IFS measures on generalized Bratteli diagrams

Sergey Bezuglyi and Palle E.T. Jorgensen

Dedicated to the memory of Robert Strichartz

Abstract

The purpose of the paper is a general analysis of path space measures. Our focus is a certain path space analysis on generalized Bratteli diagrams. We use this in a systematic study of systems of self-similar measures (the term “IFS measures” is used in the paper) for both types of such diagrams, discrete and continuous. In special cases, such measures arise in the study of iterated function systems (IFS). In the literature, similarity may be defined by, e.g., systems of affine maps (Sierpinski), or systems of conformal maps (Julia). We study new classes of semi-branching function systems related to stationary Bratteli diagrams. The latter plays a big role in our understanding of new forms of harmonic analysis on fractals. The measures considered here arise in classes of discrete-time, multi-level dynamical systems where similarity is specified between levels. These structures are made precise by prescribed systems of functions which in turn serve to define self-similarity, i.e., the similarity of large scales, and small scales. For path space systems, in our main result, we give a necessary and sufficient condition for the existence of such generalized IFS measures. For the corresponding semi-branching function systems, we further identify the measures which are also shift-invariant.

Keywords— Bratteli diagrams, invariant path space measures, self-similar measures, iterated and semibranching function systems, IFS measures, Perron-Frobenius.

AMS Mathematics Subject Classification: 28A33, 28A80, 37B10, 46G12, 60J22

Contents

1 Introduction 2

2 Basics on generalized Bratteli diagrams 3
2.1 Main definitions ........................................ 3
2.2 Measures on the path space of a Bratteli diagram .......... 5
2.3 Semibranching function systems on Bratteli diagrams .... 7
2.4 Shift invariant measures on stationary Bratteli diagrams ... 10

3 IFS measures on discrete generalized Bratteli diagrams 12
3.1 Iterated function systems and measures .................. 12
3.2 IFS measures on generalized stationary Bratteli diagrams 14
1 Introduction

We present new results in the study of self-similar measures on path space for infinite graph models, here called generalized Bratteli diagrams. By this, we mean particular graph systems $B$ with the property that the corresponding sets $V$ of vertices, and $E$ of edges, admit discrete level-structures, defined below. This means that $V$ is a disjoint union of the sets $V_n$ and $E$ is a disjoint union of the sets $E_n$. We emphasize that the class of generalized Bratteli diagrams include two cases: (1) all the sets $V_n, E_n$ are countable (if all $V_n$’s are finite, then we have a standard Bratteli diagram); (2) all the set $V_n, E_n$ are standard Borel spaces. In case (2), we will say that $B$ is a measurable (or continuous) Bratteli diagram.

Our present framework is motivated by, but more versatile than, the more familiar setting in earlier studies of Bratteli diagrams. We use the discrete levels in order to identify a class of self-similar path space measures, called iterated function system (IFS) measures. The latter in turn are inspired by earlier studies of IFS systems arising the analysis of fractals, such as various Sierpinski constructs, or conformal attractors. However, by contrast our analysis centers around a new path space analysis for discrete-time random walk models in generalized Bratteli diagrams. We also discuss IFS measures for the case when the system of levels making up $B$ are instead standard measure spaces, so non-discrete.

We recall that a measure $\mu$ on a standard Borel space $X$ is called self-similar with respect to an iterated function system $f_i : X \to X, i \in I$, if

$$\mu = \sum_{i \in I} p_i \mu \circ f_i^{-1},$$

where $p = (p_i)_{i \in I}$ is a probability vector and $I$ is finite or countable.

A main aim of our paper is an identification of, and an analysis of, iterated function system (IFS) measures on the path space $X_B$ of generalized Bratteli diagrams. This entails two tasks, (a) an analysis of IFS structure of the path spaces, and (b) a study of the particular IFS measures on them.

Our main results consist of finding an explicit construction of IFS measures for both discrete and measurable generalized Bratteli diagrams.

In order to motivate, and to place this in context, we add the following three comments here: (i) Previously, IFS measures have been considered in special geometries of self-similar fractals (see e.g., [Jor18] and the papers cited there). These standard self-similar fractals may be realized in finite-dimensional Euclidean space. By contrast, there are no previous studies of IFS measures in path space, i.e., IFS measures realized in the class of path space structures considered here, standard and generalized Bratteli diagrams. (ii) While there are earlier results on other, different but related, classes of measures on path space of generalized Bratteli diagrams, e.g., tail invariant measures, and Markov measures, our present identification of IFS measures on path space is new. (iii) In order to prepare the reader for the IFS path space measures, it will be necessary for us to begin with an account of tail invariant measures, and Markov measures. The tools involved there for generalized Bratteli diagrams are also needed in our introduction of the new IFS measures. But this means our main results for IFS measures will be postponed to sections 3 and 4 below, after the necessary preliminaries have been presented.
Generalized Bratteli diagrams considered here arise in various areas and have many applications. We mention Cantor and Borel dynamics where they are used to construct models of transformations, see [HPS92], [GPS95], [Dur10], [BK16], [BDK06]. Measurable Bratteli diagrams can be met in the theory of Markov chains, we refer to [Num84] as an example of such literature. Several other applications come to mind, (1) models from financial mathematics, and (2) neural networks. In each case, the role of the IFS measures must be specified. In some applications, there are important limit theorems, for example for financial derivative models both discrete and continuous pricing formulas are important. And it may be stated in the setting of generalized Bratteli diagrams, measurable setting. We mean first of all financial derivative models, e.g., binomial models; vs continuous/measurable (cm), e.g., pricing of options via Ito calculus, see e.g., [BS13], [KS98]. In these applications, we also have theorems to the effect that the continuous models are limits of discrete counterparts. Typically, the limit arguments involve the Central Limit Theorem from probability. Another case of discrete models take the form of deep neural network models, deep means a “large number of levels”, so many steps through the levels in our diagrams.

The organization of the paper is the following. Section 2 contains the basic definitions related to the concept of generalized Bratteli diagram (discrete case), and the description of various classes of measures on the path space of a generalized Bratteli diagrams. We consider tail invariant, shift-invariant, and Markov measures and their relations to a semibranching function system generated by a stationary generalized Bratteli diagram. In section 3, we consider a semibranching function system \( \{ \tau_e \} \) defined on the path space of a stationary generalized Bratteli diagram and indexed by the edge set \( E \). We prove there one of the main results by giving necessary and sufficient conditions on the existence of an IFS measure for \( \{ \tau_e \} \), see Theorem 3.4. Section 4 focuses on measurable Bratteli diagrams. Since this notion is relatively new, we give detailed definitions and discuss the properties of such diagrams. Then we prove a measurable version of the main theorem about the existence of an IFS measure, Theorem 4.7.

At the end of this introduction, we mention the literature that may be interesting for the reader. The literature on standard Bratteli diagrams their applications in dynamics is very extensive. We mention here the following pioneering papers [HPS92], [GPS95], [DHS99], a recent book [Put18], and surveys [Dur10], [BK16], [BK20]. Generalized Bratteli diagrams are less studied. The stationary case uses the Perron-Frobenius theory for infinite matrices. We refer to the book [Kit98] and the literature there. These diagrams are discussed in [BJ22] and [BJS22]. More references and numerous connections with other areas can be found therein. In particular, the following papers on IFS measures and fractals are related to the current paper [BJ99], [BJ02], [DJ09], [DJ10], [DJ14b], [Jor06], [Jor18], [RS16], [ARCG+20], [CHIQ+21].

## 2 Basics on generalized Bratteli diagrams

We consider the fundamentals of path space for generalized Bratteli diagrams in this section. This notion was first introduced in [BDK06] under the name of Borel-Bratteli diagram. More detailed exposition of this concept can be found in [BJ22]. For the reader’s convenience we give a concise version here.

### 2.1 Main definitions

In the introduction, we described the notion of a Bratteli diagram as an infinite graded graph. A natural extension of this concept consists of consideration of diagrams with countably infinite levels.
Definition 2.1. (Generalized Bratteli diagrams, vertices, edges, incidence matrices) Let $V_0$ be a countable set (which can be identified with either $\mathbb{N}$ or $\mathbb{Z}$ if necessary). Set $V_i = V_0$ for all $i \geq 1$, and $V = \bigsqcup_{i=0}^{\infty} V_i$. A countable graded graph $B = (V, E)$ is called a generalized Bratteli diagram if it has the following properties.

(i) The set of edges $E$ of $B$ is represented as $\bigsqcup_{i=0}^{\infty} E_i$ where $E_i$ is the set of edges between the vertices of levels $V_i$ and $V_{i+1}$, $i \geq 0$.

(ii) The set $E(w, v)$ of edges $e$ between the vertices $w \in V_i$ and $v \in V_{i+1}$ is finite (or empty). Let $f_{v,w}^{(i)} = |E(w, v)|$ where $|\cdot|$ denotes the cardinality of a set. It defines a sequence of finite (countable-by-countable) incidence matrices $(F_i : i \in \mathbb{N}_0)$ with entries $F_i = (f_{v,w}^{(i)} : v \in V_{i+1}, w \in V_i)$, $f_{v,w}^{(i)} \in \mathbb{N}_0$.

(iii) It is required that the matrices $F_i$ have at most finitely many non-zero entries in each row. In general, we do not impose any restrictions on the columns of $F_i$.

(iv) The maps $r, s : E \to V$ are defined on the diagram $B$: for every $e \in E$, there are $w, v$ such that $e \in E(w, v)$; then $s(e) = w$ and $r(e) = v$. They are called the range $(r)$ and source $(s)$ maps.

(v) For every $w \in V_i$, $i \geq 0$, there exist an edge $e \in E_i$ such that $s(e) = w$; for every $v \in V_i$, $i > 1$, there exists an edge $e' \in E_{i-1}$ such that $r(e') = w$. In other words, every row and every column of the incidence matrix $F_i$ has non-zero entries.

(vi) If all entries of incidence matrices $F_n$ are zero or ones, the corresponding generalized Bratteli diagram is called a 0-1 diagram.

Remark 2.2. (1) It follows from Definition 2.1 that the structure of every generalized Bratteli diagram is completely determined by a sequence of matrices $(F_n)$ such that every matrix $F_n$ satisfies (iii) and (iv). Indeed, the entry $f_{v,w}^{(n)}$ indicates the number of edges between the vertex $w \in V_n$ and vertex $v \in V_{n+1}$. It defines the set $E(w, v)$; then one takes

$$E_n = \bigcup_{w \in V_n, v \in V_{n+1}} E(w, v).$$

In this case, we write $B = B(F_n)$. If all $F_n = F$, the corresponding generalized Bratteli diagram $B(F)$ is called stationary.

(2) If $V_0$ is a singleton, and each $V_n$ is a finite set, then we obtain the standard definition of a Bratteli diagram originated in [Bra72]. Later it was used in the theory of $C^*$-algebras and dynamical systems for solving important classification problems of Cantor dynamics and construction of various models of homeomorphisms of a Cantor set (for references, see Introduction).

Definition 2.3. (Path space and cylinder sets) A finite or infinite path in a generalized Bratteli diagram $B = (V, E)$ is a sequence of edges $(e_i : i \geq 0)$ such that $r(e_i) = s(e_{i+1})$ for all $i \geq 0$. Denote by $X_B$ the set of all infinite paths. Every finite path $\bar{e} = (e_0, \ldots, e_n)$ determines a cylinder subset $[\bar{e}]$ of $X_B$:

$$[\bar{e}] := \{x = (x_i) \in X_B : x_0 = e_0, \ldots, x_n = e_n\}.$$

The collection of all cylinder subsets forms a base of neighborhoods for a topology on $X_B$. In this topology, $X_B$ is a Polish zero-dimensional space, and every cylinder set is clopen. In general, $X_B$ is not locally compact. But if the set $s^{-1}(v)$ is finite for every vertex $v \in V$, then the path space $X_B$ is locally compact.

In the following remark, we formulate several statements about properties of generalized Bratteli diagrams and their path spaces.
Remark 2.4. (1) If $x = (x_i)$ is a point in $X_B$, then it is obviously represented as the intersection of clopen sets:

$$\{ x \} = \bigcap_{n \geq 0} [\mathcal{E}]_n$$  \hspace{1cm} (2.1)

where $[\mathcal{E}]_n = [x_0, \ldots, x_n]$. 

(2) Define a metric on $X_B$ compatible with the clopen topology: for $x = (x_i)$ and $y = (y_i)$ from $X_B$, we set

$$\text{dist}(x, y) = \frac{1}{2^N}, \quad N = \min\{i \in \mathbb{N}_0 : x_i \neq y_i\}.$$ 

(3) We will assume that the diagram $B$ is chosen so that the space $X_B$ has no isolated points. This means that every column of the incidence matrix $F_n, n \in \mathbb{N}_0$, has more than one non-zero entry.

(4) Let $\mathbf{1} = (\ldots, 1, 1, \ldots)$ be the vector indexed by $v \in V_0$ such that every entry equals 1. Define $H^{(n)} := F_{n-1} \cdots F_0 \mathbf{1}$. Let $E(V_0, v), v \in V_n$, denote the set of all finite paths between $V_0$ and a fixed vertex $v \in V_n$. Then $H_v^{(n)} = |E(V_0, v)|$.

(5) Let $X_v^{(n)}$ be a subset of $X_B$ such that

$$X_v^{(n)} = \bigcup_{v_0 \in V_0} \bigcup_{\pi \in E(v_0, v)} [\pi].$$ \hspace{1cm} (2.2)

For every level $V_n$, the collection $\{X_v^{(n)} : v \in V_n\}$ forms a partition $\zeta_n$ of $X_B$ into disjoint clopen sets. Every set $X_v^{(n)}$ is a finite union of cylinder sets. The number of the cylinder sets here is exactly $H_v^{(n)}$. The sequence of partitions $(\zeta_n)$ is refining. According to (2.2), the cylinder sets from all $X_v^{(n)}$ generate the topology (and Borel $\sigma$-algebra) on $X_B$.

For a generalized Bratteli diagram $B$, define the tail equivalence relation $\mathcal{R}$ on the path space $X_B$.

**Definition 2.5.** (Tail equivalence relation) It is said that two infinite paths $x = (x_i)$ and $y = (y_i)$ are tail equivalent if there exists $m \in \mathbb{N}$ such that $x_i = y_i$ for all $i \geq m$. Let $[x]_{\mathcal{R}} := \{ y \in X_B : (x, y) \in \mathcal{R} \}$ be the set of points tail equivalent to $x$. We say that a point $x$ is periodic if $|[x]_{\mathcal{R}}| < \infty$. If there is no periodic points, then the tail equivalence relation is called aperiodic.

Without loss of generality, we will consider generalized Bratteli diagrams with aperiodic $\mathcal{R}$. Clearly, $\mathcal{R}$ is a hyperfinite countable Borel equivalence relation, see [DJK94] for definitions.

**Definition 2.6.** It is said that a generalized Bratteli diagram $B$ is irreducible if, for any two vertices $v$ and $w$ and any $n \in \mathbb{N}_0$, there exists a level $V_m (m > n)$ such that $w \in V_n$ and $v \in V_m$ are connected by a finite path. This is equivalent to the property that, for any fixed $v, w$, there exists $m \in \mathbb{N}$ such that the product of matrices $F_{m-1} \cdots F_n$ has a non-zero $(v, w)$-entry.

### 2.2 Measures on the path space of a Bratteli diagram

In this subsection, we will consider two classes of Borel measures on the path space $X_B$ of a generalized Bratteli diagram. They are tail invariant measures and Markov measures.

**Definition 2.7.** (Tail equivalent measures) Let $B = (\mathcal{V}, \mathcal{E})$ be a generalized Bratteli diagram, and $X_B$ the path space of $B$. A Borel measure $\mu$ on $X_B$ (finite or $\sigma$-finite) is called tail invariant if, for any two finite paths $\pi$ and $\pi'$ such that $r(\pi) = r(\pi')$, one has

$$\mu([\pi]) = \mu([\pi']),$$ \hspace{1cm} (2.3)

where $[\pi]$ and $[\pi']$ denote the corresponding cylinder sets.
If \( \mu(X_B) = 1 \), then the property of tail invariance means that the probability to arrive at a vertex \( v \in V_n \) does not depend on a starting point \( w \in V_0 \) and does not depend on the path connecting \( w \) and \( v \).

Let \( \mu \) be a Borel tail invariant measure on \( X_B \). Relation (2.3) defines a sequence of non-negative vectors \((\mu^{(n)})\) where \( \mu^{(n)} = (\mu^{(n)}_v : v \in V_n) \):

\[
\mu^{(n)}_v = \mu([\tau]), \quad \tau \in E(V_0, v), \ v \in V_n. \tag{2.4}
\]

Because \( \mu \) is tail invariant the value \( \mu^{(n)}_v \) does not depend on the choice of \( \tau \in E(V_0, v) \).

The following theorem is a key tool in the study of tail invariant measures, see [BKMS10], [Dur10], [BK16], [BJ22].

**Theorem 2.8.** Let \( B = (\mathcal{V}, \mathcal{E}) \) be a generalized Bratteli diagram defined by a sequence \((F_n)\) of incidence matrices. Let \( \mu \) be a Borel probability measure on the path space \( X_B \) of \( B \) which is tail invariant. Then the corresponding sequence of vectors \( \mu^{(n)} \) (defined as in (2.4)) satisfies the property

\[
A_n\mu^{(n+1)} = \mu^{(n)}, \tag{2.5}
\]

where \( A_n = F_n^T \) is the transpose of \( F_n \).

Conversely, if a sequence of vectors \( \mu^{(n)} \) satisfies (2.5), then it defines a unique tail invariant measure \( \mu \).

The theorem remains true for \( \sigma \)-finite measures \( \nu \) satisfying the property \( \nu([\tau]) < \infty \) for every cylinder set \([\tau]\).

The other interesting class of measures on Bratteli diagrams is Markov measures. In the context of Bratteli diagrams, these measures were considered in [DH03], [Ren18], [BJ22] and some other papers.

**Definition 2.9.** (Markov measures) Let \( B = (\mathcal{V}, \mathcal{E}) \) be a generalized Bratteli diagram constructed by a sequence of incidence matrices \((F_n)\). Let \( q = (q_v) \) be a strictly positive vector, \( q_v > 0, v \in V_0 \), and let \((P_n)\) be a sequence of non-negative infinite matrices with entries \((p^{(n)}_{v,e})\) where \( v \in V_n, e \in E_n, n = 0, 1, 2, \ldots \). To define a Markov measure \( m \), we require that the sequence \((P_n)\) satisfies the following properties:

\[
(a) \quad p^{(n)}_{v,e} > 0 \iff (s(e) = v); \quad (b) \quad \sum_{e : s(e) = v} p^{(n)}_{v,e} = 1. \tag{2.6}
\]

Condition (2.6)(a) shows that \( p^{(n)}_{v,e} \) is positive only on the edges outgoing from the vertex \( v \), and therefore the matrices \( P_n \) and \( A_n = F_n^T \) share the same set of zero entries. For any cylinder set \([\tau] = [(e_0, e_1, \ldots, e_n)]\) generated by the path \( \tau \) with \( v = s(e_0) \in V_0 \), we set

\[
m([\tau]) = q_{s(e_0)}P_{s(e_0),e_0}^{(0)} \cdots p^{(n)}_{s(e_n),e_n}. \tag{2.7}
\]

Relation (2.7) defines the value of the measure \( m \) of the set \([\tau]\). By (2.6)(b), this measure satisfies the Kolmogorov consistency condition and can be extended to the \( \sigma \)-algebra of Borel sets. To emphasize that \( m \) is generated by a sequence of stochastic matrices, we will also write \( m = m(P_n) \).

If all stochastic matrices \( P_n \) are equal to a matrix \( P \), then the corresponding measure \( m(P) \) is called stationary Markov measure.

We refer to [BJ22] for a detailed study of Markov measures. We mention here only the following result.
Theorem 2.10. Let $\nu$ be a tail invariant probability measure on the path space $X_B$ of a generalized Bratteli diagram $B = (V,E)$. Then there exists a sequence of Markov matrices $(P_n)$ such that $\nu = m(P_n)$.

For stationary generalized Bratteli diagrams, we can find explicit formulas for tail invariant measures. In the following statement, we use the terminology from the Perron-Frobenius theory, see [Kit98] for details.

Theorem 2.11. [BJ22] (1) Let $B = B(F)$ be a stationary Bratteli diagram such that the incidence matrix $F$ (and therefore $A = F^T$) is irreducible, aperiodic, and recurrent. Let $t = (t_v : v \in V_0)$ be a right eigenvector corresponding to the Perron eigenvalue $\lambda$ for $A$, $At = \lambda t$. Then there exists a tail invariant measure $\mu$ on the path space $X_B$ whose values on cylinder sets are determined by the following rule: for every finite path $\bar{e}(w,v)$ that begins at $w \in V_0$ and terminates at $v \in V_n$, $n \in \mathbb{N}_0$, we set

$$\mu_n(v) = \mu([\bar{e}(w,v)]) = \frac{t_v}{\lambda^n}. \quad (2.8)$$

(2) The measure $\mu$ is finite if and only if the right eigenvector $t = (t_v)$ has the property $\sum_v t_v < \infty$.

In particular, $\mu(X_0(w)) = t_w, w \in V_0$, and

$$\mu(X_n(v)) = H_v^{(n)} \frac{t_v}{\lambda^n}. \quad (2.9)$$

2.3 Semibranching function systems on Bratteli diagrams

Here we give the definition of a semibranching function system following [MP11] and [BJ15]. This notion was used in the literature, in particular, for the construction of representations of Cuntz-Krieger algebras, see [BJ15], [FGJ+18a], [FGJ+18b].

Definition 2.12. (Semibranching function systems and coding maps)

(1) Let $(X,\mu)$ be a probability measure space with non-atomic measure $\mu$. We consider a family $\{\sigma_i : i \in \Lambda\}$ of one-to-one $\mu$-measurable maps indexed by a finite (or countable) set $\Lambda$. The family $\{\sigma_i\}$ is called a semibranching function system (s.f.s.) if the following conditions hold:

(i) $\sigma_i$ is defined on a subset $D_i$ of $X$ and takes values in $R_i = \sigma_i(D_i)$ such that $\mu(R_i \cap R_j) = 0$ for $i \neq j$ and $\mu(X \setminus \bigcup_{i \in \Lambda} R_i) = 0$;

(ii) the measure $\mu \circ \sigma_i$ is equivalent to $\mu$ and, i.e.,

$$\rho_\mu(x, \sigma_i) := \frac{d\mu \circ \sigma_i}{d\mu}(x) > 0 \text{ for } \mu\text{-a.e. } x \in D_i;$$

(iii) there exists an endomorphism $\sigma : X \to X$ (called a coding map) such that $\sigma \circ \sigma_i(x) = x$ for $\mu$-a.e. $x \in D_i$, $i \in \Lambda$.

If, additionally to properties (i) - (iii), we have $\bigcup_{i \in \Lambda} D_i = X$ ($\mu$-a.e.), then the s.f.s. $\{\sigma_i : i \in \Lambda\}$ is called saturated.

(2) It is said that a saturated s.f.s. satisfies condition $C$-$K$ \footnote{C-K stands for Cuntz-Krieger.} if for any $i \in \Lambda$ there exists a subset $\Lambda_i \subset \Lambda$ such that up to a set of measure zero

$$D_i = \bigcup_{j \in \Lambda_i} R_j.$$
In this case, condition C-K defines a 0-1 matrix \( \tilde{A} \) by the rule:

\[
\tilde{a}_{i,j} = 1 \iff j \in \Lambda_i, \ i \in \Lambda.
\]  
(2.10)

Then the matrix \( \tilde{A} \) is of the size \(|\Lambda| \times |\Lambda|\).

**Semibranching function system associated with a generalized stationary Bratteli diagram.**

Let \( B \) be a generalized stationary 0-1 Bratteli diagram. We construct an s.f.s. \( \Sigma \) which is defined on the path space \( X_B \) endowed with a Markov measure \( m \). As we will see below, this measure must have some additional properties to satisfy Definition 2.12. The role of the index set \( \Lambda \) for this s.f.s. is played by the edge set \( E \) which is the set of edges between any two consecutive levels of \( B \). For any \( e \in E \), we denote

\[
D_e = \{ y = (y_i) \in X_B : s(y) = s(y_0) = r(e) \},
\]
(2.11)

\[
R_e = \{ y = (y_i) \in X_B : y_0 = e \}.
\]
(2.12)

We see that \( D_e \) depends on \( r(e) \) only so that \( D_e = D_{e'} \) if and only if \( r(e) = r(e') \).

The collection of maps \( \{ \tau_e : e \in E \}, \tau_e : D_e \to R_e \), is defined by the formula

\[
\tau_e(y) := (e, y_0, y_1, \ldots), \ y = (y_i).
\]
(2.13)

Since \( s(y_0) = r(e) \), the map \( \tau_e \) is well defined on \( D_e \).

**Remark 2.13.** We defined the metric dist in Remark 2.4. It follows from the definition of \( \tau_e \) that

\[
\text{dist}(\tau_e(x), \tau_e(y)) = \frac{1}{2}\text{dist}(x, y), \ e \in E,
\]

that is \( \tau_e \) is a contractive map for every \( e \).

**Proposition 2.14.** The system \( \{ D_e, R_e, \tau_e : e \in E \} \) is a saturated s.f.s. on the path space \( X_B \) of a generalized stationary Bratteli diagram satisfying conditions (i), (iii) of Definition 2.12 and the C-K condition.

**Proof.** Let \( \sigma : X_B \to X_B \) be defined as follows: for any \( x = (x_i)_{i \geq 0} \in X_B \),

\[
\sigma(x) := (x_1, x_2, \ldots)
\]
(2.14)

It follows from (2.13) and (2.14) that the map \( \sigma \) is onto and

\[
\sigma \circ \tau_e(x) = x, \ x \in D_e;
\]

Hence \( \sigma \) is a coding map.

We deduce from (2.12) that \( \{ R_e : e \in E \} \) constitutes a partition of \( X_B \) into clopen sets. Relation (2.11) implies that \( \{ \tau_e : e \in E \} \) is a saturated s.f.s. Moreover, we claim that it satisfies condition C-K. Indeed,

\[
D_e = \bigcup_{f : s(f) = r(e)} R_f, \ e \in E,
\]
(2.15)

because \( y = (y_i) \in D_e \iff s(y_0) = r(e) \iff \exists f = y_0 \text{ such that } y = (f, y_1, \ldots) \iff y \in \bigcup_{f : s(f) = r(e)} R_f. \) Thus, \( \Lambda_e = \{ f : s(f) = r(e) \} \), see (2.10).

Relation (2.15) shows that the non-zero entries of the 0-1 matrix \( \tilde{A} \) from Definition 2.12 are defined by the rule:

\[
(\tilde{a}_{e,f} = 1) \iff (s(f) = r(e)).
\]
(2.16)
We observe that the matrix $\tilde{A}$ has the following property: there are finitely many nonzero entries in every column of $\tilde{A}$, but the rows of $\tilde{A}$ may contain infinitely many nonzero entries.

Next, we observe that $\sigma : X_B \to X_B$ is a finite-to-one continuous map. Indeed,

$$|\sigma^{-1}(x)| = |r^{-1}(r(x))| = \sum_{u \in V_0} f_{v,u}. $$

The latter is finite by Definition 2.1.

Thus, it remains to find out under what conditions property (ii) of Definition 2.12 holds. We consider here two classes: tail invariant measures and Markov measures.

Let $m$ be a Borel probability measure on $X_B$. Since $X_B$ is naturally partitioned into a refining sequence of clopen partitions $Q_n$ formed by cylinder sets of length $n$, we can apply de Possel’s theorem (see, for instance, [SG77]). We have that for $m$-a.a. $x$,

$$\rho_m(x, \tau_e) = \lim_{n \to \infty} \frac{m(\tau_e([\bar{e}(n)]))}{m(\bar{e}(n))}$$

(2.17)

where $\{x\} = \cap_n [\bar{e}(n)]$.

*Tail invariant measure.* We first consider the case when $m$ is the tail invariant measure $\mu$ determined in Theorem 2.11. Let $At = \lambda t$ where $t = (t_v)$. If $\bar{f} = (f_0, f_1, \ldots, f_n) \in B_e$, then $r(\bar{f}) = r(f_n)$ and $\tau_e(\bar{f}) = (e, f_0, \ldots, f_n)$. By (2.8), we have

$$\mu([\bar{f}]) = \frac{t_r(\bar{f})}{\lambda^n}, \quad \mu(\tau_e([\bar{f}])) = \frac{t_r(\bar{f})}{\lambda^{n+1}},$$

and therefore

$$\rho_\mu(x, \tau_e) = \lambda^{-1}. $$

(2.18)

*Stationary Markov measure.* Let $m$ be a stationary Markov measure determined by a stochastic matrix $P$, $m = m(P)$ as in (2.7). If $\{x\} = \cap_n [f(n)] \in B_e$, then

$$m([\bar{f}(n)]) = q_s(f_0)p_{s(f_0),f_0} \cdots p_{s(f_n),f_n},$$

$$m(\tau_e([\bar{f}(n)])) = q_s(e)p_{s(e),e}p_{s(f_0),f_0} \cdots p_{s(f_n),f_n},$$

and the Radon-Nikodym derivative can be found by

$$\rho_m(x, \tau_e) = \frac{q_s(e)p_{s(e),e}}{q_s(f_0)}. $$

(2.19)

It follows from (2.18) and (2.19) that $\rho_m$ is positive on $B_e$ if and only if all entries of the vector $q = (q_v)$ are positive. The latter means that the support of $m$ is the entire space $X_B$.

*Non-stationary Markov measure.* In this case, we need some additional conditions to guarantee that the Radon-Nikodym derivative $\rho_m(x, \tau_e)$ is positive on $B_e$. As above, we represent $x = (x_i) \in B_e$ by means of the sequence $[\bar{f}(n)] = [(f_0, f_1, \ldots, f_n)]$ such that $x_i = f_i, i = 0, 1, \ldots, n$, and find

$$m([f(n)]) = q_{s(f_0)}p_{s(f_0),f_0}^{(0)} \cdots p_{s(f_n),f_n}^{(n)}$$

and

$$m(\tau_e([f(n)])) = q_{s(e)}p_{s(e),e}^{(0)}p_{s(f_0),f_0}^{(1)} \cdots p_{s(f_n),f_n}^{(n+1)}.$$
Lemma 2.15. Let $m$ be a Markov measure on the path space of a generalized stationary 0-1 Bratteli diagram. Then $\rho_m(x,\tau_e) > 0$ on $D_e$ if and only if

$$0 < \prod_{i=1}^{\infty} P_{s(f_i),f_i}^{(i)} < \infty$$

(2.20)

for any $x = \bigcap_n [\mathcal{F}(n)] \in D_e$.

A Markov measure satisfying (2.20) is called a quasi-stationary measure. Condition (2.20) appeared first in [DJ14a], [DJ15] in a different context.

We summarize the above results in the following theorem.

Theorem 2.16. Given a generalized stationary 0-1 Bratteli diagram $B$ with the edge set $E$, the collection of maps $\{\tau_e : D_e \to R_e\}$, $e \in E$, defined in (2.13) on the space $(X_B,m)$, forms a saturated s.f.s. $\Sigma$ satisfying C-K condition where the Markov measure $m$ is either the tail invariant measure $\mu$, or a stationary Markov measure $m(P)$ of full support, or a quasi-stationary measure Markov measure of full support.

Remark 2.17. Let $B$ be a generalized stationary Bratteli diagram of bounded size, see [BJKS] for the definition. In particular, if the incidence matrix $A$ is banded, then $B$ is of bounded size. In this case, the definition of the Cuntz-Krieger algebra $O_A$ can be given similar to the case of finite matrices. Then we can use the methods developed in [BJ15, Theorem 4.12] to construct a representation of the Cuntz-Krieger algebra $O_A$ generated by the s.f.s. defined in Theorem 2.16. We omit the details.

2.4 Shift invariant measures on stationary Bratteli diagrams

Let $B = (V,E)$ be a generalized stationary Bratteli diagram defined by the incidence infinite matrix $F$ and $A = F^T$. In this subsection, we discuss $\sigma$-invariant measures on $X_B$. We will consider two cases, tail invariant measures and Markov measures.

Recall that, for every $v \in V_1$, the row sum $H^{(1)}_e = \sum_{w \in V_0} f_{e,w}$ is finite, see Remark 2.4 for notation. In (2.14), we defined a finite-to-one endomorphism $\sigma$ acting on the path space $X_B$. For $x = (e_0,e_1,...)$, we have

$$\sigma^{-1}(x) = \{ y = (y_i) \in X_B : r(y_0) = s(e_0), y_i = e_{i-1}, i \geq 1 \}$$

(2.21)

and $|\sigma^{-1}(x)| = H^{(1)}_{s(e_0)}$.

Tail invariant measures. There is a special case of a generalized stationary Bratteli diagram such that the tail invariant measure is also shift-invariant.

Proposition 2.18. Let the matrix $A$ has a Perron-Frobenius eigenpair $(\xi,\lambda)$. Let $\mu$ be the tail invariant measure on $X_B$ defined as in Theorem 2.11. Then $\mu$ is $\sigma$-invariant if and only if $H^{(1)}_e = \lambda$ for all $e \in V_0$.

Proof. For a cylinder set $[\bar{v}] = [e_0,e_1,...,e_n]$, calculate $m([\bar{v}])$ and $m(\sigma^{-1}([\bar{v}]))$. it follows from (2.21) that

$$\sigma^{-1}([\bar{v}]) = \bigcup_{f : r(f) = s(e_0)} [f,e_0,...,e_n] = \bigcup_{f : r(f) = s(e_0)} [f,\bar{v}]$$

(2.22)

and this union is disjoint. Let $\lambda$ and $\xi$ be a Perron-Frobenius eigenpair, $A \xi = \lambda \xi$. By Theorem 2.11, we have

$$\mu([\bar{v}]) = \frac{\xi_v}{\lambda^n}, \quad \mu([f,\bar{v}]) = \frac{\xi_v}{\lambda^{n+1}}, \quad v = r(\bar{v})$$
where $w = s(e_0) \in V_1$. By de Possel’s theorem (2.17),
\[
\frac{d \mu \circ \sigma^{-1}}{d \mu}(x) = \lim_{n \to \infty} \frac{\mu \circ \sigma^{-1}(\bar{\sigma}_n)}{\mu(\bar{\sigma}_n)} = H_w^{(1)} \lambda^{-1}
\]
where $x = \bigcap_n [\bar{\sigma}]_n$. Therefore,
\[
\mu \circ \sigma^{-1} = \mu \iff H_w^{(1)} = \sum_{w \in V_0} f_{w,n} = \lambda, \ \forall w \in V_0.
\]

**Remark 2.19.** If $\max\{H_w^{(1)} : v \in V_1\} = M < \infty$, then
\[
\lambda^{-1} \leq \frac{d \mu \circ \sigma^{-1}}{d \mu}(x) \leq M \lambda^{-1}.
\]

This means that $\mu$ is equivalent to a $\sigma$-invariant measure $\mu'$. In particular, this is true for bounded size diagrams.

**Markov measures.** Consider first a stationary Markov measure on the path space of a generalized stationary Bratteli diagram, see Definition 2.9. For simplicity we will work with a $0 - 1$ diagram; the general case is considered similarly.

Let $q = (q_v : v \in V_0)$ be a positive probability vector (called initial distribution). Let $P$ be a Markov matrix. For a cylinder set $[\bar{\sigma}] = [e_0, e_1, \ldots, e_n]$, we use (2.7) to determine the value of the corresponding Markov measure $m(P)$.

**Theorem 2.20.** For $B, q, P, m(P)$ as above, the measure $m = m(P)$ is $\sigma$-invariant if and only if $qP = P$.

**Proof.** Suppose that $qP = q$. We know that $m([\bar{\sigma}]_n) = q_{s(e_0)} p_{s(e_1),r(e_1)} \cdots p_{s(e_n),r(e_n)}$ and
\[
m(\sigma^{-1}[\bar{\sigma}]) = \sum_{f : r(f) = s(e_0)} m([f, \bar{\sigma}])
\]
\[
= \sum_{f : r(f) = s(e_0)} q_{r(f)} p_{s(f),r(f)} p_{s(e_0),r(e_0)} \cdots p_{s(e_n),r(e_n)}
\]
(2.23)
because $q_{r(f)} = q_{s(e_0)}$ and $q$ is $P$-invariant. Since $\bar{\sigma}$ is arbitrary, we see that $m$ is $\sigma$-invariant.

The converse statement follows from (2.23): the equality $m([\bar{\sigma}]) = m(\sigma^{-1}[\bar{\sigma}])$ implies $qP = q$. \hfill \Box

It remains to consider a non-stationary Markov measure $m$ defined on the path space of a generalized stationary Bratteli diagram $B$. In this case, the measure $m$ is defined by a sequence of Markov matrices $(P_n)$ and an initial distribution $q = (q_v)$:
\[
m([e_0, \ldots, e_n]) = q_{s(e_0)} p_{s(e_0),r(e_0)}^{(0)} \cdots p_{s(e_n),r(e_n)}^{(n)}.
\]
Proposition 2.21. Suppose that $qP_0 = q$. Then the measures $m = m(P_n)$ is $\sigma$-invariant if and only if
$$\prod_{i=1}^{\infty} \frac{p_s(e_i), r(e_i)}{p_s(i), r(e_i)} = 1.$$ 

Proof. Using the same technique, we can show that, under the condition $qP = q$,
$$d\mu \circ \sigma^{-1} - d\mu(x) = \prod_{i=1}^{\infty} \frac{p_s(e_i), r(e_i)}{p_s(i), r(e_i)},$$
where $x = \bigcap_n [e_0, ..., e_n]$. We omit the details. \hfill \Box

3 IFS measures on discrete generalized Bratteli diagrams

In this section, we consider the notion of iterated function system (IFS). This concept have been discussed in many books and papers. We mention here only several related references such as [Hut96], [HR00], [Jor06], [Bar06], [Jor18], [DJ07], [DJ09], [JKS11], [MS21]. Our goal is to describe IFS measures defined on the path space of a generalized Bratteli diagram.

3.1 Iterated function systems and measures

By an endomorphism $\sigma$ we mean a finite-to-one (or countable-to-one) Borel map of a standard Borel space $(X, B)$ onto itself. In this case, there exists a family of one-to-one maps $\{\tau_i\}_{i \in \Lambda}$ such that $\tau_i : X \to X$ and $\sigma \circ \tau_i = \text{id}_X$ where $\Lambda$ is at most countable. The maps $\tau_i$ are called the inverse branches for $\sigma$. The collection of maps $\{\tau_i : 1 \in \Lambda\}$ gives an example of iterated function system (IFS). In general, an IFS is defined by a collection of maps $\{\tau_i : i \in \Lambda\}$ of a complete metric space (or a compact space) such that $\tau_i$’s are continuous (or even contractions). In this work, we focus on the following features of the family $\{\tau_i : i \in \Lambda\}$: (i) each $\tau_i$ is a one-to-one map defined on a subset $D_i$ of $X$, and (ii) there exists an endomorphism $\sigma$ of $X$ such that $\sigma \circ \tau_i = \text{id}_X$ for each $i$.

The general theory of infinite iterated function systems is more detailed and requires additional assumptions (see, for example, the expository article [Mau95]). As an example, one can consider the Gauss map $x \mapsto \{1/x\}$ of the unit interval with the piecewise monotone inverse branches $\tau_i : 1/(i + x) \text{ defined on } (1/(i + 1), 1/i)$.

Definition 3.1. (IFS measures) Suppose that $\{\tau_i : i \in \Lambda\}$ is a given IFS on a Borel space $(X, B)$. Let $p = (p_i : i \in \Lambda)$ be a strictly positive vector indexed by a countable set $\Lambda$. A measure $\mu_p$ on $(X, B)$ is called an IFS measure for $\{\tau_i\}$ if
$$\mu_p = \sum_{i \in \Lambda} p_i \mu_p \circ \tau_i^{-1}, \quad (3.1)$$
or, equivalently,
$$\int_X f(x) \, d\mu_p(x) = \sum_{i \in \Lambda} p_i \int_X f(\tau_i(x)) \, d\mu_p(x), \quad f \in L^1(\mu_p).$$
The main tool in the study of an IFS \((\tau_i : i \in \Lambda)\) on a space \(X\) is a realization of the IFS as the full one-sided shift \(S\) on a symbolic product space \(\Omega\). Then any \(S\)-invariant and ergodic product-measure \(P\) on \(\Omega\) can be pulled back to \(X\). This construction gives ergodic invariant measures for IFSs.

In the next subsection, we will find an explicit method for construction of an IFS measure on the path space of a generalized stationary Bratteli diagram.

We discuss here a method that leads to construction of IFS measures. This approach works perfectly for many specific applications under some additional conditions on \(X\) and maps \(\tau_i\).

Suppose that \((X; \tau_i, i \in \Lambda)\) is a given IFS. Let \(\Omega\) be the infinite direct product

\[
\Omega = \prod_{i \in \mathbb{N}_0} \Lambda_i, \quad \Lambda_i = \Lambda.
\]

For \(\omega = (\omega_1, \omega_2, \ldots) \in \Omega\), let \(\omega|_n\) denote the finite word \((\omega_0, \ldots, \omega_n)\. Then, \(\omega|_n\) defines a map \(\tau_{\omega|_n}\) acting on \(X\) by the formula:

\[
\tau_{\omega|_n}(x) := \tau_{\omega_0} \cdots \tau_{\omega_n}(x), \quad x \in X, \ \omega \in \Omega, \ n \in \mathbb{N}_0.
\]

It is said that \(\Omega\) is an encoding space if, for every \(\omega \in \Omega\),

\[
F(\omega) = \bigcap_{n \geq 1} \tau_{\omega|_n}(X)
\] (3.2)

is a singleton. Relation (3.2) defines a Borel map \(F: \Omega \to F(\Omega)\). If each \(\tau_i\) is a contraction and \(X\) is a complete metric, then a coding map \(F: \Omega \to X\) always exists.

Let \(S\) be the left shift on \(\Omega:\)

\[
S(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots).
\]

The inverse branches \(s_i, i \in \Lambda\), of \(S\) are

\[
s_i(\omega_0, \omega_1, \ldots) = (i, \omega_0, \omega_1, \ldots).
\]

Clearly,

\[
s_i(\Omega) = C(i) = \{\omega \in \Omega : \omega_0 = i\},
\]

and the space \(\Omega\) is partitioned into the sets \(C(i), i \in \Lambda\).

Let \(p = (p_i : i \in \Lambda)\) be a positive probability vector. It defines the product measure \(P = p \times p \times \cdots\) on \(\Omega\). We observe that the maps \((s_i : i \in \Lambda)\) constitute an IFS on \(\Omega\) such that \(P\) is an IFS measure:

\[
P = \sum_{i \in \Lambda} p_i \circ s_i^{-1}.
\] (3.3)

The following result shows how IFS measures arise.

**Proposition 3.2.** Suppose that \((X; \tau_i, i \in \Lambda)\) is an IFS that admits a coding map \(F: \Omega \to X\). Let \(p = (p_i)\) be a probability vector generating the product measure \(P = p \times p \times \cdots\). Then the measure \(\mu := P \circ F^{-1}\) is an IFS measure satisfying

\[
\mu = \sum_{i=1}^N p_i \mu \circ \tau_i^{-1}.
\]

Moreover, if \(F\) is continuous, then \(\mu\) has full support.

The proof can be found in [BJ18].

In Section 4, we will consider a analogue of the above construction for measurable Bratteli diagrams.
3.2 IFS measures on generalized stationary Bratteli diagrams

In this subsection we discuss an explicit formula for an IFS measure on the path space of a generalized stationary 0-1 Bratteli diagram \( B = (V, \mathcal{E}) \).

In the following remark, we show that the requirement to be a 0-1 Bratteli diagram is not restrictive.

**Remark 3.3.** Every generalized stationary Bratteli diagram \( B = (V, \mathcal{E}) \) can be represented as a 0-1 Bratteli diagram. Indeed, we can use the matrix \( \tilde{A} \) defined in (2.16) to built a new generalized Bratteli diagram \( \tilde{B} = (\tilde{V}, \tilde{E}) \). Note that \( \tilde{A} \) is a 0-1 matrix such that every row of \( \tilde{F} = \tilde{A}^T \) has finitely many non-zero entries so that \( \tilde{F} \) can be viewed as an incidence matrix of \( \tilde{B} \). In this case, the set of vertices \( \tilde{V} \) for each level is coincides with \( \tilde{E} \) and two vertices \( \tilde{v} = f \in \tilde{E} \) and \( \tilde{w} = f \in E \) are connected by a single edge if and only if \( r(e) = s(f) \). Moreover, the path spaces \( X_{\tilde{B}} \) coincides with \( X_B \).

The above argument can be applied to any non-stationary generalized Bratteli diagram.

We can now apply the results of Subsection 2.3 and construct an s.f.s. \( \Psi = (X_B; \tau_e, e \in E) \) where \( \tau_e : D_e \to R_e \) is defined in (2.13).

**Theorem 3.4.** Let \( B = (V, \mathcal{E}) \) be a generalized stationary 0-1 Bratteli diagram, \( p = (p_e : e \in E) \) a probability vector, and \( \Psi \) an s.f.s. defined by \( \tau_e : D_e \to R_e, e \in E \). We identify every edge \( e \in E \) with the pair of vertices \( (s(e), r(e)) \). Let \( \nu \) be a Borel probability full measure on \( X_B \); we define a probability positive vector \( q = (q_v : v \in V_0) \) by setting \( q_v = \nu([w]) \) where \( [w] = \{x \in X_B : s(x) = w\} \). Then

\[
\nu = \sum_{e \in E} p_e \nu \circ \tau_e^{-1}
\]

if and only if

\[
q_w = \sum_{w \in V_0} p_{w,v}q_v = \sum_{e : s(e) = w} p_{s(e),r(e)}q_{r(e)},
\]

that is \( Pq = q \) where the matrix \( P \) has the entries \( (p_{s(e),r(e)} : e \in E) \).

**Proof.** We use in the proof notation from (2.11) - (2.15). We note that relation (3.4) will be proved if we show that it holds for any cylinder set \( C \subset X_B \). Recall that it follows from the definition of \( \tau_e \) that \( \tau_e^{-1} \) is uniquely determined on \( R_e \) and \( \tau_e^{-1}(e, y_1, y_2,...) = (y_1, y_2,...) \).

Let \( q = (q_w) > 0 \) be a solution to \( q = Pq \). Define \( \nu([w]) = q_w, w \in V_0 \). For any edge \( f \in E \) and the corresponding cylinder set \( [f] \), we set

\[
\nu([f]) = q_{r(f)}p_f.
\]

Check that this definition of \( \nu \) satisfies (3.4) for the cylinder sets \( [f] \). Indeed, since \( \tau_e^{-1} \) is defined only on \( R_e = [e] \) and \( \tau_f^{-1}([f]) = [r(f)] \), we have

\[
\sum_{e \in E} p_e \nu \circ \tau_e^{-1}([f]) = p_f \nu \circ \tau_f^{-1}([f]) = p_f \nu([r(f)]) = q_{r(f)}p_f = \nu([f]).
\]

Then, by induction, we define the values of \( \nu \) on all cylinder sets of length \( n \):

\[
\nu([f_0, ..., f_{n-1}]) := p_{f_0} \cdots p_{f_{n-1}} q_{r(f_{n-1})}.
\]

Verify that this definition satisfies the Kolmogorov extension theorem. Because

\[
[f_0, ..., f_{n-1}] = \bigcup_{e : s(e) = r(f_{n-1})} [f_0, ..., f_{n-1}, e],
\]
we find that
\[ \sum_{e : s(e) = r(f_{n-1})} \nu([f_0, \ldots, f_{n-1}, e]) = \sum_{e : s(e) = r(f_{n-1})} p_{f_0} \cdots p_{f_{n-1}} p_e q_r(e) \]
\[ = p_{f_0} \cdots p_{f_{n-1}} \sum_{e : s(e) = r(f_{n-1})} p_e q_r(e) \]
\[ = p_{f_0} \cdots p_{f_{n-1}} q_r(f_{n-1}) \]
\[ = \nu([f_0, \ldots, f_{n-1}]) \]

(we used here (3.5)).

It remains to show that (3.4) holds for any cylinder \([f_0, \ldots, f_n]\). Indeed,
\[ \sum_{e \in E} p_e \nu \circ \tau_e^{-1}([f_0, \ldots, f_n]) = p_{f_0} \nu([f_1, \ldots, f_n]) \]
\[ = p_{f_0} p_{f_1} \cdots p_{f_n} q_r(f_n) \]
\[ = \nu([f_0, \ldots, f_n]). \]

Remark that the condition \(Pq = q\) is used to check that the measure \(\nu\) defined inductively on cylinder sets can be extended to all Borel sets.

**Theorem 3.5.** Let \(B\) be an irreducible generalized stationary Bratteli diagram, and let \(\nu\) be the IFS measure defined in Theorem 3.4. Then:
(i) \(\nu\) is not tail invariant;
(ii) \(\nu\) is shift-invariant.

**Proof.** (i) Suppose that \(\nu\) is tail invariant. Let \(f, e \in E\) be two edges with \(r(f) = r(e)\), i.e., \(f, e\) are tail-equivalent. Then it follows from (3.6) that \(\nu([f]) = \nu([e])\) or
\[ q_r(f) p_f = q_r(e) p_e. \]
This means that \(p_f = p_e\). In other words, the matrix
\[ P = (p_{w,v}) : e = (w, v) \in E \]
has constant columns.

Consider two cylinder sets \([f_0, f_1]\) and \([e_0, e_1]\) such that \(r(f_1) = r(e_1)\) and there exist edges \(g, h\) satisfying the properties: \(s(g) = s(h), r(g) = r_{f_0}, \) and \(r(h) = r_{e_0}\). From the tail invariance of \(\nu\) and the case considered above, we obtain that \(p_{f_1} = p_{e_1}\) and
\[ q_r(f_1) p_{f_0} p_{f_1} = q_r(e_1) p_{e_0} p_{e_1}. \]
Hence, \(p_{f_0} = p_{e_0}\) and therefore \(p_f = p_e\). This proves that the rows of the matrix \(P\) are constant. But the vector \((p_e : e \in E)\) is probability, contradiction.

(ii) Let \(\varphi(x)\) be a bounded positive Borel function on the path space \(X_B\). For the left shift \(\sigma\) on \(X_B\), compute
\[
\int_{X_B} \varphi(x) \, d\nu \circ \sigma^{-1}(x) = \int_{X_B} \varphi(\sigma x) \, d\nu(x)
\]
\[
= \sum_{e \in E_i} \int_{X_B} p_e \varphi(\sigma x) \, d\nu \circ \tau_e^{-1}(x)
\]
\[
= \sum_{e \in E_i} \int_{X_B} p_e \varphi(\sigma(\tau_e x)) \, d\nu(x)
\]
\[
= \sum_{e \in E_i} \int_{X_B} p_e \varphi(x) \, d\nu(x)
\]
\[
= \int_{X_B} \varphi(x) \, d\nu(x).
\]

This proves that \( \nu \circ \sigma^{-1} = \nu. \)

\(\Box\)

4 Measurable Bratteli diagrams and IFS measures

In this section, we discuss a measurable analogue of generalized Bratteli diagrams. The principal differences between discrete and measurable generalized Bratteli diagrams are: (a) the levels \( V_n \) of a measurable Bratteli diagram are formed by standard Borel spaces \( (X_n, \mathcal{A}_n) \), and (b) the sets of edges \( E_n \) are Borel subsets of \( X_n \times X_{n+1} \). Because these objects are non-discrete, we need to use new methods and techniques.

4.1 Measurable Bratteli diagrams and path space measures

We give the definitions of main objects in this subsection.

**Definition 4.1.** (Measurable Bratteli diagrams) Let \( \{(X_n, \mathcal{A}_n) : n \in \mathbb{N}_0\} \) be a sequence of standard Borel spaces. Let \( \{E_n : n \in \mathbb{N}_0\} \) be a sequence of Borel subsets such that \( E_n \subseteq X_n \times X_{n+1} \). Denote by \( \mathcal{E} = \bigsqcup_{i \geq 0} E_i \) and \( \mathcal{X} = \bigsqcup_{i \geq 0} X_i \) the sets of “edges” and “vertices”, respectively. For \( e = (x, y) \in E_i \), the maps \( s_i(x, y) = x \) and \( r_i(x, y) = y \) are onto projections of \( E_i \) to \( X_i \) and \( X_{i+1} \). Define \( s, r \) on \( \mathcal{E} \) by setting \( s = s_i, r = r_i \) on \( E_i \). Then we call \( \mathcal{B} = (\mathcal{X}, \mathcal{E}) \) a measurable Bratteli diagram. The pair \( (X_n, E_n) \) is called the \( n \)-th level of the measurable Bratteli diagram \( \mathcal{B} \).

If all \( E_n = E \), then the measurable Bratteli diagram \( \mathcal{B} \) is called stationary.

**Remark 4.2.** (1) We will identify the standard Borel spaces \( \{(X_n, \mathcal{A}_n) : n \in \mathbb{N}_0\} \) with an uncountable standard Borel space \( (X, \mathcal{A}) \). This means that a paint \( x \) can be seen in all levels \( (X_n, \mathcal{A}_n) \). This fact explains why we do not require that all levels \( X_n = X_0 \) in the definition of a stationary Bratteli diagram. Nevertheless, we will keep using subindices to indicate the level of a measurable Bratteli diagram.

(2) The set of edges \( E_i, i \geq 0 \), can be represented as follows:

\[
E_i = \bigcup_{x \in X_i} s^{-1}(x) = \bigcup_{x \in X_{i+1}} r^{-1}(x).
\]

This means that \( E_i \) can be seen as the union of “vertical” and “horizontal” sections. The fact that \( r \) is an onto map says that \( \forall y \in X_{i+1} \exists x \in X_i \) such that \( (x, y) \in E_i \). A similar property holds for the map \( s \).
(3) To define the path space $X_B$ of a measurable Bratteli diagram $B = (\mathcal{X}, \mathcal{E})$, we take a sequence $\bar{\imath} = \{e_i = (x_i, x_{i+1})\}, e_i \in E_i$, such that $r(e_i) = s(e_{i+1})$ for all $i$. Equivalently, $\bar{\imath} = (x_0, x_1, x_2, \ldots)$ where every pair $(x_i, x_{i+1})$ is in $E_i$, $i \in \mathbb{N}_0$. Then $s(\bar{\imath}) = x_0$.

(4) We denote
$$X_B(w) = \{\bar{\imath} \in X_B : s(\bar{\imath}) = w\}. \quad (4.1)$$
Clearly, the sets $X_B(w)$ form a partition $\eta$ of $X_B$, and $X_0$ is the quotient space with respect to this partition.

We will consider below measurable stationary Bratteli diagrams, i.e., $E_i = E$. For every $e = (w, v) \in E$, we set
$$D_e = \{\bar{\imath} \in X_B : s(\bar{\imath}) = r(e) = v\},$$
$$R_e = \{\bar{\imath} \in X_B : (x_0, x_1) = e\}.$$
For $\bar{\imath} = (x_0, x_1, \ldots) \in D_e$, define the map $\tau_e : D_e \to R_e$ by setting
$$\tau_e(\bar{\imath}) = (w, v, x_1, x_2, \ldots).$$
We note that $\bar{\imath} \in D_e$ means that $x_0 = v$ so that $\tau_e(\bar{\imath})$ is well defined and belongs to $R_e$. Moreover, the map $\tau_e$ is one-to-one on its domain and the map $\tau_e^{-1} : R_e \to D_e$ is defined by
$$\tau_e^{-1}(e, \bar{y}) = \bar{y}.$$

We can consider the metric dist on the path space $X_B$ similarly to the case of generalized Bratteli diagrams. As shown in Remark 2.13, the maps $\tau_e$ are contractive for all $e \in E$.

Let $B$ be a measurable Bratteli diagram with the path space $X_B$. By a measure $\mu$ on $X_B$, we mean a Borel positive (finite or sigma-finite) measure.

Consider a sequence of Borel sets $C_i \subset X_i : i \in \mathbb{N}_0$. Then $(C_0 \times C_1 \times \cdots \times C_N) \cap X_B$ is called a cylinder subset of $X_B$ of length $N$ and denoted by $[C_0, \ldots, C_N]$. In particular, we can consider the set $R_e$ as a cylinder set $[e]$ defined by the edge $e$, $e \in E$. It defines the partition
$$\xi = \{[e] : e \in E\} \quad (4.2)$$
of $X_B$. We remark that the quotient $X_B/\xi$ is isomorphic to the set $E$. Since $D_e = X_B(r(e))$ for every $e \in E$, the partition $\eta$ (see Remark 4.2 (4)) is an enlargement of $\xi$ because
$$D_e = \bigcup_{f : s(f) = r(e)} [f].$$

**Observation.** The collection of cylinder sets $[C_0, \ldots, C_N]$ of any finite length generates the Borel sigma-algebra of $X_B$. A measure $\mu$ on $X_B$ is completely determined by its values on cylinder sets. These facts are obvious.

### 4.2 IFS measures on measurable Bratteli diagrams

Let $B = (\mathcal{X}, \mathcal{E})$ be a stationary measurable Bratteli diagram. Recall that $E$ denotes a Borel subset of $X_0 \times X_1$ (in fact, $E$ can be viewed as a subset of $X_0 \times X_0$). Suppose that $p$ is a Borel measure on $E$, then $(E, p)$ becomes a standard measure space. We consider both cases, finite and sigma-finite measures $p$.

The key tool of this subsection is the disintegration theorem for a finite or sigma-finite measure with respect to a measurable partition of a measure space. The literature on this subject is very extensive. We refer to the original paper by Rokhlin [Roh49] and more recent paper [Sim12] (see also [BJ18]).

---

1. Observation.
2. Remark.
3. Example.
4. Definition.
5. Lemma.
6. Theorem.
7. Corollary.
8. Proposition.
9. Claim.
10. Conclusion.
11. Proof.
12. Construction.
13. Illustration.
14. Remark.
15. Example.
16. Definition.
17. Lemma.
18. Theorem.
19. Corollary.
20. Claim.
21. Conclusion.
22. Proof.
23. Construction.
24. Illustration.
25. Remark.
26. Example.
27. Definition.
28. Lemma.
29. Theorem.
30. Corollary.
31. Claim.
32. Conclusion.
Definition 4.3. Suppose that \((Z, \mathcal{C}, \mu)\) and \((Y, \mathcal{D}, \nu)\) are standard \(\sigma\)-finite measure spaces. Let \(\pi: Z \to Y\) be a measurable function. A system of conditional measures for \(\mu\) with respect to \(\pi\) is a collection of measures \(\{\mu_y : y \in Y\}\) such that
(i) \(\mu_y\) is a Borel measure on \(\pi^{-1}(y)\),
(ii) for every \(B \in \mathcal{C}\), the function \(y \mapsto \mu_y(B)\) is measurable and \(\mu(B) = \int_Y \mu_y(B) \, d\nu(y)\).

Theorem 4.4 ([Sim12]). Let \((Z, \mathcal{C}, \mu)\) and \((Y, \mathcal{D}, \nu)\) be as above. For any measurable function \(\pi: Z \to Y\) such that \(\mu \circ \pi^{-1} \ll \nu\), there exists a uniquely determined system of conditional measures \(\{\mu_y\}_{y \in Y}\) which disintegrates the measure \(\mu\), i.e.

\[
\mu = \int_Y \mu_y d\nu(y).
\]

Remark 4.5. For the edge set \(E\) of a stationary measurable Bratteli diagram \(\mathcal{B}\) and a Borel measure \(p\) on \(E\), consider the partition of \(E\) into the sets \(r(s^{-1}(x)), x \in X_0\). This partition is measurable in the sense of [Roh49]. We can apply the disintegration theorem cited above. More precisely, the sets \(r(s^{-1}(x)) = \{(x, y) \in E : y \in X_1\}\), where \(x \in X_0\), are vertical sections of the set \(E\). Setting \(\bar{p}(\cdot) = p(s^{-1}(\cdot))\), we have the projection of the measure \(p\) onto \(X_0\). Denoting by \(x \mapsto p(x, \cdot)\) the corresponding system of conditional measures, we have

\[
p = \int_{X_0} p(x, dy) \, d\bar{p}(x)
\]

where the measure \(p(x, dy)\) is supported by the set \(r(s^{-1}(x))\).

We will consider several versions of disintegration theorem related to measures on the path space \(X_\mathcal{B}\) of a measurable Bratteli diagram \(\mathcal{B}\). We assume that the measures will satisfy conditions (i) and (ii) formulated below.

(i) For a given measure \(m\) on \(X_\mathcal{B}\) and the set \(X_\mathcal{B}(y)\) defined in (4.1), the measurable function

\[
q_m(y) = m(X_\mathcal{B}(y)), \quad y \in X_0,
\]

takes finite values.

(ii) The partition \(\xi\) defined in (4.2) is obviously measurable, and the measure \(m\) on \(X_\mathcal{B}\) can be disintegrated with respect to \(\xi\). Denote by \(\hat{m}\) the projection of \(m\) onto \(E = X/\xi\). Let \(p\) be a measure on \(E\). Assuming that \(\hat{m} \ll p\) and applying Theorem 4.4, we have

\[
m = \int_E m_e \, dp(e), \quad (4.3)
\]

where \(e \mapsto m_e\) is the system of conditional measures of \(m\) with respect to \((E, p)\). We note that the conditional measure \(m_e\) is supported by the set \([e], e \in E\).

The class of IFS measures is determined by a special form of measures \(m_e\) defined in (4.3). We will assume that the measures considered on the path space \(X_\mathcal{B}\) satisfy conditions (i) and (ii).

Definition 4.6. (IFS measures and disintegration) Let \(\mathcal{B}\) be a stationary measurable Bratteli diagram, and let \(\{\tau_e, e \in E\}\) be the system of contractive maps defined in subsection 4.1. Suppose \(p\) is a fixed probability measure on the set \(E\). A Borel measure \(\mu\) is called an IFS measure with respect to \(p\) if

\[
\mu = \int_E \mu \circ \tau_e^{-1} \, dp(e). \quad (4.4)
\]
Our goal is to find conditions under which a measure $m$ defined on the path space of a stationary measurable Bratteli diagram is an IFS measure.

**Theorem 4.7.** Let $p$ be a Borel probability measure on $E$ where $E$ is the edge set of a stationary measurable Bratteli diagram $\mathcal{B} = (\mathcal{X}, \mathcal{E})$. Let $\mu$ be a measure on $\mathcal{X}_B$ and $q(x) = \mu(\mathcal{X}_B(x))$. Then $\mu$ is an IFS measure if and only if the following condition holds:

$$\int_{X_1} p(x, dy)q(y) = q(x).$$

(4.5)

**Proof.** In the proof of the theorem, we apply the idea used in Theorem 3.4.

We will construct an IFS measure $\mu$ by defining its values on cylinder sets of consequently increasing length. This construction will be based on the application of (4.4) so that the measure $\mu$ will be an IFS measure automatically. Relation (4.5) is used to satisfy the Kolmogorov extension theorem.

As mentioned in Remark 4.5, the projection of $p$ onto $X_0$ defines the measure $\tilde{\mu}$. This allows us to define $\mu$ on cylinder subsets of $\mathcal{X}_B$ generated by Borel sets on $X_0$:

$$\mu([C_0]) = \int_{C_0} q(x_0) \, d\tilde{\mu}(x_0),$$

where $C_0$ is a Borel subset of $X_0$.

For a cylinder set $[C_0, C_1] = \{x = (x_i) \in \mathcal{X}_B : x_0 \in C_0, x_1 \in C_1\}$ of length two, we define

$$\mu([C_0, C_1]) = \int_{C_0} \left( \int_{C_1 \cap r(s^{-1}(x_0))} p(x_0, dy)q(y) \right) \, d\tilde{\mu}(x_0).$$

(4.6)

Show that this definition of the measure $\mu$ satisfies (4.4):

$$\int_{E} \mu \circ \tau_e^{-1}([C_0, C_1]) \, dp(e) = \int_{E} \mu \circ \tau_e^{-1}([C_0, C_1] \cap [e]) \, dp(e)$$

$$= \int_{C_0} \int_{C_1 \cap r(s^{-1}(x_0))} \mu(\mathcal{X}_B(y)) p(x_0, dy) \, d\tilde{\mu}(x_0)$$

$$= \int_{C_0} \int_{C_1 \cap r(s^{-1}(x_0))} q(y) p(x_0, dy) \, d\tilde{\mu}(x_0)$$

$$= \mu([C_0, C_1])$$

In general, if we defined the measure $\mu$ on cylinder sets of the form $[C_0, ..., C_k], k = 1, ..., n - 1,$ then we set

$$\mu([C_0, ..., C_n]) = \int_{E} \mu \circ \tau_e^{-1}([C_0, ..., C_{n-1}]) \, dp(e).$$

This means that this definition of $\mu$ shows that relation (4.4) holds automatically.

It remains to show that the measure $\mu$ is well defined, that is it satisfies the Kolmogorov extension theorem. We check this property for the cylinder sets of length two. Indeed, in this case we take $C_1 = X_1$ and compute

$$\mu([C_0, X_1]) = \int_{C_0} \left( \int_{X_1 \cap r(s^{-1}(x_0))} q(y)p(x_0, dy) \right) \, d\tilde{\mu}(y)$$

$$= \int_{C_0} \left( \int_{r(s^{-1}(x_0))} q(y)p(x_0, dy) \right) \, d\tilde{\mu}(y)$$

$$= \int_{C_0} q(x_0) \, d\tilde{\mu}(x_0)$$

$$= \mu([C_0]).$$
The general case is proved analogously.

**Remark 4.8.** We note that the existence of an IFS measure on \(X_B\) can be proved by using a fixed point theorem following [Hut81]. We consider the metric dist on \(X_B\) defined in Remark 2.4. Let

\[
\text{Lip}_1 = \{ f : X_B \rightarrow \mathbb{R} : |f(x) - f(y)| \leq \text{dist}(x, y) \}.
\]

For a measure \(\nu\) on \(X_B\), set

\[
L(\nu) = \int_E \nu \circ \tau_e^{-1} \, dp(e).
\]

Since \(\tau_e\) is a contractive map for every \(e\), one can show that \(\rho(L(\nu), L(\mu)) \leq \rho(\nu, \mu)\) where the metric \(\rho\) is defined by

\[
\rho(\nu, \mu) = \sup\{ |\int f \, d\nu - \int f \, d\mu| : f \in \text{Lip}_1 \}.
\]

Then we conclude that there exists a measure \(\mu_0\) such that \(L(\mu_0) = \mu_0\).

**Remark 4.9.** The IFS measure \(\mu\) defined in Theorem 4.7 is shift invariant. The proof of this fact is similar to that giving in Theorem 3.5.

**Acknowledgements.** The authors are pleased to thank our colleagues and collaborators, especially, R. Curto, H. Karpel, P. Muhly, W. Polyzou, S. Sanadhya. We are thankful to the members of the seminars in Mathematical Physics and Operator Theory at the University of Iowa for many helpful conversations.

**Declaration.** The authors declare that they have no conflict of interest.

**References**

[ARCG+20] P. Alonso Ruiz, Y. Chen, H. Gu, R. S. Strichartz, and Z. Zhou. Analysis on hybrid fractals. *Commun. Pure Appl. Anal.*, 19(1):47–84, 2020.

[Bar06] Michael Fielding Barnsley. *Superfractals*. Cambridge University Press, Cambridge, 2006.

[BDK06] Sergey Bezuglyi, Anthony H. Dooley, and Jan Kwiatkowski. Topologies on the group of Borel automorphisms of a standard Borel space. *Topol. Methods Nonlinear Anal.*, 27(2):333–385, 2006.

[BJ99] Ola Bratteli and Palle E. T. Jorgensen. Iterated function systems and permutation representations of the Cuntz algebra. *Mem. Amer. Math. Soc.*, 139(663):x+89, 1999.

[BJ02] Ola Bratteli and Palle Jorgensen. *Wavelets through a looking glass*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2002. The world of the spectrum.

[BJ15] S. Bezuglyi and Palle E. T. Jorgensen. Representations of Cuntz-Krieger relations, dynamics on Bratteli diagrams, and path-space measures. In *Trends in harmonic analysis and its applications*, volume 650 of *Contemp. Math.*, pages 57–88. Amer. Math. Soc., Providence, RI, 2015.

[BJ18] Sergey Bezuglyi and Palle E. T. Jorgensen. *Transfer operators, endomorphisms, and measurable partitions*, volume 2217 of *Lecture Notes in Mathematics*. Springer, Cham, 2018.
Sergey Bezuglyi and Palle E. T. Jorgensen. Harmonic analysis on graphs via Bratteli diagrams and path-space measures. *Dissertationes Math.*, 574:74, 2022.

Sergey Bezuglyi, Palle E. T. Jorgensen, Olena Karpel, and Shrey Sanadhya. Dynamics on the path space of generalized Bratteli diagrams, *preprint*, 2022.

Sergey Bezuglyi, Palle E. T. Jorgensen, and Shrey Sanadhya. Invariant measures and generalized bratteli diagrams for substitutions on infinite alphabets, *arXiv:2203.14127v2*. *ArXiv*, 2022.

S. Bezuglyi and O. Karpel. Bratteli diagrams: structure, measures, dynamics. In *Dynamics and numbers*, volume 669 of *Contemp. Math.*, pages 1–36. Amer. Math. Soc., Providence, RI, 2016.

Sergey Bezuglyi and Olena Karpel. Invariant measures for Cantor dynamical systems. In *Dynamics: topology and numbers*, volume 744 of *Contemp. Math.*, pages 259–295. Amer. Math. Soc., Providence, RI, [2020] ©2020.

S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak. Invariant measures on stationary Bratteli diagrams. *Ergodic Theory Dynam. Systems*, 30(4):973–1007, 2010.

O. Bratteli. Inductive limits of finite dimensional $C^*$-algebras. *Trans. Amer. Math. Soc.*, 171:195–234, 1972.

Gerard Brunick and Steven Shreve. Mimicking an Itô process by a solution of a stochastic differential equation. *Ann. Appl. Probab.*, 23(4):1584–1628, 2013.

Shiping Cao, Malte S. Haßler, Hua Qiu, Ely Sandine, and Robert S. Strichartz. Existence and uniqueness of diffusions on the Julia sets of Misiurewicz-Sierpinski maps. *Adv. Math.*, 389:Paper No. 107922, 41, 2021.

A. H. Dooley and Toshihiro Hamachi. Nonsingular dynamical systems, Bratteli diagrams and Markov odometers. *Israel J. Math.*, 138:93–123, 2003.

F. Durand, B. Host, and C. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergodic Theory Dynam. Systems*, 19(4):953–993, 1999.

Dorin Ervin Dutkay and Palle E. T. Jorgensen. Harmonic analysis and dynamics for affine iterated function systems. *Houston J. Math.*, 33(3):877–905, 2007.

Dorin Ervin Dutkay and Palle E. T. Jorgensen. Probability and Fourier duality for affine iterated function systems. *Acta Appl. Math.*, 107(1-3):293–311, 2009.

Dorin Ervin Dutkay and Palle E. T. Jorgensen. Spectral theory for discrete Laplacians. *Complex Anal. Oper. Theory*, 4(1):1–38, 2010.

Dorin Ervin Dutkay and Palle E. T. Jorgensen. Monic representations of the Cuntz algebra and Markov measures. *J. Funct. Anal.*, 267(4):1011–1034, 2014.

Dorin Ervin Dutkay and Palle E. T. Jorgensen. The role of transfer operators and shifts in the study of fractals: encoding-models, analysis and geometry, commutative and non-commutative. In *Geometry and analysis of fractals*, volume 88 of *Springer Proc. Math. Stat.*, pages 65–95. Springer, Heidelberg, 2014.
[DJ15] Dorin Ervin Dutkay and Palle E. T. Jorgensen. Representations of Cuntz algebras associated to quasi-stationary Markov measures. *Ergodic Theory Dynam. Systems*, 35(7):2080–2093, 2015.

[DJK94] R. Dougherty, S. Jackson, and A. S. Kechris. The structure of hyperfinite Borel equivalence relations. *Trans. Amer. Math. Soc.*, 341(1):193–225, 1994.

[Dur10] Fabien Durand. Combinatorics on Bratteli diagrams and dynamical systems. In *Combinatorics, automata and number theory*, volume 135 of *Encyclopedia Math. Appl.*, pages 324–372. Cambridge Univ. Press, Cambridge, 2010.

[FGJ+18a] Carla Farsi, Elizabeth Gillaspy, Palle Jorgensen, Sooran Kang, and Judith Packer. Purely atomic representations of higher-rank graph C*-algebras. *Integral Equations Operator Theory*, 90(6):Art. 67, 26, 2018.

[FGJ+18b] Carla Farsi, Elizabeth Gillaspy, Palle Jorgensen, Sooran Kang, and Judith Packer. Representations of higher-rank graph C*-algebras associated to A-semibranching function systems. *J. Math. Anal. Appl.*, 468(2):766–798, 2018.

[GPS95] Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Topological orbit equivalence and C*-crossed products. *J. Reine Angew. Math.*, 469:51–111, 1995.

[HPS92] Richard H. Herman, Ian F. Putnam, and Christian F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.*, 3(6):827–864, 1992.

[HR00] John E. Hutchinson and Ludger Rüschendorf. Selfsimilar fractals and selfsimilar random fractals. In *Fractal geometry and stochastics, II (Greifswald/Koserow, 1998)*, volume 46 of *Progr. Probab.*, pages 109–123. Birkhäuser, Basel, 2000.

[Hut81] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[Hut96] John E. Hutchinson. Elliptic systems. In *Instructional Workshop on Analysis and Geometry, Part I (Canberra, 1995)*, volume 34 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 111–120. Austral. Nat. Univ., Canberra, 1996.

[JKS11] Palle E. T. Jorgensen, Keri A. Kornelson, and Karen L. Shuman. Iterated function systems, moments, and transformations of infinite matrices. *Mem. Amer. Math. Soc.*, 213(1003):x+105, 2011.

[Jor06] Palle E. T. Jorgensen. *Analysis and probability: wavelets, signals, fractals*, volume 234 of *Graduate Texts in Mathematics*. Springer, New York, 2006.

[Jor18] Palle E. T. Jorgensen. *Harmonic analysis*, volume 128 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 2018. Smooth and non-smooth, Published for the Conference Board of the Mathematical Sciences.

[Kit98] Bruce P. Kitchens. *Symbolic dynamics*. Universitext. Springer-Verlag, Berlin, 1998. One-sided, two-sided and countable state Markov shifts.

[KS98] Ioannis Karatzas and Steven E. Shreve. *Methods of mathematical finance*, volume 39 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1998.
[Mau95] R. Daniel Mauldin. Infinite iterated function systems: theory and applications. In *Fractal geometry and stochastics (Finsterbergen, 1994)*, volume 37 of *Progr. Probab.*, pages 91–110. Birkhäuser, Basel, 1995.

[MP11] Matilde Marcolli and Anna Maria Paolucci. Cuntz-Krieger algebras and wavelets on fractals. *Complex Anal. Oper. Theory*, 5(1):41–81, 2011.

[MS21] Ian D. Morris and Cagri Sert. A strongly irreducible affine iterated function system with two invariant measures of maximal dimension. *Ergodic Theory Dynam. Systems*, 41(11):3417–3438, 2021.

[Num84] Esa Nummelin. *General irreducible Markov chains and nonnegative operators*, volume 83 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1984.

[Put18] Ian F. Putnam. *Cantor minimal systems*, volume 70 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2018.

[Ren18] Jean Renault. Random walks on Bratteli diagrams. In *Operator theory: themes and variations*, volume 20 of *Theta Ser. Adv. Math.*, pages 187–204. Theta, Bucharest, 2018.

[Roh49] V. A. Rohlin. On the fundamental ideas of measure theory. *Mat. Sbornik N.S.*, 25(67):107–150, 1949.

[RS16] Robert J. Ravier and Robert S. Strichartz. Sampling theory with average values on the Sierpinski gasket. *Constr. Approx.*, 44(2):159–194, 2016.

[SG77] G. E. Shilov and B. L. Gurevich. *Integral, measure and derivative: a unified approach*. Dover Books on Advanced Mathematics. Dover Publications, Inc., New York, english edition, 1977. Translated from the Russian and edited by Richard A. Silverman.

[Sim12] David Simmons. Conditional measures and conditional expectation; Rohlin’s disintegration theorem. *Discrete Contin. Dyn. Syst.*, 32(7):2565–2582, 2012.