Dependence of the density of states on the probability distribution for discrete random Schrödinger operators

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Christoph Marx
joint with Peter Hislop (Univ. of Kentucky)

Oberlin College, Department of Mathematics

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Basic question: Can we recover the density of states for the Bernoulli-Anderson model from regular approximations?
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We answer this question positively by:

1. proving that the density of states is weak-star continuous in the underlying probability measure
2. quantifying the modulus of continuity
I. Introduction & motivation
Random lattice Schrödinger operators

Consider discrete Anderson model on \( \mathbb{Z}^d, d \in \mathbb{N} \):

\[
H_\omega = \Delta^{(d)} + \sum_{j \in \mathbb{Z}^d} \omega_j |j\rangle \langle j| \quad \text{on } \ell^2(\mathbb{Z}^d)
\]

- \( \omega_j \) iid random potentials, \( \omega_j \in [-V, V] \), some \( V > 0 \)
- common **single-site probability measure** \( \nu \), \( \text{supp} \nu \subseteq [-V, V] \)
- **Probability space**: \( \Omega = [-V, V]^{\mathbb{Z}^d} \), \( \mathbb{P} = \bigotimes_{j \in \mathbb{Z}^d} \nu =: \nu^{(\infty)} \)
Random lattice Schrödinger operators

Consider discrete Anderson model on $\mathbb{Z}^d$, $d \in \mathbb{N}$:

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- Probability space: $\Omega = [-V, V]^{\mathbb{Z}^d}$, $\mathbb{P} = \bigotimes_{j \in \mathbb{Z}^d} \nu =: \nu^{(\infty)}$

Our results are not limited to the discrete Anderson model. Indeed, our framework also applies to e.g.

- finite rank potentials: finite-range Anderson model, random polymer models
- random Schrödinger operators on graphs, e.g. the Bethe lattice
- models on the strip: long-range Jacobi, Anderson model on the Bethe-strip
The density of states

... carries information about the averaged spectral properties of ergodic family \( \{ H_\omega, \omega \in \Omega \} \)

Probability space: \( \Omega = [-V, V]^{\mathbb{Z}^d} \), \( \mathbb{P} = \bigotimes_{j \in \mathbb{Z}^d} \nu =: \nu^{(\infty)} \)
where \( \nu \) ... single site probability measure, \( \text{supp} \ \nu \subseteq [-V, V] \)

Define:

- density of states measure (DOSm)
  \[ n(f) := \mathbb{E}_{\nu^{(\infty)}} \langle 0 | f(H_\omega) | 0 \rangle, \text{where } f \in \mathcal{C}_c(\mathbb{R}) \]

- integrated density of states (IDS):
  \[ N(E) := n((-\infty, E)) \]
Known continuity in the energy

\[ H_\omega = \Delta^{(d)} + \sum_{j \in \mathbb{Z}^d} \omega_j |j \times j|, \omega_j \in [-V, V] \]

**Theorem (Craig-Simon; Bourgain- A. Klein)**

*The IDS is log-Hölder continuous in the energy, i.e. there exists a constant \( C_I = C_I(d, V) \) such that for all \( E \in \mathbb{R} \) and \( 0 < \epsilon \leq \frac{1}{2} \):

\[
|N_\nu(E) - N_\nu(E + \epsilon)| = n_\nu^{(\infty)}([E, E + \epsilon]) \leq \frac{C_I}{\log \left( \frac{1}{\epsilon} \right)}.
\]

(i) Mild modulus of continuity is optimal in general and takes into account that single-site measure \( \nu \) could be highly singular, e.g. Bernoulli-Anderson model (Carmona, Klein, Martinelli; Simon, Taylor; . . .)

(ii) For more regular \( \nu \), modulus of continuity improves, e.g. Wegner estimate: If \( d\nu(\omega_j) = \phi(\omega_j) d\omega_j \) for some \( 0 \leq \phi \in L^1 \cap L^\infty([-V, V]) \), \( \|\phi\|_1 = 1 \), then \( n_\nu^{(\infty)}([E, E + \epsilon]) \leq \|\phi\|_\infty \cdot \epsilon \).
II. Main result
Basic hypothesis: Given

- a random lattice Schrödinger operator with finite rank-potentials.
- single-site probability measures \((\nu_\alpha)\) and \(\nu\) compactly supported in \([-V, V]\) such that

\[\nu_\alpha \xrightarrow{w^*} \nu.\]
Basic hypothesis: Given

- a random lattice Schrödinger operator with finite rank-potentials.
- single-site probability measures \((\nu_\alpha)\) and \(\nu\) compactly supported in \([-V, V]\) such that

\[
\nu_\alpha \overset{w^*}{\longrightarrow} \nu .
\]

**Theorem (Qualitative continuity, P. Hislop, CM ’18 - [1] )**

If \(\nu_\alpha \overset{w^*}{\longrightarrow} \nu\), then so do the density of states measures (DOSm), i.e.

\[
n^{(\infty)}_{\nu_\alpha} \overset{w^*}{\longrightarrow} n^{(\infty)}_{\nu} .
\]

Moreover, the integrated density of states (IDS) converges point-wise:

\[
\lim_{\alpha \to \infty} N_{\nu_\alpha}(E) = N_{\nu}(E) , \forall E \in \mathbb{R} .
\]
Quantitative continuity

... to quantify modulus of continuity of the map

\[ \nu \mapsto n^{(\infty)}_\nu \] useful to work with metric
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Consider the space:

- $\mathcal{P}([a, b])$ ... Borel probability measures on $[a, b] \subset \mathbb{R}$
- $w^*$-topology on $\mathcal{P}([a, b])$ is metrizable by metric derived from the Lipschitz dual norm (e.g. R. M. Dudley, 1966), i.e.

$$d_w(\mu, \nu) = \sup \{|\mu(f) - \nu(f)| : f \in \text{Lip}([a, b]) \text{ with } \|f\|_{\text{Lip}} \leq 1\}$$,

where $\text{Lip}([a, b])$ are the Lipschitz functions on $[a, b]$ with the norm

$$\|f\|_{\text{Lip}} := \|f\|_\infty + \sup_{x \neq y \in [a, b]} \frac{|f(x) - f(y)|}{|x - y|}$$

While there are other common metrics that induce $w^*$-convergence on $\mathcal{P}([a, b])$ (e.g. Prokhorov or Wasserstein metric), the metric above will be most natural in view of our applications.
Theorem (Modulus of continuity, P. Hislop, CM ’18 - [1])

For $\nu \in \mathcal{P}([-V, V])$ and $E \in \mathbb{R}$, the modulus of continuity of the maps

$$\nu \mapsto n^{(\infty)}_{\nu}, \quad \nu \mapsto N_{\nu}(E),$$

in the weak-* topology is quantified by the following: there exist constants $\gamma > 0$, $C > 0$, and $0 < \rho < 1$, only depending on $d$ and $V$, such that for all single-site measures $\mu, \nu \in \mathcal{P}([-V, V])$ with $d_w(\mu, \nu) < \rho$ one has

$$d_w(n^{(\infty)}_{\mu}, n^{(\infty)}_{\nu}) \leq \gamma d_w(\mu, \nu)^{\frac{1}{1+2d}},$$

and, for all $E \in \mathbb{R}$,

$$|N_{\mu}(E) - N_{\nu}(E)| \leq \frac{C}{\log \left(\frac{1}{d_w(\mu, \nu)}\right)}.$$

Remarks:

- The log-Hölder modulus of the IDS in the probability distribution is a consequence of the general log-Hölder continuity of the IDS in the energy.
- It can be improved to Hölder locally about every measure $\nu \in \mathcal{P}([-V, V])$ where $E \mapsto N_{\nu}(E)$ is known to be Hölder.
Our results are not limited to the discrete Anderson model on $\mathbb{Z}^d$.

as we will explain later, our approach extends to random operators on graphs with a certain finite range structure.

e.g. to the Anderson model on the Bethe-lattice $\mathcal{B}$ with coordination number (degree) $k \geq 3$.

\[
 H_\omega = \Delta_{\mathcal{B}} + \sum_{x \in \mathcal{B}} \omega_x |x\rangle\langle x|
\]

\[
 (\Delta_{\mathcal{B}}\psi)(x) := \sum_{y \in \mathcal{B}: y \sim x} \psi(y) \quad \text{Laplacian on } \mathcal{B}
\]

Coordination number $k = 3$. 

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Given the Anderson model on the Bethe lattice with coordination number $k \geq 3$. Then, for $\nu \in \mathcal{P}([-V, V])$ and $E \in \mathbb{R}$, the modulus of continuity of the maps

$$
\nu \mapsto n^{(\infty)}_{\nu}, \quad \nu \mapsto N_{\nu}(E),
$$

in the weak-* topology is quantified by the following: there exist constants $\gamma_B > 0$ and $0 < \rho_B < 1$, only depending on $k$ and $V$, such that for all single-site measures $\mu, \nu \in \mathcal{P}([-V, V])$ with $d_w(\mu, \nu) < \rho_B$ one has

$$
d_w(n^{(\infty)}_{\mu}, n^{(\infty)}_{\nu}) \leq \frac{\gamma_B}{\sqrt{\log \left( \frac{1}{d_w(\mu, \nu)} \right)}}.
$$
Extensions: In a recent work in progress [2], we generalize this quantitative continuity result for the DOS measure to:

(a) discrete lattice operators with non-compactly supported single-site measures

(b) continuum Schrödinger operators on $\mathbb{R}^d$ with compactly supported single-site measures and

$$H_\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} \omega_n \phi(\cdot - n), \text{ for } \phi \in \mathcal{C}^d_c(\mathbb{R}^d; \mathbb{R})$$

(c) continuum Schrödinger operators with $\delta$-potentials and compactly supported single-site measures
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$$H_\omega = -\Delta + \sum_{n \in \mathbb{Z}^d} \omega_n \phi(\cdot - n), \text{ for } \phi \in C_c^d(\mathbb{R}^d; \mathbb{R})$$

(c) continuum Schrödinger operators with $\delta$-potentials and compactly supported
    single-site measures

These extensions however, in general, require higher regularity assumptions (than
Lipschits) on the class of functions, i.e. for $\nu_\alpha \overset{w^*}{\to} \nu$

$$|n^{(\infty)}_{\nu_\alpha}(f) - n^{(\infty)}(f)| \leq \gamma \|f\|_{C_c^m} d_{w}(\nu_\alpha, \nu)^{\kappa},$$

for all $f \in C_c^m(\mathbb{R})$ with $m = m(d)$. 
III. Application: weak disorder limit of the DOS
Weak disorder limit of the DOSm and IDS

\[ H_\omega = \Delta^{(d)} + \lambda \cdot \sum_{j \in \mathbb{Z}^d} \omega_j |j \times j|, \quad 0 \leq \lambda \quad \ldots \quad \text{disorder parameter} \]

with a \textit{general (not necessarily ac)} single-site measure \( \nu \in \mathcal{P}([-1, 1]) \)

\[ \text{Theorem (Weak disorder continuity of the DOSm, P. Hislop, CM '18 - [1])} \]

The DOSm is \( \omega \)-continuous as \( \lambda \rightarrow 0 \), that is

\[ n_\lambda \rightarrow n_0. \]

Moreover, there exists \( C_1 > C_{1P} \) and \( \lambda_0 < 2^{-p/2}d \) such that for every \( f \in \text{Lip} \) one has

\[ |n_\lambda f - n_0 f| \leq C_1 \| f \|_{\text{Lip}} \lambda^{p/2}d, \quad \text{for all} \quad 0 \leq \lambda \leq \lambda_0. \]
Weak disorder limit of the DOSm and IDS

\[ H_\omega = \Delta^{(d)} + \lambda \cdot \sum_{j \in \mathbb{Z}^d} \omega_j |j \times j| , \quad 0 \leq \lambda \ldots \text{disorder parameter} \]

with a \textbf{general (not necessarily ac)} single-site measure \( \nu \in \mathcal{P}([-1, 1]) \)

\[ \Rightarrow \text{rescaled single-site measure:} \]

\[ \mathcal{P}([-\lambda, \lambda]) \ni d\nu\left(\frac{x}{\lambda}\right) \xrightarrow{w^*} \delta(x) , \text{as } \lambda \to 0^+ \]

\section*{Theorem (Weak disorder continuity of the DOSm, P. Hislop, CM '18 - [1])}

The DOSm \( n_\lambda \) is \( w^* \)-continuous as \( \lambda \to 0^+ \), that is \( n_\lambda \xrightarrow{w^*} n_{\lambda=0} \).

Moreover, there exists \( C_1 = C_1(d) > 0 \) and \( \lambda_0 = 2^{-(1+2d)} \) such that for every \( f \in \text{Lip}_c(\mathbb{R}) \) one has

\[ |n_\lambda(f) - n_{\lambda=0}(f)| \leq C_1 \|f\|_{\text{Lip}} \cdot \lambda^{\frac{1}{1+2d}} , \text{for all } 0 \leq \lambda \leq \lambda_0. \]
DOSm for free case ($\lambda = 0$) is given explicitly \(dn_{\lambda=0}(E) = \rho_{\lambda=0}^{(d)}(E)dE\) with
\[
\rho_{\lambda=0}^{(1)}(E) = \frac{1}{2\pi} \frac{1}{\sqrt{1 - \left(\frac{E}{2}\right)^2}} \chi(-2,2)(E)
\]
\[
\rho_{\lambda=0}^{(d)}(E) = \left(\rho_{\lambda=0}^{(1)} * \cdots * \rho_{\lambda=0}^{(1)}\right)(E), \text{ for } d \geq 2.
\]
d-times

⇒ Hölder-continuity of IDS for free case ($\lambda = 0$):
\[
N_{\lambda=0}(E + \epsilon) - N_{\lambda=0}(E) = n_{\lambda=0}([E, E + \epsilon]) \leq c_0 \epsilon^\delta,
\]
where we take the constants \(c_0, \delta > 0\) to be uniform in \(E\), i.e. only depending on the dimension \(d\). For instance for \(d = 1\): \(\delta = \frac{1}{2}\) ("van Hove singularity"), while for \(d \geq 2\), one can take \(\delta = 1\).

**Theorem (Weak disorder continuity of the IDS, P. Hislop, CM ’18 - [1])**

There exists a constant \(C_2 = C_2(d, V) > 0\) such that for all \(E \in \mathbb{R}\) and \(0 \leq \lambda \leq \lambda_0 = 2^{-(1+2d)}\), one has
\[
|N_\lambda(E) - N_{\lambda=0}(E)| \leq C_2 \lambda^{\left(\frac{\delta}{1+\delta}\right)} \left(\frac{1}{1+2d}\right).
\]
Context: known continuity results of the IDS in the disorder (fixed energy!) for the weak-disorder regime
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- most known results restricted to dimension $d = 1$
  - the IDS and the usual Anderson model
  - with appropriate decay of the Fourier-transform of the single-site measure (Speis; Campanino, Klein; Bovier, Klein; Pastur Figotin; ...), thus in particular, exclude Bernoulli-Anderson!
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Specifically, using the super-symmetric replica method:

- Campanino, Klein (1990): for a dense set of energies $E \in (-2, 2)$, the map $\lambda \mapsto N_\lambda(E)$ is $C^\infty$ about $\lambda = 0$
- Speis (1991): for all energies $E \in (-2, 2)$, $\lambda \mapsto N_\lambda(E)$ is continuous about $\lambda = 0$
• **very few results for** $d > 1$: *Schenker* (2004) and *Hislop, Klopp, and Schenker* (2005) prove Hölder continuity of $\lambda \mapsto N_\lambda(E)$ IF the single-site measure is absolutely continuous (ac) with bounded density
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**Our approach:**

• works in any dimension and works for a general (i.e. not necessarily ac, in particular for Bernoulli) single-site measure of compact support

• yields quantitative information about the DOSm, not just the IDS
IV. Ideas of the proof
of the main result
Necessary features of the model

- the theory is **not restricted** to the lattice operators considered so far
- in fact, our framework applies to all discrete random operators \( H_\omega \) where

  1. the DOSm is well-defined as a spectral average:

\[
n(f) := \frac{1}{m} \mathbb{E}\{\operatorname{Tr}(Pf(H_\omega)P)\}, \ f \in C_c(\mathbb{R}),
\]

for some fixed finite-rank projector \( P \), \( \operatorname{rk}P = m \), some \( m \in \mathbb{N} \)

  2. the **operator has a “finite-range structure,”** that is for each \( n \in \mathbb{N} \), the function

\[
\omega = (\omega_j) \mapsto \operatorname{Tr}(P(H_\omega)^nP)
\]

depends on only finitely many variables \( \omega_j \) whose number is bounded above by some counting function \( \Gamma : \mathbb{N} \to \mathbb{N} \)

The “finite-range structure” allows to only consider the effects of varying the probability distribution at **finitely** many sites

  typically \( \Gamma(n) \sim \text{volume of ball of radius } n \)
Our proof of the qualitative and quantitative continuity of the DOSm has two key steps:

step 1 - finite range reduction:

- Given $\Omega = [-V, V]^d$ and prob. measures $(\nu_\alpha)$ and $\nu$ supported on $[-V, V]$, i.e.

  $$\nu_\alpha^{(\infty)} := \bigotimes_{j \in \mathbb{Z}^d} \nu_\alpha, \quad \nu^{(\infty)} := \bigotimes_{j \in \mathbb{Z}^d} \nu \quad \ldots \quad \text{prob. meas. on } \Omega$$
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**step 1 - finite range reduction:**

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  $$
  \nu^{(\infty)}_\alpha := \bigotimes_{j \in \mathbb{Z}^d} \nu_\alpha, \quad \nu^{(\infty)} := \bigotimes_{j \in \mathbb{Z}^d} \nu \quad \text{... prob. meas. on } \Omega
  $$

- **Basic question:** For $F \in C(\Omega)$, estimate
  
  $$
  |\nu^{(\infty)}_\alpha(F) - \nu^{(\infty)}(F)|
  $$

- **Problem:** product measures differ in **infinitely** many factors

- here: $F(\omega) = \text{Tr}(Pf(H_\omega)P)$, for arbitrary $f \in C_c(\mathbb{R})$
Problem: product measures differ in \textbf{infinitely} many factors

Solution: reduce to a situation where product measures differ on only finitely many factors: replace $\nu_\alpha^{(\infty)}$ by

$$
\nu_\alpha^{(M)} := \left( \bigotimes_{j \in \mathbb{Z}^d; \|j\|_\infty \leq M} \nu_\alpha \right) \otimes \left( \bigotimes_{j \in \mathbb{Z}^d; \|j\|_\infty > M} \nu \right), \quad \text{for some } M \in \mathbb{N}
$$

Lemma ("finite range reduction")

For any $F \in \mathcal{C}(\Omega)$ and $\epsilon > 0$, there exists an integer $M = M(F, \epsilon) \in \mathbb{N}$ such that

$$
|\nu_\alpha^{(M)}(F) - \nu_\alpha^{(\infty)}(F)| < \epsilon, \quad \forall \alpha \in \mathbb{N}.
$$

we show that $M = M(F, \epsilon)$ can be determined explicitly by finding $\tilde{F} \in \mathcal{C}(\Omega)$ depending on only finitely many variables

$$
\{\omega_j : \|j\|_\infty \leq M\} \quad \text{such that } \|F - \tilde{F}\|_\infty < \frac{\epsilon}{2}.
$$

specifically, for $F(\omega) = \text{Tr}(Pf(H_\omega)P)$ and $f \in \mathcal{C}_c(\mathbb{R})$ we use approximation of $f$ by Bernstein polynomials to determine $\tilde{F}$

relies on the operator having a "finite-range structure"
step 2 - single site variations:

- Step 1 reduces to a situation where we vary the measure at only finitely many sites.
- Vary one site at a time leads to a family of finite rank perturbations; more generally, consider

\[ \lambda \mapsto H_\lambda := H^{(0)} + \lambda T_1, \]  

where \( T_1 \in S_1 \) and \( H^{(0)} \) bounded s.a.

---

**Lemma ("Lipschits property")**

Let \( T_2 \in S_2 \) be given. Then, for every \( f \in \text{Lip}_c(\mathbb{R}) \) with Lipschitz constant \( L_f \), the map \[ \mathbb{R} \ni \lambda \mapsto \text{Tr}(T_2 f (H_\lambda) T_2) \] is Lipschitz with Lipschitz constant \[ \leq 2 \| T_2 \|_{S_2}^2 \cdot L_f. \]

- Proof uses quasi-analytic extensions & Helffer-Sjöstrand functional calculus.
step 2 - single site variations:

- step 1 reduces to a situation where we vary the measure at only finitely many sites.
- vary one site at a time leads to a family of finite rank perturbations; more generally, consider

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where \( T_1 \in S_1 \) and \( H^{(0)} \) bounded s.a.

Lemma ("Lipschitz property")

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is Lipschitz with

Lipschitz constant \( \leq 2 \| T_2 \|^2_{S_2} \cdot L_f \).

- proof uses quasi-analytic extensions & Helffer-Sjöstrand functional calculus
- extends a similar lemma by B. Simon for \( C^\infty \) functions (Proc. AMS 1998)
- known results on operator Lipschitz functions (generalizing results on the spectral shift function), in general, require higher regularity in \( f \)
V. Application: Continuity of the Lyapunov exponent in the probability distribution
Continuity of the Lyapunov exponent in the probability distribution

- continuity statement of the DOSm on the underlying probability distribution
- derive respective continuity statements for integral transforms of the DOSm

  e.g. for random Schrödinger operators on \( \mathbb{Z} \)

- Lyapunov exponent related to DOSm via the Thouless formula

\[
L_\nu(E) = \int_{\mathbb{R}} \log |E' - E| \ dn^{(\infty)}_\nu(E') \in [0, \infty), \ E \in \mathbb{C}.
\]
We use our quantitative continuity theorem for the DOSm to prove:

**Theorem**

(i) **Qualitative continuity:** For each fixed $E \in \mathbb{C}$, the map

$$\nu \mapsto L_\nu(E)$$

is continuous in the $w^*$-topology on $\mathcal{P}([-V, V])$.

(ii) **Assume that $E \in \mathbb{R}$ satisfies:** there exists a constant $0 < D < \infty$ and an exponent $0 < \beta \leq 1$ such that for all $\epsilon > 0$:

$$|N_{\nu_\alpha}(E + \epsilon) - N_{\nu_\alpha}(E - \epsilon)| \leq D\epsilon^\beta, \forall \alpha \in \mathbb{N},$$

$$|N_\nu(E + \epsilon) - N_\nu(E - \epsilon)| \leq D\epsilon^\beta$$

(*equi-$\beta$-Hölder continuity at $E$*)

Then, there exists $\alpha_0 \in \mathbb{N}$ and $C_L = C_L(D, \beta)$ such that for all $\alpha \geq \alpha_0$,

$$|L_{\nu_\alpha}(E) - L_\nu(E)| \leq C_L \cdot [d_w(\nu_\alpha, \nu)]^{\frac{1}{3}} \left(\frac{\beta}{\beta + 1}\right).$$
Remarks:

- result extends to random Schrödinger operators on the strip: continuity for the sum of all non-negative Lyapunov exponents
- recovers recent results by Avila, Eskin, Viana (announced); Viana, Bocker (ETDS 2017) on (qualitative) continuity of Lyapunov exponents in the probability distribution for products of random matrices
- in addition, in part (ii), we obtain quantitative continuity results for the case of Schrödinger cocycles at energies where the IDS is equi-Hölder continuous
- The theorem extends to random Schrödinger operators on the strip: continuity for the sum of all non-negative Lyapunov exponents
- Within our framework, the Hölder condition on the IDS in part (ii) cannot be relaxed
continuity theorem for the DOSm for \((E + i\epsilon) \in \mathbb{H}^+\) yields

\[|L_{\nu\alpha}(E + i\epsilon) - L_\nu(E + i\epsilon)| \leq C_3 \frac{1}{\epsilon} \cdot d_W(\nu_\alpha, \nu)^{\frac{1}{3}}, \text{ for all } \alpha \geq \alpha_0.\]

2. use successive approximation to obtain bound on the rate of convergence of:

\[|L_{\nu\alpha}(E) - L_\nu(E)| \leq |L_{\nu\alpha}(E) - L_{\nu\alpha}(E + i\epsilon)| + |L_{\nu\alpha}(E + i\epsilon) - L_\nu(E + i\epsilon)| + |L_\nu(E + i\epsilon) - L_\nu(E)|\]

3. quantify modulus of continuity for \(\epsilon \mapsto L_\nu(E + i\epsilon)\) as \(\epsilon \to 0^+\) (for fixed \(\nu\)):

\[
\limsup_{\epsilon \to 0^+} \left| \frac{L_\nu(E + i\epsilon) - L_\nu(E)}{\epsilon^\beta} \right| = \frac{1}{\beta} \limsup_{\epsilon \to 0^+} \left\{ \epsilon^{1-\beta} P_{n_\nu}^{(\infty)}(E + i\epsilon) \right\} \quad (1)
\]

with \(P_{n_\nu}^{(\infty)}\) ... Poisson transform of the DOSm

- (1) relates fractional derivatives of the Lyapunov exponent to continuity properties of the DOSm (extension of the starting point for Kotani theory where \(\beta = 1\))
- (1) is origin of equi-\(\beta\)-Hölder continuity condition in quantitative continuity result for the LE in the probability measure
Thank you!
[1] Peter D. Hislop, C. A. Marx, *Dependence of the Density of States on the Probability Distribution for Discrete Random Schrödinger Operators*, to appear in International Mathematics Research Notices, rny156, https://doi.org/10.1093/imrn/rny156
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[2] Peter D. Hislop, C. A. Marx, *Dependence of the density of states on the probability distribution - part II: Schrödinger operators on $\mathbb{R}^d$ and non-compactly supported probability measures*, in progress.