}(N_1)) \simeq \text{Rips}(\partial N_1). \)

**Theorem 5.3.** (Footprint Detection Framework) Suppose \( X \) is a metric space, \( G \) is a group, \( k \in \mathbb{N}, 0 < a \leq b, \) and \( Z \subset X \) is \( DC((a, b); k, G) \) isolated. Then, \( \{ H_k(\text{Rips}(Z, r), G) \}_{r \in (a, b)} \) is a direct summand of \( \{ H_k(\text{Rips}(X, r), G) \}_{r \in (a, b)} \) via the inclusion-induced map.

**Proof.** The proof follows the same structure as that of Theorem 4.5, using Proposition 5.2 and the conditions of Definition 5.1 in the corresponding Mayer–Vietoris sequence for a decomposition of \( \text{Rips}(X, r) \) into \( A = \text{Rips}(N_2, r) \) and \( B = \text{Rips}(X \setminus \text{Int}(N_1), r). \)

In our future work, we intend to tackle the problems of obtaining persistences of simple spaces (such as spheres) and controlling the homology of a neighborhood. With such results, we could provide more convenient conditions for specific footprint detection situations introduced here.

### 6. Nullhomologies of Loops

In this technical section, we provide nice nullhomologies of samples of loops, which will be used in Sect. 7. Definition 6.1 introduces a cyclic order of points on a loop.

**Definition 6.1.** Choose an orientation on \( S^1. \) Notation \( x_0 < x_1 < \ldots < x_k < x_0 \) means that points \( x_i \) appear on \( S^1 \) in the suggested order along the chosen orientation of \( S^1. \) In particular, this means that for each \( i \in \mathbb{N} \mod (k + 1), \)
the interval \((x_i, x_{i+1})\) on \(S^1\) along the chosen orientation contains no point \(x_j\). Equivalently, the collection of the mentioned intervals \((x_i, x_{i+1})\) on \(S^1\) is disjoint.

If \(\alpha: S^1 \to X\) is a loop, then \(\alpha(t_0) < \alpha(t_1) < \ldots < \alpha(t_k) < \alpha(t_0)\) means that \(t_0 < t_1 < \ldots < t_k < t_0\) along the chosen orientation.

Throughout this section, we will be referring to a loop \((A, d)\) as a (not necessarily geodesic) metric space homeomorphic to \(S^1\). In this context, we will also be using the derived metric space \((A', d')\), which denotes set \(A\) equipped with the geodesic metric, i.e., a metric where the distance between two points is the length of the shortest segment in \((A, d)\) between the points.

As a technical prerequisite, we will assume that the circumference of \((A, d)\) is finite, so that the distance \(d\) between two points is the length of the shortest segment in \((A, d)\) equipped with the geodesic metric, i.e., a metric where the distance between two points is the length of the shortest segment in \((A, d)\) between the points.

**Definition 6.2.** Suppose \(A\) is a loop. Three points \(x_0, x_1, x_2 \in A\) are equidistant on \(A\), if for some chosen orientation on \(A\), \(x_0 < x_1 < x_2 < x_0\) and the lengths of the closed intervals \([x_0, x_1], [x_1, x_2]\) and \([x_2, x_0]\) along \(A\) are the same.

It is clear that the definition of equidistant points is independent of the choice of an orientation. The following lemma shows that three equidistant points, or even their appropriate approximations, may be used to obtain nice nullhomotopies of loops in Rips complexes.

**Lemma 6.3.** Suppose \((A, d)\) is an oriented loop of circumference 1 equipped with a (not necessarily geodesic) metric \(d\), and choose \(r > 1/3\). Let \(L\) be an \(r\)-sample of \(A\) given by \((x_0, x_1, \ldots, x_k, x_{k+1} = x_0)\). Suppose there exist \(t_0 < t_1 < t_2\), so that \(x_{t_0} < x_{t_1} < x_{t_2} < x_{t_0}\) with the lengths of segments of the closed intervals \([x_{t_0}, x_{t_1}], [x_{t_1}, x_{t_2}], [x_{t_2}, x_{t_0}]\) along \(A\) being less than \(r\).

Then, there exist pairwise different triangles \(\sigma_i\) in \(\text{Rips}(L, r)\), so that:

- \(L = \text{loop of circumference } \sum_{i=1}^{k} \sigma_i\), and
- if \(x_p \in L\) and \(x_q \in L\) are contained in some \(\sigma_i\), then the length of the shortest segment on \(A\) between \(x_p\) and \(x_q\) is less than \(r\).

Furthermore, if \(L = \partial \sum_{j=1}^{m} \sigma_j'\) is another such decomposition, then \(\sum_{i=1}^{k} \sigma_i = \sum_{j=1}^{m} \sigma_j'\) is a boundary in \(\text{Rips}(A, r)\).

**Proof.** The triangles \(\sigma_i\) are depicted in Fig. 4. The shaded triangle is \((x_{t_0}, x_{t_1}, x_{t_2})\). For all \(l \in \mathbb{Z}(\text{mod } 3)\) and for all \(t_l < p < t_{l+1}(\text{mod } 3)\) add a cone (with the appropriate orientation) over the segment \((x_{p-1}, x_p)\) with apex \(x_{t_l}\). Hence, we have satisfied the two bullet points.

The construction works even if two of the points \(x_{t_l}\) coincide: in such a case, we remove those triangles \(\sigma_i\) in the expression above which become degenerate.

To prove the second part, let \((A', d')\) denote set \(A\) equipped with the geodesic metric, i.e., a metric where the distance between two points is the length of the shortest segment in \((A, d)\) between the points. Each \(\sigma_i\) mentioned above is also contained in \(\text{Rips}(A', r)\), as only lengths of the connecting path segments were used in the argument and no special properties of the metric were
Figure 4. Sketch of proof of Lemma 6.3

assumed. The second homology group of $\text{Rips}(A', r)$ is trivial for all $r$ and all coefficients by Theorem 2.1; hence, $\sum_{i=1}^{k} \sigma_i - \sum_{j=1}^{m} \sigma_j$ is a boundary in $\text{Rips}(A', r)$. Since the identity $(A', d') \to (A, d)$ is a contraction, the induced map $\text{Rips}(A', r) \to \text{Rips}(A, r)$ is an inclusion; hence, $\sum_{i=1}^{k} \sigma_i - \sum_{j=1}^{m} \sigma_j$ is a boundary in $\text{Rips}(A, r)$, as well. □

Corollary 6.4. Suppose $(A, d)$ is a loop of circumference 1 equipped with a (not necessary geodesic) metric $d$, and choose $r > 1/3$. Let $L$ be a ($r - 1/3$)-sample of $A$ given by $(x_0, x_1, \ldots, x_k, x_{k+1} = x_0)$. Then, there exist pairwise different triangles $\sigma_i$ in $\text{Rips}(L, r)$, so that $L = \partial \sum_{i=1}^{k} \sigma_i$. Furthermore, if $L = \partial \sum_{j=1}^{m} \sigma'_j$ is another such decomposition, then $\sum_{i=1}^{k} \sigma_i - \sum_{j=1}^{m} \sigma'_j$ is a boundary in $\text{Rips}(A, r)$.

Proof. Fixing an orientation of $A$, choose $t_0 < t_1 < t_2$, so that $x_{t_0} < x_{t_1} < x_{t_2} < x_{t_0}$ with the lengths of the segments of the closed intervals $[x_{t_0}, x_{t_1}],[x_{t_1}, x_{t_2}]$ and $[x_{t_2}, x_{t_0}]$ along $A$ being less than $r$. For example, we can set $t_0 = 0$, choose $t_1$, so that the length of $A$ along the chosen orientation to $x_{t_1}$ is between $1/3$ and $r$ (such $x_{t_1}$ exists, since $L$ is a ($r - 1/3$)-sample), etc. □

7. Two-Dimensional Footprint

In this section, we prove the most technical result of this paper, describing an appearance of two-dimensional footprints. A sufficiently DC isolated geodesic circle in a compact geodesic space, which is also a member of a shortest homology basis, generates a one-dimensional footprint by [27]. If such a circle is not a member of a minimal homology basis, then the result of this section shows that it in fact induces a two-dimensional footprint. In contrast to the results of the previous sections, this footprint does not appear in the persistence of the circle itself but arises from the geometry of the entire space. As described below, such a footprint can be used to separate geodesic circles, detectable by higher dimensional footprints as described in the previous sections, into two groups: members of a shortest homology basis, and the rest
of the geodesic circles. As an added benefit, the mentioned two-dimensional footprint can provide a good approximation for homotopy height.

The results in this section focus on geodesic circles on surfaces. We intend to describe more general results in a forthcoming paper.

**Theorem 7.1.** Suppose $X$ is a geodesic surface, $\alpha$ is an $DC(D, 3D/2)$ isolated loop for some $D > |\alpha|/3$, and $G$ is a group. Assume $\alpha$ is homologous in $H_1(X, G)$ to a $G$-combination of loops $\beta_1, \beta_2, \ldots, \beta_k$ of length at most $|\alpha|$, none of which intersects $N_1$. Then, the following holds.

1. For each $r \in (|\alpha|/3, D]$, there exists a non-trivial $Q_r \in H_2(\text{Rips}(X, r), G)$, so that:
   
   a. For each pair $q_1 < q_2$ of parameters from $(|\alpha|/3, D]$, we have $i_{q_1,q_2}^G(Q_{q_1}) = Q_{q_2}$.
   
   b. For any $q \in (|\alpha|/3, D]$, there exists no $q_0 \leq |\alpha|/3$, for which $Q_q$ is in the image of $i_{q_0,q}^G$.
   
   c. If $\alpha$ is homotopic to some shorter geodesic circle $\beta$ in $X$ and $3q_3$ is larger than the homotopy height between $\alpha$ and $\beta$, then $i_{q,q_3}^G(Q_q)$ is trivial for any $q \in (|\alpha|/3, \min(D, q_3))$.

2. If $G$ is a field and $\{H_2(\text{Rips}(X, r), G)\}_{r>0}$ is $q$-tame, then the persistence $\{H_2(\text{Rips}(X, r), G)\}_{r<D}$ contains $G_{(|\alpha|/3,w'/3)}$ as a direct summand for some $w \in (|\alpha|/3, \min(D, q_3))$.

**Proof.** **Proof of (1).** We will be using the singular homology representation in $X$: there exist singular 2-simplices $\hat{\Delta}_j$, $\tilde{h}_j, g_i \in G$ in $X$, so that

$$[\alpha]_G = \sum_{i=1}^k g_i[\beta_i]_G + \partial \sum_{j=1}^{k'} \tilde{h}_j[\hat{\Delta}_j]_G.$$ 

We now subdivide singular 2-simplices $\hat{\Delta}_j$ into $\Delta_j$, so that for some $h_j \in G$

$$L\alpha = \sum_{i=1}^k g_iL_i + \partial \sum_{j=1}^{k'} h_j\Delta_j$$

holds with the following conditions:

- the diameter of each singular simplex $\Delta_j$ is less than $|\alpha|/3$;
- $L\alpha$ and $L_i$ are subdivided loops $\alpha$ and $\beta_i$ correspondingly, with their vertices forming $(|\alpha|/3)$-samples of $\alpha$ and $\beta_i$ correspondingly;
- each $(|\alpha|/3)$-sample above contains three equidistant points on the corresponding loop (this will allow us to apply Lemma 6.3 below).

Using the condition on the diameter of simplices, we may abuse the notation and consider $L_\alpha$ and $L_i$ to be either subdivided singular loops in $X$ (in which case they correspond to $\alpha$ and $\beta_i$) or, by retaining the vertices, $(|\alpha|/3)$-loops in $\text{Rips}(X, |\alpha|/3)$. The same goes for each $\Delta_j$: its three vertices form a simplex in $\text{Rips}(X, |\alpha|/3)$, which we also denote by $\Delta_j$.

Fix $r \in (|\alpha|/3, D]$. Using Lemma 6.3, we can express each $L_i$ as a chain $L_i = \partial \sum_{p=1}^{k_p} \tau_{i,p}$, where each $\tau_{i,p}$ is a 2-simplex in $\text{Rips}(\beta_i, r)$. Hence, we
Figure 5. An explanation of Eq. 7.1. The left side of the figure represents a portion of a surface containing a geodesic circle $\alpha$ and a shorter homotopic geodesic circle $\beta$. The right side demonstrates the geometric objects, whose triangulations are parts of Eq. 7.1. Simplices $\Delta_j$ triangulate the portion of the surface between the loops while simplices $\sigma_l$ and $\tau_{i,1}$ triangulate the “lids” arising from nullhomotopies of $\alpha$ and $\beta$ in the Rips complex (as described in Lemma 6.3), turning the total expression into a 2-cycle

obtain an expression of a chain in $\operatorname{Rips}(X, r)$

$$L_\alpha = \partial \sum_{i=1}^{k} \sum_{p=1}^{k_p} g_i \tau_{i,p} + \partial \sum_{j=1}^{k'} h_j \Delta_j.$$  

We use Lemma 6.3 once more to obtain an expression of a chain in $\operatorname{Rips}(\alpha, r)$: $L_\alpha = \partial \sum_{l=1}^{k_\alpha} \sigma_l$, implying

$$\partial \sum_{l=1}^{k_\alpha} \sigma_l = \partial \sum_{i=1}^{k} \sum_{p=1}^{k_p} g_i \tau_{i,p} + \partial \sum_{j=1}^{k'} h_j \Delta_j.$$  

(7.1)

See Fig. 5 for a demonstration of such an expression.

This means that

$$C_r = \sum_{l=1}^{k_\alpha} \sigma_l - \sum_{l=1}^{k} \sum_{p=1}^{k_p} g_i \tau_{i,p} - \sum_{j=1}^{k'} h_j \Delta_j$$

is a cycle and we define $Q_r$ to be the homology class represented by $C_r$. By standard arguments, we see that a subdivision of singular 2-simplices $\Delta_j$ above results in a different (finer) representation $C_r$, but does not change $Q_r$ (a subdivision replacing each $\Delta_j$ by the corresponding homologous sum of contained triangles would induce finer samples $L_\alpha$ and $L_i$ in the expression of $\partial \sum_{j=1}^{k'} h_j \Delta_j$). These finer samples are equivalent to the original choice by [27, Proposition 3.2(4)], and so are their nullhomologies by Lemma 6.3.

We next prove that $Q_r \in H_2(\operatorname{Rips}(X, r), G)$ is non-trivial. Consider the following excerpt of the Mayer–Vietoris sequence set up in Theorem 4.5:

$$H_2(\operatorname{Rips}(X, r)) \to H_1(A \cap B).$$  

(7.2)

We will be making use of the notation set up by the proof of Theorem 4.5 and Definition 4.1. To prove $Q_r$ is non-trivial, we will show that its image via the boundary map of Eq. 7.2 is non-trivial.
We first decompose $C_r$ into two parts $C_r = L_A + L_B$ as follows. $L_A$ consists of $\sum_{l=1}^{k_\alpha} \sigma_l$ plus $\tilde{L}_A$. $\tilde{L}_A$ consists of all the summands $-h_j\Delta_j$, for which the singular simplex in $X$ corresponding to $\Delta_j$ intersects $N_1$. $L_B$ consists of all other summands of $C_r$. Observe that $L_A$ is contained in $A$, and $L_B$ is contained in $B$ as $D > r$. We will also be interested in the boundary of $L_A$. The boundary of $\sum_{l=1}^{k_\alpha} \sigma_l$ is $L_\alpha$. The boundary of $\tilde{L}_A$ is nullhomologous in $A$ (recall it arises from $\alpha$ being homologous to a combination of shorter loops in $X$ and each involved singular 2-simplex $\Delta_j$ is contractible in the tube $N_2$ as it is of diameter less than $|\alpha|/3$, hence so is its counterpart in $A$ by [27, Proposition 3.2(7)]). Putting them together, we see that the bound-
ary of $L_A$ has winding number $\pm 1$ (the sign being dependent on the chosen orientation) in $A$. This boundary represents the sum of the components (in $H_1(A \cap B) = G \oplus G$) of the image of $Q_r$ via the boundary map of Eq. 7.2; hence, $Q_r$ is non-trivial.

**Proof of statement (a).**

The statement holds as:

- parts $\partial \sum_{j=1}^{k_1} h_j\Delta_j$ of $C_{q_1}$ and $C_{q_2}$ are the same;
- the remaining parts (nullhomologies of loops $L_\alpha$ via Lemma 6.3) can also be chosen to be the same, and are in general homologous by Lemma 6.3.

**Proof of statement (b).**

Consider the following diagram:

\[
\begin{align*}
H_2(\text{Rips}(X,q)) & \rightarrow H_1(\text{Rips}(N_2,q) \cap \text{Rips}(X \setminus \text{Int}(N_1),q)) \\
H_2(\text{Rips}(X,q_0)) & \rightarrow H_1(\text{Rips}(N_2,q_0) \cap \text{Rips}(X \setminus \text{Int}(N_1),q_0)),
\end{align*}
\]

(7.3)

where the horizontal maps are excerpts from the Mayer–Vietoris sequence setup in Theorem 4.5 and the vertical maps are inclusion induced. It is easy to verify that the diagram commutes. As was mentioned before, $\text{Rips}(N_2,q) \cap \text{Rips}(X \setminus \text{Int}(N_1),q)$ and $\text{Rips}(N_2,q_0) \cap \text{Rips}(X \setminus \text{Int}(N_1),q_0)$ are both homotopy equivalent to a disjoint union of two copies of $S^1$. The difference we are about to utilise is that while $q_0$-samples of $\alpha_i$ are not contractible in $\text{Rips}(N_2,q_0)$, they are contractible in $\text{Rips}(N_2,q)$. Hence, for example, a $q_0$-sample of $\alpha_1$ may appear as a boundary of a 2-chain in $\text{Rips}(N_2,q)$, a fact we used earlier in the proof, but not in $\text{Rips}(N_2,q_0)$, which we will use here.

Suppose $K_{q_0}$ is a 2-chain in $\text{Rips}(X,q_0)$. Keeping in mind the diagram above, we decompose $K_{q_0} = K_{q_0}^1 + K_{q_0}^2$, with $K_{q_0}^1$ being a chain in $\text{Rips}(N_2,q_0)$ and $K_{q_0}^2$ being a chain in $\text{Rips}(X \setminus \text{Int}(N_1),q_0)$, in the following way. Let $K_{q_0}^1$ consist of all summands $g\Delta$ of $K_{q_0}$ with $g \in G$, $\Delta \in \text{Rips}(X,q_0)$, for which at least one vertex of $\Delta$ is contained in $N_1$. Take the three vertices of such $\Delta$ and connect them pairwise by geodesics to obtain a triangle $S$. Since $3D/2 > 3q_0/2$ and at least one vertex is in $N_1$, the obtained triangle is contained in $N_2$. Furthermore, as the circumference of $S$ is less than $|\alpha|$ and $N_2 \xrightarrow{DC} \alpha$, we
conclude that $S$ is contractible in $N_2$. This easily implies (for an argument, see for example [27, Proposition 3.2, (7)]) that $\partial \Delta$ is contractible in $\operatorname{Rips}(N_2, q_0)$; hence, the entire boundary of $K^1_{q_0}$ is nullhomologous in $H_1(\operatorname{Rips}(N_2, q_0))$. This boundary represents the sum of the components (in $H_1(A \cap B) = G \oplus G$) of the image of $[K_{q_0}]_G \in H_2(\operatorname{Rips}(X, q_0))$ via the lower horizontal map $h$ of the diagram (7.3) above. While its image in

$$H_1(\operatorname{Rips}(N_2, q_0) \cap \operatorname{Rips}(X \setminus \operatorname{Int}(N_1), q_0)) \cong G \oplus G$$

may be non-trivial, the two components of $h([K_{q_0}]_G)$ have, up to the sign depending on the chosen orientation, the same winding number, as they have to add up to 0. This contrasts the property of $Q_r$ above, where we proved that the difference between the corresponding winding numbers is $\pm 1$. Hence, $Q_r$ cannot be an image of any element of $H_2(\operatorname{Rips}(X, q_0))$ via the natural inclusion-induced map, which proves (b).

**Proof of statement (c).**

Assume there is a homotopy $H: I \times I \to X$ (we will use the notation $H_t = H|_{\{t\} \times I}$) realizing the homotopy height $w$ between $\alpha$ and $\beta$:

- $H_0 = \alpha$;
- $H_1 = \beta$;
- $H(t, 1) = H(t, 0), \forall t \in I$;
- $|H_t(I)| \leq w, \forall t \in I$.

If the homotopy height $w$ cannot be realized precisely, we choose $\tilde{w} \in (w, q_3)$ and use a similar homotopy $H$ with the condition $|H_t(I)| \leq w$ being replaced by $|H_t(I)| \leq \tilde{w}, \forall t \in I$. In this case, the argument below also provides the same conclusion (c). For the sake of simplicity, we may now assume that the homotopy height is realized precisely by $H$.

Define $\varepsilon = q_3/3 - w > 0$. Subdivide $I \times I$ into a grid of size $M \times M$ for some even $M \in \mathbb{N}$, consisting of squares $S_{m,n} = [\frac{m}{M}, \frac{m+1}{M}] \times [\frac{n}{M}, \frac{n+1}{M}]$, so that the diameter of each $H(S_{m,n})$ is less than $\frac{\varepsilon}{20}$. To obtain a proper triangulation $T$ of $I \times I$, add diagonals to squares $S_{m,n}$: if $m$ is odd, add a diagonal from $(\frac{m}{M}, \frac{n}{M})$ to $(\frac{m+1}{M}, \frac{n+1}{M})$; if $m$ is even, add a diagonal from $(\frac{m}{M}, \frac{n}{M})$ to
For each $i$, $M$-gon.

Consider a cylinder, obtained by identifying points $(x,0)$ \sim $(x,1)$ in $[M, M+2] \times I$. Let $T'_m$ denote a copy of $T_m$ attached along the domain of $H_m$ in such a cylinder, and let $T''_m$ denote a copy of $T_m$ attached along the domain of $H_{m+2}$. Form a triangulation of a thin cylinder

\[ U'_m = U'_m \cup U''_m, \]

where

\[ U'_m = \{ T^{u}_{m,n} \}_{n=1}^{M} \cup \{ T^{l}_{m,n} \}_{n=1}^{M} \cup T'_m \]
Figure 7. Sketch of proof of part (c) of Theorem 7.1. Triangulation \( U_m \) of a cylinder is obtained by taking a vertical strip of triangles of \( T \) (the right part) and attaching a copy of a triangulation \( T_r \) of a disc (shown on left) on each slice, so that the labels of the vertices coincide

and

\[
U_m^R = \{ T^{u}_{m+1,n} \}_{n=1}^M \cup \{ T^{l}_{m+1,n} \}_{n=1}^M \cup T^R_m.
\]

Because of alternating diagonals in the construction of \( T \), \( U^L_m \) and \( U^R_m \) are isomorphic triangulations of a disc glued together along their boundaries corresponding to \( \{ m+1 \} \times I \subset I \times I \). Applying \( H \) to the vertices of \( U_m \), we thus obtain two maps, which are contiguous in Rips(\( X, q_3 \)) as \( T \) is fine enough. Thus, \( \sum_{\sigma \in U_m} H(\sigma) \) is homologically trivial in Rips(\( X, q_3 \)).

For \( m \in \{ 2, 4, \ldots, M-2 \} \), define also a triangulation (simplicial complex) \( V_m = T^L_m \cup T^R_{m-2} \). By Lemma 6.3, the chain \( \sum_{\sigma \in V_m} H(\sigma) \) is homologically trivial in Rips(\( X, q_3 \)).

Observe now that the equality of chains

\[
L = \sum_{m \in \{ 0, 2, 4, \ldots, M-2 \}} \sum_{\sigma \in U_m} H(\sigma) - \sum_{m \in \{ 2, 4, \ldots, M-2 \}} \sum_{\sigma \in V_m} H(\sigma).
\]

Since each involved summand \( \sum_{\sigma \in U_m} H(\sigma) \) and \( \sum_{\sigma \in V_m} H(\sigma) \) is trivial in Rips(\( X, q_3 \)), so is \( L \).

This concludes the proof of (1). Statement (2) follows directly from (1). \( \square \)

The following proposition explains the condition of Theorem 7.1 requiring loops being disjoint with \( N_1 \). It essentially excludes the case where \( \alpha \) would have a small cylindric neighborhood, meaning that there would be loops of length \( |\alpha| \) arbitrarily close to and homotopic to \( \alpha \). Technically speaking, any such loop is an expression of \( \alpha \) by a loop of length at most \( \alpha \). However, in such a case, the two-dimensional homology class constructed in Theorem
7.1 could be trivial. To avoid such a situation, we require an empty intersection with \( N_1 \). The following proposition explains that this is indeed the case in a generic situation.

**Proposition 7.2.** Suppose \( X \) is a geodesic surface, \( \alpha \) is an \( DC(D_1, D_2) \) isolated loop for some \( D_1 > 0, D_2 \geq |\alpha|/2, \) and \( G \) is a group. Assume \( \alpha \) is homologous in \( H_1(X, G) \) to a finite \( G \)-combination of loops \( \beta_i \) of lengths at most \( |\alpha| \), none of which are equal to \( \alpha \) (this holds, for example, if \( \alpha \) is not a member of any lexicographically shortest homology basis). Then, \( \beta_i \) can be chosen to be geodesic circles. If all loops \( \beta_i \) are shorter than \( \alpha \) or \( \alpha \) is \( SDC(D_1, D_2) \) isolated, then none of loops \( \beta_i \) intersect \( N_1 \).

**Proof.** The first part is contained in Theorem 8.4 in [27].

If some \( \beta_i \) intersected \( N_1 \), then by the condition on \( D_2 \), it would be contained in \( N_2 \) (see Definition 4.1 for notation), which is homeomorphic to a tube. The shortest generator of \( H_1(N_2, G) \) is \( \alpha \). If \( |\beta_i| < |\alpha| \), then \( \beta_i \) is contractible and obsolete. If \( \alpha \) is \( SDC(D_1, D_2) \) isolated and \( \beta_i \neq \alpha \), then the strict deformation retraction of \( \beta_i \) is shorter than \( \alpha \) and homotopic to \( \beta_i \), thus contractible, and \( \beta_i \) is again obsolete. \( \square \)

8. Footprint Detection Using Čech Complexes and Closed Filtrations

8.1. Čech Complexes

An approach, equivalent to the one based on the Rips complexes in the previous sections, could also be developed for Čech complexes. Since the arguments are almost the same, we will only state the versions of required statements and the most general main results for the Čech complexes, and comment on the few modifications to the results.

**Definition 8.1.** Suppose \( A \) is a loop. Four points \( x_0, x_1, x_2, x_3 \in A \) are in a square-like formation on \( A \), if for some chosen orientation on \( A \), \( x_0 < x_1 < x_2 < x_3 < x_0 \) and the length segments of the closed intervals \([x_0, x_1], [x_1, x_2], [x_2, x_3]\) and \([x_3, x_0]\) along \( A \) are the same.

**Lemma 8.2.** (A version of Lemma 6.3). Suppose \( (A, d) \) is a loop of circumference 1 equipped with a (not necessary geodesic) metric \( d \), and choose \( r > 1/4 \). Let \( L \) be an \( r \)-sample of \( A \) given by \((x_0, x_1, \ldots, x_k, x_{k+1} = x_0)\). Suppose there exist \( t_0 < t_1 < t_2 < t_3 \), so that \( x_{t_0} < x_{t_1} < x_{t_2} < x_{t_3} < x_{t_0} \) with the length of segments of the closed intervals \([x_{t_0}, x_{t_1}], [x_{t_1}, x_{t_2}], [x_{t_2}, x_{t_3}]\) and \([x_{t_3}, x_{t_0}]\) along \( A \) being less than \( r \).

Then, there exist pairwise different triangles \( \sigma_i \) in \( Čech_L(L, r) \), so that:

- \( L = \partial \sum_{i=1}^{k} \sigma_i \), and
- if \( x_p \in L \) and \( x_q \in L \) are contained in some \( \sigma_i \), then the length of the shortest segment on \( A \) between \( x_p \) and \( x_q \) is less than \( 1/4 \).

Furthermore, if \( L = \partial \sum_{j=1}^{m} \sigma_j \) is another such decomposition, then \( \sum_{i=1}^{k} \sigma_i - \sum_{j=1}^{m} \sigma_j \) is a boundary in \( Čech_A(A, r) \).
Figure 8. Sketch of proof of Lemma 8.2

Proof. The decomposition is given by Fig. 8. □

Definition 8.3. (A version of Definition 4.1). Suppose $0 < D_1 \leq D_2$. Loop $\alpha \subset X$ is Čech $DC(D_1, D_2)$ isolated if there exist two closed nested neighborhoods $N_1 \subset N_2$ of $\alpha$, so that:

1. $N_1$ and $N_2$ are homeomorphic to $S^1 \times [0, 1]$;
2. $N_2 \supset N(N_1, D_2)$;
3. $\partial N_1$ consists of loops $\alpha_1$ and $\alpha_2$, which are at least $2D_1$ apart from each other;
4. Čech$(\alpha_i) \simeq S^1$, $\forall i, \forall r < D_1$;
5. $N_X(N_2, D_2) \setminus \text{Int}(N_1) \xrightarrow{DC} \partial N_1$ and $N_X(N_1, D_2) \xrightarrow{DC} \alpha$.

Loop $\alpha$ is Čech $DC(D)$ isolated, if it is Čech $DC(D, D)$ isolated.

Proposition 8.4. (A version of Proposition 4.4). Suppose $D > 0$ and loop $\alpha$ is Čech $DC(D)$ isolated. Then:

1. Čech$_X(N_2, D) \simeq$ Čech$_X(N_1, D) \simeq$ Čech$_X(\alpha, D) = $ Čech$_\alpha(\alpha, D)$,
2. Čech$_X(N_2 \setminus \text{Int}(N_1)) \simeq$ Čech$_X(\partial N_1)$, and
3. Čech$_X(\partial N_1) = $ Čech$_X(\alpha_1, D) \sqcup $ Čech$_X(\alpha_2, D)$.

Proof. Follows by definitions, and Propositions 3.2 and 3.5. □

Theorem 8.5. (A version of Theorem 4.5). Suppose $X$ is a geodesic surface, $\alpha$ is a Čech $DC(D)$ isolated loop for some $D > |\alpha|/4$, and $G$ is a group. Then, $\{H_k(\text{Čech}_\alpha(\alpha, r), G)\}_{r < D}$ is a direct summand of $\{H_k(\text{Čech}_X(X, r), G)\}_{r < D}$ via the inclusion-induced map for all $k \geq 2$.

Proof. The proof is the same as that of Theorem 4.5. □

Definition 8.6. (A version of Definition 5.1). Suppose $0 < a \leq b$, $G$ is a group and $k \in \mathbb{N}$. Subspace $Z$ of a metric space $X$ is Čech $DC(\langle a, b \rangle; k, G)$ isolated if there exist two closed nested neighborhoods $N_1 \subset N_2$ of $Z$, so that:

1. $N_2 \supset N(N_1, r), \forall r \in \langle a, b \rangle$;
(2) for each \( r \in \langle a, b \rangle \), condition \( H_k(\text{Čech}_X(Z, r), G) \neq 0 \) implies that the following maps are trivial:

(a) the inclusion-induced \( H_k(\text{Čech}_X(\partial N_1, r), G) \rightarrow H_k(\text{Čech}_X(N_1, r), G) \)
and \( H_k(\text{Čech}_X(\partial N_1, r), G) \rightarrow H_k(\text{Čech}_X(X \setminus \text{Int}(N_1), r), G) \);
(b) the boundary map
\[ H_k(\text{Čech}_X(X, r), G) \rightarrow H_{k+1}(\text{Čech}_X(N_2 \setminus \text{Int}(N_1), r)) \]

 arising from the Mayer–Vietoris long exact sequence for a decomposition of \( \text{Čech}_X(X, r) \) into \( A = \text{Čech}_X(N_2, r) \) and \( B = \text{Čech}_X(X \setminus \text{Int}(N_1), r) \);

(3) \( N_X(N_2, R) \setminus \text{Int}(N_1) \xrightarrow{DC} \partial N_1 \) and \( N_X(N_1, R) \xrightarrow{DC} \alpha \) for some \( R \) larger that all elements of \( \langle a, b \rangle \).

**Theorem 8.7.** (A version of Theorem 5.3). Suppose \( X \) is a metric space, \( G \) is a group, \( 0 < a \leq b, k \in \mathbb{Z} \), and \( Z \subset X \) is Čech \( DC((a, b); G) \) isolated. Then, \( \{H_k(\text{Čech}_Z(Z, r), G)\}_{r \in \langle a, b \rangle} \) is a direct summand of \( \{H_k(\text{Čech}_X(X, r), G)\}_{r \in \langle a, b \rangle} \) via the inclusion-induced map.

**Theorem 8.8.** (A version of Theorem 7.1). Suppose \( X \) is a geodesic surface, \( \alpha \) is a Čech \( DC(D_1, D_2) \) isolated loop for some \( D_1 > |\alpha|/4, D_2 \geq |\alpha|/2 \), and \( G \) is a group. Assume \( \alpha \) is homologous in \( H_1(X, G) \) to a \( G \)-combination of loops of length at most \( |\alpha| \), none of which intersects \( N_1 \). Then, the following holds:

1. for each \( r \in (|\alpha|/4, D) \), there exists a non-trivial \( Q_r \in H_2(\text{Čech}_X(X, r), G) \), so that:
   (a) for each pair \( q_1 < q_2 \) of parameters from \((|\alpha|/4, D)\), we have \( i_{q_1, q_2}^G(Q_{q_1}) = Q_{q_2} \);
   (b) for any \( q \in (|\alpha|/4, D) \), there exists \( q_0 \leq |\alpha|/4 \), for which \( Q_q \) is in the image of \( i_{q_0, q}^G \);
   (c) If \( \alpha \) is homotopic to some shorter geodesic circle \( \beta \) in \( X \) and \( 4q_3 \) is larger than the homotopy height between \( \alpha \) and \( \beta \), then \( i_{q, q_3}^G(Q_q) \) is trivial for any \( q \in (|\alpha|/4, D) \).
2. If \( G \) is a field and \( \{H_2(\text{Čech}_X(X, r), G)\}_{r>0} \) is \( q \)-tame, then the persistence \( \{H_2(\text{Čech}_X(X, r), G)\}_{r<D} \) contains as a direct summand \( G_{(|\alpha|/4, w'/4)} \).

**Proof.** The proof is essentially the same as that of Theorem 7.1, using Lemma 8.2 and square-like formations of four points on loops instead of Lemma 6.3 and three equidistant points.

### 8.2. Closed Filtrations

A technical complication arising in the case of closed filtration is the fact that deformation contractions (and even strict deformation contractions) do not necessarily induce homotopy equivalences on the corresponding closed Rips or Čech complexes, as demonstrated by Fig. 2. Nonetheless, we can still detect almost the same footprint with the use of interleaving.

Open and closed filtrations are \( \varepsilon \)-interleaved for each \( \varepsilon > 0 \). Hence, by the standard stability results, PDs have the same barcodes in both cases, with a possible change in the interval endpoint types (for example, such a
change occurs in Theorem 2.1 and [27]). This means that the following holds in the case when a PD exists (i.e., if $G$ is a field and the persistent homology is q-tame):

1. the main footprint detection results of this paper (Theorems 4.5, 5.3, 4.5, Corollary 4.6 and their Čech versions) hold for closed filtrations modulo the endpoints, i.e., the endpoints of the detected intervals may change.

2. closed filtrations may see an emergence or disappearance of ephemeral summands, i.e., intervals of length 0, as in Theorem 2.1.

While detecting ephemeral summands may be complicated in general, the understanding of geometric background allows us to detect them in the case of geodesic circles as described by Proposition 8.10.

Definition 8.9. Let $X$ be a metric space and $A \subset X$ a closed subspace. A map $f : X \to A$ is a strict contraction if

- $f(a) = a, \forall a \in A$;
- $d(x, y) > d(f(x), f(y)), \forall x, y \in X \setminus A$.

Proposition 8.10. Suppose $G$ is a graph, $X$ is a geodesic space, and $\alpha \subset X$ is a geodesic circle. Suppose there exists a strict contraction $f : \overline{N}(\alpha, |\alpha| \frac{l}{2l+1}) \to \alpha$. Then, the inclusion $\alpha \to X$ induces an injection

$$H_{2l}(\bigvee_{\mathbb{R}} S^{2l}, G) \cong H_{2l} \left( \overline{\text{Rips}} \left( \alpha, |\alpha| \frac{l}{2l+1} \right), G \right) \to H_{2l} \left( \overline{\text{Rips}} \left( X, |\alpha| \frac{l}{2l+1} \right), G \right).$$

Proof. By Theorem 2.1, $\bigvee_{\mathbb{R}} S^{2l} \cong \overline{\text{Rips}}(\alpha, |\alpha| \frac{l}{2l+1})$. We parameterize $\alpha$ with complex coordinates by isometrically identifying $\alpha$ with $\{z \in \mathbb{C} \mid |z| = |\alpha|/(2\pi)\}$. For each $t \in [0, \frac{1}{2l+1})$, let $\beta_t = \{\exp(i(t + \frac{j}{2l+1})) \mid j \in \{0, 1, \ldots, 2l\}\}$ as defined in [1].

In [1, Sections 7 and 8], the basis of $H_{2l} \left( \overline{\text{Rips}} \left( \alpha, |\alpha| \frac{l}{2l+1} \right), G \right)$ is provided in terms of cross-polytopal spheres, generated by collections of points of the form $\beta_0 \cup \beta_t$. An argument in [1] demonstrating that such a sphere is not nullhomologous in $H_{2l} \left( \overline{\text{Rips}} \left( \alpha, |\alpha| \frac{l}{2l+1} \right), G \right)$ is based on an observation that the simplex spanned by $\beta_t$ is a maximal simplex in $\overline{\text{Rips}} \left( \alpha, |\alpha| \frac{l}{2l+1} \right)$ not appearing in any other such sphere. In particular, a basic geometric observation shows that if for some point $x \in \alpha$ and for some $t$ condition

$$d_\alpha(x, y) \leq |\alpha| \frac{l}{2l+1}, \forall y \in \beta_t$$

holds, then $x \in \beta_t$. This means that no choice of a point $x \in \alpha$ and $t$ satisfies

$$d_\alpha(x, y) < |\alpha| \frac{l}{2l+1}, \forall y \in \beta_t.$$  

This implies that $\beta_t$ is also a maximal simplex in $\overline{\text{Rips}} \left( X, |\alpha| \frac{l}{2l+1} \right)$: if it was in a boundary of a simplex containing $z \notin \beta_t$, then $z \notin \alpha$ as it satisfies the condition of Eq. 8.1, and thus, $f(z)$ satisfies Eq. 8.2, a contradiction. Hence, none of the $\beta_t$ is nullhomologous in $H_{2l} \left( \overline{\text{Rips}} \left( X, |\alpha| \frac{l}{2l+1} \right), G \right)$. □
Proposition 8.11. (A version of Theorem 4.5 for closed filtrations) Suppose $X$ is a geodesic space, $G$ is a field, $0 < a < b$, $k \in \mathbb{N}$, and $\alpha$ is an SDC($((a, b); k, G)$ isolated loop in $X$. Then, each interval of $\{H_k(\text{Rips}(\alpha, r), G)\}_{r \in (a, b)}$ appears in $\{H_k(\text{Rips}(X, r), G)\}_{r \in (a, b)}$ via the inclusion-induced map, with the only potential modification being the left endpoints of the non-trivial intervals, which may be added.

Proof. By Proposition 8.10, the statement holds for the ephemeral summands of $H_k(\text{Rips}(\alpha, r), G)$, i.e., for the case when $k$ is even. If $k$ is odd, then by Theorem 2.1, $\{H_k(\text{Rips}(\alpha, r), G)\}_{r \in (a, b)}$ contains precisely one interval, namely

$$I = \left(\frac{k - 1}{2k}, \frac{k + 1}{2(k + 2)}\right) \cap (a, b).$$

Since $I$ is an open interval, we can use the interleaving argument above for each value of $r \in (a, b)$, to conclude that the inclusion $\alpha \to X$ induces the same interval in $\{H_k(\text{Rips}(X, r), G)\}_{r \in (a, b)}$, with the possible exception of changed endpoints. The right endpoint of $I$ cannot be added, since the triviality of a homology element in $\text{Rips}(X, r)$, which is born before $r$, implies its triviality in $\text{Rips}(X, r)$. □

We conjecture that the left endpoints in Proposition 8.11 do not get added either.

At the first sight, detecting the ephemeral summands as in Proposition 8.10 seems to be a theoretical curiosity. However, results of [1] suggest that when approximating persistence of a geodesic circle by that of a finite subset, the ephemeral summands tend to prolong themselves to non-trivial intervals [see (5) in Sect. 9]. We plan to explore this phenomenon in geodesic spaces in future work. On the other hand, we do not expect our inability to detect the nature of the left endpoints in Proposition 8.11 to have a significant effect on the subsequent computations.

9. An Example of Interpretation

In this section, we present an interpretation of a computational example. From a two-dimensional sphere of radius 1, remove an open 2-dimensional ball of radius $1/4$ to obtain a contractible space $X$, which we equip with the geodesic metric (see the left side of Fig. 9). Alternatively, $X$ can be thought of as a closed portion of a sphere of radius 1 below the upper parallel $\alpha$ of radius $1/4$. Note that $\alpha$ is a geodesic circle in $X$ of length $\pi/2$. We select a subset $S \subset X$ consisting of 4000 points uniformly at random, approximate the geodesic metric on $S$, and use Ripser [7] to compute the PD of the Rips filtration of a random subsample of $S$ consisting of 400 points. The PD up to dimension 3 was computed by Matija Čufar as a part of his master’s thesis [13] and is shown in Fig. 9. It can be interpreted as follows:

(1) Since $X$ is contractible, the PD should have no bars for small $r$ by [21]. However, since we start with a discrete set, we get certain 1-dimensional bars, which are limited to small values of $r$. See [28] for specific bounds on the initial 1-dimensional bars with respect to the density of $S$. 
(2) At approximately $|\alpha|/3 = \pi/6$, two bars are born:

- A short 3-dimensional bar by Theorem 4.5 (see the topmost feature of Fig. 9), which dies at about $2|\alpha|/5 = \pi/5$. The loop $\alpha$ can be located by connecting the vertices of the homology class generating this bar.

- A long 2-dimensional bar by Theorem 7.1, which dies at about $2\pi/3$, which is the length of the equator divided by 3 and thus equals the nullhomotopy height of $\alpha$.

(3) Note that the long 2-dimensional bar above is born slightly earlier than the 3-dimensional bar. This is always the case, as generating the two-dimensional bar only requires a 2-dimensional portion of the generator of the 3-dimensional bar, that spans the sample of $\alpha$.

(4) A pairing of a 3-dimensional bar with 2-dimensional bar indicates that $\alpha$ is contractible in $X$.

(5) We speculate the other short 3-dimensional bars are induced by other geodesic circles (i.e., equator and its rotations) in $X$. We will delve deeper into them in our future work.

Note that, except for small values of $r$, there is essentially no noise in the PD. We are able to interpret almost all of the bars. Initial 1-dimensional bars are unavoidable as we always start with a finite sample (discrete subset). They shorten as the density of our sample increases. The only other unmentioned bar is the short 2-dimensional bar appearing at about the same time as the long 2-dimensional bar. It can be explained by the effect of discretisation and the structure of the 3-dimensional bar born at about the same time.

During our experimentation, we have generated several instances of the PD using the mentioned procedure. The obtained diagrams are qualitatively the same in all instances (and aligned with the interpretation above) with
the only exception being the short isolated 3-dimensional bar, which did not appear in all attempts due to its short length.

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