New proof to Somos’s Dedekind eta-function identities of level 10

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Abstract
Michael Somos used PARI/GP script to generate several Dedekind eta-function identities by using computer. In the present work, we prove two new Dedekind eta-function identities of level 10 discovered by Somos in two different methods. Also during this process, we give an alternate method to Somos’s Dedekind eta-function identities of level 10 proved by B. R. Srivatsa Kumar and D. Anu Radha. As an application of this, we establish colored partition identities.

Keywords Dedekind eta-functions · Modular equations · Colored partitions

1 Introduction
The Dedekind eta-function \( \eta(\tau) \) is defined by the formula

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}
\]

where \( \tau \) belongs to the upper complex half-plane. Here and all through the paper, we assume \( |q| < 1 \) and employ the standard notation

\[
(a; q)_{\infty} := \prod_{n=0}^{\infty} (1 - aq^n).
\]

For \( |xy| < 1 \), Ramanujan’s theta function \( \{x, y\} \) is defined as

\[
\{x, y\} := \sum_{n=-\infty}^{\infty} x^{n(3n-1)/2} y^{n(n-1)/2}.
\]

Also in Ramanujan’s notation, Jacobi’s triple-product identity (Berndt 1991, p. 35) is given by

\[
\{x, y\} = (-x; y)_{\infty} (-y; x)_{\infty} (xy; xy)_{\infty}.
\]

The important special cases of \( \{x, y\} \) (Berndt 1991, p. 36) are

\[
\varphi(q) := \{q, q\} = \sum_{n=-\infty}^{\infty} q^n = (-q; q^2)_{\infty} (q^2; q^2)_{\infty}.
\]

\[
\psi(q) := \{q, q^3\} = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \left( q^2; q^2 \right)_{\infty}.
\]

\[
f(-q) := \{-q, -q^2\} = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n^2+1)/2} = (q; q)_{\infty}.
\]

Following Ramanujan’s notations, for \( q = e^{2\pi i \tau} \), we set

\[
f(-q) = q^{-1/24} \eta(\tau).
\]

Also after Ramanujan, define

\[
\chi(q) := (-q; q^2)_{\infty}.
\]

For convenience, we write \( f_n = f(-q^n) \). A theta function identity which relates \( f_1, f_2, f_n \) and \( f_{2n} \) is called the theta function identity of level 2n. Ramanujan documented many modular equations which involve quotients of the function \( f_1 \) at different arguments. For example, if (Berndt 1996, p. 206)
After the publication of Berndt (1991), many authors including (Adiga et al. 2002, 2004, 2016; Baruah 2002, 2003; Naika 2006; Saikia 2011; Vasuki and Sreeramamurthy 2005; Vasuki 2006; Vasuki and Veeresha 2017; Yi 2004) and many more. Before concluding this section, we define a modular equation as defined by Ramanujan. The Gauss ordinary hypergeometric series is defined by

$$\frac{\varphi^2(-q)}{q \chi(-q) \chi(-q^5) \psi^2(q^5)} + 20q \frac{\chi(-q) \chi(-q^5) \psi^2(q^5)}{\varphi^2(-q)} - \frac{\chi^6(-q^5)}{q \chi^6(-q)} + 9 = 0.$$  

and many more.
2 Proof of Somos’s identities of level 10

**Proof of (3)** The following modular equations of degree 5 are recorded by Ramanujan on page 236 of his second notebook (Ramanujan 1957) and (Berndt 1991, pp. 280–288, Entry 13(ix) and (xiv)):

\[
1 + 4^{1/3} \left( \frac{\beta^5 (1 - \beta)^5}{\alpha (1 - \alpha)} \right)^{1/12} = \frac{m}{2} \left( 1 + (\alpha \beta)^{1/2} + [(1 - \alpha)(1 - \beta)]^{1/2} \right),
\]

(10)

\[
1 + 4^{1/3} \left( \frac{\alpha^5 (1 - \alpha)^5}{\beta (1 - \beta)} \right)^{1/12} = \frac{5}{2m} \left( 1 + (\alpha \beta)^{1/2} + [(1 - \alpha)(1 - \beta)]^{1/2} \right),
\]

(11)

and if

\[
P := [16\alpha \beta (1 - \alpha)(1 - \beta)]^{1/12}
\]

\[
Q := \left[ \frac{\beta (1 - \beta)}{\alpha (1 - \alpha)} \right]^{1/8}
\]

then

\[
Q + \frac{1}{Q} + 2 \left( P - \frac{1}{P} \right) = 0,
\]

(12)

where \( \beta \) has degree 5 over \( \alpha \) and \( m \) is the multiplier. From (10) and (11), we have

\[
m^2 = \frac{1 + 4^{1/3} \left( \frac{\beta^5 (1 - \beta)^5}{\alpha (1 - \alpha)} \right)^{1/12}}{1 + 4^{1/3} \left( \frac{\alpha^5 (1 - \alpha)^5}{\beta (1 - \beta)} \right)^{1/12}}.
\]

(13)

For \( q = e^{-\gamma} \) from (8), we can write

\[
\chi(q) = 2^{1/6} \left( \frac{q}{\alpha (1 - \alpha)} \right)^{1/24}
\]

and

\[
\chi(q^5) = 2^{1/6} \left( \frac{q^5}{\beta (1 - \beta)} \right)^{1/24}.
\]

From the above, we deduce

\[
2^{4/3} q^{2} \frac{\chi^2(q)}{\chi^5(q^5)} \left( \frac{\beta^5 (1 - \beta)^5}{\alpha (1 - \alpha)} \right)^{1/12}
\]

and

\[
2^{4/3} q^{2} \frac{\chi^2(q^5)}{\chi^5(q)} \left( \frac{\alpha^5 (1 - \alpha)^5}{\beta (1 - \beta)} \right)^{1/12}.
\]

On using these in (13), also by making use of (9) and (7), we obtain

\[
\frac{\psi^5(q)}{5 \psi^5(q^5)} = \frac{1 + 4 \frac{q^2}{\chi^5(q^5)}}{1 + 4 \frac{q^2}{\chi^5(q)}}.
\]

(14)

Similarly, on transcribing (12) into theta function, we obtain

\[
\frac{x^3 + y^3 + 4}{x^2 y^2 - x^2 y^2} = 0,
\]

(15)

where

\[
x := x(q) = q^{-1/24} \chi(q) \quad \text{and} \quad y := y(q) = q^{-5/24} \chi(q^5).
\]

Now on multiplying (15) throughout by \((4x^{12} + 5x^{10}y^{10} + 32x^6y^6 + 80x^2y^2 + 4y^{12})(x^6 - 4xy + x^5y^5 + y^6)\), we obtain

\[
\frac{x^{20}}{y^4} - \frac{x^{22}}{y^{14}} + \frac{18x^{16}}{y^8} - \frac{x^{10}}{y^2} - \frac{4x^{24}}{5y^{24}} - \frac{72x^{18}}{5y^{18}} - \frac{168x^{12}}{5y^{12}} - \frac{72x^6}{5y^6} - \frac{16x^{14}}{5y^{14}} - \frac{288x^8}{5y^{16}} - \frac{16x^2}{5y^{10}} - \frac{256x^4}{y^{20}} - \frac{4}{5} = 0,
\]

which can be rewritten as

\[
\left[ 1 + \frac{4x^2}{y^{10}} + \frac{4x^2}{y^{10}} \left( 1 + \frac{4y^2}{x^{10}} \right) \right]^{2}
\]

\[
= \frac{1}{5} \left( \frac{y}{x^5} + \frac{9x}{y^5} \right)^2 \left( 1 + \frac{4y^2}{x^{10}} \right) \left( 1 + \frac{4x^2}{y^{10}} \right).
\]

Employing (14) in the above, we see that

\[
1 + 20 \frac{x^2}{y^{10}} \frac{\psi^4(q^5)}{\psi^4(q)} = \left( \frac{y}{x^5} + \frac{9x}{y^5} \right) \frac{\psi^2(q^5)}{\psi^2(q)}.
\]

(16)

Also without difficulty, we observe that

\[
\frac{\psi(q)}{\psi(q^5)} = \frac{f_2^2}{f_1^2 f_4^2}, \quad \frac{\psi(-q)}{\psi(-q^5)} = \frac{f_1^2}{f_2^2}, \quad \psi(q) = \frac{f_2^2}{f_1^2}, \quad \chi(q) = \frac{f_2^2}{f_1 f_4^2} \quad \text{and} \quad \chi(-q) = \frac{f_1}{f_2}.
\]

(17)

From (17), we have

\[
\frac{\psi(q)}{\psi(q^5)} = \frac{x^2 f_2}{q^{1/3} y^2 f_10}.
\]
Using the above in (16), we have

\[ 1 + \frac{20q^{4/3}}{x^6 y^2} \left( \frac{f_{10}}{f_2} \right)^4 = q^{2/3} \left( \frac{y^5}{x^3 y^4} + 9 \frac{f_{10}}{f_2} \right)^2. \]

Letting \( q \to -q \) in the above, rewriting \( x(-q) \) and \( y(-q) \) in terms of \( f_n \) by employing (17) and then simplifying, we deduce the result.

**Second proof of (3)** On using (17) in (3) and then dividing throughout by \( f_1 f_2 f_3 \), we obtain

\[ 1 + 20q^2 f_2^3 f_{10}^4 f_5^2 f_6^2 + 9q f_2 f_3 f_{10}^3 f_5 - \frac{f_2^3 f_5^3}{f_1^3 f_{10}^3} = 0. \]  \hfill (18)

On using \( P, Q, A \) and \( B \) as defined as in (1) and (2), (18) reduces to

\[ \frac{20}{(AB)^4 (PQ)^2} + \frac{1}{(AB)^2} \left( \frac{9}{PQ} - \frac{Q^5}{P^5} \right) + 1 = 0 \]

equivalently

\[ (AB)^2 = \frac{40P^5}{Q^7 - 9P^6 Q \pm \sqrt{(9P^6 Q - Q^7)^2 - 80P^{12}Q^2}}. \]  \hfill (19)

Using (19) in (2) and then factorizing, we obtain

\[ L(P, Q)M(P, Q) = 0, \]

where

\[ L(P, Q) = P^6 - 5P^2 Q^2 - P^4 Q^4 + Q^6 \]

and

\[ M(P, Q) = 400P^{10} Q^2 (5P^{12} - 4P^{10} Q^4 - 20P^8 Q^2 - 14P^6 Q^6 + Q^{12}). \]

But \( L(P, Q) \) is nothing but (1), and it verifies (3). \hfill \Box

We omit the proof of (4), as the proof is similar to the previous one.

**Proof of (5)** Srivatsa Kumar and Anu Radha (2018) On dividing (5) by \( f_2 f_5 \), we obtain

\[ 1 + q f_1 f_5 f_6 f_{10} f_{12} f_2 f_5^3 + 5q^2 f_2^3 f_5 f_{10} f_{12} f_2 f_5 - \frac{f_2^3 f_{10}}{f_1 f_5} = 0. \]  \hfill (20)

On using \( P, Q, A \) and \( B \) as defined as in (1) and (2), (20) becomes

\[ \frac{1}{(AB)^2} \left( \frac{5P^5}{Q^7} - PQ \right) + \frac{P^6}{Q^6} + 1 = 0 \]

equivalently

\[ (AB)^2 = \frac{PQ^8 - 5P^5}{Q^7 + P^6 Q}. \]  \hfill (21)

Using (21) in (2) and then factorizing, we deduce

\[ L(P, Q)M(P, Q) = 0, \]

where

\[ L(P, Q) = P^6 - 5P^2 Q^2 - P^4 Q^4 + Q^6 \]

and

\[ M(P, Q) = 5P^{10} + 4P^8 + 10P^2 + Q^{14}. \]

But \( L(P, Q) \) is nothing but (1), and it verifies (5). \hfill \Box

**Proof of (6)** Srivatsa Kumar and Anu Radha (2018) On dividing (6) by \( f_2^8 f_5 \), we obtain

\[ \frac{f_2^4}{f_5^4} f_1^4 f_2^2 + 16q f_2^2 f_5 f_{10} f_{12} f_5 + 4 f_2^3 f_5 f_{10} f_{12} f_2 f_5 - 5 = 0. \]  \hfill (22)

On using \( P, Q, A \) and \( B \) as defined as in (1) and (2), (22) becomes

\[ \frac{16P^4}{Q^2 (AB)^4} + \frac{4P^5}{Q (AB)^2} + \frac{P^4}{Q^2} - 5 = 0 \]

equivalently

\[ (AB)^2 = \frac{8P^4}{-P^5 Q \pm \sqrt{P^{10} - 4P^8 + 20P^4 Q^2}}. \]  \hfill (23)

Using (23) in (2) and then factorizing, we obtain

\[ L(P, Q)M(P, Q) = 0, \]

where

\[ L(P, Q) = P^6 - 5P^2 Q^2 - P^4 Q^4 + Q^6 \]

and

\[ M(P, Q) = 4P^4 (P^{10} - 4P^8 Q^4 + 9P^4 Q^6 - 5Q^8). \]

But \( L(P, Q) \) is nothing but (1), and it verifies (6). \hfill \Box
Remark: Using the same method, we can prove the remaining Somos’s Dedekind eta-function identities of level 10 which are listed in the previous section.

3 Applications to colored partition

The identities proved in Section 2 have applications to the theory of partitions. In this section, we demonstrate colored partitions for (3). Similarly for the remaining identities mentioned in section 1, we can establish the same concept. For simplicity, we adopt the standard notation

\[(x_1, x_2, \ldots, x_m; q)_\infty := \prod_{j=1}^{m} (x_j; q)_\infty.\]

“A positive integer \(n\) has \(l\) colors if there are \(l\) copies of \(n\) available colors and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are called colored partitions.” As an example, if 1, 2 and 3 are assigned with two colors, then possible partitions of 3 are \(3_1, 3_2, 2_1 + 1_1, 2_1 + 1_2, 2_2 + 1_1, 1_1 + 1_2 + 1_1, 1_1 + 1_2 + 1_2, \) and 1_1 + 1_2 + 1_2, where we used the indices 1 (indigo) and 2 (violet) to differentiate two colors of 1, 2 and 3. Also, the generating function for the number of partitions of \(n\) is defined as

\[
\frac{1}{(q^a; q^b)_\infty^k},
\]

with \(k\) colors, and all the parts are congruent to \(a \pmod{b}\).

**Theorem 1** If \(\alpha(n)\) represent the number of partitions of \(n\) being divided into parts that are congruent to \(\pm 1, \pm 3 \pmod{10}\) with six colors, \(\pm 2, \pm 4 \pmod{10}\) with four colors and \(\pm 5\) with eight colors, respectively. If \(\beta(n)\) is chosen to represent the number of partitions of \(n\) into many parts that are congruent to \(\pm 1, \pm 3 \pmod{10}\) with three colors, \(\pm 2, \pm 4 \pmod{10}\) with two colors and \(\pm 5\) with four colors, respectively. If \(\gamma(n)\) indicates the number of partitions of \(n\) being split into parts congruent to \(\pm 1, \pm 3 \pmod{10}\) with nine colors, \(\pm 2, \pm 4 \pmod{10}\) with two colors and \(\pm 5\) with four colors, respectively. Then, the following relation holds true:

\[
20\alpha(n - 2) + 9\beta(n - 1) - \gamma(n) = 0, \quad n \geq 2.
\]

**Proof** Using (17) in (3), then dividing throughout by \(f_1, f_2, f_3, f_5, f_9\), and simplifying to the common base \(q^{10}\), we have

\[
1 + \frac{20q^2}{(q_6^1, q_4^2, q_6^3, q_4^4, q_5^5, q_4^6, q_6^7, q_4^8, q_6^9, q^{10}_1)_\infty} + \frac{9q}{(q_3^1, q_2^2, q_3^4, q_2^5, q_6^6, q_2^7, q_6^8, q_2^9, q^{10}_1)_\infty} = 0. \quad (25)
\]

For simplicity, we write

\[
(q^{a_1}; q_{b_1})_\infty := (q^{a_1}_k, q^{b_1-a_1}_k; q^{b_1})_\infty \quad a < b, \quad a, b \in \mathbb{Z}^+.
\]

Using the above identity in (25), we obtain

\[
1 + \frac{20q^2}{(q_6^1, q_4^2, q_6^3, q_4^4, q_5^5, q_4^6, q_6^7, q_4^8, q_6^9, q^{10}_1)_\infty} + \frac{9q}{(q_3^1, q_2^2, q_3^4, q_2^5, q_6^6, q_2^7, q_6^8, q_2^9, q^{10}_1)_\infty} = 0. \quad (26)
\]

On using (24) in (26), we obtain the three generating functions, namely \(\alpha(n), \beta(n)\) and \(\gamma(n)\), respectively. Now (26) reduces to

\[
1 + 20q^2 \sum_{n=0}^{\infty} \alpha(n)q^n + 9q \sum_{n=0}^{\infty} \beta(n)q^n - \sum_{n=0}^{\infty} \gamma(n)q^n = 0,
\]

where we set \(\alpha(0) = \beta(0) = \gamma(0) = 1\). Extracting the coefficients of \(q^n\), we obtain the required result.

For \(n = 2\), the following table verifies the above theorem.\(\square\)

\[\begin{array}{c|c|c}
\hline
n \alpha(n) \beta(1) = 3; & \gamma(2) = 47; & 1, 1, 1, 1, 1, \text{ and 7 others of the same type.} \\
2 & 1, 1, 1, 1, 1, \text{ and 34 others of the same type.} \\
\hline
\end{array}\]

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Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This study does not involve any human participants or animals performed by any of the authors.
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