The framework of the enclosure method with dynamical data and its applications

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Abstract
The aim of this paper is to establish the framework of the enclosure method for some class of inverse problems whose governing equations are given by parabolic equations with discontinuous coefficients. The framework is given by considering a concrete inverse initial boundary value problem for a parabolic equation with discontinuous coefficients. The problem is to extract information about the location and shape of unknown inclusions embedded in a known isotropic heat conductive body from a set of the input heat flux across the boundary of the body and output temperature on the same boundary. In this framework, the original inverse problem is reduced to an inverse problem whose governing equation has a large parameter. A list of requirements which enables one to apply the enclosure method to the reduced inverse problem is given. Two new results which can be considered as the applications of the framework are given. In the first result, the background conductive body is assumed to be homogeneous and a family of explicit complex exponential solutions are employed. Second, an application of the framework to inclusions in an isotropic inhomogeneous heat conductive body is given. The main problem is the construction of the special solution of the governing equation with a large parameter for the background inhomogeneous body required by the framework. It is shown that, introducing another parameter which is called the virtual slowness and making it sufficiently large, one can construct the required solution which yields an extraction formula of the convex hull of unknown inclusions in a known isotropic inhomogeneous conductive body.

1. Introduction
The aim of this paper is to establish the framework of the enclosure method [9] for possible application to some class of inverse problems whose governing equations are given by parabolic equations with discontinuous coefficients.
The framework is given by considering a concrete inverse initial boundary value problem for a parabolic equation with discontinuous coefficients. The problem is to extract information about the location and shape of unknown inclusions embedded in a known isotropic heat conductive body from a set of the input heat flux across the boundary of the body and output temperature on the same boundary.

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n, n = 2, 3 \), with a smooth boundary. We denote the unit outward normal vectors to \( \partial \Omega \) by the symbol \( \nu \). Let \( T \) be an arbitrary fixed positive number. We put \( W(0, T ; H^1(\Omega), (H^1(\Omega))^\prime) = \{ u \in L^2(0, T; H^1(\Omega)) \mid u' \in L^2(0, T; (H^1(\Omega))^\prime) \} \).

Given \( f \in L^2(0, T; H^{-1/2}(\partial \Omega)) \), let \( u = u_f \in W(0, T; H^1(\Omega), (H^1(\Omega))^\prime) \) be the weak solution of the initial boundary value problem for the parabolic equation:

\[
\begin{align*}
\partial_t u - \nabla \cdot \gamma \nabla u &= 0 \quad \text{in} \quad \Omega \times [0, T], \\
\gamma \nabla u \cdot \nu &= f \quad \text{on} \quad \partial \Omega \times [0, T], \\
u u(x, 0) &= 0 \quad \text{in} \quad \Omega,
\end{align*}
\]

where \( \gamma = \gamma(x) = (\gamma_{ij}(x)) \) satisfies the following conditions.

1. For each \( i, j = 1, \ldots, n \), \( \gamma_{ij}(x) \) is real, belongs to \( L^\infty(\Omega) \) and satisfies \( \gamma_{ij}(x) = \gamma_{ji}(x) \);
2. There exists a positive constant \( C \) such that \( \gamma(x) \xi \cdot \xi \geq C|\xi|^2 \) for all \( \xi \in \mathbb{R}^n \) and a.e. \( x \in \Omega \).

See [3] for the notion of the weak solution and subsection 2.1 in [14] for the detailed description based on [3]. This paper is concerned with the extraction of information about ‘discontinuity’ of \( \gamma \) from \( u \) and \( \nabla u \cdot \nu \) on \( \partial \Omega \times [0, T] \) for some \( f \) and an arbitrary fixed \( T < \infty \). However, we do not consider completely general \( \gamma \). Instead we assume that there exists an open set \( D \) with a smooth boundary such that \( \overline{D} \subset \Omega \) and \( \gamma(x) \) a.e. \( x \in \Omega \setminus D \) coincides with the \( n \times n \) identity matrix \( I_n \) multiplied by a smooth positive function \( \gamma_0(x) \) of \( x \in \overline{\Omega} \) and satisfies one of the following two conditions:

A1) There exists a positive constant \( C' \) such that \(- (\gamma(x) - \gamma_0(x) I_n) \xi \cdot \xi \geq C'|\xi|^2 \) for all \( \xi \in \mathbb{R}^n \) and a.e. \( x \in D \);
A2) There exists a positive constant \( C' \) such that \((\gamma(x) - \gamma_0(x) I_n) \xi \cdot \xi \geq C'|\xi|^2 \) for all \( \xi \in \mathbb{R}^n \) and a.e. \( x \in D \).

Write \( h(x) = \gamma(x) - \gamma_0(x) I_n \) a.e. \( x \in D \). Roughly speaking, condition (A1)/(A2) means that \( D \) has a lower/higher conductivity from known reference conductivity \( \gamma_0 I_n \).

We consider

Inverse Problem 1.1. Fix a \( T > 0 \). Assume that both \( D \) and \( h \) are unknown and that \( \gamma_0 \) is known. Extract information about the location and shape of \( D \) from a set of the pair of temperature \( u_f(x, t) \) and heat flux \( f(x, t) \) for \( (x, t) \in \partial \Omega \times [0, T] \).

The \( D \) is a model of the union of unknown inclusions where the heat conductivity is anisotropic, different from that of the surrounding inhomogeneous isotropic conductive medium. The problem is a mathematical formulation of a typical inverse problem in thermal imaging. Note that in [4] a uniqueness theorem with infinitely many \( f \) for inverse problem 1.1 has been established provided \( h \) has the form \( b I_n \) with a smooth function \( b \) on \( \overline{D} \). Thus, the point is to give a concrete procedure or formula which yields information about the location and shape of \( D \). Note that when \( n = 1 \), there are some results: the procedure in [1, 2] with infinitely many \( f \) and the formula of theorem 2.1 in [10] with a single \( f \).

In [14] we considered inverse problem 1.1 in the case when \( \gamma_0 \equiv 1 \) and \( n = 3 \) and gave four extraction formulae of some information including the convex hull of \( D \). In subsection 1.3, we will reconsider three results of those which employ infinitely many \( f \), from the view point of the framework given here.
Note that in this paper we do not consider the single-input case. Formulae with a single $f$ see theorem 1.1 in [14] and [15] for a cavity which is the extremal case $\gamma \equiv 0$ in $D$. Those can be considered as some kind of extension of the enclosure method for elliptic equations with a single Cauchy data started in [8] to the parabolic equations.

1.1. A reduction to an inverse boundary value problem with a parameter-A general framework

Define

$$
wf(x, \tau) = \int_0^T e^{-\tau t} u_f(x, t) \, dt, \quad x \in \Omega, \quad \tau > 0.
$$

$w = w_f$ satisfies

$$
\begin{align*}
(\nabla \cdot \gamma \nabla - \tau)w &= e^{-\tau T} u_f(x, T) \quad \text{in } \Omega, \\
\gamma \nabla w \cdot \nu &= \int_0^T e^{-\tau t} f(x, t) \, dt \quad \text{for } x \in \partial \Omega.
\end{align*}
$$

(1.2)

This motivates a formulation of the reduced problem given below.

Given $F(\cdot, \tau) \in L^2(\Omega)$ and $g(\cdot, \tau) \in H^{-1/2}(\partial \Omega)$ with $\tau > 0$, let $w = w(\cdot, \tau) \in H^1(\Omega)$ be the weak solution of

$$
\begin{align*}
(\nabla \cdot \gamma \nabla - \tau)w &= e^{-\tau T} F(x, \tau) \quad \text{in } \Omega, \\
\gamma \nabla w \cdot \nu &= g(x, \tau) \quad \text{on } \partial \Omega.
\end{align*}
$$

(1.3)

Inverse Problem 1.2. Assume that $F(x, \tau), D$ and $h$ are all unknown and that $\gamma_0$ is known. Extract information about the location and shape of $D$ from a set of the pair of $w(x, \tau)$ and $g(x, \tau)$ for $x \in \partial \Omega, \tau > 0$.

For this problem we propose the following general framework which reduces the problem to construct a family of special solutions of an equation coming from the background body.

Theorem 1.1. Assume that there exist constants $C_1$ and $\mu_1$ such that, as $\tau \to \infty$

$$
\|F(\cdot, \tau)\|_{L^2(\Omega)} = O(e^{C_1 \tau^{\mu_1}}),
$$

(1.4)

we have a family $(v_\tau)$ indexed with $\tau \geq \tau_0 > 0$ of $H^1(\Omega)$ solutions of the equation

$$
\begin{align*}
\nabla \cdot \gamma_0 \nabla v - \tau v &= 0 \quad \text{in } \Omega \\
\gamma_0 \nabla v \cdot \nu &= g(x, \tau) \quad \text{on } \partial \Omega.
\end{align*}
$$

(1.5)

satisfying the conditions, for some constants $\mu_2, \mu_3, \mu_4, C_2, C_3$ and positive constant $C'$

$$
\begin{align*}
\|\nabla v_\tau\|_{L^2(\Omega)} &= O(e^{C_2 \tau^{\mu_2}}), \\
\|\nabla v_\tau\|_{L^2(\partial \Omega)} &\geq C'' e^{C_2 \tau^{\mu_3}}, \\
\|v_\tau\|_{H^1(\Omega)} &= O(e^{C_1 \tau^{\mu_4}}).
\end{align*}
$$

(1.6) (1.7) (1.8)

Let $g = g(x, \tau)$ be a function of $x \in \partial \Omega$ having the form

$$
g = \Psi(\tau) \gamma_0 \frac{\partial v_\tau}{\partial \nu} |_{\partial \Omega},
$$

where $\Psi$ satisfies the conditions, for constants $\mu$ and $\mu'$,

$$
\liminf_{\tau \to \infty} \tau^{\mu} |\Psi(\tau)| > 0
$$

(1.9)

and

$$
|\Psi(\tau)| = O(\tau^{\mu'}),
$$

(1.10)
Let \( w \) be the solution of (1.3). If \( T \) satisfies
\[
T > C_1 + C_2 - 2C_2, \tag{1.11}
\]
then
\[
\lim_{\tau \to \infty} \frac{1}{2\tau} \log \left| \int_{\partial \Omega} \left( g \nabla \tau \ - \ w \nabla_0 \frac{\partial \nu}{\partial v} \right) \ dS \right| = C_2. \tag{1.12}
\]

**Remark 1.1.** It follows from (1.6) and (1.7) that
\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log \| \nabla v_\tau \|_{L^1(D)} = C_2.
\]
This is the meaning of \( C_2 \) which is uniquely determined by \( \| \nabla v_\tau \|_{L^1(D)} \) with all \( \tau \gg \tau_0 \).

**Remark 1.2.** It follows from (1.7) and (1.8) that \( C_3 \geq C_2 \).

**Remark 1.3.** From the proof one also obtains the order of the convergence of the formula (1.12):
\[
\frac{1}{2\tau} \log \left| \int_{\partial \Omega} \left( g \nabla \tau \ - \ w \nabla_0 \frac{\partial \nu}{\partial v} \right) \ dS \right| = C_2 + O \left( \frac{\log \tau}{\tau} \right).
\]
This is important for a suitable choice of \( \tau \) in the case when the data are noisy.

Here, we present an application of theorem 1.1 to the case when \( \gamma_0 \equiv 1 \). Let \( c > 0 \) and \( \tau \geq \tau_0 = c^{-2} \). Let \( \omega, \omega^\perp \in S^{n-1}, n \geq 2 \) and satisfy \( \omega \cdot \omega^\perp = 0 \). Set
\[
z = c \tau \left( \omega + i \sqrt{1 - \frac{1}{c^2} \omega \cdot \omega^\perp} \right). \tag{1.13}
\]
z satisfies
\[
z \cdot z = \tau. \tag{1.14}
\]
We observe that
\[
(\Delta - \tau) e^{z} = 0. \tag{1.15}
\]
Thus, we take the family \( (v_\tau)_{\tau \geq \tau_0} \) in theorem 1.1
\[
v_\tau(x) = v(x; z) = e^{z}. \tag{1.16}
\]

Note that

- it follows from (1.15) that the function \( e^{-\tau t}v(x; z) \) of \( (x, t) \) satisfies the *backward heat equation*
\[
(\Delta + \partial_t)(e^{-\tau t}v(x; z)) = 0,
\]
- the absolute value of \( e^{-\tau t}v(x; z) \) coincides with \( e^{-\tau(t-ct \cdot \omega)} \) and this is a solution of the *wave equation* with the propagation speed \( 1/c \). By this reason we call this \( c \) the *virtual slowness*. See also [10] for this interpretation.

Recall the support functions of \( D \) and \( \Omega \):
\[
h_D(\omega) = \sup_{x \in D} x \cdot \omega, \quad h_\Omega(\omega) = \sup_{x \in \Omega} x \cdot \omega.
\]
We have \( h_D(\omega) < h_\Omega(\omega) \) for all \( \omega \) since \( \bar{D} \subset \Omega \).

Applying theorem 1.1 to (1.2), we obtain the following corollary.
Corollary 1.1. Assume that $\gamma_0 \equiv 1$. Let $f$ be the function of $(x, t) \in \partial \Omega \times \mathbb{R}$, also depending on a parameter $\tau > 0$, defined by the equation

$$f(x, t) = \frac{\partial v_\tau}{\partial \nu}(x)\phi(t),$$

(1.16)

where a real-valued function $\phi \in L^2(0, T)$ satisfying the condition that there exists $\mu \in \mathbb{R}$ such that

$$\liminf_{\tau \to \infty} \tau^\mu \left| \int_0^T e^{-\tau t} \phi(t) \, dt \right| > 0.$$  

(1.17)

Let $u_f = u_f(x, t)$ be the weak solution of (1.1) for $f = f(x, t)$. If $T$ satisfies

$$T > 2c(h_{\Omega}(\omega) - h_D(\omega)),$$

(1.18)

then

$$\lim_{\tau \to \infty} \frac{1}{2\tau} \log \left| \int_{\partial \Omega} \int_0^T e^{-\tau t} \left( -v_\tau(x)f(x, t; \tau) + u_f(x, t)\frac{\partial v_\tau}{\partial \nu}(x) \right) \, dt \, dS \right| = c h_D(\omega).$$

(1.19)

Remark 1.4. There is no restriction on the position of centre of coordinates relative to $\Omega$ and $D$ which affects the sign of $h_D(\omega)$.

Since $-h_{\Omega}(-\omega) = \inf_{x \in \Omega} x \cdot \omega < \inf_{x \in D} x \cdot \omega \leq h_D(\omega)$, we have

$$h_{\Omega}(\omega) - h_D(\omega) < h_D(\omega).$$

Thus, $T$ in corollary 1.1 can be arbitrary small by choosing a small known $c$ in such a way that, for example,

$$c \leq \frac{T}{2(h_{\Omega}(\omega) + h_D(-\omega))}$$

since from this one obtains (1.18). Therefore, if one wants to estimate $D$ from the direction $\omega$ by the formula (1.19) and the ‘size’ of $\Omega$ at the $\omega$ direction, that is the quantity $h_{\Omega}(\omega) + h_D(-\omega)$, is too large compared with $T$ (this is the most difficult case), the virtual slowness $c$ in (1.13) should be chosen very small. This is one of the two roles of virtual slowness. In the next subsection we give another role of virtual slowness.

Let us explain how to deduce (1.19) from theorem 1.1. Comparing (1.2) with (1.3), one knows that the $g$ and $F$ in theorem 1.1 have the form

$$g(x, \tau) = \int_0^T e^{-\tau t} \phi(t) \, dt \frac{\partial v_\tau}{\partial \nu}(x), \quad x \in \partial \Omega, \quad \tau > 0,$$

(1.20)

and

$$F(x, \tau) = u_f(x, T).$$

(1.21)

Thus, (1.9) is satisfied with the same $\mu$ as (1.17); (1.10) is satisfied with $\mu' = -1$. We have to also know an estimation of $C_1$ in (1.4) from above. From [3] it follows that

$$\|u(., T)\|_{L^2(\Omega)} = O(\|f\|_{L^2(0, T; H^{-1/2}(\partial \Omega))})$$

(1.22)

and thus one can choose

$$C_1 = ch_{\Omega}(\omega).$$

We also have

$$C_2 = ch_D(\omega) < ch_{\Omega}(\omega) = C_3.$$  

Since $C_1 + C_3 - 2C_2 = 2c(h_{\Omega}(\omega) - h_D(\omega))$, (1.11) becomes (1.18).
Remark 1.5. Note that the choice \((1.13)\) of \(z\) satisfying \((1.14)\) goes back to [11] in which an application of the enclosure method for inverse source problems for the heat equations are given. The point of the choice is that the growing orders of \(|z|\) and \(z \cdot z\) as \(\tau \to \infty\) are the same. See also [12] for an application to the so-called inverse heat conduction problem. In the one-space-dimensional case, in [10] instead of \((1.13)\) the \(z\) having the form

\[
z = c\tau \left(1 + i\sqrt{1 - \frac{1}{c^2\tau}}\right)
\]

has been used. In this case \(\text{Re} (z \cdot z) = \tau\). Formula \((1.19)\) can be considered as an extension of a result in the one-space-dimensional case [10]. It means that if one could always control the initial temperature in the process of all possible measurements at the boundary to be zero, then one can extract the convex hull of unknown inclusions.

Since \(f\) is complex-valued, \(u_\gamma\) on \(\partial\Omega \times ]0, T[\) cannot be directly measured and should be computed from real data via the formula

\[
u = u\text{Re} f + iu\text{Im} f.
\]

This is a consequence of the zero initial data. Thus, the zero initial data are essential for this procedure. Note also that since both \(\text{Re} f\) and \(\text{Im} f\) are highly oscillatory as \(\tau \to \infty\) with respect to the space variables, it will be difficult to prescribe those fluxes on the boundary directly. Instead, one has to make use of the principle of superposition to compute the right-hand side of \((1.23)\) on \(\partial\Omega\) from experimental data which are generated by finite numbers of independent simpler input fluxes on \(\partial\Omega\). Needless to say, for this procedure the zero initial data are also essential.

1.2. The case when \(\gamma_0\) is not necessarily a constant

It is possible to extend corollary 1.1 to the case when \(\gamma_0\) is not necessarily a constant. For simplicity of the description, assume that \(\gamma_0 - 1 \in C_0^\infty(\mathbb{R}^n)\). We construct a special solution of the equation of \((1.5)\) which has the form

\[
v(x) \sim \frac{e^{\epsilon z}}{\sqrt{\gamma_0}}
\]
as \(\tau \to \infty\), where \(z\) is given by \((1.13)\).

Following [20], we make use of the change of the dependent variable formula (the Liouville transform):

\[
\frac{1}{\sqrt{\gamma_0}} \nabla \cdot \gamma_0 \nabla \left(\frac{1}{\sqrt{\gamma_0}}\right) = \Delta - V,
\]

where

\[
V = \Delta \sqrt{\gamma_0} \sqrt{\gamma_0}.
\]

We find the special solution of \((1.5)\) having the form

\[
v = \frac{e^{\epsilon z}}{\sqrt{\gamma_0}} (1 + \epsilon z),
\]

where \(\epsilon\) is a new unknown function. It follows from \((1.24)\) and \((1.14)\) that the equation for \(\epsilon\) becomes

\[
\left\{\Delta + 2z \cdot \nabla - \tau \left(\frac{1}{\gamma_0} - 1\right) - V\right\} \epsilon = \tau \left(\frac{1}{\gamma_0} - 1\right) + V.
\]
Thus, the problem is to construct a solution of equation (1.26) such that \( \epsilon_\tau \approx 0 \) in \( \Omega \) as \( \tau \to \infty \). In general, this is not an easy task because of the growing factor \( \tau \) on the zeroth-order term in (1.26). However, we found that if the virtual slowness \( c \) in (1.13) is sufficiently large and fixed, then one can construct such a solution for all large \( \tau \gg 1 \) with an arbitrary small \( \epsilon_\tau \) by using a combination of the Fourier transform and perturbation methods in [20] for the construction of the so-called complex geometrical optics solutions of the equation \( \nabla \cdot \gamma \nabla v = 0 \). Its precise description is the following second result.

Let \( L^2_s = L^2_s(\mathbb{R}^n) \) with \( s \in \mathbb{R} \) denote the set of all tempered distributions \( f \) in \( \mathbb{R}^n \) such that \( (1 + |x|^2)^{s/2} f \in \mathbb{L}^2(\mathbb{R}^n) \) equipped with the norm \( \| f \|_{L^2_s(\mathbb{R}^n)} = \| (1 + |x|^2)^{s/2} f \|_{L^2(\mathbb{R}^n)} \).

**Theorem 1.2.** Let \( -1 < \delta < 0 \) and \( a, b \in C_0^\infty(\mathbb{R}^n) \). Given \( \eta > 0 \), there exist positive constants \( C_j = C_j(\Omega, \delta, \eta), j = 1, 2 \) such that if \( c \geq C_1 \) and \( \tau \geq C_2 \), then \( c^2 \tau > 1 \) and there exists a unique \( \epsilon_\tau \in L^2_s(\mathbb{R}^n) \) with \( \epsilon \) given by (1.13) such that

\[
(\Delta + 2z \cdot \nabla - \tau a - b)\epsilon_\tau = \tau a + b \quad \text{in} \quad \mathbb{R}^n.
\]

Moreover, \( \epsilon_\tau \mid_{\Omega} \) can be identified with a function in \( C^1(\overline{\Omega}) \) and

\[
\|\epsilon_\tau\|_{L^\infty(\Omega)} + \|\nabla \epsilon_\tau\|_{L^\infty(\Omega)} \leq \eta.
\]

This theorem indicates the important role of the virtual slowness \( c \) when \( \gamma_0 \) is not necessarily a constant. It is not an accessible! To the best knowledge of the author, this idea, that is, choosing a large \( c \) and fix, has never been pointed out.

Having this theorem, we obtain a result which corresponds to corollary 1.1.

Let \( 0 < \eta \ll 1 \) and fix a \( c \geq C_1 \) in theorem 1.2. Let \( a = (1/\gamma_0 - 1) \) and \( b \) be given by (1.25). Let \( \epsilon_\tau \) be the solution of (1.27) constructed in theorem 1.2. Define

\[
v_\tau(x) = \frac{e^{t\tau}}{\sqrt{\gamma_0(x)}}(1 + \epsilon_\tau(x)), \quad x \in \Omega, \quad \tau \geq C_2.
\]

The function \( e^{-t\tau}v_\tau(x) \) satisfies the backward parabolic equation

\[
(\nabla \cdot \gamma_0 \nabla + \partial_t)(e^{-t\tau}v_\tau(x)) = 0
\]

and its absolute value has the form

\[
|e^{-t\tau}v_\tau(x)| \approx e^{-t(f-cx)}.
\]

This again supports the name virtual slowness of \( c \).

It is easy to see that the family \( (v_\tau)_{\tau \geq C_2} \) satisfies (1.5), (1.6) and (1.7) with \( C_2 = ch_\Omega(\omega) \), (1.8) \( C_3 = ch_\Omega(\omega) \) and (1.4) \( C_1 = ch_{\Omega}(\omega) \). Thus, applying theorem 1.1 to this case, we obtain the following corollary.

**Corollary 1.2.** Assume that \( \gamma_0 - 1 \in C_0^\infty(\mathbb{R}^n) \). Fix the virtual slowness as \( c = C_1 \), where \( C_1 \) is just the same as theorem 1.2. Let \( f \) be the function of \((x, t) \in \partial \Omega \times [0, T] \), also depending on a parameter \( \tau \to 0 \), defined by the equation

\[
f(x, t) = \frac{\partial v}{\partial x}(x)\varphi(t),
\]

where \( v = v_\tau \) is given by (1.29) and a real-valued function \( \varphi \in L^2(0, T) \) satisfying condition (1.17) for a \( \mu \in \mathbb{R} \). Let \( u_\tau = u_\tau(x, t) \) be the weak solution of (1.1) for \( f = f(x, t; \tau) \). If \( T \) satisfies

\[
T > 2\epsilon(h_\Omega - h_D(\omega)),
\]

then

\[
\|\nabla \epsilon_\tau\|_{L^\infty(\Omega)} \leq \eta.
\]
then
\[
\lim_{\tau \to \infty} \frac{1}{2\tau} \log \left| \int_{\Omega} \int_{0}^{T} e^{-\sqrt{\tau} t} \left( -v_\tau(x) f(x, t; \tau) + u f(x, t) \frac{\partial \bar{v}_\tau}{\partial v}(x) \right) \, dt \, dS \right| = c h_D(\omega).
\]

In corollary 1.2, \( c = C_1 \) and thus (1.30) should be considered as a restriction on the length of the time for data collection. A sufficient condition to ensure (1.30) is
\[
T \geq 2C_1(h_\Omega(\omega) + h_D(-\omega)).
\]

Note also that the centre of coordinates in theorem 1.2 and corollary 1.2 is free from \( D \) and \( \Omega \).

1.3. Real versus complex

Replace conditions (1.4), (1.6), (1.7) and (1.8) with the following ones, respectively:
\[
\|F(\cdot, \tau)\|_{L^1(\Omega)} = O(e^{C_1 \sqrt{\tau} \mu_1}); \quad \text{(1.31)}
\]
\[
\|\nabla v_\tau\|_{L^1(\Omega)} = O(e^{C_2 \sqrt{\tau} \mu_2}); \quad \text{(1.32)}
\]
\[
\|\nabla v_\tau\|_{L^1(\Omega)} \geq e^{C_3} e^{C_2 \sqrt{\tau} \mu_3}; \quad \text{(1.33)}
\]
\[
\|v_\tau\|_{H^1(\Omega)} = O(e^{C_4 \sqrt{\tau} \mu_4}). \quad \text{(1.34)}
\]

Then, instead of (1.12) we have, for any fixed \( T > 0 \) without (1.11) and exactly same \( g \) satisfying (1.9) and (1.10) for some \( \mu, \mu' \in \mathbb{R} \),
\[
\lim_{\tau \to \infty} \frac{1}{2\sqrt{\tau}} \log \left| \int_{\Omega} \left( g(u_\tau - w) \frac{\partial \bar{v}_\tau}{\partial v} \right) \, dS \right| = C_2. \quad \text{(1.35)}
\]

Since the proof is simpler than that of theorem 1.1, we omit its description as a theorem.

Explicit examples of \( v \) satisfying (1.32), (1.33) and (1.34) in the case when \( \gamma_0 \equiv 1 \) and \( n = 3 \) are the following:
\[
v(x; \tau, \omega) = e^{\sqrt{\tau} x \cdot \omega}, \quad x \in \mathbb{R}^3, \quad \omega \in S^2;
\]
\[
v(x; \tau, p) = \frac{e^{-\sqrt{\tau} |x-p|}}{|x-p|}, \quad x \in \mathbb{R}^3 \setminus \{p\}, \quad p \in \mathbb{R}^3 \setminus \overline{\Omega};
\]
\[
v(x; \tau, y) = \frac{e^{\sqrt{\tau} x \cdot y} - e^{-\sqrt{\tau} x \cdot y}}{|x-y|}, \quad x \in \mathbb{R}^3 \setminus \{y\}, \quad v(y; \tau, y) = 2\tau, \quad y \in \mathbb{R}^3.
\]

These are all real-valued functions and not oscillatory as \( \tau \to \infty \). We think that this non-oscillatory character is an advantage in computing the left-hand side of (1.12).

If \( g \) and \( F \) are coming from (1.20) and (1.21) for \( f \) given by (1.16) for a \( \varphi \) satisfying (1.17) for \( \mu \in \mathbb{R} \), then one can choose
\[
C_2 = \begin{cases} 
    h_D(\omega), & \text{if } v = v(x; \tau, \omega), \\
    -d_D(p), & \text{if } v = v(x; \tau, p), \\
    R_D(\gamma), & \text{if } v = v(x; \tau, y),
\end{cases}
\]
where \( d_*(p) \) and \( R_*(y) \) denote the distance of \( p \) to \( * \) and minimum radius of the open ball that contains \( * \) and centred at \( y \). See lemma 3.1 in [14] and proposition 3.2 in [13] for these facts.

By virtue of (1.22) one can choose
\[
C_1 = \begin{cases} 
    h_\Omega(\omega), & \text{if } v = v(x; \tau, \omega), \\
    -d_\Omega(p), & \text{if } v = v(x; \tau, p), \\
    R_\Omega(\gamma), & \text{if } v = v(x; \tau, y).
\end{cases}
\]

Then, formula (1.35) reproduces theorems 1.2–1.4 in [14].
However, when $\gamma_0$ is not necessarily a constant, to construct a suitable $v$ one has to solve the eikonal equation

$$\gamma_0 \nabla \varphi \cdot \nabla \varphi = 1 \quad \text{in} \quad \Omega$$

and corresponding transport equations. However, this is not a simple matter in general because of the complicated behaviour of the characteristic curve $x = x(t)$ under suitable initial conditions on $(x(t), \xi(t))$:

$$\frac{dx}{dt} = 2\xi,$$

$$\frac{d\xi}{dt} = \frac{2}{\sqrt[4]{\gamma_0(x)}} \nabla \left( \frac{1}{\sqrt[4]{\gamma_0(x)}} \right).$$

See section 3 in [21] for an assumption for $\gamma_0$ in on the solvability of the eikonal equation globally in a neighbourhood of $\Omega$. Note that corollary 1.2 which is derived from theorems 1.1 and 1.2 does not require any other condition on $\gamma_0$, except for the regularity.

Thus, corollary 1.2 suggests that when the background body is isotropic, however, not necessary homogeneous, the use of complex geometrical optics solutions is better than that of geometrical optics solutions.

Finally, we note that theorem 1.1 is valid also for general inhomogeneous anisotropic background conductivity instead of $\gamma_0 I_n$. However, it is just a framework and the problem is to construct the complex geometrical optics solution for general inhomogeneous anisotropic conductivity which plays the role of $v_\tau$ in corollary 1.2. In two dimensions, it may be possible to construct the solution by a combination of the idea of isothermal coordinates as used in [19] for the Calderón problem and a choice of the large virtual slowness developed in this paper. However, in three dimensions, we do not know whether one can construct the solution at the present time.

2. Proof of theorem 1.1

In what follows for simplicity we write $v_\tau = v$ and $\gamma_0 I_n = \gamma_0$.

Let $R_\gamma(\tau)$ and $R_{\gamma_0}(\tau)$ denote the Neumann-to-Dirichlet maps on $\partial \Omega$ for the operators $\nabla \cdot \gamma \nabla - \tau$ and $\nabla \cdot \gamma_0 \nabla - \tau$, respectively. We have

$$R_{\gamma_0}(\tau) \left( \frac{\partial \overline{v}}{\partial \nu} \bigg|_{\partial \Omega} \right) = \overline{v}|_{\partial \Omega}, \quad R_\gamma(\tau) g = p|_{\partial \Omega},$$

where $p$ solves

$$\begin{cases}
(\nabla \cdot \gamma \nabla - \tau) p = 0 & \text{in} \quad \Omega, \\
\gamma \nabla p \cdot \nu = g(x, \tau) & \text{on} \quad \partial \Omega.
\end{cases} \quad (2.1)$$

Our starting point is the following identity which is an easy consequence of equations (1.3), (1.5) and integration by parts:

$$\int_{\partial \Omega} \left( g\overline{v} - \gamma_0 \frac{\partial \overline{v}}{\partial \nu} \right) dS = \int_{\Omega} \left( g(R_{\gamma_0}(\tau) - R_\gamma(\tau)) \left( \gamma_0 \frac{\partial \overline{v}}{\partial \nu} \bigg|_{\partial \Omega} \right) \right) dS$$

$$+ \int_{\Omega} (\gamma - \gamma_0) \nabla \overline{v} \cdot \nabla (w - p) \, dx + e^{-\tau T} \int_{\Omega} F(x, \tau) \overline{v}(x) \, dx. \quad (2.2)$$

Define $\epsilon = w - p$. It follows from (1.3) and (2.1) that $\epsilon$ solves

$$(\nabla \cdot \gamma \nabla - \tau) \epsilon = e^{-\tau T} F(x, \tau) \quad \text{in} \quad \Omega$$

$$\gamma \nabla \epsilon \cdot \nu = 0 \quad \text{on} \quad \partial \Omega.$$
Since $\tau > 0$, it is easy to see that
\[ \| \nabla v \|_{L^2(\Omega)} \leq C e^{-\tau T} \tau^{-1/2} \| F(\cdot, \tau) \|_{L^2(\Omega)} \]
and from (1.4) one obtains
\[ \| \nabla v \|_{L^2(\Omega)} = O(e^{-\tau(T-C_1)\tau^{1/2}}). \]
From this together with (1.4), (1.6) and (1.8) we obtain
\[ \int_\Omega (\gamma - \gamma_0) \nabla v \cdot \nabla v \, dx + e^{-\tau T} \int_\Omega F(x, \tau) \overline{v(x)} \, dx \]
\[ = O(e^{C_2 \tau} \tau^{1/2} e^{-\tau(T-C_1)\tau^{1/2}}) + O(e^{-\tau T} e^{C_3 \tau} \tau^{1/2} e^{C_4 \tau}) \]
\[ = O(e^{-\tau(T-C_1)\tau^{1/2}}) + O(e^{-\tau(T-C_1)\tau^{1/2}}). \]  
(2.3)
A combination of (2.2) and (2.3) gives
\[ \int_{\partial S} \left( g \frac{\partial v}{\partial \nu} - w_0 \frac{\partial \Psi}{\partial v} \right) \, dS = \Psi(\tau) \int_{\partial S} \gamma_0 \frac{\partial v}{\partial \nu} (R_{\gamma_0}(\tau) - R_v(\tau)) \left( \gamma_0 \frac{\partial \Psi}{\partial v} \right) \, dS \]
\[ + O(e^{-\tau(T-C_1)\tau^{1/2}}) + O(e^{-\tau(T-C_1)\tau^{1/2}}). \]  
(2.4)

The following type of estimates now are well known and it is a consequence of proposition 2.1 in [14] which goes back to [7] and one of assumptions (A1) and (A2). See also [16] when $\gamma_0 = 1$ and $h = (k-l)\epsilon$ with a positive constant $k$.

**Lemma 2.1.** There exist $C > 0$ and $C' > 0$ such that for all $v$ satisfying (1.5)
\[ C \| \nabla v \|^2_{L^2(\Omega)} \leq \int_{\partial S} \gamma_0 \frac{\partial v}{\partial \nu} (R_{\gamma_0}(\tau) - R_v(\tau)) \left( \gamma_0 \frac{\partial \Psi}{\partial v} \right) \, dS \leq C' \| \nabla v \|^2_{L^2(\Omega)}. \]  
(2.5)

Note also that
- (1.9) is equivalent to the following statement: there exists $C_0 > 0$ such that, as $\tau \to \infty$
\[ C_0 \leq \tau^h |\Psi(\tau)|. \]  
(2.6)

From the right-hand side of (2.5), (1.6), (2.4), (1.10) and (2.3) we have
\[ \left| \int_{\partial S} \left( g \frac{\partial v}{\partial \nu} - w_0 \frac{\partial \Psi}{\partial v} \right) \, dS \right| \]
\[ = O(\tau^{1/2} e^{2C_2 \tau} \tau^{1/2}) + O(e^{-\tau(T-C_1)\tau^{1/2}}) + O(e^{-\tau(T-C_1)\tau^{1/2}}) \]
\[ = O(e^{2C_2 \tau} \tau^{1/2} + 1 + e^{-\tau(T-C_1)\tau^{1/2}} \tau^{1/2} \tau^{1/2}) \]
\[ = O(e^{-\tau(T-C_1)\tau^{1/2}}). \]  
(2.7)

On the other hand, from the left-hand side of (2.5), (2.6) and (1.7) we obtain
\[ \left| \int_{\partial S} \left( g \frac{\partial v}{\partial \nu} - w_0 \frac{\partial \Psi}{\partial v} \right) \, dS \right| \]
\[ \geq C C_0(C_0)^2 \tau^2 e^{2C_2 \tau} \tau^2 + O(e^{-\tau(T-C_1)\tau^{1/2}}) \]
\[ = O(e^{-\tau(T-C_1)\tau^{1/2}}). \]  
(2.8)
where $C'' = C C_0(C_0)^2 > 0$.

Now formula (1.12) is a consequence of (1.11), (2.7) and (2.8) provided
\[ T > \max(C_1 - C_2, C_1 + C_3 - 2C_2). \]  
(2.9)
However, by remark 1.2 we have $(C_1 + C_3 - 2C_2) = (C_1 - C_2) = C_3 - C_2 \geq 0$ and thus (2.9) is nothing but (1.11).
3. Proof of theorem 1.2

3.1. A special fundamental solution

Given $F$, we construct a solution of the inhomogeneous modified Helmholtz equation

$$(-\Delta + \tau)v + F = 0 \quad \text{in} \quad \mathbb{R}^n \quad (3.1)$$

that has the form

$$v(x) = e^{ix \cdot \Psi(x)} \quad (3.2)$$

where the complex vector $z$ is given by (1.13).

Write

$$F(x) = e^{ix \cdot \Psi_1(x)} f(x). \quad (3.3)$$

Then, if $\Psi_1$ satisfies the equation

$$-\Delta \Psi_1 - 2z \cdot \nabla \Psi_1 + f = 0 \quad \text{in} \quad \mathbb{R}^n; \quad (3.4)$$

then $v$ given by (3.2) satisfies (3.1) with $F$ given by (3.3).

We construct a solution of (3.4) by using a special fundamental solution of (3.4). Set

$$Q_z(\xi) = |\xi|^2 - 2iz \cdot \xi, \quad \xi \in \mathbb{R}^n. \quad (3.5)$$

We have

$$\text{Re} Q_z(\xi) = \left| \frac{\xi + c\tau \sqrt{1 - \frac{1}{c^2 \tau} \omega \cdot \xi}}{c^2 \tau} \right|^2 - \frac{1}{c^2 \tau} \left( 1 - \frac{1}{c^2 \tau} \right)$$

and

$$\text{Im} Q_z(\xi) = -2c\tau \omega \cdot \xi.$$

Thus, the set $Q_z^{-1}(0) = \{ \xi \in \mathbb{R}^n | Q_z(\xi) = 0 \}$ consists of the circle on the plane $\omega \cdot \xi = 0$ centred at $-c\tau \sqrt{1 - (c^2 \tau)^{-1} \omega \cdot \xi}$ with radius $c\tau \sqrt{1 - (c^2 \tau)^{-1}}$. Moreover, $\nabla \text{Re} Q_z(\xi)$ and $\nabla \text{Im} Q_z(\xi)$ on $Q_z^{-1}(0)$ are linearly independent. Thus, $-1/Q_z(\xi)$ is locally integrable on $\mathbb{R}^n$ and can be identified with a unique tempered distribution and its inverse Fourier transform

$$g_z(x) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \frac{d\xi}{Q_z(\xi)} \quad (3.5)$$

is well defined. This $g = g_z$ satisfies, in the sense of tempered distribution

$$-\Delta g - 2z \cdot \nabla g + \delta(x) = 0 \quad \text{in} \quad \mathbb{R}^n$$

and thus, in the sense of the Schwartz distribution

$$(-\Delta + \tau)(e^{ix \cdot g_z} + \delta(x)) = 0 \quad \text{in} \quad \mathbb{R}^n.$$
Thus, the operator: $f \mapsto -\Psi_1$ has the unique continuous extension as a bounded linear operator of $L^2_{\delta+1}(\mathbb{R}^n)$ to $L^2_\delta(\mathbb{R}^n)$. We still denote the operator by the same symbol $g_\tau * f$ which yields the unique solution of equation (3.4) in $L^2_\delta(\mathbb{R}^n)$ for all $f \in L^2_{\delta+1}(\mathbb{R}^n)$.

However, for our purpose, one has to clarify the behaviour of $C(\alpha, \delta)$ in (3.7) as $\tau \to \infty$. This is not the well-known case $z \cdot z = -k^2$ with $k \geq 0$ which appeared in inverse scattering/boundary value problems (cf [17, 18, 20]) since we have (1.14) and thus $z \cdot z \to \infty$ as $\tau \to \infty$. In the next subsection we study more about $C(\alpha, \delta)$ by carefully checking a proof of (3.7).

3.2. Asymptotic behaviour

Define

$$\lambda(c, \tau) = \sqrt{1 - \frac{1}{c^2 \tau}}.$$  

The aim of this subsection is to give a proof of the following estimates.

**Proposition 3.1.** Let $R > 0$ and $-1 < \delta < 0$. We have

$$\|D^\alpha g_\tau * f\|_{L^2_\delta(\mathbb{R}^n)} \leq (c\tau\lambda)^{|\alpha| - 1}C_{\delta, R} \|f\|_{L^2_{\delta+1}(\mathbb{R}^n)}, \quad |\alpha| \leq 2, \quad c\tau \lambda \geq R. \quad (3.8)$$

A change of variables yields

$$g_\tau(x) = (c\tau \lambda(c, \tau))^{n-2}h_\lambda(y; \omega, \omega^\perp)|y = c\tau \lambda(c, \tau)x, \lambda = \lambda(c, \tau) \quad (3.9)$$

where

$$h_\lambda(y; \omega, \omega^\perp) = -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy \cdot \xi} d\xi |\xi + \omega^\perp|^{2} - 1 - 2i\lambda^{-1}\omega \cdot \xi, \quad 0 < \lambda \leq 1. \quad (3.10)$$

Given an arbitrary rapidly decreasing function $f$, we give an estimate of $h_\lambda * f$. Using a rotation in $\xi$-space, it suffices to consider the case when $\omega = (0, 1, 0, \ldots, 0)^T$ and $\omega^\perp = (1, 0, \ldots, 0)^T$.

Define

$$\Sigma = \{\xi \in \mathbb{R}^n| |\xi + \omega^\perp|^2 = 1, \omega \cdot \xi = 0\}.$$  

$\Sigma$ is the set of all real zero points of the complex-valued polynomial

$$h(\xi; \lambda) = |\xi + \omega^\perp|^2 - 1 - 2i\lambda^{-1}\omega \cdot \xi;$$

however, $\Sigma$ itself is independent of $\lambda$.

**Lemma 3.1.** Given $\epsilon > 0$, there exists a positive constant $C_\epsilon$ independent of $\lambda$ such that for all $\xi \in \mathbb{R}^n$ with $\text{dist}_\Sigma(\xi) \geq \epsilon$, we have

$$|h(\xi; \lambda)| \geq C_\epsilon|\xi|^2.$$  

**Proof.** Since $0 < \lambda \leq 1$, we have

$$|h(\xi; \lambda)| \geq |h(\xi; 1)|.$$  

Let $|\xi| \geq 2$. We have

$$|h(\xi; 1)| = (|\xi|^2 + 2\omega^\perp \cdot \xi)^2 + 4(\omega \cdot \xi)^2$$

$$= |\xi|^4 + 4(\omega^\perp \cdot \xi)|\xi|^2 + 4(\omega \cdot \xi)^2 + 4(\omega \cdot \xi)^2$$

$$\geq |\xi|^4 - 4|\xi|^2 = |\xi|^4 \left(1 - \frac{1}{|\xi|} \right) \geq \frac{|\xi|^4}{2}.$$
Since $\xi \neq 0$ and $|h(\xi; 1)| > 0$ for $\xi$ with $\text{dist}_\Sigma(\xi) \geq \epsilon$, the function $|\xi|^{-2}|h(\xi; 1)|$ is continuous and positive on the nonempty compact set $K_\epsilon = \{\xi \in \mathbb{R}^n \mid |\xi| \leq 2\text{ and dist}_\Sigma(\xi) \geq \epsilon\}$. Thus, $m_\epsilon \equiv \inf_{\xi \in K_\epsilon} |\xi|^{-2}|h(\xi; 1)| > 0$ and choosing $C_\epsilon = \min\left(1, \sqrt{2}, m_\epsilon\right)$, we obtain the desired estimate.

For the treatment of $h(\xi; \lambda)$ in a neighbourhood of $\Sigma_1$, we start with the following fundamental fact:

$$\frac{-1}{2\pi i} \int \frac{e^{ix \cdot \eta}}{\eta_1 + i\eta_2} \, d\eta.$$  

This yields

$$\frac{-1}{2\pi i} \int \frac{e^{ix \cdot \eta}}{\eta_1 + i\lambda^{-1}\eta_2} \, d\eta.$$  

(Lemma 3.2) Let $-1 < \delta < 0$ and $0 < \lambda \leq 1$. Then

$$Z_\delta f = \left(\frac{\frac{\xi}{\delta}}{\eta_1 + i\lambda^{-1}\eta_2}\right)$$

defines a bounded map from $L^2_{\delta+1}(\mathbb{R}^n)$ to $L^2_\delta(\mathbb{R}^n)$ and its operator norm is bounded from above with respect to $\lambda$.

Proof. From (3.11) we have

$$Z_\delta f = -\frac{1}{2\pi i} \left\{ \frac{1}{\lambda^{-1}x_1 + ix_2} \right\} * f,$$

where $*$ denotes convolution with respect to variables $(x_1, x_2)$. Let $g$ be an arbitrary rapidly decreasing function. Since $0 < \lambda \leq 1$, we have $|\lambda^{-1}x_1 + ix_2| \geq |x|$. This gives

$$(2\pi)^2|\langle Z_\delta f, g \rangle|^2 \leq \left( \int \int \left| \frac{g(x)f(y)}{\lambda^{-1}(x_1 - y_1) + i(x_2 - y_2)} \right| \, dx \, dy \right)^2 \leq \left( \int \int \left| \frac{g(x)f(y)}{(x_1 - y_1) + i(x_2 - y_2)} \right| \, dx \, dy \right)^2.$$  

Thus, everything is reduced to the case when $\lambda = 1$ and it is nothing but lemma 3.1 in [20].

Having lemmas 3.1–3.2 and using a rotation in $\xi$-space, localization and a change of variable which are exactly same as Sylvester–Uhlmann’s argument [20] for the case $z \cdot \bar{z} = 0$, we obtain

$$\|D^\alpha h_\lambda * f\|_L^2(\mathbb{R}^n) \leq C_\delta \|f\|_{L^2_\delta(\mathbb{R}^n)}, \quad |\alpha| \leq 2.$$  

(3.12)

The constant $C_\delta$ is independent of $\lambda$.

To deduce (3.8) from (3.12) we employ a scaling argument [6]. It follows from (3.9) that

$$g_\epsilon * f = (\epsilon \tau \lambda)^{-2} \left( h_\lambda * f(\epsilon \tau \lambda^{-1}) \right)_{\epsilon \tau \lambda}, \quad \lambda = \lambda(c, \tau),$$  

(3.13)
where
\[ f_\eta(x) = f(\eta x), \quad \eta > 0. \]

Let \( s > 0 \). We have
\[ \|f_\eta\|_{L^2_s(\mathbb{R}^n)} \leq \eta^{s/n} \max(1, R^{-s}) \| f \|_{L^2_s(\mathbb{R}^n)}, \quad 0 < R \leq \eta. \]

Now from this and (3.13) we have
\[
|\langle g_* \ast f, g \rangle| = (c_t \lambda)^{-2} |\langle (h_* \ast f_{(c_t \lambda)^{-1}} \circ c_t \lambda, g \rangle | \\
= (c_t \lambda)^{-2(2n+1)} |\langle h_* \ast f_{(c_t \lambda)^{-1}} \circ c_t \lambda, \mathcal{S}(c_t \lambda)^{-1} g \rangle | \\
\leq (c_t \lambda)^{-3(2n+1)} \| h_* \ast f_{(c_t \lambda)^{-1}} \|_{L^2(\mathbb{R}^n)} \| \mathcal{S}(c_t \lambda)^{-1} g \|_{L^2(\mathbb{R}^n)} \\
\leq (c_t \lambda)^{-3(2n+1)} C_b \| f_{(c_t \lambda)^{-1}} \|_{L^2(\mathbb{R}^n)} \| \mathcal{S}(c_t \lambda)^{-1} g \|_{L^2(\mathbb{R}^n)} \\
\leq (c_t \lambda)^{-3(2n+1)} C_b (c_t \lambda)^{3+1/n/2} \max(1, R^{-n/2}) \| f \|_{L^2(\mathbb{R}^n)} (c_t \lambda)^{-3/n}. \\
\times \max(1, R^n) \| g \|_{L^2(\mathbb{R}^n)} \\
= (c_t \lambda)^{-3} C(\delta, R) \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)}. \\
\]

This together with the same argument for \( D^{\alpha} g_* \ast f \) yields (3.8).

### 3.3. Uniqueness and construction of \( \varepsilon_z \)

**Write** (1.27) as
\[
(-\Delta - 2z \cdot \nabla) \varepsilon_z = -(\tau a + b) \varepsilon_z - (\tau a + b) \quad \text{in} \quad \mathbb{R}^n.
\]

Since the solution of (3.4) is unique in \( L^2(\mathbb{R}^n) \) and has the form (3.6) for \( f \in L^2_{\delta,1}(\mathbb{R}^n) \), we have
\[
\varepsilon_z = -(\tau g_* \ast (a \varepsilon_z)) - (\tau g_* \ast a - g_* \ast a) - g_* \ast b. \tag{3.14}
\]

Define
\[
K_z(a) : L^2(\mathbb{R}^n) \ni h \mapsto -\tau g_* \ast (ah) \in L^2(\mathbb{R}^n)
\]
and
\[
L_z(b) : L^2(\mathbb{R}^n) \ni h \mapsto g_* \ast (bh) \in L^2(\mathbb{R}^n).
\]

It follows from (3.7) that both \( K_z \) and \( L_z \) define bounded linear operators in \( L^2(\mathbb{R}^n) \). **Rewrite** (3.14) as
\[
(I - K_z(a) - L_z(b)) \varepsilon_z = -\tau g_* \ast a - g_* \ast b.
\]

It follows from (3.8) for \( |a| = 0 \) that
\[
\| K_z(a) h \|_{L^2(\mathbb{R}^n)} \leq (c_t \lambda)^{-1} C_{\delta, R} \| ah \|_{L^2(\mathbb{R}^n)}
\]
provided \( c\lambda \geq R > 0 \). Since \( \| ah \|_{L^2(\mathbb{R}^n)} \leq \| \langle x \rangle a \|_{L^2(\mathbb{R}^n)} \| h \|_{L^2(\mathbb{R}^n)} \), one obtains
\[
\| K_z(a) \| \leq (c_t \lambda)^{-1} C_{\delta, R} \| \langle x \rangle a \|_{L^2(\mathbb{R}^n)}
\]
and similarly
\[
\| L_z(b) \| \leq (c_t \lambda)^{-1} C_{\delta, R} \| \langle x \rangle b \|_{L^2(\mathbb{R}^n)}.
\]

From these we obtain
\[
\| K_z(a) + L_z(b) \| \leq (c_t \lambda)^{-1} C(\delta, R)(\| \langle x \rangle a \|_{L^2(\mathbb{R}^n)} + \tau^{-1} \| \langle x \rangle b \|_{L^2(\mathbb{R}^n)}). \tag{3.15}
\]
Let $c_1 \geq c_2 > 0$ and $R > 0$. Let $c$ and $\tau$ satisfy
\[ c \geq \sqrt{c_1^2 + R^2} \]
and
\[ \tau \geq \frac{1}{c_2}. \]
We have $c^2 \tau > 1$ and $c\lambda \geq R$. Now given $-1 < \delta < 0$, $R > 0$ and $\eta > 0$ choose a large $c_1$ in such a way that
\[ \eta \sqrt{c_1^2 - c_2^2} \geq C(\delta, R)(\|\langle x \rangle a\|_{L^\infty(\mathbb{R}^n)} + c_2^2 \|\langle x \rangle b\|_{L^\infty(\mathbb{R}^n)}), \tag{3.16} \]
Since $c\lambda \geq \sqrt{c_1^2 - c_2^2}$, a combination of (3.15) and (3.16) gives
\[ \|Kz(a) + Lz(b)\| \leq \eta. \tag{3.17} \]
Choosing $\eta < 1$ and
\[ C_1 = \sqrt{c_1^2 + R^2}, \quad C_2 = \frac{1}{c_2^2} \]
for $c_1$ satisfying (3.16), specially chosen $c_2 \leq c_1$ and $R$, we obtain the uniqueness and existence of the solution of (3.12) and thus those of (1.27), too.
\[ \epsilon_z \in L^2_{\text{loc}}(\mathbb{R}^n) \]
takes the form
\[ \epsilon_z = -\sum_{n=0}^{\infty} (Kz(a) + Lz(b))^n (\tau g_z * a + g_z * b). \]
It follows from (3.17) and (3.8) for $|a| = 0$ that
\[ \|\epsilon_z\|_{L^2_{\text{loc}}(\mathbb{R}^n)} \leq \sum_{n=0}^{\infty} \eta^n (c\lambda)^{-1} C(\delta, R) \left( \|a\|_{L^2_{\text{loc}}(\mathbb{R}^n)} + c_2^2 \|b\|_{L^2_{\text{loc}}(\mathbb{R}^n)} \right) \]
\[ = \frac{1}{1 - \eta} (c\lambda)^{-1} C(\delta, R) \left( \|a\|_{L^2_{\text{loc}}(\mathbb{R}^n)} + c_2^2 \|b\|_{L^2_{\text{loc}}(\mathbb{R}^n)} \right) = O((c\lambda)^{-1}). \tag{3.18} \]
Differentiating both sides of (3.14) in the sense of distribution, we have
\[ (I - Kz(a) - Kz(b)) \partial_j \epsilon_z = -\tau g_z * \partial_j a - Kz(\partial_j a) \epsilon_z - g_z * \partial_j b - Lz(\partial_j b) \epsilon_z. \]
A similar argument for the derivation of (3.18) yields
\[ \|\partial_j \epsilon_z\|_{L^2_{\text{loc}}(\mathbb{R}^n)} = O((c\lambda)^{-1}). \]
Continue this procedure for higher order derivatives of $\epsilon_z$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ and apply the Sobolev imbedding theorem to the resulted estimates. Then, one concludes that $\epsilon_z|_{\Omega_1}$ can be identified with a function in $C^1(\overline{\Omega})$ and
\[ \|\epsilon_z\|_{L^\infty(\Omega)} + \|\nabla \epsilon_z\|_{L^\infty(\Omega)} = O((c\lambda)^{-1}). \]
Thus, choosing again a large $c_1$, we obtain (1.28). This completes the proof of theorem 1.2.

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