Violations of the String Hypothesis

in the

Solutions of the Bethe Ansatz Equations in the XXX-Heisenberg Model

Karl Isler*

Instituut voor Theoretische Fysica

Rijksuniversiteit Utrecht

Princetonplein 5

Utrecht, Netherlands

and

M. B. Paranjape,

Institut für Theoretische Physik,

Universität Innsbruck, Technikerstrasse 25, Innsbruck, Austria, A-6020

and

Laboratoire de physique nucléaire†, Département de physique

Université de Montréal, C. P. 6128 succ. “A”

Montréal, Québec, Canada, H3C 3J7

Abstract

We study the equations for the quasi-momenta which characterize the wave-functions in the Bethe ansatz for the XXX-Heisenberg model. We show in a simple analytical fashion, that the usual “string hypothesis” incorrectly predicts the number of real solutions and the
number of complex solutions for \( N > 21 \) in the sector with two spins flipped, confirming the work of Essler et al. Two complex pair solutions drop out and form two additional real pair solutions. We also introduce a new set of variables which allows the equations to be written as a single polynomial equation in one variable. We consider in some detail the case of three spins flipped.

The XXX-Heisenberg model appears to be fundamental to many of the recent interesting developments in modern quantum field theory and mathematical physics. Integrable models, conformal field theories and quantum groups are seen to arise from various limits, extensions and generalizations\(^1\). It is, in fact, the original model considered by Bethe\(^2\), which gave rise to the celebrated “Bethe ansatz” solutions. Recently, there appeared an article\(^3\), studying the Bethe ansatz equations in the two particle sector of the spin one-half model. It was shown that, as the number of lattice sites, \( N \), increases past 21, two new real pair solutions appear. These solutions do not fit into the conventional scheme of classification of the solutions to the Bethe ansatz equations which causes some anxiety as to the verity of the completeness of the full set of \( SU(2) \) extended Bethe ansatz states. The full set of states, just by counting, should number \( 2^N \). This, however, is not a problem. As the real pair solutions appear, complex pair solutions disappear, conserving the total number of states. We analyze the Bethe ansatz equations analytically and confirm their result; two complex pair solutions disappear simultaneously giving two new real pair solutions.

The spin \( \frac{1}{2} \) XXX-Heisenberg model with \( N \) spins corresponds to the Hamiltonian

\[
H = \frac{J}{4} \sum_{i=1}^{N} (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - 1)
\]
with $\vec{\sigma}_{N+1} = \vec{\sigma}_1$, and the $\vec{\sigma}_i$ are just the Pauli matrices for each $i$. The Bethe ansatz in the $M$ particle sector corresponds to eigenfunctions of the form

$$|\Psi\rangle = \sum_{x_1 < x_2 < \cdots < x_M} \psi(x_1, \cdots, x_M) \sigma_{x_1}^- \sigma_{x_2}^- \sigma_{x_3}^- \cdots \sigma_{x_M}^- |0\rangle$$

with

$$\psi(x_1, \cdots, x_M) = \sum_{P \in S_M} e^{\{i \sum_{j=1}^M k_{P(j)}x_j + i \sum_{j<l} P(j) > P(l) \phi_{P(j), P(l)}\}}.$$ 

$|0\rangle$ is the ferromagnetic state with all spins up, $P$ is a permutation on $M$ objects and $\phi_{i,j}$ satisfies

$$2 \cot\left(\frac{\phi_{i,j}}{2}\right) = \cot\left(\frac{k_i}{2}\right) - \cot\left(\frac{k_j}{2}\right).$$

$k_1 \cdots k_M \in [0, 2\pi]$ are the quasi-momenta or spectral parameters. The wave-function is symmetric under interchange of any two coordinates $x_j$.

Periodic boundary conditions

$$\psi(x_1, \cdots, x_{M-1}, N+1) = \psi(1, x_1, \cdots, x_{M-1})$$

imply the $M$ coupled equations

$$e^{ik_j N} = \prod_{l=1}^M e^{i\phi_{j,l}} \quad j = 1, 2, \cdots, M.$$ 

The usual set of variables considered are $\Lambda_j = \cot\left(\frac{k_j}{2}\right)$ which give the equivalent form for the equations

$$(e(\Lambda_j))^N = \prod_{l=1}^M e^{\left(\frac{\Lambda_j - \Lambda_l}{2}\right)} \quad j = 1, 2, \cdots, M,$$

with

$$e(\Lambda) = \frac{\Lambda + i}{\Lambda - i}.$$
The energy of the state is then given by

\[ E(\Lambda_1, \cdots, \Lambda_M) = \sum_{j=1}^{M} \frac{-2J}{\Lambda_j^2 + 1}. \tag{9} \]

In the large \( N \) limit, the “string hypothesis”\(^4\) states that, for fixed \( M \), any solution \( \Lambda_1, \cdots, \Lambda_M \) consists of “strings” of the form

\[ \Lambda_{\alpha}^{n,j} = \Lambda_{\alpha}^{n} + i(n + 1 - 2j) + o(e^{-\delta N}) \quad j = 1, \cdots, n \tag{10} \]

where \( n \geq 1 \) gives the length of the string, \( \alpha \) labels strings of a given length, \( j \) specifies the imaginary part of \( \Lambda \) and \( \delta > 0 \). With such a hypothesis, the Bethe ansatz equations only involve the real parts, \( \Lambda_{\alpha}^{n} \). The solutions are parametrized by (half-odd) integer numbers \( I_{\alpha}^{n} \) for \( N - M_n \) (even) odd, where \( M_n \) is the total number of strings of length \( n \). Clearly \( \sum_{n=1}^{\infty} nM_n = M \). It is generally believed that there is a \( 1-1 \) correspondence between solutions of the Bethe ansatz equations and sets of independent, non-repeating integers \( I_{\alpha}^{n} \), \( (I_{\alpha}^{n} \neq I_{\beta}^{n} \text{ for } \alpha \neq \beta, \text{i.e., no two strings of the same length contain the same integer, within one solution set of } \Lambda_1 \cdots \Lambda_M) \).

The Bethe ansatz equations for \( M = 2 \) are

\[ (e(\Lambda_1))^N = e(\frac{\Lambda_1 - \Lambda_2}{2}) \]
\[ (e(\Lambda_2))^N = e(\frac{\Lambda_2 - \Lambda_1}{2}). \tag{11} \]

This set of equations has the following symmetries, \( \Lambda_i \to \Lambda_i^* \), \( \Lambda_i \to -\Lambda_i \) and \( \Lambda_1 \leftrightarrow \Lambda_2 \).

Now

\[ e(\frac{\Lambda_1 - \Lambda_2}{2}) = \frac{\Lambda_1 - \Lambda_2}{\frac{\Lambda_1 - \Lambda_2}{2} - i} = \frac{\Lambda_1 + i - (\Lambda_2 - i)}{\Lambda_1 - i - (\Lambda_2 + i)}. \tag{12} \]
Replacing for \( \Lambda_i \) with \( \Lambda_i = \frac{i(e(\Lambda_i)+1)}{e(\Lambda_i)-1} \) gives
\[
e\left(\frac{\Lambda_1 - \Lambda_2}{2}\right) = -\left(\frac{e(\Lambda_1)e(\Lambda_2) - 2e(\Lambda_1) + 1}{e(\Lambda_1)e(\Lambda_2) - 2e(\Lambda_2) + 1}\right).
\] (13)

We choose to work with the variables \( X_i = e(\Lambda_i) \). In terms of these variables, real solutions (in the \( \Lambda_i \)) are mapped to the unit circle while complex conjugate pairs are mapped to complex pairs which are related by, \((z, \frac{1}{z^*})\). Then we get
\[
X_1^N = -\left(\frac{X_1X_2 - 2X_1 + 1}{X_1X_2 - 2X_2 + 1}\right) \quad \text{and} \quad X_2^N = -\left(\frac{X_1X_2 - 2X_2 + 1}{X_1X_2 - 2X_1 + 1}\right).
\] (14a, b)

Multiplying these equations together gives,
\[
(X_1X_2)^N = 1, \quad \text{hence} \quad X_1X_2 = \omega
\] (15)
where \( \omega \) is an \( N \)th root of unity. Thus replacing \( X_2 = \frac{\omega}{X_1} \) in (14a), yields
\[
X_1^N = -\left(\frac{\omega - 2X_1 + 1}{\omega - 2\frac{\omega}{X_1} + 1}\right) = -X_1^N \left(\frac{\omega - 2X_1 + 1}{\omegaX_1 - 2\omega + X_1}\right).
\] (16)

Assuming \( X_1 \neq \frac{2\omega}{\omega + 1}, \) or 0, we get
\[
X_1^{(N-1)}((\omega + 1)X_1 - 2\omega) + \omega - 2X_1 + 1 = 0
\] (17)

ie.
\[
(\omega + 1)X_1^N - 2\omega X_1^{(N-1)} - 2X_1 + (\omega + 1) = 0.
\] (18)

We will be interested in the real roots in terms of \( \Lambda_1 \). These are mapped to roots on the unit circle in terms of \( X_1 \). Furthermore
\[
\left|\frac{2\omega}{(\omega + 1)}\right|^2 = \frac{4}{(\omega + 1)(\omega^* + 1)} = \frac{4}{2 + 2\text{Re}\omega} = \frac{2}{1 + \text{Re}\omega} > 1
\] (19)
for $\omega \neq 1$. Thus the denominator by which we multiplied may only vanish for $\omega = 1$ and $X_1 = 1$, which gives $X_2 = 1$. The corresponding wave function vanishes identically as the Bethe ansatz wave functions respect the Pauli principle. We want the roots of equation (18), for each choice of $\omega$. This gives $N$ polynomials, each of order $N$, which is a total of $N^2$ roots. $X_1 = \sqrt{\omega}$, however, is always a solution with $X_2 = \sqrt{\omega}$. These wave functions also vanish because of the Pauli principle. Furthermore, if $x$ is a root, then $\frac{x}{\omega}$ is also a root. Each one gives the same Bethe ansatz wave function, only the roles of $X_1$ and $X_2$ are exchanged. Therefore, we get a total of $\frac{N^2 - N}{2} = \binom{N}{2}$ different wave functions. This is exactly the dimension of the subspace of states with two spins flipped. There are exactly $\binom{N}{2}$ independent ways of flipping two spins among $N$.

The equation is cast in a more symmetric form with the replacement $X_1 \rightarrow \sqrt{\omega} \tilde{X}_1$ and correspondingly $Y_1 \rightarrow \sqrt{\omega} \tilde{Y}_1$. These satisfy the symmetric relation $\tilde{Y}_1 = \frac{1}{\tilde{X}_1}$. We first consider the case $N$ odd, where it is always possible to take $\sqrt{\omega}^N = -1$. This means with $\omega = e^{i \frac{2\pi m}{N}}$, for $m$ odd we take $\sqrt{\omega} = e^{i \frac{\pi m}{N}}$ but for $m$ even we must take $\sqrt{\omega} = e^{-i \frac{\pi (N-m)}{N}}$.

The set of $\{\sqrt{\omega}\}$ for $m$ odd are exactly the inverses (complex conjugates) of the set for $m$ even, except for $\omega = 1$, which is not so paired. Then taking into account $\sqrt{\omega}^N = -1$, we get

$$0 = (\omega + 1)\sqrt{\omega}^N \tilde{X}_1^N - 2\omega \sqrt{\omega}^{N-1} \tilde{X}_1^{(N-1)} - 2\sqrt{\omega} \tilde{X}_1 + (\omega + 1)$$

$$= -(\omega + 1)\tilde{X}_1^N + 2\sqrt{\omega} \tilde{X}_1^{(N-1)} - 2\sqrt{\omega} \tilde{X}_1 + (\omega + 1)$$

$$= -\sqrt{\omega} \left( (\sqrt{\omega} + \frac{1}{\sqrt{\omega}}) \tilde{X}_1^N - 2\tilde{X}_1^{N-1} + 2\tilde{X}_1 - (\sqrt{\omega} + \frac{1}{\sqrt{\omega}}) \right)$$

which yields

$$\cos\left(\frac{\theta}{2}\right) \tilde{X}_1^N - \tilde{X}_1^{(N-1)} + \tilde{X}_1 - \cos\left(\frac{\theta}{2}\right) = 0,$$
with the definition $\sqrt{\omega} = e^{i(\frac{\theta}{2})}$. The symmetries of equation (11) now translate into $(\tilde{X}_1 \rightarrow \frac{1}{\tilde{X}_1}, \omega \rightarrow \omega)$, $(\tilde{X}_1 \rightarrow \frac{1}{\tilde{X}_1}, \omega \rightarrow \omega^*)$ and $(\tilde{X}_1 \rightarrow \frac{1}{\tilde{X}_1}, \omega \rightarrow \omega)$.

The roots of Equation (21) come in pairs, if $z$ is a root, so is $\frac{1}{z}$. Furthermore, equation (21) has real coefficients, thus complex solutions come in complex conjugate pairs. The natural grouping of complex solutions is actually in quartets, $(z, z^*, \frac{1}{z}, \frac{1}{z^*})$. Real solutions in terms of the $\Lambda$ variables are mapped to solutions on the unit circle, thus the quartets degenerate into pairs for these. Complex solutions in terms of the $\Lambda$ variables are then mapped off the unit circle, and should generally come in quartets. Therefore, if there is truly only one complex pair in terms of the $\Lambda$ variables, it must be mapped to a real pair, $r, \frac{1}{r}$ in terms of the $\tilde{X}$ variables. $\tilde{X}_1 = r$ fixes the solution for the corresponding $\Lambda = i\frac{\sqrt{\omega - 1}}{r\sqrt{\omega + 1}} = -2r^2 \sin \theta + i\frac{r^2 - 1}{r^2 + 1}$. It would be interesting to find true quartet solutions.

Below we only find pairs. Equation (21) is the same for $\sqrt{\omega}$ and for $\frac{1}{\sqrt{\omega}}$. Although the equations, and consequently the solutions, are the same in the tilde variables, the solutions in terms of the original $X$ variables are of course different, giving rise to different Bethe ansatz wave functions.

Now we look for solutions on the unit circle, $\tilde{X}_1 = e^{i\alpha}$. Equation (21) becomes

$$\cos(\frac{\theta}{2})e^{iN\alpha} - e^{i(N-1)\alpha} + e^{i\alpha} - \cos(\frac{\theta}{2}) = 0,$$

which simplifies magically to

$$0 = e^{-\frac{iN\alpha}{2}} \cos(\frac{\theta}{2})(e^{\frac{iN\alpha}{2}} - e^{-\frac{iN\alpha}{2}}) - e^{i\alpha} e^{\frac{i(N-2)\alpha}{2}}(e^{\frac{i(N-2)\alpha}{2}} - e^{-\frac{i(N-2)\alpha}{2}})$$

$$= e^{-\frac{iN\alpha}{2}} 2i \left( \cos(\frac{\theta}{2}) \sin(\frac{N\alpha}{2}) - \sin(\frac{(N-2)\alpha}{2}) \right).$$
Thus

\[ \cos(\frac{\theta}{2}) = \frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})}. \] (24)

This equation is easily studied graphically, see figure (1). \( \alpha \in [0, 2\pi] \). The right hand side has zeros at \( \alpha = \frac{2k\pi}{N-2} \) for \( k = 1, 2, 3, \ldots, N - 3 \) and poles at \( \alpha = \frac{2k\pi}{N} \) for \( k = 1, 2, 3, \ldots, N - 1 \). The value at \( \alpha = 0 \) is \( \frac{N-2}{N} \) but at \( \alpha = \pi \) it is \(-1\). We get a root for each intersection of a horizontal line with \( y = \cos(\frac{\theta}{2}) \) with the curve \( y = \frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})} \).

![Figure 1](image-url)

**Figure 1:** \( f(\alpha) = \frac{\sin(\frac{(N-2)\alpha}{2})}{\sin(\frac{N\alpha}{2})}, \ N = 11 \)

By counting the intersections it is evident from figure (1), that there is a root for each zero. There are always \( N - 3 \) such roots. An additional pair of real roots can appear if

\[ \cos(\frac{\theta}{2}) \geq \frac{N - 2}{N}, \] (25)
giving $N - 1$ real roots, but $X_1 = 1$, $(X_1 = \sqrt{\omega})$, is also a root of the original equation, giving totally $N$ roots. This exhausts all roots of the polynomial and there are no complex conjugate pairs (for $\Lambda$). Thus we find

$$\cos\left(\frac{\theta_{\text{critical}}}{2}\right) = \frac{N - 2}{N}. \quad \text{(26)}$$

For large $N$, this will have solutions for $\frac{\theta}{2}$ near 0 or $2\pi$, but because of reflection symmetry about $\alpha = \pi$ we need only search near $\alpha = 0$. Assuming $\theta$ is small and making an expansion in $\theta$ and $\frac{1}{N}$, we get

$$1 - \frac{1}{2}\left(\frac{\theta_{\text{critical}}}{2}\right)^2 + \cdots = 1 - \frac{2}{N} \quad \text{(27)}$$

giving

$$\theta_{\text{critical}} = \frac{4}{\sqrt{N}}. \quad \text{(28)}$$

The values taken by $\frac{\theta}{2}$ (the condition that $\sqrt{\omega^N} = -1$ must be satisfied) give

$$\frac{\theta}{2} = \frac{\pi m}{N} \quad \text{for} \quad m = 1, 3, 5, \ldots, N - 2, \quad \text{(29)}$$

or

$$\frac{\theta}{2} = \pi - \frac{\pi m}{N} \quad \text{for} \quad m = 2, 4, 6, \ldots, N - 1, \quad \text{(30)}$$

($N$ is odd). Hence, for the odd series in $m$, we get

$$m < \frac{2}{\pi} \sqrt{N} \quad \text{(31)}$$

and for the even series in $m$,

$$|N - m| < \frac{2}{\pi} \sqrt{N}. \quad \text{(32)}$$

The number of new solutions behaves like $\sqrt{N}$. 
The first $N$ for which we have a new solution, (to equation (25) actually), is $N = 23$, when $m = 3$ is allowed, next at $N = 63$, $m = 5$ is allowed and so on. (The solution at $m = 1$ is accounted for in the string hypothesis set of solutions. It corresponds to the action of the lowering operator applied to the corresponding state in the $M = 1$ sector, in the context of the $SU(2)$ extended Bethe ansatz.)

Analyzing in more detail the complex pair of roots, we insert $\tilde{X}_1 = r$ into equation (21), giving

$$
\cos\left(\frac{\theta}{2}\right) = \frac{r(r^{N-2} - 1)}{r^N - 1} = f(r).
$$

(Fig. 2) $f(r) = \frac{r(r^{N-2} - 1)}{r^N - 1}$, $N = 11$

It is easy to see that $f(r)$ is monotone decreasing for $r > 1$. Therefore, it is monotone
increasing for $0 < r < 1$, as $f(\frac{1}{r}) = f(r)$. The maximum occurs at $r = 1$, $f(1) = \frac{N-2}{N}$.

Evidently there are two roots for each $0 < \cos(\theta) < \frac{N-2}{N}$, as we can see from figure (2). This is exactly the same condition as equation (25), as expected. We see that for $\frac{\theta}{2}$ near $\frac{T}{2}$ the solution $(r, \frac{1}{r})$ is $(\infty, 0)$, which implies $\Lambda \approx -2\frac{r\sin\theta}{r^2 + 1} \pm i$, as predicted by the string hypothesis. For $\frac{\theta}{2}$ near zero, however, the imaginary part becomes arbitrarily small and the string hypothesis is grossly violated, finally to the extreme that the complex solutions drop out all together.

For $r < 0$,

$$f(r) = -|r|(\frac{|r|^N - 2 + 1}{|r|^N + 1})$$

(34)

It is again easy to see that $f(r)$ decreases monotonically from 0 to $-1$ as $r$ varies from $-\infty$ to $-1$ after which it rises monotonically to 0, since always $f(\frac{1}{r}) = f(r)$. Thus, we always get exactly one complex pair for $\frac{\theta}{2} \in (\frac{\pi}{2}, \frac{3\pi}{2})$, as we can again see from inspection of figure (2).

Finally we note that actually the complex pair solutions come in pairs. This is because of the pairing of $\sqrt{\omega}$ for a particular $m$ odd with $\frac{1}{\sqrt{\omega}}$ for the corresponding $m$ even. These yield identical equations (21) and (33). The corresponding pairs of complex pair solutions are, however, not the quartets to which we had referred earlier. Interestingly enough though, clearly both pairs drop out simultaneously when the critical condition equation (25) is satisfied. Thus actually two complex pair solutions of the “string hypothesis” drop out simultaneously and become two real pair solutions.

We present below in somewhat less detail, the case when $N$ even. First of all, we have
\((\sqrt{\omega})^N = (-1)^m\), for either choice for the square root, \(\sqrt{\omega} = \pm e^{i \pi m/2}\). Thus, for the case \(m\) odd, we get the condition as above,

\[
\cos\left(\frac{\theta}{2}\right) = \frac{\sin\left(\frac{(N-2)\alpha}{2}\right)}{\sin\left(\frac{N\alpha}{2}\right)},
\]

(35)

but here \(N = 2n\). Thus,

\[
\cos\left(\frac{\theta}{2}\right) = \frac{\sin((n-1)\alpha)}{\sin(n\alpha)}.
\]

(36)

The R.H.S. is a function which has zeros at \(\alpha = \frac{k\pi}{(n-1)}\), \(k = 1, 2, 3, \ldots, 2n - 3\), excluding \(k = n - 1\), i.e. \(2n - 4\) zeros, and poles at \(\alpha = \frac{k\pi}{n}\), \(k = 1, 2, \ldots, 2n - 1\), excluding \(k = n\).

The value at \(\alpha = 0\) is \(\frac{n-1}{n}\). The situation is as before except at \(\alpha = \pi\), the value is \(-\frac{n-1}{n}\), not \(-1\), and the function “turns over”. The function is reflection symmetric about \(\alpha = \pi\).

We therefore obtain for each \(\omega\) at least \(2n - 4\) real roots (the number of zeros), but we get two additional real roots if

\[
|\cos\left(\frac{\theta}{2}\right)| \geq \frac{n-1}{n},
\]

(37)

making a total of \(2n - 2 = N - 2\). This condition can be satisfied for \(\frac{\theta}{2}\) near zero or near \(\pi\). We also of course get the reflections of these about \(\pi\) hence we can restrict \(\frac{\theta}{2} \in (0, \pi)\).

At \(N = 22\), \(m = 3\) is allowed, and then at \(N = 62\), \(m = 5\) is allowed. The condition becomes \(m < \frac{2}{\pi}\sqrt{N}\) and \(|N - m| < \frac{2}{\pi}\sqrt{N}\) for large \(N\). Thus for the case \(N\) even, but \(m\) odd there can be two solutions which violate the string hypothesis correspondence. We also always have solutions of the original polynomial corresponding to \(\tilde{X}_1 = \pm 1\), \((N\) even\) which gives totally \(N\) roots, in which case, there are no complex roots for values of \(m\) satisfying equation (37). We notice that equation (37) is satisfied simultaneously by two values, \(m_1, m_2\) such that \(m_1 + m_2 = N\). \((\text{N.B. } m_i \text{ are odd, } N \text{ is even.})\) Hence we again lose two complex pair solutions, giving two additional real pair solutions.
For $m$ even, the equation for $\tilde{X}_1$ becomes

$$0 = (\omega + 1)\sqrt{\omega} \tilde{X}_1^N - 2\omega \sqrt{\omega}^{N-1} \tilde{X}_1^{(N-1)} - 2\sqrt{\omega} \tilde{X}_1 + (\omega + 1)$$

$$= (\omega + 1)\tilde{X}_1^N - 2\omega \tilde{X}_1^{(N-1)} - 2\sqrt{\omega} \tilde{X}_1 + (\omega + 1)$$

$$= \sqrt{\omega} \left( (\sqrt{\omega} + \frac{1}{\sqrt{\omega}})\tilde{X}_1^N - 2\tilde{X}_1^{N-1} - 2\tilde{X}_1 + (\sqrt{\omega} + \frac{1}{\sqrt{\omega}}) \right).$$

(38)

Replacing $\tilde{X}_1 = e^{i\alpha}$ yields

$$\cos(\frac{\theta}{2}) e^{iN\alpha} - e^{i(N-1)\alpha} - e^{i\alpha} + \cos(\theta) = 0,$$

(39)

(with $\sqrt{\omega} = e^{i\frac{\theta}{4}}$) which simplifies to

$$0 = e^{i\frac{N\alpha}{2}} \cos(\frac{\theta}{2}) (e^{i\frac{N\alpha}{2}} + e^{-i\frac{N\alpha}{2}}) - e^{i\alpha} e^{i(N-2)\alpha} (e^{i\frac{(N-2)\alpha}{2}} + e^{-i\frac{(N-2)\alpha}{2}})$$

$$= e^{i\frac{N\alpha}{2}} 2 \left( \cos(\frac{\theta}{2}) \cos(\frac{N\alpha}{2}) - \cos(\frac{(N-2)\alpha}{2}) \right),$$

(40)

yielding

$$\cos(\frac{\theta}{2}) = \frac{\cos(\frac{(N-2)\alpha}{2})}{\cos(\frac{N\alpha}{2})}.$$  

(41)

This equation is also easily studied graphically, see figure (3). The R.H.S. has zeros at $\alpha = \frac{k\pi}{(N-2)}$ for $k = 1, 3, 5, \ldots, 2N - 5$ and poles at $\alpha = \frac{k\pi}{N}$ for $k = 1, 3, 5, \ldots, 2N - 1$. We can check that at $\alpha = 0$ the function is 1 and at $\alpha = \pi$ it is $-1$. Graphically we get a root for each intersection of a horizontal line $y = \cos(\frac{\theta}{2})$ with the curve $y = \frac{\cos(\frac{(N-2)\alpha}{2})}{\cos(\frac{N\alpha}{2})}$.

Counting the intersections it is evident from figure (3), that there is a root for each zero. There are $N - 2$ zeros, thus there are always only $N - 2$ real roots, $\tilde{X}_1 = \pm 1$ are not roots in this case, in fact it is easy to see that $\cos(\frac{N\alpha}{2})$ and $\cos(\frac{(N-2)\alpha}{2})$ never vanish simultaneously. Hence there are always two complex roots and no new real roots which violate the string hypothesis correspondence.
Figure 3: $f(\alpha) = \frac{\cos\left(\frac{(N-2)\alpha}{\sqrt{N}}\right)}{\cos\left(\frac{\alpha}{\sqrt{N}}\right)}$, $N = 10$

We summarize in the following way. For $N$ sufficiently large, we have $2 \left(\frac{2\pi}{\sqrt{N}}\right)$ additional real pair solutions with respect to the prediction of the string hypothesis. For $N$ odd, they are distributed evenly between $m$ odd and $m$ even, however, they occur only near $\tilde{X}_1 = 1$. For $N$ even, they occur only for $m$ odd, however, now they are distributed evenly near $\tilde{X}_1 = \pm 1$. We give simple, exact expressions for the critical values $N$, equations (25) and (37), which do not appear in Reference (3).

Consider now the sector with $M$ spins flipped. Here the equations (7), rewritten in terms
of the variables $X_i = e(\Lambda_i)$ are

$$X_i^N = (-1)^{M-1} \prod_{\substack{l=1 \atop l \neq i}}^{M} \left( \frac{X_i X_l - 2X_l + 1}{X_i X_l - 2X_l + 1} \right), \quad i = 1, \ldots, M.$$  \hfill (42)

Multiplying the $M$ equations together gives

$$(X_1 X_2 \cdots X_M)^N = 1.$$  \hfill (43)

Thus

$$X_M = \frac{\omega}{X_1 X_2 \cdots X_{M-1}}$$  \hfill (44)

with $\omega^N = 1$. Multiplying through with the denominator we get the coupled polynomial system

$$X_i \prod_{\substack{l=1 \atop l \neq i}}^{M} (X_i X_l - 2X_l + 1) + (-1)^M \prod_{\substack{l=1 \atop l \neq i}}^{M} (X_i X_l - 2X_l + 1) = 0 \quad i = 1, \ldots, M.$$  \hfill (45)

Using equation (44) we can eliminate $X_M$ from the first $M - 1$ equations and multiplying through with the denominator yields the system of $M - 1$ polynomial equations

$$X_i \prod_{\substack{l=1 \atop l \neq i}}^{M-1} (X_i X_l - 2X_l + 1)((X_i - 2)\omega + X_1 X_2 \cdots X_{M-1})$$

$$+ (-1)^{M-1} \prod_{\substack{l=1 \atop l \neq i}}^{M-1} (X_i X_l - 2X_l + 1)(X_i \omega - (2X_i - 1)X_1 X_2 \cdots X_{M-1}) = 0 \quad i = 1, \ldots, M - 1.$$  \hfill (46)

Each equation is just a permutation of the variables in any other one. Removing an overall factor of $X_i$, the $i$th equation is of order $N + M - 2$ in $X_i$, but only quadratic in all the other variables.
Specializing to $M = 3$ we find the system

$$0 = X_1^N(X_1X_2 - 2X_2 + 1)((X_1 - 2)^2 + X_1X_2)$$

$$- (X_1X_2 - 2X_1 + 1)(X_1 - 2)^2 + X_1X_2)$$

$$0 = X_2^N(X_1X_2 - 2X_1 + 1)((X_2 - 2)^2 + X_1X_2)$$

$$- (X_1X_2 - 2X_2 + 1)(X_2 - 2)^2 + X_1X_2).$$

Simplifying we get

$$0 = X_2^2(X_1^{N+1} - 2X_1^N + 2X_1^2 - X_1) + X_2(\omega(X_1^{N+1} - 4X_1^N + 4X_1^{N-1} - X_1)$$

$$+ (X_1^N - 4X_1^2 + 4X_1 - 1)) + \omega(X_1^N - 2X_1^{N-1} + 2X_1 - 1)$$

$$X_1 \leftrightarrow X_2.$$ 

There are two ways to proceed, both give the same result up to trivial factors. The simplest and straightforward method is to solve the quadratic equation (48a) for $X_2$

$$A(X_1)X_2^2 + B(X_1)X_2 + C(X_1) = 0$$

i.e.

$$X_2 = -\frac{B(X_1) \pm \sqrt{B^2(X_1) - 4A(X_1)C(X_1)}}{2A(X_1)}$$

as an algebraic function of $X_1$. Then replace this in the corresponding equation (48b) for $X_1$

$$A(X_2)X_1^2 + B(X_2)X_1 + C(X_2) = 0.$$ 

Expanding the powers of $X_2$ we can isolate the odd powers of the square root to one side of the equation, the other side then only involves rational polynomial expressions in $X_1$. Finally squaring both sides and multiplying through by the denominator gives a single polynomial equation for $X_1$. Generalizing from calculations with Mathematica for low $N$,
we obtain the form of the polynomial as $(X_1 - 1)^{N+4}X_1^{N-2}A(X_1)^{N+1}P_N(X_1)$, where $P_N$ is of degree $N(N - 1) + 1$. It is not illuminating to present the explicit form of $P_N$ for a few explicit value of $N$, we do not have a closed form for it. The roots of the polynomial along with the equivalent of equation (44) should give rise to all solutions of the Bethe ansatz equations in this sector.

The second way to proceed affords generalization to higher values of $M$, although this is not straightforward. We consider the quadratic equation in $X_2$ as giving the first step of a recurrence relation

$$X_2^2 = -\frac{B(X_1)}{A(X_1)}X_2 - \frac{C(X_1)}{A(X_1)}.$$  

(52)

Then if $X_2^n = \alpha_nX_2 + \beta_n$ we have

$$\alpha_nX_2 + \beta_n = X_2^n = X_2X_2^{n-1} = X_2(\alpha_{n-1}X_2 + \beta_{n-1})$$

$$= (\alpha_{n-1}X_2^2 + \beta_{n-1}X_2)$$

$$= (\alpha_{n-1}(-\frac{B(X_1)}{A(X_1)}X_2 - \frac{C(X_1)}{A(X_1)}) + \beta_{n-1}X_2)$$

$$= (\alpha_{n-1}(-\frac{B(X_1)}{A(X_1)}) + \beta_{n-1})X_2 + \alpha_{n-1}(-\frac{C(X_1)}{A(X_1)}).$$

(53)

Thus

$$\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \frac{1}{A(X_1)} \begin{pmatrix} -B(X_1) & A(X_1) \\ -C(X_1) & 0 \end{pmatrix} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix},$$

(54)

which is trivial to solve as

$$\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \mathcal{M}^{n-2} \begin{pmatrix} -\frac{B(X_1)}{A(X_1)} \\ -\frac{C(X_1)}{A(X_1)} \end{pmatrix}.$$  

(55)

Then replacing in equation (51) for $X_2^n$ we obtain a linear equation in $X_2$ that is a rational polynomial expression in $X_1$ and we solve for $X_2$ as a rational polynomial in $X_1$. Replacing this back in equation (49) and multiplying through by the denominator again gives a
single polynomial equation in $X_1$. We obtain from Mathematica the same polynomial as
the above, modulo some trivial factors, $(X_1 - 1)^N X_1^N A(X_1)^N P_N(X_1)$. However, since
we have no closed form expression for $P_N$, whose degree is of order $N^2$, it is actually
quite unfeasible to go much beyond $N = 20$. We are presently engaged in numerical and
analytical analyses of the roots of this polynomial to see if there are any new violations of
the string hypothesis.

We find it is a dramatic simplification to deal with even, coupled polynomial equations
(46) than the original transcendental equations. We hope to extend our analysis to other
models where the Bethe ansatz has proven useful.

We thank T. Gisiger, Y. Saint-Aubin, R. MacKenzie and V. Spiridonov for useful discus-
sions. This work supported in part by NSERC of Canada, FCAR du Québec and FOM of
the Netherlands. We also thank Gebhard Grübl and the Institut für Theoretische Physik,
Innsbruck, Austria for hospitality, where some of this work was done.

References

* address after November 1st, 1992, Sonnmatt 8, Bäch, Switzerland.
† permanent address

1.) L. Faddeev and Takhtadjan, Cargese lectures,1987.
2.) H. Bethe, Z. Phys. 71, (1931)205.
3.) F.H.L. Essler, V.E. Korepin and K. Schoutens, J. Phys.A25,4115,(1992).
4.) M. Takahashi, Prog. Theo. Phys.46, (1971) 401.