Geometry of the Quantum States of Light in a Mach-Zehnder Interferometer

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Abstract. A geometric approach to the quantum states of light that is injected into a Mach-Zehnder interferometer is discussed. It is based on the Jordan-Schwinger representation of angular momentum and permits the identification of the photon states with cones in the three-dimensional space that are rotated by the action of the optical devices. Some cases of the evolution loops, dynamical processes for which any state evolves cyclically, are presented.

1. Introduction
Diverse techniques for manipulating the state of single quantum systems have been developed in recent years, mainly addressed to precision measurements. This is because the controlled evolution of the quantum state is as relevant as the measurement by itself. In the theoretical scenery, for instance, the so called inverse method (see e.g. [1,2]) has been used to find a way of controlling the free evolution of non-relativistic charged particles by time-independent magnetic fields [3, 4] (see also [5,6]). The method has been also applied to determine the evolution loops [7,8], dynamical processes for which any state evolves cyclically, associated with two-level quantum systems [9,10]. On the other hand, using the above ideas, the experimental possibilities of steering an energy state to a given destination have been discussed [11]. In this paper we present preliminary results of our research addressed to control the quantum states of light in a Mach-Zehnder interferometer [12]. The main motivation of such a trend is that our theoretical predictions can be verified in the optical bench of any well equipped laboratory of quantum optics. With this in mind, in Section 2 we revisit the properties of the space of states associated with the conventional two-channel optical devices that integrate the interferometer. Then, in Section 2.1 we use the Jordan-Schwinger approach [13,14] to get a set of angular momentum operators in term of the photon states in Heisenberg representation. The description in terms of the Stokes parameters is then used in Section 3 to identify the photon states with cones in the three-dimensional space that are rotated by the action of the optical devices. Some implementations to get evolution loops are discussed in Section 3.1 and final remarks are given in Section 4.

2. Vector space of states
Let us consider the Mach-Zehnder interferometer depicted in Figure 1. We are interested in the case when $n_1+n_2$ photons are sent towards the interferometer, with $n_1$ ($n_2$) photons propagating along the horizontal (vertical) direction defined in the figure. As usual, the interferometer is
composited by two beam splitters and two mirrors that define two possible optical paths for each photon that may differ in a quantity $\Delta \ell$. This last can be corrected by using phase shifters in both the theoretical calculations and the experimental setup. The initial state is then defined by the tensor product $|n_1, n_2\rangle = |n_1\rangle \otimes |n_2\rangle$, where $|n_{1,2}\rangle$ is a Fock state belonging to the Hilbert space $\mathcal{H}_{1,2}$. To represent the interchange of photons between the vertical and the horizontal configurations as the action of an optical device we shall use the boson operators

$$[a_i, a_j] = 0, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2,$$

which are defined to act on $\mathcal{H}_i$ only. Therefore, we promote them to act on the entire space $\mathcal{H} = \text{span}\{|n_1\rangle \otimes |n_2\rangle\}_{n_1, n_2 \geq 0}$ as follows [15]:

$$a_1 \leftrightarrow a_1 \otimes I_2, \quad a_1^\dagger \leftrightarrow a_1^\dagger \otimes I_2, \quad a_2 \leftrightarrow I_1 \otimes a_2, \quad a_2^\dagger \leftrightarrow I_1 \otimes a_2^\dagger,$$

where $I_i$ stands for the identity operator in $\mathcal{H}_i$. For instance, the combined action of the annihilator operators is given by

$$(a_1 \otimes a_2) |n_1, n_2\rangle = (a_1 |n_1\rangle) \otimes (a_2 |n_2\rangle) = \sqrt{n_1 n_2} |n_1 - 1, n_2 - 1\rangle.$$

For the sake of simplicity, in the sequel we shall omit the symbol `\otimes`, so that we will use the simpler notation $A_1 \otimes A_2 = A_1 A_2$ for the direct product of the operators $A_1$ and $A_2$.

![Figure 1](image1.png)

**Figure 1.** The Mach-Zehnder interferometer. Dotted arrows represent incoming and outgoing beams.

The majority of the optical devices are two-channel in the sense that two incoming signals can be processed simultaneously to produce either a single or a double output signal. In the Heisenberg picture the signal is represented by the boson operators rather than by the Fock vectors, so that the matrix array

$$\begin{pmatrix} a_1^\prime \\ a_2^\prime \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

represents the most general transformation that is suffered by the incoming signal. That is, we shall work with the operator-like vectors $|\hat{a}\rangle = (a_1, a_2)^T$ that are transformed by the operators $U = [u_{ij}]$. To ensure the symmetry properties of the optical devices we know that the matrices $[u_{ij}]$ are the unitary representation of the operators $U$ in the space of the vectors $|\hat{a}\rangle$. For instance, to represent the beam splitters appearing in Figure 1 and the phase shifters that are necessary to correct the optical path difference $\Delta \ell$ we use respectively the matrices

$$D(\alpha) = \begin{pmatrix} \cos \frac{\alpha}{2} & -i \sin \frac{\alpha}{2} \\ -i \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}, \quad F(\phi_1, \phi_2) = \begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}.$$

Another useful matrix, leading to essentially the same outputs as $D(\alpha)$ but with an additional phase in the reflected beam, is as follows

$$E(\alpha) = \begin{pmatrix} \cos \frac{\alpha}{2} & -\sin \frac{\alpha}{2} \\ \sin \frac{\alpha}{2} & \cos \frac{\alpha}{2} \end{pmatrix}.$$
In particular, the action of the beam splitter \( D(\alpha)F(\phi_1, \phi_2)D(\gamma) \) on the initial state \( |\text{in}\rangle = |\hat{a}\rangle \). In particular, for \( \gamma = -\alpha = \pi/2 \) and \( \phi_2 = -\phi_1 \) the output state \( |\text{out}\rangle = |\hat{a}'\rangle \) is given by \( |\text{out}\rangle = M_z(\phi_1) |\text{in}\rangle \), where

\[
M_z(\phi_1) = \begin{pmatrix} \cos \phi_1 & -\sin \phi_1 \\ \sin \phi_1 & \cos \phi_1 \end{pmatrix}.
\] (7)

2.1. Jordan-Schwinger representation of angular momentum

In the Jordan-Schwinger representation [13,14], we can use the boson operators to construct the components of the angular momentum operator

\[
J_x = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1), \quad J_y = -\frac{i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1), \quad J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2),
\] (8)
as well as the number operator

\[
N = a_1^\dagger a_1 + a_2^\dagger a_2 = N_1 + N_2.
\] (9)

In the above expressions we are adhering to the notation of Ref. [16]. It is easy to verify that \( N \) commutes with all the \( J \)-operators as well as the following commutation rules

\[
[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.
\] (10)

In turn, the operator \( J^2 = J_x^2 + J_y^2 + J_z^2 \) can be written as

\[
J^2 = \frac{N}{2}(\frac{N}{2} + 1), \quad J_z = \frac{1}{2}(N_1 - N_2).
\] (11)

Thus, the total number of photons \( N = N_1 + N_2 \) gives information about the total angular momentum while the difference in the number of incoming photons \( N_1 - N_2 \) is associated with the \( z \)-projection of \( \mathbf{J} \). After a straightforward calculation we get

\[
\langle J_x \rangle_{n_1,n_2} = \langle J_y \rangle_{n_1,n_2} = 0, \quad \langle J_z \rangle_{n_1,n_2} = \frac{1}{2}(n_1 - n_2), \quad \langle J^2 \rangle_{n_1,n_2} = \frac{n}{2}(\frac{n}{2} + 1).
\] (12)

This last permits a geometrical description of the state \( |n_1, n_2\rangle \) in \( \mathbb{R}^3 \), see Figure 2(a). Indeed, depending on \( n_1 \) and \( n_2 \), the expectation values (12) represent a cone of height \( h = |\langle J_z \rangle| \) and generatrix \( s = \sqrt{|\langle J^2 \rangle|} \) that is aligned with the vertical. The positiveness of \( \langle J_z \rangle \) determines the orientation of the cone.

3. Geometric action of the optical devices

In the description of the quantum Stokes parameters [17] (see also [18]), the operators \( J_i \) can be identified with the cartesian coordinates that localize the axis of the cone:

\[
J_x = \frac{1}{2} \langle \hat{a} | \sigma_x | \hat{a} \rangle.
\] (13)

After the transformation generated by the unitary operator \( U \) one gets [16]:

\[
J'_i = \langle \hat{a} | \sigma_i | \hat{a} \rangle = \langle \hat{a} | U^\dagger \sigma_i U | \hat{a} \rangle,
\] (14)

In particular, the action of the beam splitter \( D(\alpha) \) on the components of \( \mathbf{J} \) gives

\[
J'_z = \cos \alpha J_z + \sin \alpha J_y, \quad J'_y = \cos \alpha J_y - \sin \alpha J_z, \quad J'_x = J_x.
\] (15)
Figure 2. (a) The quantum state of $n_1$ photons in the horizontal arm and $n_2$ photons in the vertical arm of a Mach-Zehnder interferometer can be represented as a cone (oriented along the $z$-axis) of height $h = \frac{1}{2}|n_1 - n_2|$ and generatrix $s = \frac{1}{2}(n_1 + n_2)$. (b) The action of the beam splitter $D(\alpha)$ produces rotations of the cone by an angle $\alpha$ around the $x$-axis. (c) The action of the phase shifter $F(\phi_1, \phi_2)$ produces rotations by an angle $\phi = \phi_2 - \phi_1$ around the $z$-axis.

Figure 3. (a) The net effect of the optical devices that integrate the Mach-Zehnder interferometer is reduced to a finite rotation around the $y$-axis. (b) As the generatrix $s$ is a constant (the number of photons $n = n_1 + n_2$ is preserved), the axis of the cone describes a path on a sphere of radius $h$.

Or, in matrix form, $J' = R_z(\alpha)J$. Thus, the action of the beam splitter produces rotations of the vector $J$ by an angle $\alpha$ around the $x$-axis, see Figure 2(b). In turn, the phase shifter $F(\phi_1, \phi_2)$ produces rotations by an angle $\phi = \phi_2 - \phi_1$ around the $z$-axis: $J' = R_z(\phi)J$, see Figure 2(b).

In this form, the entire circuit of the interferometer is associated to the operator

$$U_{\text{global}} = D(\pi/2)F(\phi_1, \phi_2 = -\phi_1)D(-\pi/2) = M_z(\phi_1),$$

(16)

and gives $J' = R_y(2\phi_1)J$. In other words, the net effect of the entire circuit produces rotations of $J$ by an angle $2\phi_1$ around the $y$-axis (see Figure 3). This last leads to the interference term $\langle J'_y \rangle = \frac{1}{2}(n_1 - n_2)\cos 2\phi_1$, which clearly accounts for the difference of photons in the vertical and horizontal outputs of the interferometer. It can be shown that the net rotation (16) can be also produced by acting $E(\alpha)$ on the initial state with $\alpha = 2\phi_1$ [12].

The axis of the cone describes a path on a sphere of radius $h$ because the generatrix $s$ is a constant, as it is shown in Figure 3(b). It is well known that the points on such a surface (i.e. the points on the 2-sphere $S^2$) are in a one-to-one correspondence with the pure states of any qubit (see e.g. [19] and references quoted therein). The dynamics of the two-level pure states is in this form reduced to rotations and reflections of the 2-sphere [20]. Therefore, the path depicted in Figure 3(b) should correspond to the successive transitions of pure states that are
required to connect the signal outputs (points $B$, $C$ and $D$ on the sphere) associated with the optical devices involved.

### 3.1. Manipulating the quantum states in the interferometer

The above approach is useful in sculpturing the final quantum state of the light that is injected into the interferometer. For instance, making the rotations indicated in Figure 3 in the reverse order we arrive at another state. Namely, if we rotate around $y$, $z$ and $y$ by $\frac{x}{2}$, $\phi_1$ and $\frac{-x}{2}$ respectively then the net effect is a rotation by $\phi_1$ around the $x$-axis, see Figure 4(a). To produce such a circuit in the optical bench we only require the substitution of the beam splitters $D$ by the ones of type $E$. The combination of these results is useful to produce close paths on the $h$-sphere. For instance, the net operator

$$U_c(\phi_1) = E\left(-\frac{x}{2}\right)F(\phi_1, \phi_1 + \frac{x}{2})D\left(-\frac{x}{2}\right) = e^{i\phi_1}I \quad (17)$$

produces the adding of a global phase on any given initial state, as it is shown in Figure 4(b). However, notice that even for $\phi_1 = 0$, this last result does not mean that the initial and final states coincide. The difference lies on the fact that the tangent vectors in the paths from the, and arriving at, the point $N$ are not the same. A different situation arises if one uses

$$U_{\text{loop}}(\phi_1) = D\left(-\frac{x}{2}\right)F(\phi_1, \phi_1 - \pi)d\left(-\frac{x}{2}\right) = I \quad (18)$$

because this produces close paths in which the initial and final tangent vectors are the same, no matter the initial state, see Figure 4(c). This kind of path on the 2-sphere is known as *evolution loop* [9,10] and lead, in natural form, to the notion of *geometric phases* [21].

### 4. Concluding remarks

We have presented preliminary results of our research addressed to control the quantum states of light in a Mach-Zehnder interferometer. Using a geometric approach, we described the action of the optical devices on the photon states as paths described on a sphere of radius $h = \frac{1}{2}|n_1 - n_2|$, with $n_1$ ($n_2$) the number of photons that is injected into the horizontal (vertical) arm of the interferometer as initial condition. Similar geometric descriptions have been already reported in connection with the dynamical manipulation of spin-1/2 systems by using magnetic fields [20,22] and with linear combinations of $j = 1/2$ angular momentum eigenkets (mesoscopic kittens) that manifest quantum interference effects analogous to those occurring for linear combinations of...
coherent light waves [23]. In our case, the paths on the 2-sphere correspond to the successive transitions of pure states that are required to connect the signal outputs associated with the optical devices involved. In particular, close paths in which the initial and final states coincide have been found to describe the so called evolution loops [8, 9]. The calculation of the corresponding geometric phase is in progress and will be reported elsewhere.

Acknowledgements
The author is grateful to Prof. Oscar Rosas-Ortiz for useful comments and suggestions. The support of CONACyT is acknowledged.

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