Bistable lattice equations: Symmetry groups of periodic stationary solutions

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Abstract

We study spatially periodic stationary solutions of the Nagumo lattice differential equation in a regime with small diffusion intensity. The stationary solutions with a given period can be grouped into classes of equivalent solutions based on symmetries present in the equation. Our aim is to derive counting formulas determining the number of inequivalent solutions. Considering a balanced case, we extend the already known results by including a value permutation – a symmetry which enables an exchange of the two stable states. The case of stationary solutions with a given primitive period is additionally treated in this setting. To derive the desired formulas, we define groups of symmetries for the solutions and use existing results in the theory of enumeration and derive new ones. Possible applications of the ideas presented here to a wider class of graph structures and dynamics are discussed.

Keywords: reaction-diffusion equation; lattice differential equation; graph differential equation; stationary solutions; enumeration; symmetry groups

1 Introduction

In this paper, we consider the Nagumo lattice differential equation (LDE)

\[ u'_i(t) = d(u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)) + f(u_i(t); a), \]

for \( i \in \mathbb{Z}, t > 0 \) with \( d > 0 \) where the nonlinear term \( f \) is given by

\[ f(u; a) = u(1 - u)(u - a), \]

with \( a \in (0, 1) \). We are interested in symmetry properties of periodic stationary solutions of the LDE [1]. Let \( u = (u_i)_{i \in \mathbb{Z}} \) be a periodic stationary solution, then the index shift of the solution \( u_i \rightarrow u_{i-1} \) and the reflection \( u_i \rightarrow u_{-i} \) form an another doubly-infinite sequence \( v \) which is also a solution of the LDE [1]. The solutions \( u \) and \( v \) are then called equivalent and our particular aim is to count inequivalent solutions with respect to those transformations. The counting formulas determining the number of inequivalent solutions with respect to the index shift and the reflection, and other properties of periodic stationary solutions of the LDE [1] were studied in [15, 16]. We extend the results from [16] by including a new symmetry: the value permutation \( u_i \rightarrow 1 - u_i \) which creates an another solution provided \( a = 1/2 \). To obtain the formulas, we define symmetry groups acting on the solutions, use known results from the theory of enumeration [7, 8] and also derive new ones. A separate treatment is given to the case of periodic stationary solutions with a given primitive period. E.g., the constant solution \( u = (\ldots, a, a, a, \ldots) \) is \( n \)-periodic with arbitrary \( n \in \mathbb{N} \) but its primitive period is 1.

The LDE [1] is a special case of a reaction-diffusion differential equation on a lattice. The graph and lattice reaction-diffusion differential equations are used in modelling of dynamical systems whose spatial structure is not continuous but can be described by individual vertices (possibly infinitely many) and their interactions via edges. Such models arise in population dynamics [1], image processing [20], chemistry [19], epidemiology [18] and other fields. An alternative focus lies in the numerical analysis where the graph differential equations describe spatial discretizations of partial differential equations [10, 13]. Mathematically, the interaction between analytic and graph theoretic properties represent new and interesting challenges. The graph and lattice reaction-diffusion differential equations exhibit behaviour which can not be observed in their partial differential equation counterparts such as a rich structure of stationary solutions [24], or other phenomena described in the forthcoming text such as pinning and multichromatic waves.

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The LDE (1) is used as a prototype bistable equation arising from the modelling of a nerve impulse propagation in a myelinated axon [3]. It is known to possess travelling wave solutions of the form

\[ u_i(t) = \varphi(i - ct), \]

\[ \lim_{s \to -\infty} \varphi(s) = 0, \quad \lim_{s \to +\infty} \varphi(s) = 1. \] (2)

As the authors in [17] and [27] have shown, there are nontrivial parameter \((a, d)\)-regimes preventing the solutions of the type (2) from travelling \((c = 0)\) creating the so-called pinning region. This propagation failure phenomenon can be partially clarified by the existence of countably many stable stationary solutions (including periodic ones) of (1) which inhabit mainly the pinning region, see Figure 1. This pinning phenomenon was observed in other lattice systems [25], experimentally in chemistry [19] and also hinted in systems of coupled oscillators [4].

![Figure 1: Numerically computed regions in the \((a, d)\)-plane in which the waves of the type (2) travel (the regions above the two dot-dashed curves) and the pinning region (the region between the \(a\)-axis and the two dot-dashed curves). To better illustrate significance of the heterogeneous \(n\)-periodic stationary solutions of the LDE (1), we include the existence regions for two-periodic stable stationary solutions (dotted edge), three-periodic stable stationary solutions (dashed edge) and four-periodic stable stationary solutions (solid edge).](image)

Besides the significance of the periodic stationary solutions of the LDE (1) in the pinning phenomenon their study is of major importance also in a relation to travelling waves. The waves of the type (2) can be understood as solutions connecting the two stable homogeneous stationary states 0 and 1. The so-called multichromatic waves connect in general two stable stationary solutions of the LDE (1) which are not necessarily homogeneous. These waves are consequently nowhere monotone and have no counterpart in the world of partial differential equations and continuous waves. The bichromatic waves connecting homogeneous and two-periodic solutions were examined in [14]. The tri- and quadrichromatic waves incorporating three- and four-periodic solutions were studied in detail in [15]. The stable periodic stationary solutions of the LDE (1) can be thus understood as building blocks of basic multichromatic waves.

Our aim is to derive counting formulas for inequivalent periodic stationary solutions of the LDE (1) with respect to various symmetries. The notion of an equivalence is a straightforward one to define. Let \( u = (u_i)_{i \in \mathbb{Z}} \) be some \(n\)-periodic stationary solution of the LDE (1) then the index shift \( u_i \to u_{i-1} \) creates an another \(n\)-periodic stationary solution \(v\) which we call equivalent. If we further consider \(a = 1/2\), the value permutation \( u_i \to 1 - u_i \) also creates an another equivalent solution.
This still does not provide a complete insight into a structure and the number of the \( n \)-periodic stationary solutions of the LDE \((1)\) for given \( n \in \mathbb{N} \).

We thus employ the idea from \([16]\) where we have shown a one-to-one correspondence of the LDE \((1)\) \( n \)-periodic stationary solutions and stationary solutions of the Nagumo graph differential equation (GDE) on an \( n \)-vertex cycle

\[
\begin{align*}
\begin{cases}
    u'_1(t) &= d(u_n(t) - 2u_1(t) + u_2(t)) + f(u_1; a), \\
    u'_2(t) &= d(u_1(t) - 2u_2(t) + u_3(t)) + f(u_2; a), \\
    &\vdots \\
    u'_n(t) &= d(u_{n-1}(t) - 2u_n(t) + u_1(t)) + f(u_n; a), \\
    &\vdots \\
    u'_{n}(t) &= d(u_{n-1}(t) - 2u_n(t) + u_1(t)) + f(u_n; a),
\end{cases}
\end{align*}
\]

and, subsequently, with \( n \)-length vectors with elements from the three letter alphabet \( A_3 = \{o, a, 1\} \), also called the *words*. The words encode the origin of the bifurcation branches for \( d = 0 \) which can be extended using the implicit function theorem for \( d > 0 \) small enough. Moreover, the implicit function theorem also implies that the solutions preserve their stability and the asymptotically stable solutions can be thus identified with words created with the two letter alphabet \( A_2 = \{o, 1\} \).

The idea is that we use known concepts from the group theory and the theory of enumeration for groups acting on the word sets \( A_3^d, A_2^d \). Subsequently, we derive claims which are true for the stationary solutions of the GDE \((3)\) and the periodic stationary solutions of the LDE \((1)\). Further consequences for other systems are discussed in §5. Special attention is given to the words which are aperiodic, i.e., the words representing stationary solutions of the LDE \((1)\) with a primitive period of given length. Some formulas are obtained by the direct application of the results from \([7, 8]\), other are obtained by deriving enumeration results which are, to our best knowledge, original. The main tools are the Burnside’s lemma (Theorem 2.3) and the Möbius inversion formula (Theorem 2.4).

The paper is organized as follows. In §2, we comment on the relation of the \( n \)-periodic stationary solutions of the LDE \((1)\), the stationary solutions of the GDE \((3)\) and the words of length \( n \). The word transformations (equivalently, the transformations of the stationary solutions) are formally introduced and motivated therein. The first part of §3 provides counting formulas for the words composed of the three symbol alphabet \( A_3 \) and its two symbol restriction \( A_2 \) regardless of their period. The second part considers the words with primitive period of a given length. We conclude by formulation of a theorem relating the beforehand derived counting formulas to the periodic stationary solutions of the LDE \((1)\) and the stationary solutions of the GDE \((3)\) in §4. Finally, possible extensions and generalizations are discussed in §5.

2 Preliminaries

We start with a presentation of ideas and concepts important for a discussion of known results as well as new results presented in this paper.

2.1 Periodic stationary solutions of the Nagumo LDE

A line of argumentation in this paragraph reproduces the one from \([13, 16]\), further details and comments can be found therein. We also adopt notation where doubly-infinite sequences (e.g., solutions of the LDE \((1)\)) are denoted by italic letters and roman letters are used for vectors (e.g., solutions of the GDE \((3)\) or the words in \( A_3^d, A_2^d \)).

We focus on the periodic stationary solutions of the LDE \((1)\). Searching for a general stationary solution requires solving a countable system of nonlinear analytic equations. The restriction to periodic stationary solutions simplifies the case to a finite-dimensional problem. Indeed, the problem is thus reduced to finding stationary solutions of the GDE \((3)\).

Lemma 2.1 ([16 Lemma 1]). Let \( n \geq 3 \). The vector \( u = (u_1, u_2, \ldots, u_n) \) is a stationary solution of the GDE \((3)\) if and only if its periodic extension \( u \) is an \( n \)-periodic stationary solution of the LDE \((1)\).
Moreover, $u$ is an asymptotically stable solution of the GDE \[ (3) \] if and only if $u$ is an asymptotically stable solution of the LDE \[ (1) \] with respect to the $l^\infty$-norm.

If $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ is a vector then the periodic extension $(u_i)_{i \in \mathbb{Z}} \in \ell^\infty$ of $u$ satisfies $u_i = u_{i + \text{mod}(i,n)}$ for all $i \in \mathbb{Z}$. In the further text, the function $\text{mod}(a, b)$ denotes the remainder of the integer division of $b/a$ for $a, b \in \mathbb{N}$.

Let us denote the function in the right-hand side of the GDE \[ (3) \] by $h(\cdot; a, d)$.

The problem of finding a stationary solution of the GDE \[ (3) \] can be now reformulated as

$$ h(u; a, d) = 0. \tag{5} $$

The solution branch emanating from the root $v_w$ is asymptotically stable if and only if $w \in A_3^0 = \{0, a, 1\}^n$ if it lies on a solution branch emanating from the root $v_w$ of \[ (6) \]

Additionally, the solution $u_w$ of the type $w$ is asymptotically stable if and only if $w \in A_3^0 = \{0, 1\}^n$. The solutions of the GDE \[ (3) \] can be subsequently identified with the words $w \in A_3^0$ thanks to the notion of solution type. Note that the correspondence of
(a) A two-periodic stationary solution of the type 01.

(b) A three-periodic stationary solution of the type 0a2.

(c) A four-periodic stationary solution of the type 0a11.

Figure 2: Examples of two-, three- and four-periodic stationary solutions of the LDE (1). The parameters $a = 0.475$, $d = 0.025$ are set to be identical in all three cases.

The $n$-periodic stationary solutions of the LDE (1) (equivalently the stationary solutions of the GDE (3) on a cycle graph with $n$ vertices) and the words of length $n$ is one-to-one. Indeed, the system (5) consists of $n$ polynomial equations of order 3 which has exactly $3^n$ solutions in our setting, e.g., [26]. Also, see Figure 2 for examples of periodic stationary states of the LDE (1).

Let us note that this naming scheme is considered with $a \in (0, 1)$ fixed. More general naming scheme was developed in [15] which lets $a \in (0, 1)$ variable as well as $d > 0$ since some of the stationary solutions can disappear and reappear while tuning the diffusion intensity $d$ only. We also further specified estimates on critical combinations of the diffusion intensity $d$ and the parameter $a$ for which the particular roots exist.

2.2 Symmetries of stationary solutions

We examine symmetries present in the set of the periodic stationary solutions of the LDE (1) or the stationary solutions of the GDE (3), respectively. The previous introduction of the naming scheme allows us to formalize these symmetry properties in the terms of finite groups acting on the set of the words $\mathcal{A}_3^n$.

Let $u = (u_1, u_2, \ldots, u_n)$ be a stationary solution of the GDE (3) then the rotated vector $v = (u_2, u_3, \ldots, u_n, u_1)$ is also a stationary solution of the same equation. An equivalent claim is valid for their periodic extensions $u$ and $v$ as $n$-periodic stationary solutions of the LDE (1). This motivates the definition of the rotation operator $r_1 : \mathcal{A}_3^n \to \mathcal{A}_3^n$ by

$$ (r_1(w))_i = w_{1 + \text{mod}(i, n)}, $$

for $i = 1, 2, \ldots, n$. Note that if $u \in \mathbb{R}^n$ is the GDE (3) solution of the type $w \in \mathcal{A}_3^n$, then $r_1(u)$ is also a solution to the same equation of the type $r_1(w)$ while similar claim holds for the solutions $u$ of the LDE (1).

The operator $r_1$ generates a finite cyclic group which we denote by

$$ C_n = \langle \{r_0, r_1, \ldots, r_{n-1}\}, \circ \rangle. $$
The group elements are defined by
\[(r_j(w))_i = w_{1+\text{mod}(i+j-1,n)},\]
for \(i = 1,\ldots,n\) and \(j = 0,\ldots,n-1\) and the group operation \(\circ\) is composition of the rotations \(r_i \circ r_j = r_{\text{mod}(i+j,n)}\). Let us note that if \(i\) and \(n\) are coprime, i.e., \(\text{mod}(i,n) = 1\), then \(r_i\) is a generator of the group \(C_n\). This can be clarified by the fact that the group \(C_n\) is isomorphic to the group of modulo \(n\) integers endowed with addition, \(\mathbb{Z}/n\mathbb{Z}\). For example, let \(n = 4\), then the repetitive composition of \(r_3\) gives the sequence \(r_3 \rightarrow r_2 \rightarrow r_1 \rightarrow r_0 \rightarrow r_3 \rightarrow \ldots\) which covers the whole element set of \(C_4\). On the other hand, the composition of \(r_2\) gives \(r_2 \rightarrow r_0 \rightarrow r_2 \rightarrow \ldots\) which does not span the whole element set of \(C_4\).

A second symmetry present in the set of stationary solutions is the reflection. Let \(u = (u_1, u_2, \ldots, u_n)\) be a stationary solution of the GDE (3). The reflected vector \(v = (u_{C_i})\) of \(u\), where \(u_{C_i} \in \mathbb{R}\), is a stationary solution of the GDE (3). The reflected vector \(v = (u_{C_i})\) of \(u\) is also a solution of the same equation. We thus define \(s : \mathcal{A}_3^n \rightarrow \mathcal{A}_3^n\) which is defined by
\[(s(w))_i = w_{n-i+1}.\]
If \(u \in \mathbb{R}^n\) is the GDE (3) solution of the type \(w \in \mathcal{A}_3^n\) then \(s(u)\) is a solution of the type \(s(w)\). A similar statement holds for solutions of the LDE (1).

Adding the reflection \(s\) to the cyclic group \(C_n\) results in a construction of the dihedral group \(D_n\) which is generated by the transformations \(r_1\) and \(s\) via composition. Let us denote the composition of the rotation \(r_i\) and the reflection \(s\) by \(rs_i = r_i \circ s\), i.e., the operation \(rs_i\) is the reflection with the subsequent rotation by \(i\) places. Note that the operations are not commutative in general. For the sake of consistency, we also set \(rs_0 = s\). This allows us to define the group
\[D_n = \{\{r_0, r_1, \ldots, r_{n-1}, rs_0, rs_1, \ldots, rs_{n-1}\}, \circ\}\].

There is however an additional symmetry present in the system. A straightforward computation shows that
\[h(u; a, d) = h(1 - u; 1 - a, d),\]
where \(u \in [0,1]^n\) and the subtraction \(1 - u\) is component-wise. Let us consider \(a = 1/2\). If \(u\) is a stationary solution of the GDE (3), then \(1 - u\) is also a solution of the same equation. We thus define the symbol permutation \(\pi : \mathcal{A}_3^n \rightarrow \mathcal{A}_3^n\) by
\[(\pi(w))_i = \begin{cases} 1, & w_i = \text{o,} \\ a, & w_i = a, \\ \text{o,} & w_i = 1. \end{cases}\]
Note the qualitative difference here, the rotations \(r_i\) and the reflection \(s\) are in fact permutations of the words’ letters. The operation \(\pi\) is not, since it changes the value of the letters instead of their position. As in the case of the rotation \(r_1\) and the reflection \(s\) the symbol permutation \(\pi\) and the solution transformation \(u_i \mapsto 1 - u_i\) are corresponding operations on the stationary solutions of the GDE (3), periodic stationary solutions of the LDE (1) and the words \(\mathcal{A}_2^n\), respectively, provided \(a = 1/2\).

The operation \(\pi\) generates the two element permutation group
\[\Pi = \{\{e, \pi\}, \circ\},\]
where \(e\) is the identity element. The group \(\Pi\) can be also restricted to operate on the set of all word composed with the two letter alphabet \(\mathcal{A}_2\) by
\[(\pi(w))_i = \begin{cases} 1, & w_i = \text{o,} \\ \text{o,} & w_i = 1. \end{cases}\]
To enlighten the notation, we denote the symbol permutation group by the letter \(\Pi\) regardless of the used alphabet.
The main tool to achieve the objective of this paper is to study groups which incorporate the operation \( \pi \) from \( \Pi \) and \( C_n \) or \( D_n \). In the virtue of the previous notation, let us define \( r\pi_i = r_i \circ \pi \) and \( rs\pi_i = r_i \circ s \circ \pi \) and the group \( C_n^\Pi \) by
\[
C_n^\Pi = \left\{ \{ r_0, r_1, \ldots, r_{n-1}, r_s r_0, r_s r_1, \ldots, r_s r_{n-1} \}, \circ \right\}.
\]
Note that the group \( C_n^\Pi \) contains elements from \( C_n \) and the elements from \( C_n \) composed with the symbol permutation \( \pi \). Equivalently, we define the group \( D_n^\Pi \) by
\[
D_n^\Pi = \left\{ \{ r_0, r_1, \ldots, r_{n-1}, r_s r_0, r_s r_1, \ldots, r_s r_{n-1}, r_s^2 r_0, r_s^2 r_1, \ldots, r_s^2 r_{n-1} \}, \circ \right\}.
\]

Let us also emphasize that the action of the groups \( C_n^\Pi \) and \( D_n^\Pi \) preserves the stability of the corresponding solutions.

### 2.3 Group orbits

Based on the group actions, we define equivalence relation which divides the sets of the words \( A_2^n, A_3^n \) into equivalence classes; each of them containing words with similar properties. The Burnside’s lemma (Theorem 2.3) and the Möbius inversion formula (Theorem 2.4) are the main tools for determining the number of the classes and the classes representing words with a given primitive period, respectively.

Action of a group divides its underlying set into subsets called orbits. The orbits are minimal subsets invariant to the group action. In our setting, the orbits contain the words representing the equivalent periodic stationary solutions of the LDE (1) or the equivalent stationary solutions of the GDE (3). These orbits can also be called equivalence classes to highlight the likeness of the solutions’ properties. Indeed, two words \( w_1, w_2 \) are called equivalent if there exists some group operation \( g \) such that \( g(w_1) = w_2 \).

There is also another point of view on how to perceive the action of the groups on the words \( A_2^n \) and the stationary solutions. Let us for example consider the GDE (3). If the underlying cycle graph is not labelled in the sense, that it is free to move and rotate in the plane, the stationary solutions equivalent with respect to the action of the group \( C_n \) are in fact indistinguishable. If the graph is also allowed to rotate in the space, then the solutions equivalent with respect to the action of the group \( D_n \) are indistinguishable. The same holds if one considers the periodic stationary solutions of the LDE (1) when the position of the zero index is not fixed or the positive and negative directions in the lattice are indistinguishable.

**Example 2.2.** There are 27 words of length \( n = 3 \) made with the alphabet \( A_3 = \{a, o, 1\} \):
\[
W_{A_3}(3) = \{000, 00a, 001, 0oa, 0oa, 0aa, 0aa, 10a, 101, 1oa, 1oa, 1aa, 1aa, 110, 110, 11a, 11a, 111\}.
\]

Taking into account the action of the group \( C_3 \), there are 11 equivalence classes:
\[
W_{C_3}^A(3) = \{ \{000\}, \{aaa\}, \{111\}, \{00a, 0oa, 0oo\}, \{001, 01a, 100\}, \{0aa, 0oa, 0ao\}, \{0a1, 10a, 1oa\}, \{10a, 1aa, 001\}, \{aaa, 1aa, 10a\}, \{11a, 11a, 111\}\},
\]

while the action of the group \( D_3 \) merges two classes:
\[
W_{D_3}^A(3) = \{ \{000\}, \{aaa\}, \{111\}, \{00a, 0oa, 0oo\}, \{001, 01a, 100\}, \{0aa, 0oa, 0ao\}, \{0a1, 10a, 1oa, 001\}, \{aaa, 1aa, 10a\}, \{111, 11a, 11a\}\}.
\]

The action of the groups \( C_3^\Pi \) and \( D_3^\Pi \) divides the set of the words into the same system of 6 equivalence classes
\[
W_{C_3}^\Pi(3) = W_{D_3}^\Pi(3) = \{ \{000, 111\}, \{aaa\}, \{00a, 0oa, 000, 11a, 111\}, \{001, 01a, 001, 100, 101, 011\}, \{0aa, 0oa, 0oa, 1aa, 1aa\}, \{0aa, 11a, 11a, 01a\}\}.
\]

See Figure 3 for a graphical illustration of the equivalence classes.
The crucial question is whether we can determine the number of equivalence classes in a systematic manner. A useful tool for this is the Burnside’s lemma [5].

**Theorem 2.3** (Burnside’s lemma). Let \( G \) be a finite group operating on a finite set \( S \). Let \( I(g) \) be the number of set elements such that the group operation \( g \in G \) leaves them invariant. Then the number of group orbits \( O \in \mathbb{N} \) is given by the formula

\[
O = \frac{1}{|G|} \sum_{g \in G} I(g).
\]

The power of the Burnside’s lemma lies in the fact that one counts fixed points of the group operations instead of the orbits themselves. This can be much simpler in many cases as can be seen in the forthcoming sections.

The number of the orbits induced by the action of the group \( C_n \) is usually called the number of the necklaces made with \( n \) beads in two (the alphabet \( A_2 \)) or three (the alphabet \( A_3 \)) colors. The bracelets are induced by the action of the dihedral group \( D_n \). Due to the lack of a common terminology, we call the classes induced by the action of the groups \( C_n^{\Pi} \) and \( D_n^{\Pi} \) the permuted necklaces and the permuted bracelets, respectively.

When interpreting the periodic stationary solutions of the LDE (1) as the periodic extensions of the stationary solutions of the GDE (3) there is one more important subtlety to be considered. In this case, one should be considering the words whose primitive period is equal to their length. For example, the periodic extensions of the words \( w_1 = 01 \) and \( w_2 = 0101 \) form the same doubly infinite sequence \( w = (\ldots 010101 \ldots ) \). The number of equivalence classes of length \( n \) words which represent the words with
The primitive period $n$ can be determined through the Möbius inversion formula [21]. The assumption of the primitive period of a given length together with the action of the cyclic group $C_n$ create classes which are called the Lyndon necklaces. The Lyndon bracelets are a natural counterpart resulting from the action of the dihedral group $D_n$ together with the assumption of the primitive period of given length. The classes representing words with the primitive period of given length without specification of the group are called the Lyndon words. We emphasize that the terminology is not fully unified in the literature but the one presented here suits our purpose best without rising any unnecessary confusion.

**Theorem 2.4** (Möbius inversion formula). Let $f, g : \mathbb{N} \to \mathbb{R}$ be two arithmetic functions such that

$$f(n) = \sum_{d|n} g(d),$$

holds for all $n \in \mathbb{N}$. Then the values of the latter function $g$ can be expressed as

$$g(n) = \sum_{d|n} \mu \left( \frac{n}{d} \right) f(d),$$

where $\mu$ is the Möbius function.

The Möbius function $\mu$ was first introduced in [21] as

$$\mu(n) = \begin{cases} (-1)^{P(n)}, & \text{each prime factor of } n \text{ is present at most once,} \\ 0, & \text{otherwise,} \end{cases}$$

where $P(n)$ is the prime factors’ number of $n$.

Use of the Möbius inversion formula is straightforward. Let us assume, that we know the number $f(n)$ of the equivalence classes induced by the action of one of the above defined groups $(C_n, D_n, C_{\Pi}, D_{\Pi})$ for each $n \in \mathbb{N}$. Then for each $n$, this number $f(n)$ is given by the sum of the equivalence classes’ count representing the words with primitive period of length $d$ dividing $n$. Also note that the group actions preserve the length of the primitive period of each of the words.

In the further text, we extensively exploit two crucial properties of the Möbius inversion formula. Firstly, the formula is linear in the sense, that

$$\sum_{i=1}^{m} \alpha_i f_i(n) = \sum_{d|n} g(d)$$

implies

$$g(n) = \sum_{d|n} \mu \left( \frac{n}{d} \right) \sum_{i=1}^{m} \alpha_i f_i(d) = \sum_{i=1}^{m} \alpha_i \sum_{d|n} \mu \left( \frac{n}{d} \right) f_i(d),$$

and thus, each $f_i$ can be treated separately. Secondly, the product of the indices $n/d, d$ in the sum always results in $n = n/d \cdot d$. We can therefore freely exchange the indices in the following manner

$$g(n) = \sum_{d|n} \mu \left( \frac{n}{d} \right) f(d) = \sum_{d|n} \mu(d) f \left( \frac{n}{d} \right).$$

**Example 2.5.** We complement Example 2.2 with the list of equivalence classes of the length $n = 4$ words made with the alphabet $A_2 = \{0, 1\}$. There exist words of length 4 with the primitive period 2 and thus the set of the equivalence classes and the set of the Lyndon words will differ not by only the trivial constant words $\{0000\}, \{1111\}$. There are 16 words of length 4

$$W_{A_2}(4) = \{0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}.$$ 

We next include the equivalence classes induced by the action of the groups $C_4, D_4, C_{\Pi}, D_{\Pi}$. The words with primitive period of length lesser than 4 are highlighted by a grey color

$$W_{A_2}^{C_4}(4) = W_{A_2}^{D_4}(4) = \{0000, 0011, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\},$$

$$W_{A_2}^{C_{\Pi}}(4) = W_{A_2}^{D_{\Pi}}(4) = \{0000, 0010, 0011, 0100, 0101, 0110, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}. $$
2.4 Known results

Here, we summarize known results relevant to the focus of this paper.

The number of equivalence classes with respect to the action of the groups $C_n$ and $D_n$ and the connection to the stationary solutions of the GDE \((\text{3})\) and the LDE \((\text{1})\) were studied in the paper \([16]\). The results considered all stationary solutions (words from $A_n^3$) as well as the stable solutions (words from $A_n^2$). The Möbius inversion formula was used therein to determine the numbers of the Lyndon necklaces and the Lyndon bracelets.

Fine in \([7]\) considered a more general case of the group $C_{\Pi}n$ which acted on the set of words created with a given number of symbols not necessarily less or equal to 3. The author also simplified the counting formulas for the permuted necklaces and the permuted Lyndon necklaces for the case of two symbols, i.e., the alphabet $A_2$, to the form which also appears in this paper, Lemma 3.3 and 3.10. However, none of the presented results could be directly applied to the case of the transformation $\pi$ acting on the words from $A_n^3$. Formally, the group studied by Fine in \([7]\) was the group product of the cyclic group $C_n$ and a symmetric group $S_k$ (the group of all permutations of $k$ symbols). This coincides with our case only if $k = 2$, i.e., the words are created with the two symbol alphabet $A_2$. If $k = 3$, then the group $C_{\Pi}n$ is isomorphic to the group product $C_n \times G$ where $G$ is only a specific subgroup of $S_3$. Let us also mention that the problem was studied from the combinatorial point of view.

Finally, Gilbert and Riordan in \([8]\) were among other results able to derive a general counting formula for the permuted bracelets and the permuted Lyndon bracelets of words created with an arbitrary number of symbols. As in the case of the necklaces in \([7]\), the results cover the case of the reduced alphabet $A_2$ only. The generality of the presented formulas however comes with a cost of their complexity. Taking an advantage of our more specific setting, we utilize an alternative approach which enables us to further simplify the formulas for the case of the words from $A_n^2$. Moreover, an analogical approach can be used in the derivation of the expressions considering the words created with the full alphabet $A_3$. Also, the focus of the work lied mainly in clarifying certain combinatorial concepts.

3 Counting of equivalence classes

We continue with listing and deriving auxiliary counting formulas as well as those which are directly used to prove the main theorem, Theorem 4.1. We start with the counting of the equivalence classes of the words made with the alphabets $A_2$ and $A_3$ respectively. Further, we extend the results to the counting of the Lyndon words.

In the text, $(m,n)$ denotes the greatest common divisor of $m, n \in \mathbb{N}$.

3.1 Counting of the non-Lyndon words

We start with the counting of the necklaces of length $n$ made with $k$ symbols.

**Lemma 3.1** \((\text{22 p. 162})\). Let $n \in \mathbb{N}$ be given. The number of equivalence classes induced by the action of the group $C_n$ acting on the set of all words of length $n$ made with a $k$-symbol alphabet is

$$N_k(n) = \frac{1}{n} \sum_{d | n} \varphi(d) k^{n/d}. \quad (10)$$

The function $\varphi(n)$ is the Euler totient function which counts relatively coprime numbers to $n$, see \([2]\). For example, there are two numbers coprime to the number 4 (1 and 3) and thus $\varphi(4) = 2$.

Another classical result concerns the number of the bracelets of length $n$ made with $k$ symbols.

**Lemma 3.2** \((\text{22 p. 150})\). Let $n \in \mathbb{N}$ be given. The number of the equivalence classes induced by the action of the group $D_n$ acting on the set of all words of length $n$ made with a $k$-symbol alphabet is

$$B_k(n) = \begin{cases} \frac{1}{2} \left[ N_k(n) + k + \frac{1}{2} k^{n/2} \right], & n \text{ even,} \\ \frac{1}{2} \left[ N_k(n) + k^{n/2+1} \right], & n \text{ odd.} \end{cases} \quad (11)$$
The counting formulas for the necklaces and the bracelets can be derived for a general number of symbols $k$. If we take the symbol permutation $\pi$ into the account, the formulas regarding the alphabets $A_2$ and $A_3$ are slightly different and thus, we treat both cases separately. The main difference is that there are no invariant words with respect to the permutation $\pi : \emptyset \leftrightarrow 1$ with the alphabet $A_2$ and $n$ odd. Indeed, the necessary condition for the invariance is that the word has an equal number of $\emptyset$'s and 1's. This can be bypassed by the use of the symbol $a$ from the alphabet $A_3$.

**Lemma 3.3** ([1], p. 300). Let $n \in \mathbb{N}$ be given. The number of equivalence classes induced by the action of the group $C_n^3$ acting on the set of all words of length $n$ made with the alphabet $A_2$ is

$$N^\pi_{A_2}(n) = \frac{1}{2n} \left[ \sum_{d|n, d \text{ odd}} \varphi(d) 2^{3\pi} + 2 \sum_{d|n, d \text{ even}} \varphi(d) 2^{\pi} \right]. \tag{12}$$

A somewhat similar formula can be derived for the necklaces made with the three-letter alphabet $A_3$.

**Lemma 3.4.** Let $n \in \mathbb{N}$ be given. The number of equivalence classes induced by the action of the group $C_n^3$ acting on the set of all words of length $n$ made with the alphabet $A_3$ is

$$N^\pi_{A_3}(n) = \frac{1}{2n} \left[ \sum_{d|n, d \text{ odd}} \varphi(d) \left(1 + 3^{3\pi}\right) + 2 \sum_{d|n, d \text{ even}} \varphi(d) 3^{3\pi} \right]. \tag{13}$$

**Proof.** The group $C_n^3$ contains the pure rotations $r_i$ and the rotations with the symbol permutations $r_\pi_i$, totalling $2n$ operations. The direct application of the Burnside’s lemma (Theorem 2.3) yields

$$N^\pi_{A_3}(n) = \frac{1}{2n} \left[ \sum_{l=0}^{n-1} I(r_l) + \sum_{l=0}^{n-1} I(r_\pi_l) \right].$$

The expression [10] in the context of Lemma 3.1 shows that

$$\sum_{l=0}^{n-1} I(r_l) = \sum_{d|n} \varphi(d) 3^{3\pi}.$$

Let $l = 0, 1, \ldots, n-1$ be given. Our aim is to express a general form of a word $w$ fixed by the operation $r_\pi_l$. A rotation by $l$ positions induces a permutation of the word’s $w$ letters with $(n, l)$ cycles of length $n/(n, l)$. The word $w$ is then divided into $n/(n, l)$ disjoint subwords of length $(n, l)$. Assume that the first $(n, l)$ letters of the word $w$ are given. The repeated application of the operation $r_\pi_l$ then determines the form of all the remaining subwords of length $(n, l)$. Indeed, the rotation by $l$ positions applied on a word of length $n$ induces a rotation by $l/(n, l)$ positions of the $n/(n, l)$ subwords because $l/(n, l)$ and $n/(n, l)$ are relatively coprime. Here, the parity of the subwords’ number $n/(n, l)$ must be considered. If $n/(n, l)$ is odd, then the only possible word fixed by $r_\pi_l$ is constant $a$’s. The even $n/(n, l)$ allows $3^{(n,l)}$ possible words fixed by $r_\pi_l$.

Let us pick an arbitrary divisor $d$ of $n$. Then, surely $d = n/(n, l)$ for some $l = 0, 1, \ldots, n-1$. The cyclic group $C_d$ with $d$ elements can be generated by $\varphi(d)$ different values each of them being relatively coprime to $d$.

The argumentation above results in

$$N^\pi_{A_3}(n) = \frac{1}{2n} \left[ \sum_{l=0}^{n-1} I(r_l) + \sum_{l=0}^{n-1} I(r_\pi_l) \right],$$

$$= \frac{1}{2n} \left[ \sum_{d|n} \varphi(d) 3^{3\pi} + \sum_{d|n, d \text{ odd}} \varphi(d) + \sum_{d|n, d \text{ even}} \varphi(d) 3^{3\pi} \right].$$

$$= \frac{1}{2n} \left[ \sum_{d|n, d \text{ odd}} \varphi(d) \left(1 + 3^{3\pi}\right) + 2 \sum_{d|n, d \text{ even}} \varphi(d) 3^{3\pi} \right].$$
We now approach to the formulas regarding the group $D_Π^\Pi_n$, i.e., the permuted bracelets. As in the previous text, we treat the cases of the alphabets $A_2$ and $A_3$ separately. A general counting formula regarding the alphabet $A_2$ as a special case was derived in [8]. We present an alternative proof which can be generalized to the case of the alphabet $A_3$.

**Lemma 3.5.** Let $n \in \mathbb{N}$ be given. The number of equivalence classes induced by the action of the group $D_Π^\Pi_n$ acting on the set of all words of length $n$ made with the alphabet $A_2$ is

$$B_{A_2}^\Pi(n) = \begin{cases} \frac{1}{2} \left[ N_{A_2}^\Pi(n) + 2 \frac{n}{2} \right], & n \text{ even}, \\ \frac{1}{2} \left[ N_{A_2}^\Pi(n) + 2 \frac{n - 1}{2} \right], & n \text{ odd.} \end{cases} \quad (14)$$

**Proof.** The group $D_Π^\Pi_n$ contains the rotations $r_i$, the rotations with the reflection $rs_i$, the rotations with the symbol permutation $rπ_i$ and the rotations with the reflection and the symbol permutation $rsπ_i$. The Burnside’s lemma (Theorem 2.3) then yields

$$N_{A_2}^\Pi(n) = \frac{1}{4n} \left[ \sum_{l=0}^{n-1} I(r_l) + \sum_{l=0}^{n-1} I(rπ_l) + \sum_{l=0}^{n-1} I(rs_l) + \sum_{l=0}^{n-1} I(rsπ_l) \right].$$

The equivalence classes induced by the transformations $r_l$ and $rπ_l$ are enumerated in the expression (12) of Lemma 3.3. The second part of each line in formula (11) counts the number of orbits with respect to the rotation with reflection $rs_l$.

We now focus on the operations of the form $rsπ_l$. First, we clarify certain general concepts valid for the operation $rs_l$ and subsequently apply them to the case of $rsπ_l$. The composition of the rotation and the reflection is not commutative in general, but $rs_l = r_l \circ rs_0 = rs_0 \circ r_{n-l}$ holds for $l = 0, 1, \ldots, n - 1$. This formula and the group associativity yields $rs_l \circ rs_l = (r_l \circ rs_0) \circ (rs_0 \circ r_{n-l}) = r_l \circ r_{n-l} = r_0$. Thus, the induced permutation of the word letters has cycles of the length 1 or 2 only.

Let $l = 0, 1, \ldots, n - 1$ be given. Then

$$(rs_l(w))_i = w_{n-l-i+1}$$

for $i \leq \lceil l/2 \rceil$. Due to the composition formula, the positions from $n - l + 1$ to $n$ transform accordingly. This induces the division of the word $w$ into two subwords, see Figure 4 for an illustration. The combined

\[ w: \begin{array}{cccccccc}
1 & 2 & \ldots & n - l - 1 & n - l & n - l + 1 & n - l + 2 & \ldots & n - 1 & n \\
\end{array} \]

Figure 4: An illustration of an operation of the group transformation $rs_l$ on the word $w$ of length $n$. The transformation $rs_l$ divides the word $w$ into two subwords whose elements starting from the edges map to each other.

The parities of $n$ and $l$ determine the parity of the subwords’ length and thus whether there is a middle letter mapped to itself. For any subword of an odd length, there is exactly one loop. All possible combinations are

| $n \backslash l$ | even | odd |
|----------------|------|-----|
| even           | even, even | odd, odd |
| odd            | odd, even | even, odd |

Let us now assume the operation $rsπ_l$. If $n$ is odd, then one of the subwords induced by the action of $rs_l$ is always odd and thus there are no words fixed by $rsπ_l$. If $n$ is even the only possibility for the word
w to be fixed is when \( l \) is also even. There are then \( n/2 \) cycles of length 2 leading to \( n/2 \cdot 2^{n/2} \) words fixed by an operation of the form \( r_s \).

The summing of all cases results in

\[
B^*_{\mathcal{A}_3}(n)_{n \text{ even}} = \frac{1}{4n} \left[ 2n \cdot N^*_{\mathcal{A}_3}(n) + \frac{3n}{2} \cdot 2^\frac{n}{2} + \frac{n}{2} \cdot 2^\frac{n}{2} \right],
\]

\[
= \frac{1}{2} \left[ N^*_{\mathcal{A}_3}(n) + 2^\frac{n}{2} \right],
\]

\[
B^*_{\mathcal{A}_3}(n)_{n \text{ odd}} = \frac{1}{4n} \left[ 2n \cdot N^*_{\mathcal{A}_3}(n) + n \cdot 2^{\frac{n+1}{2}} \right]
\]

\[
= \frac{1}{2} \left[ N^*_{\mathcal{A}_3}(n) + 2^{\frac{n+1}{2}} \right].
\]

\[\square\]

A general idea presented in the proof of Lemma 3.5 can be applied to the case of the three letter alphabet \( \mathcal{A}_3 \).

Lemma 3.6. Let \( n \in \mathbb{N} \) be given. The number of equivalence classes induced by the action of the group \( D_n^\mathbb{N} \) acting on the set of all words of length \( n \) made with the alphabet \( \mathcal{A}_3 \) is

\[
B^*_{\mathcal{A}_3}(n) = \begin{cases} 
\frac{1}{2} \left[ N^*_{\mathcal{A}_3}(n) + \frac{4}{3} \cdot 3^\frac{n}{2} \right], & n \text{ is even,} \\
\frac{1}{2} \left[ N^*_{\mathcal{A}_3}(n) + 2 \cdot 3^{\frac{n-1}{2}} \right], & n \text{ is odd.}
\end{cases}
\]

Proof. As in the proof of Lemma 3.5, the only operations to be considered in detail are of the form \( r_s \).

For the sake of completeness, we note that there are \( 2n \cdot 3^{n/2} \) (for \( n \) even) and \( n \cdot 3^{(n+1)/2} \) (odd) words invariant to the action of transformations of the form \( r_s \), see (11).

Let \( l = 0, 1, \ldots, n - 1 \) be given. The operation \( r_s \) induces a letter permutation with cycles of length 1 or 2. Positions in the cycle of length 1 can contain the letter \( a \) only in order for the word \( w \) to be fixed by the operation \( r_s \).

If \( n \) is odd, then there are \( (n - 1)/2 \) cycles of length 2 leading to \( n \cdot 3^{(n-1)/2} \) fixed words. If \( n \) is even, then there are two cycles of length 1 only if \( l \) is odd. Summing over all \( l = 0, \ldots, n - 1 \) leads to \( n/2 \cdot (3^{n/2-1} + 3^{n/2}) \).

The summary of the results gives

\[
B^*_{\mathcal{A}_3}(n)_{n \text{ even}} = \frac{1}{4n} \left[ 2n \cdot N^*_{\mathcal{A}_3}(n) + 2n \cdot 3^{\frac{n}{2}} + \frac{n}{2} \cdot (3^{\frac{n}{2}-1} + 3^{\frac{n}{2}}) \right],
\]

\[
= \frac{1}{2} \left[ N^*_{\mathcal{A}_3}(n) + \frac{4}{3} \cdot 3^{\frac{n}{2}} \right],
\]

\[
B^*_{\mathcal{A}_3}(n)_{n \text{ odd}} = \frac{1}{4n} \left[ 2n \cdot N^*_{\mathcal{A}_3}(n) + n \cdot 3^{\frac{n+1}{2}} + n \cdot 3^{\frac{n-1}{2}} \right],
\]

\[
= \frac{1}{2} \left[ N^*_{\mathcal{A}_3}(n) + 2 \cdot 3^{\frac{n-1}{2}} \right].
\]

\[\square\]

3.2 Counting of the Lyndon words

To derive forthcoming formulas, we use a special property of the Möbius function \( \mu \) and the Euler totient function \( \varphi \). These functions are multiplicative. An arithmetic function \( \psi : \mathbb{N} \to \mathbb{R} \) is multiplicative if and only if \( \psi(1) = 1 \) and \( \psi(ab) = \psi(a)\psi(b) \) provided \( a \) and \( b \) are coprime. Equality of two multiplicative functions \( \psi_1, \psi_2 \) is sufficiently proved if \( \psi_1(p^\alpha) = \psi_2(p^\alpha) \) for all \( p \) prime and \( \alpha \in \mathbb{N} \). For further information about multiplicative functions see, e.g., [2].

We start with a technical lemma which is used later.

Lemma 3.7. The identity

\[
\sum_{d|n} \mu\left(\frac{n}{d}\right) \frac{n}{d} \varphi(d) = \mu(n),
\]

(16)
holds for any \( n \in \mathbb{N} \). Furthermore, the following identities hold for any \( n \) even,

\[
\sum_{d|n, \text{ even}} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) \bigg|_{n \text{ even}} = -\mu(n), \\
\sum_{d|n, \text{ odd}} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) \bigg|_{n \text{ even}} = 2\mu(n).
\]  (17)  (18)

Proof. The expression \((16)\) is an equality of two multiplicative functions. It is sufficient to verify the formula for \( n = p^\alpha \), where \( p \) is a prime and \( \alpha \in \mathbb{N} \). If \( \alpha \geq 2 \) then \( \mu(p^\alpha) = 0 \) and

\[
\sum_{d|n} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) = \mu(p) p \varphi(p^{\alpha - 1}) + \mu(1) \varphi(p^\alpha) = -p^{\alpha - 1}(p-1) + p^{\alpha - 1}(p-1) = 0,
\]

if \( \alpha = 1 \), then \( \mu(p^\alpha) = -1 \) and

\[
\sum_{d|n} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) = \mu(p) p \varphi(1) + \mu(1) \varphi(p) = -p + p - 1 = -1,
\]

if \( \alpha = 0 \), then \( \mu(p^\alpha) = 1 \) and

\[
\sum_{d|n} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) = \mu(1) 1 \varphi(1) = 1.
\]

This proves \((16)\).

The identity \((17)\) follows from \((16)\) and \((18)\) since

\[
\sum_{d|n} f(d) = \sum_{d|n, \text{ even}} f(d) + \sum_{d|n, \text{ odd}} f(d),
\]

holds for any \( n \in \mathbb{N} \).

Let us assume, that the even integer \( n \in \mathbb{N} \) has the form \( n = 2^\beta P \), where \( P \) is a product of odd primes. We can now rewrite \((18)\) as

\[
\sum_{d|n, \text{ odd}} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) = \sum_{d|n/2^\beta} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d).
\]

If \( \beta \geq 1 \), then the fraction \( n/d \) always contains a squared prime factor and thus \( \mu(n/d) = 0 \) which corresponds to \( \mu(2^\beta P) = 0 \). Suppose \( \beta = 1 \). We can now use the substitution \( m = n/2 \) together with the formula \((16)\)

\[
\sum_{d|n/2} \mu \left( \frac{n}{d} \right) \frac{n}{d} \varphi(d) = \sum_{d|m} \mu \left( \frac{2m}{d} \right) \frac{2m}{d} \varphi(d),
\]

\[
= -2 \sum_{d|m} \mu \left( \frac{m}{d} \right) \frac{m}{d} \varphi(d),
\]

\[
= -2\mu(m) = -2\mu \left( \frac{n}{2} \right) = 2\mu(n).
\]

The first sign change is possible due to the fact that the fraction \( m/d \) is an odd integer and thus 2 is not part of its prime factorization. The second one utilizes the same idea. This concludes the proof of \((18)\). \( \square \)

The counting formula for Lyndon necklaces can be considered a standard one in the theory of enumeration. It can be derived by a direct argument as in [9] but we choose a more technical approach whose idea is useful in the later proofs.
Lemma 3.8. Let $n \in \mathbb{N}$ be given. The number of the Lyndon necklaces (the group $C_n$) with period $n$ on the set of all words of length $n$ made with $k$ symbols is

$$NL_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d. \tag{19}$$

Proof. Since

$$L_k(n) = \sum_{d|n} NL_k(n)$$

holds for all $n \in \mathbb{N}$ the use of the Möbius inversion formula (Theorem 2.4) and the subsequent substitution $d = ml$ yields

$$NL_k(n) = \sum_{m|n} \mu(m) L_k\left(\frac{n}{m}\right),$$

$$= \sum_{m|n} \mu(m) \frac{n}{m} \sum_{l|m/n} \varphi(l) k^{m/n},$$

$$= \frac{1}{n} \sum_{d|n} k^{m/n} \sum_{l|d} \mu\left(\frac{d}{l}\right) \frac{d}{l} \varphi(l),$$

$$= \frac{1}{n} \sum_{d|n} k^{m/n} \mu(d).$$

The last step uses (16).

The counting formula for the Lyndon bracelets is a direct consequence of the Möbius inversion formula (Theorem 2.4) and Lemmas 3.2 and 3.8.

Lemma 3.9. Let $n \in \mathbb{N}$ be given. The number of the Lyndon bracelets (the group $D_n$) with period $n$ on the set of all words of length $n$ made with $k$ symbols is

$$BL_k(n) = \frac{1}{2} \left[NL_k(n) + \sum_{d|n} \mu\left(\frac{n}{d}\right) X_{BL,k}(d)\right], \tag{20}$$

where

$$X_{BL,k}(d) = \begin{cases} 
\frac{k+1}{2} k^{\frac{m}{d}}, & d \text{ is even}, \\
\frac{d+1}{2} k^{\frac{m}{d}}, & d \text{ is odd}.
\end{cases}$$

We continue with the counting formulas for the permuted Lyndon necklaces.

Lemma 3.10 ([7, p. 301]). Let $n \in \mathbb{N}$ be given. The number of the permuted the Lyndon necklaces (the group $C_{\Pi}^n$) with period $n$ on the set of all words of length $n$ made with the alphabet $A_2$ is

$$NL_{A_2}(n) = \frac{1}{2n} \sum_{d|n, d \text{ odd}} \mu(d) 2^{\frac{m}{d}}. \tag{21}$$

As previously mentioned, the statement of Lemma 3.10 cannot be generalized to the case of the three-letter alphabet $A_3$ in a straightforward manner.

Lemma 3.11. Let $n \in \mathbb{N}$ be given. The number of the permuted the Lyndon necklaces (the group $C_{\Pi}^n$) with period $n$ on the set of all words of length $n$ made with the alphabet $A_3$ is

$$NL_{A_3}(n) = \frac{1}{2n} \left[\sum_{d|n, d \text{ odd}} \mu(d) 3^{\frac{m}{d}} + X_{NL}(n)\right], \tag{22}$$
where
\[
X_{NL}(n) = \begin{cases} 
1, & n = 1, \\
-1, & n = 2^\alpha, \alpha \in \mathbb{N}, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** We directly apply the M"obius inversion formula (Theorem 2.4) to (13) in an adjusted form
\[
N_{A_3}^\pi(n) = \frac{1}{2n} \left[ \sum_{d|n} \varphi(d) \frac{3^d}{d} + \sum_{d|n, \text{d even}} \varphi(d) \frac{3^d}{d} + \sum_{d|n, \text{d odd}} \varphi(d) \right].
\]

Thanks to the linearity of the M"obius inversion formula, we may threat the expression summand-wise. For the sake of simplicity, the first summand is readily rewritten in the virtue of Lemma 3.8
\[
\sum_{m|n} \mu(m) N_{A_3}^\pi \left( \frac{n}{m} \right),
\]

We now want to show that
\[
\sum_{m|n} \mu(m) \frac{m}{2n} \sum_{l|n/m, \text{l even}} \varphi(l) \frac{3^d}{d} = -\frac{1}{2n} \sum_{d|n, \text{d even}} \mu(d) \frac{3^d}{d},
\]

which proves the first part of (22). Indeed, the use of the substitution \( d = ml \) in the virtue of the proof of Lemma 3.8 and (17) yields
\[
\sum_{m|n} \mu(m) \frac{m}{2n} \sum_{l|n/m, \text{l even}} \varphi(l) \frac{3^d}{d} = -\frac{1}{2n} \sum_{d|n, \text{d even}} \mu(d) \frac{3^d}{d}. \tag{24}
\]

The rest of the proof is concluded by the evaluation of
\[
\sum_{d|n} \mu(d) \frac{d}{2n} \sum_{l|n/d, \text{l odd}} \varphi(l).
\]

Any number \( m \in \mathbb{N} \) can be expressed as \( m = 2^\alpha P \) where \( \alpha \in \mathbb{N}_0 \) and \( P \) is a product of odd primes. Then
\[
\frac{1}{m} \sum_{d|m, \text{d odd}} \varphi(d) = \frac{P}{m} = \frac{1}{2^\alpha}, \tag{23}
\]

since
\[
\sum_{d|m} \varphi(d) = m,
\]

holds in general [2].

Assume now that \( n \in \mathbb{N} \) can be expressed as \( n = 2^\beta Q \) where \( \beta \in \mathbb{N}_0 \) and \( Q \) is a product of odd primes. Let us turn our attention to the equality
\[
\frac{1}{n} X_{NL}(n) = \sum_{d|n} \mu(d) \frac{d}{n} \sum_{l|n/d, \text{l odd}} \varphi(l).
\]

Any \( d|n \) can be expressed as \( 2^\gamma R \) where \( 0 \leq \gamma \leq \beta \) and \( R \) is a product of odd primes. Decomposing the expression by the exponent \( \gamma \) and using (23) leads to
\[
\frac{1}{n} X_{NL}(n) = \sum_{\gamma=0}^{\beta} \sum_{R|Q} \mu(2^\gamma R) \frac{1}{2^{\beta-\gamma}} = \sum_{\gamma=0}^{\beta} \frac{1}{2^{\beta-\gamma}} \sum_{R|Q} \mu(2^\gamma R). \tag{24}
\]

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Assume that \( \beta = 0 \) and \( Q = 1 \). A straightforward computation gives

\[
\left. \frac{1}{n} X_{NL(n)} \right|_{n=1} = \mu(1) \cdot 1 = 1.
\]

Assume that \( \beta > 1 \) and \( Q = 1 \). If we consider \( \gamma \geq 2 \) in (24), then \( \mu(2^\gamma R) = 0 \). The sum can be now evaluated

\[
\left. \frac{1}{n} X_{NL(n)} \right|_{n=2^\beta} = \mu(1) \frac{1}{2^\beta} + \mu(2) \frac{1}{2^\beta - 1} = -\frac{1}{2^\beta} = -\frac{1}{n}.
\]

Assume that \( Q > 1 \). Let us fix \( 0 \leq \gamma < \beta \). Without loss of generality, we can assume that \( \gamma \leq 1 \) and each prime factor in \( R \) is present at most once. Indeed, \( \mu(2^\gamma R) = 0 \) otherwise. The sign of the nonzero expression \( \mu(2^\gamma R) \) is now dependent on the number of prime factors present in \( R \). If there are \( m \) prime factors in \( Q \), then \( R \) with \( l \) factors can be chosen in \( \binom{m}{l} \) possible ways. The expression \( \mu(2^\gamma R) \) switches signs as \( l \) increases and we have

\[
\sum_{l=0}^{m} (-1)^l \binom{m}{l} = 0.
\]

Indeed,

\[
(1 + x)^m = \sum_{l=0}^{m} \binom{m}{l} x^l,
\]

\[
0 = (1 + x)^m \bigg|_{x=-1} = \sum_{l=0}^{m} (-1)^l \binom{m}{l}.
\]

This results in

\[
\left. \frac{1}{n} X_{NL(n)} \right|_{n=2^\beta Q} = 0.
\]

We conclude this section with two lemmas whose statements are direct consequences of the Möbius inversion formula (Theorem 2.4), Lemma 3.5 (3.6, respectively) and Lemma 3.8.

**Lemma 3.12.** Let \( n \in \mathbb{N} \) be given. The number of the permuted Lyndon bracelets (the group \( D_{\Pi}^H \)) with period \( n \) on the set of all words of length \( n \) made with the alphabet \( A_2 \) is

\[
BL_{A_2}(n) = \frac{1}{2} \left[ NL_{A_2}(n) + \sum_{d|n} \mu \left( \frac{n}{d} \right) X_{BL,2}(d) \right],
\]

where

\[
X_{BL,2}(d) = \begin{cases} 2^{\frac{d}{2}}, & d \text{ is even}, \\ 2^{\frac{d-1}{2}}, & d \text{ is odd}. \end{cases}
\]

**Lemma 3.13.** Let \( n \in \mathbb{N} \) be given. The number of the permuted Lyndon bracelets (the group \( D_{\Pi}^H \)) with period \( n \) on the set of all words of length \( n \) made with the alphabet \( A_3 \) is

\[
BL_{A_3}(n) = \frac{1}{2} \left[ NL_{A_3}(n) + \sum_{d|n} \mu \left( \frac{n}{d} \right) X_{BL,3}(d) \right],
\]

where

\[
X_{BL,3}(d) = \begin{cases} 4 \cdot 3^{\frac{d}{2}}, & d \text{ is even}, \\ 2 \cdot 3^{\frac{d-1}{2}}, & d \text{ is odd}. \end{cases}
\]
4 Conclusion

The counting formulas for equivalence classes induced by the action of the groups without the symbol permutation $\pi$ are summarized in [16]. We formulate a similar theorem regarding the groups containing the symbol permutation $\pi$, namely Lemmas 3.3 – 3.6 and 3.10 – 3.13.

**Theorem 4.1.** Let $n \in \mathbb{N}$, $a = 1/2$ and $d > 0$ sufficiently small be given. The number of inequivalent $n$-periodic LDE (1) stationary solutions or, equivalently, the number of inequivalent GDE (3) stationary solutions with respect to the action of the symbol permutation $\pi$ composed with

1. the translations (rotations) (constituting the group $C_\Pi$) is $N_\pi(A_3(n))$ of which $NL_\pi(A_3(n))$ are asymptotically stable solutions (formulas (13) and (12), respectively). Moreover, there are $NL_\pi(A_3(n))$ stationary solutions with primitive period of length $n$ of which $NL_\pi(A_3(n))$ are asymptotically stable (formulas (22) and (21), respectively),

2. the translations (rotations) and the reflection (constituting the group $D_\Pi$) is $B_\pi(A_3(n))$ of which $BL_\pi(A_3(n))$ represent asymptotically stable solutions (formulas (15) and (14), respectively). Moreover, there are $BL_\pi(A_3(n))$ stationary solutions with primitive period of length $n$ of which $BL_\pi(A_3(n))$ are asymptotically stable (formulas (26) and (25), respectively).

**Proof.** The implicit function theorem ensures that there are $3^n$ stationary solutions of the GDE (3) provided $d > 0$ is sufficiently small, see §2.1. This implies the existence of $3^n$ $n$-periodic stationary solutions of the LDE (1) via Lemma 2.1. Let $u_w$ be a stationary solution of the type $w$ of the GDE (3). The assumption $a = 1/2$ ensures, that the action of any of the group operations $r_l$, $rs_l$, $r_{l\pi}$ or $rs_{l\pi}$ for $l = 0, 1, \ldots, n - 1$ (here denoted by $g$) on the solution $u_w$ results in the vector $v := g(u_w)$, which is also a stationary solution of the GDE (3). Moreover, the one-to-one correspondence ensures that $v$ is of the type $g(w)$.

The enumeration of the equivalence classes on the set of the words with $n$ letters is thus equivalent to the enumeration on the set of stationary solutions of the GDE (3) and the $n$-periodic stationary solutions of the LDE (1).

The counting formulas for stable solutions result from the fact, that the stable solutions correspond to words made with the reduced two-letter alphabet $A_2$.

**Example 4.2.** We include lists of the formulas’ values, Table 1 and Table 2. Note that the numbers of the sets in Examples 2.2 and 2.5 are in the accordance with the tables.

| $n$ | $3^n$ | $N_3(n)$ | $NL_3(n)$ | $N_\Pi(A_3(n))$ | $NL_\Pi(A_3(n))$ | $B_3(n)$ | $BL_3(n)$ | $B_\Pi(A_3(n))$ | $BL_\Pi(A_3(n))$ |
|-----|------|--------|--------|----------------|----------------|--------|--------|----------------|----------------|
| 1   | 3    | 3      | 3      | 2              | 2              | 3      | 3      | 2              | 2              |
| 2   | 9    | 6      | 3      | 4              | 2              | 6      | 3      | 4              | 2              |
| 3   | 27   | 11     | 8      | 6              | 4              | 10     | 7      | 6              | 4              |
| 4   | 81   | 24     | 18     | 14             | 10             | 21     | 15     | 13             | 9              |
| 5   | 243  | 51     | 48     | 26             | 24             | 39     | 36     | 22             | 20             |
| 6   | 729  | 130    | 116    | 68             | 60             | 92     | 79     | 52             | 44             |
| 7   | 2187 | 315    | 312    | 158            | 156            | 198    | 195    | 106            | 104            |
| 8   | 6561 | 834    | 810    | 424            | 410            | 498    | 477    | 266            | 253            |
| 9   | 19683| 2195   | 2184   | 1098           | 1092           | 1219   | 1209   | 630            | 624            |
| 10  | 59049| 5934   | 5880   | 2980           | 2952           | 3210   | 3168   | 1652           | 1628           |

Table 1: The numbers of equivalence classes for the words made with the three-symbol alphabet $A_3$. See Lemmas 3.1 – 3.4 and the expressions therein.
| $n$ | $2^n$ | $N_2(n)$ | $NL_2(n)$ | $N^\Pi_{A_2}(n)$ | $NL^\Pi_{A_2}(n)$ | $B_2(n)$ | $BL_2(n)$ | $B^\Pi_{A_2}(n)$ | $BL^\Pi_{A_2}(n)$ |
|-----|------|--------|--------|------------------|------------------|-------|--------|------------------|------------------|
| 1   | 2    | 2      | 2      | 1                | 1                | 2     | 2      | 1                | 1                |
| 2   | 4    | 3      | 1      | 2                | 1                | 3     | 1      | 2                | 1                |
| 3   | 8    | 4      | 2      | 2                | 1                | 4     | 2      | 2                | 1                |
| 4   | 16   | 6      | 3      | 4                | 2                | 6     | 3      | 4                | 2                |
| 5   | 32   | 8      | 6      | 4                | 3                | 8     | 6      | 4                | 3                |
| 6   | 64   | 14     | 9      | 8                | 5                | 13    | 8      | 5                |                   |
| 7   | 128  | 20     | 18     | 10               | 9                | 18    | 16     | 9                | 8                |
| 8   | 256  | 36     | 30     | 20               | 16               | 30    | 24     | 18               | 14               |
| 9   | 512  | 60     | 56     | 30               | 28               | 46    | 42     | 23               | 21               |
| 10  | 1024 | 108    | 99     | 56               | 51               | 78    | 69     | 44               | 39               |

Table 2: The numbers of equivalence classes for the words made with the two-symbol alphabet $A_2$. See Lemmas 3.1, 3.2, 3.3, 3.5, 3.8, 3.9, 3.10, 3.12 and the expressions therein.

Let us consider Example 2.2 first:

\[ |W_{A_3}(3)| = 3^3 = 27, \]
\[ |W_{C_3}^{A_3}(3)| = N_3(3) = 11, \]
\[ |W_{D_3}^{A_3}(3)| = B_3(3) = 10, \]
\[ |W_{C_3}^{A_3}(3)| = |W_{D_3}^{A_3}(3)| = N^\Pi_{A_3}(3) = B^\Pi_{A_3}(3) = 6. \]

Further, the difference

\[ N_3(3) - NL_3(3) = B_3(3) - BL_3(3) = 3 \]

holds. It is caused by the 3 classes of words \{000\}, \{aaa\}, \{111\} with primitive period of length 1. A similar claim holds for the number of equivalence classes considering also the symbol permutations

\[ N^\Pi_{A_3}(3) - NL^\Pi_{A_3}(3) = B^\Pi_{A_3}(3) - BL^\Pi_{A_3}(3) = 2, \]

which is caused by the classes \{000, 111\}, \{aaa\}.

In Example 2.3, we have

\[ |W_{A_2}(4)| = 2^4 = 16, \]
\[ |W_{C_2}^{A_2}(4)| = |W_{D_2}^{A_2}(4)| = N_2(4) = B_2(4) = 6, \]
\[ |W_{C_2}^{A_2}(4)| = |W_{D_2}^{A_2}(4)| = N^\Pi_{A_2}(4) = B^\Pi_{A_2}(4) = 4, \]

and

\[ NL_2(4) = BL_2(4) = 3, \]
\[ NL^\Pi_{A_2}(4) = BL^\Pi_{A_2}(4) = 2. \]

5 Extensions and open questions

The approach presented in this paper can be used to obtain similar results in other or more general settings. The two main extension directions are the change of dynamics and the change of a spatial structure. The extensions can be possibly combined but for the sake of clarity, we present them separately.
5.1 Change of dynamics

5.1.1 Polynomial nonlinearity of higher order

This paper focused on the model (1) with the cubic bistable nonlinearity

\[ f(u; a) = u(1 - u)(u - a). \]

The idea presented in §2.1 can be extended to a general polynomial nonlinearity provided it allows a spatially nonhomogeneous steady state of the LDE (1) or the GDE (3)

\[ f_{\text{ext}}(u; a_1, \ldots, a_q) = u(1 - u) \prod_{i=1}^{q} (u - a_i) \]

for \( q \geq 3 \) odd, \( a_i \in (0, 1) \) and \( a_i \neq a_j \) for all \( i, j = 1, \ldots, q \). The counting formulas for the necklaces (10), the bracelets (11) and the Lyndon words (19), (20) can be straightforwardly applied with \( k = q + 2 \) for all solutions and \( k = 2 + (q - 1)/2 \) for asymptotically stable solutions. A symmetry of the values \( a_i \) around \( 1/2 \) is required in order to be reasonable to consider a symbol permutation. One should also expect the relevant counting formulas to be more intricate.

5.1.2 General bistable nonlinearity

A system with a general bistable nonlinearity \( f_{\text{gen}} \) as considered in [17]

1. \( f_{\text{gen}}(0) = f_{\text{gen}}(a) = f_{\text{gen}}(1) = 0, 0 < a < 1 \) and \( f_{\text{gen}}(x) \neq 0 \) for \( x \neq 0, a, 1 \),
2. \( f_{\text{gen}}(x) < 0 \) for \( 0 < x < a \) and \( f_{\text{gen}}(x) > 0 \) for \( a < x < 1 \),
3. \( f'_{\text{gen}}(x_0) = f'_{\text{gen}}(x_1) = 0, 0 < x_0 < a < x_1 < 1 \) and \( f'_{\text{gen}}(x) \neq 0 \) for \( x \neq x_0, x_1 \),

can be also treated since it enables the use of the implicit function theorem, see §2.1.

5.1.3 Multi-dimensional local dynamics

The local dynamics at an isolated vertex of models (1) and (3) are one-dimensional since the behaviour at a single vertex can be described by one ODE. This is not always the case in many reaction-diffusion models. For example, the Lotka-Volterra competition model on a graph as in [23]

\[ u_i'(t) = d_u \sum_{j \in N(i)} (u_j(t) - u_i(t)) + \rho_u u_i(t)(1 - u_i(t) - \alpha v_i(t)), \]
\[ v_i'(t) = d_v \sum_{j \in N(i)} (v_j(t) - v_i(t)) + \rho_v v_i(t)(1 - \beta u_i(t) - v_i(t)), \]

where \( N(i) \) is the set of all neighbours of the vertex \( i \), locally possesses two asymptotically stable stationary solutions and one unstable nontrivial steady state at each separated vertex provided \( \alpha, \beta > 1 \). The stationary solutions of the system with zero diffusion intensity can be thus continued. However, the application of our results is not straightforward since some of the continued solutions are not component-wise positive.

5.2 Change of a Spatial Structure

5.2.1 Graphs with nontrivial automorphism

An underlying structure of the GDE (3) is a cycle graph. If suitable reaction diffusion dynamics is applied upon any graph with a nontrivial automorphism (a nontrivial self map of the graph which preserves the edge-vertex connectivity) such as, e.g., Petersen graph or a wheel graph, the approach presented here can be used in a similar manner.
5.2.2 Multi-dimensional square lattices

A spatial structure of the LDE (1) is a lattice, an infinite path graph. An examination of bistable reaction diffusion systems on multi-dimensional square lattices is continuously being carried out, see e.g., [6, 12, 11]. For example, let us have a bistable reaction-diffusion system on a two-dimensional square lattice

\[
    u'_{i,j}(t) = d (u_{i-1,j}(t) + u_{i+1,j}(t) + u_{i,j-1}(t) + u_{i,j+1}(t) - 4u_{i,j}(t)) + f(u_{i,j}(t); a),
\]

for \(i, j \in \mathbb{Z}\). A reproduction of the proof of Lemma 2.1 shows that the stationary solutions of the LDE (27) in the form of a repeated 2x2 pattern are equivalent to the stationary solutions of the GDE (3) on 4 vertices with the doubled diffusion coefficient \(d\), see Figure 5 for an illustration.

![Figure 5: A graphical illustration of a possible extension of Lemma 2.1 for patterns on two-dimensional lattices. A general idea is that the edges crossing the dashed line are wrapped back inside from the opposite sides.](image)

5.3 Open Questions

The example in §5.2 with a two-dimensional square lattice and 2x2 pattern applied the idea from Lemma 2.1 to a specific multi-dimensional case. Can there be derived a more general lemma in the virtue of Lemma 2.1 for a general class of lattices, e.g., hexagonal lattice, triangular lattice, etc., and an arbitrary pattern which covers the whole lattice?

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