Topological Markov chains of given entropy and period with or without measure of maximal entropy

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Abstract

We show that, for every positive real number $h$ and every positive integer $p$, there exist oriented graphs $G, G'$ (with countably many vertices) that are strongly connected, of period $p$, of Gurevich entropy $h$, such that $G$ is positive recurrent (thus the topological Markov chain on $G$ admits a measure of maximal entropy) and $G'$ is transient (thus the topological Markov chain on $G'$ admits no measure of maximal entropy).

1 Vere-Jones classification of graphs

In this paper, all the graphs are oriented, have a finite or countable set of vertices and, if $u, v$ are two vertices, there is at most one arrow $u \rightarrow v$. A path of length $n$ in the graph $G$ is a sequence of vertices $(u_0, u_1, \ldots, u_n)$ such that $u_i \rightarrow u_{i+1}$ in $G$ for all $i \in [0, n-1]$. This path is called a loop if $u_0 = u_n$.

Definition 1 Let $G$ be an oriented graph and let $u, v$ be two vertices in $G$. We define the following quantities.

- $p_{uv}^G(n)$ is the number of paths $(u_0, u_1, \ldots, u_n)$ such that $u_0 = u$ and $u_n = v$; $R_{uv}(G)$ is the radius of convergence of the series $\sum p_{uv}^G(n)z^n$.
- $f_{uv}^G(n)$ is the number of paths $(u_0, u_1, \ldots, u_n)$ such that $u_0 = u, u_n = v$ and $u_i \neq v$ for all $0 < i < n$; $L_{uv}(G)$ is the radius of convergence of the series $\sum f_{uv}^G(n)z^n$.

Definition 2 Let $G$ be an oriented graph and $V$ its set of vertices. The graph $G$ is strongly connected if for all $u, v \in V$, there exists a path from $u$ to $v$ in $G$. The period of a strongly connected graph $G$ is the greatest common divisor of $(p_{uv}^G(n))_{u \in V, n \geq 0}$. The graph $G$ is aperiodic if its period is 1.

Proposition 3 (Vere-Jones [8]) Let $G$ be an oriented graph. If $G$ is strongly connected, $R_{uv}(G)$ does not depend on $u$ and $v$; it is denoted by $R(G)$.

If there is no confusion, $R(G)$ and $L_{uv}(G)$ will be written $R$ and $L_{uv}$.

In [8] Vere-Jones gives a classification of strongly connected graphs as transient, null recurrent or positive recurrent. These definitions are lines 1 and 2 in Table 1. The other lines of Table 1 state properties of the series $\sum p_{uv}^G(n)z^n$, which give alternative definitions (lines 3 and 4 are in [8], the last line is Proposition 4).

Proposition 4 (Salama [7]) Let $G$ be a strongly connected oriented graph. If $G$ is transient or null recurrent, then $R = L_{uu}$ for all vertices $u$. Equivalently, if there exists a vertex $u$ such that $R < L_{uu}$, then $G$ is positive recurrent.
Table 1: properties of the series associated to a transient, null recurrent or positive recurrent graph $G$ (G is strongly connected); these properties do not depend on the vertices $u, v$.

## 2 Topological Markov chains and Gurevich entropy

Let $G$ be an oriented graph and $V$ its set of vertices. We define $\Gamma_G$ as the set of two-sided infinite paths in $G$, that is,

$$\Gamma_G := \{(v_n)_{n \in \mathbb{Z}} \mid \forall n \in \mathbb{Z}, v_n \rightarrow v_{n+1} \text{ in } G\} \subset V^\mathbb{Z}.$$ 

The map $\sigma$ is the shift on $\Gamma_G$. The topological Markov chain on the graph $G$ is the dynamical system $(\Gamma_G, \sigma)$.

The set $V$ is endowed with the discrete topology and $\Gamma_G$ is endowed with the induced topology of $V^\mathbb{Z}$. The space $\Gamma_G$ is not compact unless $G$ is finite.

The topological Markov chain $(\Gamma_G, \sigma)$ is transitive if and only if the graph $G$ is strongly connected. It is topologically mixing if and only if the graph $G$ is strongly connected and aperiodic.

If $G$ is a finite graph, $\Gamma_G$ is compact and the topological entropy $h_{\text{top}}(\Gamma_G, \sigma)$ is well defined (see e.g. [2] for the definition of the topological entropy). If $G$ is a countable graph, the Gurevich entropy [3] of the graph $G$ (or of the topological Markov chain $\Gamma_G$) is given by

$$h(G) := \sup \{h_{\text{top}}(\Gamma_H, \sigma) \mid H \subset G, H \text{ finite}\}.$$ 

This entropy can also be computed in a combinatorial way, as the exponential growth of the number of paths with fixed endpoints.

**Proposition 5 (Gurevich [4])** Let $G$ be a strongly connected oriented graph. Then for all vertices $u, v$

$$h(G) = \lim_{n \rightarrow +\infty} \frac{1}{n} \log p_{uv}^G(n) = -\log R(G).$$

Moreover, the variational principle is still valid for topological Markov chains.

**Theorem 6 (Gurevich [3])** Let $G$ be an oriented graph. Then

$$h(G) = \sup \{h_\mu(\Gamma_G) \mid \mu \text{-invariant probability measure}\}.$$ 

In this variational principle, the supremum is not necessarily reached. The next theorem gives a necessary and sufficient condition for the existence of a measure of maximal entropy (that is, a probability measure $\mu$ such that $h(G) = h_\mu(\Gamma_G)$) when the graph is strongly connected.
Theorem 7 (Gurevich [4]) Let $G$ be a strongly connected oriented graph of finite positive entropy. Then the topological Markov chain on $G$ admits a measure of maximal entropy if and only if the graph $G$ is positive recurrent. Moreover, such a measure is unique if it exists.

3 Construction of graphs of given entropy and given period that are either positive recurrent or transient

Lemma 8 Let $\beta \in (1, +\infty)$. There exist a sequence of non negative integers $(a(n))_{n \geq 1}$ and positive constants $c, M$ such that

- $a(1) = 1$,
- $\sum_{n \geq 1} a(n) \frac{1}{\beta^n} = 1$,
- $\forall n \geq 2, c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + M$,
- $\forall n \geq 1, 0 \leq a(n) \leq M$ if $n$ is not a square.

These properties imply that the radius of convergence of $\sum_{n \geq 1} a(n) z^n$ is $L = \frac{1}{\beta}$ and that $\sum_{n \geq 1} n a(n) L^n < +\infty$.

Proof. First we look for a constant $c > 0$ such that

$$\frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1. \quad (1)$$

We have

$$\sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = \sum_{n \geq 2} \beta^{-n} = \frac{1}{\beta(\beta - 1)}. \quad (1)$$

Thus

$$\frac{1}{\beta} + \frac{c}{\beta(\beta - 1)} = 1 \iff c = (\beta - 1)^2.$$

Since $\beta > 1$, the constant $c := (\beta - 1)^2$ is positive. We define the sequence $(b(n))_{n \geq 1}$ by:

- $b(1) := 1$,
- $b(n^2) := \lfloor c \beta^{n^2-n} \rfloor$ for all $n \geq 2$,
- $b(n) := 0$ for all $n \geq 2$ such that $n$ is not a square.

Then

$$\sum_{n \geq 1} b(n) \frac{1}{\beta^n} \leq \frac{1}{\beta} + c \sum_{n \geq 2} \beta^{n^2-n} \frac{1}{\beta^{n^2}} = 1.$$

We set $\delta := 1 - \sum_{n \geq 1} b(n) \frac{1}{\beta^n} \in [0, 1)$ and $k := \lfloor \beta^2 \delta \rfloor$. Then $k \leq \beta^2 \delta < k + 1 < k + \beta$, which implies that $0 \leq \delta - \frac{k}{\beta^2} < \frac{1}{\beta}$. We write the $\beta$-expansion of $\delta - \frac{k}{\beta^2}$ (see e.g. [1] p 51) for the definition: there exist integers $d(n) \in \{0, \ldots, \lfloor \beta \rfloor \}$ such that $\delta - \frac{k}{\beta^2} = \sum_{n \geq 1} d(n) \frac{1}{\beta^n}$. Moreover, $d(1) = 0$ because $\delta - \frac{k}{\beta^2} < \frac{1}{\beta}$. Thus we can write

$$\delta = \sum_{n \geq 2} d'(n) \frac{1}{\beta^n}$$

where $d'(2) := d(2) + k$ and $d'(n) := d(n)$ for all $n \geq 3$.

We set $a(1) := b(1)$ and $a(n) := b(n) + d'(n)$ for all $n \geq 2$. Let $M := \beta + k$. We then have:
• \(a(1) = 1\),
• \(\sum_{n \geq 1} a(n) \frac{1}{\beta^n} = 1\),
• \(\forall n \geq 2, c \cdot \beta^{n^2-n} \leq a(n^2) \leq c \cdot \beta^{n^2-n} + \beta \leq c \cdot \beta^{n^2-n} + M\),
• \(0 \leq a(2) \leq \beta + k = M\),
• \(\forall n \geq 3, 0 \leq a(n) \leq \beta \leq M\) if \(n\) is not a square.

The radius of convergence \(L\) of \(\sum_{n \geq 1} a(n)x^n\) satisfies
\[-\log L = \limsup_{n \to +\infty} \frac{1}{n} \log a(n) = \lim_{n \to +\infty} \frac{1}{n} \log a(n^2) = \log \beta \quad \text{because} \quad a(n^2) \sim c\beta^{n^2-n}.
\]
Thus \(L = \frac{1}{\beta}\). Moreover,
\[
\sum_{n \geq 1} na(n) \frac{1}{\beta^n} \leq M \sum_{n \geq 1} n \frac{1}{\beta^n} + c \sum_{n \geq 1} n^2 \beta^{n^2-n} \frac{1}{\beta^n} = M \sum_{n \geq 1} \frac{n}{\beta^n} + c \sum_{n \geq 1} \frac{n^2}{\beta^n} < +\infty.
\]

Lemma 9 ([5], Lemma 2.4) Let \(G\) be a strongly connected oriented graph and \(u\) a vertex.

i) \(R < L_{uu}\) if and only if \(\sum_{n \geq 1} f_{uu}^n(n) L_{uu}^n > 1\).

ii) If \(G\) is recurrent, then \(R\) is the unique positive number \(x\) such that \(\sum_{n \geq 1} f_{uu}^n(n) x^n = 1\).

Proof. For (i) and (ii), use Table 1 and the fact that \(F(x) = \sum_{n \geq 1} f_{uu}^n(n) x^n\) is increasing for \(x \in [0, +\infty]\). \(\square\)

Proposition 10 Let \(\beta \in (1, +\infty)\). There exist aperiodic strongly connected graphs \(G'(\beta) \subset G(\beta)\) such that \(h(G(\beta)) = h(G'(\beta)) = \log \beta\), \(G(\beta)\) is positive recurrent and \(G'(\beta)\) is transient.

Remark: Salama proved the part of this proposition concerning positive recurrent graphs in [6, Theorem 3.9].

Proof. This is a variant of the proof of [5, Example 2.9].

Let \(u\) be a vertex and let \((a(n))_{n \geq 1}\) be the sequence given by Lemma 8 for \(\beta\). The graph \(G(\beta)\) is composed of \(a(n)\) loops of length \(n\) based at the vertex \(u\) for all \(n \geq 1\) (see Figure 1). More precisely, define the set of vertices of \(G(\beta)\) as
\[V := \{u\} \cup \bigcup_{n=1}^{+\infty} \{v_k^{n,i} \mid i \in [1, a(n)], k \in [1, n-1]\},\]
where the vertices \(v_k^{n,i}\) above are distinct. Let \(v_0^{n,i} = v_n^{n,i} = u\) for all \(i \in [1, a(n)]\). There is an arrow \(v_k^{n,i} \to v_{k+1}^{n,i}\) for all \(k \in [0, n-1], i \in [1, a(n)], n \geq 2\); there is an arrow \(u \to u\); and there is no other arrow in \(G(\beta)\). The graph \(G(\beta)\) is strongly connected and \(f_{uu}^{G(\beta)}(n) = a(n)\) for all \(n \geq 1\).

By Lemma 8, the sequence \((a(n))_{n \geq 1}\) is defined such that \(L = \frac{1}{\beta}\) and
\[
\sum_{n \geq 1} a(n)L^n = 1,
\]
(2)
where $L = L_{uu}(G(\beta))$ is the radius of convergence of the series $\sum a(n)z^n$. If $G(\beta)$ is transient, then $R(G(\beta)) = L_{uu}(G(\beta))$ by Proposition 4. But Equation (2) contradicts the definition of transient (see the first line of Table 1). Thus $G(\beta)$ is recurrent, and $R(G(\beta)) = L$ by Equation (2) and Lemma 9(ii). Moreover

$$\sum_{n \geq 1} na(n)L^n < +\infty$$

by Lemma 8 and thus the graph $G(\beta)$ is positive recurrent (see Table 1). By Proposition 5, $h(G(\beta)) = -\log R(G(\beta)) = \log \beta$.

The graph $G'(\beta)$ is obtained from $G(\beta)$ by deleting a loop starting at $u$ of length $n_0$ for some $n_0 \geq 2$ such that $a(n_0) \geq 1$ (such an integer $n_0$ exists because $L < +\infty$). Obviously one has $L_{uu}(G'(\beta)) = L$ and

$$\sum_{n \geq 1} f_{uu}(G'(\beta))(n)L^n = 1 - L^{n_0} < 1.$$

Since $R(G'(\beta)) \leq L_{uu}(G'(\beta))$, this implies that $G'(\beta)$ is transient. Moreover $R(G'(\beta)) = L_{uu}(G'(\beta))$ by Proposition 4, so $R(G'(\beta)) = R(G(\beta))$, and hence $h(G'(\beta)) = h(G(\beta))$ by Proposition 5. Finally, both $G(\beta)$ and $G'(\beta)$ are of period 1 because of the arrow $u \rightarrow u$. □

**Corollary 11** Let $p$ be a positive integer and $h \in (0, +\infty)$. There exist strongly connected graphs $G, G'$ of period $p$ such that $h(G) = h(G') = h$, $G$ is positive recurrent and $G'$ is transient.

**Proof.** For $G$, we start from the graph $G(\beta)$ given by Proposition 10 with $\beta = e^{hp}$. Let $V$ denote the set of vertices of $G(\beta)$. The set of vertices of $G$ is $V \times \llbracket 1, p \rrbracket$, and the arrows in $G$ are:

- $(v, i) \rightarrow (v, i + 1)$ if $v \in V, i \in \llbracket 1, p - 1 \rrbracket$,
- $(v, p) \rightarrow (w, 1)$ if $v, w \in V$ and $v \rightarrow w$ is an arrow in $G(\beta)$.

According to the properties of $G(\beta)$, $G$ is strongly connected, of period $p$ and positive recurrent. Moreover, $h(G) = \frac{1}{p} h(G(\beta)) = \frac{1}{p} \log \beta = h$.

For $G'$, we do the same starting with $G'(\beta)$. □

According to Theorem 7, the graphs of Corollary 11 satisfy that the topological Markov chain on $G$ admits a measure of maximal entropy whereas the topological Markov chain on $G'$ admits no measure of maximal entropy; both are transitive, of Gurevich entropy $h$ and supported by a graph of period $p$. □
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