ARITHMETIC PROPERTIES OF THE FROBENIUS TRACES DEFINED BY A RATIONAL ABELIAN VARIETY

ALINA CARMEN COJOCARU, RACHEL DAVIS, ALICE SILVERBERG, AND KATHERINE E. STANGE

Abstract. Let $A$ be an abelian variety over $\mathbb{Q}$. Under suitable hypotheses, we formulate a conjecture about the asymptotic behaviour of the Frobenius traces $a_{1,p}$ of $A$ reduced modulo varying primes $p$. This generalizes a well-known conjecture of S. Lang and H. Trotter from 1976 about elliptic curves. We prove upper bounds for the counting function $\#\{p \leq x : a_{1,p} = t\}$ and we investigate the normal order of the number of prime factors of $a_{1,p}$.

1. Introduction

Given an abelian variety $A/\mathbb{Q}$, its reductions $A_p/\mathbb{F}_p$ modulo primes encode deep arithmetic global information. A primary question related to these reductions concerns their $p$-Weil polynomials, in particular the coefficients of these polynomials.

In the simplest case when $A$ has dimension 1, that is, when $A$ is an elliptic curve over $\mathbb{Q}$, for each prime $p$ of good reduction the $p$-Weil polynomial is $P_{A,p}(X) = X^2 - a_pX + p \in \mathbb{Z}[X]$, where $a_p := p + 1 - |A_p(\mathbb{F}_p)|$. The coefficient $a_p$ satisfies the Weil bound $|a_p| \leq 2\sqrt{p}$ and is of major significance in number theory. For example, it appears as the $p$-th Fourier coefficient in the expansion of the weight 2 newform associated to $A$.

The study of $a_p$ comes in several flavours, some having led to well-known problems in arithmetic geometry, such as the Sato-Tate Conjecture from the 1960s (now a theorem) and the Lang-Trotter Conjecture on Frobenius traces from the 1970s (still open).

Briefly, the Lang-Trotter Conjecture [LaTr] on the behaviour of $a_p$ predicts that for every elliptic curve $A/\mathbb{Q}$ and every integer $t \in \mathbb{Z}$, if $\text{End}_{\mathbb{Q}}(A) \simeq \mathbb{Z}$ or $t \neq 0$, and if we write $N_A$ for the product of the primes of bad reduction for $A$, then either there are at most finitely many primes $p$ such that $a_p = t$, or there exists a constant $c(A,t) > 0$ such that, as $x \to \infty$,

$$\pi_A(x,t) := \#\{p \leq x : p \nmid N_A, a_p = t\} \sim c(A,t) \frac{x^{\frac{1}{2}}}{\log x}. \quad (1)$$

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The constant \( c(A,t) \) has a precise heuristical description derived from the Chebotarev Density Theorem, combined with the Sato-Tate Conjecture when \( \text{End}_{\mathbb{Q}}(A) \simeq \mathbb{Z} \) and with a prime distribution law arising from works of M. Deuring and E. Hecke when \( \text{End}_{\mathbb{Q}}(A) \not\simeq \mathbb{Z} \) (the CM case) and \( t \neq 0 \), upper bounds of the right order of magnitude can be proven using sieve methods. When \( \text{End}_{\mathbb{Q}}(A) \simeq \mathbb{Z} \) and \( t \neq 0 \), weaker upper bounds, unconditional or conditional (upon the Generalized Riemann Hypothesis, GRH), can be proven using effective versions of the Chebotarev Density Theorem; such bounds were first obtained by J-P. Serre [Se81, Theorem 20]. The currently best unconditional upper bound, \( \pi_A(x,t) \ll_A \frac{x \left( \log \log x \right)^2 \left( \log \log \log x \right)^2 \left( \log x \right)^2}{(\log x)^2} \), was obtained by V.K. Murty [Mu1, Theorem 5.1], while the currently best upper bound under GRH, \( \pi_A(x,t) \ll_A \frac{x \frac{1}{2} \left( \log x \right)^3}{(\log x)^2} \), was obtained by M.R. Murty, V.K. Murty & N. Saradha [MuMuSa, Theorem 4.2]. When \( \text{End}_{\mathbb{Q}}(A) \simeq \mathbb{Z} \) and \( t = 0 \), stronger results are known: in particular, the unconditional bounds \( \frac{\log \log \log x}{(\log \log \log \log x)^{1+\varepsilon}} \ll \pi_A(x,0) \ll x^{1+\varepsilon} \) were obtained by É. Fouvry & M.R. Murty [FoMu, Theorem 1] and, respectively, by N.D. Elkies [El] using, as a key tool, M. Deuring’s characterization of supersingular primes [De].

The main goals of our paper are to formulate a generalization of (1) for the Frobenius traces of a higher dimensional abelian variety \( A/\mathbb{Q} \) and to prove partial upper bounds related to this generalized conjecture. Given the central role in these investigations played by the divisibility properties of the Frobenius traces, an additional goal is to determine the normal order of the sequence defined by the prime divisor function of the Frobenius traces.

Our main setting and notation are as follows. Let \( A/\mathbb{Q} \) be a principally polarized abelian variety of dimension \( g \). Let \( \overline{\mathbb{Q}} \) denote an algebraic closure of \( \mathbb{Q} \), and let \( \text{End}_{\overline{\mathbb{Q}}}(A) \) denote the endomorphism ring of \( A \) over \( \overline{\mathbb{Q}} \). Let \( N_A \) be the product of primes of bad reduction for \( A \).

We denote by

\[ \rho_A : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GSp}_{2g}(\hat{\mathbb{Z}}) \]

the absolute Galois representation defined by the inverse limit of the representations

\[ \tilde{\rho}_{A,m} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GSp}_{2g}(\mathbb{Z}/m\mathbb{Z}) \]

of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the \( m \)-torsion \( A[m] \subset A(\overline{\mathbb{Q}}) \) for each integer \( m \geq 1 \). For each prime \( \ell \) we denote by

\[ \rho_{A,\ell} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{GSp}_{2g}(\mathbb{Z}_\ell) \]

the \( \ell \)-adic representation, i.e., the representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the \( \ell \)-adic Tate module \( \text{lim}_n A[\ell^n] \).

For each prime \( p \nmid N_A \), consider the \( p \)-Weil polynomial \( P_{A,p}(X) \) of \( A \), which is uniquely determined by the property that

\[ P_{A,p}(X) = \det (XI_{2g} - \rho_{A,\ell}(\text{Frob}_p)) \] (2)
for any prime $\ell \neq p$. In particular, we have

$$P_{A,p}(X) \equiv \det (XI_{2g} - \bar{\rho}_{A,m}(\text{Frob}_p)) \pmod{m} \quad (3)$$

for any integer $m$ coprime to $p$. We write

$$P_{A,p}(X) = X^{2g} + a_{1,p} X^{2g-1} + \ldots + a_{g,p} X^g + p a_{g-1,p} X^{g-1} + \ldots + p^{g-1} a_{1,p} X + p^g \in \mathbb{Z}[X].$$

For any integer $t \in \mathbb{Z}$, we consider the function

$$\pi_A(x,t) := \# \{ p \leq x : p \nmid N_A, a_{1,p} = t \}.$$

We set

$$G(m) := \text{Im} \bar{\rho}_{A,m},$$
$$C(m,t) := \{ M \in G(m) : \text{tr} M \equiv t (\text{mod} m) \},$$

and

$$G_\ell := \text{Im} \rho_{A,\ell},$$
$$C_\ell(t) := \{ M \in G_\ell : \text{tr} M = t \}.$$

In our conjecture, we consider the following hypothesis about $A$:

**Hypothesis (ST).** There exists a bounded, integrable function

$$\Phi_A : [-1,1] \rightarrow [0,\infty),$$

continuous at 0, such that for every interval $I \subseteq [-1,1]$ we have

$$\lim_{x \rightarrow \infty} \frac{\# \{ p \leq x : p \nmid N_A, a_{1,p} \in I \}}{\pi(x)} = \int_I \Phi_A(t) \, dt. \quad (4)$$

In the case that $A$ is an elliptic curve with $\text{End}_{\mathbb{Q}}(A) \simeq \mathbb{Z}$, the Sato-Tate conjecture is the assertion that Hypothesis (ST) holds with $\Phi_A(x) = \frac{2}{\pi} \sqrt{1-x^2}$. Remarkably, this is now a theorem (see [C] for details and references). In the case that $A$ is an elliptic curve with $\text{End}_{\mathbb{Q}}(A) \not\simeq \mathbb{Z}$, a variation of Hypothesis (ST) has been known for much longer, thanks to work of Hecke and Deuring, with $\Phi_A(x) = \frac{1}{2\pi \sqrt{1-x^2}} + \frac{1}{2} \delta_0(x)$, where $\delta_0(\cdot)$ is the Dirac delta function. Note that, in this case, the function $\Phi_A(\cdot)$ is unbounded and discontinuous at 0. In the present paper, we focus on the generic situation, hence the rationale behind the assumptions on $\Phi_A$ in our Hypothesis (ST).

We propose the following generalization of (1):

**Conjecture 1.** Let $A/\mathbb{Q}$ be a principally polarized abelian variety of dimension $g$ and let $t \in \mathbb{Z}$. Assume that $\text{Im} \rho_A$ is open in $\text{GSp}_{2g}(\hat{\mathbb{Z}})$ and that Hypothesis (ST) holds. Then as $x \rightarrow \infty$ we have

$$\pi_A(x,t) \sim \frac{\Phi_A(0)}{g} \cdot \frac{m_A |C(m_A,t)|}{|G(m_A)|} \cdot \prod_{\ell | m_A} \ell \cdot \frac{|\{ M \in \text{GSp}_{2g}(\hat{\mathbb{Z}}/\ell \mathbb{Z}) : \text{tr} M \equiv t (\text{mod} \ell) \}|}{|\text{GSp}_{2g}(\hat{\mathbb{Z}}/\ell \mathbb{Z})|} \cdot \sqrt{x} \log x.$$
where $m_A$ is the smallest positive integer $m$ such that

$$\rho_A(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = \Pi^{-1}(G(m)),$$

with $\Pi : \text{GSp}_{2g}(\mathbb{Z}) \to \text{GSp}_{2g}(\mathbb{Z}/m\mathbb{Z})$ the natural projection. If the constant in front of $\sqrt{x} \log x$ is zero, we interpret the asymptotic as saying that there are at most finitely many primes $p$ such that $a_{1,p} = t$.

In Section 3, we provide heuristical reasoning for this conjecture and address some connections with existing works.

Generalizations of the Lang-Trotter Conjecture [1] have been previously considered by other authors. For example, in [Mu2], V.K. Murty addresses weaker generalizations in the setting of modular forms, while in [Ka, pp. 421 – 423], N. Katz addresses weaker generalizations in the setting of abelian varieties. Our conjecture encompasses a generic class of abelian varieties $A$ and is precise in terms of both the growth in $x$ and the constant depending on $A$ and $t$. The potential vanishing of the constant is an important open problem in itself. In [Ka, p. 420], for instance, Katz discusses a general mechanism that leads to congruence obstructions for realizing $a_{1,p} = t$. We plan to return to this aspect of our conjecture in a future project.

We prove the following results, generalizing work of Serre [Se81] and of Murty & Murty [MuMu].

**Theorem 2.** Let $A/\mathbb{Q}$ be a principally polarized abelian variety of dimension $g$ and let $t \in \mathbb{Z}$. Assume that $\text{Im} \rho_A$ is open in $\text{GSp}_{2g}(\hat{\mathbb{Z}})$. Define

$$\alpha := \frac{1}{2g^2 + g + 1}, \quad \beta := \frac{1}{2g^2 + g + \frac{1}{2}}, \quad \gamma := \frac{1}{2g^2 + g - \frac{1}{2}}.$$

Then for any $\varepsilon > 0$ we have:

(i) unconditionally,

$$\pi_A(x, t) \ll_{A, \varepsilon} x \frac{1}{(\log x)^{1 + \alpha - \varepsilon}};$$

(ii) under GRH,

$$\pi_A(x, t) \ll_{A, \varepsilon} x^{1 - \frac{g^2}{2} + \varepsilon};$$

(iii) if $t \neq \pm 2g$, then (i1) and (i2) hold with $\alpha$ replaced by $\beta$;

(iv) if $t = 0$, then (i1) and (i2) hold with $\alpha$ replaced by $\gamma$.

Note that we will actually prove a more general result, stated as Theorem 17 below. Also note that, when $g = 1$, we recover [Se81, Thm. 20, p. 189] in the cases (241) and (242), in the special cases that $t \neq \pm 2$ or further that $t = 0$, our proof deals with the group GSp_{2g} for arbitrary $g$ and hence does not appeal to the special feature of GL_{2} that is employed in the proof of these additional parts at the bottom of [Se81, p. 189].

Recall that $\nu(n)$ denotes the number of distinct prime factors of a positive integer $n$ and that an arithmetic function $f(\cdot)$ is said to have normal order $F(\cdot)$ if for all $\varepsilon > 0$ we have $(1 - \varepsilon)F(n) < f(n) < (1 + \varepsilon)F(n)$ for all but a zero density subset of positive integers $n$. 
Theorem 3. Let \( A/\mathbb{Q} \) be a principally polarized abelian variety of dimension \( g \). Assume that \( \text{Im} \, \rho_A \) is open in \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \). Then under GRH we have

\[
\sum_{\substack{\nu \leq \sqrt{x} \\ p \nmid N_A}} (\nu(a_{1,p}) - \log \log p)^2 \ll_A \pi(x) \log \log x.
\]

Consequently, \( (\nu(a_{1,p}))_p \) has normal order \( \log \log p \).

This result vastly generalizes to abelian varieties the main result of [MuMu].

We remark that the openness assumption on \( \text{Im} \, \rho_A \) holds for a large class of abelian varieties. Indeed, in [Se72, pp. 1–2], [Se86bis, Thm. 3, p. 97], Serre proves it for all principally polarized abelian varieties \( A/\mathbb{Q} \) with \( \text{End}_\mathbb{Q}(A) \cong \mathbb{Z} \) and of dimension \( g \) equal to 1, 2, 6, or an arbitrary odd number. More recently, Serre’s result was generalized to the case of a principally polarized abelian variety \( A/\mathbb{Q} \) with \( \text{End}_\mathbb{Q}(A) \cong \mathbb{Z} \) and for which there exists a number field \( K \) such that the Néron model of \( A/K \) over the ring of integers of \( K \) has a semistable fibre of toric dimension 1; see [Ha]. As pointed out in [Ha, p. 704], for \( g \geq 2 \) the openness of \( \text{Im} \, \rho_A \) in \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \) holds for most abelian \( g \)-folds that arise as Jacobians of hyperelliptic curves defined by \( y^2 = f(x) \) with the degree \( n \) of the monic polynomial \( f \in \mathbb{Z}[x] \) equal to \( 2g + 1 \) or \( 2g + 2 \); specifically, the hypotheses in Hall’s theorem are satisfied if the Galois group of \( f \) is \( S_n \), or if there exists a rational prime \( p \) for which \( f(\mod p) \) has \( n - 1 \) distinct zeroes over an algebraic closure, one of which is a double zero; see [Ha] and [Za].

2. Generalities

2.1. Basic notation. Along with the standard analytic notation \( O, \ll, \gg, o, \sim, \pi(x), \text{li} \, x \), we will be using \( p \) and \( \ell \) to denote rational primes. We write \( n|m^{\infty} \) to mean that all the prime divisors of \( n \) occur among the prime divisors of \( m \), possibly with higher multiplicities.

For a commutative, unitary ring \( R \) and a positive integer \( g \), we denote by \( R^\times \) its group of units and by \( I_g \in M_g(R) \) the identity matrix, and we let

\[
J_{2g} := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \in M_{2g}(R).
\]

Recall that the general symplectic group on \( R \) is defined by

\[
\text{GSp}_{2g}(R) := \{ M \in \text{GL}_{2g}(R) : M^tJ_{2g}M = \mu J_{2g} \text{ for some } \mu \in R^\times \},
\]

has center \( \{ \mu J_{2g} : \mu \in R^\times \} \), and, as an algebraic group, has dimension \( 2g^2 + g + 1 \). Note that \( \text{GSp}_2(R) = \text{GL}_2(R) \).

2.2. The Chebotarev Density Theorem.
2.2.1. Finite extensions of a number field. Let $L/K$ be a finite Galois extension of number fields and let $G$ be its Galois group. Let $C$ be a non-empty subset of $G$ that is stable under conjugation. For any $x > 0$, let

\[ \pi_C(x, L/K) := \# \{ \mathfrak{p} \text{ a place of } K, \text{ unramified in } L/K : N_{K/Q}(\mathfrak{p}) \leq x, \text{Frob}_\mathfrak{p} \subseteq C \}. \]

The Chebotarev Density Theorem states that

\[ \pi_C(x, L/K) \sim \frac{|C|}{|G|} \pi(x). \]

We will use the following conditional effective version of this theorem:

**Theorem 4.** (J. Lagarias and A. Odlyzko [LaOd]; for this version see [Se81, Thm. 4, p. 133])

Keep the above setting and assume GRH. Then there exists an absolute constant $c > 0$ such that

\[ \left| \pi_C(x, L/K) - \frac{|C|}{|G|}\pi(x) \right| \leq c \frac{|C|}{|G|}x^{\frac{1}{2}} \left( \log \left| \text{disc}(L/Q) \right| + |L : Q| \log x \right). \]

In order to apply this theorem, the following variation of a result of Hensel [He], proven in [Se81], is useful:

**Proposition 5.** (J-P. Serre, [Se81, Prop. 5, p. 129])

Keep the above setting. Then

\[ \log N_{K/Q}(\left| \text{disc}(L/K) \right|) \leq (|L : Q| - |K : Q|) \left( \sum_{p \in \mathcal{P}(L/K)} \log p \right) + |L : Q| \log |L : K|, \]

where

\[ \mathcal{P}(L/K) := \{ p : \text{there is a place } \mathfrak{p} \text{ of } K, \text{ramified in } L/K, \text{with } \mathfrak{p}|p \}. \]

2.2.2. $\ell$-adic extensions of a number field. In [Se81], Serre used the effective versions of the Chebotarev Density Theorem of Lagarias & Odlyzko [LaOd] to deduce upper bounds for $\pi_C(x, L/K)$ in the case of an $\ell$-adic Galois extension $L/K$ of a number field $K$. We recall his main results below.

Let $K$ be a number field. Let $\ell$ be a rational prime and $G$ a compact $\ell$-adic Lie group of dimension $D$. Denote by $Z(G)$ the center of $G$. Let $C \subseteq G$ be a non-empty closed subset of $G$ that is stable under conjugation. We denote its Minkowski dimension (see [Se81, Section 3] for a definition) by $\dim_M C$. Let $L/K$ be an infinite Galois extension, with Galois group $G$. For any $x > 0$, let

\[ \pi_C(x, L/K) := \# \{ \mathfrak{p} \text{ a place of } K, \text{ unramified in } L/K : N_{K/Q}(\mathfrak{p}) \leq x, \text{Frob}_\mathfrak{p} \subseteq C \}. \]

Following Serre [Se81, p.151], define

\[ \epsilon(x) := \frac{\log x}{(\log \log x)^2 (\log \log \log x)} \]

and

\[ \epsilon_R(x) := \frac{x^{\frac{3}{2}}}{(\log x)^2}. \]
Theorem 6. (J-P. Serre, [Se81] Thm. 10, p. 151)
Keep the above setting. Let $0 \leq d < D$ be such that the Minkowski dimension of $C$ satisfies

$$\dim_M C \leq d.$$ 

Define

$$\alpha := \frac{D - d}{D}.$$ 

Then

(i) Unconditionally,

$$\pi_C(x, L/K) \ll_{K, L, C} \frac{\mathfrak{li}_x}{\epsilon(x)^\alpha}.$$ 

In particular, for any $\varepsilon > 0$,

$$\pi_C(x, L/K) \ll_{K, L, C, \varepsilon} \frac{x}{(\log x)^{1+\alpha-\varepsilon}}.$$ 

(ii) Under GRH,

$$\pi_C(x, L/K) \ll_{K, L, C} \frac{\mathfrak{li}_x}{\epsilon_R(x)^\alpha}.$$ 

In particular, for any $\varepsilon > 0$,

$$\pi_C(x, L/K) \ll_{K, L, C, \varepsilon} x^{1-\frac{\alpha}{2} + \varepsilon}.$$ 

In special cases, Serre obtains the following improvement:

Theorem 7. (J-P. Serre, [Se81] Theorem 12, p. 157)
Keep the above setting. Let $0 \leq d < D$ be such that the Minkowski dimension of $C$ satisfies

$$\dim_M C \leq d.$$ 

Define

$$r := \inf_{M \in C} \dim \frac{G}{Z_G(M)},$$

where $Z_G(M)$ denotes the centralizer of $M$ in $G$. Define

$$\beta := \frac{D - d}{D - \frac{r}{2}}.$$ 

Then items (i) and (ii) of Theorem 6 are satisfied with $\beta$ in place of $\alpha$.

Note that $r \geq 0$, hence $\beta \geq \alpha$ and so Theorem 7 is Theorem 6 when $\beta = \alpha$. When $r \geq 1$, hence $\beta > \alpha$, Theorem 7 improves upon Theorem 6. This happens when $C \cap Z(G) = \emptyset$. 


2.3. Abelian varieties. Let \( A/\mathbb{Q} \) be a principally polarized abelian variety of dimension \( g \) and let \( p \) be a prime of good reduction. Recall that for any root \( \pi \in \mathbb{C} \) of \( P_{A,p}(X) \) we have \( |\pi| = \sqrt{p} \), hence

\[
|a_{1,p}| \leq 2g\sqrt{p}. \tag{5}
\]

Property (2) links the \( p \)-Weil polynomial \( P_{A,p}(X) \) to the division fields of \( A \), in particular to the Galois representation defining \( \rho_A \). We recall their main properties:

- by the Néron-Ogg-Shafarevich criterion, for any integer \( m \geq 1 \),
  
  the extension \( \mathbb{Q}(A[m])/\mathbb{Q} \) is unramified outside \( mN_A \); \( \tag{6} \)

- by the injectivity of the restriction of \( \tilde{\rho}_{A,m} \) to \( \text{Gal}(\mathbb{Q}(A[m])/\mathbb{Q}) \),

\[
|G(m)| \leq |\text{GSp}_{2g}(\mathbb{Z}/m\mathbb{Z})| \leq m^{2g^2+g+1}. \tag{7}
\]

In many cases, the image of the representation \( \rho_A \) is well understood and the groups \( G_\ell \) and \( G(m) \) are known much more precisely. For example, as already mentioned at the end of Section 1, for several classes of abelian varieties \( A/\mathbb{Q} \) with a trivial endomorphism ring, \( \text{Im} \rho_A \) is open in \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \). In particular, for such \( A \) we have that

- \( G_\ell \) is open in \( \text{GSp}_{2g}(\mathbb{Z}_\ell) \) for all rational primes \( \ell \);
- \( G(\ell) \cong \text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z}) \) for all but finitely many rational primes \( \ell \).

Below are some consequences to the openness of \( \text{Im} \rho_A \) in \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \):

**Lemma 8.** Let \( A/\mathbb{Q} \) be a principally polarized abelian variety of dimension \( g \) such that \( \text{Im} \rho_A \) is open in \( \text{GSp}_{2g}(\hat{\mathbb{Z}}) \).

(i) There exists an integer \( m \geq 1 \) such that \( \rho_A(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})) = \Pi^{-1}(G(m)) \), where we recall that

\[
\Pi : \text{GSp}_{2g}(\hat{\mathbb{Z}}) \longrightarrow \text{GSp}_{2g}(\mathbb{Z}/m\mathbb{Z})
\]

is the natural projection. Denote by \( m_A \) the least such integer.

(ii) For all positive integers \( m_1, m_2 \) with \( m_1|m_A^\infty \) and \( (m_2, m_A) = 1 \), we have

\[
G(m_1m_2) \cong G(m_1) \times \text{GSp}_{2g}(\mathbb{Z}/m_2\mathbb{Z}).
\]

(iii) For all \( t \in \mathbb{Z} \) we have

\[
\prod_{\ell|m_A} \frac{\ell \cdot |\{ M \in \text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z}) : \text{tr} M \equiv t(\text{mod} \ell)\}|}{|\text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} < \infty
\]

and

\[
\lim_{m \to \infty} m|C(m,t)|/|G(m)| = m_A|C(m_A,t)|/|G(m_A)| \cdot \prod_{\ell|m_A} \frac{\ell \cdot |\{ M \in \text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z}) : \text{tr} M \equiv t(\text{mod} \ell)\}|}{|\text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|},
\]

here, for a sequence \( (s_n)_n \), \( \lim_{m \to \infty} s_m \) denotes \( \lim_{n \to \infty} s_{m_n} \) with \( m_n := \prod_{\ell \leq n} \ell^n \).
Proof. Parts (i) and (ii) are clear from the openness assumption on $\operatorname{Im} \rho_A$. For part (iii), let $\ell$ and $t \in \mathbb{Z}$ be fixed. First, we will show that
\[
\frac{\ell}{|C(\ell, t)|} = 1 + O\left(\frac{1}{\ell}\right),
\] (8)

Recall that the multiplicator of $\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ is the character of $\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ with kernel $\operatorname{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$; we denote it by mult. For $\gamma \in (\mathbb{Z}/\ell\mathbb{Z})^\times$, define
\[
\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})^\gamma := \text{mult}^{-1}(\gamma),
\]
\[
C(\ell, t)^\gamma := C(\ell, t) \cap \operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})^\gamma,
\]
\[
\mathcal{G}(\ell)^\gamma := \{\operatorname{char}(M) : M \in \operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})\gamma\},
\]
\[
C(\ell, t)^\gamma := \{M \in \mathcal{G}(\ell)^\gamma : \operatorname{tr} M = t\}.
\]
Here, $\operatorname{char}(M)$ denotes the characteristic polynomial of $M$.

By [AcHo] Lemma 2.4, p. 631,
\[
\left(\frac{\ell}{\ell + 1}\right)^{2g^2 + g} \frac{|C(\ell, t)^\gamma|}{|\mathcal{G}(\ell)^\gamma|} \leq \frac{|C(\ell, t)^\gamma|}{|\operatorname{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} \leq \left(\frac{\ell}{\ell - 1}\right)^{2g^2 + g} \frac{|C(\ell, t)^\gamma|}{|\mathcal{G}(\ell)^\gamma|}.
\]

Noting that $|C(\ell, t)^\gamma| = \ell^{g-1}$ and $|\mathcal{G}(\ell)^\gamma| = \ell^g$, we deduce that
\[
\left(\frac{\ell}{\ell + 1}\right)^{2g^2 + g} \frac{1}{\ell} \leq \frac{|C(\ell, t)^\gamma|}{|\operatorname{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} \leq \left(\frac{\ell}{\ell - 1}\right)^{2g^2 + g} \frac{1}{\ell}.
\]

Combining the above inequalities for each $\gamma \in (\mathbb{Z}/\ell\mathbb{Z})^\times$ and multiplying by $\ell$ gives
\[
\left(\frac{\ell}{\ell + 1}\right)^{2g^2 + g} \frac{\ell}{|\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} \leq \left(\frac{\ell}{\ell - 1}\right)^{2g^2 + g} \frac{\ell}{|\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|},
\]
which completes the proof of (\ref{eq:9}).

Next, assuming that $t \neq 0$, we will prove that
\[
\frac{\ell}{|\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} = 1 + O\left(\frac{1}{\ell^2}\right).
\] (9)

For this, observe that for any $t_1, t_2 \in \mathbb{Z}$ we have
\[
t_1 \equiv t_2 (\text{mod } \ell) \Rightarrow C(\ell, t_1) = C(\ell, t_2)
\]
and
\[
t_1 \not\equiv 0 (\text{mod } \ell), \quad t_2 \not\equiv 0 (\text{mod } \ell) \Rightarrow |C(\ell, t_1)| = |C(\ell, t_2)|.
\]

Indeed, the first assertion is trivial, while the second assertion follows by noting that, if $t_1 \not\equiv 0 (\text{mod } \ell)$ and $t_2 \not\equiv 0 (\text{mod } \ell)$, then the endomorphism $[t_2 t_1^{-1}]$ of $\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ defined by multiplication by $t_2 t_1^{-1}$ is a bijection satisfying that $[t_2 t_1^{-1}] (C(\ell, t_1)) = C(\ell, t_2)$.

From the above observations,
\[
|\operatorname{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})| = |C(\ell, 0)| + (\ell - 1) |C(\ell, t)|.
\]

It is easy to show that \((\ref{eq:9})\) now follows from this along with \((\ref{eq:8})\) for $|C(\ell, 0)|$.\]
This proves the convergence of the infinite product

$$\prod_{\ell \in \mathbb{Z}_\ell} \ell \cdot |\{M \in \text{GSp}_{2g}(\mathbb{Z}/\ell \mathbb{Z}) : \text{tr } M \equiv t \text{ (mod } \ell)\}|$$

(10)

for \( t \neq 0 \). It remains to prove the convergence for \( t = 0 \).

In [K], Theorem 5.3, p. 170, it is shown that

$$|C(\ell, t)| = g(\ell) + \left\{ \begin{array}{ll} -\ell^{-1}f(\ell) & \text{if } t \neq 0, \\
\ell^{-1}(\ell - 1)f(\ell) & \text{if } t = 0,
\end{array} \right. \quad (11)$$

where \( g(\ell) \) is given explicitly and has a leading term of degree \( D = 2g^2 + g \), and where \( f(\ell) \) is also given explicitly. The expression for \( f(\ell) \) is rather delicate, but we can determine its leading term as follows.

According to [K], it is:

$$f(\ell) = \ell^{g^2-1} \sum_{b=0}^{[g/2]} \ell^{b(b+1)} \prod_{m=0}^{2b-1} \left( \frac{\ell^{g^2-m} - 1}{\ell^{2b-m} - 1} \right) \prod_{j=1}^{b} (\ell^{j-1} - 1) \sum_{k=1}^{\lfloor (g-2b+2)/2 \rfloor} \ell^k \times \sum_{\alpha \in \mathbb{F}_\ell^\times} K(\alpha) \eta^{2b+2-2k} \sum_{j_1, \ldots, j_{k-1}} \prod_{v=1}^{k-1} (\ell^{j_v - 2v} - 1), \quad (12)$$

where \( K(\alpha) \) are Kloosterman sums.

First we deal with the factor

$$\sum_{\alpha \in \mathbb{F}_\ell^\times} K(\alpha)^r. \quad (13)$$

Kim remarks that when \( r \geq 2 \),

$$\sum_{\alpha \in \mathbb{F}_\ell^\times} K(\alpha)^r = \ell^2M_{r-1} - (\ell - 1)^{r-1} + 2(-1)^{r-1}$$

where \( M_s \) is the number of \((\alpha_1, \ldots, \alpha_s) \in (\mathbb{F}_\ell^\times)^s\) satisfying \( \alpha_1 + \cdots + \alpha_s = 1 \) and \( \alpha_1^{-1} + \cdots + \alpha_s^{-1} = 1 \). We have \( M_0 = 0 \) and \( M_1 = 1 \), so let us assume \( s \geq 2 \). Then the first of the two conditions gives \( \alpha_1 \) linearly in terms of the other \( \alpha_i \) and the second gives \( \alpha_2 \) as a root of a quadratic in the remaining terms. Thus, \( M_s \leq 2(\ell - 1)^{s-2} \). Thus, for \( r \geq 3 \), the leading term of \((13)\) is at most \( r - 1 \). In the case \( r = 2 \), \((13)\) is bounded by an expression of leading degree at most 2 in \( \ell \) (by direct computation using \( M_1 \)). For \( r = 1 \), we use a standard estimate on Kloosterman sums, that \( |K(\alpha)| \leq 2\sqrt{\ell} \), to find that \((13)\) is bounded by \( \ell^2 \). For \( r = 0 \), the sum is simply \((\ell - 1)\). Therefore, we can take \((13)\) to be bounded by an expression of leading degree at most \( r + 1 \) in general.

Using this estimate, it remains to compute the leading term of Kim’s expression \((12)\) for \( f(\ell) \). We find that it is of degree \( d \) in \( \ell \) satisfying

$$d \leq \max \left\{ g^2 + 2bg - 2b^2 + b + k + kg - 2bk - k^2 + 1 : 0 \leq b \leq \left\lfloor \frac{g}{2} \right\rfloor, 1 \leq k \leq \left\lfloor \frac{g - 2b + 2}{2} \right\rfloor \right\}. \quad (14)$$
The quadratic function shown is maximized when \( b = |g/2| \) and \( k = |(g-2b+2)/2| = 1 \), when it becomes
\[
d \leq \frac{3g^2}{2} + \frac{g}{2} + 1.
\]
From (11), we have \( f(\ell) = |C(\ell,0)| - |C(\ell,t)| \). Since we know (10) for \( t \neq 0 \), it suffices to show that
\[
\frac{\ell f(\ell)}{|GSp_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} = O\left(\frac{1}{\ell^2}\right).
\]
For this, it suffices that \( D - d \geq 2 \). We have
\[
D - d = \frac{g^2}{2} + \frac{g}{2} - 1.
\]
For \( t = 0, g \geq 2 \), this confirms the convergence of (10). It remains to consider \( t = 0, g = 1 \), where our estimates above were not tight enough. Fortunately, this case can be dealt with directly. A straightforward exercise shows that \( |C(\ell,0)| = \ell^3 - \ell^2 \), hence
\[
\frac{\ell f(\ell)}{|GSp_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} = \ell^3(t-1) = \frac{\ell^2}{\ell - 1} = 1 + O\left(\frac{1}{\ell}\right).
\]
This confirms the convergence of the infinite product (10) for \( t = 0, g = 1 \).

The second half of part (iii) follows from part (ii).

**Remark 9.** It is possible to give an alternative proof of the convergence of (10) in the case \( t = 0, g = 2 \) that does not rely on [Ki]. This uses [CaFoHuSu], which gives accessible exact counts for \( |C(\ell,t)| \) (these will be used in Remark 10).

Define
\[
N_{\ell,t} := \# \left\{(x,y,\delta) \in ((\mathbb{Z}/\ell\mathbb{Z})^3 : y \neq -\delta, \left(x + \frac{y}{x}\right)\left(1 + \frac{\delta}{y}\right) = t\right\}.
\]
It follows from the proof of [CaFoHuSu] Theorem 12 that the number of matrices in \( GSp_4(\mathbb{Z}/\ell\mathbb{Z}) \) with trace \( t \) is exactly
\[
\ell^4 (\ell^2 - 2\ell + 1) + \ell^4(\ell - 1)(\ell^2 - 1)^2 + \ell^5(\ell - 1)^2(\ell - 1) + \begin{cases} \ell^7 - \ell^4 & \text{if } t = 0, \\ 0 & \text{if } t \neq 0. \end{cases}
\]
We have
\[
\# \left\{(x,y,\delta) \in ((\mathbb{Z}/\ell\mathbb{Z})^3 : y \neq -\delta\right\} = (\ell - 1)^2(\ell - 2).
\]
Setting first \( t = 0 \), consider the size of the set
\[
N_{\ell,0} = \# \left\{(x,y,\delta) \in ((\mathbb{Z}/\ell\mathbb{Z})^3 : y \neq -\delta, \left(x + \frac{y}{x}\right)\left(1 + \frac{\delta}{y}\right) = 0\right\}.
\]
Taking the value of \( x \) to be free, the conditions determine \( y \) and require only that \( \delta \neq -y \). Therefore
\[
N_{\ell,0} = (\ell - 1)(\ell - 2).
\]
Therefore
\[
\# \left\{(x,y,\delta) \in ((\mathbb{Z}/\ell\mathbb{Z})^3 : y \neq -\delta, \left(x + \frac{y}{x}\right)\left(1 + \frac{\delta}{y}\right) \neq 0\right\} = (\ell - 1)(\ell - 2)^2.
\]
Dividing by \( \ell - 1 \), we discover that for fixed nonzero \( t \), \( N_{\ell,t} = (\ell - 2)^2 \).

Therefore,
\[
N_{\ell,t} = \begin{cases} (\ell - 1)(\ell - 2) & \text{if } t = 0, \\ (\ell - 2)^2 & \text{if } t \neq 0. \end{cases}
\]
We deduce the formula
\begin{equation}
|C(\ell, t)| = \begin{cases} 
\ell^5(\ell - 1)(\ell^4 - \ell - 1) & \text{if } t = 0, \\
\ell^4(\ell^6 - \ell^5 - \ell^4 + \ell + 1) & \text{if } t \neq 0.
\end{cases}
\end{equation}
(14)

These formulae can also be deduced by direct computation from Kim [K]. It suffices to combine with the fact that

\[ |\text{GSp}_4(\mathbb{Z}/\ell\mathbb{Z})| = \ell^4(\ell^6 - \ell^5 - \ell^4 + \ell + 1), \]

to see that

\[ \frac{\ell|C(\ell, 0)|}{|\text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})|} = 1 + O\left(\frac{1}{\ell^2}\right). \]

3. Heuristical reasoning for Conjecture

Let \( A/\mathbb{Q} \) be a principally polarized abelian variety of dimension \( g \) and let \( t \in \mathbb{Z} \). We devote this section to arguing heuristically towards Conjecture

\[ \text{Definition 10. Assume that Hypothesis (ST) holds for } A. \text{ For each integer } m \geq 1 \text{ and prime } p, \text{ define } c_{p, m} \in (0, \infty) \text{ by} \]

\[ c_{p, m} = \frac{|G(m)|}{m \sum_{|\tau| \leq 2g\sqrt{p}} \Phi_A\left(\frac{\tau}{2g\sqrt{p}}\right) |C(m, \tau)|} \]

and define the function

\[ f_p^{(m)} : \mathbb{Z} \rightarrow [0, \infty), \]

\[ f_p^{(m)}(\tau) := \begin{cases} 
\Phi_A\left(\frac{\tau}{2g\sqrt{p}}\right) \cdot \frac{m|C(m, \tau)|}{|G(m)|} \cdot c_{p, m} & \text{if } |\tau| \leq 2g\sqrt{p}, \\
0 & \text{else.}
\end{cases} \]

Note that

\[ \sum_{\tau \in \mathbb{Z}} f_p^{(m)}(\tau) = 1. \]

\[ \text{Lemma 11. Assume that Hypothesis (ST) holds. Then for all integers } m \geq 1 \text{ and } \tau_0 \in \mathbb{Z} \text{ we have} \]

\[ \lim_{p \rightarrow \infty} m \sum_{|\tau| \leq 2g\sqrt{p}} \Phi_A\left(\frac{\tau}{2g\sqrt{p}}\right) = 1. \]

\[ \text{Proof. This follows by viewing the expression inside the limit as a Riemann sum approximation of the integral} \]

\[ \int_{-1}^{1} \Phi_A(\tau) d\tau = 1. \text{ For more details, see [LaTr] pp. 31–32}. \]

\[ \text{Lemma 12. Assume that Hypothesis (ST) holds for } A. \text{ Then for all integers } m \geq 1 \text{ we have} \]

\[ \lim_{p \rightarrow \infty} 2g\sqrt{p} c_{p, m} = 1. \]
Proof. By the definition of $c_{p,m}$ and Lemma 11

\[
\lim_{p \to \infty} \frac{1}{2g\sqrt{p}} c_{p,m} = \lim_{p \to \infty} \frac{1}{2g\sqrt{p}} \sum_{\tau_0=0}^{m-1} \sum_{\tau \equiv \tau_0 \mod m} \Phi_A \left( \frac{\tau}{2g\sqrt{p}} \right) m \frac{|C(m,\tau)|}{|G(m)|}
\]

\[
= \lim_{p \to \infty} \sum_{\tau_0=0}^{m-1} \frac{|C(m,\tau_0)|}{|G(m)|} \left( \frac{m}{2g\sqrt{p}} \sum_{\tau \equiv \tau_0 \mod m} \Phi_A \left( \frac{\tau}{2g\sqrt{p}} \right) \right)
\]

\[
= \sum_{\tau_0=0}^{m-1} \frac{|C(m,\tau_0)|}{|G(m)|}
\]

\[
= 1.
\]

\[
\square
\]

Now assume that $\Im \rho$ is open in $GSp_{2g} \hat(Z)$ and that $A$ satisfies Hypothesis (ST). We assume that $\lim_{m \to \infty} f_p^{(m)}(t)$ models the likelihood of the event $a_{1,p} = t$, as guided by the Chebotarev law for all $m$-division fields and by the behaviour of $\frac{a_{1,p}}{2\sqrt{p}}$ in the interval $[-1, 1]$. In particular, we assume that the Chebotarev data and the Sato-Tate data are statistically independent. Then, recalling part (iii) of Lemma 8 and Lemma 12 we reason heuristically as follows:

\[
\lim_{x \to \infty} \# \{ p \leq x : p \mid N_A, a_{1,p} = t \} \approx \lim_{x \to \infty} \lim_{m \to \infty} \sum_{p \leq x} f_p^{(m)}(t)
\]

\[
= \lim_{x \to \infty} \lim_{m \to \infty} \sum_{p \leq x} \Phi_A \left( \frac{t}{2g\sqrt{p}} \right) \cdot \frac{m |C(m,t)|}{|G(m)|} \cdot c_{p,m}
\]

\[
\approx \left( \lim_{m \to \infty} \frac{m |C(m,t)|}{|G(m)|} \right) \lim_{x \to \infty} \sum_{p \leq x} \Phi_A \left( \frac{t}{2g\sqrt{p}} \right) \cdot \frac{1}{2g\sqrt{p}}
\]

\[
= \frac{m_A |C(m_A,t)|}{g|G(m_A)|} \cdot \prod_{\ell \mid m_A} \frac{\ell \cdot |\{ M \in GSp_{2g}(\mathbb{Z}/\ell\mathbb{Z}) : tr M \equiv t \mod \ell \}|}{|GSp_{2g}(\mathbb{Z}/\ell\mathbb{Z})|}
\]

\[
\cdot \lim_{x \to \infty} \sum_{p \leq x} \Phi_A \left( \frac{t}{2g\sqrt{p}} \right) \cdot \frac{1}{2g\sqrt{p}}
\]

Here, the symbol $\approx$ means equality deduced purely heuristically.

For the last limit, we use the assumptions on the function $\Phi_A$. For any $\varepsilon > 0$, by the continuity of $\Phi_A$ at 0, there exists a $\delta > 0$ such that

\[
\left| \frac{t}{2g\sqrt{p}} \right| < \delta \Rightarrow \left| \Phi_A \left( \frac{t}{2g\sqrt{p}} \right) - \Phi_A(0) \right| < \varepsilon.
\]

(15)

We thus split the sum over $p \leq x$ according to the above $\delta$-interval. By the boundedness of $\Phi_A$, we obtain

\[
\left| \sum_{p \leq x} \left( \Phi_A \left( \frac{t}{2g\sqrt{p}} \right) - \Phi_A(0) \right) \cdot \frac{1}{2g\sqrt{p}} \right| \ll_{t,\varepsilon,g} 1.
\]
By (15) and by noting that \( \sum_{p \leq x} \frac{1}{2\sqrt{p}} \sim \frac{\sqrt{x}}{\log x} \), we obtain

\[
\left| \sum_{\nu^{1/2} < p \leq x} \left( \Phi_A \left( \frac{t}{2g\sqrt{p}} \right) - \Phi_A(0) \right) \frac{1}{2\sqrt{p}} \right| \ll \varepsilon \frac{\sqrt{x}}{\log x}.
\]

Taking \( \varepsilon \to 0 \) and returning to our heuristics, we predict that

\[
\# \{ p \leq x : p \nmid N_A, a_{1,p} = t \} \sim \frac{\Phi_A(0)}{g} \cdot \frac{m_A |C(m_A,t)|}{|G(m_A)|} \cdot \prod_{\ell \nmid m_A} \ell \cdot \left| \{ M \in \text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z}) : \text{tr} M \equiv t(\text{mod } \ell) \} \right| \cdot \frac{\sqrt{x}}{\log x}. \tag{16}
\]

We conclude this section with several remarks about the above conjecture.

Remark 13. For \( g = 1 \), (16) coincides with the formulation of Conjecture 1 of Lang and Trotter [LaTr]. A more refined version of their conjecture was proposed in [BaJo]. For higher \( g \), similar refinements shall be addressed in an upcoming paper by Cojocaru.

Remark 14. For \( g = 2 \), the paper [FiKeRoSu] provides a precise (variation of) Hypothesis (ST) for each possible \( \text{End}_\mathbb{Q}(A) \), while for arbitrary \( g \), the monograph [Se11] provides a general potential Hypothesis (ST).

Similarly to the case \( g = 1 \), for a variety of non-generic abelian surfaces \( A/\mathbb{Q} \) (i.e. such that \( \text{End}_\mathbb{Q}(A) \not\simeq \mathbb{Z} \)) variations of (ST) with the function \( \Phi_A(\cdot) \) as predicted in [FiKeRoSu] were proven; see [Go] and [Jo]. For abelian surfaces \( A/\mathbb{Q} \) such that \( \text{End}_\mathbb{Q}(A) \simeq \mathbb{Z} \), (ST) remains open.

Remark 15. For abelian surfaces such that \( \text{End}_\mathbb{Q}(A) \simeq \mathbb{Z} \), the function \( \Phi_A(\cdot) \) predicted by [FiKeRoSu] may be calculated as follows. Let \( L_p(A,T) = T^{\pi} P_{A,p}(\frac{t}{\pi}) \) be the \( p \)-Euler factor in the \( L \)-function of \( A \) and let \( \bar{L}_p(A,T) = L_p \left( A, \frac{T}{\sqrt{p}} \right) \) be its normalization. Let

\[
S := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 2x_1 - 2, x_2 \geq -2x_1 - 2, x_2 \leq \frac{x_1^2 + x_2^2 + 2}{4} \right\}
\]

and let \( R(x_1) \) be the defining interval of \( x_2 \) imposed by the constraints of \( S \). Using the terminology introduced in [FiKeRoSu], the conjectured Sato-Tate group associated to \( A \) is \( \text{USp}_4 \). In particular, the conjectured joint density function of the normalized coefficients \( \bar{a}_{1,p} \) and \( \bar{a}_{2,p} \) is

\[
\frac{1}{4\pi^2} \sqrt{\max\{\rho(\bar{a}_{1,p}, \bar{a}_{2,p}), 0\}},
\]

where

\[
\rho(x_1, x_2) := (x_1^2 - 4x_2 + 8)(x_2 - 2x_1 + 2)(x_2 + 2x_1 + 2),
\]

with support in the region \( S \) where \( \rho \) is non-negative. Consequently, for any interval \( I \subseteq [-4, 4] \), the set

\[
\{ p : \bar{a}_{1,p} \in I \}
\]
is expected to have natural density
\[ \int_{I} \int_{R(x)} \frac{1}{4\pi^2} \sqrt{\max{\{\rho(x_1, x_2), 0\}}} \, dx_2 \, dx_1. \]

Thus, in our terminology,
\[ \Phi_A(x) = \frac{1}{4\pi^2} \int_{R(x)} \sqrt{\max{\{\rho(x, x_2), 0\}}} \, dx_2. \]

In particular, \( R(0) = [-2, 2] \) and
\[ \Phi_A(0) = \frac{64}{15\pi^2}. \]

Remark 16. For abelian surfaces such that \( \text{End}_\mathbb{Q}(A) \simeq \mathbb{Z} \), the constant in (16) may be described more explicitly:
\[ \pi_A(x, t) \sim \frac{32}{15\pi^2} \frac{m_A |C(m_A, t)|}{|G(m_A)|} \cdot \prod_{\ell|m_A, \ell \nmid t} \frac{\ell^2(\ell^4 - \ell^2 - 1)}{(\ell^2 - 1)(\ell^4 - 1)} \cdot \prod_{\ell|m_A, \ell \mid t} \frac{\ell^2(\ell^6 - \ell^5 - \ell^4 + \ell + 1)}{(\ell - 1)(\ell^2 - 1)(\ell^4 - 1)} \cdot \frac{x}{\log x}. \]

This is obtained by combining (14) with the value of \( \Phi_A(0) \) from the previous remark, and the equality
\[ |G_{Sp}(\mathbb{Z}/\ell\mathbb{Z})| = \ell^4(\ell^2 - 1)(\ell^4 - 1). \]

4. Proof of Theorem 2

We will deduce Theorem 2 from the following more general result:

Theorem 17. Let \( A/\mathbb{Q} \) be a principally polarized abelian variety of dimension \( g \) and let \( t \in \mathbb{Z} \). For primes \( \ell \), let \( G_\ell := GSp_{2g}(\mathbb{Z}/\ell\mathbb{Z}) \).

(i) Assume that there exists a prime \( \ell \) for which \( G_\ell \) is open in \( G_\ell \), and assume
\[ \exists 0 \leq d < \dim G_\ell \text{ such that } \dim_{\mathcal{M}} C_\ell(t) \leq d. \] \hspace{1cm} (17)

Define
\[ \alpha := \frac{\dim G_\ell - d}{\dim G_\ell}. \]

Then for any \( \varepsilon > 0 \) we have:

(i1) unconditionally,
\[ \pi_A(x, t) \ll_{A, t, \varepsilon} \frac{x}{(\log x)^{1+\alpha-\varepsilon}}; \] \hspace{1cm} (18)

(i2) under GRH,
\[ \pi_A(x, t) \ll_{A, t, \varepsilon} x^{1/2 + \varepsilon}. \] \hspace{1cm} (19)

(ii) If \( t \neq \pm 2g \), assume that there exists a prime \( \ell \) such that \( v_\ell(\frac{t}{2g}) \neq 0 \), \( G_\ell \) is open in \( G_\ell \), and (17) holds. Define
\[ \beta := \frac{\dim G_\ell - d}{\dim G_\ell - \frac{1}{2}}. \]

Then (18) and (19) hold with \( \alpha \) replaced by \( \beta \).
(iii) Suppose \( t = 0 \). Write \( P G_\ell \) for the quotient of \( G_\ell \) by its center and let \( C'_\ell(0) \) be the image of \( C_\ell(0) \) under this quotient. Assume that there exists a prime \( \ell \) such that \( G_\ell \) is open in \( G_\ell \) and assume

\[
\exists 0 \leq d < \dim P G_\ell \text{ such that } \dim_{\mathcal{M}} C'_\ell(0) \leq d. \tag{20}
\]

Define

\[
\gamma := \frac{\dim G_\ell - 1 - d}{\dim G_\ell - \frac{3}{2}}.
\]

Then (18) and (19) hold with \( \alpha \) replaced by \( \gamma \).

Proof. Let \( x > 0 \), to be thought of as approaching \( \infty \). Observe that, by (2), for any rational prime \( \ell \) we have

\[
\pi_A(x, t) \leq \pi_C(\ell)(x, L/\mathbb{Q}),
\]

where

\[
L := \mathbb{Q}^{\ker \rho A, \ell}.
\]

It remains to estimate \( \pi_C(\ell)(x, L/\mathbb{Q}) \), which we do by following the method of Serre [Se81, Section 8].

(i) We choose \( \ell \) such that \( G_\ell \) is open in \( G_\ell \) and (17) holds. We apply Theorem 6 to the extension \( L/\mathbb{Q} \) and the conjugacy set \( C_\ell(t) \), with \( D = \dim G_\ell \). For this, observe that the assumption that \( G_\ell \) is open in \( G_\ell \) implies that \( \dim G_\ell = \dim G_\ell \).

(ii) If \( t \neq \pm 2g \), we choose \( \ell \) such that \( v_\ell\left(\frac{1}{2g}\right) \neq 0 \), \( G_\ell \) is open in \( G_\ell \), and (17) holds. As before, the assumption that \( G_\ell \) is open in \( G_\ell \) implies that \( \dim G_\ell = \dim G_\ell \). For the moment, write

\[
C_\ell(t) := \{ M \in G_\ell : \text{tr}(M) = t \}.
\]

First, we claim that \( C_\ell(t) \cap Z(G_\ell) = \emptyset \), for, otherwise, recalling that

\[
Z(G_\ell) = \{ \mu I_{2g} : \mu \in \mathbb{Z}_\ell^\times \},
\]

we would have that the \( \ell \)-adic valuation of \( \frac{1}{2g} \) satisfies \( v_\ell\left(\frac{1}{2g}\right) = 0 \), a contradiction. Therefore, the centralizer \( Z_{G_\ell}(M) \) of an element \( M \in C_\ell(t) \) satisfies \( Z_{G_\ell}(M) \neq G_\ell \). Centralizers are closed subgroups, hence Lie subgroups, and \( Z_{G_\ell}(M) \) has a well-defined dimension. Since \( \text{GSp}_{2g} \) is connected as an algebraic group, the properness of \( Z_{G_\ell}(M) \) implies \( \dim Z_{G_\ell}(M) < \dim G_\ell = \dim G_\ell \). Since \( Z_{G_\ell}(M) \subset Z_{G_\ell}(M) \), we may conclude that

\[
r := \inf_{M \in C_\ell(t)} \dim \frac{G_\ell}{Z_{G_\ell}(M)} \geq 1.
\]

We apply Theorem 7 to the extension \( L/\mathbb{Q} \) and the conjugacy set \( C_\ell(t) \), with \( D = \dim G_\ell \).

(iii) If \( t = 0 \), we choose \( \ell \) such that \( G_\ell \) is open in \( G_\ell \) and (20) holds. We set

\[
PG_\ell := G_\ell/Z(G_\ell),
\]

\[
\Pi : G_\ell \to PG_\ell \text{ the canonical projection},
\]

\[
G'_\ell := \Pi(G_\ell), \quad C'_\ell(t) := \Pi(C_\ell(t)),
\]
and
\[ \hat{\rho}_{A,\ell} = \Pi \circ \rho_{A,\ell}. \]
(Note: we abuse notation and denote by $\Pi$ various natural projections.) Observe that the property of having zero trace (or non-zero trace) is meaningful under this projection. Therefore we may define
\[ C'_\ell(0) := \{ M \in P\mathcal{G}_\ell : \text{tr}(M) = 0 \}. \]
Define
\[ L' := \mathbb{Q}^{\text{Ker} \hat{\rho}_{A,\ell}}, \]
a Galois extension of $\mathbb{Q}$ with Galois group $G'_\ell$, and observe that
\[ \pi_A(x, 0) \leq \pi_{C'_\ell(0)}(x, L'/\mathbb{Q}); \]
it remains to estimate the right hand side.

Since $G_\ell$ is open in $G_\ell$, we have that $G'_\ell$ is open in $P\mathcal{G}_\ell$ and so $\dim G'_\ell = \dim P\mathcal{G}_\ell = \dim \mathcal{G}_\ell - 1$. By (20) we have $\dim_M C'_\ell(0) < \dim \mathcal{G}_\ell - 1$. Since $Z(P\mathcal{G}_\ell) = \{ I \}$, we have $C'_\ell(0) \cap Z(P\mathcal{G}_\ell) = \emptyset$. Therefore, as in the previous case, the centralizer $Z_{P\mathcal{G}_\ell}(M)$ of $M \in C'_\ell(0)$ is a proper closed subgroup, and so $\dim Z_{P\mathcal{G}_\ell}(M) < \dim P\mathcal{G}_\ell$. As before, this implies
\[ r := \inf_{M \in C'_\ell(0)} \frac{\dim G'_\ell}{\dim Z_{G'_\ell}(M)} \geq 1. \]
We apply Theorem 7 to the extension $L'/\mathbb{Q}$ and the conjugacy set $C'_\ell(0)$, with $D = \dim \mathcal{G}_\ell - 1$. This completes the proof of Theorem 17.

Proof of Theorem 2 In our setting, by the openness assumption on $\text{Im} \rho_A$, we may apply Theorem 17 with any $\ell$.

It remains to verify conditions (17) and (20) and compute the values of $\alpha$, $\beta$ and $\gamma$. To verify condition (17), observe that the matrices of trace $t$ form a closed subvariety of the algebraic group $G_{2g}$. Hence
\[ C_\ell(t) := \{ M \in \mathcal{G}_\ell : \text{tr}(M) = t \} \]
has a well-defined dimension strictly smaller than $\dim \mathcal{G}_\ell$ and (17) follows with $d = \dim \mathcal{G}_\ell - 1$.

Condition (20) follows similarly from the observation that $C'_\ell(0) := \{ M \in P\mathcal{G}_\ell : \text{tr}(M) = 0 \}$ is a well-defined, closed subvariety. Here $d = \dim \mathcal{G}_\ell - 2$.

The computations of $\alpha$, $\beta$ and $\gamma$ follow from the observation that $\dim \mathcal{G}_\ell = 2g^2 + g + 1$.

5. Proof of Theorem 3

Let $A/\mathbb{Q}$ be a principally polarized abelian variety of dimension $g$ satisfying the hypotheses of Theorem 3. We use the normal order method, similarly to the approach in [MuMu, Sections 3 and 5].
The core ingredient in the proof is the following application of \([6]-[7]\), Proposition \([5]\) and Theorem \([4]\) for any integer \(m\), we have
\[
\pi_{C(m,0)}(x, \mathbb{Q}[A[m]])/\mathbb{Q} = \frac{|C(m,0)|}{|G(m)|} \pi(x) + O \left( |C(m,0)| x^{\frac{2}{2}} \log(mN_A x) \right). \tag{21}
\]

Crucial to the method is also the following simple observation. Let \(x > 0\) and \(y := x^\delta\) for some fixed \(0 \leq \delta < 1\). For any integer \(n \geq 1\), we have
\[
|\nu(n) - \nu_y(n)| \leq \frac{\log n}{\delta \log x}, \tag{22}
\]
where \(\nu_y(n)\) denotes the number of distinct prime divisors \(\ell \leq y\) of \(n\). For our proof, we choose
\[
\delta < \frac{1}{8g^2 + 4g + 8} \tag{23}
\]
and proceed as follows.

Using \((22)\), \((5)\), and \((3)\), we obtain:
\[
\sum_{\substack{\ell \leq y \\ell A}} \nu(a_{1,\ell}) = \sum_{\substack{\ell \leq y \\ell A}} \nu_y(a_{1,\ell}) + O_{g,\delta}(\pi(x))
\]
\[
= \sum_{\ell \leq y} \pi_{C(\ell,0)}(x, \mathbb{Q}[A[\ell]]) / \mathbb{Q} + O(\pi(y) \pi_A(x,0)) + O_{g,\delta}(\pi(x));
\]
\[
= \sum_{\ell_1, \ell_2 \leq y} \pi_{C(\ell_1,\ell_2,0)}(x, \mathbb{Q}[A[\ell_1,\ell_2]]) / \mathbb{Q} + O(\pi(y)^2 \pi_A(x,0)) + O_{g,\delta}(\pi(x)).
\]

Then by \((21)\) (under GRH) we obtain:
\[
\sum_{\substack{\ell \leq y \\ell A}} \nu(a_{1,\ell}) = \sum_{\substack{\ell \leq y \\ell A}} \frac{|C(\ell,0)|}{|G(\ell)|} \pi(x) + O \left( \sum_{\substack{\ell \leq y \\ell A}} |C(\ell,0)| x^{\frac{2}{2}} \log(N_A x) \right)
\]
\[
+ O(\pi(y) \pi_A(x,0)) + O_{g,\delta}(\pi(x));
\]
\[
\sum_{\substack{\ell \leq y \\ell A}} \nu(a_{1,\ell})^2 = \sum_{\substack{\ell \leq y \\ell A}} \frac{|C(\ell,0)|}{|G(\ell)|} \pi(x) + O \left( \sum_{\substack{\ell \leq y \\ell A}} |C(\ell,0)| x^{\frac{2}{2}} \log(N_A x) \right)
\]
\[
+ O_{g,\delta} \left( \sum_{\substack{\ell \leq y \\ell A}} |C(\ell,0)| \pi(x) \right) + O_{g,\delta} \left( \sum_{\substack{\ell \leq y \\ell A}} |C(\ell,0)| x^{\frac{2}{2}} \log(N_A x) \right)
\]
\[
+ O(\pi(y)^2 \pi_A(x,0)) + O_{g,\delta}(\pi(x)).
\]
By the openness assumption of $\text{Im} \rho_A$ in $\text{GSp}_{2g}(\hat{\mathbb{Z}})$ (in particular, by the consequence to this assumption that $G(\ell) \simeq \text{GSp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$ for all but finitely many rational primes $\ell$) and by 8 from the proof of part (iii) of Lemma 8, we have

$$\frac{|C(\ell, 0)|}{|G(\ell)|} = \frac{1}{\ell} + O \left( \frac{1}{\ell^2} \right)$$

for all $\ell \nmid m_A$. In particular,

$$\sum_{\ell \leq y} \frac{|C(\ell, 0)|}{|G(\ell)|} = \log \log y + O_A(1),$$

$$\sum_{\ell_1, \ell_2 \leq y \atop \ell_1 \neq \ell_2} \frac{|C(\ell_1 \ell_2, 0)|}{|G(\ell_1 \ell_2)|} = (\log \log y)^2 + O_A(\log \log y).$$

Then using (24) and (7) we have

$$\sum_{\ell \leq y} |C(\ell, 0)| \ll \frac{y^{2g^2 + g + 2}}{\log y},$$

$$\sum_{\ell_1, \ell_2 \leq y \atop \ell_1 \neq \ell_2} |C(\ell_1 \ell_2, 0)| \ll \frac{y^{4g^2 + 2g + 4}}{(\log y)^2}.$$

We now put everything together, appealing also to part (iii) of Theorem 2 (under GRH):

$$\sum_{\substack{p \leq x \atop p \nmid N_A \atop a_1, p \neq 0}} (\nu(a_1, p) - \log \log x)^2$$

$$= \sum_{\substack{p \leq x \atop p \nmid N_A \atop a_1, p \neq 0}} \nu(a_1, p)^2 - 2(\log \log x) \sum_{\substack{p \leq x \atop p \nmid N_A \atop a_1, p \neq 0}} \nu(a_1, p) + (\log \log x)^2 \# \{ p \leq x : p \nmid N_A, a_1, p \neq 0 \}

= \sum_{\ell_1, \ell_2 \leq y \atop \ell_1 \neq \ell_2} \frac{|C(\ell_1 \ell_2, 0)|}{|G(\ell_1 \ell_2)|} \pi(x) + O \left( \sum_{\ell_1, \ell_2 \leq y} |C(\ell_1 \ell_2, 0)|^{1/2} \log(N_A x) \right) + O_g, \delta \left( \sum_{\ell \leq y} |C(\ell, 0)|^{1/2} \log(N_A x) \right)

+ O \left( \pi(y)^2 \pi_A(x, 0) \right) + O_g, \delta(\pi(x)) - 2(\log \log x) \sum_{\ell \leq y} |C(\ell, 0)| \pi(x) + O \left( \sum_{\ell \leq y} |C(\ell, 0)|^{1/2} \log(N_A x) \right) (\log \log x)

+ O(\pi(y) \pi_A(x, 0) \log \log x) + O_g(\pi(x) \log \log x) + (\log \log x)^2 \# \{ p \leq x : p \nmid N_A, a_1, p \neq 0 \}

= \pi(x) (\log \log x)^2 + 2\pi(x) (\log \delta)(\log \log x) + \pi(x) (\log \delta)^2 + O_A(\pi(x) \log \log x)

+ O_g \left( \frac{4s^2 + 2s + 4}{x^{s^2 + s + 1}} \log(x)^{-2} x^\frac{1}{2} \log(N_A x) \right) + O_g (\pi(x) \log \log x) + O_A(\pi(x))

+ O_g \left( \frac{2s^2 + s + 2}{x^{s^2 + s + 1}} \log(x)^{-1} x^\frac{1}{2} \log(N_A x) \right) + O_g \left( \frac{2s^2 + s + 2}{x^{s^2 + s + 1}} (\log x)^{1/2} \pi_A(x, 0) \right) + O_g(\pi(x)) - 2\pi(x) (\log \log x)^2

- 2\pi(x)(\log \delta)(\log \log x) + O_A(\pi(x) \log x) + O \left( \frac{2s^2 + s + 2}{x^{s^2 + s + 1}} \log(x)^{-1} x^\frac{1}{2} \log(N_A x) \right) (\log \log x)

+ O_g \left( \frac{1}{x^{s^2 + s + 1}} \log(x)^{-1} \pi_A(x, 0) \log \log x \right) + O_g(\pi(x) \log \log x) + \pi(x)(\log \log x)^2 + O \left( \pi_A(x, 0)(\log \log x)^2 \right)

= O_A(\pi(x) \log \log x).
Note that the cancellation of the $\pi(x)(\log \log x)^2$ terms is essential and that the choice of $\delta$ as given in (23) ensures that the largest emerging $O$-term depending on $y = x^\delta$,

$$O\left(\frac{4y^2+2y+4}{x^y^2+4y+8} (\log x)^{-2} x^{\frac{1}{2}} \log(N_A x)\right),$$

is sufficiently small (precisely, it is $\ll_A \pi(x) \ll_A \pi(x) \log \log x$).

In summary, we proved that

$$\sum_{\substack{p \leq x \\ p \nmid N_A}} (\nu(a_1,p) - \log \log p)^2 \ll_A \pi(x) \log \log x. \tag{25}$$

To complete the proof of the theorem, we remark that

$$\sum_{\substack{p \leq x \\ p \nmid N_A}} (\nu(a_1,p) - \log \log p)^2 \ll \sum_{\substack{p \leq x \\ p \nmid N_A, a_1,p \neq 0}} (\nu(a_1,p) - \log \log x)^2 + \sum_{\substack{p \leq x \\ p \nmid N_A, a_1,p \neq 0}} \left(\frac{\log x}{\log p}\right)^2$$

$$\ll_A \pi(x) \log \log x,$$

after using (25) in the first sum and splitting the last sum over $p$ into a sum over $p \leq \sqrt{x}$ and one over $\sqrt{x} < p \leq x$, followed by elementary estimates.

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