GLOBAL WELL-POSEDNESS TO THE CAUCHY PROBLEM OF
TWO-DIMENSIONAL DENSITY-DEPENDENT BOUSSINESQ
EQUATIONS WITH LARGE INITIAL DATA AND VACUUM

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Abstract. This paper concerns the Cauchy problem of the two-dimensional
density-dependent Boussinesq equations on the whole space \( \mathbb{R}^2 \) with zero den-
sity at infinity. We prove that there exists a unique global strong solution
provided the initial density and the initial temperature decay not too slow
at infinity. In particular, the initial data can be arbitrarily large and the
initial density may contain vacuum states and even have compact support.
Moreover, there is no need to require any Cho-Choe-Kim type compatibility
conditions. Our proof relies on the delicate weighted estimates and a lemma
due to Coifman-Lions-Meyer-Semmes [J. Math. Pures Appl., 72 (1993), pp.
247–286].

1. Introduction. Consider the two-dimensional (2D for short) density-dependent
incompressible Boussinesq equations (see [17]):

\[
\begin{align*}
\rho_t + \nabla (\rho u) &= 0, \\
(\rho u)_t + \nabla (\rho u \otimes u) - \mu \Delta u + \nabla P &= \rho \theta e_2, \\
\theta_t + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div } u &= 0,
\end{align*}
\]

(1)

where \( t \geq 0 \) is time, \( x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2 \) is the spatial coordinate, and \( \rho = \rho(x, t), \ u = (u^1, u^2)(x, t), \ \theta = \theta(x, t), \ \text{and } P = P(x, t) \) denote the density, velocity,
temperature, and pressure of the fluid, respectively; \( \mu \) and \( \kappa \) are both positive
constants. \( e_2 = (0, 1)^T \), where \( T \) is the transpose.

Let \( \Omega = \mathbb{R}^2 \) and we consider the Cauchy problem for (1) with \( (\rho, u, \theta) \) vanishing
at infinity (in some weak sense) and the initial conditions:

\[
\rho(x, 0) = \rho_0(x), \ \rho u(x, 0) = \rho_0 u_0(x), \ \theta(x, 0) = \theta_0(x), \ x \in \mathbb{R}^2,
\]

(2)

for given initial data \( \rho_0, u_0 \) and \( \theta_0 \).

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The Boussinesq system describes the motion of lighter or denser incompressible fluid under the influence of gravitational forces, and has important roles in the atmospheric sciences [20], as well as a model in many geophysical applications [22]. In recent years, the two-dimensional Boussinesq equations (1) with \( \rho = 1 \) have attracted significant attention. When \( \Omega = \mathbb{R}^2 \), Cannon and DiBenedetto [3] studied the Cauchy problem for the Boussinesq equations with full viscosity
\[
\begin{align*}
\dot{u} + u \cdot \nabla u - \mu \Delta u + \nabla P &= \theta e_2, \\
\theta_t + u \cdot \nabla \theta - \kappa \Delta \theta &= 0, \\
\text{div} \, u &= 0,
\end{align*}
\]
(3)
which describe the flow of a viscous incompressible fluid subject to convective heat transfer, where \( \mu > 0 \), \( \kappa > 0 \) are constants. They found a unique, global in time, weak solution. Furthermore, they improved the regularity of the solution when initial data is smooth. Recently, the result of global existence of smooth solutions to (3) has been generalized to cases of “partial viscosity” (that is, either \( \mu > 0 \) and \( \kappa = 0 \), or \( \mu = 0 \) and \( \kappa > 0 \)) by Hou and Li [12] and Chae [5] independently. In [12], Hou and Li proved the global well-posedness of the Cauchy problem of viscous Boussinesq equations. They showed that solutions with initial data in \( H^m(m \geq 3) \) do not develop finite-time singularities. In [5], Chae considered the Boussinesq system for incompressible fluid in \( \mathbb{R}^2 \) with either zero diffusion (\( \kappa = 0 \)) or zero viscosity (\( \mu = 0 \)). He proved global-in-time regularity in both cases. For more information, we refer the readers to [4, 15] for studies in this direction. On the other hand, the two-dimensional initial-boundary value problems of (3) have been analyzed to great extent, please see [13, 14] and references therein.

Recently, the density-dependent viscous Boussinesq equations have attracted much attention. The authors [10, 26] studied regularity criteria for three-dimensional incompressible density-dependent Boussinesq equations. Qiu and Yao [23] showed the local existence and uniqueness of strong solutions of multi-dimensional incompressible density-dependent Boussinesq equations with \( \kappa = 0 \) in Besov spaces. A blow-up criterion was also shown in [23]. We should point out here the above results always require the initial density is bounded away from zero. By contrast, however, there are few results concerning strong (or classical) solvability of the density-dependent Boussinesq equations with initial data permitting vacuum. Recently, Zhong [27] proved the local existence of strong solutions to the Cauchy problem of the system (1) with \( \kappa = 0 \) and nonnegative density. Naturally, for a partial differential system, one key consideration is to derive the local and global existence of solutions. It should be noticed that, for two-dimensional Cauchy problems, when the far field density is vacuum, it seems difficult to bound the \( L^p \)-norm of \( u \) by \( \| \sqrt{\rho} u \|_{L^2} \) and \( \| \nabla u \|_{L^2} \) for any \( p \geq 1 \), hence the question of the global-in-time existence of solutions to the two-dimensional Cauchy problem of Boussinesq equations, such as (1), is much subtle and remains open. In fact, this is the main aim of this paper. More precisely, we are going to establish the global well-posedness of strong solutions to the Boussinesq system (1) in \( \mathbb{R}^2 \), which will generalize the study of [3] to the case of variable density.

Now, we wish to define precisely what we mean by strong solutions.

**Definition 1.1.** If all derivatives involved in (1) for \( (\rho, u, P, \theta) \) are regular distributions, and equations (1) hold almost everywhere in \( \mathbb{R}^2 \times (0, T) \) and (2) almost everywhere in \( \mathbb{R}^2 \), then \( (\rho, u, P, \theta) \) is called a strong solution to (1).
For $1 \leq r \leq \infty$ and $k \geq 1$, we denote the standard Lebesgue and Sobolev spaces as follows:

$$L^r = L^r(\mathbb{R}^2), \ W^{k,r} = W^{k,r}(\mathbb{R}^2), \ H^k = W^{k,2}.$$  

Without loss of generality, we assume that the initial density $\rho_0$ satisfies

$$\int_{\mathbb{R}^2} \rho_0 dx = 1,$$

which implies that there exists a positive constant $N_0$ such that

$$\int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int \rho_0 dx = \frac{1}{2},$$

where $B_R := \{ x \in \mathbb{R}^2 \mid |x| < R \}$ for $R > 0$.

Our main result can be stated as follows:

**Theorem 1.2.** For constants $q > 2$ and $a > 1$, in addition to (4) and (5), we assume that the initial data $(\rho_0 \geq 0, u_0, \theta_0)$ satisfy

$$\left\{ \begin{array}{l}
\rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q}, \ \theta_0 \bar{x}^2 \in L^2, \ \nabla \theta_0 \in L^2, \\
\sqrt{\rho_0} u_0 \in L^2, \ \nabla u_0 \in L^2, \ \text{div} u_0 = 0,
\end{array} \right.$$  

where

$$\bar{x} := (e + |x|^2)^{\frac{1}{2}} \log^2(e + |x|^2).$$

Then the Cauchy problem (1)–(2) has a unique global strong solution $(\rho \geq 0, u, P, \theta)$ satisfying that for any $0 < T < \infty$,

$$\left\{ \begin{array}{l}
\rho \in C([0,T]; L^1 \cap H^1 \cap W^{1,q}), \\
\rho \bar{x}^a \in L^\infty(0,T; L^1 \cap H^1 \cap W^{1,q}), \\
\sqrt{\rho_0} u, \ \sqrt{\rho_0} \bar{x} u, \ \sqrt{\rho_0} \nabla P, \ \sqrt{\rho_0} \nabla^2 u \in L^\infty(0,T; L^2), \\
\theta, \ \theta \bar{x}^2, \ \nabla \theta, \ \sqrt{\rho_0} \nabla^2 \theta \in L^\infty(0,T; L^2), \\
\nabla u \in L^2(0,T; H^1) \cap L^{\frac{2q+1}{q}}(0,T; W^{1,q}), \\
\nabla P \in L^2(0,T; L^2) \cap L^{\frac{2q+1}{q}}(0,T; L^q), \\
\nabla \theta \in L^2(0,T; H^1), \ \theta_t, \ \nabla \theta \bar{x}^2 \in L^2(0,T; L^2), \\
\sqrt{\rho_0} u \in L^2(0,T; W^{1,q}), \\
\sqrt{\rho_0} u_t, \ \sqrt{\rho_0} \nabla \theta, \ \sqrt{\rho_0} \nabla u_t, \ \sqrt{\rho_0} \nabla \theta_t, \ \sqrt{\rho_0} \nabla \bar{x} u_t \in L^2(\mathbb{R}^2 \times (0,T))
\end{array} \right.$$  

and

$$\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho(x,t) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx,$$  

for some positive constant $N_1$ depending only on $\|\rho_0\|_{L^1}, \|\sqrt{\rho_0} u_0\|_{L^2}, N_0$, and $T$.

**Remark 1.** It should be noted that the initial density may contain vacuum states and even have compact support. Moreover, the initial data can be arbitrarily large.

**Remark 2.** It is worth mentioning that no Cho-Choe-Kim type compatibility condition (see [6, 8]) on the initial data is required in Theorem 1.2 for the global existence and uniqueness of strong solutions.
We now make some comments on the analysis of the present paper. Using some key ideas in \[16\], where Li and Liang dealt with the local well-posedness of classical solutions to the Cauchy problem of two-dimensional compressible Navier-Stokes equations, we first establish that if \((\rho_0, u_0, \theta_0)\) satisfies (4) and (6), then there exists a small \(T_0 > 0\) such that the Cauchy problem (1)–(2) admits a unique strong solution \((\rho, u, P, \theta)\) in \(\mathbb{R}^2 \times (0, T_0]\) satisfying (8) and (9) (see Theorem 3.1). Thus, to prove Theorem 1.2, we only need to give some global a priori estimates on the strong solutions to system (1)–(2) in suitable higher norms.

It should be pointed out that the crucial techniques of proofs in \[10\] cannot be adapted to the situation treated here, since it seems difficult to bound the \(L^p(\mathbb{R}^2)\)-norm \((p > 2)\) of \(u\) just in terms of \(\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}\) and \(\|\nabla u\|_{L^2(\mathbb{R}^2)}\). To this end, we try to adapt some basic ideas in \[19\], where the authors investigated the global existence of strong solutions to the 2D Cauchy problem of density-dependent Navier-Stokes equations. However, compared with \[19\], for the system (1)–(2) treated here, the strong coupling between the velocity field and the temperature, such as \(u \cdot \nabla \theta\), will bring out some new difficulties.

To overcome these difficulties mentioned above, some new ideas are needed. To deal with the difficulty caused by the lack of Sobolev’s inequality, we observe that, in equations (1)–2, the velocity \(u\) is always accompanied by \(\rho\). Motivated by \[16\], by introducing a weighted function to the density, as well as the Hardy-type inequality in \[17\] by Lions, the \(\|\rho^\eta u\|_{L^r(\Omega)}\) \((r > 2, \eta > 0 \text{ and } \Omega = B_R \text{ or } \mathbb{R}^2)\) is controlled in terms of \(\|\sqrt{\rho}u\|_{L^2(\Omega)}\) and \(\|\nabla u\|_{L^2(\Omega)}\) (see (34) and (152)). After some spatial estimates on \(\nabla \theta\) (i.e., \(\nabla \theta \tilde{x}^\gamma\), see (59)), and suitable a priori estimates, we then construct approximate solutions to (1), that is, for density strictly away from vacuum initially, we consider a initial boundary value problem of (1) in any bounded ball \(B_R\) with radius \(R > 0\). Combining all these ideas stated above, we derive some desired bounds on the gradients of the velocity and the spatial weighted ones on both the density and its gradients where all these bounds are independent of both the radius of the balls \(B_R\) and the lower bound of the initial density, and then obtain the local existence and uniqueness of solution (see subsection 3.2).

Base on the local existence result (see Theorem 3.1), we attempt to give some appropriate a priori estimates which are needed to obtain the global existence of strong solutions. First, we try to obtain the estimates on the \(L^\infty(0, T; L^2(\mathbb{R}^2))\)-norm of the gradients of velocity and temperature. On the one hand, motivated by \[19\], we use material derivatives \(\dot{u} := u_t + u \cdot \nabla u\) instead of the usual \(u_t\), and apply some facts on Hardy and BMO spaces (see Lemma 2.5) to bound the key term \(\int_{\mathbb{R}^2} |P|\|\nabla u\|^2\,dx\) (see the estimates of \(I_2\) of (118)). Next, after some careful analysis, we derive the desired \(L^1(0, T; L^\infty(\mathbb{R}^2))\) bound of the gradient of the velocity \(\nabla u\) (see (155)), which in particular implies the bound on the \(L^\infty(0, T; L^q(\mathbb{R}^2))\)-norm \((q > 2)\) of the gradient of the density. Moreover, some useful spatial weighted estimates on \(\rho, \theta, \nabla \theta\) are derived (see Lemmas 4.6 and 4.7). With the a priori estimates stated above in hand, we can estimate the higher order derivatives of the solution \((\rho, u, P, \theta)\) (see Lemma 4.8) to obtain the desired results.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities which will be needed in later analysis. Sections 3 is devoted to establishing the local existence and uniqueness of strong solutions. The main result Theorem 1.2 is proved in Section 4.
2. Preliminaries. In this section, we will recall some known facts and elementary inequalities which will be used frequently later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls can be shown by similar arguments as in [6, 8].

**Lemma 2.1.** For $R > 0$ and $B_R = \{ x \in \mathbb{R}^2 | |x| < R \}$, assume that $(\rho_0, u_0, \theta_0 \geq 0)$ satisfies

$$(\rho_0, u_0, \theta_0) \in H^2(B_R), \quad \inf_{x \in B_R} \rho_0(x) > 0, \quad \text{div} u_0 = 0. \quad (10)$$

Then there exist a small time $T_R > 0$ such that the equations (1) with the following initial-boundary-value conditions

$$\begin{cases} (\rho, u, \theta)(x, t = 0) = (\rho_0, u_0, \theta_0), & x \in B_R, \\ u(x, t) = 0, \quad \theta(x, t) = 0, & x \in \partial B_R, \; t > 0, \end{cases} \quad (11)$$

has a unique classical solution $(\rho, u, P, \theta)$ on $B_R \times (0, T_R]$ satisfying

$$\begin{cases} \rho \in C \left( [0, T_R]; H^2 \right), \\ (u, \theta) \in C \left( [0, T_R]; H^2 \right) \cap L^2 \left( 0, T_R; H^3 \right), \\ P \in C \left( [0, T_R]; H^1 \right) \cap L^2 \left( 0, T_R; H^2 \right), \end{cases} \quad (12)$$

where we denote $H^k = H^k(B_R)$ for positive integer $k$.

Next, for $\Omega \subset \mathbb{R}^2$, the following weighted $L^m$-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) := \{ v \in H^1_{loc}(\Omega) | \nabla v \in L^2(\Omega) \}$ can be found in [17, Theorem B.1].

**Lemma 2.2.** For $m \in [2, \infty)$ and $\theta \in (1 + m/2, \infty)$, there exists a positive constant $C$ such that for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$ and for any $v \in \tilde{D}^{1,2}(\Omega)$,

$$\left( \int_{\Omega} \frac{|v|^m}{e + |x|^2} (\log(e + |x|^2))^{-\theta} \; dx \right)^{\frac{1}{m}} \leq C \| v \|_{L^2(B_1)} + C \| \nabla v \|_{L^2(\Omega)} \quad (13)$$

A useful consequence of Lemma 2.2 is the following crucial weighted bounds for elements of $\bar{D}^{1,2}(\Omega)$, which have been proved in [16, Lemma 2.4].

**Lemma 2.3.** Let $\bar{x}$ be as in (7) and $\Omega$ be as in Lemma 2.2. Assume that $\rho \in L^1(\Omega) \cap L^\infty(\Omega)$ is a non-negative function such that

$$\int_{B_N} \rho \; dx \geq M_1, \quad \| \rho \|_{L^1(\Omega) \cap L^\infty(\Omega)} \leq M_2, \quad (14)$$

for positive constants $M_1, M_2$, and $N_1 \geq 1$ with $B_{N_1} \subset \Omega$. Then for $\varepsilon > 0$ and $\eta > 0$, there is a positive constant $C$ depending only on $\varepsilon, \eta, M_1, M_2$, and $N_1$ such that every $v \in \bar{D}^{1,2}(\Omega)$ satisfies

$$\| v^{\bar{x}^{-\eta}} \|_{L^{(2+\varepsilon)/\eta}(\Omega)} \leq C \| \sqrt{\rho} v \|_{L^2(\Omega)} + C \| \nabla v \|_{L^2(\Omega)} \quad (15)$$

with $\eta = \min\{1, \eta\}$.

Next, the following $L^p$-bound for elliptic systems, whose proof is similar to that of [7, Lemma 12], is a direct result of the combination of the well-known elliptic theory [2] and a standard scaling procedure.

**Lemma 2.4.** For $p > 1$ and $k \geq 0$, there exists a positive constant $C$ depending only on $p$ and $k$ such that

$$\| \nabla^{k+2} v \|_{L^p(B_R)} \leq C \| \Delta v \|_{W^{k,p}(B_R)}, \quad (16)$$
for every \( v \in W^{k+2,p}(B_{R}) \) satisfying

\[ v = 0 \quad \text{on} \quad B_{R}. \]

Finally, let \( H^{1}(\mathbb{R}^2) \) and \( BMO(\mathbb{R}^2) \) stand for the usual Hardy and \( BMO \) spaces (see [24, Chapter IV]). Then the following well-known facts play a key role in the proof of Lemma 4.2 in Section 4.

**Lemma 2.5.** (a) There is a positive constant \( C \) such that

\[
\|E \cdot B\|_{H^{1}(B_{R})} \leq C\|E\|_{L^{2}(B_{R})}\|B\|_{L^{2}(B_{R})},
\]

for all \( E \in L^{2}(\mathbb{R}^2) \) and \( B \in L^{2}(\mathbb{R}^2) \) satisfying \( \text{div} \ E = 0, \text{curl} \ B = 0 \) in \( \mathcal{D}'(\mathbb{R}^2) \).

(b) There is a positive constant \( C \) such that

\[
\|v\|_{BMO(B_{R})} \leq C\|\nabla v\|_{L^{2}(B_{R})},
\]

for all \( v \in D^{1}(\mathbb{R}^2) \).

**Proof.** (a) For the detailed proof, please see [9, Theorem II.1].

(b) It follows from the Poincaré inequality that for any ball \( B \subset \mathbb{R}^2 \)

\[
\frac{1}{|B|} \int_{B} |v(x)| - \frac{1}{|B|} \int_{B} v(y)dy \, dx \leq C \left( \int_{B} |\nabla v|^2 \, dx \right)^{1/2},
\]

which directly gives (17). \(\square\)

### 3. Local existence and uniqueness of solutions.

In this section, we shall prove the following local existence and uniqueness of strong solutions to the Cauchy problem (1)–(2).

**Theorem 3.1.** Assume that \((\rho_{0}, u_{0}, \theta_{0})\) satisfies (4) and (6). Then there exist a small positive time \( T_{0} > 0 \) and a unique strong solution \((\rho, u, P, \theta)\) to the Cauchy problem of system (1)–(2) in \( \mathbb{R}^2 \times (0, T_{0}] \) satisfying (8) with \( T = T_{0} \).

#### 3.1. A priori estimates.

Throughout this subsection, for \( r \in [1, \infty] \) and \( k \geq 0 \), we denote

\[
\int_{B_{R}} \rho u^r \, dx = \int_{B_{R}} \rho \, dx, \quad L^{r}(B_{R}) = W^{k,r}(B_{R}), \quad W^{k,r}(B_{R}), \quad H^{k} = W^{k,2}.
\]

Moreover, for \( R > 4N_{0} \geq 4 \), assume that \((\rho_{0}, u_{0}, \theta_{0})\) satisfies, in addition to (10), that

\[
\frac{1}{2} \leq \int_{B_{R}}\rho_{0}(x)dx \leq \int_{B_{R}}\rho_{0}(x)dx \leq 1. \tag{18}
\]

Thus Lemma 2.1 yields that there exists some \( T_{R} > 0 \) such that the initial-boundary-value problem (1) and (11) has a unique classical solution \((\rho, u, P, \theta)\) on \( B_{R} \times [0, T_{R}] \) satisfying (12).

Let \( \bar{x}, a, \) and \( q \) be as in Theorem 1.2, the main aim of this section is to derive the following key a priori estimate on \( \psi \) defined by

\[
\psi(t) := 1 + \|\sqrt{\rho} u\|_{L^{2}} + \|\nabla u\|_{L^{2}} + \|\nabla \theta\|_{L^{2}} + \|\bar{x}^{2} \theta\|_{L^{2}} + \|\bar{x}^{2} \rho\|_{L^{1} \cap H^{1} \cap W^{1,q}}. \tag{19}
\]
Proposition 1. Assume that \((\rho_0, u_0, \theta_0)\) satisfies (10) and (18). Let \((\rho, u, P, \theta)\) be the solution to the initial-boundary-value problem (1) and (11) on \(B_R \times (0, T]\) obtained by Lemma 2.1. Then there exist positive constants \(T_0\) and \(M\) both depending only on \(\mu, \kappa, q, a, N_0,\) and \(E_0\) such that

\[
\sup_{0 \leq t \leq T_0} \left( \psi(t) + \sqrt{t} \|\sqrt{\rho} u\|_{L^2} + \sqrt{t} \|\theta\|_{L^2} + \sqrt{t} \|\nabla^2 u\|_{L^2} + \sqrt{t} \|\nabla P\|_{L^2} + \sqrt{t} \|\nabla^2 \theta\|_{L^2} \right) \\
+ \int_0^{T_0} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 + \|\nabla \theta e_2\|_{L^2}^2 \right) dt \\
+ \int_0^{T_0} \left( \|\nabla u\|_{L^2}^2 + t \|\nabla^2 u\|_{L^2}^2 + t \|\nabla \theta\|_{L^2}^2 \right) + \int_0^{T_0} \left( \|\nabla u\|_{L^2}^2 + t \|\nabla \theta\|_{L^2}^2 \right) dt \leq M,
\]

where

\[
E_0 := \|\sqrt{\rho_0} u_0\|_{L^2} + \|\nabla u_0\|_{L^2} + \|\nabla \theta_0\|_{L^2} + \|\nabla^2 \theta_0\|_{L^2} + \|\nabla^2 \theta_{01}\|_{L^2} + \|\bar{\rho}_0\|_{L^1} + \|\nabla \theta_{01}\|_{L^2} + \|\nabla^2 \theta_{01}\|_{L^2}.
\]

To show Proposition 1, whose proof will be postponed to the end of this subsection, we begin with the following standard energy estimate for \((\rho, u, P, \theta)\) and the estimate on the \(L^p\)-norm of the density.

Lemma 3.2. Under the conditions of Proposition 1, let \((\rho, u, P, \theta)\) be a smooth solution to the initial-boundary-value problem (1) and (11). Then for \(T_1\) such as in Lemma 3.3 and \(t \in (0, T]\),

\[
\sup_{0 \leq s \leq t} \left( \|\rho\|_{L^1 \cap L^\infty} + \|\sqrt{\rho} u\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right) + \int_0^t \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) ds \leq C,
\]

where (and in what follows) \(C\) denotes a generic positive constant depending only on \(\mu, \kappa, q, a, N_0,\) and \(E_0\).

Proof. First, since \(\text{div} u = 0\), it is easy to deduce from (1) that (see [17, Theorem 2.1]),

\[
\sup_{0 \leq s \leq t} \|\rho\|_{L^1 \cap L^\infty} \leq C.
\]

Next, multiplying (1)\(_2\) and (1)\(_3\) by \(u\) and \(\theta\) respectively, then adding the two resulting equations together, and integrating over \(B_R\), we have

\[
\frac{1}{2} \frac{d}{dt} \int (\rho |u|^2 + |\theta|^2) \, dx + \int [\mu |\nabla u|^2 + \kappa |\nabla \theta|^2] \, dx = \int \rho \theta e_2 \cdot u \, dx \\
\leq \|\rho\|_\infty^2 \|\sqrt{\rho} u\|_{L^2} \|\theta\|_{L^2} \\
\leq C (\|\sqrt{\rho} u\|_{L^2}^2 + \|\theta\|_{L^2}^2).
\]

Thus, Gronwall’s inequality leads to

\[
\sup_{0 \leq s \leq t} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right) + \int_0^t \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) ds \leq C,
\]

which together with (22) yields (21) and completes the proof of Lemma 3.2.

Next, we will give some spatial weighted estimates on the density.
Lemma 3.3. Under the conditions of Proposition 1, let \((\rho, u, P, \theta)\) be a smooth solution to the initial-boundary-value problem (1) and (11). Then there exists a \(T_1 = T_1(N_0, E_0) > 0\) such that for all \(t \in (0, T_1]\),

\[
\sup_{0 \leq s \leq t} (\|\rho \overline{x}^a\|_{L^1} + \|\theta \overline{x}^a\|_{L^2}^2) + \int_0^t \|\nabla \theta \overline{x}^a\|_{L^2}^2 ds \leq C. \tag{24}
\]

Proof. 1. For \(N > 1\), let \(\varphi_N \in C_0^\infty(B_N)\) satisfy

\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = 1, \text{ if } |x| \leq \frac{N}{2} \quad |\nabla \varphi_N| \leq CN^{-1}. \tag{25}
\]

It follows from (1) and (21) that

\[
\frac{d}{dt} \int \rho \varphi_2 N_0 dx = \int \rho u \cdot \nabla \varphi_2 N_0 dx \
\geq -NC_0^{-1} \left( \int \rho dx \right)^{\frac{1}{2}} \left( \int \rho|u|^2 dx \right)^{\frac{1}{2}} \geq -\tilde{C}(E_0). \tag{26}
\]

Integrating (26) and using (18) give rise to

\[
\inf_{0 \leq s \leq T_1} \int_{B_2 N_0} \rho dx \geq \inf_{0 \leq s \leq T_1} \int \rho \varphi_2 N_0 dx \geq \int \rho_0 \varphi_2 N_0 dx - \tilde{C} T_1 \geq \frac{1}{4}. \tag{27}
\]

Here, \(T_1 := \min\{1, (4\tilde{C})^{-1}\}\). From now on, we will always assume that \(t \leq T_1\). The combination of (27), (21), and (15) implies that for \(\varepsilon > 0\) and \(\eta > 0\), every \(v \in D^{1,2}(B_R)\) satisfies

\[
\|v \overline{x}^{-\eta}\|_{L^2(2e+\varepsilon)/\eta} \leq C(\varepsilon, \eta) \|\sqrt{\rho v}\|_{L^2}^2 + C(\varepsilon, \eta) \|\nabla v\|_{L^2}^2, \tag{28}
\]

with \(\tilde{\eta} = \min\{1, \eta\}\).

2. Multiplying (1)_1 by \(\overline{x}^a\) and integrating by parts imply that

\[
\frac{d}{dt} \|\rho \overline{x}^a\|_{L^1} \leq C \int \rho |u| \overline{x}^{a-1} \log^2 (e + |x|^2) dx \
\leq C \|\rho \overline{x}^{a-1} + \frac{\overline{x}^a}{\overline{x}^{a}}\|_{L^\infty} \|\overline{x}^a - \frac{\overline{x}^a}{\overline{x}^{a}}\|_{L^\infty} \
\leq C \|\rho\|_{L^\infty} \|\overline{x}^a\|_{L^1} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \
\leq C (1 + \|\rho \overline{x}^a\|_{L^1}) (1 + \|\nabla u\|_{L^2}^2) 
\]

due to (21) and (28). This combined with Gronwall’s inequality and (21) leads to

\[
\sup_{0 \leq s \leq t} \|\rho \overline{x}^a\|_{L^1} \leq C. \tag{29}
\]

3. Multiplying (1)_3 by \(\theta \overline{x}^a\) and integrating by parts yield

\[
\frac{1}{2} \frac{d}{dt} \int \theta^2 \overline{x}^a dx + \kappa \int |\nabla \theta|^2 \overline{x}^a dx \
= \frac{\kappa}{2} \int |\theta|^2 \Delta \overline{x}^a dx + \frac{1}{2} \int |\theta|^2 u \cdot \nabla \overline{x}^a dx 
\]

\[
:= I_1 + I_2, \tag{30}
\]
Lemma 3.4. Let

where

First, it follows from (21), (24), and (28) that for any 

which together with (29) gives (24) and finishes the proof of Lemma 3.3.

Proof. 1. Multiplying (1) by and integrating by parts, one has

where due to Gagliardo-Nirenberg inequality (see [21, Theorem]), (21), and (28). Putting (31) into (30), we get after using Gronwall’s inequality and (21) that

which combined with Hölder’s and Gagliardo-Nirenberg inequalities yields

where (and in what follows) we use \( \alpha > 1 \) to denote a genetic constant, which may be different from line to line. For the second term on the right-hand side of (33),
we get from Cauchy-Schwarz inequality and (21) that
\[
\int \rho \theta e_2 \cdot u_i \, dx \leq \| \rho \|_{L^\infty}^2 \| \sqrt{\rho} u_i \|_{L^2} \| \theta \|_{L^2} \leq \frac{1}{2} \| \sqrt{\rho} u_i \|_{L^2}^2 + C. \tag{37}
\]
Inserting (36)–(37) into (33) gives rise to
\[
\mu \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \frac{1}{2} \| \sqrt{\rho} u_i \|_{L^2}^2 \leq C \psi^\alpha + \varepsilon \| \nabla^2 u \|_{L^2}^2. \tag{38}
\]

2. To estimate the second term on the right-hand side of (38), we see that \((\rho, u, P, \theta)\) satisfies the following Stokes system
\[
\begin{aligned}
-\mu \Delta u + \nabla P &= -\rho u_i - \rho u \cdot \nabla u + \rho \theta e_2, &x \in B_R, \\
\text{div} u &= 0, &x \in B_R, \\
u(x) &= 0, &x \in \partial B_R, 
\end{aligned}
\tag{39}
\]
applying the standard \(L^p\)-estimate to (39) (see [25]) yields that for any \(p \in [2, \infty)\),
\[
\| \nabla^2 u \|_{L^p}^2 + \| \nabla \theta \|_{L^p}^2 \leq C \| \rho u_i \|_{L^p} + C \| \rho u \cdot \nabla u \|_{L^p} + C \| \rho \theta \|_{L^p}. \tag{40}
\]
Then, it follows from (40), (21), (35), and Gagliardo-Nirenberg inequality that
\[
\| \nabla^2 u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2 \leq C \| \rho u_i \|_{L^2}^2 + C \| \rho u \cdot \nabla u \|_{L^2}^2 + C \| \rho \theta \|_{L^2}^2
\leq C \sqrt{\rho} u_i \|_{L^2}^2 + C \| \rho u \|_{L^2} \| \nabla u \|_{L^2} + C \| \rho \theta \|_{L^2}^2
\leq C \sqrt{\rho} u_i \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla \theta \|_{H^1} + C
\leq C \sqrt{\rho} u_i \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \nabla \theta \|_{H^1} + C \psi^\alpha. \tag{41}
\]
Consequently, substituting (41) into (38) and choosing \(\varepsilon\) suitably small, one has
\[
\mu \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \frac{1}{4} \| \sqrt{\rho} u_i \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 \leq C \psi^\alpha.
\]
Integrating the above inequality over \((0, t)\) yields
\[
\sup_{0 \leq s \leq t} \| \nabla u \|_{L^2}^2 + \int_0^t (\| \sqrt{\rho} u_i \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2) \, ds \leq C + C \int_0^t \psi^\alpha(s) \, ds. \tag{42}
\]

3. Multiplying (1) by \(\theta\) and then adding \(1)\) multiplied by \(-\kappa \Delta \theta\), we obtain from integration by parts that
\[
\kappa \frac{d}{dt} \| \nabla \theta \|_{L^2}^2 + \| \theta \|_{H^1}^2 + \kappa^2 \| \Delta \theta \|_{L^2}^2 \leq C \| u \| \| \nabla \theta \|_{L^2}^2
\leq C \| u \| \| \Delta \theta \|_{L^2}^2 + C \| \nabla \theta \|_{L^2}^2
\leq \frac{\kappa^2}{2} \| \Delta \theta \|_{L^2}^2 + C \psi^\alpha + C \| \nabla \theta \|_{L^2}^2 \tag{43}
\]
due to (16), (35), and Gagliardo-Nirenberg inequality. Thus we derive from (43), Gronwall’s inequality, (24), and (16) that
\[
\sup_{0 \leq s \leq t} \| \nabla \theta \|_{L^2}^2 + \int_0^t (\| \theta \|_{H^1}^2 + \| \nabla^2 \theta \|_{L^2}^2) \, ds \leq C + C \int_0^t \psi^\alpha(s) \, ds. \tag{44}
\]
This together with (42) leads to (32) and finishes the proof of Lemma 3.4. \(\Box\)
Lemma 3.5. Let \( T_1 \) be as in Lemma 3.3. Then there exists a positive constant \( \alpha > 1 \) such that for all \( t \in (0, T_1) \),

\[
\text{sup}_{0 \leq s \leq t} \left( s\|\overline{\rho}u_s\|_{L^2}^2 + s\|\theta_s\|_{L^2}^2 \right) + \int_0^t \left( s\|\nabla u_s\|_{L^2}^2 + s\|\nabla \theta_s\|_{L^2}^2 \right) ds \\
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}.
\]

(45)

Proof. 1. Differentiating (1) \( _2 \) with respect to \( t \) gives

\[
\rho u_t + \rho u \cdot \nabla u_t - \mu \Delta u_t = -\rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla P_t + (\rho \theta e_2)_t.
\]

(46) Multiplying (46) by \( u_t \) and integrating the resulting equality by parts over \( B_R \), we obtain after using (1) \( _1 \) and (1) \( _4 \) that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx \\
\leq C \int \rho |u| |u_t| \left( |\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u| \right) dx + C \int \rho |u^2|^2 |\nabla u| |\nabla u_t| dx \\
+ C \int \rho |u_t|^2 |\nabla u| dx + \int \rho \theta e_2 \cdot u_t dx + \int \rho \theta e_2 \cdot u_t dx := \sum_{i=1}^5 I_i.
\]

(47)

We estimate each term on the right-hand side of (47) as follows.

2. First, it follows from (34), (35), and Gagliardo-Nirenberg inequality that

\[
I_1 \leq C \|\sqrt{\rho} u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6} \left( \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^4} \right) \\
+ C \|\rho^\frac{3}{2} u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \\
\leq C (1 + \|\nabla u\|_{L^2}^2) \|\nabla u_t\|_{L^2} \left( \|\sqrt{\rho} u_t\|_{L^2} + \|\nabla u_t\|_{L^2} \right) \frac{1}{2} \\
\cdot \left( \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^2} \right) \\
\leq \frac{\mu}{6} \|\nabla u\|_{L^2}^2 + C \psi^\alpha \|\nabla u\|_{L^2}^2 + C \psi^\alpha + C \left( 1 + \|\nabla u\|_{L^2}^2 \right) \|\nabla^2 u\|_{L^2}^2.
\]

(48)

Then, Hölder’s inequality combined with (34) and (35) leads to

\[
I_2 + I_3 \leq C \|\sqrt{\rho} u\|_{L^6} \|\nabla u_t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6} \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \psi^\alpha \|\nabla u_t\|_{L^2}^2 + C \left( \psi^\alpha + \|\nabla^2 u\|_{L^2}^2 \right).
\]

(49)

Integration by parts together with (1) \( _1 \), (1) \( _4 \), Hölder’s and Gagliardo-Nirenberg inequalities indicates that

\[
I_4 = \int \rho u \cdot (\nabla (\theta e_2) - u_t) dx \\
\leq \int \rho |u| |\nabla \theta| |u_t| dx + \int \rho |u| |\nabla u_t| dx \\
\leq \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^6} \|\nabla \theta\|_{L^4} + \|\nabla u_t\|_{L^2} \|\rho u\|_{L^4} \|\theta\|_{L^4} \\
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \psi^\alpha \|\sqrt{\rho} u_t\|_{L^2}^2 + C \left( \psi^\alpha + \|\nabla^2 \theta\|_{L^2}^2 \right).
\]

(50)

Next, we get from Hölder’s inequality and (21) that

\[
I_5 \leq \|\nabla u_t\|_{L^2}^2 \|\theta_t\|_{L^2} \leq C \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\theta_t\|_{L^2}^2.
\]

(51)
Substituting (48)–(51) into (47), we obtain after using (41) that
\[
\frac{d}{dt} \| \sqrt{\rho_1} u \|^2_{L^2} + \mu \| \nabla u \|^2_{L^2} \\
\leq C \psi^\alpha \| \sqrt{\rho_1} u \|^2_{L^2} + C \psi^\alpha + \| \nabla^2 u \|^2_{L^2} + \| \nabla^2 \theta \|^2_{L^2} + \| \theta \|^2_{L^2}).
\] (52)

3. Differentiating (1)\textsubscript{3} with respect to \( t \) shows
\[
\theta_t + u_t \cdot \nabla \theta + u \cdot \nabla \theta_t = \kappa \Delta \theta_t.
\] (53)

Multiplying (53) by \( \theta_t \) and integrating the resulting equality over \( B_R \) yield that
\[
\frac{1}{2} \frac{d}{dt} \int \theta_t^2 \, dx + \kappa \int |\nabla \theta_t|^2 \, dx = -\int \theta_t u_t \cdot \nabla \theta \, dx - \int \theta_t u_t \cdot \nabla \theta_t \, dx \\
= \int \nabla \theta_t \cdot u_t \, dx - \frac{1}{2} \int u \cdot \nabla \theta_t^2 \, dx \\
= \int \nabla \theta_t \cdot u_t \, dx \\
\leq \frac{\kappa}{2} \| \nabla \theta_t \|^2_{L^2} + C \| u_t \| \theta t \|^2_{L^2} \\
\leq \frac{\kappa}{2} \| \nabla \theta_t \|^2_{L^2} + C \| u_t \|^2_{L^2} \| \theta \|^2_{L^2} \\
\leq \frac{\kappa}{2} \| \nabla \theta_t \|^2_{L^2} + C \| \nabla u \|^2_{L^2} + C \| \sqrt{\rho_1} u \|^2_{L^2},
\] (54)

where one has used (24), (28), and the following estimate
\[
\sup_{0 \leq s \leq t} \| \theta \|^4_{L^2} + \int_0^t \int |\nabla \theta|^2 \theta^2 \, dx \, ds \leq C.
\] (55)

Indeed, multiplying (1)\textsubscript{3} by \( 4\theta^3 \) and integrating by parts lead to
\[
\frac{d}{dt} \int \theta^4 \, dx + 12 \kappa \int |\nabla \theta|^2 \theta^2 \, dx = 0.
\] (56)

Integrating (56) with respect to \( t \) gives (55). Hence we obtain from (54) that
\[
\frac{d}{dt} \| \theta_t \|^2_{L^2} + \kappa \| \nabla \theta_t \|^2_{L^2} \leq C_1 \| \nabla u \|^2_{L^2} + C \| \sqrt{\rho_1} u \|^2_{L^2}.
\] (57)

4. Multiplying (52) by \( \mu^{-1}(C_1 + 1) \) and adding the resulting inequality to (57), we get
\[
\frac{d}{dt} \left( \mu^{-1}(C_1 + 1) \| \sqrt{\rho_1} u \|^2_{L^2} + \| \theta_t \|^2_{L^2} \right) + \| \nabla u \|^2_{L^2} + \frac{\kappa}{2} \| \nabla \theta_t \|^2_{L^2} \\
\leq C \psi^\alpha \left( 1 + \| \nabla u \|^2_{L^2} + C \| \nabla^2 u \|^2_{L^2} + \| \nabla^2 \theta \|^2_{L^2} + \| \theta_t \|^2_{L^2} \right).\]

Multiplying (58) by \( t \), we obtain (45) after using Gronwall’s inequality and (32). The proof of Lemma 3.5 is finished.

**Lemma 3.6.** Let \( T_1 \) be as in Lemma 3.3. Then there exists a positive constant \( \alpha > 1 \) such that for all \( t \in (0, T_1) \),
\[
\sup_{0 \leq s \leq t} \left( s \| \nabla^2 u \|^2_{L^2} + s \| \nabla^2 \theta \|^2_{L^2} + s \| \nabla x \|^2_{L^2} + s \| \nabla^2 \theta \|^2_{L^2} \right) + \int_0^t \| \Delta \theta x \|^2_{L^2} \, ds \\
\leq C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha \, ds \right\} \right\}.
\] (59)
Proof. 1. Multiplying (1)_3 by $\Delta \theta x^a$ and integrating by parts lead to

$$
\frac{1}{2} \frac{d}{dt} \int |\nabla \theta|^2 x^a dx + \kappa \int |\Delta \theta|^2 x^a dx
\leq C \int |\nabla \theta|^2 |u| |\nabla x^a| dx + C \int |\nabla \theta| |\Delta \theta| |\nabla x^a| dx + C \int |\nabla u| |\nabla \theta|^2 x^a dx
:= \sum_{i=1}^{3} J_i. \tag{60}
$$

Using (28), Hölder’s and Gagliardo-Nirenberg inequalities, one gets by some direct calculations that

$$
J_1 \leq C \left( \|\nabla \theta\|^2 \frac{x^a}{L^a} \|u x^{-\frac{1}{2}}\|_{L^\infty} \right) \|\nabla \theta\|^2_{\frown L^a}
\leq C \psi^\alpha \|\nabla \theta x^\frown L^a\|_{L^\infty}^{\frac{1}{2}} \|\nabla \theta\|^2_{\frown L^a} 
\leq C \psi^\alpha \|\nabla \theta x^\frown L^a\|_{L^\infty}^{\frac{1}{2}} \|\nabla \theta\|^2_{\frown L^a} + C \|\nabla \theta\|^2_{\frown L^a},
$$

$$
J_2 \leq \frac{\kappa}{4} \|\Delta \theta x^\frown L^a\|_{L^2}^2 + C \|\nabla \theta x^\frown L^a\|_{L^2}^2,
$$

$$
J_3 \leq C \|\nabla u\|_{L^\infty} \|\nabla \theta x^\frown L^a\|_{L^2}^2 \leq C \left( \psi^\alpha + \|\nabla^2 u\|_{L^4}^{\frac{2}{3}} \right) \|\nabla \theta x^\frown L^a\|_{L^2}^2.
$$

Substituting the above estimates into (60) gives

$$
\frac{d}{dt} \int |\nabla \theta|^2 x^a dx + \kappa \int |\Delta \theta|^2 x^a dx \leq C \left( \psi^\alpha + \|\nabla^2 u\|_{L^4}^{\frac{2}{3}} \right) \|\nabla \theta x^\frown L^a\|_{L^2}^2. \tag{61}
$$

2. We now claim that

$$
\int_0^t \left( \|\nabla^2 u\|_{L^4}^{\frac{2}{3}} + \|\nabla P\|_{L^4}^{\frac{2}{3}} + s \|\nabla^2 u\|_{L^4}^2 + s \|\nabla P\|_{L^4}^2 \right) ds
\leq C \exp \left\{ C \int_0^t \psi^\alpha (s) ds \right\}, \tag{62}
$$

whose proof will be given at the end of this proof. Thus, multiplying (61) by $t$, we infer from (62), (24), and Gronwall’s inequality that

$$
\sup_{0 \leq s \leq t} \left( s \|\nabla \theta x^\frown L^a\|_{L^2}^2 \right) + \int_0^t s \|\Delta \theta x^\frown L^a\|_{L^2}^2 ds \leq C \exp \left\{ C \int_0^t \psi^\alpha (s) ds \right\}. \tag{63}
$$

3. It follows from (1)_3, (16), (21), (28), Hölder’s and Gagliardo-Nirenberg inequalities that

$$
\|\nabla^2 \theta\|_{L^2}^2 \leq C \|\theta\|_{L^2}^2 + C \|\theta\|_{L^4} \|\nabla \theta\|_{L^2}^2
\leq C \|\theta\|_{L^2}^2 + C \|u x^{-\frac{1}{2}}\|_{L^\infty} \|\nabla \theta x^\frown L^a\|_{L^2} \|\nabla \theta\|_{L^4}
\leq C \|\theta\|_{L^2}^2 + C \|\nabla \theta x^\frown L^a\|_{L^2}^2 + C \|u x^{-\frac{1}{2}}\|_{L^\infty} \|\nabla \theta\|_{L^2}^2
\leq C \|\theta\|_{L^2}^2 + C \|\nabla \theta x^\frown L^a\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 \theta\|_{L^2}^2 + C (1 + \|\nabla u\|_{L^2}^4) (1 + \|\nabla \theta\|_{L^2}^2) \tag{64},
$$

which together with (41) gives that

$$
\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 \leq C \left( \|\nabla u\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \|\nabla \theta x^\frown L^a\|_{L^2}^2 \right)
+ C (1 + \|\nabla u\|_{L^2}^4) (1 + \|\nabla \theta\|_{L^2}^2). \tag{65}
$$
Then, multiplying (65) by \( s \), one gets from (45) and (63) that
\[
\sup_{0 \leq s \leq t} (s||\nabla^2 u||_{L^2}^2 + s||\nabla P||_{L^2}^2 + s||\nabla^2 \theta||_{L^2}^2)
\leq C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} + C \left( 1 + \int_0^t \psi^\alpha(s) ds \right)^{10} \right\}
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\},
\]
which combined with (63) implies (59).

4. To finish the proof of Lemma 3.6, it suffices to show (62). Indeed, choosing \( p = q \) in (40), we deduce from (21), (35), and Gagliardo-Nirenberg inequality that
\[
||\nabla^2 u||_{L^q} + ||\nabla P||_{L^q} \leq C (||\rho u||_{L^q} + ||\rho u \cdot \nabla u||_{L^q} + ||\rho \theta||_{L^q})
\leq C (||\rho u||_{L^q} + ||\rho u||_{L^{q^*}} ||\nabla u||_{L^{q^*}} + ||\rho||_{L^{q^*}} ||\theta||_{H^1})
\leq C ||\rho u||_{L^q}^{\frac{2(q-1)}{q^2-2}} ||\rho u||_{L^{q^*}}^{\frac{q^2-2q}{q^2-2}} + C \psi^\alpha \left( 1 + ||\nabla^2 u||_{L^2}^{1 + \frac{q}{q^*}} \right)
\leq C \left( \sqrt{\rho u_t} ||\nabla u_t||_{L^q}^{\frac{2(q-1)}{q^2-2}} ||\nabla^2 u_t||_{L^q}^{\frac{q^2-2q}{q^2-2}} + ||\rho u_t||_{L^q} \right)
+ C \psi^\alpha \left( 1 + ||\nabla^2 u||_{L^2}^{1 + \frac{q}{q^*}} \right),
\]
which together with (32) and (45) implies that
\[
\int_0^t \left( ||\nabla^2 u||_{L^q}^{\frac{q+1}{q}} + ||\nabla P||_{L^q}^{\frac{q+1}{q}} \right) \, ds
\leq C \int_0^t s^{-\frac{q+1}{q^*}} (s||\sqrt{\rho u_t}||_{L^2}^2) \frac{q^2-1}{2(q^2-2)} (s||\nabla u_t||_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}} \, ds
+ C \int_0^t ||\sqrt{\rho u_t}||_{L^2}^{\frac{q+1}{q}} \, ds + C \int_0^t \psi^\alpha \left( 1 + ||\nabla^2 u||_{L^2}^{\frac{q^2-1}{q^*}} \right) \, ds
\leq C \sup_{0 \leq s \leq t} (s||\sqrt{\rho u_t}||_{L^2}^2) \frac{q^2-1}{2(q^2-2)} \int_0^t s^{-\frac{q+1}{q^*}} (s||\nabla u_t||_{L^2}^2)^{\frac{(q-2)(q+1)}{2(q^2-2)}} \, ds
+ C \int_0^t \left( \psi^\alpha + ||\sqrt{\rho u_t}||_{L^2}^2 + ||\nabla^2 u||_{L^2}^2 \right) \, ds
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \left( 1 + \int_0^t \left( s^{-\frac{q^2+q^2-2q-2}{q^2-2}} + s||\nabla u_t||_{L^2}^2 \right) ds \right)
\leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}
\]
and
\[
\int_0^t (s||\nabla^2 u||_{L^q}^2 + s||\nabla P||_{L^q}^2) \, ds
\leq C \int_0^t s||\sqrt{\rho u_t}||_{L^2}^2 ds + C \int_0^t (s||\sqrt{\rho u_t}||_{L^2}^2) \frac{2(q-1)}{q^2-2} (s||\nabla u_t||_{L^2}^2)^{\frac{q^2-2q}{q^2-2}} \, ds
\]
One derives from (1). It follows from (21), (24), (32), and (70) that

\[
\text{Proof of Proposition 1. Standard arguments yield that for } \alpha > 1 \text{ is thus completed.}
\]

where in the last inequalities one has used (29).

One thus obtains (62) from (68) and (69).

\[
\text{Lemma 3.7. Let } T_1 \text{ be as in Lemma 3.3. Then there exists a positive constant } \alpha > 1 \text{ such that for all } t \in (0, T_1],
\]

\[
\sup_{0 \leq s \leq t} \|\rho \bar{x}^\alpha\|_{H^1 \cap W^{1,q}} \leq \exp \left( C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right). \tag{70}
\]

\[
\text{Proof. First, it follows from Sobolev’s inequality and (35) that for } 0 < \delta < 1,
\]

\[
\|\bar{x}^\delta\|_{L^\infty} \leq C(\delta) \left( \|\bar{x}^\delta\|_{L^4} + \|\nabla(\bar{x}^\delta)\|_{L^3} \right)
\]

\[
\leq C(\delta) \left( \|\bar{x}^\delta\|_{L^4} + \|\nabla u\|_{L^3} + \|\bar{x}^\delta\|_{L^4} \|\bar{x}^{-\delta} \nabla \bar{x}\|_{L^\frac{12}{5}}} \right)
\]

\[
\leq C(\delta) \left( \psi^\alpha + \|\nabla^2 u\|_{L^2} \right). \tag{71}
\]

One derives from (1) and (1) that \(\rho \bar{x}^\alpha\) satisfies

\[
\partial_t (\rho \bar{x}^\alpha) + u \cdot \nabla (\rho \bar{x}^\alpha) - a \rho \bar{x}^\alpha u \cdot \nabla \log \bar{x} = 0, \tag{72}
\]

which along with (71) gives that for any \(r \in [2, q],\)

\[
\frac{d}{dt} \|\nabla (\rho \bar{x}^\alpha)\|_{L^r} \leq C \left( 1 + \|\nabla u\|_{L^\infty} + \| u \cdot \nabla \log \bar{x}\|_{L^\infty} \right) \|\nabla (\rho \bar{x}^\alpha)\|_{L^r}
\]

\[
+ C \|\rho \bar{x}^\alpha\|_{L^\infty} \left( \|\nabla u\|_{L^r} + \| u \|_{L^\infty} \|\nabla^2 \log \bar{x}\|_{L^r} \right)
\]

\[
\leq C \left( \psi^\alpha + \|\nabla^2 u\|_{L^2 \cap L^q} \right) \|\nabla (\rho \bar{x}^\alpha)\|_{L^r}
\]

\[
+ C \|\rho \bar{x}^\alpha\|_{L^\infty} \left( \|\nabla u\|_{L^r} + \| u \|_{L^2} \|\bar{x}^{-\frac{\alpha}{2}} \|_{L^\frac{1}{5}} \right)
\]

\[
\leq C \left( \psi^\alpha + \|\nabla^2 u\|_{L^2 \cap L^q} \right) \left( 1 + \|\nabla (\rho \bar{x}^\alpha)\|_{L^r} + \|\nabla (\rho \bar{x}^\alpha)\|_{L^2} \right), \tag{73}
\]

where in the last inequalities one has used (29).

Finally, using (32), (62), (73), and Gronwall’s inequality, one thus gets (70) and completes the proof of Lemma 3.7.

Now, Proposition 1 is a direct consequence of Lemmas 3.2–3.7.

\[
\text{Proof of Proposition 1. It follows from (21), (24), (32), and (70) that}
\]

\[
\psi(t) \leq \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\}.
\]

Standard arguments yield that for \( M := e^{C\varepsilon} \) and \( T_0 := \min\{T_1, (CM^\alpha)^{-1}\}, \)

\[
\sup_{0 \leq t \leq T_0} \psi(t) \leq M,
\]

which together with (24), (32), (45), and (59) gives (62). The proof of Proposition 1 is thus completed.
3.2. Proof of Theorem 3.1. With the a priori estimates in subsection 3.1 at hand, it is a position to prove Theorem 3.1.

Proof of Theorem 3.1. Let \((\rho_0, \mathbf{u}_0, \theta_0)\) be as in Theorem 3.1, then it follows from (4) that there exists a positive constant \(N_0\) such that
\[
\int_{B_{N_0}} \rho_0 \, dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 \, dx = \frac{3}{4}.
\]
(74)

We construct \(\rho_0^R = \rho_0^R + R^{-1} e^{-|x|^2}\), where \(0 \leq \rho_0^R \in C_0^\infty(\mathbb{R}^2)\) satisfies
\[
\left\{ \begin{array}{ll}
\int_{B_{N_0}} \rho_0^R \, dx \geq 1/2, \\
\bar{x}^a \rho_0^R \rightarrow \bar{x}^a \rho_0 & \text{in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2), \\
\end{array} \right. \quad \text{as } R \rightarrow \infty.
\]
(75)

Noting that \(\theta_0 \bar{\theta} \tilde{\mathbf{z}} \in L^2(\mathbb{R}^2)\) and \(\nabla \theta_0 \in L^2(\mathbb{R}^2)\), we choose \(\theta_0^R \in C_0^\infty(B_R)\) satisfying
\[
\theta_0^R \tilde{\mathbf{z}} \rightarrow \theta_0 \tilde{\mathbf{z}} \quad \text{in } L^2(\mathbb{R}^2), \quad \text{as } R \rightarrow \infty.
\]
(76)

Since \(\nabla \mathbf{u}_0 \in L^2(\mathbb{R}^2)\), we select \(\mathbf{v}_0^R \in C_0^\infty(B_R)\) \((i = 1, 2)\) such that for \(i = 1, 2\),
\[
\lim_{R \rightarrow \infty} \| \mathbf{v}_0^R - \partial_i \mathbf{u}_0 \|_{L^2(\mathbb{R}^2)} = 0.
\]
(77)

We consider the unique smooth solution \(\mathbf{u}_0^R\) of the following elliptic problem:
\[
\left\{ \begin{array}{ll}
-\Delta \mathbf{u}_0^R + \rho_0^R \mathbf{u}_0^R + \nabla \mathbf{P}_0 = \sqrt{\rho_0^R} \mathbf{h}^R - \partial_i \mathbf{v}_0^R, & \text{in } B_R, \\
\text{div } \mathbf{u}_0^R = 0, & \text{in } B_R, \\
\mathbf{u}_0^R = 0, & \text{on } \partial B_R,
\end{array} \right.
\]
(78)

where \(\mathbf{h}^R = (\sqrt{\rho_0^R} \mathbf{u}_0) \ast j_{1/R}\) with \(j_\delta\) being the standard mollifying kernel of width \(\delta\).

Extending \(\mathbf{u}_0^R\) to \(\mathbb{R}^2\) by defining \(0\) outside \(B_R\) and denoting it by \(\tilde{\mathbf{u}}_0^R\), we claim that
\[
\lim_{R \rightarrow \infty} \left( \| \nabla (\tilde{\mathbf{u}}_0^R - \mathbf{u}_0) \|_{L^2(\mathbb{R}^2)} + \| \sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R - \sqrt{\rho_0^R} \mathbf{u}_0 \|_{L^2(\mathbb{R}^2)} \right) = 0.
\]
(79)

In fact, it is easy to find that \(\tilde{\mathbf{u}}_0^R\) is also a solution of (78) in \(\mathbb{R}^2\). Multiplying (78) by \(\tilde{\mathbf{u}}_0^R\) and integrating the resulting equation over \(\mathbb{R}^2\) lead to
\[
\int_{\mathbb{R}^2} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 \, dx + \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^R|^2 \, dx
\leq \| \sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R \|_{L^2(B_R)} \| \mathbf{h}^R \|_{L^2(B_R)} + C \| \mathbf{v}_0^R \|_{L^2(B_R)} \| \partial_i \tilde{\mathbf{u}}_0^R \|_{L^2(B_R)}
\leq \frac{1}{2} \| \nabla \tilde{\mathbf{u}}_0^R \|_{L^2(B_R)}^2 + \frac{1}{2} \int_{B_R} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 \, dx + C \left( \| \mathbf{h}^R \|_{L^2(B_R)}^2 + \| \mathbf{v}_0^R \|_{L^2(B_R)}^2 \right),
\]
which implies
\[
\int_{\mathbb{R}^2} \rho_0^R |\tilde{\mathbf{u}}_0^R|^2 \, dx + \int_{\mathbb{R}^2} |\nabla \tilde{\mathbf{u}}_0^R|^2 \, dx \leq C
\]
(80)

for some \(C\) independent of \(R\). This together with (75) yields that there exist a subsequence \(R_j \rightarrow \infty\) and a function \(\tilde{\mathbf{u}}_0 \in \{ \mathbf{u}_0 \in H^1_{\text{loc}}(\mathbb{R}^2) | \sqrt{\rho_0} \mathbf{u}_0 \in L^2(\mathbb{R}^2), \nabla \mathbf{u}_0 \in L^2(\mathbb{R}^2) \} \) such that
\[
\left\{ \begin{array}{ll}
\sqrt{\rho_0^R} \tilde{\mathbf{u}}_0^R_j \rightharpoonup \sqrt{\rho_0} \tilde{\mathbf{u}}_0 & \text{weakly in } L^2(\mathbb{R}^2), \\
\nabla \tilde{\mathbf{u}}_0^R_j \rightharpoonup \nabla \tilde{\mathbf{u}}_0 & \text{weakly in } L^2(\mathbb{R}^2).
\end{array} \right.
\]
(81)
Next, we will show
\[ \tilde{u}_0 = u_0. \] (82)

Indeed, multiplying (78) by a test function \( \pi \in C_0^\infty(\mathbb{R}^2) \) with \( \text{div} \, \pi = 0 \), it holds that
\[ \int_{\mathbb{R}^2} \partial_t (\tilde{u}_0^R - u_0) \cdot \partial_t \pi \, dx + \int_{\mathbb{R}^2} \sqrt{\rho_0^R} (\sqrt{\rho_0^R} \tilde{u}_0^R - h^R) \cdot \pi \, dx = 0. \] (83)

Let \( R_j \to \infty \), it follows from (75), (77), and (81) that
\[ \int_{\mathbb{R}^2} \partial_t (\tilde{u}_0 - u_0) \cdot \partial_t \pi \, dx + \int_{\mathbb{R}^2} \rho_0 (\tilde{u}_0 - u_0) \cdot \pi \, dx = 0, \] (84)
which implies (82).

Furthermore, multiplying (78) by \( \tilde{u}_0^R \) and integrating the resulting equation over \( \mathbb{R}^2 \), by the same arguments as (84), we have
\[ \lim_{R_j \to \infty} \int_{\mathbb{R}^2} (|\nabla \tilde{u}_0^R|^2 + \rho_0^R |\tilde{u}_0^R|^2) \, dx = \int_{\mathbb{R}^2} (|\nabla u_0|^2 + \rho_0 |u_0|^2) \, dx, \]
which combined with (81) leads to
\[ \lim_{R_j \to \infty} \int_{\mathbb{R}^2} |\nabla \tilde{u}_0^R|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u_0|^2 \, dx, \quad \lim_{R_j \to \infty} \int_{\mathbb{R}^2} \rho_0^R |\tilde{u}_0^R|^2 \, dx = \int_{\mathbb{R}^2} \rho_0 |u_0|^2 \, dx. \]

This, along with (82) and (81), gives (79).

Hence, by virtue of Lemma 2.1, the initial-boundary-value problem (1) and (11) with the initial data \( (\rho_0^R, u_0^R, \theta_0^R) \) has a classical solution \( (\rho^R, u^R, P^R, \theta^R) \) on \( B_R \times [0, T_R] \). Moreover, Proposition 1 shows that there exists a \( T_0 \) independent of \( R \) such that (20) holds for \( (\rho^R, u^R, P^R, \theta^R) \).

For simplicity, in what follows, we denote
\[ L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2). \]

Extending \( (\rho^R, u^R, P^R, \theta^R) \) by zero on \( \mathbb{R}^2 \setminus B_R \) and denoting it by
\[ \left( \tilde{\rho}^R := \varphi_R \rho^R, \tilde{u}^R, \tilde{P}^R, \tilde{\theta}^R \right) \]
with \( \varphi_R \) satisfying (25). First, (20) leads to
\[ \sup_{0 \leq t \leq T_0} \left( \| \sqrt{\tilde{\rho}^R} \tilde{u}^R \|_{L^2} + \| \nabla \tilde{u}^R \|_{L^2} + \| \nabla \tilde{\theta}^R \|_{L^2} + \| \tilde{\theta}^R \tilde{x} \|_{L^2} \right) \]
\[ \leq \sup_{0 \leq t \leq T_0} \left( \| \sqrt{\rho^R} u^R \|_{L^2(B_R)} + \| \nabla u^R \|_{L^2(B_R)} + \| \nabla \theta^R \|_{L^2(B_R)} + \| \theta^R \tilde{x} \|_{L^2(B_R)} \right) \]
\[ \leq C, \] (85)
and
\[ \sup_{0 \leq t \leq T_0} \| \tilde{\rho}^R \tilde{x}^a \|_{L^1 \cap L^\infty} \leq C. \] (86)
Similarly, it follows from (20) that for $q > 2$,
\[
\sup_{0 \leq t \leq T_0} \sqrt{t} \left( \|\sqrt{\rho^R} \tilde{u}_t^R\|_{L^2}^2 + \|\nabla^2 \tilde{u}_t^R\|_{L^2}^2 + \|\nabla^2 \tilde{R}\|_{L^2}^2 + \|\tilde{R}_t\|_{L^2}^2 \right) \\
+ \int_0^{T_0} \left( \|\sqrt{\rho^R} \tilde{u}_t^R\|_{L^2}^2 + \|\nabla^2 \tilde{u}_t^R\|_{L^2}^2 + \|\tilde{R}_t\|_{L^2}^2 + \|\Delta \tilde{R}\|_{L^2}^2 + \|\nabla \tilde{R}_t^2\|_{L^2}^2 \right) dt \\
+ \int_0^{T_0} \left( \|\nabla^2 \tilde{u}_t^R\|_{L^2}^2 + \|\nabla^2 \tilde{R}\|_{L^2}^2 \right) dt \\
\leq C. \tag{87}
\]

Next, for $p \in [2, q]$, we obtain from (20) and (70) that
\[
\sup_{0 \leq t \leq T_0} \|\nabla (\rho^R \tilde{x}_t^a)\|_{L^p} \leq C \sup_{0 \leq t \leq T_0} \left( \|\nabla (\rho^R \tilde{x}_t^a)\|_{L^p(B_0)} + \|R^{-1}\rho^R \tilde{x}_t^a\|_{L^p(B_0)} \right) \\
\leq C \sup_{0 \leq t \leq T_0} \|\rho^R \tilde{x}_t^a\|_{H^1(B_0) \cap W^{1, p}(B_0)} \leq C, \tag{88}
\]

which together with (71) and (20) yields
\[
\int_0^{T_0} \|\tilde{x}_t^a\|_{L^2}^2 dt \leq C \int_0^{T_0} \|\tilde{x}_t^a\|_{L^2} \|\nabla \tilde{x}_t^a\|_{L^2} dt \\
\leq C \int_0^{T_0} \|\tilde{x}_t^a\|_{L^2} \|\rho^R \tilde{x}_t^a\|_{L^2} dt \\
\leq C. \tag{89}
\]

By virtue of the same arguments as those of (59) and (62), one gets
\[
\sup_{0 \leq t \leq T_0} \sqrt{t}\|\nabla \tilde{R}_t^a\|_{L^2} + \int_0^{T_0} \left( \|\nabla \tilde{R}_t^a\|_{L^2} + \|\nabla \tilde{R}_t^a\|_{L^2} \right) dt \leq C. \tag{90}
\]

With the estimates (85)--(90) at hand, we find that the sequence $(\tilde{\rho}^R, \tilde{u}_t^R, \tilde{P}_t, \tilde{\theta}_t)$ converges, up to the extraction of subsequences, to some limit $(\rho, u, P, \theta)$ in the obvious weak sense, that is, as $R \to \infty$, we have
\[
\tilde{\rho}^R \tilde{x} \to \rho \tilde{x}, \quad C(\tilde{B}_N \times [0, T_0]), \text{ for any } N > 0, \tag{91}
\]
\[
\tilde{\rho}^R \tilde{x}_t^a \to \rho \tilde{x}_t^a, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; H^1 \cap W^{1, q}), \tag{92}
\]
\[
\tilde{\theta}_t^2 \tilde{\theta}_t \to \theta \tilde{\theta}_t^2, \quad \text{weakly } \ast \text{ in } L^\infty(0, T_0; L^2), \tag{93}
\]
\[
\nabla^2 \tilde{u}_t^R \to \nabla^2 u, \quad \nabla \tilde{P}_t \to \nabla P, \quad \text{weakly in } L^{q+1} \cap L^2(\mathbb{R}^2 \times (0, T_0)), \tag{94}
\]
\[
\tilde{\theta}_t \to \theta_t, \quad \nabla \tilde{\theta}_t^2 \to \nabla \theta_t^2, \quad \text{weakly in } L^2(\mathbb{R}^2 \times (0, T_0)), \tag{95}
\]
\[
\nabla \tilde{u}_t^R \to \nabla u, \quad \text{weakly in } L^2(0, T_0; L^2), \tag{96}
\]
\[
\sqrt{t} \|\tilde{\rho}^R \tilde{u}_t^R\|_{L^2} \to \sqrt{t} \|\rho \tilde{u}_t\|_{L^2}, \quad \text{weakly in } C(\tilde{B}_N \times [0, T_0]), \text{ for any } N > 0. \tag{97}
\]

with
\[
\rho \tilde{x}_t^a \in L^\infty(0, T_0; L^1), \quad \inf_{0 \leq t \leq T_0} \int_{B_{2N_0}} \rho(x, t) dx \geq \frac{1}{4}. \tag{101}
\]
Then letting $R \to \infty$, standard arguments together with (91)--(101) show that $(\rho, u, P, \theta)$ is a strong solution of (1)--(2) on $\mathbb{R}^2 \times (0, T_0]$ satisfying (8) and (9). Indeed, the existence of a pressure $P$ follows immediately from the (1)$_2$ and (1)$_4$ by a classical consideration. The proof of the existence part of Theorem 3.1 is finished.

It remains only to prove the uniqueness of the strong solutions satisfying (8) and (9). Let $(\rho, u, P, \theta)$ and $(\tilde{\rho}, \tilde{u}, \tilde{P}, \tilde{\theta})$ be two strong solutions satisfying (8) and (9) with the same initial data, and denote

$$\Theta := \rho - \tilde{\rho}, \quad U := u - \tilde{u}, \quad \Phi := \theta - \tilde{\theta}.$$  

First, subtracting the mass equation satisfied by $(\rho, u, P, \theta)$ and $(\tilde{\rho}, \tilde{u}, \tilde{P}, \tilde{\theta})$ gives

$$\Theta_t + u \cdot \nabla \Theta + U \cdot \nabla \rho = 0.$$  

Multiplying (102) by $2\Theta r$ for $r \in (1, \tilde{a})$ with $\tilde{a} = \min\{2, a\}$, and integrating by parts yield

$$\frac{d}{dt} \int |\Theta r|^2 dx \leq C \int u\Theta r^2 dx + C \int |\Theta r|^2 |U\Theta r| dx \leq C \int (1 + |\nabla \tilde{u}|) |\Theta r|^2 dx + C \int |\Theta r|^2 (\|\nabla U\|_{L^2} + \|\sqrt{\rho} U\|_{L^2})$$

due to Sobolev's inequality, (9), (28), and (71). This combined with Gronwall's inequality shows that for all $0 \leq t \leq T_0$,

$$\|\Theta r\|_{L^2} \leq C \int_0^t (\|\nabla U\|_{L^2} + \|\sqrt{\rho} U\|_{L^2}) ds.$$  

Next, subtracting the momentum and temperature equations satisfied by $(\rho, u, P, \theta)$ and $(\tilde{\rho}, \tilde{u}, \tilde{P}, \tilde{\theta})$ leads to

$$\rho U_t + \rho u \cdot \nabla U - \mu \Delta U = -\rho U \cdot \nabla \tilde{u} - \Theta (\tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u}) - \nabla (P - \tilde{P}) + \Theta \theta e_2 + \tilde{\rho} \Phi e_2,$$  

and

$$\Phi_t - \kappa \Delta \Phi = -u \cdot \nabla \Phi - U \cdot \nabla \theta.$$  

Multiplying (104) and (105) by $U$ and $\Phi$, respectively, and adding the resulting equations together, we obtain after integration by parts that

$$\frac{d}{dt} \int (|U|^2 + |\Phi|^2) dx + \int (\mu |\nabla U|^2 + \kappa |\nabla \Phi|^2) dx$$

$$\leq C \int |\nabla \tilde{u}| |U|^2 dx + C \int |\Theta U| (|\tilde{u}_t| + |\tilde{u} | \nabla \tilde{u}) dx$$

$$+ C \int |U| (|\Theta \theta + \tilde{\rho} \Phi|) dx - \int U \cdot \nabla \Theta \cdot \Phi dx$$

$$:= C \int |\nabla \tilde{u}| |U|^2 dx + \int (|U|^2 + |\Phi|^2) dx + \sum_{i=1}^3 K_i.$$  

(106)
We first estimate \( K_1 \). H"older’s inequality combined with (9), (15), (20), and (103) yields that for \( r \in (1, a) \),
\[
K_1 \leq C \| \Theta \bar{x}^r \| _{L^2} \| \bar{u} \| _{L^4} \left( \| \bar{u} \| _{L^4} + \| \nabla \bar{u} \| _{L^\infty} \| \bar{u} \| _{L^4} \right) \\
\leq C \varepsilon \left( \| \sqrt{\rho} \bar{u} \| _{L^2}^2 + \| \nabla \bar{u} \| _{L^2}^2 + \| \sqrt{\rho} \| _{L^\infty} \| \Theta \bar{x}^r \| _{L^2}^2 \right) \\
+ \varepsilon \left( \| \sqrt{\rho} \bar{U} \| _{L^2}^2 + \| \nabla \bar{U} \| _{L^2}^2 \right) \\
\leq C(\varepsilon) \left(1 + t \| \nabla \bar{u} \| _{L^2}^2 + t \| \nabla^2 \bar{u} \| _{L^2}^2 \right) \int_0^t \left( \| \nabla \bar{U} \| _{L^2}^2 + \| \sqrt{\rho} \bar{U} \| _{L^2}^2 \right) ds \\
+ \varepsilon \left( \| \sqrt{\rho} \bar{U} \| _{L^2}^2 + \| \nabla \bar{U} \| _{L^2}^2 \right). \quad (107)
\]

For the term \( K_2 \), we derive from H"older’s inequality, (20), and (55) that
\[
K_2 \leq C \| \Theta \bar{x}^r \| _{L^2} \| \bar{u} \| _{L^4} \| \theta \| _{L^4} \| \bar{x}^r \| _{L^{\infty}} \leq \varepsilon \left( \| \sqrt{\rho} \bar{U} \| _{L^2}^2 + \| \nabla \bar{U} \| _{L^2}^2 \right) + C(\varepsilon) \| \Theta \bar{x}^r \| _{L^2} \\
\leq \varepsilon \left( \| \sqrt{\rho} \bar{U} \| _{L^2}^2 + \| \nabla \bar{U} \| _{L^2}^2 \right) + C(\varepsilon) \int_0^t \left( \| \nabla \bar{U} \| _{L^2}^2 + \| \sqrt{\rho} \bar{U} \| _{L^2}^2 \right) ds. \quad (108)
\]

The last term \( K_3 \) can be estimated as follows
\[
K_3 \leq C \| \bar{u} \| _{L^4} \| \nabla \theta \bar{x}^0 \| _{L^4} \| \nabla \theta \bar{x}^0 \| _{L^4} \| \Phi \| _{L^4} \\
\leq C \left( \| \sqrt{\rho} \bar{U} \| _{L^2} + \| \nabla \bar{U} \| _{L^2} \right) \left( \| \bar{x}^0 \| _{L^4} \| \Phi \| _{L^4} \right) \\
\leq \varepsilon \left( \| \sqrt{\rho} \bar{U} \| _{L^2}^2 + \| \nabla \bar{U} \| _{L^2}^2 \right) + C(\varepsilon) \left( \| \bar{x}^0 \| _{L^4}^2 \| \Phi \| _{L^4}^2 \right) \quad (109)
\]

owing to (9), (15), and (20).

Denoting
\[
G(t) := \| \sqrt{\rho} \bar{U} \| _{L^2}^2 + \| \Phi \| _{L^2}^2 + \int_0^t \left( \| \nabla \bar{U} \| _{L^2}^2 + \| \nabla \Phi \| _{L^2}^2 + \| \sqrt{\rho} \bar{U} \| _{L^2}^2 \right) ds,
\]
then substituting (107)–(109) into (106) and choosing \( \varepsilon \) suitably small lead to
\[
G'(t) \leq C \left(1 + \| \nabla \bar{u} \| _{L^\infty} + \| \nabla \bar{u} \| _{L^2}^2 + t \| \nabla \bar{u} \| _{L^2}^2 + t \| \nabla^2 \bar{u} \| _{L^2}^2 \right) G(t),
\]
which together with Gronwall’s inequality and (8) implies \( G(t) = 0 \). Hence, \( \bar{U}(x, t) = \bar{\Phi}(x, t) = 0 \) for almost everywhere \( (x, t) \in \mathbb{R}^2 \times (0, T) \). Finally, one can deduce from (103) that \( \Theta = 0 \) for almost everywhere \( (x, t) \in \mathbb{R}^2 \times (0, T) \). The proof of Theorem 3.1 is completed. 

4. Proof of Theorem 1.2. This section is devoted to the proof of Theorem 1.2.

4.1. A priori estimates. In this subsection, we will establish some necessary a priori bounds for strong solutions \( (\rho, u, P, \theta) \) to the Cauchy problem (1)–(2) in order to extend the local strong solutions obtained by Theorem 3.1. Thus, let \( T > 0 \) be a fixed time and \( (\rho, u, P, \theta) \) be the strong solution to (1)- (2) on \( \mathbb{R}^2 \times (0, T) \) with initial data \( (\rho_0, u_0, \theta_0) \) satisfying (4) and (6).

In what follows, we will use the convention that \( C \) denotes a generic positive constant depending on \( \mu, \kappa, a, q, \) and the initial data, and use \( C(\alpha) \) to emphasize that \( C \) depends on \( \alpha \).

First, since \( \text{div} \ u = 0 \), we have the following estimate on the \( L^\infty(0, T; L^r) \)-norm of the density.
Lemma 4.1. There exists a positive constant $C$ depending only on $\|\rho_0\|_{L^1 \cap L^\infty}$ such that
\[
\sup_{t \in [0,T]} \|\rho\|_{L^1 \cap L^\infty} \leq C. \tag{110}
\]

Proof. See [17, Theorem 2.1]. \hfill \square

The following lemma concerns the $L^\infty(0,T;L^2)$-norm of the gradients of the velocity and the temperature.

Lemma 4.2. There exists a positive constant $C$ depending on $\mu$, $\kappa$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0} u_0\|_{L^2}$, $\|\nabla u_0\|_{L^2}$, $T$, and $\|\theta_0\|_{H^1}$ such that
\[
\sup_{t \in [0,T]} (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) + \int_0^T (\|\sqrt{\rho} \dot{u}\|^2_{L^2} + \|\Delta \theta\|^2_{L^2}) \, dt \leq C. \tag{111}
\]

Here $\dot{u} := u_t + u \cdot \nabla u$. Furthermore, one has
\[
\sup_{t \in [0,T]} t (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) + \int_0^T t (\|\sqrt{\rho} \dot{u}\|^2_{L^2} + \|\Delta \theta\|^2_{L^2}) \, dt \leq C. \tag{112}
\]

Proof. 1. Multiplying (1)_3 by $\theta$ and integrating over $\mathbb{R}^2$, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int |\theta|^2 \, dx + \kappa \int |\nabla \theta|^2 \, dx = 0. \tag{113}
\]
Thus integrating (113) in $t$ leads to
\[
\sup_{t \in [0,T]} \|\theta\|^2_{L^2} + \kappa \int_0^T \|\nabla \theta\|^2_{L^2} \, dt \leq \|\theta_0\|^2_{L^2}. \tag{114}
\]

Multiplying (1)_2 by $u$ and integrating over $\mathbb{R}^2$, we obtain from Cauchy-Schwarz inequality that
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 \, dx + \mu \int |\nabla u|^2 \, dx = \int \rho \theta e_2 \cdot u \, dx \leq \|\rho\|_{L^\infty} |\sqrt{\rho} u|_{L^2} \|\theta\|_{L^2}. \tag{115}
\]
Then integrating (115) in $t$ together with (110) and (114) implies
\[
\sup_{t \in [0,T]} \|\sqrt{\rho} u\|^2_{L^2} + \int_0^T \|\nabla u\|^2_{L^2} \, dt \leq C. \tag{116}
\]
Combining (114) and (116) yields
\[
\sup_{t \in [0,T]} (\|\sqrt{\rho} u\|^2_{L^2} + \|\theta\|^2_{L^2}) + \int_0^T (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) \, dt \leq C. \tag{117}
\]

2. Multiplying (1)_2 by $\dot{u}$ and integrating the resulting equality over $\mathbb{R}^2$ give rise to
\[
\int \rho |\dot{u}|^2 \, dx = \int \mu \Delta u \cdot \dot{u} \, dx - \int \nabla P \cdot \dot{u} \, dx + \int \rho \theta e_2 \cdot \dot{u} \, dx := I_1 + I_2 + I_3. \tag{118}
\]
It follows from integration by parts and Gagliardo-Nirenberg inequality that

$$I_1 = \int \mu \Delta u \cdot (u_t + u \cdot \nabla u) \, dx$$

$$= -\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 - \mu \int \partial_i u^j \partial_i (u^k \partial_k u^j) \, dx$$

$$\leq -\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^3}^3$$

$$\leq -\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}. \quad (119)$$

Integration by parts together with (1) yields

$$I_2 = -\int \nabla P \cdot (u_t + u \cdot \nabla u) \, dx$$

$$= \int P \partial_j u^t \partial_i u^j \, dx$$

$$\leq C \|P\|_{BMO} \|\partial_j u^t \partial_i u^j\|_{H^1}, \quad (120)$$

where one has used the duality of $H^1$ and $BMO$ (see [24, Chapter IV]) in the last inequality. Since $\text{div}(\partial_j u) = \partial_j \text{div} u = 0$ and $\text{curl}(\nabla u) = 0$, we then derive from Lemma 2.5 and (120) that

$$I_2 \leq \left| \int P \partial_j u^t \partial_i u^j \, dx \right| \leq C \|\nabla P\|_{L^2} \|\nabla u\|_{L^2}^2. \quad (121)$$

For the term $I_3$, by Cauchy-Schwarz inequality, (110), and (117), one has

$$I_3 \leq C \|\rho\|_{L^\infty} \frac{1}{2} \|\sqrt{\rho} u\|_{L^2} \|\theta\|_{L^2} \leq \frac{1}{2} \|\sqrt{\rho} u\|_{L^2}^2 + C. \quad (122)$$

Hence, inserting (119), (121) and (122) into (118) indicates that

$$\mu \frac{d}{dt} \|\nabla u\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^2}^2 \leq C \left( \|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \right) \|\nabla u\|_{L^2}^2 + C. \quad (123)$$

3. Multiplying (1)_3 by $\Delta \theta$ and integrating the resulting equality by parts over $\mathbb{R}^2$, it follows from H"older’s and Gagliardo-Nirenberg inequalities that

$$\frac{d}{dt} \int |\nabla \theta|^2 \, dx + 2\kappa \int |\Delta \theta|^2 \, dx$$

$$\leq C \int |\nabla u| |\nabla \theta|^2 \, dx$$

$$\leq C \|\nabla u\|_{L^3} \|\nabla \theta\|_{L^2} \|\Delta \theta\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} + C (1 + \|\theta\|_{L^2}^2) \|\nabla \theta\|_{L^2}^2 + \kappa \|\Delta \theta\|_{L^2}^2$$

$$\leq C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2} + C \|\nabla \theta\|_{L^2}^2 + \kappa \|\Delta \theta\|_{L^2}^2. \quad (124)$$

which together with (123) gives

$$\frac{d}{dt} \left( \mu \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) + \|\sqrt{\rho} u\|_{L^2}^2 + \kappa \|\Delta \theta\|_{L^2}^2$$

$$\leq C \|\nabla \theta\|_{L^2}^2 + C \left( \|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \right) \|\nabla u\|_{L^2}^2. \quad (125)$$
4. Since \((\rho, u, P, \theta)\) satisfies the following Stokes system
\[
\begin{cases}
-\mu \Delta u + \nabla P = -\rho \dot{u} + \rho \theta e_2, & x \in \mathbb{R}^2, \\
\text{div} u = 0, & x \in \mathbb{R}^2, \\
\mathbf{u}(x) \to 0, & |x| \to \infty,
\end{cases}
\tag{126}
\]
applying the standard \(L^r\)-estimate to (126) yields that for any \(r \geq 2\),
\[
\|\nabla^2 u\|_{L^r} + \|\nabla P\|_{L^r} \leq C\|\rho \dot{u}\|_{L^r} + C\|\rho \theta\|_{L^r},
\tag{127}
\]
where in the last inequality one has used (110).

Then it follows from (125), (127), (110), and (117) that
\[
\frac{d}{dt} (\mu \|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) + \|\sqrt{\rho} \dot{u}\|^2_{L^2} + \kappa \|\Delta \theta\|^2_{L^2}
\leq C\|\nabla \theta\|^4_{L^2} + C\|\nabla u\|^4_{L^2}
\tag{128}
\]
This combined with (117) and Gronwall’s inequality yields (111).

Finally, applying Gronwall’s inequality to (128) multiplied by \(t\), together with (117) gives (112) and finishes the proof of Lemma 4.2.

The following spatial weighted estimate on the density plays an important role in deriving the bounds on the higher order derivatives of the solutions \((\rho, u, P, \theta)\).

**Lemma 4.3.** There exists a positive constant \(C\) depending on \(T\) such that
\[
\sup_{t \in [0, T]} \|\rho \bar{a}^\alpha\|_{L^1} \leq C(T).
\tag{129}
\]

**Proof.** 1. For \(N > 1\), let \(\varphi_N \in C_0^\infty(B_N)\) satisfy
\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = \begin{cases} 1, & |x| \leq N/2, \\
0, & |x| \geq N,
\end{cases} \quad |\nabla \varphi_N| \leq CN^{-1}.
\tag{130}
\]
It follows from (1)_1 that
\[
\frac{d}{dt} \int \rho \varphi_N dx = \int \rho u \cdot \nabla \varphi_N dx
\geq -CN^{-1} \left( \int \rho dx \right)^{\frac{1}{2}} \left( \int |u|^2 dx \right)^{\frac{1}{2}} \geq -\tilde{C}N^{-1}
\tag{131}
\]
owing to (110) and (117). Integrating (131) and choosing \(N = N_1 := 2N_0 + 4\tilde{C}T\), we obtain after using (5) that
\[
\inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho dx \geq \inf_{0 \leq t \leq T} \int_{B_{N_1}} \rho \varphi_{N_1} dx
\geq \int \rho_0 \varphi_{N_1} dx - \tilde{C}N_1^{-1} T
\]
\[
\int_{B_{R_0}} \rho_0 dx \geq \frac{\dot{C}T}{2N_0 + 4\dot{C}} \geq \frac{1}{4}.
\] (132)

Hence, it follows from (132), (110) and (15) that for any \( v \in \tilde{D}^{1,2}(B_R) \),
\[
\| v \tilde{x}^{-\eta} \|_{L^\frac{1}{\eta}} \leq C(\eta, s)(\| \sqrt{\rho} v \|_{L^2} + \| \nabla v \|_{L^2}),
\] (133)
where \( \eta \in (0, 1] \) and \( s > 2 \). In particular, we deduce from (133), (117), and (111) that
\[
\| u \tilde{x}^{-\eta} \|_{L^\frac{1}{\eta}} \leq C(\| \sqrt{\rho} u \|_{L^2} + \| \nabla u \|_{L^2}) \leq C.
\] (134)

2. Multiplying (1)_1 by \( \tilde{x}^2 \) and integrating the resulting equation by parts over \( \mathbb{R}^2 \) yield that
\[
\frac{d}{dt} \int \rho \tilde{x}^a dx \leq C \int \rho |\tilde{x}^{a-1}\log^2(e + |x|^2)| dx
\leq C\int \rho |\tilde{x}^{a-1+\frac{2}{s+a}}\|_{L^{s+a+2}} \| u \tilde{x}^{-\frac{1}{s+a}} \|_{L^{s+a}}
\leq C \int \rho \tilde{x}^a dx + C,
\]
which along with Gronwall’s inequality gives (129) and finishes the proof of Lemma 4.3.

Lemma 4.4. There exists a positive constant \( C \) depending on \( T \) such that
\[
\sup_{t \in [0, T]} t \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \int_0^T t \| \nabla \dot{u} \|_{L^2}^2 dt \leq C(T),
\] (135)
and
\[
\sup_{t \in [0, T]} t \left( \| \nabla^2 u \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 \right) \leq C(T).
\] (136)

Proof. 1. Motivated by [11], operating \( \partial_t + u \cdot \nabla \) to (1)_2, one gets by some simple calculations that
\[
\partial_t(\rho \dot{u}^j) + \text{div}(\rho u \dot{u}^j) - \mu \Delta \dot{u}^j = -\mu \partial_i(\partial_i u \cdot \nabla \dot{u}^j) - \mu \text{div}(\partial_i u \partial_i \dot{u}^j) - \partial_j \partial_i P
- (u \cdot \nabla) \partial_j P + \partial_i(\rho \theta e_i^j) + u \cdot \nabla(\rho \theta e_i^j),
\] (137)
which multiplied by \( \dot{u}^j \), together with integration by parts and (1)_4, leads to
\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx + \mu \int |\nabla \dot{u}|^2 dx
= -\int \mu \partial_i(\partial_i u \cdot \nabla \dot{u}^j) \dot{u}^j dx - \int \mu \text{div}(\partial_i u \partial_i \dot{u}^j) \dot{u}^j dx
- \int (\dot{u} \cdot \nabla) \partial_j P + \dot{u}^j (u \cdot \nabla) \partial_j P) dx
+ \int \dot{u}^j \left( \partial_i(\rho \theta e_i^j) + u \cdot \nabla(\rho \theta e_i^j) \right) dx := \sum_{i=1}^4 J_i.
\] (138)
We estimate each term on the right-hand side of (138) as follows.
2. By the same arguments as in [19, Lemma 3.3], one has

\[ \sum_{i=1}^{3} J_i \leq \frac{d}{dt} \int P \partial_j u^i \partial_i u^j dx + C (\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) + \frac{\mu}{4} \|\nabla \dot{u}\|_{L^2}^2. \]  

(139)

Next, it follows from integration by parts, (1)_{1}, (1)_{3}, (1)_{4}, (110), (111), and (117) that

\[ J_4 = \int \dot{u}_t (-\text{div}(\rho u)\theta e^j_2) dx + \int \dot{u}_t \rho (-u \cdot \nabla \theta + \kappa \Delta \theta) e^j_2 dx - \int \partial_t \dot{u}_t u^i \rho \theta e^j_2 dx \]

\[ = \int \nabla \dot{u}_t \cdot \rho u \theta e^j_2 dx + \kappa \int \rho \dot{u}_t \Delta \theta e^j_2 dx - \int \partial_t \dot{u}_t u^i \rho \theta e^j_2 dx \]

\[ \leq C \int \rho |\nabla \dot{u}_t| |u| \theta dx + C \int \rho |\nabla \Delta \theta| dx \]

\[ \leq C \|\rho\|_{L^2} \|\nabla \dot{u}_t\|_{L^2} \|\nabla u\|_{L^2} \|\theta\|_{L^\infty} + C \|\rho\|_{L^2} \|\nabla \dot{u}_t\|_{L^2} \|\Delta \theta\|_{L^2} \]

\[ \leq \frac{\mu}{4} \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\Delta \theta\|_{L^2}^2 + C. \]  

(140)

Substituting (139) and (140) into (138) gives

\[ \frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}_t|^2 dx + \frac{\mu}{2} \int \|\nabla \dot{u}_t\|^2 dx \leq \frac{d}{dt} \int P \partial_j u^i \partial_i u^j dx + C (\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) \]

\[ + C \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\Delta \theta\|_{L^2}^2 + C. \]  

(141)

Thus, we arrive at

\[ \frac{d}{dt} F(t) + \frac{\mu}{2} \|\nabla \dot{u}_t\|_{L^2}^2 \leq C (\|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4) + C \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\Delta \theta\|_{L^2}^2 + C, \]  

(142)

where

\[ F(t) := \frac{1}{2} \|\nabla \dot{u}_t\|_{L^2}^2 - \int P \partial_j u^i \partial_i u^j dx \]  

(143)

satisfies

\[ \frac{1}{4} \|\nabla \dot{u}_t\|_{L^2}^2 - C \|\nabla u\|_{L^4}^4 - C \|\nabla u\|_{L^4}^2 \leq F(t) \leq C \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \]  

(144)

owing to the following estimate

\[ \left| \int P \partial_j u^i \partial_i u^j dx \right| \leq C (\|\nabla \dot{u}_t\|_{L^2} + \|\rho \theta\|_{L^2}) \|\nabla u\|_{L^2}^2 \]

\[ \leq \frac{1}{4} \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2, \]  

(145)

which is deduced from (121), (127), (110), (117) and Young’s inequality.

3. We next estimate the first term on the right-hand side of (142). It follows from Sobolev’s inequality, (127), Hölder’s inequality, (110), and (117) that

\[ \|P\|_{L^4}^4 + \|\nabla u\|_{L^4}^4 \leq C \left( \|\nabla P\|_{L^2}^4 + \|\nabla^2 u\|_{L^4}^4 \right) \]

\[ \leq C \left( \|\rho \dot{u}_t\|_{L^4}^4 + \|\rho \theta\|_{L^4}^4 \right) \]

\[ \leq C \|\rho\|_{L^2}^2 \|\nabla \dot{u}_t\|_{L^2}^2 + C \|\rho\|_{L^2}^4 \|\theta\|_{L^2}^4 \]

\[ \leq C \|\nabla \dot{u}_t\|_{L^2}^4 + C. \]  

(146)
Thus, putting (146) into (142), together with (144), one has
\[
\frac{d}{dt} F(t) + \frac{\mu}{2} \| \nabla \tilde{u} \|_{L^2}^2 \leq C (\| \sqrt{\rho} \tilde{u} \|_{L^2}^2 + 1) (F(t) + \| \nabla u \|_{L^2}^4 + 1) + C \| \Delta \theta \|_{L^2}^2. \tag{147}
\]
Then, applying Gronwall’s inequality to (147) multiplied by \( t \), it follows from (144), (111), (112), and (117) that
\[
\sup_{t \in [0,T]} (tF(t)) + \int_0^T t \| \nabla \tilde{u} \|_{L^2}^2 dt \\
\leq C \int_0^T F(t) dt + C \int_0^T t (\| \sqrt{\rho} \tilde{u} \|_{L^2}^2 + 1) (\| \nabla u \|_{L^2}^4 + 1) dt + C \int_0^T t \| \Delta \theta \|_{L^2}^2 dt \\
\leq C \int_0^T (\| \sqrt{\rho} \tilde{u} \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) dt + C \sup_{t \in [0,T]} \| \nabla u \|_{L^2}^2 \int_0^T t \| \nabla \tilde{u} \|_{L^2}^2 dt \\
+ C \int_0^T (t \| \sqrt{\rho} \tilde{u} \|_{L^2}^2 + t) dt + C \left( \sup_{t \in [0,T]} \| \nabla u \|_{L^2}^2 \right)^2 \int_0^T t \| \sqrt{\rho} \tilde{u} \|_{L^2}^2 dt \\
+ C \sup_{t \in [0,T]} (t \| \nabla u \|_{L^2}^2) \int_0^T \| \nabla u \|_{L^2}^2 dt + C \int_0^T t \| \Delta \theta \|_{L^2}^2 dt \\
\leq C(T). \tag{148}
\]
This together with (144) and (112) yields the desired result (135), which combined with (127) implies (136). The proof of Lemma 4.4 is completed. □

**Lemma 4.5.** There exists a positive constant \( C \) depending on \( T \) such that
\[
\sup_{t \in [0,T]} \| \rho \|_{H^1 \cap W^{1,q}} + \int_0^T \left( \| \nabla^2 u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^{4+1} + t \| \nabla^2 u \|_{L^2 \cap L^q}^2 \right) dt \\
+ \int_0^T \left( \| \nabla P \|_{L^2}^2 + \| \nabla P \|_{L^2 \cap L^q}^{4+1} + t \| \nabla P \|_{L^2 \cap L^q}^2 \right) dt \leq C(T). \tag{149}
\]

**Proof.** It follows from the mass equation (1)_1 and the incompressibility (1)_4 that \( \nabla \rho \) satisfies for any \( r \geq 2 \),
\[
\frac{d}{dt} \| \nabla \rho \|_{L^r} \leq C \| \nabla u \|_{L^\infty} \| \nabla \rho \|_{L^r}. \tag{150}
\]
One gets from Gagliardo-Nirenberg inequality, (111), and (127) that for \( q > 2 \),
\[
\| \nabla u \|_{L^\infty} \leq C(q) \| \nabla u \|_{L^q}^{q-2} \| \nabla^2 u \|_{L^q}^{q \delta - 1} \|
\leq C \left( \| \rho \tilde{u} \|_{L^q}^{q \delta - 1} + \| \rho \theta \|_{L^q}^{q \delta - 1} \right) \\
\leq C \| \rho \tilde{u} \|_{L^q}^{q \delta - 1} + C. \tag{151}
\]
From (110), (133), and (129), one easily deduces that for any \( \eta \in (0,1) \) and any \( s > 2 \),
\[
\| \rho^\eta v \|_{L^{\frac{4s}{3}}} \leq C \| \rho^\eta \tilde{x}^{\frac{4s}{3}} \|_{L^{\frac{4s}{3}}} \| v \tilde{x}^{-\frac{4s}{3}} \|_{L^{\frac{4s}{3}}} \| \tilde{x} \|_{L^s} \\
\leq C \| \rho \|_{L^{4s-3\eta}} \| \rho \tilde{x}^\eta \|_{L^s} \left( \| \sqrt{\rho} v \|_{L^2} + \| v \|_{L^2} \right) \\
\leq C \left( \| \sqrt{\rho} v \|_{L^2} + \| v \|_{L^2} \right), \tag{152}
\]
which together with Gagliardo-Nirenberg inequality yields that
\[
\|\rho \dot{u}\|_{L^q} \leq C \|ho \dot{u}\|_{L^2}^{\frac{2(q-1)}{q-2}} \|ho \dot{u}\|_{L^2}^{\frac{q-1}{q-2}}
\]
\[
\leq C \|ho \dot{u}\|_{L^2}^{\frac{2(q-1)}{q-2}} \left( \|\sqrt{\rho} \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2} \right)^{\frac{q-2}{q-2}}
\]
\[
\leq C \|\sqrt{\rho} \dot{u}\|_{L^2} + C \|ho \dot{u}\|_{L^2}^{\frac{2(q-1)}{q-2}} \|\nabla \dot{u}\|_{L^2}^{\frac{q-2}{q-2}}. 
\] (153)

This along with (111) and (135) implies that
\[
\int_0^T \left( \|\rho \dot{u}\|_{L^q}^{\frac{q+1}{q}} + t\|\rho \dot{u}\|_{L^q}^2 \right) dt
\]
\[
\leq C \int_0^T \left( \|\sqrt{\rho} \dot{u}\|_{L^2}^2 + t\|\nabla \dot{u}\|_{L^2}^2 + t \frac{q^3-q^2-2q-1}{q^3-q^2-2q^2+1} \right) dt
\]
\[
\leq C. 
\] (154)

The combination of (151) and (154) gives
\[
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C. 
\] (155)

Thus, applying Gronwall’s inequality to (150) yields
\[
\sup_{t \in [0,T]} \|\nabla \rho\|_{L^2 \cap L^q} \leq C(T). 
\] (156)

Finally, it is easy to infer from (127), (154), (111), and (112) that
\[
\int_0^T \left( \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla^2 u\|_{L^2 \cap L^q}^2 \right) dt
\]
\[
+ \int_0^T \left( \|\nabla P\|_{L^2}^2 + \|\nabla P\|_{L^q}^{\frac{q+1}{q}} + t\|\nabla P\|_{L^2 \cap L^q}^2 \right) dt \leq C, 
\] (157)
which along with (110) and (156) gives (149), and finishes the proof of Lemma 4.5. 

Now, we give the following spatial weighted estimate on the gradient of the density, which has been proved in [19, Lemma 3.6]. We omit the detailed proof here for simplicity.

**Lemma 4.6.** There exists a positive constant C depending on T such that
\[
\sup_{t \in [0,T]} \|\rho \dot{x}\|_{H^1 \cap W^{1,q}} \leq C(T). 
\] (158)

Next, we shall show the following spatial weighted estimates of \( \theta \) and \( \nabla \theta \), which are crucial to derive the estimates on the gradients of both \( u \) and \( \theta \).

**Lemma 4.7.** There exists a positive constant C depending on T such that
\[
\sup_{t \in [0,T]} \|\theta \dot{x}\|_{L^2} + \int_0^T \|\nabla \theta \dot{x}\|_{L^2}^2 dt \leq C(T), 
\] (159)

and
\[
\sup_{t \in [0,T]} t\|\nabla \theta \dot{x}\|_{L^2}^2 + \int_0^T t\|\Delta \theta \dot{x}\|_{L^2}^2 dt \leq C(T). 
\] (160)
Proof. 1. Multiplying (1)_3 by $\theta \bar{x}^a$ and integrating the resulting equality by parts over $\mathbb{R}^2$ indicate that
\[
\frac{1}{2} \frac{d}{dt} \|\theta \bar{x}^2\|_{L^2}^2 + \kappa \|\nabla \theta \bar{x}^2\|_{L^2}^2 = \frac{\kappa}{2} \int |\theta|^2 \Delta \bar{x}^a \, dx + \frac{1}{2} \int |\theta|^2 u \cdot \nabla \bar{x}^a \, dx
\]
\[
:= \tilde{J}_1 + \tilde{J}_2, \tag{161}
\]
where
\[
\tilde{J}_1 \leq C \int |\theta|^2 \bar{x}^a \bar{x}^{-2} \log^4 (e + |x|^2) \, dx \leq C \|\theta \bar{x}^2\|_{L^2}^2,
\]
\[
\tilde{J}_2 \leq C \|\theta \bar{x}^2\|_{L^4} \|\theta \bar{x}^2\|_{L^2} \|\bar{x}^{-\frac{2}{3}}\|_{L^4} \leq C \|\theta \bar{x}^2\|_{L^2}^2 + \frac{\kappa}{2} \|\nabla \theta \bar{x}^2\|_{L^2}^2,
\]
due to Gagliardo-Nirenberg inequality, (111), and (134). Then, substituting the above estimates into (161), together with Gronwall’s inequality, gives (159).

2. Multiplying (1)_3 by $\Delta \theta \bar{x}^a$ and integrating the resultant equality by parts over $\mathbb{R}^2$ lead to
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla \theta|^2 \bar{x}^a \, dx + \kappa \int |\Delta \theta|^2 \bar{x}^a \, dx
\]
\[
\leq C \int |\nabla \theta|^2 |u| |\nabla \bar{x}^a| \, dx + C \int |\nabla \theta| |\Delta \theta| \bar{x}^a \, dx + C \int |\nabla u| |\nabla \theta|^2 \bar{x}^a \, dx
\]
\[
:= \sum_{i=1}^{3} \tilde{J}_i. \tag{162}
\]
Applying Gagliardo-Nirenberg inequality, (111), (134), and (159), it holds
\[
\tilde{J}_1 \leq C \|\nabla \theta\|^{\frac{2}{3}} \|\bar{x}^{-\frac{1}{3}}\|_{L^6} \|\bar{x}^{-\frac{2}{3}}\|_{L^6} \|\nabla \theta\|_{L^6} \|\nabla \theta\|_{L^6},
\]
\[
\leq C \|\nabla \theta\|^{\frac{2}{3}} \|\bar{x}^{-\frac{1}{3}}\|_{L^6} \|\nabla \theta\|_{L^6}^2 + C \|\nabla \theta\|_{L^6}^2,
\]
\[
\tilde{J}_2 \leq C \|\Delta \theta\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2,
\]
\[
\tilde{J}_3 \leq C \|\nabla u\|_{L^\infty} \|\nabla \theta\|_{L^2}^2 \leq C \left(1 + \|\nabla^2 u\|_{L^2}^{\frac{q+1}{q}}\right) \|\nabla \theta\|_{L^2}^2.
\]
Inserting the above estimates into (162) implies that
\[
\frac{d}{dt} \|\nabla \theta \tilde{x}^2\|_{L^2}^2 + \kappa \|\Delta \theta \tilde{x}^2\|_{L^2}^2 \leq C \left(1 + \|\nabla^2 u\|_{L^q}^{\frac{q+1}{q}}\right) \|\nabla \theta \tilde{x}^2\|_{L^2}^2, \tag{163}
\]
which multiplied by $t$, together with Gronwall’s inequality, (149), and (159) yields (160). The proof of Lemma 4.7 is finished.

**Lemma 4.8.** There exists a positive constant $C$ depending on $T$ such that
\[
\sup_{t \in [0, T]} t \left(\|\sqrt{\rho} u\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 + \int_0^T (t\|\nabla u_t\|_{L^2}^2 + t\|\nabla \theta_t\|_{L^2}^2) \, dt\right)
\]
\[
\leq C(T). \tag{164}
\]
**Proof.** 1. It follows from (152), (117), (111), and (134) that for any $\eta \in (0, 1]$ and any $s > 2$,
\[
\|\rho^\eta u\|_{L^{s/\eta}} + \|u \tilde{x}^{-\eta}\|_{L^{s/\eta}} \leq C. \tag{165}
\]
Next, we will prove the following estimate

$$\sup_{t \in [0,T]} (\|u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 + \|\Delta \theta\|_{L^2}^2) \, dt \leq C. \tag{166}$$

With (134) at hand, we need only to show

$$\int_0^T (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) \, dt \leq C. \tag{167}$$

Indeed, due to Gagliardo-Nirenberg inequality and (165), one has

$$\|\sqrt{\rho} u_t\|_{L^2}^2 \leq \|\sqrt{\rho} u\|_{L^2}^2 + \|\sqrt{\rho} u\|_{L^2}^2 \\| \nabla u\|_{L^2}^2 \leq \|\sqrt{\rho} u\|_{L^2}^2 + C \|\sqrt{\rho} u\|_{L^2}^2 \| \nabla u\|_{L^2}^2 \leq \|\sqrt{\rho} u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \|\nabla^2 u\|_{L^2}^2. \tag{168}$$

On the other hand, (13)_3 combined with Gagliardo-Nirenberg inequality leads to

$$\|\theta_t\|_{L^2}^2 \leq C \|\Delta \theta\|_{L^2}^2 + \|u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \leq C \|\Delta \theta\|_{L^2}^2 + C \|\nabla \theta \|_{L^2}^2, \tag{169}$$

where in the last inequality one has used

$$\|u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \leq C \|u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \leq C \|u\|_{L^2} \|\nabla \theta\|_{L^2}^2 \leq C \|\nabla^2 \theta\|_{L^2}^2 + C \|\nabla \theta \|_{L^2}^2 \tag{170}$$

owing to (165), Gagliardo-Nirenberg inequality, and (111). Hence, (167) is a direct consequence of (168), (169), (111), (149), and (159).

2. Differentiating (1)_2 with respect to $t$ gives

$$\rho u_t + \rho u \cdot \nabla u - \mu \Delta u_t + \nabla P_t = -\rho u_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u + (\rho \theta e_2)_t. \tag{171}$$

Multiplying (171) by $u_i$ and integrating the resulting equality by parts over $\mathbb{R}^2$, we obtain after using (1)_1 and (1)_4 that

$$\frac{1}{2} \int \frac{d}{dt} \rho |u_t|^2 \, dx + \mu \int |\nabla u_t|^2 \, dx \leq C \int \rho |u||u_t|(|\nabla u_t| + |\nabla u|) + |u||\nabla^2 u|) \, dx \tag{172}$$

$$+ C \int \rho |u_t|^2 |\nabla u| \, dx + \int \rho \theta e_2 \cdot u_t \, dx + \int \rho \theta e_2 \cdot u_t \, dx := \sum_{i=1}^5 J_i.$$

We estimate each term on the right-hand side of (172) as follows.

First, it follows from Hölder’s inequality, (165), (152), Garliardo-Nirenberg inequality, and (111) that

$$J_1 \leq C \|\sqrt{\rho} u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}$$

$$+ C \|\rho u\|_{L^6} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla^2 u\|_{L^2} \leq C \|\sqrt{\rho} u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla u\|_{L^2} + C \left(1 + \|\sqrt{\rho} u_t\|_{L^2} + \|\nabla^2 u\|_{L^2} \right). \tag{173}$$
Next, Hölder’s inequality, (165), and (111) imply
\[
\bar{J}_2 + \bar{J}_3 \leq C\|\sqrt{\rho}u\|_{L^2}^2 \|\nabla u\|_{L^4} \|\nabla u_i\|_{L^2} + \|\nabla u\|_{L^2} \|\sqrt{\rho}u_i\|_{L^2} + \|\nabla u\|_{L^2} \|\sqrt{\rho}u_i\|_{L^2} \frac{3}{2} \|\nabla u\|_{L^2} \frac{1}{2} \\
\leq \frac{\mu}{6} \|\nabla u_i\|_{L^2}^2 + C (1 + \|\sqrt{\rho}u_i\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2).
\] (174)

For the term $\bar{J}_4$, (1) together with integration by parts, (165), (111), (117), and Garliardo-Nirenberg inequality, we obtain that
\[
\bar{J}_4 = \int \rho u \cdot \nabla (\theta e_2 \cdot u_i) \, dx \\
\leq \int \rho |u| |\nabla \theta| |u_i| \, dx + \int \rho |u| \theta |\nabla u_i| \, dx \\
\leq \|\sqrt{\rho}u_i\|_{L^2} \|\nabla u_i\|_{L^4} \|\theta\|_{L^4} + \|\nabla u_i\|_{L^2} \|\rho u\|_{L^4} \|\theta\|_{L^4} \\
\leq \frac{\mu}{6} \|\nabla u_i\|_{L^2}^2 + C (1 + \|\sqrt{\rho}u_i\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2).
\] (175)

The last term $\bar{J}_5$ can be bounded from Hölder’s inequality and (110),
\[
\bar{J}_5 \leq \|\rho\|_{L^\infty} \|\sqrt{\rho}u_i\|_{L^2} \|\theta_i\|_{L^2} \leq C \|\sqrt{\rho}u_i\|_{L^2}^2 + C \|\theta_i\|_{L^2}^2. 
\] (176)

Substituting (173)–(176) into (172) gives
\[
\frac{d}{dt} \|\sqrt{\rho}u_i\|_{L^2}^2 + \mu \|\nabla u_i\|_{L^2}^2 \\
\leq C (\|\sqrt{\rho}u_i\|_{L^2}^2 + \|\theta_i\|_{L^2}^2) + C (1 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2).
\] (177)

3. Differentiating (1) with respect to $t$ shows
\[
\theta_{tt} + u_t \cdot \nabla \theta + u \cdot \nabla \theta_t = \kappa \Delta \theta_t.
\] (178)

Multiplying (178) by $\theta_t$ and integrating the resulting equality by parts over $\mathbb{R}^2$, it follows from (1) by Hölder’s inequality, (133), (159), (111), and (117) that
\[
\frac{1}{2} \frac{d}{dt} \int \theta_t^2 \, dx + \kappa \int |\nabla \theta_t|^2 \, dx = - \int \theta_t u_i \cdot \nabla \theta dx - \int \theta_t u \cdot \nabla \theta_t dx \\
= \int \nabla \theta_t \cdot u_t \theta dx - \frac{1}{2} \int u \cdot \nabla \theta_t^2 dx \\
= \int \nabla \theta_t \cdot u_t \theta dx \\
\leq \frac{\kappa}{2} \|\nabla \theta_t\|_{L^2}^2 + C \|u_i\|_{L^2} \|\theta\|_{L^2}^2 \\
\leq \frac{\kappa}{2} \|\nabla \theta_t\|_{L^2}^2 + C \|u_i x^{-\frac{3}{2}}\|_{L^2}^2 \|\theta x^{\frac{3}{2}}\|_{L^2} \\
\leq \frac{\kappa}{2} \|\nabla \theta_t\|_{L^2}^2 + C \|\nabla u_i\|_{L^2}^2 + C \|\sqrt{\rho}u_i\|_{L^2}^2,
\]
which gives
\[
\frac{d}{dt} \|\theta_t\|_{L^2}^2 + \kappa \|\nabla \theta_t\|_{L^2}^2 \leq C_1 \|\nabla u_i\|_{L^2}^2 + C \|\sqrt{\rho}u_i\|_{L^2}^2.
\] (179)
Moreover, it follows from (129), (149), (158), and [17, Lemma 2.3] that where one has used the standard embedding (136), and (164) that for any $q > 0$.

First, for any $0 < \tau < T$, we have

$$\frac{d}{dt}(\mu^{-1}(C_1 + 1)\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \kappa\|\nabla \theta_t\|_{L^2}^2 \leq C (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) + C \left(1 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2\right),$$

which multiplied by $t$, together with Gronwall’s inequality, (166), (149), and (160) leads to

$$\sup_{t \in [0, T]} t (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2) + \int_0^T (t\|\nabla u_t\|_{L^2}^2 + t\|\nabla \theta_t\|_{L^2}^2) dt \leq C(T).$$

Finally, it follows from (1) and (170) that

$$\|\nabla^2 \theta\|_{L^2}^2 \leq C\|\theta_t\|_{L^2}^2 + C\|u\|\nabla \theta\|_{L^2}^2 \leq C\|\theta_t\|_{L^2}^2 + \frac{1}{2}\|\nabla^2 \theta\|_{L^2}^2 + C\|\nabla \theta x \parallel_{L^2}^2.$$

Thus, by (160) and (181), we get

$$\sup_{t \in [0, T]} t\|\nabla^2 \theta\|_{L^2}^2 \leq C \sup_{t \in [0, T]} (t\|\theta_t\|_{L^2}^2 + t\|\nabla \theta x \parallel_{L^2}^2) \leq C,$$

which combined with (181) indicates (164) and finishes the proof of Lemma 4.8. \hfill $\Box$

4.2. Proof of Theorem 1.2. With all the a priori estimates in subsection 4.1 at hand, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. By Theorem 3.1, there exists a $T_0 > 0$ such that the problem (1)-(2) has a unique strong solution $(\rho, u, P, \theta)$ on $\mathbb{R}^2 \times (0, T_0]$. Now, we will extend the local solution to all time.

Set

$$T^* = \sup \{T \mid (\rho, u, P, \theta) \text{ is a strong solution on } \mathbb{R}^2 \times (0, T] \}. \hfill (183)$$

First, for any $0 < \tau < T_0 < T \leq T^*$ with $T$ finite, one deduces from (111), (117), (136), and (164) that for any $q \geq 2$,

$$\nabla u, \nabla \theta, \theta \in C([\tau, T]; L^2 \cap L^q), \hfill (184)$$

where one has used the standard embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C(\tau, T; L^q) \text{ for any } q \in [2, \infty).$$

Moreover, it follows from (129), (149), (158), and [17, Lemma 2.3] that

$$\rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}). \hfill (185)$$

Finally, we claim that

$$T^* = \infty. \hfill (186)$$

Otherwise, if $T^* < \infty$, it follows from (184), (185), (111), (117), (158), and (159) that

$$(\rho, u, \theta)(x, T^*) = \lim_{t \to T^*} (\rho, u, \theta)(x, t)$$

satisfies the initial conditions (6) at $t = T^*$. Thus, taking $(\rho, u, \theta)(x, T^*)$ as the initial data, Theorem 3.1 implies that one could extend the local strong solutions
beyond $T^*$. This contradicts the assumption of $T^*$ in (183). The proof of Theorem 1.2 is completed.

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