Production of kinks in an inhomogenous medium.

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Abstract

The purpose of this report is presentation of the main modifications of the standard Kibble-Zurek formalism caused by the existence of unperfections in the system. We know that the distribution of kinks created during a second order phase transition in pure systems is determined solely by the correlation length at freeze-out time. The correlation length at that instant of time intuitively describes the size of the defect and therefore the number density of defects is limited by the possibility of holding kinks in a unit volume. On the other hand if the system is populated by the impurities then kinks emerge mainly in knots of the force distribution which correspond to extremes of the impurity potential i.e. positions of imperfections. The purpose of this report is to show that, due to existence of the strong gradients of the impurity potential, kinks can be created mainly in the close vicinity of the impurities. It seems that this simple mechanism can be responsible for occurrence, in the number density formula, the additional length scale describing the impurity distribution. We know that in pure systems, as a consequence of kink-antikink annihilation, the number density of kinks decrease in time. In contradiction to pure systems, kinks produced in the systems populated by impurities could be confined by the impurity centers and therefore they may not dis-
appear from the system and may remain above the level established by thermal nucleation of pairs.

1 Introduction

Last years topological defects attract attention of many researchers. The motivation of these studies comes from the fact that they can be seen as macroscopic manifestations of underlying physical processes. On the other hand they can help to study the nature of critical dynamics.

The theory describing the dynamics of the second order phase transition was proposed by Kibble and Zurek [1]. The key point of the Kibble-Zurek mechanism is an observation that the order parameter evolves adiabatically through a sequence of nearly equilibrium configurations up to the freeze-in time. At that instant the system loses capacity to respond for the changes of the external parameters. From that time to the freeze-out time the field configuration remains almost unchanged. The dynamical evolution restarts below the critical temperature at freeze-out time. At that instant the system regains ability to respond for the changes of the external parameters but it is too late to undo non-trivial arrangements of the order parameter from above the critical point. This paradigm works for the overdamped and underdamped systems as well. The main prediction of this scenario is the dependence of the number density of produced defects on correlation length \( n \sim \xi^{-d} \) at freeze-out time or its dependence on quench time \( n \sim \tau^{-d/4} \), where \( d \) denotes the number of space dimensions. This scenario was well verified in a series of numerical experiments [2].

The defect network density obtained at freeze-out time is an initial condition for dynamics which is determined by the defect - antidefect interactions. Due to annihilation of defects and antidefects the initial density of defect network first is quickly reduced in time and then is stabilized on the level determined by the Boltzman factor which describes the probability of thermal nucleation of the kink-antikink pairs [3].

In real life experiments researchers use mainly stable coherent quantum systems. So far, experiments were performed in Helium-3 on symmetric phase \(^3\)He-B which is more simple to experimental and theoretical treatment. The results of experimental studies confirms the dependence of the number density of produced vortices on quench time [4]. Importance of studies of the
transitions in Helium-3, follows from the fact that due to nontrivial structure of the order parameter it allows for experimental verifications of ideas concerning the structure of vacuum of the quantum field theory.

Researchers performed also experiments on liquid crystals. The creation of disclinations of different types produced during a quench from disordered to nematic phase in liquid crystals was examined and the results were to some degree consistent with the Kibble-Zurek predictions [5].

There were also more controversial experiments made in superfluid helium-4 where almost no vortices of topological origin were observed [6].

Lately there are also attempts to study the creation of vortices in optically cooled alkali atom clouds during formation of the Bose-Einstein condensate.

In this report I would like to concentrate on influence of impurities on creation of topological defects. It is difficult to imagine free of imperfections liquid crystal or even superconductor. The population of the superconductors and liquid crystals by the impurities and admixtures seems to be an inevitable outcome of their preparation. On the other hand quantum liquids are one of the purest substances in the nature. Although the solubility of foreign materials in liquid helium is almost zero there exists some artificial techniques, like aerogel technique [7], which allow to introduce impurities even into quantum liquid.

Prevailed part of the obtained hitherto results concern homogenous medium. On the other hand, the presence of the impurities can significantly change properties of the system.

This report aims in presentation of the main modifications of the standard Kibble-Zurek formalism caused by the existence of imperfections in the system.

## 2 An influence of inhomogenities on production of kinks

First let us recall K-Z formalism applied to description of homogenous systems. A pure, overdamped \( \phi^4 \) system is described by the following equation of motion

\[
\gamma \partial_t \phi(t, x) = \partial_x^2 \phi(t, x) - a(t) \phi(t, x) - \lambda \phi^3(t, x) + \eta(t, x),
\]

(1)
where $\eta(t, x)$ is a temperature white gaussian noise defined by the correlators

\[
\langle \eta(t, x) \rangle = 0, \\
\langle \eta(t, x) \eta(t', x') \rangle = \frac{2\pi\gamma}{\beta} \delta(x - x') \delta(t - t').
\]

The explicit time dependence of the "mass" parameter $a(t)$ allows for modelling the phase transition in the system. Depending on the sign of this parameter we can find a system in the phase with one trivial or two nontrivial ground states.

The number of kinks produced during a phase transition at freeze-out time is calculated from the Liu-Mazenko-Halperin formula \[8\]

\[
n = \frac{1}{\pi} \sqrt{\frac{\langle \phi'^2 \rangle}{\langle \phi^2 \rangle}}.
\]

In fact this formula can be expressed with the use of the power spectrum which is defined by the equal time correlator of the order parameter. The cut-off in this formula separates the stable from unstable modes of the system. We integrate only over unstable modes because only they can grow to form stable kink structures. As a result of this calculation one could obtain a Kibble-Zurek critical exponent $1/4$ which describes the dependence of the number density of produced kinks on quench time $\tau$

\[
n \sim \frac{1}{\tau^{1/4}}.
\]

In this context usually a linear quench is presumed. The critical exponent depends only on number of spatial dimensions.

On the other hand, many physical systems are dense populated by imperfections of different types. The presence of impurities in the system can be taken into account by introducing a deterministic force distribution $D(t, x)$ into the equation of motion

\[
\gamma \partial_t \phi(t, x) = \partial_x^2 \phi(t, x) - a(t) \phi(t, x) - \lambda \phi^3(t, x) + \eta(t, x) + D(t, x).
\]

First we reconsider the way of counting zeros of the order parameter. The number density of produced zeros can be defined as a ratio of zeros located
in some interval of space to the length of this interval. On the other hand
the number of zeros can be calculated as a sum of arbitrary quantity divided
by itself over all points where the scalar field disappears. One of the possible
choices of this quantity is $\phi'$

$$n(t, x) = \lim_{L \to 0} \frac{\langle N \rangle}{2L} = \lim_{L \to 0} \frac{1}{2L} \langle \sum_i |\phi'(t, x_i)| \rangle. \quad (6)$$

We know that integration of the delta function can be replaced by the sum-
mation over zeros of delta argument

$$\int dx f(x) \delta[g(x)] = \sum_i \frac{f(x_i)}{|g'(x_i)|} \text{ where } x_i \text{ are defined by the equation } g(x_i) = 0.$$

If we identify in this lemma function $f$ with $\phi'$ and $g$ with $\phi$ then we replace
the sum over all zeros by the average of some integral

$$n(x) = \lim_{L \to 0} \frac{1}{2L} \langle \int_{x-L}^{x+L} d\tilde{x} |\phi'(t, \tilde{x})| \delta[\phi(t, \tilde{x})] \rangle. \quad (7)$$

In zero $L$ limit this expression reduces to the average over all realizations of
the noise of some combination of the delta and sign functions

$$n(x) = \langle \text{sign}[\phi'(t, x)] \phi'(t, x) \delta[\phi(t, x)] \rangle. \quad (8)$$

In the next step we use integral representations of the signum and delta
functions and then divide the scalar field on two components $\phi(t, x) =
\psi(t, x) + u(t, x)$. One which carries deterministic part of the evolution $u$
and the second $\psi$ which carries the stochastic part of the evolution of the
scalar field $\phi$. We known that above critical point the field fluctuations around
trivial ground state. If the amplitude of the noise is small i.e. if the temper-
ature of the system is low then fluctuations of the field are also small and
its value is close to zero. As the field fluctuate around zero value its aver-
ge magnitude is small and therefore cubic term in the equation of motion
is negligible. As the system evolves through a sequence of almost equili-
rium states this remains true up to freeze-in time. Identification of those two
components in linear approximation is straightforward.

Then we use theorems which allow to replace $n$-th order correlators by
the average and correlator of second order and therefore we are able to replace
the averages of some functions of random variable by the functions of the second order correlator of this variable.

Finally we obtain a formula which in the absence of impurities reduces to the well known Halperin-Liu-Mazenko formula (3)

\[
n(t, x) = \frac{1}{\pi} \sqrt{\frac{\langle \psi'^2 \rangle}{\langle \psi^2 \rangle}} e^{-\frac{\langle \psi'^2 \rangle}{2 \langle \psi^2 \rangle}} + \frac{u'}{\sqrt{2 \pi \langle \psi^2 \rangle}} e^{-\frac{u'^2}{2 \langle \psi^2 \rangle}} \text{Erf}\left(\frac{u'}{\sqrt{2 \langle \psi^2 \rangle}}\right),
\]

where \( \text{Erf} \) is the error function. The analytical and numerical studies of the kink distribution shows that kinks are created mainly in the vicinity of knots of the force distribution which corresponds to extremes of the impurity potential [9].

In most of the physical systems the shape and the distribution of the impurities is random and therefore we have to allow the force distribution to be a random type with some length scale which characterizes the average distance between impurities. This time the equation of motion contains two random forces. First represents thermal fluctuations in the system \( \eta \) and the second one which describes distribution of impurities \( D \).

The angle bracket represents the average with respect to all realizations of the thermal noise and the new bracket \( \{\ldots\} \) represents an average with respect to all possible distributions of the impurities in the system.

Equation of motion in this new situation, at first sight, seems to be identical with equation (5)

\[
\dot{\gamma} \partial_t \phi(t, x) = \partial_x^2 \phi(t, x) - a(t) \phi(t, x) - \lambda \phi^3(t, x) + \eta(t, x) + D(t, x).
\]

In fact \( \dot{\gamma} \) in this equation is not a simple constant but it is an integral operator. The existence of this term is an inevitable if we restrict our studies to stationary processes. The explicit form of this operator is the following:

\[
\dot{\gamma} \partial_t \phi(t, x) \equiv \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dx' \gamma(t, t'; x - x') \partial_t \phi(t', x').
\]

If the impurity force distribution \( D \) has a form of the gaussian white noise then \( \gamma(t, t'; x - x') = \gamma \delta(t - t') \delta(x - x') \), and this integral reduces to the damping constant multiplied by the time derivative of the order parameter. In generic situation we expect a dependence of \( D \) correlators on some length...
scale (e.g. average distance between impurity centers) and therefore the force distribution is defined as follows

\[
\{D(t, x)\} = 0,
\]
\[
\{D(t, x)D(t', x')\} = \frac{1}{\beta} W(|x - x'|) \delta(t - t'). \tag{11}
\]

To better understand the source of this complication let us consider a simple mechanical analogy which is a Brownian motion theory.

The erratic motion of a Brownian particle is caused by collisions with the molecules of the fluid in which it moves. These collisions allow an exchange of the energy between the fluid and the Brownian particle. If the Brownian particle is much more massive than the molecules of the fluid then the influence of the molecules on the observed particle can be approximated by a Gaussian white noise \(\eta_G(t)\):

\[
m\ddot{x}(t) + \gamma \dot{x}(t) = \eta_G(t), \tag{12}
\]

where \(x(t)\) is the position of the Brownian particle. The generalization of Brownian motion theory to the random motion of a particle which is not necessarily heavier than the molecules of the fluid was proposed by Kubo [10]. In this case the time scale of molecular motion is no longer very much shorter than that of the motion of the particle under observation, so that the random force \(\eta(t)\) can not be of Gaussian type. To describe an influence of the molecules on the observed particle we have to introduce a color noise characterized by some time scale. This time scale may describe the average time interval between two subsequent collisions of the molecules with the observed particle. In addition, if we consider a stationary process we have to abandon the assumption of a constant friction and to introduce generally a frequency-dependent friction

\[
m\ddot{x}(t) + \int_{t_0}^{t} dt' \gamma(t - t') \dot{x}(t') = \eta(t). \tag{13}
\]

In case of \(\phi^4\) model the distribution of impurities is not generally described by the white gaussian noise. In generic situation the distribution of impurities is characterized by some length scale which describes the average separation of impurity centers and therefore, in similar way as it was in case of Brownian particle, we introduce retardation to \(\phi^4\) model.
Next we have to make further generalization of the Liu-Mazenko-Halperin formula. This generalization is achieved by averaging the formula (9) with respect to possible distributions of the impurity centers

\[ n = \{n(t, x)\} = \frac{1}{\pi} \sqrt{\frac{\langle \psi'^2 \rangle + \{u'^2\}}{\langle \psi^2 \rangle + \{u^2\}}} \] (14)

The further understanding can be made for particular choice of the noise amplitude. The most representative one is Ornstein-Uhlenbeck amplitude \( W(|x|) = A e^{-\frac{|x|}{L}} \) which interpolates between constant distribution and gaussian white noise. In this model the number density of produced kinks depends on quench time and on characteristic length scale of the impurity distribution as well [11]

\[ n \approx \frac{1}{\pi} \sqrt{\frac{0.43b + 0.34c}{1.81b + 0.83c}}, \] (15)

where \( b = \frac{1}{\tau} \) and \( c = \frac{1}{\sqrt{\tau}} \). Let us notice that in case of week imperfections the usual scaling is recovered

\[ n \sim \frac{1}{\tau^{\frac{1}{4}}}. \] (16)

3 Remarks

We show that there are two components which determine the number density of kinks produced in the inhomogenous system during the second order phase transition. First component follows from the K-Z formalism for pure systems and is determined by the quench time. The second component is determined by the characteristic length which describes the distribution of impurities in the system. Due to existence of impurities and admixtures the kinks are created mainly in the knots of the impurity force distribution. One could even find exact solutions which describe the kinks confined by some particular impurity potentials [12]. An examples of those solutions are squeezed kink or squeezed anti-kink. One could also check linear stability of those solutions. The other solution obtained for some particular force distribution is a static kink-antikink solution which illustrates that, in contradiction to pure systems, the configuration of this type have no tendency to annihilate. This is also a reason why at late time after transition in the inhomogenous
system we still can find the number density of kinks substantially larger than estimated from the probability of thermal nucleation of kink-antikink pairs.

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References

[1] T.W.B. Kibble, J.Phys. A 9 1387 (1976).
    T.W.B. Kibble, Phys. Rep. 67, 183 (1980).
    W.H. Zurek, Acta Phys.Polon. B 24, 1301 (1993).
    W.H. Zurek, Nature 317 505 (1993).
    W.H. Zurek, Phys.Rep. 276, 177 (1996).

[2] T. Vachaspati, and A. Vilenkin, Phys.Rev.D 30, 2036, (1984).
    A. Bray, Adv.Phys. 43, 357 (1994).
    M. Hindmarsh and T. Kibble, Rep.Progr.Phys. 58, 477 (1995).
    N.D. Antunes, L.M.A. Bettencourt and W.H. Zurek, Phys.Rev.Lett.82, 2824 (1999).

[3] M. Buttiker and R. Landauer, Phys.Rev.Lett. 43, 1453 (1979).
    M. Buttiker and T. Christen, Phys.Rev.Lett. 75, 1895 (1995).
    T. Christen and M. Buttiker, Phys. Rev. E 58, 1533 (1998).
    S. Habib and G. Lythe, Phys.Rev.Lett. 84, 1070 (2000).

[4] V.M.H. Ruutu, V.B. Eltsov, A.J. Gill, T.W.B. Kibble, M. Krusius, Y.G. Makhlin, B. Placais, G.E. Volovik and Wen Xu, Nature 382, 334 (1996).
    V.M. Ruutu, V.B. Eltsov, M. Krusius, Yu.G. Makhlin, B. Placais, G.E. Volovik, Phys. Rev. Lett. 80, 1465 (1998).
    D.I. Bradley, S.N. Fisher and W.M. Hayes, J. Low Temp.Phys. 113, 687 (1998).
C. Bäuerle, Yu.M. Bunkov, S.N. Fisher, H. Godfrin and G.R. Pickett, Nature \textbf{382}, 332 (1996).
C. Bäuerle, Yu.M. Bunkov, S.N. Fisher, H. Godfrin and G.R. Pickett, J. Low Temp.Phys. \textbf{110}, 13 (1998).

[5] M.J. Bowick, L. Chandar, E.A. Schiff and A.M. Srivastava, Science \textbf{263} (1994).
L. Chandar, E.A. Schiff and A.M. Srivastava, Science \textbf{263} (1994).

[6] M.E. Dodd, P.C. Hendry, N.S. Lawson, P.V.E. McClintock and C.D.H. Williams, Phys.Rev.Lett. \textbf{81}, 3703 (1998).
G. Karra, and R.J. Rivers, Phys.Rev.Lett. \textbf{81}, 3707 (1998).
R.J. Rivers, Phys.Rev.Lett. \textbf{84}, 1248 (2000).

[7] G. Lawes, S.C.J. Kingsley, N. Mulders, and J.M. Parpia, Phys.Rev.Lett. \textbf{84}, 4148 (2000).

[8] B.I. Halperin, in \textit{Physics of Defects}, edited by R. Balian, M. Kleman, and J.P. Poirier (North-Holland, New York, 1981).
A. Weinrib and B.I. Halperin, Phys.Rev. \textbf{B 26}, 1362 (1982).
F. Liu and G.F. Mazenko, \textit{ibid}. \textbf{46}, 5963 (1992).

[9] T. Dobrowolski, Phys.Rev.\textbf{E 65}, 036136 (2002).

[10] R. Kubo, Rep.Prog.Phys.\textbf{29}, 255 (1966).

[11] T. Dobrowolski, Phys.Rev.\textbf{E 65}, 046133 (2002).
T. Dobrowolski, Europhys.Jour. \textbf{B 29}, 269 (2002).

[12] T. Dobrowolski, Phys.Rev.\textbf{E 66}, 066112 (2002).