The Dynamics Theorem for properly embedded minimal surfaces

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January 10, 2014

Abstract

In this paper we prove two theorems. The first one is a structure result that describes the extrinsic geometry of an embedded surface with constant mean curvature (possibly zero) in a homogeneously regular Riemannian three-manifold, in any small neighborhood of a point of large almost-maximal curvature. We next apply this theorem and the Quadratic Curvature Decay Theorem in [14] to deduce compactness, descriptive and dynamics-type results concerning the space $D(M)$ of non-flat limits under dilations of any given properly embedded minimal surface $M$ in $\mathbb{R}^3$.

Mathematics Subject Classification: Primary 53A10, Secondary 49Q05, 53C42

Key words and phrases: Minimal surface, constant mean curvature, stability, curvature estimates, finite total curvature, minimal lamination, dynamics theorem.

1 Introduction.

In this paper we will describe the local extrinsic geometry of a complete embedded surface $M$ of constant mean curvature around a point of large Gaussian curvature in a homogeneously regular Riemannian three-manifold (see Theorem 1.1). We then obtain some consequences among which we highlight the Dynamics Theorem (Theorem 1.3) concerning the space $D(M)$ of non-flat limits under a divergent sequence of dilations of any given properly embedded minimal surface $M$ in $\mathbb{R}^3$; by a dilation we mean any diffeomorphism of $\mathbb{R}^3$ into itself given by composition of a translation with a homothety. An important consequence of the Dynamics Theorem is that every properly embedded minimal surface in $\mathbb{R}^3$ with infinite total curvature has a surprising amount of internal dynamical periodicity; see Theorem 1.3 and Proposition 6.1 below for this interpretive consequence.

*This material is based upon work for the NSF under Award No. DMS - 1309236. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

†The second and third authors were supported in part by the MEC/FEDER grant no. MTM2011-22547, and by the regional J. Andalucía grant no. P06-FQM-01642.
In order to state the main results, it is worth setting some specific notation to be used throughout the paper. Given a Riemannian three-manifold $N$ and a point $p \in N$, we denote by $B_N(p, r), \overline{B}_N(p, r), S^2_N(p, r)$ the open metric ball of center $p$ and radius $r > 0$, its closure and boundary sphere, respectively. In the case $N = \mathbb{R}^3$, we use the notation $B(p, r) = B_{\mathbb{R}^3}(p, r), S^2(p, r) = S^2_{\mathbb{R}^3}(p, r)$ and $\mathbb{B}(r) = \mathbb{B}(\vec{0}, r), S^2(\vec{r}) = S^2(\vec{0}, r)$, where $\vec{0} = (0, 0, 0)$. For a surface $M$ embedded in $N$, we denote by $\|\sigma_M\|$ the norm of the second fundamental form of $M$. Finally, for a surface $M \subset \mathbb{R}^3$, $K_M$ denotes its Gaussian curvature function. We will call a Riemannian three-manifold $N$ homogeneously regular if there exists an $\varepsilon > 0$ such that $\varepsilon$-balls in $N$ are uniformly close to $\varepsilon$-balls in $\mathbb{R}^3$ in the $C^2$-norm. In particular, if $N$ is compact, then $N$ is homogeneously regular.

**Theorem 1.1 (Local Picture on the Scale of Curvature)** Suppose $M$ is a complete, embedded, constant mean curvature surface (here we include minimal surfaces as being those with constant mean curvature zero) with unbounded second fundamental form in a homogeneously regular three-manifold $N$. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$, such that the following statements hold.

1. For all $n$, the component $M_n$ of $B_N(p_n, \varepsilon_n) \cap M$ that contains $p_n$ is compact with boundary $\partial M_n \subset \partial B_N(p_n, \varepsilon_n)$.

2. Let $\lambda_n = |\sigma_{M_n}(p_n)|$. Then, $\lim_{n \to \infty} \varepsilon_n \lambda_n = \infty$ and $\frac{|\sigma_{M_n}|}{\varepsilon_n} \leq 1 + \frac{1}{n}$ on $M_n$.

3. The metric balls $l_n B_N(p_n, \varepsilon_n)$ of radius $l_n \varepsilon_n$ converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify $p_n$ with $\vec{0}$ for all $n$), and, for any $k \in \mathbb{N}$, the surfaces $l_n M_n$ converge $C^k$ on compact subsets of $\mathbb{R}^3$ and with multiplicity one to a connected, properly embedded minimal surface $M_\infty$ in $\mathbb{R}^3$ with $\vec{0} \in M_\infty$, $|\sigma_{M_\infty}| \leq 1$ on $M_\infty$ and $|\sigma_{M_\infty}|(\vec{0}) = 1$.

Every complete, embedded minimal surface in $\mathbb{R}^3$ with bounded curvature is properly embedded, see Meeks and Rosenberg [20]. The key idea in the proof of Theorem 1.1 is to exploit this fact, together with a careful blow-up argument. Note that a direct consequence of Theorem 1.1 is that every complete embedded surface with constant mean curvature in $\mathbb{R}^3$ which is not properly embedded, has natural limits under a sequence of dilations, which are properly embedded non-flat minimal surfaces.

Theorem 1.1 together with the Local Removable Singularity Theorem in [13] can be used to understand the structure of the collection of limits of a non-flat, properly embedded minimal surface in $\mathbb{R}^3$ under any divergent sequence of dilations, which is the purpose of the next theorem. In order to clarify its statement, we need some definitions.

**Definition 1.2** Let $M \subset \mathbb{R}^3$ be a non-flat, properly embedded minimal surface. Then:

1. $M$ is periodic if it is invariant under a non-trivial translation or screw motion symmetry.
2. \( M \) is \textit{quasi-translation-periodic} if there exists a divergent sequence \( \{p_n\}_n \subset \mathbb{R}^3 \) such that \( \{M - p_n\}_n \) converges \( C^2 \) on compact subsets of \( \mathbb{R}^3 \) to \( M \); note that every periodic surface is also quasi-translation-periodic, even in the case the surface is invariant under a screw motion symmetry.

3. \( M \) is \textit{quasi-dilation-periodic} if there exists a sequence \( \{l_n\}_n \subset \mathbb{R}^+ \) and a divergent sequence \( \{p_n\}_n \subset \mathbb{R}^3 \) such that the sequence \( \{l_n(M - p_n)\}_n \) of dilations of \( M \) converges in a \( C^2 \)-manner on compact subsets of \( \mathbb{R}^3 \) to \( M \). Since \( M \) is not flat, then it is not stable and, it can be proved that the convergence of such a sequence \( \{h_n(M - p_n)\}_n \) to \( M \) has multiplicity one (see Lemma 4.1 below for a similar result about multiplicity one convergence).

4. Let \( D(M) \) be the set of non-flat, properly embedded minimal surfaces in \( \mathbb{R}^3 \) which are obtained as \( C^2 \)-limits of a divergent sequence \( \{l_n(M - p_n)\}_n \) of dilations of \( M \) (i.e., the translational part \( \{p_n\}_n \) of the dilations diverges). A non-empty subset \( \Delta \subset D(M) \) is called \( D \)-invariant, if for any \( \Sigma \in \Delta \), then \( D(\Sigma) \subset \Delta \). A \( D \)-invariant subset \( \Delta \subset D(M) \) is called a minimal \( D \)-invariant set, if it contains no proper, non-empty \( D \)-invariant subsets. We say that \( \Sigma \in D(M) \) is a minimal element, if \( \Sigma \) is an element of a minimal \( D \)-invariant subset of \( D(M) \).

\textbf{Theorem 1.3 (Dynamics Theorem)} Let \( M \subset \mathbb{R}^3 \) be a properly embedded, non-flat minimal surface. Then, \( D(M) = \emptyset \) if and only if \( M \) has finite total curvature. Now assume that \( M \) has infinite total curvature, and consider \( D(M) \) endowed with the topology of \( C^k \)-convergence on compact sets of \( \mathbb{R}^3 \) for all \( k \). Then:

1. \( D_1(M) = \{ \Sigma \in D(M) \mid \vec{0} \in \Sigma, |K_\Sigma| \leq 1, |K_\Sigma|(\vec{0}) = 1 \} \) is a non-empty compact subspace of \( D(M) \).

2. For any \( \Sigma \in D(M) \), \( D(\Sigma) \) is a closed \( D \)-invariant set of \( D(M) \). If \( \Delta \subset D(M) \) is a \( D \)-invariant set, then its closure \( \bar{\Delta} \) in \( D(M) \) is also \( D \)-invariant.

3. Suppose that \( \Delta \subset D(M) \) is a non-empty minimal \( D \)-invariant set which does not consist of exactly one surface of finite total curvature. If \( \Sigma \in \Delta \), then \( D(\Sigma) = \Delta \) and the closure in \( D(M) \) of the path connected subspace \( \{(\Sigma - p) \mid p \in \mathbb{R}^3, l > 0 \} \) of all dilations of \( \Sigma \) equals \( \Delta \). In particular, any minimal \( D \)-invariant set is connected and closed in \( D(M) \).

4. Every non-empty \( D \)-invariant subset of \( D(M) \) contains minimal elements. In particular, since \( D(M) \) is \( D \)-invariant, then \( D(M) \) always contains some minimal element.

5. Let \( \Delta \subset D(M) \) be a non-empty \( D \)-invariant subset. If no \( \Sigma \in \Delta \) has finite total curvature, then \( \Delta_1 = \{ \Sigma \in \Delta \mid \vec{0} \in \Sigma, |K_\Sigma| \leq 1, |K_\Sigma|(\vec{0}) = 1 \} \) contains a minimal element \( \Sigma' \), and every such surface \( \Sigma' \) satisfies that \( \Sigma' \in D(\Sigma') \) (in particular, \( \Sigma' \) is a quasi-dilation-periodic surface of bounded curvature).
6. If a minimal element Σ of \( D(M) \) has finite genus, then either Σ has finite total curvature, Σ is a helicoid, or Σ is a Riemann minimal example.

7. If a minimal element Σ of \( D(M) \) has more than one end, then every middle end of Σ is smoothly asymptotic to the end of a plane or catenoid.

2 Preliminaries.

When proving the results stated in the introduction, we will make use of three results from our previous paper [14]. For the reader’s convenience, we collect here these results and the definitions necessary in order to state them.

Definition 2.1 A codimension one lamination \( \mathcal{L} \) of a Riemannian three-manifold \( N \) is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces called leaves of \( \mathcal{L} \), with a certain local product structure. More precisely, it is a pair \((\mathcal{L}, \mathcal{A})\) satisfying:

1. \( \mathcal{L} \) is a closed subset of \( N \);

2. \( \mathcal{A} = \{\varphi_\beta : \mathbb{D} \times (0, 1) \to U_\beta\} \) is an atlas of coordinate charts of \( N \) (here \( \mathbb{D} \) is the open unit disk in \( \mathbb{R}^2 \), \( (0, 1) \) is the open unit interval and \( U_\beta \) is an open subset of \( N \)); note that although \( N \) is assumed to be smooth, we only require that the regularity of the atlas (i.e., that of its change of coordinates) is of class \( C^0 \), i.e., \( \mathcal{A} \) is an atlas for the topological structure of \( N \).

3. For each \( \beta \), there exists a closed subset \( C_\beta \) of \( (0, 1) \) such that \( \varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta \).

A leaf \( L \) of \( \mathcal{L} \) is called a limit leaf if for some (or every) point \( p \in L \), there exists a coordinate chart \( \varphi_\beta : \mathbb{D} \times (0, 1) \to U_\beta \) as in Definition 2.1 such that \( p \in U_\beta \) and \( \varphi_\beta^{-1}(p) = (x, t) \) with \( t \) belonging to the accumulation set of \( C_\beta \).

We will simply denote laminations by \( \mathcal{L} \), omitting the charts \( \varphi_\beta \) in \( \mathcal{A} \). Every lamination \( \mathcal{L} \) naturally decomposes into a collection of disjoint, connected topological surfaces (locally given by \( \varphi_\beta(\mathbb{D} \times \{t\}), t \in C_\beta \), with the notation above), called the leaves of \( \mathcal{L} \). As usual, the regularity of \( \mathcal{L} \) requires the corresponding regularity on the change of coordinate charts \( \varphi_\beta \). A lamination \( \mathcal{L} \) of \( N \) is called a foliation of \( N \) if \( \mathcal{L} = N \). A lamination \( \mathcal{L} \) of \( N \) is said to be a minimal lamination if all its leaves are smooth with zero mean curvature. Since the leaves of \( \mathcal{L} \) are pairwise disjoint, we can consider the second fundamental form \( |\sigma_\mathcal{L}| \) of \( \mathcal{L} \), which is the function defined at any point \( p \in \mathcal{L} \) as \( |\sigma_\mathcal{L}|(p) \), where \( L \) is the unique leaf of \( \mathcal{L} \) passing through \( p \).

There are three key results that we will need from [14], and which we list below for the reader’s convenience.
Theorem 2.2 (Local Removable Singularity Theorem [14]) A minimal lamination $\mathcal{L}$ of a punctured ball $B_N(p,r) - \{p\}$ in a Riemannian three-manifold $N$ extends to a minimal lamination of $B_N(p,r)$ if and only if there exists a positive constant $C$ such that $|\sigma_\mathcal{L}|d_N(p,\cdot) \leq C$ in some subball.

Definition 2.3 In the sequel, we will denote by $R = \sqrt{x_1^2 + x_2^2 + x_3^2}$ the radial distance to the origin $\vec{0} \in \mathbb{R}^3$. A surface $M \subset \mathbb{R}^3$ has quadratic decay of curvature if there exists $C > 0$ such that $|K_M| R^2 \leq C$ on $M$. Analogously, a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ or of $\mathbb{R}^3 - \{0\}$ has quadratic decay of curvature if $|K_\mathcal{L}| R^2$ is bounded on $\mathcal{L}$, where $|K_\mathcal{L}|$ is the function that associates to each point $p \in \mathcal{L}$ the absolute Gaussian curvature of the unique leaf of $\mathcal{L}$ passing through $p$.

Theorem 2.4 (Quadratic Curvature Decay Theorem [14]) A complete, embedded minimal surface in $\mathbb{R}^3$ with compact boundary (possibly empty) has quadratic decay of curvature if and only if it has finite total curvature. In particular, a complete, connected embedded minimal surface $M \subset \mathbb{R}^3$ with compact boundary and quadratic decay of curvature is properly embedded in $\mathbb{R}^3$.

Proposition 2.5 (Corollary 6.3 in [14]) Let $\mathcal{L}$ be a non-flat minimal lamination of $\mathbb{R}^3 - \{0\}$. If $\mathcal{L}$ has quadratic decay of curvature, then $\mathcal{L}$ consists of a single leaf, which extends across $0$ to a properly embedded minimal surface with finite total curvature in $\mathbb{R}^3$.

3 Proof of the Local Picture Theorem on the Scale of Curvature.

The proof of Theorem 1.1 stated in the introduction uses a blow-up technique, where the scaling factors are given by the norm of the second fundamental form of $M$ at points of large almost-maximal curvature, a concept which we develop below. After the blowing-up process, we will find a limit which is a complete minimal surface with bounded Gaussian curvature in $\mathbb{R}^3$, conditions which are known to imply properness for the limit [20]. This properness property will lead to the conclusions of Theorem 1.1.

Proof of Theorem 1.1 Since $N$ is homogeneously regular, after a fixed constant scaling of the metric of $N$ we may assume that the injectivity radius of $N$ is greater than 1. The first step in the proof is to obtain special points $p'_n \in M$, called blow-up points or points of almost-maximal curvature. First consider an arbitrary sequence of points $q_n \in M$ such that $|\sigma_M|(q_n) \geq n$, which exists since $|\sigma_M|$ is unbounded. Let $p'_n \in B_M(q_n,1)$ be a maximum of the function $h_n = |\sigma_M|d_M(\cdot, \partial B_M(q_n,1))$, where $d_M$ stands for the intrinsic distance on $M$. We define $l'_n = |\sigma_M|(p'_n)$. Note that for each $n \in \mathbb{N}$,

$$l'_n \geq l'_n d_M(p'_n, \partial B_M(q_n,1)) = h_n(p'_n) \geq h_n(q_n) = |\sigma_M|(q_n) \geq n.$$

Fix $t > 0$. Since $l'_n \to \infty$ as $n \to \infty$, the sequence $\{l'_n B_N(p'_n,\frac{t}{l'_n})\}_n$ converges to the open ball $B(t)$ of $\mathbb{R}^3$ with its usual metric, where we have used geodesic coordinates in $N$ centered
at \( p'_n \) and identified \( p'_n \) with \( \tilde{0} \). Similarly, we can consider \( \{ l'_n B_M(p'_n, \frac{1}{l'_n}) \} \) to be a sequence of embedded, constant mean curvature surfaces with non-empty topological boundary, all passing through \( \tilde{0} \) with norm of their second fundamental forms 1 at this point. We claim that the sequence of second fundamental forms of \( l'_n B_M(p'_n, \frac{1}{l'_n}) \) is uniformly bounded. To see this, pick a point \( z_n \in B_M(p'_n, \frac{1}{l'_n}) \). Note that for \( n \) large enough, \( z_n \) lies in \( B_M(q_n, 1) \). Then,

\[
\frac{\sigma_M(z_n)}{l'_n} = \frac{h_n(z_n)}{l'_n d_M(z_n, \partial B_M(q_n, 1))} \leq \frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))}.
\]

(1)

By the triangle inequality, \( d_M(p'_n, \partial B_M(q_n, 1)) \leq \frac{2}{l'_n} + d_M(z_n, \partial B_M(q_n, 1)) \), and so,

\[
\frac{d_M(p'_n, \partial B_M(q_n, 1))}{d_M(z_n, \partial B_M(q_n, 1))} \leq 1 + \frac{t}{l'_n d_M(z_n, \partial B_M(q_n, 1))} \leq 1 + \frac{t}{n - t},
\]

(2)

which tends to 1 as \( n \to \infty \).

It follows that after extracting a subsequence, the surfaces \( l'_n B_M(p'_n, \frac{1}{l'_n}) \) converge (possibly with non-constant integer or infinite multiplicity) to an embedded minimal surface \( M_\infty(t) \) contained in \( \mathbb{B}(t) \) with bounded Gaussian curvature, that passes through \( \tilde{0} \) and with norm of its second fundamental form 1 at the origin; note that the topological boundary \( \partial M_\infty(t) \) of \( M_\infty(t) \) need not be either smooth or contained in \( \mathbb{S}^2(t) \). Consider the surface \( M_\infty(1) \) together with the surfaces \( l'_n B_M(p'_n, \frac{1}{l'_n}) \) that converge to it (after passing to a subsequence). Note that \( M_\infty(1) \) is contained in \( M_\infty = \bigcup_{t \geq 1} M_\infty(t) \), which is a complete, injectively immersed minimal surface in \( \mathbb{R}^3 \), with \( \tilde{0} \in M_\infty \) and \( |\sigma_\infty(\tilde{0})| = 1 \). By construction, \( M_\infty \) has bounded Gaussian curvature, so it is properly embedded in \( \mathbb{R}^3 \) [20].

The next result describes how the surfaces \( l'_n B_M(p'_n, \frac{1}{l'_n}) \) that give rise to the limit \( M_\infty(t) \) sit nicely in space with respect to the surface \( M_\infty(t + 1) \).

**Lemma 3.1** Given \( t > 0 \), there exist \( k \in \mathbb{N} \) such that if \( n \geq k \), then \( l'_n B_M(p'_n, \frac{1}{l'_n}) \) is contained in a small regular neighborhood of \( M_\infty(t + 1) \) in \( \mathbb{R}^3 \). Furthermore, \( l'_n B_M(p'_n, \frac{1}{l'_n}) \) is a small normal graph over its projection to \( M_\infty(t + 1) \).

**Proof.** Let \( \pi: \tilde{M}_\infty \to M_\infty \) be the universal cover of \( M_\infty \). Choose a point \( \tilde{x} \in \pi^{-1}(\{0\}) \). Since \( M_\infty \) is not flat, then \( \tilde{M}_\infty \) is not stable [6] and thus, there exists \( R > 0 \) such that the intrinsic ball \( B_{\tilde{M}_\infty}(\tilde{x}, R) \) centered at \( \tilde{x} \) with radius \( R \) is unstable.

Choose \( t > 0 \). Since the closure \( \overline{M}_\infty(t) \) of \( M_\infty(t) \) is compact (because \( M_\infty(t) \subseteq \mathbb{B}(t) \)), then there exists \( \delta_t > 0 \) such that \( \overline{M}_\infty(t) \) admits a regular neighborhood \( U(t) \subseteq \mathbb{R}^3 \) of radius
\( \delta_t > 0 \); in particular, \( U(t) \) is diffeomorphic to \( \overline{M}_\infty(t) \times (-\delta_t, \delta_t) \) and we have a related normal projection \( \Pi_t: U(t) \to \overline{M}_\infty(t) \). Let \( \pi_t: \overline{M}_\infty(t) \to M_\infty(t) \) be the universal cover of \( M_\infty(t) \) and let \( \pi_t \times \text{Id}: \overline{U}(t) \equiv \overline{M}_\infty(t) \times (-\delta_t, \delta_t) \to M_\infty(t) \times (-\delta_t, \delta_t) \) be the universal cover of \( U(t) \), each one endowed with the pulled back metric. Therefore, we also have a normal projection \( \overline{\Pi}_t: \overline{U}(t) \to \overline{M}_\infty(t) \) such that \( \pi_t \circ \overline{\Pi}_t = \Pi_t \circ (\pi_t \times \text{Id}) \).

Since the sequence \( \{l_n B_M(p_n, \frac{t}{l_n})\}_n \) converges to \( M_\infty(t) \) as \( n \to \infty \), there exists \( n_0 = n_0(t) \in \mathbb{N} \) such that for every \( n \geq n_0 \), we have \( l_n B_M(p_n, \frac{t}{l_n}) \subset U(t + 1) \), which clearly implies the first sentence in the statement of Lemma 3.1. To prove the second statement, we argue by contradiction. Suppose that for some \( t > 0 \), there exists a sequence of integer numbers \( n(m) \geq n_0(t) \) tending to \( \infty \) such that for each \( m \), \( l_{n(m)} B_M(p_{n(m)}, \frac{t}{l_{n(m)}}) \) fails to be a normal graph over its projection to \( M_\infty(t + 1) \). To keep the notation simple, we will relabel \( n(m) \) as \( n \). This means that for each \( n \geq n_0(t) \), there exist two distinct points \( q_n(1), q_n(2) \in l_n \times l_n B_M(p_n, \frac{t}{l_n}) \) such that \( \Pi_{t+1}(q_n(1)) = \Pi_{t+1}(q_n(2)) \). As the sequence \( \{\Pi_{t+1}(q_n(1))\}_n \) lies in the compact set \( \overline{M}_\infty(t + 1) \), after extracting a subsequence we may assume that the sequence \( \Pi_{t+1}(q_n(1)) = \Pi_{t+1}(q_n(2)) \) converge as \( n \to \infty \) to a point \( q_\infty \in M_\infty(t) \). Therefore, there exists some \( \varepsilon = \varepsilon(t) > 0 \) small such that for each \( n \geq n_0(t) \), \( l_{n} B_M(p_{n}, \frac{t}{l_n}) \) contains two disjoint disks \( D_1(n), D_2(n) \) in such a way that

\[
(\Pi_{t+1})|_{D_i(n)}: D_i(n) \to B_{M_\infty}(q_\infty, \varepsilon) \quad \text{is a diffeomorphism}, \quad i = 1, 2,
\]

where \( B_{M_\infty}(q_\infty, \varepsilon) \subset M_\infty(t + 1) \) denotes the geodesic disk in \( M_\infty \) centered at \( q_\infty \) with radius \( \varepsilon \). Consider the universal covering

\[
\pi = \pi_{t+R+2}: \overline{M}_\infty(t + R + 2) \to \overline{M}_\infty(t + R + 2).
\]

Choose a point \( \overline{q}_\infty \in \pi^{-1}(\{q_\infty\}) \subset \overline{M}_\infty(t + R + 2) \) such that the distance from \( \overline{x} \) to \( \overline{q} \) is less than or equal to \( t \) (this can be done since \( q_\infty \in \overline{B}_{M_\infty}(\overline{x}, R) \)). We will find the desired contradiction by constructing a positive Jacobi function on the closed intrinsic ball \( \overline{B}_{M_\infty(t+R+2)}(\overline{q}_\infty, t + R) \), which is impossible since this last closed ball contains the unstable domain \( B_{\overline{M}_\infty}(\overline{x}, R) \) by the triangle inequality.

Take \( n_0 = n_0(t) \) large such that for all \( n \geq n_0 \), \( l_n B_M(p_n, \frac{t+R+1}{l_n}) \) lies in \( U(t + R + 2) \). Hence, we can lift \( l_n B_M(p_n, \frac{t+R+1}{l_n}) \) to \( \overline{U}(t + R + 2) \) via the covering \( \pi \times \text{Id} \). Note that

\[
V_n(t, R) := (\pi \times \text{Id})^{-1}\left[ l_n B_M(p_n, \frac{t+R+1}{l_n}) \right]
\]

is a possibly disconnected, non-compact surface in \( \overline{U}(t + R + 2) \). As \( B_{M_\infty}(q_\infty, \varepsilon) \subset M_\infty(t + 1) \) is a disk and \( \pi \) is a Riemannian covering, then \( B_{M_\infty}(q_\infty, \varepsilon) \) lifts to the geodesic disk \( B_{\overline{M}_\infty(t+R+2)}(\overline{q}_\infty, \varepsilon) \) in \( \overline{M}_\infty(t + R + 2) \) with center \( \overline{q}_\infty \) and radius \( \varepsilon \). Using \( \{\overline{q}_n\}_n \), we can lift \( D_1(n), D_2(n) \) to small disks \( \overline{D}_1(n), \overline{D}_2(n) \subset V_n(t, R) \) such that

\[
(\overline{\Pi}_{t+R+2})|_{\overline{D}_i(n)}: \overline{D}_i(n) \to B_{\overline{M}_\infty(t+R+2)}(\overline{q}_\infty, \varepsilon) \quad \text{is a diffeomorphism}, \quad i = 1, 2.
\]
As the closed intrinsic metric ball $\overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, R)$ is compact and the sequence of geodesic disks $\{l'_n B_M(p'_n, \frac{t+R+1}{l'_n})\}_n$ converges smoothly to $M_\infty(t + R + 1) \subset M_\infty(t + R + 2)$, then given $\mu > 0$ small there exists $n_1 = n_1(t, \mu) \in \mathbb{N}$ (we may assume $n_1 \geq n_0(t)$) such that for every $n \geq n_1$, the normal projection $\tilde{\Pi}_{t+R+2}$ restricts to $V_n(t, R)$ as a $\mu$-quasi-isometry, in the sense that the ratio between the length of tangent vectors at points in $V_n(t, R)$ and their images through the differential of $\tilde{\Pi}_{t+R+2}$ lies in the range $[1 - \mu, 1 + \mu]$. In particular, every radial geodesic arc $\gamma \subset \overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R)$ starting at $\bar{q}_\infty$ and ending at $\partial \overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R)$ can be uniquely lifted to a pair of disjoint embedded arcs $\gamma_1(n), \gamma_2(n) \subset V_n(t, R)$ starting at the points $\tilde{q}_i(\infty) := \left(\tilde{\Pi}_{t+R+2} \tilde{D}_i(n)\right)^{-1}(\bar{q}_\infty) \in \tilde{D}_i(n)$, and these arcs have length less than or equal to $t + R + 1$, for $i = 1, 2$. When $\gamma$ varies in the set of radial geodesic arcs starting at $\bar{q}_\infty$ and ending at $\partial \overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R)$, the union of the related lifted arcs $\gamma_1(n), \gamma_2(n)$ give rise to closed, disjoint topological disks $\overline{D}_{i,R}(n), \overline{D}_{2,R}(n) \subset V_n(t, R)$, with the property that

$$\tilde{\Pi}_{t+R+2}|_{\overline{D}_{i,R}(n)}: \overline{D}_{i,R}(n) \to \overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R) \text{ is a } \mu\text{-quasi-isometry, } i = 1, 2.$$ (5)

Property (5) implies that $\overline{D}_{i,R}(n)$ can be expressed as the graph of a function

$$u_{i,n}: \overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R) \to \mathbb{R}, \quad i = 1, 2.$$ 

As the sequence of disks $\{\overline{D}_{i,R}(n)\}_n$ converges as $n \to \infty$ to $\overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R)$ for $i = 1, 2$, then a subsequence of the functions $\frac{u_{1,n} - u_{2,n}}{u_{1,n} - u_{2,n}(\bar{q}_\infty)}$ converges as $n \to \infty$ a non-zero Jacobi function on $\overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R)$, which has non-zero sign since $\overline{D}_{1,R}(n), \overline{D}_{2,R}(n)$ are disjoint. This contradicts the unstability of $\overline{B}_{M_\infty(t+R+2)}(\bar{q}_\infty, t + R)$ and finishes the proof. \hfill $\Box$

**Lemma 3.2** For all $R > 0$, there exist $t > 0$ and $k \in \mathbb{N}$ such that if $m \geq k$, then the component of $\left[\left(l'_m B_M(p'_m, \frac{t+R+1}{l'_m})\right) \cap \overline{B}(R)\right]$ that passes through $\bar{0}$ is compact has its boundary on $\mathbb{S}^2(R)$.

**Proof.** We fix $R > 0$ and suppose that $M_\infty$ intersects transversely the sphere $\mathbb{S}^2(R)$ (this transversality property holds for almost every $R$; we will prove the lemma assuming this transversality property, and the lemma will hold for every $R$ after a continuity argument). As $M_\infty$ intersects transversely $\mathbb{S}^2(R)$, there exists $\varepsilon = \varepsilon(R) > 0$ small such that $M_\infty$ intersects transversely $\mathbb{S}^2(R')$ for all $R' \in [R, R + \varepsilon]$. Given $R' \in [R, R + \varepsilon]$, let $\Delta(R')$ be the component of $M_\infty \cap \overline{B}(R')$ that contains $\bar{0}$. Note that $\Delta(R + \varepsilon)$ is compact and contained in the interior of some intrinsic geodesic disk $B_{M_\infty}(\bar{0}, R_1)$, $R_1 = R_1(R) \geq R + \varepsilon$. Also note that $B_{M_\infty}(\bar{0}, R_1)$ is the limit as $n \to \infty$ of the intrinsic geodesic disks $l'_m B_M(p'_m, \frac{R+1}{l'_m})$.

Applying Lemma 3.1 to $t = R_1 + 1$, we obtain an integer number $k = k(R)$ such that for each $m \geq k$, the surface $l'_m B_M(p'_m, \frac{R_1+1}{l'_m})$ is contained in a small regular neighborhood $U(R_1 + 2)$
Remark 3.3 Note that Lemmas 3.1 and 3.2 remain valid under the hypotheses that \( M_\infty \) is not flat and the Gaussian curvature of \( \{ \Omega(t) \} \) is uniformly bounded for each \( t > 0 \) (with the bound depending on \( t \)).

We next finish the proof of Theorem 1.1. Applying Lemma 3.2 to \( R_n = l'_n, k_n \in \mathbb{N} \) such that if \( M_n \) denotes the component of \( B_M \left( p'_k, \frac{l(n)}{k(n)}, \frac{l(n)'}{k(n)'}, \frac{l_n}{k(n)} \right) \) that contains \( p'_k \), then \( M_n \) is compact and its boundary satisfies \( \partial M_n \subset \partial B_N \left( p'_k, \frac{t_n}{k(n)} \right) \). Clearly this compactness property remains valid if we increase the value of \( k(n) \). Hence, we may assume without loss of generality that

\[
t(n)(n+1) < k(n) \quad \text{for all } n, \quad \frac{l_n}{k(n)} \to 0 \quad \text{as } n \to \infty.
\]

We now define \( p_n = p'_k, \ v_n = \frac{l_n}{k(n)} \) and \( l_n = l'_n \). Then it is easy to check that the \( p_n, v_n, l_n \) and \( M_n \) satisfy the conclusions stated in Theorem 1.1 (in order to prove item 2 in the
statement of Theorem 1.1 simply note that equations (1) and (2) imply that 
\[ |\sigma_{M_n}| \ln l \leq |\sigma_{M_n}| \ln l' (n) \leq 1 + t(n)(n + 1) < k(n) \]. This finishes the proof of Theorem 1.1. 

\[ \square \]

Remark 3.4 If the surface \( M \subset N \) in Theorem 1.1 were properly embedded, then the arguments needed to carry out its proof could be formulated in a more standard manner by using the techniques developed in the papers [16, 20]. It is the non-properness of \( M \) that necessitates being more careful in defining the limit surface \( M_\infty \) and in proving additional properties of how it arises as a limit surface of compact embedded minimal surfaces that appear in the blow-up procedure in the proof of Theorem 1.1.

4 Applications of Theorem 1.1.

In any flat three-torus \( T^3 \), there exists a sequence \( \{M_n\}_n \) of embedded, compact minimal surfaces of genus three, such that the areas of these surfaces diverge to infinity [11] (a similar result holds for any genus \( g \geq 3 \), see Traizet [27]). After choosing a subsequence, these surfaces converge to a minimal foliation of \( T^3 \) and the convergence is smooth away from two points. Since by the Gauss-Bonnet formula, these surfaces have absolute total curvature \( 8\pi \), this example demonstrates a special case of Theorem 4.2 below. Before stating this result, we recall a somewhat standard result concerning limits of minimal surfaces. A similar statement can be found in item 5 of Lemma A.1 in Meeks and Rosenberg [21].

Lemma 4.1 Suppose that \( \{M_n\}_n \) is a sequence of complete embedded minimal surfaces in a Riemannian three-manifold \( N \), which converge to minimal lamination \( L \) of \( N \). Let \( L \) be a leaf of \( L \) which is either a limit leaf of \( L \) or it is an isolated leaf and in this case, the convergence of the sequence \( \{M_n\}_n \) to \( L \) has multiplicity greater than 1. Then, the two-sided cover of \( L \) is stable.

Proof. If \( L \) is a limit leaf of \( L \), then the main theorem in [19] insures that the two-sided cover of \( L \) is stable. Next suppose that \( L \) is an isolated leaf of \( L \) and that the convergence of the \( M_n \) to \( L \) has multiplicity greater than 1. Consider a compact subdomain \( D \subset L \) and let \( D_\delta \) be a regular neighborhood of \( D \) in \( N \) of small radius \( \delta > 0 \). After possibly lifting to a two-sheeted cover of \( D_\delta \), we may assume that \( D \) is two-sided. Thus \( D_\delta \) is diffeomorphic to \( D \times [-\delta, \delta] \). Since \( L \) is isolated as a leaf of \( L \), then the ‘top’ and ‘bottom’ sides \( D \times \{-\delta\} \) and \( D \times \{\delta\} \) of \( D_\delta \) can be assumed to be disjoint from \( L \) and, since they are compact, they are also disjoint from the surfaces \( M_n \) for \( n \) sufficiently large. Another consequence of the convergence of the \( M_n \) to \( L \) and of the compactness of \( D \) is that we may assume that the \( \bigcap M_n \cap D_\delta \) are locally graphs over their projections to \( D \). Consider the sequence of minimal laminations \( \{M_n \cap D_\delta\}_n \), which converges to \( D \). Note that for \( n \) large, each normal unit speed geodesic \( \gamma_x \) in \( D_\delta \) starting at a point
\[ x \in D \text{ intersects the lamination } \overline{M_n \cap D_b} \text{ in a closed set which has a highest point } \gamma_x(t_n(x)) \text{ and a lowest point } \gamma_x(s_n(x)), \text{ for some real numbers } s_n(x) \leq t_n(x). \text{ As the multiplicity of the limit } M_n \to L \text{ is greater than one, then } s_n(x) < t_n(x) \text{ for each } n \in \mathbb{N}. \text{ Consider the function } u_n(x) = t_n(x) - s_n(x), \text{ for all } x \in D. \text{ Since the lamination } \overline{M_n \cap D_b} \text{ is minimal for each } n \in \mathbb{N}, \text{ then after normalizing } u_n \text{ to be 1 at some point } p \in \text{Int}(D), \text{ a standard argument shows that these normalized functions converge to a positive Jacobi function of } D, \text{ which implies that } D \text{ is stable. Finally, } L \text{ is stable as every compact subdomain } D \subset L \text{ is stable.} \quad \Box

\textbf{Theorem 4.2} Suppose \( \{M_n\}_n \) is a sequence of complete, embedded minimal surfaces in a Riemannian three-manifold \( N \), such that there exists an open covering of \( N \) and \( \int_{M_n \cap B} |\sigma_n|^2 \) is uniformly bounded for any open set \( B \) in this covering (here \( \sigma_n \) denotes the second fundamental form of \( M_n \)). Then, there exists a subsequence of \( \{M_n\}_n \) that converges to a minimal lamination \( \mathcal{L} \) of \( N \), and the singular set of convergence of the \( M_n \) to \( \mathcal{L} \), defined as

\[ S(\mathcal{L}) = \left\{ p \in \mathcal{L} \mid \text{the sequence } \{|\sigma_{M_n}|\}_n \text{ is not uniformly bounded in any neighborhood of } p \right\}, \tag{6} \]

is closed and discrete. Furthermore:

1. If \( L \) is a limit leaf of \( \mathcal{L} \) or a leaf with infinite multiplicity as a limit of the surfaces \( M_n \), then the two-sided cover of \( L \) is stable and \( L \) is totally geodesic.

2. If each \( M_n \) is connected and \( N \) is compact, then \( \mathcal{L} \) is compact and connected in the subspace topology.

\textbf{Proof.} We will distinguish between \textit{good} and \textit{bad points of} \( N \), depending on whether or not the surfaces \( M_n \) have a good behavior around the point to take limits; the set \( A \subset N \) of bad points will be then proven to be discrete and closed in \( N \), and we will produce a limit minimal lamination \( \mathcal{L} \) of the \( \{M_n\}_n \) in \( N - A \). The final step in the proof of the first statement in the theorem will be to show that \( \mathcal{L} \) extends as a minimal lamination across \( A \).

Let \( q \) be a point in \( N \). We will say that \( q \) is a bad point for the sequence \( \{M_n\}_n \) if there exists a subsequence \( \{M_{n_k}\}_k \subset \{M_n\}_n \) such that

\[ \int_{M_{n_k} \cap B_N(q, \frac{1}{k})} |\sigma_{M_{n_k}}|^2 \geq 2\pi, \quad \text{for all } k \in \mathbb{N}. \]

First note that we can replace the covering in the statement by a countable open covering of \( N \) by balls \( B_i, i \in \mathbb{N} \). Assume for the moment that \( B_1 \) contains a bad point \( q_1 \) for \( \{M_n\}_n \). We claim that \( B_1 \) has a finite number of bad points after replacing \( \{M_n\}_n \) by a subsequence. To see this, since \( q_1 \) is a bad point for \( \{M_n\}_n \), there exists a subsequence \( \{M'_{k} = M_{n_k}\}_k \subset \{M_n\}_n \) such that the total curvature of every \( M'_k \) in \( B_N(q_1, \frac{1}{k}) \) is at least \( 2\pi \). Suppose that \( q_2 \in B_1 \) is another bad point for \( \{M'_k\}_k \). Then we find a subsequence \( \{M''_j = M'_{k_j}\}_j \subset \{M'_k\}_k \) such that
the total curvature of every $M''_j$ in $B_N(q_2, \frac{1}{j})$ is at least $2\pi$. In particular, for $j$ large, there are disjoint neighborhoods of $q_1$ and $q_2$ in $B_1$, each with total curvature of $M''_j$ at least $2\pi$. Since \( \{\int_{M_n \cap B_1} |\sigma_n|^2\}_n \) is uniformly bounded, this process of finding bad points and subsequences in $B_1$ stops after a finite number of steps, which proves our claim. A standard diagonal argument then shows that after replacing the $M_n$ by a subsequence, the set of bad points $A \subset N$ for $\{M_n\}_n$ is a discrete closed set in $N$.

Suppose that $q \in N - A$. We claim that $\{M_n\}_n$ has pointwise bounded second fundamental form in some neighborhood of $q$. Arguing by contradiction, suppose there exist points $p_n \in M_n$ converging to $q$ and such that $|\sigma_{M_n}|(p_n) \to \infty$ as $n \to \infty$. Let $\varepsilon_q = \frac{1}{2}d_N(q, A) > 0$. By the Local Picture Theorem on the Scale of Curvature, there exists a blow-up limit of the $M_n$ that converges to a non-flat, properly embedded minimal surface in $\mathbb{R}^3$; as the total curvature of a non-flat, complete minimal surface in $\mathbb{R}^3$ is at least $4\pi$, and the $L^2$-norm of the second fundamental form is invariant under rescalings of the ambient metric, then we may assume that for $n$ large,

$$\int_{M_n \cap B_N(q, r_n)} |\sigma_n|^2 > 2\pi,$$

where $r_n \searrow 0$ satisfies $d_N(q, p_n) < r_n < \frac{\varepsilon_q}{2}$. This clearly contradicts that $q \in N - A$, and so, we conclude that $\{M_n\}_n$ has pointwise bounded second fundamental form in some neighborhood $U_q$ of $q$. Therefore, there exists a minimal lamination $L_q$ of $U_q$ such that a subsequence of the $M_n$ converges as $n \to \infty$ to $L_q$ in $U_q$. Another standard diagonal argument proves that after extracting a subsequence, the $M_n$ converge to a minimal lamination $L$ of $N - A$.

Next we show that $L$ extends across $A$ to a minimal lamination of $N$. Consider the second fundamental form $|\sigma_L|$ of $L$. We claim that $|\sigma_L|$ does not grow faster than linearly at any point $q \in A$ in terms of the inverse of the extrinsic distance function to $q$: otherwise, there exists a sequence of blow-up points $p_n \in L$ converging to a point $q \in A$ with $|\sigma_{L_n}|(p_n)d_N(p_n, q)$ unbounded, where $L_n$ is the leaf of $L$ passing through $p_n$. Using again the Local Picture Theorem on the Scale of Curvature, we deduce that there exist disjoint small neighborhoods $V(p_n)$ of $p_n$ in $L_n$, such that

$$\int_{V(p_n)} |\sigma_{L_n}|^2 > 2\pi,$$

for all $n \in N$.

Since $M_n$ converges to $L$, this contradicts the hypothesis that $\int_{M_n \cap B} |\sigma_n|^2$ is uniformly bounded for the open set $B$ in the covering which contains $q$. Once we know that $|\sigma_L|$ does not grow faster than linearly at any point of the discrete closed set $A$, Theorem 2.2 implies that $L$ extends across $A$ to a minimal lamination of $N$. Observe that by construction, the singular set of convergence $S(L)$ defined in (6) coincides with the set $A$ of bad points. This proves the first sentence of the theorem.

We next prove item 1. Let $L$ be a limit leaf of $L$. By the main result in [19], the two-sided cover of $L$ is stable. If $L$ is not totally geodesic, then there exists $q \in L - S(L)$ such that $|\sigma_L|(q) = 4\varepsilon > 0$ (recall that $S(L)$ is closed and discrete). Then, there exists some open set
$U \subset [L - S(L)] \cap B$ such that $|\sigma_L| \geq 2\varepsilon$ in $U$, where $B$ is an open set in the covering that appears in the statement of the theorem, such that $q \in B$. As $L$ is a limit leaf of $\mathcal{L}$ and $L$ is the limit in $N - S(L)$ of the $M_n$, then there exist pairwise disjoint domains $U_n \subset M_n \cap B$ such that $|\sigma_n| \geq \varepsilon$ in $U_n$ for all $n$. This clearly contradicts that $\int_{M_n \cap B} |\sigma_n|$ is bounded. This proves item 1 of the theorem when $L$ is a limit leaf of $L$.

If $L$ is not a limit leaf of $\mathcal{L}$ but the multiplicity of the limit $\{M_n\}_n \rightarrow L$ is infinity, then Lemma 4.1 insures that the two-sided cover of $L - S(\mathcal{L})$ is stable. Given a compact subdomain $D$ of the two-sided cover $\hat{L}$ of $L$, the fact that $S(\mathcal{L})$ is closed and discrete implies that $D \cap S(\mathcal{L})$ is finite. After enlarging slightly $D$, we can assume that $D \cap S(\mathcal{L})$ lies in the interior of $D$. As $\hat{L} - S(\mathcal{L})$ is stable, then $D - S(\mathcal{L})$ is also stable. A standard argument in elliptic theory then shows that $D$ is also stable, and thus $\hat{L}$ is also stable, as desired. The arguments in the last paragraph to show that $L$ is totally geodesic can be adapted easily to this case, since the multiplicity of the limit $\{M_n\}_n \rightarrow L$ is infinity. Now item 1 of the theorem is proved.

The proof of item 2 of the theorem is straightforward and we leave it for the reader. □

**Remark 4.3** Under the hypotheses and notation of Theorem 4.2, we cannot conclude that $L$ is path-connected in the subspace topology when $M$ is connected and $N$ is compact: a counterexample can be found for geodesic laminations in Example 3.5 of [18], and this example can be adapted to produce a minimal lamination counterexample simply by taking products with $S^1$.

We now give another application of Theorem 1.1 which is a partial positive answer to Conjecture 1.7 in [17]. Given a two-sided minimal surface $M$ in a flat three-manifold $N$ and given $a > 0$, we say that $M$ is $a$-stable if for any compactly supported smooth function $u \in C^\infty_0(M)$, we have

$$\int_M (|\nabla u|^2 + aKu^2) \geq 0,$$

where $\nabla u$ stands for the gradient of $u$ and $K$ is the Gaussian curvature of $M$ (the usual stability condition for the area functional corresponds to the case $a = 2$). More generally, we say that $M$ has finite index of $a$-stability if there is a bound on the number of negative eigenvalues (counted with multiplicities) of the operator $\Delta - aK$ acting on smooth functions defined on compact subdomains of $M$.

The authors conjectured in [17] (Conjecture 1.7) that if $a > 0$ and $M$ is a two-sided, complete, embedded, $a$-stable minimal surface in a complete flat three-manifold $N$, then $M$ is totally geodesic (flat). Do Carmo and Peng [6], Fischer-Colbrie and Schoen [8] and Pogorelov [24] proved independently that if $M$ is a complete, orientable $a$-stable minimal surface immersed in $\mathbb{R}^3$, for $a \geq 1$, then $M$ is a plane. This result was later improved by Kawai [10] to $a > 1/4$, see also Ros [25]. In [17], the authors proved the conjecture (for every $a > 0$ and in any complete flat three-manifold $N$) under the additional hypotheses that $M$ is embedded and it has finite genus. On the other hand, for small values of $a > 0$, there exist complete, non-flat, immersed,
$a$-stable minimal surfaces in $\mathbb{R}^3$: for instance, apply Lemma 6.3 in [17] to the universal cover of any doubly periodic Scherk minimal surface. In the next result we will prove the conjecture for $a > 1/8$ assuming solely that $M$ is embedded. It should be also mentioned that Fischer-Colbrie [7] proved item 3 of the next result in the case $a \geq 1$, independently of whether or not $M$ is embedded.

**Theorem 4.4** Let $a \in (1/8, \infty)$ and $M$ be a two-sided, complete, embedded, minimal surface with compact boundary $\partial M$ in a complete, flat three-manifold $N$. Then:

1. If $\partial M = \emptyset$ and $M$ is $a$-stable, then $M$ is totally geodesic.

2. There exists $C > 0$ (independent of $M, N$) such that if $\partial M \neq \emptyset$ and $M$ is $a$-stable, then $|\sigma_M|_{d_M}(\cdot, \partial M) \leq C$ and $M$ has finite total curvature.

3. If $N = \mathbb{R}^3$ and $M$ has finite index of $a$-stability, then $M$ has finite total curvature.

**Proof.** Suppose that $M \subset N$ has empty boundary and is $a$-stable. After lifting $M$ to $\mathbb{R}^3$ and applying Theorem 1.1 (note that $a$-stability is preserved after lifting to a covering, rescaling and taking smooth limits), we can assume that $M$ has bounded Gaussian curvature and $N = \mathbb{R}^3$, in particular $M$ is proper [20]. A straightforward application of the maximum principle at infinity implies that $M$ has an embedded regular neighborhood of fixed positive radius and so, $M$ has at most cubical extrinsic area growth, see Meeks and Rosenberg [22]. The following applications of previous results show that $M$ is homeomorphic to $\mathbb{C}$ or to $\mathbb{C} - \{0\}$:

- If $a > 1/4$, then apply Theorem 2.9 in [18] (see also Castillon [3]).

- If $a = 1/4$, then apply part (ii) of Theorem 1.2 in Berard and Castillon [1].

- If $1/8 < a < 1/4$, then the cubical extrinsic area growth property of $M$ implies that the intrinsic area growth of $M$ is $k_a$-subpolynomial, where $k_a = 2 + \frac{4a}{1 - 4a}$. This means that the limit as $r \to \infty$ of the area of the intrinsic ball in $M$ of radius $r$ (centered at any fixed point) divided by $r^{k_a}$ is zero. Now apply part (iii) of Theorem 1.2 in [1].

As $M$ is a properly embedded minimal surface in $\mathbb{R}^3$ homeomorphic to $\mathbb{C}$ or to $\mathbb{C} - \{0\}$, then $M$ is either a plane, a catenoid [26] or a helicoid [20]. As both the catenoid and the helicoid are $a$-unstable for all $a > 0$ by Proposition 1.5 in [17], we deduce item 1 of the theorem.

Item 2 follows from item 1 by a rescaling argument on the scale of curvature that is given in the proof of Theorem 1.1 (also see the proof of Theorem 15 in [12] for a similar argument).

To see item 3, suppose that $M \subset \mathbb{R}^3$ has finite index of $a$-stability. By Proposition 1 in Fischer-Colbrie [7], there exists a compact domain $\Omega \subset M$ such that $M - \text{Int}(\Omega)$ is $a$-stable. In this situation, the already proven item 2 of this theorem implies $M$ has quadratic decay of curvature. Then, Theorem 2.4 implies that $M$ has finite total curvature. $\square$
5 Proof of the Dynamics Theorem.

Our next goal is to prove the Dynamics Theorem (Theorem 1.3) stated in the introduction. Regarding the notions introduced in Definition 1.2 we make the following observations.

(i) If $\Sigma \in D(M)$, then $D(\Sigma) \subset D(M)$ (this follows by considering double limits).

This property allows us to consider $D$ as an operator $D: D(M) \rightarrow \mathcal{P}(D(M))$, where $\mathcal{P}(D(M))$ denotes the power set of $D(M)$.

(ii) If $\Sigma \in D(M)$ and $D(\Sigma) = \emptyset$, then $\{\Sigma\}$ is a minimal $D$-invariant set.

(iii) $\Sigma \in D(M)$ is quasi-dilation-periodic if and only if $\Sigma \in D(\Sigma)$.

(iv) Any minimal element $\Sigma \in D(M)$ is contained in a unique minimal $D$-invariant set.

(v) If $\Delta \subset D(M)$ is a minimal $D$-invariant set and $\Sigma \in \Delta$ satisfies $D(\Delta) = \emptyset$ (otherwise $D(\Sigma)$ would be a proper non-empty $D$-invariant subset of $\Delta$). In particular, $\Sigma$ is quasi-dilation-periodic.

(vi) If $\Delta \subset D(M)$ is a $D$-invariant set and $\Sigma \in \Delta$ is a minimal element, then the unique minimal $D$-invariant subset $\Delta'$ of $D(M)$ which contains $\Sigma$ satisfies $\Delta' \cap \Delta = \emptyset$ (otherwise $\Delta' \cap \Delta$ would be a proper non-empty $D$-invariant subset of $\Delta'$).

Proof of Theorem 1.3. First assume that $M$ has finite total curvature. Then, its total curvature outside of some ball in space is less than $2\pi$, and so, any $\Sigma \in D(M)$ must have total curvature less than $2\pi$, which implies $\Sigma$ is flat. This gives that $D(M) = \emptyset$.

Reciprocally, assume that $D(M) = \emptyset$ and $M$ does not have finite total curvature. By Theorem 2.4, $M$ does not have quadratic decay of curvature, and so, there exists a divergent sequence of points $z_n \in M$ with $|K_M(z_n)|z_n| \to \infty$. Let $p_n \in B(z_n, |z_n|/2)$ be a maximum of the function $h_n = |K_M|d_{\mathbb{R}^3}(\cdot, \partial B(z_n, |z_n|/2))^2$. Note that $\{p_n\}_n$ diverges in $\mathbb{R}^3$ (because $|p_n| \geq |z_n|$). We define $l_n = \sqrt{|K_M|p_n(p_n)}$. By similar arguments as those in the proof of Theorem 1.1 applied to $M \cap B(p_n, \sqrt{h_n(p_n)/2})$, the sequence $\{l_n(M - p_n)\}_n$ converges (after passing to a subsequence) to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with a non-flat leaf $L$ which passes through $\hat{0}$ and satisfies $|K_L|(\hat{0}) = 1$. Furthermore, the curvature function $K_L$ of $\mathcal{L}$ satisfies $|K_L| \leq 1$ and so, the leaf $L$ of $\mathcal{L}$ passing through $\hat{0}$ is properly embedded in $\mathbb{R}^3$, as are all the leaves of $\mathcal{L}$; see [22] for this properness result. By the Strong Half-space Theorem, $\mathcal{L}$ consists just of $L$, and the convergence of the surfaces $l_n(M - p_n)$ to $L$ has multiplicity one (by Lemma 4.1 since the two-sided cover of $L$ is not stable as $L$ is not flat). Therefore, $L \in D_1(M)$, which contradicts that $D(M) = \emptyset$. This proves the equivalence stated in Theorem 1.3.
Assume now that $D(M) \neq \emptyset$. Hence, $M$ does not have finite total curvature and the arguments in the last paragraph show that $D_1(M) \neq \emptyset$. To conclude the proof of item 1 of the theorem it remains to analyze the topology of $D_1(M)$.

We will now define a metric space structure on $D_1(M)$ which generates a topology that coincides with the topology of uniform $C^k$-convergence on compact subsets of $\mathbb{R}^3$ for any $k \in \mathbb{N}$ (in particular, compactness of $D_1(M)$ will follow from sequential compactness). To do this, we first prove that there exists some $\varepsilon > 0$ such that $\overline{B}(\varepsilon)$ intersects every surface $\Sigma \subset D_1(M)$ in a unique component which is a graphical disk over its projection to the tangent space to $\Sigma$ at $\tilde{0}$ and with gradient less than 1. Otherwise, there exists a sequence $\{\Sigma_n\}_n \subset D_1(M)$ such that this property fails in the ball $\overline{B}(\frac{1}{n})$ for every $n \in \mathbb{N}$. As the Gaussian curvature of $\Sigma_n$ is not greater than 1, the uniform graph lemma [23] implies that around every point $p_n \in \Sigma_n$, this surface can be locally expressed as a graph over a disk in the tangent space $T_{p_n} \Sigma_n$ of uniform radius. Therefore, there exists $\delta > 0$ such that for $n$ large, $\Sigma_n$ intersects $\overline{B}(\delta)$ in at least two components, one of which passes through $\tilde{0}$ and the other one intersects $\overline{B}(\frac{1}{n})$, and such that both components are graphical over domains in the tangent space of $\Sigma_n$ at $\tilde{0}$ with small gradient. Hence, a subsequence of these $\Sigma_n$ (denoted in the same way) converges smoothly to a minimal lamination $L_1$ of $\mathbb{R}^3$ with a leaf $L_1 \in L_1$ passing through $\tilde{0}$ such that the multiplicity of the limit $\{\Sigma_n\}_n \to L_1$ is greater than one. This last property implies that the two-sided cover $\hat{L}_1$ of $L_1$ is stable, by Lemma 4.4. As $\hat{L}_1$ is complete and stable then $\hat{L}_1$ is a plane (and $L_1 = \hat{L}_1$), which contradicts that the convergence $\{\Sigma_n\}_n \to L_1$ is smooth and the curvature of the $\Sigma_n$ is $-1$ at $\tilde{0}$ for every $n$. This proves our claim on the existence of $\varepsilon$.

With the above $\varepsilon > 0$ at hand, we define the distance between any two surfaces $\Sigma_1, \Sigma_2 \in D_1(M)$ as

$$d(\Sigma_1, \Sigma_2) = d_H(\Sigma_1 \cap \overline{B}(\varepsilon/2), \Sigma_2 \cap \overline{B}(\varepsilon/2)),$$

where $d_H$ denotes the Hausdorff distance. Standard elliptic theory implies that the metric topology on $D_1(M)$ associated to the distance $d$ agrees with the topology of the uniform $C^k$-convergence on compact sets of $\mathbb{R}^3$ for any $k$.

Next we prove that $D_1(M)$ is sequentially compact (hence compact). Every sequence $\{\Sigma_n\}_n \subset D_1(M)$ contains a subsequence which converges to a minimal lamination $\mathcal{L}$ of $\mathbb{R}^3$ with bounded Gaussian curvature $K_{\mathcal{L}}$ and $K_{\mathcal{L}}(\tilde{0}) = -1$. The same arguments given in the second paragraph of this proof imply that $\mathcal{L}$ consists just of the leaf $L$ passing through $\tilde{0}$, which is a properly embedded minimal surface in $\mathbb{R}^3$. Clearly $L \in D_1(M)$, which proves item 1 of the theorem.

We now prove item 2. Using the definition of $D$-invariance, it is elementary to show that for any $\Sigma \in D(M)$, $D(\Sigma)$ is a closed set in $D(M)$; essentially, this is because the set of limit points of a set in a topological space forms a closed set. That $D(\Sigma)$ is $D$-invariant follows from property (i) stated at the beginning of this section. Similar techniques show that if $\Delta \subset D(M)$ is a $D$-invariant subset, then its closure in $D(M)$ is also $D$-invariant, and item 2 of the theorem is proved.
Now assume that $\Delta$ is a minimal $D$-invariant set in $D(M)$. If $\Delta$ contains a surface of finite total curvature, then the minimality of $\Delta$ implies $\Delta$ consists only of this surface, and so, it is connected and closed in $D(M)$. Suppose now that $\Delta$ does not consist of exactly one surface of finite total curvature, or equivalently, $\Delta$ contains no surfaces of finite total curvature. Then, property (v) above implies that for any $\Sigma \in \Delta$, $D(\Sigma) = \Delta$. Since $D(\Sigma)$ is closed by item 2, then $\Delta$ is closed as well. Since $D(\Sigma) = \Delta$, then $\Delta$ also contains the path connected subset $S \subset D(M)$ of all dilations of $\Sigma$. Since $\Delta$ is a closed set in $D(M)$, then the closure of $S$ in $D(M)$ is contained in $\Delta$. Reciprocally, if $\Sigma_1 \in \Delta = D(\Sigma)$ then $\Sigma_1$ is a non-flat, properly embedded minimal surface in $\mathbb{R}^3$ which is the $C^2$-limit of a sequence $\mu_n(\Sigma - p_n)$ for some $\{\mu_n\}_n \subset \mathbb{R}^+$ and $\{p_n\}_n \subset \mathbb{R}^3$, $p_n \to \infty$. As $\mu_n(\Sigma - p_n) \in S$ for each $n \in \mathbb{N}$, then $\Sigma_1$ lies in the closure of $S$ in $D(M)$, and so, $\Delta$ equals the closure of $S$ in $D(M)$ (in particular, $\Delta$ is connected as $S$ is path connected). This proves item 3 of the theorem.

Next we prove item 4. Suppose $\Delta \subset D(M)$ is a non-empty, $D$-invariant set. One possibility is that $\Delta$ contains a surface $\Sigma$ of finite total curvature. By the main statement of this theorem, $D(\Sigma) = \emptyset$ and by property (ii) above, $\Sigma$ is a minimal element in $\Delta$. Now assume $\Delta$ contains no surfaces of finite total curvature. Consider the collection

$$\Lambda = \{\Delta' \subset \Delta \mid \Delta' \text{ is non-empty, closed and } D\text{-invariant}\}.$$ 

Note that $\Lambda$ is non-empty, since for any $\Sigma \in \Delta$, the set $D(\Sigma) \subset \Delta$ is such a closed, non-empty $D$-invariant set by the first statement in item 2. $\Lambda$ has a partial ordering induced by inclusion. We just need to check that any linearly ordered subset in $\Lambda$ has a lower bound, and then apply Zorn’s Lemma to obtain item 4 of the theorem. Suppose $\Lambda' \subset \Lambda$ is a non-empty linearly ordered subset. We must check that the intersection $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is an element of $\Lambda$. In our case, this means we only need to prove that such an intersection is non-empty, because the intersection of closed (resp. $D$-invariant) sets is closed (resp. $D$-invariant).

Given $\Delta' \in \Lambda'$, consider the collection of surfaces $\Delta'_1 = \{\Sigma \in \Delta' \mid \exists \bar{0} \in \Sigma, |K_{\Sigma}| \leq 1, |K_{\Sigma}|(\bar{0}) = 1\}$. Note that $\Delta'_1$ is a closed subset of $D(M)$, since $\Delta'$ and $D_1(M)$ are closed in $D(M)$. The set $\Delta'_1$ is non-empty by the following argument. Let $\Sigma \in \Delta'$. Since $\Sigma$ does not have finite total curvature and $\Delta'$ is $D$-invariant, $D(\Sigma)$ is a non-empty subset of $\Delta'$. By item 1, $D_1(\Sigma)$ is a non-empty subset of $\Delta'_1$, and so, $\Delta'_1$ is non-empty. Now define $\Lambda'_1 = \{\Delta'_1 \mid \Delta' \in \Lambda'\}$. As $\bigcap_{\Delta' \in \Lambda'} \Delta'_1 = \bigcap_{\Delta' \in \Lambda'} \Delta'_1 = \bigcap_{\Delta' \in \Lambda'} (\Delta' \cap D_1(M)) = (\bigcap_{\Delta' \in \Lambda'} \Delta') \cap D_1(M)$, in order to check that $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is non-empty, it suffices to show that $\bigcap_{\Delta' \in \Lambda'} \Delta'_1$ is non-empty. But this is clear since each element of $\Lambda'_1$ is a closed subset of the compact metric space $D_1(M)$, and the finite intersection property holds for the collection $\Lambda'_1$.

Next we prove item 5. Let $\Delta \subset D(M)$ be a non-empty $D$-invariant subset which contains no surfaces of finite total curvature. By item 4, there exists a minimal element $\Sigma \in \Delta$. Since none of the surfaces of $\Delta$ have finite total curvature, it follows that $D(\Sigma) \neq \emptyset$. As $\Sigma$ is a minimal element, there exists a minimal $D$-invariant subset $\Delta' \subset D(M)$ such that $\Sigma \in \Delta'$. By property (v) above, $D(\Sigma) = \Delta'$. Note that $\Delta'_1 = \Delta' \cap D_1(M)$ contains $D_1(\Sigma)$, which is
non-empty since $D(\Sigma) \neq \emptyset$ (by item 1 of this theorem). Then there exists a surface $\Sigma_1 \in \Delta_1$, which in particular is a minimal element (any element of $\Delta'$ is), and lies in $\Delta_1$ (because $\Delta' \subset \Delta$ by property (vi)). Finally, $\Sigma_1$ is dilation-periodic by property (v), thereby proving item 5 of the theorem.

To prove item 6, suppose $\Sigma \in D(M)$ is a minimal element with finite genus, and assume also that $\Sigma$ has infinite total curvature. Since $\Sigma \in D(\Sigma)$ (by property (v) applied to $\Delta = D(\Sigma)$) and $\Sigma$ has finite genus, then the genus of $\Sigma$ must be zero. In this setting, the classification results in [1, 20, 15] imply that $\Sigma$ is a helicoid or a Riemann minimal example.

Finally we prove item 7. Consider a minimal element $\Sigma$ of $D(M)$ with more than one end. By the Ordering Theorem [2], after possibly a rotation in $\mathbb{R}^3$ so that the limit tangent plane at infinity for $\Sigma$ is horizontal (see [2] for this notion of limit tangent plane at infinity), $\Sigma$ has a middle end $e$ in the natural ordering of the ends of $\Sigma$ by their relative heights. The results in [5] imply that $e$ is a simple end and $e$ admits an end representative $E \subset \Sigma$ with the following properties:

- $E$ is a proper, non-compact subdomain of $\Sigma$ with compact boundary and only one end.
- $E$ is contained in the open region $W \subset \mathbb{R}^3 - B(R)$ between two graphical, disjoint vertical half-catenoids or horizontal planes, where $R > 0$ is sufficiently large. Furthermore, we can assume that $\Sigma \cap W = E$.

We now discuss two cases for $E$. First assume that $E$ has quadratic decay of curvature. By Theorem 2.4, $E$ has finite total curvature, in which case item 7 is known to hold. So, assume that $E$ has infinite total curvature and we will find a contradiction. On one hand, the proof of the first statement of this theorem applies to $E$ to produce the collection $D(E)$ of properly embedded minimal surfaces in $\mathbb{R}^3$ which are limits of $E$ under a sequence of dilations with divergent translational part (note that surfaces in $D(E)$ have empty boundary). Notice that under every divergent sequence of dilations which give rise to a surface in $D(E)$, the dilated regions $W_n$ related to $W$ converge to all of $\mathbb{R}^3$, by the Half-space Theorem. This implies that $D(E)$ is a subset of $D(\Sigma)$. Since $\Sigma$ is a minimal element, then $D(E) = D(\Sigma)$. On the other hand, the results in [5] imply that $E$ has quadratic area growth, and hence, the monotonicity formula gives that every surface in $D(E)$ has at most the same quadratic area growth as $E$. This discussion applies to $\Sigma$ since $\Sigma \in D(\Sigma)$. But since $\Sigma$ has other ends different from $E$, the quadratic area growth of $\Sigma$ is strictly greater than the one of $E$. This contradiction proves item 7, thereby finishing the proof of Theorem 1.3.
6 Internal dynamical periodicity of properly embedded minimal surfaces with infinite total curvature.

Let $M \subset \mathbb{R}^3$ be a properly embedded minimal surface with infinite total curvature and let $\Sigma \in D(M)$ be a minimal element, which exists by item 4 of Theorem 1.3. Assume that $\Sigma$ also has infinite total curvature. We claim that each compact subdomain of $\Sigma$ can be approximated with arbitrarily high precision (under dilation) by an infinite collection of pairwise disjoint compact subdomains of $\Sigma$, and these approximations can be chosen not too far from each other. This property can be considered to be a weak notion of periodicity; next we describe a more precise statement for this ‘weak periodicity’ property for $\Sigma$.

As $\Sigma \in D(M)$ is a minimal element, there exists a minimal $D$-invariant subset $\Delta \subset D(M)$ such that $\Sigma \in \Delta$. Since $\Sigma$ has infinite total curvature, then $D(\Sigma) = \Delta$, in particular $\Sigma \in D(\Sigma)$ (i.e., $\Sigma$ is quasi-dilation-periodic). Given $R > 0$ such that $\Sigma$ intersects $S^2(R)$ transversely, let $\Sigma(R) = \Sigma \cap B(R)$. Since $\Sigma \in D(\Sigma)$, then for every small $\varepsilon > 0$ there exists a collection $\{\mathcal{B}_n = B(p_n, R_n)\}_n$ of disjoint closed balls such that the surfaces $\Sigma_n = \frac{R}{R_n}(\Sigma \cap \mathcal{B}_n) - p_n$ can be parameterized by $\Sigma(R)$ in such a way that as mappings, they are $\varepsilon$-close to $\Sigma(R)$ in the $C^2$-norm. By Zorn’s lemma, any such collection of balls $\{\mathcal{B}_n\}_n$ is contained in a maximal collection $\text{Max}(\Sigma, R, \varepsilon)$ of closed round balls $\mathcal{B} = B(p, r)$ so that $\frac{1}{r}[(\Sigma \cap \mathcal{B}) - p]$ can be parameterized by $\Sigma(R)$ in such a way that as mappings, they are $\varepsilon$-close in the $C^2$-norm. After denoting the elements of $\text{Max}(\Sigma, R, \varepsilon)$ by $\mathcal{B}_n$, $n \in \mathbb{N}$, we define for every $n \in \mathbb{N}$ the positive number

$$d(n; R, \varepsilon) = \inf_{m \neq n} \left\{ \frac{1}{R_n} \text{dist}_{\mathbb{R}^3}(\mathcal{B}_n, \mathcal{B}_m) \mid \mathcal{B}_m \in \text{Max}(\Sigma, R, \varepsilon), \ m \neq n \right\},$$

(8)

where $\mathcal{B}_n, \mathcal{B}_m \in \text{Max}(\Sigma, R, \varepsilon)$. Hence, $d(n; R, \varepsilon)$ measures the minimum relative distance from the ball $\mathcal{B}_n \in \text{Max}(\Sigma, R, \varepsilon)$ to any other ball $\mathcal{B}_m$ in the collection $\text{Max}(\Sigma, R, \varepsilon)$, where by relative we mean that $\mathcal{B}_n$ is normalized to have radius 1; this is the task of dividing by $R_n$ in (8). In this situation we will prove the following property, which expresses in a precise way the ‘weak periodicity’ mentioned in the first paragraph of this section.

**Proposition 6.1** If $\Sigma(R)$ is not an $\mathfrak{A}$-graph over a disk in a plane, then the sequence $\{d(n; R, \varepsilon)\}_n \subset (0, \infty)$ is bounded from above (the bound depends on $\Sigma, R, \varepsilon$).

**Proof.** We argue by contradiction. Otherwise, there exists a sequence of integers $\{n(i)\}_{i \in \mathbb{N}}$ such that $d(n(i); R, \varepsilon) \geq i$ for all $i \in \mathbb{N}$. We define $\hat{\Sigma}(i) = \frac{R}{R_{n(i)}}(\Sigma - p_{n(i)})$, $i \in \mathbb{N}$. Observe that the following two properties hold:

(P1) The Gaussian curvature of $\{\hat{\Sigma}(i) \cap B(R)\}_i$ is uniformly bounded (by the defining properties of the balls in the family $\text{Max}(\Sigma, R, \varepsilon)$).

(P2) Given $\mathcal{B}_m \in \text{Max}(\Sigma, R, \varepsilon)$ with $m \neq n(i)$, the closed ball $\mathcal{B}_m' = \frac{R}{R_{n(i)}}(\mathcal{B}_m - p_{n(i)})$ is at distance at least $iR$ from $B(R)$.
We claim that the sequence of surfaces $\tilde{\Sigma}(i)$ has locally bounded Gaussian curvature outside of $\mathbb{B}(\Sigma)$; otherwise, the Gaussian curvature $K_{\tilde{\Sigma}(i)}$ blows-up around a point $P \in \mathbb{R}^3 - \mathbb{B}(\Sigma)$, and in this case we can blow-up $\tilde{\Sigma}(i)$ on the scale of curvature around points of almost-maximal curvature tending to $P$, thereby obtaining a new limit surface $\Sigma'$ of rescaled copies of portions of $\tilde{\Sigma}(i)$ in small extrinsic balls around $P$ (in particular, these small balls are disjoint from the balls corresponding to elements in $\operatorname{Max}(\Sigma, R, \varepsilon)$). Hence, $\Sigma'$ lies in $D(\Sigma)$; since $\Sigma$ is a minimal element, then $\Sigma \in D(\Sigma')$, which contradicts the maximality of the collection $\operatorname{Max}(\Sigma, R, \varepsilon)$. This contradiction proves our claim.

As $\{\tilde{\Sigma}(i) \cap [\mathbb{R}^3 - \mathbb{B}(\Sigma)]\}_i$ has locally bounded Gaussian curvature, we conclude that after replacing by a subsequence, $\{\tilde{\Sigma}(i) \cap [\mathbb{R}^3 - \mathbb{B}(2R)]\}_i$ converges as $n \to \infty$ to a minimal lamination $\mathcal{L}'(2R)$ of $\mathbb{R}^3 - \mathbb{B}(2R)$. This lamination $\mathcal{L}'(2R)$ has quadratic decay of curvature (otherwise we again contradict the minimality of $\Sigma$ and the maximality of $\operatorname{Max}(\Sigma, R, \varepsilon)$ as before), and so, $\mathcal{L}'(2R)$ has bounded curvature in $\mathbb{R}^3 - \mathbb{B}(3R)$.

Next we claim that the surfaces $\tilde{\Sigma}(i) \cap \mathbb{B}(3R)$ do not have uniformly bounded curvature. Arguing again by contradiction, the failure of our claim together with the arguments in the last paragraph imply that $\{\tilde{\Sigma}(i)\}_i$ converges to a lamination $\mathcal{L}'$ of $\mathbb{R}^3$ with quadratic curvature decay. Since we are assuming that $\Sigma(R)$ is not an $\mathcal{T}$-graph over a flat disk, then $\mathcal{L}'$ cannot be flat. Hence, Proposition 2.5 implies that $\mathcal{L}'$ consists of a single leaf which is a properly embedded minimal surface $\Sigma'$ with finite total curvature. As before, this implies that $\Sigma' \in D(\Sigma)$. This is absurd, since then $\Sigma \in D(\Sigma') = \emptyset$. This contradiction shows that the $\tilde{\Sigma}(i) \cap \mathbb{B}(3R)$ do not have uniformly bounded curvature.

Therefore, after extracting a subsequence we can find for each $i \in \mathbb{N}$ a point $\tilde{p}(i) \in \tilde{\Sigma}(i) \cap \mathbb{B}(3R)$ such that $|K_{\tilde{\Sigma}(i)}(\tilde{p}(i))| \to \infty$ as $i \to \infty$. We can also assume that $|K_{\tilde{\Sigma}(i)}|$ attains its maximum value in the compact set $\tilde{\Sigma}(i) \cap \mathbb{B}(3R)$ at $\tilde{p}(i)$, for all $i$. After translating by $-\tilde{p}(i)$, rescaling by $\sqrt{|K_{\tilde{\Sigma}(i)}(\tilde{p}(i))|}$ and extracting another subsequence, we obtain a new limit surface $\Sigma'' \in D_1(\Sigma)$. We now have two possibilities, depending on whether or not the sequence of open balls $\{\sqrt{|K_{\tilde{\Sigma}(i)}(\tilde{p}(i))(\mathbb{B}(R) - \tilde{p}(i))|}\}_i$ eventually leaves every compact set of $\mathbb{R}^3$.

Firstly suppose that the sequence of open balls $\{\sqrt{|K_{\tilde{\Sigma}(i)}(\tilde{p}(i))(\mathbb{B}(R) - \tilde{p}(i))|}\}_i$ fails to leave every compact set of $\mathbb{R}^3$. Then, after choosing a subsequence these balls converge to a closed halfspace $H^+$ and by property (P1) above, $\Sigma'' \cap H^+$ is flat. As $\Sigma''$ is not flat, then we conclude that $\Sigma''$ cannot intersect $H^+$, or equivalently $\Sigma''$ is contained in a halfspace. This contradicts the fact that the Gaussian curvature of the non-flat surface $\Sigma''$ is bounded, see [20]. Therefore, the sequence of balls $\{\sqrt{|K_{\tilde{\Sigma}(i)}(\tilde{p}(i))(\mathbb{B}(R) - \tilde{p}(i))|}\}_i$ leaves every compact set of $\mathbb{R}^3$. Since $\Sigma'' \in D(\Sigma)$ and $\Sigma$ is a minimal element, then $\Sigma \in D(\Sigma'')$, which by the same arguments as before contradicts the maximality of the collection $\operatorname{Max}(\Sigma, R, \varepsilon)$. Now the proposition is proved. \qed
Remark 6.2 We remark that in general there exist properly embedded minimal surfaces $M$ with infinite total curvature, such that $D(M)$ contains more than one minimal $D$-invariant set, see some examples in the discussion before Theorem 11.0.13 in [13].

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21
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