Neutrino Mixing based on Mass Matrices 
with a $2 \leftrightarrow 3$ Symmetry

Yoshio Koide and Eiichi Takasugi

Institute for Higher Education Research and Practice, Osaka University,
1-16 Machikaneyama, Toyonaka, Osaka 560-0043, Japan

Abstract

Under an assumption that the $2 \leftrightarrow 3$ symmetry is broken only through phases, we give a systematical investigation of possible lepton mass matrix forms without referring to explicit parameter values. Two types of the $2 \leftrightarrow 3$ symmetry are investigated: one is that the left- and right-handed fields ($f_L, f_R$) obey the symmetry, and another one is that only $f_L$ obeys the symmetry. In latter case, in spite of no $2 \leftrightarrow 3$ symmetry in the Majorana mass matrix $M_R$ for $\nu_R$, the neutrino seesaw mass matrix still obeys the $2 \leftrightarrow 3$ symmetry. Possible phenomenology is discussed.

1 Introduction

We usually consider that the quarks and leptons should be understood by a unification theory. Then, the concept of “symmetry” will become important in the understanding of “flavor”. It is well-known that the requirement of the $2 \leftrightarrow 3$ symmetry [1] for the neutrino mass matrix leads to the maximal mixing between the $\nu_2$ and $\nu_3$ components. The idea of the $2 \leftrightarrow 3$ symmetry is very promising for understanding the observed neutrino mixing.

When a mass matrix $M$ satisfies the relation

$$T_{23}MT_{23}^T = M, \quad (1.1)$$

where $T_{23}$ is defined as

$$T_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (1.2)$$

the matrix $M$ is called “$2 \leftrightarrow 3$ symmetric”. The matrix form of $M$ is explicitly expressed as

$$M = \begin{pmatrix} d & a & a \\ a & b & c \\ a & c & b \end{pmatrix}. \quad (1.3)$$

First, we would like to notice that the mass matrix which satisfies Eq.(1.1) is considered a consequence of the invariance of the mass matrix under the field transformation. Explicitly, for the Dirac mass matrix $\bar{f}_L M f_R$, Eq.(1.1) is derived by requiring the invariance under the transformation, $f_L \rightarrow T_{23}^{\dagger} f_L$ and $f_R \rightarrow T_{23} f_R$. This is true if the neutrino mass matrix is derived by the seesaw mechanism, $M_\nu = m_D M_R^{-1} m_D^T$ because this matrix $M_\nu$ is invariant under $f_R \rightarrow T f_R$ with any $T$. Next, we focus on the transformation of $\nu_L$ and $e_L$. Since they forms
a doublet of the electroweak symmetry, the transformation for them should be the same. That makes a big trouble to realize the reasonable neutrino mixing as we see in the next section.

Now we extend the $2 \leftrightarrow 3$ symmetry according to multiplets under the electroweak symmetry. In general, the transformation between $(\nu_L, e_L)$ and $\nu_R$ are different. This is true even we consider the SU(5) GUT. On the other hand, in the SO(10) GUT, $(\nu_L, e_L)$ and $(\nu_R, e_R)$ will be transformed under the same operator $T_{23}$. According to this classification, two types of $2 \leftrightarrow 3$ symmetry arise. The one (we call it Type I) is that both $f_L$ and $f_R$ obey the $2 \leftrightarrow 3$ symmetry.

Since, in the neutrino sector, the Dirac neutrino mass matrix $M_L$ and the Majorana mass matrix $M_R$ for $\nu_R$ satisfy the relations $T_{23}M_LT_{23}^\dagger = M_L$ and $T_{23}M_RT_{23} = M_R$, we will find that the neutrino mass matrix $M_\nu = M_L M_R^{-1} (M_L)^T$ is also given by the form (1.3).

The other one (we call it Type II) is a case where only $f_L$ obeys the $2 \leftrightarrow 3$ symmetry. Then, we find for the Dirac mass matrix $M_f^L$ (we define a Dirac mass matrix $M_f^L$ as $\bar{f}_L M_f^L f_R$)

$$T_{23} M_f^L = M_f^L.$$

and the explicit form of the mass matrix $M_f^L$ is given by

$$M_f^L = \begin{pmatrix} a_1 & b_1 & c_1 \\ a & b & c \\ a & b & c \end{pmatrix}.$$ (1.5)

The neutrino mass matrix $M_\nu = M_f^L M_R^{-1} (M_f^L)^T$ is given as a special case of Eq.(1.3) by taking $b = c$ as we shall see later.

Note that, in the both types I and II, the Hermitian matrix defined by $H_f = M_f M_f^\dagger$ satisfies the constraint

$$T_{23} H_f T_{23}^\dagger = H_f,$$ (1.6)

independently whether the mass matrix has the form in Eq.(1.3) or (1.5).

Now the neutrino mixing matrix $U$ is given by

$$U = U_{Le} U_{L\nu},$$ (1.7)

where $U_{Lf}$ are defined by

$$U_{Lf}^\dagger H_f U_{Lf} = \text{diag}(m_{f1}^2, m_{f2}^2, m_{f3}^2) = D_f^2.$$ (1.8)

From the argument given above, we learned that as far as the mixing matrix $U$ is concerned, the structure of the neutrino mixing matrix is independent of the mass matrices of Types I or II. Only difference arises in the mass spectrum.

The purposes of the present paper is to investigate the general properties of the models with the $2 \leftrightarrow 3$ symmetry, paying attention to the difference between types I and II, and taking relations to the grand unification (GUT) scenarios into consideration. Although we investigate the masses and mixings in the lepton sectors, the formulation in this paper is also applicable
to the quark sectors. Since, in the quark sectors, there is essentially no complexity about the mass spectrum such as the inverse hierarchy as in the neutrino sector, the application is more straightforward. Therefore, we will investigate only the lepton sectors in this paper.

2 Extended 2 ↔ 3 symmetry and the neutrino mixing

In this section, we will demonstrate that the 2 ↔ 3 symmetry in the exact meaning cannot explain the observed neutrino mixing. For the convenience of the discussion in later, let us introduce the so-called extended 2 ↔ 3 operator $T_{23}(2\delta)$ [2]

$$T_{23}(2\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{i2\delta} \\ 0 & e^{-i2\delta} & 0 \end{pmatrix}, \quad (2.1)$$

instead of the operator (1.2). The operator $T_{23}(2\delta)$ is unitary and Hermitian. We obtain the constraint

$$T_{23}(2\delta)MM^\dagger T_{23}^\dagger(2\delta) = MM^\dagger, \quad (2.2)$$

for the Hermitian matrix $MM^\dagger$ irrespective of Type I or II. Note that we can express the operator (2.1) as

$$T_{23}(2\delta) = P_{23}(2\delta)T_{23} = P_{23}(\delta)T_{23}P_{23}^\dagger(\delta) = T_{23}P_{23}^\dagger(2\delta), \quad (2.3)$$

where $T_{23} = T_{23}(0)$ and

$$P_{23}(\delta) = \text{diag}(1, e^{i\delta}, e^{-i\delta}). \quad (2.4)$$

Therefore, we can express the constraint (2.2) as

$$MM^\dagger = P_{23}(\delta)T_{23}P_{23}^\dagger(\delta)MM^\dagger P_{23}(\delta)T_{23}P_{23}^\dagger(\delta). \quad (2.5)$$

Now we define

$$H = P_{23}^\dagger(\delta)MM^\dagger P_{23}(\delta), \quad (2.6)$$

then we find

$$H = T_{23}HT_{23}, \quad (2.7)$$

where $H$ is a Hermitian matrix. In general, the Hermitian matrix $H$ which satisfies the constraint (2.6) can be expressed by the form

$$H = \begin{pmatrix} D & Ae^{i\phi} & Ae^{i\phi} \\ Ae^{-i\phi} & B & C \\ Ae^{-i\phi} & C & B \end{pmatrix}, \quad (2.8)$$

where $A, B, C$ and $D$ are real, so that $H$ can be transformed to a real matrix $\tilde{H}$ as

$$P_{1}^\dagger(\phi)HP_{1}(\phi) = \tilde{H}, \quad (2.9)$$
where
\[ P_1(\phi) = \text{diag}(e^{i\phi}, 1, 1). \] (2.10)

It is also well-known that the 2 ↔ 3 symmetric real matrix \( \tilde{H} \) is diagonalized by a rotation \( R(\theta) \) as
\[ R^T(\theta)\tilde{H}R(\theta) = \tilde{H}_D \equiv \text{diag}(m_1^2, m_2^2, m_3^2), \] (2.11)
where
\[ R(\theta) = \begin{pmatrix} c_\theta & s_\theta & 0 \\ -\frac{1}{\sqrt{2}} s_\theta & \frac{1}{\sqrt{2}} c_\theta & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} c_\theta & \frac{1}{\sqrt{2}} s_\theta & \frac{1}{\sqrt{2}} \end{pmatrix}, \] (2.12)
and
\[ s_\theta \equiv \sin \theta = \sqrt{\frac{D - m_1^2}{m_2^2 - m_1^2}}, \quad c_\theta \equiv \cos \theta = \sqrt{\frac{m_2^2 - D}{m_2^2 - m_1^2}}, \] (2.13)

\[ m_1^2 = \frac{1}{2} \left( B + C + D - \sqrt{8A^2 + (B + C - D)^2} \right), \]
\[ m_2^2 = \frac{1}{2} \left( B + C + D + \sqrt{8A^2 + (B + C - D)^2} \right), \]
\[ m_3^2 = B - C. \] (2.14)

As a result, the Hermitian matrix \( MM^\dagger \) is diagonalized by
\[ U = P_{23}(\delta)P_1(\phi)R(\theta), \] (2.15)
as
\[ U^\dagger MM^\dagger U = \tilde{H}_D. \] (2.16)

Although we did not consider the size of masses, the ordering of them is needed. Therefore, the unitary matrix to diagonalize the mass matrix in an proper mass ordering is given by \( UT \), where \( T \) is the matrix to exchange the mass ordering. Then, we find the neutrino mixing matrix defined by (1.7) as
\[ U = U_e^\dagger U_\nu = T_e^T R^T(\theta_\nu)P_\nu^\dagger P_{\nu \ell} R(\theta_\ell)T_\ell, \] (2.17)
where
\[ P_\ell = P_{23}(\delta_\ell)P_1(\phi_\ell) = \text{diag}(e^{i\delta_\ell}, e^{i\delta_\ell}, e^{-i\delta_\ell}). \] (2.18)

Here, we recall that the operation (2.1) must be the same for \( \nu_L \) and \( e_L \); so that, in the expression, \( \delta_e \) is exactly equal to \( \delta_\nu \). Therefore, we obtain
\[ U = T_e^T U_\nu T_\ell \equiv T_e^T \begin{pmatrix} s_\ell s_\nu + c_\ell c_\nu e^{i\phi} & -s_\ell c_\nu + c_\ell s_\nu e^{i\phi} & 0 \\ -c_\ell s_\nu + s_\ell c_\nu e^{i\phi} & c_\ell c_\nu + s_\ell s_\nu e^{i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} T_\nu, \] (2.19)

where \( \phi = \phi_\nu - \phi_\ell \). Obviously, the mixing matrix (2.19) cannot give the observed values \([3, 4]\] \( \tan^2 \theta_{12} \simeq 1/2 \) and \( \sin^2 2\theta_{23} \simeq 1 \) simultaneously. (It is a general feature for any flavor symmetry
with a transformation $f_L \rightarrow U_X f_L$ that we obtain only a family-mixing between two families. See Ref.[5].

3 Extended $2 \leftrightarrow 3$ symmetry with the breaking term

We saw in the previous section that the $2 \leftrightarrow 3$ symmetry which arises as a consequence of the transformation for fields cannot reproduce the observed neutrino mixing. However, we consider that the $2 \leftrightarrow 3$ symmetry is still useful from the phenomenological point of view. Therefore, from the phenomenological point of view, we assume [6] that the $2 \leftrightarrow 3$ symmetry is broken only through the phase parameters. Hereafter, we will use the extended $2 \leftrightarrow 3$ symmetry operator (2.1) in the phenomenological meaning, and we will consider the case $\delta_e \neq \delta_\nu$ in the left-handed sectors.

3.1) Charged lepton mass spectrum

First, we investigate the $2 \leftrightarrow 3$ symmetry of Type II. The mass matrix $M^e_L$ for the charged leptons must also satisfy the relation

$$T_{23}(2\delta)M^e_L = M^e_L,$$

where, for convenience, we have dropped the index “e” from $\delta_e$. Then, the explicit form of $M^e_L$ is also given by

$$M^e_L = \begin{pmatrix} a_1 & b_1 & c_1 \\ a e^{i\delta} & b e^{i\delta} & c e^{i\delta} \\ a e^{-i\delta} & b e^{-i\delta} & c e^{-i\delta} \end{pmatrix} = P_{23}(\delta) \begin{pmatrix} a_1 & b_1 & c_1 \\ a & b & c \\ a & b & c \end{pmatrix},$$

where the parameters $a, b, \cdots$ in $M^e_L$ can be complex. Therefore, we obtain the Hermitian matrix

$$M^e_L(M^e_L)^\dagger = P_{23}(\delta) \begin{pmatrix} D & A e^{i\phi} & A e^{i\phi} \\ A e^{-i\phi} & B & B \\ A e^{-i\phi} & B & B \end{pmatrix} P_{23}(\delta)^\dagger,$$

where

$$A = |a a_1^* + b b_1^* + c c_1^*|,$$

$$B = |a|^2 + |b|^2 + |c|^2,$$

$$D = |a_1|^2 + |b_1|^2 + |c_1|^2.$$ (3.4)

Then, we can obtain a real matrix $\tilde{H}_e$ as

$$\tilde{H}_e = P_1^\dagger(\phi)P_{23}^\dagger(\delta)M^e_L(M^e_L)^\dagger P_{23}(\delta)P_1(\phi).$$ (3.5)

From the formula (2.14), we obtain

$$m_{e3} = 0,$$ (3.6)

because of $B = C$ in this case. Therefore, Type II transformation in charged lepton sector cannot give a realistic mass spectrum.
Next, we investigate the case of Type I, i.e.

$$\ell_L \rightarrow T_{23}(2\delta_L)\ell_L, \quad \epsilon_R \rightarrow T_{23}(2\delta_R)\epsilon_R. \quad (3.7)$$

The case (3.7) may be realized in an SU(5)-GUT model\(^1\). In this case, instead of the constraint (3.1), we have the constraint

$$T_{23}(2\delta_L)M_L^\epsilon T_{23}(2\delta_R) = M_L^\epsilon. \quad (3.8)$$

The explicit form of $M_L^\epsilon$ is given by

$$M_L^\epsilon = \begin{pmatrix} d & a'e^{-i\delta_R} & a'e^{i\delta_R} \\ ae^{i\delta_L} & be^{i(\delta_L - \delta_R)} & ce^{-i(\delta_L + \delta_R)} \\ ae^{-i\delta_L} & ce^{i(\delta_L + \delta_R)} & be^{-i(\delta_L - \delta_R)} \end{pmatrix} = \begin{pmatrix} d & a' & a' \\ a & b & c \\ a & c & b \end{pmatrix} P_{23}^\dagger(\delta_R), \quad (3.9)$$

so that we obtain

$$M_L^\epsilon(M_L^\epsilon)^\dagger = P_{23}(\delta_L) \begin{pmatrix} D & A e^{i\phi} & A e^{i\phi} \\ A e^{-i\phi} & B & C \\ A e^{-i\phi} & C & B \end{pmatrix} P_{23}^\dagger(\delta_L), \quad (3.10)$$

where

$$A = |ad' + (b + c)a'\epsilon|, \quad B = |a|^2 + |b|^2 + |c|^2, \quad C = |a|^2 + 2|b||c|\cos(\beta - \gamma), \quad D = |d|^2 + 2|a'|^2, \quad (3.11)$$

where $\beta$ and $\gamma$ are defined by $b = |b|e^{i\beta}$ and $c = |c|e^{i\gamma}$, respectively. Therefore, since

$$m_{e3}^2 = B - C = |b|^2 + |c|^2 - 2|b||c|\cos(\beta - \gamma) = |b - c|^2, \quad (3.12)$$

we can obtain $m_{e3} \neq 0$ when $b \neq c$.

In both cases, Types I and II, the Hermitian matrix $M_L^\epsilon(M_L^\epsilon)^\dagger$ is diagonalized by a unitary matrix

$$U_e = P_{23}(\delta_e)P_1(\phi_e)R(\theta_e), \quad (3.13)$$

as

$$U_e^\dagger M_L^\epsilon(M_L^\epsilon)^\dagger U_e = D_\epsilon^2 \equiv (m_{e1}^2, m_{e2}^2, m_{e3}^2). \quad (3.14)$$

### 3.2 Neutrino mass spectrum

\(^1\)In SU(5) GUT models, the matter fields $(\nu_L, \epsilon_L)$ and $\nu_R^c$ are assigned to $\bar{5}_L$ and $10_L$ of SU(5), respectively. Therefore, in general, flavor symmetries for $\bar{5}_L$ and $10_L$ can be different from each other. However, in this paper, we consider a specific case in which $\bar{5}_L$ and $10_L$ obey the same flavor symmetry. Hereafter, we call it an SU(5) $\bar{5}_L + 10_L$ model for convenience.
We consider that the neutrino masses are generated by the seesaw mechanism

\[ M_\nu = M'_L M_R^{-1} (M'_L)^T, \]  

(3.15)

where \( M'_L \) and \( M_R \) are defined by \( \bar{\nu}_L M'_L \nu_R \) and \( \bar{\nu}_R^c M_R \nu_R \) \((\nu_R^c \equiv C \bar{\nu}_R^T)\), respectively. The Dirac mass matrix \( M'_L \) is given by the form similar to (3.9) or (3.2) according as Type-I or Type-II. In Type-II, we obtain the neutrino mass matrix form

\[ M_\nu = P_{23}(\delta) \begin{pmatrix} D & A & A \\ A & B & B \\ A & B & B \end{pmatrix} P_{23}(\delta), \]  

(3.16)

where

\begin{align*}
A &= a a_1 d_R^{-1} + b b_1 b_R^{-1} c_c b_R^{-1} + (a b_1 + a_1 b) a_R^{-1} + (a c_1 + a_1 c) a_R^{-1} + (b_1 c_1 + b c_1) c_R, \\
B &= b^2 b_R^{-1} + c^2 b_R^{-1} + a^2 d_R + 2 b c c_R^{-1} + 2 a b a_R^{-1} + 2 a c a_R^{-1}, \\
D &= b^2 b_R^{-1} + c^2 b_R^{-1} + a^2 d_R + 2 b_1 c_1 c_R^{-1} + 2 a_1 b_1 a_R^{-1} + 2 a_1 c_2 a_R^{-1},
\end{align*}

(3.17)

Since the neutrino masses \( m_{\nu_i} \) in Type-II are given by

\begin{align*}
m_{\nu_1} &= \frac{1}{2} \left( B + C + D - \sqrt{8 A^2 + (B + C - D)^2} \right), \\
m_{\nu_2} &= \frac{1}{2} \left( B + C + D + \sqrt{8 A^2 + (B + C - D)^2} \right), \\
m_{\nu_3} &= B - C,
\end{align*}

(3.19)

with \( C = B \), we obtain

\[ m_{\nu_3} = 0. \]  

(3.20)

On the other, in Type I, such the constraint (3.20) does not appear.

In both cases, Types I and II, the Hermitian matrix \( M_\nu M_\nu^T \) is diagonalized by a unitary matrix

\[ U_\nu = P_{23}(\delta_\nu) P_1(\phi_\nu) R(\theta_\nu), \]  

(3.21)

as

\[ U_\nu^T M_\nu M_\nu^T U_\nu = D_\nu^2 \equiv (m_{\nu_1}^2, m_{\nu_2}^2, m_{\nu_3}^2), \]  

(3.22)

where \( R(\theta_\nu) \) is defined by Eq.(2.12) with

\begin{align*}
s_\nu &\equiv \sin \theta_\nu = \sqrt{\frac{D - m_{\nu_1}}{m_{\nu_2} - m_{\nu_1}}}, \\
c_\nu &\equiv \cos \theta_\nu = \sqrt{\frac{m_{\nu_2} - D}{m_{\nu_2} - m_{\nu_1}}},
\end{align*}

(3.23)
3.3) Neutrino mixing matrix

So far, we have used the notation \((f_1, f_2, f_3)\) for the mass eigenstates of the fundamental fermions \(f\), whose masses \(m_{f_i}\) have been defined by Eq.(2.14). Hereafter, in order to distinguish the mass-eigenstates \((e, \mu, \tau)\) and \((\nu_1, \nu_2, \nu_3)\) in the conventional notations from the mass-eigenstates whose masses \(m_i\) are defined by Eq.(2.14), we denote the states whose masses are defined by Eq.(2.14) as \(f_0\). The states \((\nu_1, \nu_2, \nu_3)\) and \((\nu_e, \nu_\mu, \nu_\tau)\), which is the SU(2)\(_L\) partner of the charged lepton state \((e, \mu, \tau)\), are related by

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\end{pmatrix} = U 
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\end{pmatrix}, \tag{3.24}
\]

with the neutrino mixing matrix \(U\) in the conventional notation. Here, the neutrino mixing matrix \(U\) in Eq.(3.24) is given by

\[
U = U_{\nu}^\dagger U_{\nu}. \tag{3.25}
\]

On the other hand, as seen in Secs.2 and 3, the mass matrices \(M_\nu M_\nu^\dagger\) and \(M_e L_e^\dagger\) are diagonalized by unitary matrices (3.21) and (3.13) (we denote them \(U_0\)) and \(U_{0e}\), respectively. When we define the mixing matrix

\[
U_0 = U_{0\nu} U_{0e} = R^T(\theta_{\nu}) PR(\theta_{\nu}), \tag{3.26}
\]

where

\[
P = \text{diag}(e^{i\phi}, e^{i\delta}, e^{-i\delta}), \tag{3.27}
\]

\(\phi = \phi_{\nu} - \phi_{e}\) and \(\delta = \delta_{\nu} - \delta_{e}\). the mixing matrix \(U_0\) does not always denote the observed neutrino mixing matrix \(U\). When we define the observed fermions \((e, \mu, \tau)\) and \((\nu_1, \nu_2, \nu_3)\) as

\[
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\end{pmatrix} = T_{ijk} \begin{pmatrix}
\nu_{01} \\
\nu_{02} \\
\nu_{03} \\
\end{pmatrix}, \quad \begin{pmatrix}
e \\
\mu \\
\tau \\
\end{pmatrix} = T_{lmn} \begin{pmatrix}
\nu_{01} \\
\nu_{02} \\
\nu_{03} \\
\end{pmatrix}, \tag{3.28}
\]

the observed neutrino mixing matrix \(U\) is given by

\[
U = T_{lmn} U_0 T_{ijk}^T, \tag{3.29}
\]

where \(T_{ijk}\) denotes the exchange operator \((f_{i1}^0, f_{i2}^0, f_{i3}^0) \rightarrow (f_{j1}^0, f_{j2}^0, f_{j3}^0)\). However, as we discuss below, the possible choices of \(T_{ijk}\) are not so many.

The explicit form of the matrix \(U_0\) is given by

\[
U_0 = \begin{pmatrix}
c_e c_{\nu} e^{i\phi} + s_e s_{\nu} \cos \delta & c_e s_{\nu} e^{i\phi} - s_e c_{\nu} \cos \delta & i s_e \sin \delta \\
 s_e c_{\nu} e^{i\phi} - c_e s_{\nu} \cos \delta & s_e s_{\nu} e^{i\phi} + c_e c_{\nu} \cos \delta & -i c_e \sin \delta \\
is_{\nu} \sin \delta & -i c_{\nu} \sin \delta & \cos \delta \\
\end{pmatrix}. \tag{3.30}
\]
Table 1: Possible constraints on the Dirac mass matrices $m^f_L$: Models A, B, C, and D are defined according as the constraint types.

| Type     | Type II for $M^f_L$                  | Type I for $M^f_L$                  |
|----------|-------------------------------------|-------------------------------------|
| Type II for $M^f_L$ | Model A: non-GUT type               | Model D: unrealistic                 |
|          | $m_{e3} = m_{\nu_3} = 0$            | $m_{e3} = 0 \& m_{\nu_3} \neq 0$   |
| Type I for $M^f_L$ | Model B: SU(5) 5_L + 10_L model     | Model C: SO(10)-GUT type            |
|          | $m_{e3} \neq 0 \& m_{\nu_3} = 0$   | $m_{e3} \neq 0 \& m_{\nu_3} \neq 0$ |

Obviously, the cases (3.29) with $\delta = 0$ are ruled out as we have already discussed in Sec.2.

For convenient, we name Models A, B, C and D for combinations of Types I and II for $M^f_L$ and $M^f_L$ as shown in Table 1. In Model A, since only the left-handed fields $f_L$ obey the $2 \leftrightarrow 3$ symmetry, the model cannot be embedded into a GUT scenario. In Model B, the fields $\ell_L = (\nu_L, e_L)$ and $e_R$ obey the $2 \leftrightarrow 3$ symmetry, but the field $\nu_R$ is free from the symmetry, so that the model can be embedded into SU(5) GUT [see a footnote below Eq.(3.71)]. In Model C, all fields $\ell_L = (\nu_L, e_L)$, $e_R$ and $\nu_R$ obey the $2 \leftrightarrow 3$ symmetry, so that the model can be embedded into SO(10) GUT. Model D is unlikely, so that we will not investigate this case.

In Models A and D with Type-II symmetry in the charged lepton sector, we obtain $m_{e3} = 0$, so that the cases are ruled out.

In Model B (an SU(5) 5_L + 10_L model), we can obtain $m_{e3} \simeq 0$ (but $m_{e3} \neq 0$) because of $b \simeq c$. (In Model B, although we can, in principle, consider any value of $m_{e3}$, we have assumed $b \simeq c$ because the case $b \simeq c$ can reasonably be realized in most practical models with $2 \leftrightarrow 3$ symmetry.) Therefore, we may suppose a case $m^2_{e2} > m^2_{e1} > m^2_{e3}$ in the model. Such the case means the assignment

$$
\begin{pmatrix}
  e \\
  \mu \\
  \tau
\end{pmatrix}_L =
\begin{pmatrix}
  e^0_3 \\
  e^0_1 \\
  e^0_2
\end{pmatrix}_L = \begin{pmatrix}
  0 & 0 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix} \begin{pmatrix}
  e^0_1 \\
  e^0_2 \\
  e^0_3
\end{pmatrix}_L \equiv T_{312} \begin{pmatrix}
  e^0_1 \\
  e^0_2 \\
  e^0_3
\end{pmatrix}_L .
$$

(3.31)

Then, from the relation $U_e = U_{e0} T_{312}^T$, the observed neutrino mixing matrix $U$ is described by

$$
U = T_{312} U_0 = \begin{pmatrix}
  i s_\nu \sin \delta & -i c_\nu \sin \delta & \cos \delta \\
  c_\nu c_\nu e^{i\phi} + s_\nu s_\nu \cos \delta & c_\nu s_\nu e^{i\phi} - s_\nu c_\nu \cos \delta & i s_\nu \sin \delta \\
  s_\nu c_\nu e^{i\phi} - c_\nu s_\nu \cos \delta & s_\nu s_\nu e^{i\phi} + c_\nu c_\nu \cos \delta & -i c_\nu \sin \delta
\end{pmatrix},
$$

(3.32)

if we regard the observed neutrino states $(\nu_1, \nu_2, \nu_3)$ as $(\nu^0_1, \nu^0_2, \nu^0_3)$ with $m_{\nu_3} = 0$, whose case corresponds to the inverse hierarchy. (Such an inverted assignment between up- and down-sectors was first proposed by Matsuda and Nishiura [7].) The case (3.32) predicts

$$
\tan^2 \theta_{\text{solar}} = \frac{|U_{12}|^2}{|U_{11}|^2} = \frac{m_{\nu_2}^2 - m_{\nu_1}^2}{s^2_{\nu} D_{\nu} - m_{\nu_1}^2},
$$

(3.33)
\[
\sin^2 2\theta_{\text{atm}} = 4|U_{23}|^2|U_{33}|^2 = \sin^2 2\theta_e \sin^4 \delta = \sin^2 2\theta_e (1 - |U_{13}|^2)^2, \tag{3.34}
\]
where \(s_e\) and \(c_e\) are given by Eq.(2.13). In order to give \(|U_{13}|^2 \approx 0\), the condition \(\cos \delta \approx 0\) (\(\delta \approx \pi/2\)) is required. In order to \(\sin^2 2\theta_e = 1\) (\(s_e^2 = c_e^2 = 1/2\)), the relation \(2D_e = m_{e1}^2 + m_{e2}^2\) (i.e. \(D_e = B_e + C_e\)) is required from Eq.(2.14). Then, the masses (2.14) are given by

\[
m_{e3}^2 = B_e - C_e = |b_e - c_e|^2, \\
m_{e1}^2 = D_e - \sqrt{2}A_e, \\
m_{e2}^2 = D_e + \sqrt{2}A_e. \tag{3.35}
\]

Therefore, a suitable choice of the parameter values of \(M_
u^c\) can give \(\sin^2 2\theta_e = 1\) keeping \(m_{e2}^2 > m_{e1}^2 > m_{e3}^2\). Also, a suitable choice of the parameter values of \(M_\nu\) can give a reasonable value of (3.33). If these conditions are satisfied, the model B is preferable. However, note that the parameter value \(\delta \approx \pi/2\) cannot be realized unless \(\text{SU}(2)_L\) is broken.

By the way, the case \(m_{e3} = 0\) does not always mean the inverse hierarchy of neutrino masses. At present, as far as the observed neutrino masses \(m_\nu\) satisfy the relation \((m_{e2}^2 - m_{e1}^2)/(m_{e3}^2 - m_{e2}^2) \approx 10^{-2}\), we may consider any cases \(U = T_{312}U_0T_{312}^T\). Therefore, even the case \(m_{e3} = 0\), we can consider a case of the normal hierarchy: \((\nu_1, \nu_2, \nu_3) = (\nu_0^1, \nu_0^2, \nu_0^3)\). Then, in Model B with \(c_e \simeq b_e\), the neutrino mixing matrix \(U\) is given by

\[
U = T_{312}U_0T_{312}^T = \begin{pmatrix}
\cos \delta & i s_\nu \sin \delta & -i c_\nu \sin \delta \\
- i c_\nu \sin \delta & c_\nu c_\rho \cos \delta & s_\nu s_\rho e^{i \phi} - c_\nu c_\rho \cos \delta \\
- i s_\nu \sin \delta & s_\nu c_\rho e^{i \phi} - c_\nu s_\rho \cos \delta & s_\nu s_\rho e^{i \phi} + c_\nu c_\rho \cos \delta
\end{pmatrix}. \tag{3.36}
\]

In order to give \(\tan^2 \theta_{\text{solar}} \approx 1/2\) and \(\sin^2 2\theta_{\text{atm}} \approx 1\), we have to consider \(c_\nu \approx 0\). From the expression (3.23), the limit of \(c_\nu = 0\) requires \(m_{e2} = D_\nu\), which leads \(A_\nu = 0\) and gives the mass spectrum \(m_{e1} = D_\nu\), \(m_{e2} = 2B_\nu\) and \(m_{e3} = 0\). If we choose \(B_\nu^2 \gg D_\nu^2\) in the neutrino sector, we can give a reasonable value of \(R = \Delta m_{\text{solar}}^2/\Delta m_{\text{atm}}^2\) because of \(R = (m_2^2 - m_1^2)/(m_2^2 - m_3^2) = D_\nu^2/(4B_\nu^2 - D_\nu^2)\) in the normal mass hierarchy. Therefore, we cannot rule out this case (Model B with \(m_{e2}^2 \gg m_{e1}^2 \gg m_{e3}^2\) and \(m_{e2}^2 \gg m_{e1}^2 \gg m_{e3}^2\) in a normal hierarchy). However, we must accept a phenomenological value \(\tan^2 \delta \approx 1/2\) (\(\delta \approx 35.3^\circ\)) in order to understand \(\tan^2 \theta_{\text{solar}} \approx 1/2\).

So far, we have consider the case with \(c_e \simeq b_e\) (i.e. \(m_{e2}^2 \ll m_{e1}^2 \ll m_{e3}^2\)) for the charged lepton masses in Model B. We can also consider the case \(m_{e1}^2 \ll m_{e2}^2 \ll m_{e3}^2\) in Model B. In Model B, the neutrino masses are still given by \(m_{e3}^2 = 0 < m_{e1}^2 < m_{e2}^2\), so that the cases \(U = T_{123}U_0T_{123}^T\) and \(U = T_{123}U_0T_{123}^T\) correspond to the normal and inverse hierarchies, respectively. The explicit form of \(U\) for the case \(U = T_{123}U_0T_{123}^T\) has been given in (3.30) because \(U = T_{123}U_0T_{123}^T = U_0\). The explicit form of the case \(U = T_{123}U_0T_{312}^T\) is given by

\[
U = T_{123}U_0T_{312}^T = \begin{pmatrix}
 i s_\nu \sin \delta & c_\nu c_\rho e^{i \phi} + s_\nu s_\rho \cos \delta & c_\nu s_\rho e^{i \phi} - c_\nu c_\rho \cos \delta \\
- i c_\nu \sin \delta & s_\nu c_\rho e^{i \phi} - c_\nu s_\rho \cos \delta & s_\nu s_\rho e^{i \phi} + c_\nu c_\rho \cos \delta \\
\cos \delta & i s_\nu \sin \delta & -i c_\nu \sin \delta
\end{pmatrix}. \tag{3.37}
\]
Table 2: Possible neutrino mixing matrix form in Model B.

| $m_{\nu 0i}$ | $m^2_{\nu 03} = 0 < m^2_{\nu 01} < m^2_{\nu 02}$ | $m^2_{\nu 01} < m^2_{\nu 02} < m^2_{\nu 03}$ |
|--------------|------------------------------------------|------------------------------------------|
| Hierarchy    | Normal                                   | Inverse                                  |
| $U$          | $T_{312}U_0T^T_{312}$                     | $T_{123}U_0T^T_{123}$                     |
| Limit of     | $\tan^2\delta = 1/2$                     | $\tan^2\delta = 5$                      |
| $\sin^22\theta_{23} = 1$ | $s^2_\nu = 1/2$                         | $s^2_\nu = 4/5$                         |
| $&: \tan^2\theta_{12} = 1/2$ | $s^2_\nu = 2/3$                         | $s^2_\nu = 2/5$                         |

In order to see whether those cases cannot be ruled out or not, it is convenient to see whether we can take or not possible parameter values in the limit of $\tan^2\theta_{\text{solar}} = 1/2$, $\sin^22\theta_{\text{atm}} = 1$ and $|U_{13}|^2 = 0$, without contradicting with the observed neutrino mass hierarchy. The results are listed in Table 2. All cases are acceptable if we neglect the problem whether such a set of the parameter values is natural or not, although we think that the case with $U = T_{123}U_0T^T_{123}$ is unlikely.

In Model C, since we can take any order of $m^2_1$, we cannot say any definite conclusion (predictions) without giving the explicit mass matrix parameters. Therefore, for the case C, we do not give a table such as Table 2.

4 Summary

In conclusion, we have systematically investigated possible lepton mass matrix forms and mixings under the expended $2 \leftrightarrow 3$ symmetry.

As we discussed in Sec.2, if we require an exact (unbroken) $2 \leftrightarrow 3$ symmetry, which corresponds to the case $\delta \equiv \delta_{\nu} - \delta_{e} = 0$, we cannot obtain realistic neutrino mixing matrix. Therefore, from the phenomenological point of view, we have assumed that the $2 \leftrightarrow 3$ symmetry is broken only through the phase parameters, i.e. by the parameter $\delta \neq 0$. In this paper, we did not argue the origin of $\delta \neq 0$. We consider that the phase difference value $\delta = \delta_{\nu} - \delta_{e}$ is not always small. Rather, we consider that it is likely that the value $\delta$ takes a specific (not always small) value as $\delta = \pi/2$, $\delta = \pi/4$, and so on.

We have investigated two types of the $2 \leftrightarrow 3$ symmetry: one (Type I) is that the left- and right-handed fields ($f_L, f_R$) obey the symmetry, and another one (Type II) is that only $f_L$ obeys the symmetry. Note that even in Type II, in spite of no $2 \leftrightarrow 3$ symmetry in the Majorana mass matrix $M_R$ for $\nu_R$, the neutrino seesaw mass matrix still obey the $2 \leftrightarrow 3$ symmetry. However, we have concluded that the fermion mass $m_3$ is always zero in Type II. Therefore, the possibility that the charged lepton sector obeys the $2 \leftrightarrow 3$ symmetry of Type II is ruled out. We have been interested in the case B classified in Table 1, where the neutrino sector obeys the $2 \leftrightarrow 3$ symmetry of Type II, because we consider a model with an SU(5)-GUT type scenario [8]. In this case, we have only four cases of the neutrino mixing matrix. The results are summarized in Table 2.

We are also interested in a model with an SO(10)-type scenario. In this case (Model C), the
right-handed neutrino $\nu_R$ is also transformed as $\nu_R \rightarrow T_{23} \nu_R$, so that we can consider any value of $m_{\nu_3} \neq 0$ and any mixing matrix form (3.29). However, in the SO(10)-GUT model, a more strict constraint on the neutrino mass matrix appears because the neutrino mass matrix form is strictly related to the quark and charged lepton mass matrices, so that most naive SO(10) models have, at present, not succeeded [9] in giving reasonable fits for all the masses and mixings in the quark and lepton sectors, even without the $2 \leftrightarrow 3$ symmetry.

In the practical point of view, we think that there is a possibility to build a realistic model (SU(5) $\bar{5}_L + 10_L$ model) based on SU(5)-GUT rather than SO(10). In Model B, we are interested in the case of an inverse neutrino mass hierarchy, because the case $\delta = \pi/2$ is likely. The case predicts the effective electron neutrino mass $\langle m_{\nu_e} \rangle$ is of the order of $\sqrt{\Delta m_{\text{atm}}^2} \approx 0.05$ eV, which is within the reach of the next generation experiments of the neutrinoless double beta decay.

We hope that the present investigation will be helpful to investigate more explicit model based on a GUT scenario.

Acknowledgment
The one of the authors (YK) is supported by the Grant-in-Aid for Scientific Research, Ministry of Education, Science and Culture, Japan (No.18540284).

References
[1] T. Fukuyama and H. Nishiura, hep-ph/9702253, in Proceedings of the International Workshop on Masses and Mixings of Quarks and Leptons, Shizuoka, Japan, 1997, edited by Y. Koide (World Scientific, Singapore, 1998), p. 252; R. N. Mohapatra and S. Nussinov, Phys. Rev. D60, 013002, (1999); E. Ma and M. Raidal, Phys. Rev. Lett. 87, 011802 (2001); C. S. Lam, Phys. Lett. B507, 214 (2001); K. R. S. Balaji, W. Grimus and T. Schwetz, Phys. Lett. B508, 301 (2001); W. Grimus and L. Lavoura, Acta Phys. Pol. B32, 3719 (2001). For a recent work, for example, see, R. N. Mohapatra, JHEP 0410, 027 (2004); W. Grimus, A. S. Joshipura, S. Kaneko, L. Lavoura, H. Sawanaka and M. Tanimoto, Nucl. Phys. B713, 151 (2005).

[2] Y. Koide, Phys. Rev. D69, 093001 (2004).

[3] D. G. Michael et al. MINOS Collaboration, Phys. Rev. Lett. 97, 191801 (2006). Also see, J. Hosaka et al. the Super-Kamiokande Collaboration, arXiv: hep-ex/0604011.

[4] B. Aharmim et al. SNO Collaboration, Phys. Rev. C72, 055502 (2005). T. Araki et al. KamLAND Collaboration, Phys. Rev. Lett. 94, 081801 (2005).

[5] Y. Koide, Phys. Rev. D71, 016010 (2005).

[6] Y. Koide, H. Nishiura, K. Matsuda, T. Kikuchi and T. Fukuyama, Phys. Rev. D66, 093006 (2002).

[7] K. Matsuda and H. Nishiura, Phys. Rev. D69, 053005 (2004).
[8] For example, R. N. Mohapatra, S. Nussinov and H. B. Yu, Phys. Lett. B636, 114 (2006).

[9] For recent works after the establish of the large $\nu_e-\nu_\mu$ mixing, for example, see, K. Matsuda, T. Fukuyama and H. Nishiura, Phys. Rev. D61, 053001 (2000); K. Matsuda, Y. Koide, T. Fukuyama and H. Nishiura, Phys. Rev. D65, 033008 (2002); Erratum-ibid. D65, 079904 (2002); W. Grimus and H. Hühböck, Phys. Lett. B643, 182 (2006); T. Fukuyama, T. Kikuchi and N. Okada, Int. J. Mod. Phys. E16, 1489 (2007).