The Amalgamation Property in Classical Lebesgue-Riesz Spaces, Banach Spaces with Almost Transitive Norm and Projection Constants

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Dedicated to the memory of S. Banach.

Abstract. This implies that the relative projection constant of a subspace $A \hookrightarrow L_p[0, 1]$ ($p \neq 4, 6, 8, \ldots$) does not depend on the position of $A$ in $L_p$. By the way the characterization of 2-dimensional subspaces of $l_1$ is obtained.

1. Introduction

The amalgamation property for universal relational systems was introduced by B. Jónsson in 1956 [1]. It was used to study partial ordered sets and lattices, Boolean algebras and general algebraic relational systems.

V.I. Gurarii [2] independently discovered the same property in the isomorphic theory of Banach spaces and showed that the set $\mathcal{M}$ of all finite-dimensional Banach spaces enjoys it. This result (which is known under the name the Gurarii's lemma on gluing of embeddings) was used in [2] to built the space $G$ of almost universal disposition with respect to $\mathcal{M}$.

In this paper we study the isometric version of the mentioned Gurarii's lemma - i.e. the amalgamation property in the isometric theory of Banach spaces.

It will be shown that if a set $H(X)$ of all different finite dimensional subspaces of a given Banach space $X$ (isometric subspaces in $H(X)$ are identified) has the amalgamation property (all definitions will be given later), then there exists a separable Banach space $G_X$, unique up to almost isometry, which is finitely equivalent to $X$ and has properties that are similar to those of the classical Gurarii space $G$:

- Let $A$ and $B$ be isometric finite-dimensional subspaces of $X$; $i: A \to B$ be the corresponding isometry. Then for every $\varepsilon > 0$ there exists an automorphism $u: X \to X$ such that $\|u\| \cdot \|u^{-1}\| \leq 1 + \varepsilon$, which extends $i$: $u|_{A} = i$. In particular, the norm of $G_X$ is almost transitive.

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• $G_X$ is an approximative envelope: every separable Banach space $Y$, which is finitely representable in $G_X$ is $(1 + \varepsilon)$-isomorphic to a subspace of $G_X$ for every $\varepsilon > 0$.

It will be convenient to regard the amalgamation property as a property of the whole class $X^f$ of finite equivalence, generated by a given Banach space $X$.

In this paper it will be shown that the class $(L_1)^f$ has the amalgamation property; that classes $(L_p)^f$ does not have the amalgamation property for $p = 4, 6, 8, ...$ (shortly: for $p \in 2\mathbb{N}\setminus\{2\}$). As it follows from [3], one W. Rudin’s result [4] implies that classes $(L_p)^f$ has the amalgamation property for $p \in (1, \infty)\setminus(2\mathbb{N}\setminus\{2\})$. The case $p = \infty$ is covered by the mentioned above Gurarii’s lemma. Worthy of note that in [4] was considered the complex case. For real scalars the similar result was obtained in [5].

At the same time, in a general case the space $G_X$ (as above) is not of almost universal disposition (with respect to the set $H(G_X)$) in the sense of [2]. This fact about $G_{L_1}$ may be deduced from one J. Pisier’s result [6]. The same result concerning $G_{L_p}$ will be obtained in this paper.

Some of results of this paper were announced in [7]; the tragical death of the first named author terminated the publication of complete proofs.

2. Definitions and notations

**Definition 1.** Let $X, Y$ be Banach spaces. $X$ is finitely representable in $Y$ (in symbols: $X <_f Y$) if for each $\varepsilon > 0$ and for every finite dimensional subspace $A$ of $X$ there exists a subspace $B$ of $Y$ and an isomorphism $u : A \to B$ such that $\|u\|\|u^{-1}\| \leq 1 + \varepsilon$.

Spaces $X$ and $Y$ are said to be finitely equivalent, shortly: $X \sim_f Y$, if $X <_f Y$ and $Y <_f X$.

Any Banach space $X$ generates classes $X^f = \{Y \in \mathcal{B} : X \sim_f Y\}$ and $X^<_f = \{Y \in \mathcal{B} : Y <_f X\}$

For any two Banach spaces $X, Y$ their Banach-Mazur distance is given by $d(X, Y) = \inf\{\|u\|\|u^{-1}\| : u : X \to Y\}$, where $u$ runs all isomorphisms between $X$ and $Y$ and is assumed, as usual, that $\inf \emptyset = \infty$.

It is well known that $\log d(X, Y)$ defines a metric on each class of isomorphic Banach spaces, where almost isometric Banach spaces are identified.

Recall that Banach spaces $X$ and $Y$ are almost isometric if $d(X, Y) = 1$. Certainly, any almost isometric finite-dimensional Banach spaces are isometric.

A set $\mathcal{M}_n$ of all $n$-dimensional Banach spaces, equipped with this metric, is a compact metric space, called the Minkowski compact $\mathcal{M}_n$.

A disjoint union $\bigcup\{\mathcal{M}_n : n < \infty\} = \mathcal{M}$ is a separable metric space, which is called the Minkowski space.

Consider a Banach space $X$. Let $H(X)$ be a set of all its different finite dimensional subspaces (isometric finite dimensional subspaces of $X$ in $H(X)$ are identified). Thus, $H(X)$ may be regarded as a subset of $\mathcal{M}$, equipped with the restriction of the metric topology of $\mathcal{M}$.

Of course, $H(X)$ need not to be a closed subset of $\mathcal{M}$. Its closure in $\mathcal{M}$ will be denoted $H(X)$. 
From definitions it follows that $X \sim_f Y$ if and only if $\overline{H(X)} \subseteq \overline{H(Y)}$. Therefore, $X \sim_f Y$ if and only if $\overline{H(X)} = \overline{H(Y)}$.

There exists a one to one correspondence between classes of finite equivalence $X^{f}$ and closed subsets of $\mathcal{M}$ of kind $\overline{H(X)}$.

Indeed, all spaces $Y$ from $X^{f}$ have the same set $\overline{H(X)}$. This set, uniquely determined by $X$ (or, equivalently, by $X^{f}$), will be denoted by $\mathfrak{M}(X^{f})$ and will be referred to as the Minkowski’s base of the class $X^{f}$.

It will be convenient to introduce some new terminology.

A fifth $v = \langle A, B_{1}, B_{2}, i_{1}, i_{2} \rangle$, where $A, B_{1}, B_{2} \in \mathfrak{M}(X^{f})$; $i_{1} : A \to B_{1}$ and $i_{2} : A \to B_{2}$ are isometric embeddings, will be called the $V$-formation over $\mathfrak{M}(X^{f})$. The space $A$ will be called the root of the $V$-formation $v$. If there exists a triple $t = \langle j_{1}, j_{2}, F \rangle$, where $F \in \mathfrak{M}(X^{f})$; $j_{1} : B_{1} \to F$ and $j_{2} : B_{2} \to F$ are isometric embeddings such that $j_{1} \circ i_{1} = j_{2} \circ i_{2}$, then the $V$-formation $v$ is said to be amalgamated in $\mathfrak{M}(X^{f})$, and the triple $t$ is said to be its amalgam.

Let $\text{Amalg}(\mathfrak{M}(X^{f}))$ be a set of all spaces $A \in \mathfrak{M}(X^{f})$ with the property:

Any $V$-formation $v$, which root is $A$, is amalgamated in $\mathfrak{M}(X^{f})$.

**Definition 2.** Let $X \in \mathcal{B}$ generates a class $X^{f}$ with a Minkowski’s base $\mathfrak{M}(X^{f})$. It will be said that $\mathfrak{M}(X^{f})$ (and the class $X^{f}$) has the amalgamation property if

$$\mathfrak{M}(X^{f}) = \text{Amalg}(\mathfrak{M}(X^{f})).$$

**Definition 3.** For a Banach space $X$ its $l_{p}$-spectrum $S(X)$ is given by

$$S(X) = \{p \in [0, \infty) : l_{p} <_{f} X\}.$$  

Certainly, if $X \sim_f Y$ then $S(X) = S(Y)$. So, the $l_{p}$-spectrum $S(X)$ may be regarded as a property of the whole class $X^{f}$.

3. The amalgamation property in the class $(L_{1})^{f}$

To show that a set $\mathfrak{M}((L_{1})^{f})$ has the amalgamation property it will be needed some additional constructions.

**Definition 4.** Let $X$ be a Banach space with a basis $(e_{n})$. This basis is said to be Besselian if there exists such a constant $C > 0$ that for any $x = \sum x_{n}e_{n} \in X$

$$\|x\| \geq C \left( \sum (x_{n})^{2} \right)^{1/2}.$$  

**Theorem 1.** Let $\langle X, (e_{n}) \rangle$ be a Banach space with a Besselian bases $(e_{n})$; $Y \hookrightarrow X$ be a finite dimensional subspace of $X$ of dimension $\dim(Y) = m$.

There exists a sequence of points $Y_{0} = (y_{n})_{n=1}^{\infty} \subset \mathbb{R}^{m} \text{ such that } X$ may be represented as a Banach space $\overline{X}$ of all rear-valued functions $f$, defined on $Y_{0}$ with a norm

$$\|f\| \stackrel{\text{def}}{=} \|f\|_{\overline{X}} = \left\| \sum f(y_{i}) e_{i} \right\|_{X} < \infty;$$

$Y$ may be represented as a subspace $\overline{Y}$ of $\overline{X}$ that consists of all linear functions on $Y_{0}$ (equipped with the corresponding restriction of the norm of $\overline{X}$). In other words, there exists an isometry $T : X \to \overline{X}$ such that its restriction $T|_{Y} : Y \to \overline{Y}$ is also an isometry.
PROOF. Let \( u = \sum u_ne_n \) and \( v = \sum v_ne_n \) be elements of \( X \). Since \( (e_n) \) is Besselian, it may be defined a scalar product \( \langle u, v \rangle = \sum u_nv_n \). Since \( Y \) is of finite dimension, there exists a projection \( P : X \to Y \) such that \( \langle Px, y \rangle = \langle x, y \rangle \). Let \( y_i = Pe_i \).

Consider a linear space \( X \) of all formal sums \( f = \sum f_i y_i \) and equip it with a norm

\[
\|f\|_X = \left\| \sum f_i y_i \right\|_X = \left\| \sum f_i e_i \right\|_X.
\]

Let a map \( T : X \to Y \) be given by

\[
T(\sum u_ne_n) = \sum u_n y_n \quad \text{for all } u = \sum u_ne_n \in X.
\]

For \( u \in Y \) it may be computed its \( i \)’th coordinate

\[
(Tu)_i = u_i = \langle u, e_i \rangle = \langle u, Pe_i \rangle = \langle u, y_i \rangle.
\]

Hence, \( T \) maps \( Y \) onto a linear space \( Y \) of all linear functions, defined on \( Y_0 \).

So, \( X, Y, Y_0 \) and \( T \) have desired properties. \( \square \)

REMARK 1. A set \( Y_0 = (y_i) \subset \mathbb{R}^m \) will be called an incarnating set for a pair \( Y \hookrightarrow X \). A pair \( [X, Y] \) will be called an incarnation pair for \( Y \hookrightarrow X \). In what follows pairs \( Y \hookrightarrow X \) and \( [X, Y] \) may be identified.

REMARK 2. If a basis \( (e_n) \) of \( X \) is, in addition, a symmetric one, a set \( Y_0 \) may be symmetrized: it may be considered instead \( Y_0 \) a central symmetric set

\[
K = K(Y \hookrightarrow X) = Y_0 \cup (-Y_0) = \{ \pm y \in \mathbb{R}^m : y \in Y_0 \}
\]

as an incarnating set for a pair \( Y \hookrightarrow X \) provided that \( X \) (resp., \( Y \)) will be considered as the set of all odd (resp., all odd linear) functions on \( K \) with the corresponding norm.

Consider a space \( \mathbb{R}^m \). Let \( (e_i)_{i=1}^m \) be a basis of \( \mathbb{R}^m \) that will be assumed orthogonal (with respect to the \( l_2 \)-norm). Define on \( \mathbb{R}^m \) an other norm, namely, put

\[
\| \| u \| \| = \left| \left| \sum_{i=1}^m u_ne_n \right| \right| = \sum_{i=1}^m |(u, e_i)| = \sum_{i=1}^m |u_i|.
\]

THEOREM 2. Let \( m \in \mathbb{N}, K = \{ cy_i : i \in \mathbb{N}, c \in \{+,-\} \} \subset \mathbb{R}^m \) be a central symmetric set. \( K \) is an incarnating set for a pair \( Y \hookrightarrow l_1 \) where \( Y \) is \( m \)-dimensional subspace of \( l_1 \) if and only if \( K \) is complete in \( \mathbb{R}^m \) and \( \sum \| y_i \| : y_i \in K \} < \infty \).

PROOF. Certainly, the completeness of \( K \) in \( \mathbb{R}^m \) is necessary.

The condition \( \sum \| y_i \| : y_i \in K \} < \infty \) is also necessary. Indeed,

\[
\sum \| y_i \| = \sum_{y_i \in K} \sum_{j=1}^m |(y_i, e_j)| = \sum_{j=1}^m \sum_{y_i \in K} |(y_i, e_j)| = \sum_{j=1}^m \| e_j \| = m < \infty.
\]

The sufficiency follows from a fact that a set \( L(K) \) of all linear functionals on \( K \) may be embedded in \( l_1 \). Indeed, let \( x \in L(K); x = \sum \{ x_i y_i : y_i \in K \} \). Then

\[
\| x \|_{l_1} = \sum_{y_i \in K} |(x, y_i)| = \sum_{y_i \in K} \left| \sum_{y_j \in K} x_j \langle y_i, y_j \rangle \right| \leq \sum_{y_i \in K} \max |x_j| \sum_{y_j \in K} |\langle y_j, y_i \rangle| = \max |x_j| \sum_{y_i \in K} \| y_i \|. \]

\( \square \)
Remark 3. Obviously, if linearly congruent sets $K$ and $K_1$ are incarnating sets for pairs $Y \hookrightarrow l_1$ and $Z \hookrightarrow l_1$, then the linear congruence $U : K \rightarrow K_1$ generates an isometric automorphism $u : l_1 \rightarrow l_1$, which restriction $u|_Y$ to $Y$ is an isometry between $Y$ and $Z$.

The following result shows how to reconstruct a pair $Y \hookrightarrow l_1$ by a given set $K = \{\pm y_i : i \in \mathbb{N}\} \subset \mathbb{R}^m$, which is complete in $\mathbb{R}^m$ and satisfies the condition $\sum \{||y_i|| : y_i \in K\} < \infty$.

Put $\varsigma K = \cup \{\sum \{y_i : \langle y_i, u \rangle > 0\} : u \in \mathbb{R}^m\}$, $K' = \text{conv}(\varsigma K)$.

So, $K'$ denotes the closure of a convex hull conv$(\varsigma K)$ of $\varsigma K$.

Theorem 3. A central symmetric, complete in $\mathbb{R}^m$, set $K = \{\pm y_i : i \in \mathbb{N}\} \subset \mathbb{R}^m$ is an incarnating set for a pair $Y \hookrightarrow l_1$ if and only if $K'$ is congruent with the unit ball $B(Y*) \triangleq \{y' \in Y^* : ||y'|| \leq 1\}$ of the conjugate space $Y^*$.

Proof. Necessity.

$$||x||_Y = \frac{1}{2} \sum_{y_i \in K} |\langle x, y_i \rangle| = \max \{(x, z) : z \in K'\}.$$ Hence, $K' = B(Y*)$.

Sufficiency. $K$ is an incarnating set for a some pair $Z \hookrightarrow l_1$. Hence, as it was shown before, $K' = B(Z^*)$. However, by conditions of the theorem, $K' = B(Y*)$.

Thus, $Y^* = Z^*$ and, hence, $Y = Z$. \hfill \square

Corollary 1. There exists an one-to-one correspondence between pairs $Z \hookrightarrow l_1$ and certain central symmetric complete subsets of $\mathbb{R}^m$.

The next result describe this correspondence in detail.

Theorem 4. Let $K$ be an incarnating set for a pair $Y \hookrightarrow l_1$. Vectors $(y_i)$ of $K$ are parallel to edges of the unit ball $B(Y*)$. A length of a sum of all vectors that are parallel to a given edge $r$ of $B(Y*)$ is equal to the length of the edge $r$.

Proof. This result may be obtained as a consequence of [8]. Below it will be presented its direct proof by induction on dimension.

If dim$(K') = 1$ then $K' = [-y, y]$ for some $y \in \mathbb{R}$; $K = \{-y, y\}$ and the theorem is true trivially.

Assume that the theorem is true for dim$(K') = n$. Let $K \in \mathbb{R}^{n+1}$; dim$(K') = n + 1$.

An $n$-dimensional subspace $E \rightarrow \mathbb{R}^{n+1}$ is said to be a support subspace (for $K'$) if $E \cap K$ is complete in $E$. A set of all support subspaces will be denoted by $E(K)$.

Let $E \in E(K)$; $r_E \in \mathbb{R}^{n+1}$; $r_E \neq 0$. Let $r_E$ be orthogonal to $E$. Let

$$x_E = \sum \{y_i \in K : \langle y_i, r_E \rangle > 0\}.$$ Then

$$V_E = (E \cap K)' + x_E = \{z + x_E : z \in (E \cap K)\}'$$ is a facet of $K'$.
Indeed, \( \dim(K') = n + 1 \); \( \dim(V_E) = n \); \( V_E \subset K \). At the same time \( E + x_E \) is an affine manifold of the minimal dimension that contains \( V_E \). By the definition of \( K' \), \( (E + x_E) \cap K' = V_E \). Hence, \( V_E \) is one of maximal \( n \)-dimensional subsets of \( K' \). Moreover,

\[
V_E = \{ z \in K' : \langle z, r_E \rangle = \langle x_E, r_E \rangle = \sup_{x \in K'} \langle x, r_E \rangle \}.
\]

So, to any support subspace \( E \in E(K) \) corresponds a facet \( V_E = (E \cap K') + x_E \).

Conversely, let \( V \) be a facet of \( K' \); \( r_V \) be an external perpendicular to \( V \). Let

\[
x_V = \sum \{ y \in K : \langle y, r_V \rangle > 0 \}; \\
E_V = \{ x \in \mathbb{R}^{n+1} : \langle x, r_V \rangle = 0 \}.
\]

Then \( V = (E_V \cap K') + x_V \) and \( E_V \) is a support subspace.

Let \( E \in E(K) \). By the induction supposition, \( E \cap K \) may be reconstructed by the facet \( V_E = (E \cap K') + x_V \).

Since \( K = \bigcup \{ E \cap K : E \in E(K) \} \), then \( K \) also may be reconstructed by facets of \( K' \). Notice that every edge belongs to some facet of \( K \). \( \square \)

Now we are ready to prove the amalgamation property for \( \mathfrak{M}((l_1)^I) \).

**Theorem 5.** The set \( \mathfrak{M}((l_1)^I) \) has the amalgamation property.

**Proof.** We show that \( H(l_1) \) has the amalgamation property. Obviously, a closure \( \overline{H(l_1)} = \mathfrak{M}((l_1)^I) \) also has this property.

Let \( Y \twoheadrightarrow l_1 \) and \( Z \twoheadrightarrow l_1 \). According to preceding results these pairs determine corresponding incarnating sets \( K_Y = (y_i) \) and \( K_Z = (z_i) \). Let \( X \) be isometrically embedded into \( Y \) (by an operator \( i : X \to Y \)) and into \( Z \) (by \( j : X \to Z \)).

Consider a conjugate \( X^* \) and for every edge of its unit ball \( B(X^*) \) choose subsets \( \langle y_i^* \rangle \subset K_Y = (y_i) \) and \( \langle z_j^* \rangle \subset K_Z = (z_j) \) that contains vectors parallel to \( r \). Consider a set \( K_{XY} = \{ x_{ij} = \{ c_{ij} \} \} \), where \( c_{ij} = y_i \| z_j \| / \| r \| \) denotes the length of the edge \( r \). Clearly, \( K_{XY} \) is an incarnating set for a subspace \( W \) of \( l_1 \) that contains isometric images of both \( Y \) and \( Z \) (say, \( Y' \) and \( Z' \)) such that their intersection \( Y' \cap Z' = X' \) is isometric to \( X \); moreover, corresponding isometries \( u_Y : Y \to Y' \) and \( u_Z : Z \to Z' \) transform embeddings \( i : X \to Y \) and \( j : X \to Z \) to identical embeddings of \( X' \) into \( Y' \) and into \( Z' \) respectively. Clearly, this proves the theorem. \( \square \)

**4. Almost \( \omega \)-homogeneous Banach spaces**

**Definition 5.** Let \( X \) be a Banach space; \( K \) be a class of Banach spaces. \( X \) is said to be almost \( \omega \)-homogeneous with respect to \( K \) if for any pair of spaces \( A \), \( B \) of \( K \) such that \( A \) is a subspace of \( B \) (\( A \hookrightarrow B \)), every \( \varepsilon > 0 \) and every isometric embedding \( i : A \to X \) there exists an isomorphic embedding \( i : B \to X \) which extends \( i \) (i.e., \( \overset{i}{i} : A = i \)) and such, then

\[
\| i \| \| i^{-1} \| \leq (1 + \varepsilon).
\]

If \( X \) is almost \( \omega \)-homogeneous with respect to \( H(X) \) it will be simply referred to as an almost \( \omega \)-homogeneous space.
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Theorem 6. Any class $X^f$ of finite equivalence, whose Minkowski’s set $\mathfrak{M}(X^f)$ has the amalgamation property contains a separable almost $\omega$-homogeneous space $G_X$.

This space is unique up to almost isometry and is almost isotropic (in an equivalent terminology, has an almost transitive norm).

This space is an approximative envelope of a class $X^f$: for any $\varepsilon > 0$ every separable Banach space which is finitely representable in $X^f$ is $(1 + \varepsilon)$-isomorphic to a subspace of $E_X$.

Proof. The proof of the theorem also literally repeats the Gurarii’s one. We present a sketch of proof for a sake of completeness (below it will be presented another proof of this fact, based on a different idea).

Let us starting with any space $X_0$ of $X^f$. Consider a dense countable subset $\{x_i : i < \infty; \|x_i\| = 1\} \subset X_0$. Any finite subset $N$ of $\mathbb{N}$ defines a finite dimensional subspace $U_N = \text{span}\{x_i : i < \infty\} \hookrightarrow X_0$. Clearly, there are only countable number of spaces of kind $U_N$. Identifying isometric subspaces from

$$\{U_N : N \subset \mathbb{N}; \quad \text{card}(N) < \infty\}$$

it will be obtained a dense countable subset $(Z_i)_{i < \infty} \subset H(X_0)$. Consider a set $\mathcal{F}$ of all triples $(A, B, i)$ where $A, B \in \mathfrak{M}(X^f)$ and there exists an isometric embedding $i : A \rightarrow B$.

Let $n < m < \infty$ and let $\mathcal{F}(n, m) \subset \mathcal{F}$ be a subset of $\mathcal{F}$, which consists of such triples $(A, B, i)$ that $\text{dim}(A) = n$; $\text{dim}(B) = m$.

$\mathcal{F}(n, m)$ may be equipped with a metric

$$\rho((A, B, i), (A_1, B_1, i_1)) = \log d^{n,m}((A, B, i), (A_1, B_1, i_1)),$$

where

$$d^{n,m}((A, B, i), (A_1, B_1, i_1)) = \inf\{\|u\| \|u^{-1}\| : u : B \rightarrow B_1; \quad i \circ u = i_1\}$$

is a generalized Banach-Mazur distance. It is known [2] that $(\mathcal{F}(n, m), \rho)$ is a compact metric space. Hence, $(\mathcal{F}, \rho) = \bigcup_{n, m} (\mathcal{F}(n, m), \rho)$ is a separable metric space and it may be chosen a countable dense subset $\mathcal{F}^0$ of $\mathcal{F}$. Without loss of generality it may be assumed that all spaces that are presented in triples from $\mathcal{F}^0$ are exactly those that belong to a defined before subset $(Z_i) \subset H(X_0)$.

Using the amalgamation property for $\mathfrak{M}(X^f)$, for any pair $A \hookrightarrow B \rightarrow X_0$ of subspaces of $X_0$ and any isometric embedding $i : A \rightarrow C$, where $C \in \mathfrak{M}(X^f)$, it may be constructed an extension of $X_0$, say, $X'_0(A \hookrightarrow B; i : A \rightarrow C)$ – a separable Banach space that contains $X_0$ and contains a triple $(A, B, C')$, where $A \hookrightarrow B$; $A \hookrightarrow C'$ and a pair $A \hookrightarrow C'$ is isometric to the pair $iA \hookrightarrow C$ in a sense of the aforementioned metric $\rho$. It will be said that $X'_0(A \hookrightarrow B; i : A \rightarrow C)$ amalgamates over $X_0$ the $V$-formation $(A \hookrightarrow B; i : A \rightarrow C)$.

Now, proceed by induction.

Let $(f_i)_{i < \infty}$ be a numeration of all triples from $\mathcal{F}^0$. Construct a sequence of Banach spaces $(X_i)_{i < \infty}$ where it will be presented the space $X_0$ itself as the first step of induction.

Let spaces $(X_i)_{i < \infty}$ be already constructed.

As $X_{n+1}$ it will be chosen the space $X'_n(A \hookrightarrow B; i : A \rightarrow C)$, where $A, B \in (U_N)_{N \subset \mathbb{N}}$;

$A \hookrightarrow B \rightarrow X_n$ be the $n$’th triple $f_n$;
(i : A → C) = f_{m+1}, where m is the least number of a triple from \( F^3 \) of kind 
(i : A → C) (for a fixed A and arbitrary i and C) and be such that \( X_n \) does not 
alagamate the V-formation \( (A \mapsto B; i : A \mapsto C) \).

Clearly, \( X_0 \leadsto X_1 \leadsto X_2 \leadsto .. \leadsto X_n \leadsto .... \). Let \( X_\infty^{(1)} = \bigcup X_i = \lim X_i. \) Now 
induction will be continued, starting with \( X_\infty^{(1)} \) instead of \( X_0 \). It will be sequentially 
constructed spaces \( X_\infty^{(2)} \leadsto X_\infty^{(3)} \leadsto ... \leadsto X_\infty^{(n)} \leadsto .... \)

Their direct limit \( \lim X_\infty^{(n)} = G_X \) is the desired space.

Proofs of the uniqueness of \( G_X \) up to almost isometry and the property of it 
to be almost isotropic are the same as in [2].

**Definition 6.** ([9]). Let \( X, Y \) be Banach spaces; \( Y \mapsto X \). \( Y \) is said to be 
a reflecting subspace of \( X \), shortly: \( Y \prec_u X \), if for every \( \varepsilon > 0 \) and every finite 
dimensional subspace \( A \mapsto X \) there exists an isomorphic embedding \( u : A \mapsto Y \) 
such that \( \|u\| \|u^{-1}\| \leq 1 + \varepsilon \), which is identical on the intersection \( Y \cap A \):

\[
u |_{Y \cap A} = Id_{Y \cap A}.\]

Clearly, \( Y \prec_u X \) implies that \( Y \sim_f X \).

**Definition 7.** ([10]). A Banach space \( E \) is said to be existentially closed in a 
class \( X^f \) if for any isometric embedding \( i : E \mapsto Z \) into an arbitrary space \( Z \in X^f \) 
its image \( iE \) is a reflecting subspace of \( Z \): \( iY \prec_u Z \).

A class of all spaces \( E \) that are existentially closed in \( X^f \) is denoted by \( \mathcal{E} (X^f) \). 
In [10] it was shown that for any Banach space \( X \) the corresponding class \( \mathcal{E} (X^f) \) 
is non-empty: moreover, any \( Y <_f X^f \) may be isometrically embedded into some \( E \in \mathcal{E} (X^f) \) of the dimension \( \dim(E) = \max\{\dim(Y), \omega\} \).

**Theorem 7.** For any class \( X^f \) such that \( \mathcal{M} (X^f) \) has the amalgamation property \( E \in \mathcal{E} (X^f) \) if and only if \( E \in X^f \) and \( E \) is almost \( \omega \)-homogeneous.

**Proof.** Let \( \mapsto E \in X^f \) and \( E \) be almost \( \omega \)-homogeneous. Let \( A \mapsto Z \) 
be a finite dimensional subspace; \( E \cap A = B \) and \( \varepsilon > 0 \). Consider the identical 
embedding \( id_B : B \mapsto E \). Since \( B \mapsto A \), \( id_B \) may be extended to an embedding 
\( u : A \mapsto E \) with \( \|u\| \|u^{-1}\| \leq 1 + \varepsilon \). Thus, \( E \in \mathcal{E} (X^f) \).

Conversely, let \( E \in \mathcal{E} (X^f) \); \( B \mapsto E \); \( B \mapsto A \) and \( A \in \mathcal{M} (X^f) \).

Consider a space \( Z \) such that \( E \mapsto Z \); \( B \mapsto A \mapsto Z \). Such space exists because 
of the amalgamation property of \( \mathcal{M} (X^f) \). Since \( E \in \mathcal{E} (X^f) \), \( E \prec_u Z \), i.e. there 
is an embedding \( u : A \mapsto E \) such that \( \|u\| \|u^{-1}\| \leq 1 + \varepsilon \), which is identical on 
the intersection \( E \cap A \): \( u |_{E \cap A} = Id_{E \cap A} \). Since \( B \mapsto A \) and \( B \mapsto E \), \( B \mapsto E \cap A \). 
Clearly, \( u \) extends the identical embedding \( id_B \). Since \( A \mapsto B \) and \( \varepsilon \) are arbitrary, 
\( E \) is almost \( \omega \)-homogeneous.

**Remark 4.** This theorem gives an alternative proof of the first part of theorem 6.

**Corollary 2.** If \( X^f \) is superreflexive and enjoys the amalgamation property 
then \( G_X \) is a norm one complemented subspace of any space \( Z \) from the class \( X^f \) 
that it contains.
5. Projection constants

Let $X$ be a Banach space; $Y \hookrightarrow X$. A projection constant $\lambda(A \hookrightarrow X)$ is given by

$$\lambda(A \hookrightarrow X) = \inf\{\|P\| : P : X \rightarrow A; P^2 = P\},$$

or, by words, $P$ runs all projections from $X$ onto $A$.

Relative projection constants $\lambda(A, X)$ and $\lambda(A, K)$, where $K$ is a class of Banach spaces, are defined as follows:

$$\lambda(A, X) = \sup\{\lambda(iA \hookrightarrow X) : i : A \rightarrow X\},$$

where $i$ runs all isometric embeddings of $A$ into $X$;

$$\lambda(A, K) = \sup\{\lambda(A, X) : X \in K\}.$$

The absolute projection constant $\lambda(A)$ is just $\lambda(A, B)$, where $B$ denotes the class of all Banach spaces.

**Theorem 8.** Let $G$ be a separable almost $\omega$-homogeneous space; $A \hookrightarrow G$. Then

$$\lambda(A \hookrightarrow G) = \lambda(iA \hookrightarrow G) = \lambda(A, G),$$

where $i$ is an arbitrary isometric embedding.

**Proof.** Obviously, any almost $\omega$-homogeneous space has the property:

- For any pair $A, B$ of isometric subspaces of $G$ under an isometry $j : A \rightarrow B$ and every $\varepsilon > 0$ there exists an isomorphic automorphism $u : G \rightarrow G$ such that $u |A = j$ and $\|u\|\|u^{-1}\| \leq 1 + \varepsilon$.

Clearly, this property implies the theorem. $\square$

In [12] it was shown how to compute absolute projection constants of some Banach spaces. Here a method, of computing relative projection constants will be presented.

Let $X$ be an $N$-dimensional Banach space with a symmetric basis $(e_i)_{i=1}^{2N}; Y$ be a subspace of $X$ of dimension $\dim(Y) = m$. Let $K = K(Y \hookrightarrow X) = (y_i)_{i=1}^{2N} \subset \mathbb{R}^m$ be an incarnating (central symmetric) set for a pair $Y \hookrightarrow X$. As $\overline{X}$ (resp., as $\overline{Y}$) it will be considered the set of all odd (resp., all odd linear) functions on $K$ with the corresponding norm. As in the previous section, a pair $[\overline{X}, \overline{Y}]$ will be identified with $Y \hookrightarrow X$.

Let $\mathfrak{G}(X)$ be a group of all isometries of $X$; $\mathfrak{G}_0$ be a given group of (non degenerated) automorphisms of $\mathbb{R}^m$.

Assume that $K = (y_i)_{i=1}^{2N} \subset \mathbb{R}^m$ is invariant under $\mathfrak{G}_0$, i.e., $gK = K$ for all $g \in \mathfrak{G}_0$. Then $\mathfrak{G}_0$ may be identified with a subgroup $\mathfrak{G}_0'$ of $\mathfrak{G}(X)$ under a group-isomorphism $\vartheta : \mathfrak{G}_0 \rightarrow \mathfrak{G}_0'$: for every $g \in \mathfrak{G}_0$ its image $g' = \vartheta g \in \mathfrak{G}_0' \subset \mathfrak{G}(X)$ is given by

$$\{g' f(y_i) = f(g^{-1} y_i) : i = 1, 2, ..., 2N\}.$$

In the future groups $\mathfrak{G}_0$ and $\mathfrak{G}_0'$ will not be distinguished.

**Definition 8.** Let $\mathfrak{G}$ be a subgroup of $\mathfrak{G}(X)$. A projection $P : X \rightarrow Y$ is said to be invariant under $\mathfrak{G}$ if for all $g \in \mathfrak{G}$ and $f \in X$ it satisfies $P(gf) = gP(f)$.

A set of all projections that are invariant under $\mathfrak{G}$ will be denoted by $\text{Inv}(\mathfrak{G})$.

Let $\mathfrak{G}$ be a subgroup of the group $\mathfrak{O}_n$ of all orthogonal matrices on $\mathbb{R}^n$. Let $m(g)$ be a normed Haar measure on $\mathfrak{G}$.
Definition 9. A subgroup $\mathcal{G}$ of the group $O_n$ is said to be an ample group if for all $u, v, x \in \mathbb{R}^n$

$$\int_{\mathcal{G}} gu(x, gv) dm(g) = n^{-1}(u, v)x,$$

where $(u, v)$ is the usual scalar product.

Theorem 9. Let $\mathcal{G}$ be a subgroup of the group $O_n$; $\Gamma$ be a subgroup of $\mathcal{G}$. If $\Gamma$ is an ample group then $\mathcal{G}$ is also ample.

Proof. Let $m(g)$ and $m'(g)$ be normed Haar measures on $\mathcal{G}$ and on $\Gamma$ respectively. Then

$$\int_{\mathcal{G}} gu(x, gv) dm(g) = \int_{\mathcal{G}} \left( \int_{\Gamma} g' gu(x, g' gv) dm'(g) \right) dm(g)$$

$$= \int_{\mathcal{G}} n^{-1}(gu, gv)x dm(g) = n^{-1}(u, v)x.$$

□

There may be presented a pair of important examples of ample groups.

Example 1. Let $\mathcal{G}_1$ be a subgroup of $O_n$, which consists of all operators that rearrange elements of a chosen basis $(e_i)$ and change their signs.

More exactly, let $\sigma$ be a rearrangement of $(1, 2, ..., n); \epsilon = (\epsilon_i)_{i=1}^n; \epsilon_i \in \{+, -\}$. Then every element $g_{\epsilon}^\sigma$ of $\mathcal{G}_1$ acts on the element $x = \sum_{i=1}^n x_i e_i$ as follows:

$$g_{\epsilon}^\sigma(x) = \sum_{i=1}^n \epsilon_i x_i e_{\sigma i} = u_{\epsilon}^\sigma.$$

Theorem 10. $\mathcal{G}_1$ is an ample group.

Proof. Certainly,$$
\sum_{g \in \mathcal{G}_1} gu(x, gv) = \sum_{\sigma, \epsilon} u_{\epsilon}^\sigma (x, v_{\epsilon}^\sigma) = \sum_{\sigma} \sum_{\epsilon} u_{\epsilon}^\sigma (x, v_{\epsilon}^\sigma).
$$

Let

$$\sum_{\sigma, \epsilon} u_{\epsilon}^\sigma (x, v_{\epsilon}^\sigma) = \sum_{i=1}^n z_i \epsilon_i.$$

Then

$$z_i = \sum_{\epsilon} \epsilon_i u_{j i} \left( \epsilon_i v_{j i} x_i + \sum_{l \neq i} \epsilon_l v_{j l} x_l \right)$$

$$= 2^n u_{j i} v_{j i} x_i + u_{j i} \sum_{l \neq i} v_{j l} x_l \left( \sum_{\epsilon} \epsilon_i \epsilon_l \right).$$

Since $\sum_{\epsilon} \epsilon_j \epsilon_l = 0$ for $j \neq l$,

$$z_i = 2^n u_{j i} v_{j i} x_i.$$

So,

$$\sum_{g \in \mathcal{G}_1} gu(x, gv) = 2^n \sum_{\sigma} \sum_{i=1}^n u_{j i} v_{j i} x_i \epsilon_i = 2^n (n-1)! (u, v)x.$$

Since $\text{card} (\mathcal{G}_1) = 2^n n!$,

$$\sum_{g \in \mathcal{G}_1} gu(x, gv) = n^{-1} \text{card} (\mathcal{G}_1) (u, v)x,$$

i.e. $\mathcal{G}_1$ is an ample group. □
Example 2. Let the group \( \mathfrak{G}_2 \) acts on a subspace of \( \mathbb{R}^{n+1} \), which is formed by such \( x = \sum_{i=1}^{m+1} x_i e_i \in \mathbb{R}^{n+1} \) that \( \sum_{i=1}^{m+1} x_i = 0 \). Every \( g_\pi \in \mathfrak{G}_2 \) maps \( x \) to \( y = \sum_{i=1}^{m+1} x_i e_i \) where \( \pi \) is a rearrangement of \( (1, 2, ..., n + 1) \). So, \( \text{card} (\mathfrak{G}_2) = (n + 1)! \).

Theorem 11. \( \mathfrak{G}_2 \) is an ample group.

Proof. Let \( \sum_{g \in \mathfrak{G}_2} gu(x, gv) = \sum_{i=1}^{n+1} z_ie_i \). Then

\[
z_j = \sum_{k=1}^{n+1} u_k \left( n! x_j v_k + \sum_{i \neq j} x_i (n-1)! \left( \sum_{j \neq k} v_i \right) \right)
= n! (u, v) x_j + (n - 1)! \left( \sum_{i \neq j} x_i \right) \left( \sum_{k=1}^{n+1} u_k \sum_{i \neq k} v_i \right).
\]

Since \( \sum_{i \neq j} x_i = -x_j \) \( \sum_{i \neq k} v_i = -v_k \),

\[
z_j = (n! + (n - 1)!)(u, v) x_j = n^{-1} \text{card} (\mathfrak{G}_2) (u, v) x_j.
\]

Hence,

\[
\sum_{g \in \mathfrak{G}_2} gu(x, gv) = n^{-1} \text{card} (\mathfrak{G}_2) (u, v) x.
\]

\( \square \)

Remark 5. Certainly, the group \( \mathfrak{G}_1 \) is a subgroup of the group of all isometries of every \( n \)-dimensional Banach space with a symmetric basis \( (e_i)_{i=1}^n \).

Let \( E \) be a Banach space ; \( \mathfrak{G}(E) \) be the corresponding group of isometries of \( E \); \( (\mathfrak{G}(E))^\# \) be a set of all (bounded, linear) automorphisms of \( E \) that commute with all elements of \( \mathfrak{G}(E) \).

As in [12] it will be said that \( E \) is sufficiently symmetric if \( (\mathfrak{G}(E))^\# = \{ \lambda \text{Id}_E \}_{\lambda \in \mathbb{R}} \).

Theorem 12. Let \( E \) be a finite dimensional Banach space. \( E \) is sufficiently symmetric if and only if \( \mathfrak{G}(E) \) is an ample group.

Proof. Let \( \mathfrak{G}(E) \) be an ample group. Let \( T : E \to E \) commutes with all elements of \( \mathfrak{G}(E) \). Then

\[
n^{-1} (u, Tv) x = \int_{\mathfrak{G}(E)} gu(gTv, x) \, dm(g)
= \int_{\mathfrak{G}(E)} gu(T^*x, gv) \, dm(g) = n^{-1} (u, v) T^*x.
\]

Put \( u = v \). Then \( T^*x = (v, Tv) \|v\|^{-2} x \) for all \( x \in E \). Hence, \( T^*x = \lambda x \) and \( T^* = \lambda \text{Id}_{E^*} \).

Conversely, let \( E \) be sufficiently symmetric. Fix \( u, v \in E \) and put for \( x \in E \)

\[
T x = \int_{\mathfrak{G}(E)} gu(x, gv) \, dm(g).
\]

Let \( g_1 \in \mathfrak{G}(E) \). Then

\[
Tg_1 x = \int_{\mathfrak{G}(E)} gu(g_1 x, gv) \, dm(g) = g_1 \int_{\mathfrak{G}(E)} g_1^{-1} gu(x, g_1^{-1} gv) \, dm(g)
= g_1 \int_{\mathfrak{G}(E)} g_1^{-1} gu(x, g_1^{-1} gv) \, dm(g) = g_1 T x.
\]
Hence, there exists a constant \( c = c(x, y) \) such that \( T = c(x, y) \text{Id}_E \). Indeed,
\[
\int_{\Theta(E)} (gu, y) (x, gv) \, dm(g) = \int_{\Theta(E)} (u, gy) (gx, v) \, dm(g) = c(x, y) (u, v).
\]
At the same time,
\[
\int_{\Theta(E)} (gu, y) (x, gv) \, dm(g) = (Tx, y) = c(u, v) (x, y).
\]
Obviously, \( c(x, y) = (\text{dim}(E))^{-1} = n^{-1} \). So, \( \int_{\Theta(E)} gu (x, gv) \, dm(g) = n^{-1} (u, v) x \).

Hence, \( \Theta(E) \) is an ample group. \( \square \)

Now it will be presented a formula for computing some relative projection constants. Let \( Y \rightarrow X \) be realized as an incarnation pair \([X, Y]\) with an incarnating central symmetric set \( K = (y_i)_{i=1}^{2n} \subset \mathbb{R}^n \), invariant under an ample group \( \Theta \).

Let \( M \) be a set of all mappings \( \mu : K \rightarrow \mathbb{R}^n \) such that \( \mu(gu) = g\mu(u) \) for any \( g \in \Theta \), \( u \in K \).

THEOREM 13. Let \( P : X \rightarrow Y \) be a linear mapping. \( P \in \text{Inv}(\Theta) \) is and only if \( P \) has a representation
\[
P(f) = n \left( \sum_{u \in K} (u, \mu(u)) \right)^{-1} \sum_{u \in K} f(u) \mu(u) + c \sum_{u \in K} \int_{\Theta} (gu, y) \mu(gu) \, dm(g).
\]

Proof. Certainly, if \( P \) has such representation then \( P \) is a linear map. Put \( c = n \left( \sum_{u \in K} (u, \mu(u)) \right)^{-1} \). Let \( m(g) \) be a normed Haar measure on \( \Theta \).

Then for \( y \in Y \)
\[
Py = \sum_{u \in K} (u, y) \mu(u) = c \sum_{u \in K} \int_{\Theta} (gu, y) \mu(gu) \, dm(g)
= c \sum_{u \in K} \int_{\Theta} (gu, y) \mu(u) \, dm(g) = n^{-1} c \sum_{u \in K} (u, \mu(u)) y = y.
\]

Let us show that \( P \) is invariant under \( \Theta \).
\[
P(gf) = P fg^{-1} = c \sum_{u \in K} f(g^{-1}u) \mu(u) = c \sum_{u \in K} f(v) \mu(gv)
= g \left( c \sum_{v \in K} f(v) \mu(v) \right) = gP(f).
\]

To show the necessity of the conditions of the theorem assume that \( P \in \text{Inv}(\Theta) \). Let \( \{e_1, e_2, ..., e_n\} \) be a basis of \( \mathbb{R}^n \).

Consider linear functionals \( \{\varphi_i\}_{i=1}^{n} \) over \( X \), which are given by
\[
\varphi_i(f) = (Pf, e_i) = \sum_{u \in K} f(u) \mu_i(u),
\]
where \( \mu_i \in X \). In a vector form this equality may be written as
\[
P(f) = \sum_{u \in K} f(u) \mu(u); \quad \mu = \{\mu_1, \mu_2, ..., \mu_n\}.
\]

Let \( g \in \Theta \). Then
\[
gP(f) = P(gf) = P fg^{-1} = \sum_{u \in K} f(g^{-1}u) \mu(u) = \sum_{u \in K} f(u) \mu(gu).
\]

Hence,
\[
\sum_{u \in K} f(u) \mu(u) = \sum_{u \in K} f(u) \mu(gu)
\]
for all \( f \in X \). Since \( X \) consists of odd functions, \( \mu(-u) = \mu(u) \). Hence, \( \mu \in M \).
Let \( y \in Y \). Since \( \mu(gu) = g\mu(u) \) and \( \mathcal{G} \) is an ample group,

\[
P(y) = \sum_{u \in K} (u, y) \mu (u) = \sum_{u \in K} \int_{\mathcal{G}} (y, gu) \mu(gu) dm(g) =
\]

\[
= \sum_{u \in K} \int_{\mathcal{G}} (y, gu) g\mu(u) dm(g) = n^{-1} \sum_{u \in K} (u, \mu(u)) y.
\]

Hence, \( n^{-1} \sum_{u \in K} (u, \mu(u)) = 1 \) and, consequently,

\[
P(f) = n \left( \sum_{u \in K} (u, \mu(u)) \right)^{-1} \sum_{u \in K} f(u) \mu(u).
\]

\( \square \)

An easy consequence of this result is a formula for computing projection constants.

**Corollary 3.** Let \( Y \hookrightarrow X \) be realized as an incarnation pair \([X, Y]\) with an incarnating central symmetric set \( K = (y_i)_{i=1}^{2N} \subset \mathbb{R}^n \), which is invariant under an ample group \( \mathcal{G} \). Let \( \mathcal{M} \) be a set of all mappings \( \mu : K \to \mathbb{R}^n \) such that \( \mu(gu) = g\mu(u) \) for any \( g \in \mathcal{G}, u \in K \). Let \( P : X \to Y \) be a linear projection. Then the projection constant \( \lambda(Y \hookrightarrow X) \) is equal to

\[
\lambda(Y \hookrightarrow X) = \inf_{\mu \in \mathcal{M}} \left\{ n \left( \sum_{u \in K} (u, \mu(u)) \right)^{-1} \sup_{f \in \mathcal{X}} \left\{ \sum_{u \in K} f(u) \mu(u) \right\} \right\}.
\]

**Proof.** It is known (see e.g. [13]) that if \( K \) is invariant under \( \mathcal{G} \) then

\[
\lambda(Y \hookrightarrow X) = \inf \{ \|P\| : P \in \text{Inv}(\mathcal{G}); \ P : X \to Y; \ P^2 = P \}.
\]

So, the desired result follows from the representation of \( P \) as above. \( \square \)

**6. Amalgamation property in classes \((L_{2n})^f\) for a natural \( n > 1 \)**

**Theorem 14.** If \( p \in 2\mathbb{N} \setminus \{2\} = \{4, 6, 8, \ldots \} \) then the set \( \mathfrak{M}((L_p)^f) \) does not have the amalgamation property.

**Proof.** Assume that \( \mathfrak{M}((L_p)^f) \) has the amalgamation property. Then, by the theorem 6, there exists a separable almost \( \omega \)-homogeneous space \( W \), which belongs to \((L_p)^f\). By the theorem 7, \( W \) must be existentially closed in \((L_p)^f\). According to [11], \( L_p \) is an envelope of \((L_p)^f\) and, hence, contains a subspace that is isomorphic to \( W \). By the corollary 2, \( W \) is 1-complemented subspace of \( L_p \). From the theorem 2 it follows that \( L_p \) is almost isometric to a complemented subspace of \( W \). Since every isometric image of \( L_p \) in \( L_p \) is orthogonal complemented, it is clear that for any pair \( A, B \) of isometric finite dimensional subspaces of \( L_p \) their projection constants must be equal: \( \lambda(A \hookrightarrow L_p) = \lambda(B \hookrightarrow L_p) \). Let us show that if \( p \in 2\mathbb{N} \setminus \{2\} \) then \( L_p \) contains a pair of Euclidean 2-dimensional subspaces that have different projection constants. Notice, that it is enough to find such a pair in some \( j_p^{(n)} \).

Let \( p = 2n; n \geq 2 \). Let \( K = (y_k)_{k=1}^{2n+2} \) be a set of vertices of symmetric polygon. Let \( (y_k)_{k=1}^{2n+2} \) has polar coordinates

\[
\{(\varphi_k, R_k) = (\pi k / (n + 1); 1) : k = 0, 1, \ldots, 2n + 1 \}.
\]

Let \( L(K) \) be a set of all odd linear functions on \( K \). The norm of an element \( u = (u_1, u_2) \in L(K) \) in the space \( L^{(2n+2)} \) (that is considered as a space of all odd
functions on $K$ with the corresponding norm) is calculated by

$$\|u\|^{2n} = \sum_{n=0}^{2n+1} \left( u_1 \cos \left( \frac{k\pi}{n+1} \right) + u_2 \sin \left( \frac{k\pi}{n+1} \right) \right)^{2n}$$

$$= (\|u\|_2^{2n}) \sum_{n=0}^{2n+1} \left( \cos \left( \frac{k\pi}{n+1} \right) \sin \phi + \sin \left( \frac{k\pi}{n+1} \right) \cos \phi \right)^{2n}$$

$$= (\|u\|_2^{2n}) \sum_{n=0}^{2n+1} \left[ \sin \left( \frac{k\pi}{n+1} + \phi \right) \right]^{2n},$$

where $\|u\|_2 = \sqrt{|u_1|^2 + |u_2|^2}$. Regard $\|u\|^{2n}$ as an function of $\phi$; $\|u\|^{2n} = F(\phi)$. Immediately, the derivative $F'(\phi) = 0$. So, the sum is a constant $c$ and $\|u\| = c \|u\|_2$, i.e. $L(K)$ is isometric to the 2-dimensional Euclidean space.

Let $\Delta_\varphi$ be a linear operator of rotation of the plane in the positive direction by the angle $\varphi$. Put $K(\varphi) = \Delta_\varphi(K)$; $K^{[m]} = \sum_{i=0}^{m-1} K(\pi i/nm)$. Then

$$\|u\|^{2n}_{L(K^{[m]})} = \sum_{i=0}^{m-1} \|\Delta_{\pi i/nm}(u)\|^{2n}_{L(K^{[m]})} = m \|u\|^{2n}_{L(K)}.$$

Hence 2-dimensional subspace $L(K^{[m]})$ of $l_2^{(2n+2)m}$ is also isometric to the Euclidean one.

Let us show that

$$\lambda(L(K^{[m]})) \to l_2^{(2n+2)m}(K^{[m]}) < \lambda(L(K)) \to l_2^{(2n+2)}(K)).$$

Certainly, this would imply the needed result.

For simplicity of calculations assume that $n = 2$; the result in a general case is obtained in a similar way.

Notice that $K$ is invariant under the apple group $\Phi_1$. Hence, it may be used the corollary 3.

$$\lambda(L(K^{[m]})) \to l_4^{(6m)}(K^{[m]})) =$$

$$= 2 \left[ \sum_{u \in K^{[m]}} (u, u) \right]^{-1} \sup_{f \in l_4^{(6m)}(K^{[m]})} \{ \|f\|^{-1} \left\| \sum_{u \in K^{[m]}} (u, f u) \right\|_{l_4} \}$$

$$= \frac{6}{n} \sup_{A_n \to L(K)} \{ A_n(f) \} = \frac{6}{n} \sup_{A_n \to L(K)} \{ A_n(f) \} \text{, where } A_n(f) = \sum_{u \in K^{[m]}} (f(u) u).$$

The adjoint operator $A_n^* : L^*(K^{[m]}) \to l_4^{(3)}(K^{[m]})$ is given by $A_n^*(v) = g$, where $g(y) = (v, y)$; $y \in K^{[m]}$. Since $\|A_n\| = \|A_n^*\|,$

$$\lambda(L(K^{[m]})) \to l_4^{(6m)}(K^{[m]}))$$

$$= \sup_{\|v\|^{-1} \left\| \sum_{i=0}^{m-1} \|\Delta_{\pi i/3n}(v)\|_{l_4^{(3)}} \right\|^{3/4}} \left\{ \frac{3\sqrt{n}}{2\sqrt{2}} \right\}^{1/2} \left( \sum_{i=0}^{m-1} \sup_{v} \left[ (v, v)^{-1/2} \left\| \Delta_{\pi i/3n}(v) \right\|_{l_4^{(3)}} \right]^4 \right)^{3/4}$$

$$\leq \left( \frac{1}{6\sqrt{2}} \right)^{1/2} \sup_{v} \left[ (v, v)^{-1/2} \left\| \sum_{i=0}^{m-1} \|\Delta_{\pi i/3n}(v)\|_{l_4^{(3)}} \right\|^{3/4} \right] \lambda(L(K)) \to l_2^{(2n+2)}(K)).$$
The equality may take place only if the exact upper bound in the expression
\( \sup_v (v, v)^{-1/2} \left\| \Delta_{\pi_i/3n} (v) \right\|_{l_{4/3}(K^{[n]})} \) is attained for all \( i = 0, 1, \ldots, n - 1 \) on the same vector \( v \).

However it is easy to check that
\[
\left\| \left( 1/2, \sqrt{2}/2 \right) \right\|_{l_{4/3}(K)} > \left\| \left( \sqrt{2}/2, \sqrt{2}/2 \right) \right\|_{l_{4/3}(K)}.
\]

Hence, if \( n \) is sufficiently large, it will take place the strong inequality
\[
\lambda(L(K^{[m]}) \hookrightarrow l_4^{(6m)}(K^{[m]})) < \lambda(L(K) \hookrightarrow l_4^{(6)}(K))
\]
and, consequently, the set \( \mathfrak{M}((L_p)^f) \) does not have the amalgamation property. \( \square \)

7. Properties of spaces \( L_1 [0, 1] \) and \( l_1 \)

**Theorem 15.** Space \( L_1 [0, 1] \) is almost \( \omega \)-homogeneous.

**Proof.** Since \( (G_X)^{**} \) is a complemented subspace of \( (L_1 [0, 1])^{**} \), \( G_X \) is a \( L_{1,1+0} \)-space in a sense of J. Lindenstrauss and A. Pelczynski [11] and, hence, is a separable \( L_1 (\mu) \)-space. All pairwise non-isometric separable \( L_1 (\mu) \)-spaces may be listed. They are:

\[
l_1 \colon L_1 [0, 1] \oplus_1 L_1 [0, 1],
\]
\[
\{l_1^{(n)} \oplus_1 L_1 [0, 1] : n \in \mathbb{N} \} \text{ and } l_1 (L_1 [0, 1]) \overset{\text{def}}{=} \left( \sum \oplus L_1 [0, 1] \right)_{l_1}
\]

It is easy to check that only the space \( L_1 [0, 1] \) is existentially closed. Any other space cannot be isometrically embedded in \( L_1 [0, 1] \) in a such way that its image will be a reflecting subspace of \( L_1 [0, 1] \). \( \square \)

**Corollary 4.** For any isometric finite dimensional subspaces \( A, B \) of \( L_1 [0, 1] \) their relative projection constants are equal:
\[
\lambda(A \hookrightarrow L_1 [0, 1]) = \lambda(B \hookrightarrow L_1 [0, 1]).
\]

**Proof.** This follows from the preceding theorem and the theorem 8. \( \square \)

The next result describes two-dimensional subspaces of \( L_1 [0, 1] \). Before its formulation recall that a point \( z \) of the unit sphere \( \partial B(X) = \{ z \in X : \| z \| = 1 \} \) of a Banach space \( X \) is said to be extreme if for any pair \( z_1, z_2 \in \partial B(X) \) the condition \( (z_1 + z_2)/2 = z \) implies that \( z_1 = z_2 = z \).

The set of all extreme points of \( \partial B(X) \) is denoted by \( \text{ext}(X) \).

**Theorem 16.** Let \( Z \) be 2-dimensional Banach space. The following conditions are equivalent:

1. \( Z^* \) is isometric to a 2-dimensional subspace of \( l_1 \).
2. \( \text{ext}(Z) \) is of linear zero measure \( \text{mes ext}(Z) \) on the boundary \( \partial B(Z) \) of the unit cell of \( Z \).

**Proof.** (2 \( \Rightarrow \) 1). Let \( V = \text{ext} (Z) \).

Since \( \text{mes}(V) = 0 \) on \( \partial B(Z) \), and \( V \) is a closed subset of \( \partial B(Z) \), \( V \) is nowhere dense in \( \partial B(Z) \).

Hence, \( \partial B(Z) \) is a closure of the union of edges of \( \partial B(Z) \). Let \( \{r_i : i \in I\} \) be their numeration. Let \( |r_i| \) be the length of the edge \( r_i \).

Since \( \sum_{i \in I} |r_i| = \text{mes} (\partial B(Z)) < \infty \), the set \( I \) is at most countable.
Consider a edge \( r_i \) and denote its ends by \( x_{2i-1} \) and \( x_{2i} \) (the numeration of this ends is formed by the positive direction of path-tracing on \( \partial B(Z) \)).

Let \( K = \{ y_i = (x_{2i} - x_{2i-1})/2 : i < \infty \} \). Let us show that \( K \) is an incarnation set for a pair \( Z^* \hookrightarrow l_1 \). It is sufficient to show that \( K' = B(Z) \) (notations are as in the section 3).

Let \( x \in V \). Let \( R_x \) be a set of all edges of the unit circle \( B(Z) \) that are appeared on the way (along the unit circumference \( \partial B(Z) \)) in the positive direction from \(-x\) up to \( x \). It will be shown that \( \sum \{ y_i : r_i \in R_x \} = x \).

Fix \( \varepsilon > 0 \). Since \( V \) is a metric compact of measure 0, there exists a finite number of pairwise disjoint arcs of \( \partial B(Z) \) that cover \( V \) and whose total length is less than \( \varepsilon \). Let \( z_1, z_2, ..., z_N \) be ends of these arcs (in the positive direction of the path-tracing).

Certainly, \( (z_1 + x)/2 + (z_2 - z_1)/2 + ... + (x - z_N)/2 = x \) and
\[
\|((z_1 + x)/2) + ((z_2 - z_1)/2) + ... + ((x - z_N)/2)\| \leq \varepsilon.
\]

By the construction,
\[
\|\sum \{ y_i : r_i \in R_x \} - (z_1 + x)/2 + (z_2 - z_1)/2 + ... + (x - z_N)/2\| \leq \varepsilon.
\]

Hence,
\[
\|x - \sum \{ y_i : r_i \in R_x \}\| \leq 2\varepsilon.
\]

Since \( \varepsilon \) is arbitrary, \( x = \sum \{ y_i : r_i \in R_x \} \).

Recall that \( x \in \partial B(Z) \) is said to be an exposed point if there exists such \( f \in Z^* \) that \( f(x') < f(x) \) for all \( x' \neq x, x' \in \partial B(Z) \).

Let \( \Pi \) be a supporting line for \( B(Z) \) such that \( \Pi \cap B(Z) = \{ x \} \).

Let \( u \) be such that \( (u, y) > 0 \) for all \( y \in \Pi \). Then
\[
\{ y_i : y_i \in K : (y_i, u) > 0 \} = \{ y_i : r_i \in R_x \}.
\]

Hence,
\[
x = \sum \{ y_i : r_i \in R_x \} = \sum \{ y_i : y_i \in K : (y_i, u) > 0 \} \in K'.
\]

According to the known S. Straszewicz theorem [14], the set of all exposed points of \( B(Z) \) is dense in \( \text{ext}(Z) = V \). So, \( V \subset K' \) and, consequently, \( B(Z) \subset K' \).

Conversely, let \( x \in K' \); \( x = \sum \{ y_i : y_i \in K : (y_i, u) > 0 \} \). Let \( \Pi \) be a supporting line for \( B(Z) \) such that \( (u, y) > 0 \) for all \( y \in \Pi \).

Let \( \Pi \cap B(Z) = \text{conv}\{u_1, u_2\} \), where \( u_1, u_2 \in V \) (recall that \( \text{conv}\{A\} \) denotes the convex hull of \( A \)). Then
\[
\sum \{ y_i : y_i \in K : (y_i, u) > 0 \} = (u_1 + u_2)/2.
\]

Hence, \( K' \subset B(Z) \). Since \( B(Z) \) is a closed convex set, \( K' = B(Z) \).

(1 \( \Rightarrow \) 2). Let \( Z^* \hookrightarrow B(Z) \) be a 2-dimensional subspace. According to the theorem 3, \( B(Z) = K' \). By the theorem 4 the incarnating set \( K \) for a pair \( Z^* \hookrightarrow B(Z) \) is situated on beams (or, in other terminology, on semi-axes) of straight lines that are parallel to edges of \( K' \). Notice that every such beam contains just one vector \( r \) from \( K \). Its length \( |r| \) is equal to one half of the length of a edge of \( K' \) that is parallel to \( r \).

Let \( x \in V \) be an exposed point; \( \Pi \) be a support line such that \( \Pi \cap B(Z) = \{ x \} \). Let \( X \) be the one-dimensional subspace of \( Z \), spanned by \( x \): \( X = \{ \lambda x : \lambda \in \mathbb{R} \} \). Let \( P : Z \rightarrow X \) be a projection, parallel to \( \Pi \). Let \( \Gamma_x \) be an arc of the curve \( \partial B(Z) \).
from $-x$ up to $x$ in the positive direction. Certainly, the mapping $P : \Gamma_x \to [-x, x]$ is one-to-one. By the theorem 4,

$$\sum \{ y_i : y_i \in K; r_i \in R_x \} = x.$$ 

This implies that

$$-x + \sum_{r_i \in R_x} (x_{2i} - x_{2i-1}) = x.$$ 

Hence,

$$-x + \sum_{r_i \in R_x} P(x_{2i} - x_{2i-1}) = x.$$ 

The last equality implies that the measure of a set $P(\Gamma_x \cap V)$ on $[-x, x]$ is equal to 0.

Consider $\Gamma_x$ as a graph of a concave function on $[-x, x]$. It is clear that the image of a 0-measure set under the mapping $P^{-1} : [-x, x] \to \Gamma_x$ also is of zero measure. Hence, $V \cup \Gamma_x$ is of linear measure 0. □

**Remark 6.** It is of interest to compare this result with the following of J. Lindenstrauss [15]:

• Let $x = \sum a_i e_i$ and $y = \sum b_i e_i$ be two elements of $l_1$. Assume that the set

$$\{ a_i/b_i : i < \infty \}$$

is dense in the real line. Then the subspace span$\{x, y\} \hookrightarrow l_1$ is strictly convex.

Using the previous result, a subspace of $l_1$ with the same property may be constructed in other way.

Namely, consider the circle $C = \{(\rho, \varphi) : \rho = 1\}$ (in polar coordinates) and put

$$C_0 = \{(1, \varphi) : \varphi = \pi \sum_{k < \infty} \alpha_k 3^{-k}; \alpha_k \in \{0, 1\} \text{ for all } k < \infty \}.$$ 

Clearly, $C_0$ is the Cantor set on $C$. It will be a set of extreme points of the unit cell $B(W)$ of the space $W$, if $B(W)$ be defined as $\text{conv} C_0$. By the previous theorem, $W$ is conjugate to a subspace $W^*$ of $l_1$. Certainly, $\partial W$ is smooth and, hence, $W^*$ is strictly convex.

8. **Properties of spaces** $L_p[0, 1]$ $(1 \leq p < \infty)$

Since the class $(L_p)^f$ for $p \in 2\mathbb{N}\setminus\{2\}$ does not have the amalgamation property, it does not contain an almost $\omega$-homogeneous space.

**Definition 10.** Let $k < \omega$; $X \in B$. The space $X$ is said to be almost $k$-isotropic if for any pair of isometric subspaces $A, B$ of $X$ of dimension $\leq k$ and every $\varepsilon > 0$ there exists an isomorphic automorphism $u : X \to X$ such that $uA = B$ and $\|u\| \|u^{-1}\| \leq 1 + \varepsilon$.

$X$ is said to be almost $\omega$-isotropic if $X$ is almost $k$-isotropic for all $k < \omega$.

Certainly, every almost $\omega$-isotropic Banach space $X$ is almost $\omega$-homogeneous with respect to the set $H(X)$ of its finite dimensional subspaces, and, hence, respect to the set $\mathfrak{M}(X^f)$. 

Remark 7. It may be proved that a class \( \mathcal{X} \) contains an almost isotropic space (=almost 1-isotropic) if and only if \( \mathcal{M}(\mathcal{X}) \) has the 1-amalgamation property, i.e. if any \( V \)-formation in \( \mathcal{M}(\mathcal{X}) \), which root is 1-dimensional, is amalgamated in \( \mathcal{M}(\mathcal{X}) \). It is known that all \( L_p \) (\( 1 \leq p < \infty \)) are almost isotropic ([16]), i.e. each \( \mathcal{M}(L_p^f) \) has the 1-amalgamation property. Notice, that from results of the previous section it follows that \( \mathcal{M}(L_p^f) \) does not have the 2-amalgamation property (i.e. for \( V \)-formations with the 2-dimensional root) for \( p = 4, 6, 8, \ldots \).

Theorem 17. The class \( (L_p)^f \) for \( p \in [1, \infty] \setminus (2N \setminus \{2\}) \) has the amalgamation property. For such \( p \) the space \( L_p[0,1] \) is almost \( \omega \)-homogeneous.

Proof. As it was noted by W. Lusky [3], from one W. Rudin’s result [4] (in a case of complex scalars) it follows that the space \( L_p[0,1] \) is almost \( \omega \)-isotropic for \( p \neq 4, 6, 8, \ldots \). The same result as in [4] was obtained by W. Linde [5] in the real case. Hence, \( L_p[0,1] \) is almost \( \omega \)-homogeneous with respect to the set \( \mathcal{M}(L_p)^f \). So, the class \( (L_p)^f \) for \( p \in [1, \infty] \setminus (2N \setminus \{2\}) \) has the amalgamation property. \( \square \)

Corollary 5. Let \( p \in [1, \infty] \setminus (2N \setminus \{2\}) \). For any isometric finite dimensional subspaces \( A, B \) of \( L_p[0,1] \) their relative projection constants are equal:

\[
\lambda(A \hookrightarrow L_p[0,1]) = \lambda(B \hookrightarrow L_p[0,1]).
\]

Proof. This follows from the preceding theorem and the theorem 8. \( \square \)

Recall the original Gurarii’s definition [2].

Definition 11. Let \( X \) be a Banach space; \( \mathcal{K} \) be a class of Banach spaces. \( X \) is said to be a space of almost universal disposition with respect to \( \mathcal{K} \) if for any pair of spaces \( A, B \) of \( \mathcal{K} \) such that \( A \) is a subspace of \( B \) (\( A \hookrightarrow B \)), every \( \varepsilon > 0 \) and every isomorphic embedding \( i : A \to X \) there exists an isomorphic embedding \( \hat{i} : B \to X \), which extends \( i \) (i.e., \( i|_A = \hat{i} \)) and such, then

\[
\|i\| \|\hat{i}^{-1}\| \leq (1 + \varepsilon)\|i\|\|\hat{i}^{-1}\|.
\]

Let us show that spaces \( L_p[0,1] \) (\( 1 \leq p < \infty \)) are not spaces of almost universal disposition with respect to \( \mathcal{M}(L_p)^f \). The proof is different for cases \( 1 < p < 2 \); \( p > 2 \) (and \( p \neq 4, 6, 8, \ldots \)) and for \( p = 1 \).

Definition 12. Let \( 1 \leq p \leq \infty \). An operator \( u \in B(X, Y) \) is said to be

- \( p \)-absolutely summing, if there is a constant \( \lambda > 0 \) such that

\[
(\sum_{j<n} \|u(x_j)\|^p)^{1/p} \leq \lambda(\sum_{j<n} |\langle x_j, f \rangle|^p)^{1/p}
\]

for any \( f \in X^* \) and any finite set \( \{x_i : i < n; n < \infty \} \subset X \).

Its \( p \)-absolutely summing norm \( \pi_p(u) \) is the smallest constant \( \lambda \).

- \( p \)-integral, if there exists a such probability measure \( \mu \) and such operators \( v \in B(X, L_\infty(\mu)), w \in B(L_p(\mu), Y^{**}) \) that \( k_Y \circ u = w \circ \varphi \circ v \), where \( \varphi \) is an inclusion of \( L_\infty(\mu) \) into \( L_p(\mu) \).

Its \( p \)-integral norm is given by

\[
i_p(u) = \inf\{\|v\|\|\varphi\|\|w\| : k_Y \circ u = w \circ \varphi \circ v\}.
\]
Theorem 18. Let $X^f$ be a class of finite equivalence, which $l_p$-spectrum $S(X^f)$ is contained in $[1, 2]$. Let $W \in X^f$ be a space of almost universal disposition with respect to $\mathfrak{M}(X^f)$. Then for every $v \in S(X^f)$ there exists a constant $c_r < \infty$ such that for every Banach space $Z$ and each finite rank operator $T : Z \to W$ its $r$-integral norm $\iota_r(T)$ is estimated by

$$
\iota_r(T) \leq c_r \pi_r(T).
$$

Proof. Let $T : Z \to G$ be a finite rank operator with $\pi_r(T) = 1$. Let $S$ be the unit ball of $Z^*$, endowed with weak* (i.e. with $\sigma(Z^*, Z)$-) topology. Let $j : Z \to C(S)$ be the canonical embedding: $jz(z') = z'(z)$ for $z' \in S$. Let $i_{\mu, p} : C(S) \to L_p(S, \mu)$ be the natural (i.e. identical) embedding; $\mu$ be some measure on $S$. Let $Z = jZ \hookrightarrow C(S); i_{\mu, p}(Z) = A_0$. The image $A_0$ is a vector subspace of $L_p(S, \mu)$. Let $A$ be the closure of $A_0$ in the $L_p(S, \mu)$-metric. Let $w_0 : A_0 \to G$ be given by $w_0 \circ (i_{\mu, p} \circ j) z = T(z)$. According to A. Pietsch [17], the measure $\mu$ may be chosen in a such way that $w_0 : A_0 \to W$ is continuous in $L_p(S, \mu)$-metric, and, hence, admits a unique extension to $w : A \to W$.

Since $r \in S(X^f)$, $L_p(S, \mu)$ is finite representable in $W$ and, because of $T$ is finite rank operator, $A \in \mathfrak{M}(X^f)$. Thus, by the definition 11, $w : A \to W$ may be extended to a finite rank operator $\tilde{w} : L_p(S, \mu) \to W$.

So, $T : Z \to W$ admits a factorization $T = \tilde{w} \circ i_{\mu, p} \circ j$. By the definition 12, its $r$-integral norm is uniformly bounded. □

Corollary 6. For $1 < p < 2$ the space $L_p$ is not a space of almost universal disposition.

Proof. According to [18] if $1 < p < 2$ then there exists a Banach space $X$ and a sequence of finite dimensional operators $u_n : X \to L_p$ such that $\sup \pi_p(u_n) < \infty$ and $\sup \iota_p(u_n) = \infty$. Since $p \in S(L_p)$, this contradicts to the previous theorem. □

To consider a case $p > 2$ it will be needed a characterization of classes $X^f$ that contain a space of almost universal disposition.

Definition 13. A class $X^f$ will be called quotient-closed if for every $A \in \mathfrak{M}(X^f)$ and its subspace $B$ the quotient $A/B$ belongs to $\mathfrak{M}(X^f)$.

Theorem 19. Every class $X^f$ that contains a space of almost universal disposition (say, $W$) is quotient-closed.

Proof. Let $G < f X; F_0 \hookrightarrow G; Z \hookrightarrow G/F_0$ be a finite dimensional space. Let us show that for every $\varepsilon > 0$ every operator $v : Z \to W$ may be extended to an operator $\tilde{v} : G/F_0 \to W$ of norm $\|\tilde{v}\| \leq (1 + \varepsilon)\|v\|$. This would imply that for any pair $A, B \in \mathfrak{M}(X^f)$, $B \hookrightarrow A$, $A/B \in \mathfrak{M}(X^f)$.

Let $G_0 \hookrightarrow G; V : G \to E = G/F_0$ be a quotient map; $Z \hookrightarrow E$ and $v : Z \to W$. Let $T : G \to W$. Put $F = V^{-1}(Z)$. Then the operator $T \circ v \circ (V|_F)$ may be extended to $w : G \to W$ with a norm $\|w\| \leq (1 + \varepsilon)\|v\|$.

Since $V^{-1}(0) \subset F$, $w(V^{-1}(0)) = 0$ and, hence, $w = v_1 \circ V$, where $v_1 : E \to W; \|v_1\| \leq (1 + \varepsilon)\|v\|$. Certainly, $v_1$ is the desired extension of $v$. □

Corollary 7. For $2 < p < \infty$ the space $L_p$ is not a space of almost universal disposition.
Proof. It is obvious that for every quotient-closed class $X^f$ its $l_p$-spectrum is:

$$S(X^f) = \left[ \inf S(X^f); \sup S(X^f) \right].$$

Since $S(L_p) = \{2, p\}$ for $p \in (2, \infty)$, the class $(L_p)^f$ cannot be quotient closed. □

It is remain the case $p = 1$.

Theorem 20. $L_1 [0, 1]$ is not the space of almost universal disposition.

Proof. If the class $X^f$ is quotient closed and $l_1$ is finitely representable in $X$ then $X^f = (l_{\infty})^f$. So, the previous theorem yields the needed result. Another proof follows from [6], prop. 1.11. □

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