Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant

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An analytical expression for the gravitational field of a homogeneous cylinder is derived. The phase of the atom interferometer produced by the gravity field of the set of cylinders has been calculated. The optimal values of the initial positions and velocities of atomic clouds were obtained. It is shown that at equal sizes of the atomic cloud in the vertical and transverse directions, as well as at equal atomic vertical and transverse temperatures, systematic errors due to the finite size and temperature of the cloud disappear. To overcome the influence of the Earth gravitational field on the accuracy of the phase double difference measurement, it is proposed to use the technique of eliminating gravity-gradient terms. After eliminating, one can use extreme values of the atomic positions and velocities. Nonlinear dependences of the phase on the uncertainties of atomic positions and velocities near those extreme values required us to modify the expression for the standard phase deviation. Moreover, such dependences lead to a phase shift, which was also calculated. The relative accuracies of measurements of the Newtonian gravitational constant $10^{-4}$ and $2 \cdot 10^{-5}$ are predicted for sets of 24 and 630 cylinders, respectively.

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I. INTRODUCTION.

Atom interferometry \cite{1} is now used to measure Newtonian gravity constant $G$ \cite{2, 4}. Searches for new schemes and options promise to increase the accuracy of these measurements. Previously, it was shown \cite{2} that, in principle, the current state-of-art in atom interferometry would allow one to measure $G$ with an accuracy of 200ppb. To achieve such a goal one has to use simultaneously the largest time delay between pulses $T = 1.15$ s \cite{2}, temperature 115pK and radius of the atom cloud 170 μ (which are larger than those observed in \cite{2}), the beam splitter with an effective wave vector $k = 8.25 \cdot 10^8$m$^{-1}$ \cite{5}, the source mass 1080kg \cite{1} and phase noise $\phi_{err} = 10^{-4}$rad. For the parameters achieved in \cite{2, 4} at present, the optimal preparation of the atomic clouds and proper positioning of the gravity sources can also lead to an increase in the accuracy of the $G$-measurement.

The following procedure was used \cite{3, 5, 10}. The source mass consists of two halves, which are placed in two different configurations C and F. We accept the notation "C and F," which was previously used in articles \cite{2, 4}. The atomic gradiometer \cite{11} measures the phase difference of two atomic interferometers (AIs) 1 and 2

$$\Delta \phi^{(C,F)} = \phi^{(C,F)}(z_1, v_{1z}) - \phi^{(C,F)}(z_2, v_{2z}),$$

(1)

where $\phi^{(C,F)}(z_j, v_j)$ is the phase of AI $j$, in which the atoms are launched vertically from point $x_j = (0, 0, z_j)$ at velocity $v_j = (0, 0, v_{jz})$. Phase difference (1) consists of two parts, the one that is induced by the gravitational field of the Earth and inertial terms and the one that is associated with the gravitational field of the source mass. One expects \cite{2, 4} that the phase double difference (PDD)

$$\Delta^{(2)} \phi = \Delta \phi^{(C)} - \Delta \phi^{(F)}$$

(2)

will depend only on the AI phase $\phi_s^{(C,F)}(z_j, v_j)$ produced only by the field of source mass, and therefore can be used to measure the Newtonian gravitational constant $G$. Despite the fact that the gravitational field of the Earth does not affect the PDD, the gradient of this field affects \cite{3, 4} on the accuracy of the PDD measurement. In article \cite{3, 4}, to reduce this influence, the mutual position of the source mass and atomic clouds are selected so that at the point of apogee of the atomic trajectories gradients of the Earth’s field and the field of the source mass cancel each other. Below in Sec. IV we will see that this technique only partially reduces the influence of the gravitational field of the Earth on the accuracy of the $G$ measurement.

To resolve this problem we suggest using the method of eliminating the sensitivity of the phase to the atom position and velocity \cite{12}, which is achieved by changing the wave vector of the second Raman pulse, so that

$$k_2 = k + \frac{1}{2} \Gamma_E T^2 k,$$

(3)
where $\Gamma_E$ is gravity-gradient tensor of the Earth field. The possibility of using the technique of \[12\] for measuring $G$ was considered by G. Rosi \[13\]. The difference from what is proposed here is that we propose to eliminate only the gradient of the gravitational field of the Earth. Since the gravity gradient tensor is measured with some accuracy was considered by G. Rosi \[13\]. The difference from what is proposed here is that we propose to eliminate only the PDD, $\sigma_r (\Delta^{(2)} \phi)$, is sufficiently large,

$$
\sigma_r (\Delta^{(2)} \phi) > \frac{k |\delta \Gamma_E| T^2}{\Delta^{(2)} \phi} \left\{ \sum_{j=1,2} \sum_{l=C,F} \left[ \sigma^2 (z_{j,l}) + T^2 \sigma^2 (v_{z_{j,l}}) \right] \right\}^{1/2},
$$

where $\sigma (z_{j,l})$ and $\sigma (v_{z_{j,l}})$ are the standard deviations (SD) of the initial atomic position and velocity. Although, starting with article \[11\], methods for measuring the gravity gradient using AIs have been studied in many articles, I know only three publications \[14-16\], in which the values of $\Gamma_{E33}$ and $\delta \Gamma_{E33}$ were published. The error $\delta \Gamma_{E33} = 10E$ was reported \[16\]. For the values obtained in \[3, 4\], $\sigma (z_{j,l}) \sim 10^{-4}m,$ $\sigma (v_{z_{j,l}}) \sim 3 \cdot 10^{-3}m/s,$ $k \approx 1.61 \cdot 10^7m^1,$

$$
\Delta^{(2)} \phi = 0.547870(63)rad,
$$

one gets

$$
\sigma_r (\Delta^{(2)} \phi) > 7ppm.
$$

We will make sure that the restriction \[6\] can be neglected with an accuracy of no more than 10%.

Since the magnitude of the gravity-gradient tensor $\Gamma_E$ is small, the change in the effective wave vector in \[8\] can be considered as a perturbation. Another small perturbation here is the gravity field of the source mass \[10\]. Since we are not going to consider the simultaneous action of these two perturbations, we can calculate the parts of the phases $\phi_s^{(C,F)}$ assuming that all three Raman pulses have the same unperturbed effective wave vector $k$.

In order to achieve the maximum value of the first order difference $\Delta \phi_s^{(C)}$ in $C$-configuration, one can choose points \{z$_1$, v$_{z1}$\} and \{z$_2$, v$_{z2}$\} at the maximum and minimum of the dependence $\phi_s^{(C)} (z, v)$ \[3, 10\]. The choice of extreme points leads to a quadratic dependence of the phase $\phi_s^{(C)} (z, v)$ on small deviations \{δz$_j$, δv$_{zj}$\} in the vicinities of these points.

Let us consider now the contribution to the phase double difference from the first-order phase difference in $F$-configuration. In principle, one can find the positions of the halves of the source mass, at which points $z_1$ and $z_2$ are extrema of the dependence $\phi_s^{(F)} (z, v)$ \[13\]. However, in this case, points in the velocity space $v_{z1}$ and $v_{z2}$, which were extreme for the $C$-configuration, become non-extreme. To avoid this difficulty one can choose points \{z$_1$, v$_{z1}$\} at which the sources mass sufficiently far so that even the linear dependences of the $\phi_s^{(F)} (z, v)$ on the deviations near the points $\{\delta z_j, \delta v_{zj}\}$ do not affect significantly the error $\sigma_s^{(C)}$ of the phase double differences \[5, 10\].

We performed \[8, 10\] calculations, determined the optimal geometry of the gravitational field, positions and velocities of atomic clouds for the source mass of a cuboid shape. The choice of this shape is convenient for calculations since one has an analytical expression for the potential of the cuboid \[19\]. Despite this, it is preferable to use the source mass in a cylindrical shape to perform high-precision measurements of $G$ \[20\]. Cylindrical source masses were used to measure $G$ with an accuracy of 150ppm \[3, 4\]. The hollow cylinder source mass has been proposed to achieve an accuracy of 10ppm \[13\]. The analytical expression for the gravitational field along the $z$-axis of the hollow cylinder was explored \[13\], but outside this axis, the potential expansion into spherical harmonics was used \[3, 4\]. Expressions for the field of the cylinders have been derived in the articles \[21, 22\]. Alternatively, the technique for calculating the gravitational field without calculating the gravitational potential was proposed in the book \[20\], but the final expression for the cylinder field is given in \[20\] without derivation. Following technique \[20\], we calculated the field and arrived at expressions \[A17, A22\]. Our expressions do not coincide with those given in \[20, 22\]. Both the derivations and final results are presented in this article. Following the derivations in the articles \[21, 22\], we are going to find out analytically the reason of the discrepancies between different expressions and publish it elsewhere.

To estimate the expected accuracy of the $G$-measurement, one has to analyze the error $\phi_s^{(C)}$. In precision gravity experiments, one calculates or measures the SD $\sigma$ of the response $f$ (such as the AI phase or phase difference) using the expression

$$
\sigma (f) = \left( \sum_{m=1}^{n} \sigma^2_m \right)^{1/2},
$$

where $\sigma^2_m$ is the error of measurement for the $m$-th point.
where \( n \) is the number of variables \( \{q_1, \ldots, q_m\} \), included in the error budget, \( \sigma_m = |\partial f / \partial q_m| \sigma(q_m) \), and \( \sigma(q_m) \) is a SD of small uncertainty in the variable \( q_m \). We assume that variables \( \{q_1, \ldots, q_m\} \) are statistically independent. See examples of such budgets in [2–4, 13, 14, 18, 23]. The situation changes when one considers uncertainties near the extreme points \( \{x_m, v_m\} \) and the signal’s uncertainty becomes a quadratic function of the uncertainties of the atomic position and velocity \( \{\delta x_m, \delta v_m\} \). There are several examples in which measurements were carried out (or proposed to be carried out) near extreme points. Extreme atomic coordinates were selected in the experiments [18]. Extreme atomic coordinates and velocities were found in the articles [2–4, 13, 14, 18, 23]. The difficulties of using extreme points are noted in the article [13], where an alternative approach was proposed, based on the elimination of the dependence of the AI phase on the atomic position and velocity proposed in [12]. However, even in this case, one eliminates only the dependence on the vertical coordinates and velocities, while the transverse coordinates \( \{x_m, y_m\} = \{0, 0\} \) remain extreme. This is because the vertical component of the gravitational field of the hollow cylinder \( \delta g_3(x) \) is axially symmetric, and the expansion of both the field and the field gradient in transverse coordinates begins with quadratic terms. We see that in all the cases listed above [5, 10, 13, 18], the use of the expression \( \sigma \) is unjustified. Revision of this expression is required. Moreover, the quadratic dependence on the uncertainties \( \{\delta x_m, \delta v_m\} \) leads to a shift in the signal [24]. Here, we expressed both the SD and the shift of the phase double difference [2] in terms of the first and second derivatives of the phases \( \phi^{(C,F)} \) at the found extreme points.

The article is arranged as follows. Standard deviation and shift are obtained in the next section. Section III is devoted to the AI phase and phase derivatives calculations, PDD and error budget for source mass consisting of 24 cylinders are obtained in the Sec. [IV]. An optimization procedure in respect to atomic positions and velocities considered in the Sec. V. The case of the 630 cylinders source mass is studied in the Sec. [VI] while the derivation of the cylinder gravitational field is presented in the Appendix.

II. SD AND SHIFT.

Let us consider the variation of the double difference [2]

\[
\delta \Delta^{(2)} \phi = \delta \phi^{(C)} (\delta x_1, \delta v_1) - \delta \phi^{(C)} (\delta x_2, \delta v_2) - \delta \phi^{(F)} (\delta x_1, \delta v_1) - \delta \phi^{(F)} (\delta x_2, \delta v_2),
\]

(8)

where \( \{\delta x_j, \delta v_j\} \) is the uncertainty of the launching position and velocity of the cloud \( j \) \( (j = 1 \text{ or } 2) \) for the source mass configuration \( I \) \( (I = C \text{ or } F) \), \( \delta \phi^{(I)} (\delta x_j, \delta v_j) \) is the variation of the AI j phase, produced when the source mass gravity field is in the \( I \)-configuration. For the shift \( s \) and standard deviation \( \sigma \) defined as

\[
s \left( \Delta^{(2)} \phi \right) = \left\langle \left( \phi^{(C)} (\delta x_1, \delta v_1) - \phi^{(C)} (\delta x_2, \delta v_2) - \phi^{(F)} (\delta x_1, \delta v_1) - \phi^{(F)} (\delta x_2, \delta v_2) \right)^2 \right\rangle - s^2 \left( \Delta^{(2)} \phi \right),
\]

(9a)

\[
\sigma \left( \Delta^{(2)} \phi \right) = \left\{ \left[ \left( \phi^{(C)} (\delta x_1, \delta v_1) - \phi^{(C)} (\delta x_2, \delta v_2) - \phi^{(F)} (\delta x_1, \delta v_1) - \phi^{(F)} (\delta x_2, \delta v_2) \right)^2 \right] \right\}^{1/2},
\]

(9b)

one finds

\[
s \left( \Delta^{(2)} \phi \right) = s \left( \phi^{(C)} (\delta x_1, \delta v_1) - s \left( \phi^{(C)} (\delta x_2, \delta v_2) - s \left( \phi^{(F)} (\delta x_1, \delta v_1) + s \left( \phi^{(F)} (\delta x_2, \delta v_2) \right) \right) \right) \right),
\]

(10a)

\[
\sigma \left( \Delta^{(2)} \phi \right) = \left\{ \left( \sum_{I = C,F} \sum_{j=1,2} \sigma^2 \left( \phi^{(I)} (\delta x_j, \delta v_j) \right) \right)^{1/2},
\]

(10b)

\[
s \left( \phi^{(I)} (\delta x_j, \delta v_j) \right) = \left( \phi^{(I)} (\delta x_j, \delta v_j) \right),
\]

(10c)

\[
\sigma \left( \phi^{(I)} (\delta x_j, \delta v_j) \right) = \left\{ \left( \sigma \left( \phi^{(I)} (\delta x_j, \delta v_j) \right)^2 \right)^{1/2} \right\}
\]

(10d)

One sees that the problem is reduced to the calculation of the shift \( s \) and SD \( \sigma \) of a variation \( \delta \phi [\delta x, \delta v] \). The phase of the given AI at the given configuration of the source mass comprises two parts

\[
\phi \left( x, v \right) = \phi_E \left( x, v \right) + \phi_s \left( x, v \right),
\]

(11)
where for the phase induced by the Earth’s field, under some simplifying assumptions (see, for example, [25]), one gets
\[\phi_E^{(l)}(x_j, v_j) = k \cdot gT^2 + k \Gamma_E T^2 (x + vT) + k \Gamma_E g T^2 \left( \frac{T}{12} T^2 + TT_1 + \frac{1}{2} T_1^2 \right),\]
where \(T_1\) is the time delay between the moment the atoms are launched and the 1st Raman pulse. For the vertical wave vector \(k = (0, 0, k)\), expanding Eq. (11) to the second order terms one gets
\[
\delta \phi \left( \delta x, \delta v \right) = \left( \tilde{\gamma}_{xm} + \frac{\partial \phi_s}{\partial x_m} \right) \delta x_m + \left( \tilde{\gamma}_{vm} + \frac{\partial \phi_s}{\partial v_m} \right) \delta v_m + \frac{1}{2} \frac{\partial^2 \phi_s}{\partial x_m \partial x_n} \delta x_m \delta x_n + \frac{1}{2} \frac{\partial^2 \phi_s}{\partial v_m \partial v_n} \delta v_m \delta v_n + \frac{\partial^2 \phi_s}{\partial x_m \partial v_n} \delta x_m \delta v_n,
\]
where
\[
\tilde{\gamma}_{xm} = k \Gamma_E s m T^2; \quad \tilde{\gamma}_{vm} = T \tilde{\gamma}_{xm} \quad (14a)
\]
A summation convention implicit in Eq. (13) will be used in all subsequent equations. Repeated indices and symbols appearing on the right-hand-side (rhs) of an equation are to be summed over, unless they also appear on the left-hand-side (lhs) of that equation. Let assume that the distribution functions of the uncertainties are sufficiently symmetric, and all odd moments are equal 0. The moments of the second and fourth orders are given by
\[
\langle \delta q_m \delta q_n \rangle = \delta_{mn} \sigma^2(q_m), \quad (15a)
\]
\[
\langle \delta q_m \delta q_n \delta q_m, \delta q_n, \delta q_m \rangle = \delta_{mn} \delta_{mn'} \sigma^2(q_m) \sigma^2(q_m') + (\delta_{mm'} \delta_{nn'} + \delta_{mm} \delta_{nn}) \sigma^2(q_m) \sigma^2(q_n) + \delta_{mm} \delta_{mn} \delta_{nn'} \kappa(q_m) \sigma^4(q_m), \quad (15b)
\]
where \(q_i\) is either a position \(x_i\) or a velocity \(v_i\), \(\sigma(q_i)\) is SD of the uncertainty \(\delta q_i\), \(\delta_{mn}\) is Kronecker symbol, \(\kappa(q_m)\) is a cumulant of the given uncertainty \(\delta q_m\), defined as
\[
\kappa(q_m) = \frac{\langle \delta q_m^4 \rangle}{\sigma^4(q_m)} - 3. \quad (16)
\]
Using the moments (15) one arrives at the following expressions for the SD and shift
\[
\sigma \left[ \phi \left( \delta x, \delta v \right) \right] = \left\{ \left( \tilde{\gamma}_{xm} + \frac{\partial \phi_s}{\partial x_m} \right)^2 \sigma^2(x_m) + \left( \tilde{\gamma}_{vm} + \frac{\partial \phi_s}{\partial v_m} \right)^2 \sigma^2(v_m) \right\}^{1/2}
\]
\[
+ \frac{1}{2} \left[ \left( \frac{\partial^2 \phi_s}{\partial x_m \partial x_n} \right)^2 \sigma^2(x_m) \sigma^2(x_n) + \left( \frac{\partial^2 \phi_s}{\partial v_m \partial v_n} \right)^2 \sigma^2(v_m) \sigma^2(v_n) \right]
\]
\[
+ \left( \frac{\partial^2 \phi_s}{\partial x_m \partial v_n} \right)^2 \sigma^2(x_m) \sigma^2(v_n) + \frac{1}{4} \left[ \left( \frac{\partial^2 \phi_s}{\partial x_m^2} \right)^2 \kappa(x_m) \sigma^4(x_m) + \left( \frac{\partial^2 \phi_s}{\partial v_m^2} \right)^2 \kappa(v_m) \sigma^4(v_m) \right] \quad (17a)
\]
\[
s \left[ \phi \left( \delta x, \delta v \right) \right] = \frac{1}{2} \left( \frac{\partial^2 \phi_s}{\partial x_m^2} \sigma^2(x_m) + \frac{\partial^2 \phi_s}{\partial v_m^2} \sigma^2(v_m) \right), \quad (17b)
\]
One sees that, even for the symmetric uncertainties distribution, the knowledge of the uncertainties’ SDs is not sufficient. One has to know also uncertainties’ cumulants [16]. The exclusion here is Gaussian distributions, for which the cumulants
\[
\kappa(x_m) = \kappa(v_m) = 0. \quad (18)
\]
Further calculations will be performed only for these distributions.
For the each case considered below we are going to calculate the double difference [2] and relative contributions to the shift [2a] and SD [9b] from the each of two atom clouds at the each of two source mass configurations.

### III. THE PHASE AND PHASE DERIVATIVES OF THE ATOM INTERFEROMETER

To calculate the phase \(\phi^{(l)}_s\) produced by the gravitational field of the source mass, we use the results obtained in the article [26]. It is necessary to distinguish three contributions to the phase, classical, quantum, and Q-term (see
Eqs. (62c, 64, 60c), (62d, 71, 60c), (89) in [26] for these three terms). For Q-term an estimate was obtained

\[ \frac{\phi}{\phi_s^{(1)}} \sim \frac{1}{24} \left( \frac{\hbar T \Gamma_{\mu}}{L M_a} \right)^2, \]

where \( M_a \) is the atom mass, \( L \) is the characteristic distance over which the gravitational potential of the test mass changes. For \(^{87}\text{Rb} \), at \( L > 0.3 \text{m} \), the relative weight of the Q-term does not exceed 2ppb, and we neglect it. For the remaining terms and the vertical effective wave vector, \( k = \{0, 0, k\} \), one gets

\[ \phi_s^{(1)}(x, v) = k \int_0^T dt \left[ (T-t)^3 \left( a(T+t) + 3 \delta g_3(a(t)) \right) + t \delta g_3(a(t)) \right], \]

where

\[ a(t) = x + v(T_1 + t) + \frac{1}{2} g(T_1 + t)^2 + v_r t, \]

the recoil velocity is given by

\[ v_r = \hbar k / 2M_a. \]

\( \delta g_3(x) \) is the vertical component of the gravitational field of the source mass. The derivatives of this phase of the first and second order are given by

\[
\frac{\partial \phi_s^{(1)}(x, v)}{\partial x_m} = k \int_0^T dt \left[ (T-t) \Gamma_{x3m} \left( a(T+t) + t \Gamma_{x3m} \left( a(t) \right) \right) \right], \\
\frac{\partial \phi_s^{(1)}(x, v)}{\partial v_n} = k \int_0^T dt \left[ (T-t) \chi_{x3mn} \left( a(T+t) + t \chi_{x3mn} \left( a(t) \right) \right) \right],
\]

where \( \Gamma_{x3m}(x) = \frac{\partial \delta g_3(x)}{\partial x_m} \) is the 3m-component of the gravity-gradient tensor of the source mass field, and

\[ \chi_{x3mn}(x) = \frac{\partial^2 \delta g_3(x)}{\partial x_m \partial x_n} \]

is the 3mn-component of the curvature tensor of this field.

**IV. ERROR BUDGET**

We applied the formula for the cylinder field [17] to calculate the phases produced by different sets of cylinders. In this section, we consider the field geometry chosen in the article [3, 4], see Fig. 11.

Two halves of the source mass, each including 12 tungsten alloy cylinders, move in a vertical direction from \( C \)-configuration to \( F \)-configuration, in each of which one measures the phase difference of the first order [1], and then PDD [4]. The following system parameters are important for calculation: cylinder density \( \rho = 18263\text{kg/m}^3 \), cylinder radius and height \( R = 0.0495\text{m} \) and \( h = 0.15011\text{m} \), Newtonian gravitational constant \( G = 6.67408 \times 10^{-11}\text{kg}^2\text{m}^{-1}\text{s}^{-2} \) [27], the Earth’s gravitational field \( g = 9.80492\text{m/s}^2 \) [28], the delay between impulses \( T = 160\text{ms} \), the time \( T_1 = 0 \), the effective wave vector \( k = 1.61058 \times 10^7\text{m}^{-1} \), the mass of the \(^{87}\text{Rb} \) \( M_a = 86.9092\text{a.u.} \) [29], atomic velocity at the moment of the first impulse action \( v = 1.62702\text{m/s} \). With respect to the apogee of the atomic trajectory in the lower interferometer, the \( z \)-coordinate of the centers of the halves of the source mass are equal to 0.04m and 0.261m in the \( C \)-configuration and \( -0.074\text{m} \) and 0.377m in the \( F \)-configuration, \( z \)-coordinate of the atomic trajectory apogee in the upper interferometer is equal to 0.328m (see Fig. 11f). Using Eq. (20) we got for PDD

\[ \Delta^{(2)} \phi = 0.530535\text{rad}, \]
FIG. 1: The mutual positioning of the source mass halves 1 and 2, and atomic clouds. Top view (a), cross-sections x=0 for C-configuration (c) and F-configuration (f). Trajectories of atoms are shown in red.

which is less than the value \( 5 \) obtained in the article \([3, 4]\) by 3.2%. The difference seems to be related to the fact that in these calculations, the contributions from platforms and other sources of gravity were not taken into account. Table S.I \([30]\) contains relative contributions to the PDD from two configurations, besides the phase values, linear and quadratic terms in the relative phase variations, due to the uncertainties of atomic coordinates and velocities, obtained using Eqs. \([13, 14, 20]\), are also given. We used the value of the \(zz\) component of the gravity gradient tensor of the Earth field, \(\Gamma_E^{33} = 3.11 \cdot 10^{-10} \text{s}^{-2}\), measured in the article \([16]\). Using data from the Table S.I and Eqs. \([10a, 10b, 17, 18]\), we obtained Eqs. \([S.1a, S.1b]\) for the RSD and shift. For SDs achieved in \([3, 4]\)

\[\sigma (x_{ji}) = \sigma (y_{ji}) = 10^{-3} \text{m}, \]
\[\sigma (z_{ji}) = 10^{-4} \text{m}, \]
\[\sigma (v_{x_{ji}}) = \sigma (v_{y_{ji}}) = 6 \cdot 10^{-3} \text{m/s}, \]
\[\sigma (v_{z_{ji}}) = 3 \cdot 10^{-3} \text{m/s}, \]

one arrives to the RSD and the shift

\[\sigma \left( \delta \Delta \phi \right) = 275 \text{ppm} \left[ 1 + 6.14 \cdot 10^{13} (\Gamma_{E31}^2 + \Gamma_{E32}^2) \right]^{1/2}, \]
\[s \left( \delta \Delta \phi \right) = 199 \text{ppm}. \]

The non-diagonal matrix elements of the gradient tensor of the Earth’s field consist of three contributions arising from the fact that the Geoid is not spherical, from the rotation of the Earth, and from the anomalous part of the field. The first two contributions were taken into account exactly \([17]\), and they are 3 orders of magnitude smaller than the diagonal element \(\Gamma_{E33}\). We have not been able to find any information about the anomalous part of the Earth’s gravitational field. However, it is seen that the non-diagonal elements of the tensor can be neglected with an accuracy of not more than 10% if

\[\sqrt{\Gamma_{E31}^2 + \Gamma_{E32}^2} < 58.5E. \]

V. PHASE DOUBLE DIFFERENCE AND ERROR BUDGET AT OPTIMAL CONDITIONS

Let us apply now for the system of 24 cylinders the optimization procedure proposed in \([3, 10]\). It should be noted that this procedure can be implemented only if the technique of eliminating the gradient of the Earth’s gravitational field \([12]\) has been previously applied. According to \([18]\) and unlike \([3, 4]\) we have chosen for calculations in this section the distance between the lower and upper set of cylinders \(dh = 0.05 \text{m}\). Our first task is to find the points of maximum
FIG. 2: The same as Fig. 1 but with different assignments for the halves 1 and 2. (a) The cross-section $y = 0$ for the $C$—configuration, (c) and (f) top views for $C$— and $F$—configurations.

and minimum (in the space of coordinates and velocities $\{z, v_z\}$) of the phase produced by the source mass field in $C$—configuration, $\phi_s^{(C)}$. Putting $T_1 = 0$, we found for these points (see Fig. 2)

$$\begin{align*}
\{z_{\text{max}}, v_{\text{max}}, \phi_s^{(C)}(z_{\text{max}}, v_{z_{\text{max}}})\} &= \{-0.124 \text{m}, 1.563 \text{m/s}, 0.215086\}, \\
\{z_{\text{min}}, v_{\text{min}}, \phi_s^{(C)}(z_{\text{min}}, v_{z_{\text{min}}})\} &= \{0.261 \text{m}, 1.563 \text{m/s}, -0.212213\}. 
\end{align*}$$

(29a, 29b)

The phase difference of the first order will be equal to

$$\Delta_s \phi^{(C)} = 0.427299 \text{rad}. \quad (30)$$

At this stage, before performing the calculations in the $F$—configuration, we can calculate the shift and the RSD only with respect to the phase difference (30). Table S.II contains linear and quadratic terms in the relative variation of the phase $\Delta_s \phi^{(C)}$, due to the uncertainties of atomic coordinates and velocities, obtained using Eqs. (13, 23). One sees that despite the choice of extreme points, linear dependences in phase variation do not completely disappear. This is because extrema were found not exactly but approximately. If $\{z, v_z\}$ is a given approximate extreme point of the function $f(z, v_z)$, then using the iteration formulas

$$\begin{align*}
\{z, v_z\} &\to \{z + \delta z, v_z + \delta v_z\}, \\
\delta z &= \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial v_z^2} - \frac{\partial f}{\partial v_z} \frac{\partial^2 f}{\partial z^2} \left\{ \left[ \frac{\partial^2 f}{\partial z \partial v_z} \right]^2 - \frac{\partial^2 f}{\partial z^2} \frac{\partial^2 f}{\partial v_z^2} \right\}, \\
\delta v_z &= \frac{\partial f}{\partial v_z} \frac{\partial^2 f}{\partial z^2} - \frac{\partial f}{\partial z} \frac{\partial^2 f}{\partial v_z^2} \left\{ \left[ \frac{\partial^2 f}{\partial z \partial v_z} \right]^2 - \frac{\partial^2 f}{\partial z^2} \frac{\partial^2 f}{\partial v_z^2} \right\},
\end{align*}$$

(31a, 31b, 31c)

one can find out the extremum with any arbitrarily high accuracy. Repeating iterations (31) three times we determined the velocities $v_{z_{\text{max}}}$ and $v_{z_{\text{min}}}$ with an accuracy of $10^{-7} \text{m/s}$. These velocities match up to the 5th digit. They are also close to the velocity of the atomic fountain $v = gT$. Differing from it only in the third digit,

$$\delta v = v_{z_{\text{max}}} - gT \approx -6 \cdot 10^{-3} \text{m/s} \quad (32)$$

This difference, however, is sufficient to exclude the parasitic signal (32), which occurs when atoms interact with a Raman pulse having an opposite sign of the effective wave vector. Indeed, the Raman frequency detuning for the parasitic signal $\delta = 2k\delta v \approx -2 \cdot 10^5 \text{s}^{-1}$. If the duration of the $\pi$—pulse $\tau \sim 60 \mu\text{s}$, then the absolute value of the detuning $\delta$ is an order of magnitude greater than the inverse pulse duration, and the probability of excitation of atoms by a parasitic Raman field is negligible, is estimated to be about 4%.

In addition, the velocity (32) is twice as large as the thermal velocity in the atomic cloud, $v = 3 \cdot 10^{-3} \text{m/s}$ (3, 4), and therefore the portion of atoms having the opposite velocity $-\delta v$ and being in resonance with the parasitic Raman pulse is also exponentially small, is no more than $10^{-8}$.
FIG. 3: To determine the distance $L$ by which one should move the halves of the source mass in the $F$−configuration.

Using the data from Table S.II and Eqs. (10a, 10b, 17, 18) we got Eqs. (S.2a, S.2b) for RSD and relative shift. At the uncertainties (26) RSD and shift are given by

$$\sigma(\Delta_s^{(C)}\phi) = 93 \text{ppm}, \quad (33a)$$
$$s(\Delta_s^{(C)}\phi) = 116 \text{ppm}. \quad (33b)$$

Now consider the $F$−configuration. In this configuration, points $\{z_{1F}, v_{z1F}\}$ and $\{z_{2F}, v_{z2F}\}$ found above are not extreme, and therefore the main contribution to the variation of the phase difference of the first order, $\delta \Delta_s^{(F)}\phi$, arises from the linear terms. Including only these terms, one receives for RSD from Eqs. (10b, 17b)

$$\sigma(\Delta_s^{(F)}\phi) \approx \left\{ \frac{\partial \phi^{(F)}(z_{1F}, v_{z1F})}{\partial z_{1F}} \frac{\sigma(z_{1F})}{\Delta_s^{(C)}\phi} \right\}^2 + \left\{ \frac{\partial \phi^{(F)}(z_{1F}, v_{z1F})}{\partial v_{z1F}} \frac{\sigma(v_{z1F})}{\Delta_s^{(C)}\phi} \right\}^2$$
$$+ \left\{ \frac{\partial \phi^{(F)}(z_{2F}, v_{z2F})}{\partial z_{2F}} \frac{\sigma(z_{2F})}{\Delta_s^{(C)}\phi} \right\}^2 + \left\{ \frac{\partial \phi^{(F)}(z_{2F}, v_{z2F})}{\partial v_{z2F}} \frac{\sigma(v_{z2F})}{\Delta_s^{(C)}\phi} \right\}^2 \right\}^{1/2} \quad (34)$$

The distance $2L$ between the halves of the source mass should be chosen so large that an increase in RSD (33a) would be insignificant. Specifically, we demanded a full RSD did not exceed (33a) on the amount more than 10%, which means that

$$\sigma(\Delta_s^{(F)}\phi) \leq \sqrt{0.21} \sigma(\Delta_s^{(C)}\phi). \quad (35)$$

One can guarantee the fulfillment of this condition if the absolute value of the each of four terms in square brackets in (34) is less than $\left(\sqrt{0.21}/2\right) \sigma(\Delta_s^{(C)}\phi)$. The most dangerous here is the last term in (34). From the Fig. 3 one can be sure that the inequality...
\[
\left| \frac{\partial \phi^{(F)}(z_{2F}, v_{z2F})}{\partial v_{z2F}} \right| \leq \alpha, \quad (36)
\]

where
\[
\alpha = \frac{\sqrt{0.21 \Delta s \phi^{(C)}}}{2 \sigma (v_{z2F})} \sigma \left( \Delta s \phi \right) = 3.05 \cdot 10^{-3}, \quad (37)
\]
is satisfied starting from the value
\[
L = 0.738 \text{m}. \quad (38)
\]

After the halves of the source mass are moved apart by distances of \( \pm L \), we get for the phase difference
\[
\Delta s^{(F)} \phi = 8.49006 \text{mrad}, \quad (39)
\]
and therefore the PDD decreases to the value
\[
\Delta^{(2)} \phi = 0.418809. \quad (40)
\]

Calculating [with the use of Eqs. (23)] derivatives of the interferometers’ phases, one arrives to the relative contributions to the PDD presented in the Table S.III. With the data from this table one arrives to the final expressions (S.3a, S.3b) for the RSD and shift. For the uncertainties (20) in the atomic positions and velocities, which we choose here following achievements in [3, 4], one gets
\[
\sigma (\Delta^{(2)} s \phi) = 100 \text{ppm}, \quad (41a)
\]
\[
s (\Delta^{(2)} s \phi) = 118 \text{ppm}. \quad (41b)
\]

In this section, we considered the movements of the halves of the source mass in the horizontal plane from the \( C \)-configuration to the \( F \)-configuration. Another option was considered in the article [13], in which it was assumed that the source mass as a whole moves in the vertical direction. To determine the distance \( L \) of the displacement in the vertical direction, it is necessary to return to the Eq. (34) and inequality (36). The calculation showed that in this case the greatest danger is again the last term in braces in the equation (34). So we return to inequality (36). Solving it numerically, one obtains that the displacement must be not less than
\[
L = 1.33 \text{m}. \quad (42)
\]

Since this distance is 1.8 times greater than the distance (38), one concludes that horizontal displacement is preferable than vertical displacement. I would also like to emphasize that the horizontal displacement of the source mass was implemented in the article [9].

VI. 13 TONS SOURCE MASS

We have already mentioned above that G. Rosi proposed and studied [13] a new approach to the measurement of \( G \) with an accuracy of 10 ppm, based on the technique of eliminating the gravity-gradient terms [12]. In addition to the new technique, estimates have been performed for the source mass weight increased to the 13 tons, time separation between Raman pulses increased to
\[
T = 243 \text{ms}, \quad (43)
\]
and the uncertainty of the velocity of atomic clouds reduced to
\[
\sigma (v_{xI}) = \sigma (v_{yI}) = 2 \text{mm/s}, \quad (44a)
\]
\[
\sigma (v_{zI}) = 0.3 \text{mm/s}. \quad (44b)
\]
In this section we show that the optimization of PDD applied to set of cylinders used in \[3, 4\], with a total weight of about 13 tons and uncertainties of the atomic initial positions and velocities \[26a, 26b\] and \[44\], can also lead to accuracy of PDD measurement of the order of 10 ppm. Specifically, we assume that the cylinders are located on a 5-storey structure with a distance between the floors of 0.20011m. Distance between floors includes the height of the cylinder \(h = 0.15011m\) \[3, 4\] and distance \(dh = 5cm\) between cylinders tops and bottoms. Each of the floors has 126 cylinders as shown in Fig. 4c,f. This arrangement of the cylinders is a natural generalization of the geometry chosen in the \[3, 4\].

Maximum and minimum points of the function \(\phi(C)(z,v)\),

\[
\begin{align*}
\{z_{\text{max}}, v_{z_{\text{max}}}, \phi_s^{(C)}(z_{\text{max}}, v_{z_{\text{max}}})\} &= \{-0.281m, 2.38m/s, 1.84859\text{rad}\}, \\
\{z_{\text{min}}, v_{z_{\text{min}}}, \phi_s^{(C)}(z_{\text{min}}, v_{z_{\text{min}}})\} &= \{0.727m, 2.38m/s, -1.81252\text{rad}\},
\end{align*}
\]

(45a) \hspace{1cm} (45b)

one selects for the initial coordinates and velocities of the atomic clouds in the first and second interferometer. The phase difference of the first order in this case is

\[
\Delta_s^{(C)} \phi = 3.66111\text{rad}.
\]

(46)

Linear and quadratic terms in relative variations \(\delta \Delta_s^{(C)} \phi / \Delta_s^{(C)} \phi\) are pieced together in the Table S.IV. Substituting these data into the equations (17), one obtains for RSD and shift Eqs. (S.4a, S.4b). For uncertainties \[26a, 26b, 44\] RSD and shift are equal to

\[
\begin{align*}
\sigma(\Delta_s^{(C)} \phi) &= 14\text{ppm}, \\
\sigma(\Delta_s^{(C)} \phi) &= 20\text{ppm}.
\end{align*}
\]

(47a) \hspace{1cm} (47b)

Let us now consider the contribution from the \(F\)–configuration of the source mass. One determines the distance \(L\) of the halves of the source mass displacement in the horizontal direction from the inequality (35). The calculation showed that the greatest danger comes from the third term in braces in Eq. (34). Then one guarantees the inequality (35) fulfilment if

\[
\left| \frac{\partial \phi^{(F)}(z_{2F}, v_{z2F})}{\partial z_{2F}} \right| \leq \alpha,
\]

(48)

where

\[
\alpha = \frac{\sqrt{0.21\Delta_s^{(C)} \phi}}{2\sigma(z_{2F})} \sigma(\Delta_s^{(C)} \phi) = 0.1195,
\]

(49)

From Fig. 5 one sees that the halves of the source mass should be moved apart by a distance.

\[
L = 1.39\text{m}.
\]

(50)
Then, by computing the first order phase difference in the \( F \)-configuration, \( \Delta_s \phi^{(F)} \), one obtains for the PDD

\[
\Delta^{(2)} \phi = 3.51021 \text{rad}
\]  

(51)

With this signal, the various relative contributions to the PDD are presented in the table S.V. Calculating from these data the first and second order derivatives with respect to the coordinates and velocities of the atomic clouds, and substituting these derivatives in the equations \([17] \), one arrives at the Eqs. \([5.5a \quad 5.5b] \), from which at the uncertainties \([26a \quad 26b \quad 44] \), RSD and shift are equal to

\[
\sigma \left( \Delta^{(2)} \phi \right) = 16 \text{ppm},
\]

(52a)

\[
s \left( \Delta^{(2)} \phi \right) = 20 \text{ppm}.
\]

(52b)

Note also that we have considered the displacement of the source mass as a whole in the vertical direction, and found that the minimal distance at this displacement is equal to

\[
L = 2.92 \text{m}.
\]

(53)

This displacement is only 25% shorter than the displacement of the source mass by 4m required for the technique \([13] \). However, it is more than two times longer than the horizontal displacement \([50] \). Therefore, in the case of a 13-ton source mass, horizontal displacement is preferable.

VII. CONCLUSION

This article is devoted to the calculation of the error budget in the measurement of the Newtonian gravitational constant \( G \) by atomic interferometry methods. Using the technique \([20] \), we obtained expressions for the gravitational field of the cylinder, which is used in these measurements.

Despite the compensation of the gradient of the Earth gravitational field at the points of apogees of the atomic trajectories achieved in the article \([3, 4] \), an absence of this compensation along the entire trajectory leads to the
influence of the Earth’s field on the $G$ measurement accuracy. To overcome this influence, we propose to use the method of eliminating the gradient of the gravitational field of the Earth [12].

The main attention in this article is paid to the calculation of standard deviation (SD) and the shift of the PDD due to the uncertainties of the mean values of the initial coordinates and the velocities of atomic clouds $\{\delta x, \delta v\}$. We propose to include in the error budget new terms. They are originated from the quadratic dependence of the variation of the AI phases on $\{\delta x, \delta v\}$. The shift arises only after including those terms. At the conditions realized in the article [3, 4], calculations brings us to the shift (27b) and to the opposite relative correction $\Delta (58)$ is true, but the contribution of each term is this, however, is not precisely correct. The calculation shows that for independent extreme variables, the expression $\kappa (q_m)$ is larger than corrections considered in [3, 4]. After including this correction, the value of the gravitational constant $G$ should be shifted to

$$ G = 6.67058m^3kg^{-1}s^{-2} \quad (54) $$

from the value $G = 6.67191m^3kg^{-1}s^{-2}$ measured in [3, 4]. We would like to note that Monte Carlo simulation in [3, 4] did not bring to any shift caused by uncertainties of the atom position and velocity.

After eliminating the gradient of the Earth field, one can use optimization technique proposed in [5, 10]. While the PDD in this case becomes smaller [compare signals (25) and (30)], one should be able to measure $G$ with 2.75 times better accuracy [compare RSDs (27a) and (41a)].

Another application of our formulas is the calculation of the systematic error due to the finite size of atomic clouds and their finite temperature [5, 10]. Let us now assume that $\delta x$ is the deviation of the atom from the center of the cloud and $\delta v$ is the deviation from the center of the atomic velocity distribution. If the temperatures are small enough to ignore the Doppler frequency shift, and the aperture of the optical field is large enough to assume that the areas of the Raman pulses do not depend on the position of atoms in the cloud, then the only reason for the dependence of the PDD on $\{\delta x, \delta v\}$ is that the gravitational field $\delta g [x(t)]$ is not the same for different atoms in the cloud. Averaging the phase of the AI over an atomic distribution one will receive from the equation (13)

$$ \langle \delta \phi \rangle = \frac{1}{4} \left( a_m^2 \frac{\partial^2 \phi}{\partial x_m^2} + v_{0m}^2 \frac{\partial^2 \phi}{\partial v_m^2} \right), \quad (55) $$

where $a_m = \sqrt{2 \langle \delta x_m^2 \rangle}$ and $v_{0m} = \sqrt{2 \langle \delta v_m^2 \rangle}$ are the radius and thermal velocity of the atomic cloud along the $m-$axis. Here we pay attention to the fact that at equal radii, $a_x = a_y = a_z$, and temperatures, $v_{0x} = v_{0y} = v_{0z}$, the systematic error (55) disappears. This follows from the Eqs. (23a, 23c) and from the fact that the gravitational field obeys the Laplace equation and, therefore, the trace of the gravitational field curvature tensor (21) $X_{3mm} = 0$.

In the absence of linear terms, one can diagonalize the quadratic shape in phase variation (16), so that

$$ \delta \phi = \sum_{m=1}^{6} a_m \delta q_m^2. \quad (56) $$

By analogy with the error budget in [2, 4, 13, 14, 18, 22], if each term $m$ in Eq. (56) is independent of the others, one might expect that his contribution to the SD is

$$ \sigma_m (\phi) = |a_m| \sigma^2 (q_m), \quad (57) $$

and

$$ \sigma (\phi) = \left[ \sum_{m=1}^{6} \sigma_m^2 (\phi) \right]^{1/2}. \quad (58) $$

This, however, is not precisely correct. The calculation shows that for independent extreme variables, the expression (55) is true, but the contribution of each term is

$$ \sigma_m (\phi) = \sqrt{2} |a_m| \sigma^2 (q_m) \left( 1 + \frac{1}{2} \kappa (q_m) \right)^{1/2}, \quad (59) $$

where $\kappa (q_m)$ is a cumulant (16), i.e. even for normal distribution, when $\kappa (q_m) = 0$, the contribution from each term is $\sqrt{2}$ times greater than in (57).

Instead of using tables, we propose to use for error budget expressions for RSD (S.1a - S.5a) and relative shift (S.1b - S.5a) [obtained by means of general Eqs. (17)], in which one should substitute the coordinates and velocities uncertainties achieved or assumed to be achieved in the experiment.
In this article we have only considered error budget related to atomic variables and did not consider errors associated with the properties of the source mass. Examples of calculation errors associated with the uncertainties of positioning and orientation of the various components of the source mass can be found in the article [10]. We would only like to emphasize that, as in the article [13], it is preferable to use source masses consisting of as many components as possible. So if one uses N cylinders, the signal \( f \) consists of the contributions \( f_m \) of the each cylinder \((m = 1, \ldots N)\).

For RSD one gets \( \sigma_r (f) = \left[ \sum_m f_m^2 \sigma_r^2 (f_m) \right]^{1/2} / f \). If one accepts for estimates that the contributions of the various cylinders and their RSDs are of the same order, \( f_m \sim f_m' \sim f/N \) and \( \sigma_r (f_m) \sim \sigma_r (f_m') \), then we get that the RSD of the entire system of cylinders decreases as \( N^{-1/2} \),

\[
\sigma_r (f) \sim N^{-1/2} \sigma_r (f_m).
\] (60)

Finally, we would like to note that, following the statement [13], that 13-ton source mass can be implemented in the experiment, we increased the number of cylinders to 630 (more than 26 times). At the same time, the optimal signal increased only 8.4 times \([\text{compare Eqs. (51) and (40)}]\), and this increase is partly due to an increase in the delay time between the Raman pulses \( T \).

This example shows that an increase in the weight of the source mass does not even lead to a proportional signal increase. More promising here is an increase in the signal due to the large values of \( T \), the effective wave vector \( k \) and the optimal aspect ratio of the source mass. Due to these factors we predicted [5] PDD \( \Delta^{(2)} s \phi = 386.527 \text{rad} \) even for a source mass \( M = 1080 \text{kg} \).

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Appendix A: Gravity field of the homogeneous cylinder

1. Axial component

It is convenient [20] to explore the following expression for the potential of the gravitational field of a homogeneous cylinder \( \Phi (x) \)

\[
\Phi (r, z) = -2G \rho \int_0^R dy \int_{r-\sqrt{R^2-y^2}}^{r+\sqrt{R^2-y^2}} d\xi \int_{z-h}^{z}\frac{d\zeta}{\sqrt{y^2 + \xi^2 + \zeta^2}}
\] (A1)

where \( \rho, R, \) and \( h \) are the density, radius, and height of the cylinder, \((r, z, \psi = 0)\) are the cylindrical coordinates of the vector \( x \). For an axial component of the gravitational field, \( \delta g_3 (r, z) = -\partial_z \Phi (r, z) \), one gets

\[
\delta g_3 (r, z) = 2G \rho g_3 (r, \zeta)_{\zeta = z-h},
\] (A2)

where the function

\[
g_3 (r, \zeta) = \int_0^R dy \int_{r-\sqrt{R^2-y^2}}^{r+\sqrt{R^2-y^2}} \frac{d\xi}{\sqrt{y^2 + \xi^2 + \zeta^2}}
\] (A3)

can be represented as

\[
g_3 (r, \zeta) = \int_0^R dy \ln \frac{t_+ (y)}{t_- (y)}
= - \int_0^R y \left( \frac{dt_+}{t_+} - \frac{dt_-}{t_-} \right),
\] (A4a)

\[
t_\pm (y) = r \pm \sqrt{R^2-y^2} + \left( \zeta^2 + r^2 + R^2 \pm 2r \sqrt{R^2-y^2} \right)^{1/2}
\] (A4b)
Since \( t_+ (R) = t_- (R) \equiv t (R) \leq t_\pm (0) \) one can write
\[
g_3 (r, \zeta) = \int_{t (R)}^{t_+ (0)} \frac{dt}{t} y_+ (t) + \int_{t_- (0)}^{t (R)} \frac{dt}{t} y_- (t),
\]
where \( y_\pm (t) \) is the root of the equation \( t_\pm (y) = t \). To find this root, consider the functions \( x_\pm(t) = \sqrt{R^2 - y_\pm^2 (t)} \).

\[
0 < x_\pm (t) < R.
\]

For them one gets
\[
x_\pm (t) = \pm t + \sqrt{\zeta^2 + R^2 + 2tr} \quad \text{or} \quad \pm t - \sqrt{\zeta^2 + R^2 + 2tr}
\]
Since \( t + \sqrt{\zeta^2 + R^2 + 2tr} > R \), then one should choose \( x_+ (t) = t - \sqrt{\zeta^2 + R^2 + 2tr} \). Since \( t_- (0) > r - R + |r - R| > 0 \), \(-t - \sqrt{\zeta^2 + R^2 + 2tr} < 0 \), hence \( x_- (t) = \sqrt{\zeta^2 + R^2 + 2tr} - t \) or
\[
x_\pm (t) = \pm \left( t - \sqrt{\zeta^2 + R^2 + 2tr} \right).
\]

Therefore, one concludes that the functions \( y_\pm (t) \) are coincident and equal to
\[
y_+ (t) = y_- (t) = y(t) = \left[ 2t \left( \sqrt{\zeta^2 + R^2 + 2tr} - r \right) - t^2 - \zeta^2 \right]^{1/2}
\]
and.
\[
g_3 (r, \zeta) = \int_{t_- (0)}^{t_+ (0)} \frac{dt}{t} y (t).
\]

Introducing new variable,
\[
u = \sqrt{\zeta^2 + R^2 + 2tr} - r,
\]
for which
\[
u [t_\pm (0)] \equiv u_\pm = \sqrt{\zeta^2 + (r \pm R)^2},
\]
\[
y(t) = \frac{\sqrt{q (u^2)}}{2r},
\]
\[
q (\eta) = \{u_+^2 - \eta\} \{\eta - u_+^2\},
\]
\[
dt = \frac{u + r}{r} du
\]
and so
\[
g_3 (r, \zeta) = I + I',
\]
\[
I = \int_{u_-}^{u_+} \frac{du}{\sqrt{q (u^2)}} J (u),
\]
\[
J (u) = \frac{q (u^2) \left( u^2 - \zeta^2 - R^2 - u^2 \right)}{w (u^2)},
\]
\[
I' = \frac{1}{2r} \int_{u_-}^{u_+} d\eta J' (\eta),
\]
\[
J' (\eta) = \frac{(\eta - \zeta^2 - R^2 - r^2)}{\sqrt{q (\eta) w (\eta)}},
\]
\[
w (\eta) = (\eta - \eta_1) (\eta - \eta_2),
\]
\[
\eta_{1,2} = \left( r \pm \sqrt{\zeta^2 + R^2} \right)^2.
\]
Using equality
\[ w (\eta) + q (\eta) = -4r^2 \zeta^2, \quad (A14) \]
one can show that the integrand \( J'(\eta) \) is an antisymmetric function with respect to the middle point \( \eta = \{ u^2 [t_+(0)] + u^2 [t_-(0)] \} / 2 \), and, therefore, the term \( \text{(A13d)} \) is equal 0. At the same time, expanding \( J(u) \) into partial fractions, one obtains
\[ g_3 (r, \zeta) = (R^2 + \zeta^2 - r^2) I_1 + I_2 + I_{3+} + I_{3-}, \quad (A15a) \]
\[ I_1 = \int_{u_-}^{u_+} \frac{du}{\sqrt{q(u^2)}} \quad (A15b) \]
\[ I_2 = \int_{u_-}^{u_+} \frac{duu^2}{\sqrt{q(u^2)}} \quad (A15c) \]
\[ I_{3\pm} = 2r\zeta^2 \left( r \pm \sqrt{\zeta^2 + R^2} \right) \int_{u_-}^{u_+} \frac{du}{\sqrt{q(u^2)} (u^2 - \eta_{1,2})} \quad (A15d) \]
The integrals \( \text{(A15)} \), one can compute using the substitution
\[ u = \sqrt{u^2_+ - (u^2_+ - u^2_-) \sin^2 \phi} \quad (A16) \]
Finally, one arrives at the following expression for the axial component of the cylinder’s field
\[ \begin{align*}
\delta g_3 (r, z) &= 2G \rho g_3 (r, \zeta) \zeta^2_{z-h} \\
g_3 (r, \zeta) &= \frac{(\zeta^2 + R^2 - r^2)}{\sqrt{\zeta^2 + (r + R)^2}} K(k) + \sqrt{\zeta^2 + (r + R)^2} E(k) \\
&+ \frac{\zeta^2}{\sqrt{\zeta^2 + (r + R)^2}} \sum_{j=\pm 1} \left[ \frac{r + j \sqrt{\zeta^2 + R^2}}{R - j \sqrt{\zeta^2 + R^2}} I \left( \frac{2R}{R - j \sqrt{\zeta^2 + R^2}} k \right) \right], \quad (A17b) \\
\left(\frac{4\zeta R}{\zeta^2 + (r + R)^2} \right)^2 &= \left(\frac{4\zeta R}{\zeta^2 + (r + R)^2} \right)^2, \quad (A17c) \\
\end{align*} \]
where \( K(k) \), \( E(k) \) and \( \Pi (\alpha | k) \) are the complete elliptic integrals of the first, second and third order respectively.

### 2. Radial component

For the radial component of the gravitational field \( \delta g_r (r, z) = -\partial_r \Phi (r, z) \) one obtains from \( \text{(A1)} \)
\[ \begin{align*}
\delta g_r (r, z) &= 2G \rho g_r (r, \zeta) \zeta^2_{z-h} \\
g_r (r, \zeta) &= -\int_0^R y \left( \frac{dt_+}{t_+} - \frac{dt_-}{t_-} \right), \quad (A18b) \\
t_\pm (y) &= \zeta + \left[ \zeta^2 + r^2 + R^2 \pm 2r \sqrt{R^2 - y^2} \right]^{1/2} \quad (A18c) \\
\end{align*} \]
Since still \( t_+ (R) = t_- (R) \leq t_\pm (0) \), one gets,
\[ g_r (r, \zeta) = \int_{t_- (0)}^{t_+(R)} \frac{dt}{t} y_- (t) + \int_{t_- (R)}^{t_+(0)} \frac{dt}{t} y_+ (t), \quad (A19) \]
where \( y_\pm (t) \) are functions inverse to \( \text{(A18c)} \). Since these functions are the same
\[ y_+ (t) = y_- (t) \equiv y (t) = \frac{1}{2r} \left[ 4r^2 R^2 - (t^2 - 2\zeta t - r^2 - R^2)^2 \right]^{1/2}, \quad (A20) \]
then, choosing as an integration variable \( u = t - \zeta \), one finds that

\[
g_r (r, \zeta) = I + I',
\]

\[
I = -\frac{\zeta}{2r} \int_{u_+}^{u_-} \frac{du \eta (u^2)}{(u^2 - \zeta^2)^2 \sqrt{\eta}} \sqrt{q (u^2)},
\]

\[
I' = \frac{1}{4r} \int_{u_+}^{u_-} \frac{du \sqrt{q (\eta)}}{(\eta - \zeta^2)},
\]

where \( u_+ \) and \( q (\eta) \) are given by Eqs. (A12a) and (A12c). Because \( u_+^2 - \zeta^2 \) and \( q (\eta + \zeta^2) \) are independent of \( \zeta \), the term \( I' \) gives no contribution to the acceleration (A18a) and can be omitted. While using the substitution (A16), one reduces the integral in (A21b) to elliptic integrals, which brings us to the next final result

\[
delta g_r (r, z) = 2G \rho g_r (r, \zeta = z - h), \]

\[
g_r (r, \zeta) = \frac{\zeta}{2r \sqrt{\zeta^2 + (r + R)^2}} \left[ -\left( \zeta^2 + 2r^2 + 2R^2 \right) K(k) + \left( \zeta^2 + (r + R)^2 \right) E(k) + \frac{(r^2 - R^2)^2}{(r + R)^2} \Pi \left( \frac{4rR}{(r + R)^2}; \right) \right],
\]

where \( k \) is given by Eq. (A17c).

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Supplemental Material

Linear and non-linear terms in the variations of the phase differences and expressions for the relative standard deviations (RSD) and relative shifts, obtained numerically are pieced together in the following Tables and expressions.

### TABLE S.I: Relative contributions to the phase double difference (PDD) and error budget for two configurations of source mass

| Term                          | C-configuration | F-configuration |
|-------------------------------|-----------------|-----------------|
| $\pm \Delta \phi / \Delta(\phi^2)$ | 0.685376        | 0.314624        |
| Linear in position            | $0.322\delta z_{1C} + 0.111\delta z_{2C} + 7.77 \cdot 10^7$ | $0.132\delta z_{1F} + 0.518\delta z_{2F} - 7.77 \cdot 10^7$ |
|                               | $[\Gamma_{E31}(\delta x_{1C} - \delta x_{2C}) + \Gamma_{E32}(\delta y_{1C} - \delta y_{2C})]$ | $[\Gamma_{E31}(\delta x_{1F} - \delta x_{2F}) + \Gamma_{E32}(\delta y_{1F} - \delta y_{2F})]$ |
| Linear in velocity            | $0.0377\delta v_{1C} + 0.0150\delta v_{2C} + 1.24 \cdot 10^7$ | $0.0132\delta v_{1F} + 0.0683\delta v_{2F} - 1.24 \cdot 10^7$ |
|                               | $[\Gamma_{E31}(\delta v_{1C} - \delta v_{2C}) + \Gamma_{E32}(\delta y_{1C} - \delta y_{2C})]$ | $[\Gamma_{E31}(\delta v_{1F} - \delta v_{2F}) + \Gamma_{E32}(\delta y_{1F} - \delta y_{2F})]$ |
| Nonlinear in position         | 12.3 $(\delta x_{1C}^2 + \delta y_{1C}^2) - 24.7\delta z_{1C}^2$ | 15.5 $(\delta x_{1F}^2 + \delta y_{1F}^2) - 30.9\delta z_{1F}^2$ |
|                               | $+12.2(\delta x_{2C}^2 + \delta y_{2C}^2) - 24.3\delta z_{2C}^2$ | $+15.7(\delta x_{2F}^2 + \delta y_{2F}^2) - 31.3\delta z_{2F}^2$ |
| Nonlinear in velocity         | 0.375 $(\delta v_{1C}^2 + \delta v_{2C}^2) - 0.750\delta v_{1C}^2$ | 0.451 $(\delta v_{1F}^2 + \delta v_{2F}^2) - 0.901\delta v_{1F}^2$ |
|                               | $+0.351(\delta v_{2C}^2 + \delta v_{2C}^2) - 0.702\delta v_{2C}^2$ | $+0.456(\delta v_{2F}^2 + \delta v_{2F}^2) - 0.937\delta v_{2F}^2$ |
| Position-velocity cross term  | 3.99 $(\delta v_{1C}\delta x_{1C} + \delta v_{1C}\delta y_{1C}) - 7.91\delta v_{1C}\delta z_{1C}$ | 5.10 $(\delta v_{1F}\delta x_{1F} + \delta v_{1F}\delta y_{1F}) - 10.2\delta v_{1F}\delta z_{1F}$ |
|                               | $+4.05[\delta v_{2C}\delta x_{2C} + \delta v_{2C}\delta y_{2C}] - 8.10\delta v_{2C}\delta z_{2C}$ | $+5.08[\delta v_{2F}\delta x_{2F} + \delta v_{2F}\delta y_{2F}] - 10.2\delta v_{2F}\delta z_{2F}$ |

\[
\sigma \left( \Delta(\phi^2) \right) = \begin{cases} 
0.104\sigma^2(z_{1C}) + 0.0137\sigma^2(z_{2C}) + 1.42 \cdot 10^{-3}\sigma^2(v_{1C}) + 2.24 \cdot 10^{-4}\sigma^2(v_{2C}) \\
+ 0.0173\sigma^2(z_{1F}) + 0.269\sigma^2(z_{2F}) + 1.75 \cdot 10^{-4}\sigma^2(v_{1F}) + 4.66 \cdot 10^{-3}\sigma^2(v_{2F}) \\
+ \sum_{j=1,2} \sum_{t=C,F} [6.04 \cdot 10^{11}(\Gamma_{E31}^2\sigma^2(x_{jt}) + \Gamma_{E32}^2\sigma^2(y_{jt})) + 1.55 \cdot 10^{10}(\Gamma_{E31}^2\sigma^2(x_{jt}) + \Gamma_{E32}^2\sigma^2(y_{jt}))]
\end{cases}
\]
\[
+ 305[\sigma^4(x_{1C}) + \sigma^4(y_{1C})] + 1220\sigma^4(z_{1C}) + 296[\sigma^4(x_{2C}) + \sigma^4(y_{2C})] + 1180\sigma^4(z_{2C}) \\
+ 0.282[\sigma^4(v_{1C}) + \sigma^4(v_{1C})] + 1.13\sigma^4(v_{1C}) + 0.247[\sigma^4(v_{2C}) + \sigma^4(v_{2C})] + 0.987\sigma^4(v_{2C}) \\
+ 15.9[\sigma^2(x_{1C})\sigma^2(v_{1C}) + \sigma^2(v_{1C})\sigma^2(v_{1C})] + 63.5\sigma^2(z_{1C})\sigma^2(z_{1C}) \\
+ 16.4[\sigma^2(x_{2C})\sigma^2(v_{2C}) + \sigma^2(v_{2C})\sigma^2(v_{2C})] + 65.6\sigma^2(z_{2C})\sigma^2(z_{2C}) \\
+ 478[\sigma^2(x_{1F}) + \sigma^2(y_{1F})] + 1910\sigma^2(z_{1F}) + 490[\sigma^2(x_{2F}) + \sigma^4(y_{2F})] + 1960\sigma^4(z_{2F}) + \\
+ 0.406[\sigma^4(v_{1F}) + \sigma^4(v_{1F})] + 1.63\sigma^4(v_{1F}) + 0.439[\sigma^4(v_{2F}) + \sigma^4(v_{2F})] + 1.76\sigma^4(v_{2F}) \\
+ 26.0[\sigma^2(x_{1F})\sigma^2(v_{1F}) + \sigma^2(v_{1F})\sigma^2(v_{1F})] + 104\sigma^2(z_{1F})\sigma^2(z_{1F}) \\
+ 25.8[\sigma^2(x_{2F})\sigma^2(v_{2F}) + \sigma^2(v_{2F})\sigma^2(v_{2F})] + 103\sigma^2(z_{2F})\sigma^2(z_{2F})]^{1/2},
\]
\[
s \left( \Delta(\phi^2) \right) = 12.3[\sigma^2(x_{1C}) + \sigma^2(y_{1C})] - 24.7\sigma^2(z_{1C}) + 12.2[\sigma^2(x_{2C}) + \sigma^2(y_{2C})] - 24.3\sigma^2(v_{2C}) \\
+ 0.375[\sigma^2(v_{1C}) + \sigma^2(v_{1C})] - 0.750\sigma^2(v_{1C}) + 0.351[\sigma^2(v_{2C}) + \sigma^2(v_{2C})] - 0.702\sigma^2(v_{2C}) \\
+ 15.5[\sigma^2(x_{1F}) + \sigma^2(y_{1F})] - 30.9\sigma^2(z_{1F}) + 15.7[\sigma^2(x_{2F}) + \sigma^2(y_{2F})] - 31.3\sigma^2(z_{2F}) \\
+ 0.451[\sigma^2(v_{1F}) + \sigma^2(v_{1F})] - 0.901\sigma^2(v_{1F}) + 0.468[\sigma^2(v_{2F}) + \sigma^2(v_{2F})] - 0.937\sigma^2(v_{2F}).
\]

### TABLE S.II: The same as Table S.I, but at the optimal choice of the atomic positions and velocities in the C-configuration.

| Term                          | C-configuration |
|-------------------------------|-----------------|
| Linear in position            | 5.82 \cdot 10^{-7}\delta z_{1C} - 9.34 \cdot 10^{-4}\delta z_{2C} |
| Linear in velocity            | 1.30 \cdot 10^{-3}\delta v_{1C} - 1.62 \cdot 10^{-4}\delta v_{2C} |
| Nonlinear in position         | -17.7 (\delta x_{1C}^2 + \delta y_{1C}^2) + 35.35\delta z_{1C}^2 - 15.1 (\delta x_{2C}^2 + \delta y_{2C}^2) + 30.1\delta z_{2C}^2 |
| Nonlinear in velocity         | -0.49 (\delta v_{1C}^2 + \delta v_{1C}^2) + 0.977\delta v_{1C}^2 - 0.451 (\delta v_{2C}^2 + \delta v_{2C}^2) + 0.902\delta v_{2C}^2 |
| Position-velocity cross term  | -5.66 (\delta v_{1C}\delta x_{1C} + \delta v_{1C}\delta y_{1C}) + 11.3\delta v_{1C}\delta z_{1C} - 4.82 (\delta v_{2C}\delta x_{2C} + \delta v_{2C}\delta y_{2C}) + 9.65\delta v_{2C}\delta z_{2C} |
\[ \sigma \left( \Delta_1^{(C)} \phi \right) = \{ 625 \sigma^4 (x_1C) + \sigma^4 (y_1C) \} + 2600 \sigma^4 (z_1C) + 473 \sigma^4 (x_2C) + \sigma^4 (y_2C) \} + 1890 \sigma^4 (z_2C) + 0.497 \sigma^4 (v_1C) + \sigma^4 (y_1C) + 1.99 \sigma^4 (v_1C) + 0.424 \sigma^4 (v_2C) + \sigma^4 (y_2C) \} + 1.69 \sigma^4 (v_2C) + 33.3 \sigma^2 (x_1C) \sigma^2 (v_1C) + \sigma^2 (y_1C) \sigma^2 (v_1C) \} + 133 \sigma^2 (z_1C) \sigma^2 (v_1C) \} + 24.2 \sigma^2 (x_2C) \sigma^2 (v_2C) + \sigma^2 (y_2C) \sigma^2 (v_2C) \} + 96.8 \sigma^2 (z_2C) \sigma^2 (v_2C) \} + [2.03 \sigma^2 (z_1F) + 2.07 \sigma^2 (z_2F)] \cdot 10^{-3} + [5.20 \sigma^2 (v_1F) + 5.30 \sigma^2 (v_2F)] \cdot 10^{-5} + [1050 \sigma^4 (x_1F) + 73.4 \sigma^4 (y_1F) + 9.85 \sigma^2 (x_1F) \sigma^2 (y_1F) + 570 \sigma^4 (z_1F) + 1020 \sigma^4 (x_2F) + 70.9 \sigma^4 (y_2F) + 9.56 \sigma^2 (x_2F) \sigma^2 (y_2F) + 554 \sigma^4 (z_2F)] \cdot 10^{-5} + [963 \sigma^4 (v_1F) + 67.5 \sigma^4 (v_1F) + 1.0 \sigma^2 (v_1F) \sigma^2 (v_1F) + 520 \sigma^4 (v_1F) + 881 \sigma^4 (v_2F) + 60.9 \sigma^4 (v_2F) + 8.25 \sigma^2 (v_2F) \sigma^2 (v_2F) + 479 \sigma^4 (v_2F)] \cdot 10^{-8} + [539 \sigma^2 (x_1F) \sigma^2 (v_1F) + 2.52 \sigma^2 (z_1F) \sigma^2 (v_1F) + 37.6 \sigma^2 (y_1F) \sigma^2 (v_1F) + 292 \sigma^2 (z_1F) \sigma^2 (v_1F) + 523 \sigma^2 (x_2F) \sigma^2 (v_2F) + 36.3 \sigma^2 (y_2F) \sigma^2 (v_2F) + 2.45 \sigma^2 (x_2F) \sigma^2 (v_2F) + 284 \sigma^2 (z_2F) \sigma^2 (v_2F)] \cdot 10^{-6} \}^{1/2}, \] (S.3a)

\[ \sigma \left( \Delta_2^{(C)} \phi \right) = -18.0 \sigma^2 (x_1C) + \sigma^2 (y_1C) + 36.1 \sigma^2 (z_1C) - 15.4 \sigma^2 (x_2C) + \sigma^2 (y_2C) + 30.74 \sigma^2 (v_2C) - 0.498 \sigma^2 (v_1C) + \sigma^2 (v_1C) - 0.977 \sigma^2 (v_1C) - 0.451 \sigma^2 (v_2C) + \sigma^2 (v_2C) + 0.902 \sigma^2 (v_2C). \] (S.3b)

TABLE S. III: The same as in Table S. I, but at the optimal choice of the atomic positions and velocities in the C-configuration, and after horizontal moving apart the halves of the source mass in the opposite directions by the distance \( \Delta z \) in the \( F \)-configuration.

| Term | C-configuration | F-configuration |
|------|----------------|----------------|
| \( \Delta_{1}^{(C)} \phi / \Delta_{2}^{(C)} \phi \) | 1.020272 | -0.02072 |
| Linear in position | \( 5.94 \cdot 10^{-5} \delta z_{1C} - 9.53 \cdot 10^{-7} \delta z_{2C} \) | \( [4.51 \delta z_{1F} - 4.55 \delta z_{2F}] \cdot 10^{-2} \) |
| Linear in velocity | \( 1.33 \cdot 10^{-3} \delta v_{1C} - 1.66 \cdot 10^{-7} \delta v_{2C} \) | \( [7.21 \delta v_{1F} - 7.28 \delta v_{2F}] \cdot 10^{-3} \) |
| Nonlinear in position | \( -18.0 \delta x_{2C} + 36.1 \delta z_{2C} - 15.4 \delta x_{2C} + 30.7 \delta z_{2C} \) | \( -72.6 \delta x_{1F} + 19.2 \delta y_{1F} + 53.4 \delta z_{1F} \) |
| | \( +0.997 \delta y_{2C} + 0.997 \delta y_{2C} \) | \( -9.93 \delta x_{1F} \delta y_{1F} \) |
| | \( -0.460 \delta v_{2C} + 30.7 \delta z_{2C} \) | \( -71.5 \delta x_{2F} + 18.8 \delta y_{2F} - 52.6 \delta z_{2F} \) |
| | \( +0.921 \delta v_{2C} \) | \( -9.8 \delta x_{2F} \delta y_{2F} \cdot 10^{-3} \) |
| Nonlinear in velocity | \( -0.498 \delta v_{1C} + \delta y_{1C} \) | \( -21.9 \delta v_{1F} + 5.81 \delta y_{1F} \) |
| | \( +0.997 \delta y_{1C} \) | \( +16.1 \delta x_{1F} - 3.0 \delta v_{1F} \delta y_{1F} \) |
| | \( -0.460 \delta v_{2C} + \delta y_{2C} \) | \( -21.0 \delta v_{2F} + 5.5 \delta y_{2F} \) |
| | \( +0.921 \delta v_{2C} \) | \( +15.5 \delta v_{2F} - 2.8 \delta x_{2F} \delta y_{2F} \cdot 10^{-4} \) |
| position-velocity cross term | \( -5.77 \delta v_{1C} \delta x_{1C} + \delta y_{1C} \delta y_{1C} \) | \( -23.2 \delta v_{1F} \delta x_{1F} + 6.13 \delta v_{1F} \delta y_{1F} \) |
| | \( +11.5 \delta v_{1C} \delta z_{1C} \) | \( +17.1 \delta v_{1F} \delta z_{1F} - 1.59 \delta v_{1F} \delta x_{1F} + \delta v_{1F} \delta y_{1F} \) |
| | \( -4.92 \delta v_{2C} \delta x_{2C} + \delta y_{2C} \delta y_{2C} \) | \( -22.9 \delta v_{2F} \delta x_{2F} + 6.02 \delta v_{2F} \delta y_{2F} + 16.8 \delta v_{2F} \delta z_{2F} \) |
| | \( +9.84 \delta v_{2C} \delta z_{2C} \) | \( -1.56 \delta v_{2F} \delta x_{2F} + \delta v_{2F} \delta y_{2F} \cdot 10^{-3} \) |
TABLE S.IV: The same as Table S.II, but for the 630 cylinders.

| Term                                | C-configuration                                                                 |
|-------------------------------------|---------------------------------------------------------------------------------|
| Linear in position                  | $-9.25 \cdot 10^{-8} \delta z_{1C} + 1.28 \cdot 10^{-7} \delta z_{2C}$          |
| Linear in velocity                  | $-1.79 \cdot 10^{-8} \delta v_{z1C} + 1.18 \cdot 10^{-8} \delta v_{z2C}$          |
| Nonlinear in position               | $4.92 (\delta x_{1C} + \delta y_{1C}) - 9.84 \delta z_{1C} + 2.97 (\delta x_{2C} + \delta y_{2C}) - 5.95 \delta z_{2C}$ |
| Nonlinear in velocity               | $0.306 (\delta v_{x1C}^2 + \delta v_{y1C}^2) - 0.612 \delta v_{z1C}^2 + 0.197 (\delta v_{x2C}^2 + \delta v_{y2C}^2) - 0.393 \delta v_{z2C}^2$ |
| Position-velocity cross term        | $2.39 (\delta v_{x1C} \delta x_{1C} + \delta v_{y1C} \delta y_{1C}) - 4.78 \delta v_{z1C} \delta z_{1C} + 1.45 (\delta v_{x2C} \delta x_{2C} + \delta v_{y2C} \delta y_{2C}) - 2.89 \delta v_{z2C} \delta z_{2C}$ |

\[
\sigma \left( \Delta_s^{(C)} \phi \right) = \{ 48.4 [\sigma^4 (x_{1C}) + \sigma^4 (y_{1C})] + 193 \sigma^4 (z_{1C}) + 17.7 [\sigma^4 (x_{2C}) + \sigma^4 (y_{2C})] + 70.8 \sigma^4 (z_{2C}) \\
+ 0.187 \sigma^4 (v_{x1C}) + \sigma^4 (v_{y1C}) \} + 0.749 \sigma^4 (v_{x1C}) + 0.0773 \sigma^4 (v_{x2C}) + \sigma^4 (v_{y2C}) + 0.309 \sigma^4 (v_{z1C}) + 5.71 \sigma^4 (v_{z1C}) \sigma^2 (v_{x1C}) \sigma^2 (v_{y1C}) + 22.8 \sigma^2 (z_{1C}) \sigma^2 (v_{x1C}) + 20.9 \sigma^2 (z_{1C}) \sigma^2 (v_{y1C}) + 8.36 \sigma^2 (z_{2C}) \sigma^2 (v_{x1C}) \}^{1/2},
\]

\[
s \left( \Delta_s^{(C)} \phi \right) = 4.92 [\sigma^2 (x_{1C}) + \sigma^2 (y_{1C})] - 9.84 \sigma^2 (z_{1C}) + 2.97 [\sigma^2 (x_{2C}) + \sigma^2 (y_{2C})] - 5.95 \sigma^2 (z_{2C}) + 0.306 [\sigma^2 (v_{x1C}) + \sigma^2 (v_{y1C})] - 0.612 \sigma^2 (v_{z1C}) + 0.197 [\sigma^2 (v_{x2C}) + \sigma^2 (v_{y2C})] - 0.393 \sigma^2 (v_{z2C}),
\]

TABLE S.V: The same as Table S.III, but for the 630 cylinders.

| Term                                | C-configuration                                                                 | F-configuration                                                                 |
|-------------------------------------|---------------------------------------------------------------------------------|---------------------------------------------------------------------------------|
| $\phi_s^{(1)} / \Delta_s^{(2)} \phi$ | 1.042989                                                                        | $(3.30 \delta z_{1F} - 3.41 \delta z_{2F}) \cdot 10^{-2}$                      |
| Linear in position                  | $-9.65 \cdot 10^{-8} \delta z_{1C} + 1.33 \cdot 10^{-7} \delta z_{2C}$         | $(8.03 \delta v_{z1F} - 8.28 \delta v_{z2F}) \cdot 10^{-3}$                   |
| Linear in velocity                  | $-1.86 \cdot 10^{-8} \delta v_{z1C} + 1.23 \cdot 10^{-7} \delta v_{z2C}$       | $(-33.5 \delta x_{1F}^2 + 9.01 \delta y_{1F}^2 - 1.63 \delta v_{1F} \delta y_{1F}) + 24.5 \delta z_{1F}^2 - 32.7 \delta x_{2F}^2 + 8.75 \delta y_{2F}^2 + 1.60 \delta v_{1F} \delta y_{2F} + 24.0 \delta z_{2F}^2) \cdot 10^{-3}$ |
| Nonlinear in position               | $5.13 (\delta x_{1C}^2 + \delta y_{1C}^2) - 10.3 \delta z_{1C}^2 + 3.10 (\delta x_{2C}^2 + \delta y_{2C}^2) - 6.21 \delta z_{2C}^2$ | $(23.2 \delta x_{1F}^2 + 6.28 \delta y_{1F}^2 + 17.0 \delta v_{1F}^2 + 22.3 \delta x_{2F}^2 + 5.94 \delta y_{2F}^2 + 1.09 \delta v_{1F} \delta y_{2F} + 16.3 \delta z_{2F}^2) \cdot 10^{-4}$ |
| Nonlinear in velocity               | $0.319 (\delta v_{x1C}^2 + \delta v_{y1C}^2) - 0.638 \delta v_{z1C}^2 + 0.025 (\delta v_{x2C}^2 + \delta v_{y2C}^2) - 0.410 \delta v_{z2C}^2$ | $(-163 \delta v_{1F} \delta x_{1F} + 43.8 \delta v_{1F} \delta y_{1F}) - 3.97 (\delta x_{1F} \delta v_{1F} + \delta y_{1F} \delta v_{1F}) + 119 \delta v_{z1F} \delta z_{1F} - 159 \delta v_{z1F} \delta x_{2F} + 42.5 \delta v_{1F} \delta y_{2F} + 3.88 (\delta v_{x2F} \delta y_{2F} + \delta v_{y2F} \delta x_{2F}) + 117 \delta v_{1F} \delta z_{2F}) \cdot 10^{-4}$ |
| Position-velocity cross term        | $2.49 (\delta v_{x1C} \delta x_{1C} + \delta v_{y1C} \delta y_{1C}) - 4.99 \delta v_{z1C} \delta z_{1C} + 1.51 (\delta v_{x2C} \delta x_{2C} + \delta v_{y2C} \delta y_{2C}) - 3.02 \delta v_{z2C} \delta z_{2C}$ | $(-163 \delta v_{1F} \delta x_{1F} + 43.8 \delta v_{1F} \delta y_{1F}) - 3.97 (\delta x_{1F} \delta v_{1F} + \delta y_{1F} \delta v_{1F}) + 119 \delta v_{z1F} \delta z_{1F} - 159 \delta v_{z1F} \delta x_{2F} + 42.5 \delta v_{1F} \delta y_{2F} + 3.88 (\delta v_{x2F} \delta y_{2F} + \delta v_{y2F} \delta x_{2F}) + 117 \delta v_{1F} \delta z_{2F}) \cdot 10^{-4}$ |
\( \sigma \left( \Delta_s^{(2)} \phi \right) = \left\{ 52.6 \left[ \sigma^4(x_{1C}) + \sigma^4(y_{1C}) \right] + 210\sigma^4(z_{1C}) + 19.3 \left[ \sigma^4(x_{2C}) + \sigma^4(y_{2C}) \right] + 77.0\sigma^4(z_{2C}) \\
+ 0.204 \left[ \sigma^4(v_{11C}) + \sigma^4(v_{12C}) \right] + 0.814\sigma^4(v_{21C}) + 0.0841 \left[ \sigma^4(v_{21C}) + \sigma^4(v_{22C}) \right] \\
+ 0.336\sigma^4(v_{22C}) + 6.21 \left[ \sigma^2(x_{1C}) \sigma^2(v_{11C}) + \sigma^2(y_{1C}) \sigma^2(v_{12C}) \right] + 24.9\sigma^2(z_{1C}) \sigma^2(v_{12C}) \\
+ 2.27 \left[ \sigma^2(x_{2C}) \sigma^2(v_{21C}) + \sigma^2(y_{2C}) \sigma^2(v_{22C}) \right] + 9.10\sigma^2(z_{2C}) \sigma^2(v_{22C}) \\
+ [1.09\sigma^2(z_{1F}) + 1.16\sigma^2(z_{2F})] \cdot 10^{-3} + [6.44\sigma^2(v_{11F}) + 6.85\sigma^2(v_{22F})] \cdot 10^{-5} \\
+ [2240\sigma^2(x_{1F}) + 162\sigma^2(y_{1F}) + 2.66\sigma^2(x_{1F}) \sigma^2(y_{1F}) + 1200\sigma^4(z_{1F}) \\
+ 2140\sigma^4(x_{2F}) + 153\sigma^4(y_{2F}) + 2.55\sigma^2(x_{2F}) \sigma^2(y_{2F}) + 1150\sigma^4(z_{2F})] \cdot 10^{-6} \\
+ [1080\sigma^4(v_{11F}) + 78.8\sigma^4(v_{12F}) + 1.28\sigma^2(v_{11F}) \sigma^2(v_{12F}) + 575\sigma^4(v_{12F}) \\
+ 993\sigma^4(v_{12F}) + 70.6\sigma^4(v_{12F}) + 1.18\sigma^2(v_{12F}) \sigma^2(v_{12F}) + 534\sigma^4(v_{12F})] \cdot 10^{-8} \\
+ [2650\sigma^2(x_{1F}) \sigma^2(v_{11F}) + 192\sigma^2(y_{1F}) \sigma^2(v_{12F}) + 1.57 \left[ \sigma^2(y_{1F}) \sigma^2(v_{11F}) + \sigma^2(x_{1F}) \sigma^2(v_{12F}) \right] \\
+ 1410\sigma^2(z_{1F}) \sigma^2(v_{12F}) + 2530\sigma^2(x_{2F}) \sigma^2(v_{22F}) + 1181\sigma^2(y_{2F}) \sigma^2(v_{22F}) \\
+ 1.51 \left[ \sigma^2(x_{2F}) \sigma^2(v_{22F}) + \sigma^2(y_{2F}) \sigma^2(v_{22F}) \right] + 1360\sigma^2(z_{2F}) \sigma^2(v_{22F}) \right] \cdot 10^{-7} \right\}^{1/2}, \tag{5a} \]

\begin{align} 
\sigma \left( \Delta_s^{(2)} \phi \right) &= 5.13 \left[ \sigma^2(x_{1C}) + \sigma^2(y_{1C}) \right] - 10.2\sigma^2(z_{1C}) + 3.10 \left[ \sigma^2(x_{2C}) + \sigma^2(y_{2C}) \right] - 6.22\sigma^2(z_{2C}) \\
+ 0.319 \left[ \sigma^2(v_{11C}) + \sigma^2(v_{12C}) \right] - 0.639\sigma^2(v_{21C}) + 0.205 \left[ \sigma^2(v_{21C}) + \sigma^2(v_{22C}) \right] - 0.410\sigma^2(v_{22C}) \\
+ \left[ -33.5\sigma^2(x_{1F}) + 9.01\sigma^2(y_{1F}) + 24.5\sigma^2(z_{1F}) - 32.7\sigma^2(x_{1F}) + 8.75\sigma^2(y_{2F}) + 24.0\sigma^2(z_{2F}) \right] \cdot 10^{-3} \\
+ \left[ -23.2\sigma^2(v_{11F}) + 6.28\sigma^2(v_{12F}) + 17.0\sigma^2(v_{12F}) - 22.3\sigma^2(v_{12F}) + 5.94\sigma^2(v_{22F}) + 16.3\sigma^2(v_{22F}) \right] \cdot 10^{-4} \tag{5b}\end{align} 

Appendix B: Manuscript consideration

1. Date: Tuesday, July 16, 2019 8:53:52 AM

From: pra@aps.org
To: bdubetsky@gmail.com
Subject: Your manuscript AG11947 Dubetsky
Re: AG11947

Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant by B. Dubetsky

Dear Dr. Dubetsky,

The Physical Review editors attempt to accept only papers that are scientifically sound, important to the field, and contain significant new results in physics. We judge that these acceptance criteria are not met by your manuscript. We regret that consequently we cannot accept the paper for publication in the Physical Review.

Yours sincerely,
Doerte Blume
Associate Editor

2. Date: Friday, September 6, 2019 7:12:08 PM

From: Boris Dubetsky
To: pra@aps.org
Subject: Re: Your manuscript AG11947 Dubetsky

Attachments: image.png
cylinders3_short..pdf

Dear Dr. Blume,

I examined the Physical Review A criteria before I dared to submit my article. They are strict, but they are not quantitative. You wrote about two parameters: scientific importance to the field and significant new results in physics. My manuscript AG11947 is devoted to the important scientific problem, measurement of the Newton gravity
constant G, which is measured worse than all other fundamental constant. It also contains new results: restriction of the measurement accuracy, caused by the gradient of the Earth gravity field, the necessity to change the approach to the error budget, caused by the point that measurements have been ALWAISE performed near extremal values of the atomic position and velocity, the new systematic error arising from the same reason, inventing of the regime, where errors caused by the atomic finite temperature and finite size of atomic clouds disappear.

I see only one way to estimate scientific importance and significance of the new results fairly - by comparing my article with others published in your Journal. Please, understand me correctly, I am not complaining, not claiming, why did you accept those articles and reject my one. I indeed don’t know another way to estimate of my article. If you are aware of another way, please let me know it.

Last two theoretical articles devoted to the atom interferometry are A. Bertoldi et.al, Phys. Rev. A 99, 033619 (2019), Ya-Lie Wang et.al, Phys. Rev. A 98, 053604 (2018). Both articles consider only one systematic error caused by the Raman pulses finite duration. I consider the whole class of the errors caused by the use of extreme variables and show that one has to change totally the approach of including those variables in the error budget. I considered both systematic and statistical errors. If statistical errors were considered previously in [2-4, 14, 18] just incorrectly, then systematic errors were totally missed.

It was accepted in the articles [2,3] that the Earth gravity field plays no role in the G measurement. I showed that it is incorrect. Earth gravity field gradient does affect on the error budget. Now look, in the articles [2,3] published in the journals Science and Nature it was missed part of the error budget, which is larger than all other parts, I found it, but according to your conclusion, for the journal Physical Review A it is insignificant.

Would you please to reconsider your conclusion about the manuscript.

I prepared a new version of the article. To emphasize the significance of my findings I’ve changed the abstract and add the following paragraph on page 2:

"We think that the restriction (8) also played a certain role in the experiments [2,18] and was one of the reasons for the low accuracy of the measurement of $G$. In the article [2] for the following values of parameters $\Delta^2\phi \approx 0.1$rad, $k = 1.47 \cdot 10^7$m$^{-1}$, $|\Gamma_{Ezz}| = 2.93 \cdot 10^{-6}$s$^{-2}$ [16], $\sigma (z_{1j}) \sim 3 \cdot 10^{-4}$m, $\sigma (v_{z1j}) \sim 2 \cdot 10^{-3}$m/s, $T = 150$ms one gets from (8) $\sigma_r (\Delta^2\phi) > 6 \cdot 10^{-3}$. On the order of magnitude, this restriction is close to the measurement accuracy $4 \cdot 10^{-3}$ in [2]. In the work [18] at $\Delta^2\phi = 0.503$rad, $\sigma (z_{1j}) \sim 1.2 \cdot 10^{-4}$m, $\sigma (v_{z1}) \sim 1.3 \cdot 10^{-4}$m, $\sigma (v_{z1j}) \sim 5 \cdot 10^{-3}$m/s, $|\Gamma_{Ezz}| = 3.11 \cdot 10^{-6}$s$^{-2}$, $k \approx 1.61 \cdot 10^7$m$^{-1}$, $T = 150$ms one finds $\sigma_r (\Delta^2\phi) > 2.41 \cdot 10^{-3}$. This restriction 2.8 times larger than the RSD in the error $\phi^{(C)}_G$ in Table 5.1, but it an order of magnitude smaller than the accuracy of the measurement of $G$ in [18]. It should be emphasized that the influence of the gradient of Earth gravity field on the accuracy of the measurement of the Newton gravitational constant was not discussed clearly in all preceding articles [2-4,14,18]."

To make the manuscript better readable, I withdraw tables and lengthy expressions with numerical data, but I saved references on those tables and equations, which are still presented in the online version of the article arXiv:1907.03352v3 [physics.atom-ph].

Please, find enclosed the new version.

Yours sincerely,
Boris Dubetsky

3. Date: Wednesday, September 18, 2019 11:04:07 PM

From: Boris Dubetsky
To: pra@aps.org
Subject: Re: Publication Rights AG11947 Dubetsky
Attachments: cylinders4_short.pdf
Dear Physical Review A Journal Services representative,
I am really sorry, but unfortunately, I found an error in my calculations. The error happened owing to the wrong assumption that contributions to the atom interferometer phase produced by the Earth field and source mass field are statistically independent. After calculations have been corrected, I have changed Abstract, Sections I, IV, and VII. Would you please, find enclosed the manuscript after those changes. If it is not impossible, please resend manuscript to Referee.
Sincerely yours,
Boris Dubetsky
Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant
by B. Dubetsky

Dear Dr. Dubetsky,

The above manuscript has been reviewed by two of our referees. Comments from the reports appear below.

These comments suggest that the present manuscript is not suitable for publication in the Physical Review.

Yours sincerely,

Xiangyu Yin
Assistant Editor

Report of the First Referee – AG11947/Dubetsky

In this manuscript the author presents a couple of results:

First, directly quoting the abstract, "An analytical expression for the gravitational field of a homogeneous cylinder is derived."

Second, the results of two experiments, one completed in 2014, the other, presently at the proposal stage, both measuring the gravitational constant G with cold atom interferometers and cylindrical source masses, are carefully analyzed in order to optimize and quantify the expected uncertainty on G as a function of the source masses’ positions.

I think that the material is potentially interesting but in its present form it cannot be considered for publication. The note [28] states:

"Tables and lengthy equations containing coefficients obtained numerically one can find only in the online version of this article B. Dubetsky arXiv:1907.03352v4 [physics.atom-ph]. Starting from the Sec. IV the references on those tables and equations are put in braces."

The article submitted is then not complete so, i.e., tables {I} to {V} cited in the text are nowhere to be found in the manuscript and are available only on arXiv. Phys. Rev. A offers the possibility to include supplementary contents. The author should take advantage of this opportunity and submit a full, self-contained, manuscript.

Since this is a matter that, I assume, can be easily managed by the author, I went through the reading anyway, hoping to save some time, eventually, to the review process. I have two main comments and some minor ones.

1) I checked and I have found a previous article, recently published actually, reporting the gravitational field of a cylinder:

"Closed-form expressions for the non-axial component of the gravitational field of an arbitrary cylinder segment" by David Miles Journal of Applied Geophysics 159 (2018) 621–630 see https://doi.org/10.1016/j.jappgeo.2018.10.006

Apparently, according to a reference in it, also the axial component has been published before. See Nabighian, N., 1962. "The Gravitational attraction of a vertical circular cylinder at points external to it."

Geofis. Pura e Appl. 53, 45–51.

2) The author correctly points out that when a function F of a random variable X for which \( \mu = E[X] \) (here E[] denotes he expectation value), has maximum or a minimum in F(\( \mu \)) then E[F(X)] is not, approximately, F(\( \mu \)) but there is a substantial bias.

Then he proceeds to evaluate the correction, assuming a known probability density p(X) for X, and eventually suggests, in the conclusions, close to eq. (50), that a shift of the order of -200 ppm should be applied to the value of G published in [3].

A less elegant but just as accurate procedure to obtain the unbiased value for E[F(X)] is to run a Montecarlo i.e. by sampling X over p(X) and evaluating an average of F(X). This is actually the procedure used in [3], as stated in the Methods section, so I think that the correction mentioned proposed by the author is not needed.

Minor points

- Some choices of numerical parameters seem peculiar. Can for example the author motivate the choice of "a distance between the floors of 0.20011 m" (why the extra 110 um?) or "temperature 115 pK and radius of the atom cloud 170 um (which are larger than those observed in [7])" (why then not use then the numbers given in [7] in an estimate).
- Choosing a given notation is a subjective matter so I take it as perfectly fine if the author does not agree with my comments, but I would recommend to simplify some formulas in order to improve readability. As an example consider eq.(20) and eq. (21a-e):

1) \(\hbar k/M\) is the recoil velocity so \(v_r/2 = \hbar k/2M\) could be defined and used. By the way, to erase the Earth’s gradient \(k\) should change at every pulse. Which \(k\) should be taken here? Please clarify.

2) \(\Delta g, \gamma_{zm}, \chi_{zmn}\) are meant to be evaluated in the integrands along two trajectories in space:

\[
a(t) = x + \nu^*(T_1 + T + t) + g/2*(T_1 + T + t)^2 + v_r/2*(T + t)
b(t) = x + \nu^*(T_1 + t) + g/2*(T_1 + t)^2 + v_r*t
\]

Does that really need to be repeated explicitly every time? Also I find misleading to use \(f[x(t)]\) instead of \(f(x(t))\) i.e. using \(\[]\) to indicate the argument of a function. It is easy to assume that it is a multiplicative factor.

Other scattered examples: why the double difference in eq.˜(2) is \(\Delta^2\) and not, i.e. \(\Delta 2\), so it won’t be confused with a square? Why the vertical component of Earth gradient is \(\Gamma\{E33\}\) while for the source masses \(\gamma_{zm}\) is use instead of, following the first notation, with something like \(\Gamma\{S3m\}\)?

- eq(7) assumes that the covariance matrix is diagonal i.e. that \(\{q_1..q_n\}\) are statistically independent. It should be explicitly stated.

- the moments in eq. (15a,b) are only about positions. Mentioning that the obvious extensions for velocities and mixed terms could be useful. Maybe just indicate with \(\|\) that position \(x\) or a velocity \(v\).

- Eq. (17a) is a general result however samples of cold atoms from a MOT are usually modeled with Gaussians distributions for velocities while for positions the density distributions are either Gaussians (at long times after release from the MOT) or uniform ellipsoids at short times. I will not discuss here what "long" and "short" mean in this contest. I just want to point out that taking \(\kappa=0\) should be a good approximation for most experimental cases.

- Fig. 1: Using yellow for the trajectories is not the best choice for contrast.

- at page 8 the author writes "...up to the third digit, they coincide with the velocity of the atomic fountain [29] \(v = gT\)." If I got it right, this means that \(v_0\) will hold at the second pulse. This situation is always avoided in Raman interferometers because it cannot select the direction in which momentum is absorbed i.e. \(+v_r\) and \(-v_r\) recoils are equally probable, so half of the signal is lost. Recently, however, a proposal for implementing an interferometer with \(v_0\) at the pi pulse was published as arXiv:1907.04403v1 and maybe this point can be discussed by the author.

- refs. [3] and [4] are actually the same: [4] is the arXiv copy of [3] so, probably, [4] should be removed.

"Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant", by B. Dubetsky, discusses a geometry for the determination of G. The proposed geometry is a variant on the one used in 2014 by G. Tino and coworkers. The article seems reasonable, careful, and correct. However, the extremely technical nature of the manuscript and its limited target audience make it inappropriate for Physical Review A. I recommend that the author consider a more specialized journal for the work.

5. Date: Sunday, November 17, 2019 8:05:22 PM

From: Boris Dubetsky
To: pra@aps.org
Subject: Re: Your manuscript AG11947 Dubetsky
Attachments: cylinders5_short.pdf

Dear Dr. Yin,
I have studied carefully the Referees’ reports and modified the manuscript along with their recommendations. Please, see below my answers to the Referees conclusions and suggestions, where I also listed the modifications.

I disagree with Referees’ conclusions and ask you to consider the attached new version of the manuscript for publication.

Yours sincerely,
Boris Dubetsky

---

Answer on the Report of the First Referee

1. Referee concluded: "I think that the material is potentially interesting but in its present form it cannot be considered for publication."
I've changed the form and content of the manuscript along with Referee recommendations. I hope after those changes Referee would change his opinion.

2. Referee recommended writing supplementary contents.

I wrote the Supplemental Material and replaced reference on the online version of the article with the reference on the Supplemental Material.

3. I am appreciated to the Referee findings of the previous expressions of the cylinder gravity field derived by Drs. Miles and Nabighian. In contrast to my calculations, Drs. David Miles and N.Nabighian do not use the technique proposed in the textbook [20], and, therefore, in my opinion, previous derivations lengthier. I wrote the code for the Drs. Miles and Nabighian expressions and found numerically that previous expressions do not coincide with my expressions. Moreover, I tested numerically previous and my expressions for several limiting cases (large distance, an axial field component at the plane z=h/2 and at x=y=0) and found that only my expressions converge to the expected results.

I wrote following email to Dr. Miles:

"Dear Dr. Miles:

Regarding your really interesting article Journal of Applied Geophysics 159 (2018) 621–630, I would like to bring to your attention my recent preprint arXiv:1907.03352v4 [physics.atom-ph]. In the appendix of the preprint, I also derived the cylinder gravitational field. I compared numerically your expression (21) and my expression (A22). To write code for your expression I used Eqs. (13,17-19,22-25,A-9,A-10,A-11) from your article. I also tested successfully my Eqs. (A17, A22) for the large distance, where cylinder can be considered as a point source of gravity.

Unfortunately, your expression (21) and my expression (A22) do not coincide numerically.

I'd be really appreciated it if you help to resolve this discrepancy.

With best regards, Boris"

I've got no answer from Dr. Miles.

I've added references found by Referee (see references [21,22]), the previous text

"The technique for calculating the gravitational field without calculating the gravitational potential was proposed in book [20], but the final expression for the cylinder field is given in [20] without derivation. Following technique [20], we calculated the field and arrived at expressions (A17, A22) that do not coincide with those given in [20]. Both the derivations and final results are presented in this article."

was replaced on page 2 with the following text

"Expressions for the field of the cylinders have been derived in the articles [21,22]. Alternatively, the technique for calculating the gravitational field without calculating the gravitational potential was proposed in the book [20], but the final expression for the cylinder field is given in [20] without derivation. Following technique [20], we calculated the field and arrived at expressions (A17, A22). Our expressions do not coincide with those given in [20-22]. Both the derivations and final results are presented in this article. Following the derivations in the articles [21,22], we are going to find out analytically the reason of the discrepancies between different expressions and publish it elsewhere."

4. Referee disagree "that a shift of the order of -200 ppm should be applied to the value of G published in [3]". It is because the use of a Montecarlo did not bring to any shift. Montecarlo is just an alternative method of generating the error budget. Evidently, two different methods, Montecarlo and expectation values’ calculations in my manuscript should bring to the same error budget. I failed to find in [3] the information necessary to verify the Montecarlo in [3], while all details of my calculations are presented in the Eqs. (10, 17), Table S1, which brought me to the Eqs. (S.1), which in turn for Standard deviations (24), which I took from [3], lead to the shift (25b) and it changes the gravity constant G to the value (51).

Referee agrees with me that the use of maximal or minimal points leads to the "substantial bias". I've calculated this bias (200 ppm). I did not get from [3] a piece of information about bias in the Montecarlo. Sorry but, logically, if Referee agrees with a bias then he should agree with the consequence of the bias, i.e. with the shift of G.

In addition, I would like to pay Referee attention to the phrase in [3]: "Extracting the value of G from the data involved the following steps: calculation of the gravitational potential produced by the source masses; calculation of the phase shift for single-atom trajectories; Monte Carlo simulation of the atomic cloud; and calculation of the corrections for the effects not included in the Monte Carlo simulation (Table 1)." Uncertainties to the atomic cloud position are included in the Table 1. Then from the cited text, I can conclude that uncertainties in atomic position (and, maybe, velocity) were not included in the Montecarlo simulation in [3].

Regarding this point I added the following text on page 11:

"We would like to note that Monte Carlo simulation in [3, 4] did not bring to any shift caused by uncertainties of the atom position and velocity."

5. Referee wrote: "Some choices of numerical parameters seem peculiar. Can for example the author motivate the choice of "a distance between the floors of 0.20011 m" (why the extra 110 um?) or "temperature 115 pK and radius of the atom cloud 170 um (which are larger than those observed in [7])" (why then not use then the numbers given in [7] in an estimate)." I had the following motivations:
- distance between the floors of 0.20011 m

I performed calculations for the cylinders chosen in [3,4], they have a peculiar height h=15.011cm. I added non-
peculiar distance dh=5cm between cylinders tops and bottoms (see Fig. 4a) and got the distance between floors of
0.20011 m. Regarding this choice, I added a sentence on page 9 "Distance between floors includes the height of the
cylinder h = 0.15011m [3, 4] and distance dh = 5cm between cylinders tops and bottoms."

- temperature 115 pK and radius of the atom cloud 170 um

These parameters were not chosen for the calculations in the manuscript. I showed in [5] that at these values of
parameters one can in principle measure G with an accuracy 200ppb. I referred to the article [7] only to justify that
parameters required for 200 ppb are achievable because even better parameters have been observed in [7]. To underline
this point I added a sentence on page 1 "For the parameters achieved in [2-4] at present, the optimal preparation of
the atomic clouds and proper positioning of the gravity sources can also lead to an increase in the accuracy of the
G-measurement"

6. Referee wrote " but I would recommend to simplify some formulas in order to improve readability. As an example
consider eq.(20) and eq. (21a-e)"

I followed this recommendation, see Eqs. (20 - 23e)

7. I introduced recoil velocity v\_r, see Eq. (22).

8. Referee wrote "By the way, to erase the Earth’s gradient k should change at every pulse. Which k should be
taken here? "

To clarify I added on the page 2 following text:

"Since the magnitude of the gravity-gradient tensor \( \Gamma_{\gamma} \) is small, the change in the effective wave vector in (3)
can be considered as a perturbation. Another small perturbation here is the gravity field of the source mass [10].
Since we are not going to consider the simultaneous action of these two perturbations, we can calculate the parts of
the phases \( \phi_{2}(t) \) \( \{C,F\} \) assuming that all three Raman pulses have the same unperturbed effective wave vector \( k' \)."

9. I replaced \( |\gamma| \gamma_{\text{in}} \) with \( |\gamma| \gamma_{\text{in}} \) assuming that \( |\gamma| \gamma_{\text{in}} \) and \( g_{2} \) with \( g_{2} \)

10. Subscript "2" is used for the second atom cloud so that instead of replacing \( \Delta \gamma_{2} \) with \( \Delta \gamma_{2} \), I replaced

\( \Delta \gamma_{2} \) with \( \Delta \gamma_{2} \)

11. I added text "We assume that variables \( \{q_{1}, \ldots, q_{m}\} \) are statistically independent."

12. I replaced x with q in Eqs. (15, 16) and wrote "\( q_{m} \) is either a position \( x_{\gamma} \) or a velocity \( v_{\gamma} \)."

13. Yellow color for atomic trajectories was chosen in [3,4]. But, nevertheless, I changed it to the red color now.

14. I am appreciated to the Referee for the point that \( v_{\gamma} \) should not be equal 0 at the time of pulse action. I
was not familiar with this restriction. Fortunately, the small value of \( v_{\gamma} \) at the second pulse action, \( v_{\gamma} \sim 6 \)mm/s,
leads to the detuning \( 2k_{\gamma}v \sim 2.5 \cdot 10^{-3} \) from Raman resonance for not desirable effective wave vector -k. If Pi-pulse
has duration \( \tau \sim 60 \)us, then the detuning is an order of magnitude larger than 1/\( \tan \), then the probability of not
desirable excitation with recoil \( v_{\gamma} \) is \( 10^{-3} \). In addition, \( v_{\gamma} \sim 6 \)mm/s is twice larger than the thermal velocity in
[3,4], 3mm/s, i.e. portion of atoms with velocity \( v_{\gamma} \sim 6 \)mm/s is exponentially small, <1.e-8. Regarding this point I
added the following text on page 7:

"Repeating iterations (31) three times we determined the velocities \( v_{\gamma}(z_{\text{max}}) \) and \( v_{\gamma}(z_{\text{min}}) \) with an accuracy of
10^{-7}m/s. These velocities match up to the 5th digit. They are also close to the velocity of the atomic fountain [31]
v=0.115 \mu m. differing from it only in the third digit,
\[ \delta v = v_{\gamma}(z_{\text{max}}) - gT \approx -6.10^{-3} \text{m/s} \]

This difference, however, is sufficient to exclude the parasitic signal [32], which occurs when atoms interact with a
Raman pulse having an opposite sign of the effective wave vector. Indeed, the Raman frequency detuning for the
parasitic signal \( \delta = 2k_{\gamma}v \approx 2 \cdot 10^{5} \text{s}^{-1} \). If the duration of the \( \tau \)-pulse \( \tau \sim 60 \mu \text{s} \), then the absolute value of the detuning
\( \delta \) is an order of magnitude greater than the inverse pulse duration, and the probability of excitation of atoms by a
parasitic Raman field is negligible, is estimated to be about 4%.

In addition, the velocity (32) is twice as large as the thermal velocity in the atomic cloud, \( v=3 \cdot 10^{-3} \mu m/s [3,4] \), and
therefore the portion of atoms having the opposite velocity \( -\delta v \) and being in resonance with the parasitic Raman
pulse is also exponentially small, is more than order of 10^{-8}.

15. Referee recommends that ref. [4] should be removed. Unfortunately, at the important for me point, refs [3]
and [4] do not coincide, ref. [3] says "The central pi pulse occurs about 6 ms after the atoms reach the apogees
of their trajectories", while the ref. [4] says "The central pi pulse occurs about 6 ms before the atoms reach the apogees
of their trajectories". Ref. [4] has been published later than ref. [3]. That is why to get the time of apogee, and
launching velocity, I used a sentence from ref. [4] and inserted ref. [4] in the manuscript.

Answer on the Report of the Second Referee

Second Referee negative conclusion based on "the extremely technical nature of the manuscript and its limited
target audience."
It is pretty common that gravitational constant is measured much worse than any other fundamental constants. My manuscript proposes a new technique to measure this constant, and, therefore, it could be interesting to some degree wide audience. I found for the first time that the error budget has to be revised and developed the revision. This point also could be interesting to some degree wide audience.

To minimize technical things I took out all tables and lengthy equations and put them into Supplemental material. But, evidently, if I wrote about revision then I have to demonstrate the revision consequences.

I am sorry, but the Report of the Second Referee does not contain any specific points, and, therefore, I did not make any changes following this Report.

6. Date: Friday, November 22, 2019 10:28:37 AM

From: pra@aps.org
To: bdubetsky@gmail.com
Subject: Your manuscript AG11947 Dubetsy
Re: AG11947
Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant by B. Dubetsy

Dear Dr. Dubetsy,

The above manuscript has been reviewed by two of our referees. Comments from the reports appear below. Although the referees gave conflicting advice, we agree with the assessment of the second referee. We feel that the manuscript is a very specialized study that may not be of general interest to our broad audience in the AMO community.

We regret that in view of these comments we cannot accept the paper for publication in the Physical Review.

Yours sincerely,
Xiangyu Yin
Assistant Editor

Second Report of the First Referee – AG11947/Dubetsy

After the author’s revision, the manuscript is self contained so I am in favor of publication in Phys. Rev. A.

Keeping the original numbering in the author’s response I have some comments only on the two following points:

4. On this point I feel that the author and I do agree on the heart of the matter but there is a misunderstanding on a crucial point. I apologize therefore with the editor and the author for entering, in the following, in a detailed trivial example to try to clarify the issue.

Take a standard gaussian variable $x \sim \mathcal{N}(0,1)$ and consider $y = x^2$. We actually know the distribution of $y$ i.e. a chisquare with one degree of freedom. Of course estimating $E[y]$ as $(E[x])^2$ is a bad idea since we can be sure that $E[y] > 0$. Actually we know that $E[y] = 1$.

We could obtain a better estimate of $E[y]$ either using $E[x] = 0$ and adding the shift given in eq. (17b) or by running a Montecarlo simulation, generating say $10^5$ normally distributed variables $x_i$ computing $y_i = x_i^2$ and taking the average of all the $y_i$. Note that in the second case I do not even need to know $E[x]$. The average of the $y_i$ is an unbiased estimate of $E[y]$.

The author writes: "I did not get from [3] a piece of information about bias in the Montecarlo...".

As I pointed out above, it’s because, as I understand it, it’s not usually provided since it is not useful to state the result as $E[y] = (E[x])^2 + \text{bias}$ since the Montecarlo provides directly $E[y]$.

In an experimental situation however the a-priori distribution of $x$ is not known so $x \sim \mathcal{N}(0,1)$ should be understood as: a normal distribution is assumed and mean($x$) = 0 and var($x$) = 1 are actually experimental values with an estimated or measured uncertainty.

As an example let’s assume var($x$) = 1.0 +/- 0.1.

Running a single Montecarlo is therefore not enough and I could try to see what happens to $E[y]$ when picking for var($x$) [0.9, 1.0, 1.1] for a simple example. From the three values for $E[y]$, I obtain estimates both for bias and uncertainty for $E[y]$ due to the uncertainty on var($x$). Here actually, since we know that $E[y] = \text{var}(x)$, there will be no bias but an estimate for $E[y]$ of 1.0 +/- 0.1.

In the spirit of this somehow futile example I would add to Table 1 in [3] a row that reads var($x$) 0.1 +/- 0.1 meaning that the uncertainty of 0.1 on var($x$) will not add a bias to the result of the Montecarlo but will contribute with 0.1 to the uncertainty of the results. Of course with many variables I should be very careful in assuming statistical independence and adding the uncertainties in quadrature but this is another matter.
At this point I might have succeeded or failed in convincing the author that the correction that he evaluates for the value of G in [3] is probably already in place, but this has no effect on the validity of the manuscript. The author can choose to keep or remove the claim that the correction should be applied, according to his judgment.

15. "...refs [3] and [4] do not coincide, ref. [3] says "The central pi pulse occurs about 6 ms after the atoms reach the apogees of their trajectories", while the ref. [4] says "The central pi pulse occurs about 6 ms before the atoms reach the apogees of their trajectories". Ref. [4] has been published later than ref. [3]. That is why to get the time of apogee, and launching velocity, I used a sentence from ref. [4] and inserted ref. [4] in the manuscript."

Fair enough. However I would suggest that, for better clarity, instead of always citing [3,4] together, without explaining the difference mentioned above, a difference that will be easily missed even by the most dedicated reader of the article, that [3] is cited and a footnote is added explaining the discrepancy between [3] and its successive arXiv copy, [4], while stating that the latest published value was used.

A misprint at the end of sec. II: "For the each case considered."

Second Report of the Second Referee – AG11947/Dubetsky

In my original report, I found this manuscript to be too narrow in its audience to be appropriate for Physical Review A. My opinion is unchanged by the iterative changes made to the manuscript since its first submission.

The references cited give strong support to my assessment. The majority of the 32 references fall in four categories: 1, Citations to work by two experimental groups, the Kasevich group and the Tino group. [10 citations] 2, Self-citations to the author’s previous work [7 citations] 3, Papers published 20 or more years ago and textbooks [7 references] 4, Unpublished work [6 citations]

I also note that there are only two PRA papers, one of which is from 1986. In the introductory paragraph of this manuscript, where one normally discusses the broad appeal of a scientific question, only papers by the author himself, and the two experimental groups mentioned above, are cited. (ie, groups 1 and 2 above.)

To me, this is an indication that the topic is no longer of current or broad interest to the PRA community. I do not say that the manuscript is incorrect or misleading, but rather suggest the author submit it to Metrologia, an appropriate journal (of comparable impact factor as PRA) that is dedicated to the kind of detailed analysis presented here. Citations 13 and 14 are from that journal, for instance.

7. Date: Friday, November 22, 2019 2:23:17 PM

From: Boris Dubetsky
To: pra@aps.org
Subject: Re: Your manuscript AG11947 Dubetsky
Dear Dr. Yin,

In my answer to the First Report of the Second Referee, to prove that the manuscript could be of common interest, I wrote that it is devoted to the measurement of the gravitational constant, i.e. to the topic which does have a wide auditorium. I would like to pay your attention that Second Referee totally ignored this my argument. Instead, Second Referee performed an analysis of the bibliography. I do not know why the bibliography weights more than the topic of the manuscript. I thought that the elaboration of new methods and revision of the inappropriate approach inside important topic should overcome a bibliography.

But even regarding bibliography, I do not know why Kasevich group, Tino group, Steven Chu group, A. Roura sent their articles to the non-specialized journals Physical Review Letters, Science and Nature, and did not publish those articles in the Physical Review A. But, I am sure that articles published in those journals are of a great interest for the Physical Review A community. I mean 10 References [2,3,6-8,11,12,23,29]. Together with references [9, 31], about which Second Referee wrote, we have 12 references, which are definitely interested in the Physical Review A community. Five articles [3,7-9,12] are less than 5 years old. This brief study shows that from the bibliography point of view my manuscript could be also not rejected.

Would you please to reconsider your conclusion about the manuscript.

Sincerely yours,
Boris Dubetsky
Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant by B. Dubetsky

Dear Dr. Dubetsky,

Your paper has been rejected. Further consideration can only be given if you decide to exercise the option, available under this journal’s Editorial Policies (copy attached), of appealing the decision to reject the manuscript. Adjudication of such an appeal is based on the version of the manuscript that was rejected; no revisions can be introduced at this stage.

Yours sincerely,

Xiangyu Yin
Assistant Editor

From: pra@aps.org
To: bdubetsky@gmail.com
Subject: Your manuscript AG11947 Dubetsky
Attachments: ag11947_report_eb.pdf
Re: AG11947
Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant by B. Dubetsky

Dear Dr. Dubetsky,

Your formal appeal of this manuscript has been evaluated by an Editorial Board Member. The EBM advises us not to publish in Physical Review A, and we accept this advice. Your appeal has been considered, and our decision to reject is maintained. This concludes the scientific review of your manuscript.

Yours sincerely,

Xiangyu Yin
Assistant Editor

a. Report | Editorial Board

Re: AG11947
Optimization of the atom interferometer phase produced by the set of cylindrical source masses to measure the Newtonian gravity constant by B. Dubetsky

The manuscript written by B. Dubetsky deals with the calculation of the gravitational field of a homogeneous cylinder and the measurement of the Newtonian gravitational constant G by atomic interferometry methods (phase estimates). The author uses a technique that aims at eliminating Earth’s gravity-gradient. The main claim is that higher order terms in the expansion tend to be important and cannot be neglected, and that textbook (Chen, Cook, Ref. [20]) calculations of the gravity field of the cylinder do not take important factors into account. The author also claims that there are discrepancies among his calculation and those performed in Refs. [21,22]. One of the main results of the ms is Eq. (54), where a different value is obtained for the gravitational constant (as compared to that reported in Refs. [3,4]).

The two referees offered conflicting advice. Referee 1 made a number of important technical observations, and is clearly very interested in the topic. After the second round of refereeing, she/he concluded that the manuscript can be accepted for publication. On the other hand, Referee 2 wrote that ”the extremely technical nature of the manuscript and its limited target audience make it inappropriate for Physical Review A.” Referee 2 also suggested a possible alternative venue (Metrologia), where the ms can be submitted.

Although the opinions of both referees are sensible, I agree with the standpoint of Referee 2. The manuscript is very technical and difficult to read. The calculation can be of interest for a limited audience, such as experimental
groups working on this topic (Tino’s and possibly Kasevich’s) and possibly some other researchers involved in similar analyses. It is not suitable for a wider readership, such as that of Physical Review A. Moreover, as one Editor wrote, an article submitted to Physical Review A should be well organized, and clearly written in scientific English. This manuscript is unfortunately far from meeting these criteria.

Three Editors were involved with the present manuscript: Associate Editor of Physical Review A Doerte Blume, Managing Editor of Physical Review Research Juan-Jose Lietor-Santos, and Assistant Editor of Physical Review A Xiangyu Yin. All of them concluded that the manuscript is not suitable for publication in Physical Review A. I kindly ask Dr. Dubetsky to consider that the Editors have evaluated this case with care and their consolidated editorial expertise.

In conclusion, I uphold the rejection of the Editors of Physical Review A and Physical Review Research and, regretfully, do not recommend publication in Physical Review A.

Saverio Pascazio
Editorial Board
Physical Review A