A black hole with torsion in 5D Lovelock gravity

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Abstract
We analyze static spherically symmetric solutions of five dimensional (5D) Lovelock gravity in the first order formulation. In the Riemannian sector, when torsion vanishes, the Boulware–Deser black hole represents a unique static spherically symmetric black hole solution for the generic choice of the Lagrangian parameters. We show that a special choice of the Lagrangian parameters, different from the Lovelock Chern–Simons gravity, leads to the existence of a static black hole solution with torsion, the metric of which is asymptotically anti-de Sitter (AdS). We calculate the conserved charges and thermodynamical quantities of this black hole solution.

Keywords: Lovelock gravity, torsion, black holes

1. Introduction

Lovelock gravity [1] represents an intriguing generalization of general relativity, since it is a unique, ghost-free higher derivative extension of Einstein’s theory that possesses second order equations of motion. As a higher curvature theory, Lovelock gravity has a considerable number of black hole solutions—see [2–10] and references therein. Many of these possess exotic properties, such as zero mass, peculiar topology of the event horizon etc.

This leads us to an old problem of black hole uniqueness—namely, solutions of general relativity are highly constrained, but the situation changes drastically in the case of higher dimensions. There are new black hole solutions with non-spherical event horizon topology, namely black string, black ring and black brane [11]. Often, these exotic black objects suffer from various instabilities—for example, black strings and branes have Gregory–Laflamme instability [12], and will decay into black holes with spherical horizons. Thus, gravity in higher dimensions represents an interesting area of research, full of surprising discoveries, whose importance stems from its numerous applications.

Lovelock gravity can be also studied within the framework of Poincaré gauge theory (PGT), formulated by Sciamma [13] and Kibble [14] more than half a century ago. PGT is the first
modern, gauge-field-theoretic approach to gravity obtained by gauging the Poincaré group of space-time symmetries, the semidirect product of translations and Lorentz transformations. It represents a natural extension of the gauge principle, originally formulated by Weyl within electrodynamics and further developed in the works of Yang, Mills and Utiyama, to the space-time symmetries. The gauge procedure adopted leads directly to a new, Riemann–Cartan geometry of space-time, since torsion and curvature are recovered as the Poincaré gauge field strengths. The Lagrangian in PGT contains a gravitational part, which is a function of the field strengths, the curvature and the torsion, and a suitable matter field Lagrangian.

In the context of Lovelock gravity, this more general setting contains torsionless theory as a limit, and represents a starting point for canonical analysis, coupling with matter fields, supersymmetric extensions of the theory and holographic applications. Interestingly, unlike in the case of Einstein–Cartan theory (first order formulation of general relativity) where all solutions of the equations of motion in vacuum are torsion free, the structure of the vacuum solutions of the Lovelock gravity is more complicated, because there exist solutions with non-vanishing torsion. However, it turns out that exact solutions with torsion are extremely difficult to find, since consistency conditions usually lead to an over-constrained system of equations. Solutions with non-trivial totally antisymmetric torsion have been studied in [8], [15–19]. In this paper, we continue our analysis of the exact solutions of 5D Lovelock gravity solutions with torsion, started in [8], and find a new static, spherically symmetric black hole solution with torsion with zero mass and entropy. The torsion of the solution possesses both tensorial and antisymmetric part. It, unlike the Riemannian Boulware–Deser black hole [20], exists for a specific choice of action parameters. This fine tuning of action parameters was first noticed by Canfora et al in their paper [15], and represents a different sector from the highly degenerate Lovelock Chern–Simons gravity.

The paper is organized in the following way. In the second section, we review basics of Poincaré gauge theory and Lovelock gravity in the first order formulation. In section 3 we find the black hole solution of 5D Lovelock gravity with torsion, and analyze its properties. In particular, we find that the quadratic torsional invariant is singular at \( r \to 0 \). In section 4, we explore the thermodynamics of the previously obtained solution. The appendices contain additional technical details.

We use the following conventions: the Lorentz signature is mostly negative; local Lorentz indices are denoted by the middle letters of the Latin alphabet, while space-time indices are denoted by the letters of the Greek alphabet. Throughout the paper, we mostly use differential forms instead of coordinate notation, and the wedge product is omitted for simplicity.

## 2. Lovelock gravity

Since the work of Sciamma and Kibble, it has been known that gravity in the first order formulation has the structure of Poincaré gauge theory (PGT)—see [21, 22] for a comprehensive account. For the reader’s convenience, we briefly review basics of the PGT.

### 2.1. PGT in brief

The basic dynamical variables in PGT, playing the role of gauge potentials, are the vielbein \( e^I \) 1-form and the spin connection \( \omega^I = -\omega^I_\mu dx^\mu \) 1-form. In local coordinates \( x^\mu \), we can expand the vielbein and the connection 1-forms as \( e^I = e^I_\mu dx^\mu \), \( \omega^I = \omega^I_\mu dx^\mu \). Gauge symmetries of the theory are local translations (diffeomorphisms) and local Lorentz rotations, parametrized by \( \xi^\mu \) and \( \epsilon^I \) respectively.
From the gauge potentials, we can construct field strengths, namely torsion $T^i$ and curvature $R^{ij}$ (2-forms), which are given as

$$T^i = \nabla e^i \equiv \frac{1}{2} R^i_{\mu
u} dx^\mu \wedge dx^\nu,$$

$$R^{ij} = d\omega^{ij} + \omega^i \wedge \omega^j = \frac{1}{2} R^{ij}_{\mu
u} dx^\mu \wedge dx^\nu,$$

where $\nabla = dx^\mu \nabla_\mu$ is the exterior covariant derivative.

A metric tensor can be constructed from the vielbein and flat metrics:

$$g_{ij} = \eta_{ij} e^i \otimes e^j = g_{\mu\nu} dx^\mu \otimes dx^\nu,$$

$$g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu,$$

$$\eta_{ij} = (+, -, -, -).$$

The antisymmetry of $\omega^{ij}$ in PGT is equivalent to the so-called metricity condition, $\nabla g = 0$. A geometry whose connection is restricted by the metricity condition (metric-compatible connection) is called a Riemann–Cartan geometry.

The connection $\omega^{ij}$ determines the parallel transport in the local Lorentz basis. Because parallel transport is a geometric operation, it is independent of the basis. This property is encoded into PGT via the so-called vielbein postulate, which implies

$$\omega^{ijk} = \Delta^{ijk} + K^{ijk},$$

where $\Delta$ is Levi-Civita connection, and $K^{ijk} = -\frac{1}{2}(T^{ijk} - T^{kij} + T^{jki})$ is the contortion.

### 2.2. Action and equations of motion

The Lovelock gravity Lagrangian in the first order formulation can be constructed as the linear combination of the dimensionally continued Euler densities $L_p$, which in $D$ dimensions are defined as

$$L_p = \varepsilon_{i_1 i_2 \ldots i_p} R^{i_1 i_2} \ldots R^{i_{p-1} i_p} e^{i_{p+1}} \ldots e^{i_D}.$$

In 5D, there are three Euler densities and the general form of the action of Lovelock gravity [1] is

$$I = \varepsilon_{ijklm} \left( \frac{\alpha_0}{2} e^i e^j e^k e^l e^m + \frac{\alpha_1}{3} R^{ij} e^i e^j e^m + \alpha_2 R^{ij} R^{kl} e^m e^n \right).$$

Variation of the action with respect to vielbein $e^i$ and spin connection $\omega^{ij}$ yields the gravitational field equations

$$\varepsilon_{ijklm} \left( \alpha_0 e^i e^j e^k e^l e^m + \alpha_1 R^{ij} e^i e^j e^m + \alpha_2 R^{ij} R^{kl} e^m e^n \right) = 0,$$

and

$$\varepsilon_{ijklm} \left( \alpha_1 e^i e^j e^l + 2\alpha_2 R^{ij} \right) T^m = 0.$$

### 3. Spherically symmetric solution

#### 3.1. Ansatz

We are looking for a static solution with $SO(4)$ symmetry, which orbits are three-spheres. The most general metric which fulfills these requirements in Schwarzschild-like coordinates $x^i = (t, r, \psi, \theta, \varphi)$ is given by

$$ds^2 = -dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$
\[ ds^2 = N^2 dr^2 - B^{-2} dr^2 - r^2 (d\psi^2 + \sin^2 \psi d\theta^2 + \sin^2 \psi \sin^2 \theta d\varphi^2), \]  

(3.1)

where functions \( N \) and \( B \) depend solely on \( r \), and \( r \in [0, \infty) \), \( \psi \in [0, \pi) \), \( \theta \in [0, \pi) \) and \( \varphi \in [0, 2\pi) \). The metric (3.1) possesses seven Killing vectors (see appendix A).

The vielbeins \( e^i \) are chosen in a simple diagonal form

\[ e^0 = N dr, \quad e^1 = B^{-1} dr, \quad e^2 = r d\psi, \quad e^3 = r \sin \psi d\theta, \quad e^4 = r \sin \psi \sin \theta d\varphi. \]  

(3.2)

The most general form of the spin connection compatible with Killing vectors (see appendix A) is given by

\[ \omega^{01} = A_0 dr + A_1 dr, \quad \omega^{02} = A_2 d\psi, \]
\[ \omega^{03} = A_2 \sin \psi d\theta, \quad \omega^{04} = A_2 \sin \psi \sin \theta d\varphi, \]
\[ \omega^{12} = A_3 d\psi, \quad \omega^{13} = A_3 \sin \psi \sin \theta d\varphi, \]
\[ \omega^{14} = A_3 \sin \psi \sin \theta d\varphi, \quad \omega^{23} = \cos \psi d\theta + A_4 \sin \psi \sin \theta d\varphi, \]
\[ \omega^{24} = -A_4 \sin \psi \sin \theta d\varphi + \cos \psi \sin \theta d\varphi, \]
\[ \omega^{34} = A_4 d\psi + \cos \theta d\varphi. \]  

(3.3)

where \( A_i \) are arbitrary functions of radial coordinate.

### 3.2. Solution

The sector with vanishing torsion equations of motion for spherically symmetric ansatz has a well-known solution, the Boulware–Deser black hole [20], which exists for the generic choice of action parameters. Another solution, which we construct in this paper, possesses non-vanishing torsion and is given by the following ansatz:

\[ A_0 \neq 0, \quad A_1 = A_2 = A_3 = 0, \quad A_4 \neq 0 \]
\[ N = B. \]  

(3.4)

By using the adopted ansatz we get that the equations (2.2) reduce to

\[ i = 0, 1: \quad 2\alpha_0 r^2 - \alpha_1 + \alpha_1 A_4^2 = 0, \]  

(3.5a)

\[ i = 2, 3, 4: \quad (2\alpha_2 - 2\alpha_2 A_4^2 - \alpha_1 r^2) A_4^2 + 6\alpha_0 r^2 + \alpha_1 (A_4^2 - 1) = 0. \]  

(3.5b)

The non-vanishing field equations (2.3) take the form

\[ ij = 01: \quad \alpha_1 r^2 + 2\alpha_2 A_4^2 - 2\alpha_2 + 4\alpha_2 r A_4 A_4' = 0, \]  

(3.6a)

\[ ij = 12, 13: \quad (\alpha_1 r^2 + 2\alpha_2 A_4^2 - 2\alpha_2) (NN' + A_0) + 2\alpha_1 r N^2 = 0, \]  

(3.6b)

\[ ij = 23, 24, 34: \quad -2\alpha_2 A_4' + \alpha_1 = 0. \]  

(3.6c)

From (3.5a) and (3.6c) we get

\[ A_4 = \sqrt{1 - \frac{2\alpha_0}{\alpha_1} r^2}, \quad A_0 = \frac{\alpha_1}{2\alpha_2} r, \]  

(3.7)

where the integration constant in \( A_0 \) is taken to be zero for simplicity. Equation (3.5b) in conjunction with (3.6c) yields to the following constraint between coupling constants:
\[ \alpha_1^2 - 12 \alpha_0 \alpha_2 = 0. \tag{3.8} \]

We consequently get that (3.6a) is identically satisfied, while the (3.6b) takes the form

\[ NN'' + \frac{3N^2}{r} - \frac{\alpha_1}{2 \alpha_2} r = 0, \]

and can be easily solved for \( N \):

\[ N = \sqrt{-\frac{\alpha_1}{8 \alpha_2} \left( r^2 - \frac{r_0^8}{r^6} \right)}. \tag{3.9} \]

From (3.8), we conclude that the solution exists in the sector different from the Lovelock Chern–Simons gravity. This is exactly the same fine tuning of parameters found by Canfora et al in their paper [15], where the solutions that have the structure of a direct product of a 2D Lorentzian with a 3D Euclidean constant curvature manifold are constructed.

The explicit form of torsion and curvature is given in appendix C. Let us note that both tensorial and antisymmetric part of torsion are non-vanishing unlike in the case of the solution found by Canfora et al [16], for which only totally antisymmetric part of torsion is non-vanishing.

Let us now introduce the (anti)-de Sitter ((A)dS) radius \( \ell \)

\[ \frac{\alpha_1}{8 \alpha_2} = - \frac{\sigma}{\ell^2}, \quad \sigma = \pm 1. \tag{3.10} \]

By substituting previous relation into (3.7) and (3.9), we get

\[ A_4 = \sqrt{1 + \frac{4\sigma r^2}{3 \ell^2}}, \quad N = \sqrt{\sigma \left( \frac{r^2}{\ell^2} - \frac{r_0^8}{\ell^2 r^6} \right)}. \tag{3.11} \]

Note that for the solution to describe a black hole, the following condition must hold:

\[ \frac{\alpha_1}{\alpha_2} < 0 \iff \sigma = +1 \tag{3.12} \]

with an event horizon located at \( r = r_+ \).

From the constraint (3.8), it follows that the sign of the ratio \( \frac{\alpha_0}{\alpha_1} \) is the same as the sign of \( \frac{\alpha_1}{\alpha_2} \)

\[ \text{sgn} \left( \frac{\alpha_0}{\alpha_1} \right) = \text{sgn} \left( \frac{\alpha_1}{\alpha_2} \right). \tag{3.13} \]

If the ratio is positive, the expression for \( A_4 \) implies that we have the maximum value of the radial coordinate, the so called cosmological horizon

\[ r_0 = \frac{\ell \sqrt{3}}{2}. \tag{3.14} \]

Meanwhile, if the ratio is negative, we have no restriction on the value of the radial coordinate, except that it is positive, and in maximally extended space-time goes to infinity. In this case, the black hole space-time metric is asymptotically AdS.

3.2.1. Invariants. From expressions for curvature and torsion, given in appendix C, we see that quadratic torsional invariant reads
\[ T^i \wedge ^* T_i = -\frac{12\sigma}{\ell^2} \left(1 - \frac{r_+}{r}\right) \hat{\epsilon}, \]  
(3.15)

which is obviously divergent in \( r = 0 \) for \( r_+ \) different from zero. Hence, there is a singularity of torsion at \( r \to 0 \). Scalar Cartan curvature is constant,
\[ R = \frac{16\sigma}{\ell^2}, \]  
(3.16)

while Riemannian scalar curvature is
\[ \tilde{R} = \frac{4\sigma}{\ell^2} \left(5 - \frac{3\sigma\ell^2}{2r^2} - \frac{3r_+^2}{r^2}\right), \]  
(3.17)

and is divergent for \( r \to 0 \). The quadratic Cartan and Riemannian curvature invariants both vanish:
\[ R_{ij} \wedge ^* R_{ij} = 0, \]  
\[ \tilde{R}_{ij} \wedge ^* \tilde{R}_{ij} = 0. \]  
(3.18)

We can conclude that the black hole obtained in this article is not of the regular type, and that it possesses singularity at \( r = 0 \). It is worth noting that solution [16] also possesses singularity of torsion and Riemannian curvature at \( r = 0 \).

Solving equations of motion (2.2) and (2.3) with seven arbitrary functions is an extremely tedious task, which is facilitated by Mathematica and xAct packages.

### 3.3. Conserved charges

Conserved charges can be calculated in a number of ways, we decided to make use of Nester’s formula [23], the application of which is quite simple in this particular case. In this section, we shall restrict the analysis to the asymptotically AdS case, which corresponds to the black hole.

The covariant momenta stemming from the Lovelock action (2.1) are given by
\[ \tau_i := \frac{\partial L}{\partial T^i} = 0, \]  
(3.19)

\[ \rho_{ij} = \frac{\partial L}{\partial R_{ij}} = 2\varepsilon_{ijklm} \left(\frac{\alpha_1}{3} e^{(l} e^{j)} + 2\alpha_2 R^{(l} e_{j)k}\right) e^m. \]  
(3.20)

Let us denote the difference between any variable \( X \) and its reference value \( \bar{X} \) by \( \Delta X = X - \bar{X} \). Reference space-time, in respect to which we measure conserved charges, is given for the zero radius of the event horizon \( r_+ = 0 \). Conserved charges \( Q_\xi \) associated to the Killing vector \( \xi \) are given by quasi-local surface integrals
\[ Q_\xi = \int_{\partial C} B, \]

where the boundary \( \partial C \) is located at infinity. With a suitable asymptotic behavior of the fields, the proper boundary term reads [23]
\[ B = (\xi \mid e^i) \Delta \tau_i + \Delta e^i(\xi \mid \bar{\tau}_i) + \frac{1}{2}(\xi \mid \omega_j)\Delta \rho_{ij} + \frac{1}{2}\Delta \omega_{ij}(\xi \mid \bar{\rho}_{ij}), \]  
(3.21)

where \( \mid \) denotes contraction.

For solution (3.9), by making use of the the results of appendix C, we get the covariant momenta
\[ \rho_{01} = \frac{4 \left( \alpha_1^2 - 12 \alpha_0 \alpha_2 \right)}{\alpha_1} e^2 e^3 e^4 = 0, \quad \rho_{02} = -\frac{8 \alpha_1}{3} e^1 e^3 e^4, \quad \rho_{03} = \frac{8 \alpha_1}{3} e^1 e^3 e^4, \]
\[ \rho_{23} = \frac{8 \alpha_1}{3} e^1 e^2 e^3, \quad \rho_{12} = \frac{8 \alpha_1}{3} e^0 e^3 e^4 - \frac{4 \alpha_1 N}{3} e^0 e^1 e^2, \]
\[ \rho_{13} = -\frac{8 \alpha_1}{3} e^0 e^2 e^3 - \frac{4 \alpha_1 N}{3} e^0 e^1 e^3, \quad \rho_{14} = \frac{8 \alpha_1}{3} e^0 e^2 e^3 - \frac{4 \alpha_1 N}{3} e^0 e^1 e^4, \]
\[ \rho_{24} = 0, \quad \rho_{23} = 0, \quad \rho_{34} = 0. \tag{3.22} \]

From (3.9), we conclude that the connection takes the same form on the background and for \( r^+ \neq 0, \omega^i = \omega^j \). Therefore, formula (3.21) takes the following simpler form:

\[ B = \frac{1}{2} \left( \xi_i \omega^i \right) \Delta \rho^j. \]

For the seven Killing vectors \( \xi_i \) (see appendix A) the conserved charges are given by

\[ Q_{(0)} = \int_{\partial \Sigma} \omega^0 \Delta \rho_{01} = 0, \]
\[ Q_{(1)} = \int_{\partial \Sigma} \cot \psi \sin \theta \left( \omega^{23} \omega_2 \Delta \rho_{23} + \omega^{24} \omega_2 \Delta \rho_{24} \right) = 0, \]
\[ Q_{(2)} = \int_{\partial \Sigma} \cot \psi \cos \theta \cos \varphi \left( \omega^{23} \omega_2 \Delta \rho_{23} + \omega^{24} \omega_2 \Delta \rho_{24} \right) \]
\[ - \cot \frac{\psi}{\sin \theta} \sin \varphi \left( \omega^{14} \varphi \Delta \rho_{14} + \omega^{23} \varphi \Delta \rho_{23} + \omega^{24} \varphi \Delta \rho_{24} + \omega^{34} \varphi \Delta \rho_{34} \right) = 0, \]
\[ Q_{(3)} = \int_{\partial \Sigma} \cot \psi \cos \theta \sin \varphi \left( \omega^{23} \omega_2 \Delta \rho_{23} + \omega^{24} \omega_2 \Delta \rho_{24} \right) \]
\[ + \cot \frac{\psi}{\sin \theta} \cos \varphi \left( \omega^{14} \varphi \Delta \rho_{14} + \omega^{23} \varphi \Delta \rho_{23} + \omega^{24} \varphi \Delta \rho_{24} + \omega^{34} \varphi \Delta \rho_{34} \right) = 0, \]
\[ Q_{(4)} = \int_{\partial \Sigma} \cos \varphi \left( \omega^{23} \omega_2 \Delta \rho_{23} + \omega^{24} \omega_2 \Delta \rho_{24} \right) \]
\[ - \cot \theta \sin \varphi \left( \omega^{14} \varphi \Delta \rho_{14} + \omega^{23} \varphi \Delta \rho_{23} + \omega^{24} \varphi \Delta \rho_{24} + \omega^{34} \varphi \Delta \rho_{34} \right) = 0, \]
\[ Q_{(5)} = \int_{\partial \Sigma} \sin \varphi \left( \omega^{23} \omega_2 \Delta \rho_{23} + \omega^{24} \omega_2 \Delta \rho_{24} \right) \]
\[ + \cot \theta \cos \varphi \left( \omega^{14} \varphi \Delta \rho_{14} + \omega^{23} \varphi \Delta \rho_{23} + \omega^{24} \varphi \Delta \rho_{24} + \omega^{34} \varphi \Delta \rho_{34} \right) = 0, \]
\[ Q_{(6)} = \int_{\partial \Sigma} \omega^{14} \varphi \Delta \rho_{14} + \omega^{23} \varphi \Delta \rho_{23} + \omega^{24} \varphi \Delta \rho_{24} + \omega^{34} \varphi \Delta \rho_{34} = 0. \tag{3.23} \]

Therefore, we conclude that conserved charges for the black hole with torsion (3.9) vanish. In particular, conserved charge \( Q_{(0)} \), which corresponds to the energy \( E \) of the solution, vanishes due to the specific choice of the parameters \( \alpha_1^2 = 12 \alpha_0 \alpha_2 \).

**4. Thermodynamics**

By demanding that Euclidean continuation of the black hole has no conical singularity, we obtain the standard formula for the black hole temperature
\[ T = \frac{(N^2)'}{4\pi}. \]  
(4.1)

In the particular case of the solution (3.9) we get
\[ T = \frac{2r_+}{\pi T^2}. \]  
(4.2)

The temperature is positive because solution (3.9) describes black hole iff condition (3.12) is satisfied. Let us note that this type of relation between temperature and the radius of the event horizon is unusual for black holes with spherical horizons. The relation (4.2) is standard in the case of planar black holes (black branes) or black holes in three space-time dimensions.

4.1. Euclidean action

Using the equation of motion (2.2), on-shell Euclidean action takes the form
\[ I_E = \varepsilon_{ijklm} \int \left( \frac{2\alpha_1}{3} R^{ijklm} e^i e^j e^k e^l + \frac{4\alpha_0}{5} e^i e^j e^k e^l \right). \]  
(4.3)

After substituting the solution (3.9), we get
\[ I_E = \int_0^\beta d\beta \int_{r_+}^\infty dr \int d\psi d\theta d\varphi \frac{4(\alpha_1^2 - 12\alpha_0\alpha_2)}{\alpha_2} r^3 \sin^2 \psi \sin \theta, \]  
(4.4)

where the integration over time is performed in the interval \([0, \beta := 1/T]\). By using the constraint on the parameters (3.8), we conclude that
\[ I_E = 0. \]  
(4.5)

From the well-known formula for the entropy
\[ S = (\beta \partial_\beta - 1)I_E, \]  
(4.6)

we obtain
\[ S = 0. \]  
(4.7)

This value of entropy is surprising, but it is not uncommon for Lovelock black holes—see for instance [24], where black holes with zero mass and entropy are obtained. From Euclidean action we can, also, calculate the energy
\[ E = \partial_\beta I_E, \]  
(4.8)

and obtain
\[ E = 0, \]  
(4.9)

in accordance with the results of the previous section.

5. Concluding remarks

We have analyzed static spherically symmetric solutions of Lovelock gravity in five dimensions. For the generic values of the Lagrangian parameters, the theory possesses a well-known solution, the Boulware–Deser black hole, while in the sector \( \alpha_1^2 = 12\alpha_0\alpha_2 \) we have discovered a new black hole solution with torsion.
We analyzed basic properties of the obtained solution, which torsion possesses non-vanishing tensorial and totally antisymmetric part. The solution has a singularity of torsion and Riemannian curvature for \( r \to 0 \), while the conserved charges, as well as the entropy, vanish.

It is worth stressing that the black hole metric is asymptotically AdS, which is a crucial condition for holographic investigation. The solution that describes the space-time which is asymptotically dS, with the cosmological horizon located at \( r_0 = \frac{\alpha_1}{\alpha_0} \), is not a black hole.

An interesting property of the solution in the asymptotically AdS case is that, in the semi-classical approximation, its entropy is zero. This means that its number of micro-states is ‘small’ i.e. it is of order one instead of the expected \( \mathcal{O}\left(\frac{1}{\Lambda_{\text{Planck}}}\right) \). It would be interesting to see what kind of consequences this result has on dual interpretation via gauge/gravity duality.

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Appendix A. Killing vectors for metric (3.1)

In addition to the \( \partial_t \) Killing vector static and spherically symmetric metric (3.1) possesses six Killing vectors, due to the \( SO(4) \) spherical symmetry. The complete set of Killing vectors \( \xi^\mu_{(i)} \) of the metric (3.1) is given by:

\[
\begin{align*}
\xi_{(0)} &= \partial_t, \\
\xi_{(1)} &= \cos \theta \partial_\psi - \cot \psi \sin \theta \partial_\theta, \\
\xi_{(2)} &= \sin \theta \cos \varphi \partial_\psi + \cot \psi \cos \theta \cos \varphi \partial_\theta - \frac{\cot \psi}{\sin \theta} \sin \varphi \partial_\varphi, \\
\xi_{(3)} &= \sin \theta \sin \varphi \partial_\psi + \cot \psi \cos \theta \sin \varphi \partial_\theta + \frac{\cot \psi}{\sin \theta} \cos \varphi \partial_\varphi, \\
\xi_{(4)} &= \cos \varphi \partial_\theta - \cot \theta \sin \varphi \partial_\varphi, \\
\xi_{(5)} &= \sin \varphi \partial_\theta + \cot \theta \cos \varphi \partial_\varphi, \\
\xi_{(6)} &= \partial_\varphi. 
\end{align*}
\]

(A.1)

The independent Killing vectors are \( \xi_{(0)}, \xi_{(1)}, \xi_{(4)} \) and \( \xi_{(6)} \), while the others are obtained as their commutators. The invariance conditions of the vielbein under Killing vectors and local Lorentz transformations with parameters \( \epsilon^i_j \) are

\[
\delta_0 e^\mu_\mu = L_\xi e^\mu_\mu + \epsilon^i_j e^\mu_\mu = 0,
\]

(A.2)

where the Lie derivative with respect to \( \xi \) is denoted as \( L_\xi \), giving that the only non-zero parameters of the local Lorentz symmetry are

\[
\epsilon^{23} = -\frac{\sin \theta}{\sin \psi}, \quad \epsilon^{34} = -\frac{\sin \varphi}{\sin \theta}.
\]

(A.3)

Using this and the transformation law for spin connection,

\[
\delta_0 \omega_{\mu}^{ij} = L_\xi \omega_{\mu}^{ij} + \epsilon^i_k \omega_{\mu}^{kj} + \epsilon^j_k \omega_{\mu}^{ik} = 0,
\]

(A.4)

we can derive the most general form of the spherically symmetric spin connection which is given in the main text, formula (3.3).
Appendix B. Irreducible decomposition of the field strengths

We present here formulas for the irreducible decomposition of the PGT field strengths in a 5D Riemann–Cartan space-time [25].

The torsion 2-form has three irreducible pieces:

\begin{align*}
(2) T^i &= \frac{1}{4} b^i \wedge (h_m \lrcorner T^m), \\
(3) T^i &= \frac{1}{3} h^i \lrcorner (T^m \wedge b_m), \\
(1) T^i &= T^i - (2) T^i - (3) T^i. \tag{B.1}
\end{align*}

The RC curvature 2-form can be decomposed into six irreducible pieces:

\begin{align*}
(2) R^{ij} &= - \ast(b^i \wedge \Phi^j), \\
(3) R^{ij} &= - \frac{1}{12} X^* (b^i \wedge b^j), \\
(6) R^{ij} &= \frac{1}{20} F b^i \wedge b^j, \\
(1) R^{ij} &= R^{ij} - \sum_{a=2}^{6} (a) R^{ij}. \tag{B.2a}
\end{align*}

where

\begin{align*}
F^i &:= h_m \lrcorner R^{mi} = (Ric)^i, \\
F &:= h_i \lrcorner F^i = R, \\
X^i &:= \ast(R^k \wedge b_k), \\
X &:= h_i \lrcorner X^i. \tag{B.2b}
\end{align*}

and

\begin{align*}
\Phi_i &:= F_i - \frac{1}{4} b_i F - \frac{1}{2} h_i \lrcorner (b^m \wedge F_m), \\
\Psi_i &:= X_i - \frac{1}{4} b_i X - \frac{1}{2} h_i \lrcorner (b^m \wedge X_m). \tag{B.2c}
\end{align*}

The above formulas differ from those in [25] in two minor details: the definitions of $F^i$ and $X^i$ are taken with an additional minus sign, but at the same time, the overall signs of all the irreducible curvature parts are also changed, leaving their final content unchanged.

Appendix C. Torsion and curvature for the solution (3.9)

In this appendix, we give values of torsion and curvature for the black hole solution.

C.1. Riemannian connection and curvature

The non-vanishing components of the Riemannian connection are given by

\begin{align*}
\tilde{\omega}^{01} &= - \frac{\sigma}{r^2} \left( \frac{r}{N} + \frac{3r^8}{N r^7} \right) e^0, \\
\tilde{\omega}^{12} &= \frac{N}{r} e^2, \\
\tilde{\omega}^{13} &= \frac{N}{r} e^3, \\
\tilde{\omega}^{23} &= \cot \psi e^3, \\
\tilde{\omega}^{14} &= \frac{N}{r} e^4, \\
\tilde{\omega}^{24} &= \frac{\cot \psi}{r} e^4, \\
\tilde{\omega}^{34} &= \frac{\cot \theta}{r \sin \psi} e^4. \tag{C.1}
\end{align*}

Riemannian curvature reads
Non-zero components of the (Cartan) curvature are given by

\[ R^{01} = \frac{\sigma}{\ell^2} \left( 1 - \frac{2r_0^2}{r^8} \right) e^0 e^1, \quad R^{02} = \frac{\sigma}{\ell^2} \left( 1 + \frac{3r_0^2}{r^8} \right) e^0 e^2, \]
\[ R^{03} = \frac{\sigma}{\ell^2} \left( 1 + \frac{3r_0^2}{r^8} \right) e^0 e^3, \quad R^{04} = \frac{\sigma}{\ell^2} \left( 1 + \frac{3r_0^2}{r^8} \right) e^0 e^4, \]
\[ R^{12} = \frac{\sigma}{\ell^2} \left( 1 + \frac{3r_0^2}{r^8} \right) e^1 e^2, \quad R^{13} = \frac{\sigma}{\ell^2} \left( 1 + \frac{3r_0^2}{r^8} \right) e^1 e^3, \]
\[ R^{14} = \frac{\sigma}{\ell^2} \left( 1 + \frac{3r_0^2}{r^8} \right) e^1 e^4, \]
\[ R^{23} = \frac{\sigma}{\ell^2} \left( 1 - \frac{\sigma\ell^2 - r_0^2}{r^8} \right) e^2 e^3, \quad R^{24} = \frac{\sigma}{\ell^2} \left( 1 - \frac{\sigma\ell^2 - r_0^2}{r^8} \right) e^2 e^4, \]
\[ R^{34} = \frac{\sigma}{\ell^2} \left( 1 - \frac{\sigma\ell^2 - r_0^2}{r^8} \right) e^3 e^4. \]  

(C.2)

Riemannian scalar curvature is

\[ \bar{R} = -\frac{4\sigma}{\ell^2} \left( -5 + \frac{3\sigma\ell^2}{2r^2} + \frac{3r_0^2}{r^8} \right). \]  

(C.3a)

The quadratic Riemannian curvature invariant vanishes

\[ \bar{R}_{ij} \wedge \ast \bar{R}^{ij} = 0. \]  

(C.3b)

### C.1. Torsion and its irreducible decomposition

The non-vanishing components of torsion are given by

\[ T^0 = \frac{3N}{r} e^0 e^1, \quad T^2 = \frac{N}{r} e^1 e^2 + \frac{2A_4}{r} e^3 e^4, \]
\[ T^3 = \frac{N}{r} e^1 e^3 - \frac{2A_4}{r} e^2 e^4, \quad T^4 = \frac{N}{r} e^1 e^4 + \frac{2A_4}{r} e^2 e^3. \]  

(C.4)

The non-vanishing irreducible components of torsion are

\[ (1) T^0 = \frac{3N}{r} e^0 e^1, \quad (1) T^2 = \frac{N}{r} e^1 e^2, \]
\[ (1) T^3 = \frac{N}{r} e^1 e^3, \quad (1) T^4 = \frac{N}{r} e^1 e^4, \]
\[ (2) T^2 = \frac{2A_4}{r} e^3 e^4, \quad (3) T^3 = -\frac{2A_4}{r} e^2 e^4, \quad (3) T^4 = \frac{2A_4}{r} e^2 e^3. \]  

(C.5)

The 2nd irreducible component of torsion vanishes as in the case of any solution of Lovelock gravity, excluding Lovelock Chern–Simons [8]. Quadratic torsional invariant reads

\[ T^{\ell} \wedge \ast T_\ell = -\frac{12\sigma}{\ell^2} \left( 1 - \frac{r_0^2}{r^8} \right) \hat{e}. \]  

(C.6)

Non-zero components of the (Cartan) curvature are

\[ R^{01} = \frac{4\sigma}{\ell^2} e^0 e^1, \quad R^{23} = \frac{4\sigma}{3\ell^2} \frac{N}{A_4} e^3 e^4 + \frac{4\sigma}{3\ell^2} e^2 e^3, \]
\[ R^{24} = \frac{4\sigma}{3\ell^2} \frac{N}{A_4} e^3 e^4 + \frac{4\sigma}{3\ell^2} e^2 e^3, \quad R^{34} = \frac{4\sigma}{3\ell^2} \frac{N}{A_4} e^1 e^2 + \frac{4\sigma}{3\ell^2} e^3 e^4. \]  

(C.7)
Scalar Cartan curvature is constant:

\[ R = \frac{16\sigma}{\ell^2}. \]  
(C.8)

Quadratic Cartan curvature invariant vanishes:

\[ R_{ij} \wedge *R^i = 0. \]  
(C.9)

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References

[1] Lovelock D 1971 The Einstein tensor and its generalizations  
J. Math. Phys. 12 498–501
Lanczos C 1972 The four-dimensionality of space and the Einstein tensor  
J. Math. Phys. 13 874
[2] Maeda H, Willison S and Ray S 2011 Lovelock black holes with maximally symmetric horizons  
Class. Quantum Grav. 28 165005
[3] Camanho X O and Edelstein J D 2013 A Lovelock black hole bestiary  
Class. Quantum Grav. 30 035009
[4] Cai R-G and Ohta N 2006 Black holes in pure Lovelock gravities  
Phys. Rev. D 74 064001
[5] Kastor D and Mann R B 2006 On black strings and branes in Lovelock gravity  
J. High Energy Phys. JHEP04(2006)048
[6] Garraffo C and Giribet G 2008 The Lovelock black holes  
Mod. Phys. Lett. A 23 1801–18
[7] Aros R, Troncoso R and Zanelli J 2001 Black holes with topologically nontrivial AdS asymptotics  
Phys. Rev. D 63 084015
[8] Cvetković B and Simić D 2016 5D Lovelock gravity: new exact solutions with torsion  
Phys. Rev. D 94 084037
[9] Dotti G, Oliva J and Troncoso R 2007 Exact solutions for the Einstein–Gauss–Bonnet theory in five dimensions: black holes, wormholes and spacetime horns  
Phys. Rev. D 76 064038
[10] Ray S 2015 Birkhoff's theorem in Lovelock gravity for general base manifolds  
Class. Quantum Grav. 32 195022
[11] Horowitz G T and Strominger A 1991 Black strings and P-branes  
Nucl. Phys. B 360 197–209
Emparan R and Reall H S 2002 A Rotating black ring solution in five-dimensions  
Phys. Rev. Lett. 88 101101
Emparan R and Reall H S 2008 Black holes in higher dimensions  
Living Rev. Relativ. 11 6
[12] Gregory R and Laflamme R 1993 Black strings and p-branes are unstable  
Phys. Rev. Lett. 70 2837
[13] Sciama D W 1962 The analogy between charge and spin in general relativity  
Recent Developments in General Relativity, Festschrift für Infeld (Warsaw: Pergamon) pp 415–39
[14] Kibble T W B 1961 Lorentz invariance and the gravitational field  
J. Math. Phys. 2 212–21
[15] Canfora F, Giacomini A and Willison S 2007 Some exact solutions with torsion in 5D Einstein–Gauss–Bonnet gravity  
Phys. Rev. D 76 044021
[16] Canfora F, Giacomini A and Troncoso R 2008 Black holes parallelizable horizons and half-BPS states for the Einstein–Gauss–Bonnet theory in five dimensions  
Phys. Rev. D 77 024002
[17] Canfora F and Giacomini A 2008 Vacuum static compactified wormholes in eight-dimensional Lovelock theory  
Phys. Rev. D 78 084034
[18] Canfora F and Giacomini A 2010 BTZ-like black holes in even dimensional Lovelock theories  
Phys. Rev. D 82 024022
[19] Anabalon A, Canfora F, Giacomini A and Oliva J 2011 Black holes with gravitational hair in higher dimensions  
Phys. Rev. D 84 084015
[20] Boulware D G and Deser S 1985 String-generated gravity models  
Phys. Rev. Lett. 55 2656
[21] Hehl F W, McCrea J D, Mielke E W and Neeman Y 1995 Metric-affine gauge theory of gravity: field equations, noether identities, world spinors, and breaking of dilation invariance Phys. Rep. 258 1–171
[22] Blagojević M 2002 Gravitation and Gauge Symmetries (Bristol: Institute of Physics)
[23] Nester J M 1991 A covariant Hamiltonian for gravity theories Mod. Phys. Lett. A 6 2655
[24] Caia R-G, Caob L-M and Ohta N 2010 Black holes without mass and entropy in Lovelock gravity Phys. Rev. D 81 024018
[25] Obukhov Y N 2006 Poincaré gauge gravity: selected topics Int. J. Geom. Methods Mod. Phys. 3 95–138