Exact solutions of supersymmetric Burgers equation with Bosonization procedure

Abstract: Using bosonization approach, the $\mathcal{N} = 1$ supersymmetric Burgers (SB) system is changed to a system of coupled bosonic equations. The difficulties caused by intractable anticommuting fermionic fields can be effectively avoided with the approach. By solving the coupled bosonic equations, the traveling wave solutions of the SB system can be obtained with the mapping and deformation method. Besides, the richness of the localized excitations of the supersymmetric integrable system is discovered. In the meanwhile, the similarity reduction solutions of the SB system are also studied with the Lie point symmetries theory.

Keywords: Supersymmetric Burgers equation, Bosonization approach, Symmetry reduction, Exact solution

MSC: 35Q51, 70G65

1 Introduction

Super extensions of classical integrable systems lead to super integrable systems with introduction of Grassmann variables and these have been extensive developed in the past few years. Super systems provide more prolific subjects for mathematical and physical researchers [1, 2]. In such way, a number of well known mathematical physical equations have been generalized into the supersymmetric analogues, such as supersymmetric versions of sine-Gordon, Korteweg-de Vries (KdV), Burgers, Kadomtsev-Petviashvili (KP) hierarchy, Boussinesq, modified Korteweg-de Vries (mKdV), etc [3–9]. Though the supersymmetric integrable systems have been studied by many authors in the both quantum and classical levels, many important problems are still open [10]. For instance, how to find a proper bosonization procedure is an important problem for quantum and classical supersymmetric integrable models [11]. Recently, a simple bosonization approach to treat the super integrable systems is proposed [12, 13]. One essential advantage of the method is that it can effectively avoid difficulties caused by intractable fermionic fields which are anticommuting [12–14].

On the other hand, solving nonlinear problems plays an important role in nonlinear sciences. Many effective methods have been proposed to obtain explicit exact solutions. Especially, the Lie group technique is an important method in the nonlinear system fields. The classical Lie group approach [15], the non-classical Lie group approach [16] and the Clarkson-Kruskal (CK) direct method [17] are powerful and effective methods to obtain the explicit exact solutions [18–20].

In this paper, we will use the bosonization approach in the $\mathcal{N} = 1$ supersymmetric Burgers (SB) system. Then, the exact solutions of the usual pure bosonic systems are obtained with the mapping and deformation method and Lie point symmetries theory.
As we know, the Burgers equation is

$$u_t + uu_x + u_{xx} = 0,$$

which was first proposed by Burgers as a one dimensional turbulence, acoustics and hydrodynamics model [21]. This equation is also applied to other phenomenon such as physics, chemistry, mathematical biology, etc. In view of the fractional calculus theory, the fractional Burgers equation was proposed and several properties were discovered [22–25]. The $\mathcal{N} = 1$ supersymmetric version is established with usual temporal variable $t$ and super spatial variables $(x, \theta)$, where $\theta$ is a Grassmann variable satisfying $\theta^2 = 0$, and the field $u$ leads to a fermionic superfield

$$\Phi(\theta, x, t) = \xi(x, t) + \theta u(x, t).$$

Then, we get a nontrivial fermionic extension result

$$\Phi_t + \Phi D \Phi_x + \Phi_{xx} = 0,$$

where $D = \partial_\theta + \theta \partial_x$ is the covariant derivative. The component version of (3) reads [5, 6]

$$u_t + uu_x + u_{xx} + \xi_{xx} = 0,$$

where $u$ and $\xi$ are bosonic and fermionic component fields respectively. The usual classical Burgers equation remains while the fermionic variable is absent.

The structure of this paper is organized as follows. In Sections 2, we use the bosonization approach on the SB equation with introducing two fermionic parameters. The exact solutions of the model are found by using the mapping and deformation method and Lie point symmetries theory. In addition, some special types of nontraveling wave solutions are also found by the exact solutions of the Burgers equation and its symmetries. In Sections 3, we extend the bosonization approach on the SB system to the case of $N$ fermionic parameters. At the same time, the exact solutions are obtained with the same methods. The last section is a simple summary and discussion.

2 **Two fermionic parameters bosionization and its solutions**

The $\mathcal{N} = 1$ supersymmetric versions of the Burgers equation is an important system not only in mathematics, but also in applications of various areas of modern theoretical physics [6]. Therefore, investigating their properties and searching for their exact solutions are important and interest problem.

2.1 **Bosonization approach with two fermionic parameters**

According to the bosonization approach, we expand the component fields $\xi$ and $u$ as the following form with adding to the two fermionic parameters [12–14]

$$\xi(x, t) = p_1 \xi_1 + p_2 \xi_2,$$

$$u(x, t) = u_0 + u_{12} \xi_1 \xi_2,$$

where $\xi_1$ and $\xi_2$ are two Grassmann parameters, while the coefficients $p_1$, $p_2$, $u_0$ and $u_{12}$ are four usual real or complex functions with respect to the space-time variables $x$ and $t$. Then, substituting (5) into the SB system (4) gets

$$u_{0,t} + u_{0,xx} + u_0 u_{0,x} = 0,$$

$$p_{1,t} + p_{1,xx} + p_1 u_{0,x} = 0,$$

$$p_{2,t} + p_{2,xx} + p_2 u_{0,x} = 0.$$
The way used above is just the bosonic procedure for the SB system (4) with two fermionic parameters. Equation (6a) is exactly the usual Burgers equation which has been widely studied. Equations (6b) and (6c) are linear homogeneous in \( p_1 \) and \( p_2 \) respectively. Equation (6d) is linear nonhomogeneous in \( u_{12} \). Thereby, these equations which are usual pure bosonic systems can be easily solved in principle. This is just one of the advantages of the bosonization approach.

### 2.2 Traveling wave solutions with mapping and deformation method

Introducing the traveling wave variable \( X = kx + \omega t + c_0 \) with constants \( k, \omega \) and \( c_0 \), equation (7) is transformed to the ordinary differential equations (ODEs)

\[
\begin{align*}
k^2 u_{0,XX} + (\omega + ku_0)u_{0,X} &= 0, \\
k^2 p_{1,XX} + \omega p_{1,X} + kp_1 u_{0,X} &= 0, \\
k^2 p_{2,XX} + \omega p_{2,X} + kp_2 u_{0,X} &= 0, \\
k^2 u_{12,XX} + \omega u_{12,X} + k(u_0 u_{12}) &= k^2(p_{2p1,XX} - p_{1p2,XX}).
\end{align*}
\]  

(7d)

**Notation:** The traveling wave in the superspace, \( \Phi(x,t,\phi) = \Phi(kx + \omega t + c_0 + \zeta \theta) \), with Grassmann constant \( \zeta \) are different from those of in the usual spacetime \( \{x,t\} \). Therefore, the traveling waves we discuss are only in the usual spacetime \( \{x,t\} \) but not in the superspace \( \{x,t,\theta\} \).

Due to the well know exact solutions of (7a), we try to build the mapping and deformation relationship between the traveling wave solutions of the classical Burgers equation and the SB equation, then to construct the exact solutions of the SB equation with the known solutions of Burgers equation.

At first, \( u_{0,X} \) reads as the following form by solving (7a)

\[
u_{0,X} = -\frac{k u_0^2 + 2\omega u_0 + 2c_1}{2k^2}.
\]  

(8)

To integrate inhomogeneous equation (7d) once, it becomes

\[
k^2 u_{12,XX} + (\omega + ku_0)u_{12} = k^2(p_{2p1,XX} - p_{1p2,XX}) + c_2.
\]  

(9)

where \( c_1 \) and \( c_2 \) are the integral constants. In order to get the mapping relationship between \( p_1, p_2, u_{12} \) and \( u_0 \), the variable transformations are introduced

\[
p_1(X) = P_1(u_0(X)), \quad p_2(X) = P_2(u_0(X)), \quad u_{12}(X) = U_{12}(u_0(X)).
\]  

(10)

Using the transformation (10) and vanishing \( u_{0,X} \) via (8), the linear ODEs (7b)–(7c) as well as (9) become

\[
\begin{align*}
(ku_0^2 + 2\omega u_0 + 2c_1) \frac{d^2 P_1(u_0)}{du_0^2} + 2ku_0 \frac{dP_1(u_0)}{du_0} - 2kP_1(u_0) &= 0, \\
(ku_0^2 + 2\omega u_0 + 2c_1) \frac{d^2 P_2(u_0)}{du_0^2} + 2ku_0 \frac{dP_2(u_0)}{du_0} - 2kP_2(u_0) &= 0, \\
\left( ku_0^2 + \omega u_0 + c_1 \right) \frac{dU_{12}(u_0)}{du_0} - ku_0 U_{12}(u_0) - \omega U_{12}(u_0) &= F(u_0),
\end{align*}
\]

(11)

where

\[
F(u_0) = \left( \frac{dP_1(u_0)}{du_0} P_2(u_0) - \frac{dP_2(u_0)}{du_0} P_1(u_0) \right) \left( ku_0^2 + \omega u_0 + c_1 \right) + c_2.
\]

The mapping and deformation relations thus are constructed by solving (11)

\[
\begin{align*}
P_1(u_0) &= c_3 u_0 - c_4 (ku_0 + 2\omega)(\exp(iA_1))^A_2, \\
P_2(u_0) &= c_5 u_0 - c_6 (ku_0 + 2\omega)(\exp(iA_1))^A_2.
\end{align*}
\]

(12)
The solution of SB system will be obtained by means of (13).

$$U_{12}(u_0) = \left( ku_0^2 + 2\omega u_0 + 2c_1 \right) \int \frac{2F(y)}{(ky^2 + 2\omega y + 2c_1)^2} dy + c_7. \quad (12c)$$

where $c_i \ (i = 3, 4, ..., 7)$ are arbitrary constants and

$$A_1 = \arctan \left( \frac{\omega + ku_0}{k^2u_0^2 + 2\omega ku_0 + 2c_1k} \right), \quad A_2 = \frac{\omega}{\sqrt{-2c_1k + \omega^2}}.$$

If we know the solution of $u_0$, the solution of $p_1$, $p_2$ and $u_{12}$ will be obtained with the help of (12). The solution of two fermionic parameters SB system writes

$$u = u_0 + \left( ku_0^2 + 2\omega u_0 + 2c_1 \right) \int \frac{2F(y)}{(ky^2 + 2\omega y + 2c_1)^2} dy \zeta_1 \zeta_2, \quad (13a)$$

$$\xi = \zeta_1 [c_3u_0 - c_4(ku_0 + 2\omega)(\exp(iA_1))^A] + \zeta_2 [c_5u_0 - c_6(ku_0 + 2\omega)(\exp(iA_1))^A]. \quad (13b)$$

Here, we list one solution as an example. The solution of $u_0$ can be expressed as the following form by solving (7a)

$$u_0 = \sqrt{2c_8k} \tan \left( \frac{\sqrt{2c_8k}(X + c_9)}{2k} \right) - \frac{\omega}{k}. \quad (14)$$

The solution of SB system will be obtained by means of (13).

Besides, we can get the special exact solution by using (12)

$$P_1(u_0) = c_3u_0, \quad (15a)$$

$$P_2(u_0) = c_5u_0, \quad (15b)$$

$$U_{12}(u_0) = A_3u_{0, x}, \quad (15c)$$

where $A_3 = -2c_7k^2$ and (15c) is due to the relation (8). The expression of $U_{12}$ in (15c) is an ordinary type of the symmetry of the traveling wave (7a). For any given solution $u_0$ of the usual Burgers equation, a certain type of solutions the bosonic equation (7) can be constructed

$$p_1 = c_3u_0, \quad (16a)$$

$$p_2 = c_5u_0, \quad (16b)$$

$$u_{12} = \sigma(u_0), \quad (16c)$$

where $\sigma(u_0)$ represents any symmetry of the usual Burgers equation (6a).

While $p_1$ and $p_2$ describe as the form of (16), $u_0$ can be chosen as any solution of the Burgers equation. The first three equations of the bosonic-looking equations (7) will be satisfied automatically. The right hand side of the nonhomogeneous equation (6d) equals to zero considering with (16a) and (16b). In such circumstance, $u_{12}$ from (6d) exactly satisfies the symmetry equation of the usual Burgers system. It means that we have much freedom to choose $u_0$ so as to construct solutions of the SB equations. Furthermore, it is not restricted to the traveling wave solutions of $u_0$. It is worth mentioning that the Burgers equation possesses infinitely many symmetries, infinitely many solution of $u_{12}$ can thus be generated. All in all, we can construct not only traveling wave solutions but also some novel types of solutions of the SB system by using the solutions and symmetries of the Burgers equation.

### 2.3 Similarity reduction solutions with symmetry reduction approach

In order to get exact solutions of (7) with Lie group theory, we first construct its Lie point symmetries, then give the corresponding symmetry reductions. A symmetry is characterized by an infinitesimal transformation which leaves the given differential equation invariant under the transformation of all independent and dependent variables. We assume the corresponding Lie point symmetry has the following vector form

$$V = X \frac{\partial}{\partial x} + T \frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial u_0} + U_{12} \frac{\partial}{\partial u_{12}} + P_1 \frac{\partial}{\partial p_1} + P_2 \frac{\partial}{\partial p_2}. \quad (17)$$
where \( X, T, U_0, U_{12}, P_1 \) and \( P_2 \) are functions with respect to \( x, t, u_0, u_{12}, p_1 \) and \( p_2 \) which means that the system of (7) is invariant under the following transformations

\[
\{x, t, u_0, u_{12}, p_1, p_2\} \Rightarrow \{x + \epsilon X, t + \epsilon T, u_0 + \epsilon U_0, u_{12} + \epsilon U_{12}, \ldots\},
\]

with an infinitesimal parameter \( \epsilon \). The symmetry can be supposed to have the following form

\[
\sigma_0 = Xu_0, x + Tu_0, t - U_0, \quad \sigma_{12} = Xu_{12}, x + Tu_{12}, t - U_{12}, \ldots,
\]

where “\( \ldots \)” indicate other allowed terms about \( p_1 \) and \( p_2 \). Consider with the notation (19), \( \sigma_k \) is the solution of the linearized equations for (7)

\[
\begin{align*}
\sigma_{0,t} + \sigma_{0,xx} + u_0 \sigma_{0,x} + \sigma_{0u_0,x} &= 0, \\
\sigma_{1,t} + \sigma_{1,xx} + p_1 \sigma_{1,x} + \sigma_{1u_0,x} &= 0, \\
\sigma_{2,t} + \sigma_{2,xx} + p_2 \sigma_{2,x} + \sigma_{2u_0,x} &= 0, \\
\sigma_{12,t} + \sigma_{12,xx} + (u_0 \sigma_{12})_x + (\sigma_{0u_12})_x &= p_2 \sigma_{1,xx} - \sigma_1 p_{2,xx} + \sigma_{21,xx} - p_1 \sigma_{2,xx}.
\end{align*}
\]

Substituting (19) into the symmetry equations (20) with \( u_0, u_{12}, p_1 \) and \( p_2 \) satisfying (7), we obtain the determining equations by identifying all coefficients of derivatives of \( u_0, u_{12}, p_1 \) and \( p_2 \). The solutions of the functions \( X, T, U_0, U_{12}, P_1 \) and \( P_2 \) can be concluded using the determining equations

\[
\begin{align*}
T &= C_1 t + C_2, \\
X &= \frac{C_1 x}{2} + C_3, \\
U_0 &= -\frac{C_1}{2} u_0, \\
P_1 &= C_4 p_1 + C_5 p_2, \\
P_2 &= C_7 p_2 + C_6 p_1, \\
U_{12} &= (C_4 + C_7) u_{12}.
\end{align*}
\]

where \( C_i \ (i = 1, 2, \ldots, 7) \) are arbitrary constants.

Then, one can solve the characteristic equations to obtain similarity solutions

\[
\frac{dx}{X} = \frac{dt}{T}, \quad \frac{du_0}{U_0} = \frac{dt}{T}, \quad \frac{dp_1}{P_1} = \frac{dt}{T}, \quad \frac{dp_2}{P_2} = \frac{dt}{T}, \quad \frac{du_{12}}{U_{12}} = \frac{dt}{T},
\]

where \( X, T, U_0, U_{12}, P_1 \) and \( P_2 \) are given by (21). Two subcases are distinguished concerning the solutions of (7) in the following.

**Case I.** \( C_1 \neq 0 \). We make the transformations \( C_3 \mapsto \frac{1}{2} C_1 C_3, \ C_2 \mapsto C_1 C_2 \) and \( C_5 = C_6 = 0 \) for simplicity, the similarity variable form reads

\[
\xi = \frac{x + C_3}{\sqrt{t + C_2}}
\]

and the similarity solutions for \( u_0, u_{12}, p_1 \) and \( p_2 \) express as the following form using characteristic equations

\[
\begin{align*}
u_0 &= \frac{U_0 (\xi)}{\sqrt{t + C_2}}, \\
p_1 &= P_1 (\xi) (t + C_2)^{\frac{C_1}{2}}, \\
p_2 &= P_2 (\xi) (t + C_2)^{\frac{C_7}{2}}, \\
u_{12} &= U_{12} (\xi) (t + C_2)^{\frac{C_4 + C_7}{C_1}}.
\end{align*}
\]

Substituting (25) into (7), the reduction equations lead to

\[
\begin{align*}
2U_{0,\xi\xi} + (2U_0 - \xi)U_{0,\xi} - U_0 &= 0, \\
2P_{1,\xi\xi} - \xi P_{1,\xi} + \left(2U_{0,\xi} + \frac{2C_4}{C_1}\right)P_1 &= 0, \\
2P_{2,\xi\xi} - \xi P_{2,\xi} + \left(2U_{0,\xi} + \frac{2C_7}{C_1}\right)P_2 &= 0, \\
2U_{12,\xi\xi} - \xi U_{12,\xi} + \left(\frac{2C_4 + 2C_7}{C_1}\right)U_{12} + 2(U_0 U_{12})\xi - 2P_2 P_{1,\xi} + 2P_1 P_{2,\xi} &= 0.
\end{align*}
\]

These reduction equations are linear ODEs while the previous functions are known, we can theoretically solve \( U_0, P_1, P_2 \) and \( U_{12} \) one after another.
Case II. $C_1 = 0$. We can find the general similarity solutions after solving out the characteristic equations while $C_4 = C_5 = C_6 = C_7 = 0$

$$u_0 = U_0(\xi), \quad u_{12} = U_{12}(\xi), \quad p_1 = P_1(\xi), \quad p_2 = P_2(\xi).$$

(26)

with the similarity variable $\xi = t - (C_2/C_3)x$. We redefine the similarity variable as $\xi = x + ct$ with $c$ an arbitrary velocity constant. Substituting (26) into (7), the invariant functions $U_0$, $P_1$, $P_2$ and $U_{12}$ satisfy the following reduction systems

$$U_{0,\xi} + (U_0 + c)U_{0,\xi} = 0, \quad \text{(27a)}$$

$$P_{1,\xi} + cP_{1,\xi} + P_1U_{0,\xi} = 0, \quad \text{(27b)}$$

$$P_{2,\xi} + cP_{2,\xi} + P_2U_{0,\xi} = 0, \quad \text{(27c)}$$

$$U_{12,\xi} + cU_{12,\xi} + (U_{12}U_0)\xi + P_1P_{2,\xi} - P_2P_{1,\xi} = 0. \quad \text{(27d)}$$

The exact solution of (27a) can be expressed by

$$U_0 = -c + \frac{\sqrt{2}}{B_1} \tanh\left(\frac{\xi + B_2}{2B_1}\right). \quad \text{(28)}$$

where $B_1$ and $B_2$ are the arbitrary constants. In the same procedures, (27b)-(27d) can theoretically be solved one after another.

### 3 $N$ fermionic parameters bosonization

Like the last section, we introduce $N$ fermionic parameters to expand the SB system. For $N \geq 2$ fermionic parameters $\xi_i$ ($i = 1, 2, \ldots, N$) instance, the component fields $u$ and $\xi$ can be expanded as

$$\xi(x, t) = \sum_{n=1}^{\lfloor N/2 \rfloor} \sum_{1 \leq i_1 < \cdots < i_{2n-1} \leq N} p_{i_1i_2 \cdots i_{2n-1}} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2n-1}}, \quad \text{(29a)}$$

$$u(x, t) = u_0 + \sum_{n=1}^{\lfloor N/2 \rfloor} \sum_{1 \leq i_1 < \cdots < i_{2n} \leq N} u_{i_1i_2 \cdots i_{2n}} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2n}}, \quad \text{(29b)}$$

where the coefficients $u_0, u_{i_1i_2 \cdots i_{2n}}$ ($1 \leq i_1 < \cdots < i_{2n} \leq N$) and $p_{i_1i_2 \cdots i_{2n-1}}$ ($1 \leq i_1 < \cdots < i_{2n-1} \leq N$) are $2^N$ real or complex bosonic functions. Substituting (29) into the SB system (4), the following bosonic system of $2^N$ equations obtains

$$u_{0,t} + u_{0,xx} + u_0 u_{0,xx} = 0, \quad \text{(30a)}$$

$$\hat{L}_o p_{i_1i_2 \cdots i_{2n-1}} = \left\{ \begin{array}{ll}
0, & n = 1; \\
\sum W_1 (-1)^{\tau(j_1, j_2, \ldots, j_{2n-1})} p_{j_1j_2 \cdots j_{2n-1}} u_{j_1j_2+1j_2+2 \cdots j_{2n-1}, x}, & n = 2, 3, \ldots, \lfloor N/2 \rfloor, \\
-\sum W_2 (-1)^{\tau(j_1, j_2, \ldots, j_{2n-1})} p_{j_1j_2 \cdots j_{2n-1}} u_{j_1j_2+1j_2+2 \cdots j_{2n-1}, x}, & n = 2, 3, \ldots, \lfloor N/2 \rfloor,
\end{array} \right. \quad \text{(30b)}$$

$$\hat{L}_e u_{i_1i_2 \cdots i_{2n}} = \left\{ \begin{array}{ll}
0, & n = 1; \\
\sum W_3 (-1)^{\tau(j_1, j_2, \ldots, j_{2n})} u_{i_1i_2 \cdots i_{2n}} u_{i_1i_2+1i_2+2 \cdots i_{2n}, x}, & n = 2, 3, \ldots, \lfloor N/2 \rfloor,
\end{array} \right. \quad \text{(30c)}$$

where

$$\tau(j_1, j_2, \ldots, j_N) = \left\{ \begin{array}{ll}
0, & j_1, j_2, \ldots, j_N \text{ is even permutation}; \\
1, & j_1, j_2, \ldots, j_N \text{ is odd permutation},
\end{array} \right.$$

$$W_1 = \{j_1, j_2, \ldots, j_{2n-1}\} | 1 \leq j_1 < j_2 < \cdots < j_{2n-1} \leq 2n - 1, 1 \leq j_{2l+1} < j_{2l+2} < \cdots < j_{2n-1} \leq 2n - 1, 1 \leq l \leq n - 1, j_{h_1} \neq j_{h_2} (h_1 \neq h_2) \}$$. 

\[ W_2 = \{(j_1, j_2, \ldots, j_{2n})| 1 \leq j_1 < j_2 < \cdots < j_{2l-1} \leq 2n, 1 \leq j_{2l} < j_{2l+1} < \cdots < j_{2n} \leq 2n, 1 \leq l \leq n, j_{2l} \neq j_{2l+1}(h_1 \neq h_2) \}, \]

\[ W_3 = \{(j_1, j_2, \ldots, j_{2n})| 1 \leq j_1 < j_2 < \cdots < j_{2l} \leq 2n, 1 \leq j_{2l+1} < j_{2l+2} < \cdots < j_{2n} \leq 2n, 1 \leq l \leq n-1, j_{2l} \neq j_{2l+2}(h_1 \neq h_2) \}. \]

Similar to the last section, the traveling wave solution of the component field \( u \) can be written

\[ u(x, t) = u_0 + \sum_{n=1}^{N-1} \sum_{i_1 < \cdots < i_{2n} < \mathcal{N}} u_{i_1 i_2 \cdots i_{2n}}(y) \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{2n}}, \tag{31} \]

where

\[ u_{i_1 i_2 \cdots i_{2n}}(y) = \left( k u_0^2 + 2 c_0 u_0 + 2 c_1 \right) \left[ \int \frac{2 F_{i_1 i_2 \cdots i_{2n}}(y)}{(k y^2 + 2 c_0 y + 2 c_1)^2} dy + c_{i_1 i_2 \cdots i_{2n}} \right], \]

with

\[ F_{i_1 i_2 \cdots i_{2n}} = \begin{cases} 2 k^2 u_0, & n=1; \\ \sum W_1 \left( \begin{array}{c} -1 \end{array} \right) \tau_{(j_1, j_2, \ldots, j_{2n-1})} P_{j_1} (P_{j_2}) u_0 + b_{i_1 i_2}, \\ \sum W_3 \left( \begin{array}{c} -1 \end{array} \right) \tau_{(j_1, j_2, \ldots, j_{2n})} \left( P_{j_1} \right)_{i_1 i_2 \cdots i_{2n}} U_{i_1 i_2 \cdots i_{2n}} - 2 k^2 u_0, & n=2,3,\ldots, \left[ \frac{N+1}{2} \right]. \end{cases} \]

where \( u_0 \) represent the solution of the usual Burgers equation, \( b_{i_1 i_2, \cdots, i_{2n}} \), and \( c_{i_1 i_2, \cdots, i_{2n}} \) are arbitrary constants.

The similarity solutions and reduction equations of (30) can be obtained with similar procedure. Here we just list case II as an example

\[ \dot{U}_{0,\xi} + (U_0 + c) U_{0,\xi} = 0, \tag{32a} \]

\[ \dot{L}_1 P_{i_1 i_2 \cdots i_{2n-1}} = \begin{cases} 0, & n=1; \\ \sum W_1 \left( \begin{array}{c} -1 \end{array} \right) \tau_{(j_1, j_2, \ldots, j_{2n-1})} P_{j_1} (P_{j_2}) U_{i_1 i_2 \cdots i_{2n-1}}, & n=2,3,\ldots, \left[ \frac{N+1}{2} \right], \end{cases} \tag{32b} \]

\[ \dot{L}_2 U_{i_1 i_2 \cdots i_{2n}} = \begin{cases} \sum W_1 \left( \begin{array}{c} -1 \end{array} \right) \tau_{(j_1, j_2, \ldots, j_{2n})} \left( P_{j_1} \right)_{i_1 i_2 \cdots i_{2n}} U_{i_1 i_2 \cdots i_{2n}}, & n=2,3,\ldots, \left[ \frac{N}{2} \right]. \end{cases} \tag{32c} \]

where two operators are

\[ \dot{L}_1 = c \partial_{\xi} + \partial_{\xi} \xi + U_{0,\xi} \]

\[ \dot{L}_2 = c \partial_{\xi} + \partial_{\xi} \xi + U_{0,\xi} + U_{0,\xi}. \]

These reduction equations are linear ODEs while the previous functions are known, we can solve (32) one after another in principle.

\section{4 Conclusion}

In summary, the \( \mathcal{N} = 1 \) supersymmetric Burgers system is changed to a system of coupled bosonic equations by means of a simple bosonization approach. The coupled bosonic systems are obtained with introducing two and \( N \) fermionic parameters. The systems are just usual Burgers equation together with several linear differential equations. Therefore, the approach can avoid the difficulties caused by intractable anticommuting fermionic fields. Using the mapping and deformation method, the traveling wave solutions of the bosonized systems are obtained. Besides, some special types of exact solutions can be obtained straightforwardly through the exact solutions of the Burgers equation and its symmetries. In addition, the similarity reduction solutions of the model are derived using the Lie point symmetries theory.

The solutions obtained via the bosonization procedure are completely different from those obtained via other methods [26]. The bosonization procedure should be effectively applied to not only the supersymmetric integrable systems but also other not integrable supersymmetric systems. In this paper, we only consider the \( \mathcal{N} = 1 \) SB system.

There are certain interesting applications in the calculation for the case of \( \mathcal{N} \geq 2 \) supersymmetric systems. The future work on these aspects is worthy of studying.
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