POISSON OVERLAPPING MICROBALLS: SELF-SIMILARITY AND X-RAY IMAGES

HERMINE BIERMÉ AND ANNE ESTRADE

Abstract. We study a random field obtained by counting the number of balls containing each point, when overlapping balls are thrown at random according to a Poisson random measure. We are particularly interested in the local asymptotical self-similarity (lass) properties of the field, as well as the action of X-ray transforms. We discover two different lass properties when considering the asymptotic either in law or on the second order moment and prove a relationship between the lass behavior of the field and the lass behavior of its X-ray transform. We also describe a microscopic process which leads to a multifractional behavior. These results can be used to model and analyze porous media, images or connection networks.

1. Introduction

The purpose of this paper is the study of a random field obtained by throwing overlapping balls. Such a field is particularly well-adapted for modeling 3D porous or heterogeneous media. In fact we consider a collection of balls in $\mathbb{R}^3$, whose centers and radii are chosen at random according to a Poisson random measure on $\mathbb{R}^3 \times \mathbb{R}^+$. Equivalently, we consider a germ-grain model where the germs are Poisson distributed and the grains are balls of random radius.

The field under study is the mass density defined as the number of balls containing each point: the more one point is covered by balls, the higher is the mass density at this point. From a mathematical point of view, the dimension three does not yield any specific behavior, so the study will be carried out in dimension $d$ ($d \geq 1$). Moreover for $d = 2$, the number of balls covering each point defines the discretized gray level of each pixel in a black and white picture. A one dimensional ($d = 1$) germ-grain model is also relevant for modeling communication networks: the germs stand for the starting time of the individual ON periods (calls) and the grains stand for the ‘half-ball’ intervals of duration. The obtained process is a counter which delivers at each time the number of active connections in the network.

We have in mind a microscopic model which yields to self-similar macroscopic properties. In order to get this scaling behavior, we introduce some

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power law behavior in the radius distribution and consider Poisson random measures on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity of the following type

$$d\nu(\xi, r) = Cr^{-d+\theta} drd\xi$$

for some $\theta$, which may depend on the location $\xi$. The origin of the ‘microballs’ described in this paper can be found in the ‘micropulses’ introduced by Ciosek-Georges and Mandelbrot [6] with a fixed power $\theta$ in the intensity measure. The idea is not new and appeared eighty years ago when S.D. Wicksell [19] introduced a first model, the famous ‘corpuscles’, made of random 3D spheres defined as above. The aim of his study was a stereological question. Since then, this kind of model has been extensively deepened and extended. We address to [18] or [17] for many examples of random models based on Poisson point process and germ-grain models. Let us also mention two recent papers dealing with similar questions. A one dimensional germ-grain model with locations (arrival times) uniformly distributed on the time axis and intervals lengths (call durations) given by a power law is considered by Cohen and Taqqu in [7]. A mixed moving average is performed that sums the height of connections and the so-called Poissonized Telecom process is obtained. Also similar is the model recently studied by Kaj et al. [10]: the germs are uniformly chosen at random in $\mathbb{R}^d$ and the grains are obtained by random dilation of a fixed bounded set. In contrast to the quoted models, let us point out that inhomogeneity is allowed in our model by choosing a non stationary intensity measure (1) with a non constant power $\theta = \theta(\xi)$.

This paper is not only concerned with the presentation of a model for random media. We also investigate the analysis of the random media mass intensity following two ways. On one hand, self-similarity properties are explored. More precisely, we focus on a parameter that is supposed to contain tangible information on the structure of the media, the local asymptotical self-similar index, lass index in short. On the other hand, the action of an X-ray transform on the field is explored. This transform is the mathematical interpretation for a radiographic process. These techniques are inspired from the ones created for Gaussian fields and are still valuable in the Poisson context. More specifically we turn to [5] where anisotropic Gaussian fields are analyzed by performing X-ray transforms and evaluating lass indices. The fundamental aim of these methods is to make a 3D parameter directly tractable from X-ray images of the media.

The notion of local asymptotic self-similarity has been introduced in [3] in a Gaussian context and extended to the non-Gaussian realm in [12], or [4] where a general presentation is performed for fields with stationary increments. The lass index can also be related to other parameters of interest as roughness index [2] or Hausdorff dimension [1]. In the area of network modeling as well, the notion of self-similarity, at small or large scales, is fundamental and it is highly connected to long-range dependence. The usual
self-similarity property requires a scale invariance valid for all scales. This is quite restrictive and we will deal with self-similarity properties that are fulfilled ‘at small scales’ only. We introduce a light refinement of the lass property of [3].

**Definition 1.1.** Let \( X = \{X(x); x \in \mathbb{R}^d\} \) be a random field and \( x_0 \in \mathbb{R}^d \). We call the distribution lass index of \( X \) at point \( x_0 \) (fdd-lass index for short) the following supremum

\[
H_{fdd}(X, x_0) = \sup \left\{ \alpha \mid \frac{\Delta_{x_0} X(\lambda x) - E(\Delta_{x_0} X(\lambda x))}{\lambda^\alpha} \overset{fdd}{\to} 0 \quad \text{as} \quad \lambda \to 0^+ \right\}
\]

where \( \Delta_{x_0} X \) denotes the field of increments at \( x_0 \):

\[
\Delta_{x_0} X(x) = X(x_0 + x) - X(x_0)
\]

and \( \overset{fdd}{\to} \) means the convergence of the finite dimensional distributions.

Moreover, when \( H = H_{fdd}(X, x_0) \) is finite and when the finite dimensional distributions of the centered and renormalized increments \( \lambda^{-H}[\Delta_{x_0} X(\lambda x) - E(\Delta_{x_0} X(\lambda x))] \) converge to the finite dimensional distributions of a non-vanishing field as \( \lambda \to 0^+ \), the limit field is called tangent field at point \( x_0 \) (see [3]).

When one deals with real data, it is almost impossible to see whether such a limit exists in distribution. Therefore we introduce another asymptotic self-similarity property, which only uses the second order moment.

**Definition 1.2.** Let \( X = \{X(x); x \in \mathbb{R}^d\} \) be a random field and \( x_0 \in \mathbb{R}^d \). We call the covariance lass index of \( X \) at point \( x_0 \) (cov-lass index for short) the following supremum

\[
H_{cov}(X, x_0) = \sup \left\{ \alpha \mid \text{Cov} \left( \frac{\Delta_{x_0} X(\lambda x)}{\lambda^\alpha}, \frac{\Delta_{x_0} X(\lambda x')}{\lambda^\alpha} \right) \overset{\lambda \to 0^+}{\to} 0 \quad \forall x, x' \in \mathbb{R}^d \right\}
\]

By analogy with the fdd situation, when the covariance function of \( \lambda^{-H} \Delta_{x_0} X(\lambda x) \) for \( H = H_{cov} \) converges to a non vanishing covariance function as \( \lambda \to 0^+ \), the limit covariance will be called the tangent covariance at point \( x_0 \).

Note that the above self-similarity indices are equal for Gaussian fields but not in a general setting. Also note that the existence of a tangent covariance does not imply existence of the tangent field, and neither the converse implication. Actually if \( H_{cov} \) is the cov-lass index for \( X \) at point \( x_0 \), then for all \( H < H_{cov} \) the covariance function of \( \lambda^{-H} \Delta_{x_0} X(\lambda x) \) converges to 0 as \( \lambda \to 0^+ \). Thus, the finite dimensional distributions of its centered version converges also to 0 as \( \lambda \to 0^+ \), and the fdd-lass index \( H_{fdd} \) for \( X \) at point \( x_0 \) - if exists - satisfy \( H_{fdd} \geq H_{cov} \).

Our main results can be summarized as follows:

- the proposed models provide microscopic descriptions of macroscopic asymptotical self-similar fields which look like fractional -or multifractional- Brownian motions, depending on the involved intensity measure;
in contrast to the Gaussian case, the covariance lass index and distribution lass index are not equal: the first one can be finite whereas the second one is infinite, or they can both be finite but with different values;

- we present explicit formulas that link the lass indices of a field and the lass indices of the X-ray transform of it. In particular, when inhomogeneity or anisotropy is introduced in the model, it can be recovered through the lass indices.

The paper is organized as follows: the microball model, i.e. the field that counts the number of balls covering each point, is introduced in Section 2. The intensities of the Poisson random measures we will use are specified in Section 2.2. A constant power \( \theta \) in (1) will yield a fractional microball model which is stationary and isotropic. A non-constant power \( \theta \) will yield the multifractional model. We also introduce in Section 2.3 the X-ray transform. Section 3 is devoted to the self-similarity properties of the microball model and its X-ray transform. Theorem 3.1 deals with the fractional model and Theorems 3.4 and 3.6 deal with the multifractional model, where \( \theta = \theta(\xi) \) is a smooth, or a singular function respectively. We also compare our results to homogenization results in Section 3.3. The proofs of Theorems 3.4 and 3.6 are detailed at the end of the paper. The covariance lass properties are proved in Section 5 and the distribution lass properties in Section 6.

2. The microball model and its X-ray transforms

2.1. The random grain model. The model is built by considering the superposition of balls \( B(\xi, r) \), where \( \xi \) is a point in \( \mathbb{R}^d \) and \( r > 0 \) is the radius of the ball. As in [10], we want to study the mass distribution generated by a family of balls \( B(\xi_j, r_j) \), with random location \( \xi_j \) and random radius \( r_j \). We assume that \( (\xi_j, r_j) \) are given by a Poisson point process with intensity \( \nu(d\xi, dr) \), where \( \nu \) is a non-negative \( \sigma \)-finite measure on \( \mathbb{R}^d \times \mathbb{R}^+ \).

For each \( x \in \mathbb{R}^d \) we are naturally interested in the number of balls \( B(\xi, r) \) that contain the point \( x \), given by

\[
\# \{ j; x \in B(\xi_j, r_j) \} = \sum_j 1_{B(\xi, r)}(x).
\]

Such a field is well defined as soon as

\[
\forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi, r)}(x) \nu(d\xi, dr) < +\infty.
\]

Moreover, in that case, we can represent the field through a stochastic integral with respect to a Poisson measure \( N \) with intensity \( \nu \), as

\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi, r)}(x) N(d\xi, dr).
\]

Let us also describe the intuitive scenario we have in mind for the one dimensional case \( (d = 1) \). We look at the ‘half-ball’-interval \([\xi, \xi + r] \) as the ON period of a single call and the number of connected users at time \( x \) is equal
to the one dimensional integral -if exists- \( \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{[\xi, \xi + r]}(x) N(d\xi, dr) \). Since the integrand \( 1_{[\xi, \xi + r]} \) behaves like \( 1_{B(\xi, r)} \), we will not distinguish anymore the special case \( d = 1 \).

2.2. The microball model. As in [6], we assume that the radii of such a random grain model follow a power law in \((0, 1)\). The exponent of the power law can either be constant (fractional case) or can depend on the center of the ball \( \xi \) (multifractional case). More precisely we consider intensity measures on \( \mathbb{R}^d \times \mathbb{R}^+ \) with special shapes \( \nu_m \) and \( \nu_h \) described below.

For \( m > 0 \) we define the fractional intensity measure \( \nu_m \) on \( \mathbb{R}^d \times \mathbb{R}^+ \) as

\[
\nu_m(d\xi, dr) = r^{-d-1+2m}1_{(0,1)}(r)d\xi dr.
\]

For a function \( h \) on \( \mathbb{R}^d \) such that \( \text{essinf} \ h > 0 \), we define the multifractional intensity measure \( \nu_h \) on \( \mathbb{R}^d \times \mathbb{R}^+ \) as

\[
\nu_h(d\xi, dr) = r^{-d-1+2h(\xi)}1_{(0,1)}(r)d\xi dr.
\]

It is straightforward to see that these kind of measures satisfy (2). This allows the following definition.

Definition 2.1. Let \( h \) be a function on \( \mathbb{R}^d \) such that \( \text{essinf} \ h > 0 \), and let \( N_h \) be a Poisson random measure with intensity \( \nu_h \). The field defined on \( \mathbb{R}^d \) as

\[
X(x) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi, r)}(x) N_h(d\xi, dr)
\]

is called a microball model with index \( h \).

Note that the microball model has moments of all order. In particular, its mean value is given, for each \( x \in \mathbb{R}^d \), by

\[
E(X(x)) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi, r)}(x) \nu_h(d\xi, dr).
\]

Moreover, by the isometry of the Poisson measure, its covariance function is equal to

\[
\text{Cov}(X(x), X(x')) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi, r)}(x) 1_{B(\xi, r)}(x') \nu_h(d\xi, dr),
\]

for all \( x, x' \in \mathbb{R}^d \).

The self-similarity properties of the microball model that we will study in the sequel deal with the local behavior of \( X \). We now compute the increments of \( X \) and analyse their moments in the following lemma. In what follows we write for \( x_0, x \in \mathbb{R}^d \)

\[
\Delta_{x_0} X(x) = X(x_0 + x) - X(x_0) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \psi(x, \xi - x_0, r) N_h(d\xi, dr),
\]

where

\[
\psi(x, \xi, r) = 1_{B(\xi, r)}(x) - 1_{B(\xi, r)}(0) = 1_{|x - \xi| < r \leq |\xi|} - 1_{|\xi| < r \leq |x - \xi|},
\]

and \( |.| \) denotes the usual Euclidean norm.
Lemma 2.2. Let $m \in (0, 1/2)$. There exists a constant $C(m) \in (0, +\infty)$ such that, for all $p \in (0, +\infty)$ and $x \in \mathbb{R}^d$,\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} |\psi(x, \xi, r)|^p \ r^{-d-1+2m} \ dr \ d\xi = C(m)|x|^{2m}.
\]

Proof. Note that $|\psi(x, \xi, r)|^p = |\psi(x, \xi, r)| = 1_{|x-\xi| < r} 1_{|\xi| < r} 1_{|\xi| < |x-\xi|}$. Hence, for all $x, \xi \in \mathbb{R}^d$,\[
\int_{\mathbb{R}^+} |\psi(x, \xi, r)|^p \ r^{-d-1+2m} \ dr = 2^{d-2m} 1_{|\xi| < |x-\xi|} \left(|\xi|^{-d+2m} - |x-\xi|^{-d+2m}\right)
\]
and for $0 < m < 1/2$, the function $\xi \mapsto |\xi|^{-d+2m} - |x-\xi|^{-d+2m}$ is integrable on $\mathbb{R}^d$. The conclusion is obtained by rotation invariance and homogeneity. \[\square\]

2.3. X-Ray transform. One motivation for this paper is to describe, model and analyze heterogeneous media. We have in mind the possibility to estimate a macroscopic 3D parameter through X-ray images. By this method, it will be possible to get an analysis of the media without entering the media (non-invasive method). In this section the mathematical tool associated with X-ray images is presented and tested on the microball model. We assume that $d \geq 2$.

Following the usual notation (see [15] for instance), for a direction $\alpha \in S^{d-1} = \{x \in \mathbb{R}^d; |x| = 1\}$, the X-ray transform in the direction $\alpha$ of any function $f \in L^1(\mathbb{R}^d)$ is given by
\[
y \in <\alpha> \mapsto \int_{\mathbb{R}} f(y + p\alpha) \ dp
\]
where $<\alpha> := \{x \in \mathbb{R}^d; x \cdot \alpha = 0\}$, and $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$. We want to define, in the same way, the X-ray transform of a microball model $X$. Unfortunately, the realizations $x \in \mathbb{R}^d \mapsto X(x, \omega)$ do not belong to $L^1(\mathbb{R}^d)$. We will therefore work with the windowed X-ray transform defined through a fixed window $\rho$. We assume that $\rho$ is a continuous function on $\mathbb{R}$ with fast decay, namely
\[
\forall N \in \mathbb{N}, \exists C_N, \forall p \in \mathbb{R}, \ |\rho(p)| \leq C_N \ (1 + |p|)^{-N}.
\]
For any function $f \in L^1_{loc}(\mathbb{R}^d)$ with slow growth, we define the windowed X-ray transform of $f$ in the direction $\alpha$ to be the map
\[
y \in <\alpha> \mapsto \mathcal{P}_\alpha f(y) := \int_{\mathbb{R}} f(y + p\alpha) \rho(p) \ dp.
\]

Our aim is to apply such a transformation to a microball model. We use the properties of the Poisson random measure and the fact that one can define the field
\[
\left\{ \int_{\mathbb{R}^d \times \mathbb{R}^+} \left( \int_{\mathbb{R}} 1_{B(\xi, r)}(y + p\alpha) \rho(p) \ dp \right) N_h(d\xi, dr); \ y \in <\alpha> \right\},
\]
as soon as the integrand

\[ P_\alpha \left( 1_{B(\xi,r)}(\cdot) \right)(y) = \int_\mathbb{R} 1_{B(\xi,r)}(y + p\alpha)\rho(p)dp \]

is integrable with respect to \( \nu_h(d\xi,dr) \) for each \( y \in \alpha_{\bot} \). But this is straightforward using (2) and (7). This allows us to state the following definition.

**Definition 2.3.** Let \( h \) be a function on \( \mathbb{R}^d \) such that \( \text{essinf } h > 0 \) and let \( N_h \) be a Poisson random measure with intensity \( \nu_h \). Let \( X \) be the microball model with index \( h \), \( X(\cdot) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi,r)}(\cdot)N_h(d\xi,dr) \).

Let \( \rho \) be a continuous window that satisfies (7). For \( \alpha \in S^{d-1} \), we call the windowed X-ray transform of \( X \) in the direction \( \alpha \), the field defined on \( \alpha_{\bot} \) by

\[ P_\alpha X(y) = \int_{\mathbb{R}^d \times \mathbb{R}^+} P_\alpha \left( 1_{B(\xi,r)}(\cdot) \right)(y)N_h(d\xi,dr). \]

Note that for \( (\xi, r) \) in \( \mathbb{R}^d \times \mathbb{R}^+ \), Cauchy-Schwarz inequality leads to

\[ |P_\alpha \left( 1_{B(\xi,r)}(\cdot) \right)(y)|^2 \leq 2(r^2 - |y - \pi_{\alpha_{\bot}}(\xi)|^2)^{1/2} \int_\mathbb{R} 1_{B(\xi,r)}(y + p\alpha)\rho^2(p)dp \]

\[ \leq 2r \int_\mathbb{R} 1_{B(\xi,r)}(y + p\alpha)\rho^2(p)dp , \]

where \( \pi_{\alpha_{\bot}} \) denotes the orthogonal projection on \( \alpha_{\bot} \). Thus, for all \( y \in \alpha_{\bot} \), the function \( P_\alpha \left( 1_{B(\xi,r)}(\cdot) \right)(y) \) belongs to \( L^2(\mathbb{R}^d \times \mathbb{R}^+, \nu_h(d\xi,dr)) \) and \( P_\alpha X \) admits a second order moment.

As for the microball model itself, we are particularly interested in the local behavior of the X-ray transform and we need to estimate its increments. For \( y_0, y \in \alpha_{\bot} \), using (5) we get

(9) \[ \Delta_{y_0} P_\alpha X(y) = P_\alpha X(y_0 + y) - P_\alpha X(y_0) = \int_{\mathbb{R}^d \times \mathbb{R}^+} G_\rho(y,\xi - y_0, r)N_h(d\xi,dr) \]

where

(10) \[ G_\rho(y,\xi, r) = \int_\mathbb{R} G(y,\xi, r, \cdot)\rho(d\eta) \]

Note that the above integral is well defined for any bounded function \( \rho \). In the special case where \( \rho \equiv 1 \), we write \( G \) instead of \( G_1 \) and for \( y, \gamma \) in \( \alpha_{\bot} \) and \( r \) in \( \mathbb{R}^+ \), a simple computation gives

(11) \[ G(y,\gamma, r) = G_1(y,\gamma, r) = (r^2 - |y - \gamma|^2)^{1/2} - (r^2 - |\gamma|^2)^{1/2} , \]

where, as usual, \( t^+_+ := \max(0,t) \) for all \( t \in \mathbb{R} \). The next lemma provides upper-bounds for the integral of \( G(y,\cdot) \).
Lemma 2.4. For \( m \in (0, 1/2) \) and \( y \in < \alpha , 1 \),
\[
G(y, .) \in L^2(< \alpha > \times \mathbb{R}^+, r^{-d-1+2m}d\gamma dr).
\]

Proof. For \( m \in (0, 1/2) \), \( y \in < \alpha , 1 \) and \( \lambda > 0 \), on one hand
\[
\int_{<\alpha>\perp} G(y, \gamma, r)^2 d\gamma \leq r^2 \int_{<\alpha>\perp} (1_{|y-\gamma|<r} + 1_{|\gamma|<r}) d\gamma
\]
(12)
On the other hand, a change of variable gives, for \( y \neq 0 \) and \( r > 0 \),
\[
\left(\frac{|y|}{2}\right)^{(d+1)} \int_{<\alpha>\perp} G(y, \gamma, r)^2 \, d\gamma
\]
\[
= \int_{<\alpha>\perp} \left(\left(\frac{2r}{|y|}\right)^2 - |\gamma - \frac{y}{|y|}|^2\right)^{1/2} - \left(\left(\frac{2r}{|y|}\right)^2 - |\gamma + \frac{y}{|y|}|^2\right)^{1/2}\right)^2 d\gamma.
\]
The next lemma provides an upper bound for the last quantity, which leads to
\[
\int_{<\alpha>\perp} G(y, \gamma, r)^2 \, d\gamma \leq C|y|^2 r^{d-1} \ln(2 + \frac{2r}{|y|}).
\]
Since \( m \in (0, 1/2) \), inequalities (12) and (13) conclude for the proof. \( \square \)

Lemma 2.5. Let \( n \in \mathbb{N}^* \). There exists a constant \( C > 0 \) such that for all direction \( e \in S^{n-1} \) and all \( r > 0 \),
\[
\int_{\mathbb{R}^n} \left(r^2 - |x - e|^2\right)^{1/2} - (r^2 - |x + e|^2)^{1/2}\right)^2 \, dx \leq Cr^n \ln(2 + r).
\]

Proof. For \( n = 1 \), we have to prove that there exists a constant \( C \) such that, for \( r > 0 \),
\[
\int_{0}^{r+1} \left(r^2 - (x - 1)^2\right)^{1/2} - (r^2 - (x + 1)^2)^{1/2}\right)^2 \, dx \leq Cr \ln(r + 2).
\]
This is an easy consequence of the fact that the function that we integrate is bounded by \( 4r \) for \( x \in [r - 1, r + 1] \), and by \( 16r^2((r-1)(r-x+1))^{-1} \) for \( x \in [0, r-1] \) when \( r > 1 \).
In the general case \( (n > 1) \) we write \( x = x' + x''e \) with \( x' = \pi_{<e>\perp}(x) \) and \( x'' \in \mathbb{R} \). From the one-dimensional case, for \( x' \in < e > \perp \),
\[
\int_{\mathbb{R}} \left(r^2 - |x'|^2 - |x'' - 1|^2\right)^{1/2} - (r^2 - |x'|^2 - |x'' + 1|^2)^{1/2}\right)^2 \, dx''
\]
\[
\leq C(r^2 - |x'|^2\right)^{1/2} \ln(r + 2).
\]
But
\[
\int_{<e>\perp} (r^2 - |x'|^2)^{1/2} \, dx' = r^n |S^{n-2}| \int_{0}^{1} (1 - t^2)^{1/2} t^{-n+2} \, dt.
\]
Finally, we can change the constant $C$ such that

$$
\int_{\mathbb{R}^n} \left( (r^2 - |x-e|^2)^{1/2} - (r^2 - |x+e|^2)^{1/2} \right)^2 \, dx \leq C r^n \ln(2 + r).
$$

\[\square\]

3. LASS PROPERTIES

The section is devoted to the study of self-similarity properties of the microball models and their X-ray transforms.

3.1. The fractional microball model. Let $d \geq 1$. Let $m > 0$ and recall that the fractional intensity measure $\nu_m$ on $\mathbb{R}^d \times \mathbb{R}^+$ is given by

$$
\nu_m(d\xi, dr) = r^{-d-1+2m} 1_{(0,1)}(r) d\xi dr
$$

and the fractional microball model by

$$
X = \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi,r)}(x) N_m(d\xi, dr); \ x \in \mathbb{R}^d \right\},
$$

where $N_m$ is a random Poisson measure of intensity $\nu_m$.

First, let us remark that the choice of $\nu_m$ as intensity means that the center and the radius are thrown independently. Hence, since the centers are uniformly distributed on the state space $\mathbb{R}^d$, the fractional microball model $X$ is isotropic and stationary, i.e. for every rotation $R$ centered at 0 in $\mathbb{R}^d$ and for all $x_0 \in \mathbb{R}^d$,

$$
X \circ R \overset{\text{fdd}}{=} X \quad \text{and} \quad X(x_0 + .) \overset{\text{fdd}}{=} X.
$$

In the following, we will assume that $m < 1/2$ so that Lemma 2.2 applies. Then, using [15] and the stationarity of $X$ we get for $x, x' \in \mathbb{R}^d$,

$$
\mathbb{E} \left( (X(x) - X(x'))^2 \right) \leq C(m) |x - x'|^{2m}.
$$

Thus, the field $X$ is mean square continuous.

Furthermore, by the correlation theory of stationary random fields (see [20] for example), there exists a finite positive Radon measure $\sigma$ such that

$$
\text{Cov} \left( X(x), X(0) \right) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} d\sigma(\xi).
$$

By computing the inverse Fourier transform we obtain that $\sigma$ is absolutely continuous with respect to the Lebesgue’s measure. Moreover, the spectral density of the fractional microball model $X$ is given by

$$
(2\pi)^{d/2} |\xi|^{-2m-d} \int_0^{||\xi||} J_{d/2}^2(s) s^{-d-1+2m} ds,
$$

where $J_{d/2}$ is the Bessel function (see [15] p. 406 for example). Let us recall that

$$
J_{d/2}(s) = \frac{s^{d/2}}{2^{d/2}\Gamma(1+d/2)} \left( 1 + O \left( |s|^2 \right) \right), \ s \to 0,
$$
and

\[ J_{d/2}(s) = \sqrt{\frac{2}{\pi s}} \left( \cos \left( s - \frac{(d + 1) \pi}{4} \right) + O(s^{-1}) \right), \quad s \to +\infty. \]

Hence, the previous spectral density has a power law behavior and loss properties for the associated field are expected. This point of view is detailed in [4] section 2.2.2.

We first investigate the covariance loss property of \( X \). Recall that \( X \) is stationary and therefore the behavior of the increments of \( X \) around any \( x_0 \in \mathbb{R}^d \) does not depend on \( x_0 \). We have to deal, for \( x, x' \in \mathbb{R}^d \) and \( \lambda \to 0^+ \), with

\[
\text{Cov}(\Delta_{x_0} X(\lambda x), \Delta_{x_0} X(\lambda x')) = \int_{\mathbb{R}^d \times (0,1)} \psi(\lambda x, \xi, r) \psi(\lambda x', \xi, r) r^{-d-2m} drd\xi,
\]

where \( \psi \) is given by (6). By homogeneity and integrability of \( \psi \) (see Lemma 2.2) it is easy to establish that, for all \( x, x' \in \mathbb{R}^d \),

\[
\lim_{\lambda \to 0^+} \lambda^{-2m} \int_{\mathbb{R}^d \times (0,1)} \psi(\lambda x, \xi, r) \psi(\lambda x', \xi, r) r^{-d-2m} drd\xi = \int_{\mathbb{R}^d \times \mathbb{R}^+} \psi(x, \xi, r) \psi(x', \xi, r) r^{-d-2m} drd\xi.
\]

The cov-loss property of \( X \) follows from (14).

Now, let us assume that \( d \geq 2 \) and choose \( \alpha \in S^{d-1} \). We look for the cov-loss property of \( \mathcal{P}_\alpha X \), the windowed X-ray transform of \( X \). Let us recall (9) and (10): for any \( y_0 \) and \( y \) in \( \langle \alpha \rangle \perp \),

\[
\Delta_{y_0} \mathcal{P}_\alpha X(y) = \int_{\mathbb{R}^d \times \mathbb{R}^+} G_\rho(y, \xi - y_0, r) N_{m}(d\xi, dr)
\]

where

\[
G_\rho(y, \xi, r) = \int_\mathbb{R} \psi(y, \xi - p\alpha, r) \rho(p) dp.
\]

It is straightforward to see that \( \mathcal{P}_\alpha X \) is also stationary so that the cov-loss properties will not depend on the point \( y_0 \).

Writing any \( \xi \in \mathbb{R}^d \) as \( \xi = \gamma + t\alpha \) with \( \gamma \in \langle \alpha \rangle \) and \( t \in \mathbb{R} \) and performing a change of variable (translation-dilation) in the integral that defines \( G_\rho \), we obtain

\[
\text{Cov}(\Delta_{y_0} \mathcal{P}_\alpha X(\lambda y), \Delta_{y_0} \mathcal{P}_\alpha X(\lambda y'))
= \lambda^{1+2m} \int_{\langle \alpha \rangle \perp \times \mathbb{R} \times (0,\lambda^{-1})} G_\rho(t + \lambda \gamma, y, r) G_\rho(t + \lambda \gamma', y', r) r^{-d-1+2m} drdtdtr,
\]

with \( \rho(t + \lambda \cdot) \) denoting the window \( p \mapsto \rho(t + \lambda p) \). Note that for \( r \in \mathbb{R}^+ \), \( t \in \mathbb{R} \) and \( y, \gamma \in \langle \alpha \rangle \),

\[
G_\rho(t + \lambda \gamma, y, r) \xrightarrow{\lambda \to 0^+} G(y, \gamma, r) \rho(t).
\]
Assumption 7 on $\rho$ and Lemma 2.4 allow to conclude that
\[ \lim_{\lambda \to 0^+} \lambda^{-1-2m} \text{Cov}(\Delta_{y_0} P_\alpha X(\lambda y), \Delta_{y_0} P_\alpha X(\lambda y')) = \left( \int_{\mathbb{R}} \rho(t)^2 dt \right) \int_{\alpha > \perp} G(y, \gamma, r) G(y', \gamma, r) r^{-d+1+2m} d\gamma dr. \]

We now state a theorem including the notions of local asymptotical self-similarity index given in Definitions 1.1 and 1.2. The fdd-lass properties are proved in Section 6, as a consequence of Theorem 3.4 in the special case where the multifractional index $h$ is constant equal to $m$.

**Theorem 3.1.** Let $d \geq 2$. Let $X$ be the fractional microball model with index $m \in (0, 1/2)$ and let $P_\alpha X$ be its $X$-ray transform in the direction $\alpha \in S^{d-1}$.

- At any point $x_0 \in \mathbb{R}^d$, the cov-lass index of $X$ is equal to $m$ and the fdd-lass index of $X$ is equal to $+\infty$. Moreover the covariance of $\lambda^{-m} \Delta_{x_0} X(\lambda \cdot)$ converges, up to a multiplicative constant, to the covariance of a fractional Brownian motion of index $m$.
- At any point $y_0 \in \alpha > \perp$, the cov-lass index and the fdd-lass index of $P_\alpha X$ are equal to $m+1/2$. Moreover the covariance and the finite dimensional distributions of $\lambda^{-m-1/2} (\Delta_{y_0} (P_\alpha X)(\lambda \cdot) - E (\Delta_{y_0} (P_\alpha X)(\lambda \cdot)))$ converge, up to a multiplicative constant, to the corresponding ones of a fractional Brownian motion of index $m + 1/2$.

**Remark 3.2.** The first point of Theorem 3.1 is still true in the one-dimensional case ($d = 1$).

Let us comment this result.

In one dimension, this result describes the small scale behavior of the number of active connections in a communication network: the covariance is locally asymptotically self similar and behaves like a fractional Brownian motion covariance. More generally, the same is observed in the multi-dimensional case. Hence the fractional microball model provides a microscopic description of a random media which behaves, up to the second-order moment, like a fractional Brownian motion.

The second point of Theorem 3.1 is very interesting from a practical point of view. The one-to-one correspondence between the lass index of $X$ and the lass index of $P_\alpha X$ allows the estimation of the 3D lass index through the analysis of the media radiographic images. The fractional microball model is thus relevant to model isotropic and stationary media.

Following a widespread idea ([14], [3], [5]), we now consider the multifractional microball model, where $m$ is replaced by a function that depends on the ball location.

### 3.2. The multifractional microball model

In this section we study the lass properties of the multifractional microball model, associated with the
multifractional intensity measure \( \nu_h \), given on \( \mathbb{R}^d \times \mathbb{R}^+ \) by
\[
\nu_h(d\xi, dr) = r^{-d-1+2h(\xi)} \mathbf{1}_{(0,1)}(r) d\xi dr,
\]
where \( h \) is a function on \( \mathbb{R}^d \) such that \( 0 < h(\xi) < 1/2 \).

First, let us remark that the multifractional microball model is not stationary nor isotropic when \( h \) is not constant.

Let us deal with the cov-class properties of the multifractional microball. One has to study, for all \( x_0, x, x' \in \mathbb{R}^d \), and \( \lambda \to 0^+ \), the asymptotic behavior of \( \text{Cov}(\Delta_{x_0}X(\lambda x), \Delta_{x_0}X(\lambda x')) \). By a change of variables, with \( \psi \) given by (16),
\[
\text{Cov}(\Delta_{x_0}X(\lambda x), \Delta_{x_0}X(\lambda x')) = \int_{\mathbb{R}^d \times (0, \lambda^{-1})} \lambda^{2h(x_0+\lambda \xi)} \psi(x, \xi, r) \psi(x', \xi, r) r^{-d-1+2h(x_0+\lambda \xi)} d\xi dr.
\]
Similarly, when \( d \geq 2 \) and \( \alpha \in S^{d-1} \), for all \( y_0, y, y' \in \langle \alpha >_1 \), (16) becomes
\[
\text{Cov}(\Delta_{y_0}P_\alpha X(\lambda y), \Delta_{y_0}P_\alpha X(\lambda y')) = \lambda \int_{\lambda \alpha \times \mathbb{R} \times (0, \lambda^{-1})} G_{\rho(t+\lambda \gamma)}(y, \gamma, r) G_{\rho(t+\lambda \gamma)}(y', \gamma, r)
\]
\[\times (\lambda r)^{2h(y_0+\gamma t+\lambda \gamma)} r^{-d-1} d\gamma dt dr.
\]
In order to get cov-class properties for both \( X \) and its windowed X-ray transform, further assumptions on \( h \) have to be made. We are mainly interested in two kinds of functions. The first kind deals with the case where \( h \) is smooth on \( \mathbb{R}^d \). It is linked with the multifractional Brownian motion \( \text{[14]} \), \( \text{[3]} \), obtained by substituting the Hurst parameter \( H \) by a Lipschitz function on the state space. The second class of function is when \( h \) is singular at the point 0, defined on \( \mathbb{R}^d \setminus \{0\} \) by homogeneity, ie \( h(\lambda \xi) = h(\xi) \) for all \( \lambda \in \mathbb{R}^* \). This follows the point of view of \( \text{[5]} \) to get anisotropic generalizations of the fractional Brownian motion.

Let us recall the definition of a \( \beta \)-Lipschitz function.

**Definition 3.3.** Let \((M, d_M)\) be a metric space and \( \beta \in (0, 1] \). A function \( f : M \to \mathbb{R} \) is called \( \beta \)-Lipschitz on \( M \) if there exists \( C > 0 \) such that
\[
\forall x, y \in M, \ d_M(x, y) \leq 1 \Rightarrow |f(x) - f(y)| \leq Cd_M(x, y)^\beta.
\]

**3.2.1. The smooth case:** let us first study the case of a \( \beta \)-Lipschitz function on \( \mathbb{R}^d \). By continuity of \( h \) around \( x_0 \in \mathbb{R}^d \), we get from (16) that \( H_{\text{cov}}(X, x_0) = h(x_0) \), using Lemma 2.2. A rigorous proof of this statement is given in Section 3. Moreover, the continuity of \( h \) around \( y_0 + t\alpha \), for \( y_0 \in \langle \alpha >_1 \) and \( t \in \mathbb{R} \), applied in (17) and Lemma 2.4 will imply that \( H_{\text{cov}}(P_\alpha X, y_0) = m(\alpha, y_0) + 1/2 \), where
\[
m(\alpha, y_0) := \inf_{t \in \mathbb{R}} h(y_0 + t\alpha).
\]
These observations yield to the next theorem. A detailed proof is given in Sections 5 and 6. We denote \( \text{meas} \) the Lebesgue’s measure.

**Theorem 3.4.** Let \( d \geq 2 \). Let \( h \) be a \( \beta \)-Lipschitz function on \( \mathbb{R}^d \) such that \( 0 < h < 1/2 \). Let \( X \) be the multifractional microball model with index \( h \) and let \( P_\alpha X \) be its X-ray transform in the direction \( \alpha \in S^{d-1} \).

- At any point \( x_0 \in \mathbb{R}^d \), the cov-lass index of \( X \) is equal to \( h(x_0) \) and the fdd-lass index of \( X \) is greater than \( \min(1, \beta + 2h(x_0)) \). Moreover, the covariance of \( \lambda^{-h(x_0)} \Delta_{x_0} X(\lambda) \) converges, up to a multiplicative constant, to the covariance of a fractional Brownian motion of index \( h(x_0) \).
- At any point \( y_0 \in < \alpha > \), the cov-lass index and the fdd-lass index of \( P_\alpha X \) are equal to \( m(\alpha, y_0) + 1/2 \). Moreover, when \( \text{meas}(\{t \in \mathbb{R}; h(y_0+t\alpha) = m(\alpha, y_0)\}) > 0 \), the covariance and the finite dimensional distributions of \( \lambda^{-m(\alpha, y_0)+1/2} (\Delta_{y_0}(P_\alpha X)(\lambda) - E(\Delta_{y_0}(P_\alpha X)(\lambda))) \) converge, up to a multiplicative constant, to the corresponding ones of a fractional Brownian motion of index \( m(\alpha, y_0) + 1/2 \).

**Remark 3.5.** The first point of Theorem 3.4 is still true in the one-dimensional case \( (d = 1) \).

We remark that the cov-lass index of \( X \) at point \( x_0 \in \mathbb{R}^d \) is equal to \( h(x_0) \), as for the multifractional Brownian motion with index \( h \), and this justifies the name multifractional. Also let us just point out that these results apply to the fractional case, where \( h \) is constant. In this case we have shown that the fdd-lass index for the microball is equal to \( +\infty \). This is a consequence of the stationarity of the model. The first order moment of the increments is then equal to 0 and this is no longer the case when \( h \) is not constant. From a certain point of view, for the microball model, the fdd-lass index deals with the regularity of the first order moment and the cov-lass index with those of the second order moment. Finally, compared to the fractional case, we still have an additive factor of 1/2 for the lass indices of the windowed X-ray transforms, but here only the infimum of \( h \) along straight lines can be recovered.

3.2.2. The singular case: let us now consider \( h \) to be an even, \( \beta \)-Lipschitz function on the sphere, defined on \( \mathbb{R}^d \setminus \{0\} \) by homogeneity. Of course, this case is only relevant when \( d \geq 2 \) because otherwise it turns to be the fractional case, with \( h \) constant. The singularity of \( h \) at point 0 makes this point a very special one. The balls are thrown from 0 and their numbers and sizes only depend on the direction along which they are thrown. To distinguish the microball model associated with such a singular \( h \) from the multifractional one, we will call it the star microball model. Let us remark that the \( \beta \)-Lipschitz assumption on \( h \) means that there exists \( C > 0 \) such
that for all \(x_0 \in \mathbb{R}^d \setminus \{0\}\) and \(\xi \in \mathbb{R}^d\), when \(|\xi| \leq 1\)

\[
|h(x_0 + \xi) - h(x_0)| \leq C|x_0|^{-\beta}|\xi|^\beta.
\]

Then, the lass properties of the star microball model, respectively its X-ray transform in the direction \(\alpha \in S^{d-1}\), at any point \(x_0 \in \mathbb{R}^d \setminus \{0\}\), respectively \(y_0 \in <\alpha>^{1}\setminus \{0\}\), are the same as for the multifractional microball model, respectively its X-ray transform in the direction \(\alpha\), given in Theorem 3.4. Thus, the next theorem will only deal with the lass properties around \(0\). Let us remark that, in that case, by homogeneity of \(h\), the exponent in (16) is equal to \(\lambda^\alpha(\xi)\) for all \(\xi \neq 0\). Then, denoting by \(m \in (0,1/2)\) the minimum of \(h\) on \(S^{d-1}\), by Lemma 2.2 we will get \(H_{\text{cov}}(X,0) = m\). Moreover, in (17), since \(h\) is continuous around \(t_0\) and \(h(t_0) = (h(\alpha))\), for \(t \in \mathbb{R}^*\), using Lemma 2.4 we prove that \(H_{\text{cov}}(\mathcal{P}_\alpha X,0) = h(\alpha) + 1/2\).

Let us state the different lass properties of the star microball model.

**Theorem 3.6.** Let \(h\) be an even \(\beta\)-Lipschitz, non constant, function on \(S^{d-1}\) such that \(0 < h < 1/2\). Let \(X\) be the star microball model with index \(h\) and let \(\mathcal{P}_\alpha X\) be its X-ray transform in the direction \(\alpha \in S^{d-1}\). We consider lass properties at point \(0\).

- The cov-lass index of \(X\) is equal to \(m\) and the fdd-lass index of \(X\) is equal to \(2m\). Moreover, when \(\text{meas}\{\{h = m\}\} > 0\), the covariance of \(\lambda^{-m}\Delta_0 X(\lambda.)\) converges to \(\gamma_m\) with
  \[
  \gamma_m(x,x') = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{h(\xi) = m} \psi(x,\xi,r) \psi(x',\xi,r) r^{-d-1+2m} d\xi dr,
  \]
  for \(x, x' \in \mathbb{R}^d\), while the finite dimensional distributions of \(\lambda^{-2m} (\Delta_0 X(\lambda.) - \mathbb{E}(\Delta_0 X(\lambda.)))\) converge to the deterministic field \(Z_m\), with
  \[
  Z_m = \left\{-|x|^{2m} \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{h(\xi) = m} \psi(x,|x|,\xi,r) r^{-d-1+2m} d\xi dr ; \ x \in \mathbb{R}^d\right\},
  \]
  where \(\psi\) is given by (10).

- The cov-lass index and the fdd-lass index of \(\mathcal{P}_\alpha X\) are equal to \(h(\alpha) + 1/2\). Moreover, the covariance and the finite dimensional distributions of \(\lambda^{-h(\alpha)+1/2} (\Delta_0(\mathcal{P}_\alpha X)(\lambda.) - \mathbb{E}(\Delta_0(\mathcal{P}_\alpha X)(\lambda.)))\) converge, up to a multiplicative constant, to the corresponding ones of a fractional Brownian motion of index \(h(\alpha) + 1/2\).

The proof is given in Sections 5 and 6. Let us remark that for the star microball model, there exists both a cov-lass index and a fdd-lass index and that the later equals the double of the former. This multiplicative factor is typical for the Poisson structure proved by the following exercise.

**Remark 3.7.** Let \((X_n)\) be a sequence of Poisson random variables. Suppose there exists some \(H > 0\) and \(v > 0\) such that \(\text{Var}(n^H(X_n - E(X_n))) \to v\) when \(n\) tends to \(+\infty\). Then \(n^{2H}(X_n - E(X_n))\) tends in distribution to \(-v\).
Moreover when \( \{ h = m \} \) has positive measure, the tangent field at 0 is
deterministic and not zero, hence does not have stationary increments. This
is worth to be noticed and linked to a result of Falconer [8], which states
that at almost all points the tangent field -if it exists- must have stationary
increments. Hence the point 0 appears as an ‘exceptional point’ (see [13]
for other examples of exceptional points).

Finally, let us point out that the tangent field of the X-ray transform,
when it exists, is Gaussian, even a fractional Brownian motion, whereas the
tangent field of the star microball model was deterministic. This justifies,
from a mathematical point of view, modeling radiographic images by fBm,
even when the media under study is far from being of this type (see [9]
for an experimental study).

3.3. Comparison with homogenization results. There are different ways
to consider self-similarity at small scales, depending on which part of the
signal the scaling acts. Instead of performing a scaling on the increments lag,
as it is performed in Sections 3.1 and 3.2 we act on the radius of the
balls as follows. Suppose we zoom and consider the balls \( B(\xi, r/\varepsilon) \) instead
of the balls \( B(\xi, r) \), where the \((\xi, r)\) are randomly chosen by the Poisson
random measure \( N_h \), and we let \( \varepsilon \) tend to 0. Denoting by \( X^\varepsilon \) the associated
field
\[
X^\varepsilon(x) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{B(\xi, r/\varepsilon)}(x) N_h(d\xi, dr) \ , \ x \in \mathbb{R}^d ,
\]
we look for normalization terms \( n(x_0) \) such that \( e^{n(x_0)}(\Delta x_0 X^\varepsilon - E(\Delta x_0 X^\varepsilon)) \)
converge in distribution to a non degenerate field. Note that the field \( X^\varepsilon \)
can also be considered as a microball model (see Definition 2.1) associated
with a Poisson measure with intensity
\[
\nu^\varepsilon_h(d\xi, dr) = \varepsilon^{-d+2h(\xi)} r_{-d-1+2h(\xi)} 1_{(0,\varepsilon^{-1})}(r) d\xi dr .
\]
Actually this procedure is nothing but homogenization and is close to the
thermodynamical limit investigated in [6] or the scaling limit in [10]. Similar
computations as for the previous theorems yield
\begin{itemize}
  \item if \( 0 < m := \min h < 1/2 \) and the set \( \{ \xi; h(\xi) = m \} \) has positive measure,
then the normalization term \( n(x_0) \) is equal to \( d/2 - m \) for all \( x_0 \in \mathbb{R}^d \)
  \item moreover if \( h \) is constant equal to \( m \), then the limit field is a fractional
Brownian motion with index \( m \).
\end{itemize}

4. Conclusion

We propose to model -from a microscopic point of view- the mass intensity
of a porous media or the number of connected customers in a network with a
non-Gaussian field, which presents a macroscopic (multi)fractional behavior.
The rich structure of Poisson point processes allows us to reach this goal and
also to perform explicit computations as in the Gaussian case. In order to
keep the model as intuitive as possible, we did not try to produce more
general fields. The Poisson structure can obviously be exploited further on
by considering more general integrators with respect to the Poisson measure. Replacing the indicator function \(1_{B(\xi,r)}\) in Definition 2.1 by a more general one will, for instance, give the possibility to model granular media with non spherical grains. Another model for porous media can also be built up from a collection of random spheres which correspond no more to grains, but to pores or bubbles. By this way, one will get a \(\{0,1\}\)-valued field and leave the linear context.

5. Proofs of the cov-lass properties

Let \(h\) be a function defined on \(\mathbb{R}^d\) or \(\mathbb{R}^d \setminus \{0\}\), with \(0 < h < 1/2\). In this section we will give rigorous proofs for the cov-lass properties of the microball models and their windowed X-ray transforms.

5.1. Cov-lass properties of the microball model. Let \(x_0 \in \mathbb{R}^d\). For \(H \in (0,1)\) and \(x, x' \in \mathbb{R}^d\), recall from (16) that

\[
\Gamma^H_{\lambda}(x_0, x, x') := \text{Cov}\left(\frac{\Delta_{x_0}X(\lambda x)}{\lambda^H}, \frac{\Delta_{x_0}X(\lambda x')}{\lambda^H}\right)
\]

\[
= \int_{\mathbb{R}^d \times (0,\lambda^{-1})} \lambda^{-2(\lambda - h(x_0 + \lambda \xi))} \psi(x, \xi, r) \psi(x', \xi, r) r^{-d-1+2h(x_0+\lambda \xi)} d\xi dr.
\]

a. The smooth case: we assume that there exists \(C > 0\) such that, for \(|\xi| < 1\)

\[
|h(x_0 + \xi) - h(x_0)| \leq C|\xi|^\beta.
\]

We first establish that

\[
\Gamma^h_{\lambda}(x_0, x, x') \xrightarrow{\lambda \to 0^+} \Gamma^h(x_0)(x, x'),
\]

where

\[
\Gamma^h(x_0)(x, x') = \int_{\mathbb{R}^d \times \mathbb{R}^+} \psi(x, \xi, r) \psi(x', \xi, r) r^{-d-1+2h(x_0)} d\xi dr,
\]

and then prove that \(\Gamma^h(x_0)\) is the covariance function of a fractional Brownian motion with Hurst index \(h(x_0)\).

Since \(h(x_0) < 1/2\), Lemma 2.2 indicates that the function \(\psi(x, \cdot)\) belongs to \(L^2(\mathbb{R}^d \times \mathbb{R}^+, r^{-d-1+2h(x_0)} d\xi dr)\). By Cauchy-Schwarz inequality, in order to prove (21), it is enough to prove that the difference \(I(\lambda)\), given by

\[
I(\lambda) = \int_{\mathbb{R}^d \times (0,\lambda^{-1})} \psi(x, \xi, r)^2 \left| (\lambda r)^{-2(h(x_0)-h(x_0+\lambda \xi))} - 1 \right| r^{-d-1+2h(x_0)} d\xi dr,
\]

tends to 0 as \(\lambda\) tends to 0\(^+\). Let us remark that, by (20), for some positive \(p\) and \(q \in (0,1)\), to be fixed later, when \(\lambda^{p+1} < \lambda r < 1\) and \(\lambda|\xi| < \lambda^{1-q} < 1\), one can find \(C > 0\) such that

\[
\left| (\lambda r)^{-2(h(x_0)-h(x_0+\lambda \xi))} - 1 \right| \leq C (\lambda |\xi|)^\beta |\ln \lambda|.
\]
Moreover, for $\xi = \lambda r$, we have $\theta = \beta < 1 - 2h(x_0)$. In this case we will prove that, for $\lambda$ small enough compared to $x$,

$$I(\lambda) \leq C\lambda^{\beta} |\ln \lambda|,$$

with $C > 0$. We split the integral into

$$\int_{\mathbb{R}^d \times (0, \lambda^{-1})} = \int_{B(0, \lambda^{-\eta}) \times (\lambda^p, \lambda^{-1})} + \int_{\mathbb{R}^d \times (0, \lambda^p)} + \int_{B(0, \lambda^{-\eta}) \times (0, \lambda^{-1})}.$$

By (23), the same kind of arguments as in the proof of Lemma 2.2 yield to

$$\left| \int_{B(0, \lambda^{-\eta}) \times (\lambda^p, \lambda^{-1})} \right| \leq C\lambda^{\beta} |x|^{\beta + 2h(x_0)} |\ln \lambda|.$$

Moreover, for $\xi \in B(0, \lambda^{-\eta})$ or $r \in (0, \lambda^p)$, we use the following inequality

$$\left| (\lambda r)^{-2(h(x_0) - h(x_0 + \lambda t))} - 1 \right| \leq 2(\lambda r)^{2(m - h(x_0))},$$

which holds since $h(x_0 + \lambda t) \geq m$ and $\lambda r \leq 1$. Then, on one hand,

$$\left| \int_{\mathbb{R}^d \times (0, \lambda^p)} \right| \leq 2\lambda^{2(m - h(x_0))} \int_{\mathbb{R}^d \times (0, \lambda^p)} 1_{|\xi| < r} r^{-d - 1 + 2m} dr$$

$$\leq C\lambda^{2(m - h(x_0)) + 2mp}.$$

On the other hand, for $\lambda \leq (4(1 + |x|))^{-1/2}$,

$$\left| \int_{B(0, \lambda^{-\eta}) \times (0, \lambda^{-1})} \right| \leq C|x|\lambda^{2(m - h(x_0))} \int_{|\xi| > \lambda^{-\eta}} |\xi|^{-d - 1 + 2m} d\xi$$

$$\leq C|x|\lambda^{2(m - h(x_0)) + q(1 - 2m)}.$$

Finally, it suffices to choose $p$ and $q$ such that

$$p > \frac{\beta + 2(h(x_0) - m)}{2m} \quad \text{and} \quad \frac{\beta + 2(h(x_0) - m)}{1 - 2m} < q < 1.$$
b. The singular case: we now assume that $h$ is given by an even $\beta$-Lipschitz function on the sphere, defined on $\mathbb{R}^d \setminus \{0\}$ by homogeneity, and deal with the case $x_0 = 0$.

b-i) Assume that $\{\xi \in \mathbb{R}^d \setminus \{0\}; h(\xi) = m\}$ has positive measure. 
Thus, using Lemma 2.2, Lebesgue’s Theorem gives that $\Gamma^m_\lambda(0, x, x')$ tends to 
\begin{equation}
\int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{h(\xi) = m} \psi(x, \xi, r) \psi(x', \xi, r) r^{-d-1+2m} d\xi dr := \gamma^m(x, x'),
\end{equation}
which does not vanish by assumption.

b-ii) Assume now that $\{\xi \in \mathbb{R}^d \setminus \{0\}; h(\xi) = m\}$ has measure 0. 
Since $\gamma^m(x, x') = 0$ by (25), the covariance $\Gamma^m_\lambda(0, x, x')$ tends to 0 for $H \leq m$. 
On the other hand, for $H = m + 2\epsilon$ with $\epsilon > 0$ and for $\lambda$ in $(0, 1)$,
\[ \Gamma^H_\lambda(x, x) \geq \lambda^{-\epsilon} \int_{\mathbb{R}^d \times (0, 1)} 1_{h(\xi) < m + \epsilon} \psi(x, \xi, r)^2 r^{-d-1+2H(\xi)} d\xi dr, \]
where $\{\xi \in \mathbb{R}^d \setminus \{0\}; h(\xi) < m + \epsilon\}$ has positive measure. So the above quantity tends to infinity when $\lambda$ tends to $0^+$. Hence the exponent $m$ is proved to be the cov-lass index for $X$ at $x_0 = 0$.

The cov-lass properties asserted in the first part of Theorem 3.6 are now proved.

5.2. Cov-lass properties of the windowed X-ray transforms. Let $\alpha \in S^{d-1}$ and $y_0 \in < \alpha > ^{+}$. Let us denote by $\Gamma^H_\lambda(y_0, .)$ the covariance function of the field $\lambda^{-H} \Delta_{y_0} \mathcal{P}_\alpha X(\lambda)$. Then for $y$ and $y'$ in $< \alpha > ^{+}$, by (17)
\[ \Gamma^H_\lambda(y_0, y, y') = \lambda^{-2H+1} \int_{< \alpha > ^{+} \times \mathbb{R} \times (0, \lambda^{-1})} G_{\rho(t+\lambda)}(y, \gamma, r) G_{\rho(t+\lambda)}(y', \gamma, r) \]
\[ \times (\lambda r)^{2h(\gamma + \alpha + y_0)} r^{-d-1} d\gamma dt dr. \]
Let us write $G_{\rho(t+\lambda)}$ as the integral given by (10)
\[ \Gamma^H_\lambda(y_0, y, y') = \lambda^{-2H+1} \int_{< \alpha > ^{+} \times \mathbb{R} \times (0, \lambda^{-1}) \times \mathbb{R}} \psi(y, \gamma - p\alpha, r) \rho(t + \lambda p)(y, \gamma, r) \]
\[ \times G_{\rho(t+\lambda)}(y', \gamma, r)(\lambda r)^{2h(\gamma + \alpha + y_0)} r^{-d-1} d\gamma dt dr dp. \]
Two changes of variables allow to write $\Gamma^H_\lambda(y_0, y, y')$ as
\[ \lambda^{-2H+1} \int_{\mathbb{R} \times (0, \lambda^{-1}) \times \mathbb{R}^d} \psi(y, \xi, r) \rho(t) G_{\rho(t+\lambda)}(y', \xi, r) (\lambda r)^{2h(\lambda \xi + \alpha + y_0)} r^{-d-1} d\xi dt dr. \]

a. The smooth case: we assume that $h$ is $\beta$-Lipschitz. By the same arguments as in the proof of (24), using the fact that $G_{\rho} \leq 2\|\rho\|_{\infty} |G|$, for $\lambda$ small enough compared to $y$ and $y'$,
\[ \int_{\mathbb{R}^d \times (0, \lambda^{-1})} \psi(y, \xi, r) G_{\rho(t+\lambda)}(y', \xi, r) (\lambda r)^{2h(\lambda \xi + \alpha + y_0) - 2h(\alpha + y_0)} - 1 \]
(26) \[\times r^{-d-1+2h(t\alpha+y_0)} \, d\xi \, dr \leq C \lambda^\beta |\ln(\lambda)|,\]

with \(C > 0\) and \(\beta < 1 - 2h(t\alpha + y_0)\). Thus we write \((\lambda r)^{2h(\lambda^\xi + t\alpha + y_0)}\) as

\[(\lambda r)^{2h(\lambda^\xi + t\alpha + y_0) - 2h(t\alpha+y_0)} \times (\lambda r)^{-2m(\alpha,y_0)} \times (\lambda r)^{2m(\alpha,y_0)},\]

where \(m(\alpha, y_0)\) is given by (18). For almost every \(t \in \mathbb{R}\), the second factor tends to \(1h(\lambda^\xi + t\alpha) = m(\alpha,y_0)\). Since \(m(\alpha,y_0) < 1/2\), let us recall that by Lemma 2.4 the function \(G(y,.)\) belongs to \(L^2(<\alpha> \times \mathbb{R}^+, r^{-d-1+2m(\alpha,y_0)} \, d\gamma \, dr)\).

We use Lebesgue’s Theorem and (26), to obtain the following asymptotics:

\[
\Gamma^{m(\alpha,y_0)+1/2}(y_0, y, y') \quad \xrightarrow{\lambda} \quad \Gamma^{m(\alpha,y_0)+1/2}(y, y'),
\]

where

\[
(27) \quad \Gamma^H(y, y') := \int_{<\alpha> \times \mathbb{R}^+} G(y, \gamma, r)G(y', \gamma, r) r^{-d-2+H} \, d\gamma \, dr.
\]

The identification of \(\Gamma^H\) as the covariance of a fractional Brownian motion defined on \(<\alpha>\) with Hurst index \(H = m(\alpha, y_0) + 1/2\) is straightforward following the same arguments as in the part a. of 5.1.

If \(\{t \in \mathbb{R}; h(y_0 + t\alpha) = m(\alpha,y_0)\}\) has positive measure, then we have finished with the proof. Otherwise, we proceed in the same way as in part b-ii) of 5.1. The cov-latt property of the second part of Theorem 3.4 is established.

b. The singular case: we assume that \(h\) is \(\beta\)-Lipschitz on \(S^{d-1}\) with \(0 < \beta \leq 1\) and look at the cov-latt property around \(0\). By homogeneity, the \(\beta\)-Lipschitz condition is replaced by

\[|h(t\alpha + x) - h(\alpha)| \leq C t^{-\beta} |x|^{\beta}.\]

Hence the upper bound of (26) is now given by

\[C \left(|t|^{-(1-2m)} + |t|^{-(1-2h(\alpha))}\right) \lambda^\beta |\ln(\lambda)|.\]

Lebesgue’s theorem still applies to get the result enounced in the second part of Theorem 3.6.

6. FDD-LATT PROPERTIES

We now prove the finite dimensional distribution lattice properties.
6.1. Fdd-lass properties of the microball model. Let us denote \( \tilde{X} = X - \mathbb{E}(X) \) the centered version of \( X \). For notational sake of simplicity, we will only consider the limit in distribution of \( \lambda^{-H} \Delta_{x_0} \tilde{X}(\lambda x) \) for a fixed \( x \) in \( \mathbb{R}^d \) instead of a random vector \( (\lambda^{-H} \Delta_{x_0} \tilde{X}(\lambda x_j))_{1 \leq j \leq n} \). The general case follows along the same lines.

For \( H > 0, \; x \in \mathbb{R}^d \) and \( t \in \mathbb{R} \), let
\[
\mathbb{E} \exp \left( \frac{it \Delta_{x_0} \tilde{X}(\lambda x)}{\lambda^H} \right) = \exp \Phi(H, \lambda, x_0, x, t)
\]
where \( \Phi(H, \lambda, x_0, x, t) \) is given by
\[
\int_{\mathbb{R}^d \times \mathbb{R}^+} \left( e^{it\lambda^{-H} \psi(x, \xi - x_0, r)} - 1 - it\lambda^{-H} \psi(x, \xi - x_0, r) \right) d\nu_h(\xi, r) .
\]

A change of variable yields
\[
\Phi(H, \lambda, x_0, x, t) = \int_{\mathbb{R}^d \times (0, \lambda^{-1})} \left( e^{it\lambda^{-H} \psi(x, \xi, r)} - 1 - it\lambda^{-H} \psi(x, \xi, r) \right) \lambda^{2h(x_0 + \lambda \xi)} r^{-d-1+2h(x_0 + \lambda \xi)} d\xi dr .
\]

Lemma 2.2 allows us to split the integral into \( \Phi = \Phi_1 + (\Phi - \Phi_1) \), where \( \Phi_1(H, \lambda, x_0, x, t) \) is equal to
\[
\int_{\mathbb{R}^d \times (0, \lambda^{-1})} \left( e^{it\lambda^{-H} \psi(x, \xi, r)} - 1 \right) \lambda^{2h(x_0 + \lambda \xi)} r^{-d-1+2h(x_0 + \lambda \xi)} d\xi dr .
\]

Then,
\[
|\Phi_1(H, \lambda, x_0, x, t)| \leq \lambda^{2m} \int_{\mathbb{R}^d \times \mathbb{R}^+} |e^{it\lambda^{-H} \psi(x, \xi, r)} - 1| r^{-d-1+2m} d\xi dr .
\]

We notice that
\[
|e^{it\lambda^{-H} \psi(x, \cdot)} - 1| \leq 2 \mathbb{1}_{t \psi(x, \cdot) \neq 0} \leq 2 |\psi(x, \cdot)|
\]
and recall that \( \psi(x, \cdot) \) belongs to \( L^1(\mathbb{R}^d \times \mathbb{R}^+, r^{-d-1+2m} d\xi dr) \) by Lemma 2.2, so that
\[
\lim_{\lambda \to 0^+} \Phi_1(H, \lambda, x_0, x, t) = 0 .
\]

The second term \( \Phi_2 := \Phi - \Phi_1 \) is given by
\[
(28) \quad -it \int_{\mathbb{R}^d \times (0, \lambda^{-1})} \lambda^{-H+2h(x_0 + \lambda \xi)} \psi(x, \xi, r) r^{-d-1+2h(x_0 + \lambda \xi)} d\xi dr .
\]

We will now analyse on this expression.

First, let us note that in the case where \( h \) is constant, then the increments of \( X \) are equal to the increments of \( \tilde{X} \) by stationarity and hence \( \Phi_2 = 0 \). That concludes the proof for the fractional microball model, i.e. the first part of Theorem 3.1.

Now we deal with the multifractional model (case \( h \) non constant).

a. The smooth case: We assume that (20) holds and we prove that the
critical fdd-lass index of $X$ at $x_0$ is $\geq \min(1, \beta + 2h(x_0))$. Without loss of generality, we can assume that $\beta + 2h(x_0) \leq 1$. For $H < \beta + 2h(x_0)$, we will establish that $\Phi_2(H, \lambda, x_0, x, t)$ tends to 0 when $\lambda \to 0^+$. Recall that by equation (28) $\Phi_2(H, \lambda, x_0, x, t)$ is equal to

$$-it \lambda^{-H+2h(x_0)} \int_{\mathbb{R}^d \times (0, \lambda^{-1})} (\lambda r)^{2(h(x_0+\lambda x)-h(x_0))} \psi(x, \xi, r) r^{-d-1+2h(x_0)} d\xi dr.$$  

Since $\int_{\mathbb{R}^d \times (0, \lambda^{-1})} \psi(x, \xi, r) r^{-d-1+2h(x_0)} d\xi dr$ vanishes, we get

$$\Phi_2(H, \lambda, x_0, x, t) = -it \lambda^{-H+2h(x_0)}$$

\[\times \int_{\mathbb{R}^d \times (0, \lambda^{-1})} (\lambda r)^{2(h(x_0+\lambda x)-h(x_0))} - 1) \psi(x, \xi, r) r^{-d-1+2h(x_0)} d\xi dr.\]

By the $\beta$-Lipschitz assumption on $h$, for $\lambda$ small enough compared to $x$, using (24) and since $|\psi| = \psi^2$, we get

$$\int_{\mathbb{R}^d \times (0, \lambda^{-1})} (\lambda r)^{2(h(x_0+\lambda x)-2h(x_0))} - 1) |\psi(x, \xi, r)| r^{-d-1+2h(x_0)} d\xi dr \leq C(x)\lambda^\beta.$$  

This implies the lower bound for the fdd-lass index.

b. The singular case: We now assume that $h$ is $\beta$-Lipschitz on the sphere and only deal with the case $x_0 = 0$.

b-i) Assume that $\{\xi; h(\xi) = m\}$ has positive measure. For $H = 2m$, using Lemma 2.2 again, we get

(29)

$$\lim_{\lambda \to 0^+} \Phi_2(2m, \lambda, 0, x, t) = -it \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{h(\xi) = m}\psi(x, \xi, r)r^{-d-1+2m} d\xi dr .$$

Hence the finite dimensional distributions of $\lambda^{-2m}\Delta_0 X(\lambda)$ converge to the finite dimensional distributions of the deterministic field $Z^m$ given by

$$Z^m(x) = -\int_{\mathbb{R}^d \times \mathbb{R}^+} 1_{h(\xi) = m}\psi(x, \xi, r)r^{-d-1+2m} d\xi dr , \ x \in \mathbb{R}^d.$$  

It remains to show that $Z^m$ is not zero. This follows from the next lemma, since $\{\xi; h(\xi) \neq m\}$ contains a ball by continuity of $h$.

Lemma 6.1. Let $m \in (0, 1/2)$. For all Borel sets $E \subset \mathbb{R}^d$ with positive measure, if

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} 1_E(\xi)\psi(., \xi, r)r^{-d-1+2m} d\xi dr = 0$$  

in $\mathbb{R}^d$, then the set $E^c$ does not contain any open ball.

Proof. Note that for all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^+} \psi(x, \xi, r)r^{-d-1+2m} d\xi dr = (2m - d)^{-1} (|x - \xi|^{-d+2m} - |\xi|^{-d+2m})$$

This completes the proof of Lemma 6.1.
and hence, for all Borel sets \( E \subset \mathbb{R}^d \),
\[
I^m_E(x) : = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_E(\xi) \psi(x, \xi, r)r^{-d-1+2m}d\xi dr
\]
\[
= \int_{\mathbb{R}^d} 1_E(\xi)(2m-d)^{-1} \left( |x - \xi|^{-d+2m} - |\xi|^{-d+2m} \right) d\xi .
\]

Let us suppose that we can find an open nonempty ball \( B \subset E^c \). Then \( I^m_E \) is smooth on \( B \). The Laplacian of \( I^m_E \) can easily be computed
\[
\Delta I^m_E(x) = -2(1-m) \int_{\mathbb{R}^d} 1_E(\xi)|x - \xi|^{-d+2m-2}d\xi ,
\]
and proved to be negative on \( B \). Thus, \( I^m_E \) does not vanish on \( B \). This completes the proof of Lemma 6.1 \( \square \)

b-ii) Assume that \( \{\xi; h(\xi) = m\} \) is of measure 0. we will establish that the fdd-llass index of \( X \) at 0 is still equal to \( 2m \). For \( H \leq 2m \), from (29), \( \Phi_2(H, \lambda, 0, x, t) \) tends to 0. On the other hand, for \( H \in (2m, 1) \) and \( H < 2 \max h \), we have to prove that there exists at least one \( x \in \mathbb{R}^d \) such that \( \Phi_2(H, \lambda, 0, x, t) \) does not tend to 0. First, let us remark that, for all \( x \in \mathbb{R}^d \), \( \Phi_2(H, \lambda, 0, x, t) \) may be written as
\[
- \tilde{\Phi}_2(H, \lambda, x) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda^{-H+2h(\xi)} \psi(x, \xi, r) r^{-d-1+2h(\xi)} d\xi dr + O(\lambda^{-H}) .
\]

Thus it is sufficient to consider
\[
\tilde{\Phi}_2(H, \lambda, x) : = \int_{\mathbb{R}^d \times \mathbb{R}^+} \lambda^{-H+2h(\xi)} \psi(x, \xi, r) r^{-d-1+2h(\xi)} d\xi dr
\]
\[
= \int_{\mathbb{R}^d} (2h(\xi) - d)^{-1} \lambda^{-H+2h(\xi)} f(x, \xi) d\xi ,
\]
with
\[
f(x, \xi) = |x - \xi|^{-d+2h(\xi)} - |\xi|^{-d+2h(\xi)} .
\]

The function \( f(\cdot, \xi) \) is smooth on \( \mathbb{R}^d \setminus \{\xi\} \) and its Laplacian given by
\[
\Delta f(x, \xi) = (2h(\xi) - d)(h(\xi) - 1)|x - \xi|^{-d+2h(\xi)} - 2 ,
\]
is negative. Then, approximating the Laplacian by the second order increments, one can find \( C \in (0, \frac{1}{2}) \) such that, whenever \( |x - \xi| \neq 0 \) and \( \delta \leq C|x - \xi| \),
\[
(30) \sum_{1 \leq j \leq d} f(x + \delta e_j, \xi) + f(x - \delta e_j, \xi) - 2f(x, \xi) \leq \frac{\delta^2}{2} \Delta f(x, \xi).
\]

Let us take \( H \) such that \( 2m < H < 2 \max h \) and \( H < 1 \) and note that the sets \( \{\xi; H \leq 2h(\xi)\} \) and \( \{\xi; H > 2h(\xi)\} \) have positive measure. Then, by continuity of \( h \), there exists a nonempty open ball \( B \subset \{\xi; H > 2h(\xi)\} \). For every \( x \in \mathbb{R}^d \) we introduce
\[
\tilde{\Phi}_\lambda(x) := \int_{\mathbb{R}^d} 1_{2h(\xi) \leq H} (2h(\xi) - d)^{-1} \lambda^{-H+2h(\xi)} f(x, \xi) d\xi ,
\]
such that
$$\tilde{\Phi}_2(H, \lambda, x) = \tilde{\Phi}_\lambda(x) + \left( \tilde{\Phi}_2(H, \lambda, x) - \tilde{\Phi}_\lambda(x) \right).$$

By Lebesgue’s Theorem, the second term tends to 0 with \( \lambda \). Suppose that \( \tilde{\Phi}_\lambda(x) \) tends to 0 with \( \lambda \) for every \( x \) in \( \mathbb{R}^d \). Then for all \( x \in \mathbb{R}^d \) and all \( \delta \in \mathbb{R} \),

$$\Delta^{(2)}_{\delta} \tilde{\Phi}_\lambda(x) := \sum_{1 \leq j \leq d} \left( \tilde{\Phi}_\lambda(x + \delta e_j) + \tilde{\Phi}_\lambda(x - \delta e_j) - 2\tilde{\Phi}_\lambda(x) \right) \to 0.$$  \( \text{(31)} \)

For \( x \in B \), and \( \delta > 0 \) such that \( B \left( x, \frac{\delta}{\tau} \right) \subset B \), according to (\text{30}),

$$\Delta^{(2)}_{\delta} \tilde{\Phi}_\lambda(x) \leq \frac{\delta^2}{2} \int_{\mathbb{R}^d} (h(\xi) - 1) \mathbf{1}_{2h(\xi) \leq H} |x - \xi|^{-d+2h(\xi)-2} \, d\xi \leq 0.$$

Then (\text{31}) implies that \( \{ \xi; 2h(\xi) \leq H \} \) has measure zero, which contradicts the assumption \( H > 2m \).

The proof of the first part of Theorem 3.4 is now complete.

6.2. Fdd-llass properties for the X-ray transforms. Finally we consider the fdd-llass property at point \( y_0 \) for the X-ray transform. As previously, we restrict the computation to the one-dimensional distribution and denote \( \overline{P}_\alpha X = P_\alpha X - E(P_\alpha X) \) the centered version of \( P_\alpha X \). For any \( y \in \langle \alpha \rangle^+ \), \( t \in \mathbb{R} \) and \( H \in (0, 1) \), we write

$$E \exp \left( it\lambda^{-H} \left( \overline{P}_\alpha X(y_0 + \lambda y) - \overline{P}_\alpha X(y_0) \right) \right) = \exp \Phi(H, \lambda, y_0, y, t),$$

where \( \Phi(H, \lambda, y_0, y, t) \) is given by

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} \left( e^{it\lambda^{-H} G_{\rho(\lambda y, \xi - y_0, r)} - 1 - it\lambda^{-H} G_{\rho(\lambda y, \xi - y_0, r)}} \right) \, d\nu_h(\xi, r).$$

By the same change of variable as in the covariance part of the proof, \( \Phi(H, \lambda, y_0, y, t) \) is equal to

$$\int_{\langle \alpha \rangle^+ \times \mathbb{R} \times (0, \lambda^{-1})} \lambda^{-1} \left( e^{it\lambda^{-H} G_{\rho(\lambda y, \gamma, r)} - 1 - it\lambda^{-H} G_{\rho(\lambda y, \gamma, r)}} \right) \times (\lambda r)^{2h(\lambda y + \rho(\gamma + y_0), r)} p^{-d-1} \, d\gamma dp r.$$

Let us remark that

$$\lambda^{-1} G_{\rho(\lambda y, \gamma, r)} \xrightarrow{\lambda \to 0^+} 0$$

and so

$$\lambda^{-1} \left( e^{it\lambda^{-H} G_{\rho(\lambda y, \gamma, r)}} - 1 - it\lambda^{-H} G_{\rho(\lambda y, \gamma, r)} \right) \xrightarrow{\lambda \to 0^+} -\frac{1}{2} i^2 \lambda^{-2H+1} G^2_{\rho^2},$$

where \( f(\lambda) \xrightarrow{\lambda \to 0^+} g(\lambda) \) if \( \frac{f(\lambda)}{g(\lambda)} \xrightarrow{\lambda \to 0^+} 1 \). Consequently we can argue along the same lines as in the covariance part of the proof to get, for \( H = m(\alpha, y_0) + \)}
\( \Phi(H, \lambda, y_0, y, t) \xrightarrow{\lambda \to 0^+} -\frac{1}{2} t^2 \left( \int_{\mathbb{R}} 1_{h(y_0+pa)=m(a,y_0)} \rho(p)^2 dp \right) \Gamma^{m(a,y_0)+1/2}(y, y), \)

which concludes the proof.

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MAP5-UMR 8145, Université René Descartes, 45, rue des Saints-Pères F 75270 PARIS cedex 06 FRANCE, hermine.bierme@univ-orleans.fr

MAP5-UMR 8145, Université René Descartes, 45, rue des Saints-Pères F 75270 PARIS cedex 06 FRANCE, anne.estrade@univ-paris5.fr