How rigid the finite ultrametric spaces can be?

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Abstract. A metric space $X$ is rigid if the isometry group of $X$ is trivial. The finite ultrametric spaces $X$ with $|X| \geq 2$ are not rigid since for every such $X$ there is a self-isometry having exactly $|X| - 2$ fixed points. Using the representing trees we characterize the finite ultrametric spaces $X$ for which every self-isometry has at least $|X| - 2$ fixed points. Some other extremal properties of such spaces and related graph theoretical characterizations are also obtained.

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1. Introduction

Recall some definitions from the theory of metric spaces. A metric on a set $X$ is a function $d: X \times X \to \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, such that for all $x, y, z \in X$:

(i) $d(x, y) = d(y, x)$,
(ii) $(d(x, y) = 0) \iff (x = y)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

A metric space $(X, d)$ is ultrametric if the strong triangle inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}$$

holds for all $x, y, z \in X$. In this case $d$ is called an ultrametric on $X$ and $(X, d)$ is an ultrametric space. The spectrum of a metric space $(X, d)$ is the set

$$\text{Sp}(X) = \{d(x, y) : x, y \in X\}.$$

In 2001 at the Workshop on General Algebra [15] the attention of experts on the theory of lattices was guided to the following problem of I. M. Gelfand: using graph theory describe up to isometry all finite ultrametric spaces. An
appropriate representation of ultrametric spaces by rooted trees was proposed in [10], [11], [13], [16].

An application of the representation from [10], [11] is a structural characteristic of all finite ultrametric spaces $X$ for which the Gomory-Hu inequality

$$|\text{Sp}(X)| \leq |X|$$

becomes an equality (see [7] and [17]). The purpose of this papers is to describe the structure of finite ultrametric spaces which have maximum rigidity.

Recall that a graph is a pair $(V,E)$ consisting of a nonempty set $V$ and a (probably empty) set $E$ elements of which are unordered pairs of different points from $V$. For a graph $G = (V,E)$, the sets $V = V(G)$ and $E = E(G)$ are called the set of vertices and the set of edges, respectively. A graph is complete if $\{x,y\} \in E(G)$ for all distinct $x,y \in V(G)$. A path in a graph $G$ is a subgraph $P$ of $G$ for which

$$V(P) = \{x_0, x_1, \ldots, x_k\}, \quad E(P) = \{\{x_0, x_1\}, \ldots, \{x_{k-1}, x_k\}\},$$

where all $x_i$ are distinct. A graph $G$ is connected if any two distinct vertices of $G$ can be joined by a path. A finite graph $C$ is a cycle if $|V(C)| \geq 3$ and there exists an enumeration $(v_1, \ldots, v_n)$ of its vertices such that

$$\{\{v_i, v_j\} \in E(C) \iff (|i - j| = 1 \text{ or } |i - j| = n - 1).$$

A connected graph without cycles is called a tree. A tree $T$ may have a distinguished vertex called the root; in this case $T$ is called a rooted tree. For a rooted tree $T$, we denote by $L_T$ the set of leaves of $T$. We denote by $(G,v,l)$ a graph $G$ with a distinguished vertex $v \in V(G)$ and a labeling function $l: V(G) \to L$. In what follows we usually suppose that the set $L$ coincides with $\mathbb{R}^+$. Let $k \geq 2$. A graph $G$ is called complete $k$-partite if its vertices can be divided into $k$ disjoint nonempty sets $X_1, \ldots, X_k$ so that there are no edges joining the vertices of the same set $X_i$ and any two vertices from different $X_i, X_j, 1 \leq i, j \leq k$ are adjacent. In this case we write $G = G[X_1, \ldots, X_k]$. We shall say that $G$ is a complete multipartite graph if there exists $k \geq 2$ such that $G$ is complete $k$-partite, cf. [3].

2. The representing trees of finite ultrametric spaces

For every metric space $(X,d)$ we write

$$\text{diam } X = \sup\{d(x,y) : x, y \in X\}.$$ 

Definition 2.1 ([4]). Let $(X,d)$ be a finite ultrametric space, $|X| \geq 2$. Define the diametrical graph $G_X$ as follows $V(G_X) = X$ and, for all $u, v \in X$,

$$\{\{u,v\} \in E(G_X) \iff (d(u,v) = \text{diam } X).$$

Lemma 2.2 ([4]). Let $(X,d)$ be a finite ultrametric space, $|X| \geq 2$. Then $G_X = G_X[X_1, \ldots, X_k]$ for some $k \geq 2$. 

With a finite nonempty ultrametric space \((X, d)\), we can associate a labeled rooted tree \(T_X\) by the following rule. The root of \(T_X\) is, by definition, the set \(X\). If \(X = \{x\}\) is a one-point set, then \(T_X\) is a tree consisting of one node \(\{x\}\) which has the label 0. Let \(|X| \geq 2\). According to Lemma 2.2 we have \(G_X = G_X [X_1, \ldots, X_k]\). In this case, we set that, the root of the tree \(T_X\) is labeled by \(\text{diam } X\) and \(T_X\) has \(k\) nodes \(X_1, \ldots, X_k\) of the first level with the labels

\[
l(X_i) := \text{diam } X_i \quad (2.1)
\]

\(i = 1, \ldots, k\). The nodes of the first level indicated by zero are leaves, and those indicated by positive numbers are internal nodes of \(T_X\). If the first level has no internal nodes, then the tree \(T_X\) is constructed. Otherwise, by repeating the above-described procedure with \(X_i\) corresponding to internal nodes of the first level instead of \(X\), we obtain the nodes of the second level, etc. Since \(X\) is finite, all vertices on some level will be leaves, and the construction of \(T_X\) is completed.

The above-constructed labeled rooted tree \(T_X\) is called the representing tree of the ultrametric space \((X, d)\).

Let \((T, v^*)\) be a rooted tree. For every node \(u^*\) of \((T, v^*)\) define a rooted subtree \(T_{u^*}\) of \((T, v^*)\) as follows: \(u^*\) is the root of \(T_{u^*}\) and a vertex \(w \in T\) belongs to \(V(T_{u^*})\) if and only if \(u^*\) lies on the path joining \(v^*\) and \(w\) in \(T\), moreover

\[
\{u, v\} \in E(T_{u^*}) \iff \{u, v\} \in E(T).
\]

for all \(u, v \in V(T_{u^*})\).

The following lemma was formulated in [17] for finite ultrametric spaces \(X\) satisfying the equality \(|X| = |\text{Sp}(X)|\) but its proof is also true for arbitrary finite ultrametric spaces.

**Lemma 2.3.** Let \((X, d)\) be a finite ultrametric space, \(|X| \geq 2\), and let \(a\) and \(b\) be two different leaves of the tree \(T_X\). If \((x_1, x_2, \ldots, x_n)\), \(x_1 = a\), \(x_n = b\), is the path joining \(a\) and \(b\) in \(T_X\), then

\[
d(a, b) = \max_{1 \leq i \leq n} l(x_i). \quad (2.2)
\]

Let \((X, d)\) be a metric space. Recall that a subset \(B\) of \(X\) is a ball in \((X, d)\) if there is \(r \geq 0\) and \(t \in X\) such that

\[
B = \{x \in X: d(x, t) \leq r\}.
\]

In this case we write \(B = B_r(t)\). By \(\mathbf{B}_X\) we denote the set of all balls in \((X, d)\).

The proof of the next lemma can be found in [18] but we reproduce it here for the convenience of the reader.

**Lemma 2.4.** Let \((X, d)\) be a finite ultrametric space with representing tree \(T_X\), \(|X| \geq 2\). Then

(i) \(L_{T_v} \in \mathbf{B}_X\) holds for every node \(v \in V(T_X)\).

(ii) For every \(B \in \mathbf{B}_X\) there exists a node \(v\) such that \(L_{T_v} = B\).
Proof. (i) Let $v \in V(T_X)$ and $t \in L_{T_v}$. Consider the ball

$$B_{l(v)}(t) = \{x \in X : d(x, t) \leq l(v)\}.$$ 

Let $t_1 \in L_{T_v}$ such that $t_1 \neq t$. Since $T_v$ contains a path joining $t$ and $t_1$, according to Lemma $2.3$ we have $d(t, t_1) \leq l(v)$. The inclusion $L_{T_v} \subseteq B_{l(v)}(t)$ is proved. Conversely, suppose there exists $t_0 \in B_{l(v)}(t)$ such that $t_0 \notin L_{T_v}$. Let us consider the path $(t_0, v_1, ..., v_n, t)$. From $t_0 \notin L_{T_v}$ it follows that

$$\max_{1 \leq i \leq n} l(v_i) > l(v),$$

i.e. $d(t_0, t) > l(v)$, we have a contradiction.

(ii) Let $t \in X$ and $B = B_{r}(t)$, where $r = \text{diam } B$. Let $x, y \in B$ with $d(x, y) = r$. Let us consider the path $(v_1, ..., v_n)$ with $v_1 = x$ and $v_n = y$ in the tree $T_X$. According to Lemma $2.3$ we have $d(x, y) = \max_{1 \leq i \leq n} l(v_i)$. Let $i$ be an index such that max here is attained. The proof of the equality $L_{T_{v_i}} = B$ is analogous to the proof of (i). \hfill \square

Let $(G^i, v^i, l^i)$, $i = 1, 2$, be a labeled graphs with the distinguished vertices $v^1, v^2$ and a common set $L$ of labels. A bijective function $f : V(G^1) \to V(G^2)$ is an isomorphism of $G^1$ and $G^2$ if

$$(\{x, y\} \in E(G^1)) \iff (\{f(x), f(y)\} \in E(G^2))$$

for all $x, y \in V(G^1)$. If, in addition, we have $f(v^1) = v^2$, then $f$ is an isomorphism of $(G^1, v^1)$ and $(G^2, v^2)$. The isomorphism $f$ of $(G^1, v^1)$ and $(G^2, v^2)$ is called an isomorphism of the $(G^1, v^1, l^1)$ and $(G^2, v^2, l^2)$ if $l^1(v) = l^2(f(v))$ for every $v \in V(G^1)$.

Definition 2.5. Let $(X, d)$ and $(Y, \rho)$ be metric spaces. A bijective function $f : X \to Y$ is an isometry if

$$d(x, y) = \rho(f(x), f(y))$$

holds for all $x, y \in X$.

Two metric spaces $(X, d)$ and $(Y, \rho)$ are isometric if there is an isometry $f : X \to Y$.

Theorem 2.6. Let $(X, d)$ and $(Y, \rho)$ be finite nonempty ultrametric spaces. Then $(X, d)$ and $(Y, \rho)$ are isometric if and only if the labeled rooted trees $T_X$ and $T_Y$ are isomorphic.

Proof. An isomorphism of $T_X$ and $T_Y$ for isometric $X$ and $Y$ can be inductively constructed if we use of the definition of the representing trees given above. The converse statement follows from Lemma $2.3$. \hfill \square

For the relationships between ultrametric spaces and the leaves or the ends of certain trees see also $[2, 8, 10, 13, 15, 16]$.

Let $(X, d)$ be a finite metric space and let $B_1$, $B_2$ and $B$ be some balls in $(X, d)$. We shall say that $B$ lies between $B_1$ and $B_2$ if

$$B_1 \subseteq B \subseteq B_2 \text{ or } B_2 \subseteq B \subseteq B_1.$$ 

If $B$ lies between $B_1$ and $B_2$, we write $B \in [B_1, B_2]$. 
Definition 2.7. Let us define a graph $\Gamma_X$ by the rule: $V(\Gamma_X) = B_X$ and, for all $B_1, B_2 \in B_X$, $\{B_1, B_2\} \in E(\Gamma_X)$ if and only if
1. $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, and
2. $(B \in \{B_1, B_2\}) \Rightarrow (B = B_1$ or $B = B_2)$ for every $B \in B_X$.

For any labeled rooted tree $T_X = (T_X, X, l)$ we write $T_X = (T_X, X)$.

Theorem 2.8. Let $(X, d)$ be a finite nonempty metric space. Then the graph $\Gamma_X$ is a tree if and only if $(X, d)$ is ultrametric. If $(X, d)$ is ultrametric, then $(\Gamma_X, X)$ is isomorphic to the rooted tree $T_X$ with the isomorphism $V(T_X) \ni u \mapsto L_{T_u} \in V(\Gamma_X)$.

Proof. Suppose that $(X, d)$ is not an ultrametric space. Then there are distinct $x_1, x_2, x_3 \in X$ such that

$$d(x_1, x_2) > \max\{d(x_1, x_3), d(x_3, x_2)\}.$$  \hfill (2.3)

Let $B^1 = B_{r_1}(x_1)$ with $r_1 = d(x_1, x_3)$ and let $B^2 = B_{r_2}(x_2)$ with $r_2 = d(x_2, x_3)$. It is evident that

$$\{x_3\} \subseteq B^1 \cap B^2 \subseteq B^1 \cup B^2 \subseteq X.$$  \hfill (2.4)

Using \hfill (2.4) \hfill and the finiteness of $B_X$ we can find $B^0 \in B_X$ such that $B^0 \subseteq B^1 \cap B^2$ and

$$(B^0 \subseteq B \subseteq B^1 \cap B^2) \Rightarrow (B^0 = B)$$

for every $B \in B_X$. Similarly there exists $B^3 \in B_X$ such that

$$B^3 \supseteq B^1 \cup B^2$$

and

$$(B^1 \cup B^2 \subseteq B \subseteq B^3) \Rightarrow (B = B^3)$$

for every $B \in B_X$.

For all distinct balls $C, D \in B_X$ satisfying $C \subseteq D$, there is a chain

$$L(C, D) = \{A_1, \ldots, A_n\} \subseteq B_X$$

such that

$$A_1 = C, \quad A_n = D, \quad A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$$

and, for every $B \in B_X$ and $i = 1, \ldots, n - 1$,

$$(A_i \subseteq B \subseteq A_{i+1}) \Rightarrow (B = A_i$ or $B = A_{i+1}).$$  \hfill (2.5)

Using \hfill (2.3) \hfill we see that $x_1 \notin B^2$ and $x_2 \notin B^1$. Consequently the balls $B^0$, $B^1$, $B^2$, $B^3$ are pairwise distinct, so that there exist the chains $L(B^0, B^1)$, $L(B^0, B^2)$, $L(B^1, B^3)$ and $L(B^2, B^3)$. Let $L(B^0, B^1) = (A_1, \ldots, A_n)$ and $L(B^1, B^3) = (A_n, \ldots, A_{n+m})$. We claim that $L(B^0, B^1, B^3) := (A_1, \ldots, A_n, \ldots, A_{n+m})$ is a path in the graph $\Gamma_X$ joining $B^0 = A_1$ and $B^3 = A_{n+m}$.

Indeed, from the definition of $L(B^0, B^1)$ and $L(B^1, B^3)$ it follows that $A_i \neq A_j$ for any distinct $i, j \in \{1, \ldots, n + m\}$. Moreover, \hfill (2.5) \hfill implies that \hfill $\{A_i, A_{i+1}\} \in E(\Gamma_X)$ for every $i \in \{1, \ldots, n+m-1\}$. Hence $L(B^0, B^1, B^3)$ is a
path in $\Gamma_X$. Inequality (2.3) implies that $x_1 \notin B^2$ and $x_2 \notin B^1$, consequently $B^1 \not\subseteq B^2$ and $B^2 \not\subseteq B^1$. Since for every vertex $B \in V(L(B^0, B^1, B^3))$ we have

$$B \subset B^1 \text{ or } B^1 \subset B,$$

the ball $B^2$ is not a vertex of the path $L(B^0, B^1, B^3)$. Similarly we can construct a path $L(B^0, B^2, B^3)$ such that $B^1 \notin V(L(B^0, B^2, B^3))$. Consequently the vertices $B^0$ and $B^3$ can be joined by two different paths in $\Gamma_X$. Hence $\Gamma_X$ is not a tree.

Suppose now that $(X, d)$ is an ultrametric space. We must show that $\Gamma_X$ is a tree and that $(\Gamma_X, X)$ is isomorphic to the rooted tree $T_X$. By Lemma 2.4 for every $v \in V(\Gamma_X)$, we have $L_{T_v} \in B_X$. Consequently the mapping

$$V(T_X) \ni v \mapsto L_{T_v} \in V(\Gamma_X) \quad (2.6)$$

is correctly defined. Hence it is sufficient to show that this mapping is an isomorphism of the rooted tree $T_X$ and $(\Gamma_X, X)$.

Using Lemma 2.4 again we obtain that (2.6) is bijective. It is also clear that the root of $T_X$ corresponds to $X$ under mapping (2.6). It still remains to show that

$$(\{v_1, v_2\} \in E(T_X)) \Leftrightarrow (\{L_{T_{v_1}}, L_{T_{v_2}}\} \in E(\Gamma_X)) \quad (2.7)$$

holds for all $v_1, v_2 \in V(T_X)$.

Write $B_1 = L_{T_{v_1}}$ and $B_2 = L_{T_{v_2}}$. Then it follows from the definition of the rooted subtrees that $B_1 \subseteq B_2$ if and only if $v_1$ is a node of the tree $T_{v_2}$. Moreover if $B_1 \subseteq B_2$, then

$$(B_1 \subseteq B \subseteq B_2) \Rightarrow (B_2 = B \text{ or } B_1 = B)$$

holds for all $B \in B_X$ if and only if $v_1$ is a direct successor of $v_2$. Statement (2.7) follows.

\[\Box\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{For all nonultrametric metric triangles $X$ their graphs $\Gamma_X$ are isomorphic to the depicted graph.}
\end{figure}

Since a connected graph $G$ is a tree if and only if

$$|V(G)| = |E(G)| + 1$$

(see, for example, \cite[Corollary 1.5.3]{3}), Theorem 2.8 implies the following
Corollary 2.9. Let $X$ be a finite nonempty metric space. Then $X$ is an ultrametric space if and only if

$$|V(\Gamma_X)| = |E(\Gamma_X)| + 1.$$ 

Recall that a graph $H$ is a spanning subgraph of a graph $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A graph is connected if and only if it has a spanning tree [1, Theorem 4.6].

Corollary 2.10. Let $X$ be a finite nonempty metric space and $Y$ be a finite nonempty ultrametric space. If $|B_X| = |B_Y|$, then the inequality

$$|E(\Gamma_X)| \geq |E(\Gamma_Y)|$$

(2.8)

holds. Furthermore, equality in (2.8) occurs if and only if $X$ is ultrametric.

Proof. It is easy to see that $\Gamma_X$ is connected. Let $T$ be a spanning tree of $\Gamma_X$. Then we have

$$|E(\Gamma_X)| \geq |E(T)|.$$ (2.9)

Since $|V(T)| = |V(\Gamma_Y)| = |V(\Gamma_X)| = |B_X| = |B_Y|$ and $|V(T)| = |E(T)| + 1$ and $|V(\Gamma_Y)| = |E(\Gamma_Y)| + 1$, inequality (2.9) implies (2.8). Now using Corollary 2.9 we obtain that

$$|E(\Gamma_X)| = |E(\Gamma_Y)|$$

holds if and only if $X$ is an ultrametric space. □

3. Finite ultrametric spaces having maximum rigidity

Let $(X, d)$ be a metric space and let $\text{Iso}(X)$ be the group of isometries of $(X, d)$. We say that $(X, d)$ is rigid if $|\text{Iso}(X)| = 1$. It is clear that $(X, d)$ is rigid if and only if $g(x) = x$ for every $x \in X$ and every $g \in \text{Iso}(X)$.

For every self-map $f: X \to X$ we denote by $\text{Fix}(f)$ the set of fixed points of $f$. Using this denotation we obtain that a finite metric space $(X, d)$ is rigid if and only if

$$\min_{g \in \text{Iso}(X)} |\text{Fix}(g)| = |X|.$$ (3.1)

Proposition 3.1. Let $(X, d)$ be a finite ultrametric space with $|X| \geq 2$. Then $(X, d)$ is nonrigid.

Proof. It is sufficient to construct $g \in \text{Iso}(X)$ such that

$$|\text{Fix}(g)| \leq |X| - 2.$$ (3.1)

Since $X$ is finite, the representing tree $T_X$ contains an internal node $v$ such that the direct successors of $v$ are leaves of $T_v$. We have the inequality

$$|L_{T_v}| \geq 2$$

because $v$ is internal. Hence there is a fixed point free bijection $\psi: L_{T_v} \to L_{T_v}$, i.e.

$$|\text{Fix}(\psi)| = 0$$ (3.2)
holds. The leaves of $T_X$ is the one-point subsets of $X$. Identifying the leaves with their respective points of $X$ we can define a bijection $g : X \to X$ as

$$g(x) = \begin{cases} 
\psi(x) & \text{if } x \in L_{T_v} \\
x & \text{if } x \in X \setminus L_{T_v}.
\end{cases} \quad (3.3)$$

Lemma 2.3 implies that $g \in \text{Iso}(X)$. From (3.2) and (3.3) we obtain the equality

$$|\text{Fix}(g)| = |X \setminus L_{T_v}| = |X| - |L_{T_v}|.$$

Since $|L_{T_v}| \geq 2$, inequality (3.1) follows. □

Remark 3.2. The Fibonacci space is an interesting example of a compact infinite rigid ultrametric space. (See [13] and [14] for some interesting properties of the Fibonacci space.)

If a metric space $(X, d)$ is finite, nonempty and nonrigid, then the inequality

$$\min_{g \in \text{Iso}(X)} |\text{Fix}(g)| \leq |X| - 2 \quad (3.4)$$

holds, because the existence of $|X| - 1$ fixed points for $g \in \text{Iso}(X)$ implies that $g$ is identical.

The quantity $\min_{g \in \text{Iso}(X)} |\text{Fix}(g)|$ can be considered as a measure for “rigidness” for finite metric spaces $(X, d)$. Thus the finite ultrametric spaces satisfying the equality

$$\min_{g \in \text{Iso}(X)} |\text{Fix}(g)| = |X| - 2, \quad (3.5)$$

are as rigid as possible. Let us denote by $\mathcal{R}$ the class of all finite ultrametric spaces $(X, d)$ which satisfy this equality.

Lemma 3.3. Let $(X, d)$ be a nonempty finite ultrametric space. Then for every set of partial self-isometries

$$S = \{\psi_i : B_i \to B_i : i \in I\},$$

where $I$ is an index set, such that

$$B_i \in \mathcal{B}_X, \quad B_i \cap B_j = \emptyset$$

for all distinct $i, j \in I$, there exists an isometry $\psi \in \text{Iso}(X)$ for which the restriction $\psi|_{B_i}$ equals $\psi_i$ for every $i \in I$.

Proof. Let us define a required $\psi$ by the rule

$$\psi(x) = \begin{cases} 
\psi_i(x), & \text{if } x \in B_i, i \in I, \\
x, & \text{if } x \in X \setminus \left( \bigcup_{i \in I} B_i \right).
\end{cases}$$

It follows from Lemma 2.2 that $\psi \in \text{Iso}(X)$. □

Recall that a rooted tree is strictly $n$-ary if every internal node has exactly $n$ children. In the case $n = 2$ such tree is called strictly binary. A level of a vertex of a rooted tree $(T, v^*)$ can be defined by the following inductive rule: The level of the root $v^*$ is zero and if $u \in V(T)$ has a level $x$, then every direct successor of $u$ has the level $x + 1$. 
Theorem 3.4. Let \((X,d)\) be a finite ultrametric space with \(|X| \geq 2\). Then the following statements are equivalent:

(i) \((X,d) \in \mathcal{R}\);
(ii) \(|\text{Iso}(X)| = 2\);
(iii) \(T_X\) is strictly binary with exactly one inner node at each level except the last level.

Proof. (i)\(\Rightarrow\)(ii). Let (i) hold and

\[|\text{Iso}(X)| \neq 2.\]

By Proposition 3.1, the ultrametric space \((X,d)\) is nonrigid. Consequently there exist \(\psi_1, \psi_2 \in \text{Iso}(X)\) such that

\[\psi_1 \neq \psi_2 \quad \text{and} \quad \psi_1 \neq \text{id}_X \neq \psi_2,\]

where \(\text{id}_X\) is the identical mapping of \(X\). Since \((X,d) \in \mathcal{R}\), we can find the sets \(\{x_1^1, x_2^1\}\) and \(\{x_1^2, x_2^2\}\) such that \(\psi_i(x_1^1) = x_2^2, \psi_i(x_2^1) = x_1^1, i = 1, 2\) and

\[\{x_1^1, x_2^1\} \neq \{x_1^2, x_2^2\}. \quad (3.6)\]

Note that if \(\{x_1^1, x_2^1\} = \{x_1^2, x_2^2\}\), then \(\psi_1 = \psi_2\) because all points of the set \(X\setminus (\{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2\})\) are fixed points of \(\psi_1\) and \(\psi_2\). A short calculation shows that the set \(\{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2\}\) contains no fixed points of the composition \(\psi_1 \circ \psi_2\) and that

\[\psi_1 \circ \psi_2(x) = x\]

for every \(x \in X\setminus (\{x_1^1, x_2^1\} \cup \{x_1^2, x_2^2\})\). Hence we have

\[|\text{Fix}(\psi_1 \circ \psi_2)| = |X| - 3 \quad \text{or} \quad |\text{Fix}(\psi_1 \circ \psi_2)| = |X| - 4,\]

that contradicts equality (3.5).

(ii)\(\Rightarrow\)(iii). Let \(|\text{Iso}(X)| = 2\). We must prove (iii). First, we prove that:

\((s_1)\) For every inner node \(v\) of \(T_X\), the set \(\text{Ch}(v)\) of children of \(v\) contains at most one inner node and at most two leaves.

Let \(v\) be an inner node of \(T_X\) and let \(v_1, v_2 \in \text{Ch}(v)\) be distinct inner nodes. Then the balls \(B_1 = L_{T,v_1}\) and \(B_2 = L_{T,v_2}\) are disjoint and the inequalities

\[|B_1| \geq 2, \quad |B_2| \geq 2\]

hold. By Proposition 3.1, the metric spaces \((B_1,d)\) and \((B_2,d)\) are nonrigid. Hence there are \(\psi_1 \in \text{Iso}(B_1)\) and \(\psi_2 \in \text{Iso}(B_2)\) such that

\[\psi_1 \neq \text{id}_{B_1}, \quad \text{and} \quad \psi_2 \neq \text{id}_{B_2},\]

where \(\text{id}_{B_i}\) is the identical mapping of the set \(B_i, i = 1, 2\). By Lemma 3.3 there are \(\psi^1, \psi^2 \in \text{Iso}(X)\) such that

\[\psi^1|_{B_1} = \psi_1, \quad \psi^1|_{B_2} = \text{id}_{B_2}, \quad \psi^2|_{B_1} = \text{id}_{B_1}, \quad \psi^2|_{B_2} = \psi_2.\]

Since \(\psi^1, \psi^2 \in \text{Iso}(X)\) and \(\psi^1 \neq \psi^2\) and \(\psi^1 \neq \text{id}_X \neq \psi^2\), we have

\[|\text{Iso}(X)| \geq 3,\]

contrary to \(|\text{Iso}(X)| = 2\). The first part of \((s_1)\) is proved. In what follows we write \(B = L_{T,v}\) for short. Let \(G_B = G_B[X_1, \ldots, X_k]\) be the diametrical
graph of the ball $B$. A leaf $\{x\}$ of $T_v$ is a child of $v$ if and only if there is
$i \in \{1, \ldots, k\}$ such that $X_i = \{x\}$. Suppose that there are some three distinct
leaves among the children of $v$. For certainty, we can assume that

$$X_1 = \{x_1\}, \quad X_2 = \{x_2\} \quad \text{and} \quad X_3 = \{x_3\}. $$

Let $S = \{x_1, x_2, x_3\}$. Using the definition of the diametrical graphs we can
easily prove that every bijection $\alpha : S \to S$ can be extended to an isometry
of $B$. Consequently, by Lemma [3.3] there is an extension of $\alpha$ to an isometry
of $S$. Since $\text{Sym}(S)$ (the group of symmetries of $S$) has the order 6, we have
$|\text{Iso}(X)| \geq 6$, contrary to $|\text{Iso}(X)| = 2$. Statement $(s_1)$ follows.

To finished the proof of $(iii)$, it suffices to show that $T_X$ does not contain
any node with exactly 3 children. Suppose contrary that $v$ is an node of $T_X$
with 3 children. Write $B = L_{T_v}$. Then we have $G_B = G_B[X_1, X_2, X_3]$. Using
$(s_1)$ we can suppose that

$$|X_1| = |X_2| = 1 \quad \text{and} \quad |X_3| \geq 2. $$

Let $\{x_1\} = X_1$ and $\{x_2\} = X_2$, $x_1, x_2 \in X$, and let $S = \{x_1, x_2\}$. Let us
consider $\alpha \in \text{Sym}(S)$ and $\beta \in \text{Iso}(X_3)$. Define $\psi : B \to B$ as

$$\psi(x) = \begin{cases} 
\alpha(x) & \text{if } x \in S, \\
\beta(x) & \text{if } x \in X_3. 
\end{cases} $$

It follows directly from Definition [2.1] that $\psi \in \text{Iso}(B)$. Hence, $|\text{Iso}(B)| \geq 4$.
This inequality and Lemma [3.3] imply the inequality $|\text{Iso}(X)| \geq 4$, contrary to $|\text{Iso}(X)| = 2$.

$(iii) \Rightarrow (i)$. Let $(iii)$ hold. By Theorem [2.6] to prove $(i)$ it suffices to show
that every self-isomorphism $f : V(T_X) \to V(T_X)$ of representing tree $T_X$
has at most two points which are not fixed points. Let us consider a self-
isomorphism $f : V(T_X) \to V(T_X)$. The root $X$ is evidently a fixed point of $f$.
If $v'_1$ and $v'_2$ are the nodes of the second level and $v'_2$ is inner, then according to
$(iii)$ $v'_2$ is a leaf and $f(v'_1) = v'_1$ and $f(v'_2) = v'_2$ because $f$ preserves the levels,
the inner nodes and the labels. Similarly we can prove that all vertices on an
arbitrary level are fixed if it is not the last level of $T_X$. Now it is sufficient to
note that the last level contains exactly two vertices, because $T_X$ is strictly
binary and the previous level contains exactly one inner node of $T_X$. \hfill \Box

Using Theorem [3.4] we can obtain the following extremal property of
ultrametric spaces belonging to $\mathfrak{R}$.

**Corollary 3.5.** Let $X$ and $Y$ be finite ultrametric spaces with $|X| = |Y|$. If $Y \in \mathfrak{R}$, then the inequality

$$|\text{Sp}(X)| \leq |\text{Sp}(Y)|$$

holds.

**Proof.** Indeed, it was proved by E. R. Gomory and T. C. Hu in [9] that for
every finite ultrametric space $X$ we have the inequality

$$|\text{Sp}(X)| \leq |X|.$$
As was shown in [17] the equality $|\text{Sp}(X)| = |X|$ holds if and only if $T_X$ is strictly binary and the labels of different internal nodes are different. Note that if $Z$ is a finite ultrametric space with $u, v \in V(T_Z)$ and $u$ is a child of $v$, then Definition 2.1, Lemma 2.2 and the definition of $T_Z$ imply the strict inequality

$$l(u) < l(v). \quad (3.8)$$

Since, for every $Y \in \mathcal{R}$, the representing tree $T_Y$ has exactly one inner node on each level except the last level, inequality (3.8) shows that the labels of different internal nodes are different. Hence (3.7) holds if $Y \in \mathcal{R}$. □

**Remark 3.6.** If $X$ is a finite ultrametric space and $|X| = |\text{Sp}(X)|$, then $X$ is generally not an element of $\mathcal{R}$ (see Figure 3).
4. Rigidness, stars and weak similarities

Following [1] denote by $K_{m,n} = K_{m,n}[X,Y]$ a complete bipartite graph such that $|X| = m$ and $|Y| = n$. In the case when $m = 1$ or $n = 1$ such graphs are called stars.

The following proposition gives us the first characterization of the class $\mathcal{R}$ by stars.

**Proposition 4.1.** Let $(X,d)$ be a finite ultrametric space with $|X| \geq 2$. Then $X \in \mathcal{R}$ if and only if the diametrical graphs $G_B$ are stars for all $B \in B_X$ with $\text{diam } B > 0$.

**Proof.** From Lemma 2.4 and the definition of strictly binary trees it follows that the proposition is a reformulation of the logic equivalence (i)$\iff$(iii) of Theorem 3.4. □

**Definition 4.2.** Let $(X,d)$ be a metric space with a spectrum $\text{Sp}(X)$ and let $r \in \text{Sp}(X)$ be positive. Denote by $G_{r,X}$ a graph for which $V(G_{r,X}) = X$ and

$$ (\{u,v\} \in E(G_{r,X})) \iff (d(u,v) = r), \quad u,v \in X. $$

For $r = \text{diam } X$ it is clear that $G_{r,X}$ is the diametrical graph of $X$.

Let $G = (V,E)$ be a nonempty graph, and let $V_0$ be the set (possibly empty) of all isolated vertices of the graph $G$. Denote by $G'$ the subgraph of the graph $G$, generated by the set $V \setminus V_0$.

**Proposition 4.3.** Let $(X,d)$ be a finite ultrametric space with $|X| \geq 2$. Then $(X,d) \in \mathcal{R}$ if and only if for every positive $r \in \text{Sp}(X)$ the graph $G'_{r,X}$ is isometric to the star $K_{1,n-p}$, where $p$ is the level of a node of $T_X$ labeled by $r$ and $n = |X| - 1$.

**Proof.** Let $(X,d) \in \mathcal{R}$ and let $r \in \text{Sp}(X)$ be positive. Let $x_n$ be a leaf of the last level $n$. Consider the path $(x_n, x_{n-1}, \ldots, x_0)$ from $x_n$ to the root $x_0 = X$ of $T_X$. Using statement (iii) of Theorem 3.4 and properties of representing trees we conclude that $x_n, \ldots, x_0$ are the only possible inner nodes of $T_X$ and all labels of $x_n, x_{n-1}, \ldots, x_0$ are different. Hence there is a unique node, say $x_p$, $0 \leq p \leq n - 1$ labeled by $r$. Let $x'$ be a direct successor of $x_p$ which is leaf and $x''$ be another direct successor of $x_p$. According to Lemma 2.3 the equality $d(x,y) = r$ is possible only if $x = x'$ and $y \in L_{x''}$. Since $|L_{x''}| = n - p$, the graph $G'_{r,X}$ is isomorphic to $K_{1,n-p}$.

The converse follows from Definition 4.2, statement (iii) of Theorem 3.4 and the definition of representing tree $T_X$. □

The following lemma is a reformulation of Theorem 4.1 from [1].

**Lemma 4.4.** Let $(X,d)$ be a finite ultrametric space with $|X| \geq 2$ and let $G_X$ be the diametrical graph of $X$. Then the inequality

$$ |E(G_X)| \geq |X| - 1 \quad (4.1) $$

holds. The equality in (4.1) occurs if and only if $G_X$ is isomorphic to a star.
Lemma 4.5. Let \((X, d)\) be a finite ultrametric space with \(|X| \geq 2\). If \((X, d) \in \mathcal{R}\) then for every \(Y \subseteq X, |Y| \geq 2\), we have \((Y, d) \in \mathcal{R}\).

Proof. Let \(n = |X|\). It is sufficient to prove that \((Y, d) \in \mathcal{R}\) for the case

\[ Y = X \setminus \{x_i\}, \quad x_i \in X. \]

Taking into consideration Lemma 2.4 the space \(X = \{x_1, ..., x_n\}\) for which \(T_X\) fulfils statement (iii) of Theorem 3.4 can be the uniquely presented by sequence of nested balls \(B_1 \subset B_2 \subset ... \subset B_{n-1} \subset B_n\), where \(B_1 = \{x_1\}, B_i = B_{i-1} \cup \{x_i\}, i = 2, ..., n\). Let us consider the set \(B_Y\). It is clear that the following relations hold

\[ B_1 \subset B_2 \subset ... \subset B_{n-2} \subset B_{n-1} \]

where \(B_1 = B_1, ... , B_{i-1} = B_{i-1}, B_i = B_{i+1} \setminus \{x_i\}, ... , B_{n-1} = B_n \setminus \{x_i\}\).

Relations (4.2) and Lemma 2.4 imply that \(T_Y\) fulfils statement (iii) of Theorem 3.4. \(\square\)

The next proposition gives us a new characteristic extremal property of spaces \(X \in \mathcal{R}\).

Proposition 4.6. Let \((X, d)\) be a finite ultrametric space with \(|X| \geq 2\). Then the following statements are equivalent.

(i) \(X \in \mathcal{R}\);

(ii) The inequality

\[ |E(G_Y)| \leq |E(G_Z)| \]

holds for all \(Y \subseteq X\) and all ultrametric spaces \(Z\) which satisfy the condition \(|Y| = |Z| \geq 2\).

Proof. Let \(X \in \mathcal{R}\). Then by Lemma 4.5 we have \(Y \in \mathcal{R}\) for every \(Y \subseteq X\) with \(|Y| \geq 2\). Hence \(G_Y\) is a star, so that

\[ |E(G_Y)| = |Y| - 1. \]

Lemma 4.4 implies

\[ |E(G_Z)| \geq |Z| - 1 \]

if \(Z\) is a finite ultrametric space with \(|Z| \geq 2\). Now (4.3) follows from (4.4), (4.5) and the equality \(|Y| = |Z|\).

Let (ii) hold and let \(B \in \mathcal{B}_X\). Condition (ii) and Lemma 4.4 imply that the diametrical graph \(G_B\) is a star. Hence \(X \in \mathcal{R}\) by Proposition 4.1. \(\square\)

A graph \(G = (V, E)\) together with a function \(\omega : E \rightarrow \mathbb{R}^+\) is called a weighted graph with the weight \(\omega\). The weighted graph will be denoted as \((G, \omega)\). In the following we identify a finite ultrametric space \((X, d)\) with a complete weighted graph \((G, \omega_d)\) such that \(V(G) = X\) and

\[ \omega_d(\{x, y\}) = d(x, y) \]

for all distinct \(x, y \in X\).
Lemma 4.7. Let \((X, d)\) be an ultrametric space. Then for any cycle \(C\) in \((G, \omega_d)\) there exist at least two different edges \(e_1, e_2 \in E(C)\) such that
\[
\omega_d(e_1) = \omega_d(e_2) = \max_{e \in E(C)} \omega_d(e).
\]

If \(|E(C)| = 3\), then Lemma 4.7 is a reformulation of the strong triangle inequality. For \(|E(C)| \geq 4\), it can be proved by induction on \(|E(C)|\). (For details see [6, Lemma 1].)

For a graph \(G = (V, E)\) a Hamiltonian path is a path in \(G\) that visits every vertex of \(G\) exactly once. It is clear that a path \(P \subseteq G\) is Hamiltonian if and only if \(P\) is a spanning tree of \(G\). The following theorem gives us some characterizations of ultrametric spaces \((X, d) \in \mathcal{R}\) via Hamiltonian paths and spanning stars of \((G, \omega_d)\).

Theorem 4.8. Let \((X, d)\) be a finite ultrametric space with \(|X| \geq 2\). Then the following statements are equivalent.

(i) \((X, d) \in \mathcal{R}\).
(ii) The graph \((G, \omega_d)\) has a Hamiltonian path \(P = (x_1, \ldots, x_n)\) such that
\[
\omega_d(\{x_k, x_{k+1}\}) > \omega_d(\{x_{k+1}, x_{k+2}\}) \tag{4.6}
\]
for \(k = 1, \ldots, n - 2\), where \(n = |X|\).
(iii) The graph \((G, \omega_d)\) has a spanning star \(S\) with
\[
E(S) = \{\{y_0, y_1\}, \ldots, \{y_0, y_{n-1}\}\}
\]
such that
\[
\omega_d(\{y_0, y_i\}) \neq \omega_d(\{y_0, y_j\}) \tag{4.7}
\]
for distinct \(i, j \in \{1, \ldots, n - 1\}\), where \(n = |X|\).

Proof. (i)⇒(ii). Let \((X, d) \in \mathcal{R}\) and let \(n = |X|\). By statement (iii) of Theorem 3.4 the representing tree \(T_X\) has exactly one inner node at each level expect the last level and, moreover, the last level contains exactly two leaves. Hence the number of the levels of \(T_X\) is \(n\) and we can enumerate the points of \(X\) in the sequence \((x_1, \ldots, x_n)\) such that, for \(k = 1, \ldots, n - 2\), \(\{x_k\}\) is the leaf on the \(k\)-th level and \(\{x_{n-1}\}, \{x_n\}\) are the leaves of the last level. From the definition of the diametrical graphs it follows that
\[
\omega_d(\{x_1, x_2\}) = d(x_1, x_2) = l(X) = \text{diam } X.
\]
(Recall that \(X\) is the root of \(T_X\).)

Similarly, for every \(k \in \{2, \ldots, n - 1\}\), we have
\[
\omega_d(\{x_k, x_{k+1}\}) = d(x_k, x_{k+1}) = l(v_k),
\]
where \(v_k\) is the unique inner node at the \(k\)-th level. Note now that if \(v, u\) are the nodes of \(T_X\) and \(v\) is a child of \(u\), then the inequality
\[
l(v) < l(u)
\]
holds. It follows directly from statement (iii) of Theorem 3.4 that \(v_{k+1}\) is a child of \(v_k\) for \(k = 1, \ldots, n - 2\) and that \(v_1\) is a child of \(X\). Inequality (4.7) follows.
(ii) ⇒ (iii). Let $P = (x_1, \ldots, x_n)$ be a Hamiltonian path in $(G, \omega_d)$ such that (4.6) holds for $k = 1, \ldots, n - 2$. Let us define $y_i = x_{n-i}$ for $i = 0, \ldots, n - 1$. Using Lemma 4.7 with the cycles $C = C_i$ and $C = C_{i+1}$ such that

$$E(C_i) = \{\{y_{0}, y_{1}\}, \{y_{1}, y_{2}\}, \ldots, \{y_{i-1}, y_{i}\}, \{y_{i}, y_{0}\}\}$$

and

$$E(C_{i+1}) = \{\{y_{0}, y_{1}\}, \{y_{1}, y_{2}\}, \ldots, \{y_{i-1}, y_{i}\}, \{y_{i}, y_{i+1}\}, \{y_{i+1}, y_{0}\}\}$$

we obtain

$$\omega_d(\{y_{0}, y_{i}\}) < \omega_d(\{y_{0}, y_{i+1}\})$$

for $i = 0, \ldots, n - 1$. Statement (iii) follows.

(iii) ⇒ (i). Let (iii) hold. Then, without loss of generality, we can set $X = \{x_1, x_2, \ldots, x_n\}$ and

$$d(x_n, x_1) > d(x_n, x_2) > \ldots > d(x_n, x_{n-1}).$$

![Figure 4. A spanning star in $(G, \omega_d)$ for $(X,d) \in \mathcal{R}$ with $|X| = 8$.](image)

Using the strong triangle inequality we obtain that

$$d(x_1, x_i) = \text{diam } X \quad \text{and} \quad d(x_i, x_j) < \text{diam } X$$

for all distinct $i, j \in \{2, \ldots, n\}$. Similarly we see that

$$d(x_k, x_i) = \text{diam} \{x_k, x_{k+1}, \ldots, x_n\} \quad \text{and} \quad d(x_i, x_j) < d(x_k, x_i)$$

if $i, j \in \{k + 1, \ldots, n\}$.

Hence statement (iii) of Theorem 3.4 holds. The implication (iii) ⇒ (i) follows. □

Recall that a cycle $C$ in a graph $G$ is Hamiltonian if $V(C) = V(G)$.

**Corollary 4.9.** Let $(X,d)$ be a finite ultrametric space with $|X| \geq 3$. Then $(X,d) \in \mathcal{R}$ if and only if the weighted graph $(G,\omega_d)$ contains a Hamiltonian cycle $(x_1, \ldots, x_n)$ such that

$$\omega_d(\{x_1, x_2\}) = \omega_d(\{x_n, x_1\}) = \max_{e \in E(C)} \omega_d(e)$$
and
\[ \omega_d(\{x_k, x_{k+1}\}) > \omega_d(\{x_{k+1}, x_{k+2}\}) \]
for \( k = 1, \ldots, n - 2 \).

The proof is immediate from statement (ii) of Theorem 4.8 and Lemma 4.7. The next lemma is a particular case of Theorem 7 from [5].

**Lemma 4.10.** Let \((S, \omega)\) be a weighted star with \(\omega(e) > 0\) for every \(e \in E(S)\). Then the following conditions are equivalent

1. There is a unique ultrametric space \((X, d)\) such that \(X = V(S)\) and \(d(x, y) = \omega(\{x, y\})\) for every \(\{x, y\} \in E(S)\).
2. The weight \(\omega : E(S) \to \mathbb{R}^+\) is an injective function.

This lemma and statement (iii) of Theorem 4.8 give us the following.

**Corollary 4.11.** Let \((X, d)\) be a finite ultrametric space with \(|X| \geq 2\). Then \((X, d) \in \mathcal{R}\) if and only if \((G, \omega_d)\) contains a spanning star \(S\) such that, for every ultrametric \(\rho : X \times X \to \mathbb{R}\), we have
\[(\forall e \in E(C) : \omega_d(e) = \omega_{\rho}(e)) \Rightarrow (\rho = d).\]

**Remark 4.12.** Using Theorem 2.5 from [7] and Theorem 7 from [5] we can show that the statement

“There is a path \(P\) is \((G, \omega_d)\) such that
\[(\forall e \in E(P) : \omega_d(e) = \omega_{\rho}(e)) \Rightarrow (\rho = d)\]

holds for every ultrametric \(\rho : X \times X \to \mathbb{R}\)”

is equivalent to
\[|X| = |\text{Sp}(X)|.\]

Hence we can not use any Hamiltonian path instead of star in Corollary 4.11.

Recall that a function \(\Phi\) from a metric space \((X, d)\) to a metric space \((Y, \rho)\) is a similarity if there is \(\lambda > 0\) such that
\[\lambda(d(x, y)) = \rho(\Phi(x), \Phi(y))\]
for all \(x, y \in X\).

**Definition 4.13.** Let \((X, d)\) and \((Y, \rho)\) be metric spaces. A bijective mapping \(\Phi : X \to Y\) is a weak similarity if there is a strictly increasing bijection \(f : \text{Sp}(X) \to \text{Sp}(Y)\) such that the equality
\[f(d(x, y)) = \rho(\Phi(x), \Phi(y))\]
holds for all \(x, y \in X\).

If \(\Phi : X \to Y\) is a weak similarity we say that \(X\) and \(Y\) are weakly similar. If \((X, d)\) is a finite metric space, then every weak similarity \(\Phi : X \to X\) is an isometry. The notion of weak similarity was introduced in [6] for more general case of semimetric spaces in a slightly different form.

**Proposition 4.14.** Let \((X, d) \in \mathcal{R}\) and let \((Y, \rho)\) be a metric space. If \((X, d)\) and \((Y, \rho)\) are weakly similar, then \((Y, \rho) \in \mathcal{R}\).
Proof. If $\Phi : X \to Y$ is a weak similarity, then $B_Y = \{ \Phi(B) : B \in B_X \}$ and the mapping

$$B_X \ni B \mapsto \Phi(B) \in B_Y$$

is a bijection. It is clear also that

$$(A \subseteq C) \iff (\Phi(A) \subseteq \Phi(C))$$

holds for all $A \subseteq X$ and $C \subseteq X$. Hence the graphs $(\Gamma_X, X)$ and $(\Gamma_Y, Y)$ are isomorphic. Using Theorem 2.8 we obtain that $Y$ is ultrametric and $T_Y$ is isomorphic to $T_X$. Now $Y \in R$ follows from statement (iii) of Theorem 3.4.

□

Proposition 4.15. Let $X, Y \in R$. Then the following statements are equivalent.

(i) The trees $T_X$ and $T_Y$ are isomorphic as rooted trees.
(ii) $X$ and $Y$ are weakly similar.
(iii) The equality $|X| = |Y|$ holds.

Proof. The implication (i)$\Rightarrow$(iii) is immediate. Analysis similar to that in the proof of Proposition 4.14 shows that (ii)$\Rightarrow$(i) holds.

Let us prove (iii)$\Rightarrow$(i). Let $|X| = |Y|$. Write $n = |X| = |Y|$. Statement (ii) of Theorem 4.8 implies that there is a Hamiltonian path $(x_1, \ldots, x_n) \subseteq (G, \omega_d)$ and a Hamiltonian path $(y_1, \ldots, y_n) \subseteq (G, \omega_\rho)$ such that

$$\omega_d(\{x_k, x_{k+1}\}) > \omega_d(\{x_{k+1}, x_{k+2}\}) \tag{4.9}$$

and

$$\omega_\rho(\{x_k, x_{k+1}\}) > \omega_\rho(\{x_{k+1}, x_{k+2}\}) \tag{4.10}$$

for $k = 1, \ldots, n - 2$. The Gomory-Hu inequality implies that

$$\text{Sp}(X) = \{d(x_k, x_{k+1}) : k = 1, \ldots, n - 1\} \cup \{0\}$$

and

$$\text{Sp}(Y) = \{\rho(y_k, y_{k+1}) : k = 1, \ldots, n - 1\} \cup \{0\}.$$ 

Let us define the functions $\Phi : X \to Y$ and $f : \text{Sp}(X) \to \text{Sp}(Y)$ such that

$$\Phi(x_i) = y_i \quad \text{and} \quad f(0) = 0 \quad \text{and} \quad f(d(x_k, x_{k+1})) = \rho(x_k, x_{k+1})$$

for $k = 1, \ldots, n - 1$.

Inequality (4.9) and (4.10) imply that $f$ is strictly increasing. Moreover, it is clear that $\Phi$ and $f$ are bijective. Now using Lemma 4.7 we obtain that equality (4.8) holds for all $x, y \in X$. The implication (iii)$\Rightarrow$(ii) follows. □

Remark 4.16. Let $a, b > 0$. If $(X, d)$ and $(Y, \rho)$ are ultrametric spaces for which

$$d(x_1, x_2) = a \quad \text{and} \quad \rho(y_1, y_2) = b$$

for all distinct $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then $(X, d)$ and $(Y, \rho)$ are weakly similar if and only if $|X| = |Y|$. For these spaces we have also $(X, d) \notin R$ and $(Y, \rho) \notin R$ if $|X|, |Y| \geq 3$. 

How rigid the finite ultrametric spaces can be?
It seems to be interesting to find a “representing tree description” of the classes of finite ultrametric spaces $X$, $Y$ for which the conditions $|X| = |Y|$, implies that $X$ and $Y$ are weakly similar.

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