Characterization of curves in $C^{(2)}$

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Abstract

In this paper we characterize the irreducible curves lying in $C^{(2)}$. We prove that a curve $B$ has a degree one morphism to $C^{(2)}$ with image a curve of degree $d$ with irreducible preimage in $C \times C$ if and only if there exists an irreducible smooth curve $D$ and morphisms from $D$ to $C$ and $B$ of degrees $d$ and 2 respectively forming a diagram which does not reduce.

Keywords: Symmetric product, curve, irregular surface, curves in surfaces.

1 Introduction

Given a curve $B \subset C^{(2)}$, we define the degree of $B$ as the integer $d$ such that $C_P \cdot B = d$ where $C_P$ denotes the coordinate curve in $C^{(2)}$ with base point $P$.

The curves of degree one in $C^{(2)}$ are completely characterized by the two results in [ACGH85, Pg. 310, D-10] and [Cih83], where it is proven that a curve $B$ of degree one in $C^{(2)}$ different from a coordinate curve is smooth and it exists if and only if there exists a degree two morphism $f : C \to B$. Moreover, $B = \{f^{-1}(q) \mid q \in B\} \subset C^{(2)}$.

In [Cha08] a different proof of this result is given. From this proof we remark that considering the curve $B \subset C^{(2)}$ as before, then, the preimage of $B$ by $\pi_C : C \times C \to C^{(2)}$ is isomorphic to $C$ through the projection onto the first factor.

Let $\tilde{B}$ be an irreducible curve in $C^{(2)}$ different from a coordinate divisor. Let $B$ be its normalization and assume that there is no degree two morphism from $C$ to $B$. Then, since $C_P$ is ample in $C^{(2)}$, from the characterization of degree one curves we deduce that $\tilde{B} \cdot C_P \geq 2$. In this paper we present a characterization of curves with any degree. First of all we need the following definition:

Definition 1.1. We say that a diagram of morphisms of curves

\[
\begin{array}{ccc}
D & \xrightarrow{(e:1)} & B \\
\downarrow{(d:1)} & & \downarrow \\
C & & 
\end{array}
\]

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reduces if there exist curves $F$ and $H$ such that there exists a diagram

\[
\begin{array}{c}
\begin{array}{c}
D \xrightarrow{(d:1)} B \\
\downarrow_{(k:1)} \\
\downarrow_{(k:1)} \\
F \xrightarrow{(e:1)} H \\
\downarrow \\
C
\end{array}
\end{array}
\]

with $k > 1$, the upper square being a commutative diagram and the left vertical arrows giving a factorization of the original degree $d$ morphism.

When $k = d$ we will say that the diagram completes, and we will obtain a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
D \xrightarrow{(e:1)} B \\
\downarrow_{(d:1)} \\
\downarrow_{(d:1)} \\
C \xrightarrow{(e:1)} H.
\end{array}
\end{array}
\]

Notice that when $d$ is a prime number both definitions are equivalent.

In this paper we prove

**Theorem 1.2.** Let $B$ be an irreducible smooth curve such that there are no non-trivial morphisms $B \to C$. A morphism of degree one from the curve $B$ to the surface $C^{(2)}$ exists, with image $\tilde{B}$ of degree $d$ if, and only if, there exists a smooth irreducible curve $D$ and a diagram

\[
\begin{array}{c}
\begin{array}{c}
D \xrightarrow{(2:1)} B \\
\downarrow_{(d:1)} \\
C
\end{array}
\end{array}
\]

which does not reduce.

If we consider the case $d = 1$ we recover the results for degree one.

We prove the theorem in two steps, giving a separated proof for each implication (see **Theorem 2.2** and **Theorem 2.3**). First, given a diagram

\[
\begin{array}{c}
\begin{array}{c}
D \xrightarrow{(2:1)} B \\
\downarrow_{(d:1)} \\
C
\end{array}
\end{array}
\]

which does not reduce we find a curve in $C^{(2)}$ defined by it as the image by $g^{(2)}$ of the immersion of $B$ in $D^{(2)}$ given by $f$. We prove that $g^{(2)}|_{B}$ has degree one, and
hence the curve in \( C^{(2)} \) has normalization \( B \) and the normalization map is precisely \( g(2)|_B \). Second, given a curve lying in \( C^{(2)} \) we find a diagram defined by the curve, its preimage by \( \pi_C \) and the projection on one factor of \( C \times C \). We compute the degrees of the different maps and prove that this diagram does not reduce.

In a following paper we are going to use this result to study and classify curves of degree two and some of degree three.

**Notation:** We work over the complex numbers. By curve we mean a complex projective reduced algebraic curve. Let \( C \) be a smooth curve of genus \( g \geq 2 \), we put \( C^{(2)} \) for its 2nd symmetric product. We denote by \( \pi_C : C \times C \to C^{(2)} \) the natural map, and \( C_P \subset C^{(2)} \) a coordinate curve with base point \( P \in C \). We put \( \Delta_C \) for the main diagonal in \( C^{(2)} \), and \( \Delta_{C \times C} \) denotes the diagonal of the Cartesian product \( C \times C \).

## 2 Characterization

We begin with a lemma that will simplify the rest of the exposition.

**Lemma 2.1.** We consider a diagram of morphisms of smooth irreducible curves

\[
\begin{array}{ccc}
D & \xrightarrow{f} & B \\
\downarrow^{(2:1)} & & \downarrow^{(2:1)} \\
C & \xrightarrow{g} & C
\end{array}
\]

The image of \( B \subset D^{(2)} \) (with the immersion given by the fibers of \( f \)) by the morphism \( g^{(2)} \) is the diagonal \( \Delta_C \subset C^{(2)} \) if and only if the morphism \( g \) factorizes through the curve \( B \) by \( f \).

**Proof.** Let \( i \) be the involution on \( D \) that defines \( f \), that is, the change of sheet. Since \( B = \{ x + y \mid f(x) = f(y) \} = \{ x + i(x) \} \subset D^{(2)} \), then \( \text{Im}(g^{(2)}|_B) = \{ g(x) + g(i(x)) \} \). It is contained in the diagonal \( \Delta_C \) if and only if \( g(x) = g(i(x)) \) for all \( x \in D \), that is, if and only if \( g \) factorizes through \( B \) by \( f \).

In the following theorem, given a diagram that does not reduce we deduce the existence of a curve in \( C^{(2)} \) naturally attached to it.

**Theorem 2.2.** Assume that there exists a diagram of morphisms of smooth irreducible curves

\[
\begin{array}{ccc}
D & \xrightarrow{f} & B \\
\downarrow^{(2:1)} & & \downarrow^{(2:1)} \\
C & \xrightarrow{g} & C
\end{array}
\]

which does not reduce and such that the morphism \( g \) does not factorize through \( B \) by \( f \). Then, \( g^{(2)} \) gives a degree one map \( B \to C^{(2)} \) with reduced image a curve \( \tilde{B} \) of degree precisely \( d \).
Proof. Consider a diagram as above and look at the induced morphism $D^{(2)} \xrightarrow{g^{(2)}} C^{(2)}$. As we have seen in the Introduction, we have an immersion $B \subset D^{(2)}$ as the set of pairs of points in $D$ with the same image by $f$. Then, we consider $D$ inside $D \times D$ as $\pi_D^{-1}(B) \cong D$, that is, ordered pairs of points with the same image by $f$.

Let $\tilde{B} = g^{(2)}(B)_{\text{red}}$, the reduced image curve in $C^{(2)}$, and consider the map $B \xrightarrow{(k;1)} \tilde{B}$ induced by $g^{(2)}$. We want to see that $k = 1$.

Notice that by Lemma 2.1 we can assume that $\tilde{B}$ is not $\Delta_C$. We know that $B \cdot D_P = 1$, hence,

$$1 = g^{(2)}_*(B \cdot D_P) = g^{(2)}_*(B) \cdot \left(\frac{1}{d}C_P\right) \Rightarrow g^{(2)}_*(B) \cdot C_P = d.$$ 

In addition, since the map $B \xrightarrow{(k;1)} \tilde{B}$ is $g^{(2)}|_B$, we obtain that $d = (k\tilde{B}) \cdot C_P$, and thus $\tilde{B} \cdot C_P = \frac{d}{k}$, that is, $k$ divides $d$.

Assume by contradiction that $k > 1$.

Let $F$ be the preimage of $\tilde{B}$ by the morphism $\pi_C : C \times C \to C^{(2)}$. Then $F \to \tilde{B}$ has degree two and thus we obtain a diagram

$$D \times D \xrightarrow{\pi_D} D^{(2)}$$

Observe that the exterior arrows form a commutative diagram, and hence, also the interior arrows give a commutative diagram. Thus, the morphism $D \to F$ has degree $k$. Now, the restriction to $D$ of $g \times g$ followed by the projection onto one factor of $C \times C$ is precisely $g : D \to C$ by construction. That is, we obtain the diagram

$$D \xrightarrow{(2:1)} B$$

Hence, the original diagram reduces, contradicting our hypothesis.
Consequently, $k = 1$ and thus we deduce that the curve $\tilde{B}$ has normalization $B$.

Moreover, looking at diagram (1) we deduce that $D \xrightarrow{(1:1)} F$, that is, the preimage of $\tilde{B}$ by $\pi_C$ has normalization $D$, and we will denote it by $\tilde{D}$. So we have:

$$D \times D \xrightarrow{\pi_D} D^{(2)}$$

$$\xrightarrow{D \xrightarrow{f} B}$$

$$\xrightarrow{\tilde{D} \xrightarrow{g} B^{(2)}}$$

$$\xrightarrow{g \times g}$$

$$\xrightarrow{\tilde{D} \times C \xrightarrow{\pi_C} C^{(2)}}$$

$$\xrightarrow{\ell} C$$

where the dashed arrows show the original diagram. \( \square \)

Conversely, we have also a theorem in the opposite direction, from the existence of curves in $C^{(2)}$ we deduce the existence of diagrams which do not reduce.

**Theorem 2.3.** Given an irreducible curve $\tilde{B}$ lying in $C^{(2)}$ with degree $d$, let $B$ be its normalization, and assume that there are no non trivial morphisms $B \rightarrow C$. Then, there exists a smooth irreducible curve $D$ and a diagram

$$D \xrightarrow{(2:1)} B$$

$$\xrightarrow{D \xrightarrow{(d:1)} C}$$

which does not reduce.

**Proof.** First of all, we observe that $\tilde{B}$ is not the diagonal in $C^{(2)}$ because we are assuming that there are no morphisms from $B$ to $C$.

Let $\tilde{D} = \pi_C^*(\tilde{B}) \in Div(C \times C)$ and $D$ its normalization. We notice that with our hypothesis $\tilde{D}$ is irreducible. Indeed, otherwise, one of its components would have as normalization the curve $B$, because we have a $(2 : 1)$ morphism from $\tilde{D}$ to $B$, and since $\tilde{D} \subset C \times C$ we would obtain a non trivial morphism from $B$ to $C$ contradicting our hypothesis.

Now, we are going to compute the degree of $\tilde{D} \rightarrow C$, given by the projection onto one factor:

$$\tilde{D} \cdot (C \times P + P \times C) = \pi_{C^*}(\pi_C^*(\tilde{B}) \cdot \pi_C^*(C_P)) = \tilde{B} \cdot \pi_{C^*}(C_P) = 2\tilde{B} \cdot C_P = 2d.$$
And therefore, since $\tilde{D}$ is symmetric with respect to the involution $(x, y) \rightarrow (y, x)$ by construction, $\tilde{D} \cdot (C \times P) = d$, and so, the degree of the morphism on $C$ is precisely $d$. In this way, we have a diagram

$$
\begin{array}{c}
\tilde{D} \xrightarrow{(2:1)} \tilde{B} \\
\downarrow^{(d:1)} \\
C
\end{array}
\quad \text{and taking their normalizations}
\quad \begin{array}{c}
\text{we obtain a diagram of} \\
\text{morphisms of smooth curves}
\end{array}
\quad \begin{array}{c}
D \xrightarrow{(2:1)} B \\
\downarrow^{(d:1)} \\
C
\end{array}
$$

We call $f : D \rightarrow B$ the map coming from $\pi_C|_{\tilde{D}}$ and $g : D \rightarrow C$ the map coming from the projection onto one factor of $C \times C$.

Let $\alpha$ be the degree one morphism induced in $B$ by the immersion of $\tilde{B}$ in $C^{(2)}$. Since we have $D \xrightarrow{(2:1)} B$, as we have seen in the Introduction there exists an immersion of $B$ in $D^{(2)}$ as pairs of points with the same image by this morphism.

Since $D \xrightarrow{(1:1)} \tilde{D} \subset C \times C$ we can consider that a general point in $D$ is a pair $(x, y)$ with $x, y \in C$. Moreover, since $D \rightarrow B$ is induced by $\pi_C|_{\tilde{D}}$, a general fiber of $D \rightarrow B$ will be two points $(x, y)$ and $(y, x)$. Hence, we can write a general point of $B \subset D^{(2)}$ as $(x, y) + (y, x)$.

Now, we consider the restriction to $B \subset D^{(2)}$ of $g^{(2)}$. By construction this morphism is precisely $\alpha$ and therefore the image is the original $\tilde{B}$. In particular, $g^{(2)}|_{B}$ is generically of degree one.

We are going to see that the diagram does not reduce by contradiction: Assume that there exist curves $F$ and $H$, and a diagram

$$
\begin{array}{c}
D \xrightarrow{f} B \\
\downarrow^{g} \\
F \xrightarrow{s} H \\
\downarrow^{h} \\
C
\end{array}
\quad \begin{array}{c}
\text{as in Definition 1.1. Then, as we have seen in the Introduction, the fibers of} s \text{ give} \\
\text{a curve isomorphic to} H \text{ inside} F^{(2)}. \text{ Hence, we have}
\end{array}
$$

\begin{align*}
\begin{array}{c}
D^{(2)} \xrightarrow{g^{(2)}} C^{(2)} \\
\downarrow^{h^{(2)}} \\
H \xrightarrow{f^{(2)}} \end{array}
\quad \begin{array}{c}
\text{and}
\end{array}
\begin{array}{c}
\tilde{B} \\
\downarrow^{\alpha} \\
B \xleftarrow{\alpha} D^{(2)} \xrightarrow{g^{(2)}} C^{(2)} \\
\downarrow^{h^{(2)}} \\
H \xrightarrow{f^{(2)}} \end{array}
\end{align*}
By definition, the image of \( B \subset D^{(2)} \) by \( h^{(2)} \) is \( H \subset F^{(2)} \), that is, the embedding of \( H \) in \( F^{(2)} \) given by \( s \), and we know that \( l \circ h = g \) so \( l^{(2)} \circ h^{(2)} = g^{(2)} \), hence

\[
g^{(2)}|_B : B \xrightarrow{h^{(2)} = r} H \xrightarrow{(2)} \tilde{B}
\]

thus \( r \), as well as \( h \), have degree one. Consequently, our diagram does not reduce (see Definition 1.1).

We observe that we could change the hypothesis of the non existence of morphisms from \( B \) to \( C \) by assuming that \( \tilde{B} \) is not the diagonal and that \( \pi^{-1}_C(B) \) is irreducible.

Putting these two theorems together we find the characterization of curves in the symmetric square \( C^{(2)} \) previously stated in Theorem 1.2.

References

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