CRITERIA FOR SMOOTHNESS OF POSITROID VARIETIES VIA PATTERN AVOIDANCE, JOHNSON GRAPHS, AND SPIROGRAPHS

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Abstract. Positroids are certain representable matroids originally studied by Postnikov in connection with the totally nonnegative Grassmannian and now used widely in algebraic combinatorics. The positroids give rise to determinantal equations defining positroid varieties as subvarieties of the Grassmannian variety. Rietsch, Knutson-Lam-Speyer, and Pawlowski studied geometric and cohomological properties of these varieties. In this paper, we continue the study of the geometric properties of positroid varieties by establishing several equivalent conditions characterizing smooth positroid varieties using a variation of pattern avoidance defined on decorated permutations, which are in bijection with positroids. This allows us to give two formulas for counting the number of smooth positroids along with two $q$-analogs. Furthermore, we give a combinatorial method for determining the dimension of the tangent space of a positroid variety at key points using an induced subgraph of the Johnson graph. We also give a Bruhat interval characterization of positroids.

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1. Introduction

Positroids are an important family of realizable matroids originally defined by Postnikov in \cite{Pos06} in the context of the totally nonnegative part of the Grassmannian variety. These matroids and the totally positive part of the Grassmannian variety have played a critical role in the theory of cluster algebras and soliton solutions to the KP equations and have connections to statistical physics, integrable systems, and scattering amplitudes \cite{AHBC+16, BGY06, FWZ22, Lus98, Rie06, Wil07, Wil}. Positroids are closed under restriction, contraction, duality, and cyclic shift of the ground set, and furthermore they have particularly elegant matroid polytopes \cite{ARW16}.

Positroid varieties were studied by Knutson, Lam, and Speyer in \cite{KLS13}, building on the work of Lusztig, Postnikov and Rietsch. They are homogeneous subvarieties of the complex Grassmannian variety $Gr(k, n)$ which are defined by determinantal equations determined by the nonbases of a positroid. They can also be described as projections of Richardson varieties in the complete flag manifold to $Gr(k, n)$. These varieties have beautiful geometric, representation theory, and combinatorial connections \cite{KLS14, Paw18}. See the background section for notation and further background.
Positroids and positroid varieties can be bijectively associated with many different combinatorial objects \cite{Oh11, Pos06}. For the purposes of this paper, we will need to use the bijections associating each of the following types of objects with each other:

1. positroids $\mathcal{M}$ of rank $k$ on a ground set of size $n$,
2. decorated permutations $w^\circ$ on $n$ elements with $k$ anti-exceedances,
3. Grassmann necklaces $\mathcal{N} = (I_1, \ldots, I_n) \in \left(\binom{n}{k}\right)^n$, and
4. Grassmann intervals $[u, v]$ in $Gl(k, n)$.

Here, a decorated permutation $w^\circ$ on $n$ elements is a permutation $w \in S_n$ together with an orientation clockwise or counterclockwise, denoted $\rightarrow$ or $\leftarrow$ respectively, assigned to each fixed point of $w$. A Grassmann interval $[u, v] \in Gl(k, n)$ is an interval in Bruhat order on permutations in $S_n$ such that $v$ has at most one descent, specifically in position $k$. In Section 2, we will sketch the relevant bijections and remaining terminology. In addition to these, there are bijections to juggling sequences, $J$-diagrams, equivalence classes of plabic graphs, and bounded affine permutations \cite{ARW16, KLS13, Pos06}.

Many of the properties of positroid varieties can be “read off” from one or more of these bijectively equivalent definitions. Thus, we will index a positroid variety

\begin{equation}
\Pi_\mathcal{M} = \Pi_{w^\circ} = \Pi_\mathcal{N} = \Pi_{[u, v]}
\end{equation}

using any of the associated objects, depending on the relevant context. For example, the codimension is easy to read off from the decorated permutation using the notions of the chord diagram and its alignments.

Let $S_{n,k}^\circ$ be the set of decorated permutations on $n$ elements with $k$ anti-exceedances. The chord diagram $D(w^\circ)$ of $w^\circ \in S_{n,k}^\circ$ is constructed by placing the numbers $1, 2, \ldots, n$ on $n$ vertices around a circle in clockwise order, and then, for each $i$, drawing a directed arc from $i$ to $w(i)$ with a minimal number of crossings between distinct arcs while staying completely inside the circle. The arcs beginning at fixed points should be drawn clockwise or counterclockwise according to their orientation in $w^\circ$.

An alignment in $D(w^\circ)$ is a pair of directed edges $(i \mapsto w(i), j \mapsto w(j))$ which can be drawn as distinct noncrossing arcs oriented in the same direction. A pair of directed edges $(i \mapsto w(i), j \mapsto w(j))$ which can be drawn as distinct noncrossing arcs oriented in opposite directions is called a misalignment. A pair of directed edges which must cross if both are drawn inside the cycle is called a crossing \cite[Sect. 5]{Pos06}. Let $\text{Alignments}(w^\circ)$ denote the set of alignments of $w^\circ$.

**Example 1.1.** Let $w^\circ = 895 \leftarrow 76 132$ be the decorated permutation with a counterclockwise fixed point at 4 and a clockwise fixed point at 6. The chord diagram for $w^\circ$ is the following.
Here for example, \((9 \mapsto 2, 8 \mapsto 3)\) highlighted in yellow is an alignment, \((2 \mapsto 9, 8 \mapsto 3)\) is a misalignment, and both \((7 \mapsto 1, 8 \mapsto 3)\) and \((7 \mapsto 1, 5 \mapsto 7)\) are crossings. Note, \((7 \mapsto 1, 6 \mapsto 6)\) is an alignment and \((7 \mapsto 1, 4 \mapsto 4)\) is a misalignment.

**Theorem 1.2.** \cite{KLS13, Pos06} For any decorated permutation \(w^\alpha \in S_n^\alpha\) and associated Grassmann interval \([u, v]\), the codimension of \(\Pi_{w^\alpha}\) in \(Gr(k, n)\) is

\[
\text{codim}(\Pi_{w^\alpha}) = \#\text{Alignments}(w^\alpha) = k(n - k) - [\ell(v) - \ell(u)].
\]

We use the explicit equations defining a positroid variety in \(Gr(k, n)\) to determine whether the variety is smooth or singular. In general, a variety \(X\) defined by polynomials \(f_1, \ldots, f_s\) is **singular** if there exists a point \(x \in X\) such that the Jacobian matrix, \(Jac\), of partial derivatives of the \(f_i\) satisfies \(\text{rank}(Jac|_x) \neq \text{codim} X\). It is **smooth** if no such point exists. The value \(\text{rank}(Jac|_x)\) is the codimension of the tangent space to \(X\) at the point \(x\). Thus, \(\text{rank}(Jac|_x) < \text{codim} X\) implies the dimension of the tangent space to the variety \(X\) at \(x\) is strictly larger than the dimension of the variety \(X\), hence \(x\) is a singularity like a cusp on a curve. In the case of a positroid variety \(\Pi_{w^\alpha}\), Theorem 1.2 implies that a point \(x \in \Pi_{w^\alpha}\) is a singularity of \(\Pi_{w^\alpha}\) if

\[
\text{rank}(Jac|_x) < \text{codim} \Pi_{w^\alpha} = \#\text{Alignments}(w^\alpha).
\]

Our first main theorem reduces the problem of finding singular points in a positroid variety to checking the rank of the Jacobian only at a finite number of \(T\)-fixed points. For any \(J = \{j_1, \ldots, j_k\} \subseteq \{n\}\), let \(A_J\) be the element in \(Gr(k, n)\) spanned by the elementary row vectors \(e_i\) with \(i \in J\), or equivalently the subspace represented by a \(k \times n\) matrix with a 1 in cell \((i, j_i)\) for each \(j_i \in J\) and zeros everywhere else. These are the \(T\)-fixed points of \(Gr(k, n)\), where \(T\) is the set of invertible diagonal matrices over \(\mathbb{C}\). The reduction follows from the decomposition of \(\Pi_{[u, v]}\) as a projected Richardson variety. Every point \(A \in \Pi_{[u, v]}\) lies in the projection of some intersection of a Schubert cell with an opposite Schubert variety \(C_y \cap X^v\) for \(y \in [u, v]\). In particular, if \(y = y_1y_2\ldots y_n \in [u, v]\) in one-line notation and we define \(y[k] := \{y_1, y_2, \ldots, y_k\}\) to be an initial set of \(y\), then \(A_{y[k]}\) is in the projection of \(C_y \cap X^v\).

**Theorem 1.3.** Assume \(A \in \Pi_{[u, v]}\) is the image of a point in \(C_y \cap X^v\) projected to \(Gr(k, n)\) for some \(y \in [u, v]\). Then the codimension of the tangent space to \(\Pi_{[u, v]}\) at \(A\) is bounded below by \(\text{rank}(Jac|_{A_{y[k]}})\).
Theorem 1.3 indicates that the $T$-fixed points of the form $A_{y[k]}$ such that $y \in [u, v]$ are key to understanding the singularities of $\Pi_{[u, v]}$. In fact, the equations determining $\Pi_{[u, v]}$ and the bases of the positroid $\mathcal{M}$ associated to the interval $[u, v]$ can be determined from the permutations in the interval by the following theorem. Our proof of the following theorem depends on Knutson-Lam-Speyer’s Theorem/Definition 2.21 of a positroid variety as the projection of a Richardson variety and Theorem 2.6. It also follows from [KW15, Lemma 3.11]. Both groups, Knutson-Lam-Speyer and Kodama-Williams, were aware of this result in the context of positroid varieties for Grassmannians, but it does not appear to be in the literature in the form we needed, hence we prove the result in Section 3.

**Theorem 1.4.** Let $w^\circ \in S^\circ_{n,k}$ with associated Grassmann interval $[u, v]$ and positroid $\mathcal{M}$. Then $\mathcal{M}$ is exactly the collection of initial sets of permutations in the Grassmann interval $[u, v]$, 

\[ \mathcal{M} = \{ y[k] : y \in [u, v] \}. \]

Our next theorem provides a method to compute the rank of the Jacobian of $\Pi_{[u, v]}$ explicitly at the $T$-fixed points. Therefore, we can also compute the dimension of the tangent space of a positroid variety at those points. Comparing that with the number of alignments gives a test for singularity of points in positroid varieties by (1.3).

**Theorem 1.5.** Let $w^\circ \in S^\circ_{n,k}$ with associated Grassmann interval $[u, v]$ and positroid $\mathcal{M}$. For $y \in [u, v]$, the codimension of the tangent space to $\Pi_{[u,v]} \subseteq Gr(k,n)$ at $A_{y[k]}$ is

(1.4) \[
\text{rank}(\text{Jac}|_{A_{y[k]}}) = \#\{ I \in \binom{[n]}{k} \setminus \mathcal{M} : |I \cap y[k]| = k-1 \}.
\]

The formula in (1.4) is reminiscent of the Johnson graph $J(k,n)$ with vertices given by the $k$-subsets of $[n]$ such that two $k$-subsets $I,J$ are connected by an edge precisely if $|I \cap J| = k-1$. For a positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $J(\mathcal{M})$ denote the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. We call $J(\mathcal{M})$ the Johnson graph of $\mathcal{M}$. Note, the Johnson graph is closely related to bases of matroids by the Basis Exchange Property. Theorem 1.5 implies $J(\mathcal{M})$ encodes aspects of the geometry of the positroid varieties like the Bruhat graph in the theory of Schubert varieties [Car94].

To state our main theorem characterizing smoothness of positroid varieties, we need to define two types of patterns that may occur in a chord diagram. First, given an alignment $(i \mapsto w(i), j \mapsto w(j))$ in $D(w^\circ)$, if there exists a third arc $(h \mapsto w(h))$ which forms a crossing with both $(i \mapsto w(i))$ and $(j \mapsto w(j))$, we say $(i \mapsto w(i), j \mapsto w(j))$ is a crossed alignment of $w^\circ$. In the example above, $(9 \mapsto 2,8 \mapsto 3)$ is a crossed alignment; this alignment is crossed for instance by $(7 \mapsto 1)$, highlighted in blue.

The second type of pattern is related to the Spirograph™ toy, designed by Denys Fisher and trademarked by Hasbro to draw a variety of curves inside a circle which meet the circle in a finite number of discrete points. See Figure 1. Once oriented and vertices are added, such curves each determine a chord diagram from a special class,
which we will call spirographs. We think of alignments, crossings, crossed alignments, and spirographs as \textit{subgraph patterns} for decorated permutations.

![Figure 1. Spirographs made by the Spirograph Maker app for the iphone.](image)

\textbf{Definition 1.6.} A decorated permutation \(w^\circ = (w, co) \in S_{n,k}^\circ\) is a \textit{spirograph permutation} if there exists a positive integer \(m\) such that \(w(i) = i + m \pmod{n}\) for all \(i\) and \(w^\circ\) has at most one fixed point. The chord diagram of a spirograph permutation will be called a \textit{spirograph}.

\textbf{Theorem 1.7.} Let \(w^\circ \in S_{n,k}^\circ\) with associated Grassmann interval \([u, v]\) and positroid \(\mathcal{M} = \{y[k] : y \in [u, v]\}\). Then, the following are equivalent.

1. The positroid variety \(\Pi_{w^\circ} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}\) is smooth.
2. For every \(y \in [u, v]\), \(#\{I \in \mathcal{M} : |I \cap y[k]| = k - 1\} = \ell(v) - \ell(u)\).
3. For every \(J \in \mathcal{M}\), \(#\{I \in \mathcal{M} : |I \cap J| = k - 1\} = k(n - k) - \#\text{Alignments}(w^\circ)\).
4. The graph \(J(\mathcal{M})\) is regular, and each vertex has degree \(\ell(v) - \ell(u)\).
5. The decorated permutation \(w^\circ\) has no crossed alignments.
6. The chord diagram \(D(w^\circ)\) is a disjoint union of spirographs.
7. The positroid \(\mathcal{M}\) is a direct sum of uniform matroids.

Schubert varieties in the flag variety are indexed by permutations and are closely related to positroid varieties, as explained in Section 2. Smoothness of Schubert varieties is completely characterized by pattern avoidance of the corresponding permutations \([LS90]\). When studying the partially asymmetric exclusion process and its surprising connection to the Grassmannian, Sylvie Corteel posed the idea of considering patterns in decorated permutations \([Cor07]\). This suggestion was the foundation for the present work. Some of the results in this paper were announced in \([BW22]\).

In Section 2, we provide background material from the literature and define our notation. In Section 3, we study initial sets for Grassmann intervals and prove Theorem 1.4 as well as the equalities (2) \(\Leftrightarrow\) (3) \(\Leftrightarrow\) (4) of Theorem 1.7. In Section 4, we prove Theorem 1.3 and Theorem 1.5. Theorem 1.3 and Theorem 1.5 together yield the equality (1) \(\Leftrightarrow\) (2) in Theorem 1.7. In Section 5, we consider several rigid transformations of chord diagrams and the associated transformations of the related objects. Furthermore, we show in Lemma 5.6 that applying these transformations to the objects indexing positroid varieties preserves the property of being smooth. In Section 6, we utilize a
decomposition of $\mathcal{M}$ into connected components on a non-crossing partition and show that a positroid variety is smooth if and only if all of the components of the decomposition correspond to smooth positroid varieties. In Section 7, we identify the special properties of positroid varieties indexed by spirograph permutations, which leads to the implication (6) $\Rightarrow$ (1) and the equalities (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) in Theorem 1.7. In Section 8, we complete the proof of Theorem 1.7 by showing that (1) $\Rightarrow$ (5). This implication is accomplished by constructing an injective map from the anti-exchange pairs for a particular set $J \in \mathcal{M}$ to Alignments$(w^o)$. In Section 9 we provide some enumerative results for smooth positroid varieties. We conclude with some open questions in Section 10.

2. Background

We begin by giving notation and some background on several combinatorial objects and theorems from the literature. These objects will be used to index the varieties discussed throughout the paper. We will then introduce notation for several geometrical objects, including Grassmannian varieties, flag varieties, Schubert varieties, Richardson varieties, and positroid varieties.

2.1. Combinatorial objects. For integers $i \leq j$, let $[i, j]$ denote the set $\{i, i+1, \ldots, j\}$, and write $[n] := [1, n]$ for a positive integer $n$. Let $\binom{[n]}{k}$ be the set of size $k$ subsets of $[n]$ for $k \in [0, n]$. Call $J \in \binom{[n]}{k}$ a $k$-subset of $[n]$.

Define the Gale partial order, $\preceq$, on $k$-subsets of $[n]$ as follows. Let $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$. Then $I \preceq J$ if and only if $i_h \leq j_h$ for all $h \in [k]$. This partial order is known by many other names; we are following [ARW16] for consistency. Gale studied this partial order in the context of matroids in the 1960s [Gal68].

For any $k \times n$ matrix $A$ and any set $J \in \binom{[n]}{k}$, define $\Delta_J(A)$ to be the determinant of the $k \times k$ submatrix of $A$ lying in column set $J$. The minors $\Delta_I(A)$ for $I \in \binom{[n]}{k}$ are called the Plücker coordinate of $A$. We think of $\Delta_J$ as a polynomial function on the set of all $k \times n$ matrices over a chosen field using variables of the form $x_{ij}$ indexed by row $i \in [k]$ and column $j \in J$.

**Definition 2.1.** Let $S$ be a partition $[n] = B_1 \sqcup \cdots \sqcup B_t$ of $[n]$ into pairwise disjoint, nonempty subsets. We say that $S$ is a non-crossing partition if there are no distinct $a, b, c, d$ in cyclic order such that $a, c \in B_i$ and $b, d \in B_j$ for some $i \neq j$. Equivalently, place the numbers $1, 2, \ldots, n$ on $n$ vertices around a circle in clockwise order, and then for each $B_i$, draw a polygon on the corresponding vertices. If no two of these polygons intersect, then $S$ is a non-crossing partition of $[n]$.

A matroid of rank $k$ on $[n]$, defined by its bases, is a nonempty subset $\mathcal{M} \subseteq \binom{[n]}{k}$ satisfying the following Basis Exchange Property: if $I, J \in \mathcal{M}$ such that $I \neq J$ and $a \in I \setminus J$, then there exists some $b \in J \setminus I$ such that $(I \setminus \{a\}) \cup \{b\} \in \mathcal{M}$. Compare the
notion of matroid basis exchange to basis exchange in linear algebra. We call the sets in $\binom{[n]}{k} \setminus \mathcal{M}$ the nonbases of $\mathcal{M}$, and we denote this collection of sets by $Q(\mathcal{M})$. The set $\binom{[n]}{k}$ is the uniform matroid of rank $k$ on $[n]$. For more on matroids, see [Ard15, Oxl11].

**Example 2.2.** A notable family of matroids called representable matroids comes from matrices. Let $A$ be a full rank $k \times n$ matrix. The matroid of $A$ is the set

$$\mathcal{M}_A := \left\{ J \in \binom{[n]}{k} : \Delta_J(A) \neq 0 \right\}.$$ 

The matroid of

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

is

$$\mathcal{M}_A = \{\{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,6\}, \{5,6\}\} \subseteq \binom{[6]}{2}.$$ 

Recall, the Johnson graph $J(k,n)$ with vertices given by the $k$-subsets of $[n]$ such that two $k$-subsets $I, J$ are connected by an edge precisely if $|I \cap J| = k - 1$. For a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$, $J(\mathcal{M})$ is the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. The Basis Exchange Property for $\mathcal{M}$ implies that $J(\mathcal{M})$ is connected, and furthermore, between any two vertices $I, J$ in $J(\mathcal{M})$, there exists a path in $J(\mathcal{M})$ which is a minimal length path between $I$ and $J$ in $J(k,n)$. The Johnson graph $J(\mathcal{M}_A)$ from Example 2.2 is

$$\begin{array}{c}
\{4,6\} \\
\{3,6\} \\
\{3,5\} \\
\{2,5\} \\
\{2,6\} \\
\{2,4\} \\
\{6,5\} \\
\end{array}$$

The direct sum of two matroids on disjoint ground sets, denoted $M_1 \oplus M_2$, is the matroid with bases given by $\{I \cup J : I \in M_1, J \in M_2\}$ on the ground set which is the disjoint union of the ground sets of the matroids $M_1$ and $M_2$. A matroid $M$ on ground set $[n]$ is connected if $M$ cannot be expressed as the direct sum of two matroids. Every matroid can be decomposed into the direct sum of its connected components.

Given a matroid $\mathcal{M}$ on ground set $[n]$, the dual matroid $\mathcal{M}^*$ is the matroid on $[n]$ with bases $\{\{I \subseteq [n] : I \in \mathcal{M}\}$. If $\mathcal{M}$ has rank $k$, then its dual matroid has rank $n-k$.

Let $S_n$ be the set of permutations of $[n]$, where we think of a permutation as a bijection from a set to itself. For $w \in S_n$, let $w_i = w(i)$, and write $w$ in one-line notation.
as $w = w_1 w_2 \cdots w_n$. For an interval of indices $[i, j]$, we denote the image of $[i, j]$ under $w$ by $w[i, j] = \{w_i, w_{i+1}, \ldots, w_j\}$. The permutation matrix $M_w$ of $w$ is the $n \times n$ matrix that has a 1 in cell $(i, w_i)$ for each $i \in [n]$ and zeros elsewhere. The length of $w \in S_n$ is

$$\ell(w) := \#\{(i, j) : i < j \text{ and } w(i) > w(j)\}. $$

The permutation in $S_n$ of maximal length is $w_0 := n(n - 1) \cdots 1$.

**Example 2.3.** For $w = 3124$, the length of $w = 3124$ is $\ell(w) = 2$, and $M_w$ is the matrix

$$M_{3124} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the permutation matrix of $w^{-1}$ is $M_w^T$. Furthermore, permutation multiplication is given by function composition so that if $wv = u$, then $w(v(i)) = u(i)$. By this definition, $M_w^T M_v^T = M_w v$.

**Definition 2.4.** [BB05a Chapter 2] For $u, v$ in $S_n$, $u \leq v$ in Bruhat order if $u[i] \leq v[i]$ for all $i \in [n]$. Equivalently, Bruhat order is the ranked poset defined as the transitive closure of the relation $u < ut_{ij}$ whenever $u_i < u_j$, where $t_{ij}$ is the permutation transposing $i$ and $j$ and fixing all other values. For each $u \leq v$ in Bruhat order, the interval $[u, v]$ is defined to be

$$[u, v] := \{y \in S_n : u \leq y \leq v\}. \quad (2.1)$$

For $0 \leq k \leq n$, write $S_k \times S_{n-k}$ for the subgroup of $S_n$ consisting of permutations that send $[k]$ to $[k]$ and $[k+1, n]$ to $[k+1, n]$. If $k = 0$, consider $[k]$ to be the empty set. A permutation $w \in S_n$ is $k$-Grassmannian if $w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_n$. This is equivalent to saying that $w$ is the minimal length element of its coset $w \cdot (S_k \times S_{n-k})$. For example, the permutation $w = 3124$ is 1-Grassmannian. The set of Grassmannian permutations in $S_n$ is the union over $k \in [0, n]$ of all $k$-Grassmannian permutations.

**Definition 2.5.** Assume $u \leq v$ in Bruhat order on $S_n$. Then, the interval $[u, v]$ is a Grassmann interval provided $v$ is a $k$-Grassmannian permutation for some $k \in [0, n]$. Denote by $Gi(k, n)$ the set of all Grassmann intervals $[u, v]$ in $S_n$, where $v$ is $k$-Grassmannian.

The Grassmann intervals $[u, v]$ are key objects for this paper. Note that $u$ need not be a Grassmannian permutation. In the case where $v$ is Grassmannian, there is a simpler criterion for Bruhat order that follows closely from work of Bergeron-Sottile [BS98 Theorem A]. This criterion also appears in [Pos06, Lemma 20.2]. We will use this criterion extensively.

**Theorem 2.6.** Let $u, v \in S_n$, where $v$ is $k$-Grassmannian. Then $u \leq v$ if and only if

(i) for every $1 \leq j \leq k$, we have $u(j) \leq v(j)$, and

(ii) for every $k < m \leq n$, we have $u(m) \geq v(m)$. 

2.2. Grassmannian, Flag, and Richardson Varieties. Schubert varieties in both the flag variety and the Grassmannian are a well studied class of varieties that have found importance in algebraic geometry, representation theory, and combinatorics. Singularities of Schubert varieties, in particular, have been investigated thoroughly. See [BL00, Ful97, Kum02] for further background on these varieties.

For $0 \leq k \leq n$, the points in the Grassmannian variety, $Gr(k,n)$, are the $k$-dimensional subspaces of $\mathbb{C}^n$. Up to left multiplication by a matrix in $GL_k$, we may represent $V \in Gr(k,n)$ by a full rank $k \times n$ matrix $A_V$ such that $V$ is the row span of $A_V$. Let $Mat_{kn}$ be the set of full rank $k \times n$ matrices. The points in $Gr(k,n)$ can be bijectively identified with the cosets $GL_k \setminus Mat_{kn}$. The Grassmannian varieties are smooth manifolds via the Plücker coordinate embedding of $Gr(k,n)$ into projective space. This includes the case when $k = n = 0$, in which case $Gr(k,n)$ consists of one point, which is the 0-dimensional vector space in $\mathbb{C}^0$.

Let $F\ell(n)$ be the complete flag variety of nested subspaces of $\mathbb{C}^n$. A complete flag $V_* = (0 \subset V_1 \subset \cdots \subset V_n)$ can be represented as an invertible $n \times n$ matrix where the row span of the first $k$ rows is the subspace $V_k$ in the flag. Throughout the paper, we will often identify a full rank matrix with the point it represents in $Gr(k,n)$ or $F\ell(n)$. For a subset $J \subseteq [n]$, let $Proj_J : \mathbb{C}^n \to \mathbb{C}^{|J|}$ be the projection map onto the indices specified by $J$. Then, for every permutation $w \in S_n$, there is a Schubert cell $C_w$ and an opposite Schubert cell $C^w$ in $F\ell(n)$ defined by

$$C_w = \{ V_* \in F\ell(n) : \dim(Proj_j(V_i)) = |w[i] \cap [j]| \text{ for all } i, j \},$$

$$C^w = \{ V_* \in F\ell(n) : \dim(Proj_{[n-j+1,n]}(V_i)) = |w[i] \cap [n-j+1,n]| \text{ for all } i, j \}.$$

By row elimination and rescaling, we can find a canonical matrix representative $A_{V_*} := (a_{i,j})$ for each $V_* \in C_w$ such that $a_{i,w_i} = 1$ for $i \in [n]$, $a_{i,j} = 0$ for all $1 \leq j < w_i$, and $a_{h,w_i} = 0$ for $h > i$. Canonical matrices for $C^w$ can be found similarly, but so that every leading 1 has all zeros to the right instead of the left. For example, the canonical matrices for $C_{3124}$ have the form below, where entries labeled $*$ can be replaced by any element of $\mathbb{C}$.

$$\begin{bmatrix} 0 & 0 & 1 & \ast \\ 1 & \ast & 0 & \ast \\ 0 & 1 & 0 & \ast \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The Schubert variety $X_w$ is the closure of $C_w$ in the Zariski topology on $F\ell(n)$, and similarly, the opposite Schubert variety $X^w$ is the closure of $C^w$. Specifically, $X_w$ and $X^w$ can be defined by

$$X_w = \{ V_* \in F\ell(n) : \dim(Proj_j(V_i)) \leq |w[i] \cap [j]| \text{ for all } i, j \},$$

$$X^w = \{ V_* \in F\ell(n) : \dim(Proj_{[n-j+1,n]}(V_i)) \leq |w[i] \cap [n-j+1,n]| \text{ for all } i, j \}.$$

Bruhat order determines which Schubert cells are in a Schubert variety,

$$X_w = \bigsqcup_{y \leq w} C_y \quad \text{and} \quad X^w = \bigsqcup_{w \leq y} C^y.$$
Schubert varieties in the Grassmannian manifold $Gr(k, n)$ are indexed by the sets in $\binom{[n]}{k}$ and can be defined as the projections of Schubert varieties in $\mathcal{F}(n)$. Let
\begin{equation}
\pi_k : \mathcal{F}(n) \to Gr(k, n)
\end{equation}
be the projection map which sends a flag $V_\bullet = (0 \subset V_1 \subset \cdots \subset V_n)$ to the $k$-dimensional subspace $V_k$. Identifying a full rank $n \times n$ matrix $M$ with the point it represents in $\mathcal{F}(n)$, then $\pi_k(M)$ denotes the span of the top $k$ rows of $M$. For each $J \in \binom{[n]}{k}$, there exists a $k$-Grassmannian permutation $v$ such that $J = v[k] := \{v_1, v_2, \ldots, v_k\}$. The Grassmannian Schubert variety indexed by $J$ is the projection $\pi_k(X_v)$.

For permutations $u$ and $v$ in $S_n$, with $u \leq v$, the Richardson variety is a nonempty variety in $\mathcal{F}(n)$ and is defined as the intersection $X_u^v := X_u \cap X_v$. Then $\dim X_u^v = \ell(v) - \ell(u)$. The decompositions of $X_u$ and $X_v$ into Schubert cells and opposite Schubert cells in (2.2) yield
\begin{equation}
X_u^v = \bigsqcup_{u \leq y \leq v} (C_y \cap X_u^v) = \left(\bigsqcup_{y \leq u} C_y \right) \cap \left(\bigsqcup_{v \leq y} C^\ell_y \right).
\end{equation}

By Equation (2.2), one can observe that the permutation matrix $M_y \in (C_y \cap X_u^v) \subset X_u^v$ for each $y \in [u, v]$.

In [LS90], Lakshmibai and Sandhya prove that $X_w$ is smooth if and only if $w$ avoids 1324 and 2143 as permutation patterns. Related work was also done by Ryan [Rya87], Wolper [Wol89], and Haiman [Hai92]. In [BP05, Thm 2.4], Billey and Postnikov extend this result by giving a pattern avoidance criterion for smoothness of Schubert varieties for all Weyl groups. Given a singular Schubert variety, $X_w$, [BW03, Thm 1], [KLR03, Thm 1.3], [Man01, Thm 2], and [Cor03] developed criteria based on permutation patterns to determine the singular points of $X_w$. These theorems characterizing smooth versus singular points in Schubert varieties using permutation patterns motivated this work.

Singularities in Richardson varieties and their projections have also been studied in the literature [Bri05, BCT12, KLS14, KL04]. The characterizations of smooth versus singular Richardson varieties described there are not based on pattern avoidance but rely on computations in the associated cohomology rings.

2.3. Decorated Permutations. A decorated permutation $w^\omega$ is defined by a permutation $w$ together with a circular orientation map called $co$ from the fixed points of $w$ to the set of clockwise or counterclockwise orientations, denoted by $\{\circlearrowright, \circlearrowleft\}$. Therefore, we will sometimes describe a decorated permutation as a pair $(w, co)$.

Postnikov made the following definitions in [Pos06, Sect 16]. Given a decorated permutation $w^\omega$, call $i \in [n]$ an anti-exceedance of $w^\omega = (w, co)$ if $i < w^{-1}(i)$ or if $w^\omega(i) = i$ is a clockwise fixed point. If $i \in [n]$ is not an anti-exceedance, it is an exceedance. Let $I_1(w^\omega) = I_1$ be the set of anti-exceedances of $w^\omega$. For an arc $(i \mapsto w(i))$ in the chord diagram of $w^\omega$, we say the arc is an anti-exceedance arc or exceedance arc depending on whether $w(i)$ is in $I_1(w^\omega)$ or not. Let $k(w^\omega) := |I_1(w^\omega)|$. Recall that $S^\omega_{n,k}$ is the set of decorated permutations with anti-exceedance set of size $k$. 

More generally, let \(<_r\) be the shifted linear order on \([n]\) given by \(r <_r (r + 1) <_r \cdots <_r n <_r 1 <_r \cdots <_r (r - 1)\) for \(r \in [n]\). The shifted anti-exceedance set \(I_r(w^o)\) of \(w^o\) is the anti-exceedance set of \(w^o\) with respect to the shifted linear order \(<_r\) on \([n]\),

\[
I_r(w^o) = \{ i \in [n] : i <_r w^{-1}(i) \text{ or } w^o(i) = \overset{\rightarrow}{i} \}.
\]

An element \(i \in I_r(w^o)\) is called an \(r\)-anti-exceedance, and an element \(i \notin I_r(w^o)\) is called an \(r\)-exceedance. Note from the construction that either \(I_{r+1}(w^o) = I_r(w^o)\) or \(I_{r+1}(w^o) = I_r(w^o) \setminus \{r\} \cup \{w(r)\}\), so \(|I_1(w^o)| = \cdots = |I_n(w^o)| = k(w^o)\). Furthermore, \(I_{r+1}(w^o) = I_r(w^o)\) if and only if \(r\) is a fixed point of \(w\), so clockwise fixed points will be in all of the shifted anti-exceedance sets and counterclockwise fixed points will be in none. Therefore, \(w^o\) is easily recovered from \((I_1(w^o), \ldots, I_n(w^o))\). The sequence of \(k(w^o)\)-subsets \((I_1(w^o), \ldots, I_n(w^o))\) is called the Grassmann necklace associated to \(w^o\).

Given a decorated permutation \(w^o = (w, co) \in S^o_{n,k}\) and its anti-exceedance set, we can also easily identify the Grassmann interval associated to it. The \(k\)-Grassmannian permutation \(v\) is determined by \(v[k] = w^{-1}(I_1(w^o))\), and then \(u\) is determined by \(u = wv\), and hence \(I_1(w^o) = u[k]\) by this construction. Let \(u(w^o) = u\) and \(v(w^o) = v\). The interval \([u, v] \in G(k, n)\) is the Grassmann interval associated to \(w^o \in S^o_{n,k}\). To identify a decorated permutation from a Grassmann interval \([u, v] \in G(k, n)\), simply let

\[
(2.4) \quad w = uv^{-1} \quad \text{with} \quad co(j) := \begin{cases} \circ & \text{if } j \in u[k] \\ \circ & \text{if } j \notin u[k]. \end{cases}
\]

We visualize the bijection from decorated permutations to Grassmann intervals as a shuffling algorithm.

1. Write \(w^o\) in two-line notation with the numbers \(1, 2, \ldots, n\) on the first row and \(w(1), w(2), \ldots, w(n)\) on the second row, including orientations on fixed points.
2. Highlight the columns \(i_1 < i_2 < \cdots < i_k\) such that \(i_j > w(i_j)\) or \(w^o(i_j) = \overset{\rightarrow}{i_j}\), corresponding with anti-exceedances.
3. Keeping the columns intact, reorder the columns so that the highlighted columns \(i_1 < i_2 < \cdots < i_k\) come first followed by the non-highlighted columns maintaining their relative order within the highlighted and non-highlighted blocks. Then, drop any orientation arrows to obtain a \(2 \times n\) array of positive integers \([u \atop v]\) with the one-line notation for \(u\) determined by the lower row and the one-line notation for \(v\) determined by the upper row.

From this shuffling algorithm, note that \(v\) is a \(k\)-Grassmannian permutation by construction and \(v[k] = w^{-1} \cdot I_1(w^o)\). Furthermore, \(u \in S_n\) and its initial set \(u[k] = I_1(w^o)\). Also observe that \(u_i \leq v_i\) for all \(1 \leq i \leq k\), and \(u_i \geq v_i\) for all \(k + 1 \leq i \leq n\). Hence, \(u \leq v\) in Bruhat order by Theorem 2.6. Thus, \([u, v]\) is the Grassmann interval associated to \(w^o\), since the shuffling algorithm above is equivalent to the permutation multiplication \(wv = u\).
Conversely, to go from \([u, v]\) to \(w^\ominus = (w, co)\), simply take the two line array \(\begin{bmatrix} u & v \end{bmatrix}\), sort the columns by the top row to obtain \(w\). Observe that \(j\) is a fixed point of \(w\) if and only if \(u^{-1}(j) = v^{-1}(j)\). If \(j\) is a fixed point of \(w\), then \(co(j) = \emptyset\) if \(j \in u[k]\), and \(co(j) = \emptyset\) otherwise.

**Remark 2.7.** Observe that the chord diagram is just as easily obtained from \([u, v]\) as it is from \(w^\ominus\). The chord diagram arcs \(\{(i \mapsto w(i)) : i \in [n]\}\) are determined by the columns in the two line notation of \(w^\ominus\), so \(\{(i \mapsto w(i)) : i \in [n]\} = \{(v_j \mapsto u_j) : j \in [n]\}\)
with fixed points oriented appropriately. Therefore, the graphical patterns determining properties for decorated permutations also determine patterns for Grassmann intervals.

**Remark 2.8.** Note that the map above between decorated permutations in \(S_{n,k}^\alpha\) and Grassmann intervals in \(Gi(k, n)\) follows the work of [KLS13]. In [Pos06 Sect. 20], a slightly different map is given. In particular, in Postnikov’s work, the decorated permutation \(w^\ominus\) obtained from a Grassmann interval \([u, v]\) is computed as \(w = w_0vuw^{-1}w_0\), with a fixed point \(i\) assigned a clockwise orientation if \(n+1-i\) is in \(u[k]\). Here \(w_0 = n\ldots21 \in S_n\). We will return to this involution on decorated permutations and associated objects in Remark 5.5. Our reason for prioritizing the Knutson-Lam-Speyer bijection is the direct connection to the corresponding positroid given in Theorem 1.4.

**Example 2.9.** For the decorated permutation \(w^\ominus = 541276983\), the anti-exceedance set is \(I_1(w^\ominus) = \{1, 2, 6, 3\}\). These values occur in positions \(\{3, 4, 6, 9\}\) in \(w^\ominus\). The Grassmann necklace for \(w^\ominus\) is

\[(I_1, \ldots, I_9) = (\{1236\}, \{2356\}, \{3456\}, \{1456\}, \{1256\}, \{1267\}, \{1267\}, \{1269\}, \{1269\})\].

Write the two-line notation, highlighting the anti-exceedances and then shuffling the anti-exceedances to the front to identify the associated Grassmann interval \([u, v] = [126354798, 346912578]\),

\[w^\ominus = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \backslash 6 & 7 & 8 & 9 \\ 5 & 4 & 1 & 2 & 7\end{bmatrix} \Rightarrow \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 & 9 \backslash 1 & 2 & 6 & 3 \end{bmatrix} \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 & 9 \backslash 1 & 2 & 6 & 3 \end{bmatrix} \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 & 7 & 8 \end{bmatrix} \]

Using Postnikov’s map, the decorated permutation corresponding to the interval \([u, v] = [126354798, 346912578]\) is \(3 \overline{2} 5 4 \overline{9} 8167\). Postnikov’s inverse map would associate the Grassmann interval \([416732598, 478912356]\) to the original \(w^\ominus = 541276983\).

A key artifact of a decorated permutation in our work is its set of alignments. To formally define alignments, we first establish the following notation for a cyclic interval of elements in \([n]\) for a fixed integer \(n\).

**Definition 2.10.** Let \(a, b \in [n]\). Then

\[(a, b)_{\text{cf}} := \begin{cases} (a, b) & \text{if } a \leq b \\ [a, n] \cup [1, b] & \text{if } a > b \end{cases}\]

**Definition 2.11.** An alignment of \(w^\ominus = (w, co) \in S_n^\alpha\) is a pair of arcs \((p \mapsto w(p))\) and \((s \mapsto w(s))\) in \(D(w^\ominus)\) which can be drawn as distinct noncrossing arcs such that
(1) both \( w(p) \in [p, w(s) - 1]^{\text{yc}} \) and \( w(s) \in [w(p) + 1, s]^{\text{yc}} \),
(2) if \( w(s) = s \), then \( \text{co}(s) = \varnothing \), and
(3) if \( w(p) = p \), then \( \text{co}(p) = \varnothing \).

In this case, we denote the alignment by \((p \mapsto w(p), s \mapsto w(s))\) or \(A(p, s)\) if \(w^o\) is understood from context. We say the arc \((p \mapsto w(p))\) is the port side and the arc \((s \mapsto w(s))\) is the starboard side of the alignment. See Figure 2. We use \(\text{Alignments}(w^o)\) to denote the set of all alignments of \(w^o\).

![Figure 2. Alignment with port side \((p \mapsto w(p))\) and starboard side \((s \mapsto w(s))\).](image)

**Example 2.12.** Recall the chord diagram in Example 1.1 for \(w^o = 895476132\), the alignment \((9 \mapsto 2, 8 \mapsto 3)\) highlighted in yellow has port side \((9 \mapsto 2)\) and starboard side \((8 \mapsto 3)\) as if they were two sides of a boat with its bow pointing to the right. Furthermore, \(w^o\) has 13 alignments,

\[
\text{Alignments}(w^o) = \{A(3,1), A(3,2), A(3,6), A(4,1), A(4,2), A(4,3), A(4,6), A(5,1), A(5,2), A(7,6), A(8,6), A(9,6), A(9,8)\}.
\]

**Definition 2.13.** A crossed alignment \(A(p, s, x)\) consist of an alignment \(A(p, s)\) of \(w^o\) and an additional arc \((x \mapsto w(x))\) crossing both \((p \mapsto w(p))\) and \((s \mapsto w(s))\). We partition the set of crossed alignments according to which side of the alignment the crossing arc intersects first as it passes from \(x\) to \(w(x)\).

(1) If \(x \in [w(s), s]^{\text{yc}}\) and \(w(x) \in [p, w(p)]^{\text{yc}}\), we say \(A(p, s, x)\) is a starboard tacking crossed alignment.
(2) If \(x \in [p, w(p)]^{\text{yc}}\) and \(w(x) \in [w(s), s]^{\text{yc}}\), we say \(A(p, s, x)\) is a port tacking crossed alignment.

Note that if \(A(p, s, x)\) is a crossed alignment, then \(p\) and \(s\) cannot be fixed points. The highlighted crossed alignment in Example 1.1 \(A(9,8,7)\) is an example of a starboard tacking crossed alignment, while \(A(9,8,1)\) is a port tacking crossed alignment.

The following family of (decorated) permutations are key to this work. As a subset of \(S_n\), these permutations have nice enumerative properties \([\text{Cal}04], \text{OEIS A075834}\).

**Definition 2.14.** \([\text{Cal}04]\) A permutation \(w \in S_n\) is stabilized-interval-free provided no proper nonempty interval \([a, b] \subseteq [n]\) exists such that \(w[a, b] = [a, b]\).
Note, the only stabilized-interval-free permutation with a fixed point is the identity permutation in $S_1$. Thus, the definition of stabilized-interval-free permutations extends easily to \textit{decorated stabilized-interval-free permutations}. Both decorated permutations in $S_1^{\circ}$ are stabilized-interval-free, and for $n \geq 2$ the SIF permutations and the decorated SIF permutations are the same.

2.4. \textbf{Positroids and Positroid Varieties.} Postnikov and Rietsch considered an important cell decomposition of the totally nonnegative Grassmannian \cite{Pos06, Rie06}. The term \textit{positroid} does not appear in either paper, but has become the name for the matroids that index the nonempty matroid strata in that cell decomposition. They also individually considered the closures of those cells, which determines an analog of Bruhat order. The cohomology classes for these cell closures was investigated by Knutson, Lam, and Speyer \cite{KLS13, KLS14} and Pawlowski \cite{Paw18}. Further geometrical properties of positroid varieties connected to Hodge structure and cluster algebras can be found in \cite{GL21, Lam19}.

\textbf{Definition 2.15.} A real valued $k \times n$ matrix $A$ is \textit{totally nonnegative} (tnn) if each maximal minor $\Delta_I(A)$ satisfies $\Delta_I(A) \geq 0$ for $I \in \binom{[n]}{k}$. Let $Gr(k,n)^{tnn}$ be the points in $Gr(k,n)$ that can be represented by totally nonnegative matrices.

\textbf{Definition 2.16.} A \textit{positroid} of rank $k$ on the ground set $[n]$ is a matroid of the form $M_A$ for some matrix $A \in Gr(k,n)^{tnn}$. More generally, a positroid can be defined on any ordered ground set $B = \{b_1 < \cdots < b_n\}$.

Positroids are an especially nice class of realizable matroids. For example, they are closed under the matroid operations of restriction, contraction, and duality, as well as a cyclic shift of the ground set \cite{ARW16}. The following theorem is nice connection between positroids and non-crossing partitions as explored by Ardila, Rincón and Williams. See also \cite{For15} for one direction of this theorem.

\textbf{Theorem 2.17.} \cite[Thm. 7.6]{ARW16} The connected components of a positroid on ground set $[n]$ give rise to a non-crossing partition of $[n]$. Conversely, each positroid $M$ on $[n]$ can be uniquely constructed by choosing a non-crossing partition $B_1 \sqcup \cdots \sqcup B_t$ of $[n]$, and then putting the structure of a connected positroid $M_i$ on each block $B_i$, so $M = M_1 \oplus \cdots \oplus M_t$.

The connected components appearing in the decomposition of $M$ above each correspond to decorated SIF permutations. This relationship is given in the following corollary.

\textbf{Corollary 2.18.} The decomposition of a positroid $M = M_1 \oplus \cdots \oplus M_t$ into connected components gives rise to a noncrossing partition $[n] = B_1 \sqcup \cdots \sqcup B_t$ and a decomposition of the associated decorated permutation $w^0 = w^{(1)} \oplus \cdots \oplus w^{(t)}$, where each $w^{(i)}$ is a decorated stabilized-interval-free permutation on $B_i$, and the converse holds as well.

As mentioned in the introduction, positroids are in bijection with decorated permutations. The following bijection goes by way of the associated Grassmann necklace. The
bijection depends on the shifted Gale order $\prec_r$ on $\binom{[n]}{k}$ using the shifted linear order $\preceq_r$ on $[n]$. Specifically, if $I$ and $J$ are $k$-sets that can be written under $\prec_r$ as $I = \{i_1 \prec_r \cdots \prec_r i_k\}$ and $J = \{j_1 \prec_r \cdots \prec_r j_k\}$, then $I \preceq_r J$ if $i_h \preceq_r j_h$ for all $h \in [k]$.

**Theorem 2.19.** [Oh11] [Pos06] For $w^\omega \in S_{n,k}^\odot$, the set

$$
\mathcal{M}(w^\omega) := \left\{ I \in \binom{[n]}{k} : I_r(w^\omega) \preceq_r I \text{ for all } r \in [n] \right\}
$$

is a positroid of rank $k$ on ground set $[n]$. Conversely, for every positroid $\mathcal{M}$ of rank $k$ on ground set $[n]$, there exists a unique decorated permutation $w^\omega \in S_{n,k}^\odot$ such that the sequence of minimal elements in the shifted Gale order on the subsets in $\mathcal{M}$ is the Grassmann necklace of $w^\omega$.

**Corollary 2.20.** For $w^\omega = (w, co) \in S_{n,k}^\odot$ with associated positroid $\mathcal{M}(w^\omega)$, every shifted anti-exceedance set $I_r(w^\omega)$ is in $\mathcal{M}(w^\omega)$.

**Theorem/Definition 2.21.** [KLS13] Thm 5.1] Given a decorated permutation $w^\omega \in S_{n,k}^\odot$ along with its associated Grassmann interval $[u, v]$ and positroid $\mathcal{M} = \mathcal{M}(w^\omega) \subseteq \binom{[n]}{k}$, the following are equivalent definitions of the positroid variety $\Pi_w = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$.

1. The positroid variety $\Pi_{\mathcal{M}}$ is the homogeneous subvariety of $Gr(k, n)$ whose vanishing ideal is generated by the Plücker coordinates indexed by the nonbases of $\mathcal{M}$, $\{\Delta_I : I \in Q(\mathcal{M})\}$.
2. The positroid variety $\Pi_{[u,v]}$ is the projection of the Richardson variety $X_u^v \subseteq F\ell(n)$ to $Gr(k, n)$, so $\Pi_{[u,v]} = \pi_k(X_u^v)$.

**Example 2.22.** In Example 2.2, the matrix $A$ has all nonnegative $2 \times 2$ minors, so the associated matroid is a positroid. The minimal elements in shifted Gale order are $\{24\} \cup \{24\} \cup \{34\} \cup \{46\} \cup \{56\} \cup \{26\}$, which is the Grassmann necklace for the decorated permutation $136524$. The associated Grassmann interval is $[241365, 561234]$. The set of nonbases of $\mathcal{M}$ is

$$
Q = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{4, 5\}\}.
$$

Therefore, the points in the positroid variety $\Pi_{w^\omega}$ are represented by the full rank complex matrices of the form

$$
\begin{bmatrix}
0 & a_{12} & ca_{12} & a_{14} & da_{14} & a_{16} \\
0 & a_{22} & ca_{22} & a_{24} & da_{24} & a_{26}
\end{bmatrix}.
$$

As mentioned in the introduction, there are many other objects in bijection with positroids and decorated permutations. We refer the reader to [ARW16] for a nice survey of many other explicit bijections.

### 3. Positroid Characterization Using Initial Sets

The definition of positroid varieties in Theorem/Definition 2.21 is given in terms of positroids/positroid complements and the Grassmann intervals. In this section, we show...
how the initial sets of permutations in Grassmann intervals determine the positroids, proving Theorem 1.4. This theorem completes the commutative diagram below.

\[ S_{n,k}^\alpha \xrightarrow{\text{shuffle}} GI(k, n) \xrightarrow{\text{initial}} \]

Grassmann necklaces in \( \binom{n}{k} \) \( \xrightarrow{\text{shifted Gale}} \) Positroids in \( \binom{n}{k} \)

3.1. Canonical Representatives. For a k-set \( I = \{i_1 < i_2 < \cdots < i_k\} \) such that \( i_j \leq v_j \) for all \( 1 \leq j \leq k \), we will show there exists a unique maximal element below \( v \) in each \( S_k \times S_{n-k} \) coset with initial set \( I \). Let \( u(I, v) \leq v \) be this maximal element when it exists. We think of \( u(I, v) \) as the canonical representative of \( I \) in \([id, v]\).

**Lemma 3.1.** Let \( u \leq v \) be two permutations in \( S_n \), and assume \( v \) is \( k \)-Grassmannian. Then there exists a unique permutation \( u' \in [u, v] \) such that \( u' \) is maximal in Bruhat order among all permutations in the interval \([u, v]\) with \([u'_1, \ldots, u'_k] = [u_1, \ldots, u_k]\) fixed as an ordered list. Similarly, there exists a unique permutation \( u' \in [u, v] \) such that \( u' \) is maximal in Bruhat order among all permutations in the interval \([u, v]\) with \([u'_{k+1}, \ldots, u'_n] = [u_{k+1}, \ldots, u_n]\) fixed.

**Proof.** Let \( u' = u'_1, \ldots, u'_n \) be defined by \( u'_a := u_a \) for \( a \leq k \) and for \( b = n, n-1, \ldots, k+1 \) by

\[
(3.1) \quad u'(b) = \min \{v(b), n\} \setminus (u'[k] \cup u'[b+1, n])
\]

We claim that \( u' \) is well defined. To see this, recall that \( u \leq v \) implies \( u(b) \geq v(b) \) for all \( b > k \) by Theorem 2.6, since \( v \) is \( k \)-Grassmannian. Hence, for all \( b' \) such that \( k < b \leq b' \leq n \), we have \( u(b') \geq v(b') \geq v(b) \), so \([v(b), n] \cap u[k+1, n] \geq n-b+1 \). Then, at each step, the set \([v(b), n] \setminus (u'[k] \cup u'[b+1, n]) = ([v(b), n] \cap u[k+1, n]) \setminus u'[b+1, n] \) is nonempty.

By construction, \( u'(a) = u(a) \leq v(a) \) for all \( a \leq k \) and \( u'(b) \geq v(b) \) for all \( b > k \), so again by Theorem 2.6, we know \( u' \leq v \). To see that \( u \leq u' \), observe that we can obtain \( u' \) from \( u \) by applying a sequence of transpositions which increase length at each step. Since \( u'(n) = u(b) \) for some \( b \in [k+1, n] \), then \( u(b) \leq u(n) \) by (3.1). Therefore,

\[
(3.1) \quad u < ut_{bn} \quad \text{by Definition 2.4.}
\]

Next, \( u'(n-1) = ut_{bn}(c) \) for some \( c \in [k+1, n-1] \) such that \( ut_{bn}(c) \leq ut_{bn}(n-1) \) by (3.1). Hence \( u < ut_{bn} < ut_{bn}c(n-1) \), etc. for \( b = n-2, \ldots, k+1 \).

Consider the permutation matrix \( M_{u'} \) below row \( k \). For each \( k < b \leq n \), we see from the construction of \( u' \) that the rectangle northeast of \((b, v(b))\) and southwest of \((k+1, n)\) contains a decreasing sequence of 1’s ending at \( u'(b) \geq v(b) \). Therefore, there is no \( a < b \) such that \( u' < u't_{ab} \leq v \) unless \( a \leq k \) which would change the first \( k \) values of \( u' \). So, \( u' \) is maximal among all elements below \( v \) with prefix \([u_1, \ldots, u_k]\).

It remains to show that \( u' \) is the unique maximal element below \( v \) with prefix \([u_1, \ldots, u_k]\). Say \( x \in S_n \) is another element such that \( u \leq x \leq v \) and \( u_i = x_i \) for all \( 1 \leq i \leq k \). If \( u' \neq x \),
then there exists a maximal \( j \) such that \( u'(j) \neq x(j) \). Since \( x \leq v \), we know \( x(j) \in [v(j), n] \) by Theorem \( 2.6 \) again. By construction \( u'(j) \) is the minimal element in

\[
[v(j), n] \setminus (u'[k] \cup u'[j + 1, n]) = [v(j), n] \setminus (u'[k] \cup x[j + 1, n]),
\]
so \( u'(j) < x(j) \). Since \( u, u' \), and \( x \) are bijections, there exists some \( k < i < j \) such that \( x(i) = u'(j) \). Therefore, \( x < xt_{ij}, x(i) \geq v(j) \), and \( x(j) \geq v(i) \), so \( xt_{ij} \leq v \) by Theorem \( 2.6 \). Hence, \( x \) is not a maximal element below \( v \) with initial values \([u(1), \ldots, u(k)]\).

The proof for the second claim is very similar. We leave the details to the reader. \( \square \)

**Definition 3.2.** Given a \( k \)-Grassmannian permutation \( v \in S_n \) and a \( k \)-subset \( I = \{i_1 < i_2 < \cdots < i_k\} \) such that \( i_j \leq v_j \) for all \( 1 \leq j \leq k \), let \( u' = u(I, v) \) be the permutation obtained as follows.

- Loop for \( j \) from 1 to \( k \), assuming \( u'(1), u'(2), \ldots, u'(j - 1) \) are defined. Let
  \[
  u'(j) = \max \{i \in I : i \leq v_j\} \setminus u'[j - 1].
  \]
  \[ (3.2) \]

- Loop for \( j \) from \( n \) down to \( k + 1 \), assuming \( u'(n), u'(n - 1), \ldots, u'(j + 1) \) are defined. Let
  \[
  u'(j) = \min [v(j), n] \setminus (u'[k] \cup u'[j + 1, n]).
  \]
  \[ (3.3) \]

**Corollary 3.3.** Let \( v \) be a \( k \)-Grassmannian permutation, and let \( u \leq v \). If \( I = \{u_1, \ldots, u_k\} \) then \( u \leq u(I, v) \leq v \), and \( u(I, v) \) is the unique maximal element in the \( S_k \times S_{n-k} \) right coset containing \( u \).

**Proof.** Use Lemma \( 3.1 \) to find the maximal permutation \( u \leq u' \leq v \) with the prefix determined by \( u \). Then use Lemma \( 3.1 \) again to find the maximal permutation below \( v \) with suffix determined by \( u' \) to obtain \( u(I, v) \). \( \square \)

**Remark 3.4.** In preparation of this manuscript, we have learned that the restriction to \( k \)-Grassmannian permutations is unnecessary in Corollary \( 3.3 \). In fact, Oh and Richmond have proved a substantially more general statement. Their results imply that for any elements \( u \leq v \) in any Coxeter group \( W \) and any parabolic subgroup \( W_J \), there exists a unique maximal element in \([u, v] \cap uW_J\), see [OR22] Thm. 2.1.

3.2. **Intervals and Positroids.** For \( I \subseteq [n] \), define \( w_0 \cdot I := \{n + 1 - i : i \in I\} \). For any integer \( s \), we also establish the notation

\[
I^{+s} := \{i + s : i \in I\},
\]
\[
I^{-s} := \{i - s : i \in I\},
\]
where all values are taken mod \( n \) in the range \([n]\).

**Lemma 3.5.** Let \( w^\circ = (w, co) \) be a decorated permutation in \( S_{n,k}^\circ \). Let \( \mathcal{M} \) and \([u, v]\) be the positroid and Grassmann interval corresponding to \( w^\circ \). Then every \( I \in \mathcal{M} \) satisfies

\[
I_1(w^\circ) = u[k] \leq I \leq v[k] = w^{-1}(I_1(w^\circ)).
\]
Furthermore, for any \( r \in [n] \), \( I \) satisfies
\[
I_r(w^o) \preceq_r I \preceq_r w^{-1}(I_r(w^o)).
\]

**Proof.** Recall the equalities \( u[k] = I_1(w^o) \) and \( v[k] = w^{-1}(I_1(w^o)) \) from the bijection between decorated permutations and Grassmann intervals given in Section 2.3. To see the lower bound, use the characterization of \( \mathcal{M} \) from Theorem 2.19. Any \( I \in \mathcal{M} \) must satisfy \( I_r(w^o) \preceq_r I \) for every shifted anti-exceedance set \( I_r(w^o) \). In particular, for \( r = 1 \), \( I \) must satisfy \( u[k] = I_1(w^o) \preceq I \).

To prove the upper bound \( I \preceq v[k] \), let \( I = \{i_1 < \cdots < i_k\} \), and suppose that \( I \neq v[k] = \{v_1 < \cdots < v_k\} \). Then there is some maximal index \( r \) for which \( i_r > v_r \). In particular, we have \( v[k] \cap [i_r, n] < k - r + 1 \). Consider the positroid variety \( \Pi_{w^o} = \pi_k(\Pi_u \cap X^v) \). Recall that the opposite Schubert variety \( X^v \) is defined by
\[
X^v = \{ V \in \mathcal{F} \ell(n) : \dim(\text{Proj}_{[n-j+1,n]}(V_i)) \leq |v[i] \cap [n-j+1, n]| \text{ for all } i, j \}.
\]
Any point in \( \Pi_{w^o} \) is the projection of a point in \( X^v \). Thus, for any matrix \( A \) representing a point in \( \Pi_{w^o} \), the rank of columns \([i_r, n]\) of \( A \) must be strictly less than \( k - r + 1 \). In particular, the rank of \( A \) in the column set \( \{i_r, \ldots, i_k\} \) is strictly less than \( k - r + 1 \) so \( \text{rank}(A_i) < k \), which implies \( \Delta_I(M) = 0 \). Since this argument holds for every matrix representing a point in \( \Pi_{w^o} \), then by Theorem/Definition 2.21 it follows that \( I \notin \mathcal{M} \).

Thus, \( I \in \mathcal{M} \) must also satisfy \( I \preceq v[k] \).

For general \( r \), we have already seen above from the definition of \( \mathcal{M}(w^o) \) that \( I \) must satisfy \( I_r(w^o) \preceq_r I \). Thus again, it remains to show that \( I \preceq_r w^{-1}(I_r(w^o)) \). Consider the action of rotating \( D(w^o) \) in the counterclockwise direction by \( r - 1 \) positions. As discussed further in Section 3, this transformation of \( D(w^o) \) has the action of cyclically shifting the anti-exceedance sets of \( w^o \) as well as the ground set of \( \mathcal{M}(w^o) \). Let \( z^o = (z, co^o) \) be the decorated permutation whose chord diagram is obtained by applying this rotation to \( D(w^o) \). Then \( I_1(z^o) = I_1(w^o)^{(r-1)} \), \( z^{-1}(I_1(z^o)) = w^{-1}(I_1(w^o))^{-(r-1)} \), and \( \mathcal{M}(z^o) = \mathcal{M}(w^o)^{(r-1)} \). Applying the result above yields that \( I^{-(r-1)} \preceq_r z^{-1}(I_1(z^o)) = w^{-1}(I_r(w^o))^{-(r-1)} \). Cyclically shifting the values back up by \( r - 1 \) positions, it follows that \( I = (I^{-(r-1)})^{(r-1)} \preceq_r w^{-1}(I_r(w^o))^{-(r-1)} = w^{-1}(I_r(w^o)) \), as desired. \( \square \)

The result of Lemma 3.3 gives a constraint on the elements in a positroid. From this constraint, we prove the characterization of a positroid as a collection of initial sets of the corresponding Grassmann interval.

**Proof of Theorem 1.4.** Let \( S = \{y[k] : y \in [u, v]\} \) be the collection of initial sets for \([u, v]\). For \( y \in [u, v] \), the point represented by the permutation matrix \( M_y \) is in the Richardson variety \( X^v_u \). Let \( A_y = A_{y[k]} \) be the submatrix of \( M_y \) given by the top \( k \) rows. Then \( A_y \) represents a point in the positroid variety \( \pi_k(X^v_u) = \Pi_{[u, v]} \), by Part (1) of Theorem/Definition 2.21. The restriction of \( A_y \) to the columns indexed by \( I = y[k] \) is a \( k \times k \) permutation matrix. Therefore, \( \Delta_I \) does not vanish at \( A_y \in \Pi_{[u, v]} \). Since \( \Delta_I \) is not in
the vanishing ideal of $\Pi_{[u,v]}$, we must have $I \in M$ by Part (2) of Theorem/Definition 2.21. Thus, we have the inclusion $S \subseteq M$.

For the reverse inclusion, let $I \in M$. By Lemma 3.5, $I$ must satisfy $u[k] \preceq I \preceq v[k]$. In particular, since $I \preceq v[k]$, we may use Definition 3.2 to define a permutation $y = u(I,v)$ with initial set $y[k] = I$. By Corollary 3.3, $y \preceq v$ under Bruhat order, so to prove $I \in S$ it remains only to show that this $y$ satisfies $u \preceq y$.

By [BB05b, Ex. 8, Ch 2], $u \leq y$ if and only if $u[j] \leq y[j]$ for all $j \in [k]$ and $u[n-j+1,n] \geq y[n-j+1,n]$ for all $j \in [n-k]$. This exercise follows from Definition 2.4. We will prove that $u[j] \leq y[j]$ for $j \in [k]$ by induction on $j$.

Let $w^\omega$ be the decorated permutation associated to $[u,v]$. Since $I \in M$, we know by Theorem 2.19 that $I_r(w^\omega) \preceq I$ for all $r \in [n]$ where $I_r(w^\omega)$ is the shifted anti-exceedance set for $w^\omega$. By the shuffling algorithm described in Section 2 mapping $w^\omega$ to $[u,v]$, one can observe for all $1 \leq j \leq k$ that $u_j \in I_r(w^\omega)$ unless $u_j < r \leq v_j$. In particular, $u[j] \in I_r(w^\omega)$ for all $r > v_j$ since $v_1 < \cdots < v_j$ and $u \preceq v$. Therefore,

$$ |u[j] \cap [i,v_j]| \leq |I_r(w^\omega) \cap [i,v_j]| \quad \forall \ r > v_j. \tag{3.4} $$

For the case $j = 1$, fix $r = v_1+1$ (or if $v_1 = n$ fix $r = 1$) so $v_1$ is maximal under the order $<$. By Definition of $I \cap [u_1,v_1]$ must also be nonempty. By construction of $y = u(I,v)$ and Definition 3.2, $y_1$ is the maximal element of $I \cap [u_1,v_1]$. Thus $u_1 \leq y_1$.

For the inductive step, fix $j \in [2,k-1]$, and assume that

$$ u[j-1] = \{a_1 < \cdots < a_{j-1}\} \leq y[j-1] = \{b_1 < \cdots < b_{j-1}\}. \tag{3.5} $$

We note that all $a_i, b_i \in [v_j]$ since $v$ is $k$-Grassmannian and $u, y \preceq v$ in Bruhat order by Theorem 2.6. If $u_j \leq y_j$, then no matter where $u_j$ and $y_j$ fit in among the increasing sequence of $a_i$'s and $b_i$'s, we will have $u[j] \leq y[j]$ by (3.5). Therefore, assume $1 \leq y_j < u_j \leq v_j$. By the induction hypothesis, $1 \leq a_i \leq b_i \leq v_i$ for each $1 \leq i < j$, so if $m$ is the largest index such that $b_m < y_j$, then

$$ u[j] \cap [1,a_m] \leq y[j] \cap [1,b_m]. $$

So, to prove $u[j] \preceq v[j]$, it suffices to show that for all $i \in [y_j,v_j]$,

$$ |u[j] \cap [i,v_j]| \leq |y[j] \cap [i,v_j]|. \tag{3.6} $$

Fix $r = v_j + 1$ modulo $n$ so $v_j$ is maximal in the shifted $\leq_r$ order. By definition of $y = u(I,v)$, we know $y_j$ is maximal in $(I \setminus y[j-1]) \cap [v_j]$ under $\leq_r$ and the usual order on the integers. Therefore, for all $i \in [y_j,v_j]$ we have $y[j] \cap [i,v_j] = I \cap [i,v_j]$. Hence, since $I_r(w^\omega) \preceq_r I$, we have

$$ |I_r(w^\omega) \cap [i,v_j]| \leq |I \cap [i,v_j]| = |y[j] \cap [i,v_j]|. \tag{3.7} $$

By (3.4),

$$ |u[j] \cap [i,v_j]| \leq |I_r(w^\omega) \cap [i,v_j]|, \tag{3.8} $$

and $u \preceq y$. Therefore, we have $u \preceq y$ for all $j \in [k]$ by induction on $j$. Thus, $u \preceq y$.
so (3.7) and (3.8) together imply (3.6) for all $i \in [y_j, v_j]$. Hence, $u[j] \leq y[j]$ for all $j \in [k]$.

Observe that the map on permutations in $S_n$ sending $x \rightarrow x' = w_0 x w_0$ rotates the permutation matrices 180 degrees. Using Theorem 2.6 we have $[u, v] \in G_i(k, n)$ if and only if $[w_0 u w_0, w_0 v w_0] \in G_i(n - k, n)$. By the symmetry in Definition 3.2 we observe that $u(y[k + 1, n], w_0 v w_0) = w_0 y w_0$. Therefore, by applying the argument above to $y' \in [u', v']$ with $I' = w_0 \cdot y[k + 1, n]$, we have $u'[j] \leq y'[j]$ for all $j \in [n - k]$ which implies $u[n - j + 1, n] \geq y[n - j + 1, n]$ for $j \in [n - k]$ as needed to complete the proof. □

Proof of Theorem 1.7. (2) $\iff$ (3) $\iff$ (4). By Theorem 1.4, we have the equality $\mathcal{M} = \{y[k] : y \in [u, v]\}$. By Theorem 1.2 we have $\#\text{Alignments}(w^o) = k(n - k) - [\ell(v) - \ell(u)]$. Hence, $\ell(v) - \ell(u) = k(n - k) - \#\text{Alignments}(w^o)$. Together, these facts yield (2) $\iff$ (3).

The equality (3) $\iff$ (4) comes from the definition of the matroid Johnson graph $J(\mathcal{M})$. For a set $J \in \mathcal{M}$, the vertices adjacent to $J$ in $J(\mathcal{M})$ are exactly the sets $I \in \mathcal{M}$ such that $|I \cap J| = k - 1$. So again, the equality $\ell(v) - \ell(u) = k(n - k) - \#\text{Alignments}(w^o)$ from Theorem 1.2 implies (3) $\iff$ (4). □

4. Reduction to $T$-fixed points

To determine whether a variety is singular, in general one would have to test every point in the variety. In the results below, we show that for a positroid variety $\Pi_{[u,v]}$, one need only check whether $\Pi_{[u,v]}$ is singular at the $T$-fixed points $A_{y[k]}$ for $y \in [u, v]$. In particular, we will complete the proofs of Theorem 1.3 and Theorem 1.5. Together these two theorems along with Theorem 1.4 prove the equality between (1) and (2) in Theorem 1.7 as mentioned in the introduction.

First, we recall some specifics for the definition of the Jacobian matrix of a positroid variety. By Theorem/Definition 2.21 Part (1), the polynomials generating the vanishing ideal of $\Pi_{[u,v]}$ are exactly the determinants $\Delta_I$ for $I \in \mathcal{Q}_{[u,v]}$ where

\[(4.1) \quad \mathcal{Q}_{[u,v]} := \begin{pmatrix} [n] \end{pmatrix}_k \setminus \mathcal{M} = \left\{ I \in \begin{pmatrix} [n] \end{pmatrix}_k : I \neq y[k] \forall y \in [u, v] \right\} \]

is the set of nonbases of the positroid $\mathcal{M}$ corresponding to $[u, v] \in G_i(k, n)$. The second equality in (4.1) holds by Theorem 1.4. Each determinant $\Delta_I$ is a polynomial function using variables of the form $x_{ij}$ indexed by row $i \in [k]$ and column $j \in [n]$. Let $Jac_{[u,v]}$, or just $Jac$ if $[u,v]$ is understood, be the Jacobian of $\Pi_{[u,v]}$. Similarly, we will suppress the subscript $\mathcal{Q} = \mathcal{Q}_{[u,v]}$. Then, the rows of $Jac$ are indexed by the sets $I \in \mathcal{Q}$, the columns are indexed by variables $x_{ij}$, and the $(I, x_{ij})$ entry of $Jac$ is $\frac{\partial \Delta_I}{\partial x_{ij}}$. If $A$ is a $k \times n$ complex matrix representing a point in $\Pi_{[u,v]} \subseteq Gr(k, n)$, then $Jac|_A$ is the matrix with entries in the complex numbers obtained from $Jac$ by evaluating each entry at $A$,

\[(4.2) \quad Jac|_A(I, x_{ij}) = \frac{\partial \Delta_I}{\partial x_{ij}}(A).\]
By Theorem/Definition 2.21 Part (2), we can write $\Pi_{[u,v]}$ as the union

\[(4.3) \quad \Pi_{[u,v]} = \pi_k(X_u^v) = \pi_k\left( \bigcup_{u \leq y < v} (C_y \cap X^v) \right) = \bigcup_{u \leq y < v} \pi_k(C_y \cap X^v).\]

For any $y \in [u,v]$, recall $A_y[k]$ is the projection of the permutation matrix $M_y$ to the top $k$ rows.

**Lemma 4.1.** For $[u,v] \in G_i(k,n)$ and $y \in [u,v]$, the matrix $\text{Jac}_{[u,v]}|_{A_y[k]}$ is a partial permutation matrix, up to the signs of the entries. Furthermore, the nonzero entries occur exactly in the entries $(I, x_{st})$, where $I \in \mathcal{Q}_{[u,v]}$ satisfies $|I \cap y[k]| = k - 1$, the value $s \in [k]$ is determined by $y[k] \setminus I = \{y_s\}$, and $t \in [n]$ is determined by $I \setminus y[k] = \{t\}$.

**Proof.** Write $A_y[k]|_{I}$ for the restriction of $A_y[k]$ to column set $I$. By expanding $\Delta_I$ along row $i \in [k]$ of the partial permutation matrix $A_y[k]$, observe from (4.2) that entry $(I, x_{ij})$ of $\text{Jac}|_{A_y[k]}$ is

1. $(j \in I)$ up to sign, the $(k - 1) \times (k - 1)$ minor of $A_y[k]|_I$ in column set $I \setminus \{j\}$ and row set $[k] \setminus \{i\}$, or
2. $(j \notin I)$ 0, since $\Delta_I$ does not contain $x_{ij}$.

Consider any $I \in \mathcal{Q}$. By Theorem 1.4, $y[k] \in \mathcal{M}$, so $I \neq y[k]$. Since $I$ and $y[k]$ are distinct $k$-sets, then $|I \cap y[k]|$ is at most $k - 1$. If $|I \cap y[k]| \leq k - 2$, then $A_y[k]$ has at most $k - 2$ ones in $A_y[k]|_I$, and the remaining entries of $A_y[k]|_I$ are zeros. In that case, every $(k - 1) \times (k - 1)$ minor of $A_y[k]|_I$ is zero. Therefore, row $I$ of $\text{Jac}|_{A_y[k]}$ contains only zeros.

Otherwise, $I \in \mathcal{Q}$ satisfies $|I \cap y[k]| = k - 1$. Say $I \setminus y[k] = \{t\}$ and $y[k] \setminus I = \{y_s\}$ for some $s \in [k]$. Then column $t$ of $A_y[k]$ contains all zeros, row $s$ of $A_y[k]|_I$ contains all zeros, and the submatrix of $A_y[k]|_I$ obtained by removing row $s$ and column $t$ is a permutation matrix. Therefore, by cofactor expansion of the determinant

$$\text{Jac}|_{A_y[k]}(I, x_{st}) = \frac{\partial \Delta_I}{\partial x_{st}}(A_y[k]) = \pm 1,$$

and $x_{st}$ is the unique variable such that $\frac{\partial \Delta_I}{\partial x_{ij}}(A_y[k])$ is nonzero.

Conversely, we claim column $x_{st}$ contains a unique nonzero entry in row $I$. If $I' \in \mathcal{Q}$ is another set such that $|I' \cap y[k]| = k - 1$, let $I' \setminus y[k] = \{t'\}$ and $y[k] \setminus I' = \{y_{s'}\}$. Then, the only nonzero entry of $\text{Jac}|_{A_y[k]}$ in row $I'$ occurs in column $x_{s't'}$ by the same reasoning as above. If $t \neq t'$, then $x_{st} \neq x_{s't'}$. If $t = t'$, either $s = s'$ and $I' = I$ or else $y_s \in I'$ so $s' \neq s$ since $y$ is a bijection, in which case $x_{st} \neq x_{s't'}$. Therefore, $\text{Jac}|_{A_y[k]}$ is a partial permutation matrix up to signs as stated.

**Proof of Theorem 1.5.** From Lemma 4.1, $\text{Jac}|_{A_y[k]}$ is a partial permutation matrix, up to the signs of its entries. Thus, the rank of $\text{Jac}|_{A_y[k]}$ is exactly the number of its nonzero entries, which are exactly in the rows indexed by the sets $I \in \mathcal{Q}$ satisfying $|I \cap y[k]| = k - 1$. 


Therefore, the rank of $\text{Jac}|_{A_y[k]}$ is equal to the number of these sets,

$$\text{rank}(\text{Jac}|_{A_y[k]}) = \# \{ I \in \binom{[n]}{k} \setminus \mathcal{M} : |I \cap [k]| = k - 1 \}.$$  

Next, consider a flag $V_v \in C_y \cap X^v$ and its canonical matrix representative $A_{V_v}$ as defined in Section 2. Let $A$ be the projection of $A_{V_v}$ to $Gr(k, n)$. We will see that the rank of the Jacobian matrix evaluated at $A$ is at least the rank of the Jacobian evaluated at $A_y[k]$.

Proof of Theorem 1.3. Without loss of generality, we can assume $A \in \pi_k(C_y \cap X^v)$ is a canonical matrix with leading ones in exactly the same entries as $A_y[k]$. Therefore, permuting the columns of $A$ and $A_y[k]$ in the same way, we can assume the first $k$ columns of both have the form of an upper triangular matrix with ones on the diagonal. If we apply the same permutation to the values in the $k$-sets in $Q$ and the variables $x_{ij}$, then such a rearrangement of columns will not change the rank of $\text{Jac}|_{A_y[k]}$ or $\text{Jac}|_A$. After permuting, we can then assume $y$ is the identity permutation, so $A_y[k] = A[k]$. Let $N = \text{rank}(\text{Jac}|_A)$. By Theorem 1.5, we must show that

$$\text{rank}(\text{Jac}|_A) \geq N = \# \{ I \in Q : |I \cap [k]| = k - 1 \}.$$  

Let $R = \{ I_1, \ldots, I_N \}$ and $C = \{ (s_1, t_1), \ldots, (s_N, t_N) \}$ be the unique row and column sets such that the $N \times N$ minor of $\text{Jac}|_A$ in rows $R$ and columns $C$ is nonzero, as determined in Lemma 4.1. Then, for each $I_j \in R$, we know $I_j = ([k] \setminus \{s_j\}) \cup \{t_j\}$, where $s \in [k]$ and $t_j > k$ after permuting values. Since the first $k$ columns of $A$ form an upper triangular matrix with ones along the diagonal, we observe that for all $j \in [N]$,

$$\text{Jac}|_A(I_j, x_{s_j t_j}) = \frac{\partial \Delta_{I_j}}{\partial x_{s_j t_j}}(A) = \pm 1.$$  

Fix $j \in [N]$. Assume that the sets in $R$ are ordered lexicographically as sorted lists, and let $1 \leq h < j \leq N$. Either $t_h \neq t_j$ or $t_h = t_j$ and $s_h > s_j$ by lex order. If $t_h \neq t_j$, then $I_h$ does not contain $t_j$, by construction of $I_h$. Hence, $\Delta_{I_h}$ does not depend on the variable $x_{s_j t_j}$, so $\frac{\partial \Delta_{I_h}}{\partial x_{s_j t_j}}(A) = 0$. Otherwise, $s_h > s_j$, and $\frac{\partial \Delta_{I_h}}{\partial x_{s_j t_j}}(A)$ is the determinant of an upper triangular matrix with a zero on its diagonal. Hence, in either case

$$\text{Jac}|_A(I_h, x_{s_j t_j}) = \frac{\partial \Delta_{I_h}}{\partial x_{s_j t_j}}(A) = 0.$$  

Arrange the rows and columns of $\text{Jac}$ so that $I_1, \ldots, I_N$ are the top $N$ rows listed in order and $x_{s_1 t_1}, x_{s_2 t_2}, \ldots, x_{s_N t_N}$ are the first $N$ columns listed in order. Then, the $N \times N$ upper left submatrix of $\text{Jac}|_A$ is lower triangular with plus or minus one entries along the diagonal. Thus $\text{Jac}|_A$ contains a rank $N$ submatrix, so $\text{rank}(\text{Jac}|_A) \geq N$. 

□
Example 4.2. Rearranging the columns of the matrix in Example 2.2 so that $y[k] = [k]$, consider the matrix

$$
A = \begin{bmatrix}
1 & 2 & 3 & 0 & 4 & 0 \\
0 & 1 & 0 & 0 & 2 & 1
\end{bmatrix}.
$$

The nonbases of $\mathcal{M}_A$ are

$$
Q = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,4\}, \{4,5\}, \{4,6\}\} \subseteq \begin{pmatrix} 6 \\ 2 \end{pmatrix}.
$$

With $k = 2$, the sets $I \in Q$ satisfying $[k] \cap I = k - 1$, ordered lexicographically, are $I_1 = \{1,3\}$, $I_2 = \{1,4\}$, $I_3 = \{2,4\}$, and $I_4 = \{2,5\}$. The corresponding $x_{st}$ so that $I_j = ([k] \times \{s_j\}) \cup \{t_j\}$ are $x_{23}$, $x_{24}$, $x_{14}$, and $x_{15}$, respectively. Arranging the rows and columns of $Jac|_A$ as described in the proof above, the upper left submatrix of $Jac|_A$ is a lower triangular matrix with plus or minus one entries on the diagonal, as shown below.

$$
\begin{array}{cccc}
\{1,3\} & 1 & 0 & 0 \\
\{1,4\} & 0 & 1 & 0 \\
\{2,4\} & 0 & 2 & -1 \\
\{2,5\} & 0 & 0 & 1
\end{array}
$$

Corollary 4.3. Let $[u, v] \in Gi(k, n)$ and $y \in [u, v]$. Then $\pi_k(C_y \cap X_v) \subseteq \Pi_{[u, v]}$ contains a singular point of $\Pi_{[u, v]}$ if and only if $\Pi_{[u, v]}$ is singular at $A_{y[k]}$. Furthermore, $A_{y[k]}$ is a singularity of $\Pi_{[u, v]}$ if and only if

$$
\# \{I \in Q_{[u, v]} : |I \cap y[k]| = k - 1\} < k(n-k) - [\ell(v) - \ell(u)].
$$

Proof. Recall from (1.3) that $A \in \Pi_{[u, v]}$ is a singular point of $\Pi_{[u, v]}$ if and only if $\text{rank}(Jac|_A) < \text{codim} \Pi_{[u, v]}$. By Theorem 1.3, for any $A \in \pi_k(C_y \cap X_v)$, $\text{rank}(Jac|_{A_{y[k]}}) \leq \text{rank}(Jac|_A)$. Therefore, if there exists some $A \in \pi_k(C_y \cap X_v)$ that is a singular point of $\Pi_{[u, v]}$, then

$$
\text{rank}(Jac|_{A_{y[k]}}) \leq \text{rank}(Jac|_A) < \text{codim} \Pi_{[u, v]}
$$

implies that $A_{y[k]}$ is also a singular point of $\Pi_{[u, v]}$. Conversely, note that since the permutation matrix $M_y$ is in $C_y \cap X_v$, then $A_{y[k]} = \pi_k(M_y) \in \pi_k(C_y \cap X_v)$. Thus, if $A_{y[k]}$ is a singular point of $\Pi_{[u, v]}$, then $A_{y[k]}$ is already a singular point of $\Pi_{[u, v]}$ in $\pi_k(C_y \cap X_v)$. This proves the first statement.

For the second statement, recall from Theorem 1.2 that

$$
\text{codim} \Pi_{[u, v]} = k(n-k) - [\ell(v) - \ell(u)].
$$

By Theorem 1.5, $\text{rank}(Jac|_{A_{y[k]}}) = \# \{I \in Q : |I \cap y[k]| = k - 1\}$. Therefore, $A_{y[k]}$ is a singular point of $\Pi_{[u, v]}$ if and only if

$$
\# \{I \in Q : |I \cap y[k]| = k - 1\} = \text{rank}(Jac|_{A_{y[k]}}) < \text{codim} \Pi_{[u, v]} = k(n-k) - [\ell(v) - \ell(u)].
$$

□
Proof of Theorem 1.7. (1) $\Leftrightarrow$ (2). From the definition of a smooth variety and the decomposition
\[
\Pi_{[u,v]} = \pi_k(X_u^v) = \pi_k\left( \bigcup_{y \leq v} (C_y \cap X^v) \right) = \bigcup_{y \leq v} \pi_k(C_y \cap X^v),
\]
it follows that $\Pi_{[u,v]}$ is smooth if and only if, for every $y \in [u, v]$, every point in $\pi_k(C_y \cap X^v)$ is a smooth point of $\Pi_{[u,v]}$. By Corollary 4.3, every point of $\pi_k(C_y \cap X^v)$ is a smooth point of $\Pi_{[u,v]}$ if and only if $A_y[k]$ is a smooth point of $\Pi_{[u,v]}$, which occurs if and only if
\[
(4.10) \quad \#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1 \} \geq k(n-k) - [\ell(v) - \ell(u)].
\]
Since $\text{rank}(\text{Jac}_{|A_y[k]})$ is the codimension of the tangent space to $\Pi_{[u,v]}$ at $A_y[k]$, then $\text{rank}(\text{Jac}_{|A_y[k]})$ is bounded above by $\text{codim}\Pi_{[u,v]}$. Thus, the inequality in (4.6) can never be strict. Therefore, $A_y[k]$ is a smooth point of $\Pi_{[u,v]}$ if and only if
\[
(4.11) \quad \#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1 \} = \text{rank}(\text{Jac}_{|A_y[k]}) = \text{codim}\Pi_{[u,v]} = k(n-k) - [\ell(v) - \ell(u)].
\]

We can compute the left side of (4.11) as follows. For any $j \in y[k]$ and any element $i \in [n]\setminus y[k]$, observe that there are $k(n-k)$ sets $I \in \binom{[n]}{k}$ of the form $I = (y[k]\setminus\{j\}) \cup \{i\}$ so that $|I \cap y[k]| = k-1$. Since $\mathcal{Q} = \binom{[n]}{k} \setminus \mathcal{M}$, then
\[
(4.8) \quad \#\{I \in \mathcal{M} : |I \cap y[k]| = k-1\} + \#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1\} = k(n-k).
\]
Substituting the result of (4.8) into (4.11) yields that $A_y[k]$ is a smooth point of $\Pi_{[u,v]}$ if and only if
\[
(4.9) \quad \#\{I \in \mathcal{M} : |I \cap y[k]| = k-1\} = \ell(v) - \ell(u).
\]
Thus, $\Pi_{[u,v]}$ contains only smooth points if and only if every $y \in [u, v]$ satisfies (4.9). \qed

Finally, in this section, we relate the vertex degree conditions in the Johnson graph to the sets of nonbases of positroids similar to Corollary 1.3. If a positroid $\mathcal{M}$ corresponds with the decorated permutation $w^\omega$, let $\text{Alignments}(\mathcal{M}) = \text{Alignments}(w^\omega)$.

Corollary 4.4. Given a rank $k$ positroid $\mathcal{M}$ on ground set $[n]$, let $\mathcal{Q} := \binom{[n]}{k} \setminus \mathcal{M}$ be the corresponding set of nonbases. For any $J \in \mathcal{M}$, the codimension of the tangent space to $\Pi_{\mathcal{M}}$ at $A_J$ is
\[
(4.10) \quad \#\{I \in \mathcal{Q} : |I \cap J| = k-1\} \leq \#\text{Alignments}(\mathcal{M}) = \text{codim}(\Pi_{\mathcal{M}}).
\]
Furthermore, $A_J$ is a singular point in $\Pi_{\mathcal{M}}$ if and only if
\[
(4.11) \quad \#\{I \in \mathcal{Q} : |I \cap J| = k-1\} < \#\text{Alignments}(\mathcal{M}).
\]

Proof. Equation (4.10) follows from Equation (1.3), Theorem 1.4 and Theorem 1.5. The second claim now follows by Corollary 1.3 and Theorem 1.4. \qed
5. Rigid Transformations

The authors of [ARW16] show that the set of positroids is closed under restriction, contraction, duality, and a cyclic shift of the ground set. We add to this list reversal of the ground set. The fact that positroids are closed under duality, cyclic shift of the ground set, and reversal of the ground set can be obtained by considering rigid transformations of the chord diagram of the associated decorated permutation. We consider three types of rigid transformations on chord diagrams: arc reversal, reflection, and rotation. We associate these transformations with the symmetric group operations of taking the inverse, conjugation by \(w_0\), and conjugation by a cycle. Because these rigid transformations are bijections on the set of chord diagrams, they generate a group of transformations on Grassmann intervals, Grassmann necklaces, and positroids. The results are collected in Proposition 5.2.

Throughout this section, fix \(w^\circ = (w, co) \in S_{n,k}^\circ\), and let \(z^\circ = (z, co')\) be the decorated permutation whose chord diagram \(D(z^\circ)\) is obtained from \(D(w^\circ)\) by a rigid transformation. Denote by \(F(w^\circ)\) the set of fixed points of \(w^\circ\). Define the map \(\text{flip} : \{\bigcirc, \bigcirc\} \to \{\bigcirc, \bigcirc\}\) to be the involution on \(\{\bigcirc, \bigcirc\}\) that reverses the orientation. Let \(\chi \in S_n\) be the cycle with \(\chi(i) = i + 1 \mod n\).

To give the maps on decorated permutations corresponding to the chord diagram transformations, we will consider the transformations on the arcs of the chord diagrams. The transformations of arcs then lead to transformations of the two-line notation for decorated permutations.

First, consider arc reversal, where \(D(z^\circ)\) is obtained from \(D(w^\circ)\) by reversing all the arcs in \(D(w^\circ)\). So, an arc \(i \mapsto w(i)\) becomes \(w(i) \mapsto i\). When \(i \in F(w^\circ)\), then \(i\) becomes a fixed point in \(z^\circ\) with opposite orientation of \(i\). Then \(z = w^{-1}\), and \(F(z^\circ) = F(w^\circ)\) with all fixed point orientations reversed. Therefore, \(co' = \text{flip} \circ co\). With these observations in mind, we define

\[
(w^\circ)^{-1} := (w^{-1}, \text{flip} \circ co).
\]

The two-line notation for \((w^\circ)^{-1}\) is obtained from the two-line notation of \(w^\circ\) by swapping the rows, reversing arrows labeling fixed points, and reordering the columns so that the entries of the top line appear in increasing order.

Next, consider reflection, where \(D(z^\circ)\) is obtained from \(D(w^\circ)\) by reflecting \(D(w^\circ)\) across the vertical axis. So, an arc \(i \mapsto w(i)\) becomes \(w(i) \mapsto w_0(w(i))\). When \(i \in F(w^\circ)\), then \(w_0(i)\) becomes a fixed point of \(z^\circ\) with opposite orientation of \(i\). Then \(z = w_0 w w_0\), and \(F(z^\circ) = w_0 \cdot F(w^\circ)\) with all fixed point orientations reversed after applying \(w_0\) to the value of a fixed point in \(F(z^\circ)\). Therefore, \(co' = \text{flip} \circ co \circ w_0\), and we define

\[
w_0 \cdot w^\circ := (w_0 w w_0, \text{flip} \circ co \circ w_0).
\]

The two-line notation for \(z^\circ\) is obtained from the two-line notation of \(w^\circ\) by replacing \(i\) with \(w_0(i)\) in both lines, reversing all arrows labeling fixed points, and reversing the order of the columns.
Finally, consider rotation, where $D(z^\circ)$ is obtained from $D(w^\circ)$ by rotating $D(w^\circ)$ by $s$ units in the clockwise direction. So, an arc $i \mapsto w(i)$ becomes $i + s \mapsto w(i) + s$, taken modulo $n$. When $i \in F(w^\circ)$, then $i + s$ becomes a fixed point of $z^\circ$ with the same orientation as $i$. Then $z = \chi^s w \chi^{-s}$, and $F(z^\circ) = F(w^\circ)^{rs}$ with all fixed point labels preserved after applying $\chi^{-s}$ to the value of a fixed point in $F(z^\circ)$. Therefore, $co' = co \circ \chi^{-s}$, and we define

$$\chi^s \cdot w^\circ := (\chi^s w \chi^{-s}, co \circ \chi^{-s}).$$

The two-line notation for $z^\circ$ is obtained from the two-line notation of $w^\circ$ by replacing $i$ with $i + s \mod n$ in both lines and cyclically shifting all columns $s$ units to the right.

Associated with the maps listed above are maps on Grassmann intervals. Let $[u, v] \in Gi(k, n)$ be the Grassmann interval corresponding to $w^\circ$, and let $[u', v']$ be the Grassmann interval corresponding to $z^\circ$. As we know from the shuffling algorithm described in Section 2.3, $u'$ and $v'$ are easily extracted from the two-line notation for $z^\circ$ by reordering the columns so that the highlighted columns, corresponding to the anti-exceedances of $z^\circ$, appear on the left and the top line has the form of a $k(z^\circ)$-Grassmannian permutation. This new array is the two-line array $[v' u']$.

We have already described how the two-line notation for $z^\circ$ is obtained from the two-line notation of $w^\circ$ under the three rigid transformations of $D(w^\circ)$. From these descriptions, we obtain the following maps $[v' u'] \mapsto [v' u']$.

1. For the arc reversal map, $[v' u']$ is obtained from $[u' v]$ by swapping the rows, swapping the left $k$-column block with the right $(n-k)$-column block, and reordering the columns within the two blocks so that the top row gives an $(n-k)$-Grassmannian permutation. Define $[u, v]^{-1} \in Gi(n-k, n)$ to be the Grassmann interval whose two-line array is obtained in this way from $[u' v]$.

2. For the reflection map, $[v' u']$ is obtained from $[u' v]$ by replacing every $i$ with $w_0(i)$ in both rows and reversing all columns. It follows that $u' = w_0 u w_0$ and $v' = w_0 v w_0$. Define $w_0 \cdot [u, v] := [w_0 u w_0, w_0 v w_0] \in Gi(n-k, n)$. Note that this operation on $[u, v]$ of conjugation by $w_0$ was used at the end of the proof of Theorem 1.4.

3. For the rotation map, $[v' u']$ is obtained from $[u' v]$ by replacing every $i$ with $i + s$, highlighting all columns corresponding to anti-exceedances, then reordering the columns into a highlighted and an unhighlighted block so that the elements of the top row are increasing within each block. Define $\chi^s \cdot [u, v] \in Gi(k, n)$ to be the Grassmann interval obtained from $[u, v]$ in this way.

There are also naturally associated maps on the positroids corresponding to $w^\circ$ and $z^\circ$. These maps on positroids, listed in Proposition 5.2, can be seen from the following facts.

1. The dual map on positroids corresponds to arc reversal, as shown in the following lemma.
(2) The map corresponding to $w^o \mapsto w_0 w^o w_0$ on Grassmann intervals is $[u, v] \mapsto [w_0 w_0 w_0, w_0 v w_0] \in G_I(n-k, n)$, and $[w_0 w_0 w_0, w_0 v w_0] = \{ w_0 y w_0 : y \in [u, v] \}$. In particular, the map $y \mapsto w_0 y w_0$ is an interval isomorphism between $[u, v]$ and $[w_0 w_0, w_0 v w_0]$. The related map on positroids then follows from Theorem 1.4.

(3) Every $r$-anti-exceedance of $w^o$ translates to an $(r+s)$-anti-exceedance of $z^o$ so that $I_r(w^o)^{+s} = I_{r+s}(z^o)$. Also, for sets $I, J \in \binom{[n]}{k}$, $I \preceq_r J$ if and only if $I^{+s} \preceq_{r+s} J^{+s}$.

Lemma 5.1. Let $w^o \in S^o_{n,k}$ have associated positroid $M(w^o) \subseteq \binom{[n]}{k}$, and let $z^o = (w^o)^{-1} \in S^o_{n,n-k}$. The positroid associated with $z^o$ is the dual of $M(w^o)$,

$$\mathcal{M}(z^o) = \{ [n] \setminus I : I \in M(w^o) \} \subseteq \binom{[n]}{n-k}.$$  

Proof. Since arc reversal is an involution, to show that $\mathcal{M}(z^o)$ is the dual of $\mathcal{M}(w^o)$, as in (5.4), it suffices to show that $I \in M(w^o)$ implies that $[n] \setminus I \in \mathcal{M}(z^o)$. In particular, we will show that $I \in M(w^o)$ implies that $I_r(z^o) \preceq_r [n] \setminus I$ for all $r \in [n]$.

For $r \in [n]$, let $J_r(w^o)$ be the set of $r$-exceedances of $w^o$, $J_r(w^o) = [n] \setminus I_r(w^o)$. Fix $r \in [n]$ and $I \in M(w^o)$. By definition, $D(z^o)$ is obtained from $D(w^o)$ by reversing every arc. Then every arc $a \mapsto w(a)$ corresponding an $r$-anti-exceedance $w(a) \in I_r(w^o)$ yields an arc $w(a) \mapsto a$ in $D(z^o)$, which corresponds to an $r$-exceedance of $z^o$, so that $a \in J_r(z^o)$. Similarly, $r$-exceedances $w(a) \in J_r(w^o)$ yield $r$-anti-exceedances $a \in I_r(z^o)$. Therefore, we have

$$J_r(z^o) = w^{-1}(I_r(w^o))$$
$$I_r(z^o) = w^{-1}(J_r(w^o)) = [n] \setminus w^{-1}(I_r(w^o)).$$

Since $I \in \mathcal{M}$, then by Lemma 3.5, $I$ must satisfy $I_r(w^o) \preceq_r I \preceq_r w^{-1}(I_r(w^o))$. In particular, $I \preceq_r w^{-1}(I_r(w^o))$ implies that $[n] \setminus I \preceq_r [n] \setminus w^{-1}(I_r(w^o)) = I_r(z^o)$, as desired. Therefore, $[n] \setminus I$ is in $\mathcal{M}(z^o)$ by Theorem 2.19. □

The transformations of arc reversal, reflection, and rotation can now be applied to any of the bijectively equivalent objects. These transformations are summarized in the following proposition.

Proposition 5.2. Fix a decorated permutation $w^o = (w, co) \in S^o_{n,k}$ with associated Grassmann interval $[u, v]$, Grassmann necklace $(I_1(w^o), \ldots, I_n(w^o))$, and positroid $M(w^o)$. Let $z^o$ be a decorated permutation whose chord diagram is obtained from $D(w^o)$ by (1) arc reversal, (2) reflection, or (3) rotation. Let $[u', v']$ be the Grassmann interval associated with $z^o$. The table below describes $z^o$ and its associated objects and values.

| Trans \ Obj | $z^o$ | $[u', v']$ | $I_r(z^o)$ | $M(w^o) \mapsto M(z^o)$ | $k(z^o)$ |
|-------------|-------|-----------|------------|------------------------|--------|
| arc reversal | $(w^o)^{-1}$ | $[u, v]^{-1}$ | $[n] \setminus w^{-1}(I_r(w^o))$ | $I \mapsto [n] \setminus I$ | $n-k$ |
| reflection  | $w_0 \cdot w^o$ | $w_0 \cdot [u, v]$ | $w_0 \cdot ([n] \setminus I_{w_0(r)}(w^o))$ | $I \mapsto w_0 \cdot ([n] \setminus I)$ | $n-k$ |
| rotation    | $\chi^s \cdot w^o$ | $\chi^s \cdot [u, v]$ | $I_{r-s}(w^o)^{+s}$ | $I \mapsto I^{+s}$ | $k$ |
Corollary 5.3. The set of positroids is closed under reversal of the ground set $[n]$.

Proof. Let $\mathcal{M} \subseteq \binom{[n]}{k}$ be a positroid, and let $w^\circ$ be the decorated permutation associated with $\mathcal{M}$ so that $\mathcal{M} = \mathcal{M}(w^\circ)$. Let $w_0 \cdot \mathcal{M} = \{ w_0 \cdot I : I \in \mathcal{M} \}$ be the matroid obtained from $\mathcal{M}$ by reversal of the ground set $[n]$. We show that $w_0 \cdot \mathcal{M}$ is a positroid.

Let $z^\circ$ be the decorated permutation whose chord diagram is obtained from $D(w^\circ)$ by arc reversal followed by a reflection across the vertical axis. By Proposition 5.2, $z^\circ = w_0 \cdot (w^\circ)^{-1}$, and in particular, $\mathcal{M}(z^\circ)$ is obtained from $\mathcal{M}(w^\circ)$ via the sequence of maps corresponding with arc reversal followed by reflection on the positroid,

$$I \mapsto [n] \setminus I \mapsto w_0 \cdot ([n] \setminus ([n] \setminus I)) = w_0 \cdot I.$$ 

Therefore, $\mathcal{M}(z^\circ)$ is exactly $w_0 \cdot \mathcal{M}(w^\circ)$, so $w_0 \cdot \mathcal{M}(w^\circ) = w_0 \cdot \mathcal{M}$ is a positroid by Theorem 2.19. $\square$

Remark 5.4. The closure of the set of positroids under reversal of the ground set can also be obtained using the fact that positroids are the matroids of totally nonnegative matrices. In particular, suppose $A$ is a totally nonnegative $k \times n$ matrix with positroid $\mathcal{M}(A)$. Let $A'$ be obtained by reversing the columns of $A$ and multiplying the bottom row by $(-1)^{\binom{k}{2}}$. Then $A'$ is totally nonnegative, and $\mathcal{M}(A') = \{ w_0 \cdot I : I \in \mathcal{M}(A) \}$.

Remark 5.5. Using the notation of rigid transformations, Postnikov’s bijection from decorated permutations to Grassmann intervals mentioned in Remark 2.8 maps $w^\circ$ to $w_0 \cdot [u, v]^{-1}$, which preserves the size of the anti-exceedance set. This map is similar to the reversal of ground set involution on positroids.

Observe that each of the three transformations in Proposition 5.2 induces a bijection from $\text{Alignments}(w^\circ)$ to $\text{Alignments}(z^\circ)$. Then $\# \text{Alignments}(w^\circ) = \# \text{Alignments}(z^\circ)$, so it follows from Theorem 1.2 that $\text{codim } \Pi_{w^\circ} = \text{codim } \Pi_{z^\circ}$. Furthermore, as we have seen from the results of Section 4, one may determine whether a positroid variety $\Pi_{\mathcal{M}}$, corresponding to a positroid $\mathcal{M}$, is smooth or singular by performing certain computations involving the sets in $\mathcal{M}$. This fact and the maps given in Proposition 5.2 imply the following relationship between $\Pi_{w^\circ}$ and $\Pi_{z^\circ}$.

Lemma 5.6. Let $w \in S^\circ_{n,k}$, and let $z^\circ$ be a decorated permutation whose chord diagram is obtained from $D(w^\circ)$ by (1) arc reversal, (2) reflection, or (3) rotation. Then, in any of these three cases, $\Pi_{w^\circ}$ is smooth if and only if $\Pi_{z^\circ}$ is smooth.

Proof. We know from the equivalence of Parts (1) and (3) of Theorem 1.7 that $\Pi_{w^\circ} \subseteq \text{Gr}(k, n)$ is smooth if and only if, for every $J \in \mathcal{M}(w^\circ)$, $J$ satisfies

$$\# \{ I \in \mathcal{M}(w^\circ) : |I \cap J| = k - 1 \} = k(n - k) - \# \text{Alignments}(w^\circ).$$

As noted above, in all three cases, $\# \text{Alignments}(z^\circ) = \# \text{Alignments}(w^\circ)$. Observe that $\Pi_{z^\circ} \subseteq \text{Gr}(k(z^\circ), n)$, where $k(z^\circ) = n - k$ in the cases of arc reversal and reflection,
and \( k(z^\omega) = k \) in the case of rotation. Therefore, \( \Pi_2^\omega \) is smooth if and only if every \( J \in \mathcal{M}(z^\omega) \) satisfies
\[
\# \{ I \in \mathcal{M}(z^\omega) : |I \cap J| = k(z^\omega) - 1 \} = k(n - k) - \# \text{Alignments}(w^\omega).
\]

For the cases of arc reversal and reflection, recall from Proposition 5.2 that the positroids \( \mathcal{M}(z^\omega) \) and their nonbases in these cases are obtained by the maps \( I \mapsto [n] \setminus I \) and \( I \mapsto w_0 \cdot ([n] \setminus I) \), respectively. Let \( I, J \in \binom{n}{k} \). Then \( |I \cap J| = k - 1 \) if and only if \( |([n] \setminus I) \cap ([n] \setminus J)| = (n-k) - 1 \) if and only if \( w_0 \cdot ([n] \setminus I) \cap w_0 \cdot ([n] \setminus J) = (n-k) - 1 \). Hence, \( I \in \mathcal{M}(w^\omega) \) contributes to the set in (5.5) if and only if \( [n] \setminus I \in \mathcal{M}((w^\omega)^{-1}) \) contributes to the set in (5.6) for the case of arc reversal if and only if \( w_0 \cdot ([n] \setminus I) \in \mathcal{M}(w_0 \cdot w^\omega) \) contributes to the set in (5.6) for the case of reflection. Therefore, the result follows in these cases.

For case of rotation, recall from Proposition 5.2 that \( \mathcal{M}(z^\omega) = \mathcal{M}(w^*)^s \). For \( I, J \in \binom{n}{k} \), \( |I \cap J| = k - 1 \) if and only if \( |I^s \cap J^s| = k - 1 \). Therefore \( I \in \mathcal{M}(w^\omega) \) contributes to the set in (5.5) if and only if \( I^s \in \mathcal{M}(z^\omega) \) contributes to the set in (5.6), so the result also follows for this case.

\[
\text{Remark 5.7.} \quad \text{In} \ [\text{Pos06}] \text{, Postnikov defined a partial order on decorated permutations called circular Bruhat order. This order determines the containment relations on positroid varieties just as Bruhat order determines the containment relation on Schubert varieties. The covering relations in circular Bruhat order on} \ S_{n,k}^\omega \text{ are determined by exchanging a simple crossing with a simple alignment by} \ [\text{Pos06}] \text{ Thm. 17.8. Since the rigid transformations preserve simple crossings and simple alignments, we see these operations are order preserving under circular Bruhat order.}
\]

6. Reduction to Connected Positroids

Recall that by Theorem 2.17, each positroid \( \mathcal{M} \) on \([n]\) can be uniquely constructed by choosing a non-crossing partition \( B_1 \sqcup \cdots \sqcup B_t \) of \([n]\), and then putting the structure of a connected positroid \( \mathcal{M}_t \) on each block \( B_i \), so \( \mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t \). The noncrossing partition also determines a decomposition of the chord diagram of the associated decorated permutation into connected components as a union of directed arcs inscribed in the plane. We will show that a positroid variety is smooth if and only if the positroid varieties corresponding with each connected component are smooth. By direct analysis of the Jacobian matrix at a \( T \)-fixed point, we show that this matrix can be decomposed into a block diagonal matrix corresponding with the connected components of the associated positroid. We will need a slight refinement of Corollary 2.18.

\[
\text{Lemma 6.1.} \quad \text{Let} \ \mathcal{M} \text{ be a positroid on ground set} \ [n]. \ \text{If} \ \mathcal{M} \text{ is not connected, then up to a possible cyclic shift, it has a decomposition of the form}
\]
\[
\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{\ast n_1} = \{ I \cup J^{+n_1} : I \in \mathcal{M}_1, J \in \mathcal{M}_2 \},
\]
\[
\text{where} \ n = n_1 + n_2, \ \mathcal{M}_1 \text{ and} \ \mathcal{M}_2 \text{ are positroids on ground sets} \ [n_1] \text{ and} \ [n_2] \text{ respectively.}
\]
Proof. By [ARW16, Prop 7.4], if $\mathcal{M}$ is not connected then we can assume it is the direct sum of two positroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on disjoint cyclic intervals. Using Lemma 5.6 and the fact that positroids are closed under the transformation of rotating the set $[n]$, we can assume that $\mathcal{M}_1$ has ground set $[1,n]$ and $\mathcal{M}_2$ has ground set $[n_1+1,n]$.

As discussed in Section 4, for a set $I \in \mathcal{M}$, one may classify the point $A_I \in \Pi_\mathcal{M}$ as a smooth or singular point by computing the rank of the Jacobian matrix, $\text{Jac}(\mathcal{M})$, for $\Pi_\mathcal{M}$ evaluated at $A_I$. In the regime where $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{n_1}$, as in Lemma 6.1, $\text{Jac}(\mathcal{M})|_{A_I}$ can be written as a block diagonal matrix as follows.

**Lemma 6.2.** Let $\mathcal{M}_i \subseteq \binom{[n]}{k_i}$ be positroids for $i \in \{1,2\}$, and let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{n_1}$. Let $I_i \in \mathcal{M}_i$ for $i \in \{1,2\}$, and let $I = I_1 \sqcup I_2^{n_1} \in \mathcal{M}$. Then the Jacobian matrix $\text{Jac}(\mathcal{M})|_{A_I}$ can be written as a block diagonal matrix whose first and second blocks are, up to signs of entries, the matrices $\text{Jac}(\mathcal{M}_i)|_{A_{I_i}}$ for $i \in \{1,2\}$, and whose third block has rank $k_1(n_2-k_2)+k_2(n_1-k_1)$.

**Proof.** Recall that for a positroid $\mathcal{M} \subset \binom{[n]}{k}$, the rows of $\text{Jac}(\mathcal{M})$ are indexed by the nonbases in $\mathcal{Q}(\mathcal{M}) = \binom{[n]}{k} \setminus \mathcal{M}$, and the columns of $\text{Jac}(\mathcal{M})$ are indexed by variables $x_{ij}$, where $i \in [k]$ and $j \in [n]$. Consider $I \in \mathcal{M}$. By Theorem 1.4, if $[u,v] \in G\text{i}(k,n)$ is the Grassmann interval corresponding to $\mathcal{M}$, then there is some $y \in [u,v]$ such that $I = y[k]$. By Lemma 4.1, $\text{Jac}(\mathcal{M})|_{A_I}$ is, up to the signs of the entries, a partial permutation matrix whose nonzero entries occur exactly in the cells $(j,x_{st})$, where $J \in \mathcal{Q}(\mathcal{M})$ satisfies $|J \cap J'| = k-1$, $I \setminus J = \{y_s\}$, and $J \setminus I = \{t\}$.

Set $n = n_1 + n_2$, $k = k_1 + k_2$, $\text{Jac} = \text{Jac}(\mathcal{M})$, $\text{Jac}_i = \text{Jac}(\mathcal{M}_i)|_{A_{I_i}}$, and $\mathcal{Q}_i = \mathcal{Q}(\mathcal{M}_i) \subset \binom{[n]}{k_i}$ for $i \in \{1,2\}$. Let $[u^{(i)},v^{(i)}]$ be the Grassmann interval corresponding to $\mathcal{M}_i$, and let $y^{(i)} = u(I_i,v) \in [u^{(i)},v^{(i)}]$, as in Definition 3.2. By considering the decorated permutations corresponding to $\mathcal{M}$ and the $\mathcal{M}_i$ and the associated decomposition, as in Corollary 2.18, $v_j = v_j^{(1)}$ for $j \in [k_1]$ and $v_{j+k_1} = v_j^{(2)}+n_1$ for $j \in [k_2]$. Since every element of $I_2^{n_1}$ is greater than every element of $v[k_1]$, then the construction of $y = u(I,v)$ replicates the constructions of the $y^{(i)}$ so that $y_j = y_j^{(1)}$ for $j \in [k_1]$ and $y_{j+k_1} = y_j^{(2)}+n_1$ for $j \in [k_2]$.

By definition, $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{n_1}$. Therefore, the collection of nonbases can be partitioned as $\mathcal{Q} = B \cup C \cup D \cup E \cup F$, where the sets in the partitioned are defined as

$B = \mathcal{Q}_1 \oplus \{I_2^{n_1}\}$, $\quad C = \{I_1\} \oplus \mathcal{Q}_2^{n_1}$,

$D = \left\{ J \in \binom{[n]}{k} : |J \cap [n_1]| > k_1 \text{ or } |J \cap [n_1+1,n]| > k_2 \right\}$,

$E = \mathcal{Q}_1 \oplus \left( \binom{[n_1+1,n]}{k_2} \setminus \{I_2^{n_1}\} \right)$, and $\quad F = (\mathcal{M}_1 \setminus \{I_1\}) \oplus \mathcal{Q}_2^{n_1}$.

We also partition the set $[k] \times [n] = (I) \cup (II) \cup (III)$ in the following way.

- (I) $= [k_1] \times [n_1]$
The partitions above yield partitionings of the row and column sets of $\text{Jac}|_{A_t}$. From these partitions, $\text{Jac}|_{A_t}$ can be decomposed as a block diagonal matrix in the following way.

**Block 1:** $(J \in B)$ The rows of the first block will be indexed by the sets $J \in B$, and the columns of this block will be indexed by $x_{rc}$ with pairs $(r, c)$ in (I). The maps $J_1 \mapsto J_1 \cup I_2^{n_1}$ for $J_1 \in Q_1$ and $(r, c) \mapsto (r, c)$ for $(r, c) \in [k_1] \times [n_1]$ together give a bijection between the cells in $\text{Jac}|_{A_t}$ and the cells in this upper left block of $\text{Jac}|_{A_t}$.

Since $I = I_1 \cup I_2^{n_1}$, then a set $J = J_1 \cup I_2^{n_1} \in Q_1 \oplus \{I_2^{n_1}\}$ satisfies $|I \cap J| = k - 1$ if and only if $|I_1 \cap J_1| = k_1 - 1$. By Lemma 4.1 and Theorem 4.4 for any such $J_1 \in Q_1$, the unique nonzero entry of $\text{Jac}|_{A_t}$ in row $J_1$ is in column $x_{st}$, where $I_1 \setminus J_1 = \{y_s^{(1)}\}$ and $J_1 \setminus I_1 = \{t\}$. Since $I = y[k] = y^{(1)}[k_1] \cup y^{(2)}[k_2]^{n_1}$, then $I \setminus J = \{y_s\}$ and $J \setminus I = \{t\}$. Hence, for this same pair $(s, t)$, entry $(J, x_{st})$ of $\text{Jac}|_{A_t}$ will be nonzero. Note also that this unique pair is in (I).

Therefore, this upper left block with rows indexed by $B$ and columns indexed by (I) looks, up to sign, like $\text{Jac}|_{A_1}$. Furthermore, all entries in the rows indexed by $B$, but outside of the columns indexed by (I), will be zeros.

**Block 2:** $(J \in C)$ This case is similar to the previous case. The rows of the second block will be indexed by the sets $J \in C$, and the columns of this block will be indexed by $x_{rc}$ with $(r, c)$ in (II). The maps $J_2 \mapsto I_1 \cup I_2^{n_1}$ for $J_2 \in Q_2$ and $(r, c) \mapsto (r + k_1, c + n_1)$ for $(r, c) \in [k_2] \times [n_2]$ together give a bijection between the cells in $\text{Jac}|_{A_t}$ and the cells in the second block of $\text{Jac}|_{A_t}$. A similar argument to that of the Block 1 case with elements and indices shifted by these maps shows that this second block looks, up to sign, like $\text{Jac}|_{A_2}$ and that all entries in the rows indexed by $C$, but outside of the columns indexed by (II), will be zeros.

**Block 3:** $(J \in D \cup E \cup F)$ The rows of the third block will be indexed by the sets $J \in D \cup E \cup F$, and the columns of this block will be indexed by pairs $(r, c)$ in set (III). First, consider $J = J_1 \cup J_2 \in D$, where $J_1 = J \cap [n_1]$ and $J_2 = J \cap [n_1 + 1, n]$. Then $J$ satisfies $|I \cap J| = k - 1$ if and only if either

(i) $J_1 = I_1 \setminus \{y_s\}$ for some $y_s \in I_1$ and $J_2 = I_2^{n_1} \cup \{t\} \in \binom{[n_1 + 1, n]}{k_2 + 1}$, or

(ii) $J_1 = I_1 \cup \{t\} \in \binom{[n_1]}{k_1 + 1}$ and $J_2 = I_2^{n_1} \setminus \{y_s\}$ for some $y_s \in I_2^{n_1}$.

In case (i), the column pair $(s, t)$ satisfying Lemma 4.1 has $s \in [k_1]$ since $y_s \in [n_1]$ and $t \in [n_1 + 1, n]$. There are $k_1$ choices for $s$ and $n_2 - k_2$ choices for $t$. In case (ii), the pair $(s, t)$ satisfying Lemma 4.1 has $s \in [k_1 + 1, k]$ since $y_s \in [n_1 + 1, n]$ and $t \in [n_1]$. There are $n_1 - k_1$ choices for $t$ and $k_2$ choices for $s$. Thus, the rank of this block restricted to the rows in $D$ is $k_1(n_2 - k_2) + k_2(n_1 - k_1)$. Furthermore, all nonzero entries in the
rows indexed by \( D \) occur in the columns in (III) by arguments similar to the proof of Lemma 4.1.

We claim that for \( J \in E \cup F, |I \cap J| < k - 1 \). Hence, \( \text{Jac}_{\mathcal{A}_I} \) has all zeros in row \( J \) by Lemma 4.1. To prove the claim, consider \( J = J_1 \cup J_2 \in E \), where \( J_1 \in \mathcal{Q}_1 \) and \( J_2 \in \left( [n_1+1, n] \right) \setminus \{J_2^{n_1}\} \). Since \( J_1 \) is in \( \mathcal{Q}_1 \), so \( J_1 \neq I_1 \), then \( |I_1 \cap J_1| \leq k_1 - 1 \). Similarly, since \( J_2 \neq J_2^{n_1} \), then \( |J_2^{n_1} \cap J_2| \leq k_2 - 1 \). Therefore, \( |I \cap J| \leq (k_1 - 1) + (k_2 - 1) = k - 2 \). Next, consider \( J = J_1 \cup J_2^{n_1} \in F \), where \( J_1 \in \mathcal{M}_1 \setminus \{I_1\} \) and \( J_2 \in \mathcal{Q}_2 \). Since \( J_1 \neq I_1 \), then \( |I_1 \cap J_1| \leq k_1 - 1 \). Similarly, since \( J_2 \) is in \( \mathcal{Q}_2 \), and therefore is not \( I_2 \), then \( |I_2 \cap J_2| = |J_2^{n_1} \cap J_2^{n_1}| \leq k_2 - 1 \). Therefore, \( |I \cap J| \leq (k_1 - 1) + (k_2 - 1) = k - 2 \).

Lemma 6.3. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be positroids on grounds sets \([n_1]\) and \([n_2]\). Let \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{n_1} \) and let \( \omega^0 = w^{(1)} \oplus w^{(2)} \) be the associated decorated permutation. Let \( k_i \) be the number of anti-exceedances of \( w(i) \). Then,
\[
\#\text{Alignments}(\omega^0) = \#\text{Alignments}(w^{(1)}) + \#\text{Alignments}(w^{(2)}) + k_1(n_2 - k_2) + k_2(n_1 - k_1).
\]

Proof. Consider the chord diagram for \( \omega^0 \) partitioned by a line separating the connected components of the noncrossing partition corresponding with \( w^{(1)} \) and \( w^{(2)} \). See Figure 3. Every alignment of \( w^{(i)} \) remains an alignment of \( \omega^0 \). In addition, every anti-exceedence arc of \( w^{(i)} \) forms an alignment with every exceedance arc of \( w^{(i')} \), where \( i' = \frac{3 + (-1)^{i-1}}{2} \). Since \( w^{(i)} \) has \( k_i \) anti-exceedances and \( n_i - k_i \) exceedances, the result follows.

Lemma 6.4. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be positroids on grounds sets \([n_1]\) and \([n_2]\) of rank \( k_1 \) and \( k_2 \), respectively. Let \( \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{n_1} \). For any \( I = I_1 \cup I_2^{n_1} \in \mathcal{M}, A_I \) is a smooth point of \( \Pi_{\mathcal{M}} \) if and only if the \( A_{I_1} \) and \( A_{I_2} \) are both smooth points of \( \Pi_{\mathcal{M}_1} \) and \( \Pi_{\mathcal{M}_2} \), respectively.

Proof. By Theorem 2.17 and the construction, \( \mathcal{M} \) is a positroid of rank \( k = k_1 + k_2 \) on ground set \([n]\) for \( n = n_1 + n_2 \). Let \( \omega^0 = w^{(1)} \oplus w^{(2)} \) be the corresponding decomposition of the associated decorated permutation, as in Lemma 6.3. Let \( \text{Jac} = \text{Jac}(\mathcal{M})|_{A_I} \) and \( \text{Jac}_i = \text{Jac}(\mathcal{M}_i)|_{A_I} \) for \( i \in \{1, 2\} \). Since \( \text{rank}(\text{Jac}) \) is bounded above by \( \#\text{Alignments}(\omega^0) \), then by [1.3], \( A_I \) is a smooth point of \( \Pi_{\mathcal{M}} \) if and only if \( \text{rank}(\text{Jac}_i) = \#\text{Alignments}(w^{(i)}) \). Similarly, \( A_{I_i} \) is a smooth point of \( \Pi_{\mathcal{M}_i} \) if and only if \( \text{rank}(\text{Jac}_i) = \#\text{Alignments}(w^{(i)}) \).
By Lemma 6.2, \( \text{rank}(Jac) = \text{rank}(Jac_1) + \text{rank}(Jac_2) + k_1(n_2 - k_2) + k_2(n_1 - k_1) \). It follows from Lemma 6.3 that \( A_I \) is a smooth point of \( \Pi_M \) if and only if
\[
(6.1) \quad \text{rank}(Jac_1) + \text{rank}(Jac_2) = \# \text{Alignments}(w^{(1)}) + \# \text{Alignments}(w^{(2)}).
\]
Once again, since \( \text{rank}(Jac_i) \leq \# \text{Alignments}(w^{(i)}) \), then (6.1) holds if and only if \( \text{rank}(Jac_i) = \# \text{Alignments}(w^{(i)}) \) for both \( i = 1, 2 \).

**Corollary 6.5.** Let \( M = M_1 \oplus \cdots \oplus M_t \) be a positroid decomposed into its connected components. Then \( \Pi_M \) is smooth if and only if \( \Pi_{M_i} \) is smooth for each \( i \in [t] \).

**Proof.** If \( M \) is connected the statement holds, so assume \( M \) is not connected. We know that \( \Pi_M \) is smooth if and only if all cyclic rotations of \( M \) correspond with smooth positroid varieties by Lemma 5.6. Therefore, by Lemma 6.1, we can assume it has a decomposition of the form \( M = M_1 \oplus M_2^{*n} \) where \( n = n_1 + n_2 \) for some \( 1 \leq n_1, n_2 < n \), and \( M_1 \) and \( M_2 \) are positroids on ground sets \([n_1]\) and \([n_2]\), respectively. By Lemma 6.4, \( \Pi_M \) has a singular \( T \)-fixed point \( A_I \) for \( I \in M \) if and only if either \( \Pi_{M_1} \) or \( \Pi_{M_2} \) has a singular \( T \)-fixed point. Therefore, the result holds by Theorem 1.4, Corollary 4.3, and induction on \( t \).

**Remark 6.6.** By Corollary 2.18 the connected components of positroids are associated with decorated SIF permutations. In order to complete the proof of Theorem 1.7 it remains to characterize the decorated SIF permutations indexing smooth positroid varieties by Corollary 6.5. We will show that these are in bijection with the spirograph permutations, discussed further in the next section.

### 7. Crossed Alignments and Spirograph Permutations

Recall the class of spirograph permutations introduced in Section 1. It is straightforward to show that spirograph permutations always index smooth positroid varieties. We will also show in this section that direct sums of spirograph permutations on a non-crossing partition are exactly the same as decorated permutations that have no crossed alignments. In particular, we prove the equivalence of Parts (5), (6), and (7) in Theorem 1.7.

By Definition 1.6, the spirograph permutations are a subset of the decorated SIF permutations. The chord diagram and the positroid associated to a spirograph permutation is always connected. The two decorated permutations in \( S_i^{*n} \) are spirographs, and for \( n > 1 \), there are \( n - 1 \) distinct spirograph permutations corresponding with \( m \in [n - 1] \). Therefore, the generating function for spirograph permutations in \( S_n \) for \( n \geq 1 \) is
\[
(7.1) \quad S(x) = 2x + x^2 + 2x^3 + 3x^4 + \cdots = \frac{2x - 3x^2 + 2x^3}{(1 - x)^2}.
\]

**Remark 7.1.** Each \( S_{n,k}^2 \) has a unique spirograph permutation, denoted \( \pi_{n,k} \). This spirograph permutation is defined so that \( \pi_{n,k}(i) = i + k \) for all \( i \) and is the unique maximal
is the unique the spirograph permutation in $S_{n,k}^0$ for $n > 1$. For $k = n = 1$, $\pi_{1,1}$ is the unique decorated permutation in $S_{n,k}^0$, which consists of a single clockwise fixed point. For $k = 0, n = 1$, $\pi_{1,0}$ is the unique decorated permutation in $S_{n,k}^0$, which has a counterclockwise fixed point.

**Lemma 7.2.** If $w^\circ$ is the unique the spirograph permutation in $S_{n,k}^0$, then $\Pi_{w^\circ}$ is $Gr(k, n)$, which is a smooth variety.

**Proof.** Observe that the chord diagram of a spirograph permutation $w^\circ$ has no alignments, so the codimension of $\Pi_{w^\circ}$ as a subvariety of $Gr(k, n)$ is zero by Theorem 1.2. Hence, $\Pi_{w^\circ} = Gr(k, n)$.

**Proof of Theorem 1.7.** (6) $\Rightarrow$ (1). Let $D(w^\circ)$ be a disjoint union of spirographs corresponding to the decomposition $w^\circ = w^{(1)} \oplus \cdots \oplus w^{(l)}$ of $w^\circ$ into decorated SIF permutations with $\mathcal{M}(w^\circ) = \mathcal{M}(w^{(1)}) \oplus \cdots \oplus \mathcal{M}(w^{(l)})$ using Corollary 2.18. By Lemma 7.2, the positroid varieties $\Pi_{w^{(i)}}$ are all smooth. Hence, $\Pi_{w^\circ} = \Pi_{\mathcal{M}(w^\circ)}$ is smooth by Corollary 6.5.

(5) $\Leftrightarrow$ (6). By definition of a chord diagram $D(w^\circ)$, arcs from distinct components of the associated noncrossing partition must be drawn so that they do not intersect. It follows that, if $w^\circ$ contains a crossed alignment, then all arcs involved in the crossed alignment must be contained in the same connected component of $D(w^\circ)$.

A spirograph permutation has no crossed alignments since it has no alignments. By the observation above, if every connected component of $D(w^\circ)$ is a spirograph, then $w^\circ$ has no crossed alignments.

For the converse, observe that the property of containing a crossed alignment is invariant under rotation of the chord diagram. Hence, by Corollary 2.18 and the fact that a crossed alignment is contained in a single connected component of $D(w^\circ)$, the argument may be completed by assuming that $w^\circ$ is a decorated SIF permutation that is not a spirograph and showing that $w^\circ$ has a crossed alignment. For $n < 4$, $w^\circ$ is either a spirograph permutation or has more than one connected component. Therefore, we must have $n \geq 4$, in which case we will denote $w^\circ$ simply by $w$ since it has no fixed points.

Since $w$ has no fixed points, $m_i := w(i) - i \pmod{n} \in [n - 1]$ for each $i \in [n]$. Since $w$ is not a spirograph permutation, the $m_i$ are not all equal. Therefore, taking indices mod $n$, there must exist some $i \in [n]$ such that $m_i < m_{i+1}$. Observe from the chord diagram that $m_i < m_{i+1}$ implies that the arcs $(i \leftrightarrow w(i))$ and $(i + 1 \leftrightarrow w(i + 1))$ form a crossing. Use the crossing arcs to create three disjoint cyclic intervals. Let $A = [w(i+1), w(i+1)-1]^{cyc}$, $B = [w(i+1), i-1]^{cyc} \setminus [i, w(i+1)-1]^{cyc}$, and $C = [i+2, w(i)]^{cyc} \setminus [w(i+1)+1, i+1]^{cyc}$. See Figure 4.

Since $w$ is a SIF permutation, the arcs incident to elements of $A$ cannot form an isolated connected component. Thus, there must be an arc mapping some $j \in [n] \setminus (A \cup (i, i+1))$ to $w(j) \in A$. If $j \notin B$, then $(i \leftrightarrow w(i), j \leftrightarrow w(j))$ is an alignment crossed by
Figure 4. Chord diagrams with \((i \mapsto w(i))\) and \((i + 1 \mapsto w(i + 1))\) crossing.

\[
(i + 1 \mapsto w(i + 1)). \text{ If } j \in C, \text{ then } (j \mapsto w(j), i + 1 \mapsto w(i + 1)) \text{ is an alignment crossed by } (i \mapsto w(i)). \text{ Therefore, } w \text{ contains a crossed alignment.}
\]

Proof of 1.7, (6) \(\iff\) (7). The unique spirograph permutation \(\pi_{n,k}\) in \(S_{n,k}^0\), mentioned in Remark 7.1, has shifted anti-exceedance sets \(I_r(\pi_{n,k}) = \binom{[n]}{k} \ominus (r-1)\) for all \(r \in [n]\). Since \(\binom{[n]}{k} \ominus (r-1)\) is the minimal element of \(\binom{[n]}{k}\) under \(\prec_r\), it follows from Theorem 2.19 that \(\mathcal{M}(\pi_{n,k}) = \binom{[n]}{k}\). Hence, spirograph permutations correspond exactly to uniform matroids. By the correspondence in Corollary 2.18, a positroid \(\Pi_{w^\circ} = \mathcal{M}(w^\circ)\) is a direct sum of uniform matroids if and only if \(w^\circ\) is a direct sum of spirograph permutations. \(\square\)

8. Anti-Exchange Pairs and Crossed Alignments

In this section, we relate decorated permutations with crossed alignments to singular positroids varieties as determined by the vertex degrees of the matroid Johnson graphs in Parts (1)-(4) in Theorem 1.7. By the end of Section 7, we proved Parts (5) \(\iff\) (6) \(\iff\) (7) and (6) \(\Rightarrow\) (1) in Theorem 1.7. To finish the proof of Theorem 1.7, it suffices to prove (1) \(\Rightarrow\) (5), namely if \(\Pi_{w^\circ}\) is smooth, then \(w^\circ\) has no crossed alignments. This final step will be proved using the characterization of smoothness from Corollary 4.4 utilizing the nonbases in \(Q(w^\circ)\). Specifically, for any \(w^\circ \in S_{n,k}^0\), \(\Pi_{w^\circ}\) is singular if and only if there exists a \(J \in M(w^\circ)\) such that

\[
\#\{I \in Q(w^\circ) : |I \cap J| = k - 1\} < \#\text{Alignments}(w).
\]

Thus, we now carefully study the sets \(I\) that occur on the left side of this equation. The following vocabulary refers back to the Basis Exchange Property of matroids from Section 2.

**Definition 8.1.** For a set \(J\) in a positroid \(\mathcal{M} \subseteq \binom{[n]}{k}\), let \(a \in J\) and \(b \in [n] \setminus J\). If \((J \setminus \{a\}) \cup \{b\} \in \mathcal{M}\), we say the pair \((a, b)\) is an exchange pair for \(J\) and that the values \(a\) and \(b\) are exchangeable. Otherwise, \((J \setminus \{a\}) \cup \{b\} \notin \mathcal{M} = \binom{[n]}{k} \setminus \mathcal{M}\), in which case we say \((a, b)\) is an anti-exchange pair for \(J\) and that the values \(a\) and \(b\) are not exchangeable.

For a set \(J \in \mathcal{M} \subseteq \binom{[n]}{k}\), pairs \((a, b)\) with \(a \in J\) and \(b \notin J\) are in bijection with the vertices in the full Johnson graph \(J(k, n)\) adjacent to \(J\). Exchange pairs for \(J\) are in
bijection with the vertices adjacent to $J$ in the matroid Johnson graph $J(\mathcal{M})$. Anti-exceedance pairs for $J$ are in bijection with the vertices adjacent to $J$ in $J(k,n)$ that do not appear in $J(\mathcal{M})$.

8.1. Characterizing Exchange Pairs in the Johnson Graph. For a positroid $\mathcal{M}(w^o)$, we will focus on the set $J = I_1(w^o)$, which is in $\mathcal{M}(w^o)$ by Corollary 2.20. The following lemma characterizes exchange pairs for $I_1(w^o)$.

Lemma 8.2. For $w^o = (w,co) \in S^o_{n,k}$, suppose that $a \in I_1(w^o)$ and $b \in [n] \setminus I_1(w^o)$. Then $I = (I_1(w^o) \setminus \{a\}) \cup \{b\}$ is in $\mathcal{M}(w^o)$ if and only if $a < b$ and for every $r \in [a+1,b]$, both of the following conditions hold:

1. there exists $x \in [a,r-1]$ such that $w^{-1}(x) \in [r,n]$, and
2. there exists $y \in [r,b]$ such that $w^{-1}(y) \in [1,r-1]$.

Proof. Let $I_r = I_r(w^o)$. Recall from Theorem 2.19 that $I$ is in $\mathcal{M} = \mathcal{M}(w^o)$ if and only if $I_r \preceq_r I$ for all $r \in [n]$. We use this characterization of $\mathcal{M}$ in terms of the elements of the Grassmann necklace to derive the conditions of the lemma.

Consider the case when $a > b$. Then $I$ is obtained by replacing an element of $I_1$ with a smaller element. By definition of the Gale order, $I$ cannot satisfy $I_1 \preceq_I$, so $I \not\in \mathcal{M}$.

For the remainder of the proof, assume $a < b$. For $r \in [b+1,a]^{\text{cy}}$, the inequality $a <_r b$ holds, which implies that $I$ is obtained by replacing an element of $I_1$ with an element that is larger under $<_r$. Thus, $I_1 <_r I$. By Corollary 2.20, $I_1$ is in $\mathcal{M}$, so by Theorem 2.19 $I_1$ must satisfy $I_r \preceq_r I_1$. Hence, the sequence of inequalities $I_r \preceq_r I_1 <_r I$ holds for every $r \in [b+1,a]^{\text{cy}}$. Therefore, $I \in \mathcal{M}$ if and only if $I_r \preceq_r I$ for every $r \in [a+1,b]$.

Fix some $r \in [a+1,b]$, so $a >_r b$. An arc $(i \leftrightarrow j)$ with $i \in [r,n]$ and $j \in [a,r-1]$ is an anti-exceedance arc, but not an $r$-anti-exceedance arc, so $j \in I_1 \setminus I_r$. Similarly, an arc $(i \leftrightarrow j)$ with $i \in [1,r-1]$ and $j \in [r,b]$ is an $r$-anti-exceedance arc, but not an anti-exceedance arc, so $j \in I_r \setminus I_1$. Therefore, the following statements hold.

\begin{align*}
(8.2) & \quad \text{For } x \in [a,r-1], \; w^{-1}(x) \in [r,n] \iff x \in I_1 \setminus I_r. \\
(8.3) & \quad \text{For } y \in [r,b], \; w^{-1}(y) \in [1,r-1] \iff y \in I_r \setminus I_1.
\end{align*}

See Figure 5 for these two cases. From these observations, for the fixed $r \in [a+1,b]$, Conditions (1) and (2) in the statement of the lemma are equivalent to the following:

1. $I_1 \setminus I_r \cap [a,r-1] \neq \emptyset$, and
2. $(I_r \setminus I_1) \cap [r,b] \neq \emptyset$.

We must now compare the sets $I_r$ and $I$ under the shifted Gale order $<_r$ for $r \in [a+1,b]$. Recall that $I$ is obtained from $I_1$ by exchange of the elements $a$ and $b$, where $a \in I_1$ and $b \not\in I_1$. Thus we begin by writing $I_1 = \{j_1 <_r \cdots <_r j_k\}$ increasing under the $<_r$ order.
Setting $f = |I_1 \cap [r, b]| + 1$, $g = |I \cap [r, n]| + 1$, and $h = |I \cap [r, a]|$, we refine the expression of $I_1$ in the following way,

$$I_1 = \left\{ j_1 <_r \cdots <_r j_{f-1} <_r j_f <_r \cdots <_r j_{g-1} <_r j_g <_r \cdots <_r j_{h-1} <_r a <_r j_{h+1} <_r \cdots <_r j_k \right\}.\]

Here, $a = j_h$. Note that since $b \notin I_1$, then $j_{f-1} <_r b$. Write $I_r$ and $I$ under $<_r$ as

$$I_r = \left\{ i_1 <_r \cdots <_r i_{f-1} <_r i_f <_r i_{f+1} <_r \cdots <_r i_g <_r i_{g+1} <_r \cdots <_r i_h <_r i_{h+1} <_r \cdots <_r i_k \right\}$$

$$I = \left\{ j_1 <_r \cdots <_r j_{f-1} <_r b <_r j_f <_r \cdots <_r j_{g-1} <_r j_g <_r \cdots <_r j_{h-1} <_r j_{h+1} <_r \cdots <_r j_k \right\}.$$  

Then $I_r \preceq_r I$ if and only if all three of the following hold:

(i) $i_\ell \preceq_r j_\ell$ for all $\ell \in [1, f - 1] \cup [h + 1, k]$;
(ii) $i_f \preceq_r b$ and $i_{f+1} \preceq_r j_\ell$ for all $\ell \in [f, g - 1]$; and
(iii) $i_\ell \preceq_r j_\ell$ for all $\ell \in [g, h - 1]$.

For $\ell \in [1, f - 1] \cup [h + 1, k]$, $I_r \preceq_r I_1$ already implies that $i_\ell \preceq_r j_\ell$, so (i) is always satisfied. Thus, $I_r \preceq_r I$ if and only if (ii) and (iii) are satisfied.

Consider Condition (ii) above. We will show that (ii) is satisfied if and only if $(I_r \setminus I_1) \cap [r, b]$ is nonempty, as in Condition (2').

We require the following observations about the elements $j_1, \ldots, j_{g-1}$. By construction, \{j_1, \ldots, j_{g-1}\} = I_1 \cap [r, n]$. Each of these $j_\ell$ has $w^{-1}(j_\ell) \in [j_\ell, n]$, as in Figure 6a. But then $j_\ell$ and $w^{-1}(j_\ell)$ also satisfy $r \preceq_r j_\ell \preceq_r w^{-1}(j_\ell) \preceq_r n$. In particular, this implies that every such $j_\ell$ is also an element of $I_r$ so that $\{j_1, \ldots, j_{g-1}\} \subseteq I_r$. Furthermore, $j_1 <_r \cdots <_r j_{g-1}$ appear in $I_r$ in the same relative order as they do in $I_1$, but there may be additional elements of $I_r$ that are interspersed among them. Therefore, for every $\ell \in [1, g - 1]$, there is some $\ell' \geq \ell$ such that $i_{\ell'} = j_\ell$.

For the only if direction, (ii) implies (2'), assume $(I_r \setminus I_1) \cap [r, b]$ is empty. By construction, $I_1 \cap [r, b] = \{j_1, \ldots, j_{f-1}\}$. Since $\{j_1, \ldots, j_{f-1}\} \subseteq I_r$, by the previous paragraph and $(I_r \setminus I_1) \cap [r, b] = \emptyset$, then $I_r \cap [r, b] = \{j_1, \ldots, j_{f-1}\} = \{i_1, \ldots, i_{f-1}\}$. In particular, $i_f \notin [r, b]$, so $b <_r i_f$, violating Condition (ii).
Consider any $i_\ell \in I_\ell \cap [1, r-1]$, as in Figure 6b. Since $i_\ell \in I_\ell$, then $w^{-1}(i_\ell)$ must be in $[i_\ell, r-1]$ so that $1 \leq r i_\ell \leq w^{-1}(i_\ell) \leq r-1$. Hence, $i_\ell$ is also in $I_1$, so $I_\ell \cap [1, r-1] \subseteq I_1 \cap [1, r-1]$. Thus, these $i_\ell \in I_\ell \cap [1, r-1]$ appear among the elements $j_g, \ldots, j_k$, with possibly some additional elements. So, each of these $i_\ell$ has some $j_\ell'$ with $\ell' \leq \ell$ for which $i_\ell = j_\ell'$. After possibly deleting $a$, all of these elements are in $I$.

For the only if direction in this case, suppose $[a, r-1] \cap (I_1 \setminus I_\ell)$ is empty. By construction, $I_1 \cap [a, r-1] = \{j_h, \ldots, j_k\}$. Then $I_1 \cap [a, r-1] = I_\ell \cap [a, r-1]$, so $i_\ell = j_\ell$ for every $\ell \in [h, k]$. In particular, $i_h = j_h = a$. But then we have $i_h = a >_r j_h-1$, which violates Condition (iii) for $\ell = h-1$.

Conversely, suppose $(I_1 \setminus I_\ell) \cap [a, r-1]$ is nonempty. Then $|I_1 \cap [a, r-1]| > |I_\ell \cap [a, r-1]|$. Since $I_1 \cap [a, r-1] = \{j_h, \ldots, j_k\}$, then $i_h$ cannot be in $[a, r-1]$, so $i_h <_r a$, and thus $i_\ell <_r a$ for all $\ell \in [g+1, h]$. Furthermore, the existence of at least one extra element in $I_1 \cap [a, r-1]$ implies that the elements of $I_\ell \cap [1, a-1]$ all appear shifted to the left in $I_1$. Specifically, for $i_\ell \in \{i_{g+1}, \ldots, i_h\} \cap [1, a-1]$, there is some $\ell' < \ell$ for which $i_\ell = j_\ell'$. Then, for these $i_\ell$, we have $i_\ell = j_\ell' \leq_r i_{\ell-1}$. The remaining $i_\ell \in \{i_{g+1}, \ldots, i_h\}$ are in $[r, n]$, so $i_\ell <_r j_{\ell-1} \in [1, a-1]$. Hence, $i_\ell \leq_r j_{\ell-1}$ for all $\ell \in [g+1, h]$, which is equivalent to Condition (iii).

**Figure 6.** Elements in both $I_1$ and $I_\ell$. □
8.2. Mapping Anti-Exchange Pairs to Alignments. For a fixed \( w^\circ \in S_n^\circ \) whose chord diagram contains a crossed alignment, we will show that (8.1) holds for \( J = I_1(w^\circ) \). This will be achieved using an injective map \( \Psi_{w^\circ} \), which maps anti-exchange pairs for \( I_1(w^\circ) \) to alignments in \( D(w^\circ) \) and is defined below for any \( w^\circ \in S_n^\circ \) using Lemma 8.2.

Given \( w^\circ \in S_n^\circ \), let \( \text{AE}(w^\circ) \) be the set of all anti-exchange pairs \((a, b)\) for \( I_1(w^\circ) \), so \( a \in I_1(w^\circ) \), \( b \notin I_1(w^\circ) \), and \((I_1(w^\circ) \setminus \{a\}) \cup \{b\} \notin M(w^\circ) \). For each \((a, b) \in \text{AE}(w^\circ) \), either \( a > b \) or \( a < b \) and there must exist some \( r \in [a+1, b] \) for which Condition (1) or (2) of Lemma 8.2 fails. In the \( a < b \) case, we say \( r \) is a witness for \((a, b)\) to be in \( \text{AE}(w^\circ) \). Let \( \text{AE}_1(w^\circ) \) be all the anti-exchange pairs with \( a > b \). Let \( \text{AE}_2(w^\circ) \) be all the anti-exchange pairs with \( a < b \) such that Condition (1) fails for some \( r \in [a+1, b] \). Let \( \text{AE}_2(w^\circ) \) be the anti-exchange pairs with \( a < b \) such that Condition (1) holds for all \( r \in [a+1, b] \). These three sets form a partition of the anti-exchange pairs for \( I_1(w^\circ) \),

\[
\text{AE}(w^\circ) = \text{AE}_1(w^\circ) \cup \text{AE}_2(w^\circ) \cup \text{AE}_3(w^\circ).
\]

**Lemma 8.3.** Fix \( w^\circ = (w, co) \in S_n^\circ \) and \( a \in I_1(w^\circ) \). Let \( \{b_1 < \cdots < b_s\} \) be the set of all elements in \([a+1, n] \setminus I_1(w^\circ)\) such that for each \( 1 \leq i \leq s \), there exists some minimal \( r_i \in [a+1, b_i] \) for which Condition (1) of Lemma 8.2 fails, so \((a, b_i) \in \text{AE}_1(w^\circ) \). If \( \{b_1 < \cdots < b_s\} \) is nonempty, then defining \( \bar{r}(a, w^\circ) := r_1 \) we have

\[
\{b_1 < \cdots < b_s\} = [\bar{r}(a, w^\circ), n] \setminus I_1,
\]

and \( \bar{r}(a, w^\circ) \in [a+1, b_i] \) is a witness for Condition (1) failing for \((a, b_i)\) for each \( i \).

**Proof.** Assume \( \{b_1 < \cdots < b_s\} \) is nonempty. By construction, \( r_1 \in [a+1, b_1] \subseteq [a+1, b_i] \) for all \( i \). Furthermore, \( r_1 \) is chosen so that Condition (1) of Lemma 8.2 fails for the pair \((a, b_1)\), and hence there is no \( x \in [a, r_1 - 1] \) such that \( w^{-1}(x) \in [r_1, n] \). This last condition only depends on \( r_1 \), so Condition (1) still fails for \( r_1 \) when determining whether or not \( a \) is exchangeable with any given \( b \in [r_1, n] \setminus I_1 \), according to Lemma 8.2. Hence, \( \{b_1 < \cdots < b_s\} = [r_1, n] \setminus I_1 \).

**Corollary 8.4.** If \( \{b_1 < \cdots < b_s\} \) from Lemma 8.3 is nonempty, then \( b < \bar{r}(a, w^\circ) \) for any \((a, b) \in \text{AE}_2(w^\circ) \).

**Proof.** For \((a, b) \in \text{AE}_2(w^\circ) \), Condition (1) of Lemma 8.2 is satisfied for every \( r \in [a+1, b] \). In particular, \( b \) is not one of the \( b_i \)s as in Lemma 8.3, so \( b \) is not in \([\bar{r}(a, w^\circ), n] \setminus I_1 \). Since \( b \notin I_1 \) by definition of an anti-exchange pair, \( b \) must be in \([1, \bar{r}(a, w^\circ) - 1] \).

**Lemma 8.5.** Fix \( w^\circ = (w, co) \in S_n^\circ \) and \( b \notin I_1(w^\circ) \). Let \( \{a_1 < \cdots < a_s\} \) be the elements of \([b] \cap I_1(w^\circ)\) such that there exists some maximal \( r_i \in [a_i + 1, b] \) for which Condition (2) of Lemma 8.2 fails, so \((a_i, b) \in \text{AE}(w^\circ) \). If \( \{a_1 < \cdots < a_s\} \) is not empty, then defining \( \bar{r}(b, w^\circ) := r_s \) we have

\[
\{a_1 < \cdots < a_s\} = [1, \bar{r}(b, w^\circ) - 1] \cap I_1(w^\circ),
\]

and \( \bar{r}(b, w^\circ) \in [a_i + 1, b] \) is a witness for Condition (2) failing for \((a_i, b_i)\) for each \( i \).
Proof. The proof is similar to the proof of Lemma 8.3 by reflecting \( D(w^\circ) \) across the vertical axis.

For each \( w^\circ \in S_n^\circ \), we construct a map \( \Psi_{w^\circ} \) from the anti-exchange pairs for \( I_1(w^\circ) \) partitioned according to (8.4) to ordered pairs of the form \((p \mapsto w(p), s \mapsto w(s))\) for \( p \neq s \).

We will show in the lemma that follows that the range of \( \Psi_{w^\circ} \) is contained in the set of alignments for \( w^\circ \), as defined in Definition 2.11.

**Definition 8.6.** For \( w^\circ = (w, co) \in S_n^\circ \), let \( \Psi_{w^\circ} \) be defined by the following algorithm.

\[
\begin{align*}
\text{Input} & : (a, b) \in AE(w^\circ) = AE_s(w^\circ) \cup AE_1(w^\circ) \cup AE_2(w^\circ) \\
\text{Output} & : (p \mapsto w(p), s \mapsto w(s)) \in \text{Alignments}(w^\circ)\\
1 & \text{set } p \leftarrow w^{-1}(b) \text{ and } s \leftarrow w^{-1}(a). \\
2 & \text{if } (a, b) \in AE_1(w) \text{ then} \\
3 & \quad \text{while } p \notin [\bar{r}(a, w^\circ), n], \text{ do} \\
4 & \quad \quad \text{update } p \leftarrow w^{-1}(p). \\
5 & \quad \text{end} \\
6 & \text{end} \\
7 & \text{if } (a, b) \in AE_2(w) \text{ then} \\
8 & \quad \text{while } s \notin [1, \underline{r}(b, w^\circ) - 1], \text{ do} \\
9 & \quad \quad \text{update } s \leftarrow w^{-1}(s). \\
10 & \quad \text{end} \\
11 & \text{end} \\
12 & \text{return } (p \mapsto w(p), s \mapsto w(s)).
\end{align*}
\]

**Lemma 8.7.** For \( w^\circ \in S_n^\circ \) and \((a, b) \in AE(w^\circ)\), the algorithm defined in Definition 8.6 terminates in finitely many steps, and the image \( \Psi_{w^\circ}(a, b) \) is in \( \text{Alignments}(w^\circ) \).

Proof. First, consider the case when \((a, b) \in AE_s(w^\circ)\), so \( a > b \). Since \( a \in I_1(w^\circ) \) and \( b \notin I_1(w^\circ) \), we have the inequalities \( w^{-1}(b) \leq b < a \leq w^{-1}(a) \), as indicated in Figure 7. Thus, \( \Psi_{w^\circ}(a, b) = (w^{-1}(b) \mapsto b, w^{-1}(a) \mapsto a) \) is an alignment. Note that \( a \) and \( b \) could be fixed points with co\((a) = \circ\) and co\((b) = \circ\), according to the definition of anti-exceedances from Section 2.3.

![Figure 7](image-url) The image of \((a, b)\) when \( a > b \) is \((w^{-1}(a) \mapsto a, w^{-1}(b) \mapsto b)\).
Next, consider the case when \((a, b) \in \text{AE}_1(w^\circ)\), and let \(\bar{r} = r(a, w^\circ)\). Then \(\Psi_{w^\circ}(a, b) = (p \mapsto w(p), w^{-1}(a) \mapsto a)\), where \(p\) is in \([\bar{r}, n]\). By definition of \(\text{AE}_1(w^\circ)\), Condition (1) of Lemma \(8.2\) fails for some \(r \in [a + 1, b]\). It follows from Lemma \(8.3\) that \(b \geq \bar{r}\), and furthermore, that Condition (1) fails for this \(\bar{r}\). In particular, there is no \(x \in [a, \bar{r} - 1]\) with preimage in \([\bar{r}, n]\). Since \(a \in I_1(w^\circ)\), then \(w^{-1}(a) \in [a, n]\). But \(a \in [a, \bar{r} - 1]\) implies that \(w^{-1}(a) \notin [\bar{r}, n]\). Hence, \(w^{-1}(a) \in [a, \bar{r} - 1]\), as drawn in Figure 8.

The termination of the while loop in Line 3 of Definition \(8.6\) comes from the fact that \(p\) is obtained by tracing in reverse the cycle containing \(b\). By Lemma \(8.3\), \(b \in [\bar{r}, n]\). If \(w^{-1}(b) \in [\bar{r}, n]\), then \(p = w^{-1}(b)\), and the loop ends immediately. In this case, since \(b\) is an exceedance by the assumption that \((a, b) \in \text{AE}_1(w^\circ)\), then \(w^{-1}(b) \in [\bar{r}, b]\). See Figure 8a. However, even if \(w^{-1}(b)\) is not in \([\bar{r}, n]\) so that the reverse cycle immediately leaves the interval \([\bar{r}, n]\), it must eventually return to \([\bar{r}, n]\), since \(b \in [\bar{r}, n]\). In this case, the first element at which the reverse cycle returns to \([\bar{r}, n]\) is \(p\), which implies that \(w(p) \in [1, \bar{r} - 1]\). However, the fact that Condition (1) is not satisfied for \(\bar{r}\) then implies that \(w(p) \notin [a, \bar{r} - 1]\), and thus \(w(p) \in [1, a - 1]\). See Figure 8b. In both cases, \((p \mapsto w(p), w^{-1}(a) \mapsto a)\) is an alignment with port side \((p \mapsto w(p))\) and starboard side \((w^{-1}(a) \mapsto a)\).

![Figure 8](image)

**Figure 8.** The red alignment is the image of \((a, b) \in \text{AE}_1\).

Finally, consider the case where \((a, b) \in \text{AE}_2(w^\circ)\), and let \(\bar{r} = r(b, w^\circ)\). Again, this case is symmetric to the \(\text{AE}_1(w^\circ)\) case above. From the algorithm, \(\Psi_{w^\circ}(a, b) = (w^{-1}(b) \mapsto b, s \mapsto w(s))\), where \(s \in [1, \bar{r} - 1]\). By Lemma \(8.5\), \(a < \bar{r}\), and Condition (2) is not satisfied for \(\bar{r}\). Thus, no \(y \in [\bar{r}, b]\) has preimage in \([1, \bar{r} - 1]\). Since \(b \notin I_1(w^\circ)\), we must have \(w^{-1}(b) \in [\bar{r}, b]\), as drawn in Figure 9.

The termination of the while loop in Line 8 of Definition \(8.6\) is again seen to terminate by tracing the cycle containing \(a\) in reverse. Since \(a \in I_1(w^\circ)\), then \(a \leq w^{-1}(a)\). If \(w^{-1}(a) \in [a, \bar{r} - 1]\), then \(s = w^{-1}(a)\), as in Figure 9a. If \(w^{-1}(a) \in [\bar{r}, n]\), then \(s\) is the first element at which the reverse cycle returns to the interval \([1, \bar{r} - 1]\), as in Figure 9b. Thus \(w(s) \in [\bar{r}, n]\), and the fact that Condition (2) of Lemma \(8.2\) fails for \(r\) implies that \(w(s) \in [b + 1, n]\). So, in both cases, \((w^{-1}(b) \mapsto b, s \mapsto w(s))\) is an alignment with port side \((w^{-1}(b) \mapsto b)\) and starboard side \((s \mapsto w(s))\).
Lemma 8.8. Fix \( w^\alpha = (w, co) \in S^\alpha_n \). The map \( \Psi_w^\alpha : \text{AE}(w^\alpha) \rightarrow \text{Alignments}(w^\alpha) \) is injective.

Proof. Let \((a_1, b_1)\) and \((a_2, b_2)\) be two distinct anti-exchange pairs for \( I_1(w^\alpha) \). We need to show \( \Psi_w(a_1, b_1) \neq \Psi_w(a_2, b_2) \). We again utilize the partition of \( \text{AE}(w^\alpha) \) from (8.4) and consider several cases.

**Case 1:** Assume at least one of \((a_1, b_1)\) and \((a_2, b_2)\) is in \( \text{AE}_s(w^\alpha) \). Observe from Definition 8.6 that anti-exchange pairs \((a, b) \in \text{AE}_s(w^\alpha)\) are the only pairs assigned alignments \( \Psi_w(a, b) = (p \mapsto w(p), s \mapsto w(s)) \) with \( p \leq w(p) < s \leq w(s) \). Thus, \( \Psi_w(a_1, b_1) \neq \Psi_w(a_2, b_2) \) unless both \((a_1, b_1)\) and \((a_2, b_2)\) are in \( \text{AE}_s(w^\alpha) \). In this case, \( \Psi_w(a_i, b_i) = (w^{-1}(b_i) \mapsto b_i, w^{-1}(a_i) \mapsto a_i) \) for both \( i \) by definition. Therefore, since \((a_1, b_1)\) and \((a_2, b_2)\) are distinct, we must have that \( \Psi_w(a_1, b_1) \neq \Psi_w(a_2, b_2) \).

**Case 2:** Assume both \((a_1, b_1)\) and \((a_2, b_2)\) are in \( \text{AE}_2(w^\alpha) \) and \( \Psi_w(a_1, b_1) = (p_i \mapsto w(p_i), w^{-1}(a_i) \mapsto a_i) \). If \( a_1 \neq a_2 \), then the two alignments have distinct starboard sides. If \( a_1 = a_2 = a \), then \( b_1 \neq b_2 \). Since the \( p_i \) are determined uniquely by tracing the cycle containing \( b_i \) backwards and finding the first element in the interval \([r(a, w), n]\), then \( b_1 \neq b_2 \) implies \( p_1 \neq p_2 \), so the alignments have distinct port sides. Either way, we have \( \Psi_w(a_1, b_1) \neq \Psi_w(a_2, b_2) \).

**Case 3:** Assume both \((a_1, b_1)\) and \((a_2, b_2)\) are in \( \text{AE}_2(w^\alpha) \) and \( \Psi_w(a_1, b_1) = (w^{-1}(b_i) \mapsto b_i, s_i \mapsto w(s_i)) \). If \( b_1 \neq b_2 \), then the two alignments have distinct port sides. If \( b_1 = b_2 = b \), then \( a_1 \neq a_2 \). Since the \( s_i \) are determined uniquely by tracing the cycle containing \( a_i \) backwards and finding the first element in the interval \([1, r(b, w^\alpha) - 1]\), then \( a_1 \neq a_2 \) implies \( s_1 \neq s_2 \). Either way, \( \Psi_w(a_1, b_1) \neq \Psi_w(a_2, b_2) \).

**Case 4:** Assume one of the \((a, b)\) is in \( \text{AE}_1(w^\alpha) \) and the other anti-exchange pair is in \( \text{AE}_2(w^\alpha) \). Without loss of generality, we may assume that \((a_1, b_1) \in \text{AE}_1(w^\alpha)\) with \( r(a_1, w^\alpha) \in [a_1 + 1, b_1] \) and \((a_2, b_2) \in \text{AE}_2(w^\alpha)\) with \( r(b_2, w^\alpha) \in [a_2 + 1, b_2] \). Then \( \Psi_w(a_1, b_1) = (p \mapsto w(p), w^{-1}(a_1) \mapsto a_1) \) for some \( p \geq r(a_1, w^\alpha) \), and \( \Psi_w(a_2, b_2) = (w^{-1}(b_2) \mapsto b_2, s \mapsto w(s)) \) for some \( s < r(b_2, w^\alpha) \).
If \( a_1 = a_2 = a \), then Corollary 8.4 shows that \( b_2 < \tilde{r}(a, w^\circ) \) since \( (a, b_2) \in \text{AE}_2(w^\circ) \) and \( (a, b_1) \in \text{AE}_1(w^\circ) \). Since \( b_2 \notin I_1(w^\circ) \), \( w^{-1}(b_2) \leq b_2 < \tilde{r}(a, w^\circ) \leq p \). Therefore, once again, the two alignments cannot have the same port sides, so \( \Psi_{w^\circ}(a_1, b_1) \neq \Psi_{w^\circ}(a_2, b_2) \).

If \( a_1 \neq a_2 \), observe from Definition 8.6 that \( (s \mapsto w(s)) \) is either the arc \( (w^{-1}(a_2) \mapsto a_2) \), or it is an arc with \( s \prec r(b_2, w) \) and \( w(s) \geq r(b_2, w^\circ) \), in which case it is an exceedance arc. See Figure 9 for these two cases. In either case, \( (s \mapsto w(s)) \) is not the anti-exceedance arc \( (w^{-1}(a_1) \mapsto a_1) \). Hence, \( \Psi_{w^\circ}(a_1, b_1) \neq \Psi_{w^\circ}(a_2, b_2) \). \[ \square \]

8.3. The Case of Crossed Alignments. Recall the definition of a starboard tacking crossed alignment from Definition 2.13. Figure 10 depicts two examples of a starboard tacking crossed alignment where the tail of the crossing arc is 1.

![Figure 10](a) \( d \neq 1 \)

![Figure 10](b) \( d = 1 \)

**Figure 10.** Two cases for starboard tacking crossing arc with tail at 1.

**Lemma 8.9.** If \( w^\circ = (w, co) \in S_n^\circ \) has a starboard tacking crossed alignment \( (p \mapsto w(p), s \mapsto w(s)) \), where 1 is the tail of the crossing arc, then there is no element of \( \text{AE}(w^\circ) \) mapping to \( (p \mapsto w(p), s \mapsto w(s)) \) under \( \Psi_{w^\circ} \). Hence, \( \Psi_{w^\circ} \) is not surjective.

**Proof.** Assume by way of contradiction that there is some anti-exchange pair \( (a, b) \in \text{AE}(w^\circ) \) such that \( \Psi_{w^\circ}(a, b) = (p \mapsto w(p), s \mapsto w(s)) \). Since fixed point loops cannot cross other arcs, then none of the arcs involved in a crossed alignment are loops. Therefore, none of 1, \( p \), or \( s \) is a fixed point. There are two cases to consider, depending on whether \( w(s) = 1 \).

If \( w(s) \neq 1 \), as in Figure 10a, then \( w(p) \) and \( w(s) \) are both exceedances. By considering the various cases in the algorithm defined in Definition 8.6, \( w(p) \) and \( w(s) \) can both be exceedances only if \( a < b \) and \( (a, b) \in \text{AE}_2(w^\circ) \), as in Figure 9c. In this case, we must have that \( w(p) = b \). Hence, the arc \( (1 \mapsto w(1)) \) satisfies Condition (2) of Lemma 8.2 for all \( r \in [2, w(1)] \), and the arc \( (p \mapsto w(p) = b) \) satisfies Condition (2) for \( r \in [p + 1, w(p)] \). Since the alignment is crossed by \( (1 \mapsto w(1)) \), then \( w(1) \geq p \). It follows that Condition (2) is satisfied for all \( r \in [2, w(p)] \) and thus for all \( r \) in \([a + 1, w(p)]\). This contradicts that \( (a, b) \in \text{AE}_2(w^\circ) \).
If \( w(s) = 1 \), as in Figure 10, then the arrangement of the arcs in \((p \mapsto w(p), s \mapsto w(s))\) implies that \( a < b \). Thus, \((a, b) \in \text{AE}_1(w^\circ) \cup \text{AE}_2(w^\circ)\). If \((a, b) \in \text{AE}_2(w^\circ)\), then we must have \( w(p) = b \). Again, looking at the arcs \((1 \mapsto w(1)) \) and \((p \mapsto w(p))\) contradicts the assumption that Condition (2) fails for some \( r \in [a + 1, w(p)] \). Otherwise, \((a, b) \in \text{AE}_1(w^\circ)\), and the starboard side being \((s \mapsto w(s))\) implies that \( a = 1 \). The port side being an exceedance implies that \( w(p) = b \), as in Figure 8a. Thus, Condition (1) fails for some \( r \in [2, w(p)] \). If \( r \in [2, w(1)] \), let \( x = 1 \), and if \( r \in [w(1) + 1, w(p)] \), let \( x = p \). In either case, the arc \((x \mapsto w(x))\) maps from \([1, r - 1]\) to \([r, n]\). The cycle containing the arc \((x \mapsto w(x))\) must then also contain an arc \((y \mapsto w(y))\) from \([r, n]\) to \([1, r - 1]\). The existence of the arc \((y \mapsto w(y))\) contradicts the fact that Condition (1) fails for \( r \). Therefore, \((a, b) \notin \text{AE}_1(w^\circ)\).

For these two cases of \( w(s) \neq 1 \) and \( w(s) = 1 \), we have obtained contradictions that \((a, b)\) is an anti-exchange pair. Therefore, there can be no such anti-exchange pair mapping to \((p \mapsto w(p), s \mapsto w(s))\) under \( \Psi_{w^\circ} \).

**Proof of Theorem 1.7**. \((1) \Rightarrow (5)\). We prove the implication \((1) \Rightarrow (5)\) using the contrapositive. Thus, assume that \( w^\circ \) is a decorated permutation whose chord diagram contains a crossed alignment. The chord diagram \( D(w^\circ) \) may be rotated and reflected as necessary to produce a new chord diagram with a starboard tacking crossed alignment whose crossing arc has its tail at 1. By Lemma 5.6, any such operations on \( D(w^\circ) \) preserve the property of being smooth or singular. Therefore, we may assume that \( D(w^\circ) \) has a starboard tacking crossed alignment whose crossing arc has its tail at 1. It follows from Lemma 8.9 that \( \# \text{AE}(w^\circ) < \# \text{Alignments}(w^\circ) \). Since \( I_1(w) \in \mathcal{M} \), then Corollary 4.4 and Definition 8.1 of anti-exchange pairs imply that \( \Pi_{w^\circ} \) is singular.

9. Enumeration of Smooth Positroids

Let \( w^\circ \) be a decorated permutation in \( S_n^\circ \) with associated positroid \( \mathcal{M} \) of rank \( k \) on ground set \([n]\). We say \( w^\circ \) is a smooth decorated permutation and \( \mathcal{M} \) is a smooth positroid if \( \Pi_{w^\circ} = \Pi_{\mathcal{M}} \) is a smooth positroid variety.

**Definition 9.1.** Let \( s(n) \) be the number of smooth positroids on ground set \([n]\), and let \( s_c(n) \) be the number of connected smooth positroids on ground set \([n]\).

By Theorem 1.7, each connected smooth positroid can be bijectively associated with a spirograph permutation. Thus, by (7.1), we have

\[
s_c(n) = \begin{cases} 
2 & n = 1 \\
\frac{n - 1}{n} & n > 1.
\end{cases}
\]

In analogy with the enumeration of smooth Schubert varieties studied by Haiman, Bona, Bousquet-Mélo and Butler [B98, BM07, Hai92], the enumeration of smooth positroids on \([n]\) gives rise to several interesting sequences. See [OEIS], A349413, A349456,
Theorem 9.2. The number of smooth positroids on ground set \([n]\) is the coefficient
\[
(9.2) \quad s(n) = [x^n] \frac{1}{n+1} \left(1 + 2x + \sum_{i=2}^{\infty} (i-1)x^i\right)^{n+1} = [x^n] \frac{1}{n+1} \left(1 + 2x + \sum_{i=2}^{n} (i-1)x^i\right)^{n+1}.
\]

Proof. Let \(\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t\) be a positroid decomposed into its connected components on a non-crossing partition. Then, \(\Pi_{\mathcal{M}}\) is smooth if and only if \(\Pi_{\mathcal{M}_i}\) is smooth for each \(i \in [t]\) by Corollary 6.5. Each connected smooth positroid can be bijectively associated with a spirograph permutation by Theorem 1.7. Therefore, every smooth positroid on \([n]\) can be uniquely determined by a non-crossing partition of \([n]\) along with a spirograph permutation on each block. Hence, the number of smooth positroids on ground set \([n]\) is given by
\[
(9.3) \quad s(n) = \sum s_c(\#B_1) s_c(\#B_2) \cdots s_c(\#B_t)
\]
where the sum is over all non-crossing partitions \(B_1 \cup \cdots \cup B_t\) of \([n]\). Equation (9.2) now follows directly from [Spe94, Corollary 0] and (9.1). \(\square\)

The partial Bell polynomial, \(B_{n,k}(x_1, \ldots, x_{n-k+1})\) introduced in [Bel27], is defined as
\[
(9.4) \quad B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{B_1 \cup \cdots \cup B_k} \prod_{i=1}^{k} x_{|B_i|},
\]
where the sum is taken over all set partitions of \([n]\) into \(k\) blocks. The following formula of Faá di Bruno expresses the \(n\)th derivative of a composition of functions in terms of the partial Bell polynomials [Joh02]
\[
(9.5) \quad \frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)).
\]

Define the triangle of numbers \(b_{n,k}\) using the Bell polynomials evaluated at \(x_i = s_c(i) \cdot i!\) for all \(i\), so
\[
(9.6) \quad b_{n,k} = B_{n,k}(2 \cdot 1!, 1 \cdot 2!, 2 \cdot 3!, \ldots, (n-k) \cdot (n-k+1)!) = \sum_{i=1}^{n-k+1} \binom{n-i}{i-1} \cdot s_c(i) \cdot i! \cdot b_{n-i,k-1}
\]
for \(1 \leq k \leq n\) along with initial condition \(b_{0,0} = 1\), and \(b_{0,k} = b_{n,0} = 0\) if \(n > 0\) or \(k > 0\), see [OEIS, A353131]. Also, let \((n)_k\) denote the falling factorial,
\[
(n)_k := n(n-1) \cdots (n-k+1).
\]
Corollary 9.3. The number of smooth positroids on ground set \([n]\) is

\[
s(n) = \frac{1}{(n + 1)!} \sum_{k=1}^{n} (n + 1)_k \cdot b_{n,k} = \sum_{k=1}^{n} \frac{b_{n,k}}{(n - k + 1)!}.
\]

Proof. Set \(f(x) = x^{n+1}\) and \(g(x) = 1 + 2x + \sum_{i=2}^{n} (i - 1)x^i\). By Theorem 9.2 \(s(n) = \frac{1}{n+1} \cdot [x^n]f(g(x))\). Furthermore, the coefficient of \(x^n\) in the composition \(f(g(x))\) can be computed as \([x^n]f(g(x)) = \frac{1}{n!} \frac{d^n}{dx^n} f(g(x))\) \(\big|_{x=0}\). Then, by Faà di Bruno’s formula (9.5),

\[
s(n) = \frac{1}{(n+1)!} \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)) \big|_{x=0}.
\]

The \(k\)th derivative of \(f(x) = x^{n+1}\) is \(f^{(k)}(x) = (n+1)_k \cdot x^{n-k+1}\), so

\[
f^{(k)}(g(x)) \big|_{x=0} = (n+1)_k \left(1 + 2x + \sum_{i=1}^{n} (i - 1)x^i \right)^{n-k+1} \big|_{x=0} = (n+1)_k.
\]

The derivatives of \(g\) are \(g'(x) = 2 + \sum_{i=1}^{n-1} i(i+1)x^i\) and \(g^{(j)}(x) = \sum_{i=0}^{n-j} (i+j-1)! \cdot (i+j)x^i\) for \(j > 1\). Hence, \(g^{(j)}(x) \big|_{x=0} = s_{c_j} \cdot j!\) for all \(j \geq 1\). Therefore, the formula in (9.7) follows from (9.6) and (9.9).

Note that the value \(k\) in the derivation above is different from the value \(k = k(w^o)\) used elsewhere in the paper. These distinct values may be used to refine the enumeration of \(s(n)\). To this end, we define

\[
s_1(n,k) := \# \{\text{smooth positroid varieties in } Gr(k,n)\} \quad \text{for } 0 \leq k \leq n
\]

(9.11) \(= \# \{\text{smooth decorated permutations in } S^o_{n,k}\},\)

(9.12)

\[
s_2(n,k) := \# \{\text{smooth decorated permutations in } S^o_n \text{ with exactly } k \text{ components in its SIF decomposition}\} \quad \text{for } 1 \leq k \leq n, \text{ and}
\]

(9.13)

\[
s_3(n,k) := \frac{b_{n,k}}{(n - k + 1)!} \quad \text{for } 1 \leq k \leq n.
\]

The terms \(s_i(n,k)\) are displayed in Figure 11 and Figure 12 for \(1 \leq n \leq 10\).

For each \(i \in \{1,2,3\}\), we may define a \(q\)-analog of \(s(n)\) as

\[
s^{(i)}(n;q) = \sum_{k \leq n} s_i(n,k)q^k.
\]

(9.14)

In fact, the \(q\)-analogs for \(i = 2\) and \(i = 3\) coincide, leading to the following theorem. This theorem was recently proved by the second author. The proof will appear in her PhD thesis.
Theorem 9.4. [Wea22] For any $1 \leq k \leq n$, the number of smooth decorated permutations in $S_n^\Delta$ with exactly $k$ components in its SIF decomposition is $\frac{b_{n,k}}{(n-k+1)!}$.

Corollary 9.5. The numbers $\frac{b_{n,k}}{(n-k+1)!}$ are integers for any $1 \leq k \leq n$.

| $n$ | $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|---|---|---|---|---|---|---|---|---|---|----|
| 1   | 1   | 1 | 1 |   |   |   |   |   |   |   |   |    |
| 2   | 1   | 3 | 1 |   |   |   |   |   |   |   |   |    |
| 3   | 1   | 7 | 7 | 1 |   |   |   |   |   |   |   |    |
| 4   | 1   | 15| 29| 15| 1 |   |   |   |   |   |   |    |
| 5   | 1   | 31| 96| 96| 31| 1 |   |   |   |   |   |    |
| 6   | 1   | 63| 282|440|282|63|1   |   |   |   |   |    |
| 7   | 1   | 127|771|1688|1688|771|127|1   |   |   |   |    |
| 8   | 1   | 255|2011|5803|8089|5803|2011|255|1   |   |   |    |
| 9   | 1   | 511|5074|18520|33721|33721|18520|5074|511|1   |   |    |
| 10  | 1   | 1023|12488|55998|127698|166325|127698|55998|12488|1023|1   |

Figure 11. Table of $s_1(n, k)$.

| $n$ | $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|---|---|---|---|---|---|---|---|---|----|
| 1   | 1   | 2 |   |   |   |   |   |   |   |   |    |
| 2   | 1   | 4 |   |   |   |   |   |   |   |   |    |
| 3   | 2   | 6 | 8 |   |   |   |   |   |   |   |    |
| 4   | 3   | 18| 24|16 |   |   |   |   |   |   |    |
| 5   | 4   | 40| 100|80|32 |   |   |   |   |   |    |
| 6   | 5   | 78| 305|440|240|64 |   |   |   |   |    |
| 7   | 6   | 140|798|1750|1680|672|128 |   |   |   |    |
| 8   | 7   | 236|1876|5838|8400|5824|1792|256 |   |   |    |
| 9   | 8   | 378|4056|17136|34524|35616|18816|4608|512 |   |    |
| 10  | 9   | 580|8190|45480|122682|175896|137760|57600|11520|1024 |

Figure 12. Table of $s_2(n, k) = s_3(n, k)$.

It is interesting to consider the asymptotic growth function for the number $s(n)$ of smooth positroids on $[n]$. We use the second formula in (9.2) to obtain the following data:
\[ s(51)/s(50) \approx 5.4489775, \]
\[ s(101)/s(100) \approx 5.528236, \]
\[ s(151)/s(150) \approx 5.555362, \]
\[ s(201)/s(200) \approx 5.569062, \]
\[ s(251)/s(250) \approx 5.5773263. \]

Based on this data, we conjecture the growth function is of the order \( O(c^n) \) for some constant \( c < 6 \).

10. Future Directions

There are many parallels between Schubert varieties and positroid varieties that have yet to be fully examined in the literature. In particular, the Johnson graph of a positroid \( J(\mathcal{M}) \) is shown to play a key role in the geometry of positroid varieties by Theorem 1.7, much like the Bruhat graph controls the geometry of Schubert varieties [Car94].

In 1998, Goresky, Kottwitz, and MacPherson [GKM98] developed a very general theory of equivariant cohomology for spaces with a torus action and a moment map corresponding with a simple graph embedded in the space with vertices given by fixed points of the torus action and edges given by 1-dimensional orbits connecting fixed points. The connection between these GKM graphs and equivariant cohomology has been extensively developed over the past 24 years by Goldin [Gol99], Guillemin-Holm-Zara [GZ01, GHZ06], Harada-Henriques-Holm [HHH05], and Fukukawa-Ishida-Masuda [FIM14].

For positroid varieties, we have a natural way to obtain an analog of the Bruhat order and the Bruhat graph. Recall the Johnson graph \( J(\mathcal{M}) \) has vertices indexed by bases in \( I \in \mathcal{M} \), each of which correspond bijectively with the \( T \)-fixed points in \( \Pi_\mathcal{M} \). If \( I, J \in \mathcal{M} \) are two bases that are connected via an edge in \( J(\mathcal{M}) \), then by definition we can assume there is some exchange pair \( i < j \) such that \( I \setminus \{i\} \cup \{j\} = J \). Orient the edge between them \( I \rightarrow J \). We call this the directed Johnson graph of \( \mathcal{M} \). This graph is closely related to the Gale partial order on \( k \)-subsets in Section 2.

**Problem 10.1.** Study the geometry and equivariant cohomology of a positroid variety via the directed Johnson graph \( J(\mathcal{M}) \).

**Problem 10.2.** Identify the singular locus of a positroid variety.

**Problem 10.3.** Study the enumeration and coset structure of the group operations of flip, inverse, rotation for perms, or for derangements, or SIF perms.

**Problem 10.4.** Study the combinatorics of the directed Johnson graphs of positroids.
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