On the Coulomb corrections to the total cross section of the interaction of the $\pi^+\pi^-$ atom with ordinary atoms at high energy.

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Abstract:

The size of $\pi^+\pi^-$ atom in the low lying states is considerably smaller than the radius of atomic screening. Due to that we can neglect this screening calculating the contribution of multi-photon exchanges. We obtain the analytic formula for Coulomb corrections which works with a very good accuracy for the ground state of $\pi^+\pi^-$ atom.
1. The proposed accurate measurement of the $\pi^+\pi^-$ atom (dimesoatom) lifetime in the experiment DIRAC would give important information about the low energy QCD dynamics. The process of the breakup (ionization) of the dimesoatom due to its electromagnetic interaction with the target matter (ordinary atoms) is important for this experiment.

The cross section of this process is related to the imaginary part of the forward scattering amplitude. At high energy it is given by the diagrams of Fig.1 with the even number $N = 2n$ of photon exchanges in the t-channel.

![Feynman diagrams](image)

Figure 1: Feynman diagrams relevant for the electromagnetic breakup or relativistic dimesoatom: a) Born approximation, b) multiphoton contributions.

In the Born approximation this process was considered in [2]. The contributions of the diagrams with multiphoton exchanges $n > 1$, Fig.1(b), is important for an atom having large $Z$ due to the strong Coulomb field of the nucleus. The expansion parameter $\nu = Z\alpha$ is not small in the case of the atom with large charge $Z$, therefore the contribution of all diagrams at Fig.1 should be summed to achieve the exact result.

When our paper was under preparation we learned about the paper [3] in which a similar ideas about the dominance of small transverse distances region to Coulomb corrections were proposed. Technically our method is different from that one used in [3], it looks for us more straightforward. Here we reproduce the result of [3], see our eqs.(12-15). Also we calculate the first correction to this result, see eq.(29). It gives a possibility to estimate the accuracy of analytical approach and expand its applicability region from ground to low lying states of dimesoatom.

Let us divide the total cross section into two parts

$$\sigma = \sigma^{Born} + \sigma^{Coulomb},$$

(1)
where $\sigma^{\text{Born}}$ is given by the contribution of the lowest order two photon exchange amplitude only, see Fig.1(a), $\sigma^{\text{Coulomb}}$ describes the contribution of the diagrams Fig.1(b) with $n > 1$ and their interference with the Born amplitude, Fig.1(a).

It is known that the sum of type Fig.1 diagrams in high energy kinematics is equivalent to the eikonal approximation. Recently this process was considered in the eikonal approach in [4]. In our note we would like to emphasize the physical picture underlying this process. Namely we discuss the hierarchy of transverse distances relevant for various contributions to the cross section: $\sigma^{\text{Born}}$ and $\sigma^{\text{Coulomb}}$. In contrast to Born part, Coulomb correction receives the main contribution from the small distances where the electromagnetic field of atom is determined by the Coulomb field of nuclei. Therefore $\sigma^{\text{Coulomb}}$ can be calculated with high precision since the details of atomic screening in this case are of a small importance. Our consideration will be very similar to that in [5], where the closely related problem, lepton pair photoproduction in a strong Coulomb field, was considered. We derive the analytical formula which gives with good accuracy $\sigma^{\text{Coulomb}}$ for the ground state and describes qualitatively low lying dimesoatom states.

Electromagnetic field of atom consists of the field of nucleus and the field of electron shell. The nucleus field is screened by the electron shell at large distances

$$r \sim r_A = \frac{1}{m_e \alpha Z^{1/3}}.$$  \hspace{1cm} (2)

At very small distances

$$r \leq r_N = \frac{1}{\Lambda}, \Lambda \approx 30 \text{ MeV},$$  \hspace{1cm} (3)

electromagnetic field depends on the distribution of electric charge inside nucleus. There is very broad region

$$r_N < r < r_A$$  \hspace{1cm} (4)

where electromagnetic field of atom coincides with the Coulomb law $Z^2/r$.

The other important for our problem dimensional parameter is the distance between $\pi^+$ and $\pi^-$ in dimesoatom, which is in a good approximation positronium–like weakly bound state,

$$r_{2\pi} \sim \frac{2}{m_{\pi}\alpha}.$$  \hspace{1cm} (5)

It is important to note that this parameter lies in between of the parameters describing atomic and nuclear screening

$$r_N << r_{2\pi} << r_A.$$  \hspace{1cm} (6)

Let us discuss now the relevant transverse momenta $<k_i>$ in the integrals describing the diagrams at Fig.1, and therefore, the important for our process
transverse distances \( r \sim 1/ < k_i > \) between nucleus and high energy dimesoatom. These \( < k_i > \) are different for the various contributions to the cross section. Calculating Born part \( \sigma^{\text{Born}} \) we meet the logarithmic-type integral collecting from the region

\[
\frac{1}{r_A} \leq < k_{1,2} > \leq \frac{1}{r_{2\pi}} .
\]

Therefore large distances, \( r \sim r_A \), where the Coulomb field of nucleus is screened by the electron shell give sizeable contribution to the \( \sigma^{\text{Born}} \). On the other hand \( < k_i > \sim \frac{1}{r_{2\pi}} \) in the case of Coulomb contribution to the cross section and hence the typical transverse distances for \( \sigma^{\text{Coulomb}} \) are of order of the dimesoatom size \( r_{2\pi} \). Since there is a large gap between this size and \( r_A \) and \( r_N \) in calculation of \( \sigma^{\text{Coulomb}} \) we can safely neglect the nuclear screening and with a good approximation the atomic screening. The accuracy of this approximation will be discussed later.

2. We start with eq.(1) of [4], see also similar equations (48),(52) in [5],

\[
\sigma_{nlm} = 2 \Re \int d^2b d^3r |\psi_{nlm}(\vec{r})|^2 \left[ 1 - \exp \left( i\chi(\vec{b} - \vec{s}/2) - i\chi(\vec{b} + \vec{s}/2) \right) \right].
\]

(8)

Here \( \vec{s} = \vec{r}_\perp \) is the projection of the vector \( \vec{r} \) on the plane perpendicular to the collision axis, the impact parameter of dimesoatom is \( \vec{b} \), \( \psi_{nlm}(\vec{r}) \) is the wave function of \( \pi^+ \pi^- \) atom in the state with principal, orbital and magnetic quantum numbers \( n, l \) and \( m \) respectively. The phase shift \( \chi(\vec{b}) \) is expressed via potential of the target atom:

\[
\chi(\vec{b}) = \int \limits_{-\infty}^{\infty} U(\sqrt{b^2 + z^2}) \, dz .
\]

(9)

The difference between the phase shifts of \( \pi^+ \) and \( \pi^- \) can be easily calculated in the case of Coulomb potential. Therefore

\[
\sigma_{nlm} = 2 \Re \int d^2b d^3r |\psi_{nlm}(\vec{r})|^2 \left[ 1 - \left( \frac{(\vec{b} - \vec{s}/2)^2}{(\vec{b} + \vec{s}/2)^2} \right)^{iv} \right].
\]

(10)

This integral becomes convergent after the subtraction of the Born contribution which is divergent in the case of unscreened Coulomb potential.

\[
\sigma_{nlm}^{\text{Coulomb}} = 2 \Re \int d^2b d^3r |\psi_{nlm}(\vec{r})|^2 \left[ 1 - \left( \frac{(\vec{b} - \vec{s}/2)^2}{(\vec{b} + \vec{s}/2)^2} \right)^{iv} - \frac{v^2}{2} \ln^2 \left( \frac{(\vec{b} - \vec{s}/2)^2}{(\vec{b} + \vec{s}/2)^2} \right) \right].
\]

(11)

After the substitution \( \vec{b} \rightarrow s(\vec{R} - \vec{n}) \), where \( \vec{n} = \frac{\vec{s}}{s} \), the integral in (11) factorizes, see also [5].

\[
\sigma_{nlm}^{\text{Coulomb}} = < s^2 > I_\nu ,
\]

(12)
\[ I_\nu = \int d^2 \vec{R} \left\{ 2 - \left( \frac{R^2}{(\vec{R} - \vec{n})^2} \right)^{i\nu} - \left( \frac{R^2}{(\vec{R} - \vec{n})^2} \right)^{-i\nu} - \nu^2 \ln \left( \frac{R^2}{(\vec{R} - \vec{n})^2} \right) \right\} \]  

(13)

and

\[ <s^2> = \int d^3 r r^2 \sin^2 \Theta |\psi_{nlm}(\vec{r})|^2 = <r^2>_{(n,l)} <\sin^2 \Theta>_{(l,m)} \]  

(14)

The integral \( I_\nu \) was calculated in [5]

\[ I_\nu = -4\pi \nu^2 f(\nu) , \quad f(\nu) = \frac{1}{2}\left[ \Psi(1 - i\nu) + \Psi(1 + i\nu) - 2\Psi(1) \right] , \]  

(15)

where \( \Psi(z) = d(\ln \Gamma(z))/dz \). Note that the dependence of the \( \sigma^{\text{Coulomb}} \) on \( Z \) factorizes from the variables describing the state of \( \pi^+\pi^- \) atom. It is given by the universal function \( f(\nu) \).

\[ <r^2>_{(n,l)} \) for the positronium–like states is given by [6]

\[ <r^2>_{(n,l)} = \left( \frac{2}{m_\pi \alpha} \right)^2 \frac{n^2}{2} \left[ 5n^2 + 1 - 3l(l + 1) \right] \]  

(16)

We will consider for simplicity the cross section averaged over the magnet quantum number:

\[ \sigma_{nl} = \frac{1}{2l + 1} \sum_m \sigma_{nlm} . \]  

(17)

In this case

\[ <\sin^2 \Theta> = 2/3 . \]  

(18)

Taking into account all factors we find the result

\[ \sigma_{nl} = \sigma^{\text{Born}}_{nl} + \sigma^{\text{Coulomb}}_{nl} \]

\[ \sigma^{\text{Coulomb}}_{nl} = -\frac{16\pi \nu^2 f(\nu)}{m_\pi^2 \alpha^2} \frac{n^2}{3} \left[ 5n^2 + 1 - 3l(l + 1) \right] . \]  

(19)

3. Note that \( \sigma^{\text{Coulomb}} \) is proportional to \( r_{2\pi}^2 \), the mean square of the distance between \( \pi^+ \) and \( \pi^- \) in the dimesoatom. \( r_{2\pi}^2 \) grows rapidly, \( \sim n^4 \), with increasing \( n \) for weakly bounded dimesoatom. At \( n \sim 4 \) the distance between \( \pi^+\pi^- \) becomes of the order of the radius of atomic screening \( r_A \) for atom with large \( Z \). Therefore our approach based on the large difference between \( r_{2\pi} \) and \( r_A \) can not be applied to the highly excited states of dimesoatom.

The first correction related to appearance of the atomic screening can also be calculated analytically. In order to evaluate it let us replace the denominator of the photon propagator by

\[ \frac{1}{k^2} \rightarrow \frac{1}{k^2 + \mu^2} , \]  

(20)

5
where $\mu$ is the inverse of the radius of atomic screening $r_A$. This replacement leads to the analog of the formula (11) having now the form

$$
\sigma^\text{Cond.+Atom.Scr.}_{nlm} = 2 \text{Re} \int d^3b \ d^3r |\psi_{nlm}(\vec{r})|^2 \left[ 1 - e^{2i\nu[K_0(\mu|\vec{b} - \vec{s}|) - K_0(\mu|\vec{b} + \vec{s}|)]} \right] - 2\nu^2[K_0(\mu|\vec{b} - \vec{s}/2|) - K_0(\mu|\vec{b} + \vec{s}/2|)]^2,
$$

(21)

where $K_0(z)$ is the modified Bessel function.

The first correction due to the atomic screening can be obtained from eq.(21) by taking into account that for small values of $\mu$

$$
K_0(\mu|\vec{b} - \vec{s}/2|) - K_0(\mu|\vec{b} + \vec{s}/2|) = \ln \frac{|\vec{b} - \vec{s}/2|}{|\vec{b} + \vec{s}/2|} + \mu^2 4 \left[ (|\vec{b} - \vec{s}/2|^2 - |\vec{b} + \vec{s}/2|^2)\Psi(2) + |\vec{b} + \vec{s}/2|^2 \ln \frac{\mu |\vec{b} - \vec{s}/2|}{2} - |\vec{b} - \vec{s}/2|^2 \ln \frac{\mu |\vec{b} + \vec{s}/2|}{2} \right],
$$

(22)

and keeping terms proportional to $\mu^2$. In this way we obtain

$$
\Delta \sigma^\text{Atom.Scr.}_{nlm} = -2\mu^2 \text{Re} \int d^3b \ d^3r |\psi_{nlm}(\vec{r})|^2 \left[ \frac{\nu}{2} e^{2i\nu \ln \frac{|\vec{b} + \vec{s}/2|}{|\vec{b} - \vec{s}/2|}} + \frac{\mu^2}{2} \ln \frac{\mu |\vec{b} + \vec{s}/2|}{|\vec{b} - \vec{s}/2|} \right] \cdot

\left[ (|\vec{b} - \vec{s}/2|^2 - |\vec{b} + \vec{s}/2|^2)\Psi(2) + |\vec{b} + \vec{s}/2|^2 \ln \frac{\mu |\vec{b} - \vec{s}/2|}{2} - |\vec{b} - \vec{s}/2|^2 \ln \frac{\mu |\vec{b} + \vec{s}/2|}{2} \right].
$$

(23)

The dominant contribution to this expression comes from the region of values of \( \vec{b} \) being much larger than the size of the dimesoatom described by \( \vec{s} = \vec{r}_\perp \). In this limit we obtain

$$
\Delta \sigma^\text{Atom.Scr.}_{nlm} \approx \frac{8}{3} \mu^2 \nu^4 \int d^3r \int d^3b |\psi_{nlm}(\vec{r})|^2 \frac{(\vec{b} \vec{s})^4}{(|\vec{b}|^2)} \left[ \Psi(2) - \frac{1}{2} \ln \frac{\mu^2 |\vec{b}|^2}{4} - \frac{1}{2} \right],
$$

(24)

and consequently with the logarithmic accuracy we can write

$$
\Delta \sigma^\text{Atom.Scr.}_{nlm} \approx \pi \mu^2 \nu^4 \int d^3r |\psi_{nlm}(\vec{r})|^2 r^4 \sin^4 \Theta \ln^2(\mu r).
$$

(25)

This result can be generalized to the case in which there appear a sum of several potentials, i.e. when instead of eq.(21) we perform the substitution

$$
\frac{1}{k^2} \rightarrow \sum_{i} \frac{c_i}{k^2 + \mu_i^2}, \quad \sum_{i} c_i = 1.
$$

(26)
This generalization includes, in particular, the Molière [7] parametrization of the Thomas-Fermi potential. In this case the eq.(25) generalizes to the formula
\[
\Delta \sigma_{\text{Atom.Scr.}}^{nlm} \approx \pi \nu^4 \int d^3r \left| \psi_{nlm}(\vec{r}) \right|^2 r^4 \sin^4 \Theta \sum_{i} \frac{1}{4} c_i \mu_i^2 \ln^2(\mu_i^2 r^2).
\]
(27)

Since the derivation of eq.(25) was performed with the logarithmic accuracy we are free to put in the argument of logarithmic function in eq.(27) some average value \(\bar{\mu}^2\) of the square of masses \(\mu_i\). We choose as \(\bar{\mu}^2\) the quantity
\[
\bar{\mu}^2 = \frac{1}{N} \sum_{i} c_i \mu_i^2.
\]
(28)

In this way we arrive to the final form of correction related to the atomic screening averaged over the magnet quantum number \(m\) (see eq.(17))
\[
\Delta \sigma_{\text{Atom.Scr.}}^{nl} \approx \frac{1}{4\pi \nu^4} \left( \sum_{i} c_i \mu_i^2 \right) \frac{1}{2l+1} \sum_{m=-l}^{l} \int d^3r \left| \psi_{nlm}(\vec{r}) \right|^2 r^4 \ln^2(\bar{\mu}^2 r^2) \sin^4 \Theta
\]
\[
\approx \frac{2}{15} \pi \nu^4 \left( \sum_{i} c_i \mu_i^2 \right) <r^4>_{n,l} \ln^2(\bar{\mu}^2 <r^2>_{n,l})
\]
(29)

The cross section under consideration is given by the sum of terms
\[
\sigma_{nl} = \sigma_{nl}^{\text{Born}} + \sigma_{nl}^{\text{Coulomb}} + \Delta \sigma_{\text{Atom.Scr.}}^{nl},
\]
(30)

where \(\sigma_{nl}^{\text{Coulomb}}\) is given by eqs. (19) and (15), and \(\Delta \sigma_{\text{Atom.Scr.}}^{nl}\) is given by eq. (29).

Let us now compare the results of numerical calculations for \((\sigma_{nl}^{\text{Born}} - \sigma_{nl})/\sigma_{nl}^{\text{Born}}\) in the case of Tantalum (\(Z = 73\)) presented on Fig.2 of Ref. [4] with the analytic results obtained above. The values of \(\sigma_{nl}^{\text{Born}}\) for different values of \(n\) and \(l = 0\) we take from the first column of the Table in Ref. [4]. The values of constants \(c_i\) and masses \(\mu_i\) are equal to those in Ref. [4] and read
\[
c_1 = 0.35 \quad c_2 = 0.55 \quad c_3 = 0.1
\]
\[
\mu_1 = 0.3 \mu_0 \quad \mu_2 = 1.2 \mu_0 \quad \mu_3 = 6 \mu_0.
\]
(31)

Table 1 shows the result of this comparison.

We see that for the ground state of dimesoatom the first atomic screening correction reduces practically to zero the difference between result of the numerical calculation of Ref. [4] and our analytic result (19). For the exited states this correction leads to decreasing of this difference but the precision is low. Consequently, for those states the analytic approach based on eqs. (19) or (30) works only qualitatively.
Table 1: Comparison of the numerical results of Ref.[4] with the analytic results.

| $n$, $l$ | Fig.2 of Ref.[4] | eq.(19) | eq.(30) |
|----------|-------------------|---------|---------|
| $n=1$, $l=0$ | 0.082 | 0.0851 | 0.08212 |
| $n=2$, $l=0$ | 0.109 | 0.1321 | 0.1265 |
| $n=3$, $l=0$ | 0.128 | 0.1962 | 0.1961 |
| $n=4$, $l=0$ | 0.138 | 0.2846 | 0.1812 |

4. We want still to add two additional remarks. First, let us note that the nuclear screening can be really safely neglected since the corresponding relative accuracy is very high

$$\sim \left(\frac{r_N}{r_{2\pi}}\right)^2 < 10^{-4}.$$ (33)

Secondly, due to the fact that the values of muon and pion masses are close to each other, our qualitative consideration and its consequence given by eqs. (19), (29) are directly applicable to the process of the breakup of $\mu^+\mu^-$ atom.

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