Isotopy invariants for closed braids and almost closed braids via loops in stratified spaces

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Abstract

Let \( \phi : S^1 \times D^2 \to S^1 \) be the natural projection. An oriented knot \( K \hookrightarrow V = S^1 \times D^2 \) is called an almost closed braid if the restriction of \( \phi \) to \( K \) has exactly two (non-degenerate) critical points (and \( K \) is a closed braid if the restriction of \( \phi \) has no critical points at all).

We introduce new isotopy invariants for closed braids and almost closed braids in the solid torus \( V \). These invariants refine finite type invariants. They are still calculable with polynomial complexity with respect to the number of crossings of \( K \). Let the solid torus \( V \) be standardly embedded in the 3-sphere and let \( A \) be the axis of the complementary solid torus \( S^3 \setminus V \). We give examples which show that our invariants can detect non-invertibility of 2-component links \( K \cup A \hookrightarrow S^3 \). Notice that all quantum link invariants fail to do so and that it is not known whether there are finite type invariants which can detect non-invertibility of 2-component links.\(^1\)

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1 Introduction

This paper is a new and much shortened version of the preprint [10]. As well known, the isotopy problem for closed braids in the solid torus reduces to the conjugacy problem in braid groups (see [13]). The latter problem is solved, but in general only with exponential complexity with respect to the braid length (see [5] and references therein). It is therefore interesting to construct invariants which distinguish conjugacy classes of braids and which are calculable with polynomial complexity. Finite type invariants for knots in the solid torus are an example of such invariants (see [16], [17], [2], [12], [7] and references therein).

In this paper we construct another class of calculable invariants. We give now a brief outline of our approach: let \( \hat{\beta} \subset V \) be a closed braid (i.e. the restriction of \( \phi \) to \( \hat{\beta} \) has no critical points) and such that \( \hat{\beta} \) is a knot. We fix a projection \( pr : V \rightarrow S^1 \times I \). Let \( M(\hat{\beta}) \) be the infinite dimensional) space of all closed braids which are isotopic to \( \hat{\beta} \) in \( V \). \( M(\hat{\beta}) \) has a natural stratification. The strata \( \sum^{(1)} \) of codimension 1 are just the braid diagrams which have in the projection \( pr \) either exactly one ordinary triple point or exactly one ordinary autotangency. We call the corresponding strata \( \sum^{(1)}(tri) \) and respectively \( \sum^{(1)}(tan) \). First, we associate to a closed braid in a canonical way a loop in \( M(\hat{\beta}) \), called the \textit{canonical loop} and denoted \( rot(\hat{\beta}) \). Next, we associate to the canonical loop an oriented singular link in a thickened
torus. This link is called the trace graph and it is denoted by $TL(\hat{\beta})$. All singularities of $TL(\hat{\beta})$ are ordinary triple points. These triple points correspond exactly to the intersections of $\text{rot}(\hat{\beta})$ with $\sum^{(1)}(\text{tri})$. There is a natural coorientation on $\sum^{(1)}(\text{tri})$ and, hence, each triple point in $TL(\hat{\beta})$ has a sign. To each triple point corresponds a knot diagram which has just ordinary crossings and exactly one triple crossing. We use the position of the ordinary crossings with respect to the triple crossing in the Gauss diagram in order to construct weights for the triple points. We associate to the loop $\text{rot}(\hat{\beta})$ then weighted intersection numbers with $\sum^{(1)}(\text{tri})$. These intersection numbers turn out to be finite type invariants. We call them one-cocycle invariants. As all finite type invariants the one-cocycle invariants have a natural degree. However, they are rather special finite type invariants, as shows the following proposition.

**Proposition 1** Let $\hat{\beta}$ be a closed $n$-braid which is a knot and let $c$ be the word lenght of $\beta \in B_n$ (with respect to the standard generators of $B_n$). Then all one-cocycle invariants of degree $d$ vanish for $\hat{\beta}$ if $d \geq c + n^2 - n - 1$.

One has to compare this proposition with the following well known fact: the trefoil has non-trivial finite type invariants of arbitrary high degree. Consequently, the one-cocycle invariants define a natural filtered subspace in the filtered space of all finite type invariants for those knots in the solid torus, which are closed braids. There is a very simple procedure, coming from singularity theory, in order to construct all one-cocycle invariants for closed braids, without solving a big system of equations. They verify automatically the marked 4T-relations (compare \cite{12} and also \cite{7}).

However, the one-cocycle invariants can be refined considerably. When we deform $\hat{\beta}$ in $V$ by a generic isotopy then $\text{rot}(\hat{\beta})$ in $M(\hat{\beta})$ deforms by a generic homotopy. The following lemma is our key observation.

**Lemma 1** The loop $\text{rot}(\hat{\beta})$ is never tangential to $\sum^{(1)}(\text{tan})$.

It follows from this lemma that the connected components of the natural resolution of $TL(\hat{\beta})$ (i.e. the abstract union of circles where the branches in the triple points are separated) are isotopy invariants of $\hat{\beta} \hookrightarrow V$. We apply now our theory of one-cocycle invariants but only to those triple points in $TL(\hat{\beta})$ where three given components of $TL(\hat{\beta})$ intersect. The resulting invariants are called character invariants. They are no longer of finite type but they are still calculable with polynomial complexity with respect to the braid length.

The set of finite type invariants (in particular, the set of one-cocycle invariants) can be seen as a trivial local system on $M(\hat{\beta})$. In contrast to
this, the set of character invariants is in general a non-trivial local system on $M(\hat{\beta})$.

Alexander Stoimenow has written a computer program in order to calculate character invariants. It turns out that already character invariants of linear complexity can sometimes detect non-invertibility of closed braids (i.e. the closed braid together with the axis of the complementary solid torus in $S^3$ is a non-invertible link in $S^3$). This implies in particular that character invariants are in general not of finite type.

We observe that the most simple character invariants are still well defined for almost closed braids.

The basic notions of our one parameter approach to knot theory, namely the space of non-singular knots, its discriminant, the stratification of the discriminant, the coorientation of strata of low codimension, the canonical loop, the trace graph, the equivalence relation for trace graphs, are worked out in all details in our joint work with Vitaliy Kurlin [11]. Therefore we concentrate in this paper only on the construction of the new invariants.

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2 Basic notions of one parameter knot theory

In this section we recall briefly the basic notions of our theory. All details with complete proofs can be found in [11] (even in a much more general setting).

2.1 The space of closed braids and its discriminant

We work in the smooth category and all orientable manifolds are actually oriented. We fix once for all a coordinate system in $\mathbb{R}^3 : (\phi, \rho, z)$. Here, $(\phi, \rho) \in S^1 \times \mathbb{R}^+$ are polar coordinates of the plane $\mathbb{R}^2 = \{z = 0\}$. A closed n-braid $\hat{\beta}$ is a knot in the solid torus $V = \mathbb{R}^3 \setminus z-axes$, such that $\phi : \hat{\beta} \to S^1$ is non-singular and $[\hat{\beta}] = n \in H_1(V)$. Let $M(\hat{\beta})$ be the infinite dimensional space of all closed braids (with respect to $\phi$) which are isotopic to $\hat{\beta}$ in V. Let $M_n$ be the union of all spaces $M(\hat{\beta})$. A well known theorem of Artin (see e.g. [12]) says that two closed braids in the solid torus are isotopic as links in the solid torus if and only if they are isotopic as closed braids.
Therefore it is enough to consider only isotopies through closed braids. Let $pr : \mathbb{R}^3 \setminus \text{axes} \rightarrow \mathbb{R}^2 \setminus 0$ be the canonical projection $(\phi, \rho, z) \rightarrow (\phi, \rho)$. Each closed braid is then represented by a knot diagram with respect to $pr$. A generic closed braid $\beta$ has only ordinary double points as singularities of $pr(\beta)$. Let $\sum$ be the discriminant in $M(\beta)$ which consists of all non-generic diagrams of closed braids isotopic to $\beta$.

The discriminant $\sum$ has a natural stratification: $\sum = \sum^{(1)} \cup \sum^{(2)} \cup \ldots$, where $\sum^{(i)}$ are the union of all strata of codimension $i$ in $M(\beta)$.

**Theorem 1** (Reidemeisters theorem for closed braids)

$\sum^{(1)} = \sum(\text{tri}) \cup \sum(\text{tan})$,
where $\sum(\text{tri})$ is the union of all strata which correspond to diagrams with exactly one ordinary triple point (besides ordinary double points) and $\sum(\text{tan})$ is the union of all strata which correspond to diagrams with exactly one ordinary autotangency.

In the sequel we need also the description of $\sum^{(2)}$.

**Theorem 2** $\sum^{(2)} = \sum_1 \cup \sum_2 \cup \sum_3 \cup \sum_4$,
where $\sum_1$ is the union of all strata which correspond to diagrams with exactly one ordinary quadruple point, $\sum_2$ is the union of all strata which correspond to diagrams with exactly one ordinary autotangency through which passes another branch transversally, $\sum_3$ corresponds to the union of all strata of diagrams with an autotangency in an ordinary flex, $\sum_4$ is the union of all transverse intersections of strata from $\sum^{(1)}$.

### 2.2 The canonical loop

We identify $\mathbb{R}^3 \setminus \text{axes}$ with the standard solid torus $V = S^1 \times D^2 \hookrightarrow \mathbb{R}^3 \setminus \text{axes}$. We identify the core of $V$ with the unit circle in $\mathbb{R}^2$.

Let $rot(V)$ denote the $S^1$-parameter family of diffeomorphismes of $V$ which is defined in the following way: we rotate the solid torus monotonically and with constant speed around its core by the angle $t$, $t \in [0, 2\pi]$, i.e. all discs $(\phi = \text{const}) \times D^2$ stay invariant and are rotated simulaneously around their centre.

Let $\tilde{\beta}$ be a closed braid.

**Definition 1** The canonical loop $rot(\tilde{\beta}) \in M(\tilde{\beta})$ is the oriented loop induced by $rot(V)$. 
Notice that the whole loop $\text{rot}(\hat{\beta})$ is completely determined by an arbitrary point in it.

The following lemma is an immediate corollary of the definition of the canonical loop.

**Lemma 2** Let $\hat{\beta}_s, s \in [0, 1]$, be an isotopy of closed braids in the solid torus. Then $\text{rot}(\hat{\beta}_s), s \in [0, 1]$, is a homotopy of loops in $M(\hat{\beta})$.

Evidently, the canonical loop can be defined for an arbitrary link in $V$ in exactly the same way. However, in the case of closed braids we can give an alternative combinatorial definition, which makes concrete calculations much easier.

Let $\Delta \in B_n$ be Garside’s element, i.e. $\Delta^2$ is a generator of the centre of $B_n$ (see [3]). Geometrically, $\Delta^2$ is the full twist of the $n$ strings.

**Definition 2** Let $\gamma \in B_n$ be a braid with closure isotopic to $\hat{\beta}$. Then the combinatorial canonical loop $\text{rot}(\gamma)$ is defined by the following sequence of braids:

\[ \gamma \to \Delta \Delta^{-1} \gamma \to \Delta^{-1} \gamma \Delta \to \cdots \to \Delta^{-1} \Delta \gamma' \to \gamma' \to \Delta \Delta^{-1} \gamma' \to \Delta^{-1} \gamma' \Delta \to \cdots \to \Delta^{-1} \Delta \gamma \to \gamma \]

Here, the first arrow consists only of Reidemeister II moves, the second arrow is a cyclic permutation of the braid word (which corresponds to an isotopy of the braid diagram in the solid torus) and the following arrows consist of “pushing $\Delta$ monotonically from the right to the left through the braid $\gamma$”. We obtain a braid $\tilde{\gamma}$ and we start again.

We give below a precise definition in the case $n = 3$. The general case is a straightforward generalization which is left to the reader. $\Delta = \sigma_1 \sigma_2 \sigma_1$ for $n = 3$. We have just to consider the following four cases:

\[
\begin{align*}
\sigma_1 \Delta &= \sigma_1 (\sigma_1 \sigma_2 \sigma_1) \\
\sigma_1^{-1} \sigma_2 \sigma_2 \Delta &= (\sigma_2 \sigma_1 \sigma_2) \sigma_1 \\
\sigma_2^{-1} \sigma_1^{-1} \Delta &= (\sigma_1 \sigma_2 \sigma_1) \sigma_1 \\
\sigma_2^{-1} \Delta &= (\sigma_1 \sigma_2 \sigma_1) \\
\end{align*}
\]

Notice, that the sequence is canonical in the case of a generator and almost canonical in the case of an inverse generator. Indeed, we could replace the above sequence $\sigma_1^{-1} \Delta \to \Delta \sigma_2^{-1}$ by

\[ \sigma_1^{-1} (\sigma_1 \sigma_2 \sigma_1) \to \sigma_2 \sigma_1 \to \sigma_2 \sigma_1 \sigma_2 \sigma_2^{-1} \to (\sigma_1 \sigma_2 \sigma_1) \sigma_2^{-1} \].

But it turns out that the corresponding canonical loops in $M(\hat{\beta})$ differ just by a homotopy which passes once transversally through a stratum of $\Sigma^{(2)}$.

Let $c$ be the word length of $\gamma$. Then we use exactly $2c(n - 2)$ braid relations (or Reidemeister III moves) in the combinatorial canonical loop.
This means that the corresponding loop in \(M(\hat{\beta})\) cuts \(\sum^{(1)}(\text{tri})\) transversally in exactly \(2c(n - 2)\) points.

One easily sees that the combinatorial canonical loop \(\text{rot}(\gamma)\) from Definition 1 is homotopic in \(M(\hat{\beta})\) without touching \(\sum^{(1)}(\text{tan})\) (i.e. we never make in the one parameter family a Reidemeister II move forwards and just after that the same move backwards) to the geometrical canonical loop \(\text{rot}(\hat{\beta})\) from Definition 2.

2.3 The trace graph

The trace graph \(\text{TL}(\hat{\beta})\) is our main combinatorial object. It is an oriented singular link in a thickened torus. All its singularities are ordinary triple points.

Let \(\hat{\beta}_t, t \in S^1\), be the (oriented) family of closed braids corresponding to the canonical loop \(\text{rot}(\hat{\beta})\). We assume that the loop \(\text{rot}(\hat{\beta})\) is a generic loop. Let \(\{p_1^{(t)}, p_2^{(t)}, \ldots, p_n^{(t)}\}\) be the set of double points of \(\text{pr}(\hat{\beta}_t) \subset S^1_\phi \times \mathbb{R}_\rho\). The union of all these crossings for all \(t \in S^1\) forms a link \(\text{TL}(\hat{\beta}) \subset (S^1_\phi \times \mathbb{R}_\rho^+) \times S^1_t\) (i.e. we forget the coordinate \(z(p_i^{(t)}))\). \(\text{TL}(\hat{\beta})\) is non-singular besides ordinary triple points which correspond exactly to the triple points in the family \(\text{pr}(\hat{\beta}_t)\). A generic point of \(\text{TL}(\hat{\beta})\) corresponds just to an ordinary crossing \(p_i^{(t)}\) of some closed braid \(\hat{\beta}_t\). Let \(t : \text{TL}(\hat{\beta}) \rightarrow S^1_t\) be the natural projection. We orient the set of all generic points in \(\text{TL}(\hat{\beta})\) (which is a disjoint union of embedded arcs) in such a way that the local mapping degree of \(t\) at \(p_i^{(t)}\) is +1 if and only if \(p_i^{(t)}\) is a positive crossing (i.e. it corresponds to a generator of \(B_n\), or equivalently, its writhe \(w(p_i^{(t)}) = +1\)).

The arcs of generic points come together in the triple points and in points corresponding to an ordinary autotangency in some \(\text{pr}(\hat{\beta})\). But one easily sees that the above defined orientations fit together to define an orientation on the natural resolution \(\text{TL}(\hat{\beta})\) of \(\text{TL}(\hat{\beta})\) (compare also [8]). \(\text{TL}(\hat{\beta})\) is a union of oriented circles, called trace circles. We can attach stickers \(i \in \{1, 2, \ldots, n - 1\}\) to the edges of \(\text{TL}(\hat{\beta})\) in the following way: each edge of \(\text{TL}(\hat{\beta})\) corresponds to a letter in a braid word. Indeed, each generic point in an edge corresponds to an ordinary crossing of a braid projection and, hence, to some \(\sigma_i\) or some \(\sigma_i^{-1}\). We attach to this edge the number \(i\). The information about the exponent +1 or −1 is contained in the orientation of the edge.

We identify \(H_1(V)\) with \(\mathbb{Z}\) by sending the core of \(V\) to the generator +1. If \(\hat{\beta}\) is a knot then we can attach to each trace circle a homological marking \(a \in H_1(V)\) in the following way: let \(p\) be a crossing corresponding
to a generic point in the trace circle. We smooth \( p \) with respect to the orientation of the closed braid. The result is an oriented 2-component link. The component of this link which contains the undercross which goes to the overcross at \( p \) is called \( p^+ \). We associate now to \( p \) the homology class \( a = [p^+] \in H_1(V) \) (compare also [6]). One easily sees that \( a \in \{1, 2, \ldots, n-1\} \) and that the class \( a \) does not depend on the choice of the generic point in the trace circle. Indeed, the two crossings involved in a Reidemeister II move have the same homological marking and a Reidemeister III move does not change the homological marking of any of the three involved crossings.

Evidently, crossings with different homological markings belong to different trace circles. Surprisingly, the inverse is also true. The following lemma is essentially due to Stepan Orevkov.

**Lemma 3** Let \( TL(\hat{\beta}) \) be the trace graph of the closure of a braid \( \beta \in B_n \), such that \( \hat{\beta} \) is a knot. Then \( TL(\hat{\beta}) \) splits into exactly \( n-1 \) trace circles. They have pairwise different homological markings.

Consequently, the trace circles are characterised by their homological markings. Notice, that the set of homological markings is independent of the word length of the braid.

### 2.4 A higher order Reidemeister theorem for trace graphs of closed braids

**Definition 3** A trihedron is a 1-dimensional subcomplex of \( TL(\hat{\beta}) \) which is contractible in the thickened torus and which has the form as shown in Figure 1.

**Definition 4** A tetrahedron is a 1-dimensional subcomplex of \( TL(\hat{\beta}) \) which is contractible in the thickened torus and which has the form as shown in Figure 2.

**Definition 5** A trihedron move is shown in Figure 3.

**Definition 6** A tetrahedron move is shown in Figure 4.

The rest of \( TL(\hat{\beta}) \hookrightarrow S^1 \times S^1 \times \mathbb{R}^+ \) is unchanged under the moves. The stickers on the edges change in the canonical way.

Notice, that a trihedron move corresponds to a generic homotopy of the canonical loop which passes once through an ordinary tangency with a stratum of \( \sum^{(1)}(tri) \). A tetrahedron move corresponds to a generic homotopy of the canonical loop which passes transversally once through a stratum of \( \sum_1^{(2)} \), i.e. corresponding to an ordinary quadruple point.
**Definition 7** The equivalence relation for trace graphs $TL(\hat{\beta})$ is generated by the following three operations:

1. isotopy in the thickened torus
2. trihedron moves
3. tetrahedron moves

The following important Reidemeister type theorem is a particular case of Theorem 1.10. in [11].

**Theorem 3** Two closed braids (which are knots) are isotopic in the solid torus if and only if their trace graphs in the thickened torus are equivalent.

**Remark 1** Notice, that not all representatives of an equivalence class of a trace graph correspond to the canonical loop (which is very rigid) of some closed braid. However, one easily sees that all representatives correspond to loops in the space $M(\hat{\beta})$.

**Remark 2** A generic homotopy of a loop in $M(\hat{\beta})$ could of course be tangent at some point to $\sum^{(1)}(\text{tan})$. One easily sees that this would imply a Morse modification of the trace graph and, hence, change the components of its natural resolution. The cornerstone of the theory developed in this paper is the fact, that this does not happen for the (very rigid) homotopies of $\text{rot}(\hat{\beta})$ which are induced by generic isotopies of $\hat{\beta}$ in $V$. Consequently, the trace circles are isotopy invariants for closed braids (compare Lemma 3.4. in [11]).

**Definition 8** A homotopy of loops $\gamma_s, s \in [0,1]$, in $M(\hat{\beta})$ is called a tan-transverse homotopy if no loop $\gamma_s$ is tangential to $\sum^{(1)}(\text{tan})$. Let $S$ be any loop in $M(\hat{\beta})$. Its tan-transverse homotopy class is denoted by $[S]_{\text{t-t}}$.

**Remark 3** There can be loops $\gamma_s$ in a tan-transverse homotopy which are tangential to $\sum^{(1)}(\text{tri})$. For the trace graphs associated to the loops $\gamma_s$ this corresponds to a trihedron move.

In order to define our invariants we need more precise information about isotopies of trace graphs.

**Definition 9** A time section in the thickened torus $(S^{1}_\phi \times \mathbb{R}^+_{\rho}) \times S^{1}_t$ is an annulus of the form $(S^{1}_\phi \times \mathbb{R}^+_{\rho}) \times \{t = \text{const}\}$. 
The intersection of $TL(\hat{\beta})$ with a generic time section corresponds to the crossings of the closed braid $\hat{\beta}_t$. Using the orientation and the stickers on $TL(\hat{\beta})$ we can read off a cyclic braid word for $\hat{\beta}$ in each generic time section. The tangent points of $TL(\hat{\beta})$ with time sections correspond exactly to the Reidemeister II moves in the one parameter family of diagrams $\hat{\beta}_t, t \in S^1$. A triple point in $TL(\hat{\beta})$ slides over such a tangent point if and only if the canonical loop passes in a homotopy transversally through a stratum of $\Sigma^{(2)}_2$. We illustrate this in Figure 5.

When the canonical loop passes transversally through a stratum of $\Sigma^{(2)}_3$ then the trace graph changes as shown in Figure 6.

Finally, when the canonical loop passes transversally through a stratum of $\Sigma^{(2)}_4$ then the $t$-values of triple points or tangencies with time sections are interchanged. We show an example in Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{Figure 5:}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Figure 6:}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{Figure 7:}
\end{figure}
3 One-cocycle invariants

In this section we introduce our one-cocycle invariants. They are a special class of finite type invariants for knots in the solid torus.

3.1 Gauss diagrams for closed braids with a triple crossing

Let $f : S^1 \to \hat{\beta}$ be a generic orientation preserving diffeomorphism. Let $p$ be any crossing of $\hat{\beta}$. We connect $f^{-1}(p) \in S^1$ by an oriented chord, which goes from the undercross to the overcross and we decorate it by the writhe $w(p)$. Moreover, we attach to the chord the homological marking $[p+]$. The result is called a Gauss diagram for $\hat{\beta}$ (compare e.g. [14] and [7]).

One easily sees that $\hat{\beta}$ up to isotopy is determined by its Gauss diagram and the number $n = [\hat{\beta}] \in H_1(V)$. (Notice, that the trace graph of a closed braid is always transverse to the $\phi$-sections.)

A Gauss sum of degree $d$ is an expression assigned to a diagram of a closed braid which is of the following form:

$$\sum \text{function} (\text{writhes of the crossings})$$

where the sum is taken over all possible choices of $d$ (unordered) different crossings in the knot diagram such that the chords without the writhes arising from these crossings build a given subdiagram with given homological markings. The marked subdiagrams (without the writhes) are called configurations. If the function is the product of the writhes, then we will denote the sum shortly by the configuration itself. We need to define Gauss diagrams for knots with an ordinary triple point too. The triple point corresponds to a triangle in the Gauss diagram of the knot. Notice, that the preimage of a triple point has a natural ordering coming from the orientation of the $\mathbb{R}^+$-factor. One easily sees that this order is completely determined by the arrows in the triangle.

We provide each stratum of $\sum^{(1)}(\text{tri})$ with a coorientation which depends only on the non-oriented underlying curves $pr(\hat{\beta})$ in $S^1 \times \mathbb{R}^+$. Consequently, for the definition of the coorientation we can replace the arrows in the Gauss diagram simply by chords.

Definition 10 The coorientation of the strata in $\sum^{(1)}(\text{tri})$ is given in Figure 8.

Notice that the second line in Figure 8 does not occur for closed braids, but it does occur for almost closed braids. The two coorientations are chosen in such a way that they fit together in $\sum^{(2)}$. 

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There are exactly two types of triple points without markings. We show them in Figure 9.

We attach now the homological markings to the three chords. Let $a, b \in \{1, 2, \ldots, n-1\}$ be fixed. Then the markings are as shown in Figure 10. We encode the types of the marked triple points by $(a, b)^-$ and $(a, b)^+$. The union of the corresponding strata of $\sum^{(1)}(tri)$ are encoded in the same way.

3.2 One-cocycle invariants of degree one

We will construct in a canonical way one-cocycles on the space $M_n$ (the space of all closed n-braids which are knots). We obtain numerical invariants
when we evaluate these cocycles on the homology class represented by the canonical loop.

Let \( n, d \in \mathbb{N}^* \) be fixed. Let \((a, b)^\pm\) be a fixed type of marked triple point as shown in Figure 10. Here \( a, b, a + b \in \mathbb{Z}/n\mathbb{Z} \).

**Definition 11** A configuration \( I \) of degree \( d \) is an abstract Gauss diagram without writhes which contains exactly \( d - 1 \) arrows marked in \( \mathbb{Z}/n\mathbb{Z} \) besides the sub-diagram \((a, b)^\pm\).

Let \( \{ I_i \} \) be the finite set of all configurations of degree \( d \) with respect to \((a, b)^\pm\). Let \( \Gamma_{(a,b)^\pm}(S) = \sum_i \epsilon_i I_i \) be a linear combination with each \( \epsilon_i \in \{0, +1, -1\} \). (The type of the triple point is always fixed in any cochain \( \Gamma \)).

**Definition 12** \( \Gamma_{(a,b)^\pm} \) gives rise to a 1-cochain of degree \( d \) by assigning to each oriented generic loop \( S \subset M_n \) an integer \( \Gamma_{(a,b)^\pm}(S) \) in the following way:

\[
\Gamma_{(a,b)^\pm}(S) = \sum_{s_i \in S \cap \Sigma^{(1)}(tri) \text{ of type } (a, b)^\pm} w(s_i)(\sum_i \epsilon_i(\prod_j w(p_j)))
\]

where \( D_i \) is the set of unordered \((d-1)\)-tuples \( p_1, \ldots, p_{d-1} \) of arrows which enter in \( I_i \) in the Gauss diagram of \( s_i \).

**Lemma 4** If \( \Gamma_{(a,b)^\pm}(S) \) is invariant under each generic deformation of \( S \) through a stratum of \( \Sigma^{(2)} \), then \( \Gamma_{(a,b)^\pm} \) is a 1-cocycle.

**Proof:** In this case, \( \Gamma_{(a,b)^\pm}(S) \) is invariant under homotopy of \( S \). Indeed, tangent points of \( S \) with \( \Sigma^{(1)}(tri) \) correspond just to trihedron moves. The two triple points give the same contribution to \( \Gamma_{(a,b)^\pm}(S) \) but with different
signs. A tangency with \( \Sigma_1(\tan) \) does not change the contribution of the triple points at all. This implies the invariance under homology of \( S \) because the contributions to \( \Gamma_{(a,b)}^\pm(S) \) of different diagrams with triple points are not related to each other. \( \square \)

**Definition 13** A cohomology class in \( H^1(M_n; \mathbb{Z}) \) is of degree \( d \) if it can be represented by some 1-cocycle \( \Gamma_{(a,b)}^\pm \) of degree at most \( d \).

**Remark 4** The above definition induces a filtration on a part of \( H^1(M_n; \mathbb{Z}) \).

Let \( M \) be the (disconnected) space of all embeddings \( f: S^1 \hookrightarrow \mathbb{R}^3 \). Vassiliev [16] has introduced a filtration on a part of \( H^1(M; \mathbb{Z}) \) using the discriminant \( \Sigma_{\text{sing}} \). It is not difficult to see that the space of all (unparametrized) differentiable maps of the circle into the solid torus is contractible. Indeed, there is an obvious canonical homotopy of each (perhaps singular) knot to a multiple of the core of the solid torus.

The core of the solid torus is invariant under \( \text{rot}_{S^1}(V) \times \text{rot}_{D^2}(V) \). Thus, the above space is star-like. Therefore, Alexander duality could be applied and Vassiliev’s approach could be generalized for knots in the solid torus too. It would be interesting to compare his filtration with our filtration.

In the next sections, we will construct 1-cocycles \( \Gamma_{(a,b)}^\pm \) in an explicit way.

Let \( \beta \in B_n \) be such that its closure \( \hat{\beta} \hookrightarrow V \) is a knot.

**Proposition 2** The space of finite type invariants of degree 1 is of dimension \( \lfloor n/2 \rfloor \) (here \( \lfloor \cdot \rfloor \) is the integer part). It is generated by the Gauss diagram invariants \( W_a(\hat{\beta}) = \sum w(p) \), where \( a \in \{1, 2, \ldots, \lfloor n/2 \rfloor \} \). The sum is over all crossings with fixed homological marking \( a \).

**Proof:** It follows from Goryunov’s [12] generalization of finite type invariants for knots in the solid torus that the invariants of degree 1 correspond just to marked chord diagrams with only one chord. Obviously, all these invariants can be expressed as Gauss diagram invariants:

\[
W_a(\hat{\beta}) = \sum w(p), \ a \in \{1, \ldots, n-1\}
\]

(see also [3], and Section 2.2 in [7].)

Let us define \( V_a(\hat{\beta}) := W_a(\hat{\beta}) - W_{n-a}(\hat{\beta}) \) for all \( a \in \{1, \ldots, n-1\} \). We observe that \( V_a(\hat{\beta}) \) is invariant under switching crossings of \( \hat{\beta} \). Indeed, if the marking of the crossing \( p \) was \( [p] = a \), then the switched crossing \( p^{-1} \) has marking \( [p^{-1}] = n - a \), but \( w(p) = -w(p^{-1}) \). But every braid \( \beta \in B_n \) is homotopic to \( \gamma = \prod_{i=1}^{n-1} \sigma_i \). A direct calculation for \( \gamma \) shows that \( V_a(\hat{\gamma}) \equiv 0 \).
It is easily shown by examples that $W_a, a \in \{1, \ldots, [n/2]\}$ (seen as invariants in $\mathbb{Q}$) are linearly independent. □

**Lemma 5** Let $a, b \in \mathbb{Z}/n\mathbb{Z}$ be fixed. Consider the union of all cooriented strata of $\Sigma^{(1)}$ which correspond to triple points of type either $(a,b)^-$ or $(a,b)^+$. The closure in $M_n$ of each of these sets form integer cycles of codimension 1 in $M_n$.

**Remark 5** Otherwise stated, $\Gamma(a,b)^+$ and $\Gamma(a,b)^-$ both define integer 1-cocycles of degree 1. $\Gamma(a,b)^\pm(S)$ is in this case by definition just the algebraic intersection number of $S$ with the corresponding union of strata of $\Sigma^{(1)}(\text{tri})$.

**Proof:** According to Section 2, we have to prove that the cooriented strata fit together in $\Sigma^{(2)}_1$ and $\Sigma^{(2)}_2$. The first is evident, because at a stratum of $\Sigma^{(2)}_1$ just four strata of $\Sigma^{(1)}(\text{tri})$ intersect pairwise transversally. For the second, we have to distinguish 24 cases. Three of them are illustrated in Figure 11. The whole picture in a normal disc of $\Sigma^{(2)}_2$ is then obtained from Figure 12.

All other cases are obtained from these three by inverting the orientation of the vertical branch, by taking the mirror image (i.e. switching all crossings), and by choosing one of two possible closings of the 3-tangle (in order to obtain an oriented knot). In all cases, one easily sees that the two adjacent triple points are always of the same marked type and that the coorientations fit together. □

![Figure 11](image_url)

**Proposition 3** $\Gamma(a,b)^+$ defines a non-trivial 1-cohomology class of degree 1 if and only if $a \neq b$ and $a+b \leq n-1$.

$\Gamma(a,b)^-$ defines a non-trivial 1-cohomology class of degree 1 if and only if $a \neq b$ and $a+b \geq n+1$.

The following identities hold:

\[(*)\quad \Gamma(a,b)^+ + \Gamma(b,a)^+ \equiv 0\]

\[(*)\quad \Gamma(a,b)^- + \Gamma(b,a)^- \equiv 0\]
Proof: For closed braids, the markings are all in \( \{1, \ldots, n-1\} \). Therefore, if \( a + b > n - 1 \) in \( \Gamma_{(a,b)}^+ \) or \( a + b < n + 1 \) in \( \Gamma_{(a,b)}^- \), then there is no such triple point at all and the 1-cocycle is trivial. \( \Gamma_{(a,a)}^+ \equiv 0 \) is a special case of the identities.

Examples show that all the remaining 1-cocycles are non-trivial (we will see lots of examples later).

In order to prove the identities, we use the following Gauss diagram sums (see also section 1.6 in [7]):

\[
I_{(a,b)}^+ = \sum w(p)w(q), \quad I_{(a,b)}^- = \sum w(p)w(q)
\]

Here, the first sum is over all couples of crossings which form a subconfiguration as shown in Figure 13. The second sum is over all couples of crossings which form a subconfiguration as shown in Figure 14. (Here, \( a \) and \( b \) are the homological markings.) These sums applied to diagrams of \( \hat{\beta} \) are not invariants. Let \( S \subset M(\hat{\beta}) \) be a generic loop. Then \( I_{(a,b)}^\pm \) is constant except when \( S \) crosses \( \Sigma^{(1)}(tri) \) in strata of type \( (a,b)^\pm \) or \( (b,a)^\pm \). At each such intersection in positive (resp., negative) direction, \( I_{(a,b)}^\pm \) changes exactly by \(-1\) (resp., +1). Indeed, the configurations of the three crossings which come together in a triple point are shown in Figure 15. In each of the four cases, exactly one pair \( p, q \) of crossings contributes to one of the sums \( I \).

After drawing all possible triple points, it is easily seen that \( p, q \) must verify: \( w(p)w(q) = -1 \) for the first two cases and \( w(p)w(q) = +1 \) for the last.
two cases. Notice that the type of the triple point is completely determined by the sub-configuration shown in Figure 13 and 14. Thus, the sums $I$ are constant by passing all types of triple points except those shown in Figure 15. The generic loop $S$ intersects $\Sigma$ only in strata that correspond to triple points or to autotangencies. An autotangency adds to the Gauss diagram always one of the sub-diagrams shown in Figure 16. The two arrows do not enter together in the configurations shown in Figure 13 and 14. If one of them contributes to such a configuration, then the other contributes to the same configuration but with an opposite sign.

Therefore, for any $\hat{\beta}_1, \hat{\beta}_2 \in M(\hat{\beta}) \setminus \Sigma$, the difference $I_{(a,b)}^\pm(\hat{\beta}_1) - I_{(a,b)}^\pm(\hat{\beta}_2)$ is just the algebraic intersection number of an oriented arc from $\hat{\beta}_1$ to $\hat{\beta}_2$ with the union of the cycles of codimension one $(a,b)^+ \cup (b,a)^+$ (resp., $(a,b)^- \cup (b,a)^-$). Hence, for each loop $S$, these numbers are 0. The identities (*) follow and the proof of the proposition is complete. □

Remark 6 Obviously, for closed 2-braids, there are no 1-cocycles of any degree $d$ at all (because there are never triple points). The previous proposition implies that there are no non-trivial 1-cocycles of degree one for closed
Figure 15:
3-braids, but that in general there are $2((n-3) + (n-5) + (n-7) + \ldots)$ such cocycles.

We will see later that if we apply a 1-cocycle of degree one to the canonical class $[\text{rot}(\hat{\beta})]$ then we obtain a finite type invariant of degree one for $\hat{\beta}$. Proposition 3 says that there are exactly $[n/2]$ such invariants which are independent. But the number of non-trivial 1-cocycles of degree one is quadratic in $n$. Therefore, there are lots of relations between them when they are restricted to the canonical class. The following question seems to be interesting: Are the relations (*) from Proposition 3 the only relations in general between the 1-cocycles of degree one?

**Example 1** Let $\hat{\beta}$ be the closure of the 4-braid $\beta = \sigma_1 \sigma_2^{-1} \sigma_3^{-1}$. We consider $\Gamma_{(1,2)^-} \text{ and } \Gamma_{(2,1)^-}$

$\Gamma_{(2,3)^+} \text{ and } \Gamma_{(3,2)^+}$

A calculation by hand gives:

$$\Gamma_{(1,2)^-}(\text{rot}(\hat{\beta})) = \Gamma_{(2,3)^+}(\text{rot}(\hat{\beta})) = -1$$

and

$$\Gamma_{(2,1)^-}(\text{rot}(\hat{\beta})) = \Gamma_{(3,2)^+}(\text{rot}(\hat{\beta})) = +1$$

Therefore, all four 1-cocycles of degree 1 are non-trivial.

### 3.3 One-cocycles of degree two

Let $\beta \in B_n$ such that $\hat{\beta} \hookrightarrow V$ is a knot and let $a \in \mathbb{Z}/n\mathbb{Z}$ be fixed. Let $I$ be a configuration of degree 2, i.e. besides the marked triple point there is exactly one marked arrow.
Definition 14 An adjacent configuration of $I$ is any configuration which is obtained by sliding exactly one of the end points of the arrow over exactly one of the vertices of the triangle and which preserves the markings. An example is shown in Figure 17. A chain of adjacent configurations is a sequence of configurations such that any two consecutive configurations are adjacent.

Definition 15 A configuration $I$ of degree 2 is called braid impossible if it never occurs as a subconfiguration of a Gauss diagram of a closed $n$-braid which is a knot. Otherwise it is called braid possible.

Lemma 6 Any configuration $I$ which contains one of the sub-configurations in Figure 18 is braid impossible.

Proof: This follows immediately from the proof of Proposition 4.2 in [9].

Remark 7 There are other configurations $I$ which are braid impossible. For example, it follows from Proposition 4.3 in [9] that $I$ is braid impossible for $n = 3$ if it contains one of the sub-configurations drawn in Figure 19.

Definition 16 A configuration $I$ is called rigid if all adjacent configurations are braid impossible. Otherwise it is called flexible.
Lemma 7 Let $a \in \{1, 2, \ldots, [(n-1)/2]\}$ be fixed. The configurations shown in Figure 20 are all braid possible and rigid.

Proof: It is easy to show that the above configurations are braid possible by just taking examples. Let us show that e.g. the first of them is rigid. Indeed, all four adjacent configurations contain the third sub-configuration of Figure 18. Thus, by Lemma 6, they are not braid possible. The proof in the other cases is analogous. □

Remark 8 Let $I$ be a braid possible and rigid configuration. Let $I'$ be any configuration obtained from $I$ by reversing some arrows and replacing for each of these arrows its marking $x$ by $n - x$. We say that $I'$ is derived from $I$. If the triangle of $I'$ corresponds still to a Reidemeister III move, then $I'$ is also braid possible and rigid. This is evident from the fact that the set of closed braids is invariant under arbitrary changings of crossings. Therefore, the above construction allows to get lots of braid possible rigid configurations starting from the three cases in Figure 20 (examples are shown in Figure 21).

Let $\Gamma(S)$ be a 1-cochain of degree 2. Obviously, $\Gamma(S)$ is invariant under deformation of $S$ through all strata in $\Sigma^{(2)}$ besides $\Sigma_1^{(2)}$ and $\Sigma_2^{(2)}$ (see the proof of Proposition 3). Let us consider quadruple points. We have to guarantee that $\Gamma(S_m) = 0$ if $S_m$ is the boundary of a small normal disc.
of a stratum in $\Sigma^{(2)}$. Each triple point occurs exactly twice in $S_m$, say at $s_1, s_2 \in S_m$, and with different signs $w(s_1) = -w(s_2)$.

The Gauss diagrams of the closed braids are the same at $s_1$ and $s_2$ besides exactly three crossings $p_1, p_2, p_3$ which are now in another position with respect to the triple point (and to each other). We illustrate this in Figure 22. Let $I$ be a configuration which contains the triple point and only one of the three crossings, say $p_1$. One easily sees that the triple point together with $p'_1$ is then an adjacent configuration. For example, $p_1$ and $p'_1$ are in the same position with respect to the crossing between the branches 2 and 3 (see Figure 22). Thus, if $n > 3$, then for each configuration $\epsilon_i I_i$ in $\Gamma$ (see Definition 12), $\Gamma$ has to contain all adjacent configurations of $I_i$ and they have to enter all with the same coefficient $\epsilon_i$. We call this condition on $\Gamma$ the quadruple-condition. If $\Gamma$ verifies the quadruple-condition, then $\Gamma(S_m) = 0$. In particular, this implies that the triple point together with $p_1$ or $p_2$ or $p_3$ is never a rigid configuration! It remains to study a loop $S_m$ which is the boundary of a normal disc for a stratum in $\Sigma^{(2)}$ (see Figure 12). The two triple points in $S_m$ are of the same type and have different signs as already explained in the proof of Lemma 5. But their Gauss diagrams differ by exactly one crossing as shown in Figure 23. But now $p$ and $p'$ are different crossings. The homological markings $[p] = [p']$ coincide but $w(p) = -w(p')$.

Therefore, the configurations on the left-hand side and on the right-hand side of Figure 23 should enter in $\Gamma$ with opposite coefficients $\epsilon_i$. Notice that the marking $[p]$ of $p$ coincides with at least one of the markings of the triple point.

**Figure 20:**

**Definition 17** A configuration $I$ is called t-invariant if the marking of the arrow is different from all three markings of the triple point.
Let $\Gamma' \subset \Gamma$ be the t-invariant part of $\Gamma$. Then, evidently, $\Gamma'(S_m) = 0$.

Let us now consider configurations which are not t-invariant. If the two configurations in Figure 23 could be related by a chain of adjacent configurations then our method would break down. Indeed, in order to guarantee invariance under passing a quadruple point, they should enter in $\Gamma$ with the same coefficient $\epsilon$. But in order to guarantee invariance under passing an autotangency with a transverse branch they should enter in $\Gamma$ with opposite coefficients. The following surprising lemma implies that this does not occur for some types of triple points.

**Lemma 8** Let the type of the triple point in Figure 23 be one of the types shown in Figure 20 (or one obtained from them as explained in Remark 8). Then at least one of the two configurations in Figure 23 is rigid.

Proof: We have to distinguish two cases for the markings.

*Case 1* All three markings of the triple point are different. This is the case in $II$ and $III$ of Figure 20 if $3a \neq n$.

*Case 2* There are exactly two markings which are equal, hence we are in $I$ of Figure 20 or in $II$ or $III$ with $3a = n$. (Remember that markings of braid possible configurations are always non-zero).

The crossings $p$ and $p'$ form together a sub-configuration as shown in Figure 18. The crossings $p$ and $p'$ interchange the place in the triangle (corresponding to the triple point) and one easily sees that both arrows $p$ and $p'$ always move in the same direction on the circle. Therefore, in Case 1, we have the couples of configurations as shown in Figure 24. We see that the configurations on the left-hand side are always rigid and those on the right-hand side are always flexible. In Case 2, we have the couples of configurations shown in Figure 25.

The cases $I$ and $III$ with $3a = n$ are equivalent to the cases shown in Figure 25. We see that both configurations are always rigid. 

Thus, for each configuration $\epsilon_i I_i$ in $\Gamma$ which is shown on the right-hand side of Figure 24 or 25, $\Gamma$ has to contain also the corresponding rigid configuration on the left-hand side of Figure 24 or 25. This configuration has to enter in $\Gamma$ with the coefficient $-\epsilon_i$. We call this condition on $\Gamma$ the t-condition.

**Theorem 4** Let $\Gamma$ be a 1-cochain of degree 2 that satisfies the quadruple-condition and the t-condition. Then, $\Gamma$ is a 1-cocycle of degree 2.

Proof: We have proven that $\Gamma(S)$ is invariant under all homotopies of $S$. The rest of the proof follows from Lemma 8. □
Figure 24:

Figure 25:
Example 2 The case of closed 3-braids is very special because there do not appear any quadruple points in isotopies. Therefore we do not need the quadruple-condition. It follows easily from the proof of Lemma 8 that the 1-cochain in Figure 26 defines a 1-cocycle of degree 2. (For closed 3-braids, the homological markings of the triple point are determined by the arrows.) An easy calculation yields:

$$\Gamma(\text{rot}(\sigma_1 \sigma_2^{-1})) = +2$$

Therefore, $\Gamma$ defines a non-trivial 1-cohomology class of degree 2.

It follows from Theorem 4 that the $\Gamma$ shown in Figure 27 defines a 1-cocycle of degree 2 for closed 4-braids. A calculation yields:

$$\Gamma(\text{rot}(\sigma_1 \sigma_2^{-1} \sigma_3^{-1})) = -1$$

Therefore, $\Gamma$ is a non-trivial class of degree 2.

\[\]
Observation: Let \( a \in \{1, \ldots, \lfloor n/2 \rfloor - 1 \} \) be fixed and let \( I_{a}^{(1)} \) be the configuration shown in Figure 28. This configuration is derived from the configuration \( I \) in Figure 20 (see Remark 8). Therefore, the configuration \( I_{a}^{(1)} \) is a rigid braid-possible configuration. But moreover, \( I_{a}^{(1)} \) is a \( t \)-invariant configuration and hence, \( I_{a}^{(1)} \) already defines a 1-cocycle of degree 2. In the next section, we will see how to generalize \( I_{a}^{(1)} \) in order to obtain non-trivial 1-cocycles of arbitrary degree in a very simple way.

3.4 One-cocycles of arbitrary degree

For the degree \( d > 2 \), other strata in \( \Sigma^{(2)} \) also impose conditions on \( \Gamma \).

Definition 18 \( \Gamma \) verifies the tan-condition if no configuration \( I \) in \( \Gamma \) contains any sub-configuration as shown in Figure 16.

If \( \Gamma \) verifies the tan-condition then \( \Gamma(S) \) is invariant under deformation of \( S \) through points in \( \Sigma^{(1)}(\text{tri}) \cap \Sigma^{(1)}(\text{tan}) \). The next definition is a straightforward generalization of Definition 14.

Definition 19 An adjacent configuration of a braid possible configuration \( I \) is any configuration that is obtained in the following way: one chooses an arrow among the \( (d-1) \) that are not part of the triangle, and one slides an endpoint of this arrow over exactly one endpoint of another arrow (the latter may belong to the triangle). All markings of arrows are preserved. The resulting configuration has to be also braid possible.
Definition 20. \( \Gamma \) verifies the tri-condition if for each (braid possible) configuration \( \varepsilon I \) in \( \Gamma \), all adjacent configurations of \( I \) are also contained in \( \Gamma \), with the same coefficient \( \varepsilon \).

If \( \Gamma \) verifies the tri-condition, then \( \Gamma(S) \) is invariant under deformation of \( S \) through points in \( \Sigma^{(1)}(\text{tri}) \cap \Sigma^{(1)}(\text{tri}) \). The quadruple-condition for degree 2 is generalized for arbitrary degree in the obvious way. Notice that if \( \Gamma \) satisfies the tri-condition then it satisfies automatically the quadruple-condition.

Obviously, the strata of \( \Sigma_3^{(2)} \) do not impose any condition on \( \Gamma \). The t-condition for degree 2 has to be generalized in the following way.

Definition 21. The configurations on the same line in Figure 29 are called associated configurations if besides the shown sub-configuration, the rest of the configurations are identical. Notice that in the small arcs of the circle there are no other endpoints of arrows.

(Of course we extend all definitions to the configurations derived from those of Figure 29 by the Remark 8). If a configuration \( I \) is different from all configurations in Figure 29 (and of their derived configurations), then the associated configuration will be the empty configuration.
**Definition 22** \( \Gamma \) satisfies the t-condition if each of its configurations \( \epsilon I \) occurs together with \(-\epsilon I' \) where \( I' \) is the associated configuration.

**Theorem 5** Let \( \Gamma \) be a 1-cochain of degree \( d \) that satisfies the t, tri, tan-conditions. Then \( \Gamma \) is a 1-cocycle of degree \( d \).

**Proof:** It is completely similar to the proof of Theorem 4. □

**Remark 9** Let us take a 1-cocycle of degree 1, and consider the sum of all diagrams that are obtained from this 1-cocycle by adding one single arrow with a new marking, in any possible position with respect to the triangle. This sum is a 1-cocycle of degree 2. But this 1-cocycle is not interesting because it is just the product of the 1-cocycle of degree 1 with an invariant of degree 1 (see Proposition 2). In order to get a 1-cocycle \( \Gamma \) which do not decompose into products of 1-cocycles of lower degrees we need that not all possible positions of an arrow with a fixed marking enter into \( \Gamma \) with the same coefficient.

**Observation:** Let \( \beta \in B_n \) and let \( a \in \{1, \ldots, n-1\} \) be fixed. Then the sub-configurations shown in Figure 30 are locally rigid, i.e. none of the two arrows can slide over the other one in the small pictured arcs, because the resulting configuration would not be braid possible. Moreover, they verify the tan-condition. Using this observation, it is easy to construct non-decomposing 1-cocycles of arbitrary degrees. We give an example in the following proposition.

**Proposition 4** The configuration in Figure 31 defines a 1-cocycle of odd degree \( d \) if \( 3a = n \).

**Proof:** Obviously, the whole configuration is rigid. Moreover, no arrow with marking \( n-a \) can become an arrow of the triangle (by passing \( \Sigma^{(2)}_2 \)), because it is always separated from the triangle by an arrow with marking \( a \). Evidently, no arrow with marking \( a \) can become an arrow of the triangle (by passing \( \Sigma^{(2)}_2 \)). No arrow can slide over the triangle, because \( 3a = n \). □

We generalize now the 1-cocycle \( I^{(1)}_a \) from the end of the previous section.

**Definition 23** Let \( d \in \mathbb{N}^* \) be odd and let \( a \in \{1, 2, \ldots, [n/2] - 1\} \) be fixed. The configuration \( I^{(d)}_a \) is defined in Figure 32 (in the actual picture, we took \( d = 5 \)). The arrows with markings \( a \) and \( n-a \) are alternating in the figure.

**Proposition 5** \( I^{(d)}_a \) defines a 1-cocycle of degree \( d + 1 \).
Proof: It is completely analogous to the proof of Proposition 4. □

Definition 24 Let \( d \in \mathbb{N}^* \) be even, let \( n \) be divisible by 3 and let \( a = n/3 \). The 1-cochain \( \Gamma^{(d)}_{n/3} \) is defined in Figure 33 (in the actual picture, we took \( d = 4 \)).

![Figure 30](image1)

![Figure 31](image2)

Proposition 6 \( \Gamma^{(d)}_{n/3} \) defines a 1-cocycle of degree \( d + 1 \).

Proof: The first configuration in \( \Gamma^{(d)}_{n/3} \) is still rigid. But exactly one of the arrows with marking \( n/3 \) can be interchanged now with exactly one of the arrows in the triangle (by passing \( \Sigma^{(2)}_2 \)). The result is the second configuration in the figure. This configuration is no longer rigid. The remaining configurations in Figure 33 are just all the adjacent configurations. We need that \( 3a = n \) in order to guarantee that in the last configuration the arrow \( n/3 \) cannot slide further over some of the remaining two vertices of the triangle. □
Figure 32:

Figure 33:
Remark 10 For example, the braid possible sub-configuration II in Figure 20 with \( a = n/3 \) is not contained in \( \Gamma_{n/3}^{(d)} \). Such considerations imply easily that in fact \( \Gamma_{n/3}^{(d)} \) is not a product of 1-cocycles of lower degrees.

If \( n \) is not divisible by 3 then we replace \( \hat{\beta} \) by a \( 3k \)-cable, \( k \in \mathbb{N}^* \), and take \( a = kn \).

The mirror image \( \Gamma_{n/3}^{(d)} \) of \( \Gamma_{n/3}^{(d)} \) (obtained by reversing all arrows, including those of the triangles, and replacing all markings by their opposites) is of course also a 1-cocycle of degree \( d + 1 \).

Example 3 Let \( \beta = \sigma_1^{-1} \sigma_3^2 \in B_3 \). Then, \( \Gamma_1^{(2)}([\text{rot}(\hat{\beta})]) = -1 \). This shows that the above 1-cocycles are not always trivial.

Theorem 6 Let \( K \hookrightarrow V \) be a closed braid which is a knot and let \( \Gamma \) be a 1-dimensional cohomology class of degree \( d \). Let \( [\text{rot}(K)] \) be its canonical class. Then, \( \Gamma([\text{rot}(K)]) \) is a \( \mathbb{Z} \)-valued finite type invariant of degree at most \( d \) for \( K \hookrightarrow V \).

Proof: We have shown that \( \Gamma([\text{rot}(K)]) \) depends only on the isotopy type of \( K \hookrightarrow V \). The cocycle invariant \( \Gamma([\text{rot}(K)]) \) is calculated as some sum \( \sum_{s_i} \) over triple points \( s_i \) in \( \text{rot}(K) \). Therefore, it suffices to prove that this sum \( \sum_{s_i} \) for each triple point \( s_i \) is of finite type (even if it is not invariant). If \( \Gamma \) is of degree \( d \) then \( \sum_{s_i} \) depends only on the triple point and of configurations of \( d - 1 \) other crossings. This means that in order to calculate a summand in \( \sum_{s_i} \), we can switch all other crossings besides the triple point and the fixed \( d - 1 \) crossings. The result will not change. This implies immediately that each \( \sum_{s_i} \) is of degree \( d \) (see [14] and also [7]). □

Definition 25 We call \( \Gamma([\text{rot}(K)]) \) a 1-cocycle invariant of degree \( d \).

Example 4 Let \( K \hookrightarrow V \) be a closed 4-braid. Using Theorem 6, Proposition 2, Proposition 3 and the examples of the next section, one easily calculates that

\[
\Gamma_{(2,1)}([\text{rot}(K)]) \equiv \Gamma_{(3,2)}([\text{rot}(K)]) \equiv W_1(K) - W_2(K)
\]

The 1-cocycle invariants have the following nice property, which can be used to estimate from below the length of conjugacy classes of braids.

Proposition 1 Let the knot \( K = \hat{\beta} \hookrightarrow V \) be a closed \( n \)-braid and let \( c(K) \) be its minimal crossing number, i.e. its minimal word length in \( B_n \). Then all 1-cocycle invariants of degree \( d \) vanish for

\[
d \geq c(K) + n^2 - n - 1.
\]
Proof: Assume that the word length of $β$ is equal to $c(K)$. We can represent $[rot(K)]$ by the following isotopy which uses shorter braids.

$β → ΔΔ^{-1}β → Δ^{-1}βΔ → Δ^{-1}Δβ' → β' → ΔΔ^{-1}β' → Δ^{-1}βΔ → Δ^{-1}Δβ → β$

Here, $β'$ is the result of rotating $β$ by $π$ i.e. each $σ_i^{±1}$ is replaced by $σ_i^{±1}_{n-1}$.

Obviously, $c(Δ) = \frac{n(n-1)}{2}$. Thus, each Gauss diagram which appears in the isotopy has no more than $c(K) + n^2 - n$ arrows. Indeed, we create a couple of crossings by pushing $Δ$ through $β$ only after having eliminated a couple of crossings before (see the previous section). Therefore, for each diagram with a triple point there are at most $c(K) + n^2 - n - 3$ other arrows, and hence, each summand in a 1-cocycle of degree $d$ is already zero if $d ≥ c(K) + n^2 - n - 1$.

3.5 The invariants are not functorial under cabling

Bar-Natan, Thang Le, Dylan Thurston [3] and independently S. Willerton [18] have given a formula for the Kontsevich integral of the cable of a knot in $\mathbb{R}^3$. This formula has the following corollary:

Corollary 1 (Bar-Natan, Thang Le, D. Thurston - Willerton) Let $K, K' \hookrightarrow \mathbb{R}^3$ be knots which have the same Vassiliev invariants up to a fixed degree $d$. Then, all (the same) cables of $K$ and $K'$ have the same Vasiliev invariants up to degree $d$.

In other words, it is useless to cable knots in purpose to distinguish them by Vassiliev invariants.

Our 1-cocycle invariants of degree $d$ form a subset of all finite type invariants of degree $d$ for knots in the solid torus. For example, as already mentioned, there are no non-trivial 1-cocycle invariants at all for closed 2-braids and there are no non-trivial 1-cocycle invariants of degree 1 for closed 3-braids. However, it turns out that cabling is a usefull operation for the subset of 1-cocycle invariants of degree $d$.

Proposition 7 The closed 2-braids $\hat{σ}_1, \hat{σ}_1^{-1}$ can not be distinguished by any 1-cocycle invariants. However, $Cab_2(\hat{σ}_1)$ and $Cab_2(\hat{σ}_1^{-1})$ can be distinguished by a 1-cocycle invariant of degree 1.

Proof: $Cab_2(\hat{σ}_1)$ and $Cab_2(\hat{σ}_1^{-1})$ can be represented respectively by the 4-braids $β = σ_3σ_2σ_1σ_2$ and $β' = σ_3^{-1}σ_2σ_1^{-1}σ_2^{-2}$. A calculation by hand shows:

$$\Gamma_{(2,1)} - ([rot(β)]) = \Gamma_{(3,2)} + ([rot(β)]) = +1$$
\[
\Gamma_{(2,1)^-}(\text{rot}(\tilde{\beta}')) = \Gamma_{(3,2)^+}(\text{rot}(\tilde{\beta}')) = -3
\]

\[\square\]

**Remark 11** Obviously, closed 2-braids are classified by the unique invariant \(W_1\) of degree 1. The braid \(\hat{\sigma}_1\) is obtained from \(\hat{\sigma}_1^{-1}\) by multiplying \(\sigma_1^{-1}\) with \(\sigma_2^{-1}\). From this, one easily concludes that for all \(k \in \mathbb{Z}\),

\[
\Gamma_{(2,1)^-}(\text{rot}(\text{Cab}_2(\sigma_1^{2k+1}))) \equiv \Gamma_{(3,2)^+}(\text{rot}(\text{Cab}_2(\sigma_1^{2k+1}))) \equiv 1 + 4k
\]

Therefore, the unique non-trivial 1-cocycle of degree 1 for the 2-cable of 2-braids classifies also closed 2-braids.

Do all finite type invariants of closed braids and of local knots arise as linear combinations of 1-cocycle invariants of appropriate cables?

### 3.6 Homological estimates for the number of braid relations in one-parameter families of closed braids

Our 1-cocycles can be used in order to obtain information about one-parameter families of closed braids which are knots. Let \(K\) be a closed \(n\)-braid which is a knot. Let \(S\) be a generic loop in \(M(K)\).

**Definition 26** The *-*length \(b([S])\) of \([S] \in H_1(M(K)\}; \mathbb{Z})\) is the minimal number of triple points in \(S\) among all unions of generic loops \(S\) in \(M(K)\) which represent \([S]\).

**Theorem 7** Let \(a, b \in \{0, 1, \ldots, n\}\) with \(a < b\). Then

\[
b([S]) \geq 2 \sum_{(a,b)^+} |\Gamma_{(a,b)^+}(\{[S]\})| + 2 \sum_{(a,b)^-} |\Gamma_{(a,b)^-}(\{[S]\})|
\]

**Proof:** Each \(\Gamma_{(a,b)^+}, \Gamma_{(a,b)^-}\) is a (in general non-trivial) 1-cocycle of degree 1 (see Proposition 3). Each triple point in \(S\) contributes by \(\pm 1\) to the value of such a cocycle. The inequality follows now from the relations (*) in Proposition 3.\(\square\)

**Example 5** Let \(\beta = \sigma_1\sigma_2^{-1}\sigma_3^{-1} \in B_4\). For all \(m \in \mathbb{Z}\), we have \(b(m[\text{rot}(\tilde{\beta})]) \geq 4|m|\). This follows immediately from Example 2.
Theorem 7 does not contain any information in the case of closed 3-braids (because all 1-cocycles of degree 1 are trivial). Therefore, we use the 1-cocycle $\Gamma$ of degree 2 from Example 2. Let $\beta = \sigma_1\sigma_2^{-1} \in B_3$. One has $\Gamma(\text{rot}(\hat{\beta})) = +2$. It is easily seen that this implies $\Gamma!(\text{rot}(\hat{\beta})) = +2$ too, where $\Gamma!$ is the mirror image of $\Gamma$ (see Remark 10). Thus, for all $m \in \mathbb{Z} \setminus 0$, $m(\text{rot}(\hat{\beta}))$ intersects both types of strata in $\Sigma^{(1)}(\text{tri})$ (compare Figure 9). But the closure of the union of all strata of the same type in $\Sigma^{(1)}(\text{tri})$ are trivial cycles of codimension 1 in $M(\hat{\beta})$ and hence, $m(\text{rot}(\hat{\beta}))$ intersects each of these two strata in at least two points. It follows that $b(m[\text{rot}(\hat{\beta})]) \geq 4$. The canonical loop shows that this inequality is sharp for $m = \pm 1$.

4 Character invariants

In this section we introduce new easily calculable isotopy invariants for closed braids.

Remember, that a generic isotopy $\hat{\beta}_s, s \in [0, 1]$, induces a tan-transverse homotopy $\text{rot}(\hat{\beta}_s), s \in [0, 1]$, of the canonical loops. The trace circles for different parameter $s$ are in a natural one-to-one correspondence. Consequently, we can give names $x_i$ to the circles of $TL(\hat{\beta}_0)$ and extend these names in a unique way on the whole family of trace circles.

Let $\{x_1, x_2, \ldots\}$ be the set of named trace circles. Obviously, for each circle $x_i$ there is a well defined homological marking $h_i \in H_1(V)$. Let $[x_i] \in H_1(T^2)$ be the homology class represented by the circle $x_i$ (with its natural orientation induced from the orientation of the trace graph).

4.1 Character invariants of degree one

In this section, we use the named cycles, i.e. the trace circles, in order to refine the 1-cocycles of degree 1 which were defined in Section 3.2.

Let $K$ be a closed $n$-braid which is a knot. Let $S \subset M(K)$ be a generic loop and let $X = \{x_1, \ldots, x_m\}$ be the corresponding set of cycles of named crossings. Let $x_{i_1}, x_{i_2}, x_{i_3} \in X$ be fixed. We do not assume that they are necessarily different. Let $h_{i_1}, h_{i_2}, h_{i_3}$ be the corresponding homological markings.

**Definition 27** A character of degree 1 of $S$, denoted by

$$C(h_{i_1}, h_{i_2}) \pm (x_{i_1}, x_{i_2}, x_{i_3})(S)$$

or sometimes shortly $C(S)$, is the algebraic intersection number of $S$ with the strata $(h_{i_1}, h_{i_2}) \pm$ in $\Sigma^{(1)}(\text{tri})$ and such that the crossings of the triple point...
belong to the named cycles as shown in Figure 34. We call the unordered set \( \{x_{i_1}, x_{i_2}, x_{i_3}\} \) the support of the character \( \mathcal{C} \).

\[
\begin{align*}
\left( h_{i_1}, h_{i_2} \right)^+ \quad & \quad \left( h_{i_1}, h_{i_2} \right)^- \\
\end{align*}
\]

Figure 34:

**Remark 12** Evidently, in order to obtain a non-trivial intersection number we need that \( h_{i_3} = h_{i_1} + h_{i_2} - n \) in \( (h_{i_1}, h_{i_2})^+ \) and that \( h_{i_3} = h_{i_1} + h_{i_2} \) in \( (h_{i_1}, h_{i_2})^- \) (see Figure 10). It follows also that \( x_{i_1} = x_{i_2} = x_{i_3} \) implies that necessarily \( h_{i_1} = h_{i_2} = h_{i_3} = 0 \) or \( h_{i_1} = h_{i_2} = h_{i_3} = n \).

Notice that for characters of degree 1 the relations (*) from Proposition 3 are no longer valid. For example, \( C(h_{i_1}, h_{i_2})^+(x_{i_1}, x_{i_2}, x_{i_3}) (S) \) can be non-trivial for \( h_{i_1} = h_{i_2} \) and even for \( x_{i_1} = x_{i_2} \).

**Theorem 8** Let \( S, S' \subset M(K) \) be generic loops and let \( \{x_1, \ldots, x_m\}, \{x'_1, \ldots, x'_m\} \) be the corresponding sets of named cycles. If \( S \) and \( S' \) are tan-transverse homotopic, then \( m = m' \) and there is a bijection \( \sigma : \{x_1, \ldots, x_m\} \to \{x'_1, \ldots, x'_m\} \) which preserves the homological markings \( h_i \) as well as the homology classes \( [x_i] \) and such that

\[
C(h_i, h_j)^\pm(x_{i_1}, x_{i_2}, x_{i_3})(S) = C(h_i, h_j)^\pm(\sigma(x_{i_1}), \sigma(x_{i_2}), \sigma(x_{i_3}))(S')
\]

for all triples \( (x_{i_1}, x_{i_2}, x_{i_3}) \).

**Proof:** This is an immediate consequence of Lemma 5 and the fact that the trace circles are isotopy invariants. □

In other words, characters \( C(S) \) of degree one are invariants of \( [S]_{t-1} \).
4.2 Character invariants of arbitrary degree for loops of closed braids

We refine the results of the sections 3.3 and 3.4. in a straightforward way. Let $S \subset M(K)$ be an oriented generic loop and let $X = \{x_1, \ldots, x_m\}$ be the corresponding set of named cycles.

Let $I$ be a configuration of degree $d$ (see Definition 11) and let $(x_{i_1}, \ldots, x_{i_{d+2}})$ be a fixed $(d+2)$-tuple of elements in $X$ (not necessarily distinct). A named configuration $I(x_{i_1}, \ldots, x_{i_{d+2}})$ is the configuration $I$ together with a given bijection $\phi$ of $(x_{i_1}, \ldots, x_{i_{d+2}})$ with the $d+2$ arrows in $I$ and such that $x_{i_1}, x_{i_2}, x_{i_3}$ are the arrows of the triangle exactly as in the previous section. Of course, different bijections give in general different named configurations.

A character chain $\Gamma$ of degree $d$ is a linear combination $\Gamma = \sum \epsilon_i I_i, \epsilon_i \in \{-1, 1\}$ of named configurations $I_i$ of degree $d$ which are all defined with the same $d+2$-tuple $(x_{i_1}, \ldots, x_{i_{d+2}})$ where $x_{i_1}, x_{i_2}, x_{i_3}$ are the arrows of the triangle as shown in Figure 34. We refine the definitions and conditions of Sections 3.3. and 3.4. in the obvious way. Adjacent named configurations are obtained by sliding the arrows and preserving the names.

In associated named configurations, the two arrows which interchange have the same name. We show an example in Figure 35. The named quadruple and tri-conditions are now: $\Gamma$ contains all named adjacent configurations and they enter $\Gamma$ with the same sign.

The $t$-condition is now: $\Gamma$ contains all named associated configurations and they enter $\Gamma$ with different signs.

The tan-condition is now: no named configuration $I_i$ in $\Gamma$ contains any sub-configuration as shown in Figure 36. Notice that sub-configurations as shown in Figure 36 are allowed if the arrows have different names $x_i \neq x_j$, even if they have the same homological markings $h_i = h_j$.

**Theorem 9** Let $S, S' \subset M(K)$ be generic loops and let $\{x_1, \ldots, x_m\}, \{x'_1, \ldots, x'_m\}$ be the corresponding sets of named cycles. If $S$ and $S'$ are tan-transverse homotopic, then there is a bijection

$$\sigma : \{x_1, \ldots, x_m\} \rightarrow \{x'_1, \ldots, x'_m\}$$

which preserves the homological markings $h_i$ as well as the homology classes $[x_i]$ and such that

$$C_{(x_{i_1}, \ldots, x_{i_{d+2}})}(S) = C_{(\sigma(x_{i_1}), \ldots, \sigma(x_{i_{d+2}}))}(S')$$

for all character chains of degree $d$ which satisfy the named quadruple, tri, $t$, tan-conditions.
Proof: This is completely analogous to the proofs of Theorems 5 and 8. □

Definition 28 The character chains $C(S)$ of the above theorem are called characters of degree $d$. They are invariants of $[S]_{t-t}$.

Example 6 Here, all names $x_i$ are different.

The named configuration shown in Figure 37 defines a character of degree 5 for closed $n$-braids if $h_1 = h_2 = h_4 \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Indeed, the configuration is braid possible and rigid (see Lemmas 6 and 7) and neither $x_2$ nor $x_4$ can become an arrow of the triangle by passing $\Sigma^{(2)}_2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure35.png}
\caption{Figure 35:}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure36.png}
\caption{Figure 36:}
\end{figure}

Let $h_i, h_j$ and a type of triple point, e.g. $(h_i, h_j)^+$, be fixed. It follows immediately from the definitions that

\[ (** ) \Gamma_{(h_i, h_j)^+} = \sum_{(x_k, x_l, x_m)} C_{(h_i, h_j)^+}(x_k, x_l, x_m) \]

where $h(x_k) = h_i$ and $h(x_l) = h_j$. 

39
Figure 37:

Hence, character invariants define splittings of one-cocycle invariants. However, the set of character invariants on the right hand side of (***) is not an ordered set. Therefore we have to consider the set of character invariants as a local system on $M(\hat{\beta})$.

It follows from Lemma 3 that the names $x_i$ are determined by their homological markings $h_i$, and that there are exactly $n-1$ trace circles.

However, this is in general no longer true in the case of multiples of the canonical loop.

Let $l \in \mathbb{N}$ be fixed and let $l \text{rot}(\hat{\beta})$ be the loop which is defined by going $l$ times along the canonical loop. Let $TL(l\hat{\beta})$ denote the corresponding trace graph (the t-coordinate in the thickened torus covers now $l$ times the t-circle).

We will show in a simple example that the local system of character invariants is in general non trivial if $l > 1$.

**Example 7** Let $\beta = \sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1} \in B_3$. We will write shortly $\beta = 2\overline{1}2\overline{1}$.

The combinatorial canonical loop for $l = 2$ is given by the following sequence (where we write the names of the crossings just below the crossings).

$2\overline{1}2\overline{1} \rightarrow \overline{1}2\overline{1}2(\overline{1}1)21 \rightarrow 1\overline{2}1\overline{2}2\overline{1} \rightarrow \overline{1}2\overline{1}21(\overline{1}2)1 \ast_1 \rightarrow$

$abcd \quad \text{zyxabcdxyz} \quad \text{zyxacyz} \quad \text{zyxbcu}_1u_1yz$

$1\overline{2}1\overline{1}(212)1 \ast_2 \rightarrow \overline{1}2\overline{1}2(\overline{1}1)21\overline{1}2 \rightarrow \overline{1}2\overline{1}22\overline{1}2 \rightarrow \overline{1}2\overline{1}21(\overline{1}2)1 \ast_3 \rightarrow$

$\text{zyxbcu}_1zyu_1 \quad \text{zyxabzcu}_1cyu_1 \quad \text{zyxaucy}_1u_1 \quad \text{zyxaucu}_1u_2cyu_1$

$1\overline{2}\overline{1}(212)1\overline{2} \ast_4 \rightarrow (1\overline{2}\overline{1}121)1\overline{2} \rightarrow 1\overline{2}1 \rightarrow \overline{1}2\overline{1}21(\overline{1}212)1 \ast_5 \rightarrow$

$\text{zyxaucu}_1u_2yu_1 \quad \text{zyxau}_1u_2cyu_1 \quad u_1u_2yu_1 \quad u_1u_2yu_1 \quad z_1y_1x_1u_1u_2yu_1x_1y_1z_1$
The following examples are calculated by Alexander Stoimenow using his program in C++. His program is available by request (see [15]).

4.3 The examples

The examples are calculated by Alexander Stoimenow using his program in C++. His program is available by request (see [15]).

Let $\beta = 121212(11) \to 1211212 \to 121121(121) \ast_6 \to 1211(212)121$

$$z_1y_1x_1u_1u_2yy_1x_1u_1z_1 \to z_1y_1x_1u_1u_2yy_1x_1v_1v_1 \to z_1y_1x_1u_1u_2yy_1x_1y_1v_1$$

$$\ast_7 \to 121112(11)21 \to 12111221 \to 12111(121)121 \ast_8 \to (121121)2121$$

$$z_1y_1x_1u_1v_1y_1y_1v_1 \to z_1y_1x_1u_1v_1y_1y_1v_1 \to z_1y_1x_1u_1v_1y_1v_1v_1 \to z_1y_1x_1u_1v_2vy_1v_1$$

$$\to 2121$$

$$v_1v_2y_1v_1$$

We have the identifications: $d = x$, $b = z$, $x = c$, $y = u_2$, $z = a$, $u_1 = z_1$,

$$u_2 = x_1$, $x_1 = u_1$, $y_1 = v_2$, $z_1 = y$. This gives us:

$$2121 \to 2121$$

$aacc$ $v_1v_2v_2v_1$

$\text{together with the names } u_1 = z_1 = x_1 = u_2 = y.$

The second rotation gives us:

$$2121 \to 2121 \to 2121$$

$aacc$ $v_1v_2v_2v_1$ $v_3v_4v_3$

$\text{together with the identification } v_1 = v_2, y_3 = v_1, z_2 = v_2 \text{ and with the names } u_1 = z_1 = x_1 = u_2 = y, u_3 = z_3 = x_3 = u_4 = y_2$. The monodromy (i.e. how the set of crossings is mapped to itself after the rotation) implies now: $a = v_3 = v_4 = c$.

Therefore, we have exactly four named cycles: $a, v_1, u_1, u_3$ for $2121 \to 2121$ with $l = 2$.

Consequently, we have $2121 \to 2121 \to 2121$ with the names $aacc \to v_1v_1v_1v_1 \to aaaa$.

Hence, rot($\hat{\beta}$) acts by interchanging $a$ and $v_1$ (as well as $u_1$ and $u_3$). Consequently, the local system is non trivial in this example.

We want to show that the link $\hat{\beta} \cup$ (core of complementary solid torus) is not invertible in $S^3$. This is equivalent to show that $\beta$ is not conjugate to $\beta_{\text{inverse}} = 2121$. (compare [7]).

Because $\beta$ is a 3-braid, the homological markings are in $\{1, 2\}$.

Character invariants of degree one for $l = 1$ and $l = 2$ do not distinguish $\hat{\beta}$ from $\hat{\beta}_{\text{inverse}}$. However, for $l = 3$ we obtain three different named cycles.
$x_1, x_2, x_3$ of homological marking 1 and three different named cycles $y_1, y_2, y_3$ of marking 2.

We consider the set of nine character invariants of degree one which are of the following form (see Figure 38), where $i, j \in \{1, 2, 3\}$. For $\hat{\beta}$

![Figure 38](image)

we obtain the set $\{-1, -1, -1, -1, -1, 2, 2, 2\}$ and for $\hat{\beta}_{\text{inverse}}$ we obtain the set $\{1, 1, 1, 1, 1, -2, -2, -2\}$. Evidently, even without knowing the local system, there is no bijection of the trace circles for $\hat{\beta}$ and those for $\hat{\beta}_{\text{inverse}}$ which identifies the above sets. Consequently, $\hat{\beta}$ and $\hat{\beta}_{\text{inverse}}$ are not isotopic.

The knot 9_5 can be represented as a 8-braid with 33 crossings. Character invariants of degree one for $l = 2$ show that the braid is not invertible in the same way as in the previous example.

The knot 8_6 can be represented as a 5-braid with 14 crossings. Character invariants of degree one for $l = 2, 4, 6$ show that it is not invertible as a 5-braid. (It does not work for for $l = 1, 3, 5$.)

The knot 8_17 is not invertible as a 3-braid, which is shown with $l = 4$. (It does not work with $l = 1, 2, 3$.)

Let $b \in P_5$ be Bigelow’s braid (see [1]). It has trivial Burau representation. Let $s = \sigma_1\sigma_2\sigma_3\sigma_4$. The braids $s$ and $bs$ have the same Burau representation. This is still true for their 2-cables, i.e. we replace each strand by two parallel strands. Character invariants of degree one for $l = 2$ show that the (once positively half-twisted) 2-cables of the above braids are not conjugate, and, consequently, the braids $s$ and $bs$ are not conjugate either.
4.4 A refinement of character invariants of degree one

The number of triple points in a trace graph can only change by trihedron moves, as follows from Theorem 3.

**Definition 29** A generalized trihedron is a trihedron which might have other triple points on the edges.

Figure 39 shows a tetrahedron move which transforms a trihedron into a generalized trihedron. The generalized trihedron has still exactly two vertices. Evidently, the number of generalized trihedrons does not change under tetrahedron moves. Let $E$ be the set of all triple points in the trace graph $TL(\hat{\beta})$ which are not vertices of generalized trihedrons. The following lemma is an immediate consequence of Theorem 3.

**Lemma 9** The set $E$, and, hence $\text{card}(E)$, is an isotopy invariant of closed braids $\hat{\beta}$.

Moreover, for each element of $E$ we have the additional structure defined before: type, sign, markings, names.

It follows from Theorem 3 and the geometric interpretation of generalized trihedrons in [11] that the two vertices of a generalized trihedron have always different signs. Consequently, character invariants of degree one count just the algebraic number of elements in $E$ which have a given type and given names. But already the geometric number of such elements in $E$ is an invariant as shows Lemma 9.

**Definition 30** Let $C_{(h_i,h_j)}^{+(-)}(x_k,x_l,x_m)$ be the number of all positive (respectively negative) triple points in $E$ of given type $(h_i,h_j)^{+(-)}$ and with given names $x_k,x_l,x_m$. We call these the positive (respectively negative) character invariants.
The following proposition is now an immediate consequence of Lemma 9 and Definition 30.

**Proposition 8** The positive and the negative character invariants are isotopy invariants of closed braids.

**Example 8** Let us consider $\beta = \sigma_2\sigma_1^{-1} \in B_3$. Its trace graph $TL(\beta)$ is shown in Figure 40. One easily sees that it does not contain any generalized trihedrons. Consequently, all four triple points are in $E$. There are exactly two names $x_1$ and $x_2$. They correspond to the homological markings $h_1 = 1$ and $h_2 = 2$.

One easily calculates that two of the triple points are of type $(1,1)^-$ and they have different signs. The other two are of type $(2,2)^+$ and they have different signs too. Consequently, all character invariants of degree one are zero.

However, we have $C_{(1,1)^-}^+(x_1, x_1, x_2) = C_{(1,1)^-}^-(x_1, x_1, x_2) = 1$, and $C_{(2,2)^+}^+(x_2, x_2, x_1) = C_{(2,2)^+}^-(x_2, x_2, x_1) = 1$. Consequently, the positive and negative character invariants contain in this example more information than the character invariants of degree one (for $l = 1$).

There is not yet a computer program available in order to calculate these invariants in more sophisticated examples.

### 4.5 Homotopical estimates for the number of braid relations in one parameter families of closed braids

In section 3.6, we have used the 1-cocycles in order to estimate the $\ast$-length $b([S])$ for classes $[S] \in H_1(M(K); \mathbb{Z})$.

Let $S \subset M(K)$ be a generic loop.

**Definition 31** The $\ast$-length $b([S]_{t-t})$ of the tan-transvers homotopy class is the minimal number of triple points in $S$ amongst all generic loops in $M(K)$ which represent $[S]_{t-t}$.

**Theorem 10** Let $\{x_1, \ldots, x_m\}$ be the set of essential named cycles of $S$. Let $C_{(x_{i_1}, x_{i_2}, x_{i_3})}(S)$, $\{i_1, i_2, i_3\} \subset \{1, \ldots, m\}$ be the set of all characters of degree one for $S$. Then

$$b([S]_{t-t}) \geq \sum_{\{i_1, i_2, i_3\} \subset \{1, \ldots, m\}} |C_{(x_{i_1}, x_{i_2}, x_{i_3})}(S)|$$
Figure 40:

Proof: Different triples \((x_{i_1}, x_{i_2}, x_{i_3})\) correspond to different strata of \(\Sigma^{(1)}(tri)\). Locally, each intersection index of \(S\) with such a stratum is equal to \(\pm 1\). The result follows. \(\square\)

Example 9 One easily calculates that for \(S = 2\text{rot}(\hat{\sigma}_2\hat{\sigma}_1^{-1})\) there are exactly eight non-trivial characters of degree one. Each of them is equal to \(\pm 1\). This can be generalized straightforwardly for arbitrary \(l\) with \(|l| \geq 2\). We obtain the following proposition:

**Proposition 9** For all integers \(l\) such that \(|l| \geq 2\), we have:

\[
b([l\text{rot}(\hat{\sigma}_2\hat{\sigma}_1^{-1})]_{l-1}) = 4|l|
\]

Proof: This follows immediately from Theorem 10 together with a direct calculation which shows that

\[
b([l\text{rot}(\hat{\sigma}_2\hat{\sigma}_1^{-1})]_{l-1}) \leq 4|l|
\]

\(\square\)

Characters of higher degree can be used, of course, in the same way as 1-cocycles of higher degree were used to estimate \(b([S])\) (see section 3.6.).
5 Character invariants of degree one for almost closed braids

Let $K \hookrightarrow V = \mathbb{R}^3 \setminus z - axes$ be an oriented knot such that the restriction of $\phi$ to $K$ has exactly two critical points. In this case $K$ is called an almost closed braid. Necessarily, one of the critical points is a local maximum and the other is a local minimum. In analogy to the case of closed braids we consider almost closed braids up to isotopy through almost closed braids.

We consider the (geometric) canonical loop $rot(K)$ and the corresponding trace graph $TL(K)$. $TL(K)$ is an oriented link with triple points and exactly four boundary points (corresponding to the oriented tangencies at the critical points of $\phi$).

There are more types of moves for $TL(K)$ as in the case of closed braids (see [11]). But it turns out that only two of these additional moves are relevant for the construction of the invariants.

**Definition 32** A band move is shown in Figure 41.

**Definition 33** A unknot move is shown in Figure 42.

![Figure 41:](image1)

![Figure 42:](image2)

A band move corresponds to a branch which passes transversally through an ordinary cusp and an unknot move corresponds to the case that the maximum and the minimum of $\phi$ have the same critical value. Notice, that there are no extrem pair moves, i.e. two maxima or two minima of $\phi$ with the same critical value (for all this compare [11]). An extreme pair move induces
a Morse modification of index 1 of the trace graph. Such modifications are
difficult to control. We restrict ourselfs to almost closed braids in order to
avoid them.

The unknot component $x_i$ of $TL(K)$ which was created by an unknot
move, has always $[x_i] = 0$ in $H_1(T^2)$. Notice, that in a band move there is al-
ways involved one component $x_i$ which has boundary. Let $X = \{x_1, x_2, \ldots\}$
be the set of all closed trace circles of $TL(K)$ which represent non-trivial
homology classes in $H_1(T^2)$.

The above observations together with Theorem 8 and Theorem 1.10 from
[11] imply immediately the following theorem.

**Theorem 11** Each character invariant of degree one with names $x_l, x_k, x_m \in
X$ is an isotopy invariant for almost closed braids.

Evidently, we can apply the above theorem to $lrot(K)$ with arbitrary
$l \in \mathbb{N}$ exactly as in the case of closed braids.

**References**

[1] Bigelow S. : The Burau representation is not faithful for $n = 5$ , Geom.
Topol. 3 (1999) , 397-404.

[2] Bar-Natan D. , On the Vassiliev knot invariants, Topology 34 (1995),
423-472.

[3] Bar-Natan D. , Thang T. Q. Le , Thurston D. : Two applications of
elementary knot theory to Lie algebras and Vassiliev invariants , Geom.
Topol. 7 (2003) , 1-31.

[4] Birman J . : Braids , Links and Mapping class groups , Annals of
Mathematics Studies 82 , Princeton University Press (1974).

[5] Birman J. , Brendle T. : Braids : A survey , mathGT/0409205 (2004).

[6] Fiedler T. : A small state sum for knots , Topology 32 (1993) , 281-294.

[7] Fiedler T. : Gauss diagram invariants for Knots and Links , Math-
ematics and Its Applications 532 ,Kluwer Academic Publishers(2001).

[8] Fiedler T. : Isotopy invariants for smooth tori in 4-manifolds , Topology
40 (2001) , 1415-1435.
[9] Fiedler T. : Gauss diagram invariants for knots which are not closed braids , Math. Proc. Camb. Phil. Soc. 135(2003) , 335-348.

[10] Fiedler T. : One parameter knot theory , preprint 262, Lab. Math. Emile Picard ,UPS ,(2003).

[11] Fiedler T. , Kurlin V. : A one-parameter approach to knot theory , math.GT/0606381

[12] Goryunov V. : Finite order invariants of framed knots in a solid torus and in Arnold’s $J^+$-theory of plane curves , ”Geometry and Physics”, Lect. Notes in Pure and Appl. Math. (1996), 549-556.

[13] Morton H. : Infinitely many fibered knots having the same Alexander polynomial , Topology 17 (1978) ,101-104.

[14] Polyak M. ,Viro O. : Gauss diagram formulas for Vassiliev invariants , Internat. Math. Res. Notes 11 (1994), 445-453.

[15] Stoimenow A. : www.kurims.kyoto-u.ac.jp/ stoimeno/

[16] Vassiliev V. : Cohomology of knot spaces , Adv. in Sov. Math. , Theory of Singularities and its Appl. , A.M.S. Providence, R.I. (1990) , 23-69.

[17] Vassiliev V. : Combinatorial formulas of cohomology of knot spaces , Moscow Math. Journal 1 (2001) , 91-123.

[18] Willerton S. : The Kontsevich integral and algebraic structures on the space of diagrams , ”Knots in Hellas 98” , Series on Knots and Everything 24 , World Scientific (2000), 530-546.

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