GROUPOID REPRESENTATIONS AND MODULES OVER THE
CONVOLUTION ALGEBRAS

J. KALIŠNIK

Abstract. The classical Serre-Swan’s theorem defines a bijective correspondence between vector bundles and finitely generated projective modules over the algebra of continuous functions on some compact Hausdorff topological space. We extend these results to obtain a correspondence between the category of representations of an étale Lie groupoid and the category of modules over its convolution algebra that are of finite type and of constant rank. Both of these constructions are functorially defined on the Morita bicategory of étale Lie groupoids and the given correspondence represents a natural equivalence between them.

1. Introduction

There are many phenomena in different areas of mathematics and physics that are most naturally described in the language of smooth manifolds and smooth maps between them. However, some natural constructions, coming from the theory of foliations or from Lie group actions, result in slightly more singular spaces and require a different approach. The Morita category of Lie groupoids and principal bundles provides a natural framework in which to study many such singular spaces like spaces of leaves of foliations, spaces of orbits of Lie group actions or for example orbifolds. A Lie groupoid can be considered as an atlas for the given singular space. It turns out that different Lie groupoids represent the same geometric space precisely when they are Morita equivalent, i.e. when they are isomorphic in the Morita category of Lie groupoids. For this reason we are primarily interested in those algebraic invariants of Lie groupoids that are functorially defined on the Morita category of Lie groupoids.

Tangent bundles, bundles of higher order tensors, line bundles and other vector bundles play a central role in the study of smooth manifolds. The theory of representations of Lie groupoids naturally extends all these notions to the category of étale Lie groupoids and shows their close connection to the theory of Lie groups representations. It encompasses many well known constructions like equivariant vector bundles, orbibundles over orbifolds, foliated and transversal vector bundles over spaces of leaves of foliations, as well as ordinary vector bundles over manifolds or representations of discrete groups. The construction of the category of representations of a Lie groupoid is invariant under the Morita equivalence and thus represents one of the basic algebraic invariants of Lie groupoids.

The Connes convolution algebra of smooth functions with compact support on an étale Lie groupoid is another example of such an invariant. Smooth functions with compact support on a smooth manifold and group algebras of discrete groups are all special examples of convolution algebras. Finitely generated projective modules over the algebra of smooth functions on a compact manifold are closely connected to smooth vector bundles over that manifold by a theorem due to Serre and Swan. Representations of a discrete group Γ on complex vector...
spaces on the other hand correspond bijectively to modules over the group algebra $\mathbb{C}[\Gamma]$ of the group $\Gamma$. One is thus lead to believe that both of these examples represent a similar type of objects, namely a correspondence between representations of an étale Lie groupoid and a certain class of modules over its convolution algebra.

We show in this paper (Theorem 3.2) how to extend the classical Serre-Swan’s correspondence to the category of étale Lie groupoids. The functor of smooth sections with compact support defines an equivalence between the category of representations of an étale Lie groupoid $G$ and the category of modules over the convolution algebra of $G$ which are of finite type and of constant rank. These modules generalize the well known finitely generated projective modules over the algebras of functions and coincide with them if the manifold of objects of $G$ is compact and connected. For example, such a correspondence makes it possible to study the category of orbibundles over a compact orbifold with the tools of noncommutative geometry, applied to the convolution algebra of the corresponding orbifold groupoid.

The constructions of the categories of representations and of the modules of finite type and of constant rank extend to morphisms from the Morita bicategory of étale Lie groupoids to the 2-category of additive categories. In this more general context, the given correspondence can be described (Theorem 4.3) as a natural equivalence between these two morphisms.

2. Basic definitions and examples

2.1. The Morita category of Lie groupoids. For the convenience of the reader, and to fix the notations, we begin by summarizing some basic definitions and results concerning Lie groupoids that will be used throughout this paper. We refer the reader to one of the books [13, 17, 18] for a more detailed exposition and further examples.

A Lie groupoid over a smooth, second countable, Hausdorff manifold $M$ is given by a smooth manifold $G$ and a structure of a category on $G$ with objects $M$, in which all the arrows are invertible and all the structure maps

$$G \times^s t_M \xrightarrow{\text{mlt}} G \xrightarrow{\text{inv}} G \xrightarrow{s} M \xrightarrow{\text{uni}} G$$

are smooth, with the source map $s$ a submersion. We allow the manifold $G$ to be non-Hausdorff, but we assume that the fibers of the source map are Hausdorff. We write $G(x, y) = s^{-1}(x) \cap t^{-1}(y)$ for the set of arrows from $x \in M$ to $y \in M$ and $G_x = G(x, x)$ for the isotropy group of the groupoid $G$ at $x$. The set $G(x, y)$ is a submanifold of $G$ and the isotropy group $G_x$ is a Lie group. If $g \in G$ is any arrow from $x$ to $y$, and $g' \in G$ is an arrow from $y$ to $z$, then the product $g'g = \text{mlt}(g', g)$ is an arrow from $x$ to $z$. The map uni assigns to each $x \in M$ the identity arrow $1_x = \text{uni}(x)$ in $G$, and we often identify $M$ with its image $\text{uni}(M)$ in $G$. The map inv maps each arrow $g \in G$ to its inverse $g^{-1}$.

A Lie groupoid is étale if all of its structure maps are local diffeomorphisms. A bisection of an étale Lie groupoid $G$ is an open subset $U$ of $G$ such that both $s|_U$ and $t|_U$ are injective. To any such bisection $U$ corresponds a diffeomorphism $\tau_U : s(U) \to t(U)$ defined by $\tau_U = t|_U \circ (s|_U)^{-1}$. Bisctions of the groupoid $G$ form a basis for the topology on $G$, so in particular they can be chosen arbitrarily small.

Generalised morphisms [15, 20] turn out to be the right notion of a map between Lie groupoids in the representation theory of Lie groupoids. They are closely connected to groupoid actions and principal bundles, which we briefly describe.

A smooth left action of a Lie groupoid $G$ on a smooth manifold $P$ along a smooth map $\pi : P \to M$ is a smooth map $\mu : G \times^s t_M P \to P$, $(g, p) \mapsto g \cdot p$, which satisfies $\pi(g \cdot p) = \tau(g)$, $1_{\pi(p)} \cdot p = p$ and $g' \cdot (g \cdot p) = (g'g) \cdot p$, for all $g', g \in G$ and $p \in P$ with
s(g') = t(g) and s(g) = π(p). We define right actions of Lie groupoids on smooth
manifolds in a similar way.

Let G be a Lie groupoid over M and let H be a Lie groupoid over N. A principal
H-bundle over G is a smooth manifold P, equipped with a left action \( \mu \) of G along
a smooth submersion \( \pi : P \to M \) and a right action \( \eta \) of H along a smooth map
\( \phi : P \to N \), such that (i) \( \phi \) is G-invariant, \( \pi \) is H-invariant and both actions
commute: \( \phi(g \cdot p) = \phi(p) \), \( \pi(p \cdot h) = \pi(p) \) and \( g \cdot (p \cdot h) = (g \cdot p) \cdot h \)
for every \( g \in G \), \( p \in P \) and \( h \in H \) with \( s(g) = \pi(p) \) and \( \phi(p) = t(h) \), (ii) \( \pi : P \to M \) is a principal
right H-bundle: \( (\text{pr}_1, \eta) : P \times^H_N H \to P \times^H_M P \) is a diffeomorphism.

A map \( f : P \to P' \) between principal H-bundles \( P \) and \( P' \) over G is equivariant
if it satisfies \( \pi'(f(p)) = \pi(p), \phi'(f(p)) = \phi(p) \) and \( f(g \cdot p \cdot h) = g \cdot f(p) \cdot h \), for
every \( g \in G \), \( p \in P \) and \( h \in H \) with \( s(g) = \pi(p) \) and \( \phi(p) = t(h) \). Any such
map is automatically a diffeomorphism. Principal H-bundles \( P \) and \( P' \) over G
are isomorphic if there exists an equivariant diffeomorphism between them. There
is a natural structure of a category (actually a groupoid) on the set of principal
H-bundles over G for any two Lie groupoids \( G \) and \( H \). It has principal H-bundles
over G as objects and equivariant diffeomorphisms as morphisms between them.

To any smooth functor \( \psi : G \to H \) there corresponds a principal H-bundle
\( P(\psi) = M \times^H_N H \) over G with the actions given by the maps \( g \cdot (x, h) = (t(g), \psi(g)h) \)
for \( g \in G(x, y) \) and \( (x, h) \cdot h' = (x, hh') \) for \( h, h' \in H \) such that \( s(h) = t(h') \). A
principal bundle \( P \) is isomorphic to one induced by a functor if and only if it is
trivial, i.e. if there exists a global smooth section of the bundle \( P \).

If \( P \) is a principal H-bundle over G and if \( P' \) is a principal K-bundle over H,
for another Lie groupoid \( K \), one can define the composition \( P \otimes_H P' \) \([18, 19, 20]\),
which is a principal K-bundle over G. It is the quotient of \( P \times_N^H P' \) with respect
to the diagonal action of the groupoid \( H \). So defined composition is associative
only up to a natural isomorphism.

The Morita category \( \text{GPD} \) of Lie groupoids consists of Lie groupoids as objects
and isomorphism classes of principal bundles as morphisms between them. The
morphisms in \( \text{GPD} \) are sometimes referred to as Hilsum-Skandalis maps or generalised
morphisms between Lie groupoids. Two Lie groupoids are Morita equivalent
if they are isomorphic in the category \( \text{GPD} \). The Morita category \( \text{EtGPD} \) of \( \text{étale} \)
Lie groupoids is the full subcategory of the category \( \text{GPD} \) with \( \text{étale} \) Lie groupoids
as objects. If \( G \) and \( H \) are \( \text{étale} \) Lie groupoids and if \( P \) is a principal H-bundle over G,
then the corresponding map \( \pi : P \to M \) is automatically a local diffeomorphism.
We will be primarily interested in the Morita bicategory of Lie groupoids which we
describe later on in the paper.

2.2. Representations of Lie groupoids. Lie groupoids admit a twofold inter-
pretation. They can be used to describe symmetries of fibre bundles in a similar
way as Lie groups are used to study symmetries of topological spaces. However,
the extra transversal part of the structure, which is encoded in the manifold of objects,
makes them a convenient model for singular geometric spaces such as orbifolds,
spaces of leaves of foliations or spaces of orbits of Lie group actions. The theory
of representations of Lie groupoids provides a unified framework for the study of
vector bundles on such geometric spaces and shows their intimate connection to
representations of Lie groups.

Let \( G \) be a Lie groupoid over a smooth manifold \( M \) and let \( E \) be a smooth
complex vector bundle of rank \( k \) over \( M \). A representation of the groupoid \( G \) on
\( E \) is a smooth left action \( \rho : G \times_M E \to E \), denoted by \( \rho(g, v) = g \cdot v \), of \( G \) on \( E \)
along the bundle projection \( p : E \to M \) \([17]\), such that for any arrow \( g \in G(x, y) \)
the induced map \( g_* : E_x \to E_y, v \mapsto g \cdot v \), is a linear isomorphism.
Example 2.1. (i) Representations of a unit groupoid associated to a smooth manifold $M$ correspond precisely to smooth complex vector bundles over $M$.

(ii) Let $G$ be a point groupoid with only one object, i.e. $G$ is a Lie group $K$. The representation theory of $G$ then coincides with the representation theory of the Lie group $K$ on finite dimensional complex vector spaces.

(iii) The pair groupoid $G = M \times M$ over a smooth manifold $M$ has both projections as source and target maps and multiplication defined in a natural way. Every representation of $G$ on a vector bundle $E$ over $M$ amounts to a natural identification of all the fibers of $E$ and is thus isomorphic to a trivial representation.

(iv) Let $G = K \ltimes M$ be the translation groupoid of a smooth left action of a Lie group $K$ on a manifold $M$. In this case the representations of the groupoid $G$ correspond to $K$-equivariant vector bundles over $M$.

(v) Any étale Lie groupoid $G$ over a manifold $M$ has a natural representation on the complexified tangent bundle of the manifold $M$, where the action of any arrow is defined by the differential of the local diffeomorphism corresponding to some bisection through that arrow. The cotangent bundle and tensor bundles all inherit this natural representation, so it makes sense to speak of vector fields, differential forms or riemannian metrics on étale Lie groupoids.

(vi) Let $G$ be an orbifold groupoid (i.e. a proper étale Lie groupoid) over $M$. Such groupoids are used as models for orbifolds. Representations of such groupoids correspond to orbibundles as defined in.

(vii) Let $(M, \mathcal{F})$ be a foliated manifold. Representations of the holonomy groupoid $\text{Hol}(M, \mathcal{F})$ are sometimes referred to as transversal vector bundles, while those of the monodromy groupoid $\text{Mon}(M, \mathcal{F})$ are referred to as foliated vector bundles.

A morphism between representations $E$ and $F$ of the groupoid $G$ over $M$ is a $G$-equivariant morphism $\phi : E \to F$ of vector bundles. More precisely, $\phi : E \to F$ is a fiberwise linear smooth map which commutes with bundle projections and satisfies $\phi(g \cdot e) = g \cdot \phi(e)$ for all $g \in G$ and all $e \in E_{\phi(g)}$. Representations of a Lie groupoid $G$ together with $G$-equivariant morphisms between them form a category $\text{Rep}(G)$ of representations of $G$. Categories of (equivariant) vector bundles over a manifold, categories of orbibundles over an orbifold or categories of representations of Lie groups are some examples of categories of representations of Lie groupoids that arise naturally in various contexts. Direct sums, tensor products, duals and other operations on vector bundles generalize to representations of Lie groupoids and turn the category $\text{Rep}(G)$ into an additive category for every groupoid $G$.

Generalised maps between groupoids can be used to pull back representations in the same sense as vector bundles can be pulled back along smooth maps. Let $G$ and $H$ be Lie groupoids over $M$ respectively $N$ and let $P$ be a principal $H$-bundle over $G$. For any representation $E$ of the groupoid $H$ we get the pull back representation $P \otimes_H E$ as follows (see [9] for details). The pull back bundle $\phi^*E = P \times_N E$ has a natural structure of a vector bundle over $P$ with projection onto the first factor as the projection map. Groupoid $H$ acts diagonally from the right on the space $\phi^*E$ along the fibers of the projection onto $M$ and it is easy to see that the natural map $P \otimes_H E = \phi^*E/H \to M$ is well defined, smooth and makes $P \otimes_H E$ a vector bundle over $M$. Finally, the action of the groupoid $G$ on the space $P$ induces a representation of the groupoid $G$ on the bundle $P \otimes_H E$ by acting on the first factor.

The construction of pulling back representations along a principal bundle $P$ extends to a functor from the category of representations of $H$ to the category of representations of $G$. Define a representation $\text{Rep}(P)(E) = P \otimes_H E$ of $G$ for any representation $E$ of $H$ and a morphism $\text{Rep}(P)(\phi) : P \otimes_H E \to P \otimes_H F$ of representations of $G$ by $\text{Rep}(P)(\phi)(p \otimes v) = p \otimes \phi(v)$ for any morphism $\phi : E \to F$. 
of representations of the groupoid $H$. We thus obtain a covariant functor

$$\text{Rep}(P) : \text{Rep}(H) \rightarrow \text{Rep}(G).$$

One can use an alternative description of the above operation in the case of trivial bundles, i.e. when the principal bundle comes from a smooth functor. Suppose that $\psi : G \rightarrow H$ is a smooth functor between Lie groupoids and let $E$ be a representation of the groupoid $H$. One defines a representation $\psi^* E$ of the groupoid $G$ on the vector bundle $\psi_0^* E$ over $M$ with the action $g : (x,v) = (t(g), \psi(g)v)$ for $g \in G(x,y)$ and $v \in E_{\psi_0(x)}$. So defined representation is naturally isomorphic to the representation $P(\psi) \otimes_H E$ of $G$ via the isomorphism $f : P(\psi) \otimes_H E \rightarrow \psi^* E$, which sends the element $(x,h) \otimes v$ to the element $(x,hv)$.

2.3. Convolution algebras and principal bimodules. A smooth manifold $M$ is closely connected with the commutative algebra $C_\infty^c(M)$ of smooth functions with compact support on $M$. By replacing a manifold $M$ with an étale Lie groupoid $G$ over $M$ and by defining a proper notion of the convolution product on the space of functions one obtains the convolution algebra $C^\infty_c(G)$ of the groupoid $G$. It is in general noncommutative but it contains the algebra $C^\infty_c(M)$ as a commutative subalgebra. The above construction naturally extends to a covariant functor from the Morita category of étale Lie groupoids to the Morita category of algebras, i.e. for each principal bundle between groupoids one naturally constructs a bimodule between corresponding algebras. In this subsection we briefly recall the definition of the convolution algebra $[4,5,20,22]$ assigned to an étale, not necessarily Hausdorff, Lie groupoid and of the principal bimodule $[10,20]$ assigned to a principal bundle.

We first recall the definition of the convolution product on the vector space $C^\infty_c(G)$ of smooth functions with compact support on a Hausdorff étale Lie groupoid $G$. Define a bilinear operation on the space $C^\infty_c(G)$ by the formula

$$\left(ab\right)(g) = \sum_{g = g'g''} a(g')b(g''),$$

for any $a,b \in C^\infty_c(G)$. Equipped with this product the space $C^\infty_c(G)$ becomes an associative algebra called the convolution algebra $[4]$ of the étale Lie groupoid $G$.

In the case of a general étale Lie groupoid a suitable notion of a smooth function with compact support on a non-Hausdorff manifold, as given in [5], is needed. Considering that smooth functions on a Hausdorff manifold $M$ correspond precisely to the continuous sections of the sheaf of germs of smooth complex valued functions on $M$ it makes sense to use this alternative approach to define smooth functions with compact support on an arbitrary manifold $P$. One first considers the vector space of all (not-necessarily continuous) sections of the sheaf of germs of smooth functions on $P$. The trivial extension of any smooth function with a compact support in a Hausdorff open subset of $P$ naturally represents a section of that sheaf. The vector space $C^\infty_c(P)$ of smooth functions with compact support on $P$ is then defined to be the subspace of the space of all sections, generated by such sections. This definition of the vector space $C^\infty_c(P)$ agrees with the classical one if $P$ is Hausdorff, but in general there exists no natural multiplicative structure on the space $C^\infty_c(P)$. The support, i.e. the set where the values of the section are nontrivial, of any function in $C^\infty_c(P)$ is always a compact subset of $P$ but not necessarily closed if $P$ is a non-Hausdorff manifold.

The stalk of the sheaf of germs of smooth functions on $P$ at any point of $P$ is a commutative algebra with identity, which enables us to perform pointwise operations such as addition, multiplication or pullbacks along smooth maps. In particular, for any non-Hausdorff étale Lie groupoid $G$ a formula analogous to the
formula \([\text{\[1\]}}\) can be used to define the convolution algebra \(C_c^\infty(G)\) of the étale Lie groupoid \(G\). We refer the reader to \([\text{\[22\]}}\) for details.

**Example 2.2.** (i) The convolution product on the space \(C_c^\infty(G)\) coincides with the ordinary pointwise product of functions in \(C_c^\infty(M)\) if \(G\) is the unit groupoid associated to a smooth manifold \(M\). On the other hand, if \(G\) is the point groupoid of a discrete group \(\Gamma\), it follows \(C_c^\infty(G) = \mathbb{C}[\Gamma]\), where \(\mathbb{C}[\Gamma]\) is the group algebra of the group \(\Gamma\). The subalgebra \(C_c^\infty(M) = \{\lambda \cdot 1_\Gamma | \lambda \in \mathbb{C}\}\) is isomorphic to complex numbers and is central in \(\mathbb{C}[\Gamma]\).

(ii) The convolution algebra \(C_c^\infty(G)\) of the pair groupoid on \(n\) points coincides with the algebra of \(n \times n\) complex matrices and contains the subalgebra \(C_c^\infty(M)\) of diagonal matrices. This example shows that \(C_c^\infty(M)\) need not lie in the center of the algebra \(C_c^\infty(G)\).

(iii) Let \(G\) be the translation groupoid of a smooth action of a discrete group \(\Gamma\) on a manifold \(M\). The convolution algebra \(C_c^\infty(G)\) of \(G\) is known in the literature \([\text{\[1\]}}\) as the crossed product algebra \(\Gamma \rtimes C_e^\infty(M)\).

One can naturally extend the convolution algebra construction to a functor from the Morita category of étale Lie groupoids to the Morita category of algebras. Let \(G\) and \(H\) be étale Lie groupoids and let \(P\) be a principal \(H\)-bundle over \(G\). One can define convolution actions of the algebras \(C_c^\infty(G)\) and \(C_c^\infty(H)\) on the vector space \(C_c^\infty(P)\) to turn it into a \(C_c^\infty(G)\)-\(C_c^\infty(H)\)-bimodule, which is called the principal bimodule associated to the principal \(H\)-bundle \(P\) over \(G\). The functor \(C_c^\infty\) from the Morita category of étale Lie groupoids to the Morita category of algebras assigns to every étale Lie groupoid its convolution algebra and to an isomorphism class of a principal bundle the isomorphism class of the associated principal bimodule \([\text{\[20\]}}\) (see also \([\text{\[10\]}}\) for a treatment of the non-Hausdorff case).

Throughout the rest of the paper we will restrict ourselves to Hausdorff groupoids for simplicity, although essentially the same formulas apply in the non-Hausdorff case as well.

### 3. Groupoid representations and modules over convolution algebras

The vector space of sections of a smooth vector bundle \(E\) over a manifold \(M\) admits a natural action of the algebra of smooth functions on \(M\). Additional structure of a representation of an étale Lie groupoid \(G\) on the bundle \(E\) allows a natural extension of that action to the action of the convolution algebra of the groupoid \(G\). In this section we characterize the modules over the convolution algebra of the groupoid \(G\) that arise in this fashion from sections of representations of \(G\).

Results of this type were first considered by Serre \([\text{\[25\]}}\) in the the category of algebraic varieties and Swan in the category of compact Hausdorff topological spaces \([\text{\[20\]}}\). All our vector bundles will be assumed to be of globally constant rank, a condition which is automatically satisfied if the manifold of objects of the groupoid is connected. Similar results hold however in the case of vector bundles of globally bounded rank as well.

#### 3.1. Module of sections of a representation.

Let \(M\) be a smooth, Hausdorff and second countable manifold and denote by \(C_c^\infty(M)\) the algebra of smooth functions with compact support on \(M\). We will denote by \(\text{\text{Rep}}(M)\) the category of smooth vector bundles over \(M\) since it coincides with the category of representations of the unit groupoid associated to the manifold \(M\). For any vector bundle \(E\) over the manifold \(M\), the vector space \(\Gamma_c^\infty(E)\) of smooth sections of \(E\) with compact support admits a natural structure of a left \(C_c^\infty(M)\)-module given by \((fu)(x) = f(x)u(x)\) for any \(f \in C_c^\infty(M)\) and any \(u \in \Gamma_c^\infty(E)\). Every morphism \(\phi : E \to F\) of vector bundles over \(M\) induces a homomorphism \(\Gamma_c^\infty(\phi) : \Gamma_c^\infty(E) \to \Gamma_c^\infty(F)\) of left
$C_c^\infty(M)$-modules by composing with $\phi$, i.e. $\Gamma_c^\infty(\phi)(u) = \phi \circ u$. As a result we obtain the covariant functor

$$\Gamma_c^\infty = (\Gamma_c^\infty)_M : \text{Rep}(M) \to \text{MMod}$$

from the category of smooth vector bundles over the manifold $M$ to the category of left modules over the commutative algebra $C_c^\infty(M)$.

Now let $G$ be an étale Lie groupoid over $M$. We will associate an action of the convolution algebra $C_c^\infty(G)$ on the space of sections $\Gamma_c^\infty(E)$ to any representation $E$ of the groupoid $G$. Define a bilinear map

$$C_c^\infty(G) \times \Gamma_c^\infty(E) \to \Gamma_c^\infty(E)$$

by the formula

$$(au)(x) = \sum_{t(g) = x} a(g)(g \cdot u(s(g))),$$

for $a \in C_c^\infty(G)$ and $u \in \Gamma_c^\infty(E)$. Since the function $a \in C_c^\infty(G)$ has a compact support, there are only finitely many $g \in t^{-1}(x)$ with $a(g) \neq 0$ for each $x \in M$, hence $au$ is a well defined section of the vector bundle $E$. It remains to be proven that $au$ belongs to the space $\Gamma_c^\infty(E)$ and that the above map really defines an action of the algebra $C_c^\infty(G)$ on the space $\Gamma_c^\infty(E)$.

Strictly speaking, the above formula only holds for Hausdorff Lie groupoids. However, by evaluating the sections of the sheaf of germs of smooth functions, one can use virtually the same formula for non-Hausdorff groupoids as well.

**Proposition 3.1.** The vector space of sections $\Gamma_c^\infty(E)$ has a natural structure of a left module over the convolution algebra $C_c^\infty(G)$.

**Proof.** First we show that $au$ represents a smooth section of the vector bundle $E$ for any $a \in C_c^\infty(G)$ and any $u \in \Gamma_c^\infty(E)$. We can decompose any function $a \in C_c^\infty(G)$ as a sum $a = \sum a_j$ of functions, each of which has support contained in some bisection, so we can assume right from the start that the support of $a$ is contained in some bisection $U$. Let us denote $W = t(U)$ and $V = s(U)$. We then have the following commuting diagram

$$\begin{array}{ccc}
E|_W & \xleftarrow{\mu} & U \times_V E \\
\downarrow & & \downarrow \text{pr}_E \\
W & \xleftarrow{t|_U} & U \\
\downarrow \text{pr}_U & & \downarrow s|_U \\
V & \xrightarrow{s|_U} & V
\end{array}$$

of maps of vector bundles. The maps $\mu$ and $\text{pr}_E$ are isomorphisms of vector bundles covering the diffeomorphisms $t|_U : U \to W$ respectively $s|_U : U \to V$. Let us denote by $\sigma_U = (t|_U) \circ (s|_U)^{-1} : V \to W$ the diffeomorphism corresponding to the bisection $U$, and by $\tau_U = \mu \circ (\text{pr}_E)^{-1} : E|_V \to E|_W$ the corresponding isomorphism of vector bundles. The smooth section $u|_V$ of the bundle $E|_V$ gets mapped by the above isomorphism to the smooth section $u' = \tau_U \circ u \circ (\sigma_U)^{-1}$ of the bundle $E|_W$. Furthermore, since $a$ has compact support in $U$, the function $a \circ (t|_U)^{-1}$ has compact support in $W$. One can now express the section $au$ of the bundle $E$ as $(au)(x) = (a \circ (t|_U)^{-1})(x)u'(x)$ to prove that it is a smooth section of $E$ with compact support in $W$.

To see that the space of sections $\Gamma_c^\infty(E)$ is a module over the algebra $C_c^\infty(G)$, the equality $a(bu) = (ab)u$ must hold for all $a, b \in C_c^\infty(G)$ and all $u \in \Gamma_c^\infty(E)$. To
this effect we compute
\[
((ab)u)(x) = \sum_{t(g) = x} (ab)(g \cdot u(s(g)))
\]
\[
= \sum_{t(g) = x} \left( \sum_{g = g'g''} a(g') b(g'') \right)(g \cdot u(s(g)))
\]
\[
= \sum_{t(g') = x, s(g') = t(g'')} a(g') b(g'') \left( (g'g'') \cdot u(s(g'')) \right).
\]
On the other hand we have
\[
(a(bu))(x) = \sum_{t(g) = x} a(g)(b \cdot (bu)(s(g)))
\]
\[
= \sum_{t(g) = x} a(g)(g \cdot \left( \sum_{t(g') = s(g)} b(g')(g' \cdot u(s(g'))\right))
\]
\[
= \sum_{t(g) = x, s(g) = t(g'')} a(g) b(g')(g \cdot (g' \cdot u(s(g'))).
\]
In the last line we have used the linearity of the map \(g \cdot : E_x \to E_y\) for each \(g \in G(x, y)\). Since \(E\) is a representation of \(G\), the equality \(g \cdot (g' \cdot e) = gg' \cdot e\) holds for all pairs of composable arrows \(g, g' \in G\) and all \(e \in E_s(g')\), thus \((ab)u = a(bu)\).

By the above procedure we obtain a left module of sections \(\Gamma_c^\infty(E)\) over the convolution algebra \(C^\infty_c(G)\) for any representation \(E\) of the groupoid \(G\). Any morphism \(\phi : E \to F\) of representations of \(G\) is in particular a morphism of vector bundles and therefore produces a homomorphism \(\Gamma_c^\infty(\phi) : \Gamma_c^\infty(E) \to \Gamma_c^\infty(F)\) of \(C^\infty_c(M)\)-modules. Considering that the map \(\phi\) is fiberwise linear and \(G\)-equivariant we get the equalities
\[
(\Gamma_c^\infty(\phi)(au))(x) = \phi((au)(x))
\]
\[
= \phi \left( \sum_{t(g) = x} a(g)(g \cdot u(s(g))) \right)
\]
\[
= \sum_{t(g) = x} a(g)(g \cdot \phi(u(s(g))))
\]
\[
= (a\Gamma_c^\infty(\phi)(u))(x),
\]
for \(a \in C^\infty_c(G)\) and \(u \in \Gamma_c^\infty(E)\). The homomorphism \(\Gamma_c^\infty(\phi)\) is therefore a homomorphism of \(C^\infty_c(G)\)-modules so we have the covariant functor
\[
\Gamma_c^\infty = (\Gamma_c^\infty)_G : \text{Rep}(G) \to \text{CMod}
\]
from the category of representations of the groupoid \(G\) to the category of left modules over the convolution algebra \(C^\infty_c(G)\) of \(G\).

3.2. Modules of finite type and of constant rank. According to the previous subsection we can consider a module over the convolution algebra of an étale Lie groupoid \(G\) as a module of sections of some representation of \(G\). However, not every \(C^\infty_c(G)\)-module is of this kind and it is not too hard to find counterexamples. In the following subsection we define and explain the conditions that characterize the modules of sections of representations of the groupoid \(G\).

Let \(M\) be a smooth manifold and denote by \(C^\infty_c(M)\) the algebra of smooth functions with compact support on \(M\). There is a standard bijective correspondence between nontrivial homomorphisms \(\eta : C^\infty_c(M) \to \mathbb{C}\) of complex algebras and the points of the manifold \(M\). To any \(x \in M\) one associates the evaluation
\[ \epsilon_x : C^\infty_c(M) \to \mathbb{C} \] at the point \( x \) given by \( \epsilon_x(f) = f(x) \) for \( f \in C^\infty_c(M) \). Conversely, the kernel of any nontrivial homomorphism \( \eta : C^\infty_c(M) \to \mathbb{C} \) is a maximal ideal of the form \( \ker(\eta) = \{ f \in C^\infty_c(M) | f(x) = 0 \} \) for a unique point \( x \in M \), thus \( \eta = \epsilon_x \).

We will use the notation \( \Lambda C^\infty_c(M) = \{ f \in C^\infty_c(M) | f(x) = 0 \} \) for the maximal ideal of functions that vanish at \( x \) and \( C^\infty_c(M)(x) = C^\infty_c(M)/\Lambda C^\infty_c(M) \) for the quotient algebra. Evaluation at the point \( x \) induces a canonical isomorphism between the algebra \( C^\infty_c(M)(x) \) and the field of complex numbers.

Now let \( G \) be an étale Lie groupoid over \( M \) and let \( \mathcal{M} \) be a left \( C^\infty_c(G) \)-module. It follows that \( \mathcal{M} \) is a \( C^\infty_c(M) \)-module as well since \( C^\infty_c(M) \) is a subalgebra of the convolution algebra \( C^\infty_c(G) \). The \( C^\infty_c(G) \)-module \( \mathcal{M} \) is of \textit{finite type} if it is isomorphic, as a \( C^\infty_c(M) \)-module, to some submodule of the module \( C^\infty_c(M)^k \) for some natural number \( k \). The \( C^\infty_c(M) \)-modules of the form \( C^\infty_c(M)^k \) correspond precisely to the modules of sections of trivial vector bundles \( M \times \mathbb{C}^k \), so one can roughly think of modules of finite type as corresponding to subfamilies of trivial vector bundles.

Now choose an arbitrary point \( x \in M \). The \( C^\infty_c(M) \)-module \( I_x \mathcal{M} = I_x C^\infty_c(M) \cdot \mathcal{M} \) is then a \( C^\infty_c(M) \)-submodule of \( \mathcal{M} \) and we denote by \( \mathcal{M}(x) = \mathcal{M}/I_x \mathcal{M} \) the quotient \( C^\infty_c(M)(x) \)-module and consider it as a complex vector space. Suppose now that the \( C^\infty_c(G) \)-module \( \mathcal{M} \) is of finite type and let \( \Phi : \mathcal{M} \to C^\infty_c(M)^k \) be an injective homomorphism of \( C^\infty_c(M) \)-modules. For each \( x \in M \) we obtain an injective complex linear map \( \Phi(x) : \mathcal{M}(x) \to C^\infty_c(M)(x)^k \cong \mathbb{C}^k \), which shows that \( \mathcal{M}(x) \) is a finite dimensional complex vector space for each \( x \in M \). We denote by \( \text{rank}_x \mathcal{M} = \dim_{\mathbb{C}} \mathcal{M}(x) \) the rank of the module \( \mathcal{M} \) at the point \( x \in M \). The \( C^\infty_c(G) \)-module \( \mathcal{M} \) of finite type is of \textit{constant rank} if the function \( x \mapsto \text{rank}_x \mathcal{M} \) is a constant function. One can similarly define the notions of modules of locally constant rank and of modules of globally bounded rank.

Suppose now that \( M \) is a smooth, Hausdorff and paracompact manifold and let \( E \) be a vector bundle over \( M \). The module \( \Gamma^\infty_c(E) \) of sections of the bundle \( E \) is a basic example of a module of finite type and of constant rank. One can see that as follows. Since \( M \) is finite dimensional and paracompact, there exists a vector bundle \( F \) over \( M \) such that the bundle \( E \oplus F \) is isomorphic to some trivial vector bundle \( M \times \mathbb{C}^k \) over \( M \); vector bundles with this property are said to be of finite type. This property basically follows from the proof of Lemma 5.9 in [13]. As a result we obtain the isomorphism \( \Gamma^\infty_c(E) \oplus \Gamma^\infty_c(F) \cong C^\infty_c(M)^k \), i.e. the module \( \Gamma^\infty_c(E) \) is of finite type. Furthermore, there is a natural isomorphism \( \Gamma^\infty_c(E)(x) \to E_x \) of complex vector spaces for every \( x \in M \), induced by the evaluation at the point \( x \), which shows that the module \( \Gamma^\infty_c(E) \) is of constant rank.

Our notions of modules of finite type and of constant rank are closely connected with the classical notions in the Serre-Swan’s theorem. The algebra \( C^\infty_c(M) \) is unital precisely when the manifold \( M \) is compact and in this case it makes sense to speak of free and projective modules over the algebra \( C^\infty_c(M) \). Finitely generated, projective \( C^\infty_c(M) \)-modules correspond in this case to the modules of finite type and of constant rank if the manifold \( M \) is connected and to the modules of finite type and of locally constant rank in general.

3.3. \textbf{Equivalence between the categories of representations and of modules of finite type and of constant rank}. We will denote by \( \text{Mod}(G) \) the full subcategory of the category of left modules over the convolution algebra of the groupoid \( G \) consisting of modules of finite type and of constant rank. Since every module of sections of a representation is such a module, we have the functor

\[ (\Gamma^\infty_c)_G : \text{Rep}(G) \to \text{Mod}(G) \]

which represents a natural equivalence between the given categories.
Theorem 3.2. Let $G$ be an étale Lie groupoid over a smooth manifold $M$. The functor $(\Gamma^\infty_G)_c : \text{Rep}(G) \to \text{Mod}(G)$ is an equivalence between the category $\text{Rep}(G)$ of representations of $G$ and the category $\text{Mod}(G)$ of modules over the convolution algebra $C^\infty_c(G)$ of the groupoid $G$ which are of finite type and of constant rank.

Before we begin with the proof of Theorem 3.2, we briefly recall the classical version of the Serre-Swan’s theorem in the setting of smooth manifolds and modules over the algebras of smooth functions with compact support.

Theorem 3.3. The functor $(\Gamma^\infty_G)_M : \text{Rep}(M) \to \text{Mod}(M)$ is an equivalence of categories for any smooth, Hausdorff and paracompact manifold $M$.

Proof. The crucial point in the proof of the theorem is the observation that every vector bundle over a paracompact manifold is of finite type, i.e. a subbundle of some trivial bundle. Taking this into account, basically the same proof as in the Swan’s original paper [29] goes through. □

Let $G$ be an étale Lie groupoid over $M$. We will prove Theorem 3.2 by constructing a quasi-inverse

$$R_G : \text{Mod}(G) \to \text{Rep}(G)$$

to the functor $\Gamma^\infty_G$ to show that it is an equivalence of categories. For any $C^\infty_c(G)$-module $\mathcal{M}$ of finite type and of constant rank we define a vector bundle $R_G(\mathcal{M})$ over $M$ as follows. As a set, the bundle $R_G(\mathcal{M})$ is defined as a disjoint union of the spaces $\mathcal{M}(x)$ for $x \in M$

$$R_G(\mathcal{M}) = \bigsqcup_{x \in M} \mathcal{M}(x),$$

together with a natural projection onto the manifold $M$. To define a topology and a smooth structure on the space $R_G(\mathcal{M})$ we first choose a vector bundle $E$ over $M$ and an isomorphism $\Phi : \Gamma^\infty_G(E) \to \mathcal{M}$ of left $C^\infty_c(M)$-modules. Such an isomorphism exists due to the classical version of Serre-Swan’s Theorem 3.3. The induced map $\Phi(x) : E_x \to \mathcal{M}(x)$ is then an isomorphism of complex vector spaces for each $x$, so we can use the fiberwise linear bijection

$$\phi = \bigsqcup \Phi(x) : E \to R_G(\mathcal{M})$$

to define a structure of a smooth vector bundle over $M$ on the space $R_G(\mathcal{M})$. So defined vector bundle structure on the space $R_G(\mathcal{M})$ is well defined. Namely, if $E'$ is another vector bundle over $M$ and if $\Phi' : \Gamma^\infty_G(E') \to \mathcal{M}$ is an isomorphism of $C^\infty_c(M)$-modules, we obtain the isomorphism $(\Phi')^{-1} \circ \Phi : \Gamma^\infty_c(E) \to \Gamma^\infty_c(E')$ of $C^\infty_c(M)$-modules. Bundles $E$ and $E'$ are therefore isomorphic by Theorem 3.3 and in turn they define the same vector bundle structure on the space $R_G(\mathcal{M})$.

We will next use the extra structure of a $C^\infty_c(G)$-module on the space $\mathcal{M}$ to define a representation of the groupoid $G$ on the vector bundle $R_G(\mathcal{M})$. Choose any arrow $g \in G(x, y)$ and any vector $v \in \mathcal{M}(x)$. We can find an element $m \in \mathcal{M}$ such that $v = m(x)$ and a function $a \in C^\infty_c(G)$ with compact support in some bisection $U$ such that $a(g) = 1$. Since $\mathcal{M}$ is a left $C^\infty_c(G)$-module, the element $am \in \mathcal{M}$ is well defined and we define

$$g \cdot m(x) = am(y) \in \mathcal{M}(y).$$

We will denote by

$$\mu_\mathcal{M} : G \times_M R_G(\mathcal{M}) \to R_G(\mathcal{M})$$

the map defined by the above formula.

Proposition 3.4. The map $\mu_\mathcal{M}$ defines a representation of the Lie groupoid $G$ on the vector bundle $R_G(\mathcal{M})$ over $M$. 
Proof. Before we start with the proof of the proposition, we list some properties of the $C^\infty_c(M)$-modules $M(x)$ for $x \in M$ and of the convolution algebra $C^\infty_c(G)$.

(i) For any $m(x) \in M(x)$ and any function $f \in C^\infty_c(M)$ the action of $f$ on $m(x)$ is just the multiplication by $f(x)$ in the vector space $M(x)$. In particular, $fm(x) = m(x)$ if $f(x) = 1$ and $fm(x) = 0 \in M(x)$ if $f(x) = 0$.

(ii) Let $a \in C^\infty_c(G)$ be a smooth function with compact support in the bisection $U$ of the groupoid $G$ and denote $W = t(U)$ respectively $V = s(U)$. For any function $f \in C^\infty_c(M)$ with compact support in $V$ we have the formula $af = (f \circ (\sigma_U)^{-1})a$, where $\sigma_U : V \to W$ is the diffeomorphism corresponding to the bisection $U$ and $f \circ (\sigma_U)^{-1} \in C^\infty_c(M)$ is a smooth function with compact support in $W$. Moreover, $af = a$ for every function $f$ which is identically equal to 1 on the set $s(supp(a)) \subset V$. Suppose now that $g \in G(x,y)$ is an arrow from $x$ to $y$ and let $m \in M(x)$ be an arbitrary vector. We will first show that the element $\mu_M(g,v)$ is well defined and independent of $g$ and $v$. To this effect choose an arbitrary function $a \in C^\infty_c(G)$ with support in a bisection $U$ such that $a(g) = 1$ and let $m,m' \in M$ both satisfy $m(x) = m(v) = v$. We then have $(m - m')(x) = 0$, so we can find a function $f \in I_2C^\infty_c(M)$ and an element $m'' \in M$ with $m - m' = fm''$. Now choose a smooth function $f' \in C^\infty_c(M)$ with compact support in $V = s(U)$, such that $f' = a$ as in (ii). The support of the function $f'f$ is then contained in $V$, so by (i) it follows

$$a(m - m')(y) = afm''(y) = af'fm''(y) = ((f'f) \circ (\sigma_U)^{-1})am''(y) = 0$$

since $((f'f) \circ (\sigma_U)^{-1})(y) = (f'f)(x) = 0$. We obtain the equality $am(y) = am'(y)$ which proves the independence of $\mu_M(g,v)$ of the choice of the representative for the element $v$. Next we prove a similar statement for the choice of the representative for the arrow $g$. Suppose $a' \in C^\infty_c(G)$ is another function with $a'(g) = 1$ and with support in a bisection $U'$. One can then find functions $f,f' \in C^\infty_c(M)$ with $f(y) = f'(y) = 1$, such that $fa = f'a'$ is a function with compact support in the bisection $U \cap U'$. The equalities

$$am(y) = fam(y) = f'a'm(y) = a'm(y)$$

then show that the element $\mu_M(g,v) \in R_G(M)$ is well defined.

We next show that $\mu_M$ defines an action of the groupoid $G$ on the vector bundle $R_G(M)$ over $M$. This will be true if we show that the equalities

$$(g'g) \cdot v = g' \cdot (g \cdot v) \quad \text{and} \quad 1_x \cdot v = v$$

hold for all arrows $g \in G(x,y)$, $g' \in G(y,z)$ and all $v \in M(x)$. If $a \in C^\infty_c(G)$ with $a(g) = 1$ has support in a bisection $U$ and if $a' \in C^\infty_c(G)$ with $a'(g') = 1$ has support in a bisection $U'$, then $a'a$ has support in the bisection $U' \times_M U$ and $(a'a)(g'g) = 1$. We then have

$$(g'g) \cdot v = a'am(z) = a' \cdot am(y) = g' \cdot (g \cdot v).$$

where $m \in M$ is an arbitrary element such that $m(x) = v$. To prove the second claim, it is enough to observe that any function $f \in C^\infty_c(M) \subset C^\infty_c(G)$ with $f(x) = 1$ represents the identity arrow $1_x$ at the point $x \in M$. The result then follows directly from the definition of the action of the algebra $C^\infty_c(M)$ on $M(x)$.

It remains to be proven that $\mu_M : G \times_M R_G(M) \to R_G(M)$ is a smooth map. Choose any element $(g,v) \in G \times_M R_G(M)$ and a function $a \in C^\infty_c(G)$ with compact support in a bisection $U'$ such that $a|_{U'} \equiv 1$ for some small neighbourhood $U \subset U'$ of the arrow $g$. There exist elements $m^1, \ldots, m^k \in M$ with the property that the vectors $\{m^1(x), \ldots, m^k(x)\}$ form a basis of the vector space $M(x)$ for all $x \in s(U)$. As a result we obtain smooth functions $\lambda_1, \ldots, \lambda_k : R_G(M)|_{s(U')} \to \mathbb{C}$, implicitly defined by the formula $w = \sum_{i=1}^k \lambda_i(w)m^i(x)$ for any $w \in M(x)$ where $x \in s(U)$. 


Locally, on a neighbourhood $U \times_{\mathcal{M}} R_G(\mathcal{M})$ of the point $(g, v)$, we have

$$\mu_M(h, w) = \sum_{i=1}^{k} \lambda_i(w)am^i(t(h)),$$

where $am^i$ are smooth sections of the vector bundle $R_G(\mathcal{M})$. This concludes the proof of the proposition. \hfill \Box

Now choose left $C^\infty_c(G)$-modules of finite type and of constant rank $\mathcal{M}$ and $\mathcal{N}$ and let $\Phi : \mathcal{M} \to \mathcal{N}$ be a homomorphism between them. For each $x \in M$ we have the induced linear maps $\Phi(x) : \mathcal{M}(x) \to \mathcal{N}(x)$ that define a fiberwise linear map

$$R_G(\Phi) = \prod_{x \in M} \Phi(x) : R_G(\mathcal{M}) \to R_G(\mathcal{N}).$$

The map $R_G(\Phi)$ is a smooth morphism of vector bundles, since it transforms smooth sections of the bundle $R_G(\mathcal{M})$ to smooth sections of the bundle $R_G(\mathcal{N})$. We claim that $R_G(\Phi)$ defines a $G$-equivariant morphism of representations of the groupoid $G$ on $R_G(\mathcal{M})$ respectively $R_G(\mathcal{N})$ as described above. To see this, choose an arrow $g \in G(x, y)$ and a vector $v \in R_G(\mathcal{M})_x = \mathcal{M}(x)$. Let $a \in C^\infty_c(G)$ be a function with support in a bisection $U$ such that $a(g) = 1$ and suppose $m \in \mathcal{M}$ satisfies $m(x) = v$. The statement then follows from the equalities

$$R_G(\Phi)(g \cdot v) = \Phi(y)(am(y)) = \Phi(am)(y) = a\Phi(m)(y) = g \cdot R_G(\Phi)(v).$$

The functoriality of the assignment $\Phi \mapsto \Phi(x)$ for each $x \in M$ extends to the functoriality of the map $R_G$, so we have the functor

$$R_G : \text{Mod}(G) \to \text{Rep}(G).$$

Proof of Theorem \ref{thm:equivalence}. We will prove the theorem by showing that the functors $R_G : \text{Mod}(G) \to \text{Rep}(G)$ and $\Gamma^\infty_c : \text{Rep}(G) \to \text{Mod}(G)$ represent mutual inverses.

We can naturally identify the $C^\infty_c(G)$-module $\mathcal{M}$ with the space of sections $\Gamma^\infty_c(R_G(\mathcal{M}))$ of the representation $R_G(\mathcal{M})$ of $G$ by assigning a section $x \mapsto m(x)$ of the bundle $R_G(\mathcal{M})$ to the element $m \in \mathcal{M}$. Denote by $\epsilon : \mathcal{M} \to \Gamma^\infty_c(R_G(\mathcal{M}))$ the corresponding isomorphism of $C^\infty_c(G)$-modules and let

$$\epsilon : \text{Id}_{\text{Mod}(G)} \Rightarrow \Gamma^\infty_c \circ R_G$$

be the corresponding natural equivalence of functors.

Let $E$ be a representation of the groupoid $G$. There is a natural isomorphism $\Gamma^\infty_c(E)(x) \to E_x$ of complex vector spaces for every $x \in M$, induced by the evaluation at the point $x$, which induces an isomorphism $\eta_E : R_G(\Gamma^\infty_c(E)) \to E$ of representations of the groupoid $G$. The natural equivalence of functors

$$\eta : R_G \circ \Gamma^\infty_c \Rightarrow \text{Id}_{\text{Rep}(G)}$$

together with the equivalence $\epsilon$ shows that $\Gamma^\infty_c$ is an equivalence of categories. \hfill \Box

4. Equivalence of morphisms between bicategories

According to the previous sections we can associate to any étale Lie groupoid the additive categories of representations and of modules of finite type and of constant rank. We would like to extend those constructions to define functors $\text{Rep}$ and $\text{Mod}$ from the category of étale Lie groupoids and principal bundles to the category of additive categories and functors such that their values at the groupoid $G$ are $\text{Rep}(G)$ respectively $\text{Mod}(G)$. Furthermore, the family of functors $(\Gamma^\infty_c)_G$ from Theorem \ref{thm:equivalence} should represent a natural equivalence between these two functors.

If one wants to work in the framework of categories, it is necessary to work with isomorphism classes of representations and of modules and forget the extra categorical structure on these objects. A more convenient way of describing our
13

statement uses the language of bicategories and morphisms between them which we now briefly summarize.

4.1. **Bicategories and Morphisms.** We first recall the definition and some examples of bicategories and morphisms between them as given in [3, 11] (see also [12] for a very concise treatment). A bicategory $\mathcal{C}$ consists of

1. A collection $\mathcal{C}_0$ of objects of $\mathcal{C}$.
2. Categories $\mathcal{C}(A, B)$ for each pair $A, B$ of objects of $\mathcal{C}$. The objects and arrows of the categories $\mathcal{C}(A, B)$ are called 1-cells respectively 2-cells of the bicategory $\mathcal{C}$.
3. For all $A, B, C \in \mathcal{C}_0$ there exist bifunctors $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ which define composition of 1-cells of $\mathcal{C}$ and horizontal composition of 2-cells of $\mathcal{C}$. Furthermore, for each $A \in \mathcal{C}_0$ there is the identity 1-cell $1_A \in \mathcal{C}(A, A)$.

Composition of 1-cells in a bicategory $\mathcal{C}$ resembles the usual composition of arrows in ordinary categories, although it is in general associative only up to natural associativity coherence isomorphisms, which are given as a part of the structure of $\mathcal{C}$. Similarly, the identity 1-cells $1_A$ for $A \in \mathcal{C}_0$ act as units only up to predefined unit coherence isomorphisms. All these coherence isomorphisms have to satisfy some natural pentagon and triangle axioms (see [12] for details) as to insure that all the diagrams constructed out of coherence isomorphisms commute. Apart from the horizontal composition of 2-cells in $\mathcal{C}$ there exists a vertical composition of 2-cells, defined by the composition in the categories $\mathcal{C}(A, B)$. Strict bicategories or 2-categories are bicategories in which the composition of 1-cells is strictly associative with strict identities.

**Example 4.1.** (i) The bicategory $\text{Cat}$ consists of categories as objects, functors between them as 1-cells and natural transformations between functors as 2-cells. For any category $A \in \text{Cat}_0$ the 1-cell $1_A$ is represented by the identity functor on $A$. All the coherence isomorphisms are identities so $\text{Cat}$ is in fact a 2-category. We will denote by $\text{AdCat}$ the full sub 2-category consisting of additive categories.

(ii) Next we describe the Morita bicategory $\text{GPD}$ of Lie groupoids. For any two Lie groupoids $G$ and $H$ there exists a category $\text{GPD}(G, H)$ with principal $H$-bundles over $G$ as objects and equivariant diffeomorphisms as morphisms between them. Composition of 1-cells $P$ and $Q$ is defined by the tensor product construction $P \otimes_H Q$ together with the natural associativity coherence isomorphisms $a_{P,Q,R} : (P \otimes_H Q) \otimes_K R \to P \otimes_H (Q \otimes_K R)$ for $P \in \text{GPD}(G, H)$, $Q \in \text{GPD}(H, K)$ and $R \in \text{GPD}(K, L)$. For any $G \in \text{GPD}_0$ the identity 1-cell $1_G$ is simply the groupoid $G$ viewed as a principal $G$-bundle over $G$. The unit coherence isomorphisms $l_P : G \otimes_G P \to P$ and $r_P : P \otimes_H H \to P$ are induced by the action maps of the groupoids on the principal bundles. If $f : P \to P'$ and $g : Q \to Q'$ are equivariant diffeomorphisms, their horizontal composition is naturally defined as $f \otimes g : P \otimes_H Q \to P' \otimes_H Q'$. We will denote by $\text{EtGPD}$ the full subbicategory of $\text{GPD}$ consisting of étale Lie groupoids.

Now let $C$ and $D$ be two bicategories. A morphism $F : C \to D$ consists of

1. A function $F : C_0 \to D_0$.
2. Functors $F_{AB} : C(A, B) \to D(F(A), F(B))$ for each pair of objects of $C$.

A morphism of bicategories is functorial only up to a family of natural isomorphisms $\phi_{f,g} : F(f) \circ F(g) \to F(f \circ g)$ and $\phi_A : 1_{F(A)} \to F(1_A)$.
for each pair of composable 1-cells of $C$ and for each object of $C$. These natural isomorphisms have to satisfy some further natural coherence axioms. Morphisms of bicategories are also referred to as Lax functors in the literature. Contravariant versions of morphisms between bicategories can be defined analogously.

Our main examples of morphisms between bicategories come from the correspondence between representations of a Lie groupoid and modules of finite type and of constant rank over its convolution algebra.

We will first describe the contravariant morphism

$$\text{Rep} : \text{EtGPD} \to \text{AdCat}$$

from the Morita bicategory of étale Lie groupoids to the 2-category of additive categories. To any étale Lie groupoid $G$ we assign the additive category $\text{Rep}(G)$ of representations of $G$. If $H$ is another étale Lie groupoid and if $P$ is a principal $H$-bundle over $G$, then the functor $\text{Rep}(P) : \text{Rep}(H) \to \text{Rep}(G)$ is defined by

$$\text{Rep}(P)(E) = P \otimes_H E \quad \text{and} \quad \text{Rep}(P)(\phi) : P \otimes_H E \to P \otimes_H F$$

for any $E \in \text{Rep}(H)$ respectively any $\phi \in \text{Rep}(H)(E,F)$. The morphism $\text{Rep}(P)(\phi)$ of representations of $G$ is explicitly defined by $\text{Rep}(P)(\phi)(p \otimes v) = p \otimes \phi(v)$. Furthermore, if $f : P \to P'$ is an isomorphism of principal $H$-bundles over $G$, we define a natural transformation $\text{Rep}(f) : \text{Rep}(P) \Rightarrow \text{Rep}(P')$ by assigning the morphism

$$\text{Rep}(f)_E = f \otimes \text{id} : P \otimes_H E \to P' \otimes_H E$$

describing the natural isomorphisms of representations of the groupoid $G$ to the representation $E \in \text{Rep}(H)$. So defined morphism of bicategories is functorial up to the family of natural isomorphisms $\phi_{P,Q} : \text{Rep}(P) \circ \text{Rep}(Q) \to \text{Rep}(P \otimes_H Q)$ defined by the natural maps

$$\phi_{P,Q}(E) : P \otimes_H (Q \otimes_K E) \to (P \otimes_H Q) \otimes_K E$$

for any $E \in \text{Rep}(K)$. Natural transformations for the identity 1-cells can be defined in a similar fashion by applying the representation maps.

Our second example is the contravariant morphism

$$\text{Mod} : \text{EtGPD} \to \text{AdCat}$$

from the Morita bicategory of étale Lie groupoids to the 2-category of additive categories. For each étale Lie groupoid $G$ let $\text{Mod}(G)$ denote the category of modules over the convolution algebra $C^\infty_c(G)$ that are of finite type and of constant rank. If $P$ is a principal $H$-bundle over $G$ the functor $\text{Mod}(P) : \text{Mod}(H) \to \text{Mod}(G)$ is simply the restriction of the functor of tensoring by $C^\infty_c(P)$

$$C^\infty_c(P) \otimes_{C^\infty_c(H)} - : H \text{Mod} \to G \text{Mod}$$

Tensoring by the right $C^\infty_c(H)$-module $C^\infty_c(P)$ maps the category of left $C^\infty_c(H)$-modules to the category of left $C^\infty_c(G)$-modules since $C^\infty_c(P)$ is a $C^\infty_c(G)$-$C^\infty_c(H)$-bimodule. If $M \in \text{Mod}(H)$ is a $C^\infty_c(H)$-module of finite type and of constant rank, then $C^\infty_c(P) \otimes_{C^\infty_c(H)} M$ is a $C^\infty_c(G)$-module of finite type and of constant rank by Proposition 12 which shows that $\text{Mod}(P)$ is a well defined functor. Natural isomorphisms $\phi_{P,Q} : \text{Mod}(P) \circ \text{Mod}(Q) \to \text{Mod}(P \otimes_H Q)$ are defined by the maps

$$\Omega_{P,Q} \otimes \text{id} : C^\infty_c(P) \otimes (C^\infty_c(Q) \otimes \Gamma_c^\infty(E)) \to C^\infty_c(P \otimes_H Q) \otimes \Gamma_c^\infty(E),$$

where $\Omega_{P,Q} : C^\infty_c(P) \otimes_{C^\infty_c(H)} C^\infty_c(Q) \to C^\infty_c(P \otimes_H Q)$ is the isomorphism of $C^\infty_c(G)$-$C^\infty_c(K)$-bimodules as defined in [20], see also [10]. The identity coherence isomorphisms are provided by the actions of convolution algebras on the modules.

It is straightforward to check that so defined families of functors and natural isomorphisms satisfy the axioms for the morphisms between bicategories.
4.2. Natural equivalence of the morphisms $\text{Rep}$ and $\text{Mod}$. We will show in this section how one can interpret the Serre-Swan's correspondence as a natural transformation between the contravariant morphisms $\text{Rep}$ and $\text{Mod}$ from the Morita bicategory of étale Lie groupoids to the 2-category of additive categories.

To this effect we first review the definition of a natural transformation between two morphisms of bicategories \[11\] \[12\]. Let $C$ and $D$ be two bicategories and let $F,G : C \to D$ be morphisms between them. A natural transformation $\sigma : F \to G$ consists of

1. For each $A \in C_0$ a 1-cell $\sigma_A : F(A) \to G(A)$.
2. For each 1-cell $f : A \to B$ in $C$ a 2-cell $\sigma_f : G(f) \circ \sigma_A \Rightarrow \sigma_B \circ F(f)$

\[
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
\sigma_A & \downarrow & \downarrow \sigma_B \\
G(A) & \xrightarrow{G(f)} & G(B)
\end{array}
\]

such that $\sigma_f$ is natural in $f$ and satisfies coherence axioms as in $\[12\]$. We begin with a proposition that is of some interest independently of our discussion.

**Proposition 4.2.** Let $G$ and $H$ be étale Lie groupoids and let $P$ be a principal $H$-bundle over $G$. For any representation $E$ of $H$ there exists a natural isomorphism

\[
\sigma_P(E) : C_c^\infty(P) \otimes_{C_c^\infty(H)} \Gamma_c^\infty(E) \to \Gamma_c^\infty(P \otimes_H E)
\]

of $C_c^\infty(G)$-modules.

**Proof.** We first define a bilinear map

\[
\sigma_P(E) : C_c^\infty(P) \times \Gamma_c^\infty(E) \to \Gamma_c^\infty(P \otimes_H E)
\]

by the formula

\[
(\sigma_P(E)(f,u))(x) = \sum_{\pi(p)=x} f(p)(p \otimes u(\phi(p)))
\]

for $f \in C_c^\infty(P)$ and $u \in \Gamma_c^\infty(E)$. It is not too hard to check that the map $\sigma_P(E)$ is well defined and that it induces a homomorphism

\[
\sigma_P(E) : C_c^\infty(P) \otimes_{C_c^\infty(H)} \Gamma_c^\infty(E) \to \Gamma_c^\infty(P \otimes_H E)
\]

of $C_c^\infty(G)$-modules, which we claim to be an isomorphism. It suffices to show that $\sigma_P(E)$ is a bijective map.

We first consider the case when the bundle $P$ is trivial, i.e. $P = P(\psi)$ for some smooth functor $\psi : G \to H$. The representation $P \otimes_H E$ of $G$ is then isomorphic to the representation $\psi^* E$ of $G$ via the isomorphism $f : P \otimes_H E \to \psi^* E$. In particular, $f$ is an isomorphism of vector bundles over $M$, so we obtain the isomorphism

\[
\Gamma_c^\infty(f) : \Gamma_c^\infty(P \otimes_H E) \to \Gamma_c^\infty(\psi^* E)
\]

of $C_c^\infty(M)$-modules. Now consider the representation $E$ of $H$ as a vector bundle over $N$. It is a classical result (see $\[15\]$) that the map

\[
\sigma_{\psi^*}(E) : C_c^\infty(M) \otimes_{C_c^\infty(N)} \Gamma_c^\infty(E) \to \Gamma_c^\infty(\psi^* E),
\]

defined analogously as the map $\sigma_P(E)$, is an isomorphism of $C_c^\infty(M)$-modules. Finally, since $P = M \times_{N}^{\psi} H$ is a trivial bundle, we have by $\[16\]$ the isomorphism $\Omega_{M,H} : C_c^\infty(P) \cong C_c^\infty(M) \otimes_{C_c^\infty(N)} C_c^\infty(H)$ of $C_c^\infty(M)$-$C_c^\infty(H)$-bimodules which gives us an isomorphism

\[
i : C_c^\infty(P) \otimes_{C_c^\infty(H)} \Gamma_c^\infty(E) \to C_c^\infty(M) \otimes_{C_c^\infty(N)} \Gamma_c^\infty(E)
\]
of $C^\infty(M)$-modules. We can collect all these isomorphisms into the following commutative diagram of homomorphisms of $C^\infty_c(M)$-modules

$$
\begin{array}{ccc}
C^\infty_c(P) \otimes C^\infty_c(H) \Gamma^\infty_c(E) & \xrightarrow{\sigma_P(E)} & \Gamma^\infty_c(P \otimes_H E) \\
\downarrow \cong & & \downarrow \cong \\
C^\infty_c(M) \otimes C^\infty_c(N) \Gamma^\infty_c(E) & \xrightarrow{\sigma_{P|U}(E)} & \Gamma^\infty_c(\psi_0^* E)
\end{array}
$$

Since the remaining three maps are bijective, the map $\sigma_P(E)$ is bijective as well.

A principal $H$-bundle $P$ over $G$ is in general only locally trivial \[18]. Let $U$ be an open subset of $M$ such that $P|_U$ is a trivial $H$-bundle. We then have a natural injective homomorphism $i : C^\infty_c(P|_U) \to C^\infty_c(P)$ of right $C^\infty_c(H)$-modules which induces an injective homomorphism

$$
i \otimes \text{id} : C^\infty_c(P|_U) \otimes C^\infty_c(H) \Gamma^\infty_c(E) \to C^\infty_c(P) \otimes C^\infty_c(H) \Gamma^\infty_c(E)
$$

of abelian groups. Injectivity of $i \otimes \text{id}$ follows from the fact that $C^\infty_c(P)$ is a locally unital $C^\infty_c(M)$-module. Namely, for any $w = \sum f_i \otimes u_i \in C^\infty_c(P|_U) \otimes C^\infty_c(H) \Gamma^\infty_c(E)$ there exists a function $f \in C^\infty_c(M)$ with support in $U$ such that $f f_i = f_i$ for each $i$. Left action $\mu_E : C^\infty_c(P) \otimes C^\infty_c(H) \Gamma^\infty_c(E) \to C^\infty_c(P|_U) \otimes C^\infty_c(H) \Gamma^\infty_c(E)$ by $f$ is then a homomorphism of abelian groups such that $\mu_E(i \otimes \text{id})(w) = w$. We now have the following commuting square of homomorphisms of abelian groups

$$
\begin{array}{ccc}
C^\infty_c(P|_U) \otimes C^\infty_c(H) \Gamma^\infty_c(E) & \xrightarrow{\sigma_{P|U}(E)} & \Gamma^\infty_c(P|_U \otimes_H E) \\
\downarrow \cong & & \downarrow \cong \\
C^\infty_c(P) \otimes C^\infty_c(H) \Gamma^\infty_c(E) & \xrightarrow{\sigma_P(E)} & \Gamma^\infty_c(P \otimes_H E)
\end{array}
$$

where the right vertical map is defined by trivially extending the sections outside of the set $U$. Note that both the vertical maps are injective. The injectivity of the map $\sigma_P(E)$ will now follow from the injectivity of the map $\sigma_{P|_U}(E)$. Indeed, suppose that $\eta \in C^\infty_c(P) \otimes C^\infty_c(H) \Gamma^\infty_c(E)$ is such that $\sigma_P(E)(\eta) = 0$. Since the module $C^\infty_c(P)$ is locally unital, there exists a function $f \in C^\infty_c(M)$ such that $f \eta = \eta$ and a decomposition $f = \sum f_i$ into functions with supports contained in open sets $U_i$ such that the bundle $P|_{U_i}$ is trivial for each $i$. It now follows from the $C^\infty_c(M)$-linearity of the map $\sigma_P(E)$ and from the preceding diagram that

$$
\sigma_{P|_{U_i}}(E)(f_i \eta) = \sigma_P(E)(f_i \eta) = f_i \sigma_P(E)(\eta) = 0.
$$

The injectivity of the maps $\sigma_{P|_{U_i}}(E)$ now implies that $f_i \eta = 0$ for each $i$ and thus $\eta = 0$, which proves that $\sigma_P(E)$ is injective. Surjectivity can be proven by using similar arguments and by noting that the sets $C^\infty_c(P|_U) \otimes C^\infty_c(H) \Gamma^\infty_c(E)$, as $U$ varies over some trivializing cover of the bundle $P$, generate the $C^\infty_c(G)$-module $C^\infty_c(P) \otimes C^\infty_c(H) \Gamma^\infty_c(E)$.

Let us now return to our morphisms $\text{Rep}$ and $\text{Mod}$ of bicategories. Define a natural transformation $\sigma : \text{Rep} \Rightarrow \text{Mod}$ by defining a functor

$$
\sigma_G = (\Gamma^\infty_c)_G
$$

for each étale Lie groupoid $G$ and a natural transformation

$$
\sigma_P : \text{Mod}(P) \circ \sigma_H \Rightarrow \sigma_G \circ \text{Rep}(P)
$$

for each 1-cell $P \in \text{EtGPD}(G, H)$ of the bicategory $\text{EtGPD}$. Taking into account Proposition \[12\] it is now straightforward to verify the following theorem.
Theorem 4.3. The morphisms $\text{Rep}$ and $\text{Mod}$ from the Morita bicategory of Lie groupoids to the bicategory of additive categories are naturally equivalent. Natural equivalence is given by the family of functors $(\Gamma^\infty_c)_G : \text{Rep}(G) \to \text{Mod}(G)$ for any groupoid $G$ and the family of transformations $\sigma_P : \text{Mod}(P) \circ \sigma_H \Rightarrow \sigma_G \circ \text{Rep}(P)$ for any principal $H$-bundle $P$ over $G$.

Acknowledgements. I would like to thank J. Mrčun for many helpful discussions and advice related to the paper.

References

[1] A. Adem, J. Leida, Y. Ruan, Orbifolds and Stringy Topology. Cambridge Tracts in Mathematics 171, Cambridge (2007).
[2] M. F. Atiyah, K-Theory. W.A. Benjamin, Inc. New York (1967).
[3] J. Benabou, Introduction to bicategories. Lecture Notes in Mathematics 47, Springer (1967) 1-77.
[4] A. Connes, Noncommutative Geometry. Academic Press, San Diego (1994).
[5] M. Crainic, I. Moerdijk, A homology theory for étale groupoids. J. Reine Angew. Math. 521 (2000) 25–46.
[6] W. Greub, S. Halperin, R. Vanstone, Connections, curvature, and cohomology. Pure and applied mathematics 47 Volume 1, Academic Press, New York, (1978).
[7] A. Haefliger, Structures feuilletées et cohomologie à valeur dans un faisceau de groupoides. Comment. Math. Helv. 32 (1958) 248–329.
[8] M. Hilsum, G. Skandalis, Morphismes K-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes). Ann. Sci. École Norm. Sup. 20 (1987) 325–390.
[9] J. Kalisnik, Representations of orbifold groupoids. Topology and its Applications 155:11 (2008) 1201–1206.
[10] J. Kalisnik, J. Mrčun, Equivalence between the Morita categories of étale Lie groupoids and of locally grouplike Hopf algebroids. Preprint arXiv: math/0703374v1 (2007).
[11] T. Leinster, Higher Operads, Higher Categories. London Mathematical Society Lecture Note Series 298, Cambridge University Press, Cambridge (2004).
[12] T. Leinster, Basic Bicategories. Preprint arXiv: math/9810017 (1998).
[13] K. C. H. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids. London Math. Soc. Lecture Note Ser. 213, Cambridge University Press, Cambridge (2005).
[14] J. W. Milnor, J. D. Stasheff, Characteristic classes. Annals of Mathematics Studies, No. 76, Princeton University Press, Princeton, New Jersey (1974).
[15] I. Moerdijk, Classifying toposes and foliations. Ann. Inst. Fourier (Grenoble) 41 (1991) 189–299.
[16] I. Moerdijk, Orbifolds as groupoids: an introduction. Contemp. Math. 310 (2002) 205-222.
[17] I. Moerdijk, J. Mrčun, Introduction to Foliations and Lie Groupoids. Cambridge Studies in Advanced Mathematics 91, Cambridge University Press, Cambridge, 2003.
[18] I. Moerdijk, J. Mrčun, Lie groupoids, sheaves and cohomology. Poisson Geometry, Deformation Quantisation and Group Representations, London Math. Soc. Lecture Note Ser. 323, Cambridge University Press, Cambridge, (2005) 145–272.
[19] J. Mrčun, Stability and invariants of Hilsum-Skandalis maps. PhD Thesis, Utrecht University, (1996) (arXiv: math.DG/0506484).
[20] J. Mrčun, Functoriality of the bimodule associated to a Hilsum-Skandalis map. K-Theory 18 (1999) 235–253.
[21] J. Mrčun, On spectral representation of coalgebras and Hopf algebroids. Preprint arXiv: math.QA/0208199 (2002).
[22] J. Mrčun, On duality between étale groupoids and Hopf algebroids. J. Pure Appl. Algebra 210 (2007) 267–282.
[23] J. Renault, A Groupoid Approach to C*-algebras. Lecture Notes in Math. 793, Springer, New York (1980).
[24] Y. Ruan, Stringy geometry and topology of orbifolds. Symposium in Honor of C. H. Clemens, Contemp. Math., Providence, RI: American Mathematical Society, 312 (2002) 187–233.
[25] Y. Ruan, Stringy orbifolds. Orbifolds in mathematics and physics, Contemp. Math., Providence, RI: American Mathematical Society, 310 (2002) 259–299.
[26] I. Satake, On a generalization of the notion of manifold. Proc. Nat. Acad. Sci. USA 42 (1956) 359–363.
[27] G. Segal, Equivariant K-theory. Publications Mathmatiques de l’IHS. 34 (1968) 129–151.
[28] J. P. Serre, Faisceaux algébriques cohérents. *Ann. of Math.* 61 (1955) 197–278.
[29] R. G. Swan, Vector bundles and projective modules. *Trans. Amer. Math. Soc.* 105 (1962) 264–277.

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: jure.kalisnik@fmf.uni-lj.si