Equivalence between Sobolev spaces of first-order dominating mixed smoothness and unanchored ANOVA spaces on $\mathbb{R}^d$

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Abstract

We prove that a variant of the classical Sobolev space of first-order dominating mixed smoothness is equivalent (under a certain condition) to the unanchored ANOVA space on $\mathbb{R}^d$, for $d \geq 1$. Both spaces are Hilbert spaces involving weight functions, which determine the behaviour as different variables tend to $\pm \infty$, and weight parameters, which represent the influence of different subsets of variables. The unanchored ANOVA space on $\mathbb{R}^d$ was initially introduced by Nichols & Kuo in 2014 to analyse the error of quasi-Monte Carlo (QMC) approximations for integrals on unbounded domains; whereas the classical Sobolev space of dominating mixed smoothness was used as the setting in a series of papers by Griebel, Kuo & Sloan on the smoothing effect of integration, in an effort to develop a rigorous theory on why QMC methods work so well for certain non-smooth integrands with kinks or jumps coming from option pricing problems. In this same setting, Griewank, Kuo, Leövey & Sloan in 2018 subsequently extended these ideas by developing a practical smoothing by preintegration technique to approximate integrals of such functions with kinks or jumps.

We first prove the equivalence in one dimension (itself a non-trivial task), before following a similar, but more complicated, strategy to prove the equivalence for general dimensions. As a consequence of this equivalence, we analyse applying QMC combined with a preintegration step to approximate the fair price of an Asian option, and prove that the error of such an approximation using $N$ points converges at a rate close to $1/N$.

1 Introduction

In this paper we establish equivalence between a variant of the classical Sobolev (Hilbert) space $\mathcal{H} = \mathcal{H}_d$ of real-valued functions with first-order dominating mixed smoothness on $\mathbb{R}^d$, and a reproducing kernel Hilbert space $\mathcal{W} = \mathcal{W}_d$ introduced in [31]. Throughout this paper we will refer to $\mathcal{H}$ as a “Sobolev space”, and refer to $\mathcal{W}$ as an “ANOVA space” due to an intimate connection with the ANOVA decomposition of functions. (Specifically, the ANOVA decomposition of a function in $\mathcal{W}$, or $\mathcal{H}$, is an orthogonal decomposition with respect to the inner product in $\mathcal{W}$, see [62] below). Both spaces involve weight functions (see $\rho_j$ and $\psi_j$ below) to control the behaviour of the functions and their mixed derivatives as the $j$th variable $x_j$ goes to $\pm \infty$. Both spaces also involve weight parameters (see $\gamma_u$ below) that moderate the relative importance of subsets of variables. We prove that the spaces $\mathcal{H}$ and $\mathcal{W}$ are equivalent provided that the weight functions satisfy a certain condition (see [5] below).

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Motivation

The motivation for this work requires a side trip in this introduction. Since the mid 1990s, Quasi-Monte Carlo (QMC) methods \cite{8, 9, 32, 36} have been a powerful tool for practitioners, but presented a challenge to theorists, arising from the unexpected success of QMC methods in tackling practical high dimensional integrals from mathematical finance—a popular example being the pricing of path-dependent options. Many research papers, e.g., \cite{4, 6, 29, 34, 35, 38}, demonstrated empirically that applying QMC methods to a range of finance problems gave significantly faster convergence rates than the commonly used Monte Carlo (MC) simulations. Yet it is surprising that QMC beats MC for these problems, since both the classical and more recent QMC theory cannot be applied, because the integrands typically involve “kinks” or “jumps” (i.e., integrands and/or their first partial derivatives are not continuous) and so fail to satisfy the smoothness requirements of the theory.

One approach to explaining the success of QMC for finance problems was by way of the concept of effective dimension \cite{7, 37}. In principle any \( d \)-variate function can be expressed in terms of its unique ANOVA (ANalysis Of VAriance) decomposition

\[
    f = \sum_{u \subseteq D} f_u,
\]

where \( D := \{1, 2, \ldots, d\} \), each term \( f_u \) depends only on the variables \( x_j \) with indices \( j \in u \), and the \( 2^d \) terms are \( L^2 \) orthogonal (with respect to a weight function; see below). It is generally accepted that QMC works well when \( f \) has a low truncation dimension (i.e., \( f \) is dominated by the contributions from ANOVA terms involving only a small number of early variables) or a low superposition dimension (i.e., \( f \) is dominated by a few ANOVA terms, each involving only a small number of variables). Although in those papers an explicit ANOVA decomposition was not carried out in practice, the handwaving justification for the QMC success was that, behind the scenes, QMC (for some initially unexplained reason) approximates well the low-dimensional contributions to the integral, while the remaining high-dimensional contributions collectively make a negligible contribution. Recent work has developed rigorous definitions of truncation \cite{26} and superposition \cite{10} dimensions in certain function space settings, but for finance applications the problem remains that typical integrands fail to satisfy the smoothness assumptions.

A series of papers \cite{16, 17, 18, 19} provided partial justification for the “low effective dimension” argument, by proving that most of the ANOVA terms of the option pricing integrands are smooth. Specifically, it was proved in \cite{18} that, with the single exception of the very last term with \( u = D \), all \( 2^d - 1 \) other ANOVA terms \( f_u \) belong to the Sobolev space \( \mathcal{H} \) (details to be given below). A subsequent paper \cite{20} took the theory one step further by developing a practical smoothing by preintegration technique for cubature over \( \mathbb{R}^d \), whereby a strategically chosen coordinate is integrated out first (either analytically or numerically using a high precision 1-dimensional quadrature rule) to yield a new function involving the remaining \( d - 1 \) variables, and a high-dimensional cubature rule can then be applied to the resulting “preintegrated” function, which by the theory in \cite{20} was shown to belong to the Sobolev space \( \mathcal{H}_{d-1} \). However, the paper \cite{20} could not at that time be used to guarantee the success of QMC combined with preintegration because the necessary QMC analysis had been carried out not in \( \mathcal{H} \) but in \( \mathcal{W} \).

Since QMC theory has the unit cube as its natural setting, its extension to the unbounded region \( \mathbb{R}^d \) has necessitated development of a new theoretical setting. In particular, the paper \cite{31} derived a new reproducing kernel Hilbert space (RKHS) \( \mathcal{W} \) and proved that the generating vector for a randomly shifted lattice rule can be constructed using a component-by-component algorithm to achieve the optimal rate of convergence.
In summary, on the one hand much progress has been made in justifying the “low effective dimension” argument [16, 17, 18, 19] and then in developing from it a practical preintegration technique [20], all in the setting of the Sobolev space $H$; while on the other hand a constructive QMC theory has been developed in the setting of $W$ [31]. But this QMC methodology could not validly be used with preintegration without knowledge of the relationship between the two spaces.

This issue is completely resolved in the present paper. We show that the two spaces $H$ and $W$ are indeed equivalent, under an appropriate condition on the weight functions, with embedding constants expressed explicitly in terms of the weight functions and weight parameters. As a result there is now available a complete QMC-based strategy, with solid theoretical foundations, for tackling the high-dimensional integrals arising from option pricing.

Before moving on, we note that the basic idea of smoothing by preintegration is not original to [20], and is a special case of conditioning or conditional sampling, see, e.g., [11, Sec. 7.2.3]. Indeed, several conditional sampling methods using different quadrature rules (such as MC, QMC and sparse grids) have previously been applied to option pricing problems in, e.g., [2, 3, 5, 12, 20, 25, 39]. The contribution of [20] was to formalise the notion of preintegration, and prove that the $(d-1)$-dimensional preintegrated function will be sufficiently smooth.

Having provided the background motivation for this work, in the remainder of this introduction we summarise our key results, and discuss how our present paper relates to recent work [13, 14, 15, 21, 22, 24, 27] on embeddings of similar spaces.

The 1-dimensional case

Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a probability density function defined on $\mathbb{R}$, and let $\psi : \mathbb{R} \to \mathbb{R}_+$ be a locally integrable function such that $1/\psi$ is also locally integrable. Let $\gamma > 0$ be a weight parameter (it plays little role in one dimension). Starting from the set of all locally integrable functions on $\mathbb{R}$, the Sobolev space $H$ and the unanchored ANOVA space $W$ each contains those functions for which the respective norm is finite:

\begin{align}
\|f\|_H^2 &= \int_{-\infty}^{\infty} |f(x)|^2 \rho(x) \, dx + \frac{1}{\gamma} \int_{-\infty}^{\infty} |f'(x)|^2 \psi(x) \, dx, \\
\|f\|_W^2 &= \left| \int_{-\infty}^{\infty} f(x) \, \rho(x) \, dx \right|^2 + \frac{1}{\gamma} \int_{-\infty}^{\infty} |f'(x)|^2 \psi(x) \, dx.
\end{align}

Here $f'$ is the weak derivative, which is defined to be the locally integrable function satisfying

$$
\int_{-\infty}^{\infty} f'(x)v(x) \, dx = -\int_{-\infty}^{\infty} f(x) \frac{d}{dx}v(x) \, dx
$$

for all smooth functions $v$ with compact support. Thus the functions in $H$ are square-integrable against the weight function $\rho$, and their first derivatives are square-integrable against the weight function $\psi$. On the other hand, the functions in $W$ only needs to be integrable against the weight function $\rho$. We summarise this as

$$
f \in H \iff f \in L^2_{\rho} \quad \text{and} \quad f' \in L^2_{\psi},
$$

$$
f \in W \iff f \in L^1_{\rho} \quad \text{and} \quad f' \in L^2_{\psi}.
$$

It follows from the Cauchy–Schwarz inequality that $\|f\|_W^2 \leq \|f\|_H^2$ and so $H$ is embedded in $W$. Obviously $H$ is embedded in $L^2_{\rho}$, but $W$ may or may not be embedded in $L^2_{\rho}$.
Since trivially \( \|f\|^2_W \leq \|f\|^2_H \leq \|f\|^2_{L^2}\rho + \|f\|^2_W \), we see that the two spaces \( H \) and \( W \) are equivalent if and only if \( W \) is embedded in \( L^2_{\rho^*} \).

We remark that the two norms (1) and (2) differ in just their first terms, but to establish the norm equivalence we need to make use of their common second term involving the derivative of \( f \). This hinges upon the interplay between the two weight functions \( \rho \) and \( \psi \).

Let

\[
\Phi(x) := \int_{-\infty}^{x} \rho(t) \, dt
\]

(3)

denote the distribution function corresponding to the density \( \rho \). Throughout we consider two different conditions on the relationship between the weight functions \( \rho \) and \( \psi \). In Section 2 we show that \( W \) is a RKHS if the pair of weight functions \( (\rho, \psi) \) satisfies the weaker condition

\[
\int_{-\infty}^{c} \frac{(\Phi(t))^2}{\psi(t)} \, dt < \infty \quad \text{and} \quad \int_{c}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \, dt < \infty \quad \text{for all finite } c.
\]

(4)

Furthermore, \( W \) is embedded in \( L^2_{\rho} \) if the pair \( (\rho, \psi) \) satisfies the stronger condition

\[
\int_{-\infty}^{c} \frac{\Phi(t)}{\psi(t)} \, dt < \infty \quad \text{and} \quad \int_{c}^{\infty} \frac{1 - \Phi(t)}{\psi(t)} \, dt < \infty \quad \text{for all finite } c.
\]

(5)

This allows us in this paper under the stronger condition to establish the norm equivalence

\[
\|f\|^2_W \leq \|f\|^2_H \leq (1 + \gamma C(\rho, \psi)) \|f\|^2_W,
\]

(6)

with

\[
C(\rho, \psi) := \int_{-\infty}^{\infty} \frac{\Phi(t)(1 - \Phi(t))}{\psi(t)} \, dt < \infty,
\]

(7)

where finiteness is due to (5). Hence we conclude that the two spaces \( H \) and \( W \) are equivalent under the stronger condition (5).

The reproducing kernel for \( W \) was derived in [31] under the assumption that the stronger condition (5) holds from the outset, and so \( W \) is embedded in \( L^2_{\rho} \) by assumption. The question of whether the reproducing property exists under the weaker condition (4) was not considered in that paper. Moreover, the results in [31] were not proved in the generality claimed there and this is repaired in the current work. Note additionally that in this paper we write \( \psi(x) \) instead of \( (\psi(x))^2 \), which was the notation in [31].

The condition (4) on its own is not sufficient to establish the equivalence of \( W \) and \( H \). This can be seen by choosing \( \rho \) to be the standard normal density and \( \psi(x) := \exp(-\frac{3}{4}x^2) \), noting that with this choice (4) is satisfied but (5) is not; yet the function \( f(x) = 1/\sqrt{\rho(x)} \) belongs to \( W \), but not to \( H \), making \( W \) strictly larger than \( H \). This example also shows that the equivalence shown in this paper is not a trivial consequence of standard embeddings.

The \( d \)-dimensional case

Consider now a potentially different pair of weight functions \( (\rho_j, \psi_j) \) for each coordinate index \( j = 1, \ldots, d \), and a weight parameter \( \gamma_u \) for every subset \( u \subseteq D \). As in [31] we denote the distribution function of \( \rho_j \) by \( \Phi_j \) for each \( j \).
In [31] the ANOVA space \( \mathcal{W} \) was extended to \( d \) dimensions by defining its reproducing kernel to be a particular sum of products of 1-dimensional kernels. This particular representation gives an impression that the resulting function space \( \mathcal{W}_d \) may not include all functions in the classical Sobolev space \( \mathcal{H}_d \). To close this loophole, in this paper we will not define \( \mathcal{W}_d \) in terms of the reproducing kernel, and additionally do not assume any product structure. Instead we will define the spaces in complete analogy to our one dimensional case, as follows.

Starting from the set of all locally integrable functions on \( \mathbb{R}^d \), the Sobolev space \( \mathcal{H}_d \) and the unanchored ANOVA space \( \mathcal{W}_d \) each contains those functions for which the respective norm is finite:

\[
\|f\|_{\mathcal{H}_d}^2 := \sum_{u \subseteq D} \frac{1}{\gamma_u} \int_{\mathbb{R}^d} |\partial^\mu f(x)|^2 \psi_u(x_u) \rho_{D\setminus u}(x_{D\setminus u}) \, dx,
\]

\[
\|f\|_{\mathcal{W}_d}^2 := \sum_{u \subseteq D} \frac{1}{\gamma_u} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d-|u|}} |\partial^\mu f(x_u, x_{D\setminus u})|^2 \rho_{D\setminus u}(x_{D\setminus u}) \, dx_{D\setminus u} \, dx_u,
\]

where \( \partial^\mu f \) denotes the weak derivative (see [11] below) of \( f \) respect to the “active variables” (the ones being differentiated), \( x_u := \{x_j : j \in u\} \), which in turn are weighted by the product \( \psi_u(x_u) := \prod_{j \in u} \psi_j(x_j) \), while the “inactive variables” are weighted by the product \( \rho_{D\setminus u}(x_{D\setminus u}) := \prod_{j \in D\setminus u} \rho_j(x_j) \). Since each \( \rho_j \) is a probability density function, the Cauchy–Schwarz inequality implies that \( \|f\|_{\mathcal{W}_d}^2 \leq \|f\|_{\mathcal{H}_d}^2 \) and therefore \( \mathcal{H}_d \) is embedded in \( \mathcal{W}_d \). We also know that \( \mathcal{H}_d \) is embedded in \( \mathcal{L}_\rho^2 \), with \( \rho(x) := \prod_{j=1}^d \rho_j(x_j) \). The question is again whether or not \( \mathcal{W}_d \) is embedded in \( \mathcal{L}_\rho^2 \).

We prove in Section 3 that \( \mathcal{W}_d \) is a reproducing kernel Hilbert space if the weaker condition [(1)] holds for all pairs of weight functions \( (\rho_j, \psi_j) \), and furthermore that \( \mathcal{W}_d \) is indeed embedded in \( \mathcal{L}_\rho^2 \) if the stronger condition [(2)] holds for all pairs \( (\rho_j, \psi_j) \). In turn, with the condition [(2)] we prove the norm equivalence

\[
\|f\|_{\mathcal{W}_d}^2 \leq \|f\|_{\mathcal{H}_d}^2 \leq \left( \max_{v \subseteq D} \sum_{u \subseteq v} \gamma_u \prod_{j \in v \setminus u} C(\rho_j, \psi_j) \right) \|f\|_{\mathcal{W}_d}^2,
\]

where \( C(\rho_j, \psi_j) \) is defined as in [(7)] for each \( j \). In the special case of product weights, i.e., there is a weight parameter \( \gamma_j \) associated with each coordinate \( x_j \), and \( \gamma_u := \prod_{j \in u} \gamma_j \), the embedding constant (squared) in [(10)] is precisely

\[
\prod_{j=1}^d (1 + \gamma_j C(\rho_j, \psi_j)),
\]

which is simply the product of the constant in [(6)], and is bounded independently of \( d \) provided that \( \sum_{j=1}^\infty \gamma_j C(\rho_j, \psi_j) < \infty \).

Knowing an explicit and simple formula for the reproducing kernel of \( \mathcal{W}_d \) (see Theorem [11] below) allowed the development of QMC theory in [31], namely, the construction of randomly shifted lattice rules that achieve the optimal rate of convergence. Note that if [(2)] holds then, because \( \mathcal{H}_d \) with inner product corresponding to the norm [(8)] is equivalent to \( \mathcal{W}_d \), we conclude that \( \mathcal{H}_d \) is also a reproducing kernel Hilbert space, but with a kernel that is unknown as well as likely more complicated, hence our preference for working with \( \mathcal{W}_d \).

**Implication for smoothing by preintegration applied to option pricing problems**

As discussed earlier in this introduction, there is a gap in the analysis of QMC methods combined with preintegration. The theory on smoothing by preintegration from [20] exists
for the space $H_d$, whereas the error analysis of QMC methods giving a root-mean-square (RMS) error close to $O(1/N)$, where $N$ is the number of function evaluations, assumes that the integrand belongs to the space $W_d$ (see [31, Theorem 8]). The equivalence of the two spaces $H_d$ and $W_d$ bridges this gap, and an important consequence is we can now show that QMC methods combined with preintegration can achieve a RMS error close to $O(1/N)$ for some option pricing problems; explicit details are given in Section 4.

Other embedding and related results

In a series of related papers [13, 14, 15, 21, 22, 24, 27] different combinations of authors established, in a variety of settings, continuous embeddings between ANOVA spaces and so-called “anchored” spaces. While all of these papers considered functions defined on $d$-dimensional spaces, it is easier to explain the concept in the case $d = 1$. In this case the anchored equivalent (with anchor at zero) of the squared norms defined in (1) and (2) is

$$
\|f\|_\text{anch}^2 := |f(0)|^2 + \frac{1}{\gamma} \int_{-\infty}^{\infty} |f'(x)|^2 \psi(x) \, dx.
$$

In the present paper we do not consider anchored spaces. Note also that by continuous embedding we mean that the identity mapping from one space into the other is a bounded linear operator. Often we will simply use the term embedding, which should be understood as a continuous embedding.

The first such paper [21] studies embeddings of tensor products of 1-dimensional Hilbert spaces with product weights $\{\gamma_j\}$. The setting in one dimension is quite general, but explicit examples cover only the bounded domain $[0, 1]$. It is possible to put our 1-dimensional spaces $H$ and $W$ into the setting of [21], however due to the tensor product structure used there it is not possible to extend those results to our $d$-dimensional spaces. Furthermore, the constants arising from the general theory in [21] are not as sharp as those we obtain in (6), see also Remark 8. The paper [22] extends [21] to Banach spaces involving $L^p$ norms on $[0, 1]^d$ with general weights $\{\gamma_u\}$, and provides embedding constants between the ANOVA and anchored spaces for $p = 1$ and $p = \infty$. The paper [21] extends [22] to general $p \in [1, \infty]$ by interpolation. The paper [27] provides lower bounds on the norm of the embedding operator, again for bounded domains and ANOVA and anchored spaces. The paper [14] extends [21] to higher order derivatives and $d = \infty$. The paper [15] considers spaces with increasing smoothness and $d = \infty$.

Apart from [21], the other paper close to the present work is [14], in that it deals with general weights and unbounded domain, though with integrals restricted to $\mathbb{R}_d^+$. The function spaces are also defined in a different way, through convolutions with integral kernels rather than derivatives, and $\psi$ is restricted to be equal (in our notation) to $\rho$. The condition on $\rho$ assumed in that paper can be stated as

$$
\left\| \frac{1 - \Phi}{\rho} \right\|_{L^\infty(\mathbb{R}_d^+)} < \infty \quad \text{and} \quad \int_{\mathbb{R}_d^+} x \rho(x) \, dx < \infty.
$$

It can easily be seen that this condition (when adapted to the whole real line) is stronger than our condition [14] but weaker than [15]. Clearly, embedding theory for Sobolev-type spaces in high dimensions is an active area of research, however, it would seem not possible to infer the results in the present paper from the equivalence results for ANOVA and anchored spaces.

Another possible explanation for the success of QMC for option pricing was proposed in [23], where it was suggested that Besov spaces are more suitable for the analysis of functions with kinks. However, [23] deals only with Besov spaces of periodic functions on
the unit cube and only considers products of simple kink functions on $[0, 1]$. As such, the analysis there does not apply to real-world option pricing problems.

## 2 The 1-dimensional case

Let $\rho : \mathbb{R} \to \mathbb{R}_+$ be a strictly positive probability density function defined on $\mathbb{R}$, and let $\psi : \mathbb{R} \to \mathbb{R}_+$ be a locally integrable strictly positive function such that $1/\psi$ is also locally integrable. For any locally integrable function $f$ on $\mathbb{R}$, we define the $\rho$-weighted integral

$$I_\rho(f) := \int_{-\infty}^{\infty} f(x) \rho(x) \, dx,$$

along with the $L^2_\rho$ and $L^2_\psi$ norms

$$\|f\|_{L^2_\rho} := \int_{-\infty}^{\infty} |f(x)|^2 \rho(x) \, dx, \quad \|f\|_{L^2_\psi} := \int_{-\infty}^{\infty} |f(x)|^2 \psi(x) \, dx.$$

The Sobolev space $\mathcal{H}$ and the unanchored ANOVA space $\mathcal{W}$ are the restriction of the set of locally integrable functions on $\mathbb{R}$ for which the norms (11) and (12), respectively, are finite. Equivalently, the norms can be written as

$$\|f\|^2_\mathcal{H} = \|f\|_{L^2_\rho}^2 + \frac{1}{\gamma} \|f'\|_{L^2_\psi}^2, \quad (11)$$

$$\|f\|^2_\mathcal{W} = |I_\rho(f)|^2 + \frac{1}{\gamma} \|f'\|_{L^2_\psi}^2. \quad (12)$$

Thus for $f \in \mathcal{H}$ we have $f \in L^2_\rho$ and $f' \in L^2_\psi$, and the norm in $\mathcal{H}$ is a weighted $L^2$-norm involving first derivatives, but with differentiated functions weighted differently from undifferentiated functions. On the other hand, for $f \in \mathcal{W}$ we have $f \in L^1_\rho$ and $f' \in L^2_\psi$. It is well-known that $\mathcal{H}$ is complete, see, e.g., [30, Section 1.1.12]. Since we could not find a proof that $\mathcal{W}$ is complete in the literature, we provide one in the following lemma.

**Lemma 1.** If the condition (11) holds, then the space $\mathcal{W}$ is complete.

**Proof.** Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $\mathcal{W}$. Since, for any $k, \ell \in \mathbb{N}$,

$$|I_\rho(f_k) - I_\rho(f_\ell)| \leq \|f_k - f_\ell\|_\mathcal{W} \quad \text{and} \quad \|f'_k - f'_\ell\|_{L^2_\psi} \leq \sqrt{7} \|f_k - f_\ell\|_\mathcal{W},$$

it follows that $\{I_\rho(f_k)\}$ is a Cauchy sequence in $\mathbb{R}$ and $\{f'_k\}$ is a Cauchy sequence in $L^2_\psi(\mathbb{R})$. We denote the respective limits by

$$\lim_{k \to \infty} I_\rho(f_k) =: I \in \mathbb{R} \quad \text{and} \quad \lim_{k \to \infty} f'_k =: g \in L^2_\psi(\mathbb{R}). \quad (13)$$

We prove that $f_k$ converges to some $f \in \mathcal{W}$. Let $a \in \mathbb{R}$ and define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \int_a^x g(t) \, dt + B, \quad \text{with} \quad B := I - \int_{-\infty}^{\infty} \left( \int_a^x g(t) \, dt \right) \rho(x) \, dx,$$

implying $I_\rho(f) = I$ because $B$ is finite, as we will now show. To that end, consider

$$\int_{-\infty}^{\infty} \left( \int_a^x g(t) \, dt \right) \rho(x) \, dx = \int_{-\infty}^{\infty} \left( a - \int_a^x g(t) \, dt \right) \rho(x) \, dx + \int_a^{\infty} \left( \int_a^x g(t) \, dt \right) \rho(x) \, dx,$$
where we have split the outer integral based on whether \( x \leq a \) or \( x > a \) and swapped the limits for the inner integral in the first term. Taking the absolute value then using the triangle inequality and Fubini’s Theorem we obtain

\[
\left| \int_{-\infty}^{\infty} \left( \int_{a}^{t} g(t) \, dt \right) \rho(x) \, dx \right| \\
\leq \left| \int_{-\infty}^{a} g(t) \left( \int_{-\infty}^{t} \rho(x) \, dx \right) \, dt \right| + \left| \int_{a}^{\infty} g(t) \left( \int_{t}^{\infty} \rho(x) \, dx \right) \, dt \right| \\
= \left| \int_{-\infty}^{a} g(t) \Phi(t) \, dt \right| + \left| \int_{a}^{\infty} g(t)(1 - \Phi(t)) \, dt \right| \\
\leq \left( \int_{-\infty}^{a} \frac{\Phi(t)^2}{\psi(t)} \, dt \right)^{1/2} \left( \int_{-\infty}^{a} |g(t)|^2 \psi(t) \, dt \right)^{1/2} \\
+ \left( \int_{a}^{\infty} \frac{|1 - \Phi(t)|^2}{\psi(t)} \, dt \right)^{1/2} \left( \int_{a}^{\infty} |g(t)|^2 \psi(t) \, dt \right)^{1/2} \\
\leq \|g\|_{\mathcal{L}_\psi^2} \left[ \left( \int_{-\infty}^{a} \frac{\Phi(t)^2}{\psi(t)} \, dt \right)^{1/2} + \left( \int_{a}^{\infty} \frac{|1 - \Phi(t)|^2}{\psi(t)} \, dt \right)^{1/2} \right] < \infty,
\]

the finiteness following because \( g \in \mathcal{L}_\psi^2(\mathbb{R}) \) and by assumption (i). Hence, \( |B| < \infty \).

Next we show \( f \in \mathcal{W} \). Since \( g \in \mathcal{L}_\psi^2(\mathbb{R}) \), it follows that \( g \) is also locally integrable. Indeed, let \( \Omega \subset \mathbb{R} \) be compact then because \( 1/\psi \) is locally integrable

\[
\int_{\Omega} g(t) \, dt \leq \left( \int_{\Omega} |g(t)|^2 \psi(t) \, dt \right)^{1/2} \left( \int_{\Omega} \frac{1}{\psi(t)} \, dt \right)^{1/2} \\
\leq \|g\|_{\mathcal{L}_\psi^2} \left( \int_{\Omega} \frac{1}{\psi(t)} \, dt \right)^{1/2} < \infty.
\]

Since \( g \) is locally integrable, it follows that \( f \) is absolutely continuous, and by the Fundamental Theorem of Calculus the classical derivative of \( f \) is equal to \( g \) almost everywhere. Since the classical derivative exists, the weak derivative exists as well and is equal to the classical derivative almost everywhere. This implies that \( f' = g \) almost everywhere in \( \mathbb{R} \). Then using [40] Sec. 1.1.2, Theorem], \( f' \) locally integrable implies that \( f \) is also locally integrable.

Hence, \( f \in \mathcal{W} \) since

\[
\|f\|_{\mathcal{W}}^2 = |I_\rho(f)|^2 + \|f'|_{\mathcal{L}_\psi^2}^2 = |I|^2 + \|g\|_{\mathcal{L}_\psi^2}^2 < \infty.
\]

Finally, by construction \( f_k \to f \) in \( \mathcal{W} \) because (13) implies

\[
\|f - f_k\|_{\mathcal{W}}^2 = |I_\rho(f - f_k)|^2 + \|f' - f'_k\|_{\mathcal{L}_\psi^2}^2 = |I - I_\rho(f_k)|^2 + \|g - f'_k\|_{\mathcal{L}_\psi^2}^2 \to 0,
\]
as \( k \to \infty \). Thus, every Cauchy sequence in \( \mathcal{W} \) converges and so \( \mathcal{W} \) is complete. \( \square \)

It follows from the definition of the norms that the functions in \( \mathcal{H} \) and \( \mathcal{W} \) are continuous and absolutely continuous. The spaces \( \mathcal{H} \) and \( \mathcal{W} \) are both Hilbert spaces, with the respective inner products

\[
\langle f, \tilde{f} \rangle_{\mathcal{H}} : = \int_{-\infty}^{\infty} f(x) \tilde{f}(x) \rho(x) \, dx + \frac{1}{\gamma} \int_{-\infty}^{\infty} f'(x) \tilde{f}'(x) \psi(x) \, dx,
\]

\[
\langle f, \tilde{f} \rangle_{\mathcal{W}} : = \left( \int_{-\infty}^{\infty} f(x) \rho(x) \, dx \right) \left( \int_{-\infty}^{\infty} \tilde{f}(x) \rho(x) \, dx \right) + \frac{1}{\gamma} \int_{-\infty}^{\infty} f'(x) \tilde{f}'(x) \psi(x) \, dx.
\]

\[8\]
The question to be addressed is whether the spaces are identical; a question whose answer is not obvious even in this 1-dimensional case. In one direction, it is immediately clear that $\mathcal{H}$ is a subset of $\mathcal{W}$, in that by the Cauchy–Schwarz inequality we have

$$\|I_\rho(f)\|^2 = \left| \int_{-\infty}^{\infty} f(x) \rho(x) \, dx \right|^2 \leq \left( \int_{-\infty}^{\infty} |f(x)|^2 \rho(x) \, dx \right) \left( \int_{-\infty}^{\infty} \rho(x) \, dx \right) = \|f\|^2_{L_\rho^2}.$$  

We conclude easily that $\|f\|^2_{\mathcal{W}} \leq \|f\|^2_{\mathcal{H}}$, and therefore the space $\mathcal{H}$ is embedded in $\mathcal{W}$. The central purpose of this section is to prove that embedding holds in the opposite direction under an equivalent condition, see (5).

The parameter $\gamma > 0$ in (1)–(2) and (11)–(15) is a weight parameter that controls the contribution of $\|f'\|^2_{L_\psi^2}$ relative to $\|f\|^2_{L_\rho^2}$ or $(I_\rho(f))^2$ for functions in the unit ball of $\mathcal{H}$ or $\mathcal{W}$. A small $\gamma$ means that $\|f'\|^2_{L_\psi^2}$ must be small. In the limiting case where $\gamma = 0$, we assume that $f' \equiv 0$ so the spaces contain constant functions. The weight parameter $\gamma$ does not play an essential role in the 1-dimensional setting but it will become significant later when we extend the setting to higher dimensions.

Let $\Phi$ denote the distribution function corresponding to the density $\rho$, see (3). In Subsection 2.1 we will show that $\mathcal{W}$ is a RKHS if the pair of weight functions $(\rho, \psi)$ satisfies the weaker condition (4). In Subsection 2.2 we will show that $\mathcal{W}$ is embedded in $L_\psi^2$ if the pair $(\rho, \psi)$ satisfies the stronger condition (5). In turn we will establish the norm equivalence between $\mathcal{H}$ and $\mathcal{W}$.

### 2.1 The reproducing kernel for $\mathcal{W}$ exists under the weaker condition (4)

Before we proceed to find the reproducing kernel for $\mathcal{W}$, we give a couple of remarks on the conditions (4), (3) and (7).

**Remark 2.** Since $0 \leq \Phi(t) \leq 1$, the condition (3) is clearly stronger than the condition (4). We now verify that (7) and (4) are equivalent. For any finite $c$

$$\int_{-\infty}^{\infty} \frac{\Phi(t) (1 - \Phi(t))}{\psi(t)} \, dt = \int_{-\infty}^{c} \frac{\Phi(t) (1 - \Phi(t))}{\psi(t)} \, dt + \int_{c}^{\infty} \frac{\Phi(t) (1 - \Phi(t))}{\psi(t)} \, dt$$

$$\leq \int_{-\infty}^{c} \frac{\Phi(t)}{\psi(t)} \, dt + \int_{c}^{\infty} \frac{1 - \Phi(t)}{\psi(t)} \, dt,$$

which shows that (4) implies (7). Since $\Phi$ is the cumulative distribution function of the probability density $\rho$, there exists some $a \leq c$ such that $\Phi(a) < 1$. Using

$$\Phi(x) = \frac{\Phi(x) (1 - \Phi(x))}{1 - \Phi(a)} \leq \frac{\Phi(x) (1 - \Phi(x))}{1 - \Phi(a)}$$

for all $x \in (-\infty, a]$,

we obtain

$$\int_{-\infty}^{c} \frac{\Phi(x)}{\psi(x)} \, dx = \int_{-\infty}^{a} \frac{\Phi(x)}{\psi(x)} \, dx + \int_{a}^{c} \frac{\Phi(x)}{\psi(x)} \, dx$$

$$\leq \frac{1}{1 - \Phi(a)} \int_{-\infty}^{c} \frac{\Phi(x) (1 - \Phi(x))}{\psi(x)} \, dx + \int_{a}^{c} \frac{1}{\psi(t)} \, dx,$$

where the first term is finite if (7) holds and the second is finite because $1/\psi$ is locally integrable. Finiteness of the second integral in (4) can be shown in a similar way. This shows that (7) implies (4) and hence they are equivalent.
By the reverse triangle inequality, we can bound this from below as follows:

\[ \frac{1}{4} (1 - \frac{1}{t^2}) \exp \left( - \frac{1}{2} t^2 \right) \leq \int_{t}^{\infty} \exp \left( - \frac{1}{2} x^2 \right) \, dx \leq \frac{1}{4} \exp \left( - \frac{1}{2} t^2 \right) \quad \text{for all } t > 0, \]

which are obtained using asymptotic expansions from, e.g., [1]. Indeed, if \( \psi(x) = \exp \left( - \frac{1}{2a^2} x^2 \right) \) then \([3]\) holds if and only if \( \alpha \geq 1/2 \), but \([4]\) holds if and only if \( \alpha > 1 \).

We first establish the following technical lemma. This result was used implicitly in \([31]\) without proof.

**Lemma 4.** Let \( f : \mathbb{R} \to \mathbb{R} \) be locally integrable, with \( f \in \mathcal{L}^1_{\rho} \) and \( f' \in \mathcal{L}^2_{\psi} \). If the condition \([3]\) holds, then

\[
\begin{align*}
    f(x) \Phi(x) & \to 0 & x & \to -\infty, \quad \text{and} \quad (16) \\
    f(x) (1 - \Phi(x)) & \to 0 & x & \to +\infty. \quad (17)
\end{align*}
\]

**Proof.** We prove the first limit (16) by contradiction; the proof for the second limit (17) follows analogously.

To simplify the notation, we define \( F(x) := f(x) \Phi(x) \) so that we must now show that \( F(x) \to 0 \) as \( x \to -\infty \). Note \( F \) is absolutely continuous because both \( f \) and \( \Phi \) are absolutely continuous, and since \( \Phi \) is the distribution function of \( \rho \), the derivative of \( F \) is given by

\[
F'(x) = f'(x) \Phi(x) + f(x) \rho(x).
\]

We first show that \( F' \) is integrable on \((-\infty, c]\) for any finite \( c \in \mathbb{R} \). By the triangle inequality and the Cauchy–Schwarz inequality, we can bound the integral by

\[
\int_{-\infty}^{c} |F'(x)| \, dx \leq \int_{-\infty}^{c} |f'(x)| \Phi(x) \, dx + \int_{-\infty}^{c} |f(x)| \rho(x) \, dx \\
\leq \left( \int_{-\infty}^{c} |f'(x)|^2 \psi(x) \, dx \right)^{1/2} \left( \int_{-\infty}^{c} \left( \Phi(x) \right)^2 \frac{1}{\psi(x)} \, dx \right)^{1/2} + \int_{-\infty}^{c} |f(x)| \rho(x) \, dx \\
\leq \|f'\|_{\mathcal{L}^2_{\psi}} \left( \int_{-\infty}^{c} \left( \frac{\Phi(x)}{\psi(x)} \right)^2 \, dx \right)^{1/2} + \|f\|_{\mathcal{L}^1_{\rho}} < \infty,
\]

where the finiteness follows from the assumptions \( f \in \mathcal{L}^1_{\rho} \), \( f' \in \mathcal{L}^2_{\psi} \), and \([3]\). Hence, \( F' \) is integrable on the interval \((-\infty, c]\).

For a contradiction, suppose that (16) does not hold, in which case there exists \( \delta > 0 \), \( M \in \mathbb{N} \) and a sequence \( \{t_m\} \) with \( t_m \to -\infty \) as \( m \to \infty \), such that \( t_m \leq c \) and \( |F(t_m)| = |f(t_m)| \Phi(t_m) \geq \delta \) for all \( m \geq M \).

For any \( m \geq M \), let \( x \in (-\infty, t_m) \) be arbitrary. Then by the Fundamental Theorem of Calculus we can write

\[
F(x) = F(t_m) - \int_{x}^{t_m} F'(t) \, dt.
\]

By the reverse triangle inequality, we can bound this from below as follows

\[
|F(x)| \geq |F(t_m)| - \left| \int_{x}^{t_m} F'(t) \, dt \right| \geq \delta - \int_{x}^{t_m} |F'(t)| \, dt \geq \delta - \int_{-\infty}^{t_m} |F'(t)| \, dt.
\]
Since $F'$ is integrable on $(-\infty, c]$, we now can choose $m \geq M$ large enough such that
\[ \int_{-\infty}^{t_m} |F'(t)| \, dt \leq \frac{\delta}{2}, \]
in which case we have the bound
\[ |F(x)| \geq \frac{\delta}{2} \text{ for all } x \leq t_m. \]
This in turn implies that
\[ |f(x)| \geq \frac{\delta}{2 \Phi(x)} \text{ for all } x \leq t_m. \]
Thus
\[ \|f\|_{L^1_\rho} \geq \int_{-\infty}^{t_m} |f(x)| \rho(x) \, dx \geq \frac{\delta}{2} \int_{-\infty}^{t_m} \frac{\rho(x)}{\Phi(x)} \, dx = \frac{\delta}{2} \int_{-\infty}^{t_m} \frac{d}{dx} \left[ \log(\Phi(x)) \right] \, dx. \]
The last integral is divergent, as can be seen by
\[ \int_{-R}^{t_m} \frac{d}{dx} \left[ \log(\Phi(x)) \right] \, dx = \log(\Phi(t_m)) - \log(\Phi(-R)) = \log(\Phi(t_m)) + \log \left( \frac{1}{\Phi(-R)} \right) \to \infty \text{ as } R \to \infty, \quad (18) \]
because $\Phi(-R) \to 0$ as $R \to \infty$. This contradicts the fact that $f \in L^1_\rho$ and so the result (16) must hold.

We now arrive at the main result of this subsection, which proves that the reproducing kernel exists for $\mathcal{W}$ under the weaker condition (4). This result seems not previously known because in [31] the stronger condition (5) was assumed from the outset.

**Theorem 5.** If the condition (4) holds, then $\mathcal{W}$ is a reproducing kernel Hilbert space with kernel
\[ K(x, y) := 1 + \gamma \eta(x, y), \quad (19) \]
where
\[ \eta(x, y) := \int_{-\infty}^{\min(x,y)} \frac{\Phi(t)^2}{\psi(t)} \, dt + \int_{-\infty}^{\max(x,y)} \frac{1 - \Phi(t)^2}{\psi(t)} \, dt - \int_{-\infty}^{\min(x,y)} \frac{\Phi(t)(1 - \Phi(t))}{\psi(t)} \, dt \quad (20) \]

**Proof.** First we observe that $K$ is symmetric, and can be written in two other ways:
\[ \eta(x, y) = \int_{-\infty}^{\max(x,y)} \frac{\Phi(t)^2}{\psi(t)} \, dt + \int_{\max(x,y)}^{\infty} \frac{1 - \Phi(t)^2}{\psi(t)} \, dt - \int_{\min(x,y)}^{\max(x,y)} \frac{\Phi(t)}{\psi(t)} \, dt \]
\[ \leq \eta(\max(x, y), \max(x, y)), \quad (21) \]
and
\[ \eta(x, y) = \int_{-\infty}^{\min(x,y)} \frac{\Phi(t)^2}{\psi(t)} \, dt + \int_{\min(x,y)}^{\infty} \frac{1 - \Phi(t)^2}{\psi(t)} \, dt - \int_{\max(x,y)}^{\min(x,y)} \frac{1 - \Phi(t)}{\psi(t)} \, dt \]
\[ \leq \eta(\min(x, y), \min(x, y)). \quad (22) \]
Moreover, we have
\[
0 \leq \eta(y, y) = \int_{-\infty}^{y} \frac{(\Phi(t))^2}{\psi(t)} \, dt + \int_{y}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \, dt < \infty,
\]
where the integrals are finite due to (11). Thus \( K(x, y) < \infty \) for all \( x, y \in \mathbb{R} \).

To verify that \( \mathcal{W} \) is a RKHS with kernel given by (13), we have to show that (i) \( K(\cdot, y) \in \mathcal{W} \) for all \( y \in \mathbb{R} \), and (ii) \( f(y) = \langle f, K(\cdot, y) \rangle_{\mathcal{W}} \) for all \( f \in \mathcal{W} \) and \( y \in \mathbb{R} \).

For any \( y \in \mathbb{R} \) we use (21) for \( x \leq y \) and (22) for \( x > y \) to write
\[
\int_{-\infty}^{\infty} \eta(x, y) \rho(x) \, dx
= \int_{-\infty}^{y} \left( \int_{-\infty}^{y} \frac{(\Phi(t))^2}{\psi(t)} \, dt + \int_{y}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \, dt - \int_{y}^{x} \frac{\Phi(t)}{\psi(t)} \, dt \right) \rho(x) \, dx
+ \int_{y}^{\infty} \left( \int_{-\infty}^{y} \frac{(\Phi(t))^2}{\psi(t)} \, dt + \int_{y}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \, dt - \int_{y}^{x} \frac{1 - \Phi(t)}{\psi(t)} \, dt \right) \rho(x) \, dx
= \int_{-\infty}^{y} \frac{(\Phi(t))^2}{\psi(t)} \, dt + \int_{y}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \, dt
- \int_{y}^{\infty} \frac{\Phi(t)}{\psi(t)} \left( \int_{-\infty}^{t} \rho(x) \, dx \right) \, dt - \int_{y}^{\infty} \frac{1 - \Phi(t)}{\psi(t)} \left( \int_{t}^{\infty} \rho(x) \, dx \right) \, dt
= 0,
\]
where in the second step we used the Fubini theorem to change the order of integration. Thus
\[
\int_{-\infty}^{\infty} K(x, y) \rho(x) \, dx = 1 + \gamma \int_{-\infty}^{\infty} \eta(x, y) \rho(x) \, dx = 1.
\] (24)

Now we have
\[
\frac{\partial}{\partial x} K(x, y) = \gamma \frac{\partial}{\partial x} \eta(x, y), \quad \text{with}
\]
\[
\frac{\partial}{\partial x} \eta(x, y) = 1_{(-\infty, y)}(x) \frac{\Phi(x)}{\psi(x)} - 1_{(y, \infty)}(x) \frac{1 - \Phi(x)}{\psi(x)},
\] (26)
which follows easily by differentiating (21) for \( x \leq y \) and (22) for \( x > y \). Thus
\[
\int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} \eta(x, y) \right|^2 \psi(x) \, dx = \int_{-\infty}^{y} \frac{(\Phi(x))^2}{\psi(x)} \, dx + \int_{y}^{\infty} \frac{(1 - \Phi(x))^2}{\psi(x)} \, dx
= \eta(y, y),
\] (27)
and so
\[
\left\| \frac{\partial}{\partial x} K(\cdot, y) \right\|_{L^2_\psi}^2 = \int_{-\infty}^{\infty} \left| \frac{\partial}{\partial x} K(x, y) \right|^2 \psi(x) \, dx = \gamma^2 \eta(y, y).
\] (28)

Hence we conclude that
\[
\|K(\cdot, y)\|^2_{\mathcal{W}} = 1 + \gamma \eta(y, y) < \infty,
\] (29)
and \( K(\cdot, y) \in \mathcal{W} \) for all \( y \in \mathbb{R} \) as required for (i).

For the reproducing property (ii), we have from (24) that for any \( f \in \mathcal{W} \) and \( y \in \mathbb{R} \),
\[
\langle f, K(\cdot, y) \rangle_{\mathcal{W}} = \int_{-\infty}^{\infty} f(x) \rho(x) \, dx + \frac{1}{\gamma} \int_{-\infty}^{\infty} f'(x) \left( \frac{\partial}{\partial x} K(x, y) \right) \psi(x) \, dx.
\] (30)
Using (25) and (26), then applying integration by parts, we obtain
\[
\frac{1}{\gamma} \int_{-\infty}^{\infty} f'(x) \left( \frac{\partial}{\partial x} K(x, y) \right) \psi(x) \, dx
\]
\[
= \lim_{R \to \infty} \left( \int_{-R}^{y} f'(x) \Phi(x) \, dx - \int_{y}^{R} f'(x) (1 - \Phi(x)) \, dx \right)
\]
\[
= \lim_{R \to \infty} \left( f(y) \Phi(y) - f(-R) \Phi(-R) - \int_{y}^{R} f(x) \rho(x) \, dx \right)
\]
\[
- \lim_{R \to \infty} \left( f(R) (1 - \Phi(R)) + f(y) (1 - \Phi(y)) - \int_{y}^{R} f(x) \rho(x) \, dx \right)
\]
\[
= f(y) - \int_{-\infty}^{\infty} f(x) \rho(x) \, dx,
\]
where the boundary terms vanish as \( R \to \infty \) by Lemma [I]. Substituting this into (30) gives the reproducing property (ii).

Point evaluation is now clearly bounded, since for all \( f \in W \) and \( y \in \mathbb{R} \) we have from (29)
\[
(f(y))^2 = \langle f, K(\cdot, y) \rangle_W^2 \leq \|f\|^2_W \|K(\cdot, y)\|^2_W = \|f\|^2_W (1 + \gamma \eta(y, y)) < \infty.
\]
This completes the proof. \( \square \)

Before leaving the reproducing kernel properties it is convenient to observe that the subspace \( \mathcal{V} \subset W \) defined by
\[
\mathcal{V} := \left\{ f \in W : \int_{-\infty}^{\infty} f(t) \rho(t) \, dt = 0 \right\}
\]
is a RKHS when equipped with the inner product
\[
\langle f, \tilde{f} \rangle_{\mathcal{V}} := \int_{-\infty}^{\infty} f'(x) \tilde{f}'(x) \psi(x) \, dx, \quad f, \tilde{f} \in \mathcal{V}.
\]
Note first that for \( f, \tilde{f} \in \mathcal{V} \) we have, by definition of the \( W \) inner product,
\[
\langle f, \tilde{f} \rangle_W = \frac{1}{\gamma} \langle f, \tilde{f} \rangle_{\mathcal{V}}.
\]
Thus for \( f \in \mathcal{V} \subset W \) we have
\[
f(y) = \langle f, K(\cdot, y) \rangle_W = \gamma \langle f, \eta(\cdot, y) \rangle_W = \langle f, \eta(\cdot, y) \rangle_{\mathcal{V}},
\]
from which it follows that \( \eta(x, y) \) is the reproducing kernel in \( \mathcal{V} \) (noting that we have already proved in (23) that \( \eta(\cdot, y) \in \mathcal{V} \)).

In fact, the space \( W \) is the direct sum of \( \mathcal{V} \) and the space of constant functions, since an arbitrary \( f \in W \) can be written uniquely as
\[
f = f_{\theta} + f_{\{1\}},
\]
where \( f_{\{1\}} \in \mathcal{V} \) and \( f_{\theta} \) is a constant function given by
\[
f_{\theta} = \int_{-\infty}^{\infty} f(x) \rho(x) \, dx.
\]
(We shall see in Section 3 that (33) is a special case of the ANOVA decomposition. We are anticipating here the notation to be used in Section 3.)
2.2 Norm equivalence in $\mathcal{H}$ and $\mathcal{W}$ under the stronger condition \((5)\)

In this Subsection we assume the stronger condition \((5)\). Under this condition it is easily seen that the kernel $\eta$ defined in \((20)\) can be rewritten as

$$
\eta(x, y) = \int_{-\infty}^{\min(x, y)} \frac{\Phi(t)}{\psi(t)} \, dt + \int_{\max(x, y)}^{\infty} \frac{1 - \Phi(t)}{\psi(t)} \, dt - C(\rho, \psi),
$$

\((34)\)

where $C(\rho, \psi)$ is defined in \((7)\).

We will now show that the norms in $\mathcal{H}$ and $\mathcal{W}$ are equivalent. The embedding constant in \((35)\) will be improved in Theorem \((7)\).

**Lemma 6.** If the condition \((5)\) holds then

$$
\|f\|_W^2 \leq (1 + \gamma C(\rho, \psi)) \|f\|_V^2 \quad \text{for all } f \in \mathcal{W}, \text{ and}
$$

$$
\|f\|_H^2 \leq (2 + \gamma C(\rho, \psi)) \|f\|_V^2 \quad \text{for all } f \in \mathcal{H},
$$

\((35)\)

where $C(\rho, \psi)$ is defined in \((4)\).

**Proof.** For any $f \in \mathcal{W}$, we use the reproducing property to write

$$
\|f\|_W^2 = \int_{-\infty}^{\infty} |f(y)|^2 \rho(y) \, dy = \left(\int_{-\infty}^{\infty} |f, K(\cdot, y)\rangle_W|^2 \rho(y) \, dy\right) d\mu(y)
$$

where we used the symmetry and the reproducing property of the kernel to write $\|K(\cdot, y)\|_W^2 = \langle K(\cdot, y), K(\cdot, y)\rangle_W = K(y, y)$.

Starting from the formula \((20)\), we use the Fubini theorem to change the order of integration, to obtain

$$
\int_{-\infty}^{\infty} \eta(y, y) \rho(y) \, dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{y} \frac{(\Phi(t))^2}{\psi(t)} \, dt + \int_{y}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \, dt\right) \rho(y) \, dy
$$

$$
= \int_{-\infty}^{\infty} \frac{(\Phi(t))^2}{\psi(t)} \left(\int_{-\infty}^{t} \rho(y) \, dy\right) \, dt + \int_{-\infty}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \left(\int_{-\infty}^{t} \rho(y) \, dy\right) \, dt
$$

$$
= \int_{-\infty}^{\infty} \frac{(\Phi(t))^2}{\psi(t)} (1 - \Phi(t)) \, dt + \int_{-\infty}^{\infty} \frac{(1 - \Phi(t))^2}{\psi(t)} \Phi(t) \, dt
$$

$$
= \int_{-\infty}^{\infty} \frac{\Phi(t)}{\psi(t)} (1 - \Phi(t)) \, dt
$$

$$
= C(\rho, \psi),
$$

\((36)\)

which is finite under the assumption \((7)\), or equivalently \((5)\). Hence

$$
\int_{-\infty}^{\infty} K(y, y) \rho(y) \, dy = 1 + \gamma C(\rho, \psi),
$$

which leads to the first bound. The second bound then follows from $\|f\|_H^2 \leq \|f\|_W^2 + \|f\|_V^2$, which follows from the definitions \((11)\) and \((12)\).

Since we have now proved the norm equivalence of $\mathcal{H}$ and $\mathcal{W}$ under the stronger condition \((5)\), and since $\mathcal{W}$ is a RKHS, it follows that $\mathcal{H}$ under the same condition is another RKHS, but not one with a known simple kernel that corresponds to the inner product \((14)\). Since the spaces $\mathcal{H}$ and $\mathcal{W}$ are equivalent, it makes sense that from now on
we choose to work with the inner product \([14]\), and use interchangeably the names \(\mathcal{H}\) and \(\mathcal{W}\) for the space itself.

Finally we restate the main result of this subsection, that the \(\mathcal{H}\) and \(\mathcal{W}\) norms are equivalent under the condition \([8]\), but now with an improved embedding constant. We do not know whether the embedding property holds (with a different embedding constant) under the weaker condition \([4]\) or some other intermediate condition.

**Theorem 7.** Under the condition \([5]\), the spaces \(\mathcal{H}\) and \(\mathcal{W}\) are equivalent, with

\[
\|f\|_{\mathcal{W}}^2 \leq \|f\|_{\mathcal{H}}^2 \leq (1 + \gamma C(\rho, \psi)) \|f\|_{\mathcal{W}}^2 \quad \text{for all } f \in \mathcal{H},
\]

where \(C(\rho, \psi)\) is defined in \([7]\). This space is a reproducing kernel Hilbert space and, when equipped with the inner product \([14]\), its kernel is given by \([19]\).

**Proof.** As discussed at the start of this section the first inequality follows by the Cauchy–Schwarz inequality. Together with \([35]\) this implies that the spaces \(\mathcal{H}\) and \(\mathcal{W}\) are equivalent. We also know from the preceding subsection that for every \(f \in \mathcal{W}\), or equivalently \(f \in \mathcal{H}\), we can write uniquely

\[
f = f_0 + f_{(1)},
\]

where

\[
f_0 := I_\rho(f) = \int_{-\infty}^{\infty} f(x) \rho(x) \, dx \quad \text{and} \quad f_{(1)} := f - f_0 \in \mathcal{V}.
\]

It is easily seen that the two terms \(f_{(1)}\) and \(f_0\) are orthogonal in \(L_\rho^2\) (by virtue of \([31]\)) and also orthogonal in \(\mathcal{W}\), from which it follows that

\[
\|f\|_{\mathcal{W}}^2 = \|f_0\|_{L_\rho^2}^2 + \|f_{(1)}\|_{L_\rho^2}^2 = f_0^2 + \|f_{(1)}\|_{L_\rho^2}^2,
\]

\[
\|f\|_{\mathcal{H}}^2 = \|f_0\|_{L_\rho^2}^2 + \|f_{(1)}\|_{L_\rho^2}^2 = f_0^2 + \frac{1}{\gamma} \|f_{(1)}\|_{L_\rho^2}^2.
\]

and therefore

\[
\|f\|_{\mathcal{H}}^2 = \|f\|_{\mathcal{W}}^2 + \frac{1}{\gamma} \|f'\|_{L_\rho^2}^2 = f_0^2 + \|f_{(1)}\|_{L_\rho^2}^2 + \frac{1}{\gamma} \|f_{(1)}'\|_{L_\rho^2}^2
\]

\[
= \|f\|_{\mathcal{W}}^2 + \|f_{(1)}\|_{L_\rho^2}^2.
\]

(We shall meet this double orthogonality property in more general form in Section \([3]\).)

From the reproducing property \([32]\) in \(\mathcal{V}\) we have

\[
|f_{(1)}(y)|^2 = \langle f_{(1)}, \eta(\cdot, y) \rangle_{\mathcal{V}} \leq \|f_{(1)}\|_{\mathcal{V}}^2 \|\eta(\cdot, y)\|_{\mathcal{V}}^2 = \|f_{(1)}\|_{\mathcal{V}}^2 \eta(y, y) \leq \gamma \|f\|_{\mathcal{W}}^2 \eta(y, y),
\]

and therefore

\[
\|f_{(1)}\|_{L_\rho^2}^2 = \int_{-\infty}^{\infty} |f_{(1)}(y)|^2 \rho(y) \, dy
\]

\[
\leq \gamma \|f\|_{\mathcal{W}}^2 \int_{-\infty}^{\infty} \eta(y, y) \rho(y) \, dy = \gamma C(\rho, \psi) \|f\|_{\mathcal{W}}^2.
\]

Inserting \([39]\) into \([38]\) gives the required result. 

**Remark 8.** Although they do not consider the spaces \(\mathcal{H}\) and \(\mathcal{W}\) explicitly, in \([21]\) the setting is general enough that it covers the equivalence \([37]\). However, the constants in \([21]\) are \((1 + \sqrt{\gamma} C(\rho, \psi) + \gamma (C(\rho, \psi))^2)^{-1}\) on the left and \(1 + \sqrt{\gamma} C(\rho, \psi) + \gamma (C(\rho, \psi))^2\) on the right, which are not as sharp as our constants of 1 and \(1 + \gamma C(\rho, \psi)\), respectively. Also, in \(d\) dimensions they only consider tensor product spaces and so the remainder of this paper is not covered by \([21]\).
3 The $d$-dimensional case

Let $d \geq 1$. In this section we will define the Sobolev space $\mathcal{H}_d$ and the ANOVA space $\mathcal{W}_d$ in $d$ dimensions, and then show that their norms are equivalent under the condition \( \psi \). Starting with the set of locally integrable functions on $\mathbb{R}^d$, we define the spaces $\mathcal{H}_d$ and $\mathcal{W}_d$ to be the restriction of such functions for which the norms $\| \cdot \|_H$ and $\| \cdot \|_W$ are finite, respectively. Both $\mathcal{H}_d$ and $\mathcal{W}_d$ are Hilbert spaces, and rather than restating the norms, here we give the corresponding inner products:

\[
\langle f, \tilde{f} \rangle_{\mathcal{H}_d} := \sum_{u \subseteq D} \frac{1}{|u|} \int_{\mathbb{R}^d} \partial^u f(x) \partial^u \tilde{f}(x) \psi_u(x) \rho_{D \setminus u}(x_{D \setminus u}) \, dx,
\]

\[
\langle f, \tilde{f} \rangle_{\mathcal{W}_d} := \sum_{u \subseteq D} \frac{1}{|u|} \int_{\mathbb{R}^{|u|}} \left( \int_{\mathbb{R}^{d-|u|}} \partial^u f(x_u, x_{D \setminus u}) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} \right) \cdot \left( \int_{\mathbb{R}^{d-|u|}} \partial^u \tilde{f}(x_u, x_{D \setminus u}) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} \right) \psi_u(x_u) \, dx_u.
\] (40)

In the following, we explain the various ingredients in detail. First, we give the notation for mixed partial derivatives of first order. Let

\[
\partial^i := \frac{\partial}{\partial x_j},
\]

and for a subset $u \subseteq D := \{1, 2, \ldots, d\}$, let $\partial^u$ denote the first-order mixed partial derivative with respect to the variables $x_j$ for $j \in u$, given by

\[
\partial^u := \prod_{j \in u} \frac{\partial}{\partial x_j} = \prod_{j \in u} \partial^i.
\]

These derivatives should be understood as weak derivatives: for example the $u$-th weak derivative of $f$ is the locally integrable function $\partial^u f : \mathbb{R}^d \to \mathbb{R}$ satisfying

\[
\int_{\mathbb{R}^d} \partial^u f(x) \, v(x) \, dx = (-1)^{|u|} \int_{\mathbb{R}^d} f(x) \partial^u v(x) \, dx \quad \text{for all } v \in C_0^\infty(\mathbb{R}^d),
\] (41)

where $|u|$ is the cardinality of $u$, and $C_0^\infty(\mathbb{R}^d)$ is the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support. Since the weak derivative, if it exists, is assumed to be locally integrable, it follows that it is equivalent to a function for which point evaluation is well defined almost everywhere on $\mathbb{R}^d$. Hence, in each term in the $\mathcal{W}$ norm (40) and inner product the inner integral over $\mathbb{R}^{d-|u|}$ is well defined for almost all $x_u \in \mathbb{R}^{|u|}$. Then since we eventually integrate with respect to $x_u \in \mathbb{R}^{|u|}$, both the norm and inner product in $\mathcal{W}$ are well defined.

For each $j = 1, \ldots, d$, let $\rho_j : \mathbb{R} \to \mathbb{R}_+$ be a strictly positive probability density function defined on $\mathbb{R}$ with the corresponding distribution function denoted by $\Phi_j$, and let $\psi_j : \mathbb{R} \to \mathbb{R}_+$ be a locally integrable strictly positive function on $\mathbb{R}$ such that $1/\psi_j$ is also locally integrable. Both spaces $\mathcal{H}_d$ and $\mathcal{W}_d$ involve weak derivatives and weight functions, with differentiated variables weighted differently to undifferentiated variables. For each $u \subseteq D$, the “active variables” (the differentiated variables) are weighted by the product

\[
\psi_u(x_u) := \prod_{j \in u} \psi_j(x_j), \quad \text{with } x_u := \{x_j : j \in u\},
\]

while the “inactive variables” are weighted by

\[
\rho_{D \setminus u}(x_{D \setminus u}) := \prod_{j \in D \setminus u} \rho(x_j), \quad \text{with } x_{D \setminus u} := \{x_j : j \in D \setminus u\}.
\]

Hence, in each term in the $\mathcal{W}$ norm (40) and inner product the inner integral over $\mathbb{R}^{d-|u|}$ is well defined for almost all $x_u \in \mathbb{R}^{|u|}$. Then since we eventually integrate with respect to $x_u \in \mathbb{R}^{|u|}$, both the norm and inner product in $\mathcal{W}$ are well defined.
Both $H_d$ and $W_d$ also involve weight parameters: there is a weight parameter $\gamma_u > 0$ for each $u \subseteq D$, and together they moderate the relative contribution of the $2^d$ derivatives to the norm. For a function $f$ in the unit ball, if $\gamma_u$ is small then the derivative $\partial^u f$ must contribute less to the norm. In the limiting case where $\gamma_u = 0$, we assume that $\partial^u f \equiv 0$.

Using the Cauchy–Schwarz inequality and the fact that each $\rho_j$ is a probability density, we have

$$\left| \int_{\mathbb{R}^d} \partial^u f(x_u, x_{D \setminus u}) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} \right|^2 \leq \int_{\mathbb{R}^d} \left| \partial^u f(x_u, x_{D \setminus u}) \right|^2 \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u}.$$ 

Thus $\|f\|_{W_d}^2 \leq \|f\|_{H_d}^2$ and we conclude trivially that $H_d$ is embedded in $W_d$.

The $u = \emptyset$ terms in the norm for $H_d$ and $W_d$ correspond to, respectively,

$$\|f\|_{L_\rho^2}^2 := \int_{\mathbb{R}^d} |f(x)|^2 \rho(x) \, dx \quad \text{and} \quad \|I_\rho(f)\|^2 := \int_{\mathbb{R}^d} f(x)^2 \rho(x) \, dx,$$

with $\rho(x) := \prod_{j=1}^d \rho_j(x_j)$. This is consistent with the representations (11)–(12) in one dimension. Trivially we know that $H_d$ is embedded in $L_\rho^2$. But $W_d$ may or may not be embedded in $L_\rho^2$.

We follow the general strategy for the 1-dimensional case to obtain norm equivalence for $H_d$ and $W_d$. In Subsection 3.1 we verify that the reproducing kernel for $W_d$ exists under the weaker condition (4) for the weight functions $(\rho_j, \psi_j)$ for all $j = 1, \ldots, d$. Again this case was not previously considered in [31], and moreover, some technical details were not adequately addressed in [31]. In Subsection 3.2 we discuss the ANOVA decomposition which provides the crucial steps to prove norm equivalence in Subsection 3.3 under the the stronger condition (5) for all $(\rho_j, \psi_j)$.

### 3.1 The reproducing kernel for $W_d$ exists under the weaker condition (4)

To verify the reproducing property of the kernel for $W_d$, we need a multivariate extension of the property of absolute continuity in one dimension. An important property of the classical Sobolev spaces is absolute continuity along lines, which is the property that functions in the first-order space $W^{1,1}(\Omega) := \{f \in L^1(\Omega) : \partial^k f \in L^1(\Omega) \text{ for all } k = 1, \ldots, d\}$, for $\Omega \subseteq \mathbb{R}^d$ (with no weight functions or weight parameters), are absolutely continuous along almost all line segments parallel to the coordinate axes (see, e.g., [31] Theorem 1.1.3(1)).

We show here that this absolute continuity along lines property also holds for functions in $W_d$. We write $x_{-k} := x_{D \setminus \{k\}}$ for brevity.

**Lemma 9.** Suppose that $f \in W_d$ and the condition (4) holds for all pairs of weight functions $(\rho_j, \psi_j)$ for $j = 1, 2, \ldots, d$. Let $k \in D$. Then for almost all $x_{-k} \in \mathbb{R}^{d-1}$ the univariate function $f(\cdot, x_{-k}) : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on any compact interval.

**Proof.** The proof follows the proof of [31] Sec. 1.1.3, Theorem 1] for classical first order Sobolev spaces $W^{1,1}$ almost exactly. We include it here for completeness, and because of the added difficulties caused by $W_d$ involving weight functions and dealing explicitly with an unbounded domain.

Let $[a, b] \subseteq \mathbb{R}$ be a compact interval (i.e., $-\infty < a < b < \infty$). We first show that $f \in W_d$ implies that the univariate function $\partial^k f(\cdot, x_{-k})$ is integrable on $[a, b]$ for almost all $x_{-k} \in \mathbb{R}^{d-1}$. First consider the following integral, which we bound using Fubini’s
Theorem and the Cauchy–Schwarz inequality,

\[
\left| \int_a^b \int_{\mathbb{R}^{d-1}} \partial^k f(x_k, x_{-k}) \rho_{-k}(x_{-k}) \, dx_{-k} \, dx_k \right| \\
\leq \left( \int_a^b \frac{dx_k}{\psi_k(x_k)} \right)^{1/2} \left( \int_a^b \left| \int_{\mathbb{R}^{d-1}} \partial^k f(x_k, x_{-k}) \rho_{-k}(x_{-k}) \, dx_{-k} \right|^2 \psi_k(x_k) \, dx_k \right)^{1/2} \\
\leq \left( \int_a^b \frac{dx_k}{\psi_k(x_k)} \right)^{1/2} \left( \int_{-\infty}^\infty \int_{\mathbb{R}^{d-1}} \partial^k f(x_k, x_{-k}) \rho_{-k}(x_{-k}) \, dx_{-k} \right)^2 \psi_k(x_k) \, dx_k \right)^{1/2} \\
\leq \left( \int_a^b \frac{dx_k}{\psi_k(x_k)} \right)^{1/2} \gamma^{1/2} \|f\|_{\mathcal{W}_d} < \infty,
\]

where the integral over \([a, b]\) is finite because \(1/\psi_k\) is locally integrable.
Equivalently, we also have absolute integrability over \(\mathbb{R}^{d-1} \times [a, b]\)

\[
\int_a^b \int_{\mathbb{R}^{d-1}} |\partial^k f(x_k, x_{-k})| \rho_{-k}(x_{-k}) \, dx_{-k} \, dx_k < \infty.
\]

Then, by Fubini’s Theorem again

\[
\int_a^b \int_{\mathbb{R}^{d-1}} |\partial^k f(x_k, x_{-k})| \rho_{-k}(x_{-k}) \, dx_{-k} \, dx_k < \infty \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1},
\]

which, since \(\rho_{-k}(x_{-k}) > 0\) is independent of \(x_k\), in turn implies that

\[
\int_a^b |\partial^k f(x_k, x_{-k})| \, dx_k < \infty \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1}.
\]

Thus, \(\partial^k f(\cdot, x_{-k})\) is integrable on \([a, b]\) for almost all \(x_{-k} \in \mathbb{R}^{d-1}\).

Now for \(x_{-k} \in \mathbb{R}^{d-1}\) define the function \(h(\cdot, x_{-k}) \in \mathcal{L}^1[a, b]\) by

\[
h(t, x_{-k}) := \int_t^b \partial^k f(x_k, x_{-k}) \, dx_k,
\]

which for almost all \(x_{-k} \in \mathbb{R}^{d-1}\) is absolutely continuous on \([a, b]\) because \(\partial^k f(\cdot, x_{-k})\) is integrable on \([a, b]\). Differentiating \(h(\cdot, x_{-k})\), we see that the classical derivative \(\partial/\partial t h(\cdot, x_{-k})\) is equal to the weak derivative \(\partial^k f(\cdot, x_{-k})\) almost everywhere on \([a, b]\).

In the remainder of the proof we show that \(h(\cdot, x_{-k})\) and \(f(\cdot, x_{-k})\) differ only by a constant on \([a, b]\), which proves the result. Let \(\{\chi_{\ell}\}_{\ell=1}^\infty\) be a sequence in \(C_0^\infty[a, b]\), which shall be specified later. Since \(h(\cdot, x_{-k})\) is absolutely continuous and each \(\chi_{\ell}\) has compact support in \([a, b]\), by integration by parts

\[
\int_a^b h(t, x_{-k}) \frac{d}{dt} \chi_{\ell}(t) \, dt = - \int_a^b \frac{\partial}{\partial t} h(t, x_{-k}) \chi_{\ell}(t) \, dt.
\]

Multiplying by an arbitrary \(\xi \in C_0^\infty(\mathbb{R}^{d-1})\) then integrating over \(\mathbb{R}^{d-1}\) gives

\[
\int_{\mathbb{R}^{d-1}} \int_a^b h(t, x_{-k}) \frac{d}{dt} \chi_{\ell}(t) \, dt \, \xi(x_{-k}) \, dx_{-k} = - \int_{\mathbb{R}^{d-1}} \int_a^b \frac{\partial}{\partial t} h(t, x_{-k}) \chi_{\ell}(t) \xi(x_{-k}) \, dt \, dx_{-k},
\]

which is finite since \(\xi\) has compact support.
By the definition of the weak derivative \( \text{II} \) we also have
\[
\int_{\mathbb{R}^d} f(x) \frac{d}{dx_k} \chi(t(x_k) \xi(x_k)) dx = -\int_{\mathbb{R}^d} \partial^k f(x) \chi(t(x_k) \xi(x_k)) dx,
\]
which, since \( \chi \) has compact support in \([a,b]\), is equivalent to
\[
\int_{\mathbb{R}^d} \int_{[a,b]} f(t, x_{-k}) \frac{d}{dt} \chi(t) \xi(x_{-k}) dt dx_{-k} = -\int_{\mathbb{R}^d} \int_{[a,b]} \frac{\partial}{\partial t} h(t, x_{-k}) \chi(t(x_k)) \xi(x_{-k}) dt dx_{-k},
\]
where we have relabelled \( x_k \) as \( t \), and on the right hand side we have also used the property that \( (\partial/\partial t) h(\cdot, x_{-k}) = \partial^k f(\cdot, x_{-k}) \) for almost all \( x_{-k} \in \mathbb{R}^{d-1} \).

Subtracting \( \|f\| \) from \( \|\| \) and using the fact that \( \xi \) was arbitrary we have
\[
\int_{[a,b]} (f(t, x_{-k}) - h(t, x_{-k})) \frac{d}{dt} \chi(t) dt = 0 \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1}.
\]

Then, applying \( \|3\| \) Sec. 1.1.3, Lemma], scaled to the interval \([a,b]\) and choosing \( \{\chi_k\} \) as in this lemma, we have that for almost all \( x_{-k} \in \mathbb{R}^{d-1} \) there is a constant \( c(\xi_{-k}) \in \mathbb{R} \) such that
\[
f(t, x_{-k}) = h(t, x_{-k}) + c(\xi_{-k}) \quad \text{for almost all } t \in [a,b].
\]
Hence, \( f(\cdot, x_{-k}) \) is absolutely continuous on \([a,b]\), (or more specifically, is equivalent to an absolutely continuous function on \([a,b]\)). \( \square \)

The next result we need is that the integrability of functions in \( \mathcal{W}_d \) necessarily implies the following limits as one variable tends to \( \pm \infty \). The proof relies on the absolute continuity property above.

**Lemma 10.** Suppose that \( f \in \mathcal{W}_d \) and the condition \( \|4\| \) holds for all pairs of weight functions \( (\rho_j, \psi_j) \) for \( j = 1, \ldots, d \). Then for any \( k \in \mathcal{D} \),
\[
\lim_{x_k \to -\infty} f(x) \Phi_k(x_k) = 0 \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1}, \quad \text{and} \quad (44)
\]
\[
\lim_{x_k \to +\infty} f(x) (1 - \Phi_k(x_k)) = 0 \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1}. \quad (45)
\]

**Proof.** Let \( k \in \mathcal{D} \). For notational convenience, we define \( F_k : \mathbb{R}^d \to \mathbb{R} \) by
\[
F_k(x) := f(x) \Phi_k(x_k).
\]
By Lemma \( \|3\| \) for almost all \( x_{-k} \in \mathbb{R}^{d-1} \), \( f(\cdot, x_{-k}) \) is absolutely continuous on any compact interval. Clearly, \( \Phi_k \) is absolutely continuous, and so it follows that \( F_k(\cdot, x_{-k}) \) is absolutely continuous on any compact interval for almost all \( x_{-k} \in \mathbb{R}^{d-1} \).

Now we follow a similar strategy to the proof of the one-dimensional case in Lemma \( \|4\| \) and prove the first limit \( \|44\| \) by contradiction. The second limit \( \|45\| \) then follows by an analogous argument.

First, we show that for any finite \( c \in \mathbb{R} \) the function \( \partial^k F_k(\cdot, x_{-k}) \) is integrable on \(( -\infty, c] \) for almost all \( x_{-k} \in \mathbb{R}^{d-1} \). Then using the fact that \( F_k(\cdot, x_{-k}) \) is absolutely continuous, we show that the premise that \( \lim_{x_k \to -\infty} F_k(x) = 0 \) leads to a contradiction.

Since \( \Phi_k \) is the distribution function of \( \rho_k \), the derivative of \( F_k \) with respect to \( x_k \) is
\[
\partial^k F_k(x) = f(x) \rho_k(x_k) + \partial^k f(x) \Phi_k(x_k).
\]
Consider first the following integral over $\mathbb{R}^{d-1} \times (-\infty, c]$

$$I := \left| \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{c} \partial^k F_k(x) \rho_{-k}(x_{-k}) \, dx_k \, dx_{-k} \right|$$

$$\leq \left| \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{c} f(x) \rho_{-k}(x_{-k}) \, dx_k \, dx_{-k} \right| + \left| \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{c} \partial^k f(x) \Phi_k(x_{-k}) \rho_{-k}(x_{-k}) \, dx_k \, dx_{-k} \right|,$$

where we have used the triangle inequality. Since $\mathbb{R}^{d-1} \times (-\infty, c] \subset \mathbb{R}^d$, the finiteness of $|I_p(f)|$ also implies that $I_1 < \infty$. For $I_2$ we can swap the order of the integrals, multiply and divide by $(\psi_k(x_{-k}))^{1/2}$, and use the Cauchy–Schwarz inequality to give the bound

$$I_2 \leq \int_{-\infty}^{c} \left| \int_{\mathbb{R}^{d-1}} \partial^k f(x) \rho_{-k}(x_{-k}) \, dx_{-k} \right| \left( \psi_k(x_{-k}) \right)^{1/2} \, dx_k \, dx_{-k} \left( \frac{\Phi_k(x_{-k})}{(\psi_k(x_{-k}))^{1/2}} \right)^{1/2} \left( \int_{-\infty}^{c} \frac{\Phi_k(x_{-k})}{(\psi_k(x_{-k}))^{1/2}} \, dx_{-k} \right)^{1/2} < \infty,$$

where finiteness for the two factors follows from $f \in \mathcal{W}_d$ and the condition $[\mathbf{1}]$, respectively.

Hence $I < \infty$, and since $\rho_{-k} > 0$ we also have

$$\int_{\mathbb{R}^{d-1}} \int_{-\infty}^{c} |\partial^k F_k(x)| \rho_{-k}(x_{-k}) \, dx_k \, dx_{-k} < \infty.$$  

It then follows by Fubini’s Theorem that

$$\int_{-\infty}^{c} |\partial^k F_k(x)| \rho_{-k}(x_{-k}) \, dx_{-k} \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1},$$

and since $\rho_{-k} > 0$ is independent of $x_k$ we also have

$$\int_{-\infty}^{c} |\partial^k F_k(x)| \, dx_k < \infty \quad \text{for almost all } x_{-k} \in \mathbb{R}^{d-1},$$

so that $\partial^k F_k(\cdot, x_{-k})$ is integrable on $(-\infty, c]$.

Let now $x_{-k} \in \mathbb{R}^{d-1}$ be such that $\partial^k F_k(\cdot, x_{-k})$ is integrable and $F_k(\cdot, x_{-k})$ is absolutely continuous, and suppose for a contradiction that $\lim_{t \to -\infty} F_k(t, x_{-k}) \neq 0$. Then there exists $\delta > 0$, $M \in \mathbb{N}$ and a sequence $\{t_m\} \subset \mathbb{R}$ with $t_m \to -\infty$ as $m \to \infty$, such that

$$|F_k(t_m, x_{-k})| \geq \delta \quad \text{for all } m \geq M.$$  

Assume also that $t_m \leq c$ for all $m \geq M$.

For any $m \geq M$, let $x_k \in (-\infty, t_m)$ be arbitrary. Then since $F_k(\cdot, x_{-k})$ is absolutely continuous on $[x_k, t_m]$, by the Fundamental Theorem of Calculus

$$F_k(x) = F_k(t_m, x_{-k}) - \int_{x_k}^{t_m} \partial^k F_k(t, x_{-k}) \, dt.$$  

The reverse triangle inequality then gives the lower bound

$$|F_k(x)| \geq |F_k(t_m, x_{-k})| - \left| \int_{x_k}^{t_m} \partial^k F_k(t, x_{-k}) \, dt \right| \geq \delta - \int_{-\infty}^{t_m} |\partial^k F_k(t, x_{-k})| \, dt.$$  

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Since \( \partial^k F_k(\cdot, x_k) \) is integrable on \((-\infty, c]\), we now choose \( m \geq M \) such that
\[
\int_{-\infty}^{t_m} |\partial^k F_k(t, x_{-k})| \, dt \leq \frac{\delta}{2}.
\]
Hence, we have the lower bound \(|F_k(x)| \geq \delta/2\) for all \( x_k \leq t_m \), from which it follows that
\[
|f(x)| \geq \frac{\delta}{2\Phi_k(x_k)} \quad \text{for all} \quad x_k \leq t_m.
\]

Since
\[
\left| \int_{\mathbb{R}^d} f(x) \rho(x) \, dx \right| \leq \sqrt{\gamma_0} \|f\|_{W_d} < \infty,
\]
we have \( f \in \mathcal{L}_p^1 \), and hence
\[
\|f\|_{\mathcal{L}_p^1} \geq \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{t_m} |f(x)| \rho(x) \, dx_k \, dx_{-k}
\]
\[
\geq \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{t_m} \frac{\delta}{2\Phi_k(x_k)} \rho(x) \, dx_k \, dx_{-k}
\]
\[
= \frac{\delta}{2} \int_{-\infty}^{t_m} \frac{\rho_k(x_k)}{\Phi_k(x_k)} \, dx_k = \frac{\delta}{2} \int_{-\infty}^{t_m} \frac{d}{dx_k} [\log(\Phi_k(x_k))] \, dx_k.
\]

However, the last integral is again divergent, as can be seen from (18).

This contradicts the fact that \( f \in \mathcal{L}_p^1 \). Hence, we must have that
\[
\lim_{x_k \to -\infty} F_k(x) = 0,
\]
as required.

In the theorem below we will show that the space \( \mathcal{W}_d \) is a RKHS if the condition (11) holds for all pairs of weight functions \((\rho_j, \psi_j)\). Note that the norm (11) and inner product (10) remain well defined whether or not the condition (11) holds, but that the reproducing kernel is not well defined if the condition (11) fails.

**Theorem 11.** If the condition (11) holds all pairs of weight functions \((\rho_j, \psi_j)\) for \( j = 1, \ldots, d \), then the space \( \mathcal{W}_d \) with inner product (10) is a reproducing kernel Hilbert space with kernel
\[
K_d(x, y) := \sum_{u \subseteq D} \gamma_u \prod_{j \in u} \eta_j(x_j, y_j),
\]
where
\[
\eta_j(x, y) := \int_{-\infty}^{\min(x,y)} \frac{\Phi_j(t)^2}{\psi_j(t)} \, dt + \int_{\max(x,y)}^{\infty} \frac{(1 - \Phi_j(t))^2}{\psi_j(t)} \, dt
\]
\[
- \int_{\max(x,y)}^{\min(x,y)} \frac{\Phi_j(t)(1 - \Phi_j(t))}{\psi_j(t)} \, dt.
\]

**Proof.** The kernel is clearly symmetric, and it is bounded due to (11).

To verify that \( \mathcal{W}_d \) is a RKHS with kernel given by (10), we have to show that (i) \( K_d(\cdot, y) \in \mathcal{W}_d \) for all \( y \in \mathbb{R}^d \), and (ii) \( f(y) = \langle f, K_d(\cdot, y) \rangle_{\mathcal{W}_d} \) for all \( f \in \mathcal{W}_d \) and \( y \in \mathbb{R}^d \).

First we consider the norm of \( K_d(\cdot, y) \) for \( y \in \mathbb{R}^d \):
\[
\| K_d(\cdot, y) \|_{\mathcal{W}_d}^2 = \frac{1}{\gamma_u} \int_{\mathbb{R}^{|u|}} \int_{\mathbb{R}^{d - |u|}} \frac{\partial^{|u|}}{\partial x_u} K_d(x, y) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} \psi_u(x_u) \, dx_u,
\]

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\[21\]
where we have from (40) (with \( u \) replaced by \( v \))

\[
K_d(x, y) = \sum_{u \subseteq D} \gamma_u \prod_{j \in u} \eta_j(x_j, y_j).
\]

For \( u \subseteq D \), the weak derivative of \( K_d(x, y) \) with respect to \( x_u \) is

\[
\frac{\partial |u|}{\partial x_u} K_d(x, y) = \sum_{u \subseteq v \subseteq D} \gamma_v \left( \prod_{j \in u} \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \right) \left( \prod_{j \in v \setminus u} \eta_j(x_j, y_j) \right),
\]

since if \( u \not\subseteq v \) then the differentiation makes the term vanish. In turn we have

\[
\int_{\mathbb{R}^d - |u|} \frac{\partial |u|}{\partial x_u} K_d(x, y) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} = \gamma_u \prod_{j \in u} \frac{\partial}{\partial x_j} \eta_j(x_j, y_j),
\]

since \( \rho_j \) is a probability density, and if \( v \not= u \) then the property (23) makes the term vanish. This leads to

\[
\|K_d(\cdot, y)\|_{\mathcal{W}_d}^2 = \sum_{u \subseteq D} \gamma_u \prod_{j \in u} \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \right)^2 \psi_j(x_j) \, dx_j
\]

\[
= \sum_{u \subseteq D} \gamma_u \prod_{j \in u} \eta_j(y_j, y_j) < \infty,
\]

where we used (27). Hence \( K_d(\cdot, y) \in \mathcal{W}_d \). This completes the proof of (i).

For the reproducing property (ii), consider

\[
\langle f, K_d(\cdot, y) \rangle_{\mathcal{W}_d} = \sum_{u \subseteq D} \frac{1}{\gamma_u} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{d - |u|}} \frac{\partial^{|u|}}{\partial x_u} f(x) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} \right) \cdot \left( \int_{\mathbb{R}^{d - |u|}} \frac{\partial |u|}{\partial x_u} K_d(x, y) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{D \setminus u} \right) \psi_u(x_u) \, dx_u
\]

\[
= \sum_{u \subseteq D} \int_{\mathbb{R}^d} \frac{\partial^{|u|}}{\partial x_u} f(x) \left( \prod_{j \in u} \left( \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \psi_j(x_j) \right) \right) \rho_{D \setminus u}(x_{D \setminus u}) \, dx,
\]

where we used (18).

Consider now any \( u \not= \emptyset \) and suppose \( k \in u \). By Fubini’s Theorem and Leibniz’s Theorem, we can interchange the order of integrals and derivatives to write

\[
J_u := \int_{\mathbb{R}^d} \frac{\partial^{|u|}}{\partial x_u} f(x) \left( \prod_{j \in u} \left( \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \psi_j(x_j) \right) \right) \rho_{D \setminus u}(x_{D \setminus u}) \, dx
\]

\[
= \int_{\mathbb{R}^{d-1}} \frac{\partial^{|u\setminus\{k\}|}}{\partial x_u \setminus \{k\}} I_k(x_{-k}) \left( \prod_{j \in u \setminus \{k\}} \left( \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \psi_j(x_j) \right) \right) \rho_{D \setminus u}(x_{D \setminus u}) \, dx_{-k},
\]

where

\[
I_k(x_{-k}) := \lim_{R \to \infty} \int_{-R}^{R} \frac{\partial^{|k|}}{\partial x_k} f(x_k, x_{-k}) \frac{\partial}{\partial x_k} \eta_k(x_k, y_k) \psi_k(x_k) \, dx_k.
\]
By Lemma [5] for almost all $x_{-k} \in \mathbb{R}^{d-1}$, the function $f(\cdot, x_{-k})$ is absolutely continuous on $[-R, R]$. Using (26) and integration by parts gives

$$I_k(x_{-k}) = \lim_{{R \to \infty}} \left( \int_{{-R}}^{y_k} \partial^k f(x_k, x_{-k}) \Phi_k(x_k) \, dx_k + \int_{{y_k}}^{R} \partial^k f(x_k, x_{-k}) (\Phi_k(x_k) - 1) \, dx_k \right) = \lim_{{R \to \infty}} \left( f(y_k, x_{-k}) - f(-R, x_{-k}) - \int_{{-R}}^{y_k} f(x_k, x_{-k}) \rho_k(x_k) \, dx_k + f(R, x_{-k}) (\Phi_k(R) - 1) - \int_{{y_k}}^{R} f(x_k, x_{-k}) \rho_k(x_k) \, dx_k \right) = f(y_k, x_{-k}) - \lim_{{R \to \infty}} \int_{{-R}}^{R} f(x_k, x_{-k}) \rho_k(x_k) \, dx_k,$$

where the boundary terms vanish as $R \to \infty$ due to Lemma [10]. Hence, we can write

$$J_u = \int_{{\mathbb{R}^{d-1}}} \partial^\mu \{k\} f(y_k, x_{-k}) \left( \prod_{{j \in u \setminus \{k\}}} \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \psi_j(x_j) \right) \rho(x_{D \setminus u}) \, dx_{-k} - J_u \setminus \{k\}. \quad (50)$$

Returning to the inner product (19), we split the sum over whether $d \in u$ to write

$$\langle f, K_d(\cdot, y) \rangle_{W_d} = \sum_{{u \subseteq D}} J_u = \sum_{{u \subseteq \{1, \ldots, d-1\}}} J_u \cap \{d\} + \sum_{{u \subseteq \{1, \ldots, d-1\}}} J_u = \sum_{{u \subseteq \{1, \ldots, d-1\}}} (J_u \cap \{d\} + J_u).$$

Using the recursive formula (50) with $u$ replaced by $u \cup \{d\}$ and $k$ replaced by $d$ gives

$$\langle f, K_d(\cdot, y) \rangle_{W_d} = \sum_{{u \subseteq \{1, \ldots, d-1\}}} \int_{{\mathbb{R}^{d-1}}} \partial^\mu f(y_d, x_{-d}) \left( \prod_{{j \in u \cup \{d\}}} \frac{\partial}{\partial x_j} \eta_j(x_j, y_j) \psi_j(x_j) \right) \rho(x_{D \setminus u}) \, dx_{-d}.$$

Hence we have “reproduced” the variable $y_d$.

Applying this procedure iteratively, we can see that the integral terms will always cancel, until we have “reproduced” all of the variables $y_d, y_{d-1}, \ldots, y_1$, and are left only with

$$\langle f, K_d(\cdot, y) \rangle_{W_d} = f(y),$$

which holds for almost all $y \in \mathbb{R}^d$. \qed

3.2 The ANOVA decomposition

Every $d$-variate function on $\mathbb{R}^d$ can be written as a sum of $2^d$ terms of the form

$$f(x) = \sum_{{u \subseteq D}} f_u(x_u),$$

where each term $f_u$ depends only on the variables $x_u$. Obviously there are infinitely many ways to do this. One way to ensure uniqueness of the decomposition for functions $f \in L^1_\rho$ is to impose the “annihilating condition” that

$$\int_{{-\infty}}^{\infty} f_u(x_u) \rho_j(x_j) = 0 \quad \text{for all } j \in u \text{ whenever } u \neq \emptyset,$$

...
which leads to the “ANOVA decomposition”, see e.g., [28]. The ANOVA terms can be expressed using a recursive formula, or an explicit formula [28]

\[ f_u(x_u) = \sum_{v \leq u} (-1)^{|u|-|v|} \int_{R^d \setminus D\setminus v} f(x) \rho_{D\setminus v}(x_{D\setminus v}) \, dx_{D\setminus v}. \] (51)

ANOVA stands for “ANalysis Of VAriance” and is traditionally considered for \( L^2_{\rho} \) functions. If \( f \in L^2_{\rho} \), then the ANOVA terms are orthogonal

\[ \int_{R^d} f_u(x_u) f_v(x_u) \, d\rho(x) = \delta_{u,v}, \]

and there is a nice decomposition for the \( L^2_{\rho} \) norm and the variance of \( f \)

\[ \| f \|_{L^2_{\rho}}^2 = \sum_{u \subseteq D} \| f_u \|_{L^2_{\rho}}^2 \quad \text{and} \quad \sigma^2(f) = \sum_{u \subseteq D} \sigma^2(f_u), \]

with \( \sigma^2(f) := I_\rho(f^2) - (I_\rho(f))^2 \) and \( \sigma^2(f_u) = I_\rho(f_u^2) \) for \( u \neq \emptyset \).

In the space \( \mathcal{W}_d \) the functions are in \( L^1_{\rho} \) but not necessarily in \( L^2_{\rho} \). However, there is a nice decomposition for the \( \mathcal{W}_d \) norm

\[ \| f \|_{\mathcal{W}_d}^2 = \sum_{u \subseteq D} \| f_u \|_{\mathcal{W}_d}^2, \] (52)

because the ANOVA terms are orthogonal with respect to the inner product in \( \mathcal{W}_d \). This can be verified directly from the explicit formula (51), without the need for condition \( (4) \).

It also follows from the general result in [28] using the annihilating condition [23] of the kernel, which is well defined under condition \( (4) \).

**Lemma 12.** If the condition \( (5) \) holds for all pairs of weight functions \((\rho_j, \psi_j)\) for \( j = 1, \ldots, d \), then the space \( \mathcal{W}_d \) is embedded in \( L^2_{\rho} \), with

\[ \| f \|_{L^2_{\rho}}^2 \leq \left( \sum_{u \subseteq D} \gamma_u \prod_{j \in u} C(\rho_j, \psi_j) \right) \| f \|_{\mathcal{W}_d}^2 \quad \text{for all} \quad f \in \mathcal{W}_d, \]

where \( C(\rho_j, \psi_j) \) is defined in \( (7) \).

**Proof.** For any \( f \in \mathcal{W}_d \), we use the reproducing property to write

\[ \| f \|_{L^2_{\rho}}^2 = \int_{R^d} |f(y)|^2 \rho(y) \, dy = \int_{R^d} |\langle f, K_d(\cdot, y) \rangle_{\mathcal{W}_d}|^2 \rho(y) \, dy \leq \int_{R^d} \| f \|_{\mathcal{W}_d}^2 \| K_d(\cdot, y) \|_{\mathcal{W}_d}^2 \rho(y) \, dy, \]

where

\[ \| K_d(\cdot, y) \|_{\mathcal{W}_d}^2 = \langle K_d(\cdot, y), K_d(\cdot, y) \rangle_{\mathcal{W}_d} = K_d(y, y) = \sum_{u \subseteq D} \gamma_u \prod_{j \in u} \eta_j(y_j, y_j). \]

The result now follows from \( (56) \). \( \square \)

In the next subsection we will establish the norm equivalence between \( \mathcal{W}_d \) and \( \mathcal{H}_d \). As in the one dimensional case we will make use of the reproducing property. However, for each term in \( (8) \) we only want to “reproduce” the variables that are not differentiated.
Moreover, the property (55) implies that the decomposition terms also satisfy the useful property

\[ \langle g, \tilde{g} \rangle_{\mathcal{W}_u} := \sum_{v \subseteq u} \frac{1}{\gamma_v} \int_{\mathbb{R}^{|v|}} \left( \int_{\mathbb{R}^{|u| - |v|}} \partial^p g(x_v, x_{u\setminus v}) \rho_{u\setminus v}(x_{u\setminus v}) \, dx_{u\setminus v} \right) \cdot \left( \int_{\mathbb{R}^{|u| - |v|}} \partial^p \tilde{g}(x_v, x_{u\setminus v}) \rho_{u\setminus v}(x_{u\setminus v}) \, dx_{u\setminus v} \right) \psi_v(x_v) \, dx_v, \]

induced norm

\[ \|g\|_{\mathcal{W}_u}^2 = \sum_{v \subseteq u} \frac{1}{\gamma_v} \int_{\mathbb{R}^{|v|}} \left( \int_{\mathbb{R}^{|u| - |v|}} \partial^p g(x_v, x_{u\setminus v}) \rho_{u\setminus v}(x_{u\setminus v}) \, dx_{u\setminus v} \right)^2 \psi_v(x_v) \, dx_v, \]

and kernel \( K_u : \mathbb{R}^{|u|} \times \mathbb{R}^{|u|} \to \mathbb{R} \) given by

\[ K_u(x_u, y_u) := \sum_{v \subseteq u} \gamma_v \prod_{j \in v} \eta_j(x_j, y_j). \]

Note that the case \( u = D \) gives our original space \( \mathcal{W}_d \).

Analogous to the space \( \mathcal{W}_d \), every \( g \in \mathcal{W}_u \) admits an ANOVA decomposition

\[ g(x_u) = \sum_{v \subseteq u} g_v(x_v), \]

where each \( g_v \in \mathcal{W}_v \) depends only on the variables \( x_v \), and is given explicitly by

\[ g_v(x_v) = \sum_{w \subseteq v} (-1)^{|w| - |v|} \int_{\mathbb{R}^{|u| - |w|}} g(x_w, x_{u\setminus w}) \rho_{u\setminus w}(x_{u\setminus w}) \, dx_{u\setminus w}. \]  

(53)

The ANOVA decomposition is orthogonal in \( \mathcal{W}_u \), and orthogonal in \( L^2_{\rho_u} \) provided \( \mathcal{W}_u \) is embedded in \( L^2_{\rho_u} \) (which is the case if \[ \Box \] holds for all pairs \((\rho_j, \psi_j)\) for \( j \in u \):

\[ \|g\|_{\mathcal{W}_u}^2 = \sum_{v \subseteq u} \|g_v\|_{\mathcal{W}_v}^2, \quad \|g\|_{L^2_{\rho_u}}^2 = \sum_{v \subseteq u} \|g_v\|_{L^2_{\rho_v}}^2, \]

(54)

and the decomposition terms also satisfy the useful property

\[ \int_{-\infty}^{\infty} g_v(x_v) \rho_j(x_v) \, dx_v = 0 \quad \text{for any } j \in v \neq 0. \]  

(55)

Moreover, the property \[ \Box \] implies that

\[ \|g_v\|_{\mathcal{W}_v}^2 = \|g_v\|_{\mathcal{W}_v}^2 = \frac{1}{\gamma_v} \int_{\mathbb{R}^{|v|}} \left| \partial^p g_v(x_v) \right|^2 \psi_v(x_v) \, dx_v, \]

(56)

and

\[ g_v(y_v) = \left( g_v, \frac{\gamma_v}{\gamma_{\emptyset}} \prod_{j \in v} \eta_j(\cdot, y_j) \right)_{\mathcal{W}_v}, \]  

(57)

and we have also

\[ \left\| \prod_{j \in v} \eta_j(\cdot, y_j) \right\|_{\mathcal{W}_v}^2 = \frac{1}{\gamma_{\emptyset}} \prod_{j \in v} \eta_j(y_j, y_j). \]  

(58)
3.3 Norm equivalence in $H_d$ and $W_d$ under the stronger condition (5)

To prove the general norm equivalence we require the following technical lemma. The basic idea is that if we differentiate some function $f \in W_d$ with respect to the variables $x_u$ and then treat those $|u|$ variables as fixed, then that function also belongs to the weighted unanchored ANOVA space in $(d - |u|)$ dimensions.

**Lemma 13.** If $f \in W_d$ then for each $u \subseteq D$ and almost all $x_u \in \mathbb{R}^{|u|}$, $\partial^\mu f(x_u, \cdot) \in W_{D \setminus u}$.

**Proof.** For $f \in W_d$, $u \subseteq D$ and $x_u \in \mathbb{R}^{|u|}$, we write

$$\|\partial^\mu f(x_u, \cdot)\|_{W_{D \setminus u}}^2 = \sum_{v \subseteq D \setminus u} \frac{1}{\gamma_v} h_v(x_u), \tag{59}$$

where for each $v \subseteq D \setminus u$ we define the function $h_v : \mathbb{R}^{|u|} \to \mathbb{R}$ by

$$h_v(x_u) := \int_{\mathbb{R}^{|v|}} \int_{\mathbb{R}^{d - |u| - |v|}} \partial^\mu \mathbb{J}_v f(x_u, x_v, x_{D \setminus \{u,v\}}) \rho_D(x_{D \setminus \{u,v\}}) (x_{D \setminus \{u,v\}}) \, dx_{D \setminus \{u,v\}} \, \psi_v(x_v) \, dx_v.$$

We have

$$\int_{\mathbb{R}^{|u|}} |h_v(x_u) \psi_u(x_u)| \, dx_u$$

$$= \int_{\mathbb{R}^{|u|+|v|}} \int_{\mathbb{R}^{d - |u| - |v|}} \partial^\mu \mathbb{J}_v f(x_u, x_v, x_{D \setminus \{u,v\}}) \rho_D(x_{D \setminus \{u,v\}}) (x_{D \setminus \{u,v\}}) \, dx_{D \setminus \{u,v\}} \, \psi_v(x_v) \, dx_{D \setminus \{u,v\}}$$

$$\leq \gamma_{u \cup v} \|f\|_{W_d}^2 < \infty,$$

where we recognised that the integral expression corresponds to one of the terms in the norm $\|f\|_{W_d}^2$, see (5). Thus $h_v \psi_u$ is integrable on $\mathbb{R}^{|u|}$. It then follows that $h_v \psi_u = h_v \psi_u |u| < \infty$ almost everywhere on $\mathbb{R}^{|u|}$, and since each $\psi_j$ is strictly positive, we also have that $h_v = h_v |u| < \infty$ almost everywhere. Hence by (59), we have that $\|\partial^\mu f(x_u, \cdot)\|_{W_{D \setminus u}}$ is finite for almost all $x_u \in \mathbb{R}^{|u|}$.

**Theorem 14.** If the condition (5) holds for all pairs of weight functions $(\rho_j, \psi_j)$ for $j = 1, \ldots, d$, then the spaces $H_d$ and $W_d$ are equivalent, and

$$\|f\|_{W_d}^2 \leq \|f\|_{H_d}^2 \leq \left( \max_{v \subseteq D} \sum_{u \subseteq v} \gamma_u \prod_{j \in v} C(\rho_j, \psi_j) \right) \|f\|_{W_d}^2, \tag{60}$$

where $C(\rho_j, \psi_j)$ is defined in (7).

**Proof.** As we explained before, $\|f\|_{W_d}^2 \leq \|f\|_{H_d}^2$ follows easily from the Cauchy–Schwarz inequality.

To prove the second inequality in (60) we use Lemma 13 to reproduce the variables that are not differentiated. First, for $f \in W_d$, $u \subseteq D$ and $x_u \in \mathbb{R}^{|u|}$, define the function $g^u : \mathbb{R}^{d - |u|} \to \mathbb{R}$ by

$$g^u(x_{D \setminus u}) := \partial^\mu f(x_u, x_{D \setminus u}),$$

26
where to ease the notation we omit the dependence of $g^u$ on $x_u$. Then the $\mathcal{H}_d$ norm \[8\] of $f$ can be written as

$$
\|f\|_{\mathcal{H}_d}^2 = \sum_{u \subseteq D} \frac{1}{\gamma_u} \int_{[|u|]} \left( \int_{\mathbb{R}^{d-|u|}} |\partial^u f(x_u, x_{D\setminus u})|^2 \rho_{D\setminus u}(x_{D\setminus u}) \right) \psi_u(x_u) \, dx_u
$$

$$
= \sum_{u \subseteq D} \frac{1}{\gamma_u} \int_{[|u|]} \|g^u\|_{L_{\rho_{D\setminus u}}}^2 \psi_u(x_u) \, dx_u,
$$

(61)

where $\| \cdot \|_{L_{\rho_{D\setminus u}}}^2$ is the $L^2$-norm with respect to the variables $x_{D\setminus u}$ and weight function $\rho_{D\setminus u}$.

By Lemma \[13\] for almost all $x_u \in \mathbb{R}^{[u]}$, we have $g^u \in \mathcal{W}_{D\setminus u}$. Each $g^u$ admits an ANOVA decomposition

$$
g^u = \sum_{v \subseteq D \setminus u} (g^u)_v,
$$

where $(g^u)_v \in \mathcal{W}_v$ and which, by \[53\], can be written explicitly in terms of $g^u$, and thus in terms of $f$, as

$$
(g^u)_v(x_v) = \sum_{w \subseteq v} (-1)^{|v|-|w|} \int_{\mathbb{R}^{d-|w|}} \partial^w f(x) \rho_{D\setminus (u \cup w)}(x_{D\setminus (u \cup w)}) \, dx_{D\setminus (u \cup w)}.
$$

(62)

Since \[6\] holds, the ANOVA decomposition is orthogonal in the $L^2$-norm (see \[1\]), and so we can write

$$
\int_{[|u|]} \|g^u\|_{L_{\rho_{D\setminus u}}}^2 \psi_u(x_u) \, dx_u = \sum_{v \subseteq D \setminus u} \int_{[|v|]} \|g^u\|_{L_{\rho_{D\setminus u}}}^2 \psi_u(x_u) \, dx_u.
$$

(63)

Now, using the reproducing property \[57\] in $\mathcal{W}_v$ as well as \[58\], for all $x_v \in \mathbb{R}^{[v]}$ with $v \subseteq D \setminus u$, we have

$$
|(g^u)_v(x_v)|^2 = \left( (g^u)_v, \gamma_v \prod_{j \in v} \eta_j(\cdot, y_j) \right)_\mathcal{W}_v^2 \leq \gamma_v^2 \|g^u\|_{\mathcal{W}_v}^2 \left\| \prod_{j \in v} \eta_j(\cdot, y_j) \right\|_{\mathcal{W}_v}^2
$$

$$
= \gamma_v \|g^u\|_{\mathcal{W}_v}^2 \prod_{j \in v} \eta_j(y_j, y_j).
$$

Hence, the $L^2_{\rho_v}$-norm of $(g^u)_v$ is bounded by

$$
\|g^u\|_{L_{\rho_v}}^2 \leq \gamma_v \|g^u\|_{\mathcal{W}_v}^2 \prod_{j \in v} \int_{-\infty}^{\infty} \eta_j(y_j, y_j) \rho_j(y_j) \, dy_j
$$

$$
= \gamma_v \|g^u\|_{\mathcal{W}_v}^2 \prod_{j \in v} C(\rho_j, \psi_j),
$$

(64)

where we have used \[30\].

Next we use property \[56\] for the $\mathcal{W}_v$-norm of $(g^u)_v$ and substitute in the explicit formula \[62\] for $(g^u)_v$, to obtain

$$
\|g^u\|_{\mathcal{W}_v}^2 = \frac{1}{\gamma_v} \int_{[|v|]} \left| \partial^w f(x_v) \right|^2 \psi_v(x_v) \, dx_v
$$

$$
= \frac{1}{\gamma_v} \int_{[|v|]} \left| \partial^w f(x) \right|^2 \rho_{D\setminus (u \cup w)}(x_{D\setminus (u \cup w)}) \, dx_{D\setminus (u \cup w)} \right|^2 \psi_v(x_v) \, dx_v
$$

$$
= \frac{1}{\gamma_v} \int_{[|v|]} \left| \partial^w f(x) \right|^2 \rho_{D\setminus (u \cup w)}(x_{D\setminus (u \cup w)}) \, dx_{D\setminus (u \cup w)} \right|^2 \psi_v(x_v) \, dx_v,
$$

(65)
where in the last step we applied the Leibniz rule for differentiation under the integral from [20, Theorem 4].

Substituting (65) into (64), and in turn into (63) and then (61), we obtain

$$
\|f\|_{\hat{H}_d}^2 \leq \sum_{u \subseteq D} \frac{1}{\gamma_u} \sum_{v \subseteq D \setminus u} \left( \prod_{j \in v} C(\rho_j, \psi_j) \right) \\
\cdot \int_{|u|} \left| \int_{R^{d-|u|-|v|}} \partial^{u \cup v} f(x) \rho_{D \setminus (u \cup v)}(x) dx \right|^2 \psi_{u \cup v}(x) dx u \cup v.
$$

Substituting \(w = u \cup v\) and then rearranging the sums, we can write this as

$$
\|f\|_{\hat{H}_d}^2 \leq \sum_{m \subseteq D} \frac{1}{\gamma_m} \left( \sum_{v \subseteq m} \frac{1}{\gamma_w \setminus v} \prod_{j \in v} C(\rho_j, \psi_j) \right) \\
\cdot \int_{|v|} \left| \int_{R^{d-|w|}} \partial^{v \setminus w} f(x) \rho_{D \setminus w}(x) dx \right|^2 \psi_{v \setminus w}(x) dx v \setminus w
\leq \left( \max_{m \subseteq D} \sum_{v \subseteq m} \frac{1}{\gamma_m \setminus v} \prod_{j \in v} C(\rho_j, \psi_j) \right) \|f\|_{V_d}^2,
$$

where in the first step we have multiplied and divided each term by \(\gamma_m\) to give the correct weights. Finally interchanging the labels for \(v\) and \(w\) gives the required result.  

4 Analysis of QMC with preintegration for option pricing

To conclude this paper we return to the application that served as our initial motivation, namely, we use the equivalence from Theorem 14 to obtain a rigorous error bound for a QMC method combined with preintegration technique for approximating the fair price of an option.

As a concrete example we consider the arithmetic-average Asian call option analysed in, e.g., [16, 17, 18, 20]. Without going into the details here we simply recall that the problem can be expressed, after an appropriate change of variables, as an integral of the form

$$
I_d(f) = \int_{R^d} f(x) \rho(x) dx, \quad f(x) = \max(\phi(x), 0),
$$

(66)

with \(\rho(x)\) being a product of standard normal density \(\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\), and

$$
\phi(x) = \frac{1}{d} \sum_{t=1}^d S_0 \exp \left( (r - \frac{1}{2} \sigma^2) \frac{tT}{d} + \sigma A_t x \right) - K,
$$

(67)

where \(S_0\) is the initial asset price, \(K\) is the strike price, \(r\) is the risk-free interest rate, \(\sigma\) is the volatility, \(d\) is the number of equal time steps with final time \(T\), and \(A_t\) are the rows of a matrix \(A\) arising from some factorisation of the covariance matrix \(\Sigma = [\min(iT/d, jT/d)]_{i,j \in D} = AA^T\). See, e.g., [16] for three factorisation methods (Cholesky a.k.a. standard construction, Brownian bridge construction, principal components construction (PCA)) that lead to different matrices \(A\). The maximum appears in (66) since an option is worthless when its value is negative. This gives rise to a kink in the integrand \(f\) even though \(\phi\) is smooth.
An \( N \)-point randomly shifted lattice rule (see e.g., [8]) approximates \( I_d(f) \) by

\[
Q_{d,N}(f) = \frac{1}{N} \sum_{n=0}^{N-1} f(t_n), \quad \text{with} \quad t_n := \Phi^{-1}\left(\left\{ \frac{nz}{N} + \Delta \right\}\right),
\]

where \( z \in \mathbb{N}^d \) is the generating vector, \( \Delta \in [0, 1)^d \) is a uniformly distributed random shift, \( \{\cdot\} \) denotes taking the fractional part of each component in a vector and \( \Phi^{-1} \) denotes applying the inverse cumulative normal distribution function \( \Phi^{-1} \) to each coordinate. If the integrand \( f \) belongs to the ANOVA space \( \mathcal{W}_d \), then a generating vector \( z \) can be constructed to achieve a RMS error bound close to \( \mathcal{O}(1/N) \), see [31, Theorem 8]. Unfortunately, the kink means that our \( f \) lacks the smoothness requirement to be in \( \mathcal{W}_d \).

The smoothing by preintegration technique from [20] goes as follows. First we integrate out one strategically chosen variable, say, \( x_1 \):

\[
P_1f(x_{2:d}) := \int_{-\infty}^{\infty} f(x_1, x_{2:d}) \rho(x_1) \, dx_1,
\]

which is either computed analytically or numerically by a 1-dimensional quadrature rule to high accuracy. Then we apply a randomly shifted lattice rule to the resulting \((d - 1)\)-dimensional function \( P_1f \). It has been established in [17] [20] that \( P_1f \) belongs to the Sobolev space \( \mathcal{H}_{d-1} \) for all three factorisation methods mentioned earlier. Indeed, the general theory in these papers applies to functions of the form \( f = \max(\phi, 0) \), with a number of specific conditions on the generic function \( \phi \) and the density \( \rho \), including \((\partial^{d-2}/\partial x_1^{d-2})\phi \in \mathcal{H}_d \) and \( \partial^1\phi > 0 \). All of these conditions have been verified for our specific function (67), see [20 Thm 3 and Sec. 6]. For standard and Brownian bridge constructions these assumptions have also been verified for any choice of preintegration variable \( x_j \), whereas for principal components construction these are only guaranteed for the variable \( x_1 \).

So, on the one hand we know that \( P_1f \in \mathcal{H}_{d-1} \), and on the other hand we have established in this paper that \( \mathcal{H}_{d-1} \) is equivalent to \( \mathcal{W}_{d-1} \). We therefore conclude that \( P_1f \in \mathcal{W}_{d-1} \), and hence we can apply [31, Theorem 8] to bound the error.

**Theorem 15.** For the Asian option pricing problem (66)–(67), an \( N \)-point randomly shifted lattice rule can be constructed for the preintegrated \((d - 1)\)-dimensional function \( P_1f \) to achieve the RMS error bound

\[
\sqrt{\mathbb{E}_\Delta \left[ |I_d(f) - Q_{d-1,N}(P_1f)|^2 \right]} \leq CN^{-1+\delta} \quad \text{for} \ \delta > 0,
\]

where \( C \) depends on \( \delta \) and the norm \( \|P_1f\|_{\mathcal{W}_{d-1}} < \infty \).

To our knowledge this is the first rigorous error bound giving close to \( \mathcal{O}(1/N) \) convergence for a QMC rule applied to an option pricing problem.

Practical details on how to efficiently construct a lattice generating vector for such option pricing problems, along with a full error analysis that is explicit in how the constant depends on the dimension (including how to choose the weight parameters \( \{\gamma_u\} \)) will be studied in a future paper.

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