Equivelar Toroids with Few Flag-Orbits

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Abstract

An \((n + 1)\)-toroid is a quotient of a tessellation of the \(n\)-dimensional Euclidean space with a lattice group. Toroids are generalisations of maps on the torus to higher dimensions and also provide examples of abstract polytopes. Equivelar toroids are those that are induced by regular tessellations. In this paper we present a classification of equivelar \((n + 1)\)-toroids with at most \(n\) flag-orbits; in particular, we discuss a classification of 2-orbit toroids of arbitrary dimension.

Keywords Symmetries of toroids · Maps · Polytopes · Regular tessellation

Mathematics Subject Classification 51B15 · 52C22 · 51F15 · 51M20

1 Introduction

The study of symmetric discrete objects in both combinatorial and geometrical sense has been of interest in recent years. In particular, symmetric maps on surfaces have been widely studied (see [2,3,11,12]).

In [5] Coxeter and Moser presented a classification of regular (reflexible) and chiral (regular irreflexible) maps on the torus. All such maps arise as quotients of a regular tessellation of the Euclidean plane. Several results regarding a classification of highly symmetric maps on surfaces of small genus have been obtained since. See [2,3]. When looking for generalisations of maps to higher dimensions the approach of abstract...
polytopes has been one of the most studied. The number of recent contributions to the theory of symmetric abstract polytopes is large. Many of such results can be found in [15].

One concept that generalises maps in a combinatorial way while keeping the topological idea behind them is that of tessellations of space forms (see [15, Chap. 6]). Euclidean space forms are probably the most studied in the setting of symmetric tessellations. Among those, the $n$-dimensional torus is probably the most well-understood.

When talking about symmetric structures on the $n$-dimensional torus, much of the work follows the ideas introduced by Coxeter and Moser in [5]. Toroids are generalisations to higher dimensions of maps on the torus and may be regarded as tessellations of the $n$-dimensional torus.

The degree of symmetry of a toroid can be understood through the action of its automorphism group on a set of certain substructures called flags. In this context, the most symmetric toroids (called regular toroids) are precisely those whose automorphism group induces one orbit on flags. Chiral toroids are a particular class of the so-called 2-orbit toroids (toroids with 2 flag-orbits).

Several classification results of highly symmetric toroids have been developed in the recent years. In [14] McMullen, and Schulte classify regular toroids of arbitrary rank. They also show that there are no chiral toroids of rank higher than 3. In [8] Hartley, McMullen, and Schulte extend this result and show that the only Euclidean space form that admits chiral tessellations is the $n$-torus and prove that this is only possible when $n = 2$.

In [1] Brehm and Kühnel classify the equivelar maps on the two-dimensional torus. This classification is also achieved by Hubard et al. in [10], where they also classify equivelar tessellations of the three-dimensional torus (rank-4 toroids).

As a consequence of their results in [10], Hubard et al. find that there are no equivelar 2-orbit $(3 + 1)$-toroids. Therefore, they mention that even though a classification of $(n + 1)$-toroids for $n > 4$ seems to be too hard to achieve with their techniques, it would be of interest to obtain a classification of 2-orbit toroids of rank $n + 1$ for $n \geq 4$.

In this paper we give a classification of equivelar $(n + 1)$-toroids with at most $n$ flag-orbits. In particular, 2-orbit $(n + 1)$-toroids are classified for arbitrary $n$. If $n \leq 3$ the classification is a consequence of the results in [5] and [10]. The main results of this article are Theorems 3.7, 3.9, 3.12, 3.16, 4.2, 4.3, 5.2, and 5.3. These results can be summarised in the following theorem.

**Theorem 1.1** Let $n \geq 4$. The classification up to duality of equivelar $(n + 1)$-toroids with at most $n$ flag-orbits is explained in Table 1.

### 2 Basic Notions

#### 2.1 Tessellations of the Euclidean Space

In this section we introduce basic concepts about Euclidean tessellations and toroids. We focus mainly on those with high degree of symmetry. Readers interested in further details are referred to [15, Chap. 6] and [10].
Table 1 Classification of equivelar few-orbit toroids

| $n = 4$ | | $n \geq 4$ |
|---------|---------|
| $\{4, 3^{n-2}, 4\}$ | Three families of regular toroids | Three families of regular toroids |
| | One family of 2-orbit toroids in class $2_{1,2,3}$ | If $n$ is odd, there are no 2-orbit toroids. If $n$ is even there is one family of toroids in class $2_{1,2,...,n-1}$ |
| | One family of 3-orbit toroids | No $k$-orbit toroids if $2 < k < n$ |
| | Five infinite families of toroids with 4 flag-orbits, all with the same symmetry type | Five infinite families of toroids with $n$ flag-orbits, all with the same symmetry type |
| $\{3, 3, 4, 3\}$ | Two families of regular toroids | |
| | No 2-orbit toroids | |
| | Two families of 3-orbit toroids with different symmetry type | |
| | No 4-orbit toroids | |

A convex $n$-polytope $\mathcal{P}$ is the convex hull of a finite set of points of $\mathbb{E}^n$ such that the interior of $\mathcal{P}$ is non-empty. If $\mathcal{P}$ is a convex $n$-polytope, then $F \subset \mathcal{P}$ is a face of $\mathcal{P}$ if $F = \mathcal{P} \cap \Pi$ for some hyperplane $\Pi$ that leaves $\mathcal{P}$ contained in one of the closed half spaces determined by $\Pi$. If the affine dimension of a face $F$ is $i$ for some $i \in \{0, \ldots, n-1\}$, then $F$ is an $i$-face. The 0-faces, 1-faces, and $(n-1)$-faces of a convex $n$-polytope are also called vertices, edges, and facets, respectively. We usually consider a convex $n$-polytope $\mathcal{P}$ itself as its (unique) $n$-face. Readers interested in more details concerning basic notions of convex polytopes are referred to [13, Chap. 5].

A tessellation of the Euclidean space $\mathbb{E}^n$ (or a Euclidean tessellation) is a family $\mathcal{U}$ of convex $n$-polytopes that is locally finite, meaning that every compact set of $\mathbb{E}^n$ meets only finitely many members of $\mathcal{U}$. We also require that $\mathcal{U}$ covers the space and tile it in a face-to-face manner. In other words, if two members of $\mathcal{U}$ have non-empty intersection, then they have disjoint interiors, and they meet in a common $i$-face for some $i \in \{0, \ldots, n-1\}$. If $\mathcal{U}$ is a Euclidean tessellation, the elements of $\mathcal{U}$ are called cells.

A flag in a convex $n$-polytope is an $(n+1)$-tuple of incident faces containing exactly one face of each dimension, including the polytope itself. This definition extends naturally to tessellations of $\mathbb{E}^n$. A flag of a tessellation $\mathcal{U}$ is a flag of any cell of $\mathcal{U}$. It is sometimes useful to identify a flag $\Phi$ of $\mathcal{U}$ with the non-regular $n$-simplex induced by the centroids of the faces of $\Phi$. Observe that given a flag $\Phi$ of $\mathcal{U}$ and $i \in \{0, \ldots, n\}$, there exists exactly one flag $\Phi^i$ of $\mathcal{U}$ that differs from $\Phi$ only in the face of dimension $i$. In this situation we say that $\Phi$ and $\Phi^i$ are adjacent (or $i$-adjacent if we want to emphasise on $i$).

A symmetry of a tessellation $\mathcal{U}$ is an isometry of $\mathbb{E}^n$ that preserves $\mathcal{U}$. The group of symmetries of $\mathcal{U}$ is denoted by $G(\mathcal{U})$. It is not hard to see that $G(\mathcal{U})$ acts freely on the set of flags of $\mathcal{U}$. A tessellation $\mathcal{U}$ is regular if the action of $G(\mathcal{U})$ on the flags of $\mathcal{U}$ is transitive.
The dual tessellation of a regular Euclidean tessellation \( \mathcal{U} \), usually denoted by \( \mathcal{U}^* \), is the tessellation whose cells are the polytopes given by the convex hull of the centroids of the cells of \( \mathcal{U} \) incident to a common vertex of \( \mathcal{U} \). A tessellation is self-dual if it is similar to its dual.

The Schläfli type (or type, for short) of a convex \( n \)-polytope \( \mathcal{P} \) is defined recursively as follows. If \( n = 2 \) then \( \mathcal{P} \) is a convex \( p \)-gon for some \( p \) and we say that \( \mathcal{P} \) has Schläfli type \( \{ p \} \). For \( n \geq 3 \), whenever all the facets of a convex \( n \)-polytope have type \( \{ p_1, \ldots, p_{n-2} \} \) and there are exactly \( p_{n-1} \) facets around each \(( n - 3 )\)-face of \( \mathcal{P} \), we say that \( \mathcal{P} \) has Schläfli type \( \{ p_1, \ldots, p_{n-1} \} \). Observe that not every convex polytope has a well-defined Schläfli type, however all regular convex polytopes do. The notion of Schläfli type extends to tessellations in a natural way. We say that \( \mathcal{U} \) has (Schläfli) type \( \{ p_1, \ldots, p_n \} \) if each cell of \( \mathcal{U} \) has type \( \{ p_1, \ldots, p_{n-1} \} \) and the number of cells around each \(( n - 2 )\)-face of \( \mathcal{U} \) is \( p_n \).

Euclidean regular tessellations are well known. For every \( n \geq 2 \), there exists a self-dual regular tessellation with cubes on \( \mathbb{E}^n \) with type \( \{ 4, 3^{n-2}, 4 \} \). Here \( 3^{n-2} \) denotes a sequence of \( 3 \) with length \( n - 2 \); if there is no possible confusion, in this work we will use exponents to denote a sequence of equal symbols. In \( \mathbb{E}^2 \) there exists a regular tessellation with equilateral triangles and type \( \{ 3, 6 \} \), and a tessellation with regular hexagons and type \( \{ 6, 3 \} \). Those two are dual of each other. In \( \mathbb{E}^4 \) there is another pair of regular tessellations, one with 24-cells as facets and type \( \{ 3, 4, 3, 3 \} \), and its dual of type \( \{ 3, 3, 4, 3 \} \) whose cells are four-dimensional cross-polytopes. These tessellations are unique up to similarity and they complete the list of regular tessellations of the Euclidean \( n \)-space [4, Table II]. In Table 2 we give explicit coordinates for the vertex set of one of each pair of dual regular tessellations. Those coordinates determine uniquely the tessellation up to similarity, therefore throughout this work we will assume that those coordinates are fixed.

If \( \mathcal{U} \) is a regular tessellation of \( \mathbb{E}^n \) and \( \Phi \) is a fixed base flag, there exist \( R_0, \ldots, R_n \) symmetries of \( \mathcal{U} \) such that \( \Phi R_i = \Phi^i \). The symmetries \( R_0, \ldots, R_n \) are the reflections in the planes containing the facets of the \( n \)-simplex induced by \( \Phi \) and generate the group \( G(\mathcal{U}) \). If \( \mathcal{U} \) has type \( \{ p_1, \ldots, p_n \} \), the group \( G(\mathcal{U}) \) with the generators \( R_0, \ldots, R_n \) is the string Coxeter group \( [ p_1, \ldots, p_n ] \) meaning that

\[
G(\mathcal{U}) = \langle R_0, \ldots, R_n : (R_i R_j)^{p_{i,j}} = \text{id} \rangle
\]

is a presentation for the group, where \( p_{i,j} = p_{j,i} \), \( p_{i,i} = 2 \), \( p_{i,j} = 2 \) if \( |i - j| > 1 \), and \( p_{i-1,i} = p_i \). Moreover, the group of symmetries of the cell contained in the base flag is \( \langle R_0, \ldots, R_{n-1} \rangle \cong \langle p_1, \ldots, p_{n-1} \rangle \) and the stabiliser of the vertex of \( \Phi \) is \( \langle R_1, \ldots, R_n \rangle \cong \langle p_2, \ldots, p_n \rangle \) (see [15, Sect. 3A] for details).

Note that if \( R_0, \ldots, R_n \) denote the distinguished generators of \( \mathcal{U} \) with respect to a base flag \( \Phi \), then \( R_n, \ldots, R_0 \) act as distinguished generators of \( \mathcal{U}^* \) with respect to the dual flag of \( \Phi \). A consequence of this is that \( G(\mathcal{U}) = G(\mathcal{U}^*) \). In Table 2 we give coordinates and explicit expressions for \( R_0, \ldots, R_n \) for one of each dual pair of regular tessellations.
Table 2 Vertex sets and distinguished generators of regular tessellations

| \(\mathcal{U}\) | Vertex set of \(\mathcal{U}\) | Generators of \(G(\mathcal{U})\), \(R_i \cdot (x_1, \ldots, x_n) \mapsto\) |
|---|---|---|
| \([3, 6]\) | \(\{a(1, 0) + b(1/2, \sqrt{3}/2) : a, b \in \mathbb{Z}\}\) | \((1 - x_1, x_2)\) if \(i = 0\), \((x_1/2 + x_2\sqrt{3}/2, x_1\sqrt{3}/2 - x_2/2)\) if \(i = 1\), \((x_1, -x_2)\) if \(i = 2\). |
| \([4, 3^n-2, 4]\) | \(\mathbb{Z}^n\) | \((1 - x_1, x_2, \ldots, x_n)\) if \(i = 0\), \((x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)\) if \(1 \leq i < n\), \((x_1, \ldots, -x_n)\) if \(i = n\). |
| \([3, 3, 4, 3]\) | \(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 : x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod{2}\}\) | \((2 - x_1, x_2, x_3, x_4)\) if \(i = 0\), \((x, x - x_3 - x_4, x - x_2 - x_4, x - x_2 - x_3)\) where \(x = (x_1 + x_2 + x_3 + x_4)/2\) if \(i = 1\), \((x_1, x_2, x_3, -x_4)\) if \(i = 2\), \((x_1, x_2, x_4, x_3)\) if \(i = 3\), \((x_1, x_3, x_2, x_4)\) if \(i = 4\). |

2.2 Equivelar Toroids

Toroids generalise maps in the two-dimensional torus to higher dimensions. In this section we will discuss the basic results about toroids and their symmetries, with special emphasis on those induced by regular tessellations of \(\mathbb{E}^n\).

Let \(0 \leq d \leq n\). A rank-\(d\) lattice group in \(\mathbb{E}^n\) is a group generated by \(d\) independent translations. If \(\Lambda = \langle t_1, \ldots, t_d \rangle\) is a lattice group and \(v_i\) is the translation vector of \(t_i\), then the lattice \(\Lambda\) induced by \(\Lambda\) is the orbit of the origin \(o\) under \(\Lambda\), that is

\[\Lambda = o\Lambda = \{a_1v_1 + \cdots + a_dv_d : a_1, \ldots, a_d \in \mathbb{Z}\}.\]

In this case we say that \(\{v_1, \ldots, v_d\}\) is a basis for \(\Lambda\). We denote by \(\Lambda_{(1,0^{n-1})}\) the cubic lattice, which consists of all the points of \(\mathbb{E}^n\) with integer coordinates with respect to the standard basis \(\{e_1, \ldots, e_n\}\). Observe that \(\{e_1, \ldots, e_n\}\) is also a basis for \(\Lambda_{(1,0^{n-1})}\). The lattice \(\Lambda_{(1,1,0^{n-2})}\) is the rank-\(n\) lattice consisting of the points of integer coordinates of \(\mathbb{E}^n\) whose coordinate sum is even. A basis for \(\Lambda_{(1,1,0^{n-2})}\) is given by \(\{2e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}\}\). We denote by \(\Lambda_{(1^n)}\) the lattice consisting of the points whose coordinates are all integers having the same parity. A basis for this lattice is \(\{2e_1, \ldots, 2e_{n-1}, e_1 + \cdots + e_n\}\). The corresponding lattice groups are denoted by \(\Lambda_{(1,0^{n-1})}, \Lambda_{(1,1,0^{n-2})}, \text{and} \Lambda_{(1^n)}\). It is important to remark that this notation is slightly different from that of [15]. Note also that \(\Lambda_{(1,0^{n-1})}, \Lambda_{(1,1,0^{n-2})}, \text{and} \Lambda_{(1^n)}\) are contained in the vertex set of \([4, 3^n-2, 4]\), and \(\Lambda_{(1,1,1,1)}\) is precisely the vertex set of \([3, 3, 4, 3]\).

Let \(T(\mathcal{U})\) denote the group of translations of a tessellation \(\mathcal{U}\) of \(\mathbb{E}^n\). A toroid of rank \(n + 1\), or \((n + 1)\)-toroid, is the quotient of a tessellation \(\mathcal{U}\) of \(\mathbb{E}^n\) by a rank-\(n\) lattice group \(\Lambda \leq T(\mathcal{U})\). We say that \(\Lambda\) induces the toroid, and denote the latter by \(\mathcal{U}/\Lambda\). If \(\mathcal{U}\) is regular of type \(\{p_1, \ldots, p_n\}\) we say that the toroid is equivelar of (Schläfli) type \(\{p_1, \ldots, p_n\}\); in this situation we also denote the toroid induced by
Λ by \( \{p_1, \ldots, p_n\}_\Lambda \) (cf. [5, Chap. 7] and [15, Chap. 6]). An \((n + 1)\)-toroid may be regarded as a tessellation of the \(n\)-dimensional torus \( \mathbb{E}^n/\Lambda \).

For all regular tessellations \( \mathcal{U} \) except \{6, 3\} and \{3, 4, 3, 3\} the group of translations \( T(\mathcal{U}) \) acts transitively on the vertex set of \( \mathcal{U} \). Therefore the vertex set of \( \mathcal{U} \) may be identified with the lattice associated to \( T(\mathcal{U}) \) and the group of symmetries \( G(\mathcal{U}) \) is of the form \( T(\mathcal{U}) \rtimes G_\Lambda(\mathcal{U}) \), where \( G_\Lambda(\mathcal{U}) \) denotes the stabiliser of the origin \( o \) (see for example [15, Chap. 6]). From now on, we restrict our study to regular tessellations whose vertex set is a lattice. The results regarding toroids of type \{6, 3\} and \{3, 4, 3, 3\} may be recovered by duality.

Let \( \text{Isom}(\mathbb{E}^n) \) denote the group of isometries of \( \mathbb{E}^n \). If \( t_o \) is the translation by a vector \( v \) and \( S \in \text{Isom}(\mathbb{E}^n) \) fixes the origin \( o \), then \( S^{-1} t_o S = t_o S \), the translation by \( v S \). In other words, if \( \Lambda \) is the lattice associated to \( \Lambda \), then \( \Lambda S \) is the lattice associated to \( S^{-1} \Lambda S \). Therefore, if there exists an isometry mapping a lattice \( \Lambda \) to another lattice \( \Lambda' \), then there exists an isometry \( S \) that fixes \( o \) that maps \( \Lambda \) to \( \Lambda' \). In this case the corresponding tori \( \mathbb{E}^n/\Lambda \) and \( \mathbb{E}^n/\Lambda' \) are isometric. Geometrically this means that \( S \) maps fundamental regions of \( \Lambda \) to fundamental regions of \( \Lambda' \).

With the notation given above, when \( \Lambda = \Lambda' \), an isometry \( S \) of \( \mathbb{E}^n \) induces an isometry \( S' \) of \( \mathbb{E}^n/\Lambda \) that makes the diagram (1) commutative if and only if \( S \) normalises \( \Lambda \). Furthermore, two isometries of \( \mathbb{E}^n \) induce the same isometry of \( \mathbb{E}^n/\Lambda \) if and only if they differ by an element of \( \Lambda \). In particular, all the elements of \( \Lambda \) induce a trivial isometry of \( \mathbb{E}^n/\Lambda \). This implies that the group \( \text{Norm}_{\text{Isom}(\mathbb{E}^n)}(\Lambda)/\Lambda \) acts as a group of isometries of \( \mathbb{E}^n/\Lambda \). It can be proved that every isometry of \( \mathbb{E}^n/\Lambda \) is given this way, that is, \( \text{Isom}(\mathbb{E}^n/\Lambda) \cong \text{Norm}_{\text{Isom}(\mathbb{E}^n)}(\Lambda)/\Lambda \) (see [17, p. 336] and [15, Sect. 6A]).

\[
\begin{array}{ccc}
\mathbb{E}^n & \xrightarrow{S} & \mathbb{E}^n \\
\downarrow & & \downarrow \\
\mathbb{E}^n/\Lambda & \xrightarrow{S' \Lambda} & \mathbb{E}^n/\Lambda
\end{array}
\]

With the previous discussion in mind, it makes sense to define the group of automorphisms of a toroid \( \mathcal{U}/\Lambda \), denoted by \( \text{Aut}(\mathcal{U}/\Lambda) \), as the quotient \( \text{Norm}_{G(\mathcal{U})}(\Lambda)/\Lambda \). Intuitively speaking, \( \text{Norm}_{G(\mathcal{U})}(\Lambda) \) denotes the symmetries of \( \mathcal{U} \) that are compatible with the quotient by \( \Lambda \). In the same sense, two toroids \( \mathcal{U}/\Lambda \) and \( \mathcal{U}/\Lambda' \) are isomorphic if \( \Lambda \) and \( \Lambda' \) are conjugates in \( G(\mathcal{U}) \).

If an isometry \( S \) normalises \( \Lambda \) then we say that \( S \) induces or projects to an automorphism of \( \mathcal{U}/\Lambda \) (namely, to the automorphism \( \Lambda S \in \text{Norm}_{G(\mathcal{U})}(\Lambda)/\Lambda \)). If \( S' \in G_\Lambda(\mathcal{U}) \), then \( S' \) normalises \( \Lambda \) if and only if \( S' \) preserves \( \Lambda \). Since every element \( S \in G(\mathcal{U}) \) may be written as a product \( t S' \) with \( t \in T(\mathcal{U}) \) and \( S' \in G_\Lambda(\mathcal{U}) \), and every translation normalises \( \Lambda \), an isometry \( S \) induces an automorphism of \( \mathcal{U}/\Lambda \) if and only if \( S' \) preserves \( \Lambda \). Therefore we may restrict our analysis to elements of \( G_\Lambda(\mathcal{U}) \).

The \( i \)-faces of a toroid \( \mathcal{U}/\Lambda \) are the orbits of the \( i \)-faces of \( \mathcal{U} \) under \( \Lambda \). Whenever all the vertices on each cell of \( \mathcal{U} \) are different under the action of \( \Lambda \), the set of faces of \( \mathcal{U}/\Lambda \) has the structure of an abstract polytope (in the sense of [15]). In this case, the symmetry properties of \( \mathcal{U}/\Lambda \) as an abstract polytope coincide with those as a toroid. However, even when \( \mathcal{U}/\Lambda \) is not an abstract polytope, we may define the set of flags of \( \mathcal{U}/\Lambda \) as the set of orbits of flags of \( \mathcal{U} \) under \( \Lambda \). The group \( \text{Aut}(\mathcal{U}/\Lambda) \) has a well-
defined action on the set of flags of $\mathcal{U}/\Lambda$ by $(\Phi \Lambda) \Lambda S = \Phi S \Lambda$ with $\Phi$ a flag of $\mathcal{U}$ and $\Lambda S \in \text{Aut}(\mathcal{U}/\Lambda)$. In what follows, we shall slightly abuse notation and write simply $\Phi \Lambda S = \Phi S \Lambda$. We say that a toroid $\mathcal{U}/\Lambda$ is a $k$-orbit toroid if $\text{Aut}(\mathcal{U}/\Lambda)$ has $k$ orbits on flags. Following [15], regular toroids are precisely 1-orbit toroids.

Observe that every translation of $\mathcal{U}$ induces an automorphism of $\mathcal{U}/\Lambda$, and the translations of $\Lambda$ induce the trivial automorphism. Also, the central inversion of $\mathbb{E}^n$, $\chi : x \mapsto -x$, is always an automorphism of $\mathcal{U}$ and preserves every lattice $\Lambda$, so it projects to an automorphism of every toroid (see [10, Table 1] for an expression of $\chi$ in terms of the generating reflections). This implies that $(\mathcal{T}(\mathcal{U}), \chi) \leq \text{Norm}_{\mathcal{G}(\mathcal{U})}(\Lambda)$. Furthermore, because $\chi$ normalises $\mathcal{T}(\mathcal{U})$ and $\mathcal{T}(\mathcal{U}) \cap \langle \chi \rangle = \{ \text{id} \}$ it follows that $(\mathcal{T}(\mathcal{U}), \chi) = \mathcal{T}(\mathcal{U}) \rtimes \langle \chi \rangle$. Therefore, groups of automorphisms of toroids are induced by groups $K$ such that $\mathcal{T}(\mathcal{U}) \rtimes \langle \chi \rangle \leq K \leq \mathcal{G}(\mathcal{U})$.

By the correspondence theorem for groups, those groups $K$ with $\mathcal{T}(\mathcal{U}) \rtimes \langle \chi \rangle \leq K \leq \mathcal{G}(\mathcal{U})$ are in one-to-one correspondence with groups $K'$ such that $\langle \chi \rangle \leq K' \leq \mathcal{G}_o(\mathcal{U})$. In this correspondence, the group $K'$ corresponds to the group $\mathcal{T}(\mathcal{U}) \rtimes K'$.

Recall that if $\Lambda = S^{-1} \Lambda' S$ for some $S \in \mathcal{G}(\mathcal{U})$ then the toroids $\mathcal{U}/\Lambda$ and $\mathcal{U}/\Lambda'$ are isomorphic. Furthermore, the corresponding automorphism groups are $K/\Lambda$ and $(S^{-1}KS)/\Lambda'$, for some group $\mathcal{T}(\mathcal{U}) \rtimes \langle \chi \rangle \leq K \leq \mathcal{G}(\mathcal{U})$. Hence, in order to classify toroids up to isomorphism it is sufficient to determine their automorphism groups up to conjugacy. According to the discussion above, we only need to find conjugacy classes of groups $K'$ such that $\langle \chi \rangle \leq K' \leq \mathcal{G}_o(\mathcal{U})$. Furthermore, according to [16] the number of flag-orbits of a toroid $\mathcal{U}/\Lambda$ under $\text{Aut}(\mathcal{U}/\Lambda)$ is the same as the index of $K$ in $\mathcal{G}(\mathcal{U})$, which is the same as the index of $K'$ in $\mathcal{G}_o(\mathcal{U})$.

We summarise the discussion above in the following lemma. This is essentially [10, Lem. 6].

**Lemma 2.1** With the notation given above, the following statements hold:

- A symmetry $S \in \mathcal{G}_o(\mathcal{U})$ projects to an automorphism of $\mathcal{U}/\Lambda$ if and only if $S$ normalises $\Lambda$. This occurs if and only if $\Lambda S = \Lambda$.
- Since all lattices are centrally symmetric, $\chi : x \mapsto -x$ always projects to an automorphism of $\mathcal{U}/\Lambda$.
- The automorphism group $\text{Aut}(\mathcal{U}/\Lambda)$ of $\mathcal{U}/\Lambda$ is isomorphic to

$$
(K' \rtimes \mathcal{T}(\mathcal{U}))/\Lambda \cong K' \rtimes (\mathcal{T}(\mathcal{U}))/\Lambda,
$$

where $K' = \{ S \in \mathcal{G}_o(\mathcal{U}) : S^{-1} \Lambda S = \Lambda \} = \{ S \in \mathcal{G}_o(\mathcal{U}) : \Lambda S = \Lambda \}$. In particular $\langle \chi \rangle \leq K' \leq \mathcal{G}_o(\mathcal{U})$. The group $\text{Aut}(\mathcal{U}/\Lambda)$ has $k$ orbits on the set of flags of $\mathcal{U}/\Lambda$ if and only if the index of $K'$ in $\mathcal{G}_o(\mathcal{U})$ is $k$.
- The toroids $\mathcal{U}/\Lambda$ and $\mathcal{U}/\Lambda'$ are isomorphic if and only if $\Lambda$ and $\Lambda'$ are conjugate in $\mathcal{G}(\mathcal{U})$. This in turn is true if and only if there exists $S \in \mathcal{G}_o(\mathcal{U})$ such that $\Lambda S = \Lambda'$.

An important class of lattices are those preserved by a reflection on a hyperplane. Assume that $\Pi$ is a hyperplane that contains $o$, a lattice $\Lambda$ is a vertical translation lattice with respect to $\Pi$ if

$$
\Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + ku),
$$

where $\Lambda_0$ is a lattice and the coefficients $u$ are not necessarily vectors of $\mathbb{Z}^n$. In this case, the set of flags of $\mathcal{U}/\Lambda$ is then the set of flags of $\mathcal{U}/\Lambda_0$.
where \( \Lambda_0 = \Lambda \cap \Pi \) and \( u \in \Pi^\perp \). Vertical translation lattices with respect to a hyperplane \( \Pi \) are preserved by the reflection in \( \Pi \). However, even those lattices invariant under a plane reflection that are not vertical translation are provided with an interesting structure, which is described in Lemma 2.2. This result is a slightly more detailed version of [10, Lem. 11], where the lemma is proved for dimension 3.

**Lemma 2.2** Let \( \Lambda \) be a rank-\( n \) lattice on \( \mathbb{R}^n \). Let \( R \) be the reflection on a hyperplane \( \Pi \) that contains \( o \) and assume that \( \Lambda \) is preserved by \( R \), that is \( \Lambda R = \Lambda \). Then the following statements hold:

1. If \( \Lambda_0 = \Lambda \cap \Pi \), then \( \Lambda_0 \) is a rank-(\( n - 1 \)) lattice and

\[
\Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + kw)
\]

for any \( w \in \Lambda \setminus \Pi \) with minimum (positive) distance \( d \) to \( \Pi \).

2. The point \( w \) may be chosen in \( \Pi^\perp \) if and only if \( \Lambda \) is a vertical translation lattice with respect to \( \Pi \).

3. If \( \Lambda \) is not a vertical translation lattice and \( \{v_1, \ldots, v_{n-1}\} \) is a basis for \( \Lambda_0 \), then \( w \) may be chosen of the form \((\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + u)/2 \) where \( u \in \Pi^\perp \setminus \{o\} \), \( |u| = 2d \), and \( \alpha_1, \ldots, \alpha_{n-1} \in \{0, 1\} \) not all zero. Furthermore, the choice of such \( \alpha_1, \ldots, \alpha_{n-1} \) is unique.

**Proof** Items (1) and (2) follow the same proof as that given in [10, Lem. 11]. We will only prove the third one.

Assume that \( \Lambda \) is not a vertical translation lattice. Let \( \{v_1, \ldots, v_{n-1}\} \) be a basis for \( \Lambda_0 \) and \( w \in \Lambda \setminus \Pi \) such that \( d = d(w, \Pi) \) is minimum among the points of \( \Lambda \setminus \Pi \). Let \( u = w - wR \) and note that \( |u| = 2d \), hence \( u \) is a closest point of \( \Lambda \cap \Pi^\perp \) to \( o \) other than \( o \) itself. Observe that \( 2w - u \in \Lambda_0 \), thus there exist \( m_1, \ldots, m_{n-1} \in \mathbb{Z} \) such that \( 2w - u = m_1 v_1 + \cdots + m_{n-1} v_{n-1} \). Let \( k_i \in \mathbb{Z} \) for \( 1 \leq i \leq n-1 \) be even and \( \alpha_i := m_i - k_i \in \{0, 1\} \). Define

\[
w_1 = \frac{\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} + u}{2}
\]

and note that \( \Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + kw) \) since \( w \) and \( w_1 \) differ by an element in \( \Lambda_0 \), and hence \( d(\Pi, w_1) = d \). Now, observe that if \( \alpha_i = 0 \) for every \( 1 \leq i \leq n-1 \), then \( m_1, \ldots, m_{n-1} \) are all even and this will imply that \( u/2 \in \Lambda \), contradicting that \( \Lambda \) is not a vertical translation lattice. Finally, suppose there exist \( \alpha_1', \ldots, \alpha_{n-1}' \in \{0, 1\} \) such that if \( w_2 = (\alpha_1' v_1 + \cdots + \alpha_{n-1}' v_{n-1} + u)/2 \) satisfies that \( \Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + kw) \). Let \( i \in \{1, \ldots, n-1\} \) be such that \( \alpha_i \neq \alpha_i' \). On one hand, \( w_1 + w_2 - u \) is a point in \( \Lambda_0 \), on the other hand, in the (unique) expression of \( w_1 + w_2 - u \) as a linear combination of \( \{v_1, \ldots, v_{n-1}\} \), the coefficient of \( v_i \) is 1/2, contradicting that \( \{v_1, \ldots, v_{n-1}\} \) is a basis for \( \Lambda_0 \).

\( \square \)
3 Few-Orbit Cubic Toroids

Now we start the study of equivelar few-orbit toroids, which are equivelar \((n + 1)\)-toroids with at most \(n\) flag-orbits. In this section we discuss few-orbit cubic toroids, meaning equivelar few-orbit \((n + 1)\)-toroids of type \(\{4, 3^{n-2}, 4\}\). These toroids are quotients of the tessellation \(\mathcal{U} = \{4, 3^{n-2}, 4\}\) of \(\mathbb{E}^n\) by unitary \(n\)-cubes.

Since the vertex set of \(\mathcal{U}\) is a lattice, according to the discussion in Sect. 2.2, the symmetry group of \(\mathcal{U}\) is the semidirect product \(T(\mathcal{U}) \rtimes G_0\), where \(T(\mathcal{U})\) is the translation group of \(\mathcal{U}\) and \(G_0\) is the stabiliser of the base vertex \(o\), that is, the group of the vertex figure at \(o\) of \(\mathcal{U}\).

The group \(G_0\) is the string Coxeter group \([3^{n-2}, 4, 2]\), generated by reflections \(R_1, \ldots, R_n\). The group \(S_n = \langle R_1, \ldots, R_{n-1}\rangle\) is isomorphic to the symmetric group in \(n\) symbols, acting on the points of \(\mathbb{E}^n\) by permutation of coordinates. Observe that \(R_n\) together with its conjugates under \(S_n\) generate a group isomorphic to the group \(C_n^2\); we denote this group by \(C_n^2\). Notice that the vector \((1^{n-1}, -1, 1^{n-i}) \in C_n^2\) may be identified with the reflection \(E_i\) of \(C_n^2\) through the coordinate plane \(x_i = 0\), given by \((x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)\). It is not hard to see that \(G_0 = C_n^2 \rtimes S_n\). Finally, observe that the action of conjugacy of \(S_n\) on \(C_n^2\) is precisely the same as the action of \(S_n\) permuting coordinates in \(C_n^2\). Let \(A_n\) denote the rotational subgroup of \(S_n\), this is, the group consisting of the orientation preserving isometries in \(S_n\). With the identification of the elements of \(S_n\) with permutations mentioned above, the group \(A_n\) is in correspondence with the alternating group \(A_n\) and the conjugacy action of \(A_n\) on elements of \(C_n^2\) is the permutation action of \(A_n\) on the coordinates of the corresponding elements of \(C_n^2\). Similarly, let \((C_n^2)^+\) denote the rotational subgroup of \(C_n^2\). The group \((C_n^2)^+\) is generated by the products \(E_i E_j\) for \(i, j \in \{1, \ldots, n\}\).

We will use the structure of the groups described above to prove some results regarding the conjugacy classes of subgroups of \(G_0\). During the discussion we will identify the elements of \(S_n\) with permutations and the elements of \(C_n^2\) with vectors with entries \(\pm 1\) as described before.

**Lemma 3.1** Let \(n \geq 3\) and let \(H\) be a subgroup of \(C_n^2\). If \(H\) is normalised by \(A_n\), then one of the following holds:

\[ H = \{1\}, \quad H = \langle \chi \rangle, \quad H = (C_n^2)^+, \quad H = C_n^2, \]

where \(\chi\) denotes the central inversion \(x \mapsto -x\).

**Proof** We use the structure discussed above for the group \(C_n^2 \rtimes S_n\). In this context, we only have to classify groups \(H \leq C_n^2\) preserved by the action of \(A_n\) on the coordinates of its elements. Suppose that \(H \neq \{1\}\) and define \(m\) to be the minimum positive number of \(-1\) entries in a non-trivial vector of \(H\). Let \(A \in H\) be the transformation given by the vector \((a_1, \ldots, a_n)\) and assume that \((a_1, \ldots, a_n)\) has precisely \(m\) entries equal to \(-1\).

If \(m = 1\), then \(A = E_i\), the reflection given by the vector that has \(1\) in all its coordinates except the \(i\)-th one. Since \(A_n\) acts transitively on the coordinates of the elements of \(H\), for every \(k \in \{1, \ldots, n\}\) there exists \(S_k \in A_n\) such that \(S_k^{-1} E_i S_k = E_k\).

Since \(H\) is normalised by \(A_n\) and the set \(\{E_k : 1 \leq k \leq n\}\) generates \(C_n^2\), \(H\) must be \(C_n^2\).
If $m = 2$ then $A = E_i E_j$, the transformation given by vector whose $i$-th and $j$-th coordinates are $-1$. Since $A_n$ acts 2-transitively on the coordinates of the elements of $H$, we may show that $\{E_k E_l : 1 \leq k < l \leq n\} \subset H$, which implies that $(C_n^2)^+ \leq H$. However, by the minimality of $m$ we must have that $H = (C_n^2)^+$.

If $2 < m < n$, without loss of generality we may assume that $a_i = -1$ if and only if $1 \leq i \leq m$, that is, $A$ is the transformation induced by the vector $(-1^m, 1^{n-m})$. Consider the transformation $S$ of $A_n$ induced by the permutation $(1 m + 1)(2 3) \in A_n$. Since $H$ is normalised by $A_n$, the transformation $A' = S^{-1} A S$ given by $(1, -1^m, 1^{n-m-1})$ belongs to $H$. Therefore, $AA' \in H$ is given by the vector $(-1, 1^{m-1}, -1, 1^{n-m-1})$, which contradicts the minimality of $m$. Finally, if $m = n$ then $H = \langle \chi \rangle$.

The following result states a series of basic properties of semidirect products of groups that will be useful throughout this paper.

**Lemma 3.2** Let $G$ be the semidirect product $N \rtimes H$. Let $\eta : G \to H$ be the natural mapping with kernel $N$. Let $K \leq G$, then the following hold:

1. The restriction of $\eta$ to $K$ has kernel $K_0 = N \cap K$.
2. If $K_0 \triangleleft N$, then the image of $K$ under $\eta$ normalises $K_0$.
3. If $G$ is finite, then $[G : K] \geq [H : \eta(K)]$.

**Proof.** Item (1) is a basic property of group homomorphisms. To prove (2) take $k \in K_0$ and $h \in \eta(K)$. Since $h \in \eta(K)$, there exists $n \in N$ such that $nh \in K$. Now, since $K_0 \triangleleft N$, $nkn^{-1} \in K_0$. Finally,

$$h^{-1}kh = h^{-1}n^{-1}nkn^{-1}nh = (nh)^{-1}(nkn^{-1})(nh) \in K.$$ 

To prove (3) just observe that

$$[G : K] = \frac{|G|}{|K|} = \frac{|N| \cdot |H|}{|K|} \geq \frac{|N \cap K| \cdot |H|}{|K|} = \frac{|H|}{|\eta(K)|} = [H : \eta(K)].$$

According to Lemma 2.1, in order to classify few-orbit cubic toroids, we need to classify lattices preserved by subgroups of $C_n^2 \rtimes S_n$ of small index. The following lemma is a very basic result in this direction. Its proof is a direct consequence of the work in [15, p. 166].

**Lemma 3.3** Let $\Lambda$ be a rank-$n$ lattice preserved by the group $C_n^2 \rtimes A_n$. Then $\Lambda$ is an integer multiple of one of the following lattices:

$$\Lambda(1, 0^{n-1}), \quad \Lambda(1, 1, 0^{n-2}), \quad \Lambda(1^n).$$

### 3.1 Cubic Toroids with 2-Orbits

In this section we classify 2-orbit toroids of type $\{4, 3^{n-2}, 4\}$. According to Lemma 2.1, to do so we need to find lattices preserved by index-2 subgroups of $[4, 3^{n-2}, 4]$. The following results enumerate such groups.
**Lemma 3.4** Let \( n \geq 3 \). If \( K \leq C_2^n \rtimes S_n \) is an index-2 subgroup, then \((C_2^n)^+ \rtimes A_n \leq K\).

**Proof** Let \( \eta : C_2^n \rtimes S_n \rightarrow S_n \) be the natural mapping. By Lemma 3.2, \([S_n : \eta(K)] \leq 2\), therefore \( A_n \leq \eta(K) \leq S_n \). In particular, \(|\eta(K)| \in \{n!/2, n!\}\). Since

\[
2^{n-1}n! = |K| = |K \cap C_2^n| \cdot |\eta(K)|,
\]

we must have \(|K \cap C_2^n| \geq 2^{n-1}\). Since \( K \) and \( C_2^n \) are normal subgroups, \( K \cap C_2^n \) must be preserved by conjugation under \( A_n \), and by Lemma 3.1, \((C_2^n)^+ \leq K\).

Since \( A_n \leq \eta(K) \), for every 3-cycle \( S \) in \( A_n \) there exists \( A \in C_2^n \) such that \( AS \in K \). This implies \((AS)^2 = ASAS = ASAS^{-1}S^2 \in K \). Now observe that \( A(SAS^{-1}) \in (C_2^n)^+ \leq K \), which implies that \( S_2 = S^{-1} \in K \). Since this holds for every 3-cycle \( S \), then \( A_n \leq K \), therefore \((C_2^n)^+ \rtimes A_n \leq K \), as desired. \( \square \)

**Corollary 3.5** If \( K \) is an index-2 subgroup of \( C_2^n \rtimes S_n \), then \( K \) is one of the following:

\[ (C_2^n \rtimes S_n)^+, \ \text{the rotational subgroup of } C_2^n \rtimes S_n; \quad (C_2^n)^+ \rtimes S_n; \quad C_2^n \rtimes A_n. \]

**Proof** It is clear that those three groups are different. By Lemma 3.4, \( K \) must contain \((C_2^n)^+ \rtimes A_n\) which is a normal subgroup of \( C_2^n \rtimes S_n \) of index 4. By the correspondence theorem for groups there are at most three index-2 subgroups containing \((C_2^n)^+ \rtimes A_n\). \( \square \)

Corollary 3.5 determines all the subgroups of \( C_2^n \rtimes S_n \) of index 2. According to Lemma 2.1, by classifying the lattices preserved by such groups we obtain a classification of 2-orbit toroids. By Lemma 3.4, every lattice preserved by an index-2 subgroup of \( C_2^n \rtimes S_n \) is also invariant under \((C_2^n)^+ \rtimes A_n\). Therefore it is useful to know all those lattices.

Consider the vectors \( v_1 := (1, 1, \ldots, 1) \), \( v_i := (-1, 1^{i-2}, -1, 1^{n-i}) \) for \( 2 \leq i \leq n \) and \( w_i := (1^{i-1}, -1, 1^{n-i}) \) for \( 1 \leq i \leq n \). Let

\[
\begin{align*}
\Lambda_0 &= \{a_1v_1 + \cdots + a_nv_n : a_1, \ldots, a_n \in \mathbb{Z}\} \quad \text{and} \\
\Lambda_1 &= \{a_1w_1 + \cdots + a_nw_n : a_1, \ldots, a_n \in \mathbb{Z}\}.
\end{align*}
\]

In other words, \( \Lambda_0 \) and \( \Lambda_1 \) are the lattices whose bases are \( \{v_i : 1 \leq i \leq n\} \) and \( \{w_i : 1 \leq i \leq n\} \), respectively. First observe that \( \Lambda_0 \) and \( \Lambda_1 \) are isometric lattices. In fact, if \( R \) is the reflection through the hyperplane \( x_1 = 0 \), then \( v_i = w_iR \).

Notice that for every \( s \in \mathbb{N} \), we have

\[
4s\Lambda_{(1,0^{n-1})} \subset 2s\Lambda_{(1,1,0^{n-2})} \subset s(\Lambda_0 \cap \Lambda_1).
\]

In particular, \( 2e_j + 2e_k \in \Lambda_0 \cap \Lambda_1 \) for \( j, k \in \{1, \ldots, n\} \), \( j \neq k \). This implies that \( \Lambda_0 \) and \( \Lambda_1 \) are preserved by \((C_2^n)^+\), since \( v_i - v_i(E_jE_k) \in 2\Lambda_{(1,1,0^{n-2})} \) and \( w_i - w_i(E_jE_k) \in 2\Lambda_{(1,1,0^{n-2})} \) for \( i, j, k \in \{1, \ldots, n\} \), \( j < k \). Similar arguments prove that \( \Lambda_0 \) and \( \Lambda_1 \) are also preserved by \( A_n \). Now it is easy to classify the lattices preserved by \((C_2^n)^+ \rtimes A_n\).
Lemma 3.6 If \( n \geq 4 \) and \( \Lambda \) is a rank-\( n \) integer lattice preserved by the group \((\mathbb{C}_2^n)^+ \rtimes \mathbb{A}_n\), then \( \Lambda \) is an integer multiple of one of the following:

\[
\Lambda_{(1^k,0^{n-k})} \text{ for } k \in \{1,2,n\}, \quad \Lambda_0, \text{ or } \Lambda_1,
\]

where \( \Lambda_0 \) and \( \Lambda_1 \) are the lattices defined in (2).

Proof Let \( s \) be the smallest positive integer among the coordinates of the vectors of \( \Lambda \). As before, since \( \mathbb{A}_n \) acts transitively on the coordinates of \( \mathbb{E}^n \), \( s \) divides every entry of every point of \( \Lambda \). By permuting coordinates, we may assume that \((s, s_2, \ldots, s_n) \in \Lambda \) for some \( s_2, \ldots, s_n \in \mathbb{Z} \). By adding some elements in \( \Lambda \) and using elements of \((\mathbb{C}_2^n)^+ \rtimes \mathbb{A}_n\) we deduce the existence of some vectors in \( \Lambda \) as follows:

\[
(s, s_2, \ldots, s_n) \in \Lambda \Rightarrow (s, s_2, \ldots, s_n) - (s, -s_2, \ldots, s_n) = (2s, 2s_2, 0, \ldots, 0) \in \Lambda
\]

\[
\Rightarrow (2s, 2s_2, 0, \ldots, 0) \in \Lambda \Rightarrow (2s, -2s_2, 0, \ldots, 0) \in \Lambda
\]

This implies that \( 4s \Lambda_{(1,0^{n-1})} \subset \Lambda \). Therefore we may write every point of \( \Lambda \) as \((s_1, \ldots, s_n) + x \) where \( s_1, \ldots, s_n \in \{-s, 0, s, 2s\} \) and \( x \in 4s \Lambda_{(1,0^{n-1})} \).

Assume that there exists a point of the form \((s, s_2, \ldots, s_n) + x \) with \( s_2, \ldots, s_n \in \{0, 2s\} \) and \( x \in 4s \Lambda_{(1,0^{n-1})} \), such that for some \( i \in \{2, \ldots, n\} \), \( s_i \in \{0, 2s\} \). By permuting coordinates, we may assume that \( i = 2 \). As seen before, \((2s, 2s_2, 0, \ldots, 0) \in \Lambda \), but \( 4s \) divides \((0, 4s, 0, \ldots, 0) \in \Lambda \), then \((2s, 0, \ldots, 0) \in \Lambda \). Hence \((0, s, 0^{n-1}) \in \Lambda \) for all \( 1 \leq j \leq n \), which implies that \( \Lambda \) is preserved not only by \((\mathbb{C}_2^n)^+ \rtimes \mathbb{A}_n\) but by \( \mathbb{C}_2^n \rtimes \mathbb{A}_n \). This implies that \( \Lambda \) is an integer multiple of one of the following:

\[
\Lambda_{(1^k,0^{n-k})} \text{ for } k \in \{1,2,n\} \text{ (see Lemma 3.3)}.
\]

If every point of \( \Lambda \) is of the form \((\pm s, \ldots, \pm s) + x \) with \( x \in 4s \Lambda_{(1,0^{n-1})} \), then there are several cases to consider. Assume that there exist points \( p, q \in \Lambda \) with \( p = (p_1, \ldots, p_n) + x_p, q = (q_1, \ldots, q_n) + x_q \) where \( |p_i| = |q_i| = s \) for \( i \in \{1, \ldots, n\} \), \( x_p, x_q \in 4s \Lambda_{(1,0^{n-1})} \), such that the number of entries of \((p_1, \ldots, p_n)\) equal to \( -s \) is even and the number of entries of \((q_1, \ldots, q_n)\) equal to \( -s \) is odd. By performing some permutation of coordinates and an even number of sign changes (if needed) we may assume that \( p = (s, s, \ldots, s) + x_p \) and \( q = (-s, s, \ldots, s) + x_q \). In this situation \( p - q = (2s, 0, \ldots, 0) + (x_p - x_q) \) and since \( 2se_i + 2se_i \in \Lambda \) for every \( i \in \{0, \ldots, n\} \), then \( 2se_i \in \Lambda \). This implies that \( \Lambda \) is preserved by \( \mathbb{C}_2^n \rtimes \mathbb{A}_n \). As before, Lemma 3.3 implies that \( \Lambda \) is an integer multiple of one of the following:

\[
\Lambda_{(1^k,0^{n-k})} \text{ for } k \in \{1,2,n\} \text{ (see Lemma 3.3)}.
\]

Now, with the notation used above, the other possibility is that for every point \( p = (p_1, \ldots, p_n) + x_p \in \Lambda \), the parity of the number of entries of \((p_1, \ldots, p_n)\) equal to \( -s \) does not depend on \( p \). Recall that \( 2s \Lambda_{(1,0^{n-2})} \subset s \Lambda_0 \cap s \Lambda_1 \). Hence, if the parity is even then all such points \((p_1, \ldots, p_n)\) belong to \( s \Lambda_0 \), while if the parity is odd then all belong to \( s \Lambda_1 \). Furthermore, since \( 4s \Lambda_{(1,0^{n-1})} \subset s \Lambda_0 \cap s \Lambda_1 \), then \( \Lambda \subset s \Lambda_i \) for exactly one \( i \in \{0,1\} \). The other inclusion follows from the fact that
2s (e_i + e_j) \in \Lambda$ for every $i, j \in \{1, \ldots, n\}$, which implies that $\Lambda$ contains either $\{sv_i : 1 \leq i \leq n\}$ or $\{sw_i : 1 \leq i \leq n\}$.

Following [9], we say that a 2-orbit $(n + 1)$-toroid belongs to class $2_I$ for $I \subseteq \{0, \ldots, n\}$ if, given a flag $\Phi$ and $i \in \{0, \ldots, n\}$, the $i$-adjacent flag $\Phi^i$ belongs to the same orbit as $\Phi$ if and only if $i \in I$. In particular, chiral toroids are those in class $2_0$.

The following result follows almost immediately from Lemma 3.6 and together with [10, Thm. 7] completes a classification of 2-orbit toroids of type $\{4, 3^{n-2}, 4\}$.

**Theorem 3.7** Let $n \geq 3$. If $n$ is odd, there are no equivelar $(n + 1)$-toroids of type $\{4, 3^{n-2}, 4\}$ with 2 flag-orbits. If $n$ is even, every equivelar $(n + 1)$-toroid of type $\{4, 3^{n-2}, 4\}$ with two flag-orbits may be described as a toroid $\{4, 3^{n-2}, 4\}_\Lambda$ where the lattice associated to $\Lambda$ is an integer multiple of the lattice $\Lambda_1$ described above. Furthermore, such toroids belong to class $2_{\{1,2,\ldots,n-1\}}$.

**Proof** According to Lemma 2.1, every 2-orbit cubic toroid must be given by a lattice $\Lambda$ preserved by an index-2 subgroup $K$ of $G_0$ containing $\chi$. By Corollary 3.5, the only possibilities for $K$ are $(C_2^n \times S_n)^+$, $C_2^n \rtimes A_n$, and $(C_2^n)^+ \times S_n$. However, $K$ cannot be $(C_2^n \times S_n)^+$, since this would induce a chiral $(n + 1)$-toroid and they are known not to exist if $n \geq 3$ (see [14, Thm. 9.1] and [15, Sect. 6H]). If $\Lambda$ is a lattice preserved by $C_2^n \times A_n$ then $\Lambda$ is an integer multiple of a lattice $\Lambda_{(1,k,0^{n-k})}$ for $k \in \{1, 2, n\}$ (see Lemma 3.3), but such lattices induce regular toroids. Therefore, the only possibility for $K$ is $(C_2^n)^+ \times S_n$.

Note that if $n$ is odd, then $\chi \notin (C_2^n)^+ \times S_n$. This implies that there are no two-orbit $(n + 1)$-toroids whenever $n$ is odd. If $n$ is even then $\Lambda$ must be preserved by $(C_2^n)^+ \times S_n$. In particular, $\Lambda$ must be preserved by $\gamma_i$ and those lattices are classified in Lemma 3.6. Since the lattices $\gamma_i$ induce regular toroids for all $i \in \mathbb{Z}^+$, then $\Lambda$ must be an integer multiple of $\Lambda_0$ or an integer multiple of $\Lambda_1$. Finally, observe that both $\Lambda_0$ and $\Lambda_1$ are preserved by $S_n$, and since they are isometric lattices, then they produce isomorphic toroids. Consequently, every two-orbit toroid is induced by an integer multiple of $\Lambda_1$, as stated.

To see that those toroids belong to class $2_{\{1,2,\ldots,n-1\}}$, note that if $R_0, \ldots, R_n$ are the distinguished generators of $G(\mathcal{U})$ then $R_1, \ldots, R_{n-1} \in S_n$ and hence preserve $\Lambda_1$. These generators induce automorphisms $\tilde{R}_1, \ldots, \tilde{R}_{n-1}$ of $\mathcal{U}/\Lambda$ such that if $\Phi \Lambda$ is the base flag of $\mathcal{U}/\Lambda$, then $\Phi \Lambda \tilde{R}_i = \Phi R_i \Lambda = \Phi^i \Lambda = (\Phi \Lambda)^i$, for $1 \leq i \leq n - 1$. Moreover, there is no automorphism of $\mathcal{U}/\Lambda$ mapping $\Phi \Lambda$ to $(\Phi \Lambda)^i$ for $i \in \{0, n\}$ since this would imply that $\Lambda$ is preserved by $C_2^n$ and hence by $C_2^n \times A_n$, forcing $\mathcal{U}/\Lambda$ to be regular (see Lemma 3.3). Therefore, $\mathcal{U}/\Lambda$ belongs to class $2_{\{1,2,\ldots,n-1\}}$.

### 3.2 Cubic $(n + 1)$-Toroids with k Flag-Orbits for $2 < k < n$

Now we proceed to classify toroids of type $\{4, 3^{n-2}, 4\}$ with $k$ orbits for $2 < k < n$. The following lemma is the key of such classification.

**Lemma 3.8** Let $n \geq 5$. If $K \leq C_2^n \rtimes S_n$ has index $k$ for some $2 < k < n$ then $K = (C_2^n)^+ \times A_n$. 

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Lemma 3.11 Assume that \( n \) is an integer multiple of one of the following:

- \( \Lambda(1,1) \times \Lambda(1,1) \), the lattice generated by the vectors \( (1, 0, 1, 0), (1, 0, -1, 0), (0, 1, 0, 1), \) and \( (0, 1, 0, -1) \).

Proof It is clear that all those lattices are preserved by \( \mathbf{C}_2^4 \times \mathbf{D}_4 \). Assume that \( \Lambda \) is a lattice preserved by \( \mathbf{C}_2^4 \times \mathbf{D}_4 \). Let \( s \) be the minimum positive value among all the coordinates of the points in \( \Lambda \) and take \( s(s_2, s_3, s_4) \in \Lambda \). Since \( \Lambda \) is preserved by \( \mathbf{C}_2^4 \) and \( \mathbf{D}_4 \) acts transitively on the coordinates of the elements in \( \Lambda \), proceeding as in the…
proof of Lemma 3.6, we may conclude that \((0^{i-1}, 2s, 0^{n-j}) \in \Lambda\) for all \(1 \leq j \leq n\). Therefore, we may assume that \(s_i \in \{0, s\}\) for \(i \in \{2, 3, 4\}\).

Among all such points, we take \(x \in \Lambda\) with minimum number \(k\) of non-zero entries. If \(k = 1\) then \(\Lambda = s\Lambda_{(1,0,0,0,0)}\). If \(k = 2\), up to permutation of coordinates with an element of \(D_4\) there are two possibilities: \(x = (s, s, 0, 0)\) in which case \(\Lambda = s\Lambda_{(1,1,0,0,0)}\); or \(x = (s, 0, s, 0)\) which implies that \(\Lambda = \Lambda_{(1,1)} \times \Lambda_{(1,1)}\). The case \(k = 3\) is impossible since up to the action of \(D_4\), the only possibility is \(x = (s, s, s, 0)\) which implies that \((0, s, s, s) \in \Lambda\) and hence \((s, 0, 0, -s) \in \Lambda\), contradicting the choice of \(k\). The only remaining possibility is \(k = 4\) which implies that \(\Lambda = s\Lambda_{(1,1,1,1)}\).

\(\square\)

**Theorem 3.12** Every equivelar rank-5 toroid of type \(4, 3, 3, 4\) with three flag-orbits may be described as a toroid \([4, 3, 3, 4]_{s_\Lambda}\), where \(s \in \mathbb{Z}\) and \(\Lambda\) is the lattice group generated by the translations with respect to vectors \((1, 0, 1, 0), (1, 0, -1, 0), (0, 1, 0, 1),\) and \((0, 1, 0, -1)\).

**Proof** It follows from Lemma 3.11, since the integer multiples of the lattices \(\Lambda_{(1,0^{i-1})}\) for \(k \in \{1, 2, 4\}\) induce regular toroids. \(\square\)

### 3.3 Cubic \((n + 1)\)-Toroids with \(n\) Flag-Orbits

Now we proceed to classify \((n + 1)\)-cubic toroids with \(n\) flag-orbits for \(n \geq 4\). The strategy is essentially the same that we have used before: first we determine the \(n\)-index subgroups of \(C_2^n \times S_n\) and then we determine the lattices preserved by each of them. In order to classify subgroups of \(C_2^n \times S_n\) of index \(n\) we use the following result regarding permutation groups. This is a consequence of [7, Thm. 5.2B].

**Theorem 3.13** Let \(n \geq 4\) and let \(S_n\) be the group of permutations of \(\{1, \ldots, n\}\). If \(G\) is a subgroup of \(S_n\) of index \(n\) then, up to conjugacy, one of the following holds:

- The group \(G\) is the stabiliser of \(\{n\}\) and hence is isomorphic to \(S_{n-1}\).
- \(n = 6\) and \(G\) acts on \(\{1, 2, \ldots, 6\}\) as the group \(PGL_2(5)\) acts on the points of the projective line over \(GF(5)\), the field with 5 elements.

Now we may classify the subgroups of \(C_2^n \times S_n\) of index \(n\).

**Lemma 3.14** Let \(n \geq 4\). If \(K\) is a subgroup of \(C_2^n \times S_n\) of index \(n\) then, up to conjugacy, one of the following holds:

- \(K = C_2^n \times S_{n-1}\);
- \(n = 4\) and \(K = (C_2^4)^+ \rtimes A_4\);
- \(n = 6\) and \(K = C_2^6 \rtimes PGL_2(5)\).

Here \(S_{n-1}\) denotes the stabiliser of the last coordinate in \(S_n\) and \(PGL_2(5)\) denotes the subgroup of \(S_6\) that acts on the coordinates of \(\mathbb{P}^6\) as \(PGL_2(5)\).

**Proof** The result is a consequence of Theorem 3.13. By Lemma 3.2, \([S_n : \eta(K)] \leq n\), which implies that \(\eta(K)\) is \(S_n, A_n\), or one of the groups described in Theorem 3.13. Let \(K_0 = K \cap C_2^n\) and recall that \(|K| = |\eta(K)| \cdot |K_0|\).
If $\eta(K) = S_n$, then $2^n(n-1)! = |K| = |\eta(K)| \cdot |K_0| = n!|K_0|$, which implies that $2^n/n = |K_0|$. However, if $n \geq 4$ we have $2 < 2^n/n < 2^{n-1}$, but this is impossible since $K_0$ is a subgroup of $C_2^n$ normalised by $S_n$ and, by Lemma 3.1, must be one among $C_2^n$, $(C_2^3)^+$, and $\langle \chi \rangle$.

Proceeding in a similar way we may see that if $\eta(K) = A_n$, the unique possibility for $K_0$ is $(C_2^3)^+$ and this is only possible if $n = 4$. With similar arguments to those used in the proof of Lemma 3.4 we may conclude that $K = A_4 \rtimes (C_2^3)^+$.

Finally, if $\eta(K)$ has index $n$ in $S_n$, then we have $|K| = 2^n(n - 1)! = |\eta(K)| \cdot |K_0| = (n - 1)!|K_0|$. Therefore $K_0$ must be $C_2^n$ and $K = C_2^n \rtimes S_{n-1}$ or $n = 6$ and $K = C_2^n \rtimes PGL_2(5)$. \qed

Now we proceed to classify lattices preserved by subgroups of $C_2^n \rtimes S_n$ of index $n$. Lattices preserved by $(C_2^3)^+ \rtimes A_4$ are described in Lemma 3.6. It only remains to determine those rank-$n$ lattices preserved by $C_2^n \rtimes S_{n-1}$ and those rank-$6$ lattices preserved by $C_2^6 \rtimes PGL_2(5)$. Such lattices are described in the following results.

**Lemma 3.15** If $\Lambda$ is an integer rank-$6$ lattice preserved by $C_2^n \rtimes PGL_2(5)$, then $\Lambda = s\Lambda_{(1k, 06-k)}$ for some $s \in \mathbb{Z}$ and $k \in \{1, 2, 6\}$.

**Proof** Let $\Lambda$ be an integer rank-$6$ lattice preserved by $C_2^n \rtimes PGL_2(5)$. Let $s \in \mathbb{Z}$ be the minimum positive integer among the entries of all the vectors in $\Lambda$. As before, $PGL_2(5)$ acts transitively on the coordinates of $\mathbb{E}^6$, all the entries of any vector of $\Lambda$ must be multiples of $s$. Let $(s_1, \ldots, s_6) \in \Lambda$ be such that $s_1 = s$. Since $\Lambda$ is preserved by $C_2^6$ this implies that $(-s, s_2, \ldots, s_6) \in \Lambda$. Hence $2se_1 \in \Lambda$ and therefore $2se_i \in \Lambda$ for $i \in \{1, \ldots, 6\}$.

We may take $(s_1, \ldots, s_6) \in \Lambda$ such that all its entries are either $0$ or $s$. Furthermore, since the action of $PGL_2(5)$ is 3-transitive on the coordinates of $\mathbb{E}^6$, we may assume that $(s_1, \ldots, s_6)$ is of the form $(s^k, 0^{6-k})$. Among all non-zero points in $\Lambda$ with this form take one of those where $k$ is minimum.

If $k = 1$ then $se_i \in \Lambda$ for $i \in \{1, \ldots, 6\}$ which implies that $s\Lambda_{(1, 05)} = \Lambda$. Suppose that $k = 2$. Since $\Lambda$ is preserved by $PGL_2(5)$, then $s\Lambda_{(1, 1, 04)} \subset \Lambda$. On the other hand, by the observations made above, every point of $\Lambda$ may be written as the sum of a point in $s\Lambda_{(1, 1, 04)}$ and a point in $2s\Lambda_{(1, 05)}$. Since $2s\Lambda_{(1, 05)} \leq s\Lambda_{(1, 1, 04)}$ we must have $\Lambda = s\Lambda_{(1, 1, 04)}$.

Now assume that $k \in \{3, 4, 5\}$ and take $(s^k, 0^{6-k}) \in \Lambda$. Given that $\Lambda$ is preserved by $PGL_2(5)$, we must have $(0, s^k, 0^{6-k-1})$ is a point of $\Lambda$. This implies that $(s, 0^{k-1}, s, 0^{6-k-1}) \in \Lambda$, hence $(s, s, 0, 0, 0, 0) \in \Lambda$ which contradicts the choice of $k$. Finally, if $k = 6$, proceeding in a similar way as when $k = 2$, we may conclude that $\Lambda = s\Lambda_{(16)}$. \qed

The previous lemma implies that all $(6 + 1)$-toroids induced by lattices preserved by $C_2^n \rtimes PGL_2(5)$ are regular toroids. Since all $(4 + 1)$-toroids induced by lattices preserved by $(C_2^3)^+ \rtimes A_4$ are regular or 2-orbit, Lemma 3.14 implies that if there exist $(n + 1)$-toroids with $n$ flag-orbits, they must be induced by lattices preserved by $C_2^n \rtimes S_{n-1}$ that do not induce regular toroids. We now proceed to classify those lattices.

We use Lemma 2.2 to classify lattices preserved by $C_2^n \rtimes S_{n-1}$. We shall assume that $S_{n-1}$ is precisely the stabiliser of the last coordinate on $\mathbb{E}^n$. Let $\Pi$ be the...
hyperplane $x_n = 0$ and $R$ the reflection through $\Pi$, that is, $R: (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n)$. If $\Lambda$ is a lattice preserved by $C_2^n \times S_{n-1}$, then $\Lambda$ is a lattice preserved by $R$. Since $S_{n-1}$ preserves $\Pi$, if $\Lambda_0 = \Lambda \cap \Pi$, then $\Lambda_0$ must be a rank-$(n-1)$ lattice preserved by the restriction of the action of $C_2^n \times S_{n-1}$ to $\Pi$ and therefore $\Lambda_0$ must be one among $s\Lambda_{(1,0^{n-2})}$, $s\Lambda_{(1,1,0^{n-3})}$, and $s\Lambda_{(1^{n-1})}$ for some positive integer $s$.

If $\Lambda$ is a vertical translation lattice then $\Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + k(den))$ for some $d \in \mathbb{Z}^+$. All the lattices of this form are preserved by $C_2^n \times S_{n-1}$, we only need to determine those values of $d$ which induce regular lattices. Observe that among the lattices $\Lambda_{(1^k,0^{n-k})}$, only those of the form $s\Lambda_{(1,0^{n-1})}$ for $s \in \mathbb{N}$ are vertical translation lattices. This implies that if $\Lambda_0 = s\Lambda_{(1,0^{n-2})}$, then $d = s$ is the only value of $d$ such that $\Lambda$ induces a regular toroid. If $\Lambda_0$ is $s\Lambda_{(1,1,0^{n-3})}$ or $s\Lambda_{(1^{n-1})}$, then any value of $d$ gives a lattice that induces an $n$-orbit toroid. Therefore, for the following discussion we shall assume that $\Lambda$ is not a vertical translation lattice.

Assume that $\Lambda_0 = s\Lambda_{(1,0^{n-2})}$. Let $u$ be the point of $\Lambda \cap \Pi^\perp \setminus \{0\}$ closest to $o$, say $u = den$. By Lemma 2.2 there exist unique $\alpha_1, \ldots, \alpha_{n-1} \in [0, 1)$, not all zero, such that if $w = s(\alpha_1 e_1 + \cdots + \alpha_{n-1} e_{n-1})/2 + den/2$, then $\Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + kw)$. Since $\Lambda$ is preserved by $S_{n-1}$, the only possible choice of the numbers $\alpha_1, \ldots, \alpha_{n-1}$ is $\alpha_1 = \cdots = \alpha_{n-1} = 1$. Since $w$ must be a point with integer coordinates, $s$ and $d$ must be even. If $d = s$, then $\Lambda = s\Lambda_{(1^n)}/2$. In any other case $\Lambda$ induces an $n$-orbit toroid.

Now assume that $\Lambda_0 = s\Lambda_{(1,1,0^{n-3})}$. As before, let $u \in \Lambda \setminus \Pi$ be the closest point of $\Pi^\perp$ to $o$, say $u = den$ for some integer $d$. Recall that $\{2se_1, s(e_2 - e_1), s(e_3 - e_2), \ldots, s(e_{n-1} - e_{n-2})\}$ is a basis for $\Lambda_0$. Take $\alpha_1, \ldots, \alpha_{n-1} \in [0, 1)$, not all zero, such that

$$w = \frac{s}{2}(\alpha_1(2e_1) + \alpha_2(e_2 - e_1) + \cdots + \alpha_{n-1}(e_{n-1} - e_{n-2})) + \frac{d}{2}en = \frac{s}{2}((2\alpha_1 - \alpha_2)e_1 + (\alpha_2 - \alpha_3)e_2 + \cdots + (\alpha_{n-2} - \alpha_{n-1})e_{n-2} + \alpha_{n-1}e_{n-1}) + \frac{d}{2}en$$

satisfies $\Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + kw)$. Let $w_i$ be the image of $w$ under the reflection through the plane $x_i = 0$ for $1 \leq i \leq n-1$. Since $\Lambda$ is preserved by $C_2^n$, $w_i \in \Lambda$ for all $i \in [1, \ldots, n]$. If $\alpha_{n-1} = 1$ then $w - w_{n-1} = se_{n-1}$. This contradicts that $\Lambda_0 = s\Lambda_{(1,1,0^{n-3})}$; if $\alpha_{n-1} = 0$ similarly if $\alpha_{n-2} = 0$ and $\alpha_{n-2} = 1$, then $w - w_{n-2} = se_{n-2}$, hence $\alpha_{n-2} = 0$. Proceeding in a similar way, we may conclude that $\alpha_2 = \cdots = \alpha_{n-1} = 0$, which implies that $\alpha_1 = 1$ and $w = se_1 + den/2$. Again, since $w$ must have integer coordinates, $d$ must be even. If $d = 2s$, then $\Lambda = s\Lambda_{(1,1,0^{n-2})}$. In any other case $\Lambda$ is a lattice that induces an $n$-orbit toroid.

Finally, assume that $\Lambda_0 = s\Lambda_{(1^{n-1})}$. As before, take $u \in (\Lambda \cap \Pi^\perp) \setminus \{0\}$ closest to $o$, say $u = den$ for some $d \in \mathbb{Z}^+$. Since $\{2e_1, \ldots, 2e_{n-2}, e_1 + \cdots + e_{n-1}\}$ is a basis for $\Lambda_0$, by Lemma 2.2 there exist $\alpha_1, \ldots, \alpha_{n-1} \in [0, 1]$ such that

$$w = \frac{s}{2}(\alpha_1(2e_1) + \cdots + \alpha_{n-2}(2e_{n-2}) + \alpha_{n-1}(e_1 + \cdots + e_{n-1})) + \frac{d}{2}en$$
satisfies that \( \Lambda = \bigcup_{k \in \mathbb{Z}} (\Lambda_0 + kw) \). For \( i \in \{1, \ldots, n-1\} \) let \( w_i \) the image of \( w \) under the reflection through the plane \( x_i = 0 \). If \( \alpha_{n-1} = 1 \) then \( w - w_{n-1} = sw_{n-1} \), which contradicts that \( \Lambda_0 = s\Lambda_{(1^{n-1})} \); then \( \alpha_{n-1} = 0 \). Let \( i \in \{1, \ldots, n-2\} \) be such that \( \alpha_i = 1 \) and let \( S_i \) be the reflection \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-2}, x_i, x_{n}) \). Then \( w - w_i S_i = s(e_i + e_{n-1}) \) which for \( n \geq 4 \) does not belong to \( s\Lambda_{(1^{n-1})} \). This is a contradiction. Therefore if \( \Lambda_0 = s\Lambda_{(1^{n-1})} \), then \( \Lambda \) must be a vertical translation lattice.

The previous discussion completes the classification of cubic toroids with \( n \) flag-orbits. We summarise it with the following result.

**Theorem 3.16** Let \( n \geq 4 \). Every equivelar cubic \( n \)-orbit toroid of rank \( n + 1 \) may be described as a toroid \( \{4, 3^{n-2}, 4\}_\Lambda \) where \( \Lambda \) is the lattice group associated to one of the following lattices:

1. \( \bigcup_{k \in \mathbb{Z}} (s\Lambda_{(1,0^{n-2})} + k(den)) \) for some \( s, d \in \mathbb{Z}^+, s \neq d \);
2. \( \bigcup_{k \in \mathbb{Z}} (s\Lambda_{(1,1,0^{n-3})} + k(den)) \) for some \( s, d \in \mathbb{Z}^+ \);
3. \( \bigcup_{k \in \mathbb{Z}} (s\Lambda_{(1,1,1,0^{n-4})} + k(den)) \) for some \( s, d \in \mathbb{Z}^+ \);
4. \( \bigcup_{k \in \mathbb{Z}} (s\Lambda_{(1,1,1,0^{n-4})} + k(se_1/2 + se_2/2 + \cdots + se_{n-1}/2 + den/2)) \) for some \( s, d \in \mathbb{Z}^+, s \) and \( d \) even, and \( d \neq s \);
5. \( \bigcup_{k \in \mathbb{Z}} (s\Lambda_{(1,1,1,0^{n-4})} + k(se_1 + den/2)) \) for some \( s, d \in \mathbb{Z}^+, d \) even, and \( d \neq 2s \).

### 4 Few-Orbit Toroids of Type \{3, 3, 4, 3\} and \{3, 4, 3, 3\}

So far we have classified the cubic few-orbit toroids arising from the tessellation \{4, 3^{n-2}, 4\} of the Euclidean \( n \)-space. If \( n \notin \{2, 4\} \), every equivelar toroid is a quotient of such tiling. The equivelar toroids for \( n = 2, 3 \) are fully described in [10]. Nevertheless, for \( n = 4 \) the Euclidean space admits a tessellation of type \{3, 4, 3, 3\} by 24-cells, along with its dual, of type \{3, 3, 4, 3\}, by regular 4-cross polytopes.

In this section we classify the equivelar few-orbit toroids that are quotients of the regular tessellation of \( \mathbb{E}^4 \) of type \{3, 3, 4, 3\}. We will follow the same ideas we used in the classification of the cubic toroids. Recall that the vertex set of \{3, 3, 4, 3\} is the lattice \( \Lambda_{(1,1,1,1)} \), and therefore its automorphism group has the form \( T(U) \rtimes G_0 = \langle R_0, \ldots, R_4 \rangle \), where \( T(U) \) is the translation subgroup of \( U \) and \( G_0 \) is the stabiliser of the base vertex \( o \). The group \( G_0 = \langle R_1, \ldots, R_4 \rangle \), with \( R_1, R_2, R_3, R_4 \) defined as in Table 2, is isomorphic to \{3, 4, 3\}, the automorphism group of the 24-cell \{3, 4, 3\}. To proceed as in the case of the cubic toroids, we need some facts about the structure of \{3, 4, 3\} which we present in the following paragraphs.

First, observe that \((\pm 2, 0, 0, 0), \ldots, (0, 0, 0, \pm 2)\) and \((\pm 1, \pm 1, \pm 1, \pm 1)\) are the vertices of a 24-cell whose automorphism group is precisely the group \( \langle R_1, R_2, R_3, R_4 \rangle = \{3, 4, 3\} \). Furthermore, they are partitioned into the sets \( \mathcal{O}_0 \), consisting of the points \((\pm 2, 0, 0, 0), (0, \pm 2, 0, 0), (0, 0, \pm 2, 0), \) and \((0, 0, 0, \pm 2)\); \( \mathcal{O}_1 \), the set of points of the form \((\pm 1, \pm 1, \pm 1, \pm 1)\) with an even number of \(-1\) entries; and \( \mathcal{O}_{-1} \), the set of points of the form \((\pm 1, \pm 1, \pm 1, \pm 1)\) with an odd number of entries equal to \(-1\).
The vertex set of the 24-cell described above is preserved by $C_2^4 \rtimes S_4$, implying that $C_2^4 \rtimes S_4 \leq [3, 4, 3]$. Furthermore, the partition $\{O_0, O_1, O_{-1}\}$ is preserved by every element of $[3, 4, 3]$. Also note that the elements of $[3, 4, 3]$ that fix every element of the particle are precisely those in $(C_2^4)^+ \rtimes S_4$. Hence $(C_2^4)^+ \rtimes S_4$ is a normal subgroup of $[3, 4, 3]$. Moreover, $\langle R_1, R_2 \rangle \cap ((C_2^4)^+ \rtimes S_4) = \{1\}$ and $|((C_2^4)^+ \rtimes S_4)| = 2^3 \cdot 4! = |[3, 4, 3]|/|\langle R_1, R_2 \rangle|$. Therefore, $[3, 4, 3] = ((C_2^4)^+ \rtimes S_4) \rtimes \langle R_1, R_2 \rangle$.

Finally, it is not hard to see that $\langle R_1, R_2 \rangle$ acts as the full symmetric group on the elements of the partition $\{O_0, O_1, O_{-1}\}$; this observation will prove to be very useful in computing the symmetry types of the 3-orbit non-cubic toroids.

A straightforward implementation of the ConjugacyClassesSubgroups procedure in GAP System [18] can be used to find representatives of conjugacy classes of subgroups of index 2, 3, and 4 that contain $\chi$. Those representatives are listed in Table 3.

Observe that if $G$ is one of the index-2 subgroups of $[3, 4, 3]$ of Table 3, then $G$ contains the index-4 subgroup $((C_2^4)^+ \rtimes A_4) \rtimes \langle R_1, R_2 \rangle$. Any lattice preserved by $G$ must be preserved by $((C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$. Similarly, since $((C_2^4)^+ \rtimes A_4) \leq ((C_2^4)^+ \rtimes S_4) \rtimes \langle R_1, R_2 \rangle$, any lattice preserved by $G$ must be preserved by $(C_2^4)^+ \rtimes A_4$. These lattices were classified in Lemma 3.6. It is very easy to extend this result to obtain a classification of the lattices preserved by $(C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$. This classification is presented in the following result.

**Lemma 4.1** Let $\Lambda$ be a rank-4 lattice contained in $\Lambda_{(1,1,1,1)}$ such that $\Lambda$ is preserved by $((C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$. Then $\Lambda$ is one of the following:

\[
s \Lambda_{(1,1,0,0)} \quad \text{for some even } s; \quad s \Lambda_{(1,1,1,1)} \quad \text{for some } s \in \mathbb{Z}^+.
\]

**Proof** Let $\Lambda \subseteq \Lambda_{(1,1,1,1)}$ be a lattice preserved by $((C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$. By Lemma 3.6, $\Lambda$ is of the form $s \Lambda_i$, $i \in \{0, 1\}$ (see (2)), or $s \Lambda_{(i,0^4-k)}$, for $k \in \{1, 2, 4\}$ and $s \in \mathbb{Z}^+$. Observe that the lattices $s \Lambda_0$, $s \Lambda_1$, and $s \Lambda_{(1,1,1,1)}$ are subsets of $\Lambda_{(1,1,1,1)}$ for every $s \in \mathbb{Z}^+$, but $s \Lambda_{(1,0^4-k)}$, $k \in \{1, 2\}$, are contained in $\Lambda_{(1,1,1,1)}$ if and only if $s$ is even. Furthermore, $\Lambda_{(1,0,0,0)}$ is not invariant under $((C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$, since $(1, 0, 0, 0) R_1 R_2 = (1/2, 1/2, 1/2, -1/2) \notin \Lambda_{(1,0,0,0)}$. Note also that

---

**Table 3** Representatives of conjugacy classes of small index subgroups in $[3, 4, 3]$

| Index | Representatives |
|-------|-----------------|
| 2     | $[3, 4, 3]^+$, the rotational subgroup of $[3, 4, 3]$ |
|       | $((C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$ |
|       | $((C_2^4)^+ \rtimes S_4) \times \langle R_1, R_2 \rangle$ |
| 3     | $((C_2^4)^+ \rtimes S_4) \times \langle R \rangle$ |
|       | $((C_2^4)^+ \rtimes D_4) \times \langle R_1, R_2 \rangle$ |
| 4     | $((C_2^4)^+ \rtimes A_4) \times \langle R_1, R_2 \rangle$ |
\[(1, 1, 1, 1) R_1 R_2 = (2, 0, 0, 0) \notin \Lambda_0 \text{ and } (-1, 1, 1, 1) R_1 R_2 = (1, -1, -1, 1) \notin \Lambda_1.\]

It follows that \( R_1 R_2 \) does not preserve the lattices \( \Lambda_0 \) and \( \Lambda_1 \).

Finally, recall that for every \( s \in \mathbb{Z}^+ \), the lattices \( s\Lambda(1,1,1,1) \) and \( 2s\Lambda(1,1,0,0) \) are invariant under the full group \([3, 4, 3]\) (see [15, Sect. 6E]).

From the previous result we derive the following theorem.

**Theorem 4.2** Let \( k \in \{2, 4\} \). There are no equivale\( r \) (4 + 1)-toroids of type \([3, 3, 4, 3]\) with \( k \) flag-orbits. Dually, there are no (4 + 1)-toroids of type \([3, 4, 3, 3]\) with \( k \) orbits.

*Proof* It follows directly from Lemma 4.1. If \( U/\Lambda \) is a toroid of type \([4, 4, 3, 4]\) with \( k \) orbits, for \( k \in \{2, 4\} \), then the lattice \( \Lambda \) must be of the form \( s\Lambda(1,1,0,0) \) or of the form \( s\Lambda(1,1,1,1) \) for some \( s \in \mathbb{Z}^+ \). However, it is known that such lattices induce regular toroids (see [15, Sect. 6E]).

For the first index-3 subgroup, observe that \( ((C_2^4)^{+} \rtimes S_4) \rtimes \langle R_2 \rangle = C_2^4 \rtimes S_4 \), and the only lattices invariant under \( ((C_2^4)^{+} \rtimes S_4) \rtimes \langle R_2 \rangle \) are of the form \( s\Lambda(1,0,4,-4) \), for \( k \in \{1, 2, 4\} \). From the previous analysis we derive that \( 2s\Lambda(1,0,0,0) \) is the only family of lattices preserved by \( ((C_2^4)^{+} \rtimes S_4) \rtimes \langle R_2 \rangle \) that is not preserved by \([3, 4, 3]\).

For the other representative of the index-3 subgroups, we will proceed in a similar way as before. Note that every lattice preserved by \( ((C_2^4)^{+} \rtimes D_4) \rtimes \langle R_1, R_2 \rangle \) should be preserved by \( (C_2^4)^{+} \rtimes D_4 \). Those lattices are classified in Lemma 3.11. The only instances of those not preserved also by \([3, 4, 3]\) are of the form \( s\Lambda(1,0,0,0) \) or of the form \( s(\Lambda(1,1) \rtimes \Lambda(1,1)) \). Moreover, such lattices are contained in \( \Lambda(1,1,1,1) \) if and only if \( s \) is even. However, we already know that \( \Lambda(1,0,0,0) \) is not preserved by \( R_1 R_2 \). Finally, observe that both \( R_1 \) and \( R_2 \) permute the vectors of the basis of \( \Lambda(1,1) \rtimes \Lambda(1,1) \) given in Lemma 3.11. Thus, we have proved the following result.

**Theorem 4.3** The non-cubic equivale\( r \) toroids with 3 flag-orbits are exactly those of the form \([3, 3, 4, 3]/\Lambda \) or \([3, 4, 3, 3]/\Lambda \), with \( \Lambda \) either an even integer multiple of the lattice group \( \Lambda(1,0,0,0) \) or an even integer multiple of the lattice group generated by the translations with respect to the vectors \((1, 0, 1, 0), (1, 0, -1, 0), (0, 1, 0, 1), \) and \((0, 1, 0, -1)\).

**5 Symmetry Type of Few-Orbit Toroids**

In this section we determine the symmetry type of each family of few-orbit toroids classified in Sects. 3 and 4. Following [6], the symmetry type graph \( T(U/\Lambda) \) of a toroid \( U/\Lambda \), is the labelled pre-graph (that is, semi-edges and multiple edges are allowed) whose vertex set is the set of orbits of flags of \( U/\Lambda \) and such that there is an edge labelled \( i \) between the orbits \( O_1 \) and \( O_2 \) if and only if \( O_1 \neq O_2 \) and there exists a flag \( \Phi \) such that \( \Phi \in O_1 \) and \( \Phi^i \in O_2 \). Observe that since \( (\Phi^i)S = (\Phi S)^i \) for every flag \( \Phi \) and every automorphism \( S \), the symmetry type graph does not depend on the representatives of the flag-orbits. We shall agree on using a semi-edge labelled \( i \) whenever \( \Phi \) and \( \Phi^i \) belong to the same orbit. Note that this definition is slightly different than that of [6], however it is easy to see that both are equivalent.
According to this definition, the symmetry type graph of a regular $n$-toroid consists of only one vertex and $n$ semi-edges labelled with $\{0, \ldots, n-1\}$. If an $n$-toroid is in the class $2_I$ for $I \subseteq \{0, \ldots, n-1\}$, then its symmetry type graph is composed of only two vertices with edges whose labels lie in $\{0, \ldots, n-1\} \setminus I$ and semi-edges on each vertex labelled with the elements of $I$. In this sense, the symmetry type graph generalises the notion of toroids in class $2_I$ for toroids with more than two flag-orbits. Observe that if $T(\mathcal{U}/\Lambda)$ is the symmetry type graph of a toroid $\mathcal{U}/\Lambda$, then the symmetry type graph of $(\mathcal{U}/\Lambda)^*$ can be obtained from $T(\mathcal{U}/\Lambda)$ using the same vertex set and an edge labelled $n - i - 1$ for every $i$-labelled edge in $T(\mathcal{U}/\Lambda)$.

Symmetry type graphs describe not only the number of flag-orbits of a toroid, but also the local arrangement of the orbits. In order to determine the symmetry graph of few-orbit toroids we use the following result, which is essentially [6, Prop. 1] in the language of toroids.

Lemma 5.1 Let $\mathcal{U}/\Lambda$ be a toroid with symmetry type graph $T(\mathcal{U}/\Lambda)$. Let $T^i(\mathcal{U}/\Lambda)$ be the subgraph of $T(\mathcal{U}/\Lambda)$ obtained by erasing the edges labelled $i$ from $T(\mathcal{U}/\Lambda)$. Then $\mathcal{U}/\Lambda$ is $i$-face-transitive if and only if $T^i(\mathcal{U}/\Lambda)$ is connected.

Recall that the group of automorphisms of a toroid $\mathcal{U}/\Lambda$ is $\text{Norm}_{\text{G}(\mathcal{U})}(\Lambda)/\Lambda$. If $\mathcal{U}$ is a regular tessellation of $\mathbb{E}^n$ then, up to duality, $T(\mathcal{U})$ acts transitively on the vertices of $\mathcal{U}$. This implies that all the flag-orbits of $\mathcal{U}/\Lambda$ occur on the base vertex of $\mathcal{U}/\Lambda$. Furthermore, with the correspondence introduced in Lemma 2.1 between $\text{Norm}_{\text{G}(\mathcal{U})}(\Lambda)$ and a subgroup $K$ of the vertex stabiliser of $G(\mathcal{U})$, the configuration of flag-orbits of $\mathcal{U}/\Lambda$ around the base vertex is the same as the configuration of orbits of flags of $\mathcal{U}$ containing the base vertex under the action of $K$. We use Lemma 5.1 and the previous observation, together with the results of [6] on symmetry type graphs of highly symmetric maniplexes, to determine the symmetry type graphs of the few-orbit toroids.

5.1 Cubic Toroids

Symmetry type graphs of regular and 2-orbit toroids were already described above. Since $k$-orbit $(n + 1)$-toroids do not exist for $2 < k < n$ unless $n = 4$ we only need to determine the symmetry type graphs of 3-orbit $(4 + 1)$-toroids and $n$-orbit $(n + 1)$-toroids for $n \geq 4$.

Recall that if $\mathcal{U}/\Lambda$ is a cubic toroid, then $\text{Aut}(\mathcal{U}/\Lambda) = \text{Norm}_{\text{G}(\mathcal{U})}(\Lambda)/\Lambda = (T(\mathcal{U}) \rtimes K)/\Lambda$ for a certain group $K$ with $K \leq G_0(\mathcal{U})$, where $G_0(\mathcal{U})$ denotes the stabiliser of the vertex $o$ under $G(\mathcal{U})$. Note that the symmetry type graph depends only on $K$, in particular all $(n + 1)$-toroids with $n$ orbits share the same symmetry type graph.

To determine the symmetry type graph of 3-orbit toroids we use the results of [6], where symmetry type graphs of 3-orbit maniplexes are classified. In particular, as a consequence of [6, Prop. 4], the symmetry type graph of an equivlar 3-orbit toroid $\mathcal{U}/\Lambda$ is completely determined by the set $J$ defined by the property that $\text{Aut}(\mathcal{U}/\Lambda)$ is $j$-face-transitive if and only if $j \in J$. Now we determine the sets $J$ for every class of 3-orbit toroids.
Let $\mathcal{U}/\mathcal{A}$ be a cubic $(4 + 1)$-toroid with 3 flag-orbits. According to Lemma 3.10, the group $K$ is the group $\mathbb{C}_2^4 \rtimes \mathbb{D}_4$, where $\mathbb{D}_4$ acts on the coordinates of $\mathbb{R}^4$ as the dihedral group. Note that since $\text{Tr}(\mathcal{U}) \leq \text{Norm}_G(\mathcal{U})$, Aut $(\mathcal{U}/\mathcal{A})$ acts transitively on vertices and on cells of $\mathcal{U}/\mathcal{A}$. Vertex-transitivity of $\mathcal{U}/\mathcal{A}$ implies that every edge of $\mathcal{U}$ is in the same orbit as one joining the vertex $o$ with the vertex $\pm e_i$ for some $i \in \{1, 2, 3, 4\}$. Since $\mathbb{C}_2^4 \leq K$ we may assume that the sign is positive. However, $\mathbb{D}_4$ acts transitively on points $e_i$ for $i \in \{1, 2, 3, 4\}$. This implies that $\mathcal{U}/\mathcal{A}$ is edge-transitive. A similar argument can be used to prove that $\mathcal{U}/\mathcal{A}$ is transitive on rank-3 faces. Since there is no 3-orbit $i$-face transitive toroid for every $i \in \{0, \ldots, 4\}$ (see [6, Thm. 1]), then the symmetry type graph of $\mathcal{U}/\mathcal{A}$ must be that of Fig. 1a.

To determine the symmetry type graph of an $n$-orbit $(n + 1)$-toroid we shall change slightly the approach. Instead of looking at the action of the vertex-stabalisier of Aut $(\mathcal{U}/\mathcal{A})$ we will use the stabiliser of a cell. One way to imagine this is to think of $o$ not as the vertex of $\mathcal{U}/\mathcal{A}$ but as the center of a cell. Alternatively we may just take the dual of $\mathcal{U}/\mathcal{A}$, which is isomorphic to $\mathcal{U}/\mathcal{A}$ since $\mathcal{U}$ is self-dual. With this in mind we may think that the group $K = \mathbb{C}_2^n \rtimes \mathbb{S}_{n-1}$ acts on a cell.

Note that even though our results regarding cubic $n$-orbit $(n + 1)$-toroids are for $n \geq 4$, the ideas apply as well when $n = 3$. In particular, these ideas give a way to classify 4-toroids on class 3 listed in Tables 2 and 3 of [10]. Observe that the symmetry type graph of such toroids is that of Fig. 2 for $n = 3$ (see [10, Fig. 4]).

The observation in the previous paragraph allows us to give the following inductive argument to determine the symmetry type graph of any $(n + 1)$-toroid with $n$ orbits. Let $n \geq 3$ and let $\mathcal{U}/\mathcal{A}$ be an $(n + 1)$-toroid with $n$ orbits. We will show that the symmetry type graph of $\mathcal{U}/\mathcal{A}$ is that of Fig. 2. The case $n = 3$ is explained above; assume then that $n \geq 4$. As said before, we shall think of $o$ as the center of a cell $C_o$ whose edges are segments of length 1. Observe that the group $K = \mathbb{C}_2^n \rtimes \mathbb{S}_{n-1}$ acts transitively on the flags containing either $F$ or $-F$ with $F$ the facet of $C_o$ in the hyperplane $x_n = 1/2$. Moreover, $K$ acts transitively on the set of facets of $C_o$ contained in the hyperplanes $x_i = \pm 1/2$ for $i \neq n$. This implies that every flag not containing $F$ or $-F$ belongs to the same orbit as one flag containing the facet $FR_{n-1}$ of $C_o$. This facet is contained in the hyperplane $x_{n-1} = 1/2$. However, the stabiliser of such face under the action of $K$ is precisely $\mathbb{C}_2^{n-1} \rtimes \mathbb{S}_{n-2}$, where $\mathbb{C}_2^{n-1}$ denotes the group generated by the reflections through all the coordinate hyperplanes $x_i = 0$ for $i \neq n - 1$, and $\mathbb{S}_{n-2}$ denotes the point-wise stabiliser in $\mathbb{S}_n$ of the $(n - 1)$-th and $n$-th coordinates. By inductive hypothesis, the arrangement of flag-orbits of $FR_{n-1}$ induces the graph in Fig. 2 with $n - 1$ vertices. By the observations at the beginning of this paragraph, there exists a flag $\Phi$ containing $F$ such that its $(n - 1)$-adjacent flag belongs to $FR_{n-1}$. The flag $\Phi$ is a representative of the orbit consisting of the flags that contain $F$ or $-F$. The choice of $\Phi$ implies that the vertex $v$ representing its orbit has an edge labelled with $n - 1$ to one of the vertices represented by a flag incident to $FR_{n-1}$. Moreover, since for every $j \in \{0, \ldots, n - 2\}$, the flags $\Phi$ and $\Phi^j$ belong to the same orbit, then the edge of $v$ labelled with $j$ is actually a semi-edge at $v$. We have proved that the symmetry type graph of $\mathcal{U}/\mathcal{A}$ is that of Fig. 2. These results are summarised in the following theorem.
Theorem 5.2. The symmetry type graph of a cubic \((4 + 1)\)-toroid with three flag-orbits is the graph shown in Fig. 1a. If \(n \geq 3\), the symmetry type graph of a cubic \((n + 1)\)-toroid with \(n\) flag-orbits is the graph with \(n\) vertices shown in Fig. 2.

5.2 Non-Cubic Toroids

Now we will determine the symmetry type graphs of the few-orbit toroids of type \(\{3, 3, 4, 3\}\) and \(\{3, 4, 3, 3\}\). As mentioned before, we can confine ourselves to toroids of type \(\{3, 3, 4, 3\}\); the symmetry type graphs of the toroids of type \(\{3, 4, 3, 3\}\) may be obtained by duality, namely, by interchanging the labels 0 and 4 as well as 1 and 3 of the symmetry type graphs of the toroids of type \(\{3, 3, 4, 3\}\).

From Theorem 4.2 we know that there are no \((4 + 1)\)-toroids of type \(\{3, 3, 4, 3\}\) with 2 or 4 orbits. We only need to describe the symmetry type graphs of 3-orbit toroids. Just as in the case of cubic toroids, we will determine the transitivity of the automorphism group on the set of \(i\)-faces of each toroid and use the results of [6] to determine its symmetry type.

Recall that the vertex set of the tessellation \(U = \{3, 3, 4, 3\}\) is a lattice. Since every translation of \(G(U)\) induces an automorphism of any toroid \(U/\Lambda\), the group \(\text{Aut}(U/\Lambda)\) acts transitively on vertices.

As before, since \(\text{Aut}(U/\Lambda) = \text{Norm}_{G(U)}(\Lambda)/\Lambda = (T(U) \rtimes K)/\Lambda\) for some group \(K \leq G_0(U)\), to determine whether or not \(\text{Aut}(U/\Lambda)\) acts transitively on the \(i\)-faces of \(U/\Lambda\) for \(i \in \{1, 2, 3, 4\}\), it is enough to determine if \(K\) acts transitively on the \(i\)-faces of \(U\) that contain the vertex \(o\). The \(i\)-faces of \(U\) that contain \(o\) are in correspondence with the \((i - 1)\)-faces of the convex polytope \(\{3, 4, 3\}\) with vertices \((\pm 2, 0, 0, 0), \ldots, (0, 0, 0, \pm 2),\) and \((\pm 1, \pm 1, \pm 1, \pm 1)\). We will use this correspondence in the following paragraphs.

Following the notation of Table 3, the groups \(K\) for the two families of 3-orbit toroids are \(((C_4^d) + \times S_4) \times \langle R_2 \rangle = C_2^4 \rtimes S_4\) and \(((C_4^d) + \times D_4) \times \langle R_1, R_2 \rangle\).

Let \(U/\Lambda\) be a toroid of type \(\{3, 3, 4, 3\}\) such that the group \(K\) described above is \(C_2^4 \rtimes S_4\). The edges of \(U\) containing \(o\) are line segments \(op\) where \(p\) is one of the vertices of the polytope \(\{3, 4, 3\}\) mentioned above. Note that \(K\) maps points of the form \((\pm 1, \pm 1, \pm 1, \pm 1)\) to points of the form \((\pm 1, \pm 1, \pm 1)\), implying that \(U/\Lambda\) is not edge-transitive. The same argument proves that the edge of \(\{3, 4, 3\}\) determined by \((2, 0, 0, 0)\) and \((1, 1, 1, 1)\), and the edge of \(\{3, 4, 3\}\) determined by \((1, 1, 1, 1)\) and \((-1, 1, 1, 1)\) belong to different orbits under the action of \(K\), hence \(U/\Lambda\) is not 2-face-transitive. Therefore, the symmetry type graph of \(U/\Lambda\) is that of Fig. 1b (see [6, Prop. 3]).

Note that the group \(((C_4^d) + \times D_4) \times \langle R_1, R_2 \rangle\) acts transitively on the vertices of \(\{3, 4, 3\}\). This implies that if \(U/\Lambda\) is a toroid such that the group \(K\) is \(((C_4^d) + \times D_4) \times \langle R_1, R_2 \rangle\), then \(U/\Lambda\) is edge-transitive.

The stabiliser of \((2, 0, 0, 0)\) under the action of \(K\) has size \(|K|/24 = 16\) and contains the subgroup \(H = ((E_2, E_3, E_4) + \times (S_{2, 4})) \times \langle R_2 \rangle = \langle E_2, E_3, E_4 \rangle \times (S_{2, 4})\), where \(E_i\) denotes the reflection through the hyperplane \(x_i = 0\) and \(S_{2, 4}\) denotes the isometry that interchanges the second and the fourth coordinates. Note that \(|H| = 16\) and thus the stabiliser of \((2, 0, 0, 0)\) is precisely \(H\).
Next we will prove that $\mathcal{U}/\Lambda$ is 2-face-transitive. As before, we use the correspondence between the 2-faces of $\mathcal{U}/\Lambda$ containing the vertex $o$ and the edges of the polytope $\{3, 4, 3\}$ mentioned above. Since the group $((C^4_2)^+ \rtimes D_4) \rtimes (R_1, R_2)$ acts transitively on the vertices of $\{3, 4, 3\}$, then it is enough to show that the group $H$ mentioned above acts transitively on the edges of $\{3, 4, 3\}$ incident to the vertex $(2, 0, 0, 0)$. The other vertex of each of these edges is a point of the form $(1, \pm 1, \pm 1, \pm 1)$. Clearly the group $H$, containing the reflection $E_2$, $E_3$, and $E_4$, acts transitively on those points. It follows that the group $K$ acts transitively on the edges of $\{3, 4, 3\}$. Therefore, $\mathcal{U}/\Lambda$ is 2-face-transitive.

We will prove now that $\mathcal{U}/\Lambda$ is not 3-face-transitive. Again, it is enough to prove that $K$ does not act transitively on the set of 2-faces of the polytope $\{3, 4, 3\}$ defined above. Let $T$ be the triangle defined by the vertices $(2, 0, 0, 0)$, $(1, 1, 1, 1)$, and $(1, 1, 1, -1)$. Note that

$$(2, 0, 0, 0) R_1 R_2 = (1, 1, 1, -1),$$
$$(1, 1, 1, -1) R_1 R_2 = (1, 1, 1, 1),$$
$$(1, 1, 1, 1) R_1 R_2 = (2, 0, 0, 0).$$

Let $K_T$ denote the stabiliser of $T$ under the action of $K$. The previous computations imply that $\langle R_1 R_2 \rangle \leq K_T$. In particular, $3 \mid |K_T|$. If $T \cdot K$ denotes the orbit of $T$ under the action of $K$, then we have

$$2^7 3 = |K| = |T \cdot K| : |K_T|.$$ 

It follows that $|T \cdot K|$ is a power of 2. In particular, the orbit of $T$ under the action of $K$ cannot have size 96 and therefore, $K$ does not act transitively on the set of 2-faces of the polytope $\{3, 4, 3\}$. As a consequence, $\mathcal{U}/\Lambda$ cannot be 3-face-transitive.

In a very similar way we can prove that the group $K$ does not act transitively on the 3-faces of $\{3, 4, 3\}$. Just observe that the group $\langle R_1 R_2 \rangle$, of size 3, stabilises the octahedral 3-face $O$ determined by the vertices $(2, 0, 0, 0)$, $(0, 0, 2, 0)$, $(1, \pm 1, 1, \pm 1)$. Again, the orbit-stabiliser theorem implies that 3 does not divide the length of the orbit of $O$ under the action of $K$; in particular, this orbit cannot contain the 24 octahedral 3-faces of $\{3, 4, 3\}$.

The analysis of the previous paragraphs is enough to determine that the symmetry type graph of a toroid whose group $K$ is $((C^4_2)^+ \rtimes D_4) \rtimes (R_1, R_2)$ is the one in Fig. 1c. The discussion of this section is summarised in the following result.

**Theorem 5.3** Let $\mathcal{U}/\Lambda$ be an equivelar $(4 + 1)$-toroid of type $\{3, 3, 4, 3\}$ with 3 flag-orbits. Assume that $\text{Aut}(\mathcal{U}/\Lambda) = (\text{T}(\mathcal{U}) \rtimes K)/\Lambda$ with $K$ an index-3 subgroup of $\{3, 4, 3\}$. Then

- If $K = C^4_2 \rtimes S_4$, then the symmetry type graph of $\mathcal{U}/\Lambda$ is that of Fig. 1b.
- If $K = ((C^4_2)^+ \rtimes D_4) \rtimes (R_1, R_2)$, then the symmetry type graph of $\mathcal{U}/\Lambda$ is that of Fig. 1c.
Fig. 1 Symmetry type graphs of 3-orbit 4-toroids

Fig. 2 Symmetry type graph of cubic n-orbit \((n+1)\)-toroids

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