On Alberseron irregularity measure of graphs

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Abstract: Let $G = (V,E)$, $V = \{1,2,\ldots,n\}$ be a simple connected graph with $n$ vertices, $m$ edges and a sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$. The irregularity measure of graph is defined as $\text{irr}(G) = \sum_{i\sim j} |d_i - d_j|$, where $i \sim j$ denotes adjacency of vertices $i$ and $j$. New upper bounds for $\text{irr}(G)$ are obtained.

Keywords: Irregularity of graph, Zagreb indices, inverse sum indeg index, Alberseron index

1 Introduction

Let $G = (V,E)$, $V = \{1,2,\ldots,n\}$ be a simple connected graph with $n = |V|$ vertices and $m = |E|$ edges. Denote by $i \sim j$ an edge connecting vertices $i$ and $j$. Further, let $d_i = d(i)$ be the degree of a vertex $i$, and $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$ the sequence of vertex degrees. A graph $G$ is said to be regular if and only if there exists an integer $k, 1 \leq k \leq n-1$, so that $d_1 = d_2 = \cdots = d_n = k$, otherwise it is irregular. A union of disjonted components of graph, that is $G = G_1 \cup G_2 \cup \cdots \cup G_r$ is regular by components if every component $G_i$ is a regular graph. Without loss of generality we will assume that $G$ is connected. A graph invariant $I(G)$ is measure of irregularity of graph $G$ with the property $I(G) = 0$ if and only if $G$ is regular, and $I(G) > 0$ otherwise. A number of different irregularity measures have been defined in the literature (see for example [26, 13, 14, 4, 2, 8, 1, 10]). Here we will mention only two that are of interest for the present consideration.

In [4] Bell suggested a variance of vertex degrees,

$$\text{VAR}(G) = \frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} d_i \right)^2,$$

to be taken as a measure of irregularity of $G$. 

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Denote by $e = ij$ an arbitrary edge of $G$ which is incident to the vertices $i$ and $j$. In [2] Albertson defined the imbalance of an edge $e$ as $imb(e) = |d_i - d_j|$, and used it to introduce another irregularity measure

$$irr(G) = \sum_{i,j} |d_i - d_j|,$$

which is sometimes referred to as Albertson index [39, 40] or the third Zagreb index [10].

In this paper we are interested in determining upper bounds for $irr(G)$ in terms of some basic graph parameters and some other graph invariants. In what follows we outline graph invariants that will be used in the paper.

A single number that can be used to characterize some property of the graph is called a topological index for that graph. Obviously, the number of vertices and the number of edges are topological indices.

Two vertex-degree based topological indices, the first and the second Zagreb index, $M_1$ and $M_2$, are defined as (see [15, 16])

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i,j} d_i d_j.$$

The Zagreb indices are among the oldest and most studied molecular structure descriptors and found significant applications in chemistry.

An alternative expressions for the first Zagreb index is [24]

$$M_1(G) = \sum_{i,j} (d_i + d_j).$$

A modification of the first Zagreb index, $F$, defined as the sum of third powers of vertex degrees, that is

$$F = F(G) = \sum_{i=1}^{n} d_i^3,$$

was first time encountered in 1972, in the paper [15], but was eventually disregarded. Recently, it was re-considered in [12] and named the forgotten index.

Nowadays, there exist hundreds of papers on Zagreb indices and related matter [24, 20, 5, 3, 17].

A family of Adriatic indices was introduced in [28, 29]. An especially interesting subclass of these descriptors consists of 148 discrete Adriatic indices. A so called inverse sum indeg index, $ISI(G)$, was selected in [28] as a significant predictor of total surface area of octane isomers. The inverse sum indeg index is defined as

$$ISI(G) = \sum_{i,j} \frac{d_i d_j}{d_i + d_j}.$$
More on mathematical properties of this topological index can be found in [9, 27, 19].

In [29] a topological index named symmetric division deg, \( SDD(G) \), was defined as

\[
SDD(G) = \sum_{i \neq j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right).
\]

2 Preliminaries

In this section we recall some results for the upper bounds of \( irr(G) \). We will compare these results to the new ones derived in this paper.

In [30] Zhou and Liu proved the inequality

\[
irr(G) \leq \sqrt{m(M_1(G) - 4m^2)}.
\]  

(1)

Since

\[
VAR(G) = \frac{1}{n} \sum_{i=1}^{n} d_i^2 - \left( \frac{1}{n} \sum_{i=1}^{n} d_i \right)^2 = \frac{1}{n^2} (nM_1(G) - 4m^2),
\]

the inequality (1) can be rewritten as

\[
irr(G) \leq \sqrt{nmVAR(G)}.
\]

The above inequality establishes a relation between two irregularity measures, that is between \( irr(G) \) and \( VAR(G) \).

Goldberg [11] proved the following inequality

\[
irr(G) \leq \sqrt{\frac{m\mu_1(nM_1(G) - 4m^2)}{n}} = \sqrt{nm\mu_1VAR(G)},
\]  

(2)

where \( \mu_1 \) is the Laplacian spectral radius of \( G \). Since \( \mu_1 \leq n \), the inequality (2) is stronger than (1).

In [7] Chen at all, proved the following

\[
irr(G) \leq \frac{n\mu_1(\Delta - \delta)}{4}.
\]  

(3)

Che and Chen [6] proved that

\[
irr(G) \leq \sqrt{m(F(G) - 2M_2(G))},
\]  

(4)

and

\[
irr(G) \leq \sqrt{2mF(G) - M_1(G)^2}.
\]  

(5)

In [38] the following inequality was proven

\[
0 \leq irr(G) + irr(\bar{G}) \leq \frac{1}{6} (n - 1)(n + 1)(2n - 3),
\]

where \( \bar{G} \) is the complement of \( G \). Equality holds if and only if \( G \) is a regular graph.
3 Main results

In this section we will prove some new inequalities that establish upper bounds for the \( \text{irr}(G) \). But, first recall one analytical inequality for positive real number sequences proved in [25].

Lemma 1. [25] Let \( x = (x_i) \) and \( a = (a_i) \), \( i = 1, 2, \ldots, m \), be two positive real number sequences. Then for any real \( r \geq 0 \), holds
\[
\sum_{i=1}^{m} \frac{x_i^{r+1}}{a_i} \geq \left( \frac{\sum_{i=1}^{m} x_i}{\sum_{i=1}^{m} a_i} \right)^{r+1}. \tag{6}
\]

Equality holds if and only if \( r = 0 \) or \( \frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_m}{a_m} \).

In the next theorem we establish an upper bound for \( \text{irr}(G) \) in terms of indices \( M_1(G) \) and \( ISI(G) \).

**Theorem 1.** Let \( G \) be a simple connected graph with \( n \) vertices and \( m \) edges. Then
\[
\text{irr}(G) \leq \sqrt{M_1(G)(M_1(G) - 4\text{ISI}(G))}. \tag{7}
\]

Equality holds if and only if \( \frac{|d_i - d_j|}{d_i + d_j} \) is constant for each edge of \( G \).

**Proof.** For \( r = 1 \), \( x_i := |d_i - d_j| \), \( a_i := d_i + d_j \), where the summation is performed over all edges of \( G \), the inequality (6) becomes
\[
\sum_{i \sim j} \frac{(d_i - d_j)^2}{d_i + d_j} \geq \left( \frac{\sum_{i \sim j} |d_i - d_j|}{\sum_{i \sim j} (d_i + d_j)} \right)^2 = \frac{\text{irr}(G)^2}{M_1(G)}. \tag{8}
\]

Since
\[
0 \leq \sum_{i \sim j} \frac{(d_i - d_j)^2}{d_i + d_j} = \sum_{i \sim j} (d_i + d_j) - 4 \sum_{i \sim j} \frac{d_i d_j}{d_i + d_j} = M_1(G) - 4\text{ISI}(G),
\]
from the above and (8) we arrive at (7).

For \( r = 1 \) equality in (6) holds if and only if \( \frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_m}{a_m} \), which implies that equality in (8), that is (7), holds if and only if \( \frac{|d_i - d_j|}{d_i + d_j} \) constant for each edge of \( G \). \( \square \)

**Remark 1.** Equality in (7) holds, for example, if \( G \) is regular or semiregular bipartite graph.

Since \( \text{ISI}(G) \geq \frac{m^2}{n} \) (see for example [9]), we have the following corollary of Theorem 1.
Corollary 1. Let $G$ be a simple connected graph of order $n$ and size $m$. Then

$$\text{irr}(G) \leq \sqrt{\frac{M_1(G)(nM_1(G) - 4m^2)}{n}} = \sqrt{nM_1(G)\text{VAR}(G)}. \quad (9)$$

Equality holds if and only if $G$ is regular or semiregular bipartite graph.

In [21] (see also [22, 18]) the following inequality was proved

$$M_1(G) \leq \frac{4m^2}{n} + n\alpha(n)(\Delta - \delta)^2, \quad (10)$$

where

$$\alpha(n) = \frac{1}{n} \left| \frac{n}{2} \right| \left( 1 - \frac{1}{n} \right) = \frac{1}{4} \left( 1 - \frac{(-1)^{n+1} + 1}{2n^2} \right) \leq \frac{1}{4}.$$ 

Therefore we have the following corollary of Theorem 1.

Corollary 2. Let $G$ be a simple connected graph with $n$ vertices and $m$ edges. Then

$$\text{irr}(G) \leq \sqrt{n\alpha(n)M_1(G)(\Delta - \delta)}.$$ 

Equality holds if and only if $G$ is regular.

Remark 2. According to (10) we have

$$\text{VAR}(G) \leq \alpha(n)(\Delta - \delta)^2.$$ 

Since $\alpha(n) \leq \frac{1}{4}$, the above inequality is stronger than

$$\text{VAR}(G) \leq \frac{(\Delta - \delta)^2}{4},$$

which was proved in [23].

Corollary 3. Let $G$ be a simple connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$\text{irr}(G) \leq \frac{m}{n-1} \sqrt{\frac{(n-2)((n(n-1) - 2m)(2m + (n-1)(n-2))}{n}}.$$

Equality holds if and only if either $G \cong K_n$, or $G \cong K_{1,n-1} \quad (11)$

Proof. In [31] it was proven that

$$M_1(G) \leq m \left( \frac{2m}{n-1} + n - 2 \right),$$

with equality if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$ [32]. From the above and inequality (9) we obtain (11).
Based on (11) we obtain the following result.

**Corollary 4.** Let $T$ be a tree with $n \geq 2$ vertices. Then

$$\text{irr}(T) \leq (n-1)(n-2).$$

(12)

Equality holds if and only if $T \cong K_{1,n-1}$.

**Remark 3.** The inequality (12) was proven in [33].

In the next theorem we establish an upper bound for $\text{irr}(G)$ in terms of parameter $m$ and invariants $M_2(G)$ and $\text{SDD}(G)$.

**Theorem 2.** Let $G$ be a simple connected graph with $m$ edges. Then

$$\text{irr}(G) \leq \sqrt{M_2(G)(\text{SDD}(G) - 2m)}.$$

(13)

Equality holds if and only if $\frac{|d_i - d_j|}{d_i d_j}$ is constant for each edge of $G$.

**Proof.** For $r = 1$, $x_i := |d_i - d_j|$ and $a_i := d_i d_j$, where summation is performed over all edges of $G$, the inequality (6) transforms into

$$\sum_{i \sim j} \frac{(d_i - d_j)^2}{d_i d_j} \geq \left( \sum_{i \sim j} |d_i - d_j| \right)^2 / \sum_{i \sim j} d_i d_j = \frac{\text{irr}(G)^2}{M_2(G)}.$$}

(14)

Since

$$0 \leq \sum_{i \sim j} \frac{(d_i - d_j)^2}{d_i d_j} = \sum_{i \sim j} \left( \frac{d_i}{d_j} + \frac{d_j}{d_i} \right) - 2 \sum_{i \sim j} \frac{d_i d_j}{d_i d_j} = \text{SDD} - 2m,$$

from the above and inequality (14) we obtain (13).

For $r = 1$, the equality in (6) holds if and only if $\frac{a_1}{a_2} = \frac{a_2}{a_3} = \cdots = \frac{a_m}{a_n}$. Therefore we conclude that equality in (14), i.e. in (13), holds if and only if $\frac{|d_i - d_j|}{d_i d_j}$ is constant for each edge of $G$.



4. Equality in (13) holds, for example, if $G$ is regular or semiregular bipartite graph.

**Remark 5.** The inequalities (7) and (13) are, mainly, incomparable with (1), (2), (3), (4) and (5). Thus, for example, for $G = C_{n-1} + e$, the inequality (7) is stronger than (1), (2), (3), (4) and (5), and (13) from (1), (2), (3) and (5). For $G = K_n - e$, inequalities (1), (2) and (4) are stronger than (7) and (13). By testing for large $n$ we didn’t find a graph for which (13) is stronger than (4) and (7), as well as a graph for which (5) is stronger than (4) and (7).
Corollary 5. Let $G$ be a simple connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$\text{irr}(G) \leq \frac{\Delta - \delta}{\sqrt{\Delta \delta}} \sqrt{mM_2(G)}.$$  \hfill (15)

Equality holds if and only if $G$ is regular or semiregular bipartite graph.

Proof. The function $f(x) = x + \frac{1}{x}$ is monotone increasing for $x \geq 1$. On the other hand, for every $d_i \geq d_j$ it holds $\frac{\Delta}{\delta} \geq \frac{d_i}{d_j} \geq 1$, so we have

$$\sum_{i,j} \frac{(d_i - d_j)^2}{d_i d_j} = \sum_{i,j} \left( \sqrt{\frac{d_i}{d_j}} - \sqrt{\frac{d_j}{d_i}} \right)^2 \leq m \left( \sqrt{\frac{\Delta}{\delta}} - \sqrt{\frac{\delta}{\Delta}} \right)^2 = \frac{m(\Delta - \delta)^2}{\Delta \delta}.$$  

Now, from the above and inequality (14) we obtain (15). \hfill \Box

Based to the (15) we have the next result.

Corollary 6. Let $G$ be a simple connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$\text{irr}(G) \leq \frac{n - 2}{\sqrt{n - 1}} \sqrt{mM_2(G)}.$$  \hfill (16)

Equality holds if and only if $G \cong K_{1,n-1}$.

Remark 6. In [34] it was proven than for any tree $T$ holds

$$SDD(T) \leq (n - 1)^2 + 1,$$

with equality if and only if $T \cong K_{1,n-1}$.

In [35] (see also [36]) it was proven that

$$M_2(T) \leq (n - 1)^2,$$  \hfill (17)

with equality if and only if $T \cong K_{1,n-1}$. It can be easily verified that inequality (12) can be obtained from the above inequalities and (13), as well as and from (15), (16) and (17).

Remark 7. The irregularity measure similar to the Albertson irregularity measure, named the sigma index was defined in [37] as

$$\sigma(G) = \sum_{i,j} (d_i - d_j)^2.$$ 

It is not difficult to see that for these two topological indices the following relation is valid

$$\sqrt{\sigma(G)} \leq \text{irr}(G) \leq \sqrt{m\sigma(G)},$$

with equalities if and only if $G$ is regular.
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