On the $L^p$-estimates for Beurling-Ahlfors and Riesz transforms on Riemannian manifolds

Xiang-Dong Li*

Academy of Mathematics and Systems Science, Chinese Academy of Sciences
55, Zhongguancun East Road, Beijing, 100190, P. R. China
E-mail: xdli@amt.ac.cn

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Abstract

In our previous papers [6, 9], we proved some martingale transform representation formulas for the Riesz transforms and the Beurling-Ahlfors transforms on complete Riemannian manifolds, and proved some explicit $L^p$-norm estimates for these operators on complete Riemannian manifolds with suitable curvature conditions. In this paper we correct a gap contained in [6, 9] and prove that the $L^p$-norm of the Riesz transforms $R_a(L) = \nabla(a - L)^{-1/2}$ can be explicitly bounded by $C(p^* - 1)^{3/2}$ if $Ric + \nabla^2 \phi \geq -a$ for $a \geq 0$, and the $L^p$-norm of the Riesz transform $R_0(L) = \nabla(-L)^{-1/2}$ is bounded by $2(p^* - 1)$ if $Ric + \nabla^2 \phi = 0$. We also prove that the $L^p$-norm estimates for the Beurling-Ahlfors transforms obtained in [9] remain valid. Moreover, we prove the time reversal martingale transform representation formulas for the Riesz transforms and the Beurling-Ahlfors transforms on complete Riemannian manifolds.

1 Introduction

In our previous paper [6], the author obtained a martingale transform representation formula for the Riesz transforms on complete Riemannian manifolds. More precisely, by the formula (24) in Theorem 3.2 in [6], the probabilistic representation formula of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ acting on a nice function $f$ was given by

$$\frac{-1}{2} R_a(L)f(x) = \lim_{y \to +\infty} E_y \left[ \int_0^\tau e^{a(s - \tau)} M_{s} M_{s}^{-1} dQ_a f(X_s, B_s) dB_s \bigg| X_{\tau} = x \right].$$

Recently, R. Bañuelos and F. Baudoin [2] pointed out that, since $e^{-a\tau} M_{\tau}$ is not adapted to the filtration $\mathcal{F}_t = \sigma(X_s, B_s, s \leq t)$, the above probabilistic representation formula should

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be corrected as follows

\[-\frac{1}{2}R_\phi(L)f(x) = \lim_{y \to +\infty} E_y \left[ e^{-\phi \tau} M_\tau \int_0^\tau e^{\phi s} M_s^{-1} dQ_s f(X_s, B_s) dB_s \bigg| X_\tau = x \right]. \quad (1)\]

Indeed, a careful check of the original proof of the formula (24) in Theorem 3.8 in [6] indicates that the correct probabilistic representation formula of $R_\phi(L)f$ should be given by (1). See Section 2 below. By the above observation, R. Bañuelos and F. Baudoin [2] pointed out that there is a gap in the proof of the $L^p$-norm estimates of the Riesz transforms in [6] and they proved a new martingale inequality which can be used to correct this gap. In this paper, we correct the above gap and prove that the $L^p$-norm of the Riesz transform $R_\phi(L)$ is bounded above by $C(p^*-1)^{3/2}$ if $\text{Ric} + \nabla^2 \phi \geq -a$ for $a \geq 0$, and the $L^p$-norm of the Riesz transform $R_0(L)$ is bounded by $2(p^*-1)$ if $\text{Ric} + \nabla^2 \phi = 0$. See Theorem 2.4 below. We also correct the gap contained in [9] (due to the same reason as above) and prove that the main results on the $L^p$-norm estimates of the Beurling-Ahlfors transforms obtained in [9] remain valid. See Theorem 4.4 and Remark 4.5 below. Moreover, we prove the time reversal martingale transform representation formulas for the Riesz transforms and the Beurling-Ahlfors transforms on complete Riemannian manifolds.

## 2 Riesz transforms on functions

Let $(M, g)$ be a complete Riemannian manifold, $\nabla$ the gradient operator on $M$, $\Delta$ the Laplace-Beltrami operator on $M$. Let $\phi \in C^2(M)$, and $d\mu = e^{-\phi} dv$, where $dv$ is the standard Riemannian volume measure on $M$. Let $L^2_0(M, \mu) = L^2(M, \mu)$ if $\mu(M) = \infty$, and $L^2_0(M, \mu) = \{ f \in L^2(M, \mu) : f d\mu = 0 \}$ if $\mu(M) < \infty$.

Let $L = \Delta - \nabla \phi \cdot \nabla$. Let $d$ be the exterior differential operator, $d_\phi^*$ be its $L^2$-adjoint with respect to the weighted volume measure $d\mu = e^{-\phi} dv$. Let $\square_\phi = dd_\phi^* + d_\phi^* d$ be the Witten-Laplacian acting on forms over $(M, g)$ with respect to the weighted volume measure $d\mu = e^{-\phi} dv$.

Let $B_t$ be one dimensional Brownian motion on $\mathbb{R}$ starting from $B_0 = y > 0$ and with infinitesimal generator $\frac{d^2}{2 dy^2}$. Let

$$\tau = \inf \{ t > 0 : B_t = 0 \}.$$ 

Let $X_t$ be the $L$-diffusion process on $M$. Let $\text{Ric}$ be the Ricci curvature on $(M, g)$, $\nabla^2 \phi$ be the Hessian of the potential function $\phi$. Let $M_t \in \text{End}(T_{X_t}M, T_{X_t}M)$ is the unique solution to the covariant SDE along the trajectory of $(X_t)$:

$$\frac{\partial}{\partial t} M_t = -(\text{Ric} + \nabla^2 \phi)(X_t) M_t, \quad M_0 = \text{Id}_{T_{X_0}M}.$$ 

In particular, in the case where $\text{Ric} + \nabla^2 \phi = -a$, we have

$$M_t = e^{at} U_t, \quad \forall t \geq 0,$$

where $U_t : T_{X_0}M \to T_{X_t}M$ denotes the stochastic parallel transport along $X_t$.

The following result is the correct reformulation of Lemma 3.7 in [6].
Lemma 2.1 For all $\eta \in C_0^\infty(M, \Lambda^1 T^* M)$, and $\eta_a(x, y) = e^{-y\sqrt{\alpha + \frac{1}{2}}} \eta(x)$, we have
\[
\eta(X_\tau) = e^{\alpha \tau} M^*_{\tau, k} \eta_a(X_0, B_0) + e^{\alpha \tau} M^*_{\tau, k} \int_0^\tau e^{-\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) \cdot (U_s dW_s, dB_s). \tag{2}
\]
Proof. By Itô's calculus, we have (see p.266 line 16 in [6])
\[
\frac{\partial}{\partial t} \left( e^{-\alpha t} M^*_t \eta_a(X_t, B_t) \right) = e^{-\alpha t} M^*_t \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_t, B_t) \cdot (U_t dW_t, dB_t).
\]
Integrating from $t = 0$ to $t = \tau$, we complete the proof of Lemma 2.1.

The following result is the correct reformulation of Theorem 3.8 in [6].

Theorem 2.2 Let $\omega \in C_0^\infty(M < \Lambda^1 T^* M)$, and $\omega_a(x, y) = e^{-y\sqrt{\alpha + \frac{1}{2}}} \omega(x)$. Then
\[
\frac{1}{2} \omega(x) = \lim_{y \to \infty} E_y \left[ e^{-\alpha \tau} M^*_\tau \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s \bigg| X_\tau = x \right]. \tag{3}
\]
Proof. The proof is indeed a small modification of the original proof of Theorem 3.8 given in [6]. For the completeness of the paper, we produce the details here. Let $Z_t = (X_t, B_t)$, $\eta \in C_0^\infty(\Lambda^k T^* M)$. By (2) in Lemma 2.1, we have
\[
\eta(X_\tau) = e^{\alpha \tau} M^*_\tau \eta_a(Z_0) + e^{\alpha \tau} M^*_\tau \int_0^\tau e^{-\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(Z_s) \cdot (U_s dW_s, dB_s).
\]
Hence
\[
\int_M \left\langle E_y \left[ e^{-\alpha \tau} M^*_\tau \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s \bigg| X_\tau = x \right], \eta(x) \right\rangle d\mu(x)
\]
\[
= E_y \left[ e^{-\alpha \tau} M^*_\tau \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s, \eta(X_\tau) \right]
\]
\[
= I_1 + I_2,
\]
where
\[
I_1 = E_y \left[ \left\langle e^{-\alpha \tau} M^*_\tau \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s \right|, e^{\alpha \tau} M^*_{\tau, k} \eta_a(X_0, B_0) \right],
\]
\[
I_2 = E_y \left[ \left\langle e^{-\alpha \tau} M^*_\tau \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s, e^{\alpha \tau} M^*_{\tau, k} \eta_a(X_0, B_0) \right|, e^{\alpha \tau} M^*_{\tau, k} \eta_a(X_0, B_0) \right].
\]
Using the martingale property of the Itô integral, we have
\[
I_1 = E_y \left[ \left\langle \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s, \eta_a(X_0, B_0) \right| \right]
\]
\[
= E_y \left[ E \left[ \int_0^\tau e^{\alpha s} M^*_s \left( \nabla, \frac{\partial}{\partial y} \right) \eta_a(X_s, B_s) dB_s \bigg| (X_0, B_0) \right], \eta_a(X_0, B_0) \right]
\]
\[
= 0.
\]
On the other hand, using the $L^2$-isometry of the Itô integral, we have

\[
I_2 = E_y \left[ \left\langle \int_0^\tau e^{as}M_s^{-1} \frac{\partial}{\partial y} \omega(a, B_s) dB_s, \int_0^\tau e^{-as}M_s \left( \nabla, \frac{\partial}{\partial y} \eta(a, B_s) \cdot (U_s dW_s, dB_s) \right) \right\rangle \right]
\]

\[
= E_y \left[ \int_0^\tau \left\langle e^{as}M_s^{-1} \frac{\partial}{\partial y} \omega(a, B_s), e^{-as}M_s \frac{\partial}{\partial y} \eta(a, B_s) \right\rangle ds \right]
\]

\[
= E_y \left[ \int_0^\tau \left\langle \frac{\partial}{\partial y} \omega(a, B_s), \frac{\partial}{\partial y} \eta(a, B_s) \right\rangle ds \right].
\]

The Green function of the background radiation process is given by $2(y \wedge z)$. Thus

\[
E_y \left[ \int_0^\tau \left\langle \frac{\partial}{\partial y} \omega(a, B_s), \frac{\partial}{\partial y} \eta(a, B_s) \right\rangle ds \right] = 2 \int_M \int_0^\infty (y \wedge z) \left\langle \frac{\partial}{\partial z} \omega(a, x, z), \frac{\partial}{\partial z} \eta(a, x, z) \right\rangle d\mu(x).
\]

By spectral decomposition, we have the Littlewood-Paley identity

\[
\lim_{y \to \infty} \int_M \int_0^\infty (y \wedge z) \left\langle \frac{\partial}{\partial z} \omega(a, x, z), \frac{\partial}{\partial z} \eta(a, x, z) \right\rangle d\mu(x) = \int_M \langle \omega(x), \eta(x) \rangle d\mu(x).
\]

Thus

\[
\langle \omega, \eta \rangle_{L^2(\mu)} = 2 \lim_{y \to \infty} \int_M \left\langle E_y \left[ e^{-as}M_s \int_0^\tau e^{as}M_s^{-1} \frac{\partial}{\partial y} \omega(a, B_s) dB_s \bigg| X_\tau = x \right] \eta(a, x, z) \right\rangle d\mu(x).
\]

This completes the proof of Theorem 2.2.

The following martingale transform representation formula of the Riesz transforms on complete Riemannian manifolds, which is the extension of the Gundy-Varopoulos representation formula of the Riesz transforms on Euclidean space [5], is the correct reformulation of the one that we obtained in Theorem 3.2 in [6].

**Theorem 2.3** Let $R_\alpha(L) = \nabla (a - L)^{-1/2}$. Then, for all $f \in C^\infty_0(M)$, we have

\[
R_\alpha(L)f(x) = -2 \lim_{y \to +\infty} E_y \left[ e^{-as}M_s \int_0^\tau e^{as}M_s^{-1} \frac{\partial}{\partial y} \omega(a, B_s) dB_s \bigg| X_\tau = x \right].
\]

In particular, in the case where $\text{Ric} + \nabla^2 \phi = -a$, we have

\[
R_\alpha(L)f(x) = -2 \lim_{y \to +\infty} E_y \left[ U_s \int_0^\tau U_s^{-1} dQ_s f(X_s, B_s) dB_s \bigg| X_\tau = x \right].
\]

**Proof.** Applying Theorem 2.2 to $\omega = d(a - L)^{-1/2} f$, the proof of Theorem 2.3 is as the same as the one of Theorem 3.2 given in [6]. $\Box$

We now state the $L^p$-norm estimates of the Riesz transforms on complete Riemannian manifolds. Throughout this paper, for any $p \in (1, \infty)$, let

\[
p^* = \max \left\{ p, \frac{p}{p - 1} \right\}.
\]

The following result is a correction of Theorem 1.4 in [6].
Theorem 2.4 Let $M$ be a complete Riemannian manifold, and $\phi \in C^2(M)$. Then

(i) for all $f \in C^\infty_0(M)$,

\[ \|\nabla(a - L)^{-1/2}f\|_2 \leq \|f\|_2, \]

(ii) if $\text{Ric} + \nabla^2\phi \equiv 0$, then for all $p \in (1, \infty)$,

\[ \|\nabla(-L)^{-1/2}f\|_p \leq 2(p^* - 1)\|f\|_p, \quad \forall f \in C^\infty_0(M), \]

if $\text{Ric} + \nabla^2\phi \equiv -a$, where $a > 0$ is a constant, then for all $p \in (1, \infty)$,

\[ \|\nabla(a - L)^{-1/2}f\|_p \leq 2(p^* - 1)(1 + 4\|T_1\|_p)\|f\|_p, \quad \forall f \in C^\infty_0(M), \]

where $T_1$ is the first exiting time of the standard $3$-dimensional Brownian motion from the unit ball $B(0,1) = \{ x \in \mathbb{R}^3 : \|x\| = 1 \}$.

(iii) if $\text{Ric} + \nabla^2\phi \geq -a$, where $a \geq 0$ is a constant, then there is a numerical constant $C > 0$ such that for all $p > 1$,

\[ \|\nabla(a - L)^{-1/2}f\|_p \leq C(p^* - 1)^{3/2}\|f\|_p, \quad \forall f \in C^\infty_0(M). \]

Proof. The case (i) for $p = 2$ is well known, cf. [6, 7]. By [6], for any fixed $x \in M$, there exists a bounded operator $A(x) \in \text{End}(T_xM)$ such that $\text{d} \omega(x) = A\nabla \omega(x)$ and $\|A(x)\|_{op} \leq 1$. In the case $\text{Ric} + \nabla^2\phi = -a$, we have

\[ \nabla(a - L)^{-1/2}f(x) = -2 \lim_{y \to +\infty} E_y \left[ U_x \int_0^\tau U_s^{-1} A\nabla Q_\alpha f(X_s, B_s)dB_s \bigg| X_\tau = x \right]. \]

The stochastic integral in the above formula is a subordination of martingale transforms. By Burkholder’s sharp $L^p$-inequality for martingale transforms [3] we obtain

\[ \|\nabla(a - L)^{-1/2}f\|_p \leq \sup_{s \in [0,\tau]} \|A(x_s)\|_{op} \left\| \int_0^\tau (\nabla, \partial_y)Q_\alpha(f)(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p, \]

where $\|A(x_s)\|_{op}$ denotes the operator norm of $A(x_s)$ on $T_{X_s}M$. Note that

\[ \sup_{s \in [0,\tau]} \|A(x_s)\|_{op} \leq 1. \]

This yields

\[ \|\nabla(a - L)^{-1/2}f\|_p \leq 2(p^* - 1) \left\| \int_0^\tau (\nabla, \partial_y)Q_\alpha(f)(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p, \]

In [6], we have proved that, for all $1 < p < \infty$, it holds

\[ \left\| \int_0^\tau (\nabla, \partial_y)Q_\alpha(f)(X_s, B_s) \cdot (U_s dW_s, dB_s) \right\|_p \leq (1 + 4\|T_1\|_p 1_{a > 0})\|f\|_p. \]

Combining this with the previous inequality, we obtain

\[ \|\nabla(a - L)^{-1/2}f\|_p \leq 2(p^* - 1)(1 + 4\|T_1\|_{p 1_{a > 0}})\|f\|_p. \]
where for all \( p \),

\[
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\]
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By Theorem 2.6 due to Bañuelos and Baudoin in [2], under the condition

By Proposition 6.2 in our previous paper [7], for all \( p \)

This proves the case of (ii).

In general case \( \text{Ric} + \nabla^2 \phi \geq -a \), we have

\[
\nabla (a - L)^{-1/2} f(x) = 2 \lim_{y \to +\infty} E_y \left[ e^{-a \tau} M_{\tau} \int_0^\tau e^{as} M_s^{-1} A \nabla Q_a f(X_s, B_s) dB_s \mid X_\tau = x \right].
\]

By the \( L^p \)-contractivity of conditional expectation, see [6], we have

\[
\| \nabla (a - L)^{-1/2} f \|_p \leq 2 \lim_{y \to +\infty} \inf \| e^{-a \tau} M_{\tau} \int_0^\tau e^{as} M_s^{-1} A \nabla Q_a f(X_s, B_s) dB_s \|_p.
\]

Let

\[
J_y = \left\{ \int_0^\tau |\nabla Q_a f(X_s, B_s)|^2 \, ds \right\}^{1/2}.
\]

By Theorem 2.6 due to Bañuelos and Baudoin in [2], under the condition \( \text{Ric} + \nabla^2 \phi \geq -a \), we can prove that

\[
\left\| e^{-a \tau} M_{\tau} \int_0^\tau e^{as} M_s^{-1} d Q_a f(X_s, B_s) dB_s \right\|_p \leq 3 \sqrt[p]{2p-1} \| J_y \|_p.
\]

By Proposition 6.2 in our previous paper [7], for all \( p \in (1, \infty) \), we proved that

\[
\| J_y \|_p \leq B_p \| f \|_p,
\]

where for all \( p \in (1, 2) \), \( B_p = (2p)^{1/2}(p - 1)^{-3/2} \), \( B_2 = 1 \), and for all \( p \in (2, \infty) \), \( B_p = \frac{p^2}{\sqrt{2(p-2)}} \). From the above estimates, for all \( p \in (1, 2) \), we can obtain

\[
\| \nabla (a - L)^{-1/2} f \|_p \leq 6 \sqrt[2p]{2p}^{3/2}(2p - 1)^{1/2}(p - 1)^{-3/2} \| f \|_p
\]

\[
\leq 12 \sqrt[6]{6}(p - 1)^{-3/2} \| f \|_p,
\]

and for \( p > 2 \),

\[
\| \nabla (a - L)^{-1/2} f \|_p \leq 3 \sqrt[2p]{2p}^{3/2}(2p - 1)^{1/2}(p - 2)^{-1/2} \| f \|_p
\]

\[
\leq 6(p - 1)^{3/2}(1 + O(1/p)) \| f \|_p.
\]

The proof of Theorem 2.4 is completed.

\[\square\]

**Remark 2.5** The above proof corrects a gap in the proof of Theorem 1.4 given in [6] (p.270 line 9 to line 12 in [6]), where we used the Burkholder sharp \( L^p \)-inequality for martingale transforms. As \( e^{-a \tau} M_{\tau} \) is not adapted with respect to the filtration \( F_s = \sigma(X_u, B_u, u \in [0, s]) \), \( s < \tau \), the proof given in [6] is valid only in the case \( e^{-a \tau} M_{\tau} \) is independent of \( (X_s : s \in [0, \tau]) \), which only happens if \( \text{Ric} + \nabla^2 \phi \equiv -a \) for some constant \( a \geq 0 \).

The following result is the correction of Corollary 1.5 in [6].

**Corollary 2.6** Let \( M \) be a complete Riemannian manifold with non-negative Ricci curvature. Then there exists a numerical constant \( C > 0 \) such that for all \( p > 1 \),

\[
\| \nabla (\Delta)^{-1/2} f \|_p \leq C(p^* - 1)^{3/2} \| f \|_p.
\]

In particular, if \( \text{Ric} = 0 \), i.e., if \( M \) is a Ricci flat Riemannian manifold, then for all \( 1 < p < \infty \),

\[
\| \nabla (\Delta)^{-1/2} f \|_p \leq 2(p^* - 1) \| f \|_p.
\]
In view of Theorem 2.4 and Corollary 2.6, we need to reformulate Conjecture 1.7 in [6] as follows.

**Conjecture 2.7** Let $M$ be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $\text{Ric}(L) = \text{Ric} + \nabla^2 \phi = 0$. Then there exists a constant $c > 0$ such that for all $p > 1$, we have

$$c(p^*-1)(1+o(1)) \leq \|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^*-1).$$

In particular, on any complete Riemannian manifold $M$ with flat Ricci curvature, for all $p > 1$, we have

$$c(p^*-1)(1+o(1)) \leq \|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^*-1).$$

**Remark 2.8** Using the Bellman function technique, Carbonaro and Dragičević [4] proved that if $\text{Ric} + \nabla^2 \phi \geq -a$, then for all $p \in (1,\infty)$,

$$\|\nabla(a-L)^{-1/2}f\|_p \leq 12(p^*-1)f\|_p, \quad \forall f \in C^\infty_0(M).$$

It would be nice if one can find a probabilistic proof of this result.

## 3 Riesz transforms on Gaussian spaces

In this section, we give the proof of Corollary 1.6 in [6]. Let $G$ be a compact Lie group endowed with a bi-invariant Riemannian metric, $\mathcal{G}$ its Lie algebra, and $n = \dim G$. Let $X_1, \ldots, X_n$ be an orthonormal basis of $\mathcal{G}$, and $\Delta_G = \sum_{i=1}^n X_i^2$ the Laplace-Beltrami operator on $G$. In [1], Arcozzi proved that, the $L^p$-norm of the Riesz transform $R^G := \sum_{i=1}^n R_{X_i}X_i$ on $G$ satisfies $\|R^G\|_p \leq 2(p^*-1)$ for all $p \in (1,\infty)$, where $R_{X_i} = X_i(-\Delta_G)^{-1/2}$ is the Riesz transform on $G$ in the direction $X_i$. As the unit sphere $S^{n-1}$ can be identified as $S^{n-1} = SO(n)/SO(n-1)$, where $SO(n)$ is the rotation group of $\mathbb{R}^n$, Arcozzi proved that the $L^p$-norm of the Riesz transform $R^{S^{n-1}} = \nabla S^{n-1}(-\Delta_{S^{n-1}})^{-1/2}$ on $S^{n-1}$ satisfies $\|R^{S^{n-1}}\|_p \leq 2(p^*-1)$ for all $p \in (1,\infty)$. Let $S^{n-1}(\sqrt{n})$ be the $(n-1)$-dimensional sphere of radius $\sqrt{n}$. Then the $L^p$-norm of the Riesz transform $R^{S^{n-1}(\sqrt{n})}$ satisfies $\|R^{S^{n-1}(\sqrt{n})}\|_p \leq 2(p^*-1)$. By the Poincaré limit, as $n \to \infty$, $S^{n-1}(\sqrt{n})$ endowed with the normalized volume measure converges in a proper way to the infinite dimensional Wiener space $\mathbb{R}^N$ endowed with the Wiener measure, and the Laplace-Beltrami operator on $S^{n-1}(\sqrt{n})$ converges to the Ornstein-Uhlenbeck operator on $\mathbb{R}^N$. From this, Arcozzi derived that the Riesz transform associated with the Ornstein-Uhlenbeck operator $L = \Delta - \nabla \cdot \nabla$ on the Wiener space satisfies $\|\nabla(-L)^{-1/2}\|_p \leq 2(p^*-1)$ for all $p \in (1,\infty)$.

In general, let $A \in M(n,\mathbb{R})$ be a positive definite symmetric matrix on $\mathbb{R}^n$, and let $\langle x, y \rangle_A = \langle x, Ay \rangle$, $\forall x, y \in \mathbb{R}^n$. Then $\sqrt{A} : (\mathbb{R}^n, \langle \cdot, \cdot \rangle) \to (\mathbb{R}^n, \langle \cdot, \cdot \rangle_A)$ is an isometry. Let $SO(n, A)$ be the rotation group on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_A)$, and $S^{n-1}_A$ be the $(n-1)$-dimensional sphere in $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_A)$. Then $S^{n-1}_A = SO(n, A)/SO(n-1, A)$. By the same argument as used by Arcozzi [1], we can prove that the $L^p$-norm of the Riesz transform on $SO(n, A)$ satisfies $\|R^{SO(n, A)}\|_p \leq 2(p^*-1)$, and the $L^p$-norm of the Riesz transform on $S^{n-1}_A$ satisfies $\|R^{S^{n-1}_A}\|_p \leq 2(p^*-1)$. Similarly, we have $\|R^{S^{n-1}_A}\|_p \leq 2(p^*-1)$. Thus, we have proved the following
Theorem 3.1 Let $A \in M(n, \mathbb{R})$ be a positive definite symmetric matrix on $\mathbb{R}^n$, and let
\[ L_A = \Delta - Ax \cdot \nabla \]
be the Ornstein-Uhlenbeck operator on the Gaussian space $(\mathbb{R}^n, \mu_A)$, where
\[ d\mu_A(x) = \frac{1}{(2\pi \det A)^{n/2}} e^{-(x,Ax)} dx. \]
Then, for all $1 < p < \infty$, the $L^p$-norm of the Riesz transform $R = \nabla (-L_A)^{-1/2}$ on $(\mathbb{R}^n, \mu_A)$ satisfies
\[ \|\nabla (-L_A)^{-1/2}\|_p \leq 2(p^* - 1). \]

Using the Poincaré limit, we can derive the following result from Theorem 3.1.

Theorem 3.2 (i.e., Corollary 1.6 in [6]) Let $(W, H, \mu_A)$ be an abstract Wiener space, where $W$ is a real separable Banach space, $H$ is a real separable Hilbert space which is densely embedded in $W$, $A \in \mathcal{L}(H)$ be a self-adjoint positive definite operator with finite Hilbert-Schmidt norm, and $\mu$ the Gaussian measure on $W$ with mean zero and with covariance $A$.

Let $L_A = \Delta - Ax \cdot \nabla$ be the generalized Ornstein-Uhlenbeck operator on $(W, H, \mu_A)$. Then, for all $1 < p < \infty$, the $L^p$-norm of the Riesz transform $R = \nabla (-L_A)^{-1/2}$ on $(W, H, \mu_A)$ satisfies
\[ \|\nabla (-L_A)^{-1/2}\|_p \leq 2(p^* - 1). \]

4 Beurling-Ahlfors transforms

Throughout this section, let $M$ be a complete and stochastically complete Riemannian manifold, $n = \text{dim} M$. Let $X_t$ be Brownian motion on $M$, $W_k$ the $k$-th Weitzenböck curvature operator. Let $A_i \in \text{End}(\Lambda^k T^* M)$, $i = 1, 2$, be the bounded endomorphism which, in a local normal coordinate $(e_1, \ldots, e_n)$ at any fixed point $x$, is defined by
\[ A_1 = (a_i a_j^*)_{n \times n}, \quad A_2 = (a_i^* a_j)_{n \times n}, \]
where $a_i$ is int $e_i$ is the inner multiplication by $e_i$, and $a^*_j = e_j \wedge$ is the exterior multiplication by $e_j, i, j = 1, \ldots, n$. For details, see [9].

Let $M_t \in \text{End}(\Lambda^k T_{X_0}^* M, \Lambda^k T_{X_t}^* M)$ be defined by
\[ \frac{\nabla M_t}{\partial t} = -W_k(X_t) M_t, \quad M_0 = \text{Id}_{\Lambda^k T_{X_0}^* M}. \]
For any fixed $T > 0$, the backward heat semigroup generated by the Hodge Laplacian $\square$ on $k$-forms is defined by
\[ \omega(x, T - s) = e^{-(T-s)\square} \omega(x), \quad \forall x \in M, s \in [0, T], \quad \omega \in C_0^\infty(\Lambda^k T^* M). \]
Recall that, the Weitzenböck formula reads as follows
\[ \square = -T \nabla^2 + W_k. \]

We now state the martingale transform representation formula for the Beurling-Ahlfors transforms on $k$-forms over complete Riemannian manifolds.
Theorem 4.1: Let $M$ be a complete and stochastically complete Riemannian manifold. Suppose that $W_k \geq -a$, where $a \geq 0$ is a constant. Then, for all $\omega, \eta \in C_0^\infty(\Lambda^k T^* M)$, we have

$$
\langle dd^* (a + \Box)^{-1} \omega, \eta \rangle = 2 \lim_{T \to \infty} \int_M \langle S_A^T \omega, \eta \rangle dx,
$$

$$
\langle d^*(a + \Box)^{-1} \omega, \eta \rangle = 2 \lim_{T \to \infty} \int_M \langle S_A^T \omega, \eta \rangle dx,
$$

where, for a.s. $x \in M$,

$$
S_A^T \omega(x) = E \left[ M_T e^{-aT} \int_0^T e^{at} M_t^{-1} A_i \nabla \omega_a(X_t, T - t) dX_t \bigg| X_T = x \right], \quad i = 1, 2.
$$

In particular, the Beurling-Ahlfors transform

$$
S_B \omega := (d^* d - dd^*) (a + \Box)^{-1} \omega,
$$

has the following martingale transform representation: for a.s. $x \in M$,

$$
S_B \omega(x) = 2 \lim_{T \to \infty} E \left[ M_T e^{-aT} \int_0^T e^{at} M_t^{-1} B \nabla \omega_a(X_t, T - t) dX_t \bigg| X_T = x \right],
$$

where

$$
B = A_1 - A_2.
$$

Remark 4.2: The martingale transform representation formulas in Theorem 4.1 are the correct reformulation of the formulas that we obtained in Theorem 3.4 in [9], where the martingale transform representation formulas of $S_A$ and $S_B$ were given in the following way

$$
S_A^T \omega(x) = E \left[ \int_0^T e^{a(t-T)} M_T M_t^{-1} A_i \nabla \omega_a(X_t, T - t) dX_t \bigg| X_T = x \right], \quad i = 1, 2,
$$

and

$$
S_B \omega(x) = 2 \lim_{T \to \infty} E \left[ \int_0^T e^{a(t-T)} M_T M_t^{-1} B \nabla \omega_a(X_t, T - t) dX_t \bigg| X_T = x \right].
$$

The same correction should also be made for Theorem 3.5 in [9], where $a = 0$. The reason is that, as pointed out by Bañuelos and Baudoin in [2], $M_T$ is not adapted with respect to the filtration $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$, $t < T$. Moreover, in the proof of Theorem 1.2 in [9] (p.135, line 7 to line 8), we used the Burkholder-Davis-Gundy inequality to derive that

$$
\| S_A^T \omega \|_p \leq C_p \sup_{0 \leq t \leq T} \| e^{a(t-T)} M_T M_t^{-1} A_i \|_{op} \left\| \int_0^T \| \nabla \omega_a(X_t, T - t) \|_{op} dt \right\|_{L_p}^{1/2},
$$

where $\| \cdot \|_{op}$ denotes the operator norm, and $C_p$ is a constant. However, except that $M_T$ is independent of the $(X_t : t \in [0, T])$, one cannot use the Burkholder-Davis-Gundy inequality in above way, due to the fact that $M_T$ is not adapted with respect to the filtration $\mathcal{F}_t = \sigma(X_s : s \in [0, t]), t < T$.
Proof of Theorem 4.1. By Remark 4.2, we need only to correct the martingale transform representation formulas appeared in Theorem 3.4 and Theorem 3.5 in [9] in the right way stated in Theorem 4.1. Thus, the original proof given in [9] for these formulas remain valid after a small modification. To save the length of the paper, we omit it here. □

Proposition 4.3 For all constant \(a \geq 0\) and \(\omega \in C_0^\infty(\Lambda^k T^* M)\), we have
\[
\|dd^* (a + \Box)^{-1} \omega\|_2^2 + \|d^* d(a + \Box)^{-1} \omega\|_2^2 = \|\Box(a + \Box)^{-1} \omega\|_2^2,
\]
Moreover,
\[
\|dd^* (a + \Box)^{-1} \omega\|_2 \leq \|\omega\|_2,
\]
\[
\|d^* d(a + \Box)^{-1} \omega\|_2 \leq \|\omega\|_2,
\]
and
\[
\|(d^* d - dd^*)(a + \Box)^{-1} \omega\|_2 \leq 2\|\omega\|_2.
\]

Proof. By Gaffney’s integration by parts formula, we have
\[
\|dd^* (a + \Box)^{-1} \omega\|_2^2 = \int_M \langle dd^* (a + \Box)^{-1} \omega, dd^* (a + \Box)^{-1} \omega \rangle dv
\]
\[
= \int_M \langle (a + \Box)^{-1} \omega, dd^* dd^* (a + \Box)^{-1} \omega \rangle dv.
\]
Similarly, we can prove
\[
\|d^* d(a + \Box)^{-1} \omega\|_2^2 = \int_M \langle (a + \Box)^{-1} \omega, d^* dd^* (a + \Box)^{-1} \omega \rangle dv.
\]
Using the fact that \(dd^* dd^* + d^* dd^* d = \Box^2\), we get
\[
\|dd^* (a + \Box)^{-1} \omega\|_2^2 + \|d^* d(a + \Box)^{-1} \omega\|_2^2 = \int_M \langle (a + \Box)^{-1} \omega, \Box^2 (a + \Box)^{-1} \omega \rangle dv.
\]
This proves the identity (10). Again, integration by parts yields
\[
\|(a + \Box) \omega\|_2^2 = \|\Box \omega\|_2^2 + 2a \langle \omega, \Box \omega \rangle + a^2 \|\omega\|_2^2
\]
\[
= \|\Box \omega\|_2^2 + 2a \|d\omega\|_2^2 + 2a \|d^* \omega\|_2^2 + a^2 \|\omega\|_2^2
\]
\[
\geq \|\Box \omega\|_2^2,
\]
which implies that
\[
\|\Box(a + \Box)^{-1} \omega\|_2 \leq \|\omega\|_2.
\]
Combining (10) with (11), we obtain
\[
\|dd^* (a + \Box)^{-1} \omega\|_2^2 + \|d^* d(a + \Box)^{-1} \omega\|_2^2 \leq \|\omega\|_2^2.
\]
This finishes the proof of Proposition 4.3. □

We now state the \(L^p\)-norm estimates of the Beurling-Ahlfors transforms on complete Riemannian manifolds. The following result is the restatement of Theorem 1.2 and Theorem 5.1 in [9]. Here, as in [9], \(\| \cdot \|_{op}\) denotes the operator norm.
Theorem 4.4 Suppose that there exists a constant $a \geq 0$ such that

$$W_k \geq -a.$$ 

Then, there exists a universal constant $C > 0$ such that for all $1 < p < \infty$, and for all $\omega \in C_0^\infty(\mathcal{A}^k T^* M)$,

$$\|S_{A_i} \omega\|_p \leq C(p^* - 1)^{3/2}\|A_i\|_{op}\|\omega\|_p,$$

and

$$\|S_{B} \omega\|_p \leq C(p^* - 1)^{3/2}\|B\|_{op}\|\omega\|_p.$$ 

In particular, in the case where $W_k \equiv -a$, we have

$$\|S_{A_i} \omega\|_p \leq 2(p^* - 1)\|A_i\|_{op}\|\omega\|_p,$$

and

$$\|S_{B} \omega\|_p \leq 2(p^* - 1)\|B\|_{op}\|\omega\|_p.$$ 

Proof. By Proposition 4.3, we need only to study the case $p \neq 2$. For simplicity, we only consider the case $W_k \geq 0$. The general case $W_k \geq -a$ can be similarly proved. Let

$$Z_i^t = M_t \int_0^t M_s^{-1} A_i \nabla \omega(X_s, t - s) dX_s, \quad i = 1, 2.$$ 

By Theorem 2.6 due to Bañuelos and Baudoin in [2], for all $p \in (1, \infty)$, we have

$$\|Z_i^t\|_p \leq 3\sqrt{p(2p - 1)} \left( \int_0^T |A_i \nabla \omega(X_t, T - t)|^2 dt \right)^{1/2}.$$ 

Obviously, we have

$$\left\| \left( \int_0^T |A_i \nabla \omega(X_t, T - t)|^2 dt \right)^{1/2} \right\|_p \leq \|A_i\|_{op} \left( \int_0^T |\nabla \omega(X_t, T - t)|^2 dt \right)^{1/2}.$$ 

By the same argument as used in the proofs of Proposition 6.2 and Proposition 6.3 in [7], for all $1 < p < \infty$, we can prove that

$$\left\| \left( \int_0^T |\nabla \omega(X_t, T - t)|^2 dt \right)^{1/2} \right\|_p \leq B_p \|\omega\|_p,$$

where $B_p = (2p)^{1/2}(p - 1)^{-3/2}$ for $p \in (1, 2)$, $B_p = 1$ for $p = 2$, and $B_p = \frac{p}{\sqrt{2(p - 2)}}$ if $p > 2$. Hence, for $1 < p < 2$,

$$\|S_{A_i} \omega\|_p \leq 3\sqrt{p(2p - 1)}\|A_i\|_{op} \frac{(2p)^{1/2}}{(p - 1)^{3/2}}\|\omega\|_p \leq 6\sqrt{6}(p - 1)^{-3/2}\|A_i\|_{op}\|\omega\|_p.$$
and for $p > 2$, 
\[ \|S^T_{A_i}\omega\|_p \leq 3(p - 1)^{3/2}(1 + O((p - 1)^{-1}))\|A_i\|_{\text{op}}\|\omega\|_p. \]

Indeed, by duality argument as used in [9], for all $p > 2$, we have 
\[ \|S^T_{A_i}\|_{p,p} = \|S^T_{A_i}\|_{q,q}, \]
which yields for $p > 2$,
\[ \|S^T_{A_i}\omega\|_p \leq 6\sqrt{6}(p - 1)^{3/2}\|A_i\|_{\text{op}}\|\omega\|_p. \]

In summary, for all $1 < p < \infty$, we have proved that 
\[ \|S^T_{A_i}\omega\|_p \leq 6\sqrt{6}(p - 1)^{3/2}\|A_i\|_{\text{op}}\|\omega\|_p. \]

Similarly, for all $1 < p < \infty$, we can prove 
\[ \|S^T_{B_i}\omega\|_p \leq 6\sqrt{6}(p - 1)^{3/2}\|B\|_{\text{op}}\|\omega\|_p. \]

In the particular case where $W_k \equiv -a$, we have 
\[ S^T_{A_i}\omega(x) = E\left[U_T\int_0^T U_t^{-1} A_i \nabla \omega_a(X_t, T - t) dX_t \mid X_T = x\right], \quad i = 1, 2, \text{ a.s.} x \in M. \]

The $L^p$-contractiveness of the conditional expectation yields 
\[\begin{align*}
\|S^T_{A_i}\omega\| & = \left\| U_T\int_0^T U_t^{-1} A_i \nabla \omega_a(X_t, T - t) dX_t \right\|_p \\
& = \left\| \int_0^T U_t^{-1} A_i \nabla \omega_a(X_t, T - t) dX_t \right\|_p.
\end{align*}\]

Using the Burkholder sharp $L^p$-inequality for martingale transforms, for all $p > 1$, we deduce that 
\[ \|S^T_{A_i}\omega\| \leq (p^* - 1) \sup_{0 \leq t \leq T} \|U_t^{-1} A_i U_t\|_{\text{op}} \left\| \int_0^T U_t^{-1} \nabla \omega_a(X_t, T - t) dX_t \right\|_p. \quad (12) \]

By Itô’s formula, we can prove that (see Eq. (49) in [9]) 
\[ \omega(X_T) - U_T \omega_a(X_0, T) = U_T \int_0^T U_t^{-1} \nabla \omega_a(X_t, T - t) dX_t. \quad (13) \]

Substituting (13) into (12), we have 
\[ \|S^T_{A_i}\omega\|_p \leq (p^* - 1)\|A_i\|_{\text{op}}\|\omega(X_T) - U_T \omega_a(X_0, T)\|_p. \]

Using the argument in [9], we obtain 
\[ \|S^T_{A_i}\omega\|_p \leq (p^* - 1) \left(1 + e^{-2\min\left(\frac{1}{p}, \frac{1}{p'}\right)}a_T\right)\|\omega\|_p. \]
Hence
\[ \|dd^*(a + \Box)^{-1}\omega\|_p \leq 2 \lim_{T \to \infty} \|S^T_A \omega\|_p \leq 2(p^* - 1)\|A_1\|_{op}\|\omega\|_p, \]
\[ \|d^*d(a + \Box)^{-1}\omega\|_p \leq 2 \lim_{T \to \infty} \|S^T_A \omega\|_p \leq 2(p^* - 1)\|A_2\|_{op}\|\omega\|_p, \]
and
\[ \|(dd^* - d^*d)(a + \Box)^{-1}\omega\|_p \leq 2 \lim_{T \to \infty} \|S^T_B \omega\|_p \leq 2(p^* - 1)\|B\|_{op}\|\omega\|_p. \]

The proof of Theorem 4.4 is completed. \(\square\)

**Remark 4.5** The above proof corrects a gap contained in [9]. In summary, the \(L^p\)-norm estimates in Theorem 4.4 indicates that the results in Theorem 1.2, Theorem 1.3 and Theorem 1.4, Theorem 5.1 and Corollary 5.2 obtained in [9] remain valid. As a consequence, the main theorems proved in [9] remain valid. In particular, see Theorem 1.3 in [9], on complete and stochastically complete Riemannian manifolds non-negative Weitzenböck curvature operator \(Wk \geq 0\), where \(1 \leq k \leq n = \dim M\), the Weak \(L^p\)-Hodge decomposition theorem holds for \(k\)-forms, the De Rham projection \(P_1 = dd^*\Box^{-1}\), the Leray projection \(P_2 = d^*d\Box^{-1}\) and the Beurling-Ahlfors transform \(B_k = (d^*d - dd^*)\Box^{-1}\) on \(k\)-form is bounded in \(L^p\) for all \(1 < p < \infty\).

## 5 Time reversal martingale transformation representation formula for the Riesz transforms

In this section, we prove a time reversal martingale transformation representation formula for the Riesz transforms on complete Riemannian manifolds.

First, we prove the following time reversal martingale transformation representation formula for one forms.

**Theorem 5.1** Let \(\hat{X}_t = X_{\tau - t}\), \(\hat{B}_t = B_{\tau - t}\), \(t \in [0, \tau]\). Let \(\hat{M}_t\) be the solution to the covariant SDE
\[
\frac{\nabla}{\partial t}\hat{M}_t = -\hat{M}_t(Ric + \nabla^2 \phi)(\hat{X}_t),
\]
\[ \hat{M}_0 = \text{Id}|_{\hat{X}_0 = M}. \]

For any \(\omega \in C^\infty_0(\Lambda^1 T^* M)\), let \(\omega_\alpha(x, y) = e^{-y\sqrt{a + \Box_\alpha}}\omega(x), \forall x \in M, y \geq 0\). Then, for a.s. \(x \in M\),
\[ \frac{1}{2}\omega(x) = \lim_{y \to +\infty} E_y \left[ \hat{Z}_\tau \big| \hat{X}_0 = x \right], \]
where
\[ \hat{Z}_\tau = \int_0^\tau e^{-at}\hat{M}_t \partial_y \omega_\alpha(\hat{X}_t, \hat{B}_t) d\hat{B}_t - \int_0^\tau e^{-at}\hat{M}_t \partial_y^2 \omega(\hat{X}_t, \hat{B}_t) dt. \]
Proof. By Theorem 2.2, we have
\[
\frac{1}{2} \omega(x) = \lim_{y \to +\infty} E_y [Z_\tau | X_\tau = x],
\]
where
\[
Z_\tau = e^{-\alpha \tau} M_\tau \int_0^\tau e^{\alpha s} M_s^{-1} \nabla_y \omega_a(X_s, B_s) dB_s.
\]
Taking \(0 = s_0 < s_1 < \ldots < s_n < s_{n+1} = \tau\) be a partition of \([0, \tau]\), then
\[
Z_{\tau,n} := e^{-\alpha \tau} M_\tau \sum_{i=1}^N e^{\alpha s_i} M_{s_i}^{-1} \nabla_y \omega(X_{s_i}, B_{s_i}) (B_{s_i+1} - B_{s_i})
\]
converges in \(L^2\) and in probability to \(Z_\tau\). We can rewrite \(Z_{\tau,n}\) as follows
\[
Z_{\tau,n} = \sum_{i=1}^N e^{-\alpha(\tau-s_i)} M_{\tau} M_{s_i}^{-1} \nabla_y \omega(X_{s_i}, B_{s_i}) (B_{s_i+1} - B_{s_i}).
\]
Note that
\[
\partial_s M_{\tau-s} = \frac{\partial_s \hat{M}_{\tau-s} \text{Ric}(L)(\hat{X}_{\tau-s})}{\hat{M}_{\tau-s} \text{Ric}(L)(X_s)},
\]
and
\[
\partial_s (M_\tau M_{s_i}^{-1}) = -M_\tau M_{s_i}^{-1} \partial_s M_{s_i} M_{s_i}^{-1} = M_\tau (M_{\tau} M_{s_i}^{-1} \text{Ric}(L)(X_s)) M_{s_i} M_{s_i}^{-1} = (M_{\tau} M_{s_i}^{-1} \text{Ric}(L)(X_s)).
\]
By the uniqueness of the solution to ODE, as \(\frac{\partial_s \hat{M}_{\tau-s}}{\hat{M}_{\tau-s}} \bigg|_{s=\tau} = M_\tau M_{s_i}^{-1} \bigg|_{s=\tau} = \text{Id}_{T_0 M},\) we have
\[
M_\tau M_{s_i}^{-1} = \frac{\partial_s \hat{M}_{\tau-s}}{\hat{M}_{\tau-s}}.
\]
Therefore
\[
Z_{\tau,n} = \sum_{i=1}^N e^{-\alpha(\tau-s_i)} M_{\tau-s_i} \nabla_y \omega(\hat{X}_{\tau-s_i}, \hat{B}_{\tau-s_i}) (\hat{B}_{\tau-s_i+1} - \hat{B}_{\tau-s_i}).
\]
Let \(t_i = \tau - s_i\). Then \(\tau = t_0 > t_1 > \ldots > t_n > t_{n+1} = 0\), and
\[
Z_{\tau,n} = \sum_{i=1}^N e^{-\alpha t_i} \hat{M}_{t_i} \nabla_y \omega(\hat{X}_{t_i}, \hat{B}_{t_i}) (\hat{B}_{t_{i+1}} - \hat{B}_{t_{i+1}}).
\]
By Taylor’s formula, we have
\[
\omega(\hat{X}_{t_i}, \hat{B}_{t_i}) = \omega(\hat{X}_{t_{i+1}}, \hat{B}_{t_{i+1}}) - \nabla_y \omega(\hat{X}_{t_{i+1}}, \hat{B}_{t_{i+1}}) (\hat{B}_{t_{i+1}} - \hat{B}_{t_{i+1}}) + O((\hat{B}_{t_{i+1}} - \hat{B}_{t_i})^2).
\]

Hence
\[
Z_{\tau,n} = \sum_{i=1}^{N} e^{-at} \hat{M}_t \nabla_y \omega(\hat{X}_{t+1}, \hat{B}_{t+1}, \hat{B}_{t+1} - \hat{B}_t) \\
- \sum_{i=1}^{N} e^{-at} \hat{M}_t \nabla_y^2 \omega(\hat{X}_{t+1}, \hat{B}_{t+1}, \hat{B}_{t+1} - \hat{B}_t)^2 + O((\hat{B}_{t+1} - \hat{B}_t)^3)
\]
which converges in $L^2$ and in probability to the following limit
\[
\hat{Z}_\tau = \int_0^{\tau} e^{-as} \hat{M}_s \nabla_y \omega(\hat{X}_s, \hat{B}_s) d\hat{B}_s - \int_0^{\tau} e^{-as} \hat{M}_s \nabla_y^2 \omega(\hat{X}_s, \hat{B}_s) ds.
\]
The proof of Theorem 5.1 is completed. □

By Theorem 5.1, we can prove the following time reversal martingale transformation representation formula for the Riesz transforms on complete Riemannian manifolds.

**Theorem 5.2** Let $R_a(L) = \nabla (a - L)^{-1/2}$. Then, for $f \in C_0^\infty(M)$, we have
\[
R_a(L)f(x) = -2 \lim_{y \to +\infty} E_y \left[ \hat{Z}_\tau \big| \hat{X}_0 = x \right],
\]
where
\[
\hat{Z}_\tau = \int_0^{\tau} e^{-as} \hat{M}_s dQ_a f(\hat{X}_s, \hat{B}_s) d\hat{B}_s - \int_0^{\tau} e^{-as} \hat{M}_s \partial_y dQ_a f(\hat{X}_s, \hat{B}_s) ds.
\]

**Remark 5.3** As noticed in [6], there exists a standard one dimensional Brownian motion $\beta_t$ such that
\[
d\hat{B}_t = d\beta_t + \frac{dt}{B_t}, \quad t \in (0, \tau).
\]

### 6 Time reversal martingale transforms representation formula for the Beurling-Ahlfors transforms

Similarly to the proof of Theorem 5.1, we prove a time reversal martingale transformation representation formula for the Beurling-Ahlfors transforms on complete Riemannian manifolds.

**Theorem 6.1** Let $\hat{X}_t = X_{T-t}$, $t \in [0, T]$. Let $\hat{M}_t$ be the solution to the covariant equation
\[
\frac{\nabla \hat{M}_t}{\partial t} = -\hat{M}_t W_k(\hat{X}_t), \quad \hat{M}_0 = \text{Id}_{\Lambda^k T^* M}.
\]
Then, for any $\omega \in C_0^\infty(\Lambda^1 T^* M)$, the Beurling-Ahlfors transform
\[
S_B \omega := (d^* d - dd^*) (a + \Box)^{-1} \omega
\]
has the following time reversal martingale transform representation: for a.s. $x \in M$,
\[
S_B \omega(x) = 2 \lim_{T \to \infty} E \left[ \hat{Z}_T \big| \hat{X}_0 = x \right],
\]
where

\[
\tilde{Z}_T = \int_0^T e^{-as} \tilde{M}_s B \nabla \omega_{a}(\tilde{X}_s, s) d\tilde{X}_s - \int_0^T e^{-as} \tilde{M}_s BT \nabla^2 \omega_{a}(\tilde{X}_s, s) ds.
\]

To end this paper, let us mention that, in a forthcoming paper [10], we will prove a martingale transform representation formula for the Riesz transforms associated with the Dirac operator acting on Hermitian vector bundles over complete Riemannian manifolds and for the Riesz transforms associated with the ∂-operator acting on holomorphic Hermitian vector bundles over complete Kähler manifolds. By the same argument as used in this paper, we can prove some explicit dimension free \(L^p\)-norm estimates of these Riesz transforms on complete Riemannian or Kähler manifolds with suitable curvature conditions. See also [8].

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