Some applications of matrix inequalities in Rényi entropy

Hadi Reisizadeh\textsuperscript{1,*} and S. Mahmoud Manjegani\textsuperscript{2,†}

\textsuperscript{1}Department of Electrical and Computer Engineering, 
\textsuperscript{2}Department of Mathematical Sciences, 
Isfahan University of Technology

Abstract

The Rényi entropy is one of the important information measures that generalizes Shannon’s entropy. The quantum Rényi entropy has a fundamental role in quantum information theory, therefore, bounding this quantity is of vital importance. Another important quantity is Rényi relative entropy on which Rényi generalization of the conditional entropy, and mutual information are defined based. Thus, finding lower bound for Rényi relative entropy is our goal in this paper. We use matrix inequalities to prove new bounds on the entropy of type $\beta$, Rényi entropy.

1 Introduction

There are several entropic quantities belonging to the family of $\beta$-entropies that have been shown to be useful. The entropies of type $\beta$ are defined by means of information functions \textsuperscript{3}. Recall that a real function $f$ defined on $[0,1]$ is an information function if it satisfies the boundary conditions

$$f(0) = f(1); \quad f\left(\frac{1}{2}\right) = 1,$$  

(1.1)

and the functional equation

$$f(x) + (1-x)f\left(\frac{y}{1-x}\right) = f(y) + (1-y)f\left(\frac{x}{1-y}\right).$$

\textsuperscript{*}hadi.reisizadeh@gmail.com 
\textsuperscript{†}manjgani@cc.iut.ac.ir
for all \((x, y) \in D\), where
\[
D = \{(x, y); 0 \leq x \leq 1, 0 \leq y \leq 1, \text{ and } x + y \leq 1\}.
\]

Let \(f\) be an information function and \((p_1, p_2, \cdots , p_n)\) be an finite discrete probability distribution. Then in \([3]\) the entropy of the distribution \((p_1, p_2, \cdots , p_n)\) with respect to \(f\) is defined by
\[
H_{\beta}^f(p_1, p_2, \cdots , p_n) = \sum_{i=2}^{n} s_i f\left(\frac{p_i}{s_i}\right), \quad s_i = p_1 + \cdots + p_i; \quad i = 2, \cdots , n.
\]

**Definition 1.1.** \([3]\) Let \(\beta\) be a positive number. We call the real function defined in \([0, 1]\) an **information function of type \(\beta\)** if it satisfies the boundary condition (1.1) and the function equation
\[
f(x) + (1 - x)^\beta f\left(\frac{y}{1 - x}\right) = f(y) + (1 - y)^\beta f\left(\frac{x}{1 - y}\right),
\]
for all \((x, y) \in D\).

The **entropy of type \(\beta\)** of a probability distribution \((p_1, p_2, \cdots , p_n)\) is defined by
\[
H_{\beta}^f(p_1, p_2, \cdots , p_n) = \sum_{i=2}^{n} s_i^\beta f\left(\frac{p_i}{s_i}\right), \quad s_i = p_1 + \cdots + p_i; \quad i = 2, \cdots , n.
\]

where \(f\) is an information function of type \(\beta\).

**Theorem 1.2.** \([3]\) Let \(f\) be an information function of type \(\beta\) with \(\beta \neq 1\). Then
\[
f(x) = (2^{1-\beta} - 1)^{-1}\left[x^\beta + (1 - x)^\beta - 1\right], \quad \text{for all } x \in [0, 1].
\]

**Theorem 1.3.** \([3]\) Let \(\beta\) be a positive number \(\beta\) with \(\beta \neq 1\). Then we have for the entropy of type \(\beta\) of a probability distribution \((p_1, p_2, \cdots , p_n)\),
\[
H_{\beta}^f(p_1, p_2, \cdots , p_n) = (2^{1-\beta} - 1)^{-1}\left(\sum_{i=1}^{n} p_i^\beta - 1\right). \quad (1.2)
\]

The Shannon’s entropy \(H_n(p_1, p_2, \cdots , p_n)\) is the limit function of \(H_{\beta}^f(p_1, p_2, \cdots , p_n)\), when \(\beta \to 1\).

For a positive number \(\beta\) with \(\beta \neq 1\), Rényi \([4]\) has extended the concept of Shannon’s entropy by defining the **entropy of order \(\beta\)** of a probability distribution \((p_1, p_2, \cdots , p_n)\) as
\[
\beta H_n(p_1, p_2, \cdots , p_n) = (1 - \beta)^{-1}\log_2 \sum_{i=1}^{n} p_i^\beta. \quad (1.3)
\]
Let $H$ be a complex Hilbert space with dimension $n$. The set of linear operators on $H$ is denoted by $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the set (the $C^*$-algebra) of all $n$ by $n$ complex matrices. The conjugate transpose of matrix $A \in M_n(\mathbb{C})$ is denoted by $A^*$ or $A^\dagger$. If $A^\dagger = A$, then $A$ is called Hermitian matrix. A Hermitian or self-adjoint matrix $A \in M_n(\mathbb{C})$ is called positive semi definite (resp. positive definite) if $\langle Ax, x \rangle \geq 0$ (resp. $\langle Ax, x \rangle > 0$) for each $x \in \mathbb{C}^n$. The set $M^+_n(\mathbb{C})$ of all positive semi definite matrices is then a closed convex cone in $M_n(\mathbb{C})$ and makes the set of all Hermitian matrices partially ordered: for Hermitian matrices $A$ and $B$, $A \leq B$ if and only if $B - A \in M^+_n(\mathbb{C})$ [2].

**Lemma 1.4.** If $A \geq 0$, then for all $r \in \mathbb{R}$; $A^r \geq 0$.

*Proof.* It follows from definition of positive semidefinite matrix. \(\square\)

**Lemma 1.5.** If $A$ and $B$ are positive semidefinite matrices, then

$$0 \leq tr(AB) \leq tr(A)tr(B).$$

*Proof.* This is an immediate result of the Von Neumann’s trace inequality [5]. \(\square\)

**Lemma 1.6.** Let $A$ and $B$ be $n$ by $n$ positive semidefinite matrices, then

$$n(\det A \det B)^{\frac{1}{n}} \leq tr(AB). \tag{1.4}$$

Equality holds if and only if $B^{\frac{1}{2}}AB^{\frac{1}{2}} = cI$; $c \in \mathbb{R}^+.$

*Proof.* $B$ is a positive semidefinite matrix, therefore it has a unique square root. Also, for square matrices $X$ and $Y$, the trace of $XY$ is equal to the trace of $YX$. It follows that the inequality \((1.4)\) can be rewritten as

$$(\det(B^{\frac{1}{2}}AB^{\frac{1}{2}}))^{\frac{1}{n}} \leq \frac{tr(B^{\frac{1}{2}}AB^{\frac{1}{2}})}{n}.$$ 

Assuming that $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}^+_0$ are the eigenvalues of $B^{\frac{1}{2}}AB^{\frac{1}{2}}$, by arithmetic-geometric mean inequality we have

$$(\det(B^{\frac{1}{2}}AB^{\frac{1}{2}}))^{\frac{1}{n}} = \sqrt[\lambda_1 \lambda_2 \cdots \lambda_n} \leq \frac{\sum_{i=1}^{n} \lambda_i}{n} = \frac{tr(B^{\frac{1}{2}}AB^{\frac{1}{2}})}{n}.$$ 

Moreover, equality holds in the arithmetic-geometric mean inequality if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ which implies equality holds in \((1.4)\), if $B^{\frac{1}{2}}AB^{\frac{1}{2}} = \lambda_1 I$. It is clear that $B^{\frac{1}{2}}AB^{\frac{1}{2}} = cI$ for some $c \in \mathbb{R}^+$, then $n(\det A \det B)^{\frac{1}{n}} = tr(AB).$ \(\square\)
Lemma 1.7. If $A$ is a positive definite matrix, then
\[ \text{tr}(I - A^{-1}) \leq \log \det(A) \leq \text{tr}(A - I). \] (1.5)

Equality holds if and only if $A = I$.

Proof. Let $\lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}_0^+$ be the eigenvalues of $A$. Then by functional calculus theorem, the inequality (1.5) can be rewritten as
\[ \sum_{i=1}^{n}(1 - \frac{1}{\lambda_i}) \leq \sum_{i=1}^{n} \log(\lambda_i) \leq \sum_{i=1}^{n}(\lambda_i - 1), \]
or, by setting $\lambda_i = e^{u_i}$,
\[ \sum_{i=1}^{n}(1 - e^{-u_i}) \leq \sum_{i=1}^{n} u_i \leq \sum_{i=1}^{n}(e^{u_i} - 1), \]
that follows from the convexity of the exponential function. Equality holds if and only if $u_i = 0$ for every $1 \leq i \leq n$ implying $\lambda_i = 1$ for every $1 \leq i \leq n$. Thus, equality holds in (1.5) if and only if $A = I$. \qed

Definition 1.8. An operator $\rho$ on a finite dimensional Hilbert space $H$, is called density operator if it satisfies the following three requirements:

(i) $\rho$ is Hermitian,
(ii) $\text{tr} \rho = 1$, and
(iii) $\rho$ is a positive semi-definite operator.

2 Entropy of type $\beta$ and Rényi entropy

First, we prove some results about the Rényi entropy of order $\beta$. For $n \geq 1$, we denote
\[ \Delta_n = \left\{ (p_1, p_2, \ldots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\}. \]

Theorem 2.1. For all $(p_1, p_2, \ldots, p_n) \in \Delta_n$ and positive real number $\beta$,

1. if $0 < \beta < 1$, then
\[ \beta H_n(p_1, p_2, \ldots, p_n) \geq (1 - \beta)^{-1} \left[ \log_2(n - n_0) + \frac{\beta}{n - n_0} \sum_{i=1}^{n} \log_2 p_i' \right]. \]
2. if $\beta > 1$, then

$$
\beta H_n(p_1, p_2, \ldots, p_n) \leq (1 - \beta)^{-1} \left( \log_2(n - n_0) + \frac{\beta}{n - n_0} \sum_{i=1}^{n-n_0} \log_2 p_i' \right),
$$

where $p_i' \in \{ p_i \in \Delta_n | p_i > 0 \}$ and $n_0$ is the number of $p_i$ that $p_i = 0$.

Proof. We have

$$
\beta H_n(p_1, p_2, \ldots, p_n) = (1 - \beta)^{-1} \log_2 \sum_{i=1}^{n} p_i^\beta
$$

$$
= (1 - \beta)^{-1} \left( \log_2(n - n_0) \sum_{i=1}^{n-n_0} \frac{p_i'^\beta}{n-n_0} \right)
$$

$$
= (1 - \beta)^{-1} \left( \log_2(n - n_0) + \log_2 \sum_{i=1}^{n-n_0} \frac{p_i'^\beta}{n-n_0} \right).
$$

Therefore, for $0 < \beta < 1$, we get

$$
\beta H_n(p_1, p_2, \ldots, p_n) \geq (1 - \beta)^{-1} \left( \log_2(n - n_0) + \sum_{i=1}^{n-n_0} \frac{1}{n-n_0} \log_2 p_i'^\beta \right)
$$

$$
= (1 - \beta)^{-1} \left( \log_2(n - n_0) + \frac{\beta}{n-n_0} \sum_{i=1}^{n-n_0} \log_2 p_i' \right),
$$

and for $\beta > 1$, we get

$$
\beta H_n(p_1, p_2, \ldots, p_n) \leq (1 - \beta)^{-1} \left( \log_2(n - n_0) + \sum_{i=1}^{n-n_0} \frac{1}{n-n_0} \log_2 p_i'^\beta \right)
$$

$$
= (1 - \beta)^{-1} \left( \log_2(n - n_0) + \frac{\beta}{n-n_0} \sum_{i=1}^{n-n_0} \log_2 p_i' \right).
$$

From (1.2) and (1.3) we have the following relations between the entropy of order $\beta$ and the entropy of type $\beta$ [3]:

$$
\beta H_n = (1 - \beta)^{-1} \log_2 \left[ (2^{1-\beta} - 1) H_n^\beta + 1 \right] \quad (2.1)
$$

Theorem 2.2. If $0 < \beta < 1$, then for all $(p_1, p_2, \ldots, p_n) \in \Delta_n$ we have

$$
0 \leq H_n^\beta(p_1, p_2, \ldots, p_n) \leq (2^{1-\beta} - 1)^{-1} \left[ (n - n_0) \left( \prod_{i=1}^{n-n_0} p_i' \right)^{\frac{\beta}{n-n_0}} - 1 \right],
$$

where $p_i' \in \{ p_i \in \Delta_n | p_i > 0 \}$ and $n_0$ is the number of $p_i$ that $p_i = 0$. 
Proof. For $0 < \beta < 1$ from (1.3), we have:

$$(1 - \beta)^{-1} \log_2 \left[ (2^{1-\beta} - 1) H_n^\beta + 1 \right] \leq (1 - \beta)^{-1} \left( \log_2 (n - n_0) + \frac{\beta}{n - n_0} \sum_{i=1}^{n-n_0} \log_2 p_i \right)$$

$$\log_2 \left[ (2^{1-\beta} - 1) H_n^\beta + 1 \right] \leq \log_2 \left[ (n - n_0) \left( \prod_{i=1}^{n-n_0} p_i \right)^{\frac{\beta}{n-n_0}} \right],$$

which implies

$$(2^{1-\beta} - 1) H_n^\beta + 1 \leq (n - n_0) \left( \prod_{i=1}^{n-n_0} p_i \right)^{\frac{\beta}{n-n_0}}.$$ 

Thus,

$$H_n^\beta(p_1, p_2, ..., p_n) \leq (2^{1-\beta} - 1)^{-1} \left[ (n - n_0) \left( \prod_{i=1}^{n-n_0} p_i \right)^{\frac{\beta}{n-n_0}} - 1 \right].$$

Product of non-zero probabilities of distribution identifies upper bound, so it is an important criteria for limiting value of entropy of type $\beta$.

## 3 Some bounds on the quantum Rényi entropy

In this section, we’re going to extend Theorem 2.1 to quantum setting. In [1] the Rényi entropy in quantum setting of order $\alpha \in (0, 1) \cup (1, \infty)$ is given as

$$H_\alpha(\rho) = \frac{1}{1 - \alpha} \log \text{tr} \rho^\alpha.$$ 

**Theorem 3.1.** Let $\rho \in S$ with $\text{rank}(\rho) = d$. Then,

1. if $0 < \alpha < 1$,

$$H_\alpha(\rho) \geq \frac{1}{1 - \alpha} \left( \log(d - d_0) + \frac{\alpha}{d - d_0} \log \prod_{i=1}^{d-d_0} \lambda_i' \right),$$

2. if $\alpha > 1$,

$$H_\alpha(\rho) \leq \frac{1}{1 - \alpha} \left( \log(d - d_0) + \frac{\alpha}{d - d_0} \log \prod_{i=1}^{d-d_0} \lambda_i' \right),$$

where $\lambda_i' \in \{ \lambda_i \in \sigma(\rho) | \lambda_i > 0 \}$ and $d_0$ is the number of $\lambda_i$ that $\lambda_i = 0$. 
Proof. For any density matrix $\rho$, there exist unitary matrix $U$ and diagonal matrix $D$ such that $D = U\rho U^\dagger$. Also, $D \in S$. By property of quantum entropy, we have

$$H_\alpha(D) = H_\alpha(U\rho U^\dagger) = H_\alpha(\rho),$$

$$H_\alpha(D) = \frac{1}{1-\alpha} \log \text{tr} D^\alpha = \frac{1}{1-\alpha} \log \sum_{i=1}^d \lambda_i^\alpha,$$

where

$$D = \begin{pmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_d
\end{pmatrix}$$

and $\lambda_i \in \sigma(\rho)$.

By Theorem 2.1, we get

$$\frac{1}{1-\alpha} \log \sum_{i=1}^d \lambda_i^\alpha \geq \frac{1}{1-\alpha} \left( \log(d - d_0) + \frac{\alpha}{d - d_0} \sum_{i=1}^{d-d_0} \log \lambda_i' \right) \quad \text{if} \quad 0 < \alpha < 1,$$

$$\frac{1}{1-\alpha} \log \sum_{i=1}^d \lambda_i^\alpha \leq \frac{1}{1-\alpha} \left( \log(d - d_0) + \frac{\alpha}{d - d_0} \sum_{i=1}^{d-d_0} \log \lambda_i' \right) \quad \text{if} \quad \alpha > 1.$$

Thus,

$$H_\alpha(D) \geq \frac{1}{1-\alpha} \left( \log(d - d_0) + \frac{\alpha}{d - d_0} \sum_{i=1}^{d-d_0} \log \lambda_i' \right) \quad \text{if} \quad 0 < \alpha < 1,$$

$$H_\alpha(D) \leq \frac{1}{1-\alpha} \left( \log(d - d_0) + \frac{\alpha}{d - d_0} \sum_{i=1}^{d-d_0} \log \lambda_i' \right) \quad \text{if} \quad \alpha > 1.$$

Also, $H_\alpha(D) = H_\alpha(\rho)$.

Theorem 3.2. Let $\rho \in S$ with $\text{rank}(\rho) = d$. Then,

$$H_\alpha(\rho) \leq \log d.$$

Proof. By the convexity of $t^\alpha(\alpha > 1)$, we obtain

$$\sum_{i=1}^d \lambda_i^\alpha = d \left( \frac{1}{d} \sum_{i=1}^d \lambda_i^\alpha \right) \geq d \left( \frac{1}{d} \sum_{i=1}^d \lambda_i \right)^\alpha = d^{1-\alpha}.$$

Thus, for $\alpha > 1$,

$$H_\alpha(\rho) = H_\alpha(D) = \frac{1}{1-\alpha} \log \sum_{i=1}^d \lambda_i^\alpha \leq \frac{1}{1-\alpha} \log d^{1-\alpha} = \log d$$
From concavity of $t^\alpha(0 < \alpha < 1)$, we obtain
\[
\sum_{i=1}^{d} \lambda_i^\alpha = d\left(\frac{1}{d} \sum_{i=1}^{d} \lambda_i^\alpha\right) \leq d\left(\frac{1}{d} \sum_{i=1}^{d} \lambda_i\right)^\alpha = d^{1-\alpha}
\]
Thus, for $0 < \alpha < 1$,
\[
H_\alpha(\rho) = H_\alpha(D) = \frac{1}{1-\alpha} \log \sum_{i=1}^{d} \lambda_i^\alpha \leq \frac{1}{1-\alpha} \log d^{1-\alpha} = \log d.
\]

Corollary 3.3. Let $\rho \in S$ with $\text{rank}(\rho) = d$ and $0 < \alpha < 1$. Then,
\[
\frac{1}{1-\alpha}\left(\log(d - d_0) + \frac{\alpha}{d - d_0} \log \prod_{i=1}^{d-d_0} \lambda_i'\right) \leq H_\alpha(\rho) \leq \log d,
\]
where $\lambda_i' \in \{\lambda_i \in \sigma(\rho)|\lambda_i > 0\}$ and $d_0$ is the number of $\lambda_i$ that $\lambda_i = 0$.

This theorem provides a lower bound which does not depend on the off-diagonal elements (called as coherences in [4]), and it also shows that product of non-zero probabilities of finding the system in the respective states gives us certain least value of quantum Rényi entropy.

4 The Rényi relative entropy, conditional entropy, and mutual information

Definition 4.1. [6] The Rényi relative entropy of $\alpha \geq 0$ is defined as
\[
D_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \text{tr} \left[\rho^\alpha \sigma^{1-\alpha}\right].
\]

Theorem 4.2. For Rényi relative entropy of $\alpha > 1$, we have
\[
D_\alpha(\rho||\sigma) \geq \frac{1}{\alpha - 1} \left(\log d + \frac{\alpha}{d} \log \det(\rho) + \frac{1-\alpha}{d} \log \det(\sigma)\right),
\]
where $d$ is the dimension of $\mathcal{H}(\cdot)$.

Proof. Since $\rho$ and $\sigma$ are positive definite matrices, by Lemma [4], $\rho^\alpha$ and $\sigma^{1-\alpha}$ are also positive definite.
\[
\text{tr} (\rho^\alpha \sigma^{1-\alpha}) \geq d \left(\det(\rho^\alpha) \det(\sigma^{1-\alpha})\right)^{\frac{1}{d}}
\]
\[
\log \text{tr} (\rho^\alpha \sigma^{1-\alpha}) \geq \log d + \frac{1}{d} \log \det(\rho^\alpha) + \frac{1}{d} \log \det(\sigma^{1-\alpha}).
\]
Since $\alpha > 1$,$$
\frac{1}{\alpha - 1} \log \text{tr}(\rho^\alpha \sigma^{1-\alpha}) \geq \frac{1}{\alpha - 1} \left( \log d + \frac{1}{d} \log \det(\rho^\alpha) + \frac{1}{d} \log \det(\sigma^{1-\alpha}) \right).
$$
Thus,$$D_\alpha(\rho||\sigma) \geq \frac{1}{\alpha - 1} \left( \log d + \frac{\alpha}{d} \log \det(\rho) + \frac{1-\alpha}{d} \log \det(\sigma) \right).
$$

**Definition 4.3.** [1] From Rényi relative entropy, we define Rényi generalization of entropy, conditional entropy, and mutual information in analogy respectively with the below formulations;

\[ H_\alpha(A)_{\rho} = -D_\alpha(\rho_A||I_A), \]
\[ H_\alpha(A|B)_{\rho} := \log(dimH_A) - \min_{\sigma_B} D_\alpha(\rho_{AB}||\mu_A \otimes \sigma_B), \]
\[ I_\alpha(A;B)_{\rho} := \min_{\sigma_B} D_\alpha(\rho_{AB}||\rho_A \otimes \sigma_B), \]

where $\mu_A = \frac{1}{\dim H_A} I_A$.

**Theorem 4.4.** For Rényi generalization of conditional entropy and mutual information $\alpha > 1$ , if there are $c_1, c_2 \in \mathbb{R}^+$ and $\sigma_B \in H_B$ that $\mu_A^{1-\alpha} \otimes \sigma_B^{1-\alpha} = c_1 \rho_{AB}^\alpha$, $\rho_A^{1-\alpha} \otimes \sigma_B^{1-\alpha} = c_2 \rho_{AB}^\alpha$, then we have

\[ H_\alpha(A|B)_{\rho} = \log d_A - \frac{1}{\alpha - 1} \left( \log d_A d_B + \frac{2\alpha}{d_A d_B} \log \det(\rho_{AB}) + \log c_1 \right), \]
\[ I_\alpha(A;B)_{\rho} = \frac{1}{\alpha - 1} \left( \log d_A d_B + \frac{2\alpha}{d_A d_B} \log \det(\rho_{AB}) + \log c_2 \right). \]

**Proof.** From Theorem 4.2

\[ D_\alpha(\rho||\sigma) \geq \frac{1}{\alpha - 1} \left( \log d + \frac{\alpha}{d} \log \det(\rho) + \frac{1-\alpha}{d} \log \det(\sigma) \right). \]

Equality holds if and only if $\sigma^{1-\alpha} = c \rho^\alpha$; $c \in \mathbb{R}^+$.

So, if there are $c_1, c_2 \in \mathbb{R}^+$ and $\sigma_B \in H_B$ that $\mu_A^{1-\alpha} \otimes \sigma_B^{1-\alpha} = c_1 \rho_{AB}^\alpha$, $\rho_A^{1-\alpha} \otimes \sigma_B^{1-\alpha} = c_2 \rho_{AB}^\alpha$ for $\alpha > 1$, then

\[ \min_{\sigma_B} D_\alpha(\rho_{AB}||\mu_A \otimes \sigma_B) = \frac{1}{\alpha - 1} \left( \log d_A d_B + \frac{\alpha}{d_A d_B} \log \det(\rho_{AB}) + \frac{1}{d_A d_B} \log \det(c \rho_{AB}^\alpha) \right) = \frac{1}{\alpha - 1} \left( \log d_A d_B + \frac{2\alpha}{d_A d_B} \log \det(\rho_{AB}) + \log c \right). \]

Therefore,

\[ H_\alpha(A|B)_{\rho} = \log d_A - \frac{1}{\alpha - 1} \left( \log d_A d_B + \frac{2\alpha}{d_A d_B} \log \det(\rho_{AB}) + \log c_1 \right). \]
Similarly, we have these conclusion for \( I_\alpha(A;B)_\rho \):

\[
I_\alpha(A;B)_\rho = \frac{1}{\alpha - 1} \left( \log d_A d_B + \frac{2\alpha}{d_A d_B} \log \det(\rho_{AB}) + \log c_2 \right),
\]

where \( \rho_A^{1-\alpha} \otimes \sigma_B^{1-\alpha} = c_2 \rho_{\alpha AB} \), \( c_2 \in \mathbb{R}^+ \), and \( \alpha > 1 \).

Now, we are going to survey previous theorem in a wider sense.

**Theorem 4.5.** For Rényi generalization of mutual information \( \alpha > 1 \), we have

\[
I_\alpha(A;B)_\rho \geq \frac{\alpha}{\alpha - 1} \left( \log d_A d_B + \frac{1}{d_A d_B} \log \det(\rho_{AB}) \right)
\]

**Proof.**

\[
\det(\sigma) = \frac{1}{d^d} \det(d\sigma).
\]

From Lemma 1.7 and for \( \alpha > 1 \),

\[
\frac{1 - \alpha}{d} \log \det(\sigma) \geq \frac{1 - \alpha}{d} \left( \text{tr}(d\sigma - \mathbb{I}) - d \log d \right)
\]

\[
= (\alpha - 1) \log d
\]

From Theorem 4.2

\[
D_\alpha(\rho||\sigma) \geq \frac{1}{\alpha - 1} \left( \log d + \frac{\alpha}{d} \log \det(\rho) + \frac{1 - \alpha}{d} \log \det(\sigma) \right)
\]

\[
\geq \frac{1}{\alpha - 1} \left( \log d + (\alpha - 1) \log d + \frac{1}{d} \log \det(\rho) \right)
\]

\[
= \frac{\alpha}{\alpha - 1} \left( \log d + \frac{1}{d} \log \det(\rho) \right).
\]

Hence,

\[
I_\alpha(A;B)_\rho \geq \frac{\alpha}{\alpha - 1} \left( \log d_A d_B + \frac{1}{d_A d_B} \log \det(\rho_{AB}) \right)
\]

Therefore, we know the least Rényi generalization of mutual information for \( \alpha > 1 \), before finding optimum \( \sigma_B \).

For instance, we suppose \( \rho_{AB} = \frac{1}{d_A d_B} \mathbb{I}_{AB} \), so

\[
c_1 = c_2 = \left( \frac{1}{d_A d_B} \right)^{1-2\alpha}, \quad \rho_A = \text{tr} B(\rho_{AB}) = \frac{1}{d_A} \mathbb{I}_A
\]
and optimum $\sigma_B$ is $\frac{1}{d_B}I_B$, we have below conclusions:

$$H_\alpha(A|B) = \log d_A - \frac{1}{\alpha - 1} \left( \log d_Ad_B + \frac{2\alpha}{d_A} \log(\frac{1}{d_A}I_{AB}) + \log(\frac{1}{d_A}d_B) \right)$$

$$= \log d_A - \frac{1}{\alpha - 1} \left( \log d_Ad_B + \frac{2\alpha}{d_A} \log(\frac{1}{d_A}d_B) + \log(\frac{1}{d_A}d_B \log(\frac{1}{d_A}I_{AB}) + \log(\frac{1}{d_A}d_B) \right)$$

$$= \log d_A - \frac{1}{\alpha - 1} \left( \log d_Ad_B + 2\alpha \log(\frac{1}{d_A}d_B) + (1 - 2\alpha) \log(\frac{1}{d_A}d_B) \right)$$

$$= \log d_A - \frac{1}{\alpha - 1} \left( \log d_Ad_B + \log(\frac{1}{d_A}d_B) \right)$$

$$= \log d_A = H_\alpha(A)$$

and also, $I_\alpha(A;B) = 0$.

The last theorem shows relationship between Renyi relative entropy of $\rho$, $\sigma$ and identity for $\alpha > 1$.

**Theorem 4.6.** For Rényi relative entropy of $\alpha > 1$,

$$D_\alpha(\rho||\sigma) \leq D_\alpha(\rho||I) + D_\alpha(I||\sigma).$$

**Proof.** Using Lemma 1.6, we get

$$\text{tr} (\rho^\alpha \sigma^{1-\alpha}) \leq \text{tr} (\rho^\alpha) \text{tr} (\sigma^{1-\alpha})$$

$$\log \text{tr} (\rho^\alpha \sigma^{1-\alpha}) \leq \log \text{tr} (\rho^\alpha) + \log \text{tr} (\sigma^{1-\alpha})$$

$$\frac{1}{\alpha - 1} \log \text{tr} (\rho^\alpha \sigma^{1-\alpha}) \leq \frac{1}{\alpha - 1} \log \text{tr} (\rho^\alpha) + \frac{1}{\alpha - 1} \log \text{tr} (\sigma^{1-\alpha}) \text{ for } \alpha > 1.$$  

So,

$$D_\alpha(\rho||\sigma) \leq D_\alpha(\rho||I) + D_\alpha(I||\sigma).$$

**References**

[1] M. Berta, K. Seshadreesan, M. M Wilde, Renyi generalizations of the conditional quantum mutual information. J. Math. Phys. 56, 022205 (2015) [arXiv:1403.6102]
[2] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.

[3] Z. Daróczy, Generalized Information Functions, Information and Control, 16, 36-51 (1970).

[4] D. McMahon, Quantum computing explained, John Wiley & Sons, Inc, Hoboken, New Jersey, 2007.

[5] L. Mirsky, A trace inequality of John von Neumann”. Monatshefte für Mathematik 79 (4): 303306 (1975).

[6] M. Müller-Lennert, F. Dupuis, Frédéric, O. Szehr, S. Fehr, Serge, and M. Tomamichel, On quantum Rnyi entropies: A new generalization and some properties. Journal of Mathematical Physics, 54(12), 122203-12220320 (2013).

[7] A. Rényi, On Measure of Entropy and Information, Vol. I, p. 547-561, Proc. 4-th Berkeley Symp. Math. Statist. and Probability 1960, Univ. of Calif. Press, Berkeley, Calif. 1961.