Universal Inequalities for Eigenvalues of the Buckling Problem of Arbitrary Order

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We investigate the eigenvalues of the buckling problem of arbitrary order on compact domains in Euclidean spaces and spheres. We obtain universal bounds for the kth eigenvalue in terms of the lower eigenvalues independently of the particular geometry of the domain.

Keywords Buckling problem; Eigenvalues; Euclidean space; Sphere; Universal bounds.

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1. Introduction

Let $\Omega$ be a connected bounded domain with smooth boundary in an $n \ (\geq 2)$-dimensional Euclidean space $\mathbb{R}^n$ and let $\nu$ be the outward unit normal vector field of $\partial \Omega$. Denote by $\Delta$ the Laplacian operator on $\mathbb{R}^n$. Let us consider the following well-known eigenvalue problems:

\begin{align*}
\Delta u &= -\lambda u \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega, \\
\Delta^2 u &= \eta u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega, \\
\Delta^3 u &= -\Lambda \Delta u \quad \text{in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega.
\end{align*}

They are called the fixed membrane problem; the clamped plate problem and the buckling problem, respectively. Let

\begin{align*}
0 &< \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \\
0 &< \eta_1 \leq \eta_2 \leq \lambda_3 \leq \cdots,
\end{align*}

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denote the successive eigenvalues for (1.1), (1.2) and (1.3), respectively. Here each eigenvalue is repeated according to its multiplicity. Deriving bounds for these (and other) eigenvalues is an important theme of mathematical analysis. In most cases, eigenvalues are controlled by the geometry of the underlying domain, the \( n \)-dimensional ball often representing an extremal case. On the other hand, it has been found that one can also control higher eigenvalues in terms of lower ones, completely independently of the geometry of the domain (apart from its dimension). Such eigenvalue bounds are called universal. Universal bounds for the eigenvalues \( \lambda_{k+1}, \eta_{k+1} \) and \( \Lambda_{k+1} \) have been derived by many mathematicians, and we shall now recall the pertinent results. Payne et al. [19, 20] proved the bound

\[
\lambda_{k+1} - \lambda_k \leq \frac{2}{k} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots, \quad (1.4)
\]

for \( \Omega \subseteq \mathbb{R}^2 \). This result easily extends to \( \Omega \subseteq \mathbb{R}^n \) as

\[
\lambda_{k+1} - \lambda_k \leq \frac{4}{kn} \sum_{i=1}^{k} \lambda_i, \quad k = 1, 2, \ldots, \quad (1.5)
\]

In 1980, Hile and Protter [15] proved

\[
\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{kn}{4}, \quad \text{for } k = 1, 2, \ldots, \quad (1.6)
\]

In 1991, Yang proved the following much stronger inequality [23]:

\[
\sum_{i=1}^{k} \left( \lambda_{k+1} - \lambda_i \right) \left( \lambda_{k+1} - \left( 1 + \frac{4}{n} \right) \lambda_i \right) \leq 0, \quad \text{for } k = 1, 2, \ldots. \quad (1.7)
\]

The inequality (1.7) is the strongest of the classical inequalities that are derived following the scheme devised by Payne–Pólya–Weinberger. Yang’s inequality provided a marked improvement for eigenvalues of large index. It should be also mentioned that the development of Yang’s inequality came to fruition only thanks to the work of Ashbaugh [2] and that of Harrell and Stubbe [13]. In fact, it was Harrell and Stubbe who first explained the key commutator facts behind the “trick” introduced by Yang in the traditional Payne–Pólya–Weinberger scheme and introduced the Yang inequality to the mathematical physics and geometry community. This trick was explained in further work of Ashbaugh (and later in the work of Ashbaugh and Hermi [3, 4]) as an instance of the use of the “optimal Cauchy–Schwarz” inequality. It was Ashbaugh who dubbed it the “Yang inequality”. The optimal Cauchy–Schwarz trick is what enabled Cheng and Yang [6] and Wang and Xia [21] to extend the earlier work of Yang to the case of the clamped plate problem for bounded domains of Euclidean space and of minimal submanifolds of the same space, respectively. This is the trick that makes all extensions à la Yang. The arguments around this trick were later generalized by Harrell [10], Harrell and Michel [11, 12], and Levitin and Parnovski [18], following the commutator method via Rayleigh–Ritz.
Consider now the problem (1.3) which is used to describe the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary. In 1956, Payne et al. [20] proposed the following

**Problem 1.** Can one obtain a universal inequality for the eigenvalues of the buckling problem (1.3) that is similar to the universal inequalities for the eigenvalues of the fixed membrane problem (1.1)?

Ashbaugh [1] mentioned this problem again. With respect to the above problem, Payne et al. proved

\[ \frac{\Lambda_2}{\Lambda_1} < 3 \quad \text{for } \Omega \subset \mathbb{R}^2. \]

For \( \Omega \subset \mathbb{R}^n \) this reads

\[ \frac{\Lambda_2}{\Lambda_1} < 1 + \frac{4}{n}. \]

Subsequently Hile and Yeh [16] reconsidered this problem obtaining the improved bound

\[ \frac{\Lambda_2}{\Lambda_1} \leq \frac{n^2 + 8n + 20}{(n + 2)^2} \quad \text{for } \Omega \subset \mathbb{R}^n. \]

Ashbaugh [1] proved:

\[ \sum_{i=1}^{n} \Lambda_{i+1} \leq (n + 4)\Lambda_1. \]  

(1.8)

This inequality has been improved to the following form [17]:

\[ \sum_{i=1}^{n} \Lambda_{i+1} + \frac{4(\Lambda_2 - \Lambda_1)}{n+4} \leq (n + 4)\Lambda_1. \]

Recently, Cheng and Yang introduced a new method to construct trial functions for the problem (1.3) and obtained the following universal inequality [7]:

\[ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + 2)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Lambda_i. \]  

(1.9)

It has been proved in [22] that for the problem (1.3) if \( \Omega \) is a bounded connected domain in an \( n \)-dimensional unit sphere, then the following inequality holds

\[ 2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \delta \Lambda_i + \frac{\delta^2 (\Lambda_i - (n - 2))}{4(\delta \Lambda_i + n - 2)} \right) \]

\[ + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n - 2)^2}{4} \right), \]  

(1.10)

where \( \delta \) is any positive constant.
In this paper, we will investigate the eigenvalues of the buckling problem of higher order:

\[ (-\Delta)^l u = -\Lambda \Delta u \quad \text{in } \Omega, \]
\[ u|_{\partial\Omega} = \frac{\partial u}{\partial v}|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial v^{l-1}}|_{\partial\Omega} = 0, \]  

where \( \Omega \) is a connected bounded domain in a Euclidean space or a unit sphere and \( l \) is any integer no less than 2.

For the eigenvalues of the problem (1.11), Chen and Qian [8] obtained some upper bounds on the \( k \)th eigenvalue in terms of the lower ones when \( k \) is small and \( \Omega \) is contained in a Euclidean space. It has been shown in [17] that the first \((l+1)\) eigenvalues of the problem (1.11) when \( \Omega \subset \mathbb{R}^n \) satisfy

\[ \sum_{i=1}^n \frac{k}{2l+k} (\Lambda_{n+2-k} - \Lambda_1) < 4(l-1)\Lambda_1. \]

To the authors’ knowledge, there are no universal inequalities on \( \Lambda_k \) in terms of \( \Lambda_1 \) for general \( k \). The purpose of this paper is to prove such inequalities. Namely, we will prove

**Theorem 1.1.** Let \( l \geq 2 \) and let \( \Lambda_i \) be the \( i \)th eigenvalue of the following eigenvalue problem:

\[ (-\Delta)^l u = -\Lambda \Delta u \quad \text{in } \Omega, \]
\[ u|_{\partial\Omega} = \frac{\partial u}{\partial v}|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial v^{l-1}}|_{\partial\Omega} = 0, \]

where \( \Omega \) is a connected bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial\Omega \) and \( v \) is the unit outward normal vector field of \( \partial\Omega \). Then for \( k = 1, \ldots, \), we have

\[ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{2(2l^2 + (n-4)l + 2 - n)^{1/2}}{n} \left\{ \frac{k}{n} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)} \right\}^{1/2} \times \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)} \right\}^{1/2}. \]  

**Remark.** If we take \( l = 2 \) in Theorem 1.1, then we obtain Cheng and Yang’s inequality (1.9).

From Theorem 1, we can obtain more explicit inequalities which are weaker than (1.13):

**Corollary 1.1.** Under the same assumptions as in Theorem 1.1, we have

\[ \Lambda_{k+1} \leq \frac{1}{k} \sum_{i=1}^k \Lambda_i + \frac{2(2l^2 + (n-4)l + 2 - n)}{k^2 n^2} \left( \sum_{i=1}^k \Lambda_i^{(l-2)/(l-1)} \right) \left( \sum_{i=1}^k \Lambda_i^{1/(l-1)} \right). \]
Let \( \delta > 0 \) be any positive number and \( k \) be a positive integer. Then we have

\[
\sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \left( 2 + \frac{n-2}{\Lambda_i^{1/(l-1)} - (n-2)} \right) \\
\leq \delta \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 H_i + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i) \left( \Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right),
\]

where \( l \geq 2 \) and \( \Lambda_i \) be the \( i \)th eigenvalue of the following eigenvalue problem:

\[
(-\Delta)^l u = -\Lambda u \quad \text{in } \Omega,
\]

\[
\frac{\partial u}{\partial \nu}|_{\partial \Omega} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}|_{\partial \Omega} = 0,
\]

where \( \Omega \) is a connected bounded domain in an \( n \)-dimensional Euclidean sphere with smooth boundary \( \partial \Omega \) and \( \nu \) is the unit outward normal vector field of \( \partial \Omega \). For each \( q = 0, 1, \ldots, \) define the polynomials \( F_q \) and \( G_q \) inductively by

\[
F_0 = G_0 = 1, \quad F_1(t) = t - (n + 2), \quad G_1(t) = 3t + n - 2,
\]

\[
F_q(t) = (2t - 2)F_{q-1}(t) - (t^2 + 2t - n(n - 2))F_{q-2}(t),
\]

\[
G_q(t) = (2t - 2)G_{q-1}(t) - (t^2 + 2t - n(n - 2))G_{q-2}(t), \quad q = 2, \ldots
\]

Set

\[
tF_{l-2}(t) - G_{l-2}(t) = t^{l-1} + a_{l-2}t^{l-2} + \cdots + a_1 t + a_0.
\]

Let \( \delta \) be any positive number and \( k \) be a positive integer. Then we have

\[
\sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \left( 2 + \frac{n-2}{\Lambda_i^{1/(l-1)} - (n-2)} \right) \\
\leq \delta \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 H_i + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i) \left( \Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4} \right),
\]
where

\[ H_i = \Lambda_i^{1/(l-1)} \left(1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n-2)}\right) + \sum_{j=0}^{l-2} |a_j| \Lambda_i^{1/(l-1)}. \] (1.22)

**Remark.** When \( l = 2 \), it is easy to see that

\[ H_i = 1 + \Lambda_i \left(1 - \frac{1}{\Lambda_i - (n-2)}\right) \]

and so the inequality (1.21) in this case can be written as

\[ 2 \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left(\delta + \delta \Lambda_i \left(1 - \frac{1}{\Lambda_i - (n-2)}\right) - \frac{n-2}{\Lambda_i - (n-2)}\right) + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4}\right). \] (1.23)

Observe that (1.23) is stronger than (1.10) since for any \( \delta > 0 \), we have

\[ \delta + \delta \Lambda_i \left(1 - \frac{1}{\Lambda_i - (n-2)}\right) - \frac{n-2}{\Lambda_i - (n-2)} \leq \delta \Lambda_i + \frac{\delta^2 (\Lambda_i - (n-2))}{4(\delta \Lambda_i + n-2)}. \]

From Theorem 1.2, we can obtain an explicit upper bound on \( \Lambda_{k+1} \) in terms of \( \Lambda_1, \ldots, \Lambda_k \) which is weaker than (1.21).

**Corollary 1.2.** Let the assumptions and the notations be as in Theorem 1.2. It holds

\[ \Lambda_{k+1} \leq A_{k+1} + \sqrt{B_{k+1}}, \] \( \text{where} \)

\[ A_{k+1} = \frac{1}{k} \sum_{i=1}^{k} \Lambda_i + \frac{2}{k} \sum_{i=1}^{k} T_i, \quad B_{k+1} = \frac{1}{k} \sum_{i=1}^{k} \Lambda_i^2 + \frac{4}{k} \sum_{i=1}^{k} T_i \Lambda_i, \] \( \text{and} \)

\[ S_k = 2 + \frac{n-2}{\Lambda_k^{1/(l-1)} - (n-2)}, \quad T_i = H_i \left(\Lambda_i^{1/(l-1)} + \frac{(n-2)^2}{4}\right). \] (1.26)

**2. Proof of the Results**

Before proving our results, let us recall a method of constructing trial functions developed by Cheng and Yang (cf. [7, 22]). We will state it in a quite general form since we believe that it could be useful for studying eigenvalues of the buckling problem of high orders on compact domains of complete submanifolds in a Euclidean space. Let \( M \) be an \( n \)-dimensional complete submanifold in an \( m \)-dimensional Euclidean space \( \mathbb{R}^m \). Denote by \( \langle ., . \rangle \) the canonical metric on \( \mathbb{R}^m \) as well as that induced on \( M \). Denote by \( \Delta \) and \( \nabla \) the Laplacian and the gradient operator of \( M \), respectively. Let \( \Omega \) be a bounded connected domain of \( M \) with
smooth boundary $\partial \Omega$ and let $\nu$ be the outward unit normal vector field of $\partial \Omega$. For functions $f$ and $g$ on $\Omega$, the Dirichlet inner product $(f, g)_D$ of $f$ and $g$ is given by

$$(f, g)_D = \int_\Omega \langle \nabla f, \nabla g \rangle.$$ 

The Dirichlet norm of a function $f$ is defined by

$$\|f\|_D = \{(f, f)_D\}^{1/2} = \left( \int_\Omega |\nabla f|^2 \right)^{1/2}.$$ 

Consider the eigenvalue problem

$$(-\Delta)^l u = -\Lambda u \quad \text{in } \Omega,$$

$$u|_{\partial \Omega} = \frac{\partial^1 u}{\partial v}|_{\partial \Omega} = \cdots = \frac{\partial^{l-1} u}{\partial v^{l-1}}|_{\partial \Omega} = 0.$$ 

Let

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots,$$

denote the successive eigenvalues, where each eigenvalue is repeated according to its multiplicity.

Let $u_i$ be the $i$th orthonormal eigenfunction of the problem (2.1) corresponding to the eigenvalue $\Lambda_i$, $i = 1, 2, \ldots$, that is, $u_i$ satisfies

$$(-\Delta)^l u_i = -\Lambda_i u_i \quad \text{in } \Omega,$$

$$u_i|_{\partial \Omega} = \frac{\partial^1 u_i}{\partial v}|_{\partial \Omega} = \cdots = \frac{\partial^{l-1} u_i}{\partial v^{l-1}}|_{\partial \Omega} = 0,$$

$$(u_i, u_j)_D = \int_\Omega \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}, \quad \forall i, j.$$ 

For $k = 1, \ldots, l$, let $\nabla^k$ denote the $k$th covariant derivative operator on $M$, defined in the usual weak sense via an integration by parts formula. For a function $f$ on $\Omega$, the squared norm of $\nabla^k f$ is defined as (cf. [14])

$$|\nabla^k f|^2 = \sum_{i_1, \ldots, i_n=1}^n (\nabla^k f(e_{i_1}, \ldots, e_{i_n}))^2,$$

where $e_1, \ldots, e_n$ are orthonormal vector fields locally defined on $\Omega$. Define the Sobolev space $H^2_l(\Omega)$ by

$$H^2_l(\Omega) = \{ f : \| f \|, |\nabla f|, \ldots, |\nabla^l f| \in L^2(\Omega) \}.$$ 

Then $H^2_l(\Omega)$ is a Hilbert space with respect to the norm $\| \cdot \|_{l, 2}$:

$$\|f\|_{l, 2} = \left( \int_\Omega \left( \sum_{k=0}^l |\nabla^k f|^2 \right) \right)^{1/2}.$$ 

Consider the subspace $H^2_{L,D}(\Omega)$ of $H^2(\Omega)$ defined by

$$H^2_{L,D}(\Omega) = \left\{ f \in H^2(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial v}|_{\partial\Omega} = \cdots = \frac{\partial^{\ell-1} f}{\partial v^{\ell-1}}|_{\partial\Omega} = 0 \right\}.$$ 

The operator $(-\Delta)^{\ell}$ defines a self-adjoint operator acting on $H^2_{L,D}(\Omega)$ with discrete eigenvalues $0 < \Lambda_1 \leq \cdots \leq \Lambda_k \leq \cdots$ for the buckling problem (2.1) and the eigenfunctions $\{u_j\}_{j=1}^\infty$ defined in (2.2) form a complete orthonormal basis for the Hilbert space $H^2_{L,D}(\Omega)$. If $\phi \in H^2_{L,D}(\Omega)$ satisfies $(\phi, u_j)_D = 0$, $\forall j = 1, \ldots, k$, then the Rayleigh–Ritz inequality tells us that

$$\Lambda_{k+1} \|\phi\|_D^2 \leq \int_\Omega \phi(-\Delta)^\ell \phi. \tag{2.5}$$

For vector-valued functions $F = (f_1, f_2, \ldots, f_m)$, $G = (g_1, g_2, \ldots, g_m) : \Omega \to \mathbb{R}^m$, we define an inner product $(F, G)$ by

$$(F, G) \equiv \int_\Omega \langle F, G \rangle = \int_\Omega \sum_{j=1}^m f_j g_j.$$ 

The norm of $F$ is given by

$$\|F\| = (F, F)^{1/2} = \left\{ \int_\Omega \sum_{j=1}^m f_j^2 \right\}^{1/2}.$$ 

Let $H^2_1(\Omega)$ be the Hilbert space of vector-valued functions given by

$$H^2_1(\Omega) = \{ F = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m; f_z, |\nabla f_z| \in L^2(\Omega), \text{ for } z = 1, \ldots, m \}$$

with norm

$$\|F\|_1 = \left( \|F\|^2 + \int_\Omega \sum_{z=1}^m |\nabla f_z|^2 \right)^{1/2}.$$ 

Observe that a vector field on $\Omega$ can be regarded as a vector-valued function from $\Omega$ to $\mathbb{R}^m$. Let $H^2_{L,D}(\Omega) \subset H^2_1(\Omega)$ be a subspace of $H^2_1(\Omega)$ spanned by the vector-valued functions $\{\nabla u_j\}_{j=1}^\infty$, which form a complete orthonormal basis of $H^2_{L,D}(\Omega)$. For any $f \in H^2_{L,D}(\Omega)$, we have $\nabla f \in H^2_{L,D}(\Omega)$ and for any $X \in H^2_{L,D}(\Omega)$, there exists a function $f \in H^2_{L,D}(\Omega)$ such that $X = \nabla f$.

**Lemma 2.1.** Let $u_i$ and $\Lambda_i$, $i = 1, 2, \ldots$, be as in (2.2), then

$$0 \leq \int_M u_i (-\Delta)^k u_i \leq \Lambda_i^{(k-1)(l-1)}, \quad k = 1, \ldots, l - 1. \tag{2.6}$$

**Proof of Lemma 2.1.** When $k \in \{1, \ldots, l - 1\}$ is even, we have

$$\int_M u_i (-\Delta)^k u_i = \int_M u_i \Delta^k u_i = \int_M (\Delta^{k/2} u_i)^2 \geq 0. \tag{2.7}$$
On the other hand, if $k \in \{1, \ldots, l-1\}$ is odd, it holds

\[
\int_M u_i(-\Delta)^k u_i = -\int_M u_i \Delta^k u_i = -\int_M \Delta^{(k-1)/2} u_i \Delta (\Delta^{(k-1)/2} u_i) = \int_M |\nabla (\Delta^{(k-1)/2} u_i)|^2 \geq 0.
\]

Thus the inequality at the left hand side of (2.6) holds. Observe that when $k$ is even, we have

\[
\int_M u_i(-\Delta)^k u_i = \int_M \Delta^{k/2-1} u_i \Delta (\Delta^{k/2} u_i) \\
\leq \left( \int_M |\nabla (\Delta^{k/2-1} u_i)|^2 \right)^{1/2} \left( \int_M |\nabla (\Delta^{k/2} u_i)|^2 \right)^{1/2} \\
= \left( -\int_M \Delta^{k/2-1} u_i \Delta^{k/2} u_i \right)^{1/2} \left( -\int_M \Delta^{k/2} u_i \Delta^{k/2+1} u_i \right)^{1/2} \\
= \left( \int_M u_i(-\Delta)^{k-1} u_i \right)^{1/2} \left( \int_M u_i(-\Delta)^{k+1} u_i \right)^{1/2}.
\]

(2.8)

On the other hand, when $n$ is odd, it holds

\[
\int_M u_i(-\Delta)^k u_i = \int_M (-\Delta)^{(k-1)/2} u_i (-\Delta)^{(k+1)/2} u_i \\
\leq \left( \int_M ((-\Delta)^{(k-1)/2} u_i)^2 \right)^{1/2} \left( \int_M (-\Delta)^{(k+1)/2} u_i)^2 \right)^{1/2} \\
= \left( \int_M u_i(-\Delta)^{k-1} u_i \right)^{1/2} \left( \int_M u_i(-\Delta)^{k+1} u_i \right)^{1/2}.
\]

(2.9)

Thus we always have

\[
\int_M u_i(-\Delta)^k u_i \leq \left( \int_M u_i(-\Delta)^{k-1} u_i \right)^{1/2} \left( \int_M u_i(-\Delta)^{k+1} u_i \right)^{1/2}.
\]

(2.10)

When $k = 1$ or $l = 2$, the right hand side of (2.6) holds obviously. Now we consider the case that $l > 2$ and $k \geq 2$. We claim now that for any $k = 2, \ldots, l-1$, it holds

\[
\left( \int_M u_i(-\Delta)^k u_i \right)^k \leq \left( \int_M u_i(-\Delta)^{k+1} u_i \right)^{k-1}.
\]

(2.11)

Since

\[
\int_M u_i \Delta^2 u_i = \int_M \Delta u_i \Delta u_i = -\int_M \nabla \Delta u_i \nabla u_i,
\]
we have from Schwarz inequality that
\[
\left( \int_{\Omega} u_i \Delta^2 u_i \right)^2 \leq \left( \int_{\Omega} |\nabla u_i|^2 \right) \left( \int_{\Omega} |\nabla u_i|^2 \right)
\]
\[
= - \int_{\Omega} \Delta u_i \Delta^2 u_i = \int_{\Omega} u_i (-\Delta^2 u_i).
\]  
(2.12)

Hence (2.11) holds when \( k = 2 \). Suppose that (2.11) holds for \( k - 1 \), that is
\[
\left( \int_{\Omega} u_i (-\Delta)^{k-1} u_i \right)^{k-1} \leq \left( \int_{\Omega} u_i (-\Delta)^{k-2} u_i \right)^{k-2}.
\]  
(2.13)

Substituting (2.13) into (2.10), we know that (2.6) is true for \( k \). Using (2.6) repeatedly, we get
\[
\int_{\Omega} u_i (-\Delta)^k u_i \leq \left( \int_{\Omega} u_i (-\Delta)^{k+1} u_i \right)^{(k-1)/k} \leq \cdots \leq \left( \int_{\Omega} u_i (-\Delta)^{k/(i-1)} u_i \right)^{(i-1)/(i-1)} = \Lambda^i_{(i-1)/(i-1)}.
\]

This completes the proof of Lemma 2.1.

**Lemma 2.2.** Let \( \{a_i\}_{i=1}^m \), \( \{b_i\}_{i=1}^m \) and \( \{c_i\}_{i=1}^m \) be three sequences of non-negative real numbers with \( \{a_i\} \) decreasing and \( \{b_i\} \) and \( \{c_i\}_{i=1}^m \) increasing. Then the following inequality holds:
\[
\left( \sum_{i=1}^m a_i^2 b_i \right) \left( \sum_{i=1}^m a_i c_i \right) \leq \left( \sum_{i=1}^m a_i^2 \right) \left( \sum_{i=1}^m a_i b_i c_i \right).
\]  
(2.14)

**Proof.** When \( m = 1 \), (2.14) holds trivially. Suppose that (2.14) holds when \( m = k \), that is
\[
\left( \sum_{i=1}^k a_i^2 b_i \right) \left( \sum_{i=1}^k a_i c_i \right) \leq \left( \sum_{i=1}^k a_i^2 \right) \left( \sum_{i=1}^k a_i b_i c_i \right).
\]  
(2.15)

Then when \( m = k + 1 \), we have from (2.15) that
\[
\left( \sum_{i=1}^{k+1} a_i^2 \right) \left( \sum_{i=1}^{k+1} a_i b_i c_i \right) - \left( \sum_{i=1}^k a_i^2 b_i \right) \left( \sum_{i=1}^k a_i c_i \right)
\]
\[
= \left( \sum_{i=1}^k a_i^2 \right) \left( \sum_{i=1}^k a_i b_i c_i \right) - \left( \sum_{i=1}^k a_i^2 b_i \right) \left( \sum_{i=1}^k a_i c_i \right) + a_{k+1}^2 \sum_{i=1}^k a_i b_i c_i
\]
\[
- a_{k+1}^2 b_{k+1} \sum_{i=1}^k a_i c_i + a_{k+1} b_{k+1} c_{k+1} \sum_{i=1}^k a_i^2 - a_{k+1} c_{k+1} \sum_{i=1}^k a_i^2 b_i
\]
\[
\geq a_{k+1}^2 \sum_{i=1}^k a_i b_i c_i - a_{k+1}^2 b_{k+1} \sum_{i=1}^k a_i c_i + a_{k+1} b_{k+1} c_{k+1} \sum_{i=1}^k a_i^2 - a_{k+1} c_{k+1} \sum_{i=1}^k a_i^2 b_i
\]
Universal Inequalities for Eigenvalues

\[-a_2^k \sum_{i=1}^k (b_{k+1} - b_i) a_i c_i + a_{k+1} c_{k+1} \sum_{i=1}^k a_i^2 (b_{k+1} - b_i)\]

\[= \sum_{j=1}^k a_{k+1} a_j (b_{k+1} - b_j) (c_{k+1} a_i - a_{k+1} c_i) \geq 0. \quad (2.16)\]

Where in the last inequality we have used the fact that

\[a_{k+1} a_j (b_{k+1} - b_j) (c_{k+1} a_i - a_{k+1} c_i) \geq 0, \quad i = 1, \ldots, k.\]

Thus (2.14) holds for \(m = k + 1\). This completes the proof of Lemma 2.2.

The following result is the so-called **Reverse Chebyshev Inequality** (cf. [9]).

**Lemma 2.3.** Suppose \(\{a_i\}_{i=1}^m\) and \(\{b_i\}_{i=1}^m\) are two real sequences with \(\{a_i\}\) increasing and \(\{b_i\}\) decreasing. Then the following inequality holds:

\[\sum_{i=1}^m a_i b_i \leq \frac{1}{m} \left( \sum_{i=1}^m a_i \right) \left( \sum_{i=1}^m b_i \right). \quad (2.17)\]

We are now ready to prove the main results in this paper.

**Proof of Theorem 1.1.** With the notations as above, we consider now the special case that \(\Omega\) is a connected bounded domain in \(\mathbb{R}^n\). Denote by \(x_1, \ldots, x_n\) the coordinate functions on \(\mathbb{R}^n\) and let us decompose the vector-valued functions \(x_2 \nabla u_i\) as

\[x_2 \nabla u_i = \nabla h_{x_i} + W_{x_i}, \quad (2.18)\]

where \(h_{x_i} \in H^1_{l,D}(\Omega)\), \(\nabla h_{x_i}\) is the projection of \(x_2 \nabla u_i\) in \(H^1_{l,D}(\Omega)\) and \(W_{x_i} \perp H^1_{l,D}(\Omega)\).

Thus we have

\[W_{x_i}|_{\Omega} = 0, \quad \text{and} \quad (W_{x_i}, \nabla u) = \int_{\Omega} \langle W_{x_i}, \nabla u \rangle = 0, \quad \text{for any} \ u \in H^2_{l,D}(\Omega) \quad (2.19)\]

and from the discussions in [7, 22] we know that

\[\text{div} W_{x_i} = 0, \quad (2.20)\]

where for a vector field \(Z\) on \(\Omega\), \(\text{div} Z\) denotes the divergence of \(Z\).

For each \(x = 1, \ldots, n, \ i = 1, \ldots, k\), consider the functions \(\phi_{x_i} : \Omega \to \mathbb{R}\), given by

\[\phi_{x_i} = h_{x_i} - \sum_{j=1}^k a_{x_i j} u_j, \quad (2.21)\]

where

\[a_{x_i j} = \int_{\Omega} x_2 \langle \nabla u_j, \nabla u_i \rangle = a_{jix}. \quad (2.22)\]
We have
\[
\phi_{\alpha i}|_{\Omega} = \frac{\partial \phi_{\alpha j}}{\partial \nu} \bigg|_{\Omega} = \cdots = \frac{\partial^{i-1} \phi_{\alpha i}}{\partial \nu^{i-1}} \bigg|_{\Omega} = 0, \quad (2.23)
\]
\[
(\phi_{\alpha i}, u_j)_D = \int_\Omega \langle \nabla \phi_{\alpha i}, \nabla u_j \rangle = 0, \quad \forall j = 1, \ldots, k. \quad (2.24)
\]

It then follows from the Rayleigh–Ritz inequality for \( \Lambda_{k+1} \) that
\[
\Lambda_{k+1} \int_\Omega |\nabla \phi_{\alpha i}|^2 \leq \int_\Omega \phi_{\alpha i}(-\Delta)^i \phi_{\alpha i}, \quad \forall x = 1, \ldots, n, \ i = 1, \ldots, k. \quad (2.25)
\]

Since \( \text{div} W_{\alpha i} = 0 \), we have from (2.18) and (2.21) that
\[
\Delta \phi_{\alpha i} = \Delta h_{\alpha i} - \sum_{j=1}^k a_{\alpha ij} \Delta u_j
\]
\[
= \text{div}(x_i \nabla u_j) - \sum_{j=1}^k a_{\alpha ij} \Delta u_j
\]
\[
= u_{i,x} + x_i \Delta u_j - \sum_{j=1}^k a_{\alpha ij} \Delta u_j,
\]
where \( u_{i,x} = \frac{\partial u_i}{\partial x_j} \). Thus we have
\[
(-\Delta)^i \phi_{\alpha i} = (-1)^i \Delta^{i-1} \left( u_{i,x} + x_i \Delta u_j \right) + \sum_{j=1}^k a_{\alpha ij} \Delta u_j. \quad (2.26)
\]

Since
\[
\int_\Omega \phi_{\alpha i} \Delta u_j = - \int_\Omega \langle \nabla \phi_{\alpha i}, \nabla u_j \rangle = 0,
\]
\[
\Delta^{i-2}(x_i \Delta u_i) = 2(l-2)(\Delta^{i-2}u_i)_x + x_i \Delta^{i-1} u_i,
\]

We have
\[
\int_\Omega \phi_{\alpha i}(-\Delta)^i \phi_{\alpha i}
\]
\[
= \int_\Omega \phi_{\alpha i}(-1)^i \Delta^{i-1} \left( u_{i,x} + x_i \Delta u_j \right)
\]
\[
= \int_\Omega h_{\alpha i}(-1)^i \Delta^{i-1} \left( u_{i,x} + x_i \Delta u_j \right) - \sum_{j=1}^k a_{\alpha ij} \int_\Omega u_j(-\Delta)^i h_{\alpha i}
\]
\[
= \int_\Omega \Delta h_{\alpha i}(-1)^i \Delta^{i-2} \left( u_{i,x} + x_i \Delta u_j \right) - \sum_{j=1}^k a_{\alpha ij} \int_\Omega h_{\alpha i}(-\Delta)^i u_j
\]
\[
= \int_\Omega \Delta h_{\alpha i}(-1)^i ((\Delta^{i-2} u_i)_x + \Delta^{i-2}(x_i \Delta u_i)) + \sum_{j=1}^k \Lambda_j a_{\alpha ij} \int_\Omega h_{\alpha i} \Delta u_j
\]
Let us make some calculations. Since

\begin{equation}
\Delta^{l-1}(x_i)u_i = 2(l-1)(\Delta^{l-2}u_i)_{,z} + x_i\Delta^{l-1}u_i,
\end{equation}

we have

\begin{align*}
\int_{\Omega} x_i u_i (\Delta^{l-1}u_i)_{,z} &= \int_{\Omega} x_i u_i \Delta^{l-1}u_i_{,z} \\
&= \int_{\Omega} \Delta^{l-1}(x_i)u_i_{,z} \\
&= \int_{\Omega} \left(2(l-1)(\Delta^{l-2}u_i)_{,z} + x_i\Delta^{l-1}u_i\right)u_i_{,z}.
\end{align*}

(2.28)

On the other hand, it holds

\begin{equation}
\int_{\Omega} x_i u_i (\Delta^{l-1}u_i)_{,z} = -\int_{\Omega} \Delta^{l-1}u_i(u_i + x_i u_{i,z}).
\end{equation}

(2.29)

Combining (2.28) and (2.29), we obtain

\begin{equation}
\int_{\Omega} x_i u_i (\Delta^{l-1}u_i)_{,z} = \int_{\Omega} \left\{ (l-1)(\Delta^{l-2}u_i)_{,z}u_i_{,z} - \frac{1}{2}u_i\Delta^{l-1}u_i \right\}.
\end{equation}

(2.30)

Hence

\begin{align*}
\int_{\Omega} x_i u_i, u_i z \Delta^{l-1}u_i &= -\int_{\Omega} u_i (\Delta^{l-1}u_i + x_i (\Delta^{l-1}u_i)_{,z}) \\
&= -\int_{\Omega} \left\{ (l-1)(\Delta^{l-2}u_i)_{,z}u_i_{,z} + \frac{1}{2}u_i\Delta^{l-1}u_i \right\}
\end{align*}

(2.31)

and consequently, we have

\begin{align*}
\int_{\Omega} x_i u_i (\Delta^{l-2}u_i)_{,z} &= \int_{\Omega} x_i u_i \Delta^{l-2}u_i_{,z} \\
&= \int_{\Omega} \Delta^{l-2}(x_i u_i)u_i_{,z} \\
&= \int_{\Omega} u_i (2(l-2)(\Delta^{l-2}u_i)_{,z} + x_i\Delta^{l-1}u_i) \\
&= \int_{\Omega} \left\{ (l-3)(\Delta^{l-2}u_i)_{,z}u_i_{,z} - \frac{1}{2}u_i\Delta^{l-1}u_i \right\}.
\end{align*}

(2.32)
Also, one has
\[
\int_{\Omega} u_i x_i^2 \Delta u_i = - \int_{\Omega} x_i^2 |\nabla u_i|^2 - 2 \int_{\Omega} x_i u_i u_{i,zz} \\
= - \int_{\Omega} x_i^2 |\nabla u_i|^2 + \int_{\Omega} u_i^2, \tag{2.33}
\]
\[
\int_{\Omega} x_i^2 \Delta u_i \Delta^{l-1} u_i = \int_{\Omega} u_i \Delta (x_i^2 \Delta^{l-1} u_i) \\
= \int_{\Omega} u_i (2 \Delta^{l-1} u_i + x_i^2 \Delta^{l-1} u_i + 4 x_i (\Delta^{l-1} u_i)_{,zz}) \\
= \int_{\Omega} u_i (2 \Delta^{l-1} u_i + (-1)^{l-1} A_i x_i^2 \Delta u_i + 4 x_i (\Delta^{l-1} u_i)_{,zz}). \tag{2.34}
\]
Combining (2.30), (2.33) and (2.34), we get
\[
\int_{\Omega} x_i^2 \Delta u_i \Delta^{l-1} u_i = 4(l - 1) \int_{\Omega} (\Delta^{l-2} u_i)_{,zz} u_{i,zz} \\
+ (-1)^{l-1} A_i \left\{ - \int_{\Omega} x_i^2 |\nabla u_i|^2 + \int_{\Omega} u_i^2 \right\}. \tag{2.35}
\]
Substituting (2.32), (2.33) and (2.35) into (2.27), one gets
\[
\int_{\Omega} \phi_{,i} (-\Delta)^l \phi_{,i} = \int_{\Omega} (-1)^l \left\{ (-l + 1) u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{,zz} u_{i,zz} \right\} \\
+ A_i \left\{ \int_{\Omega} x_i^2 |\nabla u_i|^2 - \int_{\Omega} u_i^2 \right\} - \sum_{j=1}^{k} A_i a_{ij}^2. \tag{2.36}
\]
It is easy to see that
\[
\|x_i \nabla u_i\|^2 = \|\nabla h_{,i}\|^2 + \|W_{,i}\|^2, \quad \|\nabla h_{,i}\|^2 = \|\nabla \phi_{,i}\|^2 + \sum_{j=1}^{k} a_{ij}^2, \tag{2.37}
\]
where for a vector field $Z$ on $\Omega$, $\|Z\|^2 = \int_{\Omega} |Z|^2$. Combining (2.25), (2.36) and (2.37), we infer
\[
(A_{k+1} - A_i) \|\nabla \phi_{,i}\|^2 \leq \int_{\Omega} (-1)^l \left\{ (-l + 1) u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{,zz} u_{i,zz} \right\} \\
- A_i (\|u_i\|^2 + \|W_{,i}\|^2) + \sum_{j=1}^{k} (A_i - A_i) a_{ij}^2 \tag{2.38}
\]
Observe that $\nabla(x_i u_i) = u_i \nabla x_i + x_i \nabla u_i \in H^1_{i,0}(\Omega)$. For $A_{si} = \nabla(x_i u_i - h_{si})$, we have
\[
u_{si} \nabla x_i = A_{si} - W_{si} \tag{2.39}
\]
and so
\[
\|u_i\|^2 = \|nu_i \nabla x_i\|^2 = \|W_{si}\|^2 + \|A_{si}\|^2.
\]
Because of $(\nabla u_{i,z}, W_{u}) = 0$, it follows that
\[
2\|u_{i,z}\|^2 = -2 \int_{\Omega} \langle u_i \nabla x_{z}, \nabla u_{i,z} \rangle = -2 \int_{\Omega} \langle \Lambda_{u}, \nabla u_{i,z} \rangle \\
\leq \Lambda_{j}^{1/(l-1)} \|A_{u}\|^2 + \frac{1}{\Lambda_{j}^{1/(l-1)}} \|\nabla u_{i,z}\|^2
\]
which gives
\[
-\Lambda_{i} \|A_{u}\|^2 \leq -2\Lambda_{j}^{(l-2)/(l-1)} \|u_{i,z}\|^2 + \Lambda_{j}^{(l-3)/(l-1)} \|\nabla u_{i,z}\|^2. \tag{2.40}
\]
Introducing (2.40) into (2.38), we get
\[
(\Lambda_{k+1} - \Lambda_{j}) \|\nabla \phi_{u}\|^2 \leq \int_{\Omega} (-1)^l \{(1-l)u_{\Delta}^{l-1}u_{i} + (2l^2 - 4l + 3)(\Delta^{l-2}u_{i})_{,u}u_{i,z} \}
\]\[
- 2\Lambda_{j}^{(l-2)/(l-1)} \|u_{i,z}\|^2 + \Lambda_{j}^{(l-3)/(l-1)} \|\nabla u_{i,z}\|^2 + \sum_{j=1}^{k} (\Lambda_{i} - \Lambda_{j}) a_{ij}^2. \tag{2.41}
\]
Since
\[
-2 \int_{\Omega} x_{z} \langle \nabla u_{i}, \nabla u_{i,z} \rangle = 2 \int_{\Omega} u_{i,z}^2 + 2 \int_{\Omega} x_{z} u_{i,z} \Delta u_{i}
\]\[
= 2 \int_{\Omega} u_{i,z}^2 + 2 \int_{\Omega} u_{i}\Delta (x_{z} u_{i,z})
\]\[
= 2 \int_{\Omega} u_{i,z}^2 + 2 \int_{\Omega} u_{i} x_{z} (\Delta u_{i})_{,z} + 4 \int_{\Omega} \langle u_{i} \nabla x_{z}, \nabla u_{i,z} \rangle
\]\[
= 2 \int_{\Omega} u_{i,z}^2 - 2 \int_{\Omega} \Delta u_{i} (u_{i} + x_{z} u_{i,z}) - 4 \int_{\Omega} u_{i,z} \text{div}(u_{i} \nabla x_{z})
\]\[
= 2 \int_{\Omega} u_{i,z}^2 + 2 - 2 \int_{\Omega} x_{z} u_{i,z} \Delta u_{i} - 4 \int_{\Omega} u_{i,z}^2
\]\[
= -2 \int_{\Omega} u_{i,z}^2 + 2 - 2 \int_{\Omega} \langle \nabla u_{i}, \nabla (x_{z} u_{i,z}) \rangle
\]\[
= 2 + 2 \int_{\Omega} x_{z} \langle \nabla u_{i}, \nabla u_{i,z} \rangle,
\]
we have
\[
-2 \int_{\Omega} x_{z} \langle \nabla u_{i}, \nabla u_{i,z} \rangle = 1. \tag{2.42}
\]
Set
\[
d_{ij} = \int_{\Omega} \langle \nabla u_{i,z}, \nabla u_{j} \rangle;
\]
then $d_{ij} = -d_{ji}$ and we have from (2.18), (2.20) and (2.21) that
\[
1 = -2 \int_{\Omega} x_{z} \langle \nabla u_{i}, \nabla u_{i,z} \rangle
\]
\[\sum_{j=1}^{k} a_{ij}d_{xj} \]

Thus, we have

\[(\Lambda_{k+1} - \Lambda_i)^2 \left( 1 + 2 \sum_{j=1}^{k} a_{ij}d_{xj} \right)\]

\[= (\Lambda_{k+1} - \Lambda_i)^2 \left( -2\nabla \phi_{xj}, \left( \nabla u_{i,x} - \sum_{j=1}^{k} d_{xj} \nabla u_j \right) \right)\]

\[\leq \delta(\Lambda_{k+1} - \Lambda_i)^2 \| \nabla \phi_{xj} \|^2 + \frac{1}{\delta}(\Lambda_{k+1} - \Lambda_i) \left( \| \nabla u_{i,x} \|^2 - \sum_{j=1}^{k} d_{xj}^2 \right).\] (2.44)

where \(\delta\) is any positive constant. Substituting (2.41) into (2.44), we get

\[(\Lambda_{k+1} - \Lambda_i)^2 \left( 1 + 2 \sum_{j=1}^{k} a_{ij}d_{xj} \right)\]

\[\leq \delta(\Lambda_{k+1} - \Lambda_i)^2 \left( \int_{\Omega} (-1)^l \{ (-l+1)u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{x,x} \} \right)\]

\[- 2\Lambda_i^{(l-2)/(l-1)} \| u_{i,x} \|^2 + \Lambda_i^{(l-3)/(l-1)} \| \nabla u_{i,x} \|^2 + \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) a_{xj}^2 + \sum_{j=1}^{k} d_{xj}^2 \] (2.45)

Summing on \(i\) from 1 to \(k\) and noticing the fact that \(a_{xj} = a_{xji}, d_{xj} = -d_{xji}\), we infer

\[\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 - 2 \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j)a_{xj}d_{xj}\]

\[\leq \delta \left( \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \int_{\Omega} (-1)^l \{ (-l+1)u_i \Delta^{l-1} u_i + (2l^2 - 4l + 3)(\Delta^{l-2} u_i)_{x,x} \} \right)\right)\]

\[- 2\Lambda_i^{(l-2)/(l-1)} \| u_{i,x} \|^2 + \Lambda_i^{(l-3)/(l-1)} \| \nabla u_{i,x} \|^2 - \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j) a_{xj}^2 \] (2.45)

which gives

\[\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left( \int_{\Omega} (-1)^l \{ (-l+1)u_i \Delta^{l-1} u_i \right)\]
Let us consider the sum for $\ell$ from 1 to $n$, we get

\begin{align}
\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 & \leq \delta \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \left( \int_{\Omega} (-1)^{\ell} \left\{ n(-l+1)u_1{\Delta}^{l-1}u_1 + (2\ell^2 - 4\ell + 3)(\nabla{\Delta}^{l-2}u_1, \nabla u_1) \right\} 
- 2\lambda_j^{(l-2)/(l-1)} + \lambda_j^{(l-3)/(l-1)} \sum_{i=1}^{n} \|\nabla u_{i,x,l}\|^2 \right) + \frac{1}{\delta} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \sum_{i=1}^{n} \|\nabla u_{i,x,l}\|^2 \\
& = \delta \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \left( -2\lambda_j^{(l-2)/(l-1)} + \lambda_j^{(l-3)/(l-1)} \sum_{i=1}^{n} \|\nabla u_{i,x,l}\|^2 \right) \\
& \quad + (2\ell^2 + (n-4)l + 3 - n) \int_{\Omega} u_1(-\Delta)^{l-1}u_1 \right) + \frac{1}{\delta} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \sum_{i=1}^{n} \|\nabla u_{i,x,l}\|^2.
\end{align}

But

\begin{align}
\sum_{i=1}^{k} \|\nabla u_{i,x,l}\|^2 &= \int_{\Omega} \sum_{i=1}^{k} u_{i,x,l} \Delta u_{i,x,l} \\
& = \int_{\Omega} \sum_{i=1}^{k} u_{i,x,l} (\Delta u_i)_{,x} \\
& = \int_{\Omega} \sum_{i=1}^{k} u_{i,xx} \Delta u_i \\
& = \int_{\Omega} \Delta u_i^2 \\
& = \int_{\Omega} u_i \Delta^2 u_i,
\end{align}

where $u_{i,xx} = \frac{\partial^2 u_i}{\partial x^2}$. Thus, we have

\begin{align}
\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 & \leq \delta \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \left( -2\lambda_j^{(l-2)/(l-1)} + \lambda_j^{(l-3)/(l-1)} \int_{\Omega} u_1 \Delta^2 u_i \\
& \quad + (2\ell^2 + (n-4)l + 3 - n) \int_{\Omega} u_1(-\Delta)^{l-1}u_1 \right) \\
& \quad + \frac{1}{\delta} \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i) \int_{\Omega} u_i \Delta^2 u_i.
\end{align}
Taking $k = 2$ and $k = l - 1$ in (2.6), respectively, one gets

$$\int_{\Omega} u_i (-\Delta)^{l-1} u_i \leq \Lambda_j^{(l-2)/(l-1)} \int_{\Omega} u_i \Delta^j u_j \leq \Lambda_j^{1/(l-1)}$$

which, combining with (2.47) implies that

$$n \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \delta (2l^2 + (n - 4)l + 2 - n) \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_j^{(l-2)/(l-1)} + \frac{1}{\delta} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_j^{1/(l-1)}.$$

Taking

$$\delta = \frac{\left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_j^{1/(l-1)} \right\}^{1/2}}{(2l^2 + (n - 4)l + 2 - n) \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_j^{(l-2)/(l-1)}}^{1/2},$$

we get (1.13). This completes the proof of Theorem 1.1.

Proof of Corollary 1.1. It follows from (2.17) that

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_j^{1/(l-1)} \leq \frac{1}{k} \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \right) \left( \sum_{i=1}^k \Lambda_j^{1/(l-1)} \right) \tag{2.48}$$

and

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_j^{(l-2)/(l-1)} \leq \frac{1}{k} \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \right) \left( \sum_{i=1}^k \Lambda_j^{(l-2)/(l-1)} \right). \tag{2.49}$$

Introducing (2.48) and (2.49) into (1.13), we infer

$$\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4l^2 + (n - 4)l + 2 - n}{k^2 n^2} \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \right) \times \left( \sum_{i=1}^k \Lambda_j^{1/(l-1)} \right) \left( \sum_{i=1}^k \Lambda_j^{(l-2)/(l-1)} \right).$$

Solving this quadratic polynomial about $\Lambda_{k+1}$, one gets (1.14). From (2.14), we have

$$\left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_j^{(l-2)/(l-1)} \right\} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_j^{1/(l-1)} \right\} \leq \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \right\} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_j \right\}.$$
It then follows from (1.13) that
\[ k \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(2l^2 + (n - 4)l + 2 - n)}{n^2} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \Lambda_i, \]
which implies (1.15). This completes the proof of Corollary 1.1.

**Proof of Theorem 1.2.** We use the same notations as in the beginning of this section and take \( M \) to be the unit \( n \)-sphere \( S^n(1) \). Let \( x_1, x_2, \ldots, x_{n+1} \) be the standard coordinate functions of the Euclidean space \( \mathbb{R}^{n+1} \); then
\[ S^n(1) = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}; \sum_{z=1}^{n+1} x_z^2 = 1 \right\}. \]
It is well known that
\[ \Delta x_z = -nx_z, \quad z = 1, \ldots, n+1. \quad (2.50) \]
As in the proof of Theorem 1.1, we decompose the vector-valued functions \( x_z \nabla u_i \) as
\[ x_z \nabla u_i = \nabla h_{zi} + W_{zi}, \quad (2.51) \]
where \( h_{zi} \in H^2_{1,D}(\Omega) \), \( \nabla h_{zi} \) is the projection of \( x_z \nabla u_i \) in \( H^2_{1,D}(\Omega) \), \( W_{zi} \perp H^2_{1,D}(\Omega) \) and
\[ W_{zi}|_{\partial \Omega} = 0, \quad \text{div} \ W_{zi} = 0. \quad (2.52) \]
We also consider the functions \( \phi_{zi} : \Omega \to \mathbb{R} \), given by
\[ \phi_{zi} = h_{zi} - \sum_{j=1}^{k} b_{zj} u_j, \quad b_{zj} = \int_{\Omega} x_z \langle \nabla u_i, \nabla u_j \rangle = b_{zj}. \quad (2.53) \]
Then
\[ \phi_{zi}|_{\partial \Omega} = \frac{\partial \phi_{zi}}{\partial v} \bigg|_{\partial \Omega} = \cdots = \frac{\partial^{d-1} \phi_{zi}}{\partial v^{d-1}} \bigg|_{\partial \Omega} = 0, \]
\[ (\phi_{zi}, u_j)_D = \int_{\Omega} \langle \nabla \phi_{zi}, \nabla u_j \rangle = 0, \quad \forall j = 1, \ldots, k \]
and we have the basic Rayleigh–Ritz inequality for \( \Lambda_{k+1} \):
\[ \Lambda_{k+1} \int_{\Omega} |\nabla \phi_{zi}|^2 \leq \int_{D} \phi_{zi} (-\Delta)' \phi_{zi}, \quad \forall z = 1, \ldots, n, \quad i = 1, \ldots, k. \quad (2.54) \]
We have
\[ \Delta \phi_{zi} = \langle \nabla x_z, \nabla u_i \rangle + x_z \Delta u_i - \sum_{j=1}^{k} b_{zj} \Delta u_j \quad (2.55) \]
and as in the proof of (2.27),
\[
\int_{\Omega} \phi_a(-\Delta)^k \phi_a = \int_{\Omega} (-1)^k \langle \nabla x, \nabla u_j \rangle x_j \Delta u_j - \sum_{j=1}^2 \Delta \beta_{ai}^j.
\] (2.56)
For a function \( g \) on \( \Omega \), we have (cf. (2.31) in [22])
\[
\Delta \langle \nabla x, \nabla g \rangle = -2x_1 \Delta g + \langle \nabla x, \nabla((-\Delta + n - 2)g) \rangle. \] (2.57)
For each \( q = 0, 1, \ldots \), thanks to (2.50) and (2.57), there are polynomials \( F_q \) and \( G_q \) of degree \( q \) such that
\[
\Delta^q(\langle \nabla x, \nabla u_j \rangle + x_j \Delta u_j) = x_j F_q(\Delta) \Delta u_j + \langle \nabla x, \nabla(G_q(\Delta) u_j) \rangle. \quad (2.58)
\]
It is obvious that
\[
F_0 = 1, \quad G_0 = 1. \quad (2.59)
\]
It follows from (2.50) and (2.57) that
\[
\Delta(x_1 \Delta u_1 + \langle \nabla x, \nabla u_j \rangle) = x_1(\Delta - (n + 2)) \Delta u_1 + \langle \nabla x, \nabla((3\Delta + n - 2)u_1) \rangle \quad (2.60)
\]
which gives
\[
F_1(t) = t - (n + 2), \quad G_1(t) = 3t + n - 2. \quad (2.61)
\]
Also, when \( q \geq 2 \), we have
\[
\Delta^q(\langle \nabla x, \nabla u_j \rangle + x_j \Delta u_j) = \Delta(\Delta^{q-1}(\langle \nabla x, \nabla u_j \rangle + x_j \Delta u_j))
\]
\[
= \Delta(x_j F_{q-1}(\Delta) \Delta u_j + \langle \nabla x, \nabla(G_{q-1}(\Delta) u_j) \rangle)
\]
\[
= x_j((\Delta - n) F_{q-1}(\Delta) - 2G_{q-1}(\Delta)) \Delta u_j
\]
\[
+ \langle \nabla x, \nabla((\Delta + n - 2)G_{q-1}(\Delta) + 2\Delta F_{q-1}(\Delta))u_j \rangle
\] (2.62)
which, combining with (2.58), implies that
\[
F_q(\Delta) = (\Delta - n) F_{q-1}(\Delta) - 2G_{q-1}(\Delta), \quad q = 2, \ldots, \quad (2.63)
\]
\[
G_q(\Delta) = (\Delta + n - 2) G_{q-1}(\Delta) + 2\Delta F_{q-1}(\Delta), \quad q = 2, \ldots. \quad (2.64)
\]
It then follows from (2.63) and (2.64) that
\[
F_q(\Delta) = (\Delta - n) F_{q-1}(\Delta) - 2((\Delta + n - 2)G_{q-2}(\Delta) + 2\Delta F_{q-2}(\Delta))
\]
\[
= (\Delta - n) F_{q-1}(\Delta) + (\Delta + n - 2)(F_{q-1}(\Delta) - (\Delta - n) F_{q-2}(\Delta)) - 4\Delta F_{q-2}(\Delta)
\]
\[
= (2\Delta - 2) F_{q-1}(\Delta) - (\Delta^2 + 2\Delta - n(n - 2)) F_{q-2}(\Delta)
\]
and

\[ G_q(\Delta) = (\Delta + n - 2)G_{q-1}(\Delta) + 2\Delta((\Delta - n)F_{q-2}(\Delta) - 2G_{q-2}(\Delta)) \]
\[ = (\Delta + n - 2)G_{q-1}(\Delta) + (\Delta - n)(G_{q-1}(\Delta) - (\Delta + n - 2)G_{q-2}(\Delta)) - 4\Delta G_{q-2}(\Delta) \]
\[ = (2\Delta - 2)G_{q-1}(\Delta) - (\Delta^2 + 2\Delta - n(n - 2))G_{q-2}(\Delta). \]

Thus, we have

\[ F_q(t) = (2t - 2)F_{q-1}(t) - (t^2 + 2t - n(n - 2))F_{q-2}(t), \quad \text{for} \quad q = 2, \ldots \]

(2.65)

That is, the polynomials \( F_q \) and \( G_q \) are defined inductively by (1.17)–(1.19).

Substituting

\[ \Delta^{l-2}(\nabla x_i, \nabla u_j) + x_i \Delta u_j = x_i F_{l-2}(\Delta) \Delta u_j + \langle \nabla x_j, \nabla (G_{l-2}(\Delta)u_j) \rangle \]

(2.67)

into (2.56), we get

\[ \int_{\Omega} \phi u (-\Delta)^j \phi u = \int_{\Omega} (-1)^j \langle \nabla x_j, \nabla u_j, \nabla (G_{l-2}(\Delta)u_j) \rangle 
+ \langle x_i \nabla x_i, \Delta u_i \nabla (G_{l-2}(\Delta)u_i) + (F_{l-2}(\Delta) \Delta u_i) \nabla u_i \rangle 
\]
\[ + \int_{\Omega} (-1)^j x_i^2 \Delta u_i F_{l-2}(\Delta)(\Delta u_i) - \sum_{j=1}^{k} \Lambda_j b_{2ij}^2. \]

(2.68)

Summing over \( x \) and noticing

\[ \sum_{z=1}^{n+1} x_i^2 = 1, \quad \sum_{z=1}^{n+1} \langle \nabla x_j, \nabla u_j, \nabla (G_{l-2}(\Delta)u_j) \rangle = \langle \nabla u_i, \nabla (G_{l-2}(\Delta)u_i) \rangle, \]

(2.69)

we get from (1.20) and (2.6) that

\[ \sum_{x=1}^{n+1} \int_{\Omega} \phi x \phi x = \int_{\Omega} (-1)^j \langle \nabla u_i, \nabla (G_{l-2}(\Delta)u_i) \rangle + \int_{\Omega} (-1)^j \Delta u_i F_{l-2}(\Delta) (\Delta u_i) - \sum_{x=1}^{n+1} \sum_{j=1}^{k} \Lambda_j b_{2ij}^2 
\]
\[ = \int_{\Omega} (-1)^j u_i \Delta (G_{l-2}(\Delta)u_i) + \int_{\Omega} (-1)^j u_i \Delta (F_{l-2}(\Delta)u_i) - \sum_{x=1}^{n+1} \sum_{j=1}^{k} \Lambda_j b_{2ij}^2 
\]
\[ = \int_{\Omega} (-1)^j u_i (\Delta F_{l-2}(\Delta) - (G_{l-2}(\Delta)) (\Delta u_i) - \sum_{x=1}^{n+1} \sum_{j=1}^{k} \Lambda_j b_{2ij}^2 
\]
\[ = \int_{\Omega} (-1)^j u_i (\Delta^{l-1} + a_{l-2} \Delta^{l-2} + \cdots + a_1 \Delta + a_0) (\Delta u_i) - \sum_{x=1}^{n+1} \sum_{j=1}^{k} \Lambda_j b_{2ij}^2 \]
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\[
\begin{align*}
&= \Lambda_i + \int \Omega (-1)^l u (a_{l-2} \Delta^{l-2} + \cdots + a_1 \Delta + a_0) (\Delta u_i) - \sum_{j=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{aj}^2 \\
&\leq \Lambda_i + \sum_{j=0}^{l-2} |a_j| \int \Omega u \Delta^{l-j-1} u_i - \sum_{j=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{aj}^2 \\
&\leq \Lambda_i + \sum_{j=0}^{l-2} |a_j| \Lambda_i^{l-j-1} - \sum_{j=1}^{n+1} \sum_{j=1}^k \Lambda_j b_{aj}^2. 
\end{align*}
\]

(2.70)

Observe from (2.51) and (2.53) that

\[
\| x \Delta u_i \|^2 = \| \nabla h_{zj} \|^2 + \| W_{zj} \|^2 = \| \nabla \phi_{zj} \|^2 + \| W_{zj} \|^2 + \sum_{j=1}^k b_{aj}^2. 
\]

(2.71)

Summing over \( z \), one gets

\[
1 = \sum_{z=1}^{n+1} \left( \| \nabla \phi_{zj} \|^2 + \| W_{zj} \|^2 + \sum_{j=1}^k b_{aj}^2 \right). 
\]

(2.72)

Set

\[
Z_{zj} = \nabla \langle \nabla x_z, \nabla u_i \rangle - \frac{n-2}{2} x_z \nabla u_i, \qquad c_{zj} = \int \Omega \langle \nabla u_j, Z_{zj} \rangle; 
\]

(2.73)

then \( c_{zj} = -c_{zj} \) (cf. Lemma in [22]). By using the same arguments as in the proof of (2.37) in [22], we have

\[
\begin{align*}
&\Lambda_{k+1} - \Lambda_i \left( 2 \| \nabla x_z, \nabla u_i \|^2 + \int \Omega \langle \nabla x_z, \Delta u_i \rangle + (n - 2) \| x_z \nabla u_i \|^2 + 2 \sum_{j=1}^k b_{aj}^2 c_{zj} \right) \\
&\quad \leq \frac{\delta (\Lambda_{k+1} - \Lambda_i)^3}{2} \| \nabla \phi_{zj} \|^2 + \frac{\Lambda_{k+1} - \Lambda_i}{\delta} \left( \| Z_{zj} \|^2 - \sum_{j=1}^k c_{zj}^2 \right) \\
&\quad + (n - 2) (\Lambda_{k+1} - \Lambda_i)^2 \| W_{zj} \|^2 
\end{align*}
\]

(2.74)

where \( \delta \) is any positive constant. Since

\[
\sum_{z=1}^{n+1} \| \langle \nabla x_z, \nabla u_i \rangle \|^2 = \int \Omega |\nabla u_i|^2 = 1, 
\]

(2.75)

we have by summing over \( z \) in (2.74) from 1 to \( n+1 \) that

\[
\begin{align*}
&\Lambda_{k+1} - \Lambda_i \left( n + 2 \sum_{z=1}^{n+1} \sum_{j=1}^k b_{aj}^2 c_{zj} \right) \\
&\quad \leq \frac{\delta n \sum_{z=1}^{n+1} (\Lambda_{k+1} - \Lambda_i)^3}{2} \| \nabla \phi_{zj} \|^2 + \frac{n \sum_{z=1}^{n+1} (\Lambda_{k+1} - \Lambda_i) \sum_{j=1}^k (\| Z_{zj} \|^2 - \sum_{j=1}^k c_{zj}^2)}{\delta} \\
&\quad + (n - 2) \sum_{z=1}^{n+1} (\Lambda_{k+1} - \Lambda_i)^2 \| W_{zj} \|^2. 
\end{align*}
\]

(2.76)
It follows from (2.6) and (2.57) that
\[
\sum_{z=1}^{n+1} \| \nabla (\nabla_x z, \nabla u_i) \|^2 = - \sum_{z=1}^{n+1} \int_{\Omega} \langle \nabla_x z, \nabla u_i \rangle \Delta \langle \nabla_x z, \nabla u_i \rangle \\
= - \sum_{z=1}^{n+1} \int_{\Omega} \langle \nabla_x z, \nabla u_i \rangle \left( - 2x_z \Delta u_i + \langle \nabla x_z, \nabla (\Delta u_i) \rangle \\
+ (n-2) \langle \nabla x_z, \nabla u_i \rangle \right) \\
= - \int_{\Omega} \langle \nabla u_i, \nabla (\Delta u_i) \rangle - (n-2) \| \nabla u_i \|^2 \\
= \int_{\Omega} u_i \Delta^2 u_i - (n-2) \\
\leq \Lambda_1^{1/(i-1)} - (n-2)
\] (2.77)

and so
\[
\sum_{z=1}^{n+1} \| Z_{x_z} \|^2 = \int_{\Omega} \left| \nabla \langle \nabla_x z, \nabla u_i \rangle - \frac{n-2}{2} x_z \nabla u_i \right|^2 \\
= \sum_{z=1}^{n+1} \left( \| \nabla \langle \nabla_x z, \nabla u_i \rangle \|^2 - (n-2) \int_{\Omega} \langle \nabla \langle \nabla_x z, \nabla u_i \rangle, x_z \nabla u_i \rangle \\
+ \frac{(n-2)^2}{4} \| x_z \nabla u_i \|^2 \right) \\
\leq \Lambda_1^{1/(i-1)} - (n-2) + (n-2) + \frac{(n-2)^2}{4} = \Lambda_1^{1/(i-1)} + \frac{(n-2)^2}{4}.
\] (2.78)

Since
\[
\int_{\Omega} \langle \nabla x_z, \nabla u_i \rangle^2 = \int_{\Omega} \left( \langle \nabla x_z, \nabla u_i \rangle \nabla u_i, \nabla x_z \right) \\
= - \int_{\Omega} x_z \text{div}(\langle \nabla x_z, \nabla u_i \rangle \nabla u_i) \\
= - \int_{\Omega} \langle x_z \nabla u_i, \nabla \langle \nabla x_z, \nabla u_i \rangle \rangle - \int_{\Omega} x_z \langle \nabla x_z, \nabla u_i \rangle \Delta u_i \\
= - \int_{\Omega} \langle \nabla h_{x_z} + W_{x_z}, \nabla \langle \nabla x_z, \nabla u_i \rangle \rangle - \int_{\Omega} x_z \langle \nabla x_z, \nabla u_i \rangle \Delta u_i \\
= - \int_{\Omega} \langle \nabla h_{x_z}, \nabla \langle \nabla x_z, \nabla u_i \rangle \rangle - \frac{1}{2} \int_{\Omega} \langle \nabla x_z^2, \nabla u_i \rangle \Delta u_i,
\]

we have
\[
1 = \sum_{z=1}^{n+1} \int_{\Omega} \langle \nabla x_z, \nabla u_i \rangle^2 = - \sum_{z=1}^{n+1} \int_{\Omega} \langle \nabla h_{x_z}, \nabla \langle \nabla x_z, \nabla u_i \rangle \rangle
\]
which gives

$$\sum_{z=1}^{n+1} \|\nabla(\nabla x_z, \nabla u_i)\|^2 > 0.$$ 

It then follows from (2.77) that \(\Lambda_i^{1/(l-1)} - (n - 2) > 0\) and

$$1 = -\sum_{z=1}^{n+1} \int_{\Omega} \langle \nabla h_{2z}, \nabla(\nabla x_z, \nabla u_i) \rangle$$

$$\leq \frac{1}{2} \sum_{z=1}^{n+1} \left( (\Lambda_i^{1/(l-1)} - (n - 2)) \|\nabla h_{2z}\|^2 + \frac{1}{\Lambda_i^{1/(l-1)} - (n - 2)} \|\nabla(\nabla x_z, \nabla u_i)\|^2 \right)$$

$$= \frac{1}{2} + \frac{1}{2} \sum_{z=1}^{n+1} (\Lambda_i^{1/(l-1)} - (n - 2)) \|\nabla h_{2z}\|^2.$$ 

Thus, we have

$$-\sum_{z=1}^{n+1} \|\nabla h_{2z}\|^2 \leq -\frac{1}{\Lambda_i^{1/(l-1)} - (n - 2)}$$

and consequently, one has

$$\sum_{z=1}^{n+1} \|W_{2i}\|^2 = \sum_{z=1}^{n+1} \left( \|x_z \nabla u_i\|^2 - \|\nabla h_{2z}\|^2 \right) \leq 1 - \frac{1}{\Lambda_i^{1/(l-1)} - (n - 2)}.$$ 

From \(b_{2ij} = b_{2ji}, \ c_{2ij} = -c_{2ji}\), we have

$$2 \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 b_{2ij} c_{2ij} = -2 \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j) b_{2ij} c_{2ij}$$

$$\delta \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 (\Lambda_i - \Lambda_j) b_{2ij}^2 = \delta \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j) b_{2ij}^2.$$ 

Combining (2.54), (2.70) and (2.72), we get

$$\sum_{z=1}^{n+1} (\Lambda_{k+1} - \Lambda_i) \|\nabla \phi_{2z}\|^2 \leq \sum_{j=0}^{l-2} |a_j| \Lambda_i^{j/(l-1)} \sum_{z=1}^{n+1} \|W_{2i}\|^2 + \sum_{z=1}^{n+1} \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{2ij}^2.$$ 

We have by substituting (2.83) into (2.76) that

$$(\Lambda_{k+1} - \Lambda_i)^2 \left( n + 2 \sum_{z=1}^{n+1} \sum_{j=1}^{k} b_{2ij} c_{2ij} \right)$$

$$\leq \delta (\Lambda_{k+1} - \Lambda_i)^2 \left( \sum_{j=0}^{l-2} |a_j| \Lambda_i^{j/(l-1)} + \sum_{z=1}^{n+1} \sum_{j=1}^{k} (\Lambda_i - \Lambda_j) b_{2ij}^2 \right).$$
\[\frac{\Lambda_{k+1} - \Lambda_{j}}{\delta} \sum_{z=1}^{n+1} \left( \|Z_{z}\|^2 - \sum_{j=1}^{k} c_{zj}^2 \right) \]

\[+ \sum_{z=1}^{n+1} (\Lambda_{k+1} - \Lambda_{j})^2 (\delta \Lambda_{j} + n - 2)\|W_{z}\|^2. \tag{2.84}\]

Hence, by summing over \(i\) from 1 to \(k\) and noticing (2.78), (2.80), (2.81) and (2.82), we infer

\[n \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 \leq \delta (\Lambda_{k+1} - \Lambda_{j})^2 \sum_{j=0}^{l-2} |a_j| A_{j}^{l/(l-1)} + \sum_{i=1}^{k} \frac{\Lambda_{k+1} - \Lambda_{j}}{\delta} \sum_{z=1}^{n+1} \|Z_{z}\|^2 \]

\[+ \sum_{i=1}^{k} \sum_{z=1}^{n+1} (\Lambda_{k+1} - \Lambda_{j})^2 (\delta \Lambda_{j} + n - 2)\|W_{z}\|^2 \]

\[\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 \sum_{j=0}^{l-2} |a_j| A_{j}^{l/(l-1)} \]

\[+ \sum_{i=1}^{k} \frac{\Lambda_{k+1} - \Lambda_{j}}{\delta} \left( A_{j}^{1/(l-1)} + \frac{(n - 2)^2}{4} \right) \]

\[+ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 (\delta \Lambda_{j} + n - 2) \left( 1 - \frac{1}{A_{j}^{1/(l-1)} - (n - 2)} \right). \tag{2.85}\]

That is

\[\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 \left( 2 + \frac{n - 2}{A_{j}^{1/(l-1)} - (n - 2)} \right) \]

\[\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 H_{i} + \frac{1}{\delta} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j}) \left( A_{j}^{1/(l-1)} + \frac{(n - 2)^2}{4} \right). \tag{2.85}\]

where \(H_{i}\) is given by (1.22). This completes the proof of Theorem 1.2.

**Proof of Corollary 1.2.** Taking

\[\delta = \left\{ \frac{\sum_{j=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 (A_{j}^{1/(l-1)} + \frac{(n - 2)^2}{4})^{1/2}}{\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 H_{i}^{1/2}} \right\}^{1/2}, \]

in (1.21), we have

\[\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 \left( 2 + \frac{n - 2}{A_{j}^{1/(l-1)} - (n - 2)} \right) \]

\[\leq 2 \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j})^2 H_{i}^{1/2} \right\}^{1/2} \times \left\{ \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_{j}) \left( A_{j}^{1/(l-1)} + \frac{(n - 2)^2}{4} \right)^{1/2} \right\}^{1/2}. \tag{2.86}\]
Since
\[ 2 + \frac{n - 2}{\Lambda_i^{1/(t-1)} - (n - 2)} \geq 2 + \frac{n - 2}{\Lambda_i^{1/(t-1)} - (n - 2)} = S_k, \quad i = 1, \ldots, k, \]
we have
\[ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \left( 2 + \frac{n - 2}{\Lambda_i^{1/(t-1)} - (n - 2)} \right) \geq S_k \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \quad (2.87) \]
and we infer from Lemma 2.1 that
\[
\left\{ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 H_i \right\} \times \left\{ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i) \left( \Lambda_i^{1/(t-1)} + \frac{(n - 2)^2}{4} \right) \right\} \\
\leq \left\{ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \right\} \times \left\{ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i) T_i \right\} \\
= \left\{ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \right\} \times \left\{ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i) T_i \right\}, \quad (2.88)
\]
where \( S_k \) and \( T_i \) are defined as in (1.26). Substituting (2.87) and (2.88) into (2.86), one gets
\[ \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i)^2 \leq \frac{4}{S_k} \sum_{i=1}^{k} (\Lambda_{i+1} - \Lambda_i) T_i, \]
where \( A_{k+1} \) and \( B_{k+1} \) are given by (1.25). Solving this quadratic polynomial about \( \Lambda_{k+1} \), we get (1.24).

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