Viscosity Solutions to Second Order Path-Dependent Hamilton-Jacobi-Bellman Equations in Hilbert Spaces

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Abstract

In this article, a notion of viscosity solutions is introduced for second order path-dependent Hamilton-Jacobi-Bellman (PHJB) equations associated with optimal control problems for path-dependent stochastic evolution equations in Hilbert spaces. We identify the value functional of optimal control problems as unique viscosity solution to the associated PHJB equations. We also show that our notion of viscosity solutions is consistent with the corresponding notion of classical solutions, and satisfies a stability property.

Key Words: Path-dependent Hamilton-Jacobi-Bellman equations; Viscosity solutions; Optimal control; Path-dependent stochastic evolution equations

AMS Subject Classification: 49L20; 49L25; 93C23; 93C25; 93E20.

1 Introduction

Let us consider a controlled path-dependent stochastic evolution equation (PSEE):

\[
\begin{aligned}
    dX^{x,u}(s) &= AX^{x,u}(s)ds + F(X^{x,u}(s), u(s)) ds + G(X^{x,u}(s), u(s)) dW(s), \quad s \in [t, T], \\
    X^{x,u}_t &= \gamma_t \in \Lambda_t,
\end{aligned}
\]

for an unknown process \(X^{x,u}\) in a Hilbert space \(H\). Here \(T > 0\) is an arbitrarily fixed finite time horizon; denote by \(X^{x,u}(s)\) the value of \(X^{x,u}\) at time \(s\), and \(X^{x,u}_s\) the whole history path of \(X^{x,u}\) from time \(0\) to \(s\); \(\{W(t), t \geq 0\}\) is a cylindrical Wiener process on \((\Omega, \mathcal{F}, P)\) with values in a Hilbert space \(\Xi\); \(u(\cdot) = (u(s))_{s \in [t,T]}\) is \(\mathcal{F}^t\)-progressively measurable and take values in some metric space \((U, d)\) (we will say that \(u(\cdot) \in \mathcal{U}[t,T]\) and \(\mathcal{F}^t\) denotes the natural filtration of \(W(s) - W(t), s \in [t,T]\), augmented with the family \(\mathcal{N}\) of \(P\)-null of \(\mathcal{F}\). \(A\) is the generator of a strongly continuous semigroup of bounded linear operators \(\{e^{tA}, t \geq 0\}\) in Hilbert space \(H\); \(\Lambda_t\) denotes the set of all continuous \(H\)-valued functions defined over \([0,t]\) and \(\Lambda = \bigcup_{t \in [0,T]} \Lambda_t\); \(\gamma_t\) is an element of \(\Lambda_t\) and denote by \(\gamma_t(s)\) the value of \(\gamma_t\) at time \(s\). We define a norm on \(\Lambda_t\) and a metric on \(\Lambda\) as follows: for any \(0 \leq t \leq s \leq T\) and \(\gamma_t, \eta_s \in \Lambda_t\),

\[
||\gamma_t||_0 := \sup_{0 \leq s \leq t} |\gamma_t(s)|, \quad d_\infty(\gamma_t, \eta_s) := |t - s| + \sup_{0 \leq s \leq T} \left| e^{((\sigma-t)\vee 0)A} \gamma_t(\sigma \wedge t) - e^{((\sigma-s)\vee 0)A} \eta_s(\sigma \wedge s) \right|.
\]

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We assume the coefficients $F : \Lambda \times U \to H$ and $G : \Lambda \times U \to L_2(\Xi, H)$ satisfy Lipschitz condition under $|| \cdot ||_0$ with respect to the path function.

One tries to maximize a cost functional of the form:

$$J(\gamma_t, u(\cdot)) := Y^{\gamma_t,u}(t), \quad (t, \gamma_t) \in [0, T] \times \Lambda,$$

(1.2)

over $U[t, T]$, where the process $Y^{\gamma_t,u}$ is defined by backward stochastic differential equation (BSDE):

$$Y^{\gamma_t,u}(s) = \phi(X_T^{\gamma_t,u}) + \int_s^T q(X_s^{\gamma_t,u}, Y_s^{\gamma_t,u}(\sigma), Z_s^{\gamma_t,u}(\sigma), u(\sigma))d\sigma$$

$$- \int_s^T Z_s^{\gamma_t,u}(\sigma)dW(\sigma), \quad a.s., \quad all \ s \in [t, T].$$

(1.3)

Here $q$ and $\phi$ are given real functionals on $\Lambda \times \mathbb{R} \times \Xi \times U$ and $\Lambda_T$, respectively, and satisfy Lipschitz condition under $|| \cdot ||_0$ with respect to the path function. We define the value functional of the optimal control problem as follows:

$$V(\gamma_t) := \sup_{u(\cdot) \in U[t,T]} Y^{\gamma_t,u}(t), \quad (t, \gamma_t) \in [0, T] \times \Lambda.$$  

(1.4)

The goal of this article is to characterize this value functional $V$. We consider the following second order path-dependent Hamilton-Jacobi-Bellman (PHJB) equation:

$$\begin{cases}
\partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t)) + H(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)) = 0, & (t, \gamma_t) \in [0, T] \times \Lambda, \\
V(\gamma_T) = \phi(\gamma_T), & \gamma_T \in \Lambda_T;
\end{cases}$$

(1.5)

where

$$H(\gamma_t, r, p, l) = \sup_{u \in U} \left[ (p, F(\gamma_t, u))_H + \frac{1}{2} \text{Tr}[lG(\gamma_t, u)G^*(\gamma_t, u)] \\
+ q(\gamma_t, r, pG(\gamma_t, u), u), \right], \quad (t, \gamma_t, r, p, l) \in [0, T] \times \Lambda \times \mathbb{R} \times H \times \mathcal{S}(H).$$

Here we let $A^*$ the adjoint operator of $A$, $G^*$ the adjoint operator of $G$, $\mathcal{S}(H)$ the space of bounded, self-adjoint operators on $H$ equipped with the operator norm, $\langle \cdot, \cdot \rangle_H$ the scalar product of $H$ and $\partial_t, \partial_x$ and $\partial_{xx}$ the so-called pathwise (or functional or Dupire; see [11, 3, 4]) derivatives, where $\partial_t$ is known as horizontal derivative, while $\partial_x$ and $\partial_{xx}$ are first order and second order vertical derivatives, respectively.

In this paper we will develop a concept of viscosity solutions to PHJB equations on the space of $H$-valued continuous paths and show that the value functional $V$ defined in (1.4) is unique viscosity solution to the PHJB equation (1.5). In order to focus on our main well-posedness objective, here we address the Lipschitz case under $|| \cdot ||_0$.

A definition of viscosity solutions for second order Hamilton-Jacobi-Bellman (HJB) equations in Hilbert spaces has been introduced in Lions [20, 22] for the case without unbounded term, and in Święch [30] for the case with unbounded linear term. For the latter case, it was based on the notion of the so-called $B$-continuous viscosity solutions which was introduced for first-order equations by Crandall and Lions in [7, 8]. In earlier paper Lions [21], a specific second-order HJB equation for an optimal control of a Zakai equation was studied by also using some ideas of the $B$-continuous viscosity solutions. Based on the maximum principle of Lions [22], the first comparison theorem for $B$-continuous viscosity sub/supersolutions was proved in Święch [30]. We refer to the monograph of Fabbri, Gozzi, and Święch [17] for a detailed account for the theory of viscosity solutions. One of the structural assumptions is that the state space is a separable Hilbert space, excluding for
instance the metric space $\Lambda$ (notice that in this paper we do not directly generalize those results to $\Lambda$, as we adopt pathwise, rather than Fréchet, derivatives on $\Lambda$).

Fully nonlinear first order path-dependent Hamilton-Jacobi equations in finite dimensional spaces were deeply studied by Lukoyanov \[23\]. By formulating the maximum/minimum condition on the subset of absolutely continuous paths, the existence and uniqueness theorems of viscosity solutions were obtained when Hamilton function $H$ is $d_p$-locally Lipschitz continuous in the path function. In our paper \[34\], we extended the results in \[23\] to Lipschitz continuous case under $||\cdot||_0$. For minimax solutions, Bayraktar and Keller \[1\] proposed the notion for a class of fully nonlinear PHJB equations with nonlinear, monotone, and coercive operators on Hilbert spaces and proved the existence, uniqueness and stability.

Concerning the second order case, In \[24\], Peng made the first attempt to extend Crandall-Lions framework to path-dependent case in finite dimensional spaces. By the left frozen maximization principle, a comparison principle was proved under a technical condition (16) in \[24\] which is not easy to be satisfied in practice. Tang and Zhang \[31\] improved Peng \[24\] and identified the value functional of optimal control problems as a viscosity solution to the path-dependent Bellman equations by restricting semi-jets on a space of $\alpha$-Hölder continuous paths. However, the uniqueness result was not given. Our paper \[33\] studied a class of infinite-horizon optimal control problems for stochastic differential equations with delays and identified the value functional as a unique viscosity solution to the associated second order HJB equation. The Crandall-Lions definition was not further investigated, as the fact that the supremum norm $||\cdot||_0$ is not Gâteaux differentiable was perceived as an almost insurmountable obstacle.

At the same time, when the space $H$ is finit-dimensionaional, a new concept of viscosity solutions was introduced by Ekren, Keller, Touzi and Zhang \[13\] in the semilinear context, and further extended to fully nonlinear parabolic equations by Ekren, Touzi, and Zhang \[14, 15\], elliptic equations by Ren \[26\], obstacle problems by Ekren \[12\], and degenerate second-order equations by Ren, Touzi, and Zhang \[27\] and Ren and Rosestolato \[28\]. Cosso, Federico, Gozzi, Rosestolato and Touzi \[5\] studied a class of semilinear second order PHJB equations with a linear unbounded operator on Hilbert spaces. This new notion adopted is different from the Crandall-Lions definition as the tangency condition is not pointwise but in the sense of expectation with respect to an appropriate class of probability measures. This modification simplifies a lot the proof of uniqueness and is successfully applied to the semilinear case, as this does not require anymore the passage through the maximum principle of Lions \[22\]. For instance, the comparison theorem in Cosso, Federico, Gozzi, Rosestolato and Touzi \[5\], even in the Markovian case, extends the result in Crandall-Lions framework. However, in the nonlinear case, this new notion faces some obstacles, which limits its applications. The comparison theorems in Ekren, Touzi & Zhang \[14, 15\], Ekren \[12\] and Ren \[26\], in particular, require that the Hamilton function $H$ is uniformly nondegenerate. In Ren, Touzi and Zhang \[27\] and Ren and Rosestolato \[28\], the degenerate case was studied, but in order to apply these results one has to require that the coefficients $F$, $G$, $q$ and $\phi$ are $d_p$-uniformly continuous with respect to the path function. Our context does not fall into the latter framework as in our case the coefficients are required to have continuity properties under supremum norm $||\cdot||_0$. 

At present, some authors turn to study viscosity solutions of PHJB equations in Crandall-Lions framework by defining a smooth functional and applying the Borwein-Preiss variational principle. Cosso and Russo \[6\] studied the existence and comparison theorem of viscosity solutions for the path-dependent heat equation. Our paper \[35\] and \[36\] introduced notions of viscosity solutions for second order PHJB equations in finite dimensional spaces and first order PHJB equations in Hilbert spaces, respectively, and identified the value functional as unique viscosity solution to the associated PHJB equations. None of the results above are directly applicable to our situation, as in our case, it is needed to adapt the maximum principle to the infinite-dimensional context.
As mentioned earlier, in the Markovian case, the core of Crandall-Lions viscosity solutions is $B$-continuity. Specifically, in order to overcome the difficulties caused by unbounded operator $A$, it is necessary to assume that the coefficients satisfy $B$-continuity. Then ones can study the $B$-continuity of value function and obtain the comparison theorem. The main objective of this paper is to extend Crandall-Lions viscosity solutions to our infinite-dimensional path-dependent context without $B$-continuity assumption on the coefficients.

Our core results are as follows. We define a functional $\Upsilon^3: \Lambda \to \mathbb{R}$ by

$$\Upsilon^3(\gamma_t) = S(\gamma_t) + 3|\gamma_t(t)|^6, \quad \gamma_t \in \Lambda,$$

and

$$S(\gamma_t) = \begin{cases} \frac{(||\gamma_t||^6 - |\gamma_t(t)|^6)^3}{||\gamma_t||^6}, & ||\gamma_t||_0 \neq 0; \\ 0, & ||\gamma_t||_0 = 0. \end{cases} \quad \gamma_t \in \Lambda.$$

We show that it is equivalent to $|| \cdot ||_0^6$ and study its regularity in the horizontal/vertical sense mentioned above. This key functional is the starting point for the proof of stability and uniqueness results. The uniqueness property is derived from the comparison theorem. We list some points about the proof of the comparison theorem.

(a) For every fixed $(t, \gamma_t) \in [0, T) \times A_t$, $f(\eta_t) := \Upsilon^3(\eta_t - \gamma_{t,s,A})$, $(s, \eta_t) \in [t, T] \times A$ can be used as a test function in our definition of viscosity solutions as we show that $f$ satisfies a functional Itô inequality. This is important as functional $\Upsilon^3$ is equivalent to $|| \cdot ||_0^6$. Then we can define an auxiliary function $\Psi$ which includes the functional $f$. By this, we only need to study the continuity under $|| \cdot ||_0$ rather than $B$-continuity of value functional. In particular, the comparison theorem is established when the coefficients satisfy Lipschitz assumption under $|| \cdot ||_0$. We emphasize that, with respect to the standard viscosity solution theory in infinite dimensional spaces, the $B$-continuity assumption on the coefficients is bypassed in our framework.

(b) We use $\Upsilon^3$ to define a smooth gauge-type function $\Upsilon^3: \Lambda \times \Lambda \to \mathbb{R}$ by

$$\Upsilon^3(\gamma_t, \eta_s) = \Upsilon^3(\eta_s - \gamma_{t,s,A}) + |s - t|^2, \quad 0 \leq t \leq s \leq T, \quad \gamma_t, \eta_s \in \hat{\Lambda},$$

where $\gamma_{t,s,A} \in \Lambda$, and $\gamma_{t,s,A}(\sigma) = \gamma_t(\sigma)$, $\sigma \in [0, t]$ and $\gamma_{t,s,A}(\sigma) = e^{(\sigma-t)A}\gamma_t(t)$, $\sigma \in (t, s]$. Then we can apply a modification of Borwein-Preiss variational principle (see Theorem 2.5.2 in Borwein & Zhu [2]) to get a maximum of a perturbation of the auxiliary function $\Psi$.

(c) Unfortunately, the second order vertical derivative $\partial_{x^2} S$ is not equal to 0. To apply the maximum principle (see Theorem 8.3 of [9]), a strong convergence property of auxiliary functional is needed. By doing more detailed calculations, we obtain the expected convergence property of auxiliary functional and prove the comparison theorem.

An important consequence of (a) is that our notion of viscosity solutions is meaningful even in the Markovian case. More precisely, in the Markovian case, the functional $f$ defined in (a) reduces to $f(s, y) := |y - e^{(s-t)A}x|^2_H$, $(s, y) \in [t, T] \times H$, for every fixed $(t, x) \in [0, T] \times H$. The $B$-continuity assumption on the coefficients can be bypassed with this functional.

Regarding existence, we show that the value functional $V$ defined in [1.4] is a viscosity solution to the PHJJB equation given in [1.5] by functional Itô formula, functional Itô inequality and dynamic programming principle. Such a formula was firstly provided in Dupire [11] (see Cont and Fourniè [3, 4] for a more general and systematic research). In this paper we extend the functional Itô formula to infinite dimensional spaces. We also provide a functional Itô inequality for $f$ defined in (a) from functional Itô formula.
Finally, following Dupire [11], we study PHJB equation (1.5) in the metric space \((\Lambda, d_\infty)\) in the present paper, while some literatures (for example, [5], [6]) study PHJB equations in a complete pseudometric space. The reason we do this is that it is convenient to define \(\gamma_{t-s,A}\) for every \((t, \gamma_t) \in [0, T) \times \Lambda\) in our framework, which is useful in defining metric \(d_\infty\) and test functional \(f\).

The outline of this article is as follows. In the following section, we introduce the notations used throughout the paper and review the background for BSDEs. We give a functional Itô formula which will be used to prove the existence of viscosity solutions to PHJB equation (1.5). We also present a modification of Borwein-Preiss variational principle and a smooth functional \(S\) which are the key to prove the stability and uniqueness results of viscosity solutions. Section 3 is devoted to studying PSEE (1.1). A functional Itô inequality for \(f\) defined in (a) is also provided by functional Itô formula. In Section 4, we introduce preliminary results on path-dependent stochastic optimal control problems. We give the dynamic programming principle, which will be used in the following sections. In Section 5, we define classical and viscosity solutions to PHJB equations (1.5) and prove that the value functional \(V\) defined by (1.4) is a viscosity solution. We also show the consistency with the notion of classical solutions and the stability result. Finally, the uniqueness of viscosity solutions to (1.5) is proven in section 6.

2 Preliminary work

2.1. Notations and Spaces. We list some notations that are used in this paper. Let \(\Xi, K\) and \(H\) denote real separable Hilbert spaces, with scalar products \((\cdot, \cdot)_\Xi\), \((\cdot, \cdot)_K\) and \((\cdot, \cdot)_H\), respectively. We use the symbol \(|\cdot|\) to denote the norm in various spaces, with a subscript if necessary. \(L(\Xi, H)\) denotes the space of all bounded linear operators from \(\Xi\) into \(H\); the subspace of Hilbert-Schmidt operators, with the Hilbert-Schmidt norm, is denoted by \(L_2(\Xi, H)\). Let \(\hat{\Lambda}\) denote \(\hat{\Lambda}\) of all bounded linear operators from \(\hat{\Lambda}\) to \(H\) into itself. Denote by \(S(H)\) the Banach space of bounded and self-adjoint operators in the Hilbert space \(H\) endowed with the operator norm. The operator \(A\) is the generator of a strongly continuous semigroup \(\{e^{tA}, t \geq 0\}\) of bounded linear operators in the Hilbert space \(H\). The domain of the operator \(A\) is denoted by \(\mathcal{D}(A)\). \(A^*\) denotes the adjoint operator of \(A\) with domain \(\mathcal{D}(A^*)\). Let \(T > 0\) be a fixed number. For each \(t \in [0, T]\), define \(\hat{\Lambda}_t := D([0, t]; H)\) as the set of càdlàg \(H\)-valued functions on \([0, t]\). We denote \(\hat{\Lambda}^t = \bigcup_{s \in [t, T]} \hat{\Lambda}_s\) and let \(\hat{\Lambda}\) denote \(\hat{\Lambda}^0\).

As in Dupire [11], we will denote elements of \(\hat{\Lambda}\) by lower case letters and often the final time of its domain will be subscripted, e.g. \(\gamma \in \hat{\Lambda}_t \subset \hat{\Lambda}\) will be denoted by \(\gamma_t\). Note that, for any \(\gamma \in \hat{\Lambda}\), there exists only one \(t\) such that \(\gamma \in \Lambda_t\). For any \(0 \leq s \leq t\), the value of \(\gamma_t\) at time \(s\) will be denoted by \(\gamma_t(s)\). Moreover, if a path \(\gamma_t\) is fixed, the path \(\gamma_t|[0,s]\), for \(0 \leq s \leq t\), will denote the restriction of the path \(\gamma_t\) to the interval \([0, s]\).

Following Dupire [11], for \(x \in H, \gamma_t \in \hat{\Lambda}_t, 0 \leq t \leq \bar{t} \leq T\), we define \(\gamma^x_t \in \hat{\Lambda}_t\) and \(\gamma_{t,\bar{t}}, \gamma_{t,\bar{t},A} \in \hat{\Lambda}_t\) as

\[
\begin{align*}
\gamma^x_t(s) &= \gamma_t(s), \quad s \in [0, t]; \quad \gamma^x_t(t) = \gamma_t(t) + x; \\
\gamma_{t,\bar{t}}(s) &= \gamma_t(s), \quad s \in [0, t]; \quad \gamma_{t,\bar{t}}(t) = \gamma_t(t), \quad s \in (t, \bar{t}); \\
\gamma_{t,\bar{t},A}(s) &= \gamma_t(s), \quad s \in [0, t]; \quad \gamma_{t,\bar{t},A}(s) = e^{(s-t)A}\gamma_t(t), \quad s \in (t, \bar{t}).
\end{align*}
\]

We define a norm on \(\hat{\Lambda}_t\) and a metric on \(\hat{\Lambda}\) as follows: for any \(0 \leq t \leq s \leq T\) and \(\gamma_t, \eta_s \in \hat{\Lambda}\),

\[
||\gamma_t||_0 := \sup_{0 \leq s \leq t} |\gamma_t(s)|, \quad d_\infty(\gamma_t, \eta_s) := |t - s| + ||\gamma_{t,T,A} - \eta_{s,T,A}||_0.
\]

Then \((\hat{\Lambda}_t, ||\cdot||_0)\) is a Banach space, and \((\hat{\Lambda}^t, d_\infty)\) is a complete metric space by Lemma 5.1 in [36].

5
Now we define the pathwise derivatives of Dupire [11].

**Definition 2.1.** (Pathwise derivatives) Let $t \in [0, T)$ and $f : \hat{\Lambda}^t \to \mathbb{R}$.

(i) Given $(s, \gamma_s) \in [t, T] \times \hat{\Lambda}^t$, the horizontal derivative of $f$ at $\gamma_s$ (if the corresponding limit exists and is finite) is defined as

$$\partial_t f(\gamma_s) := \lim_{h \to 0, h > 0} \frac{1}{h} [f(\gamma_s, s+h) - f(\gamma_s)]. \quad (2.2)$$

For the final time $T$, the horizontal derivative of $f$ at $\gamma_T \in \hat{\Lambda}^t$ (if the corresponding limit exists and is finite) is defined as

$$\partial_t f(\gamma_T) := \lim_{s \to T, s > T} \partial_t f(\gamma_T|_{[0,s]}).$$

If the above limit exists and is finite for every $(s, \gamma_s) \in [t, T] \times \Lambda^t$, the functional $\partial_t f : \hat{\Lambda}^t \to \mathbb{R}$ is called the horizontal derivative of $f$ with domain $\hat{\Lambda}^t$.

(ii) Given $(s, \gamma_s) \in [t, T] \times \hat{\Lambda}^t$, if there exists a $B \in H$ such that

$$\lim_{|h| \to 0} \frac{|f(\gamma_s^h) - f(\gamma_s) - (B, h)_H|}{|h|} = 0,$$

we say $\partial_x f(\gamma_t) := B$ as the first order vertical derivative of $f$ at $\gamma_s$. If $\partial_x f$ exists for every $(s, \gamma_s) \in [t, T] \times \Lambda^t$, the map $\partial_x f : \hat{\Lambda}^t \to H$ is called the first order vertical derivative of $f$ with domain $\hat{\Lambda}^t$.

If there exists a $B_1 \in S(H)$ such that

$$\lim_{|h| \to 0} \frac{|\partial_x f(\gamma_s^h) - \partial_x f(\gamma_t) - B_1 h|}{|h|} = 0,$$

we say $\partial_x x f(\gamma_t) := B_1$ as the second order vertical derivative of $f$ at $\gamma_s$. If $\partial_{xx} f$ exists for every $(s, \gamma_s) \in [t, T] \times \Lambda^t$, the map $\partial_{xx} f : \hat{\Lambda}^t \to H$ is called the second order vertical derivative of $f$ with domain $\hat{\Lambda}^t$.

**Definition 2.2.** Let $t \in [0, T)$ and $f : \hat{\Lambda}^t \to \mathbb{R}$ be given.

(i) We say $f \in C^0(\hat{\Lambda}^t)$ if $f$ is continuous in $\gamma_s$ on $\hat{\Lambda}^t$ under $d_\infty$.

(ii) We say $f \in C^p_\infty(\hat{\Lambda}^t) \subset C^0(\hat{\Lambda}^t)$ if $f$ grows in a polynomial way.

(iii) We say $f \in C^{1,2}(\hat{\Lambda}^t) \subset C^{0,2}(\hat{\Lambda}^t)$ if $\partial_t f$ exists and is continuous in $\gamma_s$ on $\hat{\Lambda}^t$ under $d_\infty$.

(iv) We say $f \in C^{p,2}(\hat{\Lambda}^t) \subset C^{1,2}(\hat{\Lambda}^t)$ if $f$ and all of its derivatives grow in a polynomial way.

Let $\Lambda_t := C([0, t], H)$ be the set of all continuous $H$-valued functions defined over $[0, t]$. We denote $\Lambda_t = \bigcup_{s \in [t, T]} \Lambda_s$ and let $\Lambda$ denote $\Lambda^0$. Clearly, $\Lambda := \bigcup_{t \in [0, T]} \Lambda_t \subset \hat{\Lambda}$, and each $\gamma \in \Lambda$ can also be viewed as an element of $\hat{\Lambda}$. $(\Lambda_t, \| \cdot \|_0)$ is a Banach space, and $(\Lambda^t, d_\infty)$ is a complete metric space by Lemma 5.1 in [36]. $f : \Lambda^t \to R$ and $\hat{f} : \hat{\Lambda}^t \to R$ are called consistent on $\Lambda^t$ if $f$ is the restriction of $\hat{f}$ on $\Lambda^t$.

**Definition 2.3.** Let $t \in [0, T)$ and $f : \Lambda^t \to \mathbb{R}$ be given.
(i) We say \( f \in C^0(\Lambda^t) \) if \( f \) is continuous in \( \gamma_s \) on \( \Lambda^t \) under \( d_\infty \).

(ii) We say \( f \in C^0_p(\Lambda^t) \subset C^0(\Lambda^t) \) if \( f \) grows in a polynomial way.

(iii) We say \( f \in C^{1,2}_p(\Lambda^t) \) if there exists \( \hat{f} \in C^{1,2}_p(\Lambda^t) \) which is consistent with \( f \) on \( \Lambda^t \).

By a cylindrical Wiener process defined on a complete probability space \((\Omega, \mathcal{F}, P)\), and with values in a Hilbert space \( \Xi \), we mean a family \( \{W(t), t \geq 0\} \) of linear mappings \( \Xi \to L^2(\Omega) \) such that for every \( \xi, \eta \in \Xi \), \( \{W(t)\xi, t \geq 0\} \) is a real Wiener process and \( E(W(t)\xi \cdot W(t)\eta) = (\xi, \eta)\xi t. \)

\( \mathcal{F}_t \), \( t \in [0, T] \), will denote the natural filtration of \( W \), augmented with the family \( \mathcal{N} \) of \( P \)-null of \( \mathcal{F} \):

\[
\mathcal{F}_t = \sigma(W(s) : s \in [0, t]) \cup \mathcal{N}.
\]

The filtration \( \{\mathcal{F}_t\}_{0 \leq t \leq T} \) satisfies the usual condition. For every \([t_1, t_2] \subset [0, T] \), we also use the notation:

\[
\mathcal{F}_{t_2}^{t_1} = \sigma(W(s) - W(t_1) : s \in [t_1, t_2]) \cup \mathcal{N}.
\]

We also write \( \mathcal{F}_t \) for \( \{\mathcal{F}_t, t \leq s \leq T\} \).

By \( \mathcal{P} \) we denote the predictable \( \sigma \)-algebra generated by predictable processes and by \( \mathcal{B}(\Theta) \) we denote, the Borel \( \sigma \)-algebra of any topological space \( \Theta \).

Next we define several classes of stochastic processes with in a Hilbert space \( K \).

- \( L^2_p(\Omega \times [0, T]; K) \) denotes the space of equivalence classes of processes \( y \in L^2(\Omega \times [0, T]; K) \), admitting a predictable version. \( L^2_p(\Omega \times [0, T]; K) \) is endowed with the norm

\[
|y|^2 = E \int_0^T |y(t)|^2 dt.
\]

- \( L^p_p(\Omega; L^2([0, T]; K)) \), defined for \( p \in [1, \infty) \), denotes the space of equivalence classes of processes \( \{y(t), t \in [0, T]\} \), with values in \( K \) such that the norm

\[
|y|^p = E \left( \int_0^T |y(t)|^2 dt \right)^{p/2}
\]

is finite, and \( \{y(t), t \in [0, T]\} \) admits a predictable version.

- \( L^p_p(\Omega; C([0, t]; K)) \), defined for \( p \in [1, \infty) \) and \( t \in (0, T] \), denotes the space of predictable processes \( \{y(s), s \in [0, t]\} \) with continuous paths in \( K \), such that the norm

\[
|y|^p = E \sup_{s \in [0, t]} |y(s)|^p
\]

is finite. Elements of \( L^p_p(\Omega; C([0, t]; K)) \) are identified up to indistinguishability.

2.2. Functional Itô formula. Assume that \( \vartheta, \varphi, \varphi_\gamma, \gamma_\theta \in \mathcal{D}_p(\Omega \times [0, T]; H) \), \( \varphi_\gamma \in L^p_p(\Omega \times [0, T]; L_2(\Xi, H)) \) for some \( p > 2 \), and \((\theta, \gamma_\theta) \in [0, T] \times \Lambda \), then the following process

\[
X(s) = e^{(s-\gamma_\theta)A} \gamma_\theta(\theta) + \int_{\theta}^{s} e^{(s-\sigma)A} \varphi_\gamma(\sigma) d\sigma + \int_{\theta}^{s} e^{(s-\sigma)A} \varphi(\sigma) dW(\sigma), \quad s \in [\theta, T],
\]

and \( X(s) = \gamma_\theta(s), \quad s \in [0, \theta] \) is well defined and \( E \sup_{s \in [0, T]} |X(s)|^p < \infty \) (see Proposition 7.3 in [10]).
Lemma 2.4. Suppose $f \in C^{1,2}_p(\hat{\Lambda}^t)$ and $A^*\partial_x f \in C^0_p(\hat{\Lambda}^t)$ for some $t \in [\theta, T)$. Then, under the above conditions, $P$-a.s., for all $s \in [t,T]$:

$$f(X_s) = f(X_t) + \int_t^s [\partial_t f(X_\sigma) + (A^*\partial_x f(X_\sigma, X(\sigma))_H + (\partial_x f(X_\sigma), \vartheta(\sigma))_H$$

$$+ \frac{1}{2} \text{Tr}(\partial_{xx} f(X_\sigma)\varpi(\sigma)\varpi^*(\sigma))] d\sigma + \int_t^s (\partial_x f(X_\sigma), \varpi(\sigma)dW(\sigma))_H.$$  

(2.4)

Here and in the following, for every $s \in [0,T]$, $X(s)$ denotes the value of $X$ at time $s$, and $X_s$ the whole history path of $X$ from time $0$ to $s$.

The proof is similar to Theorem 4.1 in Cont & Fournie [1] (see also Dupire [11]). For the convenience of readers, here we give its proof.

**Proof.** We can assume that the process $X(s)$, $s \in [0,T]$ is bounded. This can be shown by localization. Namely for arbitrary constant $C > ||\gamma||_0$ define a stopping time $\tau_C$:

$$\tau_C = \inf\{s \in [\theta, T] : |X(s)| \geq C\}$$

with the convention that $T_C = T$ if this set is empty. If one defines

$$\vartheta_C(s) = 1_{[0,\tau_C]}\vartheta(s), \quad \varpi_C(s) = 1_{[0,\tau_C]}\varpi(s), \quad s \in [0,T],$$

and $X^C(s) = e^{(s-\tau_C)^+}A X(s \land C)$, i.e.

$$\begin{cases}
X^C(s) = X(s) & \text{if } s \leq \tau_C, \\
X^C(s) = e^{(s-\tau_C)^+}A X(\tau_C) & \text{if } s > \tau_C.
\end{cases}$$

It follows from that

$$X^C(s) = e^{(s-\theta)^+}A \gamma_{\theta}(\theta) + \int_{\theta}^s e^{(s-\sigma)^+}A \vartheta_C(\sigma)d\sigma + \int_{\theta}^s e^{(s-\sigma)^+}A \varpi_C(\sigma)dW(\sigma), \quad s \in [\theta,T],$$

and $X^C(s) = \gamma_{\theta}(s)$, $s \in [0,\theta)$. If the formula (2.4) is true for $\vartheta_C$, $\varpi_C$ and $X^C$ for arbitrary $C > 0$, then, it is true in the general case.

For any $s \in [t,T]$, denote $X^n(s) = X1_{[0,t]}(\sigma) + \sum_{i=0}^{2^n-1} X(t_{i+1})1_{[t_i,t_{i+1})}(\sigma) + X(s)1_{s}(\sigma)$, $\sigma \in [0,\tau]$. Here $t_i = t + \frac{i(s-t)}{2^n}$. For every $(\sigma, \gamma_{\sigma}) \in [0,T] \times \hat{\Lambda}$, define $\gamma_{\sigma}$ by

$$\gamma_{\sigma} = \gamma_{\sigma} - l \gamma_{\sigma}(l), \quad l \in [0,\sigma], \quad \gamma_{\sigma} - l \gamma_{\sigma}(l).$$

We start with the decomposition

$$f(X^n_{t_i-}) - f(X^n_{t_{i+1}-}) = f(X^n_{t_i-}) - f(X^n_{t_{i+1}}), \quad f(X^n_{t_{i+1}-}) - f(X^n_{t_{i+1}}) = f(X^n_{t_i-}) - f(X^n_{t_{i+1}}), \quad i \geq 1.$$  

(2.5)

Let $\psi(\sigma) = f(X^n_{t_i,t_{i+1}})$, we have $f(X^n_{t_{i+1}}) - f(X^n_{t_i}) = \psi(h) - \psi(0)$, where $h = \frac{s-t}{2^n}$. Let $\psi_l$ denote the right derivative of $\psi$, then

$$\psi_l(l) = \lim_{\delta > 0, \delta \to 0} \frac{\psi_l(l + \delta) - \psi_l(l)}{\delta} = \lim_{\delta > 0, \delta \to 0} \frac{f(X^n_{t_i,t_{i+1}+\delta}) - f(X^n_{t_i,t_{i+1}})}{\delta} = \partial_t f(X^n_{t_i,t_{i+1}}), \quad l \in [0,h].$$

By

$$d_{\infty}(X^n_{t_i,t_{i+1}}, X^n_{t_i,t_{i+1}+\delta}, X^n_{t_i,t_{i+1}+h}) = |l_1 - l_2| + \sup_{0 \leq \sigma \leq |t_{i+1} - t_i|} |X^n(t_{i+1}) - e^\sigma A X^n(t_{i+1})|, \quad l_1, l_2 \in [0,h],$$
and $f \in C^{1,2}_p(\Lambda^t)$, we have $\psi$ and $\psi_{t+i}$ is continuous on $[0, h]$, therefore,

$$f(X_{t+i+1}^n) - f(X_t^n) = \psi(h) - \psi(0) = \int_0^h \psi_{t+i}(l)dl = \int_{t+i}^{t+i+1} \partial_t f(X_{t+i}^n)dl, \ i \geq 0.$$ 

The term $f(X_{t+i}^n) - f(X_t^n)$ in (2.5) can be written $\pi(X(t_{i+1}) - X(t_i)) - \pi(0)$, where $\pi(l) = f(X_{t_i}^n + l1_{\{t_i\}})$. Since $f \in C^{1,2}_p(\Lambda^t)$, $\pi$ is a $C^2$ function and $\nabla_x \pi(l) = \partial_x f(X_{t_i}^n + l1_{\{t_i\}})$, $\nabla_x^2 \pi(l) = \partial_{xx} f(X_{t_i}^n + l1_{\{t_i\}})$. Applying the Itô formula (see Proposition 1.165 in [17]) to $\pi$ between 0 and $h$ and the continuous process $(X(t_i + s) - X(t_i))_{s \geq 0}$, yields:

$$f(X_{t+i}^n) - f(X_t^n) = \pi(X(t_{i+1}) - X(t_i)) - \pi(0)$$

$$= \int_{t_i}^{t+i+1} \left[ (A^*\partial_x f(X_{t_i}^n) + (X(\sigma) - X(t_i))1_{\{t_i\}}, X(\sigma))_H + (\partial_x f(X_{t_i}^n) + (X(\sigma) - X(t_i))1_{\{t_i\}}, \vartheta(\sigma))_H 
+ \frac{1}{2} \text{Tr}[\partial_{xx} f(X_{t_i}^n + (X(\sigma) - X(t_i))1_{\{t_i\}})\varpi(\sigma)\varpi^*(\sigma)]d\sigma 
+ \int_{t_i}^{t+i+1} (\partial_x f(X_{t_i}^n + (X(\sigma) - X(t_i))1_{\{t_i\}}), \varpi(\sigma)dW(\sigma), \ i \geq 1. \ (2.6)$$

Summing over $i \geq 0$ and denoting $i(l)$ the index such that $l \in [t_i(l), t_i(l)+1)$, we obtain

$$f(X_{t+i}^n) - f(X_t) = f(X_t^n) - f(X_t^n)$$

$$= \int_t^{s_i} \partial_t f(X_{t_i(l)}(\sigma), \sigma)d\sigma + \int_{t_i}^{s_i+1} ((A^*\partial_x f(X_{t_i(l)}(\sigma), \sigma) + (X(\sigma) - X(t_i(l)))1_{\{t_i(l)\}}, X(\sigma))_H + (\partial_x f(X_{t_i(l)}(\sigma), \sigma) + (X(\sigma) - X(t_i(l)))1_{\{t_i(l)\}}, \vartheta(\sigma))_H 
+ \frac{1}{2} \text{Tr}[\partial_{xx} f(X_{t_i(l)}(\sigma) + (X(\sigma) - X(t_i(l)))1_{\{t_i(l)\}})\varpi(\sigma)\varpi^*(\sigma)]d\sigma 
+ \int_{t_i}^{s_i+1} (\partial_x f(X_{t_i(l)}(\sigma) + (X(\sigma) - X(t_i(l)))1_{\{t_i(l)\}}), \varpi(\sigma)dW(\sigma))_H. \ (2.7)$$

$f(X_{t+i}^n)$ converges to $f(X_s)$ almost surely. Since all approximations of $X$ appearing in the various integrals have a $|| \cdot ||_0$-distance from $X_s$ less than $||X_{t+i}^n - X_s||_0 \to 0$, $f \in C^{1,2}_p(\Lambda^t)$ and $A^*\partial_x f \in C^0_p(\Lambda^t)$ imply that the integrands appearing in the above integrals converge respectively to $\partial_t f(X_s), A^*\partial_x f(X_s), \partial_x f(X_s), \partial_{xx} f(X_s)$ as $n \to \infty$. By $X$ is bounded and $f \in C^{1,2}_p(\Lambda^t)$, the integrands in the various above integrals are bounded. The dominated convergence and Burkholder-Davis-Gundy inequalities for the stochastic integrals then ensure that the Lebesgue integrals converge almost surely, and the stochastic integral in probability, to the terms appearing in (2.7) as $n \to \infty$. $\Box$

By the above Lemma, we have the following important results.

**Lemma 2.5.** Let $f \in C^{1,2}_p(\Lambda^t)$ and $\hat{f} \in C^{1,2}_p(\Lambda^t)$ such that $\hat{f}$ is consistent with $f$ on $\Lambda^t$, then the following definition

$$\partial_t f := \partial_t \hat{f}, \ \partial_x f := \partial_x \hat{f}, \ \partial_{xx} f := \partial_{xx} \hat{f} \text{ on } \Lambda^t$$

is independent of the choice of $\hat{f}$. Namely, if there is another $\hat{f}' \in C^{1,2}_p(\Lambda^t)$ such that $\hat{f}'$ is consistent with $f$ on $\Lambda^t$, then the derivatives of $\hat{f}'$ coincide with those of $\hat{f}$ on $\Lambda^t$.

**Proof.** By the definition of the horizontal derivative, it is clear that $\partial_t \hat{f}(\gamma_s) = \partial_t \hat{f}'(\gamma_s)$ for every $(s, \gamma_s) \in [t, T] \times \Lambda^t$. Next, let $A = 0, \varpi = 0, \theta = t$ and $\vartheta = h \in H$ in (2.3), by the Lemma
\( \int_t^s (\partial_x \hat{f}(X_\sigma), h)_H d\sigma = \int_t^s (\partial_x \hat{f}'(X_\sigma), h)_H d\sigma, \ s \in [t, T]. \)

Here and in the sequel, for notational simplicity, we use 0 to denote elements, operators, or paths which are identically equal to zero. By the regularity \( \partial_x \hat{f}, \partial_x \hat{f}' \in C^0(\hat{A}; H) \) and the arbitrariness of \( h \in H \), we have \( \partial_x \hat{f}(\gamma_s) = \partial_x \hat{f}(\gamma_s) \) for every \( (s, \gamma_s) \in [t, T] \times \Lambda^t \). Finally, let \( A = 0, \vartheta = 0, \theta = t \) and \( \varpi \equiv a \in L_2(\Xi; H) \) in (2.3), by the Lemma 2.4,

\[ \int_t^s \text{Tr}(\partial_{xx} \hat{f}(X_\sigma) a a^*) d\sigma = \int_t^s \text{Tr}(\partial_{xx} \hat{f}'(X_\sigma) a a^*) d\sigma, \ s \in [t, T]. \]

By the regularity \( \partial_{xx} \hat{f}, \partial_{xx} \hat{f}' \in C^0(\hat{A}; L(H)) \) and the arbitrariness of \( a \in L_2(\Xi; H) \), we also have \( \partial_{xx} \hat{f}(\gamma_s) = \partial_{xx} \hat{f}(\gamma_s) \) for every \( (s, \gamma_s) \in [t, T] \times \Lambda^t \). \( \square \)

2.3. Backward stochastic differential equations. We consider the backward stochastic differential equations (BSDEs) in a Hilbert space \( K \):

\[ Y(t) + \int_t^T Z(\sigma) dW_\sigma = \int_t^T f(\sigma, Y(\sigma), Z(\sigma)) d\sigma + \eta, \quad 0 \leq t \leq T, \quad (2.8) \]

for \( t \) varying on the time interval \([0, T]\). The mapping \( f : \Omega \times [0, T] \times K \times L_2(\Xi, K) \to K \) is assumed to be measurable with respect to \( \mathcal{P} \times \mathcal{B}(K \times L_2(\Xi, K)) \). \( \eta : \Omega \to K \) is assumed to be \( \mathcal{F}_T \)-measurable. \( (f, \eta) \) are the parameters of (2.8).

We recall the well known results on the existence and uniqueness of the BSDEs.

Lemma 2.6. (Proposition 4.3 in [18]) Assume that: (i) there exists \( L > 0 \) such that

\[ |f(\sigma, y_1, z_1) - f(\sigma, y_2, z_2)| \leq L(|y_1 - y_2| + |z_1 - z_2|), \]

\( P \)-a.s. for every \( \sigma \in [0, T], y_1, y_2 \in K, z_1, z_2 \in L_2(\Xi, K) \);

(ii) there exists \( p \in [2, \infty) \) such that

\[ \mathbb{E}\left( \int_0^T |f(\sigma, 0, 0)|^2 d\sigma \right)^{\frac{p}{2}} < \infty, \quad \mathbb{E}|\eta|^p < \infty. \]

Then there exists a unique pair of processes \( Y, Z \) such that (2.8) holds for \( t \in [0, T] \) and

\[ \mathbb{E} \sup_{t \in [0, T]} |Y(t)|^p + \mathbb{E} \left( \int_0^T |Z(\sigma)|^2 d\sigma \right)^{\frac{p}{2}} \leq C_p \mathbb{E} \left( \int_0^T |f(\sigma, 0, 0)|^2 d\sigma \right)^{\frac{p}{2}} + C_p \mathbb{E}|\eta|^p, \quad (2.9) \]

for some constant \( C_p > 0 \) depending only on \( p, L, T \). Moreover, let two BSDEs of parameters \( (\eta^1, f^1) \) and \( (\eta^2, f^2) \) satisfy all the assumptions (i) and (ii). Then the difference of the solutions \( (Y^1, Z^1) \) and \( (Y^2, Z^2) \) of BSDE (2.8) with the data \( (\eta^1, f^1) \) and \( (\eta^2, f^2) \), respectively, satisfies the following estimate:

\[ \mathbb{E} \sup_{t \in [0, T]} |Y^1(t) - Y^2(t)|^p + \mathbb{E} \left( \int_0^T |Z^1(\sigma) - Z^2(\sigma)|^2 d\sigma \right)^{\frac{p}{2}} \leq C_p \mathbb{E} \left( \int_0^T |f^1(\sigma, Y^2(\sigma), Z^2(\sigma)) - f^2(\sigma, Y^2(\sigma), Z^2(\sigma))|^2 d\sigma \right)^{\frac{p}{2}} + C_p \mathbb{E}|\eta^1 - \eta^2|^p. \quad (2.10) \]

We also have the following comparison theorem on BSDEs in infinite dimensional spaces.
Lemma 2.7. (Theorem 2.7 in [BZ]) Let two BSDEs of parameters $(\eta^1, f^1)$ and $(\eta^2, f^2)$ satisfy all the assumptions of Lemma 2.6. Denote by $(Y^1, Z^1)$ and $(Y^2, Z^2)$ their respective adapted solutions. If

$$\eta^1 \geq \eta^2, \text{P-a.s., and } f^1(t,Y^1_t,Z^1_t) \geq f^2(t,Y^2_t,Z^2_t), \quad dP \otimes dt \text{ a.s.}$$

Then we have that $Y^1_t \geq Y^2_t$, a.s., for all $t \in [0,T]$.

Moreover, the comparison is strict: that is

$$Y^1_0 = Y^2_0 \iff \eta^1 = \eta^2, \quad f^1(t,Y^1_t,Z^1_t) = f^2(t,Y^2_t,Z^2_t), \quad dP \otimes dt \text{ a.s.}$$

2.4. Borwein-Preiss variational principle and functional $S$. In this subsection we introduce the Borwein-Preiss variational principle and functional $S$, which are the key to proving the uniqueness and stability of viscosity solutions.

Definition 2.8. Let $t \in [0,T]$ be fixed. We say that a continuous functional $\rho : \Lambda^t \times \Lambda^t \to [0, +\infty)$ is a gauge-type function provided that:

(i) $\rho(\gamma_s, \gamma_s) = 0$ for all $(s, \gamma_s) \in [t, T] \times \Lambda^t$,

(ii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\gamma_s, \eta_s \in \Lambda^t$, we have $\rho(\gamma_s, \eta_s) \leq \delta$ implies that $d_{\infty}(\gamma_s, \eta_s) < \varepsilon$.

The following lemma is a modification of Borwein-Preiss variational principle (see Theorem 2.5.2 in Borwein & Zhu [2]). It will be used to get a maximum of a perturbation of the auxiliary function in the proof of uniqueness. The proof is completely similar to the finite dimensional case (see Lemma 2.12 in [BZ]). Here we omit it.

Lemma 2.9. Let $t \in [0,T]$ be fixed and let $f : \Lambda^t \to \mathbb{R}$ be an upper semicontinuous functional bounded from above. Suppose that $\rho$ is a gauge-type function and $\{\delta_i\}_{i \geq 0}$ is a sequence of positive number, and suppose that $\varepsilon > 0$ and $(t_0, \gamma^0_{t_0}) \in [t,T] \times \Lambda^t$ satisfy

$$f(\gamma^0_{t_0}) \geq \sup_{(s, \gamma_s) \in [t,T] \times \Lambda^t} f(\gamma_s) - \varepsilon.$$

Then there exist $(\hat{t}, \hat{\gamma}_t) \in [t,T] \times \Lambda^t$ and a sequence $\{(t_i, \gamma^i_{t_i})\}_{i \geq 1} \subset [t,T] \times \Lambda^t$ such that

(i) $\rho(\gamma^0_{t_0}, \gamma^1_{t_0}) \leq \frac{\varepsilon}{20}, \rho(\gamma^1_{t_1}, \gamma^2_{t_1}) \leq \frac{\varepsilon}{20}$ and $t_i \uparrow \hat{t}$ as $i \to \infty$,

(ii) $f(\gamma^i_{t_i}) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma^i_{t_i}, \hat{\gamma}_t) \geq f(\gamma^0_{t_0})$, and

(iii) $f(\gamma_s) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma^i_{t_i}, \gamma_s) < f(\gamma^i_{t_i}) - \sum_{i=0}^{\infty} \delta_i \rho(\gamma^i_{t_i}, \hat{\gamma}_t)$ for all $(s, \gamma_s) \in [\hat{t}, T] \times \Lambda^t \setminus \{(\hat{t}, \gamma^i_{t_i})\}$.

Define $S : \hat{\Lambda} \to \mathbb{R}$ by, for every $(t, \gamma_t) \in [0,T] \times \hat{\Lambda}$,

$$S(\gamma_t) = \begin{cases} \frac{(||\gamma_t||^2_0 - ||\gamma(t)||_0^2)^2}{||\gamma(t)||_0^2}, & ||\gamma_t||_0 \neq 0; \\ 0, & ||\gamma_t||_0 = 0. \end{cases}$$

Define, for every $M \in \mathbb{R}$,

$$\Upsilon^M(\gamma_t) := S(\gamma_t) + M|\gamma(t)|_0^6, \quad \gamma_t \in \hat{\Lambda};$$

$$\Upsilon^M(\gamma_t, \eta_s) := \Upsilon^M(\eta_s, \gamma_t) := \Upsilon^M(\eta_s - \gamma(t,s,\Lambda)), \quad 0 \leq t \leq s \leq T, \quad \gamma_t, \eta_s \in \hat{\Lambda};$$

and

$$\Upsilon^M(\gamma_t, \eta_s) := \Upsilon^M(\eta_s, \gamma_t) := \Upsilon^M(\eta_s) + |s-t|^2, \quad 0 \leq t \leq s \leq T, \quad \gamma_t, \eta_s \in \hat{\Lambda}.$$
Lemma 2.10. \( S(\cdot) \in C_p^{1,2}(\hat{\Lambda}) \). Moreover, for every \( M \geq 1 \),
\[
\frac{8}{27} \|\gamma_t\|^6_0 \leq \Upsilon^M(\gamma_t) \leq (M + 1)\|\gamma_t\|^6_0, \quad (t, \gamma_t) \in [0, T] \times \hat{\Lambda}.
\]

**Proof.** First, we prove \( S \in C^0(\hat{\Lambda}) \). For any \( (t, \gamma_t), (s, \eta_s) \in [0, T] \times \hat{\Lambda} \), if \( s \geq t \),
\[
|\gamma_t(t) - \eta_s(s)| \leq |\gamma_t(t) - e^{(s-t)A}\gamma_t(t)| + |e^{(s-t)A}\gamma_t(t) - \eta_s(s)|,
\]
and
\[
\sup_{t \leq s \leq t} (e^{(t-s)A} - I)|\gamma_t(t)| + |\gamma_t(s) - \eta_s(s)|.
\]
If \( s < t \), let \( \gamma(t-) := \lim_{t \uparrow t} \gamma(t) \), we have
\[
|\gamma(t) - \eta_s(s)| \leq |\gamma(t) - e^{(t-s)A}\eta_s(s)| + |e^{(t-s)A}(\eta_s(s) - \gamma(t))| + |e^{(t-s)A}(\gamma(t) - \gamma(t-))|
\]
and
\[
\sup_{s \leq t \leq \inf t} (e^{(t-s)A} - I)|\gamma(t)| + \sup_{s \leq t \leq \inf t} |(e^{(t-s)A} - I)|\gamma(t) - \eta(s)| + |\eta(s) - \gamma(t)|. \]

Then we have \( S(\eta_s) \rightarrow S(\gamma_t) \) as \( \eta_s \rightarrow \gamma_t \) under \( d_\infty \). Thus \( S \in C^0(\hat{\Lambda}) \). Second, by the definition of \( S(\cdot) \), it is clear that \( \partial_t S(\gamma_t) = 0 \) for all \( (t, \gamma_t) \in [0, T] \times \hat{\Lambda} \). Next, we consider \( \partial_x S \). Clearly,
\[
\partial_x S(\gamma_0) = 0, \quad \gamma_0 \in A_0.
\]
For every \( (t, \gamma_t) \in (0, T] \times \hat{\Lambda} \), let \( \|\gamma_t\|_0^- = \sup_{0 \leq s < t} |\gamma_t(s)| \). Then, if \( |\gamma(t)| < \|\gamma_t\|_0^- \),
\[
\lim_{|h| \rightarrow 0} \frac{|S(\gamma_t^h) - S(\gamma_t)|}{|h|} = \lim_{|h| \rightarrow 0} \frac{|(\|\gamma_t\|_0^6 - |\gamma(t)|^6)^2|\gamma_t|^4(\gamma(t), h)|_H|}{|\gamma_t|_0^6}.
\]
Thus,
\[
\partial_x S(\gamma_t) = -\frac{18 (||\gamma_t||_0^6 - |\gamma(t)|^6)^2 |\gamma(t)|^4\gamma(t)}{||\gamma_t||_0^6}.
\]
If \( |\gamma(t)| > \|\gamma_t\|_0^- \),
\[
\partial_x S(\gamma_t) = 0;
\]
\[
\partial_x S(\gamma_t) = 0.
\]
if $|\gamma(t)| = \|\gamma\|_{0-} \neq 0$, since
\[
|\gamma^h|_0^6 - |\gamma(t)|_0^6 + h^6 = \begin{cases} 
0, & |\gamma(t)|_0^6 + h^6, \\
|\gamma(t)|_0^6 - |\gamma(t)|_0^6, & |\gamma(t)| + h \geq |\gamma(t)|, \\
& |\gamma(t)| + h < |\gamma(t)|,
\end{cases}
\]
we have
\[
0 \leq \lim_{|h| \to 0} \left| \frac{S(\gamma^h) - S(\gamma)}{|h|} \right| \leq \lim_{|h| \to 0} \left| \frac{|\gamma(t)|^6 - |\gamma(t) + h|^6|}{|h| \|\gamma^h\|_0^{12}} \right| = 0; \tag{2.16}
\]
if $|\gamma(t)| = \|\gamma\|_{0-} = 0$,
\[
\partial_x S(\gamma_t) = 0. \tag{2.17}
\]
From (2.12), (2.13), (2.14), (2.16) and (2.17) we obtain that, for all $(t, \gamma_t) \in [0, T] \times \dot{\Lambda}$,
\[
\partial_x S(\gamma_t) = \begin{cases} 
- \frac{18(\|\gamma\|_0^6 - |\gamma(t)|_0^6)^2|\gamma(t)|_0^4\gamma(t)}{\|\gamma\|_0^6}, & \|\gamma\|_0 \neq 0, \\
0, & \|\gamma\|_0 = 0.
\end{cases}
\]
It is clear that $\partial_x S \in C^0(\dot{\Lambda})$.
We now consider $\partial_{xx} S$. Clearly,
\[
\partial_{xx} S(\gamma_0) = 0, \quad \gamma_0 \in \Lambda_0. \tag{2.18}
\]
For every $(t, \gamma_t) \in (0, T] \times \dot{\Lambda}$, since
\[
\left( (\|\gamma\|_0^6 - |\gamma(t) + h|^6)^2 |\gamma(t) + h|^4 |\gamma(t) + h| + (\|\gamma\|_0^6 - |\gamma(t)|_0^6)^2 |\gamma(t)|_0^4 |\gamma(t)| \right) - \\
- \left( (\|\gamma\|_0^6 - |\gamma(t) + h|^6)^2 |\gamma(t) + h|^4 h + [4 (\|\gamma\|_0^6 - |\gamma(t) + h|^6)^2 |\gamma(t)|_0^2 |\gamma(t), h| + o(h) |\gamma(t)| \right),
\]
we have if $|\gamma(t)| < \|\gamma\|_{0-}$,
\[
\partial_{xx} S(\gamma_t) = \frac{216 (\|\gamma\|_0^6 - |\gamma(t)|_0^6) |\gamma(t)|_0^8 \gamma(t) \gamma(t) - 72 (\|\gamma\|_0^6 - |\gamma(t)|_0^6)^2 |\gamma(t)|^2 \gamma(t) \gamma(t) \gamma(t)}{\|\gamma\|_0^{12}} - \frac{18 (\|\gamma\|_0^6 - |\gamma(t)|_0^6)^2 |\gamma(t)|_0^4 I}{\|\gamma\|_0^{12}}; \tag{2.19}
\]
if $|\gamma(t)| > \|\gamma\|_{0-}$,
\[
\partial_{xx} S(\gamma_t) = 0; \tag{2.20}
\]
if $|\gamma(t)| = \|\gamma\|_{0-} \neq 0$, by (2.15), we have
\[
0 \leq \lim_{|h| \to 0} \left| \frac{\partial_x (\gamma^h) - \partial_x S(\gamma_t)}{|h|} \right| \leq \lim_{|h| \to 0} 18 \left( \left( |\gamma(t)|_0^6 - |\gamma(t) + h|^6 \right)^2 |\gamma(t) + h|^6 |\gamma(t) + h| \right) = 0; \tag{2.21}
\]
if $|\gamma(t)| = \|\gamma\|_{0-} = 0$,
\[
\partial_{xx} S(\gamma_t) = 0. \tag{2.22}
\]
Combining (2.19), (2.20), (2.21) and (2.22) we obtain, for all \((t, \gamma_t) \in [\hat{t}, T] \times \hat{\Lambda}^i\),

\[
\partial_{xx} S(\gamma_t) = \begin{cases} 
216(||\gamma_t||_0^6 - |\gamma_t(t)|^6)||\gamma_t(t)||^6(\gamma_t(t) - \gamma_t(0)) - 72(||\gamma_t||_0^6 - |\gamma_t(t)|^6)^2||\gamma_t(t)||^2(\gamma_t(t) - \gamma_t(0)) \\
-18(||\gamma_t||_0^6 - |\gamma_t(t)|^6)^2||\gamma_t(t)||^4 I, \\
0,
\end{cases}
\]

\[||\gamma_t||_0 \neq 0, \Rightarrow ||\gamma_t||_0 = 0.\]

It is clear that \(\partial_{xx} S \in C^0(\hat{\Lambda})\). By simple calculation, we can see that \(S\) and all of its derivatives grow in a polynomial way. Thus, we have show that \(S \in C^1_{p,2}(\hat{\Lambda})\).

Now we prove (2.11). It is clear that, for every \(n \geq 1\),

\[Y^M(\gamma_t) \leq (M + 1)||\gamma_t||_0^6, \quad (t, \gamma_t) \in [0, T] \times \hat{\Lambda} \quad \text{and every} \quad n \geq 1,
\]

\[Y^M(\gamma_t) \geq \frac{M}{3}||\gamma_t||_0^6, \quad \text{if} \quad ||\gamma_t||_0^6 - |\gamma_t(t)|^6 \leq \frac{2}{3}||\gamma_t||_0^6,
\]

and

\[Y^M(\gamma_t) \geq \frac{8}{27}||\gamma_t||_0^6, \quad \text{if} \quad ||\gamma_t||_0^6 - |\gamma_t(t)|^6 > \frac{2}{3}||\gamma_t||_0^6.
\]

Thus, we have (2.11) holds true. The proof is now complete. \(\square\)

In the proof of uniqueness of viscosity solutions, in order to apply theorem 8.3 in [2], we also need the following lemma. Its proof is completely similar to the finite dimensional case (see Lemma 2.13 in [35]). Here we omit it.

**Lemma 2.11.** For \(M \geq 3\), we have that

\[2^5 Y^M(\gamma_t) + 2^5 Y^M(\eta_t) \geq Y^M(\gamma_t + \eta_t), \quad (t, \gamma_t, \eta_t) \in [0, T] \times \hat{\Lambda} \times \hat{\Lambda}.\]  \(2.23\)

## 3 Path-dependent stochastic evolution equations.

In this section, we consider the controlled state equation (1.1). We introduce the admissible control. Let \(t, s\) be two deterministic times, \(0 \leq t \leq s \leq T\).

**Definition 3.1.** An admissible control process \(u(\cdot) = \{u(r), r \in [t, s]\}\) on \([t, s]\) is an \(\mathcal{F}^t\)-progressing measurable process taking values in some metric space \((U, d)\). The set of all admissible controls on \([t, s]\) is denoted by \(U[t, s]\). We identify two processes \(u(\cdot)\) and \(\tilde{u}(\cdot)\) in \(U[t, s]\) and write \(u(\cdot) \equiv \tilde{u}(\cdot)\) on \([t, s]\), if \(P(u(\cdot) = \tilde{u}(\cdot) \, \text{a.e. in} \, [t, s]) = 1\).

First, we describe some continuous properties of the solutions of state equation (1.1). We assume the following.

**Hypothesis 3.2.** (i) The operator \(A\) is the generator of a \(C_0\) semigroup \(\{e^{tA}, t \geq 0\}\) of bounded linear operator in Hilbert space \(H\).

(ii) The operator \(A\) is the generator of a \(C_0\) contraction semigroup \(\{e^{tA}, t \geq 0\}\) of bounded linear operators in the Hilbert space \(H\).

(iii) \(F: \Lambda \times U \to H\) and \(G: \Lambda \times U \to L_2(\Xi, H)\) are continuous, and there exists a constant \(L > 0\) such that, for all \((t, \gamma_t, u)\), \((s, \eta_s, u)\) \(\in [0, T] \times \Lambda \times U\),

\[
|F(\gamma_t, u)|^2 \vee |G(\gamma_t, u)|_{L_2(\Xi, H)}^2 \leq L^2(1 + ||\gamma_t||_0^2),
\]

\[
|F(\gamma_t, u) - F(\eta_t, u)| \vee |G(\gamma_t, u) - G(\eta_t, u)|_{L_2(\Xi, H)} \leq L||\gamma_t - \eta_t||_0.
\]  \(3.1\)
We say that $X$ is a mild solution of equation (1.1) with initial data $\xi_t \in L^p_\mathbb{P}(\Omega; C([0, t]; H))$ if it is a continuous, $\{\mathcal{F}_t\}_{t \geq 0}$-predictable process with values in $H$, and it satisfies: \(P\)-a.s.,

\[
X(s) = e^{(s-t)A} \xi_t(t) + \int_t^s e^{(s-\sigma)A} F(X_\sigma, u(\sigma))d\sigma + \int_t^s e^{(s-\sigma)A} G(X_\sigma, u(\sigma))dW(\sigma), \quad s \in [t, T],
\]

where $X(s) = \xi_t(s)$, $s \in [0, t)$. To emphasize dependence on initial data and control, we denote the solution by $X^{\xi_t, u}(.).$ Note that, if $u(\cdot) \in \mathcal{U}[t, T]$ and $\xi_t = \gamma_t \in \Lambda_t$, then $X^{\gamma_t, u}(s)$ is $\mathcal{F}^s_t$-measurable, hence independent of $\mathcal{F}_t$.

The following lemma is standard; see, for example, Theorem 3.6 in [29]. We include the proof for completeness and because it will be useful in the following.

**Lemma 3.3.** Assume that Hypothesis [LA] (i) and (ii) hold. Then for every $p > 2$, $u(\cdot) \in \mathcal{U}[0, T]$, $\xi_t \in L^p_\mathbb{P}(\Omega; C([0, t]; H))$, (1.1) admits a unique mild solution $X^{\xi_t, u}$. Moreover, if we let $X^{\xi_t, u}$ be the solutions of (1.1) corresponding $\xi_t \in L^p_\mathbb{P}(\Omega; C([0, t]; H))$ and $u(\cdot) \in \mathcal{U}[0, T]$. Then the following estimates hold:

\[
\mathbb{E}||X_t^{\xi_t, u}||_0^p \leq C_p(1 + \mathbb{E}||\xi_t||_0^p).
\]

The constant $C_p$ depending only on $p$, $T$, $L$ and $M_1 = \sup_{s \in [0, T]} |e^{sA}|$.

Finally, let $A_\mu = \mu A(\mu I - A)^{-1}$ be the Yosida approximation of $A$ and let $X^\mu$ be the solution of the following:

\[
X^\mu(s) = e^{(s-t)A_\mu} \xi_t(t) + \int_t^s e^{(s-\sigma)A_\mu} F(X_\sigma, u(\sigma))d\sigma + \int_t^s e^{(s-\sigma)A_\mu} G(X_\sigma, u(\sigma))dW(\sigma), \quad s \in [t, T];
\]

and $X^\mu(s) = \xi_t(s)$, $s \in [0, t)$. Then

\[
\lim_{\mu \to \infty} \mathbb{E} \sup_{s \in [t, T]} |X_t^{\xi_t, u}(s) - X^\mu(s)|^p = 0.
\]

**Proof.** We define a mapping $\Phi$ from $L^p_\mathbb{P}(\Omega; C([0, T]; H))$ to itself by the formula

\[
\Phi(X)(s) = e^{(s-t)A} \xi_t(t) + \int_t^s e^{(s-\sigma)A} F(X_\sigma, u(\sigma))d\sigma + \int_t^s e^{(s-\sigma)A} G(X_\sigma, u(\sigma))dW(\sigma), \quad s \in [t, T],
\]

\[
\Phi(X)(s) = \xi_t(s), \quad s \in [0, t),
\]

and show that it is a contraction, under an equivalent norm $||X|| = \left(\mathbb{E}\sup_{s \in [0, T]} e^{-\beta sp} |X(s)|^p\right)^{\frac{1}{p}}$, where $\beta > 0$ will be chosen later.

We will use the so called factorization method (see Theorem 5.2.5 in [10]). Let us take $\alpha \in (0, 1)$ such that

\[
\frac{1}{p} < \alpha < \frac{1}{2} \quad \text{and let} \quad c^{-1}_\alpha = \int_{\sigma}^{s} (s - l)^{\alpha - 1}(l - \sigma)^{-\alpha}dl.
\]

Then by the Fubini theorem and stochastic Fubini theorem, for $s \in [t, T]$,

\[
\Phi(X)(s) = e^{(s-t)A} \xi_t(t) + c_\alpha \int_t^s \int_{\sigma}^{s} (s - l)^{\alpha - 1}(l - \sigma)^{-\alpha} e^{(s-\sigma)A} e^{(l-\sigma)A} \mathbb{d} l F(X_\sigma, u(\sigma))d\sigma
\]

\[
\quad + c_\alpha \int_t^s \int_{\sigma}^{s} (s - l)^{\alpha - 1}(l - \sigma)^{-\alpha} e^{(s-\sigma)A} e^{(l-\sigma)A} \mathbb{d} l G(X_\sigma, u(\sigma))dW(\sigma)
\]

\[
= e^{(s-t)A} \xi_t(t) + c_\alpha \int_t^s (s - l)^{\alpha - 1} e^{(s-l)A} \mathbb{d} l Y(l)dl,
\]

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where
\[ Y(l) = \int_t^l (l - \sigma)^{-\alpha} e^{(l - \sigma)^{A} F(X_\sigma, u(\sigma))} d\sigma + \int_t^l (l - \sigma)^{-\alpha} e^{(l - \sigma)^{A} G(X_\sigma, u(\sigma))} dW(\sigma). \]

By the Hölder inequality, setting \( q = \frac{p}{p-1} \),
\[
e^{-\beta s} \left| \int_t^s (s - l)^{\alpha - 1} e^{(s - l)^{A} Y(l)} dl \right| \\
\leq \left( \int_t^s e^{-\beta(s-l)(s-l)^{\alpha-1}q dl} \right)^{\frac{1}{q}} \left( \int_t^s e^{-p\beta l} |e^{(s-l)^{A} Y(l)}|^p dl \right)^{\frac{1}{p}} \\
\leq M_1 \left( \int_t^T e^{-\beta l (\alpha-1)q} dl \right)^{\frac{1}{q}} \left( \int_t^T e^{-p\beta l} |Y(l)|^p dl \right)^{\frac{1}{p}}.
\]

Then we get
\[
||\Phi(X)|| \leq M_1 (E||\xi||_0^p)^{\frac{1}{p}} + M_1 c_0 \left( \int_0^T e^{-\beta l (\alpha-1)q} dl \right)^{\frac{1}{q}} (E \int_t^T e^{-p\beta l} |Y(l)|^p dl)^{\frac{1}{p}}.
\]

By the Burkholder-Davis-Gundy inequalities and Hypothesis\( \ref{3.2} \)(i) and (ii), there exists a constant \( c_\rho \) depending only on \( p \) and it may vary from line to line such that
\[
E|Y(l)|^p \\
\leq c_\rho E \left( \int_t^l (l - \sigma)^{-\alpha} |e^{(l - \sigma)^{A} F(X_\sigma, u(\sigma))}| d\sigma \right)^p \\
+ c_\rho \left( \int_t^l (l - \sigma)^{-2\alpha} |e^{(l - \sigma)^{A} G(X_\sigma, u(\sigma))}|^2_{L_2(\Xi, H)} d\sigma \right)^{\frac{1}{2}} \\
\leq c_\rho M_1^p L^p E \left( \int_t^l (l - \sigma)^{-\alpha} (1 + ||X_\sigma||_0) d\sigma \right)^p + c_\rho M_1^p L^p E \left( \int_t^l (l - \sigma)^{-2\alpha} (1 + ||X_\sigma||_0)^2 d\sigma \right)^{\frac{1}{2}} \\
\leq c_\rho M_1^p L^p E \sup_{\sigma \in [0,T]} [(1 + ||X_\sigma||_0)^p e^{-p\beta |\sigma|}] \left[ \left( \int_t^l (l - \sigma)^{-\alpha} e^{-\beta |\sigma|} d\sigma \right)^p + \left( \int_t^l (l - \sigma)^{-2\alpha} e^{-2\beta |\sigma|} d\sigma \right)^{\frac{1}{2}} \right],
\]

which implies
\[
e^{-\beta l (\alpha-1)q} E|Y(l)|^p \leq c_\rho M_1^p L^p (1 + ||X||^p) \left[ \left( \int_0^T \sigma^{-\alpha} e^{-\beta |\sigma|} d\sigma \right)^p + \left( \int_0^T \sigma^{-2\alpha} e^{-2\beta |\sigma|} d\sigma \right)^{\frac{1}{2}} \right].
\]

We conclude that
\[
||\Phi(X)|| \leq M_1 (E||\xi||_0^p)^{\frac{1}{p}} + M_1^2 L c_\alpha (Tc_\rho (1 + ||X||^p))^{\frac{1}{p}} \left( \int_0^T e^{-\beta l (\alpha-1)q} dl \right)^{\frac{1}{q}} \\
\times \left[ \left( \int_0^T \sigma^{-\alpha} e^{-\beta |\sigma|} d\sigma \right)^p + \left( \int_0^T \sigma^{-2\alpha} e^{-2\beta |\sigma|} d\sigma \right)^{\frac{1}{2}} \right]^{\frac{1}{p}}.
\]

This show that \( \Phi \) is a well defined mapping on \( L^p_\rho(\Omega; C([0,T]; H)) \). If \( X, X' \) are processes belonging to this space, similar passages show that
\[
||\Phi(X) - \Phi(X')|| \leq M_1^2 L c_\alpha (Tc_\rho)^{\frac{1}{2}} ||X - X'|| \left( \int_0^T e^{-\beta l (\alpha-1)q} dl \right)^{\frac{1}{q}}.
\]
By the Hölder inequality and the Burkholder-Davis-Gundy inequalities, we have
\[
\frac{1}{2}
\left[ \left( \int_0^T \sigma^{-\alpha} e^{-\beta \sigma} \, d\sigma \right)^p + \left( \int_0^T \sigma^{-2\alpha} e^{-2\beta \sigma} \, d\sigma \right)^{\frac{p}{2}} \right]^{\frac{1}{p}}.
\]

Therefore, for \( \beta \) sufficiently large, the mapping is a contraction. In particular, we obtain \( \|X_{\xi,t,u}\| \leq C_p(1 + (E|\xi_1|^p)^{\frac{1}{p}}) \) which prove the estimate (3.2).

Finally, for some constant \( c_p \) depending only on \( p \) that may be vary from line to line,
\[
\begin{align*}
\mathbb{E} \sup_{r \in [0,s]} |X_{\xi,t,u}(r) - X_{\mu}(r)|^p & \\
\leq c_p \left( \mathbb{E} \int_t^s \sup_{\theta \in [0,T]} |(e^{\theta A} - e^{\theta A_{\mu}})F(X_{\xi,t,u}^{\sigma}, u(\sigma))|^{p} \, d\sigma + LP\mu_1^p \mathbb{E} \int_t^s \sup_{\theta \in [0,s]} |X_{\xi,t,u}(\theta) - X_{\mu}(\theta)|^p \, d\sigma \\
+ \mathbb{E} \sup_{r \in [t,s]} \left| \int_t^r (e^{(r-s)A} G(X_{\xi,t,u}^{\sigma}, u(\sigma)) - e^{(r-s)A_{\mu}} G(X_{\mu}^{\sigma}, u(\sigma)))dW(\sigma) \right|^p \\
+ \mathbb{E} \sup_{r \in [0,s]} \left| (e^{rA} - e^{r A_{\mu}}) \xi_t(t) \right|^p.
\end{align*}
\]
(3.5)

On the other hand, by the stochastic Fubini theorem,
\[
\int_t^r \left( e^{(r-s)A} G(X_{\xi,t,u}^{\sigma}, u(\sigma)) - e^{(r-s)A_{\mu}} G(X_{\mu}^{\sigma}, u(\sigma)) \right) dW(\sigma) = c_\alpha \int_t^r (r - \theta)^{\alpha - 1} \left( e^{(r-\theta)A} Y(\theta) - e^{(r-\theta)A_{\mu}} Y(\theta) \right) d\theta,
\]
where
\[
Y(\theta) = \int_t^\theta (\theta - \sigma)^{-\alpha} e^{(\theta - \sigma)A} G(X_{\xi,t,u}^{\sigma}, u(\sigma)) dW(\sigma),
\]
and
\[
Y(\theta) = \int_t^\theta (\theta - \sigma)^{-\alpha} e^{(\theta - \sigma)A_{\mu}} G(X_{\mu}^{\sigma}, u(\sigma)) dW(\sigma).
\]

By the Hölder inequality and the Burkholder-Davis-Gundy inequalities, we have
\[
\begin{align*}
\begin{split}
\mathbb{E} \sup_{r \in [t,s]} \left| \int_t^r (e^{(r-s)A} G(X_{\xi,t,u}^{\sigma}, u(\sigma)) - e^{(r-s)A_{\mu}} G(X_{\mu}^{\sigma}, u(\sigma))) \right|^p d\sigma \\
\leq c_p \left( \mathbb{E} \int_0^T \left( t^{\alpha - 1} \right)^p \left( \mathbb{E} \int_{[0,T]} \left| e^{\theta A} - e^{\theta A_{\mu}} \right|^p d\theta \right) d\sigma \right) \\
+ M^p_1 \mu \mathbb{E} \int_t^s |Y(\theta) - Y(\theta)|^p d\theta \\
+ M^p_1 \mu \mathbb{E} \int_t^s \sup_{r \in [0,T]} |X_{\xi,t,u}(\sigma) - X_{\mu}(\sigma)|^p d\theta
\end{split}
\end{align*}
\]
(3.6)

From (3.5), (3.6) and applying Gronwall’s inequality and the Dominated Convergence Theorem, it follows that \( \mathbb{E} \sup_{0 \leq s \leq T} |X_{\xi,t,u}(s) - X_{\mu}(s)|^p \to 0 \) as \( \mu \to \infty \). \( \square \)

The next result contains the local boundedness and the continuity of the trajectory \( X_{\xi,t,u} \).
Lemma 3.4. Assume that Hypothesis 3.2 (i) and (ii) hold. Then, for any $p > 2$, $0 \leq t \leq \bar{t} \leq T$, $\xi_t, \xi'_t \in L^p_{\mathcal{F}}(\Omega; C([0,t]; H))$ and $u(\cdot) \in \mathcal{U}[0,T]$,

$$
\sup_{u(\cdot) \in \mathcal{U}[0,T]} \mathbb{E} \left| X^{\xi_t,u}(s) - e^{(s-t)A} \xi_t(t) \right|^p \leq C^1_p (1 + \mathbb{E} \|\xi_t\|_0^p) \left| s - t \right|^\frac{p}{2}, \quad s \in [t, T]; \tag{3.7}
$$

$$
\mathbb{E} \left| X^\xi_T - X^{\xi_t,u,A,u}_T \right|^p \leq C^1_p (1 + \mathbb{E} \|\xi'_t\|_0^p) (\bar{t} - t)^\frac{p}{2} + C^1_p \mathbb{E} \|\xi'_t - \xi_t\|_0^p; \tag{3.8}
$$

The constant $C^1_p$ depending only on $p$, $T$, $L$ and $M_1$.

Proof. For any $\xi_t \in \Lambda$, by (3.1) and (3.2), we obtain the following result:

$$
\mathbb{E} \left| X^{\xi_t,u}(s) - e^{(s-t)A} \xi_t(t) \right|^p \leq c_p M^p_1 1 + C_p (1 + \mathbb{E} \|\xi'_t\|_0^p) \left| s - t \right|^\frac{p}{2} \left( 1 + \left| s - t \right|^\frac{p}{2} \right).
$$

Here and in the rest of this proof, $c_p$ denotes a positive constant, whose value depend only on $p$ and may vary from line to line. Taking the supremum in $\mathcal{U}[0,T]$, we obtain (3.7). For any $0 \leq t \leq \bar{t} \leq T$, $\xi_t, \xi'_t \in \Lambda$ and $u(\cdot) \in \mathcal{U}[0,T]$, by (3.1) and (3.2), we have

$$
\mathbb{E} \sup_{t \leq s \leq \sigma} \left| X^{\xi_t,u}(s) - X^{\xi_t,u,A,u}(s) \right|^p \leq c_p M^p_1 \mathbb{E} \|\xi'_t(t) - \xi_t(t)\|^p + c_p \mathbb{E} \left[ \int_t^\sigma \left| e^{(s-r)A} F(X^{\xi_t,u}_r, u(r)) \right| dr \right]^p
$$

$$
+ c_p \mathbb{E} \left( \int_t^\sigma \left| (s-r)A \left( F(X^{\xi_t,u}_r, u(r)) - F(X^{\xi_t,u,A,u}_r, u(r)) \right) \right| dr \right)^p
$$

$$
+ c_p \mathbb{E} \left[ \int_t^\sigma \left| G(X^{\xi_t,u}_r, u(r)) - G(X^{\xi_t,u,A,u}_r, u(r)) \right| dr \right]^p
$$

$$
\leq c_p M^p_1 \mathbb{E} \left| \xi'_t(t) - \xi_t(t)\right|^p + c_p M^p_1 \left( 1 + C_p (1 + \mathbb{E} \|\xi'_t\|_0^p) \right) (\bar{t} - t)^\frac{p}{2} \left( 1 + (\bar{t} - t)^\frac{p}{2} \right)
$$

$$
+ c_p M^p_1 \int_t^\sigma \mathbb{E} \left( \left| X^{\xi_t,u}_r - X^{\xi_t,u,A,u}_r \right| \right)_0^p dr
$$

$$
+ c_p \mathbb{E} \left[ \int_t^\sigma \left( (s-r)A \left( F(X^{\xi_t,u}_r, u(r)) - F(X^{\xi_t,u,A,u}_r, u(r)) \right) \right) \right]^p
$$

Proceeding as in the proof of Lemma 4.2 we get for $\frac{1}{p} < \alpha < \frac{1}{2}$ and for a suitable constant $c_p$

$$
\mathbb{E} \left| \sup_{t \leq s \leq \sigma} \int_t^s (s-r)^\alpha \left( G(X^{\xi_t,u}_r, u(r)) - G(X^{\xi_t,u,A,u}_r, u(r)) \right) \right|^p dr
$$

$$
\leq c_p c^p_\alpha \left( \int_0^{T} r^{(\alpha-1)} dr \right)^\frac{p}{2} M^{2p}_1 L^p \left( \int_0^{T} r^{2\alpha} dr \right)^\frac{p}{2} \int_t^\sigma \mathbb{E} \left( \left| X^{\xi_t,u}_r - X^{\xi_t,u,A,u}_r \right| \right)_0^p dr.
$$

Thus,

$$
\mathbb{E} \left( \left| X^{\xi_t,u}_\sigma - X^{\xi_t,u,A,u}_\sigma \right| \right)_0^p \leq c_p M^p_1 \mathbb{E} \left| \xi'_t - \xi_t \right|^p + c_p L^p M^p_1 \left( 1 + T^\frac{p}{2} \right) \left( 1 + C_p (1 + \mathbb{E} \|\xi'_t\|_0^p) \right) (\bar{t} - t)^\frac{p}{2}
$$

$$
+ c_p M^p_1 \left( 1 + c^p_\alpha M^{2p}_1 \right) \int_t^\sigma \mathbb{E} \left( \left| X^{\xi_t,u}_r - X^{\xi_t,u,A,u}_r \right| \right)_0^p dr.
$$
Lemma 3.5. Suppose \( X^{\gamma_t,u} \) is a solution of (1.1) with initial data \( \gamma_t \in \Lambda_t \), \( f \in C_{p,1}^2(\Lambda_t) \) and \( A^* \partial_x f \in C_{p,1}^0(\Lambda_t) \) for some \( t \in [t,T] \). Then, \( P \)-a.s., for any \( s \in [t,T] \):

\[
\begin{align*}
 f (X^{\gamma_t,u}_s) &= f (X^{\gamma_t,u}_t) + \int_t^s \partial_\sigma f (X^{\gamma_t,u}_{\sigma}) d\sigma + \int_t^s (A^* \partial_x f (X^{\gamma_t,u}_{\sigma}), X^{\gamma_t,u}_{\sigma})_H \\
 &+ (\partial_\nu f (X^{\gamma_t,u}_t), F (X^{\gamma_t,u}_\sigma, u(\sigma)))_H + \frac{1}{2} \text{Tr}(\partial_{xx} f (X^{\gamma_t,u}_\sigma) G (X^{\gamma_t,u}_\sigma, u(\sigma)) G^* (X^{\gamma_t,u}_\sigma, u(\sigma))) d\sigma \\
 &+ \int_t^s (\partial_\nu f (X^{\gamma_t,u}_\sigma), G (X^{\gamma_t,u}_\sigma, u(\sigma)) dW(\sigma))_H. 
\end{align*}
\]  

(3.9)

The following lemma is also needed to prove the existence of viscosity solutions.

Lemma 3.6. Assume the Hypothesis 3.2 holds true. For every \( t \in [0,T] \), \( \eta_t \in \Lambda \) and \( M \geq 3 \), \( P \)-a.s., for all \( s \in [t,T] \):

\[
\begin{align*}
\Upsilon^M (X^{\gamma_t,u}_s - \eta_t, s, A) &= \Upsilon^M (X^{\gamma_t,u}_t - \eta_t) + \int_t^s (\partial_x \Upsilon^M (X^{\gamma_t,u}_{\sigma} - \eta_t, s, A), F (X^{\gamma_t,u}_{\sigma}, u(\sigma)))_H \\
&+ \frac{1}{2} \text{Tr}(\partial_{xx} \Upsilon^M (X^{\gamma_t,u}_{\sigma} - \eta_t, s, A) G (X^{\gamma_t,u}_{\sigma}, u(\sigma)) G^* (X^{\gamma_t,u}_{\sigma}, u(\sigma))) d\sigma \\
&+ \int_t^s (\partial_\nu \Upsilon^M (X^{\gamma_t,u}_{\sigma} - \eta_t, s, A), G (X^{\gamma_t,u}_{\sigma}, u(\sigma)) dW(\sigma))_H; 
\end{align*}
\]

(3.10)

and

\[
|X^{\gamma_t,u}(s) - e^{(s-t)A}\eta_t(t)|^6 = |y(s)|^6 
\leq |y(t)|^6 + 6 \int_t^s |y(s)|^4(y(s), F (X^{\gamma_t,u}_{\sigma}, u(\sigma)))_H \\
+ \frac{1}{2} \text{Tr}(24y(s)^2y(s)(y(s), \cdot)) G (X^{\gamma_t,u}_{\sigma}, u(\sigma)) G^* (X^{\gamma_t,u}_{\sigma}, u(\sigma))) d\sigma \\
+ 6 \int_t^s |y(s)|^4(y(s), G (X^{\gamma_t,u}_{\sigma}, u(\sigma)) dW(\sigma))_H,
\]

where \( y(s) = X^{\gamma_t,u}(s) - e^{A(s-t)}\eta_t(t), \ t \leq s \leq T \) and \( y(s) = \gamma_t(s) - \eta_t(s), \ 0 \leq s < t \).

Proof. Let \( X^\mu \) be the solution of equation (3.3) and define \( \mu^\mu \) by \( \mu^\mu(s) = X^\mu(s) - e^{(s-t)A^\mu}\eta_t(t), \ t \leq s \leq T \) and \( \mu^\mu(s) = \gamma_t(s) - \eta_t(s), \ 0 \leq s < t \), then \( \mu^\mu \) be the solution of the following:

\[
y^\mu(s) = e^{(s-t)A^\mu} y^\mu(t) + \int_t^s e^{(s-\sigma)A^\nu} F (X^\mu_{\sigma}, u(\sigma)) d\sigma + \int_t^s e^{(s-\sigma)A^\nu} G (X^\mu_{\sigma}, u(\sigma)) dW(\sigma), \ s \in [t,T],
\]

and \( \mu^\mu(s) = \gamma_t(s) - \eta_t(s), \ s \in [0,t] \). By Lemmas 2.4 and 2.10 we have, \( P \)-a.s., for all \( s \in [t,T] \):

\[
\Upsilon^M (\mu^\mu) = \Upsilon^M (\mu^\mu(t)) + \int_t^s (\partial_x \Upsilon^M (\mu^\mu), A^\mu \mu^\mu(\sigma) + F (X^\mu_{\sigma}, u(\sigma)))_H \\
+ \frac{1}{2} \text{Tr}(\partial_{xx} \Upsilon^M (\mu^\mu) G (X^\mu_{\sigma}, u(\sigma)) G^* (X^\mu_{\sigma}, u(\sigma))) d\sigma
\]
\[ + \int_t^s (\partial_x \Upsilon^M(y_\sigma^\mu), G(X_\sigma^\mu, u(\sigma))dW(\sigma))_H. \]

Noting that \( A \) is the infinitesimal generator of a \( C_0 \) contraction semigroup, we have, if \( \|y_\sigma^\mu\|_0^2 \neq 0, \)
\[ \left(6M|y^\mu(\sigma)|^4y^\mu(\sigma) - \frac{18(\|y_\sigma^\mu\|_0 - |y^\mu(\sigma)|)}{\|y_\sigma^\mu\|_0^4} |y^\mu(\sigma)|^4y^\mu(\sigma), A_\mu y^\mu(\sigma) \right)_H \leq 0, \text{ for } M \geq 3. \]

Thus, \( P \)-a.s., for all \( s \in [t,T] \):
\[ \Upsilon^M(y_\sigma^\mu) \leq \Upsilon^M(y_t^\mu) + \int_t^s (\partial_x \Upsilon^M(y_\sigma^\mu), F(X_\sigma^\mu, u(\sigma))_H \]
\[ + \frac{1}{2} \operatorname{Tr}(\partial_{xx} \Upsilon^M(y_\sigma^\mu)G(X_\sigma^\mu, u(\sigma))G^*(X_\sigma^\mu, u(\sigma)))d\sigma \]
\[ + \int_t^s (\partial_x \Upsilon^M(y_\sigma^\mu), G(X_\sigma^\mu, u(\sigma))dW(\sigma))_H. \]

Letting \( \mu \rightarrow \infty \), by (3.3), we obtain, \( P \)-a.s., for all \( s \in [t,T] \):
\[ \Upsilon^M(y_s) \leq \Upsilon^M(y_t) + \int_t^s (\partial_x \Upsilon^M(y_\sigma), F(X_\sigma, u(\sigma))_H d\sigma \]
\[ + \frac{1}{2} \operatorname{Tr}(\partial_{xx} \Upsilon^M(y_\sigma)G(X_\sigma, u(\sigma))G^*(X_\sigma, u(\sigma)))d\sigma \]
\[ + \int_t^s (\partial_x \Upsilon^M(y_\sigma), G(X_\sigma, u(\sigma))dW(\sigma))_H, \]

where \( y(s) = X^{\gamma_t,u}(s) - e^{A(s-t)}\eta_t(t), t \leq s \leq T \) and \( y(s) = \gamma_t(s) - \eta_t(s), 0 \leq s < t \). That is (3.10).

By the similar (or easier) process, we can show (3.11) holds true. The proof is now complete. \( \square \)

According to Lemma 3.6, the following

**Remark 3.7.** (i) Since \( \| \cdot \|_0^6 \) is not belongs to \( C_p^{1,2}(\Lambda) \), then, for every \( a_\varepsilon \in \Lambda, \| \gamma_t - a_{\varepsilon,t,A} \|_0^6 \) cannot appear as an auxiliary functional in the proof of the uniqueness and stability of viscosity solutions. However, by the above lemma, we can replace \( \| \gamma_t - a_{\varepsilon,t,A} \|_0^6 \) with its equivalent functional \( \Upsilon^3(\gamma_t - a_{\varepsilon,t,A}) \). Therefore, we can get the uniqueness result of viscosity solutions when coefficients satisfy continuity assumptions under \( \| \cdot \|_0 \).

(ii) It follows from (2.11) that \( \Upsilon^3(\cdot, \cdot) \) is a gauge-type function. We can apply it to Lemma 2.9 to get a maximum of perturbation of the auxiliary functional in the proof of uniqueness.

## 4 A DPP for optimal control problems.

In this section, we consider optimal control problem (1.1) and (1.2). First let us make the following assumptions:

**Hypothesis 4.1.** \( q : \Lambda \times \mathbb{R} \times \Xi \times U \rightarrow \mathbb{R} \) and \( \phi : \Lambda_T \rightarrow \mathbb{R} \) are continuous and there exists a constant \( L > 0 \) such that, for all \( (t, \gamma_t, \eta_t, y, z, u), (t', \gamma'_t, \eta'_t, y', z', u) \in [0,T] \times \Lambda \times \Lambda_T \times \mathbb{R} \times \Xi \times U, \)
\[ |q(\gamma_t, y, z, u)| \leq L(1 + \|\gamma_t\|_0 + |y| + |z|), \]
\[ |q(\gamma_t, y, z, u) - q(\gamma'_t, y', z', u)| \leq L(\|\gamma_t - \gamma'_t\|_0 + |y - y'| + |z - z'|), \]
\[ |\phi(\eta_T) - \phi(\eta'_T)| \leq L\|\eta_T - \eta'_T\|_0. \]
The above two equations are multiplied by 1 into account that \( \sum \) estimates hold:

where \( X \) and (3.2), we obtain inequality (4.2).

The constant \( C_p \) depending only on \( p, T \) and \( L \).

**Proof.** Existence and uniqueness of the solution of the backward equation (1.3) follows from Lemma 2.6. Using inequalities (2.10) and (3.2) we get inequality (4.1). Combing inequalities (2.9) and (3.2), we obtain inequality (4.2) \( \Box \).

The \( J(\xi_t, u(\cdot)) = J(\gamma_t, u(\cdot))|_{\gamma_t = \xi_t} \) and \( Y^{\xi_t, u}(t), (t, \xi_t) \in [0, T] \times L_\P^p(\Omega; C([0, t]; H)) \) and \( p > 2 \), are related by the following theorem.

**Theorem 4.3.** Under Hypothesis 3.2 (i), (ii) and Hypothesis 4.1 for every \( p > 2, t \in [0, T], u(\cdot) \in U(t, T) \) and \( \xi_t, \xi'_t \in L_\P^p(\Omega; C([0, t]; H)) \), we have

\[
J(\xi_t, u(\cdot)) = Y^{\xi_t, u}(t),
\]

and

\[
|Y^{\xi_t, u}(t) - Y^{\xi'_t, u}(t)| \leq C_p \frac{1}{p}||\xi - \xi'||_0.
\]

**Proof.** Let \( \{h^n\}, n \in \mathbb{N} \), be a dense subset of \( \Lambda_t \), \( B(h^n, \frac{1}{k}) \) be the open sphere in \( \Lambda_t \) with the radius equal to \( \frac{1}{k} \) and the center at the point \( h^n \). Set \( B_{n,k} := B(h^n, \frac{1}{k}) \setminus \bigcup_{m<n} B(h^m, \frac{1}{k}) \) and \( A_{n,k} := \{ \omega \in \Omega \mid \xi_t(\omega) \in B_{n,k} \} \). Then \( \bigcup_{n=1}^{\infty} A_{n,k} = \Omega \) and the sequence \( j^n_t(\omega) := \sum_{n=1}^{\infty} h^n \chi_{n,k}^{\omega}1_{A_{n,k}}(\omega) \) is \( \mathcal{F}_t \)-measurable and converges to \( \xi_t \) strongly and uniformly.

For every \( n \) and \( u(\cdot) \in U(t, T) \), we put \( (X^n(s), Y^n(s), Z^n(s)) = (X^{h^n, u}(s), Y^{h^n, u}(s), Z^{h^n, u}(s)) \).

Then \( X^n(s) \) is the solution of the PSEE

\[
X^n(s) = e^{(s-t)A}h^n(t) + \int_t^s e^{(s-\sigma)A} F(X^n_\sigma, u(\sigma)) d\sigma + \int_t^s e^{(s-\sigma)A} G(X^n_\sigma, u(\sigma)) dW(\sigma), \ s \in [t, T],
\]

where \( X^n_t = h^n_t \); and \( (Y^n(s), Z^n(s)) \) is the solution of the associated BSDE

\[
Y^n(s) = \phi(X^n_T) + \int_s^T q(X^n_\sigma, Y^n(\sigma), Z^n(\sigma), u(\sigma)) d\sigma - \int_s^T Z^n(\sigma) dW(\sigma), \ s \in [t, T].
\]

The above two equations are multiplied by \( 1_{A_{n,k}} \) and summed up with respect to \( n \). Thus, taking into account that \( \sum_{n=1}^{\infty} \varphi(h^n)1_{A_{n,k}} = \varphi \sum_{n=1}^{\infty} h^n 1_{A_{n,k}} \), we obtain

\[
\sum_{n=1}^{\infty} 1_{A_{n,k}} X^n(s) = \sum_{n=1}^{\infty} 1_{A_{n,k}} e^{(s-t)A} h^n(t) + \int_t^s e^{(s-\sigma)A} F \left( \sum_{n=1}^{\infty} 1_{A_{n,k}} X^n_\sigma, u(\sigma) \right) d\sigma
\]
Consequently, from the estimate (4.1), we get

$$\sum_{n=1}^{\infty} 1_{A_{n,k}} Y^n(s)$$

and

$$\phi \left( \sum_{n=1}^{\infty} 1_{A_{n,k}} X^n_T \right) + \int_s^T q \left( \sum_{n=1}^{\infty} 1_{A_{n,k}} X^n_\sigma, \sum_{n=1}^{\infty} 1_{A_{n,k}} Y^n(\sigma), \sum_{n=1}^{\infty} 1_{A_{n,k}} Z^n(\sigma), u(\sigma) \right) d\sigma$$

$$- \int_s^T \sum_{n=1}^{\infty} 1_{A_{n,k}} Z^n(\sigma) dW(\sigma).$$

Then the strong uniqueness property of the solution to the PSEE and the BSDE yields

$$X^{f_k, u}_t(s) = \sum_{n=1}^{\infty} 1_{A_{n,k}} X^n(s),$$

$$\left( Y^{f_k, u}_t(s), Z^{f_k, u}_t(s) \right) = \left( \sum_{n=1}^{\infty} 1_{A_{n,k}} Y^n(s), \sum_{n=1}^{\infty} 1_{A_{n,k}} Z^n(s) \right), \quad s \in [t, T]. \quad (4.5)$$

Finally, from $J(h^n_t, u(\cdot)) = Y^n(t)$, $n \geq 1$, we deduce that

$$Y^{f_k, u}_t = \sum_{n=1}^{\infty} 1_{A_{n,k}} Y^n(t) \sum_{n=1}^{\infty} 1_{A_{n,k}} J(h^n_t, u(\cdot)) = J(h^n_t, u(\cdot)). \quad (4.6)$$

Consequently, from the estimate (4.1), we get

$$\mathbb{E}|Y^{\xi_t, u}_t(t) - J(\xi_t, u(\cdot))|^p \leq 2^{p-1} \mathbb{E}|Y^{\xi_t, u}_t(t) - Y^{f_k, u}_t(t)|^p + 2^{p-1} \mathbb{E}|J(f_k, u(\cdot)) - J(\xi_t, u(\cdot))|^p$$

$$\leq 2^p C_p \mathbb{E}||\xi_t - f_k||_0^p \to 0 \text{ as } k \to \infty.$$

Now let us prove (4.7). By (4.1) and (4.3),

$$|Y^{\xi_t, u}_t(t) - Y^{\xi_t, u}_t(t)| = |J(\xi_t, u(\cdot)) - J(\xi_t, u(\cdot))| = |Y^{\gamma_t, u}_t(t)|_{\gamma_t = \xi_t} - |Y^{\gamma_t, u}_t(t)|_{\gamma_t = \xi_t}$$

$$= |(Y^{\gamma_t, u}_t(t) - Y^{\gamma_t, u}_t(t))_{\gamma_t = \xi_t, \eta = \xi_t}| \leq C_p \mathbb{E}||\xi_t - \xi_t||_0.$$

The proof is complete. \qed

From the uniqueness of the solution of (1.3), it follows that

$$Y^{\gamma_t, u}_t(t + \delta) = Y^{X^{\gamma_t, u}_t, u}_t(t + \delta) = J \left( X^{\gamma_t, u}_t, u(\cdot) \right), \quad \text{a.s.}$$

As we mention in Section 3, for every $(t, \gamma_t, u(\cdot)) \in [0, T] \times \Lambda \times \mathcal{U}[t, T]$, $X^{\gamma_t, u}_t(s)$ is $\mathcal{F}^s_t$-measurable for all $s \in [t, T]$, then $Y^{\gamma_t, u}_t(s)$ is $\mathcal{F}^s_t$-measurable for all $s \in [t, T]$. In particular, $Y^{\gamma_t, u}_t(t)$ is deterministic. Then we have

Theorem 4.4. Suppose Hypothesis 3.2 (i), (ii) and Hypothesis 4.1 hold true. Then $V$ is a deterministic functional.

The following property of the value functional $V$ which we present is an immediate consequence of Lemma 4.2.
Lemma 4.5. Assume that Hypothesis 3.2 (i), (ii) and Hypothesis 4.1 hold, then, for all \( p > 2 \), \( 0 \leq t \leq T \), \( \gamma_t, \eta_t \in \Lambda_t \),
\[
|V(\gamma_t) - V(\eta_t)| \leq C_p^\frac{1}{2} ||\gamma_t - \eta_t||_0; \quad |V(\gamma_t)| \leq C_p^\frac{1}{2} (1 + ||\gamma_t||_0).
\]

We also have that
\[
\text{Lemma 4.6. \ For all } t \in [0, T], \xi_t \in L^p_Q(\Omega, F_t, \Lambda_t) \text{ for any } p > 1 \text{, and } u(\cdot) \in U[t, T] \text{ we have}
\]
\[
V(\xi_t) \geq Y^{\xi.t,u}(t), \quad \text{a.s.},
\]
and for any \( \varepsilon > 0 \) there exists an admissible control \( u(\cdot) \in U[t, T] \) such that
\[
V(\xi_t) \leq Y^{\xi.t,u}(t) + \varepsilon, \quad \text{a.s.}
\]

Proof. By Theorem 4.3 and the definition of \( V(\gamma_t) \) we have, for any \( u(\cdot) \in U[t, T] \),
\[
V(\xi_t) = E(\xi_t) = \sup_{v(\cdot) \in U[t, T]} J(\varepsilon_t, v(\cdot)|_{\eta_t = \xi_t}) \geq J(\varepsilon_t, u(\cdot)|_{\eta_t = \xi_t}) = Y^{\xi.t,u}(t).
\]

We now prove (4.9). Let \( \{h^n_t\} \), \( n \in \mathbb{N} \), be a dense subset of \( \Lambda_t \), \( B(h^n_t, \frac{1}{n}) \) be the open sphere in \( \Lambda_t \) with the radius equal to \( \frac{1}{n} \) and the center at the point \( h^n_t \). Set \( B_n := B(h^n_t, \frac{1}{n}) \setminus \bigcup_{n=0}^{m} \overline{B}(h, \frac{1}{n}) \). Then the sequence \( f^k(\omega) := \sum_{n=1}^{\infty} h^n_11_{\{\xi_t \in B_n\}}(\omega) \) is \( F_t \)-measurable and converges to \( \xi_t \) strongly and uniformly. For \( k > 3 \frac{1 + C^1_p}{\varepsilon} \), we have
\[
\|f^k_t - \xi_t\|_0 \leq \frac{\varepsilon}{3}.
\]
Therefore, for every \( u(\cdot) \in U[t, T] \),
\[
\left| Y^f_t, u(t) - Y^{\xi.t,u}(t) \right| \leq \frac{\varepsilon}{3},
\]
\[
\left| V(f^k_t) - V(\xi_t) \right| \leq \frac{\varepsilon}{3}.
\]
Moreover, for every \( h^n_t \in \Lambda_t \), by the definition of \( V \) we can choose an admissible control \( v^n \) such that
\[
V(h^n_t) \leq Y^{h^n_t,v^n}(t) + \frac{\varepsilon}{3}, \quad \text{P-a.s.}
\]
Denote \( u(\cdot) := \sum_{n=1}^{\infty} v^n(\cdot)1_{\{\xi_t \in B_n\}} \), then
\[
Y^{\xi.t,u}(t) \geq -\left| Y^f_t, u(t) - Y^{\xi.t,u}(t) \right| + Y^f_t, u(t) \geq -\frac{\varepsilon}{3} + \sum_{n=1}^{\infty} Y^{h^n_t,v^n}(t)1_{\{\xi_t \in B_n\}}
\]
\[
\geq -\frac{2\varepsilon}{3} + \sum_{n=1}^{\infty} V(h^n_t)1_{\{\xi_t \in B_n\}} = -\frac{2\varepsilon}{3} + V(f^k_t)
\]
\[
\geq -\varepsilon + V(\xi_t), \quad \text{P-a.s.}
\]
Thus (4.9) holds. \( \square \)

We now discuss a dynamic programming principle (DPP) for the optimal control problem (1.1), (1.3) and (1.4). For this purpose, we define the family of backward semigroups associated with BSDE (1.3), following the idea of Peng [25].
From (4.8) and the comparison theorem (see Lemma 2.7) it follows that for any $u(\cdot) \in \mathcal{U}[t, t+\delta]$ and a real-valued random variable $\zeta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$, we put
\[ G_{t,s+\delta}^{\gamma_t,u}[\zeta] := \tilde{Y}_{t,s+\delta}^{\gamma_t,u}(s), \quad s \in [t, t+\delta], \tag{4.10} \]
where $(\tilde{Y}_{t,s+\delta}^{\gamma_t,u}(s), \tilde{Z}_{t,s+\delta}^{\gamma_t,u}(s))_{t\leq s\leq t+\delta}$ is the solution of the following BSDE with the time horizon $t+\delta$:
\[
\begin{cases}
d\tilde{Y}_{t,s+\delta}^{\gamma_t,u}(s) = -q(s, X_{s+\delta}^{\gamma_t,u}(s), \tilde{Y}_{t,s+\delta}^{\gamma_t,u}(s), \tilde{Z}_{t,s+\delta}^{\gamma_t,u}(s), u(s))ds + \tilde{Z}_{t,s+\delta}^{\gamma_t,u}(s)dW(s), \\
\tilde{Y}_{t,t+\delta}^{\gamma_t,u}(t+\delta) = \zeta,
\end{cases}
\tag{4.11}
\]
and $X_{t+\delta}^{\gamma_t,u}(\cdot)$ is the solution of the following SDE (1.1).

**Theorem 4.7.** Assume Hypothesis 3.2 (i), (ii) and Hypothesis 4.1 hold true, the value functional $V$ obeys the following DPP: for any $\gamma_t \in \Lambda_t$ and $0 \leq t < t+\delta \leq T$,
\[ V(\gamma_t) = \sup_{u(\cdot) \in \mathcal{U}[t,t+\delta]} G_{t,t+\delta}^{\gamma_t,u}[V(X_{t+\delta}^{\gamma_t,u})]. \tag{4.12} \]

**Proof.** By the definition of $V(\gamma_t)$ we have
\[ V(\gamma_t) = \sup_{u(\cdot) \in \mathcal{U}[t,t]} G_{t,T}^{\gamma_t,u}[\phi(X_T^{\gamma_t,u})] = \sup_{u(\cdot) \in \mathcal{U}[t,t+\delta]} G_{t,t+\delta}^{\gamma_t,u}[Y_{t+\delta}^{\gamma_t,u}(t+\delta)]. \]
From (4.9) and the comparison theorem (see Lemma 2.7) it follows that
\[ V(\gamma_t) \leq \sup_{u(\cdot) \in \mathcal{U}[t,t]} G_{t,T}^{\gamma_t,u}[V(X_T^{\gamma_t,u})]. \]
On the other hand, from (4.9), for any $\varepsilon > 0$ and $u(\cdot) \in \mathcal{U}[t, T]$ there exists an admissible control $\bar{u}(\cdot) \in \mathcal{U}[t+\delta, T]$ such that
\[ V(X_{t+\delta}^{\gamma_t,u}) \leq Y_{t+\delta}^{X_{t+\delta}^{\gamma_t,u},\bar{u}}(t+\delta) + \varepsilon \quad \text{P-a.s.} \]
For any $u(\cdot) \in \mathcal{U}[t, T]$ with $\bar{u}(\cdot) \in \mathcal{U}[t+\delta, T]$ from above, we put
\[ v(s) = \begin{cases} u(s), & s \in [t, t+\delta]; \\ \bar{u}(s), & s \in [t+\delta, T]; \end{cases} \in \mathcal{U}[t,T], \]
then we have
\[ \left| G_{t,t+\delta}^{\gamma_t,u}[Y_{t+\delta}^{X_{t+\delta}^{\gamma_t,u},\bar{u}}(t+\delta)] - G_{t,t+\delta}^{\gamma_t,u}[V(X_{t+\delta}^{\gamma_t,u})] \right| \leq C\varepsilon, \]
where the constant $C$ is independent of admissible control processes. Therefore,
\[ V(\gamma_t) \geq G_{t,t+\delta}^{\gamma_t,u}[Y_{t+\delta}^{X_{t+\delta}^{\gamma_t,u},v}(t+\delta)] = G_{t,t+\delta}^{\gamma_t,u}[Y_{t+\delta}^{X_{t+\delta}^{\gamma_t,u},\bar{u}}(t+\delta)] \]
\[ \geq G_{t,t+\delta}^{\gamma_t,u}[V(X_{t+\delta}^{\gamma_t,u})] - C\varepsilon, \quad \text{P-a.s.} \]
For the arbitrariness of $\varepsilon$, we get (4.12). \qed

With the help of Theorem 4.7 now show that the continuity property of $V(\gamma_t)$.

**Theorem 4.8.** Under Hypothesis 3.2 (i), (ii) and Hypothesis 4.1 then $V \in C^0(\Lambda)$ and there is a constant $C > 0$ such that for every $0 \leq t < s \leq T$, $\gamma_t, \eta_t \in \Lambda_t$,
\[ |V(\gamma_t) - V(s, \eta_t,s),\Lambda)| \leq C(1 + ||\gamma_t||_0 + ||\eta_t||_0)(s-t)^{1/2} + C||\gamma_t - \eta_t||_0. \tag{4.13} \]
Proof. Let \((t, \gamma_t, \eta_t) \in [0, T) \times \Lambda \times \Lambda\) and \(s \in [t, T]\). From Theorem 4.7 it follows that for any \(\varepsilon > 0\) there exists an admissible control \(u^\varepsilon \in U[t, s]\) such that

\[
EG_{t,s}^{\gamma_t,u^\varepsilon} [V(X_s^{\gamma_t,u^\varepsilon})] + \varepsilon \geq V(\gamma_t) \geq EG_{t,s}^{\gamma_t,u^\varepsilon} [V(X_s^{\gamma_t,u^\varepsilon})].
\]

Therefore,

\[
|V(\gamma_t) - V(\eta_{t,s,A})| \leq |I_1| + |I_2| + \varepsilon,
\]

where

\[
I_1 = EG_{t,s}^{\gamma_t,u^\varepsilon} [V(X_s^{\gamma_t,u^\varepsilon})] - EG_{t,s}^{\gamma_t,u^\varepsilon} [V(\eta_{t,s,A})],
\]

\[
I_2 = EG_{t,s}^{\gamma_t,u^\varepsilon} [V(\eta_{t,s,A})] - V(\eta_{t,s,A}).
\]

By Lemmas 2.4 and 3.4 and 4.5 we have that, for some suitable constant \(C\) independent of the control \(u^\varepsilon\),

\[
|I_1| \leq CE \left| V \left( X_s^{\gamma_t,u^\varepsilon} \right) - V(\eta_{t,s,A}) \right| \leq CE \left| X_s^{\gamma_t,u^\varepsilon} - \eta_{t,s,A} \right|_0 \leq C(1 + ||\gamma_t||_0)(s - t)^{\frac{1}{2}} + C||\gamma_t - \eta_t||_0.
\]

From the definition of \(G_{t,s}^{\gamma_t,u^\varepsilon} [\cdot]\) we get that the second term \(I_2\) can be written as

\[
I_2 = \mathbb{E} \left[ V(\eta_{t,s,A}) + \int_t^s q(\sigma, X_{\sigma}^{\gamma_t,u^\varepsilon}, Y_{\gamma_t,u^\varepsilon}(\sigma), Z_{\gamma_t,u^\varepsilon}(\sigma), u^\varepsilon(\sigma)) d\sigma \right.
\]

\[
- \int_t^s Z_{\gamma_t,u^\varepsilon}(\sigma) dW(\sigma) \left] - V(\eta_{t,s,A}) \right.
\]

\[
= \mathbb{E} \int_t^s q(\sigma, X_{\sigma}^{\gamma_t,u^\varepsilon}, Y_{\gamma_t,u^\varepsilon}(\sigma), Z_{\gamma_t,u^\varepsilon}(\sigma), u^\varepsilon(\sigma)) d\sigma,
\]

where \((Y_{\gamma_t,u^\varepsilon}(s), Z_{\gamma_t,u^\varepsilon}(s))_{t \leq s \leq s}\) is the solution of (4.11) with the terminal condition \(\eta = V(\eta_{t,s,A})\) and the control \(u^\varepsilon\). With the help of the Schwartz inequality, we then have, for some suitable constant \(C\) independent of the control \(u^\varepsilon\),

\[
I_2 \leq (s - t)^{\frac{1}{2}} \left[ \int_t^s \mathbb{E} \left| q(\sigma, X_{\sigma}^{\gamma_t,u^\varepsilon}, Y_{\gamma_t,u^\varepsilon}(\sigma), Z_{\gamma_t,u^\varepsilon}(\sigma), u^\varepsilon(\sigma)) \right|^2 d\sigma \right]^{\frac{1}{2}}
\]

\[
\leq C(s - t)^{\frac{1}{2}} \left[ \int_t^s \mathbb{E} \left( 1 + ||X_{\sigma}^{\gamma_t,u^\varepsilon}||_0^2 + ||Y_{\gamma_t,u^\varepsilon}(\sigma)||^2 + ||Z_{\gamma_t,u^\varepsilon}(\sigma)||^2 \right) d\sigma \right]^{\frac{1}{2}}
\]

\[
\leq C(1 + ||\gamma_t||_0 + ||\eta_t||_0)(s - t)^{\frac{1}{2}}.
\]

Hence, from (4.15),

\[
|V(\gamma_t) - V(s, \eta_{t,s,A})| \leq C(1 + ||\gamma_t||_0 + ||\eta_t||_0)(s - t)^{\frac{1}{2}} + C||\gamma_t - \eta_t||_0 + \varepsilon,
\]

and letting \(\varepsilon \downarrow 0\) we get (4.13) holds true.

Finally, let \((t, \gamma_t), (s, \gamma_s) \in [0, T] \times \Lambda\) and \(s \in [t, T]\), by (4.7) and (4.13)

\[
|V(\gamma_t) - V(\gamma_s)| \leq |V(\gamma_t) - V(\eta_{t,s,A})| + |V(\gamma_{t,s,A}) - V(\gamma_s')| \leq C(1 + ||\gamma_t||_0)(s - t)^{\frac{1}{2}} + C\frac{1}{p} ||\gamma_{t,s,A} - \gamma_s'||_0.
\]

Thus, \(V \in C^0(\Lambda)\). The proof is complete. □
5 Viscosity solutions to PHJB equations: Existence theorem.

In this section, we consider the second order path-dependent Hamilton-Jacobi-Bellman (PHJB) equation \([1.5]\). As usual, we start with classical solutions.

**Definition 5.1.** (Classical solution) A functional \( v \in C^1_p(\Lambda) \) with \( A^*\partial_x v \in C^0_p(\Lambda) \) is called a classical solution to the PHJB equation \([1.5]\) if it satisfies the PHJB equation \([1.5]\) point-wisely.

We will prove that the value functional \( V \) defined by \([1.4]\) is a viscosity solution of PHJB equation \([1.5]\). Define

\[
\Phi = \{ \varphi \in C^{1,2}_p(\Lambda) | A^*\partial_x \varphi \in C^0_p(\Lambda) \}. 
\]

\[ G_t = \left\{ g \in C^0_p(A^i) | \exists h \in C^1([0,T];\mathbb{R}), \delta_i, \delta'_i \geq 0, \gamma_i \in \Lambda, t_i \leq t, i = 0, 1, 2, \ldots, N > 0, \text{ with } h \geq 0, \right. \]

\[
\|\gamma_i\|_0 \leq N, \sum_{i=0}^\infty (\delta_i + \delta'_i) \leq N \text{ such that } \]

\[
g(s) = h(s)\psi^3(g_s) + \sum_{i=0}^\infty [\delta_1 \psi^3(g_s - \gamma_{t_i,s,A}) + \delta'_1 |\gamma_s(s) - e^{(s-t_i)}A^-\gamma_{t_i}(t_i)|^6], \forall \gamma_s \in A^i \}
\]

For notational simplicity, if \( g \in G_t \), we use \( \partial_t g(\gamma_s), \partial_x g(\gamma_s) \) and \( \partial_{xx} g(\gamma_s) \) to denote \( h_t(s)\psi^3(g_s) + 2 \sum_{i=0}^\infty \delta_i(s-t_i), h(s)\partial_x \psi^3(g_s) + \sum_{i=0}^\infty \delta_i \psi^{3}(\gamma_s - \gamma_{t_i,s,A}) + \delta'_i |\gamma_s(s) - e^{(s-t_i)}A^-\gamma_{t_i}(t_i)|^6 \) and \( h(s)\partial_{xx} \psi^3(g_s) + \sum_{i=0}^\infty \partial_{xx} \partial_x \psi^3(\gamma_s - \gamma_{t_i,s,A}) + \delta'_i |\gamma_s(s) - e^{(s-t_i)}A^-\gamma_{t_i}(t_i)|^6 \), respectively.

Now we can give the following definition for viscosity solutions.

**Definition 5.2.** \( w \in C^0(\Lambda) \) is called a viscosity subsolution (resp., supersolution) to \([1.5]\) if the terminal condition, \( w(\gamma_T) \leq \phi(\gamma_T) \) (resp., \( w(\gamma_T) \geq \phi(\gamma_T) \)), \( \gamma_T \in \Lambda_T \) is satisfied, and for any \( \varphi \in \Phi \) and \( g \in G_t \) with \( t \in [0,T) \), whenever the function \( w - \varphi - g \) (resp., \( w + \varphi + g \)) satisfies

\[
0 = (w - \varphi - g)(\gamma_t) = \sup_{(s,\eta_s) \in [t,T] \times \Lambda^i} (w - \varphi - g)(\eta_s),
\]

\[
\left(\text{resp., } 0 = (w + \varphi + g)(\gamma_t) = \inf_{(s,\eta_s) \in [t,T] \times \Lambda^i} (w + \varphi + g)(\eta_s), \right)
\]

where \( \gamma_t \in \Lambda \), we have

\[
\partial_t \varphi(\gamma_t) + \partial_x g(\gamma_t) + (A^*\partial_x \varphi(\gamma_t), \gamma_t(t))_H + H(\gamma_t, \varphi(\gamma_t) + g(\gamma_t), \partial_x \varphi(\gamma_t) + \partial_{xx} g(\gamma_t), \partial_{xx} \varphi(\gamma_t) + \partial_{xx} g(\gamma_t)) \geq 0,
\]

\[
\left(\text{resp., } -\partial_t \varphi(\gamma_t) - \partial_x g(\gamma_t) - (A^*\partial_x \varphi(\gamma_t), \gamma_t(t))_H + H(\gamma_t, -\varphi(\gamma_t) - g(\gamma_t), -\partial_x \varphi(\gamma_t) - \partial_{xx} g(\gamma_t), -\partial_{xx} \varphi(\gamma_t) - \partial_{xx} g(\gamma_t)) \leq 0. \right)
\]

\( w \in C^0(\Lambda) \) is said to be a viscosity solution to \([1.5]\) if it is both a viscosity subsolution and a viscosity supersolution.

**Theorem 5.3.** Suppose that Hypotheses \([3.2]\) and \([4.1]\) hold. Then the value functional \( V \) defined by \([1.4]\) is a viscosity solution to \([1.5]\).
Proof. First, let $\varphi \in \Phi$ and $g \in \mathcal{G}_t$ with $\hat{t} \in [0, T)$ such that

$$0 = (V - \varphi - g)(\hat{\gamma}_t) = \sup_{(s, \eta_s) \in \hat{t}, T] \times \Lambda^t} (V - \varphi - g)(\eta_s), \tag{5.1}$$

where $\hat{\gamma}_t \in \Lambda$. Thus, by the DPP (Theorem 4.17), we obtain that, for all $\hat{t} \leq \hat{t} + \delta \leq T$,

$$0 = V(\hat{\gamma}_t) - (\varphi + g)(\hat{\gamma}_t) = \sup_{u(\cdot) \in \mathcal{U}[\hat{t}, \hat{t} + \delta]} G_{\hat{t}, \hat{t} + \delta}^{\hat{\gamma}_t, u} \left[ V \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t, u} \right) \right] - (\varphi + g)(\hat{\gamma}_t). \tag{5.2}$$

Then, for any $\varepsilon > 0$ and $0 < \delta \leq T - \hat{t}$, we can find a control $u^\varepsilon(\cdot) \equiv u^\varepsilon, \delta(\cdot) \in \mathcal{U}[\hat{t}, \hat{t} + \delta]$ such that the following result holds:

$$-\varepsilon \delta \leq G_{\hat{t}, \hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \left[ V \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \right) \right] - (\varphi + g)(\hat{\gamma}_t). \tag{5.3}$$

We note that $G_{\hat{t}, \hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \left[ V \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \right) \right]$ is defined in terms of the solution of the BSDE:

$$\begin{cases}
    dY_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon}(s) = -q \left( X_{s}^{\hat{\gamma}_t, u^\varepsilon}, Y_{s}^{\hat{\gamma}_t, u^\varepsilon}(s), Z_{s}^{\hat{\gamma}_t, u^\varepsilon}(s), u^\varepsilon(s) \right) ds + Z_{s}^{\hat{\gamma}_t, u^\varepsilon}(s) dW(s), s \in [\hat{t}, \hat{t} + \delta], \\
    Y_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} = V \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \right),
\end{cases} \tag{5.4}$$

by the following formula:

$$G_{\hat{t}, \hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \left[ V \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon} \right) \right] = Y_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon}(s), \quad s \in [\hat{t}, \hat{t} + \delta].$$

Applying functional Itô formula (2.15) to $\varphi \left( X_{s}^{\hat{\gamma}_t, u^\varepsilon} \right)$ and inequality (3.10) to $g \left( X_{s}^{\hat{\gamma}_t, u^\varepsilon} \right)$, we get that

$$(\varphi + g) \left( X_{s}^{\hat{\gamma}_t, u^\varepsilon} \right) \leq Y_{s}^{1}(s) \equiv (\varphi + g)(\hat{\gamma}_t) + \int_{\hat{t}}^{s} \left( \mathcal{L}(\varphi + g) \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon}, u^\varepsilon(\sigma) \right) \right) d\sigma$$

$$- \int_{\hat{t}}^{s} q \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon}, (\varphi + g) \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon} \right), \partial_x (\varphi + g) \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon} \right) G \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon}, u^\varepsilon(\sigma) \right), u^\varepsilon(\sigma) \right) d\sigma$$

$$+ \int_{\hat{t}}^{s} \left( \partial_x (\varphi + g) \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon} \right), G \left( X_{\sigma}^{\hat{\gamma}_t, u^\varepsilon}, u^\varepsilon(\sigma) \right) dW(\sigma) \right)_H, \tag{5.5}$$

where

$$(\mathcal{L}(\varphi + g))(\gamma_t, u) = \partial_t (\varphi + g)(\gamma_t) + (A^* \partial_x \varphi(\gamma_t), \gamma_t(t))_H + (\partial_x (\varphi + g)(\gamma_t), F(\gamma_t, u)_H$$

$$+ \frac{1}{2} \text{Tr} [\partial_{xx} (\varphi + g)(\gamma_t) G(\gamma_t, u) G(\gamma_t, u)^*]$$

$$+ q(\gamma_t, (\varphi + g)(\gamma_t), (\partial_x (\varphi + g)(\gamma_t)) G(\gamma_t, u)_H, (t, \gamma_t, u) \in [0, T] \times \Lambda \times \mathcal{U}. \tag{5.6}$$

It is clear that

$$Y_{1}(\hat{t}) = (\varphi + g)(\hat{\gamma}_t).$$

Set

$$Y_{2, \hat{t}, u^\varepsilon}(s) := Y_{1}(s) - Y_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon}(s) \geq (\varphi + g) \left( X_{s}^{\hat{\gamma}_t, u^\varepsilon} \right) - Y_{\hat{t} + \delta}^{\hat{\gamma}_t, u^\varepsilon}(s), \quad s \in [\hat{t}, \hat{t} + \delta],$$

$$Y_{3}(s) := Y_{1}(s) - (\varphi + g) \left( X_{s}^{\hat{\gamma}_t, u^\varepsilon} \right), \quad s \in [\hat{t}, \hat{t} + \delta],$$
\[
Z^{\gamma_i, u^\varepsilon} (s) := \partial_x (\varphi + g) \left( X^{\gamma_i, u^\varepsilon}_s \right) G \left( X^{\gamma_i, u^\varepsilon}_s, u^\varepsilon (s) \right) - Z^{\gamma_i, u^\varepsilon} (s), \ s \in [\hat{t}, \hat{t} + \delta].
\]

Comparing (5.4) and (5.5), we have, \( P \)-a.s.,
\[
dY^{\gamma_i, u^\varepsilon} (s) = \left[ (\mathcal{L} (\varphi + g)) \left( X^{\gamma_i, u^\varepsilon}_s, u^\varepsilon (s) \right) - q \left( X^{\gamma_i, u^\varepsilon}_s, (\varphi + g) \left( X^{\gamma_i, u^\varepsilon}_s \right) \right) \right. \\
\left. \quad + q \left( X^{\gamma_i, u^\varepsilon}_s, Y^{\gamma_i, u^\varepsilon} (s), Z^{\gamma_i, u^\varepsilon} (s), u^\varepsilon (s) \right) \right] ds + Z^{\gamma_i, u^\varepsilon} (s) dW (s)
\]
where \(|A_1| \vee |A_2| \leq L\). Applying Itô formula (see also Proposition 2.2 in [16]), we obtain
\[
Y^{\gamma_i, u^\varepsilon} (t) = \mathbb{E} \left[ Y^{\gamma_i, u^\varepsilon} (t + \hat{t}) \Gamma^\hat{t} (\hat{t} + \delta) \right. \\
\left. - \int_t^{\hat{t} + \delta} \Gamma^\sigma (\mathcal{L} (\varphi + g)) \left( X^{\gamma_i, u^\varepsilon}_\sigma, u^\varepsilon (\sigma) \right) + A_1 (\sigma) Y^3 (\sigma) \right] d\sigma \left| \mathcal{F}_t \right].
\]
where \( \Gamma^\sigma (\cdot) \) solves the linear SDE
\[
d\Gamma^\sigma (s) = \Gamma^\sigma (A_1 (s) ds + A_2 (s) dW (s)), \ s \in [\hat{t}, \hat{t} + \delta]; \quad \Gamma^\sigma (\hat{t}) = 1.
\]
Obviously, \( \Gamma^\sigma \geq 0 \). Combining (5.3) and (5.7), we have
\[
-\varepsilon \leq \frac{1}{\delta} \mathbb{E} \left[ - Y^{\gamma_i, u^\varepsilon} (\hat{t} + \delta) \Gamma^\hat{t} (\hat{t} + \delta) + \int_t^{\hat{t} + \delta} \Gamma^\sigma \left( [\mathcal{L} (\varphi + g)) \left( X^{\gamma_i, u^\varepsilon}_\sigma, u^\varepsilon (\sigma) \right) + A_1 (\sigma) Y^3 (\sigma) \right] d\sigma \right]
\]
\[
= \frac{1}{\delta} \mathbb{E} \left[ Y^{\gamma_i, u^\varepsilon} (\hat{t} + \delta) \Gamma^\hat{t} (\hat{t} + \delta) \right] + \frac{1}{\delta} \mathbb{E} \left[ \int_t^{\hat{t} + \delta} (\mathcal{L} (\varphi + g)) \left( \hat{\gamma}_i, u^\varepsilon (\sigma) \right) d\sigma \right]
\]
\[
+ \frac{1}{\delta} \mathbb{E} \left[ \int_t^{\hat{t} + \delta} \left( \mathcal{L} (\varphi + g)) \left( X^{\gamma_i, u^\varepsilon}_\sigma, u^\varepsilon (\sigma) \right) - (\mathcal{L} (\varphi + g)) \left( \hat{\gamma}_i, u^\varepsilon (\sigma) \right) \right] d\sigma \right]
\]
\[
+ \frac{1}{\delta} \mathbb{E} \left[ \int_t^{\hat{t} + \delta} \Gamma^\sigma (\sigma) A_1 (\sigma) Y^3 (\sigma) d\sigma \right]
\]
\[
:= I + II + III + IV + V.
\]
(5.8)

Since the coefficients in \( \mathcal{L} \) satisfy linear growth condition, combining the regularity of \( \varphi \in \Phi \) and \( g \in \mathcal{G} \), there exist an integer \( \bar{p} \geq 1 \) and a constant \( C > 0 \) independent of \( u \in U \) such that, for all \((t, \gamma_i, u) \in [0, T] \times \Lambda \times U, \)
\[
|\langle \varphi + g \rangle \gamma_i | \vee | \langle \mathcal{L} (\varphi + g) \rangle \gamma_i, u | \leq C (1 + || \gamma_i ||_0)^{\bar{p}}.
\]
(5.9)

In view of Lemma 3.4, we also have
\[
\sup_{u \in U \left[ t, t + \delta \right]} \mathbb{E} | \Gamma^\hat{t} (\hat{t} + \delta) - 1 |^2 \leq C \delta.
\]
Thus, by (5.1) and (5.5),

\[ I = -\frac{1}{\delta} \mathbb{E} \left[ \left( Y^1(\hat{t} + \delta) - Y^{\hat{\gamma}_t; u^\varepsilon}(\hat{t} + \delta) \right) \Gamma^\varepsilon(\hat{t} + \delta) \right] \]

\[ \leq -\frac{1}{\delta} \mathbb{E} \left[ \left( (\varphi + g) \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t; u^\varepsilon} \right) - Y^{\hat{\gamma}_t; u^\varepsilon}(\hat{t} + \delta) \right) \Gamma^\varepsilon(\hat{t} + \delta) \right] \]

\[ = \frac{1}{\delta} \mathbb{E} \left[ \left( V \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t; u^\varepsilon} \right) - (\varphi + g) \left( X_{\hat{t} + \delta}^{\hat{\gamma}_t; u^\varepsilon} \right) \right) \Gamma^\varepsilon(\hat{t} + \delta) \right] \leq 0; \quad (5.10) \]

\[ II \leq \frac{1}{\delta} \int_{\hat{t}}^{\hat{t} + \delta} \sup_{u \in \mathcal{U}} \mathbb{E} \left( \mathcal{L}(\varphi + g)(\hat{\gamma}_t, u)d\sigma \right) \]

\[ = \partial_t(\varphi + g)(\hat{\gamma}_t) + (A^* \partial_x \varphi(\hat{\gamma}_t), \hat{\gamma}_t(\hat{t})) \mathcal{H} \]

\[ + \mathbb{H}(\hat{\gamma}_t, (\varphi + g)(\hat{\gamma}_t), \partial_x(\varphi + g)(\hat{\gamma}_t), \partial_{xx}(\varphi + g)(\hat{\gamma}_t)). \quad (5.11) \]

Now we estimate higher order terms III, IV and V. By (5.9) and the dominated convergence theorem, we have

\[ \lim_{\sigma \to \hat{t}} \mathbb{E} \left[ \left( \mathcal{L}(\varphi + g) \right) \left( X_{\sigma}^{\hat{\gamma}_t; u^\varepsilon}(\sigma) \right) - (\mathcal{L}(\varphi + g)(\hat{\gamma}_t, u^\varepsilon(\sigma)) \right] = 0, \]

and

\[ \lim_{\sigma \to \hat{t}} \mathbb{E} \left[ \Gamma^\varepsilon(\sigma) A_1(\sigma) Y^3(\sigma) \right] \]

\[ \leq L \lim_{\sigma \to \hat{t}} \mathbb{E} \left[ \left| \mathcal{L}(\sigma) - (\varphi + g)(\hat{\gamma}_t) \right| + \left| (\varphi + g) \left( X_{\sigma}^{\hat{\gamma}_t; u^\varepsilon} \right) - (\varphi + g)(\hat{\gamma}_t) \right| \right] = 0, \]

then

\[ \limsup_{\delta \to 0} |III| \leq \frac{1}{\delta} \int_{\hat{t}}^{\hat{t} + \delta} \mathbb{E} \left[ \left( \mathcal{L}(\varphi + g) \right) \left( X_{\sigma}^{\hat{\gamma}_t; u^\varepsilon}(\sigma) \right) - (\mathcal{L}(\varphi + g)(\hat{\gamma}_t, u^\varepsilon(\sigma)) \right] d\sigma = 0; \quad (5.12) \]

\[ \limsup_{\delta \to 0} |V| \leq \frac{1}{\delta} \int_{\hat{t}}^{\hat{t} + \delta} \mathbb{E} \left[ \Gamma^\varepsilon(\sigma) A_1(\sigma) Y^3(\sigma) \right] d\sigma = 0. \quad (5.13) \]

Moreover, for some finite constant \( C > 0, \)

\[ |IV| \leq \frac{1}{\delta} \int_{\hat{t}}^{\hat{t} + \delta} \mathbb{E} \left[ |(\mathcal{L}(\sigma) - (\varphi + g)) \left( X_{\sigma}^{\hat{\gamma}_t; u^\varepsilon}(\sigma) \right) \right] d\sigma \]

\[ \leq \frac{1}{\delta} \int_{\hat{t}}^{\hat{t} + \delta} \left( \mathbb{E} \left[ \Gamma^\varepsilon(\sigma) - 1 \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| \mathcal{L}(\varphi + g) \left( X_{\sigma}^{\hat{\gamma}_t; u^\varepsilon}(\sigma) \right) \right| \right] \right)^{\frac{1}{2}} d\sigma \]

\[ \leq C(1 + ||\hat{\gamma}_t||_0)^p \delta^{\frac{1}{2}}. \quad (5.14) \]

Substituting (5.10), (5.11) and (5.14) into (5.8), we have

\[ -\varepsilon \leq \partial_t(\varphi + g)(\hat{\gamma}_t) + (A^* \partial_x \varphi(\hat{\gamma}_t), \hat{\gamma}_t(\hat{t})) \mathcal{H} \]

\[ + H(\hat{\gamma}_t, (\varphi + g)(\hat{\gamma}_t), \partial_x(\varphi + g)(\hat{\gamma}_t), \partial_{xx}(\varphi + g)(\hat{\gamma}_t)) \]

\[ + |III| + |V| + C(1 + ||\hat{\gamma}_t||_0)^p \delta^{\frac{1}{2}}. \quad (5.15) \]

Sending \( \delta \) to 0, by (5.12) and (5.13), we have

\[ -\varepsilon \leq \partial_t(\varphi + g)(\hat{\gamma}_t) + (A^* \partial_x \varphi(\hat{\gamma}_t), \hat{\gamma}_t(\hat{t})) \mathcal{H} \]

\[ + H(\hat{\gamma}_t, (\varphi + g)(\hat{\gamma}_t), \partial_x(\varphi + g)(\hat{\gamma}_t), \partial_{xx}(\varphi + g)(\hat{\gamma}_t)). \]

By the arbitrariness of \( \varepsilon, \) we show \( V \) is a viscosity subsolution to (1.5).

In a symmetric (even easier) way, we show that \( V \) is also a viscosity supersolution to equation (1.5). This step completes the proof. \( \square \)

Now, let us give the result of classical solutions, which show the consistency of viscosity solutions.
Theorem 5.4. Let $V$ denote the value functional defined by (5.4). If $V \in C^{1,2}_p(\Lambda)$ and $A^*_x V \in C^0_p(\Lambda)$, then $V$ is a classical solution of equation (5.3).

Proof. First, using the definition of $V$ yields $V(\gamma_T) = \phi(\gamma_T)$ for all $\gamma_T \in \Lambda_T$. Next, for fixed $(t, \gamma, u) \in [0, T] \times \Lambda \times U$, from the DPP (Theorem 4.7), we obtain the following result:

$$0 \geq G_{t,t+\delta}^\gamma [V(X_t^{\gamma,u})] - V(\gamma_t), \quad 0 \leq \delta \leq T - t. \quad (5.16)$$

Thus

$$V(\gamma_t) \geq Y^{\gamma_t,u}(t) = \int_t^{t+\delta} q(X_s^{\gamma,t,u}, Y^{\gamma_t,u}(s), Z^{\gamma_t,u}(s), u) ds + V(X_t^{\gamma,u}) - \int_t^{t+\delta} Z^{\gamma_t,u}(s) dW(s).$$

Applying functional Itô formula (2.4) to $V(X_t^{\gamma,u})$, the inequality above implies that, for all $0 \leq \delta \leq T - t$,

$$0 \geq \int_t^{t+\delta} q(X_s^{\gamma,t,u}, Y^{\gamma_t,u}(s), Z^{\gamma_t,u}(s), u) ds + \int_t^{t+\delta} (\mathcal{L}V)(X_s^{\gamma,t,u}, u) ds,$$

$$- \int_t^{t+\delta} q(X_s^{\gamma,t,u}, V(X_s^{\gamma,t,u}), \partial_x V(X_s^{\gamma,t,u})G(X_s^{\gamma,t,u}, u), u) ds,

+ \int_t^{t+\delta} (\partial_x V(X_s^{\gamma,t,u})G(X_s^{\gamma,t,u}, u) - Z^{\gamma_t,u}(s)) dW(s),

= \int_t^{t+\delta} [(\mathcal{L}V)(X_s^{\gamma,t,u}, u) ds - A_1(s)Y^2(s) - (A_2(s), Z^2(s))\Xi] ds + \int_t^{t+\delta} Z^2(s) dW(s),$$

where

$$Y^2(s) := V(X_s^{\gamma,t,u}) - Y^{\gamma_t,u}(s), \quad s \in [t, t+\delta],$$

$$Z^2(s) := (\partial_x V(X_s^{\gamma,t,u}))G(X_s^{\gamma,t,u}, u) - Z^{\gamma_t,u}(s), \quad s \in [t, t+\delta],$$

and $|A_1| \vee |A_2| \leq L$. Applying Itô formula (see also Proposition 2.2 in [10]), we obtain

$$Y^2(t) = E \left[ Y^2(t+\delta) \Gamma^t(t+\delta) - \int_t^{t+\delta} \Gamma^t(\sigma)(\mathcal{L}V)(X_{s}^{\gamma,t,u}, u(\sigma)) d\sigma \right] F_t, \quad (5.17)$$

where $\Gamma^t(\cdot)$ solves the linear SDE

$$d\Gamma^t(s) = \Gamma^t(s)(A_1(s) ds + A_2(s) dW(s)), \quad s \in [t, t+\delta]; \quad \Gamma^t(t) = 1.$$

Obviously, $\Gamma^t \geq 0$. Combining (5.16) and (5.17), we obtain that

$$0 \geq \frac{1}{\delta} E \left[ \int_t^{t+\delta} (\mathcal{L}V)(\gamma_t, u) d\sigma \right] + \frac{1}{\delta} E \left[ \int_t^{t+\delta} [(\mathcal{L}V)(X_{s}^{\gamma,t,u}, u) - (\mathcal{L}V)(\gamma_t, u)] d\sigma \right]$$

$$+ \frac{1}{\delta} E \left[ \int_t^{t+\delta} (\Gamma^t(\sigma) - 1)(\mathcal{L}V)(X_{s}^{\gamma,t,u}, u) d\sigma \right].$$

By the proving process of the above theorem, letting $\delta \to 0$, we have

$$0 \geq (\mathcal{L}V)(\gamma_t, u).$$

Taking the supremum over $u \in U$,

$$0 \geq \partial_t V(\gamma_t) + (A^*_x V(\gamma_t), \gamma(t))_H + H(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)) \quad (5.18)$$

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On the other hand, let \((t, \gamma_t) \in [0, T] \times \Lambda\) be fixed. Then, by (4.12), there exists an \(\tilde{u}(\cdot) \equiv u_{\epsilon, \delta}(\cdot) \in \mathcal{U}[t, t + \delta]\) for any \(\epsilon > 0\) and \(0 \leq \delta \leq T - t\) such that

\[
-\epsilon \delta \leq G_{t, t+\delta}^{\gamma_t, \tilde{u}} [V(X_{t+\delta}^{\gamma_t, \tilde{u}})] - V(\gamma_t).
\] (5.19)

Thus we have

\[
V(\gamma_t) - \epsilon \delta \leq Y_{\gamma_t, \tilde{u}}(t) = \int_t^{t+\delta} q(X_s^{\gamma_t, \tilde{u}}, Y_s^{\gamma_t, \tilde{u}}(s), Z_s^{\gamma_t, \tilde{u}}(s), \tilde{u}(s))ds + V(X_{t+\delta}^{\gamma_t, \tilde{u}}) - \int_t^{t+\delta} Z_s^{\gamma_t, \tilde{u}}(s)dW(s).
\]

Applying functional Itô formula (2.4) to \(V(X_{t+\delta}^{\gamma_t, \tilde{u}})\), the inequality above implies that, for all \(0 \leq \delta \leq T - t\),

\[-\epsilon \delta \leq \int_t^{t+\delta} q(X_s^{\gamma_t, \tilde{u}}, Y_s^{\gamma_t, \tilde{u}}(s), Z_s^{\gamma_t, \tilde{u}}(s), \tilde{u}(s))ds + \int_t^{t+\delta} (\mathcal{L}V)(X_s^{\gamma_t, \tilde{u}}, \tilde{u}(s))ds
- \int_t^{t+\delta} q(X_s^{\gamma_t, \tilde{u}}, V(X_s^{\gamma_t, \tilde{u}}), \partial_x V(X_s^{\gamma_t, \tilde{u}})G(X_s^{\gamma_t, \tilde{u}}, \tilde{u}(s)), \tilde{u}(s))ds
+ \int_t^{t+\delta} (\partial_x V(X_s^{\gamma_t, \tilde{u}})G(X_s^{\gamma_t, \tilde{u}}, \tilde{u}(s)) - Z_s^{\gamma_t, \tilde{u}}(s))dW(s),
= \int_t^{t+\delta} (\mathcal{L}V)(X_s^{\gamma_t, \tilde{u}}, \tilde{u}(s))ds - B_1(s) \tilde{Y}^2(s) - (B_2(s), \tilde{Z}^2(s))_{\Xi}ds + \int_t^{t+\delta} \tilde{Z}^2(s)dW(s),
\]

where

\[
\tilde{Y}^2(s) := V(X_s^{\gamma_t, \tilde{u}}) - Y_s^{\gamma_t, \tilde{u}}(s), \quad s \in [t, t + \delta],
\]
\[
\tilde{Z}^2(s) := \partial_x V(X_s^{\gamma_t, \tilde{u}})G(X_s^{\gamma_t, \tilde{u}}, u) - Z_s^{\gamma_t, \tilde{u}}(s), \quad s \in [t, t + \delta],
\]

and \(|B_1| \vee |B_2| \leq L\). Applying Itô formula (see also Proposition 2.2 in [16]), we obtain

\[
\tilde{Y}^2(t) = \mathbb{E} \left[ \tilde{Y}^2(t + \delta) \hat{\Gamma}^t(t + \delta) - \int_t^{t+\delta} \hat{\Gamma}^t(\sigma)(\mathcal{L}V)(X_{\sigma}^{\gamma_t, \tilde{u}}, \tilde{u}(\sigma))d\sigma \bigg| \mathcal{F}_t \right],
\] (5.20)

where \(\hat{\Gamma}^t(\cdot)\) solves the linear SDE

\[
d\hat{\Gamma}^t(s) = \hat{\Gamma}^t(s)(B_1(s)ds + B_2(s)dW(s)), \quad s \in [t, t + \delta]; \quad \hat{\Gamma}^t(t) = 1.
\]

Obviously, \(\hat{\Gamma}^t \geq 0\). Combining (5.19) and (5.20), we get that

\[-\epsilon \leq \partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H + \mathbf{H}(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t))
+ \frac{1}{\delta} \mathbb{E} \left[ \int_t^{t+\delta} (\mathcal{L}V)(X_{\sigma}^{\gamma_t, \tilde{u}}, \tilde{u}(\sigma)) - (\mathcal{L}V)(\gamma_t, \tilde{u}(\sigma))d\sigma \right]
+ \frac{1}{\delta} \mathbb{E} \left[ \int_t^{t+\delta} (\hat{\Gamma}^t(\sigma) - 1)(\mathcal{L}V)(X_{\sigma}^{\gamma_t, \tilde{u}}, \tilde{u}(\sigma))d\sigma \right].
\]

By the proving process of the above theorem, letting \(\delta \to 0\), we obtain

\[-\epsilon \leq \partial_t V(\gamma_t) + (A^* \partial_x V(\gamma_t), \gamma_t(t))_H + \mathbf{H}(\gamma_t, V(\gamma_t), \partial_x V(\gamma_t), \partial_{xx} V(\gamma_t)).
\]

The desired result is obtained by combining the inequality given above with (5.18). \(\Box\)

We conclude this section with the stability of viscosity solutions.
Theorem 5.5. Let $F,G,q,\phi$ satisfy Hypotheses 3.2 and 4.1 and $v \in C^0(\Lambda)$. Assume

(i) for any $\varepsilon > 0$, there exist $F^\varepsilon,G^\varepsilon,q^\varepsilon,\phi^\varepsilon$ and $v^\varepsilon \in C^0(\Lambda)$ such that $F^\varepsilon,G^\varepsilon,q^\varepsilon,\phi^\varepsilon$ satisfy Hypotheses 3.2 and 4.1 and $v^\varepsilon$ is a viscosity subsolution (resp., supsolution) of PHJB equation (1.5) with generators $F^\varepsilon,G^\varepsilon,q^\varepsilon,\phi^\varepsilon$;

(ii) as $\varepsilon \to 0$, $(F^\varepsilon,G^\varepsilon,q^\varepsilon,\phi^\varepsilon,v^\varepsilon)$ converge to $(F,G,q,\phi,v)$ uniformly in the following sense:

$$
\lim_{\varepsilon \to 0} \sup_{(t,\gamma_t,x,y,u) \in [0,T] \times \Lambda \times \mathbb{R} \times \mathbb{R} \times \mathbb{U}} \left| (|F^\varepsilon - F| + |G^\varepsilon - G|)(\gamma_t, u) + |q^\varepsilon - q|(\gamma_t, x, yG(\gamma_t, u), u) + |\phi^\varepsilon - \phi|(\eta_T) + |v^\varepsilon - v|(\gamma_t) \right| = 0.
$$

Then $v$ is a viscosity subsolution (resp., supersolution) of PHJB equation (1.5) with generators $F,G,q,\phi$.

Proof. Without loss of generality, we shall only prove the viscosity subsolution property. First, from $v^\varepsilon$ is a viscosity subsolution of equation (1.5) with generators $F^\varepsilon,G^\varepsilon,q^\varepsilon,\phi^\varepsilon$, it follows that

$$
v^\varepsilon(\gamma_T) \leq \phi^\varepsilon(\gamma_T), \quad \gamma_T \in \Lambda_T.
$$

Letting $\varepsilon \to 0$, we have

$$
v(\gamma_T) \leq \phi(\gamma_T), \quad \gamma_T \in \Lambda_T.
$$

Next, Let $\varphi \in \Phi$ and $g \in G_i$ with $i \in [0,T)$ such that

$$
0 = (V - \varphi - g)(\hat{\gamma}_i) = \sup_{(s,\nu_s) \in [i,T] \times \Lambda^i} (V - \varphi - g)(\eta_s),
$$

where $\hat{\gamma}_i \in \Lambda$. Denote $g_1(\gamma_t) := g(\gamma_t) + \overline{Y}^i(\gamma_t, \hat{\gamma}_i)$ for all $(t,\gamma_t) \in [i,T] \times \Lambda$. Then we have $g_1 \in G_i$. Define a sequence of positive numbers $\{\delta_i\}_{i \geq 0}$ by $\delta_i = \frac{1}{2^i}$ for all $i \geq 0$. For every $\varepsilon > 0$, since $v^\varepsilon - \varphi - g_1$ is an upper semicontinuous functional and $\overline{Y}^i(\cdot, \cdot)$ is a gauge-type function, from Lemma 2.9 it follows that, for every $(t_0,\gamma_{t_0}) \in [i,T] \times \Lambda^i$ satisfy

$$(v^\varepsilon - \varphi - g_1)(\gamma_{t_0}) \geq \sup_{(s,\nu_s) \in [i,T] \times \Lambda^i} (v^\varepsilon - \varphi - g_1)(\gamma_s) - \varepsilon, \quad \text{and} \quad (v^\varepsilon - \varphi - g_1)(\gamma_{t_0}) \geq (v^\varepsilon - \varphi - g_1)(\hat{\gamma}_i),$$

there exist $(t_\varepsilon,\gamma_{t_\varepsilon}) \in [i,T] \times \Lambda^i$ and a sequence $\{(t_i,\gamma_{t_i})\}_{i \geq 1} \subset [i,T] \times \Lambda^i$ such that

(i) $\overline{Y}^i(\gamma_{t_0}^0,\gamma_{t_\varepsilon}^\varepsilon) \leq \varepsilon$, $\overline{Y}^i(\gamma_{t_i}^i,\gamma_{t_\varepsilon}^\varepsilon) \leq \frac{\varepsilon}{i^2}$ and $t_i \uparrow t_\varepsilon$ as $i \to \infty$,

(ii) $(v^\varepsilon - \varphi - g_1)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^\infty \frac{1}{2^i} \overline{Y}^i(\gamma_{t_i}^i,\gamma_{t_\varepsilon}^\varepsilon) \geq (v^\varepsilon - \varphi - g_1)(\gamma_{t_0}^0)$, and

(iii) $(v^\varepsilon - \varphi - g_1)(\gamma_s) - \sum_{i=0}^\infty \frac{1}{2^i} \overline{Y}^i(\gamma_{t_i}^i,\gamma_s) < (v^\varepsilon - \varphi - g_1)(\gamma_{t_\varepsilon}^\varepsilon) - \sum_{i=0}^\infty \frac{1}{2^i} \overline{Y}^i(\gamma_{t_i}^i,\gamma_{t_\varepsilon}^\varepsilon)$ for all $(s,\gamma_s) \in [t_\varepsilon,T] \times \Lambda^i \setminus \{(t_i,\gamma_{t_i})\}$.

We claim that

$$
d_{\infty}(\gamma_{t_\varepsilon}^\varepsilon,\hat{\gamma}_i) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{5.22}
$$

Indeed, if not, by (2.11) we can assume that there exists an $\nu_0 > 0$ such that

$$
\overline{Y}^i(\gamma_{t_\varepsilon}^\varepsilon,\hat{\gamma}_i) \geq \nu_0.
$$
Thus, we obtain that
\[
0 = (v - \varphi - g)(\dot{\gamma}_f) = \lim_{\varepsilon \to 0} (v^\varepsilon - \varphi - g_1)(\dot{\gamma}_f) \leq \limsup_{\varepsilon \to 0} \left[ (v^\varepsilon - \varphi - g_1)(\gamma^\varepsilon_f) - \sum_{i=0}^{\infty} \frac{1}{2i} \mathbf{Y}^3(\gamma^i_f, \gamma^\varepsilon_f) \right] \\
\leq \limsup_{\varepsilon \to 0} [(v - \varphi - g)(\gamma^\varepsilon_f) + (v^\varepsilon - v)(\gamma^\varepsilon_f)] - \nu_0 \leq (v - \varphi - g)(\dot{\gamma}_f) - \nu_0 = -\nu_0,
\]
contradicting \( \nu_0 > 0 \). We notice that, by the property (i) of \((t_\varepsilon, \gamma^\varepsilon_f)\),
\[
2 \sum_{i=0}^{\infty} \frac{1}{2i} (t_\varepsilon - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2i} \left( \frac{\varepsilon}{2i} \right)^{\frac{3}{2}} \leq 4 \varepsilon^{\frac{3}{2}};
\]
\[
|\partial_x \mathbf{Y}^3(\gamma^\varepsilon_f - \gamma^i_{t,\varepsilon}, t)\mathbf{A}| \leq 18 |e^{(t_\varepsilon - t_\varepsilon)A} \gamma^i_{t,\varepsilon} - \gamma^\varepsilon_f(t_\varepsilon)|^5;
\]
\[
|\partial_{xx} \mathbf{Y}^3(\gamma^\varepsilon_f - \gamma^i_{t,\varepsilon}, t)\mathbf{A}| \leq 306 |e^{(t_\varepsilon - t_\varepsilon)A} \gamma^i_{t,\varepsilon} - \gamma^\varepsilon_f(t_\varepsilon)|^4;
\]
\[
|\partial_x \sum_{i=0}^{\infty} \frac{1}{2i} \mathbf{Y}^3(\gamma^\varepsilon_f - \gamma^i_{t,\varepsilon}, t)\mathbf{A}| \leq 18 \sum_{i=0}^{\infty} \frac{1}{2i} |e^{(t_\varepsilon - t_\varepsilon)A} \gamma^i_{t,\varepsilon} - \gamma^\varepsilon_f(t_\varepsilon)|^5 \leq 18 \sum_{i=0}^{\infty} \frac{1}{2i} \left( \frac{\varepsilon}{2i} \right)^{\frac{5}{2}} \leq 36 \varepsilon^{\frac{5}{2}},
\]
and
\[
|\partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2i} \mathbf{Y}^3(\gamma^\varepsilon_f - \gamma^i_{t,\varepsilon}, t)\mathbf{A}| \leq 306 \sum_{i=0}^{\infty} \frac{1}{2i} |e^{(t_\varepsilon - t_\varepsilon)A} \gamma^i_{t,\varepsilon} - \gamma^\varepsilon_f(t_\varepsilon)|^4 \leq 306 \sum_{i=0}^{\infty} \frac{1}{2i} \left( \frac{\varepsilon}{2i} \right)^{\frac{4}{3}} \leq 612 \varepsilon^{\frac{4}{3}}.
\]
Then for any \( \varepsilon > 0 \), by (5.21) and (5.22), there exists \( \varepsilon > 0 \) small enough such that
\[
\dot{\varepsilon} \leq t_\varepsilon < T, \quad 2|t_\varepsilon - \varepsilon| + 2 \sum_{i=0}^{\infty} \frac{1}{2i} (t_\varepsilon - t_i) + |\partial_t \varphi(\gamma^\varepsilon_f) - \partial_t \varphi(\gamma^\varepsilon_f)| + |\partial_t g(\gamma^\varepsilon_f) - \partial_t g(\gamma^\varepsilon_f)| \leq \frac{\varepsilon}{3},
\]
\[
|\mathbf{A}^* \partial_x \varphi(\gamma^\varepsilon_f, \gamma^\varepsilon_f(t)) - (\mathbf{A}^* \partial_x \varphi(\gamma^\varepsilon_f, \gamma^\varepsilon_f(t)))_H| \leq \frac{\varepsilon}{3}, \quad \text{and} \quad |I| \leq \frac{\varepsilon}{3},
\]
where
\[
\mathbf{H}^\varepsilon(\gamma^\varepsilon_f, \partial_x \varphi(\gamma^\varepsilon_f), \partial_x g(\gamma^\varepsilon_f), \partial_{xx} \varphi(\gamma^\varepsilon_f) + \partial_{xx} g(\gamma^\varepsilon_f))
\]
\[
- \mathbf{H}(\gamma^\varepsilon_f, \partial_x \varphi(\gamma^\varepsilon_f) + \partial_x g(\gamma^\varepsilon_f), \partial_{xx} \varphi(\gamma^\varepsilon_f) + \partial_{xx} g(\gamma^\varepsilon_f)),
\]
\[
g_2(\gamma^\varepsilon_f) = g(\gamma^\varepsilon_f) + \mathbf{Y}^3(\gamma^\varepsilon_f - \gamma^i_{t,\varepsilon}, t)\mathbf{A} + \sum_{i=0}^{\infty} \frac{1}{2i} \mathbf{Y}^3(\gamma^\varepsilon_f - \gamma^i_{t,\varepsilon}, t)\mathbf{A},
\]
and
\[
\mathbf{H}^\varepsilon(\gamma^\varepsilon_f, \partial_x \varphi(\gamma^\varepsilon_f), \partial_x g(\gamma^\varepsilon_f) + \mathbf{A}^* \partial_x \varphi(\gamma^\varepsilon_f), \gamma^\varepsilon_f(t)) + \mathbf{H}^\varepsilon(\gamma^\varepsilon_f, \partial_x \varphi(\gamma^\varepsilon_f) + \partial_x g(\gamma^\varepsilon_f), \gamma^\varepsilon_f(t)) + \partial_{xx} \varphi(\gamma^\varepsilon_f) + \partial_{xx} g(\gamma^\varepsilon_f) \geq 0.
\]
Since \( v^\varepsilon \) is a viscosity subsolution of PHJB equation (1.5) with generators \( F^\varepsilon, G^\varepsilon, q^\varepsilon, \phi^\varepsilon \), we have
\[
\partial_t \varphi(\gamma^\varepsilon_f) + \partial_t g(\gamma^\varepsilon_f) + (\mathbf{A}^* \partial_x \varphi(\gamma^\varepsilon_f), \gamma^\varepsilon_f(t)) + \mathbf{H}^\varepsilon(\gamma^\varepsilon_f, \partial_x \varphi(\gamma^\varepsilon_f) + \partial_x g(\gamma^\varepsilon_f), \partial_{xx} \varphi(\gamma^\varepsilon_f) + \partial_{xx} g(\gamma^\varepsilon_f)) \geq 0.
\]
Thus

\[
0 \leq \partial_t \varphi(\gamma^e_{t\epsilon}) + \partial_t g(\gamma^e_{t\epsilon}) + 2(t_\epsilon - \hat{t}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i}(t_\epsilon - t_i) + (A^0 \partial_x \varphi(\gamma^e_{t\epsilon}, \gamma^e_{t\epsilon}(t_\epsilon))H + M(\gamma^e_{t\epsilon}, \partial_x \varphi(\gamma^e_{t\epsilon}, \gamma^e_{t\epsilon}(t_\epsilon))) = \leq \partial_t \varphi(\gamma^e_{t\epsilon}) + \partial_t g(\gamma^e_{t\epsilon}) + (A^0 \partial_x \varphi(\gamma^e_{t\epsilon}), \gamma^e_{t\epsilon}(t_\epsilon))H + M(\gamma^e_{t\epsilon}, \partial_x \varphi(\gamma^e_{t\epsilon}, \gamma^e_{t\epsilon}(t_\epsilon)) + \partial_x g(\gamma^e_{t\epsilon}) + \varphi.
\]

Letting \( g \downarrow 0 \), we show that

\[
\partial_t \varphi(\gamma^e_{t\epsilon}) + \partial_t g(\gamma^e_{t\epsilon}) + (A^0 \partial_x \varphi(\gamma^e_{t\epsilon}), \gamma^e_{t\epsilon}(t_\epsilon))H + M(\gamma^e_{t\epsilon}, \partial_x \varphi(\gamma^e_{t\epsilon}, \gamma^e_{t\epsilon}(t_\epsilon)) + \partial_x g(\gamma^e_{t\epsilon}) + \varphi \geq 0.
\]

Since \( \varphi \in \Phi \) and \( g \in G_t \) with \( t \in [0, T] \) are arbitrary, we see that \( \varphi \) is a viscosity subsolution of PHJB equation (1.5) with generators \( F, G, q, \phi \), and thus completes the proof. \( \square \)

6 Viscosity solutions to PHJB equations: Uniqueness theorem.

This section is devoted to a proof of uniqueness of viscosity solutions to (1.5). This result, together with the results from the previous section, will be used to characterize the value functional defined by (1.4).

Let \( \{e_i\}_{i \geq 1} \) be an orthonormal basis of \( H \) such that \( e_i \in D(A^0) \) for all \( i \geq 1 \). For every \( N \geq 1 \), we let \( H_N \) denote the vector space generated by \( e_1, \ldots, e_N \), and denote the orthogonal projection onto \( H_N \). Define \( Q_N = I - P_N \), we then have an orthogonal decomposition \( H = H_N + H_N^\perp \), where \( H_N^\perp = Q_N H \). We will denote by \( x_1, y_1, \ldots \) points in \( H_N \) and by \( x_\perp, y_\perp, \ldots \) points in \( H_N^\perp \), and write \( x = (x_1, x_\perp) \) for \( x \in H \). Our uniqueness result requires the following assumption of \( G \).

**Hypothesis 6.1.** For every \( (t, \gamma_t) \in [0, T] \times \Lambda \),

\[
\sup_{u \in U} |Q_N G(\gamma_t, u)|^2_{L_2(\Xi; H)} \rightarrow 0 \ 	ext{as} \ N \rightarrow \infty. \tag{6.1}
\]

We assume without loss of generality that, there exists a constant \( K > 0 \) such that, for all \( (t, \gamma_t, p, l) \in [0, T] \times \Lambda \times H \times S(H) \) and \( r_1, r_2 \in \mathbb{R} \) such that \( r_1 < r_2 \),

\[
H(\gamma_t, r_1, p, l) - H(\gamma_t, r_2, p, l) \geq K(r_2 - r_1). \tag{6.2}
\]

We now state the main result of this section.

**Theorem 6.2.** Suppose Hypotheses 3.2 and 6.1 hold. Let \( W_1 \in C^0(\Lambda) \) (resp., \( W_2 \in C^0(\Lambda) \)) be a viscosity subsolution (resp., supsolution) to equation (1.3) and let there exist constant \( L > 0 \) such that, for any \( 0 \leq t \leq s \leq T, \gamma_t, \eta_s \in \Lambda \),

\[
|W_1(\gamma_t)| \vee |W_2(\gamma_t)| \leq L(1 + ||\gamma_t||_0); \tag{6.3}
\]

\[
|W_1(\gamma_{t,s,A}) - W_1(\eta_t)| \vee |W_2(\gamma_{t,s,A}) - W_2(\eta_t)| \leq L(1 + ||\gamma_t||_0 + ||\eta_s||_0) |s - t|^{1/2} + L||\gamma_t - \eta_s||_0. \tag{6.4}
\]

Then \( W_1 \leq W_2 \).

Theorems 5.3 and 6.2 lead to the result (given below) that the viscosity solution to PHJB equation given in (1.5) corresponds to the value functional \( V \) of our optimal control problem given in (1.1), (1.3) and (1.4).
Theorem 6.3. Let Hypotheses 3.4, 4.7 and 6.1 hold. Then the value functional \( V \) defined by (1.4) is the unique viscosity solution to (1.5) in the class of functionals satisfying (6.3) and (6.4).

Proof. Theorem 6.3 shows that \( V \) is a viscosity solution to equation (1.5). Thus, our conclusion follows from Theorems 4.8 and 6.2. \( \square \)

Next, we prove Theorem 6.2. Let \( W \) be a viscosity subsolution of PHJB equation (1.5). We note that for \( \delta > 0 \), the functional defined by \( \tilde{W} := W - \frac{\rho}{t} \) is a viscosity subsolution for

\[
\begin{align*}
\tilde{W}(\gamma_t) + (A^{*} \partial_x \tilde{W}(\gamma_t), \gamma_t(t))_H &+ \mathbf{H}(\gamma_t, \tilde{W}(\gamma_t), \partial_x \tilde{W}(\gamma_t), \partial_{xx} \tilde{W}(\gamma_t)) = \frac{\rho}{t}, \quad (t, \gamma_t) \in [0, T) \times \Lambda, \\
\tilde{W}(\gamma_T) = \phi(\gamma_T), &\quad \gamma_T \in \Lambda_T.
\end{align*}
\]

As \( W_1 \leq W_2 \) follows from \( \tilde{W} \leq W \) in the limit \( \rho \downarrow 0 \), it suffices to prove \( W_1 \leq W_2 \) under the additional assumption given below:

\[
\partial_t W_1(\gamma_t) + (A^{*} \partial_x W_1(\gamma_t), \gamma_t(t))_H + \mathbf{H}(\gamma_t, W_1(\gamma_t), \partial_x W_1(\gamma_t), \partial_{xx} W_1(\gamma_t)) \geq c := \frac{\rho}{T^2}, \quad (t, \gamma_t) \in [0, T) \times \Lambda.
\]

Proof of Theorem 6.2. The proof of this theorem is rather long. Thus, we split it into several steps.

Step 1. Definitions of auxiliary functionals.

We only need to prove that \( W_1(\gamma_t) \leq W_2(\gamma_t) \) for all \((t, \gamma_t) \in (T - \bar{a}, T) \times \Lambda. \) Here,

\[
\bar{a} = \frac{1}{8(342L + 36)L} \wedge T.
\]

Then, we can repeat the same procedure for the case \( [T - i\bar{a}, T - (i - 1)\bar{a}) \). Thus, we assume the converse result that \((\tilde{t}, \tilde{\gamma}) \in [T - \bar{a}, T) \times \Lambda \). exists such that \( \tilde{m} := W_1(\tilde{\gamma}) - W_2(\tilde{\gamma}) > 0 \).

Consider that \( \varepsilon > 0 \) is a small number such that

\[
W_1(\tilde{\gamma} - \varepsilon) - W_2(\tilde{\gamma}) - 2\varepsilon \frac{\nu T - \bar{t}}{\nu T} Y(\tilde{\gamma}) > \frac{\tilde{m}}{2},
\]

and

\[
\frac{\varepsilon}{4\nu T} \leq c \frac{2}{3}, \quad (6.5)
\]

where

\[
\nu = 1 + \frac{1}{8T(342L + 36)L}.
\]

Next, we define for any \((t, \gamma_t, \eta_t) \in [T - \bar{a}, T) \times \Lambda \times \Lambda,

\[
\Psi(\gamma_t, \eta_t) = W_1(\gamma_t) - W_2(\eta_t) - \beta Y(\gamma_t, \eta_t) - \beta Y^3(\gamma_t(t) - \eta_t(t)) - \varepsilon \frac{\nu T - t}{\nu T} (Y^3(\gamma_t) + Y^3(\eta_t))
\]

Define a sequence of positive numbers \( \{\delta_i\}_{i \geq 0} \) by \( \delta_i = \frac{1}{2^i} \) for all \( i \geq 0 \). Since \( \Psi \) is a upper semicontinuous function bounded from above and \( Y^3 \) is a gauge-type function, from Lemma 2.9 it follows that, for every \((t_0, \gamma^0_0, \eta^0_0) \in [\tilde{t}, T] \times \Lambda^\tilde{t} \times \Lambda^\tilde{t}

\[
\Psi(\gamma^0_0, \eta^0_0) \geq \sup_{(s, \gamma_s, \eta_s) \in [\tilde{t}, T] \times \Lambda^\tilde{t} \times \Lambda^\tilde{t}} \Psi(\gamma_s, \eta_s) - \frac{1}{\beta^i}, \quad \text{and} \quad \Psi(\gamma^0_0, \eta^0_0) \geq \Psi(\tilde{\gamma}^i, \tilde{\gamma}^i) > \frac{\tilde{m}}{2},
\]

there exist \((\tilde{t}, \tilde{\gamma}, \tilde{\eta}) \in [\tilde{t}, T] \times \Lambda^\tilde{t} \times \Lambda^\tilde{t}\) and a sequence \(\{(t_i, \gamma^i_k, \eta^i_k)\}_{i \geq 1} \subset [\tilde{t}, T] \times \Lambda^\tilde{t} \times \Lambda^\tilde{t}\) such that
Thus, we have

$$\sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \hat{\gamma}_i) + Y^3(n_i, \hat{n}_i) + |\hat{t} - t_i|^2] \leq \frac{1}{\beta^2} \text{ and } t_i \uparrow \hat{t} \text{ as } i \to \infty,$$

(ii) $\Psi(\hat{\gamma}_i, \hat{n}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \hat{\gamma}_i) + Y^3(n_i, \hat{n}_i) + |\hat{t} - t_i|^2] \geq \Psi(\gamma_0, \eta_0)$, and

(iii) for all $(s, \gamma_s, \eta_s) \in \hat{t}, T \times \Lambda^i \times \Lambda^j \setminus \{(\hat{t}, \hat{\gamma}_i, \hat{n}_i)\}$,

$$\Psi(\gamma_s, \eta_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \gamma_s) + Y^3(n_i, \eta_s) + |s - t_i|^2] \leq \Psi(\hat{\gamma}_i, \hat{n}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \hat{\gamma}_i) + Y^3(n_i, \hat{n}_i) + |\hat{t} - t_i|^2]. \quad (6.6)$$

We should note that the point $(\hat{t}, \hat{\gamma}_i, \hat{n}_i)$ depends on $\beta$ and $\varepsilon$.

**Step 2.** There exists $M_0 > 0$ such that

$$||\hat{\gamma}_i||_0 \vee ||\hat{n}_i||_0 < M_0, \quad (6.7)$$

and the following result holds true:

$$\beta^{1/2} ||\hat{\gamma}_i - \hat{n}_i||_0^2 + \beta^{1/2} ||\hat{\gamma}_i||_0^2 \to 0 \text{ as } \beta \to +\infty. \quad (6.8)$$

Let us show the above. First, noting $\nu$ is independent of $\beta$, by the definition of $\Psi$, there exists an $M_0 > 0$ that is sufficiently large that $\Psi(\gamma_t, \eta_t) < 0$ for all $t \in [T - \bar{t}, T]$ and $||\gamma_t||_0 \vee ||\eta_t||_0 \geq M_0$. Thus, we have $||\hat{\gamma}_i||_0 \vee ||\hat{n}_i||_0 \vee ||\gamma_0||_0 \vee ||\eta_0||_0 < M_0$.

Second, by (6.6), we have

$$2\Psi(\hat{\gamma}_i, \hat{n}_i) - 2 \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \hat{\gamma}_i) + Y^3(n_i, \hat{n}_i) + |\hat{t} - t_i|^2] \geq \Psi(\hat{\gamma}_i, \hat{n}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \hat{\gamma}_i) + Y^3(n_i, \hat{n}_i) + |\hat{t} - t_i|^2]$$

$$+ \Psi(\hat{n}_i, \hat{n}_i) - \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(\gamma_i, \hat{n}_i) + Y^3(n_i, \hat{n}_i) + |\hat{t} - t_i|^2]. \quad (6.9)$$

This implies that

$$2\beta Y^3(\hat{\gamma}_i, \hat{n}_i) + 2\beta^{1/2} ||\hat{\gamma}_i||_0^2$$

$$\leq |W_1(\gamma_t) - W_1(\eta_t)| + |W_2(\gamma_t) - W_2(\eta_t)| + \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(n_i, \hat{\gamma}_i) + Y^3(\gamma_i, \hat{n}_i)]. \quad (6.10)$$

On the other hand, by Lemma [2,10]

$$\sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(n_i, \hat{\gamma}_i) + Y^3(\gamma_i, \hat{n}_i)]$$

$$\leq 2^5 \sum_{i=0}^{\infty} \frac{1}{2^i} [Y^3(n_i, \hat{\gamma}_i) + Y^3(\gamma_i, \hat{n}_i) + 2Y^3(\gamma_i, \hat{n}_i) \leq \frac{2^6}{\beta} + 2^7 Y^3(\gamma_i, \hat{n}_i). \quad (6.11)$$
Then we have
\[ (2\beta - 2^7)\Psi^3(\hat{\gamma}_i, \hat{\eta}_i) + 2\beta^4 |\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^2 \]
\[ \leq |W_1(\hat{\gamma}_i) - W_1(\hat{\eta}_i)| + |W_2(\hat{\gamma}_i) - W_2(\hat{\eta}_i)| + \frac{2^6}{\beta} \]
\[ \leq 2L(2 + \|\hat{\gamma}_i\|_0 + \|\hat{\eta}_i\|_0) + \frac{2^6}{\beta} \leq 4L(1 + M_0) + \frac{2^6}{\beta}. \tag{6.12} \]

Letting $\beta \to \infty$, we get
\[ \Psi^3(\hat{\gamma}_i, \hat{\eta}_i) \leq \frac{1}{2\beta - 2^7} \left[ 4L(1 + M_0) + \frac{2^6}{\beta} \right] \to 0 \text{ as } \beta \to +\infty. \]

From (2.11) it follows that
\[ \|\hat{\gamma}_i - \hat{\eta}_i\|_0 \to 0 \text{ as } \beta \to +\infty. \tag{6.13} \]

Combining (2.11), (6.4), (6.10), (6.11) and (6.13), we see that
\[ \beta \|\hat{\gamma}_i - \hat{\eta}_i\|^6_0 + \beta^4 |\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^2 \leq 4\beta \Psi^3(\hat{\gamma}_i, \hat{\eta}_i) + \beta^4 |\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^2 \]
\[ \leq 4L\|\hat{\gamma}_i - \hat{\eta}_i\|_0 + \frac{2^7}{\beta} + 2^{10} \|\hat{\gamma}_i - \hat{\eta}_i\|^6_0 \to 0 \text{ as } \beta \to +\infty. \tag{6.14} \]

Then we have
\[ \beta^\frac{1}{2} |\hat{\gamma}_i(t) - \hat{\eta}_i(t)|^2 \leq \beta^\frac{1}{2} \left( 4L\|\hat{\gamma}_i - \hat{\eta}_i\|_0 + \frac{2^7}{\beta} + 2^{10} \|\hat{\gamma}_i - \hat{\eta}_i\|^6_0 \right) \]
\[ = 4L\beta^\frac{1}{2} \|\hat{\gamma}_i - \hat{\eta}_i\|_0 + \frac{2^7}{\beta^2} + 2^{10} \beta^\frac{1}{2} \|\hat{\gamma}_i - \hat{\eta}_i\|^6_0 \to 0 \text{ as } \beta \to +\infty. \tag{6.15} \]

Thus, we get (6.8) holds true.

**Step 3.** There exists $N_0 > 0$ such that $\hat{t} \in [\tilde{t}, T)$ for all $\beta \geq N_0$.

By (6.13), we can let $N_0 > 0$ be a large number such that
\[ L\|\hat{\gamma}_i - \hat{\eta}_i\|_0 \leq \frac{\bar{m}}{4}, \]
for all $\beta \geq N_0$. Then we have $\hat{t} \in [\tilde{t}, T)$ for all $\beta \geq N_0$. Indeed, if say $\hat{t} = T$, we will deduce the following contradiction:
\[ \frac{\bar{m}}{2} \leq \Psi(\hat{\gamma}_i, \hat{\eta}_i) \leq \phi(\hat{\gamma}_i) - \phi(\hat{\eta}_i) \leq L\|\hat{\gamma}_i - \hat{\eta}_i\|_0 \leq \frac{\bar{m}}{4}. \]

**Step 4.** Completion of the proof.

From above all, for the fixed $N_0 > 0$ in step 3, we find $(\hat{t}, \hat{\gamma}_i), (\hat{t}, \hat{\eta}_i) \in [\tilde{t}, T] \times \Lambda^{\hat{t}}$ satisfying $\hat{t} \in [\tilde{t}, T)$ for all $\beta \geq N_0$ such that
\[ \Psi_1(\hat{\gamma}_i, \hat{\eta}_i) \geq \Psi(\hat{\gamma}_i, \hat{\gamma}_i) \text{ and } \Psi_1(\hat{\gamma}_i, \hat{\eta}_i) \geq \Psi_1(\gamma_i, \eta_i), \quad (t, \gamma_i, \eta_i) \in [\tilde{t}, T] \times \Lambda^{\hat{t}} \times \Lambda^{\hat{t}}, \tag{6.16} \]
where we define
\[ \Psi_1(\gamma_i, \eta_i) := \Psi(\gamma_i, \eta_i) - \sum_{i=0}^\infty \frac{1}{2^i} \left[ \Psi^3(\gamma_i, \gamma_i) + \Psi^3(\eta_i, \eta_i) + |t - t_i|^2 \right], \quad (t, \gamma_i, \eta_i) \in [\tilde{t}, T] \times \Lambda^{\hat{t}} \times \Lambda^{\hat{t}}. \]
We put, for \((t, \gamma_t, \eta_t) \in [\hat{t}, T] \times \Lambda \times \Lambda,\)

\[
W_1'(\gamma_t) = W_1(\gamma_t) - \frac{\nu T - t}{\nu T} \Upsilon^3(\gamma_t) - \epsilon \Upsilon^3(\gamma_t, \gamma_t) - \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\gamma_t^i, \gamma_t);
\]

\[
W_2'(\eta_t) = W_2(\eta_t) + \frac{\nu T - t}{\nu T} \Upsilon^3(\eta_t) + \epsilon \Upsilon^3(\eta_t, \hat{\eta}_t) + \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\eta_t^i, \eta_t).
\]

Now we define, for \((t, x_N, y_N, \gamma_t, \eta_t) \in [\hat{t}, T] \times H_N \times H_N \times \Lambda^\ell \times \Lambda^\ell,\)

\[
\hat{W}_1(\gamma_t, x_N) = \sup_{\xi_t \in \Lambda^\ell, (\xi_t)^N = x_N} \left[ W_1'(\xi_t) - 2^{\frac{5}{2}} \beta \Upsilon^3(\xi_t, \xi_t) - 2 \beta^{\frac{1}{2}} |(\gamma_t(t))^- - (\xi_t(t))^-|^2 \right];
\]

\[
\hat{W}_2(\eta_t, y_N) = \inf_{\xi_t \in \Lambda^\ell, (\xi_t)^N = y_N} \left[ W_2'(\xi_t) + 2^{\frac{5}{2}} \beta \Upsilon^3(\eta_t, \xi_t) + 2 \beta^{\frac{1}{2}} |(\eta_t(t))^- - (\xi_t(t))^-|^2 \right].
\]

Then by Lemma 2.10 we obtain that, for all \((t, x_N, y_N, \xi_t) \in [\hat{t}, T] \times H_N \times H_N \times \Lambda,\)

\[
\hat{W}_1(\xi_t, x_N) - \hat{W}_2(\xi_t, y_N) - \beta^{\frac{3}{2}} |x_N - y_N|^2
\]

\[
= \sup_{\gamma_t, \eta_t \in \Lambda^\ell, (\gamma_t(t))_N = x_N, (\eta_t(t))_N = y_N} \left[ W_1'(\gamma_t) - 2^{\frac{5}{2}} \beta \Upsilon^3(\gamma_t, \gamma_t) - 2 \beta^{\frac{1}{2}} |(\gamma_t(t))^- - (\xi_t(t))^-|^2 \right]

- \left[ W_2'(\eta_t) - 2^{\frac{5}{2}} \beta \Upsilon^3(\xi_t, \eta_t) - 2 \beta^{\frac{1}{2}} |(\eta_t(t))^- - (\xi_t(t))^-|^2 - 2 \beta^{\frac{1}{2}} |(\gamma_t(t))_- - (\eta_t(t))_-|^2 \right] \leq \sup_{\gamma_t, \eta_t \in \Lambda^\ell, (\gamma_t(t))_N = x_N, (\eta_t(t))_N = y_N} \left[ W_1'(\gamma_t) - W_2'(\eta_t) - \beta \Upsilon^3(\gamma_t, \eta_t) - 2 \beta^{\frac{1}{2}} |(\gamma_t(t))^- - (\eta_t(t))^-|^2 \right]

\leq W_1'(\hat{\gamma}_t) - W_2'(\hat{\eta}_t) - \beta \Upsilon^3(\hat{\gamma}_t, \hat{\eta}_t) - 2 \beta^{\frac{1}{2}} |(\hat{\gamma}_t(t))^- - (\hat{\eta}_t(t))^-|^2,
\]

where the last inequality becomes equality if and only if \(t = \hat{t}, \gamma_t = \hat{\gamma}_t, \eta_t = \hat{\eta}_t.\) The previous inequality becomes equality if \(\xi_t = \frac{\gamma_t + \eta_t}{2}.\) Then we obtain that, for all \((t, x_N, y_N, \xi_t) \in [\hat{t}, T] \times H_N \times H_N \times \Lambda,\)

\[
\hat{W}_1(\xi_t, x_N) - \hat{W}_2(\xi_t, y_N) \leq W_1'(\hat{\gamma}_t) - W_2'(\hat{\eta}_t) - \beta^{\frac{1}{2}} |(\hat{\gamma}_t(t))^- - (\hat{\eta}_t(t))^-|^2;
\]

and the equality holds at \(\hat{t}, (\hat{\gamma}_t(t))_N, (\hat{\eta}_t(t))_N, \hat{\xi}_t = \frac{\hat{\gamma}_t + \hat{\eta}_t}{2}.\)

Define, for \((t, x_N, y_N) \in [0, T] \times H_N \times H_N,\)

\[
W_1(t, x_N) = \begin{cases} W_1(\xi_t, x_N) - (\hat{t} - t) \frac{1}{2}, & t \in [0, \hat{t}), \\ W_1(\hat{\xi}_{t, T}, x_N), & t \in [\hat{t}, T]; \end{cases}
\]

\[
W_2(t, y_N) = \begin{cases} W_2(\xi_t, y_N) + (\hat{t} - t) \frac{1}{2}, & t \in [0, \hat{t}), \\ W_2(\hat{\xi}_{t, T}, y_N), & t \in [\hat{t}, T]; \end{cases}
\]

and

\[
\hat{W}_1(t, x_N) = \lim_{s \to \hat{t}} \sup_{s \to t} \hat{W}_1(s, x_N), \quad (t, x_N) \in [0, T] \times H_N;
\]

\[
\hat{W}_2(t, y_N) = \lim_{s \to \hat{t}} \inf_{s \to t} \hat{W}_2(s, y_N), \quad (t, y_N) \in [0, T] \times H_N.
\]
Thus by Lemma [6.4] \(\tilde{W}_1(t, x_N) - \tilde{W}_2(t, y_N) - \beta \frac{1}{t} |x_N - y_N|^2\) has a maximum at \((\tilde{t}, (\tilde{\gamma}_t)(\tilde{t}), (\tilde{\eta}_t)(\tilde{t}))\) on \([0, T] \times H_N \times H_N\). Then, by Lemmas 6.4 and 6.5 the Theorem 8.3 in [9] can be used to obtain sequences \((x^k_N, y^k_N) \in H_N \times H_N, l_k, s_k \in [0, T]\) such that \((l_k, x^k_N) \to (\tilde{t}, (\tilde{\gamma}_t)(\tilde{t})), (s_k, y^k_N) \to (\tilde{t}, (\tilde{\eta}_t)(\tilde{t}))\) as \(k \to +\infty\) and the sequences of functions \(\varphi_k, \psi_k \in C^{1, 2}((T - \bar{a}, T) \times H_N)\) such that

\[
\tilde{W}_1(t, x_N) - \varphi_k(t, x_N) \leq 0, \quad \tilde{W}_2(t, x_N) + \psi_k(t, x_N) \geq 0,
\]
equalities only hold true at \((l_k, x^k_N), (s_k, y^k_N)\), respectively,

\[
\begin{align*}
\nabla_x \varphi_k(l_k, x_N) &\to b_1, \\
\nabla^2 \varphi_k(l_k, x_N) &\to X_N, \\
\nabla_x \psi_k(s_k, y_N) &\to b_2, \\
\nabla^2 \psi_k(s_k, y_N) &\to Y_N,
\end{align*}
\]
where \(b_1 + b_2 = 0\) and \(X_N, Y_N\) satisfy the following inequality:

\[
-6 \beta \frac{1}{t} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X_N & 0 \\ 0 & Y_N \end{pmatrix} \leq 6 \beta \frac{1}{t} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

(6.20)

We claim that we can assume the sequences \(\{l_k\}_{k \geq 1} \in [\tilde{t}, T]\) and \(\{s_k\}_{k \geq 1} \in [\tilde{t}, T]\). Indeed, if not, for example, there exists a subsequence of \(\{l_k\}_{k \geq 1}\) still denoted by itself such that \(l_k < \tilde{t}\) for all \(k \geq 0\). Since \(\tilde{W}_1(t, x_N) - \varphi_k(t, x_N)\) has a maximum at \((l_k, x_N^k)\) on \([0, T] \times H_N\), we obtain that

\[
(\varphi_k)_t(l_k, x_N^k) = \frac{1}{2}(\tilde{t} - l_k)^{-\frac{1}{2}} \to +\infty, \quad \text{as} \; k \to +\infty,
\]

which is contradict to \((\varphi_k)_t(l_k, x_N^k) \to b_1, (\psi_k)_t(s_k, y_N^k) \to b_2\) and \(b_1 + b_2 = 0\).

We may without loss of generality assume that \(\varphi_k, \psi_k\) grow quadratically at \(0\). Now we consider the functional, for \((t, \gamma_t, (s, \eta_s)) \in [T - a, T] \times \Lambda,

\[
\Gamma_k(\gamma_t, \eta_s) = W'_1(\gamma_t) - W'_2(\eta_s) - 2^5 \beta (\Upsilon^3(\gamma_t, \xi_t(t)) - \varphi_k(t, (\gamma_t(t))_N) - \gamma_s(s, (\eta_s(s))_N) \\
-2 \beta \frac{1}{t} |(\gamma_t(t))_N - (e^{i(t - \tilde{t})} A \xi_t(t))_N|^{-2} - 2 \beta \frac{1}{t} |(\eta_s(s))_N - (e^{i(s - \tilde{t})} A \xi_t(t))_N|^{-2} - (6.21)
\]

Define a sequence of positive numbers \(\{\delta_i\}_{i \geq 0}\) by \(\delta_i = \frac{1}{2^i}\) for all \(i \geq 0\). For every \(k \) and \(\delta > 0\), from Lemma [2.1] it follows that, for every \((\tilde{t}_0, \tilde{\gamma}_{\tilde{t}_0}^0, (\tilde{s}_0, \tilde{\eta}_{\tilde{s}}^0)) \in [\tilde{t}, T] \times \Lambda,\)

\[
\Gamma_k(\tilde{\gamma}_{\tilde{t}_0}, \tilde{\eta}_{\tilde{s}}^0) \geq \sup_{(t, \tau_t, (s, \eta_s)) \in [\tilde{t}, T] \times \Lambda} \Gamma_k(\gamma_t, \eta_s) - \delta,
\]
there exist \((\tilde{t}_i, \tilde{\gamma}_{\tilde{t}_i}), (\tilde{s}_i, \tilde{\eta}_{\tilde{s}}^i) \in [\tilde{t}, T] \times \Lambda,\) and two sequences \(\{(\tilde{t}_i, \tilde{\gamma}_{\tilde{t}_i})\}_{i \geq 1}, \{(\tilde{s}_i, \tilde{\eta}_{\tilde{s}}^i)\}_{i \geq 1} \subset [\tilde{t}, T] \times \Lambda\) such that

(i) \(\Upsilon^3(\tilde{\gamma}_{\tilde{t}_0}, \tilde{\gamma}_{\tilde{t}_i}) + \Upsilon^3(\tilde{\eta}_{\tilde{s}}^0, \tilde{\eta}_{\tilde{s}}^i) \leq \delta, \Upsilon^3(\tilde{\gamma}_{\tilde{t}_i}, \tilde{\gamma}_{\tilde{t}_i}) + \Upsilon^3(\tilde{\eta}_{\tilde{s}}^i, \tilde{\eta}_{\tilde{s}}^i) \leq \frac{\delta}{2} \) and \(\tilde{t}_i \uparrow \tilde{t}, \tilde{s}_i \uparrow \tilde{s}\) as \(i \to \infty,\)

(ii) \(\Gamma_k(\tilde{\gamma}_{\tilde{t}_i}, \tilde{\eta}_{\tilde{s}}^i) \geq \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\tilde{\gamma}_{\tilde{t}_i}, \tilde{\gamma}_{\tilde{t}_i}) + \Upsilon^3(\tilde{\eta}_{\tilde{s}}^i, \tilde{\eta}_{\tilde{s}}^i) \geq \Gamma_k(\tilde{\gamma}_{\tilde{t}_0}, \tilde{\eta}_{\tilde{s}}^0),\) and

(iii) for all \((t, \gamma_t, (s, \eta_s)) \in [\tilde{t}, T] \times \Lambda, \times [\tilde{s}, T] \times \Lambda,\)

\[
\Gamma_k(\gamma_t, \eta_s) \geq \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\tilde{\gamma}_{\tilde{t}_i}, \tilde{\gamma}_{\tilde{t}_i}) + \Upsilon^3(\tilde{\eta}_{\tilde{s}}^i, \tilde{\eta}_{\tilde{s}}^i) \geq \Gamma_k(\tilde{\gamma}_{\tilde{t}_0}, \tilde{\eta}_{\tilde{s}}^0) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\tilde{\gamma}_{\tilde{t}_i}, \tilde{\gamma}_{\tilde{t}_i}) + \Upsilon^3(\tilde{\eta}_{\tilde{s}}^i, \tilde{\eta}_{\tilde{s}}^i).\]
By the following Lemma 6.6, we have

\[ \bar{t} \to l_k, \ (\gamma_{\bar{t}}(\bar{t}))_N \to x^k_N, \ \bar{s} \to s_k, \ (\hat{\eta}_s(s))_N \to y^k_N \text{ as } \delta \to 0; \]  

(6.22)

and

\[ \lim_{k \to \infty} \limsup_{\delta \to 0} [||\gamma_{\bar{t}} - \gamma_{\bar{t},l_A}||_0 + ||\hat{\eta}_s - \hat{\eta}_{\bar{t},s,A}||_0] = 0. \]

(6.23)

From \( l_k, s_k \to \bar{t} \) as \( k \to +\infty \) and \( \bar{t} < T \) for \( \beta > N_0 \), it follows that, for every fixed \( \beta > N_0 \), constant \( K_\beta > 0 \) exists such that

\[ |l_k| \vee |s_k| < T, \text{ for all } k \geq K_\beta. \]

For every fixed \( k > K_\beta \), by \( (6.22) \), there exists constant \( \Delta_{k,\beta} > 0 \) such that

\[ |\bar{t}| \vee |\bar{s}| < T, \text{ for all } 0 < \delta < \Delta_{k,\beta}. \]

Now, for every \( \beta > N_0 \), \( k > K_\beta \) and \( 0 < \delta < \Delta_{k,\beta} \), from the definition of viscosity solutions it follows that

\[
(\varphi_k)_{t}(\bar{t}, (\gamma_{\bar{t}}(\bar{t}))_N) - \frac{\varepsilon}{\nu T} \Upsilon^3(\gamma_{\bar{t}}) + 2 \sum_{i=0}^{\infty} \frac{1}{2i} [(\bar{t} - t) + (\bar{t} - t_i)] + (A^* \nabla_x (\varphi_k)(\bar{t}, (\gamma_{\bar{t}}(\bar{t}))_N, \gamma_{\bar{t}}(\bar{t}))_H
+ 2 \varepsilon (\bar{t} - \bar{t}) - 4 \beta^3 (A^* ((\gamma_{\bar{t}}(\bar{t}))_N - e^{(\bar{t} - t_i)^A} \xi_{\bar{t}}(\bar{t})))_N, \gamma_{\bar{t}}(\bar{t}))_H
+ H\left( \gamma_{\bar{t}}, W_1(\gamma_{\bar{t}}), \nabla_x (\varphi_k)(\bar{t}, (\gamma_{\bar{t}}(\bar{t}))_N) + 4 \beta^3 ((\gamma_{\bar{t}}(\bar{t}))_N - e^{(\bar{t} - t_i)^A} \xi_{\bar{t}}(\bar{t}))_N) + 2 \beta^3 (\gamma_{\bar{t}}(\bar{t}) - e^{(\bar{t} - t_i)^A} \xi_{\bar{t}}(\bar{t})), \\gamma_{\bar{t}}(\bar{t})_N
+ \varepsilon \partial_x \Upsilon^3(\gamma_{\bar{t}} - \gamma_{\bar{t},l_A}) + \frac{\nu T}{\nu T} \partial_x \Upsilon^3(\gamma_{\bar{t}}) + \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\gamma_{\bar{t}} - \gamma_{\bar{t},i,A}) + \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\gamma_{\bar{t}} - \gamma_{\bar{t},i,A}) \right],
\]

\[
\nabla^2_x (\varphi_k)(\bar{t}, (\gamma_{\bar{t}}(\bar{t}))_N) + 4 \beta^3 Q_N Q_N + 2 \beta^3 \partial_{xx} \Upsilon^3(\gamma_{\bar{t}} - \xi_{\bar{t},l_A}) + \varepsilon \partial_{xx} \Upsilon^3(\gamma_{\bar{t}} - \xi_{\bar{t},l_A})
+ \varepsilon \partial_x \Upsilon^3(\gamma_{\bar{t}} - \gamma_{\bar{t},l_A}) + \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\gamma_{\bar{t}} - \gamma_{\bar{t},i,A}) + \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\gamma_{\bar{t}} - \gamma_{\bar{t},i,A}) \right] \geq c; \]  

(6.24)

and

\[
-(\psi_k)_{t}(s, (\hat{\eta}_s(s)))_N + \frac{\varepsilon}{\nu T} \Upsilon^3(\hat{\eta}_s) - 2 \sum_{i=0}^{\infty} \frac{1}{2i} (\bar{s} - \bar{s}_i) - (A^* \nabla_x (\psi_k)(s, (\hat{\eta}_s(s)))_N, \hat{\eta}_s(s)_H
- 2 \varepsilon (\bar{s} - \bar{t}) + 4 \beta^3 (A^* ((\hat{\eta}_s(s)))_N - e^{(\bar{s} - t_i)^A} \xi_{\bar{s}}(s)))_N, \hat{\eta}_s(s)_H
+ H\left( \hat{\eta}_s, W_2(\hat{\eta}_s), \nabla_x (\psi_k)(s, (\hat{\eta}_s(s)))_N) - 4 \beta^3 ((\hat{\eta}_s(s)))_N - (e^{(\bar{s} - t_i)^A} \xi_{\bar{s}}(s))_N) - 2 \beta^3 \partial_x \Upsilon^3(\hat{\eta}_s - \hat{\xi}_{t,s,A})
- \varepsilon \partial_x \Upsilon^3(\hat{\eta}_s - \hat{\eta}_{s,l,A}) - \varepsilon \frac{\nu T - \bar{t}}{\nu T} \partial_x \Upsilon^3(\hat{\eta}_s) - \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\hat{\eta}_s - \hat{\eta}_{s,i,A}) + \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\hat{\eta}_s - \hat{\eta}_{s,i,A}) \right],
\]

\[
- \nabla^2_x (\psi_k)(s, (\hat{\eta}_s(s)))_N) - 4 \beta^3 Q_N Q_N - 2 \beta^3 \partial_{xx} \Upsilon^3(\hat{\eta}_s - \hat{\xi}_{t,s,A}) - \varepsilon \partial_{xx} \Upsilon^3(\hat{\eta}_s - \hat{\xi}_{t,s,A})
- \varepsilon \frac{\nu T - \bar{t}}{\nu T} \partial_{xx} \Upsilon^3(\hat{\eta}_s) - \partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\hat{\eta}_s - \hat{\eta}_{s,i,A}) + \sum_{i=0}^{\infty} \frac{1}{2i} \Upsilon^3(\hat{\eta}_s - \hat{\eta}_{s,i,A}) \right] \leq 0. \]

(6.25)

We notice that, by the property (i) of \((\bar{t}, \gamma_{\bar{t}}, s, \hat{\eta}_s)\),

\[ 2 \sum_{i=0}^{\infty} \frac{1}{2i} [(\bar{s} - \bar{s}_i) + (\bar{t} - t_i)] \leq 4 \sum_{i=0}^{\infty} \frac{1}{2i} \left( \frac{\delta}{2i} \right) \leq 8 \delta \frac{1}{2}. \]
\begin{align*}
|\partial_x \Upsilon^3(\gamma\hat{t} - \gamma_i;\hat{s},\hat{A})| + |\partial_x \Upsilon^3(\eta\hat{t} - \eta_i;\hat{s},\hat{A})| &\leq 18|e^{(\nu T - \hat{t})A\gamma_{\hat{t}}(\hat{t}) - \gamma_{\hat{t}}(\hat{t})}|^5 + 18|e^{(\nu T - \hat{t})A\eta_{\hat{t}}(\hat{t}) - \eta_{\hat{t}}(\hat{t})}|^5;
|\partial_{xx} \Upsilon^3(\gamma\hat{t} - \gamma_i;\hat{s},\hat{A})| + |\partial_{xx} \Upsilon^3(\eta\hat{t} - \eta_i;\hat{s},\hat{A})| &\leq 306|e^{(\nu T - \hat{t})A\gamma_{\hat{t}}(\hat{t}) - \gamma_{\hat{t}}(\hat{t})}|^4 + 306|e^{(\nu T - \hat{t})A\eta_{\hat{t}}(\hat{t}) - \eta_{\hat{t}}(\hat{t})}|^4.
\end{align*}

\begin{align*}
|\partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\gamma_{\hat{t}} - \gamma_i;\hat{s},\hat{A}) \right] | + |\partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\eta_{\hat{t}} - \eta_i;\hat{s},\hat{A}) \right] | 
&\leq 18 \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |e^{(\nu T - \hat{t})A\gamma_{\hat{t}}(\hat{t}) - \gamma_{\hat{t}}(\hat{t})}|^5 \right] + 18 \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |e^{(\nu T - \hat{t})A\eta_{\hat{t}}(\hat{t}) - \eta_{\hat{t}}(\hat{t})}|^5 \right] 
&\leq 36 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\delta}{2} \right)^{\frac{\delta}{2}} \leq 72 \delta^{\frac{\delta}{2}};
\end{align*}

and

\begin{align*}
|\partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\gamma_{\hat{t}} - \gamma_i;\hat{s},\hat{A}) \right] | + |\partial_{xx} \left[ \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\eta_{\hat{t}} - \eta_i;\hat{s},\hat{A}) \right] | 
&\leq 306 \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |e^{(\nu T - \hat{t})A\gamma_{\hat{t}}(\hat{t}) - \gamma_{\hat{t}}(\hat{t})}|^4 \right] + 306 \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |e^{(\nu T - \hat{t})A\eta_{\hat{t}}(\hat{t}) - \eta_{\hat{t}}(\hat{t})}|^4 \right] 
&\leq 612 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{\delta}{2} \right)^{\frac{\delta}{2}} \leq 1224 \delta^{\frac{\delta}{2}}.
\end{align*}

Combining (6.24) and (6.25), and letting \( \delta \to 0 \) and \( k \to \infty \), we obtain

\begin{equation}
\begin{align*}
c + \frac{\epsilon}{\nu T} (\Upsilon^3(\gamma\hat{t}) + \Upsilon^3(\eta\hat{t})) &\leq \mathbf{H}_1(W_1(\gamma\hat{t})) - \mathbf{H}_2(W_2(\eta\hat{t})) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\hat{t} - t_i),
\end{align*}
\end{equation}

where, for every \( x \in \mathbb{R} \),

\( \mathbf{H}_1(x) = H \left( \gamma\hat{t}, x; 2\beta^3(\gamma\hat{t} - \eta_{\hat{t}}(\hat{t})) + 2^5 \beta \partial_x \Upsilon^3(\gamma\hat{t} - \hat{\xi}_t) + \frac{\nu T - \hat{t}}{\nu T} \partial_x \Upsilon^3(\gamma\hat{t}) \right) \)

\begin{align*}
+ &\partial_x \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\gamma_{\hat{t}} - \gamma_i;\hat{s},\hat{A}), X_N + 4\beta^3 Q_{NQ_N} + 2^5 \beta \partial_x \Upsilon^3(\gamma_{\hat{t}} - \hat{\xi}_t) \\
+ &\frac{\nu T - \hat{t}}{\nu T} \partial_x \Upsilon^3(\gamma_{\hat{t}}) + \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\gamma_{\hat{t}} - \gamma_i;\hat{s},\hat{A});
\end{align*}

\( \mathbf{H}_2(x) = H \left( \eta\hat{t}, x; 2\beta^3(\gamma\hat{t} - \eta_{\hat{t}}(\hat{t})) - 2^5 \beta \partial_x \Upsilon^3(\eta\hat{t} - \hat{\xi}_t) - \frac{\nu T - \hat{t}}{\nu T} \partial_x \Upsilon^3(\eta\hat{t}) \right) 

- \partial_x \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\eta_{\hat{t}} - \eta_i;\hat{s},\hat{A}), -Y_N + 4\beta^3 Q_{NQ_N} - 2^5 \beta \partial_x \Upsilon^3(\eta\hat{t} - \hat{\xi}_t) 

- \frac{\nu T - \hat{t}}{\nu T} \partial_x \Upsilon^3(\eta_{\hat{t}}) - \partial_{xx} \sum_{i=0}^{\infty} \frac{1}{2^i} \Upsilon^3(\eta_{\hat{t}} - \eta_i;\hat{s},\hat{A}).
\end{align*}

On the other hand, by (6.2) and a simple calculation we obtain

\begin{equation}
\begin{align*}
\mathbf{H}_1(W_1(\gamma\hat{t})) - \mathbf{H}_2(W_2(\eta\hat{t})) &\leq \mathbf{H}_1(W_2(\eta\hat{t})) - \mathbf{H}_2(W_2(\eta\hat{t})) \leq \sup_{u \in U} (J_1 + J_2 + J_3),
\end{align*}
\end{equation}

where

\( J_1 = \left( F(\gamma\hat{t}, u), 2\beta^3(\gamma\hat{t} - \eta_{\hat{t}}(\hat{t})) + 2^5 \beta \partial_x \Upsilon^3(\gamma\hat{t} - \hat{\xi}_t) + \frac{\nu T - \hat{t}}{\nu T} \partial_x \Upsilon^3(\gamma\hat{t}) \right) \)
\[ J_2 = \frac{1}{2} \text{Tr} \left[ \left( X_N + 4\beta \frac{\hat{T}}{\nu T} Q_N Q_N + 2\beta \partial_{x^2} Y^3(\hat{\gamma}_i - \hat{\xi}_i) + \varepsilon \frac{\nu T - \hat{t}}{\nu T} \partial_{x^2} Y^3(\hat{\gamma}_i) \right) + \partial_{x^2} \left( \sum_{i=0}^{\infty} \frac{1}{2t} Y^3(\hat{\gamma}_i - \hat{\xi}_i, t_A) \right) \right] \]

\[ + \text{Tr} \left[ \left( -Y_N - 4\beta \frac{\hat{T}}{\nu T} Q_N Q_N \right) \left( \sum_{i=0}^{\infty} \frac{1}{2t} Y^3(\hat{\gamma}_i - \hat{\xi}_i, t_A) \right) \right] \]

\[ \leq 3\beta \frac{\hat{T}}{\nu T} L^2 |\hat{\gamma}_i - \hat{\xi}_i|^2 + 2\beta \frac{\hat{T}}{\nu T} |Q_N G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} + |Q_N G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} \]

\[ + 306\beta |\hat{\gamma}_i - \hat{\xi}_i|^4 |G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} + |G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} \]

\[ + 153 \frac{\nu T - \hat{t}}{\nu T} \left| e^{(i-t)A} \hat{\gamma}_i(t_i) - \hat{\gamma}_i(t) \right|^4 |G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} \]

\[ + 153 \frac{\nu T - \hat{t}}{\nu T} \left| e^{(i-t)A} \hat{\gamma}_i(t_i) - \hat{\gamma}_i(t) \right|^4 |G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} \]

\[ \leq 3\beta \frac{\hat{T}}{\nu T} L^2 |\hat{\gamma}_i - \hat{\xi}_i|^2 + 2\beta \frac{\hat{T}}{\nu T} |Q_N G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} + |Q_N G(\hat{\gamma}_i, u)|^2_{L^2(\Xi, H)} \]

\[ + 153 \left( \sum_{i=0}^{\infty} \frac{1}{2t} \left| e^{(i-t)A} \hat{\gamma}_i(t_i) - \hat{\gamma}_i(t) \right|^4 \right) L^2 (1 + ||\hat{\gamma}_i||^2_0 + ||\hat{\gamma}_i||^6_0) \]

\[ + 306\beta |\hat{\gamma}_i - \hat{\xi}_i|^4 (2 + ||\hat{\gamma}_i||^2_0 + ||\hat{\gamma}_i||^6_0) \]

\[ + 306 \frac{\nu T - \hat{t}}{\nu T} L^2 (1 + ||\hat{\gamma}_i||^6_0 + ||\hat{\gamma}_i||^6_0) \]

\[ J_3 = q \left( \hat{\gamma}_i, W_2(\hat{\xi}_i), \left( 2\beta \frac{\hat{T}}{\nu T} (\hat{\gamma}_i(t) - \hat{\xi}_i(t)) + 2\beta \partial_{x^2} Y^3(\hat{\gamma}_i - \hat{\xi}_i) + \varepsilon \frac{\nu T - \hat{t}}{\nu T} \partial_{x^2} Y^3(\hat{\gamma}_i) \right) \right) \]

\[ + \text{Tr} \left[ \left( -Y_N - 4\beta \frac{\hat{T}}{\nu T} Q_N Q_N \right) \left( \sum_{i=0}^{\infty} \frac{1}{2t} Y^3(\hat{\gamma}_i - \hat{\xi}_i, t_A) \right) \right] \]

\[ \leq L ||\hat{\gamma}_i - \hat{\xi}_i||_0 + 2\beta \frac{\hat{T}}{\nu T} L^2 ||\hat{\gamma}_i(t) - \hat{\xi}_i(t)|| \times ||\hat{\gamma}_i - \hat{\xi}_i||_0 + 18\beta L^2 \left( \hat{\gamma}_i(t) - \hat{\xi}_i(t) \right)^5 (2 + ||\hat{\gamma}_i||_0 + ||\hat{\gamma}_i||_0) \]
+18L^2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ \left| \gamma_i(t_i) - e^{(t_i - \hat{\gamma}_i(t)}A \hat{\gamma}_i(t) \right|^5 + \left| \eta_i(t_i) - e^{(t_i - \hat{\gamma}_i(t)}A \hat{\eta}_i(t) \right|^5 \right] (1 + ||\hat{\gamma}_i||_0 + ||\hat{\eta}_i||_0) \\
+36\varepsilon \frac{\nu T - \hat{\gamma}_i}{\nu T} L^2 (1 + ||\hat{\gamma}_i||_0^6 + ||\hat{\eta}_i||_0^6); \quad \text{(6.30)}

We notice that, by the property (i) of \( \hat{\gamma}_i, \hat{\eta}_i \),

\[ 2 \sum_{i=0}^{\infty} \frac{1}{2^i} (\hat{\gamma}_i - t_i) \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2\beta} \right)^{\frac{1}{4}} \leq \frac{1}{\beta^{\frac{1}{4}}} , \]

\[ \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ |e^{(t_i - t_i)}A \gamma_i(t_i) - \hat{\gamma}_i(\hat{\gamma}_i(t_i))| + |e^{(t_i - t_i)}A \eta_i(t_i) - \hat{\eta}_i(t)| \right] \leq 2 \sum_{i=0}^{\infty} \frac{1}{2^i} \left( \frac{1}{2\beta} \right) \leq 4 \left( \frac{1}{\beta} \right)^{\frac{1}{2}} , \]

and since \( \hat{\gamma}_i \) and \( \hat{\eta}_i \) are independent of \( N \), by Hypothesis 6.1,

\[ \sup_{\hat{\gamma}_i, \hat{\eta}_i} \left[ |Q_N G(\hat{\gamma}_i, u)|^2 \right] \rightarrow 0 \text{ as } N \rightarrow \infty. \]

Combining (6.26)-(6.30), and letting \( N \rightarrow \infty \) and then \( \beta \rightarrow \infty \), by (6.31) and (6.5), we obtain

\[ c \leq -\varepsilon \frac{\nu T}{\nu T} (342L + 36) \left( 1 + \frac{1}{2\beta} \right) (1 + ||\hat{\gamma}_i||_0^6 + ||\hat{\eta}_i||_0^6). \quad \text{(6.31)} \]

Recalling \( \nu = 1 + \frac{1}{8T(342L + 36)} \) and \( \tilde{a} = \frac{1}{8(342L + 36)} \), by (6.11) and (6.31), the following contradiction is induced:

\[ c \leq \frac{\varepsilon}{4\nu T} \leq \frac{c}{2}. \]

The proof is now complete. \( \square \)

To complete the previous proof, it remains to state and prove the following lemmas.

**Lemma 6.4.** The functions \( \tilde{W}_1 \) and \( \tilde{W}_2 \) defined by (6.14) are upper semicontinuous in \( (t, x_N) \in [0, T] \times H_N \), and \( \hat{W}_1(t, x_N) - \hat{W}_2(t, y_N) - \beta |x_N - y_N|^6 \) has a maximum at \( (\hat{\xi}_i(t), N), (\hat{\eta}_i(t), N) \) on \( [0, T] \times H_N \times H_N \). Moreover,

\[ \hat{W}_1(\hat{\gamma}_i(t), N) = \tilde{W}_1(\hat{\xi}_i(t), N), \quad \hat{W}_2(\hat{\eta}_i(t), N) = \tilde{W}_2(\hat{\eta}_i(t), N). \quad \text{(6.32)} \]

**Proof.** For every \( \hat{\xi}_i(t) \leq t \leq s \leq T \) and \( x_N, y_N \in H_N \), satisfying \( |x_N| \vee |y_N| \leq M_2 \) for some constant \( M_2 > 0 \), there exists a constant \( C > 0 \) depending only on \( M_2 \) such that

\[ \hat{W}_1(t, x_N) - \hat{W}_1(s, y_N) = \hat{W}_1(t, x_N) - \hat{W}_1(s, e^{(s-t)}A x_N) + \hat{W}_1(s, e^{(s-t)}A x_N) - \hat{W}_1(s, y_N) \]

\[ = \sup_{\gamma_i \in A^i, (\gamma_i(t))_N} \left[ W_1(\gamma_i) - 2^5 \beta \gamma^3(\gamma_i(t), \hat{\xi}_i(t)) - 2^5 \beta \gamma^2((\gamma_i(t))_N) - (e^{(s-t)}A \hat{\xi}_i(t))_N \right] \]

\[ - \sup_{\eta_i \in A^i, (\eta_i(s))_N} \left[ W_1(\eta_i) - 2^5 \beta \gamma^3(\eta_i, \hat{\xi}_i(t)) - 2^5 \beta \gamma^2((\eta_i(s))_N) - (e^{(s-t)}A \hat{\xi}_i(t))_N \right]. \]
Similarly, and clearly, if $0 \leq t \leq \hat{t}$, we have

$$ W_1(t, x_N) - W_1(s, y_N) \leq C|x_N - y_N|. $$

And, if $0 \leq t \leq \hat{t} \leq s \leq T$, we have

$$ W_1(t, x_N) - W_1(s, y_N) \leq W_1(\hat{t}, x_N) - W_1(s, y_N) \leq C((s - \hat{t})^{\frac{1}{2}} + |e^{(s-\hat{t})A}x_N - x_N| + |x_N - y_N| + |e^{s-\hat{t}}A\xi(\hat{t}) - \xi(\hat{t})|).$$

Then we have

$$ W_1(\hat{t}, (\gamma_{\hat{t}}(\hat{t}))_N) \leq \liminf_{s \downarrow t, y_N \to (\gamma_{\hat{t}}(\hat{t}))_N} [W_1(s, y_N) + C((s - \hat{t})^{\frac{1}{2}} + |e^{(s-\hat{t})A}(\gamma_{\hat{t}}(\hat{t}))_N - (\gamma_{\hat{t}}(\hat{t}))_N)|$$

$$ + |(\gamma_{\hat{t}}(\hat{t}))_N - y_N| + |e^{s-\hat{t}}A\xi(\hat{t}) - \xi(\hat{t})|)]$$

$$ = \liminf_{s \downarrow t, y_N \to (\gamma_{\hat{t}}(\hat{t}))_N} W_1(s, y_N);$$

(6.33)

$$ |W_1(t, y_N) - W_1(t, x_N)| \leq C|x_N - y_N|, \quad t \in [0, T]; $$

And

$$ W_1(t, x_N) = \limsup_{s \to t} W_1(s, x_N) $$

$$ \geq \limsup_{s \to t}[W_1(t, x_N) - C((s - t)^{\frac{1}{2}} + |e^{(s-t)A}x_N - x_N| + |e^{s-t}A\xi(t) - \xi(t)|)]$$

$$ = W_1(t, x_N).$$

(6.34)

Similarly,

$$ -\tilde{W}_2(\hat{t}, \tilde{\eta}(\hat{t})) \leq \liminf_{s \downarrow t, y_N \to (\gamma_{\hat{t}}(\hat{t}))_N} [-\tilde{W}_2(s, y_N)]; $$

(6.35)
\[ |\bar{W}_2(t, y_N) - \bar{W}_2(t, x_N)| \leq C|x_N - y_N|, \ t \in [0, T]; \]

and

\[-\bar{W}_2(t, y_N) \geq -\bar{W}_2(t, y_N). \tag{6.36} \]

In particular,

\[ \hat{W}_1(t, (\hat{\gamma}_t(t))_N) \geq \hat{W}_1(t, (\hat{\gamma}_t(t))_N); \quad -\hat{W}_2(t, (\hat{\eta}_t(t))_N) \geq -\hat{W}_2(t, (\hat{\eta}_t(t))_N). \tag{6.37} \]

Therefore,

\[ \hat{W}_1(t, (\hat{\gamma}_t(t))_N) - \hat{W}_2(t, (\hat{\eta}_t(t))_N) - \beta^2 |(\hat{\gamma}_t(t))_N - (\hat{\eta}_t(t))_N|^2 \geq \hat{W}_1(t, (\hat{\gamma}_t(t))_N) - \hat{W}_2(t, (\hat{\eta}_t(t))_N) - \beta^2 |(\hat{\gamma}_t(t))_N - (\hat{\eta}_t(t))_N|^2. \tag{6.38} \]

On the other hand, for every \((t, x_N, y_N) \in [0, T] \times H_N \times H_N,

\[ \hat{W}_1(t, x_N) - \hat{W}_2(t, y_N) - \beta^2 |x_N - y_N|^2 \leq \sup_{(t, x, y) \in [0, T] \times H_N \times H_N} [\hat{W}_1(t, x) - \hat{W}_2(t, y) - \beta^2 |x - y|^2]. \]

Thus we have

\[ \hat{W}_1(t, (\hat{\gamma}_t(t))_N) - \hat{W}_2(t, (\hat{\eta}_t(t))_N) - \beta^2 |(\hat{\gamma}_t(t))_N - (\hat{\eta}_t(t))_N|^2 = \sup_{(t, x_N, y_N) \in [0, T] \times H_N \times H_N} [\hat{W}_1(t, x_N) - \hat{W}_2(t, y_N) - \beta^2 |x_N - y_N|^2] \geq \sup_{(t, x_N, y_N) \in [0, T] \times H_N \times H_N} [\hat{W}_1(t, x_N) - \hat{W}_2(t, y_N) - \beta^2 |x_N - y_N|^2] \geq \hat{W}_1(t, (\hat{\gamma}_t(t))_N) - \hat{W}_2(t, (\hat{\eta}_t(t))_N) - \beta^2 |(\hat{\gamma}_t(t))_N - (\hat{\eta}_t(t))_N|^2, \]

combining with (6.37), we obtain that (6.32) holds true, and \(\hat{W}_1(t, x_N) - \hat{W}_2(t, y_N) - \beta^2 |x_N - y_N|^2\) has a maximum at \((\hat{\gamma}_t(t))_N, (\hat{\eta}_t(t))_N)\) on \([0, T] \times H_N \times H_N\).

We shall only prove that the function \(\hat{W}_1\) is an upper semicontinuous function in \((t, x_N) \in [0, T] \times H_N\), the proof for \(-\hat{W}_2\) being similar. For every \((t, x_N) \in [0, T] \times H_N\), by the definition of \(\hat{W}_1\), for every \(\varepsilon > 0\), there exists a constant \(\delta > 0\) such that

\[ \hat{W}_1(t, x_N) \geq \hat{W}_1(s, x_N) - \varepsilon \quad \text{for all} \quad s \in [0 \vee (t - \delta), (t + \delta) \wedge T]. \]

Then

\[ \hat{W}_1(t, x_N) \geq \limsup_{s \to t} \left( \sup_{l \to s} \hat{W}_1(l, x_N) \right) - \varepsilon = \limsup_{s \to t} \hat{W}_1(s, x_N) - \varepsilon. \]

By the arbitrariness of \(\varepsilon > 0\), we obtain that

\[ \hat{W}_1(t, x_N) \geq \limsup_{s \to t} \hat{W}_1(s, x_N), \quad \text{for all} \quad (t, x_N) \in [0, T] \times H_N. \]

Therefore,

\[ \limsup_{(s, y_N) \to (t, x_N)} \hat{W}_1(s, y_N) = \limsup_{(s, y_N) \to (t, x_N)} [\hat{W}_1(s, y_N) - \hat{W}_1(s, x_N) + \hat{W}_1(s, x_N)] \leq \limsup_{(s, y_N) \to (t, x_N)} C|x_N - y_N| + \hat{W}_1(t, x_N) = \hat{W}_1(t, x_N), \quad (t, x_N) \in [0, T] \times H_N. \]

The proof is now complete. \(\square\)
Lemma 6.5. The functions $\tilde{W}_1$ and $-\tilde{W}_2$ defined in [6.18] satisfy condition (8.5) of Theorem 8.3 in [9].

Proof. We only prove $\tilde{W}_1$ satisfies condition (8.5) of Theorem 8.3 in [9]. The same result for $-\tilde{W}_2$ can be obtained by a symmetric way.

Set $r = \frac{1}{2}|T - \hat{t}|$, for a given $L > 0$, let $\varphi \in C^{1,2}((T - \bar{a}, T) \times H_N)$ be a function such that $\tilde{W}_1(\hat{t}, x_N) - \varphi(\hat{t}, x_N)$ has a maximum at $(\hat{t}, x_N) \in (T - \bar{a}, T) \times H_N$, moreover, the following inequalities hold true:

$$|\hat{t} - \hat{t}| + |x_N - \hat{\gamma}(\hat{t})| < r, \quad |\tilde{W}_1(\hat{t}, x_N)| + |\nabla x \varphi(\hat{t}, x_N)| + |\nabla^2 \varphi(\hat{t}, x_N)| \leq L.$$ 

We can modify $\varphi$ such that $\varphi \in C^{1,2}_{\beta}((T - \bar{a}, T) \times H_N)$, $\tilde{W}_1(\hat{t}, x_N) - \varphi(\hat{t}, x_N)$ has a strict maximum at $(\hat{t}, x_N) \in (T - \bar{a}, T) \times H_N$ and the above two inequalities hold true. If $\hat{t} < \hat{t}$, we have $b = \varphi(\hat{t}, x_N) = \frac{1}{2}(\hat{t} - \hat{t})^{-\frac{1}{2}} \geq 0$. If $\hat{t} \geq \hat{t}$, we consider the functional

$$\Gamma(\gamma_\ell) = W_1(\gamma_\ell) - 2\gamma_\ell^2 \sum_{i=1}^{N} (\gamma_{i\ell}, \xi_i) - (\varphi(t, (\gamma_i(t))_N) - 2\gamma_\ell^2 \frac{|(\gamma_i(t))_N| - (e^{(t-\hat{t})A \xi_i(\hat{t}))_N|^2}{(t, \gamma_i) \in [\hat{t}, T] \times \Lambda}.$$ 

We may assume that $\varphi$ grow quadratically at $\infty$. By Lemma 2.4.9 we have that, for every $(t_0, \gamma_{i0}) \in [\hat{t}, T] \times \Lambda^i$ satisfy

$$\Gamma(\gamma_{i0}) \geq \sup_{(s, \gamma_s) \in [\hat{t}, T] \times \Lambda^i} \Gamma(\gamma_s) - \delta \geq \tilde{W}_1(\hat{t}, x_N) - \varphi(\hat{t}, x_N) - \delta,$$

there exist $(\hat{t}, \gamma_\ell) \in [\hat{t}, T] \times \Lambda^i$ and a sequence $\{(\hat{t}_i, \gamma_{i\ell})\}_{i \geq 1} \subset [\hat{t}, T] \times \Lambda^i$ such that

(i) $\Gamma(\gamma_{i0}) \leq \delta, \Gamma(\gamma_{i\ell}) \leq \frac{\delta}{12}$ and $t_i \uparrow \hat{t}$ as $i \to \infty$,

(ii) $\Gamma(\gamma_{i\ell}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Gamma(\gamma_{i\ell}) \geq \Gamma(\gamma_{i0})$, and

(iii) for all $(s, \gamma_s) \in \Lambda^i \setminus \{(\hat{t}, \gamma_\ell)\}$,

$$\Gamma(\gamma_s) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Gamma(\gamma_{i\ell}) \leq \Gamma(\gamma_{i\ell}) - \sum_{i=0}^{\infty} \frac{1}{2^i} \Gamma(\gamma_{i\ell}) > \Gamma(\gamma_{i0}) \geq \tilde{W}_1(\hat{t}, x_N) - \varphi(\hat{t}, x_N) - \delta.$$

Then

$$\tilde{W}_1(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N) - \varphi(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N) \geq \tilde{W}_1(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N) - \varphi(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N)$$

$$\geq W_1(\gamma_{i\ell}) - 2\beta \gamma_{i\ell}^2 \sum_{i=0}^{\infty} \frac{1}{2^i} (|(\gamma_{i\ell}(\hat{t}))_N| - (e^{(t-\hat{t})A \xi_i(\hat{t}))_N|^2 - \varphi(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N)$$

$$\geq \Gamma(\gamma_{i0}) \geq \tilde{W}_1(\hat{t}, x_N) - \varphi(\hat{t}, x_N) - \delta.$$

Letting $\delta \to 0$, we obtain

$$\hat{t} \to \hat{t}, \ (\gamma_{i\ell}(\hat{t}))_N \to \gamma_{i\ell} \text{ as } \delta \to 0;$$

(6.39)

Since $\hat{t} \leq \hat{t} + \frac{|T-\hat{t}|}{2}$, we get $\hat{t} < T$. Then, by (6.39), we have $\hat{t} < T$ provided by $\delta > 0$ be small enough. Thus, the definition of the viscosity subsolution can be used to obtain the following result:

$$\varphi(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N) - \frac{\varepsilon}{\nu T} \Gamma(\gamma_{i\ell}) + 2 \sum_{i=0}^{\infty} \frac{1}{2^i} [(\hat{t} - t_i) + (\hat{t} - \hat{t}_i)] + (A^* \nabla x \varphi(\hat{t}, (\gamma_{i\ell}(\hat{t}))_N), \gamma_{i\ell}(\hat{t}))_H$$

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+2\varepsilon(\bar{t} - \hat{t}) - 4\beta(A^*((\hat{\gamma}_t(\tilde{t}))_N) - (e^{(\bar{t} - \hat{t})A}\xi_t(\tilde{t}))_N), \gamma_t(\hat{t}))_N + H(\hat{\gamma}_t, W_1(\gamma_t),

\nabla_x \varphi_t(\tilde{t}, (\gamma_t(\hat{t}))_N) + 2\beta \frac{1}{2} \partial_x((\gamma_t(\hat{t}))_N^2 - (e^{(\bar{t} - \hat{t})A}\xi_t(\tilde{t}))_N^2)^2 + 2\beta \partial_x Y^3(\gamma_t - \xi_{\tilde{t},A}) + \varepsilon \partial_x Y^3(\gamma_t - \xi_{\tilde{t},A})

+ \varepsilon \frac{\nu T - \bar{t}}{\nu T} \partial_x Y^3(\gamma_t) + \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2\tau} Y^3(\gamma_t - \xi_{\tilde{t},i,A}) + \sum_{i=0}^{\infty} \frac{1}{2\tau} Y^3(\gamma_t - \xi_{\tilde{t},i,A}) \right],

\nabla_x \varphi_t(\tilde{t}, (\gamma_t(\hat{t}))_N)

+ 2\beta \frac{1}{2} \partial_x((\gamma_t(\hat{t}))_N^2 - (e^{(\bar{t} - \hat{t})A}\xi_t(\tilde{t}))_N^2)^2 + 2\beta \partial_x Y^3(\gamma_t - \xi_{\tilde{t},A}) + \varepsilon \partial_x Y^3(\gamma_t - \xi_{\tilde{t},A})

+ \varepsilon \frac{\nu T - \bar{t}}{\nu T} \partial_x Y^3(\gamma_t) + \partial_x \left[ \sum_{i=0}^{\infty} \frac{1}{2\tau} Y^3(\gamma_t - \xi_{\tilde{t},i,A}) + \sum_{i=0}^{\infty} \frac{1}{2\tau} Y^3(\gamma_t - \xi_{\tilde{t},i,A}) \right] \geq c. \quad (6.40)

Letting \delta \to 0, by the definition of \( H \), it follows that there exists a constant \( c \) such that \( b = \varphi_t(\bar{t}, \bar{\omega}_N) \geq c \). The proof is now complete. \( \square \)

**Lemma 6.6.** The maximum points \((\bar{t}, \bar{\gamma}_t, \bar{s}, \bar{\eta}_s)\) of \( \Gamma_k(\bar{s}, \eta_s) = \sum_{i=0}^{\infty} \frac{1}{2\tau} [\bar{T}^3(\bar{\gamma}_t^i, \gamma_s) + \bar{Y}^3(\bar{\eta}_s^i, \eta_s)] \)

defined by (6.21) in \([\bar{t}, T] \times \Lambda^t \times [\bar{s}, T] \times \Lambda^k \) satisfy conditions (6.22) and (6.33).

**Proof.** By (6.34), (6.36) and the definitions of \( \bar{W}_1 \) and \( \bar{W}_2 \), we get that

\[ \bar{W}_1(\bar{t}, (\gamma_t(\hat{t}))_N) - \bar{W}_2(\bar{s}, (\bar{\eta}_s(\hat{s}))_N) - \varphi_k(\bar{t}, (\gamma_t(\hat{t}))_N) - \psi_k(\bar{s}, (\bar{\eta}_s(\hat{s}))_N) \]

\[ \geq \bar{W}_1'(\gamma_t) - \bar{W}_2'(\eta_s) - 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) - \varphi_k(\bar{t}, (\gamma_t(\hat{t}))_N) - \psi_k(\bar{s}, (\bar{\eta}_s(\hat{s}))_N) - 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) \]

\[ \geq \Gamma_k(\gamma_t, \eta_s) \geq \sup_{(t, \gamma_t), (s, \eta_s) \in [\bar{t}, T] \times \Lambda^t} \Gamma_k(\gamma_t, \eta_s) - \delta \]

\[ \geq \bar{W}_1(\bar{t}, k, \bar{s}) - \bar{W}_2(\bar{s}, k, \bar{y}_N) - \varphi_k(\bar{t}, k, \bar{x}_N) - \psi_k(\bar{s}, k, \bar{y}_N) - \delta. \]

Letting \( \delta \to 0 \), we obtain (6.22) and

\[ \lim_{\delta \to 0} [W_1'(\gamma_t) - W_2'(\eta_s)] = 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) - \varphi_k(\bar{t}, (\gamma_t(\hat{t}))_N) - \psi_k(\bar{s}, (\bar{\eta}_s(\hat{s}))_N) - 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) \]

\[ = \bar{W}_1(\bar{t}, k, \bar{x}_N) - \bar{W}_2(\bar{s}, k, \bar{y}_N) - \varphi_k(\bar{t}, k, \bar{x}_N) - \psi_k(\bar{s}, k, \bar{y}_N). \]

Letting \( \delta \to 0 \) and \( k \to \infty \), by (6.17) and (6.33)-(6.36), we show that

\[ \liminf_{k \to \infty} \lim_{\delta \to 0} [W_1'(\gamma_t) - W_2'(\eta_s)] = 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) \]

\[ - 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) \]

\[ = \bar{W}_1(\bar{t}, k, \bar{x}_N) - \bar{W}_2(\bar{s}, k, \bar{y}_N) \geq \bar{W}_1(\bar{t}, (\gamma_t(\hat{t}))_N) - \bar{W}_2(\bar{t}, (\eta_s(\hat{s}))_N) \]

\[ = W_1'(\gamma_t) - W_2'(\eta_s) - 2\beta \bar{Y}^3(\gamma_t, \xi_t) - \bar{Y}^3(\eta_s, \xi_t) \]

\[ \leq \Psi_1(\gamma_t, \eta_s, \tilde{t}, A) - W_2(\eta_s) + W_2(\eta_s, A) - \varepsilon [\bar{T}^3(\gamma_t, \gamma_t) + \bar{Y}^3(\eta_s, \tilde{t}, A, \eta_t)] \]

On the other hand, without loss of generality, we may assume \( \bar{s} \leq \bar{t} \),

\[ W_1'(\gamma_t) - W_2'(\eta_s) - 2\beta \bar{Y}^3(\gamma_t, \xi_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) \]

\[ - 2\beta \bar{Y}^3(\gamma_t, \xi_t) + \bar{Y}^3(\eta_s, \xi_t) \]

\[ \leq \Psi_1(\gamma_t, \eta_s, \tilde{t}, A) - W_2(\eta_s) + W_2(\eta_s, A) - \varepsilon [\bar{T}^3(\gamma_t, \gamma_t) + \bar{Y}^3(\eta_s, \tilde{t}, A, \eta_t)] \]

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+2\beta^3(\langle e^{(t-s)A}\tilde{\eta}_k(s)\rangle_N - \langle e^{(t-s)A}\tilde{\xi}_k(t)\rangle_N^2 - \langle \tilde{\eta}_k(s)\rangle_N - \langle e^{(s-t)A}\tilde{\xi}_k(t)\rangle_N^2)
+\beta^4(\langle \tilde{\eta}_k(s)\rangle_N - \langle e^{(t-s)A}\tilde{\eta}_k(s)\rangle_N^2)
\leq \Psi_1(\gamma_t, \tilde{\eta}_t) + 2L(1 + ||\tilde{\eta}_k||_0)(t - s)^2 - \epsilon[\Psi^3(\gamma_t, \tilde{\eta}_t) + \Psi^3(\tilde{\eta}_k, \tilde{\eta}_t) + 2|s - t|^2]
+2\beta^3(\langle e^{(t-s)A}\tilde{\eta}_k(s)\rangle_N - \langle e^{(t-s)A}\tilde{\xi}_k(t)\rangle_N^2 - \langle \tilde{\eta}_k(s)\rangle_N - \langle e^{(s-t)A}\tilde{\xi}_k(t)\rangle_N^2)
+\beta^4(\langle \tilde{\eta}_k(s)\rangle_N - \langle e^{(t-s)A}\tilde{\eta}_k(s)\rangle_N^2).

Then letting $\delta \to 0$ and $k \to \infty$, we obtain

$$\limsup_{k \to \infty} \limsup_{\delta \to 0} \left[ W'_1(\gamma_t) - W'_2(\tilde{\eta}_k) - 2\beta(\Psi^3(\gamma_t, \tilde{\eta}_t) + \Psi^3(\tilde{\eta}_k, \tilde{\eta}_t)) + \epsilon(\Psi^3(\gamma_t, \tilde{\eta}_t) + \Psi^3(\tilde{\eta}_k, \tilde{\eta}_t)) \right]
\leq W'_1(\gamma_t) - W'_2(\tilde{\eta}_t) - \beta\Psi^3(\gamma_t, \tilde{\eta}_t) - \beta^4(\langle \gamma_t(\tilde{\eta}_t)\rangle_N - \langle \tilde{\eta}_t(\tilde{\eta}_t)\rangle_N^2).

Combining with (6.41), we get

$$\lim_{k \to \infty} \limsup_{\delta \to 0} \left[ \Psi^3(\gamma_t, \tilde{\eta}_t) + \Psi^3(\tilde{\eta}_k, \tilde{\eta}_t) \right] = 0.
$$

Noting that, by (2.11),

$$||\gamma_t - \gamma_{t,f,A}||_0 + ||\tilde{\eta}_t - \tilde{\eta}_{t,f,A}||_0 \leq ||\gamma_t - \gamma_{t,f,A}||_0 + ||\tilde{\eta}_{t,f,A} - \tilde{\eta}_{t,f,A}||_0 \leq 4\Psi^3(\gamma_t, \tilde{\eta}_t) + 4\Psi^3(\tilde{\eta}_k, \tilde{\eta}_t)
$$
we obtain (6.23) holds true. The proof is now complete. \qed

References

[1] E. Bayraktar, C. Keller, Path-dependent Hamilton-Jacobi equations in infinite dimensions, J. Funct. Anal., 275 (2018), 2096-2161.

[2] J. M. Borwein and Q. J. Zhu, Techniques of variational analysis, volume 20 of CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer-Verlag, New York, 2005.

[3] R. Cont and D.-A. Fournié, Change of variable formulas for non-anticipative functionals on path space, J. Funct. Anal., 259 (2010), no. 4, 1043-1072.

[4] R. Cont and D.-A. Fournié, Functional Itô calculus and stochastic integral representation of martingales, Ann. Probab., 41 (2013), 109-133.

[5] A. Cosso, S. Federico, F. Gozzi, M. Rosestolato, N. Touzi, Path-dependent equations and viscosity solutions in infinite dimension, Ann. Probab., 46 (2018), no. 1, 126-174.

[6] A. Cosso, and F. Russo, Crandall-Lions viscosity solutions for path-dependent PDEs: The case of heat equation, ArXiv:1911.13095v2, 2020.

[7] M. G. Crandall, and P. L. Lions, Hamilton-Jacobi equations in infinite dimensions, IV: Hamiltonians with Unbounded linear terms, J. Funct. Anal., 90 (1990), 237-283.

[8] M. G. Crandall, and P. L. Lions, Hamilton-Jacobi equations in infinite dimensions, V: Unbounded linear terms and $B$-continuous solutions, J. Funct. Anal., 97 (1991), 417-465.

[9] M. G. Crandall, H. Ishii, and P. L. Lions, Users guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc., 27 (1992), 1-67.
[10] G. Da Prato and J. Zabczyk, Ergodicity for Infinite-Dimensional Systems, Cambridge Unive., 1996.

[11] B. Dupire, Functional Itô calculus, Preprint. Available at papers.ssrn.com, (2009).

[12] I. Ekren, Viscosity solutions of obstacle problems for fully nonlinear path-dependent PDEs, Stochastic Process. Appl., 127 (2017), 3966-3996.

[13] I. Ekren, C. Keller, N. Touzi and J. Zhang, On viscosity solutions of path dependent PDEs, Ann. Probab., 42 (2014), 204-236.

[14] I. Ekren, N. Touzi and J. Zhang, Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part I, Ann. Probab., 44 (2016), 1212-1253.

[15] I. Ekren, N. Touzi and J. Zhang, Viscosity solutions of fully nonlinear parabolic path dependent PDEs: Part II, Ann. Probab., 44 (2016), 2507-2553.

[16] N. El Karoui, S. Peng, and M.C. Quenez, Backward stochastic differential equations in finance, Math. Finance, 7 (1997), No.1, 1-71.

[17] G. Fabbri, F. Gozzi, A. Święch, Stochastic Optimal Control in Infinite Dimension. Dynamic Programming and HJB Equations, in: Probability Theory and Stochastic Modelling, volume 82, Springer, 2017.

[18] M. Fuhrman, G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control, Ann. Probab., 30 (2002), no. 3, 1397-1465.

[19] I. Karatzas and S. E. Shreve, Methods of Mathematical Finance, Applications of Mathematics (New York), 39, Springer-Verlag, New York, 1998.

[20] P. L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. I. The case of bounded stochastic evolutions, Acta Math., 161 (1988), no. 3-4, 243-278.

[21] P. L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. II. Optimal control of Zakai’s equation. in: Stochastic partial differential equations and applications, II, eds. G. Da Prato, L. Tubaro, 147-170, Lecture Notes in Mathematics 1390, Springer, 1989.

[22] P. L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. III. Uniqueness of viscosity solutions for general second-order equations, J. Funct. Anal., 86 (1989), no. 1, 1-18.

[23] N. Y. Lukoyanov, On viscosity solution of functional Hamilton-Jacobi type equations for hereditary systems, Proceedings of the Steklov Institute of Mathematics, 259 (2007), S190-S200.

[24] S. Peng, Note on Viscosity Solution of Path-Dependent PDE and G-Martingales- 2nd version, arXiv:1106.1144v2, 2012.

[25] S. Peng, BSDE and stochastic optimizations, Topics in stochastic analysis, J. Yan, S. Peng, S. Fang and L. Wu, Ch.2, Science Press. Beijing (in Chinese), 1997.
[26] Z. Ren, Viscosity solutions of fully nonlinear elliptic path dependent partial differential equations, Ann. Appl. Probab., 26 (2016), 3381-3414.

[27] Z. Ren, N. Touzi, and J. Zhang, Comparison of viscosity solutions of fully nonlinear degenerate parabolic Path-dependent PDEs, SIAM J. Math. Anal., 49 (2017), 4093-4116.

[28] Z. Ren, M. Rosestolato, Viscosity solutions of path-dependent PDEs with randomized time, SIAM J. Math. Anal., 52 (2020), 1943-1979.

[29] M. Rosestolato, Path-dependent SDEs in Hilbert spaces, arXiv:1606.06321, 2016.

[30] A. Święch, ”Unbounded” second order partial differential equations in infinite-dimensional Hilbert spaces, Comm. Partial Differential Equations 19, (1994), no. 11-12, 1999-2036.

[31] S. Tang, F. Zhang, Path-dependent optimal stochastic control and viscosity solution of associated Bellman equations, Discrete Cont. Dyn. Syst. -A, 35 (2015), no. 11, 5521-5553.

[32] J. Zhou, B. Liu, Optimal control problem for stochastic evolution equations in Hilbert spaces, Int. J. Control, 83 (2010), 1771-1784.

[33] J. Zhou, A class of delay optimal control problems and viscosity solutions to associated Hamilton-Jacobi-Bellman equations, ESAIM Control Optim. Calc. Var., 24 (2018), 639-676.

[34] J. Zhou, Viscosity solutions to first order path-dependent HJB equations, arXiv:2004.02095, 2020.

[35] J. Zhou, Viscosity solutions to second order path-dependent Hamilton-Jacobi-Bellman equations and applications, arXiv:2005.05309, 2020.

[36] J. Zhou, Viscosity solutions to first order path-Dependent Hamilton-Jacobi-Bellman equations in Hilbert space, arXiv:2007.04079v1, 2020.