Mass conserved Allen-Cahn equation and volume preserving mean curvature flow

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February 20, 2009

Abstract
We consider a mass conserved Allen-Cahn equation

\[ u_t = \Delta u + \varepsilon^{-2} (f(u) - \varepsilon \lambda(t)) \]

in a bounded domain with no flux boundary condition, where \( \varepsilon \lambda(t) \) is the average of \( f(u(\cdot, t)) \) and \( -f \) is the derivative of a double equal well potential. Given a smooth hypersurface \( \gamma_0 \) contained in the domain, we show that the solution \( u^\varepsilon \) with appropriate initial data approaches, as \( \varepsilon \to 0 \), to a limit which takes only two values, with the jump occurring at the hypersurface obtained from the volume preserving mean curvature flow starting from \( \gamma_0 \).

1 Introduction.

In this paper, we study the limit, as \( \varepsilon \to 0 \), of the solution \( u^\varepsilon \) to the mass conserved Allen-Cahn equation \( (P^\varepsilon) \)

\[
(P^\varepsilon) \begin{cases}
  u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2} (f(u^\varepsilon) - \int_\Omega f(u^\varepsilon) ) & \text{in } \Omega \times \mathbb{R}^+, \\
  \partial_\nu u^\varepsilon = 0 & \text{on } \partial \Omega \times \mathbb{R}^+, \\
  u^\varepsilon(\cdot, 0) = g^\varepsilon(\cdot) & \text{on } \Omega \times \{0\},
\end{cases}
\]

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Part of this work was done during a visit to the University of Paris-Sud. The author thanks the support of the National Science Foundation Grant DMS–9971043.
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\[ \int_{\Omega} f(u^\varepsilon) = \frac{1}{|\Omega|} \int_{\Omega} f(u^\varepsilon(x,t))dx. \]

Here \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \) \((n \geq 1)\), \( \partial_\nu \) the outward normal derivative to \( \partial \Omega \), and \(-f(u)\) is the derivative of a smooth double equal well potential; more precisely,

\[ f \in C^\infty(\mathbb{R}), \quad f(\pm 1) = 0, \quad f'(\pm 1) < 0, \quad \int_{-1}^{u} f = \int_{1}^{u} f < 0 \quad \forall u \in (-1,1). \tag{2} \]

A typical example is \( f(u) = u - u^3 \). The initial data \( g^\varepsilon \) satisfies, for some smooth hypersurface \( \gamma_0 \subset \subset \Omega \),

\[ \lim_{\varepsilon \to 0} g^\varepsilon(x) = \begin{cases} -1 & \text{inside } \gamma_0 \\ +1 & \text{outside } \gamma_0 \end{cases} \quad \forall x \in \bar{\Omega} \setminus \gamma_0. \tag{3} \]

Problem (1) was proposed, along with its well–posedness, by Rubinstein and Sternberg \[19\] as a model for phase separation in binary mixture. The model is mass preserving and energy decreasing since

\[ \forall t \geq 0, \quad \frac{d}{dt} \int_{\Omega} u^\varepsilon(x,t)dx = 0 \]

and

\[ \forall t \geq 0, \quad \frac{d}{dt} \int_{\Omega} \left( \frac{\varepsilon |\nabla u^\varepsilon|^2}{2} + \frac{1}{\varepsilon} F(u^\varepsilon) \right) dx = -\varepsilon \int_{\Omega} (u^\varepsilon_t)^2 \leq 0, \]

where \( F(u) := -\int_{-1}^{u} f(s)ds \) is the double equal well potential.

Formally, one can show that, as \( \varepsilon \to 0 \), the solution \( u^\varepsilon \) to (1) and (3) tends to a limit

\[ \lim_{\varepsilon \to 0} u^\varepsilon(x,t) = \begin{cases} -1 & \text{inside } \gamma_t \\ +1 & \text{outside } \gamma_t \end{cases} \quad \forall x \in \bar{\Omega} \setminus \gamma_t \tag{4} \]

where \( \Gamma := \bigcup_{t \geq 0} (\gamma_t \times \{t\}) \) is the solution to the volume preserving mean curvature motion equation

\[ V = (n-1)K_{\gamma_t} - \frac{(n-1)}{|\gamma_t|} \int_{\gamma_t} K_{\gamma_t}dH^{n-1} \quad \text{on } \gamma_t \tag{5} \]
starting from $\gamma_0$. Here $V$ is the normal velocity of $\gamma_t$ (positive when $\gamma_t$ is shrinking) and $K_{\gamma_t}$ the mean curvature (positive at points where $\gamma_t$ is locally the boundary of a convex domain).

The local in time existence of a unique smooth solution to (5) has been first established in a two-dimensional setting in [11]. The general result in arbitrary space dimension is obtained in [13], where the large time behaviour of solutions for initial data close to a sphere was also investigated. When the initial data is convex, it is shown in [16] that (5) admits a unique global in time convex solution. Related properties of other volume-preserving curvature driven flows are established in [12].

Concerning the connection between (1) and (5), Bronsard and Stoth [3] considered a radially symmetric case with multiple interfaces (rings) and proved (4). Let us also mention [15] where a similar result is established for a different nonlocal mass conserved Allen-Cahn equation, using the method introduced in [2]. In the present paper, we shall consider general smooth initial interfaces $\gamma_0 \subset \subset \Omega$ and prove the following:

**Theorem 1** Let $\Gamma = \bigcup_{0 < t \leq T} (\gamma_t \times \{t\})$ be a smooth solution to (5) satisfying $\gamma_t \subset \subset \Omega$ for all $t \in [0, T]$. Then there exists a family of continuous functions $\{g^\varepsilon\}_{0 < \varepsilon \leq 1}$ such that the solution $u^\varepsilon$ to (1) satisfies (4) for all $t \in [0, T]$.

For the Allen-Cahn equation $u_t^\varepsilon = \Delta u^\varepsilon - \varepsilon^{-2}f(u^\varepsilon)$, (1) holds with $\Gamma$ being the solution to the motion by mean curvature flow $V = (n - 1)K_{\gamma_t}$. A simple method to verify this is to use a comparison principle and construct sub-super solutions [4, 14]. There are different notions of weak solutions such as viscosity [14] and varifold [17] which can be used to establish the global in time limit. Nevertheless, (1) does not have a comparison principle (due to the volume preserving property) and the simple method does not seem to work. Here we shall employ a method first used by de Mottoni and Schatzman [10] for the Allen-Cahn equation, and later on by Alikakos, Bates, Chen [1] for the Cahn–Hillard equation and Caginalp and Chen [6] for the phase field system.

Namely we first rewrite the equation for $u^\varepsilon$ in Problem $(P^\varepsilon)$ as

$$u_t^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-2}(f(u^\varepsilon) - \varepsilon\lambda_t(t)) \text{ in } \Omega \times \mathbb{R}^+,$$  \hspace{1cm} (6)
where we define
\[ \forall t \geq 0, \quad \lambda_\varepsilon(t) = \frac{1}{\varepsilon} \int_\Omega f(u^\varepsilon(\cdot, t)). \]  

(7)

The basic strategy of the proof goes as follows.

1. For a large enough \( k \in \mathbb{N} \), construct an approximate solution \((u^\varepsilon_k, \lambda^\varepsilon_k)\) satisfying

\[
\begin{cases}
    u^\varepsilon_{k,t} - \Delta u^\varepsilon_k - \varepsilon^{-2}(f(u^\varepsilon_k) - \varepsilon \lambda^\varepsilon_k) = \delta^\varepsilon_k & \text{in } \Omega_T := \Omega \times [0, T], \\
    \int_\Omega u^\varepsilon_{k,t} \, dx = 0 & \forall t \in [0, T], \\
    \partial_\nu u^\varepsilon_k = 0 & \text{on } \partial \Omega_T := \partial \Omega \times [0, T]
\end{cases}
\]

(8)

where \( \delta^\varepsilon_k = O(1)\varepsilon^k \). Note that, by integration,

\[ \varepsilon \lambda^\varepsilon_k = \int_\Omega f(u^\varepsilon_k) + O(1)\varepsilon^{k+2}. \]

2. For each \( t \in [0, T] \) and small positive \( \varepsilon \), estimate the lower bound of the spectrum of the self-adjoint operator \(-\Delta - \varepsilon^{-2}f'(u^\varepsilon_k(\cdot, t))\); namely, show that for some positive constant \( C^* \),

\[
\inf_{0 < t \leq T} \inf_{0 < \varepsilon \leq 1} \int_\Omega (|\nabla \phi|^2 - \varepsilon^{-2}f'(u^\varepsilon_k(\cdot, t))\phi^2) \geq -C^*. \]

(9)

3. Set \( R = u^\varepsilon - u^\varepsilon_k \) and show that \( R \) tends to 0 as \( \varepsilon \to 0 \).

The organization of this paper is as follows. In section 2, we present an error estimate required in step 3. In section 3, we recall a known spectrum estimate \[9, 5\] that can be adapted here to prove step 2 in the strategy described above. After some geometrical preliminary computations in section 4, we finally construct the approximate solutions in section 5.

## 2 Error Estimate

The error estimate relies on the following result which is proved in the appendix.
Lemma 1 Let $\Omega \subset \mathbb{R}^n$ (with $n \geq 1$) be a bounded domain, let $p = \min\{4/n, 1\}$. Then there exists $C = C_n(\Omega) > 0$ such that for every $R \in H^1(\Omega)$ with $\int_\Omega R \, dx = 0$,
\[ \|R\|^2_{L^2} + \|R\|^p_{L^p} \leq C \|\nabla R\|^2_{L^2}, \] (10)
where for any $q \geq 1$, $L^q = L^q(\Omega)$.

Rubinstein-Sternberg established in [19] $L^\infty$ bounds for the solution $u^\varepsilon$ to Problem $(P^\varepsilon)$ using invariant rectangles. Therefore we can modify $f$ outside of a compact interval and assume for simplicity that
\[ \lim_{u \to \pm\infty} f(u) = \mp\infty \]
and that there exists $M > 0$ such that
\[ \forall |u| \geq M, \ uf''(u) \leq 0. \]
Since $p \in (0, 1]$, for any $C_0 > 0$, there exists $C = C(C_0, p)$ such that for all $|u| \leq C_0$ and $R \in \mathbb{R}$,
\[ (f(u + R) - f(u) - f'(u)R)R \leq C |R|^{p+2}. \]

Indeed, note that for $R$ in a compact interval,
\[ (f(u + R) - f(u) - f'(u)R)R = \frac{f''(u + \theta R)}{2} R^3 \leq C |R|^{p+2}, \]
whereas for $|R| \to +\infty$, $f(u + R)R \to -\infty$, uniformly in $|u| \leq C_0$ so that
\[ (f(u + R) - f(u) - f'(u)R)R \leq (-f(u) - f'(u)R)R \leq CR^2 \leq C |R|^{p+2}. \]

Lemma 2 Assume that $k > \max\{4, n\}$ and $\{u_k^\varepsilon\}_{0 < \varepsilon \leq 1}$ satisfies (8) and (9) with
\[ \|\delta_k^\varepsilon\|_{L^2(\Omega_T)} \leq \varepsilon^k, \quad \|u_k^\varepsilon\|_{L^\infty(\Omega_T)} \leq 2. \]

Let $\{u^\varepsilon\}_{0 < \varepsilon \leq 1}$ be solutions to (1) with initial data $\{g^\varepsilon\}$ satisfying
\[ g^\varepsilon(\cdot) = u_k^\varepsilon(\cdot, 0) + \phi^\varepsilon(\cdot), \quad \int_\Omega \phi^\varepsilon = 0, \quad \|\phi^\varepsilon\|_{L^2(\Omega)} \leq \varepsilon^k. \]

Then for all sufficiently small positive $\varepsilon$,
\[ \sup_{0 \leq t \leq T} \|u^\varepsilon(\cdot, t) - u_k^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C(T)\varepsilon^k. \]
**Remark 1** By a bootstrap argument, one can show that other norms of $(u^\varepsilon - u_k^\varepsilon)$ tend to 0 as $\varepsilon \searrow 0$.

**Proof.** In the sequel, $C$ denotes a generic positive constant independent of $\varepsilon$. Set $p = \min\{1, 4/n\}$ and $R = u^\varepsilon - u_k^\varepsilon$. Then $\int_\Omega R(x, t) dx = 0$ for all $t \in [0, T]$. Also,

$$R\{f(u^\varepsilon) - f(u_k^\varepsilon) - f'(u_k^\varepsilon) R\} \leq C|R|^{2+p}.$$  

Multiplying by $R$ the difference of the equations for $u^\varepsilon$ and $u_k^\varepsilon$ and integrating the resulting equation over $\Omega$ gives, after integration by parts,

$$\frac{1}{2} \frac{d}{dt} \|R\|_{L^2}^2 + \int_\Omega \left\{ |\nabla R|^2 - \varepsilon^{-2} f'(u_k^\varepsilon)|R|^2 \right\} \leq \int_\Omega \left\{ C\varepsilon^{-2} |R|^{2+p} + |R\delta_k^\varepsilon| \right\}.$$  

By (9),

$$\int_\Omega \left\{ |\nabla R|^2 - \varepsilon^{-2} f'(u_k^\varepsilon)|R|^2 \right\} = \varepsilon^2 \int_\Omega + (1 - \varepsilon^2) \int_\Omega \geq \varepsilon^2 \|\nabla R\|_{L^2}^2 - C\|R\|_{L^2}^2.$$  

The interpolation (10) then yields

$$\frac{1}{2} \frac{d}{dt} \|R\|_{L^2}^2 \leq C\|\delta_k^\varepsilon\|_{L^2} \|R\|_{L^2} + C\|R\|_{L^2}^2 - \|\nabla R\|_{L^2}^2 \{\varepsilon^2 - C_1\varepsilon^{-2} \|R\|_{L^2}^p\}. \quad (11)$$

We define

$$T^\varepsilon := \sup\{t \in [0, T] \mid \|R(\cdot, \tau)\|_{L^2} \leq \varepsilon^{4/p} C_1^{-1/p} \text{ for all } \tau \in [0, t]\}.$$  

Since $k > \max\{4, n\} = 4/p$, it follows that

$$\|R(\cdot, 0)\|_{L^2} \leq \varepsilon^k < \varepsilon^{4/p} C_1^{-1/p}$$

for $\varepsilon > 0$ small enough. Therefore, $T^\varepsilon > 0$. Also, from (11), we have for all $t \in (0, T^\varepsilon]$,

$$\frac{d}{dt} \|R\|_{L^2} \leq C(\|R\|_{L^2} + \|\delta_k^\varepsilon\|_{L^2})$$

The Gronwall’s inequality then provides

$$\sup_{0 \leq t \leq T^\varepsilon} \|R(\cdot, t)\|_{L^2} \leq e^{CT}[\|R(\cdot, 0)\|_{L^2} + \int_0^T \|\delta_k^\varepsilon\|_{L^2} dt] \leq C(T)\varepsilon^k < \frac{1}{2} \varepsilon^{4/p} C_1^{-1/p}$$

if $\varepsilon$ is small enough. Thus, we must have $T^\varepsilon = T$. This completes the proof.
3 The linearized operator

3.1 A Spectrum Estimate

Assume that $f$ satisfies (2). Then there is a unique solution $\theta_0(\cdot) : \mathbb{R} \to (0, 1)$ to

$$\theta_0'' + f(\theta_0) = 0 \text{ on } \mathbb{R}, \quad \theta_0(\pm \infty) = \pm 1, \quad \theta_0(0) = 0. \quad (12)$$

The solution satisfies, for $\alpha = \min\{\sqrt{-f'(1)}, \sqrt{-f'(-1)}\}$,

$$D_m^{\rho}\{\theta_0(\rho) \mp 1\} = O(e^{-\alpha|\rho|}) \text{ as } \pm \rho \to \infty, \quad \forall m \in \mathbb{N}. \quad (13)$$

Let $\theta_1 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be any function satisfying

$$\int_{\mathbb{R}} \theta_0''(\theta_0) \theta_1 = 0.$$  

Let $\Omega^- \subset \subset \Omega$ be a subset with $C^3$ boundary $\gamma = \partial \Omega^-$. Denote by $d(x)$ the signed distance (negative in $\Omega^-$) from $x$ to $\gamma$ and by $s(x)$, for $x$ close to $\gamma$, the projection from $x$ on $\gamma$ along the normal to $\gamma$.

We look for the spectrum of the linearized operator of $-\Delta u - \varepsilon^{-2} f'(u)$ around $u = \psi^\varepsilon$ given by

$$\psi^\varepsilon(x) = \begin{cases} 
\theta_0(d(x)) & \text{if } |d(x)| \leq \sqrt{\varepsilon}, \\
\pm 1 & \text{if } \pm |d(x)| \geq \sqrt{\varepsilon}.
\end{cases} \quad (14)$$

The following spectrum estimate was first proven by de Mottoni and Schatzman [9], then by Chen [5] in a more general situation that can be used in [1, 6].

**Proposition 1** Let $\gamma \in C^3$, and $p^\varepsilon$ and $O(1)$ in [14] be bounded independently of $\varepsilon$. Then there exists a positive constant $C^*$ depending on $\|\gamma\|_{C^3}$, $\|p^\varepsilon\|_{L^\infty}$ and $\|O(1)\|_{L^\infty}$ such that for every $\varepsilon \in (0, 1]$ and $\phi \in H^1(\Omega)$,

$$\int_{\Omega} \{|
abla \phi|^2 - \varepsilon^{-2} f'(\psi^\varepsilon)\phi^2\} \geq -C^* \int_{\Omega} \phi^2.$$
We define the linearized operator around \(\theta_0(\rho)\) acting on \(v = v(\rho)\) by
\[
\mathcal{L} v := -v'' - f'(\theta_0) v
\]
(15)

In our application, \(\theta_1\) is the unique solution to
\[
\mathcal{L} \theta_1 = 1 - \sigma \theta_0' \quad \text{in } \mathbb{R},
\]
\[
\theta_1(0) = 0, \quad \sigma := 2/\int_{\mathbb{R}} \theta_0'^2.
\]

Integrating \(\theta''_0 \mathcal{L} \theta_1\) over \(\mathbb{R}\) and by parts, one can verify that (13) is satisfied; see [9, 5, 1].

We remark that the distance function \(d\) in (14) can be replaced by a “quasi-distance” function \(d^\varepsilon\) given by
\[
d^\varepsilon(x) = d(x) - \varepsilon h_1(s(x)) - \varepsilon^2 h_2(s(x)) + O(1)\varepsilon^3
\]
where \(h_1\) and \(h_2\) are smooth functions on \(s \in \gamma\).

3.2 Solvability Condition

**Lemma 3** Assume that \(f\) satisfies (2). Let \(\theta_0\) be the solution to (12), \(\alpha = \min\{\sqrt{-f'(1)}, \sqrt{-f'(-1)}\}\) and \(\mathcal{L}\) be defined in (14). Assume that a function \(h(\rho, s, t)\) satisfies, as \(\rho \to \pm\infty\),
\[
D^m_\rho D^n_s D^l_t [h(\rho, s, t) - h^\pm(t)] = O(|\rho|^i e^{-\alpha|\rho|})
\]
for some \(i \geq 0\) and all \((m, n, l) \in \mathbb{N}^3\) and \((s, t)\) in \(U \times [0, T]\). Then
\[
\mathcal{L}Q = h(\cdot, s, t) \quad \text{in } \mathbb{R}, \quad Q(0, s, t) = 0
\]
has a unique bounded solution \(Q(\rho, s, t)\) if and only if
\[
\forall (s, t) \in U \times [0, T], \quad \int_{\mathbb{R}} h(\rho, s, t) \theta_0'(\rho) d\rho = 0.
\]

If the solution exists, then it satisfies, for all \((m, n, l) \in \mathbb{N}^3\) and \((s, t) \in U \times [0, T]\),
\[
D^m_\rho D^n_s D^l_t [Q(\rho, s, t) + h^\pm(t)] = O(|\rho|^i e^{-\alpha|\rho|}) \quad \text{as } \rho \to \pm\infty.
\]

**Proof** Since \(\mathcal{L} \theta_0' = 0\), the ode \(\mathcal{L}Q = h\) can be solved explicitly. We omit the details of the proof; see [9].
4 Differential Geometry: local coordinates

4.1 Parametrization around the limit interface

Let \( \Gamma = \bigcup_{t \in [0,T]} \gamma_t \times \{t\} \subset \Omega_T \) be the smooth solution to (5) on \([0,T]\) and \(\Omega^\pm(t)\) the two domains separated by \(\gamma_t\), with \(\gamma_t = \partial \Omega^-(t)\). For each fixed \(t\), we use \(d(x,t)\) to denote the signed distance from \(x\) to \(\gamma_t\) (positive in \(\Omega^+(t)\)). Then \(d(\cdot, \cdot)\) is smooth in a tubular neighborhood of the interface. Locally we choose a parametrization of \(\gamma_t\) by \(X_0(s,t)\) with \(s \in U \subset \mathbb{R}^{n-1}\) so that

\[
\left( \frac{\partial X_0}{\partial s_1}, \ldots, \frac{\partial X_0}{\partial s_{n-1}} \right)
\]

is a basis of the tangent space to \(\gamma_t\) at \(X_0(s,t)\), for each \(s \in U\). We denote by \(n(s,t)\) the unit normal vector to \(\gamma_t\), pointing towards \(\Omega^+(t)\) so that

\[n(s,t) = \nabla d(X_0(s,t), t).\]

Up to a suitable multiplication factor \(s_1 \rightarrow \lambda s_1\), we may assume that

\[\det (n(s,t), \frac{\partial X_0}{\partial s_1}, \ldots, \frac{\partial X_0}{\partial s_{n-1}}) = 1\]  

Next for each fixed \(t \in [0,T]\), a local parametrization by coordinates \((s, r) \in U \times (-3\delta, 3\delta)\) is obtained by

\[x = X_0(s,t) + r n(s,t) = X(r,s,t),\]

which defines a local diffeomorphism from \((-3\delta, 3\delta) \times U\) onto the tubular neighborhood of \(\gamma_t\),

\[V^t_{3\delta} = \{x \in \Omega, \ |d(x,t)| < 3\delta\}.\]  

We denote the inverse by

\[r = d(x,t), \ s = S(x,t) = (S^1(x,t), S^2(x,t), \ldots, S^{n-1}(x,t)).\]

In particular, since for all fixed \(s \in U, t \in [0,T]\) and for all \(r \in (-3\delta, 3\delta)\),

\[d(X_0(s,t) + r n(s,t), t) = r,\]
it follows by differentiation with respect to $r$ that for all $r \in (-3\delta, 3\delta)$,
\[
\nabla d(X_0(s, t) + r \mathbf{n}(s, t), t) \cdot \mathbf{n}(s, t) = 1.
\]

Using that
\[
|\nabla d(x, t)| = 1 \text{ for } x \text{ close to } \gamma_t,
\]
this equality imposes that for all $(r, s) \in (-3\delta, 3\delta) \times U$,
\[
\nabla d(X_0(s, t) + r \mathbf{n}(s, t), t) = \mathbf{n}(s, t)
\]
proving that $\nabla d$ is constant along the normal lines to $\gamma_t$. Thus the projection from $x$ on $\gamma_t$ is defined by
\[
X_0(S(x, t), t) = x - d(x, t) \nabla d(x, t).
\]

It follows also from (21) that for all $i = 1, \ldots, n$ and for $x \in V^t_{3\delta}$,
\[
\sum_{j=1}^{n} \frac{\partial^2 d}{\partial x_i \partial x_j} (x, t) \frac{\partial d}{\partial x_j} (x, t) = 0.
\]

Thus the symmetric matrix $D^2_x d(x, t)$ has eigenvalues $\{\kappa_1, \cdots, \kappa_{n-1}, 0\}$ with unit eigenvectors $\{\tau_1, \cdots, \tau_{n-1}, \nabla d\}$ forming an orthonormal basis of $\mathbb{R}^n$ for $x \in V^t_{3\delta}$. In particular, for $x \in \gamma_t$, the $\tau_i$ are the principal directions and the $\kappa_i$ are the principal curvatures of $\gamma_t$. Note that $\{\tau_1, \cdots, \tau_{n-1}\}$ form a basis of the tangent hyperplane to $\gamma_t$ at $x = X_0(s, t)$. By definition, $K$ and $K_{\gamma_t}$ are respectively the sum of principal curvatures and the mean curvature of $\gamma_t$, given by
\[
K = (n - 1)K_{\gamma_t} = \Delta d(X_0(s, t), t) = \sum_{i=1}^{n-1} \kappa_i(s, t).
\]

Note that using (24), for $x \in \gamma_t$, we have that
\[
\nabla d \cdot \nabla \Delta d = \sum_{ij} \frac{\partial}{\partial x_j} \left( \frac{\partial d}{\partial x_i} \frac{\partial^2 d}{\partial x_i \partial x_j} \right) - \sum_{ij} \left( \frac{\partial^2 d}{\partial x_i \partial x_j} \right)^2
\]
\[
= - \sum_{ij} \left( \frac{\partial^2 d}{\partial x_i \partial x_j} \right)^2 = -\text{Trace}((D^2_x d)^2) = - \sum_{i=1}^{n-1} \kappa_i^2.
\]
We denote
\[
b(s, t) = -\nabla d \cdot \nabla d|\nabla X_0(s,t)| = \sum_{i=1}^{n-1} \kappa_i^2.
\] (26)

Let \(V(s, t)\) be the normal velocity of the interface at the point \(X_0(s, t)\) so that using (22),
\[
V(s, t) = X_0(s, t) \cdot n(s, t)
= X_0(s, t) \cdot \nabla d(X_0(s, t) + r \, n(s, t), t) = -d_t(X(r, s, t), t)
\] (27)
where the last equality follows from differentiating with respect to \(t\) the identity
\[
d(X_0(s, t) + r \, n(s, t), t) = r.
\]
It follows that \(d_t(x, t)\) is independent of \(r = d(x, t)\) for \(|r|\) small enough.
Changing coordinates from \((x,t)\) to \((r,s,t)\), we associate to any function \(\phi(x,t)\) the function
\[
\tilde{\phi}(r, s, t) = \phi(X_0(s, t) + r \, n(s, t), t)
\] (28)
or equivalently
\[
\phi(x, t) = \tilde{\phi}(d(x, t), S(x, t), t).
\]
By differentiation we obtain the following formulas
\[
\begin{align*}
\partial_t \phi &= (-V \partial_r + \partial_t^\Gamma) \tilde{\phi} \\
\nabla \phi &= (n \partial_r + \nabla^\Gamma) \tilde{\phi} \\
\Delta \phi &= (\partial_{rr} + \Delta d \partial_r + \Delta^\Gamma) \tilde{\phi}
\end{align*}
\] (29)
with
\[
\begin{align*}
\partial_t^\Gamma \tilde{\phi} &= (\partial_t + \sum_{i=1}^{n-1} S_i^i \partial_{s^i}) \tilde{\phi} \\
\nabla^\Gamma \tilde{\phi} &= \sum_{i=1}^{n-1} \nabla S_i^i \partial_{s^i}) \tilde{\phi} \\
\Delta^\Gamma \tilde{\phi} &= \sum_{i=1}^{n-1} \Delta S_i^i \partial_{s^i} + \sum_{i,j=1}^{n-1} \nabla S_i^i \nabla S_j^j \partial_{s^i s^j}) \tilde{\phi}
\end{align*}
\] (30)
where $\nabla S^i, S^i_t, \Delta d, d_t$ are evaluated at $x = X(r,s,t)$ and are viewed as functions of $(r,s,t)$. Note that the mixed derivatives of the form $\partial^2_{rsj} \tilde{\phi}$ do not appear eventually in (29) because for all $j = 1,2,...,n-1$,

$$\nabla S^j(x,t).\nabla d(x,t) = 0$$

(This follows from differentiating with respect to $r$ the identity

$$\forall r \in (-3\delta,3\delta), \ S^j(X_0(s,t) + r n(s,t), t) = s^j$$

which holds for all fixed $s \in U, t \in [0,T]$ and $j = 1,2,...,n-1$.)

### 4.2 The stretched variable

Following the method used in [7], we now define the stretched variable $\rho$ by considering a graph over $\gamma_t$ of the form

$$\tilde{\gamma}_t^\epsilon = \{X(r,s,t) \mid r = \epsilon h_x(s,t), s \in U\}$$

which is (formally) expected to be a representation of the 0 level set at time $t$ of the solution $u^\epsilon$ of Problem $(P^\epsilon)$.

The stretched variable $\rho$ is then defined by

$$\rho = \rho'(x,t) = \frac{d(x,t) - \epsilon h_x(S(x,t), t)}{\epsilon}$$

which represents the distance from $x$ to $\tilde{\gamma}_t^\epsilon$ in the normal direction divided by $\epsilon$. From now on, we use $(\rho, s, t)$ as independent variables for the inner expansions. The relation between the old and new variables are

$$x = \hat{X}(\rho,s,t) = X(\epsilon (\rho + h_x(s,t)), s,t) = X_0(s,t) + \epsilon (\rho + h_x(s,t)) n(s,t)$$

We associate to any function $w(x,t)$ the function

$$\hat{w}(\rho,s,t) = w(X_0(s,t) + \epsilon (\rho + h_x(s,t)) n(s,t), t)$$

or equivalently

$$w(x,t) = \hat{w}(\frac{d(x,t) - \epsilon h_x(S(x,t), t)}{\epsilon}, S(x,t), t).$$
Note that
\[ \tilde{w}(r, s, t) = \hat{w}(\frac{r - \varepsilon h_{\tau}(s, t)}{\varepsilon}, s, t). \]

By differentiation we obtain the following formulas
\[ \partial_t \tilde{w} = (-V \varepsilon^{-1} - \partial^F h_{\varepsilon}) \hat{w}_\rho + \partial^F \hat{w}, \]
\[ \nabla \tilde{w} = (n \varepsilon^{-1} - \nabla^F h_{\varepsilon}) \hat{w}_\rho + \nabla^F \hat{w}, \]
\[ \Delta \tilde{w} = (\varepsilon^{-2} + |\nabla^F h_{\varepsilon}|^2) \hat{w}_{\rho \rho} + (\Delta d \varepsilon^{-1} - \Delta^F h_{\varepsilon}) \hat{w}_\rho - 2\nabla^F h_{\varepsilon} \nabla^F \hat{w}_\rho + \Delta^F \hat{w}, \] (35)

where in the above formula for \( \Delta \tilde{w} \),
\[ \Delta d = \Delta d \bigg|_{x=X_0(s,t)+\varepsilon(\rho+h_{\varepsilon}(s,t))n(s,t)} \approx K(s,t) - \varepsilon(\rho + h_{\varepsilon}(s,t))b(s,t) + \sum_{i \geq 2} \varepsilon^i b_i(s,t)(\rho + h_{\varepsilon}(s,t))^i, \] (36)

with \( b \) defined in (26), \( K \) defined in (25) and for some given functions \( (b_i(s,t))_{i \geq 2} \) only depending on \( \gamma_t \). Therefore
\[ \varepsilon^2(\partial_t \tilde{w} - \Delta \tilde{w}) = -\hat{w}_{\rho \rho} - \varepsilon(V + \Delta d) \hat{w}_\rho \]
\[ + \varepsilon^2[(\partial^F \hat{w} - \Delta^F \hat{w}) - (\partial^F h_{\varepsilon} - \Delta^F h_{\varepsilon}) \hat{w}_\rho] \]
\[ - \varepsilon^2[|\nabla^F h_{\varepsilon}|^2 \hat{w}_{\rho \rho} - 2\nabla^F h_{\varepsilon} \nabla^F \hat{w}_\rho] \] (37)

The Jacobi For later purposes, we need to compute the Jacobi of the transformation \( \hat{X} \). In the \( (\rho, s) \) coordinates, \( dx = \varepsilon J^F(\rho, s, t)dsd\rho \) where \( ds \) is the surface element of \( \gamma_t \) and where \( \varepsilon J^F(\rho, s, t) = \partial \hat{X}(\rho, s, t)/\partial(\rho, s) \) is the Jacobi. We prove below that

**Lemma 4** For all \( \rho \in \mathbb{R}, s \in U \) and \( t \in [0, T] \),
\[ J^F(\rho, s, t) = \prod_{i=1}^{n-1}[1 + \varepsilon(\rho + h^\varepsilon(s, t))\kappa_i(s, t)]. \] (38)

**Proof.** The equality (38) is obtained in two steps. First we consider the function \( X = X(r, s, t) \) defined in (13), denote its Jacobi by \( J = J(r, s, t) \) and prove that for all \( \rho \in \mathbb{R}, s \in U \) and \( t \in [0, T] \),
\[ J^F(\rho, s, t) = J(\varepsilon(\rho + h_{\varepsilon}(s, t)), s, t). \] (39)
Second we compute $J$ and show that for all $s \in U$, for all $t \in [0, T]$, 

$J(r, s, t) = \prod_{i=1}^{n-1} [1 + r \kappa_i(s, t)]$

$$= 1 + \Delta d(X_0(s, t), t) r + \sum_{i=2}^{n-1} r^i j_i(s, t), \quad (40)$$

for some given functions $j_i$ depending on $\gamma_t$. Consequently (38) follows directly from (39) and (40).

In order to establish (39), note that by definition (33),

$$\hat{X}(\rho, s, t) = X(\varepsilon(\rho + h_\varepsilon(s, t)), s, t)$$

so that

$$\frac{\partial \hat{X}}{\partial \rho} = \varepsilon \frac{\partial X}{\partial r}$$

and for $i = 1, \ldots, n - 1$,

$$\frac{\partial \hat{X}}{\partial s_i} = \frac{\partial X}{\partial s_i} + \varepsilon \frac{\partial h_\varepsilon}{\partial s_i} \frac{\partial X}{\partial r}.$$ 

Thus for all $\rho \in \mathbb{R}$, $s \in U$ and $t \in [0, T]$,

$$\varepsilon J^\varepsilon(\rho, s, t) = \varepsilon \det \left[ \frac{\partial X}{\partial r}, \frac{\partial X}{\partial s_1} + \varepsilon \frac{\partial h_\varepsilon}{\partial s_1} \frac{\partial X}{\partial r}, \ldots, \frac{\partial X}{\partial s_{n-1}} + \varepsilon \frac{\partial h_\varepsilon}{\partial s_{n-1}} \frac{\partial X}{\partial r} \right]$$

$$= \varepsilon \det \left[ \frac{\partial X}{\partial r}, \frac{\partial X}{\partial s_1}, \ldots, \frac{\partial X}{\partial s_{n-1}} \right](\varepsilon(\rho + h_\varepsilon(s, t)), s, t) = \varepsilon J(\varepsilon(\rho + h_\varepsilon(s, t)), s, t)$$

which is (39).

In order to establish (40), we consider the Hessian matrix of $d$ on $\gamma_t$ and denote for $s \in U$ and $t \in [0, T]$ 

$$A = A(s, t) = D^2_X d(X_0(s, t), t)$$

so that (24) reads

$$A.n(s, t) = 0. \quad (41)$$

Moreover, differentiating the identity (22) at $r = 0$ with respect to $s_i$ for $i = 1, \ldots, n - 1$ yields

$$A.\frac{\partial X_0}{\partial s_i} = \frac{\partial n}{\partial s_i}. \quad (42)$$
From
\[ X(r, s, t) = X_0(s, t) + r \mathbf{n}(s, t), \]
it follows that using (41)
\[ \frac{\partial X}{\partial r} = \mathbf{n}(s, t) = (I_n + r A(s, t))(\mathbf{n}(s, t)) \]
and that, using (42) for \( i = 1, \ldots, n - 1, \)
\[ \frac{\partial X}{\partial s_i} = \frac{\partial X_0}{\partial s_i} + r \frac{\partial \mathbf{n}}{\partial s_i} = (I_n + r A(s, t))\left( \frac{\partial X_0}{\partial s_i} \right). \]
Therefore for all \( s \in U \) and \( t \in [0, T], \)
\[ J(r, s, t) = \det \left[ \frac{\partial X}{\partial r}, \frac{\partial X}{\partial s_1}, \ldots, \frac{\partial X}{\partial s_{n-1}} \right] \]
\[ = \det \left[ (I_n + r A)(\mathbf{n}), (I_n + r A)\left( \frac{\partial X_0}{\partial s_1} \right), \ldots, (I_n + r A)\left( \frac{\partial X_0}{\partial s_{n-1}} \right) \right] \]
which in view of (17) proves that
\[ J(r, s, t) = \det [I_n + r A(s, t)] \]
which yields (40), since the eigenvalues of \( A(s, t) \) are \( \kappa_1, \ldots, \kappa_{n-1}, 0. \)

5 The approximate solution

5.1 Asymptotic Expansions

Let \( k > \max\{2, n/2\} \) be a fixed integer. In the sequel, we use the sign \( \approx \) to represent an asymptotic expansion; namely, \( \phi^\varepsilon \approx \sum_{i \geq 0} \varepsilon^i \phi_i \) means that for every integer \( j \in \mathbb{N}, \phi^\varepsilon = \sum_{i=0}^{j} \varepsilon^i \phi_i + O(1)\varepsilon^{j+1} \) where \( O(1) \) is bounded independently of \( \varepsilon \in (0, 1). \) For example, since \( f \) is smooth, for any bounded
sequence \( \{b, a_0, a_1, a_2, \ldots\} \), we have the asymptotic expansion

\[
 f(b + \varepsilon \sum_{i \geq 0} \varepsilon^i a_i) \approx \sum_{j \geq 0} \varepsilon^j f^{(j)}(b) \left( \sum_{i \geq 0} \varepsilon^i a_i \right)^j / j!
\]

\[
 \approx f(b) + \varepsilon f'(b) \sum_{i \geq 0} \varepsilon^i a_i + \varepsilon^2 \sum_{i \geq 0} \varepsilon^i f_i(b, a_0, \ldots, a_i)
\]

(43)

where for any fixed \( b \), \( f_i(b, a_0, \ldots, a_i) \) is a polynomial in \( (a_0, \ldots, a_i) \) of degree \( \leq i \).

**Outer expansion**  We expand \( \lambda^\varepsilon(t) \) and \( u^\varepsilon(x, t) \) for \( |d(x, t)| \geq 3\delta \) by

\[
 \lambda^\varepsilon(t) \approx \lambda_0(t) + \varepsilon \lambda_1(t) + \varepsilon^2 \lambda_2(t) + \cdots 
\]

(44)

\[
 u^\varepsilon(x, t) \approx u_0^\varepsilon(t) := \pm 1 + \varepsilon \{u_0^\varepsilon(t) + \varepsilon u_1^\varepsilon(t) + \cdots\}.
\]

(45)

Substituting (44) and (45) into (6) gives

\[
 f(u_0^\varepsilon(t)) = \varepsilon \lambda^\varepsilon(t) + \varepsilon^2 (u_0^\varepsilon)'(t) 
\]

which yields for all \( i \geq 0, \)

\[
 u_i^\varepsilon(t) = \{\lambda_i - f_{i-1}(\pm 1, u_{i-1}^\varepsilon, \ldots, u_{i-2}^\varepsilon) - u_{i-2}^\varepsilon, \ldots, u_{i-1}^\varepsilon\} / f'(\pm 1)
\]

(46)

where \( f_{-1} = u_{-2}^\varepsilon = 0, u_{-1}^\varepsilon = \pm 1, \) and \( f_i \) \((i \geq 0)\) is as in (43). Hence, \( u_i^\varepsilon \) are determined by \( \{\lambda_0, \ldots, \lambda_i\} \).

**Inner expansion**  We shall assume that \( h^\varepsilon \) has the asymptotic expansion

\[
 \varepsilon h^\varepsilon(s, t) \approx \varepsilon h_1(s, t) + \varepsilon^2 h_2(s, t) + \cdots, \quad (s, t) \in U \times [0, T]
\]

(47)

Near the interface, we assume that the function \( \hat{u}^\varepsilon \) associated to \( u^\varepsilon \) by (34) has the asymptotic expansion

\[
 \hat{u}^\varepsilon(\rho, s, t) \approx \theta_0(\rho) + \varepsilon \{u_0(\rho, s, t) + \varepsilon u_1(\rho, s, t) + \cdots\}.
\]

(48)

In the sequel, the zero-th order expansion refers to

\[
 \{d(x, t), \lambda_0(t), u_0(\rho, s, t), u_0^\varepsilon(t)\}
\]

and the \( i \)-th order expansion refers to

\[
 \{h_i(s, t), \lambda_i(t), u_i(\rho, s, t), u_i^\varepsilon(t)\}.
\]

We shall use \((\cdots)_{i-1}\) to denote a generic function of \((\rho, s, t)\) depending only on expansions of order \( \leq i - 1 \).
Matching condition  We impose that for all $i \in \mathbb{N}$,
\[ \forall (s, t) \in U \times [0, T], \quad \lim_{\rho \to \pm \infty} u_i(\rho, s, t) = u_i^\pm(t) \quad (49) \]

Translation  We also impose for all $i \in \mathbb{N}$,
\[ \forall (s, t) \in U \times [0, T], \quad u_i(0, s, t) = 0 \quad (50) \]

5.2 The $u$-equation in the new variables

The equation (6) is
\[ -f(u) = -\varepsilon^2(u_t - \Delta u) - \varepsilon \lambda_\varepsilon(t). \]

In the new variables $(\rho, s, t)$, using (37), it becomes the following equation for the function $u = \hat{u}_\varepsilon$ associated to $u_\varepsilon$ by (34),
\[ -f(u) = u_{\rho\rho} + \varepsilon ([V(s, t) + \Delta d] u_{\rho} - \lambda_\varepsilon) \]
\[ + \varepsilon^2 [(\Delta^\Gamma u - \partial_t^\Gamma u) + (\partial_t^\Gamma h_\varepsilon - \Delta^\Gamma h_\varepsilon) u_{\rho}] \]
\[ + \varepsilon^2 [\|
abla^\Gamma h_\varepsilon\|^2 u_{\rho\rho} - 2\nabla^\Gamma h_\varepsilon \cdot \nabla^\Gamma u_{\rho}], \quad (51) \]

where $V(s, t)$ is given by (27) and $\Delta d$ is expanded from (36) as
\[ \Delta d \approx K(s, t) - \sum_{i \geq 1} \varepsilon^i [b(s, t) h_i(s, t) + \delta_{i-1}(\rho, s, t)], \quad (52) \]

with $\delta_{i-1}$ depending only on expansions of order $\leq i - 1$ (in particular, $\delta_0(\rho, s, t) = \rho b(s, t)$). Note that $\delta_{i-1}(\rho, s, t)$ is a polynomial in $\rho$ of degree $\leq i$, whose coefficients are polynomial in $(h_1, ..., h_{i-1})$ with $(s, t)$-dependent coefficients.

5.3 The recursive $i$-th equations

The zeroth order expansion  Since $\theta_0$ defined in (12) satisfies
\[ -f(\theta_0) = (\theta_0)_{\rho\rho}, \quad \theta_0(\pm \infty) = \pm 1, \quad \theta_0(0) = 0, \]
the equation (51) is satisfied at zeroth order as well as the matching and translation condition (49)-(50).
The first order expansion  At first order ($\varepsilon^1$), the equation (51) imposes
\[
\mathcal{L} u_0 = (K(s,t) + V(s,t))(\theta_0)'(\rho) - \lambda_0(t),
\]
with $\mathcal{L}$ defined in (15). The solvability condition stated in Lemma 3 reads
\[
(K(s,t) + V(s,t)) \int_{\mathbb{R}} (\theta_0')^2(z)dz = 2\lambda_0(t)
\]
which reads in view of (3.1)
\[
V(s,t) = -K(s,t) + \sigma \lambda_0(t)
\]
for $s \in U$. Note that for all non-negative $m, n, l,$
\[
D^m_\rho D^n_s D^l_t [u_0(\rho, s, t) - u_0^\pm(t)] = O(e^{-\alpha|\rho|}) \text{ as } \rho \to \pm \infty.
\]

Higher order expansion  Plugging the expansions (43), (47), (48) into (51) and using (54) and (52) leads to the following identity
\[
-f(\theta_0) - \varepsilon f'(\theta_0)(\sum_{i \geq 0} \varepsilon^i u_i) - \varepsilon^2 \sum_{i \geq 0} \varepsilon^i f_i(\theta_0, u_0, ..., u_i) = \theta_0'' + \varepsilon([\sigma \lambda_0(t) - \sum_{i \geq 1} \varepsilon^i (bh_i + \delta_{i-1})]u_0 - \sum_{i \geq 0} \varepsilon^i \lambda_i] (57)
\]
\[
+ \varepsilon^3 \sum_{i \geq 0} \varepsilon^i (\Delta^\Gamma - \partial^\Gamma_t) u_i - \varepsilon [\sum_{i \geq 1} \varepsilon^i (\Delta^\Gamma - \partial^\Gamma_i) h_i)(\theta_0' + \varepsilon \sum_{i \geq 0} \varepsilon^i (u_i)_{\rho}) (58)
\]
\[
+ [\varepsilon^2|\nabla^\Gamma h|^2 u_{\rho \rho} - 2\varepsilon(\sum_{i \geq 1} \varepsilon^i \nabla^\Gamma h_i).\nabla^\Gamma u_\rho]. (59)
\]
Define the operator $\mathcal{N}^T$ acting on functions $h = h(s, t)$ by
\[
\mathcal{N}^T h := (\partial^\Gamma h - \Delta^\Gamma h - bh) (60)
\]
We derive below the $(i + 1)$-th order expansion for $i \geq 1$ and obtain the following result.
Lemma 5. At order $\varepsilon^{i+1}$, with $i \geq 1$, the equation (51) imposes

$$\mathcal{L} u_i = \mathcal{N}^\Gamma(h_i)\theta'_0 - \lambda_i(t) + b_{12}(\nabla^\Gamma h_1, \nabla^\Gamma h_i)\theta''_0 + R_{i-1}(\rho, s, t),$$

with $R_{i-1}$ only depending on expansions of order $\leq i - 1$. Besides $R_{i-1}(\rho, s, t)$ is a polynomial in $\rho$ of degree $\leq i$ (whose coefficients are polynomial in $(h_1, ..., h_{i-1}, u_1, ..., u_{i-1})$ and in their derivatives with respect to $(\rho, s, t)$).

Proof. First note that using (54), the coefficient of order $\varepsilon^{i+1}$ in (57) is

$$(u_i)_{\rho\rho} + \sigma \lambda_i(t)(u_{i-1}) - b(s, t)h_i(s, t)\theta'_0 - \lambda_i(t) + (\cdots)_{i-1}$$

$$(u_i)_{\rho\rho} - bh_i\theta'_0 - \lambda_i(t) + (\cdots)_{i-1},$$

with $(\cdots)_{i-1}$ depending only on expansions of order $\leq i - 1$. Moreover in view of (52), it is a polynomial in $\rho$ of degree $\leq i$ (whose coefficients are polynomial in $(h_1, ..., h_{i-1}, u_1, ..., u_{i-1})$ and in their derivatives with respect to $(\rho, s, t)$).

Next, in view of (47), the coefficient of order $\varepsilon^{i+1}$ in (58) is

$$((\Delta^\Gamma - \partial_t^\Gamma)u_{i-2} + (\partial_t^\Gamma - \Delta^\Gamma)h_i\theta'_0 + (\cdots)_{i-2}$$

$$= (\partial_t^\Gamma - \Delta^\Gamma)h_i\theta'_0 + (\cdots)_{i-2}.$$

To obtain the term of order $\varepsilon^{i+1}$ in (59), note that

$$\varepsilon^2|\nabla^\Gamma h_i|^2 \approx [\varepsilon^2|\nabla^\Gamma h_1|^2 + \sum_{i \geq 3} \varepsilon^i(2\nabla^\Gamma h_1, \nabla^\Gamma h_{i-1} + (\cdots)_{i-2})][\theta''_0 + \varepsilon \sum_{i \geq 0} \varepsilon^i(u_i)_{\rho\rho}].$$

so that

$$\varepsilon^2|\nabla^\Gamma h_i|^2 \approx [\varepsilon^2|\nabla^\Gamma h_1|^2 + \sum_{i \geq 3} \varepsilon^i(2\nabla^\Gamma h_1, \nabla^\Gamma h_{i-1} + (\cdots)_{i-2})][\theta''_0 + \varepsilon \sum_{i \geq 0} \varepsilon^i(u_i)_{\rho\rho}].$$

Hence the coefficient of order $\varepsilon^{i+1}$ in $\varepsilon^2|\nabla^\Gamma h_i|^2 u_{\rho\rho}$ is

$$b_{1,2}(\nabla^\Gamma h_1, \nabla^\Gamma h_i)\theta''_0 + (\cdots)_{i-2}$$

with $b_{1,2} = 1$ or 2 for $i = 1$ or $i \geq 2$ respectively.
Similarly, the coefficient of order $\varepsilon^{i+1}$ in the term $\varepsilon^2 \nabla^t h_i, \nabla^t u_\rho$ is

$$\nabla^t h_{i-1} \cdot \nabla^t (u_0) \rho + \nabla^t h_i \cdot \nabla^t (u_1) \rho + \ldots + \nabla^t h_i \cdot \nabla^t (u_{i-2}) \rho$$

where the first term cancels out since $\nabla^t (u_0)' = 0$ in view of (56); thus it only depends on expansions of order $\leq i - 2$ and appears below in the remainder. Finally at order $\varepsilon^{i+1}$, with $i \geq 1$, the equation (51) reads

$$- f'(\theta_0) u_i - f_{i-1}(\theta_0, u_0, \ldots, u_{i-1}) = (u_i)_{\rho \rho} - \lambda_i(t)$$

$$+ (\partial_i^t h_i - \Delta^t h_i - bh_i) \theta_0'' + b_{12}(\nabla^t h_1, \nabla^t h_i) \theta_0'' + R_{i-1}(\rho, s, t),$$

with $R_{i-1}$ only depending on expansions of order $\leq i - 2$. Moreover $R_{i-1}(\rho, s, t)$ is a polynomial in $\rho$ of degree $\leq i$ as described in Lemma 5.

The solvability condition

According to Lemma 3, the equation (61) has a solution if and only if the following solvability condition is satisfied.

$$\forall (s, t) \in U \times [0, T], \quad \int [L(u_i)(\rho, s, t) \theta_0'(\rho)] d\rho = 0. \quad (64)$$

Note that

$$\int b_{12}(\nabla^t h_1, \nabla^t h_i) \theta_0''(\rho) \theta_0'(\rho) d\rho = b_{12}(\nabla^t h_1, \nabla^t h_i)(s, t) \int \frac{1}{2} [\theta_0''(\rho)]^2 d\rho = 0$$

so that the condition (64) reads

$$\mathcal{N}^t(h_i) = \sigma \lambda_i(t) + r_{i-1}(s, t), \quad (65)$$

with

$$r_{i-1}(s, t) = - \frac{\sigma}{2} \int R_{i-1}(\rho, s, t) \theta_0'(\rho) d\rho$$

only depending on expansions of order $\leq (i - 1)$. We summarize these results in the next lemma.

**Lemma 6** Let $k \geq 1$ be given. Assume that for all $i \leq k - 1$, (61) has a solution $u_i$ satisfying

$$D^m_\rho D^n_s D^l_i [u_i(\rho, s, t) - u_i^\pm(t)] = O(\rho^a e^{-a|\rho|}) \text{ as } \rho \to \pm\infty. \quad (66)$$

Also assume that for $i = k$, $\{h_i(s, t), \lambda_i(t)\}$ satisfies (62). Then for $i = k$, (61) admits a unique solution satisfying $u_i(0, s, t) = 0$ and (66).
The proof follows from Lemma 3 and an induction argument and is omitted. Just note that in the limit \( \rho \to \pm \infty \), the equation \( 0 = \varepsilon^2 (u_t^\varepsilon - \Delta u^\varepsilon) + f(u^\varepsilon) - \varepsilon \lambda^\varepsilon |_{x = \hat{X}(\rho, s, t)} \) becomes the outer expansion equation, so that \( u_i(\pm \infty, s, t) = u_i^\pm(t) \). Furthermore since \( R_i-1 \) is a polynomial in \( \rho \) of degree \( \leq i \), (66) is satisfied for each \( i \geq 0 \) and \((s, t) \in U \times [0, T]\).

### 5.4 Equation for \( \lambda^\varepsilon \).

To find \( \lambda^\varepsilon(t) \), we use an asymptotic expansion for \( 0 = \int_{\Omega} u_t^\varepsilon(x, t)dx \). We denote by \( \Omega^+_{\varepsilon}(t) \) the two domains separated by \( \hat{\gamma}^\varepsilon_i \) defined in (31), with \( \hat{\gamma}^\varepsilon_i = \partial \Omega^-_{\varepsilon}(t). \) Hence in view of (32)

\[
\Omega^+_{\varepsilon}(t) = \{ x \in \Omega \mid d(x, t) > 3\delta \} \cup \{ x \in V^\varepsilon_{3\delta} \mid [d(x, t) - \varepsilon h^\varepsilon(S(x, t), t) > 0 \}
\]

\[
= \{ x \in \Omega \mid d(x, t) > 3\delta \} \cup \{ x \in V^\varepsilon_{3\delta} \mid \rho^\varepsilon(x, t) > 0 \}
\]

and

\[
\Omega^-_{\varepsilon}(t) = \Omega \setminus \Omega^+_{\varepsilon}(t)
\]

\[
= \{ x \in \Omega \mid d(x, t) < -3\delta \} \cup \{ x \in V^\varepsilon_{3\delta} \mid \rho^\varepsilon(x, t) < 0 \}
\]

We write

\[
\int_{\Omega} u_t^\varepsilon(x, t)dx = \int_{[d(x, t) \geq 3\delta]} u_t^\varepsilon(x, t)dx + \int_{[d(x, t) < 3\delta]} u_t^\varepsilon(x, t)dx
\]

where

\[
\int_{[d(x, t) < 3\delta]} u_t^\varepsilon(x, t)dx = \int_{[\rho^\varepsilon(x, t) \geq \frac{\delta}{2}]} u_t^\varepsilon(x, t)dx + \int_{[\rho^\varepsilon(x, t) < \frac{\delta}{2}]} u_t^\varepsilon(x, t)dx
\]

In the sequel we choose \( 0 < \varepsilon \leq \varepsilon_0 \) small enough so that

\[
\forall \varepsilon \in (0, \varepsilon_0], \quad \max_{s \in U, t \in [0, T]} |\varepsilon h^\varepsilon(s, t)| \leq \frac{\delta}{2}
\]

Then it follows that

\[
|\rho^\varepsilon(x, t)| \geq \frac{\delta}{\varepsilon} \Rightarrow |d(x, t)| \geq \frac{\delta}{2}.
\]
Thus if \(|d(x,t)| \geq 3\delta \) or \(|\rho^\varepsilon(x,t)| \geq \frac{\delta}{\varepsilon} \), then \(|d(x,t)| \geq \frac{\delta}{2} \) so that at these points \((x,t)\),

\[
u^\varepsilon_i(x,t) \approx (u^\varepsilon_+)'(t) \chi_{\{d(x,t)>0\}} + (u^\varepsilon_-)'(t) \chi_{\{d(x,t)<0\}}
\]

(exponentially small terms of order \(O(e^{-\frac{\delta}{\varepsilon}})\) do not affect the asymptotic expansion in the \(\varepsilon\) power series). Therefore in view of (69)-(70)

\[
\int_\Omega u^\varepsilon_i(x,t) dx \approx \int_\Omega [(u^\varepsilon_+)'(t) \chi_{\{d(x,t)>0\}} + (u^\varepsilon_-)'(t) \chi_{\{d(x,t)<0\}}] dx 
+ \int_{|\rho^\varepsilon(x,t)|<\frac{\delta}{\varepsilon}} [u^\varepsilon_i - (u^\varepsilon_+)'(t) \chi_{\{d(x,t)>0\}} - (u^\varepsilon_-)'(t) \chi_{\{d(x,t)<0\}}] dx
\approx I_1 + \int_{|\rho^\varepsilon(x,t)|<\frac{\delta}{\varepsilon}} [u^\varepsilon_i - (u^\varepsilon_+)'(t) \chi_{\{d(x,t)>0\}} - (u^\varepsilon_-)'(t) \chi_{\{d(x,t)<0\}}] dx, \quad (73)
\]

where

\[
I_1 = (u^\varepsilon_+)'(t)|\Omega^+(t)| + (u^\varepsilon_-)'(t)|\Omega^-_+(t)|. \quad (74)
\]

In the second integral, we make the change of variables given in (33) and substitute the expression of \(u^\varepsilon_i\) in formula (35) to obtain

\[
\int_{|\rho|<\delta/\varepsilon} [u^\varepsilon_i - (u^\varepsilon_+)'(t) \chi_{\{d(x,t)>0\}} - (u^\varepsilon_-)'(t) \chi_{\{d(x,t)<0\}}] dx = \\
\int_{0<\rho<\delta/\varepsilon} \partial_t^F [u^\varepsilon_-(\rho,s,t) - (u^\varepsilon_+)'(t)] \epsilon J^\varepsilon(\rho,s,t) d\rho ds \\
+ \int_{-\delta/\varepsilon<\rho<0} \partial_t^F [u^\varepsilon_-(\rho,s,t) - (u^\varepsilon_-)'(t)] \epsilon J^\varepsilon(\rho,s,t) d\rho ds \\
+ \int_{|\rho|<\delta/\varepsilon} (-V\varepsilon^{-1} - \partial_t^F h^\varepsilon) \frac{\partial u^\varepsilon_i}{\partial \rho} \epsilon J^\varepsilon(\rho,s,t) d\rho ds \quad (75)
\]

Finally,

\[
\int_\Omega u^\varepsilon_i(x,t) dx \approx I_1 + I_2 + I_3,
\]

where

\[
I_2 = \int_{|\rho|<\delta/\varepsilon} \partial_t^F [u^\varepsilon_-(\rho,s,t) - (u^\varepsilon_+)'(t) \chi_{\{\rho>0\}} - (u^\varepsilon_-)'(t) \chi_{\{\rho<0\}}] \epsilon J^\varepsilon(\rho,s,t) d\rho ds \quad (76)
\]

and

\[
I_3 = \int_{|\rho|<\delta/\varepsilon} (-V\varepsilon^{-1} - \partial_t^F h^\varepsilon) \frac{\partial u^\varepsilon_i}{\partial \rho} \epsilon J^\varepsilon(\rho,s,t) d\rho ds. \quad (77)
\]
The calculation for $I_1$. The boundary of $\Omega^{-}(t)$ is $\tilde{\gamma}^\varepsilon_t$ which according to (31) is given in local coordinates $(r, s)$ by $r = \varepsilon h^\varepsilon(s, t)$. Therefore in view of (40), we have that

\[
|\Omega^{-}_\varepsilon(t)| = |\Omega^{-}(t)| + \int_U \int_0^{\varepsilon h^\varepsilon(s, t)} J(r, s, t) \, dr \, ds \\
\approx |\Omega^{-}(t)| + \sum_{i \geq 1} \varepsilon^i \{ \int_U h_i(s, t) \, ds + (...)_{i-1} \},
\]

where $(...)_{i-1}$ only depends on expansions of order $\leq i - 1$. Hence

\[
|\Omega^{-}_\varepsilon(t)| = |\Omega| - |\Omega^{-}_\varepsilon(t)| \\
\approx |\Omega^+(t)| - \sum_{i \geq 1} \varepsilon^i \{ \int_U h_i(s, t) \, ds + (...)_{i-1} \}.
\]

From the outer expansion,

\[
u_{\varepsilon,t}^\pm \approx \varepsilon \sum_{i \geq 0} \varepsilon^i (u_{\varepsilon,t}^\pm)'(t) \approx \sum_{i \geq 1} \varepsilon^i (u_{i-1}^\pm)'(t),
\]

with $(u_{i-1}^\pm)'(t)$ given by (46) and depending only on expansions of order $\leq i - 1$. Therefore

\[
I_1 = u_{\varepsilon,t}^+(t)|\Omega^+_\varepsilon(t)| + u_{\varepsilon,t}^-(t)|\Omega^-_\varepsilon(t)| \approx \sum_{i \geq 1} \varepsilon^i (\ldots)_{i-1}
\]

where $(\ldots)_{i-1}$ depends only on expansions of order $\leq i - 1$.

The calculation for $I_2$. Using the expression for $\partial_t^F \hat{u}^\varepsilon$ in formula (30) and (66), we compute

\[
\begin{align*}
\partial_t^F \left[ u^\varepsilon(\rho, s, t) - (u_+^\varepsilon)(t)\chi_{\{\rho>0\}} - (u_-^\varepsilon)(t)\chi_{\{\rho<0\}} \right] \\
\approx \varepsilon \sum_{i \geq 1} \varepsilon^i (\partial_t + \sum_{j=1}^{n-1} S_i^j \partial_{s_j})[u_i(\rho, s, t) - u_i^+(t)\chi_{\{\rho>0\}} - u_i^-(t)\chi_{\{\rho<0\}}]
\approx \varepsilon \sum_{i \geq 2} \varepsilon^i O(\rho^{-1} e^{-\alpha|\rho|})
\end{align*}
\]

with $O(\rho^{-1} e^{-\alpha|\rho|})$ depending only on expansions of order $\leq i - 1$. Therefore by definition of $I_2$ in (76),

\[
I_2 \approx \sum_{i \geq 3} \varepsilon^i (\ldots)_{i-2},
\]

23
where \((...)_{i-2}\) depends only on expansions of order \(\leq i - 2\).

The calculation for \(I_3\). Using the expansions

\[
\frac{\partial \hat{u}_\varepsilon}{\partial \rho} \approx \theta'_0 + \varepsilon \sum_{i \geq 0} \varepsilon^i \frac{\partial u_i}{\partial \rho},
\]

\[
(-V - \varepsilon \partial_t^F h_\varepsilon) = d_t(X_0(s,t), t) - \varepsilon \sum_{i \geq 1} \varepsilon^i \partial_t^F h_i
\]

and rewriting the expression of \(J^\varepsilon\) in (38) as

\[
J^\varepsilon(\rho, s, t) = \prod_{i=1}^{n-1} [1 + \varepsilon(\rho + h^\varepsilon(s,t))\kappa_i(s,t)]
\]

\[
\approx 1 + \Delta d(X_0(s,t), t)\varepsilon(\rho + h^\varepsilon(s,t)) + \sum_{i \geq 2} \varepsilon^i(...)_{i-1},
\]

with \((...)_{i-1}\) depending only on expansions of order \(\leq i - 1\), we obtain that

\[
(-V - \varepsilon \partial_t^F h_\varepsilon) \frac{\partial \hat{u}_\varepsilon}{\partial \rho} J^\varepsilon(\rho, s, t) \approx d_t(X_0(s,t), t)\theta'_0(\rho) + \varepsilon \sum_{i \geq 1} \varepsilon^i \theta'_0(\rho)(-\partial_t^F h_i + d_t(X_0(s,t), t)h_i\Delta d) + \varepsilon \sum_{i \geq 1} \varepsilon^i(...)_{i-1}
\]

so that

\[
I_3 \approx \int_U \int_{\mathbb{R}} \left\{ \theta'_0 d_t(s,t) + \sum_{i \geq 1} \varepsilon^i \theta'_0(\rho)(-\partial_t^F h_i + d_t(s,t)\Delta d(s,t)h_i) + (...)_{i-1} \right\} d\rho ds
\]

\[
\approx 2 \int_U d_t(s,t) ds + \sum_{i \geq 1} \varepsilon^i \left\{ 2 \int_U (-\partial_t^F h_i + (d^t\Delta d)h_i) ds + (...)_{i-1} \right\}.
\]

Finally, substituting \(d_t\) and \(\partial_t^F h_i\) by (55) and (59), and using \(\int_U \Delta^F h_i ds = 0\), we obtain

\[
\frac{1}{2} \int_{\Omega} u^\varepsilon_i \approx \int_U (\Delta d - \sigma \lambda_0) ds + \sum_{i \geq 1} \varepsilon^i \left\{ \int_U ([-b + d_t\Delta d)h_i - \sigma \lambda_i] ds + (...)_{i-1} \right\}
\]

Thus the condition \(\int_{\Omega} u^\varepsilon_i dx \approx 0\) is equivalent to

\[
\sigma \lambda_0(t) = \Delta d(\cdot, t), \quad (78)
\]

\[
\sigma \lambda_i(t) = -[b(\cdot, t) - d_t(\cdot, t)\Delta d(\cdot, t)]h_i(\cdot, t) + \Lambda_{i-1}(t), \quad i \geq 1 \quad (79)
\]
where $\Lambda_{i-1}(t)$ depends only on expansions of order $\leq i - 1$, and $\overline{\phi(\cdot)} := \frac{1}{|U|} \int_U \phi$, the average of $\phi$ over $\gamma_t$ parametrized by $U$. Hence, we obtain closed systems for $d, h_1, \cdots, h_i$, namely

$$d_t(s,t) = \Delta d(s,t) - \overline{\Delta d(s,t)}, \quad (80)$$

$$\partial_t h_i = \Delta h_i + b h_i - \overline{[b(\cdot,t) - d_t(\cdot,t)\Delta(\cdot,t)]h_i(\cdot,t)} + \Lambda_{i-1}(t) \quad (81)$$
on $U \times [0,T]$.

### 5.5 Construction of Expansions of Each Order

We can now use induction to construct each order of expansion as follows:

1) **Zeroth order.** Given a smooth initial interface $\gamma_0$. (80) is equivalent to the volume preserving mean curvature flow (5). By the result established in [13], there is a time $T > 0$ such that there is a unique smooth solution on a time interval $[0,T]$. Consequently, $\Gamma = \bigcup_{0 \leq t \leq T} (\gamma_t \times \{t\})$ and the modified distance function $d$ are well defined. Set $\lambda_0(t)$ as in (78), $u_0(\rho,s,t)$ as in (56) and $u_{0}^\pm(t) = \lambda_0/f'(\pm1)$ as in (46). We obtain the zeroth order expansion $\{d(x,t), \lambda_0(t), u_0(\rho,s,t), u_0^\pm(t)\}$.

2) **Higher order expansion.** Fix $i \geq 1$. Assume that all expansions of order $\leq i - 1$ are constructed. Then $\Lambda_{i-1}(\cdot)$ in (81) is known. Since $\gamma_t$ is a smooth hypersurface without boundary, it follows from standard parabolic PDE theory [18] that (81) admits a unique smooth solution (assuming an initial condition such as $h_i(\cdot,0) = 0$ on $U$ is given). Consequently, we can define $\lambda_i(t)$ as in (79), $u_i^\pm$ as in (46) and $u_i$ as the solution of (61) given by Lemma [5]. This gives the $i$-th order expansion $\{h_i(s,t), \lambda_i(t), u_i(\rho,s,t), u_i^\pm(t)\}$ and completes the induction.

### 5.6 Construction of the Approximate Solution

Now fix an arbitrary positive integer $k$. We construct an approximate solution $u_k^\varepsilon$ such that Lemma [2] can be applied.

Let $\delta > 0$ be a small fixed constant such that (i) $d(x,t)$ is smooth in a $3\delta$-neighborhood of $\Gamma$, and (ii) for each $t \in [0,T]$, $\gamma_t$ is at least $3\delta$ distance away
from $\partial \Omega$. We define
\[
\rho^\varepsilon_k(x, t) = \varepsilon^{-1}\{d(x, t) - \sum_{i=1}^{k+1} \varepsilon^i h_i(S(x, t), t)\},
\]
\[
u^\text{in}_{\varepsilon, k}(x, t) = \theta_0(\rho^\varepsilon_k) + \varepsilon \sum_{i=0}^{k+1} \varepsilon^i u_i(\rho^\varepsilon_k(x, t), S(x, t), t),
\]
\[
u^\text{out}_{\varepsilon, k, \pm}(t) = \pm 1 + \varepsilon \sum_{i=0}^{k+1} \varepsilon^i u^\pm_i(t).
\]
We note that $\rho^\varepsilon_k, \nu^\text{in}_{\varepsilon, k}$ are smooth in a $3\delta$ neighborhood of $\Gamma$.

Now let $\zeta(s) \in C^\infty(\mathbb{R})$ be a cut-off function (depending only on $\delta$) satisfying
\[
\zeta(s) = 1 \quad \text{if} \quad |s| \leq \delta, \quad \zeta(s) = 0 \quad \text{if} \quad |s| > 2\delta,
\]
\[
0 \leq s\zeta'(s) \leq 4 \quad \text{if} \quad \delta \leq |s| \leq 2\delta.
\]
We define the needed approximation solution $u^\varepsilon_k$ by
\[
\tilde{u}^\varepsilon_k(x, ., 0) := \zeta(d) u^\text{in}_{\varepsilon, k} + [1 - \zeta(d)] \{ u^\text{out}_{\varepsilon, k, +} \chi_{d>0} + u^\text{out}_{\varepsilon, k, -} \chi_{d<0} \},
\]
\[
u^\varepsilon_k(x, t) := \tilde{u}^\varepsilon_k(x, t) + \int \{ \tilde{u}^\varepsilon_k(., 0) - \tilde{u}^\varepsilon_k(., t) \}
\]
for all $(x, t) \in \bar{\Omega} \times [0, T]$. Then by construction $u^\varepsilon_k$ is an approximation of order $k$ as needed in Lemma 2. Here we just remark that (i) in the set $\{(x, t) \mid \delta \leq \pm d(x, t) \leq 2\delta\}$, the limiting behavior (66) guarantees that $u^\varepsilon_k(x, t) = u^\text{out}_{\varepsilon, k, \pm}(t) + O(e^{-\alpha\delta/(4\varepsilon)})$, valid also for differentiation, (ii) $\partial_n u^\varepsilon_k = 0$ on $\partial\Omega_T$ since $u^\varepsilon_k$ is a function of $t$ near $\partial\Omega_T$, and (iii) the correction
\[
\int \{ \tilde{u}^\varepsilon_k(., 0) - \tilde{u}^\varepsilon_k(., t) \} = - \int \int_{[0,T]} (\tilde{u}^\varepsilon_k)_t (y, \tau) \, d\tau \, dy
\]
is of order $O(\varepsilon^{k+1})$, valid also for differentiation.
This completes the construction of the approximating solution, and also the proof of Theorem 1.

Appendix A : Proof of Lemma 1
We first consider the case $n \geq 4$ so that $p = 4/n$. The Gagliardo-Nirenberg-Sobolev inequality (see [3], Theorem 2, p.265) states that there exists $C > 0$ such that for every $R \in H^1(\Omega)$,
\[
\|R\|_{L^{2^*}} \leq C\|R\|_{H^1},
\]
with \( 2^* = \frac{2n}{n-2} \). Using Poincaré-Wirtinger inequality (see [8], Theorem 1, p.275), it follows that there exists \( C > 0 \) such that for every \( R \in H^1(\Omega) \) with \( \int_{\Omega} R \, dx = 0 \),

\[
\| R \|_{L^{2^*}} \leq C \| \nabla R \|_{L^2}.
\]  

(82)

Writing Hölder inequality, we have that

\[
\| R \|_{L^{2^*+p}}^{2^*+p} = \int_{\Omega} |R|^2 |R|^p \leq (\int_{\Omega} |R|^{2\beta})^{1/\beta} (\int_{\Omega} |R|^{p\beta'})^{1/\beta'}
\]

and we choose

\[
\beta = \frac{n}{n-2} = \frac{2^*}{2}, \quad \beta' = \frac{n}{2}
\]

to obtain

\[
\| R \|_{L^{2^*+p}}^{2^*+p} \leq \| R \|_{L^{2^*}}^2 \| R \|_{L^2}^p.
\]

Combined with (82), this yields the inequality

\[
\| R \|_{L^{2^*+p}}^{2^*+p} \leq C \| R \|_{L^2}^p \| \nabla R \|_{L^2}^2,
\]

which is the conclusion of Lemma 1.

Next we consider the case that \( 1 \leq n \leq 3 \) so that \( p = 1 \). Schwarz’s inequality then gives that

\[
\| R \|_{L^3}^3 = \int_{\Omega} |R|^3 \leq \| R \|_{L^4}^2 \| R \|_{L^2}
\]

For \( n = 1, 2, 3 \), by Sobolev’s imbedding theorem, \( H^1 \subset L^4 \), so that there exists \( C > 0 \) such that for every \( R \in H^1(\Omega) \),

\[
\| R \|_{L^4} \leq C \| R \|_{H^1}.
\]

Using again Poincaré-Wirtinger inequality, we finally deduce that there exists \( C > 0 \) such that for every \( R \in H^1(\Omega) \) with \( \int_{\Omega} R \, dx = 0 \),

\[
\| R \|_{L^3}^3 \leq C \| \nabla R \|_{L^2}^2 \| R \|_{L^2},
\]

which concludes the proof of Lemma 1.
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