Scheduling a single machine with compressible jobs to minimize maximum lateness

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Abstract

The problem of scheduling non-simultaneously released jobs with due dates on a single machine with the objective to minimize the maximum job lateness is known to be strongly NP-hard. Here we consider an extended model in which the compression of the job processing times is allowed. The compression is accomplished at the cost of involving additional emerging resources, whose use, however, yields some cost. With a given upper limit $U$ on the total allowable cost, one wishes to minimize the maximum job lateness. It is clear that, by using the available resources, some jobs may complete earlier and the objective function value may respectively be decreased. As we show here, for minimizing the maximum job lateness, by shortening the processing time of some specially determined jobs, the objective value can be decreased. Although the generalized problem is harder than the generic non-compressible version, given a “sufficient amount” of additional resources, we can solve the problem optimally. We determine the compression rate for some specific jobs and develop an algorithm that obtains an optimal solution. Such an approach can be beneficial in practice since the manufacturer can be provided with an information about the required amount of additional resources in order to solve the problem optimally. In case the amount of the available additional resources is less than used in the above solution, i.e., it is not feasible, it is transformed to a tight minimal feasible solution.

Keywords: scheduling; single machine; release and due dates; algorithm; compressible job processing times
1 Introduction

The main part of the scheduling literature deals with deterministic problems, where all data including the processing times are fixed values. However, in many practical problems the processing time of a job can be controlled e.g. by the amount of an allocated resource used the processing time can be compressed from a standard value to a smaller value. In this paper, we consider such a problem with compressible processing times which can be described as follows.

The $n$ jobs $1, 2, \ldots, n$ from a set $J$ are to be processed on a single machine. Each job $j$ is available for processing from its release time $r_j$ and has the desired completion time or due date $d_j$. For each job $j$, we have an initially given processing time $a_j$ and the cost for the unitary compression $c_j$. If the processing time of job $j$ is compressed by $x_j(S)$ time units in the schedule $S$, then its real processing time is

$$p_j(S) = a_j - x_j(S),$$

and the cost for this compression is $x_j(S)c_j$. There is an upper limit $U \geq 0$ on the total compression cost. Hence, the total cost in a feasible schedule $S$ has to be not larger than $U$, i.e.,

$$\sum_j x_j(S)c_j \leq U. \quad (1)$$

We shall refer to an $n$-component vector $(x_1(S), \ldots, x_n(S))$ as the compression vector for a schedule $S$. The completion time of job $j$ in the schedule $S$ is

$$f_j(S) = s_j(S) + p_j(S),$$

where $s_j(S)$ is the starting time of job $j$ in that schedule. The lateness of job $j$ in the schedule $S$ is

$$L_j(S) = f_j(S) - d_j,$$

and the lateness $L(S)$ of schedule $S$ is the maximum job lateness in it, i.e.,

$$L(S) = \max_j L_j(S).$$

The aim is to find an optimal feasible schedule, i.e., a feasible schedule with the minimum lateness $L^{\text{opt}}$.

This setting is motivated by real-life scenarios, where additional resources are available for processing the jobs. In case an additional resource is used for job $j$, the processing requirement of that job reduces accordingly. The cost for incorporating an additional resource for job $j$ is reflected by the parameter $c_j$. The total budget for the additional resources is limited by a constant $U$, which makes it harder to solve the problem (similarly, as bounding the knapsack capacity makes the KNAPSACK problem hard). Note that the problem $1|r_j, \text{compressible}(U)|L_{\text{max}}$ is a special case of the generic version $1|r_j|L_{\text{max}}$ with $U = 0$. Hence, the former problem is as hard as the latter generic one which is known to be strongly NP-hard [1]. The problem can also be seen as a bi-criteria optimization problem with two contradictory objective criteria, to minimize the
maximum lateness and to minimize the total compression cost. It is not difficult to see that the corresponding Pareto optimization problem remains strongly NP-hard. Among the feasible schedules with a given total cost $U$, finding one with the minimum lateness is clearly as hard as problem $1|r_j|L_{\text{max}}$ (in particular, for $U = 0$, we have an instance of problem $1|r_j|L_{\text{max}}$). Likewise, among the feasible solutions with a given maximum job lateness, it is hard to find one with the minimum total compression cost.

There are a number of exact exponential-time implicit enumeration algorithms for problem $1|r_j|L_{\text{max}}$. However, it is not easy to arrive at an efficient enumerative method for the extended problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$ since it is not easy to enumerate all feasible ways for the compression of the job processing times; here, for each possible selection of compressed processing times, an exact algorithm for problem $1|r_j|L_{\text{max}}$ has to be invoked. Hence, it is unlikely that one can arrive at an easy and efficient exact solution method for the general setting $1|r_j,\text{compressible}(U)|L_{\text{max}}$. Given this, a polynomial-time algorithm that gives optimal solutions under some reasonable conditions and reasonable sub-optimal solutions would clearly be of an interest. In this paper, we describe such a polynomial-time method considering two alternative practical versions of the problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$.

The method that we propose provides the manufacturer with an amount of additional resources; given this amount of additional resources, the method obtains an optimal solution to the problem. If the amount of additional resources is a priori fixed and the total compression cost of our solution is greater than $U$ (i.e., it is not feasible for problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$), we transform it to a feasible solution which is optimal for the corresponding instance of the generic problem $1|r_j|L_{\text{max}}$. Our transformation delivers a tight minimal feasible solution problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$, in the sense that by increasing the processing time of a job by some amount of time, the maximum job lateness in the resultant solution will increase by the same amount. In particular, for a chosen compression vector $(x_1, \ldots, x_n)$, an instance of problem $1|r_j|L_{\text{max}}$ with the job processing times $a_1 - x_1, \ldots, a_n - x_n$ is naturally associated with the initially given instance of problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$. The two algorithms that we propose, for a given instance $I$ of problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$, find a compression vector with the associated instance $I'$ of problem $1|r_j|L_{\text{max}}$ and a solution $S$ respecting that vector, which is optimal for the instance $I'$. If the solution $S$ satisfies condition $(1)$, then it is also optimal for instance $I$. Otherwise, it is transformed to a tight feasible solution, as indicated above.

Our first algorithm is pseudo-polynomial, its running time depends linearly on the maximum job processing time $p_{\text{max}}$. The second one is polynomial and runs in time $O(n^2 \log n)$. It applies a similar strategy as the pseudo-polynomial one but the search is organized in a different way so that the dependence on $p_{\text{max}}$ is avoided.

The remainder of the paper is organized as follows. In Section 2, we give a brief overview of the related literature. Section 3 presents some necessary preliminaries. Section 4 consists of three subsections. In Section 4.1 we describe an algorithm that provides us with our initial solution. In Section 4.2, we discuss the proposed approach in details and give our pseudo-polynomial time algorithm. In Section 4.3 we describe our polynomial-time algorithm. Finally, Section 5 gives a few concluding remarks.
2 Literature Review and Applications

In this section we survey the earlier work that we found in the literature for scheduling problems using some kind of the control or compression of job processing times, where they are not constant. Since the 1980ies, a lot of papers dealing with such a scheduling environment appeared so that we mention here only a few of the most relevant ones. One of the first papers considering scheduling problems with so-called controllable processing times was given by Vickson [24] in 1980. He considered two single machine problems and presented a polynomial algorithm as well as a complexity result. While the problem of minimizing total completion time can be formulated as an assignment problem and thus be solved in $O(n^{2.5})$ time, the consideration of job weights makes the problem $NP$-hard. Such problems with controllable processing times have not only theoretical but also practical importance. Janiak [2] described a real-world problem in the context of steel production, where batches of ingots have to be preheated before they are hot-rolled in a blooming mill. Both the preheating and rolling times are indirectly proportional to the gas flow intensity. Another application arises in a machine tooling environment when the processing time of a job depends on the feed rate and the spindle speed for the particular operations. Using our model for such applications, the manufacturer can be provided by the required amounts of gas portions for an optimal production process. Alternatively, if the amount of gas is insufficient, then our method would adopt the solution to a feasible minimal solution that uses precisely the available amount of resource (gas).

The paper [3] considers single machine scheduling problems with controllable processing times and minimizing the maximum job cost. For this problem, several polynomial time results are derived. On the other hand, for the corresponding problem with minimizing the total weighted completion time of the jobs an $NP$-hardness result has been presented. In [4], the single machine problem with given release dates, the criterion of minimizing maximum lateness subject to linearly controllable processing times has been considered. For this problem, a polynomial time approximation scheme was derived with a worst case approximation ratio of $11/6$.

A very detailed excellent survey on scheduling problems with controllable processing times for the period up to 2007 has been given in [5], where 113 papers are surveyed. Most reviewed results refer to single machine problems. In separate subsections, the authors present results for the single machine problem with minimizing the maximum penalty term, total weighted completion time of the jobs, the weighted number of tardy jobs, batch scheduling problems, and due date assignment problems. In addition, for multi-machine problems, detailed tables with polynomial algorithms, approximation algorithms and complexity results are given.

Shakhlevich et al. [6] deal with bi-criteria single machine scheduling with controllable processing times, where the maximum cost depending on the job completion times and the total compression cost should be minimized. This problem is reduced to a series of linear programs defined over the intersection of a submodular polyhedron with a box, and then a greedy algorithm is applied which is faster than earlier ones. Later, the same authors [7] consider a single machine problem with controllable processing times as well as given release date and deadlines for each job subject to the minimization of the total cost for reducing the processing times. They reformu-
late the problem as a submodular optimization problem and develop a recursive decomposition algorithm.

In the paper [8], a single machine scheduling problem with controllable processing times and learning effect is investigated. The objective is to minimize a cost function, containing makespan, total completion (waiting) time, total absolute differences in the completion (waiting) times and total compression cost. The resulting problem is formulated as an assignment problem and can thus be solved in polynomial time. Shabtay and Zofi [9] consider a single machine scheduling problem with controllable processing times with an unavailability period and the objective of minimizing the makespan. For the case when the processing times are convex decreasing functions of the amount of the allocated resource, they show that the problem is NP-hard and present both a constant factor approximation algorithm as well as a fully polynomial time approximation scheme. In the paper [10], the single machine problem with controllable processing times and no inserted idle times with minimizing total tardiness and earliness is considered. After presenting a mathematical model, several heuristic approaches are suggested and compared on instances of different sizes. Recently, Luo and Zhang [11] considered a single machine problem, where in addition to controllable processing times also the setup times can be controlled. For the case of job and position-dependent workloads and minimizing the makespan, the authors show that this problem can be optimally solved in polynomial time.

A special situation happens when processing times can be compressed. Cheng et al. [12] consider a single-machine scheduling problem with common due date assignment and compressible processing times. The authors consider two variants of due-date assignment methods with the goal to determine an optimal job sequence, the optimal due dates and the optimal compressions of the processing times in order to minimize a total penalty function. Such an approach somewhat resembles ours, where we determine optimal job compression rates to minimize the maximum lateness. A single machine problem with randomly compressible processing times, which may result e.g. from the introduction of a new technology, has been considered by Qi et al. [13]. Considering the cost for the breakdown and the compressible processing times, it is proven that under certain conditions, the optimal sequence satisfies the V-shape property.

Cao et al. [14] deal with three scheduling problems, where jobs can be rejected or the processing times can be discretely compressed. All three problems considered are NP-hard. For two of these problems pseudo-polynomial dynamic programming algorithms and FPTASs are presented, and for the third problem a greedy heuristic is given. Zhang and Zhang [15] consider a scheduling problem with identical parallel machines to minimize the makespan subject to a constraint on the total compression cost. They suggest a pseudo-polynomial dynamic programming algorithm and an FPTAS. Peng et al. [16] consider a single resource scheduling problem with compressible processing times. The goal is to minimize the length of the delay time and the number of compressed tasks. For this problem, a heuristic algorithm is presented and tested.

A related class of scheduling problems are those with deterioration, where the processing time depends on the starting time of the job, i.e., the later a job starts, the larger is its processing time. In [17], a single machine problem with minimizing the makespan subject to a cumulative deterioration is considered, where the processing conditions can be restored by a single mainte-
nance. The authors investigate two versions of the problem and present fully polynomial-time approximation schemes. In [18], the problem of scheduling deteriorating jobs with given release dates on a single batching machine is considered with the objective to minimize maximum lateness. After proving NP-hardness, a 2-approximation algorithm is given for the case of negative due dates of all jobs. It is also proven that the ED rule delivers an optimal solution for the case when all jobs have agreeable release dates, due dates and deterioration rates. Miao et al. [19] consider parallel and single machine problems with step-deteriorating jobs, given release dates and minimizing the makespan. For the single machine problem, NP-hardness in the strong sense has been proven. Finally, we mention that a very recent survey about deterioration (and learning) effects in scheduling has been given by Pei et al. [20].

3 Preliminaries and basic properties

We use the existing notions, terminology and known properties that we briefly overview in this section. We refer the reader to, e.g., [21], for more details and illustrative examples.

A commonly used method for scheduling jobs with due dates was proposed by Jackson [22] for the model without release times. Later it was extended by Schrage [23] for non-simultaneously released jobs. The algorithm, iteratively, among all currently released jobs, schedules a most urgent job, one with the minimum due date; it updates the current time \( t \) and repeats the same operation for the updated current time: initially, \( t := \min_j r_j \). Then, iteratively, \( t \) is set to the minimum among the release time of a yet unscheduled job and the completion time of the last assigned job to the machine.

We will refer to a schedule generated by the heuristic as an ED-schedule, and denote the ED-schedule constructed for the original problem instance (with the original job processing times) by \( \sigma \). We may easily observe that in an ED-schedule \( S \), there will arise an idle-time interval or a gap if the release time of a yet unscheduled job is strictly larger than the completion time of the last assigned job to the machine.

One can look at the schedule \( S \) as a sequence of blocks, a consecutive sequence of jobs such that for each two neighboring jobs \( i \) and \( j \), job \( j \) starts directly at the completion time of job \( i \) behind time \( r_j \). If job \( j \) starts at time \( r_j \) then job \( j \) starts a new block; in this case, we will say that a 0-length gap between the two jobs occurs.

We will distinguish a “most critical” part in a schedule \( S \), one containing a job that realizes the value of the objective function. Let us call such a job an overflow job; i.e., if \( o \) is an overflow job in the schedule \( S \), then

\[
L_o(S) = \max_j L_j(S).
\]

In case there are two or more consecutively scheduled jobs realizing the maximum objective function value, the last one is set to be the overflow job.

A critical segment in the schedule \( S \) containing an overflow job \( o \) is called a kernel in that schedule: it is a longest consecutive sequence of jobs ending with job \( o \) such that no job from this sequence has a due date larger than \( d_o \). By definition, there may exist no gap in a kernel, and a
kernel is contained within some block. Since there may exist more than one overflow job, there will be the same amount of kernels in the schedule $S$. We use
\[
r(K) = \min_{i \in K} \{r_i\}
\]
for the minimum release time of a job of kernel $K$, and abusing the notation, we will also use $K$ for the set of jobs from the kernel $K$.

Suppose the first job of kernel $K$ starts behind time $r(K)$ in schedule $S$. Then note that the completion time of the overflow job $o \in K$ can be decreased if some job $j \in K$ gets scheduled earlier than the earliest job of kernel $K$ was scheduled in $S$. We call job $e$ scheduled immediately before the earliest scheduled job of kernel $K$ the *delaying emerging* job of kernel $K$ in schedule $S$. Note that job $e$ causes a forced right-shift for the jobs of the kernel $K$ in the schedule $S$, and that $d_e > d_j$. We will use $E(S)$ for the set of the delaying emerging jobs in the schedule $S$.

**Property 1** (*Proposition 1 in [21]*) If ED-schedule $S$ contains a kernel with no delaying emerging job, then it is optimal.

Assume a kernel $K \in S$ possesses the delaying emerging job $e \in E(S)$, and let
\[
\Delta(K) = f_e(S) - r(K)
\]
be the forced right-shift or the *delay* for kernel $K$ in the schedule $S$. Furthermore, let
\[
\Delta_{\text{min}}(S) = \min_{K \in S} \Delta(K).
\]
The following property states previously known facts (e.g., see [21]).

**Property 2** Given $K \in S$ with the delaying emerging job $e$ and the overflow job $o$, we have
\[
L_o(S) - L_{\text{OPT}} \leq \Delta_{\text{min}}(S) \leq \Delta(K) < p_{\text{max}}.
\]

### 3.1 Regularizing Kernels

Based on Property 1, it remains to consider the case, where every kernel from the ED-schedule $\sigma$ possesses the delaying emerging job. Let $e$ be the delaying emerging job for kernel $K$ with the overflow job $o$ in an ED-schedule $S$. And let us consider an auxiliary partial ED-schedule $\tilde{K}$ constructed solely for the jobs of the kernel $K$ by ED-heuristic. In this partial schedule, the earliest included job $i$ of the kernel $K$ starts at its release time $r_i = r(K)$. We shall refer to the kernel $K$ as *regular* if the overflow job in the partial schedule $\tilde{K}$ has the due date $d_o$ (we note that the kernel in the schedule $\tilde{K}$ will not be formed by all jobs of the kernel $K$ in case a gap in between them in that schedule occurs).

As it is easy to see, if a kernel $K \in S$ is *irregular* (i.e., it is not regular), then the processing order of at least one pair of jobs $(i, j)$ in the schedule $S$ is reverted in the schedule $\tilde{K}$. In other
words, there occurs an anticipating job \( j \), a job from kernel \( K \) which ordinal number in the schedule \( \bar{K} \) is less, compared to that in the schedule \( S \). Since job \( j \) is rescheduled before job \( i \) in schedule \( \bar{K} \), \( r_j \leq r_i \) (and \( r_i > r(K) \)), and also \( d_j > d_i \) as ED-heuristic included \( i \) before \( j \) in the schedule \( S \).

It follows that a former kernel job \( j \) becomes the anticipated delaying emerging job for a newly arisen kernel in the schedule \( \bar{K} \) in case \( K \) is irregular. The set of irregular kernels is similarly determined in the schedule \( \bar{K} \), and a similar construction is carried out for each of them. This results in the omission of the newly arisen anticipated delaying emerging jobs. A recursive \( O(|K|^2 \log |K|) \) time procedure fully decomposes (collapses) the kernel \( K \) into a sequence of so-called substructure components of that kernel with no irregular kernel. Note that the union of the set of jobs from these components is the set of jobs from the kernel \( K \) minus the set of all the omitted anticipated delaying emerging jobs. We refer the reader to Section 4 of [21] for a detailed description of the decomposition procedure and examples illustrating it.

Property 3 Suppose \( K \) is a kernel in a schedule \( S \). Then by increasing the processing time of the delaying emerging job \( e \) of that kernel by \( \tau \) time units, \( 0 \leq \tau \leq \Delta(K) \), the maximum job lateness of a job in kernel \( K \) will be increased by at most \( \tau \) time units. If \( K \) is a regular kernel, then by decreasing the processing time of job \( e \) by \( \tau \) time units, either the maximum job lateness of a job in kernel \( K \) will decrease by \( \tau \) time units or/and it will become a valid lower bound on the optimum job lateness.

Proof. First, we note that since a kernel may contain no emerging job, the jobs of any kernel \( K \) are processed in an ED-order in schedule \( S \), except that these jobs may be processed in a weak ED-order if there is a job \( j \) with \( d_j = d_o \) in the kernel \( K \), where \( o \) is the overflow job in that kernel: since \( j \) is a non-emerging job, it will not “split” the kernel \( K \); it may start the kernel or may be included somewhere in between some more urgent jobs of the kernel and hence the processing order will not be strictly ED. Suppose there exists such a job \( j \) and the processing time of the delaying emerging job \( e \) is increased. Then the time moment when job \( j \) will be considered is respectively increased. As a result, a more urgent job of the kernel \( K \), which was not yet released by the time moment \( s_j(S) \) may get released and hence be included by ED-heuristic ahead job \( j \). Applying the same reasoning to all jobs \( j \) with \( d_j = d_o \), we obtain that the last scheduled job of the kernel \( K \) with the due date \( d_o \) will be the overflow job in the modified schedule and will start \( \tau \) time units later than \( s_o(S) \). Similar reasoning holds for the first claim in case there exists no job \( j \) with \( d_j = d_o \), since an ED processing order of the jobs of the kernel \( K \) will be kept in the modified schedule. This shows the first claim in the property.

For the second claim, assume that the processing time of job \( e \) is decreased by \( \Delta \) time units; i.e., the first job of the kernel \( K \) starts at its release time. Note that in the resultant complete ED-schedule \( S' \), all jobs of the kernel \( K \) will be scheduled as they were scheduled in the partial schedule \( \bar{K} \). Furthermore, since the kernel \( K \) is regular, schedules \( S, \bar{K} \) and \( S' \) possess the same overflow job \( o \). Then the second claim immediately follows if there arises no gap in between the jobs of the kernel \( K \) in the schedule \( \bar{K} (S') \). Suppose there arises a gap, i.e., the kernel \( K \) decomposes into substructure components. Then there is a single kernel \( K' \) in schedule \( \bar{K} \) that
belongs to its last substructure component (see Lemma 4 in [21]). Since kernel $K$ is regular, there may exist no anticipated job and hence no emerging job for the kernel $K'$. Then the lateness of job $o$ is a lower bound on the minimum jobs lateness (see Lemma 3 in [21]). Now for any $\tau < \Delta$ the same reasoning can obviously be applied, which completes the proof.

Kernel regularization procedure. Now we describe a procedure that regularizes all irregular kernels an ED-schedule $S$. First, it carries out the decomposition procedure for every irregular kernel $K \in S$. Let $D$ be the set of the delaying emerging jobs omitted during the decomposition of all irregular kernels, and let $K^*$ be the partial schedule, formed by the substructure components in the full decomposition of the kernel $K$. Note that schedule $K^*$ initiates at time $r(K)$, and that $\cup_{K \in S} K^* \cap D = \emptyset$ and $\cup_{K \in S} K^* \cup D = J$. We compose a complete schedule $(S)^*$ from the schedule $S$ in two passes. Pass 1 first copies the schedule $S$, except its parts containing irregular kernels and the corresponding delaying emerging job, into an auxiliary partial schedule. The latter auxiliary schedule is completed by the partial schedules $K^*$ for each irregular kernel $K \in S$. Note that the resultant partial schedule of pass 1 is feasible, i.e., there will occur no overlapping with the jobs from the initial auxiliary partial schedule in that schedule. Furthermore, there will arise a gap before every of these irregular kernels in that partial schedule. This schedule is augmented with the omitted jobs from the set $D$ at pass 2. These jobs are included by the ED-heuristic, as follows. Whenever there is a yet unscheduled already released job from the set $D$, a most urgent one is included at the beginning of the earliest available gap and the following jobs are correspondingly shifted to the right (in their actual processing order).

**Theorem 1** Every kernel possessing the delaying emerging job in the schedule $(S)^*$ is regular, and this schedule can be constructed in

$$\sum_{K \in S} O(|K|^2 \log |K|) + O(n \log n)$$

time.

Proof. As to the time complexity, the decomposition of each irregular kernel $K$ takes $O(|K|^2 \log |K|)$ time (see Theorem 1 in [21]), and the insertion of the jobs from the set $D$ will take an additional time of $O(n \log n)$. As to the first claim in the theorem, observe that, all kernels in the partial schedule of pass 1 are regular. Hence it will suffice to show that the insertion of no delaying emerging job $e \in D$ will cause the rise of an irregular kernel. Suppose thus, once included, job $e$ yielded a forced right-shift for the subsequently scheduled jobs, that, in turn, provoked the rise of a new kernel $K'$ in the schedule $(S)^*$. We need to show that the kernel $K'$ is regular. By the way of contradiction, suppose it is irregular, and let $j$ be the first anticipated job of that kernel. Let $i$ be a job preceding a job $j$ in the schedule $S$, such that job $j$ was included ahead job $i$ in the schedule $(S)^*$. We now observe that, because of the right-shift caused by job $e$, job $j$ cannot start in the schedule $(S)^*$ before the time moment $s_i(S)$. Then both jobs should have been ready by the time moment, when job $j$ was included in the schedule $(S)^*$. Then the ED-heuristic could not include job $j$ before job $i$ since $d_j > d_i$, a contradiction. Thus, there may exist no anticipated job for any newly arisen kernel and hence it is regular.
4 The algorithms

4.1 The first polynomial-time algorithm

By Property 1, if there is a kernel in the schedule \((\sigma)^*\) with no delaying emerging job, then it is optimal. Hence, assume from here on that every regular kernel in schedule \((\sigma)^*\) possesses the delaying emerging job. Then by reducing the processing time of the delaying emerging job of each regular kernel of this schedule, the current maximum job lateness can be reduced: Algorithm 1 compresses the delaying emerging job of each kernel \(K \in (\sigma)^*\) by the amount \(\Delta_{\min}(\sigma)\) and shifts the jobs succeeding each compressed delaying emerging job to the left correspondingly. We denote the resultant schedule by \(\sigma_{-\Delta_{\min}}\) (for notational simplicity we omit argument \(\sigma\) in \(\Delta_{\min}\)). Note that in schedule \(\sigma_{-\Delta_{\min}}\), the former overflow job of every kernel from schedule \((\sigma)^*\) has the same lateness.

**Theorem 2** Algorithm 1 constructs the schedule \(\sigma_{-\Delta_{\min}}\) in time

\[
\sum_{K \in (\sigma)^*} O(|K|^2 \log |K|) + O(n \log n),
\]

and it is an optimal feasible solution if

(i) \[
\sum_{e \in E((\sigma)^*)} c_e \Delta_{\min}((\sigma)^*) \leq U,
\]

and

(ii) it contains the same set of kernels as the schedule \((\sigma)^*\).

Proof. As to the construction cost, the construction of the original schedule \(\sigma\) takes \(O(n \log n)\) time, and the regularization of each kernel in that schedule and the creation of the schedule \((\sigma)^*\) has an additional cost of

\[
\sum_{K \in (\sigma)^*} O(|K|^2 \log |K|) + O(n \log n)
\]

(Theorem 1). The required parameters of each of these regular kernels from the schedule \((\sigma)^*\) can be obtained in \(O(n)\) time, and the left-shift of the corresponding schedule portions after the compression of each delaying emerging job from the set \(E((\sigma)^*)\) can also be accomplished in \(O(n)\) time since the processing order of the jobs need not be changed (see again Property 3).

As to the optimality, suppose the feasibility condition (i) is satisfied. Since the condition (ii) is also satisfied, there may exist no anticipated job for a kernel in schedule \(\sigma_{-\Delta_{\min}}\) and hence the processing order of the jobs of every kernel is optimal (see the proof of Property 3). Furthermore, let \(K\) be a kernel in schedule \(\sigma_{-\Delta_{\min}}\) with

\[
\Delta(K) = \Delta_{\min}(\sigma_{-\Delta_{\min}}).
\]
The first job of every kernel $K$ with this property starts at its release time in the schedule $\sigma_{-\Delta_{\text{min}}}$. Then the maximum job lateness is minimized in that schedule since the processing order of the jobs of every kernel agrees with an optimal sequence.

4.2 The pseudo-polynomial-time algorithm

It remains to study the case when the schedule $\sigma_{-\Delta_{\text{min}}}$ does not satisfy the conditions from Theorem 2. Assume that a new kernel occurs in the schedule $\sigma_{-\Delta_{\text{min}}}$ (see condition (ii) in Theorem 2). Algorithm 2 regularizes every such kernel and then compresses the processing time of the corresponding delaying emerging jobs by just one time unit. It repeats the same operations for each newly created schedule, as long as this schedule remains feasible (i.e., it satisfies condition (I)):

Algorithm 2.

Step 0.

Create the initial ED-schedule $\sigma$; $S := (\sigma)^*$;

Step 1.

IF all kernels in the schedule $S$ possess the delaying emerging job

THEN create a modified auxiliary ED-schedule $S'$ from the schedule $S$ by compressing the processing time of the delaying emerging job of each kernel in schedule $S$ by one time unit and shifting the following jobs correspondingly to the left;

IF in the schedule $S'$ the feasibility condition (I) is not violated

THEN $S := (S')^*$ \{regularize schedule $S'$\}; repeat Step 1

ELSE return schedule $S$ \{S is a feasible schedule\}

ELSE return schedule $S$.

We will refer to a schedule $S$ as well-balanced if by dis-compressing the delaying emerging job of all kernels of that schedule by $\tau$ time units, the maximum job lateness will increase by the same amount. Note that the schedule $S$ created by Algorithm 2 has a nice property that the lateness of the overflow jobs of all kernels in it is the same. As a consequence, this schedule is well-balanced (see Property 3):
**Property 4** The schedule \( S \) of every iteration \( h \) in Algorithm 2 is well-balanced. In particular, the maximum job lateness in that schedule is simultaneously attained in all kernels from the set \( K^h \).

Let us also observe that every kernel of iteration \( h - 1 \) will remain regular at iteration \( h \) and the following iterations (again Property 3 see also Theorem 1). At the same time, since the lateness of the overflow jobs of iteration \( h - 1 \) are decreased, a new kernel may arise at iteration \( h \) in the schedule \( S' \). If a newly arisen kernel in schedule \( S' \) of iteration \( h \) is irregular, it is regularized at iteration \( h \) in the newly created schedule \( S = (S')^* \). We will use \( K^h \) for the set of all the (regularized) kernels by iteration \( h \); as we just observed, \( K^h \subseteq K^{h-1} \) holds for any \( h > 1 \) in Algorithm 2.

**Property 5** The maximum job lateness in the schedule \( S \) of each iteration \( h \) in Algorithm 2 (except possibly the last iteration) is exactly one less than that of iteration \( h - 1 \).

Proof. By the construction of step 1, at every iteration \( h > 1 \), the maximum job lateness in the schedule \( S' \) is one less than that in the regularized schedule \( S \) of iteration \( h - 1 \). Then by Property 3, it suffices to show that the maximum job lateness in the regularized schedule \( S \) of iteration \( h \) is the same as that in the schedule \( S' \) of the same iteration. There are two possible cases. If there arises no new kernel in the schedule \( S' \) of iteration \( h \), then this is obviously true. Otherwise, note that the overflow job of a newly arisen kernel will have the lateness equal to the maximum job lateness in the schedule of iteration \( h - 1 \) minus one, and the maximum job lateness in the regularized schedule \( S \) must be the same as that in the schedule \( S' \).

**Theorem 3** Algorithm 2 creates a well-balanced feasible solution \( S \) for problem 1\(|r_j|\text{compressible}(U)|L_{\text{max}}\) in less than \( p_{\text{max}} \) iterations. If it does not create an infeasible solution \( S' \) (the first ELSE condition is never entered at Step 1), then \( S \) is also optimal. Otherwise (the first ELSE condition is executed at Step 1 before the algorithm halts), the solution \( S \) is optimal for the generic problem 1\(|r_j|L_{\text{max}}\) with the compressed set of job processing times defined by the compression vector \((x_1(S), \ldots, x_n(S))\).

Proof. By Property 5 the maximum job lateness in the schedule \( S \) of each (non-terminal) iteration \( h \) in Algorithm 2 is one less than that of iteration \( h - 1 \). Then the number of iterations is bounded by \( p_{\text{max}} \) due to Property 2. The schedule \( S \) is well-balanced by Property 4 and it satisfies feasibility condition 1 by the construction of step 1.

Suppose now at step 1 the first “ELSE” statement is never entered. Then the feasibility condition 1 was never violated and the algorithm has stopped by entering the second “ELSE” statement; i.e., there is a kernel in schedule \( S \) possessing no delaying emerging job and this schedule \( S \) is optimal by Property 1.

Suppose now that the algorithm halts by entering the first “ELSE” statement, i.e., the feasibility condition 1 does not hold in the schedule \( S' \) and the schedule \( S \), regularized at the previous iteration, is returned. We have to show that \( S \) is an optimal feasible schedule. Similarly as in the
previous case, $S$ is feasible by the construction of step 1. Furthermore, since all kernels in that schedule are regular, by changing the processing order of the jobs of any kernel, the lateness of the corresponding overflow job cannot be decreased. Let us now consider the remaining non-kernel jobs of the schedule. Without loss of generality, assume that schedule $S$ consists of a single block, since our reasoning can be applied to each individual block independently. Schedule $S$ consists of kernels and alternative sequences of jobs scheduled by ED-heuristic before the first kernel, in between two neighboring kernels, and after the last kernel. Since there is no gap within any of these sequences, changing the processing order of the jobs from any of these sequences will either leave the current lateness unaltered or will increase it (due to a possible gap that may occur because of the order change). Interchanging the jobs from different sequences may again cause new gaps. As a result, if the lateness of some overflow job from the schedule $S$ decreases, that of some other overflow job will increase, see Properties 3, 4 and 5. It follows that schedule $S$ minimizes maximum job lateness for the problem instance of the generic problem $1|r_j|L_{\text{max}}$ with the compressed set of job processing times determined by Algorithm 2.

### 4.3 The second polynomial-time algorithm

In the previous sub-section, we constructed a feasible schedule which is optimal for the instance of the generic problem $1|r_j|L_{\text{max}}$ with the compressed set of job processing times determined by Algorithm 2. Since the compression at each iteration is carried out by one unit of time, the total number of iterations depends on the maximum job processing time and hence the algorithm runs in pseudo-polynomial time. At first glance, $O(p_{\text{max}})$ seems to be a natural bound on the total number of iterations since the lateness of the overflow job in a newly arisen kernel may differ just by one time unit from that of an earlier arisen kernel. Nevertheless, maintaining schedules with regular kernels is helpful and will permit us to obtain a schedule with the same property as the one delivered by Algorithm 2 in $O(n)$ iterations.

This section’s polynomial-time Algorithm 3 combines some features of Algorithms 1 and 2. At stage 1, it outputs a well-balanced schedule, which is optimal for the instance of the generic problem $1|r_j|L_{\text{max}}$ with the obtained compressed set of job processing times. If this solution is not feasible for problem $1|r_j,\text{compressible}(U)|L_{\text{max}}$, then at stage 2, the solution of stage 1 is converted to a well-balanced feasible schedule. Like Algorithm 1 (and unlike Algorithm 2), Algorithm 3 makes jumps of $\Delta_{\text{min}}(\sigma_{h-1})$ while compressing the processing times of the delaying emerging jobs at iteration $h$ at stage 1, which guarantees polynomial-time performance. As Algorithm 2, Algorithm 3 maintains the schedule $\sigma_h$ of each iteration $h$ regular and well-balanced. Since, unlike Algorithm 2, the “compression jumps” in Algorithm 3 are carried out by more than one units time, a special care is to be taken to keep the schedule of every iteration well-balanced (note that, unlike Algorithm 2, the maximum job lateness in a non-regularized and the corresponding regularized schedules will not necessarily be equal).
4.3.1 Stage 1

At the initial iteration 1, stage 1 invokes Algorithm 1, and the schedule \( \sigma_1 = \sigma - \Delta_{\text{min}} \) is obtained. For convenience, let \( \sigma_0 = \sigma \), and let

\[
\Lambda^1 = L(\sigma_0) - \Delta_{\text{min}}(\sigma_0) = L_0(\sigma_1)
\]

be the lateness of an overflow job \( o \in \sigma_0 \) in the schedule \( \sigma_1 \) (recall that schedule \( \sigma - \Delta_{\text{min}} \) is obtained from a regularized schedule \( (\sigma)^* \) by compressing the processing time of each delaying emerging job of that schedule by the amount \( \Delta_{\text{min}}(\sigma) \), so that the lateness of the overflow jobs of the schedule \( \sigma = \sigma_0 \) is decreased by \( \Delta_{\text{min}}(\sigma) \) time units in the schedule \( \sigma_1 \), see Property 3).

If \( \Lambda^1 = L(\sigma_1) \), i.e., in the schedule \( \sigma_1 \) no new kernel arises (condition (ii) in Theorem 2), stage 1 outputs this well-balanced (but not necessarily feasible) schedule. Note that \( \Lambda^1 > L(\sigma_1) \) is not possible. If now \( \Lambda^1 < L(\sigma_1) \) (at iteration 1, a new overflow job arises in the schedule \( \sigma_1 \)), stage 1 proceeds with the next iteration 2. Again, first, schedule \( \sigma_1 \) is regularized, and then the processing time of every delaying emerging job in the regularized schedule \( (\sigma_1)^* \) is compressed by the magnitude \( \Delta_{\text{min}}((\sigma_1)^*) \) and the jobs succeeding each compressed delaying emerging job are shifted to the left. This results in an auxiliary schedule \( \sigma_1 - \Delta_{\text{min}} \), in which each newly arisen kernel \( K \in K^2 \) is delayed by

\[
\Delta(K) - \Delta_{\text{min}}(\sigma_1)
\]

time units; in particular, a kernel \( K \) with

\[
\Delta(K) = \Delta_{\text{min}}(\sigma_1)
\]

starts at time \( r(K) \) without any delay. Below we describe how delaying emerging jobs are iteratively compressed and dis-compressed.

In general, for a given iteration \( h \geq 1 \), let

\[
\Lambda^h = L(\sigma_{h-1}) - \Delta_{\text{min}}(\sigma_{h-1}) = L_0(\sigma_h),
\]

where \( o \) is an overflow job in the schedule \( \sigma_{h-1} \). Stage 1 outputs the current schedule \( \sigma_h \) if \( \Lambda^h = L(\sigma_h) \), i.e., no new kernel in the schedule \( \sigma_h \) arises.

Suppose now \( \Lambda^h < L(\sigma_h) \), i.e., a new kernel in the schedule \( \sigma_h \) arises (again, \( \Lambda^h > L(\sigma_h) \) is not possible). Then stage 1 proceeds with iteration \( h + 1 \) by first creating an auxiliary schedule \( (\sigma_h)^* - \Delta_{\text{min}} \). The following two cases are distinguished: (1)

\[
L((\sigma_h)^* - \Delta_{\text{min}}) > \Lambda^h,
\]

i.e., the lateness of an overflow job in the schedule \( (\sigma_h)^* - \Delta_{\text{min}} \) is more than the maximum job lateness at iterations 1, \ldots, \( h \). And (2)

\[
L((\sigma_h)^* - \Delta_{\text{min}}) < \Lambda^h,
\]

i.e., the lateness of an overflow job in the schedule \( (\sigma_h)^* - \Delta_{\text{min}} \) is less than the maximum job lateness of iterations 1, \ldots, \( h \).
Property 6 Suppose

\[ L((\sigma_h)^* - \Delta_{\text{min}}) > \Lambda^h. \]

Then

\[ L^{\text{OPT}} \geq L((\sigma_h)^* - \Delta_{\text{min}}). \]

Hence, the delaying emerging job of every kernel in \( K_h \) can be dis-compressed by

\[ L((\sigma_h)^* - \Delta_{\text{min}}) - \Lambda^h \]

time units in the schedule \((\sigma_h)^* - \Delta_{\text{min}}\).

Proof. Note that the schedule \( \sigma_h \) is regularized at iteration \( h + 1 \) and hence all kernels in \( K_{h+1} \) are regular. In particular, there is a regular kernel \( K \) with \( \Delta(K) = \Delta_{\text{min}}(\sigma_h) \) in the schedule \((\sigma_h)^* - \Delta_{\text{min}}\). This kernel possesses no delaying emerging job and hence

\[ L^{\text{OPT}} \geq L_o((\sigma_h)^* - \Delta_{\text{min}}) = L((\sigma_h)^* - \Delta_{\text{min}}), \]

where \( o \) is the overflow job in the kernel \( K \). The second claim in the property now obviously follows.

Property 7 \( L^{\text{OPT}} \geq \Lambda^h \) holds. Hence, if

\[ L((\sigma_h)^* - \Delta_{\text{min}}) < \Lambda^h, \]

then the delaying emerging job of every (new) kernel in \( K^{h+1} \setminus K^h \) can be dis-compressed by

\[ \Lambda^h - L((\sigma_h)^* - \Delta_{\text{min}}) \]

time units in the schedule \((\sigma_h)^* - \Delta_{\text{min}}\).

Proof. Similar to that of Property 6.

Stage 1 dis-compresses delaying emerging jobs according to Properties 6 and 7, shifting to the right the jobs succeeding each dis-compressed delaying emerging job correspondingly. This results in the schedule \( \sigma_{h+1} \). In case (1) \( L(\sigma_{h+1}) = L_o((\sigma_h)^* - \Delta_{\text{min}}) \), as indicated earlier, and in case (2) \( L(\sigma_{h+1}) = \Lambda^h \) (the current maximum job lateness is kept unchanged). These operations are carried out as long as in schedule \( \sigma_{h+1} \) a new kernel/overflow job arises. Otherwise, stage 1 outputs a well-balanced schedule \( \sigma_{h+1} \).

Algorithm 3, Stage 1.

Step 0.
Create the initial ED-schedule $\sigma$ and regularize it; Create the schedule $\sigma_1 = \sigma - \Delta_{\text{min}}$; $h := 1$;

**Step 1.**

IF $\Lambda^h = L(\sigma_h)$ \{no new kernel arises in the schedule $\sigma_h$\}

THEN RETURN schedule $\sigma_h$

ELSE \{$\Lambda^h < L(\sigma_h)$\}

IF $L((\sigma_h)^* - \Delta_{\text{min}}) > \Lambda^h$ \{the current maximum job lateness is surpassed\}

THEN determine the schedule $\sigma_{h+1}$ by dis-compressing in the schedule $(\sigma_h)^* - \Delta_{\text{min}}$ the processing time of the delaying emerging job of every kernel in $K_h$ by $L((\sigma_h)^* - \Delta_{\text{min}}) - \Lambda^h$ time units

ELSE determine the schedule $\sigma_{h+1}$ by dis-compressing in the schedule $(\sigma_h)^* - \Delta_{\text{min}}$ the processing time of the delaying emerging job of every (newly arisen) kernel in $K^{h+1} \setminus K^h$ by $\Lambda^h - L((\sigma_h)^* - \Delta_{\text{min}})$ time units;

$h := h + 1$; REPEAT Step 1.

**Theorem 4** Stage 1 delivers a well-balanced schedule $\sigma^h$ possessing a kernel without the delaying emerging job in $O(n)$ iterations in $O(n^2 \log n)$ time. The schedule $\sigma^h$ is optimal for the corresponding instance of the generic problem $1|r_j|L_{\text{max}}$ with the compressed set of job processing times defined by the compression vector $(x_1(\sigma), \ldots, x_n(\sigma))$, and it is also optimal for the original instance of problem $1|r_j, \text{compressible}(U)|L_{\text{max}}$, if it does not violate the feasibility condition (7).

Proof. Initially at step 0, the cost of invoking Algorithm 1 is as in Theorem 2. Iteratively, in iteration $h > 1$ at step 1, the regularization of each kernel $K \in K^{h-1}$ from the schedule $\sigma_{h-1}$ has the cost of $O(|K|^2 \log |K|)$ (Theorem 1), and similarly as in Algorithm 1, the required parameters of each of these regular kernels can be obtained in $O(n)$ time. After each compression (dis-compression, respectively) of the delaying emerging jobs, the left-shift (right-shift, respectively) of the jobs of the corresponding kernels needs $O(n)$ time.

Assume, for now, that the total number of different kernels that may occur during the execution of stage 1 is bounded from above by $O(n)$. Then the above operations at step 1 have to be repeated a number of times bounded by $O(n)$ and the bound $O(n^2 \log n)$ follows. Below we show that in less than $n$ iterations the algorithm arrives at an iteration $h$ such that there occurs no new overflow.
job at that iteration (equivalently, $K^h \setminus K^{h-1} = \emptyset$, i.e., there occurs no new kernel, condition (ii) from Theorem 2).

Thus we wish to show that in no more than $n$ iterations, stage 1 arrives at an iteration $h$ with $K^h \setminus K^{h-1} = \emptyset$, as we claimed above. Consider an arbitrary iteration $g$ with a newly arisen kernel $K^g \in K^g$, $K^g \notin K^{g-1}$. There may clearly occur no more than $n$ such iterations if $K^g \cap K^f = \emptyset$, for any $K^f \in K^f$, $f = 1, \ldots, g - 1$. If now $K^g \cap K^f \neq \emptyset$, then the two kernels $K^f$ and $K^g$ must be identical. Indeed, let, first, $j$ be any job scheduled after the kernel $K^f$. Note that job $j$ could not have been shifted to the right by more than any job of kernel $K^f$. Then job $j$ cannot become a part of kernel $K^g$ since it did not form a part of kernel $K^f$. Suppose that $j$ is a job scheduled before the delaying emerging job $e$ of kernel $K^f$ in the schedule $\sigma_f$. If job $j$ belongs to kernel $K^g$ then job $e$ must also belong to that kernel, but no kernel may contain a (compressed) delaying emerging job. The kernel $K^g$ cannot form a sub-sequence/subset of the kernel $K^f$ since both kernels are regular. It follows that the two kernels coincide (as sequences of jobs), and they have the same overflow job $o$. But the lateness of job $o$ is optimal since this kernel is regular and its first job starts at its release time. We showed that $g = h$ and the schedule $\sigma_h$ is optimal for the instance of problem 1|rj|Lmax with the compressed set of job processing times. Note also that the schedule $\sigma_h$ is well-balanced since, by the construction, the lateness of the overflow jobs of all the kernels arisen at iterations 1, . . . , $h$ is the same. The last claim in the theorem obviously follows.

4.3.2 Stage 2

Stage 2 transforms the solution $\hat{\sigma} = \sigma^h$ of stage 1 into a feasible solution $\bar{\sigma}$ to problem 1|rj, compressible(U)|Lmax by dis-compressing the processing time of the compressed delaying emerging jobs. Observe that

$$\sum_{e \in E} x_j(\hat{\sigma}) c_j - U$$

is the excess of the total compression cost in the solution $\hat{\sigma}$. Let $k = |K^h|$ (recall that $K^h$ is the set of all (regular) kernels formed at stage 1), and let $E$ be the set of the corresponding delaying emerging jobs. By increasing the processing time of the delaying emerging job $e \in E$ of each kernel from the set $K^h$ by

$$\xi = \lceil (\sum_{e \in E} x_j(\hat{\sigma}) c_j - U)/k \rceil$$

time units and shifting the jobs of that kernel correspondingly to the right, stage 2 obtains the schedule $\bar{\sigma}$ in $O(n)$ time, in which the maximum job lateness is increased by $\xi$ time units compared to schedule $\hat{\sigma}$.

**Theorem 5** The schedule $\bar{\sigma}$ delivered by Algorithm 3 in $O(n^2 \log n)$ time is a feasible schedule for the problem 1|rj, compressible(U)|Lmax and satisfies the following equality:

$$L_{\max}(\bar{\sigma}) = L_{\max}(\hat{\sigma}) + \xi.$$  \hfill (2)
Furthermore, the schedule $\bar{\sigma}$ is well-balanced and optimal for the corresponding instance of the generic problem $1|r_j|L_{\text{max}}$ with the compressed set of job processing times in the schedule $\bar{\sigma}$.

Proof. The time complexity follows from Theorem 4 since stage 2 clearly runs in $O(n)$ time. The feasibility of the schedule $\bar{\sigma}$ follows from the definition of the parameter $\xi$. Furthermore, since every kernel $K \in \hat{\sigma}$ is regular, by increasing the processing time of the corresponding delaying emerging job by $\xi$ time units, the lateness of the corresponding overflow job will increase by the same amount (Property 3). Then Equation (2) holds since the set of kernels in the schedule $\bar{\sigma}$ is the same as that in the schedule $\hat{\sigma}$. Moreover, the schedule $\bar{\sigma}$ is well-balanced since the schedule $\hat{\sigma}$ is well-balanced (Theorem 4). Then schedule $\bar{\sigma}$ is optimal for the instance of the problem $1|r_j|L_{\text{max}}$ with the compressed set of processing times in that schedule, similarly as in Theorems 3 and 4.

5 Conclusion

We showed that by shortening the processing times of some specially determined (emerging) jobs to “requisite amounts” of time units, the maximum job lateness can be effectively reduced. Algorithm 3, in practice, provides the manufacturer with the information about the number of additional resources that are required to solve the problem $1|r_j, \text{compressed}(U)|L_{\text{max}}$ optimally. Such an approach is particularly useful in situations when the manufacturer is willing to allocate additional resources to shorten the processing of some late pending (emerging) jobs and provide in this way a non-delay completion of recently arrived more urgent jobs. In the case when this amount of additional resources is not available, stage 2 returns a well-balanced schedule that is optimal for the corresponding instance of problem $1|r_j|L_{\text{max}}$. Note that, a well-balanced schedule has an important optimal tightness property that, by dis-compressing the processing of any compressed delaying emerging job, the maximum job lateness will be increased by the same amount. More generally, in a well-balanced schedule, by increasing the processing time of any kernel or non-kernel job preceding some kernel, the maximum job lateness will be increased by the same amount. Finally, the pseudo-polynomial time algorithm of Section 4.2 may be attractive for applications where $p_{\text{max}}$ is a priori known non-large magnitude. For further research, it will be interesting to see how the proposed approach can be used for parallel machine environments.

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