CHAOS IN BOHMIAN MECHANICS OF COMMENSURATE HARMONIC OSCILLATORS

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Abstract. Bohmian mechanics is a causal interpretation of quantum mechanics in which there are well-defined particle trajectories guided by the wave function. Within this framework, it is shown that certain classical integrable systems can exhibit both periodic and chaotic behaviours. In this paper, we extend the work of Borondo et al. by investigating the behaviours of commensurate two-dimensional harmonic oscillator systems with different ratios of frequencies. We show that the system may produce chaotic behavior of Bohmian trajectories to be dependent on frequencies ratio. This result is illustrated numerically in computer experiments displaying the Bohmian trajectories.

1. Introduction
Despite the great success of standard quantum mechanics, it still remains as an indeterministic theory, which means that it generally does not predict the outcome of any measurement with certainty [1]. This fact inspired some physicists to construct a deterministic quantum theory and one of the relatively successful one is Bohmian mechanics.

Bohmian-type mechanics was first introduced by Louis de Broglie and independently developed again by David Bohm[2] in 1950s. Despite initial criticisms by Pauli(1953), Keller(1953), Kochen and Specker(1967) and others, this theory recently became the subject of intense theoretical and experimental study.

The critical idea in Bohmian mechanics is this theory is based on a well defined position and its trajectory guided by the wavefunction. It clearly gives advantage in term of intuitive concepts and ideas because the quantum trajectories provide causal connection between events in physical space time. Therefore this notion gives researcher a new way to solve problems in quantum mechanics. One of the problems well investigated is in the study of quantum chaos. Quantum chaos tries to understand the connection between two phenomena in physics, chaos theory and quantum theory. Note that many concepts of classical chaos (periodic orbits, attractors, phase space maps) rely on the classical notion of trajectory. However for microscopic system the situation is totally different because the fundamental implication of standard quantum mechanics substantially differs from classical mechanics. This is due the fact that the concept of classical mechanics like position and trajectory is not allowed because one cannot measure the position and momentum of each particle at the same time and with precision due to Heisenberg uncertainty principle. Furthermore the concept of state of physical systems in standard quantum mechanics is influenced by wave-particle duality. In terms of evolution, the wavefunction which describes the state of microscopic system obeys Schrodinger equation, and the linearity of this evolution makes the wavefunction become less sensitivity to initial condition. These factors make the notion of chaotic trajectories cannot be adopted straightforwardly in standard quantum mechanics. However physicists attempt to overcome this problem by finding quantum mechanical criteria which
is able to trace the existence of chaos in the quantum domain. They found several approaches which are based on the statistical distribution of energy eigenvalue spacing and spatial correlations of wavefunction. However the fact remains that they are not directly comparable.

In recent years, Bohmian mechanics has attracted increasing attention by researchers. A number of papers discuss chaos in Bohmian mechanics for various systems, such as cat map [3], a hydrogen atom in an oscillating electric field [4], two dimensional harmonic oscillator [5] and kick rotor [6]. Chaos in Bohmian mechanics has been found when no chaos appear in standard quantum mechanics [7]. Chaos in Bohmian mechanics may also even appear when the corresponding classical system is non chaotic [8]. Furthermore several paper have shown that Lyapunov exponent which is typical chaos indicator in classical mechanics have successfully apply to demonstrated chaos in Bohmian framework. This may be demonstrated a correlation between classical and Bohmian chaos. In 1996, Guglielmo Iacomelli and Marco Pettini apply Lyapunov exponent to study the difference in stability between classical regime and quantum regime [9]. Besides that it was shown that Lyapunov exponent indicated Bohmian trajectories in one dimension will not bechaotic but maybe chaotic in the case of two dimensional [10]. Its also demonstrated that vortices generated by Bohmian trajectories can generate chaos [11]. In this respect, Bohmian mechanics may have a clear advantage to study the correlation between a quantum system and its classical counterpart because we can compare quantitatively both orbits in these two regimes using Lyapunov exponent.

This study is thus to investigate numerically the chaotic behaviours of two dimensional commensurate (rational frequency ratio) harmonic oscillator in framework of Bohmian mechanics using Lyapunov exponent. Aparticular concern is whether or not the change of rational frequency ratio will influence the behaviour of this system.

2. Bohmian mechanics formulation
In order to derive the equation describing Bohmian trajectories, let we start with Schrodinger equation

\[ \frac{i\hbar}{\hbar} \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \psi + V(t) \psi \]  \hspace{1cm} (1)

Now we express wavefunction in polar form.

\[ \psi(x, t) = R(x, t) \exp \left( \frac{i}{\hbar} S(x, t) \right) \] \hspace{1cm} (2)

where R and S are real function. In order to get an evolution in time for R and S, we insert equation (2) into equation (1).

\[ i\hbar \frac{\partial (Re^{iS})}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 (Re^{iS}) + V(t)(Re^{iS}) \] \hspace{1cm} (3)

In this way we obtain a pair of couple equation.

\[ \frac{dR}{dt} = \frac{1}{2m} (RV^2S + 2V \nabla S \nabla S) \] \hspace{1cm} (4)

\[ \frac{dS}{dt} = -\frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2 \nabla^2 R}{2mR} \] \hspace{1cm} (5)
Equation (4) is the continuity equation where the position probability distribution and equation (5) is identical in form to the Hamiltonian-Jacobi equation

\[- \frac{\partial \mathcal{S}}{\partial t} = \frac{(\nabla \mathcal{S})^2}{2m} + V\]  

However, there exist two significant differences. First, Hamilton's principal function, $S$, have the relation $\nabla S = p(x, t)$ where $p$ is the conjugate momenta in generalized coordinate but the term $S$ in equation (5) represent the phase of quantum state, both term obviously express two different things. However, if we attach this physical meaning to $S$, $\nabla S = p(x, t)$, then we may interpret equation (5) as a modified Hamilton-Jacobi equation for real quantum particles with well-defined and trajectory. We called $\nabla S = p(x, t)$ as guidance condition where $p$ is the canonical momentum of a particle in Bohmian trajectory. Second, the extra term $\hbar\sqrt{g_2870}\sqrt{g_1844}\frac{1}{2}\sqrt{g_1865}\frac{1}{\sqrt{g_2870}}\sqrt{\psi^*\nabla\psi - \psi\nabla\psi^*}|\psi|^2$ is the only thing that makes equation (5) differ compared to usual classical Hamiltonian-Jacobi equation. This extra term is called "quantum potential". If we attach $Q$ with force, as a classical force is associated with a classical potential energy, then we get

\[F_Q = -\nabla Q\]  

Here, $F_Q$ is called quantum force. This expression makes Bohmian mechanics becomes more similar as Newtonian formulation, therefore the total (classical plus quantum) force become

\[F_t = -\nabla(V + Q)\]  

Now we ready to derive the guiding equation, first we define the velocity vector field

\[\nu = \frac{\nabla S}{m}\]  

as we known $\nabla S = p(x, t)$. Then by inserting equation (2) into equation (10) we finally get

\[\nu(t) = \frac{i}{\hbar} \frac{\psi^* \nabla \psi - \psi \nabla \psi^*}{|\psi|^2}\]  

### 3. Two-dimensional harmonic oscillator with commensurable frequencies

The focus of this research is to study the behaviour of two-dimensional commensurate harmonic oscillator. The Hamiltonian of a particle in this system is given by

\[H(x, y) = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}(\omega_1x^2 + \omega_2y^2)\]  

where $\omega_1$ and $\omega_2$ are the frequencies of the oscillator, $x$ and $y$ are the position of the particle.

Consider the combination of eigenstates of this system
\[\psi_{0,0} = \frac{\alpha}{\sqrt{\pi}} e^{\frac{i}{2}(\omega_1 x^2 + \omega_2 y^2)} e^{-\frac{t}{2}(\omega_1 + \omega_2)t}\]

\[\psi_{0,1} = \frac{2\sqrt{\pi}}{\sqrt{2\pi}} e^{\frac{i}{2}(\omega_1 x^2 + \omega_2 y^2)} e^{-\frac{t}{2}(\omega_1 + 3\omega_2)t}\]

\[\psi_{1,0} = \frac{2\sqrt{\pi}}{\sqrt{2\pi}} e^{\frac{i}{2}(\omega_1 x^2 + \omega_2 y^2)} e^{-\frac{t}{2}(3\omega_1 + \omega_2)t}\]

Here \(\alpha = (\omega_1 \omega_2)^{\frac{1}{4}}\) and the eigenstate \(\psi_{0,0}\) correspond to ground state energy and \(\psi_{0,1}, \psi_{1,0}\) correspond to the degenerate first excited states. In this way we obtain the time evolution of the resulting wavefunction is given by

\[
\left(\frac{\alpha e^{-i(\omega_1 + \omega_2)t}}{\sqrt{\pi}} + \frac{2\sqrt{\pi}}{\sqrt{2\pi}} e^{-i(\omega_1 + 3\omega_2)t} + \frac{2\sqrt{\pi}}{\sqrt{2\pi}} e^{-i(3\omega_1 + \omega_2)t}\right) e^{\frac{i}{2}(\omega_1 x^2 + \omega_2 y^2)}
\]

(13)

where \(A = a + id, B = b + ig, C = f + ic\). These complex numbers must satisfy normalization condition \(|A| + |B| + |C| = 1\). By inserting equation (13) in equation (11), we obtain the following

\[v_x = -\frac{\sqrt{2}\alpha^2 \beta_x - 2\alpha^2 \gamma_x y}{V(x, y, t)}\]

(14)

\[v_y = -\frac{\sqrt{2}\alpha^2 \beta_y - 2\alpha^2 \gamma_y x}{V(x, y, t)}\]

(15)

where

\[\beta_x = (ab + dg)\sin(\omega_2 t) - (ag - bd)\cos(\omega_2 t)\]

\[\beta_y = (af + dg)\sin(\omega_1 t) - (ag - bd)\cos(\omega_1 t)\]

\[\gamma_x = (bf + gc)\cos(\omega_1 - \omega_2)t + (bc - gf)\cos(\omega_1 - \omega_2)t\]

\[\gamma_y = (bf + gc)\cos(\omega_1 - \omega_2)t - (bc - gf)\cos(\omega_1 - \omega_2)t\]

\[V(x, y, t) = 2\sqrt{2}\alpha^2 (b^2 + g^2) + 2\alpha^2 (c^2 + f^2) + a^2 (a^2 + d^2)\]

\[+2\sqrt{2}\alpha^2 [(ab + dg)\cos(\omega_2 t) + (ag + bd)\sin(\omega_2 t)]x\]

\[+2\sqrt{2}\alpha^2 [(af + dc)\cos(\omega_1 t) + (ac + df)\sin(\omega_1 t)]y\]

\[+4\alpha^2 [(bg + gc)\cos(\omega_1 - \omega_2)t + (bc + gf)\sin(\omega_1 - \omega_2)t]xy\]

and quantum trajectories defined in Bohmian framework is the solution of this two differential form.

4. Lyapunov exponent

Lyapunov exponent provides a quantitative measure of the divergence or convergence of nearby trajectories for a dynamical system. To do computation, Benettin et al. have introduced a procedure to calculate Lyapunov exponents for bounded systems. Consider two nearby points in phase space M with very small separation.
where $x_0$ and $y_0$ are initial point of two nearby trajectories and $\| \ldots \|$ is the Euclidean norm. Then we can define the Hamiltonian flow $\{ T^t \}$ where time $t \in \mathbb{R}$ such that

$$
\begin{align*}
  x(t) &= T^t x_0 \\
  y(t) &= T^t y_0
\end{align*}
$$

Therefore the separation between these two trajectories at time $t$ is

$$
|\delta p_t| = \| x(t) - y(t) \|
$$

In order to keep the trajectories separation within the linearized flow range, given a fixed time $t$ we modify equations (17) and (18) by

$$
\begin{align*}
  x_i &= T^t x_{i-1} \\
  y_i &= T^t y_{i-1}
\end{align*}
$$

where $t \in \mathbb{R}$ and equation (19) become

$$
|\delta p_t| = \| T^t x_{i-1} - T^t y_{i-1} \|
$$

Now we can define $x_1 = T^t x_0$ and $|\delta p_1| = \| T^t x_0 - T^t y_0 \|$. Thereafter we have to rescale $y_i$ such that $|y_i - x_i| = |\delta p_0|$. Such procedure will be repeated by $n$ times to get a sequence of positive number $\{ \delta p_i \}$ and the Lyapunov exponent given by

$$
\lambda = \frac{1}{t} \sum_{i=1}^{n} \ln \frac{|\delta p_i|}{|\delta p_0|}
$$

5. Results and conclusion

From the numerical calculations, we find that there exists three types of Bohmian trajectories: regular, mixture between regular and chaotic, and chaotic which are illustrated by figures 1(a), 2(a), 3(a) respectively. Figure 1(b) represent the calculation of $\lambda$ for the trajectory of figure 1(a). It shows the logarithmic value of Lyapunov number decreases when time increases, following the power law, $\lambda \sim 1/t$. This means that the trajectory is ordered and $\lim_{t \to \infty} \lambda = 0$. Such results is also found in figure 2(b) in the early time period ($t = 500$), however for time larger than 500, the quantity $\lambda$ stabilizes at a positive value. The latter time evolution of figure 2(b) is similar to that of figure 3(b) where it shows that $\lambda$ decreases slowly and tends to stabilize to a positive value, $\lambda \equiv 0.1$ at $t = 1000$. 
Figure 1. (a) A Bohmian orbit with initial condition $x_0 = 1$, $y_0 = 0$, $a = 0.37$, $b = 0.44$, $c = 0.44$, $d = -0.02$, $e = 0.49$, $f = -0.49$ for frequencies ratio 4:1. (b) The time evolution of the Lyapunov exponent.

Figure 2. Same as Figure 1, for frequencies ratio 4:7.

Figure 3. Same as Figure 1, for frequencies ratio 7:3.

From these results, we can conclude that a system which is classically integrable (commensurate two-dimensional harmonic oscillators) may have three types of trajectories: regular, mixture between regular and chaotic, and chaotic dependent on the frequency ratio.
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