Discrimination of symmetric states in operational probabilistic theory

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Abstract—A state discrimination problem in an operational probabilistic theory (OPT) is investigated in diagrammatic terms. It is well-known that, in the case of quantum theory, if a state set has a certain symmetry, then there exists a minimum-error measurement having the same type of symmetry. However, to our knowledge, it is not yet clear whether this property also holds in a more general OPT. We show that it also holds in OPTs, i.e., for a symmetric state set, there exists a minimum-error measurement that has the same type of symmetry. It is also shown that this result can be utilized to optimize over a restricted class of measurements, such as sequential or separable measurements.

I. INTRODUCTION

Operational probabilistic theories (OPTs) and other similar theories, such as generalized probabilistic theories, provide a general framework that allows us to better understand the physical structure of quantum theory [1]–[5]. OPTs can be interpreted as a generalization of probability theory, including classical probability theory, quantum theory, and many others (such as the theory of Popescu-Rohrlich boxes [6]). One of the motivations for using OPTs is to investigate quantum processes from an operational point of view, which helps us to deeply understand quantum theory. Another motivation is that an OPT might be useful in developing new physical theories, such as a theory of quantum gravity.

One of the fundamental problems in probability theory is the state discrimination. In the case of quantum theory, a vast number of studies have been carried out to obtain an optimal measurement with respect to some criteria (e.g., [7]–[16]). Although obtaining a closed-form analytical solution for an optimal measurement is generally very difficult, it is known that if a state set has a certain symmetry, then there exists an optimal measurement having the same type of symmetry (e.g., [11]–[14], [17]–[25]). This property allows us to simplify finding an optimal measurement for a symmetric state set analytically and/or numerically. However, to our knowledge, this result has not been extended to a more general OPT. Note that it would not be surprising if this result does not hold in general since the space of states and that of effects are not symmetric in the case of a general system of OPTs, while they are highly symmetric in the case of any quantum system.

In this paper, we investigate a state discrimination problem in an OPT; we consider the case in which a state set has a certain symmetry, in which case we show that there exists an optimal measurement that has the same type of symmetry. This result can be proved without reference to specific algebraic structures such as Hilbert spaces and operator algebras. We also show that this result can be applied to the discrimination problem over a restricted class of measurements. As examples, we discuss four classes of measurements: sequential, local operations and classical communication (LOCC), separable, and partially transformable (PT). It is worth noting that, in this paper, we restrict our attention to the minimum-error strategy to simplify the discussion. However, the results given in this paper can be easily applied to other various criteria (see [16], [26]) in OPTs.

II. BRIEF SUMMARY OF OPERATIONAL PROBABILISTIC THEORIES (OPTs)

In this section, we briefly review the framework of OPTs. The framework can be explained in several ways, leading to essentially almost the same formalism. The proofs of some of the results are not presented in this paper, which can be found in, e.g., Refs. [1], [4], [27]–[29]. Note that we consider only fixed causal structure. We use diagrammatic representations that are used in Ref. [29] to represent formulae in an intuitive way, which is motivated by the work of Coecke, Abramsky, and others (see, e.g., [30]–[32]).

A. Systems and processes

An OPT consists of a collection of systems and a collection of processes. Systems and processes respectively represent a physical system (e.g., a photon) and a particular behavior of a physical process (e.g., a beam splitter). Each process has input and output systems. Let $\text{Proc}_{A \rightarrow B}$ be the set of all processes having an input system $A$ and an output system $B$, referred to as processes from $A$ to $B$. A trivial (or empty) system, denoted by $I$, is a special system. A process from $I$ to $A$, denoted like $|\psi\rangle$, is called a state of $A$. Similarly, a process from $A$ to $I$, denoted like $|\psi\rangle$, is called an effect of $A$. $\text{St}_A := \text{Proc}_{I \rightarrow A}$
and $\text{Eff}_A \coloneqq \text{Proc}_{A \to I}$ are, respectively, called the state space and effect space of system $A$. A process from $I$ to $I$ is called a scalar. Let $\text{Scalar} \coloneqq \text{Proc}_{I \to I}$.

In diagrammatic terms, a process $f \in \text{Proc}_{A \to B}$, a state $|\rho\rangle \in \text{St}_A$, an effect $|e\rangle \in \text{Eff}_A$, and a scalar $\rho \in \text{Scalar}$ are depicted as

$$
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array}
\quad
\begin{array}{c}
|e\rangle \\
\downarrow \\
e
\end{array}
\quad
\begin{array}{c}
\rho \\
\downarrow \\
p
\end{array}.
\tag{1}
$$

Labeled wires (labels are often omitted) represent systems, while boxes represent processes. Each process has an input wire at the bottom and an output wire at the top. $I$ is represented by ‘no wire’. For a scalar, the box will be omitted. Diagrammatic representations can be interpreted such as data flow diagrams, where time increases from the bottom to the top.

**Example of (fully) quantum theory** For simplicity, we consider only finite-dimensional systems in the examples of quantum theory. Let $\mathcal{C}$ be the set of all complex numbers. Also, let $\text{St}(\mathcal{C}^n)$ and $\text{Eff}(\mathcal{C}^n)$ be, respectively, the sets of all complex Hermitian matrices and all complex positive semidefinite matrices of order $n$. $\text{St}_A$ and $\text{Eff}_A$ are isomorphic to $\text{St}_A(\mathcal{C}^{N_A})$, where $N_A$ is a natural number determined by a system $A$. Note that $\text{St}_A(\mathcal{C}^{N_A})$ is a symmetric cone, whose shape is highly symmetric. In particular, $N_I = 1$ holds, i.e., $\text{St}_I = \text{Scalar} \cong \text{St}_A(\mathcal{C}) \cong \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all nonnegative real numbers. $\text{Proc}_{A \to B}$ is isomorphic to the space of all CP maps from $\text{St}(\mathcal{C}^{N_A})$ to $\text{St}(\mathcal{C}^{N_B})$. In the examples of quantum theory, we will identify a process with its corresponding CP map. Also, we will identify a state (or effect) with the corresponding positive semidefinite matrix.

**B. Sequential and parallel compositions**

Two processes can be composed sequentially whenever the output system of one and the input system of the other are the same. The sequential composition of $f \in \text{Proc}_{A \to B}$ and $g \in \text{Proc}_{B \to C}$ is also a process, denoted as $g \circ f \in \text{Proc}_{A \to C}$. When we write $g \circ f$, we always assume that the output system of $f$ and the input system of $g$ are equal. For any $|\rho\rangle \in \text{St}_A$ and $(e) \in \text{Eff}_A$, $(e) \circ |\rho\rangle \in \text{Scalar}$ is denoted by $(e|\rho\rangle)$. $g \circ f$ and $(e|\rho\rangle)$ are respectively depicted as

$$
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array}
\quad
\begin{array}{c}
|e\rangle \\
\downarrow \\
e
\end{array}
\quad
\begin{array}{c}
\rho \\
\downarrow \\
|\rho\rangle
\end{array}.
\tag{2}
$$

Any two systems and processes can be composed in parallel. The parallel composition of two systems, $A$ and $B$, is a system, denoted by $A \otimes B$. Assume that $f \otimes A = A \otimes f$ holds. The parallel composition of $f \in \text{Proc}_{A \to B}$ and $g \in \text{Proc}_{C \to D}$ is a process from $A \otimes C$ to $B \otimes D$, denoted as $f \otimes g$. $f \otimes g$ is diagrammatically depicted as

$$
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array}
\quad
\begin{array}{c}
|e\rangle \\
\downarrow \\
e
\end{array}
\quad
\begin{array}{c}
\rho \\
\downarrow \\
|\rho\rangle
\end{array}.
\tag{3}
$$

A collection of connected processes will be called a diagram. These sequential and parallel compositions are associative, e.g., $(h \circ g \circ f) = (h \circ g) \circ f$ holds for any $f \in \text{Proc}_{A \to B}$, $g \in \text{Proc}_{B \to C}$, and $h \in \text{Proc}_{C \to D}$. Assume that

$$
(g_1 \otimes g_2) \circ (f_1 \otimes f_2) = (g_1 \circ f_1) \otimes (g_2 \circ f_2),
$$

or diagrammatically

$$
\begin{array}{c}
|g_1\rangle \\
\downarrow \\
g_1
\end{array}
\quad
\begin{array}{c}
|f_1\rangle \\
\downarrow \\
f_1
\end{array}
\quad
\begin{array}{c}
|g_2\rangle \\
\downarrow \\
g_2
\end{array}
\quad
\begin{array}{c}
|f_2\rangle \\
\downarrow \\
f_2
\end{array}
\quad
\begin{array}{c}
|g_1\rangle \otimes |g_2\rangle \\
\downarrow \\
g_1 \otimes g_2
\end{array},
\tag{4}
$$

holds for any processes $f_1$, $f_2$, $g_1$, and $g_2$, where the auxiliary lines (dashed lines) are drawn to guide the eye. For any scalar $a$ and process $f$, $a \circ f$ is denoted by $a f$ or $a \cdot f$. One can see that $(a g) \circ f = a(g \circ f) = g \circ (af)$, $(af) \otimes h = a(f \otimes h) = f \otimes (ah)$, and $a \otimes b = ab$ hold for any scalars $a$ and $b$ and any processes $f$, $g$, and $h$.

**Example of quantum theory** $N_{A \otimes B} = N_A N_B$ holds for any systems $A$ and $B$. Since $\text{St}_{\mathbb{C}} = \text{St}_{\mathbb{C}^d} \cong \text{Eff}_{\mathbb{C}^d} \cong \text{St}_{\mathbb{C}^d}$ holds from $N_I = 1$, $I \otimes A$ and $A \otimes I$ can be identified with $A$. For two processes $f$ and $g$, $g \circ f$ is the CP map satisfying $(g \circ f)[|\rho\rangle] := g[f(|\rho\rangle)]$. In particular, $(e|\rho\rangle) = Tr[|e\rangle \cdot |\rho\rangle]$ holds ($\cdot$ is the matrix product). For two states $|\rho\rangle$ and $|\sigma\rangle$, $|\rho\rangle \otimes |\sigma\rangle$ is the tensor product of the matrices $|\rho\rangle$ and $|\sigma\rangle$. $f \otimes g$ is the CP map defined as $(f \otimes g)[|\rho\rangle] \otimes |\sigma\rangle] := f[|\rho\rangle] \otimes g[|\sigma\rangle]$.

**C. Identity processes and discarding effects**

An identity process on $A$, denoted by $I_A$ or simply $id_A$, is the process satisfying $(a) \circ I_A = a = I_A \circ (a)$. $id_A \otimes id_B = id_{A \otimes B}$, and $(f) \circ id_A = f = id_B \circ f$, where $A$ and $B$ are any systems and $f$ is any process from $A$ to $B$. Assume that there exists $id_A$ for each system $A$. Diagrammatically, $id_A$ is depicted as

$$
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array}.
\tag{5}
$$

$id_A$ is depicted as empty space. The above property (a) is depicted as

$$
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array} =
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array} =
\begin{array}{c}
|\rho\rangle \\
\downarrow \\
\rho
\end{array}.
\tag{6}
$$

where the auxiliary boxes indicate the identity processes. This, intuitively, implies that the length of lines does not
change diagrams. It also follows from Eq. (7) that, for any processes \( f \) and \( g \),

\[
\begin{array}{c}
g \cdot f = \begin{array}{c}
g \\
\end{array} \begin{array}{c}
f \\
\end{array} = \begin{array}{c}
g \\
\end{array} \begin{array}{c}
f \\
\end{array} = \begin{array}{c}
g \\
\end{array} \begin{array}{c}
f \\
\end{array} \\
\end{array}
\]

(8)

holds. Intuitively, this yields that the vertical shifts of processes do not affect diagrams.

Assume that, for any systems \( A \) and \( B \), there exists a process \( \times_{A \# B} \in \text{Proc}_{A \# B \rightarrow B \# A} \), called a swap process and diagrammatically depicted by

\[
\begin{array}{c}
B \\
\end{array} \begin{array}{c}
A \\
\end{array} = \begin{array}{c}
B \\
\end{array} \begin{array}{c}
A \\
\end{array},
\]

(9)

such that

\[
\begin{array}{c}
B \\
\end{array} \begin{array}{c}
A' \\
\end{array} = \begin{array}{c}
B \\
\end{array} \begin{array}{c}
A \\
\end{array} = \begin{array}{c}
B \\
\end{array} \begin{array}{c}
A' \\
\end{array}
\]

(10)

(i.e., \( \times_{A',B} = (f \otimes \text{id}_A) \circ \times_{A,B} = \text{id}_B \otimes f \)) holds for any systems \( A, A' \), and \( B \) and any process \( f \in \text{Proc}_{A \rightarrow A'} \). Also, assume that \( \times_{A,B} = \text{id}_A \), i.e.,

\[
\begin{array}{c}
A \\
\end{array} = \begin{array}{c}
A \\
\end{array}
\]

(11)

and \( \times_{A,B,C} = (\text{id}_B \otimes \times_{A,C}) \circ (\times_{A,B} \otimes \text{id}_C) \) hold for any systems \( A, B, \) and \( C \).

Let us consider a trivial measurement of system \( A \), which always gives the same outcome regardless of the input state. The effect representing an event associated to the outcome of a trivial measurement is called a discarding effect and denoted by \( \oplus_A \), or simply \( \oplus \), which is depicted as

\[
\begin{array}{c}
\oplus \\
\end{array}
\]

(12)

\( \oplus_A \) has a natural operational intuition: one performs any measurement on system \( A \) and then discards the results.

Example of quantum theory The identity process is the identity map. The discarding effect of \( A \) is the identity matrix of order \( N_A \), denoted by \( 1_{N_A} \).

D. Probabilistic behavior

Assume that \( \text{Scalar} \equiv \mathbb{R}_+ \) holds, i.e., each scalar is identified with a nonnegative real number. Let \( \text{Scalar}^\text{F} := \{ p \in \text{Scalar} : p \leq 1 \} \). A state \( \rho \in \text{St}_A \) is called feasible if \( (\oplus_A)\rho \in \text{Scalar}^\text{F} \) holds; in particular, \( \rho \) is called normalized (or deterministic) if \( (\oplus_A)\rho = 1 \) holds. Let \( \text{St}_A^\text{F} \) and \( \text{St}_A^\text{N} \) be, respectively, the sets of all feasible and normalized states of \( A \). A process \( f \in \text{Proc}_{A \rightarrow B} \) is called feasible if \( (f \otimes \text{id}_E) \circ |\sigma\rangle \in \text{St}_{B \# E}^\text{F} \) holds for any system \( E \) and \( |\sigma\rangle \) \in \text{St}_{A \# E}^\text{N}. Also, \( f \in \text{Proc}_{A \rightarrow B} \) is called deterministic if \( (f \otimes \text{id}_E) \circ |\sigma\rangle \in \text{St}_{B \# E}^\text{N} \) holds for any system \( E \) and \( |\sigma\rangle \in \text{St}_{A \# E}^\text{N}. \)

Let \( \text{Proc}_{A \rightarrow B}^\text{F} \) and \( \text{Proc}_{A \rightarrow B}^\text{D} \) be, respectively, the sets of all feasible and deterministic processes from \( A \) to \( B \). Also, let \( \text{Eff}^\text{F} := \text{Proc}_{A \rightarrow B}^\text{F} \). Assume that each scalar consisting of the sequential and/or parallel compositions of \( k \) feasible processes \( f_1, \ldots, f_k \), for example, the scalar depicted by

\[
\begin{array}{c}
f_k \\
\end{array} \begin{array}{c}
f_s \\
\end{array} = \begin{array}{c}
f_k \\
\end{array} \begin{array}{c}
f_s \\
\end{array} = \begin{array}{c}
f_k \\
\end{array} \begin{array}{c}
f_s \\
\end{array} \begin{array}{c}
f_k \\
\end{array} \begin{array}{c}
f_s \\
\end{array} \begin{array}{c}
f_k \\
\end{array} \begin{array}{c}
f_s \\
\end{array}
\]

(13)

is the probability of the joint occurrence of \( f_1, \ldots, f_5 \). In particular, for any \( p, q \in \text{Scalar}^\text{F} \), \( pq \), i.e., the probability of the joint occurrence of \( p \) and \( q \), is the product of real numbers \( p \) and \( q \). Assume \( (\oplus_A) = 1 \) and \( (\oplus_A) \in \text{Proc}_{A \rightarrow A}^\text{D} \). 1 is the unique deterministic scalar and \( \text{id}_A \) holds. It follows that any process consisting of the sequential and/or parallel compositions of deterministic (resp. feasible) processes is deterministic (resp. feasible). One can easily see \( \text{St}_A^\text{N} = \text{Proc}_{A \rightarrow A}^\text{F} \), \( \text{Scalar}^\text{F} = \text{St}_A^\text{F} = \text{Proc}_{A \rightarrow A}^\text{D} \), \( \text{St}_A^\text{N} = \text{Proc}_{A \rightarrow A}^\text{F} \), and \( \text{St}_A^\text{D} = \text{Proc}_{A \rightarrow A}^\text{D} \). Any \( \rho \in \text{St}_A^\text{F} \) is in the form \( \rho = p|\rho^N\rangle \) with \( p \in \text{Scalar}^\text{F} \) and \( |\rho^N\rangle \in \text{St}_A \) (note that \( p = (\oplus_A)\rho \) holds from \( (\oplus_A)\rho = 1 \)), which means that \( \rho \) can be identified with the process preparing the normalized state \( |\rho^N\rangle \) with probability \( p \).

From the definition, a scalar larger than 1 is unfeasible. Unfeasible scalars cannot be interpreted as probabilities and thus are not intuitive. However, it is mathematically convenient to consider unfeasible scalars, so we assume \( \text{Scalar} = \mathbb{R}_+ \). Similarly, \( \text{Proc}_{A \rightarrow B} \) is defined as

\[
\text{Proc}_{A \rightarrow B} := \{ f : a \in \text{Scalar}, f \in \text{Proc}_{A \rightarrow B}^\text{F} \}.
\]

(14)

Although unfeasible processes exist in each process space \( \text{Proc}_{A \rightarrow B} \) (i.e., \( \text{Proc}_{A \rightarrow B} \subseteq \text{Proc}_{A \rightarrow B}^\text{F} \)), Eq. (14) implies that any unfeasible process is expressed as scalar multiplication of a feasible process.

We consider the following diagram, denoted by \( u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar} \), that maps a process \( f \in \text{Proc}_{A \rightarrow B} \) to a scalar \( u(f) := \langle u_2 \rangle \circ (f \circ \text{id}_E) \circ |u_1\rangle \in \text{Scalar} \), where \( |u_1\rangle \in \text{St}_{A \# E}^\text{N} \) and \( |u_2\rangle \in \text{Eff}_{B \# E}^\text{F} \). \( u \) can be interpreted as a set of a system \( E \), a state \( |u_1\rangle \) \in \text{St}_{A \# E}^\text{N}, and an effect \( |u_2\rangle \in \text{Eff}_{B \# E}^\text{F} \), which is diagrammatically depicted as

\[
\begin{array}{c}
\text{St}_{A \# E} \\
\end{array} = \begin{array}{c}
\text{St}_{A \# E} \\
\end{array} = \begin{array}{c}
\text{St}_{A \# E} \\
\end{array} = \begin{array}{c}
\text{St}_{A \# E} \\
\end{array} = \begin{array}{c}
\text{St}_{A \# E} \\
\end{array}
\]

(15)

Any scalar that includes \( f \in \text{Proc}_{A \rightarrow B} \) is expressed in the form \( u(f) \) with some diagram \( u : \text{Proc}_{A \rightarrow B} \rightarrow \text{Scalar} \), for

\(^1\)In OPTs, a certain set of scalars is associated with a probability distribution. See, e.g., Ref. [3] for details.
where \( |u_1\rangle \) and \( |u_2\rangle \) are, respectively, the state and effect enclosed by the auxiliary boxes.

For two processes \( f, f' \in \text{Proc}_{A \rightarrow B} \), \( f = f' \) is defined as

\[
\frac{\rho}{|f\rangle} = \frac{\rho}{|f'\rangle} \quad \Leftrightarrow \quad \forall p \in \mathbb{I}, \quad \frac{e}{|f\rangle} = \frac{e}{|f'\rangle} .
\]

(17)

This means that \( f = f' \) holds if they are indistinguishable in a probabilistic sense. For \( f, f' \in \text{Proc}_{A \rightarrow B} \), \( f_{\text{local}} = f' \) is defined as

\[
\frac{\rho}{|f\rangle} = \frac{\rho}{|f'\rangle} \quad \Leftrightarrow \quad \forall p \in \mathbb{I}, \quad \frac{e}{|f\rangle} = \frac{e}{|f'\rangle} .
\]

(18)

One can easily see that \( f_{\text{local}} = f' \) holds if \( f = f' \) holds, but the converse is not necessarily true. It follows that, in the case of \( A = I \) or \( B = I \), \( f_{\text{local}} = f' \) and \( f = f' \) are always the same, which means

\[
\frac{e}{|f\rangle} = \frac{e}{|f'\rangle} \quad \Leftrightarrow \quad \forall p \in \mathbb{I}, \quad \frac{e}{|f\rangle} = \frac{e}{|f'\rangle} .
\]

(19)

\[
\frac{e}{|f\rangle} = \frac{e}{|f'\rangle} \quad \Leftrightarrow \quad \forall p \in \mathbb{I}, \quad \frac{e}{|f\rangle} = \frac{e}{|f'\rangle} .
\]

(20)

\( (\Downarrow_A) \) is the unique deterministic effect of \( A \) and that \( (\Downarrow_{A \otimes B}) := (\Downarrow_A) \otimes (\Downarrow_B) \) holds, which is depicted as

\[
\frac{\rho}{|f\rangle} = \frac{\rho}{|g\rangle} .
\]

(21)

Example of quantum theory. Since \( (\Downarrow_A)[\rho] = \text{Tr} [\rho] \) holds, \( [\rho] \in \text{St}_A \) means \( \text{Tr} [\rho] \leq 1 \). Also, \( [\rho] \in \text{St}_A \) means \( \text{Tr} [\rho] = 1 \). \( f \in \text{Proc}_{A \rightarrow B}^f \) holds if and only if \( f \) is a trace non-increasing CP map. Also, \( f \in \text{Proc}_{A \rightarrow B}^D \) means that \( f \) is a trace-preserving (TP) CP map. \( [e] \in \text{Eff}_{A \rightarrow B}^f \) means that the maximal eigenvalue of the matrix \( (e) \) is not larger than 1. For any \( f, f' \in \text{Proc}_{A \rightarrow B} \), \( f = f' \) and \( f \equiv f' \) are equivalent.

E. Process space spans vector space

Assume that, for any two feasible processes \( g_1, g_2 \in \text{Proc}_{A \rightarrow B}^f \) and any \( p \in \text{Scalar}^f \), there exists a feasible process \( h \in \text{Proc}_{A \rightarrow B}^f \) satisfying

\[
h \quad \Leftrightarrow \quad \sum_{m=1}^{N} \frac{e_m}{p} = 1 , \quad \forall p \in \text{St}_A^N
\]

(22)

for any \( u \in \text{Proc}_{A \rightarrow B} \). Such a process \( h \) is denoted by \( pg_1 + (1-p)g_2 \). This can be interpreted as a probabilistic mixture of \( g_1 \) and \( g_2 \) with probabilities \( p \) and \( 1 - p \). For any two processes \( f_1, f_2 \in \text{Proc}_{A \rightarrow B} \), the process \( f' \) satisfying

\[
\sum_{m=1}^{N} \frac{e_m}{p} = 1 , \quad \forall p \in \text{St}_A^N
\]

(23)

holds. This means that the sum of probabilities over all possible outcomes is 1 whenever one performs a measurement on a normalized state. Equation (24) is equivalent to

\[
\sum_{m=1}^{N} \frac{e_m}{p} = \sum_{p} \frac{1}{p} \in \text{St}_A^N
\]

(25)

and thus, from Eq. (20), it follows that \( \{e_m\} \in \text{Eff}_{A \rightarrow B}^f \) is a measurement if and only if

\[
\sum_{m=1}^{N} \frac{e_m}{p} = \sum_{p} \frac{1}{p} \in \text{St}_A^N
\]

(26)

We can consider the real vector space \( \mathbf{V}_{A \rightarrow B} \) spanned by \( \text{Proc}_{A \rightarrow B} \), whose elements are formal sums of the form \( \sum_{i=1}^{n} a_i f_i \) with \( a_i \in \mathbb{R} \) and \( f_i \in \text{Proc}_{A \rightarrow B} \), where the element \( f := \sum_{i=1}^{n} a_i f_i \) satisfies

\[
\sum_{j=1}^{n} a_j f_j \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_j f_j \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_j f_j
\]

(27)
over addition. \( \text{Proc}_{A \rightarrow B} \subseteq V_{A \rightarrow B} \) obviously holds. Any \( f \in V_{A \rightarrow B} \) is expressed by \( f = f_1 + f_2 \) with some \( f_1, f_2 \in V_{A \rightarrow B} \). As well as processes, extended processes can be composed sequentially and in parallel. Specifically, for any \( f \in V_{B \rightarrow C}, \) \( g \in \text{Proc}_{B \rightarrow C} \), and \( h \in \text{Proc}_{C \rightarrow D} \), \( f \circ g \in \text{Proc}_{A \rightarrow D} \) holds. Let \( V_A := V_{A \rightarrow A} \) and \( V_A^* := V_{A \rightarrow A}^* \); then, \( V_A \) can be regarded as the dual vector space of \( V_A \). We can easily verify that, for any systems \( A \) and \( B \), \( \text{Proc}_{A \rightarrow B} \) is a convex cone. In particular, \( \text{St}_A \) is a convex cone in \( V_A \). The dimension of the real vector space \( V_A \) is called the dimension of \( A \). Assume that \( \text{Proc}_{A \rightarrow B} \) is closed.

### Example of quantum theory

The sum of processes is equal to the sum of CP maps. In particular, the sum of states (or effects) is the sum of matrices. \( \Pi := \{ |e_m \rangle \in \text{Eff}[A]_{\text{meas} \mathcal{T}_m} \} \) is a measurement if and only if \( \sum_{m=1}^{M} |e_m \rangle = I_{K_m} \) holds, i.e., \( \Pi \) is a positive operator-valued measure (POVM) (note that each effect \( |e_m \rangle \) is a positive semidefinite matrix). In fully quantum theory, \( V_{A \rightarrow B} \) is isomorphic to the space of all linear maps from \( S(\mathbb{C}^N) \) to \( S(\mathbb{C}^N) \) (which are also called Hermitian-preserving maps); in particular, \( V_A \) and \( V_A^* \) are isomorphic to \( S(\mathbb{C}^N) \). \( |\psi \rangle \in \text{St}_A \) is pure if and only if \( \langle \psi | \psi \rangle \) holds for some vector \( |\psi \rangle \).

### III. OPTs with classical systems

#### A. Classical systems

We will call an \( M \)-dimensional system \( C \) classical if there exist \( M \) normalized pure states \( |1 \rangle, \ldots, |M \rangle \in \text{St}_C^\text{NP} \) and a measurement \( \{ |m \rangle \in \text{Eff}[C]_{\text{meas} \mathcal{T}_m} \} \) satisfying

\[
\begin{bmatrix}
    m \\
    n
\end{bmatrix} = \delta_{mn}
\]

and

\[
\sum_{m=1}^{M} \begin{bmatrix}
    m \\
    n
\end{bmatrix} = \begin{bmatrix}
    M \\
    M
\end{bmatrix},
\]

where \( \delta_{m,n} \) is the Kronecker delta. A classical system is depicted as the dotted line. One can easily see that any \( |\varphi \rangle \in \text{St}_C \) is expressed in the form

\[
|\varphi \rangle = \sum_{m=1}^{M} P_{m} |m \rangle,
\]

where \( P_{m} := \langle m | \rho \rangle \) is in \( \text{Scalar} \). This immediately gives that \( \text{St}_C^\text{NP} = \{ |m \rangle \}_{m=1}^{M} \) holds. Indeed, from Eq. (35), \( |\varphi \rangle \in \text{St}_C^\text{NP} \) holds if and only if \( P_{m} = \delta_{m,\varphi} \) holds for some \( i \in \mathcal{T}_M \).

In what follows, we consider an OPT that has an \( M \)-dimensional classical system \( C \). Note that a classical system is not intrinsically necessary for investigating a state discrimination problem in an OPT, but it helps us to express this problem in straightforward diagrammatic terms.

It follows from Eq. (34) that any state of \( C \otimes A \) (where \( A \) is an arbitrary system that is not classical in general) is separable. Indeed, one can easily obtain

\[
|\sigma \rangle = \sum_{m=1}^{M} \begin{bmatrix}
    m \\
    m
\end{bmatrix} \begin{bmatrix}
    P_{m} \\
    P_{m}
\end{bmatrix},
\]

with \( g_i \in \text{Proc}_{A \rightarrow C} \) and \( h_i \in \text{Proc}_{B \rightarrow D} \).
where the state enclosed by the auxiliary box is denoted by $|\psi_m\rangle$. Similarly, any state of $A \otimes C$, effect of $C \otimes A$, and effect of $A \otimes C$ are separable. Moreover, for any $f \in \text{Proc}_{C \rightarrow A}$ with a classical system $C$, $|\sigma\rangle \in \text{St}_{C \otimes E}$, and $(e) \in \text{Eff}_{A \otimes E}$, Eq. (36) yields

$$f = f' \iff f^\text{local} = f'$$

and

$$\sum_{n=1}^{M} \rho_n^m = \sum_{n=1}^{M} \frac{f^\text{local}}{m} \rho_n^m = \sum_{n=1}^{M} \frac{f^\text{local}}{m} \rho_n^m$$

(37)

where the effect enclosed by the auxiliary box is denoted by $(e_m)$. One can see from Eq. (37) that, for any $f, f' \in \text{Proc}_{C \rightarrow A}$,

$$f = f' \iff f^\text{local} = f'$$

(38)

holds. Similarly, Eq. (38) holds for any $f, f' \in \text{Proc}_{A \rightarrow C}$ with a classical system $C$.

**B. $|U\rangle$ and $(|\cap\rangle$**

We will introduce the state $|U\rangle$ and the effect $|\cap\rangle$ defined as

$$|U\rangle := \sum_{m=1}^{M} (|I\rangle I) \otimes (|I\rangle I)$$

(39)

Clearly, we have

$$|U\rangle := \sum_{m=1}^{M} (|I\rangle I) \otimes (|I\rangle I)$$

(40)

and

$$|U\rangle := \sum_{m=1}^{M} (|I\rangle I) \otimes (|I\rangle I)$$

(41)

(42)

One can also show

$$\sum_{n=1}^{M} \frac{n}{m} \rho_n^m = \delta_{m,n}$$

(43)

Intuitively, this means that a curved line consisting of ‘U’ and ‘\cap’ can be yanked. The first equality of Eq. (43) is obtained from

$$\sum_{m=1}^{M} \frac{n}{m} \rho_n^m = \sum_{m=1}^{M} \frac{n}{m} \rho_n^m = \sum_{m=1}^{M} \frac{n}{m}$$

(44)

The same is true for the second equality of Eq. (43).

$$\sum_{m=1}^{M} \left(\sum_{n=1}^{M} \frac{n}{m} \right) \rho_n^m$$

(45)

then, we have

$$\sum_{m=1}^{M} \left(\sum_{n=1}^{M} \frac{n}{m} \right) \rho_n^m$$

(46)

**C. Expression of measurements**

In an OPT with a classical system $C$, any measurement $\{\rho_{m}\}_{m \in A}$ can be expressed by the process $e := \sum_{m=1}^{M} |n\rangle \langle m| \in \text{Proc}_{A \rightarrow C}$. Diagrammatically,

$$e := \sum_{m=1}^{M} \frac{n}{m} \rho_n^m$$

(47)

Each effect $(e_m)$ with $m \in I_M$ is obtained from

$$e_m := \sum_{n=1}^{M} \frac{n}{m} \rho_n^m$$

(48)

One can easily verify that $(\chi_A) \circ e = (\chi_A)$, i.e., $e$ is deterministic from Eq. (31). Conversely, any deterministic process from $A$ to $C$ can be interpreted as a measurement since $e' \in \text{Proc}_{A \rightarrow C}$ can be depicted as

$$e' := \sum_{m=1}^{M} \frac{n}{m} \rho_n^m$$

(49)

where $(e'_m) := \langle n | e' \rangle$. For each system $A$, $\text{Proc}_{A \rightarrow C}$ is denoted by $\text{Meas}_A$ or simply $\text{Meas}_A$.

**D. State preparation**

We assume that one of $M$ normalized states of system $A$, $|\psi_1^A\rangle, \ldots, |\psi_M^A\rangle \in \text{St}_A$, is prepared with prior probabilities $\xi_1, \ldots, \xi_M$ with $\sum_{m=1}^{M} \xi_m = 1$. Let $\rho_m := \xi_m |\psi_m^A\rangle \langle \psi_m^A|$. We consider the process $\rho \in \text{Proc}_{C \rightarrow A}$ defined as

$$\rho := \sum_{m=1}^{M} \frac{n}{m} \rho_n^m = \sum_{m=1}^{M} \xi_m \rho_n^m$$

(50)
Each $|\rho_n\rangle$ is obtained from

$\rho_n = \frac{1}{n} \rho$.

(51)

Note that $\rho$ is not deterministic unless $M = 1$. We have

$\rho = \sum_{m=1}^{M} \frac{M}{m} \rho_m$.

(52)

$\sum_{m=1}^{M} |\rho_m\rangle$ is obviously normalized. We will call a process $\rho \in \text{Proc}_{C \rightarrow A}$ with $\rho \circ |\frac{1}{2}\rangle \in \text{St}_C^N$ a state preparation. Let $\text{Prep}_A$ be the set of all state preparations.

IV. MINIMUM-ERROR MEASUREMENT FOR GROUP COVARIANT STATES

A. Discrimination problem

Let us review the problem of discriminating a given set of $M$ known normalized states with given prior probabilities [33]–[35]. Here, we consider the following scenario: One party (Charlie) randomly chooses one of the $M$ states $|\psi_1^N\rangle, \ldots, |\psi_M^N\rangle \in \text{St}_C^N$ with prior probabilities $\xi_1, \ldots, \xi_M$. Such a process is expressed by the state preparation $\rho \in \text{Prep}_A$, depicted by Eq. (50). Since she knows which state she has, we can interpret that she has the following state

$\sum_{m=1}^{M} \frac{M}{m} \rho_m = \rho$.

(53)

Indeed, by performing a measurement $(|m\rangle \langle m|)_{m \in I_M}$ on the classical system $C$, she can always determine which state she has. Charlie sends the state to the other party (Alice). Alice knows the possible states $|\psi_1^N\rangle, \ldots, |\psi_M^N\rangle$ and their prior probabilities but does not know which state Charlie sent. We can interpret that Alice gets the following normalized state

$\rho = \frac{1}{n} \rho$.

(54)

What Alice has to do is to perform a measurement that will correctly discriminate between the states $|\psi_1^N\rangle, \ldots, |\psi_M^N\rangle$ with high probability. Alice performs a measurement $e := \sum_{m=1}^{M} |m\rangle \langle m| \in \text{Meas}_A$ with $e \in \text{Eff}_A$ to discriminate between the states as accurately as possible. After that, Charlie and Alice check whether Alice correctly determines the state. In this paper, we apply the strategy that maximizes the average success probability. This probability is given as a function of the measurement $e$, denoted by $P(e)$, which is expressed by

$P(e) := \frac{1}{2} \sum_{m=1}^{M} (e \circ \rho \circ |m\rangle \langle m|).

(51)

Note that $P(e)$ is also expressed by

$P(e) = \sum_{m=1}^{M} (e \circ \rho \circ |m\rangle \langle m|).

(52)

A measurement that maximizes the average success probability is called a minimum-error measurement. The problem of finding a minimum-error measurement is formulated as

maximize $P(e)$

subject to $e \in \text{Meas}_A$.

(53)

with variable $e$. If we want to optimize over a restricted class of measurements, denoted by $M_A$ with $M_A \subseteq \text{Meas}_A$, then we consider the following problem:

maximize $P(e)$

subject to $e \in M_A$.

(54)

B. Group action

We use group theory to represent the symmetric properties of a state preparation. Let $G$ be a group and $I \in G$ be its identity element. Let $\text{Aut}(Z)$ be the group of all automorphisms of an object $Z$. A map $\psi : G \ni g \rightarrow \psi_g : \text{Aut}(Z)$ is called a (right) group action of $G$ on $Z$ if $\psi_{gh}(z) = \psi_h[\psi_g(z)]$ $(\forall g, h \in G)$ and $\psi_z(z) = z$ hold for any $z \in Z$. In the following, we give two examples of group actions.

The first example is a group action, $T : G \rightarrow \text{Aut}(I_M)$, of $G$ on $I_M$. $\text{Aut}(I_M)$ is the group of all permutations of $I_M$. We will identify $I_M$ with $\{ |m\rangle \in \text{Eff}_C \}_{m \in I_M}$; then, we have

$\tau_1 = 1$.

(55)

Also, we have that, for any $g, h \in G$,

$\forall m \in I_M, \tau_g \circ \tau_h = \tau_{gh}$.

(56)
which means \( \tau_g \circ \tau_h \overset{\text{local}}{=} \tau_{gh} \). Thus, from Eq. (38) with \( A = C \), we have

\[
\begin{align*}
\tau_g \cdot \tau_b &= \tau_{gh} \\
\tau_b &= \tau_{gh}^{-1}.
\end{align*}
\]

(61)

Clearly, for any \( g \in G \), \( \tau_g \) is deterministic and \( \tau_{g^{-1}} \) is the inverse of \( \tau_g \). Note that we also have

\[
\begin{align*}
\forall \ m \in I_M, \\
\tau_g \cdot \tau_b &= \tau_{gh} \\
\tau_b &= \tau_{gh}^{-1}.
\end{align*}
\]

(62)

We can easily verify

\[
\begin{align*}
\tau_g &= \tau_{g^{-1}}
\end{align*}
\]

(63)

Indeed, from Eq. (39), we have that, for any \( m, n \in I_M \),

\[
\begin{align*}
\tau_g &= \delta_{\tau_g(m),n} = \delta_{m,\tau_g^{-1}(n)} = \tau_{g^{-1}}.
\end{align*}
\]

(64)

Since any state in \( C \otimes C \) is in the form \( \sum_{m=1}^{M} \sum_{n=1}^{M} c_{m,n}|m\otimes n| \) with \( c_{m,n} \in \mathbb{R}_+ \), Eq. (64) yields Eq. (63). Similarly, we have

\[
\begin{align*}
\tau_g &= \tau_{g^{-1}}
\end{align*}
\]

(65)

The second example is a group action, \( \bar{\pi} : G \to \text{Aut}(M_4) \), of \( G \) on \( M_4 \), where \( M_4 \) is a convex subset of Meas_4. Each automorphism of \( M_4 \) is in \( V_{A\to A} \), i.e., each element of \( \text{Aut}(M_4) \) is linear. We will say that \( M_4 \subseteq \text{Meas}_4 \) is symmetric under permutations of the measurement outcomes if, for any \( h \in \text{Aut}(I_M) \) and \( e \in M_4 \), \( h\circ e \in M_4 \) holds. This means that a process that first performs the measurement \( e \in M_4 \) and then makes permutations among the measurement results is also in \( M_4 \). \( \bar{\pi} \) satisfies

\[
\begin{align*}
\bar{\pi} &= \tau_g \cdot \rho_{\bar{\pi}}
\end{align*}
\]

(66)

and

\[
\begin{align*}
\forall \ e \in M_4, \\
\bar{\pi}_g &= \bar{\pi}_h.
\end{align*}
\]

(67)

for any \( g, h \in G \). Clearly, \( \bar{\pi}_g^{-1} \) is the inverse of \( \bar{\pi}_g \). Each \( \bar{\pi} \in \text{Aut}(M_4) \) is reversible.

An example of \( M_4 \) is Meas_4 itself; Meas_4 is obviously symmetric under permutations of the measurement outcomes. It is also easily seen that the followings are equivalent:

1) \( \bar{\pi} \in \text{Aut}(M_4) \).
2) \( \bar{\pi} \in V_{A\to A} \) is reversible and satisfies \( e \circ \bar{\pi} \in \text{Meas}_4 \) for any \( e \in \text{Meas}_4 \).
3) \( \bar{\pi} \in V_{A\to A} \) is reversible, deterministic, and positive for effects, where we will call \( \bar{\pi} \in V_{A\to A} \) positive for effects if \( (e \circ \bar{\pi}) \in \text{Eff}_{A} \) holds for any \( (e \in \text{Eff}_{A} \).

Other examples of \( M_4 \) will be shown in Subsec. IV-D. Note that \( \text{Aut}(M_4) \subseteq \text{Aut}(\text{Meas}_4) \) does not hold in general; an example will be given in the example of quantum theory stated in Subsec. IV-D.

C. Symmetric properties

We consider a set \( (G, \tau, \bar{\pi}) \), where \( \tau : G \to \text{Aut}(I_M) \) and \( \bar{\pi} : G \to \text{Aut}(M_4) \) are group actions with a group \( G \).

We will say that a state preparation \( \rho := \sum_{n=1}^{N} |\psi_n\rangle \otimes (|n\rangle \in \text{Prep}_A \) is \((G, \tau, \bar{\pi})\)-covariant (or simply \( G \)-covariant) if

\[
\begin{align*}
\tau_g \cdot \rho_{\bar{\pi}} &= \rho
\end{align*}
\]

(68)

holds for any \( g \in G \). We will also say that a measurement \( e^\circ := \sum_{n=1}^{N} |\psi_n\rangle \otimes (|e^\circ \rangle \in M_4 \) is \((G, \tau, \bar{\pi})\)-covariant if

\[
\begin{align*}
\tau_g \cdot e^\circ &= e^\circ
\end{align*}
\]

(69)

holds for any \( g \in G \). Equation (68) is the same as

\[
\begin{align*}
\bar{\pi}_g &= \bar{\pi}_h
\end{align*}
\]

(70)
since

\[
\begin{array}{c}
\text{Diagram (50)} \\
\text{Diagram (41)} \\
\text{Diagram (51)} \\
\text{Diagram (42)} \\
\end{array}
\]

(71)

and

\[
\begin{array}{c}
\text{Diagram (50)} \\
\text{Diagram (41)} \\
\text{Diagram (51)} \\
\text{Diagram (42)} \\
\end{array}
\]

(72)

Note that Eq. (68) is also the same as

\[
\begin{array}{c}
\text{Diagram (68)} \\
\text{Diagram (69)} \\
\text{Diagram (70)} \\
\text{Diagram (71)} \\
\end{array}
\]

(73)

Similar results hold for \( e \); for example, Eq. (69) is the same as

\[
\begin{array}{c}
\text{Diagram (69)} \\
\text{Diagram (70)} \\
\text{Diagram (71)} \\
\text{Diagram (72)} \\
\end{array}
\]

(74)

since

\[
\begin{array}{c}
\text{Diagram (69)} \\
\text{Diagram (70)} \\
\text{Diagram (71)} \\
\text{Diagram (72)} \\
\end{array}
\]

(75)

and

\[
\begin{array}{c}
\text{Diagram (69)} \\
\text{Diagram (70)} \\
\text{Diagram (71)} \\
\text{Diagram (72)} \\
\end{array}
\]

(76)

**Theorem 1:** Let \( \mathcal{M}_A \) be a convex subset of \( \text{Meas}_A \) that is symmetric under permutations of the measurement outcomes. Let \( \tau : \mathcal{G} \to \text{Aut}(\mathcal{I}_M) \) and \( \pi : \mathcal{G} \to \text{Aut}(\mathcal{M}_A) \) be group actions with a group \( \mathcal{G} \). If a state preparation \( \rho \in \text{Prep}_A \) is \( (\mathcal{G}, \tau, \pi) \)-covariant, then, for any measurement \( e \in \mathcal{M}_A \), there exists a \( (\mathcal{G}, \tau, \pi) \)-covariant measurement \( e^\circ \in \mathcal{M}_A \) satisfying \( P(e^\circ) = P(e) \).

**Proof.** Let

\[
\begin{array}{c}
\text{Diagram (77)} \\
\text{Diagram (78)} \\
\text{Diagram (79)} \\
\text{Diagram (80)} \\
\end{array}
\]

(77)

From the definition of \( \mathcal{M}_A \), we can easily verify \( e^\circ \in \mathcal{M}_A \). Equation (77) yields

\[
\begin{array}{c}
\text{Diagram (78)} \\
\text{Diagram (79)} \\
\text{Diagram (80)} \\
\end{array}
\]

(78)

and thus

\[
\begin{array}{c}
\text{Diagram (78)} \\
\text{Diagram (79)} \\
\text{Diagram (80)} \\
\end{array}
\]

(79)

holds for any \( g \in \mathcal{G} \), where the last equality follows from \( \{ h : h \in \mathcal{G} \} = \mathcal{G} = \{ h g : h \in \mathcal{G} \} \). Thus, \( e^\circ \) is \( (\mathcal{G}, \tau, \pi) \)-covariant. Also, we obtain

\[
\begin{array}{c}
\text{Diagram (78)} \\
\text{Diagram (79)} \\
\text{Diagram (80)} \\
\end{array}
\]

(80)

which gives \( P(e^\circ) = P(e) \). \( \square \)

Theorem 1 immediately yields the following corollary.

**Corollary 2:** Let \( \mathcal{M}_A \) be a convex subset of \( \text{Meas}_A \) that is symmetric under permutations of the measurement outcomes. Let \( \tau : \mathcal{G} \to \text{Aut}(\mathcal{I}_M) \) and \( \pi : \mathcal{G} \to \text{Aut}(\mathcal{M}_A) \) be group actions with a group \( \mathcal{G} \). If a state preparation \( \rho \in \text{Prep}_A \) is \( (\mathcal{G}, \tau, \pi) \)-covariant, then there exists a \( (\mathcal{G}, \tau, \pi) \)-covariant measurement that is optimal for Problem (58).

**Example of quantum theory** We consider the special case of \( \mathcal{M}_A = \text{Meas}_A \). Any \( \mathcal{f} \in \text{Aut}(\text{Meas}_A) \) is expressed in the form

\[
\mathcal{f} \circ \rho = U_\mathcal{f} \cdot \rho \cdot U_\mathcal{f}^\dagger,
\]

(81)
where \( T \) is a unitary or anti-unitary matrix of order \( N_A \) and \( ^\dagger \) denotes the conjugate transpose. Thus, \( \pi_k \in \text{Aut}(\text{Meas}_A) \) must be in the form

\[
\pi_k \circ |\mu_m\rangle = U_k \cdot |\mu_m\rangle \cdot U_k^\dagger,
\]

where \( \pi : G \to \text{Aut}(\text{Meas}_A) \) is a group action, \( U_1 \cdot |\mu_m\rangle \cdot U_1^\dagger = \mathbb{I}_{N_A} \) and \( U_k \cdot |\mu_m\rangle \cdot U_k^\dagger = U_{gh} \cdot |\mu_m\rangle \cdot U_{gh}^\dagger \) must hold for any \( g, h \in G \). This type of symmetry has been discussed in Ref. [25].

D. Optimization over a restricted class of measurements

Theorem 1 can be utilized to optimize over a restricted class of measurements, as we will see in this subsection. We here consider state discrimination problems in a bipartite system.

Let us introduce a three-party: Alice, Bob, and Charlie. Charlie randomly chooses one of the \( M \) states \( \rho_1^N, \ldots, \rho_M^N \in \mathcal{S}^N_{A\otimes B} \) with prior probabilities \( \xi_1, \ldots, \xi_M \), which is expressed by the state preparation \( \rho = \sum_{m=1}^M |\mu_m\rangle \circ (n) \in \text{Prep}_{A\otimes B} \) with \( |\mu_m\rangle = \xi_m |\phi_m^N \rangle \) and \( (n) \in \text{Eff}_C \). Also, let us introduce a three-party: Alice, Bob, and Charlie. In this subsection, assume that they can only perform restricted measurements. We consider four classes of measurements: sequential, LOCC, separable, and PT. Let \( D \) and \( D' \) be classical systems, which can be infinite-dimensional.

A measurement \( e \in \text{Meas}_{A\otimes B} \) is referred to as sequential if it can be expressed in the form

\[
A \quad B
\begin{array}{c}
| e \\
| a \\
| b \\
| c \\
\end{array}
= D
\]

with \( a \in \text{Meas}_A \) and \( b \in \text{Meas}_{D\otimes B} \). \( b \) can be interpreted as a classical controlled measurement. Indeed, let us define \( b' \) as a measurement of \( B \) depicted by

\[
\begin{array}{c}
\begin{array}{c}
| a \\
| b \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
| c \\
| d \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
| b' \\
| b \\
\end{array}
\end{array}
\]

then, we have

\[
\begin{array}{c}
\begin{array}{c}
| b \\
\end{array}
\end{array}
= \sum_i | i \\
\begin{array}{c}
\begin{array}{c}
| b' \\
| b \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
| b \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
| b \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
| b' \\
| b \\
\end{array}
\end{array}
\]

Thus, \( b' \) can be interpreted as a process which performs a measurement \( b' \in \text{Meas}_B \) when the state \( |i\rangle \) is inputted to the system \( D \).

A measurement \( e \in \text{Meas}_{A\otimes B} \) is referred to as LOCC if it can be expressed in the form

\[
A \quad B
\begin{array}{c}
| e \\
| a \\
| b \\
\end{array}
= D
\]

with \( a \in \text{Proc}_{A \rightarrow A_1 \otimes D} \), \( a_k \in \text{Proc}_{A_{k-1} \otimes D \rightarrow A_k \otimes D} \) \( (k \in \{2, \ldots, n-1\}) \), \( a_n \in \text{Meas}_{A_n \otimes D} \), \( b_1 \in \text{Proc}_{D \otimes B \rightarrow D_1 \otimes B_1} \), \( b_l \in \text{Proc}_{D_{l-1} \otimes B_1 \cdots \otimes D_{l-1}} \) \( (l \in \{2, \ldots, n-1\}) \), and \( b_n \in \text{Meas}_{D_n \otimes B_n} \), where \( n \) is some natural number.

\[3\]Easy proof: One can easily verify that any \( T \in \text{Aut}(\text{Meas}_A) \) maps pure effects to pure effects and thus maps normalized pure states to normalized pure states. Therefore, according to Lemma 4 of Ref. [36], \( T \) is expressed in the form of Eq. (81) or in the form \( T |\phi\rangle = [\text{Tr}(\rho)] |\phi\rangle \) with a fixed \( |\phi\rangle \in \mathcal{S}^N_A \). The latter case is ruled out since \( T \) is reversible.
A measurement \( e \in \text{Meas}_{A \oplus B} \) is referred to as separable if it can be expressed in the form

\[
\begin{array}{c}
\text{Alice :} \quad c \\
\text{Bob :} \quad D \quad a \quad D' \\
\end{array}
\]

where \( a \in \text{Proc}_{A \rightarrow D} \), \( b \in \text{Proc}_{B \rightarrow D'} \), and \( c \in \text{Proc}_{D \oplus D' \rightarrow C} \). Any \( c \in \text{Proc}_{D \oplus D' \rightarrow C} \) can be expressed by

\[
\begin{array}{c}
a \quad b \\
\end{array}
\]

obviously hold. \( \text{Meas}_{A \oplus B}^{\text{SEQ}} \subseteq \text{Meas}_{A \oplus B}^{\text{SEP}} \) follows from

\[
\begin{array}{c}
a \quad b \\
\end{array}
\]

where \( c \) and \( b' \) are the processes endowed by the upper and lower auxiliary boxes, respectively. \( \text{Meas}_{A \oplus B}^{\text{SEQ}} \subseteq \text{Meas}_{A \oplus B}^{\text{SEP}} \) can be immediately proved in the same way. The proof is completed by showing \( \text{Meas}_{A \oplus B}^{\text{SEQ}} \subseteq \text{Meas}_{A \oplus B}^{\text{PT}} \). Let \( e \in \text{Meas}_{A \oplus B}^{\text{SEQ}} \) be expressed in the form of Eq. (88). Arbitrarily choose a deterministic extended process \( \tilde{f} \in \text{V}_{A' \rightarrow A} \) that is positive for effects. Since \( e' := e \circ (\tilde{f} \otimes V_B) \) is obviously deterministic, it remains to prove that \( e' \) is a process. We have

\[
\begin{array}{c}
a \quad b' \\
\end{array}
\]

Since \( \tilde{f} \) is positive for effects, the extended effect enclosed by the auxiliary box is an effect, and thus \( e' \) is a process. Therefore, \( e \) is PT.

\( \square \)

**Proposition 4:** In quantum theory, \( \text{Meas}_{A \oplus B}^{\text{SEQ}} = \text{Meas}_{A \oplus B}^{\text{PT}} \) holds.

**Proof.** Since \( \text{Meas}_{A \oplus B}^{\text{SEQ}} \subseteq \text{Meas}_{A \oplus B}^{\text{PT}} \) holds, it suffices to show \( \text{Meas}_{A \oplus B}^{\text{SEP}} \subseteq \text{Meas}_{A \oplus B}^{\text{PT}} \), i.e., for any \( e \in \text{Meas}_{A \oplus B}^{\text{SEP}} \) and \( m \in I_M \), \( (m \circ e) \) is separable.

Assume, by contradiction, that \( (e_m) := (m \circ e) \) is entangled for some \( e \in \text{Meas}_{A \oplus B}^{\text{SEP}} \) and \( m \in I_M \). It has been shown in Ref. [37] that there exists an extended state \( |\tilde{e}\rangle \) in \( \text{V}_{A \oplus B} \) such that \( (e_m |\tilde{e}\rangle < 0 \) and \( [(b \otimes (b')] |\tilde{e}\rangle \geq 0 \) hold for any \( (b') \in \text{Eff}_A \) and \( (b') \in \text{Eff}_B \). One can easily verify that, for each \( (b') \in \text{Eff}_A \), \( |\tilde{e}'\rangle := [(b') \otimes \text{id}_B] |\tilde{e}\rangle \) is a separable state in \( \text{St}_{A \oplus B} \). It is full rank (if not, we can replace \( |\tilde{e}\rangle \) with \( |\tilde{e}\rangle + c |\phi\rangle \otimes |\pm\rangle \) with \( 0 < c \in R^+ \), \( |\phi\rangle \in \text{St}_A \), and \( |\pm\rangle := 1_{N_B} \)). Assume, without loss of generality, that \( |\tilde{e}'\rangle \) is full rank (if not, we can replace \( |\tilde{e}\rangle \) with \( |\tilde{e}\rangle + c |\phi\rangle \otimes |\pm\rangle \) with \( 0 < c \in R^+ \), \( |\phi\rangle \in \text{St}_A \), and \( |\pm\rangle := 1_{N_B} \)). It is easily seen that there exists a reversible
process \( g \in \text{Proc}_{A \to B} \) such that \( g \circ |\psi_B \rangle = |\frac{1}{\sqrt{2}} \rangle_B \). Let \( g^{-1} \) be the inverse of \( g \). It is well-known that, for each system \( A \), there exist \( |\psi \rangle_A \in \text{St}_{A\otimes B} \) and \( \langle \psi |_{B} \otimes \text{id}_{A} \in \text{Eff}_{A\otimes B} \) such that \((|\psi \rangle_{A} \otimes \text{id}_{B}) \circ (\text{id}_{A} \otimes |\psi \rangle_{A}) = |\psi \rangle_{A} \) and \((\langle \psi |_{B} \otimes |\psi \rangle_{A} \otimes \text{id}_{A}) = (\text{id}_{A}) \) (see, e.g., [32]). Let

\[
\begin{array}{c}
\text{A} \quad \text{B} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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