New method for determining singularities on enveloped surface and its application to study curvature interference theory of involute worm drive

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Abstract
In this paper, a more computationally convenient singularity condition of the enveloped surface is proposed using the theory of linear algebra. Its preconditions are only the tangential vector of the enveloping surface, the relative velocity vector, and the total differential of the meshing function. It avoids calculating the curvature parameters of the enveloping surface. It is proved that the singularity conditions of enveloped surface from different references are equivalent to each other and the relational equations among them are obtained. The curvature interference theory for the involute worm drive is established using the proposed singularity condition. The equation for the singularity trajectory is obtained. The calculation method for the singularity trajectory is proposed and its numerical result is obtained. The influence of the design parameters on the singularity trajectory is studied using the proposed curvature interference theory. The study results show that the risk of curvature interference is high when the transmission ratio is too small, especially in the case of the single-threaded worm and large modulus. The proposed singularity condition can also be applied to study the curvature interference mechanism in other types of the worm drive and to study the undercutting mechanism when machining the worm drive.

Keywords
Enveloped surface, singularity condition, curvature interference, singularity trajectory, worm drive

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Introduction
The machining and meshing processes of the gears are essentially the enveloping process. The tooth surface of the driving gear is usually considered to be the enveloping surface, and the tooth surface of the driven gear is the enveloped surface. The enveloping theory has been studied in depth in differential geometry,¹ and gear theory.²–³ These studies have shown that the occurrence of singularities on the enveloped tooth surface can lead to the occurrence of the curvature interference during the

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machining and meshing of the gear pair. The curvature interference can lead to undercutting. Generally speaking, the enveloping surface is the regular surface but the regular points on the enveloping surface can still lead to singularities on the enveloped surface. These singularities can form a smooth curve on the enveloped surface, which is generally the envelope of a family of contact lines on the enveloped surface and is also the edge of regression on the enveloped surface, as demonstrated in Litvin and Fuentes\(^2\) and Wu and Luo\(^3\). The singularity trajectory will divide the enveloped surface into two branches, which are connected along this singularity trajectory. To avoid the occurrence of singularities on the enveloped surface, it is necessary to use the singularity trajectory to limit the size of the gear blank. It is also due to this feature that the singularity trajectory on the enveloped surface is also known as the curvature interference limit line or the first kind of limit line, \(^3\)\(^4\).

According to differential geometry, \(^1\) and meshing theory of gear, \(^2\)\(^3\)\(^4\)\(^5\) at the singularity of the enveloped surface: (i) the normal vector of the enveloped surface is equal to zero; (ii) the tangential vector of the enveloped surface is equal to zero; (iii) the velocity vector of the enveloped surface is equal to zero. However, as these singularity conditions for the enveloped surface are in vector form, they are slightly inconvenient to apply. For this reason, it is common practice to convert it into the scalar form, which makes a further expansion of the singularity condition for the envelope surface.

By associating the vector equation that the velocity vector of the enveloped surface is equal to zero with the full differential of the meshing equation, Litvin and colleagues\(^2\)\(^6\) transformed the singularity condition in vector form into the problem of the existence of solutions to the overdetermined equation system, i.e. ensuring that the rank of the coefficient matrix of the overdetermined equation system is equal to 2. Then, the singularity condition in vector form can be replaced by a singularity condition in scalar form, that is, the sum of the squares of the three third-order sub-determinants in the coefficient matrix is zero. Using this singularity condition, the singularity problems for the enveloped surfaces of families of two-parameter surfaces and planar parametric curves were studied by Litvin and colleagues\(^7\)\(^8\). Using the dot product of the velocity vector on the enveloped surface with the normal vector on the action surface, a singularity condition in scalar form of the enveloped surface was obtained in Litvin et al.\(^9\), which is used to study the singularity trajectory on the face worm-gear of spiroid gear drive. A different scalar form of the singularity condition was derived\(^1\)\(^0\)\(^1\)\(^1\) using the dot product of the normal vector of the enveloped surface and the normal vector of the enveloping surface, which has been applied to the study of the curvature interference theory of machining the involute gears, the spiral bevel gears, and the hypoid gears. Analogously, using the vector equation that the normal vector of the enveloped surface is equal to zero, a similar singularity condition to that in Litvin et al.\(^6\) is obtained in Wu and Luo\(^3\) and Litvin et al.\(^1\)\(^2\) Dong\(^4\) derived the singularity condition for the enveloped surface in scalar form by associating the equation that the tangent vector of the enveloped surface is equal to zero with the full differentiation of the meshing equation and employing Euler and Bertrand formulas, \(^1\)\(^3\) and then he\(^1\)\(^5\) applied this singularity condition to the study of the curvature interference theory for different types of the toroidal worm pairs. Using Dong's singularity condition, Zhao and Meng\(^1\)\(^4\)\(^\text{16}\) investigated the curvature interference theory of conical surface enveloping the conical worm pair and the ZC1 worm pair. Sohn and Park\(^1\)\(^7\)\(^1\)\(^8\) studied the geometric interference problem of mismatched cylindrical worm pairs using the separation topology method.

However, in practice, it was found to be complicated to calculate the singularity trajectories on the enveloped surface using the three scalar forms of the singularity condition obtained. \(^6\)\(^\text{10}\)\(^\text{11}\). The numerical results and the equations of the singularity trajectories of the enveloped surfaces in the numerical examples are also not obtained in these references. Although the physical meaning of the singularity condition for the enveloped surface given by Dong\(^4\) is clear and easy to calculate, the normal curvature and the geodesic torsion of the enveloped surface need to be calculated when applying this method. Several simplified methods for calculating the normal curvature and the geodesic torsion of the enveloped surface are provided by Zhao and Zhang\(^1\)\(^9\)\(^2\)\(^0\), but there is still some difficulty in the actual calculation.

Thus, this paper proposes a simpler singularity condition of the enveloped surface, which is inspired by Litvin and Fuentes\(^2\) and Litvin et al.\(^6\). A rigorous derivation procedure is given using the theory in linear algebra.\(^2\)\(^1\) This new singularity condition of the enveloped surface avoids calculating the equation that the sum of the squares of the three third-order sub-determinants in the coefficient matrix is zero and also avoids calculating the normal curvature and the geodesic torsion of the enveloped surface.\(^4\)

On the other hand, although these singularity conditions in scalar form of enveloped surface are all derived from the singularity conditions in vector form, the final expressions for these singularity conditions in scalar form are quite different. For example, the singularity conditions presented in Litvin et al.\(^6\)\(^\text{9}\), although they both derive from the singularity condition that the normal vector of the enveloped surface is equal to zero, the final expressions of the singularity conditions in scalar form obtained by them are quite different. However, whether these singularity conditions with different
expressions are equivalent and whether there are relationships among these singularity conditions have not been studied in the existing literature in this regard. Thus, this paper studies the equivalence among these singularity conditions of enveloped surface and investigates their relationships.

To illustrate the application of the singularity condition for the enveloped surface proposed in this paper, the involute worm drive is taken as an example. For this purpose, a brief introduction to the involute worm drive is required. As reported in Crosher, the involute worm drive was patented in 1915 by F.J. Bostock, who was an assistant plant manager at David Brown. During the meshing of the involute worm drive, the involute worm tooth surface is the enveloping surface and the worm gear tooth surface is the enveloped surface. The study of the singularity trajectory of the involute worm gear tooth surface has been investigated in Litvin et al. Although the singularity trajectory on the involute worm gear tooth face was plotted in Litvin et al., no specific equations for the singularity trajectory on the involute worm tooth face have been obtained, nor have specific numerical results been obtained.

Thereupon, this paper uses the newly proposed singularity condition to establish the curvature interference theory of involute worm drive to solve the above problems. The calculation method of the singularity trajectory on the worm gear tooth surface is proposed. The nonlinear systems of equations encountered in the course of the study will be determined using the elimination method and the geometric construction to determine the existence of solutions as well as the iterative initial values. The influences of the design parameters, such as modulus, transmission ratio, and the number of worm threads, on the singularity path of the worm gear tooth surface are studied employing numerical examples.

**Theoretical study for determination method of singularities on enveloped surface**

**Notation**

As shown in Figure 1, two unit orthogonal rotating frames $\sigma_1\{O_1; \hat{e}_1, \hat{e}_2, \hat{e}_3\}$ and $\sigma_2\{O_2; \hat{e}_1, \hat{e}_2, \hat{e}_3\}$ are fixed on the driving and driven gears to indicate their current position. The unit basis vectors $\hat{e}_1$ and $\hat{e}_2$ lie along the axial lines of the driving and driven gears, respectively. The symbols $S_1$ and $S_2$ indicate the tooth surface of the driving gear and the tooth surface of the driven gear in the gear pair, respectively. Two tooth surfaces are tangent along a contact line. The contact point $P$ is a point on the contact line. The vectors $\vec{r}_1$ and $\vec{r}_2$ express the radial vectors of the two tooth surfaces, respectively.

The vector $\vec{n}_1$ refers to the unit normal vector of the tooth surface $S_1$, $\vec{v}_{12}$ stands for the relative velocity vector of the gear pair.

Generally speaking, the tooth surface $S_1$ is the regular one. In the frame $\sigma_1$, the equation of the tooth surface $S_1$ can be represented as

$$\vec{r}_1 = \vec{r}_1(u, \theta) = x_1(u, \theta) \hat{e}_1 + y_1(u, \theta) \hat{e}_2 + z_1(u, \theta) \hat{e}_3,$$

(1)

where $u$ and $\theta$ denote two curvilinear coordinates of the tooth surface $S_1$.

Differentiating equation (1) with respect to $u$ and $\theta$ yields

$$\frac{\partial \vec{r}_1}{\partial u} = \frac{\partial x_1}{\partial u} \hat{e}_1 + \frac{\partial y_1}{\partial u} \hat{e}_2 + \frac{\partial z_1}{\partial u} \hat{e}_3,$$

$$\frac{\partial \vec{r}_1}{\partial \theta} = \frac{\partial x_1}{\partial \theta} \hat{e}_1 + \frac{\partial y_1}{\partial \theta} \hat{e}_2 + \frac{\partial z_1}{\partial \theta} \hat{e}_3.$$

(2)

From equation (2), the unit normal vector $\vec{n}_1$ of the tooth surface $S_1$ can be obtained as

$$\vec{n}_1 = \frac{\vec{r}_1 \times \frac{\partial \vec{r}_1}{\partial \theta}}{D} = \frac{\vec{r}_1 \times \frac{\partial \vec{r}_1}{\partial \theta}}{\sqrt{\left(\frac{\partial x_1}{\partial \theta}\right)^2 + \left(\frac{\partial y_1}{\partial \theta}\right)^2 + \left(\frac{\partial z_1}{\partial \theta}\right)^2}},$$

(3)

where $n_{1z} = \frac{1}{D} \left(\frac{\partial x_1}{\partial \theta} \frac{\partial z_1}{\partial \theta} - \frac{\partial y_1}{\partial \theta} \frac{\partial z_1}{\partial \theta}\right), n_{1y} = -\frac{1}{D} \left(\frac{\partial x_1}{\partial \theta} \frac{\partial z_1}{\partial \theta} - \frac{\partial y_1}{\partial \theta} \frac{\partial z_1}{\partial \theta}\right), n_{1x} = \frac{1}{D} \left(\frac{\partial x_1}{\partial \theta} \frac{\partial z_1}{\partial \theta} - \frac{\partial y_1}{\partial \theta} \frac{\partial z_1}{\partial \theta}\right),$ and $D = \sqrt{\left(\frac{\partial x_1}{\partial \theta}\right)^2 + \left(\frac{\partial y_1}{\partial \theta}\right)^2 + \left(\frac{\partial z_1}{\partial \theta}\right)^2}$.

Since the tooth surface $S_1$ is the regular one, the unit normal vector $\vec{n}_1 \neq 0$. In equation (3), the coefficient $D = (EG - F^2)^{1/2}$ where the symbols $E$, $F$, and $G$ indicate the coefficients of the first fundamental form of the tooth surface $S_1$ and $E = (\partial_1 \vec{r}_1 / \partial u)^2$, $F = (\partial_1 \vec{r}_1 / \partial u) \cdot (\partial_1 \vec{r}_1 / \partial \theta)$, and $G = (\partial_1 \vec{r}_1 / \partial \theta)^2$.

The relative velocity vector of the gear pair at the contact point $P$ can be expressed as
\[ \vec{V}_{12} = V_x \vec{I} + V_y \vec{J} + V_z \vec{K}. \]  

(4)

When the driving gear rotates around its axis with the angular velocity \( |\vec{\omega}_1| = 1 \text{ rad/s} \), its tooth surface \( S_1 \) can form a family of parameter surfaces, \( \{ S_i \} \). Every point on the tooth surface \( S_2 \) of the driven gear belongs to the tooth surface \( S_1 \) of the driving gear, and the tooth surface \( S_2 \) is tangent to the tooth surface \( S_1 \) at every moment \( \varphi_1 \) along the contact line. Therefore, the tooth surface \( S_2 \) of the driven gear can be called the enveloped surface of the family of surfaces \( \{ S_i \} \). According to the meshing theory, the conjugate condition between the tooth surfaces \( S_1 \) and \( S_2 \) is \( \Phi(u, \theta, \varphi_1) = \vec{n}_1 \cdot \vec{V}_{12} = 0 \), which is called the meshing equation.\(^2\)\(^-\)\(^5\) Thereupon, the equation of the enveloped surface \( S_2 \) can be represented as

\[ \vec{r}_2 = \vec{r}_2(u, \theta, \varphi_1) = \vec{r}_1 + O_2 \partial_1, \Phi = \Phi(u, \theta, \varphi_1) = 0, \]  

(5)

where the symbols \( u, \theta \) and \( \varphi_1 \) are three non-independent variables of the enveloped surface \( S_2 \). Therefore, only two of these three variables are independent of each other so that the equation of the enveloped surface \( S_2 \) is double-parameter. After determining the value for the kinematic parameter \( \varphi_1 \) in equation (5), a contact line can be obtained on the tooth surfaces of the gear pair. Thus, the gear pair maintains a line-contact at every instant. The contact line is also the characteristic line of the family of surfaces.\(^3\)

**Singularity conditions for enveloped surface**

Since the enveloping surface \( S_1 \) is the regular one, the partial derivatives of the meshing function, \( \Phi_u \) and \( \Phi_\theta \), are not simultaneously zero at a given contact point.\(^2\)\(^-\)\(^5\) Thereupon, it may be assumed that \( \Phi_\theta \neq 0 \). Then, theoretically, \( \theta \) can be solved as a function \( \Theta(u, \varphi_1) \) about \( u \) and \( \varphi_1 \) from the second equation in equation (5). Substituting \( \Theta(u, \varphi_1) \) into the first equation of equation (5) for the equation of the enveloped surface \( S_2 \) can be rewritten as

\[ \vec{r}_2 = \vec{r}_2(u, \theta, \varphi_1) = \vec{r}_1(u, \Theta(u, \varphi_1), \varphi_1) + O_2 \partial_1(\varphi_1), \]  

(6)

which indicates that the parametric curves on the enveloped surface \( S_2 \) are the \( u \)-line and \( \varphi_1 \)-line respectively.

From equation (6), the two tangent vectors of the enveloped surface \( S_2 \) along each of these two parametric curves can be calculated as

\[ \frac{\partial \vec{r}_2}{\partial u} = \frac{\partial \vec{r}_2}{\partial u} = \frac{\partial}{\partial u}(\vec{r}_1 + O_2 \partial_1) = \frac{\partial \vec{r}_1}{\partial u} + \frac{\partial \vec{r}_1}{\partial \theta} \frac{\partial \theta}{\partial u}, \]  

(7)

\[ \frac{\partial \vec{r}_2}{\partial \varphi_1} = \frac{\partial \vec{r}_2}{\partial \varphi_1} = \frac{\partial \vec{r}_2}{\partial \varphi_1} - \vec{\omega}_2 \times \vec{r}_2 = \frac{\partial \vec{r}_1}{\partial \varphi_1} + \vec{V}_{12} = \frac{\partial \vec{r}_1}{\partial \theta} \frac{\partial \theta}{\partial \varphi_1} + \vec{V}_{12}. \]  

(8)

On the premise of \( \Phi_\theta \neq 0 \), the meshing equation in equation (5) can be rewritten as

\[ \Phi_u + \Phi_\theta \frac{\partial \theta}{\partial u} = 0, \Phi_{\varphi_1} + \Phi_\theta \frac{\partial \varphi_1}{\partial \theta} = 0. \]  

(9)

From equation (9), it can be obtained as

\[ \frac{\partial \theta}{\partial u} = -\frac{\Phi_u}{\Phi_\theta}, \frac{\partial \varphi_1}{\partial \theta} = -\frac{\Phi_{\varphi_1}}{\Phi_\theta}. \]  

(10)

After substituting equation (10) into the two tangent vectors of the enveloped surface \( S_2 \) in equation (7), a normal vector of the enveloped surface \( S_2 \) can be calculated as

\[ \vec{N}_2 = \frac{\partial \vec{r}_2}{\partial \theta} \times \frac{\partial \vec{r}_2}{\partial \varphi_1} = \left( \frac{\partial \vec{r}_1}{\partial u} + \frac{\partial \vec{r}_1}{\partial \theta} \frac{\partial \theta}{\partial \varphi_1} \right) \times \left( \frac{\partial \vec{r}_1}{\partial u} + \frac{\partial \vec{r}_1}{\partial \theta} \frac{\partial \theta}{\partial \varphi_1} \right) \]  

(11)

\[ = \frac{\Phi_u}{\Phi_\theta} \frac{\partial \vec{r}_1}{\partial u} \times \vec{V}_{12} + \frac{\partial \vec{r}_1}{\partial \theta} \frac{\partial \varphi_1}{\partial \theta} \frac{\partial \theta}{\partial \varphi_1} \]  

\[ + \Phi_{\varphi_1} \left( \frac{\partial \vec{r}_1}{\partial \theta} \frac{\partial \varphi_1}{\partial \varphi_1} \right) \]  

\[ = \frac{1}{\Phi_\theta} \left[ \Phi_u \left( \vec{V}_{12} \times \frac{\partial \vec{r}_1}{\partial \theta} \right) \right] + \Phi_{\varphi_1} \left( \frac{\partial \vec{r}_1}{\partial \theta} \right) \frac{\partial \varphi_1}{\partial \varphi_1}. \]

When the normal vector \( \vec{N}_2 \) is not zero everywhere on the enveloped surface \( S_2 \), the enveloped surface \( S_2 \) is regular. From equation (11), it can be seen that the relative velocity vector \( \vec{V}_{12} \), the partial derivatives \( \partial \vec{r}_1/\partial \varphi_1 \) and \( \partial \vec{r}_1/\partial u \) are all common tangent vectors to the enveloping surface \( S_1 \) and the enveloped surface \( S_2 \). Thereupon, all the cross products in equation (11) are the normal vectors to the common tangent plane between \( S_1 \) and \( S_2 \). Thus, their sum, \( \vec{N}_2 \), is also the normal vector to this common tangent plane.

When the normal vector \( \vec{N}_2 \) at a point on the enveloped surface \( S_2 \) is equal to 0, this point is a singularity. Therefore, the singularity condition for the enveloped surface \( S_2 \) can be written as

\[ \Phi(u, \theta, \varphi_1) = 0, \vec{N}_2 = 0. \]  

(12)

The singularity condition for the enveloped surface \( S_2 \) in equation (12) is derived on the premise of \( \Phi_\theta \neq 0 \). Due to the symmetry of the parameters \( u \) and \( \theta \) for the enveloped surface \( S_2 \), the singularity condition in equation (12) can still be derived in the context of \( \Phi_u \neq 0 \).
Next, the singularity conditions in scalar form of the enveloped surface $S_2$ derived from these singularity conditions in vector form will be discussed. Since the singularity condition for the enveloped surface $S_2$ in equation (12) contains the vector equation $\hat{N}_2 = 0$, it is slightly inconvenient to apply, and for this reason, it has been converted to a scalar equation in Litvin and Fuentes, Wu and Luo and Litvin et al. Considering the enveloping surface $S_1$ has no singularities, i.e. the cross product $\partial \hat{r}_1 / \partial u \times \partial \hat{r}_1 / \partial \theta \neq 0$, so the normal vector $\hat{N}_2 = 0$ is equivalent to the mixed product $\langle \hat{N}_2, \partial \hat{r}_1 / \partial u, \partial \hat{r}_1 / \partial \theta \rangle = 0$. Thus, using Lagrange’s identical equation^2 obtains

$$\begin{align*}
\left( \hat{N}_2, \frac{\partial \hat{r}_1}{\partial u}, \frac{\partial \hat{r}_1}{\partial \theta} \right) &= \frac{1}{\Phi_0} \left[ \Phi_0 \left( \hat{V}_12 \times \frac{\partial \hat{r}_1}{\partial \theta} \right) + \Phi_0 \left( \frac{\partial \hat{r}_1}{\partial u} \times \hat{V}_12 \right) + \Phi_0 \left( \frac{\partial \hat{r}_1}{\partial u} \times \frac{\partial \hat{r}_1}{\partial \theta} \right) \right], \\
\Phi(u, \theta, \varphi_1) &= 0, \quad \Psi = \begin{vmatrix} 
\frac{\partial \hat{r}_1}{\partial u} & \frac{\partial \hat{r}_1}{\partial \theta} & \hat{V}_12 \\
E & F & \Phi_u \\
G & \Phi_0 & \Phi_0 
\end{vmatrix},
\end{align*}$$
(13)

where the normal vector $\hat{N} = \frac{1}{\Phi} \left[ \frac{\partial \hat{r}_1}{\partial \theta} \hat{V}_12 + \frac{\partial \hat{r}_1}{\partial u} \hat{V}_12 + \Phi_{\varphi_1} \right]$, and the symbol $\Phi_{\varphi_1}$ is called as the meshing limit function. If the determinant on the right-hand side of equation (13) is equal to $\Psi$, i.e. $\Psi = \begin{vmatrix} 
\frac{\partial \hat{r}_1}{\partial u} & \frac{\partial \hat{r}_1}{\partial \theta} & \hat{V}_12 \\
E & F & \Phi_u \\
G & \Phi_0 & \Phi_0 
\end{vmatrix}$, the singularity condition for the vector form in equation (12) can be rewritten as a singularity condition in scalar form as

$$\Phi(u, \theta, \varphi_1) = 0, \quad \Psi = 0.$$  
(14)

Furthermore, since the tangent vector $d_2 \hat{r}_2$ at the singularity of the enveloped surface $S_2$ is equal to zero, a necessary singularity condition for the enveloped surface $S_2$ can also be written as

$$d_2 \hat{r}_2 = \frac{\partial \hat{r}_2}{\partial u} du + \frac{\partial \hat{r}_2}{\partial \varphi_1} d\varphi_1 = 0.$$  
(15)

Litvin and colleagues^2,6 proposed that at the singularity of the enveloped surface $S_2$, the velocity vector $\dot{V}_2$ of the enveloped surface $S_2$ is equal to zero. Since the velocity vector $\dot{V}_2 = d_2 \hat{r}_2 / dt$, the velocity vector $\dot{V}_2 = 0$ when the tangent vector $d_2 \hat{r}_2 = 0$, so they are equivalent. Similarly, since the singularity condition in equation (15) is represented in vector form, it is inconvenient to apply. Thereupon, in Dong^3, firstly, the tangent vector $d_2 \hat{r}_2$ can be expressed as $d_2 \hat{r}_2 = d_1 \hat{r}_1 + \hat{V}_12 d\varphi_1 = 0$ using the relative differentiation method. Then, by introducing the normal vector $\hat{N}$ of the contact line, after a series of derivations, a singularity condition in scalar form of the enveloped surface $S_2$ can be obtained as

$$\Phi(u, \theta, \varphi_1) = 0, \quad \Psi = \begin{vmatrix} 
\frac{\partial \hat{r}_1}{\partial u} & \frac{\partial \hat{r}_1}{\partial \theta} & \hat{V}_12 \\
E & F & \Phi_u \\
G & \Phi_0 & \Phi_0 
\end{vmatrix} = 0,$$  
(16)

and can be obtained by taking the partial derivative of the meshing function $\Phi$ with respect to $\varphi_1$. 

**Establishment of new singularity condition for enveloped surface**

After the discussion in the previous section, in practice, the singularity conditions, equations (14) and (16), inevitably require the calculation of the coefficients, $E$, $F$ and $G$, of the first fundamental form for the enveloping surface $S_1$. In particular, to simplify the calculation of the normal vector $\hat{N}$ of the contact line, it is usually necessary to calculate the curvature parameters of first in the main frame $\{P; \hat{g}_1, \hat{g}_2, \hat{n}_1\}$ or the unit orthogonal moving frame $\{P; \hat{a}_i, \hat{a}_g, \hat{n}_i\}$ according to Dong and Zhao and Zhang. This is not convenient in practical engineering applications. Thereupon, to further simplify the singularity condition, make it easier to apply in engineering practice and eliminate the need to calculate the curvature parameters of the enveloping surface $S_1$, this section proposes a new singularity condition for the enveloped surface $S_2$ from the component equations of the tangent vector $d_2 \hat{r}_2$ of the enveloped surface $S_2$ with the help of the relative differential method and the concepts from the linear algebra. The process is as follows.

First, the equation of the enveloped surface $S_2$ in equation (6) is differentiated to obtain its tangent vector $d_2 \hat{r}_2$. Since $d_2 \hat{r}_2 = 0$ at the singularity of the enveloped surface $S_2$, with the help of the relative differentiation method, the singularity condition of the envelope can
be written as \( d_2 \vec{r}_2 = d_1 \vec{r}_1 + \vec{V}_{12}d\phi_1 = 0 \). Next, rewriting this expression of the tangent vector \( d_2 \vec{r}_2 \) into the component form, and associating it with the full differentiation of the meshing equation \( \Phi(u, \theta, \phi_1) = 0 \) obtains

\[
\begin{align*}
\frac{\partial x_1}{\partial u} du + \frac{\partial x_1}{\partial \theta} d\theta + V_x d\phi_1 &= 0 \\
\frac{\partial y_1}{\partial u} du + \frac{\partial y_1}{\partial \theta} d\theta + V_y d\phi_1 &= 0 \\
\frac{\partial z_1}{\partial u} du + \frac{\partial z_1}{\partial \theta} d\theta + V_z d\phi_1 &= 0 \\
\Phi_u du + \Phi_\theta d\theta + \Phi_{\phi_1} d\phi_1 &= 0
\end{align*}
\]

which can be regarded as a homogeneous system of linear equations with three unknowns \( du, d\theta \) and \( d\phi_1 \). At the singularity of the enveloping surface \( S_2 \), this system of equations should have solutions.

From equation (17), its coefficient matrix is

\[
M = \begin{bmatrix}
\frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial \theta} & V_x \\
\frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial \theta} & V_y \\
\frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial \theta} & V_z \\
\Phi_u & \Phi_\theta & \Phi_{\phi_1}
\end{bmatrix}
\]

(18)

It can be seen from equation (17) that the rank of its augmented matrix is the rank of its coefficient matrix \( M \) in equation (18). Thus, it is only necessary to determine the rank of the coefficient matrix \( M \) to determine the number of solutions to equation (17). According to the theory of linear algebra, to ensure the existence of non-zero solutions in equation (17) to determine whether there is an isolated singularity or a singularity trajectory, it is necessary to determine the rank of the coefficient matrix in equation (17). To this end, firstly, since the coefficient matrix \( M \) is four rows and three columns, the rank of the coefficient matrix \( M \) in equation (18) should be less than or equal to 3, i.e. \( R(M) \leq 3 \), according to the definition of linear algebra.

Next, from the coefficient matrix \( M \) in equation (18), its four third-order sub-determinants are

\[
M_1 = \begin{vmatrix} V_x \\ V_y \\ V_z \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} = \begin{vmatrix} \frac{\partial y_1}{\partial u} \\ \frac{\partial y_1}{\partial \theta} \\ \frac{\partial z_1}{\partial u} \\ \frac{\partial z_1}{\partial \theta} \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} \quad (19)
\]

\[
M_2 = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial \theta} & V_x \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial \theta} & V_y \\ \frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial \theta} & V_z \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial u} \\ \frac{\partial x_1}{\partial \theta} \\ \frac{\partial z_1}{\partial u} \\ \frac{\partial z_1}{\partial \theta} \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} \quad (20)
\]

\[
M_3 = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial \theta} & V_x \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial \theta} & V_y \\ \frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial \theta} & V_z \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial u} \\ \frac{\partial x_1}{\partial \theta} \\ \frac{\partial z_1}{\partial u} \\ \frac{\partial z_1}{\partial \theta} \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} \quad (21)
\]

\[
M_4 = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial \theta} & V_x \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial \theta} & V_y \\ \frac{\partial z_1}{\partial u} & \frac{\partial z_1}{\partial \theta} & V_z \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial \theta} & V_y \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} = \begin{vmatrix} \frac{\partial x_1}{\partial u} \\ \frac{\partial x_1}{\partial \theta} \\ \frac{\partial z_1}{\partial u} \\ \frac{\partial z_1}{\partial \theta} \\ \Phi_u & \Phi_\theta & \Phi_{\phi_1} \end{vmatrix} \quad (22)
\]

From equations (19)–(22), it can be seen that the three sub-determinants \( M_1, M_2 \) and \( M_3 \) contain the components \( n_{1x}, n_{1y} \) and \( n_{1z} \) of the unit normal vector \( \vec{n}_1 \) of the enveloping surface \( S_1 \), respectively. Furthermore, after derivation, it can be seen that the third-order sub-determinant in equation (22) can be reduced to the meshing equation \( \Phi = 0 \) in equation (5). This shows that the third-order sub-determinant \( M_4 \) is constant to zero.

By multiplying \( n_{1y} \) with \( M_4 \) and with \( M_2 \) can get

\[
n_{1y} M_3 + n_{1z} M_2 = \Phi_u \left( n_{1y} \frac{\partial x_1}{\partial u} V_x + n_{1z} \frac{\partial x_1}{\partial z_1} V_z \right) - \Phi_\theta \left( n_{1y} \frac{\partial x_1}{\partial \theta} V_y + n_{1z} \frac{\partial x_1}{\partial z_1} V_z \right) - \Phi_{\phi_1} \left( n_{1y} \frac{\partial y_1}{\partial \theta} V_y + n_{1z} \frac{\partial y_1}{\partial \theta} V_z \right) = 0.
\]

(23)
After the derivation of equation (23), it can be seen that there is a linear relationship between the third-order sub-determinants \(M_2\) and \(M_3\). Similarly, by multiplying \(n_{1x}, n_{1y}\) and \(n_{1z}\) with the third-order sub-determinants \(M_1, M_2\) and \(M_3\), respectively, it can be obtained as

\[
n_{1x}M_3 - n_{1z}M_1 = \Phi \frac{\partial \gamma_1}{\partial u} \frac{\partial \gamma_1}{\partial \theta} = 0, n_{1y}M_1 + n_{1z}M_2 = 0.
\]  

(24)

which illustrates that there are linear relationships between these third-order sub-determinants \(M_1, M_2\) and \(M_3\) two by two. Thus, using equations (23) and (24), the linear relationship among \(M_1, M_2\) and \(M_3\) can be obtained as

\[
(n_{1y} - n_{1z})M_1 + (n_{1x} + n_{1z})M_2 + (n_{1x} + n_{1y})M_3 = 0,
\]  

(25)

which shows that if one of the three third-order determinants \(M_1, M_2\) and \(M_3\) is equal to zero, the other two determinants are automatically zero.

The form of existence of singularities on the enveloped surface \(S_2\) will be discussed as follows. Firstly, the singularities on the enveloped surface \(S_2\) may be an isolated point or they may form a continuous smooth curve. For example, the cone surface, which is enveloped surface of a one-parameter family of planes, has a cone vertex that is an isolated singularity.\(^1\) When the singularity of the enveloped surface \(S_2\) is an isolated singularity, equation (17) should have a unique solution, which indicates that the rank of the coefficient matrix \(M\) in equation (18) should be equal to 3. Therefore, in this case, none of these three sub-determinants \(M_1, M_2\) and \(M_3\) will be equal to zero.

Secondly, when the singularities on the enveloped surface \(S_2\) form a smooth continuous curve, equation (17) should have infinitely many solutions. Therefore, according to linear algebra,\(^2\) the rank of the coefficient matrix of equation (18) should be less than or equal to 2, i.e. \(R(M) \leq 2\). From equations (19)–(21), since there are some second-order sub-determinants in the three determinants \(M_1, M_2\) and \(M_3\), and as such

\[
\frac{\partial \gamma_1}{\partial u} \frac{\partial \gamma_1}{\partial \theta} = \frac{\partial \gamma_1}{\partial u} \frac{\partial \gamma_1}{\partial \theta} = Dn_{1x},
\]

\[
\frac{\partial \gamma_1}{\partial u} \frac{\partial \gamma_1}{\partial \theta} = - Dn_{1y},
\]

\[
\frac{\partial \gamma_1}{\partial u} \frac{\partial \gamma_1}{\partial \theta} = Dn_{1z}.
\]  

(26)

Since there are no singularities on the enveloping surface \(S_1\), its normal vector \(\vec{n}_1\) will not be zero which illustrates its three components \(n_{1x}, n_{1y}\), and \(n_{1z}\) will not be zero simultaneously. This means that the matrix \(M\) must contain some non-zero second-order sub-determinants. Therefore, according to linear algebra,\(^2\) the rank of the matrix \(M\) should be equal to two, i.e. \(R(M) = 2\). When \(R(M) = 2\), equation (17) has infinitely many solutions satisfying the tangent vector \(dS_2 = 0\). As a result, a new singularity condition of the enveloped surface \(S_2\) can be obtained as

\[
\Phi(u, \theta, \varphi_1) = 0, R(M) = 2.
\]  

(27)

Since the direct calculation of the rank of the coefficient matrix \(M\) is slightly inconvenient in the application, it can be further simplified according to the theory of linear algebra,\(^2\) The steps are as follows. Firstly, due to \(R(M) = 2\), according to linear algebra,\(^2\) if there is a non-zero second-order sub-determinant in the matrix \(M\), all of the third-order sub-determinants of \(M\) containing the non-zero second-order sub-determinant should be equal to zero. Secondly, because some non-zero second-order sub-determinants have been found in equation (27), it is only necessary to ensure that its corresponding third-order sub-determinant is equal to zero to guarantee \(R(M) = 2\).

Based on the above theory, since there are no singularities on the enveloping surface \(S_1\), the normal vector \(\vec{n}_1\) has non-zero components. Thereupon, by making sure that one of the third-order sub-determinants containing the non-zero normal vector components is equal to zero, it can ensure that the other two third-order determinants are automatically equal to zero and ensure \(R(M) = 2\) and \(dS_2 = 0\). Thus, if the normal vector \(\vec{n}_1\) is not equal to zero, the singularity condition of the enveloped surface \(S_2\) in equation (27) can be rewritten as a scalar form, i.e.

\[
M_i(u, \theta, \varphi_1) = 0, M_i(u, \theta, \varphi_1) = \Phi(u, \theta, \varphi_1) = 0, (i = 1, 2, or 3).
\]  

(28)

Similarly to this section, equation (17) has been derived in Litvin and Fuentes\(^2\) and Litvin et al.\(^6\) based on the singularity condition \(\vec{V}_{12} = 0\). The references also suggested that the rank of the coefficient matrix \(M\) needs to be equal to 2 to ensure equation (17) has solutions. However, when calculating \(R(M) = 2\), the references suggested that it is sufficient to guarantee that the sum of squares of the three third-order sub-determinants \(M_1, M_2\) and \(M_3\) is equal to zero, i.e. \(\sum_{i=1}^{3} M_i^2 = 0\). Although, as mentioned in the later Argyris et al.\(^8\) and Litvin et al.\(^9\), it is also possible to determine the singularity of the enveloped surface \(S_2\) using one of the three third-order sub-determinants \(M_1, M_2\) and \(M_3\), no specific reasons or proof process is given in these references.

By comparison, this section differs from the Litvin and colleagues\(^2,6-9\) in that: firstly, it is on the premise of the singularity condition \(dS_2 = 0\). Secondly, to
determine \( R(M) = 2 \), this section first determines that there is a linear relationship among the sub-determinants \( M_1, M_2 \) and \( M_3 \). With the help of the theory of linear algebra, a simpler singularity condition in equation (28) is obtained, which avoids calculating \( \sum_{i=1}^{3} M_i^2 = 0,2,6,7 \). Thirdly, in contrast to Litvin and colleagues' studies, this section gives a rigorous derivation and proof procedure for the singularity conditional equation (28) based on the theory of linear algebra. For these reasons, the singularity condition in equation (28) can be considered as an alternative to and development of the singularity condition obtained in Litvin and colleagues' works.

### Consistency of singularity conditions

The singularity conditions in scalar form, equations (14) and (16), of the enveloped surface \( S_2 \) have been derived using the same techniques as in Section 2.2. However, it is difficult to judge directly whether these singularity conditions are equivalent since there are significant formal differences among the functions \( \Psi \) and \( \Psi \) in equations (14) and (16), and the determinants \( M_i, (i = 1,2,3) \), in equation (28). Moreover, there is no discussion in the literature as to whether these singularity conditions are equivalent to each other. Thus, this section will study this question.

Using the properties of determinant, the function \( \Psi \) in equation (13) can be further simplified as

\[
\Psi = \begin{vmatrix}
\frac{\partial \tilde{F}_1}{\partial u} & \frac{\partial \tilde{F}_2}{\partial u} & \frac{\partial \tilde{F}_3}{\partial u} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\end{vmatrix}
\]

\[
= \frac{\partial \tilde{F}_1}{\partial u} \left( \begin{array}{c}
\tilde{F}_2 \\
\Phi_{u1} \\
\Phi_{u2} \\
\Phi_{u3}
\end{array} \right) - \frac{\partial \tilde{F}_1}{\partial u} \left( \begin{array}{c}
\tilde{F}_2 \\
\Phi_{u1} \\
\Phi_{u2} \\
\Phi_{u3}
\end{array} \right) + D^2 \Phi_{u1}
\]

\[
= \frac{\partial \tilde{F}_1}{\partial u} \left( \begin{array}{c}
\tilde{F}_2 \\
\Phi_{u1} \\
\Phi_{u2} \\
\Phi_{u3}
\end{array} \right) - \frac{\partial \tilde{F}_1}{\partial u} \left( \begin{array}{c}
\tilde{F}_2 \\
\Phi_{u1} \\
\Phi_{u2} \\
\Phi_{u3}
\end{array} \right) + D^2 \Phi_{u1}
\]

\[
= \begin{vmatrix}
E & F & \Phi_{u1} \\
F & G & \Phi_{u2} \\
G & H & \Phi_{u3}
\end{vmatrix} + D^2 \Phi_{u1}
\]

\[
= D^2 (\tilde{N} \cdot \tilde{V}_{12} + \Phi_{u1}) = D^2 \Psi.
\]

(29)

After the derivation in equation (29), it can be seen that the relationship between the functions \( \Psi \) and \( \Psi \) is \( \Psi = D^2 \Psi \). Since \( D^2 > 0 \) according to differential geometry \( ^{1} \) when \( \Psi = 0 \) in equation (14), \( \Psi = 0 \) in equation (16). Thus, they are equivalent.

Then, to investigate the relationship between the singularity condition in equation (28) and other forms of singularity conditions, firstly, the expression for the normal vector \( \tilde{N}_2 \) of the enveloped surface \( S_2 \) in equation (12) is simplified. Accordingly, substituting the partial derivative vectors \( \partial \tilde{F}_1/\partial u \) and \( \partial \tilde{F}_1/\partial \theta \) in equation (2) and the component equation for the relative velocity \( \tilde{V}_{12} \) in equation (28) into equation (12), the normal vector \( \tilde{N}_2 \) can be rewritten as

\[
\tilde{N}_2 = - \frac{1}{D} \begin{vmatrix}
\frac{\partial \tilde{F}_1}{\partial u} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\frac{\partial \tilde{F}_1}{\partial \theta} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\end{vmatrix}
\]

\[
- \frac{1}{D} \begin{vmatrix}
\frac{\partial \tilde{F}_1}{\partial u} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\frac{\partial \tilde{F}_1}{\partial \theta} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\end{vmatrix}
\]

\[
- \frac{1}{D} \begin{vmatrix}
\frac{\partial \tilde{F}_1}{\partial u} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\frac{\partial \tilde{F}_1}{\partial \theta} & \Phi_{u1} & \Phi_{u2} & \Phi_{u3} \\
\end{vmatrix}
\]

\[
= - \frac{1}{D} \Phi_{u1} (M_1 \tilde{N}_1 - M_2 \tilde{N}_1 + M_3 \tilde{N}_1).
\]

(30)

After the derivation in equation (30), it can be found that the three components of the normal vector \( \tilde{N}_2 \) can be expressed by the three third-order sub-determinants \( M_i, (i = 1,2,3) \). According to equation (30), when the singularity condition in equation (28) holds, the normal vector \( \tilde{N}_2 \) is equal to zero in equation (12). This shows that the singularity condition in equation (28) is equivalent to the singularity condition \( \tilde{N}_2 = 0 \).

Thereupon, substituting the new expression for the normal vector \( \tilde{N}_2 \) in equation (30) into equation (13), the functions \( \Psi \) and \( \Psi \) in equations (14) and (16) can be rewritten as

\[
\Psi = \frac{\Phi_{u1}}{D^2} \tilde{N}_2 = \frac{\Phi_{u1}}{D^2} \left( \begin{array}{c}
\tilde{F}_1 \\
\tilde{F}_2 \\
\tilde{F}_3
\end{array} \right)
\]

\[
= \Phi_{u1} \tilde{N}_2 = \frac{1}{D} (M_1 n_1 x - M_2 n_1 y + M_3 n_1 z),
\]

(31)

which illustrates that when the singularity condition in equation (28) holds, \( \Psi = 0 \) and \( \Psi = 0 \). Therefore, the singularity condition in equation (28) is equivalent to the singularity conditions in equations (14) and (16). With the help of the linear relationship between \( M_1, M_2 \) and \( M_3 \) in equations (23)–(25), when the subscripts \( x, y \) and \( z \) in equation (3) are one-to-one corresponding to 1, 2, and 3 in equation (28), the relationship equation (31) can be simplified as

\[
\Psi = \frac{\Phi_{u1}}{D^2} \tilde{N}_2 = (-1)^{i+1} \frac{M_i}{D n_{1i}},
\]

(32)

In summary, this section further simplifies the function \( \Psi \) based on the introduction of the normal vector \( \tilde{N} \) of the contact line and obtains the relationship.
besides, using the relationship between the normal vector \( \mathbf{N}_2 \) and clearly, we summarise these singularity conditions in Table 1.

### Curvature interference theory of involute worm drive

The worm gear of an involute worm drive is usually formed by a cylindrical hob which has the same generating surface as the tooth surface of the matching involute worm. The working process of the involute worm gear reproduces exactly the process of its formation, and therefore this study does not distinguish between the cutting engagement of the involute worm gear and the working mesh of the corresponding involute worm drive.

### Equation of involute worm tooth surface and its characteristic parameters

As shown in Figure 2, a unit orthogonal rotation frame \( \{O_1; \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1\} \) is fixed on the base cylinder of the right-hand involute helical surface \( S_1 \) and used to reflect its current position. The radius of the base cylinder for the involute helical surface is denoted as \( r_b \). The unit basis vector \( \mathbf{k}_1 \) lies along the axial line of the base cylinder of the involute helical surface. The unit basis vectors \( \mathbf{i}_1 \) and \( \mathbf{j}_1 \) are in the cross-section of the base cylinder. In Figure 2, the symbols \( \mathbf{e}_1(\theta) \) and \( \mathbf{g}_1(\theta) \) indicate the circular vector functions, and the symbols \( \dot{\mathbf{m}}(\theta - \pi/2, \pi - \beta_b) \) and \( \ddot{\mathbf{m}}(\theta - \pi/2, \pi - \beta_b) \) represent the spherical vector functions.\(^4\)

![Figure 2. Schematic of involute helical surface \( S_1 \)](image)

The cross-sectional shape, \( \mathbf{AB} \) of the end face for the involute helical surface \( S_1 \) is an involute. The angle \( \angle AO_1C = \theta \) and the length of the straight-line section \( |CD| = p \theta \) where the symbol \( p \) indicates the helix parameter of the involute helical surface \( S_1 \). The curve \( AD \) is the intersection curve between the involute helical surface \( S_1 \) and its base cylinder. The straight line \( BD \) is a tangent line of the curve \( AD \) at the point \( D \) and also the tangent line to the base cylinder at the point \( D \). Thereupon, \( \angle BDC = \beta_b \) which is the helix angle of the base cylinder for the involute helical surface \( S_1 \). The tangent line \( BD \) can be seen as the generating line of the involute helical surface \( S_1 \) and its direction is \( \ddot{\mathbf{m}}(\theta - \pi/2, \pi - \beta_b) \). The point \( P \) on the tangent line \( BD \) can be regarded as an arbitrary point on the involute helical surface \( S_1 \), and \( |DP| = u \).

According to the above geometric relationship, the radius vector of an arbitrary point \( P \) on the involute helical surface \( S_1 \) can be represented as \( \mathbf{r}_1 = O_1 \mathbf{P} = O_1 \mathbf{C} + CD + DP \). Employing the properties of the circle vector function and the sphere vector function,\(^4\) these vectors can be expressed as \( O_1 \mathbf{C} = r_b \mathbf{e}_1(\theta) \), \( CD = p \theta \mathbf{k}_1 \) and \( DP = u \dot{\mathbf{m}}(\theta - \pi/2, \pi - \beta_b) \). Thereupon, the equation of the involute helical surface \( S_1 \), i.e., the equation of the involute worm tooth surface, with the curvilinear coordinates \( u \) and \( \theta \) can be obtained as:

| Singular conditions for enveloped surface | Relationship |
|------------------------------------------|-------------|
| Meshing equation \( \Phi = 0 \)          | \( \Psi = \frac{\sqrt{\sum_{i=1}^{3} M_i^2}}{Dn_i} \) |
| Vector form                              | (\( i = 1, 2 \) or 3) |
| \( d_2 f_2 = 0 \)                        | \( \Psi = 0 \) |
| \( \mathbf{N}_2 = 0 \)                    | \( \Psi = 0 \) |
| \( \dot{V}_2 = 0 \)                       | \( R(M) = 2 \) |
| \( \mathbf{V}_2 = 0 \)                    | \( \sum_{i=1}^{3} M_i^2 = 0 \), |
|                                          | \( M_i (i = 1, 2) \) |
\[ (\mathbf{r}_i(u, \theta))_1 = r_3 \mathbf{e}_i(\theta) + p\theta \mathbf{k}_1 + u \mathbf{m} (\theta - \frac{\pi}{2}, \pi - \beta_b) \]
\[ = x_1 \mathbf{i}_1 + y_1 \mathbf{j}_1 + z_1 \mathbf{k}_1, \quad (33) \]

where \( x_1 = r_3 \cos \theta + \sin \theta \sin \beta_b, \quad y_1 = r_3 \sin \theta - u \cos \theta \sin \beta_b \) and \( z_1 = p \theta - u \cos \beta_b \). By the nature of the involute, the relationship between the radius, \( r_b \), of the base cylinder and the helix parameter, \( p \), of the involute helical surface \( S_1 \) is \( r_b = p \tan \beta_b \). The reason for using the vector function representation here is that the ensuing calculation and results can be simplified using its operational rules.

Because the involute helical surface \( S_1 \) is a ruled surface, the tangent line, \( BD \), of the base cylinder is the straight generatrix of the involute helical surface \( S_1 \). If letting \( \theta = \theta_0 = \text{const} \) in equation (33), it can be obtained the \( u \)-line of involute helical surface \( S_1 \), which is the straight generatrix \( BD \). If letting \( u = u_0 = \text{const} \) in equation (33), it can be obtained the \( \theta \)-line of involute helical surface \( S_1 \), which is a helical curve.

According to the operational rules of the vector functions, two partial derivative vectors of \( \mathbf{r}_i(u, \theta) \) concerning \( u \) and \( \theta \) can be obtained from equation (33) as
\[ \left( \frac{\partial \mathbf{r}_i}{\partial u} \right)_1 = \mathbf{m} (\theta - \frac{\pi}{2}, \pi - \beta_b), \quad \left( \frac{\partial \mathbf{r}_i}{\partial \theta} \right)_1 = \sin \beta_b \mathbf{e}_i(\theta) + r_3 \mathbf{g}_i(\theta) + p \mathbf{k}_1, \quad (34) \]

Then, from equation (34), the coefficients of the first kind of fundamental form which is also called the first kind of fundamental quantities\(^1\) of the involute helical surface \( S_1 \) can be worked out as
\[ E_1 = \left( \frac{\partial \mathbf{r}_i}{\partial u} \right)_1^2 = 1, \quad F_1 = \left( \frac{\partial \mathbf{r}_i}{\partial \theta} \right)_1, \quad (35) \]
\[ G_1 = \left( \frac{\partial \mathbf{r}_i}{\partial \theta} \right)_1^2 = F_1^2 + u^2 \sin^2 \beta_b, \]

where \( E_1 = \text{const} > 0 \), and because the \( \beta_b \) is an acute one, \( F_1 = \text{const} < 0 \). Due to \( F_1 \neq 0 \), the two parametric curves of the surface \( S_1 \), i.e., \( u \)-line and \( \theta \)-line, are not orthogonal which means that the parametric curve net of the involute helical surface \( S_1 \) is not the curvature line net.

Again, from equation (35), the unit normal vector, \( \mathbf{n}_i \), of the involute helical surface \( S_1 \) can be worked out as
\[ \mathbf{n}_i = \left( \frac{\partial \mathbf{r}_i}{\partial u} \right)_1 \times \left( \frac{\partial \mathbf{r}_i}{\partial \theta} \right)_1 = \left( \frac{\partial \mathbf{r}_i}{\partial u} \right)_1 \times \left( \frac{\partial \mathbf{r}_i}{\partial \theta} \right)_1 \]
\[ = - \mathbf{m} (\theta - \frac{\pi}{2}, \pi - \beta_b) = n_{1x} \mathbf{i}_1 + n_{1y} \mathbf{j}_1 + n_{1z} \mathbf{k}_1, \quad (36) \]

where the coefficients \( n_{1x} = \cos \beta_b \sin \theta, \quad n_{1y} = -\cos \beta_b \cos \theta, \quad n_{1z} = \sin \beta_b \) and \( D = \sqrt{EG - F^2} = \sin \beta_b \).

Equation (36) reflects that the unit normal vector \( \mathbf{n}_i \) is invariant along the straight generatrix \( BD \) of the involute helical surface \( S_1 \).

Herein, this paper specifies that the direction along with the sphere vector function \( \mathbf{m} (\theta - \frac{\pi}{2}, \pi - \beta_b) \) points to the direction of the worm entity, and its opposite direction points to the direction of the tooth groove. Thereupon, from equation (36), since the direction of the normal vector \( \mathbf{n}_i \) is the opposite direction of the vector function \( \mathbf{m} (\theta - \frac{\pi}{2}, \pi - \beta_b) \), the direction of the normal vector \( \mathbf{n}_i \) points from the involute worm entity to the tooth groove.

**Meshing equation and equation of worm gear tooth surface**

As shown in Figure 3, two unit right-handed orthogonal stationary frames, \( \sigma_{a1} \{ O_1; \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1 \} \) and \( \sigma_{a2} \{ O_2; \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2 \} \), are fixed on the initial places of the involute worm and worm gear, respectively. The unit basis vectors \( \mathbf{k}_{a1} \) and \( \mathbf{k}_{a2} \) are along the axial lines of the worm and worm gear, respectively, and perpendicular to each other. The positive direction of the unit basis vectors \( \mathbf{i}_1 \) and \( \mathbf{j}_2 \) is from \( O_2 \) to \( O_1 \). The length of the straight-line vector \( |O_2 O_1| = a \) where the symbol \( a \) denotes the center distance of the involute worm drive.

Two unit orthogonal rotating frames, \( \sigma_1 \{ O_1; \mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1 \} \) and \( \sigma_2 \{ O_2; \mathbf{i}_2, \mathbf{j}_2, \mathbf{k}_2 \} \), are set to denote the current location of the involute worm and the worm gear. The unit basis vector \( \mathbf{k}_{a1} \) and \( \mathbf{k}_{a2} \) are collinear. When the involute worm rotates the angle \( \varphi_1 \), the rotational angle of the worm gear is \( \varphi_2 \) where \( \varphi_2 = \varphi_1/i_{12} \) and \( i_{12} \) indicates the transmission ratio of the involute worm drive.

In the initial location of the worm pair, the rotational angles \( \varphi_1 = \varphi_2 = 0 \).

When the involute worm rotates around its axis \( \mathbf{k}_{a1} \), its tooth surface \( S_1 \) can form a family, \( \{ S_1 \} \), of parametric surfaces in \( \sigma_{a1} \). From equation (33) and using the rotational transformation matrix, the equation of \( \{ S_1 \} \) can be represented as
\[ (\mathbf{r}_i(u, \theta, \varphi_1))_1 = R \left( \mathbf{k}_{a1}, \varphi_1 \right) (\mathbf{r}_i)_1 \]
\[ = \begin{bmatrix} \cos \varphi_1 & -\sin \varphi_1 & 0 \\ \sin \varphi_1 & \cos \varphi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1i} \mathbf{i}_1 + y_{1i} \mathbf{j}_1 + z_{1i} \mathbf{k}_1 \end{bmatrix} \quad (37) \]

where \( x_{1i} = r_3 \cos(\theta + \varphi_1) + \sin \beta_b \sin(\theta + \varphi_1) \) and \( y_{1i} = r_3 \sin(\theta + \varphi_1) - \sin \beta_b \cos(\theta + \varphi_1) \).

In general, we can assume that the involute worm rotates around the axis \( \mathbf{k}_{a1} \) with the angular velocity \( |\omega_{1i}| = 1 \) rad/s. Therefore, the angular velocity of the
worm gear is $|\mathbf{\dot{\omega}}_2| = 1/i_{12} \text{ rad/s}$. Thereupon, the angular velocity vectors of the involute worm and worm gear in $\sigma_1$ can be worked out as

$$
(\mathbf{\dot{\omega}})_1 = \mathbf{k}_1, (\mathbf{\dot{\omega}})_2 = \frac{1}{i_{12}}(\sin \varphi_1 \mathbf{i}_1 + \cos \varphi_1 \mathbf{j}_1). \quad (38)
$$

From equation (38), the relative angular velocity vector of the worm pair can be obtained in $\sigma_1$ as

$$
(\mathbf{\dot{\omega}}_{12})_1 = (\mathbf{\dot{\omega}}_1)_1 - (\mathbf{\dot{\omega}}_2)_1 = -\frac{1}{i_{12}} \sin \varphi_1 \mathbf{i}_1 - \frac{1}{i_{12}} \cos \varphi_1 \mathbf{j}_1 + \mathbf{k}_1. \quad (39)
$$

Since the angular velocity of the worm is $|\mathbf{\dot{\omega}}_1| = l \text{ rad/s}$, the rotational angle $\varphi_1$ and the time $t$ are equivalent, i.e. $\varphi_1 = t$ and $d\varphi_1 = dt$. Thus, according to the definition and from equations (33), (38) and (39), the relative velocity vector $\mathbf{v}_{12}$ at an arbitrary contact point $P$ can be obtained in $\sigma_1$ as

$$
(\mathbf{v}_{12})_1 = (\mathbf{\dot{\omega}}_{12})_1 \times (\mathbf{\ddot{r}})_1 - (\mathbf{\dot{\omega}}_2)_1 \times (\mathbf{\mathcal{O}}_2 \mathbf{O}_1)_1 = V_x \mathbf{i}_1 + V_y \mathbf{j}_1 + V_z \mathbf{k}_1,
$$

where $V_x = -\frac{z_1}{i_{12}} \cos \varphi_1 - y_1,$

$V_y = \frac{z_1}{i_{12}} \sin \varphi_1 + x_1,$

$V_z = \frac{1}{i_{12}} [r \cos(\theta + \varphi_1) + us \sin(\theta + \varphi_1) + a].$

By definition, the meshing function of the involute worm drive can be worked out by equations (36) and (40) as

$$
\Phi(u, \theta, \varphi_1) = (\mathbf{\ddot{r}})_1 \cdot (\mathbf{v}_{12})_1 = \frac{1}{i_{12}} [A \sin(\theta + \varphi_1) + r_3 \sin \beta b \cos(\theta + \varphi_1) + B], \quad (41)
$$

where the coefficients $A = u - p \beta b \cos \beta b,$ $B = \sin \beta b - i_{12} r_3 \cos \beta b$ and $B$ is a constant.

From equations (37) and (41), the equation of the worm gear tooth surface $S_2$ can be obtained in $\sigma_2$ as

$$
(\mathbf{r}_2)_2 = R \left[ \mathbf{k}_2, -\frac{\varphi_1}{i_{12}} \left\{ R \left[ \mathbf{k}_2, \frac{\pi}{2} \right](\mathbf{\ddot{r}})_n + ad \mathbf{e}_2 \right\} \right] = x_2 \mathbf{i}_2 + y_2 \mathbf{j}_2 + z_2 \mathbf{k}_2, \Phi(u, \theta, \varphi_1) = 0, \quad (42)
$$

where the coefficients $x_2 = (x_1 + a) \cos \frac{\varphi_1}{i_{12}} + z_1 \sin \frac{\varphi_1}{i_{12}}$ and $y_2 = - (x_1 + a) \sin \frac{\varphi_1}{i_{12}} + z_1 \cos \frac{\varphi_1}{i_{12}}$.

In equation (42), there is a relationship among the three variables $u, \theta$ and $\varphi_1$. Therefore, only two of these three variables are independent of each other so that the equation of the worm gear tooth surface $S_2$ is two parameters. After determining the value for a kinematic parameter, i.e. the angle $\varphi_1$ in equation (42), a contact line can be obtained on the tooth surfaces of the involute worm and worm gear. Thus, the involute worm drive maintains a line-contact at every instant.

**Equation of singularity trajectory on worm gear tooth surface**

To determine the singularity trajectory on the worm gear tooth surface, firstly, this section will establish the singularity condition for the worm gear tooth surfaces based on the singularity condition in equation (28) proposed in Section 2.3. Accordingly, differentiating equation (41) with respect to $u$, $\theta$ and $\varphi_1$ successively yields

$$
\Phi_u = \frac{1}{i_{12}} \sin(\theta + \varphi_1), \quad (43)
$$
\[ \Phi_\theta = -\frac{1}{i_{12}} \left[ \frac{r_b}{\sin \beta_y} \sin(\theta + \varphi_1) - A \cos(\theta + \varphi_1) \right], \quad (44) \]
\[ \Phi_{\varphi_1} = -\frac{1}{i_{12}} \left[ r_b \sin \beta_y \sin(\theta + \varphi_1) - A \cos(\theta + \varphi_1) \right]. \quad (45) \]

Secondly, from equation (36), all components of the normal vector \( \vec{n}_1 \) are non-zero. Thus, according to the singularity condition in equation (28), it is sufficient to choose any one of the three third-order sub-determinants of the coefficient matrix \( M \) in equations (19)–(21). To obtain a simpler third-order sub-determinant, it can be determined by choosing the simplest component of the normal vector \( \vec{n}_1 \). From equation (36), it can be seen that the component \( n_{1z} \) is the simplest of these three components. Thus, the third-order sub-determinant \( M_3 \) is chosen as the singularity condition for the worm gear tooth surface. Substituting equations (33), (34), (40), and (43)–(45) into equation (21), the singularity condition for the worm gear tooth surface can be obtained as
\[ M_3 = \frac{1}{i_{12}} \left[ p \theta z_1 \cos \beta_y \sin^2(\theta + \varphi_1) + i_{12} r_b \cos^2 \beta_y \sin(\theta + \varphi_1) + z_1 A \right] = 0, \Phi(u, \theta, \varphi_1) = 0. \quad (46) \]

The derivation of equation (49) shows that the third-order sub-determinant \( M_3 \) obtained in equation (46) can be deduced as the function \( \Psi \) obtained in equation (48) using the relational equation in equation (32). Therefore, equation (49) illustrates that the expression of the singularity condition obtained in equation (46) for the worm gear tooth surface is correct, theoretically.

Thus, from equation (42) and (46), the equation for the singularity trajectory of the worm gear tooth surface can be obtained as
\[ \begin{align*}
(\vec{r}_2(u, \theta, \varphi_1))_x &= x_2 + y_2 + y_0 \frac{L}{2} \\
A \sin(\theta + \varphi_1) + r_b \sin \beta_y \cos(\theta + \varphi_1) + B &= 0 \\
p \theta z_1 \cos \beta_y \sin^2(\theta + \varphi_1) + i_{12} r_b \cos^2 \beta_y \sin(\theta + \varphi_1) + z_1 A &= 0
\end{align*} \]
\[ (50) \]

The last two equations of equation (50) show that there is a relationship between the three variables \( u \) and \( \varphi_1 \). Therefore, only one of these three variables is independent so that the equation for the singularity trajectory of the worm gear tooth surface is the single parameter. After determining the position coordinates of the worm gear tooth surface, the corresponding singularity can be found on the worm gear tooth surface by equation (50). By connecting the obtained singularities smoothly using the interpolation method, the singularity trajectory of the worm gear tooth surface can be obtained. The specific calculation process for singularity trajectory will be discussed in the next section.

**Calculation method of singularity trajectory on worm gear tooth surface**

In Figure 4, the coordinate \( (O_2; y_{01}, \sqrt{x_2^2 + y_2^2}) \) is established according to equation (42) to reflect the worm gear tooth surface. The original point is \( O_2 \). The vertical coordinate \( \sqrt{x_2^2 + y_2^2} \) represents the radial direction of the worm gear and the horizontal coordinate \( y_{01} \) is set along the axis of the worm gear. Based on the geometric relationships in Figure 4, the equation for the singularity trajectory of the worm gear tooth surface in equation (50) can be rewritten as
\[ \begin{align*}
y_{01} &= r_b \sin(\theta + \varphi_1) - \sin \beta_y \cos(\theta + \varphi_1) = L \\
A \sin(\theta + \varphi_1) + r_b \sin \beta_y \cos(\theta + \varphi_1) &= B \\
p \theta z_1 \cos \beta_y \sin^2(\theta + \varphi_1) + i_{12} r_b \cos^2 \beta_y \sin(\theta + \varphi_1) + z_1 A &= 0
\end{align*} \]
\[ (51) \]
where the symbol $L$ is a constant and its range is $-b_2/2 \leq L \leq b_2/2$. Equation (51) is a non-linear equation system with three variables $u$, $\theta$ and $\varphi_i$. From equation (51), only one of the three variables is independent. After determining the position coordinates on the worm gear tooth surface, i.e. the value of $L$, the corresponding singularity can be found by equation (51). By varying the value of $L$ and solving point by point, the singularity trajectory of the worm gear tooth surface can be obtained. The process is as follows.

To solve the ternary non-linear equation system in equation (51), the elimination method and the geometric construction method\textsuperscript{23} are applied in this paper.

Combining the first and second equations in equation (51), the trigonometric functions $\sin(\theta + \varphi_i)$ and $\cos(\theta + \varphi_i)$ can be obtained as

\[
\sin(\theta + \varphi_i) = \frac{-Bu + r_bL}{Au + r_b^2}, \cos(\theta + \varphi_i)
\]

\[
= \frac{AL + r_bB}{(Au + r_b^2)\sin\beta_b},
\]

where $\sin(\theta + \varphi_i)$ and $\cos(\theta + \varphi_i)$ can be regarded as the functions of two variables $u$ and $A$.

According to trigonometric identity, it can be obtained from equation (52) as

\[
\sin^2(\theta + \varphi_i) + \cos^2(\theta + \varphi_i) = \left(\frac{Bu - r_bL}{Au + r_b^2}\right)^2 + \left[\frac{AL + r_bB}{(Au + r_b^2)\sin\beta_b}\right]^2 = 1.
\]

Let $X = Au + r_b^2$, then equation (53) can be rewritten as

\[
X^2(u^2 \sin^2\beta_b - L^2) - 2r_bLX(Bu - r_bL)
\]

\[
- (Bu - r_bL)^2(Y^2 - r_b^2) = 0,
\]

which is a quadratic equation with respect to the unknown $X$.

When $u^2 \sin^2\beta_b - L^2 \neq 0$, from equation (54), the two solutions of the unknown $X$ can be found as

\[
X(u) = \frac{Bu - r_bL}{u^2 \sin^2 \beta_b - L^2}(Y \sin \beta_b + r_bL),
\]

where the coefficient $Y = \pm \sqrt{u^2 \sin^2 \beta_b + r_b^2 - L^2}$. The solution of $X(u)$ is a function with respect to the variable $u$. Due to $X \neq 0$, the numerator term $Bu - r_bL \neq 0$. Since the unknown $X$ has two solutions, it suggests that each constant $L$ will correspond to two singularities on the worm gear tooth surface.

Then, due to $X(u) = Au + r_b^2$, the coefficient $A$ can be expressed as a function with respect to $u$ as $A = [X(u) - r_b^2]/u$. Thus, from the expression of the coefficient $A$ in equation (41), the angle $\theta$ can be expressed as the function about $u$.

\[
\theta(u) = \frac{u - A}{p \cos \beta_b}.
\]

Thus from equation (56) and $z_1 = p\theta - u \cos \beta_b$ in equation (33), the coefficient $z_1$ can be expressed as a function of $u$ as

\[
z_1(u) = \frac{u \sin^2 \beta_b - A}{\cos \beta_b}.
\]

Substituting equations (52) and (55)–(57) into the third equation in equation (51) yields

\[
\left(\frac{u - A}(u^2 \sin^2 \beta_b - A)(Bu - r_bL)^2}{X^2 \cos^2 \beta_b}
\]

\[
- i_2r_b \cos \beta_b
\]

\[
\frac{Bu - r_bL}{X} + A \frac{\sin^2 \beta_b - A}{\cos^2 \beta_b}
\]

\[
= (Bu - r_bL)^2
\]

\[
\left[\left(u^2 - r_b^2\right)^2 + A_u(r_b \sin \beta_b + LY)^2\right]
\]

\[
\left[u(Y^2 - r_b^2) \sin^2 \beta_b - A_u\right]
\]

\[
- i_2r_b \cos \beta_b
\]

\[
\left(Y^2 - r_b^2\right)^3(Y \sin \beta_b + r_bL) = 0,
\]

where the coefficient $A_u = (Bu - r_bL)Y \sin \beta_b - i_2r_b \sin^3 \beta_b - BL$.

Since $X \neq 0$, $Bu - r_bL \neq 0$ and $u^2 \sin^2 \beta_b - L^2 \neq 0$, equation (58) can be reduced to a higher-order non-linear equation with respect to $u$, which is the equation of the singularity trajectory for the worm gear tooth surface as

\[
f(u) = \left[u(Y^2 - r_b^2)^3 + A_u(r_b \sin \beta_b + LY)^2\right]
\]

\[
\left[u(Y^2 - r_b^2) \sin^2 \beta_b - A_u\right]
\]

\[
- i_2r_b \cos \beta_b
\]

\[
\left(Y^2 - r_b^2\right)^3(Y \sin \beta_b + r_bL) = 0.
\]
To solve the singularity trajectory, the initial values of the iterations of the variable $u$ can be first determined by the geometric construction technique.\(^{23}\) Secondly, after iteratively solving for the value of $u$, the values of the variables $u$ and $u_1$ can be calculated from equation (52) and (56). Thus, the singularity on the worm gear tooth surface can be determined. Finally, by varying the value of $L$ and solving point by point, the singularity trajectory on the worm gear tooth surface can be obtained. Using the proposed calculating method for the singularity trajectory on the worm gear tooth surface, it is possible to further investigate the singularity trajectory on the worm gear tooth surface and to discuss how to avoid the occurrence of curvature interference.

### Numerical examples and analysis

#### Parameters of involute worm drive in numerical example

A numerical example of the right-handed involute worm drive is taken as an example to illustrate the specific calculation procedure for the singularity trajectory of the worm gear tooth surface. The numerical results of the basic parameters of the right-handed involute worm drive are listed in Table 2. To determine the basic parameters of the worm drive, the center distance, the number of the worm threads, and the transmission ratio should be determined, firstly. Then, according to the empirical formulas\(^{24}\) shown in Table 2, the pitch circle radius and the modulus of the worm can be reasonably determined according to the standard values. Finally, the other parameters can be worked out by the formulas listed in Table 2.

### Numerical computation and results of singularity trajectory on worm gear tooth surface

In this numerical example, the singularity trajectory on the worm gear tooth surface is shown in Figure 5(a). According to equation (38), the unit normal vector of the involute worm is directed from the involute worm solid to space. Based on the geometric relationships in Figure 3, this paper investigates the mesh of the left tooth face of the involute worm with the right tooth face of the worm gear. Thus, based on the coordinate settings in Figure 3, it can be determined that Figure 5(a) depicts the observation result from the gear tooth groove to the worm gear tooth surface.

As shown in Figure 5(a), there are two singularity trajectories on the worm gear tooth surface, which may be called the upper and lower singularity trajectories.
respectively. They are represented by the green and red lines, respectively. After determining the boundary dimensions of the worm gear tooth surface, the variation range of the abscissa $L$ can be defined as $-64.5 \text{mm} = -b_2/2 \leq L \leq b_2/2 = 64.5 \text{mm}$. Based on the variation range of the abscissa $L$, twelve $L$ values are selected in this study. Finally, the corresponding singularity trajectory can be obtained by interpolating the resulting singularities. To clearly observe the position of the singularity trajectory on the worm tooth surface, the conjugate lines of singularity trajectories can be projected onto the worm axial section as shown in Figure 5(b).

In this section, the corresponding singularity when the abscissa $L = 0$ is chosen as an example to illustrate the calculation method of the singularity on the worm gear tooth surface. From Figure 5, when $L = 0$, it represents the point $k = 5$. Substituting $L = 0$ into equation (59), it can be obtained as

$$f_{L-0}(u) = \left[u^3 \sin^3 \beta_b + r_b^2 (BY - r_b^2 \sin \beta_b)\right]$$

$$= u^2 \sin^3 \beta_b - BY + r_b^2 \sin \beta_b - i_{12} r_1 u^4 \sin^3 \beta_b \cos^3 \beta_b = 0,$$  \hspace{1cm} (60)

where the coefficient $Y$ in equation (55) is

$$Y_{L-0} = \pm \sqrt{u^2 \sin^2 \beta_b + r_b^2},$$  \hspace{1cm} (61)

where $Y_{L-0}$ is a function with respect to the variable $u$. Since there are two different expressions in equations (61) and equation (60) represents two non-linear equations for the variable $u$, which can be denoted separately as $f_{L-0}^{(Y>0)}(u) = 0$ and $f_{L-0}^{(Y<0)}(u) = 0$. It means that there are two different singularities on the worm gear tooth surface when $L = 0$.

To determine the iterative initial value of equation (60), firstly, using the geometric construction method, two curves $f_{L-0}^{(Y>0)}(u)$ and $f_{L-0}^{(Y<0)}(u)$ are drawn in Figure 6 where the abscissa is the variable $u$ whose variation range is $[0, 140 \text{mm}]$ and the ordinate is $f_{L-0}(u)$. In Figure 6, the red curve is $f_{L-0}^{(Y<0)}(u)$ and the green one is $f_{L-0}^{(Y>0)}(u)$.

As can be seen from Figure 6, each curve has an intersection point with the horizontal axis, which means that each equation of $f_{L-0}^{(Y<0)}(u) = 0$ has a real root. From Figure 6, the intersection point of the curve $f_{L-0}^{(Y>0)}(u)$ with the horizontal axis is approximated as $(50 \text{mm}, 0)$, and the intersection point of the curve $f_{L-0}^{(Y<0)}(u)$ with the horizontal axis is approximated as $(140 \text{mm}, 0)$. The non-linear equation $f_{L-0}^{(Y<0)}(u)$ can therefore be solved using $u = 50 \text{mm}$ as an iterative initial value. After obtaining $u$, the values of $\theta$ and $\varphi_1$ can be solved according to equations (54) and (58). The results are listed in Table 3(a). Likewise, the non-linear equation $f_{L-0}^{(Y>0)}(u)$ can be solved using $u = 140 \text{mm}$ as an initial value, and the results are shown in Table 3(b).

By modifying the value of $L$, other points $A_k$ and $B_k$ can be solved in the same method as above and their numerical results are also listed in Table 3(a) and (b), respectively.

According to equation (36), since the direction of the normal vector $\overrightarrow{n}$ of the worm tooth surface is from the worm solid to space, the value of the third-order sub-determinant $M_3$ in the singularity condition in equation (46) should be negative when the worm gear tooth surface is located at the non-interference zone. Therefore, to determine whether the worm gear tooth face is located in the non-interference zone in the current example, a point $P_1^{15}$ near the worm gear dedendum can be selected as a checkpoint as shown in Figure 5. The numerical results for the point $P_1$ are shown in Table 4. By calculating the value of the third-order sub-determinant $M_3$ at the point $P_1$, it can be seen that the value of the third-order sub-determinant $M_3$ at the point $P_1$ is $M_3 = -0.2287$, which indicates that the worm gear tooth surface is located at the non-interference zone. As a result, the curvature interference does not occur in this numerical example.

**Influence of design parameters on singularity trajectories**

In this section, nine numerical examples are given to illustrate the influence of the basic parameters on the singularity trajectory of the worm gear tooth surface, which are denoted by the symbols $\circ - \circ$, respectively. In this section, the center distance of the worm gear pair is fixed as $a = 400 \text{mm}$. Thus, according to the empirical formula, the radius of the pitch circle for the involute worm can be reasonably selected as $r_1 = 70 \text{mm}$. Then, the number of the worm threads and the transmission ratios are chosen. Finally, other design parameters are selected and calculated following the gear handbook. The results for the basic parameters of these numerical examples are included in Table 5.

As shown in Figure 5(a), the upper singularity trajectory does not cause the curvature interference.
because the upper singularity trajectory is far from the worm gear tooth surface and does not enter the worm gear tooth surface in these nine numerical examples. However, the lower singularity trajectory is closer to the worm gear dedendum, which indicates that there is a risk of curvature interference near the worm gear dedendum. On the other hand, in the numerical examples, the conjugate lines of the two singularity trajectories do not enter the worm tooth surface. Therefore, the discussion of this section concentrates on the influence of the design parameters on the lower singularity trajectory of the worm gear tooth surface.

In Table 5, Examples ①–⑩ are used to discuss the influence of the transmission ratio and the modulus on the lower singularity trajectory, respectively. The lower singularity trajectories on the worm gear tooth surface of Examples ①–⑩, ⑪, ⑫, and ⑬ are plotted separately in Figures 7(a) to (d), 8(a), (b) and 9. By observing each of the examples ①–⑩ in Figures 7(a) to (d), 8(a), (b) and 9 individually, with a constant modulus and a constant number of the worm threads, when the calculated modification coefficient of the worm gear is negative after selecting the transmission ratio, the lower singularity trajectory may appear on the worm gear tooth surface, which can lead to curvature interference. Furthermore, when the number of the worm threads is constant, increasing the modulus and decreasing the transmission ratio will cause the lower singularity trajectory to move closer to the worm gear dedendum, which may increase the risk of curvature interference. On the other hand, the horizontal comparison of Examples ①–⑩ shows that when the number of worm threads is constant and the modulus increases and the transmission ratio decreases, the lower singularity trajectory gradually moves closer to the worm gear dedendum. When the modulus is too large and the transmission ratio is too small, the lower singularity trajectory may enter the worm gear tooth surface, which can lead to the occurrence of the curvature interference during the meshing of the involute worm drive.

Examples ⑪ and ⑫ are used to discuss the influence of the number of worm threads on the lower singularity trajectory. The lower singularity trajectories of the worm gear tooth surface of Examples ⑪ and ⑫ are plotted in Figure 10(a) and (b). By observing Figure 10(a), when the modulus and the teeth number of worm gear are constant, increasing the number of worm threads can make the lower singularity trajectory be moved away from the worm gear dedendum. From Example ⑫ in Figure 10(b), when the transmission ratio is constant and the number of worm threads decreases and the modulus increases, the lower singularity trajectory will move closer to the worm gear tooth surface, which increases the risk of curvature interference in involute worm pair.

In conclusion, the influence of the basic design parameters on the lower singularity trajectory can be summarized as follows: (1) when the ratio of the worm drive is too small, especially in the case of the single-threaded worm and the large modulus, there is a greater risk of the curvature interference; (2) the risk of

Table 3. Numerical results of singularity trajectory: (a) Numerical results of points Ak, (k = 1, 2 ... 12) and (b) Numerical results of points Bk, (k = 1, 2 ... 12).

| Ak | A1 | A2 | A3 | A4 | A5 | A6 |
|----|----|----|----|----|----|----|
| u/\(\text{mm}\) | 134.9492 | 118.8232 | 100.3802 | 75.4905 | 50.1120 | 37.1442 |
| \(\theta/^\circ\) | 2429.5 | 2250.2 | 2041.4 | 1750.8 | 1438.4 | 1268.3 |
| \(\varphi_1/^\circ\) | 132.1 | 312.6 | 163.1 | 96.6 | 53.1 | 226.2 |
| \(A_k\) | A7 | A8 | A9 | A10 | A11 | A12 |
| u/\(\text{mm}\) | 23.8770 | 32.7487 | 43.6649 | 54.5811 | 65.4974 | 74.4669 |
| \(\theta/^\circ\) | 1081.3 | 952.3 | 954.3 | 1010.3 | 1093.3 | 1173.5 |
| \(\varphi_1/^\circ\) | 57.6 | 206.4 | 220.9 | 177.4 | 104 | 30.2 |
| \(B_k\) | B1 | B2 | B3 | B4 | B5 | B6 |
| u/\(\text{mm}\) | 148.3917 | 142.8058 | 137.3088 | 131.5085 | 127.6123 | 126.4075 |
| \(\theta/^\circ\) | 644.0 | 636.5 | 628.7 | 620.6 | 615.9 | 614.7 |
| \(\varphi_1/^\circ\) | 0.1 | 1.7 | 2.9 | 1.7 | 3.3 | 7 |
| \(\varphi_1/^\circ\) | 1.7 | 2.9 | 1.7 | 3.3 | 7 | 7 |

Table 4. Numerical results of checkpoint \(P_1\).

| \(P_1\) | u/\(\text{mm}\) | \(\theta/^\circ\) | \(\varphi_1/^\circ\) | \(M_3\) |
|--------|-------------|-------------|-------------|--------|
| 78.6468 | 187.6464 | 52.8215 | -0.2287 |
curvature interference is reduced when the number of worm heads is more; (3) the smaller the calculated result of the modification coefficient, especially if the modification coefficient is negative, the greater the risk of curvature interference, and conversely the less risk.

In a word, when the design parameters are reasonably selected using the gear handbook, the singularity trajectory will not enter the worm gear tooth surface, meaning that there is no curvature interference in the involute worm pair. Besides, an important finding is that the larger the absolute value of the negative modification coefficient, the greater the risk of curvature interference occurring. Thus, after the basic parameters of the involute worm gearing have been selected, the possibility of curvature interference for the involute worm pair can be initially judged by calculating the modification coefficient. If further precision is required to determine whether there is curvature interference, the singularity trajectory of the worm gear tooth surface can be solved to determine whether the entire worm gear tooth surface is in the non-interference zone according to the curvature interference theory for the involute worm drive proposed in Section 3. The singularity condition of the enveloped surface presented in Section 2 can also be used to study the curvature interference theory of other types of the worm drive. Furthermore, the singularity condition presented in Section 2 can also be used to determine the singularity trajectory when machining the worm and the worm gears as a means of determining whether undercutting has occurred or not.

Conclusions

In this paper, a simpler form and more computationally convenient singularity condition of the enveloped surface is proposed using the theory of linear algebra. The pre-conditions for this new singularity condition of the enveloped surface are the components of the tangent vector of the enveloping surface, the relative velocity vector, and the total differential of the meshing
function. It is then sufficient to calculate a third-order determinant containing these pre-conditions. It avoids calculating the curvature parameters of the enveloping surface. Furthermore, it is proved that the singularity conditions of enveloped surface proposed in different literature are all equivalent although they are of different forms. The equations for the relationship among these singularity conditions are obtained.

The curvature interference theory for the involute worm drive is adequately established using the singularity condition of enveloped surface proposed in this paper. The equations for the tooth surface for the worm and worm gear, the meshing equations, and the equations for the singularity trajectory on the worm gear tooth surface are obtained. The calculation method for the singularity trajectory on the worm gear tooth surface is proposed using the elimination method and the geometric construction method, and the numerical results of the singularity trajectory are obtained. The influence of the design parameters on the singularity trajectory of the worm gear tooth surface is studied using the developed curvature interference theory.

**Figure 7.** Influence of transmission ratio and modulus on lower singularity trajectory when worm threads \( Z_1 = 1 \). (a) modulus \( m = 8 \), (b) modulus \( m = 10 \), (c) modulus \( m = 16 \), and (d) modulus \( m = 20 \).

**Figure 8.** Influence of transmission ratio and modulus on lower singularity trajectory when worm threads \( Z_1 = 2 \). (a) modulus \( m = 10 \) and (b) modulus \( m = 16 \).

**Figure 9.** Influence of transmission ratio and modulus on lower singularity trajectory when worm threads \( Z_1 = 3 \).
The numerical results of the study show that the risk of curvature interference will be high when the transmission ratio of the involute worm drive is too small. Increasing the number of worm threads and the transmission ratio and reducing the modulus can reduce the risk of curvature interference. The risk of curvature interference is greater when the value of the modification coefficient is smaller, especially if the modification coefficient is negative, and conversely the less risk.

Therefore, this paper proposes that a preliminary judgment of whether the curvature interference occurs during the meshing of the involute worm pair can be made by calculating the modification coefficient. If an accurate determination of the singularity trajectory on the worm gear tooth surface is required, then the curvature interference theory established in this paper can be relied upon. The singularity condition proposed in this paper can also be applied to the study of curvature interference theory for other types of worm drives and the study of undercutting theory during machining the worm and worm gear.

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Figure 10. Influence of number of worm threads and transmission ratio on lower singularity trajectory: (a) constant transmission ratio and modulus and (b) constant transmission ratio.
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Appendix

Notation

\begin{align*}
\alpha_n & \quad \text{Normal profile angle of worm} \\
\xi & \quad \text{Modification coefficient of worm gear} \\
r_1 & \quad \text{Radius of worm pitch circle} \\
Z_1 & \quad \text{Number of worm threads} \\
Z_2 & \quad \text{Number of worm gear teeth} \\
r_{a1} & \quad \text{Radius of worm addendum circle} \\
r_{a2} & \quad \text{Radius of worm gear addendum circle} \\
r_b & \quad \text{Radius of worm base circle} \\
\beta_b & \quad \text{Helical angle of worm base circle} \\
h_a & \quad \text{Worm addendum} \\
p & \quad \text{Worm helix parameter} \\
\gamma_b & \quad \text{Lead angle of worm base circle} \\
\alpha_t & \quad \text{End profile angle of worm} \\
\gamma_1 & \quad \text{Lead angle of worm pitch circle} \\
\beta_h & \quad \text{Helical angle of worm pitch circle} \\
g & \quad \text{Lead angle of worm gear} \\
r_{g1} & \quad \text{Radius of worm gear addendum circle} \\
r_{g2} & \quad \text{Radius of worm gear dedendum circle} \\
r_{g3} & \quad \text{Radius of worm gear outer circle} \\
l & \quad \text{Angle of face width of worm gear} \\
a & \quad \text{Center distance} \\
m & \quad \text{Modulus} \\
b_1 & \quad \text{Worm face width} \\
b_2 & \quad \text{Worm gear face width} \\
\lambda & \quad \text{Angle of face width of worm gear}
\end{align*}