Quantization of the Maxwell field in curved spacetimes of arbitrary dimension

Michael J Pfenning

Department of Physics, United States Military Academy, West Point, NY 10996-1790, USA
E-mail: Michael.Pfenning@usma.edu

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Abstract

We quantize the massless $p$-form field that obeys the generalized Maxwell field equations in curved spacetimes of dimension $n \geq 2$. We begin by showing that the classical Cauchy problem of the generalized Maxwell field is well posed and that the field possesses the expected gauge invariance. Then the classical phase space is developed in terms of gauge-equivalent classes, first in terms of the Cauchy data and then reformulated in terms of Maxwell solutions. The latter is employed to quantize the field in the framework of Dimock. Finally, the resulting algebra of observables is shown to satisfy the wave equation with the usual canonical commutation relations.

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1. Introduction

Tensor and/or spinor fields of assorted types in curved spacetime have been studied by various authors, Buchdahl [1–6], Gibbons [7] and Higuchi [8] to name a few. It is often found that the straightforward generalization of the flat spacetime field equations to an arbitrary curved spacetime is wrought with internal inconsistencies unless specific conditions are met. For example, in his earliest work Buchdahl [2] shows that 'the natural field equations for particles of spin-$\frac{3}{2}$ are consistent if and only if the (pseudo)Riemann space(time) in which they are contemplated is an Einstein space(time)', that is, one for which the Ricci tensor satisfies $R_{ab} = \lambda g_{ab}$. Gibbons demonstrates that there is also a breakdown in the Rarita–Schwinger formulation of spin-$\frac{3}{2}$ in four-dimensional curved spacetime. Meanwhile, Higuchi derives the constraint condition, $\nabla_c R_{ab} = \left[ \frac{1}{2} (g_{ab} \partial_b + g_{ac} \partial_c) + \frac{\lambda}{2} g_{ab} \partial_c \right] R$, on the background metric for the generalization of the massive symmetric tensor field equations to curved spacetime. There are two cases where the above condition is met: the metric is a solution to the vacuum Einstein equations or the Ricci tensor $R_{ab}$ is covariantly constant.
In this paper, we study the \( p \)-form field theory (fully antisymmetric rank-\( p \) tensors) in curved spacetimes of arbitrary dimension which obey the generalization of the Maxwell equations. Unlike the above examples, the \( p \)-form field equations generalize to any dimension without inconsistencies or the need for constraints. Thus, \( p \)-form field theories seem to be a natural model for the study of quantum field theories in higher dimension curved spacetime. For example, the minimally coupled scalar field (\( p = 0 \) in any dimension) and the electromagnetic field (\( p = 1 \) in four dimensions) are two examples of this self-consistent \( p \)-form theory. Better still, the quantization of the classical theory turns out to be identical for all \( p \)-form fields, independent of the rank of form.

This paper proceeds in the following order. In section 2, we discuss the classical generalized Maxwell field. This begins with a review of electrodynamics in four dimensions and then proceeds to the generalization of the Maxwell equations into arbitrary dimension. It is at this point that we convert from the traditional notation used in relativistic physics to that of exterior differential calculus, thus giving us \( p \)-form fields. We then discuss fundamental solutions to the resulting wave equation and the initial value problem for the classical field. We shall see that there exists a gauge freedom in the field which complicates the uniqueness of solutions for a given set of initial data. Thus, the Cauchy problem is only well posed if we work with gauge-equivalent classes.

In section 3, we quantize the \( p \)-form field. From the classical field theory, we obtain a symplectic phase space consisting of a real vector space and a symplectic form. This phase space is quantized by promoting functions in the phase space to operators acting on a Hilbert space while simultaneously requiring the commutator of such operators to be \(-i\) times their classical Poisson bracket. In this way, the algebra of observables for the quantized field on a manifold is obtained. Finally, there will be discussion and conclusions.

Throughout this paper, we will use units where \( \hbar = c = 1 \). The notation \( C^\infty_0(\mathbb{R}^n) \) denotes the space of smooth, compactly supported\(^1\), complex-valued functions on \( \mathbb{R}^n \). We take \( M \) to be a smooth \( n \)-dimensional manifold (without boundary) which is connected, orientable, Hausdorff, paracompact and equipped with a smooth metric of index \( s \).\(^2\) We denote the space of smooth, complex-valued \( p \)-forms on \( M \) by \( \Omega^p(M) \); the subspace of compactly supported \( p \)-forms will be written as \( \Omega^p_0(M) \). Each \( p \)-form may be regarded as a fully antisymmetric covariant \( p \)-tensor field. Our conventions for forms are consistent with that of Abraham, Marsden and Ratiu (AMR) \[9\] and are also summarized in our earlier paper \[10\] to which we refer the reader. Therefore, we only introduce the remaining notational necessities here to make this paper sufficiently self-contained.

The exterior product between forms will be denoted by \( \wedge \), the exterior derivative on forms will be denoted by \( d \), the Hodge \(*\)-operator by \( * \) and the co-derivative by \( \delta \). The Laplace–Beltrami operator, simply called the Laplacian on a Riemannian manifold, is defined as \( \Box = -(\delta d + d\delta) \). All these are consistent with the previous paper.

The only difference between the preceding paper and present one is the definition for the symmetric pairing \( \langle \cdot, \cdot \rangle \) of \( p \)-forms under integration:

\[
\langle \mathcal{U}, \mathcal{V} \rangle_M = \int_M \mathcal{U} \wedge * \mathcal{V}
\]

for any \( \mathcal{U}, \mathcal{V} \in \Omega^p(M) \) for which the integral exists. Also, for smooth \( \mathcal{U} \in \Omega^{p-1}(M) \) and \( \mathcal{V} \in \Omega^p(M) \) we have \( d(\mathcal{U} \wedge * \mathcal{V}) = d\mathcal{U} \wedge * \mathcal{V} - \mathcal{U} \wedge \delta \mathcal{V} \); therefore, by the Stokes theorem

\[
\langle d\mathcal{U}, \mathcal{V} \rangle_M = \langle \mathcal{U}, \delta \mathcal{V} \rangle_M
\]

\(^1\) The support of a function is the closure of the set of points on which it is nonzero.

\(^2\) The index is the number of spacelike (i.e. negative norm-squared) basis vectors in any \( g \)-orthonormal frame.
whenever the supports of the forms have compact intersection. In this sense, the operators \( d \) and \( \delta \) are dual.

2. Classical analysis of the generalized Maxwell field

2.1. Classical electrodynamics in four dimensions

In Minkowski spacetime, it is most common to study the electromagnetic field in the abstract index notation \([11]\) where \( F_{ab} \) is the covariant field-strength tensor and the Maxwell equations are

\[
\partial_{[a} F_{bc]} = 0 \quad \text{and} \quad \partial^a F_{ab} = -4\pi j_b. \tag{2.1}
\]

Here \( j_b \) is the current density, \( \partial_a \) is the partial derivative, lowering or raising of indices is done with respect to the metric \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) and its inverse respectively, and \([ \ ]\) in the homogeneous Maxwell equation is shorthand for the antisymmetric permutation over the indices.

It is known that the generalization of the Maxwell equations to curved four-dimensional spacetimes is internally consistent. This is accomplished by the minimal substitution rule; replace the partial derivatives with \( \nabla_a \), the unique covariant derivative operator associated with the spacetime metric such that \( \nabla_ag_{bc} = 0 \). The Maxwell equations then become

\[
\nabla_{[a} F_{bc]} = 0 \quad \text{and} \quad \nabla^a F_{ab} = -4\pi j_b. \tag{2.2}
\]

It is also common to introduce the (co)vector potential \( A_a \) related (at least locally) to the field-strength tensor by \( F_{ab} = \nabla_{[a} A_{b]} \). Recast in \( A_a \), the homogeneous equation is trivially satisfied, as a result of the first Bianchi identity, and the inhomogeneous equation becomes

\[
\nabla^a \nabla_a A_b - \nabla_b \nabla^a A_a - R^a_b A_a = -4\pi j_b. \tag{2.3}
\]

The Maxwell equations, in both flat and curved spacetime, have a gauge freedom in that many different forms of \( A_a \) give rise to the same \( F_{ab} \). This comes from the freedom to add to \( A_a \) the gradient of any scalar function \( \Lambda \). Because the covariant derivatives, like partial derivatives, commute when acting on scalars the addition of the gradient term has no effect on the final outcome of the resulting field strength. Therefore one can choose to work in a particular gauge, say the Lorenz gauge where \( \nabla^a A_a = 0 \). Then we have considerable simplification to the globally hyperbolic equation

\[
\nabla^a \nabla_a A_b - R^a_b A_a = -4\pi j_b. \tag{2.4}
\]

The benefit of doing so is that existence and uniqueness of solutions to globally hyperbolic equations have been well studied and with a little work we can ‘extract’ solutions to our original equation. This will be covered in more detail below.

2.2. Generalized Maxwell field

The electromagnetic field is a specific example in four dimensions of a much broader theory of fully antisymmetric tensor fields in curved spacetimes of arbitrary dimension. It is mathematically natural to handle such tensors in the language of exterior differential calculus, there being a number of benefits to doing so as follows. (a) All equations are coordinate chart independent; thus, we obtain global results directly. (b) Even if a coordinate chart is specified, the differential forms are still independent of the choice of connection, thus substantially simplifying coordinate-based calculations. (c) The generalized Maxwell equations do index bookkeeping in a ‘natural’ (albeit hidden) way for forms of different ranks or in spacetimes
of varying dimensions. It is precisely this mechanism by which the generalized Maxwell equations avoid all of the consistency problems discussed above for other types of fields. No subsidiary conditions are needed on the spacetime, and the spacetime itself need not satisfy the Einstein equation. We will elaborate more on this point later.

At first glance, one could consider the fundamental object of study to be the \((p+1)\)-form field strength \(\mathcal{F}\), which can be thought of as a fully antisymmetric rank-\((0, p+1)\) tensor. Then, the generalized Maxwell equations are

\[
d\mathcal{F} = 0 \quad \text{and} \quad -\delta \mathcal{F} = J, \tag{2.5}
\]

where \(J\) is the \(p\)-form current density. Electromagnetism happens to be the case where \(\mathcal{F}\) is a 2-form in a four-dimensional spacetime and given a local coordinate chart, the above equations reduce to the conventional Maxwell equations (2.2).

However, we have chosen not to take \(\mathcal{F}\) to be our fundamental object for three reasons as follows. (a) The action in terms of \(\mathcal{F}\), as we will see below, still involves \(A\) in the interaction term with \(\mathcal{F}\); thus, \(A\) has to be defined from the relation \(\mathcal{F} = dA\). However, only on spacetimes that have trivial \((p+1)\)th cohomology group, that is, \(H_{\text{cohomology}}^{p+1}(M) = \{0\}\), can \(\mathcal{F}\) be formulated globally in terms of a \(p\)-form potential \(A \in \Omega^p(M)\) such that \(\mathcal{F} = dA\) is true everywhere. There is a topological restriction to doing this if the cohomology group is nontrivial. Thus, starting with \(A\) as the fundamental object avoids cohomological problems early on. (b) Furthermore, even if we studied the free field, dropping all the terms involving \(\mathcal{F}\), it is unclear how to then go from the action in terms of \(\mathcal{F}\) to field equations without the introduction of \(A\) again. (c) We specifically work with the massless field here, but from our previous experience with the Proca field we find that \(A\) is the fundamental object of study. Also, when \(A \in \Omega^p(M)\), the action and field equation are that of the minimally coupled massless scalar field in curved spacetime. So we have strong reason to treat \(A\) as the fundamental object here which is derivable directly from action (2.6).

This last point is rather remarkable. The minimally coupled scalar field and the electromagnetic field are but two examples of a general \(p\)-form field theory in curved spacetime. In some of our previous work, we had indications of this property. Solutions of the massless scalar field theory in four dimensions could be used to construct the gauge photon polarizations of the (co)vector potential and that information could be used to help generate one of the two physically allowed polarization states of the (co)vector potential [10]. The remaining physical state comes from the Gram–Schmidt orthogonalization. It had also been noted that the quantum inequality for the electromagnetic field was exactly twice that of the minimally coupled scalar field.

### 2.3. The generalized Maxwell field equations and fundamental solutions

Let \(M\) be a globally hyperbolic spacetime, that is, a manifold of \(\text{dim}(M) = n\) with the Lorentzian metric of signature \(s = n - 1\), i.e. the metric is of the form \((+, −, −, \ldots)\). On this spacetime, we take our fundamental object to be the field \(A \in \Omega^p(M)\) with \(0 \leq p < n\). The classical action is given by

\[
S = (-1)^{p+1} \left[ -\frac{1}{2} (dA, dA)_M - (A, \mathcal{F})_M \right] + S(J), \tag{2.6}
\]

where \(S(J)\) is the remainder of the action for the current density \(J \in \Omega^p(M)\). The only criterion that we ask of \(J\) is that it be co-closed, i.e. \(\delta J = 0\) so as to preserve charge/current conservation. Variation with respect to the field yields the generalized Maxwell equation

\[
-\delta dA = J. \tag{2.7}
\]

The field strength is then calculated from \(A\) by \(\mathcal{F} = dA\). It is easily seen from this definition of the field strength that there is a gauge freedom in that to any solution \(A\) of
equation (2.7), we may add $d\Lambda$ where $\Lambda \in \Omega^{p-1}(M)$. While this changes the value of the gauge field $A$ at every point, it leaves the field strength $F$ unchanged. We will denote any two solutions $A$ and $A'$ to be gauge equivalent by $A \sim A'$ if they differ by the exterior derivative of a $(p-1)$-form. Thus when discussing the potential, particularly for the quantum problem, we will often work in gauge-equivalent classes denoted by $[A] = A + d\Omega^{p-1}(M)$.

In practice, one often chooses not to solve the above equation directly but instead work with the constrained Klein–Gordon system,

$$\Box A = \mathcal{J} \quad \text{with} \quad \delta A = 0,$$

(2.8)
as any solution that satisfies (2.8) is also a solution to (2.7). Typically, $\delta A = 0$ is called the Lorenz gauge condition. In a given coordinate chart, this constrained Klein–Gordon system can be written in component form as [12]

$$\nabla^\beta \nabla_\beta A_{i_1\ldots i_p} - \sum_{j=1}^{p} R_{i_j}^{\;\beta} A_{i_1\ldots \beta \ldots i_p} + \sum_{j,k=1, j\neq k}^{p} R_{i_j}^{\;\beta} A_{i_1\ldots \beta \ldots i_p} = \mathcal{J}_{i_1\ldots i_p}$$

(2.9)

and

$$\nabla^\beta A_{\beta i_2\ldots i_p} = 0.$$  

(2.10)

Here $R_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ are the Ricci and Riemann tensors, respectively. Also, the notation is such that the index $\beta$ in the first summation occupies the $j$th place in the tensor component of $A$ while in the double summation the indices $\beta$ and $\gamma$ occupy the $j$th and $k$th spots in the tensor components. The Riemann and Ricci terms are not unexpected; in differential geometry, they are a result of the Weitzenböck identity.

The advantage of using the constrained Klein–Gordon system of equations is that $\Box$ is a normally hyperbolic operator [13] with principal part $g^{\mu\nu} \partial_\mu \partial_\nu$. Thus, there exists a unique advanced and retarded Green’s operator denoted by $E^\pm: \Omega^0(M) \to \Omega^p(M)$ (see corollary 3.4.3 of Bär et al [13] or Choquet-Bruhat [14], and proposition 3.3 of Sahlmann and Verch [15]) which have the properties

$$\Box E^\pm = E^\pm \Box = 1,$$

(2.11)

and for all $f \in \Omega^0(M)$, supp $(E^\pm f) \subset J^+(\text{supp } f)$. Furthermore, the map defined by the Green’s operators are sequentially continuous. All of the differential operations on forms commute with the Green’s operator.

**Proposition 2.1.** Let $f \in \Omega^0(M)$ be a test function; then the following operations involving $E^\pm$ commute: (a) $dE^\pm f = E^\pm df$ and (b) $\delta E^\pm f = E^\pm \delta f$.

**Proof.** (a) Let $f \in \Omega^0(M)$. We know that $A^\pm = E^\pm f \in \Omega^p(M)$ is the unique solution to $\Box A^\pm = f$. Likewise, we have $A^\pm = E^\pm df \in \Omega^{p+1}(M)$ as the unique solution to $\Box A^\pm = df$. Consider

$$\Box df \circ A^\pm = df \circ A^\pm = df = \Box A^\pm.$$

(2.12)

Since $A^\pm$ is unique, we deduce $dA^\pm = A^\pm$; therefore, $dE^\pm f = E^\pm df$.

(b) Let $f$ and $A^\pm$ be as defined above. Set $A^\pm = E^\pm \delta f \in \Omega^{p+1}(M)$, which is the unique solution to $\Box A^\pm = \delta f$. Then consider

$$\delta \Box A^\pm = \delta \Box A^\pm = \delta f = \Box A^\pm.$$  

(2.13)

Since $A^\pm$ is unique, we deduce $\delta A^\pm = A^\pm$; therefore, $\delta E^\pm f = E^\pm \delta f$. \hfill $\square$

We also need the advanced minus retarded propagator $E = E^- - E^+$. Since it is a linear combination of the advanced and retarded propagators, it has all of the same commutation properties above and gives the unique solutions to the homogeneous (source free) Klein–Gordon equation. We are now ready to show that the existence of a fundamental solution for the Klein–Gordon equation also gives us a solution to the generalized Maxwell equations.
2.4. Initial value formulation

The initial value problem for a gauge field in four-dimensional spacetimes has been treated in the initial sections of Dimock and for the Proca field by Furlani. We follow closely the notation and structure of both these papers in this section as we generalize their results to all $p$-form fields with $p < n$ in globally hyperbolic spacetimes of arbitrary dimension.

In order to relate initial data to solutions of the wave equation, we first need to discuss Green’s theorem for forms. Let $M$ be a globally hyperbolic spacetime and $O \subset M$ be an open region in the spacetime with boundary $\partial O$. Define the natural inclusion $i : \partial O \to O$ and $i^*$ the pullback. On $M$, let $A \in \Omega^p(M)$ and $B \in \Omega^p_0(M)$; then by the Stokes theorem, we have

$$\int_O (A \wedge * \Box B - B \wedge * \Box A) = \int_{\partial O} i^* (A \wedge * dB + \delta A \wedge * B) - \int_i (B \wedge * dA + \delta B \wedge * A),$$

(2.14)

which is called Green’s identity for $\Box$ (section 7.5 of AMR [9]). The integrals are all well defined because $B$ has compact support which does not expand under any of the derivative operations.

Next, let $\Sigma \subset M$ be a Cauchy surface in the spacetime and define $\Sigma^\pm \equiv J^\pm(\Sigma) \setminus \Sigma$. If we use $O = \Sigma^\pm$ and $\partial O = \Sigma$ in Green’s identity, we have

$$\int_{\Sigma^\pm} (A \wedge * \Box B - B \wedge * \Box A) = \mp \left[ \int_{\Sigma} i^* (A \wedge * dB + \delta A \wedge * B) - \int_i (B \wedge * dA + \delta B \wedge * A) \right],$$

(2.15)

where the sign difference on the right-hand side comes about because of the opposite orientation of the unit normal to the Cauchy surface. For smooth maps, the pullback is natural with respect to both the wedge product and the exterior derivative; thus, we may distribute it across the above terms. We define the following operations which act on $p$-forms:

$$\rho_{(0)} = i^*$$

(2.16)

$$\rho_{(d)} = (-1)^{(n-p-1)\cdot(n-1)} i^* d$$

(2.17)

$$\rho_{(\delta)} = i^* \delta$$

(2.18)

$$\rho_{(n)} = (-1)^{(n-p)\cdot(n-1)} i^* * .$$

(2.19)

The first operation is the pullback of the form onto the Cauchy surface, the second is the forward normal derivative, the third is the pullback of the divergence and the last is the forward normal [16, 17]. Note that all operations to the right of $i^*$ act on forms which are defined on the whole manifold $M$. However, operations to the left of $i^*$ are defined with respect to the Cauchy surface $\Sigma$; thus in the forward normal derivative and forward normal, the first Hodge star in each expression is with respect to the induced metric on the Cauchy surface. With these operations, the above equation reduces to

$$\int_{\Sigma^\pm} (A \wedge * \Box B - B \wedge * \Box A) = \mp \left[ \langle \rho_{(0)} A, \rho_{(d)} B \rangle_{\Sigma} + \langle \rho_{(\delta)} A, \rho_{(n)} B \rangle_{\Sigma} ight. - \left. \langle \rho_{(0)} B, \rho_{(d)} A \rangle_{\Sigma} - \langle \rho_{(\delta)} B, \rho_{(n)} A \rangle_{\Sigma} \right].$$

(2.20)

We begin the discussion of the fundamental solutions to the Klein–Gordon equation. First, we look at the mapping of $\Box$ solutions to initial data.
Proposition 2.2. Let $A \in \Omega^p(M)$ be a smooth solution of $\Box A = f$ with Cauchy data
\[
A_{(0)} \equiv \rho_{(0)} A \in \Omega^{p}(\Sigma),
A_{(\delta)} \equiv \rho_{(\delta)} A \in \Omega^{p}(\Sigma),
A_{(\theta)} \equiv \rho_{(\theta)} A \in \Omega^{p-1}(\Sigma),
A_{(\omega)} \equiv \rho_{(\omega)} A \in \Omega^{p-1}(\Sigma).
\]
Then, for any compactly supported test function $f \in \Omega^0_0(M)$, we have
\[
\int_{M} A \wedge f = \langle f, E^+ f \rangle_{\Sigma} + \langle E^-, E^+ f \rangle_{\Sigma} + \langle A_{(0)}, \rho_{(\theta)} E f \rangle_{\Sigma} + \langle A_{(\delta)}, \rho_{(\omega)} E f \rangle_{\Sigma}
- \langle A_{(\delta)}, \rho_{(0)} E f \rangle_{\Sigma} - \langle A_{(\omega)}, \rho_{(\delta)} E f \rangle_{\Sigma}.
\tag{2.21}
\]
Proof. Using equation (2.20), set $A = A$ and $B = E^\pm f$ for $\Sigma^\pm$ respectively; then
\[
\int_{\Sigma^\pm} \left( A \wedge \Box E^\pm f - E^\pm f \wedge \Box A \right) = \pm \left( \langle \rho_{(0)} A, \rho_{(\theta)} E^\pm f \rangle_{\Sigma} + \langle A_{(\delta)}, \rho_{(\omega)} E^\pm f \rangle_{\Sigma}
- \langle \rho_{(0)} E^\pm f, \rho_{(\delta)} A \rangle_{\Sigma} - \langle \rho_{(\delta)} E^\pm f, \rho_{(\omega)} A \rangle_{\Sigma} \right).
\tag{2.22}
\]
Substituting the Cauchy data, noting that $A$ is a smooth solution to the Klein–Gordon equation with source $f$ and using $\Box E^\pm = 0$, we can simplify the above to
\[
\int_{\Sigma^\pm} A \wedge f = \langle f, E^\pm f \rangle_{\Sigma} + \langle A_{(0)}, \rho_{(\theta)} E^\pm f \rangle_{\Sigma} + \langle A_{(\delta)}, \rho_{(\omega)} E^\pm f \rangle_{\Sigma}
- \langle \rho_{(0)} E^\pm f, A_{(\delta)} \rangle_{\Sigma} - \langle \rho_{(\delta)} E^\pm f, A_{(\omega)} \rangle_{\Sigma}.
\tag{2.23}
\]
When the above two equations are added for $\Sigma^\pm$, the result is equation (2.21). Furthermore, the integrals in this expression are well defined because $f$ has compact support; thus $E^\pm f$ and $E f$ have compact support on all other Cauchy surfaces.

We now address the issues of existence and uniqueness for homogenous solutions to the Klein–Gordon equation.

Proposition 2.3 (Uniqueness of homogeneous $\Box$ solutions). If $A$ is a smooth solution to $\Box A = 0$ with Cauchy data $A_{(0)} = 0$, $A_{(\delta)} = 0$, $A_{(\theta)} = 0$ and $A_{(\omega)} = 0$, then $A = 0$.

Proof. By proposition 2.2, we have $\int_{M} A \wedge f = 0$ which is true for all compactly supported $f$; therefore, $A = 0$.

Proposition 2.4 (Existence of homogeneous $\Box$ solutions). Let $A_{(0)}, A_{(\delta)} \in \Omega^0_0(\Sigma)$ and $A_{(\theta)}, A_{(\omega)} \in \Omega^{p-1}_0(\Sigma)$ specify Cauchy data on $\Sigma$. Then
\[
A' = -E\rho'_{(\theta)} A_{(0)} - E\rho'_{(\omega)} A_{(\delta)} + E\rho'_{(\omega)} A_{(\theta)} + E\rho'_{(\omega)} A_{(\delta)}
\tag{2.24}
\]
is the unique smooth solution of $\Box A' = 0$ with these data.

Proof. Let $f \in \Omega^0_0(M)$ be any compactly supported test form and consider
\[
\langle A', f \rangle_M = -\langle E\rho'_{(\theta)} A_{(0)}, f \rangle_M - \langle E\rho'_{(\omega)} A_{(\delta)}, f \rangle_M + \langle E\rho'_{(\omega)} A_{(\theta)}, f \rangle_M + \langle E\rho'_{(\omega)} A_{(\omega)}, f \rangle_M
- \langle \rho'_{(\theta)} A_{(0)}, E' f \rangle_M - \langle \rho'_{(\omega)} A_{(\delta)}, E' f \rangle_M + \langle \rho'_{(\omega)} A_{(\theta)}, E' f \rangle_M + \langle \rho'_{(\omega)} A_{(\omega)}, E' f \rangle_M.
\tag{2.25}
\]
The operator $\Box = -(d\delta - d\theta)$ is self-adjoint; therefore, the transpose operator $E' = -E$ (see [14, corollary to theorem II]). The pullback operators are all linear and continuous; thus, there exist transpose operators for $\rho$’s denoted with the primes. Thus, we have
\[
\langle A', f \rangle_M = \langle A_{(0)}, \rho_{(\theta)} E f \rangle_{\Sigma} + \langle A_{(\delta)}, \rho_{(\omega)} E f \rangle_{\Sigma} - \langle A_{(\delta)}, \rho_{(\omega)} E f \rangle_{\Sigma} - \langle A_{(\delta)}, \rho_{(\theta)} E f \rangle_{\Sigma}.
\tag{2.26}
\]
By proposition 2.2, if \( A \) is a solution to \( \Box A = 0 \) with the Cauchy data given above, then we have \((A', f)_M = (A, f)_M\) which implies \( A' = A \) in a distributional sense. Thus, \( A' \) is identified with the unique smooth solution \( A \). □

Unless otherwise stated, from this point forward we will be discussing systems whose initial data are smooth and compactly supported on the Cauchy surfaces, i.e. \((A(0), A(d), A(n), A(\delta)) \in \Omega^p(\Sigma) \oplus \Omega^q(\Sigma) \oplus \Omega^{-p}(\Sigma) \oplus \Omega^{-q}(\Sigma)\). We may now discuss the sense in which \( A' \), as defined above, varies with respect to the Cauchy data.

**Proposition 2.5.** \( A' \) is continuously dependent on the Cauchy data \((A(0), A(d), A(n), A(\delta))\).

**Proof.** The proof is a generalization of theorem 3.2.12 of Bär et al [13]. Define \( \mathcal{H}^p(M) \equiv \{ A \in \Omega^p(M) | \Box A = 0 \} \) as the space of smooth homogeneous Klein–Gordon solutions; then the mapping \( \mathcal{P} : \mathcal{H}^p(M) \to \Omega^p(\Sigma) \oplus \Omega^q(\Sigma) \oplus \Omega^{-p}(\Sigma) \oplus \Omega^{-q}(\Sigma) \) of a solution to its Cauchy data is by definition both linear and continuous. Next, let \( K \subset M \) be a compact subset of \( M \). On \( K \), we have the spaces \( \Omega^p_0(K) \subset \Omega^p(M) \) and \( \Omega^q_0(K \cap \Sigma) \subset \Omega^q(\Sigma) \) for all \( p \leq n \).

We also define the space \( \mathcal{V}^p_K = \mathcal{P}^{-1}[\Omega^p_0(K \cap \Sigma) \oplus \Omega^q_0(K \cap \Sigma) \oplus \Omega^{-p}_0(K \cap \Sigma) \oplus \Omega^{-q}_0(K \cap \Sigma)] \).

Since \( \mathcal{P} \) is continuous and \( \Omega^p_0(K \cap \Sigma) \oplus \Omega^q_0(K \cap \Sigma) \oplus \Omega^{-p}_0(K \cap \Sigma) \oplus \Omega^{-q}_0(K \cap \Sigma) \subset \Omega^p(\Sigma) \oplus \Omega^q(\Sigma) \oplus \Omega^{-p}(\Sigma) \oplus \Omega^{-q}(\Sigma) \) is a closed subset, this implies that \( \mathcal{V}^p_K \subset \mathcal{H}^p(M) \) is also a closed subset. Furthermore, both \( \Omega^p_0(K \cap \Sigma) \oplus \Omega^q_0(K \cap \Sigma) \oplus \Omega^{-p}_0(K \cap \Sigma) \oplus \Omega^{-q}_0(K \cap \Sigma) \) and \( \mathcal{V}^p_K \) are Fréchet spaces. Also, the map \( \mathcal{P} : \mathcal{V}^p_K \to \Omega^p_0(K \cap \Sigma) \oplus \Omega^q_0(K \cap \Sigma) \oplus \Omega^{-p}_0(K \cap \Sigma) \oplus \Omega^{-q}_0(K \cap \Sigma) \) is linear, continuous, bijective and by the open mapping theorem for Fréchet spaces [18, Theorem V.6] it is also an open mapping. Since a bijection that is open implies a continuous inverse, we conclude that \( \mathcal{P}^{-1} \) is continuous.

Finally, if we have a convergent sequence of Cauchy data \((A(0), i, A(d), i, A(n), i, A(\delta), i) \to (A(0), A(d), A(n), A(\delta)) \in \Omega^p_0(\Sigma) \oplus \Omega^q_0(\Sigma) \oplus \Omega^{-p}_0(\Sigma) \oplus \Omega^{-q}_0(\Sigma)\), then we can choose a compact subset \( K \subset M \) with the property that \( \text{supp}(A(0), i) \cup \text{supp}(A(d), i) \cup \text{supp}(A(n), i) \cup \text{supp}(A(\delta), i) \subset K \) for all \( i \) and \( \text{supp}(A(0)) \cup \text{supp}(A(d)) \cup \text{supp}(A(n)) \cup \text{supp}(A(\delta)) \subset K \).

Thus, \((A(0), i, A(d), i, A(n), i, A(\delta), i) \to (A(0), A(d), A(n), A(\delta)) \in \Omega^p_0(K \cap \Sigma) \oplus \Omega^q_0(K \cap \Sigma) \oplus \Omega^{-p}_0(K \cap \Sigma) \oplus \Omega^{-q}_0(K \cap \Sigma)\) and we conclude that the inverse mapping \( \mathcal{P}^{-1}(A(0), A(d), A(n), A(\delta)) \to \mathcal{P}^{-1}(A(0), A(d), A(n), A(\delta))\) is continuous.

By an appropriate restriction of the Cauchy data for the Klein–Gordon equation, we are able to identify useful subspaces of solutions. For example, of all solutions to the Klein–Gordon equation, those that also satisfy the Lorenz gauge condition have the following property.

**Proposition 2.6** (Lorenz solutions). Suppose \( A \in \Omega^p(M) \) solves \( \Box A = 0 \) with Cauchy data \((A(0), A(d), A(n), A(\delta))\); then \( \delta A = 0 \) if and only if \( \delta A(d) = 0, \delta A(n) = 0 \) and \( A(\delta) = 0 \).

**Proof.** First, suppose that \( A \) solves \( \Box A = 0 \) and \( \delta A = 0 \) and let us evaluate the above three conditions. For the forward normal derivative, we have

\[
\delta A(d) = \delta \rho(d) A = \delta \rho(d) \delta A. \tag{2.27}
\]

Using the identity for \( p \)-forms that \( \delta \rho(n) = (-1)^n p(n) \delta \), we find

\[
\delta A(d) = (-1)^n p(n) \delta \rho(d) A = (-1)^n p(n) \delta A = 0, \tag{2.28}
\]

where we have used the wave equation and the subsidiary condition in the last two steps. For the divergence of the forward normal, we find

\[
\delta A(n) = \delta \rho(n) A = (-1)^n p(n) \delta A = 0. \tag{2.29}
\]
Finally, for the pullback of the divergence we have
\[ A_{(\delta)} = \rho_{(0)} \delta A = 0. \] (2.30)

On the other hand, assume that \( A \) is a Klein–Gordon solution whose Cauchy data satisfy
\[ \delta A_{(d)} = 0 = \delta A_{(n)} \] and \( A_{(\delta)} = 0 \). Let \( f \in \Omega^0(M) \) and evaluate
\[
\langle \delta A, f \rangle_M = \langle A, df \rangle_M = \langle A_{(0)}, \rho_{(\delta)} Edf \rangle_\Sigma + \langle A_{(\delta)}, \rho_{(0)} Edf \rangle_\Sigma
\]
where we have used the fact that \( df \) commutes with \( E \) on forms of compact support. The first term vanishes because \( \rho_{(\delta)} d = 0 \). The second term trivially vanishes. For the third term, we have that \( \rho_{(0)} \) and \( d \) commute. For the fourth term, we use \( \rho_{(\delta)} = \rho_{(0)} \delta \) and the fact that \( Ef \) is a solution to the homogeneous Klein–Gordon equation to swap the order of the derivative operators. Therefore,
\[
\langle \delta A, f \rangle_M = -\langle A_{(\delta)}, \rho_{(\delta)} Ef \rangle_\Sigma + \langle A_{(\delta)}, \rho_{(0)} \delta Ef \rangle_\Sigma
\]
where we have used the conditions on the initial data. Since this is true for all \( f \), we deduce that \( \delta A = 0 \).

\[ \square \]

**Corollary 2.7 (Coulomb gauge solutions).** Solutions in the Coulomb gauge will have Cauchy data \( (A_{(0)}, A_{(\delta)}, 0, 0) \), where \( \delta A_{(\delta)} = 0 \).

**Proof.** Coulomb gauge solutions are Lorenz gauge solutions with the additional constraint \( A_{(\delta)} = 0 \).

By proposition 2.6, we can infer that the constrained Klein–Gordon system is self-consistent in curved spacetimes of arbitrary dimension. A slightly different way to see this is to begin with the evolution equation \( \Box A = J \) and take \( \delta \) of it, yielding
\[ 0 = \delta \Box A = \Box \delta A. \] (2.33)

We observe that \( \delta A \) satisfies the source free Klein–Gordon equation. If the Cauchy data for \( \delta A \) vanish on the initial Cauchy surface, then by proposition 2.3 the unique solution is \( \delta A = 0 \). The same holds for Coulomb gauge solutions. What ensures this property are the Riemann and Ricci terms in equation (2.9). They commute with the divergence in the proper way as a result of the first and second Bianchi identities. Unlike the results of Buchdahl and Higuchi for spinor and massive symmetric tensor fields, no other conditions are required of the spacetime for the p-form fields satisfying the generalized Maxwell equation. In fact, the spacetime does not need to satisfy the Einstein equations in any way. At a deeper level, what we have really done is to take the flat space field equations and first make the minimal substitution. Only afterward do we then commute the covariant derivatives. This gives rise to all of the Riemann and Ricci terms. The beauty of using exterior calculus is that all this is handled without our having to do it explicitly.

Also, note that Lorenz solutions can be gauge related to one and other when considered as solutions to the generalized Maxwell equations.

**Proposition 2.8.** (a) Let \( (A_{(0)}, A_{(\delta)}, A_{(\delta)}, 0) \) with \( \delta A_{(\delta)} = 0 = \delta A_{(n)} \) be Cauchy data on \( \Sigma \). If \( A, A' \in \Omega^0(M) \) are Lorenz solutions with these data, then \( A \sim A' \).
(b) Let \((A_{(0)}, A_{(d)}, A_{(a)})\) and \((A'_{(0)}, A'_{(d)}, A'_{(a)})\) with \(\delta A_{(d)} = 0 = \delta A'_{(d)}\) and \(\delta A_{(a)} = 0 = \delta A'_{(a)}\) be Cauchy data on a common Cauchy surface \(\Sigma\) and \(\Lambda \in \Omega^{p-1}(M)\) be a \(-\delta d\Lambda = 0\) solution. If \(A, A' \in \Omega^p(M)\) are Lorenz solutions with these data, respectively, then \(A' \sim A\) if and only if \(A'_{(0)} \sim A_{(0)}, A'_{(d)} = A_{(d)}\) and \(A'_{(a)} = A_{(a)} + \Lambda_{(a)}\).

**Proof.** (a) We need to show that there exists \(\Lambda \in \Omega^{p-1}(M)\) such that \(A' = A + d\Lambda\). Applying \(\delta\) to both sides of the above expression tells us that \(\Lambda\) must satisfy \(-\delta d\Lambda = 0\). Looking at the four pullbacks that relate a solution to its initial data we find \(\Lambda\) must also satisfy

\[
d\rho(0)\Lambda = 0 \quad \text{and} \quad \rho(d)\Lambda = 0,
\]

for \(A\) and \(A'\) to share common Cauchy data.

Choose any set of Cauchy data \((\lambda_{(0)}, 0, \lambda_{(a)})\) in \(\Omega^{p-1}_0(\Sigma) \oplus \Omega^{p-1}_0(\Sigma) \oplus \Omega^{p-2}_0(\Sigma) \oplus \Omega^{p-2}_0(\Sigma)\) with \(d\lambda_{(0)} = 0\) and \(\delta\lambda_{(a)} = 0\). By proposition 2.4, there exists \(\Lambda'\) which is the solution to \(\Box\Lambda' = 0\) with these data. Furthermore, the Cauchy data are such that by proposition 2.6 we have that \(\Lambda'\) is a Lorenz solution. It is now trivial to see that \(\Lambda = \Lambda'\) satisfies \(-\delta d\Lambda = -\delta d\Lambda' = \Box\Lambda' = 0\) and conditions (2.34). Therefore, \(\Lambda\) exists and we conclude \(A \sim A'\).

(b) If \(A' = A + d\Lambda\) where \(-\delta d\Lambda = 0\), then it immediately follows

\[
\begin{align*}
A_{(0)} &= \rho_{(0)}A' = \rho_{(0)}A + \rho_{(0)}d\Lambda = A_{(0)} + d\rho(0)\Lambda = A_{(0)} + d\Lambda_{(0)}, \\
A_{(d)} &= \rho_{(d)}A' = \rho_{(d)}A + \rho_{(d)}d\Lambda = A_{(d)}, \\
A'_{(a)} &= \rho_{(a)}A' = \rho_{(a)}A + \rho_{(a)}d\Lambda = A_{(a)} + \rho_{(a)}\Lambda = A_{(a)} + \Lambda_{(a)}, \\
A'_{(d)} &= \rho_{(d)}A' = \rho_{(d)}A + \rho_{(d)}d\Lambda = \rho_{(0)}d\Lambda = 0.
\end{align*}
\]

Furthermore, \(\delta A'_{(a)} = \delta \rho_{(a)}(A + d\Lambda) = (1)\rho_{(a)}\delta(A + d\Lambda) = 0\).

Conversely, let \(A'_{(0)} = A_{(0)} + d\lambda_{(0)}, A'_{(d)} = A_{(d)}, A'_{(a)} = A_{(a)} + \lambda_{(a)}\) and \(A'_{(d)} = A_{(d)} = 0\) where \(\lambda_{(0)}, \lambda_{(d)} \in \Omega^{p-1}_0(\Sigma)\) with \(\delta\lambda_{(a)} = 0\). Given \(A\), define \(\widetilde{A} = A + d\Lambda\) where \(\Lambda\) solves the homogenous equation \(\Box\Lambda = 0\) with Cauchy data

\[
\begin{align*}
\rho_{(0)}\Lambda &= \lambda_{(0)}, \\
\rho_{(d)}\Lambda &= \lambda_{(a)}, \\
\rho_{(a)}\Lambda &= \lambda_{(a)}, \\
\rho_{(d)}\Lambda &= 0,
\end{align*}
\]

where \(\lambda_{(a)} \in \Omega^{p-2}_0(\Sigma)\) satisfies \(\delta\lambda_{(a)} = 0\). By propositions 2.4 and 2.6, such a Lorenz solution exists and \(\Lambda\) is therefore a solution to \(-\delta d\Lambda = 0\). Next, we evaluate

\[
\begin{align*}
\Box\widetilde{A} &= \Box A + \Box d\Lambda = d\Box\Lambda = 0, \\
\delta\widetilde{A} &= \delta A + \delta d\Lambda = 0,
\end{align*}
\]

and the Cauchy data

\[
\begin{align*}
\rho_{(0)}\widetilde{A} &= \rho_{(0)}(A + d\Lambda) = \rho_{(0)}A + \rho_{(0)}d\Lambda = A_{(0)} + d\lambda_{(0)} = A'_{(0)}, \\
\rho_{(d)}\widetilde{A} &= \rho_{(d)}(A + d\Lambda) = \rho_{(d)}A + \rho_{(d)}d\Lambda = A_{(d)} = A'_{(d)}, \\
\rho_{(a)}\widetilde{A} &= \rho_{(a)}(A + d\Lambda) = \rho_{(a)}A + \rho_{(a)}d\Lambda = A_{(a)} + \lambda_{(a)} = A'_{(a)}, \\
\rho_{(d)}\widetilde{A} &= \rho_{(d)}(A + d\Lambda) = \rho_{(d)}d\Lambda = \rho_{(0)}\delta d\Lambda = 0.
\end{align*}
\]

We conclude that \(\tilde{A}\) is a Maxwell solution with Cauchy data identical to that of \(A'\). By part (a) of this proposition, we have \(\tilde{A} \sim A'\) and hence \(A \sim A'\). \(\square\)
Corollary 2.9. Let $\mathcal{A}$ be a Lorenz solution with Cauchy data $(A(\rho), A(\sigma), A(\omega), 0)$ where $\delta A(\rho) = 0 = \delta A(\omega)$; then $\mathcal{A}$ is gauge equivalent to the Coulomb solution $\mathcal{A}'$ with Cauchy data $(A(\rho), A(\sigma), 0, 0)$.

Proof. The Cauchy data for $\mathcal{A}$ and $\mathcal{A}'$ given above satisfy the requirements of proposition 2.8, so we conclude $\mathcal{A}' \sim \mathcal{A}$. □

This completes our discussion of $\square$ solutions and it is now possible to prove the existence of Maxwell solutions.

Proposition 2.10 (Existence of homogeneous Maxwell solutions). For any $(A(\rho), A(\sigma)) \in \Omega_{\rho}^0(\Sigma) \times \Omega_{\rho}^0(\Sigma)$ with $\delta A(\rho) = 0$, there exists $A \in \Omega^p(M)$ such that

$$-\delta dA = 0, \quad \rho(\rho)A = A(\rho) \quad \text{and} \quad \rho(\sigma)A = A(\sigma).$$

(2.40)

Proof. The proof is basically a generalization of Dimock’s proposition 2. To the above equations, we add two additional conditions and still find a solution. (a) First we impose the Lorenz condition $\delta A = 0$. Thus, the problem of solving for a solution is equivalent to finding one for $\square A = 0$. Moreover, this condition implies $\rho(\delta)A = 0$. (b) Second, we specify that the forward normal $\rho(\sigma)A = A(\sigma) \in \Omega_{\rho}^{p-1}(\Sigma)$ satisfies $\delta A(\sigma) = 0$. Dimock sets $A(\sigma)$ to zero and therefore solves the problem in the Coulomb gauge. He comments that $A(\sigma)$ could be any other function. This is only true for $A(\sigma) \in \Omega_{\rho}^0(\Sigma)$. For higher order forms $A(\rho)$ must be co-closed, a condition which happens to be trivially satisfied in Dimock’s case.

So we now seek a solution to $\square A = 0$ with Cauchy data $(A(\rho), A(\sigma), A(\omega), 0)$ where $\delta A(\rho) = 0$ and $\delta A(\omega) = 0$. By proposition 2.4 such a solution exists and by proposition 2.6 it is a Lorenz solution; thus it satisfies equations (2.40). □

Theorem 2.11 (Existence of inhomogeneous Maxwell solutions). Choose any Cauchy surface $\Sigma$ and suppose that $\mathcal{F} \in \Omega_{\rho}^0(M)$ obeys $\delta \mathcal{F} = 0$ on an open globally hyperbolic neighborhood $N$ of $\Sigma$. Set the following Cauchy data on $\Sigma$:

$$A(\rho) = A(\omega) = 0, \quad A(\rho) \in \Omega_{\rho}^p(\Sigma) \quad \text{and} \quad A(\sigma) = \omega,$$

(2.41)

where $\omega$ is any solution to

$$\delta \omega = (-1)^{p(p+1)+1} \rho(\sigma)\mathcal{F}.$$

(2.42)

(Solutions to this last equation will certainly exist if $H^p(\Sigma)$ is trivial, but might exist in other cases as well; see below.) Then the unique solution $\mathcal{A}$ to $\square \mathcal{A} = \mathcal{F}$ corresponding to these data satisfies $\delta A = 0$ on $N$. In particular, if $\mathcal{F}$ is co-closed on $M$, $\mathcal{A}$ is a Lorenz gauge solution to $-\delta dA = \mathcal{F}$.

Proof. We calculate

$$\rho(\rho)A = \rho(\delta)A = 0,$$

$$\rho(\delta)A = \rho(\sigma)\delta^2 A = 0,$$

$$\rho(\delta)A = (-1)^{p} \delta \rho(\omega)A = 0,$$

and also

$$\rho(\delta)A = \rho(\rho)\delta dA = -\rho(\omega)\mathcal{F} + \delta dA = -\rho(\omega)\mathcal{F} + (-1)^{p(p+1)}\delta \rho(\sigma)A = 0$$

(2.43)

by equation (2.42). As $\delta A$ is therefore a solution to

$$\square A = \delta \square A = \delta \mathcal{F} = 0$$

(2.44)

with trivial Cauchy data on $\Sigma$, it follows that $\delta A$ vanishes identically in $N$. □
If $\mathcal{J}$ is not compactly supported, but has spacelike compact support on Cauchy surfaces (see [13]), we can cut it off by multiplying by some smooth function $\chi \in \Omega^0(M)$ which equals 1 on $N$ such that $\chi \mathcal{J} \in \Omega^0(M)$ is compactly supported. On $N$ we have $\delta \chi \mathcal{J} = 0$, while on the rest of $M$ this may not be the case. Then the solution $\mathcal{A}$ given by the theorem is co-closed on $N$; as it is obtained by solving a Cauchy problem within $N$, if we repeat the procedure for a larger globally hyperbolic region $N'$ containing $N$, then the new solution (which is co-closed on all of $N'$) would agree with the old one on $N$. We may therefore remove the cutoff by expanding $N$ to cover all of $M$, thereby obtaining a global co-closed solution.

This has proved

**Theorem 2.12.** If $\mathcal{J}$ has compact support on Cauchy surfaces and there is a Cauchy surface $\Sigma$ for which $\rho_{(\alpha)} \mathcal{J}$ is co-exact (which always holds if $H^p(\Sigma)$ is trivial), then there exists a global Lorenz gauge solution $A \in \Omega^p(M)$ to $-dA = J$.

This generalizes the current proposition 2.2: if $\mathcal{J}$ is compact to the past, take $\Sigma$ to the past of the support of $\mathcal{J}$ and then $\rho(\alpha) \mathcal{J}$ vanishes. Then we may solve (2.42) with $\omega = 0$ (regardless of the cohomology of $\Sigma$) and we will of course obtain $A = \mathcal{E}^* \mathcal{J}$. The same will hold if $\mathcal{J}$ is compact to the future.

**Proposition 2.13.** (a) Let $(A(0), A(d), A_{(\alpha)}, A_{(\delta)})$ with $\delta A_{(\delta)} = 0$ be Cauchy data on $\Sigma$. If $\mathcal{A}, \mathcal{A}' \in \Omega^p(M)$ are solutions to $-dA = 0$ with these data, then $\mathcal{A} \sim \mathcal{A}'$.

(b) Let $(A(0), A_{(\alpha)}, A_{(\delta)})$ and $(A'_{(0)}, A'_{(\alpha)}, A'_{(\delta)})$ with $\delta A_{(\delta)} = 0 = \delta A'_{(\delta)}$ be Cauchy data on a common Cauchy surface $\Sigma$. If $\mathcal{A}, \mathcal{A}' \in \Omega^p(M)$ are Maxwell solutions with these data, respectively, then $\mathcal{A} \sim \mathcal{A}'$ if and only if $A'_{(0)} = A_{(0)}$ and $A'_{(\alpha)} = A_{(\alpha)}$.

**Proof.** (a) It is easiest to first show that any solution to the generalized Maxwell equation is gauge equivalent to some Coulomb solution. Let $\Lambda \in \Omega^{p-1}(M)$ be any solution to

$$-d\Lambda = -\delta A \quad \text{with} \quad \rho_{(\alpha)} \Lambda = 0, \quad \rho_{(\delta)} \Lambda = A_{(\alpha)} \quad \text{and} \quad \rho_{(\delta)} \Lambda = \rho_{(\delta)} \Lambda = 0.$$  

(2.45)

By propositions 2.11 and 2.12, we know that such $\Lambda$ exists. Next, we define $\tilde{\mathcal{A}} = A - d\Lambda$ and calculate

$$\rho_{(\alpha)} \tilde{\mathcal{A}} = \rho_{(\alpha)} (A - d\Lambda) = A_{(\alpha)} - d\rho_{(\alpha)} \Lambda = A_{(\alpha)},$$

$$\rho_{(\delta)} \tilde{\mathcal{A}} = \rho_{(\delta)} (A - d\Lambda) = A_{(\delta)} - \rho_{(\delta)} \Lambda = A_{(\delta)}.$$  

(2.46)

Furthermore, $\delta \rho_{(\delta)} \tilde{\mathcal{A}} = \delta A_{(\delta)} = 0$. By proposition 2.6, we conclude that $\tilde{\mathcal{A}}$ is a Lorenz solution. Better still, by corollary 2.7 we know that $\tilde{\mathcal{A}}$ is in fact a Coulomb gauge solution. Just to emphasize the point, we have shown that any Maxwell solution $\mathcal{A}$ with its associated Cauchy data is gauge equivalent to some Coulomb solution $\tilde{\mathcal{A}}$ with Cauchy data $(A_{(\alpha)}, A_{(\delta)}, 0, 0)$.

By a similar calculation we find $\mathcal{A}' \sim \tilde{\mathcal{A}}$ where $\tilde{\mathcal{A}}$ has the same Cauchy data as $\tilde{\mathcal{A}}$. By proposition 2.8(a), we have that $\mathcal{A} \sim \tilde{\mathcal{A}}$. Therefore, we conclude $\mathcal{A} \sim \mathcal{A}'$.

(b) Let $\Lambda \in \Omega^{p-1}(M)$ and set $\mathcal{A}' = A + d\Lambda$; then it immediately follows $A'_{(0)} \sim A_{(0)}$ and $A'_{(\alpha)} = A_{(\delta)}$.

Alternatively, in part (a) above, we proved that any Maxwell solution $\mathcal{A}$ with Cauchy data $(A_{(0)}, A_{(\delta)}, A_{(\alpha)}, A_{(\delta)})$ is gauge equivalent to some Coulomb solution $\tilde{\mathcal{A}}$ with Cauchy data $(A_{(0)}, A_{(\delta)}, 0, 0)$. Similarly, $\mathcal{A}'$ is gauge equivalent to some Coulomb solution $\tilde{\mathcal{A}}'$ with Cauchy data $(A'_{(0)}, A'_{(\alpha)}, 0, 0)$. If $A'_{(\alpha)} \sim A_{(\alpha)}$ and $A'_{(\delta)} = A_{(\delta)}$, then by proposition 2.8(b) we find $\tilde{\mathcal{A}}' \sim \tilde{\mathcal{A}}$ and we therefore conclude $\mathcal{A}' \sim \mathcal{A}$. \(\square\)
Proposition 2.14 (Fundamental solutions).

(a) Let \( J \in \Omega^0(M) \) with \( \delta J = 0 \) and supp\((J)\) compact to the past/future; then \( A^\pm = E^\pm J \) solves \(-\delta A^\pm = J\).

(b) If \( A^\pm \in \Omega^p(M) \), supp\((A^\pm)\) is compact to the past/future and \(-\delta A^\pm = J \) (so \( \delta J = 0 \) and supp\((J)\) compact to the past/future), then \( A^\pm \sim E^\pm J \).

(c) \( A \in \Omega^p(M) \) satisfies \(-\delta dA = 0\) on spacetimes with compact spacelike Cauchy surfaces if and only if \( A \sim E J \) for some \( J \in \Omega^0(M) \) with \( \delta J = 0 \).

Proof. Part (a) is proven directly in proposition 4 by Dimock [16], to which we refer the reader. Parts (b) and (c) are generalizations of the corresponding parts of proposition 4 in Dimock and we leave the proof to the reader. \( \square \)

2.5. Classical phase space

Finally, as a precursor to quantization we discuss the classical phase space for the \( p \)-form field which consists of a vector space and a non-degenerate antisymmetric bilinear form. For the most part, this section closely follows the discussion of the phase space for the classical electromagnetic field found in Dimock [16]. All of the propositions in his section 3 trivially generalize from 1-forms to \( p \)-forms, so we will therefore be very brief. In addition, to simplify further discussion, we will also assume from this point forward that the Cauchy surface is compact; thus \( \Omega^p(\Sigma) = \Omega^p_0(\Sigma) \). We conjecture that our analysis can be appropriately extended to spacetimes with noncompact Cauchy surfaces.

In a typical Hamiltonian formulation, the Cauchy data for the field are specified on some ‘constant time’ hypersurface. The field then evolves according to the flow generated by Hamilton’s equations. We could use the complete set of Cauchy data for our \( p \)-form field, but as we have seen above, the Cauchy problem is well posed on gauge-equivalent classes. Therefore, the initial formulation of the phase space can be accomplished via the vector space where points are \((A_{(0)}, A_{(d)}) \in \mathcal{P}_0(\Sigma) := \Omega^p_0(\Sigma) \times \Sigma_0^p(\Sigma)\), i.e. the part of the Cauchy data (with compact support) which cannot be gauge transformed away. Then on \( \mathcal{P}_0(\Sigma) \times \Sigma_0^p(\Sigma)\), define the antisymmetric bilinear form

\[
\sigma_{\Sigma}(A_{(0)}, A_{(d)}; B_{(0)}, B_{(d)}) = \langle A_{(0)}, B_{(d)} \rangle_{\Sigma} - \langle B_{(0)}, A_{(d)} \rangle_{\Sigma}. \tag{2.47}
\]

Unfortunately this form is degenerate because for all \( B_{(d)} \) with \( \delta B_{(d)} = 0 \), we have \( (d\chi, B_{(d)})_{\Sigma} = (\chi, \delta B_{(d)})_{\Sigma} = 0 \) even though \( d\chi \neq 0 \).

The way to remove the degeneracy is to pass to gauge-equivalent classes of Cauchy data with points being given by the pair \(([A_{(0)}], A_{(d)}) \in \mathcal{P} := \Omega^p_0(\Sigma) / \delta \Omega^p_0(\Sigma) \times \Omega^p_0(\Sigma); \) then

\[
\sigma_{\Sigma}([A_{(0)}], A_{(d)}; [B_{(0)}], B_{(d)}) = \langle [A_{(0)}], B_{(d)} \rangle_{\Sigma} - \langle [B_{(0)}], A_{(d)} \rangle_{\Sigma} \tag{2.48}
\]

is a suitable weakly non-degenerate bilinear form, as proven in Dimock’s proposition 5 [16].

Given any set of Cauchy data in \( \mathcal{P} \), we know from the preceding section that there is a unique equivalence class of solutions to the Maxwell equations with these Cauchy data. Therefore, we can reformulate our phase space to include time evolution without the specific introduction of a Hamiltonian. Define the solution space of all real-valued gauge-equivalent Maxwell solutions with Cauchy data on \( \Sigma \) as

\[
\mathcal{M}^p(M) \equiv \{A \in \Omega^p(M) \mid -\delta dA = 0\} / \delta \Omega^p_0(M). \tag{2.49}
\]

Since \( d \) and \( \delta \) are linear operators on \( \Omega^p(M) \), we have that the numerator of the above expression is a vector space. Furthermore, quotienting by the exact forms is also linear so formally \( \mathcal{M}^p(M) \) is a vector space, elements of which are gauge-equivalent classes of Maxwell solutions denoted by \([A]\).
Next, choose any Cauchy surface $\Sigma \subset M$ with $i : \Sigma \rightarrow M$ and define the antisymmetric bilinear form $\sigma$ on $\mathcal{M}^p(M) \times \mathcal{M}^p(M)$ by

$$\sigma([A], [B]) = \int_{\Sigma} i^\ast ([A] \wedge \ast dB - [B] \wedge \ast dA),$$

which is by definition gauge invariant. It is also non-degenerate and independent of the choice of Cauchy surface (see Dimock’s proposition 6). Thus, $(\mathcal{M}^p(M), \sigma)$ is a suitable symplectic phase space.

On this phase space, we also want to consider linear functions which map $\mathcal{M}^p(M) \rightarrow \mathbb{R}$ defined by $[A] \mapsto \langle [A], f \rangle_M$ for all $f \in \Omega^0_\delta(M)$ with $\delta f = 0$. Such functions are related to the symplectic form in the following sense.

**Proposition 2.15.** For $[A] \in \mathcal{M}^p(M)$ and $f \in \Omega^0_\delta(M)$ where $\delta f = 0$, we have

$$\langle [A], f \rangle_M = \sigma([A], Ef).$$

**Proof.** Choose any Cauchy surface $\Sigma$; then from equation (2.21), we have for all $[A]$ that are homogeneous solutions of the wave equation

$$\langle [A], f \rangle_M = \int_{\Sigma} i^\ast ([A] \wedge \ast dEf - Ef \wedge \ast d[A] - \delta Ef \wedge \ast [A] + \delta [A] \wedge \ast Ef)$$

$$= \sigma([A], Ef).$$

Also, recall that $Ef$ is a Lorenz solution and thus belongs to some equivalence class, therefore giving us the result. $\square$

Furthermore, the symplectic form induces a Poisson bracket operation on the functions over the phase space. (For a detailed description of how this arises, see Wald [19] and/or section 8.1 of AMR [9].) For the linear functions considered above, we calculate

$$\{\sigma([A], [Ef]), \sigma([A], [Ef'])\} = \sigma([Ef], [Ef']).$$

### 3. Quantization of the generalized Maxwell field

For electromagnetism in four dimensions, quantization is complicated by gauge freedom. Even in Minkowski space this presents serious problems: as shown by Strocchi [20, 21] in the Wightman axiomatic approach, the vector potential cannot exist as an operator-valued distribution if it is to transform correctly under the Lorentz group or even display commutativity at spacelike separations in a weak sense. We have already seen that the same gauge freedom exists for the generalized Maxwell $p$-form field, so we expect the same difficulty here. However several researchers have already addressed these issues and quantization of a massless free $p$-form field in four-dimensional curved spacetime has, to our knowledge, been discussed in three papers.

The first is by Folacci [22] who quantizes $p$-form fields in a ‘traditional’ manner by adding a gauge-breaking term to the action which then necessitates the introduction of Faddeev–Popov ghost fields to remove spurious degrees of freedom. This is similar in approach to the Gupta–Bleuler formalism for the free electromagnetic field in four-dimensional spacetimes [23–25]. However, unlike electromagnetism, which requires only a single ghost field, the generic quantized $p$-form field suffers from the phenomenon of having ‘ghosts for ghosts’; thus, there is a multiplicity of fields that need to be handled simultaneously [26, 27].

The second paper, by Furlani [28], employs the full Gupta–Bleuler method of quantization for the electromagnetic field in four-dimensional static spacetimes with compact Cauchy
surfaces. He constructs a Fock space and a representation of the field operator $A$ that satisfies the Klein–Gordon equation as an operator identity. This effectively quantizes all four components of the 1-form field. To remove the two spurious degrees of freedom requires applying the Lorenz gauge condition as a constraint on the space of states and imposing a sesquilinear form that is only positive on the ‘physical’ Fock space. In a later paper [17], Furlani also treats the quantization of the Proca field in four-dimensional globally hyperbolic spacetimes. Within this paper, he collects together many of the classical results referenced in the preceding section.

The third paper, by Dimock [16], uses a more elegant approach to quantize the free electromagnetic field in four-dimensional spacetimes which does not introduce gauge breaking terms and ghost fields. He constructs smeared field operators $\hat{A}((J))$ which may be smeared only with co-closed (divergence-free) test functions, i.e., $J \in \Omega^0_c(M)$ must satisfy $\delta J = 0$. These objects may be interpreted as smeared gauge-equivalence classes of quantum 1-form fields: formally, $[\hat{A}((J)) = \langle A, J \rangle_M$, where $A$ is a representative of the equivalence class $[A]$; since $\delta J = 0$, we have $(d\Lambda, J)_M = \langle \Lambda, \delta J \rangle_M = 0$ so this interpretation is indeed gauge independent. The resulting operators satisfy the Maxwell equations in the weak sense and have the correct canonical commutation relation (CCR). We adapt this approach to the generalized Maxwell field and quantize in the manner found in [16, 19].

### 3.1. Quantization via a Fock space

To pass from the classical world into the quantum realm requires replacing our symplectic phase space with a Hilbert space, while simultaneously promoting functions on the classical phase space to self-adjoint operators that act on elements of said Hilbert space. To maintain correspondence with the classical theory, the commutator of such operators must be $-i$ times their classical Poisson bracket. Thus we seek operators on a Hilbert space, indexed by $u \in \mathcal{M}^p(M)$ and denoted $[\hat{A}](u) \equiv \hat{\sigma}([A], [u])$, such that

$$[\hat{\sigma}([A], [u]), \hat{\sigma}([A], [u'])] = -i\sigma([u], [u']).$$  \hspace{1cm} (3.1)

We begin with the construction of our Fock space. Our symplectic phase space is $(\mathcal{M}^p(M), \sigma)$ where elements $[A]$ of $\mathcal{M}^p(M)$ are gauge-equivalent classes of real-valued $p$-form solutions to the Maxwell equation, as defined in the above section. On this space, choose any positive-definite, symmetric, bilinear map $\mu : \mathcal{M}^p(M) \times \mathcal{M}^p(M) \rightarrow \mathbb{R}$ such that for all $[A] \in \mathcal{M}^p(M)$, we have

$$\mu([A], [A]) = \frac{1}{2} \sup_{\{[B] \neq 0\}} \frac{\sigma([A], [B])}{\mu([B], [B])}. \hspace{1cm} (3.2)$$

Many such $\mu$ of this type exist: for each complex structure $J$ on $\mathcal{M}^p(M)$ which is compatible with $\sigma$ in the sense that $-\sigma([A], J[B])$ is a positive-definite inner product gives rise to such a $\mu$, although this method does not produce all such $\mu$. For further discussion on this point, see pp 41–42 of Wald [19].

We then define the norm $\|\cdot\|^2 = 2\mu(\cdot, \cdot)$ which is used to form $m$, the completion of $\mathcal{M}^p(M)$ with respect to this norm. Next, define the operator $J : m \rightarrow m$ by

$$\sigma([A], [B]) = 2\mu([A], J[B]) = ([A], J[B])_m, \hspace{1cm} (3.3)$$

where $(\cdot)_m$ defined in the above equation is the inner product on $m$. As already indicated above, $J$ endows $m$ with a complex structure. Furthermore, one can prove straightforwardly that $J$ satisfies $J^* = -J$ and $J^*J = \text{id}_m$. 


The next step is to complexify \( m \), i.e. \( m \rightarrow m^C \), and extend \( \sigma, \mu \) and \( J \) by complex linearity. The resulting complex space, endowed with the complex inner product

\[
([A], [B])_{m^C} = 2\mu([\overline{A}], [B])
\]

(3.4)

for \([A], [B] \in m^C\), is a complex Hilbert space. The operator \( J \) can be diagonalized into \( \pm i \) eigenspaces, as \( iJ \) is a bounded, self-adjoint operator on which we can apply the spectral theorem. Therefore, we can decompose \( m^C \) into two orthogonal subspaces based upon the eigenvalues of \( iJ \). Define \( \mathcal{H} \subset m^C \) to be the subspace with eigenvalue \( +i \) for the operator \( J \), which satisfies the three properties: (i) the inner product is positive definite over \( \mathcal{H} \), (ii) \( m^C \) is equal to the span of \( \mathcal{H} \) and its complex conjugate space \( \overline{\mathcal{H}} \) and (iii) all elements of \( \mathcal{H} \) are orthogonal to all elements of \( \overline{\mathcal{H}} \). We also define the orthogonal projection map \( K : m^C \rightarrow \mathcal{H} \) with respect to the complex inner product by \( K = \frac{1}{2}(\text{id}_{m^C} + iJ) \). Restricting this map to \( m \) defines a real linear map \( K : m \rightarrow \mathcal{H} \), i.e. a map from the Hilbert space of gauge-equivalent real-valued solutions of the Maxwell equation to the complex Hilbert space \( \mathcal{H} \). For any \([A_1], [A_2] \in m\), we have

\[
(K[A_1], K[A_2])_{\mathcal{H}} = -i\sigma(K[A_1], K[A_2]) = \mu([A_1], [A_2]) - \frac{i}{2}\sigma([A_1], [A_2]).
\]

(3.5)

Finally, the Hilbert space for our quantum field theory is given by the symmetric Fock space \( \mathfrak{F}_s(\mathcal{H}) \) over \( \mathcal{H} \), i.e.

\[
\mathfrak{F}_s(\mathcal{H}) = \mathbb{C} \oplus \bigoplus_{n=1}^\infty (\otimes^n_\sigma \mathcal{H})
\]

(3.6)

where \( \otimes^n_\sigma \mathcal{H} \) represents the \( n \)-th order symmetric tensor product over \( \mathcal{H} \).

Our next step is to complexify the appropriate self-adjoint operators on our Fock space. Let \([f], [g] \in \mathcal{H}\); then for states in \( \mathfrak{F}_s(\mathcal{H}) \) of a finite particle number, we define the standard annihilation and creation operators, \( \hat{a}(\mathcal{F}) \) and \( \hat{a}^*(\mathcal{G}) \), respectively, where the annihilation operator is linear in the argument for the complex conjugate space \( \overline{\mathcal{H}} \), while the creation operator is linear in the argument for elements of \( \mathcal{H} \). (See the appendix of Wald [19] for more detail.) On a dense domain of the Fock space, the operators satisfy the commutation relation

\[
[\hat{a}(\mathcal{F}), \hat{a}^*(\mathcal{G})] = ([f], [g])_{\mathcal{H}}
\]

(3.7)

for all \( [f], [g] \), with all other commutators vanishing.

From the analysis of the classical wave solutions in the preceding section, we know that \( E \) is a surjective map of all compactly supported test forms \( \mathcal{J} \in \Omega_0^q(M) \) which are co-compact into an equivalence class in \( \mathcal{M}^p(M) \), namely \([E, \mathcal{J}]\). Furthermore, \( \mathcal{M}^p(M) \subset m; \) thus, combined with the orthogonal projection \( K \), we have that \( K[E, \mathcal{J}] \in \mathcal{H} \). Therefore, we define the smeared quantum field operator for all co-compact \( \mathcal{J} \in \Omega_0^q(M) \) by

\[
[\hat{A}(\mathcal{J}) = \hat{\sigma}([A], [E, \mathcal{J}]) = i\hat{a}(\overline{K[E, \mathcal{J}]}), \quad \hat{a}^*(K[E, \mathcal{J}])] = i\hat{a}^*(K[E, \mathcal{J}]).
\]

(3.8)

Note that this is a slight abuse of the notation used earlier where the argument of \([A] \) was an element of the phase space. We now show that such an operator satisfies the generalized Maxwell equation and CCRs in the sense of distributions.

**Proposition 3.1.** For \( \mathcal{J} \in \Omega_0^q(M) \), \( \delta \mathcal{J} = 0 \), we have

(a) \( [\hat{A}]([\mathcal{J}]) \) satisfies the generalized Maxwell equation in the weak sense, i.e. \( [\hat{A}](\delta d \mathcal{J}) = 0 \).

(b) \( [[A],[J],[A],[J']] = i(\mathcal{J}, E, J')_M \). In particular, if \( \text{supp} \mathcal{J}, \text{supp} \mathcal{J}' \) are spacelike separated, then the commutator is zero.
Proof.
(a) By definition we have \( [\hat{A}] \delta \theta = i \hat{a} (K[E, \delta \theta]) - i \hat{a}^*(K[E, \delta \theta]) \), so we will show \( [E \delta \theta] = 0 \). For any \( \theta \in \Omega^0(M) \), we have \( \Box E \theta = E \Box \theta = 0 \), so \( [E \delta \theta] = -[E d \delta \theta] = 0 \), because all exact forms are in the equivalence class of zero. Since \( \mathcal{J} \in \Omega^0(M) \), we have \( [E \delta \theta] = 0 \) and \( [\hat{A}] \delta \theta = i \hat{a}(0) - i \hat{a}^*(0) = 0 \). Note that this part of the proposition does not require the co-closed condition \( \delta \mathcal{J} = 0 \).

(b) Substituting the definition of the field operator into the commutator yields

\[
[[\hat{A}](\mathcal{J}), [\hat{A}](\mathcal{J}')] = [\hat{a}(K[E, \mathcal{J}]), \hat{a}^*(K[E, \mathcal{J}'])] + [\hat{a}^*(K[E, \mathcal{J}]), \hat{a}(K[E, \mathcal{J}'])]
\]

\[
= (K[E, \mathcal{J}], K[E, \mathcal{J}'])_H - (K[E, \mathcal{J}], K[E, \mathcal{J}'])_H
\]

\[
= -i \sigma ([E, \mathcal{J}], [E, \mathcal{J}']).
\]

By proposition 2.15, we obtain the desired result. Finally, if \( \text{supp} \mathcal{J} \) and \( \text{supp} \mathcal{J}' \) are spacelike separated, then \( \text{supp} \mathcal{J} \cap \text{supp} E \mathcal{J}' = \emptyset \) and the integral in the inner product vanishes.

\[ \square \]

3.2. Algebraic/local quantum field theory

It is well known that different choices of \( \mu \) obviously lead to different constructions of the Fock space \( \mathcal{F}_\mu(\mathcal{H}) \) and hence unitarily inequivalent quantum field theories [19]. In Minkowski spacetime, Poincaré invariance picks out a ‘preferred’ \( \mu \) which leads to a Hilbert space \( \mathcal{H} \) of purely positive frequency solutions to build the Fock space from. There are also purely positive frequency solutions in curved stationary spacetimes where the time translation Killing field generates an isometry similar to that of Poincaré invariance in Minkowski spacetime. In a general curved spacetime there may be no such isometries, so purely positive frequency solutions may not exist and the notion of particles becomes somewhat ambiguous. This situation led to the development of the algebraic approach to quantization, also called local quantum field theory. For a general review of this topic, we recommend the articles by Buchholz [29], Buchholz and Haag [30] and Wald [31]. For a more thorough discussion, see [32–35]. Our notation will closely follow that found in chapter 4 of Bär et al [13].

As our final task, we would like to show that our quantized field theory can be used to generate the Weyl system commonly used in algebraic quantum field theory. The creation and annihilation operators are unbounded, so in order to work with bounded operators we introduce the unitary operators on our Fock space:

\[
W([u]) = \exp(i \sigma ([A], [u]))
\]  

(3.10)

for all \([u] \in \mathcal{M}^\mathcal{H}(M)\). From the definition of the field operator and its commutation relations, it is relatively straightforward to show that this map satisfies

\[
(i) \quad W([0]) = i d_{\mathcal{G},(\mathcal{H})},
\]

(3.11)

\[
(ii) \quad W([-u]) = W([u])^*,
\]

(3.12)

\[
(iii) \quad W([u]) \cdot W([v]) = e^{-i \sigma ([u], [v])/2} W([u] + [v]).
\]

(3.13)

(The last relation follows from the Baker–Campbell–Hausdorff formula.) The CCR algebra \( \mathfrak{A} \) is defined as the \( C^* \)-algebra generated by \( W([E, \mathcal{J}]) \) for all \( \mathcal{J} \in \Omega^0(M) \). The CCR-algebra \( \mathfrak{A} \) together with the map \( W \) forms a Weyl system for our symplectic phase space \( (\mathcal{M}^\mathcal{H}(M), \sigma) \) which satisfies the Haag–Kastler axioms as generalized by Dimock [35]. The elements of the algebra are interpreted as the observables related to the quantum field which satisfy the
generalized Maxwell equation. By theorem 4.2.9 of Bär et al, this CCR representation is essentially unique.

Lastly, we would like to indicate that two very different constructions of the Weyl system for a symplectic phase space that could be used are given in [13]. Unfortunately, both of the Hilbert space representations they construct are not considered physical because the states (vectors) in the Hilbert space are not Hadamard, i.e. the two-point function for these states is not related to a certain $p$-form Klein–Gordon bisolution of the Hadamard form. In this paper, we have given a framework for the rigorous quantization of the $p$-form field for which the issue of states being Hadamard can be addressed in due course. In the case of the 0-form field, the Maxwell equation and the Klein–Gordon equation are the same, so finding the Hadamard state is straightforward. The issue of Hadamard states for the 1-form field in four-dimensional globally hyperbolic spacetimes can be found in [10]. We will complete the discussion of Hadamard states for the general $p$-form field and develop the quantum weak energy inequality for these states in our next paper.

4. Conclusions

In this paper, we quantized the generalized Maxwell field $\mathcal{A}$ on globally hyperbolic spacetimes with compact Cauchy surfaces. We began by taking the Maxwell equations into the language of exterior differential calculus. The resulting field equation (2.7) could then be carried to any dimension. Rather remarkably, we found that a minimally coupled scalar field and the electromagnetic field are actually two examples of a single $p$-form field theory in arbitrary dimension. We then discussed fundamental solutions and the Cauchy problem for the classical $p$-form field theory where we showed that the Cauchy problem was well posed if we worked in terms of gauge-equivalent classes of solutions. This was followed by a discussion of the classical, symplectic phase space consisting of all real-valued gauge-equivalent Maxwell solutions $\mathcal{M}^p(M)$ and a non-degenerate antisymmetric bilinear form $\sigma([A], [B])$ for $A, B \in \mathcal{M}^p(M)$. The theory was then quantized by promoting functions on the phase space to operators that act on elements of a Hilbert space. The appropriately selected operators were shown in proposition 3.1 to satisfy the generalized Maxwell equation in the weak sense and have the proper canonical commutation relations. Finally, the Weyl system for our field theory was developed.

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Appendix. Cauchy problem for $\mathcal{F}$

Proposition A.1. Let $F_0(0) \in \Omega^n_0(\Sigma)$ and $F_n(0) \in \Omega^{n-1}_0(\Sigma)$ with $dF_0(0) = 0$ and $\delta F_n(0) = 0$ specify Cauchy data for the field strength $\mathcal{F} \in \Omega^p(M)$, with $0 < p \leq n$, such that

$$\rho_{(0)}\mathcal{F} = F_0(0) \quad \text{and} \quad \rho_{(n)}\mathcal{F} = F_n(0).$$

(A.1)
Given these data, there exists a smooth potential \( A \in \Omega^{p-1}(M) \) such that \( F = dA \) satisfies the generalized Maxwell equations \( dF = 0 \) and \( \delta F = 0 \), as well as conditions (A.1).

**Proof.** We know that \( F = dA \) will satisfy the Maxwell equations if \( \delta dA = 0 \). To show that such \( A \) exists, we choose as Cauchy data

\[
A_{(0)} = F_{(0)}, \quad A_{(d)} = F_{(n)}, \tag{A.2}
\]

while \( A_{(0)} \) and \( A_{(d)} \) are arbitrary up to \( dA_{(d)} = 0 \).

The first thing to address is the existence of \( A_{(0)} \). For a non-compact Cauchy surface \( \Sigma \), we could restrict to only those manifolds that are contractible. By the Poincaré lemma for contractible manifolds (theorem 6.4.18 of AMR [9]), all closed \( p \)-forms (for \( p > 0 \)) are exact. Alternatively, we could require that the compactly supported deRham cohomology group \( H_p^c(\Sigma) \) for \( p \)-forms on the Cauchy surface be of dimension zero, i.e. \( H_p^c(\Sigma) = \{0\} \). This is a restriction on the topology of the Cauchy surface. If we do have a trivial deRham cohomology group, then all closed \( p \)-forms \( F_{(0)} \) are exact. Either the contractible or cohomology condition is sufficient to allow for the existence of a suitable \( A_{(0)} \).

From the initial Cauchy data on \( F \), we have

\[
\delta A_{(d)} = \delta F_{(n)} = 0. \tag{A.3}
\]

Our Cauchy data for \( A \) have the properties necessary to use proposition 2.11. So \( A \) is a solution to \( \delta dA = 0 \) and therefore \( F = dA \) is a solution to the generalized Maxwell equations.

Now show that this also reproduces the Cauchy data. We evaluate

\[
\rho_{(0)} F = \rho_{(0)} dA = d\rho_{(0)} A = dA_{(0)} = F_{(0)}. \tag{A.4}
\]

Next, we evaluate

\[
\rho_{(n)} F = \rho_{(n)} dA = \rho_{(d)} A = A_{(n)} = F_{(n)}. \tag{A.5}
\]

The remaining two pullbacks are trivially zero since

\[
\rho_{(d)} F = \rho_{(d)} dA = 0 \tag{A.6}
\]

and

\[
\rho_{(n)} F = \rho_{(n)} dA = \rho_{(0)} dA = 0. \tag{A.7}
\]

\[\square\]

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