Fano resonance through Higgs bound states in tunneling of Nambu-Goldstone modes

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We study collective modes of superfluid Bose gases in optical lattices combined with potential barriers. We assume that the system is in the vicinity of the quantum phase transition to a Mott insulator at a commensurate filling, where emergent particle-hole symmetry gives rise to two types of collective mode, namely, a gapless Nambu-Goldstone (NG) phase mode and a gapful Higgs amplitude mode. We consider two kinds of potential barrier: one does not break the particle-hole symmetry while the other does. In the presence of the former barrier, we find Higgs bound states that have binding energies lower than the bulk Higgs gap and are localized around the barrier. We analyze tunneling properties of the NG mode incident to both barriers to show that the latter barrier couples the Higgs bound states with the NG mode, leading to Fano resonance mediated by the bound states. Thanks to the universality of the underlying field theory, it is expected that Higgs bound states may be present also in other condensed matter systems with a particle-hole symmetry and spontaneous breaking of a continuous symmetry, such as quantum dimer antiferromagnets, superconductors, and charge-density-wave materials.

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I. INTRODUCTION

The concept of elementary excitation is central to understanding various properties of quantum many-body systems, such as thermodynamics, transport, nonequilibrium dynamics, superfluidity, and phase transitions. It is of fundamental importance in modern condensed matter physics. In recent years, of particular interest are massive (or gapful) Higgs modes of systems with spontaneous breaking of a continuous symmetry that correspond to amplitude fluctuations of the order parameter, owing to their ubiquity in many condensed matter systems [1,2]. Examples include superconductors NbSe2 [3–5] and Nb1−xTi1xN [6–8], charge-density-wave (CDW) materials K0.3MoO3 [9,10] and TbTe3 [11,12], quantum dimer antiferromagnets TiCuCl3 [13] and KCuCl3 [14], superfluid (SF) 3He-B [15,16], and SF Bose gases in optical lattices [17,18]. Moreover, the studies of Higgs modes have been further stimulated by the recent report of observing the Higgs particle in elementary particle physics [19,20].

All the Higgs modes that have been studied thus far are delocalized states in the entire systems. In this paper, we study collective modes of SF Bose gases in optical lattices in the presence of potential barriers to predict bound states of Higgs mode that are localized around the barriers. Assuming the vicinity of the quantum phase transition to a Mott insulator (MI) at a commensurate filling, in which the system is nearly particle-hole symmet-
FIG. 1: Ground-state phase diagram of the Bose-Hubbard model in a homogeneous system. The phase boundaries (thick solid lines) and the line of integer filling factors (thin solid lines) are computed by means of the Gutzwiller mean-field approximation. The dashed lines represent the particle-hole symmetric lines ($K_0 = 0$), which are obtained from the analytical expression for $K_0$ given in Appendix A. The dashed-dotted line represents the contour of $|\psi|^2 a^x = 0.25$ and the gray shaded area roughly marks the region where the forth-order GL theory is validated for describing the SF state.

We consider a tunneling problem of the NG modes across the potential barriers and find Fano resonance of the NG modes mediated by the Higgs bound states. Existence of the Higgs bound states may be demonstrated through measurement of an asymmetric peak in the transmission probability characteristic to the Fano resonance.

The remainder of this paper is organized as follows. In Sec. II we introduce the Bose-Hubbard (BH) model that describes Bose gases in optical lattices, and present a brief review of important properties of the model with an emphasis on the Higgs and NG modes of the SF phase. In Sec. III we explain a way to create potential barriers in the chemical potential and the hopping amplitude by controlling external fields and develop the GL theory to include the effects of the barriers. In Sec. IV we analyze the collective modes in the presence of the hopping barrier and reveal the emergence of Higgs bound states localized around the barrier. In Sec. V we solve a tunneling problem of the NG mode scattered by the two types of potential barrier and show that the Higgs bound states induce Fano resonance of the NG mode. The results are summarized in Sec. VI.

II. BOSE-HUBBARD MODEL

We consider cold bosonic atoms in a hypercubic optical lattice. We assume a sufficiently deep lattice so that the system is well described by the tight-binding BH model

$$\mathcal{H} = -\sum_{i,j} J_{i,j} b_i^\dagger b_j + \sum_i \mu_i b_i^\dagger b_i + \frac{U}{2} \sum_i b_i^\dagger b_i^\dagger b_i b_i. \quad (1)$$

The vector $i = \sum_{\alpha=1}^d i_\alpha e_\alpha$ denotes the site index, where $i_\alpha$ is an integer, $d$ the spatial dimension, and $e_\alpha$ a unit vector in direction $\alpha$. For instance, the directions $\alpha = 1,2,3$ mean the $x,y,z$ direction, respectively. $b_i^\dagger (b_i)$ is a creation (annihilation) operator of bosons at site $i$, and $U > 0$ the on-site repulsive interaction. The local chemical potential,

$$\mu_i \equiv \mu_0 - V_i, \quad (2)$$

consists of the homogeneous contribution $\mu_0$ and the site-dependent external potential $V_i$. The hopping matrix element $J_{i,j}$ is assumed to be finite only for nearest-neighboring sites, i.e.,

$$J_{i,j} = \sum_{\alpha} \left( J_{j}^{(\alpha)} \delta_{i,j+e_\alpha} + J_{j-e_\alpha}^{(\alpha)} \delta_{i,j-e_\alpha} \right), \quad (3)$$

where $J_{j}^{(\alpha)}$ means the hopping amplitude between sites $j$ and $j+e_\alpha$. We set $\hbar = 1$ throughout the paper. Properties of ground states and low-lying elementary excitations of the BH model in a homogeneous system ($J_{j}^{(\alpha)} = J$, $V_i = 0$) have been extensively studied and well understood [22]. While we aim to reveal novel effects caused by local potential barriers in an inhomogeneous system, in this section we briefly review the properties of the homogeneous BH model in order to clarify the problem addressed in this paper.

In Fig. 1 we show the ground-state phase diagram in the $(Z J / U, \mu_0 / U)$-plane obtained by mean-field theories [30, 31], where $Z$ is the coordination number. It consists of two distinct phases: MI phase and SF phase [21]. For large interaction energy ($ZJ \ll U$) at a commensurate filling, the system is in the MI phase where integer number of bosons localize in each lattice site to avoid large energy cost of repulsive interaction. For large kinetic energy ($ZJ \gg U$), the system is in the SF phase, where bosons can move around and condense in the lowest energy state. The global U(1) symmetry is broken in the SF phase, while there is no broken symmetry in the MI phase. The quantum phase transition that involves spontaneous breaking of U(1) symmetry takes place at a certain value of $Z J / U$. This ratio between kinetic and interaction energy can be arbitrarily controlled by tuning the laser intensity of the lattice potential in a single system. A signature of the quantum phase transition was observed in the drastic change of the interference pattern of an atomic cloud released from a trapping potential [32].

Elementary excitations in the MI phase correspond to excess particles or holes [31, 33, 34]. The excitation spectrum has energy gap due to finite energy cost for adding or subtracting one particle. The SF phase in the close vicinity of the tips of the Mott lobes possesses two excitations, namely, gapless NG mode and gapful Higgs...
These excitations arise from the broken U(1) symmetry and approximate particle-hole symmetry. The former corresponds to phase fluctuations of the order parameter and the latter corresponds to amplitude fluctuations. As the system becomes far apart from the tips of the Mott lobes, the energy gap of the gapful mode rapidly increases and the mode turns into a single-particle excitation. In the deep SF regime ($ZJ \gg U$), there remains only the gapless NG mode as low-lying excitations, which is often referred to as Bogoliubov mode [38].

Since our focus in this paper is on NG and Higgs modes in the vicinity of the tips of the Mott lobes, we continue a review on these modes a little further. Near the SF-MI transition point, it is reasonable to expand the action in terms of the SF order parameter $\psi$. The definition of $\psi$ is given in Appendix A. As a result, an effective action $S_{\text{eff}}(\{\psi\})$ as well as the classical equation of motion for $\psi$ can be obtained. Taking the low-energy and continuum limit, SF dynamics in the vicinity of the quantum critical point is governed by the time-dependent Ginzburg-Landau (TDGL) equation [21, 22].

$$iK_0 \frac{\partial \psi}{\partial t} - W_0 \frac{\partial^2 \psi}{\partial t^2} = \left(-\frac{\nabla^2}{2m_*} + r_0 + u_0 |\psi|^2\right) \psi.$$  \hfill (4)

In Fig. 2, we schematically show the dispersion relations. The NG mode has a gapless linear dispersion, $\omega^2 = c^2 k^2$, and Higgs modes have a finite gap $\Delta = \sqrt{-2r_0/W_0}$ at $k = 0$. They are pure phase and amplitude modes for any $k$. When $K_0 \neq 0$, the particle-hole symmetry and the Lorentz invariance are broken so that the two modes are mixed. However, as long as $|K_0|$ is sufficiently small ($|K_0| \ll \sqrt{-W_0 r_0}$), the basic property is robust, i.e., the phase and amplitude fluctuations dominate the gapless and gapful modes.

Although Higgs modes apparently look long-lived within the linearized equations of motion [23] and [27], previous studies have dealt with higher order corrections with respect to the fluctuations and pointed out that the Higgs modes at $d < 3$ are not necessarily well-defined because of strong quantum fluctuations allowing for decay of a Higgs mode into a pair of NG modes [33, 40, 45]. In the following, to avoid the subtlety at low dimensions, we focus on the case of $d = 3$, where the use of TDGL equation is unambiguously justified (at least qualitatively) and Higgs modes are known to be long-lived.

III. EFFECT OF POTENTIAL BARRIERS IN THE TDGL EQUATION

In this section, we formulate effects of potential barriers in terms of the TDGL equation. We neglect the overall harmonic potential for simplicity. Moreover, we
assume that potential barriers are present only in the \( x \)-direction and that the system is homogeneous in the other directions except for the overall optical-lattice potential. Let us show that external potentials can introduce the local modulation of the chemical potential \( \mu_i \) and the hopping amplitude \( J_i^{(x)} \) in the BH model \( 11 \).

Specifically, we propose imposing two different types of potential barrier in addition to the overall optical-lattice potential for controlling these parameters independently. First, the shift of the lattice potential with little change in the lattice height can be realized by an optical dipole potential that leads to the shift of the chemical potential \( \mu_i = \mu_0 \rightarrow \mu_0 - V_i \) in Eq. \( 11 \). This situation is schematically illustrated in Fig. 3(a). Second, we consider an additional lattice potential in the Gaussian profile with the same lattice spacing as that of the overall lattice potential as shown in Fig. 3(b). The potential of this type can be created by focusing the optical-lattice laser into a narrow spatial region \( 16 \) and spatially modulates the height of the lattice potential, leading to the inhomogeneous hopping amplitude,

\[
J_i^{(x)} = J + J_i^{(x)} \delta_{0,1}. \tag{9}
\]

Since we regard \( V_i \) and \( J_i^{(x)} \) as potential barriers, they are anticipated to vanish at \( i \rightarrow \pm \infty \). Hence, \( \mu_0 \) and \( J \) mean the equilibrium values far away from the potential barriers.

The coefficients in the TDGL equation are modified by the potential barriers. We show approximate expressions of the coefficients in the lowest order of the perturbations \( V_i \) and \( J_i^{(x)} \), taking the continuum limit \( V_i \rightarrow V(x) \) and \( J_i^{(x)} \rightarrow J(x) \). See Appendix A for a detailed derivation of the expressions. We assume that \( K_0 = 0 \) such that there are independent NG and Higgs modes in the absence of the barriers. Here \( K_0, W_0, r_0, \) and \( u_0 \) denote the values of coefficients of the first-order time derivative term \( K \), the second-order time derivative term \( W \), the linear term \( r \), and the cubic nonlinear term \( v \) in the absence of the barriers. In the case that \( K_0 = 0 \), the shift of the chemical potential yields the leading contribution to \( K \) as

\[
K \simeq -2W_0V(x) \equiv v_K(x). \tag{10}
\]

This term breaks the particle-hole symmetry and locally couples phase and amplitude fluctuations. In contrast, under the assumption that \( V(x) \ll U \) one may ignore the contribution of \( V(x) \) in \( W \) and \( u \) such that \( W \simeq W_0 \) and \( u \simeq u_0 \). On the other hand, the local modulation of the hopping amplitude \( J(x) \) affects only \( r \) as

\[
r \simeq r_0 - 2J(x) \equiv r_0 + v_r(x). \tag{11}
\]

\( v_r(x) \) acts as a usual potential term that does not break particle-hole symmetry. The resulting TDGL equation including the effects of the potential barriers is given by

\[
iuK \frac{\partial \psi}{\partial t} - W_0 \frac{\partial^2 \psi}{\partial x^2} = \left( \frac{\nabla^2}{2m^*} + r_0 + v_r + u_0 |\psi|^2 \right) \psi. \tag{12}
\]

In order to simplify the notation, we represent the variables in a dimensionless form,

\[
\tilde{\psi} = \psi/(-r_0/u_0)^{1/2}, \quad \tilde{t} = t(-r_0/W_0)^{1/2}, \quad \tilde{x} = x/\xi,
\]

\[
\tilde{v}_r = -v_r/r_0, \quad \tilde{v}_K = v_K/(-r_0W_0)^{1/2}.
\tag{13}
\]

where \( \xi \equiv (-m^*r_0)^{-1/2} \) is the healing length. Hereafter, we omit the tilde and employ the following TDGL equation in the dimensionless form

\[
i\tilde{v}_K \frac{\partial \psi}{\partial \tilde{t}} - \frac{\partial^2 \psi}{\partial \tilde{x}^2} = \left( \frac{\nabla^2}{2} - 1 + |\psi|^2 + v_r \right) \psi. \tag{14}
\]

IV. HIGGS BOUND STATES

Within the effective 1D setting depicted in Fig. 4, we consider fluctuations of the order parameter \( \psi(x,t) \) around a static condensate \( \psi_0(x) \)

\[
\psi(x,t) = \psi_0(x) + \mathcal{U}(x)e^{-i\omega t} + \mathcal{V}^*(x)e^{i\omega t}. \tag{15}
\]
Here, we assumed spatial variation of fluctuations $\mathcal{U}$ and $\mathcal{V}$ only in the $x$-direction to deal with the tunneling problem of NG mode shown in Fig. 4. $\psi_0(x)$ satisfies the non linear equation that is identical to the static Gross Pitaevskii (GP) equation \[\psi(x) = 0.\] (16)

Phase and amplitude fluctuations $S(x) = \mathcal{U}(x) - \mathcal{V}(x)$ and $T(x) = \mathcal{U}(x) + \mathcal{V}(x)$ obey the coupled equations

$$
\left(-\frac{1}{2}\frac{d^2}{dx^2} - 1 + |\psi_0|^2 + v_r(x)\right)S(x) = \omega^2 S(x) - \omega v_K(x) T(x),
$$

(17)

$$
\left(-\frac{1}{2}\frac{d^2}{dx^2} - 1 + 3|\psi_0|^2 + v_r(x)\right)T(x) = \omega^2 T(x) - \omega v_K(x) S(x).
$$

(18)

The potential barrier $v_K(x)$ appears in the above equations in a peculiar manner: $v_K(x)$ is absent in Eq. (16), so it does not affect $\psi_0$. Meanwhile, $v_K(x)$ in the coefficients of the frequency $\omega$ in Eqs. (17) and (18) locally couples $S(x)$ and $T(x)$ at the position of the potential barrier. We will observe the crucial role played by this potential term in the resonant tunneling of NG mode in Sec. V.

In the following analysis, we assume delta-function potential barriers and set $v_r(x) = V_r \delta(x)$ and $v_K(x) = V_K \delta(x)$ for simplicity. This assumption is justified if the potential barrier spatially varies in the order of lattice spacing that is much smaller than the healing length $\xi$. Since $\xi$ becomes very large in the quantum critical region near the phase boundary with the MI phase, this assumption is reasonable when the TDGL equation is valid.

The static solution under a delta-function potential barrier $v_r(x) = V_r \delta(x)$ is given by \[\psi_0(x) = \tanh(|x| + x_0),\] (19)

where $x_0$ is determined by the boundary condition at $x = 0$

$$
\psi_0(-0) = \psi_0(0),
$$

(20)

$$
\frac{d\psi_0}{dx} \bigg|_{+0} - \frac{d\psi_0}{dx} \bigg|_{-0} = 2V_r \psi_0(0),
$$

(21)

to give

$$
\tanh(x_0) = -\frac{V_r}{2} + \sqrt{\frac{V_r^2}{4} + 1} \equiv \eta.
$$

(22)

The amplitude of the static condensate at $x = 0$, $\psi_0(0) = \eta$, monotonically decreases from $\eta(V_r = 0) = 1$ with increasing $V_r$ and has the asymptotic form $\eta(V_r \to \infty) \sim 1/V_r$.

As one can see from Eqs. (17) and (18), the diminishing $\psi_0(x)$ combined with the repulsive potential barrier $v_r(x)$ constitutes a double-well potential for the collective modes. To investigate localized bound states in the double-well potential, we assume $v_K(x) = 0$ hereafter in this section. Let us first examine a simple problem without a potential barrier ($v_r(x) = 0$) when the background static condensate has a kink solution $\psi_0(x) = \tanh(x)$ \[49\], because Eq. (19) for $x \neq 0$ is identical to the shifted kink solution. Scattering states of $S(x)$ and $T(x)$ in the presence of a static kink are given by \[48, 50\]

$$
S(x) = (\tanh x - ik) e^{ikx},
$$

(23)

$$
T(x) = (k' \tanh x - 3ik' \tanh x - (k')^2 - 1) e^{ik' x},
$$

(24)

where

$$
\omega^2 = \frac{1}{2} k^2 = \frac{1}{2} k'^2 + 2.
$$

(25)

Each of Eqs. (23) and (24) is a single plane wave propagating without reflection. These solutions show that NG and Higgs modes are not scattered by a kink, though their amplitudes are suppressed near the kink.

We note that there also exist bound state solutions of $S(x)$ and $T(x)$ localized around a kink. The solution $S(x) = 1/cosh x$ has imaginary frequencies $\omega = \pm i/\sqrt{2}$ that destabilize the kink. This is in sharp contrast with the stable 1D kink solution in the GP equation \[49\]. The solution $T(x) = \tanh x(1 - \tanh^2 x)$ has the frequency $\omega = \pm i/\sqrt{5}$. In addition to these solutions, there are trivial zero mode solutions: $S(x) = \tanh x$ and $T(x) = 1/cosh x = \frac{d}{dx} (\tanh x)$ with $\omega = 0$.

The solution (24) is valid for $\omega$ both above and below the Higgs gap $\Delta = \sqrt{2}$. For Higgs mode with energy above the gap ($\omega > \Delta$), $k' = \pm i\sqrt{2\omega^2 - 4} \equiv \pm k_{\eta}$ is real and Eq. (24) corresponds to a scattering state. If the energy is below the gap ($\omega < \Delta$), $k' = \pm i\sqrt{4 - 2\omega^2} \equiv \pm ik_{\eta}$ and thus Eq. (24) decays as $T(x) \propto e^{\pm ik_{\eta} x}$ for $x \to \mp \infty$. The bound state solutions for $T(x)$ can be obtained by connecting these decaying solutions.

We demonstrate that Eq. (15) allows bound state solutions of amplitude fluctuations below the Higgs gap that localize around the potential well. Since $\psi_0(x)$ on the left (right) side of the barrier $v_r(x)$ is identical to the kink solution shifted by $x_0 = (-x_0)$, the solution of $T(x)$ for $\omega = \sqrt{2 - k_{\eta}^2}/2(\leq \Delta)$ reads

$$
T(x) = \begin{cases}
A(3\gamma_+^2(x) + 3\kappa_\eta \gamma_-(x) + \kappa_\eta^2 - 1) e^{\kappa_\eta x}, & (x < 0), \\
B(3\gamma_+^2(x) + 3\kappa_\eta \gamma_+(x) + \kappa_\eta^2 - 1) e^{-\kappa_\eta x}, & (x > 0),
\end{cases}
$$

(26)

where $\gamma_\pm(x) = \tanh(\pm x + x_0)$. The bound state solutions of $T(x)$ satisfy the boundary condition at $x = 0$
the potential terms of static condensate proportional to existence of bound state solutions indeed derives from the difference between Eqs. (17) and (18) concerning has no unstable bound state solutions with imaginary fluctuations. that has no bound states of amplitude as well as phase states of amplitude fluctuations in the TDGL equation the localized bound states. The emergence of the bound Eq. (17) gives rise to the Higgs gap and accommodates fulfilling the condition

\[
\frac{dT}{dx}\bigg|_{+0} - \frac{dT}{dx}\bigg|_{-0} = 2V_r T(0).
\]  

(28)

Remarkably, the above equations have two solutions: \(A = B\) and \(A = -B\). If \(T(0) \neq 0\), Eq. (27) reduces to \(A = B\), while if \(T(0) = 0\) Eq. (28) reduces to \(A = -B\). The former corresponds to an even parity solution and the latter an odd parity one. We note that Eq. (17) has no unstable bound state solutions with imaginary \(\omega\). The difference between Eqs. (17) and (18) concerning to existence of bound state solutions indeed derives from the potential terms of static condensate proportional to \(\left|\psi_0\right|^2\). The deeper potential well in Eq. (18) than that in Eq. (17) gives rise to the Higgs gap and accommodates the localized bound states. The emergence of the bound states of amplitude fluctuations in the TDGL equation should be compared with the case of the GP equation that has no bound states of amplitude as well as phase fluctuations.

From Eqs. (27) and (28), the even parity bound state fulfills the condition

\[
c_1 + V_r c_2 = 0,
\]

(29)

where

\[
c_1 = \kappa_1^2 + 3\eta \kappa_1^2 + (6\eta^2 - 4)\kappa_1 + 6\eta(\eta^2 - 1),
\]

\[
c_2 = \kappa_2^2 + 3\eta \kappa_2^2 + 3\eta^2 - 1.
\]

(30)

Equation (29) has a single bound state solution \(\kappa_+\). Figure 5 shows the binding energy \(E_+ = \sqrt{2 - \kappa_+^2}/2\) as a function of \(V_r\). \(E_+ (V_r)\) becomes the Higgs gap \(E_+ \to \sqrt{2}\) when \(V_r \to 0\). The bound state reduces to the odd parity solution localized around a kink: \(E_+ \to \sqrt{3}/2\) as

\[
V_r \to \infty.
\]

In this limit, the bound state can be also considered as an edge state that is localized at the boundary where condensate vanishes.

The odd parity solution satisfies \(c_2 = 0\). We thus obtain

\[
\kappa_2 = \frac{1}{2} \left(-3\eta + \sqrt{4 - 3\eta^2}\right) \equiv \kappa_-.
\]

(31)

The energy of the odd parity solution is given by \(E_- = \sqrt{2 - \kappa_-^2}/2\). The odd parity bound state appears if the potential is large enough such that \(V_r > 2/\sqrt{3}\). It also reduces to the odd parity solution on a kink: \(E_- \to \sqrt{3}/2\) as \(V_r \to \infty\). The odd parity bound state has higher energy than the even parity one \((E_+ < E_-)\) as shown in Fig. 6.

Figure 6 shows the even and odd-parity bound states of \(T(x)\). We propose the existence of such bound states of amplitude fluctuations below the Higgs gap and call them Higgs bound states. So far, the main focus of the study of localized excitations in condensed matter systems have been on single-particle excitations including Andreev bound states in superconductors \([22]\) and edge states in quantum Hall systems \([51]\) and topological insulators \([52]\), or collective density modes such as ripples \([53]\) and Kelvin modes \([54]\) in SF systems. Hitherto, Higgs bound states as localized amplitude modes have never been found. Since the Higgs bound states are low-lying excitations, they may play a major role in various aspects of superfluid Bose gases in optical lattices at low temperatures. Moreover, given the fact that the presence of Higgs amplitude modes is a common feature among systems described effectively by a relativistic \(O(N)\) field theory with \(N \geq 2\) \([22]\), Higgs bound states is also expected to exist in other various systems involving approximate particle-hole symmetry and spontaneous breaking of a continuous symmetry, such as superconduc-
tors, CDW materials, and magnetic materials.

V. FANO RESONANCE OF TUNNELING NG MODE

In this section, we study scattering of NG mode in the presence of the potential barriers \( v_r(x) \) and \( v_K(x) \). We assume that NG mode with energy \( E \) is injected from the left \( x \to -\infty \) as shown in Fig. 4. The solutions of Eqs. (17) and (18) can be written in a linear combination of the scattering states in Eqs. (23) and (24) as

\[
S(x) = \begin{cases} 
(-\gamma_-(x) - i k_s)e^{i k_s x} + r_{ng}(-\gamma_-(x) + i k_s)e^{-i k_s x}, & (x < 0), \\
t_{ng}(\gamma_+(x) - i k_s)e^{i k_s x}, & (x > 0), 
\end{cases}
\]

\[
T(x) = \begin{cases} 
2r_{ng}3\gamma_+^2(x) - 3i k_s\gamma_+(x) - k_s^2 - 1)e^{-i k_s x}, & (x < 0), \\
2t_{ng}(3\gamma_+^2(x) - 3i k_s\gamma_+(x) - k_s^2 - 1)e^{i k_s x}, & (x > 0), 
\end{cases}
\]

where \( k_s = E \) and \( k_t = \sqrt{2E^2 - 4} \) (see Fig. 2). In Eq. (32), \( S(x < 0) \) consists of injected and reflected waves, while \( S(x > 0) \) is a transmitted wave. Since \( S(x) \) and \( T(x) \) are coupled by the potential \( v_K \), amplitude fluctuations may be induced and emitted from the potential barrier. Equation (33) thus corresponds to plane waves of Higgs mode propagating outward from the barrier for \( E > \Delta \). If injected NG mode has lower energy than the Higgs gap \( (E < \Delta) \), \( k_t \) should be substituted by \( \kappa_i = i\sqrt{4 - 2E^2} \) in Eq. (33) so that \( T(x) \) exponentially decays at \( |x| \to \infty \). In the following, we restrict ourselves within the latter case of \( E < \Delta \).

The asymptotic forms of Eqs. (32) and (33) far away from the potential barriers are given by

\[
S(x) \to \begin{cases} 
(-1 - i k_s)e^{i k_s x} + r_{ng}(-1 + i k_s)e^{-i k_s x}, & (x \to -\infty), \\
t_{ng}(1 - i k_s)e^{i k_s x}, & (x \to \infty), 
\end{cases}
\]

\[
T(x) \to \begin{cases} 
2r_{ng}(3\gamma_+^2(x) - 3i k_s\gamma_+(x) - k_s^2 - 1)e^{-i k_s x}, & (x \to -\infty), \\
t_{ng}(3\gamma_+^2(x) - 3i k_s\gamma_+(x) - k_s^2 - 1)e^{i k_s x}, & (x \to \infty). 
\end{cases}
\]

From the ratio of the amplitudes of reflected and transmitted waves with respect to that of the incident wave, the reflection and transmission probabilities of NG mode are defined as \( R \equiv |r_{ng}|^2 \) and \( T \equiv |t_{ng}|^2 \), respectively. They satisfy the conservation law \( R + T = 1 \). We derive the conservation law for NG and Higgs modes in Appendix B.

The coefficients \( r_{ng}, t_{ng}, r_h, \) and \( t_h \) are determined so as to satisfy the boundary conditions:

\[
S(-0) = S(+0), \quad T(-0) = T(+0),
\]

\[
-EV_K T(0) = -\frac{1}{2} \left( \frac{dS}{dx} \right)_{+0} - \frac{dS}{dx} \right)_{-0} + V_r S(0), \quad (37)
\]

\[
-EV_K S(0) = -\frac{1}{2} \left( \frac{dT}{dx} \right)_{+0} - \frac{dT}{dx} \right)_{-0} + V_r T(0). \quad (38)
\]

The transmission probability of NG mode \( T(E) = |t_{ng}|^2 \) can be cast in the form

\[
T(E)^{-1} = 1 + \frac{2E^2}{(2E^2 + 1)^2} V_{\text{eff}}(E)^2, \quad (39)
\]

\[
V_{\text{eff}}(E) = (1 - V_r^2 f(E)) V_r, \quad (40)
\]

\[
f(E) = \left( \frac{c_2}{c_1 + V_r c_2} \right) \left( \frac{2E^2 + \eta^2}{2V_r} \right). \quad (41)
\]

Figures 7 and 8 show \( T(E) \) and \( f(E) \) as functions of \( E \). In Fig. 7, \( T(E) \) increases as \( E \) decreases at low energy \( (E \lesssim 0.5) \), and it approaches unity at \( E \to 0 \). In fact, Eq. (39) clearly shows the perfect transmission of NG mode occurring in the low energy limit, i.e., \( T \to 1 \) at \( E \to 0 \), irrespective of the strength of the potential barriers \( V_r \) and \( V_K \). This is well-known as the anomalous tunneling of Bogoliubov mode \( \chi_5 \). The anomalous tunneling has been mainly discussed in the context of weakly interacting Bose gases based on the GP equation. Our results show that the NG mode in a strongly interacting Bose system also exhibits the anomalous tunneling property. Recently, it is proposed that the anomalous tunneling is a universal behavior of NG mode in systems with a broken continuous symmetry \( \chi_5 \).

Figure 5 also shows a peculiar asymmetric peak: \( T(E) \) is sharply enhanced after dropping to zero in the vicinity
of $E_+$ as $E$ decreases below the gap. This asymmetric peak is the main focus of the present paper.

Equation (40) shows that the interference between scattered waves of NG mode in two processes, one directly scattered by the bare $V_r$ and the other one by $V_K$ as well as by $V_+$, renormalizes $V_r$ giving the effective potential $V_{\text{eff}}(E)$. Moreover, Eq. (41) shows that the second process involves resonant excitation of the Higgs bound state through the scattering amplitude $f(E)$: Expansion of the denominator in Eq. (41) around $E_+$ gives

$$\alpha = \frac{2E_r}{\kappa_+} \left[ 3\kappa^2_+ + 2 \left( \frac{1}{\eta} \right) \kappa_+ + 3\eta^2 - 1 \right],$$

where $\kappa_+ = \sqrt{4 - 2E_r^2}$. Thus, $f(E)$ has a pole and diverges at $E_+$, as shown in Fig. 8. If the interference is destructive, $V_{\text{eff}}(E)$ vanishes and perfect transmission of incident wave occurs when $V_{\text{eff}}^2 f(E) = 1$. On the other hand, precisely at the energy of the bound state ($E = E_+$), $V_{\text{eff}}$ diverges due to the resonance with the Higgs bound state and therefore incident wave is perfectly reflected. Thus, such interference of scattered waves of NG mode produces the asymmetric peak in Fig. 7.

This phenomenon is a typical example of Fano resonance [28], in which interference between a directly scattered wave within continuum and a resonantly scattered wave involving excitation of bound states produces asymmetric peaks of scattering cross-section or transmission probability. The Fano resonance of NG mode in the present case exhibits interesting features. One remarkable feature is that the Higgs bound state is resonantly coupled with NG mode by the potential barrier of the first-order time-derivative term that arises due to the broken particle-hole symmetry. This is quite different from usual single-particle scatterings described by Schrödinger equation where scattering states and bound states are coupled by proximity of wave functions through a potential barrier.

In Eq. (40), the effect of $V_K$ vanishes and the effective potential $V_{\text{eff}}$ reduces to the bare potential $V_r$ at $E = E_-$, because of $c_3(E_-) = 0$ and $f(E_-) = 0$. Thus, in contrast with the even-parity bound state at $E_+$ that causes the resonance ($f(E_+) = \pm \infty$), the odd parity bound state $E_-$ cancels the effect of the potential barrier $V_K$, because the wave function of the odd parity bound state has a node at the position of the potential barrier $x = 0$.

If the odd parity bound state exists ($V_r > 2/\sqrt{3}$) and furthermore $V_K$ is sufficiently large such that $V_K^2 f(\Delta) > 1$, another perfect transmission in the region $E_+ < E < \Delta$ occurs when $V_K^2 f(E) = 1$ in addition to the one in $0 < E < E_+$. Figure 8 shows the second perfect transmission in $E_+ < E < \Delta$ for $(V_r, V_K) = (4, 4)$. The phase diagram in Fig. 9 shows the parameter region where perfect transmission occurs twice in the $V_r - V_K$ plane.

The observability of Higgs modes is a central issue in condensed matter systems [2, 63]. Observation of Higgs modes as well as Higgs bound states is difficult with standard techniques since they are not directly coupled with density or electromagnetic fields. Few exceptions include observation in bosonic superfluids in optical lattices with temporal modulation of the lattice potential [18], NbSe$_2$, which has coexisting CDW and superconducting order, by Raman spectroscopy [3], and terahertz transmission experiments in $s$-wave superconductors [6, 7]. Our results indicate that studying transport properties of NG mode could be a possible platform for observation of Higgs bound states. We propose detection of Higgs bound states in measuring the transmission probability of NG mode excited by Bragg pulses [28] through potential barriers [54]. Since the asymmetric peak in the transmission probability of NG mode is characteristic to the Fano resonance coupled with the Higgs bound states, detecting it provides with strong evidence for the existence of

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig7}
\caption{The transmission coefficient $T$ as a function of $E$ for $(V_r, V_K) = (4, 2)$ (red) and $(4, 4)$ (blue). The dotted and dash-dotted lines represent the energy of the Higgs bound states with parity even ($E_+$) and odd ($E_-$), respectively. The horizontal axis is in unit of $r_0$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig8}
\caption{Scattering amplitude of NG mode $V_{\text{eff}}^2(E)$ as a function of $E$. We set the potential barriers $(V_r, V_K) = (4, 2)$ (red) and $(4, 4)$ (blue). The horizontal axis is in unit of $r_0$.}
\end{figure}
FIG. 9: Phase diagram in terms of the perfect transmission of NG mode in the $V_r - V_K$ plane. In the red (green) area, perfect transmission associated with the Higgs bound states occurs only once at energy of $0 < E < E_+$ ($E_- < E < \Delta$). In the yellow area, Perfect transmission occurs twice with energies of $0 < E < E_+$ and $E_- < E < \Delta$. The axes are in unit of $r_0\xi$.

the Higgs bound states.

VI. CONCLUSIONS

We have studied collective modes of SF Bose gases in optical lattices in the presence of potential barriers. Assuming the system in the vicinity of the quantum phase transition to the MI phase with commensurate fillings, we derived the TDGL equation that includes the effect of external potentials. We considered two types of potential barriers: one of which shifts the chemical potential and breaks the particle-hole symmetry, while the other changes the hopping amplitude in the BH model, which does not break the particle-hole symmetry. We found that introducing the former potential leads to the peculiar potential term coupled with the first order time-derivative of the SF order parameter in the TDGL equation. In the presence of a potential barrier of the latter type, we have shown the existence of Higgs bound states localized around the barrier below the Higgs gap. We analyzed transport properties of NG mode through the potential barriers and found that the transmission probability of NG mode exhibits a remarkable asymmetric peak that is characteristic to the Fano resonance. We have shown that the Fano resonance of NG mode involving resonant excitation of Higgs bound states occurs due to the coupling of phase and amplitude fluctuations induced by the potential barrier of the former type. We proposed a possible way of detecting Higgs bound states in studying transport properties of NG mode excited by Bragg pulses.

In this paper, we confined our discussions within the case in which the system has a single potential barrier of two different types. Given the fact that various systems including disordered superconductors, Josephson junction arrays, and $^4$He absorbed in porous media are effectively described by the BH model with random chemical potential and/or hopping amplitude [21], it may be desirable to extend our results to the case of random potential barriers. If potential barriers that change the local hopping amplitude distribute randomly or periodically over the system, the Higgs bound states are expected to form energy bands below the Higgs gap. Such energy bands of Higgs bound states may be observable by measuring complex terahertz transmission [8].

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Appendix A: Derivation of time dependent Ginzburg-Landau equation

In this appendix, we present a detailed derivation of the TDGL equation [12] that includes effects of inhomogeneous chemical potential $\mu_i$ and hopping amplitude $J_i^{(a)}$ given by Eqs. [29] and [30]. For this purpose, we describe the BH model of Eq. [1] in the imaginary-time path-integral representation as

$$\Xi = \int \mathcal{D}b^* \mathcal{D}b \exp \left[ -S_{\text{BH}}(\{b_i\}) \right], \quad (A1)$$

where $\Xi$ denotes the grand partition function and the Euclidian action is given by

$$S_{\text{BH}}(\{b_i\}) = \int_{-\beta}^{\beta} d\tau \left[ \sum_i b_i^* \left( \frac{\partial}{\partial \tau} - \mu_i + \frac{U}{2} b_i^* b_i \right) b_i - \sum_{i,j} J_{i,j} b_i^* b_j \right]. \quad (A2)$$

We assume that $|V_i| \ll U$ and $|J_{i,i}'| \ll J$. We follow the standard procedure used in previous studies [21, 22, 66] in most part of the derivation except for the treatment of the inhomogeneous hopping term.

We introduce the auxiliary field $\Psi_i$ at site $i$ that corresponds to the SF order parameter by Stratonovich-Hubbard transformation. This transformation makes use
of the following Gaussian integral
\[
\int \mathcal{D}\Psi^* \mathcal{D}\Psi \exp \left( -\int d\tau \left( \bar{\Psi}^\dagger \cdot \bar{b} \cdot \bar{J} \right) \cdot \left( \bar{\Psi} - \bar{\Psi}_0 \right) \right) = \text{const.}
\] (A3)

Here, \( \bar{b} = ((b_i^1)^T \text{ and } \bar{\Psi} = (\{\Psi_i\})^T \). \( \bar{J} \) means the hopping matrix whose element is \( J_{i,j} \) and consists of the homogeneous part \( \bar{J}_0 \) and the inhomogeneous one \( \bar{J}_\text{bar} \) as
\[
\bar{J} = \bar{J}_0 + \bar{J}_\text{bar}.
\] (A4)

where
\[
(\bar{J}_0)_{i,j} = \sum_{\alpha=1}^d J(\delta_{i,j+\epsilon_n} + \delta_{i,j-\epsilon_n})
\] (A5)

and
\[
(\bar{J}_\text{bar})_{i,j} = \bar{J}_1 \delta_{i,j+\epsilon_n} + \bar{J}_1' \delta_{i,j-\epsilon_n}.
\] (A6)

Multiplying Eq. (A3) to Eq. (A1), the grand partition function is rewritten as
\[
\Xi = \int \mathcal{D}b^* \mathcal{D}b \mathcal{D}\Psi^* \mathcal{D}\Psi \exp [-S(\{b_i\}, \{\Psi_i\})],
\] (A7)

where
\[
S(\{b_i\}, \{\Psi_i\}) = \int d\tau \bar{\Psi}^\dagger \cdot \bar{J} \cdot \bar{\Psi} + S_0 + S_{\text{pert}},
\] (A8)

\[
S_0 = \int d\tau \sum_i b_i^\dagger \left( \frac{\partial}{\partial \tau} - \mu_i + \frac{U}{2} b_i b_i^\dagger \right) b_i,
\] (A9)

\[
S_{\text{pert}} = -\int d\tau (\bar{b}^\dagger \bar{\Psi} + \bar{\Psi} \bar{b}^\dagger \bar{\Psi}).
\] (A10)

Integrating out the bosonic fields \( b_i, b_i^\dagger \), the action is formally expressed as
\[
S = \beta F_0 + \int d\tau \bar{\Psi}^\dagger \cdot \bar{J} \cdot \bar{\Psi} - \ln (\langle -S_{\text{pert}} \rangle)_0,
\] (A11)

where
\[
\langle \mathcal{O} \rangle_0 = \int \mathcal{D}b^* \mathcal{D}b \exp \left( -S_0 \right),
\] (A12)

and \( F_0 \) denotes the free energy of the MI state. \( S_0 \) contains only local terms and is already diagonalized with the filling factor \( g \) as the good quantum number. Hence, the eigenstate of the system described by \( S_0 \) is simply a Fock state \( |g\rangle_i \) and the eigenenergy is given by
\[
E_{g,i} = -\mu_i g + \frac{U}{2} g(g-1).
\] (A13)

Performing a cumulant expansion of the last term of Eq. (A11) up to the fourth order with respect to the fields \( \Psi_i \) and \( \Psi_i^\dagger \), one obtains
\[
S(\{\Psi_i\}) = \int d\tau \left[ \sum_{i,j} (\bar{J}-1)_{i,j} \Psi_i^\dagger \Psi_j + \sum_i \left( \alpha_i^2 |\Psi_i|^2 + \beta_i^2 |\Psi_i^\dagger|^2 + \gamma_i^2 \frac{\partial |\Psi_i|^2}{\partial \tau} + \delta_i^2 |\Psi_i^\dagger|^2 \right) \right],
\] (A14)

where
\[
\alpha_i^2 = \frac{g+1}{E_{g+1,i} - E_{g,i}} - \frac{g}{E_{g-1,i} - E_{g,i}},
\] (A15)

\[
\beta_i^2 = \frac{g+1}{E_{g+1,i} - E_{g,i}} + \frac{g}{E_{g-1,i} - E_{g,i}},
\] (A16)

\[
\gamma_i^2 = \frac{g+1}{E_{g+1,i} - E_{g,i}} \cdot \frac{g}{E_{g-1,i} - E_{g,i}}.
\] (A17)

\[
\delta_i^2 = \frac{g+1}{E_{g+1,i} - E_{g,i}} \cdot \frac{g}{E_{g-1,i} - E_{g,i}}.
\] (A18)

It is obvious that the coefficients \( \alpha_i^2, \beta_i^2, \gamma_i^2, \) and \( \delta_i^2 \) reflect the inhomogeneity of the chemical potential while the first term in Eq. (A14) does that of the hopping. To clarify the latter effect, we transform the first term in Eq. (A14) under the assumption that \( |\bar{J}_\text{bar}| < J \),

\[
\int d\tau \sum_{i,j} (\bar{J}-1)_{i,j} \Psi_i^\dagger \Psi_j = \int d\tau \bar{\Psi}^\dagger \bar{J} \bar{J}_{\text{bar}} \bar{\Psi},
\] (A19)

Performing the Fourier transformation, the first and second terms in Eq. (A19) is expressed as
\[
\int d\tau \sum_{i,j} \Psi_i^\dagger (\bar{J}_{\text{bar}})^{-1} \Psi_j = \sum_{k,\omega} \hat{\Psi}(k,\omega)^\dagger \frac{1}{ZJ} \hat{\Psi}(k,\omega) + \frac{1}{ZJ} \left( 1 - \frac{(ka)^2}{Z} \right),
\] (A20)

and
\[
\int d\tau \bar{\Psi}^\dagger \bar{J} \cdot \bar{\Psi} \cdot \bar{J}_{\text{bar}} \bar{\Psi} = \sum_{\omega,k,k'} \hat{\Psi}(k,\omega)^\dagger \hat{\Psi}(k',\omega) \left( \frac{2J_{k-k'}^*}{ZJ} + \frac{2J_{k-k'}^*}{ZJ} e^{i\omega J} + e^{-i\omega J} \right),
\] (A21)
where

$$\Psi_i(\tau) = \frac{1}{\sqrt{M_\beta}} \sum_{k,\omega} \Psi(k, \omega) e^{i(k - i a \cdot \omega \tau)}$$  \hspace{1cm} (A22)

$$J'_a = \frac{1}{M} \sum_i J'_{ix} e^{i q \cdot i a}$$  \hspace{1cm} (A23)

$$\varepsilon_k = 2J \sum_{\alpha=1}^d \cos(k_\alpha a)$$  \hspace{1cm} (A24)

Here $M$ is the total number of sites. In Eqs. (A20) and (A21), the long-wavelength limit, $k \ll a^{-1}$, has been taken. While the terms up to the second order with respect to $ka$ is kept in the former equation that is of the order of $J$, we leave only the leading term in the latter because $J'_i$ is anticipated to be much smaller than $J$. Substituting Eqs. (A20) and (A21) into Eq. (A14) and taking the continuum limit $a \rightarrow x$, we obtain the effective GL action,

$$S_{\text{eff}}(\psi) = \beta F_0 + \int dx \int d^d x \left[ K(x) \psi^* \frac{\partial \psi}{\partial \tau} + W(x) \left| \frac{\partial \psi}{\partial \tau} \right|^2 + \frac{1}{2m_*} |\nabla \psi|^2 + r(x)|\psi|^2 + \frac{u(x)}{2} |\psi|^4 \right],$$  \hspace{1cm} (A25)

where the coefficients are given by

$$K(x) = (ZJ)^2 \beta(2)(x),$$  \hspace{1cm} (A26)

$$W(x) = (ZJ)^2 \gamma(2)(x),$$  \hspace{1cm} (A27)

$$r(x) = ZJ + (ZJ)^2 \alpha(2)(x) - 2J'(x),$$  \hspace{1cm} (A28)

$$m_* = \frac{1}{2Ja^2},$$  \hspace{1cm} (A29)

$$u(x) = 2a^d (ZJ)^4 \alpha(4)(x).$$  \hspace{1cm} (A30)

In Eq. (A26), we have expressed the order parameter in the dimension of the wave function as $\psi = \Psi / (d^d / 2ZJ)$.

We assume that the system has the particle-hole symmetry when there is no potential barrier, i.e., $K_0 \equiv K|_{\mu_1 = \mu_0} = 0$. In this case, the potential barrier in the chemical potential $V(x)$ gives the leading contribution to $K(x)$ as

$$K(x) \simeq K_0 - \frac{\partial K}{\partial \mu}_{\mu = \mu_0} V(x) = -2W_0 V(x) \equiv v_K(x),$$  \hspace{1cm} (A31)

In contrast, the contribution of $V(x)$ can be ignored in the coefficients $W(x)$ and $u(x)$ as long as $V(x) \ll U$, and we take $W(x) \simeq W|_{\mu_1 = \mu_0} = W_0$ and $u(x) \simeq u|_{\mu_1 = \mu_0} = u_0$. The inhomogeneous hopping affects only the coefficient $r(x)$,

$$r(x) \simeq r_0 - \frac{\partial r}{\partial \mu}_{\mu = \mu_0} V(x) - 2J'(x) = r_0 - 2J'(x) \equiv r_0 + v_r(x).$$  \hspace{1cm} (A32)

In Eq. (A32), the term including $V(x)$ vanishes due to the particle-hole symmetry ($K_0 = 0$). Notice that in Eqs. (A31) and (A32) we have used the following relations,

$$K = -\frac{\partial r}{\partial \mu},$$  \hspace{1cm} (A33)

$$W = \frac{1}{2} \frac{\partial K}{\partial \mu},$$  \hspace{1cm} (A34)

which stem from the U(1) gauge invariance of the system with respect to the transformation $\psi \rightarrow \psi e^{i \theta}$ and $\mu \rightarrow \mu + i \frac{\partial \theta}{\partial \tau}$ \hspace{1cm} (22). Thus, the potential barrier created by the inhomogeneous chemical potential leads to $v_K(x)$ while that by the inhomogeneous hopping leads to $v_r(x)$. This is consistent with the fact that $V_i$ acts differently for a particle and a hole while $J'_i$ does not break the particle-hole symmetry. Finally, replacing the imaginary time $\tau$ with the real time $t$ as $t = -i \tau$ and minimizing the effective action, we obtain the TDGL equation Eq. (12).

### Appendix B: Conservation law for collective modes

We discuss conservation law for collective modes in the effective 1D setting in Fig. 4. One can easily prove that the Wronskian of the coupled linear equations \hspace{1cm} (17) and \hspace{1cm} (B1) defined by

$$W(\phi_1(x), \phi_2(x)) = \left| \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right|,$$  \hspace{1cm} (B1)

is a constant which is independent of $x$ and thus provides a conserved quantity. Here, we assumed that $\phi_i(x) \equiv (S_i(x), T_i(x))^T$ ($i = 1, 2$) are solutions for the same energy $E$ and defined the product as

$$\phi_1 \phi_2 \equiv (S_1, T_1) \begin{pmatrix} S_2 \\ T_2 \end{pmatrix} = S_1 S_2 + T_1 T_2.$$  \hspace{1cm} (B2)

If we take

$$\phi_1 = \begin{pmatrix} S(x) \\ T(x) \end{pmatrix}, \hspace{1cm} \phi_2 = \phi_1^*,$$  \hspace{1cm} (B3)

and substitute the asymptotic forms Eqs. (A31) and (A32) into Eq. (B1), we obtain
\[ W(\phi_1, \phi_2) = \begin{cases} 2i(1 + k_x^2)k_y (|r_{ng}|^2 - 1), & (E < \Delta) \\ 2i(1 + k_x^2) k_s (|r_{ng}|^2 - 1) + k_t \left| \frac{2 - 3ik_t - k_x^2}{-1 - ik_x} r_h \right|^2, & (E > \Delta) \end{cases} \]

for \( x \to -\infty \), and

\[ W(\phi_1, \phi_2) = \begin{cases} -2i(1 + k_x^2)k_s |t_{ng}|^2, & (E < \Delta) \\ -2i(1 + k_x^2) \left( k_s |t_{ng}|^2 + k_t \right) \left| \frac{2 - 3ik_t - k_x^2}{-1 - ik_x} t_h \right|^2, & (E > \Delta) \end{cases} \]

for \( x \to \infty \). Note that we substituted \( \kappa_t = -ik_t = -i\sqrt{2E^2 - 4} \) in Eq. (35) for \( E > \Delta \). Since \( W \) is a constant, Eqs. (B4) and (B5) give the conservation of probability for NG mode: \( |r_{ng}|^2 + |t_{ng}|^2 = R + T = 1 \) for \( E < \Delta \). For \( E > \Delta \), incident NG mode could induce Higgs mode due to the coupling of phase and amplitude fluctuations introduced by \( V_k \). If we define the probability of Higgs mode reflected to the left of the potential barriers \( \mathcal{R}_h \) and emitted to the right of the barriers \( \mathcal{T}_h \) from Eqs. (34) and (35) to be

\[ \mathcal{R}_h = \frac{k_t}{k_s} \left| \frac{2 - 3ik_t - k_x^2}{-1 - ik_x} r_h \right|^2, \quad \mathcal{T}_h = \frac{k_t}{k_s} \left| \frac{2 - 3ik_t - k_x^2}{-1 - ik_x} t_h \right|^2, \]

Eqs. (B4) and (B5) give the conservation of the total probability including generated Higgs mode for \( E > \Delta \): \( \mathcal{R} + \mathcal{T} + \mathcal{R}_h + \mathcal{T}_h = 1 \).

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