Dichromatic polynomial for graph of a (2,n)-torus knot

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Abstract

In this study, we introduce the relationship between the Tutte polynomials and dichromatic polynomials of (2,n)-torus knots. For this aim, firstly we obtain the signed graph of a (2,n)-torus knot, marked with {+} signs, via the regular diagram of its. Thereupon, we compute the Tutte polynomial for this graph and find a generalization through these calculations. Finally we obtain dichromatic polynomial lying under the unmarked states of the signed graph of the (2,n)-torus knots by the generalization.

Keywords: Knot, knot graph, Tutte polynomial, dichromatic polynomial, signed graph

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1 Introduction

William Tutte devised a renewed polynomial for graphs at the year of 1954 [9]. This polynomial has great importance in mathematics, statiscal physics, biology and theoretical computer science. It is a polynomial of two variable.

In 1984, for knots and links, a new polynomial invariant was described by Jones. For the Jones polynomial, Thistlethwaite acquired a spanning tree dilation in 1987 [8]. His study indicated that a Tutte polynomial of a connected plane graph equals the Jones polynomial of an alternating oriented link. Kauffman developed this result in 1988 and he defined Kauffman bracket polynomials [4].

Jaeger indicated that the Tutte polynomial of a plane graph equals homfly polynomial of a connected link in 1988 [2]. A characterization of the Tutte polynomial for signed graphs was carried out by Kauffman in 1989. It was shown with $Q[G] = Q[G](a,b,d)$ such that $G$ is the signed graph and the lettering $a,b,d$ are three detached factors, by him. $Q[G]$ is a special form of the Tutte polynomial. Because it has a spanning tree dilation like

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the original Tutte polynomial [5]. Further, Kauffman showed the relation between the Tutte polynomial and dichromatic polynomial.

2 Preliminaries

A knot is defined as a simple closed curve. More conceptually, a knot is embedding of the circle $S^1$ into $R^3$ (or $S^3$) [6]. A torus knot can be constructed on the trivial torus without any intersection points. If this is possible it takes the name of the torus knot. As specified by Murasugi [6], “the torus knot of the sort of $(p,q)$ which hugs round the standard solid torus $T$ in the longitudinal destination $p$ times and in the meridional destination $q$ times is shown by $K(p,q)$ such that $p, q$ are relatively prime.”

A graph $G$ is expressed by pair of $(V, E)$, where the set of vertices is indicated with $V$ and the set of edges is indicated with $E \subseteq V \times V$ [1]. We will only work on undirected graphs inhere. A loop is the edge which lies on no change vertices. A bridge is the only edge that has no alternative between two vertex.

$G - e$ is edge deletion and $G/e$ is edge contraction, the Tutte polynomial of a graph $G = (V, E)$ is described as follows [3]:

$$T(G;x,y) = \begin{cases} 
1, & E(G) = \emptyset \\
xE(G/e), & e \in E \text{ and } e \text{ is a bridge} \\
yT(G-e), & e \in E \text{ and } e \text{ is a loop} \\
T(G-e;x,y) + T(G/e;x,y), & e \in E \text{ and } e \text{ is not a loop and not a bridge}
\end{cases}$$

The set of associated plane link diagrams and the set of associated signed plane graphs correspond to each other individually [5].

Fig. 1 Link diagram connected with signed graph.

$G$ is given as a signed graph. Let $e$ be an edge belonging to $G$. The sign (+ or -) belonging to $e$ is indicated by sign$(e)$. $e$ might be a bridge. $e$ might be a loop. In the graph $G$, the number of bridges which have the plus sign is indicated by $b_p = b_p(G)$, the number of bridges which have the minus sign is indicated by $b_n = b_n(G)$, the number of loops which have the plus sign is indicated by $l_p = l_p(G)$ and the number of loops which have the minus sign is indicated by $l_n = l_n(G)$. For signed graphs, a polynomial $Q[G] = Q[G](a,b,d)$ is described per utilizing deletion-contraction operators and the undermentioned equations [5]. The abbreviations $x = a + bd$, $y = ad + b$ will use inhere for convenience:

1) In the graph $G$, if $e$ is not a bridge or $e$ is not a loop, at the time

$$\text{sign}(e) < 0 \Rightarrow Q[G] = aQ[G'] + bQ[G''],$$

$$\text{sign}(e) > 0 \Rightarrow Q[G] = bQ[G'] + aQ[G''],$$

where $G$, $G'$ and $G''$ are graphs obtained via deletion-contraction.
2) If $G$ is associated and every edge in $G$ is either a bridge or a loop, at the time
\[ Q[G] = x^{b_p + b_n} y^{b_n} + y_p. \]

3) Providing $G$ is equal to the separated combination of graphs $G_1$ and $G_2$ at the time,
\[ Q[G] = dQ[G_1]Q[G_2]. \]

Tutte also described the dichromatic polynomial of a graph as a closer two-variable generalization of the chromatic polynomial. It is
\[ Q[G; u, v] = \sum_{A \subseteq E} u^{|A|} v^{|V| - c(A) + |V|}, \]
where the number of associated components of the spanning subgraph $(V, A)$ is shown by $c(A)$.

**Proposition 1.** (\cite{5}) Suppose that $G$ is a signed graph which every edges of its have positive signs. $Z[G](q, v)$ is the dichromatic polynomial for the underlying unmarked graph, $N$ is the number of vertices of $G$ and $c$ is the number of associated components in the graph $G$, then
\[ Z[G](q, v) = q^{\frac{N}{2}} Q[G](q^{-\frac{1}{2}} v, 1, q^{\frac{1}{2}}). \]

The Tutte polynomial of a graph can be reformulated to the dichromatic polynomial of a graph. For this reason, $T(G; x, y)$ is a special case of $Q[G]$.

### 3 Main results

(2,n)-torus knots indicated by $K_{2,n}$ are considered. Firstly, it is obtained the isomorphic graphs belonging to (2,n)-torus knots $K_{2,n}$ by constructing regular diagrams of them. After that, (+) or (-) signs are assigned to these isomorphic graphs belonging to (2,n)-torus knots according to a certain rule. And then their Tutte polynomials reckon explained as above. \cite{7} can be viewed for more details. We will debate over these results.

**Theorem 2.** (\cite{7}) About the Tutte polynomials of the graphs in which every edges of them have (+) signs for (2,n)-torus knots $K_{2,n}$ (see Figure 2), it is obtained the following characterization:
\[ Q[G^+] = b \left( \sum_{k=1}^{n-1} a^{k-1} x^{n-k} \right) + a^{n-1} y. \]

The graph signed with (+) of $K_{2,3}^+$ is shown with $G_{4}^+$. The graph signed with (+) of $K_{2,5}^+$ is shown with $G_{5}^+$.
By using the general formula in Theorem 2, the relation in proposition 1 and the abbreviation equations $x = a + bd$, $y = ad + b$ we can process the following operations:
\[ Q[G^*] = b \left( \sum_{k=1}^{n-1} a^{k-1} y^{n-k} \right) + a^n \]

\[ Q[G^*] = b \left( \sum_{k=1}^{n-1} d^{k-1} (a+b)^{n-k} \right) + a^n (ad + b) \]

\[ \Rightarrow Z[G^*](q,v) = q^{\frac{n-1}{2}} Q[G^*](q^{\frac{n-1}{2}} v, 1, q^{\frac{1}{2}}) \]

\[ Z[G^*](q,v) = q^{\frac{n-1}{2}} \left[ \left( \sum_{k=1}^{n-1} q^{k-1} (q^{-\frac{1}{2}} v + q^{\frac{1}{2}})^{n-k} \right) + q^{\frac{1}{2}} v^{n-1}(v+1) \right] \]

\[ Z[G^*](q,v) = q^{\frac{n-1}{2}} \left[ \left( \sum_{k=1}^{n-1} q^{k-1} (q^{-\frac{1}{2}} v + q^{\frac{1}{2}})^{n-k} \right) + q^{\frac{1}{2}} v^{n-1}(v+1) \right] \] \( (1) \)

**Corollary 3.** The dichromatic polynomials of graphs of \((2,n)\)-torus knots are obtained by relation \((1)\).

**Example 4.** For the graph \(G_4^*\) of \(K_{2,3}\), the following dichromatic polynomial is obtained:

\[ Z[G^*](q,v) = q^{\frac{1}{2}} \left[ \left( \sum_{k=1}^{3-1} q^{k-1} (q^{-\frac{1}{2}} v + q^{\frac{1}{2}})^{3-k} \right) + q^{\frac{1}{2}} v^{3-1}(v+1) \right] \]

\[ Z[G^*](q,v) = q^{2} \left[ (q^{-\frac{1}{2}} v^2 + q^{\frac{1}{2}})^2 + q^{-\frac{1}{2}} v (q^{-\frac{1}{2}} v + q^{\frac{1}{2}}) \right] + q^{-1} v^3 + q^{-1} v^2 \]

\[ Z[G^*](q,v) = q^{2} (q^{-1} v^2 + 2v + q^{-1} v^2 + v + q^{-1} v^3 + q^{-1} v^2) \]

\[ Z[G^*](q,v) = q^{2} (3q^{-1} v^2 + 3v + q^{-1} v^3) \]

\[ Z[G^*](q,v) = 3qv^2 + 3q^2 v + q^3 + qv^3. \]

**4 Conclusion**

Here we get a characterization for graphs of \((2,n)\)-torus knots. In doing so, we are establishing an organic bond between the knot and its graph. We propose an alternative way of calculating the dichromatic polynomial. We are concretizing what is done with an example. In the following stages, the relation of this study with other graph polynomials is a worthy problem.

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