How to Design Worked Examples for Learning Patterns in Mathematics

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Abstract. Students will understand how to solve problems more meaningfully by instructions that reduce unproductive cognitive load. A worked example instruction has been proved by numerous research, to be effective for increasing novice student’s skills in solving problems. However, there might be different ways to design worked examples as it may depend on the learning topic. Also less is known on how to implement the instruction in real classroom since most of worked example research uses rigid experiment procedures. This study attempts to study how to create worked examples on the topic of patterns, which is learned at senior high school, including contextual problems related to arithmetic or geometric sequence patterns such as growth, decaying, compound interest, and annuity. These are considered difficult topics for students. Designs of worked examples for learning applications of patterns in mathematics as well as the lesson plan is explained using the perspective of the cognitive load theory. The result of the study provides examples to teachers that it is possible to apply the worked example instruction in a real classroom.

1. Introduction
Problem solving ability is one of the principle of learning mathematics [1]. Moreover, problem solving ability is the center of mathematics learning [2, 3]. Hence, problem solving ability become important in mathematics learning. Almost all of the mathematics learning material contains problem solving competencies [4]. Problem solving ability are generally trained to students through a problem solving approach. However, the problem solving approach is assumed to be appropriate for students who have sufficient prior knowledge and are not suitable for novice learners [5].

Novice students meant for students who have insufficient prior knowledge. Retnowati, Ayres, & Sweller [6, 7] revealed that worked example was better than conventional problem solving approach. It seems easier to use examples than when solving problems without examples. This is also supported by the research of Renkl [8] which states that one reason for the effectiveness of worked example is students get the opportunity to acquire relate the principles in mathematics to many problem cases.

Worked example must be provided with a clear explanation, namely an explanation that gives step by step [9, 10]. Furthermore, the design should reduce extraneous cognitive load [11]. Extraneous cognitive load is a load that is not necessary and not in accordance with the learning objectives [10]. Extraneous cognitive load is related to images or words that are not related to learning objectives [12]. Extraneous cognitive load will produce students’ cognitive capacities exceed limits, so learning objectives are difficult to achieve [12]. So, a good design of worked example is a design that can reduce extraneous cognitive load for students.

Worked example is designed for students who do not have sufficient prerequisite knowledge. Design of worked example should provide clear step-by-step assistance so that explanations can help learners
learn. Sweller, Ayres and Kalyuga [10] explained that the application of worked example was related to several effects namely; 1) expertise reversal effect, 2) split-retention effect and redundancy effect, and 3) pairing problem effect. Expertise reversal effects relate to the level of prior knowledge possessed by students. Split-retention effects and redundancy effects are effects related to extraneous cognitive load or foreign cognitive load. Attention to this extraneous cognitive load will help students to focus only on the content of the material without being burdened on the presentation of the material. Pairing problem effects are similar problems given to students to solve problems. This problem is solved by students without being allowed to see examples. This strategy is given so that students can solve problems based on their understanding of examples that have been previously understood.

Solving a problem is a need for students, moreover if students solved a contextual problem [13], high school students are given the opportunity to explore contextual problems related to patterns, both related to social science (compound interest and annuity) as well as related to science (growth and decay). Patterns are one of the basic material in mathematics. Steen [1] states that mathematics is the science of patterns. Mulligan and Mitchelmore [14] also supports Steen’s statement by saying that virtually all mathematics is based on pattern and structure. Lee, Ng, Bull, Lee Pe, and Ho [15] also conducted research on the importance of patterns for students' problem solving abilities. Therefore, the application of patterns that contain contextual problems is an interesting and suitable material in training students' problem solving skills. However, problem solving skills are still not mastered by students especially for novice learners, on the other hand CLT offers one instruction that can help train students' problem solving abilities, namely worked example. The problem is that it is less known on how to design a working example based instruction, especially related to contextual problems. If there is a design that has been compiled, the next problem is how to implement the design in the class because it is less known on how to implement the instruction in real classroom since most of the research uses experimental procedures.

This article proposes to design example for learning patterns, especially application of patterns. In addition, Retnowati and Marissa [16] showed that worked example instructions that pay attention to some of the above effects could be applied in class. The implementation of the design in the class will be discussed.

2. The design of the worked example for learning applications of patterns

The worked example design must follow the principles of cognitive load theory. They are expertise reversal effect, split-retention effect and redundancy effect, and pairing problem effect. First, the principle related to expertise reversal effects, namely beginner students. It means that learners are students who do not have sufficient prerequisite knowledge. The prerequisite material for studying the applications of pattern are number pattern, arithmetic sequence, geometry sequence, exponential, logarithm, and single interest. So that the use of the prerequisite material in the steps to solve the problem of patterns application is explicited. Second, the principle regarding split-attention effects and redundancy effects. Split-attention effect relates to the division of attention of students because of the design of worked example is not focused. The split-attention effect is minimized by thickening and underlining the important information in the problem. It is also minimized by using of the same color for each variable at the completion step composes a pattern, such as brick red for the first term, green for the common ratio and blue for the number of terms. These are used for assist students in seeing the similarity of the patterns formed in each sequence of patterns and also helps in generalizing the patterns to be obtained. While, redundancy effect is related to overlapping information because there is excess information. Redundancy effect is minimized by designing questions that do not contain information that is not needed. The third, pairing problem effect at the end of the design, students are given questions similar to examples but are not allowed to see examples. This question aims to check students' understanding after exploring the worked example given previously. In addition, it also aims to familiarize students' memories of the concepts they have just learned to be stored in their long-term memory.

The application of patterns consists of four sub-material namely growth, decaying, compound interest, and annuity. Growth is the increase in the number / value of an object that follows the pattern of arithmetic or geometry. For example, the proliferation of bacteria and population growth. Decay is the reduction in the number / value of an object that follows an arithmetic or geometric pattern. For example, decreasing the selling value of cars and decreasing the number of animal populations. Compound interest is interest that arises at the end of a certain period of time (month / year) which
affects the amount of capital and interest at each time period. Compound interest has many variations and always changes in each period. Capital and interest are increasing every time period. While, annuity is a series of payments or receipts that are the same amount and must be paid or must be received at the end of each period on a loan or credit. Annuities given regularly at the end of each period have two functions, that are paying interest on debt and repaying the debt itself.

Below are examples and problems for each sub-material. The instructions, worked example, and questions in the form of similar problems are given as follows.

**The instruction:** Understand the example given carefully and solve the problem provided without seeing the example

1. **The worked example 1 (Growth).** A scientist observes the development of chlorella colonies that are formed every hour. The amount of chlorella colonies is initially 100 and each chlorella will split into **four every hour**. What is the number of chlorella colonies formed within **4 days**?

   **Answer:**
   
   **Step 1** Read and understand the problem carefully
   
   **Step 2** Write the important information on the problem
   
   What’s given:
   - the initial amount of chlorella = 100
   - the amount of splitting every chlorella per hour = 4
   
   What’s asked:
   - the amount of chlorella colonies within 4 days?
   
   (4 days = 4 x 24 hours = 96 hours)
   
   **Step 3** Arrange patterns
   
   Time (hours) | The amount of chlorella colonies | The number patterns
   -----------------|-----------------------------------|----------------------
   1                | 400                               | 100 x 4 = 100 x 4^1     |
   2                | 1600                              | 100 x 4 x 4 = 100 x 4^2 |
   3                | 64000                             | 100 x 4 x 4 x 4 = 100 x 4^3 |
   ⋮                | ⋮                                 | ⋮                     |
   n                | ⋮                                 | 100 x 4 x 4 x ⋯ x 4 = 100 x 4^n |

   It is seen that the amount of chlorella colonies has a number pattern that depends on the time (hours), that is
   
   The amount of chlorella at n hours = 100 x 4^n
   
   The amount of chlorella at n hours = the initial amount of chlorella x the amount of splitting every chlorella per hour ^n

   \[ U_n = U_0 \times r^n \]

   **Step 4** Make conclusion
   
   So the amount of chlorella colonies formed within 96 hours (4 days) is 6,2771 x 1059

   \[ U_n = U_0 \times r^n \]
   
   \[ U_{96} = 100 \times 4^{96} \]
   
   \[ U_{96} = 100 \times (6,2771 \times 10^{57}) \]
   
   \[ U_{96} = (6,2771 \times 10^{57}) \times 100 \]
   
   \[ U_{96} = 6,2771 \times 10^{57} \times 10^2 \]
   
   \[ U_{96} = 6,2771 \times 10^{59} \]

   **The problem 1.** Research on bacterial behavior results in bacteria **dividing into two every one hour**. If at first the bacterial colonies **totaled 50**, determine the number of bacteria after **72 hours**?
2. **The worked example 2 (Decaying).** A neutron can break into a proton and an electron and it happens so that if we have 100,000 neutrons, about 5% of them will change at the end of one minute. How many neutrons are there after 10 minutes?

**Answer:**

**Step 1** Read and understand the problem carefully

**Step 2** Write the important information on the problem

What’s given:
- the initial amount of neutron = 100,000
- the percentage of depreciation every minute = 5 %

What’s asked:
- the amount of neutrons left after 10 minutes?

**Step 3** Arrange patterns

| Time (minute) | The amount of neutrons left | The number patterns |
|---------------|-----------------------------|---------------------|
| 1             | 95000                       | 100000 − \( \frac{5}{100} \) × 100000 |
|               |                             | = 100000 \( 1 - \frac{5}{100} \)^1 |
| 2             | 90250                       | 100000 \( 1 - \frac{5}{100} \)^2 × 100000 \( 1 - \frac{5}{100} \)^1 |
|               |                             | = 100000 \( 1 - \frac{5}{100} \)^2 |
| 3             | 85737,5                     | 100000 \( 1 - \frac{5}{100} \)^3 × 100000 \( 1 - \frac{5}{100} \)^2 |
|               |                             | = 100000 \( 1 - \frac{5}{100} \)^3 |
| ⋮             | ⋮                           | ⋮                   |
| n             | ⋯                           | 100000 \( 1 - \frac{5}{100} \)^n |

It is seen that the number of remaining neutrons has a number pattern that depends on time (minutes) i.e.

\[ U_n = U_0 \times r^n \]

The amount of neutrons left at n minutes = 100000 \( 1 - \frac{5}{100} \)^n

The amount of neutrons left at n minutes = the initial amount of neutron \( \times \) factor of decay \( n \)

**Step 4** Make conclusion

So the amount of neutrons left after 10 minutes is 59874

\[ U_n = U_0 \times r^n \]

\[ U_{96} = 100000 \times \left( 1 - \frac{5}{100} \right)^{10} \]
\[ U_{96} = 100000 \times (1 - 0.05)^{10} \]
\[ U_{96} = 100000 \times (0.95)^{10} \]
\[ U_{96} = 100000 \times 0.59874 \]
\[ U_{96} = 59874 \]

**The problem 2.** Doctors diagnose that there are 800,000 bacteria that infect a baby’s ear. To speed up the healing process, doctors give doses of penicillin which can kill 10% of the number of bacteria every hour. How many bacteria are left after 12 hours?
3. **The worked example 3 (Compound Interest).** Yusuf, a class eleventh high school student, likes to save money in the bank. During this time he managed to save up to IDR 1,000,000 in a bank with a compound interest rate of 10% per year. How much is Joseph’s money if he saved the money for the next 4 years?

**Answer:**

**Step 1** Read and understand the problem carefully

**Step 2** Write the important information on the problem

- What’s given: the initial amount of save up = IDR 1,000,000, - the percentage of compound interest every minute = 10%

- What’s asked: the amount of save up after 4 years?

**Step 3** Arrange patterns

Number patterns for compound interest are geometric patterns, namely:

\[
U_n = U_0 \times r^n \\
M = M_0 \times (1 + i)^n
\]

with

- \( M \) = the value of savings after \( n \) years
- \( M_0 \) = the initial savings value
- \( i \) = the percentage of compound interest
- \((1+i)\) = the factor of multiplication

**Step 4** Make conclusion

So the amount of savings within 4 years is IDR 1,464,100

\[
M = IDR\ 1,000,000 \times (1 + 10\%)^4 \\
M = IDR \ 1,464,100
\]

**The problem 3.** The increase in the price of goods is called inflation. Based on the analysis, Indonesia’s economy will experience inflation of 8% per year for the next 5 years. If the current gold price is IDR 200,000 per gram, what is the price of gold for four more years?

4. **The worked example 4 (Annuity).** Rini buys a motorcycle from a dealer that uses an annuity system on credit payments. The motorbike price is IDR 10,000,000.00 using an interest rate of 4% per year. Rini plans to repay her loan with 6 annuities. Calculate the amount of annuity paid by Rini?

**Answer:**

**Step 1** Read and understand the problem carefully

**Step 2** Write the important information on the problem

- What’s given: loan amount = IDR 10,000,000, - interest rate per year = 4%

- What’s asked: annuity value for credit repayment with 6 annuities?

**Step 3** Arrange patterns

Number patterns for annuities are:

\[
A_n = A \frac{1 - (1 + i)^{-n}}{i} \leftrightarrow A = M_n \frac{i}{1 - (1 + i)^{-n}}
\]

with

- \( A \) = annuity value
- \( M_n \) = capital / total loan
- \( i \) = interest rate
- \((1+i)\) = number of annuities

**Step 4** Make conclusion

So the amount of annuity for repayment of credit with 6 annuities is IDR 1,907,619.

\[
A = M_n \frac{i}{1 - (1 + i)^{-n}} \quad \text{4%} \\
A = IDR \ 10,000,000 \frac{4\%}{1 - (1 + 4\%)^{-6}} \\
A = IDR \ 1,907,619
\]
The problem 4. Mr. Eko borrows money from a bank that uses an annuity system for payment of installments. The amount of Mr. Eko’s loan is IDR 20,000,000.00 using an interest rate of 5% per year. Mr. Eko plans to repay the loan with 12 annuities. Calculate the amount of annuity paid by Mr. Eko?

The design above is a pair of worked example and problem for each sub-material. The design is a core component of learning by using the worked example approach. While, the learning of the four sub-materials above will be explained below.

3. The implementation of the worked example based instruction

A learning plan of patterns application for this design is given for the four sub-material above. The learning steps are adapted from Retnowati [6] and Rohman [17]. The learning steps using the worked example approach for this material include, (1) the teacher gives apperception to students, (2) the teacher provides motivation for students, (3) students solve the problem of application of the patterns using the worked example strategy by first studying the examples given, (4) students present their work, (5) students conclude learning outcomes with teacher guidance. First, giving apperceptions to students in the form of questions and answers about arithmetic sequences and geometric sequences. Second, giving motivation to students in the form of examples of daily problems related to the application of the sequence. The third and fourth sections are the core part of learning, students are given worksheets that contain the above design and then students are asked to present their work in writing and verbally. The final step is students are asked to conclude learning outcomes with direction from the teacher.

The most important thing that must be considered by the teacher in the worked example learning is the material characteristics and student characteristics. Worked example is suitable for complex learning material and the learner is a beginner's problem solver. Because of beginners, the teacher must always give apperception to students about prerequisite knowledge before continuing learning with the worked example approach. At the beginning of the learning process, motivation is needed for students. This motivation can be made in the form of problems that raises questions in the minds of students, so they have more desire to learn it. For example, the teacher can provide a problem that demands a solution so they will think to find the solution. In order to they are curious and want to know more, it is better for the teacher to save the solution and it will be given at the end of learning.

It is better to do worked examples for individual students in the core learning activities. It corresponds with the results of research conducted by Retnowati, Ayres, & Sweller [6] that worked example and high complexity of tests are better studied individually than in groups. Furthermore, characteristics of worked example are students understand examples and work on similar problems without seeing examples. The teacher should always remind students not to see examples when working on questions so they can practice the automation of the concept. The end of this activity is clarification of students’ work that aims to enable students to appreciate the results of their work when correct and justify their work when wrong. At the end of learning, students can be asked again about the problems that arise in the motivational part of learning. Teachers can provide opportunities for students to express their opinions about possible solutions. Students are expected to be able to answer because they have learned the concepts related to the problems given. Then, students are directed by the teacher to conclude learning outcomes.

4. Conclusion

Solving contextual problems related to patterns might be difficult for novices. Worked example is recommended for novice students because it can minimize extraneous cognitive load. This is applied to the design by considering the three principles of cognitive load theory. First, expertise reversal effect, that is paying attention to students' prerequisite knowledge by providing detailed steps on the use of prerequisite material in the completion of the given example. The steps given to help students consist of four steps. They are 1) read and understand the problem carefully, 2) write the important information on the problem, 3) arrange patterns, and 4) make conclusion. The first part, “read and understand the problem carefully”, directs students to recognize and understand the problem. The second part, “write the important information on the problem”, focuses students on the information needed in the problem. The third, “arrange patterns” directs students to understand that the problem given is a problem of
patterns. In this part, students will be directed at finding patterns of problems asked. The last, “make conclusion” is a part of determining the value of what was asked using a pattern that was found in the previous step. Second, split-attention effect and redundancy effect, that are considering both of these effects with thickening and underlining the important information on the problem, using the same color for each variable at the completion stage of arranging patterns, designing questions that do not contain unnecessary information and giving systematic steps for resolving. Third, giving a similar problem that serves to train students to remember and can automate the concepts they have just learned. Furthermore, the learning design using the worked example approach to application of patterns are (1) the teacher gives apperception to students, (2) the teacher provides motivation for students, (3) students solve the problem of application of patterns using the worked example strategy by first studying the examples given, (4) students present their work, (5) students conclude learning outcomes with teacher guidance.

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