Abstract. For the planar elasticity equation, we prove the uniform convergence and optimality of an adaptive mixed method using the hybridized mixed finite element in [Numer. Math., 141 (2019), pp. 569-604] proposed by Gong, Wu, and Xu. The main ingredients of the proof consist of the discrete reliability and quasi-orthogonality, which are both derived using a discrete approximation result. Compared with Arnold–Winther and Hu–Zhang mixed elements, the adaptive hybridized mixed method yields nested discrete stress spaces so that the rigorous quasi-optimal convergence rate can be established.

Key words. linear elasticity, mixed finite element, hybridization, convergence, quasi-optimality

AMS subject classifications. 65N12, 65N15, 65N30, 65N50

1. Introduction. Adaptive finite element methods (AFEMs) for numerical solutions of partial differential equations has been an active research area since 1980s. Using a sequence of self-adapted graded meshes, AFEMs can achieve quasi-optimal convergence rate even for problems with singularity arising from, e.g., irregular data or domains with nonsmooth boundary. Convergence and optimality analysis of AFEMs for symmetric and positive-definite elliptic problems has now reached maturity, see, e.g., [21, 39, 8, 40, 15, 20] and references therein.

The elasticity equation is often discretized by the mixed finite element method (MFEM). As opposed to FEMs based on the primal formulation, the mixed approach is robust with respect to the Lamé coefficient $\lambda$ and thus suitable for (nearly) incompressible elastic material. In practice, the boundary of the elastic body is rarely sufficiently smooth. On the other hand, the mixed formulation with strongly imposed symmetry usually leads to higher order conforming mixed finite elements [5, 1, 30, 28], whose a priori error estimates relies on unrealistic high regularity of the exact solution. From this perspective, adaptivity in elasticity is of great importance, see, e.g., [11, 36, 12, 19] for a posteriori error estimates of MFEMs for plane elasticity. Readers are also referred to [14] for a robust quasi-optimal nonconforming AFEM using the lowest order Crouzeix–Raviart element.

Analysis of adaptive mixed finite element methods (AMFEMs) hinges on technical discrete reliability and quasi-orthogonality results, see, e.g., [13, 7, 17, 32, 23, 29, 35, 34, 27, 22]. As far as we know, there is no convergence or optimality result of adaptive mixed methods for elasticity equations in literature. In this paper, we shall develop a quasi-optimal adaptive method for the linear elasticity equation (1.1) using the higher order hybridizable mixed elements proposed in [25]. The convergence is uniform with respect to the Lamé coefficient $\lambda$. Our analysis cannot be directly extended to the Arnold–Winther [5] or Hu–Zhang mixed elements [30, 31] due to the non-nestedness of discrete stress spaces thereof.

Our AMFEM (denoted by AMFEM) is designed to reduce the stress error $\|\sigma - \sigma_h\|_A$. The framework of our analysis is similar to the convergence analysis of AMFEMs for Poisson’s equation [17], see also [32, 29]. On the other hand, our analysis is more algebraic without introducing an intermediate variable $\tilde{\sigma}$ compared with [17, 19]. To prove the aforementioned discrete reliability and quasi-orthogonality, we develop

*Department of Mathematics, Pennsylvania State University, University Park, PA 16802. Email: yuwli@psu.edu.
the discrete Helmholtz decomposition in Theorem 2.1, a local \(H^2\)-bounded interpolation onto the \(C^1\) finite element space in Theorem 4.8, and the technical discrete approximation result based on the space of piecewise rigid motions in Lemma 3.1. The proof of Lemma 3.1 relies on a modified discrete Korn’s inequality on each local triangle together with the inf-sup condition proved in [18], see Section 5 for details.

In the rest of this section, we introduce the continuous and discrete mixed formulations of the linear elasticity equation in \(\mathbb{R}^2\). Let \(\sigma\) and \(u\) denote the stress and displacement fields produced by a body force acting on a linearly elastic body that occupies the region \(\Omega \subset \mathbb{R}^2\). Then \(u\) takes value in \(\mathbb{R}^2\) and \(\sigma\) takes value in \(S\), which is the space of symmetric \(2 \times 2\) matrices. Let

\[
\varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T)
\]

be the symmetric gradient of \(u\). Let \(\text{tr}\) denote the trace of square matrices, \(\delta\) the \(2 \times 2\) identity matrix. The plane linear elasticity equation reads

\[
\begin{align*}
\sigma &= 2\mu \varepsilon(u) + \lambda \text{tr}(\varepsilon(u))\delta, \\
\text{div } \sigma &= f,
\end{align*}
\]

where \(\mu = \mu(x) > 0, \lambda = \lambda(x) > 0\) are Lamé coefficients. Given a vector space \(V\), let \(L^2(\Omega, V)\) denote the space of \(V\)-valued \(L^2\)-functions on \(\Omega\). Similarly, we use \(H^s(\Omega, V)\) to denote the usual \(V\)-valued Sobolev space. In this paper, we assume \(\Omega\) is a simply connected polygonal domain. Let \(V = L^2(\Omega, \mathbb{R}^2)\) and \(\Sigma = H(\text{div}, \Omega, \mathbb{R}^2) := \{\tau \in L^2(\Omega, \mathbb{R}^2) : \text{div } \tau \in L^2(\Omega, \mathbb{R}^2)\}\).

The compliance tensor \(A\) is given by

\[
A\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + 2\lambda} (\text{tr } \sigma) \delta \right).
\]

Here we assume \(\mu \geq \mu_0 > 0\) on \(\Omega\), where \(\mu_0\) is a constant. Let \(\langle \cdot, \cdot \rangle\) denote the \(L^2\) inner product on \(\Omega\), i.e.,

\[
\langle \xi, \zeta \rangle = \int_{\Omega} \xi : \zeta \, dx = \int_{\Omega} \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_{ij} \zeta_{ij} \, dx,
\]

where \(\xi = (\xi_{ij})\) and \(\zeta = (\zeta_{ij})\) are \(\mathbb{R}^{m \times n}\)-valued functions. Similarly \(\langle \cdot, \cdot \rangle_{\partial \Omega}\) is the \(L^2\) inner product on \(\partial \Omega\). Under the homogeneous Dirichlet boundary condition \(u|_{\partial \Omega} = 0\), the mixed variational formulation of (1.1) is to find \(\sigma \in \Sigma\) and \(u \in V\) such that

\[
\begin{align*}
\langle A\sigma, \tau \rangle + \langle \text{div } \sigma, u \rangle &= 0, \quad \tau \in \Sigma, \\
\langle \text{div } \sigma, v \rangle &= \langle f, v \rangle, \quad v \in V.
\end{align*}
\]

Let \(\mathcal{T} = \{T_h\}\) be a family of conforming triangulations of \(\Omega\) indexed by \(h\). For \(T_h, T_H \in \mathcal{T}\), we say \(T_H \leq T_h\) provided \(T_h\) is a refinement of \(T_H\). Assume the forest \(\mathcal{T}\) is shape regular, i.e., there exists a uniform constant \(C_{\text{shape}}\) such that

\[
\max_{T_h \in \mathcal{T}} \max_{T \in T_h} \frac{\rho_T}{\rho_T} < C_{\text{shape}} < \infty,
\]
where $r_T$ and $\rho_T$ are radii of circumscribed and inscribed circles of $T$, respectively. Let $\mathcal{P}_r(T, \mathbb{V})$ denote the space of $\mathbb{V}$-valued polynomials of degree $\leq r$ on $T$. For an integer $r \geq 0$, the mixed finite element spaces are

\[
\Sigma_h := \{ \tau_h \in \Sigma : \tau_h|_T \in \mathcal{P}_{r+3}(T, \mathbb{S}) \text{ for each } T \in \mathcal{T}_h \},
\]

\[
V_h := \{ v_h \in V : v_h|_T \in \mathcal{P}_{r+2}(T, \mathbb{R}^2) \text{ for each } T \in \mathcal{T}_h \}.
\]

The mixed method for (1.2) is to find $\sigma_h \in \Sigma_h$, $u_h \in V_h$ such that

(1.3a) \hspace{1cm} \langle A\sigma_h, \tau \rangle + \langle \text{div} \tau, u_h \rangle = 0, \quad \tau \in \Sigma_h,

(1.3b) \hspace{1cm} \langle \text{div} \sigma_h, v \rangle = \langle f, v \rangle, \quad v \in V_h.

Thanks to the nestedness $\Sigma_H \times V_H \subseteq \Sigma_h \times V_h$ when $\mathcal{T}_H \leq \mathcal{T}_h$, we have the Galerkin orthogonality

(1.4a) \hspace{1cm} \langle A(\sigma_h - \sigma_H), \tau \rangle + \langle \text{div} \tau, u_h - u_H \rangle = 0, \quad \tau \in \Sigma_H,

(1.4b) \hspace{1cm} \langle \text{div}(\sigma_h - \sigma_H), v \rangle = 0, \quad v \in V_H.

It has been shown in [25] that the method (1.3) satisfies the inf-sup condition

\[
\|v_h\| \leq C \sup_{\tau_h \in \Sigma_h} \frac{\langle \text{div} \tau_h, v_h \rangle}{\|\tau_h\|_{H(\text{div})}} \text{ for all } v_h \in V_h,
\]

where $C$ depends only on $r$ and the shape regularity of $\mathcal{T}_h$. Note that $\Sigma_h$ is not a standard finite element space. However, the method (1.3) can be efficiently implemented using hybridization technique and iterative solvers, see [25] and Section 6 for details.

Let $\mathcal{N}_h$ denote the set of grid vertices in $\mathcal{T}_h$. For $r \geq 0$, the classic Arnold–Winther mixed element spaces are

\[
\Sigma_h^{AW} := \{ \tau_h \in \Sigma : \tau_h|_T \in \mathcal{P}_{r+3}(T, \mathbb{S}), \text{div} \tau \in \mathcal{P}_{r+1}(T, \mathbb{R}^2) \text{ for each } T \in \mathcal{T}_h, \\
\tau_h \text{ is continuous at each } x \in \mathcal{N}_h \},
\]

\[
V_h^{AW} := \{ v_h \in V : v_h|_T \in \mathcal{P}_{r+1}(T, \mathbb{R}^2) \text{ for each } T \in \mathcal{T}_h \}.
\]

The Hu–Zhang mixed element spaces are

\[
\Sigma_h^{HZ} := \{ \tau_h \in \Sigma : \tau_h|_T \in \mathcal{P}_{r+3}(T, \mathbb{S}) \text{ for each } T \in \mathcal{T}_h, \\
\tau_h \text{ is continuous at each } x \in \mathcal{N}_h \},
\]

\[
V_h^{HZ} := V_h.
\]

Due to the continuity constraint of $\Sigma_h^{AW}$ and $\Sigma_h^{HZ}$ at each vertex, we find $\Sigma_H^{AW} \subseteq \Sigma_h^{AW}$, $\Sigma_H^{HZ} \subseteq \Sigma_h^{HZ}$. This non-nestedness is the motivation of our analysis of adaptive hybridized MFEM and a major difficulty arising from the analysis of AMFEMs based on Arnold–Winther and Hu–Zhang elements.

The rest of this paper is organized as follows. In Section 2, we introduce the continuous and discrete elasticity complexes and develop the discrete Helmholtz decomposition. In Section 3, we derive the discrete reliability and quasi-orthogonality for AMFEM. Section 4 is devoted to the convergence and optimality analysis of AMFEM. In Section 5, we give proofs of technical results used in our analysis. The numerical experiment is presented in Section 6.
2. Elasticity complex. Given a scalar-valued function \( w \) and a \( \mathbb{R}^2 \)-valued function \( \phi = (\phi_1, \phi_2) \), let
\[
\text{curl } w := (-\frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_1})^\top, \quad \text{rot } \phi := \frac{\partial \phi_2}{\partial x_1} - \frac{\partial \phi_1}{\partial x_2}.
\]
For \( \mathbb{R}^2 \)-valued \( v = (v_1, v_2)^\top \) and \( \mathbb{R}^{2 \times 2} \)-valued \( \tau = (\tau_1, \tau_2)^\top \), let
\[
\text{curl } v := (\text{curl } v_1, \text{curl } v_2)^\top, \quad \text{rot } \tau := (\text{rot } \tau_1, \text{rot } \tau_2)^\top.
\]
In what follows, we introduce the Helmholtz decomposition of \( \Sigma \) based on the theory in [4]. Let \( W = H^2(\Omega) \) and \( J \) denote the Airy stress:
\[
J = \text{curl } \text{curl } = \left( \begin{array}{cc}
\frac{\partial^2}{\partial x_2^2} & -\frac{\partial}{\partial x_1 \partial x_2} \\
-\frac{\partial}{\partial x_1 \partial x_2} & \frac{\partial^2}{\partial x_1^2}
\end{array} \right).
\]
Consider the sequence of Hilbert spaces
\[
(2.1) \quad W \xrightarrow{J} \Sigma \xrightarrow{\text{div}} V \longrightarrow 0.
\]
Here \( \Sigma \) is equipped with the weighted \( L^2 \) inner product \( \langle A, \cdot \rangle \) and \( V \) admits the usual \( L^2 \) inner product. The fact \( \text{div } \circ J = 0 \) implies \( J(W) \subseteq \ker(\text{div}) \). Moreover, (2.1) is exact, i.e., \( J(W) = \ker(\text{div}) \). In fact, given \( \tau \in \ker(\text{div}) \), there exists \( \phi \in H^1(\Omega, \mathbb{R}^2) \) such that \( \text{curl } \phi = \tau \). Due to the symmetry of \( \tau \), it holds that \( \text{div } \phi = 0 \) and thus \( \phi = \text{curl } w \) for some \( w \in H^2(\Omega) \).

Thanks to the exactness of (2.1), we obtain
\[
(2.2) \quad \Sigma = \ker(\text{div}) \oplus \ker(\text{div})^\perp = J(W) \oplus \ker(\text{div})^\perp.
\]
The operator \( \text{div} : L^2(\Omega, \mathbb{R}) \rightarrow L^2(\Omega, \mathbb{R}^2) \) can be viewed as a densely-defined, closed operator whose domain is \( \Sigma \). Define \( \varepsilon_A : L^2(\Omega, \mathbb{R}^2) \rightarrow L^2(\Omega, \mathbb{S}) \) by
\[
\varepsilon_A(v) := A^{-1} \varepsilon(v) \text{ for all } v \in H^1_0(\Omega),
\]
i.e., \( \varepsilon_A \) is also a densely-defined, closed operator with domain \( H^1_0(\Omega) \). It is readily checked that \( \varepsilon_A \) is the adjoint of the negative divergence \( -\text{div} : L^2(\Omega, \mathbb{S}) \rightarrow L^2(\Omega, \mathbb{R}^2) \). Due to the closed range theorem, it holds that \( \ker(\text{div})^\perp = \varepsilon_A(H^1_0(\Omega)) \). Hence (2.2) reduces to the standard Helmholtz decomposition
\[
(2.3) \quad \Sigma = J(W) \oplus \varepsilon_A(H^1_0(\Omega)).
\]
The decomposition (2.3) was introduced in [11] without using functional analysis. Many a posteriori error estimates of MFEMs for plane elasticity are based on (2.3), see, e.g., [11, 19, 12].

Given \( T \in T_h \), let \( |T| \) denote the area of \( T \) and \( h_T = |T|^{\frac{1}{2}} \) the size of \( T \), \( t \) the counterclockwise unit tangent to \( \partial T \), and \( n \) the outward unit normal to \( \partial T \). Let \( E_h \) denote the collection of edges in \( T_h \) and \( E_h(T) \) the set of edges contained in \( \partial T \). Let \( \| \cdot \| \) denote the \( L^2 \)-norm on \( \Omega \), \( \| \cdot \|_T \) and \( \| \cdot \|_{\partial T} \) the \( L^2 \)-norms on \( T \) and \( \partial T \), respectively.

For each edge \( e \in T_h \), we choose a unit tangent \( t_e \) and a unit normal \( n_e \). In addition, \( t_e \) is counterclockwise oriented and \( n_e \) is outward pointing provided \( e \subset \partial \Omega \). If \( e \) is an interior edge shared by two triangles \( T_+ \) and \( T_- \), let \( \left[ \phi \right]_e = (\phi|_{T_+})_e - (\phi|_{T_-})_e \) denote the jump of \( \phi \) over \( e \), where \( n_e \) is pointing from \( T_+ \) to \( T_- \). If \( e \subset \partial \Omega \), we
simply take $[\phi]_e = \phi|_e$. In this paper, we make use of the stress error estimator
\[\eta_h = \eta_h(\sigma_h) = \left( \sum_{T \in T_h} \eta_h^2(\sigma_h, T) \right)^{\frac{1}{2}},\]
where
\[\eta_h(\sigma_h, T) = (h_T^2 \| \text{rot rot} A \sigma_h \|_T^2 + h_T \sum_{e \in E_h(T)} \| te \cdot [A \sigma_h]_e \|_e^2 + h_T^3 \sum_{e \in E_h(T)} \| \partial_{te} [A \sigma_h]_e \cdot [\text{rot} A \sigma_h]_e \|_e^2)^{\frac{1}{2}}.\]

Let $P_h$ be the $L^2$-projection onto $V_h$. The data oscillation is
\[\text{osc}_h = \text{osc}_h(f) = \left( \sum_{T \in T_h} \text{osc}_h^2(f, T) \right)^{\frac{1}{2}},\]
where
\[\text{osc}_h(f, T) = h_T \| f - P_h f \|_T.\]

The expression of $\eta_h$ is the same as existing a posteriori error estimators for the MFEM using the Arnold–Winther and Hu–Zhang elements, see, e.g., [12, 19].

An indispensable ingredient of optimality analysis of AFEMs is the discrete upper bound for the finite element error. In light of a posteriori error analysis in the continuous case, an discrete exact sequence and a discrete Helmholtz decomposition must be essential for proving discrete reliability. To construct a discrete analogue of (2.1), consider the $C^1$-conforming space
\[W_h = \{ w_h \in C^1(\Omega) : w_h|_T \in P_{r+5}(T) \text{ for each } T \in T_h \}.\]

Therefore we obtain a discrete sequence:
\[(2.4) \quad W_h \xrightarrow{J} \Sigma_h \xrightarrow{\text{div}} V_h \longrightarrow 0.\]

Using the exactness of (2.1), it is easy to check that the discrete sequence (2.4) is also exact. Due to the exactness of (2.4), we immediately obtain the following discrete Helmholtz decomposition.

**Theorem 2.1 (discrete Helmholtz decomposition).**

\[\Sigma_h = J(W_h) \oplus \varepsilon_A(V_h),\]

where $\varepsilon_A : V_h \to \Sigma_h$ is the adjoint operator of $- \text{div} : \Sigma_h \to V_h$, i.e.,
\[\langle A \varepsilon_A(v_h), \tau_h \rangle = -\langle v_h, \text{div} \tau_h \rangle \text{ for all } \tau_h \in \Sigma_h.\]

**Proof.** Let $\ker(\text{div} |_{\Sigma_h})^\perp$ be the orthogonal complement of $\ker(\text{div} \Sigma_h)$ in $\Sigma_h$ with respect to the weighted inner product $\langle A \cdot, \cdot \rangle$. Elementary linear algebra shows that
\[\ker(\text{div} \Sigma_h)^\perp = \varepsilon_A(V_h).\]

Combining it with the exactness $\ker(\text{div} \Sigma_h) = J(W_h)$, we obtain
\[\Sigma_h = \ker(\text{div} \Sigma_h) \oplus \ker(\text{div} \Sigma_h)^\perp = J(W_h) \oplus \varepsilon_A(V_h),\]

which completes the proof.

**Remark 2.1.** For the Arnold–Winther and Hu–Zhang elements, the correct discrete elasticity sequences are
\[\widehat{W}_h \xrightarrow{J} \Sigma_h^{AW} \xrightarrow{\text{div}} V_h^{AW} \longrightarrow 0,\]
and
\[ \hat{W}_h \xrightarrow{J} \Sigma_h \xrightarrow{\text{div}} V_h \xrightarrow{} 0, \]
respectively, where
\[ \hat{W}_h = \{ w_h \in W : W_h|_T \in P_{r+5}(T) \text{ for each } T \in T_h, \]
\[ \nabla^2 w_h \text{ is continuous at each } x \in N_h \}. \]

When \( r = 0 \), \( \hat{W}_h \) is the well-known quintic Argyris finite element space. Note that \( \hat{W}_h \nsubseteq \hat{W}_H \) because of the extra \( C^2 \) continuity at each vertex.

\( W_h \) is not a standard finite element space. Fortunately, it has been shown in [38] that \( W_h \) admits a set of unisolvent nodal variables and locally supported dual nodal basis. Based on a slightly modified (but complicated) nodal variables and basis, Girault and Scott [24] constructed a locally defined and \( L^2 \)-bounded interpolation preserving the homogeneous boundary condition. To derive the discrete reliability of (1.3), we present an interpolation \( I_H : W_h \rightarrow W_H \), which is a slight variation of the interpolation in [24]. Throughout the rest, we say \( A \lesssim B \) provided \( A \leq CB \) for some generic constant \( C \) depending only on \( \mu, \Omega, \) and \( C_{\text{shape}} \).

**Proposition 2.2.** For \( T_h, T_H \in T \) with \( T_H \leq T_h \), let \( R_H := T_H \setminus T_h \) be the set of refinement elements and \( \tilde{R}_H := \{ T \in T_H : T \cap T' \neq \emptyset \text{ for some } T' \in R_H \} \) denote the enriched collection of refinement elements. There exists an interpolation \( I_H : W_h \rightarrow W_H \) such that for \( w_h \in W_h, \)
\begin{align*}
(2.5) & \quad w_h - I_H w_h = 0 \text{ at } x \in N_H, \\
(2.6) & \quad w_h - I_H w_h = 0 \text{ on } T \in T_H \setminus \tilde{R}_H. 
\end{align*}

In addition,
\[
\sum_{T \in T_H} h_T^{-1} \left[ \left| w_h - I_H w_h \right|^2_T + h_T^{-2} \left| w_h - I_H w_h \right|^2_{H^1(T)} \right] \\
+ h_T^{-3} \left| w_h - I_H w_h \right|^2_{\partial T} + h_T^{-1} \left\| \nabla (w_h - I_H w_h) \right\|^2_{\partial T} \lesssim \left| w_h \right|^2_{H^2(\Omega)}.
\]

The proof of Proposition 2.2 is left in Section 5.

**3. Discrete reliability and quasi-orthogonality.** Let \( \| \cdot \|_A \) denote the norm corresponding to \( \langle A, \cdot \rangle \) and \( \| \cdot \|_{A,T} \) the restricted \( A \)-norm on \( T \). For \( M \subseteq T_h \), let
\[
\eta_h(\sigma_h, M) = \left( \sum_{T \in M} \eta_h^2(\sigma_h, T) \right)^{\frac{1}{2}},
\]
\[
\text{osc}_h(\sigma, M) = \left( \sum_{T \in M} \text{osc}_h^2(\sigma, T) \right)^{\frac{1}{2}}.
\]

We shall prove the discrete reliability of the estimator \( \eta_h \) and quasi-orthogonality between \( \sigma - \sigma_h \) and \( \sigma_h - \sigma_H \). The analysis relies on the following discrete approximation result, whose proof is left in Section 5.
**Lemma 3.1.** Let $T_h, T_H \in T$ with $T_H \leq T_h$ and
$$
\mathcal{R}M(T) = \{(c_1, c_2)^T + c_3(-x_2, x_1)^T : c_1, c_2, c_3 \in \mathbb{R}\}
$$
be the space of rigid motions on $T \in T_H$. Let $Q_H$ denote the $L^2$-projection onto the space of piecewise rigid motions
$$
\mathcal{R}M_H = \{v \in L^2(\Omega, \mathbb{R}^2) : v|_T \in \mathcal{R}M(T) \text{ for all } T \in T_H\}.
$$
It holds that
$$
\left(\sum_{T \in T_H} h_T^{-2} \|v_h - Q_H v_h\|^2_T\right)^{\frac{1}{2}} \lesssim \|\varepsilon_h^A(v_h)\|_A.
$$
The space $\mathcal{R}M_H$ can be viewed as a broken rotated Raviart–Thomas finite element space. Here we are interested in $Q_H$ instead of $P_H$ because we will use the fact $\mathcal{R}M(T) \subset \ker(\varepsilon)$, see the proof of Lemma 5.2 for details.

The next lemma is used to get rid of the Lamé coefficient $\lambda$ in error bounds. The proof can be found in [3], Lemmas 3.1 and 3.2.

**Lemma 3.2.** There exists a constant $C_{rb}$ depends only on $\mu$ and $\Omega$, such that
$$
\|\tau\| \leq C_{rb} \left(\|\tau\|_A + \|\text{div } \tau\|_{H^{-1}(\Omega)}\right)
$$
for all $\tau \in \Sigma$ with $\int_\Omega \text{tr } \tau dx = 0$.

For $w \in H^1(T)$ and $\phi \in H^1(T, \mathbb{R}^2)$, we have the integration-by-parts formula:
$$
\int_T \text{curl } w \cdot \phi dx = \int_{\partial T} w \phi \cdot tds - \int_T w \text{rot } \phi dx.
$$
(3.1)

The matrix version of (3.1) is
$$
\int_T \text{curl } v : \tau dx = \int_{\partial T} v \cdot tds - \int_T v \cdot \text{rot } \tau dx,
$$
where $v \in H^1(T, \mathbb{R}^2)$ and $\tau \in H^1(T, \mathbb{S})$.

With the above preparations, we are able to prove the discrete reliability of $\mathcal{E}_h$.

**Theorem 3.3** (discrete reliability). Let $T_h, T_H \in T$ with $T_H \leq T_h$. There exists a constant $C_{drel}$ depending only on $\mu, \Omega$ and $C_{\text{shape}}$, such that
$$
\|\sigma_H - \sigma_h\|^2_A \leq C_{drel} \left(\eta_H^2(\sigma_H, \mathcal{R}_H) + \text{osc}_H^2(f, \mathcal{R}_H)\right).
$$

**Proof.** Since $\sigma_H - \sigma_h \in \Sigma_H$, the discrete Helmholtz decomposition in Theorem 2.1 gives
$$
\sigma_H - \sigma_h = J(w_h) + \varepsilon_h^A(v_h)
$$
for some $w_h \in W_h$ and $v_h \in V_h$. Taking $\tau = \delta$ in (1.4a) yields
$$
\int_\Omega \text{tr}(\sigma_H - \sigma_h)dx = 0.
$$
Recall the definition of $\varepsilon_h^A$. Direct calculation shows that
$$
\int_\Omega \frac{1}{2(\mu + \lambda)} \text{tr } \varepsilon_h^A(v_h)dx = \langle A\varepsilon_h^A(v_h), \delta \rangle = -\langle v_h, \text{div } \delta \rangle = 0.
$$
(3.5)
Then a combination of (3.3)–(3.5) yields

\[(3.6) \quad \int_{\Omega} \operatorname{tr} J(w_h) dx = 0.\]

Hence using Lemma 3.2, (3.6) and the \(A\)-orthogonality between \(Jw_h\) and \(\varepsilon^h_A(v_h)\), we obtain the following robust bound

\[(3.7) \quad \|Jw_h\| \lesssim \|Jw_h\|_A \leq \|\sigma_H - \sigma\|_A.\]

Using (1.3a) and \(\operatorname{div} \circ J = 0\), we have

\[\langle A(\sigma_H - \sigma), Jw_h \rangle = \langle A\sigma_H, Jw_h \rangle = \langle A\sigma_H, J(w_h - I_Hw_h) \rangle.\]

It then follows together with the property (2.6) that

\[(3.8) \quad \langle A(\sigma_H - \sigma), Jw_h \rangle = \sum_{T \in \mathcal{R}_H} \int_T A\sigma_H : J(E_h) dx,\]

where \(E_h = w_h - I_Hw_h\). Using the formulas (3.1) and (3.2), we have

\[(3.9) \quad \int_T A\sigma_H : J(E_h) dx = \int_{\partial T} (A\sigma_H t) \cdot \operatorname{curl} E_h ds - \int_{T} (\operatorname{rot} A\sigma_H) \cdot \operatorname{curl} E_h dx\]

\[= \int_{\partial T} (t^T(A\sigma_H) \operatorname{curl} E_h - (\operatorname{rot} A\sigma_H) \cdot tE_h) ds + \int_{T} (\operatorname{rot} \operatorname{rot} A\sigma_H) E_h dx.\]

Integrating by parts on each edge of \(T\), we have

\[(3.10) \quad \int_{\partial T} t^T(A\sigma_H) \operatorname{curl} E_h ds = \int_{\partial T} t^T(A\sigma_H) t \frac{\partial}{\partial n} E_h - n^T(A\sigma_H) t \frac{\partial}{\partial t} E_h ds\]

\[= \int_{\partial T} t^T(A\sigma_H) \frac{\partial}{\partial n} E_h + n^T(\frac{\partial}{\partial t} A\sigma_H) t E_h ds.\]

In the last equality, we use the property (2.5), i.e., \(E_h = 0\) at each vertex of \(T\). Let \(\mathcal{E}_H(\mathcal{R}_H)\) denote the collection of edges of triangles in \(\mathcal{R}_H\). Note that \(E_h\) and \(\frac{\partial}{\partial n} E_h\) are continuous over each edge in \(\mathcal{T}_H\). Combining (3.8), (3.9) and (3.10), we obtain

\[\langle A(\sigma_H - \sigma), Jw_h \rangle = \sum_{T \in \mathcal{R}_H} \int_{\partial T} t^T(A\sigma_H) t \frac{\partial}{\partial n} E_h ds\]

\[+ \int_T (\operatorname{rot} \operatorname{rot} A\sigma_H) E_h dx + \int_{\partial T} \left( n^T(\frac{\partial}{\partial t} A\sigma_H) t - (\operatorname{rot} A\sigma_H) \cdot t \right) E_h ds\]

\[= \sum_{e \in \mathcal{E}_H(\mathcal{R}_H)} \int_e t_e^T[A\sigma_H] t_e \frac{\partial}{\partial n_e} E_h ds + \sum_{T \in \mathcal{R}_H} \int_T (\operatorname{rot} \operatorname{rot} A\sigma_H) E_h dx\]

\[+ \sum_{e \in \mathcal{E}_H(\mathcal{R}_H)} \int_e \left( n_e^T \frac{\partial}{\partial t_e} [A\sigma_H] \cdot t_e - [\operatorname{rot} A\sigma_H] \cdot t_e \right) E_h ds.\]

Using the previous estimate and Cauchy–Schwarz inequality, we have

\[(3.11) \quad \langle A(\sigma_H - \sigma), Jw_h \rangle \lesssim \eta_H(\sigma_H, \mathcal{R}_H) \left( \sum_{T \in \mathcal{T}_H} h_T^{-1} \|\frac{\partial}{\partial n} E_h\|^2_{\partial T} + h_T^{-4} \|E_h\|^2 + h_T^{-3} \|E_h\|_{\partial T}^* \right).\]
It then follows from \((3.11), (2.7)\) and \((3.7)\) that
\[
\langle A(\sigma_H - \sigma_h), Jw_h \rangle \lesssim \eta_H(\sigma_H, \tilde{R}_H)|w_h|_{H^2(\Omega)} \\
\lesssim \eta_H(\sigma_H, \tilde{R}_H)\|\sigma_h - \sigma_H\|_A.
\]

On the other hand, \((1.3b)\) implies
\[
\langle A(\sigma_H - \sigma_h), \varepsilon_h^A(v_h) \rangle = -\langle \text{div}(\sigma_H - \sigma_h), v_h \rangle = \langle P_H f - P_H f, v_h \rangle \\
= \langle f - P_H f, v_h - Q_H v_h \rangle = \sum_{T \in \mathcal{R}_H} \langle f - P_H f, v_h - Q_H v_h \rangle_T.
\]

Using \((3.13), \text{Lemma 3.1}\), and \(\|\varepsilon_h^A(v_h)\|_A \leq \|\sigma_H - \sigma_h\|_A\), we obtain
\[
\langle A(\sigma_H - \sigma_h), \varepsilon_h^A(v_h) \rangle \leq \text{osc}_H(f, \mathcal{R}_H)(\sum_{T \in \mathcal{R}_H} h_T^{-2}\|v_h - Q_H v_h\|_T^2)^{\frac{1}{2}} \\
\lesssim \text{osc}_H(f, \mathcal{R}_H)\|\varepsilon_h^A(v_h)\|_A \leq \text{osc}_H(f, \mathcal{R}_H)\|\sigma_H - \sigma_h\|_A.
\]

Finally, a combination of \((3.12)\) and \((3.14)\) completes the proof.

Let \(\mathcal{T}_h\) be a uniform refinement of \(\mathcal{T}_H\) and let the maximum mesh size of \(\mathcal{T}_h\) go to 0 in Theorem 3.3. In this case, \(\tilde{R}_H = \mathcal{R}_H = \mathcal{T}_H\) and \(\sigma_H \rightarrow \sigma, u_h \rightarrow u\) in \(\Sigma \times V\). Therefore, we obtain the continuous upper bound
\[
\|\sigma - \sigma_H\|_A^2 \leq C_{\text{rel}}(\eta_H^2(\sigma_H) + \text{osc}_H^2(f)),
\]
where \(C_{\text{rel}} \in (0, C_{\text{drel}}]\) is a constant depending only on \(\mu, \Omega, C_{\text{shape}}\).

The quasi-orthogonality on the variable \(\sigma\) follows with a similar argument as in the proof of Theorem 3.3.

**Theorem 3.4 (quasi-orthogonality).** Let \(\mathcal{T}_h, \mathcal{T}_H \in \mathcal{T}\) with \(\mathcal{T}_H \subseteq \mathcal{T}_h\). For any \(\nu \in (0, 1)\), it holds that
\[
(1 - \nu)\|\sigma - \sigma_h\|_A^2 \leq \|\sigma - \sigma_H\|_A^2 - \|\sigma_h - \sigma_H\|_A^2 + C_{\sigma} \text{osc}_H^2(f, \mathcal{R}_H),
\]
where \(C_{\nu} = \nu^{-1}C_{\sigma}\) and \(C_{\sigma}\) is a constant depending only on \(\mu, \Omega, \) and \(C_{\text{shape}}\).

**Proof.** Combining \((3.3)\) and \((1.2a), (1.3a)\) yields
\[
\langle A(\sigma - \sigma_h), \sigma_H - \sigma_h \rangle = \langle A(\sigma - \sigma_h), \varepsilon_h^A(v_h) \rangle \leq \|\sigma - \sigma_h\|_A\|\varepsilon_h^A(v_h)\|_A.
\]

Following the same analysis in \((3.13)\), we have
\[
\|\varepsilon_h^A(v_h)\|_A^2 = -\langle \text{div} \varepsilon_h^A(v_h), v_h \rangle = -\langle \text{div}(\sigma_H - \sigma_h), v_h \rangle \\
= \sum_{T \in \mathcal{R}_H} \langle f - P_H f, v_h - Q_H v_h \rangle_T \leq C_{\sigma} \text{osc}_H(f, \mathcal{R}_H)\|\varepsilon_h^A(v_h)\|_A.
\]

A combination of \((3.16)\) and \((3.17)\) shows that
\[
\langle A(\sigma - \sigma_h), \sigma_H - \sigma_h \rangle \leq C_{\sigma}^\frac{3}{2}\|\sigma - \sigma_h\|_A \text{osc}_H(f, \mathcal{R}_H) \\
\leq \nu \|\sigma - \sigma_h\|_A^2 + \nu^{-1}C_{\sigma} \text{osc}_H^2(f, \mathcal{R}_H),
\]
where \(0 < \nu < 1\). Therefore
\[
\|\sigma - \sigma_h\|_A^2 = \|\sigma - \sigma_H\|_A^2 - \|\sigma_h - \sigma_H\|_A^2 + 2\langle A(\sigma - \sigma_h), \sigma_H - \sigma_h \rangle \\
\leq \|\sigma - \sigma_H\|_A^2 - \|\sigma_h - \sigma_H\|_A^2 + \nu \|\sigma - \sigma_h\|_A^2 + \nu^{-1}C_{\sigma} \text{osc}_H^2(f, \mathcal{R}_H).
\]

The proof is complete.
4. Quasi-optimality. The adaptive algorithm AMFEM is based on the standard “SOLVE → ESTIMATE → MARK → REFINE” feedback loop. Let
\[
\tilde{\eta}_h(\sigma_h, T) = (\eta_h^2(\sigma_h, T) + \text{osc}_{h_{\ell+1}}^2(f,T))^{\frac{1}{2}},
\]
\[
\tilde{\eta}_h(\sigma_h, M) = \left( \sum_{T \in M} \tilde{\eta}_h(\sigma_h, T) \right)^{\frac{1}{2}}, \quad M \subseteq T_h.
\]
In the procedure REFINE, we use the newest vertex bisection or quad-refinement with bisection closure in \(\mathbb{R}^2\) to ensure the shape regularity of \(T\), see, e.g., [37, 6, 41].

| AMFEM | Input an initial mesh \(T_{h_0}\), and \(\theta \in (0, 1)\). Set \(\ell = 0\). |
| SOLVE | Solve (1.3) on \(T_{h_\ell}\) to obtain the finite element solution \((\sigma_{h_\ell}, u_{h_\ell})\). |
| ESTIMATE | Compute error indicators \(\{\tilde{\eta}_{h_\ell}(\sigma_{h_\ell}, T)\}_{T \in T_{h_\ell}}\). |
| MARK | Select a subset \(M_\ell\) of \(T_{h_\ell}\) with \(\tilde{\eta}_{h_\ell}(\sigma_{h_\ell}, M_\ell) \geq \theta \tilde{\eta}_{h_\ell}\). |
| REFINE | Refine all elements in \(M_\ell\) and necessary neighboring elements to get a conforming mesh \(T_{h_{\ell+1}}\). Set \(\ell = \ell + 1\). Go to SOLVE. |

In the procedure ESTIMATE, the actual estimator is \((\eta_{h_\ell}^2 + \text{osc}_{h_{\ell+1}}^2)^{\frac{1}{2}}\) instead of \(\eta_{h_{\ell+1}}\). Due to this strategy, an extra marking step for data oscillation can be avoided, see, e.g., [32, 27]. Since the data oscillation \(\text{osc}_{h_{\ell}}\) is completely local, its behavior can be easily described by the following lemma, see, e.g., Lemma 5.2 in [27].

**Lemma 4.1.** For \(\ell \geq 0\), let \(R_\ell\) denote the collection of refinement elements from \(T_{h_\ell}\) to \(T_{h_{\ell+1}}\). It holds that
\[
\text{osc}_{h_{\ell+1}}^2 \leq \text{osc}_{h_\ell}^2 - \frac{1}{2} \text{osc}_{h_\ell}^2(f, R_\ell).
\]

The estimator reduction is a standard ingredient in the convergence analysis of AFEMs, see [15]. Since \(\tilde{\eta}_h\) involves data oscillation, readers are referred to Lemma 5.1 in [27] for a detailed proof. The only new ingredient in the proof of Lemma 4.2 is the robust inequality \(\|Ar\| \leq \|r\|_A\).

**Lemma 4.2.** There exists a constant \(\gamma \in (0, 1)\) and \(C_{\text{re}} > 0\) depending only on \(\mu, \Omega, C_{\text{shape}}, \theta\) such that
\[
\tilde{\eta}_{h_{\ell+1}}^2 \leq \gamma \tilde{\eta}_{h_\ell}^2 + C_{\text{re}} \|\sigma_{h_\ell} - \sigma_{h_{\ell+1}}\|_A^2.
\]
For convenience, let
\[
e_\ell = \|\sigma - \sigma_{h_\ell}\|_A, \quad E_\ell = \|\sigma_{h_\ell} - \sigma_{h_{\ell+1}}\|_A.
\]
The next theorem gives the contraction property of AMFEM, which is an important ingredient for proving quasi-optimal convergence rate.

**Theorem 4.3** (contraction of AMFEM). There exists constants \(\nu, \alpha \in (0, 1)\) depending only on \(\theta, \mu, \Omega, C_{\text{shape}}\) such that
\[
(1 - \nu)e_{\ell+1}^2 + 2C_\nu \text{osc}_{h_{\ell+1}}^2 + C_{\text{re}}^{-1} \tilde{\eta}_{h_{\ell+1}}^2 \leq \alpha \left( (1 - \nu)e_\ell^2 + 2C_\nu \text{osc}_{h_\ell}^2 + C_{\text{re}}^{-1} \tilde{\eta}_{h_\ell}^2 \right).
\]
Proof. A combination of Theorem 3.4 and Lemma 4.1 shows that

\begin{equation}
(1 - \nu)\varepsilon_{\ell+1}^2 + 2C_\nu \text{osc}_{h_{\ell+1}}^2 \leq \varepsilon_{\ell}^2 + 2C_\nu \text{osc}_{h_{\ell}}^2 - E_{h_{\ell}}^2.
\end{equation}

On the other hand, the reliability (3.15) gives

\begin{equation}
\varepsilon_{\ell}^2 \leq C_{\text{rel}}\eta_{h_{\ell}}, \quad \text{osc}_{h_{\ell}}^2 \leq \eta_{h_{\ell}}^2.
\end{equation}

Let \( \alpha \in (0, 1) \) be a constant. Using (4.1) and Lemma 4.2, we have

\begin{align*}
(1 - \nu)\varepsilon_{\ell+1}^2 + 2C_\nu \text{osc}_{h_{\ell+1}}^2 + C_{\text{re}}^{-1}\eta_{h_{\ell+1}}^2 &\leq \varepsilon_{\ell}^2 + 2C_\nu \text{osc}_{h_{\ell}}^2 + C_{\text{re}}^{-1}\gamma\eta_{h_{\ell}}^2 \\
&\leq \alpha(1 - \nu)\varepsilon_{\ell}^2 + 2\alpha C_\nu \text{osc}_{h_{\ell}}^2 + (1 - \alpha(1 - \nu))\varepsilon_{\ell}^2 + 2(1 - \alpha)C_\nu \text{osc}_{h_{\ell}}^2 + C_{\text{re}}^{-1}\gamma\eta_{h_{\ell}}^2.
\end{align*}

Combining it with (4.2) yields

\begin{equation}
(1 - \nu)\varepsilon_{\ell+1}^2 + 2C_\nu \text{osc}_{h_{\ell+1}}^2 + C_{\text{re}}^{-1}\eta_{h_{\ell+1}}^2 \leq \alpha(1 - \nu)\varepsilon_{\ell}^2 + 2\alpha C_\nu \text{osc}_{h_{\ell}}^2 \\
+ \{(1 - \alpha(1 - \nu))C_{\text{rel}} + 2(1 - \alpha)C_\nu + C_{\text{re}}^{-1}\gamma\}\eta_{h_{\ell}}^2.
\end{equation}

Let \((1 - \alpha(1 - \nu))C_{\text{rel}} + 2(1 - \alpha)C_\nu + C_{\text{re}}^{-1}\gamma = \alpha C_{\text{re}}^{-1}\), i.e.,

\begin{equation}
\alpha = \frac{C_{\text{rel}} + 2C_\nu + C_{\text{re}}^{-1}\gamma}{(1 - \nu)C_{\text{rel}} + 2C_\nu + C_{\text{re}}^{-1}}.
\end{equation}

Clearly \( \alpha < 1 \) provided \( 0 < \nu < \frac{1 - \gamma}{C_{\text{re}}C_{\text{rel}}} \). Then the contraction follows from (4.3).

Once the contraction is available, the proof of quasi-optimal convergence rate of AMFEM follows with a standard argument, see, e.g., [15, 40]. For simplicity, we assume \( \lambda, \mu \) are piecewise constants aligned with the initial mesh \( T_{h_0} \). In this case, the efficiency of \( \eta_{h_{\ell}} \) follows with the same bubble function technique in [19, 12]:

\begin{equation}
C_{\text{eff}}\eta_{h_{\ell}}^2(\sigma_h) \leq ||\sigma - \sigma_h||_A^2 + \text{osc}_{h_{\ell}}^2(f),
\end{equation}

where the constant \( C_{\text{eff}} > 0 \) depends only on \( \mu, \Omega, C_{\text{shape}}, \) and \( r \). Another ingredient of optimality proof is the following cardinality estimate

\begin{equation}
\#T_{h_{\ell}} - \#T_{h_0} \lesssim \sum_{j=0}^{\ell-1} M_j.
\end{equation}

It has been shown in [8] that (4.5) holds provided the newest vertex bisection (NVB) is used in the procedure REFINE and the newest vertices on the initial mesh \( T_{h_0} \) are suitably chosen. In fact, \( \{T_{h_\ell}\}_{\ell \geq 0} \) generated by a modified NVB satisfies (4.5) even with arbitrary choice of the initial newest vertices, see [33]. The second ingredient is the minimal refinement assumption [40]:

\begin{equation}
\text{Procedure MARK selects a subset } M_{\ell} \text{ with minimal cardinality.}
\end{equation}

In addition, the marking parameter \( \theta \) is required to be below the threshold

\[ \theta_\star = \min \left(1, \frac{C_{\text{eff}}}{3C_{\text{drel}}} \right)^{\frac{1}{2}}, \]

which can be derived in the next lemma.
LEMMA 4.4. (optimal marking) Let $\mathcal{T}_h, \mathcal{T}_H \in \mathbb{T}$ with $\mathcal{T}_H \leq \mathcal{T}_h$. Set $\mu = \frac{1}{2}(1 - \theta^2)$. If

$$\|\sigma - \sigma_h\|^2 + \text{osc}_h^2(f) \leq \mu \left\{ \|\sigma - \sigma_H\|^2 + \text{osc}_H^2(f) \right\}.$$  \hspace{1cm} (4.7)

Then the set $\bar{R}_H$ in Proposition 2.2 verifies the Dörfler marking property

$$\bar{\eta}_H(\sigma_h, \bar{R}_H) \geq \bar{\theta}\bar{\eta}_H.$$  \hspace{1cm} (4.8)

Proof. Using (4.4) and (4.7), we have

$$\nu \leq \mu \nu.$$  \hspace{1cm} (4.9)

In the last step, we use the obvious inequality

$$\text{osc}_H^2(f) - 2 \text{osc}_h^2(f) \leq \text{osc}_H^2(f) - \text{osc}_h^2(f) \leq \text{osc}_H^2(f, \mathcal{R}_H).$$

It then follows from (4.8) and Theorem 3.3 that

$$(1 - 2\mu)\nu \leq 3\nu.$$  \hspace{1cm} (4.10)

The proof is then complete by $\theta^2 \leq \frac{\nu}{\nu}$. \hspace{1cm} \hfill \Box

Under these assumptions, the convergence rate of AMFEM can be characterized by the nonlinear approximation property of $\mathcal{T}$. Recall that $\mathcal{T}$ is the collection of subtriangulations of $\mathcal{T}_h$ created by (modified) NVB. For $s > 0$, define the semi-norms

$$|\sigma|_s = \sup_{N > 0} \left\{ \min_{h \in \mathcal{H}} \min_{T_h \in \mathcal{T}_h, \#T_h \leq N} \|\sigma - \tau_h\|_A \right\},$$

$$|f|_s = \sup_{N > 0} \left\{ \min_{h \in \mathcal{H}} \min_{T_h \in \mathcal{T}_h, \#T_h \leq N} \text{osc}_h(f) \right\}.$$  \hspace{1cm} (4.11)

One can also define the coupled approximation semi-norm

$$\|(\sigma, f)\|_s := \sup_{N > 0} \left\{ \min_{h \in \mathcal{H}} \min_{T_h \in \mathcal{T}_h, \#T_h \leq N} \left( \|\sigma - \tau_h\|_A + \text{osc}_h(f) \right)^2 \right\}.$$  \hspace{1cm} (4.12)

Since $\lambda, \mu$ are constants, we have the following equivalence

$$|\sigma|_s < \infty \Leftrightarrow |\sigma|_s + |f|_s < \infty$$

as argued in [15], Lemma 5.3. The quasi-optimal convergence rate of AMFEM follows from the contraction and previous assumptions, see, e.g., [15], Lemma 5.10 and Theorems 5.1 for details.

THEOREM 4.5 (quasi-optimality). Let $\{ (\sigma_{h_i}, u_{h_i}, \mathcal{T}_{h_i}) \}_{i \geq 0}$ be a sequence of finite element solutions and meshes generated by AMFEM. Let $\lambda, \mu$ be piecewise constants aligned with $\mathcal{T}_{h_i}$. Assume $|\sigma|_s + |f|_s < \infty, \theta \in (0, \theta), \alpha \in (0, \alpha)$, and (4.6), (4.5) hold. There exists a constant $C_\text{opt}$ depending only on $\theta, \alpha, \mu, \Omega$, and $C_{\text{shape}}$, such that

$$\left( \|\sigma - \sigma_{h_{\ell}}\|^2_A + \text{osc}_{h_{\ell}}^2 \right)^{\frac{1}{2}} \leq C_\text{opt} \left( |\sigma|_s + |f|_s \right)^{\#T_{\ell} - \#T_0}.$$
5. Local interpolation and discrete approximation. In this section, we give proofs of Proposition 2.2 and Lemma 3.1. The $L^2$-bounded regularized interpolation in [24] does not satisfy the property (2.5). For our purpose, an $H^2$-bounded interpolation is enough. Hence we will not regularize the degree of freedom based on point evaluation.

Proof of Proposition 2.2. Let $x \in \mathcal{N}_H$ and $e \in \mathcal{E}_H$ be an edge containing $x$. For $w_H \in W_H$, we say $\partial_e \partial_e w_H(x) = \partial_e^2 w_H(x)$ is a second edge derivative at $x$, where $\partial_e$ is the directional derivative along $t_e$. Let $e'$ be another edge having $x$, and $e, e'$ are two edges of $T \in \mathcal{T}_H$. $\partial_e \partial_{e'} (w_H | T) (x)$ is called a cross derivative at $x$. We say $x$ is a singular vertex if all edges meeting at $x$ fall on two straight lines. The nodal variables or global degrees of freedom of $W_H$ given in [38] are described as follows.

1. the value of $v$ and $\nabla v$ at each vertex $x \in \mathcal{N}_H$;
2. the edge normal derivative at $r+1$ distinct interior points of each edge $e \in \mathcal{E}_H$;
3. the value at $r$ distinct interior points of each edge $e \in \mathcal{E}_H$;
4. the value at $r(r+1)/2$ distinct interior points of each triangle $T \in \mathcal{T}_H$, chosen to uniquely determine a polynomial in $\mathcal{P}_{r-1}(T)$;
5. one cross derivative at each vertex $x \in \mathcal{N}_H$;
6. at each vertex $x \in \mathcal{N}_H$, the second edge derivative for all the edges meeting there, with one exception: if the vertex is an interior nonsingular vertex, one of the second edge derivatives is omitted, where the omitted edge is chosen so that its two adjacent edges are not collinear.

Let $\{ w \rightarrow D_i w(a_i) \}_{i=1}^N$ denote the collection of the above nodal variables, where $a_i$ is a vertex or an interior point of an edge/triangle in $\mathcal{T}_H$, $D_i$ is a differential operator of order $|D_i| = 0$ (point evaluation), 1, or 2. $\{ a_i \}_{i=1}^N$ are not distinct since a node may be associated with multiple differential operators.

If $a_i \in \mathcal{N}_H$ and $D_i = \partial_{c_1} \partial_{c_2}$ is a second edge derivative or cross derivative, let $\kappa_i = c_1 \supset a_i$. If $a_i \in \mathcal{N}_H$ and $D_i = \partial_{c_1}$ or $\partial_{c_2}$, let $\kappa_i \supset a_i$ be any edge in $\mathcal{E}_H$. If $a_i$ is an interior point of $e \in \mathcal{E}_H$, we choose $\kappa_i = e$. By Riesz’s representation theorem, there exists a polynomial $\psi_i \in \mathcal{P}_{r+5}(\kappa_i)$, such that

$$ \int_{\kappa_i} w \psi_i b_i d\mu = w(a_i) \text{ for all } w \in \mathcal{P}_{r+5}(\kappa_i), $$

where $b_i$ is the edge bubble polynomial of unit size vanishing on $\partial \kappa_i$, $d\mu$ is the Lebesgue measure on $\kappa_i$. When $D_i = \partial_{c_1} \partial_{c_2}$ is a second edge derivative or cross derivative, using (5.1) and the integration-by-parts formula on $\kappa_i$, we have

$$ D_i w(a_i) = \int_{\kappa_i} \partial_{c_1} \partial_{c_2} w(a_i) \psi_i b_i d\mu $$
$$ = -\int_{\kappa_i} \partial_{c_2} w(a_i) \partial_{c_1} (\psi_i b_i) d\mu \text{ for all } w \in \mathcal{P}_{r+5}(\kappa_i). $$

Let $\{ \phi_i \}_{i=1}^N$ be the basis dual to the unisolvent set $\{ w \rightarrow D_i w(a_i) \}_{i=1}^N$. For $w_h \in W_h$, we define the interpolant $I_H w_h$ as

$$ I_H w_h = \sum_{|D_i| = 0} w_h(a_i) \phi_i + \sum_{|D_i| = 1} \left( \int_{\kappa_i} (D_i w_h) \psi_i b_i d\mu \right) \phi_i $$
$$ - \sum_{|D_i| = 2} \left( \int_{\kappa_i} \partial_{c_2} w_h(a_i) \partial_{c_1} (\psi_i b_i) d\mu \right) \phi_i. $$
The definition of $I_H$ clearly implies (2.5). For $T \in \mathcal{T}_H \setminus \mathcal{R}_H$, the choice of $\kappa_i$ implies $w_h|_{\kappa_i} \in P_{r+5}(\kappa_i), \nabla w_h|_{\kappa_i} \in P_{r+4}(\kappa_i)$. Using this fact and (5.1), (5.2), we obtain
\begin{equation}
D_i(I_H w_h) = D_i w_h \text{ for all } a_i \subset T.
\end{equation}
Due to (5.3) and the unisolvence of $\{D_i\}_{a_i \in T}$ (see the proof in [38]), we have $(w_h - I_H w_h)|_T = 0$ and thus verify the property (2.6). The inequality (2.7) directly follows from the same proof of Theorem 7.3 in [24] together with a trace inequality.

The rest of this section is devoted to the proof of Lemma 3.1. First we present a modified Korn’s inequality on each local triangle $T \in \mathcal{T}_h$.

**Lemma 5.1.** Given $T_h \in \mathcal{T}, T \in T_h$ and $v \in H^1(T, \mathbb{R}^2)$, it holds that
\begin{equation}
|v|_{H^1(T)} \leq C_{\text{Korn}}(\|\varepsilon(v)\|_T + h_T^{-1}(\|Q_T v\|_T)),
\end{equation}
where $Q_T v$ is the $L^2$-projection of $v$ onto $\mathcal{RM}(T), C_{\text{Korn}}$ is a constant depending only on $C_{\text{shape}}$ and $\Omega$.

**Proof.** Using the standard compactness argument, see, e.g., Theorem 11.2.16 in [10], we have
\begin{equation}
\|v\|_{H^1(T)} \leq C_T(\|\varepsilon(v)\|_T + \|Q_T v\|_T),
\end{equation}
where $C_T$ is a constant depending on $T$. It remains to estimate $C_T$ by a homogeneity argument. Consider a reference triangle $K$ and the affine mapping $F_T : K \rightarrow T$ given by $F_T(x) = B_T x + b_T$. Define
\begin{equation}
\Phi(B_T) = \sup_{v \in H^1(K, \mathbb{R}^2), \|v\|_{H^1(K)} = 1} \Phi_v(B_T),
\end{equation}
where
\begin{equation}
\Phi_v(B_T) = \frac{|v \circ F_T^{-1}|_{H^1(T)}}{\|\varepsilon(v \circ F_T^{-1})\|_T + h_T^{-1}(\|Q_T (v \circ F_T^{-1})\|_T)}.
\end{equation}
Note that $\Phi_v$ is independent of $b_T$. Due to (5.4), the function $\Phi$ is well-defined. It is straightforward to check that
\begin{equation}
\{\Phi_v(\cdot)\}_{v \in H^1(K, \mathbb{R}^2), \|v\|_{H^1(K)} = 1}
\end{equation}
is a family of equicontinuous functions on $GL(2, \mathbb{R})$. Hence $\Phi$ defined by taking the supremum of this family must be continuous on $GL(2, \mathbb{R})$. Let $\hat{T} = \{h_T^{-1} x : x \in T\}$ be the scaled triangle of unit size. Since $\mathcal{T}_h$ is shape regular, $\{B_{\hat{T}} : T \in \mathcal{T}_h\}$ is contained in a compact subset of $GL(2, \mathbb{R})$, see, e.g., [10]. Combining the continuity and compactness, we obtain
\begin{equation}
\sup_{T \in \mathcal{T}_h} \Phi(B_{\hat{T}}) = C_{\sup} < \infty,
\end{equation}
where $C_{\sup}$ depends on the shape regularity of $\mathcal{T}_h$. Therefore using a scaling transformation and (5.5), we obtain
\begin{equation}
\Phi(B_T) = \sup_{v \in H^1(K, \mathbb{R}^2), \|v\|_{H^1(K)} = 1} \frac{|v \circ F_T^{-1}|_{H^1(T)}}{\|\varepsilon(v \circ F_T^{-1})\|_T + \|Q_T (v \circ F_T^{-1})\|_T} \lesssim \Phi(B_{\hat{T}}) \leq C_{\sup}.
\end{equation}
The proof is complete. \qed
Then we present a modified discrete Korn’s inequality on a triangle $T$.

**Lemma 5.2.** Let $\mathcal{T}_h, \mathcal{T}_H \in \mathbb{T}$ with $\mathcal{T}_H \leq \mathcal{T}_h$. For any $T \in \mathcal{T}_H$, let $\mathcal{T}_h(T) = \{ T' \in \mathcal{T}_h : T' \subset T \}$ and $\mathcal{E}_h(T) = \{ e \in \mathcal{E}_h : e \subset T, e \not\subseteq \partial T \}$. Then for $v_h \in V_h|_T$, it holds that

$$h_T^{-2} \| v_h - Q_T v_h \|^2_T \lesssim \sum_{T' \in \mathcal{T}_h(T)} \| \varepsilon(v_h) \|^2_{T'}, + \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} \| [v_h] \|^2_e,$$

where $h_e$ is the length of $e$, $\| \cdot \|_e$ denotes the $L^2$-norm on $e$.

*Proof.* Let $w = v_h - Q_T v_h$ and $|w|^2_{H^1(T)} := \sum_{T' \in \mathcal{T}_h(T)} |w|^2_{H^1(T')}$. Let $\mathcal{V}_h(T) = \{ \tilde{v} \in C^0(T) : \tilde{v}|_{T'} \in P_{r+2}(T') \mbox{ for } T' \in \mathcal{T}_h(T) \}$ be the usual Lagrange element space of degree $r + 2$. Following the analysis in [9, 10, 32], we construct a continuous piecewise polynomial function $Ew \in \mathcal{V}_h(T)$ by setting the nodal value as

$$Ew(x) = \frac{1}{\# \omega_{h,x}} \sum_{T' \in \omega_{h,x}} (w|_{T'})(x),$$

where $x$ is a Lagrange node for the space $\mathcal{V}_h(T)$ and $\omega_{h,x} = \{ T' \in \mathcal{T}_h(T) : x \in T' \}$. An elementary estimate shows that

$$h_T^{-2} \| w - Ew \|^2_T + |w - Ew|^2_{H^1(T)} \lesssim \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} \| [w] \|^2_e,$$

(6.6)

see, e.g., the proof of Lemma 10.6.6 in [10] and Lemma 2.8 in [32]. For the continuous function $Ew$, the Poincaré inequality implies

$$\| Ew - Q_T Ew \|^2_T \lesssim h_T \| Ew \|_{H^1(T)},$$

(5.7)

Using (6.6), (5.7), the triangle inequality, and $Q_T w = 0$, we have

$$\| w \|^2_T \lesssim \| w - Ew \|^2_T + \| Q_T (w - Ew) \|^2_T + \| Ew - Q_T Ew \|^2_T$$

$$\lesssim h_T^2 \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} \| [w] \|^2_e + h_T^2 \| Ew \|^2_{H^1(T)},$$

(5.8)

It remains to estimate $\| Ew \|_{H^1(T)}$. Lemma 5.1 implies

$$\| Ew \|^2_{H^1(T)} \lesssim \| \varepsilon(Ew) \|^2_T + h_T^{-2} \| Q_T Ew \|^2_T.$$

(5.9)

It then follows from the triangle inequality, (5.9), $\varepsilon(Q_T v_h) = 0$, and $Q_T (w) = 0$ that

$$\| Ew \|^2_{H^1(T)} \lesssim \sum_{T' \in \mathcal{T}_h(T)} \left( \| \varepsilon(v_h) \|^2_{T'} + \| \varepsilon(w - Ew) \|^2_{T'} + h_T^{-2} \| Q_T (Ew - w) \|^2_T \right)$$

$$\leq \sum_{T' \in \mathcal{T}_h(T)} \| \varepsilon(v_h) \|^2_{T'} + \| w - Ew \|^2_{H^1(T')} + h_T^{-2} \| w - Ew \|^2_T.$$ 

(5.10)

Combining (5.8), (5.10), (5.6), and $\| w \|_e = \| v_h \|_e$ completes the proof.}

For $\tau_h \in \Sigma_h$ and $v_h \in V_h$, the mesh-dependent norms are defined by

$$\| \tau_h \|_{0,h} := \left( \| \tau_h \|^2 + \sum_{e \in \mathcal{E}_h} h_e \| \tau_h n_e \|^2_e \right)^{1/2},$$

$$| v_h |_{1,h} := \left( \sum_{T \in \mathcal{T}_h} \| \varepsilon(v_h) \|^2_T + \sum_{e \in \mathcal{E}_h} h_e^{-1} \| [v_h] \|^2_e \right)^{1/2}.$$


It has been shown in [18] that the following discrete inf-sup condition holds:

\[
|v_h|_{1,h} \lesssim \sup_{\tau_h \in \Sigma_h} \frac{\langle \text{div } \tau_h, v_h \rangle}{\| \tau_h \|_0,h} \quad \text{for all } v_h \in V_h.
\]  

(5.11)

With the above preparation, we are able to prove Lemma 3.1.

**Proof of Lemma 3.1.** Using the inf-sup condition (5.11) and the inclusion \( \Sigma_h^{\text{HZ}} \subset \Sigma_h \), we obtain

\[
|v_h|_{1,h} \lesssim \sup_{\tau_h \in \Sigma_h} \frac{\langle \text{div } \tau_h, v_h \rangle}{\| \tau_h \|_0,h} = \sup_{\tau_h \in \Sigma_h} \frac{\langle A\tau_h, \varepsilon^h_A(v_h) \rangle}{\| \tau_h \|_0,h}.
\]

(5.12)

It then follows from (5.12) and \( \| \tau_h \|_A \lesssim \| \tau_h \| \lesssim \| \tau_h \|_0,h \) that

\[
|v_h|_{1,h} \lesssim \| \varepsilon^h_A(v_h) \|_A.
\]

(5.13)

Combining it with Lemma 5.2, we have

\[
\sum_{T \in T_h} h_T^{-2} \| v_h - Q_H v_h \|^2_T \lesssim \sum_{T \in T_h} \left( \sum_{T' \in T_h(T)} \| \varepsilon(v_h) \|^2_{T'} + \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} \| v_h \|_e^2 \right) \lesssim \| v_h \|^2_{1,h} \lesssim \| \varepsilon^h_A(v_h) \|_A^2,
\]

which completes the proof.

6. Implementation and numerical experiment. The method (1.3) can be implemented using the hybridization technique. Let \( \mathcal{E}_h^i \) denote the set of interior edges in \( T_h \) and \( L^2(\mathcal{E}_h^i, \mathbb{R}^2) = \{ \mu : \mu|_e \in L^2(e, \mathbb{R}^2) \text{ for all } e \in \mathcal{E}_h^i \} \). Consider the multiplier space

\[ M_h = \{ \mu_h \in L^2(\mathcal{E}_h^i, \mathbb{R}^2) : \mu_h|_e \in \mathcal{P}_{r+3}(e, \mathbb{R}^2) \text{ for all } e \in \mathcal{E}_h^i \}, \]

and the broken discrete stress space

\[ \Sigma_h^{-1} = \{ \tau_h \in L^2(\Omega, \mathbb{S}) : \tau_h|_T \in \mathcal{P}_{r+3}(T) \text{ for all } T \in T_h \}. \]

The hybridized mixed method seeks \( (\tilde{\sigma}_h, \tilde{u}_h, \lambda_h) \in \Sigma_h^{-1} \times V_h \times M_h \) such that

\[
\langle A\tilde{\sigma}_h, \tau_h \rangle + \sum_{T \in T_h} \langle \text{div } \tau_h, \tilde{u}_h \rangle_T + \sum_{e \in \mathcal{E}_h^i} \int_e \lambda_e \cdot [\tau_h]|_e n_e ds = 0, \quad \tau \in \Sigma_h^{-1},
\]

(6.1)

\[
\sum_{T \in T_h} \langle \text{div } \tilde{\sigma}_h, v_h \rangle_T = \langle f, v_h \rangle, \quad v_h \in V_h,
\]

\[
\sum_{e \in \mathcal{E}_h^i} \int_e \mu_h \cdot [\tilde{\sigma}_h]|_e n_e ds = 0, \quad \mu_h \in M_h.
\]

In fact, (6.1) is a hybridized version of (1.3), i.e., \( \tilde{\sigma}_h = \sigma_h, \tilde{u}_h = u_h \), see, e.g., [25, 2]. In matrix notation, (6.1) reads

\[
\begin{pmatrix}
A & B \\
B^T & O
\end{pmatrix}
\begin{pmatrix}
X \\
\Lambda
\end{pmatrix}
= \begin{pmatrix}
F \\
O
\end{pmatrix},
\]

where \( O \) is a zero matrix or vector, \( X \) and \( \Lambda \) are vectors corresponding to the coordinates of \( (\tilde{\sigma}_h, u_h) \) and \( \lambda_h \), respectively. Since \( \Sigma_h^{-1} \) and \( V_h \) are both broken finite element
space without any continuity constraint, the matrix $A$ is block diagonal and easily invertible. Hence solving (6.2) is equivalent to solving the smaller Schur complement system

\[(6.3) \quad B^T A^{-1} B \Lambda = B^T A^{-1} F.\]

Here $B^T A^{-1} B$ is sparse and positive semi-definite. (6.3) may be singular at the presence of singular vertices in $T_h$. However, (6.3) can still be efficiently solved by Krylov subspace iterative methods like the preconditioned conjugate gradient method, see [25] for a detailed discussion and optimal iterative solvers for (6.3).

In the experiment, let $\Omega = [-1, 1]^2 \setminus ([0, 1] \times [-1, 0])$ be the L-shaped domain. Let $(r, \theta)$ be the polar coordinate with respect to the origin, where $0 \leq \theta \leq \omega = \frac{3\pi}{2}$. Let $\Phi_1(\theta) = \left( ((z + 2)(\lambda + \mu) + 4\mu) \sin(z\theta) - z(\lambda + \mu) \sin((z - 2)\theta) \right)$, $\Phi_2(\theta) = \left( z(\lambda + \mu)\cos(z\theta) - \cos((z - 2)\theta) \right)$, and

\[
\Phi(\theta) = \{ z(\lambda + \mu) \sin((z - 2)\omega) + ((2 - z)(\lambda + \mu) + 4\mu) \sin(z\omega) \} \Phi_1(\theta) - z(\lambda + \mu)\cos((z - 2)\omega) - \cos(z\omega)) \Phi_2(\theta).
\]

where $z \in (0, 1)$ is a root of $(\lambda + 3\mu)^2 \sin^2(z\omega) = (\lambda + \mu)^2 z^2 \sin^2(\omega)$. The most singular part of the solution to (1.1) behaves like $r^2 \Phi(\theta)$ in the neighborhood of $(0, 0)$, see, e.g., [26]. Therefore we choose

\[
u(r, \theta) = \frac{1}{(\lambda + \mu)^2} \left( x_2^2 - 1 \right) \left( x_1^2 - 1 \right) r^2 \Phi(\theta)
\]

as the exact solution in the test problem. The Lamé constants are $\lambda = 10000$ and $\mu = 1$. The method (1.3) or (6.1) is implemented using the package iFEM [16] in Matlab 2019a. We start with the initial mesh in Figure 6.1 and set the marking parameter $\theta = 0.3$. The algebraic system (6.3) is solved by the conjugate gradient method preconditioned by the incomplete Cholesky decomposition. Numerical results are presented in Figure 6.2, where $nt$ denotes the number of triangles.

It can be observed from Figure 6.1(right) that the adaptive algorithm AMFEM captures the corner singularity. Figure 6.2 shows that AMFEM has optimal and robust rate of convergence with respect to very large Lamé constant $\lambda$ starting from coarse initial grid, which validates our convergence and optimality analysis.

**7. Concluding remarks.** In this paper, we developed a robust and optimally convergent adaptive hybridized mixed finite element method for linear elasticity in $\mathbb{R}^2$. With slight modifications, our results can be adapted to the inhomogeneous Dirichlet, Neumann or mixed boundary condition. However, the analysis cannot be directly extended to 3-dimensional elasticity since the discrete elasticity complex for the hybridized mixed element is not clear and possibly very complicated in $\mathbb{R}^3$.

**Acknowledgements.** The author would like to thank Dr. Shihua Gong for generously sharing his Matlab code and comments on iterative methods.

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Fig. 6.2. Error curve

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