Computation of Topological Indices of Double and Strong Double Graphs of Circumcoronene Series of Benzenoid ($H_m$)

Muhammad Shoaib Sardar, 1 Imran Siddique, 2 Dalal Alrowaili, 3 Muhammad Asad Ali, 1 and Shehnaz Akhtar 4

1School of Mathematics, Minhaj University, Lahore, Pakistan
2Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan
3Mathematics Department, College of Science, Jouf University, P. O. Box: 2014, Sakaka, Saudi Arabia
4School of Natural Science, National University of Science and Technology, Islamabad, Pakistan

Correspondence should be addressed to Imran Siddique; imransmsrazi@gmail.com

Received 20 November 2021; Revised 21 December 2021; Accepted 3 January 2022; Published 21 January 2022

Academic Editor: M. T. Rahim

Copyright © 2022 Muhammad Shoaib Sardar et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Topological indices are very useful to assume certain physiochemical properties of the chemical compound. A molecular descriptor which changes the molecular structures into certain real numbers is said to be a topological index. In chemical graph theory, to create quantitative structure activity relationships in which properties of molecule may be linked with their chemical structures relies greatly on topological indices. The benzene molecule is a common chemical shape in chemistry, physics, and nanoscience. This molecule could be very beneficial to synthesize fragrant compounds. The circumcoronene collection of benzenoid $H_m$ is one family that generates from benzene molecules. The purpose of this study is to calculate the topological indices of the double and strong double graphs of the circumcoronene series of benzenoids ($H_m$). In addition, we also present a numerical and graphical comparison of topological indices of the double and strong double graphs of the circumcoronene series of benzenoid ($H_m$).

1. Introduction and Preliminaries

For undetermined notations and terminologies, we refer the readers to read the book [1].

Let $G(V, E)$ be a simple, finite connected graph, where the set of vertices is $V(G)$ and the set of edges is $E(G)$. For every vertex $x \in V(G)$, the edge connecting $x$ and $z$ is denoted by $xz$. In graph $G$, the total number of edges that connects to each vertex is known as the degree of vertex. The number of connected vertices to a fixed vertex is known as neighborhood. The degree of a vertex is denoted by $d_x$, where $x \in V(G)$. Hand-shaking lemma is very productive for calculating the size of a graph $G$.

**Lemma 1.** If a graph $G$ is having size $k$, then

$$\sum_{x \in V(G)} \deg(x) = 2k. \quad (1)$$

In chemical graph theory, topological indices show a significant role in assisting chemists for modeling the molecular structure of chemical compounds and studying their chemical and physical characteristics. In chemistry, discovery of the drugs commonly relies on the topological descriptors. Drugs are characterized as molecular graphs, where graphs considered are simple with no multiple edges and no cycle formation. These topological descriptors provide information of a chemical compound based on the arrangement of its atoms and their bonds. A wide range of topological indices have been studied, and some of the more
frequent forms of topological indices include degree-based, distance-based topological indices, and counting-related polynomials. In the topological indices, very famous and the oldest index is the Wiener index $W(\mathcal{G})$.

The Wiener index [2] is defined as follows:

$$W(\mathcal{G}) = \frac{1}{2} \sum_{x, z} d(x, z),$$

where $d(x, z)$ is the distance among vertices $x$ and $z$ of a graph $\mathcal{G}$.

A graph $\mathcal{G}$'s geometric arithmetic index (GA) [3] is defined as follows:

$$\text{GA}(\mathcal{G}) = \sum_{x \in E(\mathcal{G})} \frac{2\sqrt{d_x d_z}}{d_x + d_z}.$$

A graph $\mathcal{G}$'s atomic bond connectivity index (ABC) [4] is defined as follows:

$$\text{ABC}(\mathcal{G}) = \sum_{x \in E(\mathcal{G})} \sqrt{\frac{d_x + d_z - 2}{d_x d_z}}.$$

A graph $\mathcal{G}$'s forgotten index ($F$) [5] is defined as follows:

$$F(\mathcal{G}) = \sum_{x \in E(\mathcal{G})} (d_x^2 + d_z^2).$$

A graph $\mathcal{G}$'s inverse sum indeg index (ISI) [6] is defined as follows:

$$\text{ISI}(\mathcal{G}) = \sum_{x \in E(\mathcal{G})} \frac{1}{(1/d_x) + (1/d_z)}.$$

A graph $\mathcal{G}$'s general inverse sum indeg index ($\text{ISI}_{(\alpha, \beta)}$) [7] is defined as follows:

$$\text{ISI}_{(\alpha, \beta)}(\mathcal{G}) = \sum_{x \in E(\mathcal{G})} [d_x d_z]^{\alpha} [d_x + d_z]^\beta,$$

where $\alpha$ and $\beta$ are the real numbers.

A graph $\mathcal{G}$'s first multiplicative-Zagreb (PM$_1$) and second multiplicative-Zagreb indices (PM$_2$) are defined [8] as follows:

$$\text{PM}_1(\mathcal{G}) = \prod_{x \in E(\mathcal{G})} (d_x)^2,$$

$$\text{PM}_2(\mathcal{G}) = \prod_{x \in E(\mathcal{G})} (d_x \cdot d_z).$$

It is also possible to write the first multiplicative-Zagreb index ($\text{PM}_1$) [9] for $\mathcal{G}$ as follows:

$$\text{PM}_1(\mathcal{G}) = \prod_{x \in E(\mathcal{G})} (d_x + d_z).$$

Imran et al. [10–12] studied the edge Mostar index of nanostructures and chemical structures by using graph operations and also computed the eccentric connectivity polynomial of connected graphs and Mostar indices for melamin chain nanostructures. For more details about topological indices, we refer the works of Xiong et al. [13], Hong et al. [14], Alaeiyan et al. [15], Ch et al. [16], and Sardar et al. [17].

**Definition 1.** The well-known family of the benzenoid molecular graph is circumcoronene series of benzenoid $(H_m)$, where $(m \geq 1)$ [18]. This family of graph constructed exclusively from benzene $C_6$ on circumference. Certain main members of circumcoronene series of benzenoid are benzene $(H_1)$, coronene $(H_2)$, circumcoronene $(H_3)$, and circumcircumcoronene $(H_4)$ [19]. Generally, circumcoronene series of benzenoid $(H_m)$ is shown in Figure 1.

**Definition 2.** In order to make a double graph $D(H_m)$ of a graph $\mathcal{G}$, take two copies of the graph $\mathcal{G}$ and join the nodes in each copy with their neighbors in the other copy [20]. For example, the graph $(H_1)$ and its double graph $D(H_1)$ are shown in Figure 2. In double graph of circumcoronene series of benzenoid, there are $12m^2$ vertices and $4(9m^2 - 3m)$ edges, respectively. In $D(H_m)$, we have $12m$ vertices of degree 4 and $12(m^2 - m)$ vertices of degree.

**Definition 3.** Consider the two copies of graph $\mathcal{G}$, and by joining the closed neighborhoods of one graph’s vertex to the vertex in an adjacent graph, one can obtain the strong double graph $SD(\mathcal{G})$ of graph $\mathcal{G}$ [21]. For example, strong double graph of graph $H_1$ is shown in Figure 3.

This study is laid out as follows. We will examine some vertex-based topological indices of double and strong double graphs of circumcoronene series of benzenoid $(H_m)$ in Sections 2 and 4, respectively. The comparison is given in Sections 3 and 5. In Section 6, we provide final remarks for the whole study.

### 2. Degree-Based Topological Indices of Double Graph of Circumcoronene Series of Benzenoid Graph $(H_m)$

This section contains a calculation of the degree-based indices of the double graph of circumcoronene series of benzenoid $(H_m)$.

**Theorem 1.** Let $D(H_m)$ be the double graph of circumcoronene series of benzenoid graph $(H_m)$; then, the geometric arithmetic index of $D(H_m)$ is

$$\text{GA}[D(H_m)] = \frac{(96m - 96)\sqrt{6}}{5} + 36m^2 - 60m + 48. \quad (11)$$

**Proof.** In the double graph of circumcoronene series of benzenoid, there are $12m^2$ vertices and $4(9m^2 - 3m)$ edges, respectively. There are $12m$ vertices in $D(H_m)$ of degree 4 and $12(m^2 - m)$ of degree 6.

We separate the edges of $D(H_m)$ into the edges of the type $E[d_x, d_z]$, where $xz$ is an edge. In $D(H_m)$, we get edge of types $E(4,4)$ and $E(4,6)$ and $E(6,6)$. A list of their edges is given in Table 1.

By using Table 1 and equation (1), the result that we obtain is
Figure 1: Circumcoronene series of benzenoid \((H_1, H_2, H_3, \text{ and } H_m)\).

Figure 2: Circumcoronene series of benzenoid \((H_1)\) and its double graph \((D(H_1))\).

Figure 3: Circumcoronene series of benzenoid \((H_1)\) and its strong double graph \((SD(H_1))\).

Table 1: Separation of edges.

| Edge Separation | \(E_{(4,4)}\) | \(E_{(4,6)}\) | \(E_{(6,6)}\) |
|-----------------|--------------|--------------|--------------|
| Number of edges | 24           | 48 \((m - 1)\) | 36\(m^2 - 60m + 24\) |
\[ \text{GA}[G] = \sum_{x \in E(G)} 2\sqrt{d_x d_z}. \]
\[ \text{GA}[D(H_m)] = |E_{(4,6)}| \sum_{x \in E[D(H_m)]} \frac{2\sqrt{d_x d_z}}{d_x + d_z} + |E_{(6,6)}| \sum_{x \in E[D(H_m)]} \frac{2\sqrt{d_x d_z}}{d_x + d_z} \]
\[ \text{GA}[D(H_m)] = 24 \left( \frac{2\sqrt{16}}{8} + 48(m-1) \left( \frac{2\sqrt{24}}{10} + (36m^2 - 60m + 24) \right) \right) \frac{2\sqrt{36}}{12}. \]
\[ \text{GA}[D(H_m)] = 24 + 48(m-1) \left( \frac{24}{5} \right) + 36m^2 - 60m + 24. \]
\[ \text{GA}[D(H_m)] = \frac{(96m - 96)\sqrt{6}}{5} + 36m^2 - 60m + 48. \]

**Theorem 2.** Let \( D(H_m) \) be the double graph of the benzenoid graph \( (H_m) \); then, the ABC index of \( D(H_m) \) is

\[ \text{ABC}[D(H_m)] = (6 \sqrt{3} + (6m^2 - 10m + 4) \sqrt{5}) \sqrt{2} \]
\[ + 16 \sqrt{3}(m - 1). \]

**Proof.** By using Table 1 and equation (4), the result that we obtain is

\[ \text{ABC}[D(H_m)] = |E_{(4,6)}| \sum_{x \in E[D(H_m)]} \sqrt{\frac{d_x + d_z - 2}{d_x d_z}} + |E_{(6,6)}| \sum_{x \in E[D(H_m)]} \sqrt{\frac{d_x + d_z - 2}{d_x d_z}} \]
\[ = 24 \left( \frac{4 + 4 - 2}{(4)(4)} + 48(m-1) \left( \frac{4 + 6 - 2}{(4)(6)} + (36m^2 - 60m + 24) \right) \frac{6 + 6 - 2}{(6)(6)}. \right) \]
\[ = 6 \sqrt{6} + 48(m-1) \frac{1}{3} + (36m^2 - 60m + 24) \frac{5}{18}. \]
\[ \text{ABC}[D(H_m)] = (6 \sqrt{3} + (6m^2 - 10m + 4) \sqrt{5}) \sqrt{2} + 16 \sqrt{3}(m - 1). \]

**Theorem 3.** Let \( D(H_m) \) be the double graph of the benzenoid graph \( (H_m) \); then, the forgotten index of \( D(H_m) \) is

\[ F[D(H_m)] = 2592m^2 - 1824m. \]

**Proof.** By using Table 1 and equation (5), the result that we obtain is

\[ F[D(H_m)] = |E_{(4,6)}| \sum_{x \in E[D(H_m)]} \left( d_x^2 + d_z^2 \right) + |E_{(6,6)}| \sum_{x \in E[D(H_m)]} \left( d_x^2 + d_z^2 \right) \]
\[ = 24 \left( 4^2 + 4^2 \right) + 48(m-1) \left( 4^2 + (6)^2 \right) + (36m^2 - 60m + 24) \left( 6^2 + (6)^2 \right) \]
\[ = 768 + 2496(m-1) + (36m^2 - 60m + 24)(72). \]
\[ F[D(H_m)] = 2592m^2 - 1824m. \]
**Theorem 4.** Let $D[H_m]$ be the double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the inverse sum indeg index of $D(H_m)$ is

$$\text{ISI}[D(H_m)] = 108m^2 - \frac{324}{5}m + \frac{24}{5}.$$  \hfill (17)

**Proof.** By using Table 1 and equation (6), the result that we obtain is

$$\text{ISI}[D(H_m)] = |E_{(4.4)}| \frac{\sum_{x \in E[D(H_m)]} (d_x d_z)}{(d_x + d_z)} + |E_{(4.6)}| \frac{\sum_{x \in E[D(H_m)]} (d_x d_z)}{(d_x + d_z)} + |E_{(6.6)}| \frac{\sum_{x \in E[D(H_m)]} (d_x d_z)}{(d_x + d_z)} = 24 \left[ \frac{(4)(4)}{(4 + 4)} + 48(m - 1) \frac{(4)(6)}{(4 + 6)} + (36m^2 - 60m + 24) \frac{(6)(6)}{(6 + 6)} \right]$$

$$= 48 + 48(m - 1) \frac{12}{5} + (36m^2 - 60m + 24)[3],$$

$$\text{ISI}[D(H_m)] = 108m^2 - \frac{324}{5}m + \frac{24}{5}. \hfill \square$$

**Theorem 5.** Let $D[H_m]$ be the double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the general inverse sum indeg index (ISI$_{\alpha\beta}$) of $D(H_m)$ is

$$\text{ISI}_{\alpha\beta}[D(H_m)] = 4p[16]^\alpha [8]^\beta + 8p[16p]^\alpha [4(1 + p)]^\beta.$$  \hfill (19)

**Proof.** By using Table 1 and equation (7), the result that we obtain is

$$\text{ISI}_{\alpha\beta}[D(H_m)] = |E_{(4.4)}| \frac{\sum_{x \in E[D(H_m)]} [d_x d_z]^\alpha [d_x + d_z]^\beta}{(d_x + d_z)} + |E_{(4.6)}| \frac{\sum_{x \in E[D(H_m)]} [d_x d_z]^\alpha [d_x + d_z]^\beta}{(d_x + d_z)} + |E_{(6.6)}| \frac{\sum_{x \in E[D(H_m)]} [d_x d_z]^\alpha [d_x + d_z]^\beta}{(d_x + d_z)} = 24 \left[ (4)(4) \frac{[4 + 4]}{[4 + 6]} + 48(m - 1) \frac{[(4)(6)]^\alpha [4 + 6]^\beta}{[6 + 6]} + (36m^2 - 60m + 24) \frac{[(6)(6)]^\alpha [6 + 6]^\beta}{[6 + 6]} \right]$$

$$= 24[16]^\alpha [8]^\beta + 48(m - 1)[24]^\alpha [10]^\beta + (36m^2 - 60m + 24)[36]^\alpha [12]^\beta, \hfill \square$$

where $\alpha$ and $\beta$ are the real numbers.

**Theorem 6.** Let $D[H_m]$ be the double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the first multiplicative-Zagreb index of $D(H_m)$ is

$$\text{PM}_1[D(H_m)] = |E_{(4.4)}| \prod_{x \in E[D(H_m)]} (d_x + d_z) \times |E_{(4.6)}| \prod_{x \in E[D(H_m)]} (d_x + d_z) \times |E_{(6.6)}| \prod_{x \in E[D(H_m)]} (d_x + d_z).$$

**Proof.** By using Table 1 and equation (10), the result that we obtain is

$$\text{PM}_1[D(H_m)] = 24(8) \times 48(m - 1)(10) \times (36m^2 - 60m + 24)(12),$$

$$\text{PM}_1[D(H_m)] = 192 \times (480m - 480) \times (432m^2 - 720m + 288),$$

$$\text{PM}_1[D(H_m)] = (3m - 2)\left[ 13271040(m - 1)^2 \right]. \hfill \square$$
**Theorem 7.** Let $D[H_m]$ be the double graph of circumcoronene series of the benzenoid graph ($H_m$); then, the second multiplicative-Zagreb index of $D[H_m]$ is

$$\text{PM}_2[D(H_m)] = (m - \frac{2}{3})[573308928 (m - 1)^2].$$

**Proof.** By using Table 1 and equation (9), the result that we obtain is

$$\text{PM}_2[D(H_m)] = |E_{(4,4)}| \prod_{x \in E[D(H_m)]} (d_x \cdot d_z) \cdot |E_{(6,6)}| \prod_{x \in E[D(H_m)]} (d_x \cdot d_z),$$

$$\text{PM}_2[D(H_m)] = 24 (16) \times 48 (m - 1) (24) \times (36m^2 - 60m + 24) (36),$$

$$\text{PM}_2[D(H_m)] = 442368 (m - 1) \times (1296m^2 - 2160m + 864),$$

$$\text{PM}_2[D(H_m)] = (m - \frac{2}{3})[573308928 (m - 1)^2].$$

**3. Comparison**

In this section, we present a numerical and graphical comparison of topological indices that included the first multiplicative-Zagreb index ($\text{PM}_1$), general inverse sum indeg index (ISI), atom bond connectivity index (ABC), forgotten index ($F$), geometric arithmetic index ($\text{GA}$), second multiplicative-Zagreb index ($\text{PM}_2$), and inverse sum indeg index (ISI) for $m = 1, 2, 3, 4, \ldots, 10$ for the double graph of circumcoronene series of the benzenoid graph ($D(H_m)$), as given in Table 2 and Figure 4.

**4. Degree-Based Topological Indices of Strong Double Graphs of Circumcoronene Series of Benzenoid Graph ($H_m$)**

This section contains a calculation of the degree-based indices of the strong double graph of circumcoronene series of benzenoid ($H_m$). Figure 3 shows the strong double graph of ($H_1$).

$$\text{GA}[G] = \sum_{x \in E(G)} \frac{2\sqrt{d_x d_z}}{d_x + d_z}$$

$$\text{GA}[SD(H_m)] = |E_{(5,5)}| \sum_{x \in E[SD(H_m)]} \frac{2\sqrt{d_x d_z}}{d_x + d_z} + |E_{(5,7)}| \sum_{x \in E[SD(H_m)]} \frac{2\sqrt{d_x d_z}}{d_x + d_z} + |E_{(7,7)}| \sum_{x \in E[SD(H_m)]} \frac{2\sqrt{d_x d_z}}{d_x + d_z},$$

$$\text{GA}[SD(H_m)] = (6m + 24) \frac{2\sqrt{35}}{10} + 48 (m - 1) \frac{2\sqrt{35}}{12} + (42m^2 - 66m + 24) \frac{2\sqrt{49}}{14},$$

$$\text{GA}[SD(H_m)] = (6m + 24) + 48 (m - 1) \frac{\sqrt{35}}{6} + 42m^2 - 66m + 24,$$

$$\text{GA}[SD(H_m)] = (8m - 8) \frac{\sqrt{35}}{6} + 42m^2 - 60m + 48.$$

**Theorem 8.** Let $SD(H_M)$ be the double graph of circumcoronene series of the benzenoid graph ($H_m$); then, the geometric arithmetic index of $SD(H_m)$ is

$$\text{GA}[SD(H_m)] = (8m - 8) \sqrt{35} + 42m^2 - 60m + 48.$$
Theorem 9. Let $SD(H_m)$ be the strong double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the $ABC$ index of $SD(H_m)$ is

$$ABC[SD(H_m)] = \frac{((240m - 240)\sqrt{2} + 84m + 336)\sqrt{2}}{35} + \left(12m - \frac{48}{7}\right)\sqrt{3}(m - 1).$$

Proof. By using Table 3 and equation (4), the result that we obtain is

Table 2: Computation of topological indices of double graph of circumcoronene series of benzenoid $(D(H_m))$.

| $m$ | GA $(D(H_m))$ | ABC $(D(H_m))$ | $F(D(H_m))$ | ISI $(D(H_m))$ | PM$_1(D(H_m))$ | PM$_2(D(H_m))$ |
|-----|----------------|----------------|-------------|----------------|----------------|----------------|
| 1   | 24             | 14,697         | 768         | 48             | 0              | 0              |
| 2   | 119.03         | 67.710         | 6720        | 307.20         | 5.3084 $\times 10^7$ | 7.6441 $\times 10^8$ |
| 3   | 286.06         | 158.67         | 17856       | 782.40         | 3.7159 $\times 10^8$ | 5.3509 $\times 10^9$ |
| 4   | 525.09         | 287.58         | 34176       | 1473.6         | 1.1944 $\times 10^9$ | 1.7199 $\times 10^{10}$ |
| 5   | 836.12         | 454.42         | 55680       | 2380.8         | 2.7604 $\times 10^9$ | 3.9749 $\times 10^{10}$ |
| 6   | 1219.2         | 659.24         | 82360       | 3504           | 5.3084 $\times 10^9$ | 7.6441 $\times 10^{10}$ |
| 7   | 1674.2         | 901.98         | 1.1424 $\times 10^5$ | 4843.2       | 9.0774 $\times 10^9$ | 1.3071 $\times 10^{11}$ |
| 8   | 2201.2         | 1182.7         | 1.5129 $\times 10^5$ | 6398.4       | 1.4306 $\times 10^{10}$ | 2.0601 $\times 10^{11}$ |
| 9   | 2800.2         | 1501.3         | 1.9353 $\times 10^5$ | 8169.6       | 2.1234 $\times 10^{10}$ | 3.0576 $\times 10^{11}$ |
| 10  | 3471.3         | 1857.9         | 2.4096 $\times 10^5$ | 10156.8      | 3.0099 $\times 10^{10}$ | 4.3342 $\times 10^{11}$ |

Figure 4: Graphical representation of topological indices of double graph of circumcoronene series of benzenoid $(H_m)$. 

Table 3: Separation of edges.

| $E(d_x, d_z)$ | $E_{(3,5)}$ | $E_{(5,7)}$ | $E_{(7,7)}$ |
|---------------|-------------|-------------|-------------|
| Number of edges | $6m + 24$ | $48(m - 1)$ | $42m^2 - 66m + 24$ |
\[ \text{ABC}[SD(H_m)] = |E_{(5,5)}| \sum_{x \in E[SD(H_m)]} \sqrt{\frac{d_x + d_z - 2}{d_x d_z}} + |E_{(5,7)}| \sum_{x \in E[SD(H_m)]} \sqrt{\frac{d_x + d_z - 2}{d_x d_z}} + |E_{(7,7)}| \sum_{x \in E[SD(H_m)]} \sqrt{\frac{d_x + d_z - 2}{d_x d_z}} \]

\[ = (6m + 24)\sqrt{\frac{5 + 5 - 2}{(5)(5)}} + 48(m - 1)\sqrt{\frac{5 + 7 - 2}{(5)(7)}} + \left(42m^2 - 66m + 24\right)\sqrt{\frac{7 + 7 - 2}{(7)(7)}} \]

\[ = (6m + 24)\sqrt{\frac{8}{5}} + 48(m - 1)\sqrt{\frac{2}{7}} + \left(42m^2 - 66m + 24\right)\frac{\sqrt{12}}{7} \]

\[ \text{ABC}[SD(H_m)] = \frac{(240m - 240)\sqrt{7} + 84m + 336}\sqrt{2} + \left(12m - 48\right)(m - 1). \] (28)
**Theorem 12.** Let $SD(H_m)$ be the strong double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the general inverse sum indeg index ($ISI_{[\alpha,\beta]}$) of $SD(H_m)$ is

$$ISI_{[\alpha,\beta]}[SD(H_m)] = (6m + 24)[25]^\alpha [10]^\beta + 48(m - 1)[35]^\alpha [12]^\beta + (42m^2 - 66m + 24)[49]^\alpha [14]^\beta. \quad (33)$$

**Proof.** By using Table 3 and equation (7), the result that we obtain is

$$ISI_{[\alpha,\beta]}[SD(H_m)] = |E_{(5,5)}| \sum_{x \in \overline{E}[SD(H_m)]} [d_x d_z]^\alpha [d_x + d_z]^\beta + |E_{(5,7)}| \sum_{x \in \overline{E}[SD(H_m)]} [d_x d_z]^\alpha [d_x + d_z]^\beta + |E_{(7,7)}| \sum_{x \in \overline{E}[SD(H_m)]} [d_x d_z]^\alpha [d_x + d_z]^\beta \quad (34)$$

$$= (6m + 24)[(5)(5)]^\alpha [5 + 5]^\beta + 48(m - 1)[(5)(7)]^\alpha [5 + 7]^\beta + (42m^2 - 66m + 24)[(7)(7)]^\alpha [7 + 7]^\beta$$

$$= (6m + 24)[25]^\alpha [10]^\beta + 48(m - 1)[35]^\alpha [12]^\beta + (42m^2 - 66m + 24)[49]^\alpha [14]^\beta,$$

where $\alpha$ and $\beta$ are the real numbers. \qed

**Theorem 13.** Let $SD(H_m)$ be the strong double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the first multiplicative-Zagreb index of $SD(H_m)$ is

$$PM_1[SD(H_m)] = 20321280\left(m - \frac{4}{7}\right)(m + 4)(m - 1)^2. \quad (35)$$

**Proof.** By using Table 3 and equation (10), the result that we obtain is

$$PM_1[SD(H_m)] = |E_{(5,5)}| \sum_{x \in \overline{E}[SD(H_m)]} (d_x + d_z) \times |E_{(5,7)}| \sum_{x \in \overline{E}[SD(H_m)]} (d_x + d_z) \times |E_{(7,7)}| \sum_{x \in \overline{E}[SD(H_m)]} (d_x + d_z),$$

$$PM_1[SD(H_m)] = (6m + 24)(10) \times 48(m - 1)(12) \times (42m^2 - 66m + 24)(14), \quad (36)$$

$$PM_1[SD(H_m)] = (60m + 240) \times (576m - 576) \times (588m^2 - 924m + 336),$$

$$PM_1[SD(H_m)] = 20321280\left(m - \frac{4}{7}\right)(m + 4)(m - 1)^2. \quad \square$$

**Theorem 14.** Let $SD(H_m)$ be the strong double graph of circumcoronene series of the benzenoid graph $(H_m)$; then, the second multiplicative-Zagreb index of $SD(H_m)$ is

$$PM_2[SD(H_m)] = 51861600\left(m - \frac{4}{7}\right)(m + 4)(m - 1)^2. \quad (37)$$
Proof. By using Table 3 and equation (9), the result that we obtain is

$$\text{PM}_2[\text{SD}(H_m)] = |E_{(5,5)}| \prod_{x\in E[\text{SD}(H_m)]} (d_x \cdot d_z) \times |E_{(5,7)}| \prod_{x\in E[\text{SD}(H_m)]} (d_x \cdot d_z) \times |E_{(7,7)}| \prod_{x\in E[\text{SD}(H_m)]} (d_x \cdot d_z),$$

where $m = 6m + 24$ and $d_x = \frac{42m^2 - 66m + 24}{49}$.

$$\text{PM}_2[\text{SD}(H_m)] = (6m + 24)(25) \times 48(m - 1)(35) \times (42m^2 - 66m + 24)(49),$$

$$\text{PM}_2[\text{SD}(H_m)] = 252000(m + 3)(m - 1) \times (2058m^2 - 3234m + 1176),$$

$$\text{PM}_2[\text{SD}(H_m)] = 518616000 \left( m - \frac{4}{7} \right)(m + 4)(m - 1)^2.$$
5. Comparison
In this section, we present a numerical and graphical comparison of topological indices that included the first multiplicative-Zagreb index \( (PM_1) \), general inverse sum indeg index \( (ISI_{(a,b)}) \), atom bond connectivity index \( (ABC) \), forgotten index \( (F) \), geometric arithmetic index \( (GA) \), second multiplicative-Zagreb index \( (PM_2) \), and inverse sum indeg index \( (ISI) \) for \( m = 1, 2, 3, 4, \ldots, 10 \) for the strong double graph of circumcoronene series of the benzenoid graph \( (SD(H_m)) \), as given in Table 4 and Figure 5.

6. Conclusion
We have computed the closed formulae of topological indices such as the first multiplicative-Zagreb index \( (PM_1) \), general inverse sum indeg index \( (ISI_{(a,b)}) \), atom bond connectivity index \( (ABC) \), forgotten index \( (F) \), geometric arithmetic index \( (GA) \), second multiplicative-Zagreb index \( (PM_2) \), and inverse sum indeg index \( (ISI) \) of double and strong double graphs of circumcoronene series of benzenoid \( H_m (m \geq 1) \). Chemical compounds can be studied by these indices in order to understand their diverse properties. The geometric structure and comparison of obtained results are shown graphically and numerically. Those results are convenient for further study as they do not include any polynomial.

Data Availability
The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

References
[1] R. J. Wilson, *Introduction to Graph Theory*, John Wiley & Sons, New York, NY, USA, 1986.
[2] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, no. 1, pp. 17–20, 1947.
[3] B. Furtula and D. Vukicevic, "Topological index based on the ratios of geometrical and arithmetical mean of end-vertex degrees of edge," *Journal of Mathematical Chemistry*, pp. 1369–1376, 2009.
[4] K. Das, I. Gutman, and B. Furtula, "On atom–bond connectivity index," *Filomat*, vol. 26, no. 4, pp. 733–738, 2012.
[5] B. Furtula and I. Gutman, "A forgotten topological index," *Journal of Mathematical Chemistry*, vol. 53, no. 4, pp. 1184–1190, 2015.
[6] K. Pattabiraman, "Inverse sum indeg index of graphs," *AKCE International Journal of Graphs and Combinatorics*, vol. 15, no. 2, pp. 155–167, 2018.
[7] P. Ali, S. A. K. Kirmani, O. Al Rugaie, and F. Azam, "Degree-based topological indices and polynomials of hyaluronic acid-curcumin conjugates," *Saudi Pharmaceutical Journal*, vol. 28, no. 9, pp. 1093–1100, 2020.
[8] R. Kazemi, "Note on the multiplicative Zagreb indices," *Discrete Applied Mathematics*, vol. 198, pp. 147–154, 2016.
[9] M. Elasi, A. Iranmanesh, and I. Gutman, "Multiplicative versions of first zagreb index," *MATCH Communications in Mathematical Chemistry*, vol. 68, no. 1, pp. 217–230, 2012.
[10] M. Imran, S. Akhter, and Z. Iqbal, "On eccentric polynomial of F-Sum of connected graphs," *Complexity*, vol. 2020, Article ID 5061682, 9 pages, 2020.
[11] M. Imran, S. Akhter, and Z. Iqbal, "Edge mostar index of chemical structures and nanostructures using graph operations," *International Journal of Quantum Chemistry*, vol. 120, no. 15, 2020.
[12] S. Akhter, M. Imran, and Z. Iqbal, "Moster indices of SiO nanostructures and methane chain nanostructures," *International Journal of Quantum Chemistry*, vol. 121, no. 5, 2020.
[13] M. An and L. Xiong, "Some results on the inverse sum indeg index of a graph," *Information Processing Letters*, vol. 134, pp. 42–46, 2018.
[14] G. Hong, Z. Gu, M. Javid, H. M. Awais, and M. K. Siddiqui, "Degree based topological invariants of metal-organic networks," *IEEE Access*, vol. 8, pp. 68288–68300, 2020.
[15] M. Alaeiyan, M. S. Sardar, S. Zafar, and Z. Zahid, "Computation of topological indices of line graph of jahangir graph," *International Journal of Applied Mathematics*, vol. 12, 2018.
[16] D. K. Ch, M. Marjan, M. Emin, and M. Igor, "Bonds for symmetric division deg index of graphs," *Faculty of Sciences and Mathematics*, vol. 33, no. 3, pp. 638–698, 2019.
[17] M. S. Sardar, S. Zafar, and Z. Zahid, "Certain topological indices of line graph of Dutch windmill graphs," *Southeast Asian Bulletin of Mathematics*, pp. 119–129, 2020.
[18] Y. Gao, M. R. Farahani, and W. Nazeer, "On topological indices of circumcoronene series of benzenoid," *Chemical Methodologies*, pp. 39–46, 2018.
[19] Y. Gao, M. R. Farahani, M. S. Sardar, and S. Zafar, "On the sanskruti index of circumcoronene series of benzenoid," *Applied Mathematics*, vol. 08, no. 04, pp. 520–524, 2017.
[20] M. A. Ali, M. S. Sardar, I. Siddique, and D. Alrowaili, "Vertex-based topological indices of double and strong double graph of Dutch windmill graph," *Journal of Chemistry*, vol. 2021, p. 12, 2021.
[21] T. A. Chishti, H. A. Ganie, and S. Pirzada, "Properties of strong double graphs," *Journal of Discrete Mathematical Sciences and Cryptography*, vol. 17, no. 4, pp. 311–319, 2014.