TORSION-FREE SHEAVES AND MODULI OF GENERALIZED SPIN CURVES

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INTRODUCTION

This article treats the compactification of the space of higher spin curves, i.e. pairs $(X, L)$ with $L$ an $r^{th}$ root of the canonical bundle of $X$. More precisely, for positive integers $r$ and $g$, with $g > 2$, $r$ dividing $2g - 2$, and for a flat family of smooth curves $f : X \rightarrow T$, an $r$-spin structure on $X$ is a line bundle $L$ such that $L^\otimes r \cong \omega_X/T$. And an $r$-spin curve over $T$ is a flat family of smooth curves with an $r$-spin structure. Now, for a fixed base scheme $S$ over $\mathbb{Z}[1/r]$, let $\text{Spin}_{r,g}$ be the sheafification of the functor which takes an $S$-scheme $T$ to the set of isomorphism classes of $r$-spin curves over $T$. A compactification of the space of spin curves is a space (scheme or algebraic stack), which is proper over $M_g$ (the Deligne-Mumford compactification of the space of curves), and whose fibre over $M_g$ represents, at least coarsely, the functor $\text{Spin}_{r,g}$.

It is possible (see [18]) to compactify $\text{Spin}_{r,g}$ using geometric invariant theory. Namely, in the style of L. Caporaso [3], for a fixed $d >> 0$ one can choose a subscheme of the Hilbert scheme $\text{Hilb}_{P^N}^{dz-g}$ with a geometric quotient that coarsely represents $\text{Spin}_{r,g}$. And using results of Gieseker (c.f. [11], Theorems 1.0.0 and 1.0.1), one can show that the semi-stable closure of the subscheme in $\text{Hilb}_{P^N}^{dz-g}$ has a categorical quotient that provides a compactification. This compactification is actually a subscheme of Caporaso’s compactification of the relative Picard scheme over $M_g$.

The principle drawback to the GIT compactification is that it is not obviously the solution to a moduli problem, and therefore it is difficult to describe the resulting space and to make the construction work over a general base, rather than only over algebraically closed fields. Moreover, the GIT construction requires that one make some arbitrary choices, and it is not clear that the resulting compactification is completely independent of these choices. Therefore, the approach we take here is to pose a moduli problem, using torsion-free sheaves, and then show that the associated stack is actually algebraic and that it does indeed compactify $\text{Spin}_{r,g}$.

We discuss three different moduli problems that provide compactifications and describe some of their characteristics. The naive approach would be to use a rank-one torsion-free sheaf $\mathcal{E}$ with a suitable $O_X$-module homomorphism from $\mathcal{E}^\otimes r$ to the canonical bundle. But this doesn’t quite work, as the resulting space is not separated. Some additional conditions on the cokernel of the homomorphism are

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necessary to make the stack separated, and the resulting moduli space is called
the space of quasi-spin curves. The moduli of quasi-spin curves is relatively easy
to construct, but is difficult to describe, due to the presence of nilpotent elements.
Two better moduli problems are spin curves and pure spin curves. These are quasi-
spin curves with some additional local conditions. The local conditions require the
use of log structures, and thus the construction of the moduli space of spin curves
and pure spin curves is more difficult than for quasi-spin curves. But the resulting
spaces have well behaved singularities. In fact, the space of pure spin curves is
smooth over $\mathcal{M}_g$.

0.1. Previous Results. Work on this problem in the special case where the base
$S$ is $\mathbb{C}$ and $r = 2$ has been done by M. Cornalba in [4] and over a more general base
by P. Deligne in [5]. P. Sipe and C.J. Earle have studied $r$th roots of the canonical
bundle on the universal Teichmüller curve (c.f. [8, 22] and [23]). And topological
properties of the uncompactified moduli space of 2-spin curves have been studied
in many places (e.g. [15, 20]).

0.2. Overview. In the first section we present some background on torsion-free
sheaves, some geometric motivation for their use, and some results of Faltings on
the local structure of torsion-free sheaves. In the second section we define the
first moduli problem—singular quasi-spin curves—and make some local calculations
that lead naturally to another condition we impose later on quasi-spin curves to
get singular spin curves. In the third section we discuss how to move from local
to global structures using log-structures, and we use them to formally define the
spin curves and pure-spin curves. The fourth and fifth sections treat deformation
theory and isomorphisms, respectively. The sixth section covers the construction of
the different compactifications, proves that they are algebraic stacks, and discusses
the nature of their singularities. And in the last section we prove that all the
constructions are proper over $\mathcal{M}_g$, and therefore are true compactifications.

0.3. Notation and Conventions. We will use the term semi-stable curve of genus
g to mean a flat, proper morphism $X \to T$ whose geometric fibres $X_t$ are reduced,
connected, one-dimensional schemes, with only ordinary double points, and with
$\dim H^1(X_t, \mathcal{O}_{X_t}) = g$. A stable curve is a semi-stable curve of genus greater than
one, with the additional property that any irreducible component which is isomor-
phic to $\mathbb{P}^1$ meets the rest of the curve in at least three points. Irreducible com-
ponents of a semi-stable curve which are isomorphic to $\mathbb{P}^1$ but meet the curve in only
two points will be called exceptional curves. By line bundle we mean an invertible
(locally free of rank one) coherent sheaf. An $r$-spin structure on a smooth curve
$X/T$ will be a line bundle $\mathcal{L}$ such that $\mathcal{L}^{\otimes r}$ is isomorphic to the canonical bundle
$\omega_{X_T}$. A smooth $r$-spin curve will be a smooth curve $X/S$ with a spin structure.

1. Torsion-Free Sheaves

Compactifications of curve-line bundle pairs using geometric invariant theory
give boundary points that correspond to pairs $(X, L)$ with $X$ a semi-stable curve
having at most one exceptional curve (copy of $\mathbb{P}^1$ that intersects the remaining curve
in at most two points) in each chain of exceptional curves, and $L$ a line bundle of
degree one on each exceptional curve in $X$. Contracting all the exceptional curves
makes the underlying curve stable, and the direct image of $L$ is a torsion-free sheaf;
namely, it has no associated primes of height one. Furthermore the torsion-free
sheaves on stable curves don’t have the problem of having infinite automorphism groups that the line bundles on semi-stable curves have. It is therefore natural to expect that torsion-free sheaves will be well-suited to the compactification of the moduli of spin curves, and this is, in fact, the case.

To begin, we define torsion-free sheaves.

**Definition 1.0.1.** By relatively torsion-free sheaf (or just torsion-free sheaf) on a stable or semi-stable curve \( f : X \to T \), we mean a coherent sheaf \( E \) of \( \mathcal{O}_X \)-modules, which is of finite presentation and flat over \( T \), with the additional property that on each fibre \( X_t = X \times_T \text{Spec}(k(t)) \) the induced \( E_t \) has no associated primes of height one. Of course, on the open set where \( f \) is smooth, a torsion-free sheaf is locally free.

Our ultimate goal is to define and describe a notion of \( r \)-spin structure for stable curves that corresponds to the previously defined notion for smooth curves. Spin structures on a family of stable curves \( X \to T \) will be pairs \((E,b)\) of a relatively torsion-free sheaf \( E \) and a morphism \( b : E^\otimes r \to \omega_{X/T} \) of \( \mathcal{O}_X \)-modules, having certain properties that we will describe later. But before we can define spin structures, we need some general results about relatively torsion-free sheaves.

### 1.1. General Properties of Torsion-Free Sheaves.

**Proposition 1.1.1.** Given any family of semi-stable curves \( \mathcal{X}/T \) and an exact sequence

\[
0 \to E' \to E \to E'' \to 0
\]

of coherent sheaves on \( \mathcal{X} \),

1. If \( E'' \) is flat over \( T \) and of finite presentation, and if \( E \) is relatively torsion-free, then \( E' \) is also relatively torsion-free.
2. If \( E' \) and \( E'' \) are relatively torsion-free, then \( E \) is.

**Proof.** That the sheaves in question are of finite presentation is straightforward to check. It is enough to check that the sheaves are torsion-free on each fibre, and the flatness of \( E'' \) over \( T \) means that the sequence is still exact after restriction to the fibres, where the proposition is clear. \( \square \)

**Proposition 1.1.2.** For any invertible sheaf \( \mathcal{L} \) and any relatively torsion-free sheaf \( E \) on \( \mathcal{X}/T \), the sheaves \( \text{Ext}^i_X(E, \mathcal{L}) \) are zero for all \( i > 0 \).

**Proof.** By [13, 7.3.1.1] it is enough to check this on the individual fibres, i.e. we may assume that \( T \) is a field, and it is enough to check at the stalk of a closed point \( p \). But in this case

\[
\text{Ext}^i_X(E, \mathcal{L})_p = \text{Ext}^i_{\mathcal{O}_{X,p}}(E_p, \mathcal{L}_p) = \text{Ext}^i_{\mathcal{O}_{X,p}}(E_p, \mathcal{O}_p).
\]

And \( X \) is Gorenstein, so these vanish for all \( i > 1 \). And in the case \( i = 1 \), by duality theory ([14, Theorem 6.3]),

\[
\text{Ext}^1_{\mathcal{O}_p}(E_p, \mathcal{O}_p) \lesssim \text{Hom}_{\mathcal{O}_{X,p}}(H^0_{(p)}(E_p), I)
\]

for some dualizing module \( I \). But \( E_p \) is torsion-free, so it has no elements with support equal to \( \{p\} \), so \( H^0_{(p)}(E_p) = 0 \) and thus also \( \text{Ext}^1_{\mathcal{O}_{X,p}}(E_p, \mathcal{O}_{x,p}) = 0 \). \( \square \)
Proposition 1.1.3. If $E$ is relatively torsion-free, then for any invertible sheaf $L$ the sheaf $\hom_{\mathcal{O}_X}(E, L)$ is also relatively torsion free.

Proof. $\hom(-, L)$ preserves flatness over $R$ and commutes with base change, so it suffices to check the proposition over a field, where it is clear.

Proposition 1.1.4. Any relatively torsion-free sheaf $E$ is reflexive.

Proof. This follows from local duality.

1.2. Torsion-Free Sheaves on Semi-Stable Curves over a Field. It is well known that the stalk $F_p$ of a rank-one torsion-free sheaf $F$ at a singular point $p$ of a semi-stable curve $X$ is isomorphic either to $O_{X, p}$ or to the maximal ideal $m_p$, which is isomorphic to the direct image $\pi_* O_{X, p}$ of the normalization $O_{X, p}$. In particular, if the completion $\hat{F}_p$ of the stalk at $p$ is not free, then $\hat{F}_p \cong \mathbb{k}[[x]] / y$ over $O_{X, p}$, where $k$ is the residue field $O_{X, p}/m_p$.

The following are some simple but useful results which describe torsion-free sheaves in terms of line bundles on the normalization of the curve $X$. Namely, if $\pi : X' \to X$ is the normalization of $X$ at one point $p$ of the singular set of $F$ (i.e. where $F$ is not free), then the $O_{X'}$-module $\pi^* F$ has torsion elements, but modulo the torsion elements it is free near the two points of $\pi^{-1}(p)$. For any quasi-coherent sheaf $G$, define $\pi^* G$ to be the torsion-free $O_{X'}$-module $(\pi^* G / \text{torsion})$. Straightforward checking yields the following proposition.

Proposition 1.2.1. In the situation above, where $\pi$ is the normalization of $X$ at a singularity of a rank-one, torsion-free sheaf $F$, the canonical map $F \to \pi_! \pi^* F$ is an isomorphism.

As usual, define the degree of a sheaf $F$ on a curve $Y$ over an algebraically closed field to be $\deg(F) = \chi(F) - \chi(O_Y)$. We are primarily interested in the case of relatively torsion-free sheaves on stable curves, and in this case the degree of $E$ is locally constant on the fibres since $E$ is flat over the base. It is easy to see that this definition of degree corresponds to the usual definition of degree if $F$ is a line bundle.

Proposition 1.2.2. If $F$ is a rank-one torsion-free sheaf and $\pi$ is, as above, the normalization of $X$ at a singular point of $F$, then $\deg(\pi^* F) = \deg(F) - 1$.

Proof. $R^i \pi_* (F) = 0$ for all $i > 0$, so the Leray spectral sequence degenerates, and $H^q(X, F) = H^q(Y, f_* F)$ for all $q$. Thus $\chi(F) = \chi(\pi^* F)$, and $\chi(\pi_* O_{X'}) = \chi(O_{X'})$. Taking Euler-Poincaré characteristics of the exact sequence $0 \to O_X \to \pi_* O_{X'} \to k \to 0$ gives $\chi(\pi_* O_{X'}) = \chi(O_X) + 1$, and thus $\deg(F) = \deg(\pi^* F) + 1$.

Proposition 1.2.3. If $X' \to X$ is the normalization of $X$ at all the singularities of $F$, then $\pi^* F$ is invertible and $(\pi^* F)^{\otimes r} \cong p^*(F^{\otimes r})$. In fact, $p^* E_1 \otimes \cdots \otimes p^* E_n = p^*(E_1 \otimes \cdots \otimes E_n)$ for any torsion-free sheaves $E_1, E_2, \ldots, E_n$ with singularities equal to the singularities of $F$.

This is, again, straightforward to check.

Proposition 1.2.4. $\pi^*$ is a covariant functor from coherent sheaves on $X$ to torsion-free sheaves on $X'$. And if $\mathcal{O}_X$ is the category of rank-one torsion-free $O_X$-modules with singularities exactly those which are normalized by $\pi$, and $\mathcal{PIC}_v$ is
the category of invertible $\mathcal{O}_{X^\nu}$-modules, then the categories $\mathcal{TORF}_\nu$ and $\mathcal{PIC}_\nu$ are equivalent via $\pi^!$ and $\pi_*$.  

Proof. The first part is clear except perhaps the fact that $\pi^!$ commutes with composition of morphisms. But the effect of $\pi^!$ applied to a morphism $(\mathcal{E} \xrightarrow{f} \mathcal{F})$ is induced by the composition $\pi^*\mathcal{E} \xrightarrow{\pi^*f} \pi^*\mathcal{F} \rightarrow \pi^!\mathcal{F}$, which factors through $\pi^!\mathcal{E}$, since the target has no torsion. Moreover, $\pi^!(f \circ g)$ is given by the following commutative diagram.

\[
\begin{array}{ccc}
\pi^*\mathcal{E} & \xrightarrow{\pi^*f} & \pi^*\mathcal{F} \\
\Downarrow & & \Downarrow \\
\pi^!\mathcal{E} & \xrightarrow{\pi^!(f \circ g)} & \pi^!\mathcal{F} \\
\end{array}
\]

Both sets of bottom arrows $\pi^!(f \circ g)$ and $\pi^!f \circ \pi^!g$ commute in the rectangle, and the map $\pi^* \rightarrow \pi^!$ is surjective, so the bottom arrows must commute. The second part of the proposition is clear by Proposition 1.2.1.  

1.3. Boundedness of Rank-One Torsion-Free Sheaves. Since $X$ is projective, we can choose a very ample $\mathcal{O}(1)$ on $X$. In general, we will write $\mathcal{F}(m)$ for $\mathcal{F} \otimes \mathcal{O}(1)^\otimes m$. One of the more important facts about torsion-free sheaves of rank one is that they form bounded families. In other words, the following proposition holds. This will later be important in proving that the functor of spin curves has a versal deformation.

Proposition 1.3.1. If $\mathcal{F}$ is a rank-one torsion-free sheaf on $X$, there is an integer $m_0$ depending only on the degree of $\mathcal{F}$ on each irreducible component of $X$, and on the genus of $X$, such that for $m \geq m_0$ the following holds.

1. $H^1(X, \mathcal{F}(m)) = 0$.
2. $\mathcal{F}(m)$ is generated by global sections.

This is a straightforward generalization of D’Souza’s propositions in Section Three of [7].

1.4. Some Geometry. The original motivation for studying torsion-free sheaves comes from the fact that boundary points in the GIT compactification correspond approximately to pairs $(X, \mathcal{L})$ with $X$ a semi-stable curve, having no more than one exceptional curve in each chain of exceptional curves in any fibre, and $\mathcal{L}$ is a line bundle with degree one on each of the exceptional curves (c.f. [6, 11, 18]). Given a curve-bundle pair $(X, \mathcal{L})$ of this sort, contracting all of the exceptional curves, i.e. projection from $X$ to its stable model $\rho : X \rightarrow \overline{X}$ makes $\rho_*\mathcal{L}$ into a torsion-free sheaf. This contraction and a related “inverse” action of blowing up certain ideals warrant a more careful look.

Let $X/B$ be a stable curve over $B = \text{Spec}(R)$, with $R$ a complete local ring. The completion $A = \mathcal{O}_{X, p}$ of the local ring of $X$ at a point $p$ in the special fibre is of the form $A = R[[x, y]]/(xy - \pi)$ for some choice of $\pi$ in the ring $R$. If $p$ is singular in the special fibre, then $\pi$ is in the maximal ideal $m$ of $R$. Given two elements $p$ and $q$ of $R$ such that $pq = \pi$, we will construct a semi-stable curve with the properties...
mentioned above, namely with no chains of exceptional curves of length greater than one. Essentially, we want to undo the contraction to associate a suitable semi-stable curve and line-bundle to each stable curve and rank-one torsion-free sheaf.

First notice that if \( p \) is not a zero divisor, blowing up the ideal \( I = (x,p) \) in \( A \) to get \( \tilde{X}_I := \text{Proj}(A[\Xi]/(Px - p\Xi, Py - \Xi y)) \) gives

\[
\tilde{X}_I \cong \text{Proj}(A[\Xi]/(Px - P\Xi, Py - \Xi y)).
\]

Similarly, if \( q \) is not a zero divisor, blowing up the ideal \( J = (y,q) \) gives

\[
\tilde{X}_J \cong \text{Proj}(A[\Xi]/(Qy - yQ, pQ - xY)).
\]

And these are isomorphic via \( \text{Proj}(A[\Xi]) \to \text{Proj}(A[\Xi]) \) via \( \rho \), setting \( q \).

Now \( \Gamma(\tilde{X}, \mathcal{O}) = \{ P^n \mathcal{O}_U \text{ on } U \neq V, \Xi^n \mathcal{O}_V \text{ on } V \} \).

**Proposition 1.4.1.** In the above construction of \( \tilde{X} \) and \( \mathcal{L} := \mathcal{O}_{\tilde{X}}(n) \) the following hold.

1. \( n \geq -1 \) implies that \( \rho_* \mathcal{L} \) is flat over \( R \), it commutes with base change, and \( R^1 \rho_* \mathcal{L} = 0 \).
2. For \( n \geq 0 \), \( \Gamma(\tilde{X}, \mathcal{L}) \cong \tilde{A}_n \), the \( n \)-th graded piece of \( \tilde{A} \), and the natural map \( \rho^* \rho_* \mathcal{L} \to \mathcal{L} \) is surjective.
3. \( \rho_* \mathcal{O}_{\tilde{X}} = \mathcal{O}_X \).
4. \( n = 1 \) implies \( \rho_* \mathcal{L} \) is torsion-free of rank one.

**Proof.** It suffices to consider the case \( A = R[x,y]/(xy - \pi) \), and in this case \( \mathcal{O}_U \cong R[s,y]/(sy - q) \), and \( \mathcal{O}_V \cong R[t,x]/(xt - p) \), both of which are flat over \( R \), and \( \mathcal{O}_{U \cap V} \cong R[s,t]/(st - 1) \).

Now \( \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}(n)) = \{ (g,f) \in \mathcal{O}_U \oplus \mathcal{O}_V | g = s^nf \text{ on } \mathcal{O}_{U \cap V} \} \). So \( f \) and \( g \) are of the form

\[
g = sg_+(s) + g_0 + yg_-(y) \quad \text{and} \quad f = tf_-(t) + f_0 + xf_+(x)
\]

with \( g_+(s) \in R[s], g_-(y) \in R[y], f_-(t) \in R[t], \) and \( f_+(x) \in R[x] \), and

\[
s g_+(s) + g_0 + t q g_-(t q) = s^n(t f_-(t) + f_0 + sf_+(s p)).
\]

So

\[
s g_+(s) + g_0 + t q g_-(t q) = s^{n-1} f_-(t) + s^n f_0 + s^{n+1} f_+(s p),
\]
and rewriting $sg_+(s)$ as $s^{n+1}r + s^n g_n + \cdots + sg_1$, with $g_i \in R$, and $r(s) \in R[s]$, and $tf_1(x)$ as $t^{n+1}f_1 + t^n f_n - s^n f_{n-1} + \cdots + tf_1$, with $\phi(t) \in R[t]$, gives

$$s^{n+1}r + s^n g_n + \cdots + sg_1 = t\phi(t) + f_n + \cdots + s^{n-1}f_1 + s^n f_0 + s^{n+1}f_1(s)$$

Now $s^{n+1}r$ is a zero divisor, this implies that $r(s) = p_{f_0}(sp)$, because $r(s)$ is an element of $R[s] \subseteq O_U$ and $Ann((s^{n+1}) \cap R[s]) = (0)$ (i.e. $R[s] \subseteq O_U \rightarrow O_U \otimes R$ is injective). Similarly, $g_n = f_i$ for $0 \leq i \leq n$ and $\phi(t) = qq_+(tq)$.

Several things are easy to see from this formulation, namely
1. $\Gamma(X, L)$ is free over $R$, and hence $\rho_* L$ is $R$-flat as long as $n \geq -1$.
2. $\rho_* O_X = O_X$
3. If $(g, f)$ is an element of $\Gamma(X, L)$, then $g$ can be written as
   $$g = s^n x f_1 + s^n f_0 + s^{n-1} f_1 + \cdots + f_1 + f_n + yg_-(y)$$
   in $O_U$ (with $x = sp$), and $f$ can be written as
   $$f = x f_1 + f_0 + tf_1 + t^2 f_2 + \cdots + t^n f_n + t^n yg_-(y)$$
   in $O_V$ (with $y = tq$).

The third fact shows that the element

$$(f_0 + xf_1(x))z^n + f_1 z^{n-1}P + \cdots + f_{n-1} z P^{n-1} + (f_n + yg_-(y))P^n$$

in $\tilde{A}_n$ maps to $(g, f)$ in $\Gamma(X, L)$, and the natural homomorphism $\tilde{A}_n \rightarrow \Gamma(X, L)$ is surjective. Moreover, it is easy to check that if we write $\tilde{A}_n$ as $\tilde{A}_n = \{ F_i z^n + F_i z^{n-1}P \cdots + F_i z P^{n-1} | F_i \in A \}$, we can assume that $F_0$ is in $R[x]$ and $F_n$ is in $R[y]$ and all the remaining terms are in $R$ (i.e. $F_i \in R$ for $0 < i < n$). And thus the homomorphism $\tilde{A}_n \rightarrow \Gamma(X, L)$ must actually be injective, hence an isomorphism.

Now since $R^1 \rho_* L$ is right exact, to see that it vanishes, it suffices to check that it vanishes for each fibre. And $H^1(X \times X, O(n))$ is zero for all $x$ in $X$, except possibly the singular points of $X$. But over a singular point

$$H^1(O(n)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) = 0.$$

And thus $R^1 \rho_* L$ is zero, and $\rho_* L$ commutes with base change if $n \geq -1$.

To show that $\rho^* \rho_* L \rightarrow L$ is surjective, note that this map is locally (on $U$) just the map taking $\{(g, f) | g = s^n f \} \otimes_A O_U$ to $O_U$, given by $(g, f) \otimes z \mapsto gz$. So it suffices to show that there exists $(g, f) \in \rho_* L$ such that $g = 1$. But $(g, f) = (1, t^n)$ works as long as $t^n \in O_V$, i.e. if $n \geq 0$. A similar computation holds over $V$, so $\rho^* \rho_* L \rightarrow L$ is surjective.

To show that $\rho_* L$ is torsion-free, note that since it is flat and commutes with base change, it suffices to check the case where $R$ is a field and $p = q = \pi = 0$. In this case $O_U = R[x, y]/sy$ and $O_x = R[t, x]/tx$ and global sections of $O(1)$ are $(g, f) \in O_U \otimes O_V$ such that $g = sf$. Moreover, $x(g, f) = (0, xf)$ and $y(g, f) = (yg, 0)$, hence if the ideal $(x, y)$ annihilates $(g, f)$, then $x(g, f) = y(g, f) = 0$, and $xf = yg = 0$. Thus $f \in (t), g \in (s)$, and this contradicts the fact that $g = sf$; therefore, $\rho_* O(1)$ has no associated primes of height one and is torsion-free. \hfill $\square$

**Lemma 1.4.2** (Cornalba [3]). If $L$ is a line bundle on a semi-stable curve $f : X \rightarrow T$ such that $L|_E \cong O_E$ for some exceptional component $E$ of a special fibre $X_0$, then there is an étale neighborhood $T'$ of 0, such that $L|_{T'}$ on $X_{T'}$ is trivial in a neighborhood of $E$ in $X_{T'}$. 

Proof. There is an étale neighborhood $T''$ of 0 in $T$ such that for each irreducible component of the special fibre $X_0$ except $E$, there is a section of $X''/T''$ that does not intersect $E$ but which passes through the irreducible component (c.f. [4, 17.16.3]). Let $D$ be the divisor in $X$ corresponding to the sum of all these sections, and note that $L(mD)|_{X_0}$ is generated by global sections and has first cohomology group zero if $m$ is sufficiently large.

So the natural map $R^1f_*(\mathcal{L}(mD)) \otimes k \to R^1f_*(X_0, \mathcal{L}(mD) \otimes k) = 0$ is surjective, hence is an isomorphism, hence $R^1f_*(\mathcal{L}(mD))$ is zero on an open set $T'$ about 0 in $T''$ ([4, III.12.11]). This implies that on $T'$ the map $R^0f_*(\mathcal{L}(mD)) \otimes k \to \Gamma(X_0, \mathcal{L}(mD) \otimes k)$ is also surjective. And $\mathcal{L}(mD)$ is generated by global sections. Thus $\mathcal{L}(mD)$ is trivial on a neighborhood of $E$, and on a sufficiently small neighborhood of $E$, $\mathcal{L}(mD) \cong \mathcal{L}$.

As an immediate consequence of Cornalba’s lemma we have the following corollary.

**Corollary 1.4.3.** Proposition [1.4.1 holds for any line bundle $\mathcal{L}$ which has degree 1 on the exceptional curve of the special fibre of $\tilde{X}(p, q)$; namely,

1. $\rho_*\mathcal{L}$ is flat over $R$, it commutes with base change, and $R^1\rho_*\mathcal{L} = 0$.
2. $\Gamma(\tilde{X}, \mathcal{L}) \cong \mathcal{O}_1$, and $\rho^*\rho_*\mathcal{L} \to \mathcal{L}$ is surjective.
3. $\rho_*\mathcal{L}$ is torsion-free of rank one.

**1.5. Induced Maps.** If $p$ and $q$ have the additional relations that $p^u = wq^v$ with $u + v = r$ and $w \in R^X$, then there is a canonical map from $\mathcal{O}_X(r)$ to $\mathcal{O}_X$. Namely, on $U$ it is $\mathcal{O}(r) = (A[s]/(ps - x, sy - q)) \cdot P^\otimes$ maps to $A[s]/(ps - x, sy - q)$ via $P^\otimes \mapsto wy^r$. And on $V$ it is $\mathcal{O}(r) = (A[t]/(p - xt, y - qt)) \cdot (\Xi)^\otimes$ maps to $A[t]/(p - xt, y - qt)$ via $\Xi^\otimes = s^r \cdot P^\otimes \mapsto s^rwy^r = x^u$.

The canonical map $\rho^*\rho_*\mathcal{O}(1) \to \mathcal{O}(1)$ induces a map $(\rho^*\rho_*\mathcal{O}(1))^\otimes \to (\mathcal{O}(1)^\otimes = \mathcal{O}(r)$, and the canonical map $\mathcal{O}(r) \to \mathcal{O}_X$ gives a canonical map $\rho^*(\rho_*\mathcal{O}(1))^\otimes \to \mathcal{O}_X$. And this induces a map on the push-forward by adjointness

$$(\rho_*\mathcal{O}(1))^\otimes \to \rho_*\mathcal{O}_X = \mathcal{O}_X.$$ When we define various generalizations of a spin structure, the best-behaved ones will be those which are locally isomorphic to those induced from the canonical map $\mathcal{O}_X(r) \to \mathcal{O}_X$.

**1.6. Local Structure of Torsion-Free Sheaves.** As in the previous section, we work with a stable curve $X/B$, where the base $B$ is the spectrum of a complete local Noetherian ring $R$; the completion of the ring $\mathcal{O}_{X, p}$ at a singular point $p$ is isomorphic to $A := R[[x, y]]/(xy - \pi)$, and $\pi$ is an element of the maximal ideal of $R$. A torsion-free sheaf $\mathcal{E}$ corresponds to an $R$-flat $A$-module, $E$.

Locally on $X$, we can express a torsion-free sheaf obtained by the contraction $\tilde{X}(p, q) \to X$ (a pullback of a line bundle $\mathcal{L}$ of degree one on the exceptional curve) in the following way.

$E \cong \Gamma(\text{Spec}(A), \pi_*\mathcal{O}(1)) = A_1 = (A[\Xi, P]/(\Xi p - P x, \Xi y - P q))_1 = \{f\Xi + gP | f, g \in A\}$. It is easy to see that we can assume $f$ is in $R[[x]]$, and $g$ is in $R[[y]]$. And so the map $\rho_*\mathcal{O}(1) \to A^\otimes$, given by $(f\Xi + gP) \mapsto \begin{pmatrix} fx + gp \\ f q + gy \end{pmatrix}$ is a well-defined homomorphism of $A$-modules. Even if $p$ and $q$ are zero divisors, if $f$ is in $R[[x]]$ and $g$ is in $R[[y]]$, then $fx + gp = 0$ implies that $f = 0$. Similarly, $fq + gy = 0$ implies
that $g = 0$. So the map is injective. We can, therefore, express $\rho_*\mathcal{O}(1)$ as the image of the $A$-homomorphism

$$\alpha(p, q) = \left( \begin{array}{c} x \\ p \\ q \\ y \end{array} \right) : A^{\oplus 2} \to A^{\oplus 2}.$$  

A result of Faltings shows that every rank-one torsion-free sheaf is of this form. Namely, let $E(p, q)$ be the image of $\alpha(p, q) : A^{\oplus 2} \to A^{\oplus 2}$, where $\alpha$ is the two-by-two matrix $\left( \begin{array}{c} x \\ p \\ q \\ y \end{array} \right)$, and $p$ and $q$ are, as before, elements of $R$ such that $pq = \pi$. We saw above that $E(p, q)$ is $R$-flat, and torsion-free. When $p$ and $q$ are in $\mathfrak{m}$ then $E(p, q)$ is a deformation of the normalization of $A/\mathfrak{m}A$, i.e. of the unique (up to isomorphism) non-free torsion-free sheaf $k[[x]] \oplus k[[y]]$ over the ring $A/\mathfrak{m}A \cong \mathfrak{f}[[x, y]]/\mathfrak{m}$. Faltings’ result is the following.

**Theorem 1.6.1** (Faltings [9]). Any reflexive $E$ of rank 1 is isomorphic to an $E(p, q)$, for $p, q \in R$ with $pq = \pi$.

The fact that $\alpha\left( \begin{array}{c} y \\ 0 \end{array} \right) = \alpha\left( \begin{array}{c} 0 \\ q \end{array} \right)$ and $\alpha\left( \begin{array}{c} 0 \\ x \end{array} \right) = \alpha\left( \begin{array}{c} p \\ 0 \end{array} \right)$ implies that for any $\alpha\left( \begin{array}{c} f \\ g \end{array} \right) \in E(p, q)$, we can assume $f$ is in $R[[x]]$ and $g$ is in $R[[y]]$, and $E(p, q)$ is $R$-isomorphic to $R[[x]] \oplus R[[y]]$ via the obvious identification. Homomorphisms and isomorphisms of $E(p, q)$’s can be described by their lifts to $A^{\oplus 2}$. Namely, any morphism of $A$-modules $E \to F$, with $E$ relatively torsion-free, can be lifted to a morphism from $A^{\oplus 2}$ to $F$. And a homomorphism from $E$ to $E'$ with $E$ and $E'$ both torsion-free lifts to an endomorphism of $A^{\oplus 2}$. More exactly, the following holds.

**Proposition 1.6.2** (Faltings [9]). If $p \equiv q \equiv 0 \mod \mathfrak{m}$, then $E(p, q)$ is isomorphic to $E(p', q')$ if and only if there exist $u, v \in R^\times$ such that

$$p' = u pv^{-1}, \quad \text{and} \quad q' = vqu^{-1}.$$  

In this case the isomorphism is induced by the “constant” map $\left( \begin{array}{c} u \\ 0 \\ 0 \\ v \end{array} \right) : A^{\oplus 2} \to A^{\oplus 2}$. Moreover, writing a homomorphism $\Phi$ in $\text{Hom}_A(E(p, q), E(p', q'))$ as a lift to $A^{\oplus 2} \to A^{\oplus 2}$ given by $\left( \begin{array}{cc} \varphi_+ & \psi_+ \\ \psi_- & \varphi_- \end{array} \right)$, $\varphi_+$ can be taken to be in $R[[x]]$, $\varphi_-$ can be taken to be in $R[[y]]$, and the elements $\varphi_+(x)$ and $\varphi_-(y)$ completely determine $\psi_+$ and $\psi_-$ by the relations

$$\psi_+ = \frac{p}{x}(\varphi_+(x) - \varphi_+(0)) \quad \text{and} \quad \psi_- = \frac{q}{y}(\varphi_-(y) - \varphi_-(0)),$$

and are subject to the condition that

$$p'\varphi_-(0) = \varphi_+(0)p, \quad \text{and} \quad q'\varphi_+(0) = \varphi_-(0)q.$$  

These results also hold for Henselian rings. Suppose now that $R$ is the Henselisation of a local ring of finite type over a field or an excellent Dedekind domain, $\mathfrak{m}$ is the maximal ideal of $R$, $\pi \in \mathfrak{m}$, and $A$ is the Henselisation of $R[x, y]/(xy - \pi)$ at $\mathfrak{m} + (x, y)$. As before, for each pair $p, q \in R$ with $pq = \pi$, define $E(p, q)$, and the theorem is

**Theorem 1.6.3** (Faltings). 1. Any torsion-free $E$ of rank one over $A$ is isomorphic to $E(p, q)$ for $p, q \in R$ and $pq = \pi$. 
2. If \( p, q \) are in \( \mathfrak{m} \), then \( E(p, q) \) and \( E(p', q') \) are isomorphic if and only if there exist \( u, v \in R^k \) with \( p' = up^{-1}, q' = vqu^{-1} \).

3. Suppose \( I \subseteq R \) is a nilpotent ideal. Modulo constant automorphisms (given by \( \left( \begin{array}{cc} u & 0 \\ 0 & v \end{array} \right) \in GL_2(R) \) with \( up = pv, vq = qu \)) any automorphism of \( E(p, q)/IE(p, q) \) lifts to an automorphism of \( E(p, q) \).

2. Quasi-Spin Surves: Local Structure

A spin structure on a stable curve should have the property that where the torsion-free sheaf is free it is isomorphic to the canonical line bundle \( \omega \). A natural object to study, therefore, is a triple \((X, \mathcal{E}, b)\), with \( X \) a stable curve, \( \mathcal{E} \) a relatively torsion-free sheaf of rank one and degree \( 2g - 2/r \), and \( b \) is a homomorphism of \( \mathcal{O}_X \)-modules

\[
\lambda : \mathcal{E}^r \rightarrow \omega^{\chi/T},
\]

which is an isomorphism on the open set where \( \mathcal{E} \) is locally free. Note that for smooth curves, such a triple is just an \( r \)-th root of the canonical bundle with an explicit isomorphism of the \( r \)-th power of the bundle to the canonical bundle. Later we will need a few more conditions on these triples to get the “right” generalization of a spin curve, but we begin with these alone.

2.1. A-Linear Homomorphisms of Tensor Powers. To better understand these triples we need to study \( A \)-linear maps \( \lambda : E^r \rightarrow A \) of the \( r \)-th tensor power of a rank-one torsion-free \( A \)-module \( E \). As before, \( A \) is an étale neighborhood of the closed point defined by \( (x, y) + \mathfrak{m} \mathfrak{A} \) in \( \text{Spec}(R[[x, y]]/(xy - \pi)) \) over the base ring \( R \), where \( \mathfrak{m} \) is a maximal ideal of \( R \) containing \( \pi, p, \) and \( q \).

Any map \( \lambda : E^r \rightarrow A \) that is \( A \)-linear, lifts to a map \( \tilde{\lambda} \), thus

\[
\begin{array}{ccc}
A^{\otimes 2r} & \xrightarrow{\lambda} & A \\
\downarrow {\alpha}^{\otimes r} & & \downarrow \| \\
E^r & \xrightarrow{b} & A
\end{array}
\]

Over \( A[1/x] \) and over \( A[1/y] \) the module \( E \) is locally free, thus over these rings any homomorphism \( \lambda : E^r \rightarrow A \) will factor through \( \text{Sym}^r(E) \), and its lift to \( (A^{\otimes 2})^{\otimes r} \) will factor through \( \text{Sym}^r(A^{\otimes r}) \). And since \( A \) has no \((x, y)\)-torsion, this holds in general. So if \( f \) and \( g \in A^{\otimes 2} \) are defined as \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) and \( \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \) respectively, we only need to describe \( b_i := b(f^r \otimes g') \) for each \( 0 \leq i \leq r \) in order to completely describe \( \lambda \) and \( \tilde{\lambda} \). We will, therefore, denote \( \tilde{\lambda} : A^{2r} \rightarrow E_{\otimes r} \rightarrow A \) by the vector \( (b_0, b_1, \ldots, b_r) \).

Now, since \( \alpha \left( \begin{array}{c} p \\ 0 \end{array} \right) = \alpha \left( \begin{array}{c} 0 \\ x \end{array} \right) \) and \( \alpha \left( \begin{array}{c} 0 \\ q \end{array} \right) = \alpha \left( \begin{array}{c} y \\ 0 \end{array} \right) \), we must have for all \( i, 0 \leq i \leq r - 1 \)

\[
pb_i = xb_{i+1} \quad \text{and} \quad yb_i = qb_{i+1}.
\]

Now the fact that the map must be an isomorphism off of the singular locus of the underlying curve means that \( \lambda : E(p, q)^{\otimes r} \rightarrow A \) must be an isomorphism on \( A[1/x] \)
and $A[1/y]$. Over $A[1/x]$ the map $\tilde{b} = (b_0, b_1, \ldots, b_r)$ is completely determined by $b_0$, namely for any $i$ we have $x^ib_i = p^ib_0$, and thus

$$b_i = b_0p^i/x^i.$$  

For $\tilde{b}$ to be surjective over $A[1/x]$ we must have that $b_0$ is invertible in $A[1/x]$. Similarly, $b_r$ is invertible in $A[1/y]$.

On the special fibre, since $p$ and $q$ are in $m$, we have $\tilde{b} \equiv (\bar{b}_0, 0, 0, \ldots, 0, \bar{b}_r) \pmod{m}$. Here $\bar{z}$ denotes the image of $z$ modulo $m$. And $\bar{b}_0 = x^u\bar{\beta}_0$ and $\bar{b}_r = y^v\bar{\beta}_r$ for some $\bar{\beta}_0$ invertible in $(A/m\mathfrak{A})[1/y]$, but not in the ideal $(x)$, and for some $\bar{\beta}_r$ invertible in $(A/m\mathfrak{A})[1/\eta]$, but not in $(y)$. This makes the length of the cokernel of $\tilde{b}$ equal to $u + v - 1$. If $u$ (or similarly $v$) is zero, then over the ring $A[1/y]$ the fact that $b_0 = b_rq^r/y^r = 0 \pmod{m}$ implies that $\bar{\beta}_0 = \bar{b}_0 \equiv 0 \pmod{\text{Ann}(j, A/m\mathfrak{A}(y))}$, i.e. modulo $(x)$. But this implies that $\bar{\beta}_0 = 0$, and that is a contradiction. Hence we have proven the following.

**Proposition 2.1.1.** Any pair $(E, b)$ which is not free is of the form $(E(p, q), b)$ with $p$ and $q$ in $m$, and $b$ lifts to $\tilde{b} = (b_0, b_1, \ldots, b_r)$, where, modulo the ideal $m\mathfrak{A}$ we have $b_0 = x^u\bar{\beta}_0$ and $b_r = y^v\bar{\beta}_r$ for some $\bar{\beta}_0$ invertible in $(A/m\mathfrak{A})[1/y]$, but not in the ideal $(x)$, and for some $\bar{\beta}_r$ invertible in $(A/m\mathfrak{A})[1/\eta]$, but not in $(y)$. Moreover, $u$ and $v$ must both be at least one.

The final condition we will need on triples $(X, \mathcal{E}, b)$ to generalize spin curves is the condition that the two constants $u$ and $v$ in Proposition 2.1.1 must sum to $r$. In other words, the length of the cokernel of $b$ at each singular point of each fibre must be $r - 1$ if the sheaf $\mathcal{E}$ is not free there. Without this condition there would be too many possible triples for the associated stack to be separated. We now have the first generalization of spin structures on stable curves.

**Definition 2.1.2.** A quasi-spin structure on a stable curve $X/T$ is a pair $(\mathcal{E}, b)$ where $\mathcal{E}$ is relatively torsion-free of rank one, and $b$ is a homomorphism of $\mathcal{O}_X$-modules

$$b : \mathcal{E}^\oplus r \to \omega_{X/T}$$

to the canonical dualizing sheaf, such that

1. $\mathcal{E}$ has degree $(2g - 2)/r$.
2. $b$ is an isomorphism on the open set where $\mathcal{E}$ is not free.
3. For each closed point $t$ of the base $T$, and for each singular point $p$ of the fibre $X_t$ where $\mathcal{E}$ is not free, the length of the cokernel of $b$ at $p$ is $r - 1$.

**Definition 2.1.3.** A quasi-spin curve is simply a stable curve with a quasi-spin structure.

In the special case that $r = 2$, the requirement that the cokernel of the spin structure map be supported on the singular locus of $\mathcal{E}$ is enough to guarantee that the length of the cokernel is at least one at all singular points. The condition on the total degree of $\mathcal{E}$ can be seen to guarantee that the length of the cokernel is at most (and hence exactly) one at all singular points. Moreover, these conditions are equivalent to the condition that the map $b$ induce an isomorphism

$$\mathcal{E} \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_{X_t}}(\mathcal{E}, \omega_X) = \mathcal{E}^\vee \otimes \omega_X.$$
2.2. Power Series Expansions. Any $A$-linear map $b = (b_0, \ldots, b_r)$ as above, over the complete local ring $A = O_{A,x}$, has a power series expansion $b_i = \sum_{n\geq0} b_{i,n}x^n + \sum_{m>0} b_{i,-m}y^m$. And the relations $p^ib_0 = x^ib_i$ and $q^{-i}b_r = y^{-i}b_i$ imply that
\[
p^ib_{0,n+i} = b_{i,n}
\]
for $n \geq 0$, and
\[
b_{i,m} = q^{-i}b_{r,-m-(r-i)}
\]
for $m \geq 0$. And in particular
\[
p^jb_{0,j} = q^{-j}b_{r,j-r}
\]
for all $j, 0 \leq j \leq r$.

Moreover, if $b$ induces a quasi-spin structure on the central fibre, then there are $u, v \in \mathbb{Z}^+$ such that (mod $m$)
\[
\hat{b}_0 = x^\mu \hat{\beta}_0 \quad \text{and} \quad \hat{\beta}_0 \in (A_x)^\times.
\]
This implies that $\hat{b}_0 = \sum_{n \geq u} \hat{b}_{0,n}x^n$ with $\hat{b}_{0,u} \neq 0$, hence $b_{0,u}$ is not in $m$ and is invertible in $R$. Similarly, $b_{r,-v} \in R^\times$. So, in particular, $p^u = q^ub_{r,-v}/b_{0,u}$. Letting $w = b_{r,-v}/b_{0,u} \in A^\times$, we have the relation
\[
p^u = q^uw.
\]
In the special case that $\pi$ is not a zero divisor, the relations $p^ib_{0,i} = q^{-i}b_{r,i-r}$ for $0 \leq i \leq u$ imply that
\[
b_{0,i} = w^{-1} p^u q^{-i} b_{r,i-r} = \frac{\pi^{u-i}}{w} b_{r,i-r}.
\]
Similarly,
\[
b_{r,i-r} = w\pi^{i-u} b_{0,i}, \quad \text{for} \quad u \leq i \leq r.
\]
But even when $\pi$ is a zero divisor
\[
b_{0,0} = \pi^u b_{r,-r}, \quad b_{r,0} = \pi^v b_{0,r}, \quad b_{0,u} = wb_{r,-v},
\]
and
\[
b_{0,i} = \frac{\pi^{u-i}}{w} b_{r,i-r} + \sigma_i \quad \text{for} \quad 0 < i < u, \quad \text{and} \quad b_{r,i-r} = w\pi^{i-u} b_{0,i} + \sigma_i \quad \text{for} \quad u < i < r.
\]

The “bad” terms $\sigma_i$ are all nilpotent elements. On the one hand, for any prime ideal $p \in \text{Spec}(R)$ such that $p$ (and hence $q$) is in $p$, we have that $b_i \equiv 0 \mod p$ for $0 < i < r$. And $b_0 \equiv x^\mu \beta$, and $b_r \equiv y^\gamma$, with $\beta$ and $\gamma$ invertible elements of $R[[x]]$ and $R[[y]]$ respectively. Accordingly,
\[
b_{0,i} \in p \quad \text{for} \quad 0 < i < u, \quad \text{and} \quad b_{r,i-r} \in p \quad \text{for} \quad u < i < r,
\]
and thus $\sigma_i \in p$ for $0 < i < r$ ($\sigma_u$ is obviously zero). On the other hand, if $p$ (and therefore $q$) is not in $p$, then $p$ and $q$ are not zero divisors in $R_p$ and hence, as demonstrated before, $\sigma_i \in p$ for $0 < i < r$. Thus $\sigma_i$ is contained in the nilradical of $R$ for every $i$. And in particular, for any quasi-spin structure over a reduced, complete local ring the relations
\[
b_{0,i} = \frac{\pi^{u-i}}{w} b_{r,i-r} \quad \text{for} \quad 0 \leq i \leq u 
\]
\[
b_{r,i-r} = w\pi^{i-u} b_{0,i} \quad \text{for} \quad r \geq i \geq u
\]
hold.
Proposition 2.2.1. When the relations (2) hold, we can write \( b_0 \) and \( b_r \) as the following products:

\[
b_0 = ax^n \quad \text{and} \quad b_r = aw^r \text{ for some } a \in A^\times.
\]

In particular, given \( u, v, \) and \( w, \) the fact that the specified relations (2) hold means that \( b \) is completely determined by \( a \in A^\times. \)

Proof. The first map is just

\[
b_0 = \sum_{n \geq 0} b_{0,n+u} x^n + 1/w \sum_{u\geq m \geq 0} \pi^{u-m} b_{r,m-r} x^m + q^r \sum_{l>0} b_{r,l-r} y^l
\]

And thus

\[
a = \sum_{n \geq 0} b_{0,n+u} x^n + 1/w \sum_{m \geq 0} b_{r,-m-v} y^m.
\]

The calculation is similar for \( b_r. \)

Proposition 2.2.2. If the relations (2) hold on \( \tilde{b}, \) then \( b \) is actually the map induced on \( E(p,q) \otimes r = (\rho_* O_X(p,q)(1))^\otimes r \) as in Section 1.5. Moreover, the relations (2) hold for (the lift to \( A^{\circ}\otimes 2^r \)) of the map induced from \( O_X(p,q)(1) \) for any \( p \) and \( q \) in \( m_R \) with \( pq = \pi. \)

Proof. To see this out explicitly the map

\[
(A^2)^\otimes r \to A^{2r} \overset{\varphi}{\to} (E(p,q))^r \overset{\psi}{\to} (\rho_* \mathcal{O}(1))^r \to \rho_* \mathcal{O}(r) \to \mathcal{O}_X = A.
\]

The first map is just

\[
\left( \begin{array}{c} f_1 \\ g_1 \\ \vdots \\ f_r \\ g_r \end{array} \right) \mapsto \alpha \left( \begin{array}{c} f_1 \\ g_1 \\ \vdots \\ f_r \\ g_r \end{array} \right).
\]

The map \( \psi \) is given by \( \psi : \alpha \left( \begin{array}{c} f \\ g \end{array} \right) \mapsto (sf + g, f + tg), \) so

\[
\psi^\otimes r : \alpha \left( \begin{array}{c} f_1 \\ g_1 \\ \vdots \\ f_r \\ g_r \end{array} \right) \mapsto (sf_1 + g_1, f_1 + tg_1) \otimes \ldots \otimes (sf_r + g_r, f_r + tg_r).
\]

The map \( \varphi \) is given by \( \varphi : (h_1, k_1) \otimes \ldots \otimes (h_r, k_r) \mapsto (h_1 h_2 \ldots h_r, k_1 k_2 \ldots k_r) \) and \( \gamma \) is \( \gamma : (h, k) \mapsto wy^r h = x^r k \in A. \)

So the composite map is

\[
\left( \begin{array}{c} f_1 \\ g_1 \\ \vdots \\ f_r \\ g_r \end{array} \right) \mapsto x^n \prod_{1 \leq i \leq r} (f_i + tg_i),
\]

but this is just the map

\[
b = (x^n, tx^n, \ldots, t^r x^n) = (x^n, px^{n-1}, \ldots, p^n t, \ldots, p^u t^v).
\]

And \( p^u = wq^v, \) and \( qt = y, \) so

\[
b = (x^n, px^{n-1}, \ldots, p^u, wq^{v-1} y, \ldots, wy^v).
\]

This composite map depends only on the choice of isomorphism \( \Gamma(\tilde{X}, \mathcal{O}_{\tilde{X}}) \overset{\sim}{\to} A, \) and any element \( a \in A^\times \) induces an automorphism \( A, \) so any quasi-spin structure with the additional relations (2) is actually an induced map. \( \square \)
In the special case when \( r = 2 \) every singularity has \( u = 1 \), and thus \( \sigma_0 = \sigma_1 = \sigma_2 = 0 \). Therefore all quasi-spin structures are actually locally isomorphic to an induced structure.

We can always map \( E(p, q) \rightarrow E(p', q') \) with \( p' = \lambda p, q' = \lambda^{-1} q \). So if \( p^u = \omega q^v \), then \( p'^u = \lambda^u p^u = \lambda^u \omega q^v = \lambda^v \omega q^v \). So \( \omega = \lambda^v w \). And \( w \) has an \( r \)th root in \( k \) if the base ring \( R \) has its residue field \( k \) algebraically closed, hence by the step-by-step method the \( r \)th root of \( w \) will lift to all of \( R \), i.e. if \( \mathfrak{m} \mathfrak{f} = \emptyset \) there is an \( i \in I \) such that \( w + i = \lambda^x \), which implies that \( (\lambda - \sqrt[i]{\lambda^x})^i = w \). Thus we may assume that if the central fibre has residue field \( k = \bar{k} \), then \( w \) can be taken to be one, and, in general, \( R \) has an étale cover on which we can take \( w \) to be one. Accordingly, all \( b \)'s for which the relations \((2)\) hold are determined, étale locally, by \( p, q, u, v, \) and an element of \( A^\times \).

2.3. Behavior of the Cokernel Under Deformation. Because of the condition on the cokernel of the quasi-spin map \( b \), we need to understand the way that the length of the cokernel changes under deformation.

**Definition 2.3.1.** An \( \mathcal{O}_X \)-linear map \( b : \mathcal{E}^\otimes r \rightarrow \omega \) from the \( r \)th tensor power of a rank-one torsion-free sheaf \( \mathcal{E} \) to the canonical bundle of a curve \( X \) over \( k \) is said to have good cokernel if

1. the cokernel is supported on the singularities of \( X \), and
2. for each point \( p \) of the support of the cokernel \( C \) of \( b \)

\[
\text{length}_p C = r - 1.
\]

**Lemma 2.3.2.** If the cokernel of \( b \) is supported on the singular locus of \( \mathcal{E} \), then the property of having good cokernel is stable under generization.

**Proof.** It is enough to consider the case where \( R \) is a complete local ring. \( E \equiv E(p, q) \) is an \( A \)-module, with \( A = R[[x, y]]/(xy - \pi) \), and \( \hat{b} = (b_0, \ldots, b_r) : A^2 \rightarrow A \) is a lifting of the map \( b : \mathcal{E}^\otimes r \rightarrow \omega \). We can assume that on the special fibre \( \mathfrak{m}_{\mathfrak{f}} \in A/\mathfrak{m}^2 \)

\[
\text{equal to } x^2 \mathfrak{b}_{\mathfrak{f}},
\]

with \( \mathfrak{b}_{\mathfrak{f}} \) an invertible element of \( (A/\mathfrak{m})^\times \). Similarly, \( \overline{b}_r \in A/\mathfrak{m}^2 \) with \( \overline{b}_r = y/\mathfrak{b}_r \), and \( \mathfrak{b}_r \) invertible in \( A \).

Now for any map \( R \rightarrow K \) of \( R \) into a field, we have the following possible cases.

1. \( \pi \) does not map to zero in \( K \). In this case, the cokernel is actually zero because \( \text{Spec}(A \otimes K) \) is regular.
2. \( \pi \) maps to zero, but at least one of \( p \) and \( q \) does not. In this case again the cokernel of \( b \) is zero.
3. \( \pi \) and \( p \) and \( q \) all map to zero. This is the only interesting case. We have \( A \otimes K \equiv K[[x, y]]/xy \) and \( \hat{b}_K = (b_0, 0, \ldots, 0, b_r) \).

Now

\[
b_0 = x^i \beta_0 + d_0 \quad \text{and} \quad b_r = y^j \beta_r + d_r
\]

with \( d_0 = x^l \mathfrak{E}_0 \) and \( d_r = y^m \mathfrak{E}_r \), such that \( \mathfrak{E}_0 \) is in \( \mathfrak{m}^2 \) but not in \( (x) \), and \( \mathfrak{E}_r \) is in \( \mathfrak{m}^2 \) but not in \( (y) \). If, on the one hand, \( l \) is larger than \( i - 1 \), then

\[
x^l = b_0/((\beta_0 + x^{l-i} \mathfrak{E}_0)).
\]

The term in the denominator is invertible because \( \beta_0 \) is invertible, and \( d_0 \) is in the maximal ideal \( \mathfrak{m}^2 \). If, on the other hand, \( l \) is less than \( i \), then

\[
x^i = -x^l \mathfrak{E}_0/\beta_0.
\]
and similarly for $y^j$. In either case

$$K[[x, y]]/(xy, x^i, y^j) \cong K < 1, x, x^2, \ldots, x^{i-1}, y, y^2, \ldots, y^{j-1}>$$

surjects onto

$$K[[x, y]]/(xy, b_0, b_r) = A \otimes_R K/im(b).$$

So the length of the cokernel will be either zero (cases 1 and 2) or bounded above by $i + j - 1 = r - 1$ (case 3).

Thus the length of the cokernel can only decrease under generalization, but the degree of $\mathcal{E}_K$ on $\mathcal{X}_K$ must be $(2g - 2)/r = \deg \theta^* \mathcal{E} + \delta$, where $\delta$ is the number of singularities of $\mathcal{X}_K$, and $\theta : \mathcal{X}_K^\prime \to \mathcal{X}_K$ is the normalization of $\mathcal{X}_K$ at the singularities of $\mathcal{E}_K$. On the other hand, since the cokernel of $b$ is supported on the singular set of $\mathcal{E}$, we have that $\theta^* b$ factors

$$\theta^* \mathcal{E}_K^\otimes r \twoheadrightarrow \theta^* \omega \mathcal{X}_K (-\sum u_p p^+ - \sum v_p p^-) \hookrightarrow \theta^* \omega \mathcal{X}_K,$$

where the sum is taken over all $p$ in the singular set of $\mathcal{E}_K$, $\theta^{-1}(p) = \{p^\pm\}$, and for each $p$, $u_p + v_p - 1 = \text{length}_p(\text{coker}(b)) \leq r - 1$. So

$$\deg \mathcal{E}_K = (2g - 2)/r = \left(2g - 2 - \sum_p (u_p + v_p)\right)/r + \delta,$$

which will be strictly greater than $(2g - 2)/r$ unless at each singularity of $\mathcal{E}_K$ the cokernel of $b$ has length $r - 1$. Thus the property of having good cokernel is stable under generalization.

**Proposition 2.3.3.** Given $b : \mathcal{E}^\otimes r \to \omega$ on $f : \mathcal{X} \to T$ (with the cokernel of $b$ supported on the discriminant locus) the set of $t \in T$ such that $b_t$ has good cokernel is open in $T$. In other words, the functor of $T$-schemes

$$F_b(T') = \begin{cases} 
\{1\} & \text{if } b \text{ has good cokernel at every geometric point of } T' \\
\emptyset & \text{if there exists } \bar{t} \in T' \text{ where } b \text{ does not have good cokernel}
\end{cases}$$

is an open subfunctor of the trivial functor $T' \mapsto \{1\}$.

**Proof.** It suffices to show the complement of the set is closed. And since the previous lemma shows this complement is stable under specialization, it suffices to show that the complement is constructible. Let $P_m$ be the property of a geometric point $\bar{t}$ of $T$ that $C := \text{coker}(b)$ has a point of its support over $\bar{t}$ where $C$ has length $m$. The set we want to show is constructible is the set $\mathcal{T}_m := \{t \in T \mid C \text{ has length } m\}$. Actually, the set we are really looking for is

$$\bigcup_{0 < m < r - 1} \mathcal{T}_m,$$

for some very large $N$. $N$ can be taken to be finite because the degree of the sheaves is fixed, and the sum over all points in a given fibre of the length of the cokernel is bounded, and this bound is determined by the number of singular points and the degree. Moreover, the number of singular points is bounded as a function of the genus of the underlying curve, so this number $N$ can be chosen independently of the specific family.

Now to show constructibility we only need to consider one $m$ and one irreducible component of the discriminant locus, say $D_0$, of $\mathcal{X}$ and its image $\rho(D_0) = T_0$, i.e.
we only need to show that $\Sigma_m$ is constructible in $T_0$. And it is enough to assume $T_0$ is reduced and irreducible. Since $D_0$ is proper over $T_0$, the semi-continuity theorem shows that $\Sigma_m$ is constructible for any $m \neq r - 1$.

3. Local-to-Global Calculations

3.1. Log-Structures. The well-behaved quasi-spin curves, i.e. those for which the relations (2) hold locally, also give a compactification of the moduli space of spin curves, and their local moduli spaces are much easier to describe than those of general quasi-spin curves. But in order to formalize the notion of “well-behaved” we need to choose local coordinates for the whole curve in such a way that our constructions make sense globally.

From the deformation theory of stable curves, we know that the complete local ring $\hat{\mathcal{O}}_{\mathcal{X},x}$ over $\hat{\mathcal{O}}_{T,t}$ is of the form $\hat{\mathcal{O}}_{\mathcal{X},x} \cong \hat{\mathcal{O}}_{T,t}[[x,y]]/(xy - \pi)$ for some $\pi \in \hat{\mathcal{O}}_{T,t}$. And on some étale neighborhood $T'$ of $t$, the induced curve $\mathcal{X} \times_T T'$ has the following additional structure: on an étale cover $\mathcal{X}'$ of $\mathcal{X} \times_T T'$, there are sections $x$ and $y$ in $\hat{\mathcal{O}}_{\mathcal{X}'}$ such that

1. $xy = \pi \in \hat{\mathcal{O}}_{T,t}$.
2. The ideal generated by $x$ and $y$ has the discriminant locus of $\mathcal{X}/T$ as its associated closed subscheme.
3. The obvious homomorphism $(\hat{\mathcal{O}}_{T,t}[[x,y]]/(xy - \pi)) \to \hat{\mathcal{O}}_{\mathcal{X}',x}$ induces an isomorphism on the completions $(\hat{\mathcal{O}}_{T,t}[[x,y]]/(xy - \pi)) \xrightarrow{\sim} \hat{\mathcal{O}}_{\mathcal{X}',x}$.

Such a collection of data $(\mathcal{X}', T', x, y, \pi)$ is what we need locally. But this data is not uniquely determined; it is only determined up to the equivalence relation generated by the operations

1. pullback to étale covers.
2. change by units: namely $x' = \hat{u}x, y' = \hat{v}y, \pi' = \hat{w}\pi$ with $\hat{u}, \hat{v}, \hat{w} \in \mathcal{O}_{\mathcal{X}'}^*$, and $\hat{u}\hat{v} = \hat{w} \in \mathcal{O}_{T'}^*$.
3. switching branches: namely, interchanging $x$ and $y$.

A log structure is a way of choosing these local data coherently.

**Definition 3.1.1.** A log structure for $\mathcal{X}/T$ is given by étale covers $\mathcal{X}'$ and $T'$ of $\mathcal{X}$ and $T$

\[
\begin{array}{ccc}
\mathcal{X} & \leftarrow & \mathcal{X}' \\
\downarrow & & \downarrow \\
T & \leftarrow & T'
\end{array}
\]

And for each irreducible component of the singular locus of $\mathcal{X}'$ a choice of $\pi \in \mathcal{O}_{T'}$, and a choice of $x$ and $y$ in $\mathcal{O}_{\mathcal{X}'}$ with the three properties listed above, and with descent data related to the equivalence relation. Namely, on $\mathcal{X}'' = \mathcal{X}' \times_\mathcal{X} \mathcal{X}''$ over $T'' = T' \times_T T'$ with projection maps $pr_1$ and $pr_2$, there are 1-cocycles $u, v$ in $\mathcal{O}_{\mathcal{X}''}$, and $w$ in $\mathcal{O}_{T''}$ such that: $pr_2^*(x) = u pr_1^*(x), pr_2^*(y) = v pr_1^*(y)$, and $uv = w$, and $pr_2^*(\pi) = u pr_1^*(\pi)$ with the cocycle condition that on $\mathcal{X}''' = \mathcal{X}' \times_\mathcal{X} \mathcal{X}' \times_\mathcal{X} \mathcal{X}'$, $u, v, w$ are all compatible with the different projections, i.e. $pr_{12}^*(w) pr_{23}^*(u) = pr_{13}^*(w)$ and so forth.

As in the local case, we also impose the equivalence relation on the log structures generated by pullback to étale covers and by change by units compatible with the descent data; namely, two log structures $(\mathcal{X}', T', x, y, \pi)$ and $(\mathcal{X}', T', x', y', \pi')$
are equivalent if there exist $\tilde{u}, \tilde{v}$ in $O^*_X$, and $\tilde{w}$ in $O^*_T$ such that $x' = \tilde{u}x, y' = \tilde{v}x, \pi' = \tilde{w}\pi$, with $\tilde{u}w = \tilde{w}$ and if $(u, v, w)$ and $(u', v', w')$ are the cocycles corresponding to the two log structures, the units $\tilde{u}, \tilde{v}$ and $\tilde{w}$ must be compatible with them as well, namely $u' = (\text{pr}_1^* (\tilde{u})/\text{pr}_2^* (\tilde{u})) u$, and $v' = (\text{pr}_1^* (\tilde{v})/\text{pr}_2^* (\tilde{v})) v$, and $w' = (\text{pr}_1^* (\tilde{w})/\text{pr}_2^* (\tilde{w})) w$.

As discussed above, given any two log structures with distinguished branches $(x)$ and $(y)$ we will have the relations $x' = \tilde{u}x$ and $y' = \tilde{v}y$, etc. And they will be almost equivalent, namely $\text{pr}_1^* (x)(u' - (\text{pr}_1^* (\tilde{u})/\text{pr}_2^* (\tilde{u})) u) = 0$ and so forth; thus if $\pi$ is not a zero divisor (and hence $x$ and $y$ also) all log structures are equivalent.

In particular, since a versal deformation of the curve has no zero divisors it has a unique log structure.

Switching of branches $(x \mapsto y, y \mapsto x)$ and switching of double points (i.e. interchange the different $\pi_i$) results in an action of the $n$th symmetric group ($n$ is the number of double points) and the group $(\mathbb{Z}/n\mathbb{Z})^\times$ on the log structures. But for our purposes this is not a problem, namely we are interested in expressing $E$ as an $E(p, q)$ and this switching just interchanges $p$ and $q$ or the different $\pi_i$. So given a log structure on a stable curve, we can use the methods of Faltings to describe rank-one torsion-free sheaves, namely any such sheaf $E$ is isomorphic to an $E(p, q)$, and the results on homomorphisms and isomorphisms still hold.

In general the choice of a log structure is unique up to the automorphisms $x \mapsto ux, y \mapsto vy$, and $\pi \mapsto w\pi$, for $uw = w$. But locally this might not be all of the automorphisms of the henselization of the ring $R[x, y]/(xy - \pi)$. In other words, on a curve $C \to B$ we might have different log structures induced by different maps of $B$ to the versal deformation. Nevertheless, we can get around this by considering the problem globally instead. Namely let $S/R = \mathcal{M}(\text{Isom})$ be a presentation of the stack of stable curves, i.e. $S$ is étale over $\mathcal{M}(g)$, and $R$ is the étale equivalence relation $(\text{Isom})$. $R$ is smooth and has no zero divisors, so the two pullbacks to $R$ of the universal curve with its unique log structure over $S$ are canonically isomorphic. Hence any curve over any base has a canonical log structure induced by the unique log structure on the universal curve over $S$.

Note that given a choice of $p$ and $q$ in $O_T$, the descent data for the canonical log structure determine gluing data for the various blowings up. Thus the techniques of Section 4.4 yield a globally-defined semi-stable curve $\mathcal{X}(p, q)$ over $\mathcal{X}$, a rank-one torsion-free sheaf $E(p, q) = P, O(1)$ on $\mathcal{X}$, and a canonical map $b : E \to \mathcal{M}$, for some line bundle $\mathcal{M}$.

### 3.2. Spin Curves and Pure-Spin Curves

We can now define our “good” quasi-spin curves using the canonical log structure.

**Definition 3.2.1.** A spin structure on an arbitrary stable curve $\mathcal{X}/T$ is a pair $(\mathcal{E}, b)$, where $\mathcal{E}$ is relatively torsion-free of rank one with degree $(2g - 2)/\tau$, and $b$ is a morphism of $O_{\mathcal{X}}$-modules

$$b : E \to \omega_{\mathcal{X}/T},$$

which is an isomorphism on the open set where $\mathcal{E}$ is locally free, and such that via the canonical log structure on $\mathcal{X}/T$, the sheaf $\mathcal{E}$ is isomorphic to $E(p, q)$ for some $p$ and $q$ in $O_T$ with $pq = \pi$, and the homomorphism $b : E \to \omega$ is the canonical induced morphism.

An even stronger condition that we can impose on the spin curves is that $p$ and $q$ be such that $p = t^v$ and $q = t^u$ for some $t$ in $O_T$. Spin curves that have
this property will be called \textit{pure spin curves}. Pure-spin curves also compactify the smooth spin curves, and they have an especially well-behaved local structure, as we will see later.

4. Deformation Theory

Given a spin structure \((\bar{X}, \bar{b})\) on a curve \(\bar{X}\) over an Artin local ring \(\bar{R}\) with residue field \(k\), and given a deformation \(R\) of \(\bar{R}\), namely \(\bar{R} = R/I\) with \(I^2 = 0\), we want to study deformations of \((\bar{X}, \bar{E}, \bar{b})\) to spin curves and quasi-spin curves over \(R\).

First, do this locally. For \(\bar{A} := \bar{R}[[x, y]]/(xy - \pi)\) and \((\bar{E}, \bar{b}) = (E(\bar{p}, \bar{q}), \bar{b})\), a spin structure on \(\bar{A}\), we want to lift \(\bar{A}\) and \((\bar{E}, \bar{b})\). But any lift corresponds to a choice of \(\beta, P\) and \(Q\) such that \(P^* = Q^*\) and an isomorphism \((\bar{E}(P, Q), \bar{b}) \rightarrow (E(\bar{p}, \bar{q}), \bar{b})\). Here \(\bar{E}(P, Q)\) is the module \(E(P, Q)/I : E(P, Q) = E(\bar{P}, \bar{Q})\) induced by pulling back \(E(P, Q)\) along the canonical map \(\text{Spec}(R) \rightarrow \text{Spec}(\bar{R})\), and the map \(\beta\) is the canonical map induced from \(\bar{E}\) on \(\bar{E}(P, Q)\). By Faltings’ theorem, isomorphisms over \(\overline{\mathcal{R}}\) are of the form \(\Phi = \left( \begin{array}{cc} \zeta & 0 \\ 0 & \xi \end{array} \right)\) for lifts \(\zeta, \xi \in R^\times\) of \(\bar{\zeta}\) and \(\bar{\xi}\), and such lifts always exist in \(R\). Thus any local spin structure is given simply by a choice of \(P\) and \(Q\) in \(R\) such that \(P = \bar{p}\) and \(Q = \bar{q}\) and a choice of \(\beta\) such that the induced map \(\bar{\beta}\) on \(E(\bar{p}, \bar{q})^\otimes = E^\otimes\) differs from \(\bar{b}\) only by an automorphism of \(E\). In particular, \(\bar{\beta} = \bar{\alpha}r\) with \(\bar{a} \in \bar{A}^\times\), and thus \(\beta\) is uniquely determined by an element \(a \in A^\times\), i.e. \(\beta = a(x^n, px^{n-1}, \ldots, y^n)\). This describes the local deformations.

Any combination of local lifts will patch together into a global one. This is due to the fact that if \(\bar{\sigma}\) is a section of \(\mathcal{O}_{\bar{X}}^\times\) and \(\gamma\) is a section of \(\mathcal{O}_{\bar{X}}\) inducing \(\bar{\gamma}\) in \(\mathcal{O}_{\bar{X}}\) such that \(\bar{\sigma}^\gamma = \bar{\gamma}\), then \(\bar{\sigma}\) lifts uniquely to a section \(\sigma\) of \(\mathcal{O}_X\) such that \(\sigma^\gamma = \gamma\). This is easy to check. We can now use \(\text{fpqc}\) descent to lift \((\bar{X}, \bar{E}, \bar{b})\). Namely, a choice of \(P\) and \(Q\) for each singularity of \(\bar{X}\) still allows numerous choices of \(X\) deforming \(\bar{X}\). And given such an \(X\), we need to construct a pair \((E, b)\) extending \(\bar{E}\) and \(\bar{b}\). On \(U\), the complement of the discriminant locus, the line bundle \(E\) extends uniquely to a line bundle \(\mathcal{E}\) that is an \(r\)th root of \(\omega\). Given an extension \(E(P, Q)\) at each singularity (i.e. \(\text{Spec}(\mathcal{O}_{\mathcal{X}, x_i})\)), we have a covering datum induced by the unique lift of the covering datum on \(\bar{X}\) that makes \(\bar{E}\) an \(r\)th root of \(\bar{\omega}\). This datum is actually a descent datum because of the uniqueness of \(r\)th root lifts. And since all \(\text{fpqc}\) descent data for coherent sheaves are effective, we have the desired \((E, b)\) on \(X\) extending \((\bar{E}, \bar{b})\).

In fact the universal deformation of a spin curve \((X, E, \beta)\) over a field \(k\) is the obvious formal spin curve \((\bar{X}, \bar{E}, \bar{b})\). Here \(\bar{X}\) is the pullback of the universal deformation \(X \rightarrow \mathcal{O}_t[[t_1, \ldots, t_n]]\) of the curve \(X/k\) along the homomorphism

\[
\mathcal{O}_t[[t_1, \ldots, t_n]] \rightarrow \mathcal{O}_t[[\mathcal{P}_1, \mathcal{Q}_1, \ldots, \mathcal{P}_1, \mathcal{Q}_1, t_{i+1}, \ldots, t_n]]/(\mathcal{P}_1^{t_i} - \mathcal{Q}_1^{t_i})
\]

via \(t_i \mapsto P_iQ_i\) for \(i \leq l\). \(u_i\) and \(v_i\) are determined by the map \(\beta\) at each singularity of \(E\) on the central fibre.

A particularly useful corollary of this is the following.

**Proposition 4.0.2.** All quasi-spin structures over a field have a deformation to a smooth spin structure.
5. ISOMORPHISMS

5.1. ISOMORPHISMS OF SPIN STRUCTURES OVER A FIELD. Since spin structures, quasi-spin structures and pure-spin structures are all the same over a field, the study of isomorphisms over a field is fairly simple. Any two spin structures on \( X/k \), say \((\mathcal{E}, b)\) and \((\mathcal{E}', b')\), which are singular at the same points and are the same on \( X^\nu \) \( (X^\nu \overset{\theta^\nu}{\longrightarrow} X) \) is the normalization of \( X \) at the singularities of \( \mathcal{E} \) via \( \theta^\nu \), must be isomorphic on \( X \). For a given \( \mathcal{E} \) on \( X \), any two spin structure maps \( b \) and \( b' \) are the same if and only if \( \theta^\nu b = \theta^\nu b' \), and this is true if and only if \( \text{length}_p(\text{coker}(\theta^\nu b)) = \text{length}_p(\text{coker}(\theta^\nu b')) \) for all \( p \) in the inverse image under \( \theta \) of each singular point.

**Proposition 5.1.1.** Automorphisms of \((X, \mathcal{E}, b)\) that are trivial on \( X \) are of the form \( \gamma = (\zeta_1, \zeta_2, \ldots, \zeta_i) \), where for each \( i \), \( \zeta_i^\nu = 1 \), and each \( \zeta_i \) corresponds to a connected component \( X_i^\nu \) of the curve \( X^\nu \).

In other words, if \( U_r \) is the group of \( \nu \)-th-roots of unity in \( k \), and \( \Gamma(X^\nu) \) is the dual graph of \( X^\nu \), then

\[
\text{Aut}_X(\mathcal{E}, b) = H^0(\Gamma(X^\nu), U_r).
\]

**Proof.** \( \theta : X^\nu \rightarrow X \) makes \( \mathcal{E} \cong \theta_* \theta^\nu \mathcal{E} \), and \( \theta^\nu \) and \( \theta_* \) induce an equivalence of the categories of torsion-free rank-one \( \mathcal{O}_X \)-modules which are singular at the double points normalized by \( \theta \) and invertible sheaves on \( X^\nu \). Therefore, it is enough to study \( \theta^\nu(\mathcal{E}) \). But automorphisms of line bundles on \( X^\nu \) are just given by \( \nu \)-tuples of \( \zeta \in k^* \); moreover, \( \theta^\nu(\mathcal{E})^\nu = 1 \) implies that \( \zeta_i^\nu = 1 \) for all \( i \).

Note that, in general, isomorphisms of \((\mathcal{E}, b)\) over \( X \) must be induced by isomorphisms of \( \theta^\nu \mathcal{E} \), and therefore are always constant on each connected component of \( X^\nu \); namely, at a singularity they are of the form \( \Phi = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix} \), with \( \varphi_+ \) and \( \varphi_- \) in the base field \( k \).

5.2. ISOMORPHISMS OF FAMILIES OF SPIN STRUCTURES. We can also say something about isomorphisms of spin structures in general. We have seen that over a field these are all constant, i.e. at each singularity, any isomorphism of spin structures

\[
\Phi : (E(p, q), b) \rightarrow (E(p', q'), b')
\]

must be of the form

\[
\Phi = \begin{pmatrix} \varphi_+(0) & 0 \\ 0 & \varphi_-(0) \end{pmatrix}.
\]

Using the step-by-step method we can show that this is the case over any complete local ring. We just need to show that if \( m\mathfrak{M} = 0 \) for some ideal \( I \) in \( R \), and if \( \Phi \) is constant over \( R/I \), then \( \Phi \) is constant over \( R \). But this follows because \( \varphi_+ = \varphi_+(0) + xi_+(x) \), for some \( i_+(x) \in IR[[x]] \), and \( \psi_+ = p\varphi_+ = 0 \); and similarly \( \varphi_- = \varphi_-(0) + yi_-(y) \), for some \( i_-(y) \in IR[[y]] \) and \( \psi_- = 0 \). Therefore,

\[
\Phi = \begin{pmatrix} \varphi_+(0) + xi_+(x) & 0 \\ 0 & \varphi_-(0) + yi_-(y) \end{pmatrix},
\]

and \( b' = a'(x^u, p'x^{-1}, \ldots, y^v) \) is mapped to \( b \), namely

\[
b' \circ \Phi^r = a'(x^u \varphi_+^r x^{-1}, p' \varphi_+^r \varphi_-^r \varphi_-^r y^r) = b = a(x^u, x^{-1}, p, \ldots, y^v).
\]
So comparing the $b_0$ and $b_r$ terms, $a'\varphi_+^r = a = a'\varphi_-$, and
\[
\varphi_+^r = (\varphi_+(0) + xi_+(x))^r = \varphi_+(0)^r + rxi_+(x)\varphi_+(0)^{r-1}
\]
\[
= \varphi_-^r = (\varphi_-(0)^r + yi_-(y))^r = \varphi_-(0)^r + ryi_-(y)\varphi_-(0)^{r-1}.
\]

Equating terms with similar powers of $x$ and $y$, and using the fact that $\varphi_+(0)$ and $\varphi_-(0)$ are invertible, as is $r$, we have $i_+(x) = i_-(y) = 0$, and $\Phi$ is constant.

### 5.3. Automorphisms of Families of Quasi-Spin Structures.

Isomorphisms of quasi-spin curves are harder to classify than those of spin curves, but if we limit ourselves to automorphisms, we can completely classify these.

Given a quasi-spin structure $(\mathcal{E}, b)$ on $\mathcal{X}/S$, with $S$ local and complete, we want to study $\text{Aut}_X(\mathcal{E}, b)$. First, we study the local structure, namely, automorphisms of $(E(p, q), b)$ over $A = R[x, y]/pq - xy$, with $b = (b_0, \ldots, b_r)$. Here again, $R$ is a complete local ring with maximal ideal $m$. Now, given $\Phi \in \text{Aut}_A(E, b)$, we know

\[
\Phi = \left( \begin{array}{cc}
\varphi_+ & \psi_+ \\
\psi_- & \varphi_-
\end{array} \right)
\]

with $b \circ \Phi^r = b$, and

\[
* \varphi_+(0)p = \varphi_-(0)p \quad \text{and} \quad \varphi_+(0)q = \varphi_-(0)q
* \varphi_+ = \varphi_+(0) + x\gamma_+; \psi_+ = r\gamma_+, \quad \text{and} \quad \gamma_+ \in R[[x]]
* \varphi_- = \varphi_-(0) + y\gamma_-; \psi_- = q\gamma_-, \quad \text{and} \quad \gamma_- \in R[[y]].
\]

Now $b = (b_0, \ldots, b_r) \equiv (\beta_0x^r, 0, 0, \ldots, 0, \beta_r y^r)$ mod $m$, $(i+j = r)$ and $\beta_r, \beta_0 \in k^\ast$, and $\Phi \equiv \left( \begin{array}{cc} \varphi_+ & 0 \\
0 & \varphi_-
\end{array} \right)$. By the results of the previous section, $\varphi^r_+ \equiv 1 \equiv \varphi^r_-$. In fact, this will hold for the whole family, i.e. we can replace congruence with equality.

**Proposition 5.3.1.** \(\Phi = \left( \begin{array}{cc} \varphi_+ & 0 \\
0 & \varphi_-
\end{array} \right)\) with $\varphi^r_+ = 1, \varphi^r_- = 1$.

**Proof.** Using the step-by-step method we can assume the claim is true mod$I$ for some ideal $I$ with $mI = 0$. So $\varphi^r_+ = 1 + i$, which implies that $(\varphi_+ - \frac{1}{r\varphi_+})^r = 1$. So $\varphi_+ = \zeta + i_+$ for some $i_+$ in $I \cdot R[[x]]$, with $\zeta^r = 1$. Similarly, $\varphi_- = \xi + i_-$ for some $i_-$ in $I \cdot R[[y]]$, with $\xi^r = 1$. Thus $\Phi = \left( \begin{array}{cc} \varphi_+ & 0 \\
0 & \varphi_-
\end{array} \right)$ and $b \circ \Phi^r = b$ implies that

\[
b_0 = b_0(\zeta + i) = b_0(1 + r\zeta^{r-1}i_+), \quad \text{and} \quad b_r = b_r(1 + r\zeta^{r-1}i_-).
\]

This implies that $b_0r\zeta^{r-1}i_+ = 0 = b_r r\zeta^{r-1}i_-$. But $b_0 \equiv \overline{x^r\beta_0} \mod m$, with $\overline{\beta_0} \in k^\ast$, and since $i_+$ and $i_-$ annihilate $m$, this implies that $b_0r\zeta^{r-1}i_+ = x^i \beta_0(r\zeta^{r-1}i_+)$ for some $\beta_0 \in R^\ast$ lifting $\overline{\beta_0}$, and similarly for $b_r$. Since $1/r \in R$, $i_\pm = 0$, and $\varphi^r_+ = \varphi^r_- = 1$. Moreover, $\varphi_+$ and $\varphi_-$ are in $R$. \(\square\)

Now note also that $\xi p = \zeta p$, $\xi q = \zeta q$. So $p(\xi - \zeta) = q(\xi - \zeta) = 0$. But if $\gamma := \xi - \zeta$, then $(\xi + \gamma)^r = 1$, which implies that

\[
r\gamma\xi^{r-1} + \left( \begin{array}{c} r \\
r! \end{array} \right) \gamma^{2r-2} = 0.
\]

And if $(\gamma)$ is a proper ideal, then mod$(\gamma^2)$ we get $r\gamma\xi^{r-1} \equiv 0$, i.e. if $1/r \in R$, $\gamma \in (\gamma^2)$, which implies that $\gamma \in \bigcap \gamma^r \subseteq \bigcap \gamma^m = 0$. This implies that $\gamma = 0$.

So either

1. $\gamma$ is invertible, hence $p$ and $q$ are zero, or
2. $\gamma$ is zero and $\varphi_+ = \varphi_-$.  

So at each singularity with at least one of $p$ and $q$ not zero, $\text{Aut}(E(p,q),b) = U_r = \{\zeta \in R^r | \zeta^r = 1\}$. And thus all automorphisms of $(E,b)$ are also in $U_r$ if no singularities are of type $E(0,0)$. A singularity of type $(0,0)$ has automorphisms of type $(\xi,\zeta) \in U_r \times U_r$. Normalizing $X$ at each singularity of type $(0,0)$ to get $X'$ shows that $\text{Aut}(E,b)$ will be of type $(\xi_1, \ldots, \xi_m)$, where $m$ is equal to the number of connected components of $X'$.

5.4. **Properties of the Isom Functor.** For any two quasi-coherent sheaves $\mathcal{E}$ and $\mathcal{E}'$ on a curve $X/B$ the functors $\text{Hom}(\mathcal{E}, \mathcal{E}')$ and $\text{Isom}(\mathcal{E}, \mathcal{E}')$ are representable (cf. [13] 7.7.8 and 7.7.9) and [19]. For the $B$-scheme $V$ and map $\Phi : \mathcal{E}_X \to \mathcal{E}'_X$ on $X_V$ which represent the functor $\text{Isom}(\mathcal{E}, \mathcal{E}')$, the condition that $\Phi'$ commutes with $b$ and $b'$ is clearly an open condition, and thus is representable over $V$. Moreover, the scheme representing the functor $T \mapsto \text{Isom}_{X_T}((\mathcal{E}_T, b_T), (\mathcal{E}'_T, b'_T))$, is an open subsheaf of $\text{Hom}_{X_T}((\mathcal{E}_T, \mathcal{E}'_T))$, which is quasi-projective of finite type. Thus we have the following proposition.

**Proposition 5.4.1.** For any two quasi-spin structures $(\mathcal{E}, b)$ and $(\mathcal{E}', b')$ over a stable curve $X/B$, the functor $T \mapsto \text{Isom}_{X_T}((\mathcal{E}_T, b_T), (\mathcal{E}'_T, b'_T))$ is represented by a quasi-projective $B$-scheme of finite type.

The proof and proposition are also valid for automorphisms of spin structures and pure-spin structures. Moreover, because the $\text{Isom}$ functor for stable curves over $S$ is representable by a quasi-projective $S$-scheme of finite type, we actually have that for any two (quasi/pure) spin curves $\mathcal{G}/\mathcal{S}$ and $\mathcal{G}'/\mathcal{S}'$ the functor $T \mapsto \text{Isom}_{T \times T'}(\mathcal{G}, \mathcal{G}')$ is also representable by a quasi-projective $S$-scheme of finite type.

Not only is Isom representable, it is also unramified and finite, as the next two propositions show.

**Proposition 5.4.2.** For any two quasi-spin curves (or spin curves or pure-spin curves) $\mathcal{G} = (\mathcal{X}, \mathcal{E}, b)/\mathcal{S}$ and $\mathcal{G}' = (\mathcal{X}', \mathcal{E}', b')/\mathcal{S}'$, the scheme $\text{Isom}_{T \times T'}(pr_1^* \mathcal{G}, pr_2^* \mathcal{G}')$ is unramified over $T \times T'$.

**Proof.** It suffices to show that for a ring $R$ with square-zero ideal $I$ and for any two quasi-spin structures $(\mathcal{E}, b)$ and $(\mathcal{E}', b')$ on a stable curve $X$ over $R$ with two isomorphisms from $(\mathcal{E}, b)$ to $(\mathcal{E}', b')$ which agree over $R = R/I$, the two isomorphisms must then agree over $R$. (We do not need to consider isomorphisms of the underlying curve because the $\text{Isom}$ functor for stable curves is unramified.) Since Isom is a principal homogeneous $\text{Aut}$-space, we are reduced to showing that any automorphism of $(\mathcal{E}, b)$ which is the identity over $R$ is the identity over $R$. But this follows easily from the fact that all automorphisms of quasi-spin curves are constant and have $r^\text{th}$ power equal to the identity. Therefore, $\text{Isom}$ is unramified.

Since Isom is of finite type and unramified, it is quasi-finite, so we only need to check that it is proper to see that it is finite.

**Proposition 5.4.3.** For any two (quasi, pure) spin curves $\mathcal{G} = (\mathcal{X}, \mathcal{E}, b)/\mathcal{S}$ and $\mathcal{G}' = (\mathcal{X}', \mathcal{E}', b')/\mathcal{S}'$, the scheme $\text{Isom}_{T \times T'}(pr_1^* \mathcal{G}, pr_2^* \mathcal{G}')$ is proper over $T \times T'$.

**Proof.** We use the valuative criterion. We must show that if we are given two spin curves, quasi-spin curves, or pure-spin curves, $\mathcal{G} = (\mathcal{X}, \mathcal{E}, b)$ and $\mathcal{G}' = (\mathcal{X}', \mathcal{E}', b')$ both over $\text{Spec}(R)$, where $R$ is a discrete valuation ring, and given an isomorphism $\Phi_\eta : \mathcal{G}_\eta \to \mathcal{G}'_\eta$ defined on the generic fibres, then we can always extend $\Phi_\eta$ to an isomorphism $\Phi$ over all of $\text{Spec}(R)$.
We can also assume that $R$ is complete, and since for stable curves the $\text{ISOM}$ functor is proper, we can assume that $X = X'$. Now let $Y$ be the $fpqc$ cover of $X$ given by $Y = U \amalg \left( \prod_p \text{Spec}(O_{X,p}) \right)$, with the union being taken over all closed points $p$ of the singular locus of the special fibre of $X$, and $U$ the smooth locus of $X$. If $\Phi_\eta$ extends to all of $Y$, then it will in fact be constant on all intersections $\text{Spec}(O_{X,p}) \times_X U$, and these constant isomorphisms are uniquely determined by $\Phi_\eta$, hence $\Phi_Y$ will descend to an extension of $\Phi_\eta$ on $X$.

Thus we only need to consider the local situation; namely, about a singular point of the special fibre. This is the case where

$$A = R[[x,y]]/(xy - \pi), \quad \mathcal{E} = E(p, q) \quad \text{and} \quad \mathcal{E}' = E(p', q')$$

with $pq = p'q' = \pi$. And we need to show that an isomorphism $\Phi_\eta$ on the fibre over the field of fractions $K$ of $R$ extends to an isomorphism on all of $A$. $\Phi_\eta$ lifts to a map $\tilde{\Phi}_\eta : (A \otimes_R K)^{\oplus 2} \to (A \otimes_R K)^{\oplus 2}$, which induces the isomorphism $\Phi_\eta : E(p, q) \otimes_R K \to E(p', q') \otimes_R K$. Since $\Phi_\eta$ is constant, $\tilde{\Phi}_\eta$ is given as a matrix $\tilde{\Phi}_\eta = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix}$, with $\varphi_{\pm} \in K$. It suffices to show that $\varphi_+$ and $\varphi_-$ are actually in $R$. But to be an isomorphism, $\tilde{\Phi}_\eta$ must be such that $\tilde{b}' \circ \tilde{\Phi}_\eta^r = \tilde{b}$.

And since $\tilde{b} = (b_0, b_1, \ldots, b_r)$ and $\tilde{b}' = (b'_0, b'_1, \ldots, b'_r)$, we have $b'_0 \varphi_+ = b_0$ and $b'_r \varphi_- = b_r$. But as we have seen, $b_0$ and $b'_0$ are both invertible in $A[1/x]$, hence in $A[1/x]$ the constant $\varphi_+ = b_0/b'_0 \in (R[[x,y]]/(xy - \pi))[1/x]$, and thus $\varphi_+ \in R$, similarly for $\varphi_-$. And $R$ is normal, hence the $\varphi_{\pm}$ are in $R$. And so $\Phi_\eta$ extends to all of $\text{Spec}(A)$, and thus to all of $X$.

6. Construction of the Stacks

Fix $S$ to be a scheme of finite type over a field or over an excellent Dedekind domain with $r$ invertible in $S$. These conditions are necessary for us to be able to use Faltings’ theorem from Section 1.6 and to be able to use the standard theorems on algebraic stacks (see [11]).

We have two main functors to consider, namely $Q\text{SPIN}_{r,g}$ and $\overline{\text{SPIN}}_{r,g}$. $Q\text{SPIN}_{r,g}$ is the étale sheafification of the functor taking an $S$-scheme $T$ to the set of isomorphism classes of quasi-spin curves over $T$. And $\overline{\text{SPIN}}_{r,g}$ is the subfunctor of $Q\text{SPIN}_{r,g}$ induced by restricting to spin curves instead of quasi-spin curves. Note that for a quasi-spin structure the property of being a spin structure is local on the curve in the étale topology; therefore, the property of being a spin structure is independent of the choice of log structure.

We also will consider a third functor over $\overline{\text{SPIN}}_{r,g}$ given by restricting to pure-spin curves, namely those curves which, locally in the étale topology, have the form $E(p, q)$ with $p = t^v$, $q = t^u$, for some $t$ in the base, and $u + v = r$. In particular, this means that $\pi$ is $t^v$. Up to étale covers this condition is also independent of the particular choice of log structure and of the particular choice of $p$ and $q$. We call this functor $\text{PURE}_{r,g}$. Of course, spin structures, quasi-spin structures, and pure-spin structures are all the same thing if the underlying curve is smooth. And quasi-spin structures over a reduced base (or if $\pi$ is not a zero divisor) are actually spin structures.
The main result of this section is that $\overline{\text{QSpin}}_{r,g}$, $\text{Spin}_{r,g}$, and $\text{Pure}_{r,g}$ are all separated algebraic stacks, locally of finite type over $\mathcal{M}_g$, the moduli space of stable curves, and $\text{Spin}_{r,g}$, the moduli of smooth spin curves, is dense in each of these.

The fact that $\text{Spin}_{r,g}$ is dense in the stacks follows from Proposition 4.0.2. To prove that these stacks are algebraic, we need to do the following (see, for example, [10, pp. 15–23], or [19]):

1. Prove that the functors are limit preserving.
2. Provide a smooth cover $U$ of the stack.
3. Prove that for a fixed family of curves $X$ over $T$ the functor $\text{Isom}_{U_T \times U_T} (pr_1^*, pr_2^*)$ is representable by a scheme (it is clearly a groupoid).
4. Prove that the stacks are separated by showing that $\text{Isom}_{U_T \times U_T} (pr_1^*, pr_2^*)$ is actually proper and finite over $U_T \times U_T$.

The last two conditions follow from the results of Section 5.4. For the first two we begin by considering the stack $\overline{\text{QSpin}}_{r,g}$. Many results on the other two stacks follow relatively easily from this case. The fact that the stack $\overline{\text{QSpin}}_{r,g}$ is limit preserving is a straightforward consequence of the following theorem of Grothendieck and the fact that the condition on the length of the cokernel is an open condition (c.f. Proposition 2.3.3), hence limit preserving.

**Theorem 6.0.4 ([14, 8.5.2]).** Given a quasi-compact and quasi-separated scheme $S_0$, and given a projective system $\{S_\gamma\}$ of $S_0$-schemes, relatively affine over $S_0$, and quasi-coherent $O_{S_\gamma}$-modules $F_\gamma$ and $G_\gamma$, with $F_\gamma$ of finite presentation, the canonical homomorphism of groups

$$\lim_{\rightarrow} \text{Hom}_{S_\gamma} (F_\gamma, G_\gamma) \to \text{Hom}_S (F, G)$$

is an isomorphism. Here $S$, $F$, and $G$ are the obvious limit objects.

All that remains is condition (2), i.e. to provide a smooth cover.

### 6.1. A Smooth Cover of $\overline{\text{QSpin}}_{r,g}$

**Proposition 6.1.1.** Given a curve $X/B$ and an integer $N$, sufficiently large, the functor taking a $B$-scheme $T$ to the set of all triples $(E, b, (e_1, \ldots, e_n))$ where $(E, b)$ is a quasi-spin structure on $X_T$, and $(e_1, \ldots, e_n)$ is a basis for the module $\text{Hom}_{\mathcal{O}_{X_T}} (E_T \otimes \omega_X, \omega_X)$ on $X_T$ is representable.

**Proof.** Any quasi-spin structure $(E, b)$ on $X$ must have total degree $= \frac{1}{r} (2g - 2)$, and on its normalization $\theta : \tilde{X} \to X$

$$\theta^* \mathcal{E} \cong \theta^* \omega_X/k \left( - \sum (u_ip_i + v_ip_i) \right),$$

where the sum is taken over all singularities $\{p_i\}$ of $E$ and $\{p_i, p_i^\perp\}$ are the inverse images of $p_i$ via $\theta$. In particular, for any given irreducible component $X_j$ of $X$, we have

$$\text{deg}_{X_j} (\theta^* \mathcal{E}) \geq \frac{1}{r} (\text{deg}_{X_j} (\omega_X/k) - r\delta_j),$$

where $\delta_j$ is the number of singularities of $X$ in $X_j$. Since $\text{deg}_{X_j} (\omega_X/k)$ is always positive, and since the total number of singularities in a stable curve of genus $g$ is bounded by $3g - 3$, we have

$$\text{deg}_{X_j} \theta^* \mathcal{E} \geq 1 - \delta_i \geq 2 - 3g.$$
Proposition 6.1.2. The scheme moduli stack of stable curves, then $U$ all quasi-spin structures on $X$ which generate $\mathcal{E} \otimes \mathcal{O}(m)$ has vanishing higher cohomology for all $m \geq m_0$. Fix, once and for all, an integer $N$ large enough so that $\omega^N$ is very ample and $\mathcal{E} \otimes \omega^N$ has all the desired properties. Now we can represent torsion-free rank-one sheaves with bounded degree on each component by a subscheme of $\text{Quot} \mathcal{O}^n_{X/S}$ for some $n$, i.e. there exists $U_1 \rightarrow \text{Quot} \mathcal{O}^n_{X/S}$ which represents the functor

$$T \mapsto \{ \mathcal{F}, (e_1 \ldots e_n) \},$$

where $\mathcal{F}$ is a rank-one, torsion-free sheaf on $X_T$ with bounded degree on each component, and $(e_1 \ldots e_n)$ is a basis of $\Gamma(X_T, \mathcal{F} \otimes \omega^N)$ for $\omega^N$ sufficiently ample. So over $X_1/U_1$ there is a universal pair $(\mathcal{E}, (e_1 \ldots e_n))$. And to represent maps $\mathcal{E} \otimes r \rightarrow \omega_{X_1/U_1}$, take

$$V := \mathcal{V}(\mathcal{E} \otimes H \otimes \omega^N) := \text{Spec}_X(\text{Sym}_X(\mathcal{E} \otimes H \otimes \omega^N)),$$

So that $\text{Hom}_{X_1}(Y, V) = \text{Hom}_T(\mathcal{E} \otimes T, \omega_Y)$. So letting $V_T := V \times_{X_1} V_T = V \times_{U_1} T$, we get that

$$\text{Hom}_{X_1}(X_T, V) = \text{Hom}_{X_T}(X_T, V_T) = (\prod_{X/S} V/X)(T)$$

is the functor we want, and it is representable because $X$ is flat and projective over $S$ (see [12]).

Now we have a universal triple $\mathcal{E}, (e_1 \ldots e_n), b : \mathcal{E} \otimes r \rightarrow \omega$ on $X_2/U_2$ representing all maps $\mathcal{E} \otimes r \rightarrow \omega$, and the additional condition that the cokernel of $b$ is supported on the singular locus of $X_2$ is also representable; namely, it is just the condition that $b$ is an isomorphism on the complement of the discriminant locus, and this is an open condition. Finally, we need to represent the condition on the cokernel, but this condition is open on the base, as proved in Proposition 2.3.3. 

In general, for an arbitrary stable curve $X/T$ we have represented by some scheme $U$ all quasi-spin structures $(\mathcal{E}, b)$ on $X_U$ such that $\mathcal{E} \otimes \omega^N$ can be expressed as a quotient of $\mathcal{O}_X^n$, together with a basis for the module $\Gamma(X, \mathcal{E} \otimes \omega^N)$. Moreover, at any closed point of $T$ all quasi-spin structures on $X \times_T \text{Spec} \mathcal{O}_{T,t}$ can be expressed as such a quotient. Therefore, at each point $u$ of $U$ the complete local ring $\mathcal{O}_{U,u}$ is a versal deformation of the quasi-spin structure induced by $u$. In particular, if the curve $X$ we begin with is the universal curve over an étale cover $T$ of $\mathcal{M}_g$, the moduli stack of stable curves, then $U$ is a cover of the stack $\text{QSpin}_{r,g}$.

Proposition 6.1.2. The scheme $U$, which represents all quasi-spin structures $(\mathcal{E}, b)$ on the universal curve $X/T$ together with a choice $(e_1, e_2, \ldots, e_n)$ of global sections which generate $\mathcal{E} \otimes \omega^N$, is smooth over the stack of quasi-spin structures on the universal curve.

Proof. We have to show that if an affine $T$-scheme $Y = \text{Spec}(B)$ has a square-zero ideal $I \subseteq B$ and a quasi-spin structure $(\mathcal{E}, b)$ on $X \times_T Y$ such that $(\mathcal{E}, b)$ restricted to $Y_0 = \text{Spec}(B/I)$ has a basis $(e_1, \ldots, e_n)$ for $\Gamma(X \times_T Y_0, \mathcal{E} \otimes \omega^N)$, then $(e_1, \ldots, e_n)$ lifts to a basis of $\Gamma(X \times_T Y, \mathcal{E} \otimes \omega^N)$ on $Y$. Namely, it suffices to show that if $pr : X \times_T Y \rightarrow Y$ makes $pr_*(\mathcal{E} \otimes \omega^N)$ free on $Y_0$, then the locally free sheaf
pr_*(E \otimes \omega^N) is also free of the same rank as pr_*(\bar{E} \otimes \bar{\omega}^N). But this is clear because E \otimes \omega^N commutes with base change, and the exponential sequence

\[ 0 \to M_n(I) \to GL_n(O_Y) \to GL_n(O_{Y_0}) \to 0 \]

shows that the kernel of the homomorphism \( H^1(|Y_0|, GL_n(O_Y)) \to H^1(|Y_0|, GL_n(O_{Y_0})) \) is \( H^1 \) of the coherent sheaf \( M_n(I) \) on an affine scheme, hence is zero. Therefore, any rank \( n \), locally free sheaf on \( Y \) which restricts to a free sheaf on \( Y_0 \) must be free already.

This completes the proof that the stack \( QSpin_{r,g} \) is algebraic.

6.2. \( \overline{Spin}_{r,g} \) and \( Pure_{r,g} \). To construct a smooth cover of \( \overline{Spin}_{r,g} \), we first take an arbitrary smooth cover of \( QSpin_{r,g} \), say \( S : U \to QSpin_{r,g} \), together with its canonical log structure. We can also assume that \( U \) is affine. Thus all of our previous descriptions of spin structures apply, and in particular, the homomorphism \( b : E^{\otimes r} \to \omega \) is given (étale locally on \( X/U \)) as \( b = (b_0, \ldots, b_r) \). Also, we can still write the \( b_i \) as power series

\[ b_i = \sum_{n \geq 0} b_{i,n} x^n + \sum_{m > 0} b_{i,-m} y^m \]

with the same relations as before, and in particular, up to suitable base extension and isomorphism of \( E(p, q) \),

\[ p^u = q^v \]

and

\[ b_{0,i} = \pi^{u-i} b_{r+i-r} + \sigma_i \text{ for } 0 < i \leq u \quad \text{and} \quad b_{r,i} = \pi^{1-u} b_{0,i} + \sigma_i \text{ for } u < i < r. \]

Moreover, \( b \) is a spin structure if and only if \( \sigma_i = 0 \) for all \( i \). So the closed subscheme \( V \) defined by the ideal generated by the \( \sigma_i \) actually represents the condition that \( b \) is a spin-structure.

It is clear that the spin curve \( \mathcal{S}_{\mathfrak{g}} = (\mathcal{X}, \mathcal{E}, b)_{\mathfrak{g}} \) over \( V \) makes \( V \) a cover of \( \overline{Spin}_{r,g} \). To see that it is smooth over \( \overline{Spin}_{r,g} \), note that for any spin curve \( \mathcal{Z}/\mathcal{T} \) the scheme \( ISOM_{QSpin_{r,g}}(\mathcal{S}/\mathcal{U}, \mathcal{T}/\mathcal{T}) \) is isomorphic to \( ISOM_{\overline{Spin}_{r,g}}(\mathcal{S}/\mathcal{U}, \mathcal{T}/\mathcal{T}) \), which is isomorphic to \( ISOM_{Spin_{r,g}}(\mathcal{S}_{\mathfrak{g}}/\mathcal{U}, \mathcal{T}/\mathcal{T}) \). Hence smoothness of \( U \) over \( QSpin_{r,g} \) implies smoothness of \( V \) over \( \overline{Spin}_{r,g} \).

Alternately, we can consider, as in the construction of the versal deformation of \( QSpin_{r,g} \), a curve \( \mathcal{X}/T \) and a relatively torsion-free sheaf \( \mathcal{E} \), so that the pair \( (\mathcal{X}/T, \mathcal{E}) \) is versal for stable curves with rank-one, torsion-free sheaves with bounded degree on each component. Taking the canonical log structure and constructing the canonical induced map \( \mathcal{E} \to \rho_* M \), we can take the scheme representing the property that \( M \) is isomorphic to \( \omega_{X/T} \) to be our cover. Since the property of being spin is independent of choice of log structure, this is a cover of \( \overline{Spin}_{r,g} \). Moreover, because it represents all spin structure maps for \( (\mathcal{X}/T, \mathcal{E}) \), it is smooth over \( \overline{Spin}_{r,g} \).

The representability of \( ISOM \), as well as the other properties (finite and unramified), all follow from the case of \( QSpin_{r,g} \), hence \( \overline{Spin}_{r,g} \) is an algebraic stack, locally of finite type over \( S \).

Now to construct the stack \( Pure_{r,g} \), take, as above, the cover \( V \) of \( \overline{Spin}_{r,g} \) together with its canonical log-structure on the universal curve \( \mathcal{X}_V \) and isomorphisms \( \mathcal{E} \cong E(p, q) \) for \( p \) and \( q \) in \( O_V \). The condition that \( \mathcal{S}_{\mathfrak{g}} \) is pure is representable by the relatively affine \( V \)-scheme \( W := \text{Spec}_V(O_V[\tau]/(p - \tau^u, q - \tau^u)) \). Again it is
easy to verify that $W$ is a smooth cover of $\text{Pure}_{r,g}$, and that $\text{Pure}_{r,g}$ is algebraic. We have proved the following theorem.

**Theorem 6.2.1.** $\mathcal{QSpin}_{r,g}$, $\mathcal{Spin}_{r,g}$, and $\text{Pure}_{r,g}$ all form separated algebraic stacks of finite type over $\mathcal{M}_g$, and $\mathcal{Spin}_{r,g}$ is dense in each of these.

6.3. **Singularities and Smoothness of $\mathcal{Spin}_{r,g}$ and $\text{Pure}_{r,g}$.** The deformation theory done in Section 4 completely describes the local structure of $\mathcal{Spin}_{r,g}$, and in fact it shows that $\mathcal{Spin}_{r,g}$ is relatively Gorenstein (i.e. $\mathcal{Spin}_{r,g}$ is Gorenstein if the base $S$ is), namely it is enough to check the completion of the stalks for an étale cover, (c.f. [21, Theorem 18.3]), and these are of the form

$$\hat{\mathcal{O}}_{S,s}[[P_1, Q_1, \ldots, P_l, Q_l, t_1, t_2, \ldots, t_n]]/(P_i^{m_i} - Q_i^{u_i}).$$

Now it is enough to check the quotient

$$\hat{\mathcal{O}}_{S,s}/(a_1, a_2, \ldots, a_m),$$

where $\{a_i\}$ are any $\hat{\mathcal{O}}_{S,s}$-regular sequence. And taking $a_1 = P_1, a_2 = P_2, \ldots, a_l = P_l, a_{l+1} = t_{l+1}, \ldots, a_n = t_n$, we are reduced to the case of $\hat{\mathcal{O}}_{S,s}[[Q_1, Q_2, \ldots, Q_l]]/(Q_i^{v_i})$. But this is Gorenstein because $\hat{\mathcal{O}}_{S,s}[[Q_1, Q_2, \ldots, Q_l]]$ is, and the ring in question is just $\hat{\mathcal{O}}_{S,s}[[Q_1, \ldots, Q_l]]$ modulo the regular sequence $(Q_i^{v_i})$.

$\text{Pure}_{r,g}$ provides a resolution of the singularities of $\mathcal{Spin}_{r,g}$; namely, $\text{Pure}_{r,g}$ is smooth over $S$ because the completion of any of its local rings is of the form

$$\hat{\mathcal{O}}_{S,s}[[\tau_1, \tau_2, \ldots, \tau_1, t_{l+1}, \ldots, t_n]],$$

which is smooth over $\hat{\mathcal{O}}_{S,s}$.

7. **Compactness**

The goal of this section is to prove the properness of the stacks $\mathcal{QSpin}_{r,g}$, $\mathcal{Spin}_{r,g}$, and $\text{Pure}_{r,g}$. This is accomplished by studying the boundary of these stacks, i.e. the degeneration of smooth spin curves into spin structures on stable curves. To prove that the stacks are proper we will use the valuative criterion and the fact that smooth spin curves are dense (c.f. Proposition 4.0.2) to justify checking the valuative criterion only in the case that the generic fibre is smooth (c.f. [6, pg. 109]).

7.1. **Extending Spin-Structures and Line Bundles.** Given a complete, discrete valuation ring $R$ with field of quotients $K$, and a $K$-valued point $\eta$ of $\mathcal{QSpin}_{r,g}$, corresponding to $\mathcal{S}_\eta = (\mathcal{X}_\eta, E_\eta, b_\eta)$, with $\mathcal{X}_\eta$ smooth over $K$, we need to show that (up to finite extension of $K$) there exists a quasi-spin curve $\mathcal{G}$ over $R$, extending $\mathcal{S}_\eta$. To this end, we construct a semi-stable curve and line bundle which will give the desired extension when contracted to its stable model, as in Section 1.4.

To begin, since $\mathcal{M}_g$ is proper, there is a stable curve $\mathcal{X}$ extending $\mathcal{X}_\eta$ over $R$. Take a uniformizing parameter $t \in R$ and map $R$ to itself via $t \mapsto t'$. Pulling back $\mathcal{X}$ along this map yields another (singular) curve $\mathcal{X}_r$. Resolving the singularities of $\mathcal{X}_r$ by blowing up yields a semi-stable curve $\mathcal{X}$ with generic fibre $\mathcal{X}_\eta$ (up to a finite extension of $K$) and special fibre having chains of $n_i - 1$ exceptional curves over each singularity of $\mathcal{X}_r$. Here $n_i$ is the order of the corresponding singularity of $\mathcal{X}$, namely $\mathcal{X}$ has local equation $R[[x, y]]/(xy - t^{n_i})$.

Now, since $\mathcal{X}$ is regular, any line bundle on the generic fibre will extend (but not uniquely) to the entire curve. In particular, there is some line bundle $L$ on $\mathcal{X}$
which extends $\mathcal{L}_n$. It is well-known that in such a case, any two line bundles which agree on the generic fibre differ only by Cartier divisors supported on the special fibre. In other words, if $\mathcal{M}_n \cong \mathcal{N}_n$ then $\mathcal{M} \cong \mathcal{N} \otimes \mathcal{O}(\sum a_i X_i)$, where $X_i$ are the irreducible components of the special fibre of $\tilde{X}$, and $a_i$ are integers. In our case, therefore, $\mathcal{L}^{\otimes r} \cong \omega_{\tilde{X}} \otimes \mathcal{O}(\sum a_i X_i)$ for some integers $a_i$.

Of course, if $\mathcal{L}$ extends $\mathcal{L}_n$ then any line bundle of the form $\mathcal{L} \otimes \mathcal{O}(\sum b_i X_i)$ also extends $\mathcal{L}_n$. The following results show that there is a choice $\omega_{\tilde{X}} \otimes \mathcal{O}(\sum a_i X_i)$ for every $\mathcal{L}'$, which extends $\mathcal{L}_n$. Suppose that $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}(\sum b_i X_i)$ is a line bundle with degree zero on all but one exceptional curve per chain, has degree one on the one remaining exceptional curve, and there exists an $\mathcal{O}_{\tilde{X}}$-module homomorphism $\beta : \mathcal{L}^{\otimes r} \to \omega_{\tilde{X}}$, which is an isomorphism everywhere except on the exceptional curves where $\mathcal{L}'$ has degree one. Contracting all the exceptional curves of $\tilde{X}$ induces a spin curve on $X_r$, and hence an $R$-valued point of $\text{QSpin}_{r,g}$ extending $\mathcal{S}_n$.

**Proposition 7.1.1.** Given $\mathcal{L}$ on $\tilde{X}$ such that $\mathcal{L}^{\otimes r} \cong \omega_{\tilde{X}} \otimes \mathcal{O}(\sum a_i X_i)$, the coefficients $a_i$ which correspond to non-exceptional components of the special fibre can all be assumed divisible by $r$. In particular, the line bundle $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O} \left( \frac{-1}{r} \sum_{X_i \text{not exceptional}} a_i X_i \right)$ has $\mathcal{L'}^{\otimes r} \cong \omega \otimes \mathcal{O}(\sum e_j E_j)$ where all the $E_j$ are exceptional curves.

**Proof.** Basic intersection theory shows that, for any curve $X_j$, the degree of $\mathcal{O}(\sum a_i X_i)$ on $X_j$ is $-a_j + \sum a_i \delta_{ij}$, where $\delta_{ij}$ is the number of points in the intersection of $X_j$ with the rest of the special fibre, and $\delta_{ij}$ is the number of points in the intersection of $X_i$ and $X_j$. Now, on any given exceptional curve $E$ in a chain, with $E$ intersecting only two curves $C_1$ and $C_2$, we have $\deg_{E}(\omega(\sum a_i X_i)) = \deg_{E}(\mathcal{O}(\sum a_i X_i)) = -2e + c_1 + c_2$, where $e, c_1$, and $c_2$ are the coefficients in the sum $\sum a_i X_i$ of $E, C_1$, and $C_2$ respectively. Moreover, $\deg_{X_i}(\omega(\sum a_i X_i)) = r \deg_{X_i} \mathcal{L} \equiv 0 \pmod{r}$ for every $X_i$. So, in particular, $c_1 + c_2 \equiv 2e \pmod{r}$. Now, given a chain of exceptional curves $E_1, \ldots, E_{nr-1}$, and the two non-exceptional curves $C$ and $D$ that the chain joins, if their associated coefficients are $e_1, e_2, \ldots, e_{nr-1}, c$, and $d$, respectively, then we must have $e_2 \equiv 2e_1 - c, e_3 \equiv 2e_2 - e_1 \equiv 3e_1 - 2e$ and $e_i \equiv ie_1 - (i-1)c$, so that $e_{nr-1} \equiv (nr-1)e_1 - (nr-2)c$ and $d \equiv nre_1 - (nr-1)c \equiv c$. Therefore, since the special fibre is connected, and since all of the non-exceptional curves are joined by exceptional chains, all of the coefficients of the non-exceptional curves are congruent to $c$ for some choice of $c$. But since the divisor $(\sum X_i)$ is trivial, we can assume that at least one of the coefficients of a non-exceptional curve is zero, hence all of them are congruent to zero $\pmod{r}$, and thus $\mathcal{L}' := \mathcal{L} \otimes (\sum_{X_i \text{not exceptional}} (a_i/r) X_i)$ is a line bundle extending $\mathcal{L}_n$ such that $\mathcal{L'}^{\otimes r} \cong \omega(\sum e_i E_i)$ and the $E_i$ are all exceptional curves. \hfill $\Box$

**Proposition 7.1.2.** If $\mathcal{L}$ is a line bundle on $\tilde{X}$ such that $\mathcal{L}^{\otimes r} \cong \omega(\sum e_i E_i)$, with all of the $E_i$'s exceptional curves in the special fibre, then there is a choice of coefficients $\{e'_i\}$ such that $e'_i \equiv e_i \pmod{r}$ for every $i$, and the degree of $\omega(\sum e'_i E_i)$ is zero on every exceptional curve except perhaps one per chain, where it has degree $r$. In particular the bundle $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}(\sum (e'_i - e_i)/r E_i)$.
has degree zero on every exceptional curve except perhaps one per chain, where it has degree one. And \( \mathcal{L}^{\otimes r} \cong \omega(\sum e_i E_i) \).

**Proof.** Because \( \mathcal{L}^{\otimes r} \cong \omega(\sum e_i E_i) \), the degree of \( \mathcal{O}(\sum e_i E_i) \) on each \( E_i \) must be congruent to zero \( \pmod{r} \), and so for any particular chain \( E_1, \ldots, E_{nr-1} \) we have \( e_2 \equiv 2e_1, e_3 \equiv 3e_1 \), and so for each \( i \). Choose \( 0 \leq e_1' > -r \) with \( e_1' \equiv e_1 \pmod{r} \), and let \( e_i' = ie_1' \) for \( 1 \leq i \leq n(r+e_1') \). Choose \( e_i' = ie_1' + r \) for \( n(r+e_1') + 1 \leq i \leq nr - 1 \). This gives \( e_i' \equiv e_i \pmod{r} \) for all \( i \), \( \deg_{E_i} \mathcal{O}(\sum e_i' E_i) = 0 \) for all \( j \neq n(r+e_1') \) and on \( E_{nr+ne_1'} \), the degree is \(-2e_n'(r+e_1') + e_n'(n(r+e_1')-1) + e_n'(n(r+e_1')+1) = (-2(n(r+e_1')) + n(r+e_1') - 1 + n(r+e_1') + 1) + r \), which is \( r \).

Note also that all of the \( e_i' \) in the previous proposition were negative, thus there is a canonical inclusion map \( \omega(\sum e_i' E_i) \hookrightarrow \omega \).

### 7.2. The Stacks are Proper.

By the results of the previous section, we have for any complete discrete valuation ring \( R \) with a smooth spin curve \( \mathcal{S}_\eta = (\mathcal{X}_\eta, \mathcal{L}_\eta) \) over its generic point \( \eta \), an extension of the spin curve to a curve/line-bundle pair \( (\tilde{\mathcal{X}}, \tilde{\mathcal{L}}) \) over \( R \) (up to finite extension of the field of fractions) with the following special properties.

- \( \tilde{\mathcal{X}} \) is semi-stable.
- For any chain of exceptional curves in the special fibre of \( \tilde{\mathcal{X}} \), the degree of \( \tilde{\mathcal{L}} \) is zero on every exceptional curve in the chain except perhaps one, where it has degree one.
- \( \mathcal{L}^{\otimes r} \cong \omega_{\tilde{\mathcal{X}}}(\sum e_i E_i) \) with all of the \( e_i \) negative.

This means there is a natural map

\[
\omega(\sum e_i E_i) \hookrightarrow \omega,
\]

inducing a spin structure on the stable model \( \mathcal{X}_r \); namely, if \( \theta \) is the contraction \( \tilde{\mathcal{X}} \to \mathcal{X}_r \) then \( \theta_* \mathcal{L} \) is a rank-one torsion-free sheaf, and the map

\[
\mathcal{L}^{\otimes r} \longrightarrow \omega(\sum e_i E_i) \hookrightarrow \omega_{\tilde{\mathcal{X}}}
\]

induces a spin structure map

\[
(\theta_* \mathcal{L})^{\otimes r} \longrightarrow \omega_{\mathcal{X}_r}.
\]

Thus \( \mathcal{S}_\eta \) extends to a spin structure \( \mathcal{S} \) over all of \( R \), and the valuative criterion holds. We have proven the following theorem.

**Theorem 7.2.1.** *The stack \( \text{QSpin}_{r,g} \) is proper over \( S \).*

Since \( \text{QSpin}_{r,g} \) is a closed subscheme of \( \text{QSpin}_{r,g} \), and since it surjects to \( \overline{\mathcal{M}}_g \), it is also a compactification of \( \text{Spin}_{r,g} \) over \( \overline{\mathcal{M}}_g \), and since \( \text{Pure}_{r,g} \to \overline{\mathcal{M}}_g \) is surjective, and \( \text{Pure}_{r,g} \) is proper over \( \overline{\text{Spin}}_{r,g} \), the stack \( \text{Pure}_{r,g} \) is another compactification of \( \text{Spin}_{r,g} \) over \( \overline{\mathcal{M}}_g \).

**Conclusion**

We have constructed three algebraic stacks which compactify the moduli space of spin curves. The stack of quasi-spin curves, which, in some sense, is easiest to construct, is not as easy to describe as the substack of spin curves, which has nice (Gorenstein) singularities. And these singularities are resolved by the stack of pure-spin curves.
In the special case of 2-spin curves, many of the difficulties disappear. In particular, all three compactifications coincide. Moreover, this compactification of 2-spin curves can be shown to agree with those of Cornalba [4] and Deligne [5].

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