**L(n) graphs are vertex-pancyclic and Hamilton-connected**

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MS received 19 August 2021; revised 26 March 2022; accepted 21 June 2022

**Abstract.** A graph $\Gamma$ of order $n > 2$ is pancyclic if $\Gamma$ contains a cycle of length $l$ for each integer $l$ with $3 \leq l \leq n$ and it is called vertex-pancyclic if every vertex is contained in a cycle of length $l$ for every $3 \leq l \leq n$. A graph $\Gamma$ of order $n > 2$ is Hamilton-connected if for any pair of distinct vertices $u$ and $v$, there is a Hamilton $u$–$v$ path, namely, there is a $u$–$v$ path of length $n - 1$. The graph $B(n)$ is a graph with the vertex set $V = \{v \mid v \subset [n], |v| \in \{1, 2\}\}$ and the edge set $E = \{\{v, w\} \mid v, w \in V, v \subset w$ or $w \subset v\}$, where $[n] = \{1, 2, \ldots, n\}$. We denote by $L(n)$ the line graph of $B(n)$, that is, $L(n) = L(B(n))$. In this paper, we show that the graph $L(n)$ is vertex-pancyclic and Hamilton-connected whenever $n \geq 6$.

**Keywords.** $L(n)$ graph; path and cycle; vertex-pancyclic; Hamilton-connected.

**2010 Mathematics Subject Classification.** 05C38, 90B10.

1. Introduction and preliminaries

In this paper, a graph $\Gamma = (V, E)$ is considered as an undirected simple graph where $V = V(\Gamma)$ is the vertex-set and $E = E(\Gamma)$ is the edge-set. For all the terminologies and notations not defined here, we follow [3,4].

Let $n \geq 3$ be an integer and $[n] = \{1, 2, \ldots, n\}$. The graph $B(n)$ is a graph with the vertex set $V = \{v \mid v \subset [n], |v| \in \{1, 2\}\}$ and the edge set $E = \{\{v, w\} \mid v, w \in V, v \subset w$ or $w \subset v\}$. We denote by $L(n)$ the line graph of $B(n)$, that is, $L(n) = L(B(n))$. It is easy to check that $L(n)$ is a connected regular graph of regularity $n - 1$. Hence when $n = 3$, $L(n)$ is the cycle $C_6$. It can easily be seen that the graph $B(n)$ is an edge-transitive graph [8], thus $L(n)$ is a vertex-transitive graph [3,4]. The graph $L(n)$ has some interesting properties and has been studied in some aspects [6–9]. In [8], it has been shown that the graph $L(n)$ is a Cayley graph if and only if $n$ is a power of a prime. The graph $\Gamma$ is called an integral graph whenever each of the eigenvalues of its adjacency matrix is an integer. In [6], it has been proved that the graph $L(n)$ is an integral graph. In fact, the set of eigenvalues of $L(n)$ is $\{-2, -1, 0, n - 2, n - 1\}$. An interesting property of the graph $B(n)$ has appeared in [8], that is, $B(n)$ is (isomorphic to) the square root of the Johnson graph $J(n + 1, 2)$, that is, $B(n)^2$ is isomorphic to $J(n + 1, 2)$. If $v$ is a vertex of the graph $B(n)$, then $\deg(v) \in \{2, n - 1\}$. Therefore, if $n$ is an odd integer then $B(n)$...
is an eulerian graph [4]. Hence \( L(n) \) is a Hamilton graph when \( n \) is an odd integer. A graph \( \Gamma \) of order \( n > 2 \) is Hamilton-connected if for any pair of distinct vertices \( u \) and \( v \), there is a Hamilton \( u-v \) path, namely, there is a \( u-v \) path of length \( n - 1 \). Note that if the graph \( \Gamma \) is Hamilton-connected then it is hamiltonian. Also note that a hamiltonian graph may not be Hamilton-connected. For instance, the cycle \( C_6 \) is hamiltonian, but it is easy to see that \( C_6 \) is not Hamilton-connected. A graph \( \Gamma \) of order \( n > 2 \) is panconnected if for every two vertices \( u \) and \( v \), there is a \( u-v \) path of length \( l \) for every integer \( l \) with \( d(u, v) \leq l \leq n - 1 \). It is trivial that if \( \Gamma \) is a panconnected graph, then it is Hamilton-connected. Alspach [2] showed that the Johnson graph \( J(n, m) \) is Hamilton-connected. This result has been generalized in [7] where it has been shown that the Johnson graphs are panconnected. It is easy to check that the graph \( L(n) \) is not panconnected. In fact, it is not hard to show that for adjacent vertices \( v = [1, 12] \) and \( w = [2, 12] \), there is no \( v-w \) path of length 3 (or 4) in the graph \( L(n) \). In this paper, we wish to show that the graph \( L(n) \) has another interesting property, that is, if \( n \geq 6 \), then it is vertex-pancyclic and Hamilton-connected.

The group of all permutations of a set \( V \) is denoted by \( \text{Sym}(V) \) or just \( \text{Sym}(n) \) when \( |V| = n \). A permutation group \( G \) on \( V \) is a subgroup of \( \text{Sym}(V) \). In this case, we say that \( G \) acts on \( V \). If \( \Gamma \) is a graph with vertex-set \( V \), then we can view each automorphism of \( \Gamma \) as a permutation of \( V \), and so \( \text{Aut}(\Gamma) \) is a permutation group. Let the group \( G \) act on \( V \). We say that \( G \) is transitive (or \( G \) acts transitively on \( V \)) if there is just one orbit. This means that given any two element \( u \) and \( v \) of \( V \), there is an element \( \beta \) of \( G \) such that \( \beta(u) = v \).

The graph \( \Gamma \) is called vertex-transitive if \( \text{Aut}(\Gamma) \) acts transitively on \( V(\Gamma) \). The action of \( \text{Aut}(\Gamma) \) on \( V(\Gamma) \) induces an action on \( E(\Gamma) \), by the rule, \( \beta[x, y] = \{ \beta(x), \beta(y) \}, \beta \in \text{Aut}(\Gamma), \) and \( \Gamma \) is called edge-transitive if this action is transitive. The graph \( \Gamma \) is called symmetric, if for all vertices \( u, v, x, y \) of \( \Gamma \) such that \( u \) and \( v \) are adjacent, and \( x \) and \( y \) are adjacent, there is an automorphism \( \alpha \) such that \( \alpha(u) = x \) and \( \alpha(v) = y \). It is clear that a connected symmetric graph is vertex-transitive and edge-transitive. It is easy to show that the graph \( L(n) \) is not a symmetric graph.

Let \( G \) be any abstract finite group with identity \( 1 \), and suppose \( \Omega \) is a subset of \( G \) with the properties: (i) \( x \in \Omega \implies x^{-1} \in \Omega \) and (ii) \( 1 \notin \Omega \). The Cayley graph \( \Gamma = \Gamma(G; \Omega) \) is the (simple) graph whose vertex-set and edge-set are defined as follows: \( V(\Gamma) = G \), \( E(\Gamma) = \{ \{g, h\} \mid g^{-1}h \in \Omega \} \). As we have already stated, the graph \( L(n) \) is a Cayley graph if and only if \( n \) is a power of a prime integer [8].

2. Main results

DEFINITION 2.1

Let \( n \geq 3 \) be an integer and \( [n] = \{1, 2, \ldots, n\} \). The graph \( B(n) \) is a graph with the vertex set \( V = \{v \mid v \in [n], \ |v| \in \{1, 2\} \} \) and the edge set \( E = \{\{v, w\} \mid v, w \in V, v \subseteq w \text{ or } w \subseteq v \} \). We denote by \( L(n) \) the line graph of \( B(n) \), that is, \( L(n) = L(B(n)) \).

From Definition 2.1, it follows that every vertex of the graph \( L(n) \) is of the form \( \{i\}, \{i, j\} \), where \( i, j \in [n] \) and \( i \neq j \). In this paper, we denote \( \{i\}, \{i, j\} \) by \( [i], [i, j] \). Hence \( L(n) \) is the graph with the vertex-set \( V(L(n)) = V = \{[i, j], i, j \in [n], i \neq j\} \), in which two vertices \([i, j]\) and \([r, s]\) are adjacent if and only if \( i = r \) or \( i = j = [r, s] \). Thus, if \([i, j]\) is a vertex of \( L(n) \), then
hence, \( \text{deg}(\{i, ij\}) = n - 1 \). Therefore \( L(n) \) is a regular graph of valency \( n - 1 \). In fact, \( L(n) \) is a vertex-transitive graph [6,8]. By an easy argument, we can show that the graph \( L(n) \) is a connected graph with diameter 3. Also, its girth is 3 and hence it is not a bipartite graph. Figure 1 shows \( L(4) \) in the plane.

Amongst the various properties of the graph \( L(n) \), we are interested in its cycle structure and Hamilton-connectivity. Since, for \( n = 3 \), the graph \( L(n) \) is isomorphic with \( C_6 \), and the structure of this graph is simple, in the sequel, we let \( n \geq 4 \). As we can see in Figure 1, there are 4 cliques of order 3 in \( L(4) \) such that they construct a partition for the vertex-set of this graph. We can easily check that by these cliques, we can construct \( u-v \) paths of length \( m \) if \( 5 \leq m \leq 11 \) for any two vertices \( u \) and \( v \) in the graph \( L(4) \). Let \( u = [1,12] \) and \( v = [2,12] \). It is easy to check that there is no any \( u-v \) path of length 4 (or 3) in \( L(4) \). Hence \( L(4) \) is not a panconnected graph. In the graph \( L(n) \), the subgraph induced by the set \( C_i = \{ [i, ij] \mid j \in [n], j \neq i \} \), \( 1 \leq i \leq n \), is a clique of order \( n - 1 \). It is clear that \( P = \{ C_i \mid 1 \leq i \leq n \} \) is a partition for the vertex-set of \( L(n) \). Note that for any two cliques \( C_i \) and \( C_j \), there is a unique pair of adjacent vertices \( (u,v) \) such that \( u \in C_i \), \( v \in C_j \) (indeed \( u = [i, ij] \) and \( v = [j, ij] \)). Moreover, if \( v \) is a vertex in the clique \( C_i \), then there is exactly one vertex \( w \) in \( V(L(n)) - C_i \) such that \( w \) is adjacent to \( v \). In fact, if \( v = [i, ij] \in C_i \), then \( w = [j, ij] \) is the unique vertex which is not in \( C_i \) \((w \in C_j)\) and adjacent to \( v \).

**Lemma 2.2.** Let \( n \geq 4 \) be an integer and \( P = \{ C_i \mid 1 \leq i \leq n \} \) be the clique partition of the vertex-set of the graph \( L(n) \) where \( C_i = \{ [i, ij] \mid j \in [n], j \neq i \} \). Let \( C \in P \) and let \( v, w \) be two vertices in \( C \). Let \( t \) be an integer such that \( 2 \leq t \leq n - 1 \). Then there is a \( t \)-subset \( P_1 = \{ A_1, \ldots, A_t \} \subseteq P - \{ C \} \), and a \( v-w \) path \( Q : v, v_1, w_1, v_2, w_2, \ldots, v_t, w_t, w \) of length \( 2t + 1 \) such that \( v_i, w_i \in A_i \).

**Proof.** We prove the lemma by induction on \( t \). Let \( t = 2 \). Since \( L(n) \) is a vertex-transitive graph, without loss of generality, we can assume that \( v = [1,12] \) (and hence \( C = C_1 \)), \( w = [1,1i] \), \( i \neq 1,2 \). Consider the vertices \( v_1 = [2,12] \in C_2 \) and \( w_2 = [i,1i] \in C_i \). Thus there is a unique adjacent pair \( (w_1, v_2) \) such that \( w_1 \in C_2 \) and \( v_2 \in C_i \). Now, it is clear that the path \( Q : v, v_1, w_1, v_2, w_2, w \) is a desired path of length \( 5 = 2t + 1 \). Now let \( 2 \leq m < t \) and the claim is valid.
true for \( m \). By induction hypothesis, there are cliques \( A_1, A_2, \ldots, A_{r-1} \subset P - C_1 \) and a \( v-w \) path \( Q : v, v_1, w_1, v_2, w_2, \ldots, v_{r-1}, w_{r-1}, w \) of length \( 2r-1 \) such that \( v_i, w_i \in A_i \). Let \( A_i \in P - \{C_1, A_1, A_2, \ldots, A_{r-1}\} \) be an arbitrary clique. There is a unique adjacent pair \((u_{r-2}, z_{r-1})\) such that \( u_{r-2} \in A_{r-2} \) and \( z_{r-1} \in A_r \). Note that \( u_{r-2} \neq v_{r-2}, w_{r-2} \). Moreover, there is a unique adjacent pair \((u_{r-1}, z_r)\) such that \( u_{r-1} \in A_r \) and \( z_r \in A_{r-1} \). Again, note that \( z_t \neq v_{r-1}, \ w_{r-1} \). Now it is clear that the \( v-w \) path \( Q_0 : v, v_1, v_2, w, \ldots, v_{r-2}, u_{r-2}, z_{r-1}, u_{r-1}, z_r, w_{r-1}, w \) of length \( 2r+1 \) is a desired path if we rename \( A_{r-1} \) by \( A_r \) and \( A_r \) by \( A_{r-1} \).

A graph \( \Gamma \) of order \( n \) is \( k \)-pancyclic \( (k \leq n) \) if it contains cycles of every length from \( k \) to \( n \) inclusive, and \( \Gamma \) is pancyclic if it is \( g \)-pancyclic, where \( g = g(\Gamma) \) is the girth of \( \Gamma \). A graph is of pancyclicity if it is pancyclic. Not that if \( \Gamma \) is pancyclic, then \( \Gamma \) is hamiltonian. The pancyclicity is an important property to determine if a topology of a network is suitable for some applications where mapping cycles of any length into the topology of the network is required [10, 11]. The concept of pancyclicity, proposed first by Bondy [5], has been extended to vertex-pancyclicity and edge-pancyclicity [1]. A graph \( \Gamma \) of order \( n \) is vertex-pancyclic (resp. edge-pancyclic) if any vertex (resp. edge) lies on cycles of every length from \( g(\Gamma) \) to \( n \) inclusive. Obviously, an edge pancyclic graph is certainly vertex-pancyclic.

It is easy to check that \( L(4) \) has no cycles of lengths 4 and 5. Also \( L(5) \) has no cycle of length 5. But by Lemma 2.2, we can show that if \( n \geq 6 \), then \( L(n) \) is a vertex-pancyclic graph. Moreover, we can check that \( L(n) \) is not edge-pancyclic, since the edge \( e = \{[1, 12; [2, 12] \} \) does not lie on a 3-cycle (or 4-cycle). But we have the following result.

**Theorem 2.3.** Let \( n \geq 6 \) be an integer. Then \( L(n) \) is a vertex-pancyclic graph.

**Proof.** Let \( m \) be an integer such that \( n(n-1) \geq m \geq 3 \). Let \( v \) be an arbitrary vertex in the graph \( L(n) \). Let \( C \) be the \((n-1)\)-clique containing \( v \) in \( L(n) \). Hence if \( 3 \leq m \leq 5 \), then there is a cycle \( C_m \) containing \( v \) in \( C \). Now assume that \( 6 \leq m \leq 3(n-1) \). Let \( w \in C \) be such that \( w \neq v \). By Lemma 2.2, there are two \((n-1)\)-cliques, \( A_1 \) and \( A_2 \), and a path \( Q : v, v_1, w_1, v_2, w_2, \ldots, v_{r-1}, w_{r-1}, w \) of length 5 such that \( v_i, w_i \in A_i \) for each \( i \in \{1, 2\} \). Let \( 1 \leq t \leq n-3 \). It is clear that we can insert \( t \) vertices of the clique \( C \) between \( v \) and \( w \). Also, we can insert \( t \) vertices of the clique \( A_i, i \in \{1, 2\} \) between \( v_i \) and \( w_i \). Now it is clear how we can construct an \( m \)-cycle containing the vertex \( v \) in the graph \( L(n) \). Now let \( m > 3(n-1) \). By Lemma 2.2, there are cliques \( A_1, \ldots, A_{(n-1)} \) and the path \( Q : v, v_1, \ldots, v_{(n-1)}, w \) in \( L(n) \) such that \( v_i, w_i \in A_i, 1 \leq i \leq n-1 \). Note that \( 3(n-1) > 2n \). By inserting adequate number of vertices of each \( A_i \) (and \( C \)) between each pair of vertices \( v_i \) and \( w_i \) \((v \) and \( w) \), we can construct an \( m \)-cycle containing the vertex \( v \) in the graph \( L(n) \). \( \square \)

We now want to show that the graph \( L(n) \) is a Hamilton-connected graph.

**Theorem 2.4.** Let \( n \geq 4 \) be an integer. Then \( L(n) \) is a Hamilton-connected graph.

**Proof.** It is easy to check that the assertion of the theorem is true for the case \( n = 4 \), hence in the sequel, we assume that \( n \geq 5 \). Let \( v, w \) be two vertices in \( L(n) \). We show that there is a hamiltonian \( v-w \) path in \( L(n) \). Let \( P = [C_i \mid 1 \leq i \leq n] \) be the clique partition of the vertex-set of the graph \( L(n) \), where \( C_i = [[i, ij] \mid j \in [n], j \neq i] \). There are two cases, namely,
(i) $v$ and $w$ are in the same $(n-1)$-clique in the graph $L(n)$, or

(ii) $v$ and $w$ are in distinct $(n-1)$-cliques.

(i) Assume that $A_1 \in P$ and $v, w \in A_1$. Let $w_1 \in A_1$ be such that $w_1 \neq v, w$. Therefore by Lemma 2.2, for subset $P_1 = \{A_2, \ldots, A_n\} \subseteq P - \{A_1\}$, there is a $v$-$w_1$ path $Q_1: v = v_1, v_2, w_2, \ldots, v_n, w_n, w_1$ of length $2n-1$ such that $v_i, w_i \in A_i$. Hence $Q_1: v = v_1, v_2, w_2, \ldots, v_n, w_n, w_1, w$ is a $v$-$w$ path of length $2n$ in $L(n)$. For each $i, 2 \leq i \leq n$, we insert all other vertices in the clique $A_i$ between $v_i$ and $w_i$ and again obtain a $v$-$w$ path of greater length. Now by inserting all the vertices in the set $A_1 - \{v, w, w_1\}$ between the vertices $w_1$ and $w$, we obtain a hamiltonian $v$-$w$ path in the graph $L(n)$.

(ii) We now assume that $v$ and $w$ are in distinct $(n-1)$-cliques. Let $A_1, A_n \in P$ are such that $v \in A_1$ and $w \in A_n$. We know that for each vertex $x$ in $L(n)$, there is a unique $(n-1)$-clique $C_x$ in the graph $L(n)$ such that $x \notin C_x$ and $x$ is adjacent to a unique vertex in $C_x$. Let $P - \{A_1, A_n\} = \{A_2, \ldots, A_{n-1}\}$ be such that $A_2 \neq C_v$ and $A_{n-1} \neq C_w$. Therefore, $v$ is adjacent to no vertex in $A_2$ and $w$ is adjacent to no vertex in $A_{n-1}$. On the other hand, for each $i, 1 \leq i \leq n-1$, there is a unique adjacent pair $w_i, v_{i+1}$ such that $w_i \in A_i$ and $v_{i+1} \in A_{i+1}$. Note that $w_1 \neq v$ and $v_n \neq w$. Hence the path $Q_0: v = v_1, w_1, v_2, w_2, \ldots, v_n, w_n = w$ is a $v$-$w$ path in the graph $L(n)$. For each $i, 1 \leq i \leq n$, if we insert between $v_i$ and $w_i$ all other vertices in the clique $A_i$, we obtain a hamiltonian $v$-$w$ path in the graph $L(n)$. \qed

Acknowledgements

The authors are thankful to the anonymous referees for their valuable comments and suggestions.

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