Exact solution of buckling problem of the column loaded by self-weight

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Abstract. This work proposes the new method of investigation the buckling problem of the uniform column under the axial distribution load, which presented by self-weight. The new method is based on the exact solution of the appropriate differential equation for the buckling of a column. The solution is expressed with dimensionless fundamental functions and initial parameters. Due to the exact solution, the formulas in an explicit form for variables of the state of a column – deflections, angular displacement, bending moment and transverse force – were defined. They generally define the stress-strain state of the column due to the dimensionless nature of the fundamental functions the analytical form of load is defined. That has limited the problem of determination of critical load to determination the unknown non-dimensional buckling coefficient through characteristic equation. An example of determining the buckling coefficient for a hinged column was considered. The values of buckling coefficient for most popular design schemes are given. The proposed method does not require discretization of scheme and is a real alternative to applying approximate methods when solving stability problems of this class.

1. Introduction

The buckling problems of structures are widespread in scientific literature. In particular, in view of the prevalence of practical applications, the problems of the stability of the rods are the most common. After all, not only a single element but also a structure in general can be considered as a rod scheme. The columns of industrial buildings, the industrial high-rise buildings like smokestacks, water towers, multipurpose steel towers, used for electric-power transmission line and other can be presented as the rod structure.

Usually in column buckling problems, the self-weight is often are not taking into account because of small value of its load when compared to the applied axial loads. However, the buckling problem of structures under self-weight is considered to be an independent problem when investigating high-rise and massive structures.

The first in the XVIII century L. Euler studied the buckling of columns under the influence of their self-weight [1]. Later A. Greenhill [2] made some clarifications in this area, that is why in scientific literature this problem is often called by his name. However, the analytical solution of the buckling problem of the column with an arbitrary distribution of bending stiffness and axial force has not been found so far. The closed-form solution for some structural elements is present in the works of...
T. R. Karman, M. A. Biot [3], S. P. Timoshenko, J. M. Gere [4]. In paper [5] the calculations are made using software complexes for creation the rod design method with taking into account the self-weight. In general, the buckling problem of the rods is quite popular in scientific literature and reference books [6-9], where it is considered using the approximate methods. Noting the difficulty in obtaining exact solutions in [10] proposed the use of the differential transformation method for solving the differential equation of buckling problem with taking into account the self-weight.

One of the most widespread design schemes for investigation the buckling of structures is the uniform rod under the variable axial force, which presented by self-weight. The math model of this physical phenomenon is the differential variable-coefficient equation [6-8]. The results presented in this work are based on the exact solution of this equation.

2. General symbols and input equation

The buckling problem of uniform rod under the self-weight is considered. For example, Figure 1 shows the general scheme of hinged rod, and Figure 2 shows the internal forces acting on its element. Note, that all following formulas are surely applied to any boundary conditions on the column’s ends.

List of symbols:

- \( EI \) – the flexural rigidity of a rod;
- \( qx \) – the variable axial (compressive) force, where \( q \) – weight per unit length of beam;
- \( y(x) \) – the displacement of the axis point of the column with coordinate \( x \) (deflection);
- \( \varphi(x) \) – the angular displacement;
- \( M(x) \) – the bending moment;
- \( Q(x) \) – the transverse force.

The appropriate differential equation for rod equilibrium has the form [6–8]

\[
EIy''''(x) + q(x)y'(x)' = 0. \tag{1}
\]
Stress-strain state of the column is generally defined with displacements \( y(x), \varphi(x) \) and initial forces \( M(x), Q(x) \). It’s known \([7, 8]\), that they are connected by the equalities:

\[
\varphi(x) = y'(x); \quad M(x) = -E(x)I(x)\varphi'(x); \quad Q(x) = M'(x) - N\varphi(x).
\]

We will briefly call these values as variables of state of the rod.

3. Exact solution of buckling equation and the formulas for variables of rod’s state

Equation \((1)\) is the special case of known \([11-13]\) differential equation of free transverse vibrations with taking into account the self-weight.

\[
EI v'''(x) + q(x)v'(x)' - p^2mv(x) = 0,
\]

where \(p\) – the frequency of vibrations.

Equation \((3)\) for the long time in scientific literature was investigating using approximate methods \([14]\). The exact solution of this equation was determined by authors in paper \([15]\). For this the direct integration method, which was proposed and developed in \([16]\), was used.

Using results \([15]\) after transformation the obtained there formulas for our case, we can write the fundamental functions \(1, Z_1(x), Z_2(x), Z_3(x)\) of the equation \((1)\). Here last three functions are determined by the following equiconvergent series

\[
Z_n(x) = \beta_{n,0}(x) - S\beta_{n,1}(x) + S^2\beta_{n,2}(x) - S^3\beta_{n,3}(x) + \ldots \quad (n = 1, 2, 3),
\]

where

\[
\beta_{n,0}(x) = \frac{1}{n!}\left(\frac{x}{l}\right)^n, \quad \beta_{n,k}(x) = \frac{n(n+3)\ldots(n+3k-3)}{(n+3k)!}\left(\frac{x}{l}\right)^{n+3k} \quad (k = 1, 2, 3, \ldots),
\]

\[
S = \frac{ql^3}{EI}.
\]

Note that parameter \(S\) is dimensionless, which can be verified by checking. Therefore, the fundamental functions also will be dimensionless.

During the calculations the dimensionless functions \(\tilde{Z}_n(x) = lZ_n(x), \tilde{Z}_n(x) = l\tilde{Z}_n(x)\) \((n = 1, 2, 3)\) also will be used.

With fundamental solutions, it is easy to write the general integral of the input equation \((1)\), that is, to determine the displacement \(y(x)\). Then, based on \((2)\), we can write the formulas for other variables of the rod state \(\varphi(x), M(x), Q(x)\). The above formulas, expressed with the initial parameters, have form:

\[
y(x) = y(0) + \varphi(0)lZ_1(x) - M(0)\frac{l^3}{EI}Z_1(x) - Q(0)\frac{l^3}{EI}Z_3(x);
\]

\[
\varphi(x) = \varphi(0)\tilde{Z}_1(x) - M(0)\frac{l}{EI}\tilde{Z}_1(x) - Q(0)\frac{l^3}{EI}\tilde{Z}_3(x);
\]

\[
M(x) = -\varphi(0)\frac{EI}{l}\tilde{Z}_1(x) + M(0)\tilde{Z}_2(x) + Q(0)l\tilde{Z}_3(x);
\]

\[
Q(x) = Q(0).
\]

The important point is that dimension of constant coefficients near the dimensionless functions in the right part of formulas \((5)-(8)\) are equal to dimension of corresponding left part.
4. Analytical form of load
Based on formula (4) one arrives at the equality

\[ q = S \frac{EI}{l^3}, \quad (9) \]

which establishes an analytical relationship between load and other parameters of the considered mechanical system. Based on the role played of parameter \( S \) in formula (9), we call it the buckling coefficient. Since the formulas for the variables of the rod state depend on the parameter \( S \), then it will act as the unknown in the characteristic equations that we get realizing the given boundary conditions.

5. Determination of buckling coefficients
Usually in literature when investigating the effect of self-weight, the cantilever column is using. It has free top end, and clamped bottom end. The solution of the appropriated problem is well known \([6, 9]\) and it determined with the Bessel function. However, it is clear that other cases of boundary conditions occur in real construction.

Let’s consider the case of hinged column. For this case the following boundary conditions are used: \( y(0) = 0; M(0) = 0; y(l) = 0; M(l) = 0 \). After its realizing using (5), (7), we come to set of equations

\[
\begin{align*}
\varphi(0)Z_1(l) - Q(0) \frac{I^3}{EI} Z_1(l) &= 0; \\
-\varphi(0) \frac{EI}{l} \ddot{Z}_1(l) + Q(0) \dot{Z}_1(l) &= 0.
\end{align*}
\]

After writing the system compatibility condition, we get the characteristic equation

\[ Z_1(l)\ddot{Z}_1(l) - \ddot{Z}_1(l)Z_1(l) = 0, \]

or

\[ \eta_0 - \eta_1 S + \eta_2 S^2 - \eta_3 S^3 + \ldots = 0, \quad (10) \]

where

\[ \eta_0 = 1, \quad \eta_k = \sum_{j=0}^{k} (\beta_{k,j}(l) \dot{\beta}_{k,k-j}(l) - \dot{\beta}_{k,j}(l) \beta_{k,k-j}(l)) \quad (k = 1, 2, 3, \ldots), \]

\[ \beta_{n,k}(l) = \frac{n(n+3) \ldots (n+3k-3)}{(n+3k)!}, \quad \dot{\beta}_{n,k}(l) = \frac{n(n+3) \ldots (n+3k-3)}{(n+3k-2)!} \quad (n = 1, 2, 3)(k = 1, 2, 3, \ldots). \]

Solution of equation (10) is the value of buckling coefficient \( S_{cr} = 18.5687 \). Therefore, according to (9), we have the following final formula for critical load for hinged column

\[ q_{cr} = 18.5687 \frac{EI}{l^3}. \]

After making calculations for most popular design schemes \([17]\), the values of buckling coefficient for different boundary conditions are tabulated in table 1. To verify the results obtained by the author's method, a column was added to the table with the results of calculations from \([17]\). In \([17]\), the values of the buckling coefficient were obtained using generalized hypergeometric function (GHF) and were positioned as accurate. Because the results of this work are based on the exact solution of the corresponding differential equation, the determined values can also be considered accurate. And the fact that the values of the coefficients are equal confirms this.
6. Conclusions
In this work the exact solution of differential equation for the buckling of a uniform rod under the self-weight load has been attained. Therefore, the results obtained should be interpreted as accurate, unlike most reference values identified by approximate methods. The set of formulas for determination the variables of rod state is obtained. An analytical representation is obtained for the critical load, which reduces the calculation to a dimensionless buckling coefficient. In fact, this formula establishes a direct functional dependence of the critical load on the initial parameters of the problem - flexural rigidity and length. These formulas can be used for investigation the buckling problem of column for any boundary conditions; arbitrary value of flexural rigidity and length. The values of buckling coefficient for most popular case of design schemes are obtained. It is the author's opinion that presence of final analytical formulas eliminates the need of further use any approximate methods for problems of this class.

Table 1. Values of buckling coefficient for different boundary conditions.

| Boundary conditions       | The characteristic equation | $S_{cr}$ author’s method | $S_{cr}$ GHF |
|---------------------------|------------------------------|--------------------------|--------------|
| Pinned                    | $y(0) = 0; M(0) = 0$         | 18.5687                  | 18.5687      |
| Pinned                    | $y(l) = 0; M(l) = 0$         |                          |              |
| Free                      | $M(0) = 0; Q(0) = 0$         |                          |              |
| Clamped                   | $y(l) = 0; \varphi(l) = 0$  | 7.8373                   | 7.8373       |
| Clamped                   | $y(0) = 0; \varphi(0) = 0$  |                          |              |
| Clamped                   | $y(l) = 0; \varphi(l) = 0$  | 74.6286                  | 74.6286      |
| Clamped                   | $y(0) = 0; M(0) = 0$         | 52.5007                  | 52.5007      |
| Sliding restraint Clamped | $\varphi(l) = 0; Q(l) = 0$  | 18.9563                  | 18.9563      |
| Clamped                   | $y(l) = 0; \varphi(l) = 0$  |                          |              |

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