1. Introduction

The main geometric objects of study in this paper are double Bruhat cells $G_{u,v} = BuB \cap B_vB_-$ in a simply-connected connected complex semisimple group $G$; here $B$ and $B_-$ are two opposite Borel subgroups in $G$, and $u$ and $v$ any two elements of the Weyl group $W$. Double Bruhat cells were introduced and studied in [5] as a geometric framework for the study of total positivity in semisimple groups; they are also closely related to symplectic leaves in the corresponding Poisson-Lie groups (see [4, 6]). It will be convenient for us to replace $G_{u,v}$ with a reduced double Bruhat cell $L_{u,v}$ introduced in [3]. The variety $L_{u,v}$ can be identified with the quotient of $G_{u,v}$ modulo the left (or right) action of the maximal torus $H = B \cap B_-$. As shown in [5, 3], an algebraic variety $L_{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $m = \ell(u) + \ell(v)$, where $\ell(u)$ is the length of $u$ in the Coxeter group $W$. However, the smooth topology of $L_{u,v}$ can be quite complicated. A first step towards understanding this topology is enumerating the connected components of the real part $L_{u,v}(\mathbb{R})$. In the case when $G$ is simply-laced, a conjectural answer was given in [14, Conjecture 4.1]. Here we prove this conjecture and extend the result to an arbitrary semisimple group $G$. The answer is given in the following terms: as shown in [14] for $G$ simply-laced, every reduced word $i$ of $(u,v) \in W \times W$ gives rise to a subgroup $\Gamma_i(\mathbb{F}_2) \subset GL_m(\mathbb{F}_2)$ generated by symplectic transvections (here $\mathbb{F}_2$ is the 2-element field). We extend the construction of $\Gamma_i(\mathbb{F}_2)$ to an arbitrary $G$ (it is still generated by transvections but not necessarily by symplectic ones). Extending [14, Conjecture 4.1], we show that the connected components of $L_{u,v}(\mathbb{R})$ are in a natural bijection with the $\Gamma_i(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$. As explained in [14], this provides a far-reaching generalization of the results in [12, 13]; this also refines and generalizes results in [10, 11].

Our proof uses methods and results developed in [3, 8]. First, it was shown there that every reduced word $i$ of $(u,v) \in W \times W$ gives rise to a biregular isomorphism between the complex torus $\mathbb{C}^m_{x \neq 0}$ and a Zariski open subset $U_i \subset L_{u,v}$. We refine this result by showing that the complement $L_{u,v} - U_i$ is the union of $m$ divisors $\{M_{k,i} = 0\}$, where $M_{1,i}, \ldots, M_{m,i}$ are some irreducible regular functions on $L_{u,v}$. We further show that every $i$-bounded index $n \in [1, m]$ (see Section 3 for the definition) gives rise to a regular function $M'_{n,i}$ on $L_{u,v}$ such that replacing the divisor $\{M_{n,i} = 0\}$ with $\{M'_{n,i} = 0\}$ leads to another “toric chart” $U_{n,i}$ in $L_{u,v}$. Then we prove that the connected components of the real part of the union of charts $U_1 \cup \bigcup_n U_{n,i}$ are in a natural bijection with the $\Gamma_i(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$. Finally, we show that the complement in $L_{u,v}$ of this union of charts has codimension $\geq 2$,
the connected components of $L^{u,v}(\mathbb{R})$ are enumerated in the same way as those of the real part of $U_1 \cup \bigcup_n U_{n,i}$.

According to \cite{3}, each $M_{n,i}$ is a “twisted (generalized minor)” on $G$. We show that each $M'_{n,i}$ is obtained by the same twist from a regular function on $G$ which is no longer a minor but can be expressed as a sum of two Laurent monomials in minors. These regular functions are of independent interest for the study of the dual canonical bases in the ring of regular functions $\mathbb{C}[G]$ and its $q$-deformation.

The paper is organized as follows. After recalling the necessary background, we formulate our main result (Theorem 2.2) in Section 3. In Section 3 we formulate a lemma (Lemma 3.1) that plays the crucial role in our proof of Theorem 2.2, and then show how this lemma implies the theorem. The proof of Lemma 3.1 is given in Section 3. Finally, Section 5 discusses some examples and applications of the results in Sections 3 and 4.

2. Main theorem

To formulate our main result, let us recall the necessary background from \cite{4,5,6}. Let $G$ be a simply connected semisimple algebraic group with the Dynkin graph $\Pi$. Let $B$ and $B_-$ be two $\mathbb{R}$-split opposite Borel subgroups, $N$ and $N_-$ their unipotent radicals, $H = B \cap B_-$ an $\mathbb{R}$-split maximal torus of $G$, and $W = \text{Norm}_G(H)/H$ the Weyl group of $G$. Let $\mathfrak{g} = \text{Lie}(G)$ be the Lie algebra of $G$, and $\mathfrak{h} = \text{Lie}(H)$ the Cartan subalgebra of $\mathfrak{g}$. Let $\{\alpha_i : i \in \Pi\}$ be the system of simple roots in $\mathfrak{h}^*$ for which the corresponding root subgroups are contained in $N$. Let $\{\alpha_i^\vee : i \in \Pi\}$ be the corresponding system of simple coroots in $\mathfrak{h}$, and $A = (a_{ij} = \alpha_j(\alpha_i^\vee))$ be the Cartan matrix. Thus, for $i \neq j$ the indices $i$ and $j$ are adjacent in $\Pi$ if and only if $a_{ij}a_{ji} \neq 0$; we shall denote this by $\{i,j\} \in \Pi$. For every $i \in \Pi$, let $\varphi_i : SL_2 \to G$ denote the corresponding canonical $SL_2$-embedding.

The Weyl group $W$ is canonically identified with the Coxeter group $W(A)$ generated by the involutions $s_i$ for $i \in \Pi$ subject to the relations $(s_is_j)^{d_{ij}} = e$ for all $i \neq j$, where $d_{ij} = 2$ (resp. 3, 4, 6) if $a_{ij}a_{ji} = 0$ (resp. 1, 2, 3). A word $\mathbf{i} = (i_1, \ldots, i_m)$ in the alphabet $\Pi$ is a reduced word for $w \in W$ if $w = s_{i_1} \cdots s_{i_m}$, and $m$ is the smallest length of such a factorization. The length $m$ of any reduced word for $w$ is called the length of $w$ and denoted by $m = \ell(w)$. Let $R(w)$ denote the set of all reduced words for $w$. The identification $W = W(A)$ is given by $s_i = \varpi_i \mathcal{H}$, where

$$\varpi_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Norm}_G(H).$$

The representatives $\varpi_i \in G$ satisfy the braid relations $s_is_js_i \cdots = s_js_is_j \cdots$ (with $d_{ij}$ factors on each side); thus, the representative $\varpi$ can be unambiguously defined for any $w \in W$ by requiring that $\varpi w = \varpi \varpi$ whenever $\ell(uv) = \ell(u) + \ell(v)$.

The “double” group $W \times W$ is also a Coxeter group. The corresponding graph $\tilde{\Pi}$ is the union of two disconnected copies of $\Pi$. We identify the vertex set of $\tilde{\Pi}$ with $\{+1, -1\} \times \Pi$, and write a vertex $(\pm 1, i) \in \tilde{\Pi}$ simply as $\pm i$. For each $i \in \Pi$, we set $\varepsilon(\pm i) = \pm 1$ and $| \pm i | = i$. Thus, two vertices $i$ and $j$ of $\tilde{\Pi}$ are adjacent if and only if $\varepsilon(i) = \varepsilon(j)$ and $\{|i|, |j|\} \in \Pi$. In this notation, a reduced word for a pair $(u, v) \in W \times W$ is an arbitrary shuffle of a reduced word for $u$ written in the alphabet $-\Pi$ and a reduced word for $v$ written in the alphabet $\Pi$. 

The group $G$ has two Bruhat decompositions, with respect to $B$ and $B_-$:

$$G = \bigcup_{u \in W} BuB = \bigcup_{v \in W} B_- vB_-.$$  

The double Bruhat cells $G^{u,v}$ are defined by $G^{u,v} = BuB \cap B_- vB_-$.

Following [1], we define the reduced double Bruhat cell $L^{u,v} \subset G^{u,v}$ as follows:

$$L^{u,v} = N \pi N \cap B_- vB_-.$$  

The maximal torus $H$ acts freely on $G^{u,v}$ by left (or right) translations, and $L^{u,v}$ is a section of this action. Thus, $G^{u,v}$ is biregularly isomorphic to $H \times L^{u,v}$, and all properties of $G^{u,v}$ can be translated in a straightforward way into the corresponding properties of $L^{u,v}$ (and vice versa). In particular, Theorem 1.1 in [1] implies that $L^{u,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $\ell(u) + \ell(v)$.

The real part of $G$ is the subgroup $G(\mathbb{R})$ of $G$ generated by all the subgroups $\varphi_i(SL_2(\mathbb{R}))$. For any subset $L \subset G$, we define its real part by $L(\mathbb{R}) = L \cap G(\mathbb{R})$.

Now let us fix a pair $(u,v) \in W \times W$, and let $m = \ell(u) + \ell(v)$. Let $i = (i_1, \ldots, i_m) \in R(u,v)$ be any reduced word for $(u,v)$. We associate to $i$ an $m \times m$ matrix $(C_{kl})$ in the following way: set $C_{kl} = 1$ if $|i_k| = |i_l|$ and $C_{kl} = -a_{|i_k|,|i_l|}$ if $|i_k| \neq |i_l|$. The edges of $\Sigma(i)$ are now defined as follows.

**Definition 2.1.** A pair $\{k,l\} \subset [1,m]$ with $k < l$ is an edge of $\Sigma(i)$ if it satisfies one of the following three conditions:

(i) $k = l$;
(ii) $k^- < l^- < k$, $\{i_k, |i_l|\} \in \Pi$, and $\varepsilon(k^-) = \varepsilon(l^-)$;
(iii) $k^- < k < l$, $\{i_k, |i_l|\} \in \Pi$, and $\varepsilon(k^-) = -\varepsilon(l^-)$.

The edges of type (i) are called horizontal, and those of types (ii) and (iii) inclined. A horizontal (resp. inclined) edge $\{k,l\}$ with $k < l$ is directed from $k$ to $l$ if and only if $\varepsilon(k^-) = +1$ (resp. $\varepsilon(k^-) = -1$). We shall write $(k \to l) \in \Sigma(i)$ if $k \to l$ is a directed edge of $\Sigma(i)$.

We now associate to each $n \in [1,m]$ a transvection $\tau_n = \tau_{n,i} : \mathbb{Z}^m \to \mathbb{Z}^m$ defined as follows: if $\tau_n(\xi_1, \ldots, \xi_m) = (\xi_{1}', \ldots, \xi_{m}')$ then $\xi_{k}' = \xi_k$ for $k \neq n$, and

$$\xi_{n}' = \xi_n - \sum_{(k \to n) \in \Sigma(i)} C_{kn} \xi_k + \sum_{(n \to l) \in \Sigma(i)} C_{ln} \xi_l$$

(note that if $G$ is simply-laced then all the coefficients $C_{kn}$ and $C_{ln}$ in (2.2) are equal to 1, so (2.2) becomes formula (2.4) in [1]). We call an index $n \in [1,m]$ $i$-bounded if $n^- > 0$. Let $\Gamma_i$ denote the group of linear transformations of $\mathbb{Z}^m$ generated by the transvections $\tau_n$ for all $i$-bounded indices $n \in [1,m]$. Let $\Gamma_i(\mathbb{F}_2)$ denote the group of linear transformations of the $\mathbb{F}_2$-vector space $\mathbb{F}_2^m$ obtained from $\Gamma_i$ by reduction modulo 2 (recall that $\mathbb{F}_2$ is the 2-element field).

We are finally ready to formulate our main result.

**Theorem 2.2.** For every reduced word $i \in R(u,v)$, the connected components of $L^{u,v}(\mathbb{R})$ are in a natural bijection with the $\Gamma_i(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$. 
Note that in Theorem 2.2 we only need the modulo 2 reductions of transvections $\tau_n$, so the formula (2.2) could be simplified as follows:

$$\xi' = \xi_n + \sum_{(k,n) \in \Sigma(i)} C_{kn} \xi_k .$$

We prefer the form (2.2) because it is suggested by the construction of toric charts in $L^{u,v}$ which is our main ingredient in proving Theorem 2.2.

3. Main lemma

As before, let $G$ be a simply connected connected complex semisimple group with the Dynkin graph $\Pi$. We fix a pair $(u,v) \in W \times W$, let $m = \ell(u) + \ell(v)$, and fix a reduced word $i = (i_1, \ldots, i_m) \in R(u,v)$.

Lemma 3.1. There exist regular functions $M_1, \ldots, M_m$ on $L^{u,v}$ with the following properties:

1. If $k \in [1, m]$ is not $i$-bounded then $M_k$ vanishes nowhere on $L^{u,v}$.
2. The map $(M_1, \ldots, M_m) : L^{u,v} \to \mathbb{C}^m$ restricts to a biregular isomorphism $U_1 \to \mathbb{C}^n_{\neq 0}$, where $U_1$ is the locus of all $x \in L^{u,v}$ such that $M_k(x) \neq 0$ for all $k \in [1, m]$.
3. For every $i$-bounded $n \in [1, m]$, the rational function $M'_n$ defined by

$$M'_n M_n = \prod_{(k,n) \in \Sigma_i} M_k^{C_{kn}} + \prod_{(n-k) \in \Sigma_i} M_k^{C_{kn}}$$

is regular on $L^{u,v}$.
4. For every $i$-bounded $n \in [1, m]$, the map $(M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m) : L^{u,v} \to \mathbb{C}^m$ restricts to a biregular isomorphism $U_{n,1} \to \mathbb{C}^m_{\neq 0}$, where $U_{n,1}$ is the locus of all $x \in L^{u,v}$ such that $M'_n(x) \neq 0$ and $M_k(x) \neq 0$ for all $k \in [1, m] - \{n\}$.
5. The functions $M_k$ and $M'_n$ take real values on $L^{u,v}(\mathbb{R})$, and the biregular isomorphisms in (2) and (4) restrict to biregular isomorphisms $U_1(\mathbb{R}) \to \mathbb{R}^m_{\neq 0}$ and $U_{n,1}(\mathbb{R}) \to \mathbb{R}^m_{\neq 0}$.

The functions $M_k = M_{k,1}$ in Lemma 3.1 were introduced in [3, (4.13)]. We recall the definition and prove Lemma 3.1 in the next section; in the rest of this section we show that it implies Theorem 2.2. To be more precise, we shall prove that the bijection in Theorem 2.2 can be defined as follows. For every $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{F}_2^m$, let $U_1(\xi)$ denote the set of all $x \in U_1(\mathbb{R})$ such that $(-1)^{\xi_k} M_k(x) > 0$ for all $k$. For every $Y \subset L^{u,v}(\mathbb{R})$, let $\overline{Y}$ denote the closure of $Y$ in $L^{u,v}(\mathbb{R})$ in the real topology.

Theorem 3.2. The correspondence $\Omega \mapsto \bigcup_{\xi \in \Omega} \overline{U_1(\xi)}$ is a bijection between $\Gamma_1(\mathbb{F}_2)$-orbits in $\mathbb{F}_2^m$ and connected components of $L^{u,v}(\mathbb{R})$.

We split the proof of Theorem 3.2 into several lemmas. Let us abbreviate $X = L^{u,v}$, and let $\mathbb{C}[X]$ be the ring of regular functions on $X$. Since $X$ is isomorphic to a Zariski open subset of $\mathbb{C}^m$, the ring $\mathbb{C}[X]$ is a unique factorization domain. By property (1) in Lemma 3.1, if $k$ is not $i$-bounded then $M_k$ is an invertible element of $\mathbb{C}[X]$.

Lemma 3.3. A Laurent monomial $P = M_{d_1}^{\alpha_1} \cdots M_{d_m}^{\alpha_m}$ is a regular function on $X$ if and only if $d_n \geq 0$ for any $i$-bounded $n$. 


Proof. The “if” part is trivial. To prove the “only if” part, fix an \(i\)-bounded index \(n\), and consider the restriction of \(P\) to the Zariski open subset \(U_{n,i} \subset X\). By property (4) in Lemma 3.3, if \(P \in \mathbb{C}[X]\) then \(M''_n\) is a regular function on \(U_{n,i}\) and so it must be a Laurent polynomial in \(M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m\). In view of (3.1), this implies that \(d_n \geq 0\), as desired.

\[\text{Lemma 3.4.} \quad \text{For every } i\text{-bounded } n, \text{ the function } M_n \text{ is an irreducible element of } \mathbb{C}[X].\]

\[\text{Proof.} \quad \text{Notice that every } P \in \mathbb{C}[X] \text{ restricts to a regular function on the Zariski open subset } U_i \subset X. \text{ By property (2) in Lemma 3.3, } P \text{ is a Laurent polynomial in } M_1, \ldots, M_m. \text{ It follows that if } M_n \text{ is the product of two regular functions } P \text{ and } Q \text{ then both } P \text{ and } Q \text{ must be Laurent monomials in } M_1, \ldots, M_m. \text{ By Lemma 3.3, one of the factors } P \text{ and } Q \text{ must be a Laurent monomial in the variables } M_k \text{ for } k \text{ not } i\text{-bounded, hence is an invertible element of } \mathbb{C}[X]. \text{ Therefore, } M_n \text{ is irreducible.} \]

\[\text{Lemma 3.5.} \quad \text{For every } i\text{-bounded } n, \text{ the function } M'_n \text{ is equal to some irreducible element } M''_n \in \mathbb{C}[X] \text{ times a Laurent monomial in } M_1, \ldots, M_{n-1}, M_{n+1}, \ldots, M_m.\]

\[\text{Proof.} \quad \text{Let } P \in \mathbb{C}[X] \text{ be an irreducible factor of } M'_n. \text{ Restricting } P \text{ to } U_{n,i}, \text{ we conclude that } P \text{ is a Laurent monomial in } M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m. \text{ Restricting } P \text{ to } U_i \text{ and using property (2) in Lemma 3.3, we see that } P \text{ must be also a Laurent polynomial in } M_1, \ldots, M_m. \text{ By (3.1), this implies that the exponent of } M'_n \text{ in } P \text{ written as a Laurent monomial in } M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m \text{ must be nonnegative. It follows that there is an irreducible factor } M''_n \text{ of } M'_n \text{ which is equal to } M'_n \text{ times a Laurent monomial in } M_1, \ldots, M_{n-1}, M_{n+1}, \ldots, M_m, \text{ while the rest of the factors are just Laurent monomials in } M_1, \ldots, M_{n-1}, M_{n+1}, \ldots, M_m. \]

We set \(U = U_i \cup \bigcup_n U_{n,i}\).

\[\text{Lemma 3.6.} \quad \text{The complement } X - U \text{ is the locus of all } x \in X \text{ such that } M_n(x) = M_k(x) = 0 \text{ for two distinct } i\text{-bounded indices } n \text{ and } k, \text{ or } M_n(x) = M''_n(x) = 0 \text{ for some } i\text{-bounded } n. \text{ The variety } X - U \text{ has (complex) codimension } \geq 2 \text{ in } X.\]

\[\text{Proof.} \quad \text{Suppose } x \in X - U. \text{ Since } x \notin U_i, \text{ property (1) in Lemma 3.3 implies that } M_n(x) = 0 \text{ for some } i\text{-bounded } n. \text{ Since } x \notin U_{n,i}, \text{ it follows that either } M'_n(x) = 0, \text{ or } M_k(x) = 0 \text{ for some } i\text{-bounded } k \neq n. \]

The converse inclusion is obvious. Finally, the statement that \(X - U\) has codimension \(\geq 2\) in \(X\) is clear since \(X - U\) is the union of finitely many subvarieties, each given by two (distinct) irreducible equations.

Now consider the real part \(U(\mathbb{R}) = U_i(\mathbb{R}) \cup \bigcup_n U_{n,i}(\mathbb{R})\). By Lemma 3.6 and property (5) in Lemma 3.3, the complement \(X(\mathbb{R}) - U(\mathbb{R})\) has real codimension \(\geq 2\) in \(X(\mathbb{R})\). Therefore, the connected components of \(X(\mathbb{R})\) (in the real topology) are closures of the connected components of \(U(\mathbb{R})\). It remains to show that Theorem 3.2 holds with \(X(\mathbb{R})\) replaced by \(U(\mathbb{R})\). For a subset \(Y \subset U(\mathbb{R})\) we now denote by \(\bar{Y}\) the closure of \(Y\) in \(U(\mathbb{R})\). The role of transvections \(\tau_n\) is explained by the following lemma.

\[\text{Lemma 3.7.} \quad \text{Let } \xi^{(1)} \text{ and } \xi^{(2)} \text{ be two distinct vectors in } \mathbb{N}_2^n. \text{ Then } U(\xi^{(1)}) \cap U(\xi^{(2)}) \neq \emptyset \text{ if and only if } \xi^{(2)} = \tau_n(\xi^{(1)}) \text{ for some } i\text{-bounded index } n.\]
Proof. Suppose \( x \in U(\mathbb{R}) \) belongs to the intersection \( \overline{U(\xi^{(1)})} \cap \overline{U(\xi^{(2)})} \). Then \( M_k(x) = 0 \) whenever \( \xi_k^{(1)} \neq \xi_k^{(2)} \). Using Lemma 3.4, we see that there is a unique \( n \) such that \( \xi_n^{(1)} \neq \xi_n^{(2)} \); furthermore, this index \( n \) is \( l \)-bounded, and \( M'_n(x) \neq 0 \). Since any neighborhood of \( x \) intersects both \( U(\xi^{(1)}) \) and \( U(\xi^{(2)}) \), it follows that the two monomials on the right hand side of (3.1) must have opposite signs at \( x \). Let us write \( \xi_k = \xi_k^{(1)} = \xi_k^{(2)} \) for \( k \neq n \). Then we have
\[
\xi_n^{(2)} - \xi_n^{(1)} = 1 = \sum_{(n \to l) \in \Sigma(i)} C_{in} \xi_l - \sum_{(k \to n) \in \Sigma(i)} C_{kn} \xi_k.
\]
Comparing this with (2.2), we conclude that \( \xi_n^{(2)} = \tau_n(\xi^{(1)}) \), as claimed.

Conversely, suppose \( \xi_n^{(2)} = \tau_n(\xi^{(1)}) \neq \xi_n^{(1)} \), and let \( \xi_k = \xi_k^{(1)} = \xi_k^{(2)} \) for \( k \neq n \). Then
\[
\sum_{(n \to l) \in \Sigma(i)} C_{in} \xi_l \neq \sum_{(k \to n) \in \Sigma(i)} C_{kn} \xi_k.
\]
This implies that there exists a point \( x \in U_{\mathbb{R}}(\mathbb{R}) \) such that \( (-1)^{\xi_n} M_n(x) > 0 \) for all \( k \neq n \), and the right hand side of (3.1) vanishes at \( x \). Then any neighborhood of \( x \) contains points with the signs of all \( M_k \) for \( k \neq n \) unchanged and with the right hand side of (3.1) positive (as well as negative). Thus, \( x \in \overline{U(\xi^{(1)})} \cap \overline{U(\xi^{(2)})} \), and we are done.

Now we are ready to complete the proof of Theorem 3.2. Let \( \Omega \) be a \( \Gamma_i(\mathbb{F}_2) \)-orbit in \( \mathbb{P}_2^n \), and consider the corresponding closed subset \( Y_\Omega = \bigcup_{\xi \in \Omega} \overline{U_i(\xi)} \) of \( U(\mathbb{R}) \). Each \( U_i(\xi) \) is a copy of \( \mathbb{R}_{>0} \) and so is connected. Using the “if” part of Lemma 3.3, we conclude that \( Y_\Omega \) is connected (since the closure of a connected set and the union of two non-disjoint connected sets are connected as well). On the other hand, by the “only if” part of the same lemma, all the sets \( Y_\Omega \) are pairwise disjoint. Thus, they are the connected components of \( U(\mathbb{R}) \), and we are done.

4. Proof of Lemma 3.1

4.1. The functions \( M_k \). We start by recalling the definition of the functions \( M_k = M_{k,i} \) given in (4.13). First of all, recall that the weight lattice \( P \) of \( G \) can be thought of as the group of rational multiplicative characters of \( H \) written in the exponential notation: a weight \( \gamma \in P \) acts by \( a \mapsto a^\gamma \). The lattice \( P \) is also identified with the additive group of all \( \gamma \in \mathfrak{h}^* \) such that \( \gamma(\alpha_i^\vee) \in \mathbb{Z} \) for all \( i \in \Pi \). Thus, \( P \) has a \( \mathbb{Z} \)-basis \( \{ \omega_i : i \in \Pi \} \) of fundamental weights given by \( \omega_j(\alpha^\vee_i) = \delta_{i,j} \).

We now recall from (3) the definition of generalized minors. Denote by \( G_0 = N_\mathbb{R} H N \) the open subset of elements \( x \in G \) that have Gaussian decomposition; this (unique) decomposition will be written as \( x = [x]_-[x]_0 [x]_+ \). For \( u, v \in W \) and \( i \in \Pi \), the (generalized) minor \( \Delta_{u \omega_i, v \omega_i} \) is the regular function on \( G \) whose restriction to the open set \( \mathbb{P} G_0 \mathbb{P}^{-1} \) is given by
\[
\Delta_{u \omega_i, v \omega_i}(x) = (|\mathbb{P}^{-1} x \mathbb{P}|_0)^{\omega_i}.
\]
As shown in (3), \( \Delta_{u \omega_i, v \omega_i} \) depends on the weights \( u \omega_i \) and \( v \omega_i \) alone, not on the particular choice of \( u \) and \( v \). It is easy to see that the generalized minors are distinct irreducible elements of the ring \( \mathbb{C}[G] \) of regular functions on \( G \). In the special case \( G = SL_n \), the generalized minors are nothing but the ordinary minors of a matrix.

According to (3), Proposition 4.3], an element \( x \in G^{u,v} \) belongs to \( L^{u,v} \) if and only if \( |\mathbb{P}^{-1} x|_0 = 1 \), or equivalently if \( \Delta_{u \omega_i, w_i}(x) = 1 \) for any \( i \in [1, r] \).
We fix a pair \((u, v) \in W \times W\) and a double reduced word \(i = (i_1, \ldots, i_m) \in R(u, v)\). Recall from [3, Definition 4.6 and Theorem 4.7] that there is a birational isomorphism \(\psi^u,v\) between \(L^u,v\) and \(L^v,u\) given by

\[
\psi^{u,v}(x) = [(\prod x_i)^{-1}]_+ + \prod [(\prod^{-1} x_i)]^+ ;
\]

here \(x \mapsto x^t\) is the involutive antiautomorphism of \(G\) given by

\[
\varphi_1 \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right)^t = \varphi_1 \left( \begin{array}{ccc} d & b \\ c & a \end{array} \right) .
\]

Recall that the length \(m\) of \(i\) is equal to \(\ell(u) + \ell(v)\). For \(k \in [1, m]\), denote

\[
u_k = \prod_{i=1}^{k} s_{i} ; \quad \nu_{<k} = \prod_{i=1}^{k-1} s_{i} .
\]

This notation means that in the first (resp. second) product in \([4, 4]\), the index \(l\) is decreasing (resp. increasing); for example, if \(\Pi = \{1, 2, 3\}\) and \(i = (-2, 1, -3, 3, 2, -1, -2, 1, -1)\), then, say, \(u_\geq 7 = s_1 s_2\) and \(v_\leq 7 = s_1 s_3 s_2\).

Following [3, (4.13)], we define a regular function \(M_k = M_{k,i}\) on \(L^u,v\) by

\[
M_k(x) = M_{k,i}(x) = \Delta_{\nu_k \omega_i \nu_k, \nu_{<k} \omega_i \nu_k}(\psi^{u,v}(x)) .
\]

### 4.2. Properties (1), (2) and (5)

To prove property (1) in Lemma [3, 4] notice that if \(k\) is not \(i\)-bounded and \(|i_k| = i\) then \([4, 5]\) turns into \(M_{k,i}(x) = \Delta_{\omega_i, \omega_i}(\psi^{u,v}(x))\). It remains to show that \(\Delta_{\omega_i, \omega_i}(\psi^{u,v}(x))\) vanishes nowhere on \(L^v,u\). In fact, a stronger statement holds: \(\Delta_{\omega_i, \omega_i}(\psi^{u,v}(x))\) vanishes nowhere on \(B_- u B_- u^{-1} \subset G_0\) (cf. [3, Proposition 2.10]).

As for property (2) in Lemma [3, 4], it follows from the solution to the so-called factorization problem given in [3, Theorem 1.9] (or rather from its modification in [3, Theorem 4.8]). To formulate it, we need some notation.

For every \(i \in \Pi\) and \(t \in \mathbb{C}_{\neq 0}\), we denote

\[
x_i(t) = \varphi_1 \left( \begin{array}{ccc} 1 & t \\ 0 & 1 \end{array} \right) , \quad y_i(t) = \varphi_1 \left( \begin{array}{ccc} 1 & 0 \\ t & 1 \end{array} \right) , \quad t^\alpha_i = \varphi_1 \left( \begin{array}{ccc} t & 0 \\ 0 & t^{-1} \end{array} \right) ;
\]

following [3], we also denote

\[
x_{-i}(t) = y_i(t) t^{-\alpha_i} = \varphi_1 \left( \begin{array}{ccc} t^{-1} & 0 \\ 1 & t \end{array} \right) .
\]

For any word \(i = (i_1, \ldots, i_m)\) in the alphabet \(\tilde{\Pi}\), let us define the product map \(x_i : \mathbb{C}_{\neq 0}^m \to G\) by

\[
x_i(t_1, \ldots, t_m) = x_{i_1}(t_1) \cdots x_{i_m}(t_m) .
\]

For \(k \in [1, m]\), we denote \(k^+ = \min\{l : l > k, |i_l| = |i_k|\}\), so that \(k^+\) is the next occurrence of an index \(i_k\) in \(i\); if \(k\) is the last occurrence of \(i_k\) in \(i\) then we set \(k^+ = m + 1\). We also adopt the convention that \(M_{m+1}(x) = 1\).

The following reformulation of Theorem 4.8 in [3] provides a refinement of property (2) in Lemma [3, 4].

**Theorem 4.1.** Let \(i = (i_1, \ldots, i_m)\) be a double reduced word for \((u, v)\), and let \((M_1, \ldots, M_m) \in \mathbb{C}_{\neq 0}^m\). Then there is a unique \(x \in L^u,v\) such that \(M_{k,i}(x) = M_k\) for
This element $x$ has the form $x = x_{i_1}(t_1, \ldots, t_m)$, with the factorization parameters $t_k$ given by: if $i_k \in -\Pi$ then

$$t_k = M_k/M_{k+}$$

if $i_k \in \Pi$ then

$$t_k = \frac{1}{M_k M_{k+}} \prod_{l: l < k} M_l^{-a_{i_l} i_k}.$$  \hfill (4.8)

**Remark 4.2.** We see that the parameters $t_1, \ldots, t_m$ in the factorization $x = x_{i_1}(t_1, \ldots, t_m)$ are related to $M_1, \ldots, M_m$ by an invertible monomial transformation. The inverse of this monomial transformation can be computed explicitly: a direct calculation shows that

$$M_k = M_{k,i}(x) = \prod_{l \geq k} t_l^{-\varepsilon(i_l) v_{< k} \omega_{i_l} (\alpha_{i_l}^\vee)}.$$  \hfill (4.10)

Finally, property (5) in Lemma 3.1 is clear since each $M_k$ is just a Laurent monomial in the factorization parameters $t_1, \ldots, t_m$, while each $M'_k$ is the sum of two Laurent monomials; therefore they take real values when all $t_k$ are real.

**4.3. Property (3).** To prove property (3) in Lemma 3.1, we shall construct a new family of regular functions on the whole group $G$. Let $i = (i_1, \ldots, i_m)$ be a reduced word for $(u, v) \in W \times W$ such that $|i_1| = |i_m| = i$ for some $i \in \Pi$, and $|i_k| \neq i$ for $1 < k < m$. Let $E_\pm = \{k \in [2, m-1] : \varepsilon(i_k) = \pm 1\}$, and let $J_\pm = \{i\} \cup \{|i_k| : k \in E_\pm, k^+ = m+1\} \subset \Pi$. Let $\Delta' = \Delta'_i$ be the rational function on $G$ defined by one of the following four equations.

**Case 1.** If $i_1 = i_m = i$ then

$$\Delta' \Delta_{s_i, \omega_i, \omega_i} = \Delta_{\omega_i, \omega_i} \prod_{k \in E_+} \Delta_{\omega_{< k} \omega_{i_k}, u > k \omega_{i_k}}^{a_{i_k} i_i}
\prod_{k \in E_-} \Delta_{\omega_{< k} \omega_{i_k}, u < k \omega_{i_k}}^{-a_{i_k} i_i}$$

$$+ \Delta_{s_i, \omega_i, \omega_i} \prod_{k \in E_+} \Delta_{\omega_{< k} \omega_{i_k}, u > k \omega_{i_k}}^{-a_{i_k} i_i}
\prod_{k \in E_-} \Delta_{\omega_{< k} \omega_{i_k}, u < k \omega_{i_k}}^{a_{i_k} i_i}.$$  \hfill (4.11)

**Case 2.** If $i_1 = i$ and $i_m = -i$ then

$$\Delta' \Delta_{s_i, \omega_i, \omega_i} = \Delta_{\omega_i, \omega_i} \Delta_{s_i, \omega_i, \omega_i} \prod_{k \in E_+} \Delta_{\omega_{< k} \omega_{i_k}, u > k \omega_{i_k}}^{-a_{i_k} i_i}
\prod_{k \in E_-} \Delta_{\omega_{< k} \omega_{i_k}, u < k \omega_{i_k}}^{a_{i_k} i_i}$$

$$+ \prod_{k \in E_+} \Delta_{\omega_{< k} \omega_{i_k}, u > k \omega_{i_k}}^{-a_{i_k} i_i}
\prod_{k \in E_-} \Delta_{\omega_{< k} \omega_{i_k}, u < k \omega_{i_k}}^{a_{i_k} i_i},$$  \hfill (4.12)
Case 3. If \( i_1 = -i \) and \( i_m = i \) then

\[
\Delta' \Delta_{\omega_1, \omega_i} = \Delta_{\omega_1, u^{-1} \omega_i} \Delta_{v \omega_1, \omega_i} \prod_{k \in E_+} \Delta_{-a_{i k}, i} \Delta_{v, k \omega_1, u > k \omega_k} \\
+ \prod_{k \in E_-} \Delta_{v, k \omega_1, u > k \omega_k} \prod_{j \in \Pi - J} \Delta_{v, j \omega_i, \omega_j} \Delta_{\omega_1, \omega_i},
\]

Case 4. If \( i_1 = i_m = -i \) then

\[
\Delta' \Delta_{\omega_1, s_i \omega_i} = \Delta_{\omega_1, u^{-1} \omega_i} \prod_{k \in E_+} \Delta_{v, k \omega_1, u > k \omega_k} \\
+ \Delta_{\omega_1, \omega_i} \prod_{k \in E_-} \Delta_{v, k \omega_1, u > k \omega_k},
\]

Theorem 4.3. In each of the above four cases, \( \Delta' = \Delta'_1 \) is a regular function on \( G \).

Before proving Theorem 4.3, we show that it implies property (3) in Lemma 3.1. Let \( (u, v) \) be an arbitrary pair of elements of \( W \), and fix a reduced word \( i = (i_1, \ldots, i_m) \in R(u, v) \). Let \( n \) be an \( \ell \)-bounded index in \([1, m]\), and let \( i' \) denote the subword \((i_n, \ldots, i_1)\) of \( i \). We claim that the rational function \( M'_n \) on \( L^{u, v} \) defined by (3.1) is given by

\[(4.11) M'_n(x) = M'_{n, i}(x) = \Delta'_i(u_{>n}^{-1} \psi_{<n}^{u, v}(x)v_{<n}^{-1}) \]

and is regular. To see this, let us evaluate the defining equation for \( \Delta'_i \) at the point \( u_{>n}^{-1} \psi_{<n}^{u, v}(x)v_{<n}^{-1} \). Remembering the definition (4.1) of generalized minors, and the definition (4.5) of the functions \( \Delta_k \), a direct check shows that, in each of the above four cases, the corresponding equality turns into the equation (3.1) with \( M'_n \) given by (4.11).

It remains to prove Theorem 4.3. Our main tool will be the following identity established in [6, Theorem 1.17]:

\[
\Delta_{v' \omega_1, u' \omega_i, \omega_j} \Delta_{v' s_i \omega_1, u' s_i \omega_i, \omega_j} = \Delta_{v' s_i \omega_1, u' \omega_i, \omega_j} \Delta_{v' \omega_1, u' s_i \omega_i, \omega_j} = \prod_{j \in \Pi - \{i\}} \Delta_{-a_{j i}, i} \Delta_{\omega_1, \omega_i, \omega_j} \Delta_{v \omega_1, \omega_i} \Delta_{v' s_i \omega_1, \omega_j} \Delta_{v' \omega_1, s_i \omega_i, \omega_j} \Delta_{-a_{j i}, i} \Delta_{\omega_1, \omega_i, \omega_j}.
\]

for any \( u', v' \in W \) and \( i \in \Pi \) such that \( \ell(u s_i) = \ell(u') + 1 \) and \( \ell(v s_i) = \ell(v') + 1 \).

To prove that \( \Delta'_1 \) is regular on \( G \), we first consider the case when \( i \) is non-mixed, i.e., \( k < l \) for each \( k \in E_- \) and \( l \in E_+ \). Then the defining equation for \( \Delta' = \Delta'_1 \) simplifies as follows. Denote \( S' = \{i_k' : k \in E_\pm\} \subset \Pi \).

Case 1 (non-mixed). If \( i_1 = i_m = i \) then

\[
\Delta' \Delta_{s_i \omega_1, \omega_i} = \Delta_{\omega_1, \omega_i} \prod_{j \in S_+} \Delta_{-a_{j i}, i} \Delta_{\omega_1, \omega_i} + \prod_{j \in S_\pm} \Delta_{\omega_1, \omega_i} \Delta_{-a_{j i}, i} \Delta_{\omega_1, \omega_i}.
\]

Case 2 (non-mixed). If \( i_1 = i \) and \( i_m = -i \) then

\[
\Delta' \Delta_{s_i \omega_1, \omega_i} = \Delta_{\omega_1, \omega_i} \Delta_{-a_{j i}, i} \Delta_{\omega_1, \omega_i} + \prod_{j \in \Pi - \{i\}} \Delta_{\omega_1, \omega_i} \Delta_{-a_{j i}, i} \Delta_{\omega_1, \omega_i}.
\]
Case 3 (non-mixed). If \( i_1 = -i \) and \( i_m = i \) then
\[
\Delta' \Delta_{\omega_1, \omega_i} = \Delta_{\omega_1, u^{-1} \omega_i} \Delta_{\omega_i, a_j} \prod_{j \in S_+ \cap S_-} \Delta_{\omega_j, a_j}^{a_j} 
\]
\[+ \prod_{j \in S_-} \Delta_{\omega_j, u^{-1} \omega_j} \prod_{j \in \Pi \setminus (S_- \cup S_+)} \Delta_{\omega_j, a_j}^{a_j} .
\]

Case 4 (non-mixed). If \( i_1 = i_m = -i \) then
\[
\Delta' \Delta_{\omega_1, \omega_i} = \Delta_{\omega_1, u^{-1} \omega_i} \prod_{j \in S_-} \Delta_{\omega_j, u^{-1} \omega_j} + \Delta_{\omega_i, a_j} \prod_{j \in S_-} \Delta_{\omega_j, a_j}^{a_j} .
\]

By \([4.12]\), in Case 2 we have \( \Delta' = \Delta_{\omega_1, \omega_i} \). Cases 1 and 4 are equivalent to each other in view of the identity \( \Delta_{\gamma, \delta}(x^T) = \Delta_{\delta, \gamma}(x) \), where \( x \mapsto x^T \) is the involutive antiautomorphism of \( G \) given by
\[
\varphi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \varphi_i \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]
(see \([3.67]\)). It remains to show that \( \Delta' \) is regular in each of the cases 1 and 3.

Let us start with Case 3. Multiplying both monomials on the right hand side of the corresponding equation with the monomial
\[
\prod_{j \in \Pi \setminus \{i\} \cup (S_+ \cap S_-)} \Delta_{\omega_j, a_j}^{a_j} = \prod_{j \in \Pi \setminus \{i\} \setminus S_+} \Delta_{\omega_j, a_j}^{a_j} \prod_{j \in S_- \setminus S_+} \Delta_{\omega_j, a_j}^{a_j}
\]
and using \([4.12]\), we obtain
\[
\Delta_{\omega_1, u^{-1} \omega_i} \Delta_{\omega_i, a_j} \prod_{j \in \Pi \setminus \{i\} \setminus S_+} \Delta_{\omega_j, a_j}^{a_j} + \prod_{j \in \Pi \setminus \{i\} \setminus S_+} \Delta_{\omega_j, a_j}^{a_j} \prod_{j \in S_- \setminus S_+} \Delta_{\omega_j, a_j}^{a_j} 
\]
\[= \Delta_{\omega_1, u^{-1} \omega_i} \Delta_{\omega_i, a_j} \Delta_{\omega_1, a_j} \prod_{j \in \Pi \setminus \{i\}} \Delta_{\omega_j, a_j}^{a_j} + \Delta_{\omega_1, u^{-1} \omega_i} \Delta_{\omega_i, a_j} \prod_{j \in \Pi \setminus \{i\}} \Delta_{\omega_j, a_j}^{a_j} \prod_{j \in S_- \setminus S_+} \Delta_{\omega_j, a_j}^{a_j} 
\]
\[= \Delta_{\omega_1, u^{-1} \omega_i} \Delta_{\omega_i, a_j} \prod_{j \in \Pi \setminus \{i\}} \Delta_{\omega_j, a_j}^{a_j} \prod_{j \in S_- \setminus S_+} \Delta_{\omega_j, a_j}^{a_j}. 
\]

Since all “principal minors” \( \Delta_{\omega_j, \omega_i} \) are distinct irreducible elements of \( \mathbb{C}[G] \), it follows that \( \Delta_{\omega_1, \omega_i} \) is relatively prime with \( \prod_{j \in \Pi \setminus \{i\}} \Delta_{\omega_j, a_j}^{a_j} \prod_{j \in S_- \setminus S_+} \Delta_{\omega_j, a_j}^{a_j} \). Therefore,
\[
\Delta' = \begin{pmatrix} \Delta_{\omega_1, u^{-1} \omega_i} & \Delta_{\omega_1, a_j} & \Delta_{\omega_1, a_j} \Delta_{\omega_2, a_j}^{a_j} \\ \Delta_{\omega_1, a_j} & \Delta_{\omega_2, a_j}^{a_j} & \Delta_{\omega_2, a_j}^{a_j} \Delta_{\omega_3, a_j}^{a_j} \\ 0 & \Delta_{\omega_3, a_j}^{a_j} & \Delta_{\omega_3, a_j}^{a_j} \end{pmatrix} \prod_{j \in \Pi \setminus \{i\} \setminus (S_+ \cap S_-)} \Delta_{\omega_j, a_j}^{a_j} 
\]
is a regular function on \( G \), as required.

The argument in Case 1 is similar (and simpler). Let us only give the final answer: the function \( \Delta' \) is now given by
\[
\Delta' = (\Delta_{\omega_1, \omega_i} \Delta_{\omega_1, a_j} - \Delta_{\omega_1, \omega_i} \Delta_{\omega_1, a_j}) \prod_{j \in \Pi \setminus \{i\} \setminus S_+} \Delta_{\omega_j, a_j}^{a_j},
\]
and it is again a regular function on \( G \), as required.

We shall deduce the general case in Theorem 4.3 from the non-mixed case just considered. Note that every reduced word \( i \) in each of the cases 1 – 4 is obtained from the corresponding non-mixed word by a sequence of 2-moves each of which
interchanges a pair of consecutive indices $i_k$ and $i_{k+1}$ with $k \in E_-$ and $k+1 \in E_+$.
It suffices to show that if $i'$ is obtained from $i$ by such a move then the regularity of $\Delta'_j$ implies that of $\Delta'_j$. We shall only treat Case 1; the argument in the other three cases is the same.

Let $P_1$ and $P_2$ (resp. $P'_1$ and $P'_2$) be two monomials in the right hand side of the defining equation for $\Delta'_j$ (resp. for $\Delta'_j$). Thus, we assume that $P_1 + P_2$ is divisible by $\Delta_{s(i),w}$ in $\mathbb{C}[G]$, and need to show that the same is true for $P'_1 + P'_2$. Let us abbreviate $v' = v_{<k}$ and $u' = u_{>k+1}$, where $k$ and $k+1$ are two positions involved in the 2-move that turns $i$ into $i'$. It is clear from the definitions that $P'_1 = P_1$ and $P'_2 = P_2$ unless $-i_k = i_{k+1} = j$ for some $j \in \Pi$ such that $a_{ji} < 0$. In the latter case, we have

$$P'_1 = P_1 \left( \frac{\Delta^{\varepsilon_s s_j w_j, t' s_j w_j}}{\Delta^{\delta_1 s_i s_j w_j, t' s_j w_j}} \right)^{-a_{ji}}, \quad P'_2 = P_2 \left( \frac{\Delta^{1-\delta_1 s_i s_j w_j, t' s_j w_j}}{\Delta^{\delta_2 s_i s_j w_j, t' s_j w_j}} \right)^{-a_{ji}},$$

where $\delta_1 = 1$ (resp. $\delta_2 = 0$) if $k^- > 1$ and $i_k^- = j$ (resp. $(k+1)^+ < m$ and $i_{(k+1)^+} = j$), otherwise $\delta_1 = 0$ (resp. $\delta_2 = 1$). Since the common denominator $(\Delta^{\delta_1 s_i s_j w_j, t' s_j w_j}, \Delta^{\delta_2 s_i s_j w_j, t' s_j w_j}, \Delta^{s_i s_j w_j, t' s_j w_j})^{-a_{ji}}$ of $P'_1$ and $P'_2$ is relatively prime with $\Delta_{s(i),w}$, it remains to show that

$$P'_1 (\Delta^{s_i s_j w_j, t' s_j w_j})^{-a_{ji}} + P'_2 (\Delta^{s_i s_j w_j, t' s_j w_j})^{-a_{ji}}$$

is divisible by $\Delta_{s(i),w}$. Since $P_1 + P_2$ is divisible by $\Delta_{s(i),w}$, it suffices to show that

$$(\Delta^{s_i s_j w_j, t' s_j w_j})^{-a_{ji}} - (\Delta^{s_i s_j w_j, t' s_j w_j})^{-a_{ji}}$$

is divisible by $\Delta_{s(i),w}$. This in turn follows from the fact that

$$\Delta^{s_i s_j w_j, t' s_j w_j} - \Delta^{s_i s_j w_j, t' s_j w_j}$$

is divisible by $\Delta_{s(i),w}$. But the last expression can be factored according to (4.12), and one of the factors is $\Delta_{s(i),w}^{-a_{ji}}$. This completes the proofs of Theorem 4.3 and property (3) in Lemma 3.1.

4.4. Property (4). Let us fix a reduced word $i = (i_1, \ldots, i_m) \in R(u, v)$, and an $i$-bounded index $n \in [2, m]$. Let $|i_n| = i \in \Pi$. Let $\mathbb{C}^m$ denote the $m$-dimensional vector space with coordinates $M_1, \ldots, M_{n-1}, M'_1, M'_{n-1}, \ldots, M_m$. Let $t_1, \ldots, t_m$ be rational functions on $\mathbb{C}^m$ given by (4.8) and (4.9), where $M_n$ is determined from (3.1). By Theorem 4.1, the map

$$\pi : (M_1, \ldots, M_{n-1}, M'_1, M'_{n-1}, \ldots, M_m) \mapsto x_i(t_1, \ldots, t_m)$$

is a birational isomorphism $\mathbb{C}^m \rightarrow L^{u,v}$ inverse to the map

$$x \mapsto (M_1(x), \ldots, M_{n-1}(x), M'_1(x), M'_{n-1}(x), \ldots, M_m(x)).$$

To prove property (4) in Lemma 3.1, it suffices to show that $\pi$ restricts to a regular map $\mathbb{C}^m_{\neq 0} \rightarrow L^{u,v}$.

Let us first show that $\pi$ restricts to a regular map $\mathbb{C}^m_{\neq 0} \rightarrow G$. In view of (4.8) and (4.9), if $k < n^-$ or $k > n$ then $t_k$ is a Laurent monomial in the variables $M_L$ with $l \neq n$. Thus we only need to show that the product $x_{i_{n^-}}(t_{n^-}) \cdots x_{i_n}(t_n)$ is a
regular function on $\mathbb{C}^m_{\neq 0}$. Without loss of generality, we can assume that $n^- = 1$. For each $k = 2, \ldots, n$, we define $p_k \in \mathbb{C}$ and a rational map $y_k : \mathbb{C}^m \to G$ as follows:

$$p_k = t_1 \prod_{1 < i < k} t_i^{a_{(i),i}} ,$$

while $y_k = x_i (p_k) x_{i-1} (t_k) x_i (p_{k+1})^{-1}$ for $k < n$, and $y_n = x_i (p_n) x_{i-1} (t_n)$. Then we have $x_i (t_1) \cdots x_{i-1} (t_n) = y_2 \cdots y_n$. Thus it suffices to show that each $y_k$ is a regular function on $\mathbb{C}^m_{\neq 0}$. As in section 4.3, we denote $E_k = \{ l \in [2, n-1] : \varepsilon (i_l) = \pm 1 \}$. Let us first prove that $y_n$ is a regular function on $\mathbb{C}^m_{\neq 0}$. We have four cases to consider.

**Case 1:** $i_1 = i_n = i$. Then

$$y_n = x_i (p_n) x_{i-1} (t_n) = \varphi_i \left( \begin{array}{c} 1 & p_n \\ 0 & 1 \end{array} \right) \varphi_i \left( \begin{array}{c} 1 & t_n \\ 0 & 1 \end{array} \right) = \varphi_i \left( \begin{array}{c} 1 & p_n + t_n \end{array} \right).$$

It remains to show that $p_n + t_n$ is a regular function on $\mathbb{C}^m_{\neq 0}$. This follows by a direct calculation using (4.8), (4.9) and (3.1): we obtain that

$$p_n + t_n = \frac{M'_{n}}{M_{1} M_{n}^{+}} \cdot \prod_{l > n \atop l \in E_{-} \cup \{0\}} M_{l}^{C_{l}^{n}} \cdot \prod_{l \leq n \atop l \in E_{+}} M_{l}^{-C_{l}^{n}}$$

is a Laurent monomial in $M_{1}, \ldots, M_{n-1}, M'_{n}, M_{n+1}, \ldots, M_{m}$.

**Case 2:** $i_1 = i, i_n = -i$. Then

$$y_n = x_i (p_n) x_{-i} (t_n) = \varphi_i \left( \begin{array}{c} 1 & p_n \\ 0 & 1 \end{array} \right) \varphi_{i} \left( \begin{array}{c} t^{-1} & 0 \\ 1 & t_n \end{array} \right) = \varphi_{i} \left( \begin{array}{c} p_n + t^{-1} \\ p_n t_n \end{array} \right).$$

It remains to show that each of $t_n$, $p_n t_n$, and $p_n + t_n^{-1}$ is a regular function on $\mathbb{C}^m_{\neq 0}$. By a direct calculation, $p_n t_n$ and $p_n + t_n^{-1}$ are Laurent monomials in $M_{1}, \ldots, M_{n-1}, M'_{n}, M_{n+1}, \ldots, M_{m}$, while $t_n = M_{n}/M_{n}^{+}$ is the sum of two such Laurent monomials; in fact, we have

$$p_n + t_n^{-1} = \frac{M'_{n}}{M_{1}} \cdot \prod_{l \in E_{-} \atop l \leq n} M_{l}^{-C_{l}^{n}}.$$

**Case 3:** $i_1 = -i, i_n = i$. Then

$$y_n = x_{-i} (p_n) x_{i} (t_n) = \varphi_{i} \left( \begin{array}{c} p_{n}^{-1} & 0 \\ 1 & p_n \end{array} \right) \varphi_{i} \left( \begin{array}{c} 1 & t_n \\ 0 & 1 \end{array} \right) = \varphi_{i} \left( \begin{array}{c} p_{n}^{-1} t_{n} \\ p_n + t_n \end{array} \right).$$

It remains to show that each of $p_{n}^{-1}, p_{n}^{-1} t_{n},$ and $p_n + t_n$ is a regular function on $\mathbb{C}^m_{\neq 0}$. By a direct calculation, $p_{n}^{-1} t_{n}$ and $p_{n} + t_{n}$ are Laurent monomials in $M_{1}, \ldots, M_{n-1}, M'_{n}, M_{n+1}, \ldots, M_{m}$, while $p_{n}^{-1}$ is the sum of two such Laurent monomials; in fact, we have

$$p_n + t_n = \frac{M'_{n}}{M_{n}^{+}} \cdot \prod_{l > n \atop l \in E_{-}} M_{l}^{C_{l}^{n}}.$$

**Case 4:** $i_1 = i_n = -i$. Then

$$y_n = x_{-i} (p_n) x_{-i} (t_n) = \varphi_{i} \left( \begin{array}{c} p_{n}^{-1} & 0 \\ 1 & p_n \end{array} \right) \varphi_{i} \left( \begin{array}{c} t_{n}^{-1} & 0 \\ 1 & t_n \end{array} \right) = \varphi_{i} \left( \begin{array}{c} p_{n}^{-1} t_{n}^{-1} \\ p_n + t_{n}^{-1} p_n t_n \end{array} \right).$$
It remains to show that each of \((p_n t x)^{\pm 1}\) and \(p_n + t_n^{-1}\) is a regular function on \(\mathbb{C}_{\not= 0}^m\). By a direct calculation, both \(p_n t_n\) and \(p_n + t_n^{-1}\) are Laurent monomials in \(M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m\); in fact, we have
\[
p_n + t_n^{-1} = M'_n \cdot \prod_{t \in E_+ \setminus \{0\}} M_i^{-C_{1n}}.
\]

Now let \(1 < k < n\), and suppose \(|i_k| = j \in \Pi\). To show that \(y_k\) is a regular function on \(\mathbb{C}_{\not= 0}^m\), we have to consider another four cases.

**Case 1:** \(i_1 = i, i_k = j\). Then \(y_k = x_i(p_k)x_j(t_k)x_i(-p_k)\). Let \(p = p_k^{-1}, q = p_k^{-a_i/j}t_k\).

Clearly, both \(p\) and \(q\) are regular functions on \(\mathbb{C}_{\not= 0}^m\). The desired regularity of \(y_k\) becomes a consequence of the following lemma.

**Lemma 4.4.** For any two distinct \(i, j \in \Pi\), the map \(\mathbb{C}_{\not= 0}^m \times \mathbb{C} \rightarrow N\) given by \((p, q) \mapsto x_i(p^{-1})x_j(p^{-a_i/j}q)x_i(-p^{-1})\) extends to a regular map \(\mathbb{C}^2 \rightarrow N\).

In order not to interrupt the exposition, we will prove this lemma in the end of this section.

**Case 2:** \(i_1 = i, i_k = -j\). Then
\[
y_k = x_i(p_k)x_j(t_k)x_i(p_k^{-a_i/j})^{-1} = x_j(t_k) = x_j(M_k/M_{k+})
\]
(see [3, Proposition 7.2]), which is a regular function on \(\mathbb{C}_{\not= 0}^m\).

**Case 3:** \(i_1 = -i, i_k = j\). Then
\[
y_k = x_{-i}(p_k)x_j(t_k)x_{-i}(p_k)^{-1} = x_j(p_k^{-a_i/j}t_k)
\]
(see [3, Proposition 7.2]), which is a regular function on \(\mathbb{C}_{\not= 0}^m\) since \(p_k^{-a_i/j}t_k\) is a Laurent monomial in \(M_1, \ldots, M_{n-1}, M_{n+1}, \ldots, M_m\).

**Case 4:** \(i_1 = -i, i_k = -j\). Then
\[
y_k = x_{-i}(p_k)x_{-j}(t_k)x_{-i}(p_k^{-a_i/j})^{-1}
\]
Using \([3, (2.5)]\), and the commutation relation \([3, (2.5)]\), we can rewrite \(y_k\) as follows:
\[
y_k = y_i(p_k)y_j(p_k^{-a_i/j}t_k)y_i(-p_k)t_k^{-a_i/j}\).
\]
The “Cartan factor” \(t_k^{-a_i/j}\) is clearly a regular function on \(\mathbb{C}_{\not= 0}^m\). As for the first factor \(y_i(p_k)y_j(p_k^{-a_i/j}t_k)y_i(-p_k)\), after applying the automorphism \(x \mapsto x^T\) of \(G\), it becomes \(x_i(p_k)x_j(p_k^{-a_i/j}t_k)x_i(-p_k)\), and its regularity follows from Lemma 4.4 with \(p = p_k^{-1}\) and \(q = t_k\).

We have proved (modulo Lemma 4.4) that the map
\[
\pi : (M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m) \mapsto x_i(t_1, \ldots, t_m)
\]
is a regular map \(\mathbb{C}_{\not= 0}^m \rightarrow G\). To complete the proof of property (4), it remains to show that the image of \(\pi\) is contained in \(L^{u, v}\). By Theorem 4.1, this image is contained in the closure of \(L^{u, v}\). Recall from \([3]\) that \(L^{u, v}\) is determined inside its closure by the conditions \(\Delta_{\omega_j, u^{-i-1}\omega_j}(x) \neq 0\) for all \(j\). The results in \([3]\) also imply that, for any \(x \in L^{u, v}\), we have \(\Delta_{\omega_j, u^{-i-1}\omega_j}(x) = (\Delta_{\omega_{j', u^{-i-1}\omega_{j'}}}(\psi^{u, v}(x)))^{-1} = M_{k(j)}(x)^{-1}\), where \(k(j)\) is the first occurrence of the index \(\pm j\) in \(j\). It follows that
\[
\Delta_{\omega_j, u^{-i-1}\omega_j}(\pi(M_1, \ldots, M_{n-1}, M'_n, M_{n+1}, \ldots, M_m)) = M_{k(j)}^{-1} = 0
\]
on \(\mathbb{C}_{\not= 0}^m\), and we are done.
5. Some examples and applications

5.1. Cones of regular monomials. Let us again fix a pair \((u, v)\) in \(W \times W\), and a reduced word \(i = (i_1, \ldots, i_m)\) in \(R(u, v)\). In view of Theorem 4.1, a generic element \(x = x_i(t_1, \ldots, t_m)\), so the factorization parameters \(t_k\) are well-defined rational functions on \(L^{u,v}\) given by (4.8) and (4.9). Combining these formulas with Lemma 3.3 yields the following corollary.

**Proposition 5.1.** A Laurent monomial \(x_i^{a_1} \cdots x_m^{a_m}\) is a regular function on \(L^{u,v}\) if and only if

\[
-\varepsilon(i_n) a_n - a_n - \sum_{\substack{n^{-k} < k < n \varepsilon(i_k) = 1}} C_{nk} a_k \geq 0
\]

for any i-bounded \(n \in [1, m]\).

Two special cases are worth mentioning. If \(v = e\) then \(\varepsilon(i_k) = -1\) for all \(k\), and the inequalities (5.1) take the form \(a_n \geq a_n\). If \(u = e\) then \(\varepsilon(i_k) = 1\) for all \(k\), and the inequalities (5.1) take the form

\[
-a_n - a_n - \sum_{n^{-k} < k < n} C_{nk} a_k \geq 0;
\]

the cone defined by these inequalities appeared in a different context in [7], and also in [8].

5.2. Intersections of two open opposite Schubert cells. Let us illustrate Theorem 2.2 by the case when \(u = e\) and \(w = w_0\), the longest element in \(W\). In this case, \(L^{u,v}\) is biregularly isomorphic to the intersection of two open opposite Schubert cells \(C_{w_0} \cap w_0 C_{w_0}\), where \(C_{w_0} = (Bw_0B)/B\) is the open Schubert cell in the flag variety \(G/B\). These opposite cells appeared in the literature in various contexts, and were studied (in various degrees of generality) in [1][3][4][11][12][13]. Let \(C\) denote the number of connected components of \(L^{e,w_0}(\mathbb{R})\); to emphasize the
dependency on \( G \), we shall write \( C = C(X_r) \), where \( X_r = A_r, B_r, \ldots, G_2 \) is the type of \( G \) in the Cartan-Killing classification.

The numbers \( C(A_r) \) were determined in \cite{12,13}: it turns out that \( C(A_1) = 2, C(A_2) = 6, C(A_4) = 20, C(A_5) = 52, \) and \( C(A_r) = 3 \cdot 2^r \) for \( r \geq 5 \). Theorem \ref{Proposition 5.2} allows us to extend this result to all other simply-laced types.

**Proposition 5.2.** If \( X_r \) is one of the types \( A_r \) (\( r \geq 5 \)), \( D_r \) (\( r \geq 4 \)), \( E_6, E_7, \) or \( E_8 \) then \( C(X_r) = 3 \cdot 2^r \).

**Proof.** Following \cite{14, Definition 3.10], we say that a graph is \( E_6 \)-compatible if it is connected, and it contains an induced subgraph with 6 vertices isomorphic to the Dynkin graph \( E_6 \) (see Fig. 1).

![Figure 1. The Dynkin graph \( E_6 \).](image)

Combining Theorem \ref{Proposition 5.2} with \cite{14, Corollary 3.12}, we obtain the following sufficient condition for the equality \( C(X_r) = 3 \cdot 2^r \): it holds provided \( G \) is simply-laced, and there exists \( i \in R(w_0) \) such that the induced subgraph of \( \Sigma(i) \) (see Definition \ref{Definition 2.1}) on the set of all \( i \)-bounded vertices is \( E_6 \)-compatible. In \cite{13} this condition was checked for the type \( A_5 \). Therefore, it also holds for any simply-laced Dynkin graph that contains an induced subgraph of type \( A_5 \), that is, for \( A_r \) (\( r \geq 5 \)), \( D_r \) (\( r \geq 6 \)), \( E_6, E_7, \) and \( E_8 \). It remains to check this condition for the type \( D_4 \) (the statement for \( D_5 \) then follows). Let \( \Pi = \{1, 2, 3, 4\} \) with the branching vertex 3. Take the reduced word \( i = (1, 2, 3, 1, 2, 3, 4, 3, 1, 2, 3, 4) \in R(w_0) \). By inspection, the induced subgraph of \( \Sigma(i) \) with \( i \)-bounded vertices 4, 5, 9, 10, 11, and 12 is isomorphic to the Dynkin graph \( E_6 \), and we are done.

The numbers \( C(B_2) \) and \( C(G_2) \) were determined in \cite{11}: it turns out that \( C(B_2) = 8 \) and \( C(G_2) = 11 \). Theorem \ref{Proposition 5.2} gives a simpler way to prove these answers. In the case of \( B_2 \), take \( i = (j, i, j, i) \) with \( a_{ij} = -2 \) and \( a_{ji} = -1 \). Then \( \Gamma_i(\mathbb{F}_2) \) is the group of transformations of \( \mathbb{F}_2^4 \) generated by \( \tau_3 : \xi_3 \to \xi_3 + \xi_1 \) and \( \tau_4 : \xi_4 \to \xi_4 + \xi_3 + \xi_2 \). It is easy to see that the action of \( \Gamma_i(\mathbb{F}_2) \) in \( \mathbb{F}_2^4 \) has 8 orbits: four fixed points 0000, 0001, 0110, and 0111, two 2-element orbits 0010 \( \leftrightarrow \) 0011 and 0100 \( \leftrightarrow \) 0101, and two 4-element orbits 1000 \( \leftrightarrow \) 1010 \( \leftrightarrow \) 1011 \( \leftrightarrow \) 1101 \( \leftrightarrow \) 1100 \( \leftrightarrow \) 1110 \( \leftrightarrow \) 1111.

The case of \( G_2 \) is treated in a similar fashion. Take \( i = (j, i, j, i, j, i) \) with \( a_{ij} = -3 \) and \( a_{ji} = -1 \). Then \( \Gamma_i(\mathbb{F}_2) \) is the group of transformations of \( \mathbb{F}_2^6 \) generated by the transvections \( \tau_n : (3 \leq n \leq 6) \) acting by \( \tau_n : \xi_n \to \sum_{|k-n| \leq 2} \xi_k \). It is easy to see that the action of \( \Gamma_i(\mathbb{F}_2) \) in \( \mathbb{F}_2^6 \) has 11 orbits: four fixed points 000000, 001001, 001110, and 000111; six 8-element orbits, and one 12-element orbit. The 8-element orbits are depicted in Fig. 3 (one has to take the first depicted orbit together with its translates by the 3 non-zero fixed vectors; and the second depicted orbit together with its translate by the vector 001110); the 12-element orbit is depicted in Fig. 3.
Figure 2. The 8-element orbits for $G_2$.

Figure 3. The 12-element orbit for $G_2$.

Remark 5.3. Computing the numbers $C(B_r)$ and $C(C_r)$ for $r \geq 3$ seems to be a challenging problem. Since the transvections $\tau_n$ are no longer symplectic in this case, one cannot use [4, Corollary 3.12] (at least, not in a straightforward way).

5.3. Dual canonical basis for the type $B_2$. In conclusion, we briefly discuss a potential application of the above results. Let $G/N$ be the base affine space for $G$. It is well-known that the ring of regular functions $\mathbb{C}[G/N]$ (that is, regular functions on $G$ invariant under right translations by elements of $N$) is the multiplicity-free sum of all irreducible finite-dimensional representations of $G$. Let $B$ denote the dual canonical basis in $\mathbb{C}[G/N]$ (more precisely, $B$ is the “classical limit” of the dual canonical basis in the $q$-deformed ring $\mathbb{C}_q[G/N]$). Despite much progress in studying properties of the canonical bases, an explicit construction of $B$ still remains to be found. It is known that $B$ contains all “Plücker coordinates” $P_\gamma = \Delta_{\gamma, \omega_i}$ for $i \in \Pi$. 
and \( \gamma \in W. \) We suspect that \( B \) also contains all functions \( \Delta_1^i \) in Theorem 4.3 corresponding to reduced words \( \mathbf{i} \) consisting of elements of \( \Pi. \) Thus, these functions together with the Plücker coordinates \( P_\gamma \) are among the building blocks for \( B. \)

As an illustration, consider the case when \( G \) is of type \( B_2, \) i.e., \( \Pi = \{ i, j \} \) with \( a_{ij} = -2 \) and \( a_{ji} = -1. \) The basis \( B \) in this case was found in [9] (even before the “official” discovery of canonical bases). Translating the results in [9] into our present notation, we obtain the following.

There are 8 Plücker coordinates: \( P_\omega_1, P_\omega_j, P_{s_1\omega_1}, P_{s_j\omega_1}, P_{s_js_1\omega_1}, P_{s_i s_j\omega_j}, P_{w_0\omega_1}, \) and \( P_{w_0\omega_j}. \) Let us also denote \( Q_{\omega_j} = \Delta_{(i, j, i)} \) and \( Q_{2\omega_i} = \Delta'_{(j, i, j)}; \) thus, these functions are defined from the equations

\[
(5.2) \quad Q_{\omega_j} P_{s_1\omega_1} = P_{s_j s_1\omega_1} P_{\omega_1} + P_{\omega_j} P_{w_0\omega_1},
\]

and

\[
(5.3) \quad Q_{2\omega_i} P_{s_j\omega_j} = P_{s_j s_1\omega_1} P_{\omega_j} + P_{\omega_i} P_{w_0\omega_j}.
\]

The main result of [9] can be now summarized as follows.

**Proposition 5.4.** The dual canonical basis \( B \) of \( \mathbb{C}[G/N] \) consists of all monomials in 10 variables \( P_{\omega_1}, \ldots, P_{w_0\omega_1}, Q_{\omega_j}, Q_{2\omega_i}, \) with the following property: if this monomial contains variables in two vertices of the “magical hexagon” in Fig. 4 then these two vertices are adjacent.

![Figure 4. The “magical hexagon” for \( B_2. \)](image)

We see that \( B \) is the union (not disjoint!) of six families of elements corresponding to the edges of the hexagon in Fig. 4; each family consists of all monomials in six variables \( P_{\omega_1}, P_{\omega_j}, P_{s_1\omega_1}, P_{s_j\omega_1}, P_{w_0\omega_1}, P, Q, \) where \( P \) and \( Q \) lie in two adjacent vertices of the hexagon.

Note that the equations (5.2) and (5.3) can be now interpreted as expansions in the basis \( B \) of two “forbidden” monomials corresponding to diagonals of the hexagon. There are 7 more such identities corresponding to the remaining 7 diagonals:

\[
(5.4) \quad Q_{\omega_j} P_{s_j s_1\omega_1} = P_{w_0\omega_1} P_{\omega_1} + P_{s_j\omega_j} P_{w_0\omega_1};
\]

\[
(5.5) \quad Q_{2\omega_i} P_{s_j s_1\omega_1} = P_{2\omega_i} P_{w_0\omega_1} + P_{s_j\omega_j} P_{w_0\omega_1};
\]

\[
(5.6) \quad P_{s_1\omega_1} P_{s_j s_1\omega_1} = P_{\omega_i} P_{w_0\omega_1} + Q_{2\omega_i};
\]

\[
(5.7) \quad P_{s_j\omega_j} P_{s_j s_1\omega_1} = P_{\omega_j} P_{w_0\omega_1} + Q_{2\omega_j};
\]

\[
(5.8) \quad P_{s_1\omega_1} P_{s_j\omega_j} = P_{s_j s_1\omega_1} P_{\omega_j} + P_{\omega_i} Q_{\omega_j};
\]
\[ P_{\omega_i \omega_j} = P_{\omega_i \omega_j} P_{\omega_0} + P_{\omega_0} P_{\omega_j} Q_{\omega_i} \]

\[ Q_{2 \omega_i} Q_{\omega_j} = P_{\omega_i \omega_j} P_{\omega_0} P_{\omega_i} + P_{\omega_i \omega_i} P_{\omega_i} P_{\omega_j} \]

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Department of Mathematics, Northeastern University, Boston, MA 02115

E-mail address: andrei@neu.edu