ON THE ARITHMETIC OF WEIGHTED COMPLETE INTERSECTIONS OF LOW DEGREE

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Abstract. Following work of de Jong and Starr, we prove that smooth complete intersections in complex weighted projective space of low enough degrees are rationally simply connected. This implies in particular (due to work of Hassett and Pirutka) that smooth weighted complete intersections over function fields of smooth complex curves satisfy weak approximation at all places, that $R$–equivalence of rational points is trivial, and that the Chow group of zero cycles of degree zero is zero.

1. Introduction

For a variety defined over a field $K$, a basic question is whether it has points defined over $K$, and, if so, what can be said about this set of points. If $K$ is the function field of a smooth curve, these questions have nice geometric interpretations. An example is that of weak approximation. Geometrically, it can be formulated as follows.

Let $B$ be a smooth, connected curve over an algebraically closed field (say, $\mathbb{C}$). Let $K$ be the function field of $B$, and $S \subset B$ a proper, closed subscheme. Let $X$ be a smooth $K$–scheme, and $\mathcal{X}$ a (regular) model for $X$, i.e. a smooth, proper morphism $\pi : \mathcal{X} \to B$ with generic fiber $X$. Then $X$ satisfies weak approximation with respect to $B$ and $S$ if for every model and every $B$–morphism $\sigma : S \to \mathcal{X}$, $\sigma$ can be extended to a $B$–morphism defined on an open subset $U \subset B$ that contains $S$. $X$ satisfies weak approximation with respect to $B$ (or, equivalently, over $K$) if for every proper, closed subscheme $S \subset B$, $X$ satisfies weak approximation with respect to $B$ and $S$.

Kollár, Miyaoka and Mori showed in [KMM92] that if $X$ is rationally connected and a model admits a section, then this section enjoys good approximation properties. Years later, Graber, Harris and Starr proved in [GHS03] the existence of such sections. Finally, Hassett and Tschinkel proved in [HT06] that weak approximation holds at places of good reduction (i.e. at points $b \in B$ for which there is a model having smooth fiber over $b$). They also conjectured that every rationally connected variety over a function field $K$ as above satisfies weak approximation. After that, several people managed to prove weak approximation in many cases by studying explicitly singular fibers (and assuming that the singularities involved are not too wild). We refer the reader to [Has10] for a survey of the work that has been done in this direction. Here we adopt a different method.

Surprisingly, weak approximation is related to the notion of rational simple connectedness, analogous to that of simple path connectedness in topology (we will give
a precise definition of this notion in section 2). In particular, by a Theorem of Hassett ([Has10], Theorem 4.7), if for a smooth projective variety $X$ over $K := \mathbb{C}(B)$, $X_K$ is rationally simply connected, then $X$ satisfies weak approximation. By the Lefschetz principle, it is enough to check that $X_C$ is rationally simply connected.

Furthermore, it was proved by Pirutka that rational simple connectedness has other interesting consequences. Given a projective variety $X$ over a field $K$, we say that two $K$-points $x, y \in X(K)$ are directly $R$-equivalent if there is a morphism $\mathbb{P}^n_K \to X$ such that $x$ and $y$ belong to the image of $\mathbb{P}^n_K(K)$. One obtains an equivalence relation (due to Manin) called $R$-equivalence, whose set of classes is denoted $X(K)/R$. In [Pir12], Pirutka proves that if $X$ is a rationally simply connected variety over the function field $K$ of a smooth complex curve, then $X(K)/R = 1$. Moreover, she proves that with the same setup, the Chow group of zero cycles of degree zero is zero.

In [dJS06b], de Jong and Starr proved that complete intersections in projective space of low enough degree are rationally simply connected (in a stronger sense than the one intended in this paper), and therefore they satisfy all the properties mentioned above. Using the same technique, we prove a similar result for complete intersections $X_{d_1, \ldots, d_c}$ of degrees $d_1, \ldots, d_c$ in weighted projective space $\mathbb{P}_K(e_0, \ldots, e_n)$.

For convenience, we collect here the conditions on weights of the projective space and degrees of the complete intersection that will be useful throughout the paper:

**Hypothesis 1.1** (Main hypothesis). Let $X_{d_1, \ldots, d_c} \subset \mathbb{P}(e_0, \ldots, e_n)$ be a weighted complete intersection of degrees $d_1, \ldots, d_c$. We say that $X$ satisfies the main hypothesis if the following conditions are satisfied:

1. $X$ has dimension at least 3, i.e. $c \leq n - 3$;
2. $e_0 = e_1 = e_2 = e_3 = 1$;
3. $e_4 + \ldots + e_n + 3 + c - n \leq d_1 + \ldots + d_c$;
4. $d_1 + \ldots + d_c \leq e_4 + \ldots + e_n + 1$; and
5. $d_1^3 + \ldots + d_c^3 \leq 4 + e_4^3 + \ldots + e_n^3$.

Our main result is the following:

**Theorem 1.2.** Let $X_{d_1, \ldots, d_c} \subset \mathbb{P}_C(e_0, \ldots, e_n)$ be a smooth weighted complete intersection that satisfies the main hypothesis. Then $X_{d_1, \ldots, d_c}$ is rationally simply connected.

As already mentioned, by [Has10], Theorem 4.7, and [Pir12], Theorem 1.5, we have the following Corollary:

**Corollary 1.3.** Let $K$ be the function field of a smooth, complex, connected curve. Let $X_{d_1, \ldots, d_c} \subset \mathbb{P}_K(e_0, \ldots, e_n)$ be a smooth weighted complete intersection that satisfies the main hypothesis. Then:

1) $X_{d_1, \ldots, d_c}$ satisfies weak approximation at all places;
2) $X(K)/R = 1$; and
3) $\deg: CH_0(X) \to \mathbb{Z}$ is bijective.

We should remark that if $e_4 + \ldots + e_n + 3 + c - n > d_1 + \ldots + d_c$, then $X \simeq \mathbb{P}^{n-c}$ by a characterization of projective space of Cho, Miyaoka and Shepherd-Barron (see [CMSB02]). Therefore it is actually a quite natural hypothesis.
The easiest case in which the Theorem applies is that of smooth degree \( b \) covers of \( \mathbb{P}^{n-1} \) branched along a hypersurface of degree \( b \cdot e \). Namely, consider the case \( e_4 = \ldots = e_{n-1} = 1, e := e_n \geq 1, b \geq 1 \), \( X_{be} \subset \mathbb{P}_K(1, \ldots, 1, e) \) smooth weighted hypersurface of degree \( be \). Then \( X_{be} \) satisfies the properties of Corollary 1.3 if \( n \geq \text{max}\{4, be - e + 3, b^2e^2 - e^2\} \). In the case of double covers, i.e. \( b = 2 \), the condition becomes \( n \geq 3e^2 \) for \( e \geq 2 \), and \( n \geq 4 \) for \( e = 1 \).

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2. Preliminaries

We collect here some general results that are going to be used later. We firstly review some useful properties of weighted projective spaces and weighted complete intersections. We refer the reader to [Dol82] and [Mor75] for proofs and further discussions.

Let \( F \) be a field of characteristic 0. Let \( \mathbb{P}_F := \mathbb{P}_F(e_0, \ldots, e_n) \) be the *weighted projective space* of dimension \( n \) and weights \( e_0, \ldots, e_n \). This means that \( \mathbb{P}_F = \text{Proj} F[x_0, \ldots, x_n] \), where \( x_i \) has degree \( e_i \) for \( i = 0, \ldots, n \). In this paper, \( F \) will be either the field of complex numbers (in which case the subscript in \( \mathbb{P}_F \) will be omitted), or the function field \( K \) of a smooth, connected, complex curve. We can always assume that any \( n + 1 \) weights are coprime. Weighted projective spaces behave in many ways like usual projective space, and in many other ways differently.

Let \( V_k \) be the closed subset of \( \mathbb{P} \) defined by the ideal \( < x_i \mid k \nmid e_i > \). Then \( \text{Sing}(\mathbb{P}) = \cup_{k \geq 1} V_k \), and \( \mathbb{P}^o := \mathbb{P} \setminus \text{Sing}(\mathbb{P}) \) is called *weak projective space*.

The weighted projective space \( \mathbb{P} \) comes with a sheaf \( \mathcal{O}_{\mathbb{P}}(1) \) which in general is neither invertible, nor ample, although it becomes both once restricted to \( \mathbb{P}^o \).

A generalization of Euler’s sequence holds once we restrict to the weak projective space:

\[
0 \to \mathcal{O}_{\mathbb{P}^o} \to \oplus_{i=0}^n \mathcal{O}_{\mathbb{P}}(e_i) \to T_{\mathbb{P}^o} \to 0.
\]

For \( f_1, \ldots, f_c \) homogeneous elements of \( \text{Proj} \mathbb{C}[x_0, \ldots, x_n] \) of degrees \( d_1, \ldots, d_c \), that form a regular sequence, \( \text{Proj} \mathbb{C}[x_0, \ldots, x_n]/(f_1, \ldots, f_c) \) is a *complete intersection* in \( \mathbb{P} \) (also called *weighted complete intersection*) of degrees \( d_1, \ldots, d_c \) (and codimension \( c \)). Denote such a weighted complete intersection by \( X_{d_1, \ldots, d_c} \subset \mathbb{P} \). If a weighted complete intersection \( X_{d_1, \ldots, d_c} \subset \mathbb{P} \) is smooth, then \( X_{d_1, \ldots, d_c} \subset \mathbb{P}^o \).

A weaker notion than smoothness is *quasismoothness*. A closed subvariety \( Z \subset \mathbb{P} \) is *quasismooth* if its preimage under the canonical projection \( \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P} \) is smooth. This implies that if \( Z \) is quasismooth, \( Z \cap \mathbb{P}^o \) is smooth.

If \( X := X_{d_1, \ldots, d_c} \subset \mathbb{P} \) is a smooth weighted complete intersection, then \( \mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}^o}(1)|_X \) is an ample invertible sheaf, and we have the following formula for the canonical sheaf:

\[
\omega_X = \mathcal{O}_X \left( \sum d_i - \sum e_i \right).
\]

All degrees will be intended with respect to \( \mathcal{O}_X(1) \), unless otherwise specified.
With respect to cohomology, we have the following result (see [Dol82], Corollary 2.3.6, and [Dim92], Appendix B, Theorem B.22):

**Theorem 2.1.** Let $X \subset \mathbb{P}$ be a (not necessarily smooth) weighted complete intersection. Then for $k = 0, 1, \ldots, n$, we have $H^k(\mathbb{P}, \mathbb{Q}) \cong \mathbb{Q}$ if $k$ is even, and $H^k(\mathbb{P}, \mathbb{Q}) \cong 0$ if $k$ is odd.

The restriction homomorphism

$$H^k(\mathbb{P}, \mathbb{Q}) \to H^k(X, \mathbb{Q})$$

is an isomorphism for $k = 0, \ldots, \text{dim}(X) - 1$, and is injective for $k = \text{dim}(X)$.

The following computation will be useful in section 4.

**Lemma 2.2.** Let $X \coloneqq X_{d_1, \ldots, d_c} \subset \mathbb{P}(e_0, \ldots, e_n)^{\circ}$ be a smooth weighted complete intersection. Then

$$\text{ch}_2(T_X) = \frac{1}{2} \left( \sum e_i^2 - \sum d_j^2 \right) c_1(\mathcal{O}_{\mathbb{P}}(1))^2.$$

**Proof.** Euler’s generalized sequence

$$0 \to \mathcal{O}_{\mathbb{P}} \to \oplus_{j=0}^n \mathcal{O}_{\mathbb{P}}(e_j) \to T_{\mathbb{P}} \to 0$$

gives the relation

$$\text{ch}(T_{\mathbb{P}}) = \sum_{j=0}^n e_j c_1(\mathcal{O}_{\mathbb{P}}(e_j)) - 1.$$

The normal exact sequence

$$0 \to T_X \to T_{\mathbb{P}}|_X \to \oplus_{i=1}^c \mathcal{O}_{\mathbb{P}}(d_i)|_X \to 0$$

gives the relation

$$\text{ch}(T_X) = \text{ch}(T_{\mathbb{P}}|_X) - \sum_{i=1}^c e_i c_1(\mathcal{O}_{\mathbb{P}}(1)).$$

which together with the previous one gives

$$\text{ch}(T_X) = \sum_{j=0}^n e_j c_1(\mathcal{O}_{\mathbb{P}}(1)) - 1 - \sum_{i=1}^c e_i c_1(\mathcal{O}_{\mathbb{P}}(1)).$$

Looking at the part in degree 2, we get:

$$\text{ch}_2(T_X) = \frac{1}{2} \left( \sum e_i^2 c_1(\mathcal{O}_{\mathbb{P}}(1))^2 - \sum d_j^2 c_1(\mathcal{O}_{\mathbb{P}}(1))^2.\right.$$

\[ \square \]

We will use the following extendability result:

**Proposition 2.3.** Let $\mathcal{P} \coloneqq \mathbb{P}(e_0, \ldots, e_n)$, $Y \subset \mathbb{P}(e_0, \ldots, e_n)$, $\Pi$, $Y$ and $Y \cap \mathcal{P}$ quasismooth, $F_i \in \Gamma(Y, \mathcal{O}_Y(d_i))$ for $i = 1, \ldots, c$ such that $Z_i \coloneqq \mathcal{Z}(F_i)$ does not contain $Y \cap \mathcal{P}$. Assume that $\mathcal{Z}(F_1, \ldots, F_c) \cap \mathcal{P}$ is quasismooth. Then there are $G_i \in \Gamma(\Pi, \mathcal{O}_\Pi(d_i))$ such that $\mathcal{Z}(F_1 + G_1|_Y, \ldots, F_c + G_c|_Y)$ is quasismooth.
Proof. We proceed by induction. Consider the case $c = 1$ first. Then the result follows by looking at the subsystem of $[\mathcal{O}_Y(d_1)]$ spanned by $F$ and the image of $I_{\mathcal{O}} \cdot \mathcal{O}_\Pi(d_1) \to \mathcal{O}_Y(d_1)$. In fact, by Bertini’s Theorem (see [GH94], page 137), a generic element of the subsystem is quasismooth.

In particular, every irreducible component of the subsystem is quasismooth. In any event, the following Theorem (see [CK99], 7.1.4) describes the local structure of the subsystem of $\mathcal{O}_X$. We can apply the previous step to $W \cap \mathcal{O}_X$, and is smooth, and we will say that $\mathcal{O}_X$ is quasismooth. Therefore $W \cap [\mathcal{O}_X + G_c \mathcal{O}_X]$ is quasismooth.

In particular, we will apply the Proposition in the situation where $X_{d_1, \ldots, d_c}$ equals $Y \cap \mathbb{P}$ and is smooth, and we will say that $X_{d_1, \ldots, d_c}$ extends to $Y$ (which is also smooth).

A key role in what follows will be played by Kontsevich moduli space of stable maps $\overline{M}_{g,m}(X, \beta)$. We recall here its basic properties, and refer the reader to [FP97] and [CK99] for further details.

**Definition 2.4.** Let $X$ be a smooth, projective, complex variety. Let $\beta \in CH_1(X)$ be the class of a curve. An $m$–pointed, genus $g$ stable map to $X$ is the datum $(C, p_1, \ldots, p_m, f)$ of:

1) a projective, connected, reduced, at worst nodal curve $C$ of arithmetic genus $g$;
2) $m$ distinct smooth marked points $p_1, \ldots, p_m$ in $C$;
3) a morphism $f: C \to X$ satisfying the following stability condition: every contracted component of $C$ via $f$ must contain at least 3 distinguished points (i.e. nodes or marked points).

**Theorem 2.5.** Let $X$ be a smooth, projective, complex variety. Let $\beta \in CH_1(X)$ be the class of a curve. Then there exists a projective, coarse moduli scheme $\overline{M}_{g,m}(X, \beta)$ over $\mathbb{C}$ parametrizing isomorphism classes of $m$–pointed, genus $g$ stable maps to $X$ such that $f_*(\{C\}) = \beta$.

In what follows, we will be interested in the case $g = 0$ (although similar statements hold for any genus).

In general, $\overline{M}_{0,m}(X, \beta)$ is neither smooth nor irreducible, and it might have irreducible components of different dimensions. In any event, the following Theorem (see [CK99], 7.1.4) describes the local structure of $\overline{M}_{0,m}(X, \beta)$. Here, $f^*\Omega^1_X \to \Omega^1_C(\sum_{i=1}^m p_i)$ is a complex in degrees $-1$ and $0$, and $Ext^k$ for $k = 1, 2$ denotes the hyperext groups.

**Theorem 2.6.** Let $(C, p_1, \ldots, p_m, f) \in \overline{M}_{0,m}(X, \beta)$ be a stable map. Then $\overline{M}_{0,m}(X, \beta)$ is defined locally around $(C, p_1, \ldots, p_m, f)$ by $\dim Ext^2(f^*\Omega^1_X \to \mathcal{O}_C(\sum_{i=1}^m p_i), \mathcal{O}_C)$ equations in a nonsingular scheme of dimension $\dim Ext^1(f^*\Omega^1_X \to \Omega^1_C(\sum_{i=1}^m p_i), \mathcal{O}_C)$. In particular, every irreducible component of $\overline{M}_{0,m}(X, \beta)$ has dimension at least

$$\dim Ext^2(f^*\Omega^1_X \to \Omega^1_C(\sum_{i=1}^m p_i), \mathcal{O}_C) - \dim Ext^1(f^*\Omega^1_X \to \Omega^1_C(\sum_{i=1}^m p_i), \mathcal{O}_C) = -K_X \cdot \beta + \dim(X) + m - 3.$$
To prove rational simple connectedness, it will be crucial to analyze the fibers of suitable evaluation morphisms. The following Theorem collects some known properties (see [dJS06b], Lemmas 4.1, 5.1, 5.3, 5.5 for proofs or further references):

**Theorem 2.7.** Let $X$ be a smooth, projective variety, and $\mathcal{O}_X(1)$ a very ample invertible sheaf. Let $\alpha$ be a curve class of $\mathcal{O}_X(1)$–degree 1. Then:

Assume that $\text{ev}_1 : \overline{M}_{0,1}(X, \alpha) \to X$ is dominant. Then a general fiber of $\text{ev}_1$ is smooth and projective, and every connected component has dimension

$$-K_X \cdot \alpha - 2.$$

Assume that $\text{ev}_2 : \overline{M}_{0,2}(X,2\alpha) \to X \times X$ is dominant. Then a general fiber of $\text{ev}_2$ is a union of free curves. Therefore a general fiber is smooth, of the expected dimension

$$-K_X \cdot (2\alpha) - \dim(X) - 1,$$

and the intersection with the boundary is a simple normal crossing divisor. Moreover, every point in a general fiber parametrizes an automorphism-free stable map.

We will prove these properties in the case $l_0 = 2$ directly. The proof for $l \geq 2$ will follow by the existence of $1$–twisting surfaces and the following Lemma (see [dJS06b], Lemma 3.5):

**Lemma 2.9.** Let $M_{\alpha,0}$ be an irreducible component of $\overline{M}_{0,0}(X, \alpha)$ whose general point parametrizes a smooth, free curve. Denote by $M_{\alpha,1}$ the unique irreducible component of $\overline{M}_{0,1}(X, \alpha)$ dominating $M_{\alpha,0}$.

Assume that the generic fiber of the restriction $\text{ev}|_{M_{\alpha,1}} : M_{\alpha,1} \to X$ is geometrically irreducible.

Then for every positive integer $l$ there is a unique irreducible component $M_{\alpha,0}$ of $\overline{M}_{0,0}(X, l\alpha)$ whose general point parametrizes a smooth, free curve. Denote by $M_{\alpha,1}$ the unique irreducible component of $\overline{M}_{0,1}(X, l\alpha)$ dominating $M_{\alpha,0}$. Then the generic fiber of the restriction $\text{ev}|_{M_{\alpha,1}} : M_{\alpha,1} \to X$ is geometrically irreducible.

We will show in section 3 that the condition of Lemma 2.9 is satisfied. Therefore we can define $M_{\alpha,2}$ as the unique irreducible component of $\overline{M}_{0,2}(X, l\alpha)$ dominating $M_{\alpha,0}$. These components will be the ones of Definition 2.8. The conditions on the general fiber of $\text{ev}_2|_{M_{\alpha,2}} : M_{\alpha,2} \to X \times X$ will follow from Lemma 8.6 in [dJS06b].

### 3. Pointed curves of degree 1

The main goal of this section is to prove that the evaluation map

$$\text{ev}_1 : \overline{M}_{0,1}(X, \alpha) \to X$$

is dominant and it has irreducible generic fiber. Dominance of $\text{ev}_1$ follows from [Ko96], V, 4.11.2. Since by Theorem 2.7 a general fiber of $\text{ev}_1$ is smooth, to prove
irreducibility it is enough to prove connectedness. Our method traces back to Bertini, and in modern terms can be stated as follows:

**Proposition 3.1.** Let $h : M \to X$ be a projective morphism with $X$ a smooth, algebraically simply connected variety and $M$ a normal, quasi-projective scheme. If the closed subscheme $M_h$ of $M$ where $h$ is not smooth has codimension at least 2, then the geometric generic fiber of $h$ is connected.

**Proof.** Let $u : \overline{M} \to X$ be the finite part of the Stein factorization of $h$. Since $M_h$ has codimension at least 2, $\overline{M}_u$ also has codimension at least 2. Since $X$ is smooth, $u$ is étale by the Purity Theorem ([Gro05], X, section 3). Since $X$ is algebraically simply connected, $u$ is an isomorphism. Therefore $h$ has connected fibers. \qed

Now consider the case where $X$ is our weighted complete intersection, $M$ equals $\overline{M}_{0,1}(X,1)$, and $h$ is the evaluation morphism. Recall that since $X$ is rationally connected, it is algebraically simply connected (see [Deb04], Corollary 4.18). Let $N \subset M$ be the closed subset where the derivative $d(ev_1)$ of $ev_1 : \overline{M}_{0,1}(X,1) \to X$ is not surjective.

By Proposition 3.1, if we show that $\text{codim}(N,M) \geq 2$, then the generic fiber of $ev$ will be connected. To this end, we will let $X$ vary and work with a suitable incidence correspondence.

**Proposition 3.2.** Let $X := X_{d_1,...,d_c} \subset \mathbb{P}(e_0,...,e_n)$ be a smooth weighted complete intersection of degrees $d_1,...,d_c$. Assume that $e_0 = e_1 = e_2 = 1$ and $e_3 + ... + e_n + 2 + c - n \leq d_1 + ... + d_c \leq e_3 + ... + e_n$. Then the generic geometric fiber of $ev_1 : \overline{M}_{0,1}(X,\alpha) \to X$ is connected.

**Proof.** Let $H := H_{n,d_1,...,d_c}$ be the parameter space of $c$-uples $(f_1,...,f_c)$ of polynomials of degrees $d_1,...,d_c$ in $\mathbb{P}$, for some $d_1,...,d_c,e_3,...,e_n,n$ such that $d_1 + ... + d_c \leq e_3 + ... + e_n - 2$. Let $H' := H_{n,d_1,...,d_c}$ be the parameter space of pointed $c$-uples of polynomials in $\mathbb{P}$, i.e. the incidence correspondence $\{(f_1,...,f_c,x) | x \in Z(f_1,...,f_c)\} \subset H \times \mathbb{P}$. Let $I := \{(f_1,...,f_c,x,[f]) | x \in f(\mathbb{P}) \subset Z(f_1,...,f_c)\} \subset H' \times \text{Mor}_1(\mathbb{P}^1,\mathbb{P})$.

Let $p : I \to H', q : I \to \text{Mor}_1(\mathbb{P}^1,\mathbb{P})$ and $\pi : H' \to H$ be the projection morphisms, and let $W \subset I$ be the closed subset where the derivative homomorphism $dp : T_I \to p^*T_{H'}$ is not surjective. We want to show that $\text{codim}(W,I) \geq 2$.

**Claim:** If $\text{codim}(W,I) \geq 2$, then the generic geometric fiber of $ev_1 : \overline{M}_{0,1}(X,\alpha) \to X$ is connected.

**Proof.** If $X = Z(f_1,...,f_c)$, by base change to $H'_X := \pi^{-1}(f_1,...,f_c)$ we have a diagram:

\[
\begin{array}{ccc}
I_X & \longrightarrow & I \\
\downarrow{p_X} & & \downarrow{p} \\
H'_X & \longrightarrow & H'
\end{array}
\]

and the left hand side is the evaluation morphism $ev : \text{Mor}_1(\mathbb{P}^1,X) \times \mathbb{P}^1 \to X$. Since $p$ is smooth away from codimension 2, $ev$ is smooth away from codimension 2. The morphism $ev$ fits into the diagram:
where \( Q \) realizes \( \overline{\mathcal{M}_{0,1}}(X, \alpha) \) as an fpqc quotient of \( Mor_1(\mathbb{P}^1, X) \times \mathbb{P}^1 \) by \( Aut(\mathbb{P}^1, 0) \).

By [Gro67], Proposition 17.7.7, since \( ev \) is smooth away from codimension 2, therefore to show that \( M_{0,1}(\mathbb{P}^1, X) \times \mathbb{P}^1 \) is smooth away from codimension 2, it is enough to show that \( ev_1 \) is smooth away from codimension 2.

The Claim follows from Proposition 3.1 once we show that \( \overline{\mathcal{M}_{0,1}}(X, \alpha) \) is regular in codimension 1. By Theorem 2.7, a general fiber of \( ev_1 \) has the expected dimension, and hence it is a local complete intersection. Therefore \( \overline{\mathcal{M}_{0,1}}(X, \alpha) \) has the expected dimension, and hence it is a local complete intersection. Thus by Proposition 8.23 in [Har77], \( \overline{\mathcal{M}_{0,1}}(X, \alpha) \) is normal.

\[ \Box \]

To prove that \( codim(W, I) \geq 2 \), it is enough to show that \( codim(W \cap q^{-1}([f]), q^{-1}([f])) \geq 2 \) for every \([f] \in Mor_1(\mathbb{P}^1, \mathbb{P})\).

Every morphism \( f : \mathbb{P}^1 \to \mathbb{P} \) of degree 1 is of the form:

\[ f : [s : t] \mapsto [a_0s + b_0t : a_1s + b_1t : c^2(s, t) : \ldots : c^n(s, t)], \]

where \( c^k(s, t) \) is a polynomial of degree \( e_k \) in \( s \) and \( t \). Let us define \( C := f(\mathbb{P}^1) \). For a fixed morphism of degree 1, \( f \), as above, if \( a_0b_1 - b_0a_1 \neq 0 \), there is an automorphism \( \phi \) of \( \mathbb{P} \) such that \( \phi \circ f(s, t) = [s : t : \ldots : 0] \), namely:

\[ \phi : [x_0 : \ldots : x_n] \mapsto \left[ \frac{b_1x_0 - b_0x_1}{a_0b_1 - b_0a_1} : a_0x_1 - a_1x_0 : \frac{x_2 - c^2(x_0, x_1)}{a_0b_1 - b_0a_1} : \ldots : x_n - c^n(x_0, x_1) \right]. \]

In particular, \( C \) is smooth. Let us denote \( V \subset Mor_1(\mathbb{P}^1, \mathbb{P}) \) the open subset corresponding to such morphisms, and denote \( V^c \) its complement \( Mor_1(\mathbb{P}^1, \mathbb{P}) \setminus V \). Since \( e_0 = e_1 = e_2 = 1 \), we have \( codim(V^c, Mor_1(\mathbb{P}^1, \mathbb{P})) \geq 2 \), and therefore \( codim(W \cap q^{-1}(V^c), W) \geq codim(I \cap q^{-1}(V^c), I) \geq 2 \).

Therefore to show that \( codim(W \cap q^{-1}([f]), q^{-1}([f])) \geq 2 \) for every \([f] \in Mor_1(\mathbb{P}^1, \mathbb{P})\), it is enough to show that \( codim(W \cap q^{-1}([f]), q^{-1}([f])) \geq 2 \) for every \([f] \in V\).

Consider a fixed \([f] \in V\). We can choose coordinates on \( \mathbb{P} \) such that \([f] \) has equations \( x_2 = \ldots = x_n = 0 \). Then if \( X = Z(f_1, \ldots, f_c) \) is a complete intersection of multidegree \((d_1, \ldots, d_c)\) containing \( f(\mathbb{P}^1) \), \( f_k \) can be written as \( \sum_{i=2}^n x_i f_k^{(i)} \), for every \( k = 1, \ldots, c \).

Set \( V_i := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e_i - 1)) \) and \( W_k := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d_k - 1)) \), for \( i = 2, \ldots, n \) and \( k = 1, \ldots, c \). Any element \((g_k^{(i)})_{k,i} \in Hom(\bigoplus_{i=2}^n V_i, \bigoplus_{k=1}^c W_k)\) can be considered...
as the restriction to $C$ of some $f_k^{(i)}$ defined on $\mathbb{P}$, since for weighted projective spaces of the form $\mathbb{P}(1,1,e_2,...,e_n)$ we can extend polynomials defined on $x_2 = \cdots = x_n = 0$ to $\mathbb{P}(1,1,e_2,...,e_n)$. Thus we can associate to every element $(g_k^{(i)})_{k,i} \in Hom(\oplus_{i=2}^n V_i, \oplus_{k=1}^c W_k)$ a variety $Z(f_1,...,f_c) \subset \mathbb{P}$, and we have:

$$(*) \quad \mathbb{P}^1 \times Hom(\oplus_{i=2}^n V_i, \oplus_{k=1}^c W_k) \simeq q^{-1}([f])$$

The degeneracy locus

$$D_{d_1+...+d_c-1} \subset Hom(\oplus_{i=2}^n V_i, \oplus_{k=1}^c W_k)$$

(i.e. the locus of homomorphisms whose image has dimension at most $d_1 + ... + d_c - 1$) has codimension $(e_2 + ... + e_n - (\sum d_k + 1)(\sum d_k - (\sum d_k) + 1) = \sum_{i=2}^n e_i - \sum_{k=1}^c d_k + 1 \geq 2$ by hypothesis. The complement of the degeneracy locus corresponds to an open subset $U_{[f]} \subset q^{-1}([f])$ under $(*)$.

Therefore, to conclude, it is enough to prove the following Lemma.

\[ \square \]

**Lemma 3.3.** The open set $U_{[f]}$ of the fiber $q^{-1}([f])$ parametrizes complete intersections of codimension $c$, smooth along $C$, and $p$ is smooth at every point of $U_{[f]}$.

Therefore, for every $[f] \in V$, if $W \cap U_{[f]}$ is nonempty, we have $\text{codim}(W \cap U_{[f]}, U_{[f]}) = \text{codim}(W \cap q^{-1}([f]), q^{-1}([f])) \geq 2$. (On the other hand, if $W \cap U_{[f]}$ is empty, there is nothing to prove.)

**Proof.** Consider $(f_1,...,f_c,x,[f]) \in I$. Denote $X := Z(f_1,...,f_c)$ and $C := f(\mathbb{P}^1)$. Then we have:

$$0 \to T_{\mathbb{P}^1} \to f^* T_X \to f^* N_{C/X} \to 0.$$

Twisting by $-\mathcal{O}$, where $\mathcal{O} \in \mathbb{P}^1$:

$$0 \to \mathcal{O}(1) \to f^* T_X (-1) \to f^* N_{C/X} (-1) \to 0.$$

The long exact sequence gives:

$$0 \to H^1(\mathbb{P}^1, f^* T_X (-1)) \to H^1(\mathbb{P}^1, f^* N_{C/X} (-1)).$$

Thus if $H^1(\mathbb{P}^1, f^* N_{C/X} (-1)) = 0$, we have $H^1(\mathbb{P}^1, f^* T_X (-1)) = 0$, which implies that $p^{-1}(f_1,...,f_c,x) \simeq Mor_{[f]}(\mathbb{P}^1, X; 0 \to x)$ is smooth (see [Deb01], 2.9).

Restricting to the smooth locus $\mathbb{P}^n$ of $\mathbb{P}$, we have normal bundles:

$$N_{C/\mathbb{P}^n} \simeq \bigoplus_{i=2}^n \mathcal{O}_C(e_i), \quad N_{X/\mathbb{P}^n} \simeq \bigoplus_{k=1}^c \mathcal{O}_X(d_k),$$

and a short exact sequence:

$$0 \to f^* N_{C/X} (-1) \to \bigoplus_{i=2}^n \mathcal{O}_{\mathbb{P}^i}(e_i - 1) \to \bigoplus_{k=1}^c \mathcal{O}_{\mathbb{P}^1}(d_k - 1) \to 0.$$

Therefore, by considering the associated long exact sequence, the analysis above shows that $p$ is smooth at a point $(f_1,...,f_c,x,[f])$ if and only if the homomorphism:

$$\overline{(f_k^{(i)}): H^0(\mathbb{P}^1, \bigoplus_{i=2}^n \mathcal{O}_{\mathbb{P}^i}(e_i - 1)) \to H^0(\mathbb{P}^1, \bigoplus_{k=1}^c \mathcal{O}_{\mathbb{P}^1}(d_k - 1))}$$

is surjective.

\[ \square \]
4. Minimal pointed curves of degree 2

Throughout this section, $X$ is a smooth weighted complete intersection $X_{d_1, \ldots, d_c} \subset \mathbb{P} \mathbb{C}(e_0, \ldots, e_n)$ that satisfies the main hypothesis, and $\alpha$ is a curve class of degree 1. We are going to study the general fiber of

$$ev_2 : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X^2.$$ 

We will show that it is nonempty, irreducible, and uniruled by suitable rational curves. Uniruledness will be used to produce 1-twisting surfaces in the next section.

The next result is essentially the first part of Lemma 5.8 of [dJS06b].

**Proposition 4.1.** The evaluation morphism $ev_2 : \overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to X^2$ is dominant.

**Proof.** Let $(p_1, p_2) \in X^2$ be a general point. For $i = 1, 2$, $ev_{p_i} : \overline{\mathcal{M}}_{0, p_i}(X, \alpha) \to X$ is dominant, and the general fiber is connected of dimension $-K_X \cdot \alpha - 2$. The curves parametrized by such a fiber sweep out a subvariety $\Pi_i$ of $X$ of dimension $(-K_X \cdot \alpha - 2) + 1 = -K_X \cdot \alpha - 1$. Now $\Pi_1 \cap \Pi_2$ is nonempty by Theorem 2.1, which means that there are two curves $C_1$ and $C_2$ of class $\alpha$ containing $p_1$ and $p_2$ respectively, that intersect. Thus $C_1 \cup C_2$ has class $2\alpha$ and contains $p_1$ and $p_2$, which implies that $ev_2$ is dominant. \qed

By Theorem 2.7, this implies that a general fiber of $ev_2$ is smooth of the expected dimension.

The next step is to show that the generic fiber is uniruled by rational curves of degree 1 with respect to some ample divisor class $\lambda$.

This is the setup. Let $M_{p_1, p_2} := ev_2^{-1}(p_1, p_2)$ be a general fiber of $ev_2$. Then we have a diagram:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{g} & X \subset \mathbb{P} \subset \mathbb{P} \\
\downarrow \pi & & \\
M_{p_1, p_2} & & \\
\end{array}
$$

where $\mathcal{C}$ is the tautological family of $M_{p_1, p_2}$, with sections $s_1, s_2 : M_{p_1, p_2} \to \mathcal{C}$. Let $S_i$ be the divisor associated to $s_i$ for $i = 1, 2$.

**Definition-Lemma 4.2.** There is a divisor class $\lambda$ on $M_{p_1, p_2}$ that is ample and that satisfies the following relations:

1) $\Delta_{1,1} = 2\lambda$;
2) $\pi_* g^* c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = 2\lambda$;
3) $c_1(T_{M_{p_1, p_2}}) = (\sum e_i^2 - \sum d_j^2 + 2) \lambda$.

**Proof.** Let us compare $\mathcal{O}_{\mathcal{C}}(S_1 + S_2)$ and $g^* \mathcal{O}_{\mathbb{P}^2}(1)$: they are isomorphic on each irreducible component of each fiber of $\pi$. Thus by the Semicontinuity Theorem (see [Har77], III, exercise 12.4), they are isomorphic up to the pullback of some line bundle $\mathcal{L}$ on $M_{p_1, p_2}$:

$$\mathcal{O}_{\mathcal{C}}(S_1 + S_2) \otimes \pi^* \mathcal{L} \cong g^* \mathcal{O}_{\mathbb{P}^2}(1).$$
Applying $s_i^*$ to each side, we have:

$$s_i^* \mathcal{O}_C(S_i) \simeq \mathcal{L}_i,$$

or

$$\mathcal{O}_C(\pi_s(S_i \cdot S_i)) \simeq \mathcal{L}_i.$$ 

Denoting $\lambda := c_1(\mathcal{L})$ and using Lemma 5.8 of [1], we get

$$\Delta_{1,1} = 2\lambda$$

and

$$\pi_* g^* c_1(\mathcal{O}_X(1))^2 = 2\lambda.$$ 

Using Theorem 1.1 of [1], we also get

$$c_1(T_{M_{p_1,p_2}}) = \pi_* g^*(ch_2(T_X) + c_1(\mathcal{O}_X(1)^2) = (\frac{1}{2}(\sum e_i^2 - \sum d_j^2) + 1) \cdot 2\lambda =$$

$$= (\sum e_i^2 - \sum d_j^2 + 2)\lambda.$$ 

To prove that $\lambda$ is ample, we can proceed as follows. Since $\mathcal{O}_X(1)$ is ample, $\mathcal{O}_X(b)$ is very ample for some positive integer $b$. Let us fix such $b$. Then $\mathcal{O}_X(b)$ determines an embedding of $X$ into some projective space $\mathbb{P}^N$. By functoriality of Kontsevich spaces, we get an embedding $\overline{\mathcal{M}}_{0,2}(X, 2\alpha) \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^N, 2\alpha)$. On $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^N, 2\alpha)$ there is a divisor $\mathcal{H}$, given by maps whose image intersects a fixed codimension 2 linear subspace of $\mathbb{P}^N$. The restriction of $\mathcal{H}$ to $\overline{\mathcal{M}}_{0,2}(X, 2\alpha)$ is ample away from the locus of multiple covers of lines. By inequality (3) in the main hypothesis, for $p_1, p_2$ general there is no line contained in $X$ through $p_1$ and $p_2$. Therefore the restriction of $\mathcal{H}$ to $M_{p_1,p_2} \subset \overline{\mathcal{M}}_{0,2}(X, 2\alpha)$ is ample. Furthermore, the restriction of $\mathcal{H}$ to $M_{p_1,p_2}$ has divisor class $\pi_* g^* c_1(\mathcal{O}_X(b))^2$, which is equal to $2\delta^2\lambda$. Therefore $\lambda$ is ample.  

Now we can use $\lambda$ to further study the fibers of $\mathcal{E}_{p_2}$. Related to $\lambda$ is the tangency divisor $\mathcal{T}_{W,p_1}$, which will be used in what follows. Let $W \subset X$ be a closed subvariety of codimension 1 and degree 1 containing $p_1$; we define $\mathcal{T}_{W,p_1}$ as the Cartier divisor $\{[f : C \to X,t_1, t_2 \in C] \mid f(t_1) = p_1, f(t_2) = p_2, im(df_{t_1}) \in T_{p_1,W} \subset T_{p_1,X}\}$ in $M_{p_1,p_2}$. Note that

$$s_1^*(\Omega_\pi)[f : C \to X, t_1, t_2 \in C] \simeq Hom(T_{t_1}, C, \mathbb{C}),$$

and there is a global section $dg_{s_1}$ of $s_1^*(\Omega_\pi)$ defined over a point $[f : C \to X; t_1, t_2 \in C, f(t_1) = p_1, f(t_2) = p_2]$ by the morphism

$$T_{t_1}C \xrightarrow{df_{t_1}} T_{p_1}X \xrightarrow{T_{p_1}X/T_{p_1}W} \mathbb{C}.$$ 

Therefore the divisor $\mathcal{T}_{W,p_1}$ can be thought of as the divisor of zeroes of $dg_{s_1}$. Since $s_1$ is a section, its image $S_1$ is contained in the smooth locus $\mathcal{E}_{sm}$ of $\mathcal{E}$, so the relative cotangent sequence associated to $S_1 : \mathcal{E}_{sm} \to M_{p_1,p_2}$ is the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{E}}(-S_1)|S_1 \to \Omega_{\pi}|S_1 \to \Omega_{S_1/M_{p_1,p_2}} \to 0.$$ 

Note that $\mathcal{O}_{\mathcal{E}}(-S_1)|S_1 \simeq s_1^*(\mathcal{O}_{\mathcal{E}}(-S_1))$ and $\Omega_{\pi}|S_1 \simeq s_1^*\Omega_\pi$. Since $S_1 \to M_{p_1,p_2}$ is an isomorphism, $\Omega_{S_1/M_{p_1,p_2}} = 0$. Therefore we have that the divisor class of $\mathcal{T}_{W,p_1}$ is

$$c_1(s_1^*\Omega_\pi) = -\pi_*(S_1 \cdot S_1) = \lambda,$$

hence ample.

We can now show uniruledness.
Proposition 4.3. Assume that $-K_X \cdot D - 2 = \sum e_i^2 - \sum d_j^2 \geq 0$. Then a general fiber of $ev_2$ is uniruled by rational curves of $\lambda$–degree 1.

Proof. By Cor. 6.7 in [JLS06b], if $2(\sum e_i^2 - \sum d_j^2) + \dim(X) + 2K_X \cdot \alpha - 4 > 0$, then a general fiber of $ev_2$ is uniruled by rational curves of $\lambda$–degree 1.

Assume that the general fiber of $ev_2$ is uniruled by rational curves of $\lambda$–degree 1, and assume $D$ is such a curve. The dimension of the space of deformations of $D$ containing a general point is $-K_X \cdot D - 2 = \sum e_i^2 - \sum d_j^2$. The next step is to show that the inequality $-K_X \cdot D - 2 = \sum e_i^2 - \sum d_j^2 \geq 0$ is actually sufficient to guarantee that the general fiber of $ev_2$ is uniruled by rational curves of $\lambda$–degree 1.

By Proposition 2.3, we can extend $X$ to some $Y \in \mathbb{P}(e_0, \ldots, e_n, 1, \ldots, 1)$ such that $X \subset Y$, $X = Y \cap D_1 \cap \ldots \cap D_r$, $D_k$ Cartier divisor of degree 1 for every $k = 1, \ldots, r$.

By adding enough variables, we can choose $Y$ that satisfies the inequality $2(\sum e_i^2 + r - \sum d_j^2) + \dim(Y) + 2K_Y \cdot \alpha - 4 > 0$.

Let $(p_1, p_2) \in X^2$ be a general point, and let $M_X, M_Y$ be the fibers over $(p_1, p_2)$ of $ev_2(X) : \overline{M}_{0,2}(X, 2) \to X^2$ and $ev_2(Y) : \overline{M}_{0,2}(Y, 2) \to Y^2$ respectively.

By [Deb01], 2.10, the fiber of the $Y \times Y$–scheme $Mor(\mathbb{P}^1, \overline{M}_{0,2}(Y, 2))$ over a point $(q_1, q_2) \in Y \times Y$ is $Mor(\mathbb{P}^1, ev_2(Y)^{-1}(q_1, q_2))$. Therefore, by upper semicontinuity of the fiber dimension, the dimension of the space of rational curves in $ev_2(Y)^{-1}(q_1, q_2)$ of $\lambda$–degree 1 containing a general point is at least $-K_X \cdot D - 2$ for any $(q_1, q_2) \in Y \times Y$.

Let $[C]$ be a general point of $M_X$. By looking at the normal exact sequence of $X$ in $Y$ restricted to $C$, we get that $h^1(C, T_Y|_C(−2)) = 0$, i.e. $C$ deforms to a curve in $Y$ passing through 2 general points. Therefore the space of $\lambda$–degree 1 curves in $M_Y$ through $[C]$ has dimension at least $-K_Y \cdot D - 2 = r - K_X \cdot D - 2$.

Consider the rational map $\tau : M_{p_1, p_2} \to \mathbb{P}^N$ induced by $\lambda$ and defined on some open $\mathcal{W}$ that contains $[C]$ (this is possible since we assumed that $[C]$ is general). Note that maps curves of $\lambda$–degree 1 contained in $\mathcal{W}$ to lines in $\mathbb{P}^N$, and the divisors $T_{D_1, p_1}, \ldots, T_{D_r, p_1}$ are preimages under $\tau$ of hyperplanes $H_1, \ldots, H_r$ in $\mathbb{P}^N$ (since the divisor class of $T_{D_r, p_1}$ is $\lambda$).

Consider $\mathbb{P}^{N-1} := \mathbb{P}^N/\tau([C])$. The space of curves of $\lambda$–degree 1 through $[C]$ is mapped under $\tau$ to a subvariety of $\mathbb{P}^{N-1}$ of dimension $r - K_X \cdot D - 2$, and $H_1 \cap \ldots \cap H_r$ is mapped to a linear subspace of codimension $r$. By hypothesis, $-K_X \cdot D - 2 \geq 0$.

Therefore we can find a $\lambda$–degree 1 curve in $M_Y$ through $[C]$ whose image under $\tau$ is contained in $H_1 \cap \ldots \cap H_r$. Such a curve is a $\lambda$–degree 1 curve in $M_X$ through $[C]$. \hfill \Box

We can now also prove irreducibility of a general fiber of $ev_2$, a fact that will be needed to conclude weak approximation from the existence of 1–twisting surfaces.

Proposition 4.4. The general fiber of $ev_2 : \overline{M}_{0,2}(X, 2) \to X^2$ is irreducible.

Proof. By Proposition 2.2, we can extend $X$ to some $Y \in \mathbb{P}(e_0, \ldots, e_n, 1, \ldots, 1)$ such that $X \subset Y$, $X = Y \cap D_1 \cap \ldots \cap D_r$, $D_k$ Cartier divisor of degree 1 for every $k = 1, \ldots, r$.

By adding enough variables, i.e., by taking $r$ big enough, we can choose $Y$ that satisfies the inequality

$$\dim(Y) \leq -K_Y \cdot 2\alpha - 2.$$
In fact, \(\dim(Y) = \dim(X) + r\) and \(-K_Y \cdot 2\alpha - 2 = -K_X \cdot 2\alpha + 2r\). Similarly, we can also assume that
\[2 \cdot (-K_Y \cdot \alpha - 2) > n + r = \dim(\mathbb{P}(e_0, ..., e_n, 1, ..., 1)).\]

We will make use of the first inequality right away, and of the second one later on.

Let us consider the fiber \(M_{p_1, p_2}(Y)\) of \(\ev_2(Y) : \overline{\mathcal{M}}_{0, 2}(Y, 2\alpha) \to Y^2\) over a general point \((p_1, p_2) \in Y^2\). The first inequality allows us to use Bend and Break: therefore \(M_{p_1, p_2}(Y)\) contains a reducible curve, and thus intersects the boundary divisor \(\Delta_{1, 1}(Y) \subset \overline{\mathcal{M}}_{0, 2}(Y, 2)\). Thus to prove connectedness, it’s enough to show that \(\Delta_{1, 1}(Y) \cap M_{p_1, p_2}(Y)\) is connected. To do this, we interpret \(\Delta_{1, 1}(Y) \cap M_{p_1, p_2}(Y)\) as the preimage of a diagonal and use Badescu’s theorem ([Bad96]).

Consider the evaluation map \(e_2 : \overline{\mathcal{M}}_{0, 2}(Y, \alpha) \to Y \times Y\). In what follows, we denote the diagonal in \(Y \times Y\) as \(\Delta_Y\). Define \(M_{\bullet, \bullet} := e_2^{-1}(\{p_1\} \times Y)\), \(M_{\bullet, p_2} := e_2^{-1}(Y \times \{p_2\})\). \(M_{\bullet, \bullet}\) is projective, smooth and connected. This follows from the fact that a general fiber of \(\ev_2 : \overline{\mathcal{M}}_{0, 1}(Y, \alpha) \to Y\) is smooth and connected, and uses the following cartesian diagram:

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{0, 2}(Y, \alpha) & \xrightarrow{e_2} & Y \times Y \\
\downarrow & & \downarrow e_1 \\
\overline{\mathcal{M}}_{0, 1}(Y, \alpha) & \xrightarrow{\ev_1} & Y
\end{array}
\]

(Here the map on the lefthand side forgets the second marked point). Similarly, \(M_{\bullet, p_2}\) is projective, smooth and connected.

We have maps \(M_{\bullet, \bullet} \to Y\) and \(M_{\bullet, p_2} \to Y\), given by evaluation at the unspecified marked point \(\bullet\). Their product defines a map
\(e_{1, 2} : M_{\bullet, \bullet} \times M_{\bullet, p_2} \to Y \times Y\).

Since through any two distinct points there are at most finitely many lines, \(e_{1, 2}\) is generically finite (it is finite on \((Y \times Y) \setminus \Delta_Y\)). We also have \(e_{1, 2}^{-1}(\Delta_Y) = \Delta_{1, 1}(Y) \cap M_{p_1, p_2}(Y)\).

Now consider the Stein factorization:

\[
\begin{array}{ccc}
M_{p_1, \bullet, p_2} & \xrightarrow{\eta} & M_{\bullet, \bullet, p_2} \\
\downarrow & & \downarrow e_{1, 2} \\
M_{p_1, \bullet} \times M_{\bullet, p_2} & \xrightarrow{\epsilon} & Y \times Y
\end{array}
\]

Since \(\eta\) has connected fibers, if \(e_{1, 2}^{-1}(\Delta_Y)\) is connected, \(e_{1, 2}^{-1}(\Delta_Y)\) will be connected too. Therefore we can focus on the finite morphism
\(\epsilon : M_{p_1, \bullet, p_2} \to Y \times Y \subset \mathbb{P}(e_0, ..., e_n, 1, ..., 1) \times \mathbb{P}(e_0, ..., e_n, 1, ..., 1)\).

Since \(M_{p_1, \bullet}\) and \(M_{\bullet, p_2}\) are irreducible, and \(e_{1, 2}\) is generically finite, \(M_{p_1, \bullet, p_2}\) is irreducible of dimension \(\dim(M_{p_1, \bullet}) + \dim(M_{\bullet, p_2}) = 2(-K_X \cdot \alpha - 2)\). Since \(M_{p_1, \bullet, p_2}\)
is irreducible, it is \((\dim(M_{\pi_1,\ldots,p_2}) - 1)\)-connected, and by the second inequality at the beginning of the argument,

\[
\dim(M_{\pi_1,\ldots,p_2}) > \dim(P(e_0,\ldots,e_n,1,\ldots,1)).
\]

Therefore Badescu’s theorem \cite{Bad96} implies that \(e^{-1}(\Delta_{\pi_0,\ldots,e_n,1,\ldots,1}) = e^{-1}(\Delta_Y)\) is connected. Thus \(e_{1,2}^1(\Delta_Y) = \Delta_{1,1}(Y) \cap M_{\pi_1,p_2}(Y)\) is connected, and finally \(M_{\pi_1,p_2}(Y)\) is connected.

Now, by Theorem 2.7, a general fiber of \(ev_2(Y)\) is smooth (of expected dimension). Therefore a general fiber of \(ev_2(Y)\) is irreducible.

To conclude, consider the fiber \(M_{\pi_1,p_2}(X)\) of \(ev_2(X) : \overline{M}_{0,2}(X,2\alpha) \to X^2\) over a general point \((p_1,p_2) \in X \times X\). Let \(M_{\pi_1,p_2}(Y)\) denote the fiber over \((p_1,p_2)\) of \(ev_2(Y)\). We are going to show that \(M_{\pi_1,p_2}(Y)\) is irreducible.

Notice that \(M_{\pi_1,p_2}(X)\) is cut out in \(M_{\pi_1,p_2}(Y)\) by the ample divisors \(T_{Y \cap D_k,p_1}\), for \(k = 1,\ldots,r\). Therefore \(\dim(M_{\pi_1,p_2}(Y)) \leq \dim(M_{\pi_1,p_2}(X)) + r\). Since \((p_1,p_2) \in X \times X\) is general,

\[
\dim(M_{\pi_1,p_2}(X)) + r = -K_X \cdot 2\alpha - 1 + r = -K_Y \cdot 2\alpha - 1 \leq \dim(M_{0,2}(Y,2\alpha)) - 2\dim(Y) \leq \dim(M_{\pi_1,p_2}(Y)).
\]

Therefore \(\dim(M_{\pi_1,p_2}(Y))\) equals its lower bound.

Since \((p_1,p_2) \in X \times X\) is general, by Theorem 2.7 we have that \(M_{\pi_1,p_2}(X)\) is smooth. Since \(M_{\pi_1,p_2}(X)\) is cut by a regular sequence in \(M_{\pi_1,p_2}(Y)\), it follows that \(M_{\pi_1,p_2}(Y)\) is also smooth.

Now consider an integral (say, rational) curve \(T\) in \(Y \times Y\) connecting \((p_1,p_2)\) to a general point \((q_1,q_2) \in Y \times Y\). We proved above that the dimension of a general fiber of \(ev_2(Y)\) equals its lower bound. Therefore there is an open dense subset \(T^0 \subset T\) containing \((p_1,p_2)\) such that the dimension of the fibers of \(ev_T : ev_2(Y)^{-1}(T^0) \to T^0\) is the smallest possible. By \textit{Kolm}, II, 1.7.3, \(ev_T\) is flat. Since we showed above that a general fiber of \(ev_T\) is connected, all fibers of \(ev_T\) are connected by the Principle of Connectedness \cite{Har77}, III, Exercise 11.4). In particular, \(M_{\pi_1,p_2}(Y)\) is connected. Since \(M_{\pi_1,p_2}(Y)\) is also smooth, we have that \(M_{\pi_1,p_2}(Y)\) is irreducible.

As already noticed above, \(M_{\pi_1,p_2}(X)\) is cut out in \(M_{\pi_1,p_2}(Y)\) by the ample divisors \(T_{Y \cap D_{\pi_1}},\ldots,T_{Y \cap D_{\pi_1}}\). Thus, \(M_{\pi_1,p_2}(X)\) is connected (when its expected dimension is at least 1). By Theorem 2.7, a general fiber of \(ev_2(X)\) is smooth. Therefore a general fiber of \(ev_2(X)\) is irreducible.

\[
\square
\]

5. 1–twisting surfaces

We now produce our twisting surface. We follow the strategy of Lemma 6.8 in \textit{JLS06b}. Recall the definition of a 1–twisting surface:

**Definition 5.1.** A 1–twisting surface in a scheme \(X\) is a ruled surface \(\pi : \Sigma \to \mathbb{P}^1\) with a morphism \(h : \Sigma \to X\) such that:

1) \(h^*T_X\) is generated by global sections;

2) the morphism \((\pi,h) : \Sigma \to \mathbb{P}^1 \times X\) is finite and \(h^1(\Sigma,N_{\pi,h}(-E-F)) = 0\), where \(F\) is the divisor class of a fiber, and \(E\) is the divisor class of a section with \(E^2 = 0\).
Proposition 5.2. Let $X$ be a smooth weighted complete intersection satisfying the main hypothesis. Then a general point of a general fiber of $ev_2 : \overline{M}_{0,2}(X,2\alpha) \to X^2$ parametrizes a hyperplane section of a surface $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $X$.

Proof. Let $D \subset M_{p_1,p_2}$ be a free rational $\lambda$–degree 1 curve that meets the boundary divisor $\Delta_{1,1} \subset M_{p_1,p_2}$ transversely at two points. Consider the diagram:

$$
\begin{array}{ccc}
\mathcal{C}_D & \subset & \mathcal{C} \\
\downarrow \pi_D & & \downarrow \pi \\
D & \subset & M_{p_1,p_2}
\end{array}
$$

where $\mathcal{C}_D$ is the restriction of $\mathcal{C}$ over $D$. Then $\mathcal{C}_D$ is a smooth rational surface, fibered via $\pi$ over $D \cong \mathbb{P}^1$, with two sections $R_1$ and $R_2$. If $S_1$ and $S_2$ are the two sections of $\pi : \mathcal{C} \to M_{p_1,p_2}$, we have seen that $\pi_* (S_1)^2 = \pi_* (S_2)^2 = -\lambda$. Therefore $R_1$ and $R_2$ have self-intersection $-1$, and thus can be contracted by Castelnuovo’s Theorem ([Har77], V, Theorem 5.7).

Claim: Let $\phi : \mathcal{C}_D \to \Sigma$ be the contraction of $R_1$ and $R_2$. Then $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. The fibration $\pi : \mathcal{C}_D \to D$ has exactly two reducible fibers over, say, $m_1$ and $m_2$. In particular, $\pi^{-1}(m_i) = F_i \cup G_i$, $F_i \cong \mathbb{P}^1 \cong G_i$, for $i = 1, 2$. Say also that $F_1, F_2$ intersect $R_1$ but not $R_2$, and that $G_1, G_2$ intersect $R_2$ but not $R_1$. Since $F_i$ and $G_i$, for fixed $i$, are irreducible components of connected fibers, we have that $F_i^2 = G_i^2 = -1$ for $i = 1, 2$. In particular, by Castelnuovo’s Theorem ([Har77], V, Theorem 5.7), we can contract $F_1$ and $G_2$. Let $\phi' : \mathcal{C}_D \to \Sigma'$ be such contraction. Then $\Sigma'$ is a $\mathbb{P}^1$–bundle, and it is easy to check that $(\phi'(R_i))^2 = 0$ for $i = 1, 2$. Thus $\Sigma' \cong \mathbb{P}^1 \times \mathbb{P}^1$. We can conclude that $\Sigma$ is a del Pezzo surface with $K_{\Sigma}^2 = 8$ and four $(-1)$–curves. By the classification of del Pezzo surfaces (see for example [Kol90], III, 3.9), it follows that $\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Now consider the two morphisms:

$$
\begin{array}{ccc}
\mathcal{C}_D & \longrightarrow & \Sigma \\
\phi & \longmapsto & g|_{\mathcal{C}_D} \\
\downarrow & & \downarrow \\
\mathcal{C}_D & \longrightarrow & X
\end{array}
$$

Since the fibers of $\phi$ are contracted by $g|_{\mathcal{C}_D}$, the latter map factors through the former by a rigidity result (see [Deb01], Lemma 1.15).

We need to show now that the morphism $\Sigma \to X$ is an embedding.

Consider the rational map $L : \mathbb{P}^o \dashrightarrow \mathbb{P}^3$ determined by the global sections $x_0, x_1, x_2, x_3$ of $\mathcal{O}_{\mathbb{P}}(1)$. The base locus $Z$ of $L$ contains the subscheme $\{x_0 = x_1 = x_2 = 0\}$. Therefore $\text{codim}(Z, \mathbb{P}^o) \geq 3$. Since $g$ is generically finite by Bend and Break ([Deb01], Proposition 3.2), $\text{codim}(g^{-1}(Z), \mathcal{C}) = \text{codim}(Z, X) \geq 3$, and therefore

$$
\text{codim}(\pi(g^{-1}(Z)), M_{p_1,p_2}) \geq 2.
$$
Since $D$ is free, it can be chosen away from any codimension 2 set, so in particular it can be chosen away from $\pi(g^{-1}(Z))$. For such a choice of $D$, we have that $L \circ q|_{\varphi_D}$ is a morphism. Therefore we have a morphism $h : \Sigma \to \mathbb{P}^3$ defined by composition as in the following diagram:

$$
\begin{array}{ccc}
\Sigma & \xrightarrow{h} & \mathbb{P}^3 \\
\downarrow & & \\
X & \xrightarrow{L \circ g} & \mathbb{P}^3
\end{array}
$$

To prove that $\Sigma \to X$ is an embedding, it is enough to prove that $h : \Sigma \to \mathbb{P}^3$ is an embedding.

Since $p_1$ and $p_2$ are general points, we can assume that $L$ is defined at $p_1$ and $p_2$. Let $q_i := L(p_i)$, $i = 1, 2$, and let $\Sigma_d$ be the hyperplane section of $\Sigma$ corresponding to $d \in D$. If no image of $\Sigma_d$ for the various $d \in D$ is the line through $q_1$ and $q_2$, then $h$ is an embedding. In fact, $\Sigma_d$ embeds in $\mathbb{P}^3$ as a conic through $q_1$ and $q_2$ for every $d \in D$; if two such conics met at a further point or had the same tangent direction at either $q_1$ or $q_2$, then they would have to come from the same hyperplane section of $\Sigma$. Thus, to conclude, it is enough to prove the following:

**Claim:** We can choose $D$ so that $h : \Sigma_d \to \mathbb{P}^3$ is not the double cover of the line through $q_1$ and $q_2$ for any $d \in D$.

The proof proceeds as follows. Since $\Delta_{1,1} \subset M_{p_1,p_2}$ is ample, an irreducible subvariety of $M_{p_1,p_2}$ has codimension $c$ if its intersection with $\Delta_{1,1}$ has codimension $c$. Now consider $\text{Mor}_1(\mathbb{P}^1, \mathbb{P}^c; \underline{0} \mapsto p_1)$. As in the proof of Proposition 3.2, there is an open subset $V$, whose complement has codimension at least 2, such that composing with $L$ gives a morphism $V \to \text{Mor}_1(\mathbb{P}^1, \mathbb{P}^3; \underline{0} \mapsto q_1)$. In the target space of parametrized lines in $\mathbb{P}^3$ through $q_1$, to have image equal to the line through $q_1$ and $q_2$ gives 2 independent closed conditions (precisely, to lie in 2 independent planes that contain $q_1$ and $q_2$). Therefore there is a closed locus $W \subset \text{Mor}_1(\mathbb{P}^1, \mathbb{P}^c; \underline{0} \mapsto p_1)$ of codimension at least 2, away from which there cannot be curves that after composing with $L$ parametrize the line through $q_1$ and $q_2$. A fortiori, there is a closed locus $W' \subset \text{Mor}_1(\mathbb{P}^1, \mathbb{P}^c; \underline{0} \mapsto p_1, \underline{1} \mapsto p_2)$ of codimension at least 2, away from which there cannot be curves that after composing with $L$ parametrize double covers of the line through $q_1$ and $q_2$. In particular, after quotienting by $\text{Aut}(\mathbb{P}^1)$, we have a codimension 2 locus with such property in $\Delta_{1,1}$. This in turn shows that there is a codimension 2 locus $W''$ with such property in $M_{p_1,p_2}$.

Since the curve $D \subset M_{p_1,p_2}$ is free, we can choose $D$ disjoint from $W''$. Such $D$ will have the desired property.

By Lemma 7.8 in [dJS06b], since $\Sigma$ is abstractly isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\Sigma$ is $1$–twisting.

**6. Rational simple connectedness**

Rational simple connectedness of the weighted complete intersections considered in Theorem 1.1 follows as in the proof of Theorem 1.7 in [dJS06b, Theorem 1.7].
The prerequisites to the proof of Theorem 1.7 are the following:

1) The existence and canonicity of irreducible components of the spaces of stable maps (see Lemma 2.9);
2) The general properties of twisting surfaces and of stable maps to them, which are proved in [dJS06b] in a general setting and apply in particular to our case.

Then the proof of Theorem 1.7 in [dJS06b] works provided the general fibers of $ev_1$ and $ev_2$ are rationally connected.

The general fiber $M_p$ of $ev_1$ is Fano and hence rationally connected, since

$$c_1(M_p) = \pi_* g^*(\chi(T_X) + \frac{-K_X \cdot \alpha - 2}{2} c_1(O(1))^2) =$$

$$= \pi_* g^* ((\frac{1}{2} \sum c_i^2 - \sum d_j^2 + \sum c_i - \sum d_j - 2)) c_1(O(1))^2),$$

and, by (4) and (5) of the main hypothesis,

$$\sum c_i^2 - \sum d_j^2 + \sum c_i - \sum d_j - 3 \geq 0.$$

The general fiber $M_{p_1,p_2}$ of $ev_2$ is Fano and hence rationally connected, since as already computed,

$$c_1(T_{M_{p_1,p_2}}) = (\sum c_i^2 - \sum d_j^2 + 2) \lambda,$$

and by (5) of the main hypothesis,

$$\sum c_i^2 - \sum d_j^2 \geq 0.$$

References

[Băd96] Lucian Bădescu, Algebraic Barth-Lefschetz theorems, Nagoya Math. J. 142 (1996), 17–38. MR 1399466

[CK99] David A. Cox and Sheldon Katz, Mirror symmetry and algebraic geometry, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999. MR 1677117

[CMSB02] Koji Cho, Yoichi Miyaoka, and N. I. Shepherd-Barron, Characterizations of projective space and applications to complex symplectic manifolds, Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math., vol. 35, Math. Soc. Japan, Tokyo, 2002, pp. 1–88. MR 1929792

[Deb01] Olivier Debarre, Higher-dimensional algebraic geometry, Universitext, Springer-Verlag, New York, 2001. MR 1841091

[Dim92] Alexandru Dimca, Singularities and topology of hypersurfaces, Universitext, Springer-Verlag, New York, 1992. MR 1194180

[dJS06a] A. J. de Jong and Jason Michael Starr, Divisor classes and the virtual canonical bundle for genus 0 maps, 2006.

[dJS06b] ———, Low degree complete intersections are rationally simply connected, available at http://www.math.stonybrook.edu/~jstarr/papers/nk1006g.pdf, 2006.

[Dol82] Igor Dolgachev, Weighted projective varieties, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 34–71. MR 704986

[FP97] W. Fulton and R. Pandharipande, Notes on stable maps and quantum cohomology, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534

[GH94] Phillip Griffiths and Joseph Harris, Principles of algebraic geometry, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994, Reprint of the 1978 original. MR 1288523

[GHS03] Tom Graber, Joe Harris, and Jason Starr, Families of rationally connected varieties, J. Amer. Math. Soc. 16 (2003), no. 1, 57–67 (electronic). MR 1937199
