Model glasses coupled to two different heat baths

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In a $p$-spin interaction spherical spin-glass model both the spins and the couplings are allowed to change in the course of time. The spins are coupled to a heat bath with temperature $T$, while the coupling constants are coupled to a bath having temperature $T_J$. In an adiabatic limit (where relaxation time of the couplings is much larger that of the spins) we construct a generalized two-temperature thermodynamics. It involves entropies of the spins and the coupling constants. The application for spin-glass systems leads to a standard replica theory with a non-vanishing number of replicas, $n = T/T_J$. For $p > 2$ there occur at low temperatures two different glassy phases, depending on the value of $n$. The obtained first-order transitions have positive latent heat, and positive discontinuity of the total entropy. This is the essentially non-equilibrium effect. The predictions of longtime dynamics and infinite-time statics differ only for $n < 1$ and $p > 2$. For $p = 2$ correlation of the disorder (leading to a non-zero $n$) removes the known marginal stability of the spin glass phase. If the observation time is very large there occurs no finite-temperature spin glass phase. In this case there are analogies with the broken-ergodicity dynamics in the usual spin-glass models and non-equilibrium (aging) dynamics. A generalized fluctuation-dissipation relation is derived.

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I. INTRODUCTION

The spin-glass model proposed by Edwards and Anderson \cite{1,2} is a paradigm for large number of different random systems in nature. Its main assumptions, randomness and quenching of exchange couplings, reasonably reflect the crucial properties of random magnets with localized magnetic moments \cite{2}, structurally disordered materials \cite{2}, and large number of other systems (for example, artificial neural networks \cite{1}). In particular, alternation means that there are ferromagnetic as well as antiferromagnetic couplings (attractive and repulsive in the language of particle dynamics in structurally disordered compounds and alloys \cite{2}), and quenching means that there exists a large difference between relaxation times of couplings and magnetic degree of freedom (spins).

The typical example of a spin-glass is a dilution of a magnetic metal (such as Fe or Mn) in a non-magnetic host (for example Cu), where the concentration of the magnetic metal is not very large. Thus there is no direct exchange interaction between magnetic ions, but due to conduction electrons of Cu an indirect exchange interaction is possible (RKKY interaction) \cite{2}. This interaction has oscillating character with respect of the distance between a pair of ions, and because the positions of magnetic ions in the dilute are random we have the typical case of spin-glass. By a more simple and evident mechanism spin-glass is realized in a dilution of ferromagnetic metal with antiferromagnetic one.

However, in all these cases there can be some doubts about truly quenching of coupling constants. Furthermore, several independent mechanisms can be pointed out for relatively slow changing of coupling constants with time: 1) diffusion of magnetic ions; 2) the distance between a pair of ions can be changed due to external variations of pressure \cite{1}. Such reasons very naturally induce developments of spin-glass models where coupling constants are not quenched variables \textit{a priori}, but change with time, according to some slow stochastic process. It will allow to consider the different and perhaps unexpected case of a large but finite separation between relaxation times. If we start with some typical state far from equilibrium, and if the total relaxation time of the whole system is very large, then in some intermediate time regime the spins relax to a partial equilibrium with temperature $T$, while the coupling constants relax to a partial equilibrium at some different temperature $T_J$. For describing this regime, we can formally introduce two different thermostats with temperatures $T$ and $T_J$. Later we shall introduce other infinite time scales but all these times are much smaller than the relaxation time towards the total equilibrium with the unique temperature. The usual spin glass model is recovered when $T_J$ is infinite, because in this limit the coupling constants are totally independent of each other and of the spins. Moreover, with these general assumptions about time-scales and temperatures one can obtain the non-equilibrium stationary distribution in more general cases, and construct the appropriate thermodynamics.
First phenomenological spin-glass models with time-dependent coupling constants were considered by Horner [2]. The two-temperature approach was introduced by Coolen et al. [4] and Dotsenko et al. [8]. Recently one of us discussed a phenomenological two-temperature approach for describing the dynamics of model glasses in terms of a non-equilibrium thermodynamics. In this approach one temperature is the temperature of the thermal bath, while the second temperatures is self-generated and time-dependent, due to the non-equilibrium nature of the glassy state [23, 24]. Our approach in the present paper is complementary: We assume the existence of two fixed temperatures for the different components. It can be realized that different temperatures for different degrees of freedom, and large difference between the corresponding relaxation times are two necessary ingredients of so-called glassy behavior: Common action of these physical mechanisms ensures a specific, non-equilibrium regime of behavior.

We shall investigate the two-temperature dynamics and statics of the p-spin interaction spin glass model [21,22,24]. Multi-spin interaction models are not only a convenient laboratory for investigating phase transitions but also very well known in physics of magnets [3]. For example, this type of interaction is necessary for understanding properties of He3 [10] and metamagnets [6]. The statics of the model is described by one step of Replica Symmetry Breaking (1RSB). The transition is first-order-type. Near the spin glass transition point the order parameter has a discontinuity [21,24], due to the non-equilibrium nature of the glassy state with typical initial conditions) are known to be different; this fact can be traced back to omission of activated processes [25]-[29]. Our approach in the present paper is complementary: We assume the existence of two fixed temperatures and the second temperatures is self-generated and time-dependent, due to the non-equilibrium nature of the glassy state.

In section II the dynamics of the model is introduced via a set of Langevin equations. In section III we discuss the case where the coupling constants are fixed with respect of the dynamics of the spins (adiabatic statics). For this case we construct an analog of the usual thermodynamics. The considerations in this section are actually more general, and do not depend on the details of the adiabatic system. This general theory is applied to the concrete case of the mean-field spherical spin-glass model in section IV. We also discuss different phase transitions which arise in this context. The detailed discussion of thermodynamical quantities is given.

In section V the ergodic dynamic of the spin-glass model is investigated. As opposed to usual mean-field spin-glass systems this dynamic has a larger field of relevance in our case. We show in the adiabatic case that there is a difference between predictions of statics and long-time dynamics if the temperature of coupling constants is not larger than the temperature of spins. Effects connected with large observation times are discussed also. The summary of our results is represented at the last section. Some technical questions are considered in the appendices.

II. THE MULTI-SPIN INTERACTION MODEL AND ITS HEAT BATHS

In the mean-field spherical p-spin model we add a harmonic energy for the couplings, yielding

\[ \mathcal{H} = \sum_{1 \leq i_1 < \ldots < i_p \leq N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} + \frac{v}{2N} \sum_{1 \leq i_1 < \ldots < i_p \leq N} J_{i_1 \ldots i_p}^2, \]

where \( J_{N}^2 = p!J^2/(2N^{p-1}) \) is the usual normalization factor for mean-field models, with \( J \) and \( v \) being fixed energy scales. We assume that the coupling constants and the spins interact with heat baths at temperatures \( T_j \) and \( T \), respectively. The (overdamped) Langevin equations for the dynamics in this model have the following form:

\[ \Gamma \partial_t \sigma_i = -r \sigma_i - \frac{\partial \mathcal{H}}{\partial \sigma_i} + \eta_i(t), \quad \langle \eta_i(t) \eta_j(t') \rangle = 2\Gamma T \delta_{ij} \delta(t - t') \] (2.2)

\[ \tilde{\Gamma}_J \partial_t J_{i_1 \ldots i_p} = -\frac{\partial \mathcal{H}}{\partial J_{i_1 \ldots i_p}} + \eta_{i_1 \ldots i_p}(t), \quad \langle \eta_{i_1 \ldots i_p}(t) \eta_{j_1 \ldots j_p}(t') \rangle = 2\tilde{\Gamma}_J T \delta_{i_1 \ldots i_p,j_1 \ldots j_p} \delta(t - t') \] (2.3)

Here \( r(t) \) is the Lagrange multiplier for enforcing the spherical constraint \( \sum_i \sigma_i^2(t) = N \), while \( \Gamma \) and \( \tilde{\Gamma}_J \) are the damping constants. The coupling constants \( J_{i_1 \ldots i_p} \) and the noises \( \eta_{i_1 \ldots i_p} \) are symmetric with respect of interchange of the indices. In Eqs. (2.2,2.3) so called Einstein relation holds between the strength of noise and the damping constant. This means that the thermal baths themselves are in thermal equilibrium [3]. For ensuring the correct thermodynamical limit we must take: \( \tilde{\Gamma}_J = \Gamma J / \sqrt{N} \) (see Appendices A, B).
Straightforward calculations show that if \( v \sim T_J \) and the limit \( \Gamma_J \to \infty \) is taken first, followed by \( T_J \to \infty \), then the coupling constants are quenched (with respect to the spins) independent Gaussian random variables. This limit thus yields the standard \( p \)-spin model (see Appendix A).

With help of standard methods [11,36] we study the dynamics in Appendix B. We arrive at the following equations for the average dynamics of a single spin in the mean field caused by the other ones

\[
(\Gamma \partial_t + r)\sigma(t) = \frac{pJ^2}{2\Gamma_J} \int_{-\infty}^{\infty} d\bar{t}e^{-(t-\bar{t})/\tau_J}C^{p-1}(\bar{t}, t)\sigma(\bar{t}) + \frac{pT_JJ^2}{2v}(p-1) \int_{-\infty}^{t} d\bar{t}e^{-(t-\bar{t})/\tau_J}C^{p-2}(\bar{t}, t)G(t, \bar{t})\sigma(\bar{t}) + \eta(t),
\]

\[
\langle \eta(t)\eta(t') \rangle = 2\Gamma T \delta(t - t') + \frac{pT_JJ^2}{2v} \exp (-|t - t'|/\tau_J)C^{p-1}(t, t'),
\]

where

\[
\tau_J = \frac{\Gamma_J}{v} \tag{2.5}
\]

is the timescale at which the couplings change. Details of the derivation of this equation can be found in the appendix B.

Some comments about the general structure of Eq. (2.4) are at order. As it is well-known that the effective dynamical equations for spin-glass systems with quenched disorder are essentially non-markovian, i.e., they depend on the "history" of the process. Evidently, this arises due to quenching of a coupling constant with time. In our case — on account of the characteristic time (2.3) — the non-markovian property is "smoothened" by the exponential kernel (see Eq. (2.4)).

With help of (2.4) we may derive coupled equations for the correlation function

\[
C(t, t') = \frac{1}{N} \sum_i \langle \sigma_i(t)\sigma_i(t') \rangle \tag{2.6}
\]

and the response function

\[
G(t, t') = \frac{1}{N} \sum_i \frac{\delta \langle \sigma_i(t) \rangle}{\delta h_i(t')}, \tag{2.7}
\]

describing the response of spin \( \sigma_i \) to a small local field \( h_i \) imposed on an earlier moment \( t' \), via an instantaneous change of the Hamiltonian as \( \mathcal{H} \to \mathcal{H} - h_i\sigma_i \). To fix the units, we shall take \( \Gamma = 1 \) from now on. We find for \( t > t' \)

\[
(\partial_t + r)C(t, t') = \frac{pJ^2}{2\Gamma_J} \int_{-\infty}^{t} d\bar{t}e^{-(t-\bar{t})/\tau_J}C^{p-1}(\bar{t}, t)C(\bar{t}, t') + \frac{pT_JJ^2}{2v}(p-1) \int_{-\infty}^{t} d\bar{t}e^{-(t-\bar{t})/\tau_J}C^{p-2}(\bar{t}, t)G(t, \bar{t})C(\bar{t}, t') + \int_{-\infty}^{t} d\bar{t}G(t', \bar{t}) \left( 2T \delta(t - \bar{t}) + \frac{pT_JJ^2}{2v} e^{-|t - \bar{t}|/\tau_J}C^{p-1}(t, \bar{t}) \right) \tag{2.8}
\]

\[
(\partial_t + r)G(t, t') = \frac{pJ^2}{2\Gamma_J} \int_{-\infty}^{t} d\bar{t}e^{-(t-\bar{t})/\tau_J}C^{p-1}(\bar{t}, t)G(\bar{t}, t') + \frac{pT_JJ^2}{2v}(p-1) \int_{-\infty}^{t} d\bar{t}e^{-(t-\bar{t})/\tau_J}C^{p-2}(\bar{t}, t)G(t, \bar{t})G(\bar{t}, t') \tag{2.9}
\]

Generally speaking, both the relaxation toward a stationary state as well as fluctuations in that state are described by this closed pair of equations. Particularly, in the second case the time-translational invariance is expected to hold: one-time quantities do not depend of time, two-time quantities depend only on the difference of times:

\[
C(t, t') = C(t - t'), \quad G(t, t') = G(t - t') \tag{2.10}
\]

This regime only applies in the limit when the initial time \( t_0 \) goes to \( -\infty \); this was already inserted in the lower limits of integration of Eq. (2.3). Indeed, then the memory of the initial conditions is washed out, and the system relaxes toward its stationary state. It should be stressed that this infinity is taken only after the thermodynamic limit \( N \to \infty \), \( t - t' \) can be viewed as an observation time, or as the finding of some clock designed to display the temporal dynamics of fluctuations.
III. TWO-TEMPERATURE ADIABATIC THERMODYNAMICS: GENERAL STRUCTURE

Recently Coolen, Penney, and Sherrington [14] proposed a dynamical approach to the statistical mechanics of spin glass systems, where the introduction of replicas is not needed initially (though they enter later without the $n \to 0$ limit; see also [13]-[18]). This approach can be called adiabatically static, because it is a static limit obtained by taking

$$\Gamma_j \to \infty$$  \hspace{1cm} (3.1)

(i.e., $\tau_j \to \infty$ but $v$ remains finite) immediately in the initial equations of motion (recall that $\tau_j$ remains much smaller than the initial time: $\tau_j \ll |t_0|$). For times much less than $\tau_j$ the spins will still see random couplings. As opposed to the standard case, the couplings are no longer uncorrelated. The correlation of couplings is coded in a finite $T_J$, and may lead to new physics.

Equations (2.2), (2.3) can be investigated by the method of adiabatic elimination (see for example [5]). Here we go further and construct the corresponding thermodynamics. For the case of a static distribution the procedure is as follows. First equation (2.2) is solved keeping the coupling constants $J$ fixed (adiabatic following). Further, the Langevin equations in this case have the following equilibrium distribution

$$P(\sigma|J) = \frac{1}{Z(\sigma, J)} \exp[-\beta H(\sigma, J)]$$  \hspace{1cm} (3.2)

(In this section we do not write the spherical factor explicitly; the reader can consider it to be included in $H(\sigma, J)$.) The partition sum for given $J$-configuration is

$$Z_J(\sigma) = \text{Tr}_\sigma \exp[-\beta H(\sigma, J)]$$  \hspace{1cm} (3.3)

In the evolution of the $J$-subsystem the averaging over the fast variables in (2.3) can be carried out. In general this should be in a self-consistent way. At quasi-equilibrium of the $\sigma$-subsystem this average can be performed and leads to the use of (3.2). In this way we get from eqs. (2.1), (2.4) and (2.3) a related dynamics for the couplings, in which $H(\sigma, J)$ is replaced by $-T \ln Z_J(\sigma)$, which plays the role of effective hamiltonian in the corresponding dynamics. Specifically we have the effective equation of motion

$$\Gamma_J \partial_t J_{i_1...i_p} = \partial_{J_{i_1...i_p}} T \ln Z_J(\sigma) + \eta_{i_1...i_p}(t)$$  \hspace{1cm} (3.4)

As the noise is due to a bath at temperature $T_J$, see eq. (2.3), the equilibrium distribution of this equation reads

$$P(J) = \frac{Z^n_J(J)}{Z}$$  \hspace{1cm} (3.5)

where

$$Z = \int DJ Z^n_J(J)$$  \hspace{1cm} (3.6)

This approach introduces a ‘dynamical’ replica index

$$n = \frac{T}{T_J}$$  \hspace{1cm} (3.7)

The free energy follows from the usual formula

$$F = -\frac{T}{n} \ln Z = -T_J \ln Z$$  \hspace{1cm} (3.8)

Now we construct the appropriate thermodynamics for this two-temperature model using eqs. (3.5), (3.6). We first need the joint distribution of $\sigma$ and $J$. As usual we can express the unrestricted probability $P(\sigma, J)$ in terms of the conditional probability $P(\sigma|J)$ as

$$P(\sigma, J) = P(J)P(\sigma|J) = \frac{Z^{n-1}_J}{Z} \exp(-\beta H(\sigma, J))$$  \hspace{1cm} (3.9)

For the total energy $U$ we have
\[ U = \text{Tr}_{\sigma} \int DJ \mathcal{H}(\sigma, J) P(\sigma, J), \]  

(3.10)

which can be viewed also as the average energy of the \( \sigma \)-subsystem. Direct calculations show that

\[ -\frac{1}{n} \frac{\partial}{\partial \beta} \ln Z \bigg|_{n} = U, \quad -\frac{1}{n} \frac{\partial}{\partial \beta} \ln Z \bigg|_{T_{J}} = U - F_{J}, \]  

(3.11)

where

\[ F_{J} = -\int DJ T \ln Z_{\sigma}(J) P(J) \]

is the self-averaged free energy of the \( \sigma \)-subsystem or the mean effective energy.

We define entropies of total system and its subsystems by the usual Boltzmann-Gibbs-Shannon formula with help of the corresponding distributions (3.9). For example, the total entropy reads

\[ S = -\int DJ \text{Tr}_{\sigma} P(\sigma, J) \ln P(\sigma, J), \]  

(3.12)

This involves just the general, statistical definition of entropy for a macroscopic system, which is also relevant outside equilibrium [3]. Due to the decomposition (3.9) one gets two contributions,

\[ S = S_{\sigma} + S_{J} \]  

(3.13)

where

\[ S_{\sigma} = \int DJ P(J) \left\{ -\text{Tr}_{\sigma} P(\sigma|J) \ln P(\sigma|J) \right\} \]  

(3.14)

is our analog of the usual quenched average entropy of the spin motion (more precisely, it is the so-called conditional entropy [3]), while the coupling-part of the entropy reads

\[ S_{J} = -\int DJ P(J) \ln P(J) \]  

(3.15)

The analogous separation of the total entropy in two or more parts appears in other problems of statistical physics [37] [38]. In particular, it concerns the fine-graining procedure which introduces states of a statistical system relative to a fixed value of some properly chosen order parameter (a quantity under macroscopic control). Then the total entropy is also separated as in Eq. (3.13), where \( S_{\sigma} \) corresponds to average entropy of the relative states, and \( S_{J} \) corresponds to entropy of the order parameter itself. A simple example is just a piston separating a volume with a gas in two equivalent parts. Then \( S_{\sigma} \) is entropy of the gas in one part, and obviously \( S_{J} = \ln 2 \). A similar separation occurs in glassy transitions, where \( S_{J} \) corresponds to the configurational entropy or complexity [25] [26]. The famous phenomena of Maxwell’s demon [27] (proposed by Maxwell more than one century ago), and its subsequent reformulations and generalizations display that in a process of measurement \( S_{\sigma} \) can be decreased, instead \( S_{J} \) increases, and the total entropy \( S_{\sigma} + S_{J} \) can only increase.

Let us consider now a generalized thermodynamics which arises in the context of Eqs. (3.8)-(3.15). As a matter of fact, to generalize the usual thermodynamics we notice the following relations only

\[ F = F_{J} - T_{J} S_{J} \]  

(3.16)

\[ F = U - T_{J} S_{J} - T S_{\sigma} \]  

(3.17)

Eq. (3.16) is the usual thermodynamical formula for the \( J \)-subsystem. The second formula is more interesting because it gives a connection between the characteristics of the subsystems and the whole system. This agrees with the expression of the free energy for a glassy system put forward previously by one of us [23] [27] [28]. In that approach the equivalent of \( T_{J} \) is the dynamically generated effective temperature.

These results can be written in the differential form

\[ dF = -S_{\sigma} dT - S_{J} dT_{J} \]  

(3.18)
If we add an external field, it can be checked that
\[ dF = -S_\sigma \, dT - S_J \, dT_J - M \, dH \]  
(3.19)

This implies that the first law of thermodynamics takes the form
\[ dU = T \, dS_\sigma + T_J \, dS_J - M \, dH \]  
(3.20)

As the last term can be indentified with \( dW \), the work done on the system, the change in heat reads
\[ dQ = T \, dS_\sigma + T_J \, dS_J \]  
(3.21)

Eqs. (3.18, 3.19, 3.20) constitute a manifestation of a thermodynamic process in the following sense. A process is called “thermodynamic” if its characteristic time \( \tau_{th} \) is much larger than internal relaxation times of the considered system. Due to this condition it is possible to represent the process as a chain of stationary states. In our adiabatic system there are two relaxation times, \( \tau \) (an effective characteristic time of spins) and \( \tau_J \) with \( \tau_J \gg \tau \). There can thus exist two types of thermodynamic processes: A slow one with \( \tau_{th} \gg \tau \) and a relatively fast one with \( \tau_J \gg \tau_{th} \gg \tau \). In the second case the \( J \)-subsystem does not change during this process, implying, for example, \( dS_J = 0 \). We see that (3.18-3.20) represent a slow thermodynamic process where states of both subsystems are changed. This classification allows us to discuss the question about heating or cooling of a two-temperature adiabatic system. If cooling is slow enough, typically both temperatures change; for example, in the extreme cooling process we can have \( T_J \to 0 \) and \( T \to 0 \) simultaneously. In the opposite case of a fast cooling (or heating) process \( T_J \) is a constant while \( T \) varies in time.

Irreversible effects also can be included in the present scheme. In general, irreversibility means that there are additional sources to increase entropy or decrease free energy. Namely it reads:
\[ dF < -S_\sigma \, dT - S_J \, dT_J - M \, dH, \]
\[ dU < T \, dS_\sigma + T_J \, dS_J - M \, dH \]  
(3.22)

Eq. (3.17) can be easily generalized to a many-level adiabatic system where the first part of variables is slow with respect to the second part, the second part is slow with respect to third part,... For example, if we have a three-level system with the parts: \( \{ J \}, \{ \sigma_1 \}, \{ \sigma_2 \} \), having relaxation times \( \tau_J \gg \tau_{\sigma_1} \gg \tau_{\sigma_2} \)
\[ F = U - T_J S_J - T_{\sigma_1} S_{\sigma_1} - T_{\sigma_2} S_{\sigma_2} \]  
(3.23)

It should be noticed, however, that on time scales of order \( \tau_{\sigma_1} \), where interesting non-equilibrium dynamics of the \( \sigma_1 \) system occurs, the \( \sigma_2 \)-system is in equilibrium, while the \( J \)-system is fixed; on the other hand, on timescales of order \( \tau_J \) both the \( \sigma_1 \) and the \( \sigma_2 \) systems are in equilibrium. A physical realization of this scenario occurs in glass forming liquids, with their fast and slow processes, while the \( J \)-system then describes the configurational or \( \alpha \)-processes. In this context Eq. (3.23) corresponds also to recently introduced models with two-level disorder: Coupling constants are also considered as frozen variables with respect to some other set of variables [15].

IV. ADIABATIC STATICS.

In this section we investigate the adiabatic static limit of the mean field spherical spin-glass model introduced in the section II. The free energy is described by (3.8). As in ref [21] we have
\[ Z = \prod_{\alpha \beta} \left( \int \frac{d\lambda_{\alpha \beta}}{2\pi i} \right) \exp \left( -NG_n(q_{\alpha \beta}, \lambda_{\alpha \beta}) \right) \]  
(4.1)

\[ 2G_n(q_{\alpha \beta} \lambda_{\alpha \beta}) = -n \ln 2\pi - \frac{\mu}{p} \sum_{\alpha \beta} q_{\alpha \beta}^p + \frac{1}{2} \sum_{\alpha \beta} q_{\alpha \beta} \lambda_{\alpha \beta} + \ln \left( -\lambda \right) \]
\[ = \text{const.} - \frac{\mu}{p} \sum_{\alpha \beta} q_{\alpha \beta}^p - \text{tr} \ln(q), \]  
(4.2)
where $\mu = pT_J J^2 / 2vT^2$, $q_{\alpha\beta} = \langle \sigma_\alpha \sigma_\beta \rangle$ is the usual order parameter describing the spin-glass ordering, and $\lambda_{\alpha\beta}$ are Lagrange multipliers. We have three independent parameters: $T$, $T_J = T/n$, and $v$. In this paper we consider phase transitions only in the following subspace of three dimensional space of the parameters: $v = T_J$, $n = T/T_J$ is fixed, and the relevant parameter is $T$ (of course, all such regimes with $v \sim T_J$ are qualitatively equivalent). Thus for $\mu$ we have the standard expression

$$\mu = \frac{p\beta^2 J^2}{2} \quad (4.3)$$

As we have discussed in the previous section, there is some regime of cooling where $n$ indeed can be a constant.

A. Replica symmetric solution

In the investigation of a spin-glass the first step is to make the Replica Symmetry (RS) assumption for the order parameter: $q_{\alpha\beta} = q$ (for $\alpha \neq \beta$), where $q$ is the usual Edwards-Anderson parameter. In other words, one assumes that there is only one thermodynamical state (up to possible global symmetry transformations). The expression for the RS free energy $F_{rs} = F_{rs}/N$ has the following form

$$2\beta F_{rs} = -\ln(1-q) - \frac{1}{n} \ln \left( 1 + \frac{nq}{1-q} \right) - \frac{\mu}{p} (1 + (n-1)q^p) \quad (4.4)$$

where $q$ is determined by the saddle point equation

$$\mu q^{p-1} = \frac{q}{(1-q)(1+q(n-1))} \quad (4.5)$$

In section III we shall see that this equation can be obtained from the long-time statics if a slow dynamics for the coupling constants is assumed. Following (3.12)-(3.17) we get for energy and entropies

$$2\beta u_{rs} = -\frac{\mu}{p} (1 + (n-1)q^p) \quad (4.6)$$

$$2s_{rs} = \ln(1-q) + \frac{q}{1-q + nq} - \frac{\mu}{p} (1-q^p) \quad (4.7)$$

$$2s_J = \ln(1 + \frac{nq}{1-q}) - \frac{nq}{1-q + nq} + \frac{n\mu}{p} (1-q^p) \quad (4.8)$$

It is well known that from a point of view of phase transitions the set of $p$-spin models can be divided into the two main groups: $p > 2$ and $p = 2$. In the first case the qualitative phase diagram of the model does not depend on $p$ (as long as it is finite). Thus in the main part of this subsection we investigate the case $p > 2$; our results for $p = 2$ will be presented at the end.

We start with investigation of Eq. (4.3). For high temperatures there is only the paramagnetic phase with $q = 0$. The critical point $T_{1,rs}$ can be defined as the first temperature where a non-zero solution of (4.3) is possible. For $p > 2$ there occurs a first-order phase transition point with discontinuity of the order parameter

$$q_{1,rs} = \sqrt{(2-n)^2(p-1)^2 + 4p(p-2)(n-1) + (n-2)(p-1)} \quad (4.9)$$

The related onset temperature is

$$T_{1,rs} = J \sqrt{\frac{p}{2} q_{1,rs}^{p-2} (1-q_{1,rs}) (1+(n-1)q_{1,rs})} \quad (4.10)$$

1Besides (4.3) there is a contribution to the free energy which arises from the integration by the coupling constants, and has an order $O(N^p \ln N)$. Usually this contribution is omitted (see [13] for example), because it does not depend on the order parameter. An alternative point of view is to consider (4.3) as the leading finite-size effect.
Some limiting cases can be investigated; for example, if $p$ is large enough we get
\[
q_{1,rs} \sim 1 - \frac{1}{n}, \quad T_{1,rs} \sim \sqrt{\frac{n}{2e}} \tag{4.11}
\]

Any physical solution must be stable against small perturbations, therefore the analysis of linear stability for a possible replica-symmetric solutions should be performed. The eigenvalues of the corresponding Hessian for a finite $n$ were computed in [21]. There are three main sectors of fluctuations and the three corresponding eigenvalues:
\[
\Lambda_1 = -\mu(p - 1)q^{p-2} + \frac{1}{(1 - q)^2}, \quad \sum_{\beta} \delta q_{\alpha\beta} = 0 \tag{4.12}
\]
\[
\Lambda_2 = \Lambda_1 - \frac{(n - 2)q}{(1 - q)^2(1 + (n - 1)q)}, \quad \sum_{\beta} \delta q_{\alpha\beta} \neq 0, \quad \sum_{\alpha\beta} \delta q_{\alpha\beta} = 0 \tag{4.13}
\]
\[
\Lambda_3 = \Lambda_1 - \frac{(n - 1)q}{(1 - q)^2(1 + (n - 1)q)} \left( 2 - \frac{nq}{1 + (n - 1)q} \right), \quad \sum_{\alpha\beta} \delta q_{\alpha\beta} \neq 0 \tag{4.14}
\]

The first eigenvalue, the so-called “replicon” or “ergodon”, is displayed by the most coherent fluctuations (RSB is checked usually by this eigenvalue); the third eigenvalue corresponds to the most non-coherent fluctuations, and the second one takes an intermediate position. Before investigating the stability of the non-zero RS solution, we first discuss which eigenvalue is relevant for different values of $n$, so which is the smallest one. A simple analysis shows that for $n < 1$ $\Lambda_1$ is relevant, while for $n > 1$ $\Lambda_3$ is the most dangerous one. This result is important: as we have seen, the relevant sector of the fluctuations depends on $n$, implying that the whole structure of the phase space has strong dependence on $n$ also. At $n = 1$ $\Lambda_2$ becomes relevant too, but it causes no extra problem, since we have $\Lambda_1 = \Lambda_2$.

For $p > 2$ the paramagnetic solution $q = 0$ is stable everywhere (the case $p = 2$ will be discussed at the end of this section). Now we check the stability of the nonzero solution of Eq. (4.3) for $n < 1$. This solution monotonically decreases from (4.9) to zero. From positivity of $\Lambda_1$ we get
\[
q \geq \frac{p - 2}{n + p - 2} \tag{4.15}
\]

Thus the nonzero solution is stable only for
\[
T \leq T_{rs,st} = J \sqrt{\frac{n^2 p(p - 1)(p - 2)^{p-2}}{2(n + p - 2)^p}} \tag{4.16}
\]

Because $T_{rs,st} < T_{1,rs}$ we see that the solution is stable only for sufficiently low temperatures. In particular, for $n = 0$ it is unstable for every temperature, and that explains why this solution could be discarded till now. On the other hand, we shall show that if $n \geq 1$ the RS solution is stable everywhere. Indeed, for the nonzero solution of (4.3) $\Lambda_3$ is positive, and has the following form:
\[
\Lambda_3 = \frac{p(n - 1)q^2 + (2 - n)(p - 1)q + 2 - p}{(1 - q)^2(1 + (n - 1)q)^2} \tag{4.17}
\]

Further discussion about properties of replica symmetric spin-glass solution will be given after consideration of replica symmetry breaking solutions. In particular, we shall see that there are temperatures $(T_{2,rs}, T_{3,rs})$ of the true thermodynamical phase transitions.

| $n$ | $T_{1,rs}$ | $T_{2,rs}$ |
|-----|-----------|-----------|
| 1.01 | 0.0125 | 0.5892 |
| 1.10 | 0.0127 | 0.6001 |
| 1.20 | 0.0126 | 0.7533 |
| 1.30 | 0.0062 | 0.9213 |
| 1.40 | 1.2049 | 1.1140 |
| 1.50 | 1.5289 | 1.3567 |

**TABLE I.** $T_{1,rs}$ and $T_{2,rs}$ for different $n$ ($n > 1$), and $p = 3$. At $T = T_{1,rs}$ the RS spin-glass phase first appears as metastable one. The true first-order phase transition from the paramagnetic phase occurs at $T = T_{2,rs}$ (J=1).
1. The case $p = 2$

Let us now consider the special case $p = 2$. It is well known that the model with $n = 0$ is described by a replica symmetric Ansatz and has interesting properties \[39\]. For example, the $\Lambda_1$ eigenvalue vanishes everywhere in the low temperature spin-glass phase. For our solution we see (from (4.5)) that in the same model $\Lambda_1$ is always positive for nonzero $n$, since it holds that

$$\Lambda_1 = \frac{nq}{(1-q)^2(1+(n-1)q)} > 0 \quad (4.18)$$

So the correlation between the couplings removes the zero modes and stabilizes the structure of the vacuum. Since $\Lambda_3(p = 2, n > 1)$ is also a positive function we see that for $p = 2$ the RS solution is stable for all $n$, and for all $T$. It will be evident later that a consistent RSB solution is impossible in this case. In other words, the model with $p = 2$ is completely described by the RS anzatz.

We get for $p = 2$

$$q_{1,rs} = \frac{|n-2|+n-2}{4(n-1)} \quad (4.19)$$

Now if $n \leq 2$ the phase transition is second-order, and $T_{1,rs}$ coincide with true phase transition point into the spin-glass phase. For $n > 2$ the phase transition is first-order with the following scenario: besides $T_{1,rs}$, where the spin-glass phase occurs at the first time, there is also the true phase transition point $T_{2,rs} < T_{1,rs}$, that is determined by comparing free energies of paramagnet and spin-glass. These statements can be illustrated analytically in the case of positive but small $n-2$. Starting from Eq. (4.4) we get:

$$q_{1,rs} = \frac{n-2}{2}, \quad q_{2,rs} = \frac{2(n-2)}{3}, \quad (4.20)$$

$$T_{1,rs} = J \left( 1 + \frac{(n-2)^2}{8} \right), \quad T_{2,rs} = J \left( 1 + \frac{(n-2)^2}{9} \right). \quad (4.21)$$

The difference between free energies of the spin-glass phase and paramagnet is positive at the transition point $T = T_{1,rs}$:

$$\Delta f(T_{1,rs}) = J \frac{(n-2)^4}{384} > 0. \quad (4.22)$$

It means that the spin-glass phase appears as metastable one for $n > 2$. The free energies are equal at the second transition point: $\Delta f(T_{2,rs}) = 0$. Therefore this point must be considered as the temperature of the true phase transition from paramagnet to spin-glass. Indeed, for $T < T_{2,rs}$ we have $\Delta f(T) < 0$ as it should be, since the spin-glass phase is more stable. Such a type of phase transition will be extensively discussed later.

We have seen that a finite $n$ removes the marginal states for the $p = 2$ model. In the section \[\] we shall see that a similar statement holds also for the case $p > 2$. But then the condition $n > 1$ is needed for stabilizing the corresponding marginal states.

B. Replica symmetry breaking.

Now we investigate Replica Symmetry Breaking (RSB) solutions. As we have seen in the previous subsection, for $n > 1$ the stability of a RS spin-glass solution gives some hint about irrelevance of RSB in this range of $n$. We shall, however, first discuss the case $n < 1$. We consider here only the first step of RSB (1RSB) because a more general type of RSB is not possible in this model; we omit the proof because it can be found in \[\].

Taking the usual steps \[\] we get the following equations

$$2\beta f_{rsb} = -\frac{m-1}{m} \ln(1-q_1) - \frac{n-m}{nm} \ln(1-(1-m)q_1-nq_0)$$

$$-\frac{1}{n} \ln(1-(1-m)q_1+(n-m)q_0) - \frac{\mu}{p} (1-q_1^p + m(q_1^p - q_0^p) + nq_0^p) \quad (4.23)$$

9
where \( q_{\alpha\beta} \) takes the values \( q_1 \) and \( q_0 \), and \( m \) is the RSB parameter. For \( n = 0 \) we recover the usual 1RSB equations \[21\]. Following \( 3.12 \)-\( 3.17 \) we get for energy and entropies

\[
2\beta u_{rsb} = -\frac{\mu}{p}(1 - q_1^p + m(q_1^p - q_0^p) + nq_0^p) \tag{4.24}
\]

\[
2s_\sigma = \frac{m-1}{m} \ln(1 - q_1) + \frac{1}{m} \ln(1 - (1 - m)q_1 - mq_0) + \frac{q_0}{1 - (1 - m)q_1 + (n - m)q_0} - \frac{\mu}{p}(1 - q_1^p + m(q_1^p - q_0^p)) \tag{4.25}
\]

\[
2s_J = -\ln(1 - (1 - m)q_1 - mq_0) + \ln(1 - (1 - m)q_1 + (n - m)q_0) - \frac{nq_0}{1 - (1 - m)q_1 + (n - m)q_0} + \frac{n\mu}{p}(1 - q_1^p + m(q_1^p - q_0^p)) \tag{4.26}
\]

The saddle point equations are

\[
\mu(q_1^{p-1} - q_0^{p-1}) = \frac{q_1 - q_0}{(1 - q_1)(1 - (1-m)q_1 - mq_0)} \tag{4.27}
\]

\[
\mu q_0^{p-1} = \frac{q_0}{(1 - (1-m)q_1 - mq_0)(1 - (1-m)q_1 - (m-n)q_0)} \tag{4.28}
\]

On time scale \( \tau_J \) the spin system will, by definition, be in equilibrium. Therefore \( m \) is determined also by its saddle point equation,

\[
\frac{\mu}{p}(q_1^p - q_0^p) = \frac{1}{m^2} \ln \left( 1 + \frac{m(q_1 - q_0)}{1 - q_1} \right) - \frac{n - m}{nm} \frac{q_1 - q_0}{1 - (1 - m)q_1 - (m-n)q_0} - \frac{1}{n} \frac{q_1 - q_0}{1 - (1 - m)q_1 - (m-n)q_0} \tag{4.29}
\]

There is some other possibility for fixing \( m \): As was shown before \[11\], \[22\], \[24\] this parameter can be fixed also by the so called "marginality condition". The resulting theory describes metastable states, in a manner also monitored by the dynamics \[22\] (see also section \[5\]). In this paper we consider only the purely static condition \[4.29\].

First we note that there is a remarkable analogy between the RS finite-\( n \) free energy \[4.4\] and the free energy of the 1RSB solution with \( q_0 = 0 \). If we interchange \( q_1 \) with \( q \) and \( m \) with \( n \) we arrive at identical expressions. This analogy between corresponding free energies was considered in the SK model with similar but more complicated techniques including two types of frozen variables \[13\]. Recently XY spin-glass model has been investigated by the same approach \[14\]. We shall show that a similar, perhaps more informative analogy can be found in the dynamics of the present model.

As well-known, the physical interpretation of replica symmetry breaking is connected with decomposition of the phase space into pure states (ergodic components) \[1\], \[2\]. This structure is contained in the overlap \( q_{\alpha\beta} \). In particular, for 1RSB the values \( q_1, q_0 \) can be interpreted as self-overlap of a pure state and mutual overlap between two pure states. This information is coded also in the probability distribution of overlaps:

\[
P(q) = \frac{1}{n(n-1)} \sum_{\alpha\neq\beta} \delta(q - q_{\alpha\beta}) = \frac{1 - m}{1 - n} \delta(q - q_1) + \frac{m - n}{1 - n} \delta(q - q_0) \tag{4.30}
\]

It is associated with the fraction of matrix elements \( q_{\alpha\beta} \) which take the value \( q_1 \) or \( q_0 \) \[4\]. We see that for interpretation of \( (1 - m)/(1 - n) \) and \( (m - n)/(1 - n) \) as (non-negative !) probabilities we need:

\[
n < m < 1 \quad \text{for} \quad n < 1 \\
1 < m < n \quad \text{for} \quad n > 1. \tag{4.31}
\]

One could expect that the conditions \[4.31\] are satisfied automatically if other more obvious physical conditions (for example, \( q_1 > q_0 \)) are valid. However, it is not so. Later we shall show that they should be considered as additional conditions selecting the correct solution.
Let us now discuss the solution of Eqs. (4.27), (4.29) for the case \( q_0 = 0 \). Then the solution itself becomes independent on \( n \). However, the dependence on \( n \) does enter through Eqs. (4.30), (4.31). The considered solution has partly been investigated in [21]. First we note that there is a convenient parametrisation of Eqs. (4.27), (4.29) [21,34]. If we denote

\[
c = \frac{mq_1}{1 - q_1}
\]  

(4.32)

then for this quantity we get the temperature independent equation

\[
\frac{c^2}{p} = (1 + c) \ln(1 + c) - c
\]  

(4.33)

The positive solution of this equation should be selected. (The authors of [21] employ the slightly different variable \( y = 1/(c + 1) \)). Taking this into account, the equation for \( q_1 \) reads

\[
\mu q_1^{p-1} = \frac{q_1}{(1 + c)(1 - q_1)^2}
\]  

(4.34)

The highest temperature for which this equation has a non-zero solution will be denoted by \( T_{1,rsb} \):

\[
T_{1,rsb} = J \sqrt{\frac{2(1 + c)}{p} \left( \frac{p - 2}{p} \right)^{p-2}}
\]  

(4.35)

and \( q_1 \) at this point has the value

\[
q_1(T_{1,rsb}) = \frac{p - 2}{p}
\]  

(4.36)

Further, using (4.32) we get

\[
m(T_{1,rsb}) = \frac{2c}{p - 2}
\]  

(4.37)

This value is greater than 1 for all \( p > 2 \). The parameter \( m \) monotonically decreases with temperature from (4.37) at \( T = T_{1,rsb} \) to zero at \( T = 0 \) (see (4.43)). It means that for \( n > 1 \) only a part of the solution from \( m(T_{1,rsb}) \) to \( m = 1 \) can be physically permissible; otherwise we get physically meaningless results for the weights (4.30). Namely, if \( n \) is in the interval \( 1 < n < m(T_{1,rsb}) \), then the temperature where the solution appears as physical one will be determined from the condition \( m(T) = n \). In the opposite case where \( m(T_{1,rsb}) < n \), that temperature is just \( T_{1,rsb} \) itself.

On the other hand, in case \( n < 1 \) the physical part is consistent only with \( n < m < 1 \). Namely, for \( n < 1 \) the possible transition point must be determined from the condition \( m(T) = n \). The transition from paramagnet to RSB spin-glass with vanishing lower plateau with \( m = 1 \) at the critical point has been found in [21] for \( n = 0 \). It occurs at the temperature

\[
T_{2,rsb} = J \sqrt{\frac{p}{2(1 + c)} \left( \frac{1 + c}{c} \right)^{2-p}}
\]  

(4.38)

At this point \( q_1 \) jumps from zero to

\[
q_1(T_{2,rsb}) = \frac{c}{c + 1}
\]  

(4.39)

and it goes monotonously to unity when \( T \) tends to zero. The transition is intermediate between first-order and second-order: the order parameter has a jump but the energy and entropy are continuous, as we see from Eqs. (4.24-4.26).

Note that the free energy in the spin-glass phase is higher than that of the paramagnetic state (Fig. 1). It is usual for this type of phase transitions [1,3,31,2].
FIG. 1. The free energy (subtracted the paramagnetic contribution) of the 1RSB spin-glass phase with vanishing lower plateau vs. temperature \(p = 3\). For \(n > 1\) \((n < 1)\) a part of the right-hand (left-hand) branch of the presented curve should be chosen as the physically permissible one (see Eq. (4.30)). It turns out that the chosen part of the right-hand branch always corresponds to a metastable phase.

If \(p \to \infty\) then also \(c \to \infty\). More concretely, in this limit it holds that

\[
\frac{J^2}{2T^2_{2,rsb}} \sim (\ln c - 1) \exp \left(\frac{1}{\ln c - 1}\right) \to \infty
\]

(4.40)

\[
q_1 \sim 1 - \frac{1}{c}
\]

(4.41)

This behavior in the large-\(p\) limit is in the sharp contrast with the case of \(p\)-interaction Ising spin-glass [20], where phase transition point is finite when \(p \to \infty\).

Let us now consider the zero temperature behavior of the solution with fixed but not very large \(p\). A simple analysis shows that in this case

\[
1 - q_1 \sim \frac{T}{J} \sqrt{\frac{2}{p(1 + c)}},
\]

(4.42)

\[
m \sim \frac{T}{J} c \sqrt{\frac{2}{p(1 + c)}},
\]

(4.43)

\[
f(T \to 0) = -\frac{J(c + p)}{\sqrt{2p(1 + c)}}.
\]

(4.44)

Note again that the free energy of the solution remains finite in the zero-temperature limit in contrast to the paramagnetic free energy which tends to minus infinity.

The considered solution is stable and all relevant eigenvalues of the Hessian are nonzero at the phase transition point. In particular, for the relevant eigenvalue: (see [21] for the derivation)

\[
\Lambda_1 = -\mu(p - 1)q_1^{p-2} + \frac{1}{(1 - q_1)^2},
\]

(4.45)

we get from Eq. (4.34):
\[ \Lambda_1 = \frac{1}{(1 - q_1)^2} \left( 1 - \frac{p - 1}{1 + c} \right) > 0. \]  

(4.46)

The eigenvalue (4.43) describes fluctuations deep inside a pure state. As usual, critical slowing down is absent at this static first order phase transition. Later we shall show that in the long-time dynamics there occurs a marginal stability for all \( n < 1 \).

![FIG. 2. The entropies (subtracted the corresponding paramagnetic contributions; see Eqs. (4.23), (4.26)) of the 1RSB spin-glass phase with vanishing lower plateau vs. temperature \((p = 3, n = 0.5)\). (a) \( s_\sigma \), entropy of the spins; (b) \( s \), the total entropy; (c) \( s_J \), entropy of the coupling constants. The curves start at the transition point \( T = T_{2,rsb} \).](image)

It is of interest to discuss the behavior of entropies \( s_\sigma \), \( s_J \), and \( s \) in the vicinity of the transition point \( T_{2,rsb} \). In particular, we have from Eqs. (4.24-4.26)

\[ 2s_\sigma = \ln(1 - q_1) + \frac{1}{m} \ln(1 + c) + \frac{\mu}{p} (1 - (1 - m)q_1^p) \]  

(4.47)

\[ 2s_J = \frac{nm}{p} (1 - (1 - m)q_1^p) \]  

(4.48)

In spite of the jump of \( q_1 \), they change continuously from the paramagnetic phase to the spin-glass one. We shall compare their behavior in spin-glass and paramagnet at the same value of temperature. For \( T < T_{2,rsb} \) the entropy of the spins is higher in the spin-glass phase: \( s_\sigma > s_\sigma(q_1 = 0) \). On the other hand, the entropy of the coupling constants is lower: \( s_J > s_J(q_1 = 0) \), as is obvious from Eq. (4.48). Consequently, the total entropy \( s = s_\sigma + s_J \) is also higher in the spin-glass phase as compared to its value for \( q_1 = 0 \). The behavior of the entropies is presented in Fig. 2.

Further properties of the RSB spin-glass phase (in particular, for \( n > 1 \)) will be discussed later in the context of constructing the complete phase diagram.

2. Replica symmetry breaking with non-vanishing lower plateau.

A solution with \( q_0 \) emerging smoothly from \( q_0 = 0 \) is not compatible with the saddle-point equations (4.27). Therefore we investigate a continuous transition from the RS solution with \( q \equiv q_1 > 0 \) and a RSB solution with \( q_1 > 0, q_0 > 0 \) but small \( \Delta q = q_1 - q_0 \). With this assumption we have the following equations at the transition point

\[ \mu_c(p - 1)q_c^{p - 2} = \frac{1}{(1 - q_c)^2} \]  

(4.49)

\[ \mu_c q_c^{p - 2} = \frac{1}{(1 - q_c)(1 + (n - 1)q_c)} \]  

(4.50)
The first of these expresses that the replicon eigenvalue $\Lambda_1$ of a RS solution with $q = q_1$ vanishes. The transition point is thus given by

$$T = Jn \sqrt{\frac{p(p-2)p^{-2}(p-1)}{2(p-2+n)^p}} \quad (4.51)$$

In the limit $p \to \infty$ this transition point remains finite, and for any $p$ it tends to zero if $n$ tends to zero. The order parameters $q_0 = q_1$ have the following jump at the phase transition point

$$q_c = \frac{p-2}{p-2+n} \quad (4.52)$$

As we see, it is the same point where stability of the RS spin-glass solution is restored.

There is a useful parametrization of Eqs. (4.27), (4.28), (4.29): if we introduce

$$x = \frac{q_0}{q_1}, \quad c = \frac{m\Delta q}{1-q_1}, \quad (4.53)$$

then we have

$$\frac{1 - xp - px^{p-1}(1-x)}{p(1-x^{p-1})(1-x)} = \frac{(1+c)\ln(1+c) - c}{c^2} \quad (4.54)$$

Using Eqs. (4.53-4.54) we get at the transition point:

$$m_c = \frac{n}{2} \quad (4.55)$$

It can be proved that $m$ decreases with decreasing of temperature, and $m = 0$ when $T = 0$. According to our discussion at Eqs. (4.30), (4.31) it means that a 1RSB phase with non-vanishing lower plateau cannot be considered as a physical one, at least not for $n < 2$. For $n > 2$ and then $m(T) > 1$ this solution also cannot be considered as a physical one, because then the condition $q_1 > q_0$ is violated (this condition does not depend on $n$, and it is necessary for the interpretation of $q_0$ and $q_1$ as overlaps).

This behavior is in the sharp contrast with SK model [19] [18] where the main effect of finite $n$ is to introduce a non-vanishing lower plateau. This plateau increases with $n$, and replica symmetry breaking disappears at some critical value. It should be noted that in our case only the property (4.31) forbids the existence of the considered phase for $n < 2$. All other requirements are satisfied: it is stable, and has well-defined free energy. Stability can be checked by positivity of the following eigenvalues

$$\Lambda_1 = -\mu(p-1)q_1^{p-2} + \frac{1}{(1-q_1)^2}, \quad (4.56)$$

$$\Lambda_0 = -\mu(p-1)q_0^{p-2} + \left(\frac{1-q_1 + nq_0}{(1-q_1)(1-q_1 + m\Delta q + nq_0)} \right)^2 \quad (4.57)$$

We should mention the possibility of more general replica symmetry breaking solutions in our model. For $n = 0$ it was proven [21] that 1RSB solutions are the most general RSB ones, and more orders of RSB are impossible. This proof can be generalized also for $n > 0$. The physical meaning of this statement is that a finite $n$ introduces correlations between coupling constants partially removing frustrations. In other words, it cannot lead to more complicated phase space with more orders of RSB.

C. The static phase diagram.

In this subsection we construct the phase diagram of our model by considering all physically relevant solutions (it means stable and with correct probability of overlaps [4.30]): paramagnet, RS spin-glass, RSB spin-glass with vanishing lower plateau.
1. The case $n < 1$.

First we discuss the case $n < 1$. For high temperatures the system is in the paramagnetic phase. If temperature decreases, then at $T = T_{2,\text{rsb}}$ (see Eq. (4.38)) the RSB spin-glass with vanishing lower plateau appears. Its free energy is greater than the paramagnetic one (see Fig. 2) but, nevertheless, it is chosen as the relevant phase. It is the usual choice, and a possible argument is a hypothesis about a non-perturbative instability of the paramagnetic state below $T_{2,\text{rsb}}$. As far as we know, there is no convincing proof of this statement. There is only some hint gained from an analysis of finite-size corrections in Potts glass [31]. As we have mentioned already, this phase transition is first-order with respect to the order parameter but second-order with respect to derivatives of free energy. In particular, the latent heat (the difference between the energies of the high temperature and the low temperature phase at the transition point) vanishes.

At the present stage we shall go back to the RS spin-glass solution, and analyze its free energy. Recall that this phase is stable for $T < T_{\text{rs, st}}$ (see Eq. (4.11)), and for some range of $n$ we have $T_{\text{rs, st}} > T_{\text{2, rsb}}$ (see Fig. 3). In particular, the sign of

$$\Delta f_{\text{rs}} \equiv f_{\text{rs}} - f_{\text{para}} = -\frac{T}{2} \ln(1 - q) - \frac{T}{2n} \ln \left(1 + \frac{nq}{1-q}\right) + \frac{\beta J^2}{4} (1 - n)q^p$$

(4.58)

should be checked. In this respect the interval $0 < n < 1$ is divided into two subintervals. For $n_0 < n < 1$ (where $n_0$ is some positive value, to be discussed later) it holds that $\Delta f_{\text{rs}}(T_{\text{rs, st}}) < 0$. Taking into account that at $T = T_{\text{rs, st}}$ RS spin-glass first appears as a physical solution, we conclude that in this range of temperatures it should be considered as metastable with respect to the paramagnet, in spite of its lower free energy. The opposite point of view will mean that $T = T_{\text{rs, st}}$ must be considered as the point of a phase transition which is meaningless, because $\Delta f_{\text{rs}}(T_{\text{rs, st}}) \neq 0$. If temperature is decreasing further we get the point $T = T_{\text{2, rs}}$ where $\Delta f_{\text{rs}}(T_{2,\text{rs}}) = 0$, and $\Delta f_{\text{rs}}(T_{\text{rs, st}}) > 0$ for $T_{\text{2, rs}} > T$. This behavior of the free energy is presented by Figs. 3, 4. However, this point is always lower than the transition temperature from the paramagnet into the RSB spin-glass phase with vanishing lower plateau: $T_{2,\text{rsb}} > T_{2,\text{rs}}$. (On the other hand $T_{2,\text{rsb}} \leq T_{2,\text{rs}}$ for $1 \leq n$, but this case is physically different and will be discussed a bit later.) Therefore, the temperature $T = T_{2,\text{rs}}$ is not considered as a true phase transition point towards the most stable phase. Certainly, it is only the point where RS spin-glass phase becomes more stable than the paramagnetic state.

For $n < n_0$ one has $\Delta f_{\text{rs}} > 0$, so the RS spin-glass state at the beginning appears with higher free energy than the paramagnet. $n_0$ is defined by the condition $\Delta f_{\text{rs}}(T_{\text{rs, st}}) = 0$. In Table I we represent the values of $n_0$ for different $p$. In particular, we see that $n_0(p)$ decreases with increasing $p$.

| $n_0$   | $p$ |
|---------|-----|
| 0.68632| 3   |
| 0.61605| 4   |
| 0.53108| 6   |
| 0.36934| 15  |
| 0.31670| 30  |

TABLE II. $n_0$ for different $p$ ($J = 1$).

According to Eqs. (4.30), (4.31) for $n < 1$ any 1RSB spin glass cannot exist as a physical one if $m < n$. From subsection IV B 1 we know that for the RSB spin-glass with vanishing lower plateau $m$ monotonically decreases with temperature from $m = 1$ at $T = T_{2,\text{rsb}}$ to $m = 0$ at $T = 0$. It means that a phase transition should exist from a RSB spin-glass with vanishing lower plateau to a RS spin-glass at a temperature $T_{3,\text{rs}}$ defined by

$$m(T_{3,\text{rs}}) = n.$$  (4.59)

| $n$   | $T_{3,\text{rs}}$ |
|-------|------------------|
| 0.90  | 0.5575           |
| 0.80  | 0.6240           |
| 0.70  | 0.4806           |
| 0.60  | 0.4413           |
| 0.25  | 0.2292           |

TABLE III. The transition temperature $T_{3,\text{rs}}$ from the 1RSB spin-glass phase to the RS one, for different $n$ ($n < 1$), and $p = 3$ ($J = 1$).
FIG. 3. Free energies (subtracted the paramagnetic free energy) vs. temperature ($n = 0.955$, $p = 3$). Thick line: the RS spin-glass phase; normal line: the RSB spin-glass phase with vanishing lower plateau. The arrow denotes the phase transition point occurring at $T = T_{3,rs}$ from the 1RSB spin-glass phase to the RS one.

FIG. 4. Free energies (subtracted the paramagnetic contribution) vs. temperature ($n = 0.8$, $p = 3$). Thick line: the RS spin-glass phase; normal line: the RSB spin-glass phase with vanishing lower plateau. The arrow denotes the phase transition point at $T = T_{3,rs}$. 
As Eqs. (4.23)-(4.29) show, the order parameters are equal at this point: $q_1 = q$. It is also easy to show that the free energies, energies and entropies $s_J$, $s_J$ of the corresponding phases are also equal. Thus, at $T = T_{3,rs}$ the order parameter $q_0$ jumps from zero to $q_1$, ensuring the replica-symmetric behavior for $T < T_{3,rs}$. On the other hand, $q_1$ changes continuously. Both the RS spin-glass and the RSB spin-glass with vanishing lower plateau are stable at and around $T = T_{3,rs}$. In this sense the transition at $T = T_{3,rs}$ is very similar to the transition occurring from the paramagnetic state at $T = T_{2,rsb}$, where $q_0$ changes continuously (i.e., remains zero) but $q_1$ has a jump. It is worth to note that for $T < T_{3,rs}$ the free energy of the RS spin-glass is lower than the free energy (more exactly its analytical continuation) of the RSB spin-glass with vanishing lower plateau. Different values of $T_{3,rs}$ are presented in Table II.

Summarizing the presented facts and arguments, we conclude that when temperature is decreasing for fixed $n < 1$ there are only two temperatures of true phase transitions: 1) $T = T_{2,rsb}$ is the transition point from the paramagnet to the RSB spin-glass with vanishing lower plateau. 2) $T = T_{3,rs}$ is the transition point from the RSB spin-glass into the RS one. These transitions are quite close by their physical meaning. The main distinction is in the difference between free energies of the corresponding low-temperature phase and the high-temperature one (or rather its analytical continuation to lower temperatures). The temperatures $T_{rs,at}$ and $T_{2,rs}$ also have certain physical meanings, but are not considered as true transition temperatures.

![FIG. 5. Thermodynamical functions (minus the corresponding paramagnetic contributions) of the RS spin-glass ($n = 4$, $p = 3$) vs. temperature. The thick arrow denotes the phase transition point from paramagnet to the RS spin-glass phase at $T = T_{2,rs}$. The normal arrow denotes the temperature $T = T_{1,rs}$, where the RS spin-glass solution appears first as a metastable phase. Thick line: free energy; (a): $s_J$, entropy of the coupling constants; (b): $s$, the total entropy; (c): $u_{rs}$, mean energy; (d): $s_\sigma$, entropy of the spins.](image-url)
2. The case $n > 1$.

In the case $n > 1$ there are three relevant solutions: the RS spin-glass ($q > 0$), the paramagnet ($q = 0$), and the RSB spin-glass with vanishing lower plateau in the region $1 < m < n$. Recall that this last state formally exist for $T < T_{1,rsb}$ (see Eq. (4.35)), but the region of its actual existence must be chosen according to the value of $n$ as it has been done for $n < 1$. Doing so we immediately conclude that for $n > 1$ the RSB spin-glass with vanishing lower plateau cannot be considered as the most stable one, because its free energy is always higher than the free energy of the paramagnetic state (Fig. 7). It can be viewed only as a metastable one in the above-mentioned region. The behavior of the corresponding free energy is presented in Fig. 7. Therefore, our attention will be restricted only to the paramagnet and the RS spin-glass phase.

According to our discussion in subsection IV A, a RS spin-glass phase first appears at $T = T_{1,rs}$ (Eq. (1.9)) but its free energy at this point is higher than the paramagnetic one. These free energies are equal when $T = T_{2,rs}$, and for $T < T_{1,rs}$ the RS spin-glass phase has lower free energy. Thus, the temperature $T_{2,rs}$ is the true thermodynamical phase transition point. This transition is first order, and it is connected with a jump of the order parameter $q$. It is obvious from Eq. (1.6) that for $T < T_{2,rs}$ the mean energy of the spin-glass phase is also lower than the paramagnetic one. It means that the latent heat is positive as for (usual) first-order phase transitions in equilibrium systems. Values of $T_{1,rs}$ and $T_{2,rs}$ are represented in Table III. In particular, this scenario of phase transition is the usual one for multi-spin interaction ferromagnets \[8\], for some metamagnetic materials \[10\], or for phase transitions in a compressible lattice \[8\]. For such a system the jump of entropy at the transition point is negative (i.e., a low-temperature phase has lower entropy) according to the usual relation $F = E - TS$ between free energy, energy and entropy. At this stage it should be recalled again that our system is not in the usual equilibrium, and we are considering phase transitions in the nonequilibrium steady state. In particular, we have the basic relation (3.17) between free energy, energy, and entropies. At the transition point $T = T_{2,rs}$ this relation can be written as

$$\Delta u = T \Delta s_\sigma + T_J \Delta s_J,$$  \hspace{1cm} (4.60)

where $\Delta u = u_{rs} - u_{pm}$, $\Delta s_\sigma = s_{\sigma,rs} - s_{\sigma,pm}$, and $\Delta s_J = s_{J,rs} - s_{J,pm}$ are differences between the corresponding quantities of the RS spin-glass and the paramagnet, and $u_{rs}$, $s_{\sigma,rs}$, $s_{J,rs}$ are defined by Eqs. (4.6)-(4.8). Further, $T$ and $T_J$ are connected through $T = T_{2,rs}$. In particular, positive latent heat means $\Delta u < 0$, and consequently $T \Delta s_\sigma + T_J \Delta s_J < 0$. However, it is interesting to know signs of $\Delta s_\sigma$, $\Delta s_J$, and $\Delta s = \Delta s_\sigma + \Delta s_J$ separately because these quantities have independent physical meanings, extensively discussed in section III. We get

$$\Delta s > 0, \; \Delta s_J > 0, \; \Delta s_\sigma < 0.$$  \hspace{1cm} (4.61)

The behavior of $\Delta s$ is in the sharp contrast with the usual, equilibrium first-order transitions. This somewhat surprising fact should deserve further attention. The behavior of various thermodynamical quantities near $T_{2,rs}$ is presented in Fig. 7. The difference between spins entropies $\Delta s_\sigma$ becomes positive starting from some temperature lower than $T_{2,rs}$. In other words, for sufficiently low temperatures both entropies are higher in the spin-glass phase. Let us recall in this context that free energy and mean energy of the spin-glass phase are lower than the paramagnetic ones for sufficiently low temperatures.

Positivity of the latent heat, and the result (4.61) holds also for the first-order phase transitions occurring for $p = 2$, $n > 2$ (see section IV A). In the case of positive but small $n - 2$ we can get analytical expressions for $\Delta s_\sigma$, $\Delta s_J$, and $\Delta s$ using Eqs. (4.20)-(4.22). The jump $q_{2,rs}$ of the order parameter is positive but small at $T = T_{2,rs}$, and we get to leading order

$$\Delta s_\sigma = - \frac{1}{2} q_{2,rs}^2 + \mathcal{O}(q_{2,rs}^2) < 0, \; \Delta s_J = \frac{1}{2} q_{2,rs}^2 + \mathcal{O}(q_{2,rs}^2) > 0,$$

$$\Delta s = \frac{5}{24} q_{2,rs}^2 + \mathcal{O}(q_{2,rs}^2) > 0$$  \hspace{1cm} (4.62)

This is an analytical illustration of the more general result (1.61).

The special attention should be devoted to the case $n = 1$. The free energy, and the energy of the RS spin-glass coincide with their analogs for the paramagnetic phase. However, from the point of view of the order parameter, the phase transition occurs to the RS spin-glass phase (when decreasing temperature for a fixed $n = 1$). Further, $q$ increases monotonically from $q = (p - 2)/(p - 2 + n)$ at $T = T_{2,rsb} = T_{2,rs} = T_{3,rs}$ to $q = 1$ at $T = 0$. The phase transition is first-order for $p > 2$, and second-order for $p = 2$. It is interesting that $s_\sigma$ and $s_J$ depend on $q$ even for $n = 1$, however, they compensate each other so that their sum is $s = 0$. The final phase diagram is presented in Fig. 7.
FIG. 6. The static phase diagram of the model in the $n - T$ plane for $p = 2$. The paramagnetic phase is denoted by PM, and SG-2 means the RS spin-glass phase. (a) $T = J$, the line of the second-order transitions; (b) $T = T_{2,rs}(n)$, the line of the first-order transitions with a positive jump of the total entropy. The thick dot indicates the multicritical point.

FIG. 7. The static phase diagram of the model in the $n - T$ plane for $p > 2$. The paramagnetic phase is denoted by PM, and SG-1, SG-2 mean correspondingly the RSB spin-glass phase with vanishing lower plateau, and the RS spin-glass phase. (a) $T = T_{2,rsb}(p)$; (b) $T = T_{3,rs}(n, p)$; (c) $T = T_{2,rs}(n, p)$. The lines (a) and (b) indicate first-order transitions without latent heat; (c) indicates the first-order transition with the positive latent heat, and a positive jump of the total entropy. The thick dot indicates the multicritical point.
V. ERGODIC DYNAMICS

At the present section we consider the ergodic (or time-translation invariant) dynamics of our two-temperature model. As can be expected from our experience acquired from the study of statics, if \( n < 1 \) ergodicity holds at relatively short but infinite times where RSB is not relevant (of course, that restriction of times is not essential if temperatures is high enough). On the other hand, if \( n > 1 \) the assumptions connected with ergodicity are correct for any temperature. Indeed, in the last case the RS assumption adequately describes the system’s state for all temperatures. Thus, as opposed to the case of usual mean-field spin-glass systems (with \( n = 0 \)), in our case ergodicity can have a wider range of applicability. In particular, in the present section we compare the predictions of the long-time dynamics with the static ones, as well as investigate effects connected with very large observation times.

Eqs. (2.8, 2.9) have the following form in the time-translation invariant regime

\[
(\partial_t + r)C(t) = \frac{pJ^2}{2T} \int_0^\infty \tilde{e}^{-i/\tau_J} C^{p-1}(\tilde{t}) C(t - \tilde{t}) d\tilde{t} \\
+ \frac{pT_J J^2}{2v} (p-1) \int_0^\infty \tilde{e}^{-i/\tau_J} C^{p-2}(\tilde{t}) G(\tilde{t}) C(t - \tilde{t}) d\tilde{t} \\
+ \frac{pT_J J^2}{2v} \int_0^\infty \tilde{e}^{-(t+\tilde{t})/\tau_J} C^{p-1}(t + \tilde{t}) G(\tilde{t}) d\tilde{t},
\]

(5.1)

\[
(\partial_t + r)G(t) = \frac{pJ^2}{2T} \int_0^t \tilde{e}^{-(t-\tilde{t})/\tau_J} C^{p-1}(t - \tilde{t}) G(\tilde{t}) d\tilde{t} \\
+ \frac{pT_J J^2}{2v} (p-1) \int_0^t \tilde{e}^{-(t-\tilde{t})/\tau_J} C^{p-2}(t - \tilde{t}) G(t - \tilde{t}) G(\tilde{t}) d\tilde{t}.
\]

(5.2)

A. Adiabatic dynamics: not very large observation times.

We come to the adiabatic statics, which has been considered in section II from the dynamics if \( \Gamma_J \to \infty \) before \( t \to \infty \), so that the usual form of FDT

\[
\partial_t C(t) = T(-G(t) + G(-t))
\]

(5.3)
can be used. Indeed, in this case for all \( t \ll \tau_J \) the coupling constants are fixed, and the violation of the detailed balance condition (which arises in account of a difference between temperatures) cannot be effective. Thus we have the equation

\[
(\partial_t + r)C(t) = \frac{pJ^2}{2T_J} \int_0^\infty \tilde{e}^{-i/\tau_J} C^{p-1}(\tilde{t}) C(t - \tilde{t}) d\tilde{t} \\
+ \frac{pT_J J^2}{2v} (p-1) \int_0^\infty \tilde{e}^{-i/\tau_J} C^{p-2}(\tilde{t}) C(t - \tilde{t}) G(\tilde{t}) d\tilde{t} \\
+ \frac{pT_J J^2}{2v} \int_0^\infty \tilde{e}^{-i/\tau_J} G(\tilde{t}) C^{p-1}(t + \tilde{t}) d\tilde{t}.
\]

(5.4)

In the last two integrals we can take \( e^{-\tilde{t}/\tau_J} \sim 1 \) by considering the relevant domain of the integration. In the first integral we find by a changing variables

\[
\frac{1}{T_J} \int_0^\infty \tilde{e}^{-i/\tau_J} C^{p-1}(\tilde{t}) C(t - \tilde{t}) d\tilde{t} \to \frac{1}{v} q^p,
\]

(5.5)

where

\[
q = \lim_{t \to \infty} \lim_{\tau_J \to \infty} C(t)
\]

(5.6)

Now for investigating phase transitions we take our usual restriction to the parameters of the model: \( v = T_J, n = T/T_J \) is fixed, and \( T \) is varying. Thus \( \mu = p\beta^2 J^2/2 \). The static limit of (5.4) gives us the RS equation (1.3). Indeed, for
determination of $q$ we can take the two limits: $t = 0$ where $C(0) = 1$, $\partial C(t)|_{t=0} = -T$, and $t \to \infty$ where $C = q$. For these cases we have the following equations

$$\beta r - 1 = \mu nq^p + \mu(1 - q^p)$$

$$\beta rq = \mu nq^p + \mu(q - q^p) + \mu q^{p-1}(1 - q)$$

These equations reproduce eq. (4.3), but now as a result of long-time dynamics. As was shown in section IV, the relevant eigenvalue of Hessian is (4.12) (in the thermodynamic limit). The dynamical phase transition can be reflected by an other approach also [24]: As was shown in section IV, the initial time $\to -\infty$ the dynamical stability condition $\partial_t C(t) \leq 0$ is equivalent to the following two equations, which were obtained already in [22].

$$\beta r - \mu = \bar{r}(C(t))$$

$$\bar{r}'(C(t)) = 0,$$  

where

$$\bar{r}(C(t)) = \frac{1}{1 - C(t)} - \mu C^{p-1}(t)$$

These equations predict a phase transition at the temperature (assuming that it is the highest temperature where a phase transition from paramagnet occurs)

$$T^2_d = \frac{pJ^2 (p-2)^{p-2}}{2 (p-1)^{p-1}}$$

with jump

$$q_d = \frac{p-2}{p-1}$$

The standard interpretation of this phase transition is the following [22, 24, 32]. The dynamics is investigated in the limits: the initial time $\to -\infty$ after the thermodynamic limit $N \to \infty$. In contrast, in the purely static investigation by means of the Gibbs distribution, the opposite order of limits is implicated. Thus in the long-time dynamics the system can be blocked in some regions of the phase space. It was shown that, indeed, the dynamical phase transition is induced by the highest metastable TAP states [23] which are separated by each other by means of infinite barriers (in the thermodynamic limit). The dynamical phase transition can be reflected by an other approach also [24]. As we have seen in section IV, the relevant eigenvalue of Hessian is (4.12)

$$\Lambda_1 = -\mu(p-1)q^{p-2} + \frac{1}{(1 - q)^2}$$

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(which does not depend on n). By now considering a "long-time" but not purely static regime, the RSB parameter 
$m$ can be fixed by the condition $A_1 = 0$. In this approach (marginal replica theory) the phase transition point 
coincides with the dynamical one, but the thermodynamics is different compared to the one which is obtained from 
the long-time limit of the dynamical equations.

We have seen that the dynamical transition point is independent of $n$ if this temperature is larger than all other 
transition points predicted by statics. A simple comparison gives the following. For all $0 \leq n < 1$ there is a difference 
between statics and dynamics: $T_d > T_{2,rs} > T_{3,rs}$. It coincides with the fact that for this case the relevant eigenvalue 
of Hessian (4.12) does not depend on $n$. There is no such difference for $n > 1$, because in this case the statics predicts 
a phase transition at a greater temperature than the dynamics: $T_{1,rs} > T_{2,rs} > T_d$. Indeed, in this case the replica 
or ergodion eigenvalue $A_1$ is not a relevant one.

We see that the existence and probably the structure of the highest metastable TAP states strongly depends on $n$.

B. Adiabatic dynamics: very large observation times.

In this subsection we have concentrated exclusively on the case $n > 1$; it was mentioned above that in this case 
ergodicity is valid without any constraint on the temperature or the time of observation. As we have seen in previous 
subsection, there is a spin-glass phase if the observation time is not very large, i.e., if $t \ll \tau_J$. Further, the same result 
as predicted by the long-time dynamics can be obtained from the statics. However, for $t \sim \tau_J$ (i.e., when an observer 
waits long enough) the coupling constants will begin to fluctuate and relax toward their steady state. It is expected 
intuitively that at such observation times the spin-glass phase will disappear, and the resulting stationary state will 
be a paramagnet. As a matter of fact, this subsection is devoted to confirm that point of view. Furthermore, we will 
see that there are some peculiar features of this process, which probably have universal character related to other 
glassy systems. In the concrete calculations we use the methods of [22].

In the domain $t \sim \tau_J \to \infty$ the main time-scale is $\tau_J$; thus, it is natural to take the following form for the correlation 
and response functions

$$C(t) = C\left(\frac{t}{\tau_J}\right), \quad G(t) = \frac{1}{\tau_J} \mathcal{G}\left(\frac{t}{\tau_J}\right)$$ (5.14)

(the extra factor $1/\tau_J$ for $\mathcal{G}$ appears on account of the correct dimension of that quantity; furthermore its necessity 
will be evident further). Let us define also an auxiliary large time-scale $t_e$ at which the correlation function stabilizes:

$C(t \sim t_e) = q$ (where $q$ is defined by Eq. (4.3)). Let us stress that when in the previous subsection we have spoken 
about very large $t$ where $C(t) = q$, we meant — in the light of the present discussion — the case $t \sim t_e$.

Now our very large time-scales can be presented as: $\tau_J \gg t_e$; the cases $t \sim t_e$ and $t < t_e$ have been studied in the 
previous subsection, and now we are going to consider the cases $t \sim \tau_J$, $t \gg \tau_J$. The corresponding equations for $C, \mathcal{G}$ 
must be constructed from (5.1), (5.2), in accordance of a simple physical picture occurring from the large separation of 
the local relaxation times. Namely, if the temporal argument of $C(t), \mathcal{G}(t)$ is less or equal $t_e$, then the case of not very 
large observation times holds with all its consequences. For instance, the usual FDT can be used. At the same time 
the kernel $\exp(-t/\tau_J)$ can be put equal to unity, as we have done more than once in the previous subsection. If the corresponding temporal argument has at least the same order as $t_e$ we should take into account that $C(t \sim t_e) = q$, 
and if it has the same order as $\tau_J$ we use (5.14) accompanying with an evident consistency condition $C(0) = q$.

Our transformations come to dividing the domains of integration in Eq. (5.1) into two parts: from 0 to $t_e$, and 
from $t_e$ to $\infty$ (the corresponding integrals in Eq. (5.2) are divided into three parts because more accuracy is needed: 
from 0 to $t_e$, from $t_e$ to $t - t_e$, and from $t - t_e$ to $\infty$). Now every part is treated as described above. We omit all 
factors which have relatively small order when $\tau_J \to \infty$. In particular, for Eq. (5.1), (5.2) these factors are of order 
$O(1/\tau_J)$ ($O(1/\tau_J^2)$).

Finally, we get the following equations (where $s = t/\tau_J$)

$$rC(s) = \frac{pJ^2}{2v} \int_0^\infty d\bar{s} e^{-\bar{s}} C^{p-1}(\bar{s})C(s - \bar{s}) + \frac{p(p - 1)T_J J^2}{2v} \int_0^\infty d\bar{s} e^{-\bar{s}} C^{p-2}(\bar{s}) G(\bar{s}) C(s - \bar{s})$$

$$+ \frac{pT_J J^2}{2v} \int_0^\infty d\bar{s} e^{-(s+\bar{s})} C^{p-1}(s + \bar{s}) G(\bar{s}) + \frac{pT_J J^2}{2v} \beta e^{-\bar{s}} C^{p-1}(\bar{s})(1 - q)$$

$$+ \frac{pT_J J^2}{2v} \beta C(s)(1 - q^{p-1}),$$ (5.15)

$$rG(s) = \frac{pJ^2}{2v} \int_0^s d\bar{s} e^{-\bar{s}} C^{p-1}(\bar{s}) G(s - \bar{s}) + \frac{p(p - 1)T_J J^2}{2v} \int_0^s d\bar{s} e^{-\bar{s}} C^{p-2}(\bar{s}) G(s - \bar{s}) G(\bar{s})$$
\[ + \frac{pT_J J^2}{2v} \beta G(s)(1 - q^{-1}) + \frac{p(p - 1)T_J J^2}{2v} \beta e^{-s} C^{p-2}(s) G(s)(1 - q) \]
\[ + \frac{pJ^2}{2v} \beta e^{-s} C^{p-1}(s)(1 - q). \]

(5.16)

Now it is important to realize that the usual formulation of FDT does not hold in this case. Indeed, if the coupling constants fluctuate, then the heat current between the thermal baths cannot be neglected. In other words, the detailed balance condition is violated because there is a stationary current between the two heat baths, which changes its sign under time-reversal. Thus, we have a steady but non-Gibbsian state \[ \mathcal{G} \]. Generally speaking, in such a state there does not exist any simple or even closed general relation between the correlation and response functions \[ \mathcal{G} \]. However, such a relation is possible in some particular cases. It is interesting that in our case some generalized FDT exists, which is, however, non-universal in contrast to the usual one. Furthermore, we shall see that this theorem does not depend on secondary details of the model. As such it belongs to the thermodynamical picture of the glassy state, which is, however, non-universal in contrast to the usual one. Furthermore, we shall see that this theorem does not exist any simple or even closed general relation between the correlation and response functions \[ \mathcal{G} \]. However, particularly such considerations are a possible fundament for generalizing the notion of temperature.

Let us consider the generalized FDT in the form
\[ \partial_s \mathcal{C}(s) = \tilde{T}(-G(s) + \mathcal{G}(s)) \]
with some unknown coefficient \[ \tilde{T} \]. After some calculations using (5.15), (5.16) we get
\[ \tilde{T} = T_J \]

(5.17)

It is customary to call this the modified FDT the “Fluctuation-Dissipation Relation” (FDR) or just the “modified FDT”. In the our two-temperature dynamics without detailed balance this is a very simple relation between the correlation function and the susceptibility. It is important that the form of the FDR does not depend on concrete characteristics of the model.

We have obtained that the coefficient of FDR (it is \[ T_J \] in the case of Eq. (5.18), and \[ T \] in the case of (5.3)) depends crucially on the observation time: There is a direct correspondence between the time-scales which are considered and the proper temperature. Such phenomena were predicted recently for systems with “slow" dynamics. Particularly, such considerations are a possible fundament for generalizing the notion of temperature.

By the usual FDT (5.3) only an ergodic or short-time dynamics can be monitored. To take into account effects of RSB some regularization procedure is necessary. As was shown in [22] for the non-ergodic dynamics can be achieved by some other, “long-time” FDT
\[ \partial_t \mathcal{C}(t) = \frac{T}{m} (-G(t) + \mathcal{G}(-t)), \]
where \[ m \] is the RSB parameter. In the section [V] we discussed the analogy between the finite-\[ n \] RS free energy and the RSB free energy with \[ q_0 = 0 \]. In the spirit of this analogy this long-time FDT corresponds to (5.18) because via the substitution \[ m \rightarrow n \] we go to this equation. This analogy is not an accident, of course. It displays a deep connection between systems with RSB and ones where different time-scales of relaxation and different temperatures are assumed initially. It seems to us that other interesting results can be obtained in this way in future.

With help of the FDR we obtain the following formula for \[ \mathcal{G}(0) \]:
\[ \mathcal{G}(0) = \frac{\beta q(1 - q)}{q - (p - 2)\beta T_J (1 - q)} \]

(5.19)

Now the positivity of this quantity requires our old condition (4.15). As we have seen, this condition is the necessary one for the validity of the RS assumption. Since \[ \mathcal{G}(s) \] decreases from \[ \mathcal{G}(0) \] to zero we have also the trivial consistency condition that the theory developed in the present subsection is nontrivial only for \[ q > 0 \].

Let us now discuss what happens in the limit \[ t \gg \tau_J \). We rewrite Eq. (5.15) taking into account (5.17, 5.18), and the concrete value of \[ r \):
\[ (1 - q)(q^{p-2} - e^{-s} C^{p-2}(s)) \mathcal{C}(s) = T \int_0^s d\tilde{s} e^{-\tilde{s}} C^{p-1}(\tilde{s}) \mathcal{G}(s - \tilde{s}) \]

(5.20)

If we want to investigate the static limit of this equation we should consider the large-s limit
\[ \lim_{\tau_J \rightarrow \infty} (\lim_{t \rightarrow \infty} C(t)) = \lim_{s \rightarrow \infty} \mathcal{C}(s) = Q. \]

(5.21)

In this case we have \[ Q = 0 \], because in the relevant part of integral (5.20) the function \[ \mathcal{G} \] is zero: When \[ t/\tau_J \rightarrow \infty \) every correlation will vanish at large times, and the system goes into paramagnetic phase. It is just that statement which was predicted above starting from some heuristic arguments.

There is no spin-glass phase if the spins and the coupling constants have nearly equal characteristic times. It is quite obvious in the light of the present discussion.
VI. SUMMARY

This paper is devoted to a glassy system coupled to two heat baths. In section 11 we use the adiabatic assumption to generalize the usual thermodynamics. Its basic relation (1.17) involves the entropies of the spins and the coupling constants, which in the present approach have the independent and well-defined physical meaning. The developed theory has a general character, and does not depend on concrete details of the considered systems. After this, in section \[ 12 \] this theory is applied to the mean-field $p$-spin-interaction spherical model, extended to have correlated random bonds, expressed by a finite temperature $T_f$. In the limit $T_f \to \infty$, so $n = T_f/T_s \to 0$, the usual spin-glass model with totally uncorrelated bonds is recovered. As noted recently [30], this type of correlations can make radical changes in the phase structure. In this context, the $p$-spin model is the convenient laboratory for investigating phase transitions, since it belongs to different universality classes for $p > 2$ and $p = 2$. Indeed, if $n$ is large enough there are only first-order phase transitions with positive latent heat (see Figs. 3, 4). This is in the contrast to the first-order transitions without latent heat ($p > 2$) or the true second-order transitions ($p = 2$), which are more typical for spin glasses and glasses, and realized in the remaining parts of the phase diagram. The 1RSB (replica symmetry breaking) spin-glass phase can exist as a truly stable phase only for $m < n < 1$ (see Eqs. (4.30, 4.31)). Replica symmetry is always restored for sufficiently low temperatures and $n > 0$. Notice the differences compared to the SK model with infinite-order RSB, where a finite $n$ mainly modifies the existing spin-glass phase, introducing the lower plateau for the order parameter $q(x)$ [13, 18]. This plateau grows with $n$, and RSB disappears at some critical value $n_c$ ($n_c < 1$). Nearly the same behavior is introduced by an external magnetic field. In our case such a phase does not exist at all, and the 1RSB phase with vanishing lower plateau exists even for $n > 1$, but only as a metastable phase. These distinctions are connected with different structures of the phase space.

For all $p \geq 2$ the first-order phase transitions are related with an interesting effect: In spite of the fact that the jump of the mean energy $u$ at the transition point is negative (because the latent heat is positive) the jump of the total entropy is positive. This uncommon property is possible only due to our generalized thermodynamical relation (1.17), combined with the fact that the corresponding jumps of $s_T$ and $s_J$ have opposite signs: $\Delta s_T < 0$, $\Delta s_J > 0$, but the sum $s = s_T + s_J$ has a positive jump (see Fig. 3). The situation is slightly different for the first-order type phase transition without latent heat. In section IV B 1 we have seen that the total entropy increases continuously in the course of the phase transition from the paramagnet to 1RSB spin-glass (see Fig. 2). In this respect an interesting analogy exists with the process of coarse(fine)-graining (see the discussion after Eq. (3.13)). Further developments of these analogies will be quite interesting.

There is a close relation between free energies and saddle-point equations of the finite $n$ RS case and the $q_0 = 0$ case of the 1RSB equations. The RS solution with nonzero $n$ corresponds to the 1RSB solution with $q_0 = 0$ [15]. This mechanism is responsible for the transition between the 1RSB and RS spin-glass phases. The transition is second-order with respect to free energy and its derivatives, but is connected with a jump of $q_0$ from zero to $q_1$ that ensures replica-symmetric behavior (see Figs. 3, 4). This analogy exists in dynamics also: in our time-translation invariant two-temperature dynamics there is the phenomenon which is the analog of the generalized FDT in the $n = 0$ non-equilibrium dynamics [23] or longtime FDT in the corresponding non-ergodic dynamics [24]. Such an effect reflects intrinsic connections between systems where the complex structure of the phase space is self-generated and there are different time-scales for the global relaxation (ergodicity breaking), and systems where different components have different temperatures and relaxation-scales.

For $n < 1$ and $p > 2$ there is a difference between phase transitions predicted by adiabatic statics and adiabatic dynamics. This well known effect is due to the existence of the whole set of metastable states with free energies greater than the free energy of the pure states predicted by statics. It is connected with the absence of activated processes on the timescales considered in the dynamics [25], which enters due to mean-field (infinite dimension) character of the model [35]. Namely, it is the opposed sequence of limits: In the true static consideration we observe times $\to \infty$ before the thermodynamic limit $N \to \infty$, but if dynamics is investigated by means of generating functional [19] the first limit is taken after the second, which eliminates activated process that need time-scales exponential in $N$. In the corresponding finite-dimensional systems a smoothening of this effect is expected, where instead a sharp phase transition a near-critical domain of temperatures will take place. Notice also that for $n = 0$ and $p = 2$ the RS spin-glass phase is only marginally stable. A non-zero $n$ stabilizes the corresponding fluctuations.

The predictions of the adiabatic statics and dynamics can be compared only for the relatively short observation times. The spin-glass phase appears at times $\ll \tau_J$ (the characteristic time of the coupling constants), but disappears for the observation times $\gg \tau_J$. In this limit of long observations the coupling constants cannot be viewed as frozen. This is the non-equilibrium steady state without any spin-glass ordering (only a critical slowing down of the spin-spin correlation function occurs when $T_J \to 0$). In this respect it is similar to weak ergodicity breaking occurring in the non-equilibrium dynamics [23]. A generalized fluctuation-dissipation relation has been proven, which contains the temperature of the couplings instead of the temperature of the spins. This relation is also closely connected to the...
non-equilibrium generalization of the FDT [32,33], where the asymptotic long-time non-equilibrium state of the \( n = 0 \) \( p \)-spin spherical model is considered.

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APPENDIX A

In this appendix we consider how to get the usual Langevin equations of the spin-glass model with a priori random freezed coupling constants from (2.2, 2.3).

Eq. (2.3) can be solved exactly with respect of \( J_{i_1...i_p} \):

\[
J_{i_1...i_p}(t) = J_{i_1...i_p}^{(0)} e^{-(t-t_0)/\tau_J} + A_{i_1...i_p}(t) + B_{i_1...i_p}(t)
\]  
(A.1)

\[
A_{i_1...i_p}(t) = \frac{1}{\Gamma_J} \int_{t_0}^{t} d\tilde{t} e^{-(\tilde{t}-\bar{t})/\tau_J} \eta_{i_1...i_p}(\tilde{t}),
\]  
(A.2)

\[
B_{i_1...i_p}(t) = \frac{1}{\Gamma_J} \int_{t_0}^{t} d\tilde{t} e^{-(\tilde{t}-\bar{t})/\tau_J} J_N \sigma_{i_1}(\tilde{t})...\sigma_{i_p}(\tilde{t}),
\]  
(A.3)

where \( J_{i_1...i_p}^{(0)} \) are the initial conditions at the moment \( t = t_0 \), \( \tau_J = \bar{\Gamma}_J J_N^2/v \), and \( J_N^2 = p!J^2/2N^{p-1} \). The initial factors can be neglected if \( t_0 \rightarrow -\infty \), \( |t_0| \gg \tau_J \). Further we have

\[
\langle A_{i_1...i_p}(t) A_{j_1...j_p}(t') \rangle = \frac{T_J J_N^2}{v} \delta_{i_1...i_p,j_1...j_p} e^{-(t-t')/\tau_J},
\]  
(A.4)

Thus if \( \tau_J \gg t - t' \), then \( A_{i_1...i_p} \) is a quenched Gaussian noise, and if \( v \sim T_J \) and \( T_J \rightarrow \infty \) then \( B_{i_1...i_p} \) can be neglected with respect of \( A_{i_1...i_p} \). So in these limits \( J_{i_1...i_p} \) is a quenched Gaussian noise. Each coupling constant is independent of the other ones and the spins.

APPENDIX B

In this appendix we discuss the derivation of (2.4) from the initial Langevin equations. We investigate these equations by the dynamical generating functional method [36] [11].

\[
1 = Z_{\text{dyn}} = \int \prod_i D\eta_i \prod_{[i_1...i_p]} D\eta_{i_1...i_p} \exp \left( -\frac{1}{4T_J} \sum_i \int dt \eta_i^2(t) - \frac{1}{4T_J} \sum_{[i_1...i_p]} \int dt \eta_{i_1...i_p}^2(t) \right),
\]  
(B.1)

where \( [i_1...i_p] = 1 \leq i_1 < ... < i_p \leq N \), and normalization factors are included in \( D\eta_i, D\eta_{i_1...i_p} \). By means of \( Z_{\text{dyn}} \) we can compute the noise average of any quantity \( A(\{\sigma\}, \{J\}) \):

\[
\langle A(\{\sigma\}, \{J\}) \rangle = \int D\sigma D\eta A(\{\sigma\}, \{J\}) \exp \left[ -\frac{1}{4T_J} \sum_i \int dt \left( \Gamma \partial_\sigma \sigma_i + r \sigma_i + \partial H/\partial \sigma_i \right)^2 \right]
\]  
(B.2)

where \( \langle \delta\eta/\delta\sigma \rangle, \langle \delta\eta/\delta J \rangle \) are the corresponding functional Jacobians. After a simple transformation we have
Here the last exponent is the contribution of the functional Jacobians; we shall not write this expression explicitly because it is not relevant in the mean-field approximation.

We want to derive equations for spin dependent functions, therefore in eq. (B.1) we can take Gaussian integrations by \{J\}, \{\hat{J}\}. The result is

\[ Z_{\text{dyn}} = \int D\sigma D\tilde{\sigma} D\tilde{J} \exp \left( -\Gamma T \sum_i \int dt \hat{\sigma}_i^2 + i \sum_i \int dt \hat{\sigma}_i(t) \left( \Gamma \partial_t \sigma_i + r \sigma_i + \frac{\partial H}{\partial \sigma_i} \right) \right) \]

\[ \exp \left( \sum_{\{i_1...i_p\}} \left[ -\Gamma J_{i_1} \int dt \hat{J}_{i_1}(t) + i \int dt \hat{J}_{i_1}(t) \left( \Gamma \partial_t \hat{J}_{i_1} + r \hat{J}_{i_1} + \frac{\partial H}{\partial \hat{J}_{i_1}} \right) \right] \right) \exp(V_{\eta\sigma} + V_{\eta,J}) \]

(B.3)

By the standard mean-field procedure \[11\] we introduce the following order parameters:

\[ a_{i_1...i_p}(\sigma, \hat{\sigma}, t) = \sum_{s=1}^p \sigma_{i_1}(t)...\sigma_{i_{s-1}}(t)\hat{\sigma}_{i_s}(t)\sigma_{i_{s+1}}(t)...\sigma_{i_p}(t), \quad b_{i_1...i_p}(\sigma, t) = \sigma_{i_1}(t)...\sigma_{i_p}(t), \]

(B.5)

\[ \phi(t - t') = \theta(t - t')e^{-(t - t')/\tau_J}, \quad k(t - t') = e^{-|t - t'|/\tau_J}, \quad \gamma_J = \Gamma_J / \nu \]

(B.6)

and the corresponding Lagrange factors \(\lambda_s(t, t'), s = 1, ..., 4\). In this scheme \(Q_2\) is the correlation function, \(Q_3, Q_4\) are the susceptibilities; \(Q_1\) ("field-field" correlation function) should be taken zero by reasons of causality \[11\]. Now we have

\[ Z_{\text{dyn}} = \int \prod_{s=1}^4 \frac{D\lambda_s DQ_s}{2\pi i} \exp \left( N\Omega(\lambda_s, Q_s) + N \int D\sigma D\tilde{\sigma} e^{L(\sigma, \hat{\sigma})} \right) \]

\[ \Omega(\lambda_s, Q_s) = -\int dt dt' \sum_{s=1}^4 \lambda_s(t, t')Q_s(t, t') + \frac{pJ^2}{2\nu} \int dt dt' \phi(t - t')Q_3(t, t')Q_2^{-1}(t, t') \]

\[ + \frac{pT_J J^2}{4\nu} \int dt dt' \phi(t - t')[Q_1(t, t')Q_2^{-1}(t, t') + (p - 1)Q_3(t, t')Q_4(t, t')Q_2^{-2}(t, t')], \]

(B.8)

where

\[ L(\sigma, \hat{\sigma}) = -\Gamma T \sum_i \int dt \hat{\sigma}_i(t) - i \sum_i \int dt \hat{\sigma}_i(t)(\Gamma \partial_t + r)\sigma_i \]

\[ + \lambda_1(t, t')\hat{\sigma}(t)\hat{\sigma}(t') + \lambda_2(t, t')\sigma(t)\sigma(t') \]

(B.9)
By variational methods we obtain

\[ \lambda_1(t, t') = \frac{pT_J J^2}{4v} k(t - t')Q_2^{p-1}(t, t') \]

\[ \lambda_2(t, t') = \frac{p(p-1)J^2}{21J} \phi(t - t')Q_3(t, t')Q_2^{p-2}(t, t') \]

\[ + \frac{pT_J J^2}{4v} k(t - t')((p-1)Q_1(t, t')Q_2^{p-2}(t, t') + (p-1)(p-2)Q_3(t, t')Q_4(t, t')Q_2^{p-3}(t, t')) \]

\[ \lambda_3(t, t') = \frac{pJ^2}{21J} \phi(t - t')Q_2^{p-1}(t, t') + (p-1)\frac{pT_J J^2}{4v} k(t - t')Q_4(t, t')Q_2^{p-2}(t, t') \]

\[ \lambda_4(t, t') = (p-1)\frac{pT_J J^2}{4v} k(t - t')Q_3(t, t')Q_2^{p-2}(t, t'), \]

(B.10)

\( \lambda_2 \) can be adsorbed in the Jacobians. These results should be substituted to (B.3): the effective dynamics of a spin is determined by spins motion at the environment of the spin, and by motion of the coupling constants. After this lengthy calculation we arrive at (2.4).

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