ABSTRACT. In this paper, we seek an appropriate definition for a Shimura curve of Hodge type in positive characteristics, i.e. a characterization, in terms of geometry mod $p$, of curves in positive characteristics which are reduction of Shimura curve over $\mathbb{C}$. Specifically, we study the liftability of a curve in moduli space $A_{4,1,k}$ of principally polarized abelian varieties over $k$, char $k = p$. We show that some conditions on the $l$-adic monodromy over such a curve imply that this curve can be lifted to a Shimura curve.

1. INTRODUCTION

This paper is a sequel to our previous paper [15] and [16]. All the three paper aim at answering the following question:

What is an appropriate definition of Shimura curves in positive characteristics?

The answer is only known for Shimura varieties of PEL type which admit a natural moduli interpretation. In this paper, we consider Shimura curves of Hodge type and give an answer in the generic ordinary case, in terms of $l$-adic monodromy.

In [15], we start with a proper family of abelian varieties in characteristic $p$ and prove if this family admits some special crystalline cycles, then it is a reduction of a Shimura curve of Mumford type. In this paper we present a similar result. We find the conditions on the $l$-adic monodromy associated to the family, which imply the family is a reduction of a Mumford type family of abelian fourfolds.

Let $\pi : X \longrightarrow C$ be a family of principally polarized abelian fourfolds over a proper smooth curve $C$ defined over a finite field $\mathbb{F}_q$ with $q = p^f$, $p > 3$. Let $\mathcal{E}_l$ be the lisse étale $l$-adic sheaf $R^1\pi_*(\mathbb{Q}_l)$ over $C$. Choose a geometric point $\bar{\xi}$ and a closed point $c$ in $C$, and then $\mathcal{E}_l$ induces a monodromy:

$$\rho : \pi_1(C, \bar{\xi}) \longrightarrow \text{Aut}(\mathcal{E}_{l,c}) \cong GL(8, \mathbb{Q}_l).$$

Let $G_l$ be the Zariski closure of $\rho(\pi_1(C, \bar{\xi}))$ in $GL(8, \mathbb{Q}_l)$ and $G_l^{\text{geom}}$ be that of $\rho(\pi_1^{\text{geom}}(C, \bar{\xi}))$.

$$1 \longrightarrow \pi_1^{\text{geom}}(C, \bar{\xi}) \longrightarrow \pi_1(C, \bar{\xi}) \longrightarrow \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \longrightarrow 1.$$ 

Then $G_l^{\text{geom}}$ is a normal subgroup of $G_l$.

To every closed point $c \in C$, it associates a unique (up to conjugation) Frobenius element $F_c$ in $\pi_1(C, \bar{\xi})$. Its image in $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is the integer $\deg(\kappa(c) : \mathbb{F}_q)$.

For notational simplicity, let us define the following representation

$$\rho_0 : SL(2, \mathbb{Q}_l)^{\times 3} \longrightarrow \text{Aut}(\mathcal{E}_{l,c})$$

as the tensor product of three copies of the standard representation of $SL(2, \mathbb{Q}_l)$.

Fix an embedding $\mathbb{Q}_l \longrightarrow \mathbb{C}$ once for all. Our main theorem is as follows.
Theorem 1.1. If there exists a closed point \( c \in C \) such that
\begin{enumerate}
  \item \( \mathcal{G}^{geom.o} \otimes_{Q_l} C \cong \mathrm{im} \rho_0 \otimes C \),
  \item \( X_c \) is an ordinary abelian variety,
  \item In \( G_l, \rho(F_c) \) generates a maximal torus which is unramified over \( \mathbb{Q}_p \).
\end{enumerate}
Then \( X \longrightarrow C \) is a weak Mumford curve.

If further assume the Higgs field of \( X \longrightarrow C \) is maximal, then there exists a family of polarized abelian fourfolds \( Y \longrightarrow C' \) such that
\begin{enumerate}
  \item \( C' \longrightarrow C \) is a finite \( \acute{e}tal\)e covering,
  \item \( Y \longrightarrow X' \) is an isogeny between \( Y \) and the pullback family of \( X \) over \( C' \),
  \item \( Y \longrightarrow C' \) is a good reduction of a Mumford curve.
\end{enumerate}

For the definition of weak Mumford curve, see Section 2.

Remark 1.2. By [6.1] and Chebotarev density theorem, the Frobenius element over sufficiently many points in \( C \) generates the maximal torus in \( G_l \). From [7], we know that the torus generated by the Frobenius \( F_c \) is defined over \( \mathbb{Q} \). So it makes sense to require the torus is unramified over \( \mathbb{Q}_p \), which is equivalent to say that the eigenvalues of the Frobenius are unramified over \( \mathbb{Q}_p \).

We wonder if (3) can be replaced by a weaker condition, especially under the presence of (1) and (2). More are explained in Remark 6.2.

1.3. Structure of the paper. To prove 1.1, we reduce it to the main theorem in [15]. We need to compute the Frobenius eigenvalues in the \( l \)-adic setting and the connection with the crystalline cohomology.

In Section 3, we show some good reductions of Mumford curves satisfy the conditions in 1.1 so that we are not proving a vacuous theorem.

In Section 4, we compute the dimensions of the Frobenius eigenspaces in \( H^0_{et}(\overline{C/F_q}, \wedge^4 \mathcal{E}_l) \) and \( H^0_{et}(\overline{C/F_q}, \mathcal{E}_{nd}(\wedge^2 \mathcal{E}_l)) \).

\begin{align*}
  \dim H^0_{et}(\overline{C/F_q}, \wedge^4 \mathcal{E}_l)^{F-q^2} \otimes \mathbb{Q}_q &= 1 \\
  \dim H^0_{et}(\overline{C/F_q}, \mathcal{E}_{nd}(\wedge^2 \mathcal{E}_l))^F \otimes \mathbb{Q}_q &= 4.
\end{align*}

In Section 5, we translate the above result to \( p \)-adic case via comparing Lefschetz trace formulas. Therefore the Frobenius eigenspaces in crystalline cohomology spaces have the expected dimensions as in the main theorem of [15]. Furthermore, we conclude in Section 6 that
\begin{align*}
  \Gamma((C/W(k))_{cris}, \mathcal{E}_{nd}(\wedge^2 \mathcal{E}))^F \otimes \mathbb{Q}_q &\cong \mathbb{Q}_q \times^4
\end{align*}
as algebras. Then 1.1 boils down to the main result in [15].

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2. Mumford curves

In [11, Chapter 4], Mumford defines a family of Shimura curves. We briefly recall the construction.
Let $F$ be a cubic totally real field and $D$ be a quaternion division algebra over $F$ such that

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \times \mathbb{H} \times M_2(\mathbb{R}), \text{Cor}_{F/\mathbb{Q}}(D) \cong M_8(\mathbb{Q}).$$

Here $\mathbb{H}$ is the quaternion algebra.

Let $G = \{ x \in D^* | x\bar{x} = 1 \}$. Then $G$ is a $\mathbb{Q}$-simple algebraic group and it is the $\mathbb{Q}$-form of the $\mathbb{R}$-algebraic group $SU(2) \times SU(2) \times SL(2,\mathbb{R})$.

$$h : S_m(\mathbb{R}) \longrightarrow G(\mathbb{R}) \quad e^{i\theta} \mapsto I_4 \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

The pair $(G, h)$ forms a Shimura datum. And it defines Shimura curves, parameterizing abelian fourfolds. We call such curves (with its universal family) Mumford curves.

**Definition 2.1.**

1. A curve in $\mathcal{A}_{4,1,n} \otimes \mathbb{C}$ is called a special Mumford curve if it is the image of a Mumford curve in $\mathcal{A}_{4,1,n} \otimes \mathbb{C}$ induced by a universal family.

2. The family $X \longrightarrow C$ is a weak Mumford curve over $k$ if the image of $C \longrightarrow \mathcal{A}_{4,1,n}$ (induced by the family $X/C$) is (possibly an irreducible component of) a reduction of a special Mumford curve in $\mathcal{A}_{4,1,n} \otimes \mathbb{C}$.

**Remark 2.2.** The ”weakness” of the weak Mumford curve, comparing to good reductions, reflects in two aspects: firstly, the image of $C \longrightarrow \mathcal{A}_{4,1,n}$ might have singularities. Secondly, the reduction of a special Mumford curve at $k$ might be reducible and that image of $C \longrightarrow \mathcal{A}_{4,1,n}$ is just one of the irreducible components.

3. Examples

To indicate that 1.1 is not a vacuous result, we show good reductions of a Mumford curve with an ordinary fiber satisfy the conditions of 1.1.

Let $f : A \longrightarrow M$ be the universal family over a Mumford curve defined over $\mathbb{C}$ and $M$ is defined over the reflex field $K$. For every $p$ over which $K$ splits, $A \longrightarrow M$ admits a smooth and generically ordinary reduction $\pi : X \longrightarrow C$ over $p$ ([9]). By the definition of Mumford curve or [1, Proposition 2.4], the image of

$$\rho_C : \pi_1(M) \longrightarrow R^1 f_*(\mathbb{C})$$

is $SL(2,\mathbb{C}) \times 3$. By Grothendieck specialization theorem of algebraic monodromy, $\rho_C$ factors through

$$\rho : \pi_1(C, \bar{\xi}) \longrightarrow R^1 f_*(\mathbb{C})$$

with a surjection $\pi_1(M) \longrightarrow \pi_1(C, \bar{\xi})$. By the comparison of the $l$-adic cohomology and de Rham cohomology $R^1 f_*(\mathbb{C}) \cong R^1 f_*(\mathbb{Q}_l) \otimes \mathbb{C}$ for every $l \neq p$, we know conditions (1) and (2) in 1.1 hold for such $C$.

For condition (3), we choose $x$ to be a CM point on $M$. Since $A_x$ is simple, there is a degree 8 field $L \subset \text{End}^0(A_x)$. We can choose $p$ such that $L$ is unramified over $p$. Then look at the reduction $\bar{x}$ and $\mathbb{Q}[F_x]$ is the center of $\text{End}(X_{\bar{x}})$. Since $L \subset \text{End}(X_{\bar{x}})$ is the maximal commutative subalgebra, $F \in L$ and in particular, $F$ is unramified over $p$.

So with a careful choice of $p$, the resultant reduction of a Mumford curve satisfies all the conditions in 1.1.
4. Frobenius Eigenvalues on $\wedge^4 \mathcal{E}$ or $\text{End}(\wedge^2 \mathcal{E})$

In this section, we compute the dimension of Frobenius eigenspace at the target spaces and the corresponding eigenvalues.

Recall the short exact sequence

$$1 \rightarrow \pi_1^{\text{geom}}(C, \xi) \rightarrow \pi_1(C, \xi) \rightarrow \text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \rightarrow 1.$$ 

The Zariski closure $G_t^{\text{geom}}$, of $\rho(\pi_1^{\text{geom}}(C, \xi))$ is a normal subgroup of $G_t$. Since $\mathcal{E}_t$ is pure, it follows from [4, 1.3.9, 3.4.1(iii)] that $G_t^{\text{geom}}$ is semisimple. The connected component of identity $G_t^{\text{geom}, \text{t}}$ is the derived group of $G_t^{0}$.

**Remark 4.1.** If we assume $G_t^{\text{geom}} \otimes \mathbb{C}$ is entirely contained in $SL(2, \mathbb{C})^3$, then we only need to enlarge the base field $\mathbb{F}_q$ to kill the $S_3$ part.

For every closed point $c$ of $C$, the action of $F_c$ on $\mathcal{E}_{i,c} \cong H^1_t(X_c, \mathbb{Q}_l)$ is semisimple (see [2]). Therefore $F_c$ acts on $\wedge^4 \mathcal{E}_{i,c}$ and $\text{End}(\wedge^2 \mathcal{E}_{i,c})$ semisimply.

Firstly let us consider the space

$$H^0_{\text{et}}(C_{\overline{\mathbb{F}_q}}, \mathcal{E} \otimes \text{End}(\wedge^2 \mathcal{E}_i)) \cong \text{End}(\wedge^2 (\mathcal{E}_{i,c}))^{\pi_1^{\text{geom}}(C,c)}.$$ 

Since $F_c$ acts on $\text{End}(\wedge^2 \mathcal{E}_{i,c})$ semisimply, we can calculate its eigenvalues over $\mathbb{C}$.

Let $V$ be the dimension 2 standard representation of $SL(2, \mathbb{C})$. Condition (1) of Theorem 1.1 shows that $\mathcal{E}_{i,c} \otimes \mathbb{C} \cong V^\otimes 3$ is the tensor product of three standard representations of $SL(2, \mathbb{C})$. So base change to $\mathbb{C}$,

$$H^0_{\text{et}}(C_{\overline{\mathbb{F}_q}}, \mathcal{E} \otimes \text{End}(\wedge^2 \mathcal{E}_i)) \otimes \mathbb{C} \cong \text{End}(\wedge^2 (V^\otimes 3))^{SL(2,\mathbb{C})^\times 3}.$$ 

As a representation of $SL(2, \mathbb{C})^\times 3$, $\wedge^2 (V^\otimes 3)$ decomposes into four distinct irreducible components

$$\wedge^2 (V^\otimes 3) \cong \bigoplus_{i=1}^4 W_i.$$ 

There $W_1, W_2, W_3$ are all $S^2 V \otimes S^2 V$ and $W_4$ is the trivial representation of $SL(2, \mathbb{C})^\times 3$. Yet $W_1, W_2, W_3$ are pairwisely non-isomorphic $SL(2, \mathbb{C})^\times 3$ representations.

Let $p_k$ ($i_k$, resp.) be the projection from $\wedge^2 (V^\otimes 3)$ to $W_k$ (inclusion from $W_k$ to $\wedge^2 (V^\otimes 3)$, resp.). We have

$$\text{End}(\wedge^2 (V^\otimes 3))^{SL(2,\mathbb{C})^\times 3} \cong \bigoplus_{k=1}^4 (i_k \circ \text{id}_{W_k} \circ p_k).$$

Note each $\text{id}_{W_k}$ is invariant under the action of $SL(2, \mathbb{C})^\times 3$ and the scalar multiplication. Thus $\text{id}_{W_k}$ is further invariant under the action of $G_t$. In particular, the Frobenius $F_c$ fixes each $\text{id}_{V_i}$. So the invariant space $H^0_{\text{et}}(C_{\overline{\mathbb{F}_q}}, \mathcal{E} \otimes \text{End}(\wedge^2 \mathcal{E}_i))^F$ has dimension 4.

Secondly,

$$H^0_{\text{et}}(C_{\overline{\mathbb{F}_q}}, \wedge^4 \mathcal{E}_i) = \left(\wedge^4 \mathcal{E}_{i,c}\right)^{\pi_1^{\text{geom}}(C,c)}.$$ 

Similarly, base change to $\mathbb{C}$ and it is isomorphic to $\wedge^4 (V^\otimes 3)^{SL(2,\mathbb{C})^\times 3}$. One can directly compute by hand, or see the proof of Theorem 4.1 in [10] to conclude that this space only has dimension 1 which is generated by the polarization. Therefore the corresponding Frobenius eigenvalues are $q^2$.

In summary, we have the following results:

$$\dim H^0_{\text{et}}(C_{\overline{\mathbb{F}_q}}, \mathcal{E} \otimes \text{End}(\wedge^2 \mathcal{E}_i))^F = 4,$$

$$\dim H^0_{\text{et}}(C_{\overline{\mathbb{F}_q}}, \wedge^4 \mathcal{E}_i)^F - q^2 = 1.$$ 

(4.1)
5. COMPARISON OF LEFSCHETZ TRACE FORMULAS

In this section, we compare Lefschetz Trace Formulas to obtain a similar result to (4.1) in the case of crystalline cohomology.

We firstly consider $\mathcal{E}_p := R^1\pi_{\text{cris},*}(\mathcal{O}_X)$. Since $\sigma$ is the identity on $\mathbb{F}_q$, the absolute Frobenius $F$ acts linearly on $\mathcal{E}_{p,c}$. Since the local crystalline characteristic polynomial coincides with the $l$-adic one ([5, 1.3.5])

\[
\det(1 - tF|_{\mathcal{E}_{p,c}}) = \det(1 - tF|_{\mathcal{E}_{l,c}}),
\]

the eigenvalues of $F$ on $\mathcal{E}_{l,c}$ and $\mathcal{E}_{p,c}$ are identical.

Let $\mathcal{F}_l$ be either $\wedge^4\mathcal{E}_l$ or $\mathcal{E}\lhd(\wedge^2\mathcal{E}_l)$. Since $\mathcal{E}_l$ comes from geometry, by Deligne’s Weil II, the $l$-adic relative Lefschetz Trace Formula provides

\[
\prod_{c \in C} \det(1 - tF|_{\mathcal{F}_{l,c}}) = \prod_{i} \det(1 - tF|_{H_{\text{cris}}^i(C_{/\mathbb{F}_q}, \mathcal{F}_l)})(-1)^i.
\]

In the $p$-adic case, we still use $\mathcal{F}_p$ to represent either $\wedge^4\mathcal{E}_p$ or $\mathcal{E}\lhd(\wedge^2\mathcal{E}_p)$. Since $\mathcal{E}_p$ is a Dieudonné crystal, $\mathcal{F}_p$ is automatically overconvergent. By a theorem of Etesse and le Stum ([6, 2.1.2]), we also have a Lefschetz Trace Formula within crystalline cohomology setting

\[
\prod_{c \in C} \det(1 - tF|_{\mathcal{F}_{p,c}}) = \prod_{i} \det(1 - tF|_{H_{\text{cris}}^i(C_{/\mathbb{Z}_q}, \mathcal{F}_p)})(-1)^i.
\]

Combining with equality (5.1), we have

\[
\prod_{i} \det(1 - tF|_{H_{\text{cris}}^i(C_{/\mathbb{Z}_q}, \mathcal{F}_l)})(-1)^i = \prod_{i} \det(1 - tF|_{H_{\text{cris}}^i(C_{/\mathbb{Z}_q}, \mathcal{F}_p)})(-1)^i.
\]

By Deligne’s Weil II ([4], the étale cohomology groups $H_{\text{et}}^i(C_{/\mathbb{F}_q}, \mathcal{F}_l)$ is pure of weight $i + j$ where $\mathcal{F}_l$ has weight $j$. Since $\mathcal{F}_p$ is pointwisely pure, by [6, Theorem 5.3.2], $H_{\text{cris}}^i(C_{/\mathbb{Z}_q}, \mathcal{F}_p)$ has the purity which implies, on each side of equality (5.2), there is no cancellation between the numerator and the denominator. All zeros or poles have the expected complex norms. Then we have the following termwise equality from (5.2).

\[
\det(1 - tF|_{H_{\text{cris}}^i(C_{/\mathbb{Z}_q}, \mathcal{F}_l)}) = \det(1 - tF|_{H_{\text{cris}}^i(C_{/\mathbb{Z}_q}, \mathcal{F}_p)}).
\]

So the eigenvalues of $F$ on $H^0(C_{/\mathbb{Z}_q}, \wedge^4\mathcal{E}_p) \otimes \mathbb{Q}_q$ and $H^0(C_{/\mathbb{Z}_q}, \mathcal{E}\lhd(\wedge^2\mathcal{E}_p)) \otimes \mathbb{Q}_q$ are identical as on their $l$-adic counterparts. In particular,

\[
\dim H^0(C_{/\mathbb{Z}_q}, \wedge^4\mathcal{E}_p)^{F-q^2} \otimes \mathbb{Q}_q = 1,
\]
\[
\dim_{\mathbb{Q}_q} H^0(C_{/\mathbb{Z}_q}, \mathcal{E}\lhd(\wedge^2\mathcal{E}_p))^F \otimes \mathbb{Q}_q = 4.
\]

6. COMPUTE $\text{End}^0(\wedge^2\mathcal{E}_p)^F$

In order to apply the main theorem in ([15], we need to prove that $\text{End}^0(\wedge^2\mathcal{E}_p)^F \cong \mathbb{Q}_q^{\times 4}$ as algebras.
6.1. **Frobenius Torus.** By [13, Theorem 2], $\mathbb{Q}[F] \cong \prod K_i$ where $K_i$ are number fields. The multiplicative group $\mathbb{Q}[F]^\times$ defines a $\mathbb{Q}$-torus

$$T = \prod \text{Res}_{K_i/\mathbb{Q}}(\mathbb{G}_m).$$

Viewing $F$ as an element in $G_l$, $T$ can be regarded as the $\mathbb{Q}$-model of the connected component of 1 in the Zariski closure of the set $\{\rho(F)^n|n \in \mathbb{Z}\}$ in $G_l$ (cf. [1, Chapter II, Section 13, Proposition 3]). In particular, $T$ is contained in a maximal torus of $G_l$.

By [3, Theorem 3.7] and Chebotarev density theorem for the function field, generic points $c$ on $C$ satisfy that $F_c$ generates a maximal torus. For every $c$, the torus $T$ is defined over $\mathbb{Q}$. We say $T$ is unramified over $\mathbb{Q}_p$, if the splitting field of $T$ is unramified over prime $p$, and equivalently, the eigenvalues of $F_c$ are unramified over $p$.

**Remark 6.2.** Varying the prime $l$, we obtain a compatible system of $l$-adic representation as stated in [7, 6.5]. The existence of a point $c$ satisfying (3) in 1.1 requires that $G_l$ is unramified over $\mathbb{Q}_p$.

On one hand, by [7, Proposition 8.9] and [8, Proposition 1.2, Theorem 3.2], for a subset of primes $l$ of density 1 (or even $l$ large enough), $G_l$ is unramified over $\mathbb{Q}_l$. However, most results in the two paper have involved Dirichlet density restriction and hence can not be applied directly to our case.

On the other hand, we expect that if $G_l$ is unramified over $\mathbb{Q}_q$, then there always exists a closed point $c$ satisfying (3) in 1.1.

We also think generic ordinary property of $X \rightarrow C$ should also provide more information on Frobenius eigenvalues.

6.3. **Eigenvalues of $F_c$ on $\mathcal{E}_{p,c}$.** Note up to now, we have not used condition (2) and (3) in 1.1. Under the condition (2), by 6.1, we always can find $c$ such that $X_c$ ordinary and $\rho(F_c)$ a maximal torus. Further with the condition (3), there exists a closed point $c$ which satisfies the following two conditions:

1. $X_c$ is ordinary,
2. the Frobenius torus $T$ is a maximal torus in $G_l$.

Now we study the eigenvalues of the Frobenius on the fiber over $c$. Since $X_c$ is ordinary, $\mathcal{E}_{p,c}$ is the product of a unit root crystal $\mathcal{U}_c$ and its dual $\mathcal{U}_c'$. Let $\lambda_1, \cdots, \lambda_4$ be the eigenvalues of $F_c$ on $\mathcal{U}_c$. Then on $\mathcal{U}_c'$, the eigenvalues of $F_c$ are $\frac{q}{\lambda_i}$. Since $\mathcal{U}_c$ is a unit root crystal, $\lambda_i$ are all $p$-adic units. Since $\mathcal{E}_{p,c}$ has pure weight 1, $\lambda_i$ all have complex norm $q^{\frac{1}{2}}$.

By [5,1], the Frobenius $F_c$ also has eigenvalues $\lambda_1, \cdots, \lambda_4, \frac{q}{\lambda_1}, \cdots, \frac{q}{\lambda_4}$ on $\mathcal{E}_{l,c}$. Since the Frobenius torus is the maximal torus, the Frobenius eigenvalues $\lambda_i$ correspond to the weights in the $SL(2)^{\times 3}$ representation $V_{\otimes 3}$. Let $a, b, c$ be the three highest weights in the three standard representation of $SL(2, \mathbb{C})$. Then the eight weights of $V_{\otimes 3}$ are of the form $\pm a \pm b \pm c$ and they have a configuration as vertices of a cube. In this case, the four $p$-adic units $\lambda_1, \cdots, \lambda_4$ lie in the same face. Without loss of generality, we can assume that $\lambda_1$ corresponds to the highest weight $a + b + c$ and $\lambda_2, \lambda_3, \lambda_4$ correspond to $a + b - c, a + c - b$ and $a - b - c$. Then the only relation between $\lambda_1, \cdots, \lambda_4$ is $\lambda_1\lambda_4 = \lambda_2\lambda_3$. So we have the following lemma.
Lemma 6.4. Under the above choice of $c$, the eigenvalues $\lambda_i$ have no relations other than those generated by $\lambda_i q = q$ and $\lambda_1 \lambda_4 = \lambda_2 \lambda_3$.

Remark 6.5. Lemma 6.4 also follows from the arguments in [12, Section 4].

Proposition 6.6.

$$\text{End}^0(\wedge^2 \mathcal{E}_p)^F \cong \mathbb{Q}_q^4$$

as algebras.

Proof. From [15, Proposition 5.15] or basic representation theory of $SL(2)$, we know the condition 5.4 implies $\text{End}(\wedge^2 \mathcal{E}_p)^F \otimes \mathbb{Q}_q \cong \mathbb{C}^{\times 4}$ as algebras. In particular, the algebra $\text{End}(\wedge^2 \mathcal{E}_p)^F$ is commutative. Therefore $\text{End}^0(\wedge^2 \mathcal{E}_p)^F$ is a product of fields.

Note $\wedge^2 \mathcal{E}_p$ has the polarization as a direct summand. So

$$\text{End}(\wedge^2 \mathcal{E}_p)^F \cong \mathbb{Q}_q^4, \quad \mathbb{Q}_q \times \mathbb{Q}_q^2 \times L$$

where $K$ is a degree 3 field extension of $\mathbb{Q}_q$ and $L$ has degree 2. Comparing with the decomposition over $\mathbb{C}$, there exists $\eta_K \in \tilde{K}$ or $\eta_L \in \tilde{L}$ such that $\text{im} \eta_K$ and $\text{im} \eta_L$ are subcrystals in $\wedge^2 \mathcal{E}_p$. Further, rank $\text{im} \eta_K = 27$ and rank $\text{im} \eta_L = 18$.

If $K$ or $L$ is unramified over $\mathbb{Q}_q$, then by enlarging $f$ in $q = p^f$, it becomes a product of copies of $\mathbb{Q}_q$. Therefore we only need to consider the case $K$ or $L$ ramified over $\mathbb{Q}_q$. Since $p \neq 2$ or $3$, we can assume $L \cong \mathbb{Q}_q(\sqrt{p})$ and $K \cong \mathbb{Q}_q(\sqrt{p})$ and we can choose $\eta_K = \sqrt{p}$, $\eta_L = \sqrt{p}$.

Note $\mathcal{E}_{pc} \cong \mathcal{U}_c \oplus \mathcal{U}_c'$. Since the eigenvalues have distinct $p$-adic values, there is no $F_c$-invariant morphisms between $\wedge^2 \mathcal{U}_c$, $\wedge^2 \mathcal{U}_c'$ and $\mathcal{U}_c \otimes \mathcal{U}_c'$. Thus we have the decomposition

$$\text{End}(\wedge^2 \mathcal{E}_p)^F \rightarrow \text{End}(\wedge^2 \mathcal{E}_{pc})^F \cong \text{End}(\wedge^2 \mathcal{U}_c) \oplus \text{End}(\wedge^2 \mathcal{U}_c') \oplus \text{End}(\mathcal{U}_c \otimes \mathcal{U}_c')$$

The restriction of $F$ to $\text{End}(\wedge^2 \mathcal{E}_{pc})$ is just as $F_c$. Then by 6.4, all the eigenvalues of $F$ on $\wedge^2 \mathcal{U}_c$ are $\lambda_1 \lambda_2, \ldots, \lambda_3 \lambda_1$ and there is no more relations between the eigenvalues of $\wedge^2 \mathcal{U}_c$ other than $\lambda_1 \lambda_1 = \lambda_2 \lambda_3$. So each eigenspace $U_{\lambda_i \lambda_j}$ has dimension 1 except for $(1, 4)$ or $(2, 3)$. Thereby

$$\text{End}(\wedge^2 \mathcal{U}_c)^F \cong \bigoplus_{(i,j) \neq (1,4),(2,4)} \text{End}(U_{\lambda_i \lambda_j}) \oplus \text{End}(U_{\lambda_1 \lambda_4})$$

$$\cong \bigoplus_{(i,j) \neq (1,4),(2,4)} \mathbb{Q}_q(\lambda_i \lambda_j) \oplus \mathbb{Q}_q(\lambda_1 \lambda_4)).$$

Since the four eigenvalues $\lambda_i$ are all unramified over $\mathbb{Q}_q$ and $L$ or $K$ is ramified, the image of the composition

$$L \text{ or } K \rightarrow \text{End}(\wedge^2 \mathcal{E}_p)^F \rightarrow \text{End}(\wedge^2 \mathcal{U}_c)^F$$

lies only in $\text{End}(U_{\lambda_1 \lambda_4}) \cong \mathbb{Q}_q(\lambda_1 \lambda_4)$. Otherwise, it would induce an embedding $L$ or $K \hookrightarrow \mathbb{Q}_q(\lambda_1 \lambda_4)$. In particular, $\eta_K|_{\wedge^2 \mathcal{U}_c}$ or $\eta_L|_{\wedge^2 \mathcal{U}_c}$ has only rank 2.

Restricted to point $c$, the image of $\eta_K$ has dimension at most only 20. Contradiction.

For $L$, we know that $\eta_K|_{\mathcal{U}_c \otimes \mathcal{U}_c'}$ is a surjection. Note the eigenvalues of $F_c$ on $\mathcal{U}_c \otimes \mathcal{U}_c'$ have the form $\frac{q^{\lambda_i}}{\lambda_j}$. Again by 6.4, among these eigenvalues, $\frac{q^{\lambda_1}}{\lambda_4}$ has only multiplicity 1. Therefore

$$\text{End}(\mathcal{U}_c \otimes \mathcal{U}_c')^F \cong \text{End}(\mathcal{U}_{\lambda_1 \lambda_4}) \oplus \cdots \cong \mathbb{Q}_q(\frac{q^{\lambda_1}}{\lambda_4}) \oplus \cdots$$
as algebras. Since $Q_\lambda(a_1^\lambda)$ is unramified over $Q$, the image of $L$ in $\text{End}(U_c \otimes U_c^\vee)^F$ excludes $\text{End}(U_a^\lambda)$ and hence $\eta_L$ can not be a surjection. The contradiction concludes the proof. □

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