Decomposable penalty method for generalized game problems with joint constraints

I. V. Konnov

Department of System Analysis and Information Technologies, Kazan Federal University, Kazan, Russia

ABSTRACT
We consider an extension of a non-cooperative game problem where players have joint binding constraints. In this case, justification of a generalized equilibrium point needs a reasonable mechanism for attaining this state. We combine a penalty method and shares allocation of right-hand sides, which replaces the initial problem with a sequence of the usual Nash equilibrium problems together with an upper level variational inequality as a master problem. In order to obtain a completely decomposable problem at the lower level, we apply its additional equivalent transformation. Convergence of solutions of these auxiliary penalized problems to a solution of the initial game problem is established under weak coercivity conditions.

ARTICLE HISTORY
Received 15 April 2020
Accepted 3 July 2020

KEYWORDS
Non-cooperative games; joint constraints; generalized equilibrium points; decomposable penalty method; variational inequality

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
91A10; 91A40; 90C33

1. Introduction

Non-cooperative games with joint (binding) constraints date back to early works [1–5] and these problems are investigated in many recent works; see, e.g. [6–10] and the references therein. Let us consider the generalized l-person non-cooperative game, where the i-th player has its particular strategy set $X_i \subseteq \mathbb{R}^{n_i}$ and a payoff (utility) function $f_i : X_i \rightarrow \mathbb{R}$ with

$$X = X_1 \times \cdots \times X_l, \quad n = \sum_{i=1}^l n_i.$$ 

Besides, all the players together with the above utility functions and strategy sets have the joint constraint set

$$Y = \left\{ x \in \mathbb{R}^n \left| \sum_{i=1}^l h_i(x_i) \leq b \right. \right\},$$

where $x = (x_1, \ldots, x_l)^\top$, $h_i(x_i) = (h_{i1}(x_i), \ldots, h_{im}(x_i))^\top$, $h_{ij} : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$, $j = 1, \ldots, m$, $i = 1, \ldots, l$ are given functions, $b \in \mathbb{R}^m$ is a fixed vector in $\mathbb{R}^m$, as an
addition to the set $X$. That is, they have the common feasible set

$$D = X \cap Y.$$  \hfill (1)

A point $x^* = (x_1^*, \ldots, x_l^*)^\top \in D$ is said to be an equilibrium point for this game, if

$$f_i(x_{-i}^*, y_i) \leq f_i(x^*) \quad \forall (x_{-i}^*, y_i) \in D, \quad i = 1, \ldots, l;$$  \hfill (2)

where we set $(x_{-i}, y_i) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_l)$. The usual non-cooperative game corresponds to the case $D = X$, then (2) is equivalent to the well known Nash equilibrium concept. Examples of applications of generalized non-cooperative games with joint constraints can be, e.g. found in [6,7,11–13]. The main question consists in implementation of this constrained equilibrium concept within the custom non-cooperative framework, where players are independent and make their choices simultaneously. In fact, the presence of the binding constraints requires certain treaties or concordant actions of the players, thus contradicting the above assumptions. These drawbacks were noticed and discussed, e.g. in [5,8,14].

Rather recently, the right-hand side decomposition approach was suggested for variational inequalities with binding constraints in [15]. Its extension to generalized non-cooperative games was given in [13]. Within this approach, the initial problem is treated as a two-level one by using a share allocation procedure, which leads to a set-valued variational inequality as a master problem. In [13], a usual non-cooperative game problem is solved at the lower level. Further, a decomposable dual regularization (penalty) method that deals with a single-valued approximation of the master problem for each fixed share allocation was suggested. Application of this method for production problems with common pollution regulation was described in [16].

In this paper, we suggest to apply a direct decomposable penalty method to generalized non-cooperative games, which involve share allocation variables. Since the streamlined penalty method with shares allocation leads to an equilibrium problem with coupled variables, we propose its additional equivalent transformation to a completely decomposable problem. We show convergence of solutions of auxiliary penalized problems to a solution of the initial problem under weak coercivity conditions. Besides, the implementation of this method enables one to solve each auxiliary penalized problem as a two-level one with the usual Nash equilibrium problem at the lower level.

2. Decomposition via shares allocation

We first fix our basic assumptions.

(A1) Each strategy set $X_i \subseteq \mathbb{R}^{n_i}$ is convex and closed and each utility function $f_i$ is concave in its $i$-th variable $x_i$ and continuous for $i = 1, \ldots, l$. Also, $h_{ij} : \mathbb{R}^{n_i} \to \mathbb{R}$,
$j = 1, \ldots, m$, $i = 1, \ldots, l$ are convex functions, and the common feasible set $D$ is non-empty.

Following the Nikaido–Isoda approach from [17], we consider the normalized equilibrium problem (EP for short) of finding a point $x^* = (x_1^*, \ldots, x_l^*)^\top \in D$ such that

$$\Phi(x^*, y) \geq 0 \quad \forall y \in D,$$

where

$$\Phi(x, y) = \Psi(x, x) - \Psi(x, y), \quad \Psi(x, y) = \sum_{i=1}^{l} f_i(x_i, y_i);$$

its solutions are called normalized equilibrium points. From the above assumptions it follows that $\Phi : X \times X \to \mathbb{R}$ is an equilibrium bi-function, i.e. $\Phi(x, x) = 0$ for every $x \in X$, besides, $\Phi(x, \cdot)$ is convex for each $x \in X$ and $\Phi(\cdot, \cdot)$ is continuous. It should be noted that (3) implies (2). In other words, each normalized equilibrium point is a generalized Nash equilibrium point, but the reverse assertion is not true in general. But in case $D = X$, (2) and (3) become equivalent. We take a suitable coercivity condition and obtain the existence result from [18, Theorem 3.1].

For a function $\mu : \mathbb{R}^n \to \mathbb{R}$ and a number $r$, we define the level set

$$B_r = \{x \in X \mid \mu(x) \leq r\}.$$

We say that the function $\mu : \mathbb{R}^n \to \mathbb{R}$ is weakly coercive with respect to the set $X \subseteq \mathbb{R}^n$ if there exists a number $r$ such that the set $B_r$ is non-empty and bounded.

(C1) There exist a lower semicontinuous and convex function $\mu : X \to \mathbb{R}$, which is weakly coercive with respect to the set $D$, and a number $r$, such that, for any point $x \in D \setminus B_r$ there is a point $z \in D$ with

$$\min\{\Phi(x, z), \mu(z) - \mu(x)\} < 0 \quad \text{and} \quad \max\{\Phi(x, z), \mu(z) - \mu(x)\} \leq 0.$$

**Proposition 2.1:** If (A1) and (C1) are fulfilled, then problem (3) has a solution.

After proper specialization of the inequalities in (C1) we can somewhat strengthen the above assertion.

(C2) There exist a lower semicontinuous and convex function $\mu : X \to \mathbb{R}$, which is weakly coercive with respect to the set $D$, and a number $r$, such that, for any point $x \in D \setminus B_r$ there is a point $z \in D$ with

$$\mu(z) \leq \mu(x) \quad \text{and} \quad \Phi(x, z) < 0.$$

**Corollary 2.2:** If (A1) and (C2) are fulfilled, then problem (3) has a solution, and all the solutions are contained in $D \cap B_r$.

These coercivity conditions (C1) and (C2) clearly hold if $D$ (or $X$) is bounded. Then we can take $\mu(x) = \|x\|$ and choose $r$ large enough so that $D \subseteq B_r$. 
Let us now introduce the set of partitions of the right-hand side common constraint vector $b$:

$$\tilde{U} = \left\{ u \in \mathbb{R}^m \mid \sum_{i=1}^l u_i = b \right\}.$$  

where $u = (u_1, \ldots, u_l)^\top$, $u_i \in \mathbb{R}^m$, $i = 1, \ldots, l$. Here $u_i$ determines the share of the $i$-th player.

Given a partition $u \in \tilde{U}$, we can consider the parametric EP: Find a point $x(u) = (x_1(u), \ldots, x_l(u))^\top \in D(u)$ such that

$$\Phi(x(u), y) \geq 0 \quad \forall \; y \in D(u), \quad \text{(4)}$$

where

$$D(u) = D_1(u_1) \times \cdots \times D_l(u_l),$$

$D_i(u_i) = \{x_i \in X_i \mid h_i(x_i) \leq u_i\}, i = 1, \ldots, l$. Clearly, (4) is equivalent to the parametric Nash equilibrium problem (NEP):

$$f_i(x_{-i}(u), y_i) \leq f_i(x(u)) \quad \forall \; y_i \in D_i(u_i), \quad i = 1, \ldots, l. \quad \text{(5)}$$

If all the optimal shares $u_i$, $i = 1, \ldots, l$ of players are known, the constrained problems (2) and (3) reduce to NEPs. Hence, it seems worthwhile to insert an additional upper control level for finding the optimal shares. In optimization, this approach is known as the right-hand side (Kornai-Liptak) decomposition method; see [19].

Following this approach we notice that under certain regularity condition system (5) can be replaced with the corresponding system of primal-dual optimality conditions: Find a pair $(x(u), v(u)) \in X \times \mathbb{R}^m_+$ such that

$$f_i(x(u)) - f_i(x_{-i}(u), y_i) + \langle v_i(u), h_i(y_i) - h_i(x_i(u)) \rangle \geq 0 \quad \forall \; y_i \in X_i, \quad \text{(6)}$$

$$\langle u_i - h_i(x_i(u)), v_i - v_i(u) \rangle \geq 0, \quad \forall \; v_i \in \mathbb{R}^m_+, \quad \text{for } i = 1, \ldots, l; \quad \text{(7)}$$

where $v(u) = (v_1(u), \ldots, v_l(u))^\top$. In this system the first relations (6) are rewritten equivalently as

$$\Phi(x(u), y) + \sum_{i=1}^l \langle v_i(u), h_i(y_i) - h_i(x_i(u)) \rangle \geq 0 \quad \forall \; y \in X. \quad \text{(8)}$$

Clearly, if $(x(u), v(u))$ solves (6)–(7) or (8), (7), then $x(u)$ is a solution to (4) or (5).

We denote by $T(u)$ the set of all the solution points $-v(u)$, creating the image of the set-valued mapping $T$. This enables us to define the variational inequality (VI): Find a point $u^* \in \tilde{U}$ such that

$$\exists t^* \in T(u^*), \quad \langle t^*, u - u^* \rangle \geq 0, \quad \forall \; u \in \tilde{U}. \quad \text{(9)}$$

Then it was shown in [13, Theorem 4.1] that just the master VI (9) yields the optimal shares of common constraints among players.
Proposition 2.3: Suppose (A1) is fulfilled. If a point \( u^\ast \) solves VI (9), the corresponding point \( x(u^\ast) \) in (6)–(7) is a solution of problem (1), (3).

We conclude that VI (9) related to the parametric problems (4) or (5) enables us to find a solution to the initial generalized non-cooperative game. Hence, the two-level procedure gives a suitable regulation mechanism for these game problems. However, this approach has clear drawbacks: \( T(u) \) can be empty for some feasible partitions, besides, \( T \) is set-valued in general, and this fact reduces the number of methods applicable to solution of VI (9). Therefore, this approach needs certain modifications.

3. Decomposable penalty method

We start our description of the approach from the simple transformation of the joint constraint set \( Y \) by inserting auxiliary variables:

\[
Y = \left\{ x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m, \sum_{i=1}^{l} u_i = b, h_i(x_i) \leq u_i, i = 1, \ldots, l \right\},
\]

where \( u = (u_1, \ldots, u_l)^{\top} \), \( u_i \in \mathbb{R}^m, i = 1, \ldots, l \). These variables \( u_i \) as above determine a partition of the right-hand side vector \( b \), i.e. give explicit shares of players. In principle, some additional reasonable restrictions can be imposed on the shares \( u_i \), such as \( u_i \leq b \) or/and \( u_i \geq 0 \) for \( i = 1, \ldots, l \). Hence, we define the set of feasible partitions as follows:

\[
U = \left\{ u \in U_0 \mid \sum_{i=1}^{l} u_i = b \right\}.
\]

where \( U_0 \subseteq \mathbb{R}^ml \) is a set of these optional additional restrictions such that for each \( x \in D \) there exists \( u \in U, h_i(x_i) \leq u_i, i = 1, \ldots, l \).

We can now separate the constraints and first consider the auxiliary penalty problem: Find a pair \( w(\tau) = (x(\tau), u(\tau)) \in X \times U, \tau > 0 \) such that

\[
\Phi_{\tau}(w(\tau), w) = \Phi(x(\tau), x) + \tau [P(w) - P(w(\tau))] \geq 0 \quad \forall w = (x, u) \in X \times U,
\]

where

\[
P(w) = \sum_{i=1}^{l} P_i(w_i),
\]

and each \( P_i(w_i) = P_i(x_i, u_i) \) is a general penalty function for the set

\[
W_i = \left\{ w_i = (x_i, u_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^m \mid h_i(x_i) \leq u_i \right\}.
\]
We will define these functions as follows:

\[ P_i(x_i, u_i) = \varphi(h_i(x_i) - u_i), \quad i = 1, \ldots, l, \]

where \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}_+ \) is a convex differentiable and isotone function such that

\[ \varphi(v) = \begin{cases} 0, & \text{if } v \leq 0, \\ > 0, & \text{otherwise.} \end{cases} \]

It follows that

\[ P_i(x_i, u_i) = \begin{cases} 0, & \text{if } (x_i, u_i) \in W_i, \\ > 0, & \text{if } (x_i, u_i) \notin W_i; \end{cases} \]

for \( i = 1, \ldots, l \). We recall that the function \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R} \) is called isotone, if for any points \( u, v, u \geq v \) it holds that \( \varphi(u) \geq \varphi(v) \). The most popular and simple choice is

\[ \varphi(v) = 0.5\| [v]_+ \|^2, \]

where \([v]_+\) denotes the projection of \( v \) onto the non-negative orthant \( \mathbb{R}_+^m \).

Then each penalty function \( P_i \) is convex and differentiable for \( i = 1, \ldots, l \). We observe that decomposable penalty methods were suggested for separable convex optimization problems in [20,21].

However, problem (11) still involves coupled variables that prevents from application of decomposition schemes. For this reason, we now show that problem (11) can be replaced with the following: Find a pair \( w(\tau) = (x(\tau), u(\tau)) \in X \times U \), such that

\[
\Phi(x(\tau), x) + \tau \sum_{i=1}^{l} \langle \varphi'(h_i(x_i(\tau)) - u_i(\tau)), h_i(x_i) - h_i(x_i(\tau)) \rangle \\
+ \tau \sum_{i=1}^{l} \langle \varphi'(h_i(x_i(\tau)) - u_i(\tau)), u_i(\tau) - u_i \rangle \geq 0 \quad \forall \ w = (x, u) \in X \times U.
\]

(12)

Let \( w(\tau) \) solve (12). Then (11) holds due to the convexity of the function \( \varphi \). In fact, we have

\[
P_i(x_i, u_i) - P_i(x_i(\tau), u_i(\tau)) \geq \langle \varphi'(h_i(x_i(\tau)) - u_i(\tau)), h_i(x_i) - h_i(x_i(\tau)) \rangle \\
- \langle \varphi'(h_i(x_i(\tau)) - u_i(\tau)), u_i - u_i(\tau) \rangle,
\]

for \( i = 1, \ldots, l \), and (12) implies (11).

Conversely, let \( w(\tau) = (x(\tau), u(\tau)) \) be a solution of problem (11). Then we can temporarily set \( \phi(x) = \Phi(x(\tau), x) \) and obtain

\[
\phi(x) - \phi(x(\tau)) + \tau [P(w) - P(w(\tau))] \geq 0 \quad \forall \ w = (x, u) \in X \times U,
\]

i.e. \( w(\tau) \) is a solution of the optimization problem. Applying now Proposition 5 in [22] we conclude that \( w(\tau) \) solves (12).
In turn, problem (12) is clearly equivalent to the system: Find a pair \( w(\tau) = (x(\tau), u(\tau)) \in X \times U \), such that

\[
\Phi(x(\tau), x) + \tau \sum_{i=1}^{l} \langle \phi'(h_i(x_i(\tau))) - u_i(\tau), h_i(x_i) - h_i(x_i(\tau)) \rangle \geq 0 \quad \forall x \in X, \\
\sum_{i=1}^{l} \langle \phi'(h_i(x_i(\tau))) - u_i(\tau), u_i(\tau) - u_i \rangle \geq 0 \quad \forall u \in U. 
\]

(13)

(14)

We now collect the obtained properties.

**Lemma 3.1:** Let the conditions in (A1) be fulfilled. Then problems (11), (12), and (13)–(14) are equivalent.

Given a point \( u \in \mathbb{R}^{ml} \), we can solve only problem (13) in \( x \), which is to find \( x(u) \in X \) such that

\[
\Phi(x(u), x) + \tau \sum_{i=1}^{l} \langle \phi'(h_i(x_i(u))) - u_i, h_i(x_i) - h_i(x_i(u)) \rangle \geq 0 \quad \forall x \in X. 
\]

Let \( X(u) \) denote the whole solution set of this problem. For each \( x(u) \in X(u) \) we set

\[
g(u) = (g_1(u), \ldots, g_l(u))^\top, \quad \text{where } g_i(u) = -\phi'(h_i(x_i(u))) - u_i, \quad i = 1, \ldots, l.
\]

(16)

Thus we can define the mapping value

\[
G(u) = \{g(u) \mid x(u) \in X(u)\}.
\]

Bearing in mind (14), we now define the VI: Find a point \( u^* \in U \) such that

\[
\exists g(u^*) \in G(u^*), \quad \langle g(u^*), u - u^* \rangle \geq 0, \quad \forall u \in U.
\]

(17)

**Proposition 3.2:** Suppose (A1) is fulfilled.

(i) If a point \( u^* \) solves VI (17), then there exists a point \( x^* = x(u^*) \in X(u^*) \) such that \( g(u^*) \) is defined in (16) at \( u = u^* \) and that the pair \( w^* = (x^*, u^*) \) is a solution of problem (11).

(ii) If a pair \( w(\tau) = (x(\tau), u(\tau)) \) is a solution of problem (11), then the point \( u^* = u(\tau) \) solves VI (17).

The assertions follow directly from the definitions and Lemma 3.1.
The next step consists in replacing problem (15) with the following penalized EP: Find $x(u) \in X$ such that

$$\Phi(x(u), y) + \tau \sum_{i=1}^{l} \left( P_i(y_i, u_i) - P_i(x_i(u), u_i) \right) \geq 0 \quad \forall y \in X. \quad (18)$$

The equivalence of (15) and (18) is proved similarly to Lemma 3.1. However, (18) is clearly equivalent to the NEP: Find $x(u) \in X$ such that

$$\tilde{f}_i(x_{-i}(u), y_i) \leq \tilde{f}_i(x(u)), \quad \forall y_i \in X_i, \quad i = 1, \ldots, l; \quad (19)$$

where the $i$-th player has the penalized utility function

$$\tilde{f}_i(x) = f_i(x) - \tau P_i(x_i, u_i); \quad (20)$$
cf. (2). Therefore, $X(u)$ is now also the whole solution set of NEP (19)–(20) and we have obtained the basic equivalence result.

**Theorem 3.3:** Suppose (A1) is fulfilled.

(i) If a point $u^*$ solves VI (17), then there exists a solution $x^* = x(u^*)$ of NEP (19)–(20) and $g(u^*)$ is defined in (16) at $u = u^*$, such that the pair $w^* = (x^*, u^*)$ is a solution of problem (11).

(ii) If a pair $w(\tau) = (x(\tau), u(\tau))$ is a solution of problem (11), then the point $u^* = u(\tau)$ solves VI (17) and the point $x(\tau)$ is a solution of NEP (19)–(20) at $u = u(\tau)$.

We conclude that VI (17) related to the parametric NEP (19)–(20) yields a solution for the penalized game problem. Hence, we have derived another two-level decomposition method for the initial generalized non-cooperative game and have to indicate its preferences over the method of Section 2. First of all we observe that NEP (19)–(20) has a solution under rather mild assumptions. From Proposition 2.1 we obtain that this is the case if (A1) holds and the set $X$ is bounded. Also, we can take a suitable coercivity condition in the unbounded case.

**C3** There exists a point $\tilde{x} \in X$ such that

$$\Phi(x^k, \tilde{x}) \to -\infty \quad \text{as} \quad \|x^k - \tilde{x}\| \to \infty;$$

for any infinite sequence $\{x^k\} \subset X$.

**Proposition 3.4:** If (A1) and (C3) are fulfilled, then NEP (19)–(20) has a solution for each $\tau > 0$. 
**Proof:** It suffices to show that problem (18) has a solution. Set \( \mu(x) = \|x - \tilde{x}\| \) and
\[
\tilde{\Phi}(x, y) = \Phi(x, y) + \tau [P(y, u) - P(x, u)]
\]
for a fixed \( u \in U \). Then
\[
\tilde{\Phi}(x^k, \tilde{x}) \leq \Phi(x^k, \tilde{x}) + \tau P(\tilde{x}, u) \to -\infty \quad \text{as} \quad \|x^k - \tilde{x}\| \to \infty.
\]
This means that (C3) implies (C1) for problem (18) and the result follows from Proposition 2.1. \( \blacksquare \)

For this reason, \( G(u) \) is non-empty under usual assumptions even if the set \( D(u) \) is empty. The other preference is that \( G \) possesses a strengthened monotonicity property.

**Proposition 3.5:** Suppose (A1) is fulfilled, the bi-function \( \Phi \) is monotone on \( X \times X \), i.e.
\[
\Phi(x', x'') + \Phi(x'', x') \leq 0;
\]
for each pair of points \( x', x'' \in X \), and the gradient map \( \varphi' \) is co-coercive with constant \( \gamma \), i.e.
\[
\langle v' - v'', \varphi'(v') - \varphi'(v'') \rangle \geq \gamma \|\varphi'(v') - \varphi'(v'')\|^2
\]
for all \( v', v'' \in \mathbb{R}^{ml} \). Then the mapping \( G \) is co-coercive with constant \( \gamma \).

**Proof:** Take arbitrary points \( u', u'' \in \mathbb{R}^{ml} \) and set \( x' = x(u') \), \( g' = g(u') \), \( v' = h(x') - u' \) and \( x'' = x(u''), g'' = g(u''), v'' = h(x'') - u'' \). It follows from (15) that
\[
\Phi(x', x'') - \tau \sum_{i=1}^{l} \langle g_i', h_i(x''_i) - h_i(x'_i) \rangle \geq 0,
\]
\[
\Phi(x'', x') - \tau \sum_{i=1}^{l} \langle g_i'', h_i(x'_i) - h_i(x''_i) \rangle \geq 0;
\]

hence
\[
\sum_{i=1}^{l} \langle g_i' - g_i'', h_i(x'_i) - h_i(x''_i) \rangle \geq -[\Phi(x', x'') + \Phi(x'', x')] / \tau \geq 0
\]
since Φ is monotone. It follows that

\[0 \leq \sum_{i=1}^{l} \langle g'_i - g''_i, [h_i(x'_i) - u'_i] - [h_i(x''_i) - u''_i] \rangle + \sum_{i=1}^{l} \langle g'_i - g''_i, u'_i - u''_i \rangle \]

\[= -\sum_{i=1}^{l} \langle \varphi'(v'_i) - \varphi'(v''_i), v'_i - v''_i \rangle + \langle g' - g'', u' - u'' \rangle \]

\[\leq -\gamma \sum_{i=1}^{l} \| \varphi'(v'_i) - \varphi'(v''_i) \|^2 + \langle g' - g'', u' - u'' \rangle \]

\[= -\gamma \| g' - g'' \|^2 + \langle g' - g'', u' - u'' \rangle,\]

and therefore, G is co-coercive with constant γ.

It is well-known that the gradient map of a convex differentiable function is co-coercive with constant γ if it satisfies the Lipschitz condition with constant 1/γ; see, e.g. [23, Chapter I, Lemma 6.7]. It follows that the mapping G is then single-valued even if this is not the case for the mapping \( u \mapsto X(u) \). The assertion of Proposition 3.5 holds true if we replace the monotonicity of the bi-function Φ with the more general monotonicity property of the mapping \( F \) defined by taking the sub-differential in \( y \) for \( \Phi(x, y) \), i.e.

\[F(x) = \frac{\partial \Phi(x, y)}{\partial y} \bigg|_{y=x} ; \quad (21)\]

and following the lines of Proposition 7.2 in [13]. Proposition 3.5 shows that VI (17) admits more efficient solution methods in comparison with VI (9); see, e.g. [13,23].

We observe that the equivalent transformations from (11) to (12) and (17)–(18) are crucial for the decomposable penalty method. The solution concept based on (17)–(18) has a rather simple and natural interpretation. A system regulator chooses first the penalty parameter \( \tau \). Afterwards, he/she determines the right share allocation vector \( u(\tau) \) by sending some trial vectors \( u \) to players and announcing particular deviation penalty functions, the players then make proper corrections of their utility functions and determine the corresponding Nash equilibrium point for each trial vector. Then the system regulator changes the penalty parameter etc. Application of the other known iterative solution methods to the above generalized game problems with joint constraints was analysed in [13]. They include in particular the usual non-decomposable penalty methods. It appeared that implementation of these mechanisms within a non-cooperative game framework may meet serious difficulties; see [13] for more details.
4. Convergence of the penalty method

In this section, we intend to substantiate the penalty method with the auxiliary problem (11) (or (12)). First we take the following coercivity condition for EP (11).

(C4) There exist a lower semicontinuous and convex function $\eta : X \times U \to \mathbb{R}$, which is weakly coercive with respect to the set $X \times U$, and a number $r$, such that, for any point $w = (x, u) \in X \times U$ such that $\eta(w) > r$ there is a point $w' = (z, v) \in X \times U$ with

$$\eta(w') \leq \eta(w) \quad \text{and} \quad \Phi_\tau(w, w') < 0.$$  

We note that (C4) is a clear adjustment of condition (C2). For brevity, we define the level set

$$E_r = \{w = (x, u) \in X \times U \mid \eta(w) \leq r\}.$$

Lemma 4.1: Let the conditions in (A1) and (C4) be fulfilled for some $\tau > 0$. Then problem (11) has a solution, and all the solutions are contained in $(X \times U) \cap E_r$.

The assertion follows from Corollary 2.2.

However, condition (C4) is not suitable for verification (cf. (C3)) and we will deduce it from other conditions of form (C2). In general, we follow the approach from [18]. First we consider the case where the set $U$ is bounded.

(C5) There exist a lower semicontinuous and convex function $\mu : X \to \mathbb{R}$, which is weakly coercive with respect to the set $X$, and a number $r$, such that, for any point $x \in X \setminus B_r$ there is a point $z \in D$ with

$$\mu(z) \leq \mu(x) \quad \text{and} \quad \Phi(x, z) < 0.$$  

Theorem 4.2: Suppose that (A1) and (C5) are fulfilled, the set $U$ is bounded, and the sequence $\{\tau_k\}$ satisfies

$$\{\tau_k\} \nearrow +\infty.$$  

Then:

(i) EP (3) has a solution;
(ii) EP (11) has a solution for each $\tau > 0$ and all these solutions belong to $B_r \times U$;
(iii) Each sequence $\{w(\tau_k)\}$ of solutions of EP (11) has limit points, all these limit points belong to $(B_r \cap D) \times U$, and all the limit points of $\{x(\tau_k)\}$ are solutions of EP (3).

Proof: We first show that, for any $\tau > 0$, (C4) is true with $\eta(w) = \mu(x)$. Take any $w = (x, u) \in (X \times U) \setminus E_r$, then by (C5) there is $z \in D$ such that $\mu(z) \leq
\[ \Phi_{\tau}(w, w') = \Phi(x, z) + \tau [P(w') - P(w)] \leq \Phi(x, z) - \tau P(w) < 0. \]

Hence, assertion (ii) follows from Lemma 4.1.

For brevity, we set \( w^k = w(\tau_k) = (x^k, u^k) \), where \( x^k = x(\tau_k) \) and \( u^k = u(\tau_k) \). By (ii), the sequence \( \{w^k\} \) exists and is bounded. Therefore, it has limit points. Since \( B_r \) is convex and closed, all these limit points must belong to \( B_r \times U \). Let \( \bar{w} = (\bar{x}, \bar{u}) \) be an arbitrary limit point of \( \{w^k\} \), i.e. \( \{w^k\} \to \bar{w} \). Then, by definition,

\[ 0 \leq P(w^k) \leq \limsup_{s \to +\infty} \tau_{k_s}^{-1} \Phi(x^{k_s}, x) + P(x, u), \quad \forall (x, u) \in X \times U. \]

Taking \( (x, u) \in D \times U \) such that \( P(x, u) = 0 \) and using (22), we obtain

\[ 0 \leq P(\bar{w}) \leq \liminf_{s \to \infty} P(w^k) \leq \limsup_{s \to +\infty} \left[ \tau_{k_s}^{-1} \Phi(x^{k_s}, x) \right] \leq 0, \]

i.e. \( P(\bar{w}) = 0 \) and \( \bar{x} \in D \). Next, for each \( x \in D \) we can take \( u \in U \) such that \( P(w) = P(x, u) = 0 \) and obtain

\[ \Phi(x^{k_s}, x) - \tau_{k_s} P(w^{k_s}) = \Phi(x^{k_s}, x) + \tau_{k_s} [P(w) - P(w^{k_s})] \geq 0. \]

It now follows that

\[ \Phi(\bar{x}, x) \geq \limsup_{s \to +\infty} \Phi(x^{k_s}, x) \geq \limsup_{s \to +\infty} \left[ \tau_{k_s} P(w^{k_s}) \right] \geq 0. \]

Therefore \( \bar{x} \) solves EP (3) and assertion (iii) is true. Since \( \bar{x} \) exists, assertion (i) is also true. The proof is complete.

We now give similar properties in the unbounded case.

**Theorem 4.3:** Suppose that (A1) and (C5) are fulfilled, and the sequence \( \{\tau_k\} \) satisfies condition (22). Then:

(i) EP (3) has a solution;
(ii) There exists \( \tau' > 0 \) such that EP (11) has a solution for each \( \tau > \tau' \) and all these solutions belong to \( E_r \), where \( \eta(w) = \max \{\mu(x), P(w)\} \);
(iii) Each sequence \( \{w(\tau_k)\} \) of solutions of EP (11) has limit points, all these limit points belong to \( (B_r \cap D) \times U \), and all the limit points of \( \{x(\tau_k)\} \) are solutions of EP (3).

**Proof:** We first show that the function \( \eta(w) = \max \{\mu(x), P(w)\} \) is weakly coercive with respect to the set \( X \times U \). Fix a number \( r \) such that the set \( E_r \) is non-empty. Without loss of generality we can suppose that \( r > 0 \). If \( E_r \) is unbounded, there exists a sequence \( \{w^k\} \) such that \( w^k = (x^k, u^k) \in E_r \) and \( \|w^k\| \to \infty \) as \( k \to \infty \)
Since the function $\mu(x)$ is weakly coercive with respect to the set $X$, we have $\|x^k\| \leq C$ for some $C < \infty$. It follows that $\|u^k\| \to \infty$. Since $u \in U$, there exists at least one pair of indices $j$ and $t$ such that $u_{jt}^k \to -\infty$ for the corresponding subsequence $(u^k_j)$, and hence $P_j(x_{jt}^k, u_{jt}^k) \to +\infty$, which is a contradiction.

We now show that there exists $\tau' > 0$ such that $(C4)$ is true for any $\tau > \tau'$. Take any $w = (x, u) \in (X \times U) \setminus E_r$, then $\eta(w) > r$. If $\mu(x) > r$, then by $(C5)$ there is $z \in D$ such that $\mu(z) \leq \mu(x)$ and $\Phi(x, z) < 0$. Since $z \in D$, there exists $v \in U$ with $w' = (z, v)$ such that $P(w') = P(z, v) = 0$ and we have

$$\Phi_r(w, w') = \Phi(x, z) + \tau[P(w') - P(w)] \leq \Phi(x, z) - \tau P(w) < 0.$$ 

Hence, $(C4)$ holds. If $\mu(x) \leq r$, then $P(w) > r$. Fix a point $z \in D$, then there exists $v \in U$ such that $P(w') = P(z, v) = 0$. Since the set $B_r$ is bounded, we have

$$\max_{x \in B_r} \Phi(x, z) = d < \infty.$$ 

Take any $\tau' = d/r$, then

$$\Phi_r(w, w') = \Phi(x, z) + \tau[P(w') - P(w)] \leq d - \tau r < 0$$

if $\tau > \tau'$. Hence, $(C4)$ also holds and assertion (ii) follows from Lemma 4.1. Assertions (iii) and (i) are proved as in Theorem 4.2.

It should be noticed that coercivity condition $(C5)$ is rather weak, detailed discussion and comparison of such conditions can be found in [18,24]. Here we only notice that it follows from the following standard coercivity condition.

$(C3')$ There exists a point $\tilde{x} \in D$ such that

$$\Phi(x^k, \tilde{x}) \to -\infty \quad \text{as} \quad \|x^k - \tilde{x}\| \to \infty;$$

for any infinite sequence $\{x^k\} \subset X$.

In fact, it is sufficient to take $\mu(x) = \|x - \tilde{x}\|$ (cf. $(C3)$).

Next, although each auxiliary penalized EP (11) is equivalent to the two-level problem involving the usual Nash equilibrium problem at the lower level, it seems significant to indicate that Theorems 4.2 and 4.3 establish convergence of solutions of these penalized EPs just to normalized equilibrium points.

5. An illustrative example

In order to compare our approach with the usual penalty method we now give an illustrative example of applications of a non-cooperative game with joint constraints, which represents a multi-product oligopoly market with common treatment of industrial wastes and follows the lines of those in [25, Chapter II] and [16].
Consider a system of \( l \) industrial firms which produce \( s \) commodities independently, but have common plants for treatment of their wastes containing \( m \) polluted substances. Let \( x_i = (x_{i1}, \ldots, x_{is})^\top \) be the output vector of the \( i \)-th firm for some fixed time period. Next, the \( i \)-th firm needs \( t \) factors of production, with the particular endowment vector \( d_i \in \mathbb{R}^t \) and the \( s \times t \) input-output rate matrix \( B_i \), so that its production set is defined by

\[
X_i = \{x_i \in \mathbb{R}_+^s \mid B_ix_i \leq d_i\}.
\]

Also, the output \( x_i \) yields the pollution vector \( y_i = (y_{i1}, \ldots, y_{im})^\top = A_ix_i \in \mathbb{R}^m \), where \( A_i \) is an \( m \times s \) matrix and the total pollution volumes must be bounded above by the fixed vector \( b \in \mathbb{R}^m \). Then, we can define the common feasible set \( D = X \cap Y \) from (1) where

\[
Y = \left\{ x = (x_i)_{i=1}^l \in \mathbb{R}^{ls} \left| \sum_{i=1}^l A_ix_i \leq b \right. \right\},
\]

and \( n = ls \). Given an output \( x_i \), the \( i \)-th firm receives the income value

\[
\left\langle x_i, H \left( \sum_{j=1}^l x_j \right) \right\rangle,
\]

where \( H : \mathbb{R}^s \to \mathbb{R}^s \) is the price mapping whose values depend on the total output volumes. Also, it has the production cost \( \mu_{i1}(x_i) \) and the treatment cost \( \mu_{i2}(x_i) \). Then the profit function of the \( i \)-th firm is defined by

\[
f_i(x) = \left\langle x_i, H \left( \sum_{j=1}^l x_j \right) \right\rangle - \mu_{i1}(x_i) - \mu_{i2}(x_i),
\]

for \( i = 1, \ldots, l \). It follows that the model can be formulated as the generalized non-cooperative game (2). For the sake of simplicity, we consider the affine case, i.e. suppose that all the functions \( \mu_{i1} \) and \( \mu_{i2} \) and the price mapping \( H \) are affine. Then, \( \mu_{i1}(x_i) = \langle c'_i, x_i \rangle + \alpha'_i, \mu_{i2}(x_i) = \langle c''_i, A_ix_i \rangle + \alpha''_i \) for \( i = 1, \ldots, l \), and \( H(v) = q - Qv \), where \( Q \) is a \( s \times s \) matrix and \( q \) is a fixed vector in \( \mathbb{R}^s \). It follows from (23) that

\[
f_i(x) = \left\langle x_i, q - c_i - Q \left( \sum_{j=1}^l x_j \right) \right\rangle - \alpha_i,
\]

\[
c_i = c'_i + A_i^\top c''_i, \quad \alpha_i = \alpha'_i + \alpha''_i,
\]
for \( i = 1, \ldots, l \). We also take the normalized EP (3) where the bi-function \( \Phi \) is defined as follows:

\[
\Phi(x, y) = \sum_{i=1}^{l} (q - c_i, x_i - y_i)
+ \sum_{i=1}^{l} \left( y_i, Q \left( \sum_{j \neq i} x_j + y_i \right) \right) - \sum_{i=1}^{l} \left( x_i, Q \left( \sum_{j=1}^{l} x_j \right) \right).
\]

(24)

The streamlined application of the penalty method will consist in imposing penalties instead of the explicit common constraints of the set \( Y \). Take the custom quadratic penalty function

\[
\tilde{P}(x) = 0.5 \left\| \sum_{i=1}^{l} A_i x_i - b \right\|_+^2.
\]

Then, given a number \( \tau > 0 \), we consider the problem of finding a point \( x(\tau) \in X \) such that

\[
\Phi(x(\tau), x) + \tau [\tilde{P}(x) - \tilde{P}(x(\tau))] \geq 0 \quad \forall \ x \in X.
\]

However, this EP cannot be decomposed into a NEP due to the non-separability of the penalty function. If we modify each utility function as

\[
\tilde{f}_i(x) = f_i(x) - \tau \tilde{P}(x)
\]

and remove the set \( Y \) in (2), we obtain the usual NEP (see [6]), but then each firm will have additional charges after any violation of the common constraints regardless of individual contributions, which does not seem fair.

We now describe the application of the proposed decomposable penalty method. The basic penalized auxiliary problem (11) is re-written as follows: Find a pair \( w(\tau) = (x(\tau), u(\tau)) \in X \times U, \tau > 0 \) such that

\[
\Phi(x(\tau), x) + 0.5 \tau \sum_{i=1}^{l} \left\{ \left\| A_i x_i - u_i \right\|_+^2 - \left\| A_i x_i(\tau) - u_i(\tau) \right\|_+^2 \right\} \geq 0
\]

\(
\forall w = (x, u) \in X \times U,
\)

(25)

where we also take the custom quadratic penalty function for the set

\[
W_i = \left\{ w_i = (x_i, u_i) \in \mathbb{R}^s \times \mathbb{R}^m \mid A_i x_i \leq u_i \right\},
\]

for \( i = 1, \ldots, l \), and the set \( U \) is defined in (10). Problem (25) still involves coupled variables as indicated in Section 3. We replace it with the equivalent system (13)–(14) that now takes the form: Find a pair \( w(\tau) = (x(\tau), u(\tau)) \in X \times U \)
such that
\[
\Phi(x(\tau), x) + \tau \sum_{i=1}^{l} \langle [A_i x_i(\tau) - u_i(\tau)]_+, A_i(x_i - x_i(\tau)) \rangle \geq 0 \quad \forall x \in X,
\]
\[
\sum_{i=1}^{l} \langle [A_i x_i(\tau) - u_i(\tau)]_+, u_i(\tau) - u_i \rangle \geq 0 \quad \forall u \in U.
\]

This system reduces to the two-level decomposable problem. As indicated in Section 3, we can solve VI (17) in \( u \), whose mapping is defined from a solution of EP (15) that now takes the form: Find \( x(u) \in X \) such that
\[
\Phi(x(u), y) + \tau \sum_{i=1}^{l} \langle [A_i x_i(u) - u_i], A_i(y_i - x_i(u)) \rangle \geq 0 \quad \forall y \in X, \quad (26)
\]
at some given \( u \in U \). That is, if \( X(u) \) denotes the solution set of this problem, then for each \( x(u) \in X(u) \) we set
\[
g(u) = (g_1(u), \ldots, g_l(u))^T, \quad \text{where } g_i(u) = -[A_i x_i(u) - u_i]_+, \quad i = 1, \ldots, l,
\]
and define the mapping value
\[
G(u) = \{ g(u) | x(u) \in X(u) \}
\]
for the upper level VI (17). However, EP (26) is equivalent to the NEP: Find \( x(u) \in X \) such that
\[
\tilde{f}_i(x_{-i}(u), y_i) \leq \tilde{f}_i(x(u)), \quad \forall y_i \in X_i, \quad i = 1, \ldots, l;
\]
where the \( i \)-th player has the penalized utility function
\[
\tilde{f}_i(x) = f_i(x) - 0.5 \tau \| [A_i x_i - u_i]_+ \|^2, \quad i = 1, \ldots, l.
\]
In such a way we obtain a completely decomposable flexible procedure corresponding to the custom non-cooperative game framework.

In order to check the desired properties of penalized problems, we can replace the bi-function \( \Phi \) from (24) in these problems with \( \langle F(x), y - x \rangle \), where the mapping \( F \) is defined in (21) and now takes the form
\[
F(x) = (F_1(x), \ldots, F_l(x))^T,
\]
where
\[
F_i(x) = c_i - q + Q \left( \sum_{j=1}^{l} x_j \right) + Q^T x_i, \quad i = 1, \ldots, l;
\]
as suggested in Section 3.
For instance, we now evaluate the strengthened monotonicity property of the mapping $G$ for VI (17). It is natural to suppose that the matrix $Q$ in the price mapping $H$ is positive semi-definite, but non-symmetric in general. The Jacobian of $F$ is fixed, i.e. $\nabla F(x) = S$ where

$$S = \begin{pmatrix}
    Q + Q^T & Q & \cdots & Q \\
    Q & Q + Q^T & \cdots & Q \\
    \vdots & \vdots & \ddots & \vdots \\
    Q & Q & \cdots & Q + Q^T
\end{pmatrix}.$$

Since

$$(S + S^T) = \begin{pmatrix}
    2 & 1 & \cdots & 1 \\
    1 & 2 & \cdots & 1 \\
    \vdots & \vdots & \ddots & \vdots \\
    1 & 1 & \cdots & 2
\end{pmatrix} \otimes (Q + Q^T),$$

where $\otimes$ denotes the Kronecker product of matrices, the eigenvalues of $(S + S^T)$ coincide with all the products of the eigenvalues of both the factors; see Lemma 4.1.1 in [25]. Hence the matrix $S$ is positive semi-definite and $F$ is monotone. It follows from Proposition 3.5 that the mapping $G$ is then single-valued and co-coercive, as desired. These properties simplify essentially the solution of the upper level VI (17).

6. Conclusions

In this paper, a decomposable penalty method was applied to generalized non-cooperative games with joint constraints. It is based on right-hand side decomposition techniques. Since the streamlined penalty method with shares allocation leads to an equilibrium problem with coupled variables, we proposed its additional equivalent transformation to a completely decomposable problem. This approach enabled us to replace the initial problem with a sequence of the usual penalized Nash equilibrium problems together with an upper level variational inequality. Convergence of solutions of these auxiliary problems to a solution of the initial game problem was established under weak coercivity conditions.

Acknowledgments

The results of this work were obtained within the state assignment of the Ministry of Science and Education of Russia, project No. 1.460.2016/1.4. In this work, the author was also supported by Russian Foundation for Basic Research, project No. 19-01-00431. The author is grateful to referees for their valuable comments.

Disclosure statement

No potential conflict of interest was reported by the author(s).
Funding

The results of this work were obtained within the state assignment of the Ministry of Science and Education of Russia, project No. 1.460.2016/1.4. In this work, the author was also supported by Russian Foundation for Basic Research, project No. 19-01-00431.

References

[1] Debreu G. A social equilibrium existence theorem. Proc Nat Acad Sci USA. 1952;38:886–893.
[2] Rosen JB. Existence and uniqueness of equilibrium points for concave n-person games. Econometrica. 1965;33:520–534.
[3] Zukhovitskii SI, Polyak RA, Primak ME. Two methods of search for equilibrium points of n-person concave games. Soviet Mathem Doklady. 1969;10:279–282.
[4] Zukhovitskii SI, Polyak RA, Primak ME. Concave many person games. Ekon Matem Metody. 1971;7:888–900. (in Russian).
[5] Ichiishi T. Game theory for economic analysis. New York: Academic Press; 1983.
[6] Krawczyk JB, Uryasev S. Relaxation algorithms to find Nash equilibria with economic applications. Environ Model Assess. 2000;5:63–73.
[7] Facchinei F, Kanzow C. Generalized Nash equilibrium problems. 4OR. 2007;5:173–210.
[8] Fukushima M. Restricted generalized Nash equilibria and controlled penalty algorithm. Comput Manag Sci. 2011;8:201–218.
[9] Faraci F, Raciti F. On generalized Nash equilibrium in infinite dimension: the Lagrange multipliers approach. Optimization. 2015;64:321–338.
[10] Mastroeni G, Pappalardo M, Raciti F. Generalized Nash equilibrium problems and variational inequalities in Lebesgue spaces. Minimax Theory Appl. 2020;5:47–64.
[11] Contraseras J, Klusch M, Krawczyk JB. Numerical solutions to Nash-Cournot equilibria in coupled constraint electricity markets. IEEE Trans Power Syst. 2004;19:195–206.
[12] Pang J-S, Scutari G, Facchinei F, et al. Distributed power allocation with rate constraints in Gaussian parallel interference channels. IEEE Trans Inform Theory. 2008;54:3471–3489.
[13] Konnov IV. Shares allocation methods for generalized game problems with joint constraints. Set-Valued Variat Anal. 2016;24:499–516.
[14] Harker PT. Generalized Nash games and quasivariational inequalities. Eur J Oper Res. 1991;54:81–94.
[15] Konnov IV. Right-hand side decomposition for variational inequalities. J Optim Theory Appl. 2014;160:221–238.
[16] Allevi E, Gnudi A, Konnov IV, et al. Decomposition method for oligopolistic competitive models with common environmental regulation. Ann Oper Res. 2018;268:441–467.
[17] Nikaido H, Isoda K. Note on noncooperative convex games. Pacific J Math. 1955;5:807–815.
[18] Konnov IV. Regularized penalty method for general equilibrium problems in Banach spaces. J Optim Theory Appl. 2015;164:500–513.
[19] Kornai J, Liptak T. Two-level planning. Econometrica. 1965;33:141–169.
[20] Razumikhin BS. Iterative method for the solution and decomposition of linear programming problems. Autom Remote Control. 1967;29:427–443.
[21] Umno AE. The method of penalty functions in problems of large dimension. USSR Comp Maths Math Phys. 1975;15:32–45.
[22] Konnov IV. An approximate penalty method with descent for convex optimization problems. Russ Mathem (Iz. VUZ). 2019;63(7):41–55.
[23] Gol'shtein EG, Tret'yakov NV. Augmented Lagrange functions. Moscow; Nauka: 1989. (Engl. translation in John Wiley and Sons, New York, 1996).

[24] Konnov IV, Dyabilkin DA. Nonmonotone equilibrium problems: coercivity conditions and weak regularization. J Glob Optim. 2011;49:575–587.

[25] Okuguchi K, Szidarovszky F. The theory of oligopoly with multi-product firms. Berlin: Springer-Verlag; 1990.