Why the Mickelsson-Faddeev algebra lacks unitary representations

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Abstract

A simple plausibility argument is given.

Let $g$ be a finite-dimensional Lie algebra with generators $J^a$ and structure constants $f_{ab}^c$. The brackets are given by $[J^a, J^b] = f_{ab}^c J^c$. Denote the symmetric Killing metric (proportional to the quadratic Casimir operator) by $\delta^{ab} = \text{tr} J^a J^b$, and let the totally symmetric third Casimir operator be $d_{abc} = \text{tr} \{J^a, J^b\} J^c$.

The current algebra map $(M_3, g)$ is the algebra of maps from a 3-dimensional manifold $M_3$ to $g$. In local coordinates, the generators are $J^a = \int d^3x X^a(x) J^a$, where $X^a(x)$ are functions on $\mathbb{R}^3$. Define $[X, Y] = f_{ab}^c X^a Y^b J^c$. This algebra admits an abelian extension known as the Mickelsson-Faddeev (MF) algebra \cite{2, 6, 7}.

$$[J^a, J^b] = f_{ab}^c J^c,$$
$$[J^a, A_{\mu}^b] = f_{abc} A_{\mu}^c + \delta^a_{\mu} J^b,$$
$$[A_{\mu}^a, A_{\nu}^b] = 0,$$

where $\epsilon^{\mu\nu\rho}$ is the totally anti-symmetric epsilon tensor in three dimensions. If we specialize to the 3-torus $\mathbb{T}^3$, we can expand all fields in a Fourier basis. The algebra map($\mathbb{T}^3, g$) then takes the form

$$[J^a(m), J^b(n)] = f_{ab}^c J^c(m + n) + d_{abc} \epsilon^{\mu\nu\rho} m^\mu n^\nu A_{\rho}(m + n),$$
$$[J^a(m), A_{\nu}^b] = -f_{abc} A_{\nu}^c(m + n) + \delta^a_{\nu} \epsilon_0^\nu \delta(m + n),$$
$$[A_{\mu}^a(m), A_{\nu}^b] = 0.$$
Here \( \mathbf{m} = (m_\mu) \in \mathbb{Z}^3 \) is a momentum labelling the Fourier modes, and \( J^a(m) \) and \( A_{a\mu}(m) \) are the Fourier components of the algebra generators and the gauge connection, respectively.

No physically relevant representations of the MF algebra are known, and indeed a kind of no-go theorem was given by Pickrell long ago [11]: the algebra (1) has no faithful, unitary representations on a separable Hilbert space. The purpose of the present note is to give a very simple argument why this must be true. The idea is to consider the restrictions of the torus algebra (2) to various loop algebras. The restriction of a unitary representation to any subalgebra is obviously still unitary. However, it is well known that the only unitary representation of a proper loop algebra is the trivial representation. Since all restrictions of a unitary torus algebra representation to its loop subalgebras are trivial, the torus algebra representation must itself be trivial.

Let \( \mathbf{e} = (e_\mu) \) be a vector in \( \mathbb{Z}^3 \). A loop subalgebra is generated by elements of the form

\[
J^a_m = J^a(me).
\]

(3)

It is straightforward to verify that

\[
[J^a_m, J^b_n] = f^{abc} e^c J^{m+n}_b,
\]

(4)
i.e. the restriction of the MF extension to this subalgebra vanishes. The proof only uses anti-symmetry of the epsilon symbol, \( \epsilon^{\mu\nu\rho} e_\mu e_\nu \equiv 0 \). The algebra (4) is recognized as a proper loop algebra, i.e. an affine Kac-Moody algebra with zero central extension. It is well known that all non-trivial lowest-energy unitary representations of affine algebras have a positive central charge [3]. Hence the restriction of a unitary MF representation to this subalgebra is trivial. Since this must be true for every choice of vector \( \mathbf{e} \), we conclude that the unitary representation of the MF algebra must itself be trivial.

The argument has one loophole: the loop algebra representation is supposed to be of lowest-energy type. The algebra (4) certainly has unitary representations if we relax this condition, e.g. the direct sum of one lowest- and one higher-energy unitary representation with opposite central charges, or classical representations on fields valued in \( \mathfrak{g} \) modules. However, those are not the kind of representations that we expect in quantum systems, where there should be a Hamiltonian which is bounded from below. This Hamiltonian induces a grading by energy of every loop subalgebra. Hence
the restriction to every such subalgebra should be of lowest-energy type, and some should be non-trivial.

Mickelsson has studied another type of representation, where the algebra has a natural and fiberwise unitary action in the bundle of fermionic Fock spaces parametrized by external vector potentials \[8\]. In other words, these representations describe quantum chiral fermions living over a classical background gauge field. Whereas this construction is mathematically interesting, it can not fundamentally describe physics, where the gauge fields must be quantized as well.

This result implies that conventional gauge anomalies proportional to the third Casimir operator are inconsistent. Namely, the gauge generators must be represented by unitary operators. However, we have just seen that this means that the representation is trivial. Since the MF extension vanishes in the trivial representation, the anomaly must indeed be zero. This result is of course consistent with physical intuition \[11, 10\].

The current algebra map \((\mathbb{T}^N, g)\) also admits another extension, first found by Kassel \[4\]. It is usually called the central extension, although the extension does not commute with diffeomorphisms. In a Fourier basis, this extension is defined by the brackets

\[
\begin{align*}
[J^a(m), J^b(n)] &= f^{abc} J^c(m + n) + k \delta^{ab} m \rho S^\rho(m + n), \\
[J^a(m), S^\mu(n)] &= [S^\mu(m), S^\nu(n)] = 0, \\
m^\mu S^\mu(m) &\equiv 0.
\end{align*}
\] (5)

The restriction to the subalgebra generated by \(3\) reads

\[
\begin{align*}
[J^a_m, J^b_n] &= f^{abc} J^c_{m+n} + k \delta^{ab} m \delta_{m+n},
\end{align*}
\] (6)

where \(S_m \equiv e^\mu S^\mu(me) \propto \delta_m\), because the condition \(mS_m = 0\) implies that \(S_m\) is proportional to the Kronecker delta \(\delta_m\). Equation (6) is recognized as an affine Kac-Moody algebra, including the central term. Since the Kac-Moody algebra has unitary representations for positive central charge, the argument above does not apply to the algebra \(5\). Nothing prevents it from having unitary, lowest-energy representations, and hence such gauge anomalies may occur in physics. In fact, lowest-energy representations of the algebra \(5\) have been known since 1990 \[9\], and it was recently shown that this kind of gauge anomaly does arise when one quantizes the observer’s trajectory together with the fields \(5\).

To conclude, we observed that a lowest-energy representation of a torus algebra can only be unitary if all restrictions to loop algebras are so, and that
unitarity of loop algebra representations requires an extension proportional to the quadratic Casimir. This rules out the MF extension, because it is proportional to the third Casimir. The Kassel extension can have unitary representations. The result were formulated on the three-dimensional torus for convenience, but this not a critical assumption. On a general manifold, we can consider the restrictions to elementary loops instead; the number of such loops is given by the first Betti number.

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