Prepotential Recursion Relations in $\mathcal{N}=2$ Super Yang-Mills with Adjoint Matter

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Abstract

Linear recursion relations for the instanton corrections to the effective prepotential are derived for $\mathcal{N}=2$ supersymmetric gauge theories with one hypermultiplet in the adjoint representation of $SU(N)$ using the Calogero-Moser parameterization of the Seiberg-Witten spectral curves. S-duality properties of the Calogero-Moser parameterization and conjectures on the Seiberg-Witten spectral curves generalized to arbitrary simply laced classical gauge groups are also discussed.

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1. Introduction

Over the past half decade there has been great progress in understanding non-perturbative dynamics of $\mathcal{N}=2$ SUSY gauge theories [1]-[11]. Non-perturbative corrections in the weak coupling corresponding to instanton effects [12] were evaluated by field theory methods [13]-[16] and various other ways via the Seiberg-Witten ansatz by calculating the period integrals corresponding to the quantum moduli parameters representing the set of vacuum expectation values of the Higgs fields [17]-[20]. In a previous paper [21], a linear set of recursion relations for the instanton corrections to the effective prepotential $F$ was found for a class of $\mathcal{N}=2$ SUSY Yang-Mills theories with hypermultiplets in the fundamental representation of a classical gauge group $G$, which reproduced the results of previous recursion relations [22][23] and other methods [17]-[20].

Connections between Seiberg-Witten theory and integrable systems was first made for the case of pure $\mathcal{N}=2$ super Yang-Mills theory in connection with Toda lattices [24] and Whitham theory [25]. Later connections between $\mathcal{N}=2$ super Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge group, and the Hitchin [27] and Calogero-Moser [28] integrable systems was made. (There are claims that the Calogero-Moser integrable system can be derived from the Hitchin integrable system [29]). Convenient parameterizations of the Calogero-Moser integrable system useful for performing explicit Seiberg-Witten type of calculations were discovered [30] and forms the starting point of the present paper.

The Calogero-Moser construction [30] of the Seiberg-Witten solution for $\mathcal{N}=2$ super Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge group $SU(N)$ and the renormalization group like equation for the prepotential $\mathcal{F}$, led us to the discovery of a general recursion relation expressing the $n$-th order instanton correction to the prepotential $\mathcal{F}$ in terms of the $(n-1)$-th, ..., first order instanton corrections.

We start off by reviewing the Calogero-Moser construction of the Seiberg-Witten solution for $\mathcal{N}=2$ super Yang-Mills theory with one hypermultiplet in the adjoint representation of the gauge group $SU(N)$. The renormalization group type equation for the prepotential $\mathcal{F}$ is discussed next and it is shown how it can be used to determine the instanton corrections to the prepotential to arbitrary order in an efficient manner. Recursion relations for the instanton corrections are then derived and compared with previous results. S-
duality properties of the Calogero-Moser construction of the Seiberg-Witten solution are used to discuss the "dual" prepotential $F_D$ of the dual magnetic sector of the theory. Conjectures for possible Seiberg-Witten spectral curves for simply laced cases of other classical gauge groups are then discussed.

2. The Seiberg-Witten Solution for Super Yang-Mills with One Adjoint Hypermultiplet

The Seiberg-Witten (SW) ansatz gives a prescription for determining the prepotential of the effective action for $\mathcal{N}=2$ supersymmetric Yang-Mills gauge theories, as well as for determining the spectrum of BPS states.

For supersymmetric Yang-Mills theories with an asymptotic free coupling and one adjoint hypermultiplet in the adjoint representation of a classical gauge group, general arguments based on the holomorphicity of $F$, perturbative non-renormalization theorems beyond 1-loop order, the nature of instanton corrections, and restrictions of $U(1)_R$ invariance constrain $F$ to have the form

$$F(a) = \frac{\tau}{2} \sum_{i=1}^{r} a_i^2 - \frac{1}{8\pi i} \sum_{\alpha \in R(G)} \left\{ (\alpha \cdot a)^2 \log(\alpha \cdot a)^2 ight\} - (\alpha \cdot a + m)^2 \log(\alpha \cdot a + m)^2 + \sum_{n=1}^{\infty} \frac{q^n}{2\pi ni} F^{(n)}(a)$$

where $\alpha$ are the roots of the gauge group $G$. For $SU(N)$, the traceless condition $\sum_{i=1}^{N} a_i = 0$ is imposed.

The SW ansatz for determining the full prepotential $F$ is based on a choice of a fibration of spectral curves over the space of vacua, and of a meromorphic 1-form $d\lambda$ on each of these curves. The renormalized order parameters $a_k$ of the theory, their duals $a_{D,k}$, and the prepotential $F$ are given by

$$2\pi i a_k = \oint_{A_k} d\lambda, \quad 2\pi i a_{D,k} = \oint_{B_k} d\lambda, \quad a_{D,k} = \frac{\partial F}{\partial a_k}$$

with $A_k, B_k$ a suitable set of homology cycles on the spectral curves.

For $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $SU(N)$ and one hypermultiplet in the adjoint representation, a convenient parameterization for the spectral curves and meromorphic 1-forms is the Calogero-Moser
case of \[30\]

\[ f(k - \frac{m}{2}, z) = 0, \quad d\lambda = kdz \]  \hspace{1cm} (3)

where

\[ f(k, z) = \frac{1}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} \sum_{n=0}^{N} \frac{\partial^n}{\partial z^n} \vartheta_1(\frac{z}{2\omega_1}|\tau) (-m \frac{\partial}{\partial k})^n H(k|k) \]  \hspace{1cm} (4)

\[ H(x|k) = \prod_{j=1}^{N} (x - k_j) \equiv (x - k_i) H_i(x|k) \]  \hspace{1cm} (5)

\[ \vartheta_1(z|\tau) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+1)/2} e^{2\pi i (2n+1)z} \]  \hspace{1cm} (6)

along with a corresponding basis of \(A_k, B_k\) homology cycles as described in \[30\]. This particular choice of parameterization for the spectral curves has the geometry of a foliation over a base torus \(\Sigma\), where the complex modulus \(\tau\) of the torus \(\Sigma\) is related to the gauge coupling \(g\) and the \(\theta\)-angle of the gauge theory by

\[ \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \]  \hspace{1cm} (7)

Substituting (6) into (3) produces a simplified form for the spectral curves

\[ \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n-1)} e^{n \omega H(k - mn|k)} = 0 \]  \hspace{1cm} (8)

Substituting (8) into (2) using the Calogero-Moser parameterization (3) for the 1-form and performing a weak coupling expansion in powers of \(q\) similar to the methods in \[30\], the integral for the quantum order parameters \(a_i\)'s in terms of the classical order parameters \(k_i\)'s were calculated order by order in \(q\) producing a simplified expression of

\[ a_i = k_i + \sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k) \]

where

\[ \sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k) = \sum_{j=2}^{\infty} \sum_{\alpha_1, \ldots, \alpha_j = -\infty, \neq 0} \frac{(-1)^j}{j!} \left( \frac{\partial}{\partial k_i} \right)^{j-1} \prod_{l=1}^{j} \left[ \frac{H(k_i - \alpha_lm|k)}{H_i(k_i|k)} q^{\alpha_l^2/2} \right] \]  \hspace{1cm} (9)
The first few $\Delta_i$'s are

\[
\begin{align*}
\Delta_i^{(1)}(a) &= \frac{\partial}{\partial a_i} \left[ \frac{H(a_i - m|a)H(a_i + m|a)}{H_i(a_i|a)^2} \right] \\
\Delta_i^{(2)}(a) &= \frac{1}{4} \frac{\partial^3}{\partial a_i^3} \left[ \frac{H(a_i - m|a)H(a_i + m|a)^2}{H_i(a_i|a)^2} \right]^2 \\
\Delta_i^{(3)}(a) &= \frac{1}{36} \frac{\partial^5}{\partial a_i^5} \left[ \frac{H(a_i - m|a)H(a_i + m|a)^3}{H_i(a_i|a)^2} \right]^3 \\
&\quad - \frac{1}{2} \frac{\partial^2}{\partial a_i^2} \left[ \frac{H(a_i - 2m|a)H(a_i + m|a)^3}{H_i(a_i|a)^3} \right] \\
&\quad - \frac{1}{2} \frac{\partial^2}{\partial a_i^2} \left[ \frac{H(a_i - m|a)^2H(a_i + 2m|a)}{H_i(a_i|a)^3} \right] \\
\Delta_i^{(4)}(a) &= \frac{1}{576} \frac{\partial^7}{\partial a_i^7} \left[ \frac{H(a_i - m|a)H(a_i + m|a)^4}{H_i(a_i|a)^2} \right]^4 \\
&\quad + \frac{\partial}{\partial a_i} \left[ \frac{H(a_i - 2m|a)H(a_i + 2m|a)}{H_i(a_i|a)^2} \right] \\
&\quad - \frac{1}{6} \frac{\partial^4}{\partial a_i^4} \left[ \frac{H(a_i - 2m|a)H(a_i - m|a)H(a_i + m|a)^3}{H_i(a_i|a)^5} \right] \\
&\quad - \frac{1}{6} \frac{\partial^4}{\partial a_i^4} \left[ \frac{H(a_i - m|a)^3H(a_i + m|a)H(a_i + 2m|a)}{H_i(a_i|a)^5} \right]
\end{align*}
\]

This result can be derived in a more transparent manner by rewriting the spectral curves (8) as

\[
k \equiv k_i + F_i(k) \quad (10)
\]

where

\[
F_i(k) = \sum_{n \in \mathbb{Z}, n \neq 0} (-1)^{n+1} q^{\frac{1}{2}m(n-1)} e^{nz} \frac{H(k - nm|k)}{H_i(k|k)} \quad (11)
\]

An iterative solution expanded around $k = k_i$ to all orders in small $q$ is given by

\[
k = k_i + \sum_{n=1}^{\infty} y_n, \quad y_n = \frac{1}{n!} \frac{\partial^{n-1}}{\partial k^{n-1}} F_i^n(k)|_{k=k_i} \quad (12)
\]

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In a similar manner as in \[30\], \(z\) is substituted with \(w = e^z\) in the SW differential (3),

\[
dz = \frac{dw}{w}, \quad d\lambda = kdz = k \frac{dw}{w}
\]

(13)

along with the iterative solution (12). Performing the integral around the appropriate \(A_i\) cycle corresponding to \(k_i\) as prescribed in \[30\], reproduces (9) \[31\].

3. Renormalization Group Type Equations

In \[30\], a renormalization group type equation for the prepotential \(F\) was derived

\[
\frac{\partial F}{\partial \tau} = \frac{1}{4\pi i} \sum_{j=1}^{r} \oint_{A_j} k^2 dz
\]

(14)

up to an additive term independent of \(a_i\) and \(k_i\) which is physically immaterial.

Substituting the \(SU(N)\) spectral curves (8) into (14) using the Calogero-Moser parameterization (3) for the 1-form and solving the integral in the weak coupling limit of small \(q\) gives the renormalization group like equation for the prepotential \(F\) in terms of the classical order parameters \(k_i\)’s

\[
\frac{\partial F}{\partial \tau} = \frac{1}{2} \sum_{i=1}^{r} k_i^2 + \sum_{i=1}^{r} \sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k) + \sum_{i=1}^{r} \sum_{n=1}^{\infty} q^n \Omega_i^{(n)}(k)
\]

where

\[
\sum_{n=1}^{\infty} q^n \Omega_i^{(n)}(k) = \sum_{j=2}^{\infty} \sum_{\alpha_1, \ldots, \alpha_j = -\infty, \neq 0} (-1)^j \frac{(-1)^j}{j(j-2)!} \left( \frac{\partial}{\partial k_i} \right)^{j-2} \prod_{l=1}^{j} \left[ \frac{H(k_i - \alpha_l m_i | k)}{H_i(k_i | k)} q^{\alpha_l^2/2} \right]
\]

(15)

The first few \(\Omega_i\)’s are

\[
\Omega_i^{(1)}(a) = \frac{H(a_i - m | a)H(a_i + m | a)}{H_i(a_i | a)^2}
\]
\[
\Omega^{(2)}_i(a) = \frac{3}{4} \frac{\partial^2}{\partial a_i^2} \left[ \frac{H(a_i - m|a)H(a_i + m|a)}{H_i(a_i|a)^2} \right]^2
\]

\[
\Omega^{(3)}_i(a) = \frac{5}{36} \frac{\partial^4}{\partial a_i^4} \left[ \frac{H(a_i - m|a)H(a_i + m|a)}{H_i(a_i|a)^2} \right]^3
\]

\[
- \frac{\partial}{\partial a_i} \left[ \frac{H(a_i - 2m|a)H(a_i + m|a)^2}{H_i(a_i|a)^3} \right]
\]

\[
- \frac{\partial}{\partial a_i} \left[ \frac{H(a_i - m|a)^2H(a_i + 2m|a)}{H_i(a_i|a)^3} \right]
\]

\[
\Omega^{(4)}_i(a) = \frac{7}{576} \frac{\partial^6}{\partial a_i^6} \left[ \frac{H(a_i - m|a)H(a_i + m|a)}{H_i(a_i|a)^2} \right]^4
\]

\[
+ \frac{H(a_i - 2m|a)H(a_i + 2m|a)}{H_i(a_i|a)^2}
\]

\[
- \frac{2}{3} \frac{\partial^3}{\partial a_i^3} \left[ \frac{H(a_i - 2m|a)H(a_i - m|a)H(a_i + m|a)^3}{H_i(a_i|a)^5} \right]
\]

\[
- \frac{2}{3} \frac{\partial^3}{\partial a_i^3} \left[ \frac{H(a_i - m|a)^3H(a_i + m|a)H(a_i + 2m|a)}{H_i(a_i|a)^5} \right]
\]

4. Recursion Relations for the Prepotential \( \mathcal{F} \)

In [21], an efficient algorithm for deriving a set of recursion relations for the instanton corrections was discovered. Using similar methods as [21], a similar set of recursion relations for the prepotential \( \mathcal{F} \) was determined.

A very direct way of deriving the form of the instanton corrections to the prepotential \( \mathcal{F} \) involves substituting (1) and (9) into (15) to get

\[
\sum_{n=1}^{\infty} q^n \mathcal{F}^{(n)}(a) = \sum_{i=1}^{r} \sum_{n=1}^{\infty} q^n \Omega_i^{(n)}(k) - \frac{1}{2} \sum_{i=1}^{r} \left[ \sum_{n=1}^{\infty} q^n \Delta_i^{(n)}(k) \right]^2 \quad (16)
\]

Then (9) is substituted into (16) and expanded in powers of \( q \), replacing the \( k_i \)'s with \( a_i \)'s. The n-th order instanton correction to the prepotential \( \mathcal{F} \) finally takes on the form

\[
\mathcal{F}^{(n)}(a) = \sum_{i=1}^{r} \Omega_i^{(n)}(a) - \frac{1}{2} \sum_{i=1}^{r} \sum_{j,l=1}^{n} \Delta_i^{(j)}(a) \Delta_i^{(l)}(a)
\]
\[ - \sum_{i=1}^{n-1} \frac{1}{i!} \sum_{\beta_1, \ldots, \beta_{i+1}=1}^{i-1} \sum_{\alpha_1, \ldots, \alpha_i=1}^{r} \left[ \prod_{j=1}^{i} \Delta_{\alpha_j}^{(\beta_j)}(a) \right] \left( \prod_{l=1}^{i} \frac{\partial}{\partial a_{\alpha_l}} \right) F^{(\beta_{i+1})}(a) \]

(17)

The first few \( F^{(n)}(a) \)'s are

\[ F^{(1)}(a) = \sum_{i=1}^{r} \Omega_i^{(1)}(a) \]

\[ F^{(2)}(a) = \sum_{i=1}^{r} \Omega_i^{(2)}(a) - \frac{1}{2} \sum_{i=1}^{r} \left( \Delta_i^{(1)}(a) \right)^2 - \sum_{i=1}^{r} \Delta_i^{(1)}(a) \frac{\partial F^{(1)}(a)}{\partial a_i} \]

\[ F^{(3)}(a) = \sum_{i=1}^{r} \Omega_i^{(3)}(a) - \frac{1}{2} \sum_{i=1}^{r} \left[ 2\Delta_i^{(1)}(a)\Delta_i^{(2)}(a) \right] - \sum_{i=1}^{r} \left[ \Delta_i^{(1)}(a) \frac{\partial F^{(2)}(a)}{\partial a_i} + \Delta_i^{(2)}(a) \frac{\partial F^{(1)}(a)}{\partial a_i} \right] - \frac{1}{2!} \sum_{i,j=1}^{r} \Delta_i^{(1)}(a)\Delta_j^{(1)}(a) \frac{\partial^2 F^{(1)}(a)}{\partial a_i \partial a_j} \]

\[ F^{(4)}(a) = \sum_{i=1}^{r} \Omega_i^{(4)}(a) - \frac{1}{2} \sum_{i=1}^{r} \left[ 2\Delta_i^{(1)}(a)\Delta_i^{(3)}(a) + (\Delta_i^{(2)}(a))^2 \right] - \sum_{i=1}^{r} \left[ \Delta_i^{(1)}(a) \frac{\partial F^{(3)}(a)}{\partial a_i} + \Delta_i^{(2)}(a) \frac{\partial F^{(2)}(a)}{\partial a_i} + \Delta_i^{(3)}(a) \frac{\partial F^{(1)}(a)}{\partial a_i} \right] - \frac{1}{2!} \sum_{i,j=1}^{r} \left[ \Delta_i^{(1)}(a)\Delta_j^{(1)}(a) \frac{\partial^2 F^{(2)}(a)}{\partial a_i \partial a_j} + 2\Delta_i^{(1)}(a)\Delta_j^{(2)}(a) \frac{\partial^2 F^{(1)}(a)}{\partial a_i \partial a_j} \right] - \frac{1}{3!} \sum_{i,j,k=1}^{r} \Delta_i^{(1)}(a)\Delta_j^{(1)}(a)\Delta_k^{(1)}(a) \frac{\partial^3 F^{(1)}(a)}{\partial a_i \partial a_j \partial a_k} \]

(18)

5. Comparison With Previous Results

In the limit the full hypermultiplet is decoupled with \( \tau \to \infty, m \to \infty \) while keeping constant the parameters \( k_i \) and \( \Lambda \):

\[ \Lambda^{2N} = (-1)^N m^{2N} q \quad q = e^{2\pi i \tau} \]

(19)
equations (9) and (15) break down to their corresponding equations in the pure $SU(N)$ gauge theory cases [20][21][35].

In the $\mathcal{N}=4$ limit where $m \to 0$, all the $\Omega_i$ and $\Delta_i$ terms in (9) and (15) vanish and reproduces the expected prepotential

$$F(a) = \frac{\tau}{2} \sum_{i=1}^{N} a_i^2$$

(20)

for $SU(N)$.

For $SU(2)$, the existing results in the literature have the instanton expressed in term of

$$a_1 = a, \quad a_2 = -a$$

(21)

Explicit evaluations of the first three instanton corrections are

$$\begin{align*}
F^{(1)}(a) &= \frac{m^4}{2a^2} \\
F^{(2)}(a) &= \frac{-9m^6}{16a^4} + \frac{5m^8}{64a^6} \\
F^{(3)}(a) &= \frac{m^6}{a^4} + \frac{25m^8}{48a^6} - \frac{67m^{10}}{192a^8} + \frac{3m^{12}}{64a^{10}}
\end{align*}$$

(22)

which disagree with results in the literature beyond one instanton [33][1], but agrees in the limit where the full hypermultiplet decouples [21]. It turns out that performing the Seiberg-Witten elliptic function calculation in [32] to higher instanton orders reproduces the instanton calculations of [33].

On the other hand, the $SU(2)$ spectral curve from the Calogero-Moser construction [3] can be explicitly shown to be equivalent to the $SU(2)$ mass deformed $\mathcal{N}=4$ spectral curve construction [2] up to reparameterizations of the classical order parameters $k_i$’s. This spectral curve forms a crucial part of the elliptic function calculation in [32].

One possible problem with the elliptic function calculation in [32] is the assumption of Matone’s relation [22][34] holding in the presence of an adjoint hypermultiplet. Generalizations of Matone’s relation for classes of $\mathcal{N}=2$

\footnote{In the limit of decoupling the full $SU(2)$ adjoint hypermultiplet in [33], there is a discrepancy of a factor $\frac{1}{2}$ with the pure $SU(2)$ results of [17][2].}
SUSY gauge theories with fundamental hypermultiplets was proven in general \cite{35} and corresponds to a renormalization group type of equation for the prepotential $F$. For the case of one adjoint hypermultiplet, a renormalization group equation for the prepotential $F$ was derived \cite{30} and calculated to all orders \cite{15} which differs greatly from Matone’s relation and \cite{34} \cite{35}, but agrees with the latter cases in the limit the full adjoint hypermultiplet is decoupled \cite{30}.

6. S-Duality Properties

A closer examination of the Calogero-Moser parameterization of the Seiberg-Witten spectral curves and 1-form \cite{3} and \cite{4} reveals there’s an implicit S-duality present.

Using the transformation property of the theta functions

$$
\vartheta_1 \left( \frac{z}{\tau} - \frac{1}{\tau} \right) = \sqrt{-i\tau} \exp \left( \frac{iz^2}{\pi\tau} \right) \vartheta_1 (z|\tau) \tag{23}
$$

and substituting it into \cite{3} and \cite{4} shows explicitly that the form of the spectral curves and 1-form are indeed invariant under S-duality transformations up to reparameterizations of the classical order parameters $k_i$’s. Correspondingly, the roles of the A and B cycles in the Seiberg-Witten ansatz \cite{4} are interchanged under S-duality transformations.

With this explicit S-duality, the corresponding weakly coupled ”dual” theory in the magnetic sector of the theory expanded around a small ”dual” coupling constant

$$
q_D = e^{2\pi i \tau_D} \quad \tau_D = -\frac{1}{\tau} \quad \tau \to i0^+ \tag{24}
$$

will have a corresponding ”dual” prepotential $F_D(a_D)$ identical in form to the prepotential $F(a)$ in \cite{1} and \cite{17} with the corresponding substitutions of the coupling constant and quantum order parameters to their ”dual” counterparts

$$
q \to q_D \quad a_i \to a_{D,i} \tag{25}
$$

respectively. This can be interpreted as a non-perturbative expansion of the theory, where the dynamics of the strongly coupled regime in the electric
sector of the theory is described by the dynamics of a corresponding weakly coupled "dual" theory in the magnetic sector of the same theory. (Other strong coupling expansions in the same spirit were performed in [17][36][39]).

Considering there are claims that the Calogero-Moser system can be constructed explicitly from the Hitchin system [29], this S-duality is like a realization of the Donagi-Witten construction of Seiberg-Witten theory using the Hitchin system [27] where an underlying S-duality and general $SL(2, Z)$ symmetry is built into the geometry of the foliation over a base torus $\Sigma$ construction (7) from the start. In the prepotential calculations performed around small coupling $q$ or $q_D$, the S-duality is explicitly broken while the underlying spectral curve (3) is invariant under S-duality and in general an $SL(2, Z)$ symmetry [2][27].

7. Generalizations to Other Gauge Groups

Generalizations of the $SU(N)$ Calogero-Moser integrable system were investigated in [37] for various cases of twisted and untwisted gauge groups, but stopped short of producing parameterizations suitable for use as Seiberg-Witten spectral curves. Possible parameterizations to general untwisted classical gauge groups can be conjectured starting from the $SU(N)$ spectral curves and placing appropriate constraints such that decouplings of the full adjoint hypermultiplet reproduce the pure gauge theory results [6][7][9][38].

In the spirit of [9][38], one possibility is to replace the $H(k)$ polynomial with

$$H(x|k) \rightarrow H(x|k) = \prod_{j=1}^{N} (x^2 - k_j^2) \equiv (x - k_i)(x + k_i)H_i(x|k) \quad (26)$$

in the Calogero-Moser parameterization of the SW spectral curves (8).

The appropriate limits for full hypermultiplet decoupling are $\tau \rightarrow \infty, m \rightarrow \infty$ while keeping constant the parameters $k_i$ and $\Lambda$:

$$\begin{align*}
SO(2r) & \quad \Lambda^{4r-4} \equiv m^{4r-4}q \\
SO(2r+1) & \quad \Lambda^{4r-2} \equiv m^{4r-2}q \\
SO(2r) & \quad \Lambda^{4r+4} \equiv m^{4r+4}q
\end{align*} \quad (27)$$

where $q = e^{2\pi i \tau}$. 

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