VALUED FIELDS WITH A TOTAL RESIDUE MAP

KONSTANTINOS KARTAS

ABSTRACT. When $k$ is a finite field, Becker-Denef-Lipschitz (1979) observed that the total residue map $\text{res} : k((t)) \to k$, which picks out the constant term of the Laurent series, is definable in the language of rings with a parameter for $t$. Driven by this observation, we study the theory $VF_{\text{res},\iota}$ of valued fields equipped with a linear form $\text{res} : K \to k$ which restricts to the residue map on the valuation ring. We prove that $VF_{\text{res},\iota}$ does not admit a model companion. In addition, we show that $(k((t)), \text{res})$ is undecidable whenever $k$ is an infinite field. As a consequence, we get that $(C((t)), \text{Res}_0)$ is undecidable, where $\text{Res}_0 : f \mapsto \text{Res}_0(f)$ maps $f$ to its complex residue at 0.

INTRODUCTION

Let $k$ be a field. Consider the power series field $k((t))$ and let $\text{res} : k((t)) \to k$ be the total residue map

\[ \text{res} : \sum_{i=-N}^{\infty} c_i t^i \mapsto c_0 \]

Viewing $k((t))$ as a $k$-vector space, we see that $\text{res} : k((t)) \to k$ is a linear form which extends the usual residue map $\pi : k[[t]] \to k$ to all of $k((t))$ (hence the name total). The motivation for studying the model-theory of this structure is twofold:

1. In complex analysis, one defines the residue of a complex meromorphic function $f \in \mathbb{C}((t))$ at an isolated singularity $a \in \mathbb{C}$, denoted by $\text{Res}(f, a)$ or $\text{Res}_a(f)$. The map $\text{res}$ is essentially a shifted version of $\text{Res}_0 : \mathbb{C}((t)) \to \mathbb{C} : f \mapsto \text{Res}_0(f)$, namely $\text{res}(t \cdot f) = \text{Res}_0(f)$.

2. Becker-Denef-Lipschitz [BDL79] showed for $x \in \mathbb{F}_p((t))$ that $\text{res}(x) = 0$ precisely when there exist $x_0, x_1, ..., x_{p-1} \in \mathbb{F}_p((t))$ such that

\[ x = x_0^{p^0} - x_0 + t x_1^{p^1} + ... + t^{p-1} x_{p-1}^{p^{p-1}} \]

This easily implies that $\text{res} : \mathbb{F}_p((t)) \to \mathbb{F}_p$ is definable in $L_{\text{rings}}$ with a parameter for $t$. Becker-Denef-Lipschitz used this to show that $\mathbb{F}_p((t))$ is undecidable in the language of valued fields with a cross-section. This was in sharp contrast with the result by Ax-Kochen [AK66] and Ershov.

During this research, the author was funded by EPSRC grant EP/20998761 and was also supported by the Onassis Foundation - Scholarship ID: F ZP 020-1/2019-2020.
For any decidable field $k$ of characteristic 0, the power series field $k((t))$ is decidable in the language of valued fields with a cross-section.

Towards understanding the model theory of $\mathbb{F}_p((t))$, it may be instructive to isolate such definable functions (or predicates) and study them over $k((t))$, where $k$ is not necessarily finite. This approach is largely influenced by Cherlin [Che82], especially Problems 3 and 4 in §5 [Che82].

In the present paper, we study the model theory of $k((t))$ equipped with res:

$$k((t)) \rightarrow k.$$  

We also take this a step further and study valued fields—not necessarily power series fields—enriched with a total residue map res : $K \rightarrow k$. In the axiomatic setting, the total residue map res : $K \rightarrow k$ is assumed to be a linear form extending the usual residue map $\pi : \mathcal{O}_K \rightarrow k$, where $\mathcal{O}_K$ is the valuation ring of $K$. We consider the theory of equal characteristic valued fields $K$ with a total residue map res : $K \rightarrow k$ and a lift $\iota : k \rightarrow K$ of the residue field, namely $\iota$ is a field embedding such that $\pi \circ \iota = \text{id}_k$. We call $\text{VF}_{\text{res}, \iota}$ the resulting theory.

At first glance, the theory $\text{VF}_{\text{res}, \iota}$ seems like an innocent variant of $\text{VF}$. However, we will show the following:

**Theorem A.** The theory $\text{VF}_{\text{res}, \iota}$ does not admit a model companion.

This should be contrasted with the fact (due to A. Robinson) that the theory $\text{VF}$ of valued fields admits a model companion, namely ACVF, the theory of algebraically closed valued fields. The theory $\text{VF}_{\iota}$ of valued fields with a lift of the residue field still admits a model companion, namely the theory ACVF$_{\iota}$ described by Hrushovski-Kazhdan §6 [HK09] (there it is called $T_{\text{loc}}$). Indeed, they prove a quantifier-elimination result for ACVF$_{\iota}$, which in fact—according to §6.1 [HK09]—goes back to F. Delon.

We also prove:

**Theorem B.** Let $k$ be an infinite field. Then $k((t))$ is undecidable in $L_{\text{res}}$.

Here $L_{\text{res}}$ is the three-sorted language $L_{\text{val}}$ of valued fields together with a function symbol for res : $k((t)) \rightarrow k$. In fact, we prove that the $\exists \forall$-theory is undecidable—at least when $t$ is added in the language—and also that $(\mathbb{N}, +, \cdot)$ is interpretable (without parameters). The proof also applies to the Hahn series field $k((t_\Gamma))$, where $\Gamma$ is any non-trivial ordered abelian group, and also to the Puiseux series field $k\{\{t\}\} = \bigcup_{n \in \mathbb{N}} k((t^{1/n}))$. As an application, we get that $\mathbb{C}((t))$ and $\mathbb{C}\{\{t\}\}$ are undecidable in the language of valued fields together with a function symbol for Res$_0 : f \mapsto \text{Res}_0(f)$ which maps $f$ to its complex residue at 0 (see Corollary 2.2.5).
map $\pi : \mathcal{O}_K \to k$. To simplify notation, we identify $x \in k$ with its image $\iota(x)$ in $K$. It will always be clear from context where such an $x$ lives.

1.1. Axiomatization of $\text{VF}_{\text{res},\iota}$. Let $L_{\text{val}}$ be the three-sorted language of valued fields with sorts for the field, the value group and the residue field and a function symbol for $v : K \to \Gamma \cup \{\infty\}$. We call $L_{\text{res},\iota}$ the enrichment of $L_{\text{val}}$ which includes function symbols for $\iota$ and res. Consider the following set of axioms in $L_{\text{res},\iota}$:

1. $(K, v)$ is a valued field of equal characteristic and $\iota : k \to K$ is a field embedding such that $\pi \circ \iota = \text{id}_k$.
2. We have that $\text{res}|_{\mathcal{O}_K} = \pi$.
3. $\text{res} : K \to k$ is $k$-linear, i.e. $\text{res}(\lambda a + \mu b) = \lambda \text{res}(a) + \mu r(b)$, for all $\lambda, \mu \in k$.

Let $\text{VF}_{\text{res},\iota}$ be the $L_{\text{res},\iota}$-theory generated by the above axioms.

**Example 1.1.1.** Let $k$ be a field.

1. Let $\Gamma$ be an ordered abelian group and $k((t^\Gamma))$ be the Hahn series field over $k$ with value group $\Gamma$. We have $(k((t^\Gamma)), v_\Gamma, \iota) \models \text{VF}_{\text{res},\iota}$, where
\[
\text{res} : \sum_{q \in \Gamma} c_q t^q \mapsto c_0
\]
and $\iota : k \to k((t^\Gamma))$ is the obvious lift.
2. Similarly, $(k\{\{t\}\}, v_t, \iota) \models \text{VF}_{\text{res},\iota}$, where $k\{\{t\}\} = \bigcup_{n \in \mathbb{N}} k((t^{1/n}))$ is the Puiseux series field over $k$.

1.2. Extensions of $L_{\text{res},\iota}$-structures.

**Lemma 1.2.1.** Let $(K, v, \iota) \subseteq (K', v', \iota')$ be an extension of valued fields with lifts of their residue fields. If $\beta_1, ..., \beta_n \in K'$ are $k$-linearly independent and $a_1, ..., a_n \in K$, then
\[
v'(\sum_{i=1}^n \beta_i a_i) = \min_{1 \leq i \leq n} v(a_i)
\]

*Proof.* We may assume that $v(a_1) = ... = v(a_n)$. For each $1 \leq i \leq n$, there exist $\alpha_i \in k$ and $\varepsilon_i \in \mathfrak{m}$, such that $a_i = \alpha_i \cdot a_1 + \varepsilon_i \cdot a_1$. It follows that
\[
\sum_{i=1}^n \beta_i a_i = a_1 \cdot (\sum_{i=1}^n \alpha_i \beta_i + \sum_{i=1}^n \beta_i \varepsilon_i)
\]
Since the $\beta_i$'s are $k$-linearly independent, we get that $\sum_{i=1}^n \alpha_i \beta_i \neq 0$ and hence $v'(\sum_{i=1}^n \alpha_i \beta_i) = 0$. Since $v'(\sum_{i=1}^n \beta_i \varepsilon_i) > 0$, we get that
\[
v'(\beta_1 a_1 + ... + \beta_n a_n) = v(a_1) + v\left(\sum_{i=1}^n \alpha_i \beta_i + \sum_{i=1}^n \beta_i \varepsilon_i\right) = v(a_1)
\]
as needed. \qed
Given \((K, v, t) \subseteq (K', v', t')\), we denote by \(\langle K \rangle_{k'}\) the \(k'\)-linear span of \(K\) inside \(K'\). Note that \(\langle K \rangle_{k'}\) is isomorphic to \(K \otimes_k k'\) as an \(k'\)-vector space.

**Lemma 1.2.2.** Let \((K, v, t) \subseteq (K', v', t')\) be an extension of valued fields with lifts of their residue fields. Then \(\mathcal{O}_{K'} \cap \langle K \rangle_{k'} = \langle \mathcal{O}_K \rangle_{k'}\).

**Proof.** First, we prove the following:  
**Claim:** Let \(a_1, ..., a_n \in K\) be \(k\)-linearly independent over \(\mathcal{O}_K\). Then \(a_1, ..., a_n\) are also \(l\)-linearly independent over \(\mathcal{O}_{K'}\).

**Proof.** Suppose \(\beta_1', ..., \beta_n' \in k'\) are such that
\[
\sum_{i=1}^{n} \beta_i' a_i \in \mathcal{O}_{K'}
\]
Let \(\{\beta_1, ..., \beta_m\}\) be a \(k\)-linear basis of \(\langle \beta_1', ..., \beta_n' \rangle_k\) and write \(\beta_i' = \sum_{j=1}^{m} c_{ij} \cdot \beta_i\) with \(c_{ij} \in k\). We will then have that
\[
\sum_{i=1}^{n} \beta_i' a_i = \sum_{i=1}^{n} (\beta_i \sum_{j=1}^{m} c_{ij} a_j)
\]
Since the \(\beta_i\)'s are \(k\)-linearly independent, Lemma 1.2.1 implies that
\[
v'(\sum_{i=1}^{n} (\beta_i \sum_{j=1}^{m} c_{ij} a_j))) = \min_{1 \leq i \leq n} \{v(\sum_{j=1}^{m} c_{ij} a_j)\}
\]
For \(i = 1, ..., n\), it follows that
\[
\sum_{j=1}^{m} c_{ij} a_j \in \mathcal{O}_K
\]
Since the \(a_i\)'s are \(k\)-linearly independent over \(\mathcal{O}_K\), we get that \(c_{ij} = 0\) for all \(i, j\). Therefore \(\beta_i' = 0\) for all \(i\) and the \(a_i\)'s are \(k'\)-linearly independent over \(\mathcal{O}_{K'}\). \(\square\) **Claim**

Now let \(V\) be a complement of the \(k\)-vector subspace \(\mathcal{O}_K \subseteq K\), i.e., we have \(K = \mathcal{O}_K \oplus V\). We will then have that \(\langle K \rangle_{k'} = \langle \mathcal{O}_K \rangle_{k'} + V_{k'}\). By the claim, we get that \(V_{k'} \cap \mathcal{O}_{K'} = \{0\}\) and since \(\langle \mathcal{O}_K \rangle_{k'} \subseteq \mathcal{O}_{K'}\), we conclude that \(\mathcal{O}_{K'} \cap \langle K \rangle_{k'} = \langle \mathcal{O}_K \rangle_{k'}\). \(\square\)

**Lemma 1.2.3.** Let \((K, v, \text{res}, t) \models \text{VF}_{\text{res}, t}\) and \((K', v', \text{res}, t') \subseteq (K', v', \text{res}, t')\). Let \(\{e_i : i \in I\} \subseteq L\) be \(k'\)-linearly independent over \(\mathcal{O}_{K'} \cap \langle K \rangle_{k'}\). Then there exists an \(k'\)-linear map \(\text{res}' : K' \rightarrow k'\) extending \(\pi_{K'} : \mathcal{O}_{K'} \rightarrow k'\) with \(\text{res}'(e_i) = 0\) for all \(i \in I\) and such that \((K', v', \text{res}', t')\) is a model of \(\text{VF}_{\text{res}, t}\) extending \((K, v, \text{res}, t)\).

**Proof.** First we extend \(\text{res} : K \rightarrow k\) to \(\text{res}' : \langle K \rangle_{k'} \rightarrow k'\) by extension of scalars:
\[
\text{res}'(\beta \cdot a) = \beta \cdot \text{res}(a)
\]
for $\beta \in k'$ and $a \in K$. By Lemma 1.2.2 if $b \in \mathcal{O}_K \cap \langle K \rangle_{k'}$, we may write $b = \sum_{i=1}^n \beta_i a_i$, with $\beta_i \in k'$ and $a_i \in \mathcal{O}_K$. We now compute that

$$\text{res}'(b) = \sum_{i=1}^n \beta_i \cdot \text{res}'(a_i) = \sum_{i=1}^n \beta_i \cdot \text{res}(a_i) = \sum_{i=1}^n \beta_i \cdot \pi_K(a_i) = \pi_L(b)$$

We may therefore extend $\text{res} : K \to k$ to $\text{res}' : \mathcal{O}_{K'} + \langle K \rangle_{k'} \to k'$ so that it restricts to $\text{res}$ on $K$ and also to $\pi_K$ on $\mathcal{O}_{K'}$. Finally, we extend the linear map $\text{res}'$ to $K'$ by requiring that $\text{res}(c_i) = 0$ and get a model $(K', v', \text{res}', \iota')$ of $\text{VF}_{\text{res}, \iota}$ extending $(K, v, \text{res}, \iota)$, as required.

\[ \square \]

1.3. Structures on the rational function field.

1.3.1. Extension by an infinitesimal.

**Fact 1.3.2** (Corollary 2.2.3 [EP05]). Let $(K, v)$ be a valued field with value group $\Gamma$ and residue field $k$. Let $\Gamma'$ be an ordered abelian group extending $\Gamma$ and $\gamma \in \Gamma'$ be torsion-free over $\Gamma$, i.e., if $n \cdot \gamma \in \Gamma$, then $n = 0$. Then there is exactly one valuation $w$ on $K(X)$ extending $v$ with $w(X) = \gamma$. We have $\Gamma_{K(X)} = \Gamma \oplus \mathbb{Z}\gamma$, with the ordering induced by $\Gamma'$, and $k_{K(X)} = k$.

**Remark 1.3.3.**

(i) It follows that there is a unique valuation $w$ on $K(X)$ such that $wx > \Gamma$. We will have $k_{K(X)} = k$ and $\Gamma_{K(X)} = \mathbb{Z}\gamma \oplus \text{ex} \Gamma$.

(ii) A lift $\iota : k \to K$ of the valued field $(K, v)$ naturally induces a lift of $(K(X), w)$, namely $\iota_{K(X)} : k \to K(X) : \alpha \mapsto \iota(\alpha)$.

**Lemma 1.3.4.** Let $(K, v, \text{res}, \iota) \models \text{VF}_{\text{res}, \iota}$. Let $w$ be the unique valuation on $K(X)$ such that $w(X) > \Gamma$ and also let $\iota_{K(X)} : k \to K(X)$ be the induced lift. Then there is a map $\text{res}_{K(X)} : K(X) \to k$ with $\text{res}_{K(X)}(X^{-m}) = 0$ for all $m \in \mathbb{N}$, such that $(K(X); w, \text{res}_{K(X)}, \iota_{K(X)})$ is a model of $\text{VF}_{\text{res}, \iota}$ extending $(K, v, \text{res}, \iota)$.

**Proof.** By Lemma 1.2.3 it suffices to show that the elements $X^{-1}, X^{-2}, \ldots, X^{-n}$ are $k$-linearly independent over $K + \mathcal{O}_{K(X)}$. Indeed, for any $c \in K$ and $\beta_1, \ldots, \beta_n \in k$ with $\beta_n \neq 0$, we get that

$$w\left(\sum_{i=1}^n \beta_i X^{-i} + c\right) = -n \cdot w(X) < 0$$

using that $w(X) > \Gamma$.

\[ \square \]

**Corollary 1.3.5.** Let $(K, v, \text{res}, \iota)$ be an e.c. model of $\text{VF}_{\text{res}, \iota}$ and $n \in \mathbb{N}$. Then there exists $a \in \mathfrak{m} - \{0\}$ such that $\text{res}(a^{-m}) = 0$ for $m = 1, \ldots, n$.

**Proof.** Immediate from Lemma 1.3.4.

\[ \square \]
1.3.6. Gauss valuation. The valuation described below is known as the Gauss extension of \( v \) from \( K \) to \( K(X) \).

**Fact 1.3.7** (Corollary 2.2.2 [EP05]). Let \((K, v)\) be a valued field with value group \( \Gamma \) and residue field \( k \). There exists a unique valuation \( w \) on \( K(X) \) extending \( v \) such that \( w(X) = 0 \) and the residue \( x \) of \( X \) is transcendental over \( k \). This valuation is defined by the formula

\[
w(a_0 + a_1X + \ldots + a_nX^n) = \min_{1 \leq i \leq n} v(a_i)
\]

for \( a_i \in K \). We have \( k_{K(X)} = k(x) \) and \( \Gamma_{K(X)} = \Gamma \).

**Remark 1.3.8.** A lift \( \iota : k \to K \) automatically induces a lift

\[
\iota_{K(X)} : k(x) \to K(X) : f(x) \mapsto \iota(f)(X)
\]

Namely, \( \iota_{K(X)} \) agrees with \( \iota \) on \( K \) and maps \( x \) to \( X \). Thus, the image of \( \iota_{K(X)} \) equals \( k(X) \), once again identifying \( k \) with its image \( \iota(k) \subseteq K \) via \( \iota \).

**Lemma 1.3.9.** Let \((K, v, \text{res}, \iota) \models \text{VF}_{\text{res}, \iota} \) and \( a, c \in K \) with \( a \in m - \{0\} \) and \( v(c) < \mathbb{Z}v(a) \). Let \( w \) be the Gauss extension of \( v \) to \( K(X) \) and \( \iota_{K(X)} : k(x) \to K(X) \) be the induced lift. There is a map \( \text{res}_{K(X)} : K(X) \to k(x) \) with \( \text{res}_{K(X)}\left(\frac{c}{1-aX}\right) = 0 \), such that \((K(X); w, \text{res}_{K(X)}, \iota_{K(X)})\) is a model of \( \text{VF}_{\text{res}, \iota} \) extending \((K, v, \text{res}, \iota)\).

**Proof.** By Lemma 1.2.3, it suffices to show that \( \frac{c}{1-aX} \) is \( k(X) \)-linearly independent over \( \mathcal{O}_{K(X)} + \langle K \rangle_{k(X)} \). Note that \( \langle K \rangle_{k(X)} = K[X][X] \setminus \{0\} \), the latter being the localization of \( K[X] \) at \( k[X] \setminus \{0\} \subseteq K[X] \).

Suppose for a contradiction that

\[
\frac{c}{1-aX} + \frac{f(X)}{g(X)} \in \mathcal{O}_{K(X)}
\]

for some \( f(X) \in K[X] \) and \( g(X) \in k[X] \setminus \{0\} \). Moreover, choose \( f(X) \) and \( g(X) \) as above such that \( \deg(f) \) is minimum. Note that \( f(X) \neq 0 \) because \( w\left(\frac{c}{1-aX}\right) < 0 \) and hence \( \deg(f) \) is well-defined. Write \( f(X) = f_0 + \ldots + f_nX^n \) and \( g(X) = g_0 + \ldots + g_nX^n \) with \( f_i \in K \) and \( g_i \in k \). Note that

\[
w((1-aX)g(X)) = w(1-aX) + w(g(X)) = 0
\]

It follows that \( w(cg(X) + (1-aX)f(X)) \geq 0 \). By the definition of \( w \), this means that

(1) \( v(f_0 + cg_0) \geq 0 \), (2) \( v(f_m + cg_m - af_{m-1}) \geq 0 \) and (3) \( v(af_n) \geq 0 \)

for \( m = 1, \ldots, n \).

**Claim 1:** We have that \( v(f_0) < 0 \).
Proof. Suppose that \( v(f_0) \geq 0 \). We then have that
\[
\frac{c}{1 - aX} + \frac{f(X) - f_0}{g(X)} \in \mathcal{O}_{K(X)}
\]
Write \( f(X) - f_0 = X \cdot \tilde{f}(X) \). If \( g_0 = 0 \), then \( g(X) = X \tilde{g}(X) \) and therefore
\[
\frac{c}{1 - aX} + \frac{\tilde{f}(X)}{\tilde{g}(X)} \in \mathcal{O}_{K(X)}
\]
Since \( \deg(\tilde{f}) < \deg(f) \), this contradicts our minimality assumption. Therefore \( g_0 \neq 0 \). But then \( v(f_0 + cg_0) = vc < 0 \), which contradicts (1).
\( \square \)

Claim 2: For each \( m = 0, \ldots, n \), there exists \( k_m \in \mathbb{N} \) such that \( v(f_m) = v(c) + k_m \cdot v(a) \).

Proof. We proceed inductively. For \( m = 0 \): Recall from (1) that \( v(f_0 + cg_0) \geq 0 \). By Claim 1, we have \( v(f_0) < 0 \). Therefore \( g_0 \neq 0 \) and \( v(f_0) = v(cg_0) = vc \), which settles the base case. For \( m > 0 \): Recall that we have
\[
(2) \ v(f_m + cg_m - af_{m-1}) \geq 0
\]
By our induction hypothesis, we have
\[
v(f_{m-1}) = v(c) + k_{m-1} \cdot v(a)
\]
for some \( k_{m-1} \in \mathbb{N} \). Since \( vc < Zva \), this implies that \( v(af_{m-1}) < 0 \). If \( g_m = 0 \), we get from (2) that
\[
v(f_m) = v(af_{m-1}) = v(c) + (k_{m-1} + 1)v(a)
\]
and we take \( k_m = k_{m-1} + 1 \). Suppose that \( g_m \neq 0 \). Then \( v(cg_m - af_{m-1}) = v(cg_m) = v(c) \). By (2), we get that \( v(f_m) = v(c) \) and we take \( k_m = 0 \).
\( \square \)

For \( m = n \), we get that \( v(f_n) = v(c) + k_n v(a) \), for some \( k_n \in \mathbb{N} \). Therefore
\[
v(af_n) = v(c) + (k_n + 1)v(a) < 0
\]
because \( v(c) < Zv(a) \). This contradicts (3).
\( \square \)

Corollary 1.3.10. Let \((K, v, \text{res}, \iota)\) be an e.c. model of \(VF_{\text{res}, \iota} \). Let \( a \in m - \{0\} \) and \( c \in K \) be such that \( v(c) < Zv(a) \). Then there exists \( \beta \in k^{\times} \) such that
\[
\text{res} \left( \frac{c}{1 - a\beta} \right) = 0
\]
Proof. Immediate from Lemma 1.3.9.
\( \square \)

1.4. Non-existence of a model companion of \(VF_{\text{res}, \iota} \).
1.4.1. Generalities on model companions.

**Definition 1.4.2** (Definition 3.2.8 [TZ12]). Let $T$ be a theory. A theory $T^*$ is a model companion of $T$ if the following conditions are satisfied:

(i) Every model of $T$ embeds into a model of $T^*$.

(ii) Every model of $T^*$ embeds into a model of $T$.

(iii) $T^*$ is model-complete.

**Fact 1.4.3** (Theorem 3.2.9 [TZ12]). A theory $T$ has, up to equivalence, at most one model companion $T^*$.

**Definition 1.4.4.**

(i) Let $M, N$ be $L$-structures. Suppose that $N \models \phi \Rightarrow M \models \phi$ for any existential sentence $\phi \in L(M)$. Then we say that $M$ is existentially closed (or e.c.) in $N$ and write $M \preceq_\exists N$.

(ii) Let $T$ be a theory. A model $M \models T$ is said to be an existentially closed (or e.c.) model of $T$ if $M \preceq_\exists N$, for all $N \models T$.

**Fact 1.4.5** (Theorem 3.2.14 [TZ12]). For any theory $T$, the following are equivalent:

(i) $T$ has a model companion $T^*$.

(ii) The e.c. models of $T$ form an elementary class. Moreover, if $T^*$ exists, then $T^*$ is the theory of e.c. models of $T$.

1.4.6. Proof of Theorem A.

**Theorem A.** The theory $VF_{res,\iota}$ does not admit a model companion.

**Proof.** Assume otherwise and let $T$ be the model companion of $VF_{res,\iota}$. By Fact 1.4.5, the theory $T$ is precisely the theory of e.c. models of $VF_{res,\iota}$. Let $(K_0, v_0, res_0, \iota_0) \models T$ be $\aleph_1$-saturated. By Corollary 1.3.5 and $\aleph_1$-saturation, there is $a \in m_0 - \{0\}$ such that $res_0(a^{-m}) = 0$ for $m > 0$. Let

$$(K, v, res, \iota) := (K_0, v_0, res_0, \iota_0)^U$$

where $U$ is a non-principal ultrafilter on $\mathbb{N}$. By Łoś’ Theorem, we will have that $(K_0, v_0, res_0, \iota_0) \preceq (K, v, res, \iota)$. In particular, we get that $(K, v, res, \iota) \models T$ and therefore $(K, v, res, \iota)$ is an e.c. model of $VF_{res,\iota}$.

Set $a^* := \text{ulim } a^{-n}$. Since $U$ is non-principal, we get that $va^* < Zva$. By Corollary 1.3.10 there is $\beta \in k - \{0\}$ such that $res_0(a^{-m}) = 0$. By Łoś’ Theorem, we get that

$$\{n \in \mathbb{N} : K_0 \models \exists y \in k^\times (res_0(a^{-n}) = 0)\} \in U$$

In particular, there exists $n \in \mathbb{N}$ and $\beta_0 \in \iota(k_0) - \{0\}$ such that $res_0(a^{-n}) = 0$. By Łoś’ Theorem, we get that

$$\frac{a^{-n}}{1 - a\beta_0} \equiv \beta_0^n + a^{-1}\beta_0^{n-1} + ... + a^{-n} \mod m$$
Since \( \text{res} : K \to k \) is \( k \)-linear and \( \text{res}(m) = \{0\} \) and \( \text{res}(a^{-m}) = 0 \) for all \( m \in \mathbb{N} \), we compute that
\[
\text{res}\left( \frac{a^{-n}}{1 - a\beta_0} \right) = \text{res}(\beta_0^n + a^{-1}\beta_0^{n-1} + \ldots + a^{-n}) = \sum_{m=0}^{n} \beta_0^{m-i}\text{res}(a^{-m}) = \beta_0^n
\]
This forces \( \beta_0 = 0 \), which is a contradiction. It follows that \( \text{VF}_{\text{res}, \iota} \) does not admit a model companion. \( \square \)

2. Undecidability of \( k((t^\Gamma)) \) with a total residue map

Let \( k((t^\Gamma)) \) be the Hahn field with residue field \( k \) and value group \( \Gamma \). Recall that an element \( f \in k((t^\Gamma)) \) is of the form
\[
f = \sum_{q \in \Gamma} c_q t^q
\]
where \( \text{supp}(f) = \{ q \in \Gamma : c_q \neq 0 \} \) is well-ordered. Throughout, we fix some element \( 1 \in \Gamma^>0 \), thereby identifying a copy of \( \mathbb{Z} \) inside \( \Gamma \). We write \( t \) for \( t^1 \).

2.1. Definability in \( k((t^\Gamma)) \) in \( L_{\text{res}} \).

**Lemma 2.1.1.** Let \( k \) be any field and \( \Gamma \) be any ordered abelian group. The subfield \( k \subseteq k((t^\Gamma)) \) is \( \emptyset \)-definable in \( L_{\text{res}} \). In particular, the lift \( \iota : k \to k((t^\Gamma)) \) is \( \emptyset \)-definable in \( L_{\text{res}} \).

**Proof.** We claim that
\[
k = \{ x \in k((t^\Gamma)) : \forall y(\text{res}(x \cdot y) = \text{res}(x) \cdot \text{res}(y)) \}
\]
The inclusion "\( \subseteq \)" is clear. For "\( \supseteq \)", suppose that \( x \notin k \) and write \( x = \alpha + x' \) with \( \alpha \in k \), \( \text{res}(x') = 0 \) and \( x' \neq 0 \). Note that \( \nu(x') \neq 0 \). For \( q = \nu(x') \), we compute that
\[
\text{res}(x \cdot t^{-q}) = \alpha \cdot \text{res}(t^{-q}) + \text{res}(x' \cdot t^{-q}) = \text{res}(x' \cdot t^{-q}) \neq 0
\]
On the other hand, we have \( \text{res}(x) \cdot \text{res}(t^{-q}) = 0 \) and hence
\[
\text{res}(x \cdot t^{-q}) \neq \text{res}(x) \cdot \text{res}(t^{-q})
\]
Finally, for \( \iota \) simply note that \( \iota(\alpha) = a \) if and only if \( a \in \iota(k) \) and \( \text{res}(a) = \alpha \). \( \square \)

**Definition 2.1.2.** Given
\[
a = \sum_{q \in \mathcal{A}} c_q t^q \in k((t^\Gamma))
\]
we define the polynomial \( p_a(X) \in k[X] \) given by
\[
p_a(X) = \sum_{n\in\mathbb{N}} c_{-n}X^n
\]
Note that \( p_a(X) \) is indeed a polynomial because \( \text{supp}(a) \cap \mathbb{Z}^<0 \) is finite.
Lemma 2.1.3. For any $a \in k((t^\Gamma))$ and $y \in k$, we have
\[
\text{res} \left( \frac{a}{1-ty} \right) = p_a(y)
\]

Proof. Write $a = \sum_{q \in \Gamma} c_q t^q \in k((t^\Gamma))$. For each $q_0 \in \Gamma$, note that
\[
\text{res} \left( t^{q_0} \cdot \sum_{q \in A} c_q t^q \right) = c_{-q_0}
\]

We now have
\[
\frac{a}{1-ty} = (1 + ty + ... + t^ny^n + ...) \cdot \sum_{q \in \Gamma} c_q t^q \equiv \sum_{i=1}^n t^i y^i \cdot \sum_{q \in \Gamma} c_q t^q \mod m
\]

where $n$ is maximum such that $-n \in \text{supp}(a)$. By $k$-linearity of $\text{res}$ and since $\text{res}(m) = \{0\}$, we get that
\[
\text{res} \left( \frac{a}{1-ty} \right) = \sum_{i=1}^n \text{res}(t^i \cdot \sum_{q \in \Gamma} c_q t^q) y^i = \sum_{i=1}^n c_{-i} y^i = p_a(y)
\]
as needed. □

2.2. Infinite residue field. Let $k$ be an infinite field.

2.2.1. Undecidability of $k((t^\Gamma))$ in $L_{\text{res},t}$. We isolate the following elementary fact from algebra to pinpoint exactly where our proof fails when $k$ is finite:

Lemma 2.2.2. Let $k$ be any infinite field and $f(X) \in k[X]$. Then $f(X) \equiv 0$ if and only if $f(k) = \{0\}$.

Proof. Any non-zero polynomial over any field has finitely many roots. □

Fact 2.2.3 (Denef). Let $R = k[t]$, where $k$ is any field. Then Hilbert’s tenth problem over $R$ with coefficients in $\mathbb{Z}[t]$ is unsolvable.

Proof. See [Den79] and [Den78]. □

Theorem B. Let $k$ be an infinite field and $\Gamma$ be any non-trivial ordered abelian group. Then $(k((t^\Gamma)), \text{res})$ is $\exists \forall$-undecidable in $L_{\text{res},t}$. The same is true for the Puiseux series field $k\{\{t\}\}$.

Proof. Let $f_1(X_1, ..., X_m, T), ..., f_n(X_1, ..., X_m, T) \in \mathbb{Z}[X_1, ..., X_m, T]$, where $n, m \in \mathbb{N}$. The system
\[
f_1(X_1, ..., X_m, T) = .... = f_n(X_1, ..., X_m, T) = 0
\]
has a solution in $k[T]$ if and only if there exist $a_1, ..., a_m \in k((t^\Gamma))$ such that
\[
f_1(p_{a_1}(t^{-1}), ..., p_{a_m}(t^{-1}), t^{-1}) = .... = f_n(p_{a_1}(t^{-1}), ..., p_{a_m}(t^{-1}), t^{-1}) = 0
\]
By Lemma 2.2.2, this is also equivalent to the existence of \(a_1, \ldots, a_m \in k[(t\Gamma)]\) such that
\[
f_1(p_{a_1}(y), \ldots, p_{a_m}(y), y) = \ldots = f_n(p_{a_1}(y), \ldots, p_{a_m}(y), y) = 0
\]
for all \(y \in k\). This is expressible in \(L_{\text{res}, t}\) by Lemma 2.1.3 and Lemma 2.1.1 and we conclude from Fact 2.2.3. The proof works verbatim for the Puiseux series field. \(\Box\)

In §7.2.21 [Kar22], it is also shown that \(k[t] \subseteq k((t))\) is definable in \(L_{\text{res}, t}\).

2.2.4. A natural example from complex analysis. In complex analysis, one defines the residue of a complex function \(f \in \mathbb{C}((t))\) at an isolated singularity \(a \in \mathbb{C}\), denoted by \(\text{Res}(f, a)\) or \(\text{Res}_a(f)\). Numerically, if \(f = \sum_{i=-\infty}^{\infty} c_i(t - a)^i\), we have that \(\text{Res}(f, a) = c_{-1}\).

**Corollary 2.2.5.** We have that \(\mathbb{C}((t))\) is \(\exists\forall\)-undecidable in \(L_{\text{res}, 0, t}\). The same is true for the Puiseux series field \(\mathbb{C}\{t\}\).

**Proof.** Note that \(\text{Res}_0(a) = \text{res}(t \cdot a)\) and conclude from Theorem B. \(\Box\)

2.3. Eliminating \(t\). We now give a different proof of the undecidability of \((k((t\Gamma)), +, \cdot, \text{res})\), where \(k\) is an infinite field and \(\Gamma\) is an ordered abelian group of rank 1. The proof discussed here does not require a parameter for \(t\) and even shows that \((\mathbb{N}, +, \cdot)\) is interpretable without parameters.

2.3.1. **Interpreting the weak monadic second-order theory of \(k\).** Weak monadic second-order logic is the fragment of second-order logic where second-order quantification is restricted to quantification over finite subsets.

**Lemma 2.3.2.** Let \(\Gamma\) be an ordered abelian group of rank 1. Let \(a \in k((t\Gamma))\) and \(b \in \mathfrak{m}\). Then the set
\[
S_{a, b} = \{\beta \in k : \text{res}\left(\frac{a}{1 - b\beta}\right) = 0\}
\]
is either finite or equal to \(k\).

**Proof.** Since \(\Gamma\) is of rank 1, there exists \(n \in \mathbb{N}\) such that \(ab^n \in \mathfrak{m}\). We then have that
\[
\frac{a}{1 - b\beta} \equiv a(1 + b\beta + \ldots + b^{n-1}\beta^{n-1}) \mod \mathfrak{m}
\]
Since \(\text{res}\) is \(k\)-linear and \(\text{res}(\mathfrak{m}) = \{0\}\), we get that
\[
\text{res}\left(\frac{a}{1 - b\beta}\right) = 0 \iff p_{a, b}(\beta) = 0
\]
where
\[
p_{a, b}(X) = \sum_{i=0}^{n-1} \text{res}(ab^i) \cdot X^i \in k[X]
\]
If \(p_{a, b}(X) \equiv 0\), then \(S_{a, b} = k\) and otherwise \(S_{a, b}\) is finite. \(\Box\)
Lemma 2.3.3. Let $\Gamma$ be an ordered abelian group of rank 1. Then the weak monadic second-order theory of $(k, +, \cdot)$ is $\emptyset$-interpretable in $k((t^{\Gamma}))$ in $L_{\text{res}}$. The same is true for the Puiseux series field $k\{\{t\}\}$.

Proof. By Lemma 2.1.1, we have that the first-order theory $(k, +, \cdot)$ is $\emptyset$-interpretable in $k((t^{\Gamma}))$ in $L_{\text{res}}$. By Lemma 2.3.2, we have a uniformly $\emptyset$-definable family 
$$\{S_{a,b} : a \in k((t^{\Gamma})) \text{ and } b \in m\}$$
of finite subsets of $k$ together with $k$ itself. We claim that every finite $S \subseteq k$ arises as $S_{a,b}$, for some $a \in k((t^{\Gamma}))$ and $b \in m$. Given a finite $S \subseteq k$, we set
$$a = \prod_{s \in S} (t - 1 - s) \text{ and } b = t$$
By Lemma 2.1.3 we get indeed that $S_{a,b} = \{y : p_a(y) = 0\} = S$
We thus encode the weak monadic second-order theory of $(k, +, \cdot)$. The proof works verbatim for the Puiseux series field. □

Lemma 2.3.4. Let $k$ be an infinite field. Then the weak monadic second-order theory of $(k, +, \cdot)$ interprets $(\mathbb{N}, +, \cdot)$. In particular, it is undecidable.

Proof. See §1 [Che82]. □

Theorem C. Let $k$ be an infinite field and $\Gamma$ be an ordered abelian group of rank 1. Then $(k((t^{\Gamma})), \text{res})$ interprets $(\mathbb{N}, +, \cdot)$. In particular, $k((t^{\Gamma}))$ is undecidable in $L_{\text{res}}$. The same is true for the Puiseux series field $k\{\{t\}\}$.

Proof. From Lemma 2.3.3 and Lemma 2.3.4. □

Remark 2.3.5. In case $k$ is an infinite perfect field of positive characteristic, the undecidability of $(k((t)), +, -, \text{res})$ also follows from Lemma 2.1.1 and §2 [Che82] which shows that $k((t))$ is undecidable with a predicate for $k \subseteq k((t))$.

Acknowledgements. I wish to thank E. Hrushovski for suggesting the problem and for an instructive discussion on the elimination of the parameter. I also thank J. Koenigsmann for careful readings of earlier drafts.

References

[AK66] James Ax and Simon Kochen. Diophantine problems over local fields: III. Decidable fields. *Annals of Mathematics*, Second Series, Vol. 83, No. 3, pp. 437-456, 1966.

[BDL79] Joseph Becker, Jan Denef, and Leonard Lipshitz. Further remarks on the elementary theory of formal power series rings. *Model theory of algebra and arithmetic, Proceedings Karpacz, Poland, Lecture Notes in Mathematics*, Vol. 834. Berlin, Heidelberg, New York: Springer 1979, 1979.

[Che82] Gregory Cherlin. Undecidability of rational function fields in nonzero characteristic. *Logic Colloq., no. 82, North-Holland, Amsterdam*, 1982.
[Den78] Jan Denef. The Diophantine problem for polynomial rings of positive characteristic. 
Transactions of the American Mathematical Society, 1978.

[Den79] Jan Denef. The Diophantine problem for polynomial rings and rings of rational func-
tions. Logic Colloquium, North-Holland Publishing Company, 1979.

[EP05] Antonio J. Engler and Alexander Prestel. Valued Fields. Springer, Berlin, Heidelberg, 
2005.

[Ers65] Ju.L. Ershov. On elementary theories of local fields. Algebra i Logika 4, No. 2, 5-30, 
1965.

[HK09] Ehud Hrushovski and David Kazhdan. Motivic Poisson summation. Mosc. Math. J., 
Volume 9, Number 3, Pages 569-623, 2009.

[Kar22] Konstantinos Kartas. Contributions to the model theory of henselian fields. PhD thesis, 
University of Oxford, 2022.

[TZ12] Katrin Tent and Martin Ziegler. A Course in Model Theory. Lecture Notes in Logic. 
Cambridge University Press, 2012.

MATHEMATICAL INSTITUTE, WOODSTOCK ROAD, OXFORD OX2 6GG.
E-mail address: kartas@maths.ox.ac.uk