Hardness of almost embedding simplicial complexes in $\mathbb{R}^d$, II

Emil Alkin

Abstract

A map $f : K \to \mathbb{R}^d$ of a simplicial complex is an almost embedding if $f(\sigma) \cap f(\tau) = \emptyset$ whenever $\sigma, \tau$ are disjoint simplices of $K$. Fix integers $d, k \geq 2$ such that $k + 2 \leq d \leq \frac{3k}{2} + 1$. Assuming that the “preimage of a cycle is a cycle” (Conjecture 5) we prove $\text{NP}$-hardness of the algorithmic problem of recognition of almost embeddability of finite $k$-dimensional complexes in $\mathbb{R}^d$. Assuming that $\text{P} \neq \text{NP}$ (and that the “preimage of a cycle is a cycle”) we prove that the embedding obstruction is incomplete for $k$-dimensional complexes in $\mathbb{R}^d$ using configuration spaces. Our proof generalizes the Skopenkov-Tancer proof of this result for $d = \frac{3k}{2} + 1$.

Contents

1 Introduction 1
2 Proofs 2
3 Proof of Conjecture 5 3
4 Appendix 5

1 Introduction

Let $K$ be a finite simplicial complex. A map $f : K \to \mathbb{R}^d$ is an almost embedding if $f(\sigma) \cap f(\tau) = \emptyset$ whenever $\sigma, \tau$ are disjoint simplices of $K$.

Almost embeddings naturally appear in studies of embeddings. See more motivations in [ST] §1, ‘Motivation and background’ part and [SK] Remark 5.7.4.

Theorem 1. Assume that Conjecture 5 is true. The algorithmic problem of recognition of almost embeddability of finite $k$-dimensional complexes in $\mathbb{R}^d$ is $\text{NP}$-hard for $d, k \geq 2$ such that $k + 2 \leq d \leq \frac{3k}{2} + 1$.

The (simplicial) deleted product of $K$ is $\widetilde{K} := \bigcup \{\sigma \times \tau : \sigma, \tau \text{ are simplices of } K, \sigma \cap \tau = \emptyset\};$
i.e., $\widetilde{K}$ is the union of products $\sigma \times \tau$ formed by disjoint simplices of $K$.

A map $\widetilde{f} : \widetilde{K} \to S^{d-1}$ is equivariant if $\widetilde{f}(y, x) = -\widetilde{f}(x, y)$ for each pair $(x, y)$ from $\widetilde{K}$.

Theorem 2. Fix integers $d, k \geq 2$ such that $k + 2 \leq d \leq \frac{3k}{2} + 1$. Assume that $\text{P} \neq \text{NP}$. Assume that Conjecture 5 is true. Then there exists a finite $k$-dimensional complex $K$ that does not admit an almost embedding in $\mathbb{R}^d$ but for which there exists an equivariant map $\widetilde{K} \to S^{d-1}$.

The particular cases of both Theorem 1 and Theorem 2 (without assuming Conjecture 5) for $k, d$ such that $d = \frac{3k}{2} + 1$ or $d \equiv 1 \pmod{3}$ and $2 \leq k \leq \frac{3k}{2} + 1$ are proved in [ST]. Those proofs are based on [AMSW] Singular Borromean Rings Lemma 2.4.

Theorem 1 is deduced analogously (see details below) to [ST] Theorem 1(b)] from the following generalized Singular Borromean Rings Lemma 3. We prove Lemma 3 using Conjecture 5 stating that “the preimage of a cycle is a cycle” (a similar in a sense result can be known in folklore).

Theorem 2 is deduced analogously (see details below) to [ST] Theorem 1(a)] from Theorem 1.

By $\cdot$ denote some point in $S^n$ for some $n$.

*I would like to thank S. Avvakumov for helpful discussion.
Lemma 3 (Singular Borromean Rings). Assume that Conjecture 3 is true. For each \( k > l \geq 1 \) let \( T := S^l \times S^l \) be the 2l-dimensional torus with meridian \( m := S^l \times \cdot \) and parallel \( p := \cdot \times S^l \), and let \( S_p^k \) and \( S_m^k \) be copies of \( S^k \). Then there is no PL map \( f : T \sqcup S_p^k \sqcup S_m^k \to \mathbb{R}^{k+l+1} \) satisfying the following three properties:

1. the \( f \)-images of the components are pairwise disjoint;
2. \( fS_p^k \) is linked modulo 2 with \( fp \) and is not linked modulo 2 with \( fm \), and
3. \( fS_m^k \) is linked modulo 2 with \( fm \) and is not linked modulo 2 with \( fp \).

Remark 4. (a) The condition \( l \geq 1 \) is essential in Lemma 3. Indeed, there exists a PL map \( f : T \sqcup S_p^k \sqcup S_m^k \to \mathbb{R}^{k+l+1} \) satisfying the properties of Lemma 3 for \( k \geq 1 \), \( l = 0 \).

Let \( m := \{ \pm 1 \} \times \{ 1 \} \) and \( p := \{ 1 \} \times \{ \pm 1 \} \).

By \( 1_{n,m} \) denote the point in \( \mathbb{R}^m \) with \( n \)-th coordinate equals to one and the others equal to zeros.

Define the map \( f : \{ \pm 1 \} \times \{ \pm 1 \} \sqcup S_p^k \sqcup S_m^k \to \mathbb{R}^{k+1} \) by the rule

\[
f(x) = \begin{cases} 
   x \times (0)^{k-1} & \text{if } x \in T \\
   x + 1_{2,k+1} - 1_{1,k+1} & \text{if } x \in S_m^k \\
   x + 1_{1,k+1} - 1_{2,k+1} & \text{if } x \in S_p^k.
\end{cases}
\]

It can be easily checked that the map \( f \) satisfies the properties.

(b) The condition \( k > l \) is essential in Lemma 3. Indeed, there exists a PL map \( f : T \sqcup S_p^k \sqcup S_m^k \to \mathbb{R}^{k+l+1} \) satisfying the properties of Lemma 3 for \( l = k \).

Define the map \( f : T \sqcup S_p^k \sqcup S_m^k \to S^k \times D^{k+1} \sqcup D^{k+1} \times S^k \) by the rule

\[
f(x) = \begin{cases} 
   x & \text{if } x \in T \\
   (0,x) & \text{if } x \in S_m^k \\
   (x,0) & \text{if } x \in S_p^k.
\end{cases}
\]

Clearly, the map \( f \) satisfies the first property from Lemma 3. Since \( |fS_p^k \cap \text{Con}(fp)| = |S^k \times 0 \cap \cdot \times D^{k+1}| = |\{(0,0)\}| = 1 \) and \( fs_p^k \cap \text{Con}(fm) = S^k \times 0 \cap D^{k+1} \times \cdot = \emptyset \), the map \( f \) satisfies the second property. Analogously, the map \( f \) satisfies the third property.

2 Proofs

Proof of Theorem 1. Formally, Theorem 1 follows by modified version of [ST] Theorem 2 obtained by substitution the hypothesis “\( d = \frac{3k}{2} + 1 \)” with “\( k + 2 \leq d \leq \frac{3k}{2} + 1 \)”.

The proof of the modified version of [ST] Theorem 2 is obtained from the proof of [ST] Theorem 2 by:

- setting “\( l := d - k - 1 \)”;
- changing the second sentence of the second paragraph of ‘construction of \( K(\Phi) \)’ to “Take a triangulation of 2l-torus \( T \) extending triangulations of its meridian and parallel \( a \) and \( b \) as boundaries of \( (l+1) \)-simplices.”;
- adding the sentence “Since \( k \geq 2l \), \( K(\Phi) \) is a \( k \)-complex.” after the second paragraph of ‘construction of \( K(\Phi) \)”;
- changing the last sentence of the ‘only if’ part to “Since \( k \geq 2l \), all this contradicts the Singular Borromean Rings Lemma 3 applied to the restriction of \( f \) to \( S_q \sqcup S_r \sqcup T_q \).”.

Proof of Theorem 2. Theorem 2 follows by Theorem 1 and the existence of a polynomial algorithm for checking the existence of equivariant maps [CKV] . Indeed, for fixed \( d,k \) it is polynomial time decidable whether there exists an equivariant map \( K \to S^{d-1} \) [CKV]. Given that almost embeddability implies the existence of an equivariant map, Theorem 1 implies Theorem 2. 

\[ \square \]
Recall some known definitions.
A $c$-chain in $\mathbb{R}^d$ is a finite set $C$ of $c$-simplices in $\mathbb{R}^d$. By $V(C)$ denote the set of vertices of simplices from $C$.

A $c$-chain $C$ in $\mathbb{R}^d$ is called simplicial if the intersection of any two $c$-simplices in $C$ is a face of both of them whenever their intersection is not empty.

The boundary of a $c$-chain in $\mathbb{R}^d$ is the set of those $(c - 1)$-simplices in $\mathbb{R}^d$ that are faces of an odd number of the chain’s simplices.

A chain whose boundary is empty is called a cycle.

For a set $V$ of points in $\mathbb{R}^d$ denote $\left\{ \sum_{i=1}^n \alpha_i v_i \mid \sum_{i=1}^n \alpha_i = 1 \right\}$ by $\text{Aff}(V)$.

A set $V$ of points in $\mathbb{R}^d$ is in strong general position if for any collection $\{V_1, V_2, \ldots, V_r\}$ of $r$ pairwise disjoint subsets of $V$ the following holds $(\dim \bigcap \leq -\infty)$:

$$\dim \bigcap_{i=1}^r \text{Aff}(V_i) \leq \sum_{i=1}^r \dim \text{Aff}(V_i) - d(r - 1).$$

A set $W$ of points in $\mathbb{R}^d$ is in strong general position with respect to a $c$-chain $C$ in $\mathbb{R}^d$ if $W \cap V(C) = \emptyset$ and the set $W \cup V(C)$ is in strong general position.

For any PL manifold $N$ a PL map $f : N \to \mathbb{R}^d$ is in strong general position with respect to a $c$-chain $C$ in $\mathbb{R}^d$ if there exists a triangulation $T_N$ of $N$ such that

- the map $f$ is linear on each simplex of $T_N$, and
- the set of images of vertices of $T_N$ is in strong general position with respect to the chain $C$.

**Conjecture 5** (preimage of a cycle is a cycle). Let $c, n, d$ be non-negative integers such that $c < d$. For any $c$-chain $C$ in $\mathbb{R}^d$, $n$-dimensional closed PL manifold $N$, and PL map $f : N \to \mathbb{R}^d - \partial C$ in strong general position with respect to $C$, the preimage $f^{-1}C$ is the support of a $(c + n - d)$-cycle in $|T_N| \subset \mathbb{R}^m$.

Notice that the original proof of [AMSW] Singular Borromean Rings Lemma 2.4 implicitly used a simpler version of Conjecture 5 with additional hypothesis that the chain $C$ avoids self-intersection points of $f(N)$.

**Proof of Lemma 6** (see the full proof in the appendix). It is sufficient to make the following changes in proof of [AMSW] Lemma 2.4:

- the second part of the sentence after (**) should be changed to “they are cycles by Conjecture 5 and because $f(S^n_1) \cap \partial(C_T) = f(S^n_1) \cap f(T) = \emptyset = f(S^n_1) \cap f(S^n_2) = f(S^n_1) \cap \partial(C_m)$.”
- the second sentence in the text after (***) should be changed to “By Conjecture 5 and $f(T) \cap \partial(C_p) = f(T) \cap f(S^n_1) = \emptyset$, it follows that the $l$-chain $f_T^{-1}C_p$ is a cycle in $T$.”

**3 Proof of Conjecture 5**

For Conjecture 5 we need Lemma 6 and Conjecture 7.

Let us recall some known definitions.

The intersection of finite number of open half-spaces and an $n$-hyperplane is called an open $n$-polytope if this intersection is bounded and non-empty. The closure of an open $n$-polytope is called an $n$-polytope.

Let

- $\sigma, \tau$ be polytopes in $\mathbb{R}^d$;
- $[\tau : \sigma] \in \mathbb{Z}_2$ be the characteristic function of ‘$\sigma$ is a face of $\tau$’;
- $[\sigma \subset \partial \tau] \in \mathbb{Z}_2$ be the characteristic function of ‘$\sigma \subset \partial \tau$’;
- $P$ be a set of polytopes in $\mathbb{R}^d$;
- $[P : \sigma] := \sum_{\tau \in P} [\tau : \sigma]$.
Lemma 6. Let $P$ be a finite set of $c$-polytopes in $\mathbb{R}^n$. Suppose that

1. for any two different polytopes $\sigma_1, \sigma_2$ from $P$ the intersection $\sigma_1 \cap \sigma_2$ is a polytope of dimension at most $c - 1$ and is contained in $\partial \sigma_1 \cap \partial \sigma_2$;
2. $[P : \sigma]^{inc} = 0$ for any $(c - 1)$-polytope $\sigma$ in $\mathbb{R}^n$.

Then the union of $P$ is the support of a simplicial $c$-cycle in $\mathbb{R}^n$.

Proof. We prove the simpler version of Lemma 6 in which all “polytopes” are changed to “simplices” and the final sentence are changed to “Then $P$ is a simplicial $c$-cycle in $\mathbb{R}^n$. ”. The original lemma can easily be reduced to the simpler version by triangulating polytopes.

Denote by $\partial P$ the finite set of $(c - 1)$-simplices $\sigma$ in $\mathbb{R}^n$ for which

(*) the number of simplices $\tau \in P$ such that $\sigma$ is a simplex of $\partial \tau$ is odd ($\iff [P : \sigma] = 1$).

Assume the contrary, i.e., that the set $\partial P$ is non-empty.

Let $\sigma$ be a maximal element of the set $\partial P$ ordered by the inclusion. Then $\sigma$ is a $(c - 1)$-simplex such that (*) holds.

Let $L := \{ \sigma' \mid \sigma \subsetneq \sigma', \sigma' \text{ is a $(c - 1)$-simplex of } \tau \text{ for some } \tau \in P \}$.

Then

\[ [P : \sigma]^{inc} = \sum_{\tau \in P} [\sigma \subset \partial \tau] = \sum_{\sigma' \in (\sigma) \cup L} [P : \sigma'] = [P : \sigma] + \sum_{\sigma' \in L} [P : \sigma'] = 1. \]

Hence (2) does not hold. This contradiction concludes the proof.

Conjecture 7. Let $C$ be a $c$-chain in $\mathbb{R}^d$ with $c < d$. Let $U$ be a finite set of points in $\mathbb{R}^d$ in strong general position with respect to $C$. Then the union of $C$ is the support of a simplicial $c$-chain $C'$ in $\mathbb{R}^d$ such that $U$ is in general position with respect to $C'$.

Remark 8. The condition of strong general position is essential in Conjecture 7, i.e. modified version of Conjecture 7 obtained by substituting “strong general position” with “general position” is wrong. It can be shown by the following counterexample.

Let $c := 1$, $d := 2$. By $S$ denote a unit circle in $\mathbb{R}^2$ centered at the origin $O$. Let $xy, zt, uw$ be different diameters of $S$. Let $C := \{ xy, zt \}, U := \{ u, v \}$. Obviously, for any simplicial 1-chain $C'$ with $\bigcup C' = \bigcup C$ the point $O$ is a vertex of $C'$. Even though points $x, y, z, t, u, v$ are in general position, the set $U \cup \{ O \}$ is not in general position.

Proof of Conjecture 7. Let $T_N$ be a triangulation of $N$ satisfying the conditions from definition of strong general position with respect to $C$.

By Conjecture 7 there exists a simplicial $c$-chain $C'$ in $\mathbb{R}^d$ such that $\bigcup C = \bigcup C'$ and the map $f$ is in general position with respect to $C'$.

For any $(c + n - d)$-polytopes $\sigma$ in $C'$ and any $n$-simplex $\gamma$ in $T_N$ the intersection $\gamma \cap f^{-1}(\sigma)$ is either an empty set or a $(c + n - d)$-polytope. Then the preimage $f^{-1}(\sigma)$ is the union of a finite set $P_\sigma$ of $(c + n - d)$-polytopes in $[T_N] \subset \mathbb{R}^m$. Let $P$ be the disjoint union

\[ \bigcup_{\sigma \in C'} P_\sigma. \]

In the following three bullet points we prove that for any two $(c + n - d)$-polytopes $s$ and $t$ from $P$ having a common point their intersection is a polytope of dimension at most $c + n - d - 1$ and $s \cap t \subset \partial s \cap \partial t$ holds.

- Both $s$ and $t$ are contained in some $n$-simplex $\gamma$ of $T_N$. In this case $s \cap t = (\gamma \cap f^{-1}(s)) \cap (\gamma \cap f^{-1}(t)) = \gamma \cap f^{-1}(\sigma \cap \tau)$ for some $c$-simplices $\sigma, \tau$ in $C'$ is a polytope of dimension at most $n + (c - 1) - d = c + n - d - 1$. And,

\[ \gamma \cap f^{-1}(\sigma \cap \tau) = \gamma \cap f^{-1}(\partial \sigma \cap \partial \tau) \subset \bigcup \{ \gamma \cap f^{-1}(\partial \sigma) \subset \partial s \text{ s.t. } s \cap t \subset \partial s \cap \partial t. \} \]
• The polytopes $s$ and $t$ are contained in some different $n$-simplices $\gamma$ and $\delta$ of $T_N$, respectively. And $s, t \in P_\sigma$ for some $c$-simplex $\sigma$ in $C'$. In this case $s \cap t = (\gamma \cap f^{-1}(\sigma)) \cap (\delta \cap f^{-1}(\sigma)) = \gamma \cap \delta \cap f^{-1}(\sigma)$ is a polytope of dimension at most $(n - 1) + c - d = c + n - d - 1$. And,
\[
\gamma \cap \delta \cap f^{-1}(\sigma) = \partial \gamma \cap \partial \delta \cap f^{-1}(\sigma) \subset \left\{ \partial \gamma \cap f^{-1}(\sigma) \subset \partial s \right\} \implies s \cap t \subset \partial s \cap \partial t.
\]

• The polytopes $s$ and $t$ are contained in some different $n$-simplices $\gamma$ and $\delta$ of $T_N$, respectively. And $s \in P_\sigma, t \in P_\tau$ for some different $c$-simplices $\sigma, \tau$ in $C'$. In this case $s \cap t = (\gamma \cap f^{-1}(\sigma)) \cap (\delta \cap f^{-1}(\tau)) = \gamma \cap \delta \cap f^{-1}(\sigma \cap \tau)$ is a polytope of dimension at most $(n - 1) + (c - 1) - d = c + n - d - 2$. And,
\[
\gamma \cap \delta \cap f^{-1}(\sigma \cap \tau) = \partial \gamma \cap \partial \delta \cap f^{-1}(\partial \sigma \cap \partial \tau) \subset \left\{ \partial \gamma \cap f^{-1}(\sigma) \subset \partial s \right\} \implies s \cap t \subset \partial s \cap \partial t.
\]

By Lemma 6, it suffices to prove that for any $(c + n - d)$-polytope $u$ in the boundary of some $(c + n - d)$-polytope in $P$ the number of $(c + n - d)$-polytopes containing $u$ is even. Let us consider two cases.

• $u$ is contained in some $(n - 1)$-simplex $\eta$ of $T_N$. There exists only one $c$-simplex $\sigma$ in $C'$ such that $u \subset \partial (f^{-1}(\sigma) \cap \gamma)$ for some $n$-simplex $\gamma$ of $T_N$ containing $\eta$. By $\delta$ denote another $n$-simplex containing $\eta$. Since $f(u) \cap \sigma \subset f(\delta) \cap \sigma$ and $f(u) \cap \sigma = f(u)$ is not empty, the intersection $\delta \cap f^{-1}(\sigma)$ is $(c + n - d)$-polytope containing $u$. Hence there are only two $(c + n - d)$-polytopes containing $u$.

• $u$ is contained in some $n$-simplex $\gamma$ of $T_N$ and intersects the interior of $\gamma$. As in the previous case, there exists a $c$-simplex $\sigma$ such that $u \subset \partial (f^{-1}(\sigma) \cap \gamma)$. By $\kappa$ denote a $(c + 1)$-simplex containing $f(u)$. Let $\tau$ be a $c$-simplex in $C'$. Then $u \subset \gamma \cap f^{-1}(\tau)$ if and only if $\kappa \subset \tau$. Since the number of $c$-simplices containing $\kappa$ is even, we have that the number of $(c + n - d)$-polytopes containing $u$ is even.

\[\square\]

Apparently there is another way to prove Conjecture [5] similar to proofs of [Hu69] Lemma 11.4 and [Hu70] Lemma 1. This way is unlikely to be simpler because it is based on another definition of general position, which entails its own technical difficulties.

4 Appendix

Proof of the Singular Borromean Rings Lemma [3] (This proof repeats the proof of [AMSW] Singular Borromean Rings Lemma 2.4 [with minor changes]).

Assume to the contrary that the map $f$ exists. Without loss of generality, we may assume that $f$ is in general position.

We denote by $\partial$ the boundary of a chain.

We can view $f(T)$, $f(S_p^m)$, and $f(S_p^n)$ as $2l$-, $n$- and $n$-dimensional cycles in general position in $\mathbb{R}^{n+l+1}$. Denote by $C_T$, $C_p$, and $C_m$ singular cones in general position over $f(T)$, $f(S_p^m)$, and $f(S_p^n)$, respectively. We view these cones as $(2l + 1)$-, $(n + 1)$- and $(n + 1)$-dimensional chains. The contradiction is
\[
0 = |\partial (C_T \cap C_p \cap C_m)| = |\partial C_T \cap C_p \cap C_m| + |C_T \cap \partial C_p \cap C_m| + |C_T \cap C_p \cap \partial C_m| = |f(T)| = f(S_p^m) = f(S_p^n).
\]

Here (1) follows because $C_T \cap C_p \cap C_m$ is a 1-dimensional chain, so its boundary is a set of an even number of points. Equation (2) is Leibniz formula. So it remains to prove (3).

Proof of (3). For $X \in \{ T, S_p^m, S_p^n \}$ denote $f_X := f|_X$.

For the second term we have
\[
|C_T \cap f(S_p^m) \cap C_m| \overset{(\text{a})}{=} |(f_{S_p^m})^{-1}C_T \cap (f_{S_p^m})^{-1}C_m| \overset{(\text{c})}{=} 0,
\]

(*) holds because $(n + 1) + (2l + 1) + 2n < 3(n + l + 1)$, so by general position $C_T \cap C_m$ avoids self-intersection points of $f(S_p^m)$. 

5
(**) holds by the well-known higher-dimensional analogue of [Sk14, Parity Lemma 3.2.c] (which is proved analogously) because the intersecting objects are general position cycles in $S^n_p$, they are cycles by Conjecture [a] and because
\[ f(S^n_p) \cap \partial(C_T) = f(S^n_p) \cap f(T) = \emptyset = f(S^n_p) \cap f(S^n_m) = f(S^n_p) \cap \partial(C_m). \]

Analogously $|C_T \cap C_p \cap f(S^n_m)| = 0$.

For the first term we have
\[ |f(T) \cap C_p \cap C_m| \overset{(***)}{=} |(f^{-1}_T C_p) \cap (f^{-1}_T C_m)| \overset{****}{=} m \cap p = 1, \]

where (***) holds because $n \geq l \Leftrightarrow 2(n+1) + 4l < 3(n+l+1)$, so by general position $C_p \cap C_m$ avoids self-intersection points of $f(T)$.

(****) is proved as follows:

By Conjecture [a] and $f(T) \cap \partial(C_p) = f(T) \cap f(S^n_p) = \emptyset$, it follows that the $l$-chain $f^{-1}_T C_p$ is a cycle in $T$. By conditions (b) and (c) of Lemma 3 we have
\[ |p \cap f^{-1}_T C_p| = |f(p) \cap C_p| = 1 \quad \text{and} \quad |m \cap f^{-1}_T C_p| = |f(m) \cap C_p| = 0. \]

I.e. the cycle $f^{-1}_T C_p$ intersects the parallel $p$ and the meridian $m$ at 1 and 0 points modulo 2, respectively. Therefore $f^{-1}_T C_p$ is homologous to the meridian $m$. Likewise, $f^{-1}_T C_m$ is homologous to the parallel $p$. This implies (****).

\[ \square \]

References

[AMSW] S. Avvakumov, I. Mabillard, A. Skopenkov and U. Wagner. Eliminating Higher-Multiplicity Intersections, III. Codimension 2, Israel J. Math. 245 (2021) 501-534, [https://arxiv.org/abs/1511.03501](https://arxiv.org/abs/1511.03501)

[ˇCKV] M. ˇCadek, M. Krˇcal, L. Vokˇrinek, Algorithmic solvability of the lifting-extension problem, Discrete & Computational Geometry 57:4 (2017) 915–965, [http://arxiv.org/abs/1307.6444](http://arxiv.org/abs/1307.6444)

[Hu69] J. F. P. Hudson, Piecewise-Linear Topology, Benjamin, New York, Amsterdam, 1969.

[Hu70] J. F. P. Hudson, Obstruction to embedding disks, In: Topology of manifolds (1970), 407–415.

[PS] M. A. Perles and M. Sigron, Strong General Position, [https://arxiv.org/abs/1409.2899](https://arxiv.org/abs/1409.2899)

[Sk14] A. Skopenkov, Realizability of hypergraphs and Ramsey link theory, [https://arxiv.org/abs/1402.0658](https://arxiv.org/abs/1402.0658)

[Sk] A. Skopenkov, Algebraic Topology From Algorithmic Viewpoint, draft of a book, mostly in Russian, [http://www.mccme.ru/circles/oim/algor.pdf](http://www.mccme.ru/circles/oim/algor.pdf)

[ST] A. Skopenkov, M. Tancer. Hardness of almost embedding simplicial complexes in $\mathbb{R}^d$, Discr. and Comp. Geom. 61:2 (2019), 452-463, [https://arxiv.org/abs/1703.06305](https://arxiv.org/abs/1703.06305)