New PPP/VPPP Algorithms by using Multiple Antennas*

Atsushi Mouri†‡, Yoshifumi Karatsu‡, Goshi Okuda‡, Sueo Sugimoto‡, Yukihiro Kubo‡ and Masaharu Ohashi§

In this paper, we present a novel PPP algorithm by applying the double difference for GNSS observables among multiple antennas (receivers), and apply improved VPPP algorithms. First of all, the GR models for double difference observables are shown which are similar to the GR models for the relative positioning algorithms, but both antennas’ positions are unknown. Then we derive the Kalman filtering algorithms for recursive estimation of all antennas’ positions and double difference integer ambiguity of all carrier-phases in GNSS observables. Then using the geometric constraints for all antennas’ positions, we derive the algorithms of updating the estimated parameters including antennas’ positions and integer ambiguities. Finally we show the experimental results of the proposed VPPP algorithm comparing with the previous VPPP algorithm.

1. Introduction

In this paper, we present a novel precise point positioning (PPP) algorithm based on the double difference for Global Navigation Satellite System (GNSS) observables among multiple antennas, and apply the improved position update algorithms to the PPP algorithm.

There are three major causes of GNSS positioning errors [1]. First one is caused by the GNSS satellite-related error sources, e.g. clocks or orbits. Second one is caused by the GNSS signal travel path-related sources, e.g. ionospheric or tropospheric signal delays. And third one is caused by the receiver-related sources, e.g. measurements or multipath. All sources include bias or random noises, and the bias noises are dominant except for multipath. The satellite-related errors are globally caused, and the path-related errors are locally caused.

Basically, the so-called SPP (standard point positioning) utilizes C/A code pseudoranges of low cost single frequency receivers solving the nonlinear equations, and has several tens of meter positioning errors caused by the error sources [2]. On the other hand, relative positioning additionally utilizes the carrier-phase observables which provide more precise pseudoranges, and augmentation data obtained from reference stations through communication means. The data are based on the GNSS observables at reference stations, and utilized for cancelling the common bias-related errors by single (SD) or double difference (DD) methods [3] applied to the observables at receivers and the reference stations. Therefore the relative positioning has millimeter level positioning for topographic surveying.

PPP also utilizes the carrier-phase observables, and is an ultimately desirable technology in the GPS/GNSS positioning community [4]. According to [5–10], at first, we present the GNSS regression (GR) equation and the PPP algorithm which achieves the positioning accuracy in decimeter level without any external transmitted information such as from the wide area augmentation system (WAAS). Then we had derived the very precise point positioning (VPPP) algorithm with multiple antennas and with common receivers’ clock errors by applying the multiple GR models and Kalman filtering [11–14].

The geometrical distances among the multiple antennas at observation points are utilized as the constraints for fixing the integer ambiguity of carrier-phase observables [15]. We improve our previous positioning algorithms and derive novel DD-based GR models and position update algorithms at each measurement epoch from PPP estimates by using constraints of geometrical distances among antennas’ positions and common receivers’ clock errors based on the minimum mean square (MMS) methods. In these derivations, we had discovered the simplest derivation of Kalman’s measurement update equations shown in [13], as a byproduct.
This paper is organized as follows. Section 2 shows the GR models of DD-based measurement equations and state equations. Section 3 shows the update algorithm based on distance constraints between two antennas. Section 4 shows the experimental results according to the PPP/VPPP algorithms by or not by DD-based methods in a static environment. Finally the concluding remarks are given in Section 5.

2. GNSS Regression Models and Double Difference Measurement Equations

First of all, similarly to [5–10], we formulate all observed positioning data consisting of the L1 carrier-phase and pseudoranges based on C/A code, by using the GNSS regression models. The natural extensions of GNSS regression models for multiple frequencies of GPS, Galileo, Compass/BeiDou, GLONASS, and US-GPS modernization are also similarly formulated. Namely, we consider the following fundamental measurements of the pseudoranges $\rho_{CA,u}^p(t)$ based on the C/A code and L1 band carrier-phases $\varphi_{L1,u}^p(t)$ (equivalently, $\Phi_{L1,u}^p(t)$ as the unit of length), respectively, as follows [3,4,16–18]:

$$
\rho_{CA,u}^p(t) = r_u^p(t) - c(t - t_u^p) + \delta T_u^p(t) + \delta T_u^p(t) + \delta \rho_{CA}^p(t) + \delta \rho_{CA}^p(t),
$$

$$
\Phi_{L1,u}^p(t) = \lambda_1 \varphi_{L1,u}^p(t) = r_u^p(t) - c(t - t_u^p) - \delta T_u^p(t) + \delta T_u^p(t) + \delta \varphi_{L1}^p(t) + \delta \varphi_{L1}^p(t),
$$

where $\lambda_1$ is the wavelength of the carrier wave and calculated by $c/f_1$. $c = 2.99792458 \times 10^8$ [m/s] denotes the speed of light, and $f_1$ is the central frequency of the L1 carrier wave $f_1 = 2 \times 77 \times 10.23$ [MHz] = 1575.42 [MHz].

In Eqs. (1)-(2), the so-called receiver biases, $\{\delta b_{CA,u}, \delta b_{L1,u}\}$, and the satellite biases, $\{\delta b_{CA}^p, \delta b_{L1}^p\}$, are contained in the usual observed positioning data consisting of pseudorange based on the C/A code and the L1 carrier-phase [19]. $r_u^p(t, t - t_u^p)$ is the geometric distance between the receiver at the time $t$ and the satellite $p$ at the time $t - t_u^p$ ($t_u^p$ denotes the travel time from the satellite $p$ ($p = 1, \ldots, n_s$) to the receiver $u$. Namely,

$$
r_u^p(t) \equiv r_u^p(t, t - t_u^p) = \left[ (x_u(t) - x_p(t - t_u^p))^2 + (y_u(t) - y_p(t - t_u^p))^2 + (z_u(t) - z_p(t - t_u^p))^2 \right]^{1/2},
$$

where $u \equiv [x_u, y_u, z_u]^T$ and $s^p \equiv [x_p, y_p, z_p]^T$ are a user (unknown) and satellite positions, respectively. $n_s$ denotes the number of the observable satellites. $\delta T_u^p(t)$ and $\delta T_u^p(t)$ in Eqs. (1)-(2) reflect the delay or the advance associated with the transmission of the L1 signal through the ionosphere and the troposphere, respectively. $\delta \varphi_{CA}^p(t)$ and $\delta \varphi_{CA}^p(t)$ are the clock errors of the receiver $u$ at the time $t$ and the satellite $p$ at the time $t - t_u^p$. $N_u^p$ denotes integer ambiguity between the satellite $p$ and the receiver $u$, and $v_{CA,u}^p(t)$, $v_{L1,u}^p(t)$ denote measurement errors.

Eq. (3) contains the satellite orbital errors. The estimated satellite orbits are obtained from the navigation messages which are decoded from the transmitted L1 signal. Let us denote $s^p$ as the estimated position of the satellite $s^p$ at the time $t - t_u^p$.

We use the following relations of the derivatives

$$
\frac{\partial r_u^p}{\partial x_u} = \frac{(x_u - x_p)}{r_u^p}, \quad \frac{\partial r_u^p}{\partial y_u} = \frac{(y_u - y_p)}{r_u^p}, \quad \frac{\partial r_u^p}{\partial z_u} = \frac{(z_u - z_p)}{r_u^p}, \quad (p = 1, 2, \ldots, n_s),
$$

and

$$
\frac{\partial r_u^p}{\partial x} = -\frac{(x_u - x_p)}{r_u^p}, \quad \frac{\partial r_u^p}{\partial y} = -\frac{(y_u - y_p)}{r_u^p}, \quad \frac{\partial r_u^p}{\partial z} = -\frac{(z_u - z_p)}{r_u^p}, \quad (p = 1, 2, \ldots, n_s).
$$

Then we have the relation:

$$
\frac{\partial r_u^p}{\partial s^p} = -\frac{\partial r_u^p}{\partial \hat{s}^p},
$$

Thus the 1st order Taylor series approximation of Eq. (3) around the previous estimated value $u = \hat{u}$ and $s^p = \hat{s}^p$ is given by

$$
r_u^p \approx r_u^p + (g_u^p)^T [u - s^p - (\hat{u} - \hat{s}^p)]
$$

$$
= ||\hat{u} - \hat{s}^p|| + (\hat{u} - \hat{s}^p)^T [u - s^p - (\hat{u} - \hat{s}^p)]
$$

$$
= (\hat{u} - \hat{s}^p)^T (u - s^p),
$$

for $p = 1, 2, \ldots, n_s$, where gradient vectors are as follows:

$$
g_u^p \equiv \left[ \frac{\partial r_u^p}{\partial s^p} \right]_{u = \hat{u}, s^p = \hat{s}^p} = \left( \frac{\hat{u} - \hat{s}^p}{||\hat{u} - \hat{s}^p||} \right) .
$$

The GR models utilize the extended Kalman filter based on the geometric distance linearized by the gradient vectors [10], and provide appropriate positioning performances. The DD-based PPP methods employ the same estimation method as that of previous PPP methods. From Eqs. (1)-(2), therefore, we have the approximations:

$$
\rho_{CA,u}^p = (g_u^p)^T (u - s) + \delta \rho_{CA,u}^p + \delta \rho_{CA,u}^p + \delta \rho_{CA,u}^p + \delta \rho_{CA,u}^p + \delta T_u^p + \delta T_u^p + \delta \varphi_{CA,u}^p + \delta \varphi_{CA,u}^p,
$$

$$
\Phi_{L1,u}^p = (g_u^p)^T (u - s) - \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p + \delta \varphi_{L1,u}^p .
$$
+\delta b_{1,u} - \delta b_{1,u}^p + \lambda_1 N_{1,u}^p + \lambda_1 \epsilon_{1,u,1}\). (10)

Let us consider GR equations of pseudoranges based on C/A code for satellites \( p \) and the receivers (antennas) \( u_i \) and \( u_j \) \((n=2) \) which denotes the number of the receiver satellite-on-one connected to multiple antennae as follows:

\[
\rho_{CA,u_i}^p \approx (g_{\hat{u}_j})^T (u_i - s^p) + \delta I_{u_i}^p + \delta T_{u_i}^p + c(\delta t_{u_i} - \delta t^p) + \delta b_{CA,u_i} - \delta b_{1,u}^p + c(\epsilon_{CA,u_i})
\]

\[
\rho_{CA,u_j}^p \approx (g_{\hat{u}_j})^T (u_j - s^p) + \delta I_{u_j}^p + \delta T_{u_j}^p + c(\delta t_{u_j} - \delta t^p) + \delta b_{CA,u_j} - \delta b_{1,u}^p + c(\epsilon_{CA,u_j})
\] (11)

Then we subtract Eq. (11) from Eq. (12), namely, taking the single difference between the measurements of the receivers \( u_i \) and \( u_j \). The signal travelling paths to the adjacent GNSS antennae are very close, therefore we can assume

\[
\delta I_{u_i}^p \approx \delta I_{u_j}^p, \ \delta T_{u_i}^p \approx \delta T_{u_j}^p,
\]

then we have the relation:

\[
\rho_{CA,u_{i,j}}^p = \rho_{CA,u_{i,j}}^p - \rho_{CA,u_{i,j}}^p
\]

\[
\approx (g_{\hat{u}_j})^T (u_i - s^p) - (g_{\hat{u}_j})^T (u_j - s^p) + c(\delta t_{u_i} - \delta t_{u_j}) + \delta b_{CA,u_{i,j}} - \delta b_{1,u_{i,j}} + c(\epsilon_{CA,u_{i,j}})
\]

We also have the single difference relation of satellite \( q \):

\[
\rho_{CA,u_{i,j}}^q \approx (g_{\hat{u}_q})^T (u_i - s^q) - (g_{\hat{u}_q})^T (u_j - s^q) + c(\delta t_{u_i} - \delta t_{u_j}) + \delta b_{CA,u_{i,j}} - \delta b_{1,u_{i,j}} + c(\epsilon_{CA,u_{i,j}})
\]

Then, finally we have the following double difference measurements equations for pseudoranges based on C/A code by subtracting Eq. (13) from Eq. (14). The \( u_i \) is the reference antenna, and the \( p \) is the reference satellite:

\[
\rho_{CA,u_{i,j}}^{q,p} \equiv \rho_{CA,u_{i,j}}^{q,p} - \rho_{CA,u_{i,j}}^{q,p}
\]

\[
\approx (g_{\hat{u}_q})^T (u_i - s^q) - (g_{\hat{u}_q})^T (u_j - s^q) + c(\delta t_{u_i} - \delta t_{u_j}) + \delta b_{CA,u_{i,j}} - \delta b_{1,u_{i,j}} + c(\epsilon_{CA,u_{i,j}})
\]

\[
\] (15)

where

\[
g_{\hat{u}_j}^q \equiv g_{\hat{u}_j}^q - g_{\hat{u}_j}^p, \ g_{\hat{u}_j}^p \equiv g_{\hat{u}_j}^p - g_{\hat{u}_j}^q,
\]

\[
g_{\hat{u}_q}^u_i \equiv g_{\hat{u}_q}^u_i - g_{\hat{u}_j}^p, \ g_{\hat{u}_q}^u_j \equiv g_{\hat{u}_j}^u_j - g_{\hat{u}_j}^q,
\]

\[
\epsilon_{CA,u_{i,j}}^q \equiv \epsilon_{CA,u_{i,j}}^q - \epsilon_{CA,u_{i,j}}^p,
\]

\[
\epsilon_{CA,u_{i,j}}^p \equiv \epsilon_{CA,u_{i,j}}^p - \epsilon_{CA,u_{i,j}}^q,
\]

\[
\epsilon_{CA,u_{i,j}}^{q,p} \equiv \epsilon_{CA,u_{i,j}}^p - \epsilon_{CA,u_{i,j}}^q,
\]

Again let us consider GR equations of the L1 carrier-phase positioning data for satellites \( p \) and the receivers (antennas) \( u_i \) and \( u_j \) as follows:

\[
\Phi_{L1,u_{i,j}}^{\rho} \equiv (g_{\hat{u}_j}^q)^T (u_i - s^p) - \delta I_{1,u_i}^p + \delta T_{1,u_i}^p + c(\delta t_{1,u_i} - \delta t^p) + \delta b_{1,u_{i,j}}^p + \lambda_1 N_{1,u_{i,j}} + \lambda_1 \epsilon_{1,u_{i,j}}
\]

\[
\Phi_{L1,u_{i,j}}^{\rho} \equiv (g_{\hat{u}_j}^q)^T (u_j - s^p) - \delta I_{1,u_j}^p + \delta T_{1,u_j}^p + c(\delta t_{1,u_j} - \delta t^p) + \delta b_{1,u_{i,j}}^p + \lambda_1 N_{1,u_{i,j}} + \lambda_1 \epsilon_{1,u_{i,j}}
\] (16)

Then we repeat the similar manner to take the differences for the L1 carrier-phase positioning data. Finally we have the following double difference equations:

\[
\Phi_{L1,u_{i,j}}^{\rho} \equiv \Phi_{L1,u_{i,j}}^{\rho} - \Phi_{L1,u_{i,j}}^{\rho}
\]

\[
\approx (g_{\hat{u}_j}^q)^T (u_i - s^p) - (g_{\hat{u}_j}^q)^T (u_j - s^p) + \lambda_1 (N_{1,u_{i,j}}^p - N_{1,u_{i,j}}^p - N_{1,u_{i,j}}^p - N_{1,u_{i,j}}^p)
\]

\[
+ \lambda_1 (\epsilon_{1,u_{i,j}}^q - \epsilon_{1,u_{i,j}}^p - \epsilon_{1,u_{i,j}}^q - \epsilon_{1,u_{i,j}}^p)
\]

\[
\Phi_{L1,u_{i,j}}^{\rho} \equiv (g_{\hat{u}_j}^q)^T (u_i - s^p) - (g_{\hat{u}_j}^q)^T (u_j - s^p) + \lambda_1 N_{1,u_{i,j}}^p + \lambda_1 \epsilon_{1,u_{i,j}}^p
\]

\[
\] (17)

Let us assume that the estimated values: \( \hat{s}^p, p=1, \ldots , n_s \), of the satellite positions: \( s^p, p=1, \ldots , n_s \), are practically obtained by using ephemeris parameters broadcast by GPS satellites, or precise ephemeris of all satellites provided by the International GNSS Service (IGS), as follows [5—10]:

\[
\hat{s}^p = s^p + \epsilon_{s^p}, \quad p = 1, \ldots , n_s,
\]

where we assume \( \epsilon_{s^p} \) are Gaussian white noises. Then we have

\[
\hat{\rho}_{CA,u_{i,j}}^{\rho} \equiv \rho_{CA,u_{i,j}}^{\rho} - \rho_{CA,u_{i,j}}^{\rho}
\]

\[
\approx \hat{\rho}_{CA,u_{i,j}}^{\rho} - \hat{\rho}_{CA,u_{i,j}}^{\rho}
\]

\[
\approx (g_{\hat{u}_j}^q)^T (u_i - s^p) - (g_{\hat{u}_j}^q)^T (u_j - s^p) + \lambda_1 N_{1,u_{i,j}}^p + \lambda_1 \epsilon_{1,u_{i,j}}^p
\]

\[
\] (20)

\[
\Phi_{L1,u_{i,j}}^{\rho} \equiv \Phi_{L1,u_{i,j}}^{\rho} - \Phi_{L1,u_{i,j}}^{\rho}
\]

\[
\approx (g_{\hat{u}_j}^q)^T (u_i - s^p) - (g_{\hat{u}_j}^q)^T (u_j - s^p) + \lambda_1 N_{1,u_{i,j}}^p + \lambda_1 \epsilon_{1,u_{i,j}}^p
\]

\[
\] (21)

For the case of \( p = 1, q = 2, \ldots , n_s \) and \( u_i = u_1, u_j = u_2 \) \((n=2) \), we have the following measurement equation derived from Eqs. (20)—(21), \( y_{u_2,u_1} \), \( \eta_{u_2,u_1} \), and \( v_{u_2,u_1} \) are vectors for observables, unknown parameters, and measurement errors, respectively. \( C_{u_2,u_1} \) is a measurement matrix.

\[
y_{u_2,u_1} = C_{u_2,u_1} \eta_{u_2,u_1} + v_{u_2,u_1},
\]

\[
\] (22)
Var(satellites are independent signal reception or transmission processes with zero mean, because the different antennas are mutually independent white Gaussian random processes, and all elements are one, namely \( \sigma_u = 1 \). Then the covariance matrix of \( u \) is given by
\[
\begin{bmatrix}
\rho_{\text{CA},u} u_1 & \rho_{\text{CA},u} u_2 \\
\rho_{\text{CA},u} u_2 & \rho_{\text{CA},u} u_1 \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{\text{L}1,u} u_{n1} & \rho_{\text{L}1,u} u_{n2} & \cdots & \rho_{\text{L}1,u} u_{n1} \\
\end{bmatrix}
\]
and \( \rho_{\text{s}} \) is the following matrix as
\[
R_\rho \equiv \begin{bmatrix}
R_{\rho} & R_{\rho} \\
R_{\rho} & R_{\rho} \\
\end{bmatrix}
\]
where for the \((n_s - 1) \times 1 \) vector; \( g_{\text{s}u}^R \), we define
\[
\gamma_{\text{su}}^R \equiv (g_{\text{s}u}^R)^T g_{\text{s}u}^R,
\]
and \( R_{\text{CA},1} \) is a block diagonal matrix as
\[
R_{\text{CA},1} \equiv \begin{bmatrix}
R_{\text{CA}} & O \\
O & R_{L1} \\
\end{bmatrix},
\]
where
\[
R_{\text{CA}} \equiv \begin{bmatrix}
\sigma_{\text{CA}}^2 & \sigma_{\text{CA}}^2 & \cdots & \sigma_{\text{CA}}^2 \\
\sigma_{\text{CA}}^2 & \sigma_{\text{CA}}^2 & \cdots & \sigma_{\text{CA}}^2 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{\text{CA}}^2 & \sigma_{\text{CA}}^2 & \cdots & \sigma_{\text{CA}}^2 \\
\end{bmatrix}
\]
\[
R_{L1} \equiv \begin{bmatrix}
\lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 & \lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 & \cdots & \lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 \\
\lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 & \lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 & \cdots & \lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 & \lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 & \cdots & \lambda_{\text{L}1}^2 \sigma_{\text{L}1}^2 \\
\end{bmatrix}
\]
In the cases of \( n_r = 3 \) and \( n_r = 4 \), see Appendix.

### 2.1 State equations for the static case

In the static case, we utilize the state vector \( \eta_{u2u1} \) in Eq. (23) for antennas of \( u_1 \) and \( u_2 \) \((n_r = 2)\) and for \( n_s \) satellites. In order to simplify the expression, subscripts \( u_1,u_2 \) are omitted hereafter. Then the state equation is described by
\[
\eta_{t+1} = \eta_t, \quad \eta_t : n \times 1, \quad n = 5 + n_s,
\]
and the measurement equation is
\[
y_t = C_1 \eta + v_t.
\]
Therefore the positioning algorithms based on the Kalman filter for Eqs. (34) and (35) are given as follows:
\[
\begin{align*}
\tilde{\eta}_{t+1|t} &= \tilde{\eta}_{t|t} \\
\tilde{\eta}_{t|t} &= \tilde{\eta}_{t|t-1} + K_t v_t \\
v_t &= y_t - C_1 \tilde{\eta}_{t|t-1}
\end{align*}
\]
(: Innovation Process)
\[
K_t = \Sigma_{\eta,t|t-1} C_1^T \left( C_1 \Sigma_{\eta,t|t-1} C_1^T + R_t \right)^{-1}
\]
(: Kalman Gain)
\[
\Sigma_{\eta,t+1|t} = \Sigma_{\eta,t|t}
\]
where $R$ is a covariance matrix of measurement errors. We summarize the DD-based PPP method compared with PPP method in Table 1. PPP methods [5–10] corrects or estimates the parameters which cause errors, biases, or delays in Eq. (9)–(10), while DD-PPP methods cancels their parameters by DD-based observables.

| Positioning methods | DD-PPP | PPP |
|---------------------|--------|-----|
| Satellite orbit      | Broadcast |     |
| Satellite clock error| Canceled by SD | Broadcast |
| Satellite H/W bias   | Negligible | Corrected |
| Ionospheric delay    | Corrected |     |
| Tropospheric delay   | Corrected |     |
| Estimation method    | Extended Kalman filter |     |
| Antenna position     | Multiple | One |
| Receiver H/W bias    | Canceled by DD | Estimated |
| Receiver clock error | Estimated |     |
| Integer ambiguity    | Estimated |     |

### 3. Updating by Constraints

The constraint conditions are applied to update PPP/DD-PPP estimates as follows. Namely, when we obtain the filtering estimates $\hat{\eta}|_{t|21}$ and the error covariance matrix $\Sigma_{\eta|t|21}$, we apply the geometric distance constraints:

$$d_{ji} = ||u_j - u_i|| + e_{d_{ji}}, \quad (42)$$

where the measurement error $e_{d_{ji}}$ is assumed as a Gaussian white noise with $e_{d_{ji}} \sim N(0, r_{d_{ji}})$. We should note that the electrical phase center of an antenna is generally not identical to its geometrical center. The phase center can vary with the direction of arrival (azimuth and elevation) of the satellite signal and such variation can range from under a millimeter to 1-2 cm, depending upon antenna design. Usually the phase center variation is treated as measurement error [18]. In the case of $n_r = 2$, we define the distance constraint condition $d_{21,i}$ as the observable at time $t$, and consider the following relations of the conditional probability density function (CPDF):

$$p(\eta|Y', d_{21,i}) = \frac{p(\eta, Y', d_{21,i})}{p(Y', d_{21,i})} = \frac{p(\eta, d_{21,i}|Y')p(Y')}{p(Y', d_{21,i})} = \frac{p(\eta, Y', d_{21,i})p(Y')}{p(Y', d_{21,i})} = K_0(Y', d_{21,i})p(\eta|Y')p(d_{21,i}|\eta), \quad (43)$$

where

$$\Sigma_{\eta,t|t} = \Sigma_{\eta,t|t-1} - K_t C_t \Sigma_{\eta,t|t-1}$$

Init. Cond.: $\begin{cases} \hat{\eta}_{L,0} = \eta_{L,0} \\ \Sigma_{\eta,0} = \Sigma_{\eta,0} \end{cases}$

Then we have relations:

$$p(\eta|Y') = \frac{1}{(2\pi)^{n_r/2}|\Sigma_{\eta,|t|1}|^{1/2}} \exp \left\{ -\frac{1}{2} [\eta - \hat{\eta}]^T \Sigma_{\eta,|t|1}^{-1} [\eta - \hat{\eta}] \right\}, \quad (44)$$

$$p(d_{21,i}|\eta) = \frac{1}{\sqrt{2\pi r_{d_{21,i}}}} \exp \left\{ -\frac{[d_{21} - ||u_2 - u_1||]^2}{2r_{d_{21,i}}} \right\}. \quad (45)$$

Therefore, $p(\eta|Y', d_{21,i})$ in Eq. (43) is expressed as follows:

$$p(\eta|Y', d_{21,i}) = K_0(Y', d_{21,i}) \frac{1}{(2\pi)^{n_r/2}|\Sigma_{\eta,|t|1}|^{1/2}} \times \exp \left\{ -\frac{1}{2} [\eta - \hat{\eta}]^T \Sigma_{\eta,|t|1}^{-1} [\eta - \hat{\eta}] \right\} \times \frac{1}{\sqrt{2\pi r_{d_{21,i}}}} \exp \left\{ -\frac{[d_{21} - ||u_2 - u_1||]^2}{2r_{d_{21,i}}} \right\}. \quad (46)$$

Then we remark that the power term of $p(d_{21,i}|\eta)$ in Eq. (45) is expressed by the quadratic form of $\eta$ as follows:

$$\frac{(d_{21} - ||u_2 - u_1||)^2}{2r_{d_{21,i}}}$$

$$= \frac{1}{2r_{d_{21,i}}} \left\{ d_{21}^2 + ||u_2 - u_1||^2 - 2d_{21}||u_2 - u_1|| \right\}$$

$$\leq \frac{1}{2} \left\{ \frac{d_{21}^2}{r_{d_{21,i}}} + \frac{1}{r_{d_{21,i}}} u_2^T u_2 - \frac{1}{r_{d_{21,i}}} u_1^T u_1 - \frac{1}{r_{d_{21,i}}} u_1^T u_2 
+ \frac{1}{r_{d_{21,i}}} u_1^T u_2 - c_{21}^T u_2 + c_{21}^T u_1 \right\}, \quad (47)$$

where

$$||u_2 - u_1|| \leq \frac{u_2 - u_1}{u_2 - u_1} = \kappa_{21}^T (u_2 - u_1), \quad (48)$$

and

$$c_{21}^T = \frac{2d_{21} \kappa_{21}}{r_{d_{21,i}}}. \quad (49)$$

Finally, we have the expression of the quadratic form:

$$\frac{1}{2} \frac{|d_{21} - ||u_2 - u_1|||^2}{r_{d_{21,i}}}$$

$$\leq \frac{1}{2} \left\{ \eta^T M_2 \eta + c_{M_2}^T \eta + d_{M_2}^2 \right\}. \quad (50)$$

where...
Therefore, finally we have the following quadratic form for the power term of the CPDF (46):

\[
\frac{1}{2} \left( \eta - \eta \right) \Sigma^{-1}_n \left( \eta - \eta \right) + \frac{1}{2} \left[ d_{21} - \left| u_2 - u_1 \right| \right]^2 r_{d_{21}}
\]

\[
\simeq \frac{1}{2} \left\{ \eta^T \Sigma^{-1} \eta - \eta^T \Sigma^{-1} \eta - \eta^T \Sigma^{-1} \eta + \eta^T \Sigma^{-1} \eta + \eta^T \Sigma^{-1} \eta - \eta^T \Sigma^{-1} \eta + \frac{d_{21}^2}{r_{d_{21}}} \right\}
\]

\[
= \frac{1}{2} \left\{ \eta^T \left[ \left( \Sigma^{-1}_n + M_2 \right) - \eta \right] \eta - \eta^T \left[ \left( \Sigma^{-1}_n + M_2 \right) - \eta \right] \eta + \frac{d_{21}^2}{r_{d_{21}}} \right\}
\]

\[
= \frac{1}{2} \left\{ \eta^T \left( \Sigma^{-1}_n + M_2 \right) \eta - \eta^T \left( \Sigma^{-1}_n + M_2 \right) \eta + \frac{d_{21}^2}{r_{d_{21}}} \right\}
\]

Then the update estimated vector \( \hat{\eta} \) and error covariance matrix \( \hat{\Sigma}_n \) of \( \eta \) based on the minimum mean square estimate are given by

\[
\hat{\eta} = \left( \Sigma^{-1}_n + M_2 \right)^{-1} \left( \Sigma^{-1}_n \eta - \frac{1}{2} c_{M_2} \right),
\]

\[
\hat{\Sigma}_n = \left( \Sigma^{-1}_n + M_2 \right)^{-1}.
\]

These updated values are applied to Eqs. (36) and (40), respectively, as

\[
\hat{\eta}_{\text{pp}} \equiv \hat{\eta}, \quad \hat{\Sigma}_{\text{pp}} \equiv \hat{\Sigma}_n.
\]

Fig. 1 shows the concept of four positioning (a), (b), (c), (d) when two antennas ANT-1 \( (u_1) \) and 2 \( (u_2) \) are used. (a) PPP is individually applied for each antenna (receiver) to estimate the antenna position. (b) VPPP additionally utilizes the updating algorithm by geometric distance constraint (GDC) on (a). (c) DD-PPP utilizes the double difference methods by the both two antennas’ observables and satellites. (d) DD-VPPP additionally utilizes the updating algorithm by GDC on (c).

4. Experimental results

We have carried out the experiments of using the GPS data (see Table 2) from the antennas ANT-1,2,3,4 located at the corners of a square board (see Fig. 2), and the coordinates of their reference positions in the WGS84 system are listed in Table 3. These positioning results are provided by the GNSS software, the so-called “G-RitZ,” developed at Ritsumeikan University which will be opened as the free software, in the near future. The positioning error applying the
relative positioning method is less than a few cm. Receivers with ANT-1 and 2, or receivers with ANT-3 and 4 are operated by synchronized clocks, respectively.

The positioning experiments of (a) PPP and (b) VPPP for two methods; without differences of C/A code and L1 carrier-phase observables in[13]-[14] (call WOD-methods), and the presently proposed (c) DD-PPP and (d) DD-VPPP by taking double differences for observables (call DD-methods), are carried out by applying the Kalman filter formulation under static positioning conditions.

Fig. 3 shows the positioning errors for the ANT-1 (u₁) and 2 (u₂), respectively, using GPS observables in 06:00:00−06:00:59 (60 epochs) by the DD-methods with the local level axes (ENU: East, North, and Height(Up)), where the dotted black and black lines show the ENU errors of (c) DD-PPP and (d) DD-VPPP results, respectively. The errors are computed by difference between each estimated position and the corresponding position shown in Table 3. We can observe from Fig. 3 that the positioning quality is slightly improved by using VPPP (using the geometric constraints), and (d) DD-VPPP just needs several seconds after the positioning start to reach the fixed position. The RMSE (Root Mean Square Error) of ENU coordinate is defined as follows:

\[
RMSE = \sqrt{\frac{1}{n_r} \frac{1}{n_e} \sum_{i=1}^{n_r} \sum_{t=1}^{n_e} \left( E_{c,i,t}^2 + N_{c,i,t}^2 + U_{c,i,t}^2 \right) / 3}
\]

where \( n_e \) denotes the number of epochs. \( E_{c,i,t}, N_{c,i,t}, U_{c,i,t} \) means the ENU errors of antenna \( i \) at epoch \( t \), respectively. The RMSE of 60 epochs (\( n_e=60 \)) Eastward, Northward, Upward errors at the individual antenna (\( n_r=1 \)) in Fig. 3 are shown in Table 4.

Table 4 RMS errors of DD-methods

|     | RMSE[m] |
|-----|---------|
| ANT-1 |         |
| (c)  | 0.2925  |
| (d)  | 0.7908  |
| ANT-2 |         |
| (c)  | 1.5472  |
| (d)  | 1.1330  |

Fig. 4 shows the RMS errors of PPP and VPPP

ENU coordinate by WOD (a),(b) and DD-methods (c),(d) which utilize two antennas, ANT-1 and ANT-2. The vertical axis shows the RMS errors of two antennas (\( n_r=2 \)) after 10 seconds (tenth epoch) from the positioning start (\( n_e=1 \)) in Eq. (55). The horizontal axis shows sequential numbers of experiments. The experiments for the comparison of four positioning methods (a),(b),(c),(d) are repeated 10 times from 06:00:00 to 06:10:00 every 1 minute. The results show the RMS errors of estimated positions after 10 seconds from the positioning starts, which are \( 60(k-1) + 10 \) (\( k=1 \sim 10 \); the sequential number) seconds from 06:00:00. Basically, the errors of u-blox NEO-M8 GPS positioning are approximately 2.5 meters CEP (Circular Error Probability) [20], while the RMS errors of the four positioning methods are less than 1.5 meters. There are some fluctuations among estimated positions, because the priori initial estimations by SPP are used as the antennas’ position for

Fig. 5 shows the RMS errors of PPP and VPPP

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Kalman filter, however, (d) DD-VPPP has the smallest RMS errors. The total improvement ratio from (a) PPP to (d) DD-VPPP is approximately 83\%.

Fig. 5 shows the RMS errors of (c) DD-PPP and (d) DD-VPPP which utilize four antennas. The vertical axis shows the RMS errors of four antennas ($n_t = 4$) after 10 seconds from the positioning start ($n_c=1$) in Eq. (55). The positioning starts and the estimated positions are the same timing as the experiments of Fig. 4. 4-ANTs means that the positioning utilizes the observables by four antennas, and 2x2-ANTs means two pairs of the positioning which utilizes the observables by two antennas. The 4-ANTs (d) DD-VPPP utilizes six geometric distance constraints which are $d_{21}, d_{31}, d_{41}, d_{32}, d_{42}, d_{43}$, and the 2x2-ANTs (d) DD-VPPP utilizes $d_{21}$ and $d_{34}$ constraints in Eq. (42). The RMS errors of 4-ANTs (d) DD-VPPP are smaller than those of 4-ANTs (c) DD-PPP and 2x2-ANTs (d) DD-VPPP.

5. Conclusions

We have presented a novel PPP algorithm based on the double difference observables, and applied an improved VPPP algorithm based on geometrical distance constraints. GPS data are applied to the coupled GR equations for multiple antennas in the case of unknown positions.

The experiments in the static situation have been carried out. We have shown that DD-VPPP needs several seconds to reach a fixed position, and has the smallest RMS errors among four positioning methods which are PPP and VPPP of WOD and DD-methods when two antennas are utilized. The DD-VPPP by four antennas has smaller RMS errors than the DD-VPPP by two pairs of two antennas.

The experimental results show that the presently proposed DD-based algorithms for PPP and VPPP give the positioning accuracy in sub-meter level, and has the potential capability for estimating the baseline vectors which are the position differences between the reference antenna and the other antennas. In the future, we will estimate the baseline vectors by applying Kalman filter based methods for body attitudes, and apply for the kinematic environment.

Acknowledgements

In the process of revising this paper, the authors received helpful comments and suggestions from the anonymous reviewers. The authors would like to express their thankfulness for the help.

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Appendix

Let us define the \((n_s - 1) \times 1\) vectors as follows:

\[
\rho^n_{CA,ji} \equiv \begin{bmatrix} \tilde{\rho}^n_{CA,\hat{u}_i,\hat{u}_i} \\ \tilde{\rho}^n_{CA,\hat{u}_j,\hat{u}_i} \\ \vdots \\ \tilde{\rho}^n_{CA,\hat{u}_i,\hat{u}_j} \end{bmatrix}, \quad \Phi^n_{L1,ji} \equiv \begin{bmatrix} \tilde{\Phi}^n_{L1,\hat{u}_i,\hat{u}_i} \\ \tilde{\Phi}^n_{L1,\hat{u}_j,\hat{u}_i} \\ \vdots \\ \tilde{\Phi}^n_{L1,\hat{u}_i,\hat{u}_j} \end{bmatrix},
\]

\[
N^n_{L1,ji} \equiv \begin{bmatrix} N^n_{L1,\hat{u}_i,\hat{u}_i} \\ \vdots \\ N^n_{L1,\hat{u}_i,\hat{u}_j} \end{bmatrix}, \tag{A1}
\]

and define 3 \( (n_s - 1) \) matrix

\[
g^n_j = \begin{bmatrix} g^n_{\hat{u}_i} \\ \vdots \\ g^n_{\hat{u}_j} \end{bmatrix}; 3 \times (n_s - 1) \tag{A2}
\]

In the case of \( n_r = 3 \) and \( n_r = 4 \), the measurement equation for three antennas of \( u_1, u_2 \) and \( u_3 \) is as follows:

\[
y_{u3u2u1} = C_{u3u2u1} \eta_{u3u2u1} + v_{u3u2u1}, \tag{A3}
\]

where

\[
y_{u3u2u1} \equiv \begin{bmatrix} \rho^n_{CA,21} \\ \rho^n_{CA,31} \\ \rho^n_{CA,32} \end{bmatrix}, \quad \eta_{u3u2u1} \equiv \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},
\]

\[
\begin{bmatrix} -g^n_{u1} \cdot T & (g^n_{u2})^T & 0 \\ -g^n_{u1} \cdot T & 0 & (g^n_{u3})^T \end{bmatrix} \begin{bmatrix} O \end{bmatrix}
\]

\[
\begin{bmatrix} 0 & 0 \\ -g^n_{u1} \cdot T & (g^n_{u2})^T \end{bmatrix} = \begin{bmatrix} 0 \\ -g^n_{u1} \cdot T & (g^n_{u3})^T \end{bmatrix} \lambda_1 I
\]

\[
C_{u3u2u1} \equiv \begin{bmatrix} -g^n_{u1} \cdot T & (g^n_{u2})^T \\ -g^n_{u1} \cdot T & 0 \\ -g^n_{u1} \cdot T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} O \end{bmatrix}
\]

\[
M_{4(11)} = \begin{bmatrix} \frac{1}{r_{d21}} + \frac{1}{r_{d31}} + \frac{1}{r_{d31}} & -\frac{1}{r_{d21}} I \\ -\frac{1}{r_{d21}} I & \frac{1}{r_{d21}} + \frac{1}{r_{d21}} \end{bmatrix} \begin{bmatrix} \frac{1}{r_{d21}} I \\ -\frac{1}{r_{d21}} I \end{bmatrix} \begin{bmatrix} O \\ O \end{bmatrix}
\]

In this case, we have

\[
M_{4(12)} = \begin{bmatrix} \frac{1}{r_{d21}} - I & -\frac{1}{r_{d21}} I \\ -\frac{1}{r_{d21}} I & \frac{1}{r_{d21}} - I \end{bmatrix}
\]

\[
M_{4(13)} = \begin{bmatrix} \frac{1}{r_{d21}} - I & -\frac{1}{r_{d21}} I \\ -\frac{1}{r_{d21}} I & \frac{1}{r_{d21}} - I \end{bmatrix}
\]
M₄(21) ≡ \begin{bmatrix} -\frac{1}{r_{d1}} - I - \frac{1}{r_{d2}} I \\ -\frac{1}{r_{d1}} I - \frac{1}{r_{d2}} I \end{bmatrix},

M₄(22) ≡ \begin{bmatrix} \frac{1}{r_{d31}} + \frac{1}{r_{d32}} + \frac{1}{r_{d33}} I \\ -\frac{1}{r_{d31}} I - \frac{1}{r_{d32}} I \end{bmatrix} \begin{bmatrix} -\frac{1}{r_{d31}} I \\ -\frac{1}{r_{d31}} I \end{bmatrix},

M₄ ≡ \begin{bmatrix} M₄(11) & M₄(12) \\ M₄(21) & M₄(22) \end{bmatrix} O.

(A9)

c₄(11) = c_{21}^T + c_{31}^T + c_{41}^T, c₄(12) = -c_{21} + c_{32} + c_{42},

c₄(13) = -c_{31}^T - c_{32} + c_{33}, c₄(14) = -c_{31} + c_{32} - c_{33},

c₄(11) = \begin{bmatrix} c_{M₄}^{(11)} & c_{M₄}^{(12)} & c_{M₄}^{(13)} & c_{M₄}^{(14)} \end{bmatrix} = \begin{bmatrix} 0^T \cdots \ 0^T \end{bmatrix}.

(A10)

Authors

Atsushi Mouri (Member)

Atsushi Mouri received the B.S. (1987), M.S. (1989) degrees in electrical engineering from Osaka Prefecture University, Osaka, Japan. He joined LSI laboratory of Mitsubishi Electric Corp. in 1989, and has been working for Car Multimedia System Engineering Dept. of Sanda works from 2002. And he is presently enrolled in a Ph.D. program at graduate school of Science and Engineering Department, Ritsumeikan University from 2014. His research interests include GPS/GNSS signal processing and position information systems.

Yoshifumi Karatsu

Yoshifumi Karatsu received the B.S. (2014), M.S. (2016) degrees in electrical and electronic engineering from Ritsumeikan University, Shiga, Japan. His research interests include GNSS Navigation and VPPP algorithms. Since April, 2016, he has been working at Murata Machinery, Ltd., Kyoto, Japan.

Goshi Okuda

Goshi Okuda received the B.S. degrees in electrical and electronic engineering from Ritsumeikan University, Shiga, Japan in 2016. He is presently a M.S. student at graduate school of Science and Engineering, Ritsumeikan University. His research interests include VPPP algorithms.

Sueo Sugimoto (Member)

Sueo Sugimoto received the B.S. (1969) and M.S. (1971) degrees in mechanical engineering from the Kyoto Institute of Technology, Kyoto, Japan, and the Ph.D. (1974) degree in electrical engineering - system science from the Polytechnic Institute of New York (presently, NYU Polytechnic School of Engineering), New York. He is a professor, Department of Electrical and Electronic Engineering at Ritsumeikan University, Shiga and Kyoto, Japan. His research interests include stochastic systems, and statistical image-signal processing in various applications such as GPS/GNSS navigation.

Yukihiro Kubo (Member)

Yukihiro Kubo received the B.S. (1997), M.S. (1999) and Ph.D. (2002) degrees in electrical and electronic engineering from Ritsumeikan University, Shiga and Kyoto, Japan. He worked in the production section of GPS car-navigation systems at Mitsubishi Electric Corp., Sanda Works from 2002 to 2004. He joined Department of Electrical and Electronic Engineering of Ritsumeikan University in 2004, and he is presently a professor. His research interests include GPS/GNSS signal processing and INS/GNSS integration systems.

Masaharu Ohashi (Member)

Masaharu Ohashi received the B.S. (2010), M.S. (2012) and Ph.D. (2015) degrees in electrical and electronic engineering from Ritsumeikan University, Shiga, Japan. He joined Department of Electronic System Engineering at The University of the Shiga Prefecture in 2015 as an assistant professor. Since April, 2016, he has been working at Hitachi Zosen Corp., Tokyo, Japan. His research interests include GPS/GNSS signal processing and ionosphere modeling.