SERRE-GODEAUX VARIETIES AND THE ÉTALE INDEX

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Abstract. We use the Serre-Godeaux varieties of finite groups, projective representation theory, the twisted Atiyah-Segal completion theorem, and our previous work on the topological period-index problem to compute the étale index of Brauer classes $\alpha \in \text{Br}(X)$ in some specific examples. In particular, these computations show that the étale index of $\alpha$ differs from the period of $\alpha$ in general. As an application, we compute the index of unramified classes in the function fields of high-dimensional Serre-Godeaux varieties in terms of projective representation theory.

Let $X$ be a connected scheme, and let $\text{Br}'(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)_{\text{tors}}$ be its cohomological Brauer group. There is a subgroup $\text{Br}(X) \subseteq \text{Br}'(X)$ consisting of those Brauer classes that are represented by an Azumaya algebra. A class $\alpha \in \text{Br}'(X)$ is in $\text{Br}(X)$ if and only if it is in the image of the coboundary map

$$H^1_{\text{ét}}(X, \text{PGL}_n) \xrightarrow{\delta_n} H^2_{\text{ét}}(X, \mathbb{G}_m)$$

for some $n$, where the coboundary arises from the central extension

$$1 \to \mathbb{G}_m \to \text{GL}_n \to \text{PGL}_n \to 1.$$ 

The pointed cohomology set $H^1_{\text{ét}}(X, \text{PGL}_n)$ classifies $\text{PGL}_n$-torsors, $\mathbb{P}^{n-1}$-bundles, or degree-$n$ Azumaya algebras. Recall that a degree-$n$ Azumaya algebra is an algebra on $X$ which looks étale-locally like the algebra of $n \times n$ matrices over the ring of regular functions on $X$. The class $\delta_n(P)$ of a $\text{PGL}_n$-torsor $P$ is precisely the obstruction to lifting $P$ to a $\text{GL}_n$-torsor. Viewed from the perspective of Azumaya algebras, this class is the obstruction to writing an Azumaya algebra $A$ as the endomorphism algebra of a vector bundle. The index of $\alpha \in \text{Br}(X)$ is

$$\text{ind}(\alpha) = \gcd\{n|\alpha \text{ is in the image of } \delta_n\}.$$ 

When $X = \text{Spec } k$ is the spectrum of a field, the index of a class $\alpha \in \text{Br}(k)$ is the degree of the unique division algebra representing $\alpha$.

The period of $\alpha$, denoted $\text{per}(\alpha)$, is the order of $\alpha \in \text{Br}'(X)$. In general,

$$\text{per}(\alpha) \mid \text{ind}(\alpha).$$

When $X = \text{Spec } k$, the two integers have the same prime divisors. It is a major open problem in the theory of Azumaya algebras, even when $X = \text{Spec } k$, to determine the possible pairs $(\text{per}(\alpha), \text{ind}(\alpha))$ for $\alpha \in \text{Br}(X)$ where $X$ is fixed, or where the dimension of $X$ is fixed. For an introduction to what is known and for further references, see [3, Section 1].

In [2], the first author constructed an invariant $\text{eti}(\alpha)$ of cohomological Brauer classes $\alpha$, called the étale index. It is constructed as the positive generator of a rank map $K^0_{\alpha, \text{ét}}(X) \to \mathbb{Z}$ from twisted étale $K$-theory, which was first introduced in [2]. It was proved there and in [1] that, for $\alpha \in \text{Br}(X)$,

$$\text{per}(\alpha) \mid \text{eti}(\alpha) \mid \text{ind}(\alpha),$$

and that, in general, the étale index is strictly smaller than the index. The question of whether or not the étale index ever differed from the period was left open. In this paper, we present in Theorem 3.2 a class of examples arising from finite group theory for which

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the étale index is indeed different from the period. The examples are Serre-Godeaux varieties [19], which are smooth projective varieties that approximate the homotopy type of $BG \times K(\mathbb{Z}, 2)$. Our approach to these varieties comes from the presentation in the proof of [5, Proposition 6.6].

The étale index is interesting for at least the following two reasons: first it has the property that it is finite for all classes $\alpha \in Br(X)$, provided $X$ itself is of finite étale cohomological dimension, this contrasts with the index per se which may not be defined when $Br(X) \neq Br'(X)$. Second, in [2] we gave upper bounds for the étale index in terms of the period using stable homotopy theory, thus solving an analogue of the still open period-index conjecture.

If $X$ is a smooth projective complex variety, the unramified Brauer group of the function field $C(X)$ is $Br_{un}(C(X)) = Br(X)$. If $G$ is a finite group and $X$ is a Serre-Godeaux variety of sufficiently high dimension, our method allows us to compute in Theorem 3.4 the index of $\alpha$ for $\alpha \in Br_{un}(X)$ in terms of projective representation theory. This computation does not appear accessible by other methods.

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1. Definitions

When $X$ is a topological space, the cohomological Brauer group is $Br^\prime_{top}(X) = H^3(X, \mathbb{Z})_{\text{tors}}$, and the Brauer group $Br_{top}(X) \subseteq Br^\prime_{top}(X)$ is the subgroup of all classes represented by a topological Azumaya algebra. If $X$ is a finite CW-complex, then $Br_{top}(X) = Br^\prime_{top}(X)$ by an argument of Serre [10]. For a class $\alpha \in Br^\prime_{top}(X)$, we define $\text{per}_{top}(\alpha)$ to be the order of $\alpha$. There are coboundary maps

$$H^1(X, \text{PGL}_n(C)) \xrightarrow{\delta_n} H^2(X, C^*) \cong H^3(X, \mathbb{Z})_{\text{tors}},$$

and we define

$$\text{ind}_{top}(\alpha) = \gcd\{n | \alpha \text{ is in the image of } \delta_n\}$$

when $\alpha \in Br_{top}(X)$. There is an $\alpha$-twisted $K$-theory spectrum, $KU(X)_\alpha$, as defined in [6]. If $X$ is a finite CW-complex the group $KU^0(X)_\alpha$ is the Grothendieck of $\alpha$-twisted vector bundles. This group is equipped with a natural rank map $KU^0(X)_\alpha \to \mathbb{Z}$. The $K$-theoretic index $\text{ind}_{K}(\alpha)$ is defined in [3] to be the positive generator of the image of the rank map. In topology, the $K$-theoretic index plays the same role as the étale index $\text{eti}(\alpha)$ does in algebraic geometry. In general, we have the following analog of (1):

$$\text{per}_{top}(\alpha) | \text{ind}_{K}(\alpha) | \text{ind}_{top}(\alpha).$$

In the case where $X$ is a finite CW-complex, because topological twisted $K$-theory satisfies descent, we showed in [3, Lemma 2.23] that $\text{ind}_{K}(\alpha) = \text{ind}_{top}(\alpha)$.

If $X$ is a complex algebraic scheme, there is a natural map $Br^\prime(X) \to H^3(X, \mathbb{Z})_{\text{tors}}$. If $\alpha \in Br^\prime(X)$ is a cohomological Brauer class, we write $\pi$ for the image of $\alpha$ in $H^3(X, \mathbb{Z})_{\text{tors}}$. In general, $\text{per}_{top}(\pi)$ divides $\text{per}(\alpha)$, and it is easy to see that $\text{ind}_{top}(\pi)$ divides $\text{ind}(\alpha)$ as well, because any algebraic Azumaya algebra gives rise to a topological Azumaya algebra. More importantly for the purposes of this paper, a result of [3] says that if $X$ has the homotopy type of a finite CW-complex, then

$$\text{ind}_{top}(\pi) = \text{ind}_{K}(\pi) | \text{eti}(\alpha).$$

Fundamentally, this result again depends on descent for topological twisted $K$-theory.

Let $X$ be a topological space and $\alpha \in Br^\prime_{top}(X)$. There is a twisted Atiyah-Hirzebruch spectral sequence [7]

$$E_2^{p,q} = H^p(X, KU^q(\pi)) \Rightarrow KU^{p+q}(X)_\alpha,$$

where $KU^*(X)_\alpha$ denotes the $\alpha$-twisted $K$-theory of $X$. The differentials $d_r^p$ are of degree $(r, 1 - r)$, and $d_r^p = 0$ unless $r$ is odd. The first non-trivial differential $d_2^0 : H^0(X, \mathbb{Z}) \to H^3(X, \mathbb{Z})$ has the property that $d_2^0(1) = \pm \alpha \in H^3(X, \mathbb{Z})$, by [6] or [2]. This spectral
sequence is natural, in the sense that if \( f : X \to Y \) is a map and \( \alpha \in H^3(Y, \mathbb{Z})_{\text{tors}} \), then there is a morphism from the spectral sequence computing \( KU^*(Y)_\alpha \) to the spectral sequence computing \( KU^*(X)_{f_*\alpha} \).

The twisted Atiyah-Segal spectral sequence converges strongly when \( X \) is a finite CW-complex. When \( X \) is connected, the rank map \( KU^0(X)_\alpha \to \mathbb{Z} \) may be identified with the edge map \( KU^0(X)_\alpha \to H^0(X, \mathbb{Z}) \cong \mathbb{Z} \) in the spectral sequence, and the image of the rank map is then the group of permanent cycles \( E^\infty_{2,0} \subseteq H^0(X, \mathbb{Z}) \). The kernel of the rank map we denote by \( \overline{KU}^0(X)_\alpha \). In general the spectral sequence converges only conditionally, but something may still be said about the permanent cycles in special cases.

**Proposition 1.1.** Let \( X = \cup_k X_k \) be a skeletal filtration of the connected CW-complex \( X \) by finite connected CW-complexes with \( X_0 = * \), and let \( \alpha \in H^3(X, \mathbb{Z})_{\text{tors}} \). Suppose that

\[
\lim^1 KU^0(X)_\alpha = 0,
\]

where \( f_k \) is the inclusion of \( X_k \) in \( X \). Then, \( \text{ind}_K(\alpha) \) is the positive generator of the subgroup of permanent cycles in \( E^\infty_{2,0} \cong \mathbb{Z} \).

**Proof.** The natural map \( KU(X)_\alpha \to \varinjlim KU(X_k)_{f_k^*\alpha} \) is a homotopy equivalence, which induces a commutative diagram (the commutativity of the top square is deduced from the naturality of the relation between \( E^\infty \) and the filtration on the target group in a conditionally convergent spectral sequence, for which see [8, Chapters 5 & 7])

\[
\begin{array}{ccc}
KU^0(X)_\alpha & \longrightarrow & \lim_k KU^0(X_k)_{f^*_k\alpha} & \longrightarrow & KU^0(X_0) \cong \mathbb{Z} \\
\downarrow & & \downarrow h & & \downarrow \\
E^\infty_{0,0}(X) & \longrightarrow & \lim_k E^\infty_{0,0}(X_k) & \longrightarrow & \\
\downarrow & & \downarrow & & \\
E^\infty_{2,0}(X) & \longrightarrow & \lim_k E^\infty_{2,0}(X_k) & \longrightarrow & \\
\end{array}
\]

where the top-left horizontal arrow is surjective by an argument analogous to that of Milnor in the untwisted case [17].

The map \( h \) is surjective because of the hypothesis that

\[
\lim^1 KU^0(X)_\alpha = 0.
\]

It follows by a diagram-chase that \( E^\infty_{0,0}(X) \subseteq E^\infty_{2,0}(X) \) is precisely the image of the rank map. \( \square \)

When \( G \) is a topological group, there is yet another index associated to a class \( \alpha \in H^3(BG, \mathbb{Z})_{\text{tors}} \). Given a projective representation \( \pi : G \to \text{PGL}_n(\mathbb{C}) \), there is an associated class \( [\pi] \) in \( H^1(BG, \text{PGL}_n(\mathbb{C})) \). We define the representation index

\[
\text{ind}_G(\alpha) = \gcd\{n\mid \text{there exists } \pi : G \to \text{PGL}_n \text{ with } \delta_n([\pi]) = \alpha\}.
\]

There are relations

\[
\text{per}_{\text{top}}(\alpha) | \text{ind}_K(\alpha) | \text{ind}_{\text{top}}(\alpha) | \text{ind}_G(\alpha)
\]

for \( \alpha \in H^3(BG, \mathbb{Z})_{\text{tors}} \). In the next section, we prove that these three indices coincide when \( G \) is a compact Lie group.

2. Twisted Equivariant K-theory

If \( G \) is a compact Lie group and \( X \) is a finite \( G \)-CW-complex, we let \( H^2_G(X, \mathbb{Z}) = H^0(X_G, \mathbb{Z}) \), where \( X_G = (X \times EG)/G \), and where \( EG \) is a universal \( G \)-space. When \( \alpha \in H^3_G(X, \mathbb{Z}) \), there is a twisted equivariant \( K \)-theory \( KU^*_G(X)_\alpha \) on \( X \) [6] and a twisted \( K \)-theory \( KU^*(X_G)_\alpha \) on \( X_G \). These are related by functorial morphisms \( KU^*_G(X)_\alpha \to KU^*(X_G)_\alpha \). The twisted equivariant \( K \)-theory is a module over \( KU^*_G(*) \cong R(G) \). Let \( I \)
be the augmentation ideal of $R(G)$. Below, $EG^n$ denotes the filtration on $EG$ given by Milnor [16], so that $EG^n$ is the join of $(n + 1)$ copies of $G$. The group $G$ acts freely on $EG^n$ with quotient space $BG^n$. The spaces $BG^n$ form an exhaustive filtration of $BG$ by finite CW-complexes.

**Theorem 2.1** (Lahtinen [15]). If $G$ is a compact Lie group and $\alpha \in H^3(BG, \mathbb{Z}) = H_G^3(\ast, \mathbb{Z})$, then there is a natural isomorphism of pro-groups

$$\{KU_0^0(\ast_\alpha)/I^n \cdot KU_0^0(\ast_\alpha)\} \to \{KU_0^0(EG^n)_\alpha\}.$$  

**Corollary 2.2.** In the situation above, the natural map

$$KU_G^0(\ast_\alpha) \to KU^0(BG)$$

is an isomorphism, where $KU_0^0(\ast_\alpha)$ is the completion of $KU_0^0(\ast_\alpha)$ along $I$.

Recall that a sequence $\cdots \to A_n \to A_{n-1} \to \cdots$ satisfies the Mittag-Leffler condition if for every $k$, the image of $A_r \to A_k$ stabilizes for $r$ sufficiently large. To be precise, for every $r$, there exists $n$ such that if $k \geq n$, the images of $A_k \to A_r$ and $A_n \to A_r$ are the same.

**Corollary 2.3.** In the situation above, the sequence of reduced twisted $K$-theory groups

$$\{\widetilde{KU}^0(EG^n/G)_\alpha\}$$

satisfies the Mittag-Leffler condition.

**Proof.** Consider the functor $C : (\text{pro-groups}) \to (\text{topological groups})$ given by taking the inverse limit and endowing it with the limit topology and the functor $F : (\text{topological groups}) \to (\text{pro-groups})$ given by taking the directed system of quotients $G/U$ where $U$ is an open subgroup of $G$. If $\{A_n\}$ is a pro-group, then $F(C(\{A_n\})) \cong \{A_n\}$ if and only if $\{A_n\}$ satisfies the Mittag-Leffler condition. It is evident that

$$\{KU_0^0(\ast_\alpha)/I^n \cdot KU_0^0(\ast_\alpha)\}$$

satisfies the Mittag-Leffler condition because the maps in the system are all surjections. By the theorem, it follows that

$$\{KU_0^0(EG^n)_\alpha\}$$

does too. But

$$\{KU_0^0(EG^n)_\alpha\} \cong \{KU_0^0(EG^n/G)_\alpha\}$$

since the action of $G$ on $EG^n$ is free. Thus, $\{KU_0^0(EG^n/G)_\alpha\}$ satisfies the Mittag-Leffler condition. Given the fact that the reduced twisted $K$-theory pro-group is the kernel of the rank map

$$\{KU_0^0(EG^n/G)_\alpha\} \to \mathbb{Z},$$

it follows that the reduced twisted $K$-theory groups satisfies the Mittag-Leffler condition. □

We come to our main theorem on twisted equivariant $K$-theory, which we will use in the next section to construct the examples.

**Theorem 2.4.** If $G$ is a compact Lie group, then $\text{ind}_K(\alpha) = \text{ind}_C(\alpha) = \text{ind}_G(\alpha)$ for all $\alpha \in H^3(BG, \mathbb{Z})$. Moreover, $\text{ind}_K(\alpha)$ generates the subgroup $E_{2,0}^{0,0} \subseteq E_{2,0}^{0,0}$.

**Proof.** It follows from Corollary 2.2 that there is an isomorphism

$$KU_G^0(\ast_\alpha) \cong KU^0(BG)_\alpha.$$  

The group $KU_0^0(\ast)$ computes the Grothendieck group of $\alpha$-twisted $G$-equivariant vector bundles [9, Proposition 3.5.3] (see also [9, Example 1.10]), and such a bundle is precisely a projective representation of $G$ having obstruction class $\alpha$, so there is a natural isomorphism

$$R(G)_\alpha \to KU^0(BG)_\alpha.$$  

All of these groups have compatible rank maps to $\mathbb{Z}$. As the classes of the augmentation ideal $I$ all have rank 0, the two maps $R(G)_\alpha \to \mathbb{Z}$ and $R(G)_\alpha \to \mathbb{Z}$ have the same image. Thus, $\text{ind}_K(\alpha) = \text{ind}_G(\alpha)$. 

The second statement follows from Corollary 2.3 and Proposition 1.1 applied to the filtration of $BG$ given by $BG^n = EG^n/G$, since $KU^*_G(EG^n)_\alpha \cong KU^0(BG^n)_\alpha$. \hfill \qed

**Remark 2.5.** When $G$ is a finite group, all the indices in the theorem are finite, as one can see by explicitly constructing such representations from cocycles. See [14].

### 3. Theorems

**Proposition 3.1.** For any integers $m, n > 1$ which have the same prime divisors and such that $m | n$, there is a finite abelian group $G$ and a class $\alpha \in H^3(BG, \mathbb{Z})_{\text{tors}}$ such that $\text{per}(\alpha) = m$ and $\text{ind}_G(\alpha) = n$.

**Proof.** It suffices by taking products to assume that $m$ and $n$ are powers of the same prime. Suppose that $m = p^r$ and $n = p^s$ with $s \geq 1$. For any positive integer $t$, let $G_t = (\mathbb{Z}/p^t)^2$. Let $G = (G_t)_k \times G_g$. By [12, Lemma 2.2], for any subgroup $S$ of $G$ such that $G/S \cong H \times H$, that is, such that $G/S$ is of symmetric type in the terminology of [12], there exists a 2-cocycle $\alpha : G \to \mathbb{C}^*$ such that $U(G, \alpha) = S$, where $U(G, \alpha)$ is the group of elements $x$ of $G$ such that $\alpha(x, y) = \alpha(y, x)$ for all $y \in G$. Moreover, $[G : U(G, \alpha)] = d^2$, where $d$ is the degree of any irreducible projective representation with obstruction cocycle $\alpha$. Taking $S = \{0\}$ in $G$, we find a 2-cocycle $\alpha$ such that $\text{ind}_G(\alpha) = p^{kr+\theta} = p^s$.

It remains to show that $\alpha$ has the desired period. We know that $\text{per}_G(\alpha)|p^r$ since $G$ is $p^r$-torsion. Let $x$ be an element of $G$ of order $p^r$. Then, $p^r-1x$ is not in $U(G, \alpha)$ since the group is trivial, so there exists $y$ in $G$ such that $\alpha(p^r-1x, y) \neq \alpha(y, p^r-1x)$. Let $H$ be the subgroup of $G$ generated by $x$ and $y$. Then, $p^r-1x$ is not contained in $U(H, \alpha_H)$, so that $H/U(H, \alpha_H)$ contains an element $\overline{\alpha}$ of order $p^r$. Since the group $H/U(H, \alpha_H)$ is symmetric by [12, Lemma 2.2], it follows that it is isomorphic to $G$, and that $U(H, \alpha_H) = \{0\}$. Therefore, $\text{ind}_H(\alpha_H) = p^r$, and it follows from [12, Lemma 2.3] that this implies that $\text{per}_top(\alpha_H) = p^r$. Thus, $\text{per}_top(\alpha) = p^r$. \hfill \qed

The next theorem shows that the étale index defined in the first author’s thesis is indeed different, in general, from the period.

**Theorem 3.2.** For any integers $m, n > 1$ which have the same prime divisors and such that $m | n$, there is a smooth projective complex scheme $X$ and a class $\alpha \in \text{Br}(X)$ such that $\text{per}(\alpha) = m$ and $\text{eti}(\alpha) = \text{ind}(\alpha) = n$.

**Proof.** Let $G$ be a finite group with a class $\overline{\beta} \in H^3(BG, \mathbb{Z})_{\text{tors}}$ such that $\text{per}_top(\overline{\beta}) = m$ and $\text{ind}_G(\overline{\beta}) = n$. By Theorem 2.4, we have $\text{ind}_K(\overline{\beta}) = n$. Moreover, by Proposition 1.1, the group of permanent cycles in $E^2_{2s,0}$ is generated by $n$. In particular there are only finitely many non-zero differentials leaving $E^{0,0}$. Let $d^2_{2s+1}$ be the last non-zero differential leaving $E^{0,0}$.

We may view the class $\overline{\beta}$ as a class in $\text{Br}'_{top}(BG \times K(Z, 2)) = \text{Br}'_{top}(BG)$, and $\text{per}_top(\overline{\beta})$ and $\text{ind}_K(\overline{\beta})$ are the same computed on either $BG$ or $K(Z, 2) \times BG$, as one sees by splitting the projection $K(Z, 2) \times BG \to BG$.

Let $X_s$ be a $(2s+2)$-dimensional Serre-Godeaux variety associated to $G$. There is a natural morphism $X_s \to K(Z, 2) \times BG$, which is a $(2s+1)$-equivalence, and therefore induces an isomorphism in integral cohomology in degrees at most $2s$ and an inclusion in degree $2s+1$. Denote the image of $\overline{\beta}$ in $X_s$ by $\overline{\beta}_s$; it is non-zero because, for $s \geq 1$, the map $H^3(BG \times K(Z, 1), \mathbb{Z}) \to H^3(X_s, \mathbb{Z})$ is an inclusion. By direct comparison we find $\text{ind}_K(\overline{\beta}_s) | \text{ind}_K(\overline{\beta})$, whereas by our choice of $s$ and Proposition 1.1 we find $\text{ind}_K(\overline{\beta}) | \text{ind}_K(\overline{\beta}_s)$, whereupon it follows that

$$\text{ind}_K(\overline{\beta}_s) = \text{ind}_K(\overline{\beta}) = n.$$

Because the composition

$$H^2_{et}(X_s, \mu_m) \cong H^2(X_s, \mathbb{Z}/m) \to \text{Br}'(X_s) \to m H^3(X_s, \mathbb{Z})$$

is surjective, we can find a class $\alpha \in \text{Br}'(X)$ whose image in $H^3(X, \mathbb{Z})$ is $\overline{\beta}_n$ and whose period satisfies $\text{per}(\alpha) = \text{per}_{\text{top}}(\overline{\beta}_n) = m$. The relation

$$n = \text{ind}_K(\overline{\beta}_n) | \text{eti}(\alpha) | \text{ind}(\alpha)$$

then follows from (2). This already shows that the étale index is different, in general, from the period.

We show that $\text{eti}(\alpha) = \text{ind}(\alpha) = n$ as follows. Write $\mathcal{O}_{X,s}$ for the sheaf of holomorphic functions on $X_s$. Since $X_s$ is projective, it is easy to see that the natural map $\text{Br}'(X_s) \to H^2(X_s, \mathcal{O}_{X,s})_{\text{tors}}$ is an isomorphism; see, for example, [18, Proposition 1.3]. As part of the long exact sequence associated to the exponential sequence $0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_{X,s} \xrightarrow{\exp} \mathcal{O}_{X,s}^* \to 1$, we have

$$H^2(X, \mathcal{O}_{X,s}) \to H^2(X, \mathcal{O}_{X,s}^*) \to H^3(X, \mathbb{Z}) \to H^3(X, \mathcal{O}_{X,s}). \tag{3}$$

Finally, observe that $H^2(BG \times K(\mathbb{Z}, 2), \mathbb{Q}) \cong \mathbb{Q}$, which implies that $H^2(X_s, \mathcal{O}_{X,s}) = 0$ by the Hodge decomposition. Therefore, (3) shows that $\text{Br}'(X) \cong H^2(X, \mathcal{O}_{X,s})_{\text{tors}} \cong H^3(X, \mathbb{Z})_{\text{tors}}$, since $H^3(X, \mathcal{O}_{X,s})$ is a complex vector space.

Since $\text{Br}'(X_s) = H^3(X_s, \mathbb{Z})_{\text{tors}}$, and in particular the comparison map is injective, any flat projective bundle on $X_s$ with topological Brauer class $\overline{\beta}_n$ has Brauer class $\alpha$. The projective bundles on $X$ which are considered in computing $\text{ind}_G(\overline{\beta}_n)$ are the bundles arising from a projective representation of $\pi_1(X) = G$, viz. the flat bundles. The restrictions of flat bundles to $X_s$ remain flat, and are consequently algebraizable, and so $\text{ind}(\alpha)|\text{ind}_G(\overline{\beta}_n) = n$, proving that $\text{eti}(\alpha) = \text{ind}(\alpha) = n$, as desired. \hfill \Box

**Scholium 3.3.** For any integers $m, n > 1$ which have the same prime divisors and such that $m/n$, there is a smooth affine complex scheme $X$ and a class $\alpha \in \text{Br}(X)$ such that $\text{per}(\alpha) = m$ and $\text{eti}(\alpha) = \text{ind}(\alpha) = n$.

**Proof.** Let $Y$ be a smooth projective scheme $Y$ with a class $\alpha \in \text{Br}(Y)$ with $\text{per}(\alpha) = \text{per}_{\text{top}}(\overline{\alpha}) = m$ and $\text{ind}_{\text{top}}(\overline{\alpha}) = \text{eti}(\alpha) = \text{ind}(\alpha) = n$, as constructed in the proof of the theorem. By Jouanolou’s device [13, Lemme 1.5], there exists a vector bundle torsor $\pi : X \to Y$ with affine total space $X$. Topologically, $\pi$ is a homotopy equivalence, so

$$\text{per}(\pi^*\alpha) = \text{per}_{\text{top}}(\pi^*\overline{\alpha}) = m$$

and

$$n = \text{ind}_{\text{top}}(\pi^*\overline{\alpha}) \leq \text{eti}(\pi^*\alpha) \leq \text{eti}(\alpha) = n. \hfill \Box$$

In the course of the proof of the theorem, we saw that if $X$ is a Serre-Godeaux variety of dimension at least 5, then

$$\text{Br}_{\text{un}}(C(X)) = \text{Br}(X) = H^3(X, \mathbb{Z})_{\text{tors}}.$$

For such a variety and a class $\alpha \in H^3(X, \mathbb{Z})_{\text{tors}}$, we may thus speak unambiguously of the algebraic index $\text{ind}(\alpha)$, where we view $\alpha$ as a class in $\text{Br}(X)$.

**Theorem 3.4.** Let $G$ be a finite group. There exists a positive integer $n$ such that if $X$ is a Serre-Godeaux variety for $G$ of dimension at least $n$, and if $\alpha \in H^3(BG, \mathbb{Z})$ is a Brauer class, then $\text{ind}(\alpha_{C(X)}) = \text{ind}(\alpha) = \text{ind}_G(\alpha)$. That is, the index is computed by projective representations.

**Proof.** The theorem follows from the proof of Theorem 3.2. We can find a single finite $n$ because $H^3(BG, \mathbb{Z})$ is a finite abelian group. This shows that we can compute $\text{ind}(\alpha) = \text{ind}_G(\alpha)$ for $\alpha \in \text{Br}(X)$. It follows from an argument of David Saltman that $\text{ind}(\alpha_{C(X)}) = \text{ind}(\alpha)$, since $X$ is regular and noetherian; see [4, Proposition 5.5]. \hfill \Box

In the theorem, it is clear that $\text{ind}(\alpha)|\text{ind}_G(\alpha)$. The role of the étale index or the topological index is to show that the index is large enough, and that this divisibility relation is actually an equality.
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