Primes in sumsets

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Abstract. We obtain an upper bound for the number of pairs \((a, b) \in A \times B\) such that \(a + b\) is a prime number, where \(A, B \subseteq \{1, \ldots, N\}\) with
\[|A||B| \gg \frac{N^2}{(\log N)^2},\] \(N \geq 1\) an integer. This improves on a bound given by Balog, Rivat, and Sárközy.

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1. Introduction. Let \(A, B\) be subsets of \(\{1, \ldots, N\}\), \(N \geq 1\) an integer, and let us write \(P_{N;A,B}\) for the number of pairs \((a, b)\) in \(A \times B\) such that \(a + b\) is a prime number. One might expect that \(P_{N;A,B}\) is about \(\frac{|A||B|}{\log N}\), but there are subsets \(A, B\) of \(\{1, \ldots, N\}\) such that \(P_{N;A,B}\) is larger than what one expects. In [1, Section 5], A. Balog, J. Rivat, and A. Sárközy treated the problem of bounding \(P_{N;A,B}\) from above. Indeed, let us set \(R\) to be the quantity \(\frac{1000N}{|A|^{1/2}|B|^{1/2}}\). Then the following theorem is proved as Theorem 6 in [1] using the linear sieve.

Theorem 1.1. (Balog, Rivat, and Sárközy) We have \(P_{N;A,B} \ll \frac{|A||B|}{\log N} R\).

We begin by observing that this bound can be obtained using a very elementary counting argument, which we give in Subsection 2.7 below. Also note that this bound is implied by the trivial estimate \(P_{N;A,B} \leq |A||B|\) unless \(|A||B| \gg N^2/(\log N)^2\). In Section 4 we apply a method of D.S. Ramana and O. Ramaré [3], which was originally used to obtain an upper bound for additive energy of dense subsets of primes, to prove the following theorem, which is our main result.

Theorem 1.2. Let \(A, B\) be subsets of \(\{1, \ldots, N\}\), where \(N \geq 1\) is an integer and suppose further that \(|A||B| \gg \frac{N^2}{(\log N)^2}\). Then we have that \(P_{N;A,B} \ll \frac{|A||B|}{\log N} \log \log R\).
This upper bound for $P_{N;A,B}$ is optimal, up to the implied constant, in general. In fact, we have the following proposition, which corrects the conclusion of [1, Example 2, p. 36].

**Proposition 1.3.** Let $N$ be a positive integer, $k \ll \log \log N$ be an integer and $m_k = \prod_{p \leq k} p$. Then if $A = \{1 \leq a \leq N : a \equiv 0 \bmod m_k\}$ and $B = \{1 \leq b \leq N : b \equiv 1 \bmod m_k\}$, we have that

$$P_{N;A,B} \gg \frac{|A||B|}{\log N} \log \log R.$$

**Proof.** If $r_{A,B}(n)$ is the number of pairs $(a, b) \in A \times B$ such that $a + b = n$, then

$$P_{N;A,B} \geq \sum_{p \equiv 1 \bmod m_k} r_{A,B}(p). \quad (1)$$

We observe that for any integer $n \equiv 1 \bmod m_k$ and $n \geq 2m_k$, we have

$$r_{A,B}(n) \geq \left\lfloor \frac{n}{m_k} \right\rfloor \geq \frac{n}{2m_k}.$$

Using this lower bound for $r_{A,B}(p)$ in (1) when $N \geq 4m_k$, we get

$$P_{N;A,B} \geq \frac{1}{2m_k} \sum_{\substack{\frac{N}{4m_k} \leq p \leq N, \quad p \equiv 1 \bmod m_k}} p \geq \frac{N}{4m_k} \sum_{\substack{\frac{N}{m_k} \leq p \leq N, \quad p \equiv 1 \bmod m_k}} 1. \quad (2)$$

By the Chebyshev bound $\log m_k = \sum_{p \leq k} \log p \ll \log \log N$. Thus on using the Siegel–Walfisz theorem (see [2, page 419]), we have

$$\sum_{\substack{\frac{N}{m_k} \leq p \leq N, \quad p \equiv 1 \bmod m_k}} 1 \gg \frac{N}{\phi(m_k) \log N}. \quad (3)$$

Merten’s formula gives the the upper bound $\phi(m_k) \ll \frac{m_k}{\log \log m_k}$. Also, $|A| \sim \frac{N}{m_k}$, $|B| \sim \frac{N}{m_k}$, and therefore $R \sim m_k$, from the definition of $R$. From (2) and (3) we then get

$$P_{N;A,B} \gg \frac{N^2}{\phi(m_k)m_k \log N} \gg \frac{|A||B|}{\log N} \log \log R,$$

which proves the proposition. \qed

In Section 2 we record some preliminaries, mainly taken from [3], and give our proof of Theorem 1.1. In Section 3 we reduce the proof of Theorem 1.2 to the case of subsets which are well distributed to certain moduli. We then use this reduction to complete the proof of the Theorem 1.2 in Section 4, which is our final section.

Throughout this article we use $e(z)$ to denote $e^{2\pi iz}$ for any complex number $z$. Further, all constants implied by the symbols $\ll, \gg$ and the $O$ notation are absolute except when dependencies are indicated, either in words or by subscripts to these symbols. The Fourier transform of the characteristic function of a subset $A$ of $\mathbb{Z}$ is denoted by $\hat{A}$ and is defined by $\hat{A}(t) = \sum_{a \in A} e(at)$. 


Finally, the notations \([a, b], (a, b)\) etc. will denote intervals in \(\mathbb{Z}\) with end points \(a, b\) unless otherwise specified.

2. Preliminaries.

2.1. The large sieve inequality. The following is the classical large sieve inequality, which is proved on [4, page 68], for example.

Let \(N \geq 1\) be an integer and \(Q \geq 1\) be a real number. Then for any sequence of complex numbers \(\{a_n\}_{n=1}^N\) and real number \(\alpha\) if we set

\[
S(\alpha) = \sum_{1 \leq n \leq N} a_n e(n\alpha),
\]
we have

\[
\sum_{1 \leq q \leq Q} \sum_{a \bmod^* q} |S(a/q)|^2 \leq (N + Q^2) \sum_{1 \leq n \leq N} |a_n|^2.
\] (4)

2.2. The Brun–Titchmarsh inequality. If \(q, a\) are positive integers with \((a, q) = 1\), then for all \(q \leq x\), we have

\[
\pi(x; q, a) \leq \frac{2x}{\phi(q) \log (x/q)},
\] (5)

where \(\pi(x; q, a)\) denotes the number of primes not exceeding \(x\) and congruent to \(a\) modulo \(q\). For a proof see [5, page 121]. In particular, we have

\[
\pi(x; q, a) \leq \frac{4x}{\phi(q) \log x},
\] (6)

when \(q \leq x^{1/2}\).

2.3. An arithmetical function. For any integer \(q \geq 1\) and a positive real number \(L \geq 1\), let us set

\[
\omega(q, L) = - \sum_{\substack{1 \leq l \leq L, \\ l \equiv 0 \bmod q}} \frac{\mu(l) \log l}{l},
\] (7)

where \(\mu\) is the Möbius function. We then have the following estimates for \(\omega(q, L)\), proved in [3, Section 2.1]. Here \(\nu(q)\) denotes the number of prime divisors of \(q\).

Lemma 2.1. (i) For \(1 \leq q \leq L^{1/2}\), we have the asymptotic formula

\[
\omega(q, L) = \frac{\mu(q)}{\phi(q)} + O_\alpha \left( \frac{2^{\nu(q)} \log 2q}{q (\log L)^\alpha} \right),
\] (8)

for any \(\alpha > 0\) and (ii) for any \(q, L \geq 1\), we have

\[
|\omega(q, L)| \leq \frac{(\log 2L)^2}{q}.
\] (9)
2.4. An application of Davenport’s bound. Let \( N \geq 1 \) be an integer, \( L = N^{1/2} \) and set \( \Lambda^\flat(n) = - \sum_{d|n, \mu(d) \log d} \) for any integer \( n \).

The following lemma gives a uniform bound for the Fourier transform of the restriction of \( n \mapsto \Lambda^\flat(n) \) to the interval \([1, 2N]\).

**Lemma 2.2.** We have
\[
\sum_{1 \leq n \leq 2N} \Lambda^\flat(n) e(nt) \ll \frac{N}{(\log N)^{100}},
\]
for all \( t \in [0, 1] \).

*Proof.* The lemma follows from Davenport’s classical bound for \( \sum_{1 \leq n \leq x} \mu(n) e(nt) \), given by [2, Theorem 13.10 on p. 348], by an integration by parts. See [3, Section 3] for the details. \( \square \)

2.5. An optimisation principle. In our proof of the Theorem 1.2 we will use an optimization principle, a minor variant on a similar principle from [3]. We state this principle with the aid of the following notation.

Suppose that \( n, m \geq 1 \) are integers and let \( P_1, P_2, D_1, a n d \) \( D_2 \) be real numbers > 0. Further let
\[
K_1 = \left\{ (x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m x_i = P_1, \ 0 \leq x_i \leq D_1 \text{ for all } i \right\},
\]
and
\[
K_2 = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = P_2, \ 0 \leq x_i \leq D_2 \text{ for all } i \right\}.
\]

Let us also assume that \( K_1 \) and \( K_2 \) are non-empty sets. Then \( K_1, K_2 \) are compact and convex subsets of \( \mathbb{R}^m, \mathbb{R}^n \), respectively. Then we have:

**Lemma 2.3.** If \( f : \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbb{R} \) a bilinear form with real coefficients \( \alpha_{ij} \) defined by \( f(x, y) = \sum_{1 \leq i \leq m, 1 \leq j \leq n} \alpha_{ij} x_i y_j \), then

(i) there are extreme points \( x^* \) and \( y^* \) of \( K_1 \) and \( K_2 \), respectively, so that \( f(x, y) \leq f(x^*, y^*) \) for all \( x \in K_1, y \in K_2 \).

(ii) If \( x^* = (x_1^*, x_2^*, \ldots, x_m^*) \) is an extreme point of \( K_1 \), then, excepting at most one \( i \), we have either \( x_i^* = 0 \) or \( x_i^* = D_1 \) for each \( i \). Also, if \( l \) is the number of \( i \) such \( x_i^* \neq 0 \), then \( lD_1 \geq P_1 > (l - 1)D_1 \). A similar result holds for the extreme points of \( K_2 \).

*Proof.* An easy modification of the proof of [3, Proposition 2.2]. \( \square \)

2.6. A local problem. Let \( R \geq 1000 \) be real number, which we will eventually take to be \( \frac{10000N}{|A|^{1/2}|B|^{1/2}} \), as in Section 1. We then set
\[
U = \prod_{p \leq R} p, \quad M_R = \left( \frac{R \log R}{\log \log R} \right)^2 \quad \text{and} \quad Q = \log R \log \log R. \quad (10)
\]

Also, let \( I \) be the set of prime numbers not exceeding \( R \), and let \( J \) be a given subset of \( I \). We then write \( T_J(\mathcal{X}, \mathcal{Y}) \) for the number of pairs \((x, y)\) in \( \mathcal{X} \times \mathcal{Y} \)
such that \(x + y \not\equiv 0 \pmod{p}\) for each \(p\) in \(J\). The following lemma, which gives an upper bound for \(T_J(\mathcal{X}, \mathcal{Y})\), is [3, Proposition 2.3].

**Lemma 2.4.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be subsets of \(\mathbb{Z}/U\mathbb{Z}\) and \(t\) an integer satisfying \(1 \leq t \leq \min_{p \in J} p^{1/2}\). Then we have that

\[
T_J(\mathcal{X}, \mathcal{Y}) \leq |\mathcal{X}||\mathcal{Y}| \exp \left(-\sum_{p \in J} \frac{1}{p} \right) \exp \left(\frac{L(\mathcal{X}, \mathcal{Y})}{t} + t w(J)\right),
\]

where \(L(\mathcal{X}, \mathcal{Y}) = \log \left(\frac{U^2}{|\mathcal{X}||\mathcal{Y}|}\right)\) and \(w(J) = \sum_{p \in J} \frac{1}{p^2}\).

Lemma 2.4 easily leads to the proposition below, which gives an upper bound for \(T(\mathcal{X}, \mathcal{Y})\), the number of pairs \((x, y)\) in \(\mathcal{X} \times \mathcal{Y}\) such that \(x + y\) is an invertible element modulo \(U\), that is, \(x + y \not\equiv 0 \pmod{p}\) for all \(p\) in \(I\).

**Proposition 2.5.** Let \(\mathcal{X}\) and \(\mathcal{Y}\) be subsets of \(\mathbb{Z}/U\mathbb{Z}\) with \(|\mathcal{X}|, |\mathcal{Y}| \geq \frac{U}{M_R}\). Then we have

\[
T(\mathcal{X}, \mathcal{Y}) \leq \frac{\phi(P)}{P} |\mathcal{X}||\mathcal{Y}| \exp \left(\frac{36}{\log\log R}\right),
\]

where \(P = \prod_{Q^2 < p \leq R} p\).

**Proof.** Let \(J\) be the subset of \(I\) consisting of primes \(p\) such that \(Q^2 < p \leq R\). Then

\[
T(\mathcal{X}, \mathcal{Y}) \leq T_J(\mathcal{X}, \mathcal{Y})
\]

and by Lemma 2.4 applied to bound \(T_J(\mathcal{X}, \mathcal{Y})\), we see that for any integer \(1 \leq t \leq Q\) we have

\[
T(\mathcal{X}, \mathcal{Y}) \leq |\mathcal{X}||\mathcal{Y}| \exp \left(-\sum_{Q^2 < p \leq R} \frac{1}{p} \right) \exp \left(\frac{L(\mathcal{X}, \mathcal{Y})}{t} + t w(J)\right),
\]

where \(L(\mathcal{X}, \mathcal{Y}) \leq 9 \log R\) and \(w(J) = \sum_{Q^2 < p \leq R} \frac{1}{p^2} \leq \frac{2}{Q^2}\). Thus

\[
\exp \left(-\sum_{Q^2 < p \leq R} \frac{1}{p} \right) \leq \exp \left(\sum_{Q^2 < p \leq R} \frac{2}{p^2} \right) \prod_{Q^2 < p \leq R} (1-p^{-1}) \leq \frac{\phi(P)}{P} \exp \left(\frac{4}{Q^2}\right),
\]

on using \(-\log(1-x) \leq x + 2x^2\), valid for \(0 \leq x \leq 1/2\). From (13), (14), and the bounds for \(L(\mathcal{X}, \mathcal{Y})\) and \(w(J)\), we conclude that, for any integer \(1 \leq t \leq Q\), we have

\[
T(\mathcal{X}, \mathcal{Y}) \leq \frac{\phi(P)}{P} |\mathcal{X}||\mathcal{Y}| \exp \left(\frac{9 \log R}{t} + \frac{9t}{Q^2}\right),
\]

from which (12) follows on taking the integer \(t\) in the interval \([Q/2, Q]\) and recalling that \(Q = \log R \log\log R\). \(\square\)
2.7. Proof of Theorem 1.1. In this subsection we give a simple proof of the Theorem 1.1. To this end, for any \( A, B \subset \mathbb{N} \) and an integer \( n \), we set
\[
 r_{A,B}(n) = |\{(a, b) \in A \times B : a + b = n\}|, \tag{16}
\]
which for brevity we denote by \( r(n) \). Clearly, we have that
\[
 r(n) \leq \min(|A|, |B|), \tag{17}
\]
and that
\[
 P_{N;A,B} = \sum_{1 \leq p \leq 2N, p \text{ prime}} r(p). \tag{18}
\]
Using the bound (17) for \( r(n) \) and the Chebyshev bound for the number of primes not exceeding \( x \), we then get from (18) that
\[
 P_{N;A,B} \ll \frac{N}{\log N} \min(|A|, |B|) \ll \frac{N}{\log N} |A|^\frac{1}{2} |B|^\frac{1}{2} \ll \frac{|A||B|}{\log N} R, \tag{19}
\]
as required.

3. Reduction to well distributed subsets. Let \( N, A, \) and \( B \) be as in Theorem 1.2. In what follows we take \( R = \frac{1000N}{|A|^{1/2} |B|^{1/2}} \), as in Section 1, and for this \( R \), we define \( U, M_R \) and \( \Phi \) by (10). Also, for any subset \( Z \) of \( \mathbb{Z} \), we denote by \( \tilde{Z} \) the image of \( Z \) in \( Z/U \mathbb{Z} \) under the natural projection map from \( \mathbb{Z} \).

Let \( A_1 = A \cap [1, N^{1/8}] \), \( A_2 = A \cap [N^{1/8}, N] \). Then we have
\[
 P_{N;A,B} = P_{N;A_1,B} + P_{N;A_2,B} \leq |A_1||B| + P_{N;A_2,B} \ll \frac{|A||B|}{\log N} + P_{N;A_2,B}, \tag{20}
\]
when \( N \) is large enough, since \( |A_1| \leq N^{1/8}, |B| \leq N \), and \( \frac{N^2}{(\log N)^2} \ll |A||B| \).

We now estimate \( P_{N;A_2,B} \). To this end, for any \( a \) in \( Z/U \mathbb{Z} \) we define \( m(a) \) and \( n(a) \) to be, respectively, \( |\{x \in A_2 : x \equiv a \mod U\}| \), and \( |\{y \in B : y \equiv a \mod U\}| \), and then set
\[
 C_1 = \{a \in \tilde{A}_2 : m(a) \leq \frac{|A_2|}{U} \ M_R\},
\]
\[
 D_1 = \{b \in \tilde{B} : n(b) \leq \frac{|B|}{U} \ M_R\},
\]
\[
 C_2 = \{a \in \tilde{A}_2 : m(a) > \frac{|A_2|}{U} \ M_R\},
\]
\[
 D_2 = \{b \in \tilde{B} : n(b) > \frac{|B|}{U} \ M_R\}.
\]
Since \( \sum_{a \in \tilde{A}_2} m(a) = |A| \) and \( \sum_{b \in \tilde{B}} n(b) = |B| \), it follows that \( |C_2| \leq \frac{U}{M_R} \) and \( |D_2| \leq \frac{U}{M_R} \).
Let us now define
\[ A_3 = \{ x \in A_2 : x \equiv a \pmod{U} \text{ for some } a \in C_1 \}, \]
\[ B_1 = \{ y \in B : y \equiv b \pmod{U} \text{ for some } b \in D_1 \}, \]
\[ A_4 = \{ x \in A_2 : x \equiv a \pmod{U} \text{ for some } a \in C_2 \}, \]
\[ B_2 = \{ y \in B : y \equiv b \pmod{U} \text{ for some } b \in D_2 \}. \]

Then we have
\[ P_{N;A_2,B} = P_{N;A_3,B_1} + P_{N;A_3,B_2} + P_{N;A_4,B_1} + P_{N;A_4,B_2}. \] (21)

We first estimate \( P_{N;A_3,B_2} \). To do this, for any \( a \in A_3 = C_1 \) we define \( A_{3,a} \) by
\[ A_{3,a} = \{ x \in A_3 : x \equiv a \pmod{U} \}, \]
and similarly for any \( b \in B_2 = D_2 \) we define \( B_{2,b} \) by
\[ B_{2,b} = \{ y \in B_2 : y \equiv b \pmod{U} \}. \]

Then we have a partition of \( A_3 \) and \( B_2 \) as follows:
\[ A_3 = \bigcup_{a \in C_1} A_{3,a} \text{ and } B_2 = \bigcup_{b \in D_2} B_{2,b}. \]

Clearly, we then have
\[ P_{N;A_3,B_2} = \sum_{a \in C_1, b \in D_2} P_{N;A_{3,a},B_{2,b}}. \] (22)

The summand on the right of (22) can be estimated as
\[ P_{N;A_{3,a},B_{2,b}} \ll \frac{N}{\phi(U) \log N} |A_{3,a}|^{1/2} |B_{2,b}|^{1/2}. \] (23)

Indeed, if a pair \((x, y) \in A_{3,a} \times B_{2,b}\) is such that \(x + y\) is a prime \(p_{x,y}\), then \(p_{x,y} \equiv a + b \pmod{U}\). Since under the condition \(|A||B| \geq \frac{5000N^2}{(\log N)^2}\) we have \(R \leq \frac{2}{\phi(U) \log N}\) and thus \(U \leq N^{1/2}\) from (10) and the Chebyshev bounds, the Brun–Titchmarsh inequality (6) shows that there are at most \(\frac{4N}{\phi(U) \log N}\) such primes \(p_{x,y}\). Further, each such prime can be written in at most \(\min(|A_{3,a}|, |B_{2,b}|) \leq |A_{3,a}|^{1/2} |B_{2,b}|^{1/2}\) many ways as a sum \(x + y\), with \(x \in A_{3,a}, y \in B_{2,b}\). These remarks yield (23).

Using (23) in (22) we then get
\[ P_{N;A_3,B_2} \ll \frac{N}{\phi(U) \log N} \sum_{a \in C_1, b \in D_2} |A_{3,a}|^{1/2} |B_{2,b}|^{1/2}. \] (24)

By the Cauchy–Schwarz inequality applied to the sum on the right-hand side of (24), we have
\[ \sum_{a \in C_1, b \in D_2} |A_{3,a}|^{1/2} |B_{2,b}|^{1/2} \leq |C_1|^{1/2}|D_2|^{1/2} \left( \sum_{a \in A_3, b \in B_2} |A_{3,a}| |B_{2,b}| \right)^{1/2}. \] (25)
from which and (24) we get
\[ P_{N;A_3,B_2} \ll \frac{N}{\phi(U) \log N} |C_1|^{1/2} |D_2|^{1/2} |A_3|^{1/2} |B_2|^{1/2}. \]
Now using \(|C_1| \leq U, |D_2| \leq \frac{U}{M^R}\), and \(\phi(U) \gg \frac{U}{\log R}\), which follows from Mertens formula, we get that
\[ P_{N;A_3,B_2} \ll \frac{N}{\log N} \log R |A|^{1/2} |B|^{1/2}, \]
since \(A_3 \subset A, B_2 \subset B\). Now recalling the definitions \(R\) and \(M^R\), the latter from (10), we get that
\[ P_{N;A_3,B_2} \ll \frac{|A||B|}{\log N} \log \log R. \]
Similarly, we get the bounds
\[ P_{N;A_4,B_1}, P_{N;A_4,B_2} \ll \frac{|A||B|}{\log N} \log \log R. \]
Using these bounds in (21), we then see that
\[ P_{N;A_2,B} \ll P_{N;A_3,B_1} + \frac{|A||B|}{\log N} \log \log R. \quad (26) \]
Thus, from (26) and (20), we conclude that
\[ P_{N;A,B} \ll P_{N;A_3,B_1} + \frac{|A||B|}{\log N} \log \log R. \quad (27) \]
Therefore, to complete the proof of Theorem 1.2, we need to show that
\[ P_{N;A_3,B_1} \ll \frac{|A||B|}{\log N} \log \log R. \quad (28) \]
To do this, we may assume that
(i) \(|A_3| \geq \frac{2|A_2|}{U} M^R\) and \(|B_1| \geq \frac{2|B|}{U} M^R\).
Indeed, if (i) does not hold, say, \(|A_3| < \frac{2|A_2|}{U} M^R\), then using (19) and (10) we have the stronger conclusion that
\[ P_{N;A_3,B_1} \ll \frac{N}{\log N} |A_3|^{\frac{1}{2}} |B_1|^{\frac{1}{2}} \ll \frac{N}{\log N} \frac{|A_2|^{\frac{1}{2}} |B_1|^{\frac{1}{2}} M^{1/2}}{U^{1/2}} \ll \frac{|A||B|}{\log N}, \quad (29) \]
since \(U \geq e^{\frac{U}{2}}\), by the Chebyshev bound. A similar argument disposes the case \(|B_1| < \frac{2|B|}{U} M^R\) as well. By the definitions of \(A_3\) and \(B_1\) given at the beginning of this section, we also have
(ii) \(|\{x \in A_3 : x \equiv a (\text{mod } U)\}| \leq \frac{|A_3|}{U} M^R\) and \(|\{y \in B_1 : y \equiv b (\text{mod } U)\}| \leq \frac{|B|}{U} M^R\) for any \(a \in \tilde{A}_3, b \in \tilde{B}_1\),
(iii) each element of \(A_3\) is larger than \(N^{1/8}\).
These remarks bring us to our final section, where we shall prove (28) taking account of the conditions (i), (ii), and (iii) above and thereby complete the proof of the Theorem 1.2.
4. Proof Theorem 1.2. We shall prove (28) by closely following the method of [3]. We shall assume throughout that $N$ is a sufficiently large integer. We begin by noting that if $(a, b) \in A_3 \times B_1$ is such that $a + b$ is a prime number, then by (iii) above $N^{1/8} \leq a + b$ and consequently $\frac{1}{2} \log N \leq \Lambda(a + b)$, where $\Lambda$ is the Von Mangoldt function. It then follows that

$$P_{N; A_3, B_1} \log N \leq 8 \sum_{a \in A_3, b \in B_1} \Lambda(a + b).$$

We have the identity

$$\Lambda(n) = -\sum_{d \mid n, d \leq L} \mu(d) \log d.$$

We now estimate right hand side of (30). To this end, we set $L = N^{1/2}$ and write $\Lambda(n) = \Lambda^s(n) + \Lambda^b(n)$, where

$$\Lambda^s(n) = -\sum_{d \mid n, d \leq L} \mu(d) \log d, \quad \Lambda^b(n) = -\sum_{d \mid n, d > L} \mu(d) \log d.$$

Substituting $\Lambda(n) = \Lambda^s(n) + \Lambda^b(n)$ into (30), we get that

$$P_{N; A_3, B_1} \log N \ll \sum_n r(n) \Lambda^s(n) + \sum_n r(n) \Lambda^b(n),$$

where $r(n)$ is the number of pairs $(a, b) \in A_3 \times B_1$ such that $n = a + b$.

Let us first estimate the second sum on the right of above inequality. Since $r(n) = 0$ for $n$ not in the interval $[1, 2N]$, we have that

$$\sum_n r(n) \Lambda^b(n) = \int_0^1 \left( \sum_n r(n) e(-nt) \right) \left( \sum_{1 \leq n \leq 2N} \Lambda^b(n) e(nt) \right) dt,$$

by orthogonality of the functions $t \mapsto e(nt)$ on $[0, 1]$. By Lemma 2.2, we have that

$$\sum_{1 \leq n \leq 2N} \Lambda^b(n) e(nt) \ll \frac{N}{(\log N)^{100}}.$$  

From definition of $r(n)$ it immediately follows that $\sum_n r(n) e(-nt) = \widehat{A}_3(-t) \widehat{B}_1(-t)$. Consequently, we have from (32) and (33) that

$$\sum_n r(n) \Lambda^b(n) \ll \frac{N}{(\log N)^{100}} \int_0^1 |\widehat{A}_3(-t)||\widehat{B}_1(-t)| dt.$$ 

Applying the Cauchy–Schwarz inequality to the above integral and using the Parseval relation together with the fact that $|A_3| \leq |A|$, and $|B_1| \leq |B|$, we get that

$$\sum_n r(n) \Lambda^b(n) \ll \frac{N}{(\log N)^{100}} |A|^{1/2} |B|^{1/2}.$$
On recalling $R = \frac{1000N}{|A|^{1/2} |B|^{1/2}}$ and using the lower bound $|A||B| \gg \frac{N^{2}}{(\log N)^{2}}$, we obtain that
\[
\sum_{n} r(n) \Lambda^{\flat}(n) \ll |A||B|.
\] (35)

Now we estimate first term on the right of (30). On recalling the definition of $\Lambda^{\#}(n)$, we obtain
\[
\sum_{n} r(n) \Lambda^{\#}(n) = -\sum_{1 \leq d \leq L} \mu(d) \log d \sum_{n \equiv 0 \mod d} r(n),
\] (36)

after an interchange of summations. We note that
\[
\sum_{n \equiv 0 \mod d} r(n) e(an/d) = \frac{1}{d} \sum_{a \mod d} \sum_{n} r(n) e(an/q),
\] (37)

by orthogonality of characters on the group $\mathbb{Z}/d\mathbb{Z}$. On combining (37) with (36), interchanging summations and recalling definition of $\omega(q,L)$ from (7), we deduce that
\[
\sum_{n} r(n) \Lambda^{\#}(n) = \sum_{1 \leq q \leq L} \omega(q,L) \sum_{a \mod^{*} q} \widehat{A}_{3}(a/q) \widehat{B}_{1}(a/q).
\] (38)

We estimate the contribution to the sum on the right-hand side of (38) from $q$ satisfying $N^{1/8} < q \leq L$ by showing that
\[
\sum_{N^{1/8} < q \leq L} \omega(q,L) \sum_{a \mod^{*} q} \widehat{A}_{3}(a/q) \widehat{B}_{1}(a/q)
\ll N^{7/8} (\log N)^{2} |A|^{1/2} |B|^{1/2} \ll |A||B|.
\] (39)

Indeed, by (9) we have that the absolute value of the left side of (39) does not exceed
\[
\frac{(\log 2L)^{2}}{N^{1/8}} \sum_{1 \leq q \leq L} \sum_{a \mod^{*} q} |\widehat{A}_{3}(a/q)| |\widehat{B}_{1}(a/q)|
\leq \frac{(\log 2L)^{2}(N + L^{2}) |A_{3}|^{1/2} |B_{1}|^{1/2}}{N^{1/8}},
\] (40)

where we have applied the Cauchy–Schwarz inequality followed by the large sieve inequality (4) to the left-hand side of above relation. Since $L = N^{1/2}$, we see using $|A_{3}| \leq |A|, |B_{1}| \leq |B|$ and $|A||B| \gg \frac{N^{2}}{(\log N)^{2}}$ that (39) follows from (40).

We now consider the contribution to the sum on the right-hand side of (38) from $q$ in the range $1 \leq q \leq N^{1/8}$. We set
\[
T = \sum_{1 \leq q \leq N^{1/8}} \omega(q,L) \sum_{a \mod^{*} q} \widehat{A}_{3}(a/q) \widehat{B}_{1}(a/q),
\] (41)

and use asymptotic formula for $\omega(q,L)$ given by (8).
The contribution of error term of this asymptotic formula for $\omega(q,L)$ to $T$ is

$$\ll \frac{1}{(\log N)^{100}} \sum_{1 \leq q \leq N^{1/8}} \frac{2^{\nu(q)} \log 2q}{q} \sum_{a \mod^* q} |\hat{A}_3(a/q)| \, |\hat{B}_1(a/q)|,$$  \hspace{1cm} (42)

where we have used the value $L = N^{1/2}$ and $\alpha = 100$. Using the trivial bound $2^{\nu(q)} \log 2q \ll q$ we see that (42) is

$$\ll \frac{1}{(\log N)^{100}} \sum_{1 \leq q \leq N^{1/8}} \sum_{a \mod^* q} |\hat{A}_3(a/q)| \, |\hat{B}_1(a/q)|$$

$$\ll \frac{N}{(\log N)^{100}} |A|^{1/2} |B|^{1/2} \ll |A||B|,$$

where we have used the Cauchy–Schwarz inequality, large sieve inequality (4), and the bound $|A||B| \gg \frac{N^2}{(\log N)^2}$. Thus, we have

$$T = \sum_{1 \leq q \leq N^{1/8}} \frac{\mu(q)}{\phi(q)} \sum_{a \mod^* q} \hat{A}_3(a/q) \hat{B}_1(a/q) + O(|A||B|).$$  \hspace{1cm} (43)

We recall from (10) that $U = \prod_{p \leq R} p$. Then since $R = \frac{1000N}{|A|^{1/2}|B|^{1/2}}$ and $|A||B| \gg \frac{N^2}{(\log N)^2}$, we see that $U \leq N^{1/8}$ for all large $N$. We set $T(U)$ to be the sum over $q$ on the right-hand side of (43) restricted to $q|U$. Since for all other $q$ we have either $\mu(q) = 0$ or $q > R$, the triangle inequality applied to (43) shows that

$$|T - T(U)| \ll \sum_{R \leq q \leq N^{1/8}} \frac{1}{\phi(q)} \sum_{a \mod^* q} |\hat{A}_3(a/q)| \, |\hat{B}_1(a/q)| + O(|A||B|).$$  \hspace{1cm} (44)

We shall estimate the sum over $q$ in (44) by using $\frac{q}{\log \log q} \ll \phi(q) \ll q$, the Cauchy–Schwarz inequality, and the large sieve inequality, (4). Since $\frac{\log \log q}{q}$ decreases with $q$ for $q \geq 10$, we get that

$$|T - T(U)| \ll \frac{\log \log R}{R} \frac{N}{|A|^{1/2}|B|^{1/2}}$$

$$\ll |A||B| \log \log R,$$  \hspace{1cm} (45)

(46)

since $R = \frac{1000N}{|A|^{1/2}|B|^{1/2}}$.

Now we estimate $T(U)$. A simple argument using standard properties of Ramanujan sums, given below (3.23) on [3, p. 969] shows that

$$T(U) = \frac{U}{\phi(U)} |\{(a,b) \in A_3 \times B_1 : (a+b,U) = 1\}|.$$  \hspace{1cm} (47)

As before, we use $\tilde{A}_3$ to denote the image of $A_3$ under the natural projection from the set of all integers $\mathbb{Z}$ to $\mathbb{Z}/U\mathbb{Z}$ and similarly denote by $\tilde{B}_1$ the image of $B_1$. Further, for any residue class $a$ modulo $U$, let $m_{A_3}(a)$ be the number of elements of the set $A_3$ that belongs to this residue class. Similarly, we define
$m_B(b)$ for any residue class $b$ modulo $U$. Let $D_1 = \frac{|A_3|}{U} M_R$ and $D_2 = \frac{|B_1|}{U} M_R$. We then have used condition (ii) given at the end of the preceding section that

$$\sum_{a \in A_3} m_{A_3}(a) = |A_3| \quad \text{with} \quad 0 \leq m_{A_3}(a) \leq D_1,$$

and

$$\sum_{b \in B_1} m_{B_1}(b) = |B_1| \quad \text{with} \quad 0 \leq m_{B_1}(b) \leq D_2.$$

Let us set $c(a, b)$ to be 1 when $a + b$ is invertible modulo in $\mathbb{Z}/U\mathbb{Z}$ and to be 0 otherwise. Then from (47) we get that

$$T(U) = \frac{U}{\phi(U)} \sum_{(a, b) \in \tilde{A}_3 \times \tilde{B}_1} c(a, b) m_{A_3}(a) m_{B_1}(b). \quad (48)$$

We estimate the above sum with the help of the optimization principle given in Subsection 2.5. From Lemma 2.3, we then have that

$$\sum_{(a, b) \in \tilde{A}_3 \times \tilde{B}_1} c(a, b) x_a^* y_b^* \leq \sum_{(a, b) \in \tilde{A}_3 \times \tilde{B}_1} c(a, b) x_a^* y_b^*, \quad (49)$$

for some $x_a^*$ and $y_b^*$ with $a$ varying over $\tilde{A}_3$ and $b$ varying over $\tilde{B}_1$ , satisfying the following conditions. All the $x_a^*$ are either 0 or $D_1$, excepting at most one, which must lie in $(0, D_1)$ and similarly, all $y_b^*$ are either 0 or $D_2$, excepting at most one, which must lie in $(0, D_2)$. Moreover, if $\mathcal{X}$ and $\mathcal{Y}$ denote, respectively, the subsets of $\tilde{A}_3$ and $\tilde{B}_1$ for which $x_a^* \neq 0$ and $y_b^* \neq 0$ , then $|\mathcal{X}| D_1 \geq |A_3| \geq (|\mathcal{X}| - 1) D_1$ and $|\mathcal{Y}| D_2 \geq |B_1| \geq (|\mathcal{Y}| - 1) D_2$. Thus, from this we have the bounds

$$D_1 \leq \frac{A_3}{|\mathcal{X}| - 1} \leq \frac{2 |A_3|}{|\mathcal{X}|},$$

and

$$D_2 \leq \frac{B_1}{|\mathcal{Y}| - 1} \leq \frac{2 |B_1|}{|\mathcal{Y}|},$$

where we use $|\mathcal{X}| \geq \frac{|A_3|}{D_1} \geq 2$ and $|\mathcal{Y}| \geq \frac{|B_1|}{D_2} \geq 2$, valid by condition (i) given at the end of Section 3. These bounds on $D_1, D_2$ together with (48) and (49) give

$$T(U) \ll \frac{U}{\phi(U)} \frac{|A_3||B_1|}{|\mathcal{X}| |\mathcal{Y}|} \sum_{(a, b) \in \mathcal{X} \times \mathcal{Y}} c(a, b), \quad (50)$$

Note that $\mathcal{X}, \mathcal{Y}$ are subsets of $\mathbb{Z}/U\mathbb{Z}$ with $|\mathcal{X}| \geq \frac{|A_3|}{D_1} \geq \frac{U}{M_R}$ and $|\mathcal{Y}| \geq \frac{|B_1|}{D_2} \geq \frac{U}{M_R}$ and that the sum on the right of the above relation is nothing but $T(\mathcal{X}, \mathcal{Y})$ of Proposition 2.5, which gives

$$T(U) \ll \frac{U}{\phi(U)} \frac{\phi(P)}{P} |A_3||B_1| \exp \left( \frac{36}{\log \log R} \right), \quad (51)$$
where \( P = \prod_{Q^2 < p \leq R} U = \prod_{p \leq R} \) and \( Q = \log R \log \log R \). By Merten’s formula we then get

\[
T(U) \ll |A||B| \log \log R \exp \left( \frac{C}{\log \log R} \right) \ll |A||B| \log \log R.
\]

(52)

From (52), (46), (39), (35), and (31), we then conclude that

\[
P_{N; A_3, B_1} \ll \frac{|A||B|}{\log N} \log \log R,
\]

which together with (27) yields Theorem 1.2.

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