THE RATIONALITY OF THE HILBERT-KUNZ
MULTIPlicITY IN GRADED DIMENSION TWO

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Abstract. We show that the Hilbert-Kunz multiplicity is a rational
number for an $R_+-$primary homogeneous ideal $I = (f_1, \ldots, f_n)$ in a
two-dimensional graded domain $R$ of finite type over an algebraically
closed field of positive characteristic. More specific, we give a formula for
the Hilbert-Kunz multiplicity in terms of certain rational numbers com-
ing from the strong Harder-Narasimhan filtration of the syzygy bundle
Syz$(f_1, \ldots, f_n)$ on the projective curve $Y = \Proj R$.

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Introduction

Suppose that $(R, \mathfrak{m})$ is a local Noetherian Ring of dimension $d$ containing
a field $K$ of positive characteristic $p$. Let $I = (f_1, \ldots, f_n)$ be an
$m$-primary ideal, and denote by $I^{[q]} = (f_1^q, \ldots, f_n^q)$ the ideal given by the powers
$f_i^q$, where $q = p^e$. The ideal $I^{[q]}$ is the extended ideal of $I$ under the $e$-
th Frobenius homomorphism $R \rightarrow R$, hence independent of the choice of
generators. Since $I$ is $m$-primary, the length of the residue class ring $R/I^{[q]}$
is finite for every prime power $q$.

The function $e \mapsto \lambda(R/I^{[p^e]})$, where $\lambda$ denotes the length, is called the
Hilbert-Kunz function of the ideal $I$ and was introduced in [14] (see also
[15]). The Hilbert-Kunz multiplicity is defined as the limit

$$e_{HK}(I) = \lim_{e \to \infty} \frac{\lambda(R/I^{[p^e]})}{p^{ed}}.$$  

This limit exists as a positive real number, as shown by Monsky in [19] (see also
[13] Chapter 6) and [20] I.7.3). In the same paper, Monsky writes, “we
suspect, but have no idea how to prove, that $e(M)$[that is $e_{HK}$] is always
rational”. C. Huneke has put this question on his top ten list of problems
in commutative algebra [14]. The Hilbert-Kunz multiplicity of the maximal
ideal is also called the Hilbert-Kunz multiplicity $e_{HK}(R)$ of the ring itself
and gives an important invariant, but even in this case the rationality is only
known in some special cases.

Let us briefly recall what is known up to now. Most rationality results
deal only with the maximal ideal in hypersurface rings of special type. Han
and Monsky succeeded in the computation of the Hilbert-Kunz multiplicity
of a Brieskorn hypersurface $X_0^{\delta_0} + \ldots + X_N^{\delta_N}$ [7], and Conca provided a
formula for it in the case of a homogeneous binomial hypersurface of type $X_0^\delta \cdots X_m^{\delta_m} - X_{m+1}^{\delta_{m+1}} \cdots X_N^{\delta_N}$. The rationality of the Hilbert-Kunz multiplicity for cones $K[X,Y,Z]/(H)$ over a plane cubic curve $V_+(H) \subset \mathbb{P}^2$ was shown by Buchweitz, Chen and Pardue. It is equal to $9/4$ in the smooth case and $7/3$ in the singular case, see [4]. This was generalized by Fakhruddin and Trivedi in [6] to any cone over an elliptic curve. In his thesis [22], Teixeira obtained the rationality of the Hilbert-Kunz multiplicity in the case of hypersurfaces of type $H = \sum_i G_i(x_i,y_i)$, where the $G_i$ are homogeneous.

For more general ideals, it is well known that the Hilbert-Kunz multiplicity of a monomial ideal in a toric ring is rational [24]. Watanabe and Yoshida give a formula for the Hilbert-Kunz multiplicity of an integrally closed ideal in a two-dimensional Gorenstein quotient singularity in terms of data coming from a minimal resolution (see [25], [27], [26]). For other results concerning estimates for the Hilbert-Kunz multiplicity and the relationship to other ring-theoretic properties consult [1], [8], [12], [17], [29], [28].

In this paper we show that the Hilbert-Kunz multiplicity $e_{HK}(I)$ is indeed a rational number, where $R$ is a normal standard-graded two-dimensional domain of finite type over an algebraically closed field of positive characteristic, and where $I$ denotes a homogeneous $R_+-$primary ideal. The main idea is to use the short exact sequence (set $d_i = \deg(f_i)$) of locally free sheaves

$$0 \to \text{Syz}(f_1^q, \ldots, f_n^q)(m) \to \bigoplus_{i=1}^n \mathcal{O}(m-qd_i) \xrightarrow{f_1^q \cdots f_n^q} \mathcal{O}(m) \to 0$$

on the smooth projective curve $Y = \text{Proj} R$ and to compute

$$\lambda((R/I[q])_m) = h^0(\mathcal{O}(m)) - \sum_{i=1}^n h^0(\mathcal{O}(m-qd_i)) + h^0(\text{Syz}(f_1^q, \ldots, f_n^q)(m)).$$

One computes $\lambda(R/I[q])$ by summing over $m$, which is a finite sum since this alternating sum is 0 for $m \gg 0$. Of course the crucial point is to control the behavior of the global syzygies $H^0(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m))$ for different values of $q$ and $m$. Hence we are concerned with a Frobenius-Riemann-Roch problem. We have $\text{Syz}(f_1^q, \ldots, f_n^q)(0) = F^{e*}(\text{Syz}(f_1, \ldots, f_n)(0))$, where $q = p^e$ and $F : Y \to Y$ is the absolute Frobenius morphism; so we have to understand the global sections of the Frobenius pull-backs of the syzygy bundle $\text{Syz}(f_1, \ldots, f_n)$. The behavior of a locally free sheaf under the Frobenius morphism is full of surprising phenomena. For example, the Frobenius pull-back of a semistable sheaf need not be semistable.

However, a recent theorem of A. Langer [16] Theorem 2.7] shows for a locally free sheaf $\mathcal{S}$ on a smooth projective variety $Y$ that the Harder-Narasimhan filtration of some Frobenius pull-back has strongly semistable quotients. This means that for $e$ big enough there exists a filtration

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$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = F^{e*}(\mathcal{S})$$
such that the quotients $S_k/S_{k-1}$ are strongly semistable of decreasing slopes $\mu_k(F^r(S)) = \mu(S_k/S_{k-1})$. Strongly semistable means that every Frobenius pull-back is also semistable.

The existence of this strong Harder-Narasimhan filtration allows in particular to define for $k = 1, \ldots, t$ rational numbers by setting $\bar{\mu}_k(S) := \frac{\mu_k(F^r(S))}{p^k}$. These numbers, the length $t$ of the strong Harder-Narasimhan filtration and the ranks $r_k = \text{rk}(S_k/S_{k-1})$ are all independent of $q \gg 0$. Applying this to the syzygy bundle $S = \text{Syz}(f_1, \ldots, f_n)$ we get numbers which control the global syzygies for varying $q$. Therefore they enter (we set $\nu_k = -\bar{\mu}_k/\deg(Y)$) into the following simple formula for the Hilbert-Kunz multiplicity, which is our main result and gives the rationality (Theorem 3.6).

**Theorem 1.** Let $R$ denote a two-dimensional standard-graded normal domain and let $I = (f_1, \ldots, f_n)$ denote a homogeneous $R_+$-primary ideal generated by homogeneous elements $f_i$ of degree $d_i$, $i = 1, \ldots, n$. Then the Hilbert-Kunz multiplicity $e_{HK}(I)$ is given by

$$\frac{\deg(Y)}{2} \left( \sum_{k=1}^{t} r_k \nu_k^2 - \sum_{i=1}^{n} d_i^2 \right).$$

In particular, the Hilbert-Kunz multiplicity is a rational number.

The rationality is also true without the assumptions normal and standard-graded (see Corollary 3.7). As an easy corollary we get a description for the Hilbert-Kunz multiplicity for cones over plane curves (Corollary 4.6). This was only known for degree $h \leq 3$ so far. Furthermore, this corollary is independent of the result of Langer, since the existence of the strong Harder-Narasimhan filtration is clear by elementary means in this case.

**Corollary 1.** Let $C = V_+(H) \subset \mathbb{P}^2$ denote a smooth plane projective curve of degree $h$, $R = K[X, Y, Z]/(H)$. Then there exists a rational number $\frac{3}{2} \leq \nu_2 \leq 2$ such that the Hilbert-Kunz multiplicity of $R$ is

$$e_{HK}(R) = h(\nu_2^2 - 3\nu_2 + 3).$$

The rationality of the Hilbert-Kunz multiplicity for the maximal ideal in dimension two was proved independently by V. Trivedi in [23]. I thank M. Blickle and K. Watanabe for useful discussions and the referee for useful comments.

1. **Preliminaries**

We recall briefly some notions about vector bundles, see [13] for details. Let $Y$ denote a smooth projective curve over an algebraically closed field $K$. The degree of a locally free sheaf $S$ on $Y$ of rank $r$ is defined by $\deg(S) = \deg \wedge^r(S)$. The slope of $S$, written $\mu(S)$, is defined by $\deg(S)/r$. The slope has the property that $\mu(S \otimes T) = \mu(S) + \mu(T)$. Both the degree and the
there exists a Frobenius power $F^e$ of the Frobenius does not in general give the Harder-Narasimhan filtration of the absolute Frobenius $F$.

In positive characteristic, the pull-back of a semistable bundle under the absolute Frobenius $F^e : Y \to Y$ is in general not semistable. If it stays semistable for every Frobenius power, then the bundle is called strongly semistable, a notion introduced by Miyaoka in [18]. Consequently, the pull-back of the Harder-Narasimhan filtration of $S$ under the Frobenius does not in general give the Harder-Narasimhan filtration of $F^e(S)$. However, a recent result of A. Langer [16, Theorem 2.7] shows that there exists a Frobenius power $F^e$ such that the quotients in the Harder-Narasimhan filtration of the pull-back $F^e(S)$ are all strongly semistable. We call such a filtration the strong Harder-Narasimhan filtration and denote it by

$$0 \subset S^q_1 \subset \ldots \subset S^q_t = F^e(S).$$

For $e' \geq e$ the Harder-Narasimhan filtration of $F^{e+e'}(S)$ is

$$S^q_1 = F^{(e'-e)*}(S^q_1) \subset \ldots \subset S^q_t = F^{(e'-e)*}(S^q_t).$$

Using this we define rational numbers $\tilde{\mu}_k = \tilde{\mu}_k(S) = \frac{\mu(S^q_k/S^q_{k-1})}{q}$ for $q \gg 0$. The length $t$ of the strong Harder-Narasimhan filtration as well as the ranks $r_k = \text{rk}(S^q_k/S^q_{k-1})$ are independent of $q \gg 0$. For $q \gg 0$ we have $\mu(S^q_k/S^q_{k-1}) = q\tilde{\mu}_k$ and $\text{deg}(S^q_k/S^q_{k-1}) = qr_k\tilde{\mu}_k$. Furthermore note that

$$\text{rk}(S^q_k) = r_1 + \ldots + r_k \text{ and } \text{deg}(S^q_k) = q(r_1\tilde{\mu}_1 + \ldots + r_k\tilde{\mu}_k).$$

We also set $\tilde{\mu}_{\max}(S) = \tilde{\mu}_1(S)$ and $\tilde{\mu}_{\min}(S) = \tilde{\mu}_t(S)$.

We shall apply these notions and facts to syzygy bundles. Let $R$ denote a normal standard-graded\footnote{Throughout this paper the assumption standard-graded, meaning that $R$ is generated by finitely many elements of degree one, might be weaken to the property that $R$ is an $\mathbb{N}$-graded domain of finite type and that there exists finitely many elements $x_k$ of degree one such that $\text{Proj} R = \bigcup_k D_+(x_k)$. This last property is enough to ensure that $\mathcal{O}(1)$} domain over an algebraically closed field (of any
characteristic) and let \( f_1, \ldots, f_n \) denote homogeneous generators of an \( R_+ \)-primary ideal of degrees \( d_1, \ldots, d_n \). This gives rise to the short exact sequence of locally free sheaves on \( Y = \text{Proj} \, R \):

\[
0 \longrightarrow \text{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(m - d_i) \stackrel{f_1 \ldots f_n}{\longrightarrow} \mathcal{O}(m) \longrightarrow 0.
\]

For \( m = 0 \) we write also \( \text{Syz}(f_1, \ldots, f_n) \) instead of \( \text{Syz}(f_1, \ldots, f_n)(0) \). Due to this defining sequence, the rank of the syzygy bundle is \( n-1 \) and its degree is \( ((n-1)m - \sum_{i=1}^{n} d_i) \deg \mathcal{O}_Y(1) \).

Now suppose that the algebraically closed ground field \( K \) has positive characteristic. The pull-back of the short exact sequence of locally free sheaves under the \( e \)-th absolute Frobenius morphism \( F^e : Y \to Y \) yields

\[
0 \longrightarrow (F^e(\text{Syz}(f_1, \ldots, f_n)))(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(m - qd_i) \stackrel{f_1^q \ldots f_n^q}{\longrightarrow} \mathcal{O}(m) \longrightarrow 0
\]

(pull-back the sequence for \( m = 0 \) and tensor it with \( \mathcal{O}(m) \) again). Therefore \( (F^e(\text{Syz}(f_1, \ldots, f_n)))(m) = \text{Syz}(f_1^q, \ldots, f_n^q)(m) \). We want to compute \( \lambda((R/I^{[q]}))^m) \) using this exact sequence. The global sections \( \Gamma(Y, -) \) of this sequence yield (since \( R \) is assumed to be normal and standard-graded)

\[
0 \longrightarrow \Gamma(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m)) \longrightarrow \bigoplus_{i=1}^{n} R_{m-qd_i} \stackrel{f_1^q \ldots f_n^q}{\longrightarrow} R_m \longrightarrow \ldots
\]

and the cokernel of the last mapping is \( (R/I^{[q]})^m \). This is the same as the kernel of the mapping \( H^1(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m)) \to \bigoplus_{i=1}^{n} H^1(Y, \mathcal{O}(m - qd_i)) \).

Hence we compute

\[
\lambda((R/I^{[q]})_m) = h^0(\mathcal{O}(m)) - \sum_{i=1}^{n} h^0(\mathcal{O}(m - qd_i)) + h^0(\text{Syz}(f_1^q, \ldots, f_n^q)(m))
\]

and then we sum over \( m \).

\[\text{2. The case of a strongly semistable syzygy bundle}\]

In this section we prove some results about the asymptotic behavior of \( h^0(\mathcal{O}(m - qd_i)) \) and \( h^0(\mathcal{O}(m)) \) and we apply this to compute the Hilbert-Kunz multiplicity under the condition that the syzygy bundle is strongly semistable. We fix the following situation.

**Situation 2.1.** Let \( R \) denote a normal two-dimensional standard-graded domain over an algebraically closed field of positive characteristic \( p \) with corresponding smooth projective curve \( Y = \text{Proj} \, R \) of genus \( g \). Set \( \deg(Y) = \deg(\mathcal{O}_Y(1)) \). Let \( (f_1, \ldots, f_n) \) denote a homogeneous, \( R_+ \)-primary ideal given by homogeneous ideal generators of degree \( d_i \). Let \( q = p^e \) denote varying prime powers.

is an invertible sheaf and makes therefore everything work (look e.g. at the proof of Proposition II.5.12); see also Corollary 3.7 below.
We will often use the notation $O(g)$ for the asymptotic behavior of a function $f(q)$ in one variable. The equation $f = O(g)$ means that $f/g$ is bounded for $q \to \infty$. The functions we consider will be defined only for prime powers $q = p^e$, hence $f = O(g)$ means that $f(q)/q$ is bounded. Such functions are negligible in our situation, since then $f(q)/q^2 \to 0$.

**Lemma 2.2.** Suppose the situation of 2.1. Let $\nu$ denote a positive rational number. For $i = 1, \ldots, n$, if $\nu \geq d_i$, we have

$$\sum_{m=0}^{[\nu]} h^0(\mathcal{O}(m - qd_i)) = q^2 \frac{\deg(Y)}{2} (\nu - d_i)^2 + O(q).$$

and

$$\sum_{m=0}^{[\nu]} h^0(\mathcal{O}(m)) = q^2 \frac{\deg(Y)}{2} \nu^2 + O(q).$$

**Proof.** By Riemann-Roch we have

$$h^0(\mathcal{O}(m - qd_i)) = (m - qd_i) \deg(Y) + 1 - g + h^1(\mathcal{O}(m - qd_i)).$$

Therefore we have $\sum_{m=0}^{[\nu]} h^0(\mathcal{O}(m - qd_i)) = $.

$$= \sum_{m=0}^{[\nu]} h^0(\mathcal{O}(m - qd_i))$$

$$= \sum_{m=0}^{[\nu]} \left( (m - qd_i) \deg(Y) + 1 - g + h^1(\mathcal{O}(m - qd_i)) \right)$$

$$= \deg(Y) \sum_{m=0}^{[\nu]} (m - qd_i) + ([\nu] - qd_i + 1)(1 - g) + \sum_{m=0}^{[\nu]} h^1(\mathcal{O}(m - qd_i))$$

$$= \frac{\deg(Y)}{2} ([\nu] - qd_i + 1) ([\nu] - qd_i) + O(q) + O(q^0)$$

$$= \frac{\deg(Y)}{2} q^2 (\nu - d_i)^2 + O(q)$$

Here we used on the right that $H^1(Y, \mathcal{O}(m - qd_i)) = 0$ for $m - qd_i \gg 0$, and this bound is independent of $q$. The proof for the statement about $\mathcal{O}(m)$ is the same. \hfill \square

**Lemma 2.3.** Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve $Y$ with a very ample invertible sheaf $\mathcal{O}(1)$ of degree $\deg(\mathcal{O}(1)) = \deg(Y)$. Denote the pull-back of $\mathcal{S}$ under the $e$-th absolute Frobenius by $\mathcal{S}^q$, $q = p^e$. Set $\nu = -\frac{\mu_{\min}(\mathcal{S})}{\deg(Y)}$. Then

$$\sum_{m=0}^{[\nu]} h^1(Y, \mathcal{S}^q(m)) = O(q).$$

**Proof.** The minimal slope of $\mathcal{S}^q$ is $-q \deg(Y) \nu$ for $q$ big enough. By Serre duality we have

$$h^1(\mathcal{S}^q(m)) = h^0((\mathcal{S}^q)^* (-m) \otimes \omega_Y).$$
Now for $m > [qν] + \deg(ω_Y)/\deg(Y)$ we have
\[
\mu_{\max}((S^q)^{∨}(-m) \otimes ω_Y) = -μ_{\min}(S^q(m)) + μ(ω_Y) \\
= -μ_{\min}(S^q) - m \deg(Y) + \deg(ω_Y) \\
= -qμ_{\min}(S) - m \deg(Y) + \deg(ω_Y) \\
< 0.
\]
So for these $m$ we have $H^1(Y, S^q(m)) = 0$ and our sum is indeed finite running in the range $[qν] \leq m \leq [qν] + \deg(ω_Y)/\deg(Y)$. In particular, the length of this range is independent of $q$.

There exists a surjection $\bigoplus_{j ∈ J} O(α_j) → S → 0$. Pulling this back under $F^e$ we get surjections $\bigoplus_{j ∈ J} O(qα_j) → S^q → 0$ and therefore
\[
\bigoplus_{j ∈ J} H^1(Y, O(qα_j + m)) → H^1(Y, S^q(m)) → 0.
\]
For $m$ fulfilling $[qν] \leq m \leq [qν] + \deg(ω_Y)/\deg(Y)$ we see that $qα_j + m$ varies in a range between $qβ_j$ and $qβ_j + c$ ($c$ and $β_j$ independent of $q$). We have to understand the asymptotic behavior of $h^1(O(qβ_j + ℓ)), 0 ≤ ℓ ≤ c$ for $q$ large. But $h^1(O(qβ_j + ℓ)) = h^1(O(−qβ_j − ℓ) \otimes ω_Y)$ goes to $0$ for $β_j > 0$ and it is $O(q)$ for $β_j ≤ 0$. So in any case the first cohomology is $O(q)$ and the same is true for the finite sums over all $ℓ$ and $j ∈ J$.

**Corollary 2.4.** Suppose the situation [2.7]. Suppose that the syzygy bundle $\text{Syz}(f_1, \ldots, f_n)$ is strongly semistable. Then
\[
\sum_{m=[q^d_1+\ldots+d_n]}^{∞} h^1(\text{Syz}(f_1^q, \ldots, f_n^q)(m)) = O(q).
\]

**Proof.** This follows from Lemma [2.3] applied to $S = \text{Syz}(f_1, \ldots, f_n)$, since in this case $ν = d_1+\ldots+d_n/n-1$.

**Theorem 2.5.** Suppose the situation of [2.4]. Suppose that the syzygy bundle is strongly semistable. Then the Hilbert-Kunz function of $I = (f_1, \ldots, f_n)$ may be written as
\[
φ(I, q) = q^2 \frac{\deg(Y)}{2} \left( \frac{(\sum_i d_i)^2}{n-1} - \sum_i d_i^2 \right) + O(q)
\]
and the Hilbert-Kunz multiplicity is
\[
e_{HK}(I) = \frac{\deg(Y)}{2} \left( \frac{(\sum_i d_i)^2}{n-1} - \sum_i d_i^2 \right).
\]
In particular it is a rational number.

**Proof.** We have the equations $λ(R/I^q) = \sum_{m=0}^{∞} λ((R/I^q)_m) = \sum_{m=0}^{∞} λ(R/I^q)_m = \sum_{m=0}^{∞} \sum_{j ∈ J} λ((R/I^q)_m)$
Proof. This follows from Theorem 2.5, since $d = d_i$ is strongly semistable and that the degrees of the ideal generators are constant, $d = d_i$. Then the Hilbert-Kunz multiplicity of $I = (f_1, \ldots, f_n)$ is

$$e_{HK}(I) = \frac{\deg(Y) nd^2}{2n - 1}.$$ 

Proof. This follows from Theorem 2.3 since

$$\frac{(nd)^2}{n - 1} - nd^2 = \frac{n^2d^2 - n(n - 1)d^2}{n - 1} = \frac{n}{n - 1}d^2.$$ 

\[ \square \]

Corollary 2.7. Let $Y \subset \mathbb{P}^N$ denote a smooth projective curve, $Y = \text{Proj } R$, $R = K[X_0, \ldots, X_N]/\mathfrak{a}$. Suppose that the restriction of the tangent bundle $T_{\mathbb{P}^N}$ to the curve is strongly semistable. Then the Hilbert-Kunz multiplicity of $R$ is $e_{HK}(R) = \frac{\deg(Y) N + 1}{2N}$. 

Proof. We have to compute the Hilbert-Kunz multiplicity of the maximal ideal $m = (X_0, \ldots, X_N)$. The syzygy bundle $\text{Syz}(X_0, \ldots, X_N)$ on $\mathbb{P}^N$ is the same as the cotangent bundle of $\mathbb{P}^N$ due to the Euler sequence (see 9).
Hilbert-Kunz multiplicity is $3$ that there exists for every degree $h = \deg(H) = \deg(C)$. If the syzygy bundle $\text{Syz}(X, Y, Z)|C$ is strongly semistable, then we get $\epsilon_{HK}(R) = \frac{3h}{4}$. It was indeed a result of [4, Corollary 1] that there exists for every degree $h \geq 2$ a plane curve of degree $h$ whose Hilbert-Kunz multiplicity is $\frac{3h}{4}$.

3. Main results

We treat now the case of an arbitrary syzygy bundle on a curve making use of the strong Harder-Narasimhan filtration. The knowledge of the Harder-Narasimhan filtration of a locally free sheaf $S$ contains a lot of information about the behavior of the global sections $H^0(Y, S(m))$, as the following Lemma shows.

**Lemma 3.1.** Let $S$ denote a locally free sheaf on a smooth projective curve $Y$ of genus $g$ over an algebraically closed field. Let $O_Y(1)$ be a very ample invertible sheaf and set $\text{deg}(Y) = \text{deg}O_Y(1)$. Let $S_1 \subset \ldots \subset S_t = S$ be the Harder-Narasimhan filtration of $S$ and let $\mu_k(S) = \mu(S_k/S_{k-1})$ denote the slopes of the semistable quotient sheaves in this filtration and set $r_k = \text{rk}(S_k/S_{k-1})$. Then we have the following statements about the global sections and the first cohomology of $S(m)$.

(i) For $\mu_1(S(m)) < 0$ we have $H^0(Y, S(m)) = 0$

(ii) Fix $k$, $1 \leq k \leq t - 1$. Let $m$ be such that $\mu_k(S(m)) > \text{deg}(\omega_Y)$ and $\mu_{k+1}(S(m)) < 0$. Then $H^0(Y, S(m)) \cong H^0(Y, S_k(m))$ and

$$h^0(S(m)) = \text{deg}(S_k(m)) + \text{rk}(S_k)(1 - g)$$

$$= m(r_1 + \ldots + r_k) \text{deg}(Y) + r_1\mu_1 + \ldots + r_k\mu_k + \text{rk}(S_k)(1 - g)$$

(iii) For $\mu_t(S(m)) > \text{deg}(\omega_Y)$ we have $H^1(Y, S(m)) = 0$.

**Proof.** The condition in (i) means that the maximal slope of the sheaf $S(m)$ is negative, hence it cannot have non-trivial global sections. (iii). By Serre duality we have $h^1(S(m)) = h^0(S^\vee(-m) \otimes \omega_Y)$ and we have

$$\mu_{\text{max}}(S^\vee(-m) \otimes \omega_Y) = -\mu_{\text{min}}(S(m)) + \text{deg}(\omega_Y) = -\mu_t(S(m)) + \text{deg}(\omega_Y) < 0,$$

hence $H^1(Y, S(m)) = 0$.

(ii). Look at the sequence $0 \to S_k(m) \to S(m) \to S(m)/S_k(m) = Q_k(m)$. Then we have $\mu_{\text{min}}(S_k(m)) = \mu((S_k/S_{k-1})(m)) = \mu_k(S(m)) > \text{deg}(\omega_Y)$ and $\mu_{\text{max}}(Q_k(m)) = \mu((S_{k+1}/S_k)(m)) = \mu_{k+1}(S(m)) < 0$. Due to this last observation, $Q_k(m)$ does not have global non-trivial sections. Therefore

$$H^0(Y, S(m)) \cong H^0(Y, S_k(m))$$.
By Riemann Roch for locally free sheaves we have
\[ h^0(S_k(m)) = \deg(S_k(m)) + \text{rk}(S_k)(1 - g) + h^1(S_k(m)). \]
Serre duality yields
\[ h^1(S_k(m)) = h^0((S_k(m))^\vee \otimes \omega_Y). \]
Now
\[ \mu_{\text{max}}((S_k(m))^\vee \otimes \omega_Y) = -\mu_{\text{min}}(S_k(m)) + \deg(\omega_Y) < 0. \]
Hence \((S_k(m))^\vee \otimes \omega_Y\) does not have any non-trivial global section, therefore \(h^1(S_k(m)) = 0\) and we obtain the first equation. The second equation is clear due to
\[ \deg(S_k(m)) = \deg(S_k) + \text{rk}(S_k) \deg(O(m)) = r_1\mu_1 + \ldots + r_k\mu_k + m(r_1 + \ldots + r_k) \deg(Y). \]

We fix now the situation and notation for the results in this section.

**Situation 3.2.** Let \(R\) denote a two-dimensional normal standard-graded (see footnote 1) domain over an algebraically closed field \(K\) of positive characteristic \(p\) and let \(Y = \text{Proj} R\) denote the corresponding smooth projective curve of genus \(g\). Let \(\deg(Y)\) denote the degree of \(O_Y(1) = O(1)\) given by \(R\). Let \(I = (f_1, \ldots, f_n)\) denote an \(R_+\)-primary homogeneous ideal generated by homogeneous elements \(f_i\) of degree \(d_i\). Let \(\text{Syz}(f_1, \ldots, f_n)\) be the syzygy bundle on \(Y\) and let \(\bar{\mu}_k (r_k)\) denote the slopes (ranks) of the quotients in the strong Harder-Narasimhan filtration as explained in the preliminaries and let \(t\) denote its length. It is convenient to set \(\nu_k := -\bar{\mu}_k / \deg(Y)\).

**Remark 3.3.** The sum \(r_1 + \ldots + r_t = n - 1\) equals the rank of the syzygy bundle \(\text{Syz}(f_1, \ldots, f_n)\). We have the relationship \(\sum_{k=1}^{t} \nu_k r_k = \sum_{i=1}^{n} d_i\) and the estimates
\[ \min(d_i) \leq \nu_1 < \ldots < \nu_t \leq \max_{i \neq j}(d_i + d_j). \]
This follows from the defining sequence for the syzygy bundle and its Koszul resolution. If \(t = 1\), then \(\nu_1 = \frac{d_1 + \ldots + d_n}{n-1}\). Think of the rational numbers \(\nu_k\) as degree thresholds, where something happens in the behavior of the global syzygies \(H^0(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m))\), when \(m\) passes through \(\nu_k\).

**Proposition 3.4.** Suppose the situation and the notation of Sit. 3.2 Let \(e\) be big enough such that the Harder-Narasimhan filtration of \(F^e(\text{Syz}(f_1, \ldots, f_n)) = \text{Syz}(f_1^q, \ldots, f_n^q)\) is strong, \(q = p^e\). Then the global syzygies have the following description.

(i) For \(m < q \nu_1 = -q \frac{\bar{\mu}_1}{\deg(Y)} = -q \frac{\bar{\mu}_{\text{max}}(\text{Syz}(f_1, \ldots, f_n))}{\deg(Y)}\) we have
\[ H^0(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m)) = 0. \]

(ii) Fix \(k, 1 \leq k \leq t - 1\). For \(q \nu_k + \frac{\deg(\omega_Y)}{\deg(Y)} < m < q \nu_{k+1}\) we have
\[ h^0(Y, \text{Syz}(f_1^q)(m)) = q(r_1\bar{\mu}_1 + \ldots + r_k\bar{\mu}_k) + m(r_1 + \ldots + r_k) \deg(Y) + \text{rk}(S_k)(1 - g). \]
(iii) For \( m > q\nu_t + \frac{\deg(\omega_Y)}{\deg(Y)} \), we have
\[
\frac{\deg(\omega_Y)}{\deg(Y)} = -q\frac{\mu}{\deg(Y)} + \frac{\deg(\omega_Y)}{\deg(Y)} \quad \text{we have}
\]
\[
H^1(Y, \text{Syz}(f_1^q, \ldots, f_n^q)(m)) = 0.
\]

Proof. This follows from Lemma 3.1 applied to \( S = \text{Syz}(f_1^q, \ldots, f_n^q) \). One only has to observe that
\[
\mu_k(\text{Syz}(f_1^q, \ldots, f_n^q)(m)) = \mu_k(\text{Syz}(f_1^q, \ldots, f_n^q)) + \deg(O(m)) = q\mu_k + m\deg(Y).
\]
So for example the condition in the Lemma, that \( \mu_k(\text{Syz}(f_1^q, \ldots, f_n^q)(m)) > \deg(\omega_Y) \), is equivalent with \( m\deg(Y) + q\mu_k > \deg(\omega_Y) \) and hence with \( m > -q\frac{\mu_k}{\deg(Y)} + \frac{\deg(\omega_Y)}{\deg(Y)} \).

Proposition 3.5. Suppose the situation and notation of 3.2. Fix \( k, 1 \leq k \leq t - 1 \). Then we have
\[
\sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} h^0(\text{Syz}(f_1^q, \ldots, f_n^q)(m)) =
\]
\[
q^2 \deg(Y) \left( \frac{b^{2}_{k+1} - \nu_k^2}{2} (r_1 + \ldots + r_k) - (\nu_{k+1} - \nu_k)(r_1\nu_1 + \ldots + r_k\nu_k) \right) + O(q).
\]

Proof. We may assume that \( q \) is big enough such that the Harder-Narasimhan filtration of \( \text{Syz}(f_1^q, \ldots, f_n^q) \) is strong. Let \( S_k^q \) denote the \( k \)-th subbundle in the Harder-Narasimhan filtration. We have
\[
\sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} h^0(\text{Syz}(f_1^q, \ldots, f_n^q)(m))
\]
\[
= \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} h^0(S_k^q(m))
\]
\[
= \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} \left( \deg(S_k^q) + m \rk(S_k^q) \deg(Y) + \rk(S_k^q)(1 - q) + h^1(S_k^q(m)) \right)
\]
\[
= \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} \left( \deg(S_k^q) + m \rk(S_k^q) \deg(Y) \right) + O(q) + \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} h^1(S_k^q(m))
\]
\[
= \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} \left( q(r_1\nu_1 + \ldots + r_k\nu_k) + m \rk(S_k^q) \deg(Y) \right) + O(q)
\]
\[
= \deg(Y) \left( \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} m(r_1 + \ldots + r_k) - q \sum_{m = [q\nu_k]}^{[q\nu_{k+1}] - 1} (r_1\nu_1 + \ldots + r_k\nu_k) \right) + O(q)
\]
\[
= \deg(Y) \left( \frac{b^{2}_{k+1} - q\nu_k^2}{2} (r_1 + \ldots + r_k) - q(\nu_{k+1} - \nu_k)(r_1\nu_1 + \ldots + r_k\nu_k) \right) + O(q)
\]
the fourth equation look at Lemma 2.3. Fix $\bar{q}$ big enough such that the Harder-Narasimhan filtration is strong. Then $\bar{\mu}(S^q_k) = \mu(S^q_k/S^q_{k-1}) = \bar{q}\mu_k$. Applying Lemma 2.3 to $S = S^q_k$ with $\nu = \bar{q}\nu_k$ we get

$$\sum_{m=\lceil\bar{q}(\bar{q}\nu_k)\rceil}^{\infty} h^1((S^q_k)^{\bar{q}}(m)) = O(\bar{q}).$$

So for $q = \bar{q}q$ we get the needed result, since $\bar{q} \leq q$. The other equations are clear. □

We come now to the main result of this paper.

**Theorem 3.6.** Suppose the situation 3.2. The Hilbert-Kunz multiplicity $e_{HK}(I)$ is given by

$$\frac{\deg(Y)}{2} \left( \sum_{k=1}^{t} r_k\nu_k^2 - \sum_{i=1}^{n} d_i^2 \right).$$

In particular, the Hilbert-Kunz multiplicity is a rational number.

**Proof.** We compute the length as $\lambda(R/I^{[q]}) =$

$$= \sum_{m=0}^{\infty} \lambda((R/I^{[q]})_m)
= \sum_{m=0}^{\infty} \lambda((R/I^{[q]})_m) + O(q)
= \sum_{m=0}^{\infty} \left( h^0(\mathcal{O}(m)) - \sum_{i=1}^{n} h^0(\mathcal{O}(m - qd_i)) + h^0(\text{Syz}(f^q_j)(m)) \right) + O(q)
= \sum_{m=0}^{\infty} h^0(\mathcal{O}(m)) - \sum_{i=1}^{n} \left( \sum_{m=0}^{[q\nu_i]-1} h^0(\mathcal{O}(m - qd_i)) \right)
+ \sum_{m=0}^{[q\nu_i]-1} h^0(\text{Syz}(f^q_j)(m)) + O(q)
= q^2 \frac{\deg(Y)}{2} \left( \nu_i^2 - \sum_{i=1}^{n} (\nu_i - d_i)^2 \right) + \sum_{m=0}^{[q\nu_i]-1} h^0(\text{Syz}(f^q_j)(m)) + O(q).$$

Here the second equation is due to Lemma 2.3 (as in the proof of Theorem 2.5) and the last equation is due to Lemma 2.2. We may write the term on the right \(\sum_{m=0}^{[q\nu_i]-1} h^0(\text{Syz}(f^q_j)(m)) =\)

$$= \sum_{m=0}^{[q\nu_i]-1} h^0(\text{Syz}(f^q_j)(m)) + \sum_{k=1}^{t-1} \left( \sum_{m=[q\nu_k]}^{[q\nu_k+1]-1} h^0(\text{Syz}(f^q_j)(m)) \right)
= \sum_{k=1}^{t-1} \left( \sum_{m=[q\nu_k]}^{[q\nu_k+1]-1} h^0(\text{Syz}(f^q_j)(m)) \right)
= q^2 \deg(Y) \sum_{k=1}^{t-1} \left( \frac{\nu_{k+1}^2 - \nu_k^2}{2} (r_1 + \ldots + r_k) \right)$$. 


\[(\nu_{k+1} - \nu_k)(r_1 \nu_1 + \ldots + r_k \nu_k) + O(q).\]

Here the second equation is due to Proposition 3.4(i) and the last equation is due to Proposition 3.5. This yields for the Hilbert-Kunz multiplicity \(e_{HK}(I)\) the expression

\[
\frac{\deg(Y)}{2} \left( \nu_i^2 - \sum_{i=1}^n (\nu_i - d_i)^2 + \sum_{k=1}^{t-1} (\nu_{k+1} - \nu_k^2)(r_1 + \ldots + r_k) \right) \\
-2 \sum_{k=1}^{t-1} (\nu_{k+1} - \nu_k)(r_1 \nu_1 + \ldots + r_k \nu_k).
\]

Now we can simplify. We have

\[
\nu_i^2 - \sum_{i=1}^n (\nu_i - d_i)^2 = -(n-1)\nu_i^2 + 2\nu_i \sum_{i=1}^n d_i - \sum_{i=1}^n d_i^2.
\]

Furthermore we have

\[
\sum_{k=1}^{t-1} (\nu_{k+1} - \nu_k^2)(r_1 + \ldots + r_k) = -\sum_{k=1}^{t-1} r_k \nu_k^2 + \nu_i^2 (r_1 + \ldots + r_{t-1})
\]

and similarly

\[
\sum_{k=1}^{t-1} (\nu_{k+1} - \nu_k)(r_1 \nu_1 + \ldots + r_k \nu_k) = -\sum_{k=1}^{t-1} r_k \nu_k^2 + \nu_i (r_1 \nu_1 + \ldots + r_{t-1} \nu_{t-1}).
\]

Using the relations \(r_1 + \ldots + r_{t-1} = n - 1 - r_t\) and \(r_1 \nu_1 + \ldots + r_{t-1} \nu_{t-1} = \sum_{i=1}^n d_i - r_t \nu_t\) we get altogether

\[
-(n-1)\nu_i^2 + 2\nu_i \sum_{i=1}^n d_i - \sum_{i=1}^n d_i^2 - \sum_{k=1}^{t-1} r_k \nu_k^2 + \nu_i^2 (n - 1 - r_t) \\
-2\left( -\sum_{k=1}^{t-1} r_k \nu_k^2 + \nu_i (\sum_{i=1}^n d_i - r_t \nu_t) \right) \\
= -(n-1)\nu_i^2 - \sum_{i=1}^n d_i^2 + \sum_{k=1}^{t-1} r_k \nu_k^2 + \nu_i^2 (n - 1 - r_t) + 2r_t \nu_i^2 \\
= \sum_{k=1}^n r_k \nu_k^2 - \sum_{i=1}^n d_i^2
\]

This gives the result.

The rationality of the Hilbert-Kunz multiplicity does not require the conditions normal and standard-graded, as the following corollary shows. Also the condition that the ground field is algebraically closed is not essential, we only need that the ring is geometrically irreducible.

**Corollary 3.7.** Let \(R\) denote an \(\mathbb{N}\)-graded two-dimensional domain of finite type over an algebraically closed field \(K\) of positive characteristic. Let \(I\) denote a homogeneous \(R_+\)-primary ideal. Then the Hilbert-Kunz multiplicity \(e_{HK}(I)\) is rational.
Proof. By adjoining suitable roots for the algebra generators of $R$ we get a standard-graded $K$-domain $R \subseteq S$ finite over $R$ (see the proof of Theorem 4.2 in [3]). Due to [25, Theorem 2.7] we have for finite extensions the relationship $e_{HK}(I) = e_{HK}(IS)/\text{rk}(S)$, so we may assume that $R$ is a standard-graded domain. Its normalization $R \subseteq \tilde{R}$ is a graded domain (see [21, §62, Aufgabe 27]), which might not be standard-graded. However the open subsets $D_+(x)$ for elements $x \in R_1 \subseteq \tilde{R}_1$ do cover $Y = \text{Proj} \tilde{R}$, and, as remarked in footnote 1, Theorem 3.6 also holds under this assumption. □

4. REMARKS AND EXAMPLES

We gather together some corollaries of our main result and make several remarks.

Remark 4.1. How does the denominator of the Hilbert-Kunz multiplicity look like? The formula in Theorem 3.6 shows that possible factors are 2, $\deg(Y)$, the numbers $r < n$, where $n$ is the number of ideal generators, and some powers of the characteristic $p$.

Remark 4.2. Theorem 2.5 is of course a special case of Theorem 3.6. If the syzygy bundle is strongly semistable, then $t = 1$, $r_t = n - 1$, $\nu_t = \frac{d_1 + \ldots + d_n}{n-1}$.

The following corollary treats the next easiest case, namely $t = 2$, so that the syzygy bundle is an extension of two strongly semistable bundles.

Corollary 4.3. Suppose the situation 3.2 and suppose that $t = 2$. Then the Hilbert-Kunz multiplicity $e_{HK}(I)$ is given by

$$\frac{\deg(Y)}{2} \left( r_2 \nu_2^2 + \frac{(\sum_{i=1}^{n} d_i - r_2 \nu_2)^2}{n-1 - r_2} - \sum_{i=1}^{n} d_i^2 \right).$$

Proof. This follows directly from Theorem 3.6 using $r_1 = n - 1 - r_2$ and $r_1 \nu_1 = \sum_{i=1}^{n} d_i - r_2 \nu_2$. □

Corollary 4.4. Suppose the situation 3.2 and suppose that $n = 3$. Then the syzygy bundle has rank two and the following two cases may occur.

(i) The syzygy bundle is strongly semistable. Then the Hilbert-Kunz multiplicity is

$$e_{HK}(I) = \frac{\deg(Y)}{2} \left( \frac{(d_1 + d_2 + d_3)^2}{2} - d_1^2 - d_2^2 - d_3^2 \right).$$

(ii) The syzygy bundle is not strongly semistable. Then $\nu_2 > \nu_1$ and the Hilbert-Kunz multiplicity $e_{HK}(I)$ is given by

$$\deg(Y)(\nu_2^2 - \nu_2 \sum_{i=1}^{3} d_i + \sum_{i<j} d_i d_j).$$
Proof. The first statement follows from Theorem 2.5 (or Theorem 3.6). For the second statement suppose that the $e$-th pull-back is not semistable. Then there exist invertible sheaves $0 \to \mathcal{L} \to \text{Syz}(f_1^q, f_2^q, f_3^q) \to \mathcal{M} \to 0$, $q = p^e$, with $\deg(\mathcal{L}) > \deg(\mathcal{M})$. Such a filtration is strong. Therefore $\nu_1 = -\frac{\deg(\mathcal{L})}{\deg(Y)}$ and $\nu_2 = -\frac{\deg(\mathcal{M})}{\deg(Y)}$. We insert $r_1 = r_2 = 1$ and $\nu_1 + \nu_2 = d_1 + d_2 + d_3$ in the formula of Corollary 4.3 and get up to the factor $\deg(Y)/2$ the expression

\[
\nu_2^2 + \left( \sum_{i=1}^{3} d_i - \nu_2 \right)^2 - \sum_{i=1}^{3} d_i^2 = 2\nu_2^2 - 2\nu_2 \sum_{i=1}^{3} d_i + \left( \sum_{i=1}^{3} d_i \right)^2 - \sum_{i=1}^{3} d_i^2
\]

\[
= 2\nu_2^2 - 2\nu_2 \sum_{i=1}^{3} d_i + 2 \sum_{i<j} d_id_j.
\]

Multiplying with $\deg(Y)/2$ gives the result. \(\Box\)

Remark 4.5. If in the previous corollary the degrees are equal, then the formula in the second case reduces to $\frac{\deg(Y)}{2} \left( \nu_2^2 - 3\nu_2 d + 3d^2 \right)$. Corollary 4.4 is independent of Langers theorem, since in rank two it is clear that either $\mathcal{S}$ is strongly semistable or some Frobenius pull-back of it has an invertible subsheaf as in the proof of 4.4. The same is true for the following Corollary, which treats the Hilbert-Kunz multiplicity of the cone over a plane curve.

**Corollary 4.6.** Let $C = V_+(H) \subset \mathbb{P}^2$ denote a smooth plane projective curve of degree $h$, $R = K[X,Y,Z]/(H)$. Then the following hold.

(i) There exists a rational number $\frac{3}{2} \leq \nu_2 \leq 2$ such that $e_{HK}(R) = h(\nu_2^2 - 3\nu_2 + 3)$.

(ii) $\nu_2 = 3/2$ holds if and only if the restriction of the tangent bundle $T_{\mathbb{P}^2}$ to the curve $C$ is strongly semistable. In this case we have $e_{HK}(R) = \frac{3h}{4}$.

(iii) We have estimates $\frac{3}{4}h \leq e_{HK}(R) \leq h$.

Proof. (i),(ii). Note that the tangent bundle is dual to the syzygy bundle $\text{Syz}(X,Y,Z)$ of the variables. If the restriction of this bundle is strongly semistable, then (ii) holds due to Corollary 4.4 (i). Then also (i) is true for $\nu_2 = 3/2$.

The maximal slope of $\text{Syz}(X,Y,Z)|C \subset \oplus_3 \mathcal{O}(-1)$ cannot exceed $-h$, hence the minimal slope is at least $-2h$. Therefore $\nu_2 \leq 2$. The restriction is not strongly semistable if and only if $\nu_2 > 3/2$ holds. In this case the formula in Corollary 4.4 (ii) gives (i).

(iii). The quadratic polynomial takes its minimum at $\nu_2 = 3/2$. The value at $\nu_2 = 2$ is $1$.

Remark 4.7. We discuss the estimates in Corollary 4.6 (iii) and relate it to some results in the literature. The bound $3h/4 \leq e_{HK}(R)$ for cones over plane curves was obtained in [4].
In general there exist estimates \( \frac{e(I)}{d} \leq e_{HK}(I) \leq e(I) \) ([10] Lemma 6.1]), where \( e(I) \) denotes the Hilbert-Samuel multiplicity of \( I \) in a \( d \)-dimensional local ring. Recall that the Hilbert-Samuel multiplicity of the cone over a projective variety is the degree of the variety. Hence these estimates yield in our situation \( h/2 \leq e_{HK}(R) \leq h \).

Watanabe and Yoshida have shown for a 2-dimensional local Cohen-Macaulay ring of positive characteristic that \( e_{HK}(R) \geq \frac{e+1}{2} \) holds. We have \( 2h \geq \frac{h+1}{2} \), and equality holds exactly for \( h = 1, 2 \).

**Remark 4.8.** We take the expression in the formula in Theorem 3.6 as a definition for the Hilbert-Kunz multiplicity in characteristic 0. For the slopes we just have to take the slopes in the Harder-Narasimhan filtration of the syzygy bundle. This Hilbert-Kunz multiplicity is in fact independent of the ideal generators, as the following proposition shows.

**Proposition 4.9.** Suppose that \( R \) is a two-dimensional standard-graded normal domain over an algebraically closed field \( K \) of characteristic 0. Let \( I = (f_1, \ldots, f_n) \) denote a homogeneous \( R_1 \)-primary ideal. Then the Hilbert-Kunz multiplicity \( e_{HK}(f_1, \ldots, f_n) \) is independent of the ideal generators.

*Proof.* We write temporarily \( e_{HK}(f_1, \ldots, f_n) \) instead of \( e_{HK}(I) \). It is enough to show that

\[
e_{HK}(f_1, \ldots, f_n) = e_{HK}(f_1, \ldots, f_n, f),
\]

where \( f \) is a homogeneous element \( f \in (f_1, \ldots, f_n) \). Let \( S_1 \subset \ldots \subset S_t = \text{Syz}(f_1, \ldots, f_n) \) denote the Harder-Narasimhan filtration of \( \text{Syz}(f_1, \ldots, f_n) \), let \( \mu_k = \mu(S_k/S_{k-1}) \), \( r_k = \text{rk}(S_k/S_{k-1}) \) and \( \nu_k = -\mu_k/\deg(Y) \). These numbers determine \( e_{HK}(f_1, \ldots, f_n) = \frac{\deg(Y)}{2} (\sum_{k=1}^t r_k \nu_k^2 - \sum_{i=1}^n d_i^2) \). Suppose that the degree of \( f \) is \( e \) and that \( f = \sum_{i=1}^n a_i f_i \). Then we have the relationship

\[
\text{Syz}(f_1, \ldots, f_n, f) \cong \text{Syz}(f_1, \ldots, f_n) \oplus O(-e).
\]

The mappings from right to left are given by \((s_1, \ldots, s_n) \mapsto (s_1, \ldots, s_n, 0)\) and \(1 \mapsto (-a_1, \ldots, -a_n, 1)\). Let \( i \) be such that \( \nu_i \leq e < \nu_{i+1} \) or equivalently that \( \mu_i \geq \mu(O(-e)) = -e/\deg(Y) > \mu_{i+1} \) (suppose in the following that \( \nu_i < e \) holds, the case \( = \) is similar). Then the Harder-Narasimhan filtration of \( \text{Syz}(f_1, \ldots, f_n, f) \) is

\[
S_1 \subset \ldots \subset S_i \subset S_i \oplus O(-e) \subset S_{i+1} \oplus O(-e) \subset \ldots \subset S_t \oplus O(-e).
\]

The (semistable) quotient sheaves are then

\[
S_1, S_2/S_1, \ldots, S_i/S_{i-1}, O(-e), S_{i+1}/S_i, \ldots, S_t/S_{t-1},
\]

and the new ranks are \( r_1, \ldots, r_i, 1, r_{i+1}, \ldots, r_t \) and the new degree thresholds are \( \nu_1, \ldots, \nu_i, e, \nu_{i+1}, \ldots, \nu_t \). Hence \( e_{HK}(f_1, \ldots, f_n, f) = e_{HK}(f_1, \ldots, f_n) \). \( \square \)

**Remark 4.10.** It is of course tempting to conjecture a version of Theorem 3.6 in higher dimensions by replacing the degree of \( O_Y(1) \) by the top self intersection number \( (O_Y(1))^\text{dim}Y \). However it is not clear whether the
slopes carry enough information to control the intermediate cohomologies $H^i(Y, \text{Syz})$, $0 < i < \dim Y$.

**Remark 4.11.** We briefly explain the relation of our main result to tight closure. In [3] we proved using the strong Harder-Narasimhan filtration that the tight closure and the plus closure coincide for a homogeneous ideal in a two-dimensional graded domain of finite type over the algebraic closure of a finite field. In fact we showed that the containment $f \in (f_1, \ldots, f_n)^*$ is a property of the corresponding cohomology class $c = \delta(f) \in H^1(Y, \text{Syz}(f_1, \ldots, f_n)(m))$ ($m = \deg(f)$) in the strong Harder-Narasimhan filtration.

Since it is known (see [10, Theorem 5.4]) that in an analytically unramified and formally equidimensional local ring $R$ the equation $e_{HK}(I) = e_{HK}(J)$ holds if and only if $I^* = J^*$ holds true for two ideals $I \subseteq J$ which are primary to the maximal ideal, it is not surprising that the behavior of the Hilbert-Kunz function and the Hilbert-Kunz multiplicity is encoded in the strong Harder-Narasimhan filtration of a syzygy bundle of ideal generators.

In [2] we show that in characteristic zero the Hilbert-Kunz multiplicity as defined in Remark 4.10 has the same relationship to solid closure as the Hilbert-Kunz multiplicity in positive characteristic has to tight closure.

**Example 4.12.** Consider the monomial ideal $I = (X^3, XY^2, ZY^2)$ in $R = K[X, Y, Z]/(H)$, where $H$ is a homogeneous polynomial of degree $h$ such that $R$ is normal and such that $Z$ and $X$ are parameters. Denote by $C = \text{Proj} \ R$ the corresponding smooth projective curve. The triple $(0, Z, -X)$ defines a global syzygy of degree 4 (without common zero) and so we get the short exact sequence

$$0 \longrightarrow \mathcal{O}(-4) \longrightarrow \text{Syz}(X^3, XY^2, ZY^2) \longrightarrow \mathcal{O}(-5) \longrightarrow 0,$$

and the inclusion $\mathcal{O}(-4) \subset \text{Syz}(X^3, XY^2, ZY^2)$ is the Harder-Narasimhan filtration. This filtration is of course strong, and the slope numbers are $\mu_1 = -4h$ and $\mu_2 = -5h$ (hence $\nu_1 = 4$ and $\nu_2 = 5$) and the ranks are $r_1 = r_2 = 1$. The formula in Corollary 4.3 (ii) yields that the Hilbert-Kunz multiplicity is $h(25 - 5 \cdot 9 + 3 \cdot 9) = 7h$.

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