Rotational component spaces for infinite-type translation surfaces

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We study the spaces of rotational components of infinite-type translation surfaces to describe neighborhoods of isolated wild singularities. These spaces are non-Hausdorff in general, and we completely describe them in several examples. We prove that any finite topological space can arise as a space of rotational components of a translation surface.

Perhaps the first appearance of translation surfaces was in the context of dynamics in \cite{FK36}, more specifically in the context of billiards. The unfolding construction for billiards gives us the most visual way to describe translation surfaces: they can be obtained by gluing polygons in the Euclidean plane along parallel edges of the same length.

Since the seminal work \cite{Vee89}, the interest in this field of research has grown and different aspects have been studied. One aspect is the study of the geometry of moduli spaces of finite translation surfaces which can be broken down in strata (see \cite{Zor06} for a survey). For these strata, for example, the number and behaviour of connected components is known since \cite{KZ03}. Nowadays, dynamics on strata is a rich theory and its examination has recently lead to the breakthrough result on orbit closures in \cite{EMM13}.

Recently, infinite-type translation surfaces have come into the focus of research. A lot of results have been obtained for infinite coverings of finite translation surfaces (e.g. \cite{DHL11}, \cite{HHW13}, and \cite{FU14}). Furthermore, unfamiliar behaviours of the dynamics in the infinite case have been found in \cite{Hoo14}, \cite{Tre14}, and \cite{LT14}, for example.

One of the prominent aims in this young area of research is to extend the classical theory of dynamics on strata to the world of infinite-type surfaces. In the finite-type case, neighborhoods of singular points are simply covers of the pointed disk, and we can use the index of these covers to define strata. In order to define an analog to strata in the general case, it is natural to first study the possible topologies around singular points. This was first undertaken in \cite{BV13}, where the authors studied the germs of geodesic rays starting at singular points, called linear approaches. Linear approaches of a singularity $\sigma$ together form a space $\mathcal{L}(\sigma)$. If two linear approaches differ by a rotation centered at the singular point, we say they are in the same rotational component (see Section 1). Being in the same rotational component constitutes an equivalence relation $\sim$. 
In the present paper we study the set \( \tilde{\mathcal{L}}(\sigma) = \mathcal{L}(\sigma) / \sim \) of all rotational components, equipped with the quotient topology. This space captures how rotational components “fit together”. It is in general non-Hausdorff. In Theorem 4.1 we show that given any finite topological space \( F \), there is a translation surface with one singularity \( \sigma \) such that \( \tilde{\mathcal{L}}(\sigma) = F \). In particular, this implies that \( \tilde{\mathcal{L}}(\sigma) \) can be non-\( T_0 \), \( T_0 \) but not \( T_1 \), \( T_1 \) but not \( T_2 \), or \( T_2 \).

In Theorem 5.1 we also investigate how much information we lose by passing to the quotient \( \mathcal{L}(\sigma) \to \tilde{\mathcal{L}}(\sigma) \). It is possible to construct uncountably many translation surfaces with non-homeomorphic spaces \( \mathcal{L}(\sigma) \) but homeomorphic spaces \( \tilde{\mathcal{L}}(\sigma) \).

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1 Definitions

A translation surface is a two-dimensional manifold equipped with a translation structure, i.e. an atlas for which transition functions are translations. We can pull back the Euclidean metric to the translation surface via the charts. The additional points we obtain in the metric completion of the surface are called singularities.

In the case of finite translation surfaces whose metric completions are compact surfaces, the singularities are points where curvature is concentrated and which have an angle of \( k \cdot 2\pi \). These are called cone angle singularities of multiplicity \( k \), and they have a neighborhood which is a cyclic translation covering of a once punctured disk of degree \( k \). Throughout this article, we say a translation surface is of infinite type if its metric completion has singularities that are not cone angle singularities. By small abuse of notation we will often identify a translation surface with its metric completion. For example we will say a surface \( M \) is compact if its metric completion is compact. In this case, \( M \) might be non-compact, and its completion might be compact but not a surface.

In the case of infinite-type surfaces, not only can there be singularities for which the corresponding cyclic translation covering is infinite (these are called infinite angle singularities) but there can also be singularities that have no neighborhood which is a cyclic translation covering of a once punctured disk. The latter singularities are called wild singularities and contain most of the complication inherent to infinite-type translation surfaces. In the present article, even when statements are true for all types of singularities, we will mainly be interested in the wild singularity case.

In [BV13], Joshua Bowman and Ferrán Valdez initiated the classification of wild singularities by studying the space of linear approaches of a translation surface \( (X, \omega) \). We will recall the definitions now.
**Definition 1.1 (Space of linear approaches)**

Let \((X, \omega)\) be a translation surface and \(\epsilon > 0\). On the space

\[
\mathcal{L}^\epsilon(X) := \{\gamma : (0, \epsilon) \to X : \gamma \text{ is a geodesic curve}\}
\]

we define the following equivalence relation \(R\): \(\gamma_1 \in \mathcal{L}^\epsilon(X)\) and \(\gamma_2 \in \mathcal{L}^\epsilon(X)\) are called \(R\)-equivalent if \(\gamma_1(t) = \gamma_2(t)\) for all \(t \in (0, \min\{\epsilon, \epsilon'\})\). The space

\[
\mathcal{L}(X) := \bigsqcup_{\epsilon > 0} \mathcal{L}^\epsilon(X)/R
\]

is called *space of linear approaches of* \(X\) and the \(R\)-equivalence class \([\gamma]\) of \(\gamma \in \mathcal{L}^\epsilon(X)\) is called *linear approach*.

For a point \(x \in X\), we define in the same way \(\mathcal{L}^\epsilon(x) := \{\gamma \in \mathcal{L}(X) : \lim_{t \to 0} \gamma(t) = x\}\) and \(\mathcal{L}(x)\). The \(R\)-equivalence class \([\gamma]\) of \(\gamma \in \mathcal{L}^\epsilon(x)\) is called *linear approach to the point* \(x\).

On \(\mathcal{L}(x)\) we can define the *maximal length function* \(\ell : \mathcal{L}(x) \to (0, \infty]\) via

\[
\ell([\gamma]) = \sup \{\epsilon > 0 : \gamma \in \mathcal{L}^\epsilon(x), \gamma \text{ representant of } [\gamma]\}.
\]

Before describing the topology on \(\mathcal{L}(\sigma)\), we define the translation structure of a rotational component, seen as a class of linear approaches.

**Definition 1.2 (Translation structure on rotational components)**

Following [BV13], by an angular sector we mean a triple \((I, c, i_c)\) such that \(I\) is a non-empty generalized interval (i.e. a connected subset of \(\mathbb{R}\)), \(c \in \mathbb{R}\) and \(i_c\) is an isometry from \(\{x + iy : x < c, y \in I\} \subseteq \mathbb{C}\) with metric defined by \(e^zdz\) to \(X\). Given any angular sector \((I, c, i_c)\), there is a map \(f_c : I \to \mathcal{L}(x)\) which sends an element \(y \in I\) to the linear approach of the ray \(i_c(\ln(t) + iy)\) for \(t < c\). The image of \(f_c\) is always contained in a single rotational component, hence they give a chart of translation structure with boundary for every rotational component that contains more than one linear approach.

We now describe the topology with which we endow \(\mathcal{L}(X)\).

**Definition 1.3 (Topology on \(\mathcal{L}(X)\))**

We equip \(\mathcal{L}^\epsilon(X)\) with the topology induced by the metric

\[
d_\epsilon(\gamma_1, \gamma_2) = \sup_{t \in (0, \epsilon)} d(\gamma_1(t), \gamma_2(t)).
\]

For every \(\epsilon > 0\), we can embed \(\mathcal{L}^\epsilon(X)\) in \(\mathcal{L}(X)\) so that we obtain a direct system. Then we define the topology of \(\mathcal{L}(X)\) as the direct limit topology with respect to the embeddings \(\mathcal{L}^\epsilon(X) \hookrightarrow \mathcal{L}(X)\). On \(\mathcal{L}^\epsilon(x)\) and \(\mathcal{L}(x)\) for \(x \in \overline{X}\), we put the topology induced by inclusion into \(\mathcal{L}^\epsilon(X)\) and \(\mathcal{L}(X)\).

The following characterization from [BV13 Proposition 2.3] is more handsome than the definition we gave. It will be often used in computations. For every \(x \in X\), \(r > 0\), and \(t > 0\), the set

\[
B(x, r)^t = \{[\gamma] : \gamma \in \mathcal{L}^\epsilon \text{ for some } \epsilon > t, d(\gamma(t), x) < r\}
\]
is an open set. In fact, the collection \( \{ B(x, r) \} \) forms a subbasis for the previously described topology on \( \mathcal{L}(X) \).

As a direct consequence of the definitions, we have the following very useful lemma (see [BV13, Corollary 2.2]).

**Lemma 1.4 (Directions of linear approaches are varying continuously).** Let \((X, \omega)\) be a translation surface. Then the map \( \mathcal{L}(X) \rightarrow S^1 \) that associates to a linear approach its direction is continuous.

Define an equivalence relation \( \sim \) on \( \tilde{\mathcal{L}}(X) \) as follows: two linear approaches \([\gamma_1]\) and \([\gamma_2]\) are related by \( \sim \) if there is an angular sector \((I, c, i_c)\) such that \(i_{c^{-1}}\) of the images of \(\gamma_1\) and \(\gamma_2\) are two rays parallel to the real axis.

**Definition 1.5 (Space of rotational components)**
An equivalence class under \( \sim \) is called a rotational component. We write \([\gamma]\) for the equivalence class of a linear approach \([\gamma]\). Furthermore, we define the space of rotational components as \( \tilde{\mathcal{L}}(X) := \mathcal{L}(X) / \sim \) endowed with the quotient topology.

We define \( \tilde{\mathcal{L}}(x) \) for \( x \in \overline{X} \) as for \( \mathcal{L}(x) \), and give it the topology induced by the inclusion into \( \tilde{\mathcal{L}}(X) \).

## 2 Examples

Many interesting examples of spaces of rotational components arise from taking a translation surface, cutting it along slits and regluing it in a different way. By performing this operation carefully, we will be able to control some of the properties of the resulting translation surface, for example compactness or finite area.

More precisely, let \((X_1, \omega_1), (X_2, \omega_2)\) be translation surfaces (not necessarily different). A slit \( m \) on \((X_1, \omega_1)\) is a open geodesic segment in \( X_1 \) with an orientation. When we consider the metric completion of \((X_1 \setminus m, \mu)\) with \( \mu \) the path-length metric, we get two copies \( m' \) and \( m'' \) of \( m \), with endpoints identified. If \( m_1 \) is a slit on \((X_1, \omega_1)\) and \( m_2 \) is a slit on \((X_2, \omega_2)\) with the same holonomy vectors then we can “glue the slits together”. For instance, we can identify \( m_1' \) with \( m_2'' \) or \( m_1'' \) with \( m_2' \) and we will describe this visually as “gluing the upper part of \( m_1 \) to the lower part of \( m_2 \)”. If we glue both possibilities we just say that we glue \( m_1 \) and \( m_2 \) together. Another possibility is to consider one slit and divide it in more segments which we glue in a different order as they were originally.

As a first example, we decorate the plane using this construction. The space of rotational components of the resulting surface is just a singleton, so the topology is already determined.

**Example 2.1 (Harmonic series decoration).** Consider the plane \( \mathbb{R}^2 \) with a horizontal slit starting from 0 going to the right forever. We choose a sequence \((a_n)\) in \( \mathbb{R}_+ \) such that \( \lim_{n \to \infty} a_n = 0 \) and \( \sum_{n=1}^{\infty} a_n = \infty \), e.g. the harmonic series. We cut the upper and the lower part into segments, for the upper part the segments are \( I_1 \) of length \( a_1 \), \( I_2 \) of length \( a_2 \), and so on inductively, for the lower part the segments are \( J_2 \) of length \( a_2 \), \( J_1 \) of length \( a_1 \), then \( J_4, J_3, J_6 \), and so on (see Figure 2.1). Then we glue each \( I_n \) with \( J_n \).
Figure 2.1: Harmonic series decoration: Segments $I_i$ and $J_i$ are glued.

All the endpoints of the segments are identified via the gluings, so there is only one singularity $\sigma$, which is obviously not a cone angle singularity. Also, $\sigma$ cannot be an infinite angle singularity as for every $\epsilon > 0$ there exists a saddle connection of length smaller than $\epsilon$ starting and ending at $\sigma$. Thus, $\sigma$ is a wild singularity. It has exactly one rotational component $\gamma$ which is isometric to $\mathbb{R}$ as a class of linear approaches (see Definition 1.2). So $\tilde{L}(\sigma)$ is the one-point space equipped with the unique possible topology.

The following example can be found in [Bow12] and will be very useful in the proof of Theorem 5.1.

Example 2.2 (Stack of boxes). An infinite sequence of rectangles with height $h_n$ bounded away from 0 (for instance $h_n \geq 1$) and width $w_n$ with $\sum h_n w_n < \infty$ and $w_{n+1} < w_n$, are stacked together as shown in Figure 2.2. By gluing the opposite sides, all corners of the rectangles and the points $A_i$ are identified. So again, this surface has only one rotational component which is isometric to $(0, \infty)$.

Figure 2.2: Stack of boxes: opposite sides are glued.
It is worth noting that the space $L(\sigma)$ is Hausdorff in general (see [BV13, Corollary 2.9]) but in this case not $T_3$, thus $L(\sigma)$ is not metrizable. Indeed, the linear approach $[\gamma]$ starting in the right corner $A_1$ and going left toward $A_2$ is not inside the closed set

$$F = L(\sigma) \setminus (\cup_{\epsilon' > \epsilon} L'(\sigma))$$

for $\epsilon$ small enough. However, any open set containing $F$ has non-empty intersection with any other open set containing $[\gamma]$, as can be seen by looking carefully at limits of linear approaches starting at $A_n$ and going left toward $A_{n+1}$.

The following example is a refinement of Example 2.1 with non-trivial space of rotational components.

**Example 2.3 (Geometric series decoration).** Consider a translation surface with a slit of finite length. We choose a sequence $(a_n)$ in $\mathbb{R}_+$ for which the corresponding series is converging to the length of the slit, e.g. a geometric series. We divide the upper and the lower part of the slit in segments of length $a_n$ and identify the segments of same index.

We will describe two different recipes how to do the construction.

(i) First we can choose to assign a segment $I_1$ (resp. $J_1$) of length $a_1$ at the left (resp. right) of the upper (resp. lower) part. Then assign a segment $I_2$ (resp. $J_2$) of length $a_2$ at the right of $I_1$ (resp. at the left of $J_1$), and so on inductively (see Figure 2.3).

![Figure 2.3: Geometric series decoration with segments going from left to right on the upper part and going from right to left on the lower part.](image)

In this case we obtain two rotational components $[\gamma_1]$ and $[\gamma_2]$, both isometric to $(0, \infty)$. Now let $\gamma_1$ be the vertical linear approach starting on the far right of the slit and going upward. Its corresponding rotational component shall be $[\gamma_1]$. Then every vertical linear approach starting at the right endpoint of $I_{2n}$ and going upward is contained in the other rotational component $[\gamma_2]$. Thus there is a sequence of vertical linear approaches belonging to $[\gamma_2]$ converging to $\gamma_1$ in the topology on $L(\sigma)$. We can conclude that any open set in $\tilde{L}(\sigma)$ containing $[\gamma_1]$ also contains $[\gamma_2]$, and by symmetry we see that the topology of $\tilde{L}(\sigma)$ is

$$\{\emptyset, \{[\gamma_1], [\gamma_2]\}\}.$$

(ii) A different topology on a two-element space of rotational components can be obtained in the following way. Similar to Example 2.1 we first assign a segment $I_1$
of length $a_1$ at the left of the upper part of the slit and assign successive segments $I_2$ of length $a_2$ and so on inductively on its right. For the lower part, we assign a segment $J_2$ of length $a_2$ at the left, a segment $J_1$ of length $a_1$ on its right, then $J_4$ of length $a_4$, $J_3$ of length $a_3$, and so on. Then we glue the segments of the same length (see Figure 2.4).

Now we have an infinite rotational component $[\gamma_0]$ that is isometric to $(-\infty, \infty)$ and a finite rotational component $[\gamma_1]$ which has length $2\pi$. With a similar argument as in (i) we can see that there are linear approaches in $[\gamma_0]$ converging to a linear approach in $[\gamma_1]$. However, there is no sequence in $[\gamma_1]$ converging to a linear approach in $[\gamma_0]$. Therefore, the topology of $\tilde{L}(\sigma)$ is

$$\left\{ \emptyset, \{[\gamma_0]\}, \{[\gamma_0], [\gamma_1]\} \right\}.$$

Before investigating more examples of surfaces with finite space of rotational components in Section 4, we now look at examples of surfaces $(X, \omega)$ with singularities $\sigma$ for which $\tilde{L}(\sigma)$ is infinite. The first is from [Cham04] and is often called the baker’s map surface.

**Example 2.4 (Chamanara surface).** Consider a square where the sides are split in segments and glued crosswise: The right half of the top is glued to the left half of the bottom, the right half of the remaining part of the top is again glued to the left half of the remaining part of the bottom, and so on as in Figure 2.5.

The Chamanara surface has exactly one singularity $\sigma$. Because of the gluings it is obvious that every second split point is identified, so it admits at most two singular points. Now the distance of these two points in the metric completion of the surface is 0, so they are equal. The unique singularity $\sigma$ has two rotational components that are isometric to $\mathbb{R}$ and infinitely many of finite length (see [BV13]). All of the finite-length rotational components are images of each other by elements of the Veech group of the Chamanara surface, and no two linear approaches in different finite-length rotational components have the same direction in $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. This implies by Lemma 1.4 that the subspace of $\tilde{L}(\sigma)$ of all finite rotational components is endowed with the discrete topology. An argument similar to the one in Example 2.1 shows that any open set containing a finite-length rotational component contains both of the infinite-length rotational components as well (for instance, linear approaches $[\gamma_2]$ and $[\gamma_3]$ and very close to $[\gamma_1]$ in Figure 2.5). Moreover, each of the singletons containing an infinite rotational component is open. That can be directly proven from the definition of the topology on $\tilde{L}(\sigma)$. 

![Figure 2.4: Geometric series decoration with segments going in the same direction.](image-url)
We just proved the following characterization for the open sets of $\tilde{\mathcal{L}}(\sigma)$: a non-trivial set in $\tilde{\mathcal{L}}(\sigma)$ is open if and only if it is a singleton containing an infinite-length rotational component or if it contains both infinite-length rotational components.

Note that the two rotational components of length $\frac{\pi}{4}$ are readily visible in the representation of the Chamanara surface with a square, as in Figure 2.5. The other finite-length rotational components are harder to see in the figure, and in fact any linear approach in a rotational component of length less than $\frac{\pi}{4}$ intersects the sides of the square infinitely often. As we will see in the following examples, avoiding this type of behaviour simplifies the analysis of the space of rotational components.

**Definition 2.5 (Good cellulation)**

Let $X$ be a translation surface, and $S$ a union of saddle connections. We say $S$ is a good cellulation if any geodesic $\gamma \notin S$ verifies that $\{t : \gamma(t) \in S\}$ is a discrete set of the real line.

To be of interest, a good cellulation $S$ of $X$ should also verify that each component of $X \setminus S$ is a piece of the Euclidean plane or of a cylinder. As we already noted, the sides of the square in Figure 2.5 are not a good cellulation.
Example 2.6 (Star decoration). Let $(X, \omega)$ be a translation surface and $x \in X$. By a “star decoration” of $X$ we mean a translation surface with boundary obtained by cutting open all line segments starting at $x$, of length $2^{-n}$ and in the direction $\frac{m\pi}{2^n}$, such that $n \geq 1$ and $m$ is odd when $n > 1$ (see Figure 2.6). We will describe two ways to glue the branches of the star (the second will involve additional cylinders) which result in translation surfaces with non-homeomorphic spaces of rotational components.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{star_decoration.png}
\caption{Star decoration without indications on the gluing.}
\end{figure}

(i) The first kind of gluing we can consider is the following: For each branch, glue the right side to the left side of the antipodal branch and vice versa. Then all tips of the branches will be identified with the center and there is only one singularity which is wild. For every branch we have now a rotational component which is isometric to $[0, 2\pi]$. Also, for every “non-dyadic direction” we have a linear approach to the center of the star. It is contained in a rotational component which consists only of this point. The problem with this gluing is that the union of all the branches is not a good cellulation. In fact, there are additional rotational components that are not easy to see in the figure. It is possible to find such a rotational component by starting with any point not on a branch, and considering geodesics passing through that point. One can then inductively find nested open sets $U_n$ of directions for
which the geodesics intersect \( n \) smaller and smaller branches. By construction the intersection of these nested open intervals is a singleton, containing a direction for which the geodesic reaches the center of the star in finite (i.e. is a linear approach) while passing through infinitely many branches.

(ii) We now want to make the surface with the star decoration into a surface with a good cellulation. For every branch, glue the two sides to the two vertical sides of a rectangle as in Figure 2.7. This rectangle consists of \( 2^n + 1 \) squares of size \( 2^{-n} \), with the tops and bottoms of the squares glued crosswise, except for the middle square where top and bottom are identified.

![Figure 2.7](image)

Figure 2.7: Segments with the same letters are glued.

Now the metric completion of the resulting surface is compact (if the star decoration is performed on a compact surface). Let \( S \) be the union of all the branches of the star. By construction, \( S \) is a good cellulation (if the star decoration is performed on a surface with a good cellulation).

As in the previous case, we have exactly one singularity \( \sigma \) as all tips of the branches are identified with the center as well as the singularities on the rectangles. Because \( S \) is a good cellulation, we see that there are only four types of linear approaches:

a) rays \( \gamma \) starting from the center of the star going in a non-dyadic direction, and such that \( \gamma(t) \) is not inside any rectangle for small \( t \),

b) rays \( \gamma \) starting from the center of the star going in a dyadic direction, and such that \( \gamma(t) \) is not inside a rectangle for small \( t \),

c) rays \( \gamma \) starting from the tip of a branch of the star, and such that \( \gamma(t) \) is not inside a rectangle for small \( t \),

d) rays starting at a vertex of a rectangle and such that \( \gamma(t) \) stays inside this rectangle for small \( t \).

Now there are two types of rotational components:

a) singletons consisting of a linear approach of type a), or

b) finite-length rotational components isometric to \([0, (2^n + 2) \cdot 2\pi]\) via a map \( \phi \) such that \( \phi(0) \) and \( \phi((2^n + 2) \cdot 2\pi) \) are of type b) and go in the same (dyadic) direction. \( \phi(x) \) is of type c) for \( x \in [(2^n + 1) \cdot \pi, (2^n + 3) \cdot \pi] \) and of type d) otherwise.
Therefore, we can define a one-to-one map $P$ from $\tilde{L}(\sigma)$ to $S^1$ that associates to a rotational component the direction of its element of type a) or b). We will now show that $P$ is a homeomorphism.

To show that $P$ is open, we will show that the image of any neighborhood inside $L(\sigma)$ of any rotational component $c$ (seen as a class of linear approaches in $L(\sigma)$) is a neighborhood of $P(c)$. Assume first that $c$ is a singleton consisting of a linear approach $[\gamma]$ of type a). Any neighborhood of $[\gamma]$ contains some open set $B(\gamma(t), r)^t$ (recall notations from Section 1). Let $\theta$ be the direction of $[\gamma]$, that is $\gamma(t) = te^{i\theta}$. Choose $\epsilon$ and $\eta$ small enough that 

$$\{(t + \rho)e^{i(\theta + \varphi)} : \rho \in [0, \eta], \varphi \in (-\epsilon, \epsilon)\}$$

is contained in the disk $\{z : |z - te^{i\theta}| < r\}$. By decreasing $\epsilon$ if necessary, we can assume that no branch with direction in $(\theta - \epsilon, \theta + \epsilon)$ is longer than $\eta$. Then any ray of type a) with direction in $(\theta - \epsilon, \theta + \epsilon)$ is in $B(\gamma(t), r)^t$, as is any ray of type c) starting on a branch with direction $\varphi \in (\theta - \epsilon, \theta + \epsilon)$. Thus $P(B(\gamma(t), r)^t)$ contains $(\theta - \epsilon, \theta + \epsilon)$ so it is a neighborhood of $\{\theta\} = P(c)$. If $P(c)$ is a dyadic direction, we can conclude with a similar argument for the two linear approaches in $c$ of type b).

To show that $P$ map is continuous, we will show that its pre-composition with the quotient map $L(\sigma) \to \tilde{L}(\sigma)$ is continuous. But this map is just the map that sends a linear approach to its direction, so it is continuous by Lemma 1.4.

Therefore $P$ is a homeomorphism as was to be shown.

A modification of the previous example will be very helpful in Section 4 so we introduce it here. It will serve as a building block in the proof of Theorem 4.1.

**Example 2.7 (Shrinking star decoration).** Consider $\mathbb{R}^2$ with line segments $l_n$ from the origin $(0, 0)$ to $(2^{-n}\sin(\frac{\pi}{2^n}), 2^{-n}\cos(\frac{\pi}{2^n}))$ for $n \geq 1$ as in Figure 2.8. Make a slit along each $l_n$, glue the two sides to the two smaller sides of a $(1 + 2^{-n})$-by-$2^{-n}$ rectangle as in Figure 2.7 and then glue the other sides of the squares in this rectangle in a crosswise way as before.

In this case, we have one wild singularity with one rotational component, whose translation structure is isomorphic to $(0, \infty)$.

![Figure 2.8: Shrinking star decoration.](image-url)
3 Relations between the translation structure on rotational components and the topology on $\tilde{L}(\sigma)$

In this section we describe two relations between

(i) the translation structure on rotational components, seen as classes of linear approaches, and

(ii) the topology on $\tilde{L}(\sigma)$, coming from the uniform metric on the spaces $L^e(X)$.

**Proposition 3.1 (Rotational components isometric to $\mathbb{R}$ are open points)**

Let $(X, \omega)$ be a translation surface with discrete singularities and $\sigma$ be a wild singularity of $(X, \omega)$. If $c$ is a rotational component of $\sigma$ which is isometric to $\mathbb{R}$ then $\{c\}$ is open in $\tilde{L}(\sigma)$.

**Proof.** Let $c \in \tilde{L}(\sigma)$ be a rotational component which is isometric to $\mathbb{R}$. The set $\{c\}$ is open in $\tilde{L}(\sigma)$ if the set of linear approaches in $c$ is open in $L^e(\sigma)$.

To prove the statement, we find open neighborhoods for every linear approach contained in $c$ that do not contain linear approaches from other rotational components. The union of these neighborhoods is then an open set in $L^e(\sigma)$.

Because $c$ is isometric to $\mathbb{R}$, for any linear approach $[\gamma]$ in $c$ with $\gamma \in L^e(\sigma)$ a representant for a suitable $\epsilon$, there is an angular sector $([-2\pi, 2\pi], \ln \epsilon, i_{\ln \epsilon})$ such that

$$i_{\ln \epsilon}^{-1}(\gamma((0, \epsilon))) = \{(x, 0) : x < \ln \epsilon\}.$$

Now $B(\gamma(\frac{\epsilon}{4}), \frac{\epsilon}{4})^\perp$ is an open neighborhood of $[\gamma]$ in $L$. Recall from Section 1 that it is the set of all linear approaches with a representant $\gamma'$ for which $\gamma'(\frac{\epsilon}{4})$ is contained in the disk around $\gamma(\frac{\epsilon}{4})$ of radius $\frac{\epsilon}{4}$. We will now show that $B(\gamma(\frac{\epsilon}{4}), \frac{\epsilon}{4})^\perp$ is also contained in $c$.

The set

$$i_{\ln \epsilon}((\infty, \ln \epsilon) \times (-\pi, \pi))$$

is an open $\epsilon$-disk with a slit as in Figure 3.1.

Starting from the shaded region $\overline{B(\gamma(\frac{\epsilon}{4}), \frac{\epsilon}{4})}^\perp$, a geodesic ray of length $\frac{\epsilon}{4}$ can either hit the slit or stays in $i_{\ln \epsilon}((\infty, \ln \epsilon) \times (-\pi, \pi))$. In the first case, it can hit the slit only at the segment

$$i_{\ln \epsilon}((\infty, \ln \frac{\epsilon}{4}) \times \{-\pi\}) \quad \text{or} \quad i_{\ln \epsilon}((\infty, \ln \frac{\epsilon}{4}) \times \{\pi\}).$$

Both segments are dashed in the picture.

If it hits $i_{\ln \epsilon}((\infty, \ln \frac{\epsilon}{4}) \times \{-\pi\})$ or $i_{\ln \epsilon}((\infty, \ln \frac{\epsilon}{4}) \times \{\pi\})$, consider

$$i_{\ln \epsilon}((\infty, \ln \epsilon) \times (-2\pi, -\pi)) \quad \text{and} \quad i_{\ln \epsilon}((\infty, \ln \epsilon) \times (-2\pi, -\pi)),$$

which are half disks with radius $\epsilon$ as in Figure 3.2.

A geodesic ray of length less than $\frac{\epsilon}{4}$ starting on either of the dashed segments, heading inwards, will never hit the boundary, hence can be extended.
If the geodesic ray stays in $i_{\ln \epsilon}((-\infty, \ln \epsilon) \times (-\pi, \pi))$ and cannot be extended it can only define a linear approach of $\sigma$ if it ends at the center. Hence, it must represent a linear approach in the same rotational component as $[\gamma]$.

Note that the converse statement is not true in general. For instance, if a singularity has exactly one rotational component then it is certainly open. There exist translation surfaces with singularities that have one rotational component which is isometric to $(0, \infty)$, for example a torus with a shrinking star decoration as in Example 2.7 or the stack of boxes in Example 2.2. However, a slightly weaker statement is true.

**Proposition 3.2 (Open rotational components have infinite length)**

Let $(X, \omega)$ be a translation surface with discrete singularities and $\sigma$ be a wild singularity. If $c$ is a rotational component such that $\{c\}$ is open in $\tilde{L}(\sigma)$ it has to be isometric to an interval in $\mathbb{R}$ of infinite length.

**Proof.** Consider a rotational component $c$ of finite length. We show that there exists a linear approach belonging to that rotational component that is the limit of linear approaches not belonging to the rotational component. This implies that $\{c\}$ is not open in $\tilde{L}(\sigma)$.

As $c$ is of finite length, it is isometric to an interval with endpoints $a, b \in \mathbb{R}$ (the interval can be open, closed or half-open, half-closed). Choose a linear approach $[\gamma]$ which corresponds to a point in the interval that differs at most $\frac{\pi}{4}$ from $a$ or $b$. Choose a regular point $\gamma(t_0)$ on it. Then there exist $\epsilon$ and $\epsilon'$ with $\epsilon < \epsilon' < t_0$ and such that for every point in $\gamma([\epsilon', t_0])$, there is an immersed flat disk of radius $\epsilon$ centered at it (see Figure 3.3 for a sketch).

Let $t_\epsilon = \inf\{t > 0 : \text{there is an immersed disk centered at } \gamma(t) \text{ of radius } \epsilon\}$. By definition, we have $t_\epsilon \leq \epsilon'$ but also $t_\epsilon \geq \epsilon > 0$. Indeed, the disk $B(\gamma(t), \epsilon)$ can not be locally flat since then the corresponding point $[\gamma]$ in the interval $c$ would have distance at least $\frac{\pi}{4}$ from $a$ and from $b$.

Now the boundary of the disk $B(\gamma(t_\epsilon), \epsilon)$ has to contain a singularity. As the set of singularities of $X$ is discrete, for $\epsilon$ and $\epsilon'$ small enough it is the singularity $\sigma$ which must

![Figure 3.1: The set $i_{\ln \epsilon}((-\infty, \ln \epsilon) \times (-\pi, \pi))$ with $\gamma$ and the shaded region $B(\gamma(\frac{\pi}{4}), \frac{\epsilon}{4})$.](image)
be contained in the boundary of the disk $B(\gamma(t_0), \epsilon)$. This determines a geodesic ray $\gamma_1$ starting on the boundary of $B(\gamma(t_0), \epsilon)$, going in the same direction as $[\gamma]$ and defined at least for time $(0, t_0 - \epsilon')$. Therefore the distance between $[\gamma]$ and $[\gamma_1]$ in $L^{t_0 - \epsilon'}(\sigma)$ is no more than $\epsilon + \epsilon'$. Repeat this construction with $\epsilon'$ and $\epsilon$ small enough so that $[\gamma_1]$ is not contained in the $(\epsilon + \epsilon')$-neighborhood of $[\gamma]$ in $L^{t_0 - \epsilon'}(\sigma)$ anymore. We get a different linear approach $[\gamma_2]$ closer to $[\gamma]$. By repeating this construction, we iteratively obtain infinitely many different linear approaches $([\gamma_n])_n$ which all have the same direction. Only finitely many of them can be contained in $c$ as $c$ is of finite length. Thus there is a number $N \in \mathbb{N}$ for which $[\gamma_n]$ is not contained in $c$ for $n > N$, and $([\gamma_n])_{n>N}$ converges to $[\gamma]$ in $c$. Therefore $\{c\}$ is not open in $\tilde{L}(\sigma)$.

4 What kind of topologies are possible for $\tilde{L}(\sigma)$?

This section is devoted to the proof of Theorem 4.1 asserting that any finite topological space is the space of rotational components of a translation surface.

So far, we have already seen in Examples 2.1, 2.2, 2.3 and 2.7 that $\tilde{L}(\sigma)$ can be any topological space of cardinality 1 or 2. However, the space of rotational components does not have to be finite. For example, the singularity of the star decoration in Example 2.6 has a space of rotational components which is isometric to $S^1$. When $\tilde{L}(\sigma)$ is infinite, it can be either Hausdorff (cf. Example 2.6 (ii)) or not (cf. Example 2.4 and Example 2.6 (i)).

Theorem 4.1 (Every finite space occurs as $\tilde{L}(\sigma)$)

Every topological space of finite cardinality can be obtained as a space $\tilde{L}(\sigma)$ of rotational components of a wild singularity $\sigma$ on a compact translation surface $(X, \omega)$.

Proof. Let $n$ be a positive integer. Topologies on a finite set with $n$ elements are in one-to-one correspondence with preorders “$\leq$” defined by $x \leq y$ if and only if $x$ is in the closure of $y$.

Now we fix a topology on the set $\{X_1, \ldots, X_n\}$. To construct the required translation surface, we proceed as follows. First we consider $n$ copies of the shrinking star decoration...
on a torus as in Example 2.7 labeling them with 1, 2, . . . n. The i-th copy of the shrinking star decoration will correspond to $X_i$. For each of the copies, fix an infinite sequence of disjoint disks $D_n$ centered on the vertical ray that converges to the origin as shown in Figure 4.1.

For now, every star decoration labeled $i$ has one wild singularity $\sigma_i$ with one rotational component $c_i$ which is isometric to $[0, \infty)$ (see Example 2.7). We will glue the copies in such a way that all singularities $\sigma_i$ are identified to one singularity $\sigma$ and $\bar{L}(\sigma)$ is homeomorphic to $\{X_1, \ldots, X_n\}$ via $c_i \mapsto X_i$.

If $X_i \leq X_j$ and $i \neq j$, i.e. if every open set containing $X_i$ contains $X_j$, define

$$a_{k}^{i,j} = n^{2}k + in + j \text{ for all } k \in \mathbb{N}.$$ 

Recall that there is a rectangle glued to the two sides of any branch of the shrinking star. We replace the rectangle on the $a_{k}^{i,j}$-th branch of the $j$-th copy of the shrinking star by a cross as in Figure 4.2. Then we make a slit inside the $a_{k}^{i,j}$-th disk on the $i$-th copy in the direction for which we can glue the dashed edges of the cross to this slit. This construction extends the length of the rotational component $c_j$ but not the one of $c_i$.

To make sure that all the $\sigma_i$ are identified to a single singularity, construct a sequence of smaller and smaller tori, and make $n$ disjoint slits on each of them, replace the rectangle on the $kn^2 + in + i$-th slit of the $i$-th copy of the shrinking star with a cross and glue the two dashed sides with two sides of one of the slits on the $k$-th torus.

The surface obtained by this construction is of finite area, but not compact. To make it compact, replace the legs of the crosses of small side of length $2^{-n}$ with $2^{n-1}$ double

\[\gamma(t_{0}) - 2\epsilon' \quad \gamma(t_{0} - \epsilon') \quad \gamma(t_{0}) \quad \gamma(t_{0} - \epsilon') \quad \gamma' \quad \gamma(t_{\epsilon})\]
Figure 4.1: The shrinking star decoration with additional disks $D_n$. 
squares with cross-gluing, as in Figure 2.7.

Finally, we have to check that $\tilde{L}(\sigma) = \{c_1, \ldots, c_n\}$, i.e. that the gluing of the thin crosses and small tori did not create new rotational components. This is clear because the union of the branches of all the copies of the shrinking star is a good cellulation. Now by construction the topology on $\{c_1, \ldots, c_n\}$ is the same as the topology given by “≤” on $\{X_1, \ldots, X_n\}_\#$.

All in all, $\tilde{L}(\sigma)$ is homeomorphic to $\{X_1, \ldots, X_n\}$ as was to be shown.

To emphasize that we can realize topological spaces of different kinds as spaces of rotational components, we use the notion of dimension (see [Eng78] for definitions).

**Corollary 4.1 (Every Lebesgue covering dimension can be realized).** For every number $n$ we constructed a compact translation surface with one wild singularity $\sigma$ so that $\tilde{L}(\sigma)$ has Lebesgue covering dimension $n$. However, the small inductive dimension of the considered examples is always 1.

**Proof.** Take a set of points $p_1, \ldots, p_{n+1}, p$ with the topology characterized in the following way: a set is open if and only if it contains $p$. We can realize this as a space $\tilde{L}(\sigma)$.

By taking the open sets $\{p_1, p\}, \ldots, \{p_{n+1}, p\}$ we see that there is no refinement and $p$ is contained in $n + 1$ sets. So the Lebesgue covering dimension of $\tilde{L}(\sigma)$ is $n$. \[\Box\]
5 Results about the topology of $\mathcal{L}(\sigma)$

It is unsurprising that we lose some information about the space of linear approaches $\mathcal{L}(\sigma)$ when we consider the quotient $\tilde{\mathcal{L}}(\sigma)$. A natural question to raise is: how much information? In Theorem 5.1 we show that there are uncountably many possible translation surfaces with non-homeomorphic $\mathcal{L}(\sigma)$ but with homeomorphic $\tilde{\mathcal{L}}(\sigma)$.

**Theorem 5.1 ($\mathcal{L}(\sigma)$ does not determine $\mathcal{L}(\sigma)$)**

There are uncountably many compact translation surfaces $(M_r, \omega_r)$ with exactly one singularity $\sigma_r$, with non-homeomorphic space of linear approaches $\mathcal{L}(\sigma_r)$, but with the same space of rotational components $\tilde{\mathcal{L}}(\sigma_r)$.

**Proof.** Recall the stack of boxes from Example 2.2 for instance with $h_n = 1$ and $w_n = \frac{1}{2^n}$. We will modify this example by gluing in additional cylinders (actually rectangles in order to get compact surfaces). We first start by cutting vertical slits of length $\frac{1}{2^n}$ starting at each $A_n$ (see Figure 5.1). Let $\mathbb{K} = \{0, 1\}^\mathbb{N}$. Consider the equivalence relation $Q$ on $\mathbb{K}$ defined by $Q(r, r')$ if and only if some shifts of $r$ and $r'$ are equal, i.e. the tails of $r$ and $r'$ agree up to shift.

Let $r = (r_n) \in \mathbb{K}$. For each $n$ such that $r_n = 1$, we glue the two sides of the slit starting at $A_n$ to the two vertical sides of a rectangle of width 1, then we identify the top and bottom sides of the rectangle. We call the resulting translation surface $(M_r, \omega_r)$. It still has one singularity $\sigma_r$ as in Example 2.2 and one rotational component isometric to $(0, \infty)$. Hence the spaces $\tilde{\mathcal{L}}(\sigma_r)$ are homeomorphic for all $r$. Now we will show that if $r$ and $r'$ are not in the same $Q$-class, then $\mathcal{L}(\sigma_r)$ and $\tilde{\mathcal{L}}(\sigma_r')$ are not homeomorphic. This will prove the statement since there are uncountably many $Q$-classes in $\mathbb{K}$.

We will now elaborate tools to recover the tail of $r$ intrinsically from the topology of $\mathcal{L}(\sigma_r)$ for an $r \in \mathbb{K}$. We can visualize this topology by identifying $\mathcal{L}(\sigma_r)$ with $(0, \infty)$ via the translation structure on the only rotational component of $\tilde{\mathcal{L}}(\sigma_r)$. Then we equip $(0, \infty)$ with the coarsest topology that makes the map $(0, \infty) \to \mathcal{L}(\sigma_r)$ into a homeomorphism. We will use the parameter $x > 0$ to represent linear approaches in $\mathcal{L}(\sigma_r)$.

To record when a linear approach enters a cylinder, we will use three distinguished neighborhoods in $\mathcal{L}(\sigma_r)$:

(i) $U_r = B(\gamma_1(\frac{1}{2}), \rho)^\frac{1}{3}$ where $\gamma_1(t)$ is the ray represented by $x = \frac{t}{3}$,

(ii) $V_r = B(\gamma_2(\frac{1}{2}), \rho)^\frac{1}{2}$ where $\gamma_2(t)$ is the ray represented by $x = \frac{t}{2}$,

(iii) $W_r = B(\gamma_3(\frac{1}{2}), \rho)^\frac{1}{3}$ where $\gamma_3(t)$ is the ray represented by $x = \frac{2t}{3}$

for $\rho$ small enough that the disks centered at $\gamma_i(\frac{1}{2})$ and of radius $\rho$ are all disjoint and do not intersect any of the slits for $i \in \{1, 2, 3\}$ (see Figure 5.1).

Now let $U_r \subset U_r$ be a neighborhood of $x = \frac{n}{2}$ and similarly for $V_r$, $W_r$. Fix some integer $n$, and let $x_n := 2\pi \cdot 2n$ which corresponds to the horizontal linear approach starting at $A_{n+1}$ and going to the right. We see that for $x > x_n$ but close to $x_n$, $x$ is in neither of $U_r, V_r$ or $W_r$. Assuming $x_n$ is big enough, we eventually have $x \in U_r$, when $x$ gets closer to $x_n + \frac{\pi}{2}$. When $x$ increases, it eventually leaves $U_r$ to enter $V_r$ (note that
Figure 5.1: Stack of boxes with additional vertical segments.
there could be multiple reentries and releavings in $U_r$ before $x$ enters $V_r$). Now there are two cases to consider.

(i) If $r_n = 1$, $x$ will enter (from the right side) the cylinder attached to $A_n$, run for time $\pi$ and exit the cylinder by the tip of the slit above $A_n$. After running for about another $\frac{5\pi}{6}$, $x$ will reenter $U_r$. Then $V_r$, then $W_r$, to finally enter the same cylinder from the left side. After running for another $\pi$, $x$ will exit the cylinder, enter $V_r$ and $W_r$ again. When $x$ reaches $x_n + 5\pi$, it will enter the $n$-th box from the upper right corner at $B_n$.

(ii) If $r_n = 0$, $x$ will simply go through $U_r$, $V_r$, $W_r$ and then hit $B_n$ when reaching $x_n + \pi$.

By recording the successive passages in each of the three neighborhoods, writing 1 when seeing $(U_r, V_r, U_r, V_r, W_r, V_r, W_r)$ and 0 when seeing $(U_r, V_r, W_r)$, one records a sequence in $\mathbb{K}$ that eventually agrees with $r$ up to shift.

To finish the proof, let $\varphi$ be a homeomorphism between $L(\sigma_r)$ and $L(\sigma_{r'})$. Up to composition with a homeomorphism from $(0, \infty)$ to itself, we can assume that $\varphi$ sends the linear approaches of $\sigma_r$ corresponding to $x = \frac{\pi}{3}, \pi, \frac{2\pi}{3}$ to the linear approaches of $\sigma_{r'}$ corresponding to $x = \frac{\pi}{3}, \pi, \frac{2\pi}{3}$. By making $U_r$, $V_r$ and $W_r$ smaller if necessary, we can assume that $\varphi(U_r) \subset U_{r'}$ and similarly for $V_r$, $W_r$. Define $U_{r'}$ to be $\varphi(U_r)$ and similarly for $V_{r'}$, $W_{r'}$.

On the one hand, when $x$ goes through $L(\sigma_r) = (0, \infty)$, $\varphi(x)$ will go through $L(\sigma_{r'}) = (0, \infty)$ and will record a sequence $s \in \mathbb{K}$ that eventually agrees with $r$ up to shift. On the other hand, when $x$ going through $L(\sigma_{r'}) = (0, \infty)$, we will record a sequence $s' \in \mathbb{K}$ that eventually agrees with $r'$ up to shift. Since $\varphi$ is a homeomorphism, $s$ and $s'$ have to be equal. Therefore $r$ and $r'$ are in the same $Q$-class, as was to be shown.

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