SOME REMARKS ON TWIN GROUPS

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Abstract. The twin group $T_n$ is a right angled Coxeter group generated by $n - 1$ involutions and having only far commutativity relations. These groups can be thought of as planar analogues of Artin braid groups. In this note, we study some properties of twin groups whose analogues are well-known for Artin braid groups. We give an algorithm for two twins to be equivalent under individual Markov moves. Further, we show that twin groups $T_n$ have $R_\infty$-property and are not co-Hopfian for $n \geq 3$.

1. Introduction

The twin group $T_n$, $n \geq 2$, is a right angled Coxeter group with $n - 1$ generators and only far commutativity relations. These groups first appeared in the work of Shabat and Voevodsky [32] in the context of curves over number fields. Later, these groups were investigated by Khovanov [23], who referred them as twin groups and gave a geometric interpretation of these groups similar to the one for Artin braid groups $B_n$. Khovanov considered configurations of $n$ arcs in the infinite strip $\mathbb{R} \times [0, 1]$ connecting $n$ marked points on each of the parallel lines $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{0\}$ such that each arc is monotonic and no three arcs intersect at a common point. Two such configurations are considered equivalent if one can be deformed into the other by a homotopy of such configurations in $\mathbb{R} \times [0, 1]$ keeping the end points of the arcs fixed, and an equivalence class is called a twin. Analogous to braid groups, the product of two twins is defined by placing one twin on top of the other. The collection of twins on $n$ strands under this operation forms a group which is isomorphic to $T_n$.

Analogous to closure of a geometric braid in the 3-space, one can define the closure of a twin on a 2-sphere by taking the one point compactification of the plane. A doodle on a closed oriented surface is a collection of piecewise linear closed curves without triple intersections. It follows that closure of a twin on a 2-sphere is a doodle. Doodles on a 2-sphere were first introduced by Fenn and Taylor [13], and the idea was extended to immersed circles in a 2-sphere by Khovanov [23]. He proved an analogue of the classical Alexander theorem for doodles. On the other hand, an analogue of the Markov theorem for doodles on a 2-sphere has been established by Gotin [19]. Recently, Bartholomew-Fenn-Kamada-Kamada [3, 4] extended the study of doodles to immersed circles in a closed oriented surface of any genus, which can be thought of as virtual links analogue for doodles. A recent preprint [7] constructs an Alexander type invariant for oriented doodles from a deformation of the classical Tits representation of $T_n$ and prove that analogous to the Alexander polynomial for classical links, this invariant vanishes on unlinked doodles with more than one component.

2010 Mathematics Subject Classification. Primary 20F55; Secondary 57M27, 20E45.

Key words and phrases. Co-Hopfian, Doodle, $R_\infty$-property, right angled Coxeter group, split twin, twisted conjugacy class.
Recall that, in view of the classical Alexander and Markov theorems, the problem of classifying isotopy classes of links in the 3-space is equivalent to the algebraic problem of classifying Markov classes of braids in the infinite braid group. The algebraic link problem is to determine whether two braids are equivalent under Markov moves. The conjugacy problem (the first Markov equivalence) in braid groups was solved by Garside [14], whereas the case of second Markov equivalence was attempted by Humphries [20]. Since we have Alexander and Markov theorems for doodles on a 2-sphere, it is natural to formulate an analogue of the algebraic link problem for doodles. We address this problem in Theorem 3.1 of Section 3.

We also study $R^\infty$-property and (co)-Hopfianity of twin groups. A group $G$ is said to have $R^\infty$-property if it has infinitely many $\phi$-twisted conjugacy classes for each automorphism $\phi$ of $G$. Here, two elements $x, y \in G$ are said to lie in the same $\phi$-twisted conjugacy class if there exists $g \in G$ such that $x = gy\phi(g)^{-1}$. The idea of twisted conjugacy arose from the work of Reidemeister [31]. Due to deep connection of twisted conjugacy with Nielsen fixed-point theory, the study of $R^\infty$-property of groups has attracted a lot of attention. It is well-known that braid groups $B_n$ have $R^\infty$-property for all $n \geq 3$ [10]. We refer the reader to [8, 12, 15, 16, 17, 21, 26, 29, 30] for recent works on the topic. In Section 4, we show that twin groups $T_n$ have $R^\infty$-property for $n \geq 3$ (Theorem 4.4).

Recall that a group is co-Hopfian (respectively Hopfian) if every injective (respectively surjective) endomorphism is an automorphism. These properties are closely related to $R^\infty$-property, see, for example [27, Lemma 2.3]. Braid groups $B_n$ are known to be Hopfian being residually finite [22, Chapter I, Corollary 1.22] and are not co-Hopfian for $n \geq 2$ [5]. In fact, the map which sends each standard generator of $B_n$ to itself times a fixed power of the central element extends to an injective endomorphism which is not surjective. It is well-known that Coxeter groups, in particular twin groups, are Hopfian [6, Theorem C, p. 55]. In Section 5, we prove that twin groups $T_n$ are co-Hopfian only for $n = 2$ (Theorem 5.1). We compare our results with braid groups and mention some open problems for the reader.

2. Preliminaries

We begin the section by setting some notations. For elements $g, h$ of a group $G$, we denote $g^{-1}h^{-1}gh$ by $[g, h]$ and the conjugacy class of $g$ by $g^G$.

For an integer $n \geq 2$, the twin group $T_n$ is defined as the group with the presentation

$$\langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2 = 1 \text{ for } 1 \leq i \leq n-1 \text{ and } s_is_j = s_js_i \text{ for } |i-j| \geq 2 \rangle.$$ 

The generator $s_i$ can be geometrically presented by a configuration of $n$ arcs as shown in Figure 1.

![Figure 1. The generator $s_i$](image-url)
The kernel of the natural surjection from $T_n$ onto $S_n$, the symmetric group on $n$ symbols, is called the pure twin group and is denoted by $PT_n$.

2.1. Elementary transformations and cyclically reduced words. We recall some basic ideas and results from [28] on twin groups that will be used in subsequent sections. We begin by defining three elementary transformations of a word $w \in T_n$ as follows:

(i) Deletion: Replace the word $w$ by deleting subword of the form $s_is_i$.

(ii) Insertion: Replace the word $w$ by inserting a word of the form $s_is_i$ in $w$.

(iii) Flip: Replace a subword of the form $s_is_j$ by $s_js_i$ in $w$ whenever $|i - j| \geq 2$.

We say that two words $w_1$ and $w_2$ are equivalent, written as $w_1 \sim w_2$, if there is a finite chain of elementary transformations turning $w_1$ into $w_2$. It is easy to check that $\sim$ is an equivalence relation. Obviously, $w \sim w'$ if and only if both $w$ and $w'$ represent the same element in $T_n$.

For a given word $w = s_{i_1}s_{i_2}\ldots s_{i_k}$ in $T_n$, let $\ell(w) = k$ be the length of $w$. We say that the word $w$ is reduced if $\ell(w) \leq \ell(w')$ for all $w' \sim w$. By well-ordering principle, the equivalence class of each word contains a reduced word. It is possible to have more than one reduced word representing the same element. In fact, all reduced words representing the same element differ by finitely many flip transformations, for example, $s_1s_4$ and $s_4s_1$. Obviously, any two reduced words in the same equivalence class have the same length. This allows us to define the length of an element $w \in T_n$ as the length of a reduced word representing $w$. Below is a characterisation of a reduced word.

Lemma 2.1. A word $w \in T_n$ is reduced if and only if $w$ satisfies the property that whenever two $s_i$’s appear in $w$, there always exist at least one $s_{i-1}$ or $s_{i+1}$ in between them.

A cyclic permutation of a word $w = s_{i_1}s_{i_2}\ldots s_{i_k}$ (not necessarily reduced) is a word $w'$ (not necessarily distinct from $w$) of the form $s_{i_t}s_{i_{t+1}}s_{i_{t+2}}\ldots s_{i_k}s_{i_1}\ldots s_{i_{t-1}}$ for some $1 \leq t \leq k$. It is easy to see that $w' = (s_{i_t}s_{i_{t+1}}\ldots s_{i_{t-1}})^{-1}w(s_{i_1}s_{i_2}\ldots s_{i_{t-1}})$, that is, $w$ and $w'$ are conjugates of each other. A word $w$ is called cyclically reduced if each cyclic permutation of $w$ is reduced. It is immediate that a cyclically reduced word is reduced, but the converse is not true. For example, $s_1s_2s_1$ is reduced but not cyclically reduced.

Corollary 2.2. Each word in $T_n$ is conjugate to some cyclically reduced word.

Theorem 2.3. Let $w_1, w_2$ be two cyclically reduced words in $T_n$. Then, $w_1$ is conjugate to $w_2$ if and only if they are cyclic permutation of each other modulo finitely many flip transformations.

2.2. Doodles on a 2-sphere. A doodle on a 2-sphere with $m$ components is a collection of $m$ piecewise linear closed curves without triple or higher intersection points. Two doodles are said to be equivalent if one can be obtained from the other by an isotopy of the 2-sphere and a finite sequence of local moves $R_1$ and $R_2$ as shown in the Figure 2. An oriented doodle is a doodle with orientation on each of its components. By the closure of a twin $\beta$, we mean a diagram obtained by joining the end points of $\beta$ on a 2-sphere (by taking the one point compactification of the plane) as illustrated in Figure 3.
It is easy to check that the closure of a twin is a doodle. Further, an orientation on a twin gives an orientation on its closure. The analogue of the classical Alexander theorem in this setting has been established in [23, Theorem 2.1].

**Theorem 2.4.** Every oriented doodle on a 2-sphere is the closure of some twin.

For twins $\alpha$ and $\beta$ (possibly with different number of strands), we denote by $\alpha \otimes \beta$, the twin obtained by adding the diagram of $\alpha$ to the left of the diagram of $\beta$. For any positive integer $n$ and $\beta \in T_n$, define the following moves:

- $M_1 : \beta \otimes I \rightarrow I \otimes \beta$,
- $M_2 : \beta \rightarrow \alpha^{-1} \beta \alpha$,
- $M_3 : \beta \rightarrow (\beta \otimes I) s_{i} s_{i+1} s_{i+1} \ldots s_{n-1} s_{n}$,
- $M_4 : \beta \rightarrow (I \otimes \beta) s_{1} s_{2} \ldots s_{i-1} s_{i} s_{i-1} \ldots s_{2} s_{1}$,

where $I$ is a twin with only one strand, $\alpha \in T_n$ and $s_i \in T_{n+1}$.

Two twins are said to be $M$-equivalent if one can be obtained from the other by a finite sequence of moves $M_1 - M_4$ and their inverses. If $M_i(\beta)$ is the twin obtained from $\beta$ by applying the $M_i$-move, then it is easy to check that the closure of $M_i(\beta)$ is equivalent to the closure of $\beta$. For example, the closure of $(s_1 s_2)^3 \in T_3$ and the closure of $(s_2 s_3)^3 s_1 s_2 s_1 \in T_4$ are equivalent by $M_4$-move as shown in Figure 4. The converse has been established by Gotin [19, Theorem 4.1].

**Theorem 2.5.** Any two twins with equivalent closures are $M$-equivalent.
3. Algebraic doodle problem

In case of classical links and braids, the algebraic link problem asks whether given two braids are equivalent under the classical Markov moves. The algebraic doodle problem can be formulated along the similar lines, that is, to determine whether two twins are equivalent under the Markov moves $M_1 - M_4$.

It is easy to note that the $M_1$-move is equivalent to saying that whenever the reduced expression of $\alpha = s_{i_1}s_{i_2} \ldots s_{i_k} \in T_{n+1}$ does not contain $s_n$, we can replace $\alpha$ by $I \otimes \alpha = s_{i_1}s_{i_2} \ldots s_{i_k+1}$. Next, checking the equivalence of twins under the $M_2$-move is same as the conjugacy problem which is solvable in $T_n$ [24, 28]. Thus, we consider the moves $M_3$ and $M_4$ and prove the following result.

**Theorem 3.1.** Given a twin $\alpha \in T_{n+1}$, there is an algorithm to determine whether

1. $\alpha$ can be written as $(\beta \otimes I)s_n s_{n-1} \ldots s_{i+1}s_is_{i+1} \ldots s_{n-1}s_n$,
2. $\alpha$ can be written as $(I \otimes \beta)s_1s_2 \ldots s_{i-1}s_is_{i-1} \ldots s_2s_1$,

for some $\beta \in T_n$ and $1 \leq i \leq n$.

**Proof.** Case (1). We determine whether $\alpha \in T_{n+1}$ can be written as $(\beta \otimes I)s_n s_{n-1} \ldots s_{i+1}s_is_{i+1} \ldots s_{n-1}s_n$, for some $\beta \in T_n$ and $1 \leq i \leq n$. Upon applying Lemma 2.1 we get a reduced word equivalent to $\alpha$ and have the following possibilities:

(i) If there is only one $s_n$ present in the reduced expression of $\alpha$, then we can write $\alpha$ as $\alpha' s_n \alpha''$, where $\alpha', \alpha'' \in T_n$. Such an $\alpha$ can be written in the desired form if and only if there is no $s_{n-1}$ present in $\alpha''$.

(ii) Suppose that there are two $s_n$’s present in the reduced expression of $\alpha$. If the expression does not have a subword of the form $s_n s_{n-1} \ldots s_{i+1}s_is_{i+1} \ldots s_{n-1}s_n$ for any $1 \leq i \leq n-1$, then we cannot write $\alpha$ in the desired form. On the other hand, if the reduced expression of $\alpha$ can be written as $\alpha' s_n s_{n-1} \ldots s_{i+1}s_is_{i+1} \ldots s_{n-1}s_n \alpha''$, then $\alpha$ has the desired form if and only if $\alpha''$ is a word in $s_j$ for $1 \leq j \leq i-2$.

(iii) If the number of $s_n$’s present in the expression is greater than equal to 3, then we cannot write $\alpha$ in the desired form. For, if we get a subword of the form $s_n s_{n-1} \ldots s_{i+1}s_is_{i+1} \ldots s_{n-1}s_n$ for some $i$ and we move this subword to the rightmost position in the reduced expression of $\alpha$ by flip transformations, there will be an $s_n$ present in the expression of $\beta$ which is not possible since $\beta \in T_n$. 

![Figure 4. The closures of $(s_1s_2)^3$ and $(s_2s_3)^3s_1s_2s_1$ being equivalent as doodles.](image-url)
then $\alpha$

The following figure is an example of a split doodle which is the closure of a twin ($s$ split component of the doodle. We define a twin to be a 2-sphere is said to be split.

We now define split doodles and split twins analogous to split links and braids. A doodle on a 2-sphere is said to be split if it contains two disjoint open disks each containing at least one component of the doodle. We define a twin to be split if its closure is a split doodle on a 2-sphere. The following figure is an example of a split doodle which is the closure of a twin $(s_1s_2)^3(s_4s_5)^4$.

![Figure 5. A split doodle](image)

For each $1 \leq i \leq n-1$, let $T_n^i$ be the subgroup of $T_n$ generated by $\{s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{n-1}\}$. The following proposition, whose proof is immediate, gives sufficient conditions for a twin to be split.

**Proposition 3.2.** If $\alpha \in T_n$ satisfy one of the following conditions:

1. $\alpha$ is conjugate to a word in $T_n^i$ for some $1 \leq i \leq n - 1$,
2. $\alpha = (\beta \otimes 1)s_{n-1}s_{n-2}\ldots s_j s_{j+1} s_{j+1} \ldots s_{n-2} s_{n-1}$, where $\beta$ is conjugate of a word in $T_{n-1}^i$, $1 \leq i \leq n - 2$ and $1 \leq j \leq n - 1$,
3. $\alpha = (I \otimes \beta)s_1 s_2 \ldots s_{j-1} s_j s_{j+1} \ldots s_{2s_1}$, where $\beta$ is conjugate of a word in $T_{n-1}^i$, $1 \leq i \leq n - 2$ and $1 \leq j \leq n - 1$,

then $\alpha$ is a split twin.
4. $R_\infty$-property for twin groups

Let $G$ be a group and $\phi$ an automorphism of $G$. Two elements $x, y \in G$ are said to be ($\phi$-twisted) $\phi$-conjugate if there exists an element $g \in G$ such that $x = g\phi(y)g^{-1}$. The relation of $\phi$-conjugation is an equivalence relation and it divides the group into $\phi$-conjugacy classes.

Taking $\phi$ to be the identity automorphism gives the usual conjugacy classes. The number of $\phi$-conjugacy classes $R(\phi) \in \mathbb{N} \cup \{\infty\}$ is called the Reidemeister number of the automorphism $\phi$. We say that a group $G$ has $R_\infty$-property if $R(\phi) = \infty$ for each $\phi \in \text{Aut}(G)$. Obviously, finite groups (in particular $T_2$) do not satisfy $R_\infty$-property. In this section, we prove that twin groups $T_n$ have $R_\infty$-property for each $n \geq 3$. We begin by recalling a basic result on twisted conjugacy classes [11, Corollary 3.2].

**Lemma 4.1.** Let $\phi$ be an automorphism and $\hat{\phi}$ an inner automorphism of a group $G$. Then $R(\hat{\phi}) = R(\phi)$.

The following result relates twisted conjugacy with usual conjugacy.

**Lemma 4.2.** Let $G$ be a group and $\phi$ an order $k$ automorphism of $G$. If $x, y \in G$ are $\phi$-conjugates, then the elements $x\phi(x)\phi^2(x) \cdots \phi^{k-1}(x)$ and $y\phi(y)\phi^2(y) \cdots \phi^{k-1}(y)$ are conjugates (in the usual sense). The converse is not true in general.

**Proof.** Since $x, y \in G$ are $\phi$-conjugates, there exists $z \in G$ such that $x = zy\phi(z^{-1})$. Applying $\phi^i$, $1 \leq i \leq k-1$, to this equality gives

$$
\phi(x) = \phi(z\phi(y)\phi^2(z^{-1})) = \phi(z\phi(y)\phi^2(z)),
$$

$$
\phi^2(x) = \phi^2(z\phi(y)\phi^2(z^2)) = \phi^2(z\phi(y)\phi^2(z)),
$$

$$
\vdots
$$

$$
\phi^{k-1}(x) = \phi^{k-1}(z\phi^{k-1}(y)) = \phi^{k-1}(z)\phi^{k-1}(y)z^{-1}.
$$

Multiplying the preceding equalities gives $x\phi(x)\phi^2(x) \cdots \phi^{k-1}(x) = z(y\phi(y)\phi^2(y) \cdots \phi^{k-1}(y))z^{-1}$, which is the first assertion.

For the second assertion, consider the extra-special $p$-group

$$
\mathcal{G} = \langle a, b, c \mid a^p = b^p = c^p = 1, ab = bac, ac = ca, bc = cb \rangle
$$

of order $p^3$ and exponent $p$, where $p$ is an odd prime. Note that $\mathcal{G}' = Z(\mathcal{G}) = \langle c \rangle$ is of order $p$. It is easy to check that the map $\phi : \mathcal{G} \to \mathcal{G}$ given by $\phi(a) = ac$ and $\phi(b) = bc$ extends to an order $p$ automorphism of $\mathcal{G}$. Then, we have $a\phi(a)\phi^2(a) \cdots \phi^{p-1}(a) = 1 = b\phi(b)\phi^2(b) \cdots \phi^{p-1}(b)$.

Suppose that there exists $g \in \mathcal{G}$ such that $a = gb\phi(g^{-1})$. This gives $a = gb^{-1}c^l$ for some $l \in \mathbb{Z}$. Thus, $a = bb^{-1}gb^{-1}c^l = b[b, g^{-1}]c^l \in b\mathcal{G}'$, which is not possible. Hence, $a$ cannot be $\phi$-conjugate to $b$. \hfill $\Box$

The following result [28, Theorem 6.1] will be used in the proof of the main theorem.

**Theorem 4.3.** Let $T_n$ be the twin group with $n \geq 3$. Then the following hold:

1. $\text{Aut}(T_3) = \text{Inn}(T_3) \times \langle \psi \rangle \cong \text{Inn}(T_3) \times \mathbb{Z}_2$,
2. $\text{Aut}(T_4) = \text{Inn}(T_4) \times \langle \psi, \tau \rangle \cong \text{Inn}(T_4) \times S_3$,
3. $\text{Aut}(T_n) = \text{Inn}(T_n) \times \langle \psi, \kappa \rangle \cong \text{Inn}(T_n) \times D_8$ for $n \geq 5$,

where
Theorem 4.4. \( T_n \) satisfy \( R_{\infty} \)-property for all \( n \geq 3 \).

Proof. It follows from [28] Proposition 5.1 that \( T_n \) has infinitely many conjugacy classes for all \( n \geq 3 \). Hence, due to Lemma 4.1, it is enough to show that \( T_n \) has infinitely many \( \phi \)-conjugacy classes for automorphisms \( \phi \) in the groups \( \langle \psi \rangle \), \( \langle \psi, \tau \rangle \) and \( \langle \psi, \kappa \rangle \), which are all finite. Our plan is to use Lemma 4.2. We divide the proof into four cases, namely, \( n \geq 6 \), \( n = 5 \), \( n = 4 \) and \( n = 3 \).

Case \( n \geq 6 \). Consider the sequence of elements \( x_i = (s_1 s_2)^i, i \geq 1 \). We claim that for any automorphism \( \phi \in \langle \psi, \kappa \rangle \), \( x_i \) is not \( \phi \)-conjugate to \( x_j \) whenever \( i \neq j \). Let us, on the contrary, suppose that \( x_i \) is \( \phi \)-conjugate to \( x_j \) for some \( i \neq j \). Then, by Lemma 4.2, \( x_i, x_j \) are not conjugates in \( T_n \). Observe that \( x_i, x_j \) are conjugates, where \( k \) is the order of the automorphism \( \phi \).

Note that the subgroup \( H = \langle s_1, s_2, s_{n-2}, s_{n-1} \rangle \) of \( T_n \) is invariant under all automorphisms in \( \langle \psi, \kappa \rangle \). In fact, \( \psi(x) = \kappa(x) \) for all \( x \in H \). Thus, it is sufficient to show that \( x_i, x_j \) are not conjugates in \( T_n \). Observe that \( x_i x_j (s_1 s_2)^i (s_{n-1} s_{n-2})^j \) and \( x_j x_i (s_1 s_2)^j (s_{n-1} s_{n-2})^i \) are cyclically reduced, and hence by Theorem 2.3 they are not conjugates of each other for \( i \neq j \). Thus, we have infinitely many \( \phi \)-conjugacy classes in \( T_n \), \( n \geq 6 \), for any automorphism \( \phi \) of \( T_n \).

Case \( n = 5 \). For this case we consider the sequence of elements \( x_i = (s_1 s_2)^i, i \geq 1 \). We claim that for any automorphism \( \phi \in \langle \psi, \kappa \rangle \), \( x_i \) is not \( \phi \)-conjugate to \( x_j \) whenever \( i \neq j \). Note that there are two automorphisms of order 4, namely \( \kappa \) and \( \kappa^3 \), and five automorphisms of order 2, namely \( \kappa^2 \), \( \psi \kappa \), \( \psi \kappa^2 \), \( \psi \kappa^3 \) and \( \psi \).

Direct computations give

\[
x_i \kappa^2(x_i) \kappa^3(x_i) = (s_1 s_2)^i (s_1 s_3)^2i (s_2 s_3)^2i (s_4 s_1 s_3)^2i (s_4 s_3)^2i = (s_1 s_2)^i (s_4 s_3)^2i (s_1 s_2)^2i (s_4 s_3)^2i.
\]

Again, by Theorem 2.3, \( (s_1 s_2)^i (s_1 s_3)^2i (s_2 s_3)^2i (s_4 s_1 s_3)^2i (s_4 s_3)^2i \) and \( (s_1 s_2)^2i (s_4 s_3)^2i (s_1 s_2)^j (s_4 s_3)^2j \) are not conjugate for \( i \neq j \). Similarly, \( x_i \psi \kappa^2(x_i) = (s_1 s_2)^i (s_1 s_3 s_2)^2i (s_4 s_3)^2i = (s_1 s_2)^2i (s_1 s_3 s_2)^2i \). As before, \( x_i \psi \kappa^2(x_i) \) is not conjugate to \( x_j \psi \kappa^2(x_j) \) for \( i \neq j \). The remaining automorphisms can be considered in the same manner, and hence there are infinitely many \( \phi \)-conjugacy classes in \( T_5 \) for any automorphism \( \phi \).

Case \( n = 4 \). We again consider the sequence of elements \( x_i = (s_1 s_2)^i, i \geq 1 \), and prove that for any automorphism \( \phi \in \langle \psi, \tau \rangle \), \( x_i \) is not \( \phi \)-conjugate to \( x_j \) whenever \( i \neq j \). Note that there are two automorphisms of order 3, namely \( \tau \) and \( \tau^2 \), and three automorphisms of order 2, namely \( \psi \tau \), \( \psi \tau^2 \) and \( \psi \). Direct computations yield \( x_i \tau(x_i) \tau^2(x_i) = (s_1 s_2)^i (s_1 s_3 s_2)^i (s_3 s_2)^i \). Again by Theorem 2.3, \( x_i \tau(x_i) \tau^2(x_i) \) is not conjugate to \( x_j \tau(x_j) \tau^2(x_j) \) whenever \( i \neq j \). The remaining automorphisms can be dealt with similarly, and the assertion follows.

Case \( n = 3 \). Unlike the earlier cases, here we consider the sequence of elements \( x_i = (s_1 s_2)^i s_1, i \geq 1 \). In this case we need to consider only one automorphism \( \psi \) which is of order 2. We have
Suppose that $1 \neq w \in \ker(\psi_3)$. Without loss of generality we may assume that $w$ is a reduced word. Suppose that $w = w_1s_3w_2s_3 \cdots w_k$ for some integer $m$. Then, we have

$$\psi_4(w) = \psi_4(w_1)s_3\psi_4(w_2)s_3 \cdots \psi_4(w_k)s_3\psi_4(w_{k+1}) = 1.$$
Notice that all the \(w_i\)'s are non-trivial words in \(T_3\), since \(w\) is a reduced word. Also, the map \(\psi_4\) restricted to \(T_3\) is \(\psi_3\) which is injective. Thus, \(\psi_4(w_i) = \psi_3(w_i) \neq 1\) for all \(1 \leq i \leq k + 1\). For \(\psi_4(w) = 1\) to be true, all the \(s_3\)'s must get cancelled. But there will always be at least one \(s_2\) in between any two \(s_3\)'s, which is a contradiction. Therefore, the map \(\psi_4\) is injective.

Let us now assume that \(\psi_{n-1}\) is injective for \(n \geq 5\). Consider a non-trivial reduced word \(w\) in \(\text{Ker}(\psi_n)\). Let \(w = w_1s_{n-1}w_2s_{n-1} \cdots w_ks_{n-1}w_{k+1}\), where \(w_i \in T_{n-1}\) for all \(1 \leq i \leq k + 1\). This implies that

\[
\psi_n(w) = \psi_{n-1}(w_1)s_{n-1}\psi_{n-1}(w_2)s_{n-1} \cdots \psi_{n-1}(w_k)s_{n-1}\psi_{n-1}(w_{k+1}) = 1.
\]

For the above equality to be true, all the \(s_{n-1}\)'s must get cancelled. In particular, the two \(s_{n-1}\)'s in the subword \(s_{n-1}\psi_{n-1}(w_j)s_{n-1}\) must get cancelled. This implies that \(\psi_{n-1}(w_j)\) does not have \(s_{n-2}\), which means that \(w_j\) does not have \(s_{n-2}\), which contradicts the fact that \(w\) is reduced. Hence, \(\psi_n\) is injective. \(\Box\)

**Remark 5.2.** Note that the infinite cyclic group and a free product of any two non-trivial groups is not co-Hopfian. Thus, \(PT_n\) is not co-Hopfian for \(3 \leq n \leq 6\). Whether \(PT_n\) is co-Hopfian for \(n \geq 7\) remains unknown.

**Remark 5.3.** It is well-known that the braid group \(B_n\) is not co-Hopfian for \(n \geq 2\) [5]. In fact, the map \(\phi_n : B_n \to B_n, n \geq 2\), defined on the standard generators by

\[
\phi_n(\sigma_i) = \sigma_i z,
\]

where \(\langle z \rangle = Z(B_n)\), is an injective homomorphism which is not surjective. Since \(\phi_n(P_n) \subset P_n\) and \(z \in Z(B_n) = Z(P_n)\) does not have a preimage under \(\phi_n\), it follows that the restriction of \(\phi_n\) on \(P_n\) is injective but not surjective, and hence \(P_n\) is not co-Hopfian for \(n \geq 2\).

**Acknowledgement.** The authors thank the referee for many useful comments. Also, Timur Nasybullov is thanked for many comments and discussions on twisted conjugacy, in particular for Lemma 4.2, which considerably shortened the original proof of Theorem 4.4. Tushar Kanta Naik and Neha Nanda thank IISER Mohali for the Post Doctoral and the PhD Research Fellowships, respectively. Mahender Singh acknowledges support from the Swarna Jayanti Fellowship grants DST/SJF/MSA-02/2018-19 and SB/SJF/2019-20/04.

**References**

[1] Valeriy G. Bardakov, Mikhail V. Neshchadim and Mahender Singh, *Automorphisms of pure braid groups*, Monatsh. Math. 187 (2018), no. 1, 1–19.

[2] Valeriy Bardakov, Mahender Singh and Andrei Vesnin, *Structural aspects of twin and pure twin groups*, Geom. Dedicata 203 (2019), 135–154.

[3] Andrew Bartholomew, Roger Fenn, Naoko Kamada and Seiichi Kamada, *Colorings and doubled colorings of virtual doodles*, Topology Appl. 264 (2019), 290–299.

[4] Andrew Bartholomew, Roger Fenn, Naoko Kamada and Seiichi Kamada, *Doodles on surfaces*, J. Knot Theory Ramifications 27 (2018), no. 12, 1850071, 26 pp.

[5] Robert W. Bell and Dan Margalit, *Braid groups and the co-Hopfian property*, J. Algebra 303 (2006), no. 1, 275–294.

[6] Kenneth S. Brown, *Buildings*, Springer-Verlag, New York, 1989. viii+215 pp.

[7] Bruno Cisneros, Marcelo Flores, Jesús Juyumaya and Christopher Roque-Márquez, *An Alexander type invariant for doodles*, [arXiv:2005.06290](https://arxiv.org/abs/2005.06290)

[8] Charles Garnet Cox, *Twisted conjugacy in Houghton’s groups*, J. Algebra 490 (2017), 390–436.
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[9] Karel Dekimpe and Daciberg Gonçalves, R∞ property for free groups, free nilpotent groups and free solvable groups, Bull. Lond. Math. Soc. 46 (2014), no. 4, 737–746.
[10] Alexander Fel'shtyn and Daciberg L. Gonçalves, Twisted conjugacy classes in symplectic groups, mapping class groups and braid groups, Geom. Dedicata 146 (2010), 211–223.
[11] Alexander Fel'shtyn, Yuriy Leonov and Evgenij Troitsky, Twisted conjugacy classes in saturated weakly branch groups, Geom. Dedicata 134 (2008), 61–73.
[12] Alexander Fel'shtyn and Timur Nasybullov, The R∞ and S∞ properties for linear algebraic groups, J. Group Theory 19 (2016), no. 5, 901–921.
[13] Roger Fenn and Paul Taylor, Introducing doodles, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), pp. 37–43, Lecture Notes in Math., 722, Springer, Berlin, 1979.
[14] F. A. Garside, The braid groups and other groups, Quart. J. Math. (Oxford) 20 (1969), 235–254.
[15] Daciberg Lima Gonçalves and Timur Nasybullov, On groups where the twisted conjugacy class of the unit element is a subgroup, Comm. Algebra 47 (2019), no. 3, 930–944.
[16] Daciberg Gonçalves and Parameswaran Sankaran, Sigma theory and twisted conjugacy, II: Houghton groups and pure symmetric automorphism groups, Pacific J. Math. 280 (2016), no. 2, 349–369.
[17] Daciberg Lima Gonçalves and Parameswaran Sankaran, Twisted conjugacy in PL-homeomorphism groups of the circle, Geom. Dedicata 202 (2019), 311–320.
[18] Jesús González, Jośe Luis León-Medina and Christopher Roque, Linear motion planning with controlled collisions and pure planar braids, (2019), arXiv:1902.06190v2.
[19] Konstantin Gotin, Markov theorem for doodles on two-sphere, (2018), arXiv:1807.05337
[20] Stephen F. Humphries, On reducible braids and composite braids, Glasg. Math. J. 36 (1994), 197–199.
[21] A. Juhász, Twisted conjugacy in certain Artin groups, Ischia group theory 2010, 175–195, World Sci. Publ., Hackensack, NJ, 2012.
[22] Christian Kassel and Vladimir Turaev, Braid groups, Graduate Texts in Mathematics, 247, Springer, New York (2008), xii+340 pp.
[23] Mikhail Khovanov, Doodle groups, Trans. Amer. Math. Soc. 349 (1997), 2297–2315.
[24] Daan Krammer, The conjugacy problem for Coxeter groups, Groups Geom. Dyn. 3 (2009), no. 1, 71–171.
[25] Jacob Mostovoy and Christopher Roque-Márquez, Planar pure braids on six strands, J. Knot Theory Ramifications 29 (2020), no. 1, 1950097, 11 pp.
[26] T. Mubeena and P. Sankaran, Twisted conjugacy and quasi-isometric rigidity of irreducible lattices in semisimple Lie groups, Indian J. Pure Appl. Math. 50 (2019), no. 2, 403–412.
[27] T. Mubeena and P. Sankaran, Twisted conjugacy classes in abelian extensions of certain linear groups, Canad. Math. Bull. 57 (2014), no. 1, 132–140.
[28] Tushar Kanta Naik, Neha Nanda and Mahender Singh, Conjugacy classes and automorphisms of twin groups, Forum Math. to appear, arXiv:1906.06723v5.
[29] Timur Nasybullov, Reidemeister spectrum of special and general linear groups over some fields contains 1, J. Algebra Appl. 18 (2019), no. 8, 1950153, 12 pp.
[30] Timur Nasybullov, Twisted conjugacy classes in untriangular groups, J. Group Theory 22 (2019), no. 2, 253–266.

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