OPTIMAL EXPONENTS FOR HARDY–LITTLEWOOD INEQUALITIES FOR
m-LINEAR OPERATORS

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Abstract. The Hardy–Littlewood inequalities on \( \ell_p \) spaces provide optimal exponents for some classes of inequalities for bilinear forms on \( \ell_p \) spaces. In this paper we investigate in detail the exponents involved in Hardy–Littlewood type inequalities and provide several optimal results that were not achieved by the previous approaches. Our first main result asserts that for \( q_1, ..., q_m > 0 \) and an infinite-dimensional Banach space \( Y \) attaining its cotype \( \cot Y \), if

\[
\frac{1}{p_1} + \ldots + \frac{1}{p_m} < \frac{1}{\cot Y},
\]

then the following assertions are equivalent:

(a) There is a constant \( C_{p_1,\ldots,p_m}^Y \geq 1 \) such that

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^{q_m} \right)^{\frac{p_m}{q_m}} \cdots \right)^{\frac{1}{q_1}} \right) \leq C_{p_1,\ldots,p_m}^Y \|A\|
\]

for all continuous \( m \)-linear operators \( A : \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y \).

(b) The exponents \( q_1, ..., q_m \) satisfy

\[
q_1 \geq \lambda_{m,\cot Y}^{p_1,\ldots,p_m}, q_2 \geq \lambda_{m-1,\cot Y}^{p_2,\ldots,p_m}, \ldots, q_m \geq \lambda_1^{p_m},
\]

where, for \( k = 1, ..., m \),

\[
\lambda_{m-k+1,\cot Y}^{p_k,\ldots,p_m} := \frac{\cot Y}{1 - \left( \frac{1}{p_k} + \ldots + \frac{1}{p_m} \right) \cot Y}.
\]

As an application of the above result we generalize to the \( m \)-linear setting one of the classical Hardy–Littlewood inequalities for bilinear forms. Our result is sharp in a very strong sense: the constants and exponents are optimal, even if we consider mixed sums.

1. Introduction

Let \( \mathbb{K} \) be the real or complex scalar field. In 1934 Hardy and Littlewood proved three theorems (Theorems 1.1, 1.2, 1.3, below) on the summability of bilinear forms on \( \ell_p \times \ell_q \) (here, and henceforth, when \( p = \infty \) we consider \( c_0 \) instead of \( c_\infty \)). For any function \( f \) we shall consider \( f(\infty) := \lim_{s \to \infty} f(s) \) and for any \( s \geq 1 \) we denote the conjugate index of \( s \) by \( s^* \), i.e., \( \frac{1}{s} + \frac{1}{s^*} = 1 \).

For all \( p, q \in (1, \infty] \), such that \( \frac{1}{p} + \frac{1}{q} < 1 \), let us define

\[
\lambda := \frac{pq}{pq - p - q},
\]

and

\[
\mu = \frac{4pq}{3pq - 2p - 2q}.
\]

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If $p$ and $q$ are simultaneously $\infty$, then $\lambda$ and $\mu$ are 1 and $4/3$ respectively.

From now on, $(e_k)_{k=1}^{\infty}$ denotes the sequence of canonical vectors in $\ell_p$.

**Theorem 1.1.** (See Hardy and Littlewood [13, Theorem 1]) Let $p, q \in [2, \infty]$, with \( \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2} \).

There is a constant $C_{p,q} \geq 1$ such that

\[ \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{\lambda}} \leq C_{p,q} \| A \| , \]

and

\[ \left( \sum_{j_1,j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^\mu \right)^{\frac{1}{\mu}} \leq C_{p,q} \| A \| , \]

for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{K}$.

It is well known that the exponents $\lambda$ and $\mu$ are optimal. Also, in (1) the positions of the exponents 2 and $\lambda$ can be interchanged. Furthermore, 2 and $\lambda$ can be replaced by $a, b \in [\lambda, 2]$ provided that

\[ \frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left( \frac{1}{p} + \frac{1}{q} \right) . \]

**Theorem 1.2.** (See Hardy and Littlewood [13, Theorem 2]) Let $p, q \in [2, \infty]$, with $\frac{1}{2} < \frac{1}{p} + \frac{1}{q} < 1$.

There is a constant $C_{p,q} \geq 1$ such that

\[ \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{\lambda}} \leq C_{p,q} \| A \| , \]

and

\[ \left( \sum_{j_1,j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^\lambda \right)^{\frac{1}{\lambda}} \leq C_{p,q} \| A \| , \]

for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{K}$.

The exponent $\lambda$ above is also optimal. However, contrary to what happens in Theorem 1.1, now, in (3) the exponents 2 and $\lambda$ cannot be interchanged (see [10]).

**Theorem 1.3.** (See Hardy and Littlewood [13, Theorem 3]) Let $1 < q < 2 < p$, with $\frac{1}{p} + \frac{1}{q} < 1$.

There is a constant $C_{p,q} \geq 1$ such that

\[ \left( \sum_{j_1,j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^\lambda \right)^{\frac{1}{\lambda}} \leq C_{p,q} \| A \| , \]

for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{K}$.

The “optimal” exponent in (5) was improved in [16].
Theorem 1.4. (See Osikiewicz and Tonge [16]) Let \(1 < q \leq 2 < p\), with \(\frac{1}{p} + \frac{1}{q} < 1\). If \(A : \ell_p \times \ell_q \to \mathbb{K}\) is a continuous bilinear form, then

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} |A(e_{j_1}, e_{j_2})|^{q^*} \right)^{\frac{1}{q^*}} \right)^{\frac{1}{q}} \leq \|A\|.
\]

Hardy–Littlewood type inequalities were extensively investigated in recent years, but despite much progress there are still several open questions concerning the optimality of exponents and constants.

One of the main nuances on the optimality of exponents that apparently has been overlooked in the past is that results of optimality of exponents for expressions like

\[
\left( \sum_{j_1, \ldots, j_m=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{s} \right)^{\frac{1}{s}} \leq C \|A\|
\]

are in some sense sub-optimal. The main point is that the above inequality can be viewed as

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_{m-1}=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{s_m} \right)^{\frac{1}{s_m s_{m-1}}} \right)^{\frac{1}{s_{m-1}}} \right)^{\frac{1}{s}} \leq C \|A\|
\]

for \(s_1 = \ldots = s_m = s\), and this is the way that the optimality of the exponents can be investigated with more accuracy. A simple illustration of this fact is that the exponent \(\lambda\) of (4) is optimal, but a quick look at (3) shows that the optimality of (4) is just apparent. An extensive investigation of the Hardy–Littlewood inequalities in light of multiple sums like (7) was initiated in \([3, 4, 5]\), but there are still some subtle issues not encompassed by previous work. One of the main technical obstacles is to develop methods to find optimal exponents in situations in which the optimal exponents of each sum cannot be interchanged. This is the case of our first main result (for definition of cotype, see the next section):

**Theorem.** (See Theorem 2.2, below) Let \(q_1, \ldots, q_m > 0\) and \(Y\) be an infinite-dimensional Banach space attaining its cotype \(\cot Y\). If

\[
\frac{1}{p_1} + \ldots + \frac{1}{p_m} < \frac{1}{\cot Y},
\]

then the following assertions are equivalent:

(a) There is a constant \(C_{p_1, \ldots, p_m}^Y \geq 1\) such that

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \ldots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_{m-1}=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{q_{m-1}} \right)^{\frac{q_{m-1}}{q_{m}}} \ldots \right)^{\frac{q_2}{q_1}} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{p_1, \ldots, p_m}^Y \|A\|
\]

for all continuous \(m\)-linear operators \(A : \ell_{p_1} \times \ldots \times \ell_{p_m} \to Y\).

(b) The exponents \(q_1, \ldots, q_m\) satisfy

\[
q_1 \geq \lambda_{m, \cot Y}^{p_1, \ldots, p_m}, \quad q_2 \geq \lambda_{m-1, \cot Y}^{p_2, \ldots, p_m}, \quad \ldots, \quad q_{m-1} \geq \lambda_{2, \cot Y}^{p_{m-1}, p_m}, \quad q_m \geq \lambda_{1, \cot Y}^{p_m}.
\]
where, for \( k = 1, \ldots, m \),

\[
\lambda_{m-k+1, \cot Y} := \cot Y \\
1 - \left( \frac{1}{p_k} + \ldots + \frac{1}{p_m} \right) \cot Y.
\]

Despite the wide generality of the results of \([3, 4, 12]\), the results of this paper do not follow from the techniques developed in these earlier papers. We illustrate, by means of a concrete example, how the above Theorem provides more precise information than previously known results.

**Example 1.5.** Suppose that \( m = 3 \), \( p_1 = p_2 = p_3 = 10 \), and \( Y = \ell_3 \). The above Theorem implies that there is a universal constant \( C \geq 1 \) such that

\[
(8) \quad \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \left( \sum_{j_3=1}^{\infty} \|A(e_{j_1}, e_{j_2}, e_{j_3})\| q_j \right) \right) \right)^{\frac{q_2}{q_1}} \leq C \|A\|
\]

for all continuous 3-linear forms \( A : \ell_{10} \times \ell_{10} \times \ell_{10} \to \ell_3 \) if and only if

\[
\begin{align*}
q_1 &\geq 30, \\
q_2 &\geq \frac{15}{2}, \\
q_3 &\geq \frac{30}{7},
\end{align*}
\]

while the best previously known estimates (from \([12, \text{Proposition 4.3}] \) and \([3, \text{Theorem 1.5}] \)) just give that (8) is valid for \( q_j \geq 30 \) for all \( j = 1, 2, 3 \) and that we cannot have simultaneously \( q_1 = q_2 = q_3 < 30 \).

Our second main result, stated and proved in Section 3, is an application of this Theorem, generalizing Theorems 1.3 and 1.4 with optimal exponents, to the multilinear setting. In Section 4 we show that the optimal constant for the scalar-valued case is precisely 1, and finally we remark how our results can be translated to the theory of multiple summing operators.

### 2. Optimal Exponents: Vector-Valued Case

Let \( 2 \leq q < \infty \) and \( 0 < s < \infty \). Recall that (see \([1]\)) a Banach space \( X \) has *cotype* \( q \) if there is a constant \( C > 0 \) such that, no matter how we select finitely many vectors \( x_1, \ldots, x_n \in X \),

\[
(9) \quad \left( \sum_{j=1}^{n} \|x_j\|^q \right)^{\frac{1}{q}} \leq C \left( \int_{[0,1]} \left\| \sum_{j=1}^{n} r_j(t)x_j \right\|^s dt \right)^{1/s},
\]

where \( r_j \) denotes the \( j \)-th Rademacher function. It is well known that if (9) is satisfied for a certain \( s > 0 \), then it is satisfied for all \( s > 0 \). For a fixed \( s \), the smallest of these constants will be denoted by \( C_{q,s}(X) \) and the infimum of the cotypes of \( X \) is denoted by \( \cot X \). By convention we denote \( C_{q,2}(X) \) by \( C_{q}(X) \).

The following simple lemma will be useful.
Lemma 2.1. Let $Y$ be a Banach space, $m \geq 2$, $p_1, \ldots, p_m \in [1, \infty]$, and $q_1, \ldots, q_m, r_2, \ldots, r_m \in (0, \infty)$. Assume that if

$$
\left(\sum_{j_2=1}^{\infty} \left(\sum_{j_3=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \left\|A(e_{j_2}, \ldots, e_{j_m})\right\|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{1}{q_1}} \right)^{\frac{q_2}{q_2}} < \infty
$$

for all continuous $(m-1)$-linear operators $A : \ell_{p_2} \times \cdots \times \ell_{p_m} \to Y$, then $q_i \geq r_i$ for all $i \in \{2, \ldots, m\}$. Then

$$
\left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \left\|B(e_{j_1}, \ldots, e_{j_m})\right\|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{1}{q_1}} \right)^{\frac{q_2}{q_2}} < \infty
$$

for all continuous $m$-linear operators $B : \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y$ implies that $q_i \geq r_i$ for all $i \in \{2, \ldots, m\}$.

Proof. Let $A : \ell_{p_2} \times \cdots \times \ell_{p_m} \to Y$ be a continuous $(m-1)$-linear operator and consider the continuous $m$-linear operator $B_1 : \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y$ given by

$$
B_1(x^{(1)}, \ldots, x^{(m)}) = x^{(1)} A(x^{(2)}, \ldots, x^{(m)}).
$$

Clearly $\|B_1(e_1, e_{j_2}, \ldots, e_{j_m})\| = \|A(e_{j_2}, \ldots, e_{j_m})\|$, and since

$$
\left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \left\|B_1(e_{j_1}, \ldots, e_{j_m})\right\|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{1}{q_1}} \right)^{\frac{q_2}{q_2}} < \infty
$$

$$
= \left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \left\|B_1(e_1, e_{j_2}, \ldots, e_{j_m})\right\|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{1}{q_2}} \frac{1}{q_2}
$$

$$
= \left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \left\|A(e_{j_2}, \ldots, e_{j_m})\right\|^{q_m} \right)^{\frac{q_m-1}{q_m}} \cdots \right)^{\frac{1}{q_2}},
$$

the proof is done. 

From now on, let $r \geq 2$, and let $p_1, \ldots, p_m \in (r, \infty]$ be such that

$$
\frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{r}.
$$

For $k = 1, \ldots, m$, we define

$$
\lambda_{m-k+1,r}^{p_k, \ldots, p_m} := \frac{r}{1 - \left(\frac{1}{p_k} + \cdots + \frac{1}{p_m}\right) r}.
$$

For a Banach space $Y$ and $1 \leq s \leq \infty$, let $\ell_s(Y)$ be the Banach space of $Y$-valued sequences $(y_i)_{i=1}^{\infty}$ with the norm

$$
\|(y_i)_{i=1}^{\infty}\|_{\ell_s(Y)} = \left(\sum_{i=1}^{\infty} \|y_i\|_Y^s\right)^{\frac{1}{s}}.
$$
Proof. From now on, we shall denote using a short argument from the theory of absolutely summing operators, but we prefer to present cot \( Y \), then the following assertions are equivalent:

**Proposition 4.3** and **Theorem 2.2.**

Let \( m, p \) for all continuous \( \ell \)-linear operators \( A : \ell_p \times \cdots \times \ell_p \to Y \).

(b) The exponents \( q_1, ..., q_m \) satisfy

\[
q_1 \geq \lambda^{p_{1},...,p_{m}}_{m,\co Y}, \quad q_2 \geq \lambda^{p_{2},...,p_{m}}_{m-1,\co Y}, \quad ..., \quad q_{m-1} \geq \lambda^{p_{m-1},p_{m}}_{2,\co Y}, \quad q_{m} \geq \lambda_{1,\co Y}^{p_{m}}.
\]

**Proof.** From now on, we shall denote \( r = \cot Y \). The proof of the case \( m = 1 \) can be verified by using a short argument from the theory of absolutely summing operators, but we prefer to present a self contained argument. It suffices to note that

\[
\lambda_{1,\co Y}^{p} = \frac{\cot Y}{1 - \frac{p}{r}} = \frac{rp_1}{p_1 - r},
\]

and

\[
\left( \sum_{j=1}^{n} \| A (e_j) \|^{rp_1} \right)^{\frac{1}{p_1}} = \left( \sum_{j=1}^{n} A \left( \| A (e_j) \|^{\frac{r}{p_1 - r}} e_j \right) \right)^{\frac{1}{p_1}} \leq C_r (Y) \left( \int_{0}^{1} \left( \sum_{j=1}^{n} r_j (t) \| A (e_j) \|^{\frac{r}{p_1 - r}} A (e_j) \right)^{2} \frac{1}{2} dt \right) \leq C_r (Y) \sup_{t \in [0, 1]} \left( \sum_{j=1}^{n} r_j (t) \| A (e_j) \|^{\frac{r}{p_1 - r}} A (e_j) \right) \leq C_r (Y) \sup_{\varphi \in B_Y^*} \left( \sum_{j=1}^{n} | \varphi (A (e_j)) |^{\frac{r}{p_1 - r}} A (e_j) \right) \left( \sum_{j=1}^{n} | \varphi (A (e_j)) |^{p_1} \right)^{\frac{1}{p_1}} \leq C_r (Y) \left( \sum_{j=1}^{n} \| A (e_j) \|^{\frac{rp_1}{p_1 - r}} \right)^{\frac{1}{p_1}} \| A \|. \]
So, if (b) is true, then (a) holds.

Assume (a). By the Maurey-Pisier factorization result (see [15] and [11, pg. 286,287]) the infinite-dimensional Banach space \( Y \) finitely factors the formal inclusion \( \ell_r \hookrightarrow \ell_\infty \), i.e., there are constants \( C_1, C_2 > 0 \) such that for all \( n \) there are vectors \( z_1, \ldots, z_n \in Y \) satisfying

\[
C_1 \left\| (a_j)_{j=1}^n \right\|_\infty \leq \left\| \sum_{j=1}^n a_j z_j \right\| \leq C_2 \left( \sum_{j=1}^n |a_j|^r \right)^{1/r}
\]

for all sequences of scalars \( (a_j)_{j=1}^n \). Consider the continuous linear operator \( A_n : \ell_{p_1} \rightarrow Y \) given by

\[
A_n(x) = \sum_{j=1}^n x_j z_j.
\]

Since

\[
\frac{1}{p_1} + \frac{1}{\lambda_{1,r}^{p_1}} = \frac{1}{r},
\]

we have, using the Hölder inequality,

\[
\|A_n\| = \sup_{\|x\| \leq 1} \left\| \sum_{j=1}^n x_j z_j \right\| \leq C_2 n^{1/r}.
\]

On the other hand, there is a constant \( C_{p_1}^Y = C \), such that

\[
C \|A_n\| \geq \left( \sum_{j=1}^n \|A_n(e_j)\|_{q_1} \right)^{1/q_1} \geq C_1 n^{1/q_1}.
\]

Since \( n \) is arbitrary, \( q_1 \geq \lambda_{1,r}^{p_1} \) (i.e. (b) holds), and this concludes the proof of the case \( m = 1 \).

The proof of the general case is performed by induction on \( m \). We know that the result is valid for \( m = 1 \) and we shall prove that it is valid for a certain \( m \) whenever it is valid for \( m - 1 \).

(a)\( \Rightarrow \) (b). Let us suppose that

\[
\frac{1}{p_1} + \ldots + \frac{1}{p_m} < \frac{1}{r}.
\]

A fortiori,

\[
\frac{1}{p_2} + \ldots + \frac{1}{p_m} < \frac{1}{r}
\]

and, by our induction hypothesis, if there is a constant \( C_{p_2,\ldots,p_m}^Y \geq 1 \) such that

\[
\left( \sum_{j_2=1}^\infty \left( \sum_{j_3=1}^\infty \cdots \left( \sum_{j_m=1}^\infty \|A(e_{j_2}, \ldots, e_{j_m})\|_{q_{n-1}} \right)^{q_{n-1}/q_m} \cdots \right)^{q_2/q_m} \right)^{1/q_2} \leq C_{p_2,\ldots,p_m}^Y \|A\|
\]
for all continuous \((m - 1)\)-linear operators \(A : \ell_{p_2} \times \cdots \times \ell_{p_m} \to Y\), then by Lemma 2.1 we conclude that (a) implies

\[ q_2 \geq \lambda_{m-1,r}^{p_2,\ldots,p_m}, \]

\[ \vdots \]

\[ q_{m-1} \geq \lambda_{2,r}^{p_{m-1},p_m}, \]

\[ q_m \geq \lambda_{1,r}^{p_m}. \]

So, we must only show that

\[ q_1 \geq \lambda_{m,r}^{p_1,\ldots,p_m}. \]

As for the \(m = 1\) case, there are constants \(C_1, C_2 > 0\) such that for all \(n\) there are vectors \(z_1, \ldots, z_n \in Y\) satisfying

\[ C_1 \left\| (a_j)_{j=1}^n \right\|_\infty \leq \left\| \sum_{j=1}^n a_j z_j \right\| \leq C_2 \left( \sum_{j=1}^n |a_j|^r \right)^{1/r} \]

for all sequences of scalars \((a_j)_{j=1}^n\). Consider the continuous multilinear operator \(A_n : \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y\) given by

\[ A_n(x^{(1)}, \ldots, x^{(m)}) = \sum_{j=1}^n x_j^{(1)} x_j^{(2)} \cdots x_j^{(m)} z_j. \]

Since

\[ \frac{1}{\lambda_{m,r}^{p_1,\ldots,p_m}} + \sum_{k=1}^m \frac{1}{p_k} = \frac{1}{r}, \]

by the Hölder inequality we obtain

\[ \|A_n\| = \sup_{\|x^{(1)}\|, \ldots, \|x^{(m)}\| \leq 1} \left\| \sum_{j=1}^n x_j^{(1)} \cdots x_j^{(m)} z_j \right\| \leq \sup_{\|x^{(1)}\|, \ldots, \|x^{(m)}\| \leq 1} C_2 \left( \sum_{j=1}^n |x_j^{(1)} \cdots x_j^{(m)}|^r \right)^{1/r} \]

\[ \leq \sup_{\|x^{(1)}\|, \ldots, \|x^{(m)}\| \leq 1} C_2 \left( \prod_{k=1}^m \left( \sum_{j=1}^n |x_j^{(k)}|^{p_k} \right)^{1/p_k} \right)^{\frac{1}{\sum_{k=1}^m 1/p_k}} \lambda_{m,r}^{p_1,\ldots,p_m} \]

\[ \leq C_2 n \lambda_{m,r}^{p_1,\ldots,p_m}. \]

On the other hand, by (10)

\[ \left( \sum_{j_1=1}^n \left( \sum_{j_2=1}^n \left( \sum_{j_m=1}^n \|A_n(e_{j_1}, \ldots, e_{j_m})\|^{q_m} \right)^{q_{m-1}/q_m} \right)^{q_2} \right)^{1/q_1} \]

\[ = \left( \sum_{j=1}^n \|A_n(e_j, \ldots, e_j)\|^{q_1} \right)^{1/q_1} \]

\[ \geq C_1 n^{1/q_1}, \]

and, since \(n\) is arbitrary,

\[ q_1 \geq \lambda_{m,r}^{p_1,\ldots,p_m}. \]
(b)⇒(a). Let $A: \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y$ be a continuous $m$-linear operator and define, for all positive integers $n$,

$$A_{n,e}: \ell_{p_1} \times \cdots \times \ell_{p_{m-1}} \to \ell_{r \frac{p_m}{p_m-r}}(Y)$$

by

$$A_{n,e}(x^{(1)}, \ldots, x^{(m-1)}) = \left( A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right)_{j=1}^n.$$

We assert that

$$\| A_{n,e} \| \leq C_r(Y) \| A \|.$$

To see this, since $Y$ has cotype $r$ and using the Hölder inequality,

$$\left( \sum_{j=1}^n \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|^\frac{r_{p_m}}{r_{p_m-r}} \right)^\frac{1}{r_{p_m}}$$

$$= \left( \sum_{j=1}^n \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, \| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|^r \right) \right)^\frac{1}{r}$$

$$\leq C_r(Y) \left( \int_0^1 \left( \sum_{j=1}^n r_j(t) \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|^r \right)^\frac{1}{2} dt \right)$$

$$\leq C_r(Y) \sup_{t \in [0,1]} \left( \sum_{j=1}^n r_j(t) \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|^r \right)^\frac{1}{r}$$

$$\leq C_r(Y) \sup_{\varphi \in B_{Y^*}} \left( \sum_{j=1}^n \varphi \left( A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right) \right) \left( \sum_{j=1}^n \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|_{p_m} \right)^\frac{1}{p_m}$$

$$\leq C_r(Y) \left( \sum_{j=1}^n \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|_{p_m} \right)^\frac{1}{p_m} \leq C_r(Y) \| A \| \| x^{(1)} \| \cdots \| x^{(m-1)} \| \| x^{(m-1)} \| .$$

Therefore,

$$\left( \sum_{j=1}^n \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|^\frac{r_{p_m}}{r_{p_m-r}} \right)^\frac{1}{r_{p_m}} \leq C_r(Y) \| A \| \| x^{(1)} \| \cdots \| x^{(m-1)} \| \| x^{(m-1)} \| \| x^{(m-1)} \|$$

and thus

$$\| A_{n,e} \| = \sup_{\| x^{(1)} \|, \ldots, \| x^{(m-1)} \| \leq 1} \left( \sum_{j=1}^n \left\| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right\|^\frac{r_{p_m}}{r_{p_m-r}} \right)^\frac{1}{r_{p_m}} \leq C_r(Y) \| A \|,$$

as required.
On the other hand, since \( X = \ell_{\lambda_1^{p_1}}(Y) \) has cotype \( \lambda_1^{p_1} := R \) (because \( \lambda_1^{p_1} > r = \text{cot} Y \)) and
\[
\frac{1}{p_1} + \frac{1}{p_{m-1}} < \frac{1}{\text{cot} Y} - \frac{1}{p_m} = \frac{1}{\text{cot} X},
\]
we can use the induction hypothesis (with the \((m-1)\)-linear operator \(A_{n,e}\)), and conclude that if
\[
q_1 \geq \lambda_{m-1,R}^{p_1,\ldots,p_{m-1}}, q_2 \geq \lambda_{m-2,R}^{p_2,\ldots,p_{m-1}}, \ldots, q_{m-1} \geq \lambda_1^{p_1,\ldots,p_{m-1}},
\]
then
\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \cdots \left( \sum_{j_{m-1}=1}^{n} \left( \sum_{j_m=1}^{n} \|A(e_{j_1}, \ldots, e_{j_m})\|^R \right)^{\frac{q_m-2}{q_m-1}} \right) \right)^{\frac{q_m}{q_m-1}} \right) \frac{q_1}{q_2} \geq \frac{1}{q_1}
\]
\[
\leq C_X^{m-1} \|A_{n,e}\|
\]
\[
\leq C_X^{m-1} C_r(Y) \|A\|.
\]
Now, the proof is almost done, since
\[
\lambda_{m-k,R}^{p_k,\ldots,p_{m-1}} = \frac{R}{1 - R \left( \frac{1}{p_k} + \frac{1}{p_{k+1}} + \ldots + \frac{1}{p_{m-1}} \right)} = \frac{\lambda_1^{p_m}}{1 - \lambda_1^{p_1,\ldots,p_{m-1}} \left( \frac{1}{p_k} + \frac{1}{p_{k+1}} + \ldots + \frac{1}{p_{m-1}} \right)} = \frac{\lambda_1^{p_1,\ldots,p_{m-1}}}{1 - \frac{r^{p_1,\ldots,p_{m-1}}}{p_1^{p_1} + \frac{1}{p_{k+1}} + \ldots + \frac{1}{p_{m-1}}}} = \lambda_{m-k+1,R}^{p_k,\ldots,p_{m-1}}
\]
for each \( k \in \{1, \ldots, m-1\} \).
To conclude the proof we just need to remark that
\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^R \right)^{\frac{q_m-2}{q_m-1}} \right) \right)^{\frac{q_m}{q_m-1}} \right) \frac{q_1}{q_2} \geq \frac{1}{q_1}
\]
\[
\leq \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_{m-1}=1}^{\infty} \left( \sum_{j_m=1}^{\infty} \|A(e_{j_1}, \ldots, e_{j_m})\|^R \right) \right)^{\frac{q_1}{q_1}} \right) \right) \frac{q_1}{q_2} \frac{q_1}{q_1}
\]
provided \( q_m \geq R = \lambda_1^{p_m} \).
Since

\[ \lambda_{m,r}^{p_1,\ldots,p_m} \geq \lambda_{m-1,r}^{p_2,\ldots,p_m} \geq \cdots \geq \lambda_{2,r}^{p_m-1,p_m} \geq \lambda_{1,r}^{p_m} \]

and

\[ \frac{1}{\lambda_{m,r}^{p_1,\ldots,p_m}} = \frac{1}{r} \left( \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} \right) \]

the previous theorem generalizes Proposition 4.3 from [12] and Theorem 1.5 of [3], now with optimal exponents in a stronger sense.

**Corollary 2.3.** Let \( Y \) be an infinite-dimensional Banach space with cotype \( \cotype Y \) and \( p_1,\ldots,p_m > \cotype Y \), such that

\[ \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{\cotype Y}. \]

Then there is a constant \( B^Y_{p_1,\ldots,p_m} \geq 1 \) such that

\[ \left( \sum_{j_1,\ldots,j_m=1}^{\infty} \|A(e_{j_1},\ldots,e_{j_m})\|^{1/p_1,\ldots,p_m}_{\cotype Y} \right)^{1/p_1,\ldots,p_m}_{\cotype Y} \leq B^Y_{p_1,\ldots,p_m} \|A\| \]

for all continuous \( m \)-linear operators \( A : \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y \).

In the case that we do not know if \( Y \) attains the infimum of its cotypes, using the previous arguments, it is possible to prove the following:

**Theorem 2.4.** Let \( q_1,\ldots,q_m > 0 \) and \( Y \) be an infinite-dimensional Banach space with finite cotype. If

\[ \frac{1}{p_1} + \cdots + \frac{1}{p_m} < \frac{1}{\cotype Y}, \]

then the following assertions are equivalent:

(a) There is a constant \( C^Y_{q_1,\ldots,q_m} \geq 1 \) such that

\[ \left( \sum_{j_1,\ldots,j_m=1}^{\infty} \|A(e_{j_1},\ldots,e_{j_m})\|^{q_1,\ldots,q_m}_{\cotype Y} \right)^{q_1,\ldots,q_m}_{\cotype Y} \leq C^Y_{q_1,\ldots,q_m} \|A\| \]

for all continuous \( m \)-linear operators \( A : \ell_{p_1} \times \cdots \times \ell_{p_m} \to Y \), and all \( \varepsilon > 0 \).

(b) The exponents \( q_1,\ldots,q_m \) satisfy

\[ q_1 \geq \lambda_{m,\cotype Y}^{p_1,\ldots,p_m}, q_2 \geq \lambda_{m-1,\cotype Y}^{p_2,\ldots,p_m}, \ldots, q_{m-1} \geq \lambda_{2,\cotype Y}^{p_m-1,p_m}, q_m \geq \lambda_{1,\cotype Y}^{p_m}. \]

**Remark 2.5.** Analogous results obtained by permuting the indices in Theorems 2.2 and 2.4 hold with suitable modifications on the conditions for the exponents.
3. Optimal exponents: scalar-valued case

In this section we prove a (sharp) multilinear generalization of Theorems 1.3 and 1.4. Let $p_1, \ldots, p_m > 1$, such that $\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} < 1$. For all positive integers $m$ and $k = 1, \ldots, m$, let us define

$$\delta_{m-k+1}^{p_k, \ldots, p_m} := \frac{1}{1 - \left(\frac{1}{p_k} + \ldots + \frac{1}{p_m}\right)}.$$

As we will see, the proof of the following lemma is similar to the proof of (a) ⇒ (b) of Theorem 2.2. In fact, it is somewhat simpler here, since no appeal to the Maurey-Pisier factorization result is needed.

**Lemma 3.1.** Let $m$ be a positive integer, $q_1, \ldots, q_m > 0$, and $p_1, \ldots, p_m > 1$, with

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} < 1.$$

If there is a constant $C_{p_1, \ldots, p_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \ldots \left(\sum_{j_m=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^q\right)^{\frac{q_m-1}{q_m}} \ldots\right)^{\frac{q_2}{q_2}} \right)^{\frac{q_1}{q_1}} \leq C_{p_1, \ldots, p_m} ||A||$$

for all continuous $m$-linear operators $A : \ell_{p_1} \times \ldots \times \ell_{p_m} \to \mathbb{K}$, then the exponents $q_1, \ldots, q_m$ satisfy

$$q_1 \geq \delta_{m}^{p_1, \ldots, p_m}, q_2 \geq \delta_{m-1}^{p_2, \ldots, p_m}, \ldots, q_{m-1} \geq \delta_{2}^{p_{m-1}, p_m}, q_m \geq \delta_{1}^{p_m}.$$

**Proof.** Let $p > 1$ and $q > 0$. It is well known that if there is a constant $C_p \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty} |A(e_{j_1})|^q\right)^{\frac{1}{q}} \leq C_p ||A||$$

for all continuous linear operators $A : \ell_{p} \to \mathbb{K}$, then $q \geq \delta_{1}^{p}$; thus the case $m = 1$, is done.

Let us suppose the case $m - 1$ and prove the case $m$ by induction. By assumption if there is a constant $C_{p_2, \ldots, p_m} \geq 1$ such that

$$\left(\sum_{j_2=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \ldots \left(\sum_{j_m=1}^{\infty} |A(e_{j_2}, \ldots, e_{j_m})|^q\right)^{\frac{q_m-1}{q_m}} \ldots\right)^{\frac{q_2}{q_3}} \right)^{\frac{q_1}{q_2}} \leq C_{p_2, \ldots, p_m} ||A||$$

for all continuous $(m-1)$-linear forms $A : \ell_{p_2} \times \ldots \times \ell_{p_m} \to \mathbb{K}$, then by Lemma 2.1, (a) implies

$$q_2 \geq \delta_{m-1}^{p_2, \ldots, p_m}$$

$$q_{m-1} \geq \delta_{2}^{p_{m-1}, p_m}$$

$$q_m \geq \delta_{1}^{p_m}.$$

It remains to estimate $q_1$. For each $n$ consider the continuous multilinear form $A_n : \ell_{p_1} \times \ldots \times \ell_{p_m} \to \mathbb{K}$ given by

$$A_n(x^{(1)}, \ldots, x^{(m)}) = \sum_{j=1}^{n} x^{(1)}_j x^{(2)}_j \ldots x^{(m)}_j.$$
Since
\[ \frac{1}{\delta_{m}^{p_1 \cdots p_m}} + \sum_{k=1}^{m} \frac{1}{p_k} = 1, \]
we use the Hölder inequality and obtain
\[ \| A_n \| = \sup_{\|x(1)\|, \ldots, \|x(m)\| \leq 1} \left| \sum_{j=1}^{n} x_{j}^{(1)} x_{j}^{(2)} \cdots x_{j}^{(m)} \right| \]
\[ \leq \sup_{\|x(1)\|, \ldots, \|x(m)\| \leq 1} \left( \prod_{k=1}^{m} \left( \sum_{j=1}^{n} |x_{j}^{(k)}|^p_k \right)^{1/p_k} \right)^{1/p_m} \left( \sum_{j=1}^{n} \left| A_{m-1}^{q_1} \cdots A_{m}^{q_m} \right| \right)^{1/q_2} \]
\[ \leq n^{\delta_{m}^{p_1 \cdots p_m}}. \]
On the other hand
\[ \left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \cdots \left( \sum_{j_m=1}^{n} |A_n(e_{j_1}, \ldots, e_{j_m})|^{q_1} \right)^{q_2} \cdots \right) \right) \]
and, since \( n \) is arbitrary,
\[ q_1 \geq \delta_{m}^{p_1 \cdots p_m}. \]
\( \Box \)

The next theorem is the main result of this section. It is a consequence of our Theorem 2.2, and generalizes Theorems 1.3 and 1.4. The reader should note that the hypothesis \( 1 < p_m \leq 2 < p_1, \ldots, p_{m-1} \) is quite natural, along the lines of a generalization of these Theorems. In fact, if we had \( p_i, p_j \leq 2 \) for some \( i, j \), then we would have
\[ \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} \geq 1, \]
and this is not the environment of a generalization of Theorems 1.3 and 1.4.

**Theorem 3.2.** Let \( m \geq 2, q_1, \ldots, q_m > 0 \), and \( 1 < p_m \leq 2 < p_1, \ldots, p_{m-1} \), with
\[ \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} < 1. \]

The following assertions are equivalent:
\( (a) \) There is a constant \( C_{p_1, \ldots, p_m} \geq 1 \) such that
\[ \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \cdots \left( \sum_{j_m=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{q_1} \right)^{q_2} \cdots \right) \right)^{1/q_1} \leq C_{p_1, \ldots, p_m} \| A \| \]
for all continuous \( m \)-linear operators \( A : \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{K} \).
\( (b) \) The exponents \( q_1, \ldots, q_m > 0 \) satisfy
\[ q_1 \geq \delta_{m}^{p_1 \cdots p_m}, q_2 \geq \delta_{m-1}^{p_2 \cdots p_m}, \ldots, q_{m-1} \geq \delta_{2}^{p_{m-1} p_m}, q_m \geq \delta_{1}^{p_m}. \]
Proof. (a)⇒(b) is a particular case of Lemma 3.1.
(b)⇒(a). Let \( A : \ell_{p_1} \times \cdots \times \ell_{p_m} \to \mathbb{K} \) be a continuous \( m \)-linear operator and define, for all positive integers \( n \),
\[
A_{n,e} : \ell_{p_1} \times \cdots \times \ell_{p_{m-1}} \to \ell_{\delta_{1}^{p_m}}
\]
by
\[
A_{n,e}(x^{(1)}, \ldots, x^{(m-1)}) = \left( A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right)^n_{j=1}.
\]
Note that
\[
\|A_{n,e}\| \leq \|A\|.
\]
In fact, we obviously have
\[
\left( \sum_{j=1}^{n} \left| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right|^{\frac{p_m-1}{p_m}} \right)^{\frac{p_m}{p_m-1}} \leq \|A\| \left\| x^{(1)} \right\| \cdots \left\| x^{(m-1)} \right\|.
\]
Therefore
\[
\|A_{n,e}\| = \sup_{\left\| x^{(1)} \right\| \cdots \left\| x^{(m-1)} \right\| \leq 1} \left\| A_{n,e} \left( x^{(1)}, \ldots, x^{(m-1)} \right) \right\| = \sup_{\left\| x^{(1)} \right\| \cdots \left\| x^{(m-1)} \right\| \leq 1} \left( \sum_{j=1}^{n} \left| A \left( x^{(1)}, \ldots, x^{(m-1)}, e_j \right) \right|^{\frac{p_m-1}{p_m}} \right)^{\frac{p_m}{p_m-1}} \leq \|A\|.
\]
On the other hand, since \( \ell_{\delta_{1}^{p_m}} \) has cotype \( \delta_{1}^{p_m} := r \) (because \( p_m \leq 2 \)) and
\[
\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}} < 1 - \frac{1}{p_m} = \frac{1}{r},
\]
we can invoke Theorem 2.2 for \((m-1)\)-linear operators. Thus, if
\[
q_1 \geq \lambda_{m-1,r}, q_2 \geq \lambda_{m-2,r}, \ldots, q_{m-1} \geq \lambda_{1,r},
\]
we have
\[
\left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \cdots \left( \sum_{j_{m-1}=1}^{n} \left| A(e_{j_1}, \ldots, e_{j_m}) \right|^r \right)_{j_m=1}^{n} \right)_{j_{m-1}=1}^{n} \right)_{j_2=1}^{n} \leq \left( \sum_{j_1=1}^{n} \left( \sum_{j_2=1}^{n} \cdots \left( \sum_{j_{m-1}=1}^{n} \left\| A_{n,e} \left( e_{j_1}, \ldots, e_{j_{m-1}} \right) \right\|_{\ell_{\delta_{1}^{p_m}}} \right)_{j_m=1}^{n} \right)_{j_{m-1}=1}^{n} \right)_{j_2=1}^{n} \leq C_{p_1,\ldots,p_{m-1}} \|A_{n,e}\|.
\]
Since
\[ \lambda_{m-k+1}^{p_k, \ldots, p_m} = \frac{\delta_{1}^{p_m}}{1 - \delta_{1}^{p_m} \left( \frac{1}{p_k} + \frac{1}{p_{k+1}} + \ldots + \frac{1}{p_{m-1}} \right)} \]
for each \( k \in \{1, \ldots, m-1\} \), and

\[
\left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \ldots \left( \sum_{j_m=1}^{\infty} \left| A(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_m-1}{q_m}} \right)^{\frac{q_m}{q_m-1}} \right)^{\frac{1}{q_m}}
\]
\[
\leq \left( \sum_{j_1=1}^{\infty} \left( \sum_{j_2=1}^{\infty} \ldots \left( \sum_{j_m=1}^{\infty} \left| A(e_{j_1}, \ldots, e_{j_m}) \right|^{r} \right)^{\frac{q_m-1}{r}} \right)^{\frac{q_m}{q_m-1}} \right)^{\frac{1}{q_m}}
\]
provided \( q_m \geq r = \delta_{1}^{p_m} \), the proof is done.

Since
\[ \delta_{m}^{p_1, \ldots, p_m} \geq \delta_{m-1}^{p_2, \ldots, p_m} \geq \ldots \geq \delta_{2}^{p_m-1, p_m} \geq \delta_{1}^{p_m}, \]
and
\[ \delta_{m}^{p_1, \ldots, p_m} = \frac{1}{1 - \left( \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} \right)}, \]
then the previous theorem generalizes Theorems 1.3 and 1.4, with optimal exponents for the multilinear form case.

**Corollary 3.3.** Let \( m \geq 2 \), \( 1 < p_m \leq 2 < p_1, \ldots, p_{m-1} \), with
\[ \frac{1}{p_1} + \ldots + \frac{1}{p_m} < 1. \]
Then there is a constant \( C_{p_1, \ldots, p_m} \geq 1 \) such that
\[
(11) \quad \left( \sum_{j_1, \ldots, j_m=1}^{\infty} |A(e_{j_1}, \ldots, e_{j_m})|^{1 - \left( \frac{1}{p_1} + \ldots + \frac{1}{p_m} \right)^{1 - \left( \frac{1}{p_1} + \ldots + \frac{1}{p_m} \right)}} \right)^{\frac{1}{1 - \left( \frac{1}{p_1} + \ldots + \frac{1}{p_m} \right)}} \leq C_{p_1, \ldots, p_m} \|A\|
\]
for all continuous \( m \)-linear operators \( A : \ell_{p_1} \times \cdots \times \ell_{p_m} \rightarrow \mathbb{K} \).

In the final section we show that the optimal constant \( C_{p_1, \ldots, p_m} \) of Theorem 3.2 and Corollary 3.3 is precisely 1.
4. Optimal constants

The Banach spaces in this section are considered over the complex scalar field. Let us begin by recalling that the Rademacher matrices $R_n = \left( r_{ij}^{(n)} \right)$, $i = 1, ..., 2^n$, $j = 1, ..., n$, are the $2^n \times n$ matrices defined recursively as follows:

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \begin{pmatrix} 1 & R_n \\ -1 & \vdots & R_n \end{pmatrix},$$

for $n \in \mathbb{N}$. Note that $r_{ij}^{(n)} = r_j \left( \frac{2i-1}{2^n-1} \right)$, where $r_j$ denotes the $j$-th Rademacher function.

Let $1 \leq p \leq 2$ and $0 < s < \infty$. Recall that a Banach space $X$ has type $p$ (see [14]) if there is a constant $C > 0$ such that, no matter how we select finitely many vectors $x_1, \ldots, x_n \in X$,

$$\left( \int_{[0,1]} \left\| \sum_{k=1}^{n} r_k(t) x_k \right\|^s dt \right)^{1/s} \leq C \left( \sum_{k=1}^{n} \|x_k\|^p \right)^{\frac{1}{p}}, \quad (12)$$

where $r_k$ denotes the $k$-th Rademacher function. It is well known that if (12) is satisfied for a certain $s > 0$, then it is satisfied for all $s > 0$. For a fixed $s$, the smallest of all constants $C$ will be denoted by $T_{p,s}(X)$.

In the following result of [14], type and cotype properties are described via the linear operators induced by the Rademacher matrices and their transposes:

**Proposition 4.1.** (See [14, Proposition 2.3]) Let $X$ be a Banach space.

(i) Let $1 < p \leq 2$. Then $X$ has type $p$ if and only if there exist some $s$, $1 \leq s < \infty$, and a constant $M$ such that

$$\|R_n : \ell_2^n (X) \rightarrow \ell_2^{2^n} (X)\| \leq M 2^{\frac{1}{p}},$$

for all $n \in \mathbb{N}$. Moreover, $T_{p,s}(X) \leq M$.

(ii) Let $2 \leq q < \infty$, and $t_{R_n}$ be the transposed matrix of $R_n$. Then $X$ has cotype $q$ if and only if there exist some $s$, $1 \leq s < \infty$, and a constant $M$ such that

$$\|t_{R_n} : \ell_s^n (X) \rightarrow \ell_q^n (X)\| \leq M 2^{\frac{1}{q}},$$

for all $n \in \mathbb{N}$. Moreover, $C_{q,s}(X) \leq M$.

**Lemma 4.2.** (See [14, Lemma 2.3]) Let $H$ be a Hilbert space. Then

$$2^{\frac{1}{p}} = \|R_n : \ell_2^n (H) \rightarrow \ell_2^{2^n} (H)\| = \|t_{R_n} : \ell_2^n (H) \rightarrow \ell_2^n (H)\|,$$

for all $n \in \mathbb{N}$. 


Let us introduce the following notation: for $1 \leq p_1 < \infty$ and $1 \leq p_2 \leq \infty$, we denote by $\ell_{p_1}(\ell_{p_2})$ the Banach space of the sequences $x = (x_{i_1,i_2})_{i_1,i_2=1}^\infty$ such that
\[
\|x\|_{\ell_{p_1}(\ell_{p_2})} := \left( \sum_{i_1=1}^\infty \left( \sum_{i_2=1}^\infty |x_{i_1,i_2}|^{p_1} \right)^{\frac{1}{p_1}} \right)^{\frac{1}{p_2}} < +\infty
\]
and by $\ell_\infty(\ell_{p_2})$ the Banach space of the sequences $x = (x_{i_1,i_2})_{i_1,i_2=1}^\infty$ such that
\[
\|x\|_{\ell_\infty(\ell_{p_2})} := \sup_{i_1} \left( \sum_{i_2=1}^\infty |x_{i_1,i_2}|^{p_2} \right)^{\frac{1}{p_2}} < \infty,
\]
Inductively, for $p = (p_1, \ldots, p_m) \in [1, +\infty]^m$, we can define the Banach space $\ell_p$ by
\[
\ell_p := \ell_{p_1}(\ell_{p_2}(\cdots(\ell_{p_m})\cdots)).
\]
Namely, a vector $x = (x_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^\infty \in \ell_p$ if, and only if,
\[
\left( \sum_{i_1=1}^\infty \left( \sum_{i_2=1}^\infty \left( \cdots \left( \sum_{i_m=1}^\infty |x_{i_1,\ldots,i_m}|^{p_m} \right)^{\frac{1}{p_m}} \right)^{\frac{1}{p_{m-1}}} \right)^{\frac{1}{p_{m-2}}} \right)^{\ldots} < +\infty,
\]
(the usual modification is required if some $p_j = \infty$).

The next result is based on ideas borrowed from [14, Theorem 3.2]. We use standard notation and notions from interpolation theory, as presented e.g. in [7].

**Theorem 4.3.** Let $p = (p_1, \ldots, p_m) \in (1, +\infty)^m$, and let $t := \min \{p_1, \ldots, p_m, p_1^*, \ldots, p_m^*\}$. Then,
\[
(13) \quad \|R_n : \ell_{t^*} (\ell_p) \to \ell_{t_n^*} (\ell_p)\| = 2^{\frac{n}{n}}
\]
for any $s, 1 \leq s \leq t^*$ and all $n \in \mathbb{N}$. In other words, $\ell_p$ is of type $t$ and $T_{t,s}(\ell_p) = 1$, for all $1 \leq s \leq t^*$.

**Proof.** It is enough to show (13) for $s = t^*$. By Lemma 4.2 we know that for all $n \in \mathbb{N}$,
\[
(14) \quad \|R_n : \ell_{t^*} (\ell_2) \to \ell_{t_n^*} (\ell_2)\| = 2^{\frac{n}{n}},
\]
because $\ell_2 := \ell_2 (\ell_2 (\cdots (\ell_2) \cdots))$ is a Hilbert space.

Suppose that $p_1, \ldots, p_m$ are not all 2. As a first step, let us show
\[
(15) \quad \|R_n : \ell_{t^*} (\ell_p) \to \ell_{t_n^*} (\ell_p)\| \leq 2^{\frac{n}{n}}.
\]

If $t = p_k$ for some $k \in \{1, \ldots, m\}$ (obviously $p_k < 2$), put $\theta = \frac{2}{p_k} \in (0, 1)$ and
\[
\frac{1}{p_i^*} := \frac{1/p_i - 1/p_k^*}{1/p_k - 1/p_k^*}
\]
for all $i \in \{1, \ldots, m\}$, $i \neq k$.

Then, since
\[
\frac{1 - \theta}{1} + \frac{\theta}{2} = \frac{1}{p_k},
\]
\[
\frac{1 - \theta}{p_i^*} + \frac{\theta}{p_k} = \frac{1}{p_i},
\]
for all $i \in \{1, \ldots, m\}$, $i \neq k$, we have by [7, Theorem 5.1.1 and Theorem 5.1.2]
\[
(\ell_{p_0}, \ell_2)_{[\theta]} = \ell_p
\]
with equal norms, where $p_0 = (p_1^0, \ldots, p_{k-1}^0, 1, p_k^0, \ldots, p_m^0)$. 

With the same notation, since
\[\frac{1 - \theta}{\infty} + \frac{\theta}{2} = \frac{1}{p_k^*},\]
we have by [7, Theorem 4.2.1, Theorem 5.1.1 and Theorem 5.1.2]
\[(16)\] 
\[\ell_1^n (\ell_{p_0}) = \ell_2^n (\ell_2, \ell_0)\]
and
\[(17)\] 
\[\ell_2^n (\ell_{p_0}) = \ell_2^n (\ell_2, \ell_0)\]
with equal norms.
Computing,
\[(18)\] 
\[\| R_n : \ell_1^n (\ell_{p_0}) \to \ell_2^n (\ell_{p_0}) \| = 1\]
and interpolating (14) and (18) (by using 16, and 17) we have
\[\| R_n : \ell_1^n (\ell_{p_0}) \to \ell_2^n (\ell_{p_0}) \| \leq \left( \frac{2^n}{2^n} \right)^\theta = 2^n,\]
and then (15) is true if \( t = p_k^* \) for some \( k \in \{1, \ldots, m\} \).
If \( t = p_k^* \) for some \( k \in \{1, \ldots, m\} \), (obviously \( p_k > 2 \)), put \( \theta = \frac{2^n}{p_k} \in (0, 1) \) and
\[\frac{1}{p_k^*} := \frac{1}{p_k} - \frac{1}{p_k},\]
for all \( i \in \{1, \ldots, m\}, i \neq k \).
Then, since
\[\frac{1 - \theta}{\infty} + \frac{\theta}{2} = \frac{1}{p_k},\]
\[\frac{1 - \theta}{\infty} + \frac{\theta}{2} = \frac{1}{p_i},\]
for all \( i \in \{1, \ldots, m\}, i \neq k \), we have
\[(\ell_{p_1}, \ell_{p_2})_{[\theta]} = \ell_{p} \]
with equal norms, where \( p_1 = (p_1^1, \ldots, p_{k-1}^1, \infty, p_{k+1}^1, \ldots, p_m^1) \).
Keeping the notation, since
\[\frac{1 - \theta}{\infty} + \frac{\theta}{2} = \frac{1}{p_k},\]
we have by [7, Theorem 4.2.1, Theorem 5.1.1 and Theorem 5.1.2]
\[(19)\] 
\[\ell_1^n (\ell_{p_1}) = \ell_2^n (\ell_2, \ell_0)\]
and
\[(20)\] 
\[\ell_2^n (\ell_{p_1}) = \ell_2^n (\ell_2, \ell_0)\]
with equal norms, and
\[(21)\] 
\[\| R_n : \ell_1^n (\ell_{p_1}) \to \ell_2^n (\ell_{p_1}) \| = 1.\]
By using (19), (20), and interpolating (14) and (21) we have
\[\| R_n : \ell_1^n (\ell_{p_1}) \to \ell_2^n (\ell_{p_1}) \| \leq \left( \frac{2^n}{2^n} \right)^\theta = 2^n.\]
Therefore, the inequality (15) is true.
To show
\[\| R_n : \ell_1^n (\ell_{p_1}) \to \ell_2^n (\ell_{p_1}) \| = 2^n,\]
it is enough to see that the equality is attained with $(x, 0, ..., 0) \in \ell^n_t(\ell_p^r), \; 0 \neq x = (x_{i_1}, ..., x_{i_m})_{i_1, ..., i_m=1} \in \ell_p^r$, and the proof is done. \qed

The following result is a direct consequence of the above theorem and Proposition 4.1, using duality and the reflexivity of $\ell_p^r$:

**Corollary 4.4.** Let $p = (p_1, \ldots, p_m) \in (1, +\infty)^m$, and let $t := \min \{p_1, \ldots, p_m, p_1^*, \ldots, p_m^*\}$. Then, for any $s$ with $t \leq s < \infty$, we have

$$||t_{R_n} : \ell_s^{2^n} (\ell_p^r) \rightarrow \ell_t^n (\ell_p^r)|| = 2^{\frac{2s}{m}}$$

for all $n \in \mathbb{N}$. Hence, $\ell_p^r$ is of cotype $t^*$ and

$$C_{t^*, s}(\ell_p^r) = 1,$$

for all $t \leq s < \infty$.

**Remark 4.5.** The above corollary was proved by using complex interpolation, for the case of complex scalars but from the very definition of cotype it is obvious that (22) also holds for real Banach spaces.

**Remark 4.6. (Optimal constants for Theorems 2.2 and 3.2)** (1) From the previous results we conclude that in Theorem 2.2, when $Y = \ell_r$ (over the real or complex field) with $r \in [2, \infty)$, if (b) is true, then the optimal constant in the inequality (a) satisfies $C_{p_1, \ldots, p_m}^\ell_r = 1$. In fact, the case $m = 1$ is immediate. For $m = 2$, note that $\ell_{\lambda_{1,r}^{p_2}}(\ell_r)$ has cotype $\lambda_{1,r}^{p_2} := R > r \geq 2$ (by Corollary 4.4) with $C_{R, l}(\ell_r) = 1$, and thus, following the proof of Theorem 2.2, $C_{P_1, P_2}^\ell_r \leq C_{R, l}(\ell_r)C_{l}(\ell_r) = 1$. For the case $m = 3$, following the proof of Theorem 2.2, we have

$$C_{P_1, P_2, P_3, l}^\ell_r \leq C_{\lambda_{2,r}^{P_3}, \lambda_{1,r}^{P_3}} \left( \ell_{\lambda_{2,r}^{P_3}(\ell_r)} \right) C_{\lambda_{1,r}^{P_3}(\ell_r)} C_{l}(\ell_r) = 1$$

by Corollary 4.4 and the proof follows inductively.

(2) If (b) is true in Theorem 3.2 the optimal constat $C_{p_1, \ldots, p_m}$ in (a) is 1, because the $(m - 1)$-linear operator used in the argument of the proof of (b)$\Rightarrow$(a) has range $\ell_{\delta_{pm}}^* = \ell_{(p_m)^*}$ and $(p_m)^* \geq 2$. By the first item of this remark, we know that

$$C_{P_1, \ldots, P_{m-1}}^\ell_r \delta_{P_m} = 1.$$

**Remark 4.7.** All the above results can be translated to the setting of multiple summing operators (for recent results on multiple summing operators we refer to [5, 6, 8, 18, 20] and references therein). In fact, we just need to consider the more general concept of multiple summing operators introduced in [2] and recall how to translate coincidence situations like the Bohnenblust–Hille inequality and Hardy–Littlewood inequalities to multiple summing operators (see, for instance, [12] and [17, Corollary 3.20]).

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