Stochastic Hamiltonian dynamical systems

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Abstract

We use the global stochastic analysis tools introduced by P. A. Meyer and L. Schwartz to write down a stochastic generalization of the Hamilton equations on a Poisson manifold that, for exact symplectic manifolds, are characterized by a natural critical action principle similar to the one encountered in classical mechanics. Several features and examples in relation with the solution semimartingales of these equations are presented.

Keywords: stochastic Hamilton equations, stochastic variational principle, stochastic mechanics.

1 Introduction

The generalization of classical mechanics to the context of stochastic dynamics has been an active research subject ever since K. Itô introduced the theory of stochastic differential equations in the 1950s (see for instance \cite{Ne67, B81, Y81, ZY82, ZM84, TZ97, TZ97a, A03, CD06, BRO07, BRO07a}, and references therein). The motivations behind some pieces of work related to this field lay in the hope that a suitable stochastic generalization of classical mechanics should provide an explanation of the intrinsically random effects exhibited by quantum mechanics within the context of the theory of diffusions. In other instances the goal is establishing a framework adapted to the handling of mechanical systems subjected to random perturbations or whose parameters are not precisely determined and are hence modeled as realizations of a random variable.

Most of the pieces of work in the first category use a class of processes that have a stochastic derivative introduced in \cite{Ne67} and that has been subsequently refined over the years. This derivative can be used to formulate a real valued action and various associated variational principles whose extremals are the processes of interest.

The approach followed in this paper is closer to the one introduced in \cite{BS1} in which the action has its image in the space of real valued processes and the variations are taken in the space of processes with values in the phase space of the system that we are modeling. This paper can be actually seen as a generalization of some of the results in \cite{BS1} in the following directions:

(i) We make extensive use of the global stochastic analysis tools introduced by P. A. Meyer \cite{MS1, MS2} and L. Schwartz \cite{Sch82} to handle non-Euclidean phase spaces. This feature not only widens the spectrum of systems that can be handled but it is also of paramount importance at the time of reducing them with respect to the symmetries that they may eventually have (see \cite{LO07}); indeed, the orbit spaces obtained after reduction are generically non-Euclidean, even if the original phase space is.

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(ii) The stochastic dynamical components of the system are modeled by continuous semimartingales
and are not limited to Brownian motion.

(iii) We handle stochastic Hamiltonian systems on Poisson manifolds and not only on symplectic man-
ifolds.

(iv) The variational principle that we propose in Theorem 4.14 is not just satisfied by the stochastic
Hamiltonian equations (as in [BS1]) but fully characterizes them.

There are various reasons that have lead us to consider these generalized Hamiltonian systems.
First, even though the laws that govern the dynamics of classical mechanical systems are, in principle,
completely known, the finite precision of experimental measurements yields impossible the estimation of
the parameters of a particular given one with total accuracy. Second, the modeling of complex physical
systems involves most of the time simplifying assumptions or idealizations of parts of the system, some
of which could be included in the description as a stochastic component; this modeling philosophy has
been extremely successful in the social sciences [BJ76]. Third, even if the model and the parameters
of the system are known with complete accuracy, the solutions of the associated differential equations
may be of great complexity and exhibit high sensitivity to the initial conditions hence making the
probabilistic treatment and description of the solutions appropriate. Finally, we will see (Section 3.3)
how stochastic Hamiltonian modeling of microscopic systems can be used to model dissipation and
macroscopic damping.

The paper is structured as follows: in Section 2 we introduce the stochastic Hamilton equations
with phase space a given Poisson manifold and we study some of the fundamental properties of the
solution semimartingales like, for instance, the preservation of symplectic leaves or the characterization
of the conserved quantities. This section contains a discussion on two notions on non-linear stability,
almost sure Lyapunov stability and stability in probability, that reduce in the deterministic setup to
the standard definition of Lyapunov stability. We formulate criteria that generalize to the Hamiltonian
stochastic context the standard energy methods to conclude the stability of a Hamiltonian equilibrium
using existing conservation laws. More specifically, there are two different natural notions of conserved
quantity in the stochastic context that, via a stochastic Dirichlet criterion (Theorem 2.14) allow one
to conclude the different kinds of stability that we have mentioned above. Section 3 contains several
examples: in the first one we show how the systems studied by Bismut in [BS1] fall in the category
introduced in Section 2. We also see that a damped oscillator can be described as the average motion of
the solution semimartingale of a natural stochastic Hamiltonian system, and that Brownian motion in
a manifold is the projection onto the base space of very simple Hamiltonian stochastic semimartingale
defined on the cotangent bundle of the manifold or of its orthonormal frame bundle, depending on the
availability or not of a parallelization for the manifold in question. Section 4 is dedicated to showing
that the stochastic Hamilton equations are characterized by a critical action principle that generalizes
the one found in the treatment of deterministic systems. In order to make this part more readable, the
proofs of most of the technical results needed to prove the theorems in this section have been included
separately at the end of the paper.

One of the goals of this paper is conveying to the geometric mechanics community the plentitude of
global tools available to handle mechanical problems that contain a stochastic component and that do
not seem to have been exploited to the full extent of their potential. In order to facilitate the task of
understanding the paper to non-probabilists we have included an appendix that provides a self-contained
presentation of some major facts in stochastic calculus on manifolds needed for a first comprehension
of our results. Those pages are a very short and superficial presentation of a deep and technical field
of mathematics so the reader interested in a more complete account is encouraged to check with the
references quoted in the appendix and especially with the excellent monograph [E89].
Conventions: All the manifolds in this paper are finite dimensional, second-countable, locally compact, and Hausdorff (and hence paracompact).

2 The stochastic Hamilton equations

In this section we present a natural generalization of the standard Hamilton equations in the stochastic context. Even though the arguments gathered in the following paragraphs as motivation for these equations are of formal nature, we will see later on that, as it was already the case for the standard Hamilton equations, they satisfy a natural variational principle.

We recall that a symplectic manifold is a pair $(M, \omega)$, where $M$ is a manifold and $\omega \in \Omega^2(M)$ is a closed non-degenerate two-form on $M$, that is, $d\omega = 0$ and, for every $m \in M$, the map $v \in T_m M \mapsto \omega(m)(v, \cdot) \in T^*_m M$ is a linear isomorphism between the tangent space $T_m M$ to $M$ at $m$ and the cotangent space $T^*_m M$. Using the nondegeneracy of the symplectic form $\omega$, one can associate each function $h \in C^\infty(M)$ a vector field $X_h \in \mathfrak{X}(M)$, defined by the equality

$$i_{X_h} \omega = dh. \quad (2.1)$$

We will say that $X_h$ is the Hamiltonian vector field associated to the Hamiltonian function $h$. The expression $\mathfrak{X}(M)$ is referred to as the Hamilton equations.

A Poisson manifold is a pair $(M, \{\cdot, \cdot\})$, where $M$ is a manifold and $\{\cdot, \cdot\}$ is a bilinear operation on $C^\infty(M)$ such that $(C^\infty(M), \{\cdot, \cdot\})$ is a Lie algebra and $\{\cdot, \cdot\}$ is a derivation (that is, the Leibniz identity holds) in each argument. The functions in the center $C(M)$ of the Lie algebra $(C^\infty(M), \{\cdot, \cdot\})$ are called Casimir functions. From the natural isomorphism between derivations on $C^\infty(M)$ and vector fields on $M$ it follows that each $h \in C^\infty(M)$ induces a vector field on $M$ via the expression $X_h = \{\cdot, h\}$, called the Hamiltonian vector field associated to the Hamiltonian function $h$. Hamilton’s equations $\dot{z} = X_h(z)$ can be equivalently written in Poisson bracket form as $\dot{f} = \{f, h\}$, for any $f \in C^\infty(M)$.

The derivation property of the Poisson bracket implies that for any two functions $f, g \in C^\infty(M)$, the value of the bracket $\{f, g\}(z)$ at an arbitrary point $z \in M$ (and therefore $X_f(z)$ as well), depends on $f$ only through $df(z)$ which allows us to define a contravariant antisymmetric two–tensor $B \in \Lambda^2(M)$ by $B(z)(\alpha_z, \beta_z) = \{f, g\}(z)$, where $df(z) = \alpha_z \in T^*_z M$ and $dg(z) = \beta_z \in T^*_z M$. This tensor is called the Poisson tensor of $M$. The vector bundle map $B^\times : T^* M \to TM$ naturally associated to $B$ is defined by $B(z)(\alpha_z, \beta_z) = \langle \alpha_z, B(\beta_z) \rangle$.

We start by rewriting the solutions of the standard Hamilton equations in a form that we will be able to mimic in the stochastic differential equations context. All the necessary prerequisites on stochastic calculus on manifolds can be found in a short review in the appendix at the end of the paper.

Proposition 2.1 Let $(M, \omega)$ be a symplectic manifold and $h \in C^\infty(M)$. The smooth curve $\gamma : [0, T] \to M$ is an integral curve of the Hamiltonian vector field $X_h$ if and only if for any $\alpha \in \Omega(M)$ and for any $t \in [0, T]$

$$\int_{\gamma|_{[0, t]}} \alpha = -\int_0^t dh(\omega^s(\alpha)) \circ \gamma(s) ds, \quad (2.2)$$

where $\omega^s : T^* M \to TM$ is the vector bundle isomorphism induced by $\omega$. More generally, if $M$ is a Poisson manifold with bracket $\{\cdot, \cdot\}$ then the same result holds with (2.2) replaced by

$$\int_{\gamma|_{[0, t]}} \alpha = -\int_0^t dh(B^s(\alpha)) \circ \gamma(s) ds, \quad (2.3)$$

Lázaro and Ortega: Stochastic Hamiltonian dynamical systems
Proof. Since in the symplectic case \( \omega^t = B^t \), it suffices to prove (2.3). As (2.3) holds for any \( t \in [0,T] \), we can take derivatives with respect to \( t \) on both sides and we obtain the equivalent form

\[
\langle \alpha(\gamma(t)), \dot{\gamma}(t) \rangle = -\langle \delta h(\gamma(t)), B^t(\gamma(t))(\alpha(\gamma(t))) \rangle.
\] (2.4)

Let \( f \in C^\infty(M) \) be such that \( df(\gamma(t)) = \alpha(\gamma(t)) \). Then (2.4) can be rewritten as

\[
\langle df(\gamma(t)), \dot{\gamma}(t) \rangle = -\langle df(\gamma(t)), B^t(\gamma(t))(df(\gamma(t))) \rangle = \{f, h\}(\gamma(t)),
\]

which is equivalent to \( \dot{\gamma}(t) = X_h(\gamma(t)) \), as required. \( \blacksquare \)

We will now introduce the stochastic Hamilton equations by mimicking in the context of Stratonovich integration the integral expressions (2.2) and (2.3). In the next definition we will use the following notation: let \( f : M \to W \) be a differentiable function that takes values on the vector space \( W \). We define the \textbf{differential} \( df : TM \to W \) as the map given by \( df = p_2 \circ Tf \), where \( Tf : TM \to TW = W \times W \) is the tangent map of \( f \) and \( p_2 : W \times W \to W \) is the projection onto the second factor. If \( W = \mathbb{R} \) this definition coincides with the usual differential. If \( \{e_1, \ldots, e_n\} \) is a basis of \( W \) and \( f = \sum_{i=1}^n f^i e_i \) then \( df = \sum_{i=1}^n df^i \otimes e_i \).

\textbf{Definition 2.2} Let \( (M, \{\cdot, \cdot\}) \) be a Poisson manifold, \( X : \mathbb{R}_+ \times \Omega \to V \) a semimartingale that takes values on the vector space \( V \) with \( X_0 = 0 \), and \( h : M \to V^\ast \) a smooth function. Let \( \{e^1, \ldots, e^r\} \) be a basis of \( V^\ast \) and \( h = \sum_{i=1}^r h_i e^i \). The \textbf{Hamilton equations with stochastic component} \( X \), and \textbf{Hamiltonian function} \( h \) are the Stratonovich stochastic differential equation

\[
\delta \Gamma^h = H(X, \Gamma) \delta X,
\] (2.5)

defined by the \textit{Stratonovich operator} \( H(v, z) : T_v V \to T_z M \) given by

\[
H(v, z)(u) := \sum_{j=1}^r \{e^j, u\} X_{h_j}(z).
\] (2.6)

The dual Stratonovich operator \( H^\ast(v, z) : T_z M \to T_v V \) of \( H(v, z) \) is given by \( H^\ast(v, z)(\alpha_z) = -\delta h(z) \cdot B^t(z)(\alpha_z) \). Hence, the results quoted in Appendix 6.1 show that for any \( \mathcal{F}_0 \) measurable random variable \( \Gamma_0 \), there exists a unique semimartingale \( \Gamma^h \) such that \( \Gamma^h_0 = \Gamma_0 \) and a maximal stopping time \( \zeta^h \) that solve (2.5), that is, for any \( \alpha \in \Omega(M) \),

\[
\int (\alpha, \delta \Gamma^h) = -\int \langle \delta h(B^t(\alpha))(\Gamma^h), \delta X \rangle.
\] (2.7)

We will refer to \( \Gamma^h \) as the \textbf{Hamiltonian semimartingale} associated to \( h \) with initial condition \( \Gamma_0 \).

\textbf{Remark 2.3} The stochastic component \( X \) encodes the random behavior exhibited by the stochastic Hamiltonian system that we are modeling and the Hamiltonian function \( h \) specifies how it embeds in its phase space. Unlike the situation encountered in the deterministic setup we allow the Hamiltonian function to be vector valued in order to accommodate higher dimensional stochastic dynamics.

\textbf{Remark 2.4} The generalization of Hamilton’s equations proposed in Definition 2.2 by using a Stratonovich operator is inspired by one of the transfer principles presented in [E90] to provide stochastic versions of ordinary differential equations. This procedure can be also used to carry out a similar generalization of the equations induced by a Leibniz bracket (see [OP12]).
Remark 2.5 Stratonovich versus Itô integration: at the time of proposing the equations in Definition 2.2 a choice has been made, namely, we have chosen Stratonovich integration instead of Itô or other kinds of stochastic integration. The option that we took is motivated by the fact that by using Stratonovich integration, most of the geometric features underlying classical deterministic Hamiltonian mechanics are preserved in the stochastic context (see the next section). Additionally, from the mathematical point of view, this choice is the most economical one in the sense that the classical geometric ingredients of Hamiltonian mechanics plus a noise semimartingale suffice to construct the equations; had we used Itô integration we would have had to provide a Schwartz operator (see Section 6.4) and the construction of such an object via a transfer principle like in [E90] involves the choice of a connection.

The use of Itô integration in the modeling of physical phenomena is sometimes preferred because the definition of this integral is not *anticipative*, that is, it does not assume any knowledge about the behavior of the system in future times. Even though we have used Stratonovich integration to write down our equations, we also share this feature because the equations in Definition 2.2 can be naturally translated to the Itô framework (see Proposition 2.8). This is a particular case of a more general fact since given any Stratonovich stochastic differential equation there always exists an equivalent Itô stochastic differential equation, in the sense that both equations have the same solutions. Note that the converse is in general not true.

2.1 Elementary properties of the stochastic Hamilton’s equations

Proposition 2.6 Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold, \(X : \mathbb{R}_+ \times \Omega \to V\) a semimartingale that takes values on the vector space \(V\) with \(X_0 = 0\) and \(h : M \to V^*\) a smooth function. Let \(\Gamma_0\) be a \(\mathcal{F}_0\) measurable random variable and \(\Gamma^h\) the Hamiltonian semimartingale associated to \(h\) with initial condition \(\Gamma_0\). Let \(\zeta^h\) be the corresponding maximal stopping time. Then, for any stopping time \(\tau < \zeta^h\), the Hamiltonian semimartingale \(\Gamma^h\) satisfies

\[
f(\Gamma^h_{\tau}) - f(\Gamma^h_0) = \sum_{j=1}^r \int_0^\tau \{f, h_j\}(\Gamma^h) \delta X^j, \tag{2.8}
\]

where \(\{h_j\}_{j \in \{1, \ldots, r\}}\) and \(\{X^j\}_{j \in \{1, \ldots, r\}}\) are the components of \(h\) and \(X\) with respect to two given dual bases \(\{e_1, \ldots, e_r\}\) and \(\{e^1, \ldots, e^r\}\) of \(V\) and \(V^*\), respectively. Expression (2.8) can be rewritten in differential notation as

\[
\delta f(\Gamma^h) = \sum_{j=1}^r \{f, h_j\}(\Gamma^h) \delta X^j.
\]

Proof. It suffices to take \(\alpha = df\) in (2.7). Indeed, by (6.5)

\[
\int_0^\tau (df, \delta \Gamma^h) = f(\Gamma^h_{\tau}) - f(\Gamma^h_0).
\]

At the same time

\[- \int_0^\tau \langle dh(B^2(df))(\Gamma^h), \delta X \rangle = - \sum_{j=1}^r \int_0^\tau \langle (dh_j \otimes e^i(B^2(df))(\Gamma^h), \delta X \rangle = \sum_{j=1}^r \int_0^\tau \langle \{f, h_j\}(\Gamma^h)e^i, \delta X \rangle.
\]

By the second statement in (6.3) this equals \(\sum_{j=1}^r \int_0^\tau \{f, h_j\}(\Gamma^h) \delta \langle e^i, \delta X \rangle\). Given that \(\int \langle e^i, \delta X \rangle = X^i - X^i_0\), the equality follows. ■
Remark 2.7 Notice that if in Definition 2.2 we take $V^* = \mathbb{R}$, $h \in C^\infty(M)$, and $X : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ the deterministic process given by $(t, \omega) \mapsto t$, then the stochastic Hamilton equations (2.4) reduce to

$$\int \langle \alpha, \delta \Gamma^h \rangle = \int \langle \alpha, X^h \rangle (\Gamma^h) \, dt.$$  \hspace{1cm} (2.9)

A straightforward application of (2.8) shows that $\Gamma^h_t(\omega)$ is necessarily a differentiable curve, for any $\omega \in \Omega$, and hence the Riemann-Stieltjes integral in the left hand side of (2.9) reduces, when evaluated at a given $\omega \in \Omega$, to a Riemann integral identical to the one in the left hand side of (2.3), hence proving that (2.9) reduces to the standard Hamilton equations.

Indeed, let $\Gamma^h_{t_0}(\omega) \in M$ be an arbitrary point in the curve $\Gamma^h_t(\omega)$, let $U$ be a coordinate patch around $\Gamma^h_{t_0}(\omega)$ with coordinates $\{x^1, \ldots, x^n\}$, and let $x(t) = (x^1(t), \ldots, x^n(t))$ be the expression of $\Gamma^h_t(\omega)$ in these coordinates. Then by (2.8), for $h \in \mathbb{R}$ sufficiently small, and $i \in \{1, \ldots, n\}$,

$$x^i(t_0 + h) - x^i(t_0) = \int_{t_0}^{t_0 + h} \{x^i, h\}(x(t)) \, dt.$$

Hence, by the Fundamental Theorem of Calculus, $x^i(t)$ is differentiable at $t_0$, with derivative

$$\dot{x}^i(t_0) = \lim_{h \to 0} \frac{1}{h} \left( x^i(t_0 + h) - x^i(t_0) \right) = \lim_{h \to 0} \frac{1}{h} \left( \int_{t_0}^{t_0 + h} \{x^i, h\}(x(t)) \, dt \right) = \{x^i, h\}(x(t_0)),$$

as required.

The following proposition provides an equivalent expression of the Stochastic Hamilton equations in the Itô form (see Section 6.4).

Proposition 2.8 The stochastic Hamilton’s equations in Definition 2.2 admit an equivalent description using Itô integration by using the Schwartz operator $\mathcal{H}(v, m) : \tau_v V \rightarrow \tau_m M$ naturally associated to the Hamiltonian Stratonovich operator $H$ and that can be described as follows. Let $L \in \tau_v M$ be a second order vector and $f \in C^\infty(M)$ arbitrary, then

$$\mathcal{H}(v, m) \langle L \rangle [f] = \left\langle \sum_{i,j=1}^r \{f, h_j\}(m) e^j + \{f, h_j\}, h_i\right\rangle (m) e^i \cdot e^j, L \right\rangle.$$

Moreover, expression (2.8) in the Itô representation is given by

$$f(\Gamma^h_t) - f(\Gamma^h_0) = \sum_{j=1}^r \int_0^t \{f, h_j\} (\Gamma^h) \, dX^j + \frac{1}{2} \sum_{j,i=1}^r \int_0^t \{\{f, h_j\}, h_i\} (\Gamma^h) \, d[X^j, X^i].$$  \hspace{1cm} (2.10)

We will refer to $\mathcal{H}$ as the Hamiltonian Schwartz operator associated to $h$.

Proof. According to the remarks made in the Appendix 6.4, the Schwartz operator $\mathcal{H}$ naturally associated to $H$ is constructed as follows. For any second order vector $L_v \in \tau_v M$ associated to the acceleration of a curve $v(t)$ in $V$ such that $v(0) = v$ we define $\mathcal{H}(v, m) \langle L_v \rangle := L_{m(0)} \in \tau_m M$, where $m(t)$ is a curve in $M$ such that $m(0) = m$ and $m(t) = H(v(t), m(t)) \dot{v}(t)$, for $t$ in a neighborhood of
0. Consequently,
\[
\mathcal{H}(v, m)(L_v) [f] = \frac{d^2}{dt^2} \bigg|_{t=0} f(m(t)) = \frac{d}{dt} \bigg|_{t=0} df(m(t), \dot{m}(t)) = \frac{d}{dt} \bigg|_{t=0} \langle df(m(t), H(v(t), m(t)) \dot{v}(t) angle
\]
\[
= \frac{d}{dt} \bigg|_{t=0} \sum_{j=1}^{r} \langle \epsilon^j, \dot{v}(t) \rangle \langle df(m(t), X_{h_j}(m(t)) \rangle = \frac{d}{dt} \bigg|_{t=0} \sum_{j=1}^{r} \langle \epsilon^j, \dot{v}(t) \rangle \{f, h_j\}(m(t))
\]
\[
= \sum_{j=1}^{r} \langle \epsilon^j, \dot{v}(0) \rangle \{f, h_j\}(m) + \langle \epsilon^j, \dot{v}(0) \rangle \{df\{f, h_j\}(m), \dot{m}(0)\}
\]
\[
= \sum_{j=1}^{r} \langle \epsilon^j, \dot{v}(0) \rangle \{f, h_j\}(m) + \langle \epsilon^j, \dot{v}(0) \rangle \sum_{i=1}^{r} \langle \epsilon^j, \dot{v}(0) \rangle \{\{f, h_j\}, h_i\}(m)
\]
\[
= \left\langle \sum_{i,j=1}^{r} \{f, h_j\}(m) \epsilon^i + \{\{f, h_j\}, h_i\}(m) \epsilon^i \cdot \epsilon^j, L_v \right\rangle.
\]

In order to establish (2.10) we need to calculate \(\mathcal{H}^*(v, m)(d_2 f(m))\) for a second order form \(d_2 f(m) \in \tau^*_m \mathcal{M}\) at \(m \in \mathcal{M}, f \in C^\infty(\mathcal{M})\). Since \(\mathcal{H}^*(v, m)(d_2 f(m))\) is fully characterized by its action on elements of the form \(L_v \in \tau_v \mathcal{V}\) for some curve \(v(t)\) in \(\mathcal{V}\) such that \(v(0) = v\), we have
\[
\langle \mathcal{H}^*(v, m)(d_2 f(m)), L_v \rangle = \langle d_2 f(m), \mathcal{H}(v, m)(L_v) \rangle = \mathcal{H}(v, m)(L_v) [f]
\]
\[
= \left\langle \sum_{i,j=1}^{r} \{f, h_j\}(m) \epsilon^i + \{\{f, h_j\}, h_i\}(m) \epsilon^i \cdot \epsilon^j, L_v \right\rangle.
\]

Consequently, \(\mathcal{H}^*(v, m)(d_2 f(m)) = \sum_{i,j=1}^{r} \{f, h_j\}(m) \epsilon^i + \{\{f, h_j\}, h_i\}(m) \epsilon^i \cdot \epsilon^j\).

Hence, if \(\Gamma_h\) is the Hamiltonian semimartingale associated to \(h\) with initial condition \(\Gamma_0, \tau < \zeta_h\) is any stopping time, and \(f \in C^\infty(\mathcal{M})\), we have by (2.9) and (2.10)
\[
f(\Gamma_h^\tau) - f(\Gamma_0^\tau) = \int_0^\tau \langle d_2 f, d\Gamma_h \rangle = \int_0^\tau \langle \mathcal{H}^*(X, \Gamma_h^\tau)(d_2 f), dX \rangle
\]
\[
= \sum_{j=1}^{r} \int_0^\tau \langle \{f, h_j\}(\Gamma_h^\tau), dX^j \rangle + \sum_{j=1}^{r} \int_0^\tau \langle \{\{f, h_j\}, h_i\}(\Gamma_h^\tau), dX^i \cdot dX^j \rangle
\]
\[
= \sum_{j=1}^{r} \int_0^\tau \{f, h_j\}(\Gamma_h^\tau) dX^j + \frac{1}{2} \sum_{j=1}^{r} \int_0^\tau \{f, h_j\}, h_i\}(\Gamma_h^\tau) d[X^i, X^j].
\]

**Proposition 2.9 (Preservation of the symplectic leaves by Hamiltonian semimartingales)** In the setup of Definition 2.2 let \(\mathcal{L}\) be a symplectic leaf of \((\mathcal{M}, \omega)\) and \(\Gamma_h\) a Hamiltonian semimartingale with initial condition \(\Gamma_0(\omega) = Z_0\), where \(Z_0\) is a random variable such that \(Z_0(\omega) \in \mathcal{L}\) for all \(\omega \in \Omega\). Then, for any stopping time \(\tau < \zeta_h\) we have that \(\Gamma_h^\tau \in \mathcal{L}\).

**Proof.** Expression (2.10) shows that for any \(z \in \mathcal{L}\), the Stratonovich operator \(H(v, z)\) takes values in the characteristic distribution associated to the Poisson structure \((\mathcal{M}, \{\cdot, \cdot\})\), that is, in the tangent space \(T_z \mathcal{L}\) of \(\mathcal{L}\). Consequently, \(H\) induces another Stratonovich operator \(H_{\mathcal{L}}(v, z) : T_v \mathcal{V} \rightarrow T_z \mathcal{L}, v \in \mathcal{V}, z \in \mathcal{L}\), obtained from \(H\) by restriction of its range. It is clear that if \(i : \mathcal{L} \hookrightarrow \mathcal{M}\) is the inclusion then
\[
H_{\mathcal{L}}(v, z) \circ T_z i = H^*(v, z).
\]
(2.11)

Let \(\Gamma_h^\tau\) be the semimartingale in \(\mathcal{L}\) that is a solution of the Stratonovich stochastic differential equation
\[
\delta \Gamma_h^\tau = H_{\mathcal{L}}(X, \Gamma_h^\tau) dX
\]
(2.12)
with initial condition \( \Gamma_0 \). We now show that \( \overline{\Gamma} := i \circ \Gamma_h^\tau \) is a solution of
\[
\delta \overline{\Gamma} = H(X, \overline{\Gamma}) \delta X.
\]
The uniqueness of the solution of a stochastic differential equation will guarantee in that situation that \( \Gamma^h \) necessarily coincides with \( \overline{\Gamma} \), hence proving the statement. Indeed, for any \( \alpha \in \Omega(M) \),
\[
\int \langle \alpha, \delta \overline{\Gamma} \rangle = \int \langle \alpha, \delta (i \circ \Gamma_h^\tau) \rangle = \int \langle T^* \alpha \cdot \alpha, \delta \Gamma_h^\tau \rangle.
\]
Since \( \Gamma_h^\tau \) satisfies (2.12) and \( T^* \alpha \cdot \alpha \in \Omega(\mathcal{L}) \), by (2.11) this equals
\[
\int \langle H^*_\mathcal{L}(X, \Gamma_h^\tau) (T^* \alpha \cdot \alpha), \delta X \rangle = \int \langle H^*(X, i \circ \Gamma_h^\tau)(\alpha), \delta X \rangle = \int \langle H^*(X, \Gamma)(\alpha), \delta X \rangle,
\]
that is, \( \delta \overline{\Gamma} = H(X, \overline{\Gamma}) \delta X \), as required. \( \blacksquare \)

**Proposition 2.10 (The stochastic Hamilton equations in Darboux-Weinstein coordinates)**

Let \((M, \{\cdot, \cdot\})\) be a Poisson manifold and \( \Gamma^h \) be a solution of the Hamilton equations (2.5) with initial condition \( x_0 \in M \). There exists an open neighborhood \( U \) of \( x_0 \) in \( M \) and a stopping time \( \tau_U \) such that \( \Gamma^h_\tau(\omega) \in U \), for any \( \omega \in \Omega \) and any \( t \leq \tau_U(\omega) \). Moreover, \( U \) admits local Darboux coordinates \((q^1, \ldots, q^n, p_1, \ldots, p_n, z_1, \ldots, z_l)\) in which (2.8) takes the form
\[
q^j(\Gamma_h^\tau) - q^j(\Gamma_0^h) = \sum_{j=1}^r \int_0^\tau \frac{\partial h_j}{\partial q^j} \delta X^j,
\]
\[
p_i(\Gamma_h^\tau) - p_i(\Gamma_0^h) = -\sum_{j=1}^r \int_0^\tau \frac{\partial h_j}{\partial q^j} \delta X^j,
\]
\[
z_i(\Gamma_h^\tau) - z_i(\Gamma_0^h) = \sum_{j=1}^r \int_0^\tau \{z_i, h_j\}_T \delta X^j,
\]
where \( \{\cdot, \cdot\}_T \) is the transverse Poisson structure of \((M, \{\cdot, \cdot\})\) at \( x_0 \).

**Proof.** Let \( U \) be an open neighborhood of \( x_0 \) in \( M \) for which Darboux coordinates can be chosen. Define \( \tau_U = \inf_{t \geq 0} \{t \in U^c\} \) (\( \tau_U \) is the exit time of \( U \)). It is a standard fact in the theory of stochastic processes that \( \tau_U \) is a stopping time. The proposition follows by writing (2.8) for the Darboux-Weinstein coordinate functions \((q^1, \ldots, q^n, p_1, \ldots, p_n, z_1, \ldots, z_l)\).

Let \( \zeta : M \times \Omega \rightarrow [0, \infty] \) be the map such that, for any \( z \in M \), \( \zeta(z) \) is the maximal stopping time associated to the solution of the stochastic Hamilton equations (2.5) with initial condition \( \Gamma_0 = z \) a.s.. Let \( F \) be the flow of (2.5), that is, for any \( z \in M \), \( F(z) : [0, \zeta(z)] \rightarrow M \) is the solution semimartingale of (2.5) with initial condition \( z \). The map \( z \in M \mapsto F_t(z, \omega) \in M \) is a local diffeomorphism of \( M \), for each \( t \geq 0 \) and almost all \( \omega \in \Omega \) in which this map is defined (see [IW89]). In the following result, we show that, in the symplectic context, Hamiltonian flows preserve the symplectic form and hence the associated volume form \( \theta = \omega \wedge \cdots \wedge \omega \). This has already been shown for Hamiltonian diffusions (see Example 2.1 by Bismut [BS1]).

**Theorem 2.11 (Stochastic Liouville’s Theorem)** Let \((M, \omega)\) be a symplectic manifold, \( X : \mathbb{R}_+ \times \Omega \rightarrow V^* \) a semimartingale, and \( h : M \rightarrow V^* \) a Hamiltonian function. Let \( F \) be the associated Hamiltonian flow. Then, for any \( z \in M \) and any \((t, \eta) \in [0, \zeta(z)]\),
\[
F_t^*(z, \eta) \omega = \omega.
\]
Proof. By [KS1] Theorem 3.3 (see also [W80]), given an arbitrary form \( \alpha \in \Omega^k(M) \) and \( z \in M \), the process \( F(z)^* \alpha \) satisfies the following stochastic differential equation:

\[
F(z)^* \alpha = \alpha(z) + \sum_{j=1}^{r} \int F(z)^*(\mathcal{L}_{X_{h_j}} \alpha) \delta X^j.
\]

In particular, if \( \alpha = \omega \) then \( \mathcal{L}_{X_{h_j}} \omega = 0 \) for any \( j \in \{1, ..., r\} \), and hence the result follows. \( \blacksquare \)

### 2.2 Conserved quantities and stability

Conservation laws in Hamiltonian mechanics are extremely important since they make easier the integration of the systems that have them and, in some instances, provide qualitative information about the dynamics. A particular case of this is their use in concluding the nonlinear stability of certain systems that have them and, in some instances, provide qualitative information about their behavior.

**Definition 2.12** A function \( f \in C^\infty(M) \) is said to be a **strongly** (respectively, **weakly**) **conserved quantity** of the stochastic Hamiltonian system associated to \( h : M \to V^* \) if for any solution \( \Gamma^h \) of the stochastic Hamilton equations (2.5) we have that \( f(\Gamma^h) = f(\Gamma^h_0) \) (respectively, \( E[f(\Gamma^h_t)] = E[f(\Gamma^h_0)] \)), for any stopping time \( \tau \).

Notice that strongly conserved quantities are obviously weakly conserved and that the two definitions coincide for deterministic systems with the standard definition of conserved quantity. The following result provides in the stochastic setup an analogue of the classical characterization of the conserved quantities in terms of Poisson involution properties.

**Proposition 2.13** Let \( (M, \{\cdot, \cdot\}) \) be a Poisson manifold, \( X : \mathbb{R}_+ \times \Omega \to V \) a semimartingale that takes values on the vector space \( V \) such that \( X_0 = 0 \), and \( h : M \to V^* \) and \( f \in C^\infty(M) \) two smooth functions. If \( \{f, h_j\} = 0 \) for every component \( h_j \) of \( h \) then \( f \) is a strongly conserved quantity of the stochastic Hamilton equations (2.5).

Conversely, suppose that the semimartingale \( X = \sum_{j=1}^{r} X^j \epsilon_j \) is such that \( [X^i, X^j] = 0 \) if \( i \neq j \). If \( f \) is a strongly conserved quantity then \( \{f, h_j\} = 0 \), for any \( j \in \{1, ..., r\} \) such that \( [X^j, X^j] \) is an strictly increasing process at 0. The last condition means that there exists \( A \in \mathcal{F} \) and \( \delta > 0 \) with \( P(A) > 0 \) such that for any \( t < \delta \) and \( \omega \in A \) we have \( [X^j, X^j]_t(\omega) > [X^j, X^j]_0(\omega) \), for all \( j \in \{1, ..., r\} \).

**Proof.** Let \( \Gamma^h \) be the Hamiltonian semimartingale associated to \( h \) with initial condition \( \Gamma^h_0 \). As we saw in (2.11),

\[
f(\Gamma^h) = f(\Gamma^h_0) + \sum_{j=1}^{r} \int \{f, h_j\}(\Gamma^h) dX^j + \frac{1}{2} \sum_{j, i=1}^{r} \int \{\{f, h_j\}, h_i\}(\Gamma^h) d[X^j, X^i]. \tag{2.13}
\]

If \( \{f, h_j\} = 0 \) for every component \( h_j \) of \( h \) then all the integrals in the previous expression vanish and therefore \( f(\Gamma^h) = f(\Gamma^h_0) \) which implies that \( f \) is a strongly conserved quantity of the Hamiltonian stochastic equations associated to \( h \).

Conversely, suppose now that \( f \) is a strongly conserved quantity. This implies that for any initial condition \( \Gamma^h_0 \), the semimartingale \( f(\Gamma^h) \) is actually time independent and hence of finite variation. Equivalently, the (unique) decomposition of \( f(\Gamma^h) \) into two processes, one of finite variation plus a local martingale, only has the first term. In order to isolate the local martingale term of \( f(\Gamma^h) \) recall first that the quadratic variations \( [X^j, X^j] \) have finite variation and that the integral with respect to a finite variation process has finite variation (see [LeG97 Proposition 4.3]). Consequently, the last
summand in (2.13) has finite variation. As to the second summand, let \( M^j \) and \( A^j \), \( j = 1, \ldots, r \), local martingales and finite variation processes, respectively, such that \( X^j = A^j + M^j \). Then,

\[
\int \{ f, h_j \} (\Gamma^h) dX^j = \int \{ f, h_j \} (\Gamma^h) dM^j + \int \{ f, h_j \} (\Gamma^h) dA^j.
\]

Given that for each \( j \), \( \int \{ f, h_j \} (\Gamma^h) dA^j \) is a finite variation process and \( \int \{ f, h_j \} (\Gamma^h) dM^j \) is a local martingale (see [P90, Theorem 29, page 128]) we conclude that \( Z := \sum_{j=1}^r \int \{ f, h_j \} (\Gamma^h) dM^j \) is the local martingale term of \( f (\Gamma^h) \) and hence equal to zero.

We notice now that any continuous local martingale \( Z : \mathbb{R}_+ \times \Omega \to \mathbb{R} \) is also a local \( L^2 (\Omega) \)-martingale. Indeed, consider the sequence of stopping times \( \tau^n = \{ \inf t \geq 0 \mid |Z_t| = n \} \), \( n \in \mathbb{N} \). Then \( E \left[ (Z_{\tau^n})_t^2 \right] \leq E [n^2] = n^2 \), for all \( t \in \mathbb{R}_+ \). Hence, \( Z_{\tau^n} \in L^2 (\Omega) \) for any \( n \). In addition, \( E \left[ (Z_{\tau^n})_t \right] = E \left[ [Z_{\tau^n}, Z_{\tau^n}]_t \right] \) (see [P90] Corollary 3, page 73). On the other hand by Proposition 5.9,

\[
Z_{\tau^n} = \left( \sum_{j=1}^r \int \{ f, h_j \} (\Gamma^h) dM^j \right)_{\tau^n} = \sum_{j=1}^r \int 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) dM^j.
\]

Thus, by [P90] Theorem 29, page 75] and the hypothesis \( [X^i, X^j] = 0 \) if \( i \neq j \),

\[
E \left[ (Z_{\tau^n})_t^2 \right] = E \left[ [Z_{\tau^n}, Z_{\tau^n}]_t \right] = \sum_{j,i=1}^r E \left[ \int 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) dM^j \right] \int 1_{[0, \tau^n]} \{ f, h_i \} (\Gamma^h) dM^j
\]

\[
= \sum_{j,i=1}^r E \left[ \left( \int 1_{[0, \tau^n]} \{ f, h_j \} \{ f, h_i \} (\Gamma^h) d[M^j, M^i] \right) \right]
\]

\[
= \sum_{j,i=1}^r E \left[ \left( \int 1_{[0, \tau^n]} \{ f, h_j \} \{ f, h_i \} (\Gamma^h) d[X^j, X^i] \right) \right]
\]

\[
= \sum_{j=1}^r E \left[ \left( \int 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) d[X^j, X^j] \right) \right].
\]

Since \( [X^j, X^j] \) is an increasing process of finite variation then \( \int 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) d[X^j, X^j] \) is a Riemann-Stieltjes integral and hence for any \( \omega \in \Omega \)

\[
\left( \int 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) d[X^j, X^j] \right) (\omega) = \left( \int 1_{[0, \tau^n(\omega)]} \{ f, h_j \} (\Gamma^h (\omega)) d([X^j, X^j] (\omega)) \right).
\]

As \( [X^j, X^j] (\omega) \) is an increasing function of \( t \in \mathbb{R}_+ \), then for any \( j \in \{1, \ldots, r\} \)

\[
E \left[ \int 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) d[X^j, X^j] \right] \geq 0. \tag{2.14}
\]

Additionally, since \( E \left[ (Z_{\tau^n})_t^2 \right] = 0 \), we necessarily have that the inequality in (2.14) is actually an equality. Hence,

\[
\int_{\tau^n}^t 1_{[0, \tau^n]} \{ f, h_j \} (\Gamma^h) d[X^j, X^j] = 0. \tag{2.15}
\]

Suppose now that \( [X^j, X^j] \) is strictly increasing at 0 for a particular \( j \). Hence, there exists \( A \in \mathcal{F} \) with \( P(A) > 0 \), and \( \delta > 0 \) such that \( [X^j, X^j]_t (\omega) > [X^j, X^j]_0 (\omega) \) for any \( t < \delta \). Take now a
fixed $\omega \in A$. Since $\tau_n \to \infty$ a.s., we can take $n$ large enough to ensure that $\tau^n(\omega) > t$, where $t \in [0, \delta)$. Thus, we may suppose that $1_{[0,\tau^n]}(t, \omega) = 1$. As $[X^j, X^j](\omega)$ is an strictly increasing process at zero $\int_0^t (f, h_j)^2 (\Gamma^h(\omega)) d [X^j, X^j](\omega) > 0$ unless $(f, h_j)^2 (\Gamma^h(\omega)) = 0$ in a neighborhood $[0, \delta_\omega]$ of 0 contained in $[0, \delta]$. In principle $\delta_\omega > 0$ might depend on $\omega \in A$, so the values of $t \in [0, \delta)$ for which $(f, h_j)^2 (\Gamma^h(\omega)) = 0$ for any $\omega \in A$ are those verifying $0 < t \leq \inf_{\omega \in A} \delta_\omega$. In any case (2.15) allows us to conclude that $(f, h_j)^2 (\Gamma^h(\omega)) = 0$ for any $\omega \in A$. Finally, consider any $\Gamma^h$ solution to the Stochastic Hamilton equations with constant initial condition $\Gamma^h_0 = m \in M$ an arbitrary point. Then, for any $\omega \in A$,

$$0 = (f, h_j)^2 (\Gamma^h(\omega)) = (f, h_j)^2 (m).$$

Since $m \in M$ is arbitrary we can conclude that $(f, h_j) = 0$.  

We now use the conserved quantities of a system in order to formulate sufficient Dirichlet type stability criteria. Even though the statements that follow are enounced for processes that are not necessarily Hamiltonian, it is for these systems that the criteria are potentially most useful. We start by spelling out the kind of nonlinear stability that we are after.

**Definition 2.14** Let $M$ be a manifold and let

$$\delta \Gamma = e(X, \Gamma) \delta X$$

be a Stratonovich stochastic differential equation whose solutions $\Gamma : \mathbb{R} \times \Omega \to M$ take values on $M$. Given $x \in M$ and $s \in \mathbb{R}$, denote by $\Gamma_s^x$ the unique solution of (2.10) such that $\Gamma_s^x(\omega) = x$, for all $\omega \in \Omega$. Suppose that the point $z_0 \in M$ is an equilibrium of (2.10), that is, the constant process $\Gamma(\omega) := z_0$, for all $t \in \mathbb{R}$ and $\omega \in \Omega$, is a solution of (2.10). Then we say that the equilibrium $z_0$ is

(i) **Almost surely (Lyapunov) stable** when for any open neighborhood $U$ of $z_0$ there exists another neighborhood $V \subset U$ such that for any $z \in V$ we have $\Gamma^{0,x} \subset U$, a.s.

(ii) **Stable in probability.** For any $s \geq 0$ and $\epsilon > 0$

$$\lim_{x \to z_0} P \left\{ \sup_{t \geq s} d(\Gamma_t^{x,x}, z_0) > \epsilon \right\} = 0,$$

where $d : M \times M \to \mathbb{R}$ is any distance function that generates the manifold topology of $M$.

**Theorem 2.15 (Stochastic Dirichlet’s Criterion)** Suppose that we are in the setup of the previous definition and assume that there exists a function $f \in C^\infty(M)$ such that $df(z_0) = 0$ and that the quadratic form $d^2 f(z_0)$ is (positive or negative) definite. If $f$ is a strongly (respectively, weakly) conserved quantity for the solutions of (2.10) then the equilibrium $z_0$ is almost surely stable (respectively, stable in probability).

**Proof.** Since the stability of the equilibrium $z_0$ is a local statement, we can work in a chart of $M$ around $z_0$ with coordinates $(x_1, \ldots, x_n)$ in which $z_0$ is modeled by the origin. Moreover, using the Morse lemma and the hypotheses on the function $f$, and assuming without loss of generality that $f(z_0) = 0$, we choose the coordinates $(x_1, \ldots, x_n)$ so that $f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. Hence, in the definition of stability in probability, we can use the distance function $d(x, z_0) = f(x)$.

Suppose now that $f$ is a strongly conserved quantity and let $U$ be an open neighborhood of $z_0$. Let $r > 0$ be such that $V := f^{-1}(0, r) \subset U$. Let $z \in V$ with $f(z) = r$. As $f$ is a strongly conserved quantity $f(\Gamma^{0,x}) = r' \leq r$ and hence $\Gamma^{0,x} \subset U$, as required.

In order to study the case in which $f$ is a weakly conserved quantity, let $\epsilon > 0$ and let $U_\epsilon$ be the ball of radius $\epsilon$ around $z_0$. Then, for any $x \in U_\epsilon$ and $s \in \mathbb{R}_+$, let $\tau_{U_\epsilon}$ be the first exit time of $\Gamma^{s,x}$ with
respect to $U_\epsilon$. Notice first that if $\omega \in \Omega$ belongs to the set $\{ \omega \in \Omega \mid \sup_{0 \leq s < t} d (\Gamma^{s,x}_t, z_0) > \epsilon \} = \{ \omega \in \Omega \mid \sup_{0 \leq s < t} f (\Gamma^{s,x}_t) > \epsilon^2 \}$, then $\tau_{U_\epsilon} (\omega) \leq t$ and hence the stopped process $(\Gamma^{s,x})_{\tau_{U_\epsilon}}$ satisfies that
\[
 f ( (\Gamma^{s,x})_{\tau_{U_\epsilon}} (\omega) ) = f ( \Gamma^{s,x}_{\tau_{U_\epsilon}} (\omega) ) = \epsilon^2,
\]
for those values of $\omega$. This ensures that
\[
 \epsilon^2 1_{\{ \omega \in \Omega \mid \sup_{0 \leq s < t} d (\Gamma^{s,x}_t, z_0) > \epsilon \}} \leq f ( (\Gamma^{s,x})_{\tau_{U_\epsilon}} ) .
\]
Taking expectations in both sides of this inequality we obtain
\[
 P \left( \sup_{0 \leq s < t} d (\Gamma^{s,x}_t, z_0) > \epsilon \right) \leq E[ f ( (\Gamma^{s,x})_{\tau_{U_\epsilon}} ) ] / \epsilon^2 .
\]
Since by hypothesis $f$ is a weakly conserved quantity, we can rewrite the right hand side of this inequality as
\[
 E[ f ( (\Gamma^{s,x})_{\tau_{U_\epsilon}} ) ] = E \left[ \frac{f ( \Gamma^{s,x}_{\tau_{U_\epsilon}} )}{\epsilon^2} \right] = \frac{E[ f (\Gamma^{s,x}) ]}{\epsilon^2} = \frac{f (x)}{\epsilon^2} ,
\]
and we can therefore conclude that
\[
 P \left( \sup_{0 \leq s < t} d (\Gamma^{s,x}_t, z_0) > \epsilon \right) \leq \frac{f (x)}{\epsilon^2} . \tag{2.17}
\]
Taking the limit $x \to z_0$ in this expression and recalling that $f (z_0) = 0$, the result follows. ■

A careful inspection of the proof that we just carried out reveals that in order for (2.17) to hold, it would suffice to have $E[ f (\Gamma_\tau) ] \leq E[ f (\Gamma_0) ]$, for any stopping time $\tau$ and any solution $\Gamma$, instead of the equality guaranteed by the weak conservation condition. This motivates the next definition.

**Definition 2.16** Suppose that we are in the setup of Definition 2.14. Let $U$ be an open neighborhood of the equilibrium $z_0$ and let $V : U \to \mathbb{R}$ be a continuous function. We say that $V$ is a Lyapunov function for the equilibrium $z_0$ if $V(z_0) = 0$, $V(z) > 0$ for any $z \in U \setminus \{z_0\}$, and
\[
 E[ V (\Gamma_\tau) ] \leq E[ V (\Gamma_0) ] , \tag{2.18}
\]
for any stopping time $\tau$ and any solution $\Gamma$ of (2.10).

This definition generalizes to the stochastic context the standard notion of Lyapunov function that one encounters in dynamical systems theory. If (2.10) is the stochastic differential equation associated to an Itô diffusion and the Lyapunov function is twice differentiable, the inequality (2.18) can be ensured by requiring that $A[V](z) \leq 0$, for any $z \in U \setminus \{z_0\}$, where $A$ is the infinitesimal generator of the diffusion, and by using Dynkin’s formula.

**Theorem 2.17 (Stochastic Lyapunov’s Theorem)** Let $z_0 \in M$ be an equilibrium solution of the stochastic differential equation (2.10) and let $V : U \to \mathbb{R}$ be a continuous Lyapunov function for $z_0$. Then $z_0$ is stable in probability.

**Proof.** Let $U_\epsilon$ be the ball of radius $\epsilon$ around $z_0$ and let $V_\epsilon := \inf_{x \in U \setminus \{z_0\}} V(x)$. Using the same notation as in the previous theorem we denote, for any $x \in U_\epsilon$ and $s \in \mathbb{R}_+$, $\tau_{U_\epsilon}$ as the first exit time of $\Gamma^{s,x}$ with respect to $U_\epsilon$. Using the same approach as above we notice that if $\omega \in \Omega$ belongs to the set $\{ \omega \in \Omega \mid \sup_{0 \leq s < t} d (\Gamma^{s,x}_t, z_0) > \epsilon \}$, then $\tau_{U_\epsilon} (\omega) \leq t$ and hence the stopped process $(\Gamma^{s,x})_{\tau_{U_\epsilon}}$ satisfies that
\[
 V ( (\Gamma^{s,x})_{\tau_{U_\epsilon}} (\omega) ) = V ( \Gamma^{s,x}_{\tau_{U_\epsilon}} (\omega) ) \geq V_\epsilon ,
\]
for those values of $\omega$, since $\Gamma^{s,x}_{\tau_{U_\epsilon}}(\omega)$ belongs to the boundary of $U_\epsilon$. This ensures that
\[ V_\epsilon 1_{\{\omega \in [0,\epsilon] \sup_{0 \leq s < t} d(\Gamma^{s,x}_{\tau_{U_\epsilon}}, z_0) > \epsilon\}} \leq V ((\Gamma^{s,x}_{\tau_{U_\epsilon}})_t). \]
Taking expectations in both sides of this inequality we obtain
\[ P \left( \sup_{0 \leq s < t} d(\Gamma^{s,x}_{\tau_{U_\epsilon}}, z_0) > \epsilon \right) \leq \frac{E[V ((\Gamma^{s,x}_{\tau_{U_\epsilon}})_t)]}{V_\epsilon}. \]
We now use that $V$ being a Lyapunov function satisfies $(2.18)$ and hence
\[ \frac{E[V ((\Gamma^{s,x})_{\tau_{U_\epsilon}})]}{V_\epsilon} = \frac{E[V (\Gamma^{s,x}_{\tau_{U_\epsilon}})]}{V_\epsilon} \leq \frac{E[V (\Gamma^{s,x})]}{V_\epsilon} = \frac{V(x)}{V_\epsilon}. \]
We can therefore conclude that
\[ P \left( \sup_{0 \leq s < t} d(\Gamma^{s,x}_t, z_0) > \epsilon \right) \leq \frac{V(x)}{V_\epsilon}. \]
Taking the limit $x \to z_0$ in this expression and recalling that $V(z_0) = 0$, the result follows. \hfill \blacksquare

**Remark 2.18** This theorem has been proved by Gihman [G66] and Hasminskii [Ha80] for Itô diffusions.

### 3 Examples

#### 3.1 Stochastic perturbation of a Hamiltonian mechanical system and Bismut’s Hamiltonian diffusions

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and $h_j \in C^\infty (M)$, $j = 0, \ldots, r$, smooth functions. Let $h : M \to \mathbb{R}^{r+1}$ be the Hamiltonian function $m \mapsto (h_0 (m), \ldots, h_r (m))$, and consider the semimartingale $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{r+1}$ given by $(t, \omega) \mapsto (t, B_1^j (\omega), \ldots, B_r^j (\omega))$, where $B_j$, $j = 1, \ldots, r$, are $r$-independent Brownian motions, Lévy’s characterization of Brownian motion shows (see for instance [P90] Theorem 40, page 87) that $[B^j, B^k]^t = t \delta^{jk}$. In this setup, the equation $(2.8)$ reads
\[ f (\Gamma^h) - f (\Gamma^h_0) = \int_0^t \{ f, h_0 \} (\Gamma^h) \, dt + \sum_{j=1}^r \int_0^t \{ f, h_j \} (\Gamma^h) \, dB^j \]
for any $f \in C^\infty (M)$. According to $(2.10)$, the equivalent Itô version of this equation is
\[ f (\Gamma^h_0) - f (\Gamma^h) = \int_0^t \{ f, h_0 \} (\Gamma^h) \, dt + \sum_{j=1}^r \int_0^t \{ f, h_j \} (\Gamma^h) \, dB^j + \int_0^t \{ f, h_j \} (\gamma (t)) \, dt. \]

Equation $(3.1)$ may be interpreted as a stochastic perturbation of the classical Hamilton equations associated to $h_0$, that is,
\[ \frac{d(f \circ \gamma)}{dt} (t) = \{ f, h_0 \} (\gamma (t)). \]
by the $r$ Brownian motions $B^j$. These equations have been studied by Bismut in [BS1] in the particular case in which the Poisson manifold $(M, \{\cdot, \cdot\})$ is just the symplectic Euclidean space $\mathbb{R}^{2n}$ with the canonical symplectic form. He refers to these particular processes as **Hamiltonian diffusions**.

If we apply Proposition $(2.13)$ to the stochastic Hamiltonian system $(3.1)$, we obtain a generalization to Poisson manifolds of a result originally formulated by Bismut (see [BS1] Théorèmes 4.1 and 4.2, page 231) for Hamiltonian diffusions. See also [X99].
Proposition 3.1 Consider the stochastic Hamiltonian system introduced in (3.1). Then \( f \in C^\infty(M) \) is a conserved quantity if and only if
\[
\{f,h_0\} = \{f,h_1\} = \ldots = \{f,h_r\} = 0.
\] (3.2)

Proof. If (3.2) holds then \( f \) is clearly a conserved quantity by Proposition 2.13 Conversely, notice that as \([B^i,B^j] = t\delta^{ij}, i,j \in \{1, \ldots , r\}\), and \(X^0(t,\omega) = t\) is a finite variation process then \([X^i,X^j] = 0\) for any \(i,j \in \{0,1, \ldots , r\}\) such that \(i \neq j\). Consequently, by Proposition 2.13 if \( f \) is a conserved quantity then
\[
\{f,h_1\} = \ldots = \{f,h_r\} = 0.
\] (3.3)

Moreover, (3.1) reduces to
\[
\int_0^\tau \{f,h_0\}(\Gamma^h) \, dt = 0,
\]
for any Hamiltonian semimartingale \( \Gamma^h \) and any stopping time \( \tau \leq \zeta^h \). Suppose that \( \{f,h_0\}(m_0) > 0 \) for some \( m_0 \in M \). By continuity there exists a compact neighborhood \( U \) of \( m_0 \) such that \( \{f,h_0\}_U > 0 \). Take \( \Gamma^h \) the Hamiltonian semimartingale with initial condition \( \Gamma^h_0 = m_0 \), and let \( \xi \) be the first exit time of \( U \) for \( \Gamma^h \). Then, defining \( \tau := \xi \wedge \zeta \),
\[
\int_0^\tau \{f,h_0\}(\Gamma^h) \, dt \geq \int_0^\tau \min\{\{f,h_0\}(m) \mid m \in U\} \, dt > 0,
\]
which contradicts (3.3). Therefore, \( \{f,h_0\} = 0 \) also, as required. ■

Remark 3.2 Notice that, unlike what happens for standard deterministic Hamiltonian systems, the energy \( h_0 \) of a Hamiltonian diffusion does not need to be conserved if the other components of the Hamiltonian are not involution with \( h_0 \). This is a general fact about stochastic Hamiltonian systems that makes them useful in the modeling of dissipative phenomena. We see more of this in the next example.

3.2 Integrable stochastic Hamiltonian dynamical systems.

Let \((M,\omega)\) be a \(2n\)-dimensional manifold, \(X : \mathbb{R}^n \times \Omega \to V\) a semimartingale, and \( h : M \to V^* \) such that \( h = \sum_{i=1}^r h_ie^i \), with \( \{e^1, \ldots , e^r\} \) a basis of \( V^* \). Let \( H \) be the associated Stratonovich operator in (2.10).

Suppose that there exists a family of functions \( \{f_{r+1}, \ldots , f_n\} \subset C^\infty(M) \) such that the \( n \)-functions \( \{f_i := h_1, \ldots , f_r := h_r, f_{r+1}, \ldots , f_n\} \subset C^\infty(M) \) are in Poisson involution, that is, \( \{f_i,f_j\} = 0 \) for any \( i,j \in \{1, \ldots , n\} \). Moreover, assume that \( F := (f_1, \ldots , f_n) \) satisfies the hypotheses of the Liouville-Arnold Theorem [Ar89]: \( F \) has compact and connected fibers and its components are independent. In this setup, we will say that the stochastic Hamiltonian dynamical system associated to \( H \) is integrable.

As it was already the case for standard (Liouville-Arnold) integrable systems, there is a symplectomorphism that takes \((M,\omega)\) to \((\mathbb{T}^n \times \mathbb{R}^n, \sum_{i=1}^n d\theta^i \wedge dI_i)\) and for which \( F \equiv F(I_1, \ldots , I_n) \). In particular, in the action-angle coordinates \((I_1, \ldots , I_n, \theta^1, \ldots , \theta^n) \), \( h_j \equiv h_j(I_1, \ldots , I_n) \) with \( j \in \{1, \ldots , r\} \). In other words, the components of the Hamiltonian function depend only on the actions \( \mathbf{I} := (I_1, \ldots , I_n) \). Therefore, for any random variable \( \Gamma_0 \) and any \( i \in \{1, \ldots , n\} \)
\[
I_i(\Gamma) - I_i(\Gamma_0) = \sum_{j=1}^r \int \{I_i,h_j(\mathbf{I})\}(\Gamma) \, dX^j = 0
\] (3.4a)
\[
\theta^i(\Gamma) - \theta^i(\Gamma_0) = \sum_{j=1}^r \int \{\theta^i,h_j(\mathbf{I})\}(\Gamma) \, dX^j = \sum_{j=1}^r \int \frac{\partial h_j}{\partial I_i}(\Gamma) \, dX^j.
\] (3.4b)
Consequently, the tori determined by fixing $I = \text{constant}$ are left invariant by the stochastic flow associated to (3.4). In particular, as the paths of the solutions are contained in compact sets, the stochastic flow is defined for any time and the flow is complete. Moreover, the restriction of this stochastic differential equation to the torus given by say, $I_0$, yields the solution

$$\theta^i (\Gamma) - \theta^i (\Gamma_0) = \sum_{j=1}^r \omega_j (I_0) X^j,$$

where $\omega_j (I_0) := \frac{\partial h_j}{\partial I^i} (I_0)$ and where we have assumed that $X_0 = 0$. Expression (3.5) clearly resembles the integration that can be carried out for deterministic integrable systems.

Additionally, the Haar measure $d\theta^1 \wedge ... \wedge d\theta^n$ on each invariant torus is left invariant by the stochastic flow (see Theorem 2.11 and [L06]). Therefore, if we can ensure that there exists a unique invariant measure $\mu$ (for instance, if (3.5) defines a non-degenerate diffusion on the torus $T^n$, the invariant measure is unique up to a multiplicative constant by the compactness of $T^n$ (see [IW89, Proposition 4.5])) then $\mu$ coincides necessarily with the Haar measure.

### 3.3 The Langevin equation and viscous damping

Hamiltonian stochastic differential equations can be used to model dissipation phenomena. The simplest example in this context is the damping force experienced by a particle in motion in a viscous fluid. This dissipative phenomenon is usually modeled using a force in Newton’s second law that depends linearly on the velocity of the particle (see for instance [LL76, §25]). The standard microscopic description of this motion is carried out using the Langevin stochastic differential equation (also called the Orstein-Uhlenbeck equation) that says that the velocity $\dot{q}(t)$ of the particle with mass $m$ is a stochastic process that solves the stochastic differential equation

$$m \, d\dot{q}(t) = -\lambda \dot{q}(t) \, dt + b dB_t,$$

where $\lambda > 0$ is the damping coefficient, $b$ is a constant, and $B_t$ is a Brownian motion. A common physical interpretation for this equation (see [CH06]) is that the Brownian motion models random instantaneous bursts of momentum that are added to the particle by collision with lighter particles, while the mean effect of the collisions is the slowing down of the particle. This fact is mathematically described by saying that the expected value $q_e := \mathbb{E}[q]$ of the process $q$ determined by (3.6) satisfies the ordinary differential equation

$$m \ddot{q}_e(t) = -\lambda \dot{q}_e(t).$$

Even though this description is accurate it is not fully satisfactory given that it does not provide any information about the mechanism that links the presence of the Brownian perturbation to the emergence of damping in the equation. In order for the physical explanation to be complete, a relation between the coefficients $b$ and $\lambda$ should be provided in such a way that the damping vanishes when the Brownian collisions disappear, that is, $\lambda = 0$ when $b = 0$.

We now show that the motion of a particle of mass $m$ in one dimension subjected to viscous damping with coefficient $\lambda$ and to a harmonic potential with Hooke constant $k$ is a Hamiltonian stochastic differential equation. More explicitly, we will give a stochastic Hamiltonian system such that the expected value $q_e$ of its solution semimartingales satisfies the ordinary differential equation of the damped harmonic oscillator, that is,

$$m \ddot{q}_e(t) = -\lambda \dot{q}_e(t) - k q_e(t).$$

This description provides a mathematical mechanism by which the stochastic perturbations in the system generate an average damping.

Consider $\mathbb{R}^2$ with its canonical symplectic form and let $X : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ be the real semimartingale given by $X_t (\omega) = (t + \nu B_t (\omega))$ with $\nu \in \mathbb{R}$ and $B_t$ a Brownian motion. Let now $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the
energy of a harmonic oscillator, that is, \( h(q, p) := \frac{1}{2m}p^2 + \frac{1}{2}\rho q^2 \). By (2.10), the solution semimartingales \( \Gamma^h \) of the Hamiltonian stochastic equations associated to \( h \) and \( X \) satisfy

\[
q(\Gamma^h) - q(\Gamma^h_0) = \frac{1}{2m} \int (2p(\Gamma^h_t) - \nu^2 q(\Gamma^h_t)) \, dt + \frac{\nu}{m} \int p(\Gamma^h_t) \, dB_t, \tag{3.7}
\]

\[
p(\Gamma^h) - p(\Gamma^h_0) = -\frac{\rho}{2m} \int (\nu^2 p(\Gamma^h_t) + 2mq(\Gamma^h_t)) \, dt - \nu p \int q(\Gamma^h_t) \, dB_t. \tag{3.8}
\]

Given that \( E \left[ \int p(\Gamma^h_t) \, dB_t \right] = E \left[ \int q(\Gamma^h_t) \, dB_t \right] = 0 \), if we denote

\[
q_e(t) := E \left[ q(\Gamma^h_t) \right], \quad p_e(t) := E \left[ p(\Gamma^h_t) \right],
\]

Fubini’s Theorem guarantees that

\[
\dot{q}_e(t) = \frac{1}{m} p_e(t) - \frac{\nu^2 \rho}{2m} q_e(t) \quad \text{and} \quad \dot{p}_e(t) = -\frac{\nu^2 \rho}{2m} p_e(t) - \rho q_e(t). \tag{3.9}
\]

From the first of these equations we obtain that

\[
p_e(t) = m\dot{q}_e + \frac{\nu^2 \rho}{2} q_e
\]

whose time derivative is

\[
\dot{p}_e(t) = m\ddot{q}_e + \frac{\nu^2 \rho}{2} \dot{q}_e.
\]

These two equations substituted in the second equation of (4.8) yield

\[
m\ddot{q}_e(t) = -\nu^2 \rho \dot{q}_e(t) - \rho \left( \frac{\nu^4 \rho}{4m} + 1 \right) q_e(t), \tag{3.10}
\]

that is, the expected value of the position of the Hamiltonian semimartingale \( \Gamma^h \) associated to \( h \) and \( X \) satisfies the differential equation of a damped harmonic oscillator (5.8) with constants

\[
\lambda = \nu^2 \rho \quad \text{and} \quad k = \rho \left( \frac{\nu^4 \rho}{4m} + 1 \right).
\]

Notice that the dependence of the damping and elastic constants on the coefficients of the system is physically reasonable. For instance, we see that the more intense the stochastic perturbation is, that is, the higher \( \nu \) is, the stronger the damping becomes (\( \lambda = \nu^2 \rho \) increases). In particular, if there is no stochastic perturbation, that is, if \( \nu = 0 \), then the damping vanishes, \( k = \rho \) and (4.10) becomes the differential equation of a free harmonic oscillator of mass \( m \) and elastic constant \( \rho \).

**The stability of the resting solution.** It is easy to see that the constant process \( \Gamma_\omega(\omega) = (0, 0) \), for all \( t \in \mathbb{R} \) and \( \omega \in \Omega \) is an equilibrium solution of (3.7) and (3.8). One can show using the stochastic Dirichlet’s criterion (Theorem 2.11) that this equilibrium is almost surely Lyapunov stable since the Hamiltonian function \( h \) is a strongly conserved quantity (by 2.8) that exhibits a critical point at the origin with definite Hessian.

**The Langevin equation.** In the previous paragraphs we succeeded in providing a microscopic Hamiltonian description of the harmonic oscillator subjected to Brownian perturbations whose macroscopic counterpart via expectations yields the equations of the damped harmonic oscillator. In view of this, is such a stochastic Hamiltonian description available for the pure Langevin equation (3.6)? The answer is no. More specifically, it can be easily shown (proceed by contradiction) that (3.6) cannot
be written as a stochastic Hamiltonian differential equation on \( \mathbb{R}^2 \) with its canonical symplectic form with a noise semimartingale of the form \( X_t(\omega) = (f_0(t, B_t), f_1(t, B_t)) \) and a Hamiltonian function \( h(q, p) = (h_0(q, p), h_1(q, p)) \), \( f_0, f_1, h_0, h_1 \in C^\infty(\mathbb{R}) \). Nevertheless, if we put aside for a moment the stochastic Hamiltonian category and we use Itô integration, the Langevin equation can still be written in phase space, that is,

\[
dq_t = v_t dt, \quad dv_t = -\lambda v_t dt + bdB_t, \tag{3.11}
\]

as a stochastic perturbation of a deterministic system, namely, a free particle whose evolution is given by the differential equations

\[
dq_t = v_t dt \quad \text{and} \quad dv_t = 0. \tag{3.12}
\]

Let \( \{u^1, u^2\} \) be global coordinates on \( \mathbb{R}^2 \) associated to the canonical basis \( \{e_1, e_2\} \) and consider the global basis \( \{d_2 u^1, d_2 u^1 \cdot d_2 u^2\}_{i,j=1,2} \) of \( \tau^* \mathbb{R}^2 \). Define a dual Schwartz operator \( S^* (x, (q, v)) : \tau^* (q, v)^2 \rightarrow \tau^*_x \mathbb{R}^2 \) characterized by the relations

\[
d_2 q \mapsto vd_2 u^1, \quad d_2 v \mapsto bd_2 u^2 - \lambda v (d_2 u^2 \cdot d_2 u^2),
\]

where \((q, v) \in \mathbb{R}^2\) is an arbitrary point in phase space and \( x \in \mathbb{R}^2 \). If \( X : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}^2 \) is such that \( X(t, \omega) = (t, bB_t(\omega)) \), for any \((t, \omega) \in \mathbb{R}_+ \times \Omega\), it is immediate to see that the Itô equations associated to \( S^* \) and \( X \) are \((3.11)\). Moreover, if we set \( b = 0 \), that is, we switch off the Brownian perturbation then we recover \((3.12)\), as required.

### 3.4 Brownian motions on manifolds

The mathematical formulation of Brownian motions (or Wiener processes) on manifolds has been the subject of much research and it is a central topic in the study of stochastic processes on manifolds (see [IW89 Chapter 5], [E89 Chapter V], and references therein for a good general review of this subject).

In the following paragraphs we show that Brownian motions can be defined in a particularly simple way using the stochastic Hamilton equations introduced in Definition 2.2. More specifically we will show that Brownian motions on manifolds can be obtained as the projections onto the base space of very simple Hamiltonian stochastic semimartingales defined on the cotangent bundle of the manifold or of its orthonormal frame bundle, depending on the availability or not of a parallelization for the manifold in question.

We will first present the case in which the manifold in question is parallelizable or, equivalently, when the coframe bundle on the manifold admits a global section, for the construction is particularly simple in this situation. The parallelizability hypothesis is verified by many important examples. For instance, any Lie group is parallelizable; the spheres \( S^1, S^3 \), and \( S^7 \) are parallelizable too. At the end of the section we describe the general case.

The notion of manifold valued Brownian motion that we will use is the following. A \( M \)-valued process \( \Gamma \) is called a Brownian motion on \((M, g)\), with \( g \) a Riemannian metric on \( M \), whenever \( \Gamma \) is continuous and adapted and for every \( f \in C^\infty(M) \)

\[
f(\Gamma) - f(\Gamma_0) - \frac{1}{2} \int \Delta_M f(\Gamma) dt
\]

is a local martingale. We recall that the Laplacian \( \Delta_M (f) \) is defined as \( \Delta_M (f) = \text{Tr} (\text{Hess} f) \), for any \( f \in C^\infty(M) \), where \( \text{Hess} f := \nabla (\nabla f) \), with \( \nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M) \), the Levi-Civita connection of \( g \). Hess \( f \) is a symmetric \((0, 2)\)-tensor such that for any \( X, Y \in \mathfrak{X}(M) \),

\[
\text{Hess} f(X, Y) = X \left[ g(\text{grad} f, Y) \right] - g(\text{grad} f, \nabla_X Y). \tag{3.13}
\]
Brownian motions on parallelizable manifolds. Suppose that the \( n \)-dimensional manifold \((M, g)\) is parallelizable and let \(\{Y_1, ..., Y_n\}\) be a family of vector fields such that for each \(m \in M\), \(\{Y_1(m), ..., Y_n(m)\}\) forms a basis of \(T_m M\) (a parallelization). Applying the Gram-Schmidt orthonormalization procedure if necessary, we may suppose that this parallelization is orthonormal, that is, \(g(Y_i, Y_j) = \delta_{ij}\), for any \(i, j = 1, ..., n\).

Using this structure we are going to construct a stochastic Hamiltonian system on the cotangent bundle \(T^* M\) of \(M\), endowed with its canonical symplectic structure, and we will show that the projection of the solution semimartingales of this system onto \(M\) are \(M\)-valued Brownian motions in the sense specified above. Let \(X : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{n+1}\) be the semimartingale given by \(X(t, \omega) := (B^1_t(\omega), B^2_t(\omega), ..., B^n_t(\omega))\), where \(B^j, j = 1, ..., n\), are \(n\)-independent Brownian motions and let \(h = (h_0, h_1, ..., h_n) : T^* M \to \mathbb{R}^{n+1}\) be the function whose components are given by

\[
\begin{align*}
  h_0 : \; T^* M &\to \mathbb{R} \\
  \alpha_m &\mapsto -\frac{1}{2} \sum_{j=1}^n \langle \alpha_m, (\nabla Y_j)(m) \rangle \\

  h_j : \; T^* M &\to \mathbb{R} \\
  \alpha_m &\mapsto \langle \alpha_m, Y_j(m) \rangle.
\end{align*}
\]  

(3.14)

We will now study the projection onto \(M\) of Hamiltonian semimartingales \(\Gamma^h\) that have \(X\) as stochastic component and \(h\) as Hamiltonian function and will prove that they are \(M\)-valued Brownian motions. In order to do so we will be particularly interested in the **projectable functions** \(f\) of \(T^* M\), that is, the functions \(f \in C^\infty(T^* M)\) that can be written as \(f = \pi \circ \bar{f}\) with \(\bar{f} \in C^\infty(M)\) and \(\pi : T^* M \to M\) the canonical projection.

We start by proving that for any projectable function \(f = \pi \circ \bar{f} \in C^\infty(T^* M)\)

\[
\{f, h_0\} = g \left( \text{grad} \bar{f}, -\frac{1}{2} \sum_{j=1}^n \nabla Y_j \right) \quad \text{and} \quad \{f, h_j\} = g \left( \text{grad} \bar{f}, Y_j \right),
\]

(3.15)

and where \(\{\cdot, \cdot\}\) is the Poisson bracket associated to the canonical symplectic form on \(T^* M\). Indeed, let \(U\) a Darboux patch for \(T^* M\) with associated coordinates \((q^1, ..., q^n, p_1, ..., p_n)\) such that \(\{q^i, p_j\} = \delta_{ij}\). There exists functions \(f_j^k \in C^\infty(\pi(U))\), with \(k, j = 1, ..., n\) such that the vector fields may be locally written as \(Y_j = \sum_{k=1}^n f_j^k \frac{\partial}{\partial q^k}\). Moreover, \(h_j(q, p) = \sum_{k=1}^n f_j^k(q) p_k\) and

\[
\{f, h_j\} = \left\{ \bar{f} \circ \pi, \sum_{k=1}^n f_j^k p_k \right\} = \sum_{k=1}^n f_j^k \left\{ \bar{f} \circ \pi, p_k \right\} = \sum_{k=1}^n f_j^k \frac{\partial (\bar{f} \circ \pi)}{\partial q^k} \{q^k, p_k\} = \sum_{k=1}^n f_j^k \delta_{jk} \frac{\partial \bar{f}}{\partial q^j} = Y_j(\bar{f}) \circ \pi = g \left( \text{grad} \bar{f}, Y_j \right) \circ \pi,
\]

as required. The first equality in (3.15) is proved analogously. Notice that the formula that we just proved shows that if \(f\) is projectable then so is \(\{f, h_j\}, \text{ with } j \in \{1, ..., n\}\). Hence, using (3.15) again and (3.10) we obtain that

\[
\{\{f, h_0\}, h_j\} = Y_j \left[ g \left( \text{grad} \bar{f}, Y_j \right) \right] \circ \pi = \text{Hess} \bar{f}(Y_j, Y_j) \circ \pi + g \left( \text{grad} \bar{f}, \nabla Y_j \right) \circ \pi,
\]

(3.16)

for \(j \in \{1, ..., n\}\). Now, using (3.15) and (3.16) in (2.10) we have shown that for any projectable function \(f = \pi \circ \bar{f}\), the Hamiltonian semimartingale \(\Gamma^h\) satisfies that

\[
\bar{f} \circ \pi \left( \Gamma^h \right) - \bar{f} \circ \pi \left( \Gamma^h_0 \right) = \sum_{j=1}^n \int g \left( \text{grad} \bar{f}, Y_j \right) (\pi \circ \Gamma^h) dB_j^t + \frac{1}{2} \sum_{j=1}^n \int \text{Hess} \bar{f}(Y_j, Y_j) (\pi \circ \Gamma^h) dt,
\]

(3.17)

or equivalently

\[
\bar{f} \circ \pi \left( \Gamma^h \right) - \bar{f} \circ \pi \left( \Gamma^h_0 \right) - \frac{1}{2} \int \Delta_M(\bar{f}) (\pi \circ \Gamma^h) dt = \sum_{j=1}^n \int g \left( \text{grad} \bar{f}, Y_j \right) (\pi \circ \Gamma^h) dB_j^t.
\]

(3.18)
Since \( \sum_{i=1}^{n} \int g(\nabla f, Y_j) \left( \Gamma^h \right) dB^i \) is a local martingale (see [P90, Theorem 20, page 63]), \( \pi(\Gamma^h) \) is a Brownian motion.

**Brownian motions on Lie groups.** Let now \( G \) be a (finite dimensional) Lie group with Lie algebra \( \mathfrak{g} \) and assume that \( G \) admits a bi-invariant metric \( g \), for example when \( G \) is Abelian or compact. This metric induces a pairing in \( \mathfrak{g} \) invariant with respect to the adjoint representation of \( G \) on \( \mathfrak{g} \). Let \( \{\xi_1, \ldots, \xi_n\} \) be an orthonormal basis of \( \mathfrak{g} \) with respect to this invariant pairing and let \( \{\nu_1, \ldots, \nu_n\} \) be the corresponding dual basis of \( \mathfrak{g}^* \). The infinitesimal generator vector fields \( \{\xi_G, \ldots, \xi_{nG}\} \) defined by \( \xi_G(h) = T_h L_h \cdot \xi \), with \( L_h : G \to G \) the left translation map, \( h \in G \), \( i \in \{1, \ldots n\} \), are obviously an orthonormal parallelization of \( G \), that is \( g(\xi_G, \xi_{jG}) := \delta_{ij} \). Since \( g \) is bi-invariant then \( \nabla_X Y = \frac{1}{2} [X, Y] \), for any \( X, Y \in \mathfrak{X}(G) \) (see [O83, Proposition 9, page 304]), and hence \( \nabla_{\xi_G} \xi_{jG} = 0 \). Therefore, in this particular case the first component \( h_0 \) of the Hamiltonian function introduced in (3.14) is zero and we can hence take \( h_G = (h_1, ..., h_n) \) and \( X_G = (B_1^1, ..., B_n^n) \) when we consider the Hamilton equations that define the Brownian motion with respect to \( g \).

As a special case of the previous construction that serves as a particularly simple illustration, we are going to explicitly build the **Brownian motion on a circle.** Let \( S^1 = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \) be the unit circle. The stochastic Hamiltonian differential equation for the semimartingale \( \Gamma^h \) associated to \( X : \mathbb{R}_+ \times \Omega \to \mathbb{R} \), given by \( X_t(\omega) := B_t(\omega) \), and the Hamiltonian function \( h : T^1 S^1 \simeq S^1 \times \mathbb{R} \to \mathbb{R} \) given by \( h(e^{i\theta}, \lambda) := \lambda \), is simply obtained by writing \( \mathcal{X} \) down for the functions \( f_1(e^{i\theta}) := \cos \theta \) and \( f_2(e^{i\theta}) := \sin \theta \) which provide us with the equations for the projections \( X^h \) and \( Y^h \) of \( \Gamma^h \) onto the \( OX \) and \( OY \) axes, respectively. A straightforward computation yields

\[
\begin{align*}
    dX^h &= -Y^h dB - \frac{1}{2} X^h dt & \text{and} & \quad dY^h &= X^h dB - \frac{1}{2} Y^h dt, \quad (3.19)
\end{align*}
\]

which, incidentally, coincides with the equations proposed in expression (5.1.13) of [Ok03]. A solution of (3.19) is \((X_t^h, Y_t^h) = (\cos B_t, \sin B_t)\), that is, \( \Gamma_t^h = e^{iB_t} \).

**Brownian motions on arbitrary manifolds.** Let \( (M, g) \) be a not necessarily parallelizable Riemannian manifold. In this case we will reproduce the same strategy as in the previous paragraphs but replacing the cotangent bundle of the manifold by the cotangent bundle of its orthonormal frame bundle.

Let \( \mathcal{O}_x(M) \) be the set of orthonormal frames for the tangent space \( T_x M \). The orthonormal frame bundle \( \mathcal{O}(M) = \bigcup_{x \in M} \mathcal{O}_x(M) \) has a natural smooth manifold structure of dimension \( n(n+1)/2 \). We denote by \( \pi : \mathcal{O}(M) \to M \) the canonical projection. We recall that a curve \( \gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to \mathcal{O}(M) \) is called horizontal if \( \gamma_t \) is the parallel transport of \( \gamma_0 \) along the projection \( \pi(\gamma_t) \). The set of tangent vectors of horizontal curves that contain a point \( u \in \mathcal{O}(M) \) defines the horizontal subspace \( H_u \mathcal{O}(M) \subset T_u \mathcal{O}(M) \), with dimension \( n \). The projection \( \pi : \mathcal{O}(M) \to M \) induces an isomorphism \( T_u \pi : H_u \mathcal{O}(M) \to T_{\pi(u)} M \). On the orthonormal frame bundle, we have \( n \) horizontal vector fields \( Y_i, i = 1, ..., n \), defined as follows. For each \( u \in \mathcal{O}(M) \), let \( Y_i(u) \) be the unique horizontal vector in \( H_u \mathcal{O}(M) \) such that \( T_u \pi(Y_i) = u_i \), where \( u_i \) is the \( i \)th unit vector of the orthonormal frame \( u \). Now, given a smooth function \( F \in C^\infty(\mathcal{O}(M)) \), the operator

\[
\Delta_{\mathcal{O}(M)}(F) = \sum_{i=1}^{n} Y_i [Y_i [F]]
\]

is called Bochner’s horizontal Laplacian on \( \mathcal{O}(M) \). At the same time, we recall that the Laplacian \( \Delta_M(f) \), for any \( f \in C^\infty(M) \), is defined as \( \Delta_M(f) = \text{Tr}(\text{Hess} f) \). These two Laplacians are related by the relation

\[
\Delta_{\mathcal{O}(M)}(\pi^* f) = \Delta_M(f), \quad (3.20)
\]

for any \( f \in C^\infty(M) \) (see [H02]).
The Eells-Elworthy-Malliavin construction of Brownian motion can be summarized as follows. Consider the following stochastic differential equation on $\mathcal{O} (M)$ (see [IW89]):

$$\delta U_t = \sum_{i=1}^{n} Y_i (U_t) \delta B^i_t$$

(3.21)

where $B^j, j = 1, \ldots, n$, are $n$-independent Brownian motions. Using the conventions introduced in the appendix 6.4 the expression (3.21) is the Stratonovich stochastic differential equation associated to the Stratonovich operator:

$$e (v, u) : T_v \mathbb{R}^n \to T_u \mathcal{O} (M) \quad \text{where} \quad v = \sum_{i=1}^{n} v^i e_i \mapsto \sum_{i=1}^{n} v^i Y_i (u),$$

where $\{e_1, \ldots, e_n\}$ is a fixed basis for $\mathbb{R}^n$. A solution of the stochastic differential equation (3.21) is called a horizontal Brownian motion on $\mathcal{O} (M)$ since, by the Itô formula,

$$F (U) - F (U_0) = \sum_{i=1}^{n} \int Y_i [F] (U_s) \delta B^i_s = \sum_{i=1}^{n} \int Y_i [F] (U_s) dB^i_s + \frac{1}{2} \int \Delta_{\mathcal{O} (M)} F (U_s) \, ds,$$

for any $F \in C^\infty (\mathcal{O} (M))$. In particular, if $F = \pi^* (f)$ for some $f \in C^\infty (\mathcal{O})$, by (3.20)

$$f (X) - f (X_0) = \sum_{i=1}^{n} \int Y_i [\pi^* (f)] (U_s) dB^i_s + \frac{1}{2} \int \Delta_M f (X_s) \, ds,$$

where $X_t = \pi (U_t)$, which implies precisely that $X_t$ is a Brownian motion on $M$.

In order to generate (3.21) as a Hamilton equation, we introduce the functions $h_i : T^* \mathcal{O} (M) \to \mathbb{R}$, $i = 1, \ldots, n$, given by $h_i (\alpha) = \langle \alpha, Y_i \rangle$. Recall that $T^* \mathcal{O} (M)$ being a cotangent bundle it has a canonical symplectic structure. Mimicking the computations carried out in the parallelizable case it can be seen that the Hamiltonian vector field $X_h$ coincides with $Y_i$ when acting on functions of the form $F \circ \pi_{T^* \mathcal{O} (M)}$, where $F \in C^\infty (\mathcal{O} (M))$ and $\pi_{T^* \mathcal{O} (M)}$ is the canonical projection $\pi_{T^* \mathcal{O} (M)} : T^* \mathcal{O} (M) \to \mathcal{O} (M)$. By (2.8), the Hamiltonian semimartingale $\Gamma^h$ associated to $h = (h_1, \ldots, h_n)$ and to the stochastic Hamiltonian equations on $T^* \mathcal{O} (M)$ with stochastic component $X = (B^1, \ldots, B^n)$ is such that

$$F \circ \pi_{T^* \mathcal{O} (M)} (\Gamma^h) - F \circ \pi_{T^* \mathcal{O} (M)} (\Gamma^h_0)$$

$$= \sum_{i=1}^{n} \int \{ F \circ \pi_{T^* \mathcal{O} (M)}, h_i \} (\Gamma^h_s) \, dB^i_s = \sum_{i=1}^{n} \int Y_i [F] (\pi_{T^* \mathcal{O} (M)} (\Gamma^h_s)) \, dB^i_s$$

for any $F \in C^\infty (\mathcal{O} (M))$. This expression obviously implies that $U^h = \pi_{T^* \mathcal{O} (M)} (\Gamma^h)$ is a solution of (3.21) and consequently $X^h = \pi (U^h)$ is a Brownian motion on $M$.

### 3.5 The inverted pendulum with stochastically vibrating suspension point

The equation of motion for small angles of a damped inverted unit mass pendulum of length $l$ with a vertically vibrating suspension point is

$$\ddot{\phi} = \left( \frac{\dot{y}}{l} + \frac{g}{l} \right) \phi - \lambda \dot{\phi},$$

(3.22)

where $\phi$ is the angle that measures the separation of the pendulum from the vertical upright position, $y = y (t)$ is the height of the suspension point (externally controlled), $\lambda$ is the friction coefficient, and $g$
is the gravity constant. By construction, the point \((\phi, \dot{\phi}) = (0, 0)\) corresponds to the upright equilibrium position. It can be shown that if the function \(y(t)\) is of the form \(y(t) = az(\omega t)\), with \(z\) periodic, the amplitude \(a\) is sufficiently small, and the frequency \(\omega\) is sufficiently high, then this equilibrium becomes nonlinearly stable.

We now consider the case in which the external forcing of the suspension point is given by a continuous stochastic process \(\dot{z} : \mathbb{R}^+ \times \Omega \to \mathbb{R}\) such that \(\dot{z}^2\) is continuous and stationary. Under this assumptions, the equation (3.22) becomes the stochastic differential equation

\[
d\dot{\phi} = \dot{\phi}dt, \quad d\dot{\phi} = \left(\frac{\partial \phi}{\partial \phi} - \lambda \dot{\phi}\right) dt + \varepsilon^2 \omega^2 \dot{\phi} d\dot{z}, \tag{3.23}
\]

where \(\varepsilon := \sqrt{a/l}\). Observe that this equation is not Hamiltonian unless the friction term \(-\lambda \dot{\phi}\) vanishes \((\lambda = 0)\), in which case one obtains a Hamiltonian stochastic system with Hamiltonian function \(h(\phi, \dot{\phi}) = \frac{1}{2}(\dot{\phi}^2 - l\dot{\phi}^2)\), \(\frac{1}{2}(\varepsilon^2 \omega^2 \dot{\phi} d\dot{z})^2\) and noise semimartingale \(X_t = (t, \dot{z}, \dot{z})\) (the symplectic form is obviously \(\mathcal{L}^2 \dot{\phi} \wedge d\dot{\phi}\)).

The stability of the upright position of the stochastically forced pendulum has been studied in [O06], and references therein. In [O06] it is assumed that the noise has the fairly strong mixing property. We recall that a continuous, adapted, stationary process \(\Gamma : \mathbb{R}^+ \times \Omega \to \mathbb{R}\) has the fairly strong mixing property if \(E[\Gamma^2] < \infty\), there exists a real function \(c\) such that \(\int_0^\infty c(s) ds < \infty\), and for any \(t > s\)

\[
\|E[\Gamma_t - E[\Gamma_t]|\mathcal{F}_s]\|_{L^2} \leq c(t-s) \|\Gamma_s - E[\Gamma_s]\|_{L^2},
\]

where \(\|\cdot\|_{L^2}\) stands for the \(L^2\) norm. For example, if \(x\) is the unique stationary solution with zero mean of the Itô equations

\[
dx_t = y_t dt, \quad dy_t = -(x_t + y_t) dt + dB_t,
\]

where \(B_t\) is a standard Brownian motion, then \(\dot{x}_t^2 - \frac{1}{2} = y_t^2 - \frac{1}{2}\) has the fairly strong mixing property. Using this hypothesis, it can be shown [O06] Theorem 1] that if \(z : \mathbb{R}^+ \times \Omega \to \mathbb{R}\) is a continuously differentiable and stationary process such that, for any \(t \in \mathbb{R}^+, E[z_t] = 0\), \(E[\exp(\varepsilon |z_t|)] < \infty\) if \(\varepsilon = \sqrt{a/l}\) is sufficiently small, and the process \(\dot{z}^2\) has the fairly strong mixing property, then the solution \((\phi, \dot{\phi}) = (0, 0)\) of (3.23) is exponentially stable in probability, if \(\varepsilon\) is sufficiently small and \(\frac{\varepsilon}{l} < E[\dot{z}^2]\). Moreover, Ovseyevich shows in [O06] Section 4] that if we put \(\lambda = 0\) in (3.23) and we consider hence the inverted pendulum as a Hamiltonian system, then the equilibrium point \((\phi, \dot{\phi}) = (0, 0)\) is unstable.

4 Critical action principles for the stochastic Hamilton equations

Our goal in this section is showing that the stochastic Hamilton equations can be characterized by a variational principle that generalizes the one used in the classical deterministic situation. In the following pages we shall consider an exact symplectic manifold \((M, \omega)\), that is, there exist a one-form \(\theta \in \Omega(M)\) such that \(\omega = -d\theta\). The archetypical example of an exact symplectic manifold is the cotangent bundle \(T^*Q\) of any manifold \(Q\), with \(\theta\) the Liouville one-form.

In the following pages we will proceed in two stages. In the first subsection we will construct a critical action principle based on using variations of the solution semimartingale using the flow of a vector field on the manifold. Even though this approach is extremely natural and mathematically very tractable it yields a variational principle (Theorem 4.9) that does not fully characterize the stochastic Hamilton’s equations. In order to obtain such a characterization one needs to use more general variations associated to the flows of vector fields defined on the solution semimartingale, that is, they depend on \(\Omega\). This complicates considerably the formulation and will be treated separately in the second subsection.
Definition 4.1 Let \((M, \omega = -d\theta)\) be an exact symplectic manifold, \(X : \mathbb{R}_+ \times \Omega \to V\) a semimartingale taking values on the vector space \(V\), and \(h : M \to V^*\) a Hamiltonian function. We denote by \(S(M)\) and \(S(\mathbb{R})\) the sets of \(M\) and real-valued semimartingales, respectively. We define the stochastic action associated to \(h\) as the map \(S : S(M) \to S(\mathbb{R})\) given by
\[
S(\Gamma) = \int \langle \theta, \delta \Gamma \rangle - \int \langle \hat{h}(\Gamma), \delta X \rangle,
\]
where in the previous expression, \(\hat{h}(\Gamma) : \mathbb{R}_+ \times \Omega \to V \times V^*\) is given by \(\hat{h}(\Gamma)(t, \omega) := (X_t(\omega), h(\Gamma_t(\omega)))\).

4.1 Variations involving vector fields on the phase space

Definition 4.2 Let \(M\) be a manifold, \(F : S(M) \to S(\mathbb{R})\) a map, and \(\Gamma \in S(M)\). A local one-parameter group of diffeomorphisms \(\varphi : \mathcal{D} \subset \mathbb{R} \times M \to M\) is said to be complete with respect to \(\Gamma\) if there exists \(\epsilon > 0\) such that \(\varphi_{s}(\Gamma)\) is a well-defined process for any \(s \in (-\epsilon, \epsilon)\). We say that \(F\) is differentiable at \(\Gamma\) in the direction of a local one parameter group of diffeomorphisms \(\varphi\) complete with respect to \(\Gamma\), if for any sequence \(\{s_n\}_{n \in \mathbb{N}} \subset \mathbb{R}\) such that \(s_n \to 0\), the family
\[
X_n = \frac{1}{s_n}(F(\varphi_{s_n}(\Gamma)) - F(\Gamma))
\]
converges uniformly on compacts in probability (ucp) to a process that we will denote by \(\left.\frac{dF}{ds}\right|_{s=0} F(\varphi_s(\Gamma))\) and that is referred to as the directional derivative of \(F\) at \(\Gamma\) in the direction of \(\varphi_s\).

Remark 4.3 Note that global one-parameter groups of diffeomorphisms (for instance, flows of complete vector fields) are complete with respect to any semimartingale.

Let \(\Gamma : \mathbb{R}_+ \times \Omega \to M\) be a \(M\)-valued continuous and adapted stochastic process and \(A \subset M\) a set. We will denote by \(\tau_A = \inf \{t > 0 \mid \Gamma_t(\omega) \notin A\}\) the first exit time of \(\Gamma\) with respect to \(A\). We recall that \(\tau_A\) is a stopping time if \(A\) is a Borel set. Additionally, let \(\Gamma\) be a semimartingale and \(K\) a compact set such that \(\Gamma_0 \subset K\). Then, any local one-parameter group of diffeomorphisms \(\varphi\) is complete with respect to the stopped process \(\Gamma^{\tau_K}\). Note that this conclusion could also hold for certain non-compact sets.

The proof of the following proposition can be found in Section 6.1.

Proposition 4.4 Let \(M\) be a manifold, \(\alpha \in \Omega(M)\) a one-form, and \(F : S(M) \to S(\mathbb{R})\) the map defined by \(F(\Gamma) := \int \langle \alpha, \delta \Gamma \rangle\). Then \(F\) is differentiable in all directions. Moreover, if \(\Gamma : \mathbb{R}_+ \times \Omega \to M\) is a continuous semimartingale, \(\varphi\) is an arbitrary local one-parameter group of diffeomorphisms complete with respect to \(\Gamma\), and \(Y \in \mathfrak{X}(M)\) is the vector field associated to \(\varphi\), then
\[
\left.\frac{d}{ds}\right|_{s=0} F(\varphi_s(\Gamma)) = \left.\frac{d}{ds}\right|_{s=0} \int \langle \alpha, \delta (\varphi_s \circ \Gamma) \rangle = \left.\frac{d}{ds}\right|_{s=0} \int \langle \varphi_s^* \alpha, \delta \Gamma \rangle = \int \langle \mathcal{L}_Y \alpha, \delta \Gamma \rangle. \quad (4.1)
\]
The symbol \(\mathcal{L}_Y \alpha\) denotes the Lie derivative of \(\alpha\) in the direction given by \(Y\).

Corollary 4.5 In the setup of Definition 4.4 let \(\alpha = \omega^\sharp(Y) \in \Omega(M)\), with \(\omega^\sharp\) the inverse of the vector bundle isomorphism \(\omega^\sharp : T^*M \to TM\) induced by \(\omega\). Let \(\Gamma : \mathbb{R}_+ \times \Omega \to M\) be a continuous adapted semimartingale, \(\varphi\) an arbitrary local one-parameter group of diffeomorphisms complete with respect to \(\Gamma\), and \(Y \in \mathfrak{X}(M)\) the associated vector field. Then, the action \(S\) is differentiable at \(\Gamma\) in the direction of \(\varphi\) and the directional derivative is given by
\[
\left.\frac{d}{ds}\right|_{s=0} S(\varphi_s(\Gamma)) = -\int \langle \alpha, \delta \Gamma \rangle - \int \langle dh (\omega^\sharp (\alpha)) (\Gamma), \delta X \rangle + i_Y \theta (\Gamma) - i_Y \theta (\Gamma_0). \quad (4.2)
\]
Proof. It is clear from Proposition 4.4 that
\[
\frac{1}{s} \left[ \int \langle \varphi_s^* \theta - \theta, \delta \Gamma \rangle \right] \rightarrow 0
\]
in \textit{ucp}. The proof of that result can be easily adapted to show that \textit{ucp}
\[
\frac{1}{s} \left[ \int \left( \varphi_s^* \hat{h} - \hat{h} \right)(\Gamma), \delta X \right] \rightarrow 0.
\]
Thus, using (6.5) and \(\alpha = \omega^h(Y) \in \Omega(M)\),
\[
\frac{d}{ds} \bigg|_{s=0} S(\varphi_s(\Gamma)) = \int \langle \mathcal{L}_Y \theta, \delta \Gamma \rangle - \int \langle \mathcal{L}_Y \hat{h}(\Gamma), \delta X \rangle = \int \langle \mathcal{L}_Y \hat{h}(\Gamma), \delta X \rangle - \int \langle \mathcal{L}_Y \hat{h}(\Gamma), \delta X \rangle = 0.
\]

Corollary 4.6 (Noether’s theorem) In the setup of Definition 4.1, let \(\varphi : \mathbb{R} \times M \to M\) be a one parameter group of diffeomorphisms and \(Y \in \mathfrak{X}(M)\) the associated vector field. If the action \(S : \mathcal{S}(M) \to \mathcal{S}(\mathbb{R})\) is invariant by \(\varphi\), that is, \(S(\varphi_s(\Gamma)) = S(\Gamma)\), for any \(s \in \mathbb{R}\), then the function \(i_Y \theta\) is a conserved quantity of the stochastic Hamiltonian system associated to \(h : M \to V^*\).

Proof. Let \(\Gamma^h\) be the Hamiltonian semimartingale associated to \(h\) with initial condition \(\Gamma_0\). Since \(\varphi_s\) leaves invariant the action we have that
\[
\frac{d}{ds} \bigg|_{s=0} S(\varphi_s(\Gamma^h)) = 0
\]
and hence by (4.2) we have that
\[
0 = -\int \langle \alpha, \delta \Gamma^h \rangle - \int \langle \mathcal{L}_h (\omega^h(\alpha)), \delta X \rangle + (i_Y \theta)(\Gamma^h) - (i_Y \theta)(\Gamma_0).
\]
As \(\Gamma^h\) is the Hamiltonian semimartingale associated to \(h\) we have that
\[
-\int \langle \alpha, \delta \Gamma^h \rangle = \int \langle \mathcal{L}_h (\omega^h(\alpha)), \delta X \rangle
\]
and hence \(i_Y \theta(\Gamma^h) = i_Y \theta(\Gamma_0)\), as required.

Remark 4.7 The hypotheses of the previous corollary can be modified by requiring, instead of the invariance of the action by \(\varphi_s\), the existence of a function \(F \in C^\infty(M)\) such that
\[
\frac{d}{ds} \bigg|_{s=0} S(\varphi_s(\Gamma^h)) = F(\Gamma) - F(\Gamma_0).
\]
In that situation, the conserved quantity is \(i_Y \theta + F\).

Before we state the Critical Action Principle for the stochastic Hamilton equations we need one more definition.
Definition 4.8 Let $M$ be a manifold and $A$ a set. We will say that a local one parameter group of diffeomorphisms $\varphi: \mathcal{D} \times M \to M$ fixes $A$ if $\varphi_s(y) = y$ for any $y \in A$ and any $s \in \mathbb{R}$ such that $(s,y) \in \mathcal{D}$. The corresponding vector field $Y \in \mathfrak{X}(M)$ given by $Y(m) = \frac{d}{ds}\big|_{s=0} \varphi_s(m)$ satisfies that $Y|_{A} = 0$.

Theorem 4.9 (First Critical Action Principle) Let $(M,\omega = -d\theta)$ be an exact symplectic manifold, $X: \mathbb{R}_+ \times \Omega \to V$ a semimartingale taking values on the vector space $V$ such that $X_0 = 0$, and $h: M \to V^*$ a Hamiltonian function. Let $m_0 \in M$ be a point in $M$ and $\Gamma: \mathbb{R}_+ \times \Omega \to M$ a continuous semimartingale such that $\Gamma_0 = m_0$. Let $K$ be a compact set that contains the point $m_0$. If the semimartingale $\Gamma$ satisfies the stochastic Hamilton equations (2.7) (with initial condition $\Gamma_0 = m_0$) up to time $\tau_K$ then for any local one-parameter group of diffeomorphisms $\varphi$ that fixes the set $\{m_0\} \cup \partial K$ we have

\[ 1_{\{\tau_K < \infty\}} \left[ \frac{d}{ds}\big|_{s=0} S(\varphi_s(\Gamma^{\tau_K})) \right]_{\tau_K} = 0 \quad \text{a.s.} \quad (4.3) \]

Proof. We start by emphasizing that when we write that $\Gamma$ satisfies the stochastic Hamiltonian equations (2.7) up to time $\tau_K$ we mean that

\[
\left( \int \langle \beta, \delta \Gamma \rangle + \int \langle dh(\omega^*(\beta))(\Gamma), \delta X \rangle \right)_{\tau_K} = 0.
\]

For the sake of simplicity in our notation we define the linear operator $\text{Ham} : \Omega(M) \to S(\mathbb{R})$ given by

\[
\text{Ham}(\beta) := \left( \int \langle \beta, \delta \Gamma \rangle + \int \langle dh(\omega^*(\beta))(\Gamma), \delta X \rangle \right), \quad \beta \in \Omega(M).
\]

Suppose now that the semimartingale $\Gamma$ satisfies the stochastic Hamilton equations up to time $\tau_K$. Let $\varphi$ be a local one-parameter group of diffeomorphisms that fixes $\{m_0\} \cup \partial K$, and let $Y \in \mathfrak{X}(M)$ be the associated vector field. Then, taking $\alpha = \omega^*(Y)$, we have by Corollary 4.5

\[
\frac{d}{ds}\big|_{s=0} S(\varphi_s(\Gamma^{\tau_K})) = -\int \langle \alpha, \delta \Gamma^{\tau_K} \rangle - \int \langle dh(\omega^*(\alpha))(\Gamma^{\tau_K}), \delta X \rangle + iy\theta(\Gamma^{\tau_K}), \quad (4.4)
\]

since $Y(m_0) = 0$ and hence $iy\theta(\Gamma_0) = 0$. Additionally, since $\Gamma$ is continuous, $1_{\{\tau_K < \infty\}} \Gamma^{\tau_K} \in \partial K$ and $Y|_{\partial K} = 0$. Hence,

\[
1_{\{\tau_K < \infty\}} \left[ \frac{d}{ds}\big|_{s=0} S(\varphi_s(\Gamma^{\tau_K})) \right]_{\tau_K} = -1_{\{\tau_K < \infty\}} \left[ \int \langle \alpha, \delta \Gamma^{\tau_K} \rangle + \int \langle dh(\omega^*(\alpha))(\Gamma^{\tau_K}), \delta X \rangle \right]_{\tau_K}.
\]

Now, Proposition 4.5 and the hypothesis on $\Gamma$ satisfying Hamilton’s equation guarantee that the previous expression equals

\[
1_{\{\tau_K < \infty\}} \left[ \frac{d}{ds}\big|_{s=0} S(\varphi_s(\Gamma^{\tau_K})) \right]_{\tau_K} = -1_{\{\tau_K < \infty\}} \left[ \int \langle \alpha, \delta \Gamma^{\tau_K} \rangle + \int \langle dh(\omega^*(\alpha))(\Gamma^{\tau_K}), \delta X \rangle \right]_{\tau_K} = -1_{\{\tau_K < \infty\}} \left[ \int \langle \alpha, \delta \Gamma \rangle + \int \langle dh(\omega^*(\alpha))(\Gamma), \delta X \rangle \right]_{\tau_K} = -1_{\{\tau_K < \infty\}} [\text{Ham}(\alpha)]_{\tau_K} = 0 \quad \text{a.s.,}
\]

as required. \(\blacksquare\)
Remark 4.10 The relation between the Critical Action Principle stated in Theorem 4.9 and the classical one for Hamiltonian mechanics is not straightforward since the categories in which both are formulated are very much different; more specifically, the differentiability hypothesis imposed on the solutions of the deterministic principle is not a reasonable assumption in the stochastic context and this has serious consequences. For example, unlike the situation encountered in classical mechanics, Theorem 4.9 does not admit a converse within the set of hypotheses in which it is formulated.

In order to elaborate a little bit more on this question let $(M, \omega = -d\theta)$ be an exact symplectic manifold, take the Hamiltonian function $h \in C^\infty(M)$, and consider the stochastic Hamilton equations with trivial stochastic component $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ given by $X_t(\omega) = t$. As we saw in Remark 2.7, the paths of the semimartingales that solve these stochastic Hamilton equations are the smooth curves that integrate the standard Hamilton equations. In this situation the action reads

$$S(\Gamma) = \int \langle \theta, \delta \Gamma \rangle - \int h(\Gamma_s) \, ds.$$ 

If the path $\Gamma_t(\omega)$ is differentiable then the integral $\left( \int \langle \theta, \delta \Gamma \rangle \right)(\omega)$ reduces to the Riemann integral $\int_{\Gamma_t(\omega)} \theta$ and $S(\Gamma)(\omega)$ coincides with the classical action. In particular, if $\Gamma$ is a solution of the stochastic Hamilton equations then the paths $\Gamma_t(\omega)$ are necessarily differentiable (see Remark 2.7), they satisfy the standard Hamilton equations, and hence make the action critical. The following elementary example shows that the converse is not necessarily true, that is one may have semimartingales that satisfy (4.3) and that do not solve the Hamilton equations up to time $\tau_K$.

We will consider a deterministic example. Let $m_0, m_1 \in M$ be two points. Suppose there exists an integral curve $\gamma : [t_0, t_1] \to M$ of the Hamiltonian vector field $X_h$ defined on some time interval $[t_0, t_1]$ such that $\gamma(t_0) = m_0$ and $\gamma(t_1) = m_1$. Define the continuous and piecewise smooth curve $\sigma : [0, t_1] \to M$ as follows:

$$\sigma(t) = \begin{cases} m_0 & \text{if } t \in [0, t_0] \\ \gamma(t) & \text{if } t \in [t_0, t_1]. \end{cases}$$

Let $\varphi$ be a local one-parameter group of diffeomorphisms that fixes $\{m_0, m_1\}$. Then by (4.2)

$$\left[ \frac{d}{ds} \right]_{s=0} S(\varphi_s(\sigma)) = -\int_{\sigma[0, t]} \alpha + \int_0^t \langle \alpha, X_h \rangle(\sigma(t)) \, dt + \langle \theta(\sigma(t)), Y(\sigma(t)) \rangle - \langle \theta(m_0), Y(m_0) \rangle,$$

where $Y(m) = \frac{d}{ds} |_{s=0} \varphi_s(m)$, for any $m \in M$ and $\alpha = \omega^\flat(Y)$. Using that $\sigma$ satisfies the Hamilton equations on $[t_0, t_1]$ and $\alpha(m_0) = 0$, it is easy to see that

$$\left[ \frac{d}{ds} \right]_{s=0} S(\varphi_s(\sigma)) |_{t_1} = 0,$$

that is, $\sigma$ makes the action critical. However, it does not satisfy the Hamilton equations on the interval $[0, t_1]$, because they do not hold on $(0, t_0)$. This shows that the converse of the statement in Theorem 4.9 is not necessarily true. In the following subsection we will obtain such a converse by generalizing the set of variations allowed in the variational principle.

4.2 Variations involving vector fields on the solution semimartingale

We start by spelling out the variations that we will use in order to obtain a converse to Theorem 4.9.

Definition 4.11 Let $M$ be a manifold and $\Gamma$ a $M$-valued semimartingale. Let $s_0 > 0$; we say that the map $\Sigma : (-s_0, s_0) \times \mathbb{R}_+ \times \Omega \to M$ is a pathwise variation of $\Gamma$ whenever $\Sigma_0 = \Gamma_t$ for any $t \in \mathbb{R}_+$ a.s.. We say that the pathwise variation $\Sigma$ of $\Gamma$ converges uniformly to $\Gamma$ whenever the following properties are satisfied:
(i) For any \( f \in C^{\infty}(M) \), \( f(\Sigma) \rightarrow f(\Gamma) \) in \( ucp \) as \( s \rightarrow 0 \).

(ii) There exists a process \( Y: \mathbb{R}_+ \times \Omega \rightarrow TM \) over \( \Gamma \) such that, for any \( f \in C^{\infty}(M) \), the Stratonovich integral \( \int Y[f] \, dX \) exists for any continuous real semimartingale \( X \) (this is for instance guaranteed if \( Y \) is a semimartingale) and, additionally, the increments \( f(\Sigma^s) - f(\Gamma)/s \) converge in \( ucp \) to \( Y[f] \) as \( s \rightarrow 0 \). We will call such a \( Y \) the \textbf{infinitesimal generator} of \( \Sigma \).

We will say that \( \Sigma \) (respectively, \( Y \)) is \textbf{bounded} when its image lies in a compact set of \( M \) (respectively, \( TM \)).

The next proposition shows that, roughly speaking, there exist bounded pathwise variations that converge uniformly to a given semimartingale with prescribed bounded infinitesimal generator.

**Proposition 4.12** Let \( \Gamma \) be a continuous \( M \)-valued semimartingale \( \Gamma, K \subseteq M \) a compact set, and \( \tau_K \) the first exit time of \( \Gamma \) from \( K \). Let \( Y: \mathbb{R}_+ \times \Omega \rightarrow TM \) be a bounded process over \( \Gamma^{\tau_K} \) (that is, the image of \( Y \) lies in a compact subset of \( TM \)) such that \( \int Y[f] \, dX \) exists for any continuous real semimartingale \( X \) and for any \( f \in C^{\infty}(M) \). Then, there exists a bounded pathwise variation \( \Sigma \) that converges uniformly to \( \Gamma^{\tau_K} \) whose infinitesimal generator is \( Y \).

**Proof.** Let \( \{(V_k, \varphi_k)\}_{k \in \mathbb{N}} \) be a countable open covering of \( M \) by coordinate patches such that any \( V_k \) is contained in a compact set. This covering is always available by the second countability of the manifold and Lindelöf’s Lemma. Let \( \{U_k\}_{k \in \mathbb{N}} \) be an open subcovering such that, if \( U_k \subseteq V_i \) for some \( k, i \in \mathbb{N} \), then \( U_k \subseteq V_i \). Let \( \tau_m \in \mathbb{N} \) be a sequence of stopping times (available by Lemma 3.5 in [ES9]) such that, a.s., \( \tau_0 = 0 \), \( \tau_m \leq \tau_{m+1} \), \( \sup_m \tau_m = \infty \), and that, on each of the sets \( [\tau_m, \tau_{m+1}] \cap \{\tau_{m+1} > \tau_m\} \) the semimartingale \( X \) takes values in the open set \( U_k(m) \), for some \( k(m) \in \mathbb{N} \). Since \( K \) is compact, it can be covered by a finite number of these open sets, i.e. \( K \subseteq \bigcup_{j} U_{k(j)} \), where \( |J| < \infty \).

Let \( x_{k(j)} \equiv (x^1_{k(j)}, \ldots, x^n_{k(j)}) \), \( n = \dim(M) \) be a set of coordinate functions on \( U_{k(j)}(m) \) and \( (x_{k(j)}, v_{k(j)}) \equiv (x^1_{k(j)}, v^1_{k(j)}, \ldots, x^n_{k(j)}, v^n_{k(j)}) \) the corresponding adapted coordinates for \( TM \) on \( \pi^{-1}_TM(U_{k(j)}(m)) \). Since \( Y \) is bounded and covers \( \Gamma^{\tau_K} \), and on \( [\tau_m, \tau_{m+1}] \cap \{\tau_{m+1} > \tau_m\} \) the semimartingale \( X \) takes values in the open set \( U_{k(j)}(m) \), there exist a \( s_{k(j)} > 0 \) such that, on \( [\tau_m, \tau_{m+1}] \cap \{\tau_{m+1} > \tau_m\} \), the points \( (x^1_{k(j)}(Y), \ldots, x^n_{k(j)}(Y)) \) lie in the image of some coordinate patch \( V_k \) containing \( U_{k(j)}(m) \) for all \( s \in (-s_{k(j)}, s_{k(j)}) \). Let \( s_0 = \min_{j \in J} \{s_{k(j)}\} \). Now, since the sets of the form \( I_m := [\tau_m, \tau_{m+1}] \cap \{\tau_{m+1} > \tau_m\} \subseteq \mathbb{R}_+ \times \Omega, m \in \mathbb{N} \) form a disjoint partition of \( \mathbb{R}_+ \times \Omega \) we define \( \Sigma \) as the map that for any \( m \in \mathbb{N} \) satisfies

\[
\Sigma|_{I_m} : (-s_0, s_0) \times [\tau_m, \tau_{m+1}] \cap \{\tau_{m+1} > \tau_m\} \rightarrow V_{k(j)} \quad (s, t, \omega) \mapsto \varphi_k^{-1}(x_{k(j)}(\Gamma_t(\omega)) + sv_{k(j)}(Y_t(\omega))) .
\]

Observe that by construction the image of \( \Sigma \) is covered by a finite number of coordinated patches and therefore, by hypothesis, contained in a compact set. \( \Sigma \) is hence bounded. More specifically

\[
\{\Sigma^s_t(\omega) \mid (s, t, \omega) \in (-s_0, s_0) \times \mathbb{R} \times \Omega\} \subseteq \bigcup_{j \in J} V_{k(j)} . \tag{4.5}
\]

It is immediate to see that \( \Sigma \) is a pathwise variation which converges uniformly to \( \Gamma^{\tau_K} \). Indeed, if \( f \in C^{\infty}(M) \) has compact support within one of the elements in the family \( \{U_{k(j)}\}_{j \in J} \), it can be easily checked that

\[
f(\Sigma^s) \xrightarrow{ucp} f(\Gamma) \quad \text{and} \quad \frac{f(\Sigma^s) - f(\Gamma)}{s} \xrightarrow{ucp} Y[f] . \tag{4.6}
\]

If, more generally, \( f \in C^{\infty}(M) \) has not compact support contained in one of the \( \{U_{k(j)}\}_{j \in J} \), observe that, by [H5], we only need to consider the restriction of \( f \) to \( \bigcup_{j \in J} V_{k(j)} \). Take now a partition of the
unity \( \{ \phi_k \}_{k \in \mathbb{N}} \) subordinated to the covering \( \{ U_k \}_{k \in \mathbb{N}} \). Since \( \{ \text{supp} (\phi_k) \}_{k \in \mathbb{N}} \) is a locally finite family and \( \bigcup_{j \in J} V_{kj} \) is contained in a compact set because, by hypothesis, so is each \( V_{kj} \) for any \( j \in J \), then among all the \( \{ \phi_k \}_{k \in \mathbb{N}} \) only a finite number of them have their supports in \( \{ U_k \}_{j \in J} \), say \( \{ \phi_{k_i} \}_{i \in I} \) with \( |I| < \infty \). Thus,
\[
f|_{\bigcup_{j \in J} V_{kj}} = \sum_{i=1}^{|I|} \phi_{k_i} f
\]
and since each \( \phi_{k_i} f \) is a function similar to those considered in (4.6) it is straightforward to see that those implications also hold for \( f \).

The following result generalizes Proposition 4.4 to pathwise variations of a semimartingale. The proof can be found in Section 5.2.

**Proposition 4.13** Let \( \Gamma \) be a \( M \)-valued continuous semimartingale \( \Gamma, \ K \subseteq M \) a compact set, and \( \tau_K \) the first exit time of \( \Gamma \) from \( K \). Let \( \Sigma \) be a bounded pathwise variation that converges uniformly to \( \Gamma^\tau_K \) and \( Y : \mathbb{R}_+ \times \Omega \to TM \) the infinitesimal generator of \( \Sigma \) that we will also assume to be bounded. Then, for any \( \alpha \in \Omega (M) \),
\[
\lim_{s \to 0} \frac{1}{s} \left[ \int \langle \alpha, \delta \Sigma^s \rangle - \int \langle \alpha, \delta \Gamma^\tau_K \rangle \right] = \int \langle i_Y d\alpha, \delta \Gamma^\tau_K \rangle + \langle \alpha (\Gamma^\tau_K), Y \rangle - \langle \alpha (\Gamma^0), Y \rangle_{t=0}.
\]

The next theorem shows that the generalization of the Critical Action Principle in Theorem 4.9 to pathwise variations fully characterizes the stochastic Hamilton’s equations.

**Theorem 4.14 (Second Critical Action Principle)** Let \((M, \omega = -d\theta)\) be an exact symplectic manifold, \( X : \mathbb{R}_+ \times \Omega \to V \) a semimartingale that takes values in the vector space \( V \), and \( \hat{h} : M \to V^* \) a Hamiltonian function. Let \( m_0 \) be a point in \( M \) and \( \Gamma : \mathbb{R}_+ \times \Omega \to M \) a continuous adapted semimartingale defined on \([0, \xi]\) such that \( \Gamma_0 = m_0 \). Let \( K \subseteq M \) be a compact set that contains \( m_0 \) and \( \tau_K \) the first exit time of \( \Gamma \) from \( K \). Suppose that \( \tau_K < \infty \) a.s.. Then,

(i) For any bounded pathwise variation \( \Sigma \) with bounded infinitesimal generator \( Y \) which converges uniformly to \( \Gamma^\tau_K \) uniformly, the action has a directional derivative that equals
\[
\frac{d}{ds} \bigg|_{s=0} S (\Sigma^s) := \lim_{s \to 0} \frac{1}{s} [S (\Sigma^s) - S (\Gamma^\tau_K)] = \int \langle i_Y d\theta, \delta \Gamma^\tau_K \rangle - \int \langle \hat{h} [\Gamma^\tau_K], \delta X \rangle + \langle \theta (\Gamma^\tau_K), Y \rangle - \langle \theta (\Gamma^0), Y \rangle_{t=0}, \quad (4.7)
\]
where the symbol \( \hat{h} [\Gamma^\tau_K] \) is consistent with the notation introduced in Definition 4.1.

(ii) The semimartingale \( \Gamma \) satisfies the stochastic Hamiltonian equations (2.7) with initial condition \( \Gamma_0 = m_0 \) up to time \( \tau_K \) if and only if, for any bounded pathwise variation \( \Sigma : (-s_0, s_0) \times \mathbb{R}_+ \times \Omega \to M \) with bounded infinitesimal generator which converges uniformly to \( \Gamma^\tau_K \) and such that \( \Sigma_0 = m_0 \) and \( \Sigma^s_{\tau_K} = \Gamma_{\tau_K} \) a.s. for any \( s \in (-s_0, s_0) \),
\[
\left. \left\{ \frac{d}{ds} \bigg|_{s=0} S (\Sigma^s) \right\}_{\tau_K} \right) = 0 \quad a.s..
\]

**Proof.** We first show that the limit (4.7) exist. Let \( \Sigma \) be an arbitrary bounded pathwise variation converging to \( \Gamma \) uniformly and \( Y : \mathbb{R}_+ \times \Omega \to TM \) its infinitesimal generator, that we also assume to be bounded. We have
\[
\frac{1}{s} [S (\Sigma^s) - S (\Gamma^\tau_K)] = \frac{1}{s} \left[ \int \langle \theta (\Sigma^s), \delta \Sigma^s \rangle - \int \langle \theta (\Gamma^\tau_K), \delta \Gamma^\tau_K \rangle \right] - \frac{1}{s} \left[ \int \langle \hat{h} (\Sigma^s) - \hat{h} (\Gamma^\tau_K), \delta X \rangle \right].
\]
By Proposition 4.13, the first summand in the right hand side of (4.8) converges up to
\[ \int (i_Y \, d\theta, \delta \Gamma^{\tau_K}) + \langle \theta (\Gamma^{\tau_K}), Y \rangle - \langle \theta (\Gamma^{\tau_K}), Y \rangle_{t=0}, \]
as \( s \to 0 \). An argument similar to the one leading to Proposition 4.13 shows that the second summand converges to \( \int \langle Y \hat{h}(\Gamma^{\tau_K}), \delta X \rangle \). Hence,
\[ \lim_{s \to 0} \frac{1}{s} [S(\Sigma^s) - S(\Gamma^{\tau_K})] = \int (i_Y \, d\theta, \delta \Gamma^{\tau_K}) - \int \langle Y \hat{h}(\Gamma^{\tau_K}), \delta X \rangle + \langle \theta (\Gamma^{\tau_K}), Y \rangle - \langle \theta (\Gamma^{\tau_K}), Y \rangle_{t=0}. \]
If we denote by \( \eta := -i_Y \, d\theta = i_Y \omega \) the one-form over \( \Gamma^{\tau_K} \) built using the vector field \( Y \) over \( \Gamma^{\tau_K} \), the previous relation may be rewritten as
\[ \left[ \frac{d}{ds} \right]_{s=0} S(\Sigma^s) = -\int \langle \eta, \delta \Gamma^{\tau_K} \rangle - \int \langle (dh(\Gamma^{\tau_K}) (\omega^# (\eta)), \ell) X \rangle + \langle \theta (\Gamma^{\tau_K}), Y \rangle - \langle \theta (\Gamma^{\tau_K}), Y \rangle_{t=0}. \]

We are now going to prove the assertion in part (ii). Recall that the hypothesis that \( \Gamma \) satisfies the stochastic Hamilton equations up to time \( \tau_K \) means that
\[ \left( \int \langle \beta, \delta \Gamma \rangle + \int \langle (dh \cdot \omega^# (\beta)) (\Gamma), \delta X \rangle \right)^{\tau_K} = 0, \]
for any \( \beta \in \Omega(M) \). We now show that this expression is also true if we replace \( \beta \) with any process \( \eta : \mathbb{R}_+ \times \Omega \to T^* M \) such that the two Stratonovich integrals involved in (4.10) are well-defined (for instance if \( \beta \) is a semimartingale). Indeed, invoking ([E89, 7.7]) and Whitney’s embedding theorem, there exist an integer \( p \in \mathbb{N} \) such that the manifold \( M \) can be seen as an embedded submanifold of \( \mathbb{R}^p \). In this embedded picture, there exists a family of functions \( \{ f_1, \ldots, f_p \} \in C^\infty(\mathbb{R}^p) \) such that the one-form \( \eta \) may be written as
\[ \eta = \sum_{j=1}^p Z_j \, df_j, \]
where the \( Z_j : \mathbb{R}_+ \times \Omega \to \mathbb{R}, j \in \{ 1, \ldots, p \} \), are real processes. Moreover, using the properties of the Stratonovich integral (see [E89 Proposition 7.4]),
\[ \left( \int \langle \eta, \delta \Gamma \rangle + \int \langle (dh \cdot \omega^# (\eta)) (\Gamma), \delta X \rangle \right)^{\tau_K} = \left( \sum_{j=1}^p Z_j \delta \left( \int \langle df_j, \delta \Gamma \rangle + \int \langle (dh \cdot \omega^# (df_j)) (\Gamma), \delta X \rangle \right) \right)^{\tau_K} = \sum_{j=1}^p Z_j \delta \left( \int \langle df_j, \delta \Gamma \rangle + \int \langle (dh \cdot \omega^# (df_j)) (\Gamma), \delta X \rangle \right)^{\tau_K}, \]
where the last equality follows from Proposition 6.5. Therefore, since \( df_j \) is a deterministic one-form we can conclude that \( \left( \int \langle (dh \cdot \omega^# (df_j)) (\Gamma), \delta X \rangle \right)^{\tau_K} = 0 \), which justifies why (4.10) also holds if we replace \( \beta \in \Omega(M) \) by an arbitrary integrable one-form \( \eta \) over \( \Gamma \).

Suppose now that \( \Gamma \) satisfies the stochastic Hamilton equations up to \( \tau_K \) and let \( \Sigma : (s_0, s_0) \times \mathbb{R}_+ \times \Omega \to M \) be a pathwise variation like in the statement of the theorem. We want to show that
\[ \left[ \frac{d}{ds} \right]_{s=0} S(\Sigma^s) \bigg|_{\tau_K} = 0 \quad \text{a.s.} \]
Due to (4.9), we have that
\[
\left. \frac{d}{ds} \right|_{s=0} S(\Sigma^s) = - \left( \int \langle \eta, \delta \Gamma^\tau \rangle + \int \langle d h (\Gamma^\tau) (\omega^\#(\eta)), \delta X \rangle \right)_{\tau_K} + \langle \theta (\Gamma^\tau), Y \rangle_{\tau_K} - \langle \theta (\Gamma^\tau), Y \rangle_{t=0}.
\]
Since \( \Sigma_0^s = m_0 \) and \( \Sigma_{\tau_K}^s = \Gamma_{\tau_K} \) a.s. for any \( s \in (-s_0, s_0) \), then \( Y_0 = Y_{\tau_K} = 0 \) a.s. and both \( \langle \theta (\Gamma^\tau), Y \rangle_{\tau_K} \) and \( \langle \theta (\Gamma^\tau), Y \rangle_{t=0} \) vanish. Moreover,
\[
\left( \int \langle \eta, \delta \Gamma^\tau \rangle + \int \langle d h (\Gamma^\tau) (\omega^\#(\eta)), \delta X \rangle \right)_{\tau_K} = \left( \int \langle \eta, \delta \Gamma^\tau \rangle + \int \langle d h (\Gamma^\tau) (\omega^\#(\eta)), \delta X \rangle \right)_{\tau_K} = 0 \tag{4.11}
\]
which is zero because of (4.10). In the last equality we have used Proposition 5.5.

Conversely, suppose that \( \left. \frac{d}{ds} \right|_{s=0} S(\Sigma^s) \right|_{\tau_K} = 0 \) a.s. for arbitrary bounded pathwise variations tending to \( \Gamma^\tau \) uniformly, like in the statement. We want to show that (4.10) holds. Since our pathwise variations satisfy that \( Y_0 = Y_{\tau_K} = 0 \) a.s., we obtain that
\[
\left. \frac{d}{ds} \right|_{s=0} S(\Sigma^s) \right|_{\tau_K} = - \left( \int \langle \eta, \delta \Gamma^\tau \rangle + \int \langle d h (\Gamma^\tau) (\omega^\#(\eta)), \delta X \rangle \right)_{\tau_K} = 0 \tag{4.12}
\]
where \( \eta \) is an arbitrary bounded one form over \( \Gamma \). Suppose now that \( \eta \) is a semimartingale. Then \( 1_{[0, t]}: \mathbb{R}_+ \times \Omega \to T^* M \) is again bounded and expressions
\[
\int \langle 1_{[0, t]} \eta, \delta \Gamma^\tau \rangle \quad \text{and} \quad \int \langle d h (\Gamma^\tau) (\omega^\#(1_{[0, t]} \eta)), \delta X \rangle
\]
are well-defined by Proposition 5.7 because both \( \Gamma^\tau \) and \( X \) are continuous semimartingales. We already saw in (4.11) that (4.12) is equivalent to
\[
\left( \int \langle \eta, \delta \Gamma \rangle + \int \langle d h (\Gamma) (\omega^\#(1_{[0, t]} \eta)), \delta X \rangle \right)_{\tau_K} = 0.
\]
Replacing \( \eta \) by \( 1_{[0, t]} \eta \) in (4.12) and using again the Proposition 5.7, we write
\[
0 = \left( \int \langle 1_{[0, t]} \eta, \delta \Gamma \rangle + \int \langle d h (\Gamma) (\omega^\# (1_{[0, t]} \eta)), \delta X \rangle \right)_{\tau_K} = \left( \left( \int \langle \eta, \delta \Gamma \rangle + \int \langle d h (\Gamma) (\omega^\#(\eta)), \delta X \rangle \right)_{t} \right)_{\tau_K}
\]
\[
= \left( \int \langle \eta, \delta \Gamma \rangle + \int \langle d h (\Gamma) (\omega^\#(\eta)), \delta X \rangle \right)_{t \wedge \tau_K} = \left( \left( \int \langle \eta, \delta \Gamma \rangle + \int \langle d h (\Gamma) (\omega^\#(\eta)), \delta X \rangle \right)_{\tau_K} \right)_{t}.
\]
Since \( t \) is arbitrary this implies that the process \( (\int \langle \eta, \delta \Gamma \rangle + \int \langle d h (\Gamma) (\omega^\#(\eta)), \delta X \rangle)_{\tau_K} \) is identically zero, as required. ■

5 Proofs and auxiliary results

5.1 Proof of Proposition 4.4

Before proving the proposition, we recall a technical lemma dealing with the convergence of sequences in a metric space.
Lemma 5.1 Let \((E, d)\) be a metric space. Let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence of functions \(x_n : (0, \delta) \to E\) where \((0, \delta) \subset \mathbb{R}\) is an open interval of the real line. Suppose that \(x_n\) converges uniformly on \((0, \delta)\) to a function \(x\). Additionally, suppose that for any \(n\), the limits \(\lim_{s \to 0} x_n(s) = x_n^* \in E\) exist and so does \(\lim_{n \to \infty} x_n^*\). Then

\[
\lim_{s \to 0} x(s) = \lim_{n \to \infty} x_n^*.
\]

Proof. Let \(\varepsilon > 0\) be an arbitrary real number. We have

\[
d\left(x(s), \lim_{n \to \infty} x_n^*\right) \leq d\left(x(s), x_k(s)\right) + d\left(x_k(s), x_n^*\right) + d\left(x_n^*, \lim_{n \to \infty} x_n^*\right).
\]

From the definition of limit and since \(x_k(s)\) converges uniformly to \(x\) on \((0, \delta)\), we can choose \(k_0\) such that \(d\left(x_k(s), x_n^*\right) < \frac{\varepsilon}{3}\) and \(d\left(x(s), x_k(s)\right) < \frac{\varepsilon}{3}\), simultaneously for any \(k \geq k_0\). Additionally, since \(\lim_{s \to 0} x_k(s) = x_k^*\) we choose \(s_0\) small enough such that \(d\left(x_k(s), x_k^*\right) < \frac{\varepsilon}{3}\), for any \(s < s_0\). Thus,

\[
d\left(x(s), \lim_{n \to \infty} x_n^*\right) < \varepsilon
\]

for any \(s < s_0\). Since \(\varepsilon > 0\) is arbitrary, we conclude that \(\lim_{s \to 0} x(s) = \lim_{n \to \infty} x_n^*\). □

Proof of Proposition 4.4. First of all, the second equality in (4.1) is a straightforward consequence of [E89] page 93. Now, let \(\{U_k\}_{k \in \mathbb{N}}\) be a countable open covering of \(M\) by coordinate patches. By [E89] Lemma 3.5 there exists a sequence \(\{\tau_m\}_{m \in \mathbb{N}}\) of stopping times such that \(\tau_0 = 0, \tau_m \leq \tau_{m+1}\), \(\sup_m \tau_m = \infty\), a.s., and that, on each of the sets

\[
[\tau_m, \tau_{m+1}] \cap \{\tau_m < \tau_{m+1}\} := \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid \tau_{m+1}(\omega) > \tau_m(\omega)\} \quad (5.1)
\]

the semimartingale \(\Gamma\) takes its values in one of the elements of the family \(\{U_k\}_{k \in \mathbb{N}}\).

Second, the statement of the proposition is formulated in terms of Stratonovich integrals. However, the proof will be carried out in the context of Itô integration since we will use several times the notion of uniform convergence on compacts in probability (ucp) which behaves well only with respect to this integral. Regarding this point we recall that by the very definition of the Stratonovich integral of a 1-form \(\alpha\) along a semimartingale \(\Gamma\) we have that

\[
\int \langle \varphi_s^* \alpha, \delta \Gamma \rangle = \int \langle d_2(\varphi_s^* \alpha), d\Gamma \rangle \quad \text{and} \quad \int \langle \xi_{\gamma} \alpha, \delta \Gamma \rangle = \int \langle d_2(\xi_{\gamma} \alpha), d\Gamma \rangle. \quad (5.2)
\]

The proof of the proposition follows directly from Lemma 5.1 by applying it to the sequence of functions given by

\[
x_n(s) := \left(\int \left\langle \frac{1}{s} [d_2(\varphi_s^* \alpha) - d_2(\alpha)], d\Gamma \right\rangle\right)^{\tau_n}.
\]

This sequence lies in the space \(\mathcal{D}\) of càglàd processes endowed with the topology of the ucp convergence. We recall that this space is metric [P90] page 57 and hence we are in the conditions of Lemma 5.1 In the following points we verify that the rest of the hypotheses of this result are satisfied.

(i) The sequence of functions \(\{x_n(s)\}_{n \in \mathbb{N}}\) converges uniformly to

\[
x(s) := \int \left\langle \frac{1}{s} [d_2(\varphi_s^* \alpha) - d_2(\alpha)], d\Gamma \right\rangle.
\]

The pointwise convergence is a consequence of part (i) in Proposition 5.6. Moreover, in the proof of that result we saw that if \(d : \mathcal{D} \times \mathcal{D} \to \mathbb{R}_+\) is a distance function function associated to the ucp convergence,
then for any $t \in \mathbb{R}_+$ and any $s \in (0, \epsilon)$, $d(x_n(s), x(s)) \leq P(\{\tau_n < t\})$. Since the right hand side of this inequality does not depend on $s$ and $P(\{\tau_n < t\}) \to 0$ as $n \to \infty$, the uniform convergence follows.

(ii) $$\lim_{\text{ucp}} x_n(s) = \left( \int (d_2 (\mathcal{L} \alpha), d\Gamma) \right)_{\tau_n} =: x_n^\ast.$$ By the construction of the covering $\{U_k\}_{k \in \mathbb{N}}$ and of the stopping times $\{\tau_m\}_{m \in \mathbb{N}}$, there exists a $k(m) \in \mathbb{N}$ such that the semimartingale $\Gamma$ takes its values in $U_{k(m)}$ when evaluated in the stochastic interval $(\tau_n, \tau_{n+1}) \subset (\tau_n, \tau_{n+1}) \cap \{\tau_n < \tau_{n+1}\}$. Now, since $d_2$ is a linear operator and $\frac{1}{s} ((\varphi_\ast^\alpha - \alpha)(m) \xrightarrow{s \to 0} \mathcal{L} \alpha(m)$, for any $m \in M$, we have that $\frac{1}{s} (d_2 (\varphi_\ast^\alpha - d_2 \alpha) (m) \xrightarrow{s \to 0} d_2 (\mathcal{L} \alpha) (m)$. Moreover, a straightforward application of Taylor’s theorem shows that $\frac{1}{s} (d_2 (\varphi_\ast^\alpha - d_2 \alpha) |_{U_{k(m)}} ) \xrightarrow{s \to 0} \mathcal{L} \alpha |_{U_{k(m)}}$ uniformly, using a Euclidean norm in $\tau^\ast U_{k(m)}$ (we recall that $U_{k(m)}$ is a coordinate patch). This fact immediately implies that $\lim_{s \to 0} \int \mathbf{1}_{(\tau_n, \tau_{n+1})} \left( \frac{1}{s} (d_2 (\varphi_\ast^\alpha - d_2 \alpha), \mathcal{L} \alpha) \right) \xrightarrow{\text{ucp}} \int \mathbf{1}_{(\tau_n, \tau_{n+1})} \left( d_2 (\mathcal{L} \alpha), \mathcal{L} \alpha) \right).$ (5.3)

Consequently,

$$\lim_{s \to 0} \left( \int \left\langle \frac{1}{s} [d_2 (\varphi_\ast^\alpha - d_2 (\alpha)], d\Gamma \right\rangle \right)_{\tau_n} = \lim_{s \to 0} \sum_{m=0}^{n-1} \left( \int \left\langle \frac{1}{s} [d_2 (\varphi_\ast^\alpha - d_2 (\alpha)], d\Gamma \right\rangle \right)_{\tau_{m+1}} - \left( \int \left\langle \frac{1}{s} [d_2 (\varphi_\ast^\alpha - d_2 (\alpha)], d\Gamma \right\rangle \right)_{\tau_m} = \lim_{s \to 0} \sum_{m=0}^{n-1} \int \mathbf{1}_{(\tau_m, \tau_{m+1})} \left( \frac{1}{s} (d_2 (\varphi_\ast^\alpha - d_2 \alpha), \mathcal{L} \alpha) \right) = \sum_{m=0}^{n-1} \int \mathbf{1}_{(\tau_m, \tau_{m+1})} \left( d_2 (\mathcal{L} \alpha), \mathcal{L} \alpha) \right)$$

where in the second equality we have used Proposition 5.5 and the third one follows from (5.3).

(iii) $$\lim_{n \to \infty} x_n^\ast = \int \langle d_2 (\mathcal{L} \alpha), d\Gamma \rangle.$$ It is a straightforward consequence of Proposition 5.6

The equation (4.11) follows from Lemma 4.11 applied to the sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{x_n^\ast\}_{n \in \mathbb{N}}$, and using the statements in (i), (ii), and (iii). ■

5.2 Proof of Proposition 4.13

We will start the proof by introducing three preparatory results.

Lemma 5.2 Let $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ be two sequences of real valued processes converging in ucp to a couple of processes $X$ and $Y$ respectively. Suppose that, for any $t \in \mathbb{R}_+$, the random variables $\sup_{n \in \mathbb{N}} \sup_{0 \leq s \leq t} |(X_n)_s|$ and $\sup_{0 \leq s \leq t} |Y_n|_s$ are bounded (their images lie in a compact set of $\mathbb{R}$). Then, the sequence $X_n Y_n$ converges in ucp to $XY$ as $n \to \infty$. 

Proof. We need to prove that for any \( \varepsilon > 0 \) and any \( t \in \mathbb{R}_+ \),
\[
P \left( \sup_{0 \leq s \leq t} |(X_n Y_n)_s - (XY)_s| \leq \varepsilon \right) \xrightarrow[n \to \infty]{} 1.
\]

First of all, note that
\[
\sup_{0 \leq s \leq t} |(X_n Y_n)_s - (XY)_s| \leq \sup_{0 \leq s \leq t} |X_n| |Y_n - Y| + \sup_{0 \leq s \leq t} |Y| |X_n - X|.
\]

Hence, we have
\[
\left\{ \sup_{0 \leq s \leq t} |(X_n Y_n)_s - (XY)_s| \leq \varepsilon \right\} \supseteq \left\{ \sup_{0 \leq s \leq t} |X_n| |Y_n - Y| + \sup_{0 \leq s \leq t} |Y| |X_n - X| \leq \varepsilon \right\}
\]
\[
\sup_{0 \leq s \leq t} |X_n| |Y_n - Y| \leq \frac{\varepsilon}{2} \cap \left\{ \sup_{0 \leq s \leq t} |Y| |X_n - X| \leq \frac{\varepsilon}{2} \right\}.
\]

Denote
\[
A_n := \left\{ \sup_{0 \leq s \leq t} |X_n| |Y_n - Y| \leq \frac{\varepsilon}{2} \right\}, \quad \text{and} \quad B_n := \left\{ \sup_{0 \leq s \leq t} |Y| |X_n - X| \leq \frac{\varepsilon}{2} \right\},
\]
and let \( c \) be a constant such that \( \sup_{n \in \mathbb{N}} \sup_{0 \leq s \leq t} |(X_n)_s| \leq c \) and \( \sup_{0 \leq s \leq t} |Y_s| \leq c \), available by the boundedness hypothesis. Then,
\[
1 \geq P(A_n) \geq P \left( \left\{ \sup_{0 \leq s \leq t} |Y_n - Y| \leq \frac{\varepsilon}{2c} \right\} \right) \xrightarrow[n \to \infty]{} 1,
\]
\[
1 \geq P(B_n) \geq P \left( \left\{ \sup_{0 \leq s \leq t} |X_n - X| \leq \frac{\varepsilon}{2c} \right\} \right) \xrightarrow[n \to \infty]{} 1.
\]

Thus, \( P(A_n) \to 1 \) and \( P(B_n) \to 1 \) as \( n \to \infty \). But as \( P(A_n \cap B_n) = P(A_n) + P(B_n) - P(A_n \cup B_n) \), we conclude that
\[
P(A_n \cap B_n) \xrightarrow[n \to \infty]{} 1.
\]

Since \( A_n \cap B_n \subseteq \left\{ \sup_{0 \leq s \leq t} |(X_n Y_n)_s - (XY)_s| \leq \varepsilon \right\} \), we obtain
\[
P \left( \left\{ \sup_{0 \leq s \leq t} |(X_n Y_n)_s - (XY)_s| \leq \varepsilon \right\} \right) \xrightarrow[n \to \infty]{} 1. \quad \blacksquare
\]

Lemma 5.3 Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of real processes converging in ucp to a process \( X \). Let \( \tau \) be a stopping time such that \( \tau < \infty \) a.s.. Then, the sequence of random variables \( \{(X_n)_\tau\}_{n \in \mathbb{N}} \) converge in probability to \( (X)_\tau \).

Proof. First of all we show that since \( \tau < \infty \) a.s., then \( P(\{\tau > t\}) \) converges to zero as \( t \to \infty \).

By contradiction, suppose that this is not the case. Then, denoting \( A_n := \{\tau > n\} \), we have that \( A_{n+1} \subseteq A_n \), so \( P(A_n) \) forms a non-increasing sequence of real numbers in the interval \([0, 1]\). Since this sequence is bounded below, it must have a limit. This limit corresponds to the probability of the event \( \{\tau = \infty\} \). If it is strictly positive then there is a contradiction with the fact that \( \tau < \infty \) a.s. So \( P(\{\tau > t\}) \) tends to zero as \( t \to \infty \).

We now prove the statement of the lemma. Take some \( \varepsilon > 0 \) and an auxiliary \( t \in \mathbb{R}_+ \). The set \( \{(X_n)_\tau - X_\tau | > \varepsilon\} \) can be decomposed as the disjoint union of the following two events,
\[
\{(X_n)_\tau - X_\tau | > \varepsilon\} \cap \{\tau \leq t\} \cup \{(X_n)_\tau - X_\tau | > \varepsilon\} \cap \{\tau > t\}.
\]
The first one is contained in the set \( \{ \sup_{0 \leq s \leq t} |(X_n)_s - X_s| > \varepsilon \} \) whose probability, by hypothesis, converges to zero as \( n \to \infty \). Regarding the second one,

\[
P\left( \{|(X_n)_\tau - X_\tau| > \varepsilon \} \cap \{ \tau > t \} \right) \leq P\left( \{ \tau > t \} \right).
\]

But \( P\left( \{ \tau > t \} \right) \) can be made arbitrarily small by taking the auxiliary \( t \) big enough. In conclusion, for any \( \varepsilon > 0 \),

\[
P\left( \{|(X_n)_\tau - X_\tau| > \varepsilon \} \right) \xrightarrow{n \to \infty} 0
\]
in probability. ▼

**Lemma 5.4** Let \( \{X_n\}_{n \in \mathbb{N}} \) be a sequence of real processes converging in ucp to a real process \( X \) and \( \tau \) a stopping time. Then, the stopped sequence \( \{X^\tau_n\}_{n \in \mathbb{N}} \) converges in ucp to \( X^\tau \) as well.

**Proof.** We just need to observe that, for any \( t \in \mathbb{R}_+ \),

\[
\sup_{0 \leq s \leq t} |(X^\tau_n)_s - X^\tau_s| = \sup_{0 \leq s \leq t} |(X_n)_{\tau \wedge s} - X_{\tau \wedge s}| \leq \sup_{0 \leq s \leq t} |(X_n)_s - X_s|
\]

and, consequently, for any \( \varepsilon > 0 \),

\[
\left\{ \sup_{0 \leq s \leq t} |(X_n)_s - X_s| \leq \varepsilon \right\} \subseteq \left\{ \sup_{0 \leq s \leq t} |(X^\tau_n)_s - X^\tau_s| \leq \varepsilon \right\}.
\]

Hence, since by hypothesis \( P\left( \{ \sup_{0 \leq s \leq t} |(X_n)_s - X_s| \leq \varepsilon \} \right) \) converges to 1 as \( n \to \infty \), then so does \( P\left( \{ \sup_{0 \leq s \leq t} |(X^\tau_n)_s - X^\tau_s| \leq \varepsilon \} \right) \). ▶

We now proceed with the proof of the proposition. We will start by using Whitney’s Embedding Theorem and the remarks in [ES9] 7.7 to visualize \( M \) as an embedded submanifold of \( \mathbb{R}^p \), for some \( p \in \mathbb{N} \), and to write down our Stratonovich integrals as real Stratonovich integrals. Indeed, there exists a family of functions \( \{ h^1, \ldots, h^p \} \subset C^\infty(\mathbb{R}^p) \) such that, in the embedded picture, the one form \( \alpha \) can be written as \( \alpha = \sum_{j=1}^p h^j \mathrm{d}h^j \), where \( h^j \in C^\infty(\mathbb{R}^p) \) for \( j \in \{1, \ldots, p\} \). Therefore, using the properties of the Stratonovich integral (see [ES9] Proposition 7.4),

\[
\frac{1}{s} \left[ \int \langle \alpha, \delta \Sigma^s \rangle - \int \langle \alpha, \delta \Gamma^\tau \kappa \rangle \right] = \sum_{j=1}^p \frac{1}{s} \left[ \int Z_j(\Sigma^s) \delta( h^j(\Sigma^s)) - \int Z_j(\Gamma^\tau \kappa) \delta( h^j(\Gamma^\tau \kappa)) \right]. \tag{5.4}
\]

Adding and subtracting the term \( \sum_{j=1}^p \int Z_j(\Sigma^s) \delta h^j(\Gamma^\tau \kappa) \) in the right hand side of (5.4), we have

\[
\frac{1}{s} \left[ \int \langle \alpha, \delta \Sigma^s \rangle - \int \langle \alpha, \delta \Gamma^\tau \kappa \rangle \right] = \sum_{j=1}^p \frac{1}{s} \left[ \int Z_j(\Sigma^s) \delta h^j(\Sigma^s) - \int Z_j(\Sigma^s) \delta h^j(\Gamma^\tau \kappa) \right] \tag{1}
\]

\[
+ \sum_{j=1}^p \frac{1}{s} \left[ \int (Z_j(\Sigma^s) - Z_j(\Gamma^\tau \kappa)) \delta h^j(\Gamma^\tau \kappa) \right]. \tag{2}
\]

We are going to study the terms (1) and (2) separately. We start by considering

\[
\sigma_n = \{ 0 = T_0^n \leq T_1^n \leq \ldots \leq T_k_n < \infty \},
\]

a sequence of random partitions that tends to the identity (in the sense of [P90] page 64)].
The expression (1): We want to study the \( ucp \) convergence of \( \frac{1}{s} \left[ \int Z_j (\Sigma^s) \delta h^j (\Sigma^s) - \int Z_j (\Sigma^s) \delta h^j (\Gamma^{\tau K}) \right] \) as \( s \to 0 \). Define

\[
x_n (s) := \frac{1}{s} \sum_{i=0}^{k_n-1} \frac{1}{2} \left( Z_j (\Sigma^s)_{T_i+1} - Z_j (\Sigma^s)_{T_i} \right) \left( h^j (\Sigma^s)^{T_{i+1} - h^j (\Sigma^s)^{T_i}} + h^j (\Sigma^s)_{T_{i+1}} - h^j (\Sigma^s)_{T_i} \right) - \frac{1}{2} \left( Z_j (\Sigma^s)_{T_{i+1} - h^j (\Sigma^s)}_{T_i} + Z_j (\Sigma^s)_{T_{i+1}} - h^j (\Sigma^s)_{T_i} \right) \\
= \sum_{i=0}^{k_n-1} \frac{1}{2} \left( Z_j (\Sigma^s)_{T_{i+1} - h^j (\Sigma^s)}_{T_i} + Z_j (\Sigma^s)_{T_{i+1}} - h^j (\Sigma^s)_{T_i} \right) \left( \frac{h^j (\Sigma^s)^{T_{i+1} - h^j (\Sigma^s)}_{T_i}}{s} - \frac{h^j (\Sigma^s)^{T_{i+1} - h^j (\Sigma^s)}_{T_i}}{s} \right),
\]

which corresponds to the discretization of the Stratonovich integrals \( \frac{1}{s} \left[ \int Z_j (\Sigma^s) \delta h^j (\Sigma^s) - \int Z_j (\Sigma^s) \delta h^j (\Gamma^{\tau K}) \right] \) using the random partitions of \( \sigma_n \). Indeed, by [P90 Corollary 1, page 291],

\[
x_n (s) \xrightarrow{ucp, n \to \infty} \frac{1}{s} \left[ \int Z_j (\Sigma^s) \delta h^j (\Sigma^s) - \int Z_j (\Sigma^s) \delta h^j (\Gamma^{\tau K}) \right].
\]

On the other hand, as \( T_i^n < \infty \) a.s. for any \( i \in \{1, \ldots, k_n\} \), part (i) in Definition 4.11 and Lemma 5.3 imply that

\[
\frac{1}{2} \left( Z_j (\Sigma^s)_{T_{i+1} - h^j (\Sigma^s)}_{T_i} + Z_j (\Sigma^s)_{T_{i+1}} - h^j (\Sigma^s)_{T_i} \right) \xrightarrow{ucp, s \to 0} \frac{1}{2} \left( Z_j (\Gamma^{\tau K})_{T_{i+1} - h^j (\Sigma^s)} + Z_j (\Gamma^{\tau K})_{T_{i+1}} - Z_j (\Gamma^{\tau K})_{T_i} \right)
\]

The convergence above is in probability but, for convenience, we prefer to regard these random variables as trivial processes. Furthermore, part (ii) in Definition 4.11 and Lemma 5.3 imply that

\[
\frac{h^j (\Sigma^s)^{T_{i+1} - h^j (\Sigma^s)}_{T_i}}{s} = \left( \frac{h^j (\Sigma^s)_{T_{i+1} - h^j (\Sigma^s)}}{s} - \frac{h^j (\Sigma^s)_{T_i}}{s} \right) \xrightarrow{ucp, s \to 0} \frac{h^j (\Sigma^s)_{T_{i+1} - h^j (\Sigma^s)}}{s} \xrightarrow{ucp, s \to 0} Y \left[ h^j \right]_{T_{i+1}}^{T_i},
\]

by Definition 4.11 item 3 and Lemma 5.4. Now, since by hypothesis \( \Sigma \) and \( Y \) are bounded then so are \( \frac{1}{2} \left( Z_j (\Sigma^s)_{T_{i+1} - h^j (\Sigma^s)} + Z_j (\Sigma^s)_{T_i} \right) \) and \( Y \left[ h^j \right] \) is \( Y \left[ h^j \right] \) is only evaluated on the compact \( K \) since \( Y \) is a vector field over \( \Gamma^{\tau K} \) and hence by Lemma 5.2

\[
x_n (s) \xrightarrow{ucp, s \to 0} \sum_{i=0}^{k_n-1} \frac{1}{2} \left( Z_j (\Gamma^{\tau K})_{T_{i+1} - h^j (\Sigma^s)} + Z_j (\Gamma^{\tau K})_{T_i} \right) \left( Y \left[ h^j \right]_{T_{i+1} - h^j (\Sigma^s)} + Y \left[ h^j \right]_{T_i} \right) =: x_n^s
\]

In addition, by [P90 Corollary 1, page 291],

\[
x_n^s \xrightarrow{ucp, n \to \infty} \int Z_j (\Gamma^{\tau K}) \delta \left( Y \left[ h^j \right] \right).
\]

Hence, by Lemma 5.1 we conclude that

\[
\frac{1}{s} \left[ \int Z_j (\Sigma^s) \delta h^j (\Sigma^s) - \int Z_j (\Sigma^s) \delta h^j (\Gamma^{\tau K}) \right] \xrightarrow{ucp, s \to 0} \int Z_j (\Gamma^{\tau K}) \delta \left( Y \left[ h^j \right] \right).
\]

(5.6)
The expression (2): We want to study now the \textit{ucp} convergence of $\frac{1}{s} \int (Z_j (\Sigma^s) - Z_j (\Gamma^\tau)) \delta h^j (\Gamma^\tau)$ as $s \to 0$. As in the previous paragraphs, we define

$$y_n (s) := \frac{1}{s} \left( \sum_{i=0}^{k_n-1} \frac{1}{2} \left( Z_j (\Sigma^s)_{T_{i+1}} + Z_j (\Sigma^s)_{T_i} \right) \left( h^j (\Gamma^\tau)_{T_{i+1}} - h^j (\Gamma^\tau)_{T_i} \right) - \sum_{i=0}^{k_n-1} \frac{1}{2} \left( Z_j (\Gamma^\tau)_{T_{i+1}} + Z_j (\Gamma^\tau)_{T_i} \right) \left( h^j (\Gamma^\tau)_{T_{i+1}} - h^j (\Gamma^\tau)_{T_i} \right) \right)$$

as a discretization of the Stratonovich integral $\frac{1}{s} \int (Z_j (\Sigma^s) - Z_j (\Gamma^\tau)) \delta h^j (\Gamma^\tau)$ using $\sigma_n$. Then, by construction,

$$y_n (s) \xrightarrow{\text{ucp}} \frac{1}{s} \int (Z_j (\Sigma^s) - Z_j (\Gamma^\tau)) \delta h^j (\Gamma^\tau).$$

On the other hand, invoking Definition 4.11 and Lemma 5.3 we have that

$$\frac{Z_j (\Sigma^s)_{T_{i+1}} - Z_j (\Gamma^\tau)_{T_{i+1}}}{s} = \frac{Z_j (\Sigma^s) - Z_j (\Gamma^\tau)}{s} \xrightarrow{\text{ucp}} Y [Z_j]_{T_{i+1}}$$

and

$$\frac{Z_j (\Sigma^s)_{T_i} - Z_j (\Gamma^\tau)_{T_i}}{s} = \frac{Z_j (\Sigma^s) - Z_j (\Gamma^\tau)}{s} \xrightarrow{\text{ucp}} Y [Z_j]_{T_i}.$$

We now use again the boundedness of $\Sigma$ and $Y$ to guarantee the boundedness of $Y [Z_j]_{T_{i+1}} = (iy d Z_j)_{T_{i+1}}$ and $Y [Z_j]_{T_i} = (iy d Z_j)_{T_i}$ (notice that $d Z_j$ is only evaluated on the compact set $\mathcal{K}$ because $Y$ is a vector field over $\Gamma^\tau \subseteq \mathcal{K}$). Therefore, by Lemma 5.2

$$x_n (s) \xrightarrow{\text{ucp}} \sum_{i=0}^{k_n-1} \frac{1}{2} \left( Y [Z_j]_{T_{i+1}} + Y [Z_j]_{T_i} \right) \left( h^j (\Gamma^\tau)_{T_{i+1}} - h^j (\Gamma^\tau)_{T_i} \right) := x^*_n.$$

Additionally, the sequence $\{x^*_n\}_{n \in \mathbb{N}}$ obviously converge in \textit{ucp} to $\int Y [Z_j] \delta (h^j (\Gamma^\tau))$ as $n \to \infty$. Hence, by Lemma 5.1 we conclude that

$$\frac{1}{s} \left[ \int (Z_j (\Sigma^s) - Z_j (\Gamma^\tau)) \delta h^j (\Gamma^\tau) \right] \xrightarrow{\text{ucp}} \int Y [Z_j] \delta (h^j (\Gamma^\tau)).$$

To sum up, if we substitute (5.6) and (5.7) in (5.5) we obtain that

$$\frac{1}{s} \left[ \int \langle \alpha, \delta \Sigma^s \rangle - \int \langle \alpha, \delta \Gamma^\tau \rangle \right] \xrightarrow{\text{ucp}} \sum_{j=1}^{p} \int Z_j (\Gamma^\tau) \delta (Y [h^j]) + \int Y [Z_j] \delta (h^j (\Gamma^\tau)).$$

Using the integration by parts formula,

$$\int Z_j (\Gamma^\tau) \delta (Y [h^j]) = Z_j (\Gamma^\tau) Y [h^j] - \left. (Z_j (\Gamma^\tau) Y [h^j]) \right|_{t=0} - \int Y [h^j] \delta (Z_j (\Gamma^\tau))$$

$$= \langle \alpha (\Gamma^\tau), Y \rangle - \langle \alpha (\Gamma^\tau), Y \rangle|_{t=0} - \int Y [h^j] \delta (Z_j (\Gamma^\tau))$$
and, consequently,
\[
\frac{1}{2} \left[ \int \langle \alpha, \delta \Sigma \rangle - \int \langle \alpha, \delta \Gamma^\tau \rangle \right] \xrightarrow{u.p.} \int Y [Z_j] \delta (h^j (\Gamma^\tau)) - \int Y [h^j] \delta (Z_j (\Gamma^\tau)) + \langle \alpha (\Gamma^\tau), Y \rangle - \langle \alpha (\Gamma^\tau), Y \rangle_{t=0}.
\]

In order to conclude the proof, we claim that
\[
\int Y [Z_j] \delta (h^j (\Gamma^\tau)) - \int Y [h^j] \delta (Z_j (\Gamma^\tau)) = \int \langle Y \, d\alpha, \delta \Gamma^\tau \rangle.
\] (5.8)

Indeed,
\[
d\alpha = d \left( \sum_{j=1}^{p} Z_j \, dh^j \right) = \sum_{j=1}^{p} dZ_j \wedge dh^j, \quad \text{and} \quad i_Y \, d\alpha = \sum_{j=1}^{p} (Y [Z_j] \, dh^j - Y [h^j] \, dZ_j)
\]
which proofs (5.8), as required. ■

5.3 Auxiliary results about integrals and stopping times

In the following paragraphs we collect three results that are used in the paper in relation with the interplay between stopping times and integration limits.

**Proposition 5.5** Let \( X \) be a continuous semimartingale defined on \([0, \zeta_X]\) and \( \Gamma \) a continuous semimartingale. Let \( \tau, \xi \) be two stopping times such that \( \tau \leq \xi < \zeta_X \). Then,
\[
(X \cdot \Gamma)^\tau = (1_{[0,\tau]} X) \cdot \Gamma = (X \cdot \Gamma)^\tau \quad \text{and} \quad (X \cdot \Gamma)^\xi - (X \cdot \Gamma)^\tau = (1_{(\tau,\xi]} X) \cdot \Gamma.
\]

An equivalent result holds when dealing with the Stratonovich integral, namely
\[
\left( \int X \, d\Gamma \right)^\tau = \left( \int X \, d\Gamma \right)^\tau = \left( \int X \, d\Gamma \right)^\tau.
\]

**Proof.** By [P00] Theorem 12, page 60] we have that \( 1_{[0,\tau]} X \cdot \Gamma = (X \cdot \Gamma)^\tau = (X \cdot \Gamma)^\tau \). Therefore,
\[
(X \cdot \Gamma)^\xi - (X \cdot \Gamma)^\tau = 1_{[0,\xi]} X \cdot \Gamma - 1_{[0,\tau]} X \cdot \Gamma = \left( (1_{[0,\xi]} - 1_{[0,\tau]} ) \right) X \cdot \Gamma = (1_{(\tau,\xi]} X) \cdot \Gamma.
\]

As to the Stratonovich integral, since \( X \) and \( \Gamma \) are semimartingales, we can write [P00] Theorem 23, page 68] that
\[
\left( \int X \, d\Gamma \right)^\tau = (X \cdot \Gamma)^\tau + \frac{1}{2} [X, \Gamma]^\tau = (X \cdot \Gamma)^\tau + \frac{1}{2} [X, \Gamma]^\tau = \int X \, d\Gamma^\tau.
\]

Finally, observe that for any process, \((X^\tau)^\tau = X^\tau\). On the other hand, taking into account that \( 1_{[0,\tau]} X = 1_{[0,\tau]} X^\tau \) and \([\Gamma, X] = [X, \Gamma] \), we have
\[
\left( \int X \, d\Gamma \right)^\tau = 1_{[0,\tau]} X \cdot \Gamma + \frac{1}{2} [X, \Gamma]^\tau = 1_{[0,\tau]} X^\tau \cdot \Gamma + \left( \frac{1}{2} [X, \Gamma]^\tau \right)^\tau = (X^\tau \cdot \Gamma)^\tau + \left( \frac{1}{2} [X^\tau, \Gamma]^\tau \right)^\tau = \left( \int X^\tau \, d\Gamma \right)^\tau.
\] ■
Proposition 5.6 Let $X : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a real valued process. Let $\{\tau_n\}_{n \in \mathbb{N}}$ be a sequence of stopping times such that a.s. $\tau_0 = 0$, $\tau_n \leq \tau_{n+1}$, for all $n \in \mathbb{N}$, and $\sup_{n \in \mathbb{N}} \tau_n = \infty$. Then,

$$X = \lim_{n \to \infty} X^{\tau_n}.$$ 

In particular, if $\Gamma : \mathbb{R}_+ \times \Omega \to M$ is a continuous $M$-valued semimartingale and $\eta \in \Omega_2(M)$ then,

$$\int \langle \eta, d\Gamma \rangle = \lim_{ucp} \left( \int \langle \eta, d\Gamma \rangle \right)^{\tau_n} = \lim_{ucp} \sum_{k=0}^{n-1} \int_{\tau_k}^{\tau_{k+1}} \langle \eta, d\Gamma \rangle.$$ 

Proof. Let $\epsilon > 0$ and $t \in \mathbb{R}_+$. Then for any $s \in [0, t]$ one has

$$\{|X^{\tau_n} - X|_s > \epsilon\} \subseteq \{\tau_n < s\} \subseteq \{\tau_n < t\}.$$ 

Hence for any $t \in \mathbb{R}_+$

$$P \left( \{|X^{\tau_n} - X|_s > \epsilon\} \right) \leq P \left( \{\tau_n < t\} \right).$$ 

The result follows because $P \left( \{\tau_n < t\} \right) \to 0$ as $n \to \infty$ since $\tau_n \to \infty$ a.s., and hence in probability.

Let now $\Gamma$ be a $M$-valued continuous semimartingale and $\eta \in \Omega_2(M)$. Notice first that $\left( \int \langle \eta, d\Gamma \rangle \right)^{\tau_0} = 0$ because $\tau_0 = 0$. Consequently, by Proposition 5.5 we can write

$$\left( \int \langle \eta, d\Gamma \rangle \right)^{\tau_k} = \sum_{n=0}^{k-1} \left( \int \langle \eta, d\Gamma \rangle \right)^{\tau_{n+1}} - \left( \int \langle \eta, d\Gamma \rangle \right)^{\tau_n} = \sum_{n=0}^{k-1} \int_{\tau_n}^{\tau_{n+1}} \langle \eta, d\Gamma \rangle$$ 

and the result follows. 

Proposition 5.7 Let $X$ and $Y$ be two real semimartingales. Suppose that $X$ is continuous and $X_0 = 0$. Then, for any $t \in \mathbb{R}_+$, the Stratonovich integral $\int (1_{[0,t]} Y) \delta X$ is well defined and equal to $\left( \int Y \delta X \right)^t$.

Proof. If $\int (1_{[0,t]} Y) \delta X$ was well defined, it should be equal to $\int (1_{[0,t]} Y) dX + \frac{1}{2} [1_{[0,t]} Y, X]$. Since $\int (1_{[0,t]} Y) dX$ is well defined, the only thing that we need to check is that $[1_{[0,t]} Y, X]$ exists. On the other hand, recall that (Proposition 12 page 60 and Theorem 23 page 68)

$$\int Y \delta X = \int (1_{[0,t]} Y) dX + \frac{1}{2} [Y, X]^t = \int (1_{[0,t]} Y) dX + \frac{1}{2} [Y^t, X].$$ 

Hence, what we are actually going to proceed by showing that $[1_{[0,t]} Y, X]$ is equal to $[Y^t, X]$.

Let $\sigma_n = \{0 = T_0^n \leq T_1^n \leq \ldots \leq T_{k_n}^n < \infty\}$ be a sequence of random partitions tending to the identity (in the sense of Prop 90 page 64). Given two real processes $X$ and $Y$, their quadratic variation, if it exists, can be defined as the limit in ucp when $n \to \infty$ of the following sums

$$[Y, X] = \lim_{n \to \infty} \sum_{i=0}^{k_n-1} \left( Y^{T_{i+1}^n} - Y^{T_{i}^n} \right) \left( X^{T_{i+1}^n} - X^{T_{i}^n} \right).$$ 

Let now

$$H_n := \sum_{i=0}^{k_n-1} \left( Y^{T_{i+1}^n} - Y^{T_{i}^n} \right) \left( X^{T_{i+1}^n} - X^{T_{i}^n} \right),$$ 

$$G_n := \sum_{i=0}^{k_n-1} \left( (1_{[0,t]} Y)^{T_{i+1}^n} - (1_{[0,t]} Y)^{T_{i}^n} \right) \left( X^{T_{i+1}^n} - X^{T_{i}^n} \right).$$
It is clear that the sequence \( \{ H_n \}_{n \in \mathbb{N}} \) converges uniformly on compacts in probability to \([Y^t, X]\). We are going to prove that there exists such a convergence for the sequence of processes \( \{ G_n \}_{n \in \mathbb{N}} \) by showing that the elements \( \{ G_n \}_s \) coincide with \( \{ H_n \}_s \), for any \( s \in \mathbb{R}_+ \), up to a set whose probability tends to zero as \( n \to \infty \). We will consider two cases:

1. **The case** \( s \leq t \). Given a specific \( i \in \{ 0, ..., k_n - 1 \} \), and recalling that by construction \( T^n_i \leq T^n_{i+1} \) a.s., it is clear that \( (Y^t)^{T^n_i} - (Y^t)^{T^n_{i+1}} \) is again different from 0 only in the set \( \{ T^n_i < s \} \) and there it is equal to (5.9). Therefore, \( (G_n)_s = (H_n)_s \) whenever \( s \leq t \).

2. **The case** \( s > t \). In this case, \( (Y^t)^{T^n_i} - (Y^t)^{T^n_{i+1}} \) is different from 0 only in the set \( \{ T^n_i < t \} \), where it takes the value

\[
Y^{T^n_i}_{i+1} - Y^{T^n_i}_i,
\]

(5.10)

However, in this case \( (1_{[0,t]} Y)^{T^n_i} - (1_{[0,t]} Y)^{T^n_{i+1}} \) is again different from 0 only in the set \( \{ T^n_i < t \} \) (which contains \( \{ T^n_i < t \} \) since \( T^n_i \leq T^n_{i+1} \)), but differs from (5.10) in

\[
A^n_i (t) := \{ T^n_i \leq t < T^n_{i+1} \}
\]

where it takes the value \(-Y^{T^n_i}_i\). For any other \( \omega \in \Omega \) not in these sets, \( (1_{[0,t]} Y)^{T^n_i} - (1_{[0,t]} Y)^{T^n_{i+1}} \) is equal to (5.10) in the set \( \{ T^n_i \leq t \} \) (which contains \( \{ T^n_i < t \} \) since \( T^n_i \leq T^n_{i+1} \)), and \( (G_n)_s \) and \( (H_n)_s \) are different only for the \( \omega \in A^n_i (t) \). Observe that, since \( t \) is fixed, only one of the sets \( \{ A^n_i (t) \}_i \) is non-empty and, on it,

\[
(G_n)_s - (H_n)_s = Y_t (X_t - X^{T^n_i}_i).
\]

To sum up, the analysis that we just carried out shows that for any \( u \in \mathbb{R}_+ \)

\[
\sup_{0 \leq s \leq u} |(H_n)_s - (G_n)_s| = 1_{A^n_i (t)} |Y_t| \left| (X_t - X^{T^n_i}_i) \right|
\]

for some \( i \in \{ 0, ..., k_n - 1 \} \). If \( X \) is continuous, this expression tells us that \( \sup_{0 \leq s \leq u} |(H_n)_s - (G_n)_s| \to 0 \) a.s. as \( n \to \infty \) which, in turn, implies that \( \sup_{0 \leq s \leq u} |(H_n)_s - (G_n)_s| \) converges to 0 in probability as well. That is, for any \( \varepsilon > 0 \),

\[
P \left( \sup_{0 \leq s \leq u} |(H_n)_s - (G_n)_s| > \varepsilon \right) \to 0, \text{ as } n \to \infty,
\]

which is the same as saying that \( H_n - G_n \) converges to 0 in ucp. Thus, since \( G_n = H_n - (H_n - G_n) \) and the limit in ucp as \( n \to \infty \) exist for the both sequences \( \{ H_n \}_{n \in \mathbb{N}} \) and \( \{ H_n - G_n \}_{n \in \mathbb{N}} \), so does the limit of \( \{ G_n \}_{n \in \mathbb{N}} \) which, by definition, is the quadratic variation \([1_{[0,t]} Y, X]\). Moreover, as \( (H_n - G_n) \to 0 \) in ucp as \( n \to \infty \),

\[
[Y^t, X] = \lim_{n \to \infty} \underleftarrow{H_n} = \lim_{n \to \infty} \underleftarrow{G_n} = [1_{[0,t]} Y, X],
\]

which concludes the proof. ■
6 Appendices

6.1 Preliminaries on semimartingales and integration

In the following paragraphs we state a few standard definitions and results on manifold valued semimartingales and integration. Semimartingales are the natural setup for stochastic differential equations and, in particular, for the equations that we handle in this paper. For proofs and additional details the reader is encouraged to check, for instance, with \[ \text{CW90, Du96, E89, IW89, LeG97, P90}, \] and references therein.

Semimartingales. The first element in our setup for stochastic processes is a probability space \( (\Omega, \mathcal{F}, P) \) together with a filtration \( \{\mathcal{F}_t\} \) of \( \mathcal{F} \) such that \( \mathcal{F}_0 \) contains all the negligible events (complete filtration) and the map \( t \mapsto \mathcal{F}_t \) is right-continuous, that is, \( \mathcal{F}_t = \bigcap_{t < \cdot} \mathcal{F}_{t+} \).

A real-valued martingale \( \Gamma : \mathbb{R}_+ = [0, \infty) \times \Omega \to \mathbb{R} \) is a stochastic process such that for every pair \( t, s \in \mathbb{R}_+ \) such that \( s \leq t \), we have:

(i) \( \Gamma \) is \( \mathcal{F}_t \)-adapted, that is, \( \Gamma_t \) is \( \mathcal{F}_t \)-measurable.

(ii) \( \Gamma_s = E[\Gamma_t \mid \mathcal{F}_s] \).

(iii) \( \Gamma_t \) is integrable: \( E[|\Gamma_t|] < +\infty \).

For any \( p \in [1, \infty) \), \( \Gamma \) is called a \( L^p \)-martingale whenever \( \Gamma \) is a martingale and \( \Gamma_t \in L^p(\Omega) \), for each \( t \). If \( \sup_{t \in \mathbb{R}_+} E[|\Gamma_t|^p] < \infty \), we say that \( \Gamma \) is \( L^p \)-bounded. The process \( \Gamma \) is locally bounded if for any time \( t \geq 0 \), \( \sup \{|\Gamma_t(\omega)| \mid s \leq t \} < \infty \), almost surely. Every continuous process is locally bounded. Recall that a process is said to be continuous when its paths are continuous. Most processes considered in this paper will be of this kind. Given two continuous processes \( X \) and \( Y \) we will write \( X = Y \) when they are a modification of each other or when they are indistinguishable since these two concepts coincide for continuous processes.

A random variable \( \tau : \Omega \to [0, +\infty] \) is called a stopping time with respect to the filtration \( \{\mathcal{F}_t\} \) if for every \( t \geq 0 \) the set \( \{ \omega \mid \tau(\omega) \leq t \} \) belongs to \( \mathcal{F}_t \). Given a stopping time \( \tau \), we define

\[
\mathcal{F}_\tau = \{ \Lambda \in \mathcal{F} \mid \Lambda \cap \{ \tau \leq t \} \in \mathcal{F}_t \ \text{for any} \ t \in \mathbb{R}_+ \}.
\]

Given an adapted process \( \Gamma \), it can be shown that \( \Gamma_\tau \) is \( \mathcal{F}_\tau \)-measurable. Furthermore, the stopped process \( \Gamma^\tau \) is defined as

\[
\Gamma^\tau_t := \Gamma_{t \land \tau} := \Gamma_t 1_{\{t \leq \tau\}} + \Gamma_{\tau} 1_{\{t > \tau\}}.
\]

A continuous local martingale is a continuous adapted process \( \Gamma \) such that for any \( n \in \mathbb{N} \), \( \Gamma^\tau_n 1_{\{\tau_n > 0\}} \) is a martingale, where \( \tau_n \) is the stopping time \( \tau_n := \inf\{t \geq 0 \mid |\Gamma_t| = n\} \).

We say that the stochastic process \( \Gamma : \mathbb{R}_+ \times \Omega \to \mathbb{R} \) has finite variation whenever it is adapted and has bounded variation on compact subintervals of \( \mathbb{R}_+ \). This means that for each fixed \( \omega \in \Omega \), the path \( t \mapsto \Gamma_t(\omega) \) has bounded variation on compact subintervals of \( \mathbb{R}_+ \), that is, the supremum \( \sup \{ \sum_{i=1}^p |\Gamma_{t_i}(\omega) - \Gamma_{t_{i-1}}(\omega)| \} \) over all the partitions \( 0 = t_0 < t_1 < \cdots < t_p = t \) of the interval \([0, t] \) is finite.

A continuous semimartingale is the sum of a continuous local martingale and a process with finite variation. It can be proved that a given semimartingale has a unique decomposition of the form \( \Gamma = \Gamma_0 + V + \Lambda \), with \( \Gamma_0 \) the initial value of \( \Gamma \), \( V \) a finite variation process, and \( \Lambda \) a local continuous semimartingale. Both \( V \) and \( \Lambda \) are null at zero.

The Itô integral with respect to a continuous semimartingale. Let \( \Gamma : \mathbb{R}_+ \times \Omega \to \mathbb{R} \) be a continuous local martingale. It can be shown that there exists a unique increasing process with finite variation \( [\Gamma, \Gamma]_t \) such that \( \Gamma^2_t - [\Gamma, \Gamma]_t \) is a local continuous martingale. We will refer to \([\Gamma, \Gamma]_t \) as the
The Stratonovich integral and stochastic calculus. Given $\Gamma = \Gamma_0 + V + \Lambda$, $\Gamma' = \Gamma_0' + V' + \Lambda'$ two continuous local martingales we define their joint quadratic variation or quadratic covariation as

$$[\Gamma, \Gamma']_t = \frac{1}{2} \left( [\Lambda + \Lambda', \Lambda + \Lambda']_t - [\Lambda, \Lambda]_t - [\Lambda', \Lambda']_t \right).$$

Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of processes. We will say that $\{X_n\}_{n \in \mathbb{N}}$ converges uniformly on compacts in probability (abbreviated ucp) to a process $X$ if for any $\varepsilon > 0$ and any $t \in \mathbb{R}_+$,

$$P \left( \sup_{0 \leq s \leq t} |X_n - X|_s > \varepsilon \right) \longrightarrow 0,$$

as $n \rightarrow \infty$.

Following [P90], we denote by $\mathbb{L}$ the space of processes $X : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ whose paths are left-continuous and have right limits. These are usually called càdlàg processes, which are initials in French for left-continuous with right limits. We say that a process $X \in \mathbb{L}$ is elementary whenever it can be expressed as

$$X = X_01_{\{0\}} + \sum_{i=1}^{p-1} X_i1_{(\tau_i, \tau_{i+1})},$$

where $0 \leq \tau_1 < \cdots < \tau_{p-1} < \tau_p$ are stopping times, and $X_0$ and $X_i$ are $\mathcal{F}_0$ and $\mathcal{F}_{\tau_i}$ measurable random variables, respectively such that $|X_0| < \infty$ and $|X_i| < \infty$ a.s. for all $i \in \{1, \ldots, p-1\}$. $1_{(\tau_i, \tau_{i+1})}$ is the characteristic function of the set $(\tau_i, \tau_{i+1}] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid t \in (\tau_i(\omega), \tau_{i+1}(\omega))\}$ and $1_{\{0\}}$ of $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid t = 0\}$. It can be shown (see [P90] Theorem 10, page 57) that the set of elementary processes is dense in $\mathbb{L}$ in the ucp topology.

Let $\Gamma$ be a semimartingale such that $\Gamma_0 = 0$ and $X$ elementary. We define Itô’s stochastic integral of $X$ with respect to $\Gamma$ as given by

$$X \cdot \Gamma := \int X d\Gamma := \sum_{i=1}^{p-1} X_i(\Gamma_{\tau_{i+1}} - \Gamma_{\tau_i}). \tag{6.1}$$

In the sequel we will exchangeably use the symbols $X \cdot \Gamma$ and $\int X d\Gamma$ to denote the Itô stochastic integral. It is a deep result that, if $\Gamma$ is a semimartingale, the Itô stochastic integral is a continuous map from $\mathbb{L}$ to the space of processes whose paths are right-continuous and have left limits (càdlàg), usually denoted by $\mathbb{D}$, equipped also with the ucp topology. Therefore we can extend the Itô integral to the whole $\mathbb{L}$. In particular, we can integrate any continuous adapted processes with respect to any semimartingale.

Given any stopping time $\tau$ we define

$$\int_0^\tau X d\Gamma := (X \cdot \Gamma)_\tau.$$

It can be shown that $(1_{[0,\tau]}X) \cdot \Gamma = (X \cdot \Gamma)^\tau = X \cdot \Gamma^\tau$. If there exists a stopping times $\zeta$ such that the semimartingale $\Gamma$ is defined only on the stochastic intervals $[0, \zeta)$, then we may define the Itô integral of $X$ with respect to $\Gamma$ on any interval $[0, \tau]$ such that $\tau < \zeta$ by means of $X \cdot \Gamma^\tau$.

The Stratonovich integral and stochastic calculus. Given $\Gamma$ and $X$ two semimartingales we define the Stratonovich integral of $X$ along $\Gamma$ as

$$\int_0^t X d\Gamma = \int_0^t X d\Gamma + \frac{1}{2} [X, \Gamma]_t.$$
Let \( X^1, \ldots, X^p \) be \( p \) continuous semimartingales and \( f \in \mathcal{C}^2(\mathbb{R}^p) \). The celebrated \textit{Itô formula} states that

\[
 f(X^1_t, \ldots, X^p_t) = f(X^1_0, \ldots, X^p_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X^1_s, \ldots, X^p_s) dX^i_s \\
+ \frac{1}{2} \sum_{i,j=1}^p \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X^1_s, \ldots, X^p_s) d[X^i, X^j]_s
\]

The analogue of this equality for the Stratonovich integral is

\[
 f(X^1_t, \ldots, X^p_t) = f(X^1_0, \ldots, X^p_0) + \sum_{i=1}^p \int_0^t \frac{\partial f}{\partial x^i}(X^1_s, \ldots, X^p_s) \delta X^i_s.
\]

An important particular case of these relations are the \textit{integration by parts} formulas

\[
\int_0^t X d\Gamma = (X\Gamma)_t - (X\Gamma)_0 - \int_0^t \Gamma dX - \frac{1}{2} [X, \Gamma]_t, \\
\int_0^t X \delta \Gamma = (X\Gamma)_t - (X\Gamma)_0 - \int_0^t \Gamma \delta X.
\]

\textbf{Stochastic differential equations.} Let \( \Gamma = (\Gamma^1, \ldots, \Gamma^p) \) be \( p \) semimartingales with \( \Gamma_0 = 0 \) and \( f : \mathbb{R}^q \times \mathbb{R}^p \rightarrow \mathbb{R}^q \) a smooth function. A \textbf{solution} of the \textit{Itô stochastic differential equation}

\[
 dX^i = \sum_{j=1}^p f_j^i(X, \Gamma) d\Gamma^j
\]

with initial condition the random vector \( X_0 = (X^1_0, \ldots, X^p_0) \) is a stochastic process \( X_t = (X^1_t, \ldots, X^p_t) \) such that \( X^i_t - X^i_0 = \sum_{j=1}^p \int_0^t f_j^i(X, \Gamma) d\Gamma^j \). It can be shown [P90, page 310] that for any \( x \in \mathbb{R}^q \) there exists a stopping time \( \zeta : \mathbb{R}^q \times \Omega \rightarrow \mathbb{R}^+ \) and a time-continuous solution \( X(t, \omega, x) \) of (6.2) with initial condition \( x \) and defined in the time interval \( [0, \zeta(x, \omega)) \). Additionally, \( \lim_{t \rightarrow \zeta(\omega)} \| X_t(\omega) \| = \infty \) a.s. on \( \{ \zeta < \infty \} \) and \( X \) is smooth on \( x \) in the open set \( \{ x | \zeta(x, \omega) > t \} \). Finally, the solution \( X \) is a semimartingale.

\section{Second order vectors and forms}

In the paragraphs that follow we review the basic tools on second order geometry needed in the definition of the stochastic integral of a form along a manifold valued semimartingale. The reader interested in the proofs of the statements cited in this section is encouraged to check with [E89], and references therein.

Let \( M \) be a finite dimensional, second-countable, locally compact Hausdorff (and hence paracompact) manifold. Given \( m \in M \), a \textbf{tangent vector} at \( m \) of \textbf{order two} with no constant term is a differential operator \( L : C^\infty (M) \rightarrow \mathbb{R} \) that satisfies

\[
 L \left[ f^3 \right] (m) = 3 f (m) L \left[ f^2 \right] (m) - 3 f^2 (m) L \left[ f \right] (m).
\]

The vector space of tangent vectors of order two at \( m \) is denoted as \( \tau_m M \). The manifold \( \tau M := \bigcup_{m \in M} \tau_m M \) is referred to as the \textbf{second order tangent bundle} of \( M \). Notice that the (first order) tangent bundle \( TM \) of \( M \) is contained in \( \tau M \). A \textbf{vector field of order two} is a smooth section of the bundle \( \tau M \rightarrow M \). We denote the set of vector fields order two by \( \mathfrak{X}_2(M) \). If \( Y, Z \in \mathfrak{X}(M) \) then the
product $ZY \in \mathfrak{X}_2(M)$. Conversely, every second order vector field $L \in \mathfrak{X}_2(M)$ can be written as a finite sum of fields of the form $ZY$ and $W$, with $Z, Y, W \in \mathfrak{X}(M)$.

The forms of order two $\Omega_2(M)$ are the smooth sections of the cotangent bundle of order two $\tau^*M := \bigcup_{m \in M} \tau_m^* M$. For any $f, g, h \in C^\infty(M)$ and $L \in \mathfrak{X}_2(M)$ we define $d_2f \in \Omega_2(M)$ by $d_2f(L) := L[f]$, and $d_2f \cdot d_2g \in \Omega_2(M)$ as

$$d_2f \cdot d_2g[L] := \frac{1}{2} (L[f g] - fL[g] - gL[f]).$$

It is easy to show that for any $Y, Z, W \in \mathfrak{X}(M)$,

$$d_2f \cdot d_2g [ZY] = \frac{1}{2} (Z[f Y] + Z[g Y] - Z[f g]) \quad \text{and} \quad d_2f \cdot d_2g [W] = 0.$$

More generally, let $\alpha_m, \beta_m \in T^*_m M$ and choose $f, g \in C^\infty(M)$ two functions such that $d_2f(m) = \alpha_m$ and $d_2g(m) = \beta_m$. It is easy to check that $(d_2f \cdot d_2g)(m)$ does not depend on the particular choice of $f$ and $g$ above and hence we can write $\alpha_m \cdot \beta_m$ to denote $(d_2f \cdot d_2g)(m)$. If $\alpha, \beta \in \Omega(M)$ then we can define $\alpha \cdot \beta \in \Omega_2(M)$ as $g \in \Omega_2(M)$ as $(\alpha \cdot \beta)(m) := (\alpha(m)) \cdot (\beta(m))$. This product is commutative and $C^\infty(M)$-bilinear. It can be shown that every second order form can be locally written as a finite sum of forms of the type $d_2f \cdot d_2g$ and $d_2h$.

The $d_2$ operator can also be defined on forms by using a result (Theorem 7.1 in [E89]) that claims that there exists a unique linear map $d_2 : \Omega(M) \to \Omega_2(M)$ characterized by

$$d_2 (df) = d_2 f \quad \text{and} \quad d_2 (f \alpha) = df \cdot \alpha + f d_2 \alpha.$$

### 6.3 Stochastic integrals of forms along a semimartingale

Let $M$ be a manifold. A continuous $M$-valued stochastic process $X : \mathbb{R}_+ \times \Omega \to M$ is called a continuous $M$-valued semimartingale if for each smooth function $f \in C^\infty(M)$, the real valued process $f \circ X$ is a (real-valued) continuous semimartingale. We say that $X$ is locally bounded if the sets $\{X_s(\omega) \mid 0 \leq s \leq t\}$ are relatively compact in $M$ for each $t \in \mathbb{R}_+$, a.s.

Let $X$ be a $M$-valued semimartingale and $\theta : \mathbb{R}_+ \times \Omega \to \tau^*M$ be a càglâid locally bounded process over $X$, that is, $\pi \circ \theta = X$, where $\pi : \tau^*M \to M$ is the canonical projection. It can be shown (see [E89, Theorem 6.24]) that there exists a unique linear map $\theta \mapsto \int \langle \theta, dX \rangle$ that associates to each such $\theta$ a continuous real valued semimartingale and that is fully characterized by the following properties: for any $f \in C^\infty(M)$ and any locally bounded càglâid real-valued process $K$,

$$\int \langle d_2 f \circ X, dX \rangle = f(X) - f(X_0), \quad \text{and} \quad \int \langle K \theta, dX \rangle = \int K d\left( \int \langle \theta, dX \rangle \right). \quad (6.3)$$

The stochastic process $\int \langle \theta, dX \rangle$ will be called the Itô integral of $\theta$ along $X$. If $\alpha \in \Omega_2(M)$, we will write in the sequel the Itô integral of $\alpha$ along $X$, that is, $\int \langle \alpha, dX \rangle$ as $\int \langle \alpha, dX \rangle$.

The integral of a $(0,2)$-tensor $b$ on $M$ along $X$ is the image of the unique linear mapping $b \mapsto \int b(dX, dX)$ onto the space of real continuous processes with finite variation that for all $f, g, \in C^\infty(M)$ satisfies

$$\int (fb)(dX, dX) = \int (f \circ X) d\left( \int b(dX, dX) \right) \quad \text{and} \quad \int (df \otimes dg)(dX, dX) = [f \circ X, g \circ X]. \quad (6.4)$$

If $\alpha \in \Omega(M)$ and $X$ is a semimartingale on $M$, the real semimartingale $\int \langle d_2 \alpha, dX \rangle$ is called the Stratonovich integral of $\alpha$ along $X$ and is denoted by $\int \langle \alpha, \delta X \rangle$. This definition can be generalized by
taking $\beta$ a $T^*M$ valued semimartingale over $X$ and by defining the Stratonovich integral as the unique real valued semimartingale that satisfies the properties
\[
\int \langle df, \delta X \rangle = f(X) - f(X_0), \quad \text{and} \quad \int \langle \beta, \delta X \rangle = \int Z(X) \delta \left( \int \langle \beta, \delta X \rangle \right),
\] (6.5)
for any $f \in C^\infty(M)$ and any continuous real valued semimartingale $Z$. Finally, it can be shown that (see [ES9, Proposition 6.31]) for any $f, g \in C^\infty(M)$,
\[
\int \langle df \cdot dg, dX \rangle = \frac{1}{2} \{f(X), g(X)\}.
\] (6.6)

6.4 Stochastic differential equations on manifolds

The reader interested in the details of the material presented in this section is encouraged to check with the chapter 7 in [ES9].

Let $M$ and $N$ be two manifolds. A **Stratonovich operator** from $M$ to $N$ is a family $\{e(x, y)\}_{x \in M, y \in N}$ such that $e(x, y) : T_xM \to T_yN$ is a linear mapping that depends smoothly on its two entries. Let $e^*(x, y) : T^*_yN \to T^*_xM$ be the adjoint of $e(x, y)$.

Let $X$ be a $M$-valued semimartingale. We say that a $N$-valued semimartingale is a solution of the **Stratonovich stochastic differential equation**
\[
\delta Y = e(X, Y)\delta X
\] (6.7)
if for any $\alpha \in \Omega(N)$, the following equality between Stratonovich integrals holds:
\[
\int \langle \alpha, \delta Y \rangle = \int \langle e^*(X, Y)\alpha, \delta X \rangle.
\]

It can be shown [ES9, Theorem 7.21] that given a semimartingale $X$ in $M$, a $\mathcal{F}_0$ measurable random variable $Y_0$, and a Stratonovich operator $e$ from $M$ to $N$, there are a stopping time $\zeta > 0$ and a solution $Y$ of (6.7) with initial condition $Y_0$ defined on the set $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid t \in [0, \zeta(\omega))\}$ that has the following maximality and uniqueness property: if $\zeta'$ is another stopping time such that $\zeta' < \zeta$ and $Y'$ is another solution defined on $\{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid t \in [0, \zeta'(\omega))\}$, then $Y'$ and $Y$ coincide in this set. If $\zeta$ is finite then $Y$ explodes at time $\zeta$, that is, the path $Y_t$ with $t \in [0, \zeta)$ is not contained in any compact subset of $N$.

The stochastic differential equations from the Itô integration point of view require the notion of **Schwartz operator** whose construction we briefly review. The reader interested in the details of this construction is encouraged to check with [ES9]. Note first that we can associate to any element $L \in \mathfrak{X}_2(M)$ a symmetric tensor $\tilde{L} \in \mathfrak{X}(M) \otimes \mathfrak{X}(M)$. Second, given $x \in M$ and $y \in N$, a linear mapping from $\tau_xM$ into $\tau_yN$ is called a **Schwartz morphism** whenever $f(T_xM) \subset T_yN$ and $\tilde{f}(\tilde{L}) = (f|_{T_xM} \otimes f|_{T_yM})(\tilde{L})$, for any $L \in \tau_xM$. Third, let $M$ and $N$ be two manifolds; a **Schwartz operator** from $M$ to $N$ is a family $\{f(x, y)\}_{x \in M, y \in N}$ such that $f(x, y) : \tau_xM \to \tau_yN$ is a Schwartz operator that depends smoothly on its two entries. Let $f^*(x, y) : \tau^*_yN \to \tau^*_xM$ be the adjoint of $f(x, y)$. Finally, let $X$ be a $M$-valued semimartingale. We say that a $N$-valued semimartingale is a solution of the the **Itô stochastic differential equation**
\[
dY = f(X, Y)dX\] (6.8)
if for any $\alpha \in \Omega_2(N)$, the following equality between Itô integrals holds:
\[
\int \langle \alpha, dY \rangle = \int \langle f^*(X, Y)\alpha, dX \rangle.
\]
There exists an existence and uniqueness result for the solutions of these stochastic differential equations analogous to the one for Stratonovich differential equations.

Given a Stratonovich operator $e$ from $M$ to $N$, there exists a unique Schwartz operator $f : \tau M \times N \to \tau N$ defined as follows. Let $\gamma(t) = (x(t), y(t)) \in M \times N$ be a smooth curve that verifies $e(x(t), y(t))(\dot{x}(t)) = \dot{y}(t)$, for all $t$. We define $f(x(t), y(t))(L_{\dot{x}(t)}) := (L_{\dot{y}(t)})$, where the second order differential operators $(L_{\dot{x}(t)}) \in \tau_{\dot{x}(t)}M$ and $(L_{\dot{y}(t)}) \in \tau_{\dot{y}(t)}N$ are defined as $(L_{\dot{x}(t)})[h] := \frac{d^2}{dt^2}h(x(t))$ and $(L_{\dot{y}(t)})[g] := \frac{\partial}{\partial t}g(y(t))$, for any $h \in C^\infty(M)$ and $g \in C^\infty(N)$. This relation completely determines $f$ since the vectors of the form $L_{\dot{x}(t)}$ span $\tau_{\dot{x}(t)}M$. Moreover, the Itô and Stratonovich equations $\delta Y = e(X, Y)\delta X$ and $dY = f(X, Y)dX$ are equivalent, that is, they have the same solutions.

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