Homogenization of heterogeneous masonry beams

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Abstract This study presents a two-scale model to describe the out-of-plane masonry response. One-dimensional (1D) structural elements, like masonry columns or strips of long wall characterized by the periodic repetition of bricks and mortar arranged in stack bond, are considered. A damage-friction plasticity law is adopted to model the mortar joint constitutive response, while the bricks are assumed as linear elastic. A 1D beam formulation is introduced at both the structural and micromechanical scale, linking the two levels by means of a kinematic map. This expresses the microscopic beam strains in the masonry unit cell (UC) as function of the macroscopic generalized strains. The kinematic field in the UC is completed by adding an unknown periodic fluctuation term. A nonlinear homogenization procedure is developed, proposing a semi-analytical solution for the micromechanical problem, based on the fiber discretization of the mortar joints. A force-based beam-column finite element procedure is adopted at the structural scale and the solution algorithm for the element state determination is illustrated in details. Some numerical applications, showing the UC constitutive response and the behavior of masonry structural elements, are finally presented.

Keywords Masonry · Multiscale model · Damage · Plasticity · Beam force-based finite elements

1 Introduction

The development of accurate and efficient numerical procedures to study masonry structural response is a challenging task, related to the assessment of the safety and the rehabilitation and strengthening of old masonry buildings. Depending on the geometrical features of the buildings and the masonry material mechanical properties, masonry walls can experience three different collapse mechanisms, that is crumbling, out-of-plane and in-plane failure [5]. When the masonry crumbling is prevented, the out-plane mechanisms are the most frequent. Thus, developing efficient numerical models accounting for these is a very interesting issue.

Masonry material, which can be characterized by regular or random arrangement of different constituents, like blocks and mortar, shows a complex mechanical behavior due to its heterogeneity, anisotropy and strongly nonlinear constitutive response.
When subjected to typical loading conditions, a degrading behavior emerges characterized by the onset and evolution of microcracks, which spread and combine producing macrocracks. Moreover, plastic-friction mechanisms are developed due to the sliding of the mortar joints. A number of different approaches have been proposed to evaluate masonry structural response, under both static and dynamic loads. A common classification distinguishes between micromechanical, macromechanical and multiscale models [2, 23, 33], depending on the scale at which masonry is analyzed. The first class contains models separately describing masonry constituents [4, 15], blocks and mortar, and in some approaches also the interfaces connecting them [12, 17, 22]. Commonly, nonlinear constitutive laws are adopted for all the components, although some simplified formulations use a linear elastic behavior for the blocks. As the micromechanical models require high computational efforts, these can be conveniently adopted to analyze small masonry elements or reproduce laboratory tests on small assemblage of blocks and mortar. Macromechanical approaches [7, 24, 30], considering masonry as an equivalent homogenized medium, are much less time consuming and can be used to evaluate the structural response of large walls and buildings. Nonlinear damage-plastic models are usually introduced, describing the overall constitutive response of the homogenized medium. The most simplified formulations describe masonry buildings as the assemblage of macro-elements, piers, spandrels and rigid connections. Often, each macro-element is modeled as a beam able to reproduce the main nonlinear flexural and shear mechanisms typical of masonry walls [1, 10].

Multiscale techniques [16, 20] appear to be a very challenging and interesting approach. These are efficiently used to describe the mechanical response of heterogeneous materials in various fields, and, in particular, for masonry [25]. Two scale of interest are usually analyzed in this case: the macroscopic structural scale, where an equivalent homogenized medium is considered, and the microscopic scale, where the constituents are modeled in detail. To derive the constitutive response of a regular masonry at each macroscopic point, where an explicit relationship is unknown, a repetitive unit cell (UC) is selected at the microscale and linked to the point. The UC contains all the information about the constituents geometry and arrangement and describes the overall medium by its translation. By solving a properly formulated boundary value problem (BVP) on the UC, the stress field at the microscale is evaluated and homogenized to recover the macroscale stress components at the corresponding macroscopic point. Thus, multiscale approaches represent a good compromise between a detailed description of the masonry constituents geometry, arrangement and constitutive responses, and the computational burden. In the last decades, a wide spread of the so-called computational homogenization methods has occurred, where micro and macroscale exchange information at each iteration of the global solution procedure [26]. These can be computationally cumbersome, but the costs are significantly reduced, if a parallel computing strategy is adopted. Computational homogenization methods have mostly been adopted to describe the in-plane response of masonry structural elements. Recently, some proposals extending this methodology for describing the out-of-plane behavior of masonry have been done. A number of homogenized shell-like models have been developed [27, 28] to overcome the limitations of the classical 3D homogenization. These approaches usually couple a 3D UC at the microscale to a shell model at the macro structural level, assuming both linear and nonlinear constitutive behavior. Most of the proposed models rely on the Kirchhoff–Love theory for thin shells [9], but some recent works discussing the homogenization procedure for thick shells based on the Mindlin theory [8, 31], can be found.

This work proposes a computational homogenization procedure to describe the out-of-plane response of masonry walls. The study focuses on two-scale techniques, which model and couple the macroscopic structural level to the microscopic one. At both the scales a Timoshenko beam model is introduced. Adopting the 1D beam formulation has some advantages. First, due to the nature of the variables and equations governing the UC BVP problem, analytical solutions can be evaluated, assuming some simplified hypotheses on the masonry constitutive behavior.

It is worth noting that a number of masonry structural problems, related to interesting engineering applications, can be efficiently studied adopting the proposed 1D model. First, columns under axial and bending loads can be modeled by 1D beam FE s able to correctly reproduce their collapse mechanism and
limit strength, with and without strengthening [11, 21, 39]. Masonry arch collapse mechanisms have often been studied via analytical methods, starting from the pioneering paper by Heyman [18], the discrete element method [14, 32], as well as refined finite element methods [13, 29, 40]. Using the proposed 1D FE formulation can significantly reduce the computational costs, and, at the same time, permits to satisfactorily describe the structural response. Moreover, most of the out-of-plane mechanisms of unreinforced and strengthened masonry walls are related to the cylindrical bending mode. This mechanism can be simulated by a 1D beam model, also considering different loading and boundary conditions [19]. Lastly, the development of a computational homogenization procedure for 1D beam model is a useful tool to be extended to the formulation of 2D shell models.

To derive the constitutive law relating the beam generalized stress and strain components at each integration point, located along the beam axis, a nonlinear homogenization problem is formulated and solved on the UC. This is made by the periodic repetition of the two masonry components, brick and mortar, along a 1D direction. Here, the response of masonry elements characterized by a stack bond arrangement along a preferential direction, like for example columns and arches, is studied. The microscopic displacement field in the UC is represented as the additive composition of an assigned part, depending on the macroscopic beam strain components, and an unknown perturbation term, arising due to the heterogeneous nature of the medium. A linear elastic behavior is assumed for the bricks, while a damage-friction model [34] governs the response of the mortar joints. A fiber discretization of the mortar joints is introduced to solve the evolution problem of the damage and plastic variables. Considering some simplified assumptions on the damage and plasticity distribution, the differential equations governing the BVP on the UC are solved by an iterative procedure. Once the microscopic stress field has been evaluated, this is homogenized to compute the macroscopic stress components at the Gauss point linked to the analyzed UC. A finite element (FE) solution procedure is adopted at the macroscopic scale, based on the beam force-based formulation [36]. This has been proved to be more efficient and accurate than the classical displacement-based approach and has already been applied to masonry modeling [3]. The developed FE procedure and the solution algorithm are implemented in the MATLAB code. Some numerical applications are presented, illustrating the UC response under typical loading conditions and the response of a pre-stressed masonry column subjected to lateral loads. The solutions obtained with the proposed 1D model are compared with those evaluated by means of the 2D EPS formulation presented in [4], highlighting the efficiency of the first in reproducing the more accurate and cumbersome 2D results. Comparisons with experimental tests are also performed on a single UC to validate the mortar constitutive model, and on a masonry wallette in bending to prove the suitability of the developed two-scale model for reproducing masonry structural response.

The paper is organized as follows. In Sect. 2, the two-scale beam model is presented, introducing the main kinematic and static fields at both the scales, the governing equations and the nonlinear homogenization procedure. Section 3 contains the semi-analytical micromechanical solution, while in Sect. 4 the FE formulation and the solution algorithm are detailed. Finally, Sect. 5 illustrates the results of some numerical applications and Sect. 6 collects the concluding remarks.

2 Two-scale beam model

A masonry column, such that schematically illustrated in Fig. 1a, is considered. It is made by the regular superposition of bricks and mortar.

2.1 Macroscale model

The structural behavior of the column is studied adopting the classical Timoshenko beam model shown in Fig. 1b, in the framework of the small displacement and strain theory. Moreover, the beam is assumed to be made of an “equivalent” homogenized material. The Cartesian coordinate system (O, X, Y) is introduced as in Fig. 1b, with X denoting the beam axis direction and O located at the center of mass of the beam cross section.

At the macroscopic structural scale, the displacement field is introduced as:

\[ U_1 = U(X) - Y \Phi(X) \]
\[ U_2 = V(X), \]
with $U$, $V$ and $\Phi$ representing the generalized displacement components of the beam, i.e. the axial and transversal displacements of the point $X$, and the rotation of the beam cross section at $X$, respectively. The counterclockwise rotation is assumed positive.

The generalized strain components are collected in vector $E$ as:

$$E = \{E, K, \Gamma\}^T,$$

where $E$ denotes the beam axial elongation, $K$ the curvature and $\Gamma$ the shear strain, that are related to the displacement components by the compatibility equations as:

$$E = U_X \quad K = \Phi_X \quad \Gamma = V_X - \Phi.$$

The work-conjugate macroscopic stress measures are collected in the vector $S$, as:

$$S = \{N, M, T\}^T,$$

where $N$, $M$ and $T$ are the beam axial, flexural and shear stress components.

The classical equations govern the beam equilibrium at structural level, which are reported in order to set the notations:

$$N_X + f = 0 \quad T - M_X = 0 \quad T_X - q = 0,$$

where $f$ and $q$ are the distributed axial and transversal loads applied to the beam.

2.2 Microscale model

The mechanical response of the “equivalent” homogenized material constituting the beam at the structural, i.e. macroscale, level is derived considering a representative volume element of masonry, which accounts for the geometrical arrangement and the material properties of the components. Because of the assumed regular texture, the periodic UC, contained in Fig. 1c, is considered as representative volume. The UC occupying the region $\Omega$ is made by a single brick ($b$) and a mortar layer ($m$) arranged in stack bond, such that $\Omega = \Omega_m \cup \Omega_b$, being $\Omega_m$ and $\Omega_b$ the mortar and brick volumes, respectively.

The response of the UC is also derived considering the Timoshenko beam model given in Fig. 1d, in the small displacement and strain framework. A Cartesian reference system (O, x, y) is set, with $x$ denoting the beam axis direction. The volumes occupied by the mortar and brick materials are expressed as $\Omega_m = A \times \ell_m$ and $\Omega_b = A \times \ell_b$, respectively, with $A$ denoting the beam cross section, $\ell_m$ and $\ell_b$ the lengths along the beam axis of the material $m$ and $b$, so that $\ell = \ell_m + \ell_b$ is the total length of the UC.
2.2.1 Kinematics

The displacement fields at each point of the cross section are expressed as:

\[ u_1 = u(x) - y \varphi(x) \]
\[ u_2 = v(x), \tag{6} \]

with \( u, v \) and \( \varphi \) denoting the generalized displacement components of the beam, i.e. the axial and transversal displacements of the point \( x \), and the rotation of the beam cross section at \( x \), respectively. As at the macroscale, the counterclockwise rotation is assumed to be positive.

The axial and shear strain components are derived by the displacements in (6), using the compatibility conditions, as:

\[ \varepsilon = \varepsilon_0 - y \kappa \]
\[ \gamma = \gamma' - \varphi, \tag{7} \]

where the apex \( ' \) denotes the derivative with respect to \( x \) and the generalized strain parameters \( \varepsilon_0, \kappa \) and \( \gamma \), introduced in (7), are defined as:

\[ \varepsilon_0 = u' \quad \gamma = \gamma' - \varphi \quad \kappa = \varphi'. \tag{8} \]

2.2.2 Stresses

Denoting with \( \sigma_m, \sigma_b \) and \( \tau_m, \tau_b \) the normal and shear stresses in the regions \( \Omega_m \) and \( \Omega_b \), respectively, the resultant generalized stress components in the two materials are introduced as:

\[ n_m = \int_A \sigma_m \, dA \quad n_b = \int_A \sigma_b \, dA \]
\[ m_m = -\int_A y \sigma_m \, dA \quad m_b = -\int_A y \sigma_b \, dA \]
\[ t_m = \int_A \tau_m \, dA \quad t_b = \int_A \tau_b \, dA. \tag{9} \]

with \( A \) denoting the cross section shear area.

2.2.3 Constitutive equations

A nonlinear constitutive law is assumed for the mortar, while a linear elastic response is considered for the bricks.

For the mortar material (region \( \Omega_m \)), the following constitutive laws are adopted:

\[ \sigma_m = E_m (\varepsilon - \pi_{\varepsilon}) \]
\[ \tau_m = G_m (\gamma - \pi_{\gamma}), \tag{10} \]

where \( \pi_{\varepsilon} \) and \( \pi_{\gamma} \) are the normal and shear inelastic strains due to damage, plasticity or other inelastic effects. In particular, applying the constitutive model proposed in [34], it results:

\[ \pi_{\varepsilon} = DH(\varepsilon)\varepsilon \quad \pi_{\gamma} = D\dot{\gamma}_p, \tag{11} \]

where \( D \) describes the damage mechanism, with \( 0 \leq D \leq 1 \), \( H(\varepsilon)\varepsilon \) accounts for the unilateral effect, being \( H(\varepsilon) = 0 \) if \( \varepsilon \leq 0 \) and \( H(\varepsilon) = 1 \) if \( \varepsilon > 0 \), and \( \dot{\gamma}_p \) is the shear slip.

The coupling of the fracture modes I and II is introduced by properly describing the evolution of the damage variable. Thus, the parameter \( \eta \) coupling the normal and shear failure is defined as:

\[ \eta = \frac{1}{N^2} \left[ \left( \langle \varepsilon \rangle_+ \right)^2 \eta_{\varepsilon} + \gamma^2 \eta_{\gamma} \right], \tag{12} \]

with

\[ \eta_{\varepsilon} = \frac{\varepsilon_0 \sigma_0}{2G_{cI}}, \quad \eta_{\gamma} = \frac{\gamma_0 \tau_0}{2G_{cII}}, \quad N = \sqrt{\left( \langle \varepsilon \rangle_+ \right)^2 + \gamma^2}, \tag{13} \]

\( \sigma_0 \) and \( \tau_0 \) being the peak values of the normal and shear stresses, and \( G_{cI} \) and \( G_{cII} \) the fracture energies corresponding to modes I and II. Note that the Macaulay brackets \( \langle \cdot \rangle_+ \) give the positive part of the number. The quantities \( \eta_{\varepsilon} \) and \( \eta_{\gamma} \) measure the ratio between the maximum value of the normal and shear elastic energy and the corresponding fracture energy. The parameter \( \eta \) can be interpreted as a combination of the two ratios for mixed mode failure.

Then, the normalized equivalent strain is defined as:

\[ Y = \sqrt{\left( \frac{\langle \varepsilon \rangle_+}{\varepsilon_0} \right)^2 + \left( \frac{\gamma}{\gamma_0} \right)^2}. \tag{14} \]

Finally, the damage is evaluated according to the following law:
The friction effect is modeled by a classical Coulomb law. Introducing the normal and shear contact stresses as:

\[ \sigma_m = E_m (\dot{\varepsilon} - H(\dot{\varepsilon})) \] (16)
\[ \tau_m = G_m (\dot{\gamma} - \dot{\gamma}_p) , \] (17)

the yield function is set as:

\[ \phi(\sigma_m, \tau_m) = \mu \sigma_m + |\tau_m| , \] (18)

where \( \mu \) is the friction parameter. The slip flow is governed by the evolution equation and the loading-unloading Kuhn-Tucker conditions:

\[ \dot{\gamma}_p = \lambda \frac{\tau_m}{|\tau_m|} , \quad \lambda \geq 0 \quad \varphi \leq 0 , \quad \dot{\lambda} = 0 , \] (19)

where \( \lambda \) is the inelastic multiplier.

Taking into account the relationships (7) and (10), the resultant stresses defined by Eq. (9) are obtained as:

\[ n_m = E_m A \varepsilon_0 - P \]
\[ m_m = E_m I \kappa - R , \] (20)
\[ t_m = G_m A \dot{\gamma} - V \]

where \( I \) is the inertia of the beam cross section. Note that in Eq. (20) the terms involving the static momentum do not appear, as a coordinate system with the origin in the centroid of the beam cross section is selected. This assumption holds in the following. Moreover, it is:

\[ P = \int_A E_m \pi_\varepsilon \, dA \quad R = -\int_A E_m \pi_\gamma \, dA \]
\[ V = \int_A G_m \pi_\gamma \, dA . \] (21)

A linear elastic constitutive relationship is considered for the brick (region \( \Omega_b \)), setting:

\[ \sigma_b = E_b \varepsilon \]
\[ \tau_b = G_b \dot{\gamma} , \] (22)

The resultant stresses defined in Eq. (9) consequently are:

\[ n_b = E_b A \varepsilon_0 \]
\[ m_b = E_b I \kappa \]
\[ t_b = G_b A \dot{\gamma} . \] (23)

2.3 Nonlinear homogenization

The macroscale and microscale strain and stress components are linked by the following average relationships:

\[ E = \frac{1}{\ell} \int_\ell \varepsilon_0 \, dx \quad K = \frac{1}{\ell} \int_\ell \kappa \, dx \quad \Gamma = \frac{1}{\ell} \int_\ell \gamma \, dx \] (24)
\[ N = \frac{1}{\ell} \int_\ell n \, dx \quad M = \frac{1}{\ell} \int_\ell m \, dx \quad T = \frac{1}{\ell} \int_\ell t \, dx . \] (25)

Indeed, a kinematic driven homogenization problem is formulated, using the macroscopic generalized strain parameters \( E \), \( K \) and \( \Gamma \) introduced in (3) as input loading conditions for the UC.

Once the quantities \( E \), \( K \) and \( \Gamma \) are prescribed, the homogenization process consists in evaluating:

- The strain \( \varepsilon_0 \), \( \kappa \) and \( \gamma \) at the microscale, satisfying equations (24),
- The stress resultants in the mortar, \( n_m \), \( m_m \) and \( t_m \), after computing the inelastic resultants \( P \), \( R \) and \( V \),
- The stress resultants in the brick \( n_b \), \( m_b \) and \( t_b \), and,
- Eventually, the overall resultants \( N \), \( M \) and \( T \) via Eqs. (25).

Considering the periodic texture of the analyzed medium, the displacement components at the microlevel, \( u \), \( v \) and \( \varphi \), are written as the superposition of prescribed fields \( \bar{u} \), \( \bar{v} \) and \( \bar{\varphi} \), defined as function of the macroscopic deformations \( E \), \( K \) and \( \Gamma \), and unknown periodic fluctuations, \( u^* \), \( v^* \) and \( \varphi^* \), satisfying proper periodicity conditions on the UC boundary [37], namely:

\[ u(x) = \bar{u}(x) + u^*(x) \]
\[ v(x) = \bar{v}(x) + v^*(x) \quad \text{in} \quad (0, \ell) , \] (26)
\[ \varphi(x) = \bar{\varphi}(x) + \varphi^*(x) \]

with

\[ u^*(0) = u^*(\ell) \quad v^*(0) = v^*(\ell) \quad \varphi^*(0) = \varphi^*(\ell) . \] (27)
Enforcing that the average of the periodic part of the
generalized strain parameters has to be zero, Eqs. (8),
(24) and (26) lead to:

\[ 0 = \int_{\ell} (u^*) \, dx \]
\[ 0 = \int_{\ell} (\varphi^*) \, dx \]
\[ 0 = \int_{\ell} (v^*) \, dx - \int_{\ell} \varphi^* \, dx . \]  

While the first two equations of the (28) are satisfied
thanks to the periodicity conditions (27), the third equation requires that:

\[ \int_{\ell} \varphi^* \, dx = 0. \]  \hspace{1cm} (29)

Under the conditions (28) and (29), taking into account
(24), the following kinematic map is derived:

\[ \bar{u}(x) = E x \]
\[ \bar{v}(x) = \Gamma x + \frac{1}{2} K x^2 \quad \text{in} \quad (0, \ell) . \]
\[ \overline{\varphi}(x) = K x \]

By introducing the displacements expressed by (26)
and (30) into relation (8), the strain components (7) at
each cross section result as:

\[ \varepsilon_0 = E + u^* \quad \kappa = K + \varphi^* \quad \gamma = \Gamma + v^* - \varphi^* . \]  \hspace{1cm} (31)

Then, the micro stresses in \( \Omega_m \) and \( \Omega_b \) are evaluated
according to the constitutive laws (10) and (22)
introduced for the UC constituents, after solving the
evolution problem of the inelastic variables \( \pi_n \) and \( \pi_s \).
(11). The micromechanical solution is completed by
computing the resultant stresses \( n_m, m_m, l_m, n_b, m_b, t_b \)
on the basis of Eqs. (20) and (23).

The upscaling from the micro to macroscale is
eventually performed, by evaluating the average
resultant stress components in the whole UC, according
to formula (25). This, considering the splitting of
the UC volume \( \Omega = \Omega_m \cup \Omega_b \), becomes:

\[ N = \frac{1}{\ell} \left( \int_{\ell_m} n_m \, dx + \int_{l_m} n_b \, dx \right) \]
\[ M = \frac{1}{\ell} \left( \int_{\ell_m} m_m \, dx + \int_{l_m} m_b \, dx \right) \]
\[ T = \frac{1}{\ell} \left( \int_{\ell_m} t_m \, dx + \int_{l_m} t_b \, dx \right) \]  \hspace{1cm} (32)

This step concludes the homogenization process,
giving at the macroscopic point the overall resultant
stresses \( N, M, T \) conjugate to the macroscopic strain
parameters \( E, K, \Gamma \).

3 Micromechanical solution

Relying on the micromechanical model presented in
Sect. 2 and taking into account the constitutive, (20)
and (23), and compatibility, (31), equations, the
energy functional, including the null average require-
ment for the perturbation rotation (29), is introduced as:

\[ \Pi = \int_{\ell_m} \left[ \frac{1}{2} E_m A (E + u_m^*)^2 - P \right] (E + u_m^*) \, dx \]
\[ + \int_{l_m} \frac{1}{2} E_b A (E + u_m^*)^2 \, dx \]
\[ + \int_{\ell_n} \left[ \frac{1}{2} E_m I (K + \varphi_m^*) - R \right] (K + \varphi_m^*) \, dx \]
\[ + \int_{l_n} \frac{1}{2} E_b I (K + \varphi_b^*)^2 \, dx \]
\[ + \int_{\ell_n} \left[ \frac{1}{2} G_m A_s (\Gamma + \nu_m^* - \varphi_m^*) - V \right] \]
\[ (\Gamma + \nu_m^* - \varphi_m^*) \, dx + \int_{l_n} \frac{1}{2} G_b A_s (\Gamma + \nu_b^* - \varphi_b^*)^2 \, dx \]
\[ + \lambda \left( \int_{\ell_n} \varphi_m^* \, dx + \int_{l_n} \varphi_b^* \, dx \right) \]
\[ + \mu_n \left( u_m^* (\ell_n) - u_b^* (\ell_n) \right) + \mu_s \left( v_m^* (\ell_n) - \nu_m^* (\ell_n) \right) \]
\[ - \nu_b^* (\ell_n) - \mu_\phi \left( \varphi_m^* (\ell_n) - \varphi_b^* (\ell_n) \right) \]  \hspace{1cm} (33)

under the periodicity conditions (27). The inelastic
resultant stresses \( P, R \) and \( V \) are given functions
governed by the evolutive equations of the inelastic
strains. The introduced quantities \( \mu_n, \mu_s \) and \( \mu_\phi \) are
the Lagrange multipliers due to the continuity condition
of the displacement field at the mortar–brick interface,
while \( \lambda \) is the Lagrange multiplier related to the
constraint (29). Enforcing the stationary condition of
the functional (33), the following equations are derived:

- Euler equilibrium equations in \( \Omega_m \):

\[ \cdots \]
The differential equations (34) and (35) are completed with the periodicity and continuity boundary conditions, which, by eliminating the rigid body motions, result as:

$$u_m^*(0) = 0, \quad u_m^*(\ell) = u_b^*(\ell)$$
$$v_m^*(0) = 0, \quad v_m^*(\ell) = v_b^*(\ell)$$
$$\varphi_m^*(0) = \varphi_b^*(\ell)$$
$$m_m(\ell_m) = m_b(\ell_m)$$
$$n_m(\ell_m) = n_b(\ell_m)$$

It is interesting to highlight the mechanical meaning of the multiplier $\lambda$, which indeed corresponds to uniformly distributed couples per unit length, acting along the whole span of the heterogeneous beam. The presence of the couple field $\lambda$ is essential to ensure the equilibrium of the beam. To better clarify the meaning of $\lambda$, let the homogeneous case be considered, obtained for instance assuming $\ell_m = 0$ and $\ell = \ell_b$. Because of the homogeneity, it is expected that all the perturbation (periodic) fields in Eq. (35) vanish, i.e. $u_b^* = 0$, $v_b^* = 0$ and $\varphi_b^* = 0$. Consequently, while the first and third equations of (35), governing the equilibrium of the homogeneous beam, are trivially satisfied, the second becomes:

$$G_b A_\Gamma + \lambda = 0.$$  \hfill (39)

This last is satisfied thanks to the presence of the multiplier $\lambda$, which indeed results equal to the opposite of the shear force $G_b A_\Gamma$.

The nonlinear differential equations (34) and (35), with (21) and the boundary conditions (38), are solved via an iterative procedure joined with the analytical solution of the beam microscale model.

A simplifying assumption is introduced, considering the damage variable in the mortar as constant along $x$ and varying only along $y$, i.e. $D = D(y)$. Then, a fiber-based model is adopted to solve the nonlinear evolution problem related to the inelastic mechanisms activating in the mortar, that is damage, unilateral contact and friction, described in Sect. 2.2.2.3. Hence, the cross section of the heterogeneous beam is divided in small areas, where the mortar constitutive response is evaluated. By discretizing the mortar along the $y$ direction with $n_f$ fibers, the following linear interpolation is assumed for the damage variation in each fiber $f$:

$$D' \left( \xi \right) = \frac{1 - \xi}{2} D_i + \frac{1 + \xi}{2} D_j,$$  \hfill (40)

where $D_i$ and $D_j$ are the damage parameters at the ends of the fiber $f$, namely at $y_i$ and $y_j$, with $i = f$ and $j = f+1$, and $\xi$ is the dimensionless coordinate ranging from $-1$ to $1$ along the fiber height $h' = y_j - y_i$, related to the Cartesian coordinate $y$ as:

$$\xi = \frac{2}{h'} (y - y_i) - 1 \quad \text{or} \quad y = \frac{h'}{2} (\xi + 1) + y_i.$$  \hfill (41)

Relying on the above assumptions and accounting for the first of (7), the inelastic strains defined by Eq. (11) in each fiber are expressed as:

$$\pi'_i = D' H(\xi') \xi' = H(\xi') \left( \frac{1 - \xi}{2} D_i + \frac{1 + \xi}{2} D_j \right) \left( \xi_0 + \frac{h'}{2} (1 + \xi) \kappa - y_i \kappa \right) \pi'_j.$$  \hfill (42)

Taking into account the fiber discretization of the mortar and the expressions of $\pi'_i$ and $\pi'_j$ in (42), the integrals in (21) are evaluated as:
\[ P = \sum_{f=1}^{n_f} \int_{\mathcal{A}_f} E_m \pi^f \, dA' \quad R = - \sum_{f=1}^{n_f} \int_{\mathcal{A}_f} E_m y \pi^f \, dA' \]
\[ V = \sum_{f=1}^{n_f} \int_{\mathcal{A}_f} G_m \pi^f \, dA'. \] (43)

By introducing the coordinate transformation (41), the above integrals are computed in the dimensionless domain and the following expressions are obtained:
\[ P = \sum_{f=1}^{n_f} \frac{A_f^f}{2} E_m \left\{ \left[ \varepsilon_0 - \left( y_i + \frac{h'}{3} \right) \frac{\kappa}{\gamma} \right] D_i + \left[ \varepsilon_0 - \left( y_i + \frac{2h'}{3} \right) \frac{\kappa}{\gamma} \right] H(\ell') \right\} D_i, \]
\[ R = \sum_{f=1}^{n_f} \frac{A_f^f}{2} E_m \left\{ \left[ \left( y_i + \frac{h'}{3} \right) \varepsilon_0 - \left( y_i + \frac{h'}{3} \right) \frac{\kappa}{\gamma} \right] D_i + \left[ \left( y_i + \frac{h'}{3} \right) \varepsilon_0 - \left( y_i + \frac{h'}{3} \right) \frac{\kappa}{\gamma} \right] H(\ell') \right\} D_i, \]
\[ V = \sum_{f=1}^{n_f} \frac{A_f^f}{2} G_m \gamma_p^f \left( D_i + D_j \right), \] (44)

where \( \gamma_p^f \) is evaluated at the center of the fiber, \( \ell' \) is the average strain in the fiber \( f \) and \( A_f^f = b \times h' \), considering a rectangular cross section.

As a consequence of the introduced assumptions, the inelastic forces \( P, R \) and \( V \) are constant along the beam axis \( 0 \leq x \leq L_m \), so that it is \( P' = 0, R' = 0 \) and \( V' = 0 \) in Eqs. (34). Therefore, the analytical solution of Eqs. (34) can be determined as:
\[ u_m^x = C_{1m} x + C_{2m} \]
\[ v_m^x = \frac{1}{6} C_{3m} x^3 + \frac{1}{2} C_{4m} x^2 + C_{5m} x + C_{6m} \]
\[ \phi_m^x = \frac{1}{2} C_{3m} x^2 + C_{4m} x + \frac{E_m I}{G_m A_s} C_{3m} + C_{5m} + \Gamma + \frac{1}{G_m A_s} \lambda. \] (45)

Analogously, the solution of the differential equations (35) is:
\[ u_b^x = C_{1b} x + C_{2b} \]
\[ v_b^x = \frac{1}{6} C_{3b} x^3 + \frac{1}{2} C_{4b} x^2 + C_{5b} x + C_{6b} \]
\[ \phi_b^x = \frac{1}{2} C_{3b} x^2 + C_{4b} x + \frac{E_b I}{G_b A_s} C_{3b} + C_{5b} + \Gamma + \frac{1}{G_b A_s} \lambda. \] (46)

The integration constants \( C_{1m}, \ldots, C_{6m} \) and \( C_{1b}, \ldots, C_{6b} \) are determined enforcing the twelve boundary conditions (38). Moreover, the value of \( \lambda \) is computed enforcing the condition (37), which gives:
\[ \lambda = - \frac{G_m G_b A_s}{G_b \ell_m + G_m \ell_b} (\Phi_m \ell_m + \Phi_b \ell_b), \] (47)
where
\[ \Phi_m = \frac{1}{6} C_{3m} \ell_m^2 + \frac{1}{2} C_{4m} \ell_m + \frac{E_m I}{G_m A_s} C_{3m} + C_{5m} + \gamma - \frac{V}{G_m A_s}, \]
\[ \Phi_b = \frac{C_{3b}}{2} \left( \frac{1}{3} \ell_b^2 + \ell_b \ell_m + \ell_m^2 \right) + C_{4b} \left( \ell_m + \frac{1}{2} \ell_b \right) + \frac{E_b I}{G_b A_s} C_{3b} + C_{5b} + \gamma. \] (49)

### 4 Computational issues

A FE procedure is developed based on the two-scale model presented in Sects. 2 and 3 and implemented in MATLAB. To model the macrolevel structural problem, a 2-node 2D beam FE is formulated adopting the force-based approach [36] and a Gauss–Lobatto integration rule with \( X_g = X_g \) and \( W_g \) denoting location and weight of the control points. The element forces and displacements are expressed in the basic local reference system, obtained by eliminating the rigid body motions, by the following vectors:
\[ Q = \{ Q_1, Q_2, Q_3 \}^T, \quad q = \{ q_1, q_2, q_3 \}^T, \] (50)

where \( Q_1 \) is the axial force, and \( Q_2 \) and \( Q_3 \) are the bending moments at the end nodes \( i \) and \( j \). Similarly, \( q_1 \) is the axial elongation, and \( q_2 \) and \( q_3 \) are the nodal deformational rotations expressed in the basic system. The equilibrated stress resultants at each integration point, collected in the vector \( S \) introduced in (4), are expressed by the following polynomial interpolation:
\[ S = bQ + S_q, \] (51)

with the vector \( S_q = \{ N_q M_q T_q \}^T \) containing the section stresses due to the external loads distributed along the reference axis, and the equilibrium matrix \( b \) defined as:
\[
\mathbf{b} = \begin{bmatrix}
1 & 0 & 0 \\
0 & X^e/L^e - 1 & X^e/L^e \\
0 & 1/L^e & 1/L^e
\end{bmatrix},
\]

$L^e$ being the length of the beam finite element.

The beam strain and stress vectors, $\mathbf{E}$ and $\mathbf{S}$, are related by means of a generalized constitutive law, which in the force-based approach involves the section flexibility matrix $\mathbf{f}$. According to the two-scale approach, to determine the constitutive response at each integration point along the beam, the analytical solution of the homogenization problem derived in Sect. 3 is used and the stress resultants in (32) are computed.

The numerical solution of the global incremental nonlinear equilibrium equations, governing the response of the 2D frame model, follows a classical step-by-step method and a standard iterative Newton–Raphson algorithm. At the element level, the stiffness matrix $\mathbf{K}$ and the structural reaction force vector $\mathbf{P}$ are computed, performing the element state determination. The adopted methodology proposed for the force-based beam formulation in [36] is schematically illustrated in Table 1. The superscript $'k'$ denotes the value of the variables at the current Newton–Raphson iteration. According to the Newton–Raphson linearization of the global equilibrium equations, the structural tangent stiffness matrix should be computed at each iteration $k$. This would require the evaluation of the tangent stiffness matrix at the element level, by summing the contributions of each Gauss-Lobatto point. These last should result from the homogenization process of the UC. To make easier the computations, the UC initial stiffness is used during the overall step-by-step analysis, resulting as:

\[
\mathbf{k}_{UC} = \begin{bmatrix}
\frac{E_m E_b A \ell}{E_b \ell_m + E_m \ell_b} & 0 & 0 \\
0 & \frac{E_m E_b I \ell}{E_b \ell_m + E_m \ell_b} & 0 \\
0 & 0 & \frac{G_m G_b A_s \ell}{G_b \ell_m + G_m \ell_b}
\end{bmatrix}.
\]

This is used to evaluate the section flexibility $\mathbf{f}$ and, by integrating it over the element length, the element flexibility $\mathbf{F}$. By inverting this last and introducing the rigid body motions by means of the operator $\mathbf{B}$, the element stiffness matrix is computed. After evaluating the element nodal forces $\mathbf{Q}^{[k]}$ on the basis of the current element nodal displacements $\mathbf{q}^{[k]}$ and the element initial flexibility matrix $\mathbf{F}$, resulting as

\[
\mathbf{F} = \int_{L^e} \mathbf{b}^T \mathbf{k}_{UC}^{-1} \mathbf{b} \, dX^e,
\]

\[52\]

**Table 1  Element state determination procedure at the iteration $'k'$**

| \(\mathbf{k}_{UC}\) | Evaluation of the UC homogenized stiffness at the Gauss–Lobatto point |
| \(\mathbf{f} = (\mathbf{k}_{UC})^{-1}\) | Evaluation of the flexibility at the Gauss–Lobatto point |
| \(\mathbf{F} = \frac{L^e}{2} \sum_g \mathbf{b}^T \left( \mathbf{X}_g^e \right) \mathbf{f} \left( \mathbf{X}_g^e \right) \mathbf{b} \left( \mathbf{X}_g^e \right) W_g\) | Evaluation of the element flexibility |
| \(\mathbf{K} = \mathbf{B}^T (\mathbf{F})^{-1} \mathbf{B}\) | Updating of the element stiffness matrix |
| \(\mathbf{Q}^{[k]} = (\mathbf{F})^{-1} \mathbf{q}^{[k]}\) | Updating of the element nodal forces |
| \(\mathbf{S}^{[k]} = \mathbf{b} \mathbf{Q}^{[k]} + \mathbf{S}^{[k-1]}\) | Evaluation of the equilibrated section stresses |
| \(\mathbf{d}^{[k]} = \mathbf{f} \mathbf{S}^{[k]}\) | Evaluation of the section deformations |
| \(\mathbf{S}^{[k]}\) | Evaluation of the constitutive response at the Gauss-Lobatto point |
| \(\mathbf{p}^{[k]}\) | Evaluation of the section residual deformation |
| \(\mathbf{r}^{[k]} = \frac{L^e}{2} \sum_g \mathbf{b}^T \left( \mathbf{X}_g^e \right) \mathbf{d}^{[k]} \left( \mathbf{X}_g^e \right) W_g\) | Updating of the element residual nodal displacement |
| \(\mathbf{Q}^{[k]} = \mathbf{Q}^{[k]} - (\mathbf{F})^{-1} \mathbf{r}^{[k]}\) | Updating of the element nodal forces |
| \(\mathbf{S}^{[k]} = \mathbf{b} \mathbf{Q}^{[k]} - \mathbf{S}^{[k]}\) | Updating of the section residual stress |
| \(\mathbf{p}^{[k]}\) | Updating of the element nodal force vector |
the equilibrated section stresses $S^{[k]}$ are computed at each Gauss-Lobatto integration point. The section residual stress $S^{[k-1]}$, evaluated at the previous iteration and defined below, is added. Then, the section deformation $d^{[k]}$ is updated using the initial section flexibility matrix $f = k_{UC}^{-1}$. The constitutive response is derived at each Gauss point, by solving the homogenization problem, as described in Sect. 3. Due to the presence of the inelastic stresses $P$, $R$ and $V$ (44), Eqs. (37) and (38) are iteratively solved to compute the integration constants and the Lagrange multiplier in (45) and (46), requiring a further iterative inner loop at each Gauss-Lobatto integration point.

Once the perturbation displacement fields in the mortar (45) and brick (46) have been determined, the microstrain components are updated, then the stresses in the mortar (20) and brick (23). Finally, the average microstrain components are updated, then the stresses mortar (45) and brick (46) have been determined, the each Gauss-Lobatto integration point.

A section deformation residual $d^{[k]}$ is then determined, based on the difference between the equilibrated section stress $S^{[k]}$ and the value obtained by the homogenization of the UC micro stresses $S_{UC}^{[k]}$. By integrating this over the element length, the element deformation residual $r^{[k]}$ is computed. By pre-multiplying $r^{[k]}$ by the element stiffness $F^{-1}$, a residual on the element structural reaction forces is calculated, which is used to compute the updated $Q^{[k]}$. The section residual stress $S_{UC}^{[k]}$ can also be updated at each Gauss-Lobatto point, by subtracting from the equilibrated part $bQ^{[k]}$ the stress vector $S_{UC}^{[k]}$. By introducing the rigid body modes on the element stiffness matrix $F^{-1}$ and on the force vector $Q^{[k]}$, the matrix $K$ and the updated $P^{[k]}$ are obtained and passed to the global code for the assembling and solution procedures. As the Newton–Raphson global iterations proceed, the local deformation residual $S_{UC}^{[k]}$ vanishes.

5 Numerical results

To validate the two-scale procedure introduced in the previous Sections, four examples are presented. These concern:

- The validation of the shear failure model for the mortar joint, via comparisons with experimental evidences,
- An investigation on the overall constitutive response of a masonry UC subjected to axial, shear and bending loading conditions,
- The analysis of the response of a structural element, modeled by the proposed beam FE,
- The simulation of an experimental test related to a masonry wallette under out-of-plane loading.

5.1 Experimental shear test on a masonry specimen

This section shows the validation of the damage-plastic constitutive model adopted for the mortar joint, performed on the basis of the experimental outcomes obtained in typical shear tests on couplets [38]. These are deformation controlled joint shear tests combined with normal action (i.e., with confinement).

The sizes of the brick are $h = 204$ mm, $b = 98$ mm and $\ell_b = 50$ mm, while the mortar thickness is $\ell_m = 13.5$ mm. The mechanical parameters adopted in the mortar proposed model, corresponding to test series carried out in [38], have been deduced by Table 47 in Appendix B of [38]. These are here collected in Table 2, where case 1, 2 and 3 refer to the three different values of the applied normal stress, i.e. $\sigma_m = -0.1, -0.5, -1$ MPa, respectively. As for the brick, Young’s modulus is $E_b = 16700$ MPa and Poisson ratio $\nu_b = 0.15$.

Figure 2 shows the numerical curves (black solid lines) compared with the experimental results (shaded areas) for the three different values of the applied normal stress. The most relevant observed phenomena show that the strength and friction values are

| Case 1 | Case 2 | Case 3 |
|--------|--------|--------|
| $E_m$ (MPa) | 1000 | 814 | 568 |
| $v_m$ | 0.15 | 0.15 | 0.15 |
| $\tau_0$ (MPa) | 0.98 | 1.20 | 1.50 |
| $G_{ij}$ (MPa) | 0.023 | 0.040 | 0.075 |
| $\mu$ | 1.08 | 0.8 | 0.78 |
significantly influenced by the pre-compression level and the fracture energy increases for higher values of the normal stress $r_m$, denoting a slower damage progression in the mortar joint.

Although the damage-plastic constitutive model adopted for the mortar joint could be improved accounting for an exponential degradation law, the reduction of the friction coefficient during the damage evolution and the possible interlocking and dilatancy, a good match is obtained for all the cases, considering that the adopted mortar joint model considers a linear softening and constant friction coefficient.

5.2 UC tensile and shear response

The overall response of the masonry UC shown in Fig. 1c is investigated under typical strain histories. The sizes of the brick are $h = 120$ mm, $b = 240$ mm and $\ell_b = 55$ mm, while the mortar thickness is $\ell_m = 10$ mm. The mechanical parameters of the two constituents, bricks and the mortar, are contained in Table 3. A 20-fiber discretization is adopted for the mortar joint.

First, the UC is subjected to a tensile axial strain, with the applied macroscopic strain component, $E$, varying as reported in Table 4. Figure 3 shows the homogenized axial stress, $N$, versus the applied macroscopic strain, $E$. Three curves are reported corresponding to the analysis performed with the proposed 1D model (solid line) and that using the 2D EPS formulation presented by the authors in [4] (line with circles). A little discrepancy between the 1D and 2D solutions emerges, the 2D response resulting a little stiffer than the 1D during the elastic phase and showing a more severe softening post-peak behavior. The difference between the two solutions is due to the interaction between the brick and mortar cross sections in contact, related to the Poisson effect. Such
interaction is intrinsically reproduced by the 2D model, whereas it cannot be captured by the 1D beam model. Indeed, as a consequence of the kinematic assumptions, only the axial and in-plane shear strain and stress components are included in the beam formulation and, thus, no Poisson effect is taken into account. On the other hand, if the 2D analysis is performed by assuming vanishing Poisson ratios for the two constituents (line with stars), i.e. \( v_b = v_m = 0 \), a perfect agreement between the two solutions is achieved.

As concerns the tensile UC response, after the initial elastic branch, damage starts in the horizontal mortar joint undergoing mode I degrading mechanism and a linear softening phase follows. As no plastic flow emerges due to the pure tensile strain state in the mortar, the unloading branch tends to the origin. The axial stress \( N \) vanishes, when the UC is completely damaged.

The second analysis is performed by firstly applying a compressive axial strain, \( E \), kept constant, to the UC, then the macroscopic shear strain, \( \Gamma' \), varying as shown in Table 5. Three different values are considered for \( E \), setting \( c = -1 \times 10^{-4}, -5 \times 10^{-4}, -1 \times 10^{-3} \). The UC shear response is contained in Fig. 4, where the macroscopic homogenized shear, \( T \), is depicted in function of the applied macroscopic shear strain, \( \Gamma' \). The three curves refer to the different values of the compressive axial strain. For the lower value of \( E = -1 \times 10^{-4} \), mode II damage mechanisms are predominant in the post-peak phase (solid line), showing a steeper descending branch, while these are less evident for the higher values of \( E = -5 \times 10^{-4}, -1 \times 10^{-3} \) (dashed and dot line). At the end of the damage evolution, the UC residual strength is reached, whose value increases as the axial compressive strain grows. In the subsequent phase, the UC shows a perfectly-plastic behavior. This is a consequence of the model adopted for the mortar joints. The assumption of a pure frictional behavior after the decohesion is a simplified modeling of the actual mortar joint response. Of course, the model can be enhanced by introducing a degradation law for the friction coefficient and by accounting for the interlocking effect. However, the current version herein adopted satisfactorily describes the experimental shear response of the masonry UC after the first loading-unloading cycles [6, 15].

Table 5 History of the macroscopic axial, \( E \), and shear, \( \Gamma \), strains applied to the masonry UC

| \( t \) | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| \( E \) | c | c | c | 0 |
| \( \Gamma \) | 0 | 3.5 \times 10^{-3} | -2.5 \times 10^{-3} | 0 |

Fig. 4 UC shear constitutive response: macroscopic shear stress \( T \) versus macroscopic shear strain \( \Gamma \) for three values of the axial strain \( E = -1.0 \times 10^{-4}, -5.0 \times 10^{-4}, -1.0 \times 10^{-3} \).

Fig. 5 UC shear constitutive response: macroscopic shear stress \( T \) versus macroscopic shear strain \( \Gamma \) for the axial strain \( E = -5.0 \times 10^{-4} \). Comparison between the 1D (solid line) and 2D EPS (line with circles and stars) solutions.
with the 2D solution including the Poisson effect (line with circles). The little discrepancy is mainly due to the different response of the two models during the first stage of the analysis, where the compressive axial strain is applied to the UC and the Poisson effect is more relevant than in the second stage, during which the UC experiences a shear strain state.

In the last analysis, a macroscopic curvature history is applied to the UC and the response, again, monitored for different values of the axial strain, \( E \). The applied loading history is contained in Table 6. In Fig. 6 the macroscopic couple, \( M \), is plotted against the macroscopic curvature, \( K \). Three curves are shown, corresponding to the axial strain values equal to \( c = 0 \) (solid line), \( c = -1 \times 10^{-4} \) (dashed line) and \( c = -2 \times 10^{-4} \) (dot line). Only damage mechanisms are developed in the mortar joint and Fig. 7 contains the damage distribution at the final step of the analysis in the fibers located along the mortar width. Damage is more severe, when the UC is subjected to a pure flexural state (solid line), while this develops less rapidly for higher values of the compressive axial strain \( E \) (dashed and dot lines). Moreover, in the case of pure flexural strain, a symmetric behavior is observed, as expected, whereas in presence of the axial compressive pre-stress the curves become non symmetric. The comparison with the 2D EPS solution in Fig. 8 gives a perfect agreement both in presence (curve with circles) and in absence (curve with stars) of the Poisson effect, showing that this last plays a negligible role on the UC global flexural response.

### Table 6: History of the macroscopic axial, \( E \), and curvature, \( K \), strains applied to the masonry UC

| \( t \) | 0   | 1   | 2   | 3   |
|-------|-----|-----|-----|-----|
| \( E \) | \( c \) | \( c \) | \( c \) | 0   |
| \( K \) | 0   | \( 1.0 \times 10^{-5} \) | \( -1.0 \times 10^{-5} \) | 0   |

![Fig. 6](image6.png)  
**Fig. 6** UC flexural constitutive response: macroscopic couple \( M \) versus macroscopic curvature \( K \) for three values of the axial strain \( E = 0, -1.0 \times 10^{-4}, -2.0 \times 10^{-4} \)

5.3 Masonry column

The analyzed column shown in Fig. 9a is made of eight clay bricks, whose sizes are \( h = 120 \) mm, \( b = 240 \) mm and \( t_b = 55 \) mm, arranged in stack bond along the column height \( H = 520 \) mm. The mortar thickness \( l_m \) is equal to 10 mm. The UC in Fig. 9b is adopted for the micromechanical analysis.

The mechanical parameters of bricks and mortar are contained in Table 3. The column, completely restrained at the base, is firstly subjected to a distributed vertical load \( q \) applied on the top side. Subsequently, a monotonically increasing horizontal displacement \( u \) is imposed on the top side, left free to rotate. A 4 FE mesh is used, with 3 Gauss-Lobatto points in each element and 20 fibers in the mortar joints. The global response curves of the column are shown in Fig. 10, depicting the horizontal base...
reaction versus the applied horizontal top displacement.

Three different cases are analyzed, corresponding to the distributed vertical load \( q = 0.25, 0.5, 1.0 \) MPa. As the vertical load increases, the column peak load becomes higher and is moved forward. Moreover, the curves are characterized by a steeper softening branch for lower values of \( q \), corresponding to a more severe degrading behavior.

In Fig. 11 the damage distribution along the column base section is shown for two different values of the axial loading, \( q = 0.25 \) MPa (a) and \( q = 1.0 \) MPa (b), at two steps of the loading process, i.e. for the applied top displacement \( u = 0.6 \) mm and \( u = 1.2 \) mm. Note that for the same step of the analysis, the damaged portion of the section is greater in the case of lower vertical load, according to the global response curve trends.

5.4 Experimental wallette under bending

The response of the wallette specimen JO-VER, experimentally tested in [38], is simulated by the proposed 1D model. The experimental setup reproduces the four-point out-of-plane bending test illustrated in Fig. 12a, with the bricks aligned transversally to the flexure plane. The wallette sizes are: height \( H = 675 \) mm, width \( B = 520 \) mm and thickness \( T = 100 \) mm (Fig. 12a). The sizes of the clay bricks are: \( h = 100 \) mm, \( b = 200 \) mm and \( t_b = 50 \) mm, and the mortar thickness \( t_m \) is equal to 12.5 mm. The material parameters used in the numerical simulation have been deduced from [38] and are contained in Table 7. Note that the values of the tensile and shear fracture energies given in [38] have been divided by the area.
of the three bed mortar joints involved in the central beam region, where the moment is constant.

To reproduce the experimental setup and account for the symmetry of the problem, one half of the specimen is modeled by 2 beam FEIs, applying the concentrated load to the inter-elements node (Fig. 12b). Therefore, the central FE is subjected to a constant maximum moment.

Figure 13 contains the experimental (red line) and numerical (black line) response curves, depicting the moment per unit width versus the curvature. The shaded area corresponds to the experimental confidence region, where the upper and lower bound linear-parabola curves have been evaluated by adopting the upper and lower values of the confidence intervals for the parameters, also proposed in [35]. It emerges that the numerical curve is contained in the confidence area, departing a bit from the experimental curve in the nonlinear range. The numerical model, although very satisfactorily describing the initial stiffness and the collapse mechanism, underestimates the wallette strength, giving a slower evolution of the damage in the specimen with respect to the experimental evidence. This is probably due to the presence of material...
and geometrical imperfections in the experimental wallette not introduced in the numerical model.

It is worth noting that, considering the structural and UC sizes in this example, the principle of separation of macro and micro scales, on which the homogenization technique and the multiscale model are based, is not complied with. Despite this lack of theoretical support, the numerical results are reliable and in good accordance with the experimental evidences.

6 Conclusions

A two-scale model for masonry 1D structural elements has been presented, adopting beam models at both the macro and micro-scales and a damage-friction constitutive behavior for the mortar joints. The nonlinear homogenization procedure based on the periodicity assumptions imposed on the microscopic displacement field, applied to the Timoshenko beam formulation, has given rise to an additional condition on the UC average rotation field. Similar conditions have been derived in the thick shell model proposed in [31]. Moreover, by enforcing the stationarity of the variational energy functional associated to the UC micromechanical BVP, it has emerged that this additional periodicity constraint is work-conjugate to a uniformly distributed couple field. This last is essential to ensure the equilibrium of the UC, as analytically shown for the homogeneous UC.

Thanks to some simplified assumptions on the damage and plasticity distribution, a semi-analytical solution of the micromechanical problem has been proposed, based on the fiber discretization of the mortar joints. At the structural level, a beam force-based FE formulation has been adopted, requiring an iterative element state determination, when introduced in a displacement-based global FE procedure. Both the element state determination and the iterative procedure to solve the UC micromechanical problem have been proved to be efficient and accurate. In particular, the adopted force-based FE has allowed to determine the nonlinear solution with a low number of global and local iterations.

The homogenized response of the UC, obtained by solving the nonlinear homogenization problem with the proposed semi-analytical approach, has been explored in details by considering the main typical loading conditions. The comparison with a more sophisticated 2D EPS model has proved that the 1D model is able to accurately describe the UC nonlinear behavior, mainly when the Poisson interaction between brick and mortar does not significantly influence the response, as in the shear and flexural mechanisms. Moreover, the numerical results have shown that the proposed model is able to reproduce the structural response of masonry 1D elements, like columns. Indeed, both the global and local aspects of the response are satisfactorily described. In particular, the effect of the vertical loads on the response is taken into account, influencing both the peak load and the post-peak behavior, the damage and plasticity distribution along the masonry column cross sections being correctly reproduced. The comparison with experimental outcomes both for the UC shear response and for the masonry wallette tested in a four-point bending condition have
further proven that the proposed beam model is suitable for reproducing the main mechanisms involved in the masonry collapse and its global response with accuracy and very low computational burden.

Starting from the proposed model, a 2D shell formulation will be developed aiming to describe the out-of-plane response of masonry walls in more general conditions.

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Compliance with ethical standards
Conflicts of interest The authors declare that they have no conflict of interest.

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