Hecke algebras of classical type and their representation type

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Abstract. The purpose of this article is to determine representation type for all of the Hecke algebras of classical type. To do this, we combine methods from our previous work, which is used to obtain information on their Gabriel quivers, and recent advances in the theory of finite dimensional algebras. Principal computation is for Hecke algebras of type $B$ with two parameters. Then, we show that the representation type of Hecke algebras is governed by their Poincaré polynomials.

1. Introduction

1.1. Let $F$ be an algebraically closed field, $A$ an $F$–algebra which is finite dimensional as an $F$–vector space. $A – mod$ is the category of left $A$–modules which are finite dimensional over $F$. We say that $A$ is finite if the number of the isomorphism classes of indecomposable $A$–modules is finite.

Let $F[X]$ be the polynomial ring generated by the indeterminate $X$. If, $A$ is not finite and, for each number $d \in \mathbb{N}$, there are finitely many $(A, F[X])$–bimodules $M_1, \ldots, M_{n_d}$ which are free of finite rank as right $F[X]$–modules such that all but a finite number of the isomorphism classes of indecomposable $A$–modules of dimension $d$ contain an $A$–module of the form $M_i \otimes_{F[X]} F[X]/(X - \lambda)$, for some $i$ and some $\lambda \in F$, then we say that $A$ is tame.

Let $F\langle X,Y \rangle$ be the free $F$–algebra generated by two indeterminates $X$ and $Y$. We say that $A$ is wild if there is a $(A, F\langle X,Y \rangle)$–bimodule $M$ which is free of finite rank as a right $F\langle X,Y \rangle$–module such that the associated functor $F_M = M \otimes_{F\langle X,Y \rangle} : F\langle X,Y \rangle – mod \rightarrow A – mod$ respects indecomposability and isomorphism classes. A famous theorem of Drozd [Dr1] Theorem 2] (see also [CI]) asserts that $A$ is finite, tame or wild, and that these are mutually exclusive. This is the representation type of the algebra $A$.

1.2. Let $q \in F^\times$ and let $W$ be a finite Weyl group of classical type. First we consider the case where $W$ is an irreducible Weyl group. Then, for each of $W(A_{n-1}), W(B_n)$ and $W(D_n)$, we have the associated Hecke algebra. We denote them by $H_{n}^A(q)$, $H_{n}^B(q)$ and $H_{n}^D(q)$ respectively. For type $B_n$, we can choose two parameters $q,Q \in F^\times$ and the associated Hecke algebra is denoted by $H_n(q,Q)$.

In 1992, Uno gave a criterion for $H_n^A(q)$ to be finite [U], and conjectured that the criterion would be true for other Hecke algebras. This conjecture was settled affirmatively for Hecke algebras of classical type [A4]. Crucial for proving the Uno
conjecture was the result proven in \[\text{AM2}\] and \[\text{AM3}\] which tells when \(\mathcal{H}_n(q, Q)\) is finite.

The purpose of this paper is to determine representation type for all of the Hecke algebras of classical type. Main theorems are Theorem 40, T heorem 42 and Theorem 57. These combined gives Theorem 67, which is for the general case.

To give these final results, we begin by analysing the representation type of \(\mathcal{H}_n(q, Q)\). To carry out this, we need plenty of results from the theory of finite dimensional algebras. These include standard techniques from the covering theory, the theory of special biserial algebras, and recent results such as a criterion for wildness using the complexity of modules by Rickard \[\text{Ric}\] and classification of representation types of two–point algebras by Han \[\text{Ha2}\]. As we get \(\mathcal{H}_B^n(q)\) by setting \(Q = q\) and an embedding of \(\mathcal{H}_D^n(q)\) into \(\mathcal{H}_n(q, 1)\) by setting \(Q = 1\), which allows us to apply the Clifford theory, we can determine representation type for \(\mathcal{H}_B^n(q)\) and \(\mathcal{H}_D^n(q)\) also. In the Clifford theory argument used for \(\mathcal{H}_D^n(q)\) we also need a recent result of Hu \[\text{Hu2}\]. Finally, the general case follows from these results for indivisual Hecke algebras \(\mathcal{H}_X^n(q)\) with \(X = A, B\) or \(D\). Note that the case of \(\mathcal{H}_A^n(q)\) was already known by \[\text{EN}\].

We remark that a complete set of the isomorphism classes of simple modules, for each of the Hecke algebras of classical type, was already given. See \[\text{DJ1}\], \[\text{DJ2}\] for type \(A\), \[\text{AM1}\] and \[\text{A3}\] for type \(B\), and \[\text{P}\], \[\text{Hu1}\] and \[\text{Hu2}\] for type \(D\). In type \(D\), we assume that the characteristic of \(F\) is odd. Hence, it is natural to proceed further to the classification of the isomorphism classes of the indecomposable modules of these algebras. Our results show when this is achievable, as it is well–known that the wild case is the pathological case. See \[\text{Ge2}\] for a different approach for classifying simple \(\mathcal{H}_D^n(q)\)–modules.

The paper is structured as follows. In section 2 we review various results on finite dimensional algebras. Then we treat the case of \(\mathcal{H}_n(q, Q)\) in section 3. The cases of \(\mathcal{H}_A^n(q)\), \(\mathcal{H}_B^n(q)\) and \(\mathcal{H}_D^n(q)\) are handled in section 4. The final section is for the general case. In the appendix, I list corrections for \[\text{A1}\].

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Finally, I add a few words about the references. I assume that the reader is familiar with the crystal basis theory and the canonical basis. So, I list few about them in the references. On the other hand, as it is not appropriate to assume that researchers in our field know well about the theory of finite dimensional algebras, I always cite references whenever I quote a result. In fact, most of the results we use are not available in book form and they are scattered around in a vast literature. If the reader is familiar with these results, I recommend starting with section 3 and return to lemmas in section 2 when they are used.

2. Preliminaries

Let \(A\) be a finite dimensional \(F\)–algebra as before, and let \(\{P_1, \ldots, P_s\}\) be a complete set of the isomorphism classes of indecomposable projective \(A\)–modules. Then \(A\) is Morita–equivalent to \(\text{End}_A(P_1 \oplus \cdots \oplus P_s)^{opp}\), which can be written in the form \(FQ/I\) where \(Q = (Q_0, Q_1)\) is a directed graph with nodes \(Q_0\) and directed
edges $Q_1$, and $I$ is an admissible ideal of $FQ$. See [AHS] for this basic result of Gabriel. The directed graph $Q$ is called the Gabriel quiver of $A$. We identify $Q_0$ with the set of the isomorphism classes of simple $A$–modules.

If $A$ is a symmetric algebra then $Q_1$ is described as follows. Let $S$ and $T$ be two simple $A$–modules, $P(S)$ and $P(T)$ their projective covers. Then we write $a_{ST}$ arrows from $S$ to $T$ where $a_{ST} = [\text{Rad} P(S)/\text{Rad}^2 P(S) : T]$. Thus the Gabriel quiver is the directed graph whose adjacency matrix is $(a_{ST})_{S,T \in Q_0}$.

In [AM2], we used the following lemma to show that the $F$–algebras $\mathcal{H}_n(q, Q)$ with $n = e$ or $n = 2f + 4$, where $0 \leq f \leq \frac{9}{2}$ and $Q = -qI$, are not finite.

**Lemma 1.** Let $A$ be a finite dimensional local basic $F$–algebra. Then $A$ is finite if and only if $A$ is isomorphic to a truncated polynomial ring $F[X]/(X^n)$ for some positive integer $N$.

**2.2.** We say that $A$ is weakly tame if $A$ is not finite and, for each number $d \in \mathbb{N}$, there are finitely many $(A, F[X])$–bimodules $M_1, \ldots, M_n$ which are free of finite rank as right $F[X]$–modules such that all but a finite number of the isomorphism classes of indecomposable $A$–modules of dimension $d$ contain an $A$–module which is a direct summand of an $A$–module of the form $M_i \otimes_{F[X]} F[X]/(X - \lambda)$, for some $i$ and some $\lambda \in F$.

The following is a result of de la Peña [dIP1] Chap.I, Proposition 2.3].

**Proposition 2.** Let $A$ be a finite dimensional $F$–algebra. Then $A$ is weakly tame if and only if it is tame.

Using this result, Erdmann and Nakano proved the following.

**Proposition 3 ([EN Proposition 2.3]).** Let $A$ and $B$ be finite dimensional $F$–algebras and suppose that there are functors $\mathcal{F} : A\text{-mod} \rightarrow B\text{-mod}$, $\mathcal{G} : B\text{-mod} \rightarrow A\text{-mod}$ and a constant $C$ such that, for any $A$–module $M$,

1. $M$ is a direct summand of $\mathcal{G}\mathcal{F}(M)$ as an $A$–module,
2. $\dim_F \mathcal{F}(M) \leq C \dim_F M$.

If $A$ is wild then so is $B$.

**Corollary 4.**

1. If $\mathcal{H}_n(q, Q)$ is wild then so is $\mathcal{H}_m(q, Q)$, for all $m \geq n$.
2. Let $X$ be one of $A$, $B$, $D$. If $\mathcal{H}^X_n(q)$ is wild then so is $\mathcal{H}^X_m(q)$, for all $m \geq n$.

**Proof.** We prove (1). The proof of (2) is the same. Let $A = \mathcal{H}_n(q, Q)$ and $B = \mathcal{H}_m(q, Q)$ with $m \geq n$. Then, by taking $\mathcal{F}$ and $\mathcal{G}$ to be the induction and the restriction functors respectively, we can apply Proposition 3. The assumption (1) is satisfied as we have the Mackey decomposition theorem for Hecke algebras, and the assumption (2) is obvious.

**2.3.** We need finer notions for wild algebras. We say that $A$ is strictly wild if there exists a fully faithful exact functor $\mathcal{F} : F\langle X, Y \rangle\text{-mod} \rightarrow A\text{-mod}$.

All functors in this paper are assumed to be $F$–functors.
The strictly wildness is equivalent to the condition that we have a fully faithful exact functor

\[ B \mod \rightarrow A \mod, \]

for any finite dimensional \( F \)-algebra \( B \).

In fact, by [Br] Theorem 3] or [S] Proposition 14.10], we know that \( B \mod \), for any finite dimensional \( F \)-algebra \( B \), can be realized as a full subcategory of \( F\langle X,Y \rangle \mod \). Hence, strictly wildness implies this. To show the converse, we consider the functor for a particular 11-dimensional algebra \( \Lambda_6(F) \) which is the algebra of \( 6 \times 6 \) matrices with non-zero entries in the first column and the diagonal. Thus, we assume that there is a fully faithful exact functor

\[ \Lambda_6(F) \mod \rightarrow A \mod. \]

For this algebra, we have a fully faithful exact functor

\[ F\langle X,Y \rangle \mod \rightarrow \Lambda_6(F) \mod \]

by [Br] Theorem 2] or [S] 14.2 Example 11]. Hence, the composition of the two gives us the desired fully faithful exact functor which shows that \( A \) is strictly wild.

Another notion we need is the notion of controlled wildness. An \( F \)-algebra \( A \) is controlled wild if there exist a faithful exact functor \( \mathcal{F} \) and a full subcategory \( \mathcal{C} \) of \( A \mod \) which is closed under direct sums and direct summands such that, if we denote by \( \text{Hom}_A(-, -)_\mathcal{C} \) the morphisms of \( A \mod \) which factor through \( \mathcal{C} \), then for any \( F\langle X,Y \rangle \)-modules \( M \) and \( N \) we have

\[ \text{Hom}_A(\mathcal{F}(M), \mathcal{F}(N)) = \mathcal{F}(\text{Hom}_{F\langle X,Y \rangle}(M, N)) \oplus \text{Hom}_A(\mathcal{F}(M), \mathcal{F}(N))_\mathcal{C} \]

and \( \text{Hom}_A(\mathcal{F}(M), \mathcal{F}(N))_\mathcal{C} \subset \text{Rad}\text{Hom}_A(\mathcal{F}(M), \mathcal{F}(N)). \)

Note that strictly wildness implies controlled wildness. We also know that controlled wildness implies wildness [Ha1] Proposition 2.2]. This depends on the fact that flat \( F\langle X,Y \rangle \)-modules are free: [C] Theorem 2.2.4, Corollary 2.3.2] imply that \( F\langle X,Y \rangle \) is a left and right free ideal ring [C] Corollary 2.4.3]. Hence, \( F\langle X,Y \rangle \) is a semifir by [C] Theorem 1.4.1, Corollary 1.4.2]. Then [C] Proposition 1.4.5] says that an \( F\langle X,Y \rangle \)-module is flat if and only if every finitely generated \( F\langle X,Y \rangle \)-submodule is free.

Using the results from [C], we can also prove the following.

**Proposition 5.** A finite dimensional \( F \)-algebra \( A \) is strictly wild if and only if there exists an \( (A, F\langle X,Y \rangle) \)-bimodule \( M \) which is free of finite rank as a right \( F\langle X,Y \rangle \)-module such that the associated functor \( \mathcal{F}_M \) is fully faithful exact.

**Proof.** If part is obvious. So, we assume that \( A \) is strictly wild. We consider the composition of the two fully faithful exact functors

\[ \Lambda_6(F) \mod \rightarrow A \mod, \]

\[ F\langle X,Y \rangle \mod \rightarrow \Lambda_6(F) \mod. \]

Write \( B \) for \( \Lambda_6(F) \). Let \( L \) be the image of the \((B, B)\)-bimodule \( B \) under the first functor. Then the functor is of the form \( \mathcal{F}_L \) and \( L \) is flat as a right \( B \)-module. As \( B \) is right perfect, this implies that \( L \) is projective as a right \( B \)-module by a theorem of Bass; see [La] Theorem 24.25]. Further, the proof of [Br] Theorem 2] shows that the second functor is of the form \( \mathcal{F}_N \) where \( N \) is free of finite rank as a right \( F\langle X,Y \rangle \)-module. Hence, the composition of these functors is of the form \( \mathcal{F}_M \) such that, if we view \( M \) as a right \( F\langle X,Y \rangle \)-module then \( M \) is a direct summand of
a free $F(X,Y)$–module of finite rank. As $F(X,Y)$ is a semifir, [C] Theorem 1.4.1 implies that $M$ is free of finite rank as a right $F(X,Y)$–module. □

2.4. In this subsection, we review a criterion to tell the representation type of a path algebra. The statement (1) of the following theorem is due to Gabriel [Ga1] and the statement (2) is by [DF] and [N]. See [ARS] VIII.5.4, 5.5 for the finiteness result and [S] Theorem 14.15 for the remaining cases. Another proof is explained in [B2] Proposition 4.7.1, which is most recommendable.

Theorem 6. Let $A = FQ$ be a finite dimensional path algebra. Then

(1) $A$ is finite if and only if the underlying graph of the quiver $Q$ is one of the Dynkin diagrams of finite type.

(2) $A$ is tame if and only if the underlying graph of the quiver $Q$ is one of the Dynkin diagrams of affine type.

(3) $A$ is wild otherwise.

In (1) and (2) we do not mean that all of the Dynkin diagrams of finite and affine types actually occur. By the assumption that $F$ is algebraically closed, only $A_n$, $D_n$, $E_n$ occur in (1), and only $A_n^{(1)}$, $D_n^{(1)}$, $E_n^{(1)}$ occur in (2).

The following is well–known.

Proposition 7. A path algebra is wild if and only if it is strictly wild.

We say that a path algebra $A = FQ$ is minimal wild if it is wild and $A/AeA$ is tame or finite, for any non–zero idempotent $e \in A$. The list of minimal wild path algebras is given in [K1]. Note that we follow the older definition of minimal wildness here.

Remark 8. Recall that each path algebra $A = FQ$ is associated with a quadratic form

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i,j) \in Q_1} x_i x_j,$$

which is the Tits form of $A$. As is well–known, this coincides with the quadratic form associated with the Ringel–Euler form,

$$\langle X, Y \rangle = \dim \text{Hom}_A(X,Y) - \dim \text{Ext}^1_A(X,Y)$$

for $A$–modules $X$ and $Y$. We say that $q_A(x)$ is weakly nonnegative if $q_A(x) \geq 0$ for all $x$ with $x_i \geq 0$ ($i \in Q_0$). For those path algebras which we will meet below, we use Theorem 7 to prove that they are wild. However, we also know that if $A$ is tame then $q_A(x)$ is weakly nonnegative. This is a special case of a more general theorem. See [JIP2] Proposition 1.3 for example. Thus, we can apply this result instead.

In fact, the proof of Theorem 4 given in [B2] Proposition 4.7.1 uses this and results from [BGP].

2.5. Another important class of finite dimensional algebras is the class of tilted algebras. See [HR], [Bo1] and [K3].

Let $B$ be a tilted algebra. The Tits form of $B$ coincides with the quadratic form associated with the Ringel–Euler form again. As the global dimension of $B$ is smaller than or equal to 2 by [Bo1] 1.7, Corollary 1 or [HR] Theorem 5.2, the Ringel–Euler form here is of the form

$$\langle X, Y \rangle = \dim \text{Hom}_B(X,Y) - \dim \text{Ext}^1_B(X,Y) + \dim \text{Ext}^2_B(X,Y),$$
for $B$–modules $X$ and $Y$. In [K3] Theorem 6.2, it is proved that the Tits form of $B$ is weakly nonnegative if and only if $B$ is finite or tame, and that the Tits form is not weakly nonnegative if and only if $B$ is strictly wild. Hence,

**Theorem 9.** A tilted algebra is wild if and only if it is strictly wild.

We will use a particular wild tilted algebra.

**Definition 10.** Let $A$ be a finite dimensional path algebra, $M$ a tilting $A$–module. The tilted algebra $B = \text{End}_A(M)$ is called a concealed algebra if $M$ has preprojective direct summands (of the Auslander–Reiten quiver of $A$) only.

**Remark 11.** We can replace “preprojective” with “preinjective” in the definition above. See [R2] 4.3(1).

In [Un], there is a list of the concealed algebras associated with minimal wild path algebras. Among them, we need the following. See [Un] p.150 or XXVIII in the list of [R1] 1.5, Theorem 2.

**Lemma 12.** Let $Q$ be the directed graph

```
 α ---- β
 \       \   \\
 \       \   \\
 γ ---- δ
```

and let $A = FQ/I$ be the $F$–algebra defined by the relation $\beta \alpha = \delta \gamma$. Then $A$ is a wild concealed algebra. In particular, $A$ is strictly wild.

We have explained Lemma 12 in the most natural framework, but it may be proved by a more direct method. Following the argument in [R1] 2.3, we consider the following quiver.

```
 α ---- β
 \       \   \\
 \       \   \\
 γ ---- δ
```

Denote this quiver by $T$. Given a representation of the quiver $Q$ of Lemma 12, we associate a representation of $T$ by placing the pushout of the linear maps of the arrows $\alpha$ and $\gamma$ on the black node, which is located in the middle of $T$. The linear maps of the three arrows around the black node are the obvious ones. Then, the full subcategory of $A$–mod which is generated by those indecomposable $A$–modules which correspond to sincere $FT$–modules under the correspondence is equivalent to $(FT$–mod)$_s$. Thus, by constructing a fully faithful exact functor $F(X,Y)$–mod $\rightarrow (FT$–mod)$_s$ in a concrete manner, we know that $A$ is strictly wild.

**Remark 13.** Concealment is defined in a different way in [R1], but it does not matter as long as the proof of Lemma 12 is concerned.

### 2.6.

We turn to the theory of complexity. Let $A$ be a finite dimensional $F$–algebra, $M$ an $A$–module. Let

$$P^\bullet : \cdots \rightarrow P_i \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$
be the minimal projective resolution of \( M \). Then, the complexity \( c_A(M) \) of the \( A \)-module \( M \) is defined as follows.

\[
c_A(M) = \min \{ s \in \mathbb{N} \mid \exists C, \text{ a constant, s.t. } \dim_F P_i \leq C(i + 1)^{s-1}, \text{ for } \forall i. \}\]

In the theorem below, the statement (1) is well-known and easy to see. (2) is almost obvious. (3) is a rather new result proven by Rickard \([Ric]\) Theorem 2.

**Theorem 14.** Let \( A \) be a finite dimensional self-injective \( F \)-algebra.

1. An indecomposable \( A \)-module \( M \) has complexity \( c_A(M) = 0 \) if and only if \( M \) is a projective \( A \)-module. In particular, \( A \) is a semisimple algebra if and only if all indecomposable \( A \)-modules \( M \) have complexity \( c_A(M) = 0 \).
2. If there is an \( A \)-module \( M \) with \( c_A(M) \geq 2 \) then \( A \) is not finite.
3. If there is an \( A \)-module \( M \) with \( c_A(M) \geq 3 \) then \( A \) is wild.

Recall that the Hecke algebras \( \mathcal{H}_n(q, Q) \) and \( \mathcal{H}_n^X(q) \), for \( X = A, B, D \), and their block algebras are symmetric algebras. So we can apply Theorem 14 to them.

**Remark 15.** If \( A \) is a group algebra then an indecomposable \( A \)-module \( M \) has \( c_A(M) = 1 \) if and only if \( M \) is a periodic \( A \)-module which is not projective. See \([Ev]\) Proposition 8.4.4 for example. However, this is not true for general finite dimensional algebras. A counterexample is given by \([LS]\). See \([R4]\).

**Remark 16.** The converse does not hold in (2) and (3) above. To see this, we consider group algebras and use the theory of support varieties. References for this theory are \([Ev]\), \([B1]\), \([B3]\). Let \( G \) be a finite group, \( A = FG \) the group algebra. Assume that the characteristic of \( F \) is 2. Then

\[
X_G = \text{Spec}(H^*(G, F))
\]

is, by definition, the variety of the group \( G \). For each \( FG \)-module \( M \), we have the support variety \( X_G(M) \), which is a closed subvariety of \( X_G \). Then, the famous theorem of Alperin and Evens \([AE2]\), Avrunin \([Av]\) asserts that \( X_G(M) \) is covered by affine charts labelled by elementary abelian 2–subgroups \([Ev]\) Theorem 8.3.1. In particular, we have the following theorem of Quillen.

\[
\dim X_G = \max \{ \text{rank } E \mid E \text{ is an elementary 2–subgroup of } G. \}\]

An important fact relevant to us is that the dimension of \( X_G(M) \) is equal to the complexity of \( M \) \([Ev]\) Theorem 8.4.3. Hence, the complexity of any \( FG \)-module \( M \) cannot exceed the maximal rank of elementary 2–subgroups of \( G \).

Assume that \( A = FG \) is tame. Then, by \([BD]\) or \([S]\) Theorem 14.17, the Sylow 2–subgroup of \( G \) is one of dihedral, semidihedral or generalized quaternion groups. Take \( G \) to be the quaternion group \( Q \), which has order 8. Then the maximal rank of elementary 2–subgroups of \( G \) is 1. Thus, \( c_A(M) \leq 1 \) for all \( A \)-modules \( M \) and the converse of (2) fails. If we consider \( G = Q \times C_2 \) then we see that the converse of (3) also fails.

2.7. In this subsection, we collect results to show that an \( F \)-algebra is wild. We start with an easy lemma.

**Lemma 17.** Let \( A \) be a finite dimensional \( F \)-algebra, \( Q \) its Gabriel quiver. Assume that \( Q \) contains the following quiver, or the quiver with the reversed arrows, as a subquiver.
Then $A$ is wild.

**Proof.** Let $T$ be the quiver given above. Assume that $Q$ contains $T$ as a subquiver. Then, we have a natural surjection

$$FQ \longrightarrow FT.$$  

Note that $FQ$ is an infinite dimensional $F$–algebra in general.

Recall that $A$ is Morita–equivalent to the algebra $FQ/I$ such that $I$ is contained in the two–sided ideal consisting of pathes of length $\geq 2$ and contains all pathes of length $\geq t$ for some $t \geq 2$. Because of the orientation of $T$, $FT$ is a square–zero algebra. Thus, the surjection induces

$$FQ/I \longrightarrow FT.$$  

As $FT$ is wild by Theorem 6(3), so is $FQ/I$. □

A standard technique to show that an algebra is wild is the covering theory $\mathbf{Gr}$, $\mathbf{Ga3}$. See also $\mathbf{W}$ and $\mathbf{MP}$.

**Definition 18.** Let $Q = (Q_0, Q_1), \tilde{Q} = (\tilde{Q}_0, \tilde{Q}_1)$ be directed graphs. We say that $\tilde{Q}$ is a covering of $Q$ if there exist surjective maps of vertices and edges

$$\pi = (\pi_0, \pi_1) : \tilde{Q} \longrightarrow Q$$

such that, for any $\tilde{x} \in \tilde{Q}_0$, if we set $x = \pi(\tilde{x})$ then $\pi$ induces bijection between $s(\tilde{x}) = \{ \tilde{y} | (\tilde{x}, \tilde{y}) \in \tilde{Q}_1 \}$ (resp. $e(\tilde{x}) = \{ \tilde{y} | (\tilde{y}, \tilde{x}) \in \tilde{Q}_1 \}$) (resp. $e(x) = \{ y | (y, x) \in Q_1 \}$).

A covering $\tilde{Q}$ of $Q$ is a Galois covering if there exists a subgroup $G$ of the automorphism group $\text{Aut}(\tilde{Q})$ such that, for any $x \in Q_0$, $\pi^{-1}(x)$ is a fixed point free $G$–orbit.

**Definition 19.** Let $\tilde{Q}$ be a Galois covering of $Q$ and $F\tilde{Q} \longrightarrow FQ$ the induced algebra homomorphism. Suppose that $\tilde{I}$ and $I$ are admissible ideals of $F\tilde{Q}$ and $FQ$ respectively. Then $A = FQ/I$ is a Galois covering of $A = F\tilde{Q}/\tilde{I}$ if $\tilde{I}$ maps onto $I$.

In the definition above, we allow $A$ to be infinite dimensional. A useful result we use here is Han’s covering criterion $\mathbf{Hal}$. See $\mathbf{D}$ for a different proof.

**Theorem 20 (\cite[Theorem 3.3]{Hal}).** Let $Q = (Q_0, Q_1)$ be a directed graph with a finite vertex set $Q_0$, $I$ an admissible ideal of $FQ$ such that $A = FQ/I$ is a finite dimensional $F$–algebra. Assume the following two conditions.

(a) $A = F\tilde{Q}/\tilde{I} \longrightarrow A$ is a Galois covering whose Galois group $G$ is torsion free.

(b) There is a subquiver $Q'$ of $\tilde{Q}$ such that, if we define an admissible ideal $I'$ of $FQ'$ by replacing each path which is not contained in $Q'$ with zero, in each element of $\tilde{I}$, then $A' = FQ'/I'$ is finite dimensional and strictly wild.

Then, $A$ is controlled wild.
Remark 21. To know that $A$ is wild, it is not necessary to assume that the Galois group is torsion free, and it is enough to assume that $FQ'/I'$ is wild. This is because the pushdown functor of a Galois covering is a cleaving functor. See [dlP1, Chap.II, Lemma 2.1]. Then, we appeal to [dlP1, Chap.II, Proposition 2.1].

Now we apply the covering criterion and prepare results which we need in later sections. We remark that there are proofs which do not use the covering criterion in the first two lemmas; we can construct a tensor functor from $F\langle X,Y \rangle \text{mod}$ to the full subcategory $(FT - \text{mod})_s$ of sincere $FT$–modules, where $T$ is a directed graph whose underlying graph is

\[
\begin{cases}
\hat{D}_4 \text{ in Lemma 22} \\
\hat{E}_7 \text{ in Lemma 28}
\end{cases}
\]

Then, the pushdown functor respects the indecomposability and the isomorphism classes of indecomposable $FT$–modules in $(FT - \text{mod})_s$. In fact, this was the strategy in [Er, I.10.1–10.5]. However, the covering criterion gives us shorter proofs.

Lemma 22. Let $Q$ be the directed graph which is defined by the adjacency matrix $(a_{ij})_{1 \leq i,j \leq 4}$ where

\[a_{ij} = \begin{cases} 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases}\]

We denote by $J^3$ the ideal of $FQ$ generated by paths of length 3. Then $A = FQ/J^3$ is wild.

Proof. Let $\alpha, \beta, \gamma$ be the arrows $1 \to 2$, $2 \to 3$, $3 \to 4$ and $\alpha', \beta', \gamma'$ the arrows with the opposite direction.

We consider the following covering $\hat{Q}$ of $Q$, which contains the quiver on the right hand side as a subquiver.

$\hat{Q}$ consists of 3 zigzag lines, two going up and one going down, and the vertices with common $x$–coordinate is a fiber of $\hat{Q} \to Q$. Define $I$ to be the ideal generated by paths of length 3 in $\hat{Q}$, and set $\hat{A} = F\hat{Q}/I$.

Then, $\hat{A} \to A$ is a Galois covering with Galois group $\mathbb{Z}$ and the condition (a) of Theorem 20 is satisfied. Denote the subquiver by $Q'$. Then, as there is no path of length greater than or equal to 3 in $Q'$, we have $I' = 0$ and $A' = FQ'$ is finite dimensional and strictly wild by Theorem 6 and Proposition 7. Thus, the condition (b) is also satisfied. Therefore, $A$ is wild by Theorem 20.
Lemma 23. Let $A$ be a finite dimensional $F$–algebra, $Q$ the Gabriel quiver of $A$. Suppose that $Q$ contains a subquiver with the adjacency matrix $(a_{ij})_{1 \leq i,j \leq 3}$ where,

$$a_{ij} = \begin{cases} 1 & \text{if } i \geq 2 \text{ and } j \geq 2 \text{ or } i = 1 \text{ and } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A$ is wild.

Proof. Let $\tilde{Q}$ be the subquiver contained in $Q$. Define $\mathfrak{A} = FQ/J^2$ where $J^2$ is the ideal generated by paths of length 2 in $\tilde{Q}$. Thus, it is enough to show that $\mathfrak{A}$ is wild. We denote by $\rho, \mu, \nu$ the arrows $1 \rightarrow 2$, $2 \rightarrow 3$ and $2 \leftarrow 3$, and the loops on the nodes 2 and 3 by $\alpha$ and $\beta$ respectively. Then, we can find a Galois covering $\tilde{A} = \tilde{Q}/\tilde{I}$ of $\mathfrak{A}$ with Galois group $\mathbb{Z}$ such that, $\tilde{Q}$ contains the following quiver, which we define to be $Q'$, as a subquiver,

and $\tilde{I}$ maps to 0 under the surjective homomorphism $F\tilde{Q} \rightarrow FQ'$.

Thus, $A' = FQ'$ and this is finite dimensional and strictly wild. Therefore, Theorem 20 implies that $\mathfrak{A}$ is wild. \qed

In the next lemma, we consider the following algebra. This is (32) of Han’s Table W in [Ha2].

Definition 24. Let $Q$ be the directed graph defined by the adjacency matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and let $FQ$ be the associated path algebra.

We denote the loops on nodes 1, 2 by $\alpha$ and $\beta$, and the edges $1 \rightarrow 2$, $1 \leftarrow 2$ by $\mu$ and $\nu$ respectively. Then, $\Lambda_{32}$ is the $F$–algebra $FQ/I$ where the relations are given by

$$\mu \alpha = \beta \mu, \ \alpha \nu = 0, \ \nu \beta = 0,$$

$$\mu \nu \mu = 0, \ \nu \mu \nu = 0,$$

$$\alpha^2 = 0, \ \beta^2 = 0.$$

Lemma 25. $\Lambda_{32}$ is wild.

Proof. We consider the following covering $\tilde{Q}$ of $Q$, whose underlying graph is $\mathbb{Z}^2$. 

\[ x \]
We define $\tilde{I}$ by the same relations as for $\Lambda_{32} = FQ/I$, where monomials are understood as paths on $\tilde{Q}$, and $\mu x = \beta \mu$ is understood as two paths with the same endpoints are equal. Denote the subquiver on the right hand side by $Q'$. Then, $I' = (\mu x - \beta \mu)$ and $FQ'/I'$ is the concealed algebra given in Lemma 12. Hence, $FQ'/I'$ is finite dimensional and strictly wild. Therefore, the covering criterion implies that $\Lambda_{32}$ is wild.

2.8. There is a stable equivalence between an $F$–algebra with radical square zero and a path algebra. Further, given $F$–algebra with radical square zero, we can describe the directed graph of the corresponding path algebra. This is also a part of the Gabriel’s theorem. See [ARS X. Theorem 2.4] for example. We start with the following definition.

**Definition 26.** Let $Q = (Q_0, Q_1)$ be a directed graph. Then, the associated directed bipartite graph is the directed graph $\hat{Q} = (\hat{Q}_0, \hat{Q}_1)$ where $\hat{Q}_0$ is the disjoint union of two copies of $Q_0$, which we denote by $\{i', i''\}_{i \in Q_0}$, and $\hat{Q}_1$ is the set of arrows $i' \rightarrow j''$, one for each $i \rightarrow j$ in $Q_1$.

We call the underlying graph of $\hat{Q}$ the associated bipartite graph.

The following is the theorem of Gabriel.

**Theorem 27.** Let $A$ be a finite dimensional $F$–algebra, $Q$ the Gabriel quiver. Let $\hat{Q}$ be the directed bipartite graph associated with $Q$. Then, $A/ \text{Rad}^2 A$ is stably equivalent to $F\hat{Q}$.

On the other hand, we have a theorem of Krause. This is a beautiful application of Crawley–Boevey’s characterization of tameness in terms of the number of generic modules [Ca2]. See [Le, Theorem 1] for the Krause’s theorem. See also its final section for the proof and comments. The author thanks Professor Ringel for drawing his attention to [Le].

The Krause’s theorem asserts that stable equivalence preserves representation type, both tame and wild. Hence, Theorem 6 implies the following.

**Theorem 28.** Let $A$ be a finite dimensional $F$–algebra, $Q$ the Gabriel quiver. If the bipartite graph associated with $Q$ is not a Dynkin diagram of finite type, then $A$ is tame or wild. If it is not a Dynkin diagram of finite type nor affine type, then $A$ is wild.

Applying Theorem 28, we have the following lemmas. The first lemma may be proved by using the covering criterion, by following the similar arguments as in Lemma 22 and Lemma 23.

**Lemma 29.** Let $A$ be a finite dimensional $F$–algebra, and assume that the Gabriel quiver of $A$ contains the directed graph with adjacency matrix $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$ as a subquiver. Then, $A$ is wild.

**Proof.** Let $Q$ be the subquiver. Then, there is a surjective algebra homomorphism $A \rightarrow FQ/J^2$ where $J^2$ is the ideal generated by paths of length 2. As the associated bipartite graph is not a Dynkin diagram of finite type nor affine type, $FQ/J^2$ is wild. Thus, so is $A$.

**Lemma 30.** Let $A$ be a finite dimensional $F$–algebra with the property that $\text{Ext}^1_A(S, T) = 0$ if and only if $\text{Ext}^1_A(T, S) = 0$, for any pair of simple $A$–modules.
S and T. We denote by \( Q \) the underlying graph of the Gabriel quiver of \( A \), and suppose that the following two conditions hold.

(a) \( Q \) contains a cycle of length greater than or equal to 3.

(b) There is an edge of \( Q \) which is not contained in the cycle such that at least one of the endpoints of the edge belongs to the cycle.

Then \( A \) is wild.

**Proof.** Let \( n \) be the length of the cycle. Assume that there is an edge which connects a node in the cycle and a node outside the cycle. If \( n \) is even and \( n \geq 4 \) then the bipartite graph associated with \( Q \) contains two copies of \( A_{n-1}^{(1)} \) each of which is extended by an arrow. If \( n \) is odd and \( n \geq 3 \) then the bipartite graph contains \( A_{2n-1}^{(1)} \) which is extended by two arrows, whose endpoints on the cycle are distinct. Hence, the underlying graph of \( \hat{Q} \) is not a Dynkin diagram of finite type nor affine type and \( A \) is wild by Theorem 28. The proof is entirely similar in the cases where there is an edge which connects two distinct nodes in the cycle, or there is a loop around a node of the cycle. \( \square \)

2.9. Up to now, I have explained results to show that an algebra is wild. To show that an algebra is tame, we use the following result.

**Definition 31.** A finite dimensional \( F \)-algebra is called **special** if it is Morita–equivalent to a basic algebra \( FQ/I \) with the following properties.

(a1) For each \( i \in Q_0 \), the number of arrows \( i \to j \) in \( Q_1 \) is at most 2.

(a2) For each \( i \in Q_0 \), the number of arrows \( j \to i \) in \( Q_1 \) is at most 2.

(b1) For each directed edge \( \alpha \in Q_1 \), the number of arrows \( \beta \in Q_1 \) which satisfy \( \alpha \beta \notin I \) is at most 1.

(b2) For each directed edge \( \alpha \in Q_1 \), the number of arrows \( \beta \in Q_1 \) which satisfy \( \beta \alpha \notin I \) is at most 1.

If, moreover, \( I \) is given by a set of monomials, then \( A \) is called a **string algebra**.

**Theorem 32** ([WW, Corollary 2.4]). Any special algebra is finite or tame.

The notion of special algebras first appeared in [SW]. Recall that if, for any indecomposable projective \( A \)-module \( P \), \( P \) is uniserial or \( \text{Rad} \ P = N_1 \oplus N_2 / S \) with \( N_1 \) and \( N_2 \) uniserial and \( S \) simple or zero, then \( A \) is a **biserial algebra**. It is known that special algebras are biserial [SW, Lemma 1]. Thus, we call special algebras **special biserial algebras**. In fact, Theorem 32 is a special case of a more general theorem that biserial algebras are finite or tame. See [C3, Theorem A].

Theorem 32 together with Lemma 1 allow us to show that the special biserial algebras we will meet are tame.

For special biserial algebras, we know their Auslander–Reiten quivers. As a corollary, we can prove the existence of a module with complexity 2 in crucial cases. The author thanks Professor Erdmann for drawing his attention to this.

Let \( A \) be a finite dimensional basic \( F \)-algebra, \( I = \{ e_i | Ae_i \text{ is injective.} \} \). Define \( S = \bigoplus_{i \in I} \text{Soc } Ae_i \). \( S \) is a two–sided ideal of \( A \). Define \( \overline{A} = A / S \). The next lemma is standard. See [Er, I.8.11].

**Lemma 33.** Let \( A, I, S \) and \( \overline{A} \) be as above. Then, the set of the isomorphism classes of indecomposable \( A \)-modules is the disjoint union of \( \{ Ae_i | i \in I \} \) and the set of the isomorphism classes of indecomposable \( \overline{A} \)-modules. The Auslander–Reiten quiver of \( \overline{A} \) is obtained from that of \( A \) by removing \( \{ Ae_i | i \in I \} \).
Proof. Fix \( i \in I \) and write \( P_i = Ae_i \). Let \( M \) be an indecomposable \( A \)-module and assume that there is \( m \in M \) such that \((\text{Soc} P_i)m \neq 0\). Consider an \( A \)-module homomorphism

\[
\phi : P_i \longrightarrow M
\]

defined by \( ae_i \mapsto ae_im \), for \( a \in A \). As \( P_i \) is indecomposable and injective, \( \text{Soc} P_i \) is simple. Hence, if \( \phi \) is not a monomorphism then \( \text{Soc} P_i \subset \text{Ker} \phi \), contradicting to our assumption that \((\text{Soc} P_i)m \neq 0\). Hence, we have \( M \simeq P_i \). The first assertion is proved.

Let \( M \) and \( N \) be \( A \)-modules on which \( S \) act as zero. Suppose that there is an irreducible \( A \)-module homomorphism \( M \longrightarrow N \). Then, \( M,N \not\simeq Ae_i \), for \( i \in I \), implies that this homomorphism viewed as an \( A \)-module homomorphism is irreducible. Next assume that there is an irreducible \( A \)-module homomorphism \( M \longrightarrow N \). Then, there exists an irreducible \( A \)-module homomorphism \( M \longrightarrow N \) by [Ga2 Corollary 3.3]. \( \square \)

Now assume that the basic algebra \( A/S \) is a string algebra. For string algebras, we can classify their indecomposable modules. Among them, we only need the string modules for our purposes.

Let \( FQ/I \) be a string algebra. Consider a updown diagram of the following form with all the arrows directed downward.

```
✠ ❅❅ ❘ ❅❅ ❘ ✠ ❅❅ ❘ ✠ ❅❅ ❘ ✠ ❅❅ ❘ ✠ ❅❅ ❘ ✠ ···
```

We see this as a collection of subpaths directed to southwest or southeast. Then, this updown diagram is called a string if

- edges of each subpath is labelled by elements of \( Q_1 \) so that the product is a monomial which is not in \( I \),
- adjacent subpaths have a common source or a common target in \( Q_0 \) on a peak or in a deep,
- two arrows on a peak or in a deep have different labels.

Let \( C \) be a string, \( V(C) \) the set of its vertices. We define an \( F \)-vector space \( M(C) \) by

\[
M(C) = \bigoplus_{x \in V(C)} Fx.
\]

Then, \( M(C) \) becomes an \( FQ/I \)-module by the rule that \( \alpha x = y \) if there is an arrow \( x \rightarrow y \) with label \( \alpha \) in \( C \), and \( \alpha x = 0 \) otherwise.

Definition 34. Let \( C \) be a string. We say that \( C \) starts on a peak (resp. ends on a peak) if we cannot add an arrow directed to southwest (resp. southeast) to the right (resp. left) of \( C \) so that the extended diagram may be a string again.

Similarly, we say that \( C \) starts in a deep (resp. ends in a deep) if we cannot add an arrow directed to southeast (resp. southwest) to the right (resp. left) of \( C \) so that the extended diagram may be a string again.
Given $\alpha \in Q_1$, write $\alpha = i \rightarrow j$ and let $P_i$ be the indecomposable projective $FQ/I$–module corresponding to the node $i$, and let $P^j_i$ the $FQ/I$–submodule of $P_i$ generated by $\alpha$. Define $M_\alpha = P_i/P^j_i$. Note that $M_\alpha$ is uniserial. The following proposition gives an explicit rule for $\tau(M(C))$, where $\tau$ is the Auslander–Reiten translate. See [Er II.3], which is, in turn, based on [BuR].

**Proposition 35.** Assume that $M(C)$ is not projective nor isomorphic to any of $M_\alpha$ ($\alpha \in Q_1$).

1. If $C$ starts and ends in a deep then $\tau(M(C)) = M(C')$ where $C'$ is obtained from $C$ by deleting the leftmost and the rightmost

2. If $C$ starts in a deep but does not end in a deep then $\tau(M(C)) = M(C')$ where $C'$ is obtained from $C$ by deleting the rightmost and adding

3. If $C$ does not start in a deep but ends in a deep then $\tau(M(C)) = M(C')$ where $C'$ is obtained from $C$ by deleting the leftmost and adding

4. If $C$ does not start nor end in a deep then $\tau(M(C)) = M(C')$ where $C'$ is obtained from $C$ by adding
Let $Q$ be a directed graph with adjacency matrix \[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix}
\]. We denote the arrows $1 \to 2$, $1 \leftarrow 2$ by $\mu$ and $\nu$ respectively, and the loop on the node $2$ by $\beta$. Define a string algebra $\Lambda = FQ/I$ by the relations

\[
\nu\beta = 0, \quad \beta\mu = 0, \quad \beta^2 = 0, \quad (\nu\mu)^2 = 0, \quad (\nu\nu\mu) = 0.
\]

**Lemma 36.** Let $S$ be the simple $\Lambda$–module corresponding to the node $2$. Then, $\dim F \tau^2(S) = 6n + 1$ and $\dim F \tau^{2n+1}(S) = 6n + 6$, for $n \in \mathbb{N}$.

**Proof.** $\Lambda$ has the basis \[
\{e_1, e_2, \beta, \mu, \nu, \nu\mu, \mu\nu\mu, \nu\mu\nu\}.
\]

$M_\mu$ is the simple $\Lambda$–module corresponding to the node 1. Other $M_\nu$, $M_\beta$ and indecomposable projective $\Lambda$–modules are given by $M(C)$ with $C$ being one of the following.

Thus, by Proposition 35(2)(4), $\tau(S)$ and $\tau^2(S)$ are given by the following strings.

\[
\tau(S): \quad \nu \beta \mu \quad \text{and} \quad \tau^2(S): \quad \beta \nu \mu \nu \mu \nu \beta
\]

It is clear that the same pattern repeats in $\tau^{2n+1}(S)$ and $\tau^{2n+2}(S)$, for $n > 0$. \qed

**Lemma 37.** Let $A$ be a symmetric special biserial algebra. Suppose that each node of the Gabriel quiver of $A$ has two outgoing arrows and two incoming arrows. Then, any simple $A$–module $S$ has the complexity $c_A(S) = 2$.

**Proof.** As we mentioned before, $A$ is a biserial algebra. Thus, we have a description of the radical series of indecomposable projective $A$–modules. It follows that $A/S$ is a string algebra. As $A$ is self–injective, the Auslander–Reiten quiver
of $\mathcal{T}$ coincides with the stable Auslander–Reiten quiver of $A$. By repeated use of Proposition 35(4), we have the inequality
\[ \dim F\tau^n(S) \geq 2n + 1 \quad (n \in \mathbb{N}). \]
Since $A$ is a symmetric algebra, $\tau = \Omega^2$ where $\Omega$ is the Heller loop operator. Thus, this inequality implies that $c_A(S) \geq 2$. As $c_A(S) \leq 2$ by Theorem 14(3) and Theorem 32, the result follows. □

Let $A$ be a finite dimensional $F$–algebra, $\{P_1, \ldots, P_s\}$ a complete set of the isomorphism classes of indecomposable projective $A$–modules. Then $A$ is symmetric if and only if $\text{End}_A(P_1 \oplus \cdots \oplus P_s)$ is symmetric. See [Er, Lemma I.3.3] for example. This fact will be used without further notice.

3. The case of $\mathcal{H}_n(q, Q)$

3.1. Let $F$ be an algebraically closed field and let $q, Q \in F^\times$. $\mathcal{H}_n(q, Q)$ is the $F$–algebra defined by generators $T_0, \ldots, T_{n-1}$ and relations
\[
(T_0 - Q)(T_0 + 1) = 0, \quad (T_i - q)(T_i + 1) = 0 \quad (1 \leq i \leq n - 1),
\]
\[
T_0T_1T_0T_1 = T_1T_0T_1T_0, \quad T_iT_j = T_jT_i \quad (j \geq i + 2),
\]
\[
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad (i \geq 1).
\]

There are two cases to consider. The first case is the case where $-Q \notin q\mathbb{Z}$.

**Theorem 38 ([DJ2, Theorem 4.17]).** Suppose that $Q \neq -q^f$ for any $f \in \mathbb{Z}$. Then $\mathcal{H}_n(q, Q)$ is Morita–equivalent to
\[
\bigoplus_{k=0}^{n} \mathcal{H}_k^A(q) \otimes \mathcal{H}_{n-k}^A(q).
\]

Further, we already know representation type for all of the block algebras of $\mathcal{H}_n^A(q)$ by [EN]. Recall that block algebras of $\mathcal{H}_n^A(q)$ are labelled by $e$–cores $\kappa$ which satisfy the condition that $w_\kappa = \frac{n - |\kappa|}{e}$ is a non–negative integer [DJ1]. Let $B_\kappa$ be the block algebra of $\mathcal{H}_n^A(q)$ labelled by an $e$–core $\kappa$. The integer $w_\kappa$ is called the weight of $B_\kappa$. See [JK, 2.7] for the terminology. The Erdmann–Nakano theorem is as follows.

**Theorem 39 ([EN, Theorem 1.2]).** Let $e$ be the multiplicative order of $q \neq 1$ and $\kappa$ an $e$–core with $w_\kappa = \frac{n - |\kappa|}{e} \in \mathbb{Z}_{\geq 0}$. Then,

1. $B_\kappa$ is semisimple if and only if $w_\kappa = 0$.
2. $B_\kappa$ is
   - finite if and only if $w_\kappa \leq 1$,
   - tame if and only if $e = 2$ and $w_\kappa = 2$,
   - wild otherwise.

Using this, we can prove the following.

**Theorem 40.** Assume that $-Q \notin q\mathbb{Z}$ and that $q$ has the multiplicative order $e \geq 2$. Then, we have the following.

1. If $e \geq 3$ then $\mathcal{H}_n(q, Q)$ is
   - finite if and only if $n < 2e$,
   - wild otherwise.
2. If $e = 2$ then $\mathcal{H}_n(q, Q)$ is
- finite if and only if $n < 4$,
- tame if and only if $n = 4$ or 5,
- wild otherwise.

**Proof.** (1) If $n < 2e$ then one of $k$ and $n - k$ in Theorem 39 is strictly smaller than $e$. Thus, using Theorem 39 (1), we know that one of $\mathcal{H}^A_k(q)$ and $\mathcal{H}^A_{n-k}(q)$ is a semisimple algebra, which is a direct sum of matrix algebras because $F$ is algebraically closed. Using Theorem 39 (2) we also know that the other is finite. Hence $\mathcal{H}_n(q)$ is finite in this case. To show that $n \geq 2e$ implies that $\mathcal{H}_n(q)$ is wild, it is enough to prove that $\mathcal{H}_{2e}(q)$ is wild by Corollary 4. Theorem 39 (2) implies that $\mathcal{H}^A_{2e}(q)$ is wild, which implies the result by Theorem 39.

(2) By the assumption that $e = 2$, the characteristic of $F$ is odd. First of all, Theorem 39 (2) implies that $\mathcal{H}^A_2(q)$ and $\mathcal{H}^A_3(q)$ are finite. As $\mathcal{H}^A_3(q) = F$ and $\mathcal{H}^A_1(q) = F$ by definition, this implies that $\mathcal{H}^A_k(q) \otimes \mathcal{H}^A_{2-k}(q)$, for $0 \leq k \leq 2$, and $\mathcal{H}^A_k(q) \otimes \mathcal{H}^A_{3-k}(q)$, for $0 \leq k \leq 3$, are all finite, proving that $\mathcal{H}_2(q)$ and $\mathcal{H}_3(q)$ are finite.

Next observe that $\mathcal{H}^A_4(q)$ and $\mathcal{H}^A_5(q)$ are tame. In fact, if $n = 4$ then there is only one 2-core, which has weight 2, and if $n = 5$ then there are two 2-cores, one of which has weight 1, the other of which has weight 2. Thus, $\mathcal{H}^A_4(q)$ and $\mathcal{H}^A_5(q)$ are tame by Theorem 39 (2). This implies that $\mathcal{H}^A_k(q) \otimes \mathcal{H}^A_{4-k}(q)$, for $k = 0, 1, 3, 4$, and $\mathcal{H}^A_k(q) \otimes \mathcal{H}^A_{5-k}(q)$, for $k = 0, 1, 4, 5$, are tame. As $\mathcal{H}^A_4(q) = F[T_1]/(T_1 + 1)^2$ implies that $\mathcal{H}^A_2(q) \otimes \mathcal{H}^A_2(q)$ is tame because

$$\mathcal{H}^A_2(q) \otimes \mathcal{H}^A_2(q) \simeq F[X,Y]/(X^2,Y^2),$$

we have proved that $\mathcal{H}_4(q) = \mathcal{H}^A_4(q)$ is tame. On the other hand, $\mathcal{H}^A_3(q)$ is sum of two block algebras. One is isomorphic to the matrix algebra End$_F(S^{2,1})$, where $S^{2,1}$ is the Specht module labelled by $(2,1)$, which is a projective $\mathcal{H}^A_3(q)$-module in the present case. The other is isomorphic to $F[X]/(X^2)$. Hence, $\mathcal{H}^A_k(q) \otimes \mathcal{H}^A_{5-k}(q)$, for $k = 2$ and $k = 3$, is Morita–equivalent to $F[X,Y]/(X^2,Y^2) \oplus F[Z]/(Z^2)$, proving that $\mathcal{H}_5(q)$ is tame.

As the block algebra of $\mathcal{H}^A_h(q)$ labelled by the empty 2-core is wild, $\mathcal{H}^A_n(q)$ with $n \geq 6$ is wild by Corollary 4. Thus, $\mathcal{H}_n(q)$ is wild by Theorem 39. 

In the proof above, we heavily rely on Theorem 39. However, if we use the Fock space theory, which we will explain in the next subsection, we can give another proof which does not use Theorem 39.

**3.2.** The second case is the case where $-Q = qf$ for some $f \in \mathbb{Z}$. Note that $\mathcal{H}_n(1,-1)$ is isomorphic to the semidirect product of $F[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ with the group algebra of the symmetric group $S_n$, where $S_n$ acts on $\{x_1, \ldots, x_n\}$ in the natural way. Let $\ell$ be the characteristic of $F$, then.

**Proposition 41.** $\mathcal{H}_n(1,-1)$ is
- finite if $n = 1$,
- tame if $n = 2$ and $\ell \neq 2$,
- wild otherwise.

**Proof.** The case $n = 1$ is obvious. As $F[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ is wild when $n \geq 3$, which follows from Theorem 14 or Theorem 28, Proposition 3 implies that $\mathcal{H}_n(1,-1)$ is wild in this case. The remaining cases are for $n = 2$. Let $\sigma$ be the unique transposition of $S_2$. 

- finite if and only if $n < 4$,
- tame if and only if $n = 4$ or 5,
- wild otherwise.
Suppose that $\ell \neq 2$. Then we have two simple modules $D^\pm$ defined by $x_i = 0$, for $i = 1, 2$, and $\sigma = \pm 1$. As the radical of $\mathcal{H}_2(1, -1)$ is $F[x_1, x_2]/(x_1^2, x_2^2)$, an explicit computation of the regular representation implies that the radical series of the projective cover of $D^\pm$ is as follows.

$$D^\pm$$

$$D^+ \oplus D^-$$

$$D^\pm$$

Let $Q$ be the directed graph defined by the adjacency matrix $\left( \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} \right)$, and we denote the loops on nodes 1, 2 by $\alpha$ and $\beta$, the edges $1 \to 2$, $1 \leftrightarrow 2$ by $\mu$ and $\nu$. Then the Gabriel quiver of $\mathcal{H}_2(1, -1)$ is $Q$ and the relations include

$$\mu \alpha = \alpha \nu = 0, \quad \nu \beta = \beta \mu = 0.$$ 

Hence $\mathcal{H}_2(1, -1)$ is a special biserial algebra and Theorem 32 implies that $\mathcal{H}_2(1, -1)$ is finite or tame. On the other hand, Lemma 37 and Theorem 14(2) imply that $\mathcal{H}_2(1, -1)$ cannot be finite. Hence the result.

Suppose that $\ell = 2$. Then $\mathcal{H}_2(1, -1)$ is the projective cover of the unique simple module, and $\text{Rad} \mathcal{H}_2(1, -1)/\text{Rad}^2 \mathcal{H}_2(1, -1)$ constitutes of three simple modules. Hence $\mathcal{H}_2(1, -1)$ is wild by Theorem 28.

The theorem we are going to prove is the following.

**Theorem 42.** Assume that $q$ has the multiplicative order $e \geq 2$, and $Q = -q^f$, for some $0 \leq f < e - 1$. Then we have the following.

1. Suppose that $e \geq 3$. Then $\mathcal{H}_n(q, Q)$ is
   - finite if $n < \min\{e, 2f + 4, 2e - 2f + 4\}$,
   - tame if $f = 0$ and $4 \leq n < \min\{e, 9\}$,
   - wild otherwise.

2. Suppose that $e = 2$. Then $\mathcal{H}_n(q, Q)$ is
   - finite if $n = 1$.
   - tame if $n = 2$ or $n = 3$ and $f = 1$.
   - wild otherwise.

We have already proved in [AM2, Theorem 1.4] that $\mathcal{H}_n(q, -q^f)$ is finite if and only if $n < \min\{e, 2f + 4, 2e - 2f + 4\}$, under the assumption that $e \geq 3$. The case $e = 2$ is easy to handle, and the proof is given in [AM3]. We remark that the radical series given in (case 1) in the proof of [AM2, Theorem 5.25] needs correction as in (case 5a) below.

Hence, it is enough to prove the statements for tameness and wildness. To prove these, we need the Specht module theory for $\mathcal{H}_n(q, Q)$, developed by Dipper, James and Murphy [DJM], and the Fock space theory developed by the author [A1, A2 and A3], as for the proof of finiteness in [AM2, Theorem 1.4]. We redefine $T_0$ by $T_0^{new} = -T_0^{old}$ if $0 \leq f \leq 2$, and $T_0^{new} = -q^{-f}T_0^{old}$ if $e$ is finite and $\frac{e}{2} < f \leq e - 1$. Thus, we may and do assume $0 \leq f \leq \frac{e}{2}$ without loss of generality, and we assume that $T_0$ satisfies $(T_0 - 1)(T_0 - -q^f) = 0$ in the rest of the paper.

**Remark 43.** The Specht module theory was generalized to the Hecke algebras of type $(m, 1, n)$ by Dipper, James and Mathas [DJM]. This theory is now viewed as an example of the cell module theory for cellular algebras in the sense of Graham...
and Lehrer. Also, if we restrict ourselves to type A case, the Fock space theory was
generalized to q-Schur algebras by Varagnolo and Vasserot [VV].

Let us begin by explaining the Specht module theory for \( \mathcal{H}_n(q, Q) \). Let \( \mathcal{BP} \) be
the set of bipartitions \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \). The size of \( \lambda \) is denoted by \( |\lambda| \). If \( |\lambda| = n \)
then we write \( \lambda \vdash n \). Let \( \mathcal{BP}(n) \) be the set of bipartitions with \( \lambda \vdash n \). Then \( \mathcal{BP}(n) \)
is a poset with a partial order \( \sqsubseteq \), called the dominance ordering; we say that \( \lambda \sqsubseteq \mu \)
if
\[
\sum_{i=1}^{k} \lambda_{i}^{(1)} \leq \sum_{i=1}^{k} \mu_{i}^{(1)}, \text{ for all } k,
\]
and
\[
|\lambda^{(1)}| + \sum_{j=1}^{k} \lambda_{j}^{(2)} \leq |\mu^{(1)}| + \sum_{j=1}^{k} \mu_{j}^{(2)}, \text{ for all } k.
\]
If \( \lambda \sqsubseteq \mu \) and \( \lambda \neq \mu \) we write \( \lambda < \mu \).

In [DJM] it was shown that there exists a family \( \{ S^\lambda | \lambda \vdash n \} \) of \( \mathcal{H}_n(q, Q) \)–modules
such that each of which is equipped with a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) satisfying \( \langle T_i u, v \rangle = \langle u, T_i v \rangle \), for \( 0 \leq i \leq n - 1 \) and for all \( u, v \in S^\lambda \). We define an
\( \mathcal{H}_n(q, Q) \)–module \( D^\lambda \) by \( D^\lambda = S^\lambda / \text{rad}(\cdot, \cdot) S^\lambda \). If there is a need to specify the base
field \( F \), we write \( D^\lambda_F \). Then \( D^\lambda \) is absolutely irreducible or \( D^\lambda = 0 \). If \( D^\lambda \neq 0 \) then
its projective cover is denoted by \( P^\lambda \) or \( P^\lambda_F \).

The following results are fundamental in the Specht module theory.

**Theorem 44.**

1. Each \( D^\lambda \) is self–dual.
2. \( \{ D^\lambda | \lambda \vdash n, D^\lambda \neq 0 \} \) is a complete set of the isomorphism classes of simple
\( \mathcal{H}_n(q, Q) \)–modules.
3. If \( D^\lambda \neq 0 \) then \( S^\lambda \) is indecomposable with \( \text{Top} S^\lambda = D^\lambda \).
4. For each \( \mu \vdash n \) with \( D^\mu \neq 0 \), \( P^\mu \) has a Specht filtration
   \[
P^\mu = F_0 \supset F_1 \supset \cdots \supset F_N = 0
   \]
   such that \( S^\lambda \) appears \( |S^\lambda : D^\mu| \) times in \( \{ F_i/F_{i+1} | 0 \leq i < N \} \), for each \( \lambda \).
   Further, if we denote the maximal element of \( \{ \lambda | |S^\lambda : D^\mu| \neq 0 \} \) in the
dominance ordering by \( \lambda_{\text{max}} \), then \( S^{\lambda_{\text{max}}} \) appears as a submodule of \( P^\mu \).

The statement (1) implies that if there is an arrow in the Gabriel quiver of
\( \mathcal{H}_n(q, Q) \) then there is always an arrow with the same endpoints and the opposite
direction in the Gabriel quiver.

Another important remark about the Specht module theory is that if we have
enough knowledge on the decomposition numbers \( [S^\lambda : D^\mu] \), then, by using semi-normal representations, we can obtain an explicit matrix representation of \( D^\mu \).

We denote the set of bipartitions \( \lambda \in \mathcal{BP}(n) \) with \( D^\lambda \neq 0 \) by \( \mathcal{KBP}(n) \), and
the disjoint union \( \sqcup_{n \in \mathbb{N}} \mathcal{KBP}(n) \) by \( \mathcal{KBP} \). Then we can describe \( \mathcal{KBP} \) by using
the crystal graph theory. Further, the decomposition numbers \( [S^\lambda : D^\mu] \) can be
described in terms of the canonical basis in a combinatorial Fock space. This is the
theory which I called the Fock space theory. To explain this, we review the author’s
previous work [A2], which sprang from an important observation and results of
Lascoux, Leclerc and Thibon [LLT]. See chapters 12–14 of [A1] for details. Here,
we state the results only for \( \mathcal{H}_n(q, Q) \).

Assume as before that \( q \in F^\times \), which appeared as one of the parameters \( q \) and
\( Q \) of \( \mathcal{H}_n(q, Q) \), is a primitive \( e \)th root of unity with \( e \geq 2 \). Then we consider the
Kac–Moody Lie algebra \( \mathfrak{g} \) of type \( A^{(1)}_{1} \) and its quantized enveloping algebra \( U_v(\mathfrak{g}) \), where \( v \) is an indeterminate. We denote by \( V_\lambda(\Lambda) \) the irreducible integrable highest weight \( U_v(\mathfrak{g}) \)-module with highest weight \( \Lambda = \Lambda_0 + \Lambda_f \). Recall that \( f \) is such that \( Q = -q^f \) and \( 0 \leq f \leq \frac{e}{2} \).

The \( v \)-deformed combinatorial Fock space \( F_v(\Lambda) \) with hightest weight \( \Lambda \) is the infinite dimensional vector space

\[
F_v(\Lambda) = \bigoplus_{\lambda \in \mathcal{BP}} \mathbb{Q}(v)\lambda,
\]

which is a \( U_v(\mathfrak{g}) \)-module via

\[
e_i\lambda = \sum_{\mu \in \mathcal{BP} : \lambda / \mu = \square} v^{-N^a(\lambda / \mu)} \mu,
\]

\[
f_i\lambda = \sum_{\mu \in \mathcal{BP} : \mu / \lambda = \square} v^{N^b(\mu / \lambda)} \mu,
\]

\[
t_i\lambda = v^{N_i(\lambda)} \lambda, \quad v^d \lambda = v^{-W_0(\lambda)} \lambda.
\]

The definitions of \( N^a(\lambda / \mu) \), \( N^b(\mu / \lambda) \), \( N_i(\lambda) \) and \( W_0(\lambda) \) in the formulas above are as in [A1, Definition 10.8] and we omit them. The proof that this is a \( U_v(\mathfrak{g}) \)-module is given in [A1, Theorem 10.10].

The following is called the Misra–Miwa theorem. See [A1, Theorem 11.11].

**Theorem 45.** Define \( \mathcal{L}(\lambda) \) and \( \mathcal{B}(\lambda) \) by

\[
\mathcal{L}(\lambda) = \bigoplus_{\lambda \in \mathcal{BP}} \mathbb{Q}[v](v)\lambda, \quad \mathcal{B}(\lambda) = \{ \lambda + v\mathcal{L}(\lambda) | \lambda \in \mathcal{BP} \}.
\]

Then \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) is a (lower) crystal basis of \( F_v(\Lambda) \) in the sense of Kashiwara.

We identify the crystal graph of \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) with \( \mathcal{BP} \). Now we identify the \( U_v(\mathfrak{g}) \)-submodule of \( F_v(\Lambda) \) generated by the empty bipartition \( \emptyset \) with \( V_\emptyset(\Lambda) \). Then \( (\mathcal{L}(\lambda), \mathcal{B}(\lambda)) \) defines a crystal basis of \( V_\emptyset(\Lambda) \), which we denote by \( (L(\lambda), B(\lambda)) \). Note that \( V_\emptyset(\Lambda) \) is a \( U_v(\mathfrak{g}) \)-submodule of \( F_v(\Lambda) \) and \( B(\lambda) \) is a subset of \( \mathcal{BP} \) in our definition. The highest weight vector \( v_\Lambda \) is identified with the empty bipartition \( \emptyset \in F_v(\Lambda) \). By a fundamental theorem of Kashiwara and Lusztig, the crystal basis \( (L(\lambda), B(\lambda)) \) can be lifted to the canonical basis of \( V_\emptyset(\Lambda) \).

The following collects some of the main theorems of the Fock space theory. The second part was proved in [A2] Theorem 4.4; the proof requires plenty of results. I recommend reading the proof of [A1] Theorem 12.5, which not only proves the second part but also includes background materials necessary for the proof. The remaining parts were proved in [A3] Theorem 4.2 and [AM2] Corollary 3.16.

We say that \( (K, O, F) \) is a modular system with parameters if \( O \) is a discrete valuation ring and \( K \) the fraction field, \( F \) the residue field, such that there are elements \( \hat{q}, \hat{Q} \in O^\times \) which are lift of the parameters \( q, Q \in F^\times \).

**Theorem 46.**

1. \( B(\lambda) = KBP \).

2. Assume that the characteristic of \( F \) is zero. Let \( \mu \in KBP \) and denote the corresponding canonical basis element by \( G(\mu) \). If we write

\[
G(\mu) = \sum_{\lambda \in KBP} d_{\lambda\mu}(v)\lambda
\]
in \( \mathcal{F}_n(\Lambda) \) then we have
\[
d_{\lambda\mu}(1) = [S^\lambda : D^\mu].
\]

(3) Let \((K, \mathcal{O}, F)\) be a modular system with parameters and suppose that \(\mu\) is a Kleshchev bipartition such that
\[
G(\mu) = f_{i_1}^{(m_1)} \cdots f_{i_n}^{(m_n)} \nu_\Lambda
\]
for some \(m_1, \ldots, m_n \in \mathbb{N}\) and \(i_1, \ldots, i_n \in \mathbb{Z}/e\mathbb{Z}\). Then the decomposition map sends \([P_{K}^\mu]\) to \([P_{F}^\mu]\). In particular, the decomposition numbers in the column labelled by \(\mu\) do not depend on the characteristic of \(F\).

3.3. Now we start the proof of Theorem 42(1). As we assume that \(e \geq 3\) and \(0 \leq f \leq \frac{e}{2}\), we have \(e - f \geq 2\). We prove the theorem by case–by–case analysis. The cases we consider are as follows.

- (case 1a) \(n = e, e - f \geq 3\) and \(1 \leq f \leq \frac{e}{2}\).
- (case 2a) \(n = e, e - f = 2\) and \(1 \leq f \leq \frac{e}{2}\).
- (case 3a) \(n = e\) and \(f = 0\).
- (case 4a) \(n = 2f + 4, e > 2f + 4\) and \(1 \leq f \leq \frac{e}{2}\).
- (case 5a) \(4 \leq n < \min\{e, 9\}\) and \(f = 0\).
- (case 6a) \(n = 9 < e\) and \(f = 0\).

Our aim is to show that \(\mathcal{H}_n(q, Q)\) is wild in all the cases but (case 5a). In (case 5a) we show that \(\mathcal{H}_n(q, Q)\) is tame and has an indecomposable module with complexity \(2\).

3.4. The first three cases are for \(n = e\). Then, following [AM2], we define bipartitions
\[
\lambda_k = ((0), (k, 1^{e-k})) \quad (1 \leq k \leq e),
\]
\[
\mu_k = ((k, 1^{e-k}), (0)) \quad (1 \leq k \leq e),
\]
\[
\lambda_{k,l} = ((f - l, 1^{e-f-k}), (k, 1^l)) \quad (1 \leq k \leq e - f \text{ and } 0 \leq l < f).
\]
These bipartitions belong to the “principal block”, which we denote by \(B\).

**Proposition 47 ([AM2 Proposition 4.22]).** We have the following.

1. The complete set of Kleshchev bipartitions in \(B\) is
\[
\{\lambda_k | 1 \leq k < e\} \cup \{\lambda_{k,l} | 1 \leq k \leq e - f \text{ and } 0 \leq l < f\}.
\]

2. For \(1 \leq k < e\) we have
\[
[P_{\lambda_k}] = [S^{\lambda_k}] + [S^{\lambda_{k+1}]} + \begin{cases}
[S^{\lambda_{k,f-1}]} + [S^{\lambda_{k+1,f-1}]} & (k < e - f) \\
[S^{\lambda_{e-f,k}}] + [S^{\lambda_{e-f,k-1}}] & (k = e - f)
\end{cases} + \begin{cases}
[S^{\lambda_{e-f,k-l}}] + [S^{\lambda_{e-f,k-l-1}}] & (k > e - f)
\end{cases}
\]

3. For \(1 \leq k \leq e - f \) and \(0 \leq l < f\) we have
\[
[P_{\lambda_{k,l}}] = [S^{\lambda_{k,l}}] + \begin{cases}
[S^{\lambda_{k-1,0}}] + [S^{\mu_{f+k-1}}] + [S^{\mu_{f+k}}] & (k \neq 1 \text{ and } l = 0) \\
[S^{\mu_{e-1}}] + [S^{\mu_{f+1}}] & (k = 1 \text{ and } l = 0)
\end{cases} + \begin{cases}
[S^{\lambda_{k-1,l}}] + [S^{\lambda_{k-1,l-1}}] + [S^{\lambda_{k-1,l-1}}] & (k \neq 1 \text{ and } l \neq 0) \\
[S^{\lambda_{1,l-1}}] + [S^{\mu_{f-l}}] + [S^{\mu_{f-l+1}}] & (k = 1 \text{ and } l \neq 0)
\end{cases}
\]
Assume that we are in (case 1a). The argument is very similar to that in \( A5 \), which was for \( H_B(n(q)) \). By Proposition 47,

\[
\begin{align*}
[P^{\lambda_1}] &= [S^{\lambda_1}] + [S^{\lambda_2}] + [S^{\lambda_2,f-1}] + [S^{\lambda_1,f-1}], \\
[P^{\lambda_2}] &= [S^{\lambda_2}] + [S^{\lambda_3}] + [S^{\lambda_3,f-1}] + [S^{\lambda_2,f-1}],
\end{align*}
\]

and

\[
\begin{align*}
[S^{\lambda_1,f-1}] &= [D^{\lambda_1,f-1}] + [D^{\lambda_2,f-1}] + [D^{\lambda_1}], \\
[S^{\lambda_2,f-1}] &= [D^{\lambda_2,f-1}] + [D^{\lambda_3,f-1}] + [D^{\lambda_2}] + [D^{\lambda_1}].
\end{align*}
\]

Then, that \( S^{\lambda_1,f-1} \) is a submodule of \( P^{\lambda_1} \) implies that the radical series of \( S^{\lambda_1,f-1} \) is as follows.

\[
\begin{align*}
D^{\lambda_1,f-1} \\
D^{\lambda_2,f-1} \\
D^{\lambda_1}
\end{align*}
\]

On the other hand, \( S^{\lambda_2} \) has the radical series of the following form.

\[
\begin{align*}
D^{\lambda_2} \\
D^{\lambda_1}
\end{align*}
\]

Then, as in the proof of (case 1) of \( AM2 \), Theorem 4.21], we have

\[
\text{Rad} P^{\lambda_1} / \text{Rad}^2 P^{\lambda_1} = D^{\lambda_2} \oplus D^{\lambda_2,f-1}.
\]

Note that \( D^{\lambda_1,f-1} \) cannot appear in \( \text{Rad} P^{\lambda_1} / \text{Rad}^2 P^{\lambda_1} \) because if otherwise then \( D^{\lambda_1,f-1} \) would appear in \( \text{Soc}^2 P^{\lambda_1} / \text{Soc} P^{\lambda_1} \). This contradicts to \( [P^{\lambda_1} : D^{\lambda_1,f-1}] = 1 \) as \( D^{\lambda_1,f-1} \) has already appeared in \( \text{Soc}^3 P^{\lambda_1} / \text{Soc}^2 P^{\lambda_1} \).

Another property we need is \( \text{Soc} S^{\lambda_2,f-1} = D^{\lambda_2} \). This follows from the fact that \( S^{\lambda_2,f-1} \) is a submodule of \( P^{\lambda_2} \).

Now, we determine the radical series of \( S^{\lambda_2,f-1} \). If \( \text{Rad}^2 S^{\lambda_2,f-1} = 0 \) then \( \text{Soc} S^{\lambda_2,f-1} \) is the direct sum \( D^{\lambda_3,f-1} \oplus D^{\lambda_2} \oplus D^{\lambda_1} \), a contradiction.

If \( \text{Rad}^3 S^{\lambda_2,f-1} \neq 0 \) then \( S^{\lambda_2,f-1} \) is uniserial whose top is \( D^{\lambda_2,f-1} \) and whose socle is \( D^{\lambda_2} \). This implies that there exists a uniserial module of the following form.

\[
\begin{align*}
D^{\lambda_1} \\
D^{\lambda_3,f-1} \\
D^{\lambda_2}
\end{align*}
\]

This contradicts to \( \text{Rad} P^{\lambda_1} / \text{Rad}^2 P^{\lambda_1} = D^{\lambda_2} \oplus D^{\lambda_2,f-1} \). Hence, the radical length of \( S^{\lambda_2,f-1} \) is 3 and the radical series has the following form.

\[
\begin{align*}
D^{\lambda_2,f-1} \\
D^{\lambda_1} \oplus D^{\lambda_3,f-1} \\
D^{\lambda_2}
\end{align*}
\]

As a conclusion, the Gabriel quiver of \( H_n(q,Q) \) contains the following quiver as a subquiver.

\[
\begin{align*}
\lambda_2 \quad \lambda_1 & \quad \lambda_2 \\
\lambda_3 & \quad \lambda_2 \\
\lambda_3 & \quad \lambda_2
\end{align*}
\]
Therefore, Lemma 17 shows that \( \mathcal{H}_n(q, Q) \) is wild in this case.

Next assume that we are in (case 2a). Then we have either \( e = 3 \) and \( f = 1 \) or \( e = 4 \) and \( f = 2 \).

First we consider the case where \( e = 3 \) and \( f = 1 \). Then, by Proposition 46, the decomposition matrix of the block algebra \( B \) is as follows.

|   | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_{2,0} \) | \( \lambda_{1,0} \) |
|---|----------------|----------------|----------------|----------------|
| \( \lambda_1 \) | 1               | 0              | 0              | 0              |
| \( \lambda_2 \) | 1               | 1              | 0              | 0              |
| \( \lambda_3 \) | 0               | 1              | 0              | 0              |
| \( \lambda_{2,0} \) | 1               | 1              | 1              | 0              |
| \( \lambda_{1,0} \) | 1               | 0              | 1              | 1              |
| \( \mu_1 \)      | 0               | 0              | 0              | 1              |
| \( \mu_2 \)      | 0               | 0              | 1              | 1              |
| \( \mu_3 \)      | 0               | 0              | 1              | 0              |

Thus, simple \( B \)-modules are all one dimensional and they are given by

\[
\begin{align*}
D^{\lambda_1} : T_0 &\mapsto q, \quad T_1, T_2 \mapsto -1, \\
D^{\lambda_2} : T_0 &\mapsto q, \quad T_1, T_2 \mapsto q, \\
D^{\lambda_{2,0}} : T_0 &\mapsto 1, \quad T_1, T_2 \mapsto q, \\
D^{\lambda_{1,0}} : T_0 &\mapsto 1, \quad T_1, T_2 \mapsto -1.
\end{align*}
\]

This is easy to see: observe that \( D^{\lambda_2} = S^{\lambda_3}, D^{\lambda_{2,0}} = S^{\mu_3} \) and \( D^{\lambda_{1,0}} = S^{\mu_1} \).

The block algebra \( B \) admits a symmetry. To see this, we define an algebra automorphism \( \omega \) of \( \mathcal{H}_n(q, Q) \) as follows.

\[
\omega : \begin{cases}
T_0 &\mapsto 1 + q - T_0 \,(= qT_0^{-1}) \\
T_i &\mapsto q - 1 - T_i \,(= -qT_i^{-1}) \quad (\text{for } i = 1, 2)
\end{cases}
\]

Then, \( \omega \) interchanges \( D^{\lambda_1} \) and \( D^{\lambda_{2,0}}, D^{\lambda_2} \) and \( D^{\lambda_{1,0}}, \) respectively.

**Lemma 48.** \( \text{Ext}_B^1(D^{\lambda_a}, D^{\lambda_b}) \neq 0 \) if and only if \( \{ \lambda_a, \lambda_b \} \) is one of

\( \{ \lambda_1, \lambda_2 \}, \{ \lambda_1, \lambda_{2,0} \}, \{ \lambda_{1,0}, \lambda_{2,0} \} \).

**Proof.** By the self-duality of simple modules, \( \text{Ext}_B^1(D^{\lambda_a}, D^{\lambda_b}) \neq 0 \) implies \( \text{Ext}_B^1(D^{\lambda_a}, D^{\lambda_b}) \neq 0 \), for any pair \( \{ \lambda_a, \lambda_b \} \).

As \( S^{\lambda_2} \) is an indecomposable module with Top \( S^{\lambda_2} = D^{\lambda_2} \) and \( \mathrm{Soc} \, S^{\lambda_2} = D^{\lambda_1} \), we have \( \text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_2}) \neq 0 \) and \( \text{Ext}_B^1(D^{\lambda_2}, D^{\lambda_1}) \neq 0 \). Applying \( \omega \), we also have \( \text{Ext}_B^1(D^{\lambda_1,0}, D^{\lambda_{2,0}}) \neq 0 \) and \( \text{Ext}_B^1(D^{\lambda_{2,0}}, D^{\lambda_{1,0}}) \neq 0 \).

To prove that \( \text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_{2,0}}) \neq 0 \), we consider

\[
T_0 \mapsto \begin{pmatrix} 1 & 1 - q \\ 0 & q \end{pmatrix}, \quad T_1 \mapsto \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}, \quad T_2 \mapsto \begin{pmatrix} q & 1 + q \\ 0 & -1 \end{pmatrix}.
\]

Then, this defines an indecomposable representation of \( \mathcal{H}_3(q, Q) \). Hence, the result follows. To prove that \( \text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_{2,0}}) = 0 \), we consider

\[
T_0 \mapsto \begin{pmatrix} 1 & \alpha \\ 0 & q \end{pmatrix}, \quad T_1 \mapsto \begin{pmatrix} -1 & \beta \\ 0 & -1 \end{pmatrix}, \quad T_2 \mapsto \begin{pmatrix} -1 & \gamma \\ 0 & -1 \end{pmatrix}.
\]
and require that they satisfy the defining relations. Since $T_i - q$, for $i = 1, 2$, are invertible in this case, $\beta = 0$ and $\gamma = 0$ follow. Then, as $q \neq 1$, we can diagonalize $T_0$. This implies that a short exact sequence

\[ 0 \rightarrow D^{\lambda_{1,0}} \rightarrow M \rightarrow D^{\lambda_1} \rightarrow 0 \]

always splits, proving the result. Applying $\omega$ we also get $\text{Ext}^1_B(D^{\lambda_{2,0}}, D^{\lambda_2}) = 0$.

Similarly, to prove that $\text{Ext}^1_B(D^{\lambda_{1,2}}, D^{\lambda_{1,0}}) = 0$, we consider

\[ T_0 \mapsto \begin{pmatrix} 1 & \alpha \\ 0 & q \end{pmatrix}, \ T_1 \mapsto \begin{pmatrix} -1 & \beta \\ 0 & q \end{pmatrix}, \ T_2 \mapsto \begin{pmatrix} -1 & \gamma \\ 0 & q \end{pmatrix}. \]

We may assume $\beta = 0$ because $q \neq -1$. Then, by requiring the defining relations, we get $\alpha = 0$ by $(T_0T_1)^2 = (T_1T_0)^2$ and $\gamma = 0$ by $T_0T_2 = T_2T_0$.

By the same argument, $q \neq \pm 1$ implies that self–extensions are all zero. □

Let $P = P^{\lambda_1} \oplus P^{\lambda_2} \oplus P^{\lambda_{1,0}} \oplus P^{\lambda_{2,0}}$, and define an $F$–algebra $A$ by

\[ A = \text{End}_B(P/\text{Rad}^3 P). \]

Our aim is to determine the algebra structure of $A$. To do this, we consider the radical series of each indecomposable projective module. Observe that $S^{\lambda_{i,0}}$, for $i = 1, 2$, is a submodule of $P^{\lambda_i}$ respectively. Thus, the radical series of $S^{\lambda_{1,0}}$ and $S^{\lambda_{2,0}}$ are as follows.

\[
\begin{align*}
D^{\lambda_{1,0}} & \quad D^{\lambda_{2,0}} \\
D^{\lambda_{2,0}} & \quad D^{\lambda_1} \\
D^{\lambda_1} & \quad D^{\lambda_2}
\end{align*}
\]

We start with $P^{\lambda_1}$. By Lemma \( \text{Rad} P^{\lambda_1} / \text{Rad}^2 P^{\lambda_1} = D^{\lambda_2} \oplus D^{\lambda_{2,0}}. \) As $\text{Soc}^3 P^{\lambda_1} / \text{Soc}^2 P^{\lambda_1}$ contains $D^{\lambda_{1,0}}, D^{\lambda_{1,0}}$ appears in $\text{Rad}^2 P^{\lambda_1} / \text{Rad}^3 P^{\lambda_1}$. By the Specht filtration, $P^{\lambda_1}$ has a submodule $W$ with

\[ 0 \rightarrow S^{\lambda_{1,0}} \rightarrow W \rightarrow S^{\lambda_{2,0}} \rightarrow 0. \]

We denote the pullback of $\text{Soc} S^{\lambda_{2,0}}$ to $W$ by $\text{Soc} S^{\lambda_{2,0}} + S^{\lambda_{1,0}}$ and define a $B$–module $V$ by

\[ V = (\text{Rad} P^{\lambda_1})/(\text{Soc} S^{\lambda_{2,0}} + S^{\lambda_{1,0}}). \]

Then we have a short exact sequence

\[ 0 \rightarrow D^{\lambda_{2,0}} \rightarrow \text{Ext}^1_B(D^{\lambda_1}, D^{\lambda_{2,0}}) \rightarrow V \rightarrow D^{\lambda_2} \rightarrow 0 \]

and $V/\text{Rad} V = D^{\lambda_2} \oplus D^{\lambda_{2,0}}$. Thus, $\text{Rad} V = D^{\lambda_1} \oplus D^{\lambda_{1,0}}$ by $\text{Ext}^1_B(D^{\lambda_1}, D^{\lambda_{1,0}}) = 0$ and $D^{\lambda_1} \oplus D^{\lambda_{1,0}}$ appears in $\text{Rad}^2 P^{\lambda_1} / \text{Rad}^3 P^{\lambda_1}$. Taking the Specht filtration into consideration, these imply that the radical series of $P^{\lambda_1} / \text{Rad}^3 P^{\lambda_1}$ is as follows.

\[
\begin{align*}
D^{\lambda_1} & \quad D^{\lambda_{2,0}} \\
D^{\lambda_2} & \quad D^{\lambda_{1,0}} \\
D^{\lambda_{1,0}} & \quad D^{\lambda_1} \oplus D^{\lambda_{1,0}}
\end{align*}
\]

Next we consider $P^{\lambda_2}$. By Lemma \( \text{Rad} P^{\lambda_2} / \text{Rad}^2 P^{\lambda_2} = D^{\lambda_1}. \) Thus, by taking the Specht filtration into consideration again, $D^{\lambda_{2,0}} / \text{Rad}^2 P^{\lambda_{2,0}} = S^{\lambda_2}$ and $\text{Rad}^2 P^{\lambda_2}$ has a submodule $F$ such that

\[ (\text{Rad}^2 P^{\lambda_2})/F = S^{\lambda_2} \simeq D^{\lambda_2}, \quad F = S^{\lambda_{2,0}}. \]
If the radical length of $\text{Rad}^2 P^\lambda_2$ was 4 then $P^\lambda_2$ would be a uniserial $B$–module and $\text{Rad}^3 P^\lambda_2 = S^{\lambda_2,0}$. This contradicts to the self–duality of $P^\lambda_2$. Thus, the radical length of $\text{Rad}^2 P^\lambda_2$ is 3. If $(\text{Rad}^2 P^\lambda_2)/F \simeq D^{\lambda_2}$ appeared in $\text{Rad}^3 P^\lambda_2$ then $D^{\lambda_2}$ would appear as a submodule of $\text{Soc}^2 P^\lambda_2/\text{Soc} P^\lambda_2$, which contradicts to $\text{Ext}_B^1(D^{\lambda_2}, D^{\lambda_2}) = 0$. Thus, $D^{\lambda_2}$ appears in $\text{Rad}^2 P^\lambda_2/\text{Rad}^3 P^\lambda_2$. On the other hand, as $\text{Top} F = D^{\lambda_2,0}$ appears in $\text{Soc}^3 P^\lambda_2/\text{Soc}^2 P^\lambda_2$, $D^{\lambda_2,0}$ appears in $\text{Rad}^2 P^\lambda_2/\text{Rad}^3 P^\lambda_2$. As a conclusion, the radical series of $P^\lambda_2/\text{Rad}^3 P^\lambda_2$ is as follows.

\[
\begin{align*}
D^{\lambda_2} \\
D^{\lambda_1} \\
D^{\lambda_2} & \oplus D^{\lambda_2,0}
\end{align*}
\]

Applying $\omega$ to the radical structure of $P^\lambda_1/\text{Rad}^3 P^\lambda_1$, we know that the radical series of $P^\lambda_{2,0}/\text{Rad}^3 P^\lambda_{2,0}$ is as follows.

\[
\begin{align*}
D^{\lambda_{2,0}} \\
D^{\lambda_{1,0}} & \oplus D^{\lambda_{1,0}} \\
D^{\lambda_{2,0}} & \oplus D^{\lambda_{2,0}} \oplus D^{\lambda_{2,0}}
\end{align*}
\]

Similarly, the radical series of $P^\lambda_{2,0}/\text{Rad}^3 P^\lambda_{2,0}$ is as follows.

\[
\begin{align*}
D^{\lambda_{1,0}} \\
D^{\lambda_{2,0}} \\
D^{\lambda_{1,0}} & \oplus D^{\lambda_{1,0}}
\end{align*}
\]

Set $D_1 = D^{\lambda_2}$, $D_2 = D^{\lambda_1}$, $D_3 = D^{\lambda_{2,0}}$ and $D_4 = D^{\lambda_{1,0}}$. Then, the Gabriel quiver of the block algebra $B$ is the directed graph $Q$ in Lemma \[\text{Lemma 22}\]. We have a surjective algebra homomorphism

\[
FQ/J^3 \longrightarrow A^{opp},
\]

where $J^3$ is the ideal of $FQ$ generated by paths of length 3. Let us compare the dimensions of $FQ/J^3$ and $A$. The computation above shows $\dim_F A = 20$. Hence, $\dim_F FQ/J^3 = 20$ implies that $FQ/J^3 \simeq A^{opp}$. Therefore, we have determined the algebra structure of $A$ and Lemma \[\text{Lemma 22}\] implies that $A$ is wild. Since $A$ is a factor algebra of $\text{End}_B(P)$ and $\text{End}_B(P)^{opp}$ is Morita–equivalent to $B$, we have shown that $\mathcal{H}_n(q, Q)$ with $n = e = 3, f = 1$ is wild.

Next we consider the case where $e = 4$ and $f = 2$. We have, by Proposition \[\text{Proposition 17}\] that

\[
\begin{align*}
[P^{\lambda_1}] & = [S^{\lambda_1}] + [S^{\lambda_2}] + [S^{\lambda_{2,1}}] + [S^{\lambda_{1,1}}] , \\
[P^{\lambda_2}] & = [S^{\lambda_2}] + [S^{\lambda_3}] + [S^{\lambda_{2,1}}] , \\
[P^{\lambda_3}] & = [S^{\lambda_3}] + [S^{\lambda_4}] + [S^{\lambda_{2,1}}] + [S^{\lambda_{2,0}}] ,
\end{align*}
\]

and

\[
\begin{align*}
[S^{\lambda_{1,1}}] & = [D^{\lambda_{1,1}}] + [D^{\lambda_{2,1}}] + [D^{\lambda_1}] , \\
[S^{\lambda_{2,1}}] & = [D^{\lambda_{2,1}}] + [D^{\lambda_1}] + [D^{\lambda_2}] + [D^{\lambda_1}] , \\
[S^{\lambda_{2,0}}] & = [D^{\lambda_{2,0}}] + [D^{\lambda_{2,1}}] + [D^{\lambda_1}] .
\end{align*}
\]
We argue as in (case 1a). Then the radical series of $S^{\lambda_2,1}$ has the following form.

$$
D^{\lambda_2,1} \\
D^{\lambda_1} \oplus D^{\lambda_3} \\
D^{\lambda_2}
$$

As $S^{\lambda_2,0}$ is a submodule of $P^{\lambda_3}$, $\text{Soc} S^{\lambda_2,0} = D^{\lambda_3}$. Thus, $S^{\lambda_2,0}$ is uniserial of the following form.

$$
D^{\lambda_2,0} \\
D^{\lambda_2,1} \\
D^{\lambda_3}
$$

As a conclusion, the Gabriel quiver contains the following quiver as a subquiver.

![Gabriel quiver](image)

Thus, Lemma 17 implies that $\mathcal{H}_n(q,Q)$ with $n = e = 4, f = 2$ is wild.

Now, suppose that we are in (case 3a). We consider the case where $e \geq 4$ first. Then, we have simple modules $D^{\lambda_i}$, for $i = 1, 2, 3$. Considering $S^{\lambda_2}$ and $S^{\lambda_3}$, we obtain that $\text{Ext}^1_B(D^{\lambda_i}, D^{\lambda_j}) \neq 0$, for $\{i, j\} = \{1, 2\}$ and $\{2, 3\}$. In [AM2], we introduced modules $M^{\lambda_i}$ after Proposition 4.24, and showed that the self-extensions of the simple modules $D^{\lambda_i}$ are non-zero. Thus, $\text{Ext}^1_B(D^{\lambda_i}, D^{\lambda_j}) \neq 0$, for $i = 2, 3$. Therefore, we can apply Lemma 23 proving that $\mathcal{H}_n(q,Q)$ is wild in this case.

Next suppose that $e = 3$. We compute the decomposition numbers in the columns labelled by $\lambda_1$ and $\lambda_2$ as before. Then, $D^{\lambda_1} = S^{\lambda_1}$ and $D^{\lambda_2} = S^{\lambda_3}$. So, the Specht module theory implies that simple $B$–modules $D^{\lambda_1}$ and $D^{\lambda_2}$ are given by

$$
D^{\lambda_1} : T_0 \mapsto 1, \quad T_1, T_2 \mapsto -1, \\
D^{\lambda_2} : T_0 \mapsto 1, \quad T_1, T_2 \mapsto q.
$$

Note that $T_0 \mapsto 2 - T_0, T_i \mapsto q - 1 - T_i$, for $i = 1, 2$, is an algebra automorphism of $\mathcal{H}_n(q,Q)$ in this case. We denote this automorphism by $\omega$ again. Then, $\omega$ interchanges $D^{\lambda_1}$ and $D^{\lambda_2}$. Let us consider the following representation.
We denote the corresponding $B$–module by $M$. The underlying space is $F^5$, the column vectors of dimension $5$. If we write $F^4$, this means the subspace consisting of the vectors of $F^5$ whose final entry is $0$.

**Lemma 49.** Top $M = D^\lambda_1$ and Rad$^2 M = \text{Soc } M = D^\lambda_1 \oplus D^\lambda_2$.

**Proof.** We can compute Top $M$ and Soc $M$ explicitly. In the computation of Top $M = D^\lambda_1$, we obtain Rad $M = F^4$. We can also show that Rad$^2 M = \text{Soc } M$ by computing Top(Rad $M$) explicitly. \(\Box\)

**Remark 50.** Rad $M$ is indecomposable because

$$\text{End}_B(\text{Rad } M) = FI_4 \oplus FE_{13} \oplus FE_{24} \subset M(4, 4, F),$$

is a local $F$–algebra. Here, $I_4$ is the identity matrix and $E_{ij}$ are matrix units.

As Top $M = D^\lambda_1$, $M$ is a factor module of $P^\lambda_1$. By twisting $M$ by $\omega$, we know that $M^\omega$ is a factor module of $P^\lambda_2$. Define

$$A = \text{End}_B(M \oplus M^\omega).$$

Then $A$ is a factor algebra of $\text{End}_B(P^\lambda_1 \oplus P^\lambda_2)$. We shall show that $A$ is wild. As $B$ is Morita–equivalent to $\text{End}_B(P^\lambda_1 \oplus P^\lambda_2)^{opp}$, this implies that $B$ is wild. Define

$$\alpha \in \text{End}_B(M) \subset M(5, 5, F) \text{ and } \beta \in \text{End}_B(M^\omega) \subset M(5, 5, F)$$

by the matrix $-q^2 E_{13} + E_{15} + E_{45}$, and define

$$\mu \in \text{Hom}_B(M, M^\omega) \subset M(5, 5, F) \text{ and } \nu \in \text{Hom}_B(M^\omega, M) \subset M(5, 5, F)$$

by the matrix $E_{14} + E_{23} + qE_{35}$. The embeddings into $M(5, 5, F)$ are given by the natural basis of $F^5$ which is the underlying space of $M$ and $M^\omega$. That they commute with $B$ is proved by explicit computations. Finally, define $e_1$ and $e_2$ to be the projectors to $M$ and $M^\omega$ respectively.

**Proposition 51.**

1. $A$ has the basis

$$\{e_1, e_2, \alpha, \beta, \mu, \nu, \mu \nu, \nu \mu, \mu \alpha, \nu \beta\}.$$

2. The following relations hold in $A$.

$$\alpha^2 = 0, \quad \beta^2 = 0, \quad \alpha \nu = -\nu \beta, \quad \beta \mu = -\mu \alpha,$$

$$\nu \beta \mu = 0, \quad \mu \alpha \nu = 0, \quad \mu \nu \mu = 0, \quad \nu \mu \nu = 0.$$
(3) Let $Q$ be the directed graph in Lemma 25, then $A \simeq FQ/I$ where the relations are as in (2).

**Proof.** These are proved by explicit computations. \qed

Let us consider the factor algebra $\overline{A}$ of $A$ which is defined by requiring one more relation $\nu \beta = 0$. Then, $\overline{A}$ has the defining relations

\[
\alpha^2 = 0, \quad \beta^2 = 0, \quad \alpha \nu = 0, \quad \nu \beta = 0, \\
\beta \mu = -\mu \alpha, \quad \mu \nu \mu = 0, \quad \nu \mu \nu = 0.
\]

By changing the sign of $\alpha$, we know that $\overline{A} \simeq \Lambda_{32}$. Hence, $A$ is wild by Lemma 25. We have proved that $H_n(q, Q)$ is wild in this case.

### 3.5. The remaining two cases are for $n = 2f + 4$.

Assume that we are in (case 4a). Following [AM2], we define bipartitions $\lambda_1 = ((0), (2f + 2)), \lambda_2 = ((1), (2f + 1, 1)), \lambda_3 = ((1^2), (2f + 1)), \lambda_4 = ((2), (2f, 1^2))$ and $\lambda_5 = ((2, 1), (2f, 1))$. They belong to the same block, which we denote by $B$.

**Lemma 52 (AM2 Lemma 5.6).** The bipartitions $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5$ are all Kleshchev. Furthermore, $[P^{\lambda_2}] = [S^{\lambda_2}] + [S^{\lambda_3}] + [S^{\lambda_4}] + [S^{\lambda_5}]$ and the first five rows of the columns of the decomposition matrix of $B$ labelled by $\lambda_i$, for $1 \leq i \leq 5$, are as follows.

|   | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ |
|---|-------------|-------------|-------------|-------------|-------------|
| $\lambda_1$ | 1           | 0           | 0           | 0           | 0           |
| $\lambda_2$ | 1           | 1           | 0           | 0           | 0           |
| $\lambda_3$ | 0           | 1           | 1           | 0           | 0           |
| $\lambda_4$ | 0           | 1           | 0           | 1           | 0           |
| $\lambda_5$ | 1           | 1           | 1           | 1           | 1           |

Note that $\text{Soc} S^{\lambda_5} = \Lambda_{32}$. Our aim is to show that the radical series of $S^{\lambda_5}$ has the following form.

\[
\Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32}
\]

Once this is proved, then the Gabriel quiver contains the following quiver as a subquiver.

\[
\begin{array}{c}
\lambda_5 \\
\lambda_1 \\
\lambda_4 \\
\lambda_2 \quad \lambda_3
\end{array}
\]

So, Lemma 14 implies that $H_n(q, Q)$ is wild in this case.

Now we determine the radical structure of $S^{\lambda_5}$. In (case 2) of the proof of [AM2 Theorem 5.25], we showed

\[
\text{Rad} P^{\lambda_2}/\text{Rad}^2 P^{\lambda_2} = \Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32}.
\]

This implies that

\[
\text{Soc}^2 S^{\lambda_5}/\text{Soc} S^{\lambda_5} \subset \text{Soc}^2 P^{\lambda_2}/\text{Soc} P^{\lambda_2} = \Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32} \oplus \Lambda_{32}.
\]
If the inclusion is strict, then one of \( D^{\lambda_1}, D^{\lambda_2}, D^{\lambda_3} \) appears in \( S^{\lambda_2}, S^{\lambda_3} \) or \( S^{\lambda_4} \). Note that these Specht modules are uniserial of the following form.

\[
S^{\lambda_2} = \begin{cases} 
D^{\lambda_2} & \text{if } 2 \text{ appears in } Soc^2 \end{cases}, \quad 
S^{\lambda_3} = \begin{cases} 
D^{\lambda_3} & \text{if } 3 \text{ appears in } Soc^2 \end{cases}, \quad 
S^{\lambda_4} = \begin{cases} 
D^{\lambda_4} & \text{if } 4 \text{ appears in } Soc^2 \end{cases}.
\]

Assume that \( D^{\lambda_1} \) does not appear in \( Soc^2 S^{\lambda_5} / Soc S^{\lambda_6} \). Then the \( D^{\lambda_1} \) which appears in \( Soc^2 P^{\lambda_2} / Soc P^{\lambda_2} \) must come from \( Soc S^{\lambda_2} \). So the \( D^{\lambda_1} \) also appears in \( Rad P^{\lambda_2} / Rad^2 P^{\lambda_2} \). This implies that the heart \( H(P^{\lambda_2}) \) of \( P^{\lambda_2} \) has the form

\[
H(P^{\lambda_2}) = D^{\lambda_1} \oplus M,
\]

where \( M \) is some \( B \)-module. Observe that

\[
(a) \text{ } M \text{ is self–dual because } H(P^{\lambda_2}) \text{ is self–dual,}
\]

\[
(b) \text{ } \text{Top } M = D^{\lambda_3} \oplus D^{\lambda_4},
\]

\[
(c) \text{ } M \text{ has a filtration whose successive quotients are } S^{\lambda_3}, S^{\lambda_4} \text{ and } S^{\lambda_5} / Soc S^{\lambda_6}.
\]

As \( Soc M = D^{\lambda_3} \oplus D^{\lambda_4} \), both \( Soc S^{\lambda_2} = D^{\lambda_2} \) and \( Soc S^{\lambda_3} = D^{\lambda_3} \) cannot appear in \( Soc M \). Thus, \( Soc(S^{\lambda_5} / Soc S^{\lambda_6}) = D^{\lambda_5} \oplus D^{\lambda_4} \). As \( S^{\lambda_5} \) has the unique maximal submodule \( Rad S^{\lambda_5} \), the radical series of \( S^{\lambda_5} / Soc S^{\lambda_6} \) has the following form.

\[
D^{\lambda_5} \oplus D^{\lambda_4} \oplus D^{\lambda_3} \oplus D^{\lambda_1}
\]

In particular, \( H(M) \) contains a uniserial module of length 2 whose top is \( D^{\lambda_5} \) and whose socle is \( D^{\lambda_1} \). As \( H(M) \) is also self–dual, \( H(M) \) must have a uniserial module of length 2 whose top is \( D^{\lambda_1} \) and whose socle is \( D^{\lambda_5} \) as a factor module. However, this is impossible because \( [H(M) : D^{\lambda_1}] = 1 \) and \( [H(M) : D^{\lambda_5}] = 1 \).

Next assume that \( D^{\lambda_3} \) does not appear in \( Soc^2 S^{\lambda_5} / Soc S^{\lambda_6} \). Then the \( D^{\lambda_3} \) which appears in \( Soc^2 P^{\lambda_2} / Soc P^{\lambda_2} \) must come from \( S^{\lambda_3} \). Then, \( S^{\lambda_3} \subset Soc^2 P^{\lambda_2} \). Thus, both \( Soc S^{\lambda_5} \) and \( Soc S^{\lambda_6} \) appear in \( Soc P^{\lambda_2} \), that is, \( D^{\lambda_2} \oplus D^{\lambda_3} \subset Soc P^{\lambda_2} \). This is a contradiction.

Assume that \( D^{\lambda_1} \) does not appear in \( Soc^2 S^{\lambda_5} / Soc S^{\lambda_6} \). Then, by the similar reason as in the previous case, we reach a contradiction.

\( Soc^2 S^{\lambda_5} / Soc S^{\lambda_6} = D^{\lambda_1} \oplus D^{\lambda_3} \oplus D^{\lambda_4} \) implies that the radical series of \( S^{\lambda_5} \) is in the desired form, since \( Top S^{\lambda_5} = D^{\lambda_5} \) and \( Soc S^{\lambda_5} = D^{\lambda_2} \).

Assume that we are in (case 5a). In [AM2] section 6, we introduced the notion of path sequences for bipartitions. Let \( A = ( 0 ) ( 1 ), B = ( 1 ) ( 0 ), C = ( 0 ) ( 0 ), D = ( 1 ) ( 1 ) \) and define \( x_r \) and \( x_l \) for \( x \in \{ a, b, c, d \} \) as in [loc. cit.]. As \( f = 0 \), we do not have to define \( x_m \) for \( x \in \{ a, b, c, d \} \). By [AM2] (6.28),(6.29) we have

\[
|a_l + a_r| = b_l + b_r, \quad n > (a_l + a_r)^2.
\]

Our assumption \( n < 9 \) implies that the number of \( A \)'s is at most 2. On the other hand, as in (case 1) in the proof of [AM2] Theorem 6.30, \( n < e \) implies that Specht modules which belongs to a block must have the same set of contents. Further, if \( S^\lambda \) and \( S^\mu \) belong to the block then the path sequence of \( \mu \) is obtained from that of \( \lambda \) by a sequence of interchanging \( A \) and \( B \).

**Definition 53.** For a bipartition \( \lambda = (\lambda(1), \lambda(2)) \), we denote \( (\lambda(2), \lambda(1)) \) by \( \lambda^\sharp \).

Note that \( \lambda^\sharp \) has the same set of contents as \( \lambda \) because of \( f = 0 \).
Lemma 54. Assume that $4 \leq n < \min\{e, 9\}$. If $B$ is a block algebra of $\mathcal{H}_n(q, -1)$ then the number of Specht modules belonging to $B$ is 1, 2 or 6. If the number is 1 then $B$ is semisimple. If the number is 2 then $B$ is Morita-equivalent to $F[X]/(X^2)$.

Proof. Choose a Specht module $S^\lambda$ which belongs to $B$ so that $\lambda^2 \lessdot \lambda$ does not hold, and let $a = a_l + a_r$ be the number of $A$'s in the path sequence of $\lambda$.

If $a = 0$ then we cannot interchanged $A$ and $B$, which implies that $S^\lambda$ is the unique Specht module which belongs to $B$. Thus, $P^\lambda = S^\lambda = D^\lambda$ and $B$ is semisimple.

If $a = 1$ then the number of Specht modules which belongs to $B$ is at most 2. If it is 1 then $B$ is semisimple. If it is 2, then $S^\lambda$ belongs to $B$. Observe that

$$\dim_F B = 2(\dim_F S^\lambda)^2, \quad P^\lambda \neq S^\lambda = D^\lambda.$$ 

Thus we can write $[P^\lambda] = [S^\lambda] + m[S^\lambda']$, for some $m \geq 1$, and

$$(m + 1)(\dim_F S^\lambda)^2 = \dim_F P^\lambda \cdot \dim_F D^\lambda \leq \dim_F B = 2(\dim_F S^\lambda)^2$$

implies that $m = 1$. Hence, $S^\lambda = D^\lambda$ by dimension counting, and we have that $D^\lambda$ is the unique simple $B$-module and that $P^\lambda$ is uniserial of length 2. Hence $B$ is Morita-equivalent to $F[X]/(X^2)$.

If $a \geq 2$ then $a = 2$ and the number of Specht modules which belongs to $B$ is at most 6. However, in Proposition 55 below, we will list all of these possibilities and compute the decomposition numbers explicitly. From the decomposition matrix we know that these six Specht modules always constitute a single block. 

To cover the case where $a = a_l + a_r = 2$, we shall prove the following.

Proposition 55. Assume that $4 \leq n < \min\{e, 9\}$. If $B$ is a block algebra of $\mathcal{H}_n(q, -1)$ such that the number of Specht modules belonging to $B$ is not 1 nor 2, then there are Kleshchev bipartitions $\lambda_1$ and $\lambda_2$, and a bipartition $\lambda_3$ which is not Kleshchev, such that the transpose of the decomposition matrix of $B$ has the following form.

| $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_3^2$ | $\lambda_3^3$ | $\lambda_3^4$ |
|---|---|---|---|---|---|
| $\lambda_1$ | 1 | 1 | 0 | 0 | 1 |
| $\lambda_2$ | 0 | 1 | 1 | 1 | 0 |

Further, there are bipartitions $\mu_1, \mu_2$ of smaller ranks such that, for $i = 1, 2$, the following hold.

(i) $D^{\mu_i} = S^{\mu_i}, \quad P^{\mu_i}$ is uniserial of length 2.

(ii) $P^{\lambda_i} = P^{\mu_i} \uparrow^B$.

(iii) If we set $M_i = D^{\mu_i} \downarrow^B$ then

$$[M_1] = [S^{\lambda_1}] + [S^{\lambda_2}] = 2[D^{\lambda_1}] + [D^{\lambda_2}],$$

$$[M_2] = [S^{\lambda_2}] + [S^{\lambda_3}] = [D^{\lambda_1}] + 2[D^{\lambda_2}].$$

(iv) Let $B_i$ be the block algebra which $D^{\mu_i}$ belongs, for $i = 1, 2$. Then

$$D^{\lambda_1} \downarrow_{B_1} = D^{\mu_1}, \quad D^{\lambda_1} \downarrow_{B_2} = 0,$$

$$D^{\lambda_2} \downarrow_{B_2} = D^{\mu_2}, \quad D^{\lambda_2} \downarrow_{B_1} = 0.$$
The verification of the conditions for $\mu_1, \mu_2$ with $a_1 + a_r = 1$; if $a = 1$ for $\mu_1$ and $\mu_i$ is Kleshchev then Lemma 54 implies (i). Then $[P^{\mu_1} \uparrow B] = 2[M_i] = [P^{\lambda_i}]$ implies (ii).

We start with the cases where no $S^\lambda$ with $\lambda^{(1)} = \emptyset$ belongs to $B$. Then, by explicit computation, we know that $n = 7$ or $n = 8$. The possibilities for $n = 7$ are

$$\lambda_1 = ((1), (3^2)), \quad \lambda_2 = ((2), (3, 2)), \quad \lambda_3 = ((3), (2^2))$$

and

$$\lambda_1 = ((1), (2^3)), \quad \lambda_2 = ((1^2), (2^2, 1)), \quad \lambda_3 = ((1^3), (2^2)).$$

In the first case,

$$G(\lambda_1) = f_1 f_2 f_0 f_1 f_{-1} f_0^{(2)}((0), (0)) = \lambda_1 + v \lambda_2 + v^2 \lambda_3,$$

and we can check that $\lambda_1, \lambda_2$ are Kleshchev and the other four bipartitions are not. Useful criteria for this are

1. Assume that $f = 0$. Then $\lambda$ with $\lambda^{(1)} = \emptyset$ and $\lambda^{(2)} = \emptyset$ is not Kleshchev.
2. Assume that $f = 0$ and $e > n$ and that both $\lambda^{(1)}$ and $\lambda^{(2)}$ are hooks. Then $\lambda$ is Kleshchev if and only if $\lambda^{(1)} \subset \lambda^{(2)}$.
3. Assume that $f = 0$ and $e > n = s + 4$. Then $((2^2), (s))$ and $((2^2), (1^s))$, for $s \geq 0$, are not Kleshchev.
4. Assume that $f = 0$ and $e > n = s + 4$. Then $((s), (2^2))$ and $((1^s), (2^2))$, for $s \geq 3$, are not Kleshchev.

Thus, the decomposition matrix follows.

Next, the possibilities for $n = 8$ are six cases given in the table below. We always have

$$G(\lambda_1) = \lambda_1 + v \lambda_2 + v^2 \lambda_3,$$

$$G(\lambda_2) = \lambda_2 + v \lambda_3 + v^2 \lambda_2.$$
We turn to the cases where \( S^\lambda \) with \( \lambda^{(1)} = \emptyset \) belongs to \( B \). For \( n = 4 \), there is only one possibility \( \lambda^{(2)} = (2^2) \). For \( n = 5 \), \( \lambda^{(2)} = (3, 2) \) or \( (2^2, 1) \). For \( n = 6 \), \( \lambda^{(2)} \) is one of \( (4, 2) \), \( (3^2) \), \( (3, 2, 1) \), \( (2^3) \), \( (2^2, 2^2) \). For \( n = 7 \), there are 8 possibilities: \( \lambda^{(2)} \) is one of \( (5, 2), (4, 3), (4, 2, 1), (3^2, 1) \) and their transposes. For \( n = 8 \), there are 14 possibilities: \( \lambda^{(2)} \) is one of

\[
(6, 2), (5, 3), (5, 2, 1), (4^2), (4, 3, 1), (4, 2^2), (4, 2, 1^2), (3^2, 2)
\]

and their transposes. Note that the last two are symmetric partitions.

In these cases, \( \lambda^{(2)} \) is of the form

\[
(j + 1, i + 2, 2^{k-1}, 1^{l-k}) \quad (0 \leq i < j, \ 1 \leq k \leq l)
\]

That is, \( \lambda^{(2)} \) is obtained from \((j + 1, 1^l)\) by adding the hook \((i + 1, 1^{k-1})\). Let \( \lambda_1 \) be the bipartition \( \lambda = (0, \lambda^{(2)}) \) and define \( \lambda_2 \) and \( \lambda_3 \) by

\[
\lambda_2 = ((i + 1, 1^{k-1}),(j + 1, 1^l)), \quad \lambda_3 = ((j + 1, 1^{k-1}),(i + 1, 1^l)).
\]

Then,

\[
G(\lambda_1) = f_{-k+1}\cdots f_{-1}f_i\cdots f_{0}f_{-1}\cdots f_{-l}f_j\cdots f_{0}(0,0)
\]

\[
= \lambda_1 + v\lambda_2 + v^2\lambda_1^2,
\]

\[
G(\lambda_2) = f_{-1}\cdots f_{-k}f_j\cdots f_{i+1}f_{j+1}(2)\cdots f_{-2}f_{i}(2)\cdots f_{0}(2)(0,0)
\]

\[
= \lambda_2 + v\lambda_3 + v^2\lambda_2^2.
\]

We can check that the other four bipartitions are not Kleshchev.

We define \( \mu_1 \) to be the bipartition obtained from \( \lambda_1 \) by deleting the hook \((i + 1, 1^{k-1})\), and \( \mu_2 \) to be the bipartition obtained from \( \lambda_2 \) by deleting \((j - i)\) nodes from the first row. It is easy to check the conditions for \( \mu_i \).

Now, we are in a position to determine the radical structure of \( P^{\lambda_1} \) and \( P^{\lambda_2} \) for all of the blocks with six Specht modules. Let \( B \) be such a block algebra of \( \mathcal{H}_n(q, -1) \). By Proposition 55, \( P^{\lambda_1} \) has a submodule which is isomorphic to \( M_i \), which implies that \( \text{Soc} M_i = D^{\lambda_i} \). As \( M_i \) is self-dual and \([M_i] = 2[D^{\lambda_i}] + [D^{\lambda_i}]^\perp\), \( M_i \) is uniserial of length 3, whose top and bottom are \( D^{\lambda_i} \).

Let \( n_i = [\mu_i] \), for \( i = 1, 2 \). As \( \mathcal{H}_n(q, -1) \) is projective as a right \( \mathcal{H}_n(q, -1) \)-module, the Eckmann–Shapiro lemma [22, Corollary 2.8.4] implies that

\[
\text{Ext}^1_B(M_i, D^{\lambda_i}) \simeq \text{Ext}^1_{\mathcal{H}_n(q,-1)}(D^\mu_1, D^{\lambda_i} \downarrow \mathcal{H}_n(q,-1)) = \text{Ext}^1_{B_1}(D^\mu_1, D^{\lambda_i} \downarrow B_1).
\]

Thus, \( \dim F \text{Ext}^1_B(M_i, D^{\lambda_i}) = \delta_{ij} \). Since \( M_i \) and \( D^{\lambda_i} \) is self-dual, this also implies that \( \dim F \text{Ext}^1_B(D^{\lambda_i}, M_j) = \delta_{ij} \). As \( 0 = \text{Hom}_B(D^{\lambda_i}, M_1) \to \text{Hom}_B(D^{\lambda_i}, M_1/\text{Soc} M_1) \to \text{Ext}^1_B(D^{\lambda_i}, D^{\lambda_i}) \to \text{Ext}^1_B(D^{\lambda_i}, M_1) = 0 \), we conclude that \( \text{Ext}^1_B(D^{\lambda_1}, D^{\lambda_2}) \simeq \text{Ext}^1_B(D^{\lambda_2}, D^{\lambda_1}) = F \).

Nextly, we consider the long exact sequences

\[
0 = \text{Hom}_B(D^{\lambda_i}, M_i/\text{Soc} M_i) \to \text{Ext}^1_B(D^{\lambda_i}, M_i) \to \text{Ext}^2_B(D^{\lambda_i}, M_i) \to \text{Ext}^1_B(D^{\lambda_i}, M_i/\text{Soc} M_i),
\]

for \( i = 1, 2 \). We show that \( f_i = 0 \). Assume to the contrary that \( f_i \neq 0 \). Then, \( \text{Ext}^1_B(D^{\lambda_i}, M_i) = F \) implies that \( \text{Ext}^1_B(D^{\lambda_1}, D^{\lambda_i}) = 0 \). Thus, \( H(P^{\lambda_i}) \) has the
simple head and the simple socle, and the heart of $H(P_{\lambda^i})$ has composition factors $2[D_{\lambda^i}]$. As $\text{Ext}_B^1(D_{\lambda^i}, D_{\lambda^i}) = 0$, the heart of $H(P_{\lambda^i})$ must be semisimple. Hence, $\text{Rad} P_{\lambda^i} / \text{Soc}^2 P_{\lambda^i}$ has the following radical structure.

$$D_{\lambda^j}$$

$$D_{\lambda^i} \oplus D_{\lambda^i}$$

where $j = 3 - i$. However, this contradicts to $\text{Ext}_B^1(D_{\lambda^i}, D_{\lambda^i}) = F$. We have proved that $f_i = 0$. This implies that $\text{Ext}_B^1(D_{\lambda^i}, D_{\lambda^i}) = F$, for $i = 1, 2$. In particular, the bipartite graph associated with the Gabriel quiver of $B$ is not a Dynkin diagram and Theorem 28 implies that $B$ is not finite.

We shall show

$$\text{Hom}(D_{\lambda^i}, D_{\lambda^i}), \quad \text{Hom}(D_{\lambda^i}, D_{\lambda^i}) = D_{\lambda^i} \bigoplus D_{\lambda^i}.$$ 

First suppose that $H(P_{\lambda^i})$ has radical length 2. $H(P_{\lambda^i})$ has Specht filtration whose successive quotients are $S_{\lambda^2}$ and $S_{\lambda^2}^\sharp$. As $S_{\lambda^2}^\sharp$ is a submodule of $P_{\lambda^2}$, the decomposition matrix tells us that $S_{\lambda^2}^\sharp$ is uniserial of length 2 with $\text{Top} S_{\lambda^2}^\sharp = D_{\lambda^1}$ and $\text{Soc} S_{\lambda^2}^\sharp = D_{\lambda^2}$, which is the dual of $S_{\lambda^2}$. Our assumption implies that

$$H(P_{\lambda^1}) = S_{\lambda^2} \oplus S_{\lambda^2}^\sharp.$$ 

Let $\mu \in \text{Hom}_B(P_{\lambda^1}, P_{\lambda^2})$ be such that $\text{Im} \mu$ is equal to $D_{\lambda^1} \subset \text{Top}(\text{Rad} P_{\lambda^2})$ modulo $\text{Rad}^2 P_{\lambda^2}$. $\text{Im} \mu$ has the simple socle $\text{Soc} P_{\lambda^2}$. Thus, $\mu$ factors through $P_{\lambda^1} / \text{Soc} P_{\lambda^1}$ and $D_{\lambda^1} = \text{Soc} S_{\lambda^2} \subset \text{Soc}(P_{\lambda^1} / \text{Soc} P_{\lambda^1})$ must vanish. Therefore, $\text{Im} \mu$ is uniserial of the following form.

$$D_{\lambda^1} \quad D_{\lambda^1} \quad D_{\lambda^2}$$

As $H(P_{\lambda^2})$ contains the submodule $\text{Im} \mu / \text{Soc} P_{\lambda^2}$ and $H(P_{\lambda^2})$ is self-dual, this implies

$$0 \rightarrow D_{\lambda^2} \rightarrow \text{Ext}_B^1(D_{\lambda^2}, D_{\lambda^2}) \rightarrow H(P_{\lambda^2}) \rightarrow D_{\lambda^1} \rightarrow 0$$

because $\text{Ext}_B^1(D_{\lambda^2}, D_{\lambda^2}) = F$ implies that $[\text{Soc}(H(P_{\lambda^2})) : D_{\lambda^2}] = 1$. Since we have the submodule $\text{Im} \mu / \text{Soc} P_{\lambda^2}$, this exact sequence splits. This leads to the following contradiction.

$$M_2 / \text{Soc} M_2 \simeq D_{\lambda^2} \subset H(P_{\lambda^2}) \simeq D_{\lambda^1} \bigoplus D_{\lambda^2}.$$ 

Hence, the radical length of $H(P_{\lambda^1})$ is greater than 2.

Recall that $\text{Rad} P_{\lambda^1} / \text{Rad}^2 P_{\lambda^1} = D_{\lambda^1} \oplus D_{\lambda^2}$. Further, $D_{\lambda^1}$ must appear in $\text{Rad}^2 P_{\lambda^1} / \text{Rad}^3 P_{\lambda^1}$ because $S_{\lambda^2}$ is a factor module of $\text{Rad} P_{\lambda^1}$. Hence the radical
series of $P^{\lambda_1}$ is as follows.

$$
\begin{align*}
D^{\lambda_1} \\
D^{\lambda_1} \oplus D^{\lambda_2} \\
D^{\lambda_1} \\
D^{\lambda_2} \\
D^{\lambda_1}
\end{align*}
$$

Let $\nu \in \text{Hom}_B(P^{\lambda_2}, P^{\lambda_1})$ be such that $\text{Im} \nu$ is equal to $D^{\lambda_2} \subset \text{Top}(\text{Rad} P^{\lambda_1})$ modulo $\text{Rad}^2 P^{\lambda_1}$. We also choose $\alpha \in \text{End}_B(P^{\lambda_1})$ in such a way that $\text{Im} \alpha$ is equal to $D^{\lambda_1} \subset \text{Top}(\text{Rad} P^{\lambda_1})$ modulo $\text{Rad}^2 P^{\lambda_1}$.

If $\text{Rad}(\text{Im} \alpha)$ contained $\text{Rad}^2 P^{\lambda_1}$ then $\text{Rad}^2(\text{Im} \alpha)/\text{Rad}^3(\text{Im} \alpha) = D^{\lambda_2}$ would appear in $\text{Rad}^2 P^{\lambda_1}/\text{Rad}^3 P^{\lambda_1}$, a contradiction. Thus $\text{Rad}(\text{Im} \nu)$ contains $\text{Rad}^2 P^{\lambda_1}$:

$$
\begin{align*}
D^{\lambda_2} \\
D^{\lambda_1} \\
D^{\lambda_2} \\
D^{\lambda_1}
\end{align*}
$$

We have the following exact sequence.

$$
0 \rightarrow \text{Ker} \nu = \\
D^{\lambda_2} \rightarrow \text{Rad} P^{\lambda_2} ightarrow \text{Rad}(\text{Im} \nu) = D^{\lambda_2} \rightarrow 0
$$

As $\text{Top}(\text{Rad} P^{\lambda_2}) = D^{\lambda_1} \oplus D^{\lambda_2}$, this implies that $H(P^{\lambda_2}) = D^{\lambda_2} \oplus N_2$, for some submodule $N_2$. Now $N_2 \simeq \text{Rad} P^{\lambda_2}/\text{Ker} \nu$ implies that $H(P^{\lambda_2})$ has the desired form. In particular, the radical series of $P^{\lambda_2}$ is as follows.

$$
\begin{align*}
D^{\lambda_2} \\
D^{\lambda_1} \oplus D^{\lambda_2} \\
D^{\lambda_2} \\
D^{\lambda_1} \\
D^{\lambda_2}
\end{align*}
$$

Let $\beta \in \text{End}_B(P^{\lambda_2})$ be such that $\text{Im} \beta$ is equal to $D^{\lambda_2} \subset \text{Top}(\text{Rad} P^{\lambda_2})$ modulo $\text{Rad}^2 P^{\lambda_2}$. More precisely, we choose $\beta$ as follows.

$$
\text{Im} \beta / \text{Soc} P^{\lambda_2} = D^{\lambda_2} \subset D^{\lambda_2} \oplus N_2 = H(P^{\lambda_2}).
$$

As $\text{Rad}(\text{Im} \beta) = \text{Rad}^4 P^{\lambda_2}$, $\text{Rad}(\text{Im} \mu)$ contains $\text{Rad}^2 P^{\lambda_2}$:

$$
\begin{align*}
D^{\lambda_1} \\
D^{\lambda_2} \\
D^{\lambda_1} \\
D^{\lambda_2}
\end{align*}
$$
This implies that we can choose $N_2 = \text{Im} \mu / \text{Soc} P^{\lambda_2}$ and
$$H(P^{\lambda_2}) = \text{Im} \beta / \text{Soc} P^{\lambda_2} \oplus \text{Im} \mu / \text{Soc} P^{\lambda_2}.$$ 

If we consider
$$0 \longrightarrow \text{Ker} \mu = D^{\lambda_1} \longrightarrow D^{\lambda_2} \longrightarrow \text{Rad} P^{\lambda_1} \longrightarrow \text{Rad}(\text{Im} \mu) = D^{\lambda_1} \longrightarrow 0,$$
and argue in the same way as above, we can write $H(P^{\lambda_1}) = D^{\lambda_1} \oplus N_1$, for some $N_1$, and then we can choose $\alpha$ so that $\text{Im} \alpha / \text{Soc} P^{\lambda_1} = D^{\lambda_1}$ and $N_1 = \text{Im} \nu / \text{Soc} P^{\lambda_1}$. Thus, $H(P^{\lambda_1})$ has the desired form and
$$H(P^{\lambda_1}) = \text{Im} \alpha / \text{Soc} P^{\lambda_2} \oplus \text{Im} \nu / \text{Soc} P^{\lambda_1}.$$ 

It is now straightforward to check that $\text{End}_B(P^{\lambda_1} \oplus P^{\lambda_2})$ is generated by $\alpha, \beta, \mu, \nu$ subject to the relations
$$\mu \alpha = 0, \alpha \nu = 0, \beta \mu = 0, \nu \beta = 0,$$
$$\alpha^2 = (\nu \mu)^2, \beta^2 = (\mu \nu)^2.$$ 

The basis is given by
$$\{e_1, e_2, \alpha, \beta, \mu, \nu, \alpha^2, \beta^2, \mu \nu, \mu \mu, \nu \nu\}.$$ 

This is a special biserial algebra and we have already proved that it is not finite. Thus, Theorem 32 implies that $B$ is tame, and Lemma 37 says that there is an indecomposable $B$-module with complexity 2.

Finally, assume that we are in (case 6a). Define $\lambda_i$, for $1 \leq i \leq 10$, by
$$\lambda_1 = ((0), (3^3)), \lambda_2 = ((1), (3^2, 2)), \lambda_3 = ((1^2), (3^2, 1)), \lambda_4 = ((2), (3, 2^2)),$$
$$\lambda_5 = ((1^3), (3^2)), \lambda_6 = ((2, 1), (3, 2, 1)), \lambda_7 = ((3), (2^3)),$$
$$\lambda_8 = ((2, 1^2), (3, 2)), \lambda_9 = ((2^2), (3, 2^2)), \lambda_{10} = ((3, 1), (2^2, 1)).$$

Then, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_6$ are Kleshchev and the others are not. The canonical basis elements are as follows.

$$G(\lambda_1) = f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}((0), (0))$$
$$= \lambda_1 + v \lambda_2 + u \lambda_6 + v^2 \lambda_9 + v \lambda_5 + v^2 \lambda_6 + v \lambda_1^5$$
$$G(\lambda_2) = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}(2)((0), (0))$$
$$= \lambda_2 + v \lambda_3 + u \lambda_4 + v^2 \lambda_5 + v^2 \lambda_6 + v^3 \lambda_7 + v \lambda_3^2 + v^3 \lambda_9$$
$$G(\lambda_3) = f_1 f_2 f_3 f_4 f_5 f_6 f_{10}(2)((0), (0))$$
$$= \lambda_3 + v \lambda_5 + v \lambda_6 + v^2 \lambda_8 + v \lambda_3^5 + v^2 \lambda_7 + v \lambda_3^2$$
$$G(\lambda_4) = f_2 f_3 f_4 f_5 f_6 f_{10}(2)((0), (0))$$
$$= \lambda_4 + v \lambda_6 + u \lambda_7 + v^2 \lambda_10 + v^2 \lambda_10^2 + v^2 \lambda_6 + v \lambda_3^2$$
$$G(\lambda_6) = f_2 f_3 f_4 f_{10}(2)(2)((0), (0))$$
$$= \lambda_6 + v \lambda_8 + u \lambda_9 + v^2 \lambda_10 + v^2 \lambda_6 + v \lambda_3^2 + v \lambda_3^2.$$
Thus, the bipartitions \( \lambda_i \) and \( \lambda_1^3 \), for \( 1 \leq i \leq 10 \), form a block algebra of \( \mathcal{H}_9(q, -1) \), which we denote by \( B \). We have

\[
S^{\lambda_2} = D^{\lambda_2}, \quad S^{\lambda_3} = D^{\lambda_3}, \quad S^{\lambda_4} = D^{\lambda_4}.
\]

Let \( \mu_1 = ((1^2), (3^2)) \) and \( \mu_2 = ((2, 1), (3, 2)) \). These bipartitions appeared in the last line of the table in (case 5a) as \( \lambda_1 \) and \( \lambda_2 \). Thus, \( D^{\mu_1} \) and \( D^{\mu_2} \) belong to a tame block, say \( B' \), of \( \mathcal{H}_8(q, -1) \) and we have the following.

\[
S^{\mu_1} = D^{\mu_1}, \quad S^{\mu_2} = D^{\mu_2}.
\]

Hence, by \( [S^{\mu_1} \uparrow B] = [S^{\lambda_1}] + [S^{\lambda_5}] \) and \( [S^{\mu_2} \uparrow B] = [S^{\lambda_3}] + [S^{\lambda_8}] \), we have

\[
[D^{\mu_2} \uparrow B] = [D^{\lambda_1}] + [D^{\lambda_3}] + 2[D^{\lambda_6}].
\]

Now \( D^{\lambda_1} = S^{\lambda_1} \) and \( D^{\lambda_4} = S^{\lambda_4} \) imply \( D^{\lambda_1} \downarrow B = 0, D^{\lambda_4} \downarrow B = 0 \), thus

\[
\dim \text{Hom}_B(D^{\mu_2} \uparrow B, D^{\lambda_i}) \leq \dim \text{Hom}_{B'}(D^{\mu_2}, D^{\lambda_i} \downarrow \mathcal{H}_8(q, -1)) = 0,
\]

for \( i = 1, 4 \). Hence \( D^{\mu_2} \uparrow B \) is not semisimple. If the radical length was 2 then \( D^{\lambda_1} \oplus D^{\lambda_4} \) would appear in \( \text{Soc}(D^{\mu_2} \uparrow B) \). However, as \( D^{\mu_2} \uparrow B \) is self-dual, \( D^{\lambda_1} \oplus D^{\lambda_4} \) would also appear in \( \text{Top}(D^{\mu_2} \uparrow B) \), a contradiction. If \( D^{\lambda_6} \) appeared twice in \( \text{Top}(D^{\mu_2} \uparrow B) \), \( D^{\lambda_6} \oplus D^{\lambda_6} \) would be a direct summand of \( D^{\mu_2} \uparrow B \). Then \( D^{\lambda_1} \) or \( D^{\lambda_4} \) would appear in \( \text{Top}(D^{\mu_2} \uparrow B) \), a contradiction again. Therefore,

\[
\text{Top}(D^{\mu_2} \uparrow B) = D^{\lambda_6} = \text{Soc}(D^{\mu_2} \uparrow B)
\]

and the heart of \( D^{\mu_2} \uparrow B \) must be \( D^{\lambda_1} \oplus D^{\lambda_4} \) as it is self-dual. We have proved that the radical series of \( D^{\mu_2} \uparrow B \) is as follows.

\[
D^{\lambda_6} \oplus D^{\lambda_4} \oplus D^{\lambda_6}
\]

To summarize, the Gabriel quiver contains the following quiver as a subquiver.

\[
\begin{array}{c}
\lambda_6 \\
\lambda_1 \\
\lambda_4 \\
\lambda_2 \\
\lambda_3
\end{array}
\]

Thus, Lemma 17 implies that \( \mathcal{H}_9(q, -1) \) with \( e > n = 9, f = 0 \) is wild.

**3.6.** In this subsection we prove Theorem 12(2). Thus, we assume that \( e = 2 \). This implies that \( e \leq 2f + 4 \). As we also assume that \( 0 \leq f \leq \frac{e}{2} \), we have \( f = 0 \) or \( f = 1 \). The cases to consider are as follows.

- **(case 1b)** \( n = 2 \) and \( f = 0 \).
- **(case 2b)** \( n = 2 \) and \( f = 1 \).
- **(case 3b)** \( n = 3 \) and \( f = 0 \).
- **(case 4b)** \( n = 3 \) and \( f = 1 \).
- **(case 5b)** \( n = 4 \) and \( f = 1 \).
Our aim is to show that \( H_n(q, Q) \) is tame in (case 1b), (case 2b) and (case 4b), and wild in (case 3b) and (case 5b). We also prove that, in the tame cases, the tame block algebras of \( H_n(q, Q) \) are special biserial algebras. We will also show, by using Lemma 36 and Lemma 37, that there is an indecomposable \( H_n(q, Q) \)-module with complexity 2 in these cases.

Before going into the case–by–case analysis, we write the first three layers of the crystal graphs \( B(2\Lambda_0) \) and \( B(\Lambda_0 + \Lambda_1) \). As was explained before, the nodes of each of the layers of the crystal graphs parametrize simple \( H_n(-1, -1) \)-modules and simple \( H_n(-1, 1) \)-modules, for \( n = 1, 2, 3 \), respectively. Further, we can compute the full decomposition matrices in these cases.

First assume that we are in (case 1b). Then, \( H_2(q, Q) \) has two blocks. One is a semisimple block algebra \( \text{End}_F(S((1), (1))) \). We denote the other block algebra by \( B \). Let \( \lambda_1 = ((0), (1^2)) \). Then, \( B \) has the unique simple module \( D_{\lambda_1} \) and \([P_{\lambda_1} : D_{\lambda_1}] = 4\). If \( P_{\lambda_1} \) is uniserial, then \( B \) is finite by Lemma \[ \text{AM2} \]. However, \( B \) is not finite by \[ \text{AM2} \text{ Theorem 1.4} \]. Thus, the radical series of \( P_{\lambda_1} \) has the following form.

\[
D_{\lambda_1} \oplus D_{\lambda_1} \oplus D_{\lambda_1} \oplus D_{\lambda_1}
\]

End\(_B\)(\( P_{\lambda_1} \)) is the Kronecker algebra \( F[X, Y]/(X^2, Y^2) \), which is a special biserial algebra. Thus, \( B \) is tame. Applying Lemma \[ \text{AM2} \] we know that all simple \( B \)-modules have complexity 2.
Next assume that we are in (case 2b). Then $\mathcal{H}_2(q, Q)$ has one block. So we write $B$ for $\mathcal{H}_2(q, Q)$. We denote $((0), (1^2))$ and $((1), (1))$ by $\lambda_1$, $\lambda_2$ respectively. Then, the decomposition matrix of $B$ is as follows.

$$
\begin{array}{c|cc}
\lambda_1 & \lambda_2 \\
((0), (1^2)) & 1 & 0 \\
((0), (2)) & 1 & 0 \\
((1), (1)) & 1 & 1 \\
((1^2), (0)) & 0 & 1 \\
((2), (0)) & 0 & 1 \\
\end{array}
$$

Thus, $D^{\lambda_1}$ and $D^{\lambda_2}$ are given by

$$
D^{\lambda_1} : T_0 \mapsto -1, \quad T_1 \mapsto -1,
$$

$$
D^{\lambda_2} : T_0 \mapsto 1, \quad T_1 \mapsto -1,
$$

Let us consider the following three representations.

$$
\begin{align*}
T_0 & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & T_1 & \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \\
T_0 & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & T_1 & \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \\
T_0 & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & T_1 & \mapsto \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.
\end{align*}
$$

These are indecomposable representations. So, $\text{Ext}_B^1(D^{\lambda_i}, D^{\lambda_j}) \neq 0$, for all $i$ and for all $j$. Hence, the radical structure of $P^{\lambda_1}$ and $P^{\lambda_2}$ are as follows.

$$
\begin{array}{ccc}
D^{\lambda_1} & D^{\lambda_2} \\
D^{\lambda_1} \oplus D^{\lambda_2} & D^{\lambda_1} \oplus D^{\lambda_2} \\
D^{\lambda_1} & D^{\lambda_2} \\
\end{array}
$$

Observe that there is a uniserial submodule $U^i_j$ of $\text{Rad} P^{\lambda_j}$ whose top is $D^{\lambda_i}$ and whose socle is $\text{Soc} P^{\lambda_j} = D^{\lambda_j}$.

Define $\alpha \in \text{End}_B(P^{\lambda_1})$ and $\beta \in \text{End}_B(P^{\lambda_2})$ by $\text{Im} \alpha = U^1_1$, $\text{Im} \beta = U^2_2$. Then, $\text{Ker} \alpha = U^2_1$, $\text{Ker} \beta = U^1_2$.

Similarly, define $\mu \in \text{Hom}_B(P^{\lambda_1}, P^{\lambda_2})$ and $\nu \in \text{Hom}_B(P^{\lambda_2}, P^{\lambda_1})$ by $\text{Im} \mu = U^1_2$, $\text{Im} \nu = U^2_1$. Then, $\text{Ker} \mu = U^1_1$, $\text{Ker} \nu = U^2_2$.

Therefore, $\text{End}_B(P^{\lambda_1} \oplus P^{\lambda_2}) \simeq FQ/I$ where $Q$ is the directed graph with adjacency matrix \(
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\) and the relations are given by

$$
\mu \alpha = 0, \quad \beta \mu = 0, \quad \alpha \nu = 0, \quad \nu \beta = 0,
$$

$$
\alpha^2 = \nu \mu, \quad \beta^2 = \mu \nu,
$$

because both $\text{End}_B(P^{\lambda_1} \oplus P^{\lambda_2})$ and $FQ/I$ are 8–dimensional. Note that $FQ/I$ has the basis

$$
\{ e_1, e_2, \alpha, \beta, \mu, \nu, \alpha^2, \beta^2 \}.\]
Since $FQ/I$ is a special biserial algebra, $B$ is tame or finite. However, as $B$ is not finite by [AM2] Theorem 1.4, $B$ is tame. Applying Lemma 37 again, we know that all simple $B$–modules have complexity 2.

Now, we assume that we are in (case 3b). Then, $H_3(q, Q)$ has two blocks. One has $D((0),(2,1))$ as the unique simple module and $[P((0),(2,1)) : D((0),(2,1))] = 2$. Thus, $P((0),(2,1))$ is uniserial of length 2. Then, Lemma 1 implies that this block algebra is finite.

We denote the other block algebra by $B$. We write $\lambda_1$ and $\lambda_2$ for $((0),(1^3))$ and $((1),(1^2))$ respectively. Then, by the computation of the canonical basis as before, $B$ has the following decomposition matrix.

\[
\begin{array}{c|cc}
(\lambda_1) & \lambda_1 & \lambda_2 \\
((0),(1^3)) & 1 & 0 \\
((0),(3)) & 1 & 0 \\
((1),(1^2)) & 1 & 1 \\
((1),(2)) & 1 & 1 \\
((1^2),(1)) & 1 & 1 \\
((2),(1)) & 1 & 1 \\
((1^3),(0)) & 1 & 0 \\
((3),(0)) & 1 & 0 \\
\end{array}
\]

As the dual of $S^\lambda_2$ is uniserial of length 2 whose top is $D^\lambda_1$ and whose socle is $D^\lambda_2$, we have $\text{Ext}_B^1(D^\lambda_1, D^\lambda_2) \neq 0$. We shall show that $\text{Ext}_B^1(D^\lambda_1, D^\lambda_1) = F^2$. Then, the Gabriel quiver of $B$ contains the directed graph $Q$ with adjacency matrix $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$. Thus, Lemma 29 implies that $B$ is wild.

The computation of $\text{Ext}_B^1(D^\lambda_1, D^\lambda_1)$ is easy. In fact, $D^\lambda_1$ is given by

\[
T_0 \mapsto 1, \ T_1 \mapsto -1, \ T_2 \mapsto -1,
\]

and the result is a family of representations

\[
T_0 \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ T_1, T_2 \mapsto \begin{pmatrix} -1 & y \\ 0 & -1 \end{pmatrix}, \ \text{where} \ x, y \in F.
\]

Denote the corresponding module by $M(x, y)$. This is indecomposable if and only if $(x, y) \neq (0, 0)$. Assume that $M(x, y)$ and $M(x', y')$ are indecomposable. Then, $M(x, y) \cong M(x', y')$ if and only if $(x, y)$ and $(x', y')$ define the same element in the projective space $P^1(F)$.

Assume that we are in (case 4b). Then, $H_3(q, Q)$ has two blocks $B$ and $B'$, and if we denote $((0),(2,1))$ and $((1),(1^2))$ by $\lambda_1$, $\lambda_2$, and $((0),(1^3))$ and $((1^2),(1))$ by $\lambda'_1$, $\lambda'_2$, then the decomposition matrices are as follows.

\[
B : \begin{array}{c|cc}
(\lambda_1) & \lambda_1 & \lambda_2 \\
((0),(2,1)) & 1 & 0 \\
((1),(1^2)) & 1 & 1 \\
((1),(2)) & 1 & 1 \\
((1^3),(0)) & 0 & 1 \\
((3),(0)) & 0 & 1 \\
\end{array} \quad \quad B' : \begin{array}{c|cc}
(\lambda'_1) & \lambda'_1 & \lambda'_2 \\
((0),(1^3)) & 1 & 0 \\
((0),(3)) & 1 & 0 \\
((1^2),(1)) & 1 & 1 \\
((2),(1)) & 1 & 1 \\
((2,1),(0)) & 0 & 1 \\
\end{array}
\]
Thus, $D^{\lambda_2} = S((1^3),(0))$ and $D^{\lambda_1} = S((0),(1^2))$ imply that

\[
D^{\lambda_2} : T_0 \mapsto 1, \ T_1 \mapsto -1, \ T_2 \mapsto -1,
\]

\[
D^{\lambda_1} : T_0 \mapsto -1, \ T_1 \mapsto -1, \ T_2 \mapsto -1.
\]

We have an algebra automorphism defined by $T_0 \mapsto -T_0$ and $T_i \mapsto -2 - T_i$, for $i = 1, 2$. We denote this automorphism by $\omega$ again. Then $\omega$ interchanges $D^{\lambda_2}$ and $D^{\lambda_1}$, which implies that $\omega$ interchanges $B$ and $B'$. Hence, to show that $\mathcal{H}_3(q,Q)$ is tame, it is enough to consider $B$ only.

As $S((1^3),(2))$ is a submodule of $P^{\lambda_1}$, there is a uniserial module whose top is $D^{\lambda_2}$ and whose socle is $D^{\lambda_1}$. Hence, $\text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_2}) \neq 0$.

**Lemma 56.** (1) Choosing a suitable basis of $D^{\lambda_1}$, $D^{\lambda_1}$ can be represented by the following matrices.

\[
T_0 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_1 \mapsto \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad T_2 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.
\]

(2) $\text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_1}) = 0$ and $\text{Ext}_B^1(D^{\lambda_2}, D^{\lambda_2}) \neq 0$.

**Proof.** As $D^{\lambda_1} = S^{\lambda_1}$, we can obtain the matrix representation by reduction from the seminormal representation. To obtain the same matrix representation as above, we take a nonzero vector $v$ such that

\[
L_1v = qv, \quad L_2v = q^2v, \quad L_3v = v, \quad T_1v = qv,
\]

where $L_1 = T_0$, $L_2 = q^{-1}T_1T_0T_1$ and $L_3 = q^{-2}T_2T_1T_0T_1T_2$. We choose the basis \{v, $T_2v$\}. Then, after some computations, we obtain (1). Now we consider (2). As the second assertion is easy to verify, we focus on the first assertion. Verification of the details is left to the reader. Let $T_i$, for $i = 0, 1, 2$, be the matrices given in (1) and define $\tilde{T}_i$ as follows.

\[
\tilde{T}_0 = \begin{pmatrix} T_0 & X \\ 0 & T_0 \end{pmatrix}, \quad \tilde{T}_1 = \begin{pmatrix} T_1 & Y \\ 0 & T_1 \end{pmatrix}, \quad \tilde{T}_2 = \begin{pmatrix} T_2 & Z \\ 0 & T_2 \end{pmatrix},
\]

where $X, Y, Z \in M(2,2,F)$. We require the defining relations and write down the relations among $X, Y, Z$. The quadratic relations imply that $X$ is the zero matrix and $Y, Z$ are given by

\[
Y = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} c & d \\ 2c+d & -c \end{pmatrix}, \quad \text{where} \ a, b, c, d \in F.
\]

Further, the braid relation between $\tilde{T}_1$ and $\tilde{T}_2$ implies $2a + b = 2c + d$. Then, using these equalities, we compute the socle of this 4-dimensional matrix representation. Then, the 2-dimensional subspaces

\[
F \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \oplus F \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad F \begin{pmatrix} 0 \\ a \\ 1 \\ 0 \end{pmatrix} \oplus F \begin{pmatrix} c-a \\ b \\ 0 \\ 1 \end{pmatrix}
\]

are submodules, both of which are isomorphic to $D^{\lambda_1}$. This shows that a short exact sequence

\[
0 \rightarrow D^{\lambda_1} \rightarrow M \rightarrow D^{\lambda_1} \rightarrow 0
\]

always splits. Hence the result. \[\square\]
We shall determine the heart $H(P_{\lambda_i})$ of $P_{\lambda_i}$, for $i = 1, 2$. More precisely, our goal is to show

$$H(P_{\lambda_1}) = D^{\lambda_2} \oplus D^{\lambda_1} \quad \text{and} \quad H(P_{\lambda_2}) = D^{\lambda_2} \oplus D^{\lambda_1}. $$

We start with $P_{\lambda_1}$. Lemma [56] implies that

$$[\text{Rad}^2 P_{\lambda_1}] = 2[D^{\lambda_1}] + [D^{\lambda_2}].$$

As $S^{(1),(2)}$ is a submodule of $P_{\lambda_1}$, $\text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_2}) \neq 0$ and the radical length of $\text{Rad}^2 P_{\lambda_1}$ is greater than or equal to 2. If the radical length of $\text{Rad}^2 P_{\lambda_1}$ is 2, then

$$\text{Soc}^2 P_{\lambda_1} / \text{Soc} P_{\lambda_1} = D^{\lambda_1} \oplus D^{\lambda_2},$$

which implies that $D^{\lambda_1}$ appears in $\text{Rad} P_{\lambda_1} / \text{Rad}^2 P_{\lambda_1}$, a contradiction. Hence $P_{\lambda_1}$ is uniserial and $H(P_{\lambda_1})$ has the desired form.

Next consider $P_{\lambda_2}$. $H(P_{\lambda_2})$ has a Specht filtration whose successive quotients are $\text{Soc} S^{2\lambda_2} = D^{\lambda_1}$, $S^{(1),(2)}$ and $S^{(1^3),(0)} = D^{\lambda_2}$. Recalling that $S^{(1),(2)}$ is a uniserial submodule of $P^{\lambda_1}$, we know that $H(P_{\lambda_2})$ is not semisimple.

As $\text{Ext}_B^1(D^{\lambda_1}, D^{\lambda_2}) \neq 0$, there exists a surjective $B$–module homomorphism which is the composition of maps given as follows.

$$\text{Rad} P_{\lambda_2} \longrightarrow \text{Rad} P_{\lambda_2} / \text{Rad}^2 P_{\lambda_2} \longrightarrow D^{\lambda_1}$$

Hence, we have $\phi : P_{\lambda_1} \longrightarrow \text{Rad} P_{\lambda_2}$ which is a lift of this homomorphism. Taking the radical structure of $P_{\lambda_1}$ and $[H(P_{\lambda_2}) : D^{\lambda_1}] = 2$ into consideration, its image in $H(P_{\lambda_2})$ is either

$$D^{\lambda_1} \quad \text{or} \quad D^{\lambda_2} \quad \text{or} \quad D^{\lambda_1}$$

If the image is $D^{\lambda_1}$, then this appears both in the top and in the socle of $H(P_{\lambda_2})$. Thus we can write

$$H(P_{\lambda_2}) = D^{\lambda_1} \oplus M.$$ 

As $H(P_{\lambda_2})$ is self–dual and not semisimple, $[M] = [D^{\lambda_1}] + 2[D^{\lambda_2}]$ implies that $M$ is uniserial with $\text{Top} M = D^{\lambda_2}$, $\text{Soc} M = D^{\lambda_2}$ and $H(M) = D^{\lambda_1}$. However, as $D^{\lambda_2}$ appears in $\text{Rad}^3 P_{\lambda_1} / \text{Rad}^4 P_{\lambda_1}$, Landrock’s theorem [12] Theorem 1.7.8] implies that $D^{\lambda_1}$ must appear in $\text{Rad}^3 P_{\lambda_2} / \text{Rad}^4 P_{\lambda_2}$, a contradiction. We have proved that the image of $\phi$ in $H(P_{\lambda_2})$ is uniserial of length 3. Consider the surjection $\psi : P_{\lambda_2} \longrightarrow \text{Rad} P_{\lambda_1}$. This induces the surjection $H(P_{\lambda_2}) \longrightarrow \text{Rad}^2 P_{\lambda_1}$. If $D^{\lambda_2}$ does not appear in $\text{Soc} H(P_{\lambda_2})$ then $\text{Soc} H(P_{\lambda_2}) = D^{\lambda_1}$, which must vanish under this surjection. This is a contradiction because $D^{\lambda_1}$ must appear in the image twice. Therefore, $D^{\lambda_2}$ appears in $\text{Soc} H(P_{\lambda_2})$ and this $D^{\lambda_2}$ also appears in $\text{Top} H(P_{\lambda_2})$. Thus, $H(P_{\lambda_2})$ is the direct sum of $D^{\lambda_2}$ and the image of $\phi$ in $H(P_{\lambda_2})$, proving that $H(P_{\lambda_2})$ has the desired form. As this $D^{\lambda_2}$ must vanish under the map $H(P_{\lambda_2}) \longrightarrow \text{Rad}^2 P_{\lambda_1}$, we have also proved that $\text{Ker} \psi$ is uniserial of length 2 whose top and socle are $D^{\lambda_2}$.

Denote the submodule $\text{Im} \phi$ of $P_{\lambda_2}$ by $N_1$ and the submodule $\text{Ker} \psi$ of $P_{\lambda_2}$ by $N_2$. We define $\beta \in \text{End}_B(P_{\lambda_2})$ by $\text{Im} \beta = N_2$. Then $\text{Ker} \beta = N_1$. Nextly, we define
\[\mu \in \text{Hom}_B(P^{\lambda_1}, P^{\lambda_2})\text{ and } \nu \in \text{Hom}_B(P^{\lambda_2}, P^{\lambda_1})\text{ by } \mu = \phi, \nu = \psi. \text{ Thus, } \text{Im } \mu = N_1, \text{ Im } \nu = \text{Rad } P^{\lambda_1}, \text{ Ker } \mu = \text{Soc } P^{\lambda_1}\text{ and } \text{Ker } \nu = N_2. \text{ Let } Q \text{ be the directed graph with adjacency matrix } (0, 1) \text{ as in Lemma 38 and define } A = FQ/I \text{ by the relations}
\]
\[\nu \beta = 0, \beta \mu = 0, \beta^2 = (\mu \nu)^2.\]

Note that \(A\) has the following basis.
\[\{e_1, e_2, \mu, \nu, \beta, \mu \nu, \beta^2, \mu \mu \nu, \nu \mu \nu, (\nu \beta)^2\}.\]

Then, by multiplying \(\mu\) or \(\nu\) with a suitable scalar, we obtain the surjective algebra homomorphism
\[A \rightarrow \text{End}_B(P^{\lambda_1} \oplus P^{\lambda_2}),\]
which is an isomorphism because both are 11 dimensional. As \(A\) is a special biserial algebra, \(B\) is tame. Note that \(A\) is self–injective, and if we set \(S = \bigoplus_{i \in Q_0} \text{Soc } A e_i\) then \(A/S\) is the algebra \(A\) in Lemma 38. Therefore, using the self–injectivity of \(A\), we know that the simple \(A\)–module corresponding to the node 2 has complexity 2.

Finally, assume that we are in (case 5b). We consider the parabolic subalgebra \(H_2(q, Q) \otimes H_2(q, Q)\) of \(H(q, Q)\). Then, as in the proof of Corollary 4, the Mackey decomposition theorem implies that Proposition 3 applies. Hence, it is enough to prove that \(H_2(q, Q) \otimes_F H_2(q, Q)\) is a symmetric tame special biserial algebra whose Gabriel quiver satisfies the assumptions of Lemma 37. Thus, there is an \(H_2(q, Q)\)–module \(M\) with complexity 2. On the other hand, as \(H_2(q, Q)\) is not semisimple, Theorem 14(1) implies that there is an \(H_2(q, Q)\)–module \(N\) whose complexity is greater than or equal to 1. As the complexity of \(M \otimes N\) is greater than or equal to 3, Theorem 14(3) implies the result.

### 4. The case of one parameter Hecke algebras

4.1. Let \((W, S)\) be a finite Coxeter system. The Poincaré polynomial \(P_W(x)\) of \((W, S)\) is defined by
\[P_W(x) = \sum_{w \in W} x^{\ell(w)}\]
where \(\ell(w)\) is the length of \(w\). According to \(W = W(A_{n-1}), W(B_n)\) or \(W(D_n)\), we denote the Poincaré polynomial \(P_W(x)\) by \(P^A_n(x), P^B_n(x)\) and \(P^D_n(x)\) respectively.

It is well–known that the semisimplicity of a Hecke algebra is governed by its Poincaré polynomial \(\text{GU}\). Our aim in the subsequent sections is to show that the Poincaré polynomial governs other representation types also if the Hecke algebra is of classical type. In this section, we consider Hecke algebras associated with an irreducible Weyl group, and prove the following theorem. The general case where the Weyl group is not assumed to be irreducible will be considered in the next section. Note that the finiteness result, for \(H^X_n(q), X = A, B, D\), was proved in \(\text{AM3}\), and for \(H_n(q, Q)\), in \(\text{AM3}\).

**Theorem 57.** Let \(\epsilon\) be the multiplicative order of \(q \neq 1\).

1. Assume that \(\epsilon \geq 3\). Then, for \(X = A, B\) or \(D, H^X_n(q)\) is
   - finite if \((x - q)^2\) does not divide \(P^X_n(x)\).
   - wild otherwise.

2. Assume that \(\epsilon = 2\). Then, for \(X = A, B\) or \(D, H^X_n(q)\) is
   - finite if \((x - q)^2\) does not divide \(P^X_n(x)\).
   - tame if \((x - q)^2\) divides but \((x - q)^3\) does not divide \(P^X_n(x)\).
   - wild otherwise.
4.2. First we consider $\mathcal{H}_n^A(q)$. Note that the result in this case is nothing but Proposition 3.7, Theorem 3.8]. We give a proof for the sake of completeness. $P_n^A(x)$ is given by

$$P_n^A(x) = \prod_{k=1}^{n} \frac{x^k - 1}{x - 1}.$$ 

If $n \geq 2e$ then $x^e - 1$ and $x^{2e} - 1$ are divisible by $x - q$. If $n < 2e$ then $q^k \not= 1$, for $1 \leq k \leq e - 1$ and $e + 1 \leq k \leq 2e - 1$. Thus, $(x - q)^2$ does not divide $P_n^A(x)$ if and only if $n < 2e$.

Assume that $e \geq 3$. Then, Theorem 39 implies that if $n < 2e$ then $\mathcal{H}_n^A(q)$ is finite, and that if $n \geq 2e$ then $\mathcal{H}_n^A(q)$ is wild. Hence (1).

Next assume that $e = 2$. Then, the characteristic of $F$ is odd, and $P_n^A(x)$ is divisible by $(x + 1)^3$ if and only if $\left[\frac{e}{2}\right] \geq 3$, that is, $n \geq 6$. By Theorem 39 again, we have that $(x + 1)^3$ divides $P_n^A(x)$ if and only if $\mathcal{H}_n^A(q)$ is wild. Hence (2).

4.3. Next consider $\mathcal{H}_n^B(q)$, which is $\mathcal{H}_n(q,Q)$ with $Q = q$. The result in this case is essentially [A5] Theorem 2.1. Here, we prove this case as a corollary of Theorem 40 and Theorem 42. $P_n^B(x)$ is given by

$$P_n^B(x) = \prod_{k=1}^{n} \frac{x^{2k} - 1}{x - 1}.$$ 

Assume that the multiplicative order $e$ of $q$ is odd. Then $-Q \not\in q^{e}$, and $(x - q)^2$ does not divide $P_n^B(x)$ if and only if $n < 2e$. Thus, Theorem 40 implies the result.

Next assume that $e$ is even. Then $-Q = q^{e/2 + 1}$ and $(x - q)^2$ does not divide $P_n^B(x)$ if and only if $n < e$. If $e \geq 4$ then $f \not= 0$ also holds. If $e = 2$ then $f = 0$ and the characteristic of $F$ is odd. Further, $(x + 1)^3$ divides $P_n^B(x)$ if and only if $n \geq 3$. Hence, Theorem 42 implies the result.

4.4. Finally, we consider $\mathcal{H}_n^D(q)$. Note that $n \geq 4$ and $\mathcal{H}_n^D(q)$ is the $F$–algebra defined by generators $S_0$, $S_1, \ldots, S_{n-1}$ and relations

$$\begin{align*}
(S_i - q)(S_i + 1) &= 0 \quad (0 \leq i \leq n - 1), \\
S_0S_2S_0 &= S_2S_0S_2, \quad S_0S_i = S_iS_0 \quad (i \not= 2), \\
S_{i+1}S_iS_{i+1} &= S_iS_{i+1}S_i \quad (1 \leq i \leq n - 2), \\
S_iS_j &= S_jS_i \quad (1 \leq i < j - 1 \leq n - 2).
\end{align*}$$

$P_n^D(x)$ is given by

$$P_n^D(x) = \frac{x^n - 1}{x - 1} \prod_{k=1}^{n-1} \frac{x^{2k} - 1}{x - 1}.$$ 

We embed $\mathcal{H}_n^D(q)$ into $\mathcal{H}_n(q,1)$ by the injective algebra homomorphism

$$\mathcal{H}_n^D(q) \rightarrow \mathcal{H}_n(q,1)$$

defined by $S_0 \mapsto T_0T_1T_0$ and $S_i \mapsto T_i$, for $1 \leq i \leq n - 1$, and we identify $\mathcal{H}_n^D(q)$ with its image. Then

$$\mathcal{H}_n(q,1) = \mathcal{H}_n^D(q) \oplus T_0\mathcal{H}_n^D(q) \text{ and } T_0\mathcal{H}_n^D(q) = \mathcal{H}_n^D(q)T_0.$$ 

First assume that $e$ is odd. If $n < 2e$ then Theorem 40 implies that $\mathcal{H}_n(q,1)$ is finite. Thus, so is $\mathcal{H}_n^D(q)$ by [AM2] Lemma 2.5. If $n = 2e$ then $\mathcal{H}_n^D(q)$ is wild.
Then, by the Mackey decomposition again, Proposition 6 is applicable and \( \mathcal{H}^D_n(q) \) is wild. Thus, \( \mathcal{H}^D_n(q) \), for \( n \geq 2e \), is also wild by Corollary 2.

Next assume that \( e \) is even. In particular, the characteristic of \( F \) is odd again.

\((\text{case } e = 2)\) As \( P^D_n(x) \), for \( n \geq 4 \), is always divisible by \( (x + 1)^3 \), our aim is to show that \( \mathcal{H}^D_n(q) \), for \( n \geq 4 \), is wild. Recall that \( \mathcal{H}^n_2(q) \) is not semisimple. Thus, \( \mathcal{H}^n_2(q) \) is wild by Theorem 14. Applying Proposition 6 to \( \mathcal{H}^n_2(q) \), we know that \( \mathcal{H}^D_{4n}(q) \) is wild. Thus, \( \mathcal{H}^D_n(q) \), for \( n \geq 4 \), is wild by Corollary 4.

\((\text{case } e \geq 4)\) Note that \( \mathcal{H}_n(q,1) \) is of the form \( \mathcal{H}_n(q,-q^I) \) with \( f = \frac{q}{2} \).

If \( n < e \) then \( x^{2k} - 1 \) with \( k = \frac{e}{2} \) is the only term in \( P^D_n(x) \) which may be divisible by \( x - q \). On the other hand, \( n < e \) implies that \( \mathcal{H}_n(q,1) \) is finite by Theorem 12. Thus, \( \mathcal{H}^D_n(q) \) is finite by \([AM2]\) Lemma 2.5.

Next assume that \( n = e \). Then both \( x^n - 1 \) and \( x^{2k} - 1 \) with \( k = \frac{e}{2} \) are divisible by \( x - q \), and our aim is to show that \( \mathcal{H}^n_D(q) \) is wild in this case. To apply the Fock space theory, we set \( \Lambda = \Lambda_0 + \Lambda_+ \) and consider the Fock space \( \mathcal{F}_n(\Lambda) \). By Theorem 46, simple \( \mathcal{H}_n(q,1) \)--modules are those \( D^\lambda \) with \( \lambda \vdash n \) and \( \lambda \in B(\Lambda) \).

Our assumption that \( f = \frac{q}{2} \) and \( e \geq 4 \) imply that \( f \geq 2 \) and \( e - f \geq 2 \). So, we are in (case 1a) and (case 2a) of subsection (3.3). In both cases,

\[ [S^\lambda_1, \frac{q}{2} - 1] = [D^\lambda_1, \frac{q}{2} - 1] + [D^\lambda_2, \frac{q}{2} - 1] + [D^\lambda_1] \]

and \( S^\lambda_1, \frac{q}{2} - 1 \) is a submodule of \( P^\lambda_1 \). Thus, the radical series of \( S^\lambda_1, \frac{q}{2} - 1 \) has the following form.

\[
D^\lambda_1 \cdot \frac{q}{2} - 1 \\
D^\lambda_2 \cdot \frac{q}{2} - 1 \\
D^\lambda_1
\]

In particular, we have the following uniserial \( \mathcal{H}_n(q,1) \)--modules.

\[ \text{Rad } S^\lambda_1, \frac{q}{2} - 1 = \frac{D^\lambda_2 \cdot \frac{q}{2} - 1}{D^\lambda_1} \]

\[ (S^\lambda_1, \frac{q}{2} - 1) / \text{Soc } S^\lambda_1, \frac{q}{2} - 1) = D^\lambda_2 \cdot \frac{q}{2} - 1 / D^\lambda_1 \]

Define an algebra automorphism \( \tau \) of \( \mathcal{H}_n(q,1) \) by

\[ \tau(T_1) = T_0 T_1 T_0 \quad \text{and} \quad \tau(T_i) = T_i \quad \text{for } i \neq 1. \]

Then \( \tau^2 = 1 \) and \( \tau \) induces the Dynkin automorphism of \( \mathcal{H}^D_n(q) \) given by \( S_i \mapsto S_{i-1} \) for \( i = 0, 1 \) and \( S_i \mapsto S_{i+1} \) for \( i \geq 2 \).

Define another algebra automorphism \( \sigma \) of \( \mathcal{H}_n(q,1) \) by

\[ \sigma(T_0) = -T_0 \quad \text{and} \quad \sigma(T_i) = T_i \quad \text{for } i \neq 0. \]

Then \( \mathcal{H}^D_n(q) \) is the fixed point subalgebra \( \mathcal{H}_n(q,1)^\sigma \). Now we apply the Clifford theory to the pair \( \mathcal{H}^D_n(q) \) and \( \mathcal{H}_n(q,1) \). Note that \( \sigma \tau = \tau \sigma \).

Let \( \lambda \vdash n \) be a Kleshchev bipartition. We define another Kleshchev bipartition \( h(\lambda) \) by \( (D^\lambda)^\sigma = D^{h(\lambda)} \). Then the Clifford theory tells us the following. Recall that we are in the case where the characteristic of \( F \) is odd.

**Theorem 58** ([Hull 4.3, 4.4]). Assume that the characteristic of \( F \) is odd.

1. If \( h(\lambda) \neq \lambda \) then \( D^\lambda \mathcal{H}^D_n(q) \) remains irreducible and \( D^\lambda \mathcal{H}^D_n(q) \simeq D^{h(\lambda)} \mathcal{H}^D_n(q) \).
On the other hand, we have a surjective and

(2) As we have a surjective

Theorem 58 (Hu2 Theorem 1.5)]. Let \( \lambda \) be a Kleshchev bipartition. Take a

Then \( h(\lambda) \) is given by

\[
\lambda = \tilde{f}_{i_1} \cdots \tilde{f}_{i_n}\emptyset.
\]

Proof. (1) Theorem \( \Box \) implies that \( (P_+^\lambda)\tau = P_+^\lambda \) and, by the Mackey

decomposition theorem, \( P_+^\lambda \downarrow_{B_+^\lambda}(q) \) is a projective \( B_+^\lambda \)-module. As we have

\[
P_+^\lambda \downarrow_{B_+^\lambda}(q) \rightarrow D_+^\lambda,
\]

and \( D_+^\lambda \neq D_+^\lambda \), \( P_+^\lambda \) is a direct summand of \( P_+^\lambda \downarrow_{B_+^\lambda}(q) \). Thus, we have

\[
\dim_F P_+^\lambda + \dim_F P_+^\lambda \leq \dim_F P_+^\lambda.
\]

On the other hand, we have a surjective \( \mathcal{H}_n(q, 1) \)-module homomorphism

\[
P_+^\lambda \downarrow_{\mathcal{H}_n(q, 1)} \rightarrow D_+^\lambda \downarrow_{\mathcal{H}_n(q, 1)} = D_+^\lambda,
\]

and this implies that \( P_+^\lambda \) is a direct summand of \( P_+^\lambda \downarrow_{\mathcal{H}_n(q, 1)} \). Thus,

\[
\dim_F P_+^\lambda \leq 2 \dim_F P_+^\lambda.
\]

Therefore,

\[
2 \dim_F P_+^\lambda = \dim_F P_+^\lambda + \dim_F P_+^\lambda \leq \dim_F P_+^\lambda \leq 2 \dim_F P_+^\lambda,
\]

which proves the result.

(2) As we have a surjective \( \mathcal{H}_n(q, 1) \)-module homomorphism

\[
\mathcal{P}_+^\lambda \downarrow_{\mathcal{H}_n(q, 1)} \rightarrow (D_+^\lambda \downarrow_{B_+^\lambda}(q))\tau_{\mathcal{H}_n(q, 1)} = D_+^\lambda \oplus D_+^{h(\lambda)},
\]

and \( \lambda \neq h(\lambda) \), \( P_+^\lambda \oplus P_+^{h(\lambda)} \) is a direct summand of \( \mathcal{P}_+^\lambda \downarrow_{\mathcal{H}_n(q, 1)} \). As \( P_+^\lambda \) and \( P_+^{h(\lambda)} \)

have the same dimension, this implies that

\[
\dim_F P_+^\lambda \leq \dim_F \mathcal{P}_+^\lambda.
\]

On the other hand, we have a surjective \( B_+^\lambda \)-module homomorphism

\[
P_+^\lambda \downarrow_{B_+^\lambda}(q) \rightarrow D_+^\lambda \downarrow_{B_+^\lambda}(q),
\]

which implies that \( \mathcal{P}_+^\lambda \) is a direct summand of \( P_+^\lambda \downarrow_{B_+^\lambda}(q) \). The result follows. \( \Box \)
We apply Theorem [53] in our setting. For our purposes, it suffices to compute
\( h(\lambda_1) \), \( h(\lambda_2) \), \( h(\lambda_2, \triangleleft - 1) \) and \( h(\lambda_1, \triangleleft - 1) \). Then, since
\[
\lambda_1 = \tilde{f}_{\frac{1}{2}} \cdots \tilde{f}_{e-1} \tilde{f}_0 \cdots \tilde{f}_\frac{1}{2} \emptyset,
\lambda_2 = \tilde{f}_{\frac{1}{2}} \cdots \tilde{f}_{e-1} \tilde{f}_0 \tilde{f}_{\frac{1}{2}} \cdots \tilde{f}_\frac{1}{2} \emptyset,
\lambda_{2, \frac{1}{2} - 1} = \tilde{f}_1 \cdots \tilde{f}_{\frac{1}{2} - 1} \tilde{f}_{\frac{1}{2}} \tilde{f}_{\frac{1}{2} + 1} \cdots \tilde{f}_{e-1} \tilde{f}_0 \emptyset,
\lambda_{1, \frac{1}{2} - 1} = \tilde{f}_1 \cdots \tilde{f}_{e-1} \tilde{f}_0 \emptyset,
\]
we have
\[
\begin{align*}
\frac{1}{4} h(\lambda_1) &= \tilde{f}_1 \cdots \tilde{f}_{e-1} \tilde{f}_0 \emptyset = \lambda_{1, \frac{1}{2} - 1}, \\
\frac{1}{4} h(\lambda_2) &= \tilde{f}_2 \cdots \tilde{f}_{\frac{1}{2}} \tilde{f}_1 \tilde{f}_{\frac{1}{2} + 1} \cdots \tilde{f}_{e-1} \tilde{f}_0 \emptyset = \lambda_{1, \frac{1}{2} - 2}, \\
\frac{1}{4} h(\lambda_{2, \frac{1}{2} - 1}) &= \tilde{f}_{\frac{1}{2} + 1} \cdots \tilde{f}_{e-1} \tilde{f}_0 \tilde{f}_2 \cdots \tilde{f}_{\frac{1}{2} - 0} = \lambda_{2, \frac{1}{2} - 1}, \\
\frac{1}{4} h(\lambda_{1, \frac{1}{2} - 1}) &= \tilde{f}_{\frac{1}{2} + 1} \cdots \tilde{f}_{e-1} \tilde{f}_0 \tilde{f}_{\frac{1}{2} - 0} = \lambda_{1, \frac{1}{2} - 1}.
\end{align*}
\]

Therefore, Theorem [54] implies that \( D^{\lambda_1, \frac{1}{2} - 1} \), \( D^{\lambda_2, \frac{1}{2} - 1} \), \( D^{\lambda_1, \frac{1}{2} - 1} \), \( D^{\lambda_2, \frac{1}{2} - 1} \) are simple \( \mathcal{H}_{\frac{1}{2}}^D(q) \)-modules, and that
\[
D^{\lambda_2, \frac{1}{2} - 1} = D_+^{\lambda_2, \frac{1}{2} - 1} + D_-^{\lambda_2, \frac{1}{2} - 1}.
\]

Since \( \text{Rad} S^{\lambda_1, \frac{1}{2} - 1} \frac{1}{4} \) is a uniserial \( \mathcal{H}_n(q, 1) \)-module whose top is \( D^{\lambda_2, \frac{1}{2} - 1} \) and whose socle is \( D^{\lambda_1} \), there is a surjective \( \mathcal{H}_n^D(q) \)-module homomorphism
\[
P^{\lambda_2, \frac{1}{2} - 1} \frac{1}{4} \mathcal{H}_n^D(q) = P_+^{\lambda_2, \frac{1}{2} - 1} \oplus P_-^{\lambda_2, \frac{1}{2} - 1} \longrightarrow \text{Rad} S^{\lambda_1, \frac{1}{2} - 1} \frac{1}{4} \mathcal{H}_n^D(q) .
\]

\( \text{Rad} S^{\lambda_1, \frac{1}{2} - 1} \frac{1}{4} \mathcal{H}_n^D(q) \) is not semisimple: if otherwise, then the simple \( \mathcal{H}_n^D(q) \)-module \( D^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) \) would appear in \( \text{Top}(\text{Rad} S^{\lambda_1, \frac{1}{2} - 1} \frac{1}{4} \mathcal{H}_n^D(q)) \). However, this would imply that the morphism is not surjective. Hence, the radical series of \( \text{Rad} S^{\lambda_1, \frac{1}{2} - 1} \frac{1}{4} \mathcal{H}_n^D(q) \) is one of the following.
\[
D_+^{\lambda_2, \frac{1}{2} - 1} \oplus D_-^{\lambda_2, \frac{1}{2} - 1} \oplus D_+^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) \oplus D_-^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) \oplus D_+^{\lambda_2, \frac{1}{2} - 1} \oplus D_-^{\lambda_2, \frac{1}{2} - 1} .
\]

In any case, by applying \( \tau \) if necessary, we conclude that there are uniserial \( \mathcal{H}_n^D(q) \)-modules of the following form.
\[
D_+^{\lambda_2, \frac{1}{2} - 1} \quad \text{and} \quad D_-^{\lambda_2, \frac{1}{2} - 1} \\
D_+^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) \quad \text{and} \quad D_-^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) .
\]

If we consider the dual of \( S^{\lambda_1, \frac{1}{2} - 1} / \text{Soc} S^{\lambda_1, \frac{1}{2} - 1} \), and arguing in the same way, we prove the existence of \( \mathcal{H}_n^D(q) \)-modules with the following radical structure.
\[
D_+^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) \quad \text{and} \quad D_-^{\lambda_1} \frac{1}{4} \mathcal{H}_n^D(q) .
\]

Consider the \( \mathcal{H}_n(q, 1) \)-module \( S^{\lambda_2} \). \( S^{\lambda_2} \) is uniserial of length 2 whose top is \( D^{\lambda_2} \) and whose socle is \( D^{\lambda_1} \). Then, we have a surjective \( \mathcal{H}_n^D(q) \)-module homomorphism
\[
\overline{P}^{\lambda_2} = P^{\lambda_2} \frac{1}{4} \mathcal{H}_n^D(q) \rightarrow S^{\lambda_2} \frac{1}{4} \mathcal{H}_n^D(q) .
\]
By the same reasoning as above, \( S^{\lambda_2} \downarrow \mathcal{H}_n^D(q) \) is not semisimple. Thus, the radical series of \( S^{\lambda_2} \downarrow \mathcal{H}_n^D(q) \) is as follows.

\[
D^{\lambda_2} \downarrow \mathcal{H}_n^D(q) \\
D^{\lambda_1} \downarrow \mathcal{H}_n^D(q)
\]

As a consequence, by considering the dual of these simple \( \mathcal{H}_n^D(q) \)-modules, we know that the Gabriel quiver of \( \mathcal{H}_n^D(q) \) contains the following quiver as a subquiver.

\[
\begin{array}{c}
\lambda_1, \lambda_2, \lambda_3 \\
(\lambda_2, \lambda_3) \\
(\lambda_3, \lambda_1)
\end{array}
\begin{array}{c}
(\lambda_3, \lambda_2) \\
(\lambda_1, \lambda_3)
\end{array}
\]

Therefore, Lemma 17 implies that \( \mathcal{H}_n^D(q) \) is wild when \( n = e \). So, \( \mathcal{H}_n^D(q) \), for \( n \geq e \), is wild by Corollary 11.

5. The general case

We begin by analysing the case where \( \mathcal{H}_n^X(q) \), for \( X = A, B, D \), is finite but not semisimple.

**Lemma 61.** Suppose that \( e = 2 \). If a non-semisimple block algebra of \( \mathcal{H}_n^A(q) \), \( \mathcal{H}_n^B(q) \) or \( \mathcal{H}_n^D(q) \) is finite, then it is Morita-equivalent to \( F[X]/(X^2) \).

**Proof.** If \( \mathcal{H}_n^A(q) \) is finite but not semisimple, then \( n = 2 \) or \( n = 3 \) and non-semisimple blocks are Morita-equivalent to the algebra \( F[X]/(X^2) \). If \( \mathcal{H}_n(q, Q) \) is finite but not semisimple, then Theorem 8.2 and Theorem 10.2 imply that the same holds. In particular, we have the result for \( \mathcal{H}_n^B(q) \). \( \mathcal{H}_n^D(q) \) cannot be finite. Thus, the result follows.

**Lemma 62.** Suppose that \( e \geq 3 \).

1. If \( \mathcal{H}_n^A(q) \) is finite but not semisimple and \( B \) is a non-semisimple block algebra of \( \mathcal{H}_n^A(q) \), then
   - the number of pairwise non-isomorphic simple \( B \)-modules is \( e - 1 \),
   - \( B \) is a Brauer tree algebra with the tree being a straight line such that all indecomposable projective \( B \)-modules have radical length 3.

2. If \( \mathcal{H}_n(q, Q) \) is finite but not semisimple and \( B \) is a non-semisimple block algebra of \( \mathcal{H}_n(q, Q) \), then
   - if \( -Q \notin q^2 \) then the number of pairwise non-isomorphic simple \( B \)-modules is \( e - 1 \),
   - if \( -Q = q^f \), for \( 0 \leq f \leq e - 1 \), then the number of pairwise non-isomorphic simple \( B \)-modules is either \( e - f - 1 \) or \( f + 1 \),
   - \( B \) is a Brauer tree algebra with the tree being a straight line such that all indecomposable projective \( B \)-modules have radical length 3 unless \( f = 0 \) and either \( e = 3 \) and \( n = 1, 2 \) or \( e \geq 4 \) and \( n = 1, 2, 3 \).

**Proof.** (1) The statements follow from the computation of the decomposition numbers of \( B \) by using the Jantzen–Schaper sum formula. See Exercise 5.10 for example. We may prove these in another way: it is shown in Theorem 8.2 that \( B \) is Morita-equivalent to the non-semisimple block algebra of \( \mathcal{H}_e^A(q) \) when the characteristic of \( F \) is zero, and it is observed in that this result is valid without the assumption on the characteristic. Hence, the results follow from the
computation of the radical structure of indecomposable projective \( \mathcal{H}_n^A(q) \)-modules given in [U]. See also [Ge1] for a more general result in the characteristic zero case.

(2) When \( e \) is odd, then they follow from Theorem 38 and (1). When \( e \) is even, they were proved in the proof of [AM2, Theorem 6.30]. \( \square \)

The similar results hold for \( \mathcal{H}_n^D(q) \) as follows.

**Proposition 63.** Let \( e \geq 3 \) as before, and assume that \( \mathcal{H}_n^D(q) \) is finite but not semisimple. Then, the radical series of an indecomposable projective \( \mathcal{H}_n^D(q) \)-module is one of the following form.

\[
\begin{align*}
D^\lambda \\
D^\lambda & \quad \text{where } h(\lambda) \neq \lambda, \\
D^\lambda & \quad \text{where } h(\mu) \neq \mu, \\
D^\lambda \oplus D^\mu & \quad \text{where } h(\lambda) \neq \lambda, \\
D^\lambda & \quad \text{where } h(\mu) = \mu, \\
D^\lambda & \quad \text{where } h(\lambda) = \lambda, \\
D^\lambda & \quad \text{where } h(\mu) = \mu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu, \\
D^\lambda & \quad \text{where } h(\nu) = \nu.
\end{align*}
\]
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\[ \begin{align*}
D^\lambda_+ & \quad D^\lambda_- \\
D^\mu_+ & \quad D^\mu_- \\
D^\lambda_+ & \quad D^\lambda_- \\
D^\mu_+ & \quad D^\mu_- \\
D^\lambda_+ & \quad D^\lambda_- \\
\end{align*} \]

Here, we write \( D^\lambda \) instead of \( D^\lambda \downarrow_\mathcal{H}_n(q) \) for short.

**Proof.** As \( \mathcal{H}_n(q) \) is finite, \( \mathcal{H}_n(q,1) \) is also finite. So, Lemma \( \text{[54]} \) implies that if \( B \) is a block algebra of \( \mathcal{H}_n(q,1) \) then the number of pairwise non–isomorphic \( B \)–modules is greater than or equal to 2 and that if \( \lambda \) is a Kleshchev bipartition then the radical structure of \( P^\lambda \) is either

\[ \begin{align*}
D^\lambda_+ & \quad D^\lambda_- \\
D^\mu_+ & \quad D^\mu_- \\
D^\lambda_+ & \quad D^\lambda_- \\
D^\mu_+ & \quad D^\mu_- \\
D^\lambda_+ & \quad D^\lambda_- \\
\end{align*} \]

or

\[ \begin{align*}
D^\lambda_+ & \quad D^\lambda_- \\
D^\mu_+ & \quad D^\mu_- \\
D^\lambda_+ & \quad D^\lambda_- \\
D^\mu_+ & \quad D^\mu_- \\
D^\lambda_+ & \quad D^\lambda_- \\
\end{align*} \]

where \( \lambda, \mu \) and \( \nu \) are pairwise distinct.

Firstly, we consider the case where the \( \mathcal{H}_n(q,1) \)–module \( P^\lambda \) is uniserial. Assume that \( h(\lambda) \neq \lambda \). Then, Lemma \( \text{[54]} \) implies that \( P^\lambda \downarrow_\mathcal{H}_n(q) = \mathcal{P}^\lambda \). Thus, \( \mathcal{P}^\lambda \) has radical length 3 and \( H(\mathcal{P}^\lambda) = D^\mu \downarrow_\mathcal{H}_n(q) \). If \( \mu = h(\lambda) \) then we are in the first case of the list above. If \( h(\mu) \neq \mu \) and \( \mu \neq h(\lambda) \) then we are in the second case of the list. If \( h(\mu) = \mu \) then we are in the third case. Note that this case does not appear if the number of the isomorphism classes of simple \( \mathcal{H}_n(q,1) \)–modules in the block is 2. Assume that \( h(\lambda) = \lambda \). Then, \( (P^\lambda)^+ = P^\lambda \) implies that \( h(\mu) \neq \mu \) cannot happen. Thus, \( h(\mu) = \mu \) and \( \lambda, \mu, \nu \) are in the fourth case or in the fifth case.

Nextly, we consider the case where the \( \mathcal{H}_n(q,1) \)–module \( P^\lambda \) is not uniserial. Then, the similar argument gives us the remaining cases. \( \square \)

**Corollary 64.** Assume that \( e \geq 3 \) and \( \mathcal{H}^D_n(q) \) is finite but not semisimple. Then, for simple \( \mathcal{H}^D_n(q) \)–modules \( S \) and \( T \), we have \( \text{Ext}^1_{\mathcal{H}^D_n(q)}(S,T) \neq 0 \) if and only if \( \text{Ext}^1_{\mathcal{H}^D_n(q)}(T,S) \neq 0 \).

**Proposition 65.** Let \( B \) be a block algebra of \( \mathcal{H}^A_n(q) \), \( \mathcal{H}^B_n(q) \) or \( \mathcal{H}^D_n(q) \), and suppose that \( e \geq 3 \) and that \( B \) is finite but not semisimple. Then one of the following holds.

(a) The Gabriel quiver of \( B \) contains a straight line of length 2 (with 3 nodes), or a directed graph with adjacency matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \).

(b) There are indecomposable projective \( B \)–modules of the following form.

\[ \begin{align*}
S & \quad T \\
T & \quad S
\end{align*} \]

where \( S \) and \( T \) are non–isomorphic simple \( B \)–modules.
Proof. Assume that $B$ is a block algebra of $\mathcal{H}^A_n(q)$. Then Lemma 62(1) implies that the number of pairwise non-isomorphic simple $B$–modules is equal to $e-1 \geq 2$. Hence, we are in the case (a) if the number is greater than 2 and in the case (b) if the number is equal to 2, by Lemma 62(1).

Assume that $B$ is a block algebra of $\mathcal{H}^B_n(q)$. If $e$ is even then Theorem 65 implies that $B$ is Morita–equivalent to a block algebra of $\mathcal{H}^B_k(q)$, for some $k$. If $e$ is even then Lemma 62(2) implies that the number of pairwise non-isomorphic simple $B$–modules is equal to $\frac{e}{2} + 2$ or $\frac{e}{2} \geq 2$. Hence, by the same argument as above, we are either in the case (a) or in the case (b) again.

Assume that $B$ is a block algebra of $\mathcal{H}_n(q,1)$. Then, by Lemma 62(2) again, the number of pairwise non–isomorphic simple $B$–modules is equal to $\frac{e}{2} + 1 \geq 3$. Thus, the Gabriel quiver of $B$ contains a straight line of length 2. We denote the nodes of the line as follows.

$$\lambda \quad \mu \quad \nu$$

Assume that $h(\lambda) \neq \lambda$ and $h(\mu) \neq \mu$ or $h(\nu) \neq \nu$. If $h(\lambda) \neq \mu, \nu$ and $h(\mu) \neq \nu$ then Lemma 60 and Proposition 63 imply that the Gabriel quiver of $\mathcal{H}^A_k(q)$ also contains a straight line of length 2. If $h(\lambda) = \mu$ or $h(\lambda) = \nu$ then we obtain one of the other two quivers listed in the case (a).

Assume that $h(\lambda) = \lambda$ and $h(\mu) \neq \mu$ or $h(\lambda) \neq \lambda$ and $h(\mu) = \mu$. Then the Gabriel quiver of $\mathcal{H}^A_n(q)$ contains the straight line whose nodes are $\lambda, \mu$ or $\mu, \lambda$.

Assume that $h(\lambda) = \lambda$, $h(\mu) = \mu$ and $h(\nu) \neq \nu$. Then the Gabriel quiver of $\mathcal{H}^A_n(q)$ contains the straight line whose nodes are $\mu, \nu$ and either $\lambda$ or $\lambda$.

Assume that $h(\lambda) = \lambda$, $h(\mu) = \mu$ and $h(\nu) = \nu$. Then, the similar argument shows that the Gabriel quiver of $\mathcal{H}^A_n(q)$ contains a straight line of length 2. □

Lemma 66. Assume that $A$ and $B$ are block algebras of $\mathcal{H}^A_n(q)$, $\mathcal{H}^B_n(q)$, $\mathcal{H}^D_n(q)$ or $\mathcal{H}_n(q, Q)$ and that they are finite but not semisimple.

(1) Suppose that $e \geq 3$. Then $A \otimes B$ is wild.

(2) Suppose that $e = 2$. Then $A \otimes B$ is tame.

Proof. (1) Assume that the Gabriel quiver of $A$ contains one of the directed graphs listed in the case (a) of Proposition 65. As the Gabriel quiver of $B$ always contains a straight line of length 1 by Proposition 65, the Gabriel quiver of $A \otimes B$ contains the product of these quivers as a subquiver. In any of these three cases, Lemma 30 implies that $A \otimes B$ is wild.

Now assume that both $A$ and $B$ are as in the case (b) of Proposition 65. Let $S$ and $T$ be the simple $A$–modules. We denote the projective covers of $S$ and $T$ by $P(S)$, $P(T)$ respectively. Similarly, we denote the simple $B$–modules by $S'$ and $T'$, the projective covers $P(S')$ and $P(T')$. We define indecomposable projective $A \otimes B$–modules $P_1$ and $P_2$ by

$$P_1 = P(S) \otimes P(S'), \quad P_2 = P(T) \otimes P(T').$$

We shall show that $\text{End}_{A \otimes B}(P_1 + P_2)$ is wild. This implies the result because its opposite algebra may be identified with an algebra of the form $p(A \otimes B)p$ where $p$ is an idempotent of $A \otimes B$. Define a factor algebra $R$ of $\text{End}_{A \otimes B}(P_1 + P_2)$ by

$$R = \text{End}_{A \otimes B}(P_1/\text{Rad}^3 P_1 \oplus P_2/\text{Rad} P_2).$$
Note that $P_1/\text{Rad}^3P_1$ has the radical series of the following form.

\[ S \otimes S' \]
\[ S \otimes T' \oplus S \otimes T' \]
\[ S \otimes S' \oplus T \otimes T' \oplus S \otimes S' \]

Hence, $R$ has the form $FQ/I$ where $Q_0 = \{1, 2\}$ and $Q_1$ consists of two loops on the node 1 and the arrow 1 $\rightarrow$ 2. This implies that $R$ is wild by Lemma 29.

(2) By Lemma 61, we may assume that both $A$ and $B$ are isomorphic to $F[X]/(X^2)$. Thus, $A \otimes B \simeq F[X, Y]/(X^2, Y^2)$, which is tame.

Let $(W, S)$ be a finite Weyl group of classical type, and let $P_W(x)$ be its Poincaré polynomial. Fix $q \in F^\times$ and denote the Hecke algebra associated with $(W, S)$ and $q$ by $\mathcal{H}_W(q)$. We assume that $q$ is a primitive $e^{th}$ root of unity with $e \geq 2$. Note that if $q = 1$ then $\mathcal{H}_W(q) = FW$ and we can also tell the representation type.

**Theorem 67.** Let $P_W(x)$, $\mathcal{H}_W(q)$ and $e \geq 2$ be as above. Then,

1. Assume that $e \geq 3$. Then $\mathcal{H}_W(q)$ is
   - finite if $(x - q)^2$ does not divide $P_W(x)$.
   - wild otherwise.

2. Assume that $e = 2$. Then, $\mathcal{H}_W(q)$ is
   - finite if $(x - q)^2$ does not divide $P_W(x)$.
   - tame if $(x - q)^2$ divides but $(x - q)^3$ does not divide $P_W(x)$.
   - wild otherwise.

**Proof.** We write $W = W_1 \times \cdots \times W_s$ where $W_i$ are irreducible Weyl groups.

1. If $(x - q)^2$ does not divide $P_W(x)$, then $x - q$ divides at most one $P_W(x)$ and $(x - q)^2$ does not divide $P_W(x)$. Thus, $\mathcal{H}_W(q)$ is finite by Theorem 57 and all the other $\mathcal{H}_{W_i}(q)$, for $j \neq i$, are semisimple. Hence, the result follows. If $x - q$ divides $P_W(x)$ and $P_{W_j}(x)$, for $i \neq j$, then $\mathcal{H}_{W_i}(q)$ and $\mathcal{H}_{W_j}(q)$ are finite but not semisimple. Thus, Lemma 60 (1) implies the result. The result for the case where $(x - q)^2$ divides some $P_W(x)$ was proven in Theorem 57.

2. We have already proved that if $\mathcal{H}_n(q, Q)$ is tame then there is an $\mathcal{H}_n(q, Q)$-module with complexity 2. Assume that $\mathcal{H}_n^X(q)$ is tame. Thus, $e = 2$ and $n = 4$ or $n = 5$. EN Proposition A asserts that $\mathcal{H}_n^A(q)$ is Morita-equivalent to the path algebra which is considered in (case 4b) of the proof of Theorem 62. Thus, there is an $\mathcal{H}_n^A(q)$-module with complexity 2. Similarly, EN Proposition B asserts that the unique non-semisimple block algebra of $\mathcal{H}_n^A(q)$ has the directed graph $Q$ with adjacency matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as its Gabriel quiver, and if we denote the loops on the nodes 1 and 2 by $\alpha$, $\beta$, and 1 $\rightarrow$ 2, 1 $\leftarrow$ 2 by $\mu$ and $\nu$, then the relations are given by

\[ \alpha^2 = 0, \quad \beta^2 = 0, \quad \mu \alpha = 0, \quad \alpha \nu = 0, \quad \beta \mu = 0, \quad \nu \beta = 0. \]

This is a special biserial algebra which satisfies the assumptions of Lemma 61. Thus, there is an $\mathcal{H}_n^A(q)$-module with complexity 2. $\mathcal{H}_n^D(q)$ cannot be tame.

Hence, if $\mathcal{H}_n^X(q)$, for some $n$ and $X$, is tame then there is an $\mathcal{H}_n^X(q)$-module with complexity 2.

Assume that $(x - q)^2$ does not divide $P_W(x)$. Then, $\mathcal{H}_W(q)$ is semisimple for all but at most one $i$, say $i_0$. Then, $\mathcal{H}_{W_{i_0}}(q)$ is finite by Theorem 57 and so is $\mathcal{H}_W(q)$. Assume that $(x - q)^2$ divides $P_W(x)$ but $(x - q)^3$ does not divide $P_W(x)$. If $(x - q)^2$ divides $P_{W_i}(x)$, for some $i$, then all the other $\mathcal{H}_{W_j}(q)$, $j \neq i$, are semisimple and the result follows from Theorem 57. If $x - q$ divide $P_{W_i}(x)$ and $P_{W_j}(x)$ for
distinct \(i\) and \(j\), then \(\mathcal{H}_W(q)\) and \(\mathcal{H}_W(q)\) are finite and not semisimple. All the other \(\mathcal{H}_W(q), k \notin \{i, j\}\), are semisimple. Thus, Lemma \(56(2)\) implies that \(\mathcal{H}_W(q)\) is tame. Assume that \((x - q)^3\) divides \(P_W(x)\). If there are distinct \(i, j, k\) such that \(x - q\) divides \(P_W(x), P_W(x)\) and \(P_W(x)\) then \(\mathcal{H}_W(q), \mathcal{H}_W(q)\) and \(\mathcal{H}_W(q)\) are finite but not semisimple. Thus, there are modules \(M_i, M_j, M_k\) of these algebras such that they have complexity greater than or equal to 1. As \(M_i \otimes M_j \otimes M_k\) has complexity greater than or equal to 3, the result follows from Theorem 14(3). If \((x - q)^2\) divides \(P_W(x)\) but \((x - q)^3\) does not divide \(P_W(x)\) and \(x - q\) divides \(P_W(x)\), for \(j \neq i\), then, the same complexity argument works: \(\mathcal{H}_W(q)\) is tame by Theorem 57(2) and there is an \(\mathcal{H}_W(q)\)-module with complexity 2, as is remarked above. The case where \((x - q)^3\) divides \(P_W(x)\), for some \(i\), reduces to Theorem 57 as before.

\[\square\]

6. Appendix

The following is a list of corrections and remarks for [A1]. The author is grateful to Professor Kashiwara for pointing out many of these.

(p.11) In Assertion 4. If we consider \(U \to U(g)\) and use the universal property of \(U(g)\), the proof would be much shorter.

(p.20) As \(M(\lambda)\) is a \(T(V)\)-module, it is also a \(g(V)\)-module. Similarly, the algebra homomorphism \(T(V) \to U(\tilde{n})\) induced by \(V_\subset \tilde{n}_\subset \tilde{\tilde{n}}\) gives the surjection \(g(V) \to \tilde{n}\).

(p.25) In the proof of Proposition 4.5. After “We can choose \(m \neq 0\), add “in \(M_0 \otimes_K K\).” After “\(c = \pm v\)”, add “Hence, we can choose \(m\) in \(M_0\).”

(p.25) In Definition 4.6. \(vt - v^{-1}t^{-1} \to vt + v^{-1}t^{-1}\) and \(\pm \frac{j+1}{v-v^{-1}} \to \pm \frac{1+1}{v-v^{-1}}\).

(p.89) in 1.10; \(R_i(\lambda) \setminus \{y_1, \ldots, y_s, x_1, \ldots, x_t\} \to R_i(\lambda) \setminus \{x_1, \ldots, x_t\}.

(p.92) in 1.5; \(\lambda(\sigma, k, S) \setminus \{x, \tilde{x}\} \to \{\lambda(\sigma, k, S) \setminus \{x, \tilde{x}\} | S \subset N_i R_\lambda\}.

(p.98) Just above (12.1); \(g_{\mu, \mu}(v) = 1 \to g_{\mu, \mu}(v) = 1, g_{\lambda, \mu}(0) = \delta_{\lambda, \mu}\).

(p.98) in 1.5; \(Q[v, v^{-1}] \to Q[v] \oplus v^{-1}Q[v^{-1}]\) must be \(Q[v, v^{-1}] \to vQ[v] \oplus Q[v^{-1}]\).

(p.108) The denominator of (1.3); \(v_{ci} - q^{k^i}v_{ci-1} \to q^{k_i}v_{ci} - v_{ci-1}\) and the (1, 1)-entry; \((q - 1)v_{ci} \to q^{k_i}(q - 1)v_{ci}\), the (2, 2)-entry; \((1 - q)v_{ci} \to (1 - q)v_{ci-1}\).

(p.108) In the definition of the \(\lambda\)-separatedness; \(v_{ci} - q^{k^i}v_{ci-1} \to q^{k_i}v_{ci} - v_{ci-1}\).

(p.109) in 1.14; Delete the sentence “To prove this, ...”, and change “If we consider ... requirement.” to “Consider \(w \in S_n\) such that \(wt = s\). We argue by induction on \(l(w)\). Choose \(s_i = (i - 1, i)\) with \(s_i w < w\) so that \(s' = s_i s\) is standard. By the induction hypothesis, \(s' \in W\). Then \(a_i s' \in W\) because \(s\) appears in \(a_is'\) with a nonzero coefficient.”

Note that if there is no such \(s_i\) then the entry, say \(j\), in \(s\) of the node \(n\) of \(t\) must be \(w\) otherwise we could choose \(s_{j+1}\). We delete \(n\) and continue the same argument to conclude that \(s = t\).

(p.110) in 1.5; \(v_{ci} - q^{k^i}v_{ci-1} \to q^{k_i}v_{ci} - v_{ci-1}\).

(p.110) In Proposition 13.10(2); Add “and \(V^\lambda_k\) is irreducible.”

(p.112) Let \(L'' = \pi^{N-1}L' + L\). Then, we have \(\pi L'' \subset L \subset L''\) and \(\pi^{N-1}L' \subset L'' \subset L'\). Hence, we can also prove the result by induction on \(N\).

(p.113) The statement of Lemma 13.19 is not accurate. We must consider the subcategory of \(\tilde{H}_n,k - mod\) whose objects are those modules which admits an \(S\)-lattice here. In practice, we work with the category \(C\) (Corollary
13.26). As any module in $C$ admits an $S$–lattice, this inaccuracy does not affect the rest of the chapters.

(p.115) Murphy’s lemma, which says that $s \triangleright t$ if and only if $d(s) \leq d(t)$, is used to show that $n_{st} \cdot t^A \in V^A$ belongs to $\oplus_{u \geq u} \mathcal{H}u$. The lemma easily follows from Ehresmann’s characterization of the Bruhat order, which says that $w_1 \leq w_2$ if and only if $\{|i| i \leq p, w_1(i) \leq q\} \geq \{|i| i \leq p, w_2(i) \leq q\}$ for all $p$ and $q$.

(p.115) In Corollary 13.24; Replace $\mathcal{H}$ in l.-8 and l.-7; $H$ in l.5; the Grothendieck group

At the bottom of Definition 14.15; Add “where we fix a square root $\sqrt{p}$ of $p$ and we choose the square root $p^{r/2}$ of $p^r$ to be $-\sqrt{p^r}$.”

(p.129) Proof of Lemma 14.25 is absurd. To save this, we let $H$ be the multiplicative group of pure numbers, $\mathbb{Z}$ its group algebra, and change the statement of Lemma 14.25 as follows. Then the same proof (but we do not assert the “(a root of unity)”part) works.

“Then there exists an element $F_{m,m'}$ in $\mathbb{ZC}$ such that if we write $F_{m,m'} = \sum n_i [c_i]$, where $n_i \in \mathbb{Z}$ and $c_i \in C$, then the following holds for all $e$:

$p^{-r}(F_{m,m'}) = \sum n_i c_i$.”

(p.132) In Definition 14.26; Change “Let $H_A$ be” to “Let $v = -[\sqrt{p}^{-1}]$, $A = \mathbb{Z}[v, v^{-1}]$ and define $H_A$ to be”, and change “Define an $A$-bilinear map” to “Then let $H_C = H_A \otimes_A \mathbb{ZC}$ and define a $\mathbb{ZC}$-bilinear map”, and replace “$F_{m,m'}(v)$” with “$F_{m,m'}(v)$”.

“$H_A$ is an $A$-algebra.” $\Rightarrow$ “$H_C$ is a $\mathbb{ZC}$-algebra.”

At the end of the definition, add “We call $H_A$ the Hall module.”

(p.133) In Lemma 14.27; Change “$H_A$” to “$H_C$” in (1) and add “In particular, $H_A$ is a right $U_A^{-*}$-module.” to the end of (2).

As a $\mathbb{ZC}$-module, $H_C$ is nothing but the Grothendieck group of equivariant mixed Weil sheaves. By Lemma 14.27, $H_A$ is a $U_A^{-*}$-module.

Note that we only need $H_A$ at $v = 1$ to prove Theorem 12.5, and that the statements Lemma 14.27(2) and Proposition 14.28 for $H_A$ remain unchanged.

(p.137) Lemma 14.34: “... makes $\mathcal{H}$ into a unital associative algebra, which is isomorphic to $H_A$:” $\Rightarrow$ “... restricts to an isomorphism between the $A$-subalgebras of $\mathcal{H}$ and $H_A$ generated by $f^{(n)}$’s. More precisely, each object $F$ which appears in $U_A^{-*} \subset \mathcal{H}$ is given a unique mixed structure such that the identification of $[F] \in \mathcal{H}$ with $\sum_{j \in \mathbb{Z}} (-1)^j [\mathcal{H} \otimes (F^j)] \in H_C$ gives the identification of $U_A^{-*} \subset \mathcal{H}$ with $U_A^* \subset H_A$.”

(p.137) in l.-8 and l.-7; $\mathcal{H}(R_{p,\mathcal{C}}) \rightarrow p^{\mathcal{H}}(R_{p,\mathcal{C}})$.

(p.137) in l.-4; “take $F^r(F) \simeq F$” $\Rightarrow$ “take $F^r(F) \simeq F$, as in [cb-E, Theorem 5.2]”

(p.138) At the end of the proof of Lemma 14.34; “To summarize, ...” $\Rightarrow$ “Thus, if we expand the product of canonical basis elements $b_1 b_2$ into a linear combination of canonical basis elements $\sum c(b_1 b_2)(v) b_3$ in $\mathcal{H}$, [cb-E, Theorem 5.4] shows that the polynomial $b_1 b_2(v)$ in the shift $v$ corresponds to the
polynomial $\frac{d^3}{b_1 b_2 b_3} (v)$ in $v = -[\sqrt{F}]^{-1}$ under the identification. Thus we get the isomorphism of the subalgebras $U^- \subset \mathcal{H}$ and $U^- \subset \mathcal{H}_A.$

(p.139) $\mathcal{R}_n \cong \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i[-1] \rightarrow [\mathcal{R}_n \cong \sum_{i \in \mathbb{Z}} [\mathcal{H}^i(\mathcal{R}_n \cong \sum_{i \in \mathbb{Z}})][-i]].$

(p.141) After Theorem 14.41; Add “The second part follows from Lemma 14.27(2) and [CG, Theorem 8.6.23]. Recall that canonical basis elements are given mixed structure as in Lemma 14.34.”

(p.142) “specialized Hall algebra” $\rightarrow$ “specialized Hall module”

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