Study of the flux tube thickness in 3D lattice gauge theories by means of 2D spin models

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Abstract. We study the flux tube thickness in the confining phase of the (2+1) D SU(2) lattice gauge theory near the deconfining phase transition. Following the Svetitsky–Yaffe conjecture, we map the problem to the study of the \langle \sigma \sigma \rangle correlation function in the 2D spin model with Z_2 global symmetry (i.e. the 2D Ising model) in the high temperature phase. Using the form factor approach we obtain an explicit expression for this function and from it we infer the behaviour of the flux density of the original (2+1) D LGT. Remarkably, the result we obtain for the flux tube thickness agrees (apart from an overall normalization) with the effective string prediction for the same quantity.

Keywords: correlation functions, form factors, deconfinement

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1. Introduction

The distinctive feature of the interquark potential in a confining gauge theory is that the colour flux is confined into a thin flux tube, joining the quark–antiquark pair. As is well known the quantum fluctuations of this flux tube (which are assumed to be described by a suitable effective string model) lead to a logarithmic increase of the width of the flux tube as a function of the interquark distance $R$. This behaviour was discussed many years ago by Lüscher, Münster and Weisz in [1] and is one of the most stringent predictions of the effective string description of confining LGT’s. Indeed for a non-confining theory one would instead expect a linear increase of the width of the flux tube.

A natural question is what happens of this picture at the deconfinement point. One would naively expect a sudden jump of the flux tube thickness from a log to a linear dependence on the interquark distance. However we shall show in this paper that this is a misleading picture. Indeed the flux tube width also depends on the finite temperature of the theory. In the standard finite temperature setting of LGT’s in which the quarks are represented by Polyakov loops this means that the flux tube thickness (and its $R$ dependence) also depend on the lattice size in the ‘timelike’ compactified direction.

A tentative answer to this question can be obtained in the effective string framework. As we shall see in section 2.4 by using a duality transformation it is possible to show that as the temperature increases the log behaviour smoothly moves to a linear behaviour, thus excluding a log to linear transition at the deconfinement point. However this result strongly relies on the effective string approximation (even worse, on the Gaussian limit of the effective string) and it would be nice to have some kind of independent evidence.
Unfortunately the flux tube thickness (and in particular its dependence on the interquark distance) is very difficult to study by means of Monte Carlo simulations. The only existing numerical estimates are for the 3D Ising gauge model in which, thanks to the efficiency of the existing Monte Carlo algorithms for this model, large enough values of the interquark distance could be reached and unambiguous signatures of the logarithmic increase of the flux tube thickness (at zero temperature) could be observed [2].

In this paper we propose an alternative way to address the above question in the vicinity of the deconfinement transition using the Svetitsky–Yaffe conjecture which is a very powerful tool for studying the finite T behaviour of a confining LGT in the vicinity of the deconfinement point, at least for those LGT’s whose deconfinement transition is of second order. The Svetitsky–Yaffe conjecture [3] states that the deconfinement transition of a (d + 1) LGT lies in the same universality class as the magnetization transition of a d-dimensional spin model with symmetry group given by the centre of the original gauge group.

This gives us a non-trivial opportunity to check the effective string predictions. If we choose a (2+1)-dimensional LGT with a gauge group with centre $Z_2$ (like the gauge Ising model or the $SU(2)$ or $SP(2)$ LGT’s which all have continuous deconfinement transitions), the target spin model is the 2D Ising model in the high temperature symmetric phase for which several exact results are known. In particular we shall see that it is possible to study analytically the equivalent of the flux tube thickness. Remarkably, the results that we find agree with the effective string ones, thus strongly supporting the idea of a smooth transition from a log to a linear behaviour as the temperature increases.

This paper is organized as follows. Section 2 is devoted to the discussion of some background material on the effective string description of the flux tube thickness and on the Svetitsky–Yaffe conjecture. In section 3 we study the $\langle \sigma_1 \epsilon_2 \sigma_3 \rangle$ correlator in the 2D Ising model, discuss its asymptotic behaviour and extract from it a prediction for the flux tube thickness. Finally in section 4 we compare it with the effective string prediction and discuss some further features of our analysis.

2. Background

2.1. Definition of the flux tube thickness

The lattice operator which is commonly used to evaluate the flux density in the presence of a pair of Polyakov loops (or equivalently in the presence of a Wilson loop) is the following correlator:

$$\langle \phi(x_0, x_1, x_2, R) \rangle = \frac{\langle P(0, 0)P^+(0, R)U_p(x_0, x_1, x_2) \rangle}{\langle P(0, 0)P^+(0, R) \rangle} - \langle U_p \rangle \tag{1}$$

where $P(x_1, x_2)$ denotes a Polyakov loop in the spacelike position $(x_1, x_2)$ (in the above equation we have chosen for simplicity to locate the Polyakov loops at the positions $(0, 0)$ and $(0, R)$ and the $x_0$ coordinate runs in the compactified timelike direction) while $U_p(x)$ denotes a plaquette located at $x \equiv (x_0, x_1, x_2)$. The different possible orientations of the plaquette measure the different components of the flux. In the following we shall neglect this dependence (see however the comment at the end of the next section). In order to avoid boundary effects we then concentrate on the mid-point of the Polyakov loop correlator choosing a generic value of $x_0$ and fixing $x_2 = R/2$. With this choice the
flux density will only be a function of the interquark distance $R$ and of the transverse coordinate $x_1$. In the $x_1$ direction the flux density shows a Gaussian-like shape (see for instance figure 2 of [2]). The width of this Gaussian is the quantity which is usually denoted as the ‘flux tube thickness’: $w(R, N_t)$. This quantity only depends on the interquark distance $R$ and on the lattice size in the compactified timelike direction $N_t$, i.e. on the inverse temperature of the model (this dependence was implicit in the above definition). By tuning $N_t$ we can thus study the flux tube thickness in the vicinity of the deconfinement transition.

2.2. Dimensional reduction and the Svetitsky–Yaffe conjecture

The Svetitsky–Yaffe approach [3] to the study of finite temperature $(d + 1)$-dimensional gauge theories consists in the construction of a $d$-dimensional effective theory from the original one by integrating out the spacelike links and keeping as the only remaining degrees of freedom the Polyakov loops which are then treated as spins of the effective $(d - 1)$ model. Besides the standard coupling between nearest neighbour spins, the resulting effective model will contain in general couplings between far apart spins and among more than two spins. The main result of Svetitsky and Yaffe was to show that these couplings decay exponentially with the distance and thus the model remains in the same universality class of the simple nearest neighbour coupled model. Moreover they were able to show that if the original $(d + 1)$ LGT is characterized by a gauge group $G$ then the effective $d$-dimensional spin model will have as symmetry group the centre of $G$.

The S–Y approach is very general. It is well defined both for first-order phase transitions and for continuous ones. In the case of first-order deconfinement phase transitions one cannot estimate a priori how good the effective description obtained in this way is. In contrast in the case of continuous phase transitions (both for the original gauge theory and for the effective spin model) a standard renormalization group analysis shows that the critical behaviours of the two models must be the same. This is the case if we choose for instance $SU(2)$ as the gauge group for the $(2 + 1)$-dimensional LGT. Then the symmetry group of the effective 2D spin model will be $Z_2$ (i.e. it is the Ising model). Another equivalent choice would be to study the $(2 + 1)$-dimensional LGT with gauge group $Z_2$ (the Ising gauge model) whose centre obviously is again $Z_2$.

As mentioned above, the 2D Ising model can be used as an effective description for the original $SU(2)$ LGT not only at the critical point but also in the scaling region. Obviously we leave the critical point larger and larger deviations between the lattice gauge theory and the effective spin model must be expected.

In the S–Y construction the Polyakov loops of the original $SU(2)$ LGT are mapped onto the spins of the Ising model, the confining phase of the LGT into the high temperature phase of the spin model, while the combination $\sigma(N_t)N_t$ (where $\sigma(N_t)$ denotes the finite temperature value of the string tension while in the following $\sigma$ with no explicit dependence will denote by default the zero-temperature string tension) is mapped into the mass scale of the spin model (i.e. the inverse of the correlation length) and sets the scale of the deviations from the critical behaviour.

In order to describe in the dimensionally reduced model the expectation value of equation (1), one must extend this mapping also to the plaquette operator. This non-trivial problem was discussed a few years ago by Gliozzi and Provero in [4], where they...
were able to show that at the critical (deconfinement) point the plaquette operator of the \((2 + 1)\) LGT is mapped into a mixture of the energy and identity operators of the 2D spin model. This identification can be extended also outside the critical point since in the scaling region the Ising field theory can be seen as the thermal perturbation of the Ising critical point, its Hilbert space is simply a deformation of the critical one: it is composed by three conformal families only: the identity, the spin and the energy family. The Polyakov loop, being \(Z_2\) odd, must belong to the spin family, i.e. it will be a mixture of the primary spin operator and of all its descendents. The plaquette operator, being \(Z_2\) even, must be given by the most general mixture of operators belonging to the identity and the energy families and will be dominated by the primary energy operator (the next irrelevant operator in this expansion will be the energy–momentum tensor from the identity family).

As a consequence the combination of gauge invariant operators which measures the density of chromoelectric flux in a meson is mapped into a three-point function \(\langle \epsilon \sigma \bar{\sigma} \rangle\) where in the Ising case (i.e. for the LGT with a \(Z_2\) centre symmetry) \(\bar{\sigma} = \sigma\). Different components of the flux (i.e. different orientations of the plaquette) correspond to different coefficients in the linear combination which relates the plaquette operator of the LGT to the energy and identity operators of the spin model. These coefficients will play no role in the discussion and we shall neglect them in the following.

### 2.3. Agreement between 2D spin model estimates and effective string predictions

If we describe the correlator of two Polyakov loops in the high temperature regime of a confining lattice gauge theory in \(d = 3\) dimensions using the Nambu–Goto effective string we obtain exactly a collection of \(K_0\) Bessel functions. This was observed for the first time by Lüscher and Weisz [5] using a duality transformation and then derived in the covariant formalism in [6]. The expression which one obtains is

\[
\langle P(0,0)P(0,R)\rangle = \sum_{n=0}^{\infty} |v_n|^2 2R \left( \frac{\tilde{E}_n}{2\pi R} \right)^{(1/2)(d-1)} K_{(1/2)(d-3)}(\tilde{E}_n R) 
\]

(see equation (3.2) of [5]). Notice that in \(d = 3\) dimensions the \(R\) dependence in front of the Bessel functions cancels exactly.

The argument of the \(K_0\) functions is given by the product of the interquark distance \(R\) and the closed string energy levels:

\[
\tilde{E}_n = \sigma N_t \left\{ 1 + \frac{8\pi}{\sigma N_t^2} \left[ -\frac{1}{24} (d-2) + n \right] \right\}^{1/2}
\]

(see equation (C5) of [5]).

It is easy to see that in the large \(R\) limit only the lowest state \((n = 0)\) survives and we end up with a single \(K_0\) function:

\[
\lim_{R \to \infty} \langle P(0,0)P(0,R)\rangle \sim K_0(\tilde{E}_0 R)
\]

(4)

where

\[
\tilde{E}_0 = \sigma N_t \left\{ 1 - \frac{\pi}{3\sigma N_t^2} \right\}^{1/2}.
\]

(5)
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From this expression we read the Nambu–Goto prediction for the finite temperature dependence of the string tension:

\[ \sigma(N_t) \equiv \tilde{E}_0/N_t = \sigma \left(1 - \frac{\pi}{3\sigma N_t^2}\right)^{1/2}. \]  

(6)

As is well known this expression cannot be exact since it predicts mean value critical indices for the deconfinement transition; however it turns out to be a very good approximation up to rather high temperatures (we shall comment further on this below).

Looking at equation (4) it is tempting to identify the Polyakov loops correlator in the large \( R \) limit with the spin–spin correlator in the 2D Ising model which is exactly given by a \( K_0 \) Bessel function with argument \( mR \), \( m \) being the mass of the Ising model. This is the origin of the relation mentioned above between the mass of the effective Ising model \( m \) and the product \( \sigma(N_t)N_t \) of the LGT. Notice as an aside that this correspondence is more general than this particular Ising case, since any 2D spin model with a spectrum which starts with an isolated pole has a spin–spin correlator dominated at large distance by a \( K_0 \) Bessel function without prefactors.

A few comments are in order at this point:

• The experience with Polyakov loop correlators [7]–[10] shows that the Nambu–Goto action is a good approximation (indeed a very good one) for very large \( R \) (much larger than \( N_t \)) and values of \( N_t \) such that \( N_t \geq \sqrt{4/\sigma} \). For higher temperatures (i.e. smaller values of \( N_t \)) the deviations due to the ‘mean field’ nature of the Nambu–Goto approximation cannot be neglected.

• It will be useful in the following to define the combination \( t_f \equiv 1/N_t \sqrt{\sigma} \). With this definition \( t_f \) is dimensionless and has the meaning of a finite temperature. The Nambu–Goto approximation would suggest a deconfinement transition for \( t_f = \sqrt{\pi/3} \sim 1.02 \) while the bound \( N_t \geq \sqrt{4/\sigma} \) mentioned above corresponds to \( t_f < 0.5 \). It is interesting to observe that the Nambu–Goto prediction for the critical temperature is different, but not by too much, from the known Monte Carlo estimates for LGT’s: in the gauge Ising model the deconfinement transition occurs at \( t_f \sim 1.2 \) [11] while in the \( SU(2) \) case we have \( t_f \sim 1.13 \) [12].

• Looking at \( \tilde{E}_0 \) we see that it is useful to define the dimensionless coefficient

\[ \rho \equiv \left(1 - \frac{\pi t_f^2}{3}\right)^{1/2} \]  

(7)

from which we find

\[ m = \sigma N_t \rho. \]  

(8)

The region (\( t_f \leq 0.5 \)) in which we can trust the Nambu–Goto approximation corresponds to the range \( 0.8 \leq \rho \leq 1 \)

2.4. Effective string predictions for the flux tube thickness

The simultaneous dependence of the flux tube thickness on the two variables \( R \) and \( N_t \) can be evaluated exactly only in the Gaussian limit. Including higher order terms in the effective string action makes the problem too difficult (even if some recent results in the
framework of the covariant quantization suggest that some simplification could occur if
one chooses to study the whole Nambu–Goto action [6, 13]).

For the details of the calculations we refer the reader to the paper [2] (see also [14])
For our purpose we are only interested in the two asymptotic limits: large \( N_t \) and finite \( R \) (which is the zero-temperature limit where we expect a log-type behaviour) and the
opposite one: large \( R \) and small \( N_t \), which is high temperature limit.

One finds

\[
    w^2 \sim \frac{1}{2\pi\sigma} \log \left( \frac{R}{R_c} \right) \quad (N_t \gg R \gg 0)
\]

\[
    w^2 \sim \frac{1}{2\pi\sigma} \left( \frac{\pi R}{6N_t} + \log \left( \frac{N_t}{2\pi} \right) \right) \quad (R \gg N_t).
\]

As is easy to see, in the second limit the logarithmic dependence is on \( N_t \) (the inverse of
the temperature) and not on \( R \) which appears instead in the linear correction.

3. The three-point correlators in the 2D Ising model

As we have seen, the study of the width of long colour flux tubes in the \((2 + 1) \) D LGT’s
with gauge group \( \mathbb{Z}_2 \), \( SU(2) \) or \( SP(2) \) can be translated into the study of the ratio of correlators

\[
    \frac{\langle \sigma(x_1)\epsilon(x_2)\sigma(x_3) \rangle}{\langle \sigma(x_1)\sigma(x_3) \rangle}
\]

in the high temperature phase of the 2D Ising model in zero magnetic field.

Since we are interested in the large distance behaviour of such a quantity, we will
use the so-called form factors technique (see [15] for its application in the context of the 2D Ising model without a magnetic field). Form factors are defined as suitable matrix elements of an operator \( \phi \) connecting the vacuum and an arbitrary \( n \)-particle asymptotic state\(^4\)

\[
    F_{\phi_{a_1...a_n}}(\theta_1, \ldots, \theta_n) = \langle 0|\phi(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n) \rangle.
\]

When the theory is integrable, and the \( S \)-matrix is exactly known, they can be computed exactly as solutions of certain functional equations [16, 17]. In the present case the theory is free and the \( S \)-matrix is simply \( S = -1 \).

Such a technique turned out to be an effective tool for giving approximate expressions
for the large distance behaviour of the two-point correlators in Integrable QFTs, and recently it has been employed to analyse the three-spin correlator in the three-states Potts model [18]. In brief, it is possible to rewrite a generic correlator as a spectral expansion whose building blocks are just the form factors. For example, for the connected two-point correlator we have in general

\[
    \langle \phi(x)\phi(0) \rangle_c = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\theta_1 \ldots d\theta_n \frac{1}{(2\pi)^n} \left| F_{\phi}^{\phi}(\theta_1, \ldots, \theta_n) \right|^2 e^{-m|x|E_n}
\]

where \( E_n \) is the energy of the \( n \)-particle state as a function of the rapidity.

\(^4\) We use the parametrization of energy and momentum in terms of the rapidity \( \theta \): \( E = m \cosh \theta \), \( P = m \sinh \theta \).
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Figure 1. Generic configuration for the three-point function \( \langle \sigma(x_1)\epsilon(x_2)\sigma(x_3) \rangle \).

Similar expressions can be written for \( n \)-point connected correlators; in particular for the three-point correlation function we can proceed in close analogy with the analysis of [18]. Hence we can write (see figure 1)

\[
\langle \sigma(x_1)\epsilon(x_2)\sigma(x_3) \rangle = \int_{-\infty}^{\infty} d\theta_1 d\theta_2 (2\pi)^2 \left( F_{1}\sigma \, F_{2}\epsilon \right)^2 \left( \theta_{12} + i\alpha_+ \right) e^{-m(R_{12} \cosh \theta_1 + R_{23} \cosh \theta_2)} \\
+ \frac{1}{2} \int_{-\infty}^{\infty} d\theta_1 d\theta_2 d\theta_3 (2\pi)^3 \left( F_{1}\sigma \, F_{2}\epsilon \right)^2 \left( \theta_{23} \right) F_{3}\sigma \left( \theta_{32}, \theta_{21} + i\psi, \theta_{31} + i\psi \right) \\
\times e^{-m(R_{13} \cosh \theta_1 + R_{12} \cosh \theta_1 + R_{23} \cosh \theta_2 + \cosh \theta_3)} \\
+ \frac{1}{2} \int_{-\infty}^{\infty} d\theta_1 d\theta_2 d\theta_3 (2\pi)^3 \left( F_{1}\sigma \, F_{2}\epsilon \right)^2 \left( \theta_{12} \right) F_{3}\sigma \left( \theta_{21} + i\phi, \theta_{32} + i\phi \right) \\
\times e^{-m(R_{13} \cosh \theta_1 + R_{12} \cosh \theta_1 + R_{23} \cosh \theta_2 + \cosh \theta_3)} + \ldots .
\] (14)

The main ingredients which enter the previous formula are the first few form factors of the operators \( \sigma \) and \( \epsilon \). Let us briefly review their properties (see [15]).

Spin \( \sigma \) and disorder \( \mu \) operators

Since we are dealing with a theory of free Majorana fermions, the \( S \)-matrix is simply \( S = -1 \). As a consequence of the fact that we are in the high \( T \) phase of the theory, symmetry implies that the form factors of \( \sigma \) and \( \mu \) are non-zero on odd and even particle states respectively.

The VEV of \( \mu \) has been known for a long time and happens to be

\[
\langle \mu \rangle = F_{0}\mu = B|\tau|^{1/8} = \frac{B}{(2\pi)^{1/8}} m^{1/8} = C m^{1/8}, \quad B = 1.70852190 \ldots (15)
\]

where we used the exact relation between the coupling constant \( \tau \) (the reduced temperature) and the mass of the fermion \( m = 2\pi\tau \). The cluster condition [19] fixes the relative normalization of the FF’s of \( \sigma \) and \( \mu \), and implies \( F_{1}\sigma = \langle \mu \rangle \). The explicit expression for \( F_{3}\sigma \) is given by

\[
F_{3}\sigma (\theta_1, \theta_2, \theta_3) = i F_{1}\sigma \prod_{i<j=1}^{3} \tanh \frac{\theta_{ij}}{2}
\] (16)
and the normalization constant can be fixed by means of the residue equation on the annihilation poles:

$$-i \text{Res}_{\theta_1=\pm \theta_2=\pm \theta_3} F_3^\sigma (\theta_1, \theta_2, \theta_3) = (1 - S) F_1^\sigma = 2 F_1^\epsilon. \quad (17)$$

**Stress-energy tensor $\Theta$ and energy density $\epsilon$**

The FF’s of the perturbing operator $\epsilon$ can be extracted from those of the stress-energy tensor $\Theta$. Their relationship is given by (we recall that $\Delta_\epsilon = 1/2$)

$$\Theta = 4\pi (1 - \Delta_\epsilon) \tau \epsilon = 2\pi \tau \epsilon = m \epsilon. \quad (18)$$

The fact that we have $S = -1$ implies that the only non-zero form factor of the trace $\Theta$ is the two-particle one $F_2^\Theta$:

$$F_2^\Theta (\theta) = -2i\pi m^2 \sinh \frac{\theta}{2} \quad (19)$$

which has been normalized by means of the condition $F_2^\Theta (i\pi) = 2\pi m^2$. The final step is to write $F_2^\epsilon$ using the relation $\Theta = m\epsilon$:

$$F_2^\epsilon = \frac{1}{m} F_2^\Theta = -2i\pi m \sinh \frac{\theta}{2}. \quad (20)$$

All the other FF’s of $\epsilon(x)$ and $\Theta(x)$ are zero.

**3.1. The width of the flux tube**

We are now in a position to give an analytic estimate for the width of the flux tube in the limit of large distances. When the sides of the triangle are large, the leading behaviour of (14) is given by the first term in the rhs. It is interesting to note that such a term can be written in an explicit way for an arbitrary triangle. The reason lies in the simple form of (14), and we define the transverse distance as

$$\sqrt{R_{12}^2 + R_{23}^2},$$

and hence

$$\langle \sigma(x_1) \epsilon(x_2) \sigma(x_3) \rangle = (F_1^\sigma)^2 \frac{\sqrt{R_{12}^2 - (R_{12} - R_{23})^2}}{2 R_{12} R_{23}} e^{-m(R_{12} + R_{23})} + \ldots. \quad (22)$$

Since we are interested in analysing the width of the flux tube at the mid-point between $\sigma(x_1)$ and $\sigma(x_3)$ for large separations of them, we will put $R_{12} = R_{23} = L$ and $R_{13} = 2r$ in (14), and we define the transverse distance as $y = L \cos \alpha_+/2$ (we also have $r = L \sin \alpha_+/2$). Then, we define the three-point function in such a geometric configuration as $S(r, y)$

$$S(r, y) = (F_1^\sigma)^2 \frac{r}{y^2 + r^2} e^{-2m\sqrt{y^2 + r^2}}. \quad (23)$$
In order to study the flux tube shape we must study the following ratio:

\[ P(r, y) = \frac{S(r, y)}{\langle \sigma(2r)\sigma(0) \rangle} \]  

(24)

where \( \langle \sigma(2r)\sigma(0) \rangle \) is the two-point correlator of \( \sigma(x_1) \) and \( \sigma(x_3) \). In the large distance limit its leading behaviour is

\[ \langle \sigma(2r)\sigma(0) \rangle \simeq \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} (F_{1}^{\sigma})^{2} e^{-2mr \cosh \theta} = \frac{(F_{1}^{\sigma})^{2}}{\pi} K_{0}(2mr) \]

(25)

and hence the ratio can be cast in the form

\[ P(d, y) = \frac{S(r, y)}{\langle \sigma(2r)\sigma(0) \rangle} = \frac{\pi r}{y^2 + r^2} \frac{e^{-2m\sqrt{y^2 + r^2}}}{K_0(2mr)}. \]  

(26)

It is easy to see that for very large separations \( (mr \to \infty) \) the shape becomes of the Gaussian type in the transverse variable \( y \):

\[ P(r, y) \simeq \frac{\pi r}{y^2 + r^2} \frac{e^{-2mr}}{K_0(2mr)} e^{-(m/r)y^2}. \]  

(27)

Finally we can study the variance \( w^2(r) \) of \( P(r, y) \) wrt \( y \), which exactly corresponds to the width of the flux tubes which we were looking for. Let us define

\[ w^2(r) = \int_{-\infty}^{\infty} dy \ y^2 P(r, y). \]

(28)

As a consequence, setting \( x = y/r \) we obtain

\[ w^2(r) = \frac{\pi r^2}{K_0(2mr)} \int_{-\infty}^{\infty} dx \ \frac{x^2}{1 + x^2} e^{-2mr \sqrt{1 + x^2}}. \]  

(29)

Such an integration cannot be performed exactly, but we can still give its asymptotic estimate in the limit \( mr \to \infty \). With standard techniques we obtain

\[ \int_{-\infty}^{\infty} dx \ \frac{x^2}{1 + x^2} e^{-2mr \sqrt{1 + x^2}} \simeq \sqrt{\pi} e^{-2mr} \left( \frac{1}{2}(mr)^{-3/2} - \frac{9}{32}(mr)^{-5/2} + O((mr)^{-7/2}) \right) \]  

(30)

and combining it with

\[ \frac{1}{K_0(2mr)} \simeq \frac{e^{2mr}}{\sqrt{\pi}} \left( 2(mr)^{1/2} + \frac{1}{8}(mr)^{-1/2} + O((mr)^{-3/2}) \right) \]  

(31)

we finally obtain

\[ w^2(mr) \simeq \pi \frac{r}{m} \left( 1 - \frac{1}{2mr} + O \left( \frac{1}{(mr)^2} \right) \right) \]  

(32)

which states that the width of the flux tube, in the limit of very large separations between the spins, behaves linearly with the separation \( R = 2r \):

\[ w^2(mR) \simeq \frac{\pi R}{2m} - \frac{\pi}{2m^2} + \cdots. \]  

(33)

Finally, it is not difficult to calculate the asymptotic expansion of the higher order momenta:

\[ w^{(2n)}(r) = \frac{\pi r^{2n}}{K_0(2mr)} \int_{-\infty}^{\infty} dx \ \frac{x^{2n}}{1 + x^2} e^{-2mr \sqrt{1 + x^2}}. \]  

(34)
It turns out to be

\[ w_{\text{even}}(r) = \frac{\pi e^{-2mr}}{\sqrt{mr} K_0(2mr)} \left( \frac{r}{m} \right)^n \lim_{N \to \infty} \sum_{k=0}^{N} \sum_{j=0}^{N-k} \frac{(-1)^j}{4^k k!} \Gamma(2k + j + n + 1/2) (mr)^{-k-j} \]  

(35)

where the first few orders are as follows:

\[ w_{\text{even}}(r) = \sqrt{\pi} \Gamma(n + 1/2) \left( \frac{r}{m} \right)^n \left( 2 + \frac{1}{mr} (n^2/2 - n - 1/2) + O \left( \frac{1}{(mr)^2} \right) \right). \]  

(36)

4. Results

4.1. Comparison with the effective string predictions

Equation (33) is the main result of this paper. If we compare it with the effective string prediction of equation (10) we see a remarkable agreement of the two results. Both show a linear increase with the interquark distance of the flux tube thickness. The linear term appears in both equations with the same \( R/N_t \) dependence and the right dimensions given by \( 1/\sigma \). The factor in front of this correction is not the same in the effective string and in the spin model cases, but this is not strange since there is a finite renormalization in the mapping between the plaquette operator (in the LGT) and the energy operator in the spin model (and similarly between the Polyakov loop and the spin operator). Moreover one must recall that the effective string result is obtained in the framework of the purely Gaussian approximation.

4.2. Isolines of chromoelectric flux

It is interesting to plot the isolines of the chromoelectric flux as a function of the interquark distance in the approximation of the two-dimensional Ising model. These can be easily obtained from equation (23). Since in the reduced model we have only one scale \( m_l \) which combines both the finite temperature and the interquark distance of the original LGT, these plots can be interpreted in two ways.

- We may look at them as the result of keeping the temperature of the LGT fixed (i.e. \( m \) fixed); then as \( m_l \) increases we are effectively describing an increase of the interquark distance \( R \). It is nice to see that as \( m_l \) increases (from figure 2 to 5) the shape of the flux tube does indeed become more and more narrow, as one should expect for a confining theory.
- Alternatively we may think of keeping the interquark distance fixed and look at what happens as we increase the temperature and approach the deconfinement transition (i.e. as \( m \) decreases). Again it is nice to see that the flux tube smoothly moves from the narrow shape of a confining theory (figure 5) to a shape in which the flux lines become more and more delocalized in the two-dimensional surface (figure 2).

4.3. The shape of the flux tube

Finally it is interesting to notice that from the explicit expression for the \( \langle \sigma_1 \epsilon_2 \sigma_3 \rangle \) correlator one can see that the shape of the flux tube is not exactly a Gaussian (see equation (26)). This is indeed also what is visible in the Monte Carlo simulations.
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Figure 2. Isolines of chromoelectric flux for $m = 1$. The circles are centred in correspondence with the static quarks and their radius is $1/m$. Their extent gives a measure of the region where we expect large corrections to the leading behaviour that we used to produce both the picture and the estimate of the flux tube width (we recall that we expect the leading behaviour to be valid for $m R_{ij} \gg 1$ where $R_{ij}$ is a generic side of the triangle in figure 1).

Figure 3. Isolines of chromoelectric flux for $m = 2$.

Figure 4. Isolines of chromoelectric flux for $m = 5$.

Figure 5. Isolines of chromoelectric flux for $m = 10$. 

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(see figure 2 of [2] and the whole discussion at the beginning of [20]). Our analysis could suggest a tentative analytic form for these deviations which should be valid in the neighbourhood of the deconfinement transition and hence extend to this regime the functional form valid at low temperature proposed in [20]. This prediction is encoded in the higher order momenta reported in equations (35), (36). It would be nice to check this prediction by performing also in the high temperature regime a set of Monte Carlo simulations similar to those discussed in [20].

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Note added. An interesting application of form factors to the (2 + 1)D SU(2) gauge theory [21] has appeared at the same time as the present paper. In that work, the (2 + 1)D SU(2) gauge theory is generalized to an anisotropic form with two gauge couplings, and the form factors of the currents of the SU(2) principal chiral model in (1 + 1)D have been used to compute the string tension in the anisotropic regime. In this context, the mechanism of dimensional reduction is not related to the Svetitsky–Yaffe conjecture.

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