Planar Complex Numbers in Even $n$ Dimensions

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Abstract

Planar commutative $n$-complex numbers of the form $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ are introduced in an even number $n$ of dimensions, the variables $x_0, \ldots, x_{n-1}$ being real numbers. The planar $n$-complex numbers can be described by the modulus $d$, by the amplitude $\rho$, by $n/2$ azimuthal angles $\phi_k$, and by $n/2-1$ planar angles $\psi_{k-1}$. The exponential function of a planar $n$-complex number can be expanded in terms of the planar $n$-dimensional cosexponential functions $f_{nk}, k = 0, 1, \ldots, n-1$, and expressions are given for $f_{nk}$. Exponential and trigonometric forms are obtained for the planar $n$-complex numbers. The planar $n$-complex functions defined by series of powers are analytic, and the partial derivatives of the components of the planar $n$-complex functions are closely related. The integrals of planar $n$-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of a planar $n$-complex numbers depends on the cyclic variables $\phi_k$ leads to the concept of pole and residue for integrals on closed paths. The polynomials of planar $n$-complex variables can always be written as products of linear factors, although the factorization may not be unique.

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1 Introduction

A regular, two-dimensional complex number $x + iy$ can be represented geometrically by the modulus $\rho = (x^2 + y^2)^{1/2}$ and by the polar angle $\theta = \arctan(y/x)$. The modulus $\rho$ is multiplicative and the polar angle $\theta$ is additive upon the multiplication of ordinary complex numbers.

The quaternions of Hamilton are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, and many other hypercomplex systems are possible, but these hypercomplex systems do not have all the required properties of regular, two-dimensional complex numbers which rendered possible the development of the theory of functions of a complex variable.

A system of hypercomplex numbers in an even number of dimensions $n$ is described in this work, for which the multiplication is both associative and commutative, and which is rich enough in properties so that an exponential form exists and the concepts of analytic $n$-complex function, contour integration and residue can be defined. The $n$-complex numbers introduced in this work have the form $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$, the variables $x_0, ..., x_{n-1}$ being real numbers. The multiplication rules for the complex units $h_1, ..., h_{n-1}$ are $h_jh_k = h_{j+k}$ if $0 \leq j + k \leq n - 1$, and $h_jh_k = -h_{j+k-n}$ if $n \leq j + k \leq 2n - 2$, where $h_0 = 1$. The product of two $n$-complex numbers is equal to zero if both numbers are equal to zero, or if the numbers belong to certain $n$-dimensional hyperplanes described further in this work.

If the $n$-complex number $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ is represented by the point $A$ of coordinates $x_0, x_1, ..., x_{n-1}$, the position of the point $A$ can be described, in an even number of dimensions, by the modulus $d = (x_0^2 + x_1^2 + \cdots + x_{n-1}^2)^{1/2}$, by $n/2$ azimuthal angles $\phi_k$ and by $n/2 - 1$ planar angles $\psi_{k-1}$. An amplitude $\rho$ can be defined as $\rho^n = \rho_1^2 \cdots \rho_{n/2}^2$, where $\rho_k$ are radii in orthogonal two-dimensional planes defined further in this work. The amplitude $\rho$, the radii $\rho_k$ and the variables $\tan\psi_{k-1}$ are multiplicative, and the azimuthal angles $\phi_k$ are additive upon the multiplication of $n$-complex numbers. Because the description of the position of the point $A$ requires, in addition to the azimuthal angles $\phi_k$, only the planar angles $\psi_{k-1}$, the hypercomplex numbers studied in this work will be called planar $n$-complex number, to distinguish them from the polar $n$-complex numbers introduced in [6], which in
an even number of dimensions required two polar angles, and in an odd number of dimensions required one polar angle.

The exponential function of an n-complex number can be expanded in terms of the planar n-dimensional cosexponential functions $f_{nk}(y) = \sum_{p=0}^{\infty} (-1)^{p} y^{k+pn}/(k + pn)!$, $k = 0, 1, ..., n - 1$. It is shown that $f_{nk}(y) = (1/n) \sum_{l=1}^{n} \exp \{y \cos (\pi(2l - 1)/n)\}$ cos $\{y \sin (\pi(2l - 1)/n) - \pi k(2l - 1)/n\}$, $k = 0, 1, ..., n - 1$. Addition theorems and other relations are obtained for the planar n-dimensional cosexponential functions.

The exponential form of an n-complex number, which can be defined for all $x_0, ..., x_{n-1}$, is $u = \rho \exp \left\{\sum_{p=0}^{n-1} h_p \left[ - (2/n) \sum_{k=0}^{n/2} \cos (\pi(2k - 1)p/n) \ln \tan \psi_{k-1} \right] \right\} \exp \left(\sum_{k=1}^{n/2} \tilde{e}_k \phi_k \right)$, where $\tilde{e}_k = (2/n) \sum_{p=1}^{n-1} h_p \sin(\pi(2k - 1)p/n)$. A trigonometric form also exists for an n-complex number, $u = d (n/2)^{1/2} \left(1 + 1/\tan^2 \psi_1 + 1/\tan^2 \psi_2 + \cdots + 1/\tan^2 \psi_{n/2-1}\right)^{-1/2}$ $(e_1 + \sum_{k=2}^{n/2} e_k/\tan \psi_{k-1}) \exp \left(\sum_{k=1}^{n/2} \tilde{e}_k \phi_k \right)$.

Expressions are given for the elementary functions of n-complex variable. The functions $f(u)$ of n-complex variable which are defined by power series have derivatives independent of the direction of approach to the point under consideration. If the n-complex function $f(u)$ of the n-complex variable $u$ is written in terms of the real functions $P_k(x_0, ..., x_{n-1}), k = 0, ..., n - 1$, then relations of equality exist between partial derivatives of the functions $P_k$.

The integral $\int_{A}^{B} f(u) du$ of an n-complex function between two points $A, B$ is independent of the path connecting $A, B$, in regions where $f$ is regular. If $f(u)$ is an analytic n-complex function, then $\oint_{C} f(u) du/(u - u_0) = 2\pi f(u_0) \sum_{k=1}^{n/2} \tilde{e}_k \int \text{int}(u_0 \xi_k \eta_k, \Gamma \xi_k \eta_k)$, where the functional int takes the values 0 or 1 depending on the relation between $u_0 \xi_k \eta_k$ and $\Gamma \xi_k \eta_k$, which are respectively the projections of the point $u_0$ and of the loop $\Gamma$ on the plane defined by the orthogonal axes $\xi_k$ and $\eta_k$, as expained further in this work.

A polynomial $u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m$ can always be written as a product of linear factors, although the factorization may not be unique.

This paper belongs to a series of studies on commutative complex numbers in n dimensions. For $n = 2$, the n-complex numbers discussed in this paper become the usual 2-dimensional complex numbers $x + iy$. A detailed analysis for $n = 4$ and $n = 6$ of the planar n-complex numbers can be found in the corresponding studies mentioned in Ref. [7].
2 Operations with planar n-complex numbers

A hypercomplex number in \( n \) dimensions is determined by its \( n \) components \((x_0, x_1, ..., x_{n-1})\). The planar n-complex numbers and their operations discussed in this work can be represented by writing the n-complex number \((x_0, x_1, ..., x_{n-1})\) as

\[ u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1}, \]

where \(h_1, h_2, \cdots, h_{n-1}\) are bases for which the multiplication rules are

\[ h_j h_k = (-1)^{[(j+k)/n]} h_l, \quad l = j + k - n[(i+k)/n], \tag{1} \]

for \(j, k, l = 0, 1, ..., n-1\), where \(h_0 = 1\). In Eq. (1), \([ (j+k)/n ] \) denotes the integer part of \((j+k)/n\), the integer part being defined as \([a] \leq a < [a]+1\), so that \(0 \leq j+k - n[(j+k)/n] \leq n-1\). In this work, brackets larger than the regular brackets \([ ]\) do not have the meaning of integer part. The significance of the composition laws in Eq. (1) can be understood by representing the bases \(h_j, h_k\) by points on a circle at the angles \(\alpha_j = \pi j/n, \alpha_k = \pi k/n\), as shown in Fig. 1, and the product \(h_j h_k\) by the point of the circle at the angle \(\pi (j+k)/n\). If \(\pi \leq \pi (j+k)/n < 2\pi\), the point is opposite to the basis \(h_l\) of angle \(\alpha_l = \pi (j+k)/n - \pi\).

In an odd number of dimensions \(n\), a transformation of coordinates according to

\[ x_{2l} = x'_l, x_{2m-1} = -x'_{(n-1)/2+m}, \tag{2} \]

and of the bases according to

\[ h_{2l} = h'_l, h_{2m-1} = -h'_{(n-1)/2+m}, \tag{3} \]

where \(l = 0, ..., (n-1)/2, \ m = 1, ..., (n-1)/2\), leaves the expression of an n-complex number unchanged,

\[ \sum_{k=0}^{n-1} h_k x_k = \sum_{k=0}^{n-1} h'_k x'_k, \tag{4} \]

and the products of the bases \(h'_k\) are

\[ h'_j h'_k = h'_l, \quad l = j + k - n[(j+k)/n], \tag{5} \]

for \(j, k, l = 0, 1, ..., n-1\). Thus, the n-complex numbers with the rules (1) are equivalent in an odd number of dimensions to the n-complex numbers described in (4). Therefore, in this work it will be supposed that \(n\) is an even number, unless otherwise stated.
Two n-complex numbers \( u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1} \), \( u' = x'_0 + h_1' x'_1 + h_2' x'_2 + \cdots + h_{n-1}' x'_{n-1} \) are equal if and only if \( x_j = x'_j, j = 0, 1, \ldots, n - 1 \). The sum of the n-complex numbers \( u \) and \( u' \) is

\[
u + u' = x_0 + x'_0 + h_1(x_1 + x'_1) + \cdots + h_{n-1}(x_{n-1} + x'_{n-1}). \tag{6}\]

The product of the numbers \( u, u' \) is

\[
u u' = x_0 x'_0 - x_1 x'_n - x_2 x'_{n-1} - x_3 x'_{n-2} - \cdots - x_{n-1} x'_1 + h_1(x_0 x'_1 + x_1 x'_0 - x_2 x'_{n-1} - x_3 x'_{n-2} - \cdots - x_{n-1} x'_2) + h_2(x_0 x'_2 + x_1 x'_1 + x_2 x'_0 - x_3 x'_{n-1} - \cdots - x_{n-1} x'_3) + \cdots + h_{n-1}(x_0 x'_n + x_1 x'_{n-1} + x_2 x'_0 + x_3 x'_{n-2} + \cdots + x_{n-1} x'_1). \tag{7}\]

The product \( uu' \) can be written as

\[
v u' = \sum_{k=0}^{n-1} h_k \sum_{l=0}^{n-1} (-1)^{[(n-k+1+l)/n]} x_l x'_{k-l+n[(n-k+1+l)/n]} \tag{8}\]

If \( u, u', u'' \) are n-complex numbers, the multiplication is associative

\[(uu')u'' = u(u'u'')\tag{9}\]

and commutative

\[uu' = u'u,\tag{10}\]

because the product of the bases, defined in Eq. (1), is associative and commutative. The fact that the multiplication is commutative can be seen also directly from Eq. (7). The n-complex zero is \( 0 + h_1 \cdot 0 + \cdots + h_{n-1} \cdot 0 \), denoted simply 0, and the n-complex unity is \( 1 + h_1 \cdot 0 + \cdots + h_{n-1} \cdot 0 \), denoted simply 1.

The inverse of the n-complex number \( u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1} \) is the n-complex number \( u' = x'_0 + h_1' x'_1 + h_2' x'_2 + \cdots + h_{n-1}' x'_{n-1} \) having the property that

\[uu' = 1.\tag{11}\]
Written on components, the condition, Eq. (11), is

\[
\begin{align*}
  x_0x_0' &- x_1x_{n-1}' - x_2x_{n-2}' - x_3x_{n-3}' - \cdots - x_{n-1}x_1' = 1, \\
  x_0x_1' &+ x_1x_0' - x_2x_{n-1}' - x_3x_{n-2}' - \cdots - x_{n-1}x_2' = 0, \\
  x_0x_2' &+ x_1x_1' + x_2x_0' - x_3x_{n-1}' - \cdots - x_{n-1}x_3' = 0, \\
  &\vdots \\
  x_0x_{n-1}' &+ x_1x_{n-2}' + x_2x_{n-3}' + x_3x_{n-4}' + \cdots + x_{n-1}x_0' = 0.
\end{align*}
\]

The system (12) has a solution provided that the determinant of the system,

\[ \nu = \det(A), \] (13)

is not equal to zero, \( \nu \neq 0 \), where

\[
A = \begin{pmatrix}
  x_0 & -x_{n-1} & -x_{n-2} & \cdots & -x_1 \\
  x_1 & x_0 & -x_{n-1} & \cdots & -x_2 \\
  x_2 & x_1 & x_0 & \cdots & -x_3 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_0
\end{pmatrix}.
\] (14)

It will be shown that \( \nu > 0 \), and the quantity

\[ \rho = \nu^{1/n} \] (15)

will be called amplitude of the \( n \)-complex number \( u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1} \).

The quantity \( \nu \) can be written as a product of linear factors

\[ \nu = \prod_{k=1}^{n} (x_0 + \epsilon_kx_1 + \epsilon_k^2x_2 + \cdots + \epsilon_k^{n-1}x_{n-1}), \] (16)

where \( \epsilon_k = e^{i\pi(2k-1)/n}, k = 1, \ldots, n \), and \( i \) being the imaginary unit. The factors appearing in Eq. (16) are of the form

\[ x_0 + \epsilon_kx_1 + \epsilon_k^2x_2 + \cdots + \epsilon_k^{n-1}x_{n-1} = v_k + i\tilde{v}_k, \] (17)

where

\[ v_k = \sum_{p=0}^{n-1} x_p \cos \frac{\pi(2k-1)p}{n}, \] (18)

\[ \tilde{v}_k = \sum_{p=0}^{n-1} x_p \sin \frac{\pi(2k-1)p}{n}, \] (19)
for $k = 1, ..., n$. The variables $v_k, \tilde{v}_k, k = 1, ..., n/2$ will be called canonical polar n-complex variables. It can be seen that $v_k = v_{n-k+1}, \tilde{v}_k = -\tilde{v}_{n-k+1}$, for $k = 1, ..., n/2$. Therefore, the factors appear in Eq. (16) in complex-conjugate pairs of the form $v_k + i\tilde{v}_k$ and $v_{n-k+1} + i\tilde{v}_{n-k+1} = v_k - i\tilde{v}_k$, where $k = 1, ...n/2$, so that the determinant $\nu$ is a real and positive quantity, $\nu > 0$,

$$\nu = \prod_{k=1}^{n/2} \rho_k^2,$$

where

$$\rho_k^2 = v_k^2 + \tilde{v}_k^2.$$ 

Thus, an n-complex number has an inverse unless it lies on one of the nodal hypersurfaces $\rho_1 = 0$, or $\rho_2 = 0$, or ... or $\rho_{n/2} = 0$.

### 3 Geometric representation of planar n-complex numbers

The n-complex number $x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1}$ can be represented by the point $A$ of coordinates $(x_0, x_1, ..., x_{n-1})$. If $O$ is the origin of the n-dimensional space, the distance from the origin $O$ to the point $A$ of coordinates $(x_0, x_1, ..., x_{n-1})$ has the expression

$$d^2 = x_0^2 + x_1^2 + \cdots + x_{n-1}^2.$$ 

The quantity $d$ will be called modulus of the n-complex number $u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1}$. The modulus of an n-complex number $u$ will be designated by $d = |u|$.

The exponential and trigonometric forms of the n-complex number $u$ can be obtained conveniently in a rotated system of axes defined by a transformation which has the form

$$
\begin{pmatrix}
\vdots \\
\xi_k \\
\eta_k \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
\vdots \\
\sqrt{\frac{2}{n}} \cos \frac{\pi(2k-1)}{n} \\
0 \\
\sqrt{\frac{2}{n}} \sin \frac{\pi(2k-1)}{n} \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
x_0 \\
\vdots \\
x_{n-1}
\end{pmatrix},
$$

(23)
where \( k = 1, 2, \ldots, n/2 \). The lines of the matrices in Eq. (23) give the components of the \( n \) vectors of the new basis system of axes. These vectors have unit length and are orthogonal to each other. By comparing Eqs. (18)-(19) and (23) it can be seen that
\[
v_k = \sqrt{\frac{n}{2}} \xi_k, \quad \tilde{v}_k = \sqrt{\frac{n}{2}} \eta_k,
\]
i.e. the two sets of variables differ only by a scale factor.

The sum of the squares of the variables \( v_k, \tilde{v}_k \) is
\[
\sum_{k=1}^{n/2} (v_k^2 + \tilde{v}_k^2) = \frac{n}{2} d^2.
\]
The relation (25) has been obtained with the aid of the relation
\[
\sum_{k=1}^{n/2} \cos \left( \frac{\pi}{n} (2k - 1)p \right) = 0,
\]
for \( p = 1, \ldots, n - 1 \). From Eq. (25) it results that
\[
d^2 = \frac{2}{n} \sum_{k=1}^{n/2} \rho_k^2.
\]
The relation (27) shows that the square of the distance \( d \), Eq. (22), is equal to the sum of the squares of the projections \( \rho_k \sqrt{2/n} \). This is consistent with the fact that the transformation in Eq. (23) is unitary.

The position of the point \( A \) of coordinates \( (x_0, x_1, \ldots, x_{n-1}) \) can be also described with the aid of the distance \( d \), Eq. (22), and of \( n - 1 \) angles defined further. Thus, in the plane of the axes \( v_k, \tilde{v}_k \), the azimuthal angle \( \phi_k \) can be introduced by the relations
\[
\cos \phi_k = v_k / \rho_k, \quad \sin \phi_k = \tilde{v}_k / \rho_k,
\]
where \( 0 \leq \phi_k < 2\pi, \ k = 1, \ldots, n/2 \), so that there are \( n/2 \) azimuthal angles. The radial distance \( \rho_k \) in the plane of the axes \( v_k, \tilde{v}_k \) has been defined in Eq. (21). If the projection of the point \( A \) on the plane of the axes \( v_k, \tilde{v}_k \) is \( A_k \), and the projection of the point \( A \) on the 4-dimensional space defined by the axes \( v_1, \tilde{v}_1, v_k, \tilde{v}_k \) is \( A_{1k} \), the angle \( \psi_{k-1} \) between the line \( OA_{1k} \) and the 2-dimensional plane defined by the axes \( v_k, \tilde{v}_k \) is
\[
\tan \psi_{k-1} = \rho_1 / \rho_k,
\]
where \( k = 1, 2, \ldots, n/2 \). The lines of the matrices in Eq. (23) give the components of the \( n \) vectors of the new basis system of axes. These vectors have unit length and are orthogonal to each other. By comparing Eqs. (18)-(19) and (23) it can be seen that
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\[
\tan \psi_{k-1} = \rho_1 / \rho_k,
\]
where $0 \leq \psi_k \leq \pi/2$, $k = 2, \ldots, n/2$, so that there are $n/2-1$ planar angles. Thus, the position of the point $A$ is described by the distance $d$, by $n/2$ azimuthal angles and by $n/2-1$ planar angles. These angles are shown in Fig. 2.

The variables $\rho_k$ can be expressed in terms of $d$ and the planar angles $\psi_k$ as

$$
\rho_k = \frac{\rho_1}{\tan \psi_{k-1}},
$$

for $k = 2, \ldots, n/2$, where

$$
\rho_1^2 = \frac{nd^2}{2} \left( 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-1}} \right)^{-1}.
$$

If $u' = x'_0 + h_1 x'_1 + h_2 x'_2 + \cdots + h_{n-1} x'_{n-1}, u'' = x''_0 + h_1 x''_1 + h_2 x''_2 + \cdots + h_{n-1} x''_{n-1}$ are $n$-complex numbers of parameters $\rho'_k, \psi'_k, \phi'_k$ and respectively $\rho''_k, \psi''_k, \phi''_k$, then the parameters $v_+, \rho_k, \psi_k, \phi_k$ of the product $n$-complex number $u = u'u''$ are given by

$$
\rho_k = \rho'_k \rho''_k,
$$

for $k = 1, \ldots, n/2$,

$$
\tan \psi_k = \tan \psi'_k \tan \psi''_k,
$$

for $k = 1, \ldots, n/2 - 1$,

$$
\phi_k = \phi'_k + \phi''_k,
$$

for $k = 1, \ldots, n/2$. The Eqs. (32)-(34) are a consequence of the relations

$$
v_k = v'_k v''_k - \tilde{v}_k \tilde{v}'_k, \quad \tilde{v}_k = v'_k \tilde{v}''_k + \tilde{v}'_k v''_k,
$$

and of the corresponding relations of definition. Then the product $\nu$ in Eq. (20) has the property that

$$
\nu = \nu' \nu'',
$$

and the amplitude $\rho$ defined in Eq. (15) has the property that

$$
\rho = \rho' \rho''.
$$

The fact that the amplitude of the product is equal to the product of the amplitudes, as written in Eq. (37), can be demonstrated also by using a representation of the $n$-complex
numbers by matrices, in which the n-complex number \( u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1} \) is represented by the matrix

\[
U = \begin{pmatrix}
  x_0 & x_1 & x_2 & \cdots & x_{n-1} \\
-x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\
-x_{n-2} & -x_{n-1} & x_0 & \cdots & x_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-x_1 & -x_2 & -x_3 & \cdots & x_0 \\
\end{pmatrix}
\]  

(38)

The product \( u = u'u'' \) is be represented by the matrix multiplication \( U = U'U'' \). The relation is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices. The use of the representation of the n-complex numbers with matrices provides an alternative demonstration of the fact that the product of n-complex numbers is associative, as stated in Eq. (9).

According to Eqs. (25 and (21), the modulus of the product \( uu' \) is given by

\[
|uu'|^2 = 2 \frac{n}{n} \sum_{k=1}^{n/2} (\rho_k \rho'_k)^2.
\]

(39)

Thus, if the product of two n-complex numbers is zero, \( uu' = 0 \), then \( \rho_k \rho'_k = 0 \), \( k = 1, \ldots, n/2 \). This means that either \( u = 0 \), or \( u' = 0 \), or \( u, u' \) belong to orthogonal hypersurfaces in such a way that the afore-mentioned products of components should be equal to zero.

4 The planar n-dimensional cosexponential functions

The exponential function of the n-complex variable \( u \) can be defined by the series

\[
\exp u = 1 + u + u^2/2! + u^3/3! + \cdots.
\]

(40)

It can be checked by direct multiplication of the series that

\[
\exp(u + u') = \exp u \cdot \exp u'.
\]

(41)

If \( u = x_0 + h_1 x_1 + h_2 x_2 + \cdots + h_{n-1} x_{n-1} \), then \( \exp u \) can be calculated as \( \exp u = \exp x_0 \cdot \exp(h_1 x_1) \cdots \exp(h_{n-1} x_{n-1}) \).
It can be seen with the aid of the representation in Fig. 1 that

\[ h_k^{n+p} = (-1)^k h_k^p, \quad p \text{ integer}, \quad (42) \]

where \( k = 1, \ldots, n - 1 \). For \( k \) even, \( e^{h_k y} \) can be written as

\[ e^{h_k y} = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{kp-n[kp/n]} g_{np}(y), \quad (43) \]

where \( h_0 = 1 \), and where \( g_{np} \) are the polar \( n \)-dimensional cosexponential functions. For odd \( k \), \( e^{h_k y} \) is

\[ e^{h_k y} = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{kp-n[kp/n]} f_{np}(y), \quad (44) \]

where the functions \( f_{nk} \), which will be called planar cosexponential functions in \( n \) dimensions, are

\[ f_{nk}(y) = \sum_{p=0}^{\infty} (-1)^p \frac{y^{k+pn}}{(k+pn)!}, \quad (45) \]

for \( k = 0, 1, \ldots, n - 1 \).

The planar cosexponential functions of even index \( k \) are even functions, \( f_{n,2l}(-y) = f_{n,2l}(y) \), and the planar cosexponential functions of odd index are odd functions, \( f_{n,2l+1}(-y) = -f_{n,2l+1}(y), \quad l = 0, \ldots, n/2 - 1 \).

The planar \( n \)-dimensional cosexponential function \( f_{nk}(y) \) is related to the polar \( n \)-dimensional cosexponential function \( g_{nk}(y) \) discussed in [5] by the relation

\[ f_{nk}(y) = e^{-i\pi k/n} g_{nk}(e^{i\pi/n} y), \quad (46) \]

for \( k = 0, \ldots, n - 1 \). The expression of the planar \( n \)-dimensional cosexponential functions is then

\[ f_{nk}(y) = \frac{1}{n} \sum_{l=1}^{n} \exp \left[ y \cos \left( \frac{\pi(2l-1)}{n} \right) \right] \cos \left[ y \sin \left( \frac{\pi(2l-1)}{n} \right) - \frac{\pi(2l-1)k}{n} \right], \quad (47) \]

for \( k = 0, 1, \ldots, n - 1 \). The planar cosexponential function defined in Eq. (45) has the expression given in Eq. (47) for any natural value of \( n \), this result not being restricted to even values of \( n \). In order to check that the function in Eq. (47) has the series expansion written in Eq. (45), the right-hand side of Eq. (47) will be written as

\[ f_{nk}(y) = \frac{1}{n} \sum_{l=1}^{n} \text{Re} \left\{ \exp \left[ \left( \cos \left( \frac{\pi(2l-1)}{n} \right) + i \sin \left( \frac{\pi(2l-1)}{n} \right) \right) y - i \frac{\pi k(2l-1)}{n} \right] \right\}, \quad (48) \]
for $k = 0, 1, ..., n - 1$, where $\text{Re}(a + ib) = a$, with $a$ and $b$ real numbers. The part of the exponential depending on $y$ can be expanded in a series,

$$f_{nk}(y) = \frac{1}{n} \sum_{p=0}^{\infty} \sum_{l=1}^{n} \text{Re} \left\{ \frac{1}{p!} \exp \left[ \frac{i\pi(2l-1)}{n} (p - k) \right] y^p \right\},$$

(49)

for $k = 0, 1, ..., n - 1$. The expression of $f_{nk}(y)$ becomes

$$f_{nk}(y) = \frac{1}{n} \sum_{p=0}^{\infty} \sum_{l=1}^{n} \left\{ \frac{1}{p!} \cos \left[ \frac{\pi(2l-1)}{n} (p - k) \right] y^p \right\},$$

(50)

where $k = 0, 1, ..., n - 1$ and, since

$$\frac{1}{n} \sum_{l=1}^{n} \cos \left[ \frac{\pi(2l-1)}{n} (p - k) \right] = \begin{cases} 1, & \text{if } p - k \text{ is an even multiple of } n, \\ -1, & \text{if } p - k \text{ is an odd multiple of } n, \\ 0, & \text{otherwise,} \end{cases}$$

(51)

this yields indeed the expansion in Eq. (45).

It can be shown from Eq. (47) that

$$\sum_{k=0}^{n-1} f_{nk}^2(y) = \frac{1}{n} \sum_{l=1}^{n} \exp \left[ 2y \cos \left( \frac{\pi(2l-1)}{n} \right) \right].$$

(52)

It can be seen that the right-hand side of Eq. (52) does not contain oscillatory terms. If $n$ is a multiple of 4, it can be shown by replacing $y$ by $iy$ in Eq. (52) that

$$\sum_{k=0}^{n-1} (-1)^k f_{nk}^2(y) = \frac{4}{n} \sum_{l=1}^{n/4} \cos \left[ 2y \cos \left( \frac{\pi(2l-1)}{n} \right) \right],$$

(53)

which does not contain exponential terms.

For odd $n$, the planar n-dimensional cosexponential function $f_{nk}(y)$ is related to the n-dimensional cosexponential function $g_{nk}(y)$ discussed in [5] also by the relation

$$f_{nk}(y) = (-1)^k g_{nk}(-y),$$

(54)

as can be seen by comparing the series for the two classes of functions. For values of the form $n = 4p + 2$, $p = 0, 1, 2, ...$, the planar n-dimensional cosexponential function $f_{nk}(y)$ is related to the n-dimensional cosexponential function $g_{nk}(y)$ by the relation

$$f_{nk}(y) = e^{-\pi k/2} g_{nk}(iy).$$

(55)
Addition theorems for the planar n-dimensional cosexpontential functions can be obtained from the relation \( \exp h_1(y + z) = \exp h_1y \cdot \exp h_1z \), by substituting the expression of the exponentials as given in Eq. (44) for \( k = 1 \), \( e^{h_1y} = f_{n0}(y) + h_1f_{n1}(y) + \cdots + h_{n-1}f_{n,n-1}(y) \),

\[
f_{nk}(y + z) = f_{n0}(y)f_{nk}(z) + f_{n1}(y)f_{n,k-1}(z) + \cdots + f_{nk}(y)f_{n0}(z)
- f_{n,k+1}(y)f_{n,n-1}(z) - f_{n,k+2}(y)f_{n,n-2}(z) - \cdots - f_{n,n-1}(y)f_{n,k+1}(z),
\]

(56)

where \( k = 0, 1, \ldots, n - 1 \). For \( y = z \) the relations (56) take the form

\[
f_{nk}(2y) = f_{n0}(y)f_{nk}(y) + f_{n1}(y)f_{n,k-1}(y) + \cdots + f_{nk}(y)f_{n0}(y)
- f_{n,k+1}(y)f_{n,n-1}(y) - f_{n,k+2}(y)f_{n,n-2}(y) - \cdots - f_{n,n-1}(y)f_{n,k+1}(y),
\]

(57)

where \( k = 0, 1, \ldots, n - 1 \). For \( y = -z \) the relations (56) and (45) yield

\[
f_{n0}(y)f_{n0}(-y) - f_{n1}(y)f_{n,n-1}(-y) - f_{n2}(y)f_{n,n-2}(-y) - \cdots - f_{n,n-1}(y)f_{n1}(-y) = 1, \quad (58)
\]

\[
f_{n0}(y)f_{nk}(-y) + f_{n1}(y)f_{n,k-1}(-y) + \cdots + f_{nk}(y)f_{n0}(-y)
- f_{n,k+1}(y)f_{n,n-1}(-y) - f_{n,k+2}(y)f_{n,n-2}(-y) - \cdots - f_{n,n-1}(y)f_{n,k+1}(-y) = 0,
\]

(59)

where \( k = 1, \ldots, n - 1 \).

From Eq. (43) it can be shown, for even \( k \) and natural numbers \( l \), that

\[
\left( \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{k_{p-2}}[kp/n]g_{np}(y) \right)^l = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{k_{p-2}}[kp/n]g_{np}(ly),
\]

(60)

where \( k = 0, 1, \ldots, n - 1 \). For odd \( k \) and natural numbers \( l \), Eq. (44) implies

\[
\left( \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{k_{p-2}}[kp/n]f_{np}(y) \right)^l = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{k_{p-2}}[kp/n]f_{np}(ly),
\]

(61)

where \( k = 0, 1, \ldots, n - 1 \). For \( k = 1 \) the relation (61) is

\[
\{f_{n0}(y) + h_1f_{n1}(y) + \cdots + h_{n-1}f_{n,n-1}(y)\}^l = f_{n0}(ly) + h_1f_{n1}(ly) + \cdots + h_{n-1}f_{n,n-1}(ly).
\]

(62)
If
\[ a_k = \sum_{p=0}^{n-1} f_{np}(y) \cos \left( \frac{\pi(2k-1)p}{n} \right), \quad (63) \]
and
\[ b_k = \sum_{p=0}^{n-1} f_{np}(y) \sin \left( \frac{\pi(2k-1)p}{n} \right), \quad (64) \]
for \( k = 1, \ldots, n \), where \( f_{np}(y) \) are the planar cosexpontial functions in Eq. (47), it can be shown that
\[ a_k = \exp \left[ y \cos \left( \frac{\pi(2k-1)}{n} \right) \right] \cos \left[ y \sin \left( \frac{\pi(2k-1)}{n} \right) \right], \quad (65) \]
\[ b_k = \exp \left[ y \cos \left( \frac{\pi(2k-1)}{n} \right) \right] \sin \left[ y \sin \left( \frac{\pi(2k-1)}{n} \right) \right], \quad (66) \]
for \( k = 1, \ldots, n \). If
\[ G_k^2 = a_k^2 + b_k^2, \quad (67) \]
from Eqs. (63) and (66) it results that
\[ G_k^2 = \exp \left[ 2y \cos \left( \frac{\pi(2k-1)}{n} \right) \right], \quad (68) \]
for \( k = 1, \ldots, n \). Then the planar \( n \)-dimensional cosexpontial functions have the property that
\[ \prod_{p=1}^{n/2} G_p^2 = 1. \quad (69) \]

The planar \( n \)-dimensional cosexpontial functions are solutions of the \( n^{th} \)-order differential equation
\[ \frac{d^n \zeta}{du^n} = -\zeta, \quad (70) \]
whose solutions are of the form \( \zeta(u) = A_0 f_{n0}(u) + A_1 f_{n1}(u) + \cdots + A_{n-1} f_{n,n-1}(u) \). It can be checked that the derivatives of the planar cosexpontial functions are related by
\[ \frac{df_{n0}}{du} = -f_{n,n-1}, \quad \frac{df_{n1}}{du} = f_{n0}, \quad \ldots, \quad \frac{df_{n,n-2}}{du} = f_{n,n-3}, \quad \frac{df_{n,n-1}}{du} = f_{n,n-2}. \quad (71) \]
5 Exponential and trigonometric forms of planar n-complex numbers

In order to obtain the exponential and trigonometric forms of n-complex numbers, a canonical base \( e_1, \tilde{e}_1, ..., e_{n/2}, \tilde{e}_{n/2} \) for the planar n-complex numbers will be introduced by the relations

\[
\begin{pmatrix}
\vdots \\
e_k \\
\tilde{e}_k \\
\vdots 
\end{pmatrix} = \begin{pmatrix}
\frac{2}{n} & \frac{2}{n} \cos \frac{\pi(2k-1)}{n} & \cdots & \frac{2}{n} \cos \frac{\pi(2k-1)(n-2)}{n} & \frac{2}{n} \cos \frac{\pi(2k-1)(n-1)}{n} \\
0 & \frac{2}{n} \sin \frac{\pi(2k-1)}{n} & \cdots & \frac{2}{n} \sin \frac{\pi(2k-1)(n-2)}{n} & \frac{2}{n} \sin \frac{\pi(2k-1)(n-1)}{n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix} \begin{pmatrix}
1 \\
h_1 \\
\vdots \\
h_{n-1} 
\end{pmatrix}, \tag{72}
\]

where \( k = 1, 2, ..., n/2 \).

The multiplication relations for the bases \( e_k, \tilde{e}_k \) are

\[
e_k^2 = e_k, \tilde{e}_k^2 = -e_k, e_k \tilde{e}_k = \tilde{e}_k, e_k e_l = 0, e_k \tilde{e}_l = 0, \tilde{e}_k \tilde{e}_l = 0, k \neq l, \tag{73}
\]

where \( k, l = 1, ..., n/2 \). The moduli of the bases \( e_k, \tilde{e}_k \) are

\[
|e_k| = \sqrt{\frac{2}{n}}, |\tilde{e}_k| = \sqrt{\frac{2}{n}}. \tag{74}
\]

It can be shown that

\[
x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = \sum_{k=1}^{n/2} (e_k v_k + \tilde{e}_k \tilde{v}_k). \tag{75}
\]

The relation (75) gives the canonical form of a planar n-complex number.

Using the properties of the bases in Eqs. (72) it can be shown that

\[
\exp(\tilde{e}_k \phi_k) = 1 - e_k + e_k \cos \phi_k + \tilde{e}_k \sin \phi_k, \tag{76}
\]

\[
\exp(e_k \ln \rho_k) = 1 - e_k + e_k \rho_k, \tag{77}
\]

By multiplying the relations (76), (77) it results that

\[
\exp \left[ \sum_{k=1}^{n/2} (e_k \ln \rho_k + \tilde{e}_k \phi_k) \right] = \sum_{k=1}^{n/2} (e_k v_k + \tilde{e}_k \tilde{v}_k), \tag{78}
\]

where the fact has ben used that

\[
\sum_{k=1}^{n/2} e_k = 1, \tag{79}
\]
the latter relation being a consequence of Eqs. (72) and (26).

By comparing Eqs. (73) and (78), it can be seen that

\[ x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = \exp \left[ \sum_{k=1}^{n/2} (e_k \ln \rho_k + \tilde{e}_k \phi_k) \right]. \]  

Using the expression of the bases in Eq. (72) yields the exponential form of the n-complex number \( u = x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} \) as

\[ u = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[ -\frac{2}{n} \sum_{k=2}^{n/2} \cos \left( \frac{\pi (2k - 1)p}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2} \tilde{e}_k \phi_k \right\}, \]  

where \( \rho \) is the amplitude defined in Eq. (15), and has according to Eq. (20) the expression

\[ \rho = \left( \rho_1^2 \cdots \rho_{n/2}^2 \right)^{1/n}. \]  

It can be checked with the aid of Eq. (74) that the n-complex number \( u \) can also be written as

\[ x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = \left( \sum_{k=1}^{n/2} e_k \rho_k \right) \exp \left( \sum_{k=1}^{n/2} \tilde{e}_k \phi_k \right). \]  

Writing in Eq. (83) the radius \( \rho_1 \), Eq. (31), as a factor and expressing the variables in terms of the planar angles with the aid of Eq. (29) yields the trigonometric form of the n-complex number \( u \) as

\[ u = d \left( \frac{n}{2} \right)^{1/2} \left( 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-1}} \right)^{-1/2} \left( e_1 + \sum_{k=2}^{n/2} \frac{e_k}{\tan \psi_{k-1}} \right) \exp \left( \sum_{k=1}^{n/2} \tilde{e}_k \phi_k \right). \]  

In Eq. (84), the n-complex number \( u \), written in trigonometric form, is the product of the modulus \( d \), of a part depending on the planar angles \( \psi_1, \ldots, \psi_{n/2-1} \), and of a factor depending on the azimuthal angles \( \phi_1, \ldots, \phi_{n/2} \). Although the modulus of a product of n-complex numbers is not equal in general to the product of the moduli of the factors, it can be checked that the modulus of the factors in Eq. (84) are

\[ \left| e_1 + \sum_{k=2}^{n/2} \frac{e_k}{\tan \psi_{k-1}} \right| = \left( \frac{2}{n} \right)^{1/2} \left( 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-1}} \right)^{1/2}, \]  

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and
\[
\exp \left( \frac{n}{2} \sum_{k=1}^{\tilde{e}_k \phi_k} \right) = 1.
\] (86)

The modulus \( d \) in Eqs. (84) can be expressed in terms of the amplitude \( \rho \) as
\[
d = \rho \frac{2^{(n-2)/2n}}{\sqrt{n}} \left( \tan \psi_1 \cdots \tan \psi_{n/2-1} \right)^{2/n} \left( 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-1}} \right)^{1/2}.
\] (87)

6 Elementary functions of a planar n-complex variable

The logarithm \( u_1 \) of the n-complex number \( u \), \( u_1 = \ln u \), can be defined as the solution of the equation
\[
u = e^{u_1}.
\] (88)

The relation (78) shows that \( \ln u \) exists as an n-complex function with real components for all values of \( x_0, \ldots, x_{n-1} \) for which \( \rho \neq 0 \). The expression of the logarithm, obtained from Eq. (80) is
\[
\ln u = \frac{n}{2} \sum_{k=1}^{\tilde{e}_k \phi_k} (e_k \ln \rho_k + \tilde{e}_k \phi_k).
\] (89)

An expression of the logarithm depending on the amplitude \( \rho \) can be obtained from the exponential forms in Eq. (81) as
\[
\ln u = \ln \rho + \sum_{p=1}^{n-1} h_p \left[ -\frac{2}{n} \sum_{k=2}^{n/2} \cos \left( \frac{\pi (2k-1)p}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2} \tilde{e}_k \phi_k.
\] (90)

The function \( \ln u \) is multivalued because of the presence of the terms \( \tilde{e}_k \phi_k \). It can be inferred from Eqs. (92)–(94) and (97) that
\[
\ln(uu') = \ln u + \ln u',
\] (91)

up to integer multiples of \( 2\pi \tilde{e}_k, k = 1, \ldots, n/2 \).

The power function \( u^m \) can be defined for real values of \( m \) as
\[
u^m = e^{m \ln u}.
\] (92)
Using the expression of ln \( u \) in Eq. (89) yields

\[
    u^m = \sum_{k=1}^{n/2} p_k^m (e_k \cos m\phi_k + \tilde{e}_k \sin m\phi_k). 
\]

(93)

The power function is multivalued unless \( m \) is an integer. For integer \( m \), it can be inferred from Eq. (91) that

\[
    (uu')^m = u^m u'^m. 
\]

(94)

The trigonometric functions \( \cos u \) and \( \sin u \) of an \( n \)-complex variable \( u \) are defined by the series

\[
    \cos u = 1 - u^2/2! + u^4/4! + \cdots, 
\]

(95)

\[
    \sin u = u - u^3/3! + u^5/5! + \cdots. 
\]

(96)

It can be checked by series multiplication that the usual addition theorems hold for the \( n \)-complex numbers \( u, u' \),

\[
    \cos(u + u') = \cos u \cos u' - \sin u \sin u', 
\]

(97)

\[
    \sin(u + u') = \sin u \cos u' + \cos u \sin u'. 
\]

(98)

In order to obtain expressions for the trigonometric functions of \( n \)-complex variables, these will be expressed with the aid of the imaginary unit \( i \) as

\[
    \cos u = \frac{1}{2}(e^{iu} + e^{-iu}), \quad \sin u = \frac{1}{2i}(e^{iu} - e^{-iu}). 
\]

(99)

The imaginary unit \( i \) is used for the convenience of notations, and it does not appear in the final results. The validity of Eq. (99) can be checked by comparing the series for the two sides of the relations. Since the expression of the exponential function \( e^{h_k y} \) in terms of the units \( 1, h_1, \ldots, h_{n-1} \) given in Eq. (14) depends on the planar cosexponential functions \( f_{np}(y) \), the expression of the trigonometric functions will depend on the functions

\[
    f_{p+}^{(c)}(y) = (1/2)[f_{np}(iy) + f_{np}(-iy)] \quad \text{and} \quad f_{p-}^{(c)}(y) = (1/2i)[f_{np}(iy) - f_{np}(-iy)], 
\]

\[
    \cos(h_k y) = \sum_{p=0}^{n-1} (-1)^{[kp/n]h_{kp-n}[kp/n]} f_{p+}^{(c)}(y), 
\]

(100)
\[
\sin(h_k y) = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{kp-n[kp/n]} f_{p+}^{(c)}(y),
\]

where

\[
f_{p+}^{(c)}(y) = \frac{1}{n} \sum_{l=1}^{n} \left\{ \cos \left[ y \cos \left( \frac{(2l-1)}{n} \right) \right] \cosh \left[ y \sin \left( \frac{(2l-1)}{n} \right) \right] \cos \left( \frac{(2l-1)p}{n} \right) \\
- \sin \left[ y \cos \left( \frac{(2l-1)}{n} \right) \right] \sinh \left[ y \sin \left( \frac{(2l-1)}{n} \right) \right] \sin \left( \frac{(2l-1)p}{n} \right) \right\}, \tag{102}
\]

\[
f_{p-}^{(c)}(y) = \frac{1}{n} \sum_{l=1}^{n} \left\{ \sin \left[ y \cos \left( \frac{(2l-1)}{n} \right) \right] \cosh \left[ y \sin \left( \frac{(2l-1)}{n} \right) \right] \cos \left( \frac{(2l-1)p}{n} \right) \\
+ \cos \left[ y \cos \left( \frac{(2l-1)}{n} \right) \right] \sinh \left[ y \sin \left( \frac{(2l-1)}{n} \right) \right] \sin \left( \frac{(2l-1)p}{n} \right) \right\}, \tag{103}
\]

The hyperbolic functions \( \cosh u \) and \( \sinh u \) of the \( n \)-complex variable \( u \) can be defined by the series

\[
cosh u = 1 + \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots, \tag{104}
\]

\[
\sinh u = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots. \tag{105}
\]

It can be checked by series multiplication that the usual addition theorems hold for the \( n \)-complex numbers \( u, u' \),

\[
cosh(u + u') = \cosh u \cosh u' + \sinh u \sinh u', \tag{106}
\]

\[
\sinh(u + u') = \sinh u \cosh u' + \cosh u \sinh u'. \tag{107}
\]

In order to obtain expressions for the hyperbolic functions of \( n \)-complex variables, these will be expressed as

\[
cosh u = \frac{1}{2}(e^u + e^{-u}), \quad \sinh u = \frac{1}{2}(e^u - e^{-u}). \tag{108}
\]

The validity of Eq. (108) can be checked by comparing the series for the two sides of the relations. Since the expression of the exponential function \( e^{h_k y} \) in terms of the units 1, \( h_1, \ldots, h_{n-1} \) given in Eq. (44) depends on the planar cosexpontential functions \( f_{np}(y) \), the expression of the hyperbolic functions will depend on the even part \( f_{p+}(y) = (1/2)[f_{np}(y) + f_{np}(-y)] \) and on the odd part \( f_{p-}(y) = (1/2)[f_{np}(y) - f_{np}(-y)] \) of \( f_{np} \),

\[
cosh(h_k y) = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{kp-n[kp/n]} f_{p+}(y), \tag{109}
\]
\[
\sinh(h_k y) = \sum_{p=0}^{n-1} (-1)^{[kp/n]} h_{kp-n[kp/n]} f_p - (y),
\]

where
\[
f_{p+}(y) = \frac{1}{n} \sum_{(2l-1)=1}^{n} \left\{ \cosh \left[ y \cos \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \cos \left[ y \sin \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \cos \left( \pi \left( \frac{2l-1}{n} \right) p \right) \\
+ \sinh \left[ y \cos \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \sin \left[ y \sin \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \sin \left( \pi \left( \frac{2l-1}{n} \right) p \right) \right\},
\]
\[
f_{p-}(y) = \frac{1}{n} \sum_{(2l-1)=1}^{n} \left\{ \sinh \left[ y \cos \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \cos \left[ y \sin \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \cos \left( \pi \left( \frac{2l-1}{n} \right) p \right) \\
+ \cosh \left[ y \cos \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \sin \left[ y \sin \left( \pi \left( \frac{2l-1}{n} \right) \right) \right] \sin \left( \pi \left( \frac{2l-1}{n} \right) p \right) \right\}.
\]

The exponential, trigonometric and hyperbolic functions can also be expressed with the aid of the bases introduced in Eq. (72). Using the expression of the n-complex number in Eq. (72) yields for the exponential of the n-complex variable \(u\)
\[
e^u = \sum_{k=1}^{n/2} e^{v_k} (e_k \cos \tilde{v}_k + \tilde{e}_k \sin \tilde{v}_k).
\]

The trigonometric functions can be obtained from Eq. (113) with the aid of Eqs. (99).

The trigonometric functions of the n-complex variable \(u\) are
\[
\cos u = \sum_{k=1}^{n/2} (e_k \cos v_k \cosh \tilde{v}_k - \tilde{e}_k \sin v_k \sinh \tilde{v}_k),
\]
\[
\sin u = \sum_{k=1}^{n/2} (e_k \sin v_k \cosh \tilde{v}_k + \tilde{e}_k \cos v_k \sinh \tilde{v}_k).
\]

The hyperbolic functions can be obtained from Eq. (113) with the aid of Eqs. (108).

The hyperbolic functions of the n-complex variable \(u\) are
\[
cosh u = \sum_{k=1}^{n/2} (e_k \cos v_k \cos \tilde{v}_k + \tilde{e}_k \sin v_k \sin \tilde{v}_k),
\]
\[
\sinh u = \sum_{k=1}^{n/2} (e_k \sin v_k \cos \tilde{v}_k + \tilde{e}_k \cos v_k \sin \tilde{v}_k).
\]
7  Power series of planar n-complex numbers

An n-complex series is an infinite sum of the form

\[ a_0 + a_1 + a_2 + \cdots + a_n + \cdots, \]  

(118)

where the coefficients \( a_n \) are n-complex numbers. The convergence of the series (118) can be defined in terms of the convergence of its \( n \) real components. The convergence of a n-complex series can also be studied using n-complex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus \( d = |u| \) of an n-complex number \( u \) has been defined in Eq. (22). Since

\[ |x_0| \leq |u|, |x_1| \leq |u|, \ldots, |x_{n-1}| \leq |u|, \]

a property of absolute convergence established via a comparison theorem based on the modulus of the series (118) will ensure the absolute convergence of each real component of that series.

The modulus of the sum \( u_1 + u_2 \) of the n-complex numbers \( u_1, u_2 \) fulfils the inequality

\[ ||u'| - |u''|| \leq |u' + u''| \leq |u'| + |u''|. \]  

(119)

For the product, the relation is

\[ |u'u''| \leq \sqrt{\frac{n}{2}} |u'||u''|, \]  

(120)

as can be shown from Eq. (27). The relation (120) replaces the relation of equality extant between 2-dimensional regular complex numbers.

For \( u = u' \) Eq. (120) becomes

\[ |u^2| \leq \sqrt{\frac{n}{2}} |u|^2, \]  

(121)

and in general

\[ |u^l| \leq \left( \frac{n}{2} \right)^{(l-1)/2} |u|^l, \]  

(122)

where \( l \) is a natural number. From Eqs. (120) and (122) it results that

\[ |au^l| \leq \left( \frac{n}{2} \right)^{l/2} |a||u|^l. \]  

(123)

A power series of the n-complex variable \( u \) is a series of the form

\[ a_0 + a_1 u + a_2 u^2 + \cdots + a_l u^l + \cdots. \]  

(124)
Since
\[
|\sum_{l=0}^{\infty} a_l u^l| \leq \sum_{l=0}^{\infty} (n/2)^{l/2} |a_l| |u|^l,
\]
a sufficient condition for the absolute convergence of this series is that
\[
\lim_{l \to \infty} \frac{\sqrt{n/2} |a_{l+1}| |u|}{|a_l|} < 1.
\]
Thus the series is absolutely convergent for
\[
|u| < c,
\]
where
\[
c = \lim_{l \to \infty} \frac{|a_l|}{\sqrt{n/2} |a_{l+1}|}.
\]

The convergence of the series (124) can be also studied with the aid of the formula (93) which is valid for any values of \(x_0, \ldots, x_{n-1}\), as mentioned previously. If \(a_l = \sum_{p=0}^{n-1} h_p a_{lp}\), and
\[
A_{lk} = \sum_{p=0}^{n-1} a_{lp} \cos \frac{\pi (2k - 1)p}{n},
\]
\[
\tilde{A}_{lk} = \sum_{p=0}^{n-1} a_{lp} \sin \frac{\pi (2k - 1)p}{n},
\]
where \(k = 1, \ldots, n/2\), the series (124) can be written as
\[
\sum_{l=0}^{n/2} \left[\sum_{k=1}^{n/2} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^l\right].
\]
The series in Eq. (131) can be regarded as the sum of the \(n/2\) series obtained from each value of \(k\), so that the series in Eq. (124) is absolutely convergent for
\[
\rho_k < c_k,
\]
for \(k = 1, \ldots, n/2\), where
\[
c_k = \lim_{l \to \infty} \frac{\left[\tilde{A}_{l+1,k}^2 + \tilde{A}_{l,k}^2\right]^{1/2}}{\left[A_{l+1,k}^2 + A_{l,k}^2\right]^{1/2}}.
\]
The relations (132) show that the region of convergence of the series (124) is an \(n\)-dimensional cylinder.

It can be shown that \(c = \sqrt{2/n} \min(c_+, c_-, c_1, \ldots, c_{n/2-1})\), where \(\min\) designates the smallest of the numbers in the argument of this function. Using the expression of \(|u|\) in Eq. (27), it can be seen that the spherical region of convergence defined in Eqs. (127), (128) is a subset of the cylindrical region of convergence defined in Eqs. (132) and (133).
8 Analytic functions of planar n-complex variables

The derivative of a function \( f(u) \) of the n-complex variables \( u \) is defined as a function \( f'(u) \) having the property that

\[
|f(u) - f(u_0) - f'(u_0)(u - u_0)| \to 0 \text{ as } |u - u_0| \to 0. \tag{134}
\]

If the difference \( u - u_0 \) is not parallel to one of the nodal hypersurfaces, the definition in Eq. (134) can also be written as

\[
f'(u_0) = \lim_{u \to u_0} \frac{f(u) - f(u_0)}{u - u_0}. \tag{135}
\]

The derivative of the function \( f(u) = u^m \), with \( m \) an integer, is \( f'(u) = mu^{m-1} \), as can be seen by developing \( u^m = [u_0 + (u - u_0)]^m \) as

\[
u^m = \sum_{p=0}^{m} \frac{m!}{p!(m-p)!} u_0^{m-p}(u - u_0)^p, \tag{136}
\]

and using the definition (134).

If the function \( f'(u) \) defined in Eq. (134) is independent of the direction in space along which \( u \) is approaching \( u_0 \), the function \( f(u) \) is said to be analytic, analogously to the case of functions of regular complex variables. The function \( u^m \), with \( m \) an integer, of the n-complex variable \( u \) is analytic, because the difference \( u^m - u_0^m \) is always proportional to \( u - u_0 \), as can be seen from Eq. (136). Then series of integer powers of \( u \) will also be analytic functions of the n-complex variable \( u \), and this result holds in fact for any commutative algebra.

If an analytic function is defined by a series around a certain point, for example \( u = 0 \), as

\[
f(u) = \sum_{k=0}^{\infty} a_k u^k, \tag{137}
\]

an expansion of \( f(u) \) around a different point \( u_0 \),

\[
f(u) = \sum_{k=0}^{\infty} c_k (u - u_0)^k, \tag{138}
\]

can be obtained by substituting in Eq. (137) the expression of \( u^k \) according to Eq. (136). Assuming that the series are absolutely convergent so that the order of the terms can be
modified and ordering the terms in the resulting expression according to the increasing powers of \( u - u_0 \) yields

\[
f(u) = \sum_{k,l=0}^{\infty} \frac{(k+l)!}{k!l!} a_{k+l} u_0^l (u - u_0)^k.
\] (139)

Since the derivative of order \( k \) at \( u = u_0 \) of the function \( f(u) \), Eq. (137), is

\[
f^{(k)}(u_0) = \sum_{l=0}^{\infty} \frac{(k+l)!}{l!} a_{k+l} u_0^l,
\] (140)

the expansion of \( f(u) \) around \( u = u_0 \), Eq. (139), becomes

\[
f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(u_0) (u - u_0)^k,
\] (141)

which has the same form as the series expansion of 2-dimensional complex functions. The relation (141) shows that the coefficients in the series expansion, Eq. (138), are

\[
c_k = \frac{1}{k!} f^{(k)}(u_0).
\] (142)

The rules for obtaining the derivatives and the integrals of the basic functions can be obtained from the series of definitions and, as long as these series expansions have the same form as the corresponding series for the 2-dimensional complex functions, the rules of derivation and integration remain unchanged.

If the \( n \)-complex function \( f(u) \) of the \( n \)-complex variable \( u \) is written in terms of the real functions \( P_k(x_0, ..., x_{n-1}), k = 0, 1, ..., n-1 \) of the real variables \( x_0, x_1, ..., x_{n-1} \) as

\[
f(u) = \sum_{k=0}^{n-1} h_k P_k(x_0, ..., x_{n-1}),
\] (143)

where \( h_0 = 1 \), then relations of equality exist between the partial derivatives of the functions \( P_k \). The derivative of the function \( f \) can be written as

\[
\lim_{\Delta u \to 0} \frac{1}{\Delta u} \sum_{k=0}^{n-1} \left( h_k \sum_{l=0}^{n-1} \frac{\partial P_k}{\partial x_l} \Delta x_l \right),
\] (144)

where

\[
\Delta u = \sum_{k=0}^{n-1} h_l \Delta x_l.
\] (145)

The relations between the partials derivatives of the functions \( P_k \) are obtained by setting successively in Eq. (144) \( \Delta u = h_l \Delta x_l \), for \( l = 0, 1, ..., n-1 \), and equating the resulting expressions. The relations are

\[
\frac{\partial P_k}{\partial x_0} = \frac{\partial P_{k+1}}{\partial x_1} = \cdots = \frac{\partial P_{n-1}}{\partial x_{n-k}} = - \frac{\partial P_0}{\partial x_{n-k}} = \cdots = - \frac{\partial P_{k-1}}{\partial x_{n-1}},
\] (146)
for \( k = 0, 1, ..., n - 1 \). The relations (146) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (146) that the components \( P_k \) fulfill the second-order equations

\[
\frac{\partial^2 P_k}{\partial x_0 \partial x_l} = \frac{\partial^2 P_k}{\partial x_1 \partial x_{l-1}} = \cdots = \frac{\partial^2 P_k}{\partial x_{[l/2]} \partial x_{l-[l/2]}}
\]

\[
= - \frac{\partial^2 P_k}{\partial x_{l+1} \partial x_{n-1}} = - \frac{\partial^2 P_k}{\partial x_{l+2} \partial x_{n-2}} = \cdots = - \frac{\partial^2 P_k}{\partial x_{l+1+[(n-l-2)/2]} \partial x_{n-1-[m-l-2)/2]}}
\]

(147)

for \( k, l = 0, 1, ..., n - 1 \).

### 9 Integrals of planar n-complex functions

The singularities of n-complex functions arise from terms of the form \( 1/(u - u_0)^m \), with \( m > 0 \). Functions containing such terms are singular not only at \( u = u_0 \), but also at all points of the hypersurfaces passing through the pole \( u_0 \) and which are parallel to the nodal hypersurfaces.

The integral of an n-complex function between two points \( A, B \) along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free of singularities is zero,

\[
\oint f(u) du = 0,
\]

where it is supposed that a surface \( \Sigma \) spanning the closed loop \( \Gamma \) is not intersected by any of the hypersurfaces associated with the singularities of the function \( f(u) \). Using the expression, Eq. (143), for \( f(u) \) and the fact that

\[
\int f(u) du = \sum_{k=0}^{n-1} h_k dx_k,
\]

(149)

the explicit form of the integral in Eq. (148) is

\[
\oint f(u) du = \oint \sum_{k=0}^{n-1} h_k \int_{l=0}^{n-1} (-1)^{[(n-k-1+l)/n]} P_l dx_{k-l+n[(n-k-1+l)/n]}.
\]

If the functions \( P_k \) are regular on a surface \( \Sigma \) spanning the loop \( \Gamma \), the integral along the loop \( \Gamma \) can be transformed in an integral over the surface \( \Sigma \) of terms of the form

\[
\partial P_l / \partial x_{k-m+n[(n-k+m-1)/n]} - (-1)^s \partial P_m / \partial x_{k-l+n[(n-k+l-1)/n]},
\]

where \( s = [(n-k+m-1)/n] - [(n-k+l-1)/n] \). These terms are equal to zero by Eqs. (146), and this proves Eq. (148).
The integral of the function \((u - u_0)^m\) on a closed loop \(\Gamma\) is equal to zero for \(m\) a positive or negative integer not equal to -1,

\[ \oint_{\Gamma} (u - u_0)^m du = 0, \quad m \text{ integer, } m \neq -1. \tag{151} \]

This is due to the fact that \(\int (u - u_0)^m du = (u - u_0)^{m+1}/(m+1)\), and to the fact that the function \((u - u_0)^{m+1}\) is singlevalued for \(n\) an integer.

The integral \(\oint_{\Gamma} du/(u - u_0)\) can be calculated using the exponential form, Eq. (81), for the difference \(u - u_0\),

\[ u - u_0 = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[ -\frac{2}{n} \sum_{k=2}^{n/2} \cos \left( \frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2} \tilde{e}_k \phi_k \right\}. \tag{152} \]

Thus the quantity \(du/(u - u_0)\) is

\[ \frac{du}{u - u_0} = \frac{d\rho}{\rho} + \sum_{p=1}^{n-1} h_p \left[ -\frac{2}{n} \sum_{k=2}^{n/2} \cos \left( \frac{2\pi kp}{n} \right) d\ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2} \tilde{e}_k d\phi_k. \tag{153} \]

Since \(\rho\) and \(\ln(\tan \psi_{k-1})\) are singlevalued variables, it follows that \(\oint_{\Gamma} d\rho/\rho = 0\), and \(\oint_{\Gamma} d(\ln \tan \psi_{k-1}) = 0\). On the other hand, since \(\phi_k\) are cyclic variables, they may give contributions to the integral around the closed loop \(\Gamma\).

The expression of \(\oint_{\Gamma} du/(u - u_0)\) can be written with the aid of a functional which will be called \(\text{int}(M,C)\), defined for a point \(M\) and a closed curve \(C\) in a two-dimensional plane, such that

\[ \text{int}(M,C) = \begin{cases} 1 & \text{if } M \text{ is an interior point of } C, \\ 0 & \text{if } M \text{ is exterior to } C. \end{cases} \tag{154} \]

With this notation the result of the integration on a closed path \(\Gamma\) can be written as

\[ \oint_{\Gamma} \frac{du}{u - u_0} = \sum_{k=1}^{n/2} 2\pi \tilde{e}_k \text{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}), \tag{155} \]

where \(u_{0\xi_k\eta_k}\) and \(\Gamma_{\xi_k\eta_k}\) are respectively the projections of the point \(u_0\) and of the loop \(\Gamma\) on the plane defined by the axes \(\xi_k\) and \(\eta_k\), as shown in Fig. 3.

If \(f(u)\) is an analytic \(n\)-complex function which can be expanded in a series as written in Eq. (138), and the expansion holds on the curve \(\Gamma\) and on a surface spanning \(\Gamma\), then from Eqs. (151) and (155) it follows that

\[ \oint_{\Gamma} \frac{f(u)du}{u - u_0} = 2\pi f(u_0) \sum_{k=1}^{n/2} \tilde{e}_k \text{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}). \tag{156} \]
Substituting in the right-hand side of Eq. (156) the expression of \( f(u) \) in terms of the real components \( P_k \), Eq. (143), yields

\[
\oint_{\Gamma} f(u) du = \frac{2}{n} \sum_{k=1}^{n} \sum_{l=0}^{n-1} h_l \\
\sum_{p=1}^{n-1} (-1)^{[(l-p)/n]} \sin \left[ \frac{\pi (2k-1) p}{n} \right] P_{n-p+1-n[(n-p+l)/n]}(u_0) \text{int}(u_0 \xi_k \eta_k, \Gamma \xi_k \eta_k). \tag{157}
\]

If the integral in Eq. (157) is written as

\[
\oint_{\Gamma} f(u) du = \sum_{l=0}^{n-1} h_l I_l, \tag{158}
\]

it can be checked that

\[
\sum_{l=0}^{n-1} I_l = 0. \tag{159}
\]

If \( f(u) \) can be expanded as written in Eq. (138) on \( \Gamma \) and on a surface spanning \( \Gamma \), then from Eqs. (151) and (155) it also results that

\[
\oint_{\Gamma} f(u) du = \sum_{l=0}^{n-1} h_l I_l, \tag{158}
\]

where the fact has been used that the derivative \( f^{(n)}(u_0) \) is related to the expansion coefficient in Eq. (138) according to Eq. (142).

If a function \( f(u) \) is expanded in positive and negative powers of \( (u-u_l) \), where \( u_l \) are \( n \)-complex constants, \( l \) being an index, the integral of \( f \) on a closed loop \( \Gamma \) is determined by the terms in the expansion of \( f \) which are of the form \( r_l/(u-u_l) \),

\[
f(u) = \cdots + \sum_{l} \frac{r_l}{u-u_l} + \cdots. \tag{161}
\]

Then the integral of \( f \) on a closed loop \( \Gamma \) is

\[
\oint_{\Gamma} f(u) du = 2\pi \sum_{l} \sum_{k=1}^{n/2} \tilde{e}_k \text{int}(u_l \xi_k \eta_k, \Gamma \xi_k \eta_k) r_l. \tag{162}
\]

### 10 Factorization of planar \( n \)-complex polynomials

A polynomial of degree \( m \) of the \( n \)-complex variable \( u \) has the form

\[
P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m, \tag{163}
\]
where $a_l$, for $l = 1, \ldots, m$, are in general $n$-complex constants. If $a_l = \sum_{p=0}^{n-1} h_p a_{lp}$, and with the notations of Eqs. (123), (130) applied for $l = 1, \ldots, m$, the polynomial $P_m(u)$ can be written as

$$P_m = \frac{n}{2} \sum_{k=1}^{m/2} \left\{ (e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^{m} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} \right\},$$

(164)

where the constants $A_{lk}, \tilde{A}_{lk}$ are real numbers.

The polynomials of degree $m$ in $e_k v_k + \tilde{e}_k \tilde{v}_k$ in Eq. (164) can always be written as a product of linear factors of the form $e_k (v_k - v_{kp}) + \tilde{e}_k (\tilde{v}_k - \tilde{v}_{kp})$, where the constants $v_{kp}, \tilde{v}_{kp}$ are real,

$$(e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^{m} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk})(e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} = \prod_{p=1}^{m} \{ e_k (v_k - v_{kp}) + \tilde{e}_k (\tilde{v}_k - \tilde{v}_{kp}) \}.$$  (165)

Then the polynomial $P_m$ can be written as

$$P_m = \sum_{k=1}^{n/2} \prod_{p=1}^{m} \{ e_k (v_k - v_{kp}) + \tilde{e}_k (\tilde{v}_k - \tilde{v}_{kp}) \}.$$  (166)

Due to the relations (73), the polynomial $P_m(u)$ can be written as a product of factors of the form

$$P_m(u) = \prod_{p=1}^{m} \left\{ \sum_{k=1}^{n/2} \{ e_k (v_k - v_{kp}) + \tilde{e}_k (\tilde{v}_k - \tilde{v}_{kp}) \} \right\}.$$  (167)

This relation can be written with the aid of Eq. (73) as

$$P_m(u) = \prod_{p=1}^{m} (u - u_p),$$  (168)

where

$$u_p = \sum_{k=1}^{n/2} (e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}),$$  (169)

for $p = 1, \ldots, m$. For a given $k$, the roots $e_k v_{k1} + \tilde{e}_k \tilde{v}_{k1}, \ldots, e_k v_{km} + \tilde{e}_k \tilde{v}_{km}$ defined in Eq. (165) may be ordered arbitrarily. This means that Eq. (169) gives sets of $m$ roots $u_1, \ldots, u_m$ of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$ are ordered according to $p$ for each value of $k$. Thus, while the $n$-complex components in Eq. (165) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors.
If $P(u) = u^2 + 1$, the degree is $m = 2$, the coefficients of the polynomial are $a_1 = 0, a_2 = 1$, the $n$-complex components of $a_2$ are $a_{20} = 1, a_{21} = 0, ..., a_{2n-1} = 0$, the components $A_{2k}, \tilde{A}_{2k}$ calculated according to Eqs. (129), (130) are $A_{2k} = 1, \tilde{A}_{2k} = 0, k = 1, ..., n/2$. The left-hand side of Eq. (165) has the form $(e_kv_k + \tilde{e}_k\tilde{v}_k)^2 + e_k$, and since $e_k = -\tilde{e}_k^2$, the right-hand side of Eq. (165) is \{(e_kv_k + \tilde{e}_k(\tilde{v}_k + 1)) \{e_kv_k + \tilde{e}_k(\tilde{v}_k - 1)\}, so that $v_{kp} = 0, \tilde{v}_{kp} = \pm 1, k = 1, ..., n/2, p = 1, 2$. Then Eq. (166) has the form $u^2 + 1 = \sum_{k=1}^{n/2} \{e_kv_k + \tilde{e}_k(\tilde{v}_k + 1)\} \{e_kv_k + \tilde{e}_k(\tilde{v}_k - 1)\}$. The factorization in Eq. (168) is $u^2 + 1 = (u - u_1)(u - u_2)$, where $u_1 = \pm \tilde{e}_1 \pm \tilde{e}_2 \pm \cdots \pm \tilde{e}_{n/2}, u_2 = -u_1$, so that there are $2^{n/2-1}$ independent sets of roots $u_1, u_2$ of $u^2 + 1$. It can be checked that $(\pm \tilde{e}_1 \pm \tilde{e}_2 \pm \cdots \pm \tilde{e}_{n/2})^2 = -e_1 - e_2 - \cdots - e_{n/2} = -1$.

11 Representation of planar $n$-complex numbers by irreducible matrices

If the unitary matrix written in Eq. (23) is called $T$, it can be shown that the matrix $TUT^{-1}$ has the form

$$TUT^{-1} = \begin{pmatrix}
V_1 & 0 & \cdots & 0 \\
0 & V_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & V_{n/2}
\end{pmatrix},
$$

(170)

where $U$ is the matrix in Eq. (38) used to represent the $n$-complex number $u$. In Eq. (170), the matrices $V_k$ are the matrices

$$V_k = \begin{pmatrix}
v_k & \tilde{v}_k \\
-\tilde{v}_k & v_k
\end{pmatrix},
$$

(171)

for $k = 1, ..., n/2$, where $v_k, \tilde{v}_k$ are the variables introduced in Eqs. (18) and (19), and the symbols 0 denote the matrix

$$\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
$$

(172)

The relations between the variables $v_k, \tilde{v}_k$ for the multiplication of $n$-complex numbers have been written in Eq. (85). The matrix $TUT^{-1}$ provides an irreducible representation \cite{8} of
the $n$-complex number $u$ in terms of matrices with real coefficients. For $n = 2$, Eqs. (18) and (19) give $v_1 = x_0, \tilde{v}_1 = x_1$, and Eq. (72) gives $e_1 = 1, \tilde{e}_1 = h_1$, where according to Eq. (1) $h_1^2 = -1$, so that the matrix $V_1$, Eq. (171), is

$$v_1 = \begin{pmatrix} x_0 & x_1 \\ -x_1 & x_0 \end{pmatrix},$$

(173)

which shows that, for $n = 2$, the hypercomplex numbers $x_0 + h_1x_1$ are identical to the usual 2-dimensional complex numbers $x + iy$.

12 Conclusions

The operations of addition and multiplication of the $n$-complex numbers introduced in this work have a geometric interpretation based on the amplitude $\rho$, the modulus $d$, the planar angles $\psi_k$ and the azimuthal angles $\phi_k$. The $n$-complex numbers can be written in exponential and trigonometric forms with the aid of these variables. The $n$-complex functions defined by series of powers are analytic, and the partial derivatives of the components of the $n$-complex functions are closely related. The integrals of $n$-complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the $n$-complex numbers depends on the cyclic variables $\phi_k$ leads to the concept of pole and residue for integrals on closed paths. The polynomials of $n$-complex variables can always be written as products of linear factors, although the factorization is not unique.

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FIGURE CAPTIONS

Fig. 1. Representation of the hypercomplex bases $1, h_1, ..., h_{n-1}$ by points on a circle at the angles $\alpha_k = \pi k/n$. The product $h_j h_k$ will be represented by the point of the circle at the angle $\pi(j + k)/2n$, $j, k = 0, 1, ..., n - 1$. If $\pi \leq \pi(j + k)/2n \leq 2\pi$, the point is opposite to the basis $h_l$ of angle $\alpha_l = \pi(j + k)/n - \pi$.

Fig. 2. Radial distance $\rho_k$ and azimuthal angle $\phi_k$ in the plane of the axes $v_k, \tilde{v}_k$, and planar angle $\psi_{k-1}$ between the line $OA_{1k}$ and the 2-dimensional plane defined by the axes $v_k, \tilde{v}_k$. $A_k$ is the projection of the point $A$ on the plane of the axes $v_k, \tilde{v}_k$, and $A_{1k}$ is the projection of the point $A$ on the 4-dimensional space defined by the axes $v_1, \tilde{v}_1, v_k, \tilde{v}_k$.

Fig. 3. Integration path $\Gamma$ and pole $u_0$, and their projections $\Gamma_{\xi_k \eta_k}$ and $u_0_{\xi_k \eta_k}$ on the plane $\xi_k \eta_k$. 
Fig. 1
Fig. 2
Fig. 3