IDENTITIES OF SYMMETRY FOR \((h, q)\)–EXTENSION OF HIGHER-ORDER EULER POLYNOMIALS

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Abstract. In this paper, we study some symmetric properties of the multiple \(q\)–Euler zeta function. From these properties, we derive several identities of symmetry for the \((h, q)\)–extension of higher-order Euler polynomials, which is an answer to a part of open question in [7].

1. Introduction

Let \(\mathbb{C}\) be the complex number field. We assume that \(q \in \mathbb{C}\) with \(|q| < 1\) and the \(q\)–number is defined by \([x]_q = \frac{1-q^x}{1-q}\). Note that \(\lim_{q \to 1} [x]_q = x\). As is well known, the higher-order Euler polynomials \(E_n^{(r)}(x)\) are defined by the generating function to be

\[
F^{(r)}(x, t) = \left( \frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} , \quad \text{(see [4], [16])},
\]

where \(|t| < \pi\).

When \(x = 0, E_n^{(r)} = E_n^{(r)}(0)\) are called the Euler numbers of order \(r\). Recently, the second author defined the \((h, q)\)–extension of higher-order Euler polynomials, which is given by the generating function to be

\[
F^{(h,r)}_q(x, t) = \sum_{m_1, \ldots, m_r = 0}^{\infty} \sum_{j=1}^{r} (h-j+1) m_j \frac{(1)}{[m_1 + \cdots + m_r + x]_q} t^n,
\]

(see [6], [8]),

\[
F^{(h,r)}_q(t, x) = \sum_{m}^{\infty} E_n^{(h,r)}(x) \frac{t^n}{n!}, \quad \text{see [6], [8]),}
\]

where \(h \in \mathbb{Z}\) and \(r \in \mathbb{Z}_{\geq 0}\). Note that \(\lim_{q \to 1} F^{(h,r)}_q(x, t) = \left( \frac{2}{e^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.

By (1.2), we get

\[
F^{(h,r)}_q(t, x) = \left[ \frac{2}{e^t + 1} \right]^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!}.
\]

\[
F^{(h,r)}(t, x) = \left[ \frac{2}{e^t + 1} \right]^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(h,r)}(x) \frac{t^n}{n!} , \quad \text{(see [6], [8])},
\]
where \((x)_m^q = \frac{[x]_1[x-1]_1[x-2]_1\cdots[x-m+1]_1}{[m]_q!}\).

From (1.3), we can derive the following equation:

\[
E_{n,q}^{(h,r)}(x) = \frac{[2]^r_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} (-q^x)^l (-q^{h+r+l+1} ; q)_r,
\]

\[
= [2]^r_q \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (-q^{h+r+1})^m [m + x]_q^n, \quad \text{(see [6])},
\]

where \((x : q)_n = (1 - x)(1 - qx) \cdots (1 - qx^{n-1})\).

In [6] and [8], the second author constructed the multiple \(q\)-Euler zeta function which interpolates the \((h, q)\)-extension of higher-order Euler polynomials at negative integers as follows:

\[
\zeta_{q,r}^{(h)}(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} E_{q}^{(h,r)}(x,t) t^{s-1} dt
\]

\[
= [2]^r_q \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} q^{\sum_{j=1}^{r} (h-j+1)m_j}}{[m_1 + \cdots + m_r + x]_q^r}, \quad \text{(see [6])},
\]

where \(h, s \in \mathbb{C}, x \in \mathbb{R}\) with \(x \neq 0, -1, -2, \cdots\).

From (1.5), we have

\[
\zeta_{q,r}^{(h)}(-n, x) = E_{n,q}^{(h,r)}(x), \quad \text{(see [6], [8])}.
\]

Using the Cauchy residue theorem and Laurent series in (1.5), we obtain the following lemma.

**Lemma 1.1.** For \(n \in \mathbb{Z}_{\geq 0}\) and \(h \in \mathbb{Z}\), we have

\[
\zeta_{q,r}^{(h)}(-n, x) = E_{n,q}^{(h,r)}(0), \quad \text{(see [6], [8])}.
\]

In [7], the second author introduced many identities of symmetry for Euler and Bernoulli polynomials which are derived from the \(p\)-adic integral expression of the generating function and suggested an open problem about finding identities of symmetry for the Carlitz’s type \(q\)-Euler numbers and polynomials.

When \(x = 0\), \(E_{n,q}^{(h,r)} = E_{n,q}^{(h,r)}(0)\) are called the \((h, q)\)-Euler numbers of order \(r\).

From (1.3) and (1.4), we can derive the following equation:

\[
E_{n,q}^{(h,r)}(x) = (q^x E_{q}^{(h,r)} + [x]_q^n) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{q}^{(h,r)} [x]_q^{n-l},
\]

with the usual convention about replacing \((E_{q}^{(h,r)})^n\) by \(E_{n,q}^{(h,r)}\).

Recently, Y. Simsek introduced recurrence symmetric identities for \((h, q)\)-Euler polynomials and alternating sums of powers of consecutive \((h, q)\)-integers (see [16]).

In this paper, we investigate some symmetric properties of the multiple \(q\)-Euler zeta function. From our investigation, we give some new identities of symmetry for the \((h, q)\)-extension of higher-order Euler polynomials, which is an answer to a part of open question in [7].
2. Identities for \((h, q)\)-extension of higher-order Euler Polynomials

In this section, we assume that \(h \in \mathbb{Z}\) and \(a, b \in \mathbb{N}\) with \(a \equiv 1\) (mod 2) and \(b \equiv 1\) (mod 2). Now, we observe that

\[
\frac{1}{[2]^q} \zeta_{q^a,r}^{(h)}(s, bx + \frac{b(j_1 + \cdots + j_r)}{a}) = \sum_{m_1, \ldots, m_r=0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} q^a \sum_{j_i=1}^{r} (h-j+1) m_j}{[m_1 + \cdots + m_r + bx + \frac{b(j_1 + \cdots + j_r)}{a}]_q}.
\]

Thus, by (2.1), we get

\[
[a]_q^a \sum_{m_1, \ldots, m_r=0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} q^a \sum_{j_i=1}^{r} (h-j+1) m_j}{[m_1 + \cdots + m_r + bx + \frac{b(j_1 + \cdots + j_r)}{a}]_q} \equiv \prod_{i=1}^{r} \frac{1}{\sum_{j_i=1}^{r} (h-j+1) m_j} + \sum_{i=1}^{r} (h-j+1) m_i a q^a.
\]

By the same method as (2.2), we see that

\[
[a]_q^a \sum_{m_1, \ldots, m_r=0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} q^a \sum_{j_i=1}^{r} (h-j+1) m_j}{[m_1 + \cdots + m_r + bx + \frac{b(j_1 + \cdots + j_r)}{a}]_q} \equiv \prod_{i=1}^{r} \frac{1}{\sum_{j_i=1}^{r} (h-j+1) m_j} + \sum_{i=1}^{r} (h-j+1) m_i a q^a.
\]

Therefore, by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \(a, b \in \mathbb{N}\), with \(a \equiv 1\) (mod 2) and \(b \equiv 1\) (mod 2), we have

\[
[a]_q^a \sum_{m_1, \ldots, m_r=0}^{\infty} \frac{(-1)^{m_1 + \cdots + m_r} q^a \sum_{j_i=1}^{r} (h-j+1) m_j}{[m_1 + \cdots + m_r + bx + \frac{b(j_1 + \cdots + j_r)}{a}]_q} \equiv \prod_{i=1}^{r} \frac{1}{\sum_{j_i=1}^{r} (h-j+1) m_j} + \sum_{i=1}^{r} (h-j+1) m_i a q^a.
\]

From Lemma 1.1 and Theorem 2.1, we can derive the following theorem.
Theorem 2.2. For \( n \in \mathbb{Z}_{\geq 0} \) and \( a, b \in \mathbb{N} \), with \( a \equiv 1(\text{mod } 2) \) and \( b \equiv 1(\text{mod } 2) \), we have

\[
[2]_{q^n}^{r}[a]^{n}_{q} \sum_{j_1, \ldots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{b \sum_{i=1}^{r} (h-l+1) j_i} E_{n, q^n}^{(h, r)} \left( bx + \frac{b(j_1 + \cdots + j_r)}{a} \right) = [2]_{q^n}^{r}[b]^{n}_{q} \sum_{j_1, \ldots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{a \sum_{i=1}^{r} (h-l+1) j_i} E_{n, q^n}^{(h, r)} \left( ax + \frac{a(j_1 + \cdots + j_r)}{b} \right).
\]

By (1.4), we easily see that

\[
E_{n, q}^{(h, k)}(x + y) = (q^{x+y} E_{q}^{(h, k)} + [x + y]_q)^n = (q^{x+y} E_{q}^{(h, k)} + q^x [y]_q + [x]_q)^n = \sum_{i=0}^{n} \binom{n}{i} q^{ix} E_{i, q}^{(h, k)}(y)[x]_{q}^{n-i}. \tag{2.4}
\]

Therefore, by (2.4), we obtain the following proposition.

Proposition 2.3. For \( n \geq 0 \), we have

\[
E_{n, q}^{(h, k)}(x + y) = \sum_{i=0}^{n} \binom{n}{i} q^{ix} E_{i, q}^{(h, k)}(y)[x]_{q}^{n-i} = \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)x} E_{n-i, q}^{(h, k)}(y)[x]_{q}^{i}.
\]

From Proposition 2.3, we note that

\[
\sum_{j_1, \ldots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{b \sum_{i=1}^{r} (h-l+1) j_i} E_{n, q^n}^{(h, r)} \left( bx + \frac{b(j_1 + \cdots + j_r)}{a} \right) \times \left[ \frac{b(j_1 + \cdots + j_r)}{a} \right]^{n-i} q^{a} = \sum_{j_1, \ldots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{b \sum_{i=1}^{r} (h-l+1) j_i} \sum_{i=0}^{n} \binom{n}{i} q^{(n-i)b(j_1 + \cdots + j_r)} E_{n-i, q^n}^{(h, r)}(bx) \times \left[ \frac{b(j_1 + \cdots + j_r)}{a} \right]^{i} q^{a} = \sum_{i=0}^{n} \binom{n}{i} \left[ \frac{[b]_{q}}{[a]_{q}} \right]^{i} E_{n-i, q^n}^{(h, r)}(bx) \sum_{j_1, \ldots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{b \sum_{i=1}^{r} (h+n-l-i+1) j_i} [j_1 + \cdots + j_r]_{q}^{i} = \sum_{i=0}^{n} \binom{n}{i} \left[ \frac{[b]_{q}}{[a]_{q}} \right]^{i} E_{n-i, q^n}^{(h, r)}(bx) S_{n-i, q^n}^{(h, r)}(a),
\]

where \( S_{n-i, q^n}^{(h, r)}(a) = \sum_{j_1, \ldots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{b \sum_{i=1}^{r} (h+n-l-i+1) j_i} [j_1 + \cdots + j_r]_{q}^{i}. \tag{2.5} \)
By (2.5), we get

\[
\sum_{i=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{\sum_{l=1}^{r} (h-l+l) j_l} E_{n,q^{a}}^{(h,r)} \left( bx + \frac{b(j_1 + \cdots + j_r)}{a} \right) = 0
\]  

(2.7)

By the same method as (2.7), we see that

\[
\sum_{i=0}^{n} \binom{n}{i} [a]_q^{n-i} [b]_q^{i} E_{n-i,q^{a}}^{(h,r)} (bx) S_{n,i,q^{a}}^{(h,r)}(a).
\]  

(2.8)

Therefore, by (2.7) and (2.8), we obtain the following theorem.

**Theorem 2.4.** For \(a, b \in \mathbb{N}\) with \(a \equiv 1 \text{(mod 2)}\) and \(b \equiv 1 \text{(mod 2)}\), \(n \in \mathbb{Z}_{>0}\),

let

\[
S_{n,i,q^{a}}^{(h,r)}(a) = \sum_{j_1, \cdots, j_r=0}^{a-1} (-1)^{j_1 + \cdots + j_r} q^{\sum_{l=1}^{r} (h-n-l+i+1) j_l} \left[ j_1 + \cdots + j_r \right]_q^i.
\]

Then we have

\[
\sum_{i=0}^{n} \binom{n}{i} [a]_q^{n-i} [b]_q^{i} E_{n-i,q^{a}}^{(h,r)} (ax) S_{n,i,q^{a}}^{(h,r)}(a) = \sum_{i=0}^{n} \binom{n}{i} [b]_q^{n-i} [a]_q^{i} E_{n-i,q^{a}}^{(h,r)} (ax) S_{n,i,q^{a}}^{(h,r)}(b).
\]

It is not difficult to show that

\[
[x + y + m]_q (u + v) - [x]_q v = [x]_q u + q^r [y + m]_q (u + v).
\]  

(2.9)

From (2.9), we note that

\[
e^{[x]_q u} \sum_{m_1, \cdots, m_r=0}^{\infty} q^{\sum_{j=1}^{r} (h-j+1)m_j} \left[ (-1)^{\sum_{j=1}^{r} m_j} e^{[x+y+m_1+\cdots+m_r+y]_q (u+v)} \right]
\]

\[
e^{-[x]_q u} \sum_{m_1, \cdots, m_r=0}^{\infty} q^{\sum_{j=1}^{r} (h-j+1)m_j} \left[ (-1)^{\sum_{j=1}^{r} m_j} e^{[x+y+m_1+\cdots+m_r+y]_q (u+v)} \right].
\]  

(2.10)
The left hand side of (2.10) multiplied by \([2]^p_q\) is given by

\[
[2]^p_q e^{[x]_q u} \sum_{m_1, \ldots, m_r = 0}^\infty q^{\sum_{j=1}^r (h-j+1)m_j} (-1)^{\sum_{j=1}^r m_j} e^{[m_1 + \cdots + m_r + y]_q u} (u+v)
\]

\[
e^{[x]_q u} \sum_{n=0}^\infty q^{nx} E_{n,q}^{(h,r)} (y) \frac{1}{n!} (u+v)^n
\]

\[
= \left( \sum_{l=0}^\infty [x]_q^{l} t^l \right) \left( \sum_{n=0}^\infty q^{nx} E_{n,q}^{(h,r)} (y) \sum_{k=0}^{n} \frac{y^k}{k!(n-k)!} v^{n-k} \right)
\]

\[
= \left( \sum_{l=0}^\infty [x]_q^{l} t^l \right) \left( \sum_{n=0}^\infty \sum_{k=0}^{n} q^{(n+k)x} E_{n+k,q}^{(h,r)} (y) \frac{y^k v^n}{k! n!} \right)
\]

\[
= \sum_{m=0}^\infty \sum_{n=0}^\infty \sum_{k=0}^{m} \left( \frac{m}{k} \right) q^{(n+k)x} E_{n+k,q}^{(h,r)} (y) \frac{y^k v^n}{m! n!}
\]

The right hand side of (2.10) multiplied by \([2]^p_q\) is given by

\[
[2]^p_q e^{-[x]_q u} \sum_{m_1, \ldots, m_r = 0}^\infty q^{\sum_{j=1}^r (h-j+1)m_j} (-1)^{\sum_{j=1}^r m_j} e^{[x+y+m_1 + \cdots + m_r]_q u} (u+v)
\]

\[
e^{-[x]_q u} \sum_{n=0}^\infty q^{nx} E_{n,q}^{(h,r)} (x+y) \frac{1}{n!} (u+v)^n
\]

\[
= \left( \sum_{l=0}^\infty (-[x]_q)^l t^l \right) \left( \sum_{n=0}^\infty \sum_{m=0}^\infty E_{m+k,q}^{(h,r)} (x+y) \frac{y^m v^n}{m! k!} \right)
\]

\[
= \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^{n} \left( \frac{n}{k} \right) E_{m+k,q}^{(h,r)} (x+y) \frac{y^m v^n}{m! k!}
\]

\[
= \sum_{n=0}^\infty \sum_{m=0}^\infty \sum_{k=0}^{n} \left( \frac{n}{k} \right) \frac{E_{m+k,q}^{(h,r)} (x+y) q^{(n+k)x} [-x]_q^{n-k}}{m! n!}
\]

Therefore, by (2.10), (2.11) and (2.12), we obtain the following theorem.

**Theorem 2.5.** For \(m, n \geq 0\) we have

\[
\sum_{k=0}^{m} \left( \frac{m}{k} \right) q^{(n+k)x} E_{n+k,q}^{(h,r)} (y) [-x]_q^{m-k} = \sum_{k=0}^{n} \left( \frac{n}{k} \right) E_{m+k,q}^{(h,r)} (x+y) q^{(n-k)x} [-x]_q^{n-k}
\]

Remark. Recently, several authors have studied \((h, q)\)–extension of Bernoulli and Euler polynomials (see[1]-[5], [9]-[17]).

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