THE ALGEBRA OF THE BOX–SPLINE

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Abstract. In this paper we want to revisit results of Dahmen and
Micchelli, [13], [14], [15], which we reinterpret and make more precise.
We compare these ideas with the work of Brion, Szenes, Vergne and
others [5], [37], [38], [29], [35].

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The main purpose of this paper is to revisit a celebrated Theorem of Dahmen and Micchelli \cite{15} which we will state and prove again in a somewhat stronger form.

The theorem is on the following settings:

Start from a finite list $X := \{a_1, \ldots, a_N\}$ of non zero vectors $a_i \in \mathbb{R}^s$. 

1.1 Introduction
If \( X \) spans \( \mathbb{R}^s \), from \( X \) one builds an important function for numerical analysis, the box spline implicitly defined by the formula:

\[
\int_{\mathbb{R}^s} f(x) B_X(x) dx := \int_{0}^{1} \cdots \int_{0}^{1} f(\sum_{i=1}^{N} t_i a_i) dt_1 \cdots dt_N,
\]

where \( f(x) \) is any continuous function.

If \( 0 \) is not in the convex hull of the vectors \( a_i \) then one has a simpler function \( T_X(x) \), the multivariate spline cf \[18\] characterized by the formula:

\[
\int_{\mathbb{R}^s} f(x) T_X(x) dx = \int_{\mathbb{R}_+^N} f(\sum_{i=1}^{N} t_i a_i) dt.
\]

where \( f(x) \) has compact support.

Both \( B_X \) and \( T_X \) have a simple geometric interpretation as functions computing the volume of certain variable polytopes.

If furthermore the vectors \( a_i \) happen to be integral vectors (and \( 0 \) is not in their convex hull) one has a third function, now on \( \mathbb{Z}^n \), important for combinatorics, the partition function given by:

\[
P_X(v) := \#\{(n_1, \ldots, n_N) | \sum n_i a_i = v, n_i \in \mathbb{N}\}.
\]

One of the goals of the theory is to give computable closed formulas for all these functions and, at the same time, describe some of their qualitative behavior.

The main result of the theory is that, these three functions can be described in a combinatorial way as a finite sum over local pieces (see formulas \[15\] and \[16\]). In the case of \( B_X(x) \) and \( T_X(x) \) the local pieces span, together with their derivative, a finite dimensional space \( D(X) \) of polynomials. In the case of \( P_X(v) \) they span together with their translates, a finite dimensional space \( \nabla(X) \) of quasi polynomials (cf. Definition \[7.5\]).

The theorem we are referring to, characterizes:

- \( D(X) \) by differential equations.
- \( \nabla(X) \) by difference equations.

In particular Dahmen and Micchelli compute the dimension of both spaces. This dimension has a simple combinatorial interpretation in terms of \( X \). They also decompose \( \nabla(X) \) as a direct sum of natural spaces associated to certain special points \( P(X) \) in a suitable torus. \( D(X) \) is then the space associated to the identity.

Although this theorem originates from the theory of the box spline, nevertheless it has also an interest in commutative algebra and algebraic geometry, in particular in the theory of hyperplane arrangements and partition functions.
Here we have been inspired by the results of Orlik–Solomon on cohomology and the results of Brion, Szenes, Vergne on partition functions.

In fact a lot of work originated from the seminal paper of Khovanskiĭ, Pukhlikov [31], interpreting the counting formulas as Riemann–Roch formulas for toric varieties and of Jeffrey–Kirwan, [29] on moment maps. These topics are beyond this paper and we refer to Vergne’s survey article [38].

In the theory of hyperplane arrangements, \( X \) is viewed as a set of linear equations and the object of study is the arrangement of hyperplanes defined by these equations.

Due to the somewhat high distance between these two fields, people working in hyperplane arrangements do not seem to be fully aware of the results on the Box–spline.

On the other hand there are some methods which have been developed in this latter theory which we believe shed some light on the space of functions used to build the box spline. Therefore we feel that this presentation may be useful to make a bridge between the two theories. Thus this paper has been organized partly as a research paper and partly as a survey of some relevant aspects of these two theories.

We divide the discussion in three parts.

In the first on the differentiable case the main new results are determination of the graded dimension of the space \( D(X) \) (Theorem 6.2) in terms of the combinatorics of bases extracted from \( X \). An algorithmic characterization in terms of differential equations of a natural basis of the top degree part of \( D(X) \) (Proposition 22) from which one obtains explicit local expressions for \( T_X \) (Theorem 3.6). A duality between \( D(X) \) and a subspace of the space of polar parts relative hyperplane arrangement associated to \( X \) (Theorem 6.3). In this section we also give a simple proof of the Dahmen-Micchelli theorem on the dimension on \( D(X) \) using elementary commutative algebra (Theorem 4.6).

The second on the discrete case, contains various extensions of the results of the first part in the case in which the elements in \( X \) lie in a lattice. We develop a general approach to linear difference equations and give a method of reduction to the differentiable case (Section 9). We also give an explicit formula relating partition functions to multivariate splines (Theorem 9.22). We end with a quick survey of some of the standard applications as for example can be found in the book Box splines [18].

In the third part we explain the approach via residues (see Theorem 11.11 and Formula 60). We finish giving an overview of relations with wonderful embeddings.

**Warning:** the reader will see that we use Fourier transforms as an essentially algebraic tool.
The fact is that, for the purpose of this theory, Fourier transform is essentially a duality between polynomials and differential operators with constant coefficients.

As long as one has to perform only algebraic manipulations one can avoid difficult issues of convergence and various forms of algebraic or geometric (residues) duality is sufficient.

Thus we usually work with Laplace transforms which avoids cluttering the notations with unnecessary i’s.

Our conventions for Laplace transforms are the following. We fix a vector space $V$ and set $U := V^*$. We fix a Eucliden structure on $V$ which induces Lebesgue measures $dv, du$ on $V, U$ and all their linear subspaces. We set

$$L f(u) := \int_V e^{-\langle u | v \rangle} f(v) dv.$$  

$L$ maps functions on $V$ to functions on $U$. We have the basic properties, when $p \in U, w \in V$, writing $p, D_w$ for the linear function $\langle p | v \rangle$ and the directional derivative on $V$, (and dually on $U$):

$$L(D_w f)(u) = w Lf(u), \quad L(pf)(u) = -D_p Lf(u),$$

$$L(e^p f)(u) = Lf(u - p), \quad L(f(v + w))(u) = e^w Lf(u).$$

**Part 1. The differentiable case.**

**2. Basic definitions**

It is convenient to take a somewhat intrinsic and base free approach to our problems.

Let us fix an $s$-dimensional vector space $U$, let us denote by $V$ its dual and fix a list $X := \{a_1, \ldots, a_N\}$ of non zero elements in $V$ (we allow repetitions in the list as this is important for the applications). We identify the symmetric algebra $S[V]$ with the ring of polynomial functions on $U$ and sometimes denote it by $A$.

This algebra can also be viewed as the algebra of polynomial differential operators with constant coefficients on $V$. Similarly $S[U]$ is the ring of polynomial functions on $V$, or polynomial differential operators with constant coefficients on $U$.

Given a vector $v \in V$ we denote by $D_v$ the corresponding directional derivative. This is algebraically characterized, on $S[U]$, as the derivation which on an element $\phi \in U$ takes the value $\langle \phi | v \rangle$.

One can organize all these facts in the algebraic language of Fourier transform. Let $W(V), W(U)$ denote the two algebras of differential operators with
polynomial coefficients on $V$ and $U$ respectively. Notice that, from a purely algebraic point of view they are both generated by $V \oplus U$.

In the first case $V$ is thought of as the space of directional derivatives and then we write $D_v$ instead of $v$, and $U$ as the linear functions. In $W(U)$ the two roles are exchanged.

The relevant commutation relations are thus:

\[
[D_v, \phi] = \langle \phi | v \rangle, \quad [D_\phi, v] = \langle v | \phi \rangle.
\]

Thus we see that we have a canonical isomorphism of algebras:

\[
\mathcal{F} : W(V) \to W(U), \quad D_v \mapsto -v, \quad \phi \mapsto D_\phi.
\]

One usually writes $\hat{a}$ instead of $\mathcal{F}(a)$.

This allows us, given a module $M$ over $W(V)$, to consider its Fourier transform $\hat{M}$ as a module over $W(U)$ by $a.m := \hat{a}m$ and conversely.

2.1. Cocircuits. We come to the first basic definition of combinatorial nature. The importance of this notion will be clear once we start to study the multivariate spline. We assume that $X$ spans $V$.

**Definition 2.2.** We say that a sublist $Y \subset X$ is a cocircuit, if the elements in $X - Y$ do not span $V$.

The minimal cocircuits can thus be obtained as follows:

Fix a hyperplane $H \subset V$ spanned by elements in $X$ and consider $Y := \{x \in X | x /\in H\}$, it is immediately verified that this is a cocircuit and every cocircuit contains one of this type.

Sometimes we shall express the fact, that a sublist $Z \subset X$ consists of all the vectors in $X$ lying in a given subspace, by saying that $Z$ is complete.

Thus a minimal cocircuit is obtained by removing from $X$ a complete set spanning a hyperplane.

The set of all cocircuits will be denoted by $\mathcal{E}(X)$.

2.3. No broken circuits. The second basic combinatorial notion has been used extensively in the theory of hyperplane arrangements (cf. [11], [10], [11], [12]).

Let $\underline{a} := a_{i_1}, \ldots, a_{i_k} \in X$, $i_1 < i_2 \cdot \cdot \cdot < i_k$, be a sublist of linearly independent elements.

**Definition 2.4.** We say that $a_i$ breaks $\underline{a}$ if there is an index $1 \leq e \leq k$ such that:

- $i \leq i_e$.
- $a_i$ is linearly dependent on $a_{i_e}, \ldots, a_{i_k}$.

In particular, given any basis $\underline{b} := a_{i_1}, \ldots, a_{i_k}$ extracted from $X$, we set:

$B(\underline{b}) := \{a \in X | a \ breaks \ \underline{b}\}$ and $n(\underline{b}) = |B(\underline{b})|$ the cardinality of $B(\underline{b})$.

**Definition 2.5.** We say that $\underline{b}$ is no broken if $B(\underline{b}) = \underline{b}$ or $n(\underline{b}) = s$. 
Let us denote by $B(X)$ the set of all bases extracted from $X$. We shall consider the map $b \mapsto n(b)$ as a statistic on $B(X)$.

2.6. The box spline. Let us recall some points which are standard using the form presented in [2] or [20].

First let us recall some basic facts on splines (cf. [18]). Let $C(X) := \{ \sum_{a \in X} t_a a | 0 \leq t_a, \forall a \}$ be the cone of linear combinations of vectors in $X$ with positive coefficients.

We will assume that 0 is not in the convex hull of the vectors in $X$, i.e. that $C(X)$ does not contain lines.

We have already defined, in the introduction, the two basic functions on $V$, $B_X$ (formula (1)), and $T_X$, (formula (2)).

It is best to think of both $T_X$ and $B_X$ as tempered distributions (cf. [39]). Then the definition is valid also if $X$ does not span $V$.

$B_X$ is supported in the box

$$B(X) := \sum_{i=1}^{N} t_i a_i, \ 0 \leq t_i \leq 1, \text{ the shadow of the cube } [0,1]^N,$$

generated by $X$, $T_X$ is supported in $C(X)$.

**Basic example** Let $X = \{a_1, \ldots, a_s\}$ be a basis, $d := |\det(a_1, \ldots, a_s)|$:

$B(X)$ is the parallelepiped with edges the $a_i$, $C(X)$ is the positive quadrant generated by $X$.

$$B_X = d^{-1} \chi_{B(X)}, \quad T_X = d^{-1} \chi_{C(X)}$$

where, for any given set $A$, we denote by $\chi_A$ its characteristic function.

If $X = \{a_1, \ldots, a_k\}$, $k < s$ is only a linearly independent set, we have to consider $T_X$ and $B_X$ as measures on the subspace spanned by $X$.

$T_X$ and $B_X$ are functions as soon as $X$ spans $V$, i.e. when the support of the distribution has maximal dimension.

These functions have a nice geometric interpretation.

Let $F : \mathbb{R}^N \rightarrow V$ be defined by $F(t_1, \ldots, t_N) := \sum_{i=1}^{N} t_i a_i$. Then $B_X(w)$ is the volume of the polytope $F^{-1}(w) \cap [0,1]^N$ while $T_X(w)$ is the volume of the polytope $F^{-1}(w) \cap [0, \infty)^N$ (with a suitable normalization constant).

It is useful to generalize these notions, introducing a parametric version called $E-$splines.

Fix parameters $\mu := \{\mu_1, \ldots, \mu_N\}$ and define the functions (or tempered distributions) on $V$ by the implicit formulas:

$$\int_V f(x) B_{X,\mu}(x) dx := \int_0^1 \ldots \int_0^1 e^{-\sum_{i=1}^{N} t_i \mu_i} f(\sum_{i=1}^{N} t_i a_i) dt_1 \ldots dt_N.$$
Also these functions have a nice geometric interpretation. They represent the integral of \( e^{-\sum_{i=1}^{N} t_i \mu_i} \) on the polytope \( F^{-1}(w) \cap [0,1]^N \) or \( F^{-1}(w) \cap [0,\infty]^N \) (with the same normalization constant). Of course for \( \mu = 0 \) we recover the previous definitions.

An easy computation gives their Laplace transforms:

\[
\int_V e^{-\langle x,y \rangle} B_X(\mu)(x) \, dy = \int_0^1 \ldots \int_0^1 e^{-\sum_{i=1}^{N} t_i (\langle x,a_i \rangle + \mu_i)} \, dt_1 \ldots dt_N = \prod_{a \in X} \frac{1 - e^{-a - \mu}}{a + \mu}.
\]

and

\[
\int_V e^{-\langle x,y \rangle} T_X(\mu)(x) \, dy = \int_0^\infty \ldots \int_0^\infty e^{-\sum_{i=1}^{N} t_i (\langle x,a_i \rangle + \mu_i)} \, dt_1 \ldots dt_N = \prod_{a \in X} \frac{1}{a + \mu}.
\]

We have written shortly \( a := \langle x,a \rangle \), for the linear function on \( U \).

The use of Laplace rather than Fourier transforms is justified by the following discussion.

Define the dual cone \( \hat{C}(X) \) of \( C(X) \).

\[
\hat{C}(X) := \{ u \in U \mid \langle u \mid v \rangle \geq 0, \forall v \in C(X) \}.
\]

This cone consists thus of the linear forms that are non negative on \( C(X) \). Its interior in not empty since \( C(X) \) contains no lines.

**Proposition 2.7.** If \( T \) is a tempered distribution supported in \( C(X) \) its Fourier transform is an analytic function, of a complex variable \( z = x + iy \), \( x \in \hat{C}(X), y \in U \), on the open set where the real part \( x \) lies in the interior \( \hat{C}(X)^0 \) of \( \hat{C}(X) \).

**Proof.** This depends on the fact that, if \( u \in \hat{C}(X)^0 \) we have that \( e^{-\langle u \mid v \rangle} \) has exponential decay on \( C(X) \). \( \square \)

In fact for the \( T_X \) and all the distributions that we shall encounter we will have that their Fourier transform are not only defined in the region where the real part \( x \) lies in \( \hat{C}(X)^0 \), but in fact they extend to meromorphic functions with poles on the hyperplanes \( a_i = 0 \) (or sometimes translates of these hyperplanes).

In the course of this paper we give an idea of the general algebraic calculus involving these distributions. Under Laplace transform one can reinterpret
the calculus in terms of the structure of certain algebras of rational functions (or exponentials) as $D-$modules.

Given $a \in V$, $\mu \in \mathbb{C}$ let us introduce the notation, which will be discussed more deeply in Part 2:

$$\nabla_{\mu} f(x) := f(x) - e^{-\mu} f(x - a).$$

From the expressions of the Laplace transforms one gets that, the box spline can be obtained from the multivariate spline by a simple combinatorial formula.

**Proposition 2.8.** For every subset $S \subset X$ we set $a_S := \sum_{a \in S} a$, and $\mu_S := \sum_{a \in S} \mu_a$ then:

$$B_{X,\mu}(x) = \prod_{a \in X} \nabla_{\mu_a} T_{X,\mu}(x) = \sum_{S \subset X} (-1)^{|S|} e^{-\mu_S} T_{X,\mu}(x - a_S).$$

**Proof.** It follows from the basic rule (5) which gives the commutation relation between the Laplace transform and translations. $\square$

2.9. **The space $D(X)$.** The following definition is of central importance in the work of Dahmen and Micchelli and in this paper

**Definition 2.10.** We define the space $D(X)$ by the condition:

$$D(X) := \{ p \mid D_Y p = 0, \forall Y \in \mathcal{E}(X), \text{ the cocircuits} \}.$$  

In this definition we shall assume that $p$ is a polynomial. In fact, due to the property that the ideal generated by the elements $D_Y$ contains all large enough products of derivatives (see 4.4), one can easily see by induction that any distribution $p$ satisfying (12), is already a polynomial (cf. 26).

We will see later a generalization with parameters $\mu$ of these equations and also a discrete analogue, using difference operators rather than derivatives.

Let $m$ be the minimum number of elements in a cocircuit in $X$, assume $m \geq 2$, $m$ is characterized by the property that the basic space $D(X)$ contains all polynomials of degree $< m$. We shall see in Part 4, that $m$ controls also the smoothness of the splines that we have introduced.

2.11. **Two basic modules in correspondence.** The theory of the Laplace transform tells us how to transform some basic manipulations on distributions as an algebraic calculus. In our setting this is best seen introducing the following two $D-$modules in Fourier duality:

The first is the $D-$module $\mathcal{D}_X := W(V)T_X$ generated, in the space of tempered distributions, by $T_X$ under the action of the algebra $W(V)$ of differential operators on $V$ with polynomial coefficients (for basic facts cf. 39).

The second $D-$module is the algebra $\mathcal{R}_X := S[V][\prod_{a \in X} a^{-1}]$ obtained from the polynomials on $U$ by inverting the element $d_X := \prod_{a \in X} a$. This
is a module under $W(U)$ and it is the coordinate ring of the open set $\mathcal{A}_X$ complement of the union of the hyperplanes of $U$ of equations $a = 0$, $a \in X$.

**Theorem 2.12.** Under Laplace transform, $\mathcal{D}_X$ is mapped isomorphically onto $R_X$. In other words we get a canonical isomorphism of $\hat{\mathcal{D}}_X$ with $R_X$ as $W(U)$-modules.

$\mathcal{D}_X$ is the space of tempered distributions which are linear combinations of polynomial functions on the cones $C(A)$, $A \subset X$ a linearly independent subset and their distributional derivatives.

**Proof.** The injectivity of the Laplace transform on $\mathcal{D}_X$ is a standard fact [31].

To see the surjectivity, notice that under the Laplace transform, by definition and formula (10), the image of $\mathcal{D}_X$ under the Laplace transform is the smallest $\mathcal{D}$-module containing $\partial^{-1}X$. Since $R_X$ contains $\partial^{-1}X$ it suffices to see that $\partial^{-1}X$ generates $R_X$ as a $\mathcal{D}$-module. To see this first notice that if we take linearly dependent vectors $a_0 = \sum_{i=1}^{k} \alpha_i a_i$, in $X$, we can write:

$$\frac{1}{\prod_{i=0}^{k} a_i} = \frac{a_0}{a_0^2 \prod_{i=1}^{k} a_i} = \sum_{i=1}^{k} \frac{\alpha_i}{a_0^2 \prod_{j \neq i, j=1}^{k} a_i}$$

So, repeating this algorithm, we see that we can write a fraction $1/(\prod_i a_i^{h_i})$ as a linear combination of fractions where the elements appearing in the denominator are linearly independent. In particular $R_X$ is spanned by the functions of the form $f/(\prod_i b_i^{k_i+1})$, with $f$ a polynomial, $B = \{b_1, \ldots b_t\} \subset X$ a linearly independent subset and $k_i \geq 0$ for each $i = 1, \ldots, t$.

Now it is clear that $f/(\prod_i b_i^{k_i+1})$ lies in the $\mathcal{D}$-module generated by $1/(\prod_i b_i)$ and

$$\frac{1}{\prod_i b_i} = \frac{\prod_{a \in X-B} a}{d_X}$$

So, our first claim follows.

The second follows from our previous discussion once we remark that if $B = \{b_1, \ldots b_t\} \subset X$ is linearly independent, $1/(\prod_i b_i)$ is the Laplace transform of the measure on $V$ which is the push forward of the measure on the subspace spanned by the vectors in $B$ given by $\chi_{C(B)} \mu_B$ where $\chi_{C(B)}$ is the characteristic function of the cone $C(B)$ and $\mu_B$ is the Lebesgue measure normalized so that the parallelepiped with edges the $b_i$ has volume 1. □

Let us now introduce a filtration in $R_X$ by $D-$submodules which we will call the filtration by polar order.

This is defined algebraically as follows. One puts in filtration degree $\leq k$ all the fractions $f \prod_{a \in X} a^{-h_a}$, $h_a \geq 0$ for which the set of vectors $a$, with $h_a > 0$, spans a space of dimension $\leq k$. Denote this part of the filtration by $R_{X,k}$. Notice that by our proof of Theorem 2.12, $R_{X,s} = R_X$. 
From the proof of Theorem 2.12 one deduces that the corresponding (under inverse Laplace transform) filtration on $D_X$ can be described geometrically as follows.

We cover $C(X)$ by the positive cones $C(A)$ spanned by linearly independent subsets of $A \subset X$. We define $C(X)_k$ to be the $k$–dimensional skeleton of the induced stratification, a union of $k$–dimensional cones.

**Proposition 2.13.** $D_{X,k}$ consists of the tempered distributions in $D_X$ whose support is contained in $C(X)_k$.

2.14. **Polar parts.** Proposition 2.13 implies that the space $D_{X,s-1}$ is formed by all the distributions in $D_X$ which vanish once computed on test functions with support in the set of regular points $C(X)^0 := C(X) - C(X)_{s-1}$.

In other words we may identify $D_X/D_{X,s-1}$ with a space of distributions on the open set $C(X)^0$.

By duality we then see that the top subquotient, $R_X/R_{X,s-1}$, has a special importance in the theory [4]. Let us define it in a much more general setting which will be useful later.

Let $M$ be an analytic manifold. $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_m$ be a divisor union of the smooth irreducible components $Z_i$, $i = 1, \ldots, m$. $p$ an isolated point in $Z_1 \cap Z_2 \cap \cdots \cap Z_m$ with the property that there exist local coordinates around $p$ so that the $Z_i$ are locally given each by a linear equation $a_i = 0$.

A subset of the set of divisors $Z_i$ is said to be non spanning in $p$, if $p$ is not an isolated point of their intersection. We then define:

**Definition 2.15.** Define the module of polar parts in $p$, with respect to $Z$, and denote it by $P_{Z,p}$ (or $P_Z$ if $p$ is clear from the context) as the quotient of the space of germs in $p$ of meromorphic functions with poles in $Z$ modulo those function whose polar part is supported in a non spanning subset.

$P_{Z,p}$ is clearly a module over the ring $Z_p$ of germs of differential operators with holomorphic coefficients around $p$. In the local coordinates $x_1, \ldots, x_s$ around $p$, where $Z_i$ has an equation $a_i = 0$, $a_i$ linear this space can then be identified with the module $P_X := R_X/R_{X,s-1}$ with $X = \{a_1, \ldots, a_m\}$.

Notice the under this identification the operators induced from $Z_p$ coincide with the algebra of operators $W(U)$ where $U = T_p X$.

2.16. **Basic modules.** All the modules over Weyl algebras which appear are built out of some basic irreducible modules, in the sense that they have finite composition series in which these basic modules appear. It is thus useful to quickly discuss these basic modules. Let us take $W(V)$, the differential operators on $V$. The most basic module on $W(V)$ is the polynomial ring $S[U]$. It is the cyclic module generated by $1$ and the annihilator ideal of $1$ is generated by all the derivatives $D_v$, $v \in V$. Its Fourier transform can be identified with the module of distributions supported in $0 \in U$, a cyclic module generated by the Dirac distribution $\delta_0$. 
Given any point \( p \in U \), consider the 1-dimensional \( S[V] \) module \( C_p \) given by evaluation at \( p \) and the induced \( W(U) \) module \( N_p := W(U) \otimes_{S[V]} C_p \). \( N_p \) is easily seen to be irreducible and free of rank 1 as \( S[U] \) module.

In fact, in the language of distributions \( N_p \) is identified to the \( W(V) \) submodule generated by the Dirac distribution \( \delta_p \).

We shall need the following (easy) and standard fact ([12]).

Lemma 2.17. 1) Given a \( W(U) \) module \( M \) and a nonzero element \( u \in M \), if \( f_v = f(p)v, \forall f \in S[V] \), then \( u \) generates a submodule isomorphic to \( N_p \).

2) Given linearly independent vectors \( v_i \) satisfying the previous hypotheses, the submodules that they generate form a direct sum.

3) The module \( N_p \) has as characteristic variety the cotangent space at \( p \).

The only use we make of part 3) is its consequence that, for distinct points \( p, q \), the corresponding modules \( N_p, N_q \) are not isomorphic.

Also for a linear subspace \( A \subset V \) or a translate \( A + v \) one can define the irreducible module \( N_A \) generated by the Dirac distribution \( \delta_A \) given by:

\[
\langle \delta_A | f \rangle := \int_A f(w)dw
\]

The annihilator of \( \delta_A \) is generated by the elements \( u \in A^\perp \subset U \) vanishing on \( A \) (for \( A + v \) we have the elements \( u - u(v), u \in A^\perp \)) and the elements \( D_x, x \in A \). the fact that \( N_A \) is irreducible and the previous elements generate the annihilator of \( \delta_A \) can either be verified directly or by remarking that \( N_A \) can be obtained by twisting the polynomial ring by an automorphism (a partial Fourier transform) of \( W(V) \) defined as follows. One decomposes \( V = A \oplus Z \) for some complement \( Z \), then one has that \( W(V) = W(A) \otimes W(Z) \) and apply Fourier transform to the factor \( W(Z) \).

In explicit coordinates \( x_i \) where \( A := \{ x_i = 0, i \leq k \} \) we have that the partial Fourier transform is the identity on the variables \( x_j \) and their derivatives when \( j > k \), while

\[
x_i \mapsto \frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial x_i} \mapsto -x_i, \quad \forall i \leq k.
\]

As the polynomial ring is a free rank 1 module over the polynomials we see that \( N_A \) is thus a free rank 1 module over the partial Fourier transform of the polynomials, that is the polynomial algebra \( P_A \) in the variables \( \frac{\partial}{\partial x_i}, \forall i \leq k, \quad x_i \forall i > k \).

The module \( N_A \) appears naturally in our theory as follows. Take a basis \( \underline{c} := \{ a_1, \ldots, a_k \} \) of \( A \) and consider the open cone \( C(\underline{c}) \) generated by \( \underline{c} \). Consider next the distribution \( \delta_{C(\underline{c})} \) given by integration on this cone under a translation invariant Lebesgue measure for which the parallelepiped generated by \( \{ a_1, \ldots, a_k \} \) has volume 1. Then one easily sees that the Laplace transform of \( \delta_{C(\underline{c})} \) is \( \prod_{i=1}^k a_i^{-1} \) and \( N_A \) appears as the module of polar parts for the polynomial ring with the \( a_i \) inverted.
In particular fix some set of linear coordinates $a_1, \ldots, a_s$ for $U$. If for every subset $S \subset \{1, \ldots, s\}$ we set $A_S$ to be the subspace where the variables $a_i = 0$, $\forall i \in S$ we see, with the previous notations, that the ring of Laurent polynomials decomposes as a direct sum

$$\mathbb{C}[a_1^{\pm 1}, \ldots, a_s^{\pm 1}] = \oplus_{S \subset \{1, \ldots, s\}} P_{A_S}(\prod_{i \in S} a_i).$$

Notice that $P_{A_S}(\prod_{i \in S} a_i)$ has as basis, the monomials $\prod a_{k_i}$ with $k_i < 0$, $\forall i \in S$ and $k_i \geq 0$, $\forall i \not\in S$.

This decomposition, together with the description of $A_S$ as a polynomial algebra, gives the explicit partial fraction decomposition for Laurent polynomials, which is in any case elementary.

Following the same ideas one can develop the general partial fraction decomposition in the case in which we invert any set of linear equations as we shall see in Proposition 3.2 and Remark 3.4.

3. The function $T_X$

3.1. Local expansion. Let us extend the previous ideas to the parametric case. We take a list $\mu := \{\mu_a | a \in X\}$ of complex numbers. We introduce the ring $R_{X, \mu} := S[V|\prod_{a \in X}(a + \mu_a)^{-1}]$. It is clear that we can introduce on this algebra a filtration completely analogous to that of $R_X = R_{X, 0}$.

To a basis $b := \{b_1, \ldots, b_s\}$, from $X$, we associate the unique point $p_b \in U$ such that $b_i(p_b) = -\mu_b$ for each $i = 1, \ldots, s$. The set $P(X, \mu)$ consisting of the points $p_b$ as $b$ varies among the bases extracted from $X$ is called the set of points of the arrangement.

For generic $\mu$ all these points are distinct, while for $\mu = 0$ they all coincide with 0. In the other cases we may have various ways in which the points $P_b$ will coincide, depending on the compatibility relations among the parameters $\mu$.

Given $p \in P(X, \mu)$ we define the subset

$$X_p := \{a \in X | a(p) + \mu_a = 0\} \quad (X_p = \cup_{\mu_a = p(b)} b)$$

of the elements in $X$ such that the affine hyperplane $H_a$ of equation $a(p) + \mu_a = 0$ contains $p$. It is clear by definition that, if we restrict to the subset $X_p$, the points of this restricted arrangement reduce to $p$. Moreover a change of variables $a' := a + \mu_a$ corresponding to a translation, centers the arrangement in 0.

The divisor $Z := \cup_{a \in X} H_a$ satisfies the hypotheses of section 2.14 thus we can construct, for each point $p \in P(X, \mu)$ the corresponding module of polar parts, which we denote by $P_{Z, p}$. In order to understand these modules we need a preliminary construction.

The following Proposition allows us to reduce the computation of $T_{X, \mu}$ to that of the various $T_{X_p}$, for $p \in P(X, \mu)$. Let us denote, for a no broken basis
\( b \) by \( u_b \) the class of the element \( \prod_{a \in b} (a + \mu_a)^{-1} \) in the quotient \( R_X / R_{X,s-1} \).

Since, given \( a_0 \in b \) we clearly have that \( (a_0 + \mu_{a_0}) \prod_{a \in b} (a + \mu_a)^{-1} \in R_{X,s-1} \) we verify easily that:

\[
\text{if } u_b = f(p) u_b, \ \forall f \in \mathcal{S}.
\]

It follows that, unless \( u_b = 0 \) (which is not the case), \( u_b \) generates a submodule isomorphic to \( N_p \). In fact one has:

**Proposition 3.2.** 1) The mapping \( R_X / R_{X,s-1} \to \bigoplus_{p \in P(X, \mu)} P_{Z,p} \) is an isomorphism of \( W(U) \) modules.

2) Each \( P_{Z,p} \) is an isotypic component.

3) Each \( P_{Z,p} \) is the direct sum of as many copies of \( N_p \) as the no broken bases \( b \) in \( X_p \), each generated by a corresponding element \( u_b \).

4) \( T_{X,b} = \sum_{p \in P(X, \mu)} c_p e^p T_{X,p} \), with \( c_p \) some explicitly computable constants.

**Proof.** We can use a slightly more precise reduction. Assume we have linearly dependent vectors \( a_0 = \sum_{i=1}^k \alpha_i a_i \), in \( X \). If \( \nu := \mu_{a_0} - \sum_{i=1}^k \alpha_i \mu_{a_i} \neq 0 \) we write:

\[
\frac{1}{\prod_{i=0}^k (a_i + \mu_{a_i})} = \nu^{-1} \frac{a_0 + \mu_{a_0} - \sum_{i=1}^k \alpha_i (a_i + \mu_{a_i})}{\prod_{i=0}^k (a_i + \mu_{a_i})}
\]

and then, develop into a sum of \( k + 1 \) terms in each of which one of the elements \( a_i + \mu_{a_i} \) has disappeared. This allows us to separate the denominators with respect to the points in \( P(X, \mu) \).

If \( \nu = 0 \) we can write

\[
\frac{1}{\prod_{i=0}^k (a_i + \mu_{a_i})} = \frac{\sum_{i=1}^k \alpha_i (a_i + \mu_{a_i})}{(a_0 + \mu_{a_0}) \prod_{i=0}^k (a_i + \mu_{a_i})}.
\]

We expand and then simplify the numerators with the denominators and obtain that the element \( \prod_{i=0}^k (a_i + \mu_{a_i})^{-1} \) can be expanded as a linear combination of elements of type \( \prod_{a_i \in b} (a_i + \mu_{a_i})^{-h_i}, h_i \geq 0 \) with \( b \) a basis. Only the elements with all the \( h_i > 0 \) give non zero classes in the module of polar parts, moreover when \( h_i > 0 \) from the rules of derivatives \( \prod_{a_i \in b} (a_i + \mu_{a_i})^{-h_i} \) is obtained by applying a monomial in the derivatives to \( \prod_{a_i \in b} (a_i + \mu_{a_i})^{-1} \). This shows that the map is onto. The modules \( P_{Z,p} \) are contained in \( R_X / R_{X,s-1} \) and belong to distinct isotypic components by the previous lemma on the characteristic variety, hence the direct sum of \( R_X / R_{X,s-1} \).

A similar discussion allows us to reduce the elements appearing in the denominator to no broken sets. Thus the final point to verify, applying part 2) of Lemma 2.17 is that the elements \( u_b \) are linearly independent in \( P_{Z,p} \). This we shall show as a consequence of the theory of residues (Theorem 11.7), completing the proof of 1) and 2).

By 11) we see that \( \prod_{a \in X_p} (a - a(p))^{-1} \) is the Laplace transform of \( e^p T_{X,p} \).
Thus we are reduced to showing that, for suitable constants \( c_p \) we have:

\[
\prod_{a \in X} \frac{1}{a + \mu_a} = \sum_{p \in P(X, \mu)} c_p \prod_{a \in X_p} \frac{1}{a + \mu_a} = \sum_{p \in P(X, \mu)} c_p \prod_{a \in X_p} \frac{1}{a - a(p)}.
\]

This follows by induction applying the basic algorithm of separation of denominators.

**Remark 3.3.** Notice that the summands \( P_{Z,p} \) of \( R_X/R_{X,s-1} \) are the spaces of generalized common eigenvalues for the commuting operators induced from \( V \) (a vector \( v \in V \) has eigenvalue \( v(p) \)). Thus any \( S[V] \) submodule in \( R_X/R_{X,s-1} \) decomposes canonically into the direct sum of its intersections with the various \( P_{Z,p} \).

There is a similar description of all the pieces \( R_X/k \) based on the spaces of the arrangement generated by the hyperplanes \( a = 0, \ a \in X \) of codimension \( k \).

**Proposition 3.4.** \( R_X/k \) is a direct sum of copies of Fourier transforms of the modules \( N_W \), as \( W \) runs over the subspaces of the arrangement, and for given \( W \) the sum is made of terms indexed by the non broken bases in \( \mathbb{Z} \subset X \cap W^\perp \subset W^\perp \) and generated by the class of \( \prod_{a \in \mathbb{Z}} a^{-1} \).

**Proof.** Consider, for each \( W \), the map \( R_X \cap \mathbb{Z} \to R_X/k \). By induction the characteristic variety of all irreducible factors in \( R_X/k \) are union of conormal spaces to subspaces \( A \) of dimension \( \leq k - 1 \). Since \( R_X \cap \mathbb{Z} \to R_X/k \) is a direct sum of irreducibles with characteristic variety the conormal space to \( W \), it follows that this map induces an inclusion \( R_X \cap \mathbb{Z} \subset R_X/k \). It is easy to verify that this map is an isomorphism with the isotypic component of type \( N_W \) in \( R_X/k \), reducing to the previous case.

**3.5. Local expression for \( T_X \).** Formula (13) implies immediately that:

\[
T_{X,\mu}(x) = \sum_{p \in P(X, \mu)} c_p e^p T_{X_p}(x)
\]

This together with formula (11) gives for the box spline:

\[
B_{X,\mu}(x) = \sum_{p \in P(X, \mu)} c_p \sum_{S \subset X - X_p} (-1)^{|S|} e^{-\mu_S + p} B_{X_p}(x - a_S).
\]

Of course this is also a reformulation of the identity of Laplace transforms:

\[
\prod_{a \in X} \frac{1 - e^{-a - \mu_a}}{a + \mu_a} = \sum_{p \in P(X, \mu)} c_p \prod_{a \in X - X_p} (1 - e^{-a - \mu_a}) \prod_{a \in X_p} \frac{1 - e^{-a - \mu_a}}{a + \mu_a}
\]

As a consequence of these formulas the essential problem is the determination of \( T_X \) in the non parametric case. We are indeed ready to state and prove the main formula one can effectively use for computing the function \( T_X \).
We first need some geometry of the cone $C(X)$. Denote by $C(b)$ the positive quadrant spanned by the no broken circuit basis $b$. The positive cones $C(b)$ cover the cone $C(X)$ and induce a decomposition of $C(X)$ into polyhedral cones which happens to be independent of the order chosen and thus of the no broken circuits (this is proved in [21]). The points of $C(X)$, outside the boundaries of these cones are called regular and they are a union of open convex cones called the big cells.

The remaining singular points are the union of cones $C(A)$ generated by subsets $A \subset X$ which do not span $V$. We denote by $\mathcal{NB}$ the set of no broken bases extracted from $X$.

**Theorem 3.6.** Given a point $x$ in the closure of a big cell $c$ we have

\begin{equation}
T_X(x) = \sum_{b | c \in C(b)} \frac{\det(b)}{|b|}^{-1}p_{b,X}(-x),
\end{equation}

where for each no broken basis $b$, $p_{b,X}(x)$ is a uniquely defined homogeneous polynomial of degree $|X| - s$ lying in $D(X)$.

**Proof.** By the continuity of $T_X$ in $C(X)$, it is sufficient to show our claim in the interior on each big cell. Thus, by formula (6) and our discussion above we need to show that the identity (15) holds, with $p_{b,X}(x) \in D(X)$, in the $D$-module $D_X/D_{X,s-1}$.

If we work in the space of polar parts $P_X = R_X/R_{X,s-1}$ we have, by our previous discussion that there exist uniquely defined polynomials $p_{b,X}(x)$ homogeneous of degree $|X| - s$ with

\begin{equation}
\frac{1}{d_X} = \sum_{b \in \mathcal{NB}} p_{b,X}(\partial_x) \frac{1}{d_b}
\end{equation}

In fact this identity already holds in $R_X$.

Applying the inverse of the Laplace transform (cf. [6] [10] we obtain identity (15).

To finish notice that $Y$ is a cocircuit we have $\prod_{y \in Y} y d_X^{-1} \in R_{X,s-1}$. In other words, the polynomials $p_{b,X}$ in formula (16) lie in $D(X)$.

□

From now on, unless there is a possibility of confusion, we shall write $p_b$ instead of $p_{b,X}$.

**Remark 3.7.** The polynomials $p_b(x)$, with $b$ a no broken basis, are characterized among the polynomials in $D(X)$, by the further differential equations which will be described at a later section (see [22]). In particular we will show that the polynomials $p_b(x)$ are linearly independent.

4. The first theorem

4.1. **The basic equations.** In section 2.4 we have introduced the elements $D_Y := \prod_{x \in Y} D_x \in A, Y \subset X$, thought of as differential operators on $V$. 
We have started to see that the space
\[(17) \quad D(X) := \{ p \in S[U] \mid D_Y p = 0, \forall Y \in \mathcal{E}(X) \}, \]
of polynomials \( p \) on \( V \), which are solutions of the differential equations, \( D_Y p = 0 \), for all the cocircuits \( Y \), plays a fundamental role in the determination of \( T_X \). In [13] the authors prove that:

**Theorem 4.2.** Let \( D(X) \subset S[U] \) be the space of polynomials on \( V \), solutions of the differential equations \( D_Y f = 0 \), as \( Y \) runs over the cocircuits in \( X \).

\( D(X) \) is finite dimensional, of dimension the number \( d(X) \) of linear bases of \( V \) which one can extract from \( X \).

Since the elements \( D_Y \) are homogeneous, the space \( D(X) \) is also graded and it is interesting to compute its dimension in each degree \( D_k(X) \).

Moreover from general facts we shall see that \( D(X) \) is generated, under taking derivatives, by the homogeneous elements of top degree \( N - s \).

As usual one can arrange these dimensions in a generating function,
\[ H_X(q) := \sum_{k} \dim(D_k(X)) q^k. \]
We will show (Theorem 6.2), by exhibiting an explicit basis, that this polynomial is given by the statistic introduced in §2.1, i.e.:
\[(18) \quad H_X(q) := \sum_{b \in \mathcal{B}(X)} q^{N - n(b)}. \]
In particular the top degree polynomials in \( D(X) \) are of degree \( N - s \). They appear in the formula of the multivariate spline and thus have a very interesting geometric interpretation, since they compute the volume of certain polytopes associated to \( X \) ([20]).

### 4.3. A remarkable family.

In order to obtain the proof of 4.2 and 18, we start from the study of a purely algebraic geometric object.

For notational simplicity let us denote by \( A := S[V] \).

We want to describe the scheme defined by the ideal \( I_X \) of \( A \) generated by the elements \( M_Y := \prod_{x \in Y} x \) as \( Y \) runs over all the cocircuits.

Thus we are interested in the algebra:
\[(19) \quad A_X := A/I_X. \]
We shall soon see that \( A_X \) is finite dimensional. Formally \( A_X \) is the coordinate ring of the corresponding scheme.

The use of the word scheme may seem a bit fancy. What we really want to stress by this word is that we have a finite dimensional algebra (quotient of polynomials) which as functions vanish exactly on some point \( p \) but at least infinitesimally they are not constant. This appears clearly in the dual picture which produces solutions of differential equations.

To warm us up in the proof let us verify that this scheme is supported at 0 (which implies that \( A_X \) is finite dimensional).
For this, remark that the variety of zeros of a set of equations, each one of which is itself a product of equations, is the union of the subvarieties defined by selecting an equation out of each product. Thus what we need is the following:

**Lemma 4.4.** Take one element $x_i$ from each cocircuit $Y_i$, then the resulting set of elements $x_i$ span $V$ (hence they define $\{0\}$ as subvariety).

**Proof.** If, by contradiction, these elements do not span, their complement is a cocircuit. Since we selected an element from each cocircuit this is not possible. □

In fact it is convenient to extend the notion as follows. If $X = \{a_1, \ldots, a_N\}$ and $\mu := \{\mu_1, \ldots, \mu_N\}$ are parameters, we can define in general the ideal $I_X(\mu)$ given by the equations $\prod_{a_j \in Y}(a_j - \mu_j)$, $Y \in \mathcal{E}(X)$. This can either be viewed as an ideal in $S[V]$ depending on the parameters $\mu$ or as an ideal in the polynomial ring $S[V][\mu_1, \ldots, \mu_N]$.

4.5. **The first main theorem.** Theorem (4.2) follows easily from:

**Theorem 4.6.** For all $\mu$ the ring $A_X(\mu) := A/I_X(\mu)$ has dimension equal to the number $d(X)$ of bases which can be extracted from $X$.

**Proof.** We will use a standard procedure in commutative algebra. First we show that $\dim(A_X(\mu)) \geq d(X)$ for generic $\mu$. Then, for the special point $\mu = 0$ we show that $\dim(A_X) \leq d(X)$. Since the dimension is semicontinuous this implies the statement (we will comment after about the meaning of this theorem in terms of flatness and the Cohen Macaulay property).

As a first step we claim that, for generic $\mu$, the ideal $I_X(\mu)$ defines $d(X)$ distinct points, (where $d(X)$ is the number of bases extracted from $X$).

This implies of course that $\dim(A_X(\mu)) \geq d(X)$ where equality means that all the points are reduced.

Set theoretically, the variety described by all the equations given is the union of the varieties described by selecting, for every $Y \in \mathcal{E}(X)$ a cocircuit, an element $a \in Y$ and setting the equation $a - \mu_a = 0$.

From Lemma (4.3), the resulting list of vectors $a_i, \ldots, a_{im}$ (extracted from all the cocircuits) generates the space, thus the equations $a_{ij} - \mu_{ij} = 0$ are either incompatible or define a point of coordinates $\mu_{ij}$ in some basis extracted from $X$.

Conversely, given such a basis every cocircuit must contain at least one of the elements $a_{ij}$ hence the point of equations $a_{ij} = \mu_{ij}$ must be in the variety.

Now it is clear that, if the $\mu_i$ are generic (i.e. do not satisfy some linear compatibility equations) these $d(X)$ points are all distinct. Proving the first inequality.
We now prove that \( \dim(A_X) \leq d(X) \). To do this we proceed by induction on the cardinality of \( X \) and \( \dim(V) \).

Take \( y \in X \) and set \( Z := X - \{y\} \). Set \( V := V/Cy \) and denote by \( \pi : V \to \overline{V} \) the quotient homomorphism.

Also, for any sublist \( B \subset Z \) set \( \overline{B} \) equal to the list of non zero vectors in \( V/Cy \) which are images of vectors in \( B \).

Define finally \( \overline{A} := A/(y) = S[V/Cy] = S[\overline{V}] \).

(1) The image of \( I_X \) in \( \overline{A} \) is the ideal \( I_Z \).

The image of \( I_X \) is generated by the products \( \prod_{x \in Y} \pi(x) \) where \( Y \in \mathcal{E}(X) \) and \( y /\in Y \). Thus \( X - Y \) generates a proper subspace of \( V \) containing \( y \). This clearly implies that \( \pi(\langle X - Y \rangle) \subsetneq \overline{V} \) and that \( \overline{Z} \cap \pi(\langle X - Y \rangle) = \overline{X - Y} \).

Therefore the image \( \pi(Y) = \overline{Y} \) is a cocircuit for \( \overline{Z} \) and \( \prod_{x \in Y} \pi(x) = \prod_{x \in Y} \overline{x} \).

This proves that the image of \( I_X \) is contained in \( I_Z \) (the other generators map to 0).

On the other hand we know that \( I_Z \) is generated by the products \( \prod_{x \in Y} \overline{x} \) with \( \langle Z - Y \rangle \subsetneq V/Cy \). Setting \( Y = X - \pi^{-1}(\langle Z - \overline{Y} \rangle) \), we immediately see that \( Y \in \mathcal{E}(X) \) and \( \prod_{x \in Y} \overline{x} = \prod_{x \in Y} \pi(x) \) proving that \( I_Z \) coincides with the image of \( I_X \).

Therefore by induction \( \dim(A/(I_X + (y))) = \dim(\overline{A_Z}) = d(Z) \).

(2) The surjective map \( A \twoheadrightarrow yA_X : A \to A_X \) induces the projection \( p_Z : A \to A_Z \) followed by a (surjective) map:

\[
j : A_Z \to yA_X.
\]

To see this it suffices to show that, if \( Y \in \mathcal{E}(Z) \) then \( y \cdot \prod_{x \in Y} x \in I_X \).

We have two cases. If \( y \in \langle Z - Y \rangle \), then \( Y = X - \langle Z - Y \rangle \) and \( \prod_{x \in Y} x \in I_X \).

If \( y \notin \langle Z - Y \rangle \), then \( X \cap (Z - Y) = Z - Y \) and \( X - \langle Z - Y \rangle = Y \cup \{y\} \in \mathcal{E}(X) \) so that \( y \cdot \prod_{x \in Y} x = \prod_{x \in Y \cup \{y\}} x \in I_X \).

In particular this proves that \( \dim(A_Z) \geq \dim(yA_X) \).

Let us now use these facts to estimate \( \dim(A_X) \).

Consider the exact sequence

\[
0 \to yA_X \to A_X \to A/(I_X, y) \to 0.
\]

By what we have seen we have

\[
\dim A_X = \dim A/(I_X, y) + \dim yA_X \leq \dim(\overline{A_Z}) + \dim(A_Z).
\]
By our inductive hypothesis, \( A_Z \) has dimension equal to the number of bases of \( V \) contained in \( Z \) i.e. the number of bases in \( X \) not containing \( y \) among its elements.

Also by induction \( \dim(\overline{A_Z}) \) equals to the number of bases of \( V/\langle x \rangle \) contained in \( \overline{Z} \).

Given such a basis \( b' = \{ \pi(a_{i_1}), \ldots, \pi(a_{i_s}) \} \), the set \( \overline{b} = \{ y, a_{i_1}, \ldots, a_{i_{s-1}} \} \) as above, then \( \overline{b'} = \{ \pi(a_{i_1}), \ldots, \pi(a_{i_{s-1}}) \} \), is a basis of \( V/\langle x \rangle \) contained in \( \overline{Z} \).

Thus \( \dim(\overline{A_Z}) \) equals the number of bases of \( V \) contained in \( X \) and containing \( y \). Summarizing \( \dim A_Z + \dim(\overline{A_Z}) = d(X) \). The theorem is proved. 

Remark that, as a consequence of the proof, which we will use later, we also now have that \( j : A_Z \rightarrow yA_X \) is bijective.

As announced we want to discuss the meaning of this theorem for the variety \( V_X \) given by \( I_X(\mu) \) thought of as ideal in \( S[V][\mu_1, \ldots, \mu_N] \).

This variety is easily seen to be what is called a polygraph. It lies in \( U \times \mathbb{C}^N \) and can be described as follows. Given a basis \( \underline{b} := \{ a_{i_1}, \ldots, a_{i_s} \} \) of \( V \), extracted from \( X \), let \( a^1, \ldots, a^s \) be the associated dual basis in \( U \). Define a linear map \( \underline{i}_{\underline{b}} : \mathbb{C}^N \rightarrow U \) by:

\[
\underline{i}_{\underline{b}}(\mu_1, \ldots, \mu_N) := \sum_{j=1}^n \mu_{i_j} a^{i_j}.
\]

Let \( \Gamma_{\underline{b}} \) be its graph, then \( V_X = \cup_{\underline{b}} \Gamma_{\underline{b}} \).

\( V_X \) comes equipped with a projection map \( \rho \) to \( \mathbb{C}^N \) whose fibers are the schemes defined by the ideals \( I_X(\mu) \).

**Remark 4.7.** Theorem 4.6 implies that, \( \rho \) is flat of degree \( d(X) \) and \( I_X(\mu) \) is the full ideal of equations of \( V_X \). Furthermore \( V_X \) is Cohen Macaulay.

The fact is remarkable since it is very difficult for a polygraph to satisfy these conditions. When it does, this usually has deep combinatorial implications (see [28]).

One should make some remarks about the algebras \( A_X(\underline{\mu}) \) in general.

First some basic commutative algebra. Take an ideal \( I \subset \mathbb{C}[x_1, \ldots, x_m] \) of a polynomial ring. \( \mathbb{C}[x_1, \ldots, x_m]/I \) is finite dimensional if and only if the variety of zeroes of \( I \) is a finite set of points \( p_1, \ldots, p_k \). In this case moreover we have a canonical decomposition

\[
\mathbb{C}[x_1, \ldots, x_m]/I = \bigoplus_{i=1}^k \mathbb{C}[x_1, \ldots, x_m]/I_{p_i}
\]

where, for each \( p \in \{ p_1, \ldots, p_k \} \), the ring \( \mathbb{C}[x_1, \ldots, x_m]/I_p \) is the local ring associated to the point \( p \).
Let \( p \) have coordinates \( x_i = \mu_i \), the local ring \( \mathbb{C}[x_1, \ldots, x_m]/I_p \) is characterized, in terms of linear algebra, by the property that the elements \( x_i \) have generalized eigenvalue \( \mu_i \). Thus the previous decomposition is just the Fitting decomposition, into generalized eigenspaces, for the commuting operators \( x_i \).

In the case of the algebra \( A_{X}(\mu) \) quotient of \( S[V] \) by the ideal \( I_{X}(\mu) \) generated by the elements, \( \prod_{y \in V}(y + \mu_{y}) \) we see that, if for a point \( p \) of the resulting (finite) variety we have that \( y(p) + \mu_{y} \neq 0 \), in the local ring of \( p \) the element \( y + \mu_{y} \) is invertible and so it can be dropped from the equations. We easily deduce that:

**Proposition 4.8.** The local component \( A_{X}(\mu)(p) \) equals the algebra \( A_{X_{p}}(\mu) \) defined by the sublist \( X_{p} := \{ x \in X \mid x(p) + \mu_{x} = 0 \} \).

Furthermore by a change of variables \( X_{p}(\mu) := \{ x + \mu_{x}, \; x \in X_{p} \} \) we can even identify \( A_{X_{p}}(\mu) = A/\tau_{p}(I_{X_{p}}) \).

By \( \tau_{p}(I_{X_{p}}) \) we mean the ideal \( I_{X_{p}} \) translated at \( p \) by the automorphism of \( A \) sending \( x \mapsto x - x(p) \), \( \forall x \in V \) (hence \( x \mapsto x + \mu_{x}, \; \forall x \in X_{p} \)).

5. **Solutions of differential equations**

5.1. **Differential equations with constant coefficients.** Let us reinterpret Theorem 4.6 in terms of differential equations.

Let us consider the polynomial ring \( S[V] \), its graded dual is \( S^*[U] \).

By duality \( S[V] \) is identified with differential operators with constant coefficients on \( V \) and, using coordinates, the duality pairing can be explicitly described as follows. Given a polynomial \( p(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}) \) in the derivatives and one \( q(x_1, \ldots, x_s) \) in the variables the pairing

\[
\langle p(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_s}) \mid q(x_1, \ldots, x_s) \rangle
\]

is obtained by applying \( p \) to \( q \) and then evaluating at 0.

Of course the algebraic dual of \( S[V] \) is a rather enormous space of a rather formal nature. It can be expressed best as a space of formal power series by associating to an element \( f \in (S[V])^* \) the formal expression:

\[
\langle f \mid e^x \rangle = \sum_{k=0}^{\infty} \frac{\langle f \mid x^k \rangle}{k!}, \; x \in V.
\]

Where now \( \langle f \mid x^k \rangle/k! \) is a genuine homogeneous polynomial of degree \( k \) on \( V \).

The following facts are easy to see:

**Proposition 5.2.**

1. Given a vector \( v \in V \), the transpose of multiplication by \( v \) is the directional derivative \( D_v \) in \( (S[V])^* \).
(2) Given \( \phi \in U \), denote by \( \tau_\phi \) the automorphism of \( S[V] \) induced by translation \( x \mapsto x - x(\phi) = x - \langle \phi \mid x \rangle \), \( \forall x \in V \), and by \( \tau_\phi^* \) its transpose. We have \( \tau_\phi^* f = e^{-\langle \phi \mid x \rangle} f \).

Proof. 1) follows from the fact that for \( \phi \in U \) \( D_\phi v(\phi) = \langle \phi \mid v \rangle \). and the chain rule.

As for 2),
\[
\langle \tau_\phi^* f \mid e^x \rangle = \langle f \mid \tau_\phi(e^x) \rangle = \langle f \mid e^{x-\langle \phi \mid x \rangle} \rangle = e^{-\langle \phi \mid x \rangle} \langle f \mid e^x \rangle.
\]

Observe that, if \( J \) is an ideal of \( S[V] \) defining a subvariety \( Z \subset U \), we have that \( \tau_\phi(J) \) defines the subvariety \( Z + \phi \).

If we have a quotient \( S[V]/I \) by an ideal \( I \) we get an injection of \( i : (S[V]/I)^* \to (S[V])^* \). The image is, at least formally, the space of solutions of the differential equations given by \( I \). We denote by \( Sol(I) \) the space of \( C^\infty \) solutions of the system of differential equations given by \( I \).

Assume now that \( S[V]/I \) is finite dimensional. Denote by \( \{ \phi_1, \ldots, \phi_k \} \subset U \) the finite set of points which are the support of \( I \). Take the decomposition
\[
S[V]/I = \oplus_{i=1}^k S[V]/I(\phi_i)
\]
where \( S[V]/I(\phi_i) \) is local and supported in \( \phi_i \). Under these assumptions we get

**Theorem 5.3.** 1) If \( S[V]/I \) is finite dimensional and supported at 0. The image of \( i^* \) lies in \( S[U] \) and coincides with \( Sol(I) \).

2) If \( S[V]/I \) is finite dimensional and supported at a point \( \phi \in U \),
\[
Sol(I) = e^{\langle \phi \mid x \rangle} Sol(\tau_{-\phi}I).
\]

3) For a general finite dimensional \( S[V]/I \),
\[
Sol(I) = \oplus_{i=1}^k Sol(I(\phi_i))
\]

Proof. 1) is the duality. As for 2), clearly, for any ideal \( I \) and \( \phi \) we have \( Sol(\tau_\phi(I)) = \tau_{-\phi}^*(Sol(I)) = e^{\langle \phi \mid x \rangle} Sol(I) \). It only remains to explain the analytic nature of the statement. This follows, even in the stronger setting of tempered distributions by the fact the a tempered distribution which is annihilated by high enough derivatives is necessarily a polynomial (see [13] Prop. 3.1).

Applying this discussion to our setting, we get that, for generic \( \mu \), the solutions of the differential equations of the ideal \( I_X(\mu) \) are exactly the space with basis the functions \( e^\phi \) where the elements \( \phi \) run on the \( d(X) \) points defined previously, while for \( \mu = 0 \) we have a remarkable space of polynomial solutions. The nature of this space is explained in the next section.
In the general case we have a mixture between these two extreme cases. The set of solutions is a finite set of points and in each such point \( \phi \in U \) we have a subset \( X_\phi \) of \( X \) formed by those elements \( x_i \in X \) such that \( \langle \phi | x_i \rangle = \mu_i \).

Then one can see that this point contributes to the solutions with the functions \( e^{\phi} p \) where \( p \) are the polynomials associated to \( X_\phi \) at \( 0 \).

6. A realization of \( A_X \)

6.1. The graded dimension. Our next task is to prove that, the graded dimension of the space \( D(X) \) is given by:

**Theorem 6.2.**

\[
H_X(q) = \sum_{\underline{b} \in B(X)} q^{N-n(\underline{b})}.
\]

In order to do this, we want to realize the ring \( A_X \) as the \( S[V] \) submodule \( Q_X \), of the space of polar parts \( P_X \) generated by the class \( u_X \), of the element \( d_X^{-1} \). In doing this we shall exhibit a homogeneous basis made by elements \( u_{\underline{b}} \) as \( \underline{b} \) varies among the bases of \( X \), each of degree \( N - n(\underline{b}) \).

The spaces \( R_X \) and \( P_X \) are naturally graded (as functions). It is convenient to shift this degree by \( N \) so that \( u_X \) has degree 0 and the generators \( u_{\underline{b}} \) have degree \( N - s \). If \( \underline{b} \) is a no broken basis, these will be just the elements \( u_{\underline{b}} \) introduced in section 3.1 when \( \mu = 0 \).

With these gradation the natural map \( \pi : A \rightarrow Q_X \) defined by \( \pi(f) = fu_X \) preserves degrees.

It is clear that Theorem (6.2) follows from the following more precise result on \( Q_X \), which also describes a graded basis for \( A_X \):

**Theorem 6.3.**

(1) The annihilator of \( u_X \) is the ideal \( I_X \) generated by the elements \( M_Y = \prod_{x \in Y} x \), as \( Y \) runs over the cocircuits. Thus \( Q_X \cong A_X \) as graded \( A \)-modules.

(2) Given a basis \( \underline{b} := \{a_{i_1}, \ldots, a_{i_s}\} \) extracted from \( X \), set

\[
u_{\underline{b}} := (\prod_{a \in X-B(\underline{b})} a)u_X.
\]

The elements \( u_{\underline{b}} \), as \( \underline{b} \) runs over the bases extracted from \( X \), are a basis of \( Q_X \).

**Proof.** To see the first part, let us remark that clearly \( I_X u_X = 0 \) since, if \( Y \) is a cocircuit \( M_Y u_X = \prod_{x \in X-Y} x^{-1} \) lies in \( R_{X,s-1} \), hence it is 0 in the module \( P_X \).

Therefore, from Theorem 4.6, it is enough to see that \( \dim Q_X \geq d(X) \).

We want to proceed by induction on \( s \) and on the cardinality of \( X \).

If \( X \) consists of a basis of \( V \), clearly both \( A_X \) and \( Q_X \) are 1-dimensional and the claim is clear. Otherwise we can assume that \( X := \{Z, y\} \) and \( Z \) still spans \( V \). We need thus to compare several of the objects under analysis in the case in which we pass from \( Z \) to \( X \).
Let us consider the ring $A/(y)$, polynomial functions on the subspace of $U$ where $y$ vanishes and denote by $Z$ the set of non-zero vectors in the image of $Z$ (or $X$) in $A/(y)$.

As in Theorem (4.6) the set $B_X$ of bases extracted from $X$ can be decomposed into two disjoint sets. $B_Z$ and the bases containing $y$. This second set is in 1-1 correspondence with the bases of $V/Cy$ contained in $Z$.

We need several lemmas.

First we obviously have an inclusion $R_Z \subset R_X$ and also $R_{Z,k} \subset R_{X,k}$, $\forall k$.

**Lemma 6.4.** $R_{Z,s-1} = R_Z \cap R_{X,s-1}$ and we get an inclusion of $P_Z$ into $P_X$.

*Proof.* The two statements are equivalent and follow immediately by ordering $Z$ and adding $y$ as last element. Then all the no broken circuit bases for $Z$ are also no broken circuit bases for $X$ so that $P_Z$ (as module over the differential operators with constant coefficients) is a free direct summand of $P_X$. $\square$

As a consequence of this Lemma let us consider in $P_X$ the map multiplication by $y$. We have clearly $yu_X = u_Z$ thus we obtain an exact sequence of $A$-modules:

$$0 \to K \to Q_X \xrightarrow{y} Q_Z \to 0$$

where $K = Q_X \cap \text{Ker}(y)$.

We need to analyze $K$ and prove that $\dim(K)$ is greater or equal to the number $d_y(X)$, of bases extracted from $X$ and containing $y$. Since we already know that $\dim(K) + d(Z) \leq d(X)$ and $d(X) = d(Z) + d_y(X)$ this will prove the claim.

In order to achieve the inequality $\dim(K) \geq d_y(X)$ we will find inside $K$ a direct sum of subspaces whose dimensions add up to $d_y(X)$.

Let us first discuss a special case. Assume that $Z$ spans a subspace $V'$ of codimension 1 in $V$, so $y$ is a vector outside this subspace.

We clearly have inclusions:

$$R_Z \subset R_X \quad y^{-1}R_{Z,k-1} \subset R_{X,k} \quad \forall k.$$ 

Also the element $u_X \in P_X$ is killed by $y$ and $A/(y)$ can be identified to $S[V']$.

**Lemma 6.5.** The multiplication by $y^{-1}$ induces an isomorphism between $P_Z$ and the kernel of the multiplication by $y$ in $P_X$.

*Proof.* Since $y^{-1}R_{Z,k-1} \subset R_{X,k}$ it is clear that the multiplication by $y^{-1}$ induces a map from $P_Z = R_{Z,s-1}/R_{Z,s-2}$ to $P_X = R_{X,s}/R_{X,s-1}$. It is also clear that the image of this map lies in the kernel of the multiplication by $y$.

To see that it gives an isomorphism to this kernel, order the elements of $X$ so that $y$ is the first element. A no broken circuit basis for $X$ is of the form $\{y,c\}$ where $\{c\}$ is a no broken circuit basis for $Z$. 

Now fix a set of coordinates $x_1, \ldots, x_s$ such that $x_1 = y$ and $x_2, \ldots, x_s$ is a basis of the span of $Z$. Denote by $\partial_i$ the corresponding partial derivatives. We have that in each summand $\mathbb{C}[\partial_1, \ldots, \partial_s]u_y$, the kernel of multiplication by $x_1$ coincides with $\mathbb{C}[\partial_2, \ldots, \partial_s]u_y$.

The claim follows easily since $\mathbb{C}[\partial_2, \ldots, \partial_s]u_y$ is the image, under $y^{-1}$ of $\mathbb{C}[\partial_2, \ldots, \partial_s]u_y$. □

Notice that $y^{-1}u_Z = u_X$, so that the following lemma is immediate

**Lemma 6.6.** Under the previous hypotheses, the multiplication by $y^{-1}$ induces an isomorphism between $Q_Z$ and $Q_X$.

Let us now pass to the general case. Consider the set $S_y(X)$ of all complete sublists of $X$ which span a subspace of codimension 1 not containing $y$.

For each $Y \in S_y(X)$ we have, by Lemma 6.5, that the multiplication by $y^{-1}$ induces an inclusion $i_Y : P_Y \to P_{Y \cup \{y\}} \to P_X$ with image in the kernel $\text{Ker}(y) \subset P_X$. Thus we get a map

$$g := \oplus_{Y \in S_y(X)} i_Y : \oplus_{Y \in S_y(X)} P_Y \to \text{Ker}(y)$$

We then claim that:

**Lemma 6.7.** $g$ is an isomorphism of $\oplus_{Y \in S(X)} P_Y$ onto $\text{Ker}(y)$.

**Proof.** As before order the elements of $X$ so that $y$ is the first element. A no broken circuit basis for $X$ is of the form $\{y, \mathcal{C}\}$ where $\mathcal{C}$ is a no broken circuit basis for $Y := X \cap \mathcal{C}$.

By construction $Y \in S_y(X)$ and we obtain the direct sum decomposition

$$P_X = \oplus_{Y \in S_y(X)} P_{Y \cup \{y\}}.$$ Clearly

$$\text{Ker}(y) = \oplus_{Y \in S(X)} P_{Y \cup \{y\}} \cap \text{Ker}(y)$$

and, by Lemma 6.5, for each $Y \in S_y(X)$, $i_y$ gives an isomorphism of $P_Y$ with $P_{Y \cup \{y\}} \cap \text{Ker}(y)$. This proves the lemma. □

In order to finish the proof of Theorem 6.3 notice that by Lemma 6.6 and Lemma 6.7, we get an inclusion of $\oplus_{Y \in S_y(X)} Q_Y$ into $K = Q_X \cap \text{Ker}(y)$, so that $\dim Q_X \geq \dim Q_Z + \sum_{Y \in S_y(X)} \dim P_Y = d(X)$.

This gives the required inequality and implies also that we have a canonical exact sequence of modules:

$$(21)
0 \to \oplus_{Y \in S_y(X)} Q_Y \to Q_X \xrightarrow{y} Q_Z \to 0.$$

It remains to establish the second part of Theorem 6.3. We prove it by induction, ordering $X$ so that $y$ is the last element. Let us write $B_X$ as the disjoint union of $B_Z$ and the set $B_X^{(y)}$ of bases containing $y$. Also set $U_X = \{u_\mathcal{H} \mid \mathcal{H} \in B_X\}$ and write it as the disjoint union of $U'_X = \{u_\mathcal{H} \mid \mathcal{H} \in B_X^{(y)}\}$ and its complement $U''_X = \{u_\mathcal{H} \mid \mathcal{H} \in B_X^{(y)}\}$. Remark that by our definitions,
\[ \mathcal{U}_X' = \mathcal{U}_Z \text{ while } \mathcal{U}_X'' \text{ is the disjoint union } \bigcup_{Y \in \mathcal{S}(X)^*} i_Y(\mathcal{U}_Y). \] By induction and formula (21) this clearly implies that \( \mathcal{U}_X \) is a basis of \( Q_X \).

It follows from our theorem that in top degree \( Q_X \) has as a basis the elements \( u_{\underline{b}} \) as \( \underline{b} \) runs over the set of no broken bases.

In the parametric case, we have the decomposition \( d_X^{-1} = \sum_p c_p d_X^{-1}_p \) (cf. (13)). We set \( Q_X(\mu) \) equal to the \( S[V] \)-module generated by \( u_X \) and for each \( p \in P(X, \mu) \), \( Q_X(p) \) equal to the \( S[V] \)-module generated by \( u_X_p \).

**Proposition 6.8.**

1) \[ Q_X(\mu) = \bigoplus_{p \in P(X, \mu)} Q_X(p). \]

2) \( Q_X(\mu) \) is isomorphic to \( A_X(\mu) \) and the above decomposition coincides with the decomposition of \( A_X(\mu) \) into its local components.

**Proof.** The first part follows immediately from Remark 3.3.

As for the second since the annihilator of \( u_X \) contains \( I_X, \mu \), we have a map \( A_X(\mu) \to Q_X(\mu) \) which by Fitting decomposition induces a map of the local summands. On each such summand this map is an isomorphism by the previous theorem since we can translate \( p \) to 0. From this everything follows. \( \square \)

We can easily prove as corollary a theorem by several authors, see [1], [23], [30].

**Corollary 6.9.** Consider in \( S[V] \) the subspace \( \mathcal{P}(X) \) spanned by all the products \( M_Y := \prod_{x \in Y} x, Y \subset X \) such that \( X - Y \) spans \( V \). Then \( \mathcal{P}(X) \) is in duality with \( D(X) \).

**Proof.** Multiply by \( d_X^{-1} \). \( \mathcal{P}(X)d_X^{-1} \) is spanned by the polar parts \( \prod_{x \in Z} x^{-1}, Z \subset X \) such that \( Z \) spans \( V \). We have seen that this space of polar parts maps isomorphically to its image into \( P_X \) and its image is clearly \( Q_X \). This proves the claim. \( \square \)

**Remark 6.10.** The last theorem we have proved is equivalent to saying that, in the algebra \( R_X \) the intersection \( d_X^{-1} A \cap R_{X,s-1} = d_X^{-1} I_X \).

It is of some interest to analyze the deeper intersections \( d_X^{-1} S[V] \cap R_{X,k} := U_k \). We will sketch this point which uses the structure of the filtration by polar degree as explained in [22].

If \( X \) does not necessarily span \( V \) we define \( I_X \) as the ideal generated by the products \( \prod_{x \in Y} x \) where \( Y \subset X \) is any subset such that the span of \( X - Y \) is strictly contained in the span of \( X \). We set \( A_X(V) = A/I_X \).

Of course if we fix a decomposition \( V := \langle X \rangle \oplus T \) we have \( A_X(V) = A_X \otimes S[T] \).

Consider the set of all subspaces \( W \) spanned by elements of \( X \) (including the space \( \{0\} \) spanned by the empty set). We call any such subspace a *subspace of the arrangement* generated by \( X \). Set \( L = d_X^{-1} A, X_W := X \cap W, \) and \( L_k = L \cap R_{X,k} \). We have
Theorem 6.11. For each $k$ we have that $L_k/L_{k-1}$ is isomorphic to the direct sum $\oplus A_{X_W}(V)$ as $W$ varies on the subspaces of dimension $k$ of the arrangement.

Proof. Under the map $A \to L$ given by $f \to d^{-1}Xf$, we know that the submodule $I_X$ maps isomorphically onto $L_{s-1}$. So $L_{s-1}$ is spanned by the elements $d^{-1}_Y$ where $Y = X_W$ for the hyperplanes $W$ of the arrangement. By (a small generalization of) Lemma 6.4 we know that the filtration by polar order in $R_X$ induces the same filtration in $R_Y$ thus by induction we have that the graded associated to $d^{-1}_Y S[V]$ is the direct sum of all the pieces $A_{X_W}$ as $W$ varies on the subspaces of dimension $k$ of the arrangement generated by $Y$. This follows from Proposition 3.4. \[ \square \]

The previous theorem gives an interesting combinatorial identity once we compute the Hilbert series of $d^{-1}_X S[V]$ directly or as sum of the contributions given by the previous filtration.

$$\frac{q^{-|X|}}{(1-q)^s} = \sum_{k=0}^{s-1} \sum_{W \in W_k} \frac{q^{-|X_W|}}{(1-q)^{s-k}} H_{X_W}(q).$$

Remark 6.12. In Fourier transform the previous discussion can be translated into an analysis of the distributional derivatives of the multivariate spline, i.e. into an analysis of the various discontinuities achieved by the successive derivatives on all the strata of the singular set.

In the applications to the box spline it is interesting, given a set $X$ of vectors which we list in some way, to consider for each $k \geq 0$ the list $X^k$ in which every element $a \in X$ is repeated $k$ times. Let us make explicit the relationship between $H_X(q)$ and $H_{X^k}(q)$.

First the number of bases in the two cases is clearly related by the formula $d(X^k) = d(X)k^s$, to each basis $b := \{b_1, \ldots, b_s\}$ extracted from $X$ we associate $k^s$ bases $b(h_1, \ldots, h_s)$, indexed by $s$ numbers $h_i \leq k$ expressing the position of the corresponding $b_i$ in the list of the $k$ repeated terms in which it appears.

Now it is easy to see that:

$$n(b(h_1, \ldots, h_s)) = k(n(b) - s) + h_1 + \cdots + h_s.$$ 

Thus we deduce the explicit formula:

$$H_{X^k}(q) = \sum_{b \in B_X} \sum_{h_1, \ldots, h_s} q^{kN - kn(b) + ks - h_1 - \cdots - h_s} = H_X(q^k) \left(\frac{q^s - 1}{q - 1}\right)^k.$$
6.13. More differential equations. Our next task is to fully characterize the polynomials $p_b(x)$ appearing in formula \[10\] by differential equations. In Theorem 3.6, we have seen that these polynomials lie in $D(X)$.

For a given no broken circuit basis $b$, consider the element $D_b := \prod_{a \in b} a$.

**Proposition 6.14.** The polynomials $p_b$ satisfy the system

\[(22)\]

$$D_b p_c(x_1, \ldots, x_s) = \begin{cases} 1 & \text{if } b = c \\ 0 & \text{if } b \neq c \end{cases}$$

In particular, the polynomials $p_b$ for $b \in \mathcal{N}B$ are linearly independent and characterized, in $D(X)_{N-s}$ by the equations \[22\].

**Proof.** This follows from the identity $D_b u_X = u_b$, the linear independence of the elements $u_b$ and the fact that the dimension of $D(X)_{N-s}$ is the cardinality of $\mathcal{N}B$. □

As a consequence, using formula \[15\], we can characterize by differential equations the multivariate spline $T_X(x)$ on each big cell $c$ as the function $T$ in $D(X)_{N-s}$ satisfying the equations:

\[(23)\]

$$D_b T = \begin{cases} |\det(b)|^{-1} & \text{if } c \subset C(b) \\ 0 & \text{otherwise} \end{cases}$$

We have identified $D(X)$ to the dual of $A_X$ and hence of $Q_X$. The basis $u_b$, we found in $Q_X$, defines thus a dual basis $u^b$ in $D(X)$.

**Corollary 6.15.** When $b \in \mathcal{N}B$ we have, as polynomials $u^b = p_b$

**Proof.** This follows from the previous Proposition and the definition of the duality given in formula \[20\]. □

We shall use a rather general notion of the theory of modules. Recall that the socle $s(M)$ of a module $M$ is the sum of all its irreducible submodules. Clearly if $N \subset M$ is a submodule $s(N) \subset s(M)$ while for a direct sum $s(M_1 \oplus M_2) = s(M_1) \oplus s(M_2)$. If $M$ is finite dimensional $s(M) \neq 0$ so that, for a non–zero submodule $N$ we must have $N \cap s(M) \neq 0$.

**Proposition 6.16.** The socle of the $S[V]$-module $Q_X$ coincides with its top degree part, with basis the elements $u^b$.

**Proof.** The socle of the algebra of constant coefficients differential operators, thought of as a module over the polynomial ring, is clearly generated by 1. It follows that the socle of $P_X$ (as $S[V]$-module), has as basis the elements $u^b$. Since $u^b \in Q_X$ we have $s(Q_X) = s(P_X)$, the claim follows. □

**Theorem 6.17.** $D(X)$ is spanned by the polynomial $p_b$, as $b$ runs over the set of no broken bases, and all of their derivatives.
Proof. Everything follows once we observe:

1) $D(X)$ is in duality with $A_X$ and so with $Q_X$.
2) The polynomials $p_b, b \in \mathcal{N}B$ are in duality with the elements $u_b$.
3) The orthogonal of a proper submodule $N$ of $D(X)$ is a non-zero submodule, thus it must intersect the socle of $Q_X$ in a non trivial subspace. In particular $N$ cannot contain all the elements $p_b$. □

We finish this section observing that we can dualize the sequence (21) to get an exact sequence

$$0 \to D(Z) \xrightarrow{D_y} D(X) \to \bigoplus_{Y \in S_y(X)} D(Y) \to 0.$$ 

Here, since the map $Q_X \to Q_Z \to 0$ is given by multiplication by $y$, the inclusion $0 \to D(Z) \to D(X)$ is the inclusion of $D_y(D(Z))$.

Part 2. The discrete case

7. The discrete case

7.1. The case $X$ in a lattice. Splines are used to interpolate and approximate functions. For this purpose it is important to understand the class of smoothness of a spline. In the case of the box-spline it is easy to prove (see [18]):

Let $m$ be the minimum number of elements in a cocircuit in $X$, assume $m \geq 2$, then $B_X$ is of class $C^{m-2}$.

Thus, provided we choose the list $X$ in a suitable way, we can achieve any finite level of smoothness required.

A particularly useful case is when we have chosen a lattice $\Lambda \subset V$ spanning $V$ such that each vector in the list $X$ lies in $\Lambda$. We assume that the Lebesgue measure on $V$ is normalized in such a way that a fundamental domain for $\Lambda$ has volume 1.

In this case, if $C_{\text{sing}}(X)$ denotes the set of singular vectors of the cone $C(X)$, the set of all translates $\bigcup_{\lambda \in \Lambda} C_{\text{sing}}(X)$ is called the cut region. It is a union of a finite number of $s-1$-dimensional bounded polytopes and all their translates.

From formula (11), and the fact that $D(X)$ is stable under translation we obtain (see also [18]):

**Proposition 7.2.** The complement of the cut region is a union of all translates of a finite number of cells, each an interior of a (compact) polytope.

Over each such cell the functions $B_X(x - \lambda), \lambda \in \Lambda$ are polynomials in the space $D(X)$.

One easily proves the fundamental fact (see [18]):
Theorem 7.3. If $X$ spans $V$, the translates of $B_X$ form a partition of unity:

\[ 1 = \sum_{\lambda \in \Lambda} B_X(x - \lambda). \]

Proof. If $X$ is a basis, $B_X$ is the characteristic function of the parallelepiped with basis $X$ divided by its volume and it easily follows that the identity holds outside a set of measure 0. In general one can use the iterative description of the box spline obtained by integration:

\[ B_{X,v}(x) = \int_0^1 B_X(x - tv)dt, \]

a formula deduced immediately from the definition.

Given a function $f(\lambda)$ on the lattice $\Lambda$ define

\[ B_X \ast f(x) := \sum_{\lambda \in \Lambda} B_X(x - \lambda)f(\lambda) \]

The space of all functions obtained by this procedure is called the cardinal spline space and the space of polynomials $D(X)$ is characterized by the property of consisting exactly of the polynomials lying in the cardinal spline space (see [18]). An extensive portion of the literature on the box spline is devoted to understanding how to use this space for approximation or interpolation of functions.

7.4. The Partition functions.

Definition 7.5. We identify a function $f$ on $\Lambda$ with the distribution

\[ \sum_{\lambda \in \Lambda} f(\lambda)\delta_\lambda \]

where $\delta_v$ is the Dirac distribution supported at $v$.

Recall that we have defined the partition function on $\Lambda$ by

\[ \mathcal{P}_X(v) := \# \{(n_1, \ldots, n_N) \mid \sum n_i a_i = v, \ n_i \in \mathbb{N} \}. \]

This is then identified with the tempered distribution

\[ \mathcal{T}_X := \sum_{v \in \Lambda} \mathcal{P}_X(v)\delta_v \]
We can then compute its Laplace transform obtaining

$$LT_X = \sum_{v \in \Lambda} P_X(v)e^{-v} = \prod_{a \in X} \frac{1}{1 - e^{-a}}$$

We shall think of $T_X$ as a discrete analogue of the multivariate spline $T_X$. We also have an analogue of $B_X$. Namely, setting

$$Q_X(v) := \# \{(n_1, \ldots, n_N) | \sum n_i a_i = v, n_i \in \{0, 1\}\}.$$ 

(27) 

$$B_X := \sum_{v \in \Lambda} Q_X(v)\delta_v$$

with Laplace transform

$$LB_X = \sum_{v \in \Lambda} Q_X(v)e^{-v} = \prod_{a \in X} (1 + e^{-a}) = \prod_{a \in X} \frac{1 - e^{-2a}}{1 - e^{-a}}$$

which implies that

(29) 

$$B_X(x) = \sum_{S \subseteq X} (-1)^{|S|}T_X(x - 2a_S)$$

with $a_S = \sum_{a \in S} a$.

7.6. Some basic modules and their transforms. As in Section 2.11, we want to consider the module $L_X$ generated, in the space of tempered distribution on $V$, by the element $T_X$ under the action of a suitable algebra of operators.

It is convenient to choose the algebra $\mathcal{W}(\Lambda)$ of difference operators with polynomial coefficients.

This algebra is generated by $S[U]$, thought of as polynomials on $V$ and by the translation operators $\tau_v$, $v \in \Lambda$ defined by $\tau_v f(a) = f(a + v)$. From the basic formulas (4), (5) we get that, under Laplace transform, a polynomial becomes a differential operator with constant coefficients while the translation $\tau_v$ becomes multiplication by $e^v$.

Consider the group algebra $\mathbb{C}[\Lambda]$ with basis the formal elements $e^a$, $a \in \Lambda$. Define next the algebra $\mathcal{W}(U)$ generated by $S[U]$, thought of as differential operators with constant coefficients on $U$ and by the functions $e^v, v \in \Lambda$. Additively we have $\mathcal{W}(V) = S[U] \otimes \mathbb{C}[\Lambda]$.

The algebras $\mathcal{W}(\Lambda)$ and $\mathcal{W}(V)$ are isomorphic by the isomorphism $\phi$ defined by

$$\phi(\tau_v) = e^v, \quad \phi(u) = -Du$$

for $v \in \Lambda, u \in U$. Thus, given a module $M$ over $\mathcal{W}(\Lambda)$ we shall denote by $\hat{M}$ the same space considered as a module over $\mathcal{W}(V)$ and think of it as a formal Fourier transform.

Define $S_X := \mathbb{C}[\Lambda] \prod_{a \in X} (1 - e^{-a})^{-1}$ to be the localization of $\mathbb{C}[\Lambda]$, obtained by inverting $\delta_X := \prod_{a \in X} (1 - e^{-a})$, and consider $S_X$ as a module over $\mathcal{W}(V)$.
Given a linearly independent subset \( A \subset X \), let
\[
\Lambda_A := \{ \sum_{a \in A} n_a a, \ n_a \in \mathbb{Z} \}, \quad C(A) := \{ \sum_{a \in A} r_a a, \ n_a \in \mathbb{R}^+ \},
\]
be the sublattice and the positive cone that they generate. Set
\[
\xi_A := \sum_{v \in \Lambda_A \cap C(A)} \delta_v
\]
a tempered distribution with Laplace transform given by:
\[
L\xi_A = \prod_{a \in A} \frac{1}{1 - e^{-a}}.
\]

In analogy with Theorem 2.12, using [22], we get,

**Theorem 7.7.** Under Laplace transform, \( \mathcal{L}_X \) is mapped isomorphically onto \( S_X \). In other words we get a canonical isomorphism of \( \hat{\mathcal{L}}_X \) with \( S_X \) as \( \mathcal{W}(U) \)-modules.

\( \mathcal{L}_X \) is the space of tempered distributions which are linear combinations of polynomials times \( \xi_A \), \( A \subset X \) a linearly independent subset, and their translates under \( \Lambda \).

**Proof.** The proof is analogous to that of Theorem 2.12. We use the results on partial fractions explained in [22] to show that \( S_X \) is generated by \( \delta_X^{-1} \) as a \( \mathcal{W}(V) \)-module, together with the simple formula (30).

**Remark 7.8.** There is a similar parametric case where the Laplace transform takes values in the algebra \( S_X, \mu := \mathbb{C}[\Lambda][\prod_{a \in X} (1 - e^{-a - \mu a})^{-1}] \).

Notice that the fact that the Laplace transform of \( \mathcal{L}_X \) is an algebra means that \( \mathcal{L}_X \) is closed under convolution.

7.9. **The toric arrangement.** The algebra \( S_X \) has a precise geometric meaning. In fact \( \mathbb{C}[\Lambda] \) is the coordinate ring of an affine variety whose points are the homomorphisms \( \Lambda \to \mathbb{C}^* \) called complex characters. We shall denote this variety by \( T \).

Notice that each character is obtained as follows. One takes a vector \( \phi \in U_{\mathbb{C}} = \text{hom}(V, \mathbb{C}) \) (the complexified dual) and constructs the function \( a \mapsto e^{\langle \phi | a \rangle} \). The elements of \( \Lambda^* := \{ \phi \in U_{\mathbb{C}} | \langle \phi | a \rangle \in 2\pi i \mathbb{Z} \} \) give rise to the trivial character. Thus \( T = U_{\mathbb{C}}/\Lambda^* \) is an algebraic group isomorphic to \((\mathbb{C}^*)^\Lambda\).

The expression \( e^{\langle \phi | a \rangle} \) has to be understood as a pairing, i.e. a map
\[
e^{\langle \phi | a \rangle} : T \times \Lambda \to \mathbb{C}^*.
\]
This duality expresses the fact that \( \Lambda \) is the group of algebraic characters of \( T \).

The class of a vector \( \phi \in U_{\mathbb{C}} \) in \( T \) will be denoted by \( e^\phi \) so that the value of \( e^a \) in \( e^\phi \) is \( e^{\langle \phi | a \rangle} \).
If we fix an integral basis $e_i$ for $\Lambda$ and set $x_i := e^{e_i}$, we see that $\mathbb{C}[\Lambda] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_s^{\pm 1}]$ is the ring of Laurent polynomials, i.e. the coordinate ring of algebraic functions on the standard torus $(\mathbb{C}^\times)^s$.

The equation $\prod_{a \in X} (1 - e^{-a}) = 0$, defines an hypersurface $Y$ in $T$, the union of the kernels of all the characters $e^a, a \in X$. The algebra $S_X$ is clearly the coordinate ring of algebraic functions on the standard torus $(\mathbb{C}^\times)^s$.

Given a basis $b$ extracted from $X$, consider the lattice $\Lambda_b \subset \Lambda$ that it generates in $\Lambda$.

We have that $\Lambda/\Lambda_b$ is a finite group of order $[\Lambda : \Lambda_b] = |\text{det}(b)|$.

Its character group is the finite subgroup $T(b)$ of $T$ which is the intersection of the kernels of the functions $e^a$ as $a \in b$.

We now define the points of the arrangement

$$(31) \quad P(X) := \cup_{b \in B(X)} T(b).$$

For any point in $P(X)$ we choose once and for all a representative $\phi \in U$ so that the given point equals $e^\phi$. We will denote by $\tilde{P}(X)$ the corresponding set of representatives. We now set:

$$(32) \quad X_\phi := \{a \in X | e^{\phi|a} = 1\}.$$

The points of the arrangement form the zero dimensional pieces of the entire toric arrangement. By this we mean the finite set formed of all connected components of all the intersections of the hypersurfaces of equations $1 - e^{-a} = 0, a \in X$. For one such intersection the connected component through 1 is a subtorus $T'$, these other components are cosets of $T'$.

7.10. $S_X$ as a module. As in Section 2.11 and following [22] for every $k \leq s$, let us consider the submodule $S_{X,k} \subset S_X$ spanned by the elements

$$f \prod_i (1 - e^{-a_i})^{h_i}$$

such that the vectors $a_i$ which appear in the denominator with positive exponent span a subspace of dimension $\leq k$.

Given a point $e^\phi \in P(X)$ and a basis $b := \{a_1, \ldots, a_s\}$ extracted from $X_\phi$, we have seen that $e^\phi$ belongs to the finite group $T(b)$, in duality with $\Lambda/\Lambda_b$. The finite dimensional group algebra $\mathbb{C}[\Lambda/\Lambda_b]$ is identified with the functions on $T(b)$.

We choose a set $R_b$ of representatives in $\Lambda$ for the cosets $\Lambda/\Lambda_b$ by taking for each coset the unique representative of the form

$$\sum_{a \in b} p_a a, \quad 0 \leq p_a < 1, p_a \in \mathbb{Q}.$$

Character theory tells us that the element $e(\phi) := |\text{det}(b)|^{-1} \sum_{\lambda \in R_b} e^{-\langle \lambda | \phi \rangle} e^\lambda$ has the property of taking the value 1 on the point $e^\phi$ and 0 on all the other points of $T(b)$. Moreover:
Proposition 7.11. \( e(\phi) \prod_{a \in b} (1 - e^{-a})^{-1} \) is the Laplace transform of the distribution
\[
|\det(b)|^{-1} \sum_{v \in C(b) \cap \Lambda} e^{(\phi|v)} \delta_v.
\]

Proof. By our choice of representatives, if \( v \in C(b) \cap \Lambda \) we can write \( v = \lambda + \sum_{a \in b} n_a a, \) \( n_a \in \mathbb{N}, \lambda \in R_b. \)

In \( \Pi_X := S_X/S_{X,s-1} \) the class
\[
\omega_{b,\phi} := \left[ \frac{e(\phi)}{\prod_{a \in b} (1 - e^{-a})} \right], \text{ mod } (S_X,s-1),
\]
is clearly independent of the chosen representatives and is an eigenvector for \( \Lambda \) of eigenvalue \( e^{\phi}. \) Indeed
\[
e^{\mu}(\omega_{b,\phi}) = e^{(\mu|\phi)} \left[ \frac{|\det(b)|^{-1} \sum_{\lambda \in R_b} e^{-\langle \lambda + \mu | \phi \rangle} e^{\lambda + \mu}}{\prod_{a \in b} (1 - e^{-a})} \right] = e^{(\mu|\phi)} \omega_{b,\phi}
\]
by the independence from the chosen representatives. One can show [22]:

1. The \( W(V) \) module \( \Pi_X \) is semisimple of finite length.
2. The isotypic components of \( \Pi_X \) are indexed by the points of the arrangement.
3. Given \( e(\phi) \in P(X) \), the corresponding isotypic component \( \Pi_X(\phi) \) is the direct sum of the irreducible modules \( \Pi_{b,\phi} \) generated by the classes \( \omega_{b,\phi} \) and indexed by the no broken bases extracted from \( X_\phi. \)
4. As a module over the ring \( S[U] \) of differential operators on \( V \) with constant coefficients, \( \Pi_{b,\phi} \) is free of rank 1 with generator the class \( \omega_{b,\phi}. \)

Although we do not want to reproduce here all the details of [22], it is not too difficult to reconstruct all these statements, in particular by using the ideas of section 9.18.

In fact, since \( \mathcal{W}(V) = S[U] \otimes \mathbb{C}[\Lambda], \) we have:
\[
\Pi_{b,\phi} = \mathcal{W}(V) \otimes_{\mathbb{C}[\Lambda]} \omega_{b,\phi} = S[U] \omega_{b,\phi}
\]

Theorem 7.7 tells us that we can transport our filtration on \( S_X \) to one on \( \mathcal{L}_X. \) By formula (30), in filtration degree \( \leq k \) we have those distributions which are supported in a finite number of translates of the sets \( \Lambda \cap C(A) \) where \( A \) spans a lattice of rank \( \leq k. \)

Remark 7.12. In (22) there is a similar explicit description of the piece \( S_{X,k}/S_{X,k-1} \), for each \( k \geq 0, \) using the varieties of the arrangement, of codimension \( k. \) The proofs are in the same lines as the ones we developed in the simpler case of hyperplanes with Proposition 3.4.
7.13. **Local expression for** $T_X$. Let us now consider the element $v_X$ class of the generating function $\prod_{x \in X}(1 - e^{-x})^{-1}$ in $\Pi_X$. Decompose it uniquely as a sum of elements $v_{X,\phi}$ in $\Pi_X(\phi)$. By what we have seen in the previous section, each one of these elements is expressed as a sum

$$v_{X,\phi} = \sum_{\text{\mathcal{B} \in \mathbb{N}B_{X,\phi}}} q_{\text{\mathcal{B}},\phi} \omega_{\text{\mathcal{B}},\phi}$$

for suitable polynomials $q_{\text{\mathcal{B}},\phi}$. Thus,

(33) \[ v_X = \sum_{\phi \in \tilde{P}(X)} \sum_{\text{\mathcal{B} \in \mathbb{N}B_{X,\phi}} | \text{C}(\text{\mathcal{B}})} q_{\text{\mathcal{B}},\phi}(-x). \]

We are now ready to state but, not yet fully justify, the main formula that one can effectively use for computing the partition function $P_X$:

**Theorem 7.14.** Given a point $x$ in the closure of a big cell $\mathcal{C}$ we have

(34) \[ P_X(x) = \sum_{\phi \in \tilde{P}(X)} e^{\phi} \sum_{\text{\mathcal{B} \in \mathbb{N}B_{X,\phi}} | \text{C}(\text{\mathcal{B}})} q_{\text{\mathcal{B}},\phi}(-x). \]

**Proof.** Using Propositions 7.11 and formulas (26) and (33), we deduce that the right handside of (34) coincides with the partition function on the cone $C(X)$ minus possibly a finite number of translates of lower dimensional cones.

In particular we deduce that the partition function $P_X$ coincides locally with a quasi polynomial for the lattice $\Lambda$ on the cone $C(X)$ minus possibly a finite number of translates of lower dimensional cones. Also, since to get this result we have only used the fact that each vector in $X$ lies in $\Lambda$, we can substitute $\Lambda$ with the larger lattice $\Lambda/n$ and we also get that the partition function also coincides locally with a quasi polynomial for the lattice $\Lambda/n$ on the cone $C(X)$ minus possibly a finite number of translates of lower dimensional cones. By the continuity of quasipolynomials we deduce that our Theorem will follow from the following Proposition which will be proved in section 8.9:

**Proposition 7.15.** On the closure of each big cell $\mathcal{C}$, the partition function $P_X$ coincides with a quasi polynomial for some lattice $\Lambda/n$.

8. **Partial fractions**

8.1. **Algebraic identities.** We follow the approach of Szenes and Vergne, used in [37] (and also in [22]) to prove Proposition 7.15 and hence finish the proof of Theorem 7.14.

We do not need a very fine analysis, let us consider the set $\tilde{X}$ formed by all the positive rational multiples of the vectors in $X$.

Observe that all the cones generated by subsets of elements of $\tilde{X}$ coincide with the ones generated by subsets of $X$. In other words the geometry of $C(X)$ does not depend on the denominators that we will introduce.
The following lemma is a simple computation

**Lemma 8.2.** Let \( b \) be a set of linearly independent vectors in a lattice \( M \), \( e^{\varphi} \) a character of the lattice spanned by \( b \), \( n_c \) a positive integer for all \( c \in b \). If we expand

\[
\prod_{c \in b} (1 - e^{\langle \varphi | c \rangle} - n_c) = \sum_{y \in M \cap C(b)} Q(y) e^{-y},
\]

we have for \( y = \sum_{c \in b} k_c c \), \( k_c \geq 0 \) for all \( c \in b \)

\[
Q(y) = \prod_{c \in b} e^{\langle \varphi | y \rangle} \left( n_c + k_c - 1 \right).
\]

Notice that this lemma give immediately Proposition 7.15 in the special case in which \( X \) consists of the elements of a basis (contained in the lattice \( \Lambda \)) each repeated any number of times.

Our next task is to reduce to this case. For this we need to show

**Proposition 8.3.** Let \( \gamma = \sum_{a \in X} r_a a \) with \( r_a \in \mathbb{Q} \), \( 0 \leq r_a \leq 1 \), then the function \( e^{-\gamma} \prod_{a \in X} (1 - e^{\langle \psi | a \rangle})^{-1} \), with \( e^{\psi} \) a character, can be written as a linear combination with constant coefficients of elements of the form

\[
\prod_{c \in b} (1 - e^{\langle \varphi | c \rangle}) - n_c
\]

with \( b \) a linearly independent set of elements in \( \tilde{X} \). Furthermore if \( e^{\psi} \) is of finite order, each of the \( e^{\varphi} \) has finite order on the lattice spanned by \( \tilde{b} \).

Before giving the proof of this Proposition we need to show some simple identities.

**Lemma 8.4.**

(35) \[
\frac{n}{1 - x^n} = \sum_{i=0}^{n} \frac{1}{1 - \zeta^i x}, \quad \zeta := e^{2\pi i/n}.
\]

**Proof.** Take an auxiliary variable \( t \) and the logarithmic derivative (relative to \( t \))

\[
\frac{n t^{n-1}}{t^n - x^n} dt = d \log(t^n - x^n) = d \log(\prod_{i=0}^{n-1} (t - \zeta^i x))
\]

\[
= \sum_{i=0}^{n-1} d \log(t - \zeta^i x) = \sum_{i=0}^{n-1} \frac{1}{t - \zeta^i x} dt.
\]

Next set \( t = 1 \) in the coefficients of \( dt \). \( \square \)

**Lemma 8.5.**

(36) \[
1 - \prod_{i=1}^{r} z_i = \sum_{\emptyset \subsetneq I \subset \{1, \ldots, r\}} \prod_{i \in I} (-1)^{|I|+1} (1 - z_i)
\]
Proof. The proof is by induction on \( r \). The case \( r = 1 \) is clear. In general, using the inductive hypothesis, we have

\[
\sum_{\emptyset \subseteq I \subseteq \{1,\ldots,r\}} \prod_{i \in I} (1 - z_i) = \sum_{\emptyset \subseteq I \subseteq \{1,\ldots,r-1\}} \prod_{i \in I} (1 - z_i) - \sum_{I \subseteq \{1,\ldots,r-1\}} \prod_{i \in I} (1 - z_i)(1 - z_r) = \]

\[
= 1 - \prod_{i=1}^{r} z_i.
\]

A variation of this formula is the following:

**Lemma 8.6.** Set \( t = \prod_{i=1}^{h} z_i \prod_{i=h+1} z_i^{-1}, a \in \mathbb{C}^* \)

\[1 - t = \sum_{\emptyset \subseteq I \subseteq \{1,\ldots,h\}} \prod_{i \in I} (1 - z_i) - a^{-1} \sum_{\emptyset \subseteq I \subseteq \{h+1,\ldots,r\}} \prod_{i \in I} (1 - z_i) + \]

\[
(37) \quad a^{-1} \sum_{\emptyset \subseteq I \subseteq \{h+1,\ldots,r\}} \prod_{i \in I} (1 - z_i)(1 - at).
\]

**Proof.** This is immediate from the previous Lemma once we remark that

\[1 - t = 1 - \prod_{i=1}^{h} z_i - t(1 - \prod_{i=h+1} z_i).\]

From this we get,

**Lemma 8.7.** Set \( t = \prod_{i=1}^{h} z_i \prod_{i=h+1} z_i^{-1} \) with \( 0 \leq h \leq r \). Then

\[
\frac{1}{\prod_{i=1}^{r} (1 - z_i)} = \sum_{\emptyset \subseteq I \subseteq \{1,\ldots,h\}} \frac{(-1)^{|I|+1}}{(1 - t) \prod_{i \in I} (1 - z_i)} - \]

\[
(38) \quad - \sum_{\emptyset \subseteq I \subseteq \{h+1,\ldots,r\}} \frac{(-1)^{|I|+1}}{\prod_{i \in I} (1 - z_i)} - \frac{(-1)^{|I|+1}}{(1 - t) \prod_{i \in I} (1 - z_i)}.
\]

If \( a \in \mathbb{C}^* \) and \( a \neq 1 \)

\[
\frac{1}{\prod_{i=1}^{h} (1 - z_i)(1 - at)} = \frac{a}{a - 1} \left( \sum_{\emptyset \subseteq I \subseteq \{1,\ldots,h\}} \frac{(-1)^{|I|+1}}{(1 - at) \prod_{i \in I} (1 - z_i)} - \right.
\]

\[
(39) \quad - \frac{1}{a-1} \sum_{\emptyset \subseteq I \subseteq \{h+1,\ldots,r\}} \frac{(-1)^{|I|+1}}{\prod_{i \in I} (1 - z_i)} - \left( \frac{(-1)^{|I|+1}}{(1 - at) \prod_{i \in I} (1 - z_i)} \right) + \frac{1}{\prod_{i=1}^{r} (1 - z_i)}.
\]
Proof. The first relation follows from dividing by \((1-t)(1-z_1)(1-z_2)\cdots(1-z_r)\) and taking \(a = 1\).

The second writing

\[
1 = \frac{a}{a-1}(1-t) - \frac{1}{a-1}(1-at)
\]

and then dividing by \((1-at)(1-z_1)(1-z_2)\cdots(1-z_r)\). \qed

Lemma 8.8. Given elements \(a_i \in \tilde{X}, \ b = \sum_{i=1}^{M}(n_i/n)a_i\ 0 \leq n_j \leq n\), and a character \(e^n\) of the lattice spanned by the \(a_i\), we can write the element

\[
\prod_{i=1}^{M}(1-e^{\langle \eta|a_i\rangle-a_i}) = \prod_{i=1}^{M}e^{(n_i/n)a_i}
\]

as a linear combination with constant coefficients of elements of the form

\[
\frac{1}{(1-e^{\langle \zeta|c_1\rangle-c_1})\cdots(1-e^{\langle \zeta|c_r\rangle-c_r})}
\]

where the \(c_i \in \tilde{X}\) and \(e^\zeta\) is a character of the lattice spanned by the \(c_i\). Furthermore if \(e^n\) is of finite order, also each \(e^\zeta\) is of finite order.

Proof. Start from the identity

\[
\frac{e^{-a}}{1-e^{\langle \eta|a\rangle-a}} = \frac{e^{-\langle \eta|a\rangle} - e^{-\langle \eta|a\rangle} (1-e^{\langle \eta|a\rangle-a})}{1-e^{\langle \eta|a\rangle-a}} = \frac{e^{-\langle \eta|a\rangle} - e^{-\langle \eta|a\rangle}}{1-e^{\langle \eta|a\rangle-a}}
\]

from which we get that, if \(k \leq n\) and \(e^{\zeta_i}, i = 1,\ldots, n\) are the \(n^{th}\) roots of \(e^n\):

\[
\frac{e^{-ka/n}}{1-e^{\langle \zeta|a\rangle-a/n}} = \prod_{i=1}^{k}\frac{e^{-a/n}}{1-e^{\langle \zeta|a\rangle-a/n}} \quad \prod_{i=k+1}^{n}\frac{1}{1-e^{\langle \zeta|a\rangle-a/n}}
\]

can be expanded as a linear combination of products of elements \((1-e^{\langle \zeta|a\rangle-a/n})^{-1}\). We now apply this procedure to each term of our product. The fact that if \(e^n\) is of finite order also the \(e^\zeta\) are of finite order follows immediately from their definition. \qed

8.9. The main expansions. We can now give

Proof. (of Proposition 8.3) From Lemma 8.8 we deduce that we can assume that \(\gamma = 0\).

Also there is nothing to prove unless we have a linear dependency \(na_0 = \sum_i n_i a_i, n_i \in \mathbb{Z}\) which we write as

\[
b_0 := \frac{a_0}{\prod n_i} = \sum \pm \frac{a_i}{\prod_{j \neq i} n_j} = \sum \pm b_i.
\]

We substitute now \(a_0\) with \(b_0 \pm \prod n_i\), and for each product \(a_i \pm \prod n_i\), next apply lemma 8.4, using suitable characters \(e^\zeta\). We get a new expansion in which the \(b_i\) appear in place of the \(a_i\).

For each product \(\prod_{i=0}^{k}(1-e^{\langle \zeta|b_i\rangle-b_i})^{-1}\) we have two possibilities. If the equations \(1-e^{\langle \zeta|b_i\rangle-b_i} = 0\) are incompatible we are in position to apply the
formula \((39)\) to \(x_i = \zeta_ie^{b_i}, i \geq 1\) and to \(t = \zeta_0e^{b_0}\). If they are compatible we apply formula \((38)\). In both cases we substitute the product with a linear combination of products with less terms in the denominator. We then proceed by induction. □

We can now finish the Proof of Proposition 7.15. Using Proposition 8.3 and Lemma 8.2, we get that for any \(\gamma = \sum_{a \in X} r_a a\) with \(r_a \in \mathbb{Q}, 0 \leq r_a \leq 1\), the function \(e^{-\gamma}\prod_{a \in X}(1 - e^{-a})^{-1}\) is a quasipolynomial in the interior of the big cells. From this we deduce that

\[
\prod_{a \in X}(1 - e^{-a}) = e^{\gamma}e^{-\gamma}\prod_{a \in X}(1 - e^{-a})^{-1}
\]

is a quasipolynomial on the big cells translated by \(\gamma\). Since as \(\gamma\) varies these translates cover each big cell, our claim follows. □

8.10. \(E\)-splines. In our setting we can also consider the Euler–Maclaurin sums (cf. [9],[10]):

\[
\mathcal{P}_{X,\mu}(v) := \sum_{(n_1, \ldots, n_N) \in P(v)} e^{-\sum n_i \mu_i},
\]

\[
P(v) := \{(n_1, \ldots, n_N) \mid \sum n_i a_i = v, n_i \in \mathbb{N}\}.
\]

\(\mathcal{P}_{X,\mu}(v)\) must be understood as the distribution \(\sum_{v \in \Lambda \cup C(X)} \mathcal{P}_{X,\mu}(v)\delta_v\) supported at the points of the lattice \(\Lambda \cup C(X)\).

Its Laplace transform is

\[
\sum_{v \in \Lambda \cup C(X)} \mathcal{P}_{X,\mu}(v)e^{-\langle x | v \rangle} = \prod_{a \in X} (1 - e^{-a - \mu a})^{-1}.
\]

This leads to a similar theory in which now in general there are more points in the arrangement, in the generic case \(\delta(X)\) (see definition 44), distinct points, and correspondingly more exponential functions appear in the final formulas. We leave the details to the reader and also refer to section 9.9 where we approach the same topic from a different perspective.

A particularly interesting case is when the numbers \(e^{\mu_a} = \nu_a = \chi(a)\) for a character \(\chi\) of \(\Lambda\), in particular for a character of finite order which is 1 on a sublattice \(\Lambda_0\) of finite index. Then we have that, if \(\sum_{a \in X} n_a a = v\) we have \(e^{-\sum_{a \in X} n_a \mu_a} = \chi(v)\). In this case we denote \(\mathcal{P}_{X,\mu} = \mathcal{P}_{X,\chi}\). Thus we have that

\[
\mathcal{P}_{X,\chi} = \sum_{v \in \Lambda \cup C(X)} \chi(v)\mathcal{P}_{X}(v)\delta_v
\]

and

\[
\mathcal{P}_{X}|_{\Lambda_0} = |\Lambda/\Lambda_0|^{-1} \sum_{\chi \in \hat{\Lambda}/\hat{\Lambda}_0} \mathcal{P}_{X,\chi}
\]

where \(\mathcal{P}_{X}|_{\Lambda_0}\) is the restriction of \(\mathcal{P}_{X}\) to \(\Lambda_0\). In a similar way, by applying translations we can restrict to the other cosets.
9. Difference equations

9.1. Difference operators. Let us consider the space of complex valued functions \( f \) on the set \( \Lambda \). We identify this space with the algebraic dual \( \mathbb{C}[\Lambda]^* \) of \( \mathbb{C}[\Lambda] \) by the formula:

\[
\langle f \mid e^a \rangle := f(a).
\]

For \( v \in \Lambda \) we define the difference operator \( \nabla_v \), acting on functions \( f \in \mathbb{C}[\Lambda]^* \) as:

\[
\nabla_v f(u) := f(u) - f(u - v).
\]

Parallel to the study of \( D(X) \), we start by studying the difference equations

\[
\nabla_Y f = 0, \quad \text{where} \quad \nabla_Y := \prod_{v \in Y} \nabla_v \text{ as } Y \in \mathcal{E}(X) \text{ runs over the cocircuits.}
\]

Let us denote the space of solutions by:

\[
(41) \quad \nabla(X) := \{ f : \Lambda \to \mathbb{C}, \ | \ \nabla_Y(f) = 0, \ \forall Y \in \mathcal{E}(X) \}.
\]

As we shall see, this space is not only a formal construct, but it plays an essential role in the theory of partition functions.

We want to reformulate the fact that a function is a solution of a system of difference equations as the property for such a function to vanish on an appropriate ideal \( J_X \) of \( \mathbb{C}[\Lambda] \).

Notice that, the ideal \( I_1 \) of functions in \( \mathbb{C}[\Lambda] \) vanishing at \( 1 \in T \) has as linear basis the elements \( 1 - e^{-a}, \ a \in \Lambda, \ a \neq 0 \). If one takes another point \( e^\phi \) the ideal \( I_\phi \) of functions in \( \mathbb{C}[\Lambda] \) vanishing at \( e^\phi \), has as linear basis the elements \( 1 - e^{-a+\langle a \mid \phi \rangle}, \ a \in \Lambda, \ a \neq 0 \).

In fact, for every \( a, x \in \Lambda \) we have:

\[
\langle \nabla_a f \mid e^x \rangle = \langle f \mid (1 - e^{-a})e^x \rangle.
\]

Thus the difference operator \( \nabla_a \) is the dual of the multiplication operator by \( 1 - e^{-a} \). In this setting, we get a statement analogous to that of Theorem 5.3 on differential equations:

**Theorem 9.2.** A function \( f \) on \( \Lambda \) satisfies the difference equation

\[
p(\nabla_{a_1}, \ldots, \nabla_{a_k})f = 0 \quad \text{with } p \text{ a polynomial if and only if, thought of as element of the dual of } \mathbb{C}[\Lambda], \ f \text{ vanishes on the ideal of } \mathbb{C}[\Lambda] \text{ generated by the element } p(1 - e^{-a_1}, \ldots, 1 - e^{-a_k}).
\]

We have also the twisted difference operators \( \nabla_a^\phi, \ e^\phi \in T \) defined by

\[
(\nabla_a^\phi f)(x) := f(x) - e^{\langle a \mid \phi \rangle}f(x - a), \ \text{dual to multiplication by } 1 - e^{-a+\langle a \mid \phi \rangle}.
\]

We need some simple algebraic considerations on subschemes of \( T \) supported at a point \( e^\phi \in T \). Let \( S =: \mathbb{C}[\Lambda]/J \) be the (finite dimensional) coordinate ring of such a scheme.

In this slightly more general setting, we can repeat the discussion made in section 7.13. Given an element \( a \in \Lambda \), we have that \( e^{-\langle a \mid \phi \rangle}e^a - 1 \) is 0 in \( \phi \). Thus, in the ring \( S \), the class of \( e^{-\langle a \mid \phi \rangle}e^a - 1 \) is nilpotent. The power series
of \( \log(1 + t) = t - t/2 + t^2/3 - \ldots \) can be computed on \( t = e^{-(a \mid \phi)} e^a - 1 \) and we defines \( \underline{a} := \log(1 + (e^{-(a \mid \phi)} e^a - 1)) + \langle a \mid \phi \rangle. \)

We clearly have:

**Lemma 9.3.** The map \( i : a \to \underline{a} \) is a linear map.

\( \hat{\underline{a}} := a - \langle a \mid \phi \rangle \) is nilpotent.
\( e^{\hat{\underline{a}}} = \exp(\hat{\underline{a}}) e^{\langle a \mid \phi \rangle} \in S \) equals the class of \( e^a \).
\( i \) extends to a linear map \( i : V \to S \) and then to a surjective homomorphism \( \overline{i} : S[V] \to S. \)

**Proof.** The class \( e^{-\langle a \mid \phi \rangle + a} = \exp(\hat{\underline{a}}) = \sum_{k \geq 0} \hat{\underline{a}}^k/k! \) reduces to a finite sum lying in the image of the homomorphism \( \overline{i} \) which is therefore surjective. \( \square \)

We have thus identified \( S \) with a quotient \( S[V]/I \) by an ideal of finite codimension in \( S[V]. \)

The fact that, for each \( a \in \Lambda \) we have \( a - \langle a \mid \phi \rangle \) is nilpotent, means that \( I \) defines a unique point in which of course \( \underline{a} = \langle a \mid \phi \rangle. \) That is \( I \) defines the point \( \phi \) (notice that the choice of \( \phi \) is not unique).

We thus can identify, using this algebraic logarithm, the given scheme as a subscheme in the tangent space. This proves the following:

**Proposition 9.4.** Let \( J \subset \Lambda \) be an ideal such that \( \mathbb{C} [\Lambda]/J \) defines (set theoretically) the point \( e^\phi. \) Under the logarithm isomorphism \( \mathbb{C} [\Lambda]/J \) becomes isomorphic to a ring \( S[V]/I \) defining the point \( \phi. \)

In particular we get a canonical isomorphism of the space of solutions of the difference equations given by \( J \) with the space of solutions of the differential equations given by \( I. \)

The explicit formula is, given \( f \in S^* = (S[V]/I)^*: \)

\[
(42) \quad f(a) := \langle f \mid e^a \rangle = e^{\langle a \mid \phi \rangle} \langle f \mid \exp(\hat{\underline{a}}) \rangle.
\]

where \( \langle f \mid \exp(\hat{\underline{a}}) \rangle \) is a polynomial in \( a. \)

A special role is played by the points of finite order \( m, \) i.e. characters \( e^\phi \) on \( \Lambda \) whose kernel is a sublattice \( \Lambda_\phi \) of index \( m \) (of course this implies \( m \phi \in \Lambda^*. \))

As we have seen, a function \( f(x) \) appearing in the dual of \( \mathbb{C} [\Lambda]/J, \) when \( J \) defines (set theoretically) the finite order point \( e^\phi, \) is of the form \( f(x) := e^{\langle x \mid \phi \rangle} g(x) \) with \( g(x) \) a polynomial.

Since \( e^\phi \) is of finite order, we know \( e^{\langle x \mid \phi \rangle} \) takes constant values \( (m-\text{th roots of } 1) \) on the \( m \) cosets of the sublattice \( \Lambda_\phi. \) Thus \( f(x) \) is a polynomial only on each such coset. This is a typical example of what is called a periodic polynomial or quasi-polynomial.

Formally:

**Definition 9.5.** A function \( f \) on a lattice \( \Lambda \) is a periodic polynomial, if there exists a sublattice \( \Lambda^0 \) of finite index in \( \Lambda, \) such that \( f, \) restricted to each coset of \( \Lambda^0 \) in \( \Lambda, \) is (the restriction of) a polynomial.
We deduce the general:

**Theorem 9.6.** Let $J \subset \mathbb{C}[\Lambda]$ be an ideal such that $\mathbb{C}[\Lambda]/J$ is finite dimensional and defines (set theoretically) the points $e^{\phi_1}, \ldots, e^{\phi_k}$, all of finite order.

Then, the space $(\mathbb{C}[\Lambda]/J)^*$ of solutions of the difference equations associated to $J$ is a direct sum of spaces of periodic polynomials $e^{(a|\phi_i)} p(a)$ for the points $\phi_i$, each invariant under translations under $\Lambda$.

Although we will not need it, for completeness we now want to prove a converse. Let us thus take a finite dimensional vector space $Q$ spanned by periodic polynomials $e^{(a|\phi_i)} p(a)$ for some points $e^{\phi_i}$ of finite order) invariant under translations under $\Lambda$. We need the following:

**Lemma 9.7.** Any non zero subspace $M$ of $Q$, invariant under translations under $\Lambda$, contains one of the functions $e^{(a|\phi_i)}$.

**Proof.** Indeed, a non zero subspace invariant under translations under $\Lambda$ contains a non zero common $\Lambda$-eigenvector. This eigenvector must necessarily be a multiple of one of the $e^{(a|\phi_i)}$. 

Let $J$ be the ideal of the difference equations satisfied by $Q$.

**Theorem 9.8.** $J$ is of finite codimension and $Q$ is the space of all solutions of $J$.

**Proof.** We can easily reduce to the case of a unique point $\phi$. In fact the commuting set of linear operators $\Lambda$ on $Q$ has a canonical Fitting decomposition relative to the generalized eigenvalues which are characters of $\Lambda$. Our hypotheses imply that these points are of finite order.

For any $p \in Q$ and $b \in \Lambda$, we set $\tau_b p = p(x + b)$. Consider the function $j(p)$ on $\Lambda$ given by

\[
(j(p))(b) = p(b) = (\tau_b p)(0).
\]

By definition $j(\tau_a p)(b) = j(p)(a + b)$ so $j : Q \to \mathbb{C}[\Lambda]^*$ is equivariant. We have $j(e^{(a|\phi)}) (b) = e^{(b|\phi)}$. In particular $e^{(a|\phi)}$ is not in the kernel of $j$. Since $j$ is equivariant, Lemma 9.7 implies that $j$ is injective, and we deduce a surjective map $\mathbb{C}[\Lambda] \to Q^*$.

We give to $Q^*$ an algebra structure by defining the product of two linear forms as follows. For $f(a) = e^{(a|\phi)} p(a) \in Q$ we have that, by translation invariance, $f(a + b) = \sum_i e^{(a|\phi)} p_i(a) e^{(b|\phi)} q_i(b)$ with $e^{(a|\phi)} p_i(a), e^{(b|\phi)} q_i(b) \in Q$. This can be thought of as a *coalgebra structure* $\Delta : Q \to Q \otimes Q$ on $Q$.

By duality this induces an algebra structure for which $j^*$ is a surjective algebra homomorphism. Thus $Q^*$ is naturally isomorphic to $\mathbb{C}[\Lambda]/J$ for some ideal $J$.

Now consider the dual map $\pi : (\mathbb{C}[\Lambda]/J)^* \to Q$. By the first theorem $(\mathbb{C}[\Lambda]/J)^*$ is identified to a space of periodic polynomials. This identification coincides with the one given by $\pi$, due to the definition of $j$. 

\[\square\]
9.9. The difference Theorem. We go back to the equations \( \nabla_Y f = 0 \). As in Section 4.3, we can generalize our problem by considering, for any sequence \( \nu = \{\nu_v | v \in X\} \) of non zero complex numbers, the products \( N_Y(\nu) := \prod_{v \in Y} (1 - \nu_v e^{-v}) \) as \( Y \) runs over the cocircuits, and the ideal \( J_X(\nu) \) that they generate. We set \( J_X := J_X(1, \ldots, 1) \).

The space of functions \( \nabla(X) \), which we want to describe, is thus identified to the dual of \( \mathbb{C}[\Lambda]/J_X \).

It turns out that also this space is finite dimensional and we are going to show that its dimension is the following weighted analogue \( \delta(X) \) of \( d(X) \).

We use the notations of section 7.9. Given a basis \( b \) extracted from \( X \) consider the lattice \( \Lambda_b \subset \Lambda \) that it generates in \( \Lambda \), \( \Lambda/\Lambda_b \) is a finite group of order \( [\Lambda : \Lambda_b] = |\det(\bar{b})| \), its character group is the finite subgroup \( T(\bar{b}) \) of \( T \) which is the intersection fo the kernels of the characters \( e^a \) as \( a \in \bar{b} \).

We now define,

\[
(44) \quad \delta(X) := \sum_{h \in \mathcal{B}(X)} |\det(\bar{h})|.
\]

We have:

**Proposition 9.10.**

\[
(45) \quad \sum_{e^\phi \in P(X)} d(X_\phi) = \delta(X).
\]

**Proof.** \( \delta(X) \) counts the number of pairs \( e^\phi, \bar{h} \) such that \( e^\phi \in T(\bar{h}) \), or equivalently \( \bar{h} \subset X_\phi \). \( \square \)

**Lemma 9.11.** The variety defined by the ideal \( J_X \) is \( P(X) \).

**Proof.** The proof is identical to that of lemma 4.4 and of the first part of Theorem 4.6, the only difference being the fact, that when we extract a basis \( h \) from \( X \) the equations \( e^a - 1 = 0 \) as \( a \in \bar{h} \) define \( T(\bar{h}) \). By its definition \( P(X) = \bigcup T(\bar{b}) \). \( \square \)

Now by elementary commutative algebra we know that

\[
\mathbb{C}[\Lambda]/J_X = \bigoplus_{e^\phi \in P(X)} \mathbb{C}[\Lambda]/J_X(\phi)
\]

where \( \mathbb{C}[\Lambda]/J_X(\phi) \) is its localization at \( e^\phi \).

**Theorem 9.12.** Under the logarithm isomorphism \( \mathbb{C}[\Lambda]/J_X(\phi) \) becomes isomorphic to the ring \( A_{X_\phi} = S[V]/I_{X_\phi} \). Thus:

\[
(46) \quad \mathbb{C}[\Lambda]/J_X \cong \bigoplus_{e^\phi \in P(X)} A_{X_\phi}.
\]

In particular \( \dim(\mathbb{C}[\Lambda]/J_X) = \delta(X) \).

**Proof.** Let us see what happens to the equations \( N_Y = \prod_{v \in Y} (1 - e^{-v}) \) as \( Y \) runs over the cocircuits.

When we localize at \( e^\phi \) the elements \( 1 - e^{-v} \) where \( v \notin X_\phi \) become invertible and hence can be dropped from the equations. As for \( 1 - e^{-v} \) where \( v \in X_\phi \) we have that \( 1 - e^{-v} = v(1 - \sum_{k \geq 1} (-v)^k/(k+1)!)) \).
Obviously \((1 - \sum_{k \geq 1} (-v)^k/(k + 1)!)\) is invertible hence we can replace the equation \(NY = \prod_{v \in Y} v\). We obtain the equations defining \(AX_\phi\) and this proves the claim completely.

\[\square\]

It is convenient to single out, in this direct sum, the term relative to \(e \phi = 1\) which is \(D(X)\) and write

\[(47) \quad E(X) := \bigoplus_{e \phi \in P(X), e \phi \neq 1} e \phi D(X_\phi), \quad \nabla(X) = D(X) \oplus E(X).\]

9.13. **The parametric case.** One could treat in a similar way the parametric case.

We take a sequence \(\nu := \{\nu_a | a \in X\}\).

The main difference is that now the points of the arrangement are defined as follows.

Given a basis \(b\) extracted from \(X\) instead of the finite subgroup \(T(b)\), intersection of the kernels of the characters \(e^a\) as \(a \in b\) we have to consider the set \(T(\nu)\) where \(\nu_a e^{-a} - 1 = 0\) this is a coset of the subgroup \(T(b)\), still consisting of \(|\det(b)|\) elements.

For a generic sequence \(\nu\) the ideal \(J_X(\nu)\) defines \(\delta(X)\) distinct and reduced points, while for special values we may have less points but in each such point a similar space of quasi periodic polynomials.

So now we define (as for \((41)\))

\[P_\nu(X) := \bigcup_{b \in B(X)} T(\nu)\].

For any \(e \phi \in P_\nu(X)\) set:

\[(48) \quad X_\phi(\nu) := \{a \in X | e^{(a) \phi} = \nu_a\}\].

We have as usual the analogues of the results in the non parametric case:

**Lemma 9.14.** The variety defined by the ideal \(J_X(\nu)\) is \(P_\nu(X)\).

\[\mathbb{C}[\Lambda]/J_X(\nu) = \bigoplus_{e \phi \in P_\nu(X)} \mathbb{C}[\Lambda]/J_X(\nu)(\phi)\]

where \([\mathbb{C}[\Lambda]/J_X(\nu)](\phi)\) is its localization at \(e \phi\).

Finally

**Theorem 9.15.** Under the logarithm isomorphism \(\mathbb{C}[\Lambda]/J_X(\phi)\) becomes isomorphic to the ring \(S[V]/I_{X_\phi}\). Thus:

\[(49) \quad \mathbb{C}[\Lambda]/J_X(\nu) \cong \bigoplus_{e \phi \in P_\nu(X)} S[V]/I_{X_\phi}(\nu)\].

For all \(\nu\), we have \(\dim(\mathbb{C}[\Lambda]/J_X(\nu)) = \delta(X)\).

Of course also in this case we could perform the polygraph construction, now the base would be a torus \(\mathbb{C}^N\). We still have a flat family and the polygraph is a Cohen Macaulay variety as in Remark \((41)\).
One can use the parametric case as follows. Suppose that we pass, from the lattice $\Lambda$ to a finer lattice $\Lambda/n$ for some positive integer $n$ then, for $a \in \Lambda$ we write in $\mathbb{C}[\Lambda/n]$

$$1 - \mu e^{-a} = \prod_{\gamma \mid \gamma^n = \mu} (1 - e^{-a/n})$$

Thus if we take our list $X = \{a_1, \ldots, a_N\}$ with parameters $\nu = \{\nu_1, \ldots, \nu_N\}$ in $\Lambda$ and consider it in $\Lambda/n$, we get an ideal $J_X^{(n)}(\nu) \subset \mathbb{C}[\Lambda/n]$. The same ideal is also associated to the list

$$X^{(n)} = \{\frac{a_1}{n}, \ldots, \frac{a_1}{n}, \ldots, \frac{a_N}{n}, \ldots, \frac{a_N}{n}\}$$

with parameters $\nu^{(n)} := \{\gamma_1, \varepsilon \gamma_1, \ldots, \varepsilon^{n-1} \gamma_1, \ldots, \gamma_N, \ldots, \varepsilon^{n-1} \gamma_N\}$

where, for each $i = 1, \ldots, N$, $\gamma_i$ is an $n$-th root of $\nu_i$ and $\varepsilon = \exp(2\pi i/n)$.

One has to remark that $\Lambda/n$ is the character group of a torus $T^{1/n}$ which maps surjectively $\pi : T^{1/n} \to T$ to $T$, with kernel $K_n$ the character group of $(\Lambda/n)/\Lambda$.

The points of the arrangement associated to $X$ in $T^{1/n}$ are $\pi^{-1}(P(X))$ and a union of cosets of $K_n$.

The algebra $\mathbb{C}[\Lambda/n]/J_X^{(n)}(\nu) \mathbb{C}[\Lambda/n] = \mathbb{C}[\Lambda/n]/J_X^{(n)}(\nu^{(n)})$. Thus even if we start from the trivial parameters $= 1$, once we pass to a finer lattice we find as parameters the roots of 1.

9.16. A realization of $\mathbb{C}[\Lambda]/J_X$.

As for the box spline the functions in $D(X)$ play a basic role, so do the functions in $\nabla(X)$ in the theory of linear diophantine equations. In fact they describe combinatorially the partition function $P_{\lambda}(v)$ associated to $X$.

We go back to the space $\Pi_X := S_X/S_{X,n-1}$ of polar parts. This we have decomposed, as $\Pi_X = \bigoplus_{\phi \in P(X)} P_{\lambda_{\phi}}(\phi)$, through the points $\phi \in P(X)$ into local modules $P_{\lambda_{\phi}}(\phi)$ of polar parts for the affine arrangement centered at $\phi$.

The element $v_X$ class of the generating function $\prod_{a \in X} (1 - e^{-a})^{-1}$ in $\Pi_X$ decomposes into a sum of local elements $v_{\lambda_{\phi}}$.

We consider next the submodule $Q_X$ generated in $\Pi_X$ by $v_X$ under $\mathbb{C}[\Lambda]$ and we deduce, reducing to Theorem (6.3) and using Theorem 9.12 that:

**Theorem 9.17.** The annihilator of $v_X$ is the ideal $J_X$ generated by the elements $N_Y = \prod_{a \in Y} (1 - e^{-a})$, as $Y$ runs over the cocircuits. Thus $Q_X = \mathbb{C}[\Lambda]/J_X$. 

We obtain a canonical commutative diagram, made of isomorphisms, compatible with all the identifications made:

\[
\begin{align*}
\mathbb{C}[\Lambda]/J_X & \xrightarrow{\cong} \bigoplus_{\phi \in P(X)} A_X^\phi \\
\cong & \\
Q_X & \xrightarrow{\cong} \bigoplus_{\phi \in P(X)} Q_X^\phi
\end{align*}
\]

9.18. From volumes to partition functions. We apply the previous theory to the partition function \( P_X(a) \).

Consider the class of a product \( \prod_{a \in X} (1 - e^{-a})^{-1} \) in \( \Pi_X \). For each point \( \phi \) of the arrangement we project to the corresponding isotypic component \( \Pi_X(\phi) \) which is generated by the elements \( \omega_{b,\phi} \).

Since \( \omega_{b,\phi} \) is an eigenvector for \( \Lambda \) of eigenvalue \( e^\phi \), using the commutation relations with the derivatives, we clearly have that, for all \( a \in \Lambda \), the element \( e^a - e^{(\langle \phi | a \rangle)} \) acts locally nilpotently on the modules \( \Pi_X(\phi) \). It follows that the action of \( e^a \) on \( \Pi_X(\phi) \) is given by an operator of the form \( e^{\langle a | \phi \rangle} u_a \) with \( u_a = e^{\tilde{a}} \) with \( \tilde{a} \) locally nilpotent.

The map \( a \mapsto \tilde{a} + \langle a | \phi \rangle \) is linear. It induces a linear map of \( V = \Lambda \otimes \mathbb{Z} \mathbb{C} \) into linear operators on \( \Pi_X(\phi) \) whose image consists of mutually commuting operators. In this way we obtain a \( S[V] \) module structure on \( \Pi_X(\phi) \). The commutation relations with the operators of \( S[U] \) then give us an action of the usual Weyl algebra \( W(U) \).

**Proposition 9.19.** As a \( W(U) \) module \( \Pi_X(\phi) \) is isomorphic to the module of polar parts for the list \( X_{\phi} \) with parameters \( \phi := \{ \phi_a = -\langle \phi | a \rangle \} \) for each \( a \in X_{\phi} \) by an isomorphism \( j_{\phi} \) characterized by

\[
j_{\phi}(\omega_{b,\phi}) = u_b
\]

for each no broken basis extracted from \( X_{\phi} \).

**Proof.** The fact that the class \( \omega_{b,\phi} \) is an eigenvector is an eigenvector for \( \Lambda \) of eigenvalue \( e^\phi \) and the definition of the \( S[V] \) module structure imply that \( \omega_{b,\phi} \) is an eigenvector for \( V \) of eigenvalue \( \phi \). Thus we get a surjective map of \( W(U) \) modules

\[
\gamma_{\phi} : P_{X_{\phi}}(\phi) \rightarrow (\Pi_X)_{\phi}
\]

with \( \gamma_{\phi}(u_b) = \omega_{b,\phi} \) for each no broken basis in \( X_{\phi} \). Since both modules are free \( S[U] \)-modules and \( \gamma_{\phi} \) takes a basis for \( P_{X_{\phi}}(\phi) \) to a basis for \( (\Pi_X)_{\phi} \) we deduce that it is an isomorphism. \( j_{\phi} \) is the inverse of \( \gamma_{\phi} \). \( \square \)

By the previous discussion \( e^a \) acts on \( \Pi_X(\phi) \) by the operator \( e^{\langle \phi | a \rangle} e^{\tilde{a}} \) with \( \tilde{a} \) locally nilpotent. We distinguish two cases. If \( e^{\langle \phi | a \rangle} \neq 1 \), then \( (1 - e^{-a})^{-1} \) gives an invertible operator in \( \Pi_X(\phi) \).

If on the other hand, \( e^{\langle \phi | a \rangle} = 1 \), the operator \( \tilde{a} / (1 - e^{-\tilde{a}}) \) is invertible (its power series is written in term of Bernoulli numbers). Thus the image
under $j_\phi$ of the component of $\prod_{a \in X} (1 - e^{-a})^{-1}$ equals

$$\prod_{a \notin X_\phi} \frac{1}{1 - e^{-\tilde{a} - \langle a | \phi \rangle}} \prod_{a \in X_\phi} \frac{\tilde{a}}{1 - e^{-\tilde{a}} (a - \langle \phi | a \rangle)^{-1}} = Q_\phi \prod_{a \in X_\phi} (a - \langle \phi | a \rangle)^{-1}$$

where

$$Q_\phi = \prod_{a \notin X_\phi} \frac{1}{1 - e^{-\tilde{a} - \langle a | \phi \rangle}} \prod_{a \in X_\phi} \frac{\tilde{a}}{1 - e^{-\tilde{a}}}$$

is an invertible operator, on $P_X(\phi)$, expressed as a power series in the operators $\tilde{a}$ and hence locally given by a polynomial. We deduce

**Proposition 9.20.** Under the isomorphism $j_\phi$ of coordinates $j_\phi$:

$$j : \bigoplus_{\phi \in P(X)} \Pi_X(\phi) \to \bigoplus_{\phi \in P(X)} P_{X_\phi}(\phi)$$

we have the transformation

$$(50) \quad j : \prod_{a \in X} (1 - e^{-a})^{-1} \mapsto \sum_{\phi \in P(X)} Q_\phi \prod_{a \in X_\phi} (a - \langle \phi | a \rangle)^{-1}.$$ 

We now apply Laplace transform and deduce the final general formula expressing the partition function as a sum of transforms of the local multivariate splines. In order to justify our results let us make some remarks.

We have made an identification of $\Pi_X(\phi)$ with $P_{X_\phi}(\phi)$ as $W(U)$ modules.

In particular we have that $\Pi_X(\phi)$ is a free module, over the algebra of differential operators with polynomial coefficients, in the classes of the elements $e(\phi) \prod_{a \notin \phi}(1 - e^{-a})^{-1}$ while $\Pi_X(\phi)$ is a free module, over the same algebra, in the classes of $\prod_{a \notin \phi}(a - \langle \phi | a \rangle)^{-1}$.

Take then $M_1, M_2$ to be the two free submodules, over the algebra of differential operators with polynomial coefficients, of $S_X$ re. $R_X$ generated by these elements. Define by $j : M_2 \to M_1$ to be the module isomorphism mapping $\prod_{a \notin \phi}(a - \langle \phi | a \rangle)^{-1}$ to $e(\phi) \prod_{a \notin \phi}(1 - e^{-a})^{-1}$.

From proposition 7.11 we see that $M_1$ is the span of the Laplace transforms of distributions of type

$$\sum_{\phi \in P(X), \delta \in X_\phi} p_\phi(x) | det(\delta)|^{-1} \sum_{v \in C(\delta) \cap \Lambda} \phi(v) \delta_v$$

while $M_2$, by (5) and (10), is the span of the Laplace transforms of distributions of type

$$\sum_{\phi \in P(X), \delta \in X_\phi} p_\phi(x) | det(\delta)|^{-1} e^{\langle \phi | x \rangle} \chi_{C(\delta)}.$$

If we restrict both types of distributions on the set of regular points of $C(X)$ we see that, $M_2$ can be identified with a space of smooth functions, which are locally linear combinations of polynomials times exponentials, while $M_2$ can be identified to functions of the regular points in $\Lambda$ which are locally quasi polynomials.
Theorem 9.21. Under these identifications the map $j$ consists simply into restricting the functions to the points in $\Lambda$.

Proof. Since at the level of functions the map is a linear isomorphism compatible with the multiplication by the polynomials $S[V]$, it is enough to verify the statement on the generator $\sum_{v \in C(\mathcal{B}) \cap \Lambda} \phi(v) \delta_v$ and $e^{(\phi|_{x})X_{C(\mathcal{B})}}$ for which it is clear. □

We deduce:

Theorem 9.22. On the intersection of $\Lambda$ with the open set of regular points we have:

\[ P_X = \sum_{\phi \in P(X)} \hat{Q}_{\phi} T_{X\phi} \]

Proof. The explicit formula is a consequence of formula (50) plus the previous discussion which implies that the two sides coincide, since both functions are restrictions of quasi polynomials. □

From this formula we can deduce one valid everywhere using the method of Jeffrey–Kirwan residues.

Theorem 9.23. Given a point $x$ in the closure of a big cell $c$ we have

\[ P_X(x) = \sum_{\phi \in P(X)} \hat{Q}_{\phi} (e^{\phi} \sum_{b \in N B_{X\phi} \mid c \in C(\mathcal{B})} p_{b \cdot X\phi}(-x)). \]

Proof. Both terms of the equality are continuous on the closure of $c$, they coincide in the interior by the previous proposition, hence they are equal. □

10. Some Applications

In this short section we would like to give a streamlined presentation of some of the applications. All the results are taken from the papers of Dahmen and Micchelli, or from [18], with minor variations of the proofs.

For further details and more information the reader should look at the original literature.

The main steps are the following.

10.1. Discrete convolution. Let us first analyse the discrete convolution:

\[ B_X \ast_d p = \sum_{\lambda \in \Lambda} B_X(x + \lambda)p(-\lambda) \]

and prove that:

Theorem 10.2. When $p \in D(X)$ we have that also $B_X \ast_d p \in D(X)$.

This defines a linear isomorphism $F$ of $D(X)$ to itself, given explicitly by the invertible differential operator $F_X := \prod_{a \in X} \frac{1 - e^{-Da}}{D_a}$. 

Proof. A way to understand this convolution is by applying the Poisson summation formula to the function of $y$, $B_X(x+y)p(-y)$. Its Laplace transform is obtained from the Laplace transform $e^x \prod_{a \in X} (1 - e^{-a})/a$ of $B_X(x+y)$ by applying the polynomial $\hat{p}(x)$ as differential operator.

In our definition of Laplace transform we have

$$Lf(\xi) = (2\pi)^{n/2} \hat{f}(i\xi)$$

where $\hat{f}$ denotes the usual Fourier transform. So Poisson summation gives:

$$\sum_{\mu} L\phi(\mu) = \sum_{\lambda} \phi(\lambda)$$

where $\mu$ runs in the dual lattice $\Lambda^*$, of elements for which $\langle \mu | \lambda \rangle \in 2\pi i \mathbb{Z}$, $\forall \lambda \in \Lambda$.

Thus if we are in the situation that $L\phi(\mu) = 0, \forall \mu \neq 0, \mu \in \Lambda^*$ we have

$$L\phi(0) = \sum_{\lambda} \phi(\lambda)$$

The main observation of Dahmen and Micchelli is that:

**Lemma 10.3.** The Laplace transform of $B_X(x+y)p(-y)$ vanishes at all points $\mu \neq 0, \mu \in \Lambda^*$.

**Proof.** We may assume that $p(x)$ is homogeneous of some degree $k$. The evaluation of $p(x)$ against $\prod_{a \in X} \frac{1-e^{-a}}{a}$ can be understood as follows. Each factor $\frac{1-e^{-a}}{a}$ can be expanded in power series. We select from at most $k$ factors the homogeneous parts of some degrees $h_i > 0$ so that $\sum_i h_i = k$ and evaluate $p$ against the resulting monomial, then we multiply by the remaining factors and sum over such choices. Now, if the factors we have chosen correspond to a cocircuit the evaluation of $p$ on the monomial is 0. If instead this is not the case we have still a product $\prod_{a \in A} \frac{1-e^{-a}}{a}$ where the elements $A$ span. Thus, if $\mu \neq 0$ there is at least one $a \in A$ which does not vanish at $\mu$. But now clearly $1 - e^{-a}$ and hence $(1 - e^{-a})/a$ vanishes at $\mu$ and the Lemma follows.

We go back to the proof of our Theorem. We have shown that in our case, Poisson summation degenerates to the computation at 0. Taking a polynomial in derivatives and computing against a function and then at 0 is just duality thus:

$$p(x)(e^x \prod_{a \in X} \frac{1-e^{-a}}{a})(0) = \langle p | e^x \prod_{a \in X} \frac{1-e^{-a}}{a} \rangle = \langle F_X p | e^x \rangle = F_X p(x).$$

Since $D(X)$ is stable under derivatives and $F_X$ is clearly invertible both our claims follow.
10.4. Paving the box. Take the box $B(X) := \{\sum_{a \in X} t_a a \mid 0 \leq t_a \leq 1\}$, which is the support of the box spline.

We start by giving a nice decomposition of $B(X)$ into suitable parallelepipeds. In order to present it we need the following:

**Lemma 10.5.** If a point $v := \sum_{a \in X} v_a a$, $0 \leq v_a \leq 1$ is in the boundary of $B(X)$ the set $A := \{x \mid 0 < v_a < 1\}$ does not span $V$.

**Proof.** If $A$ spans let us extract a basis $b_1, \ldots, b_s$ from $A$, take $0 < \epsilon$ small. Then the set of points $v + \sum_{i=1}^s t_i b_i$, $|t_i| < \epsilon$ is an open ball contained in $B(X)$.

Given $\lambda \in \Lambda$ and a set of linearly independent vectors $b := \{b_1, \ldots, b_h\}$ from $X$ define:

$$\Pi_\lambda(b) := \{\lambda + \sum_{i=1}^h t_i b_i \mid 0 \leq t_i \leq 1\}.$$

**Proposition 10.6.** $B(X)$ can be paved with parallelepipeds, of the form $\Pi_\lambda(b)$ where $b$ runs on the set of all bases extracted from $X$ and $\lambda \in \Lambda$ depends on $b$.

**Proof.** Suppose that $X = \{Z, y\}$ and we have paved $B(Z)$ (by induction) so that its boundary is paved by some faces of these parallelepipeds. Consider $B(Z)_y := \{p \in B(Z) \mid p + t y \notin B(Z), \forall t > 0\}$.

We have a map $\pi : B(Z)_y \times [0, 1] \to B(X)$, $(p, t) \mapsto p + t y$. We easily see that:

1. $\pi$ is a homeomorphism to its image.
2. $B(Z) \cap \pi(B(Z)_y \times [0, 1]) = B(Z)_y$
3. $B(X) = B(Z) \cup \pi(B(Z)_y \times [0, 1])$

We are now going to pave $B(Z)_y$ by $s - 1$ dimensional parallelepipeds $\Pi_\lambda(c)$ with $c = \{c_1, \ldots, c_{s-1}\}$ running on all sets of $s - 1$ linearly independent vectors in $\mathbb{Z}$ which are together with $y$ form a basis.

Consider the set $\mathcal{H}_y$ of all hyperplanes $H$ generated by subsets of $Z$ and not containing $y$. Given such $H$ take the unique linear form $\phi_H$ vanishing on $H$ and such that $\langle \phi_H, y \rangle = 1$. Set

$$\lambda_{H,y} = \sum_{x \in Z \mid \phi_H(x) > 0} x.$$

We claim that

$$B(X)_y = \cup_{H \in \mathcal{H}_y} B(Z \cap H) + \lambda_{H,y}$$

and this is a paving. Remark that $\phi_H$ takes its maximum value on $B(Z)$ in the point $\lambda_{H,y}$. Since if $b > 0$ and $v \in B(Z \cap H) + \lambda_{H,y}$, $\phi_H(v + b y) = \phi_H(\lambda_{H,y}) + b > \phi_H(\lambda_{H,y})$ we get the inclusion $B(Z \cap H) + \lambda_{H,y} \subset B(X)_y$.

To see the converse observe that by Lemma 10.5 $B(X)_y$ is a union of polytopes of the form $B(Z \cap H) + \mu$ with $H \in \mathcal{H}_y$ and $\mu = \sum_{a \in A} \varepsilon_a a$ with
$A = Z - (Z \cap H)$ and $\varepsilon_a \in \{0,1\}$. Fix $a \in A$. Write $y = h + \phi_H(a)^{-1}a$. Take $v$ in the relative interior of $B(Z \cap H)$. We have

$$v + ty = (v + th) + (\mu - \varepsilon_a a) + (\varepsilon_a + t\phi_H(a)^{-1})a.$$ 

If $t$ is sufficiently small then $v + th \in B(Z \cap H)$. Furthermore if $\varepsilon_a = 0$ and $\phi_H(a) > 0$ or $\varepsilon_a = 1$ and $\phi_H(a) < 0$, $0 < \varepsilon_a + t\phi_H(a)^{-1} < 1$. Thus this point lies in $B(Z)$ giving a contradiction.

By induction, each $B(Z \cap H)$ is paved by $s - 1$ dimensional parallelepipeds $\Pi_\lambda(\xi)$ with $\xi = \{c_1, \ldots, c_s\}$ running on all bases of $H$ extracted from $Z \cap H$. Since $\{\xi, y\} := \{c_1, \ldots, c_{s-1}, y\}$ is a basis of $X$ and all bases containing $y$ are so obtained, we get the desired paving

$$\pi(B(Z)y \times [0,1]) = \cup_{\lambda} \Pi_\lambda(\{\xi, y\}).$$

From this our claim is immediate.

As a simple application we obtain

**Proposition 10.7.** Let $x_0$ be a regular point. Then $(B(X) - x_0) \cap \Lambda$ consists of $\delta(X)$ points.

It is easy to see that each parallelepiped $\Pi_\lambda(\bar{b})$ translated by a regular point, intersects $\Lambda$ in $|\det(\bar{b})|$ points. Summing over all parallelepipeds we obtain our claim.

10.8. **Linear independence.** We now assume that we are in the unimodular case. In this case $\delta(X) = d(X)$. Choose a regular point $x_0$ consider the $d(X)$ points $P(x_0) := (B(X) - x_0) \cap \Lambda = \{p_1, \ldots, p_{d(X)}\}$.

**Proposition 10.9.** Evaluation of polynomials in the points in $P(x_0)$ establishes a linear isomorphism between $D(X)$ and the $\mathbb{C}^{d(X)}$ (or $\mathbb{R}^{d(X)}$ if we restrict to real polynomials).

**Proof.** Since $\dim(D(X)) = d(X)$ it suffices to prove that, a polynomial $p(x) \in D(X)$ vanishing on these points is identically zero. From the formula $p(x) = \sum_{a \in \Lambda} B(x+a)p(-a)$ we see that, for given $x$ the only terms appearing in the sum are the ones where $x + a \in B(X)$ or $a \in B(X) - x$. Thus for $x$ in a small open neighborhood of the given point $x_0$ the only terms appearing are the ones in which $a \in \{p_1, \ldots, p_{d(X)}\}$. If by way of contradiction, a polynomial $p \in D(X)$ vanishes on these points we have that $p(x) = 0$ proving the Proposition.

Now we can prove the **Theorem on linear independence of the translates of the box spline**: 

**Theorem 10.10.** For $X$ unimodular and any, non identically $0$, function $f(\lambda)$ on $\Lambda$ we have:

$$\sum_{a \in \Lambda} B(x + a)f(-a) \neq 0.$$
Proof. Assume $f(a_0) \neq 0$ for some $a_0 \in \Lambda$. We can find then, a regular point $x_0$, such that $a_0 \in \mathcal{P}(x_0)$ is one of the points $p_i$ previously defined. Thus there is a non zero polynomial $p(x) \in D(X)$ coinciding, on $\mathcal{P}(x_0)$ with $f$. Then $\sum_{a \in \Lambda} B(x + a)f(-a) = p(x) \neq 0$ on the set $B(X) - x_0$. □

Unimodularity is a necessary condition for this theorem in fact one has:

**Proposition 10.11.** Discrete convolution maps $\nabla(X)$ into $D(X)$ with kernel $E(X)$ (see Formula (47)).

Proof. The first statement follows from the following identity, for $Y \subset X$ we have:

$$D_Y(B_X \ast_d a) = B_{X/Y} \ast_d \nabla_Y a$$

The second by the fact that the kernel of the discrete convolution is invariant under translation thus it suffices to verify it for the functions $e^\phi$ as $\phi$ varies over the points of the arrangement. Let $Y \subset X$ be a basis of a sub-lattice $\Lambda_0$ on which $\phi$ is 1. We have $B_X \ast_d \phi = B_{X/Y} \ast B_Y \ast_d \phi$ and $B_Y \ast_d \phi = 0$ since $\sum_{a \in \Lambda_0} \phi(a) = 0$ (the sum is over coset representatives). □

We apply the previous theory to the partition function $\mathcal{P}_X(a)$.

**Proposition 10.12.**

$$T_X(x) = \sum_{a \in \Lambda} \mathcal{P}_X(a)B_X(x - a)$$

Proof. Compute the Laplace transform $L(\sum_{a \in \Lambda} \mathcal{P}_X(a)B_X(x - a))$:

$$= \sum_{a \in \Lambda} \mathcal{P}_X(a)e^{-a}L(B_X) = \prod_{a \in X} \frac{1}{1 - e^{-a}} \prod_{a \in X} 1 - \frac{e^{-a}}{a} = \prod_{a \in X} \frac{1}{a} = LT_X.$$ □

In the unimodular case, where we have the linear independence of translates of $B_X$, we recover the results of section 9.18.

On $D(X)$, $F_X$ is invertible and its inverse is:

$$Q := \prod_{x \in X} \frac{D_x}{1 - e^{-D_x}}.$$ We have by definition $QFp = p = FQp$ on $D(X)$. Take a big cell $c$ over which $T_X$ coincides with some polynomial $p_c \in D(X)$. Set $q_c := Qp$, we have on $c$

$$T_X(x) = FQp_c = \sum_{a \in \Lambda} Qp_c(a)B(x - a)$$

since $T_X(x) = \sum_{a \in \Lambda} \mathcal{P}_X(a)B_X(x - a)$ we have by linear independence:

$$Qp_c(a) = \mathcal{P}_X(a).$$
Notice that $Q$ is like a Todd operator, its factors can be expanded using the Bernoulli numbers $B_n$ by the defining formula:

$$
\frac{D_x}{1-e^{-D_x}} = \sum_{k=0}^{\infty} \frac{B_n}{n!} (-D_x)^n
$$

Formula 53 allows us to pass from the formula for the volume to one for the partition function (a special case of formula 9.23).

Part 3. Residues

11. Residues

11.1. Cohomology. For the computations of volumes and partition functions we want to apply a cohomological method, like the usual method in one complex variable for computing definite integrals.

With the notations of section 2.11, we start from the affine algebraic variety $A_X$, with coordinate ring $R_X$. Using the De Rham’s Theorem due to Grothendieck, in order to compute the cohomology of $A_X$ with complex coefficients we can use the algebraic de Rham complex $(\Omega^*_X, d)$ (cf. [27]). Here $d$ is the usual de Rham differential while for each $0 \leq k \leq s$, the algebraic differential forms of degree $k$ are:

$$
\Omega^k_X := \{ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} f_{i_1, \ldots, i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid f_{i_1, \ldots, i_k}(x) \in R_X \}
$$

where $\{x_1, \ldots, x_s\}$ are coordinates in $U$.

Let us look more closely to what happens in degree $s$, the top degree. Define a homomorphism, uniquely determined by the choice of the volume form $dx_1 \wedge \cdots \wedge dx_s$,

$$
i_X : \Omega_X^s \to R_X
$$

by $i_X(fdx_1 \wedge \cdots \wedge dx_s) = f$. We have that $i_X(d(\Omega_X^{s-1}))$ is the space $\partial(R_X)$ spanned by all partial derivative of elements in $R_X$.

Definition 11.2. Let $H_X$ be the space spanned by the elements $M_b := \prod_{a \in b} a^{-1}$, as $b$ varies among the bases extracted from the list $X$.

From the computation of the cohomology of $A_X$ for which we refer to [32], we get the following result which also follows immediately from the discussion of the next section and the expansion in partial fractions.

**Theorem 11.3.**

1. We have the decomposition in direct sum:

$$
R_X = H_X \oplus \partial(R_X)
$$

In particular, using $i_X$, the space $H_X$ can be identified to the space $H^n(A_X)$.

2. The elements $M_b$, as $b$ varies in the set $NB(X)$ of no broken bases are a basis of $H_X$. 


If we take any basis $\mathbf{b} = \{b_1, \ldots, b_s\}$ of $V$ we have
$$db_1 \wedge \cdots \wedge db_s = \det(\mathbf{b})dx_1 \wedge \cdots \wedge dx_s$$
(this is given by the definition of $\det(\mathbf{b})$). It is then clear that
$$\det(\mathbf{b}) \prod_{i=1}^{s} \frac{1}{b_i} dx_1 \wedge \cdots \wedge dx_s = d \log(b_1) \wedge \cdots \wedge d \log(b_s).$$
Using $i_X$ we deduce:

**Proposition 11.4.** The cohomology classes of the forms
$$\omega_\mathbf{b} := d \log(a_1(x)) \wedge \cdots \wedge d \log(a_n(x)) = d \log(a_1) \wedge \cdots \wedge d \log(a_n)$$
as $a_1, \ldots, a_n$ varies in $\mathcal{N}\mathcal{B}(X)$, form a basis of the cohomology $H^\bullet(A_X)$.

11.5. Local residue. From the expansion in partial fractions follows easily that $R_X = H_X + \partial(R_X)$, thus in fact Theorem 11.3 is a consequence of Proposition 11.4. This can be proved directly using the following local computation.

Let $\mathbf{b} = \{b_1, \ldots, b_s\}$ be a basis extracted from $X$. To $\mathbf{b}$ we associate an injection
$$j_\mathbf{b} : R_X \to \mathbb{C}[u_1, \ldots, u_s][[u_1 \cdots u_s]^{-1}]$$
($\mathbb{C}[u_1, \ldots, u_s][[u_1 \cdots u_s]^{-1}]$ is the ring of formal Laurent series in the variables $u_1, \ldots, u_s$), defined by
$$j_\mathbf{b}(f(b_1, \ldots, b_s)) = f(u_1, u_1u_2, \ldots, u_1u_2 \cdots u_s).$$
This is well defined since given $a \in X$, if $k = \gamma_\mathbf{b}(a)$ is the maximum index such that $a \in \langle b_k, \ldots, b_s \rangle$, we have
$$a = \sum_{j=k}^{s} \alpha_j b_j = u_1 \cdots u_k(\alpha_k + \sum_{j=k+1}^{s} \alpha_j \prod_{i=k+1}^{j} u_i)$$
with $\alpha_k \neq 0$ so that $a^{-1}$ can be expanded as a Laurent series. Clearly this map extends at the level of differential forms.

Given a top differential form $\omega \in \Omega^s_X$ we define $\text{res}_\mathbf{b}(\omega)$ as the coefficient of $(u_1 \cdots u_s)^{-1} du_1 \wedge \cdots \wedge u_s$ in $j_\mathbf{b}(\omega)$.

The main technical result from which all the important results follow is given by

**Lemma 11.6.** Let $\mathbf{b}$ and $\mathbf{c}$ be two no broken bases extracted from $X$ then
$$\text{res}_\mathbf{b}(\omega_\mathbf{c}) = \delta_{\mathbf{b}, \mathbf{c}}.$$

**Proof.** First notice that $j_\mathbf{b}(\omega_\mathbf{b}) = d \log u_1 \wedge \cdots \wedge d \log u_s$ so that $\text{res}_\mathbf{b}(\omega_\mathbf{b}) = 1$.

Now notice that if $\mathbf{c} \neq \mathbf{b}$ there are two distinct elements $c, c' \in \mathbf{c}$ such that $\gamma_\mathbf{b}(c) = \gamma_\mathbf{b}(c')$. Indeed since in a non broken basis the first element is always the first element in the list $X$, $\gamma_\mathbf{b}(c_1) = 1$. If $\langle c_2, \ldots, c_s \rangle \neq \langle b_2, \ldots, b_s \rangle$ then there exists an index $i > 1$ with $\gamma_\mathbf{b}(c_i) = 1$. If $\langle c_2, \ldots, c_s \rangle = \langle b_2, \ldots, b_s \rangle$
then both \( c - \{c_1\} \) and \( b - \{b_1\} \) are no broken bases of \( \langle c_2, \ldots, c_s \rangle \) extracted from \( X \cap \langle c_2, \ldots, c_s \rangle \) and everything follows by induction.

We then write each \( c \) in the form \( \prod_{i=1}^{k} u_i \) with \( f(0) \neq 0 \) so that \( d \log c = d \log (\prod_{i=1}^{k} u_i) + d \log f \). Expanding the product \( j_b(\omega_b) \) we get a linear combination of forms, all terms containing only factors of type \( d \log (\prod_{i=1}^{k} u_i) \) vanish since two elements are repeated, the others are a product of a closed form by a form \( d \log (f) \) with \( f(0) \neq 0 \) which is exact, so all these terms are exact and the residue is 0.

\[ \square \]

As a first consequence we deduce:

**Theorem 11.7.** The cohomology classes of the forms \( \omega_b \), as \( b \) varies among the no broken bases are linearly independent.

For any top differential form \( \psi \), denote by \([\psi]\) its cohomology class we have:

\[ [\psi] = \sum_{b \in NB(X)} \text{res}_b(\psi) \omega_b \]

**Proof.** Since clearly the map \( j_b \) takes exact forms to exact forms we deduce that \( \text{res}_b \) factors through \( H^s(\tilde{A}_X) \). In view of this everything follows from Lemma 11.6.

**Remark 11.8.** It is not difficult to verify that we get the same cohomology if, instead of taking forms with coefficients in \( R_X \) we take coefficients of type \( f/d^k \) with \( f \) any function holomorphic around 0.

We are now going to define the total residue \( T_{\text{res}} \) and the residues \( \text{res}_b \).

With the use of \( i_X \) these operators can be defined either on algebraic differential forms of degree \( s \) or on functions.

**Definition 11.9.** Given \( f \in R_X \), \( T_{\text{res}}(f) \) is the cohomology class of the form \( f dx_1 \wedge \cdots \wedge dx_n \).

We can now reformulate formula (54) as:

\[ T_{\text{res}}(f) = \sum_{b \in NB(X)} \text{res}_b(f)[\omega_b] \]

**11.10. Residues and Laplace transform.** Recall that in formula (15) we have introduced the polynomials \( p_{b,X} \), in order to give an expression of the multivariate spline \( T_X \).

**Theorem 11.11.** For every \( b \in NB(X) \),

\[ p_{b,X}(-y) = \det(b)\text{res}_b(e^{(y|x)} d(x)) \]

with \( d(x) = \prod_{a \in X} a \).
Proof. We begin remarking that formula (55) makes sense since, if we expand $e^{\langle y \mid x \rangle}/d(x)$ with respect to the variables $y = \{y_1, \ldots, y_s\}$, we get a power series whose coefficients lie in $R_X$.

In order to prove this Theorem we need some properties of $Tres$. The first property of $Tres$, which follows from the definition is that, given a function $f$ and an index $1 \leq i \leq s$, we have $Tres(\partial f / \partial x_i) = 0$, hence for two functions $f, g$:

$$Tres(\partial f / \partial x_i g) = -Tres(f \partial g / \partial x_i).$$

In other words for a polynomial $P$:

(56) $$Tres(P(\partial_x f)) = Tres(f P(-\partial_x g)).$$

We shall use the preceding relation (56) for the function $f = e^{\langle y \mid x \rangle}$ for which we have:

(57) $$P(\partial_x) e^{\langle y \mid x \rangle} = P(y) e^{\langle y \mid x \rangle},$$

The second simple property is that, given a basis $b$ extracted from $X$ and a function $f$ regular at 0, we have:

(58) $$Tres(\frac{f}{\prod_{a \in b} a(x)}) = f(0) Tres(\frac{1}{\prod_{a \in b} a(x)}).$$

We get

$$Tres(\frac{e^{\langle y \mid x \rangle}}{d(x)}) = Tres(\sum_{b \in NB(X)} e^{\langle y \mid x \rangle} p_b X(\partial_x) \frac{1}{\prod_{a \in b} a(x)})$$

$$= \sum_{b \in NB(X)} Tres\left(\frac{p_b X(-\partial_x)(e^{\langle y \mid x \rangle})}{\prod_{a \in b} a(x)}\right) = \sum_{b \in NB(X)} p_b X(-y) Tres\left(\frac{1}{\prod_{a \in b} a(x)}\right)$$

(59) $$= \sum_{b \in NB(X)} \frac{1}{\det(b)} p_b X(-y) [\omega_b].$$

From this the theorem follows. \qed

11.12. Partition functions. The same method can be applied to partition functions. In (22) we have computed the full cohomology of the toric arrangement, nevertheless this is not strictly necessary for the residue computations which are essentially local. In the formula (34) one has contributions localized at a point of the arrangement $e^{\phi}$ and for a no broken basis $b \subset X_\phi$.

They can be computed again using the operator $res_{b, \phi}$ in this case.

This means that, we restrict a function or a form to a neighbourhood of $\phi$, use logarithmic coordinates so that the divisors $1 - e^{-a} = 1$ which appear in this neighbourhood coincide with the linear hyperplanes of the arrangement associated to $X_\phi$ and finally compute the local residue at $b$ for this local hyperplane arrangement.
By formula (33) and Proposition 9.20, we get that the class of the Laplace transform
\[ \prod_{a \in X} (1 - e^{-a})^{-1} \]
in the module \( \Pi_X \) of polar parts decomposes into a sum of local factors, corresponding to the summands \( \Pi_{X,\phi} \) and then \( \Pi_{X,\phi} = \bigoplus S[U][\prod_{a \in b} (a - \langle \phi | a \rangle)^{-1}] \) under the logarithm map. With this isomorphism \( j \) we have
\[ j(\prod_{a \in X} (1 - e^{-a})^{-1}) = \sum_{\phi \in P(X)} \sum_{b \in N_B X \phi} q_{b \phi}(y)[\prod_{a \in b} (a - \langle \phi | a \rangle)^{-1}] \]
which by Proposition 9.21 gives a formula for the partition function. We claim that we have
\[ q_{b \phi}(y) = \det(b) \text{res}_{b \phi}(e^{(y|x)} \prod_{a \in X} (1 - e^{-a(x)})^{-1}) \]
In order to prove this, we start by making a change of coordinates \( x = z + \langle y | \phi \rangle \) so that we center the point at 0 and we pick out the factor \( e^{\phi} = e^{(y|\phi)} \) and then showing, using the language of proposition 9.20, that:

**Lemma 11.13.** \( \text{res}_{b \phi}(f) = \text{res}_{b \phi}(j(f)) \) for any \( f \in M_2 \).

**Proof.** Due to the properties of \( T_{res} \) it is enough to prove it on the generators where we see that, around the point \( \phi \) we have
\[ \frac{e^{(\phi)}}{\prod_{a \in b} (1 - e^{-a})^{-1}} = \left| \det(b) \right|^{-1} \sum_{\phi \in \lambda} \phi(\lambda) e^{-\lambda} \prod_{a \in b} (1 - e^{-a})^{-1} = \left| \det(b) \right|^{-1} \sum_{\phi \in \lambda} \phi(\lambda) e^{-\lambda} \prod_{a \in b} a^{-1} h(x), \quad h(\phi) = 1. \]
Since by definition \( \phi(\lambda) = e^{(|\lambda| \phi)} \), at the point \( \phi \) we have that
\[ | \det(b) |^{-1} \sum_{\phi \in \lambda} \phi(\lambda) e^{-(|\lambda| \phi)} = 1. \]
This proves the claim by formula (59). \( \square \)

Now we go back to formula (60). We have that
\[ \text{res}_{b \phi}(j(\frac{e^{(y|x)}}{\prod_{a \in X} (1 - e^{-a(x)})})) = \text{res}_{b \phi}(e^{(y|x)} q_{b \phi}(y) \prod_{a \in b} (a - \langle \phi | a \rangle)^{-1}) \]
Change coordinates, so to center \( \phi \) at 0, getting \( x = z + \phi \) and so
\[ \text{res}_{b \phi}(e^{(y|x)} q_{b \phi}(y) \prod_{a \in b} (a - \langle \phi | a \rangle)^{-1}) = \det(b)^{-1} q_{b \phi}(y) e^{\phi} \]

**Summarizing** These formulas, together with the local computation of residues form an effective algorithm to compute the functions we have been studying.

It remains to discuss a last algorithmic point.
In order to compute the Jeffry–Kirwan residue, at a given point \( p \in C(A) \), it is necessary to determine a big cell \( c \) for which \( p \in \overline{c} \).

In general, the determination of the big cells is a very complex problem. For our computations it suffices much less.

Let us take thus simply a point \( q \) internal to \( C(A) \) and not laying on any hyperplane generated by \( n - 1 \) vectors of \( X \). This is not difficult to do, and let us consider the segment \(qp\).

This segment intersects these hyperplanes in a finite number of points, thus we can determine an \( \epsilon \) sufficiently small for which all the points \( tp + (1 - t)q, 0 < t < \epsilon \) are regular.

If we take one of these points \( q_0 \) it lays in a cell for which \( p \) is in the closure.

At this point, for every no broken basis, we must verify in simple way if \( q_0 \) lays or not in the cone generated by the basis.

12. Minimal models

12.1. Geometry of residues. For the moment our definition of residues \( \text{res}_\psi \psi \) is purely algebraic. In fact its true geometric meaning is based upon a general definition of the notion of residue in several dimensions.

This section is quite independent of the rest of the paper and can be used as an introduction to the theory developed in [19] and [20].

The first point to be understood is that, the non linear coordinates \( u_i \) used in section 11.5 represent local coordinates around a point at infinity of a suitable geometric model of a completion of the variety \( A_X \). In fact we are thinking of models proper over the space \( U \supset A_X \) in which the complement of \( A_X \) is a divisor with normal crossings. In this respect the local computation done in section 11.5 corresponds to a model in which all the subspaces of the arrangement have been blown up, but there is a subtler model which gives rise to a more intricate combinatorics but possibly to more efficient computational algorithms, due to its minimality.

13. Irreducibles and nested sets

13.1. Irreducibles and decompositions. The notions that we are about to give are of combinatorial nature (cf. [25]) but we develop them in the following context:

As usual let us consider a list \( X := \{a_1, \ldots, a_N\} \) of non zero vectors in \( V \) which in this section we assume to be a complex vector space.

Given a sublist \( A \subset X \) the list \( \overline{A} := X \cap \langle A \rangle \) will be called the completion of \( A \). Thus \( A \) is complete if \( A = \overline{A} \).

The space of vectors \( \phi \in U \) such that \( \langle a|\phi \rangle = 0 \) for every \( a \in A \) will be denoted by \( A^\perp \). Notice that clearly \( \overline{A} \) equals to the list of vectors \( a \in X \) which vanish on \( A^\perp \).
From this we see that we get a bijection between the complete sublists of \( X \) and subspaces of the arrangement defined by \( X \).

We give the main

**Definition 13.2.** Given a complete set \( A \subset X \), a **decomposition** is a decomposition \( A = A_1 \cup A_2 \) in non empty sets, such that:

\[
\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle.
\]

Clearly the two sets \( A_1, A_2 \) are necessarily complete.

We shall say that a complete set \( A \) is irreducible if it does not have a non trivial decomposition.

If \( A = A_1 \cup A_2 \) is a decomposition of a complete set and \( B \subset A \) is complete we have \( B = B_1 \cup B_2 \), where \( B_1 = A_1 \cap B, \ B_2 = A_2 \cap B \). Also \( \langle B \rangle = \langle B_1 \rangle \oplus \langle B_2 \rangle \), and we have

**Lemma 13.3.** \( B = B_1 \cup B_2 \) is a decomposition, unless one of the two sets is empty.

We deduce immediately:

**Proposition 13.4.** If \( A = A_1 \cup A_2 \) is a decomposition and \( B \subset A \) is irreducible, then \( B \subset A_1 \) or \( B \subset A_2 \)

From this get:

**Theorem 13.5.** Every set \( A \) can be decomposed as \( A = A_1 \cup A_2 \cup \cdots \cup A_k \) with the \( A_i \) irreducible and:

\[
\langle A \rangle = \langle A_1 \rangle \oplus \langle A_2 \rangle \oplus \cdots \oplus \langle A_k \rangle.
\]

This decomposition is unique up to order.

**Proof.** The existence of an irreducible decomposition follows by a simple induction.

Let \( A = B_1 \cup B_2 \cup \cdots \cup B_h \) be a second decomposition. Proposition 13.3 implies that every \( B_i \) is contained in an \( A_j \) and viceversa.

Thus the \( A_j \)'s and the \( B_i \)'s are the same up to order.

\( A = A_1 \cup A_2 \cup \cdots \cup A_k \) is called the decomposition into irreducibles of \( A \).

**Example 13.6.** An interesting example is that of the configuration space of \( s \)-ples of point in a line (or the root system \( A_s-1 \)). In this case \( X = \{ z_i - z_j | 1 \leq i < j \leq s \} \).

In this case, irreducible sets are in bijection with subsets of \( \{1, \ldots, s\} \) with least 2 elements. Indeed give one such subset \( S \) it corresponds to the irreducible \( I_S = \{ z_j - z_j | \{i, j\} \subset S \} \).

Given a complete set \( C \), the irreducible decomposition of \( C \) corresponds a sequence of disjoint subsets \( S_1, \ldots, S_k \) of \( \{1, \ldots, s\} \) with least 2 elements.
13.7. Nested sets. We define now the basic notion of nested set.

We say that two sets $A, B$ are comparable if one is contained in the other.

**Definition 13.8.** We shall say that a family $S$ of irreducibles $A_i$ is nested if given elements $A_1, \ldots, A_i \in S$ mutually incomparable we have that $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is complete and $C := A_1 \cup A_2 \cup \cdots \cup A_i$ is its decomposition into irreducibles.

**Remark 13.9.** If $A_1, \ldots, A_k$ is nested we have that $\cup A_i$ is complete. In fact this union can be obtained taking the maximal elements, that are necessarily non comparable, and then applying the definition of nested.

We are in particular interested in maximal nested sets, which we denote by MNS.

Nested sets can be inductively constructed combining the following inductive procedures.

We can use two simple inductive ways to construct nested sets. The proof is left to the reader:

1. Suppose we are given a nested set $S$, an minimal element $A \in S$ and a nested set $P$ whose elements are contained in $A$. Then we have that $S \cup P$ is nested.
2. Suppose we are given a nested set $S$, a complete set $A$ containing each element of $S$. Then if $A = A_1 \cup \cdots \cup A_k$ is the decomposition of $A$ into irreducibles, $S \cup \{A_1, \ldots, A_k\}$ is nested.

In the case of Example 13.6 one can easily verify that nested sets correspond to families of subsets in $\{1, \ldots, s\}$ each containing at least two elements and such that any two subsets of the family are either comparable or disjoint.

**Theorem 13.10.** Assume that $\langle X \rangle = V$. Let $S := \{A_1, \ldots, A_k\}$ be a MNS in $X$.

Given $A \in S$ let $B_1, \ldots, B_r$ be the elements of $S$ contained properly in $A$, and maximal with this property.

1. $C := B_1 \cup \cdots \cup B_r$ is complete and decomposed by the $B_i$.
2. $\dim \langle A \rangle = \dim \langle C \rangle + 1$.
3. $k = \dim(V)$.

**Proof.** (1) is the definition of nested set since the $B_i$, being maximal, are necessarily non comparable.

(2) Let us consider $\langle C \rangle = \bigoplus_{i=1}^r \langle B_i \rangle \subset \langle A \rangle$.

$\langle C \rangle \neq \langle A \rangle$ otherwise, since $C$ is complete, by the definition of nested set, we must have $A = C$. This is absurd since $A$ is irreducible and the $B_i$’s are properly contained in $A$.

Therefore there exists an element $a \in A \mid a \notin \langle C \rangle$. Let us denote by $A' := X \cap \langle C, a \rangle$. We have $C \subset A' \subset A$. We claim that $A = A'$.

Otherwise, as one can easily see, adding all the irreducibles that decompose $A'$ to the family $S$, we obtain a nested family that contains properly
\(S\). This contradicts the maximality of \(S\). Clearly \(A = A'\) implies that 
\(\langle A \rangle = \langle C \rangle \oplus \mathbb{C}a\) and thus \(\dim \langle A \rangle = \dim \langle C \rangle + 1\).

(3) We proceed by induction on \(s = \dim(V)\).

If \(s = 1\) there is nothing to prove, there is a unique set complete and
irreducible namely \(X\).

Let \(s > 1\). Decompose \(X = \bigcup_{i=1}^{h} X_i\) into irreducibles.

We have that a MNS in \(X\) is the union of MNS in each \(X_i\). Then \(s = \dim(\langle X \rangle) = \sum_{i=1}^{h} \dim(\langle X_i \rangle)\).

Thus we can assume that \(X\) is irreducible. In this case we have that
\(X \in S\) for every MNS \(S\).

Let \(B_1, \ldots, B_s\) be the elements of \(S\) properly contained in \(X\) and maximal
with this property.

The set \(S\) consists of \(X\) and of the subsets \(S_i := \{A \in S \mid A \subset B_i\}\).

Clearly \(S_i\) is a MNS relative to the set \(B_i\) (otherwise we could add an
element to \(S_i\) and to \(S\) contradicting the maximality of \(S\)).

By induction \(S_i\) has \(\dim(B_i)\) elements and thus by (2) the claim follows.

In the example of configuration spaces \(X\) spans the hyperplane in \(\mathbb{C}^s\)
where the sum of coordinates equals to zero. We have described irreducibles
via the corresponding subsets in \(\{1, \ldots, s\}\). Under this correspondence, one
can see easily that a maximal nested set \(S\) is formed by \(s-1\) elements and for
any \(A \in S\) with \(a > 2\) elements either \(A\) contains a unique maximal element
\(B \in S\) with necessarily \(a - 1\) elements or exactly two maximal elements
\(B_1, B_2 \in S\) with \(A = B_1 \cup B_2\).

We can present such a MNS in a convenient way as a planar binary rooted
tree with \(s\) leaves labelled by \(\{1, \ldots, s\}\). Every internal vertex of the the
corresponds to the set of numbers that appear on its leaves.

For example the graph

\[(62)\]

represents the MNS \(\{1, 2\}, \{1, 2, 3\}, \{4, 5\}, \{1, 2, 3, 4, 5\}\).

Given a MNS \(S\) let us define a map:

\[p_S : X \rightarrow S\]
as follows. Since $\bigcup_{A \in S} A = X$ every element $a \in X$ lies in at least an $A \in S$. Also if an element $a \in X$ appears in two elements $A, B \in S$ the two elements must be necessarily comparable, thus there exists a minimum among the two. It follows that for any element $a \in X$ there is a minimum element $p_S(a) \in S$.

Now a new definition:

**Definition 13.11.** We shall say that a basis $b := \{a_1, \ldots, a_s\} \subset X$ of $\mathbb{C}^s_s$, is *adapted* to the MNS $S$ if the map $a_i \mapsto p_S(a_i)$ is a bijection.

Such a basis always exists. It suffices to take, as in the proof of 13.10, for every $A \in S$ an element $a \in A - \bigcup_i B_i$, where the $B_i$ are the elements of $S$ properly contained in $A$.

Given any basis $b := \{a_1, \ldots, a_s\} \subset X$, we shall build a MNS $S_b$ to which it is adapted, in the following way. Consider for any $1 \leq i \leq s$ the complete set $A_i := X \cap \langle \{a_i, \ldots, a_s\} \rangle = \{a_i, \ldots, a_s\}$. Clearly $A_1 \supset A_2 \supset \cdots \supset A_s$.

For each $i$ consider all the irreducibles in the decomposition of $A_i$. Clearly for different $i$ we can obtain also several times the same irreducible, in any case we have:

**Theorem 13.12.** The family $S_b$ of all the (distinct) irreducibles that appear in the decompositions of the sets $A_i$ form a MNS to which the basis $b$ is adapted.

**Proof.** By induction. Decompose $X = A_1 = B_1 \cup B_2 \cup \cdots \cup B_k$ into irreducibles, by construction:

$$s = \dim \langle A_1 \rangle = \sum_{i=1}^k \dim \langle B_i \rangle.$$

We have that $A_2 = (A_2 \cap B_1) \cup (A_2 \cap B_2) \cup \cdots \cup (A_2 \cap B_k)$ is a decomposition of $A_2$, not necessarily into irreducibles.

Since $\dim \langle A_2 \rangle = s - 1$ we have:

$$s - 1 = \dim \langle A_2 \rangle = \sum_{i=1}^k \dim \langle A_2 \cap B_i \rangle.$$

Therefore $\dim \langle A_2 \cap B_i \rangle < \dim \langle B_i \rangle$ for exactly one index $i_0$. In other words we must have that $A_2 \cap B_i = B_i$ for all the $i \neq i_0$. For such an index necessarily $a_1 \in B_{i_0}$.

By induction, the family of all the (distinct) irreducibles that appear in the decompositions of the sets $A_i$, $i \geq 2$ form a MNS for $\langle A_2 \rangle$, with adapted basis $\{a_2, \ldots, a_n\}$. To this set we must thus only add $B_{i_0}$ in order to obtain $S_b$. Thus $S_b$ is a nested set with $s$ elements, hence maximal and the basis $b$ is adapted.

**Remark 13.13.** One can easily verify that, conversely, every MNS $S$ is of the form $S_b$ for some adapted basis.
13.14. **Non linear coordinates.** Now we pass to the fundamental geometric construction.

Given a MNS $S$ and a basis $b := \{a_1, \ldots, a_n\}$ adapted to $S$ let us consider the $a_i$ as a system of linear coordinates on $U$.

If $p_S(a_i) = A$ we denote also $a_i := a_A$. We build now new coordinates $z_A$, $A \in S$ using the monomial expressions:

$$ a_A := \prod_{B \in S, A \subseteq B} z_B. $$

Given $A \in S$, let $S_A := \{B \subseteq A, B \in S\}$. Clearly $S_A$ is a MNS for $A$ (in place of $X$) and the elements $a_B$ with $B \in S_A$ form a basis of $A$, adapted to $S_A$.

Take $a \in X$ with $p_S(a) = A$ and write $a = \sum_{B \in S_A} c_B a_B$, $c_B \in \mathbb{C}$. Let us substitute now the expressions (63):

$$ a_A := \sum_{B \in S_A} c_B \prod_{C \in S, B \subseteq C} z_C = \prod_{B \in S, A \subseteq B} z_B(c_A + \sum_{B \in S_A, B \neq A} c_B \prod_{C \in S, B \subseteq C} z_C). $$

Since $A$ is the minimum set of $S$ containing $a$ we must have $c_A \neq 0$.

At this point we can proceed as in Section 11.5 and define an embedding of $R_X$ into the ring of Laurent series in the variables $z_A$, $A \in S$. Thus for a top form $\psi$ we can define a local residue $\text{res}_{S_A} \psi$ (or $\text{res}_S \psi$ if $b$ is clear from the context).

13.15. **Proper nested sets.** We are going to tie the concepts of the previous section to that of no broken bases. Our goal is to define a bijection between $\mathcal{N}B(X)$ and a suitable family of MNS which we shall call *proper nested sets*. We need some preliminary steps.

Assume thus that the basis $b := \{a_1, \ldots, a_n\} \subset X$ is no broken, we get:

**Lemma 13.16.** $a_i$ is the minimum element of $p_S(a_i)$ for every $i$.

**Proof.** Let $A := p_S(a_i)$, by the definition of $S_b$, we must have that $A$ decomposes one of the sets $A_k = \langle a_k, \ldots, a_n \rangle \cap X$. Necessarily it must be $k \leq i$.

On the other hand $a_i$ belongs to one of the irreducibles of $A_i$, that therefore is contained in each irreducible $B$ of $A_k$, $k < i$ that contains $a_i$.

It follows that $A$ must be one of the irreducibles decomposing $A_i$. By definition of no broken basis, $a_i$ is the minimum element of $A_i = \langle a_i, \ldots, a_n \rangle \cap X$ hence also the minimum element of $A$.

This property suggests us to define:

**Definition 13.17.** A MNS $S$ is said to be proper, if the elements $a_S := \min a$, $a \in S | S \in S$, form a basis.
Lemma 13.18. If $S$ is proper, $p_S(a_S) = S$. Thus the elements $a_S := \min a, a \in S$ form a basis adapted to $S$.

Proof. Let $U \in S$ with $a_S \in U$. If $U \subset S$, then $a_S = \min a \in U$ and thus $a_S = a_U$.

Since the elements $a_S$ are a basis, this implies that $S = U$. $\square$

If $S$ is proper, we order its subsets $S_1, \ldots, S_n$ using the increasing order of the elements $a_S$. We then set $a_i := a_{S_i}$, and we have:

Theorem 13.19. (1) The basis $b := \{a_1, \ldots, a_n\}$ is no broken.

(2) In this way we establish a 1-1 correspondence between no broken bases and proper MNS.

Proof. (1) From the Remark setting $A_i := \bigcup_{j \geq i} S_j$, we have that $A_i$ is complete and decomposed by the $S_j$ which are maximal.

Clearly, by definition, $a_i$ is the minimum of $A_i$. It suffices to prove that $A_i = \langle a_i, \ldots, a_s \rangle \cap X$. Hence that $\langle A_i \rangle = \langle a_i, \ldots, a_s \rangle$ since $A_i$ is complete.

We prove it by induction. If $i = 1$ the maximality of $S$ implies that $A_1 = X$ so $a_1$ is the minimum element in $X$. Now $a_1 \notin A_2$ so $A_2 \neq X$ and, since $A_2$ is complete $\dim(\langle A_2 \rangle) < s$.

Clearly $\langle A_2 \rangle \supset \langle a_2, \ldots, a_s \rangle$ so $\langle A_2 \rangle = \langle a_2, \ldots, a_s \rangle$.

At this point it follows that, by induction, $\langle a_2, \ldots, a_s \rangle$ is a no broken basis in $A_2$ and thus, since $a_1$ is minimum in $X$, $\langle a_1, a_2, \ldots, a_s \rangle$ is a no broken basis.

(2) From the proof, it follows that the two constructions, of the MNS associated to a no broken basis and of the no broken basis associated to a proper MNS, are inverse of each other and thus the 1-1 correspondence is established. $\square$

14. Residues and cycles

We can now complete our analysis computing the residues. Recall that, if $b := \{b_1, \ldots, b_n\} \subset X$ is a basis, we denote with:

$$\omega_b := d \log(b_1) \wedge \cdots \wedge d \log(b_n)$$

Theorem 14.1. Given two no broken bases $b, c$ we have:

$$\text{res}_S \omega_b = \begin{cases} 1 & \text{if } b = c \\ 0 & \text{if } b \neq c \end{cases}$$

Proof. We prove first that $\text{res}_S \omega_b = 1$.

By definition $b_i = \prod_{A_i \subset B} z_B$ hence $d \log(b_i) = \sum_{A_i \subset B} d \log(z_B)$. When we expand the product we get a sum of products of type $d \log(z_{B_1}) \wedge d \log(z_{B_2}) \wedge \cdots \wedge d \log(z_{B_n})$ with $A_i \subset B_i$.

Now, the unique non decreasing and injective map, of $S_b$ in itself is the identity.
Therefore in this sum, all the monomials vanish except for the monomial
\( d \log(z_1) \wedge d \log(z_2) \wedge \cdots \wedge d \log(z_n) \), that has residue 1.

Let us pass now to the second case \( b \neq c \). This follows immediately from
the following Lemma. □

**Lemma 14.2.** 1. If \( b \neq c \), the basis \( c \) is not adapted to \( S_b \).

2. If a basis \( c \) is not adapted to \( S = \{S_1, \ldots, S_n\} \), a MNS, we have\( \text{res}_{S_b} \omega_c = 0 \).

**Proof.** (1) Let \( c = \{c_1, \ldots, c_n\} \) be adapted to \( S_b \), we want prove that we
have \( b = c \). We know that \( c_1 = a_1 = b_1 \), is the minimum element of \( X \).

Let \( A \) be the irreducible component of \( X \) containing \( a_1 \). This, by definition,
is an element of \( S_b \). We have thus \( A = S_1 \). We claim that \( p_{S_b}(a_1) = A \).

This follows from the fact that \( a_1 = b_1 \) and \( S_b \) is proper.

Set \( X' := S_2 \cup \cdots \cup S_n \), \( X' \) is complete. The set \( S_2, \ldots, S_n \) coincides with
the proper MNS \( S_b' \) associated to the no broken basis \( b' := \{b_2, \ldots, b_n\} \) of
\( \langle X' \rangle \).

Moreover clearly, \( c' = \{c_2, \ldots, c_n\} \) is adapted to \( S_b' \). Therefore \( b' = c' \) by
induction. Hence \( b = c \).

Using the first part, the proof of (2) follows the same lines as the proof
of Lemma 11.6 and we leave it to the reader. □

**Remark 14.3.** We end this section pointing out that by what we have proved
it follows that for any no broken basis \( b \) we have that
\[
\text{res}_b = \text{res}_{S_b}
\]
with \( \text{res}_b \) defined in Section 11.5

The advantage of this new definition is that one sometimes uses mono-
mial transformations of smaller degree and this could provide more efficient
algorithms.

14.4. **A minimal model.** Although we do not use it explicitly it may useful
to understand the origin of the non linear coordinates we have been using
and the entire theory of irreducibles and nested sets that we have built in
[19].

Start from a family of hyperplanes in \( U = V^* \), given by a list \( X \subset V \) of
linear equations, with complement \( A_X \subset U \).

In [19] we construct a minimal smooth variety \( Z_X \) containing \( A_X \) as an
open set with complement a normal crossings divisor, plus a proper map
\( \pi : Z_X \to U \) extending the identity of \( A_X \).

The smooth irreducible components of the boundary of \( A_X \) are indexed by
the **irreducible subsets**. To describe the intersection pattern between these
divisors, in [19] we developed the general theory of nested sets.

Maximal nested sets correspond to special points at infinity, intersections
of these boundary divisors. In the papers [34] and [37], implicitly the authors
use the points at infinity coming from complete flags which correspond, in
the philosophy of [19], to a \textit{maximal} model with normal crossings. It is thus not a surprise that by passing from a maximal to a minimal model the combinatorics gets simplified and the constructions become more canonical.

Let us recall without proofs the main construction of [19]. For each irreducible \( S \subset X \) we have an orthogonal subspace \( S^\perp \subset U \) where \( S^\perp = \{ a \in U \mid x(a) = 0, \forall x \in S \} \).

From the collections of projective spaces \( \mathbb{P}(U/S^\perp) \) of lines in \( U/S^\perp \) we deduce a map \( i: A_\Delta \to U^* \times_{S \in I} \mathbb{P}(U/S^\perp) \). Set \( Z_X \) equal to the closure of the image \( i(A_X) \) in this product. In [19] we have seen that \( Z_X \) is a smooth variety containing a copy of \( A_X \) and the complement of \( A_X \) in \( Z_X \) is a union of smooth irreducible divisors \( D_S \), having transversal intersection, indexed by the elements \( S \in I \).

Still in [19] we showed that a family \( D_{S_i} \) of divisors indexed by irreducibles \( S_i \) has non empty intersection (which is then smooth irreducible) if and only if the family is \textit{nested}. In particular a maximal nested set \( N \) identifies a special \textit{point at infinity} \( p_N \), intersection of the \( s = \dim(U) \) divisors corresponding to the irreducibles in \( N \). The non linear coordinates are indeed coordinates in a local chart around \( p_N \) in which \( p_N = 0 \) and the boundary divisors are given by the vanishing of the \( s \) coordinates.

15. Final considerations

As we have seen, the actual problem of computing explicitly the functions which we have discussed can be approached through several different ways. We have proposed three approaches, by expansion into partial fractions, by solving a system of linear equations interpreting the defining differential equations and finally with the method of residues. It is not really clear to us which is the most efficient.

A lot of computer work, aiming at computing Clebsch–Gordan coefficients using these methods has been done by several authors using the method of residues but whether this is really the fastest algorithm is yet unclear.

Due to limited space we have discussed few examples. For our motivations one of the most interesting examples is the set \( X \) of positive roots of a given root system. There are still many things to be uncovered in this case.

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