SMOOTH QUASI-PERIODIC SOLUTIONS FOR THE PERTURBED MKDV EQUATION

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Abstract. This paper aims to study the time quasi-periodic solutions for one dimensional modified KdV (mKdV, for short) equation with perturbation

\[ u_t = -u_{xxx} - 6u^2u_x + \text{perturbation}, \quad x \in \mathbb{T}. \]

We show that, for any \( n \in \mathbb{N} \) and a subset of \( \mathbb{Z}\setminus\{0\} \) like \( \{j_1 < j_2 < \cdots < j_n\} \), this equation admits a large amount of smooth \( n \)-dimensional invariant tori, along which exists a quantity of smooth quasi-periodic solutions. The proof is based on partial Birkhoff normal form and an unbounded KAM theorem established by Liu-Yuan in [Commun. Math. Phys., 307 (2011), 629-673].

1. Introduction and main result. In this paper, we consider the 1D mKdV equation with small perturbation

\[ u_t = -u_{xxx} - 6u^2u_x + \text{perturbation}, \quad (t, x) \in \mathbb{R} \times \mathbb{T}, \]

where \( u \) is a real-valued function. As we all know, the mKdV equation plays an important role in the KdV family of equations with the form

\[ u_t = u_{xxx} + P(u)_x, \quad x \in \mathbb{R}, \]

where \( P(u) \) represents a polynomial of the function \( u \). On one hand, the mKdV equation is as special as the famous KdV equation, since only these two equations in the KdV family are completely integrable, which leads to a fact that both the KdV and mKdV equation have infinitely many conserved quantities [1]. On the other hand, the mKdV equation can be treated as model equation in many nonlinear phenomena, such as transmission lines in Schottky barriers [39], traffic congestion [17], elastic media [28], acoustic waves in a certain unharmonic lattice [31], etc. Moreover, we mention that the mKdV equation can be turned into a KdV equation

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through the so-called Miura transformation, that is, $v = u_x + i u^2$. For the local and global well-posedness of the mKdV equation, we refer to [7][13] in details.

We point out that the existence of quasi-periodic solution for 1D mKdV equation is firstly proved by Shi-Xu [34], they initially turn the mKdV equation into partial normal form up to order four

$$H = \Lambda + B + h.o.t.,$$

where

$$\Lambda = \sum_{j \geq 1} (\gamma_j^6 + 6c^2\gamma_j^4)|q_j|^2,$$

$$B = 12 \sum_{j=1, l \neq n} b \gamma_n^2 \gamma_l^2 |q_{n_j}|^2 |q_l|^2 + 3 \sum_{j=1}^b (\gamma_{n_j}^4 - 4c^2)|q_{n_j}|^4;$$

then they check the non-degeneracy condition in the KAM theorem developed by Kuksin [23] (see also in Pöschel-Kappler [16]). Since the four-order term $B$ is not in diagonal form, it is tricky to solve it. Inspired by [34], in the present paper, we use an important energy conserved quantity of the mKdV equation to obtain a diagonalized normal form, which makes it easy to verify the non-degeneracy condition. Besides, the advantage of using Liu-Yuan’s KAM theorem [26] here lies in that it ensures us to study the generalized KdV equation with nonlinearity contains $u_{xx}$ other than $u_x$. Moreover, we mention that Kuksin’s KAM theorem [23] guarantees the analyticity of the quasi-periodic solution in [34], while here we obtain smooth quasi-periodic solution by Liu-Yuan’s KAM theorem [26]. Before illustrating our approach, we briefly recall some related history about the KAM theory. It is well known that, KAM theory is a powerful tool in constructing quasi-periodic solutions for semi-linear Hamiltonian PDEs, see Kuksin [18, 19, 20, 21], Wayne [35], Kuksin-Pöschel [16], Pöschel [32, 33], Bourgain [8, 9, 10, 11], Chierchia-You [12] for references. In recent years, an interesting topic attracts many mathematician’s attention, which concerns Hamiltonian PDEs with nonlinearities containing derivatives, and for the existing results in this direction, we refer to Kuksin [23, 24], Kappeler-Pöschel [16] for references. We point out that, the key point of the aforementioned papers is the so-called Kuksin’s lemma (see Kuksin [22]). Actually, Kuksin’s lemma can be applied to the KdV equations. However, when it comes to dealing with the perturbed Benjamin-Ono equation and the Schrödinger equation with nonlinearity containing derivatives, the key estimates given by Kuksin’s lemma are not sufficient to ensure us to carry out the KAM iterations, which cannot lead us to establish the existence of the according quasi-periodic solution. Under this circumstance, Liu and Yuan in [25] generalize Kuksin’s work, and successfully develop a KAM theorem suitable for critical unbounded perturbation, which leads to the existence of quasi-periodic solutions for those two equations. Moreover, based on Liu-Yuan’s work [26], a large range of Hamiltonian PDEs with nonlinearities containing derivatives are considered, see for example, Zhang-Gao-Yuan [38], Mi [29], Mi-Zhang [30], Yuan-Zhang [37], Liu-Yuan [27], Yan [36]. For the quasi-periodic solutions of nonlinear wave equation where nonlinearity contains derivatives, we refer to Berti-Biasco-Procesi [6].

We mention that all the previously listed papers deal with those semi-linear equations, for the quasi-linear and fully nonlinear equations, few result about the quasi-periodic solution is known. As far as we know, the first result may due to
Baldi-Berti-Montalto [4]. More precisely, on the basis of Nash-Moser iteration technique and reducibility theory, Baldi-Berti-Montalto in [4] establishes a general KAM theory to construct the quasi-periodic solutions for nonlinear forced perturbation of Airy equation

\[ u_t + u_{xxx} + \epsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0, \quad x \in \mathbb{T}. \]  

Later, similar results are obtained in [5] for the quasi-linear KdV equation under periodic boundary condition. The key point of these results are inspired by the work of Iooss-Plotnikov-Toland [15] in water-waves theory and Baldi [2, 3] for fully nonlinear Benjamin-Ono and forced Kirchoff equations, all of which concern the periodic solutions. For quasi-periodic solutions for fully nonlinear Schrödinger equation, we refer to Feola-Procesi [14].

Before stating our results, we need some preparations. For \( r > \frac{3}{2} \), we introduce the following phase space:

\[ \mathcal{H}_0^r := \{ u \in L^2(\mathbb{T}, \mathbb{R}) : \hat{u}(0) = 0, \| u \|_r^2 = \sum_{0 \neq j \in \mathbb{Z}} |j|^{2r} |\hat{u}(j)|^2 < \infty \}, \]

where

\[ \hat{u}(j) = \int_0^{2\pi} u(x) e^{-2\pi ijx} dx \]

denotes the \( j \)-th Fourier coefficient. In the standard inner product of \( L^2(\mathbb{T}, \mathbb{R}) \), the unperturbed mKdV equation can be written in the following Hamiltonian form

\[ \frac{\partial u}{\partial t} = -\frac{d}{dx} \frac{\partial H}{\partial u}, \]  

where the Hamiltonian function takes

\[ H(u) = \frac{1}{2} \int_\mathbb{T} (u_x^2 - u^4) dx. \]  

We summarize our main results as follows.

**Main Theorem.** We consider the small Hamiltonian perturbation for the mkdV equation, which reads,

\[ \frac{\partial u}{\partial t} = -\frac{d}{dx} \left( \frac{\partial H}{\partial u} + \epsilon \frac{\partial K}{\partial u} \right), \]  

where the Hamiltonian \( H \) takes the form (5) and the perturbation term \( K \) is real analytic in a complex neighbourhood \( W \) of the origin in \( \mathcal{H}_{0,\mathbb{C}}^r \). \( \mathcal{H}_{0,\mathbb{C}}^r \) denotes the complexification of \( \mathcal{H}_0^r \), and satisfies the following regularity conditions,

\[ \frac{\partial K}{\partial u} : W \to \mathcal{H}_{0,\mathbb{C}}^r, \| \frac{\partial K}{\partial u} \| \leq 1. \]  

Then for any given index set \( J = \{ j_1 < j_2 < \cdots < j_n \} \subset \mathbb{Z}\setminus\{0\} \), and a non-empty subset \( \Xi \) of \( \mathbb{R}_+^n \) with positive Lebesgue measure, there exists some \( \epsilon_0 > 0 \) depending on \( r \), \( J \) and \( W \), such that, for any \( \epsilon < \epsilon_0 \), there exist

- a Cantor-like subset \( \Xi_\epsilon \subset \Xi \) fulfilling,
  \[ \text{Meas}(\Xi \setminus \Xi_\epsilon) \to 0, \text{ when } \epsilon \to 0, \]

- a Lipschitz family of real analytic torus embeddings,
  \[ \Phi : \mathbb{T}^n \times \Xi_\epsilon \to W \cap \mathcal{H}_{0,\mathbb{C}}^r, \]
• a Lipschitz map
\[ \varphi : \Xi \rightarrow \mathbb{R}^n, \]
such that, for each \((\theta, I) \in T^n \times \Xi\), the curve \(u(t) = \Phi(\theta + \varphi(I)t, I)\) is a smooth quasi-periodic solution of the perturbed mKdV equation (6), winding around the invariant tori \(\Phi(T^n \times \{I\})\).

Furthermore, the frequencies of these quasi-periodic solutions are close to \((8\pi^3 j_1^3, \cdots, 8\pi^3 j_n^3)\).

2. An unbounded KAM theorem. In this section, we will state the KAM theorem developed by Liu-Yuan [26] to deal with those Hamiltonian PDEs with nonlinearity containing derivatives. As we all know, by a simple trick of rescaling, we can always treat the Hamiltonian functions of those PDEs as an integral normal form plus a small perturbation, besides, the corresponding Hamiltonian vector field of the perturbation is unbounded. In this case, we derive a small perturbation \(H = N + P\) of an infinite dimensional Hamiltonian in the parameter dependent normal form
\[
N = \sum_{j=1}^{n} \omega_j(\xi)y_j + \frac{1}{2} \sum_{j=1}^{\infty} \Omega_j(\xi)(u_j^2 + v_j^2) \tag{8}
\]
defined on a phase space
\[
P^{a,p} = \{\mathbb{R}^n \times T^n \times \ell^a \times \ell^{a,p}\} \ni (x, y, u, v),
\]
with symplectic structure \(\sum_{j=1}^{n} dy_j \wedge dx_j + \sum_{j \geq 1} du_j \wedge dv_j\), where
\[
\ell^{a,p} = \{w = (w_1, w_2, \cdots) \in \mathbb{R}^n : \|w\|_{a,p} = (\sum_{j \geq 1} e^{2a j^2 p} |w_j|^2)^{\frac{1}{2}} < \infty\},
\]
with \(a \geq 0, p \geq 0\). We should point out that, both the tangential frequencies \(\omega = (\omega_1, \cdots, \omega_n)\) and the normal frequencies \(\Omega = (\Omega_1, \Omega_2, \cdots)\) depend on parameters \(\xi \in \Xi\), where \(\Xi\) is a compact subset of \(\mathbb{R}^n\) with positive Lebesgue measure, and usually the normal frequencies admit the so-called asymptotic property, that is,
\[
\Omega_j(\xi) = j^d + \cdots.
\]

The perturbation term \(P\) is real analytic in spatial variables \((x, y, u, v)\) and Lipschitz continuous in the parameter \(\xi\). Further, we assume the regularity property for the perturbation term. More precisely, for every \(\xi \in \Xi\), the vector field \(X_P\) is well defined near the torus
\[
\mathcal{T}_0 = T^n \times \{y = 0\} \times \{u = 0\} \times \{v = 0\}
\]
and is a real analytic map
\[
X_P : P^{a,p} \rightarrow P^{a,q}.
\]
For the rest of the paper, we will adopt many notations from [32] for convenience.

Let us denote the complexification of \(P^{a,p}\) by \(P^{a,p}_c\) with \(s, r > 0\), and introduce the complex neighbourhoods in \(P^{a,p}_c\),
\[
D(s, r) : |Imx| < s, |y| < r^2, \|u\|_{a,p} + \|v\|_{a,p} < r,
\]
and weighted norm for \(W = (X, Y, U, V) \in P^{a,p}_c\),
\[
\|W\|_{r,a,q} = \|X\| + \frac{|Y|}{r^2} + \frac{\|U\|_{a,q}}{r} + \frac{\|V\|_{a,q}}{r}.
\]
Similarly, we can define Lipschitz semi-norm for the frequencies \( \omega \) as follows:

\[
\| X_p \|_{r,a,q,D(s,r) \times \Xi} = \sup_{D(s,r) \times \Xi} \| X_p \|_{r,a,q},
\]

\[
\| X_p \|_{r,a,q,D(s,r) \times \Xi}^{lip} = \sup_{\xi,\zeta \in \Xi, \xi \neq \zeta} \sup_{D(s,r)} \frac{\| \Delta_{\xi\zeta} X_P \|_{r,a,q}}{\| \xi - \zeta \|},
\]

where

\[
\Delta_{\xi\zeta} X_P = X_P(\cdot; \xi) - X_P(\cdot; \zeta).
\]

Similarly, we can define Lipschitz semi-norm for the frequencies \( \omega \) and \( \Omega \) as follows:

\[
|\omega|^{lip}_\Xi = \sup_{\xi,\zeta \in \Xi, \xi \neq \zeta} \frac{\| \Delta_{\xi\zeta} \omega \|}{\| \xi - \zeta \|},
\]

\[
|\Omega|^{lip}_{\delta,\Xi} = \sup_{\xi,\zeta \in \Xi, \xi \neq \zeta, j \geq 1} \frac{\| \Delta_{\xi\zeta} \Omega_j \|}{\| \xi - \zeta \|}.
\]

**Theorem 2.1** (Liu-Yuan [26]). Suppose the normal form part \( N \) fulfills the following conditions:

- **Assumption A**: The map \( \xi \mapsto \omega(\xi) \) is a bijective map between \( \Xi \) and \( \omega(\Xi) \), a homomorphism which is Lipschitz continuous in both directions, i.e., there exist positive constants \( M_1 \) and \( L \), such that \( |\omega|^{lip}_\Xi \leq M_1 \) and \( |\omega^{-1}|^{lip}_{\Xi} \leq L \);

- **Assumption B**: There exists some \( d > 1 \), such that the inequality

\[
|\Omega_i - \Omega_j| \geq m|d - j^d|
\]

holds true for all \( i \neq j \geq 0 \) uniformly on \( \Xi \) with some constant \( m > 0 \), here \( \Omega_0 = 0 \);

- **Assumption C**: There exists some \( \delta \leq d - 1 \), such that the map \( \xi \mapsto \frac{\Omega_j(\xi)}{\xi^\delta} \) is uniformly Lipschitz continuous on \( \Xi \) for \( j \geq 1 \), that is, there exists some positive constant \( M_2 \) such that \( |\Omega|^{lip}_{\delta,\Xi} \leq M_2 \);

- **Assumption D**: Assume further

\[
4ELM_2 \leq m,
\]

here \( E = \sup_{\xi \in \Xi} |\omega(\xi)| \).

Denote \( M = M_1 + M_2 \). Then for each \( \beta > 0 \), there exists a positive constant \( \gamma \), depending on \( n, m, s, d, \delta, \beta, \omega, \Omega \), such that, for each perturbation term \( P \) described above with

\[
\hat{d} = p - q \leq d - 1
\]

and

\[
\epsilon = \| X_P \|_{r,a,q,D(s,r) \times \Xi} + \frac{\alpha}{M} \| X_P \|_{r,a,q,D(s,r) \times \Xi}^{lip} \leq (\alpha \gamma)^{1+\beta},
\]

for some \( r > 0, 0 < \alpha < 1 \), there exist

1. a Cantor set \( \Xi_\alpha \subset \Xi \) with

\[
\text{Meas}(\Xi \setminus \Xi_\alpha) \leq c_1 \alpha,
\]

where \( \text{Meas}(\cdot) \) means Lebesgue measure and \( c_1 = c_1(n, \omega, \Omega) \) is a positive constant;

2. a Lipschitz family of smooth torus embeddings \( \Psi : T^n \times \Xi_\alpha \to \mathcal{P}^{n,p} \) satisfying:

\[
\| \partial_x^k (\Psi - \Psi_0) \|_{r,a,p,T^n \times \Xi_\alpha} + \frac{\alpha}{M} \| \partial_x^k (\Psi - \Psi_0) \|_{r,a,p,T^n \times \Xi_\alpha}^{lip} \leq c_2 \epsilon^{\frac{1}{r+1}} / \alpha,
\]
where $\partial^k = \partial^k_1 \cdots \partial^k_n$ with $|k| = |k_1| + \cdots + |k_n|$, 
$\Psi_0 : \mathbb{T}^n \times \Xi \rightarrow \mathbb{T}_0, (x, \xi) \mapsto (x, 0, 0, 0)$ 
is the trivial embedding for each $\xi$, and $c_2 = c_2(k, \gamma)$ is a positive constant; 
(3) a Lipschitz continuous map $\psi : \Psi_\alpha \rightarrow \mathbb{R}^n$ with 
$|\psi - \omega|_{\Xi_\alpha} + M |\psi - \omega|_{\text{lip}} \Xi_\alpha \leq c_3 \epsilon$, 
where $c_3 = c_3(n, m, s, d, \beta, \delta, \omega, \Omega)$ is a positive constant; 
such that, for each $\xi \in \Xi_\alpha$, the map $\Psi$ restricted to $\mathbb{T}^n \times \{\xi\}$ is a smooth embedding 
of a rotational torus with frequencies $\psi(\xi)$ for the perturbed Hamiltonian $H$ at $\xi$, 
which means, 
t $\mapsto \Psi(\theta + t\psi(\xi), \xi), t \in \mathbb{R}$ 
is a smooth quasi-periodic solution for the Hamiltonian $H$ evaluated at $\xi$ for every 
$\theta \in \mathbb{T}^n$ and $\xi \in \Xi_\alpha$.

**Remark 1.** This theorem can be applied in Hamiltonian PDEs with simple frequencies, such as DNLS equation (NLS with nonlinearity concerning derivative) subject to Dirichlet boundary conditions, KdV and Benjamin-Ono equations with periodic boundary conditions. However, we illustrate that we cannot gain the linear stability of the final invariant tori from this theorem, for this sake, Yuan-Zhang [37] manages to get the linear stability of these preserved tori by using analytic approximation technique.

3. **Partial Birkhoff normal form.** In this section, we will show that the unperturbed Hamiltonian for the mKdV equation can be transformed into partial Birkhoff normal form up to order four, so that we can deal with it as a small perturbation of some integrable system in a small neighbourhood of the origin.

We start with the Hamiltonian of the mKdV equation

$$
H(u) = \frac{1}{2} \int_{\mathbb{T}} (u_x^2 - u^4) dx
$$

Set

$$
u(t, x) = \mathcal{F}q = \sum_{j \neq 0} \gamma_j g_j(t) e^{2\pi i j x},
$$

with $\gamma_j = \sqrt{|j|}$, then we can rewrite the above Hamiltonian function in infinitely many coordinates, which reads

$$
H(q) = N(q) + P(q),
$$

with

$$
N(q) = \sum_{j \geq 1} \lambda_j |q_j|^2,
$$

$$
P(q) = -\frac{1}{2} \sum_{k+l+m+n=0} \gamma_k \gamma l \gamma m \gamma n q_k q_l q_m q_n,
$$

where $\lambda_j = (2\pi j)^3$. The coordinates are taken from Hilbert space $\mathfrak{h}_{r + \frac{1}{2}}$, which is defined by

$$
\mathfrak{h}_{r + \frac{1}{2}} = \{ q = (q_j \in \mathbb{C})_{j \neq 0} \mid \|q\|_{r + \frac{1}{2}}^2 = \sum_{j \neq 0} |q_j|^2 j^{2r + 1} < \infty, q_{-j} = \bar{q}_j \}. \quad (13)
$$
Moreover, the equations of motions in the new coordinates are determined by
\[ \dot{q}_j = -i\sigma_j \frac{\partial H}{\partial q_j}, \sigma_j = \text{sgn}(j). \] (14)

It’s easy to find that $P$ is real analytic in $h_{r+\frac{1}{2}}$, and its Hamiltonian vector field $X_P$, which is defined as follows,
\[ X_P = i \sum_{j \neq 0} \sigma_j \frac{\partial P}{\partial q_j} \frac{\partial}{\partial q_j}, \]
is real analytic as a map from $h_{r+\frac{1}{2}}$ into $h_{r-\frac{1}{2}}$, and further satisfies
\[ \|X_P\|_{r-\frac{1}{2}} = O(\|q\|^{3}_{r+\frac{1}{2}}). \] (15)

Next, we will put the Hamiltonian (11) into partial Birkhoff normal form. Firstly, note that the normal part of $P$ reads as follow,
\[ \bar{P} = -\frac{1}{2} \sum_{\text{relevant terms}} 2\pi |k|^2 |q_k|^4 - \frac{1}{2} \sum_{\text{relevant terms}} 2\pi^2 |k|m ||q_k||^2 |q_m|^2 \] (16)

As we all know, the mkdV equation fulfills many conservation laws, one of them is the energy $\int_\mathbb{T} |u|^2 dx$, which reads,
\[ \sum_{k \neq 0} 2\pi |k||q_k|^2 = C, \]
where $C$ is a constant. Hence, if we denote
\[ P_0 = \frac{1}{2} \sum_{k \neq 0} (2\pi |k|^2)|q_k|^4, \] (17)
then the normal form part $\bar{P}$ reads $P_0 - \frac{1}{2}C^2$.

Secondly, we define index sets as follows:
\[ \Upsilon_0 \triangleq \{(k, l, m, n) \in \mathbb{Z}^4 : k + l + m + n = 0, (k, l, m, n) \equiv (p, -p, r, -r)\}, \]
\[ \Upsilon_1 \triangleq \{(k, l, m, n) \in \Upsilon_0 : \text{at least two elements exist in}\{\pm 1, \pm 2, \cdots, \pm N\}\}, \]
where
\[ (k, l, m, n) \equiv (p, -p, r, -r) \]
means that $(k, l, m, n)$ takes either $(p, -p, r, -r)$ or their permutations, and $N$ denotes a given positive integer. Corresponding to the above index sets, we naturally put the non-normal form part of $P$ into two parts,
\[ P_1 + P_2, \] (18)
where
\[ P_1 = -\frac{1}{2} \sum_{(k, l, m, n) \in \Upsilon_1} \gamma_k \gamma_l \gamma_m \gamma_n q_k q_l q_m q_n, \] (19)
\[ P_2 = -\frac{1}{2} \sum_{(k, l, m, n) \in \Upsilon_0 \setminus \Upsilon_1} \gamma_k \gamma_l \gamma_m \gamma_n q_k q_l q_m q_n. \] (20)

Since a constant does not affect the dynamical behavior of Hamiltonian systems, we turn to consider the following Hamiltonian
\[ H = N + P_0 + P_1 + P_2, \] (21)
where and the Hamiltonian vector fields $X_{P_0}, X_{P_1}, X_{P_2}$ are analytic maps from $\mathfrak{h}_{r+1/2}$ into $\mathfrak{h}_{r-1/2}$, and fulfill
\[ \|X_{P_0}\|_{r-1/2}, \|X_{P_1}\|_{r-1/2}, \|X_{P_2}\|_{r-1/2} = O(\|q\|^{3}_{r+1/2}). \] These regularity properties can be derived similarly as Lemma 14.1 in [16], so we omit the details here.

The next lemma is the main result in this section.

**Lemma 3.1.** There exists a well-defined real analytic symplectic map $\Phi$, such that it transforms the above Hamiltonian into partial normal form of order four. Precisely, we have
\[ H \circ \Phi = H_0 + P_0 + P_2 + R, \]
where
\[ \|X_{R}\|_{r-1/2} = O(\|q\|^{5}_{r+1/2}). \]

**Proof.** To derive the Birkhoff normal form, we need to eliminate the term $P_1$. For this sake, let $\Phi = X_{1}^{t}$ be the time-1-map of a Hamiltonian vector field $X_{F}$, where the Hamiltonian function $F$ is to be determined. Then we have
\[ H \circ \Phi = H \circ X_{F}^{t} = H + \{N, F\} + P_0 + P_2 + R \\
= \int_{0}^{1} (1 - t)\{\{N, F\}, F\} \circ X_{F}^{t} dt + \int_{0}^{1} \{P_0 + P_1 + P_2, F\} \circ X_{F}^{t} dt \\
= \int_{0}^{1} (1 - t)\{\{N, F\}, F\} \circ X_{F}^{t} dt + \int_{0}^{1} \{P_0 + P_1 + P_2, F\} \circ X_{F}^{t} dt. \] (23)

Hence we only need to solve the following homological equation
\[ \{N, F\} + P_1 = 0, \]
where $\{\cdot, \cdot\}$ denotes the Poisson bracket with respect to the symplectic structure
\[ -i \sum_{j \geq 1} dq_j \wedge dq_{-j}. \]

Define
\[ F = \sum_{k,l,m,n} F_{klmn} q_k q_l q_m q_n, \]

note that
\[ \{N, F\} = -i \sum_{k,l,m,n} (\lambda_k + \lambda_l + \lambda_m + \lambda_n) F_{klmn} q_k q_l q_m q_n, \]
we have
\[ iF_{klmn} = \begin{cases} -\frac{1}{2} \frac{\gamma_k \gamma_l \gamma_m \gamma_n}{\lambda_k + \lambda_l + \lambda_m + \lambda_n}, & (k, l, m, n) \in \Upsilon_1. \\ 0, & \text{otherwise}. \end{cases} \]

Moreover,
\[ R = \int_{0}^{1} \{P_0 + tP_1 + P_2, F\} \circ X_{F}^{t} dt. \] (25)
Next, we will check the regularity of Hamiltonian vector field \( X_F \). When \((k, l, m, n) \in \mathbb{T}_1\), from the fact \( k + l + m + n = 0 \), we have
\[
\lambda_k + \lambda_l + \lambda_m + \lambda_n = 24\pi^3(m + l)(k + l)(k + m),
\]
When \(|k| > 3N\), at least one of \( m, l \) lies in \{\pm 1, \pm 2, \cdots, \pm N\}, then
\[
|(m + l)(k + l)(k + m)| \geq |k| - N \geq \frac{|k|}{3N}.
\]
When \(|k| \leq 3N\), since \( m + l \neq 0, k + l \neq 0, k + m \neq 0 \), then we have
\[
|(m + l)(k + l)(k + m)| \geq 1 \geq \frac{|k|}{3N}.
\]
Using the symmetry, we have the following estimate,
\[
\frac{\partial F}{\partial q_k} = 4|\sum_{l+m+n=-k} F_{klmn} q_l q_m q_n|
\]
\[
\leq \frac{1}{12\pi^3} \sum_{l+m+n=-k} \frac{\gamma_k \gamma_l \gamma_m \gamma_n}{(m + l)(k + l)(k + m)} |q_l| |q_m| |q_n|
\]
\[
\leq \frac{N}{4\pi^3|k|} \sum_{l+m+n=-k} \gamma_k \gamma_l \gamma_m \gamma_n |q_l| |q_m| |q_n|
\]
\[
\leq \frac{N}{2\pi^2 \gamma_k} \sum_{l+m+n=-k} \gamma_l \gamma_m \gamma_n |q_l| |q_m| |q_n|.
\]
Denote
\[
\tilde{q}_{-k} = \sum_{l+m+n=-k} \gamma_l \gamma_m \gamma_n |q_l| |q_m| |q_n|,
\]
then the sequence \((\tilde{q}_{-k})_{k \neq 0}\) can be treated as the following convolution
\[
\tilde{q} = v * v * v,
\]
where the sequence \( v = (v_l)_{l \neq 0} \), with \( v_l = \gamma_l |q_l| \). Thus, for any \( r > \frac{3}{2} \), we have
\[
\|X_F\|_{r+1} \leq \frac{N}{2\pi^2} \|\tilde{q}\|_{r+\frac{1}{2}} \leq \frac{AN}{2\pi^2} \|v\|_{r+\frac{1}{2}}^3 \leq \frac{AN}{2\pi^2} \|q\|_{r}^3,
\]
with \( A = A(r) > 0 \).

It remains to estimate the high order term \( R \), actually, since \( R \) is at least of six order in \( q \), similarly, we get
\[
\|X_R\|_{r-\frac{1}{2}} = O(\|q\|_{r+1/2}^5).
\]

\( \square \)

4. **Proof of the Main Theorem.** We present the proof of the Main Theorem below.

**Proof.** Denote the perturbed Hamiltonian as \( \tilde{H} \), then we have \( \tilde{H} = H + \epsilon K \), after the transformation \( \Phi \), we still denote it as \( \tilde{H} \), then
\[
\tilde{H} = N + P_0 + P_2 + R + \epsilon K \circ \Phi,
\]
and analytic in some neighbourhood \( W \) of the origin in \( \mathfrak{h}_{r+1/2} \), we refer \( N, P_0, P_2, R \) respectively to (12), (17), (20), (25), and the last term satisfies
\[
\|X_{K \circ \Phi}\|_{r-\frac{1}{2},W} \leq 2.
\]
Next, we introduce new symplectic coordinates \((x, y, z, \tilde{z})\) as follows:

\[
\begin{align*}
q_b &= \sqrt{\xi_b + y_b} e^{-ix_b}, \quad b = 1, 2, \ldots, n, \\
q_j &= z_j, \quad j \in \mathbb{Z} \setminus J,
\end{align*}
\]

where \(\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n_+\), then we have

\[
N = \sum_{1 \leq b \leq n} (2\pi j_b)^3 (\xi_b + y_b) + \sum_{j \in \mathbb{Z} \setminus J} (2\pi j)^3 |z_j|^2,
\]

\[
P_0 = \frac{1}{2} \sum_{1 \leq b \leq n} (2\pi j_b)^2 (\xi_b + y_b)^2 + \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus J} (2\pi j)^2 |z_j|^4.
\]

Hence, up to a constant, the new Hamiltonian turns to be

\[
\tilde{H} = \tilde{N} + \tilde{P} = \sum_{1 \leq b \leq n} \omega_b y_b + \sum_{j \in \mathbb{Z} \setminus J} \Omega_j z_j \tilde{z}_j + P_2 + P_4 + R + \epsilon K \circ \Phi,
\]

where

\[
\omega_b = 8\pi^3 j_b^3 + 4\pi^2 j_b^2 \xi_b,
\]

\[
\Omega_j = 8\pi^3 j^3,
\]

\[
P_4 = \sum_{1 \leq b \leq n} 2\pi^2 j_b^2 y_b^2 + \sum_{j \in \mathbb{Z} \setminus J} 2\pi^2 j^2 |z_j|^4,
\]

and the corresponding symplectic structure takes

\[
\sum_{b=1}^{n} dy_b \wedge dx_b - i \sum_{j \in \mathbb{Z} \setminus J} dq_j \wedge dq_{-j}.
\]

Define the parameter zone as follow,

\[
\Xi_r = \{ \xi \in \mathbb{R}^n_+ : |\xi| \leq r^{10^{-9}} \}.
\]

It leaves to check the conditions in KAM Theorem 2.1:

Firstly, we check assumption A: By (31) (32), we know that,

\[
\frac{\partial \omega}{\partial \xi} = \text{diag}\{4\pi^2 j_1^2, 4\pi^2 j_2^2, \ldots, 4\pi^2 j_n^2\}. \tag{34}
\]

Thus, if we denote \(M_1 = 4\pi^2 \max_{1 \leq b \leq n} j_b^2\), then we have

\[
|\omega|_{lip}^{lip} \leq M_1.
\]

By direct computation, we get

\[
\frac{\partial \xi}{\partial \omega} = \left(\frac{\partial \omega}{\partial \xi}\right)^{-1} = \text{diag}\{\frac{1}{4\pi^2 j_1^2}, \frac{1}{4\pi^2 j_2^2}, \ldots, \frac{1}{4\pi^2 j_n^2}\}.
\]

Similarly, take

\[
L = \frac{1}{4\pi^2} \max_{1 \leq b \leq n} j_b^{-2},
\]

it follows that,

\[
||\omega^{-1}|^{|lip} \leq L.
\]

Thus, the map \(\xi \mapsto \omega(\xi)\) is a homeomorphism in both directions between \(\Xi_r\) and \(\omega(\Xi_r)\), so assumption A fulfills.

Secondly, we verify assumptions B, C, D: From (33), it indicates that,

\[
|\Omega_i - \Omega_j| \geq 8\pi^3 |i^3 - j^3|, \forall i \neq j
\]
holds uniformly on $\Xi_r$, obviously, assumption B fulfills with $d = 3, m = 8\pi^3$. Furthermore, it’s not hard to find that $|f_r|_{L^p,1,\Xi_r} = 0$, so assumption C holds true for sufficiently small positive numbers $M_2$ and $\delta = 2$. From (31)(32), we know that,

$$E = |\omega|_{\Xi_r} = \sup_{\xi \in \Xi_r} |\omega(\xi)| \leq \sup_{\xi \in \Xi_r} (8\pi^3N^3 + 4\pi^2N^2r^{10}) \leq 10\pi^3N^3$$

holds true for sufficiently small $r$ (we leave the choice of $r$ later). Hence for sufficiently small $M_2 > 0$, assumption D satisfies.

Thirdly, we will check the KAM small condition: Let the phase space be,

$$D(1,r) = \{|Imx| < 1\} \times \{|y| < r^2\} \times \{\|z\|_{r-1/2} + \|\bar{z}\|_{r+1/2} < r\}$$

with $r > 0$. Now we consider the norm and Lipschitz semi-norm of the perturbation

$$\tilde{P} = P_2 + P_4 + R + \epsilon K \circ \Phi$$

on $D(1,r) \times \Xi_r$, where $p = r + \frac{1}{2}$. We choose $r = \epsilon^\frac{1}{4}$, from (33) we have

$$\|X_{P_2}\|_{r,p-1,D(1,r)\times \Xi_r} = O(r^2) = O(\epsilon_2).$$

By (20), we know

$$\|X_{P_2}\|_{r,p-1,D(1,r)\times \Xi_r} = O(r^{\frac{5}{2}}r_4^{r-2}) = O(\epsilon_2^{\frac{7}{14}}).$$

It follows from (7) that

$$\|X_{R}\|_{r,p-1,D(1,r)\times \Xi_r} = O((r^{\frac{5}{2}}r_4^{r-2}) = O(\epsilon_2^{\frac{1}{4}}).$$

Combine (35), (36), (37) and (27), we know that

$$\|X_{\tilde{P}}\|_{r,p-1,D(1,r)\times \Xi_r} = O(\epsilon_2^{\frac{1}{4}}).$$

By the fact that $X_{\tilde{P}}$ is real analytic in $\xi$, we have

$$\|X_{\tilde{P}}\|_{L^p,1,D(1,r)\times \Xi_r} = O(\epsilon_2^{\frac{1}{4}}).$$

We choose

$$\alpha = \epsilon_2^{\gamma^{-1}}, \beta = \frac{1}{14},$$

where $\gamma$ is from KAM Theorem 2.1. So, when $\epsilon$ is sufficiently small, we have

$$\tilde{\epsilon} := \|X_{\tilde{P}}\|_{r,p-1,D(1,r)\times \Xi_r} + \frac{\alpha}{M}\|X_{\tilde{P}}\|_{L^p,1,D(1,r)\times \Xi_r} = O(\epsilon_2^{\frac{1}{4}}) \leq (\alpha \gamma)^{1+\beta} = O(\epsilon_2^{\frac{1}{14}}),$$

which indicates, the KAM small condition satisfies. We complete our proof by applying KAM Theorem 2.1.

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