The optimal symmetric quasi-Banach range of the discrete Hilbert transform

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Abstract. We identify the symmetric quasi-Banach range of the discrete Calderón operator and Hilbert transform acting on a symmetric quasi-Banach sequence space. As an application, we present an example of the optimal range in the case when the domain of those operators is the weak-$\ell_1$ space of sequences.

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1. Introduction. In this paper, we consider symmetric sequence spaces (see [9, 10] and Definition 2 below). Our study deals with the behavior of the discrete Calderón operator and Hilbert transform on such spaces and is motivated by the following connected problem.

Problem 1. Given a symmetric quasi-Banach sequence space $E = E(\mathbb{Z})$, determine the least symmetric quasi-Banach sequence space $F = F(\mathbb{Z})$ such that $\mathcal{H}^d(E) \subseteq F$, where the Hilbert transform $\mathcal{H}^d$ is given by the formula

$$(\mathcal{H}^d x)(n) := \frac{1}{\pi} \sum_{k \in \mathbb{Z}, k \neq n} \frac{x(k)}{k - n}, \quad x \in E(\mathbb{Z}).$$

We shall be referring to the space $F(\mathbb{Z})$ as the optimal range space for the operator $\mathcal{H}^d$ restricted to the domain $E(\mathbb{Z}) \subset \ell_{\log}(\mathbb{Z})$, where $\ell_{\log}(\mathbb{Z})$ is the Lorentz space of sequences associated with the function $\log(1+t)$, $t > 0$. Later we will show that the maximal symmetric domain for $\mathcal{H}^d$ is the space $\ell_{\log}(\mathbb{Z})$ (see Remark 3). The class of symmetric Banach spaces $E$ of sequences with Fatou norm (that is, when the norm closed unit ball $B_E$ of $E$ is closed in $E$
with respect to the almost everywhere convergence) such that $H^d(E) \subseteq E$ was characterized by K. Andersen [1, Theorem 3] (see [3, Theorem 2.1] for the continuous case). A careful analysis of their proofs actually shows that the just cited result characterizes the class of all symmetric Banach sequence spaces $E = E(Z)$ with the Fatou norm such that the optimal range for the operator $H^d$ coincides with $E$. Nowadays, such a characterization is customarily stated in terms of the so-called Boyd indices [2,8,9] and may be viewed as a satisfactory resolution of the Problem 1 in the case of symmetric spaces $E$ possessing non-trivial Boyd indices. However, the problem remains open if we deal with a symmetric space $E$ whose (at least one) Boyd index happens to be trivial. If we restrict our attention to the subclass of symmetric Banach spaces $E$ with Fatou norm, Problem 1 reduces to a familiar problem settled by D. Boyd [3] in 1967, and K. Andersen [1] in 1976 for the discrete case. Indeed, in this special case, [1, Theorem 3] asserts that $H^d : E(Z) \rightarrow F(Z)$ if and only if $S^d : E(Z_+) \rightarrow F(Z_+)$, where the operator $S^d$, known as the Calderón operator, is defined by the formula

$$(S^d x)(n) = \frac{1}{n+1} \sum_{k=0}^{n} x(k) + \sum_{k=n+1}^{\infty} \frac{x(k)}{k}, \quad x \in \ell \log(Z_+).$$

Effectively, the problem reduces to describing the optimal receptacle of the operator $S^d$.

Consider the case when $E = \ell_p$, $1 < p < \infty$. By Hardy’s inequality, $S^d : \ell_p(Z_+) \rightarrow \ell_p(Z_+)$ and [1, Theorem 3] yields that $H^d : \ell_p(Z) \rightarrow \ell_p(Z)$. Now, if $E$ is a Banach interpolation space for the couple $(\ell_p, \ell_q)$, $1 < p < q < \infty$, then $H^d : E(Z) \rightarrow E(Z)$. A careful inspection of the proof in [1] yields that, in this case, $E(Z)$ is the optimal receptacle for $H^d$ restricted on $E(Z)$. A similar assertion for the symmetric sequence spaces with Fatou norm can be extracted from [1, Theorem 3].

In the case $p = 1$, Komori’s result [7] (Kolmogorov’s result in the continuous case [6]) asserts that the operator $H^d$ maps the symmetric Banach sequence space $\ell_1(Z)$ to the symmetric quasi-Banach sequence space $\ell_{1,\infty}(Z)$. From this, one might guess that, for $E = \ell_1(Z)$, the optimal range of $H^d$ is $\ell_{1,\infty}(Z)$, which strongly suggests that the natural setting for Problem 1 is that of symmetric quasi-Banach spaces. Addressing precisely this framework, one of our main results, Theorem 6, provides a complete description of the optimal range $F$ for a given symmetric quasi-Banach space $E$ (under a mild technical assumption that $E(Z_+) \subset \ell \log(Z_+)$. It should be pointed out that our results allow a substantial extension of Andersen’s and Komori’s results cited above thereby complementing [1, Theorem 3] and [7, Theorem].

The structure of this paper is as follows. Section 2 recalls the necessary theory on symmetric quasi-Banach sequence spaces. In Section 3, we define the optimal range for the discrete Calderón operator among the symmetric quasi-Banach sequence spaces. The same result for the discrete Hilbert transform $H^d$ is also shown in this section. Similar constructions were investigated in [12,13] for the Hardy and Hardy type operators in the continuous case. As an application of the main result, we show an example of the optimal range for
the Calderón operator and Hilbert transform when their domain is the $\ell_{1,\infty}$ space of sequences.

2. Preliminaries.

2.1. Symmetric quasi-Banach sequence spaces. Let $I = \mathbb{Z}_+$ (resp. $\mathbb{Z}$), where $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ (resp. $\mathbb{Z} = \{-\ldots, -2, -1, 0, 1, 2, \ldots\}$), and let $(I, \nu)$ be a measure space equipped with the counting measure $\nu$. Let $L(I)$ be the space of finite $\nu$-a.e., measurable sequences (with identification $\nu$ a.e.) on $I$. Define $L_0(I, \nu)$ (or $L_0(I)$) as the subalgebra of $L(I)$ which consists of all sequences $x$ such that $\nu(\{|x| > s\})$ is finite for some $s$. For $x \in L_0(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the sequence $|x|$, which is the sequence $|x| = \{|x(n)|\}_{n\geq 0}$ rearranged to be in decreasing order. If $I = \mathbb{Z}_+$ (resp. $\mathbb{Z}$) and $\nu$ is the counting measure, then $L_0(I) = \ell_\infty(I)$, where $\ell_\infty(I)$ denotes the space of all bounded sequences on $I$. Briefly we denote $\ell_\infty(I)$ by $\ell_\infty$.

Definition 2. We say that $(E, \|\cdot\|_E)$ is a symmetric (quasi-)Banach space sequence space on $I$ if the following hold:

(i) $E$ is a subset of $\ell_\infty$;
(ii) $(E, \|\cdot\|_E)$ is a (quasi-)Banach space;
(iii) If $x \in E$ and $y \in \ell_\infty$ are such that $|y| \leq |x|$, then $y \in E$ and $\|y\|_E \leq \|x\|_E$;
(iv) If $x \in E$ and $y \in \ell_\infty$ are such that $\mu(y) = \mu(x)$, then $y \in E$ and $\|y\|_E = \|x\|_E$.

For the general theory of symmetric spaces of functions and sequences, we refer the reader to [2,8,9].

The discrete dilation operators $\sigma_m$, $m \in \mathbb{N}$, on $\ell_\infty(\mathbb{Z}_+)$ is defined by

$$\sigma_m(x) := \left\{ \begin{array}{c} (x(0), x(0), \ldots, x(0)), (x(1), x(1), \ldots, x(1)), \ldots, (x(n), x(n), \ldots, x(n)), \ldots \end{array} \right\}.$$ 

It is obvious that the dilation operator $\sigma_m$ is continuous in $\ell_\infty(\mathbb{Z}_+)$ (see [8, Chapter II.3, p. 96]).

2.2. Lorentz sequence spaces. Let $\varphi$ be an increasing concave function on $\mathbb{R}_+$ such that $\lim_{t \to 0^+} \varphi(t) = 0$. The Lorentz sequence space $\ell_\varphi(I)$ is defined by setting

$$\ell_\varphi(I) := \left\{ x \in \ell_\infty(I) : \|x\|_\varphi = \sum_{n=0}^{\infty} \mu(n, x)(\varphi(n+1) - \varphi(n)) < \infty \right\},$$

and equipped with the norm

$$\|x\|_\varphi = \sum_{n=0}^{\infty} \mu(n, x)(\varphi(n+1) - \varphi(n)). \quad (2.1)$$

These spaces are examples of symmetric Banach sequence spaces. In particular, if $\varphi(t) := \log(1 + t)$, $t > 0$, then the Lorentz sequence space $\ell_{\log}(I)$ is defined as follows:

$$\ell_{\log}(I) := \left\{ x \in \ell_\infty(I) : \|x\|_{\log} = \sum_{n=0}^{\infty} \frac{\mu(n, x)}{n + 1} < \infty \right\}.$$
For more details on Lorentz spaces, we refer the reader to [2, Chapter II.5] and [8, Chapter II.5].

2.3. Weak-$\ell_1$ and $m_{1,\infty}$ spaces. The weak-$\ell_1$ sequence space $\ell_{1,\infty}$ on $I$ is defined as

$$\ell_{1,\infty}(I) := \left\{ x \in \ell_{\infty}(I) : \mu(n, x) = O\left(\frac{1}{1+n}\right) \right\},$$
equipped with the functional $\| \cdot \|_{\ell_{1,\infty}}$ defined by the formula

$$\| x \|_{\ell_{1,\infty}} := \sup_{n \in \mathbb{Z}_+} (n+1)\mu(n, x).$$

It is easy to see that $\| \cdot \|_{\ell_{1,\infty}}$ is a quasi-norm. In fact, the space $(\ell_{1,\infty}, \| \cdot \|_{\ell_{1,\infty}})$ is quasi-Banach (see [10, Example 1.2.6, p. 24]).

Define the Marcinkiewicz (or Lorentz) space $m_{1,\infty}(I)$ by setting

$$m_{1,\infty}(I) := \left\{ x \in \ell_{\infty}(I) : \sup_{n \in \mathbb{Z}_+} \frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, x) < \infty \right\}$$

equipped with the norm

$$\| x \|_{m_{1,\infty}} := \sup_{n \in \mathbb{Z}_+} \frac{1}{\log(2+n)} \sum_{k=0}^{n} \mu(k, x)$$

(see [10, Example 1.2.7, p. 24]). This is an example of a symmetric Banach sequence space. For more information on Marcinkiewicz spaces, we refer to [2, Chapter II.5] and [8, Chapter II.5].

Moreover, the space $m_{1,\infty}$ contains the quasi-Banach space $\ell_{1,\infty}$ of sequences, i.e. the following inclusion

$$\ell_{1,\infty} \subset m_{1,\infty}$$
is strict (see [10, Lemma 1.2.8 and Example 1.2.9, pp. 25-26]).

2.4. Discrete Calderón operator and Hilbert transform. Define the space $(\ell_{1,\infty} + \ell_{\infty})(I)$ as follows:

$$(\ell_{1,\infty} + \ell_{\infty})(I) := \{ x = x_1 + x_2 : x_1 \in \ell_{1,\infty}(I), x_2 \in \ell_{\infty}(I), \| x \|_{\ell_{1,\infty} + \ell_{\infty}} < \infty \},$$
where the quasi-norm is defined by

$$\| x \|_{\ell_{1,\infty} + \ell_{\infty}} := \inf \{ \| x_1 \|_{\ell_{1,\infty}} + \| x_2 \|_{\ell_{\infty}} : x_1 \in \ell_{1,\infty}(I), x_2 \in \ell_{\infty}(I) \}.$$For each $x \in \ell_{\log}(\mathbb{Z}_+)$, define the discrete Calderón operator $S : \ell_{\log}(\mathbb{Z}_+) \to (\ell_{1,\infty} + \ell_{\infty})(\mathbb{Z}_+)$ by

$$(S^d x)(n) := \frac{1}{n+1} \sum_{k=0}^{n} x(k) + \sum_{k=n+1}^{+\infty} \frac{x(k)}{k}, \quad x \in \ell_{\log}(\mathbb{Z}_+). \quad (2.3)$$It is obvious that $S^d$ is a linear operator. Next, it is easy to see that if $0 < n < n_0$, then

$$\min \left(1, \frac{m}{n_0}\right) \leq \min \left(1, \frac{m}{n}\right) \leq \frac{n_0}{n} \min \left(1, \frac{m}{n_0}\right), \quad (m > 0).$$
So, if \( x \) is non-negative, it follows from the first of these inequalities that \((S^d x)(n)\) is a decreasing function of \( n \geq 0 \). The operator \( S^d \) is often applied to the decreasing rearrangement \( \mu(x) \) of a function \( x \) defined on some other measure space. Since \( S^d \mu(x) \) is itself decreasing, it is easy to see that \( \mu(S^d \mu(x)) = S^d \mu(x) \). Let \( x \in \ell_\log(\mathbb{Z}_+) \). Since for each \( n \geq 0 \), the kernel \( K_n(k) = \frac{1}{k} \min \{ 1, \frac{k}{n+1} \} \) is a decreasing sequence of \( k > 0 \), it follows from [2, Theorem II.2.2, p. 44] that

\[
|(S^d x)(n)| \overset{(2.3)}{=} \left| \sum_{k=0}^{\infty} x(k) \min \left\{ 1, \frac{k}{n+1} \right\} \frac{1}{k} \right| \\
\leq \sum_{k=0}^{\infty} |x(k)| \min \left\{ 1, \frac{k}{n+1} \right\} \frac{1}{k} \leq \sum_{k=0}^{\infty} \mu(k, x) \min \left\{ 1, \frac{k}{n+1} \right\} \frac{1}{k} \overset{(2.3)}{=} \left( S^d \mu(x) \right)(n), \ \forall n \in \mathbb{Z}_+.
\]

For more information about these operators, we refer to [2, Chapter III] and [8, Chapter II].

If \( x \in \ell_\log(\mathbb{Z}) \), then the discrete Hilbert transform \( \mathcal{H}^d \) is defined as in [1] by

\[
(\mathcal{H}^d x)(n) := \frac{1}{\pi} \sum_{\substack{k \in \mathbb{Z} \setminus \{0\} \backslash \mathbb{Z} \setminus \{n\}}} \frac{x(k)}{n-k}, \quad x \in \ell_\log(\mathbb{Z}).
\]

**Remark 3.** Let \( x = x\chi_{[0,\infty)} \) be such that \( x \) is a non-negative decreasing sequence on \( \mathbb{Z}_+ \). Then it is easy to see that

\[
|(\mathcal{H}^d x)(-n)| \overset{(2.5)}{=} \frac{1}{\pi} \left| \sum_{k=0}^{\infty} \frac{x(k)}{-n-k} \right| \\
= \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{x(k)}{n+k} = \frac{1}{\pi} \left( \sum_{k=0}^{n} \frac{x(k)}{n+k} + \sum_{k=n+1}^{\infty} \frac{x(k)}{n+k} \right) \\
\geq \frac{1}{\pi} \left( \sum_{k=0}^{n} \frac{x(k)}{2(n+1)} + \sum_{k=n+1}^{\infty} \frac{x(k)}{2k} \right) \\
= \frac{1}{2\pi} \left( \frac{1}{n+1} \sum_{k=0}^{n} x(k) + \sum_{k=n+1}^{\infty} \frac{x(k)}{k} \right) \overset{(2.3)}{=} \frac{1}{2\pi} (S^d x)(n), \quad n \in \mathbb{Z}_+,
\]
i.e. we have

\[
\frac{1}{2\pi} (S^d \mu(x))(n) \leq |(\mathcal{H}^d x)(-n)|, \quad n \in \mathbb{Z}_+.
\]

Therefore, if \((\mathcal{H}^d x)(-n)\) exists, then it follows that \( S^d \mu(x) \) exists, and it means \( x \) belongs to the domain of \( S^d \), i.e. \( x \in \ell_\log(\mathbb{Z}_+) \) (see (2.3)). On the other hand, if \( x \in \ell_\log(\mathbb{Z}_+) \), then, by [2, Theorem III.4.8, p. 138], we have

\[
\mu(\mathcal{H}^d x) \leq c_{abs} S^d \mu(x),
\]

which shows existence of \( \mathcal{H}^d x \).
3. Optimal symmetric quasi-Banach range for the discrete Calderón operator and Hilbert transform. In this section, we describe the optimal symmetric quasi-Banach sequence range spaces for the discrete Calderón operator $S^d$ and Hilbert transform $H^d$. A continuous version of the following lemma was proved in [15], here we use the discrete version of it.

**Lemma 4.** Let $\{x_k\}_{k=1}^{\infty} \subset \ell_\infty(\mathbb{Z}_+)$, where $x_k = \{x_k(n)\}_{n \in \mathbb{Z}_+}$. If the series

$$\sum_{k=1}^{\infty} \sigma_{2^k} \mu(n, x_k)$$

converges almost everywhere (a.e.) in $\ell_\infty(\mathbb{Z}_+)$ for all $n \in \mathbb{Z}_+$, then the series $\sum_{k=1}^{\infty} x_k$ converges in measure in $\ell_\infty(\mathbb{Z}_+)$ and we have

$$\mu\left(n, \sum_{k=1}^{\infty} x_k\right) \leq \sum_{k=1}^{\infty} \sigma_{2^k} \mu(n, x_k), \ n \in \mathbb{Z}_+. \quad (3.1)$$

**Proof.** Fix $\varepsilon, \delta > 0$, and choose $N = N(\varepsilon, \delta)$ such that

$$\left(\sum_{k=N}^{\infty} \sigma_{2^k} \mu(x_k)\right) (\varepsilon) < \delta. \quad (3.2)$$

Then, for any $N_1, N_2 \geq N$ and by [8, Corollary II.2 (2.23), p. 67] and (3.2), we have

$$\mu\left(\varepsilon, \sum_{k=N}^{N_2} x_k\right) = \mu\left(\varepsilon \cdot \frac{\sum_{k=N_1+1}^{N_2} 2^{-k}}{\sum_{m=N_1+1}^{N_2} 2^{-m}}, \sum_{k=N_1+1}^{N_2} x_k\right)$$

$$\leq \sum_{k=N_1+1}^{N_2} \mu\left(\varepsilon \cdot \frac{2^{-k}}{\sum_{m=N_1+1}^{N_2} 2^{-m}}, x_k\right) \quad (2.23)$$

$$\leq \sum_{k=N_1+1}^{N_2} \mu(\varepsilon \cdot 2^{N_1-k}, x_k)$$

$$\leq \sum_{k=N}^{\infty} \mu(\varepsilon \cdot 2^{-k}, x_k) = \left(\sum_{k=N}^{\infty} \sigma_{2^k} \mu(x_k)\right) (\varepsilon) < \delta. \quad (3.3)$$

Let us denote $a_{N_1}(n) = \sum_{k=1}^{N_1} x_k(n)$ and $a_{N_2}(n) = \sum_{k=1}^{N_2} x_k$ for all $n \in \mathbb{Z}_+$. Then, by the preceding inequality, for any $N_1, N_2 \geq N$, we obtain

$$a_{N_2}(n) - a_{N_1}(n) \in U(\varepsilon, \delta) := \{x \in \ell_\infty(\mathbb{Z}_+) : \nu(\{|x| > \delta\}) < \varepsilon\},$$

which shows that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence in measure in $\ell_\infty(\mathbb{Z}_+)$. Since $\ell_\infty(\mathbb{Z}_+)$ is complete in the measure topology, it follows that the series $\sum_{k=1}^{\infty} x_k(n)$ converges in measure for all $n \in \mathbb{Z}_+$. Therefore, since the decreasing rearrangement $\mu$ is continuous from the right, it follows from (3.3) that

$$\mu\left(n, \sum_{k=1}^{\infty} x_k\right) \leq \sum_{k=1}^{\infty} \sigma_{2^k} \mu(n, x_k), \ n \in \mathbb{Z}_+. \quad \square$$
Definition 5. Let $E$ be a symmetric quasi-Banach sequence space on $\mathbb{Z}_+$. Let $E(\mathbb{Z}_+) \subset \ell_{\log}(\mathbb{Z}_+)$ and let $S^d$ be the operator defined in (2.3). Define

$$F(\mathbb{Z}_+) := \{ x \in (\ell_1, \infty + \ell_\infty)(\mathbb{Z}_+) : \exists y \in E(\mathbb{Z}_+), \mu(x) \leq S^d\mu(y) \}$$

such that

$$\|x\|_{F(\mathbb{Z}_+)} := \inf\{\|y\|_{E(\mathbb{Z}_+)} : \mu(x) \leq S^d\mu(y) \} < \infty. \quad (3.4)$$

The following is the main result of this section.

Theorem 6. Let $E$ be a symmetric quasi-Banach sequence space on $\mathbb{Z}_+$. If $E(\mathbb{Z}_+) \subset \ell_{\log}(\mathbb{Z}_+)$, then

(i) $(F(\mathbb{Z}_+), \| \cdot \|_{F(\mathbb{Z}_+)})$ is a quasi-Banach space.

(ii) Moreover, $(F(\mathbb{Z}_+), \| \cdot \|_{F(\mathbb{Z}_+)})$ is the optimal symmetric quasi-Banach range for the operator $S^d$ on $E(\mathbb{Z}_+)$.

First, we need the following Lemma.

Lemma 7. Let $E$ be a symmetric quasi-Banach sequence space on $\mathbb{Z}_+$. If $E(\mathbb{Z}_+) \subset \ell_{\log}(\mathbb{Z}_+)$, then the space $F(\mathbb{Z}_+)$ given in Definition 5 is a linear space.

Proof. For $j = 1, 2$, let $x_j \in F(\mathbb{Z}_+)$, where $x_j = \{x_j(n)\}_{n \in \mathbb{Z}_+}$, and $\alpha_j$ be any scalars from the field of complex numbers. Then, by the Definition 5 of $F(\mathbb{Z}_+)$, there exist corresponding $y_j \in E(\mathbb{Z}_+)$ such that $\mu(n, x_j) \leq (S^d\mu(y_j))(n), n \in \mathbb{Z}_+$. Therefore, for any $x_j \in F(\mathbb{Z}_+)$ and $\alpha_j$, by [2, Proposition II.1.7, p. 41], we have

$$\mu(\alpha_1 x_1 + \alpha_2 x_2) \leq \sigma_2\mu(\alpha_1 x_1) + \sigma_2\mu(\alpha_2 x_2) = |\alpha_1| \cdot \sigma_2\mu(x_1) + |\alpha_2| \cdot \sigma_2\mu(x_2) \leq |\alpha_1| \cdot \sigma_2\left(S^d(\mu(y_1))\right) + |\alpha_2| \cdot \sigma_2\left(S^d(\mu(y_2))\right)$$

$$= S^d(|\alpha_1| \cdot \sigma_2\mu(y_1)) + S^d(|\alpha_2| \cdot \sigma_2\mu(y_2))$$

$$= S^d(|\alpha_1| \cdot \sigma_2\mu(y_1) + |\alpha_2| \cdot \sigma_2\mu(y_2)).$$

Since $E(\mathbb{Z}_+)$ is a linear space and $|\alpha_1| \cdot \sigma_2\mu(y_1) + |\alpha_2| \cdot \sigma_2\mu(y_2) \in E(\mathbb{Z}_+)$, it follows that $\alpha_1 x_1 + \alpha_2 x_2 \in F(\mathbb{Z}_+)$. Thus, $F(\mathbb{Z}_+)$ is a linear space. $\square$

Lemma 8. Let $E$ be a symmetric quasi-Banach space on $\mathbb{Z}_+$. If $E(\mathbb{Z}_+) \subset \ell_{\log}(\mathbb{Z}_+)$, then the space $F(\mathbb{Z}_+)$ defined in Definition 5 is a quasi-normed space equipped with (3.4).

Proof. Let us prove that the expression

$$\|x\|_{F(\mathbb{Z}_+)} := \inf\{\|y\|_{E(\mathbb{Z}_+)} : \exists y \in E(\mathbb{Z}_+), \mu(x) \leq S^d\mu(y) \}$$

defines a quasi-norm in $F(\mathbb{Z}_+)$. Clearly, if $x = 0$, then, by (3.4), $\|x\|_{F(\mathbb{Z}_+)} = 0$, and for every scalar $\alpha$, we have $\|\alpha x\|_{F(\mathbb{Z}_+)} = |\alpha| \cdot \|x\|_{F(\mathbb{Z}_+)}$. We shall prove the non-trivial part. If $\|x\|_{F(\mathbb{Z}_+)} = 0$, then there exists $y_k$ $(k \in \mathbb{N})$ in $E(\mathbb{Z}_+)$ with $\mu(x) \leq S^d\mu(y_k)$ such that $\|y_k\|_{E(\mathbb{Z}_+)} \to 0$, as $k \to \infty$. By assumption, we have that $S^d : E(\mathbb{Z}_+) \to (\ell_1, \infty + \ell_\infty)(\mathbb{Z}_+)$ (see 2.3). Since $S^d$ is a positive operator (see [2, Chapter III, p. 134]), it follows from [11, Proposition 1.3.5, p. 27] that
$S^d : E(\mathbb{Z}_+) \rightarrow (\ell_1, \infty + \ell_\infty)(\mathbb{Z}_+)$ is bounded. Hence, $\|S^d\mu(y_k)\|_{(\ell_1, \infty + \ell_\infty)(\mathbb{Z}_+)} \rightarrow 0$, as $k \rightarrow \infty$. From the condition $\mu(x) \leq S^d\mu(y_k)$, for every $k \in \mathbb{N}$, we have

$$\|\mu(x)\|_{(\ell_1, \infty + \ell_\infty)(\mathbb{Z}_+)} \leq \|S^d\mu(y_k)\|_{(\ell_1, \infty + \ell_\infty)(\mathbb{Z}_+)} \rightarrow 0,$$

which shows that $x = 0$.

Let us prove that $\| \cdot \|_{F(\mathbb{Z}_+)}$ satisfies the quasi-triangle inequality. For $j = 1, 2$, let $x_j \in F(\mathbb{Z}_+)$, and fix $\varepsilon > 0$. Then there exist $y_j \in E(\mathbb{Z}_+)$ with $\mu(x_j) \leq S^d\mu(y_j)$ and $\|y_j\|_{E(\mathbb{Z}_+)} < \|x_j\|_{F(\mathbb{Z}_+)} + \varepsilon$. Hence, from (3.4) and since $\| \cdot \|_{E(\mathbb{Z}_+)}$ is a quasi-norm, it follows from (3.5) and [14, Remark 18] that

$$\|x_1 + x_2\|_{F(\mathbb{Z}_+)} \leq \|\sigma_2y_1\|_{E(\mathbb{Z}_+)} + \|\sigma_2y_2\|_{E(\mathbb{Z}_+)} = \|\sigma_2(\mu(y_1) + \mu(y_2))\|_{E(\mathbb{Z}_+)}$$

$$\leq 2 \cdot c_E \|\mu(y_1) + \mu(y_2)\|_{E(\mathbb{Z}_+)} \leq 2 \cdot c_E^2 (\|y_1\|_{E(\mathbb{Z}_+)} + \|y_2\|_{E(\mathbb{Z}_+)})$$

and by choice of $y_1, y_2 \in E(\mathbb{Z}_+)$,

$$\|x_1 + x_2\|_{F(\mathbb{Z}_+)} \leq 2 \cdot c_E^2 (\|x_1\|_{F(\mathbb{Z}_+)} + \|x_2\|_{F(\mathbb{Z}_+)}) + 4c_E^2 \cdot \varepsilon.$$
follows that
\[
\sum_{k=1}^{\infty} \sigma_{2^k} \mu(x_{k+1} - x_k) \leq \sum_{k=1}^{\infty} \sigma_{2^k} S^d \mu(y_k)
\]
\[
= \sum_{n=1}^{\infty} S^d(\sigma_{2^n} \mu(y_n)) = S^d \left( \sum_{k=1}^{\infty} \sigma_{2^k} \mu(y_k) \right). \tag{3.7}
\]
Hence,
\[
\left\| \sum_{k=1}^{\infty} \sigma_{2^k} \mu(y_k) \right\|^p_{E(Z_+)} \leq \sum_{k=1}^{\infty} \|\sigma_{2^k} \mu(y_k)\|^p_{E(Z_+)} \leq \sum_{k=1}^{\infty} (c_E \cdot \varepsilon)^{kp} < \infty.
\]
Therefore, the series \( \sum_{k=1}^{\infty} \sigma_{2^k} \mu(y_k) \) converges a.e. in \( E(Z_+) \). Since \( S^d \) is continuous on \( E(Z_+) \) by assumption, it follows from (3.7) that the series \( \sum_{k=1}^{\infty} \sigma_{2^k} \mu(x_{k+1} - x_k) \) belongs to \( F(Z_+) \). Then, by Lemma 4, the series \( \sum_{k=1}^{\infty} (x_{k+1} - x_k) \) converges in measure and belongs to \( F(Z_+) \), and we have
\[
\|x - x_1\|_{F(Z_+)} = \left\| \sum_{k=1}^{\infty} (x_{k+1} - x_k) \right\|_{F(Z_+)} \leq \left\| \sum_{k=1}^{\infty} \sigma_{2^k} \mu(x_{k+1} - x_k) \right\|_{F(Z_+)} < \infty.
\]
This shows that \( x \in F(Z_+) \). So, \( F(Z_+) \) is complete. On the other hand, since \( \|x\|_{F(Z_+)} = \|\mu(x)\|_{F(Z_+)} \), it follows that \( F(Z_+) \) is a symmetric space of sequences. So, the space \((F(Z_+), \|\cdot\|_{F(Z_+)})\) is a symmetric quasi-Banach sequence space.

Next, we prove the second part of the theorem. First, we need to show that \( S^d : E(Z_+) \to F(Z_+) \) is bounded. Indeed, let \( x \in E(Z_+) \). By (2.4), we obtain
\[
\|S^d x\|_{F(Z_+)} = \inf \{\|y\|_{E(Z_+)} : \mu(S^d x) \leq S^d \mu(y)\} \leq \|x\|_{E(Z_+)},
\]
Since \( x \in E(Z_+) \) is arbitrary, it follows that \( S^d : E(Z_+) \to F(Z_+) \) is bounded.

Now, suppose that \( G(Z_+) \) is another symmetric quasi-Banach sequence space such that \( S^d : E(Z_+) \to G(Z_+) \) is bounded, and let us show that \( F(Z_+) \subset G(Z_+) \). If \( x \in F(Z_+) \), then there is \( y \in E(Z_+) \) such that \( \mu(x) \leq S^d \mu(y) \). Hence, we have
\[
\|x\|_{G(Z_+)} \leq \|S^d \mu(y)\|_{G(Z_+)} \leq \|S^d\|_{E(Z_+) \to G(Z_+)} \|y\|_{E(Z_+)},
\]
Via taking infimum over all such \( y \)'s, we obtain
\[
\|x\|_{G(Z_+)} \leq \|S^d\|_{E(Z_+) \to G(Z_+)} \|x\|_{F(Z_+)},
\]
which means \( F(Z_+) \subset G(Z_+) \). Thus, \( F(Z_+) \) is the minimal space among the symmetric quasi-Banach sequence spaces. This completes the proof.

**Remark 9.** The space \( F \) given in the Definition 5 is defined similarly on \( \mathbb{Z} \), and by repeating the same method as in the Theorem 6 (i), it becomes a symmetric quasi-Banach space.

The following result provides a solution to the Problem 1.
Theorem 10. Let the assumptions of Theorem 6 hold. Then, the so defined space \( F(\mathbb{Z}) \) is the optimal range for the Hilbert transform \( \mathcal{H}^d \) on \( E(\mathbb{Z}) \).

Proof. It follows from Theorem 6 (i) and Remark 9 that \( F(\mathbb{Z}) \) is a quasi-Banach symmetric space of sequences. Let us show that \( \mathcal{H}^d : E(\mathbb{Z}) \to F(\mathbb{Z}) \) is bounded. Since \( S^d : E(\mathbb{Z}) \to F(\mathbb{Z}) \) is bounded by Theorem 6 (ii), it follows from the discrete version of [2, Theorem III. 4.8, p. 138] that \( \mathcal{H}^d : E(\mathbb{Z}) \to F(\mathbb{Z}) \) is bounded. Now, suppose that \( G(\mathbb{Z}) \) is another symmetric quasi-Banach sequence space such that \( \mathcal{H}^d : E(\mathbb{Z}) \to G(\mathbb{Z}) \) is bounded, and let us show that \( F(\mathbb{Z}) \subset G(\mathbb{Z}) \). If \( x \in E(\mathbb{Z}_+) \), then by [2, Proposition III. 4.10, p. 140] (here we again use the discrete version), there is a function \( y \in E(\mathbb{Z}) \) equimeasurable with \( x \) such that \( S^d \mu(x) \leq 2 \mu(\mathcal{H}^d y) \). Then

\[
\|S^d \mu(x)\|_{G(\mathbb{Z}_+)} \leq 2 \|\mu(\mathcal{H}y)\|_{G(\mathbb{Z}_+)} = 2 \|\mathcal{H}y\|_{G(\mathbb{Z})} 
\leq c_{abs} \|y\|_{E(\mathbb{Z})} = c_{abs} \|x\|_{E(\mathbb{Z}_+)},
\]

which shows that

\[
S^d : E(\mathbb{Z}_+) \to G(\mathbb{Z}_+)
\]

is bounded. But, since \( F(\mathbb{Z}_+) \) is the least space such that \( S^d : E(\mathbb{Z}_+) \to F(\mathbb{Z}_+) \) (see Theorem 6 (ii)), it follows that \( F(\mathbb{Z}) \subset G(\mathbb{Z}) \). This completes the proof. \( \square \)

A non-commutative extension of the following result was proved in [15, Proposition 35].

Proposition 11. Let \( E(\mathbb{Z}_+) = \ell_1, \infty(\mathbb{Z}_+) \), then

\[
F(\mathbb{Z}_+) = \left\{ a \in (\ell_1, \infty) + (\ell_\infty)(\mathbb{Z}_+) : \exists c_a, \mu(n, a) \leq c_a \frac{\log(n + 2)}{n + 1}, n \in \mathbb{Z}_+ \right\}.
\]

Indeed, if \( \mu(k, a) = \frac{1}{k+1} \), \( k \in \mathbb{Z}_+ \), then, by (2.3), we have

\[
(S^d \mu(a))(n) := \frac{1}{n + 1} \sum_{k=0}^{n} \frac{1}{k + 1} + \sum_{k=n+1}^{\infty} \frac{1}{k(k + 1)}
\]

and

\[
(S^d \mu(a))(n) \approx \frac{\log(n + 1)}{n + 1},
\]

for large \( n \), where the symbol \( \mathcal{A} \approx \mathcal{B} \) indicates that there exists universal positive constants \( c_1, c_2 \) independent of all important parameters such that \( \mathcal{A} \leq c_1 \mathcal{B} \) and \( \mathcal{B} \leq c_2 \mathcal{A} \). Therefore, if \( E(\mathbb{Z}_+) = \ell_1, \infty(\mathbb{Z}_+) \), then the optimal range for the discrete Calderón operator \( S^d \) is

\[
F(\mathbb{Z}_+) = \left\{ a \in (\ell_1, \infty) + (\ell_\infty)(\mathbb{Z}_+) : \exists c_a, \mu(n, a) \leq c_a \frac{\log(n + 2)}{n + 1}, n \in \mathbb{Z}_+ \right\},
\]

where \( c_a \) is a constant depending only on the sequence \( a \).

Remark 12. It follows from Theorem 10 and the previous result that if \( E(\mathbb{Z}) = \ell_1, \infty(\mathbb{Z}) \), then the optimal range for the discrete Hilbert transform \( \mathcal{H}^d \) is \( F(\mathbb{Z}) \), which is defined as in the Proposition 11.
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