A non-existence result for the Ginzburg–Landau equations

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Abstract
We consider the stationary Ginzburg–Landau equations in \( \mathbb{R}^d \), \( d = 2, 3 \). We exhibit a class of applied magnetic fields (including constant fields) such that the Ginzburg–Landau equations do not admit finite energy solutions. To cite this article: A. Kachmar, M. Persson, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

Résumé
Un résultat de non-existence pour les équations de Ginzburg–Landau. Nous considérons les équations de Ginzburg–Landau dans \( \mathbb{R}^d \), \( d = 2, 3 \). Nous exhibons une classe de champs magnétiques appliqués telle que les équations de Ginzburg–Landau n’admettent pas de solution d’énergie finie. Pour citer cet article : A. Kachmar, M. Persson, C. R. Acad. Sci. Paris, Ser. I 347 (2009).

1. Introduction

The aim of the present note is to study the Ginzburg–Landau system of equations in \( \mathbb{R}^2 \),

\[
\begin{align*}
-(\nabla - iA)^2 \psi &= (1 - |\psi|^2) \psi, \\
-\nabla^\perp (\text{curl} A - H) &= \text{Im}(\psi(\nabla - iA)\psi).
\end{align*}
\]

Here \( \psi \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{C}) \) is the complex order parameter, \( A \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) \) is the magnetic vector potential, \( \text{curl} A \) is the induced magnetic field

\[
B = \text{curl} A = \partial_{x_2} A_1 - \partial_{x_1} A_2,
\]

\( H \in L^2_{\text{loc}}(\mathbb{R}^2) \) is the applied magnetic field, and \( \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}) \) is the Hodge gradient.

Solutions of (1) are of particular interest in the physics literature as they do include periodic solutions with vortices distributed in a uniform lattice, named as Abrikosov’s solution. We refer the reader to [1] for the physical motivation and to [2,4] for mathematical results in that direction.
Eqs. (1) are formally the Euler–Lagrange equations of the following Ginzburg–Landau energy,

\[ \mathcal{G}(\psi, A) = \int_{\mathbb{R}^2} \left( |(\nabla - iA)\psi|^2 + \frac{1}{2} (1 - |\psi|^2)^2 + |\text{curl} A - H|^2 \right) \, dx. \]  

(3)

A solution \((\psi, A)\) of (1) is said to have finite energy if \(\mathcal{G}(\psi, A) < \infty\). When the applied magnetic field \(H \in L^2(\mathbb{R}^2)\), it is proved in [6,8] that the system (1) admits finite energy solutions. In the present note, we would like to discuss the optimality of the hypothesis \(H \in L^2(\mathbb{R}^2)\) thereby establishing negative results when this hypothesis is violated.

Our result is that if \(H\) is not allowed to decay fast at infinity (especially if it is constant), then there are no finite energy solutions to (1):

**Theorem 1.** Let \(\alpha < 1\). Assume that the applied magnetic field \(H \in L^2_{\text{loc}}(\mathbb{R}^2)\) and that there exist constants \(R_0 > 0\) and \(h > 0\) such that \(H(x) \geq \frac{h}{|x|^\alpha}\) for all \(x\) with \(|x| > R_0\). Then the Ginzburg–Landau system (1) does not admit finite energy solutions.

**Remark 2.** We note that \(\frac{1}{|x|^\alpha} \in L^2(\mathbb{R}^2 \setminus B(0, 1))\) if and only if \(\alpha > 1\), which means that the result in Theorem 1 is really complementary to the results in [6,8].

**Remark 3.** The same non-existing result still holds if we instead impose the following properties on \(H\): (1) \(H \notin L^2(\mathbb{R}^2)\), (2) there exists \(R_0 > 0\) such that for \(H(x)\) is positive for \(|x| > R_0\), and (3) there exists \(R_1 > 0\) such that the reverse Hölder-inequality

\[ \int_{B(0, R)} H(x) \, dx \geq |B(0, R)|^{1/2} \left( \int_{B(0, R)} H(x)^2 \, dx \right)^{1/2} \]  

(4)

holds for all \(R > R_1\). The proof follows the proof of Theorem 1 until the end, where the alternative properties of \(H\) are used.

We conclude by mentioning an immediate generalization to the 3-dimensional equations. Let \(H = (H_1, H_2, H_3) \in L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)\) be a given vector field. Consider the Ginzburg–Landau equations in \(\mathbb{R}^3\),

\[ \begin{cases} 
-(\nabla - iA)^2 \psi = (1 - |\psi|^2)\psi, \\
-\text{curl}(\text{curl} A - H) = \text{Im}(\psi(\nabla - iA)\psi). 
\end{cases} \]  

(5)

A solution \((\psi, A) \in H^1_{\text{loc}}(\mathbb{R}^3; \mathbb{C}) \times H^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)\) is said to have finite energy if \(\mathcal{E}(\psi, A) = \int_{\mathbb{R}^3} \left( |(\nabla - iA)\psi|^2 + \frac{1}{2} (1 - |\psi|^2)^2 + |\text{curl} A - H|^2 \right) \, dx < \infty\). We have then a similar result to Theorem 1.

**Theorem 4.** Let \(\alpha < \frac{3}{2}\). Assume that there exist \(h > 0\) and \(R_0 > 0\) such that the applied magnetic field \(H = (H_1, H_2, H_3) \in L^2_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)\) satisfies, \(H_3(x) \geq \frac{h}{|x|^\alpha}\) \(\forall x\) such that \(|x| \geq R_0\). Then the Ginzburg–Landau system (5) does not admit finite energy solutions.

**Remark 5.** Remark 3 carries over to three dimensions, but with any component \(H_j\) in place of \(H\).

The proof of Theorem 4 is exactly the same as that of Theorem 1. So, we will give details only for the proof of Theorem 1. The essential key for proving Theorem 1 is from the spectral theory of magnetic Schrödinger operators stated in Lemma 7 below.

### 2. Two auxiliary lemmas

We start with the following observation concerning the Ginzburg–Landau system (1):

**Lemma 6.** Assume that \(H \in L^2_{\text{loc}}(\mathbb{R}^2)\). Let \((\psi, A)\) be a weak solution of (1) such that \(\mathcal{G}(\psi, A) < \infty\). Then \(|\psi| \leq 1\) in \(\mathbb{R}^2\).
Proof. This result was proved by Yang [7, Lemma 3.1] for $\mathbb{R}^3$ under the assumption $H \in L^2(\mathbb{R}^3)$. The assumption on $H$ is not used in Yang’s proof but the proof only relies on the fact that the energy of $(\psi, A)$ is finite. The proof of this lemma is line-by-line the same as [7], but with $\mathbb{R}^2$ in place of $\mathbb{R}^3$. □

A key-ingredient is the following result from the spectral theory of magnetic Schrödinger operators.

Let $\chi$ be a cut-off function such that $0 \leq \chi \leq 1$, $\chi = 1$ in $[0, \frac{1}{2}]$ and $\chi = 0$ in $[1, \infty)$. For all $R > 0$, we introduce the function,

$$\chi_R(x) = \chi\left(\frac{|x|}{R}\right), \quad \forall x \in \mathbb{R}^2.$$  

(6)

Lemma 7. There exists a constant $C > 0$ such that, for all $\psi \in H^1(\mathbb{R}^2; \mathbb{C})$, $A \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ and $R > 0$, the following inequality holds,

$$\int_{B(0,R)} |(\nabla - iA)\psi|^2 \, dx \geq \frac{1}{2} \int_{B(0,R)} B(x)|\chi_R\psi|^2 \, dx - \frac{C}{R^2} \int_{B(0,R) \setminus B(0,R/2)} |\psi(x)|^2 \, dx.$$  

Here $B = \text{curl } A$ and $\chi_R$ the function from (6).

Proof. We write,

$$\int_{B(0,R)} |(\nabla - iA)\psi|^2 \, dx \geq \int_{B(0,R)} |\chi_R(\nabla - iA)\psi|^2 \, dx \geq \frac{1}{2} \int_{B(0,R)} |(\nabla - iA)(\chi_R\psi)|^2 \, dx - \int_{B(0,R)} |\psi\nabla\chi_R|^2 \, dx.$$  

To finish the proof, we just use the following well known inequality (see [3] or [5, Lemma 2.4.1]),

$$\int_{B(0,R)} |(\nabla - iA)\phi|^2 \, dx \geq \int_{B(0,R)} B(x)|\phi|^2 \, dx, \quad \forall \phi \in H^1_0(B(0, R)).$$  

3. Proof of Theorem 1

Assume that $(\psi, A)$ is a finite energy solution of (1). Thanks to Lemma 6 we have $|\psi| \leq 1$ in $\mathbb{R}^2$.

Recalling the hypothesis on the applied magnetic field $H$ that we assumed in Theorem 1, we may pick $R_0 > 0$ such that

$$H(x) \geq \frac{\hbar}{|x|^2}, \quad \forall |x| \geq R_0.$$  

(7)

Applying Lemma 7, with $(\psi, A)$ as above, a solution of (1), we obtain with $B = \text{curl } A$,

$$\int_{\mathbb{R}^2} |(\nabla - iA)\psi|^2 \, dx \geq \frac{1}{2} \int_{B(0,R)} B(x)|\chi_R\psi|^2 \, dx - \frac{C}{R^2} \int_{B(0,R) \setminus B(0,R/2)} |\psi|^2 \, dx.$$  

Let $R > 2R_0$ and $\Omega_R = \{x \in \mathbb{R}^2: \ R_0 < |x| < R\}$. Then we may write,

$$\int_{\mathbb{R}^2} |(\nabla - iA)\psi|^2 \, dx \geq \frac{1}{2} \int_{\Omega_R} B(x)|\chi_R\psi|^2 \, dx + \frac{1}{2} \int_{B(0,R_0)} B(x)|\chi_R\psi|^2 \, dx - \frac{C}{R^2} \int_{B(0,R) \setminus B(0,R/2)} |\psi|^2 \, dx.$$  

Using that $\int_{\mathbb{R}^2} |(\nabla - iA)\psi|^2 \, dx \leq \mathcal{G}(\psi, A), A \in H^1_{\text{loc}}(\mathbb{R}^2)$ and $|\chi_R\psi| \leq 1$, we get a constant $C_0$ depending on $R_0$ such that,

$$\mathcal{G}(\psi, A) \geq \frac{1}{2} \int_{\Omega_R} B(x)|\chi_R\psi|^2 \, dx - C_0.$$  

(8)

So, let us handle the first term in the right-hand side above. We write,
\[ \int_{\Omega_R} B(x) |\chi_R \psi|^2 \, dx = \int_{\Omega_R} H(x) |\chi_R \psi|^2 \, dx + \int_{\Omega_R} (B(x) - H(x)) |\chi_R \psi|^2 \, dx. \]  

(9)

In order to handle the last term on the right of (9), we apply a Cauchy–Schwarz inequality and use the fact that $|\chi_R \psi| \leq 1$. In this way we get,

\[ \left| \int_{\Omega_R} (B(x) - H(x)) |\chi_R \psi|^2 \, dx \right| \leq \left( \int_{\Omega_R} |B(x) - H(x)|^2 \, dx \right)^{1/2} \left( \int_{\Omega_R} |\psi|^4 \, dx \right)^{1/2} \leq (\mathcal{G}(\psi, A))^{1/2} |\Omega_R|^{1/2}. \]

Implementing this bound together with (7) in the right side of (8), we get the following lower bound,

\[ \int_{\Omega_R} B(x) |\chi_R \psi|^2 \, dx \geq \int_{\Omega_R} \frac{\hbar}{|x|^\alpha} |\chi_R \psi|^2 \, dx - (\mathcal{G}(\psi, A))^{1/2} |\Omega_R|^{1/2}. \]  

(10)

We need only to bound from below $\int_{\Omega_R} \frac{\hbar}{|x|^\alpha} |\chi_R \psi|^2 \, dx$. Actually, using that $\chi_R = 1$ in $B(0, R/2)$ and a Cauchy–Schwarz inequality, we obtain,

\[ \int_{\Omega_{R/2}} \frac{\hbar}{|x|^\alpha} |\chi_R \psi|^2 \, dx = \int_{\Omega_{R/2}} \frac{\hbar}{|x|^\alpha} \, dx + \int_{\Omega_{R/2}} \frac{\hbar}{|x|^\alpha} (|\psi|^2 - 1) \, dx \]

\[ \geq \frac{2\pi \hbar}{2 - \alpha} \left( (R/2)^{2-\alpha} - R_0^{2-\alpha} \right) - (\mathcal{G}(\psi, A))^{1/2} \left( \frac{2\pi}{2 - 2\alpha} \right)^{1/2} \left( (R/2)^{2-2\alpha} - R_0^{2-2\alpha} \right)^{1/2}. \]

Now we use the assumption that $\mathcal{G}(\psi, A) < \infty$. In this way, we get by implementing the right-hand side above in (10) and then by substituting the resulting lower bound into (8), a constant $C$ such that,

\[ (\mathcal{G}(\psi, A))^{1/2} \leq \frac{2^{\alpha-1} \pi \hbar}{2 - \alpha} R^{2-\alpha} - C R^{1-\alpha} - C R - C. \]  

(11)

Making $R \to \infty$ and recalling that $\alpha < 1$, we get a contradiction to the assumption that the energy $\mathcal{G}(\psi, A)$ is finite, thereby finishing the proof of Theorem 1.

Acknowledgements

The authors wish to thank the anonymous referee for valuable suggestions. A.K. is supported by a Starting Independent Researcher grant by the ERC under the FP7. M.P. is supported by the Lundbeck Foundation.

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