ROBUSTNESS OF REGULARITY FOR THE 3D CONVECTIVE
BRINKMAN–FORCHHEIMER EQUATIONS

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ABSTRACT. We prove a robustness of regularity result for the 3D convective  
Brinkman–Forchheimer equations
\[ \partial_t u - \mu \Delta u + (u \cdot \nabla)u + \nabla p + \alpha u + \beta |u|^{r-1} u = f, \]
for the range of the absorption exponent \( r \in [1,3] \) (for \( r > 3 \) there exist  
global-in-time regular solutions as shown for example in Hajduk & Robinson [14]),  
i.e. we show that strong solutions of these equations remain strong under  
small enough changes of initial condition and forcing function. We provide a  
smallness condition which is similar to the robustness conditions given for the  
3D incompressible Navier–Stokes equations by Chernyshenko et al. [5] and  
Dashti & Robinson [8].

1. INTRODUCTION

In this paper we consider strong solutions of the 3D incompressible convective  
Brinkman–Forchheimer equations (CBF, see e.g. [21])
\[ \partial_t u - \mu \Delta u + (u \cdot \nabla)u + \nabla p + \alpha u + \beta |u|^{r-1} u = f, \quad \nabla \cdot u = 0, \]
where \( u(t,x) = (u_1,u_2,u_3) \) is the velocity field, the scalar function \( p(t,x) \) is the  
pressure and \( f(t,x) = (f_1,f_2,f_3) \) are given body forces acting within the fluid.  
The constant \( \mu \) denotes the positive Brinkman coefficient (effective viscosity). The  
positive constants \( \alpha \) and \( \beta \) denote respectively the Darcy (permeability of porous  
medium) and Forchheimer (porosity of the material) coefficients. The exponent \( r \)  
can be greater or equal than 1. The equations (1.1) can be seen also as the Navier–Stokes  
equations (NSE) modified by an absorption term \( |u|^{r-1} u \) (e.g. as in [1]) or as the  
tamed Navier–Stokes equations (e.g. as in [20]). Although the motivation of  
adding the absorption term to the NSE is rather of mathematical nature, there are  
some relevant physical justifications and applications of this model (more details  
are given in [14] and references therein).

For simplicity, we often assume that \( \mu, \alpha, \beta = 1 \), but all of these coefficients can  
be taken as arbitrary nonnegative constants. In our arguments we omit the linear  
term \( \alpha u \) in (1.1) since its treatment does not cause any additional mathematical  
difficulties. Furthermore, we also neglect frequently the external forces \( f \equiv 0 \) and
consider the unforced CBF equations

\begin{equation}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p + |u|^{r-1} u = 0.
\end{equation}

It was established in [15] that when the absorption exponent \( r \) is greater than 3 there exist global-in-time regular solutions of the CBF equations on bounded domains with smooth boundary. The same result was proved in [14] for the periodic case (with \( r > 3 \)) and also for the critical exponent (\( r = 3 \)) when the viscosity and porosity coefficients are not too small, i.e. \( 4\mu\beta \geq 1 \).

In this paper we consider the equations \( 1.1 \) on a three-dimensional torus \( \mathbb{T}^3 \) with periodic boundary conditions. In this setting it is often convenient to assume zero mean-value constraint for the functions (i.e. \( \int u(t, x) \, dx = 0 \)). However, we cannot do that for the CBF equations \( 1.1 \) because the absorption term \( |u|^{r-1} u \) does not preserve this property. Therefore, we cannot use the usual Poincaré inequality \( \|u\|_{L^2} \leq c \|\nabla u\|_{L^2} \) and we have to control the full \( H^1 \)-norm instead.

We define the Sobolev spaces \( H^s(\mathbb{T}^3) \) for \( s \geq 0 \) by the Fourier expansion

\[ H^s(\mathbb{T}^3) := \left\{ u \in L^2(\mathbb{T}^3) : u(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k e^{i k \cdot x}, \quad \hat{u}_k = \overline{u}_{-k}, \quad \|u\|_{H^s(\mathbb{T}^3)} < \infty \right\}, \]

where

\[ \|u\|_{H^s(\mathbb{T}^3)}^2 = |\mathbb{T}^3| \sum_{k \in \mathbb{Z}^3} (1 + |k|^2s) |\hat{u}_k|^2 \]

and

\[ \hat{u}_k := |\mathbb{T}^3|^{-1} \int_{\mathbb{T}^3} u(x) e^{-i k \cdot x} \, dx. \]

We will also use the following function spaces:

\[ D_\sigma := \left\{ \varphi \in [C^\infty(\mathbb{T}^3)]^3 : \text{supp } \varphi \text{ is compact, } \nabla \cdot \varphi = 0 \right\}, \]

\[ L^p_\sigma := \text{closure of } D_\sigma \text{ in the } L^p \text{-norm for } p \geq 1, \]

\[ V^s := \text{closure of } D_\sigma \text{ in the } H^s \text{-norm for } s \geq 1. \]

We denote the Hilbert space \( L^2_\sigma \), which is of great importance in the theory of fluid mechanics, by \( H \). This space is endowed with the inner product induced by \( L^2(\mathbb{T}^3) \). We denote it by \( \langle \cdot, \cdot \rangle \) and the corresponding norm is denoted by \( \|\cdot\| \). When we consider the CBF equations with forcing we assume that \( f \) belongs at least to the space \( L^1(0, T; H) \).

The idea of ‘robustness of regularity’ was first introduced by Constantin [6], where it was shown that, under some conditions, regular solutions of the Euler equations are also regular solutions of the Navier–Stokes equations with small viscosity. This idea was further developed by Chernyshenko et al. [5], Dashti & Robinson [8] and by Marin-Rubio et al. [16] (for bounded sets of initial data) purely for the Navier–Stokes equations. It was also successfully applied by Blömker et al. [3] for a 1D surface growth model that has striking similarities to the NSE. Similar ideas of propagation of regularity for the Navier–Stokes equations can be found in the book by Chemin et al. [4]. In the present paper we show that the robustness of regularity of strong solutions holds for the convective Brinkman–Forchheimer equations (with \( r \in [1, 3] \)) as well.

The paper is organised as follows: in Section 2 we introduce some technical tools which will be crucial in dealing with the nonlinear term \( |u|^{r-1} u \). Section 3 is devoted to the local existence of strong solutions which is needed in the proof of
robustness of regularity result. In Section 4 we provide general properties of strong solutions of the CBF equations with the absorption exponent \( r \) in the range \([1, 3]\). We also establish there uniqueness of strong solutions in the larger class of weak solutions satisfying the energy inequality, i.e. a ‘weak-strong uniqueness’ property. In the last sections we prove the main result of the paper (Section 5) and discuss its possible applications to the convective Brinkman–Forchheimer equations (Section 6).

In our estimates we frequently use a constant \( c > 0 \), whose value can change from line to line.

2. Preliminaries

For notational convenience we denote the terms connected with the additional nonlinearity in the convective Brinkman–Forchheimer equations by \( C_r \). For \( r > 0 \) and for all functions \( u, v \in L^{r+1}_\sigma \) we define

\[
C_r(u, v) := P\left(|u|^{r-1} v\right),
\]

where \( P : L^p \to L^p_\sigma \) is the Leray projector in \( L^p \) (see e.g. [12] for details); additionally we define

\[
C_r(u) := C_r(u, u).
\]

We have the following crucial properties of the nonlinearity \( C_r \).

Lemma 2.1. For every \( r \geq 1 \) and for all functions \( u, v \in L^{r+1}_\sigma \) we have a lower bound

\[
\langle C_r(u) - C_r(v), u - v \rangle = \langle |u|^{r-1} u - |v|^{r-1} v, u - v \rangle \geq c \|u - v\|_{r+1},
\]

where \( c \) is a positive constant depending only on \( r \), and \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2 \).

It immediately follows from (2.1) that for \( r \geq 1 \) the nonlinearity \( C_r \) is monotone in the sense that

\[
\langle C_r(u) - C_r(v), u - v \rangle \geq 0
\]

for all \( u, v \in L^{r+1}_\sigma (\mathbb{T}^3) \). One can show (2.2) independently even for \( r > 0 \) by direct computation and using only Young’s inequality.

Lemma 2.1 is a consequence of properties of vectors \(|u|^{r-1} u \in \mathbb{R}^n \) (\( n \geq 1 \)). The proof of the lower bound (2.1) is taken from [9] with some minor changes.

Proof. For all \( u, v \in \mathbb{R}^n \) we observe that

\[
\left(|u|^{r-1} u - |v|^{r-1} v\right) = \int_0^1 \frac{d}{ds}\left(|su + (1-s)v|^{r-1} (su + (1-s)v)\right) \, ds
\]

and hence

\[
\left(|u|^{r-1} u - |v|^{r-1} v\right) \cdot w = \int_0^1 |su + (1-s)v|^{r-1} |w|^2 \, ds
\]

\[
+ (r-1) \int_0^1 |su + (1-s)v|^{r-3} \left(|su + (1-s)v| \cdot w\right)^2 \, ds,
\]

where \( w := u - v \). Therefore, we obtain for \( r \geq 1 \)

\[
\left(|u|^{r-1} u - |v|^{r-1} v\right) \cdot w \geq |w|^2 \int_0^1 |su + (1-s)v|^{r-1} \, ds.
\]
If $|u| \geq |v-u|$, we have

$$|su + (1-s)v| \geq ||u| - (1-s)||w|| \geq s|w|$$

and we can conclude that

$$su + (1-s)v \geq \frac{1}{r}|w|^{r+1}.$$ 

On the other hand, if $|u| < |v-u|$, we have

$$w \cdot \int_0^1 |su + (1-s)v|^{r-1} ds \geq w \cdot \int_0^1 \frac{(su + (1-s)v)^{(r+1)/2}}{(2-s)^2|w|^2} ds \geq \frac{1}{4} \left( \int_0^1 |su + (1-s)v|^2 ds \right)^{(r+1)/2} \geq c|w|^{r+1}.$$ 

Finally, we observe an equality for $u, v \in L^{r+1}_{\sigma}(\mathbb{T}^3)$

$$\langle C_r(u) - C_r(v), w \rangle = \langle \left( |u|^{-1} u - |v|^{-1} v \right), w \rangle = \int_{\mathbb{T}^3} \left( |u|^{-1} u - |v|^{-1} v \right) \cdot w \, dx$$

which ends the proof of the lemma due to monotonicity of integral and the above vector estimates. 

In what follows it will be essential to bound the difference

$$|u|^{-1} u - |v|^{-1} v$$

in terms of only $u$ and $w$, where $w := u - v$.

**Lemma 2.2.** Let $u, v \in \mathbb{R}^n$. Then for $r \geq 1$

$$|u|^{-1} u - |v|^{-1} v \leq (2^{r-2}r) \left( |u|^{-1} |w| + |w|^{-1} \right).$$

**Proof.** First, we consider the following function of one real variable $\varphi : \mathbb{R} \to \mathbb{R}^n$

$$\varphi(\lambda) := |u - \lambda w|^{-1} (u - \lambda w),$$

for $\lambda \in [0, 1]$.

It is easy to see that

$$\varphi(1) - \varphi(0) = - \left( |u|^{-1} u - |v|^{-1} v \right),$$

and that the derivative of $\varphi$ equals

$$\varphi'(\lambda) = -r |u - \lambda w|^{-1} w.$$
By the mean value theorem we estimate the difference \( (2.3) \)
\[
|u|^{r-1} u - |v|^{r-1} v = |\varphi(1) - \varphi(0)| \leq \max_{\lambda \in [0, 1]} |\varphi'(\lambda)|
\]
\[
= \max_{\lambda \in [0, 1]} \left| -r |u - \lambda w|^{r-1} w \right| \leq r |w| \max_{\lambda \in [0, 1]} |u - \lambda w|^{r-1}
\]
\[
\leq r |w| (|u| + |w|)^{r-1} \leq r |w| \left(2^{r-2} (|u|^{r-1} + |w|^{r-1}) \right)
\]
\[
\leq (2^{r-2}r) \left(|u|^{r-1} |w| + |w|^r \right).
\]

We used here the following simple fact
\[
f(x) := \frac{(1 + x)^s}{1 + x^s} \leq f(1) = 2^{s-1}
\]
which holds for all \( s \geq 0 \).

We will also make use of the following lemma, whose proof consists of integration by parts and differentiation of the absolute value function (see [19] for the proof in the periodic case or [2] in the whole space).

**Lemma 2.3.** For every \( r \geq 1 \), if \( u \in H^2(\Omega) \), where \( \Omega \) is either the whole space \( \mathbb{R}^3 \) or the three-dimensional torus \( \mathbb{T}^3 \), then
\[
\int_{\Omega} -\Delta u \cdot |u|^{r-1} u \, dx \geq \int_{\Omega} |\nabla u|^2 |u|^{r-1} \, dx.
\]

Explicitly, the left-hand side of the above equals (integrating by parts)
\[
\int_{\Omega} -\Delta u \cdot |u|^{r-1} u \, dx = \int_{\Omega} |\nabla u|^2 |u|^{r-1} \, dx + \frac{(r-1)}{4} \int_{\Omega} |u|^{r-2} |\nabla |u|^{2}|^2 \, dx.
\]

In particular, by Lemma 2.3 we can write for the absorption term \( |u|^{r-1} u \) (for \( r \geq 1 \) and for a divergence-free function \( u \in V^2 \))
\[
\int_{\Omega} |\nabla u|^2 |u|^{r-1} \, dx \leq \langle Au, C_r(u) \rangle \leq r \int_{\Omega} |\nabla u|^2 |u|^{r-1} \, dx,
\]
where \( A := -\mathbb{P} \Delta \) is the familiar Stokes operator. We recall the well know fact that the operators \( \mathbb{P} \) and \( \Delta \) commute on the domains \( \mathbb{T}^3 \) and \( \mathbb{R}^3 \) but not necessarily on an open, bounded domain \( \Omega \subset \mathbb{R}^3 \) (see e.g. [18] for examples). Therefore, we can freely use \( (2.3) \) throughout this paper.

In the proof of main result of this paper (Theorem 2.4) it will be crucial to control the \( L^6 \)-norm of the gradient of a function \( u \) by the \( L^2 \)-norm of \( Au \).

**Lemma 2.4.** Let \( u \in D(A) \) on the torus \( \mathbb{T}^3 \). Then there exists a constant \( c > 0 \) independent of \( u \) such that
\[
\|\nabla u\|_{L^6(\mathbb{T}^3)} \leq c \|Au\|.
\]

**Proof.** First, we apply the Sobolev embedding \( H^1 \hookrightarrow L^6 \)
\[
\|\nabla u\|_{L^6} \leq c \|\nabla u\|_{H^1} = c \left(\|\nabla u\|^2 + \|D^2 u\|^2 \right)^{1/2}.
\]

We can, either by direct computation or by the Poincaré inequality (noting that \( \nabla u \) has zero mean-value for a periodic function \( u \)), verify that
\[
\|\nabla u\| \leq c \|D^2 u\|.
\]
Therefore, we have the desired bound
\[
\|\nabla u\|_{L^6}^2 \leq c \|D^2 u\|^2 = c \sum_{m,n=1}^{3} \sum_{k \in \mathbb{Z}^3} k_m^2 k_n^2 |\hat{u}_k|^2 = c \sum_{k \in \mathbb{Z}^3} |k|^4 |\hat{u}_k|^2 = c \|Au\|^2.
\]

2.1. Definition of strong solutions for the CBF equations. Strong solutions of the convective Brinkman–Forchheimer equations have similar properties to those of the Navier–Stokes equations. Actually, the additional nonlinear term provides better regularity of solutions. Before explaining this in more detail, let us first define here the strong solutions for the CBF equations in a similar manner to the definition of strong solutions for the NSE (for the treatment of weak solutions for the NSE see e.g. [13] or [18]).

Definition 2.5. We say that the function \(u\) is a weak solution on the time interval \([0, T)\) of the unforced convective Brinkman–Forchheimer equations (1.2) with the initial condition \(u_0 \in H\), if
\[
u \in L^\infty(0, T; H) \cap L^{r+1}(0, T; L^2) \cap L^2(0, T; V^1)
\]
and
\[
- \int_{t_0}^{t_1} \langle u(s), \partial_t \varphi(s) \rangle \, ds + \int_{t_0}^{t_1} \langle \nabla u(s), \nabla \varphi(s) \rangle \, ds + \int_{t_0}^{t_1} \langle (u(s) \cdot \nabla) u(s), \varphi(s) \rangle \, ds
\]
\[
+ \int_{t_0}^{t_1} \langle |u(s)|^{r-1} u(s), \varphi(s) \rangle \, ds = - \langle u(t_1), \varphi(t_1) \rangle + \langle u(t_0), \varphi(t_0) \rangle
\]
(2.5)

for almost all \(t_0 \geq 0\), including 0, for all \(t_1 \in (t_0, T)\) and for all divergence-free (only in space variables) test functions \(\varphi \in D_\sigma([0, T] \times \mathbb{T}^3)\).

A function \(u\) is called a global weak solution if it is a weak solution on \([0, T)\) for every \(T > 0\).

Definition 2.6. A Leray–Hopf weak solution of the convective Brinkman–Forchheimer equations (1.2) with the initial condition \(u_0 \in H\) is a weak solution satisfying the following strong energy inequality:

\[
\|u(t_1)\|^2 + 2 \int_{t_0}^{t_1} \left( \|\nabla u(s)\|^2 + \|u(s)\|_{L^{r+1}}^{r+1} \right) \, ds \leq \|u(t_0)\|^2
\]
(2.6)

for almost all initial times \(t_0 \in [0, T)\), including zero, and all \(t_1 \in (t_0, T)\).

It is known that for every \(u_0 \in H\) there exists at least one global Leray–Hopf weak solution of (1.2) (see [1] for the proof). For the critical absorption exponent \(r = 3\) it is also known that all weak solutions satisfy (2.6) with equality (see [13] for the proof of that fact in the periodic case; the same proof but with the approximating sequence changed accordingly can be repeated also for bounded domains; for the details see the upcoming paper [11] which addresses different simultaneous approximation schemes in Lebesgue and Sobolev spaces on bounded domains).
Definition 2.7. We say that a vector field $u$ is a strong solution of the convective Brinkman–Forchheimer equations \((1.2)\) if, for the initial condition $u_0 \in V^1$, it is a weak solution and additionally it possesses higher regularity, i.e.,

$$u \in L^\infty(0, T; V^1) \cap L^2(0, T; V^2).$$

As mentioned above, strong solutions of the CBF equations are in fact even more regular than those of the NSE. As shown in \([14]\), the extra dissipative term guarantees that every strong solution $u$ belongs additionally to the spaces

$$L^{r+1}(0, T; L^2_{\text{loc}}) \quad \text{and} \quad L^{r+1}(0, T; \mathcal{N}^{2/(r+1), r+1}),$$

where $\mathcal{N}^{s-p} = B^{s,p}_\infty$ is a Nikol’skii space.

3. Local existence of strong solutions

In this section we prove local-in-time existence of strong solutions for the convective Brinkman–Forchheimer equations with $r \geq 1$.

Theorem 3.1. For every initial condition $u_0 \in V^1$ there exists a time $T > 0$ such that a Leray–Hopf weak solution $u$ starting from $u_0$ is a strong solution of the convective Brinkman–Forchheimer equations \((1.2)\) on the time interval $[0, T]$. Additionally $u$ satisfies the bound

$$\int_0^T \left( \int_{\mathbb{T}^3} |\nabla u(t)|^2 |u(t)|^{r-1} \, dx \right) \, dt < \infty. \quad (3.1)$$

Below we only show formal calculations, which can be made rigorous, for example, by the use of Galerkin approximations and passage to the limit.

Proof. We work with the equations \((1.2)\) in their functional form

$$\partial_t u + Au + B(u) + C_r(u) = 0,$$

where $B(u, v) := P(u \cdot \nabla)v$ and $B(u) := B(u, u)$ for $u, v \in V^1$. Let $u$ be a global Leray–Hopf weak solution of \((1.2)\) starting from $u_0 \in V^1$. Multiplying formally \((1.2)\) by $Au$ and integrating over the spatial domain, we obtain

$$\frac{1}{2} \frac{d}{dt} ||\nabla u||^2 + ||Au||^2 + \langle C_r(u), Au \rangle \leq |\langle B(u), Au \rangle| \quad (3.2)$$

First, we estimate the convective term $\langle B(u), Au \rangle$ using Hölder’s, Sobolev’s and Young’s inequalities and also Lemma \((2.4)\)

$$|\langle B(u), Au \rangle| \leq \int_{\mathbb{T}^3} |u| |\nabla u| |Au| \, dx \leq ||u||_{L^6} ||\nabla u||_{L^3} ||Au|| \leq c ||u||_{H^1} ||\nabla u||^{1/2}_{L^6} ||\nabla u||^{1/2}_{L^6} ||Au|| \leq c \frac{3}{2} ||u||^{3/2}_{H^1} ||Au||^{3/2} \leq c ||u||^{6/2}_{H^1} + \frac{1}{2} ||Au||^2. \quad (3.3)$$

We recall that, by Lemma \((2.8)\) we have the inequality \((2.8)\)

$$\langle C_r(u), Au \rangle \geq \int_{\mathbb{T}^3} |\nabla u|^2 |u|^{r-1} \, dx \geq 0 \quad \text{for} \quad r \geq 1.$$

Using this fact and also the estimate \((3.3)\), we obtain from \((3.2)\) a differential inequality

$$\frac{1}{2} \frac{d}{dt} ||\nabla u||^2 + \frac{1}{2} ||Au||^2 + \int_{\mathbb{T}^3} |\nabla u|^2 |u|^{r-1} \, dx \leq c ||u||^{6/2}_{H^1}.$$
Noting that \( u \) satisfies also [see energy inequality (2.6)]

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \| \nabla u \|^2 + \| u \|_{L^r}^{r+1} \leq 0,
\]

we have (adding the above inequalities and dropping some terms on the left-hand side)

\[
\frac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{1}{2} \| Au \|^2 + \int_{T^3} |\nabla u|^2 |u|^{-1} \, dx \leq c \|u\|^6_{H^1}.
\]

By setting \( X(t) := \|u(t)\|^2_{H^1} \), we rewrite the above in the form

\[
X' + \|Au\|^2 + 2 \int_{T^3} |\nabla u|^2 |u|^{-1} \, dx \leq cX^3,
\]

from which we obtain a differential problem

\[
\begin{cases}
    X' \leq cX^3, \\
    X(0) = \|u_0\|^2_{H^1}.
\end{cases}
\]

We would obtain the same differential inequality by following the above procedure for the Navier–Stokes equations (details for the NSE case can be found for example in [18]). We can conclude that \( X \) is no greater than the solution of (3.5) turned into a differential equation instead of differential inequality. The solution of this ODE blows up in finite time \( \tilde{T} = \left( \frac{4}{c} \|u_0\|^4_{H^1} \right)^{-1} \).

Therefore, for \( 0 \leq t \leq \tilde{T}/2 \)

\[
\|u(t)\|^2_{H^1} = X(t) \leq \frac{\|u_0\|^2_{H^1}}{\sqrt{1 - 2c \|u_0\|^4_{H^1} t}} \leq c \|u_0\|^2_{H^1}.
\]

Using this bound and integrating (3.4) over the time interval \([0, t]\), we obtain

\[
\|u(t)\|^2_{H^1} + \int_0^t \|Au(s)\|^2 \, ds + 2 \int_0^t \left( \int_{T^3} |\nabla u(s)|^2 |u(s)|^{-1} \, dx \right) \, ds < \infty.
\]

Hence, we can conclude the proof with \( T := \left( 4c \|u_0\|^4_{H^1} \right)^{-1} \). \( \square \)

Theorem 3.1 tells us that the time of existence of strong solutions of the unforced CBF equations (1.2) can be bounded below in terms of the initial condition

\[
T \geq \|u_0\|^{-4}_{H^1}.
\]

We recall that we have the same situation for strong solutions of the Navier–Stokes equations; however, for the CBF equations we get the additional bound (3.1).

4. Uniqueness of strong solutions

In this section we prove uniqueness of strong solutions of the convective Brinkman–Forchheimer equations for incompressible fluids. By *uniqueness* we mean here uniqueness of strong solutions in the larger class of weak solutions satisfying the energy inequality, which is often called ‘weak-strong uniqueness’. Classical uniqueness of strong solutions follows from that result since every strong solution is by definition a ‘more regular’ weak solution.

To achieve our goal we need to establish some properties of strong solutions of the CBF equations. We follow here proofs of analogous results for the 3D NSE equations, which can be found in many places, e.g. in [13] or [18].
First, we show that due to Definition 2.7 all the terms in the CBF equations are well-defined \( L^2 \) functions in the space-time domain.

**Lemma 4.1.** Let \( u \) be a strong solution of the convective Brinkman–Forchheimer equations with \( r \in [1, 3] \). Then

\[
\partial_t u, \quad \Delta u, \quad (u \cdot \nabla) u \quad \text{and} \quad |u|^{r-1} u
\]

are all elements of \( L^2(0, T; L^2) \).

**Proof.** We only need to consider the absorption term since the other terms can be dealt with in a similar way as in the analogous result for the Navier–Stokes equations. We show that \(|u|^{r-1} u\) is square integrable in the space-time domain.

We have that \( u \in L^{2r}(0, T; L^{2r}) \) using nesting of \( L^p(T^3) \) spaces and the Sobolev embedding

\[
\int_0^T \|u(t)|^{r-1} u(t)\|^2 \, dt \leq \int_0^T \|u(t)\|^{2r}_{L^{2r}} \, dt \leq c \int_0^T \|u(t)\|^{2r}_{H^r} \, dt \leq c \|u\|^{2r}_{L^{\infty}(0, T; H^r)} < \infty.
\]

(4.1)

[Note that we can extend Lemma 4.1 up to \( r \leq 5 \). Indeed, by interpolation (\( r \geq 3 \)) and Agmon’s inequality in 3D we have

\[
\|u\|^{2r}_{L^{2r}} \leq \|u\|_6^6 \|u\|^{2r-6}_{L^\infty} \leq \|u\|_6^6 \|u\|_{H^1}^{r-3} \|u\|_{H^2}^{r-3} \leq c \|u\|_{H^3}^{r+3} \|u\|_{H^2}^{r-3}
\]

and therefore we obtain

\[
\int_0^T \|u(t)\|_{L^{2r}}^{2r} \, dt \leq c \|u\|_{L^\infty(0, T; H^3)}^{r+3} \left( \int_0^T \|u(t)\|_{H^2}^2 \, dt \right)^{(r-3)/2},
\]

which is bounded for any strong solution \( u \).]

The next result states that for almost all times the Leray projection of the unforced CBF equations (1.2) is equal to zero.

**Lemma 4.2.** Let \( u \) be a strong solution of the convective Brinkman–Forchheimer equations with \( r \in [1, 3] \). Then

\[
\int_0^T \left\langle \partial_t u - \Delta u + (u \cdot \nabla) u + |u|^{r-1} u, w \right\rangle \, dt = 0
\]

(4.2)

for all \( w \in L^2(0, T; H) \).

Again, the proof follows the same lines as in the Navier–Stokes case. We omit it here completely since, due to Lemma 4.1, there are no additional problems caused by the absorption term \( |u|^{r-1} u \).

The last property which we will need to prove the main result of this section states that a strong solution of the CBF equations (actually any function with the same regularity as a strong solution) can be used as a test function in the weak formulation.

**Lemma 4.3.** Suppose that \( v \) is a weak solution of the convective Brinkman–Forchheimer equations with \( r \in [1, 3] \). If \( u \) has the regularity of a strong solution of the CBF equations, that is

\[
u \in L^2(0, T; H^2 \cap V^1) \cap L^{r+1}(0, T; L^{r+1}) \quad \& \quad \partial_t u \in L^2(0, T; L^2),
\]


then for all times \( t \in [0, T] \)
\[
- \int_0^t \langle v, \partial_t u \rangle \, ds + \int_0^t \langle \nabla v, \nabla u \rangle \, ds + \int_0^t \langle (v \cdot \nabla) v, u \rangle \, ds \\
+ \int_0^t \langle |v|^{r-1} v, u \rangle \, ds = \langle v(0), u(0) \rangle - \langle v(t), u(t) \rangle.
\]

In the proof of Lemma 4.3 we need to approximate the function \( u \) simultaneously in the Sobolev space \( H^2 \) and in the Lebesgue space \( L^{r+1} \). We need an approximation which not only converges in those spaces but which is also bounded in both of them. The natural truncation of the Fourier series
\[
u_n := \sum_{|k| \leq n} \hat{u}_k e^{ik \cdot x},
\]
behaves well in the \( L^2 \)-based spaces:
\[
\|u_n - u\|_X \to 0 \quad \text{and} \quad \|u_n\|_X \leq \|u\|_X
\]
for \( X = L^2(\mathbb{T}^3) \) or \( H^s(\mathbb{T}^3) \). However, the same does not hold in \( L^p(\mathbb{T}^3) \) for \( p \neq 2 \). There is no constant \( c_p \) such that
\[
\|u_n\|_{L^p} \leq c_p \|u\|_{L^p} \quad \text{for every} \quad u \in L^p(\mathbb{T}^3).
\]
This follows from the result of Fefferman [10] concerning the ball multiplier for the Fourier transform.

In the periodic setting we can overcome this problem by considering truncations over ‘cubes’ (\( |k_j| \leq n \)) rather than ‘balls’ (\( |k| \leq n \)) of the Fourier modes. If we define
\[
u[n] := \sum_{k \in Q_n} \hat{u}_k e^{ik \cdot x},
\]
where \( Q_n := [-n, n]^3 \cap \mathbb{Z}^3 \), then it follows from good behaviour of the truncation in the one-dimensional case that
\[
\|u[n] - u\|_{L^p} \to 0 \quad \text{and} \quad \|u[n]\|_{L^p} \leq c_p \|u\|_{L^p}
\]
(see e.g. [17] for more details). This kind of approximation was used in [14] to show that all weak solutions of the CBF equations with the absorption exponent \( r = 3 \) satisfy the energy equality in the periodic domain. Approximations with similar properties on bounded domains are discussed in [11]. We can now go to the proof of Lemma 4.3.

**Proof.** For each \( t \in [0, T] \) we take
\[
u_n(t, x) := \sum_{k \in Q_n} \hat{u}_k(t) e^{ik \cdot x},
\]
where \( Q_n \) is as defined above. From the preceding discussion we know that the sequence \( u_n \) converges in \( H^2 \) and also in \( L^{r+1}(\mathbb{T}^3) \) with
\[
\|u_n(t)\|_{L^{r+1}} \leq c \|u(t)\|_{L^{r+1}}
\]
for a.e. \( t \in [0, T] \), which is the key ingredient in adapting the proof from the Navier–Stokes case (see e.g. Lemma 6.6 in [18]). Mollifying \( u_n \) in time (see [14] for
details of a similar argument) we obtain a sequence of test functions such that

\begin{align}
(4.3) \quad & u_n \to u \quad \text{in} \quad L^2(0, T; H^2), \\
(4.4) \quad & \partial_t u_n \to \partial_t u \quad \text{in} \quad L^2(0, T; L^2), \\
(4.5) \quad & u_n \to u \quad \text{in} \quad L^{r+1}(0, T; L^{r+1}).
\end{align}

We note that \( u \in L^2(0, T; H^2) \) and \( \partial_t u \in L^2(0, T; L^2) \) implies that \( u \in C([0, T]; H^1) \) (see e.g. Proposition 1.35 in [18]) and hence we also have

\begin{equation}
(4.6) \quad u_n \to u \quad \text{in} \quad C([0, T]; H^1).
\end{equation}

Since \( v \) is a weak solution of the CBF equations we have

\begin{equation}
(4.7) \quad - \int_0^t \langle v, \partial_t u_n \rangle \, ds + \int_0^t \langle \nabla v, \nabla u_n \rangle \, ds + \int_0^t \langle (v \cdot \nabla) v, u_n \rangle \, ds \\
+ \int_0^t \langle |v|^{r-1} v, u_n \rangle \, ds = \langle v(0), u_n(0) \rangle - \langle v(t), u_n(t) \rangle
\end{equation}

for all \( t \in [0, T] \). To prove the lemma, it is sufficient to pass to the limit in (4.7).

Passing to the limit in the Navier–Stokes terms is standard and follows from (4.3), (4.4) and (4.6). Therefore, we can focus on the Brinkman–Förchheimer nonlinearity; we note that by standard estimates and (4.4) we have

\begin{align}
\left\| \int_0^t \langle |v|^{r-1} v, u - u_n \rangle \, ds \right\| \leq & \int_0^t \int_T |v|^r |u - u_n| \, ds \\
\leq & \int_0^t \|v\|_{L^{r+1}} \|u - u_n\|_{L^{r+1}} \, ds \\
\to & \|v\|_{L^{r+1}(0, T; L^{r+1})} \|u - u_n\|_{L^{r+1}(0, T; L^{r+1})} \to 0
\end{align}

as \( n \to \infty \), which ends the proof. \( \square \)

Finally, we can prove the main result of this section; we show that strong solutions are unique in the class of weak solutions satisfying the Energy Inequality (all Leray–Hopf weak solutions, not necessarily constructed via Galerkin approximation method). In the critical case when \( r = 3 \) (cubic nonlinearity \( |u|^2 v \)), since all weak solutions satisfy the Energy Equality (as shown in [14] on a periodic domain), this means that strong solutions are unique in the class of all weak solutions.

**Theorem 4.4 (Weak-strong uniqueness).** Suppose that \( u \) is a strong solution of the convective Brinkman–Förchheimer equations with \( r \in [1, 3] \) on the time interval \([0, T]\), and that \( v \) is any weak solution on \([0, T]\) arising from the same initial condition \( v(0) = u(0) \in V^1 \), that satisfies the Energy Inequality

\begin{align}
\frac{1}{2} \|v(t)\|^2 + \int_0^t \|\nabla v(s)\|^2 \, ds + \int_0^t \|v(s)\|^r \, ds \leq \frac{1}{2} \|v(0)\|^2
\end{align}

for \( t \in [0, T] \). Then \( u \equiv v \) on \([0, T]\).

**Proof.** From Lemmas 4.2 and 4.3 we have for all \( t \in [0, T] \)

\begin{align}
0 = & \int_0^t \langle \partial_t u_n, v \rangle \, ds + \int_0^t \langle \nabla u_n, \nabla v \rangle \, ds + \int_0^t \langle (u \cdot \nabla) u_n, v \rangle \, ds + \int_0^t \langle |u|^{r-1} u_n, v \rangle \, ds \\
- & \int_0^t \langle v, \partial_t u \rangle \, ds + \int_0^t \langle \nabla v, \nabla u \rangle \, ds + \int_0^t \langle (v \cdot \nabla) v, u \rangle \, ds + \int_0^t \langle |v|^{r-1} v, u \rangle \, ds \\
= & \langle v(0), u(0) \rangle - \langle v(t), u(t) \rangle.
\end{align}
Adding the above equations, we obtain

\[ 2 \int_0^t \langle \nabla u, \nabla v \rangle \, ds + \int_0^t \langle (u \cdot \nabla)u, v \rangle \, ds + \int_0^t \langle (v \cdot \nabla)v, u \rangle \, ds + \int_0^t \langle |v|^{r-1}u, v \rangle \, ds + \int_0^t \langle |v|^{r-1}v, u \rangle \, ds = \|u(0)\|_2^2 - \langle v(t), u(t) \rangle. \]

(4.8)

Our goal now is to obtain an integral inequality for the difference of the solutions \( w := v - u \). To this end, we use the following standard indentity

\[ \|a - b\|^2 = \|a\|^2 + \|b\|^2 - 2 \langle a, b \rangle \]
to deal with the linear terms. We get

\[ 2 \langle \nabla u, \nabla v \rangle = \|\nabla u\|^2 + \|\nabla v\|^2 - \|\nabla w\|^2, \]

\[ \langle v(t), u(t) \rangle = \frac{1}{2} \|u(t)\|^2 + \frac{1}{2} \|v(t)\|^2 - \frac{1}{2} \|w(t)\|^2. \]

We use the relation \( v = w + u \), and by standard properties of the convective term, we obtain

\[ \langle (u \cdot \nabla)u, v \rangle + \langle (v \cdot \nabla)v, u \rangle = \langle (w \cdot \nabla)w, u \rangle. \]

We use two different substitutions to deal with the absorption terms

\[ \langle |u|^{r-1}u, v \rangle + \langle |v|^{r-1}v, u \rangle = \langle |u|^{r-1}u, w + u \rangle + \langle |v|^{r-1}v, v - w \rangle \]

\[ = \|u\|^{r+1}_{L^{r+1}} + \|v\|^{r+1}_{L^{r+1}} - \langle |u|^{r-1}u - |v|^{r-1}v, w \rangle. \]

Hence, it follows from (4.8) that

\[ - \int_0^t \|\nabla w\|^2 \, ds + \int_0^t \|\nabla u\|^2 \, ds + \int_0^t \|\nabla v\|^2 \, ds + \int_0^t \langle (w \cdot \nabla)w, u \rangle \, ds \]

\[ + \int_0^t \|u\|^{r+1}_{L^{r+1}} \, ds + \int_0^t \|v\|^{r+1}_{L^{r+1}} \, ds - \int_0^t \langle |u|^{r-1}u - |v|^{r-1}v, w \rangle \, ds \]

\[ = \frac{1}{2} \|u(0)\|^2 + \frac{1}{2} \|v(0)\|^2 + \frac{1}{2} \|w(t)\|^2 - \frac{1}{2} \|u(t)\|^2 - \frac{1}{2} \|v(t)\|^2. \]

Rearranging the terms in the above, we obtain the following equation for the difference \( w \)

\[ (4.9) \quad \frac{1}{2} \|w(t)\|^2 + \int_0^t \|\nabla w\|^2 \, ds - \int_0^t \langle (w \cdot \nabla)w, u \rangle \, ds + I_1 = I_2 + I_3, \]

where

\[ I_1 := \int_0^t \langle |u|^{r-1}u - |v|^{r-1}v, u - v \rangle \, ds \geq 0, \]

\[ I_2 := \frac{1}{2} \|u(t)\|^2 + \int_0^t \|\nabla u\|^2 \, ds + \int_0^t \|u\|^{r+1}_{L^{r+1}} \, ds - \frac{1}{2} \|u(0)\|^2 = 0, \]

\[ I_3 := \frac{1}{2} \|v(t)\|^2 + \int_0^t \|\nabla v\|^2 \, ds + \int_0^t \|v\|^{r+1}_{L^{r+1}} \, ds - \frac{1}{2} \|v(0)\|^2 \leq 0. \]
We employed here the Energy Equality for the strong solution $u$ and the Energy Inequality for the weak solution $v$, and also the monotonicity of the absorption term (Lemma 2.1). Therefore, we can estimate (4.9) in the following way

$$
\frac{1}{2} \| w(t) \|^2 + \int_0^t \| \nabla w \|^2 \, ds \leq \left| \int_0^t ( (w \cdot \nabla) w, u ) \, ds \right| \leq \int_0^t \| u \|_{L^\infty} \| w \| \| \nabla w \| \, ds
$$

$$
\leq \frac{1}{2} \int_0^t \| u \|^2_{L^\infty} \| w \|^2 \, ds + \frac{1}{2} \int_0^t \| \nabla w \|^2 \, ds
$$

$$
\leq c \int_0^t \| u \|^2_{H^2} \| w \|^2 \, ds + \frac{1}{2} \int_0^t \| \nabla w \|^2 \, ds;
$$

we used the 3D embedding $H^2 \hookrightarrow L^\infty$ in the last line. Then, we have

$$
\| w(t) \|^2 + \int_0^t \| \nabla w \|^2 \, ds \leq c \int_0^t \| u \|^2_{H^2} \| w \|^2 \, ds
$$

and consequently

$$
\| w(t) \|^2 \leq c \int_0^t \| u(s) \|^2_{H^2} \| w(s) \|^2 \, ds.
$$

Since $u$ is a strong solution

$$
\int_0^t \| u(s) \|^2_{H^2} \, ds < \infty \quad \text{for all } \quad t \in [0, T],
$$

so application of the integral version of the Gronwall Lemma yields that $w(t) = 0$ for all $t \in [0, T]$. □

As a straightforward corollary of Theorem 4.4, we can deduce a weaker result: uniqueness of strong solutions in the class of strong solutions.

**Corollary 4.5.** Let $u$ and $v$ be two strong solutions of the convective Brinkman–Forchheimer equations (1.2) with $r \geq 0$ on the time interval $[0, T]$, starting from the same initial condition $u_0 \in V^1$. Then $u \equiv v$ for all times $t \leq T$.

This result follows from Theorem 4.4 only for the absorption exponents in the range $r \in [1, 3]$; because strong solutions are by definition weak solutions with additional regularity and they satisfy the Energy Equality, it suffices to apply Theorem 4.4 to the strong solutions $u$ and $v$. However, one can prove Corollary 4.5 independently for all exponents $r \geq 0$, following the proof for the analogous result for the Navier–Stokes equations; the only additional difficulty is in dealing with an extra nonlinear term $C_r(u)$. In this particular case we are able to eliminate the additional nonlinearity from the proof due to its properties. We provide a short sketch of this fact below.

---

1The fact that strong solutions satisfy the Energy Equality is a simple consequence of Lemma 4.2.
Proof. We set \( w := u - v \). Then, of course \( w(0) = 0 \). We subtract weak formulations of the CBF equations for the functions \( u \) and \( v \) and obtain equation for the difference

\[
- \int_0^t \langle w, \partial_t \phi \rangle \, ds + \int_0^t \langle \nabla w, \nabla \phi \rangle \, ds + \int_0^t \langle B(u) - B(v), \phi \rangle \, ds \\
+ \int_0^t \langle C_r(u) - C_r(v), \phi \rangle \, ds = - \langle w(t), \phi(t) \rangle.
\]

Using Lemma 4.3, we take as a test function \( \phi := w \) and get

\[
\frac{1}{2} \| w(t) \|^2 + \int_0^t \| \nabla w \|^2 \, ds + \int_0^t \langle C_r(u) - C_r(v), w \rangle \, ds \\
\leq \left| \int_0^t \langle B(u) - B(v), w \rangle \, ds \right|.
\] (4.10)

First, we deal with the nonlinearities connected with the operators \( C_r \). We note that we can simply drop this term on the left-hand side of (4.10). Indeed, by monotonicity (Lemma 2.1), we have

\[
\langle C_r(u) - C_r(v), w \rangle = \langle |u|^{r-1} u - |v|^{r-1} v, u - v \rangle \geq 0 \quad \text{for} \quad r \geq 0.
\]

Therefore, we can proceed as in the Navier–Stokes case to finish the proof of uniqueness. \( \square \)

5. Robustness of regularity

In this section we deal with the so called ‘robustness of regularity’ result for the solutions of the convective Brinkman–Forchheimer equations on a torus \( \mathbb{T}^3 \). It generalises the result obtained in [8] for the Navier–Stokes equations.

We take \( u_0, v_0 \in V^1 \) and fix \( T > T' > 0 \). Let \( u \) be a strong solution of the CBF equations on the time interval \( [0, T] \) with external forces \( f \) and initial condition \( u_0 \). Similarly, let \( v \) be a strong solution of the CBF equations on \( [0, T'] \) with external forces \( g \) and initial condition \( v_0 \). We will give an explicit condition, depending only on the data and on the function \( u_0 \), which allows us to extend (due to uniqueness) the function \( v \) to a strong solution on the time interval \( [0, T] \).

We consider the following system of equations

\[
\begin{align*}
\partial_t u + Au + B(u) + C_r(u) &= f, & u(0, x) &= u_0, \\
\partial_t v + Av + B(v) + C_r(v) &= g, & v(0, x) &= v_0.
\end{align*}
\]

As before, we denote the difference of solutions by \( w := u - v \). Subtracting the above equations we obtain the equation for \( w \)

\[
\partial_t w + Aw + B(u) - B(v) + C_r(u) - C_r(v) = f - g,
\]

with the initial condition

\( w(0, x) = u_0 - v_0 \).

5.1. Technical ODE lemma. In the proof of the robustness of regularity for the above equations, the following simple ODE lemma will be extremely useful. It will allow us to estimate the time of existence for solutions of certain differential inequalities in terms of coefficients of a corresponding differential equation.
Lemma 5.1. Let $T > 0$, $a > 0$ and $n \in \mathbb{N}$ ($n > 1$). Let $\delta(t)$ be a nonnegative, continuous function on the interval $[0, T]$. Let also $y$ be a nonnegative function, satisfying the following differential inequality

$$\begin{align*}
\dot{y} &\leq ay^n + \delta(t), \\
y(0) &= y_0 \geq 0.
\end{align*}$$

We define the quantity

$$\eta := y_0 + \int_0^T \delta(t) \, dt.$$

If the following condition is satisfied

$$\eta < \frac{1}{[(n-1)\alpha T]^{1/(n-1)},}$$

(1) then $y(t)$ stays bounded on the interval $[0, T]$,

(2) and $y(t) \to 0$ as $\eta \to 0$, uniformly on $[0, T]$.

For the proof of Lemma 5.1 see e.g. [6], [8] or [18].

5.2. A priori estimates. Taking into account that $u$ is a strong solution on the time interval $[0, T]$ and that we want to say the same about $v$, we need to change the form of equation (5.1) to eliminate the unknown function $v$. From the definition we have $v = u - w$, so due to bilinearity of the form $B$, we have the identity

$$B(u) - B(v) = B(u) - B(u - w) = B(u, u) - B(u - w, u - w)$$

$$= B(u, u) - B(u, u) - B(w, w) + B(u, w) + B(w, u)$$

$$= B(u, w) + B(w, u) - B(w, w).$$

Multiplying now both sides of (5.1) by $Aw$ (we assume here that the function $w$ has sufficient regularity to justify these operations) and integrating over the spatial domain, we get

$$\langle \partial_t w, Aw \rangle + \langle Aw, Aw \rangle + \langle B(u) - B(v), Aw \rangle + \langle C_r(u) - C_r(v), Aw \rangle$$

$$= \langle f - g, Aw \rangle.$$

Integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \| \nabla w \|^2 + \| Aw \|^2 \leq |\langle f - g, Aw \rangle| + |\langle B(u, w) + B(w, u) - B(w, w), Aw \rangle|$$

$$+ |\langle C_r(u) - C_r(v), Aw \rangle|.$$

(5.2)

We will now estimate all the terms on the right-hand side of the inequality (5.2). Using standard estimates for the bilinear form $B$ (cf. [7] or [8]) and Lemma 2.4, we can estimate all the terms coming from the Navier–Stokes equations (see also Chapter 9.1 in [18]).

We have

a) $$|\langle f - g, Aw \rangle| \leq |\langle f - g, |Aw| \rangle| \leq \| f - g \| \| Aw \| \leq c \| f - g \|^2 + \frac{1}{16} \| Aw \|^2,$$
b) \[
|\langle B(u, w), Aw \rangle| \leq \langle |w| |\nabla w|, |Aw| \rangle \leq \|u\|_{L^6} \|\nabla w\|_{L^3} \|Aw\| \\
\leq c \|u\|_{H^1} \|\nabla w\|^{1/2} \|\nabla w\|^{1/2} \|Aw\| \leq c \|u\|_{H^1} \|w\|^{1/2} \|Aw\|^{3/2} \\
\leq c \|u\|_{H^1}^2 \|w\|_{H^2}^2 + \frac{1}{16} \|Aw\|^2 ,
\]

c) \[
|\langle B(w, u), Aw \rangle| \leq \langle |w| |\nabla u|, |Aw| \rangle \leq \|w\|_{L^6} \|\nabla u\|_{L^3} \|Aw\| \\
\leq c \|w\|_{H^1} \|\nabla u\|^{1/2} \|\nabla u\|^{1/2} \|Aw\| \leq c \|w\|_{H^1} \|\nabla u\|^{1/2} \|Aw\|^{3/2} \\
\leq c \|w\|_{H^1}^2 \|\nabla u\| \|Aw\| + \frac{1}{16} \|Aw\|^2 ,
\]
d) \[
|\langle -B(w, w), Aw \rangle| \leq \langle |w| |\nabla w|, |Aw| \rangle \leq \|w\|_{L^6} \|\nabla w\|_{L^3} \|Aw\| \\
\leq c \|w\|_{H^1} \|\nabla w\|^{1/2} \|\nabla w\|^{1/2} \|Aw\| \leq c \|w\|_{H^1}^3 \|Aw\|^{3/2} \\
\leq c \|w\|_{H^1}^6 + \frac{1}{16} \|Aw\|^2 .
\]

Summing parts a) to d), we get
\[
|\langle f - g, Aw \rangle| + |\langle B(u, w) + B(w, u) - B(w, w), Aw \rangle| \leq c \|f - g\|^2 \\
+ c \left( \|u\|_{H^1}^4 + \|\nabla u\| \|Aw\| \right) \|w\|_{H^1}^2 + c \|w\|_{H^1}^6 + \frac{1}{4} \|Aw\|^2 .
\]

(5.3)

Note that we obtain the full $H^1$-norm of the difference $w$ on the right-hand side of (5.3) and there is only $L^2$-norm of the gradient on the left-hand side of (5.2). To circumvent that problem we can consider the energy equality for the difference
\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \|\nabla w\|^2 + \langle B(u) - B(v), w \rangle + \langle C_r(u) - C_r(v), w \rangle \\
= \langle f - g, w \rangle .
\]

(5.4)

We note again that $\langle C_r(u) - C_r(v), w \rangle \geq 0$. Substituting in (5.3) $v = u - w$, we get
\[
\langle B(u) - B(v), w \rangle = \langle B(u, u) - B(u - w, u - w), w \rangle = \langle B(u, u), w \rangle - \langle B(u, u), w \rangle \\
+ \langle B(u, w), w \rangle + \langle B(w, u), w \rangle - \langle B(w, w), w \rangle \\
= \langle B(w, u), w \rangle .
\]

We used here the fact that
\[
(u \cdot \nabla)v, w = -(u \cdot \nabla)w, v
\]
for $u \in V^1$ and $v, w \in H^1$.

Therefore, we obtain from the energy equality (5.4)
\[
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \|\nabla w\|^2 \leq \langle f - g, w \rangle + |\langle B(u, u), w \rangle| .
\]

Estimating the nonlinear term gives
\[
|\langle B(u, u), w \rangle| \leq \langle |w| |\nabla u|, |w| \rangle \leq \|w\|_{L^6} \|\nabla u\| \leq \|w\|^{1/2} \|w\|_{L^6}^{3/2} \|\nabla u\| \\
\leq c \|w\|_{H^1}^2 \|\nabla u\| ,
\]

for $u \in V^1$ and $w \in H^1$. 

from which we conclude
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|w\|^2 + \|\nabla w\|^2 \leq c \|f - g\|^2 + c \|w\|_{H^1}^2 (\|\nabla u\| + 1).
\end{equation}

To estimate the additional nonlinear terms in the right-hand side of \((5.2)\) connected with the operator \(C_r\) we use Lemma \([2.2]\)
\[ |C_r(u) - C_r(v)| \leq (2^{r-2}r) \left( |u|^{r-1} |w| + |w|^r \right) \quad \text{for } r \geq 1, \]
which gives
\begin{equation}
|\langle C_r(u) - C_r(v), Aw \rangle| \leq (2^{r-2}r) \left( \|u\|_{H^1}^{r-1} \|w\| L_{r/(r-2)} \|Aw\| + \|w\|^r \|Aw\| \right).
\end{equation}

We can estimate the first term in \((5.6)\) using Hölder’s inequality with three exponents \(6/(r-1), 6/(4-r), 2\) and Sobolev’s embedding \(H^1 \hookrightarrow L^6\)
\begin{align*}
\left\langle |u|^{r-1} |w|, |Aw| \right\rangle &\leq \|u\|_{H^1}^{r-1} \|w\|_{L^6/(4-r)} \|Aw\| \\
&\leq c \|u\|_{H^1}^{r-1} \|w\|_{H^1} \|Aw\| \\
&\leq c \|u\|_{H^1}^{2(r-1)} \|w\|^2_{H^1} + \frac{1}{8} \|Aw\|^2.
\end{align*}

Using the same bound for \(L^{2r}\)-norm as in \((4.1)\), we estimate the second term on the right-hand side of \((5.6)\)
\begin{equation}
\langle |w|^r, |Aw| \rangle \leq \|w\|_{L^{2r}}^r \|Aw\| \leq c \|w\|_{L^{2r}}^{2r} + \frac{1}{8} \|Aw\|^2
\end{equation}
\begin{equation}
\leq c \|w\|_{H^1}^{2r} + \frac{1}{8} \|Aw\|^2.
\end{equation}

Combining the inequalities \((5.7)\) and \((5.8)\) yields
\begin{equation}
|\langle C_r(u) - C_r(v), Aw \rangle| \leq c \|u\|_{H^1}^{2(r-1)} \|w\|_{H^1}^2 + c \|w\|_{H^1}^{2r} + \frac{1}{4} \|Aw\|^2.
\end{equation}

We apply \((5.3)\) and \((5.9)\) in \((5.2)\) and obtain
\begin{equation}
\frac{d}{dt} \|\nabla w\|^2 + \|Aw\|^2 \leq c_0 \|f - g\|^2 + c_1 \left( \|u\|_{H^1}^4 + \|\nabla u\| \|Au\| + \|u\|_{H^1}^{2(r-1)} \right) \|w\|^2_{H^1} + c_2 \|w\|_{H^1}^{2r} + c_3 \|w\|_{H^1}^6.
\end{equation}

Finally, we add together \((5.5)\) and \((5.10)\)
\begin{equation}
\frac{d}{dt} \|w\|^2_{H^1} + \|\nabla w\|^2 + \|Aw\|^2 \leq c_0 \|f - g\|^2 + c_2 \|w\|_{H^1}^{2r} + c_3 \|w\|_{H^1}^6,
\end{equation}
\begin{equation}
+ c_1 \left( \|u\|_{H^1}^4 + \|\nabla u\| \|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| + 1 \right) \|w\|^2_{H^1}.
\end{equation}

5.3. **Robustness of regularity.** In this section we prove the following theorem for the convective Brinkman–Forchheimer equations with \(r \in [1, 3]\) on a periodic domain \(\mathbb{T}^3\).

**Theorem 5.2.** Let \(f, g \in L^2(0, T; H)\) and \(u_0, v_0 \in V^1\). Furthermore, let \(u \in L^\infty(0, T; V^1) \cap L^2(0, T; V^2)\) be the strong solution of the convective Brinkman–Forchheimer equations \((1.2)\) on the time interval \([0, T]\), with external forces \(f\) and initial condition \(u_0\). If
\begin{equation}
\|u_0 - v_0\|^2_{H^1} + c_0 \int_0^T \|f(t) - g(t)\|^2 \, dt < R(u),
\end{equation}
then
where
\[ R(u) := c \frac{\exp(-c_2 T)}{\sqrt{T}} \exp \left( -c_1 \int_0^T \left( \|u\|_{H^1}^2 + \|\nabla u\| \|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| \right) \, dt \]
for some positive constants \( c_0, c_1, c_2, c \), then the function \( v \) solving the CBF equations \((1.2)\), with external forces \( g \) and initial condition \( v_0 \), is also a strong solution on the time interval \([0, T]\) and have the same regularity as the function \( u \).

The proof of the above theorem is similar to the proof of an analogous result for the Navier–Stokes equations (see [8] for the details).

**Proof.** Local existence of strong solutions for the CBF equations (Theorem 3.1) implies that there exists \( \tilde{T} > 0 \) such that \( v \in L^\infty(0, T'; V^1) \cap L^2(0, T'; V^2) \) for every \( T' < \tilde{T} \). We denote the maximal time of existence of the strong solution \( u \) by \( \bar{T} \), i.e.
\[ \limsup_{t \to \bar{T}^-} \|\nabla v\| = \infty. \]
This implies also that \( \|\nabla w(t)\| \to \infty \) as \( t \to \bar{T}^- \), where \( w := u - v \). We assume that \( \bar{T} \leq T \), where \( T \) is the time of existence of the strong solution \( u \), and lead to a contradiction.

The difference \( w \) satisfies
\[ \partial_t w + A w + B(u, w) + B(w, u) - B(w, w) + C_r(u) - C_r(v) = f - g \]
on the interval \((0, \bar{T})\), with the initial condition \( w(0, x) = u_0 - v_0 \). We know that \( \partial_t v \in L^2(0, T'; H) \) for every \( T' < \bar{T} \). Furthermore, \( \bar{T} \leq T \), so obviously also \( \partial_t u \in L^2(0, T'; H) \). Then, taking the inner product of \((5.13)\) with \( Aw \) in \( L^2 \) and using our a priori estimate \((5.11)\), we obtain
\[ \frac{d}{dt} \|w\|_{H^1}^2 + \|Aw\|^2 \leq c_0 \|f - g\|^2 + c_r \|w\|_{H^1}^{2r} + c_3 \|w\|_{H^1}^6, \]
\[ + c_1 \|w\|_{H^1}^2 \left( \|u\|_{H^1}^2 + \|\nabla u\| \|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| + 1 \right), \]
for appropriate values of the constants \( c_i, i \in \{0, 1, r, 3\} \).

We set the quantities
- \( X(t) := \|w(t)\|_{H^1}^2 \),
- \( \delta(t) := c_0 \|f(t) - g(t)\|^2 \),
- \( \tilde{\gamma}(t) := c_1 \left( \|u(t)\|_{H^1}^2 + \|\nabla u(t)\| \|Au(t)\| + \|u(t)\|_{H^1}^{2(r-1)} + \|\nabla u(t)\| + 1 \right) \).

Inequality \((5.14)\) gives (omitting \( \|Aw\|^2 \) on the left-hand side)
\[ X' \leq c_3 X^3 + c_r X^r + \tilde{\gamma}(t) X + \delta(t). \]
Using an inequality (valid for \( X \geq 0 \))
\[ X^p \leq X^3 + X \quad \text{for} \quad p \in [1, 3], \]
and changing the constant \( c_3 \), we get
\[ X' \leq c_3 X^3 + \gamma(t) X + \delta(t), \]
where \( \gamma(t) := \tilde{\gamma}(t) + c_r \).

We now take
\[ Y(t) := \exp \left( - \int_0^t \gamma(s) \, ds \right) X(t) \]
and multiply (5.15) by \( \exp \left( -\int_0^t \gamma(s) \, ds \right) \leq 1 \). This yields

\[
Y' \leq c_3 \exp \left( -\int_0^t \gamma(s) \, ds \right) X^3 + \delta(t) \exp \left( -\int_0^t \gamma(s) \, ds \right)
\]

\[
\leq c_3 \left[ \exp \left( \int_0^t \gamma(s) \, ds \right) \right]^2 Y^3 + \delta(t)
\]

\[
\leq c_3 \left[ \exp \left( \int_0^T \gamma(s) \, ds \right) \right]^2 Y^3 + \delta(t).
\]

Hence, we have the differential inequality on the time interval \((0, \tilde{T})\)

\[
Y' \leq K Y^3 + \delta(t),
\]

with the initial condition

\[
Y(0) = \|u_0 - v_0\|_{H^1}^2.
\]

Therefore, by Lemma 5.1 used for \( n = 3 \), the function \( Y(t) \) is uniformly bounded on the time interval \([0, T]\) provided that the following condition is satisfied

\[
Y(0) + \int_0^T \delta(t) \, dt < \frac{1}{(2KT)^{1/2}}.
\]

Substituting all the terms in the above condition, we obtain

\[
\|u_0 - v_0\|_{H^1}^2 + c_0 \int_0^T \|f(t) - g(t)\|^2 \, dt
\]

\[
\leq \frac{\exp \left( -c_1 T \right)}{\sqrt{2c_3 T}} \exp \left( -c_1 \int_0^T \left( \|u\|_{H^1}^4 + \|\nabla u\| \|Au\| + \|u\|_{H^1}^{2(r-1)} + \|\nabla u\| + 1 \right) \, dt \right),
\]

which is (up to a change of constants) the robustness condition (5.12). It follows, if this condition is satisfied, that the function \( X(t) \) is also uniformly bounded

\[
X(t) = Y(t) \exp \left( \int_0^t \gamma(s) \, ds \right) \leq Y(t) \exp \left( \int_0^T \gamma(s) \, ds \right) \leq C(T) < \infty
\]

for all \( t \in [0, \tilde{T}] \). Hence, we finally get that \( \|w(\tilde{T})\|_{H^1} \leq C(T) \) and consequently \( \|\nabla v(\tilde{T})\| < \infty \), which contradicts the maximality of the time \( \tilde{T} \). Therefore, the function \( v(t) \) does not blow up, at least on the time interval \([0, T]\). Additionally, it belongs to the space \( L^\infty(0, T; V^1) \).

Now, directly from the inequality (5.14), it follows that the function \( v(t) \) belongs also to the space \( L^2(0, T; V^2) \), which proves that it is a strong solution on \([0, T]\) and ends the proof of Theorem 5.2. \( \square \)

### 6. Conclusion

Going back to Theorem 5.2, it is natural to ask what kind of condition, if any, is required if we consider ‘robustness of regularity’ with respect to the absorption exponent \( r \). To focus our attention on the dependence on the exponents, let us take \( u_0 \equiv v_0 \in V^1 \) and \( f \equiv g \in L^2(0, T; H) \), in such a way that \( u \) is a strong solution on the time interval \([0, T]\) of the CBF equations with initial condition \( u_0 \) and the
exponent \( s \) (if \( s > 3 \) we know that it is in fact global-in-time strong solution), and let \( v \) be a weak solution of the CBF equations with initial condition \( v_0 \) and the exponent \( r \in [1,3] \), where \( r < s \). We know that \( v \) is also a strong solution on some time interval \([0,T]\). We want to find a condition for exponents \( r \) and \( s \) depending only on the function \( u \), which ensures that \( v \) remains strong at least on the time interval \([0,T]\).

The only new obstacle in the problem described above lays in estimating the difference \( C_s(u) - C_r(v) \). We observe that

\[
\left| u^{r-1} u - v^{r-1} v \right| \leq \left| u^{r-1} u - |u|^{r-1} u \right| + \left| |u|^{r-1} u - |v|^{r-1} v \right| \\
\leq |u|^r \left| |u|^{s-r} - 1 \right| + |u|^{r-1} u - |v|^{r-1} v.
\]

We have already seen how to deal with the second term on the right-hand side of (6.1) [cf. (5.6) and the following lines]. Therefore, using similar arguments to those in the proof of Theorem 5.2 we obtain the robustness condition for the absorption exponents

\[
c_0 \int_0^T \left( \int_{\mathbb{R}^3} |u|^{2r} \left| |u|^{s-r} - 1 \right|^2 \, dx \right)^{1/2} \, dt < R(u),
\]

where \( R(u) \) is the constant defined in Theorem 5.2 and it is finite for the strong solution \( u \). On the other hand, the term on the left-hand side of (6.2) tends to 0 as \( s-r \to 0^+ \) (provided that the integral is bounded). From that we infer, for example, that there exists some number \( s_0 > r \) such that, for all exponents \( s \in (r, s_0) \), the condition (6.2) is satisfied. Hence, a weak solution \( v \) is actually a strong solution on the time interval \([0,T]\) and has the same regularity as \( u \), provided that \( s \) and \( r \) are such that the condition (6.2) holds.

In the works of Chernyshenko et al. [5] and Dashti & Robinson [8] the robustness of regularity for the Navier–Stokes equations was used to construct a numerical algorithm which can verify in a finite time regularity of a given strong solution. The second ingredient required in that construction is convergence of the Galerkin approximations to the strong solution. As we showed in this article, robustness of regularity can be extended to the convective Brinkman–Forchheimer equations with the absorption exponent \( r \in [1,3] \). Using similar methods as presented here to deal with the additional nonlinearity \( |u|^{r-1} u \), it should be possible to prove also for the CBF equations that the Galerkin approximations of a strong solution converge strongly to that solution in appropriate function spaces. Consequently, it should be possible to construct a similar algorithm for numerical verification of regularity for these equations as well.

References

[1] Antontsev, S. N., and de Oliveira, H. B. The Navier–Stokes problem modified by an absorption term. Appl. Anal. 89, 12 (2010), 1805–1825.
[2] Beirão da Veiga, H. Existence and asymptotic behavior for strong solutions of the Navier–Stokes equations in the whole space. In Dynamical problems in continuum physics (Minneapolis, Minn., 1985), vol. 4 of IMA Vol. Math. Appl. Springer, New York, 1987, pp. 79–87.
[3] Blömker, D., Nolde, C., and Robinson, J. C. Rigorous numerical verification of uniqueness and smoothness in a surface growth model. J. Math. Anal. Appl. 429, 1 (2015), 311–325.
[4] Chemin, J.-Y., Desjardins, B., Gallagher, I., and Grenier, E. Mathematical geophysics, vol. 32 of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press,
Oxford University Press, Oxford, 2006. An introduction to rotating fluids and the Navier–Stokes equations.

[5] Chernyshenko, S. I., Constantin, P., Robinson, J. C., and Titi, E. S. A posteriori regularity of the three-dimensional Navier–Stokes equations from numerical computations. J. Math. Phys. 48, 6 (2007), 065204, 1–15.

[6] Constantin, P. Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. Comm. Math. Phys. 104, 2 (1986), 311–326.

[7] Constantin, P., and Foias, C. Navier–Stokes equations. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[8] Dashti, M., and Robinson, J. C. An a posteriori condition on the numerical approximations of the Navier–Stokes equations for the existence of a strong solution. SIAM J. Numer. Anal. 46, 6 (2008), 3136–3150.

[9] DiBenedetto, E. Degenerate parabolic equations. Universitext. Springer-Verlag, New York, 1993.

[10] Fefferman, C. L. The multiplier problem for the ball. Ann. of Math. (2) 94 (1971), 330–336.

[11] Fefferman, C. L., Hajduk, K. W., and Robinson, J. C. Simultaneous approximation in Lebesgue and Sobolev norms via eigenspaces. Preprint (2019).

[12] Fujikawa, D., and Morimoto, H. An $L_p$-theorem of the Helmholtz decomposition of vector fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 24, 3 (1977), 685–700.

[13] Galdi, G. P. An introduction to the Navier–Stokes initial-boundary value problem. In Fundamental directions in mathematical fluid mechanics, Adv. Math. Fluid Mech. Birkhäuser, Basel, 2000, pp. 1–70.

[14] Hajduk, K. W., and Robinson, J. C. Energy equality for the 3D critical convective Brinkman–Forchheimer equations. J. Diff. Eq. 268, 11 (2017), 7141–7161.

[15] Kalantarov, V., and Zelik, S. Smooth attractors for the Brinkman–Forchheimer equations with fast growing nonlinearities. Commun. Pure Appl. Anal. 11, 5 (2012), 2037–2054.

[16] Marín-Rubio, P., Robinson, J. C., and Sadowski, W. Solutions of the 3D Navier–Stokes equations for initial data in $H^{1/2}$: robustness of regularity and numerical verification of regularity for bounded sets of initial data in $H^1$. J. Math. Anal. Appl. 400, 1 (2013), 76–85.

[17] Muscalu, C., and Schlag, W. Classical and multilinear harmonic analysis. Vol. I, vol. 137 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2013.

[18] Robinson, J. C., Rodrigo, J. L., and Sadowski, W. The three-dimensional Navier–Stokes equations, classical theory. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016.

[19] Robinson, J. C., and Sadowski, W. A local smoothness criterion for solutions of the 3D Navier–Stokes equations. Rend. Semin. Mat. Univ. Padova 131 (2014), 159–178.

[20] Röckner, M., and Zhang, X. Tamed 3D Navier–Stokes equation: existence, uniqueness and regularity. Inf. Dimens. Anal. Quantum Probab. Relat. Top. 12, 4 (2009), 525–549.

[21] Zhao, C., and You, Y. Approximation of the incompressible convective Brinkman–Forchheimer equations. J. Evol. Equ. 12, 4 (2012), 767–788.

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