Cascades and Dissipative Anomalies in Relativistic Fluid Turbulence

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We develop first-principles theory of relativistic fluid turbulence at high Reynolds and Péclet numbers. We follow an exact approach pioneered by Onsager, which we explain as a non-perturbative application of the principle of renormalization-group invariance. We obtain results very similar to those for non-relativistic turbulence, with hydrodynamic fields in the inertial-range described as distributional or “coarse-grained” solutions of the relativistic Euler equations. These solutions do not, however, satisfy the naive conservation-laws of smooth Euler solutions but are afflicted with dissipative anomalies in the balance equations of internal energy and entropy. The anomalies are shown to be possible by exactly two mechanisms, local cascade and pressure-work defect. We derive “4/5th-law”-type expressions for the anomalies, which allow us to characterize the singularities (structure-function scaling exponents) required for their non-vanishing. We also investigate the Lorentz covariance of the inertial-range fluxes, which we find is broken by our coarse-graining regularization but which is restored in the limit that the regularization is removed, similar to relativistic lattice quantum field theory. In the formal limit as speed of light goes to infinity, we recover the results of previous non-relativistic theory. In particular, anomalous heat input to relativistic internal energy coincides in that limit with anomalous dissipation of non-relativistic kinetic energy.

I. INTRODUCTION

Relativistic hydrodynamics has a growing range of applications in current physics research, including energetic astrophysical objects such as gamma-ray bursts \textsuperscript{[1]} and pulsars \textsuperscript{[2]}, high-energy physics of the early universe and heavy-ion collisions \textsuperscript{[3]}, condensed matter physics of graphene \textsuperscript{[4,5]} and strange metals \textsuperscript{[6,7]}, and black-hole gravitational physics via the fluid-gravity correspondence in AdS/CFT \textsuperscript{[8–11]}. The ubiquity of relativistic hydrodynamics is natural, given that it represents a universal low-wavenumber description of relativistic quantum field-theories at scales much larger than the mean-free path length. When the global length-scales of such relativistic fluid systems are even larger, as measured by the dimensionless Reynolds-number, then turbulent flow is likely. There is observational evidence for relativistic turbulence in high-energy astrophysical systems, e.g. gamma-ray bursts accelerate relativistic jets to Lorentz factors $\gamma > 100$ and contain internal fluctuations with $\delta \gamma \sim 2$ \textsuperscript{[12]}. Numerical simulations of relativistic fluid models have verified the occurrence of turbulence at high Reynolds numbers \textsuperscript{[13,14]}. Relativistic turbulence is also observed in numerical solutions of conformal hydrodynamic models \textsuperscript{[15,17]} and an analogous phenomenon is seen in their dual AdS black-hole solutions \textsuperscript{[18]}. Despite the importance of relativistic fluid turbulence at high Reynolds-number for many applications, there have been only a handful of theoretical efforts to elucidate the phenomenon \textsuperscript{[10,21]}. Using a point-splitting approach, Fouxon \& Oz \textsuperscript{[19]} derived statistical relations for relativistic turbulence that in the incompressible limit reduce to the famous Kolmogorov “4/5th-law” \textsuperscript{[22,23]}. However, in the relativistic regime their relations have nothing to do with energy of the fluid. This seems to suggest a profound difference between relativistic and non-relativistic turbulence or, even more radically, an essential flaw in our current understanding of non-relativistic turbulence. As concluded by Fouxon \& Oz \textsuperscript{[12]}, “The interpretation of the Kolmogorov relation for the incompressible turbulence in terms of the energy cascade may be misleading.”

We develop here the first-principles theory of relativistic fluid turbulence at high Reynolds and Péclet numbers, which reaches a very different conclusion. We establish the existence of a relativistic energy cascade in the traditional sense and an even more fundamental entropy cascade. The appearance of thermodynamic entropy is not surprising, considering its central role in the theory of dissipative relativistic hydrodynamics \textsuperscript{[8,10,24–26]}. Our analysis follows a pioneering work of Onsager \textsuperscript{[27,28]} on incompressible fluid turbulence, who proposed that turbulent flows at very high Reynolds numbers are described by singular/distributional solutions of the incompressible Euler equations. Onsager derived in 1945 the first example of a conservation-law anomaly, showing by a point-splitting argument that the zero-viscosity limit of Navier-Stokes solutions can dissipate fluid kinetic energy for a critical 1/3 Hölder singularity of the fluid velocity \textsuperscript{[29]}. Polyakov has pointed out the formal analogy of Kolmogorov’s “4/5th-law” and its point-splitting derivation to axial anomalies in quantum gauge field theories \textsuperscript{[30,31]}. However, Onsager’s analysis is deeper than the ensemble theory of Kolmogorov \textsuperscript{[22,32,33] or “K41”}, because it applies to individual flow realizations. It is also formally exact and requires no statistical hypotheses, such as isotropic/homogeneous ensembles or mean-field arguments ignoring space-time intermittency. Onsager’s proposals were not understood at the time and he never published full proofs of his assertions. Thus, the
theory went ignored until Onsager’s 1/3 H"older condition for anomalous energy dissipation was rederived \[34\]. This triggered a stream of work in the mathematical PDE community that has improved upon the analysis, notably \[35, 37\]. More recently, concepts originating in the Nash-Kuiper theorem and Gromov’s $h$-principle have been applied to mathematically construct dissipative Euler solutions of the type conjectured by Onsager \[38, 39\]. This new circle of ideas has led to a proof that Onsager’s 1/3 criticality condition for energy dissipation is sharp \[10\].

Onsager’s theory of dissipative Euler solutions and its application to fluid turbulence is still essentially unknown to the wider physics community, however. This is unfortunate because it is the most comprehensive theoretical framework for high Reynolds turbulence and generally applicable, not only to kinetic energy dissipation in incompressible fluid turbulence, but also to cascades in magnetohydrodynamic turbulence \[41, 42\], to dissipative anomalies of Lagrangian invariants such as circulations \[43\] and magnetic fluxes \[44\], and to cascades in compressible Navier-Stokes turbulence \[45, 46\]. Furthermore, Onsager’s analysis is based on very intuitive physical ideas. As we discussed in our earlier paper on non-relativistic compressible Navier-Stokes turbulence \[47\] [hereafter, paper I] Onsager’s argument is essentially a non-perturbative application of the principle of renormalization group invariance \[10, 51\]. High-Reynolds turbulence is characterized by ultraviolet divergences of gradients of the velocity and other thermodynamic fields, referred to as a “violet catastrophe” by Onsager \[22\]. Regularizing these divergences introduces a new arbitrary length-scale $\ell$ upon which objective physics cannot depend, and exploiting this invariance yields the main conclusions of the theory on fluid singularities, inertial range, local cascades, etc.

Onsager’s unpublished work in 1945 employed a point-splitting approach \[29\], but we explore here a more powerful coarse-graining or “block-spin” regularization \[35, 52\] for relativistic fluid turbulence. Many essential steps were already taken in paper I on non-relativistic compressible turbulence, such as the identification of (neg)entropy as a key invariant and the development of appropriate non-perturbative tools of analysis, such as cumulant expansions for space-time coarse-graining and mathematical distribution theory. Relativistic turbulence brings in some completely new difficulties, however. First, kinetic energy is usually given the central role in the theory of non-relativistic energy cascade, but kinetic energy is an unnatural quantity in relativity theory. We show here that internal energy is the appropriate basis for the theory of relativistic energy cascade. Another distinction of the relativistic theory is that our non-perturbative coarse-graining regularization preserves Galilean symmetry of non-relativistic fluid models but it breaks Lorentz-symmetry. This is reminiscent of the lattice regularization of relativistic quantum field-theories \[53\], which breaks Lorentz symmetry for finite lattice spacing $a$ but recovers it in the continuum limit $a \to 0$. The situation here is similar, as we show that Lorentz symmetry is restored as our regularization parameter $\ell \to 0$, leading to a description by relativistic Euler equations. Further differences exist, such as the unit normalization of relativistic velocity vectors, which leads to new terms in flux/anomaly formulas that do not appear non-relativistically. An important caveat about the present work is that we consider only special-relativistic fluid turbulence in flat Minkowski space-time. General-relativistic (GR) fluids in curved space-times bring in additional technical difficulties. These seem tractable but it makes sense to develop the theory first in Minkowski space-time, as the simplest setting possible. For remarks on full GR, see the conclusion.

In this paper, we shall consider the $D = d + 1$ dimensional Minkowski space-time for any space dimension $d \geq 1$. This generality is motivated not only by the wider perspective it affords but also by fluid-gravity correspondence in AdS/CFT, which holds for general $D$ \[11, 54\]. We adopt signature $- + \cdots +$ of Minkowski metric $g^{\mu\nu}$. We shall follow standard relativistic notations, but we include explicit factors of speed of light $c$, e.g. space-time coordinates $x^\mu = (t^0, \mathbf{x}) = (ct, \mathbf{x})$, velocity vectors $V^\mu = \gamma(1, \mathbf{v}/c)$, etc. rather than use natural units with $c = 1$. This facilitates taking the limit $c \to \infty$ for comparison with the results of paper I.

II. RELATIVISTIC DISSIPATIVE FLUID MODELS

We consider here a relativistic fluid with conserved stress-energy tensor $T^{\mu\nu}$

$$\partial_\nu T^{\mu\nu} = 0$$

(1)

and with one conserved current $J^\mu$

$$\partial_\nu J^\nu = 0.$$  

(2)

The latter may interpreted as a particle number current (e.g. baryon number) and the fluid models that we consider reduce in the limit $c \to \infty$ and at zeroth-order in gradients to the non-relativistic compressible Euler equations. This choice allows us to compare our results here to those derived in paper I for non-relativistic compressible turbulence. However, our analysis carries over straightforwardly to other fluid systems without the additional conserved current $J^\mu$ (e.g. zero chemical potential sectors, conformal fluids) and to multicomponent systems with more than one conserved current (e.g. 2-fluid models of relativistic superfluids).

Even with the restrictions to \[11, 2\], there are many possible fluid models. Unlike the non-relativistic case, where the compressible Navier-Stokes equations have a more canonical status and are employed almost universally in the fluid regime, there are still many dissipative relativistic fluid models competing as descriptions of the same physical system (e.g. see \[55\], section 14 or \[56\].
Ch. 6). We consider a broad class of dissipative relativistic fluid theories, which includes the traditional theories of Eckart and Landau-Lifschitz [58] and the Israel-Stewart theory [24, 25], in which the number current and stress tensor have the general form

\[ J^\mu = nV^\mu + \sigma \tilde{N}^\mu \] (3)

\[ T^{\mu \nu} = p \Delta^{\mu \nu} + \epsilon V^\mu V^\nu + \Pi^{\mu \nu}, \]

\[ \Pi^{\mu \nu} = -\kappa \theta (V^\mu V^\nu) + \zeta \frac{\tilde{Q}^{\mu \nu}}{T} + 2\eta \frac{\tilde{\tau}^{\mu \nu} \tilde{\tau}^{\mu \nu}}{T}. \] (4)

Here \( n \) is number density, \( p = p(\epsilon, n) \) the pressure, and \( \epsilon = u + \rho c^2 \) the total energy density, with \( \rho = n m \) the rest-mass density for particle mass \( m \) and \( u \) the internal energy density. The velocity vector \( V^\mu \), to be specified below, is future time-like and \( V_\mu V^\mu = -1 \). The quantity \( N^\mu = \sigma \tilde{N}^\mu \) is a dissipative number current, \( Q^\mu = \sigma \tilde{Q}^\mu \) a dissipative heat current, and \( \Pi^{\mu \nu} = \tau \Delta^{\mu \nu} + \tau^{\mu \nu} \) a dissipative (viscous) stress tensor with \( \tau = \tilde{\tau} \) and \( \tau^{\mu \nu} = 2\eta \tilde{\tau}^{\mu \nu} \). Here we have defined \( \Delta^{\mu \nu} = g^{\mu \nu} + V^\mu V^\nu \) as the projection onto the space direction in the fluid rest-frame and the various dissipative terms satisfy

\[ V_\mu \tilde{N}^\mu = V_\mu \tilde{Q}^\mu = V_\mu \tilde{\tau}^{\mu \nu} = 0 \] (5)

with \( \tilde{\tau}^{\mu \nu} \) also traceless and symmetric. We have made an unconventional choice to factor out the overall dependence on particle density \( \sigma \), thermal conductivity \( \kappa \), bulk viscosity \( \zeta \), and shear viscosity \( \eta \), in order to make clearer some of our arguments below. So-called particle- or Eckart-frame theories have \( \tilde{N}^\mu = 0 \), so that \( V^\mu \) is the time-like unit vector in the \( J^\mu \)-direction and \( n = -J_\mu V^\mu \).

On the other hand, energy- or Landau-Lifschitz-frame theories have \( \tilde{Q}^\mu = 0 \), so that \( V^\mu \) and \( \epsilon \) are specified by the eigenvalue condition \( T^{\mu \nu} V^\nu = -\epsilon V^\mu \), with a time-like unit eigenvector. In the class of models that we will consider in detail, there is also an entropy current \( S^\mu \) (discussed further below) which satisfies a balance equation of the form

\[ \partial_\mu S^\mu = \sigma \frac{\tilde{N}^\mu \tilde{N}_\mu}{T^2} + \frac{\tilde{Q}^\mu \tilde{Q}_\mu}{T^2} + \frac{\zeta \tilde{\tau}^{\mu \nu} \tilde{\tau}_{\mu \nu}}{T} + \frac{2\eta \tilde{\tau}^{\mu \nu} \tilde{\tau}_{\mu \nu}}{T}, \] (6)

whose righthand side, when all of the transport coefficients \( \sigma, \kappa, \zeta, \eta \) are positive, is nonnegative as required by the second law of thermodynamics. The specific assumptions made above are mostly to simplify our proof in the next section that effective coarse-grained equations obtained in the limit \( \sigma, \kappa, \zeta, \eta \to 0 \) correspond to distributional Euler solutions. With some appropriate corresponding assumptions, our analysis will apply to any dissipative fluid model consistent with the thermodynamic second-law. In fact, our inertial-range analysis is completely general and applies to any distributional solution of the relativistic Euler equations, regardless of the dissipative model limits used to obtain the particular solution (or to even solutions constructed by other means).

Defining the energy current

\[ E^\mu = -T^{\mu \nu} V_\nu = \epsilon V^\mu + \kappa \tilde{Q}^\mu, \] (7)

and internal energy current

\[ U^\mu = E^\mu - mc^2 J^\mu = u V^\mu + \kappa \tilde{Q}^\mu, \] (8)

it is straightforward to obtain from (11, 2) for all of the class of models we consider the balance equations of total and internal energy densities as

\[ \partial_\mu E^\mu = \partial_\mu (\kappa \tilde{Q}^\mu) = -q \partial_\mu (\partial_\mu V^\mu) + Q_{\text{diss}} \]

(9)

with the dissipative “heating” of the fluid given by

\[ Q_{\text{diss}} = -\kappa \tilde{Q}^\mu A_{\mu} - \zeta \tilde{\tau} - 2\eta \tilde{\tau}^{\mu \nu} \tilde{\tau}_{\mu \nu}. \] (10)

Here \( A^\mu = DV^\mu \) is the acceleration vector with \( D = V_\nu \partial_\nu \), the material derivative for an observer moving with the fluid,

\[ \theta = \Delta^{\mu \nu} \partial_\nu V_\mu = \partial_\mu V^\mu \]

(11)

is the relativistic dilatation, and

\[ \sigma_{\mu \nu} = \partial_\mu (\mu V_\nu) \equiv \partial_\mu (\mu V_\nu) - \frac{\theta}{d} \Delta^{\mu \nu} \]

\[ = \partial_\mu (\mu V_\nu) + A_{\mu} V_\nu - \frac{\theta}{d} \Delta^{\mu \nu} \]

(12)

is the relativistic strain, for \( \partial_\mu \Delta = \Delta^{\nu \mu} \partial_\nu \). We use here standard notations for relativistic fluids [52, 53], in particular with \( C_{\mu \nu} = \frac{1}{2} (C_{\mu \nu} + C_{\nu \mu}) \) the symmetrization on \( \mu, \nu \), so that \( \sigma_{\mu \nu} \) is symmetric, traceless, and \( \sigma_{\mu \nu} V^\nu = 0 \).

The traditional theories of Eckart and Landau-Lifschitz [58] have dissipative fluxes proportional to the following tensors:

\[ \tilde{N}^\mu = -T^2 \partial_\perp \lambda \]

(13)

\[ \tilde{Q}^\mu = -\theta T A^\mu \]

(14)

\[ \tilde{\tau} = -\theta, \quad \tilde{\tau}_{\mu \nu} = -\sigma_{\mu \nu} \]

(15)

which are first-order in gradients, with particle-conductivity \( \sigma = 0 \) for Eckart and thermal-conductivity \( \kappa = 0 \) for Landau-Lifschitz, so that

\[ Q_{\text{diss}} = \kappa A^\mu \partial_\mu T + \kappa A^\mu A_\mu + \zeta \theta^2 + 2\eta \sigma_{\mu \nu} \sigma_{\mu \nu}. \] (16)

In particular, \( Q_{\text{diss}} \geq 0 \) for the Landau-Lifschitz theory. Above we have used the standard relativistic thermodynamic potentials, the temperature \( T \) (or its inverse \( \beta = 1/T \)) and \( \lambda = \mu/T \) for the chemical potential \( \mu \). For reviews of relativistic thermodynamics, see [53], also [52], section 5 or [54], §2.3.7. Here we note only that the relativistic chemical potential differs from its Newtonian counterpart \( \mu_N \) by a rest-mass contribution, \( \mu = \mu_N + mc^2 \). The entropy current of the Eckart and Landau-Lifschitz theories is defined in terms of the entropy density per volume \( s(\epsilon, n) \) and the thermodynamic potentials as

\[ S^\mu = s V^\mu + \beta Q^\mu - \lambda N^\mu. \] (17)
Using the thermodynamic second law \( ds = \beta \psi - \lambda d\mu \) and equations (12) and (14), it is then easy to check that the equation (10) holds. However, as is well-known, the Eckart and Landau-Lifschitz theories are unstable, acausal and ill-posed in both linear (30) and nonlinear (31) regimes. Thus, these theories are not useful as predictive evolutionary models of relativistic fluids.

The class of models that we consider also contain better-behaved models, however, such as the extended hydrodynamic theory of Israel-Stewart \([24, 25]\). This is itself an entire class of models, each of which uses a different definition of the off-equilibrium fluid velocity. The particle-frame and energy-frame versions have both been shown to be stable, causal, and hyperbolic in the linear (62, 63) and nonlinear (64, 65) regimes, with somewhat better stability properties in the dissipative currents. In these models the entropy current is not given by (17) but instead is modified by the addition of terms that are quadratic in the dissipative fluxes \( N^\mu, Q^\mu, \tau, \) and \( \tau^{\mu\nu} \).

The form of the entropy current may be illustrated by the expression that holds in the energy-frame Israel-Stewart theory (63, 64):

\[
S^\mu = sV^\mu - \lambda N^\mu - \frac{1}{2T} (\beta_0 \tau^2 + \beta_1 N_\alpha N^\alpha + \beta_2 \tau_{\alpha\beta} \tau^{\alpha\beta}) V^\mu + \frac{\alpha_0}{T} \tau N^\mu + \frac{\alpha_1}{T} \tau^{\mu\nu} N_\nu.
\]

The new term proportional to \( V^\mu \) can be regarded as an off-equilibrium modification of the rest-frame entropy density \( s \), and thus the coefficients \( \beta_i, i = 1, 2, 3 \) (not to be confused with \( \beta = 1/T \)! ) are required to be positive to ensure that non-vanishing gradients lower the entropy. The other two terms proportional to \( \alpha_i, i = 1, 2 \) are purely spatial in the fluid rest-frame and describe second-order contributions to dissipative entropy transport. All of the \( \alpha \) and \( \beta \) coefficients are assumed to be smooth functions of \( \epsilon, \rho \). Imposing the second law of thermodynamics in the form of (40) constrains the dissipative fluxes (62). For example, for the energy-frame Israel-Stewart theory one finds

\[
\tau = \zeta \hat{\tau} = -\zeta [\theta + \beta_0 D\tau + \cdots],
\]

\[
N^\mu = \sigma \bar{N}^\mu = -\sigma T [\hat{T}_0^\mu \lambda + \beta_1 (DN_\perp)^\mu + \cdots]
\]

\[
\tau^{\mu\nu} = 2\eta \hat{\tau}^{\mu\nu} = -2\eta [\sigma^{\mu\nu} + \beta_2 (D\tau)^{(\mu\nu)} + \cdots]
\]

with \( (DN_\perp)^\mu = \Delta^{\mu\nu} DN_\nu \) and with \( (D\tau)^{(\mu\nu)} = D\tau^{\mu\nu} + \tau^{\mu\alpha} A_\alpha V_\nu + \tau^{\nu\alpha} A_\alpha V^\mu \) the part of \( D\tau^{\mu\nu} \) symmetric, traceless, and orthogonal to \( V^\mu \). Here \((\cdots)\) indicates various terms that are second-order in gradients, involving the fluxes \( N^\mu, \tau, \tau^{\mu\nu} \) and the thermodynamic potentials. We note that in the case of the particle-frame Israel-Stewart model, nearly identical equations hold, but with \( N^\mu \rightarrow Q^\mu \) and \( T\hat{T}_0^\mu \lambda \rightarrow \hat{T}_0^\mu \lambda / T + A^\mu \). Unlike the original Eckart-Landau-Lifshitz theories, the relations (19), (20) are not simple constitutive relations for the dissipative fluxes, but are instead evolutionary equations which must be solved in time together with the conservation laws (11), (24) in order to determine both the local thermodynamic variables and the dissipative fluxes.

It is a curious fact that in the Israel-Stewart (IS) theories the “energy dissipation” \( Q_{\text{diss}} \) in (49) may possibly be negative and thus may not act to heat the fluid. Indeed, out of the entire class of models that we consider in this paper, only the (ill-posed) Landau-Lifschitz theory guarantees that \( Q_{\text{diss}} \geq 0 \). It is generally argued that negative values of \( Q_{\text{diss}} \) cannot be realized within the physical regime of validity of a fluid description. Since the dissipative fluxes in the energy-frame Israel-Stewart (IS) model differ from those in the Landau-Lifschitz (LL) theory only by terms second-order in gradients, it is plausible that for most circumstances the dissipative fluxes obtained by solving the IS model will be nearly the same as those given by the LL constitutive relations, when evaluated with the IS model solutions. More generally, Geroch \((65, 66)\) and Lindblom \((69)\) have argued that this close agreement with the Landau-Lifschitz/Eckart constitutive relations will hold in the energy/particle frame, respectively, for a wide set of extended dissipative relativistic fluid models that are hyperbolic, causal, and well-posed. We thus expect typically to have \( Q_{\text{diss}} \geq 0 \) in energy-frame fluid models. Unfortunately, the arguments of \((65, 68)\) fail in the presence of shocks with near-discontinuities extending down to lengths of the order of the mean-free-path. In fact, the IS fluid models and other broad classes of hyperbolic, causal, well-posed models of dissipative relativistic fluids do not even possess continuous solutions corresponding to strong shocks \((65, 71)\). Thus, perhaps even more than for the non-relativistic case, a better microscopic starting point for a theory of relativistic fluid turbulence might be relativistic kinetic theory or a relativistic quantum field-theory rather than a dissipative fluid model. Fortunately, our principal results do not depend upon any particular model of dissipation, but only require the general conservation laws (11), (24), a fluid description with variables given by local thermodynamic equilibrium, and the second law of thermodynamics.

In this paper we examine the hypothesis that the entropy-production is anomalous in relativistic fluid turbulence. Thus, we assume in the ideal limit \( \sigma, \kappa, \eta, \zeta \rightarrow 0 \) that distributional limits of the entropy production exist:

\[
\Sigma = D_\text{cond} + \Sigma_\text{therm} + \Sigma_\text{bulk} + \Sigma_\text{shear} > 0
\]

We shall then show that any strong limits \( \epsilon, \rho, V^\mu \) of the local equilibrium fields are weak solutions of the relativistic Euler equations, under very mild additional assumptions. The anomalous entropy production of these Euler solutions is shown to occur by a nonlinear cascade
mechanism and we characterize the type of singularities required for non-vanishing entropy cascade. As in the non-relativistic case, the ideal limit is really a limit of large Reynolds and Péclet numbers introduced by a non-dimensionalization of the fluid equations. Because the fluid velocity $V^\mu$ is already non-dimensional in natural units based on the speed of light $c$ and is assumed to be of order unity, the Reynolds numbers are $Re_\eta = \rho_0 c^2 L_0 / \eta$ and $Re_c = \rho_0 c^2 L_0 / \zeta$ as given by the shear and bulk viscosities \[71\], and the particle and thermal Péclet numbers are $Pe_\eta = \rho_0 L_0 / \sigma T_0 (mc)^2$ and $Pe_c = \rho_0 c^2 L_0 / \kappa T_0$. Here $\rho_0 c^2$ is a typical energy density, $L_0$ a length characterizing the injection scale of the flow (as well as the turnover time $L_0/c$ in natural units), and $T_0$ a temperature scale such as $T(\rho_0 c^2, \rho_0)$. There are additional dimensionless groups which multiply the terms of the dissipative fluxes that are second-order in gradients, but no assumption needs to be made in our analysis about their magnitudes.

In addition to formulating a theory of the turbulent entropy balance, we shall also derive a turbulent internal energy balance and describe with precise formulas the relativistic energy cascade. Conditions for the non-vanishing of the energy flux are very similar to those obtained in paper I for non-relativistic flow, and the relativistic energy flux reduces in the limit $c \to \infty$ to the non-relativistic kinetic energy flux. An Onsager condition for non-vanishing energy-dissipation anomaly is obtained, assuming positivity of the dissipative heating. Our main result on entropy production anomaly requires no such additional assumption and the proof requires only modest changes to that for non-relativistic fluids, as we demonstrate in detail below.

III. RELATIVISTIC COARSE-GRAINING

We employ in our analysis a coarse-graining regularization very similar to that used in our non-relativistic study in the companion paper I. Just as in the non-relativistic case, non-vanishing dissipative anomalies as in \[22\] require that gradients $\partial_\mu V_\nu$, $\partial_\mu T$, $\partial_\mu \lambda$ must diverge as $\sigma, \kappa, \eta, \zeta \to 0$ and this makes it impossible to interpret the fluid dynamical equations in the naive sense in the ideal limit. As in the non-relativistic problem, we can remove the ultraviolet divergences by space-time coarse-graining. An essential difference, however, is that coarse-graining with a spherically-symmetric filter kernel guarantees invariance of turbulent fluxes in non-relativistic flows under the full Galilean symmetry group, but there is no possible space-time coarse-graining that can preserve Lorentz symmetry. For example, consider a general space-time filtering operation of the velocity field

$$V^\mu(x) = \int d^D r \, G_\ell(r) \, V^\mu(x + r), \quad \text{(23)}$$

with $G_\ell(r) = \ell^{-D} G(r/\ell)$. Then it is easy to check that Lorentz transformations $V^\mu(x') = \Lambda^\mu_\nu V_\nu(\Lambda^{-1} x')$ for $\Lambda \in SO(1, d)$ when applied to the coarse-grained field in \[23\] correspond to a coarse-graining of the transformed field $V^\mu(x')$, but with a different kernel

$$G'(r') = G(\Lambda^{-1} r'). \quad \text{(24)}$$

The kernels in the two frames are the same if and only if the coarse-graining kernel satisfies for all $r$ in Minkowski space and all $\Lambda \in SO(1, d)$ that

$$G(\Lambda r) = G(r). \quad \text{(25)}$$

This relation requires that $G(r)$ depend upon the separation vector $r^\mu$ only through the relativistic proper-time interval $R^2 = -r_\mu r^\mu$. In that case, however, the space-time integral of the kernel must diverge

$$\int d^D r \, G(r) = \int_{-\infty}^{+\infty} dR^2 \, G(R^2) \int_{H_{R^2}} \frac{d^D r}{|\alpha|} = +\infty, \quad \text{(26)}$$

because of the non-compactness of the hyperboloids $H_{R^2} = \{ r : -r_\mu r^\mu = R^2 \}$. In contrast, in the non-relativistic case the orbits of the rotation group $SO(d)$ in its action on space are the spheres $S_\rho = \{ r : |r| = \rho \}$, which are compact and have finite area. Because of the divergence in \[26\], it is impossible to define a coarse-graining operation which commutes with Lorentz transformations and whose kernel satisfies the properties of positivity $G(r) \geq 0$ and normalization

$$\int d^D r \, G(r) = 1. \quad \text{(27)}$$

Together with rapid decay and smoothness, these properties are necessary so that coarse-graining is a regularizing operation which represents a local space-time averaging. As we shall see below, this leads to a breaking of Lorentz-covariance of the coarse-grained fluid equations at finite $\ell$ and possible observer-dependence of quantities such as turbulent cascade rates. However, we shall see that there is restoration of Lorentz symmetry in the limit $\ell \to 0$ (similar to the restoration of Lorentz invariance in lattice field-theories in the limit of lattice-spacing $a \to 0$).

The effect of Lorentz transformation on a filter kernel can be made more concrete by considering a pure boost in the $1$-direction, with rapidity $\varphi$ related to the relative velocity $w$ by $w = c \tanh \varphi$. Using standard light-front coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$ in 0-1 planes \[72\], the boost transformation becomes

$$x'^\pm = e^{\pm \varphi} x^\pm \quad \text{(28)}$$

with all other spatial variables $x^2, \ldots, x^d$ remaining unchanged. A filter kernel $G_\ell$ is thus transformed into

$$G'_\ell(r') = G_\ell(e^{-\varphi} r'^+ , e^{+\varphi} r'^-, r^2, \ldots, r^d). \quad \text{(29)}$$

Effectively, the coarse-graining scale is changed for the co-moving observer to $\ell_+ = e^{-\varphi} \ell$ in the $+$ direction, to $\ell_- = e^{+\varphi} \ell$ in the $-$ direction, and unchanged in the remaining spatial directions $2, \ldots, d$. This discussion of the
pure boost transformation underlines the fact that the notion of “scale” will be different for different observers.

While any filter kernel that is smooth and rapidly decaying in space-time can be adopted, it is also possible to use more singular kernels that will still regularize the equations of motion. For example, as in the non-relativistic case, it is possible to filter only spatially at fixed time instants $\gamma$:

$$G_\ell(r) = G_\ell(r)\delta(r^0),$$

(30)

where $G_\ell(r) = \ell^{-d}G(r/\ell)$ is a smooth kernel rapidly decaying in physical space. Such a coarse-graining does not, of course, remain instantaneous in other reference frames. For example, for an observer moving with relative velocity $v$ in the $1$-direction the kernel in (30) transforms into

$$G_\ell(r^0, r^1, \ldots, r^d) = \gamma^{-1}(w)G_\ell(r^1/\gamma(w), r^2, \ldots, r^d)\delta(r^0 + wr^1/c),$$

(31)

with Lorentz-factor $\gamma(w) = (1 - w^2/c^2)^{-1/2}$ according to the general transformation formula (23). To the relatively moving observer the filtering kernel has become non-instantaneous and, furthermore, is elongated along the $1$-direction with modified spatial scale $\ell' = \gamma(w)\ell$ in that direction. Such elongation corresponds to the well-known fact that a stationary blob of fluid at an instant in the original frame is length-contraction by the factor $1/\gamma(w)$ in the relatively moving frame but also sweeps through a distance larger by the factor $\gamma(w)$ as it moves in that frame. Once again, the notion of “scale” is seen to be different for different observers.

Another singular kernel of some interest is a spatially-weighted average over the past light-cone:

$$G_\ell(r) = G_\ell(r)\delta(r^0 + |r|).$$

(32)

This is natural as an average that can be, in principle, computed at each point independently from incoming light-signals [74]. It may also have some utility for numerical modelling of relativistic fluid turbulence by the Large-Eddy Simulation (LES) methodology [23,25] since such averages can be computed on arbitrary space-like Cauchy surfaces using only pre-computed (past) values of simulated fields. For the observer moving with relative velocity $w$ in the $1$-direction, the light-cone average transforms into another light-cone average with a different spatial kernel:

$$G_\ell(r') = \delta(r^0 + |r'|) \times \left\{ \begin{array}{ll} fG_\ell(f^1, r^2, \ldots, r^d) & \text{for } r^1 > 0 \\ f^{-1}G_\ell(f^{-1}r^1, r^2, \ldots, r^d) & \text{for } r^1 < 0 \end{array} \right.$$ (33)

for $f = \sqrt{\gamma^2 - 1} = e^{-\varphi}$. In this particular case, the spatial kernel is elongated or contracted depending upon the relative signs of $w$ and $r^1$ and an initially reflection-symmetric kernel will not remain so in a boosted frame.

A property of the space-time coarse-graining operation [23] that must be kept in mind is that the coarse-grained fluid velocity vector $\tilde{V}^\mu$, while it remains future time-like, is not generally a unit vector with respect to the Minkowski pseudometric. Under coarse-graining

$$V^\mu = \gamma(v)(1, v/c) \Rightarrow \tilde{V}^\mu = \gamma(v)(1, \tilde{v}/c),$$

(34)

where we introduced the $\gamma$-weighted space-time average

$$\tilde{v} = \gamma(v)v/\gamma(v), \quad \gamma(v) = (1 - v^2/c^2)^{-1/2}.$$ (35)

By convexity of the spatial Euclidean norm-square,

$$|\tilde{v}|^2 \leq |\tilde{v}|^2 < c^2.$$ (36)

Thus,

$$\tilde{V}_\mu\tilde{V}^\mu = -\gamma(v)^2/\gamma(\tilde{v})^2 < 0$$ (37)

with $\tilde{v} = |\tilde{v}|$ and $\tilde{V}^\mu$ remains future time-like. However, generally $\gamma(v) \neq \gamma(\tilde{v})$ and thus $\tilde{V}_\mu\tilde{V}^\mu \neq -1$. Non-unit normalization of $\tilde{V}^\mu$ introduces new terms into the coarse-grained equations of motion in the relativistic case that have no counterpart non-relativistically.

The most important feature of the space-time coarse-graining is that, for a fixed scale $\ell$, all of the dissipative transport terms in the coarse-grained conservation laws

$$\partial_\sigma T^\mu{}^\nu = 0, \quad \partial_\nu \hat{J}^\mu = 0$$ (38)

become negligible in the ideal limit $\sigma, \kappa, \eta, \zeta \to 0$. As in the non-relativistic case, this negligible direct effect of dissipation leads to the crucial concept of the “inertial-range of scales”. Because of the key importance of this result, we give here a careful demonstration for the class of dissipative fluid theories treated in this paper. For simplicity, we consider only filter kernels that are entirely smooth in space-time, as more singular kernels (such as instantaneous or light-cone averages) would introduce additional purely technical complications. See [15] for further discussion. Furthermore, we assume that the kernel is $C^\infty$, compactly supported in space-time and is thus a standard test function for space-time distributions, which further simplifies the proofs.

We illustrate the argument with the number conservation law, which contains the single dissipative term

$$\partial_\sigma N^\mu(x) = -\frac{1}{T} \int d^D r (\partial_\mu \mathcal{G})\sigma(x + r)\tilde{N}^\mu(x + r)$$ (39)

where an integration by parts has been performed. Introducing as a factor of unity $T \cdot (1/T) = 1$, the Cauchy-Schwartz inequality gives

$$|\partial_\sigma N^\mu(x)| \leq \frac{1}{T} \sqrt{\int_{\text{supp}(\mathcal{G})} d^D r (\sigma T^2)(x + r) \times \sqrt{\int d^D r \sigma T^2(x + r)}(\partial_\mu \mathcal{G})\sigma(x + r)\tilde{N}^\mu(x + r)^2}$$ (40)
The first square-root factor vanishes in the ideal limit under mild assumptions (e.g., if $\sigma$ goes to zero uniformly in space-time and if the temperature $T$ remains locally square-integrable). If we can show that the second square-root factor remains bounded in the ideal limit, then the product will also go to zero.

Because the projection tensor $\Delta^{\mu\nu}$ is symmetric and also non-negative (as seen by transforming into the fluid rest frame), it defines an inner product for which another application of Cauchy-Schwartz gives

$$\left| (\partial_\mu G)\ell (r) \tilde{N}^\mu (x + r) \right|^2 \leq (\partial_\mu G)\ell (r) \tilde{N}_\mu \tilde{N}^\mu (x + r).$$

The integral inside the second square-root in (40) is thus bounded by

$$\int d^D r \left( \partial_\mu^G \ell (r) \partial_\mu G \ell (r) \right) \cdot \frac{\sigma \tilde{N}_\mu \tilde{N}^\mu}{T^2} (x + r)$$

and the second factor in the integrand in (42) above is just the entropy production due to particle conductivity. Because this entropy production is assumed to converge distributionally to a non-vanishing measure $\Sigma_{\text{cond}}$ in the ideal limit, this integral would remain bounded if the first factor were a smooth test function. Unfortunately, the last statement is generally false, because the second factor in the integrand in (42) remains bounded in the ideal limit, if the temperature $T$ goes to zero uniformly in space-time and if the temperature $T$ remains locally square-integrable). If we can show that the second factor in the integrand in (42) remains bounded in the ideal limit, then the product will also go to zero.

The conclusion of this argument is that the coarse-grained particle conservation law in any fixed $\ell > 0$ is equivalent to their validity “weakly” or in the sense of distributions. We should emphasize the non-triviality of this result. Non-vanishing of the distributional limit $\Sigma_{\text{cond}} = \mathcal{D}$-lim$_{\sigma, \kappa, \eta, \zeta \to 0} \sigma \tilde{N}_\mu \tilde{N}^\mu / T^2$ requires that gradients of thermodynamic potentials must diverge, or $|\partial_\mu^\lambda| \to +\infty$, if the Landau-Lifschitz contribution (13) to $\tilde{N}_\mu$ is the dominant one. Nevertheless, even with such diverging gradients of fine-grained quantities (an “ultraviolet divergence”), the coarse-grained equations are regularized and any limit fields $n, V^\mu$ as $\sigma, \kappa, \eta, \zeta \to 0$ satisfy the ideal particle conservation law in the coarse-grained sense. For finite but very large values of the particle Péclet number $Pe_\sigma$ this means that there is a long “inertial-range” of scales $\ell$ where the coarse-grained ideal equation is valid.

It is worth emphasizing that the dissipation length $\ell_\sigma$ where particle conductivity $\sigma$ becomes non-negligible is presumably observer-dependent at finite $Pe_\sigma$, unlike the non-relativistic case where all observers in different Galilean frames will agree on the dissipation lengths. Notice that the upper bound in (15) is not Lorentz-invariant, because the gamma factor $\gamma (v)$ and the Euclidean norm of the kernel gradient are both frame-dependent. In fact, consider the example of a coarse-graining average over a Euclidean ball in space-time with radius $\ell$, as calculated by a certain observer. When $\ell \gg \ell_\sigma$, then the dissipative contribution to the coarse-grained particle current will be negligible to this observer. However, this same coarse-grained particle current to a co-moving observer with large relative velocity $w$ in the 1-direction will correspond to a filter kernel $\tilde{G}_{\ell_\sigma}(w)$ with dilated thickness $e^{-\sigma_\ell} \ell_\sigma$ in the + direction in the 0-1 plane and contracted thickness $e^{-\rho_\ell} \ell_\sigma$ in the - direction. As a consequence, $\partial_\ell$-gradients of coarse-grained fields become large for this observer. When $w$ is sufficiently close to $c$ so that $e^{-\rho_\ell} \ell_\sigma \simeq \ell_\sigma$, then the co-moving observer may find that dissipative particle transport is non-negligible for the coarse-grained current in his frame of reference. Of course, in the ideal limit $\sigma \to 0$ with the scale $\ell$ of the filter kernel fixed, every observer will agree that dissipative transport has vanished in the coarse-grained particle current because $\ell_\sigma \to 0$.

All of the conclusions derived above hold also for the coarse-grained equations of energy-momentum conservation, where at fixed $\ell$ in the limit $\sigma, \kappa, \eta, \zeta \to 0$, any limiting fields $e, \rho$, and $V^\mu$ will satisfy

$$\partial_\mu T^\mu \nu = \partial_\nu (\sigma g^\mu \nu + p \Delta^\mu \nu) = 0,$$

with all dissipation terms tending to zero. Just as for particle-conservation, the range of $\ell$ over which these ideal equations are valid could be observer-dependent at finite Reynolds and Péclet numbers. The proof of these statements is very similar to that given above for particle-conservation, and we thus give complete details in Appendix [13]. From the two equations (40, 13) we can
conclude that the relativistic Euler equations hold in the coarse-grained sense at fixed scale \( \ell \) in the limit of infinite Reynolds and Pécel number, that is, ideal relativistic Euler equations hold distributionally.

There is one last remark on the coarse-graining regularization which has fundamental importance in what follows. This coarse-graining is a purely passive operation which is applied \textit{a posteriori} to the fluid variables and which can effect no change whatsoever on any physical occurrence \cite{74, 75}. In prosaic terms, coarse-graining corresponds to “removing one’s spectacles” and observing the physical evolution at a reduced space-time resolution \( \ell \). The effective dynamical description is changed by regularization, of course, with coarse-grained variables satisfying much more complex equations than fine-grained fields. This is not unexpected because the coarse-grained variables are like “block-spins” in the renormalization-group theory of critical phenomena \cite{76, 80}, and such Wilson-Kadanoff RG procedures typically lead to very complicated effective descriptions. In fact, an implementation of the coarse-graining by integrating out unresolved fields in a path-integral formulation yields an effective dynamics with higher-order nonlinearity, long-time memory, and induced stochasticity \cite{81}. This is a manifestation of the “closure problem”, in which coarse-grained variables like \( \bar{V}^\mu(x) \) no longer satisfy simple closed equations of motion. As we shall see below, Onsager’s method does not solve this problem, but instead bypasses it by exploiting “4/5th-law”-type expressions for new, unclosed expressions. The essential idea is then to invoke the independence of the physics on the arbitrary coarse-graining scale \( \ell \). This simple invariance principle turns out to yield non-trivial consequences.

\section{Energy Cascade}

It is reasonable to expect that relativistic fluids at very high Reynolds and Pécel numbers should exhibit a turbulent energy cascade, just as do non-relativistic incompressible and compressible fluids. However, the familiar notion of kinetic energy cascade is not appropriate for relativistic turbulence, because kinetic energy is not a natural concept within relativity theory. On the other hand, we have seen in our discussion of non-relativistic compressible fluids in paper I that energy cascade can be understood from the coarse-grained dynamics of the internal energy. Because the concept of internal energy remains valid in relativistic thermodynamics, it provides a good basis for the theory of relativistic energy cascade.

A resolved energy current is defined most simply as

\[
\mathcal{E}^\mu = -\bar{T}^{\mu\nu}V_\nu,
\]

which (like resolved kinetic energy in non-relativistic turbulence) is a nonlinear function of coarse-grained quantities. As we have shown in some detail in Appendix \ref{A} for length-scales \( \ell \) in the inertial-range, or for all fixed \( \ell \) in the ideal limit \( \sigma, \kappa, \eta, \zeta \to 0 \),

\[
\bar{T}^{\mu\nu} = \epsilon V^\mu V^\nu + p\Delta^{\mu\nu} = \bar{p}g^{\mu\nu} + hV^\mu V^\nu,
\]

where \( h = \epsilon + p \) is the \textit{relativistic enthalpy}. If subsequent to the ideal limit \( \sigma, \kappa, \eta, \zeta \to 0 \), one considers the limit of regularization length-scale \( \ell \to 0 \), one finds that

\[
\mathcal{D}_{\lim_{\ell \to 0}} \mathcal{E}^\mu = \epsilon V^\mu,
\]

For this to hold, one needs only some modest regularity of the limiting variables \( \epsilon, p, V^\mu \), such as finite (absolute) 4th-order moments in local space-time averages. Thus, the resolved energy current converges distributionally in the “continuum limit” \( \ell \to 0 \) to the fine-grained energy current of the Euler fluid. The naive energy balance obtained by setting \( \mathcal{Q}_{\text{diss}} = 0 \) in \cite{19} does not follow, however. To obtain the correct result, we can use the balance equation for the resolved energy current

\[
\partial_\mu \tilde{\mathcal{E}}^\mu = \partial_\mu (\bar{V}_\nu \bar{T}^{\mu\nu}) = -(\partial_\nu \bar{V}_\nu) \bar{T}^{\mu\nu} = -\bar{p}(\bar{\theta}^{\mu\nu}) - (\partial_\nu \bar{V}_\nu) h V^\mu V^\nu,
\]

obtained from \( \partial_\mu \bar{T}^{\mu\nu} = 0 \) and \cite{49}. The last term in equation \cite{51} would not be present in the fine-grained internal energy balance for a smooth Euler solution, because of the orthogonality condition \( (\partial_\nu \bar{V}_\nu) V^\nu = 0 \). This term is the source of possible energy dissipation anomalies in relativistic fluid turbulence and it gives the simplest representation of turbulent energy flux.

Despite the simplicity of the above formulation, we shall follow here an alternative approach based upon a relativistic Favre-averaging, similar to that employed in paper I for non-relativistic compressible turbulence. It should be emphasized that the entire theory presented below could be developed just as easily using the equation \cite{51}. However, the relativistic Favre-averaging approach is convenient to compare with results of I in the limit \( c \to 0 \). The proper relativistic generalization of Favre-averaging is motivated by the appearance of the enthalpy in \cite{19}. Note that the “null energy condition” \( \bar{h} \geq 0 \) is a condition for stability of thermodynamic equilibrium \cite{62} and in the strict form \( \bar{h} > 0 \) is required for causality of the relativistic Euler fluid \cite{82}. We thus define the Favre-average coarse-graining for a relativistic fluid by

\[
\bar{f} = \bar{h} f/\bar{h}
\]

with enthalpy-weighting. With this definition, \cite{19} becomes

\[
\bar{T}^{\mu\nu} = \bar{p}g^{\mu\nu} + h\bar{V}_\mu \bar{V}_\nu = \bar{p}g^{\mu\nu} + h\bar{V}_\mu \bar{V}_\nu + \bar{h}\bar{\tau}(V^\mu, V^\nu)
\]

As in the non-relativistic theory, expanding in the \( \ell \)-order cumulants \( \bar{\tau}(f_1, \ldots, f_\ell) \) of the Favre-average produces only a single “unclosed” term in the coarse-grained stress-energy tensor, whereas expanding in \( \ell \)-order cumulants \( \bar{\tau}(f_1, \ldots, f_\ell) \) of the unweighted space-time coarse-graining would produce more such unclosed terms. This
is a significant advantage of the Favre-average for potential applications to “large-eddy simulation” (LES)  
modeling of relativistic fluid turbulence. Within the Favre-
averaging approach, it is convenient to define the resolved  
energy current by

\[ \mathcal{E}_\mu = -\bar{V}_\nu \bar{T}^{\mu\nu} + \frac{\bar{p}}{h} \bar{\tau}(h, V^\mu) - \frac{1}{2} \bar{\tau}(V_\nu, V^\mu) h V^\nu \]  
(54)

Alternative expressions for this current follow from \(61\), the relation

\[ \bar{V}^\mu = \bar{\nabla}^\mu + (1/\bar{h}) \bar{\tau}(h, V^\mu), \]  
(55)

and \(V^\nu V_\nu = -1\), from which one can easily derive

\[ -\bar{V}_\nu \bar{T}^{\mu\nu} + \frac{\bar{p}}{h} \bar{\tau}(h, V^\mu) = \frac{\bar{h} V^\nu + \bar{h} V_\nu V^\nu V^\mu - \bar{V}^\nu \bar{V}^\mu - \bar{P} V^\mu}{\bar{h} V^\nu + \bar{h} V_\nu V^\nu - \bar{V}^\nu \bar{V}^\mu}. \]  
(56)

Thus, one obtains that \(63\)

\[ \mathcal{E}_\mu = \bar{\nabla}^\mu + \bar{\tau}(p, V^\mu) + \frac{1}{2} \bar{\tau}(V_\nu, V^\mu) V^\nu + \bar{\tau}(V_\nu, V^\mu, V^\mu) \]

\[ = \bar{\nabla}^\mu - \bar{\tau}(h, V^\mu) + \frac{1}{2} \bar{\tau}(V_\nu, V^\mu) V^\nu + \bar{\tau}(V_\nu, V^\mu, V^\mu). \]  
(57)

Either from this expression or directly from the definition \(54\) one can see that

\[ \mathcal{D} \cdot \lim_{\ell \to 0} \mathcal{E}_\mu = \epsilon V^\mu, \]  
(58)

under the same assumptions as \(56\). Once again, however,  
the naive energy balance \(60\) for the limiting current \(E^\mu = \epsilon V^\mu\)  
need not hold with \(Q_{\text{diss}} = 0\), but instead may be modified by  
a turbulent dissipative anomaly if

\[ Q_{\text{diss}} := \mathcal{D} \cdot \lim_{\eta, \zeta, \kappa, \sigma \to 0} Q_{\text{diss}, \eta, \zeta, \kappa, \sigma} \neq 0 \]  
(59)

for \(Q_{\text{diss}, \eta, \zeta, \kappa, \sigma}^\eta\) given by \(10\). We emphasize that the vanishing  
or not of the limit in \(59\) is an objective physical fact, which cannot  
depend upon any coarse-graining.

To obtain the correct equation, one can use the inertial-
range balance equation for the energy current defined in \(60\). Using \(\partial_\mu \bar{T}^{\mu\nu} = 0\),  
one gets after some straightforward calculations that

\[ \partial_\mu \mathcal{E}_\mu = -\bar{p} (\partial^\mu V_\nu) + Q_{\text{flux}}^\ell \]  
(60)

with relativistic energy flux defined by

\[ Q_{\text{flux}}^\ell \equiv \frac{1}{h} (\partial_\mu \bar{p}) \bar{\tau}(h, V^\nu) \]

\[ - \bar{h} (\partial_\mu \bar{V}_\nu) \bar{\tau}(V_\nu, V^\mu) - \frac{1}{2} \partial_\nu h \bar{V}^\nu \bar{\tau}(V_\nu, V^\mu). \]  
(61)

The energy flux \(Q_{\text{flux}}^\ell\) can be interpreted as the “apparent  
dissipative heating” in the large-scales based only on  
measurements resolved at that scale. The first two terms  
in the energy flux \(61\) are relativistic generalizations of  
the barocyclonic work and the deformation work as defined  
by Aluie \(43, 16\) for non-relativistic compressible  
fluids, whereas the third term has no non-relativistic  
alogue. The balance equation \(60\) is formally very similar to  
the non-relativistic balance equation (I;57) for the “intrins- 
cal large-scale internal energy”, defined in (I;58). Not  
only does \(60\) resemble the non-relativistic balance equation  
derived in paper I, but we show in Appendix \(13\) that it  
reduces to it in the formal limit \(c \to 0\). In particular,  
the relativistic energy flux that we defined in \(61\) converges  
as \(c \to \infty\) to the non-relativistic expressions in \(43, 16\)  
and in (I;43).

Now let us exploit the fact that a non-zero energy dissipation anomaly in the ideal limit \(59\) cannot depend  
upon any particular choice of the regularization scale \(\ell\).  
Subsequent to the limit \(\eta, \zeta, \kappa, \sigma \to 0\) one can thus  
consider the limit \(\ell \to 0\) of the coarse-grained internal energy balance \(60\) for the relativistic Euler fluid. It follows  
from \(58\) that the left-hand side converges distributionally  
to \(\partial_\mu (\epsilon V^\mu)\), because the overall derivative \(\partial_\mu\)  
can be transferred to a test function. We also define the  
distributional product of the dilatation \(\theta = \partial_\mu V^\mu\)  
and the pressure \(p\) by a standard procedure \(83\)

\[ p \circ \theta = \mathcal{D} \cdot \lim_{\ell \to 0} \bar{P} \cdot \bar{\nabla}, \]  
(62)

just as in the non-relativistic case in paper I. Although all  
of the cumulant factors appearing in the energy flux  
\(61\) vanish as \(\ell \to 0\), the flux \(Q_{\text{flux}}^\ell\) itself need not  
vanish because the space-time gradients multiplying them  
diverge in the same limit. By taking the limit \(\ell \to 0\) of  
\(60\), one thus obtains for the relativistic Euler solutions  
the distributional energy balance

\[ \partial_\mu (\epsilon V^\mu) = -p \circ \theta + Q_{\text{flux}}^\ell, \]  
(63)

with a possible anomaly due to energy cascade given by

\[ Q_{\text{flux}}^\ell = \mathcal{D} \cdot \lim_{\ell \to 0} Q_{\text{flux}}^\ell, \]

\[ = \mathcal{D} \cdot \lim_{\ell \to 0} - (\partial_\nu \bar{V}_\mu) h V^\nu. \]  
(64)

Note that the second expression in the equation above  
arises from the corresponding \(\ell \to 0\) limit of \(51\).

A condition for the non-vanishing of the anomaly  
\(Q_{\text{flux}}^\ell\) can be obtained just as in the non-relativistic  
case (see \(60\) and section V of paper I), by deriving  
“4/5th-law”-type expressions for the turbulent energy flux.  
The key point is that the cumulants of fields with respect  
to the space-time coarse-graining can be written instead as  
cumulants of their space-time increments with respect to  
an average over displacement vectors \(r^\mu\) weighted by the  
filter kernel. That is,

\[ \bar{\tau}(f_1, \ldots, f_p)(x + a) = \langle (\delta f_1) \cdots (\delta f_p) \rangle_{\ell, a} \]  
(65)

where

\[ \delta f_i(x; r) = f_i(x + r) - f_i(x), \]  
(66)

are space-time increments and where, for any function \(h(r)\),

\[ \langle h \rangle_{\ell, a} = \int d^D r \, G_\ell(r - a) \, h(r). \]  
(67)
The superscript cum in \(55\) denotes the \(p\)-th order cumulant part of any \(p\)-th order moment. The details of the proof are given in Appendix B of [78], but the essential point is that cumulants are invariant under shifts of variables by constants and the increment \(\delta f_i(x; r)\) is the shift of \(f_i(x + r)\) by the quantity \(-f_i(x)\) which is “constant”, i.e. independent of \(r^a\). The translation by the space-time vector \(a^a\) in \(55\) is useful to derive expressions for all space-time gradients of coarse-graining cumulants in terms of increments, by differentiating with respect to \(a^a\) and then setting \(a^a = 0\). For example, for \(p = 1\) one obtains with \(\bar{f} = \tau(f)\) that
\[
\partial_{\mu} \bar{f}(x) = -\frac{1}{\ell} \int d^D r \left( \partial_{\mu} \mathcal{G}(r) \right) \delta f(x; r),
\]
and analogous expressions for all \(p > 1\) and all orders of derivatives (\(72\), Appendix B). Expanding the Favre-average cumulants into cumulants of the unweighted coarse-graining, one thus obtains expressions for all of the contributions to the energy flux in terms of space-time increments of the thermodynamic fields.

From these expressions in terms of space-time increments, we can derive necessary conditions for turbulent energy dissipation anomalies. Let us define scaling exponents of space-time structure functions by
\[
\zeta_q^p = \liminf_{|r| \to 0} \frac{\log \| \delta f(r) \|_q}{\log \| r \|_E},
\]
where \(\| \delta f(r) \|_q\) is the space-time \(L_q\) norm of the increment and \(S_q^p(r) = \| \delta f(r) \|_q^p\) is thus the \(q\)-th order (absolute) structure-function of \(f\). From the expressions in \(55, 57\) one can see that the baroplycal work term in \(61\) vanishes as \(\ell \to 0\), unless for every \(q \geq 3\)
\[
\zeta_q^p + \zeta_q^h + \zeta_q^v \leq q.
\]
Likewise, the deformation work and the third term in \(61\) vanish as \(\ell \to 0\) unless for every \(q \geq 3\) either
\[
\zeta_q^h + 2\zeta_q^v \leq q,
\]
or
\[
3\zeta_q^v \leq q.
\]
The arguments here closely parallel those in paper I for the non-relativistic case. In deriving these results we have assumed that the enthalpy \(h\) is bounded away from both zero and infinity. The inequalities \(70-72\) demonstrate that singularities of the fluid variables \(\epsilon, \rho, D\) are required in the ideal limit \(\sigma, \kappa, \eta, \zeta \to 0\) in order to obtain a non-vanishing energy dissipation anomaly from turbulent cascade. This is a scale-local cascade process as long as all of the structure-function exponents satisfy
\[
0 < \zeta_q^f < q \text{ for } f = p, h, v.
\]

The internal energy balance \(53\) of limiting Euler solutions can also be obtained from the fine-grained internal energy balance \(9\) of the dissipative fluid model, by taking directly the limit \(\sigma, \kappa, \eta, \zeta \to 0\). In particle-frame fluid models, the dissipative heat current contribution \(\partial_{\mu}(sQ^\mu)\) can be shown to vanish by arguments similar to those applied to the dissipative terms in the coarse-grained conservation laws. The details are presented in Appendix C Defining \(Q_{\text{diss}}\) as in \(59\) and defining also
\[
p * \theta = D \cdot \lim_{\sigma, \kappa, \eta, \zeta \to 0} p \cdot \theta,
\]
we then obtain the distributional balance equation
\[
\partial_{\mu} (\epsilon V^\mu) = -p * \theta + Q_{\text{diss}}.
\]
As in the non-relativistic case discussed in I, one must expect that the limit \(p * \theta\) in \(73\) is generally distinct from \(p * \theta\) in \(52\), that is, the double limits of \(\bar{p}_\theta \delta\) for \(\eta, \zeta, \kappa, \sigma \to 0\) and for \(\ell \to 0\) do not commute.
In fact, the quantities \(p * \theta\) and \(Q_{\text{diss}}\) are presumably not completely universal and may depend upon the particular sequence \(\eta_k, \zeta_k, \kappa_k, \sigma_k \to 0\) used to reach infinite Reynolds and Péclet numbers. This is known to be true in the non-relativistic limit, as verified in paper I. However, it is a consequence of \(73\) that the particular combination \(-p * \theta + Q_{\text{diss}}\) depends only upon the limiting weak solution and not upon the particular sequence of transport coefficients used to obtain it.

A comparison of \(53\) and \(74\) shows that the two balance equations can be simultaneously valid only if
\[
-p \cdot \theta + Q_{\text{flux}} = -p * \theta + Q_{\text{diss}},
\]
that is, by introducing the relativistic pressure-work defect
\[
\tau(p, \theta) \equiv p * \theta - p \cdot \theta
\]
we can then rewrite the inertial-range balance \(58\) as
\[
\partial_{\mu} (\epsilon V^\mu) = -p * \theta + Q_{\text{inert}},
\]
where the total inertial energy dissipation is defined by
\[
Q_{\text{inert}} \equiv \tau(p, \theta) + Q_{\text{flux}} = Q_{\text{diss}}.
\]
As in the non-relativistic case considered in paper I, the inertial-range energy dissipation can arise not only from energy cascade but also from pressure-work defect. Relativistic shock solutions provide explicit examples with \(\tau(p, \theta) \neq 0\) (Appendix D). Unlike the non-relativistic case, it is not known rigorously that \(Q_{\text{diss}} \geq 0\).

Another important distinction of the relativistic situation is that neither the energy flux \(Q_{\text{flux}}\) nor the pressure-work \(\bar{p}_\theta \delta\) at finite \(\ell\) are Lorentz-invariant scalars, whereas the corresponding quantities are Galilei-invariant in non-relativistic compressible turbulence. Although \(\bar{p}_\theta \delta\) and the expression \(51\) for \(Q_{\text{flux}}^\ell\) appear to define invariant scalars, they involve the kernel \(G_\ell(r)\), which is not frame-invariant. Thus, the coarse-graining regularization breaks Lorentz-symmetry, somewhat similar to lattice-regularizations in relativistic quantum field-theory with finite lattice constant \(a\). In contrast, the fine-grained dissipation \(Q_{\text{diss}}^{\eta, \zeta, \kappa, \sigma}\) and the fine-grained pressure-work \(p * \theta\) are both Lorentz-scalars, and thus their ideal limits \(Q_{\text{diss}}\) and \(p * \theta\) as \(\eta, \zeta, \kappa, \sigma \to 0\) must
be invariant as well. It may appear somewhat unsatisfactory that the energy flux $Q_{i}^{\text{flux}}$ and the resolved pressure-work $\bar{p} \tilde{\theta}_i$ at finite $\ell$ are observer-dependent. However, Lorentz-invariance is restored in the $\ell \to 0$ limit, as easily proved for the combinations $-p \circ \theta + Q_{\text{flux}}$ and, in particular, $Q_{\text{inert}} = \tau(p, \theta) + Q_{\text{flux}}$. The invariance of $-p \circ \theta + Q_{\text{flux}}$ can be seen from its equality with both $-p \circ \theta + \bar{Q}_{\text{diss}}$, which are Lorentz-scalars. Likewise, $Q_{\text{inert}} = \bar{Q}_{\text{diss}}$, which is an invariant scalar. It is reassuring that the net inertial-range dissipation is observer-independent for the limit $\ell \to 0$.

This invariance must hold, within some limits, also for $\ell$ finite but very small, at large Reynolds and Péclet numbers. The reason is that the only effect of a change of argument implies that the two distributions $Q_{\text{inert}}$, $\bar{Q}_{\text{diss}}$ are, in fact, Lorentz-invariant scalars separately and not only in combination [83]. For sufficiently small $\ell$ inside a long inertial range at large $\Re$ and $\Pe$, this invariance of the $\ell \to 0$ limiting distributions must hold approximately. On the other hand, some observer dependence presumably arises for $\ell$ small but non-zero. For example, two observers moving at sufficiently high relative velocities may disagree about the negligibility of the microscopic dissipation for the same coarse-grained fields. For one observer $Q_{\text{inert}}$ may account for all of the dissipation of resolved fields, while for the other the combination $Q_{\text{inert}} + Q_{\text{diss}}$ is necessary to account for all of the dissipation in resolved fields, where $Q_{\text{inert}}$ is the resolved viscous and conductive dissipation [51]. The observed flux contributions will then be distinct.

We have focused in this section on the large-scale/resolved internal energy balance, but there is as well a complementary budget for the unresolved/subscale energy current. In the case of an unweighted space-time coarse-graining, the unresolved current can be naturally defined by $K^\mu = -\tau(T^\mu, V^\nu)$, so that its sum with the resolved current $\mathcal{E}^\mu = -\hat{T}^\mu V^\nu$ accounts for the total energy current. Likewise within the Favre-average coarse-graining approach, the subscale internal energy current can be defined as $K^\mu = \bar{K}^\mu - \mathcal{E}^\mu$, which with [53] gives

$$K^\mu = -\bar{\tau}(V^\nu, T^\mu) + V^\nu \nabla \bar{\theta}(h, V^\nu) + \frac{1}{2} \bar{\tau}(V^\nu, V^\nu) h V^\mu.$$  

From the separate balance equations for $\mathcal{E}^\mu$ and $\mathcal{E}^\mu$ it easily follows that

$$\partial_\mu K^\mu = -\bar{\tau}(p, \theta) + \bar{Q}_{\text{diss}} - Q_{\text{flux}, \ell}.$$  

(79)

The source term on the righthand side is the difference between the true dissipative heating $\bar{Q}_{\text{diss}}$ and the “apparent dissipation” $Q_{\text{flux}, \ell}$ based on measurements at scales $> \ell$, together with the pressure-work defect $\bar{\tau}(p, \theta)$ which represents the difference between the true pressure-work $p \circ \theta$ and the apparent pressure-work $\bar{p} \cdot \bar{\theta}$ based on fields resolved also down to scales $\ell$. Using the expression [57] for $\mathcal{E}^\mu$, the subscale internal energy current can be rewritten in terms of relativistic Favre-average cumulants of the velocity. In particular, its negative becomes

$$-K^\mu = \bar{\tau} \left( \frac{1}{2} \bar{\tau}(V^\nu, V^\mu) \bar{V}^\mu + \bar{\tau}(V^\nu, V^\mu) \bar{V}^\nu + \bar{\tau}(V^\nu, V^\nu, V^\mu) \right) + \bar{\tau}(p, V^\mu).$$  

(80)

Substituting this expression into [79] yields a balance equation very similar in form to the non-relativistic subscale kinetic energy balance obtained in (I;64), and in fact formally reducing to the latter in the limit $c \to \infty$ (Appendix [13]). This identity will prove very important for the discussion in the following section.

V. ENTROPY CASCADE

Hydrodynamic turbulence, as any other macroscopic irreversible process, must be consistent with the second law of thermodynamics. In the relativistic case, in particular, positive entropy production is a primary constraint on dissipative fluid models [8, 10, 24, 26]. For non-relativistic compressible turbulence we have argued in paper I that there is a cascade of (neg)entropy, which is in addition to energy cascade and which is even more fundamental. All of these arguments carry over to relativistic fluid turbulence. The resolved pressure-work in the balance equations [31] or [60] for the large-scale internal energy current is a space-time structured source of internal energy. In relativistic thermodynamics, as in the non-relativistic case, the entropy per volume $s(\epsilon, \rho)$ is a concave function of $\epsilon$ and $\rho$, so that the creation of large-scale structure in $\epsilon$ corresponds to a decrease of entropy at large-scales. To balance this destruction, one can then expect that there will be an inverse cascade of the entropy which is injected by microscopic dissipation/entropy production. As in the non-relativistic case, we may define a “resolved entropy”

$$\bar{s} = s(\bar{\epsilon}, \bar{\rho})$$  

and an “unresolved/subscale entropy”

$$\Delta s = s(\epsilon, \rho) - s(\bar{\epsilon}, \bar{\rho}) \leq 0,$$  

(82)

whose non-positivity follows from the concavity of the entropy. It is somewhat more natural to consider the neg-entropy or information density $\iota(\epsilon, \rho) = -s(\epsilon, \rho)$, which is convex and whose unresolved/subscale contribution $\Delta \iota = -\Delta s$ is non-negative. In this equivalent picture, the pressure-work injects negentropy at large scales, which should cascade forward to small scales where it can be efficiently destroyed by dissipative transport. In order to formalize such notions, one must derive a balance equation for the large-scale entropy.

This balance is straightforward to derive after taking the limit $\eta, \zeta, \kappa, \sigma, \kappa \to 0$ for fixed positive $\ell$. Using the first
law of thermodynamics $ds = \beta de - \lambda dn$ and $\mathcal{D} = \tilde{V}_\mu \partial^\mu$, one gets
\begin{equation}
\mathcal{D}_\phi = \beta \mathcal{D}_\epsilon - \lambda \mathcal{D}_n
\end{equation}
where we employ the notation $\phi = \phi(\tau, \rho)$ for arbitrary smooth functions $\phi$ of $\epsilon$, $\rho$. The equations
\begin{equation}
\mathcal{D}_\epsilon = -\bar{e} \bar{\theta} - \partial_\mu \tau(n, V^\mu)
\end{equation}
\begin{equation}
\mathcal{D}_n = -\bar{n} \bar{\theta} - \partial_\mu \tau(n, V^\mu) + \mathcal{Q}_{\text{diss}}
\end{equation}
are direct consequences of (40) and (44). Using the Gibbs homogeneous relation $(\epsilon + p)/T = s + \lambda n$, one obtains after some straightforward calculations a balance equation of the following form:
\begin{equation}
\partial_\mu S^\mu = \Sigma^\mu_{\text{inert}}.
\end{equation}
The vector whose divergence appears on the left
\begin{equation}
S^\mu = s \tilde{V}^\mu + \beta \bar{\tau}(\epsilon, V^\mu) - \lambda \bar{\sigma}(n, V^\mu)
\end{equation}
is a natural expression for the resolved entropy current, with $s \tilde{V}^\mu$ describing the entropy transport by large-scale advection, $\beta \bar{\tau}(\epsilon, V^\mu)$ the entropy transport due to sub-scale internal energy current, and $\lambda \bar{\sigma}(n, V^\mu)$ the entropy transport due to sub-scale number current. It should be noted that entropy current due to such turbulent subscale transport will not generally be orthogonal to $\tilde{V}^\mu$ in the Minkowski pseudometric, and thus not purely spatial in the rest-frame of the coarse-grained fluid velocity.

The source on the right-hand side of (88) is the inertial-range entropy production
\begin{equation}
\Sigma^\mu_{\text{inert}} = -I^\mu_{\text{mech}} + \beta \mathcal{Q}^\mu_{\text{diss}} + \Sigma^\mu_{\text{flux}},
\end{equation}
where anomalous input of negentropy from pressure work is defined by
\begin{equation}
I^\mu_{\text{mech}} = \beta (p * \bar{\theta} - p \bar{\theta})
\end{equation}
and (forward) negentropy flux is by
\begin{equation}
\Sigma^\mu_{\text{flux}} = (\partial_\mu \beta) \bar{\tau}(\epsilon, V^\mu) - (\partial_\mu \lambda) \bar{\sigma}(n, V^\mu).
\end{equation}
The latter expression is also natural, as it represents entropy production due to subscale transport of internal energy and particle number acting against large-scale gradients of the (entropically) conjugate thermodynamic potentials. In particular, $\Sigma^\mu_{\text{flux}} > 0$ when the subscale transport vectors are “down-gradient”, or opposite to the gradients of $T$ and $\lambda$. Finally, note that one can further decompose the anomalous negentropy input as
\begin{equation}
I^\mu_{\text{mech}} = \beta \bar{\tau}(p, \theta) + I^\mu_{\text{flux}}
\end{equation}
where the first term is the contribution from the pressure-dilatation defect and the second term
\begin{equation}
I^\mu_{\text{flux}} = \bar{(p - p) \bar{\theta}}
\end{equation}
is “flux-like”, representing work of subscale pressure fluctuations against large-scale dilatation. These expressions are exactly analogous to those derived in section VI of paper I for the turbulent entropy balance of non-relativistic compressible fluid flows. In fact, as we show in Appendix B, the formal limit $c \to \infty$ recovers the previously derived non-relativistic expressions.

Now consider the case that there is a non-vanishing entropy production anomaly as in (22). If such an anomaly exists, it cannot depend upon the arbitrary coarse-graining scale $\ell$. Thus, for ideal turbulence at infinite Reynolds and Péclet numbers, we may consider the subsequent limit $\ell \to 0$ of the inertial-range entropy balance, with the coarse-graining regularization removed. This yields a fine-grained entropy balance for the relevant weak solutions of the relativistic Euler equations:
\begin{equation}
\partial_\mu (sV^\mu) = \Sigma^\mu_{\text{inert}}.
\end{equation}
Because all coarse-graining cumulants vanish distributionally as $\ell \to 0$, the resolved entropy current must converge in the sense of distributions to $sV^\mu$ under relatively mild assumptions (e.g. when $\epsilon$ and $\rho$ are bounded in space-time). The limit $\Sigma_{\text{inert}}$ of the source (88) is
\begin{equation}
\Sigma_{\text{inert}} = -I_{\text{mech}} + \Sigma_{\text{flux}} + \beta \circ \mathcal{Q}_{\text{diss}},
\end{equation}
with
\begin{equation}
\beta \circ \tau(p, \theta) = \mathcal{D}_\ell \lim_{\ell \to 0} \beta \bar{\tau}(p, \theta).
\end{equation}
and where
\begin{equation}
\beta \circ \mathcal{Q}_{\text{diss}} = \mathcal{D}_\ell \lim_{\ell \to 0} \beta \bar{Q}_{\text{diss}}.
\end{equation}
The limit source need not vanish. Although entropy is conserved for smooth solutions of relativistic Euler equations, there may be anomalous entropy production for weak solutions. Relativistic shock solutions with discontinuities in the fluid variables are, of course, a well-known example of such dissipative weak solutions (Appendix B). We shall see below, however, that even continuous solutions may exhibit anomalous entropy production. Precisely the same balance equation can be obtained by taking the limit $\eta, \zeta, \sigma, \kappa \to 0$ limit of the fine-grained entropy balance (13) for the dissipative fluid model. The limit of the dissipative entropy production is, of course, obtained directly from our fundamental hypothesis (22). The fine-grained entropy current for the dissipative fluid model also converges to $sV^\mu$ in the limit $\eta, \zeta, \sigma, \kappa \to 0$. This can be verified without great difficulty for models of the Israel-Stewart class. Recall that in such models the entropy current does not have the naive form (17) which it assumes in the Eckart-Landau-Lifschitz models, but is instead modified as in (18) by terms that are second-order in gradients. Taking the latter energy-frame expression as a concrete example, we factor out the dependence upon the transport coefficients $\eta, \zeta, \sigma$ and introduce the rescaled variables $\hat{\tau}^\mu, \hat{\tau}, N^\mu$. This yields the representation
\begin{equation}
S^\mu = sV^\mu - \sigma \lambda N^\mu - \frac{1}{2} (\zeta \hat{\beta}_0 \Sigma_\zeta + \sigma \hat{\beta}_1 \Sigma_\sigma + 2\eta \beta_2 \Sigma_\eta) V^\mu.
\end{equation}
\[ + \zeta \sigma \frac{\alpha_0}{T} \hat{N}^\mu + \eta \sigma \frac{\alpha_1}{T} \tilde{\tau}^{\mu\nu} \tilde{N}_\nu \]  

Here we have denoted as \( \Sigma_{\zeta}, \Sigma_{\sigma}, \Sigma_{\eta} \) the three terms in the fine-grained entropy production \((9)\) that are proportional to \( \zeta, \sigma, \eta \), respectively. According to our fundamental hypothesis \((22)\), these converge to positive distributions \( \Sigma_{\text{bulk}}, \Sigma_{\text{cond}}, \Sigma_{\text{shear}} \) in the limit \( \eta, \zeta, \sigma \to 0 \). Because of the remaining factors of \( \zeta, \sigma, \eta \), appearing in \((96)\), however, one should expect that the \( \beta \)-terms will all vanish in that limit. Likewise, the \( \alpha \)-terms should vanish because they are quadratic in the transport coefficients \( \zeta, \sigma, \eta \). These arguments are not rigorous because the factors involving \( \epsilon, \rho, V^\nu \) in those terms do not remain smooth in the limit. It is possible nevertheless to show by simple inequalities that these terms do vanish in the sense of distributions and, thus, \( \mathcal{D}_\text{lim}_{\eta,\zeta,\sigma,\kappa \to 0} S^\mu_{\eta,\zeta,\sigma,\kappa} = sV^\mu \). For details, see Appendix \( C \). One thus obtains finally the entropy-balance for the limiting Euler solution

\[ \partial_\mu (sV^\mu) = \Sigma_{\text{diss}}, \]  

with \( \Sigma_{\text{diss}} > 0 \) given by the limit in \((22)\). The equality

\[ \Sigma_{\text{inert}} = \Sigma_{\text{diss}} \]  

is demanded by consistency with the inertial-range limiting balance \((33)\), just as in the non-relativistic theory. Given that anomalous entropy production is possible for weak solutions, what degree of singularity of the fluid variables is required for a non-vanishing anomaly? To answer this question, we can prove an Onsager-type singularity theorem which gives necessary conditions for an anomaly. The basic idea is the same as in the non-relativistic case \((48)\) and is easy to explain. We first rewrite the resolved entropy balance \((86)\) as

\[ \partial_\mu S^\mu = \beta (\mathcal{Q}_{\text{diss}} - \tilde{\tau}(p, \theta)) - I^\mu_{\text{flux}} + \Sigma^\mu_{\text{flux}} \]  

The flux-terms \( I^\mu_{\text{flux}} \) and \( \Sigma^\mu_{\text{flux}} \) may be readily expressed in terms of space-time increments of the fluid variables, using the cumulant-expansion methods described in section \((18)\). The term which is difficult to estimate directly is the one involving \( \mathcal{Q}_{\text{diss}} - \tilde{\tau}(p, \theta) \). Note that these two quantities separately may be non-universal and may depend upon the particular sequence \( \zeta_k, \sigma_k, \eta_k \to 0 \) used to obtain the limiting Euler solution. Fortunately, exactly the same combination appears in the Favre-averaging approach \((79)\) for the subscale internal-energy current \( K^\mu \). Thus, one can define an intrinsic resolved entropy current in the Favre-averaging approach as

\[ \mathcal{S}^\mu = \tilde{S}^\mu - \beta K^\mu \]

\[ = \Delta V^\mu + \beta \tilde{\tau}(h, V^\mu) - \Delta \tilde{\tau}(n, V^\mu) \]

\[ + \Delta \tilde{\tau}(V_n, V^\nu) \tilde{V}^\nu + \tilde{\tau}(V_n, V^\nu) \tilde{V}^\nu + \tilde{\tau}(V_n, V^\nu, V^\mu) , \]

where the second equality uses \((80)\). It follows from the two balance equations \((79) \) and \((99) \) that this intrinsic entropy current satisfies the following balance:

\[ \partial_\mu \mathcal{S}^\mu = \Sigma^\mu_{\text{inert}} \]  

where net inertial-range entropy production is defined by

\[ \Sigma^\mu_{\text{inert}} = -I^\mu_{\text{flux}} + \Sigma^\mu_{\text{flux}} \]  

with the intrinsic negentropy flux

\[ \Sigma_{\text{flux}} = \Sigma_{\text{flux}} - (\partial_\mu \beta) K^\mu + \beta Q_{\text{flux}}^\mu \]

\[ = (\partial_\mu \beta) \tilde{\tau}(h, V^\mu) - (\partial_\mu \Delta) \tilde{\tau}(n, V^\mu) + \Delta \tilde{\tau}(V_n, V^\nu) \tilde{V}^\nu + \tilde{\tau}(V_n, V^\nu, V^\mu) . \]

Just as for the naive version of the resolved entropy current, \( \mathcal{D}_\text{lim}_{\eta,\zeta,\sigma,\kappa \to 0} \mathcal{S}^\mu = sV^\mu \), since all of the additional cumulant terms vanish in the limit. Furthermore, and crucially, all source terms on the right-hand side of \((101)\) are “flux-like” and are products of sub-scale cumulant terms and gradients of resolved fields, which allows us to express them in terms of space-time increments. There is a rough analogy of our entropy current modification with the Israel-Stewart correction, in that our current modification is a higher-order moment of the coarse-graining average: whereas the naive entropy current in the first line of \((100)\) involves at most 2nd-order moments of \( \epsilon, n, V^\mu \), the correction on the second line involves 3rd-order moments. Note, however, that our correction term does not have to be small relative to the naive term.

A fundamental observation is that all individual terms in the intrinsic entropy balance \((101)\) depend only upon the limiting Euler solution and not on the sequence used to obtain it. In fact, the same equation can be obtained from the distributional Euler solution directly, without considering the underlying microscopic model (dissipative fluid dynamics, kinetic equation, quantum field-theory, etc.) To see this, one can use the homogeneous Gibbs relation \( s = \beta (\tau + h) - \Delta N \) and the definition \( K^\mu = \mathcal{E}^\mu - \mathcal{E}^\mu \) to rewrite intrinsic entropy current as

\[ \mathcal{S}^\mu = \beta \mathcal{E}^\mu - \beta \mathcal{E}^\mu \]

One can then derive the intrinsic entropy balance \((101)\) directly from the inertial-range balance equation \((100)\) for \( \mathcal{E}^\mu \), the particle conservation equation \( \partial_\mu V^\mu = 0 \), and thermodynamic relation \( \partial_\mu (\beta p) = \pi(\partial_\mu \Delta) - \tau(\partial_\mu \beta) \). This crucial observation implies that our results for anomalous entropy production are universal and apply to all distributional solutions of the relativistic Euler equations, not only those obtained as ideal limits of Israel-Stewart-type dissipative fluid models.

The necessary conditions for anomalous entropy production follow directly from the intrinsic entropy balance \((101)\), exactly as for the non-relativistic case considered in \((48)\). The conclusion is that the entropy anomaly can be non-zero only if for every \( q \geq 3 \) at least one of the
following three conditions is satisfied on the structure-function scaling exponents defined in [69]:

\[
2\min(\zeta_q^c, \zeta_q^p) + c^r_q \leq q, \quad (105)
\]

\[
\min(\zeta_q^c, \zeta_q^p) + 2c^r_q \leq q, \quad (106)
\]

\[
3c^r_q \leq q. \quad (107)
\]

The first inequality is implied by (and thus replaces) the inequality shown earlier to be necessary for non-vanishing of the baroclinic work as \( \ell \to 0 \), while the inequalities (106) replace (71), (72) shown to be necessary for non-vanishing of the other two contributions to energy flux. The above inequalities would be equalities for a K41 dimensional scaling determined by mean energy flux, and the departure from the upper bound is a measure of the space-time intermittency of the solution fields. These upper bounds, even if they hold as equalities, imply that \( \epsilon, \rho, V^\mu \) must be non-smooth/singular in spacetime for the ideal limit. Roughly speaking, limit solutions with anomalous entropy production can have at most 1/3 of a derivative in a space-time \( L^1 \)-sense.

For non-relativistic fluids, the conditions analogous to (105) - (107) are known to be necessary also for an energy dissipation anomaly [48]. While \( Q_{\text{flux}} = 0 \) if none of those conditions hold, it is in principle still possible that \( \tau(p, \theta) = Q_{\text{diss}} > 0 \). When the balance equation (60) for resolved internal energy is rewritten as

\[
\partial_\mu \xi^\mu = -\bar{p} \theta + Q_{\ell}^{\text{inert}} \quad (108)
\]

with

\[
Q_{\ell}^{\text{inert}} = \bar{\tau}(p, \theta) + Q_{\ell}^{\text{flux}}, \quad (109)
\]

then it differs strikingly from the balance equation (101) for intrinsic resolved entropy, because the terms \( \bar{\tau}(p, \theta) \) and \( Q_{\ell}^{\text{inert}} \) are not determined uniquely as \( \ell \to 0 \) by the limiting weak Euler solution. Those terms in fact generally depend upon the the underlying dissipative fluid model sequence, as seen, for example, for the non-relativistic limit of shock solutions where a Prandtl-number dependence remains. In [48], vanishing energy dissipation anomaly is instead derived from the vanishing entropy production anomaly. That proof carries over to relativistic fluids whenever the dissipative fluid model satisfies the bounds

\[
\sum_{\ell} \zeta_{q,\epsilon,\kappa,\sigma} \geq Q_{\ell}^{\text{inert}} / T \geq 0. \quad (110)
\]

Amusingly, the only dissipative relativistic model in the class that we consider which guarantees (110) is the classical energy-frame Landau-Lifschitz theory [89], which is ill-posed and acausal! The result will be true, nevertheless, if the viscous transport fields \( \tau, \tau^\mu \) in the relativistic fluid model are sufficiently well approximated by the constitutive relations of the Landau-Lifschitz theory. Such results have been proved [68], [69], but need to be extended to solutions with shocks or other milder turbulent singularities in order to show that conditions (105) - (107) are necessary for anomalous energy dissipation.

VI. RELATIONS TO OTHER APPROACHES

We now briefly discuss the relation of our analysis with other approaches to relativistic fluid turbulence that have been proposed in the literature.

A. Barotropic fluid models

In paper I we have criticized non-relativistic barotropic models as being physically inapplicable to fluid turbulence, since this is a strongly dissipative process. The same criticisms carry over to relativistic barotropic models, if those are defined as in [53], [60], for example. These authors take \( \epsilon = \epsilon(\rho) \) as the condition for barotropicity, which implies that \( p = p(\epsilon, \rho) = p(\rho) \). As in the non-relativistic case, the internal energy per rest mass \( e = u/\rho \) can be obtained from the integral

\[
e = \int \frac{p \, d\rho}{\rho^2} \quad (111)
\]

if and only if the fluid is isentropic with entropy per mass \( \bar{s}_m = s/\rho \) constant in space-time (see [56], section 2.4.10). This is inconsistent with the irreversible production of entropy by turbulence. Furthermore, one obtains from (111) and \( \partial_\mu J^\mu = 0 \) that

\[
\partial_\mu (u V^\mu) = -p \theta \quad (112)
\]

which omits viscous heating. Barotropic equations of state together with formula (111) for internal energy are thus physically inconsistent, as soon as one includes dissipative terms in \( J^\mu \) and \( T^{\mu \nu} \), and are unsuitable as fluid models of turbulence. These remarks apply to the special case of polytropic equations of state with \( p(\rho) = K \rho^\Gamma \) for exponent \( \Gamma \), whenever the internal energy density is determined from the relation \( u = p(\Gamma - 1) \), as is very standard in numerical simulations with relativistic polytropic models. Such models cannot correctly represent the time-irreversible physics of relativistic fluid turbulence which is created by the spectrum of singularities that develop in the solutions. Note that barotropic fluid models in the sense of [53], [60] are already known to be physically inadequate to describe the irreversible evolution of relativistic shocks [56], section 2.4.10).

These criticisms do not apply to relativistic barotropic equations of state if those are defined instead by the alternative condition \( p = p(\epsilon) \), e.g. as in [56]. Note that such a formulation of barotropicity is more general, because it makes sense even when the constituent particles of the fluid have zero rest-mass and \( \rho = 0 \). There is no physical inconsistency of such an equation of state with irreversible entropy production by microscopic dissipation. For example, ultrarelativistic fluids with vanishingly small coldness \( mc^2/k_B T \ll 1 \) ([56], section 2.4.4) and models of hot, optically thick, radiation-pressure dominated plasmas ([54], section 2.4.8) both satisfy \( p = \frac{1}{4} \ell \) and are thus barotropic in this second
sense. Both of these models have a non-constant thermodynamic entropy, which can be made consistent with the second law of thermodynamics by addition of suitable dissipative terms to the ideal fluid equations. More generally, conformally invariant fluid models that describe low wavenumber dynamics of conformal quantum-field theories and non-conformal fluid models in the zero charge-density sector satisfy both $p = p(c)$ and dissipative second-order hydrodynamical equations similar to the Israel-Stewart models consistent with the second law of thermodynamics. In fact, the exact shock solutions considered in Appendix D are for conformal fluids. All of our conclusions apply to such models, with the simplification that hydrodynamics now reduces to the equation $\partial_\mu T^{\mu\nu} = 0$ for the stress-energy tensor alone.

B. Point-Splitting and Statistical States

In paper I we have argued that point-splitting regularizations are inadequate for non-relativistic compressible fluid turbulence and the same arguments hold for relativistic fluid turbulence. Previously, Fouxon & Oz have used a point-splitting technique in the setting of an externally forced relativistic fluid satisfying

$$\partial_\mu T^{\mu\nu} = F^\mu$$

for a Minkowski force $F^\mu$. Assuming that a statistically homogeneous and stationary state exists, those authors derived an exact statistical relation

$$\langle T_{0\mu}(0,t)T_{i\mu}(r,t)\rangle = \frac{1}{D} P_\mu r_i \quad \text{(no sum on $\mu$)}$$

with $\langle \cdot \rangle$ denoting the ensemble-average and with $\mathcal{P}_\mu = \langle T_{0\mu}(0,t)F_\mu(0,t)\rangle$ a “power input”. In the formal non-relativistic limit $c \to \infty$, this relation reduces in conformal models with sound speed $c_s = c / \sqrt{d}$ to the classical “$12/d(d+2)$th-law” for $d$-dimensional incompressible fluid turbulence (e.g., [21]), but for finite speeds of light the relation has nothing to do with energy of the fluid. As noted earlier, Fouxon & Oz concluded: “Our analysis indicates that the interpretation of the Kolmogorov relation for the incompressible turbulence in terms of the energy cascade may be misleading.”

Needless to say, our analysis contradicts this conclusion. We have already discussed the limitations of point-splitting regularizations in paper I and we shall not repeat that discussion here. We only point out that the anomalies obtained by the point-splitting arguments of [19] are for quantities such as $T_{0\mu}(x)$, which are not conserved quantities even for smooth solutions of relativistic Euler equations and which have no obvious physical significance. The specific quantities are chosen in [19] simply so that a point-splitting regularization applies. One cannot conclude that energy cascade and energy-dissipation anomaly must be absent in relativistic turbulence because a certain regularization method is insufficient to derive them. The alternative coarse-graining regularization employed by us here shows that cascades and dissipative anomalies for both energy and entropy naturally arise in relativistic fluid turbulence. Furthermore, in conformal fluid models with $c_s = c / \sqrt{d}$, the relativistic energy flux $Q^{\mu}_{\ell}^{\text{flux}}$ considered by us reduces in the non-relativistic limit to the standard kinetic energy flux for an incompressible fluid

$$\lim_{c \to \infty} c Q^{\mu}_{\ell}^{\text{flux}} = -\rho_0 \nabla \cdot \tau(v,v)$$

with constant mass density $\rho_0$, following the arguments in Appendix H. Alnie (private communication) has shown that the standard 4/5th-law of Kolmogorov, which is ordinarily derived by point-splitting, can also be obtained from [11] for incompressible Navier-Stokes. Thus, there is no unique way to extend the compressible 4/5th law to relativistic turbulence, but our extension describes energy cascade in the relativistic regime.

To underscore this point, we here briefly discuss the energy balance for forced statistical steady-states of relativistic fluid turbulence. This is a rather artificial setting quite distinct from most real-world relativistic turbulence, e.g. in astrophysics, in which there is no Minkowski force and no ensemble. We have therefore focused in this paper on freely-evolving turbulence and individual flow realizations. However, our considerations carry over directly to forced, steady-state ensembles. Note that the Minkowski force can quite generally be composed as

$$F^\mu = \frac{1}{c^2} h A^\mu_{\text{ext}} - \frac{1}{c} Q_{\text{cool}} V^\mu,$$

with $V_\mu A^\mu_{\text{ext}} = 0$. Here $A^\mu_{\text{ext}}$ is an external acceleration field with units of (length)/(time)$^2$ and $Q_{\text{cool}}$ is a cooling rate density with units of (energy)/(volume)(time). As usual, we include factors of $c$ to facilitate discussion of the non-relativistic limit. The internal energy balance in the presence of a Minkowski force becomes

$$\partial_\mu E^\mu = - p_\theta + Q_{\text{diss}} - \frac{1}{c} Q_{\text{cool}}.$$ (117)

It follows that in a statistically homogeneous and stationary state, one has the fine-grained balance

$$\frac{1}{c} \langle Q_{\text{cool}} \rangle = \langle Q_{\text{trans}} \rangle + \langle Q_{\text{diss}} \rangle$$

(118)

where $Q_{\text{trans}} = -p_\theta$ is the mechanical production of internal energy by pressure-work. Our inertial-range internal-energy balance with the addition of the Minkowski force becomes

$$\partial_\mu E^\mu = - p_\ell \theta + Q^{\mu}_{\ell}^{\text{flux}} + \dot{V}_{\ell,\mu} T^{\mu}_{\ell,\text{ext},\ell},$$

including now the coarse-graining length-scale $\ell$ explicitly. One thus has

$$\langle \dot{V}_{\ell,\mu} T^{\mu}_{\ell,\text{ext},\ell} \rangle = - \langle p_\ell \theta \rangle + \langle Q^{\mu}_{\ell}^{\text{flux}} \rangle = \langle Q_{\text{trans}} \rangle + \langle Q_{\text{diss}} \rangle,$$
where \( Q_{\text{inert}}^{\text{inert}} = Q_f^{\text{flux}} + \tau_i(p, \theta) \) is the total inertial-range effective dissipation from both energy cascade and pressure-work defect and \( Q_{\text{trans}} = -p * \theta. \) At length-scales much smaller than the scale \( L \) of the Minkowski force, \( \langle V_{\mu} T_{\text{ext}, \ell}^{\mu} \rangle \approx (1/c)\langle Q_{\text{cool}} \rangle, \) and
\[
\langle Q_{\text{inert}}^{\text{inert}} \rangle \approx \frac{1}{c} \langle Q_{\text{cool}} \rangle - \langle Q_{\text{trans}} \rangle = \langle Q_{\text{diss}} \rangle, \quad \ell \ll L.
\]
(121)

We thus find that the ideal dissipation rate has constant ensemble-average for scales \( \ell \) in the inertial-range, which equals the energy dissipation rate of the microscopic fluid model. This is formally identical to the statistical energy-balance relation that we obtained in the non-relativistic case, and reduces to it in the limit \( c \to \infty. \)

It is more traditional to expect that the effective energy dissipation rate at inertial-range lengths \( \ell \) is set by the external input of kinetic energy by the large-scale forcing, but, of course, kinetic energy is not a natural relativistic quantity. Analogous constraints arise relativistically from the conditions
\[
\langle F^\mu \rangle = 0,
\]
(122)

which are necessary if a statistically homogeneous and stationary state is to exist for the forced fluid described by \( \mu = 0 \). The \( \mu \neq 0 \) condition gives that
\[
\langle Q_{\text{cool}} \rangle = \frac{1}{c} \langle h A_{\text{ext}}^0 \rangle.
\]
(123)

In the limit \( c \to \infty \) this becomes
\[
\langle Q_{\text{cool}} \rangle \simeq c \langle \rho A_{\text{ext}}^0 \rangle = \langle \rho v \cdot A_{\text{ext}} \rangle
\]
(124)

Here we used orthogonality condition \( A_{\text{ext}}^0 = v \cdot A_{\text{ext}} / c. \) Since the equation of motion projected orthogonal to \( V^\mu \) takes the form \( D V^\mu = (1/c^2) A_{\text{ext}}^0 + \cdots \) in the presence of a Minkowski force, the limit of the spatial components as \( c \to \infty \) becomes \( D v = A_{\text{ext}} + \cdots \). Thus, \( (124) \) is equivalent to the usual non-relativistic relation that \( \langle Q_{\text{cool}} \rangle = \langle Q_{\text{in}} \rangle \), where \( Q_{\text{in}} = \rho v \cdot A_{\text{ext}} \) is the kinetic-energy injection rate per volume by the external forcing. We note in passing that the constraints \( \langle F^\mu \rangle = 0 \) from the spatial components similarly reduce in the non-relativistic limit \( c \to \infty \) to the condition \( \langle \rho A_{\text{ext}} \rangle = 0 \), or no net momentum injection by the external forcing.

In addition to energy balance, there must also be an entropy balance for homogeneous and stationary ensembles. In the presence of a Minkowski force, the fine-grained entropy balance \( (117) \) is found using \( (117) \) to be modified to
\[
\partial_\mu S^\mu = \Sigma_{\text{diss}} - \frac{1}{c} \beta Q_{\text{cool}}.
\]
(125)

Thus, for a homogeneous and stationary ensemble
\[
\langle \Sigma_{\text{diss}} \rangle = \frac{1}{c} \langle \beta Q_{\text{cool}} \rangle
\]
(126)

and microscopic entropy production is balanced by entropy removal by cooling. The inertial-range entropy balance \( (101) \) is likewise modified by a Minkowski force, with the divergence of \( (104) \) using \( (119) \) given by
\[
\partial_\mu \Sigma^\mu = \Sigma_{\text{inert}} - \langle \beta V_\mu T^{\mu}_{\text{cool}} \rangle.
\]
(127)

When the Minkowski force is supported mainly at the large scale \( L \), one obtains the inertial-range mean balance
\[
\langle \Sigma_{\text{inert}} \rangle = \langle \beta V_\mu T^{\mu}_{\text{cool}} \rangle \simeq \frac{1}{c} \langle \beta Q_{\text{cool}} \rangle, \quad \ell \ll L.
\]
(128)

This mean entropy balance is formally the same as \( (1:104) \) for the non-relativistic case and reduces to it in the limit \( c \to \infty. \) The physical picture is also the same as for non-relativistic compressible turbulence, with entropy produced at small scales inverse-cascading through the inertial range up to scales \( \ell \simeq L \) where external cooling can remove the excess entropy. Equivalently (and perhaps more naturally), the negentropy injected by a large-scale cooling will forward cascade to small-scales where irreversible microscopic transport can destroy it. If one makes the distinction in \( (102) \) between negentropy flux and anomalous negentropy input, then one can also write
\[
\langle \Sigma_{\text{inert}} \rangle = \langle \beta V_\mu T^{\mu}_{\text{cool}} \rangle \simeq \frac{1}{c} \langle \beta Q_{\text{cool}} \rangle, \quad \ell \ll L.
\]
(129)

where the negentropy flux proper is equal on average to the total negentropy input at large-scale, both from external cooling and from anomalous negentropy input.

C. Linear Wave-Mode Decompositions

In paper I we have also called into question the validity of representing turbulent solutions by decompositions into linear wave modes. This is a very popular approach in non-relativistic plasma astrophysics and has recently been developed for Poynting-dominated relativistic MHD turbulence \( (94) \). We do not consider charged plasmas in the present paper but only fluids of electrically neutral particles, so that we shall just briefly discuss here the issues with decompositions into linear wave-modes. A basic problem is that thermodynamic relations such as \( p = p(c, \rho) \) and \( s = s(c, \rho) \) impose nonlinear constraints on solutions of the fluid equations, which thus live in non-linear submanifolds of function space. Wave-modes \( \epsilon^i, \rho \) obtained by linearization of the fluid equations around a uniform equilibrium background \( \epsilon_0, \rho_0 \) only satisfy these thermodynamic constraints to linearized level. This may be an adequate representation when fluctuations are relatively small, satisfying \( \epsilon^i/\epsilon_0, \rho^i/\rho_0 \ll 1. \) However turbulence generally produces fluctuations much larger than the means, where this linear approximation to the thermodynamic relations is inadequate. Decomposition into linear wave-modes is thus clearly an approximation, with an unknown range of validity. We note that in conformal fluids with AdS gravity duals, the linear wave-mode decomposition corresponds on the gravity side to the expansion in quasinormal modes about the uniform AdS
black-hole. Expansion in such quasinormal modes has recently been independently argued \cite{16} to be inapplicable to the turbulent regime.

VII. EMPIRICAL PREDICTIONS AND EVIDENCE

High-energy astrophysical plasma flows are probably the best candidates in Nature to exhibit relativistic fluid turbulence, but remote observations of such systems poorly constrain theory. In order to confront theory with precise evidence, the only recourse at the moment is numerical simulations of turbulence for relativistic kinetic equations or dissipative fluid models. We shall here briefly discuss the relations of our work to the existing body of numerical simulations. Confining attention to electrically neutral fluids, as considered in the present work, the most relevant numerical studies have been motivated either by astrophysics \cite{13,14} or by the fluid-gravity correspondence \cite{15,17}. Numerical codes exist for simulating the particle-frame Israel-Stewart model \cite{15,14}, but we are aware of no turbulence simulations so far that exploit such codes. (The only exception is the study of \cite{14} for a very similar second-order dissipative model of conformal fluids in 2 + 1 space-time, discussed further below.) Instead, most studies have solved the relativistic Euler fluid equations using dissipative numerical schemes to remove the energy cascaded to small-scales rather than a physical viscosity.

We first discuss the astrophysically motivated simulations in 3+1 space-times with topology $T^3 \times R$. Zrake & MacFadyen \cite{13} solved the stress-energy equation \cite{15} and the equation \cite{2} for conserved particle-number. They employed a relativistic ideal-gas equation of state $p = (\Gamma - 1)u$ for $\Gamma = 4/3$, and adopted a Minkowski force

$$F^\mu = \rho A^\mu - \rho (u/u_0)^4 V^\mu, \quad (130)$$

with terms representing mechanical stirring and radiative cooling, respectively. The space-resolutions of their simulations were 256$^3$, 512$^3$, 1024$^3$, 2048$^3$ and they had a mean relativistic Mach number of about $Ma = 2.67$. Radice & Rezzolla \cite{14} instead solved only the stress-energy equation \cite{14} for a radiation-pressure dominated fluid with $p = (1/3)\varepsilon$ and with a Minkowski force

$$F^\mu = F_0(t) (0, f^i) \quad (131)$$

for $f^i$ a zero space-average, solenoidal, random vector supported at low-wavenumbers. They performed four runs with $F_0(t) = 1, 2, 5, 10 + (t/2)$, with space-resolutions 128$^3$, 256$^3$, 512$^3$, 1024$^3$, and with relativistic Mach numbers $Ma = 0.362, 0.543, 1.003, 1.759$. The simulations of both groups are consistent with a forward energy cascade, although they had at their disposal no concrete formula such as our equation \cite{61} in order to make a precise measurement of relativistic energy flux.

Both of these groups measured also the scaling exponents $\zeta_{\nu}$ of longitudinal velocity structure functions using the ESS procedure \cite{99}, and \cite{13} measured as well the exponents $\zeta_{\nu}$ for an absolute Minkowski-norm velocity structure-function. Both of these studies found $\zeta_{\nu} \leq p/3$ and $\zeta_{\nu} \leq p/3$ for $p \geq 3$, consistent with our theoretical predictions. The phenomenological model of She-Lévéque \cite{97} was found to be a reasonable approximation to the ESS results for $\zeta_{\nu}^0$, but not for $\zeta_{\nu}^v$ in \cite{13}, which took on smaller values than $\zeta_{\nu}^v$ associated to greater space-time intermittency. When $p < 3$, our analysis makes no theoretical predictions for $\zeta_{\nu}^0$ or $\zeta_{\nu}^v$, aside from the reasonable inference by concavity that $\zeta_{\nu} > p/3$. The direct (non-ESS) measurements of \cite{13} yielded $\zeta_2^0 \approx 1$ (Burgers-like), whereas \cite{14} claimed consistency with $\zeta_2^0 > 2/3$ (K41). This discrepancy could be due to the larger Mach number in the simulations of \cite{13} (see their Figure 1, which shows clear evidence of shocks). On the other hand, the spectra in Fig.2 of \cite{14} at low-wavenumbers are consistent with $\zeta_2^0 > 2/3$ and the higher wave-numbers are plausibly contaminated by bottleneck effects. In our opinion, neither of the simulations \cite{13,14} achieved a long enough inertial range to yield quantitatively reliable results for scaling exponents.

Motivated by black-hole gravitational physics through the fluid-gravity correspondence \cite{8,11}, there have also been simulations of relativistic fluid turbulence in 2 + 1 space-time dimensions, both for free-decaying \cite{13,16} and externally-forced \cite{17} cases. Here, the evolution of low-wavenumber perturbations to black-holes in a D + 1 dimensional, asymptotically AdS space-time is expected to be equivalent to a relativistic hydrodynamics on the $D = d + 1$ dimensional conformal boundary of AdS space. Thus, 3+1 dimensional black-holes correspond to relativistic hydrodynamics in 2+1 dimensions. All of our considerations are independent of the space dimension $d$ and thus apply for $d = 2$, but this case is likely to be substantially more complex than $d > 2$. Even for incompressible fluid turbulence, $d = 2$ is a much richer problem than $d > 2$. For example, freely-decaying and externally-forced incompressible turbulence appear substantially similar for $d > 2$, with both exhibiting an energy-dissipation anomaly. However, the enstrophy-dissipation anomaly predicted for $d = 2$ incompressible turbulence \cite{98,99} appears only in forced turbulence, whereas there is no enstrophy-anomaly for freely-decaying turbulence unless the initial data is very singular \cite{100,101}. Viscous energy dissipation always tends to zero in $d = 2$ incompressible turbulence, but the energy accumulates in large-scales by quite different mechanisms in the two cases: “vortex merger” \cite{28,102} for freely-decaying turbulence and “inverse energy cascade” \cite{98} for forced turbulence. The previously-mentioned simulations of 2+1 relativistic turbulence also seem to indicate that there is no energy-dissipation anomaly there, and that vortex-merger and inverse-cascade processes occur. It should be kept in
mind, however, that all of the discussed simulations are at low relativistic Mach numbers. At higher Mach numbers, shocks will surely proliferate, leading to irreversible energy dissipation and entropy production. Such behavior was observed in \[ d = 2 \] for simulations of \( d = 2 \) non-relativistic compressible turbulence, motivated by large-scale dynamics of galactic accretion disks. We thus believe that the phenomenology of relativistic turbulence will be quite non-universal, depending upon the relativistic Mach number, free-decay vs. forced, precise details of the initial-data, etc.

The simulations cited above already largely support the present work, but our theory makes a rich array of further predictions for relativistic fluid turbulence that are easily subject to empirical test. Chief among these predictions are: (1) anomalous energy dissipation both by local energy cascade and by pressure-work defect; (2) anomalous input of negentropy into the inertial-range by pressure-work, in addition to any external input by large-scale cooling mechanisms; (3) negentropy cascade to small-scales through a flux of intrinsic inertial-range entropy; and (4) singularity or “roughness” of fluid fields to sustain cascades of energy and entropy, so that at least one of the exponent inequalities \[ 105 - 107 \] must hold. The explicit formulas \[ 61 \] for energy flux and \[ 103 \] for intrinsic entropy flux provide quantitative measures of cascades rates in relativistic turbulence. Furthermore, in order to provide mean fluxes of the predicted signs, the expressions \[ 61, 103 \] require specific space-time correlations to develop, e.g. “down-gradient turbulent transport” with \( \tau(h, V^\mu) \), \( \tau(\mu, V^\mu) \) anti-correlated with the thermodynamic gradients \( \partial_x T, \partial_x \lambda \), respectively. These many predictions provide an ample field of study for future numerical investigation.

VIII. SUMMARY AND FUTURE DIRECTIONS

The theory developed in this paper is based upon the hypothesis that relativistic fluid turbulence should exhibit dissipative anomalies of energy and entropy, similar to those observed for incompressible fluids. From this hypothesis alone, we have shown that the high Reynolds- and Péclet-number limit should be governed by distributional or “coarse-grained” solutions of the relativistic Euler equations. We have also demonstrated that precisely characterized singularities or “roughness” of the fluid fields is required to permit dissipative anomalies. The argument closely follows that of Onsager\(^\text{27, 28}\) for incompressible fluids, which we have explained as a non-perturbative application of the principle of renormalization-group invariance.

One of the key open questions is certainly the extension of the present special-relativistic theory to general relativistic (GR) turbulence. There is reason to believe that much of the present theory will carry over straightforwardly to GR, since curved Lorentzian manifolds are locally diffeomorphic to Minkowski space. However, new effects may arise if curvature scales become comparable to inertial-range turbulence scales. The main technical problem in extending our theory to GR is development of suitable “coarse-graining” in curved space-times in order to regularize turbulent ultraviolet divergences. Coarse-graining operations in GR have attracted recent interest also because of problems in cosmology and in the interpretation of cosmological observations, and much of this parallel work \[ 104, 105 \] should carry over to general-relativistic turbulence. Here we may note that an Onsager singularity theorem has already been proved for incompressible fluid turbulence on general compact Riemannian manifolds, by exploiting a coarse-graining regularization defined with a heat kernel smoothing \[ 106 \].

Even in Minkowski space, there are important new directions of study opened by our work. Our quantitative formulas \[ 61 \] for energy flux and \[ 103 \] for entropy flux make possible an exploration of the physical mechanisms of relativistic turbulent cascades \[ 107, 108 \]. The vortex-stretching mechanism of Taylor\(^\text{109}\) is widely believed to drive the \( d = 3 \) incompressible energy cascade, but it is unclear whether such physics carries over to relativistic fluids. The equations of motion with the coarse-grained tensor \[ 53 \] derived in this paper also provide the mathematical foundations for Large-Eddy Simulation (LES) modeling of relativistic turbulence in Minkowski space \[ 75, 77 \]. Such LES holds promise to be an important tool in numerical investigation of local turbulence in high-energy astrophysical events, such as gamma-ray bursts. Finally, there are interesting implications of the present work for black-hole physics, because the fluid-gravity duality connects relativistic fluid-dynamics in \( d + 1 \) Minkowski space-time to Einstein’s equations in a Poincaré patch of a \( (D) + 1 \) dimensional AdS black-hole solution. Thus, when high-Reynolds-number turbulence develops in a relativistic fluid in Minkowski space, our Onsager singularity theorem implies not only that the fluid fields must become “rough”, but also that “rough” metrics must develop in the turbulent solutions of the Einstein equations in the dual gravitational description.

The “roughness” or Hölder-singularity of the turbulent velocity \( V^\mu(x) \) in particular has profound implications for relativistic fluid turbulence. It was pointed out in a landmark work of Bernard et al.\(^\text{110}\) on non-relativistic incompressible turbulence that fluid velocities with Hölder exponent \( h < 1 \) have non-unique Lagrangian particle trajectories. It was shown by those authors in a synthetic model of turbulence that the Lagrangian trajectories become “spontaneously stochastic” in the high Reynolds-number limit, with randomness of trajectories persisting even when the initial particle location and the advecting velocity become deterministic and perfectly specified. It has subsequently been shown that such “spontaneous stochasticity” of Lagrangian particle trajectories holds at Burgers shocks\(^\text{111}\) and is necessary in incompressible Navier-Stokes turbulence for anomalous dissipation of passive scalars \[ 112, 113 \]. These considerations carry over directly to relativistic fluid world-lines \( X^\mu(X_0, \tau) \)
defined by the equations
\[ dX^\mu/d\tau = V^\mu(X(\tau), \tau), \quad X^\mu(0) = X_0^\mu. \] (132)

Because of the Hölder singularities of the turbulent velocity vector predicted by our analysis, the fluid world-lines must become “spontaneously stochastic”, with a random ensemble of world-lines passing through each fixed event \( X_0 \). This implies a turbulent breakdown of the Lagrangian conservation laws that hold for smooth solutions of the relativistic Euler equations, such as the Kelvin Theorem [114, 56], section 3.7.5. Likewise, in relativistic astrophysical plasmas, the Alfven Theorem on magnetic flux-conservation for ideal MHD solutions [113, 116] must be fundamentally altered by spontaneous stochasticity effects. In non-relativistic theory, this fact leads to fast turbulent magnetic reconnection independent of collisional resistivity [44, 117, 118], and our present work implies that the same turbulent mechanisms can act in relativistic magnetic reconnection.

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Appendix A: Derivation of Coarse-Grained Relativistic Euler Equations

In this appendix we give key details of the proof of validity of the relativistic Euler equations in the coarse-grained or “weak” sense for any ideal limits of thermodynamic fields \( \epsilon, \rho, V^\mu \) as \( Pe \to \infty \), \( Re \to \infty \). Most of the argument for the particle-conservation equation has been given in section III. One final estimate was left unproved, involving the Lorentz-invariant norm defined by
\[ \Delta^{\mu\nu} A_\mu A_\nu \]
where \( \Delta^{\mu\nu} = g^{\mu\nu} + V^\mu V^\nu \) projects perpendicular to the relativistic fluid velocity vector \( V^\mu \) with respect to the Minkowski pseudometric. Lorentz-transforming into the fluid rest frame \( A_\mu \to \tilde{A}_\mu \)
\[ \Delta^{\mu\nu} A_\mu A_\nu = |\tilde{A}|^2, \]
coinciding with the standard Euclidean norm of the spatial part of the vector. The above norm is, in fact, only a semi-norm, because \( \Delta^{\mu\nu} V_\mu V_\nu = 0 \). In deriving a bound on the dissipative terms in the coarse-grained conservation laws in section III, we needed an estimate on this semi-norm from above in terms of the Euclidean norm.

To obtain this, we note that
\[ \Delta^{\mu\nu} A_\mu A_\nu = a^\top \Delta a \]
where \( a \) is the \( D \)-dimensional vector with components \( A_\mu \) of the covariant vector and \( \Delta \) is the \( D \times D \)-dimensional matrix with entries \( \Delta^{\mu\nu} \) of the contravariant tensor. We then use the standard bound
\[ |a^\top \Delta a| \leq \|\Delta\|_2 \|a\|^2 \]
(A1)

where \( \|\Delta\|_2 = \sqrt{\rho(\Delta^\top \Delta)} \) and \( \rho(M) \) is the spectral radius of the \( D \times D \) dimensional matrix \( M \) (119, section 2.3). Since \( \Delta \) is real, symmetric, one has furthermore \( \|\Delta\|_2 = \rho(\Delta) \). We thus must compute the eigenvalues of \( \Delta \). This is simply done by an orthogonal transformation, which rotates the spatial part of the vector \( V^\mu = \gamma(v)(1, v/c) \) into the 1-direction. Note that such a purely spatial rotation changes neither the Minkowski pseudonorm nor the Euclidean norm of \( A_\mu \). After this rotation, the matrix \( \Delta \) becomes block-diagonal, with an upper block which is the \( (d-1) \times (d-1) \) identity matrix and an upper block which is the \( 2 \times 2 \) matrix
\[ \Delta_2 = \frac{1}{1 - \beta_v^2} \left( \begin{array}{cc} \beta_v^2 & \beta_v \\ \beta_v & 1 \end{array} \right), \quad \beta_v = v/c. \]
The matrix \( \Delta_2 \) has an eigenvalue 0 with eigenvector \((-1, \beta_v)^\top \) (whose components are obviously those of the covariant vector \( V_\mu = g_{\mu\nu} V^\nu \)) and an eigenvalue \( \frac{1 + \beta_v^2}{1 - \beta_v^2} \) greater than 1 with eigenvector \((\beta_v, 1)^\top \). It follows that \( \rho(\Delta) = \frac{1 + \beta_v^2}{1 - \beta_v^2} \). Finally, noting that \( \|a\|^2 = |A|^2 \), we obtain the bound
\[ |\Delta^{\mu\nu} A_\mu A_\nu| \leq \frac{1 + \beta_v^2}{1 - \beta_v^2} |A|^2 \]
This upper estimate is optimal, in that it can actually be achieved for a suitable space-like vector \( A_\mu \) corresponding to the second eigenvector above. Since \( 1 + \beta_v^2 \leq 2 \), we obtain the bound stated in eq. (43) in the main text.

The dissipative terms in the coarse-grained energy-momentum conservation equation are estimated in a very similar fashion. Here we sketch briefly the bound for the shear-viscosity term, which can be written as
\[ c_\mu \partial_\nu \left( 2\eta \tilde{\tau}^{\mu\nu}(x) \right) = -\frac{1}{T} \int d^D r \ c_\mu(\partial_\nu G)(x) \cdot 2\eta(x + r)\tilde{\tau}^{\mu\nu}(x + r) \]
(A2)
and we have introduced a constant vector \( c^\mu \) which can be set to 1 for any particular component of the equation and zero for the others, in order to select that component. Cauchy-Schwartz applied to this term gives
\[ |c_\mu \partial_\nu \left( 2\eta \tilde{\tau}^{\mu\nu}(x) \right) | \leq \frac{1}{T} \sqrt{\int \text{supp}(G)(x) \ d^D r \ (2\eta T^2)(x + r)} \]
\[ \times \sqrt{\int d^D r \ 2\eta T^2(x + r) c_\mu(\partial_\nu G)(x) \cdot \tilde{\tau}^{\mu\nu}(x + r) |^2} \]
(A3)
The first square-root factor goes to zero in the ideal limit under mild assumptions on \( \eta \) and \( T \), as long as the second square-root factor remains bounded. To estimate the second term we note that for any 2nd-rank covariant tensors \( A_{\mu\nu}, B_{\mu\nu} \) the quantity
\[
\Delta^{\alpha\beta} \Delta^{\gamma\delta} A_{\alpha\gamma} B_{\beta\delta} = \sum_{ij} A'_{ij} B'_{ij}
\]
when the tensors are transformed to \( A'_{\mu\nu}, B'_{\mu\nu} \) in the rest-frame of the fluid. The expression on the right is the standard Frobenius inner product of \( d \times d \) matrices and thus the expression on the left is a degenerate inner product (vanishing whenever either tensor is a product of the form \( V_\mu C_\nu \) or \( C_\mu V_\nu \)). Employing the Cauchy-Schwartz inequality for this degenerate inner-product gives
\[
|c_\mu (\partial_\nu G)_{\ell}(r) \cdot \hat{\tau}^{\mu\nu}(x+r)|^2 \\
\leq (c_\mu^2 c_\eta^2) (\partial_\nu^2 G)_{\ell}(r) \cdot \hat{\tau}^{\mu\nu}(x+r). \tag{A4}
\]
The above inequality yields the following upper bound for the integral under the second square-root
\[
(c_\mu^2 c_\eta^2) \int d^D r \ (\partial_\nu^2 G)_{\ell}(r) \cdot \frac{2\eta \hat{\tau}^{\mu\nu}(x+r)}{T^2}. \tag{A5}
\]
Now using equation (A3) in the main text and the similar inequality
\[
0 \leq (c_\mu^2 c_\eta^2) \leq 2\gamma^2(v)|c_\eta|^2 \tag{A6}
\]
we obtain our final estimate for the integral under the second square-root
\[
4|c_\eta|^2 \gamma(v) \int d^D r \ (|\partial G|_{\ell}(r))^2 \frac{2\eta \hat{\tau}^{\mu\nu}(x+r)}{T^2}. \tag{A7}
\]
This upper estimate converges in the ideal limit to
\[
4|c_\eta|^2 \gamma(v) \int d^D r \ (|\partial G|_{\ell}(r))^2 \Sigma_\eta(x+r) \tag{A8}
\]
and thus remains bounded. We conclude that the shear-viscosity term in the coarse-grained energy momentum equation vanishes in the ideal limit.

Similar results are obtained for the bulk-viscosity term in the coarse-grained energy-momentum equation using the identity
\[
c_\mu \partial_\nu \left( \frac{\hat{\tau}^{\mu\nu}(x)}{T} \right) = -\frac{1}{T} \int d^D r \ c_\mu \ (\partial_\nu^2 G)_{\ell}(r) \cdot \hat{\zeta}(x+r) \hat{\tau}(x+r) \tag{A9}
\]
and for the thermal-conductivity term using
\[
c_\mu \partial_\nu \left( \kappa Q^{\mu}_V \right) = \\
-\frac{1}{T} \int d^D r \ k(x+r) c_\mu \ Q^{\mu}_V(x+r) \cdot (\partial_\nu G)_{\ell}(r) V^{\nu}(x+r) \\
-\frac{1}{T} \int d^D r \ c_\mu V^{\mu}_V(x+r) \cdot k(x+r) (\partial_\nu G)_{\ell}(r) Q^{\nu}(x+r). \tag{A10}
\]
The bulk-viscosity term is treated very similarly to the shear-viscosity term. For the thermal-conductivity term we need to use the standard Cauchy-Schwartz inequality \( |c_\mu V^{\mu}_V| \leq |c_\mu| |V^{\mu}_V| \) and the following estimate for the Euclidean norm of the fluid velocity vector:
\[
|V^{\mu}_V|^2 = \gamma^2(v)(1+v^2/c^2) \leq 2\gamma^2(v). \tag{A11}
\]
The details are straightforward and left to the reader.

**Appendix B: Non-Relativistic Limit**

Space-time coarse-graining kernels in relativistic theory \( G(r) = G(r^0, r) \) and in non-relativistic (Newtonian) theory \( G_N(r, \tau) \) are related by a simple change of dimensions through scaling with \( c \):
\[
G(r, r^0) = (1/c)G_N(r, r^0/c). \tag{A12}
\]
Thus,
\[
\mathcal{F}(x) = \int d^D r \ G(r) f(x+r) \\
= \int d^D r \int d\tau \ G_{N,\ell}(r, \tau) f(x+r, t+\tau) \tag{B1}
\]
and there is no need to distinguish between \( \mathcal{F} \) and \( \mathcal{F}^N \) as \( c \to \infty \). This is not true in general for more singular coarse-graining in space-time. Consider as an example the backward light-cone average with
\[
G(r) = G(r)\delta(r^0 + |r|). \tag{A13}
\]
In that case
\[
\mathcal{F}(x) = \int d^D r \ G(r)f(x+r, t-|r|/c) \tag{A14}
\]
Then in the limit \( c \to \infty \)
\[
\mathcal{F} = \mathcal{F}^N = \frac{1}{c} \int d^D r \ G_{N}(r) f(x+r, t) + O\left(\frac{1}{c^2}\right) \tag{B2}
\]
where
\[
\mathcal{F}^N(x, t) = \frac{1}{c} \int d^D r \ G_{N}(r) f(x+r, t) \tag{B3}
\]
is the non-relativistic instantaneous spatial coarse-graining. In this case, \( \mathcal{F} \) and \( \mathcal{F}^N \) are distinct. We shall assume hereafter a smooth space-time coarse-graining.

Even with smooth space-time coarse-graining, the relativistic and non-relativistic Favre-averages are distinct, because, respectively,
\[
\bar{f} := \mathcal{F}/\mathcal{H}, \quad \bar{f}^N := \mathcal{F}^N/\mathcal{P} \tag{B4}
\]
where the first is weighted by \( h = \rho c^2 + h_N \), with \( h_N = u + p \) the non-relativistic (Newtonian) enthalpy, and the
second weighted by $\rho$. Straightforward Taylor-expansion in $1/c^2$ gives

$$
\tilde{f} = \tilde{f}^N + \frac{1}{c^2 \rho^2} (f \rho - \tilde{f} \rho \tilde{h} N) + O\left(\frac{1}{c^3}\right)
$$

While the relativistic and non-relativistic Favre averages are distinct, they do agree to leading order in $1/c^2$.

With these preliminaries, we now consider the formal non-relativistic limit of $c \to \infty$. We note the standard relations:

$$
\partial_\mu = (\frac{1}{c} \partial_\nu, \nabla)
$$

$$
V^\mu = (1 + \frac{1}{2c^2} |v|^2 + O(\frac{1}{c^4}), \frac{1}{c} v + O\left(\frac{1}{c^3}\right)
$$

$$
D = V^\mu \partial_\mu = \frac{1}{c} D + O\left(\frac{1}{c^3}\right).
$$

with $D = \partial_t + v \cdot \nabla$, and

$$
\theta = \partial_\mu V^\mu = \frac{1}{c} \Theta + O\left(\frac{1}{c^3}\right),
$$

with $\Theta = \nabla \cdot v$, or, more generally,

$$
\partial_\mu (f V^\mu) = \frac{1}{c} [\partial_t f + \nabla \cdot (f v)] + O\left(\frac{1}{c^3}\right).
$$

Furthermore, because cumulants of constants vanish, we have relations such as

$$
\tilde{\tau}(V^0, f_2, f_3, ..., f_n) = \frac{1}{2c^2} \tilde{\tau}(|v|^2, f_2, f_3, ..., f_n)
$$

$$
\tilde{\tau}(V^0, V^0, f_3, ..., f_n) = \left(\frac{1}{2c^2}\right)^2 \tilde{\tau}(|v|^2, |v|^2, f_3, ..., f_n)
$$

and so forth. The same relations hold also for Favre cumulants, just replacing $\tau$ by $\tilde{\tau}$.

1. Inertial-Range Energy Balance

We consider first the energy balance [60] or

$$
\partial_\mu E^\mu = -\tilde{\tau} \tilde{U} + Q_{\tau}^{\text{flux}}
$$

Note that $\epsilon V^\mu = c^2 J^\mu + u V^\mu$, so that

$$
\partial_\mu \tilde{\epsilon} V^\mu = \partial_\mu \tilde{u} V^\mu = \frac{1}{c} [\partial_t (\vec{\tau} + \nabla \cdot (\vec{\tau} + \vec{\tau}(u, v))] + O\left(\frac{1}{c^3}\right).
$$

By the results [B3], [B7], [B8]

$$
\tilde{\tau}(p, V^\mu) = (O\left(\frac{1}{c^3}\right), \frac{1}{c} \tilde{\tau}(p, v) + O\left(\frac{1}{c^3}\right)),
$$

$$
\frac{1}{2} \tilde{\tau}(V_\nu, V^\nu) \tilde{V}^\mu =
\left(\frac{1}{2}(\frac{1}{c^2})\tilde{\tau}^N(v_i, v_i) + O\left(\frac{1}{c^3}\right), \frac{1}{c^3} \tilde{\tau}^N(v_i, v_i) v + O\left(\frac{1}{c^3}\right),
\right)
$$

$$
\tilde{\tau}(V_\nu, V^\nu, V^\mu) = \left(O\left(\frac{1}{c^3}\right), \frac{1}{c^3} \tilde{\tau}^N(v_i, v_i, v) + O\left(\frac{1}{c^3}\right),
\right)
$$

and

$$
\tilde{\tau}(V_\nu, V^\mu) \tilde{V}^\nu = \left(O\left(\frac{1}{c^3}\right), -\frac{1}{c^3} \tilde{\tau}^N(v_i, v) v + O\left(\frac{1}{c^3}\right),
\right)
$$

Putting all of these results together with the formula [57] for $\tilde{E}^\nu$ and $\tilde{\tau} = \tilde{\rho} c^2 + \tilde{\tau}^N$, gives

$$
\partial_\mu \tilde{E}^\mu \approx \frac{1}{c} \partial_t (\vec{\tau} + \frac{1}{2} \tilde{\tau}^N(v_i, v_i))
$$

$$
+ \frac{1}{c} \nabla \cdot (\vec{\tau} \vec{v} + (\vec{\tau}(h, v) + \frac{1}{c} \tilde{\tau}^N(v_i, v_i) \tilde{v}^N + \frac{1}{2} \tilde{\tau}^N(v_i, v_i, v)).
$$

Now consider relativistic energy flux given by

$$
Q_\tau^{\text{flux}} = \frac{1}{h} (\partial_\nu \vec{\rho}) \tilde{\tau}(h, V^\nu)
$$

$$
- \frac{1}{h} (\partial_\nu \vec{V}_\nu) \tilde{\tau}(V^\mu, V^\nu) - \frac{1}{c} \partial_\nu \vec{V}^\nu \tilde{\tau}(V_\mu, V^\nu).
$$

Easily from previous estimates

$$
\frac{1}{h} (\partial_\nu \vec{\rho}) \tilde{\tau}(h, V^\nu) \approx \frac{1}{c \tilde{\rho}} \nabla \cdot \tilde{\rho} \tilde{\tau}(\rho, v).
$$

Next observe that

$$
\tilde{\tau}(V^\mu, V^\nu) = \left[\frac{O\left(\frac{1}{c^2}\right)}{O\left(\frac{1}{c^2}\right)}\right] \tilde{\tau}^N(v, v)
$$

and

$$
\partial_\mu \tilde{V}^\nu = \left[\frac{O\left(\frac{1}{c^2}\right)}{O\left(\frac{1}{c^2}\right)}\right] \tilde{\tau}^N(v, v),
$$

so that

$$
\vec{h}(\partial_\mu \tilde{V}^\nu) \tilde{\tau}(V^\mu, V^\nu) \approx \frac{1}{c \tilde{\rho}} \nabla \cdot \tilde{\tau}^N(v, v)
$$

For the last term, use $h V^\nu = c^2 J^\nu + h \nu V^\nu$ to obtain

$$
\partial_\nu \vec{h} V^\nu \approx \frac{1}{c} [\partial_\nu h \nu + \nabla \cdot (h \nu v)].
$$

Since also

$$
\frac{1}{2} \tilde{\tau}(V_\nu, V^\nu) \approx \frac{1}{2c^2} \tilde{\tau}^N(v_i, v_i),
$$
we thus find
\[ \frac{1}{2} \partial_t \rho \nabla \cdot \tilde{\tau}(v, V) = O \left( \frac{1}{c^4} \right). \]  
(B23)

In conclusion,
\[ Q_{\ell}^{\text{flux}} \simeq \frac{1}{c \rho} \nabla \cdot \tilde{\rho} \tilde{\tau}(\rho, v) - \frac{1}{c} \tilde{\rho} \nabla \cdot \tilde{\tau}(\rho, v) = \frac{1}{c} Q_{\ell}^{\text{flux}} \]  
(B24)

where \( Q_{\ell}^{\text{flux}} \) is the non-relativistic energy flux.

From the results (B15), (B24), and \( \tilde{\rho} \tilde{h} \simeq (1/c) \tilde{\rho} \tilde{\Theta} \), we thus obtain as the non-relativistic limit of the inertial-range internal-energy balance for relativistic Euler that
\[ \partial_t \left( \tilde{\rho} + \frac{1}{2} \tilde{\tau}^N (v, v) \right) + \nabla \cdot \left( \tilde{\rho} \nabla \tilde{\tau}(h, v) \right) + \frac{1}{2} \tilde{\rho} \tilde{\tau}^N (v, v) \tilde{v}^N + \frac{1}{2} \tilde{\tau}^N (v, v) = -\tilde{\Theta} + Q_{\ell}^{\text{flux}}. \]  
(B25)

This is nothing other than the non-relativistic balance equation for intrinsic large-scale internal energy, obtained in equation (I:57) of paper I.

There is no natural (covariant) relativistic analogue of the large-range kinetic-energy balance (I:41) for non-relativistic compressible turbulence. On the other hand, in any fixed inertial-frame, it is easy to see that the time-component of the relativistic Euler equation when coarse-grained
\[ \partial_\tau T^{\mu \nu} = 0, \]  
(B26)
yields in the limit \( c \to \infty \) the coarse-grained conservation of mass
\[ \partial_t \tilde{\rho} + \nabla \cdot \tilde{\rho} \tilde{v} = 0 \]  
(B27)
to order \( O(c) \) and the coarse-grained balance of total non-relativistic energy
\[ \partial_t \left( \frac{1}{2} \rho |v|^2 + u + \nabla \cdot \left( \frac{1}{2} \rho |v|^2 + u + p \right) \right) v = 0 \]  
(B28)
to order \( O(1/c) \). If one subtracts (B25) from the latter equation (B28), then one obtains
\[ \partial_t \left( \frac{1}{2} \rho |\tilde{v}|^2 \right) + \nabla \cdot \left[ (\tilde{p} + \frac{1}{2} \tilde{\tau}^N) \tilde{v}^N + \tilde{\rho} \tilde{\tau}^N (v, v) \cdot \tilde{v}^N - \frac{\tilde{\rho}}{\tilde{v}} \tilde{\tau}(\rho, v) \right] = \tilde{\rho} \tilde{\Theta} - Q_{\ell}^{\text{flux}}. \]  
(B29)

In this manner, the inertial-range kinetic energy balance equation (I:41) of non-relativistic turbulence can be recovered as \( c \to \infty \) from the relativistic theory.

### 2. Inertial-Range Entropy Balance

We now consider the intrinsic inertial-range entropy current in the Favre formulation, \( S^{\mu \nu} = S^{\mu \nu} - \beta \kappa \), and its balance equation
\[ \partial_\mu S^{\mu \nu} = -I_{\ell}^{\text{flux}} + \Sigma_{\ell}^{\text{flux}, s}. \]  
(B30)
with the intrinsic negentropy flux
\[ \Sigma_{\ell}^{\text{flux}, s} = \Sigma_{\ell}^{\text{flux}} - (\partial_\mu \beta) K^{\mu} + \frac{\beta}{\mu} Q_{\ell}^{\text{flux}}. \]  
(B31)

First we note a standard difference between relativistic and Newtonian thermodynamics, due to the distinction between rest-mass and energy in the latter:
\[ \epsilon = \rho c^2 + u, \quad \lambda = \beta m c^2 + \lambda_N. \]  
(B32)

See 59 or 50, §2.3.6. Using these relations, one easily finds that
\[ \tilde{S}^{\mu \nu} = \tilde{\rho} \nabla \tilde{\tau}(u, V) - \lambda_N \tilde{\tau}(u, V) \]
\[ = \left( \tilde{\rho} + \frac{\beta}{c^2}, \frac{1}{c} \left[ \nabla \tilde{\tau}(u, v) - \lambda_N \tilde{\tau}(u, v) \right] + O \left( \frac{1}{c^3} \right) \right). \]  
(B33)

On the other hand, it follows directly from the formula (80) for \( K^{\mu} \) and the estimates in the previous subsection that
\[ \tilde{K}^{\mu} = \left( \frac{1}{2} \tilde{\rho} \tilde{\tau}^N (v, v) + O \left( \frac{1}{c^2} \right), \right. \]
\[ \frac{1}{2} \tilde{\rho} \tilde{\tau}^N (v, v) \tilde{v}^N + \frac{1}{2} \tilde{\tau}^N (v, v) \tilde{v}^N + \frac{1}{c} \tilde{\tau}(\rho, v) + O \left( \frac{1}{c^3} \right), \]  
(B34)

As an aside, we note that this last result implies that the balance equation (70) for \( K^{\mu} \) reduces in the limit \( c \to \infty \) to the non-relativistic balance equation (I:64) for the sub-scale kinetic-energy. We finally obtain that
\[ \partial_\mu S^{\mu \nu} \simeq \frac{1}{c} \partial_\tau S^{\tau} + \nabla \cdot \tilde{\tau} \tilde{S}^{\tau} \]  
(B35)
where
\[ \tilde{S}^{\tau} = \tilde{\rho} + (1/2) \tilde{\rho} \tilde{\tau}^N (v, v) \]  
(B36)
is the non-relativistic intrinsic inertial-range entropy and
\[ \tilde{S}^{\tau} = \tilde{\rho} \tilde{v} + \tilde{\tau}(h, v) - \lambda_N \tilde{\tau}(u, v) \]
\[ + \frac{\beta}{c} \left[ \tilde{\rho} \tilde{\tau}^N (v, v) \tilde{v}^N + \frac{1}{2} \tilde{\tau}^N (v, v) \tilde{v}^N \right], \]  
(B37)
is the associated spatial-current. See (I:94),(I:96).

On the other hand, using again the relation (B32) between relativistic and Newtonian thermodynamic quantities, one finds that
\[ \Sigma_{\ell}^{\text{flux}, s} = \left( \partial_\mu \tilde{\tau}(u, V) - (\partial_\mu \lambda_N) \tilde{\tau}(u, V) \right) \]
\[ \simeq \frac{1}{c} \left[ \nabla \tilde{\tau}(u, v) - \nabla \lambda_N \tilde{\tau}(u, v) \right] \]
\[ = \frac{1}{c} \Sigma_{\ell}^{\text{flux}, N}. \]  
(B38)
where \( \Sigma_{\ell}^{\text{flux}, N} \) is the (naive) entropy flux in non-relativistic compressible turbulence. Directly from (B34).
and the asymptotics for \( Q^\text{flux}_t \) in the previous subsection, one finds that
\[
\beta Q^\text{flux}_t - (\partial_\mu \tilde{E}) K^\mu \approx \frac{1}{c^2} \beta Q^\text{flux}_t + \frac{1}{2c} (\partial_\mu \beta \tilde{\rho} T^N(v_i, v_i) + \frac{1}{2c} \nabla \beta \cdot \left( \frac{1}{2} \tilde{\rho} T^N(v_i, v_i) \nabla N + \frac{1}{2} \tilde{\rho} T^N(v_i, v_i, v) + \tilde{\gamma}(p, v) \right)
\]
(B39)

This corresponds exactly to eq.(I.90) in the non-relativistic theory. Finally, the very simple equality
\[
I^\text{flux}_t = \beta (\nabla - \vartheta) \vartheta \approx \frac{1}{c^2} \beta (\nabla - \vartheta) \vartheta = \frac{1}{c^2} I^\text{flux}_t N
\]
(B40)
shows that the relativistic inertial-range entropy balance (101) reduces in the limit \( c \to \infty \) to the balance (I.95) of non-relativistic inertial-range entropy.

**Appendix C: Fine-Grained Balances of Internal Energy and Entropy in the Ideal Limit**

In this appendix we derive the balances of internal energy and entropy for the relativistic Euler solutions by considering directly the ideal limit of the fine-grained balances from the dissipative fluid model.

We begin by considering \( \partial_\mu \tilde{E}^\mu \) with the particle-frame energy current in Eq. (C7), or \( \tilde{E}^\mu = E^\mu + \kappa \tilde{Q}^\mu \). We must show that the contribution of the second term vanishes distributionally in the limit \( \kappa, \eta, \zeta \to 0 \). After smearing with a general test function \( \varphi \), a straightforward estimate by a Cauchy-Schwartz inequality gives
\[
\int d^D x \left( \partial_\mu \varphi \right) \kappa \tilde{Q}^\mu \leq \int_{\text{supp}(\varphi)} d^D x \kappa T^2 \int d^D x \left( \partial_\mu \varphi \partial_\mu \right) \kappa \tilde{Q}^\mu \frac{T^2}{T^2},
\]
(C1)
using (A6) and the definition \( \Sigma_\kappa = \kappa \tilde{Q}^\mu \tilde{\gamma}^\mu / T^2 \) to obtain the last estimate. The second integral inside the square root is bounded when \( \Sigma_\text{term} = D- \lim_{\eta, \zeta, \sigma \to 0} \Sigma_\kappa \) exists, while the first integral vanishes in the limit. We conclude that \( \partial_\mu (\kappa \tilde{Q}^{\mu}) \overset{D-}{\to} 0 \) as \( \kappa, \eta, \zeta \to 0 \).

We next consider \( \partial_\mu S^\mu \) with the entropy current given by the energy-frame Israel-Stewart formula Eq. (B6). We must show that only the term \( \partial_\mu (s V^\mu) \) survives in the ideal limit and that all of the direct dissipative contributions vanish distributionally. The easiest to treat is the \( \Sigma_\eta N^\mu \) term in \( S^\mu \), which gives a vanishing contribution by the same argument used above for \( \kappa \tilde{Q}^{\mu} \).

The terms \( \kappa \tilde{\gamma} \Sigma_{\kappa} V^\mu \), \( (1/2) \zeta \partial_\kappa \Sigma_{\zeta} V^\mu \), \( (1/2) \eta \sigma_1 \Sigma_{\sigma} V^\mu \) all give contributions to \( \partial_\mu S^\mu \) that are bounded in the same manner. We thus consider only the first. After smearing by a test function \( \varphi \), its contribution is bounded by
\[
\left| \int d^D x \left( \partial_\mu \varphi \right) V^\mu \eta \beta_2 \Sigma_\eta \right| \leq \sqrt{2} \max_{\| \gamma (v) \|_{L^\infty}} \int d^D x \left| \partial_\mu \varphi \right| \left| \eta \beta_2 \Sigma_\eta \right| \leq \sqrt{2} \max_{\| \gamma (v) \|_{L^\infty}} \int_{\text{supp}(\varphi)} d^D x \eta \beta_2 \Sigma_\eta \left( \int d^D x \left| \partial_\mu \varphi \right| \right)^2 \Sigma_\eta \)
\]
(C2)
using \( |\partial_\mu \varphi V^\mu| \leq |\partial_\mu \varphi| |V|_{L^\infty} \) and (A1) for the first inequality, and Cauchy-Schwartz for the second. Since \( \Sigma_\text{shear} = D- \lim_{\eta, \zeta, \sigma \to 0} \Sigma_\eta \), the second square-root factor is bounded. For the first square-root factor note that
\[
\int_{\text{supp}(\varphi)} d^D x \eta \beta_2 \Sigma_\eta \leq \max_{\| \gamma (v) \|_{L^\infty}} \int_{\text{supp}(\varphi)} d^D x \psi \Sigma_\eta(\psi),
\]
(C3)
follows also from \( \Sigma_\text{shear} = D- \lim_{\eta, \zeta, \sigma \to 0} \Sigma_\eta \). Finally, since \( \| \eta \beta_2 \|_{L^\infty(\text{supp}(\varphi))} \to 0 \) as \( \sigma, \eta, \zeta \to 0 \), the upper bounds (C2)-(C4) show that the entire contribution vanishes in the ideal limit.

The terms \( \eta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \), \( \zeta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \) also give contributions that are both bounded similarly and we consider only the first. After smearing with a test function, we find that
\[
\int d^D x \left( \partial_\mu \varphi \right) \eta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \leq \int_{\text{supp}(\varphi)} d^D x \left| \partial_\mu \varphi \right| \left| \eta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \right| \Sigma_\eta(\psi),
\]
(C5)
by Cauchy-Schwartz and the definitions of \( \Sigma_\eta, \Sigma_\sigma \). Then
\[
\int d^D x \left( \partial_\mu \varphi \right) \left| \partial_\mu \varphi \right| \Sigma_\sigma \leq \max_{\| \gamma (v) \|_{L^\infty}} \int d^D x \left| \partial_\mu \varphi \right| \Sigma_\sigma
\]
(C6)
using (A6) and
\[
\int_{\text{supp}(\varphi)} d^D x \eta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \leq \| \eta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \|_{L^\infty(\text{supp}(\varphi))} \Sigma_\eta(\psi).
\]
(C7)
The term \( \Sigma_\eta(\psi) \) is bounded as in (C4). In the ideal limit \( \| \eta \sigma_2 \tilde{\gamma}^\mu \tilde{N}_\nu \|_{L^\infty(\text{supp}(\varphi))} \to 0 \) and thus the bounds (C5)-(C7) imply that the contribution to \( \partial_\mu S^\mu \) vanishes distributionally as \( \sigma, \eta, \zeta \to 0 \).

We conclude that \( \partial_\mu S^\mu \overset{D-}{\to} 0 \) as \( \sigma, \eta, \zeta \to 0 \) when \( S^\mu \) is given by the energy-frame formula Eq. (B6). The argument for the entropy current of the particle-frame Israel-Stewart theory is identical, with the replacements \( \sigma \to \kappa, \tilde{N}^\mu \to \tilde{Q}^\mu \).
Appendix D: Relativistic Shock Solutions

1. Reduced Conformal Model and Shock Solution

We consider here an exact family of shock solutions for dissipative relativistic fluid models in 1+1 space-time dimensions, which were obtained in the previous work of Liu & Oz [21]. The 1+1 fluid models considered by those authors are reduced conformal fluids (RCF’s) obtained from a \( D = (d + 1) \)-dimensional conformal fluid (note that our \( D \) is instead denoted \( 2\sigma \) in [21]) and have corresponding dimensionally-reduced gravity duals [120]. We recall that the equation of state for the pressure in \( D \)-dimensional conformal fluids is given by a power of the temperature

\[
p = \alpha T^D, \tag{D1}
\]

with a dimensionless constant \( \alpha \). The tracelessness of the stress-energy tensor requires an energy density

\[
\epsilon = (D - 1)p = \alpha(D - 1)T^D. \tag{D2}
\]

There is no additional conserved current \( J^\mu \) in the RCF’s considered by [21] and consequently \( \lambda = 0 \). The resulting first law of thermodynamics \( de = Tds \) as well as the homogenous Gibbs relation \( h = \epsilon + p = sT \) imply that the entropy density is:

\[
s = \alpha DT^{D-1} - D\alpha^{1/D}p(D-1)/D. \tag{D3}
\]

In the energy frame description, the non-ideal part of the stress-tensor \( \Pi^{\mu\nu} \) is transverse to the velocity, \( \nu_{\mu}\Pi^{\mu\nu} = 0 \). As in [21], we consider only first-order terms in the gradient-expansion. Since bulk viscosity \( \zeta \equiv 0 \) for conformal fluids, the only transport coefficient at this order is shear viscosity \( \eta \) with \( \Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} \). Upon reduction to 1 + 1 dimensions, this appears as an effective bulk viscosity, so that

\[
\Pi^{\mu\nu} = -\zeta \theta \Delta^{\mu\nu} \tag{D4}
\]

with \( \zeta = \frac{1}{2 \pi} \frac{D-3}{D-2} \). However, just as in [21], we take \( \zeta := \zeta(T) \) to be an arbitrary function, since none of our results depend upon any particular choice.

Representing the two-velocity as \( V^\mu = \gamma_v(1, \beta_v) \), any stationary solution of the 1+1 viscous model satisfies:

\[
\frac{d}{dx} \left[ (Dp - \zeta \theta) \gamma_v^2 \beta_v \right] = 0, \tag{D5}
\]

\[
\frac{d}{dx} \left[ p(1 + D\gamma_v^2\beta_v^2) - \zeta \theta \gamma_v^2 \right] = 0. \tag{D6}
\]

Equations \( \text{[D5]} \) and \( \text{[D6]} \) follow from \( \nabla_{\mu} T^{\mu\nu} = 0 \) setting \( \nu = 0,1 \) and they imply:

\[
f_e = (Dp - \zeta \theta) \gamma_v^2 \beta_v \tag{D7}
\]

\[
f_p = p \left( 1 + D\gamma_v^2\beta_v^2 \right) - \zeta \theta \gamma_v^2 \tag{D8}
\]

where \( f_e \equiv T^01 \) and \( f_p \equiv T^11 \) are constant energy and momentum fluxes. Using \( \text{[D7]} \) and \( \text{[D8]} \), [21] obtained smooth viscous shock solutions by quadrature. We shall not employ these integral expressions, but only use the following important consequences of \( \text{[D7]} \), \( \text{[D8]} \):

\[
\epsilon = (D - 1)p = \frac{f_e}{\beta_v} - f_p. \tag{D9}
\]

\[
p - \zeta \theta = f_p - f_e \beta_v. \tag{D10}
\]

In particular, the representation \( \text{[D9]} \) of the pressure in terms of the velocity is analogous to the Bernoulli-type relation exploited by Becker to study shock solutions of the non-relativistic compressible Navier-Stokes equations for \( Pr = 3/4 \) [121]. Together, \( \text{[D9]} \) and \( \text{[D10]} \) completely determine \( \zeta \theta \) in terms of the velocity, yielding identical results for any choice of viscosity \( \zeta(T) \).

The viscous model solutions of interest converge in the infinite Reynolds-number limit to stationary shock solutions of the relativistic Euler equations. These are piecewise constant, with a pre-shock velocity \( \beta_0 \) to the left, and post-shock value \( \beta_1 \) to the right. The possible values are obtained by equating the two expressions for the pressure from \( \text{[D9]} \) and \( \text{[D10]} \) with \( \zeta = 0 \):

\[
f_e / \beta_v - f_p = (D - 1)p = (D - 1)[f_p - f_e \beta_v]. \tag{D11}
\]

This yields a quadratic polynomial in \( \beta_v \), with coefficients depending upon \( D \) and \( R := f_p / f_e \). The condition for two distinct real roots is \( |R| > 2(D - 1)^{1/2}/D \). The product of the roots is given by

\[
\beta_0\beta_1 = 1/(D - 1) = \beta_s^2, \tag{D12}
\]

where \( \beta_s = c_s/c \) and \( c_s = c/\sqrt{D-1} \) is the sound speed. The condition \( h = \epsilon + p = D \cdot p > 0 \) requires that both sides of \( \text{[D11]} \) be positive. Using the quadratic formula for the roots, it is easy to check that this holds if and only if \( |R| < 1 \). The simultaneous conditions

\[
1 > |R| > 2(D - 1)^{1/2}/D \tag{D13}
\]

require \( D > 2 \) in order for inviscid shock solutions to exist. A relation between pressures \( p_0, p_1 \) or temperatures \( T_0, T_1 \) on both sides of the shock can be obtained by using \( \text{[D7]} \) for \( \zeta = 0 \), which gives

\[
p_0/p_1 = (T_0/T_1)^D = (\beta_1 \gamma_1^2 / \beta_0 \gamma_0^2). \tag{D14}
\]

Equations \( \text{[D12]} \) and \( \text{[D14]} \) imply that the fluid on one side of the shock has supersonic velocity and lower temperature, whereas the other side is subsonic with higher temperature. As noted in [21], positive entropy production requires that colder, supersonic fluid flows into the shock front and hotter, subsonic fluid flows out.

We derive here all of the source terms which appear in the internal energy and entropy balances for these shock solutions, both those in the fine-grained (dissipation-range) balances as \( \zeta \to 0 \) and those in the coarse-grained (inertial-range) balances as \( \ell \to 0 \). It should be pointed out that first-order dissipative relativistic fluid models of
the type considered are acausal and have unstable solutions even at global equilibrium [60]. Thus, the viscous shock solutions obtained by [21] are expected to be unstable to small perturbations. However, they are exact stationary solutions that as \( \zeta \to 0 \) converge in \( L^p \) norms for any \( p \in [1, \infty) \) to stationary shock solutions of relativistic Euler equations, and thus provide an example for our general mathematical framework. We emphasize that the viscous model solutions are employed only to evaluate dissipation-range quantities, whereas all of our inertial-range limit results hold with complete generality for all relativistic Euler shocks with the equation of state [D1]. Inviscid solution fields are all discontinuous step-functions

\[
    f(x) = \begin{cases} 
        f_0 & x < 0 \\
        f_1 & x > 0
    \end{cases} = f_0 + (\Delta f)\theta(x). \tag{D15}
\]

where \( \Delta f = f_1 - f_0 \) and \( \theta(x) \) is the Heaviside step function. We shall also use the notation \( f_{av} = \frac{1}{2}(f_0 + f_1) \) for the average value on both sides of the shock. A fact that we shall use frequently for ideal step-function fields is

\[
    \tilde{f}(x) = f_0 + (\Delta f)\tilde{\theta}(x), \quad \tilde{g}(x) = g_0 + (\Delta g)\tilde{\theta}(x) \tag{D16}
\]

and thus

\[
    \tilde{g} = g_0 + \frac{\Delta g}{\Delta f}(\tilde{f} - f_0), \quad \partial_x \tilde{g} = \frac{\Delta g}{\Delta f} \partial_x \tilde{f}. \tag{D17}
\]

Furthermore,

\[
    \partial_x \tilde{f}(x) = (\Delta f)\tilde{\theta}(x). \tag{D18}
\]

The coarse-graining that is employed here is purely spatial, with a kernel \( G \). Because the solutions are stationary in the rest-frame of the shock, there is no need for temporal coarse-graining.

2. Energy Balance

a. Dissipation Range

It can be easily shown for stationary shocks of these RCF’s that \( Q_{\text{diss}} \) and \( p * \theta \) exist as distributions separately, not just in combination. The fine-grained energy balance equation [39] in the \( \zeta \to 0 \) limit thus reads:

\[
    \partial_x (\gamma v \beta_v) = Q_{\text{diss}} - p * \theta. \tag{D19}
\]

We now calculate the two distributions \( Q_{\text{diss}} \) and \( p * \theta \) appearing above as sources/sinks of the energy density.

\textbf{Viscous Pressure-work} \( p * \theta \): Direct differentiation yields the dilatation factor:

\[
    \theta := \partial_x (\gamma v \beta_v) = \gamma_v^3 \partial_x \beta_v. \tag{D20}
\]

and making use of the Bernoulli relation [D9] for the pressure, one obtains:

\[
    (D - 1)p\theta = \left( \frac{f_e}{\beta_v} - f_p \right) \gamma_v^3 \partial_x \beta_v. \tag{D21}
\]

It is straightforward to check that the right-hand-side of (D21) can be expressed as a total \( x \)-derivative:

\[
    (D - 1)p\theta = \frac{d}{dx} \left[ f_e \ln \left( \frac{\beta_v}{1 + \sqrt{1 - \beta_v^2}} \right) + (D - 1)\gamma_v \beta_v p \right]. \tag{D22}
\]

The distributional limit as \( \zeta \to 0 \) is thus found to be

\[
    p * \theta = \mathcal{D}. \lim_{\zeta \to 0} p\theta \tag{D23}
\]

From Eqs. (D22) and (D24), we get:

\[
    Q_{\text{diss}} = \mathcal{D}. \lim_{\zeta \to 0} \zeta \theta^2 = \left\{ \frac{f_e}{D - 1} \ln \left( \frac{\beta_v}{\beta_0} \right) + \Delta \left[ \gamma_v \beta_v p \right] \right\} \delta(x). \tag{D25}
\]

b. Inertial Range

The resolved energy in the limit \( \zeta \to 0 \) satisfies:

\[
    \partial_x (\gamma v \beta_v) = Q_{\text{flux}} - p \theta. \tag{D26}
\]

We now calculate distributional limit as \( \ell \to 0 \) of the two terms appearing above as sources/sinks.

\textbf{Inertial Pressure-work} \( p * \theta \): Since \( \gamma_v, \beta_v \) and \( p \) are all step functions in the ideal limit, [D17] gives

\[
    p \partial_x (\gamma v \beta_v) = \Delta [\gamma_v \beta_v] \partial_x \left( \frac{1}{2} \gamma v^2 \right). \tag{D27}
\]

It follows that

\[
    p * \theta = \mathcal{D}. \lim_{\ell \to 0} p \partial_x (\gamma v \beta_v) = p_{\text{av}} \Delta [\gamma_v \beta_v] \delta(x). \tag{D27}
\]

Note that, as required, this result is completely independent of the choice of the filter kernel \( G \).

\textbf{Energy Flux} \( Q_{\text{flux}} \): By definition [D4]

\[
    Q_{\text{flux}} = -\mathcal{D}. \lim_{\ell \to 0} \left( \partial_{\gamma v} \gamma v \beta_v \right) \nabla \mu \gamma v = \mathcal{D}. \lim_{\ell \to 0} \left( \partial_{\gamma v} \gamma v \beta_v \right) \nabla \mu \gamma v - \left( \partial_{\gamma v} \gamma v \beta_v \right) \nabla \mu \gamma v. \tag{D27}
\]
Enthalpy can be replaced with pressure using $h = D \cdot p$. The balance (D7) with $\zeta = 0$ for both terms then gives in the limit as $t \to 0$

$$D[p\gamma^2\beta_v] \partial_x \gamma_v = f_e \partial_x \gamma_v \frac{\partial}{\partial x} f_e \Delta \gamma_v \delta(x)$$

$$D[p\gamma^2\beta_v] \partial_x \gamma_v \beta_v = f_e \partial_x \gamma_v \beta_v = \frac{\partial}{\partial x} \Delta \gamma_v \beta_v = f_e f_e \Delta \gamma_v \beta_v \frac{\partial}{\partial x} \beta_v \beta_v \delta(x),$$

where (D7) was used for the second term. Together, these yield that:

$$Q_{\text{flux}} = f_e \left\{ \Delta \gamma_v - \beta_v^a \Delta [\gamma_v \beta_v] \right\} \delta(x). \quad \text{(D28)}$$

We see again that the limiting inertial range result is independent of the choice of filter kernel $G$. To compare this term with those previously calculated, we note that for any ideal shock solution (D9) implies

$$\Delta(\epsilon \gamma_v \beta_v) = f_e \Delta \gamma_v - f_p \Delta (\gamma_v \beta_v) \quad \text{(D29)}$$

and (D10) with $\zeta = 0$ implies

$$p_{av} = f_p - f_e \beta_v.$$

These relations can be used to rewrite the formula (D28) for $Q_{\text{flux}}$ as:

$$Q_{\text{flux}} = \left\{ \Delta [\gamma_v \beta_v \epsilon] + p_{av} \Delta [\gamma_v \beta_v] \right\} \delta(x). \quad \text{(D31)}$$

Eqns. (D27), (D31) immediately show that

$$Q_{\text{flux}} - p \circ \theta = \Delta [\gamma_v \beta_v \epsilon] \delta(x), \quad \text{(D32)}$$

as required by the limit of the balance (D26).

The relation (D28) has a further interesting implication that $Q_{\text{flux}} < 0$ for relativistic Euler shocks with the equation of state (D7). Using the relation (D12) for the product $\beta_0 \beta_1$, it is easy to show that

$$J(\beta_0, D) := \beta_{av} \frac{\Delta (\gamma_v \beta_v)}{\Delta \gamma_v} = \frac{1}{2} \left( \frac{1}{\gamma_0 \gamma_1} + \frac{D}{D - 1} \right), \quad \text{(D33)}$$

which may be regarded as a function of just one of the two velocities (say, $\beta_0$) and $D$. Using the above definition and (D28),

$$Q_{\text{flux}} = f_e \Delta \gamma_v \left[ 1 - J(\beta_0, D) \right] \delta(x). \quad \text{(D34)}$$

As noted earlier, positive entropy production at the shock requires that $\Delta \gamma_v < 0$, so that $Q_{\text{flux}} < 0$ if the second factor in (D34) is positive over the range $\beta_s < \beta_0 < 1$. Direct calculation of the derivative gives

$$\frac{\partial}{\partial \beta_0} J(\beta_0, D) = -\left( \beta_0^4 - \beta_s^4 \right) \frac{\gamma_0 \gamma_1}{\beta_0^3} < 0, \quad \text{(D35)}$$

while

$$J(\beta_s, D) = 1, \quad J(1, D) = \frac{D}{2(D - 1)} > \frac{1}{2}. \quad \text{(D36)}$$

Thus $1/2 < J(\beta_0, D) < 1$ over the permitted range of $\beta_0$, so that the second factor in (D34) remains positive and $Q_{\text{flux}} < 0$. This is a more extreme version of what occurs for shocks in a nonrelativistic, compressible Navier-Stokes fluid, where $Q_{\text{flux}} = 0$ (Appendix A of paper I). In both cases, irreversible shock-heating is not due to energy cascade, and in the relativistic case inverse energy cascade even contributes cooling rather than heating.

**Pressure-Dilation Defect:** By subtracting (D27) from (D22), we find:

$$\tau(p, \theta) \equiv p * \theta - p \circ \theta = \left\{ \Delta [\gamma_v \beta_v \epsilon] - p_{av} \Delta [\gamma_v \beta_v] + \right\} f_e \frac{D - 1}{\beta_0 - \beta_s}. \quad \text{(D37)}$$

Together with (D21), (D31) this yields

$$Q_{\text{diss}} = Q_{\text{flux}} + \tau(p, \theta). \quad \text{(D38)}$$

The latter equality can also be obtained by comparing the relations (D21) and (D22), corroborating the general result (77). Because $Q_{\text{diss}} > 0$ whereas $Q_{\text{flux}} < 0$, it follows that $\tau(p, \theta) > 0$. Just as for the non-relativistic shocks discussed in paper I, the pressure-dilation defect is responsible for the net irreversible heating at the shock.

### 3. Entropy Balance

#### a. Dissipation Range

The fine-grained entropy balance for stationary solutions is given simply by:

$$\frac{\partial}{\partial x}(\sigma \gamma_v \beta_v) = \frac{\zeta \theta^2}{T}. \quad \text{(D39)}$$

**Viscous Entropy Production:** It follow immediately from the above that, for discontinuous shock solutions,

$$\Sigma_{\text{diss}} := D \lim_{\gamma_0 \to 0} \frac{\zeta \theta^2}{T} = \Delta [\gamma_v \beta_v \epsilon] \delta(x). \quad \text{(D40)}$$

The entropy production anomaly is thus completely independent of the details of the molecular dissipation and, obviously, $\Sigma_{\text{diss}} \geq 0$. As already noted in [21], this positivity is equivalent to the condition that

$$1 < \frac{s_1 \gamma_1 \beta_1}{\beta_0 \gamma_0 \beta_0} = \left( \frac{\beta_1 \gamma_0^{D-2}}{\beta_0 \gamma_1^{D-2}} \right)^{1/D}, \quad \text{(D41)}$$

where (D41) has been used to obtain the second expression. This ratio is 1 for $\beta_0 = \beta_1 = \beta_s$ and, considered as a function of $\beta_0$ and $D$, it is shown by a straightforward calculation to have positive $\beta_0$-derivative for $\beta_0 \neq \beta_s$. This implies that $\beta_0 > \beta_s > \beta_1$ is required for positive entropy production, as earlier claimed.
b. Inertial Range

The resolved entropy equation for stationary solutions of the RCF models is:

$$\partial_x (\gamma_\ell \beta_\ell + \beta_\ell \tau(\epsilon, \gamma_\ell \beta_\ell)) = \Sigma_{\ell}^{\text{flux}}$$

$$+ \beta_\ell \left( \overline{Q_{\text{diss}}} - \tau(p, \theta) \right)$$  \hspace{1cm} (D42)

where $$\Sigma_{\ell}^{\text{flux}} = \partial_x \beta_\ell \tau(\epsilon, \gamma_\ell \beta_\ell)$$. This entropy evolution equation is considerably simpler than the general Eq. (80), since $$\lambda = 0$$ and because the pressure is proportional to the energy density so that $$I_{\ell}^{\text{flux}} \equiv 0$$.

**Inertial-Range Viscous Heating** $$\beta \circ Q_{\text{diss}}$$: From Eq. (D25), $$\overline{Q_{\text{diss}}} = q_\ast \delta(x)$$ so that

$$\overline{Q_{\text{diss}}} = q_\ast \delta(x).$$  \hspace{1cm} (D43)

From the formula (D2), we see that the inverse temperature $$\beta = 1/T$$ satisfies $$\beta = \alpha^{1/D} p^{-1/D}$$ and thus

$$\beta = \alpha^{1/D} \left( \frac{\tau}{D + 1} \right)^{-1/D} = \alpha^{1/D} p^{-1/D}.$$  \hspace{1cm} (D44)

Using (D18) to write $$\delta = \frac{\partial_x \overline{p}}{\partial p}$$, we get

$$\beta \overline{Q_{\text{diss}}} = \alpha^{1/D} \frac{D q_\ast - 1}{D - 1 \partial_p \overline{p}} \left( p^{D-1}/D \right) = q_\ast \frac{\partial_x d_\ast}{\partial \epsilon \overline{p}}.$$  \hspace{1cm} (D45)

and therefore, as $$\ell \to 0$$:

$$\beta \circ Q_{\text{diss}} = q_\ast \frac{\Delta s}{\Delta \epsilon} \delta(x).$$  \hspace{1cm} (D46)

**Pressure-Dilatation Defect** $$\beta \circ \tau(p, \theta)$$: Our earlier result Eq. (D22) that $$p \ast \theta = q_{PV} \delta(x)$$, yields, by the same argument:

$$\partial_x \tau(p, \theta) = q_{PV} \alpha^{1/D} p^{-1/D} \partial_x \gamma_{\ell \beta}.$$  \hspace{1cm} (D47)

On the other hand, using (D44), we have

$$\partial_x \overline{p} = \alpha^{1/D} p^{-1/D} \partial_x \left( \frac{\partial \overline{p}}{\partial \beta} \right) = \frac{D \alpha^{1/D} \Delta [\gamma_{\ell \beta}] \Delta \left[ p^{D-1}/D \right]}{2D - 1} \partial_x \left[ p^{D-1}/D \right].$$  \hspace{1cm} (D48)

Thus,

$$\partial_x \overline{p} = \alpha^{1/D} \frac{D \alpha^{1/D} \Delta [\gamma_{\ell \beta}] \Delta \left[ p^{D-1}/D \right]}{2D - 1} \partial_x \left[ p^{D-1}/D \right]$$  \hspace{1cm} (D49)

The following relations are useful and follow directly from (D2) and (D3):

$$\Delta \left( \frac{1}{T} \right) = \alpha^{1/D} \Delta \left[ p^{-1/D} \right] = \frac{(D - 1) (s_1 \epsilon_0 - s_0 \epsilon_1)}{D \epsilon_0 \epsilon_1}$$  \hspace{1cm} (D50)

$$\Delta \left( \frac{p^2}{T} \right) = \alpha^{1/D} \Delta \left[ p^{2D-1}/D \right] = \frac{\Delta [\epsilon]}{D (D - 1)}.$$  \hspace{1cm} (D51)

With these, we have that $$I_{\text{flux}} = \beta \circ \tau(p, \theta)$$ is given by

$$\beta \circ \tau(p, \theta) = \frac{1}{\Delta \epsilon} \left[ q_{PV} \Delta s - \frac{\Delta [\gamma_{\ell \beta}] \Delta [\epsilon]}{2D - 1} \right] \delta(x).$$  \hspace{1cm} (D52)

**Combined Contribution** $$\beta \circ Q_{\text{diss}} - \beta \circ \tau(p, \theta)$$: From Eq. (D24), we obtain that:

$$q_\ast - q_{PV} = \Delta \overline{[\gamma_{\ell \beta}]}.$$  \hspace{1cm} (D53)

Thus, the combined contribution of these terms is simply:

$$\beta \circ Q_{\text{diss}} - \beta \circ \tau(p, \theta)$$

$$= \frac{1}{\Delta \epsilon} \left[ \Delta s \Delta \overline{[\gamma_{\ell \beta}]} + \frac{\Delta [\gamma_{\ell \beta}] \Delta [\epsilon]}{2D - 1} \right] \delta(x).$$  \hspace{1cm} (D54)

**Negentropy Flux** $$\Sigma_{\ell}^{\text{flux}}$$: First, consider the contribution $$\left( \partial_x \beta \right) \epsilon_{\ell \beta}$$. From Eqn. (D14) we have:

$$\partial_x \beta = - \alpha^{1/D} D^{-1} \partial_p \left[ \frac{D+1}{D} \right] \partial_x \overline{p}.$$  \hspace{1cm} (D55)

Using (D16) to write

$$\epsilon_{\ell \beta} = \epsilon_0 \gamma_{0 \beta} + \frac{\Delta (\epsilon_{\ell \beta})}{\Delta \epsilon} (\overline{p} - \overline{p}_0),$$  \hspace{1cm} (D56)

a straightforward calculation shows

$$\left( \partial_x \beta \right) \epsilon_{\ell \beta} \rightarrow \frac{1}{D} \Delta s \Delta \overline{[\gamma_{\ell \beta}]} \delta(x)$$

$$- \frac{\Delta [\gamma_{\ell \beta}] \Delta \epsilon}{\Delta \epsilon} \epsilon_0 \alpha^{-1/D} \Delta [p^{-1/D}] \delta(x)$$

The other term is computed likewise using:

$$\left( \partial_x \beta \right) \epsilon_{\ell \beta} \rightarrow$$

$$- \alpha^{1/D} D^{-1} \frac{D - 1}{D} \partial_p \left( \gamma_{0 \beta} + \frac{\Delta [\gamma]}{\Delta \epsilon} (\overline{p} - \overline{p}_0) \right)$$

whence, after some calculation, one has:

$$\left( \partial_x \beta \right) \epsilon_{\ell \beta} \rightarrow$$

$$- \alpha^{1/D} D^{-1} \left( \frac{D - 1}{D} \right) \Delta \overline{[\gamma_{\ell \beta}]} \Delta [p^{(2D-1)/D}] \delta(x)$$

$$+ \frac{1}{D} \left( \frac{\Delta [\gamma_{\ell \beta}] \epsilon_0 - \gamma_{0 \beta} \Delta \epsilon}{\Delta \epsilon} \right) \Delta s \delta(x).$$  \hspace{1cm} (D57)

Therefore, (D57) and (D59) in combination show:

$$\Sigma_{\text{flux}} := \partial_x \gamma_{\ell \beta} \tau(\epsilon, \gamma_{\ell \beta})$$

$$= \alpha^{1/D} \frac{D - 1}{D} \Delta \overline{[\gamma_{\ell \beta}]} \Delta [p^{(2D-1)/D}] \delta(x)$$

$$- \frac{1}{D} \left( \Delta \overline{[\gamma_{\ell \beta}]} \epsilon_0 - \gamma_{0 \beta} \Delta \epsilon + \Delta [\gamma_{\ell \beta}] \Delta \epsilon \delta(x) \right)$$

$$- \alpha^{1/D} \left( \frac{\Delta [\gamma_{\ell \beta}] \epsilon_0 - \gamma_{0 \beta} \Delta \epsilon}{\Delta \epsilon} \right) \epsilon_0 \Delta [p^{-1/D}] \delta(x).$$  \hspace{1cm} (D60)

The relations (D50), (D51), (D58), (D60) can then be employed to simplify the expression for the flux to:

$$\Sigma_{\text{flux}} = \frac{1}{\Delta \epsilon} \left\{ (\epsilon_1 \epsilon_0 - \epsilon_0 \epsilon_1) \Delta \overline{[\gamma_{\ell \beta}]} - \frac{\Delta [\gamma_{\ell \beta}] \Delta [\epsilon]}{2D - 1} \right\} \delta(x).$$  \hspace{1cm} (D61)
Adding together the formulas (D54) and (D61), one has, after minor manipulation, that:
\[
\beta \circ Q_{diss} - \beta \circ \tau(p, \theta) + \Sigma_{flux} = \Delta[\gamma_4 \beta_c s] \tag{D62}
\]

in agreement with (D42) and the dissipation-range result (D41), as demanded by the general equality Eq. (G8).

A further implication of the formula (D61) for entropy flux is that \(\Sigma_{flux} > 0\) at these relativistic Euler shocks. Although not presented here, arguments like those applied to \(Q_{flux}\) show this and are confirmed by numerically plotting (D61) as a function of \(R\) for each \(D > 2\). It is interesting that \(\Sigma_{flux} > 0\) was also found for planar shock solutions of non-relativistic compressible Euler equations in paper I. In both cases, there is a forward cascade of negentropy at the shock, even though the energy flux is vanishing or negative.

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Note that the relativistic viscosities as defined in our paper are \(c\) times their non-relativistic counterparts, because \(\theta, \sigma^{\mu\nu}\) as \(c \to \infty\) are \(1/c\) times their non-relativistic analogues \(\Theta, \Sigma\) in paper I.

The fact that spatial coarse-graining alone regularizes time-derivatives is due the the field that the fields in question satisfy equations of motion that are (at least) first-order in time.

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