Invariants of coadjoint representation of regular factors

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§0. Introduction

Coadjoint orbits play an important role in the representation theory, symplectic geometry, mathematical physics. According to the orbit method of A.A.Kirillov [1] [2], for nilpotent Lie groups there exists one to one correspondence between coadjoint orbits and irreducible representations in Hilbert spaces. This gives possibility to solve problems of representation theory and harmonic analysis in geometrical terms of the orbit space. However, the problem of classification of all coadjoint orbits for specific Lie groups (such as the group of unitriangular matrices) is an open problem up today that is far from its solution. In the origin paper [2] on the orbit method the description of algebra of invariants and classification of orbits of maximal dimension was obtained.

The main result of this paper consists in construction of generators of the field of invariants for the coadjoint representations of regular factors. By regular factor we further call a Lie algebra that is a factor of unitriangular Lie algebra with respect to some regular ideal. The paper consists of three sections. In the first section we study the diagramm method, introduced in [3, 4]. In §2 we present the notion of extremal minor of characteristic matrix (see definition 2.4). We show that its highest coefficient is invariant with respect to the coadjoint representation (see theorem 2.5). The method of proof is based on the reduction of quantum minors. We state the conjecture 2.6 on structure of algebra of invariants $K[\mathcal{L}^*]^L$. In the last §3 we prove that the field of invariants $K(\mathcal{L}^*)^L$ is a field of rational functions on some system of invariants (see theorem 3.20).

Let $N = \text{UT}(n, K)$ be the group of unitriangular matrices of size $n \times n$ with units on the diagonal and with entries in the field $K$ of zero characteristic. The Lie algebra $\mathfrak{n} = \mathfrak{ut}(n, K)$ of this group consists of lower triangular matrices of size $n \times n$ with zeros on the diagonal. One can define the natural representation of the group $N$ in the conjugate space $\mathfrak{n}^*$ by the formula $\text{Ad}_g^*f(x) = f(\text{Ad}_g^{-1}x)$,
where $f \in \mathfrak{n}^*$, $x \in \mathfrak{n}$ and $g \in N$. This representation is called the coadjoint representation. We identify the symmetric algebra $S(\mathfrak{n})$ with the algebra of regular functions $K[\mathfrak{n}^*]$ on the conjugate space $\mathfrak{n}^*$. Let us also identify $\mathfrak{n}^*$ with the subspace of upper triangular matrices with zeros on the diagonal. The pairing $\mathfrak{n}$ and $\mathfrak{n}^*$ is realized due to the Killing form $(a, b) = \text{Tr}(ab)$, where $a \in \mathfrak{n}$, $b \in \mathfrak{n}^*$. After this identification the coadjoint action may be realized by the formula $\text{Ad}_g^* b = P(\text{Ad}_g b)$, where $P$ is the natural projection of the space of $n \times n$-matrices onto $\mathfrak{n}^*$.

Recall that for any Lie algebra $\mathfrak{g}$ the algebra $K[\mathfrak{g}^*]$ is a Poisson algebra with respect to the Poisson bracket such that $\{x, y\} = [x, y]$ for any $x, y \in \mathfrak{g}$. In the case $k = \mathbb{R}$ the symplectic leaves with respect to this Poisson bracket coincide with the orbits of coadjoint representation [1]. Respectively, the algebra of Casimir elements in $K[\mathfrak{g}^*]$ coincides with the algebra of invariants $K[\mathfrak{g}^*]^N$ of the coadjoint representation.

The coadjoint orbits of the group $N$ are closed with respect to the Zariski topology in $\mathfrak{n}^*$, since all orbits of a regular action of an arbitrary algebraic unipotent group in an affine algebraic variety are closed [7, 11.2.4].

To simplify language we shall give the following definition: a root is an arbitrary pair $(i, j)$, where $i, j$ are positive integers from 1 to $n$ and $i \neq j$. The permutation group $S_n$ acts on the set of roots by the formula $w(i, j) = (w(i), w(j))$.

A root $(i, j)$ is positive if $i > j$. Respectively, a root is negative if $i < j$. We denote the set positive roots by $\Delta^+$. For any root $\eta = (i, j)$ we denote by $-\eta$ the root $(j, i)$. We define the partial operation of addition in the set of positive roots: if $\eta = (i, j) \in \Delta^+$ and $\eta' = (j, m) \in \Delta^+$, then $\eta + \eta' = (i, m)$.

Consider the standard basis $\{y_{ij} : (i, j) \in \Delta^+\}$ in the algebra $\mathfrak{n}$. We shall also use the notation $y_\xi$ for $y_{ij}$, where $\xi = (i, j)$.

An ideal $\mathfrak{m}$ in the Lie algebra $\mathfrak{n}$ is called regular, if it is generated by some subsystem of vectors of the standard basis. Then $\mathfrak{m} = \text{span}\{y_\eta : \eta \in \mathcal{M}\}$, where $\mathcal{M}$ is a subset of $\Delta^+$, satisfying the following property: if in a sum of two positive roots one of summands belongs to $\mathcal{M}$, then the sum also belongs to $\mathcal{M}$.

Denote by $\mathcal{L}$ the Lie factor algebra $\mathfrak{n}/\mathfrak{m}$ (the regular factor) and by $L$ the corresponding factor group of $N$ with respect to the normal subgroup $\exp(\mathfrak{m})$. Note that the conjugate space $\mathcal{L}^*$ is a subspace in $\mathfrak{n}^*$ which consists of all $f \in \mathfrak{n}^*$ that annihilates $\mathfrak{m}$. The coadjoint $L$-orbit for $f \in \mathcal{L}^*$ coincides with its $N$-orbit.
§1. Diagram and permutation associated with Lie algebra $L$

In the paper [3] we corresponded to any regular factor $L$ the diagram $D_L$, constructed applying some formal rule of arrangement of symbols in the table. By the diagram $D_L$ one can easily calculate the index of Lie algebra $L$. Recall that the index of a Lie algebra is the minimal dimension of centralizer of a linear form on this Lie algebra. For algebraic Lie algebras the index is equal to the transcendental degree of the field of invariants of coadjoint representation. For nilpotent Lie algebras (for example, $\mathfrak{ut}(n, K)$) the field of invariants of the coadjoint representation is the pure transcendental extension of the mail field of degree being the index [7]. Respectively, by the diagram one can easily calculate the maximal dimension of coadjoint orbits (see theorem 1.2). Earlier the diagram method was used for classification of all coadjoint orbits of unitriangular group of size $n \leq 7$ [5], for description of special families of coadjoint orbits for an arbitrary $n$ (the subregular orbits [5]; the orbits, associated with involutions [6]).

Let us state the construction method of the diagram $D_L$ and formulate the main assertions of the papers [3, 4]. Consider the order $\succ$ on the set $\Delta_+$ such that

$$(n, 1) \succ (n - 1, 1) \succ \ldots \succ (2, 1) \succ (n, 2) \succ \ldots \succ (3, 2) \succ \ldots \succ (n, n - 1).$$

By means of the ideal $\mathfrak{m}$ we construct the diagram that is a $n \times n$-matrix in that all places $(i, j), \ i \leq j$, are not filled and all other places (i.e. places of $\Delta_+$) are filled by the symbols "⊗", "•", "+", and "−" according to the following rules. The places $(i, j) \in M$ are filled by the symbol "•". We shall refer the procedure of placing of "•" onto the places in $M$ as the zero step in construction of the diagram.

We put the symbol "⊗" on the greatest (in the sense of order $\succ$) place in $\Delta_+ \setminus M$. Note that this symbol will take place in the first column if the set of pairs of the form $(i, 1)$ in $\Delta_+ \setminus M$ in not empty. Suppose that we put the symbol "⊗" on the place $(k, t), \ k > t$. Further, we put the symbol "−" on all places $(k, a), \ t < a < k$, and we put the symbol "+" on all places $(b, t), \ 1 < b < k$. This procedure finishes the first step of construction of diagram.

Further, we put the symbol "⊗" on the greatest (in the sense of order $\succ$) empty place in $\Delta_+$. As above, we put the symbols "+" and "−" on empty places, taking into account the following: we put the symbols "+" and "−" in pairs; if the both places $(k, a)$ and $(a, t)$, where $k > a > t$, are empty, we put "−" on the first place and "+" on the second place; if one of these places,
(k, a) or (a, t), are already filled, then we do not fill the other place. After this procedure we finish the step which we call the second step.

Continuing the procedure further we have got the diagram. We denote this diagram by $\mathcal{D}_L$. The number of last step is equal to the number of the symbols ”⊗” in the diagram.

**Example 1.** Let $n = 7, m = Ky_{51} \oplus Ky_{61} \oplus Ky_{71} \oplus Ky_{62}$. The corresponding diagram is as follows

![Diagram](image)

We denote by $S = \{\xi_1 \succ \xi_2 \succ \ldots \succ \xi_s\}$ the set of pairs $(i, j)$, filled by the symbol ”⊗” in the diagram. For the diagram of example 1, the set $S = \{\xi_1 \succ \xi_2 \succ \xi_3 \succ \xi_4 \succ \xi_5\}$, where $\xi_1 = (4, 1), \xi_2 = (6, 2), \xi_3 = (7, 3), \xi_4 = (7, 4), \xi_5 = (5, 4)$.

Denote by $A_m$ the Poisson algebra $K[p_1, \ldots, p_m; q_1, \ldots, q_m]$ with the bracket $\{p_i, q_j\} = \delta_{ij}$.

Recall that a Poisson algebra $\mathcal{A}$ is a tensor product of two Poisson algebras $\mathcal{B}_1 \otimes \mathcal{B}_2$, if $\mathcal{A}$ is isomorphic to $\mathcal{B}_1 \otimes \mathcal{B}_2$ as commutative associative algebra and $\{\mathcal{B}_1, \mathcal{B}_2\} = 0$.

**Theorem 1.1** [3]. There exist $z_1, \ldots, z_s \in K[\mathcal{L}^*]^L$, where $s = |S|$ such that
1) any \( z_i = y_{\xi_i}Q_i + P_{>i} \), where \( Q_i \) is some product of powers of \( z_1, \ldots, z_{i-1} \) and \( P_{>i} \) is a polynomial in variables \( \{ y_\eta, \, \eta \succ \xi_i \} \);

2) denote by \( \mathcal{Z} \) the set of denominators generated by \( z_1, \ldots, z_s \); the localization \( K[\mathcal{L}^*_s]_\mathcal{Z} \) of the algebra \( K[\mathcal{L}^*_s] \) with respect to the set of denominators \( \mathcal{Z} \) is isomorphic as a Poisson algebra to the tensor product \( K[z_1^{\pm}, \ldots, z_s^{\pm}] \otimes \mathbb{A}_m \) for some \( m \).

Theorem 1.1 directly implies the following

**Theorem 1.2 [3].**

1) The field of invariants \( K(\mathcal{L}^*_s)^L \) coincides with the field \( K(z_1, \ldots, z_s) \).
2) The maximal dimension of a coadjoint orbit in \( \mathcal{L}^* \) equals to the number of symbols "+" and "−" in the diagram \( \mathcal{D}_L \).
3) The index of Lie algebra \( \mathcal{L} \) coincides with the number of symbols "⊗" in the diagram \( \mathcal{D}_L \).

To the Lie algebra \( \mathcal{L} \) we correspond the permutation, defined as follows.

**Definition 1.3 [4].** Denote by \( w = w_\mathcal{L} \) the permutation in \( S_n \) such that

1) \( w(1) = \max\{1 \leq i \leq n \mid (i, 1) \notin M\} \);
2) \( w(t) = \max\{1 \leq i \leq n \mid (i, t) \notin M, \, i \notin \{w(1), \ldots, w(t - 1)\}\} \) for all \( 2 \leq t \leq n \).

As usual, we denote by \( l(w) \) the minimal number of multipliers in decompositions of \( w \) into products of simple reflections. The number \( l(w) \) coincides with the number of inversions in the rearrangement \( (w(1), \ldots, w(n)) \).

**Theorem 1.4 [4] Theorem 2.2.** The number \( l(w) \) coincides with \( \dim \mathcal{L} \).

**Theorem 1.5 [4] Theorem 2.6.** We claim that \( w = r_{\xi_1} r_{\xi_2} \cdots r_{\xi_s} \).

Denote by \( \Delta_+^{(t)} \) the set \( \eta \in \Delta_+ \) that have the form \( (b, t) \) for \( b > t \). Let \( S^{(t)} = \Delta_+^{(t)} \cap S \). Denote

\[
S^{[t]} = S^{(1)} \sqcup \ldots \sqcup S^{(t)}. \tag{1.1}
\]

Denote by \( w^{(t)} \) (resp. \( w^{[t]} \)) the product of reflections \( r_\xi \) (arranged in the decreasing order, in the sense of \( \succ \)), where \( \xi \in S^{(t)} \) (resp. \( S^{[t]} \)). Easy to see that

\[
w^{[t]} = w^{(1)} \cdots w^{(t)}. \tag{1.2}
\]

**Theorem 1.6 [4] Theorem 2.7.** Let \( \eta \in A^{(t)} \), then

1) the place \( \eta \) is filled in the diagram \( \mathcal{D}_L \) by the symbol "−" iff \( w^{[t-1]}(\eta) < 0 \);
2) the place \( \eta \) is filled in the diagram \( \mathcal{D}_L \) by the symbol "●" iff \( w^{[t]}(\eta) > 0 \);
3) \( \eta \) is filled in the diagram \( \mathcal{D}_L \) by the symbol "+" or "⊗" iff \( w^{[t-1]}(\eta) > 0 \) \( w^{[t]}(\eta) < 0 \).

For any \( \xi \in S^{(t)} \) we denote by \( w_\xi \) the product of reflections \( r_{\xi'} \) (arranged in the decreasing, in the sense of \( \succ \), order), where \( \xi' \in S \, \xi' \succeq \xi \). Easy to see that
if $\xi \in \Delta_{+}^{(t)}$, then
\[ w_{\xi} = w^{(1)} \ldots w^{(t-1)} w^{(t)}_{\xi}, \tag{1.3} \]
where $w^{(t)}_{\xi}$ is the product of reflections $r_{\xi'}$ (arranged in the decreasing order, in the sense of $\succ$), where $\xi' \in S^{(t)}$ and $\xi' \succeq \xi$.

**Theorem 1.7.** Let $\xi = \xi_{m} \in S^{(t)}$ and $\eta \in \Delta_{+}^{(t)}$. Suppose that either a) $\xi \succ \eta$ or b) $\eta \in \Delta_{+}^{(t)}$. We claim that

1) if the place $\eta$ was empty after the $m$th step or was filled by the symbol ”•”, then $w_{\xi}(\eta) > 0$;
2) if the place $\eta$ was filled after the $m$th step by any symbol but not ”•”, then $w_{\xi}(\eta) < 0$.

**Proof.** If the place $\eta$ is filled by the symbol •, then $w_{\xi}(\eta) > 0$ (in the case b) see [4, Proposition 2.5]; in the case a) one can prove this similarly. If the place $\eta$ is empty after the $m$th step, then $w_{\xi}(\eta) > 0$ (in the both cases a) and b) see [4, Proposition 2.3(1)]). If the place $\eta$ was filled after the $m$th step by any symbol but not ”•”, then $w_{\xi}(\eta) < 0$ (in the case a) see [4, Proposition 2.3(2)]; in the case b) one can prove similarly to [4, Proposition 2.5(2)]).

**Corollary 1.8.** For any $\xi \in S$ the inequality $w_{\xi}(\xi) < 0$ holds.

**Proof.** Let $\xi = \xi_{m}$. Then
\[ w_{\xi}(\xi) = w_{\xi_{m-1}} r_{\xi}(\xi) = -w_{\xi_{m-1}}(\xi) < 0, \]

since the place $\xi = \xi_{m}$ is empty after the $(m - 1)$th step. \(\square\)

§2. Reduction of quantum minors

Recall some definitions of the theory of quantum matrices. Let $q$ be a variable. The algebra of regular functions on quantum matrices (briefly, the algebra of quantum matrices) $K_{q}[Mat(n)]$ is generated by the elements $\{a_{ij}\}_{i,j=1}^{n}$ subject to the system of relations $ab = qba$, $cd = qdc$, $ac = qca$, $bd = qdb$, $bc = cb$, $ad - da = (q - q^{-1})bc$, imposed on any $2 \times 2$-submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The algebra $K_{q}[Mat(n)]$ is a Noetherian ring without zero divisors and with the Gelfand-Kirillov dimension equals $n^{2}$ (see for instance [S]).

Denote by $B := B_{-}$ the group of lower triangular matrices and by $\mathfrak{b} := \mathfrak{b}_{-}$ its Lie algebra. One can construct the algebra of regular functions $K_{q}[B]$ on the quantum group $B$ factorizing the algebra $K_{q}[Mat(n)]$ modulo the ideal $< a_{ij} | i < j >$ and after localizing with respect to the set of denominators, generated by $a_{11}, \ldots, a_{nn}$.
The ideal \( m \) in Lie algebra \( n \) has its quantum analog – the ideal \( Q_m \) in \( K_q[B] \), generated by \( a_{ij}, (i, j) \in M \). The factor algebra of \( K_q[B] \) modulo the ideal \( Q_m \) we denote by \( K_q[L] \) and call the algebra of regular functions on the quantum group \( L \).

For any systems of columns \( J = \{j_1 < \ldots < j_m\} \) and rows \( I = \{i_1 < \ldots < i_m\} \) the element

\[
M^J_I = \sum_{F=xI} (-q)^{l(x)} a_{f_1,j_1} \ldots a_{f_m,j_m},
\]

where \( F = (f_1, \ldots, f_m) \) and \( x \in S_m \), is called the quantum minor.

By definition, the quantum universal enveloping algebra \( U_q(b) \) is generated by elements \( Y_1, \ldots, Y_{n-1}, K_1, \ldots, K_n \) subject to the relations

\[
Y_i Y_j^2 - (q + q^{-1}) Y_i Y_j Y_i + Y_j^2 Y_i = 0
\]

(the quantum Serre relations), where \( |j - i| = 1 \), and

\[
K_i Y_j = q^{-(\varepsilon_i - \varepsilon_j + 1)} Y_j K_i,
\]

where \( 1 \leq i \leq n, 1 \leq j \leq n - 1 \). By quantum algebra \( U_q(n) \) we mean the subalgebra of \( U_q(b) \), generated by \( \{Y_i\} \). The ideal \( m \) corresponds to the ideal \( \tilde{Q}_m \) in \( U_q(n) \), generated by all \( Y_{i,j} \), where \( i > j \) and \( (i, j) \in M \). We say that the quantum group \( U_q(L) \) is the factor algebra of \( U_q(n) \) modulo the ideal \( \tilde{Q}_m \).

Universal enveloping algebra \( U(n) \) is the factor algebra of \( U_q(n) \) modulo \( q - 1 \). The symmetric algebra \( S(n) \) coincides with the graded algebra \( \text{gr} U(n) \). Similarly, for the Lie algebra \( L \) we have

\[
S(L) = \text{gr} \left(U_q(L) \mod (q - 1)\right).
\]

It is obvious that the algebras \( K_q[B] \) and \( U_q(b) \) are not isomorphic (since their factors modulo \( q - 1 \) are not isomorphic). Denote by \( K'_q[B] \) and \( U'_q(b) \) the localization of \( K_q[B] \) and \( U_q(b) \) modulo \( q - 1 \).

Well known that the algebras \( K'_q[B] \) and \( U'_q(b) \) are isomorphic. This isomorphism is called the isomorphism of Drinfeld. One can construct it directly: subalgebra of \( K'_q[B] \), generated by the elements

\[
Y_{ij} = -\frac{a_{ij} q^{-1}}{q - q^{-1}}, \text{ where } i > j, \text{ and } K_i = a_{ii}^{-1}, \text{ where } 1 \leq i \leq n,
\]

is isomorphic to \( U'_q(b) \) and coincides with \( K'_q[B] \) (see for instance [9]). Briefly, one can check this as follows: we show that \( \{Y_{i+1,i}, 1 \leq i \leq n - 1\} \) satisfy
quantum Serre relations; we extend the correspondence $Y_i \rightarrow Y_{i+1,i}$ to the homomorphism of $U_q(b)$ into $K'_q[B]$; this homomorphism is an isomorphism, since it induces the isomorphism of corresponding graded algebras.

In what follows, we shall identify $U_q(b)$ and $U_q(n)$ with its images in $K'_q[B]$. Note that the system of ordered (in the sense of lexicographical order) monomials in $\{Y_{ij}| i > j\}$ forms the basis of $U_q(n)$ as a free module over $K[q,q^{-1}]$.

Consider the right action of $n$-dimensional torus on the algebra of quantum matrices by the formula $a_{ij} \cdot t = t_j a_{ij}$, where $t = (t_1, \ldots, t_n)$. We say that an element $b$ of $K_q[B]$ is homogeneous of weight $(k_1, \ldots, k_n)$, if

$$b \cdot t = t_1^{k_1} \ldots t_n^{k_n} b.$$ 

By a homogeneous element $b \in K_q[B]$ we construct the element

$$\tilde{b} = \frac{(-1)^k}{(q - q^{-1})^k} \cdot ba_{11}^{-k_1} \ldots a_{nn}^{-k_n}$$

in $U'_q(n)$, where $k = k_1 + \ldots + k_n$.

For the quantum minor $M_I^J$ of size $m$ with system of columns $I$ and rows $J$ we obtain

$$M_I^J = \sum_{F=xI, \, f_\alpha > j_\alpha \forall 1 \leq \alpha \leq m} (-q)^{l(x)} q^{\phi(x)} (q^{-1} - q)^{-\delta(F,J)} y_{f_1,j_1} \cdots y_{f_m,j_m}, \quad (2.2)$$

where $\phi(x)$ is some integer and $\delta(F,J) = \text{card}\{1 \leq \alpha \leq m| f_\alpha = j_\alpha\}$.

We construct the formal matrix $\Phi_L$ that all places in $M$ and also the places on and upper the diagonal are filled by zeros; on the other places of the form $(i,j), \, i > j$ we put elements $y_{ij}$ of the standard basis. For instance, for the Lie algebra $L$ of the example 1 we obtain the diagram $D_L$ and the matrix $\Phi_L$:

$$D_L = \begin{array}{cccccccc}
+ & + & \otimes & - & \otimes & - & \otimes & - \\
\otimes & - & + & + & + & + & + & + \\
\bullet & + & + & + & + & + & + & + \\
\bullet & \bullet & \otimes & \otimes & \otimes & \otimes & \otimes & \otimes \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}, \quad \Phi_L = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{31} & y_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\
y_{41} & y_{42} & y_{43} & 0 & 0 & 0 & 0 & 0 \\
0 & y_{52} & y_{53} & y_{54} & 0 & 0 & 0 & 0 \\
0 & y_{62} & y_{63} & y_{64} & y_{65} & 0 & 0 & 0 \\
0 & 0 & y_{73} & y_{74} & y_{75} & y_{76} & 0 & 0 \\
\end{pmatrix}.$$

Let $\lambda$ be a variable. Any minor $M_I^J$ of the matrix $\Phi_L$ is an element of $S(L) = K[L^*]$. Consider the characteristic matrix $\Phi_L - \lambda E$. The minor $M_I^J(\lambda)$ of characteristic matrix with system of rows $I$ and columns $J$ has the form:

$$M_I^J(\lambda) = \sum_{F=xI, \, f_\alpha > j_\alpha \forall 1 \leq \alpha \leq m} (-1)^{l(x)} (-\lambda)^{\delta(F,J)} y_{f_1,j_1} \cdots y_{f_m,j_m}. \quad (2.3)$$
Decomposing (2.2) (resp. (2.3)) in powers of \((q - q^{-1})^{-1}\) (resp. \(\lambda\)), we obtain

\[
\tilde{M}_I^J = \sum_{\alpha=0}^{n} M_\alpha (q - q^{-1})^{-\alpha}, \tag{2.4}
\]

\[
M_I^J(\lambda) = \sum_{\alpha=0}^{n} M_\alpha \lambda^\alpha, \tag{2.5}
\]

where \(M_\alpha \in U_q(n)\), \(M_\alpha \in U(n)\) for any \(\alpha\).

We shall say that the degree of quantum minor \(M_I^J\) (more precisely, the degree modulo \(m\)) is the greatest number \(d\) such that \(M_\alpha \notin \tilde{Q}_m\). The degree of \(M_I^J(\lambda)\) is defined in the usual way.

**Lemma 2.1.** Let \(\tilde{Q}_m\) be the ideal of \(U_q(n)\), generated by \(\tilde{Q}_m\) and \(q - 1\). We claim that

1) \(M_\alpha = \text{gr} (M_\alpha \mod \tilde{Q}_m)\),

2) the degrees of minors \(M_I^J\) and \(M_I^J(\lambda)\) coincide.

**Proof** of the statement 1) is obvious. Let us prove 2). According to PBW theorem, the monomials

\[
Y^{k_1}_{f_1,j_1} \cdots Y^{k_N}_{f_N,j_N}, \tag{2.6}
\]

where \(k_1, \ldots, k_N \in \mathbb{Z}_+\) and \((f_1, j_1) \succ \ldots \succ (f_N, j_N)\), form the basis of \(U_q(n)\) over the field \(K\). The similar system of polynomials in \(\{y_{ij}\}\) form the basis of \(S(n)\) over \(K\).

The quantum minor of (2.2) is presented as a linear combination of basic monomials of the form (2.6). Hence, \(M_\alpha\) belongs to \(\tilde{Q}_m\) if and only if every monomial, which is included as a summand in \(M_\alpha\), belongs to \(\tilde{Q}_m\). In its turn, the monomial of (2.6) belongs to \(\tilde{Q}_m\) if and only if it contains at least one element \(Y_{f,j}\) of \(\tilde{Q}_m\). Similar argumentation is true for any coefficient \(M_\alpha\) from (2.5). Hence, \(M_\alpha\) belongs to \(\tilde{Q}_m\) if and only if \(M_\alpha\) belongs to \(S(n)\).

Let \(d\) be the common degree of the minors \(M_I^J\) and \(M_I^J(\lambda)\). Then

\[
M_I^J(\lambda) = M_d\lambda^d + M_{d-1}\lambda^{d-1} + \ldots + M_0, \tag{2.7}
\]

where \(M_d \neq 0\), and the element \(\tilde{M}_I^J\), which is taken modulo \(Q_m\), decomposes:

\[
\tilde{M}_I^J = (q - q^{-1})^{-d} (M_d + (q - q^{-1})M_{d-1} + \ldots + (q - q^{-1})^d M_0), \tag{2.8}
\]

where \(M_d \neq 0 \mod \tilde{Q}_m\).

For a quantum minor \(\widetilde{M} = \widetilde{M}_I^J\) and a number \(1 \leq i < n\) we denote:

\[
\widetilde{M}^I = \begin{cases} 
M^J_{(I \setminus i) \cup i+1}, & \text{if } i \in I \text{ and } i+1 \notin I, \\
0, & \text{in all other cases},
\end{cases}
\]
Similarly, we denote the minors $M^-\lambda$ and $M^+\lambda$ for $M^J\lambda$. The commutative relations of the algebra of quantum matrices imply the following

**Lemma 2.2.** Let $M = M^J_I$ and $a = a_{i+1,i}$, then the following quantity takes place in the ring $K_q[B]$:

$$Ma - q^s aM = -(q - q^{-1})M^-a_{i+1,i+1} + (q - q^{-1})M^+a_{ii}$$  \hspace{1cm} (2.9)

for some $s \in \{0, 1, -1\}$.

**Proof** directly follows from [8, Lemmas 4.1.5, 5.1.2]. One can also prove the formula (2.9), using the formula of $R$-matrix (see [10, Chapter 7]). □

**Definition 2.3.** A nonzero minor $M^J_I\lambda$ of the characteristic matrix $\Phi_L - \lambda E$ is extremal, if for any $i$ the following inequalities hold $\deg M^-\lambda < \deg M\lambda$ and $\deg M^+\lambda < \deg M\lambda$ (this inequalities we also consider to be true if $M^-\lambda = 0$ or $M^+\lambda = 0$). Roughly speaking, a minor is extremal if its degree decreases while its rows are moving down or its columns are moving to the left.

**Remark 2.4.** Similarly, one can define an extremal minor of the algebra of quantum matrices. From lemma 2.1, a minor $M^J_I\lambda$ is extremal if and only if the corresponding quantum minor $M^J_I\lambda$ is extremal.

We denote by $P^J_I$ the highest coefficient $M_d$ in decomposition (2.7) of the minor $M^J_I\lambda$.

**Theorem 2.5.** The highest coefficient $P^J_I$ of any extremal minor is invariant with respect to the adjoint (resp. coadjoint) representation of the group $N$ in $S(n)$ (resp. $K[g^+]$).

**Proof.** It is sufficient to prove that $\text{ad}_y P^J_I = 0$, where $y = y_{i+1,i}$ and $1 \leq i \leq n - 1$. Denote

$$Y = Y_{i+1,i} = -\frac{a_{i+1,i}a_{ii}^{-1}}{(q - q^{-1})}.$$  \hspace{1cm}

The formula (2.9) implies

$$\tilde{M}Y - q^s Y\tilde{M} = \tilde{M}^- - \tilde{M}^+.$$  \hspace{1cm} (2.10)

Taking (2.10) modulo $\tilde{Q}_m$, we obtain

$$(q - q^{-1})^{-d} \left\{ [\tilde{M}_d + (q - q^{-1})\tilde{M}_{d-1} + \ldots] Y - Y [\tilde{M}_d + (q - q^{-1})\tilde{M}_{d-1} + \ldots] \right\} =$$

$$(q - q^{-1})^{-d} \left\{ [\tilde{M}_d^- + (q - q^{-1})\tilde{M}_{d-1}^+ + \ldots] - [\tilde{M}_d^+ + (q - q^{-1})\tilde{M}_{d-1}^- + \ldots] \right\}.$$
We cut down \((q - q^{-1})^{-d}\). Since \(M_d^J\) is an extremal minor (see remark 2.4), then 
\[M_d^d = M_d^d = 0 \mod q_m.\] Further, after reduction modulo \(q - 1\) we obtain that 
\(\text{ad}_y\) annihilate \(M_d \mod (q - 1) \in U(n)\). Since 
\[P_l^J = \text{gr}(M_d \mod (q - 1)),\]
then \(\text{ad}_y P_l^J = 0. \Box\)

**Conjecture 2.6.** The algebra of invariants \(K[L^*]^L\) of coadjoint representation of regular factor \(L\) is generated by the highest coefficients of extremal minors.

The conjecture is true for the case \(L = n\) (i.e. \(m = 0\)). A corner minor \(M_i\) is a minor of the matrix \(\Phi\) that is lying on the intersection of the first \(i\) columns and last \(i\) rows. The algebra of invariants \(K[n]^N\) is generated by the system of corner minors \(M_i\) (see [2]), where \(1 \leq i \leq \left[\frac{n}{2}\right]\), any of which is extremal.

§3. Field of invariants

In this section we shall correspond to any \(\xi \in S\) an extremal minor \(M_{\xi}(\lambda)\) (see definition 3.8). As it was shown in the section §2, its highest coefficient \(P_{\xi}\) is an invariant of the coadjoint representation of \(L\).

We shall prove that the field of invariants \(K[L^*]^L\) coincides with the field of rational functions in \(P_{\xi}\), where \(\xi \in S\).

Let \(\xi\) be some element of \(S\), say \(\xi = (k, t) \in S\), where \(k > t\). As above, \(w_{\xi}\) is defined from (1.3).

**Lemma 3.1.** Let \(i > t\). We claim that
1) if \(r_\eta(i) = i\) for any \(\eta \in S\) and \(\eta \succeq \xi\), then \(w_\xi(i) = i;\)
2) if \(r_\eta(i) \neq i\) for some \(\eta \in S\) and \(\eta \succeq \xi\), then \(w_\xi(i) \leq t\). In particular, \(w_\xi(i) < i;\)
3) \(w_\xi(i) \leq i.\)

**Proof** of the statement 1) is obvious, the statement 3) follows from 1) and 2).

To prove 2) we consider the sequence \(i_0 = i, i_1 = w^{(t)}(i_0)\) and \(i_\alpha = w^{(t-\alpha+1)}(i_{\alpha-1}),\) where \(2 \leq \alpha \leq t.\)

a) Under the assumption of statement 2), we shall show that there exists a number \(1 \leq \alpha \leq t\) such that \(i_\alpha = t - \alpha + 1.\)

Let \(1 \leq \alpha \leq t\) be the least number such that the root \(\eta\), which is equal to \((i, t - \alpha + 1)\), contains in \(S\) and \(\eta \succeq \xi\). Then \(i_{\alpha-1} = \ldots = i_0 = i.\)

If set \(\{\theta \in S| \theta > \eta\}\) is empty, then \(i_\alpha = r_\eta(i) = t - \alpha + 1.\) In the converse case, let \(\theta = (j, t - \alpha + 1)\) be the least, in sense of \(>,\) element of \(S| t-\alpha+1)\) such that \(\theta > \eta.\) Then \(j > i\) and \(j = w^{(t-\alpha+1)}i_{\alpha-1} = r_\eta r_\eta i_{\alpha-1} = i_\alpha.\) The positive root \(\theta\) is the sum of two positive roots \(\eta\) and \(\gamma = (j, i).\) Since
It follows from theorem 1.7(2) that the place \( \otimes \) of order \( \eta' \) before the symbol \( " - " \) appeared on the place \( \theta \). Hence, there exists at least one symbol \( \otimes \) in the row \( j = i_\alpha \) and in the columns with the numbers less or equal to \( t - \alpha + 1 \). Let \( \alpha' \) be the least number such that \( \alpha' > \alpha \) and the root \( \eta' \), which is equal to \( (i_\alpha, t - \alpha' + 1) \), contains in \( S \). Then \( i_{\alpha'-1} = \ldots = i_\alpha \).

If the set \( \{ \theta \in S^{(t-\alpha'+1)} | \theta > \eta' \} \) is empty, then

\[
i_{\alpha'} = r_{\eta'}(i_\alpha) = t - \alpha' + 1.
\]

In the converse case, the argumentation similar to above leads to existence of \( \alpha'' > \alpha' > \alpha \) such that the root \( \eta'' \), that is equal to \( (i_{\alpha'}, t - \alpha'' + 1) \), belongs to \( S \). Continuing this process further, we obtain that at some \( p \) th step the set

\[
\{ \theta \in S^{(t-\alpha^{(p)}+1)} | \theta > \eta^{(p)} \}
\]

is empty and, therefore,

\[
i_{\alpha^{(p)}} = t - \alpha^{(p)} + 1.
\]

Note that for the constructed \( \alpha = \alpha^{(p)} \) we have inequality \( i_\alpha = t - \alpha + 1 < t \) and \( i_{\alpha-1} \geq i_{\alpha-2} \geq \ldots \geq i \) at that.

b) Let us finish the proof of statement 2) using induction on the number of elements in the set \( \{ \eta \in S | \eta \succeq \xi \} \). The proof is obvious if this set contains of one element. In the general case, for \( 1 \leq \alpha \leq t \) from a) we obtain

\[
w_\xi(i) = w^{(1)} \ldots w^{(t-\alpha)} w^{(t-\alpha+1)} \ldots w^{(t-1)} w_\xi(t)(i) = w^{(t-\alpha)} i_\alpha = w^{(t-\alpha)} (t - \alpha + 1).
\]

By induction assumption, \( w^{(t-\alpha)} (t - \alpha + 1) \leq t - \alpha + 1 < t \). We conclude that \( w_\xi(i) < t \). \( \Box \)

**Corollary 3.2.** Let as above \( \xi = (k, t), \ k > t \). We claim that

1) if \( r_\eta(k) = k \) for some \( \eta \in S \) and \( \eta \succeq \xi \), then \( w_\xi(t) = k > t \);

2) if \( r_\eta(k) \neq k \) for some \( \eta \in S \) and \( \eta \succeq \xi \), then \( w_\xi(t) < t \).

**Proof.** Since \( \xi \in S \), then \( \xi = \xi_m \) for some \( 1 \leq m \leq s \) and

\[
w_\xi(t) = w_{\xi_m}(t) = w_{\xi_{m-1}} r_{\xi_m}(t) = w_{\xi_{m-1}}(k).
\]

Applying lemma 3.1 for \( \xi = \xi_{m-1} \) and \( i = k \), we have got the proof of statement of corollary. \( \Box \)

**Corollary 3.3.** Let \( \xi \) be as in previous corollary. If \( \xi \) is the least, in the sense of order \( \succ \), element of \( S^{(t)} \), then we claim that

1) if \( r_\eta(k) = k \) for some \( \eta \in S^{(t-1)} \) (i.e. there is no symbol \( " \otimes " \) in the \( k \) th row and columns with numbers \( < t \) of the diagram \( D_\xi \)), then \( w(t) = k \);
2) if \( r_\eta(k) \neq k \) for some \( \eta \in S^{[t-1]} \), then \( w(t) < t < k \).

**Proof.** For the least, in the sense of \( \succ \), root \( \xi \in S^{[t-1]} \) we have \( w_\xi = w[t] \). The statement follows from corollary 3.2 applying \( w[j] = w(j) \) for any \( 1 \leq j \leq t \).

**Corollary 3.4.** Any symbol \( \otimes \) of an arbitrary \( t \)th columns take place either \((w(t), t)\), or below it.

**Proof.** Let \( \xi \) be the least, in the sense of \( \succ \), element of \( S^{(t)} \). If \( r_\eta(k) = k \) for any \( \eta \in S^{[t-1]} \), then \((w(t), t) = \xi \). If \( r_\eta(k) \neq k \) for some \( \eta \in S^{[t-1]} \), then \( w(t) < t \), the place \((w(t), t)\) is upper the diagonal and, therefore, it is upper all symbols ”\( \otimes \)”. □

Let as above \( \xi = (k, t), \ k > t \). Denote

\[ h := w_\xi(t). \]

We shall give the definition of systems of columns \( J \) and rows \( I \) of the minor \( M_\xi \) (see definition 3.8) in each of the following cases separately: 1) \( h > t \) and 2) \( h < t \). According to corollary 3.2, the case \( h = t \) is impossible.

**Case 1.** \( h > t \). By corollary 3.2, \( h = k > t \). In this case there is no symbol ”\( \otimes \)” in the \( k \)th row on the left side of place \( \xi = (k, t) \). We put

\[ J := J(\xi) = \{ 1 \leq j \leq t : w_\xi(j) \geq h \}, \quad I := I(\xi) = wJ(\xi). \tag{3.1} \]

It is obvious that \( |I| = |J| \).

**Case 2.** \( h < t \). In this case there exists at least one symbol ”\( \otimes \)” in the \( k \)th row on the left side of the place \( \xi = (k, t) \). The system \( J := J(\xi) \) is defined as in (3.1). Denote

\[ I_* := I_*(\xi) = \{ t < i \leq n : i > t, \ w_\xi(i) < h \}, \]

\[ I := I(\xi) = [h, t] \sqcup I_* \tag{3.2} \]

Here \([h, t]\) is a segment of positive integers (see definition 3.12(1)). Note that in the case 2 the equality \( |I| = |J| \) is not obvious beforehand and will be proved in lemma 3.7.

**Remark 3.5.**

1) In both cases \( w_\xi(j) = w(j) \) for any \( 1 \leq j < t \). Hence,

\[ J = \{ 1 \leq j < t : \ w(j) < h \} \sqcup \{ t \}. \tag{3.3} \]

2) Let \( \xi = \xi_m \). The element \( i \in I_* \) (see case 2) if and only if \( w_\xi(i, t) < 0 \). By theorem 1.7, the last is equivalent to the statement that the place \((i, t)\) is filled
after the mth step by one of the symbols ”⊗”, ”+” or ”−” (but not ”•”).

**Lemma 3.6.** Let as above $\xi = \xi_m = (k, t)$. In both cases 1 and 2
1) there exists $1 \leq c \leq h$ such that $J = [c, t]$;
2) the rectangle $[h, n] \times [1, c]$ is filled by the symbols • in the diagram $D_L$ (resp. by zeros in the matrix $\Phi_L$);
3) there is no symbol ”⊗” in the rectangle $[1, h] \times J$ of the diagram $D_L$.

**Proof.** Let $w_\xi(j) < h$ for some $1 \leq j < t$. Since $h = w_\xi(t)$, then $h \notin \{w_\xi(1), \ldots, w_\xi(j - 1)\}$. By (3.3), $w_\xi(j) = w(j) < h$ and $h \notin \{w(1), \ldots, w(j - 1)\}$. The definition of $w$ implies $(h, j) \in M$ (i.e. the place $(h, j)$ is filled in the diagram by the symbol ”•”). Since $m$ is an ideal in $n$, then $[h, n] \times [1, j]$ is contained in $M$. This implies 1) and 2).

Since $w(j) > h$ for some $j \in [c, t)$, then, by corollary 3.4, all symbols ”⊗”, which lie in columns $[c, t)$, is contained in rows $[h, n]$. This proves 3). □

**Lemma 3.7.** Let $\xi$ as above and let $h < t$ (i.e. the case 2 takes place). Then $|J| = |I|$.

**Proof.** Let $a(t)$ be the greatest number of $[1, n]$ such that $(a(t), t)$ do not belong to $M$. Then $M \cap \Delta_+ = (a(t), n]$. Easy to see that $w_\xi(i) = i$ for any $i \in (a(t), n]$. Therefore, $w_\xi[1, a(t)] = [1, a(t)]$.

Consider two decompositions of the segment $[1, a(t)]$ as a union of disjoint sets:

$$[1, a(t)] = [1, t] \sqcup [t, a(t)],$$
$$[1, a(t)] = [1, h] \sqcup [h, a(t)].$$

Recall that $w_\xi(t) = h$. Let us show that

$$[1, h] = w_\xi[1, c] \sqcup w_\xi(I_*),$$  \hspace{1cm} (3.4)

$$[h, a(t)] = w_\xi(J) \sqcup w_\xi(I'_*),$$ \hspace{1cm} (3.5)

where $I'_*$ consists of all $i$ such that the place $(i, t)$ was not filled after the mth step.

Since the common number of elements of the left and right hand sets of these formulas are equal, to prove (3.4) and (3.5) it is sufficient to show that the right hand sets of these formulas are contained in the corresponding right hand sets.

Really, if $i \in I_*$, then, by remark 3.5 and theorem 1.7 (case b)), it follows $w_\xi(i, t) < 0$. Hence, $w_\xi(i) < w_\xi(t) = h$. Therefore, $w_\xi(I_*) \subset [1, h]$.

If $i \in I'_*$, then, by theorem 1.7 (case b)), it follows $w_\xi(i, t) > 0$. Hence, $w_\xi(i) > w_\xi(t) = h$. That is $w_\xi(I'_*) \subset [h, a(t)]$. 

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The inclusion $w_\xi(J) \subset [h, a^{(t)}]$ directly follows from definition of $J$. Finally, $w_\xi[1, c) \subset [1, h)$ follows from lemma 3.6(2). The formulas (3.4) and (3.5) are proved.

By the equality (3.4), we have got $|w_\xi(I_*)| = h - c$. Then $|I_*| = h - c$. Since $I = I_* \cup [h, t]$ and $J = [c, t]$, then

$$I = |I_*| + (t - h + 1) = (h - c) + (t - h + 1) = t - c + 1 = |J|. \quad \Box$$

**Definition 3.8.** In both cases 1) and 2) we denote by $M_\xi(\lambda)$ the minor of characteristic matrix $\Phi - \lambda E$ with system of rows $I = I(\xi)$ and system of columns $J = J(\xi)$. We denote by $P_\xi$ its highest coefficient.

Our next goal is to prove that in both cases $M_\xi(\lambda)$ is an extremal minor; this will be proved in proposition 3.19. Case 1 is more simple; reader interested in this case may turn directly to proposition 3.19. For the case 2 we need to prove the additional statements 3.9-3.18.

Let the case 2) takes place. As above $\xi = \xi_m = (k, t)$, where $k > t$, and $h = w_\xi(t)$, where $h < t$. As in the proof of lemma 3.7, we define $a^{(t)} := \max\{i | (i, t) \notin M\}$.

**Notations 3.9.**
1) $E := [h, a^{(t)}], \quad F_0 := I \setminus J = [c, h)$.
2) $w^{[j]}_\xi := w_\xi, \text{ if } j \geq t, \text{ and } w^{[j]}_\xi := w^{[j]}, \text{ if } 1 \leq j < t$ (for definitions of $w^{[j]}$ and $w_\xi$ see (1.2) and (1.3)).
3) Decompose $E$ into the union of disjoint sets $E = F \cup D$, where

$$F := \{a \in E | w^{[a-1]}_\xi(a) < h\}, \quad \text{(3.6)}$$

$$D := \{a \in E | w^{[a-1]}_\xi(a) \geq h\}. \quad \text{(3.7)}$$

4) $w^{(c)}_\xi := w^{(c)} \ldots w^{(t-1)}w^{(t)}_\xi$, \quad $w^{[j]}_{\xi^*} := \begin{cases} w^{(c)} \ldots w^{(j)}, & \text{if } c \leq j < t, \\ w^{[j]}_{\xi^*}, & \text{if } t < j \leq n. \end{cases}$

Note that $w_\xi = w^{[c-1]}w^{[n]}_{\xi^*}$. Since $w^{[c-1]}[1, h) \subset [1, h)$ and $w^{[c-1]}(i) = i$ for any $h \leq i \leq n$, then rewrite (3.6) and (3.7) as follows

$$F = \{j \in E | w^{[j-1]}_{\xi^*}(j) \in F_0\}, \quad \text{(3.8)}$$

$$D = \{j \in E | w^{[j-1]}_{\xi^*}(j) \in J\}. \quad \text{(3.9)}$$

**Remark 3.10.** Note that $h \in D$ (as there is no symbol ”$\otimes$” in the $h$-row, otherwise $w_\xi(t) < h$), $t \in F$ (as $w^{[t-1]}_{\xi^*}(t) = w_\xi(i_*) \in F_0$, where $i_*$ is an element of $I_*$ such that $(i_*, t)$ is the greatest, in the sense of $>$, element of $S^{(t)}$) and
I, \subset F \text{ (by definition of } I_*).$

**Lemma 3.11.** Let as above $\xi = (k, t)$, where $k > t$. Let $a \in D,$ $b \in F$ and $a > b.$ If $(b, p) \in S,$ where $p < t,$ then $(a, p) \in M.$

**Proof.** Let $\xi_l = (b, p) \in S.$ Let $q$ be the greatest number such that $q \leq m$ and $\xi_q \in \Delta_{[a-1]}^t$. Since $b < a,$ then $l \leq q.$

1) Show that $w_{\xi_l}(a) \geq h$ for any $1 \leq i \leq q.$ We shall prove by induction on $i,$ starting with the greatest number $q.$ Since $a \in D,$ then for $i = q$ we obtain

$$w_{\xi_q}(a) = w_{\xi_l}^{[a-1]}(a) \geq h.$$ 

Suppose that $w_{\xi_l}(a) \geq h$ was already proved for number $i$; let us prove for number $i - 1.$ Let $\xi_i = (a_1, b_1),$ $a_1 > b_1.$ If $a_1 \neq a,$ then $r_{\xi_i}(a) = a$ and, therefore,

$$w_{\xi_i}(a) = w_{\xi_{i-1}}r_{\xi_i}(a) = w_{\xi_{i-1}}(a).$$

Using the induction assumption, we have got $w_{\xi_{i-1}}(a) \geq h.$ If $a_1 = a,$ then

$$w_{\xi_i}(b_1) = w_{\xi_{i-1}}r_{\xi_i}(b_1) = w_{\xi_{i-1}}(a).$$

On the other hand, $w_{\xi_i}(\xi_i) < 0$ (see corollary 1.8). Since $\xi_i = (a, b_1),$ then

$$h \leq w_{\xi_i}(a) < w_{\xi_i}(b_1) = w_{\xi_{i-1}}(a),$$

this proves the statement 1).

2) Turn directly to the proof of the lemma. We may consider that $p$ is a greatest number such that $(b, p) \in S,$ where $p < t.$ Since $b \in F,$ then

$$w_{\xi_i}(b) = w_{\xi_l}^{[b-1]}(b) < h.$$ 

Introduce the notations $\eta = (a, p),$ $\gamma = (a, b).$ We obtain $\eta = \xi_l + \gamma.$

2a) Show that $w_{\xi_l}(\gamma) > 0.$ Really, by statement 1) of the proof, it follows that $w_{\xi_l}(a) \geq h.$ On the other hand, we showed above that $w_{\xi_l}(b) < h.$ We obtain

$$w_{\xi_l}(a) > w_{\xi_l}(b),$$
hence $w_{\xi_l}(\gamma) = w_{\xi_l}(a, b) > 0.$

2b) Since $w_{\xi_l}(\gamma) > 0,$ then the place $\gamma$ is either empty after the $l$th step, or filled by the symbol $\bullet$ (see theorem 1.7 (case a)). If the last case takes place, then $(a, b) \in M.$ Since $m$ is an ideal of $n,$ then $(a, p) \in M;$ this proves the statement of lemma.

Suppose that $\gamma$ empty after the $l$th step, then the place $\gamma$ is also empty after the previous $(l - 1)$th step. By theorem 1.7 (case a), $w_{\xi_{l-1}}(\gamma) > 0.$ Since the place $\xi_l$ is also empty after the $(l - 1)$th step, then $w_{\xi_{l-1}}(\xi_l) > 0.$ Since $\eta = \xi_l + \gamma,$ then $w_{\xi_{l-1}}(\eta) > 0.$ By $a > b,$ the place $\eta$ is filled after the $(l - 1)$th step. The theorem 1.7 implies that the place $\eta$ maybe filled only by the symbol $\bullet$ (i.e.
\[ \eta \in M \]. □

**Notations and definitions 3.12.**

1) By a segment \([a, b]\), \(a, b \in \mathbb{N}\), of the positive integers \(\mathbb{N}\) we call the set \(\{i \in \mathbb{N} | a \leq i \leq b\}\).

2) Let \(C \subset \mathbb{N}\). We shall say that a set \(A \subset C\) is a segment in \(C\), if \(A\) is an intersection of the positive integers with \(C\).

3) Introduce the relation \(<\) on the set of segments, according to which \(A < B\) if and only if \(i < j\) for all \(i \in A\) and \(j \in B\).

4) If \(A = [a, b]\) and \(B = [c, d]\). Introduce the relation \(A \ll B\) (resp. \(B \gg A\)) that means that \(a = c\) and \(A \subseteq B\) (resp. \(b = d\) and \(B \supseteq A\)).

Decompose \(D\) and \(F\) into segments:

\[
D = D_1 \cup D_2 \cup \ldots \cup D_l, \quad D_1 < D_2 < \ldots < D_l,
\]
\[
F = F_1 \cup F_2 \cup \ldots \cup F_l, \quad F_1 < F_2 < \ldots < F_l.
\]

Note that

\[ h \in D_1 < F_1 < D_2 < F_2 \ldots < D_l < F_l \gg [k, a(t)]. \]

Let \(i \in E\). Let \(\{i_\alpha\}\) be a sequence defined as follows. If \(i > t\), then, as in the proof of lemma 3.1, we put \(i_0 = i, \ i_1 = w^{(t)}(i_0)\) and \(i_\alpha = w^{(t-\alpha+1)}(i_{\alpha-1})\), where \(2 \leq \alpha \leq t\). If \(i \leq t\), then \(i_\alpha\) is defined similar, changing \(\xi\) by the least, with respect to \(\succ\), element of \(S^{[i-1]}\).

**Definition 3.13.** For any \(i \in E\) we denote by \(i'\) the greatest number in the sequence \(\{i_\alpha\}\), that is less than \(i\). By the proof of lemma 3.1 (see the end of a)), it follows that if \(i' = i_\alpha\), then \(i_\alpha = i - \alpha + 1 < i\) and \(i_{\alpha-1} \geq i_{\alpha-2} \geq \ldots \geq i\).

For any \(i \in I_*\) we construct a chain

\[ i > i' > i'' > \ldots > i^{(\mu(i))} = w_{\xi^*}(i) \in F_0, \]

in which \(i^{(p)} = (i^{(p-1)})'\) for any \(0 \leq p \leq \mu(i)\).

**Lemma 3.14.**

1) The chains (3.10) for different \(i \in I_*\) do not intersect.

2) For any \(b \in F_0\) there exists \(i \in I_*\) such that \(b = i^{(\mu(i))}\).

3) If \(a \in F \cap J\), then \(a = i^{(\nu)}\) for some \(i \in I_*\) and \(1 \leq \nu \leq \mu(i)\).

**Proof.** If \(a\) is a common element of chains (3.10) for \(i_1\) and \(i_2\) from \(I_*\), then

\[ w^{(c)} \cdots w^{(a-1)}(a) = w_{\xi^*}(i_1) = w_{\xi^*}(i_2). \]

Hence, \(i_1 = i_2\). This proves 1).

The statement 2) follows from \(w_{\xi^*}(I_*) \subset F_0\) and \(|I_*| = h - c = |F_0|\) (see proof of lemma 3.7 and definition of \(F_0\) from 3.9(1)).

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Finally, let $a \in F \cap J$ (i.e. $a \in F$ and $a \leq t$). Then $w_{\xi^*}^{[a-1]}(a) \in F_0$. There exists $i \in I_*$ such that

$$w_{\xi^*}^{[a-1]}(a) = w_{\xi^*}(i).$$

We have $a = w^{(a)} \cdots w^{(i)}_{\xi^*}(i)$ and, therefore, $a$ belongs to the chain (3.10) for $i$ (see the proof of lemma 3.7). □

**Definitions 3.15.**
1) For any $1 \leq a \leq l$ we denote

$$D_{[a]} = D_1 \cup \ldots \cup D_a,$$

$$F_{[a]} = F_1 \cup \ldots \cup F_a,$$

$$F'_a = \{i' \mid i \in F_a\},$$

$$F'_{[a]} = F'_1 \cup \ldots \cup F'_a.$$

2) Note that $k \in I_* \subset F_l$. Denote

$$F_{l1} = \{i \in F_l \mid i < k\}, \quad F_{l2} = \{i \in F_l \mid i \geq k\},$$

$$F'_{l1} = \{i' \mid i \in F_{l1}\}, \quad F'_{l2} = \{i' \mid i \in F_{l2}\}.$$

3) Denote $E_a = D_a \cup F_a$, for $1 \leq a < l$, and put $E_l = D_l \cup F_{l1}$. Denote

$$E_{[a]} = E_1 \cup \ldots \cup E_a.$$ 

Note that $E = E_{[l]} \cup F_{l2}$ and that $E_a, E_{[a]}$ and $E_0 \cup E_{[a]}$ are segments of positive integers, moreover $E_{[a]} < E$ and $E_0 \cup E_{[a]} < [c, n]$ for any $1 \leq a \leq l$.

4) For any $1 \leq a < l$ we denote by $J_a$ the subset $F'_{[a]} \cup D_{b(a)}$, where $b(a)$ is the least number such that $F'_{[a]} \subset F_{b(a)}$ (in the case $b(a) = 0$ we put $D_{[0]} = \emptyset$).

5) By $J_l$ we denote the subset $F'_{[l-1]} \cup F'_{l1} \cup D_{b(l)}$, where $b(l)$ is the least number such that $F'_{[p-1]} \cup F'_{l1} \subset F_{[b(l)]}$.

**Lemma 3.16.** We claim that for any $1 \leq a \leq p$

1) $F'_{[a]} \subset F_{[a-1]}$,
2) $F'_{[a]}$ is a segment in $F$,
3) $J_a$ is a segment in positive integers, moreover $J_a < E_0 \cup E_{[b(a)]}$,
4) if $i > E_{[a]}$ and $j \in J_a$, then $(i, j) \in M$,
5) the minor of matrix $\Phi_L$ with system of rows $F_a$ and columns $F'_a$, where $1 \leq a < l$, does not equal to zero,
6) the minor of matrix $\Phi_L$ with system of rows $F_{l1}$ (resp. $F_{l2}$) columns $F'_{l1}$ (resp. $F'_{l2}$) does not equal to zero.
Proof will be carried out for \( l > a = 1 \). The general case can be proved similarly, using induction by \( a \).

For \( a = 1 \) we have \( F_{[1]} = F_1 \) and \( J_1 = F'_{[1]} = F'_1 \). Let \( h_1 \) be the least element of \( D_2 \). Then \( E_{[1]} = [h, h_1] \).

Denote by \( g \) the greatest number such that \( (i, g) \in S \) for some \( i \in F_1 \). Then \( F'_1 \subset [c, g] \). By lemma 3.11, \( (h_1, g) \in M \). Since \( \mathfrak{m} \) is an ideal in \( \mathfrak{n} \), then \([h_1, n] \times [1, g] \subset M \), this proves 4).

Let us show that
\[
F'_1 = [c, g] \triangleleft F_0. \tag{3.11}
\]
This implies the statements 1), 2) and 3) for the case \( a = 1 \).

a) Show that \( g < h \) (recall that \( h \) is greatest element of \( D_1 \)). In the converse case, \( g \in D_1 \) or \( g > D_1 \).

Suppose that \( g \in D_1 \). Let \( (i, g) \) be the greatest, in the sense of \( \succ \), element of \( S^{(g)} \) such that \( i \in F_1 \). Then \( w^{(g)}(i) = g \in D_1 \) and, therefore, \( w^{[i-1]}(i) = w^{[g-1]}(g) \geq h \), this contradicts to \( i \in F \).

Suppose that \( g > D_1 \). Then \( w^{(g+1)} \cdots w^{(t-1)}w^{(t)}_\xi(t) \geq h_1 \). Since \([h_1, n] \times [1, g] \subset M \), then \( w^{(c)} \cdots w^{(g)}(p) = p \) for any \( p \geq h_1 \) and, therefore,
\[
h = w^{(c)}_\xi(t) = w^{(c)} \cdots w^{(g)}_\xi w^{(g+1)} \cdots w^{(t)}_\xi(t) \geq h_1.
\]
On the other hand, by definition \( h < h_1 \). A contradiction. We obtain \( g < h \).

b) Conclude the proof of (3.11). On one hand, \( w^{\xi}(I_*) = F_0 = [c, h] \). On the other hand,
\[
w^{(g+1)} \cdots w^{(t)}_\xi(I_*) \subset (F_0 \setminus [c, g]) \sqcup F_1 \sqcup F_{\geq 2},
\]
where \( F_{\geq 2} \) is a union of all \( F_p \), \( p \geq 2 \), and
\[
w^{(c)} \cdots w^{(g)}(i) = i
\]
for all \( i \in F_0 \) (see lemma 3.6(3)) and all \( i \in F_{\geq 2} \) (see a)). Hence,
\[
[c, g] = w^{(c)} \cdots w^{(g)}(F_1) = F'_1,
\]
this proves (3.11).

c) Let us prove the statements 5). Let \( f_1 = |F_1| \). By the formula (3.11), \( g = c + f_1 - 1 \). Let us show that there exist the systems \( \{i(p)\mid 0 \leq p \leq f_1 - 1\} \) and \( \{j(p)\mid 0 \leq p \leq f_1 - 1\} \) such that \( (i(p), j(p)) \in S \) for all \( 0 \leq p \leq f_1 - 1 \) and
\[
F_1 = \{i(0), \ldots, i(f_1-1)\}, \quad F'_1 = [c, g] = \{j(0), \ldots, j(f_1-1)\}.
\]
Note that for an arbitrary \( i \in F_1 \) and \( j \in F'_1 = [c, g] \) we have:
\[1\] \( w^{(j)}(i) = i \), if \( i \in F_0 \) (see lemma 3.6(3));
II) \( w^{(j)}(i) = j \in F_0 \), if \( i \in F_1 \) and \((i, j)\) is the greatest, in the sense of \( > \), element of \( S^{(j)} \);

III) \( w^{(j)}(i) \in F_1 \), if \( i \in F_1 \) and \( i \) does not satisfy II).

Put \( F_1 = F_{10} \). For any \( 0 \leq p \leq f_1 - 1 \) the following decomposition takes place

\[
 w^{(c+f_1-p)} \ldots w^{(c+f_1-1)}(F_1) = \{ c - f_1 - p, \ldots , c + f_1 - 1 \} \cup F_{1p},
\]

(3.12)

where \( F_{1p} \) is some subset of \( F_1 \) and \( |F_{1p}| = f_1 - p \). Since

\[
 F_1 = F_{10} \supset F_{11} \supset \ldots \supset F_{1f_1} = \emptyset,
\]

then for any \( 1 \leq p \leq f_1 - 1 \) there exists \( i(p) = F_{1p} \setminus F_{1p+1} \). Denote \( j(p) = c + f_1 - p - 1 \).

Then for \( j = j(p) \) we have either \( w^{(j)}(i(p)) = j \), or \( w^{(j)}(i(p)) \in F_{1p+1} \). In any case \((i(p), j(p)) \in S\), this proves 5). The statement 6) can be proved similarly. \( \Box \)

**Corollary 3.17.** All places \((i, j)\), where \( j \in J_a \) and \( i > E_{[a]} \), are filled by zeros in the matrix \( \Phi_L \) (resp. filled by symbols "\( \bullet \)" in the diagram \( D \)).

Denote by \( d_i \) and \( f_i \) the numbers of elements in \( D_i \) and \( F_i \) respectively. Let \( \nu \) be the greatest number such that \( D_{[\nu]} \subset J \). Denote \( d_\nu = d_1 + \ldots + d_\nu \).

**Proposition 3.18.** Let \( \xi \) satisfy the condition of case 2. We claim that

1) \( \deg M_\xi(\lambda) = d_s \);

2) \( \deg M^{(1)}_\xi(\lambda) \leq d_s - 1 \), where \( M^{(1)}_\xi(\lambda) \) is a minor that we get from \( M_\xi(\lambda) \) moving one of the rows on one line below.

**Proof.**

A) The minor \( M_\xi(\lambda) \), as a polynomial in \( \lambda \), can be decomposed in the form

\[
 M_\xi(\lambda) = \sum_{r=1}^{d} M_r \lambda^r, \quad d = \deg M_\xi(\lambda).
\]

(3.13)

The coefficient \( M_r \) is the sum of some minors of the matrix \( \Phi_L \):

\[
 M_r = \sum_{R \subseteq I \cap J, \ |R| = r} M_R, \quad M_R = M_{I \setminus R}^{J \setminus R}.
\]

(3.14)

A1) Show that \( d \leq d_s \). It is sufficient to prove that any minor \( M_R \) with \( r = |R| > d_s \) equals to zero. Denote \( R_i = R \cap E_i \) and \( r_i = |R_i| \). By assumption,

\[
 r = r_1 + \ldots + r_\nu > d_1 + \ldots + d_\nu = d_s.
\]

Let \( a \) be a least number such that

\[
 \sum_{i=1}^{a} r_i > \sum_{i=1}^{a} d_i.
\]

(3.15)
Note that $a \leq \nu$.

We conclude the proof of A1) in every of these cases separately: i) $1 \leq a < \nu$ or $a = \nu < l$ and ii) $a = \nu = l$.

i) $1 \leq a < \nu$ or $a = \nu < l$. By definitions 3.15,

$$|J_a| = \sum_{i=1}^{a} f_i + \sum_{j=1}^{b(a)} d_j,$$

$$|E_{[a]}| = \sum_{i=1}^{a} f_i + \sum_{j=1}^{a} d_j.$$

By lemma 3.16, we conclude that $1 \leq b(a) \leq a - 1$. Hence,

$$\sum_{i=1}^{b(a)} r_i \leq \sum_{i=1}^{b(a)} d_i.$$

Since $J_a \subseteq E_{[b(a)]}$, then $J_a \cap R \subseteq R_1 \sqcup \ldots \sqcup R_{b(a)}$ and, therefore,

$$|J_a \setminus R| = |J_a| - |J_a \cap R| \geq \sum_{i=1}^{a} f_i + \sum_{i=1}^{b(a)} (d_i - r_i) \geq \sum_{i=1}^{a} f_i.$$

Applying inequality (3.15), we obtain

$$|J_a \setminus R| \geq \sum_{i=1}^{a} f_i > \sum_{i=1}^{a} f_i + \sum_{i=1}^{a} (d_i - r_i) = |E_{[a]} \setminus R|.$$

By corollary 3.17, the matrix $\Phi_L$ has zeros in on all places $(i, j)$, where $j \in J_a$ and $i > E_{[a]}$. The minor $M_R$ is zero, since all its elements out of the rectangle $(E_{[a]} \setminus R) \times (J_a \setminus R)$, where $|E_{[a]} \setminus R| > |J_a \setminus R|$, are zero.

ii) $a = \nu = l$. In this case $E_{[a]}$ (resp. $J_a$) is defined in 3.14(3) (resp. 3.14(5)). The statement of A1) may be proved similarly to the case i), changing $f_i$ by $f_{l1} = |F_{l1}|$.

A2). We conclude the proof of A). It is necessary to show that $M_{d*} \neq 0$. By formula (3.14), any coefficient $M_r$ is a sum of minors $M_R$. The system of nonzero summands $\{M_R \mid |R| = r\}$ of this sum is linear independent. Hence, the coefficient $M_r \neq 0$ if and only if there exists a minor $M_R \neq 0$ in the sum (3.14).

The coefficient $M_{d*}$ contains as a summand the minor

$$M_D = \prod_{i=1}^{l-1} M_{F_{i1}}^{F_i} \cdot M_{F_{l1}}^{F_l} \cdot M_{F_{l2}}^{F_l},$$

(3.17)
that, by statements 5) and 6) of lemma 3.16, do not equal to zero. Therefore, 
\( M_{d_\ast} \neq 0 \) and \( d = d_\ast \). This proves A).

B) Let as above \( I \) (resp. \( J \)) be the system of rows (resp. columns) of the minor \( M_\xi(\lambda) \). The system of columns for the minor \( M^\perp_\xi(\lambda) \) the same as for \( M_\xi(\lambda) \), that is \( J \). There exists \( g \in I \) such that \( g + 1 \notin I \) and the system of rows \( I^\perp \) for the minor \( M^\perp_\xi(\lambda) \) coincides with \( (I \setminus \{g\}) \cap \{g + 1\} \).

By definition of the system \( I \), the number \( g \) is the greatest in some \( F_p \), where \( \nu \leq p < l \). Then \( g + 1 \in D_{p+1} \). Similarly to \( M_\xi(\lambda) \), the minor \( M^\perp_\xi(\lambda) \) is presented in the form

\[
M^\perp_\xi(\lambda) = \sum_{r=1}^{d(\perp)} M^1_r \lambda^r, \text{ where } d(\perp) = \deg M^\perp_\xi(\lambda).
\]

Each coefficient \( M^1_r \) is a sum of minors of the matrix \( \Phi_L \):

\[
M^1_r = \sum_{R \subseteq I \cap J, |R| = r} M^\perp_R
\]

where \( M^\perp_R \) is a minor of the matrix \( \Phi_L \) with system of rows \( I^\perp \setminus R \) and system of columns \( J \setminus R \). Let us show that any minor \( M^\perp_R \) equals to zero, if \( r \geq d_\ast \). This implies the last statement of proposition 3.18.

Case \( r > d_\ast \) can be considered similar to A1). Let \( r = d_\ast \). The case, when for some \( 1 \leq a \leq \nu \) the inequality (3.15) is true, also can be considered similar to A1). We have to consider the case \( d_i = r_i \) for any \( 1 \leq i \leq \nu \).

Let \( I_p \) be the subsystem of rows that consists of \( i \in I \) such that \( i \leq g \). By definition of the system \( I \),

\[
I_p = E_1 \sqcup \ldots \sqcup E_\nu \sqcup F_{\nu+1} \sqcup \ldots \sqcup F_p.
\]

As above we consider the subset \( J_p \) of columns that is a union of all \( F'_i \), \( 1 \leq i \leq p \), and all \( D_j \), \( 1 \leq j \leq b(\nu) \) (see notation 3.15(4)). We obtain

\[
|J_p \setminus R| \geq \sum_{i=1}^{p} f_i + \sum_{i=1}^{\nu} d_i - \sum_{i=1}^{\nu} r_i = \sum_{i=1}^{\nu} (f_i + d_i) + \sum_{i=\nu+1}^{p} f_i - \sum_{i=1}^{\nu} r_i = |I_p \setminus R|.
\] (3.18)

By corollary 3.17, all places \( (i, j) \), where \( j \in J_p \) and \( i > I_p \), are filled by zeros in the matrix \( \Phi_L \). Respectively, in the minor \( M_R \) all places \( (i, j) \), where \( j \in J_p \setminus R \) and \( i > I_p \setminus R \) will be filled by zeros. Hence, in the minor \( M^\perp_R \) all places \( (i, j) \), where \( j \in J_p \setminus R \) and \( i > I_p \setminus (R \cup \{g\}) \) will be filled by zeros. By (3.18),

\[
|J_p \setminus R| > |I_p \setminus (R \cup \{g\})|.
\]
Therefore, $M^1_R = 0$. □

**Proposition 3.19.**
1) For any $\xi \in S$ the minor $M^I_L(\lambda)$ is extremal.
2) For any $\xi \in S$ the element $P_\xi$ of $K[LC^*]$ is invariant with respect to the coadjoint representation of the group $L$.
3) Every $P_\xi$ can be decomposed in the form $P_\xi = y_\xi Q_\xi + R_\xi$ where $Q_\xi$ and $R_\xi$ belong to the subalgebra in $S(\mathcal{L})$, generated by $y_{ij}$, where $1 \leq j < t$, and $y_{it}$, where $(i, t) \succ \xi$.

**Remark.** Since $P_\xi$ is invariant, then $Q_\xi$ is also invariant.

**Proof.** By the theorem 2.5, the statement 2) follows from the statement 1).
In the case 1 the minor $M_\xi(\lambda)$ has zero degree. Since any root $(w(j), j)$, where $j \in J$, belongs to $S$, then $M_\xi(\lambda) = P_\xi \neq 0$. The lemma 3.6 implies that $M_\xi(\lambda)$ is extremal in the case 1. By proposition 3.18 and lemma 3.6, $M_\xi(\lambda)$ is extremal in the case 2.

In the case 1 one can prove the statement 3) decomposing the minor $P_\xi$ by its last $t$th column. To prove 3) in the case 2, it is necessary to decompose by the last $t$th column all minors $M_R$ in (3.14) for $r = d_*$ and $t \notin R$.

**Theorem 3.20.** The field of invariants of the coadjoint representation of Lie algebra $\mathcal{L}$ is isomorphic to the field of rational functions of $P_\xi$, where $\xi \in S$.

**Proof.** The theorem 1.1(2) implies that the field of invariants of the coadjoint representation of $\mathcal{L}$ is isomorphic to the field of rational functions of $z_1, \ldots, z_s$.

One can conclude the proof of theorem, using the induction by $1 \leq i \leq s$ and applying the statements of theorem 1.1(1) and proposition 3.19(3). □

**Remark 3.21.** The polynomial $z_i$ that was constructed by induction in the paper [3] do not coincide with $P_{\xi_i}$ in general.

**Conclusion of the example 1.** The generators of the filed of invariants of example 1 have the form

$$
P_{\xi_1} = y_{41}, \quad P_{\xi_2} = y_{62}, \quad P_{\xi_3} = y_{73}, \quad P_{\xi_4} = y_{74} y_{41} + y_{73} y_{31}, \quad P_{\xi_5} = \begin{vmatrix} y_{52} & y_{53} & y_{54} \\ y_{62} & y_{63} & y_{64} \\ 0 & y_{73} & y_{74} \end{vmatrix}.
$$

For all $\xi \in S$, but not for $\xi_4$, the case 1 takes place, and $P_\xi$ is a minor of $\Phi_L$.

In the case $\xi = \xi_4$, the minor of the characteristic matrix

$$
M_{\xi_4}(\lambda) = M_{2,3,4,7}^1(\lambda) = \begin{vmatrix} y_{21} - \lambda & 0 & 0 \\ y_{31} & y_{32} - \lambda & 0 \\ y_{41} & y_{42} & y_{43} - \lambda \\ 0 & 0 & y_{73} \end{vmatrix}
$$

is extremal and $P_{\xi_4}$ is its highest coefficient.
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