ABSTRACT. We introduce the $L^p$ spaces of measurable functions whose $p$-th power is summable with respect to the uniform measure over the Levi-Civita field. These spaces are the counterparts of the real $L^p$ spaces based upon the Lebesgue measure. Nevertheless, they lack some properties of the $L^p$ spaces: for instance, the $L^p$ spaces are not complete with respect to the $p$-norm. This motivates the study of the completions of the $L^p$ spaces with respect to strong convergence, denoted by $L^p_s$. It turns out that the $L^p_s$ spaces are Banach spaces and that it is possible to define an inner product over $L^2_s$, thus making it a Hilbert space. Despite these positive results, these spaces are still not rich enough to represent every real continuous function. For this reason, we settle upon the representation of real measurable functions as sequences of measurable functions in $\mathbb{R}$ that weakly converge in measure. We also define a duality between measurable functions and representatives of continuous functions. This duality enables the study of some measurable functions that represent real distributions. We focus our discussion on the representatives of the Dirac distribution and on the well-known problem of the product between the Dirac and the Heaviside distribution, and we show that the solution obtained with measurable functions over $\mathbb{R}$ is consistent with the result obtained with other nonlinear generalized functions. These results suggest that it would be possible to develop a theory of nonlinear generalized functions on the Levi-Civita field, in analogy to what has been done in other non-Archimedean extensions of the real numbers.

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1. INTRODUCTION

The Levi-Civita field $\mathbb{R}$, introduced by Levi-Civita in [18, 19] and subsequently re-discovered by many authors in the ’900, is the smallest non-Archimedean ordered field extension of the field $\mathbb{R}$ of real numbers that is both real closed and complete in the order topology. Since the early ’90s, Berz and Shamseddine have begun a fruitful development of analysis on the Levi-Civita field (see for instance [1, 3, 4, 20, 25] and references therein). The study of the differential structure of the Levi-Civita field has also led to the implementation of effective algorithms for the computation of the derivatives of real functions as differential quotients [5, 17].

Recently, an uniform measure theory inspired by the Lebesgue measure has been developed by Shamseddine and Berz in dimension 1 [26, 29], and by Shamseddine and Flynn in dimension 2 and 3 [30]. This measure theory is also relevant for the development of Nonarchimedean probabilities, as discussed in [11] and references therein.

It turns out that measurable functions on the Levi-Civita field are general enough to represent some distributions [16]. For instance, it is possible to define some measurable functions that represent the Dirac distribution, much in the spirit of the representation of distributions with functions of nonstandard analysis [7, 9, 22]. These results suggest that it would be possible to develop a theory of generalized functions on the Levi-Civita field, in analogy to what has been done in other non-Archimedean extensions of the real numbers by Benci, Bottazzi, Colombeau and Todorov. Such a theory of generalized functions would allow for the study of ill-posed partial differential equations that require a nonlinear theory of distributions. For more details, we refer to the discussion in [9], and for an example of a generalized function formulation of a nonlinear PDE we refer to [10].

In order to introduce a theory of generalized functions on the Levi-Civita field, we believe that it would be necessary to define an extension of functions from $\mathbb{R}$ to $\mathbb{R}$ in a way that some relevant properties are preserved (as argued also for the development of a surrept calculus [14]). Currently, it is possible to extend real continuous functions to locally analytic functions on the Levi-Civita field by means of the power series expansion at a point [1, 2, 3, 25]. However, if the original function is not analytic, this extension does not preserve many first-order properties [8]. As a consequence, it is well-known that many theorems of real analysis cannot be directly extended to the Levi-Civita field. For instance, in the Levi-Civita field the space of solutions of the differential equation $f' = 0$ is infinite-dimensional, while in the real case it has dimension 1 [20]. The limitations of the extension of functions based upon the power series expansion will become evident in Section 3.2, where we will prove that the extension of a function that is not locally analytic is not measurable. This result combined with an earlier theorem of Shamseddine and Berz
entails that no measurable function provides a good representation of real functions that are not locally analytic at every point in their domain.

In order to address this limitation, at first we will introduce the spaces $L^p$ of measurable functions whose $p$-th power is summable with respect to the uniform measure over the Levi-Civita field. These spaces are the counterparts of the real $L^p$ spaces based upon the Lebesgue measure. Nevertheless, the $L^p$ spaces lack some properties of the $L^p$ spaces: for instance, the former are not complete with respect to the $p$-norm. This limitation will motivate us to study the completions of the $L^p$ spaces with respect to strong convergence, denoted by $L^p_s$. It turns out that the $L^p_s$ spaces are Banach spaces and that it is possible to define an inner product over $L^2_s$, thus making it a Hilbert space. Despite these positive results, these spaces are still not rich enough to represent every real continuous function. For this reason, we settle upon the representation of real measurable functions as sequences of measurable functions in $\mathcal{R}$ that weakly converge in measure. This representation allows define a notion of duality between measurable functions and representatives of continuous functions. This duality enables the definition of measurable functions that represent the real Dirac distribution, and the study of their properties. As a significative example, we will discuss the well-known problem of the product between the Dirac and the Heaviside distribution, and we will show that the solution obtained in $\mathcal{R}$ is consistent with the result obtained with other approaches, as discussed in [8].

We have chosen to work with the uniform measure developed by Shamseddine, Berz and Flynn in order to establish a continuity with the measure theory developed so far for the Levi-Civita field. During our study, however, we have come to the conclusion that the development of a theory of generalized functions in the Levi-Civita field might benefit from a radically different notion of measure. Towards the end of the paper we will suggest that the Loeb measure of Robinson’s framework for mathematics with infinitesimals, presented for instance in [15], might allow for a better representation of real continuous functions.

1.1. Structure of the paper. Section 2 contains some basic notions on the Levi-Civita field. In particular, we will review some properties of strong and weak convergence, and the basics of the theory of power series over $\mathcal{R}$. We will also prove some technical properties that will be useful throughout the paper.

In Section 3 we will review the uniform measure on the Levi-Civita field and establish some novel results. For instance, we will provide many examples of nonmeasurable sets and we will show that it is not possible to approximate a measurable set with intervals whose measure is infinitely smaller than the measure of the set.

In our approach we chose also a slightly different characterization of simple and measurable functions. Namely, we will not require simple functions to be Lipschitz continuous and we will not assume that measurable functions are bounded. The first change will have no effect on the resulting measure theory, whereas the second will entail the existence of unbounded measurable functions. Despite this larger class of measurable functions, we will prove that it is not possible to represent real continuous functions with measurable functions on the Levi-Civita field.

The existence of unbounded measurable functions will be one of the motivations for the introduction of the $L^p$ spaces in Section 4. After having reviewed some of their basic properties, such as the existence of a $p$-norm that satisfies the familiar Hölder’s inequality, we will determine that the $L^p$ spaces are not complete with respect to strong and weak convergence in norm. However, it is possible to define completions $L^p_s$ of the $L^p$ spaces with respect to strong convergence. These completions turn out to be Banach spaces,
and $\mathcal{L}^2 \ell$ is also a Hilbert space. However, it is not possible to represent real continuous functions even in these $\mathcal{L}^p \ell$ spaces.

For this reason, in Section 5 we will settle for the representation of real continuous functions with weakly converging sequences of measurable functions. However, it is not possible to define a completion of the $\mathcal{L}^p \ell$ spaces with respect to weak convergence: this is mainly due the absence of a squeeze theorem for this notion of convergence.

Despite this negative result, the representation of real continuous functions with weakly converging sequences of measurable functions will allow us to define a class of measurable Delta functions on $\mathbb{R}$ and to study their properties. In particular, we will discuss how in the Levi-Civita field it is possible to solve some simple problems from the nonlinear theory of distributions.

Finally, in Section 6 we will present some concluding remarks and an outline for future research.

1.2. Notation. If $(\mathbb{F}, <)$ is an ordered field and $a, b \in \mathbb{F}$, we will denote by $[a, b]_{\mathbb{F}}$ the set \{x \in \mathbb{F} : a \leq x \leq b\}, and by $(a, b)_{\mathbb{F}}$ the set \{x \in \mathbb{F} : a < x < b\}. The sets $[a, b]_{\mathbb{F}}$ and $(a, b)_{\mathbb{F}}$ are defined accordingly. The above definitions are extended in the usual way if $a = -\infty$ or $b = +\infty$, with the convention that $[a, +\infty]_{\mathbb{F}} = [a, +\infty)_{\mathbb{F}}$ and that $(-\infty, b)_{\mathbb{F}} = (-\infty, b]_{\mathbb{F}}$. If $\mathbb{F} = \mathbb{R}$, we will often write $[a, b]$ instead of $[a, b]_{\mathbb{R}}$.

For all $a, b \in \mathbb{R}$ with $a < b$, we will denote by $I(a, b)$ any of the intervals $(a, b), [a, b), (a, b]_\mathbb{R}$ or $[a, b)_\mathbb{R}$. The length of an interval of the form $I(a, b)$ is denoted by $l(I(a, b))$ and is defined as $b - a$.

If $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we define $\text{supp}(f) = \{x \in A : f(x) \neq 0\}$, where the closure is in the usual topology of $\mathbb{R}$. If $f : A \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we define $\text{supp}(f) = \{x \in \mathbb{R} : f(x) \neq 0\}$.

We will denote by $C^\omega([a, b])$ the space of analytic functions over $[a, b] \subseteq \mathbb{R}$ and by $C^k([a, b])$ the space of $k$-times differentiable functions with continuous $f^{(k)}$ over $[a, b]$. If $f \in C^k([a, b])$, we will denote by $f^{(i)}$ its $i$-th derivative of order $0 \leq i \leq k$. Notice that $f^{(0)} = f$. We will sometimes write $f'$ instead of $f^{(1)}$ and $f''$ instead of $f^{(2)}$.

If $A \subseteq \mathbb{R}$ is a Lebesgue measurable set, we will denote by $m_L(A)$ its Lebesgue measure.

Consider the equivalence relation on Lebesgue measurable functions given by equality almost everywhere: two measurable functions $f$ and $g$ are equivalent if $m_L(\{x \in \Omega : f(x) \neq g(x)\}) = 0$. We will not distinguish between the function $f$ and its equivalence class, and we will say that $f = g$ whenever the functions $f$ and $g$ are equal almost everywhere.

For all $1 \leq p < \infty$, $L^p(A)$ is the set of equivalence classes of measurable functions $f : A \rightarrow \mathbb{R}$ that satisfy

$$\int_A |f|^p \, dx < \infty.$$ 

If $f \in L^p(A)$, the $L^p$ norm of $f$ is defined by

$$||f||_p = \int_A |f|^p \, dx.$$ 

$L^\infty(A)$ is the set of equivalence classes of measurable functions that are essentially bounded: we will say that $f : A \rightarrow \mathbb{R}$ belongs to $L^\infty(A)$ if there exists $y \in \mathbb{R}$ such that $m_L(\{x \in \Omega : |f(x)| > y\}) = 0$. In this case,

$$||f||_\infty = \inf\{y \in \mathbb{R} : m_L(\{x \in A : f(x) > y\}) = 0\}.$$ 

If $1 < p < \infty$, recall that $p'$ is defined as the unique solution to the equation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$
while \( l' = \infty \) and \( \infty' = 1 \).

## 2. Preliminary notions

The purpose of this section is to recall some results that will be used in the sequel of the paper. We will briefly introduce the Levi-Civita field, the notion of weak limit, power series with coefficients in \( \mathcal{R} \), and the continuations of real functions to the Levi-Civita field. The section includes also some novel results that will be used throughout the paper.

### 2.1. The Levi-Civita field

**Definition 2.1.** A set \( F \subset \mathbb{Q} \) is called left-finite if and only if for every \( q \in \mathbb{Q} \) the set \( \{ x \in F : x \leq q \} \) is finite. The Levi-Civita field is the set \( \mathcal{R} = \{ x : \mathbb{Q} \rightarrow \mathbb{R} : \{ q : x(q) \neq 0 \} \text{ is left-finite} \} \), together with the pointwise sum and the product defined by the formula

\[
(x \cdot y)(q) = \sum_{q_1+q_2=q} x(q_1) \cdot y(q_2).
\]

For a review of the algebraic and topological properties of \( \mathcal{R} \), we refer to \([1, 3, 4, 25]\) and references therein. Here we will only recall some fundamental properties and definitions of \( \mathcal{R} \) that will be useful in the sequel.

**Lemma 2.2.** An element \( x \in \mathcal{R} \) is uniquely characterized by its support, which forms an ascending (possibly finite) sequence \( \{ q_n \}_{n \in \mathbb{N}} \), and a corresponding sequence \( \{ x[q_n] \}_{n \in \mathbb{N}} \) of function values.

**Definition 2.3.** For all nonzero \( x \in \mathcal{R} \), define \( \lambda(x) = \min(\text{supp}(x)) \), and \( \lambda(0) = \infty \). If \( x, y \in \mathcal{R}, x \neq y \), we say that \( x < y \) if \( (x-y)[\lambda(x-y)] < 0 \). The relation \( x > y \) is defined as \( y < x \), and the relations \( \leq \) and \( \geq \) are defined in the usual way.

**Lemma 2.4.** The function \( \lambda \) is a valuation with range \( \mathbb{Q} \cup \{ \infty \} \). In particular, for all \( x, y \in \mathcal{R} \), \( \lambda(xy) = \lambda(x) + \lambda(y) \) and \( \lambda(x+y) \geq \min\{\lambda(x),\lambda(y)\} \). Moreover, if \( x \neq y \), \( \lambda(x+y) > \min\{\lambda(x),\lambda(y)\} \).

The Levi-Civita field contains infinite and nonzero infinitesimal elements. Moreover, the valuation \( \lambda \) allows to compare the relative size of any element of \( \mathcal{R} \).

**Definition 2.5.** Let \( x, y \in \mathcal{R}, x, y \geq 0 \). We will write \( x \ll y \) if \( \lambda(x) > \lambda(y) \). In this case, we say that \( x \) is infinitely smaller than \( y \). This relation is extended in the expected way to all \( x, y \in \mathcal{R} \). Notice that \( |x| \ll |y| \) if and only if \( n|x| < |y| \) for all \( n \in \mathbb{N} \).

Comparison of the size of two elements in \( \mathcal{R} \) can be obtained not only via the inequalities \( < \) and \( \ll \), but also by introducing different notions of “agreement on the order of magnitude” and “equality up to an infinitesimal relative error”.

**Definition 2.6.** Let \( x, y \in \mathcal{R} \). We will write \( x \sim y \) if \( \lambda(x) = \lambda(y) \), and \( x \approx y \) if \( x \sim y \) and \( x[\lambda(x)] = y[\lambda(y)] \).

In \([2]\) it is observed that \( \sim \) and \( \approx \) are equivalence relations.

We will often reference the set

\[
M_o = \{ \varepsilon \in \mathcal{R} : 0 < \lambda(\varepsilon) < \infty \} = \{ \varepsilon \in \mathcal{R} : 0 < |\varepsilon| \ll 1 \},
\]

that is the set of nonzero infinitesimal numbers in the Levi-Civita field. In analogy with the common practice of nonstandard analysis, if \( x \in \mathcal{R} \), we will refer to the set \( \mu(x) = \{ y \in \mathcal{R} : |x-y| \ll 1 \} \) as the monad of the point \( x \). Notice that monads are not intervals.
Lemma 2.7. For all $x, a, b \in \mathcal{R}$ with $a < b$, $\mu(x) \neq I(a, b)$.

Proof. If $x \notin I(a, b)$, then the assertion is trivial. If $x \in I(a, b)$, suppose at first that $b - a \neq 0$: as a consequence, either $x \neq a$ or $x \neq b$. Since there exist $z, w \in I(a, b)$ with $z \approx a$ and $w \approx b$, we conclude $I(a, b) \setminus \mu(x)$. If $b - a \approx 0$, then $x \approx a \approx b$, but we have also $x \approx a + 2(b - a) \notin \mu(x)$. However $a + 2(b - a) \notin I(a, b)$, so that $I(a, b) \subset \mu(x)$. $\square$

Finally, we find it useful to borrow the definition of standard part from Robinson’s framework [23] and define it also for elements of the Levi-Civita field.

Definition 2.8. Let $x \in \mathcal{R}$. We define the function $^o : \mathcal{R} \to \mathbb{R} \cup \{\pm \infty\}$ by posing

$$^o x = \begin{cases} x[0] & \text{if } \lambda(x) \geq 0 \\ +\infty & \text{if } \lambda(x) < 0 \text{ and } x > 0 \\ -\infty & \text{if } \lambda(x) < 0 \text{ and } x < 0. \end{cases}$$

2.2. Convergence and weak convergence. In the Levi-Civita field there are two notions of convergence. The first is induced by the metric and it is analogous to the usual definition of limit for real-valued sequences. This notion of convergence is usually called strong convergence.

Definition 2.9. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of $\mathcal{R}$ strongly converges to $l \in \mathcal{R}$ if and only if

$$\forall \varepsilon \in \mathcal{R}, \varepsilon > 0, \exists n \in \mathbb{N} : \forall m > n \sup|c_m - l| < \varepsilon.$$ We will denote strong convergence with the expression $s\lim_{n \to \infty} a_n = l$.

Many properties of strong convergence have been established by Berz and Shamseddine [1, 2, 28]. A property shared by convergence of real sequences and strong convergence in the Levi-Civita field is completeness: in fact $\mathcal{R}$ is complete with respect to strong convergence and every converging sequence in $\mathcal{R}$ is bounded. Moreover, the squeeze theorem is still satisfied by strong convergence.

Throughout the paper we will use the following property: if a series with non-negative terms strongly converges to a number $l \in \mathcal{R}$, then at least one of the terms has the same order of magnitude as $l$. The proof of this result relies on the notion of regular sequence.

Definition 2.10. A sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of $\mathcal{R}$ is regular if and only if $\bigcup_{n \in \mathbb{N}} \text{supp}(a_n)$ is a left-finite set.

In [2] it is proved that sequences that strongly converge in $\mathcal{R}$ are regular.

We are now ready to prove the result stated above on series with non-negative terms.

Lemma 2.11. Let $\{a_n\}_{n \in \mathbb{N}}$ satisfy $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} a_n = s\lim_{k \to \infty} \sum_{n \leq k} a_n = l$. If $\lambda(l) = q$, then

1. $\lambda(a_n) \geq q$ for all $n \in \mathbb{N}$
2. there exists $n \in \mathbb{N}$ such that $\lambda(a_n) = q$.

Proof. In order to prove (1), suppose that $\lambda(a_k) < q$ for some $k \in \mathbb{N}$. Taking into account the hypothesis that $a_n \geq 0$ for all $n \in \mathbb{N}$, from Lemma 2.4 we obtain $\lambda(\sum_{n < k} a_n) < q$. Since $a_n \geq 0$ for all $n \in \mathbb{N}$, the inequality $\sum_{n \leq k} a_n \leq \sum_{n \in \mathbb{N}} a_n$ entails also $\lambda(l) < q$. By contrapositive, we obtain the desired assertion.

For the proof of (2), suppose towards a contradiction that $\max_{n \in \mathbb{N}} \lambda(a_n) = p > q$ for all $n \in \mathbb{N}$. This maximum exists since strongly convergent sequences are regular. Let now $\varepsilon > 0$ satisfy $\lambda(\varepsilon) > p$, and let $k \in \mathbb{N}$ such that $\lambda(\sum_{n \leq k} a_n) = p$ and $|l - \sum_{n \leq k} a_n| < \varepsilon$. From the latter inequality we deduce that $l = \sum_{n \leq k} a_n + h$ for some $h \in \mathcal{R}$ with $h \geq 0$ and $\lambda(h) \geq$
\[ \lambda(\varepsilon) > p. \] Since \( a_n, l \) and \( h \) are non-negative, we have \( \lambda(l) = \min \{ \lambda(\sum_{n \leq k} a_n), \lambda(h) \} = p > q \), against the hypothesis \( \lambda(l) = q \).

Despite its good properties, strong convergence is very restrictive, since if \( \{a_n\}_{n \in \mathbb{N}} \) is strongly convergent, then for all \( q \in \mathbb{Q} \) the real sequence \( n \mapsto a_n[q] \) is eventually constant [3]. As a consequence, real converging sequences are not strongly convergent in \( \mathcal{R} \), unless they are eventually constant.

In order to enlarge the class of converging sequences, it has been proposed a weaker notion of convergence, usually called weak convergence.

**Definition 2.12.** A sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements of \( \mathcal{R} \) weakly converges to \( l \in \mathcal{R} \) if and only if
\[ \forall \varepsilon \in \mathbb{R} \quad \exists n \in \mathbb{N} : \forall m > n \quad \max_{q \in \mathbb{Q}, \varepsilon < 1} \| (c_m - l)[q] \| < \varepsilon. \]
We will denote weak convergence with the expression \( \operatorname{w-lim}_{n \to \infty} a_n = l \).

In [1] it is proved that strong convergence implies weak convergence, but the converse is false.

In order to determine whether a sequence is weakly convergent, it is convenient to use a weak converge criterion, stated for instance in Theorem 2.13 of [28].

**Theorem 2.13.** If \( \{a_n\}_{n \in \mathbb{N}} \) is a regular sequence of elements of \( \mathcal{R} \) and if there exists \( l \in \mathcal{R} \) such that for all \( q \in \mathbb{Q} \) \( \lim_{n \to \infty} a_n[q] = l[q] \), then \( \operatorname{w-lim}_{n \to \infty} a_n = l \).

On the other hand, if \( \{a_n\}_{n \in \mathbb{N}} \) converges weakly to \( l \in \mathcal{R} \), then for all \( q \in \mathbb{Q} \) the real sequence \( \{a_n[q]\}_{n \in \mathbb{N}} \) converges to \( l[q] \in \mathbb{R} \), and the convergence is uniform on every subset of \( \mathbb{Q} \) bounded above.

The above criterion allows to conclude that real converging sequences are weakly convergent also in \( \mathcal{R} \).

**Corollary 2.14.** If \( \{r_n\}_{n \in \mathbb{N}} \) is a sequence of real numbers such that \( \lim_{n \to \infty} r_n = r \), then \( \operatorname{w-lim}_{n \to \infty} r_n = r \).

Weak convergence does not satisfy some properties of convergence in \( \mathbb{R} \) and of strong convergence in \( \mathcal{R} \). One of them is the squeeze theorem. In fact, if two sequences \( \{a_{n}\}_{n \in \mathbb{N}} \) and \( \{b_{n}\}_{n \in \mathbb{N}} \) satisfy
- \( a_{i} \leq b_{j} \) for all \( i, j \in \mathbb{N} \) and
- \( \operatorname{w-lim}_{n \to \infty} a_{n} = l = \operatorname{w-lim}_{n \to \infty} b_{n} \),

there might exist a sequence \( \{c_{n}\}_{n \in \mathbb{N}} \) such that \( a_{i} \leq c_{j} \leq b_{k} \) for all \( i, j, k \in \mathbb{N} \), but \( \operatorname{w-lim}_{n \to \infty} c_{n} \) might be different from \( l \) or might not exist.

**Example 2.15.** Let \( a_{n} = 0 \) for all \( n \in \mathbb{N} \) and let \( b_{n} = n^{-1} \); then \( \operatorname{w-lim}_{n \to \infty} a_{n} = \operatorname{w-lim}_{n \to \infty} b_{n} = 0 \) by Corollary 2.14. Any constant sequence \( \{c_{n}\}_{n \in \mathbb{N}} \) assuming a positive infinitesimal value \( h \) satisfies \( a_{i} \leq c_{j} \leq b_{k} \) for all \( i, j, k \in \mathbb{N} \), however it converges to \( h > 0 = \operatorname{w-lim}_{n \in \mathbb{N}} b_{n} \). On the other hand, if \( c_{n} = nd \), then \( \{c_{n}\}_{n \in \mathbb{N}} \) still satisfies \( a_{i} < c_{j} < b_{k} \) for all \( i, j, k \in \mathbb{N} \), but \( \operatorname{w-lim}_{n \to \infty} c_{n} \) does not exist.

As a consequence of the previous example, we conclude that weak limits do not preserve inequalities. In particular, the weak limit of a strictly positive sequence might be negative.

**Example 2.16.** If \( a_{n} = (nd)^{-1} - 1 \), then \( a_{n} > 0 \) for all \( n \in \mathbb{N} \), but \( \operatorname{w-lim}_{n \to \infty} a_{n} = -1 < 0 \).

In addition, weak convergence is not well-behaved with respect to the equivalence relation \( \approx \).
Remark 2.17. Let \( \{a_n\}_{n \in \mathbb{N}} \) and \( \{b_n\}_{n \in \mathbb{N}} \) be two sequences of elements of \( \mathcal{A} \). If \( a_n \approx b_n \) for all \( n \in \mathbb{N} \), it might be possible that \( \{a_n\}_{n \in \mathbb{N}} \) converges (strongly or weakly), but \( \{b_n\}_{n \in \mathbb{N}} \) does not. This happens for instance if \( a_n = 1 \) for all \( n \in \mathbb{N} \) and \( b_n = 1 + nd \) for all \( n \in \mathbb{N} \).

Finally, whereas strongly converging sequences are regular, there are some weakly converging sequences that are not regular.

Remark 2.18. In [2], it is shown that strongly converging sequences are regular. On the other hand there are sequences that are not regular but that weakly converge in \( \mathcal{A} \): an example is \( \{n^{-1}d^{-n}\}_{n \in \mathbb{N}} \), whose weak limit is 0. These sequences need not be unbounded: another weakly converging sequence that is not regular is \( \{n^{-1}d^{-1+n^{-1}}\}_{n \in \mathbb{N}} \).

As a consequence, there are some sequences \( \{a_n\}_{n \in \mathbb{N}} \) that satisfy the formula
\[
\forall \varepsilon \in \mathbb{R}, \varepsilon > 0, \exists k \in \mathbb{N} : \forall m,n > k \max_{q \in \mathbb{Q}, q \leq -1} |(a_m - a_n)| < \varepsilon,
\]
that however do not converge weakly in \( \mathcal{A} \). An example is \( \{\sum_{n \leq k} n^{-1}d^{-n}\}_{k \in \mathbb{N}} \), whose limit is the function \( x: \mathbb{Q} \to \mathbb{R} \) defined by \( x(-n) = n^{-1} \) whenever \( n \in \mathbb{N} \), and \( x(q) = 0 \) otherwise. This is not an element of \( \mathcal{A} \), since \( \text{supp}(x) \) is not a left-finite set.

For a detailed study of the properties of the strong and weak convergence in the Levi-Civita field, we refer to [1, 2, 28] and references therein.

In the sequel, it will be useful to distinguish between sequences that are Cauchy with respect to the strong and weak convergence.

Definition 2.19. A sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements of \( \mathcal{A} \) is strongly Cauchy, or simply Cauchy, if and only if
\[
\forall \varepsilon \in \mathbb{R}, \varepsilon > 0, \exists k \in \mathbb{N} : \forall m,n > k |a_m - a_n| < \varepsilon.
\]
A sequence \( \{a_n\}_{n \in \mathbb{N}} \) is weakly Cauchy if and only if it satisfies formula 2.1.

2.3. Divergent sequences. Besides the usual notions of strong and weak convergence, we find it convenient to introduce also a notion of divergent sequence. In principle, in the Levi-Civita field this concept could be defined in different ways: for instance, it could be argued that all of the sequences \( \{nd\}_{n \in \mathbb{N}}, \{n\}_{n \in \mathbb{N}} \) and \( \{d^{-n}\}_{n \in \mathbb{N}} \) diverge in \( \mathcal{A} \). It would be possible to define different notions of divergence that describe each of these behaviours; however, for our purposes we find it convenient to say that a sequence of elements of \( \mathcal{A} \) diverges if it is unbounded in the strong topology of \( \mathcal{A} \).

Definition 2.20. A sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements of \( \mathcal{A} \) diverges to \( +\infty \) if and only if
\[
\forall M \in \mathcal{A}, \exists n \in \mathbb{N} : \forall m > n a_m > M.
\]

A sequence \( \{a_n\}_{n \in \mathbb{N}} \) of elements of \( \mathcal{A} \) diverges to \( -\infty \) if and only if the sequence \( \{-a_n\}_{n \in \mathbb{N}} \) diverges to \( +\infty \).

Notice that, contrarily to the real case, monotonic sequences that do not diverge in \( \mathcal{A} \) might not have a strong or a weak limit, or even a converging subsequence. In fact, if \( \{a_n\}_{n \in \mathbb{N}} \) is an increasing sequence of elements of \( \mathcal{A} \), then it might exhibit any of the following behaviours:

1. there exists \( l \in \mathcal{A} \) such that \( \text{s-lim}_{n \to \infty} a_n = l \);
2. there exists \( l \in \mathcal{A} \) such that \( \text{w-lim}_{n \to \infty} a_n = l \);
3. the sequence diverges to \( +\infty \) and converges weakly to some \( l \in \mathcal{A} \);
4. the sequence diverges to \( +\infty \) and does not converge weakly;
5. the sequence is bounded from above and does not converge weakly or strongly.
A divergent sequence \( \{a_n\}_{n \in \mathbb{N}} \) such that \( \text{w-lim}_{n \to \infty} a_n = 0 \) is obtained by defining \( a_n = n^{-1}d^{-n} \), and an increasing sequence \( \{b_n\}_{n \in \mathbb{N}} \) that is bounded from above but does not have a weak or strong limit can be obtained by posing \( b_n = nd \).

### 2.4. Power series on the Levi-Civita field.

Real power series are series of the form
\[ \sum_{n \in \mathbb{N}} a_n (x - x_0)^n, \]
where \( x_0 \in \mathbb{R} \) and \( x \) is a variable ranging in an interval \( I \) (possibly of length 0) centered at \( x_0 \); the number \( l(I)/2 \) is usually called the radius of convergence of the power series centred at \( x_0 \).

If \( a_n \in \mathcal{R} \) for all \( n \in \mathbb{N} \) and if \( x_0 \in \mathcal{R} \), it is possible to define a power series either as the strong limit or the weak limit of the sequence of the partial sums \( \sum_{n \leq k} a_n (x - x_0)^n \):

\[
\sum_{n \in \mathbb{N}} a_n (x - x_0)^n = \text{s-lim}_{k \to \infty} \sum_{n \leq k} a_n (x - x_0)^n \quad \text{or} \quad \sum_{n \in \mathbb{N}} a_n (x - x_0)^n = \text{w-lim}_{k \to \infty} \sum_{n \leq k} a_n (x - x_0)^n.
\]

Strong and weak convergence of power series has been studied in \([2, 4, 27, 28]\), where there have been proved some convergence criteria for both the strong and weak limit of equation (2.2). It has been argued by Berz and Shamseddine that the most convenient definition of power series in the Levi-Civita field is the one relying on the weak limit. With this definition, power series share many properties with their real counterpart, such as the possibility to calculate the derivative by differentiating the power series term by term, or the possibility to re-expanded the power series around any point in its domain of convergence. Other properties of the power series defined using the weak limit are discussed in detail in \([4, 28]\). From now on, the expression \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) will denote the weak limit \( \text{w-lim}_{k \to \infty} \sum_{n \leq k} a_n (x - x_0)^n \).

**Definition 2.21.** For every \( A \subseteq \mathcal{R} \), we will denote by \( \mathcal{P}(A) \) the algebra of power series that weakly converge for every \( x \in A \).

The functions defined as power series with coefficients in \( \mathcal{R} \) satisfy many properties of real continuous functions, such as an intermediate value theorem, an extreme value theorem and a mean value theorem. For these results, we refer to \([4, 27, 28]\).

There is a strong relation between real power series and power series in \( \mathcal{R} \) with real coefficients: if \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \) and if the real power series \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) has radius of convergence \( R \), the power series \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) converges weakly in \( \mathcal{R} \) whenever \( |x - x_0| < R \) and \( |x - x_0| \neq R \). As a consequence, in the region of convergence it is possible to define a function \( x \mapsto \text{w-lim}_{k \to \infty} \sum_{n \leq k} a_n (x - x_0)^n \); moreover, whenever \( x \) is real, this function agrees with the real power series \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) (see Definition 2.26 for more details).

If \( |x - x_0| < R \) and \( |x - x_0| \sim R \), however, the power series in the Levi-Civita field might not converge. This is a consequence of the following result, that does not require the additional hypothesis \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \).

**Proposition 2.22.** Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{R} \) and let \( x_0 \in \mathcal{R} \).

1. If for some \( x \in \mathcal{R} \) the power series \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) diverges, namely if the sequence \( k \mapsto \sum_{n \leq k} a_n (x - x_0)^n \) diverges in the sense of Definition 2.20, then for every \( h \in \mathcal{R} \) with \( \lambda(h) > \lambda(x - x_0) \), the power series \( \sum_{n \in \mathbb{N}} a_n (x + h - x_0)^n \) diverges.

2. If for some \( x \in \mathcal{R} \) the power series \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) does not converge, namely if the sequence \( k \mapsto \sum_{n \leq k} a_n (x - x_0)^n \) does not converge weakly in the sense of Definition 2.9, there exists \( s \in \mathbb{Q} \) such that for every \( h \) with \( \lambda(h) > s \), the power series \( \sum_{n \in \mathbb{N}} a_n (x + h - x_0)^n \) does not converge.
Proof. A consequence of Theorem 3.8 of [3] is that $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ converges weakly if and only if $\sum_{n \in \mathbb{N}} |a_n (x - x_0)^n|$ converges weakly. Thus we can assume without loss of generality that $a_n \geq 0$ for all $n \in \mathbb{N}$ and that $x - x_0 > 0$. Notice also that $\lambda(x - x_0) < \infty$, since any power series strongly converges whenever $x = x_0$.

For every $h$ satisfying the hypothesis $\lambda(h) > \lambda(x - x_0)$, we have the equality $\lambda(x + h - x_0) = \lambda(x - x_0)h$. From this equality and from Lemma 2.4 we obtain also

\begin{equation}
\lambda[(x + h - x_0)^n] = \lambda[(x - x_0)^n] = n\lambda [x - x_0]
\end{equation}

and

\begin{equation}
\sum_{n \leq k} a_n (x - x_0)^n \approx \sum_{n \leq k} a_n (x + h - x_0)^n.
\end{equation}

for all $n \in \mathbb{N}$.

(1) Since $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ diverges, for all $q \in \mathbb{Q}$ there exists $k_q \in \mathbb{N}$ such that $\sum_{n \leq k_q} a_n (x - x_0)^n > d^n$. Thanks to equation (2.4), we have also the inequality $\sum_{n \leq k_q} a_n (x + h - x_0)^n > d^n$. Thus if $M \in \mathbb{R}$ and if $k > k_{\lambda(M)}$, then $\sum_{n \leq k} a_n (x - x_0)^n > M$, so that the series $\sum_{n \in \mathbb{N}} a_n (x + h - x_0)^n$ diverges.

(2) Suppose now that $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ does not converge. If $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ diverges, then by part (1) of the proof we can choose $s = \lambda(x - x_0)$.

Under the hypothesis that $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ does not converge, there exists $p \in \mathbb{Q}$ such that $\lambda(\sum_{n \leq k} a_n (x - x_0)^n) \geq p$ for all $k \in \mathbb{N}$. Thanks to the first part of the weak convergence criterion of Theorem 2.13, then either the sequence $\{\sum_{n \leq k} a_n (x - x_0)^n\}_{k \in \mathbb{N}}$ is not regular or there exists $q \in \mathbb{Q}$ such that the real sequence $\{(\sum_{n \leq k} a_n (x - x_0)^n) \{q\}\}_{k \in \mathbb{N}}$ does not converge.

At first, suppose that $\{\sum_{n \leq k} a_n (x - x_0)^n\}_{k \in \mathbb{N}}$ is regular and chose $q \in \mathbb{Q}$ in a way that the real sequence $\{(\sum_{n \leq k} a_n (x - x_0)^n) \{q\}\}_{k \in \mathbb{N}}$ does not converge. Notice that, by definition of $p, q \geq p$. Recall the formula for the difference of two $n$-th powers:

$$(x + h - x_0)^n - (x - x_0)^n = h \sum_{j=0}^{n-1} (x + h - x_0)^{n-j} (x - x_0)^j.$$ 

Under the hypothesis that $\lambda(h) > \lambda(x - x_0)$, the previous equality and equation (2.3) entail that for all $k \in \mathbb{N}$

\begin{equation}
\lambda \left( \sum_{n \leq k} a_n (x + h - x_0)^n \right) - \lambda \left( \sum_{n \leq k} a_n (x - x_0)^n \right) = \lambda(h) + \lambda \left( \sum_{n \leq k} a_n (x - x_0)^n \right) \geq \lambda(h) + p.
\end{equation}

Hence, if $\lambda(h) > \lambda(x - x_0)$ and $\lambda(h) + p > q$, then also

$$\left( \sum_{n \leq k} a_n (x + h - x_0)^n \right) \{q\} = \left( \sum_{n \leq k} a_n (x - x_0)^n \right) \{q\},$$

so that the real sequence $\{(\sum_{n \leq k} a_n (x + h - x_0)^n) \{q\}\}_{k \in \mathbb{N}}$ does not converge. The second part of the weak convergence criterion of Theorem 2.13 ensures that this is sufficient to entail that the sequence $\{(\sum_{n \leq k} a_n (x + h - x_0)^n) \}_{k \in \mathbb{N}}$ cannot converge weakly. We obtain the desired assertion by defining $s = \max\{\lambda(x - x_0), q - p\}$.

Suppose now that the sequence $\{(\sum_{n \leq k} a_n (x - x_0)^n) \}_{k \in \mathbb{N}}$ is not regular. Then there exists $q \in \mathbb{Q}$ such that

$$\bigcup_{k \in \mathbb{N}} \text{supp} \left( \sum_{n \leq k} a_n (x - x_0)^n \right) \cap (-\infty, q] \subset \mathbb{Q}$$
is not finite. If $\lambda (h) > \lambda (x - x_0)$ and $\lambda (h) > q - p$, then by equation (2.5)

$$\bigcup_{k \in \mathbb{N}} \text{supp} \left( \sum_{n \leq k} a_n (x - x_0)^n \right) \cap (-\infty, q] = \bigcup_{k \in \mathbb{N}} \text{supp} \left( \sum_{n \leq k} a_n (x + h - x_0)^n \right) \cap (-\infty, q]_{\mathbb{Q}},$$

so that also the sequence $\{ \sum_{n \leq k} a_n (x + h - x_0)^n \}_{k \in \mathbb{N}}$ is not regular. Once again, we can define $s = \max \{ \lambda (x - x_0), q - p \}$. \hfill $\Box$

If the coefficients of the power series are real, then we can sharpen the above results.

**Corollary 2.23.** Consider the power series $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ with $x_0 \in \mathbb{R}$ and $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Then the power series $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ does not diverge for any $x \in \mathbb{R}$ such that $\lambda (x) \geq 0$. Moreover, if for some $x \in \mathbb{R}$ the series $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ does not weakly converge in $\mathcal{H}$, then for all $h \in M_0$ the series $\sum_{n \in \mathbb{N}} a_n (x + h - x_0)^n$ does not converge.

**Proof.** If $a_n = 0$ for all but finitely many $n \in \mathbb{N}$, then both assertions are trivial. Suppose then that $a_n \neq 0$ for infinitely many $n \in \mathbb{N}$.

The hypotheses over $a_n, x_0$ and $x$ entail that

1. $\lambda (a_n (x - x_0)^n) = 0$ or $\lambda (a_n (x - x_0)^n) = \infty$ for all $n \in \mathbb{N}$, and
2. $\lambda (\sum_{n \leq k} a_n (x - x_0)^n) \geq 0$ for all $k \in \mathbb{N}$.

Then the power series does not diverge as a consequence of inequality (2).

If $x \in \mathbb{R}$ and the series $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ does not converge weakly, then from our hypotheses over $a_n$, from equation (2.5) with $p = 0$ and from (1) we obtain that, whenever $\lambda (h) > 0$,

$$\left( \sum_{n \leq k} a_n (x + h - x_0)^n \right)[0] = \left( \sum_{n \leq k} a_n (x - x_0)^n \right)[0].$$

Consequently, if $h \in M_0$ then also the series $\sum_{n \in \mathbb{N}} a_n (x + h - x_0)^n$ does not converge. \hfill $\Box$

As a consequence of this corollary, there are some real power series centred at some real point $x_0$ that converge in an open interval $(x_0 - R, x_0 + R)$, but whose counterpart in $\mathcal{H}$ do not converge in any interval of the form $I(x_0 - R, x_0 + R)$.

**Example 2.24.** Consider the power series $\sum_{n \in \mathbb{N}} x^n$: it has real coefficients, so whenever $|x| < 1$ and $|x| \neq 1$ it converges to a function that can be considered as an extension to the Levi-Civita field of the real function $f(x) = (1 - x)^{-1}$ (for a more precise statement of this result, we refer to Definition 2.26 and to [8]). However, if $x \approx 1$, the series does not converge, since $\sum_{n \leq k} x^n \approx \sum_{n \leq k} 1 = k$. Similarly, if $x \approx -1$, the series does not converge, since $\sum_{n \leq k} x^n \approx \sum_{n \leq k} -1^n$.

Now consider the function $f : \mathbb{R} \setminus \{ 0 \} \to \mathbb{R}$ defined by $f(x) = x^{-1}$. The power series expansion of $f$ around any $x_0 > 0$ is $f(x_0 - x) = \sum_{n \in \mathbb{N}} x_0^n x_0^{-n} x^n$; this series weakly converges iff $\frac{|x|}{x_0} < 1$ and $\frac{|x|}{x_0} \neq 1$.

The behaviour described in the first part of the previous example can be expressed with an analogy from Robinson’s framework of analysis with infinitesimals.

**Remark 2.25.** The behaviour of power series in the Levi-Civita field can be described by introducing the notion of nearstandard point, commonly used in the topology of hyperreal fields. We will say that a point $x \in A \subseteq \mathcal{H}$ is nearstandard in $A$ iff $\exists x \in A$. With this terminology, we can say that if $\sum_{n \in \mathbb{N}} a_n (x - x_0)^n$ is a real power series with radius of convergence $R$, then its Levi-Civita counterpart weakly converges on the nearstandard points of $(x_0 - R, x_0 + R)_{\mathcal{H}}$, but it might not converge on those points that are not nearstandard in $(x_0 - R, x_0 + R)_{\mathcal{H}}$. 
2.5. Extending real functions to the Levi-Civita field. Since power series with real coefficients weakly converge also in $\mathcal{R}$, it is possible to use them to define several extensions of real continuous functions to the Levi-Civita field. These extensions are obtained from the Taylor series expansion of a function at a point.

**Definition 2.26.** Let $f \in C^\infty([a, b])$. The analytic extension of $f$ is defined as

$$\mathcal{F}_n(r + \varepsilon) = \sum_{i=0}^{n} f^{(i)}(r) \frac{\varepsilon^i}{i!}.$$  

for all $r \in [a, b]$ and for all $\varepsilon \in M_{\varepsilon}$ such that $r + \varepsilon \in [a, b]_{\mathcal{R}}$. If $f$ is analytic, then $\mathcal{F}_n$ will be called the canonical extension of $f$. The only exceptions are the canonical extensions of the exponential function and of the trigonometric functions sine and cosine, still denoted by $e^x$, $\sin(x)$ and $\cos(x)$. Notice also that these extensions are defined for all $x \in \mathcal{R}$ with $\lambda(x) \geq 0$.

Let $f \in C^k([a, b])$, possibly with $k = \infty$. The order $n$ extension of $f$, with $0 \leq n \leq k$, is defined by

$$\mathcal{F}_n(r + \varepsilon) = \sum_{i=0}^{n} f^{(i)}(r) \frac{\varepsilon^i}{i!}.$$

for all $r \in [a, b]$ and for all $\varepsilon \in M_{\varepsilon}$ such that $r + \varepsilon \in [a, b]_{\mathcal{R}}$.

**Remark 2.27.** Definition 2.26 applies also to functions whose domains are intervals of the form $(a, b)$, $[a, b)$, $(a, b)$: the only difference is that the extensions of $f$ will be defined over the nearest standard points of the corresponding interval $(a, b)_{\mathcal{R}}$, $[a, b)_{\mathcal{R}}$, $(a, b)_{\mathcal{R}}$.

It is possible to prove that each of the extensions introduced in Definition 2.26 is unique and well-defined. The extensions $\mathcal{F}_n$ and $\mathcal{F}_f$ extend the corresponding real function $f \in C^k([a, b])$ in the sense that for every $x \in [a, b]$, $f(x) = \mathcal{F}_n$ for all $n \leq k$, possibly with $k = \infty$. For more properties of the continuations of real functions to the Levi-Civita field, we refer to [1, 3, 4, 5, 8, 25] and references therein.

2.6. On the weak pointwise convergence of sequences of analytic functions. In the Levi-Civita field, weak pointwise convergence of a sequence of canonical extensions of analytic functions is possible only under very strong hypotheses.

**Proposition 2.28.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of analytic functions over $[a, b]$ and let $g_n = \lim_{n \to \infty} f_n$ for all $n \in \mathbb{N}$. Then for all $x \in [a, b]_{\mathcal{R}}$ the sequence $(g_n(x))_{n \in \mathbb{N}}$ converges weakly for all $x \in [a, b]_{\mathcal{R}}$ if and only if for all $r \in [a, b]$ and for all $p \in \mathbb{Q}$, $p \geq 0$,

$$\lim_{n \to \infty} \sum_{i \leq p} f^{(i)}(r)$$  

exists and is finite.

**Proof.** Let $x = r + h \in [a, b]_{\mathcal{R}}$ with $r \in \mathbb{R}$ and $h \in M_{\varepsilon}$. For every $q \in \mathbb{Q}$,

$$g_n(x)[q] = \sum_{i \leq q} f^{(i)}(r) \frac{h^{(i)}[q]}{i!}.$$  

Notice also that, thanks to the hypotheses over $f_n$ and $x$, $g_n(x)[q] = 0$ whenever $q < 0$.

Assume that the sequence $n \mapsto \sum_{i \leq q} f^{(i)}(r)$ has a finite limit for all $q \in \mathbb{Q}$, $q \geq 0$. Since $\lim_{n \to \infty} \frac{h^{(i)}[q]}{i!} = 0$ for all $q \in \mathbb{Q}$, the real sequence $\left\{ \frac{h^{(i)}[q]}{i!} \right\}_{i \in \mathbb{N}}$ is bounded from above.
by some $b_q \in \mathbb{R}$. Thus, for all $q \in \mathbb{Q}$

$$\left| \sum_{l \leq \frac{1}{l^2}} f_n^l(r) \frac{h^l[q]}{l^l} \right| \leq |b_q| \left| \sum_{l \leq \frac{1}{l^2}} f_n^l(r) \right|.$$ 

By our hypothesis over the sequences 2.6, we conclude that also $n \mapsto \sum_{l \leq \frac{1}{l^2}} f_n^l(r) \frac{h^l[q]}{l^l}$ has a finite limit as $n \to \infty$. As a consequence, we can define a number $g(x) \in \mathcal{S}$ by posing

$$g(x)[q] = \lim_{n \to \infty} \sum_{l \leq \frac{1}{l^2}} f_n^l(r) \frac{h^l[q]}{l^l}.$$ 

By Theorem 2.13, the sequence $\{g_n(x)\}_{n \in \mathbb{N}}$ converges weakly to $g(x)$.

Now suppose that there exists $q \in \mathbb{Q}$, $q \geq 0$, such that (2.6) does not exist or is not finite. Thanks to equality (2.7), the real sequence $g_n(x)[q]$ does not converge. By Theorem 2.13, the sequence $\{g_n(x)\}_{n \in \mathbb{N}}$ does not converge weakly for some $x \in [a,b]_\mathcal{S}$.

The condition expressed in Theorem 2.28 is rarely satisfied even by sequences of functions that converge uniformly in $\mathbb{R}$, as it is shown in the next example.

Example 2.29. Consider the sequence of analytic functions $\{f_n\}_{n \in \mathbb{N}}$, with $f_n(x) = \sin(n!x)$.

This sequence converges uniformly to 0 for all $x \in \mathbb{R}$, but $\lim_{n \to \infty} f_n^l(x) = \lim_{n \to \infty} n \cos(n^2x)$ does not converge for any $x \in \mathbb{R}$.

Consider now the sequence of the canonical extensions $\{g_n\}_{n \in \mathbb{N}}$ defined in Theorem 2.28. Thanks to Corollary 2.14, $\{g_n(r)\}_{n \in \mathbb{N}}$ converges weakly to 0 whenever $r \in \mathbb{R}$. However, $g_n(d) = n \cos(n^2d)$: since $\cos(n^2d) \approx 1$ for all $n \in \mathbb{N}$, we deduce that $\{g_n(d)\}_{n \in \mathbb{N}}$ does not converge weakly. As a consequence, it is not possible to define a weak pointwise limit for the canonical extension of the uniformly convergent sequence $\{f_n\}_{n \in \mathbb{N}}$.

3. A uniform measure on the Levi-Civita field

A uniform measure for the Levi-Civita field has been developed by Shamseddine and Berz in [26, 29] and extended to $\mathbb{R}^2$ and $\mathbb{R}^3$ by Shamseddine and Flynn [16, 30]. The underlying idea is to define a Lebesgue-like measure over subsets of $\mathcal{S}$ starting from the notion of length of an interval. Once the measurable sets are defined, it is possible to introduce a family of simple functions and finally the space of measurable functions. Besides the main definitions and results from [26, 29], we will also present some novel results and remarks that will be relevant for the sequel of the paper.

The only differences between our approach and the one proposed by Shamseddine and Berz will be in the definition of simple functions and in the definition of measurable functions. The former are not required to be Lipschitz continuous and are defined over arbitrary intervals, instead of being Lipschitz and defined over closed intervals. Thanks to Proposition 2.22, this definition turns out to be equivalent to the one proposed by Shamseddine and Berz.

In addition, measurable functions are not required to be bounded. As a consequence of avoiding this hypothesis, we will obtain a wider space of measurable functions. The richer space will provide a motivation for the introduction of the $L^p$ spaces over the Levi-Civita field, as discussed in Remark 4.8.
3.1. **Measurable sets.** In analogy with the Lebesgue measure, a set is measurable in \( R \) if it can be approximated with arbitrary precision by a countable sequence of intervals.

**Definition 3.1.** A set \( A \subseteq R \) is measurable if for every \( \varepsilon \in R \) there exist two sequences of mutually disjoint intervals \( \{I_n\}_{n \in \mathbb{N}} \) and \( \{J_n\}_{n \in \mathbb{N}} \) such that

1. \( \bigcup_{n \in \mathbb{N}} I_n \subseteq A \subseteq \bigcup_{n \in \mathbb{N}} J_n \);
2. \( \sum_{n \in \mathbb{N}} l(I_n) \) and \( \sum_{n \in \mathbb{N}} l(J_n) \) strongly converge in \( R \);
3. \( \sum_{n \in \mathbb{N}} l(J_n) - \sum_{n \in \mathbb{N}} l(I_n) \leq \varepsilon \).

By exploiting condition (3) of Definition 3.1, it is possible to define a measure for any measurable set \( A \).

**Definition 3.2.** If \( A \subseteq R \) is a measurable set, then for every \( k \in \mathbb{N} \) there exist two sequences of mutually disjoint intervals \( \{I_n^k\}_{n \in \mathbb{N}} \) and \( \{J_n^k\}_{n \in \mathbb{N}} \) satisfying properties (1)-(2) of Definition 3.1 together with the inclusions

\[
\bigcup_{n \in \mathbb{N}} I_n^k \subseteq \bigcup_{n \in \mathbb{N}} J_n^k+1 \subseteq A \subseteq \bigcup_{n \in \mathbb{N}} J_n^k \subseteq \bigcup_{n \in \mathbb{N}} I_n^k
\]

and the inequality (3) with \( \varepsilon = d^k \):

\[
\sum_{n \in \mathbb{N}} l(I_n^k) - \sum_{n \in \mathbb{N}} l(J_n^k) \leq d^k.
\]

The measure of \( A \), denoted by \( m(A) \), is defined as

\[
m(A) = \text{s-lim}_{k \to \infty} \sum_{n \in \mathbb{N}} l(I_n^k) = \text{s-lim}_{k \to \infty} \sum_{n \in \mathbb{N}} l(J_n^k).
\]

In [29] it is proved that \( m(A) \) is well-defined and that

\[
m(A) = \sup \left\{ \sum_{n \in \mathbb{N}} l(I_n) : \{I_n\}_{n \in \mathbb{N}} \text{ is a sequence of mutually disjoint intervals with } \bigcup_{n \in \mathbb{N}} I_n \subseteq A \right\}\]

\[
= \inf \left\{ \sum_{n \in \mathbb{N}} l(J_n) : \{J_n\}_{n \in \mathbb{N}} \text{ is a sequence of mutually disjoint intervals with } A \subseteq \bigcup_{n \in \mathbb{N}} J_n \right\}.
\]

**Remark 3.3.** According to Definition 3.2, measurable sets have bounded measure, i.e. there does not exist a measurable set \( A \) with \( m(A) = +\infty \). As a consequence, unbounded intervals of the form \( I(-\infty, b) = \{x \in R : x < b\} \), \( I(a, +\infty) = \{x \in R : a < x\} \) and even \( R \) itself are not measurable. We believe it would be possible to extend the definition of measure in order to include these unbounded intervals among the measurable sets, but in this paper we do not pursue this idea.

Due to the presence of infinitesimal elements in \( R \), the resulting measure turns out to be rather different from the Lebesgue measure over \( \mathbb{R} \).

The properties of measurable sets are studied in detail in [26, 29]. We recall some useful properties: if \( A \) and \( B \) are measurable, then also \( A \cup B \) and \( A \cap B \) are measurable, moreover \( m(A \cup B) = m(A) + m(B) - m(A \cap B) \). In addition, \( m \) is complete in the sense that if \( A \) is measurable and \( m(A) = 0 \), then every set \( B \subseteq A \) is measurable and \( m(B) = 0 \). Indeed, this result can be obtained as a particular case of the following property of measurable sets.

**Lemma 3.4.** Let \( B \subseteq A \subseteq R \) be measurable sets. If \( m(A) = m(B) \), then for every \( C \subseteq R \) that satisfies \( B \subseteq C \subseteq A \), \( C \) is measurable and \( m(C) = m(A) = m(B) \).
Proof. Let $A, B$ and $C$ satisfy the hypotheses of the lemma. Since $m(A) = m(B)$, for every $k \in \mathbb{N}$ there exist two sequences of mutually disjoint intervals $\{I_n^k\}_{n \in \mathbb{N}}$ and $\{J_n^k\}_{n \in \mathbb{N}}$ satisfying

- $\bigcup_{n \in \mathbb{N}} I_n^k \subseteq B \subseteq C \subseteq A \subseteq \bigcup_{n \in \mathbb{N}} J_n^k$,
- $\sum_{n \in \mathbb{N}} l(I_n^k)$ and $\sum_{n \in \mathbb{N}} l(J_n^k)$ strongly converge in $\mathcal{S}$;
- since $m(A) = m(B)$, $\sum_{n \in \mathbb{N}} l(J_n^k) - \sum_{n \in \mathbb{N}} l(I_n^k) \leq d^{-k}$.

From these properties, we deduce that $\{I_n^k\}_{n \in \mathbb{N}}$ and $\{J_n^k\}_{n \in \mathbb{N}}$ satisfy Definition 3.1 for $C$, with $\varepsilon = d^{-k}$. Thus $C$ is measurable. From Definition 3.2, we deduce that $m(C) = m(A) = m(B)$, as desired. \hfill \Box

Some of the usual theorems of the Lebesgue measure for $\mathbb{R}^k$ are not satisfied by the uniform measure over $\mathcal{S}$: for instance, the complement of a measurable set is not necessarily measurable. An example is given by $\mathbb{Q} \cap [0, 1]$, that is measurable and has measure 0, and $[0, 1] \setminus \mathbb{Q}$, that is not measurable, since it only contains intervals of an infinitesimal length. As a consequence, the family of measurable sets is not even an algebra.

This example is not isolated, and in fact in the Levi-Civita field there are many families of sets that are not measurable. An example is given by the monad at each point: this is a special case of the following result.

Lemma 3.5. For every $x \in \mathcal{S}$ and for every $q \in \mathbb{Q}$, the sets

1. $\{y \in \mathcal{S} : \lambda(x - y) > q\}$;
2. $\{y \in \mathcal{S} : \lambda(x - y) \geq q\}$;
3. $\{y \in \mathcal{S} : \lambda(x - y) < q\}$;
4. $\{y \in \mathcal{S} : \lambda(x - y) \leq q\}$;
5. $\{y \in \mathcal{S} : \lambda(x - y) = q\}$;

are not measurable.

Proof. (1) For all $x \in \mathcal{S}$ and all $q \in \mathbb{Q}$, if $I(a, b)$ and $J(c, d)$ are two intervals such that $I \subseteq \{y \in \mathcal{S} : \lambda(x - y) > q\} \subseteq J$, then $\lambda(d - c) < \lambda(b - a)$: as a consequence, property (3) of Definition 3.1 cannot be satisfied. A similar proof applies also to sets of the form (2), (3), (4) and (5). \hfill \Box

In contrast to what happens with the Lebesgue measure, countable unions of measurable sets might be non measurable.

Lemma 3.6. If $\{K_n\}_{n \in \mathbb{N}}$ is a family of mutually disjoint sets such that $\sum_{n \in \mathbb{N}} l(K_n)$ does not strongly converge, then $A = \bigcup_{n \in \mathbb{N}} K_n$ is not measurable.

Proof. Let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of mutually disjoint intervals such that $\sum_{n \in \mathbb{N}} l(I_n)$ strongly converges and $\bigcup_{n \in \mathbb{N}} I_n \subseteq A$. By strong convergence of $\sum_{n \in \mathbb{N}} l(I_n)$, there exists $l \in \mathcal{S}$ such that for every $\varepsilon \in \mathcal{S}, \varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ satisfying $l - \sum_{n \leq k} l(I_n) < \varepsilon$ for every $k > k_\varepsilon$. Since $\sum_{n \in \mathbb{N}} l(K_n)$ does not strongly converge and since $l(K_n) \geq 0$ for all $n \in \mathbb{N}$, there exists $\varepsilon \in \mathcal{S}, \varepsilon > 0$ and $j \in \mathbb{N}$ such that $\sum_{n \leq k} l(K_n) - l > \varepsilon$ for every $k > j$. Moreover, since $\bigcup_{n \in \mathbb{N}} I_n \subseteq \bigcup_{n \in \mathbb{N}} K_n$, the above inequality can be sharpened to $\sum_{n \leq k} l(K_n) - l > \varepsilon$ for every $k > j$.

Similarly, for every sequence of mutually disjoint intervals $\{J_n\}_{n \in \mathbb{N}}$ such that $\sum_{n \in \mathbb{N}} l(J_n)$ strongly converges and $A \subseteq \bigcup_{n \in \mathbb{N}} J_n$, by monotonicity of the measure $m$ we have the inequality $\sum_{n \leq k} l(K_n) < \sum_{n \in \mathbb{N}} l(J_n)$ for all $k \in \mathbb{N}$.

Putting the various inequalities together, if $k > \max\{k_\varepsilon, j\}$ then

$$\sum_{n \in \mathbb{N}} l(I_n) < \sum_{n \leq k} l(K_n) < \sum_{m \in \mathbb{N}} J_m.$$
We deduce that every sequence of mutually disjoint intervals \( \{I_n\}_{n \in \mathbb{N}} \) and \( \{J_n\}_{n \in \mathbb{N}} \) such that \( \bigcup_{n \in \mathbb{N}} I_n \subseteq A \subseteq \bigcup_{n \in \mathbb{N}} J_n \) cannot satisfy condition (3) of Definition 3.1.

The hypotheses of the above Lemma are satisfied for instance if \( l(K_n) = l \neq 0 \) for all \( n \in \mathbb{N} \). For other examples of measurable and non measurable sets and for a more detailed development of the measure theory on \( \mathcal{R} \), we refer to [21, 26, 29].

As in the real measure theory, it is convenient to introduce a notion of property that is true almost everywhere. If \( \mu \) is a real measure over \( \mathbb{R} \) and if \( P \) is a property of real numbers, then one has the equivalences

\[
\mu(\{x \in A : P(x) \text{ is true}\}) = 1 \iff \mu(\{x \in A : P(x) \text{ is false}\}) = 0
\]

However, for the uniform measure on \( \mathcal{R} \) the above equivalence is in general false. In [26] it is explicitly observed that, given two functions \( f, g : A \to \mathcal{R} \), the two properties

\begin{enumerate}
  \item \( m(\{x \in A : f(x) = g(x)\}) = m(A) \)
  \item \( m(\{x \in A : f(x) \neq g(x)\}) = 0 \)
\end{enumerate}

are not equivalent: while (1) implies (2), in general (2) does not imply (1). An example is obtained by choosing \( A = [0,1]_{\mathcal{R}}, f = \chi_{[0,1]}_{\mathcal{R}} \) and \( g = f - \chi_{[0,1]}_{\mathcal{R}} \); these functions satisfy (2), but not (1). This happens because \( \{x \in A : f(x) = g(x)\} = [0,1]_{\mathcal{R}} \setminus [0,1]_{\mathcal{Q}} \) and we have already recalled that the set \( [0,1]_{\mathcal{R}} \setminus [0,1]_{\mathcal{Q}} \) is not measurable. In [26] it is suggested that, in the case of equality, it is more convenient to use definition (1) instead of the weaker (2). Inspired by this choice, we define the notion of a property that is true almost everywhere on \( A \) as a property that is true on a measurable \( B \subseteq A \) with \( m(B) = m(A) \).

**Definition 3.7.** A property \( P \) holds almost everywhere on a measurable set \( A \subseteq \mathcal{R} \) if and only if \( T_P = \{x \in A : P(x) \text{ is true}\} \) is measurable and \( m(T_P) = m(A) \).

Notice that, as a consequence of this definition, the two assertions “\( f = g \) a.e. on \( A \)” and “\( f \neq g \) a.e. on \( A \)” are not equivalent.

Thanks to the properties of the measure, if two properties hold almost everywhere on \( A \), then both their conjunction and their disjunction hold almost everywhere on \( A \).

**Lemma 3.8.** If \( P \) and \( Q \) hold almost everywhere on a measurable set \( A \subset \mathcal{R} \), then both \( P \land Q \) and \( P \lor Q \) hold almost everywhere on \( A \).

**Proof.** Define \( T_P \) and \( T_Q \) as in Definition 3.7. Then \( T_P \) and \( T_Q \) are measurable and \( m(T_P) = m(T_Q) = m(A) \). Moreover, \( T_P \cup T_Q = T_P \cap T_Q \), and \( T_P \cup Q = T_P \cup T_Q \). Both \( T_P \land Q \) and \( T_P \lor Q \) are measurable as a consequence of Propositions 2.7 and 2.6 of [29]. Since \( m(T_P \cup T_Q) = m(T_P) + m(T_Q) - m(T_P \cap T_Q) \) and since \( m(T_P) = m(T_Q) = m(A) \), we deduce that also \( m(T_P \cap T_Q) = m(T_P \cup T_Q) = m(A) \). \( \square \)

We conclude our discussion of measurable sets by showing that if a set \( A \) is measurable and if \( \{I_n\}_{n \in \mathbb{N}} \) is a sequence of pairwise disjoint intervals such that the measure of \( \bigcup_{n \in \mathbb{N}} I_n \) is of the same magnitude as the measure of \( A \), then there must be at least an interval \( I_n \) whose measure has the same magnitude as the measure of \( A \).

**Lemma 3.9.** Let \( A \subset \mathcal{R} \) be measurable. If \( \lambda(m(A)) = q \) and if \( \{I_n\}_{n \in \mathbb{N}} \) is a sequence of pairwise disjoint intervals satisfying

\begin{enumerate}
  \item \( \bigcup_{n \in \mathbb{N}} I_n \subseteq A; \)
  \item \( \sum_{n \in \mathbb{N}} l(I_n) \) strongly converge;
  \item \( \lambda(m(A) - \sum_{n \in \mathbb{N}} l(I_n)) > q; \)
\end{enumerate}

then
(i) \( \lambda (l(I_n)) \geq q \) for all \( n \in \mathbb{N} \) and
(ii) there exists \( n \in \mathbb{N} \) such that \( \lambda (l(I_n)) = q \).

**Proof.** Both results can be obtained as a consequence of Lemma 2.11: in this case, \( a_n = l(I_n) \) and \( l = \sum_{n \in \mathbb{N}} l(I_n) \). Notice that hypothesis (3) implies that \( \lambda (\sum_{n \in \mathbb{N}} l(I_n)) = \lambda (m(A)) = q \). \( \square \)

3.2. **Measurable functions.** In analogy with the Lebesgue measure, the family of measurable functions is obtained from a family of simple functions.

For the measure on the Levi-Civita field, a meaningful choice is not the family of step functions, that are instead used for the the definition of Lebesgue measurable functions. This choice and the properties of the order topology of \( \mathcal{R} \) would lead to a narrow class of measurable functions.

Instead, it has been proposed a different notion of simple functions: these can be chosen to be any family of continuous functions that have an antiderivative and with some monotonicity requirements.

**Definition 3.10.** A family of functions defined over \( I(a, b) \) is called simple iff

1. it is an algebra over \( \mathcal{R} \) that contains the identity function;
2. every function of the family is continuous and has an antiderivative;
3. every differentiable simple function with null derivative is constant, and differentiable simple functions with non-negative derivative are nonincreasing.

It has been observed by Shamseddine and Berz that requirement (3), which is trivially satisfied by any real differentiable function, might not be satisfied for some functions defined on the Levi-Civita field. For some examples and a more detailed discussion, we refer to [29].

From Definition 3.10 it can be readily obtained that the smallest family of simple functions is the algebra of polynomials over \( I(a, b) \). However, an even richer family of simple functions, already proposed by Shamseddine and Berz, is the family of Lipschitz continuous power series that weakly converge on some closed interval \([a, b]_\mathcal{R}\).

For the remainder of the paper, we will work with the family of simple functions \( \mathcal{P} = \bigcup_{b, b \in \mathcal{R}, a < b} \mathcal{P}(I(a, b)) \); as a consequence, a function \( f \) is simple iff there exists an interval \( I \subset \mathcal{R} \) such that \( \text{supp} f = I \) and \( f \) is a power series that converges for every \( x \in I \). We remark that we do not require Lipschitz continuity, assumed in [26, 29] and used to prove that a simple function defined over an interval \((a, b)\mathcal{R}\) has a continuous extension to the interval \([a, b]_\mathcal{R}\). However, thanks to Proposition 2.22, if a function is simple on an open interval \((a, b)\mathcal{R}\), then it has a simple extension to the closed interval \([a, b]_\mathcal{R}\).

**Proposition 3.11.** If \( f \) is simple on \( I(a, b) \), then there exists a unique simple function \( g : [a, b]_\mathcal{R} \to \mathcal{R} \) such that \( g|(I(a, b)) = f \).

**Proof.** Let \( f(x) = \sum_{n \in \mathbb{N}} a_n \frac{(x-a)^n}{n!} \) for some \( a_n \in \mathcal{R} \). If \( \sum_{n \in \mathbb{N}} a_n \frac{(a-b)^n}{n!} \) does not converge, then by Proposition 2.22 there exists \( h \in M_n \), \( h > 0 \) such that \( \sum_{n \in \mathbb{N}} a_n \frac{(a+h-x)^n}{n!} \) does not converge. However, this would contradict that \( f \) is simple, i.e. analytic, on \( I(a, b) \). A similar argument applies to \( \sum_{n \in \mathbb{N}} a_n \frac{(b-x)^n}{n!} \).

Since both \( \sum_{n \in \mathbb{N}} a_n \frac{(a-x)^n}{n!} \) and \( \sum_{n \in \mathbb{N}} a_n \frac{(b-x)^n}{n!} \) converge weakly, the desired extension of \( f \) to \([a, b]_\mathcal{R}\) is

\[
g(x) = \begin{cases} f(x) & \text{if } x \in I(a, b) \\ \sum_{n \in \mathbb{N}} a_n \frac{(a-x)^n}{n!} & \text{if } x = a \\ \sum_{n \in \mathbb{N}} a_n \frac{(b-x)^n}{n!} & \text{if } x = b. \end{cases}
\]
As a consequence of this definition, $g$ is a power series, thus simple, and unique.

With a slight abuse of notation, if $f$ is simple on $I(a,b)$, we will still denote by $f$ the simple function defined on $[a,b]_{\mathbb{R}}$ that coincides with $f$ on $I(a,b)$.

From the algebra of simple functions it is possible to define the family of measurable functions. In contrast to what happens in $[26, 29]$, we will not require that measurable functions must be bounded.

**Definition 3.12.** Let $A \subset \mathbb{R}$ be measurable and let $f : A \to \mathbb{R}$. The function $f$ is measurable iff for all $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists a sequence of mutually disjoint intervals $\{I_n\}_{n \in \mathbb{N}}$ such that

1. $\bigcup_{n \in \mathbb{N}} I_n \subseteq A$;
2. $\sum_{n \in \mathbb{N}} l(I_n)$ strongly converges in $\mathbb{R}$;
3. $m(A) - \sum_{n \in \mathbb{N}} l(I_n) < \varepsilon$;
4. for all $n \in \mathbb{N}$, $f$ is simple on $I_n$.

We will denote by $\mathcal{M}(A)$ the set of measurable functions on $A$.

By removing the boundedness hypothesis from the definition of measurable functions, in principle some of the known results on measurable functions might not be valid in our development of the uniform measure over $\mathbb{R}$. However, this hypothesis is not used to prove any results of $[29]$ up to the definition of integral of a measurable function.

We recall here the main properties of measurable functions.

**Lemma 3.13.** For all measurable $A \subset \mathbb{R}$, $\mathcal{M}(A)$ is a vector space over $\mathbb{R}$.

**Proof.** See Proposition 3.9 of $[29]$. Notice that the proof of this proposition does not depend upon the hypothesis that simple functions are Lipschitz continuous or that measurable functions are bounded.

In $[29]$ it is given a necessary condition for a function $f$ to be measurable.

**Proposition 3.14.** A measurable function on $A$ is locally a simple function almost everywhere on $A$.

As a consequence of this property, measurable functions are continuous almost everywhere. Moreover, if two measurable functions are also differentiable and have the same derivative, then they differ by a constant. For the proofs of these statements, we refer to $[29]$.

Proposition 3.14 can be further sharpened by using Lemma 3.9.

**Proposition 3.15.** If $A \subset \mathbb{R}$ is a measurable set and if $f : A \to \mathbb{R}$ is measurable, then $f$ is simple on an interval $I \subseteq A$ with $\lambda(l(I)) = \lambda(m(A))$.

**Proof.** Let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of pairwise sets satisfying Definition 3.12 and hypotheses (1)-(3) of Lemma 3.9. Lemma 3.9 entails the existence of some $n \in \mathbb{N}$ such that $\lambda(l(I_n)) = \lambda(m(A))$, as desired.

A partial converse of Proposition 3.14 is that every real function that is not locally analytic on an interval cannot be obtained as the restriction of a measurable function.

**Proposition 3.16.** If $f$ is not locally analytic at any point in $[a,b]_{\mathbb{R}}$, then for every $g \in \mathcal{M}([a,b]_{\mathbb{R}})$, there exists an interval $I \subseteq [a,b]_{\mathbb{R}}$ such that $f(x) \neq g(x)$ for almost every $x \in I$.
Proof. Let \( g \in \mathcal{M}([a,b]_{\mathbb{R}}) \) and let \( \{I_n\}_{n \in \mathbb{N}} \) satisfy conditions 1-4 of Definition 3.12 for \( g \) and for some \( \varepsilon \approx 0 \). As a consequence \( \lambda(l(I_n)) \geq 0 \) for all \( n \in \mathbb{N} \). By Lemma 3.9, there is \( n \in \mathbb{N} \) such that \( \lambda(l(I_n)) = 0 \): otherwise the condition of strong convergence of \( \sum_{n \in \mathbb{N}} l(I_n) \) and the choice of \( \varepsilon \approx 0 \) would entail that \( m(\bigcup_{n \in \mathbb{N}} I_n) = 0 \), against the hypothesis (b) \( m(\bigcup_{n \in \mathbb{N}} l(I_n)) < \varepsilon \).

Let \( r \in I_n \cap \mathbb{R} \) and let \( g(x) = \sum_{n \in \mathbb{N}} a_n (x-r)^n \) whenever \( x \in I_n \). If \( \lambda(a_n) < 0 \) for some \( n \in \mathbb{N} \), then for every \( x \in I_n, x \not\approx r, \lambda(g(x)) < 0 \). Thus \( g(x) \not\approx f(x) \) for every \( x \in I_n \cap \mathbb{R}, x \not\approx r \). As a consequence, the desired interval \( I \) can be obtained as one of the connected components of \( (I_n \setminus \{r\}) \cap \mathbb{R} \).

Suppose now that \( \lambda(a_n) \geq 0 \) for all \( n \in \mathbb{N} \). We want to prove that the hypothesis \( f(x) \approx g(x) \) for all \( x \in I_n \cap \mathbb{R} \) entails that \( f \) is analytic at every point in \( I_n \cap \mathbb{R} \), against the hypotheses.

If \( a_n \in \mathbb{R} \) for all \( n \in \mathbb{N} \), then the desired result is trivially true.

If \( \lambda(a_n) \geq 0 \) for all \( n \in \mathbb{N} \), then \( a_n \not\approx a_n[0] \) for all \( n \in \mathbb{N} \). Hence, if \( x \in I_n \cap \mathbb{R} \), then \( a_0[0] \in \mathbb{R} \) and \( \sum_{n \in \mathbb{N}} a_n[0] (x-r)^n \in \mathbb{R} \), so that \( f(r) = a_0[0] \not\approx g(r) \) and \( f(x) = \sum_{n \in \mathbb{N}} a_n[0] (x-r)^n \not\approx g(x) \) for all \( x \in I_n \cap \mathbb{R} \). As a consequence, \( f \) is locally analytic at \( r \), a contradiction. \( \square \)

As observed in [26, 29], a function that is locally simple almost everywhere on \( A \) is in general not measurable. As a novel counterexample, we will prove that the order \( n \) continuation of any function that is not locally analytic is not measurable on any measurable set of non-infinitesimal length, even if it is locally a simple function at every point of its domain.

Lemma 3.17. If \( f \in C^n([a,b]_{\mathbb{R}}) \) and if \( f \) is not locally analytic at any point in \([a,b]_{\mathbb{R}}\), then \( \mathcal{F}_n \not\in \mathcal{M}(A) \) for any measurable set \( A \subseteq [a,b]_{\mathbb{R}} \) with \( m(A) \neq 0 \).

Proof. This Lemma can be obtained as a consequence of Proposition 3.16. We provide another proof that does not depend upon this result.

By definition of the order \( n \) extension of \( f \) and by the hypothesis that \( f \) is not locally analytic in \([a,b]_{\mathbb{R}}\), for every \( I \subseteq [a,b]_{\mathbb{R}} \) with \( l(I) \neq 0 \), \( f \) is not simple over \( I \). Consequently, if a family of mutually disjoint intervals \( \{I_n\}_{n \in \mathbb{N}} \) satisfies condition 4 of Definition 3.12, then \( l(I_n) \approx 0 \) for all \( n \in \mathbb{N} \).

Suppose towards a contradiction that \( \sum_{n \in \mathbb{N}} l(I_n) \) strongly converges to \( m(A) \). Lemma 3.9 entails that there exists \( m \in \mathbb{N} \) such that \( \lambda(l(I_m)) = \lambda(A) \), contradicting \( l(I_n) \approx 0 \) for all \( n \in \mathbb{N} \). Consequently, if \( \sum_{n \in \mathbb{N}} l(I_n) \) strongly converges, it does not converge to \( m(A) \). We deduce that \( \mathcal{F}_n \not\in \mathcal{M}(A) \), as desired. \( \square \)

Since the functions \( \mathcal{F}_n \) are continuous but not measurable, we deduce that some continuous functions on the Levi-Civita field are not measurable. This is in contrast with the well-known result that all real continuous functions are Lebesgue measurable.

Another class of locally simple functions that are not measurable are the reciprocal of simple functions on a neighbourhood of one of its zeroes.

Proposition 3.18. If \( f : [a,b]_{\mathbb{R}} \to \mathbb{R} \) is a simple function, if \( f(x) \neq 0 \) for all \( x \neq a \) and if \( f(a) = 0 \), then the function \( x \mapsto \frac{1}{f(x)} \) is not measurable on any interval of the form \( I(a,c) \subseteq [a,b]_{\mathbb{R}} \) with \( a < c \leq b \).

Proof. Let \( g : (a,b) \to \mathbb{R} \) be defined by \( g(x) = \frac{1}{f(x)} \). Let also \( \{I_n\}_{n \in \mathbb{N}} \) be a family of pairwise disjoint intervals such that \( g \) is simple over \( I_n \). We claim that for every \( n \in \mathbb{N} \) \( I_n \neq I(a,c) \) for any \( a < c \leq b \); otherwise there would be a power series \( \sum_{n \in \mathbb{N}} a_n (x-x_0)^n \) defined over an interval \( I(a,c) \) such that \( g(x) = \sum_{n \in \mathbb{N}} a_n (x-x_0)^n \) whenever \( x-x_0 \in I(a,c) \).
However, the hypotheses over \( f \) entail that \( \sum_{n \in \mathbb{N}} a_n (a - x_0)^n \) does not converge, so that by Proposition 2.22 there exists \( s \in \mathbb{Q} \) such that for every \( h \in \mathcal{R} \) with \( h > 0 \) and \( \lambda(h) > s \), \( \sum_{n \in \mathbb{N}} a_n (a + h - x_0)^n \) does not converge, against the hypothesis that \( \sum_{n \in \mathbb{N}} a_n (x - x_0)^n \) is convergent over \((a, c) \). \( \square \)

We conclude with an example of a nonmeasurable, locally analytic function that does not belong to the categories of nonmeasurable functions discussed above.

**Example 3.19.** Another locally analytic function that is not measurable is constructed as follows. Let \( A = [0, d^{-1}] \) and let \( f(x) = \sin \left( x_{\|0, \infty\|} \right) \). Since \( f \) is locally a composition of analytic functions, \( f \) is locally analytic \([27]\). However, \( f \) is not measurable. In order to see that this is the case, consider the sets

\[
G(r, q) = \{ y \in \mathcal{R} : \lambda(rd^q - y) \geq 0 \} = \{ y \in \mathcal{R} : y = rd^q + h \text{ for some } h \in \mathcal{R} \text{ with } \lambda(h) \geq 0 \},
\]

defined for \( r \in \mathbb{R} \setminus \{0\} \) and for \( q \in (-1, 0)_\mathbb{Q} \). They satisfy the following properties:

- \( \bigcup_{r \in \mathbb{R} \setminus \{0\}, q \in (-1, 0)_\mathbb{Q}} G(r, q) \subseteq [0, d^{-1}] \);
- if \( r, s \in \mathbb{R} \setminus \{0\} \) and \( r \neq s \), then \( G(r, p) \cap G(s, q) = \emptyset \) for all \( p, q \in (-1, 0)_\mathbb{Q} \);
- for every \( r \in \mathbb{R} \setminus \{0\} \) and \( q \in (-1, 0)_\mathbb{Q} \), \( f \) is analytic over \( G(r, q) \) and it is not analytic over any set \( S \supset G(r, q) \);
- by Lemma 3.5, each set \( G(r, q) \) is not measurable, but for every \( n \in \mathbb{N} \) it contains measurable sets of measure at least \( n \).

As a consequence of these properties, any family of disjoint intervals satisfying conditions (1) and (4) of Definition 3.12 must either include a refinement of the uncountable family \( \{G(r, q)\}_{r \in \mathbb{R} \setminus \{0\}, q \in (-1, 0)_\mathbb{Q}} \) or not satisfy condition (3) of Definition 3.12.

### 3.3. Integrals.

Since condition 2 of Definition 3.10 ensures that simple functions have an antiderivative, it is possible to define the integral of a simple function over an interval by imposing the validity of the fundamental theorem of calculus. The integral of a measurable function over a measurable set can then be obtained as a limit of the integrals of simple functions over a sequence of intervals satisfying Definition 3.12.

**Definition 3.20.** If \( f \) is a simple function over \([a, b]\) whose antiderivative is \( F \), then

\[
\int_{[a, b]} f(x) = \lim_{x \to b} F(x) - \lim_{x \to a} F(x).
\]

Notice that the two limits in the previous equality are well-defined, since \( F \) is simple on \([a, b]\) and, thanks to Proposition 3.11, \( F \) can be extended to a simple function on \([a, b]_\mathcal{R} \).

If \( A \subseteq \mathcal{R} \) is a measurable set and \( f : A \to \mathcal{R} \) is a measurable function, then define

\[
\mathcal{I}(f, A) = \left\{ I_n \subseteq A : I_n \text{ are mutually disjoint and } f \text{ is simple on } I_n \forall n \in \mathbb{N} \right\}.
\]

The integral of \( f \) over \( A \) is defined as

\[
\int_A f(x) = \lim_{\{I_n\}_{n \in \mathbb{N}} \in \mathcal{I}(f, A), \sum_{n \in \mathbb{N}} m(A) \left( \sum_{n \in \mathbb{N}} \int_{I_n} f(x) \right)} \text{ whenever the limit on the right side of the equality is defined (and possibly equal to } \pm \infty \text{ whenever the sequence } k \to \sum_{n \leq k} \int_{I_n} f(x) \text{ diverges), and it is undefined otherwise.}
\]

**Remark 3.21.** Even if there exist some measurable functions with an infinite or undefined integral, all of the results on the integrals of measurable functions discussed in \([29, 26]\)
hold for every function with a well-defined integral. For instance, if \( f \) and \( g \) have a well-defined and finite integral over the disjoint measurable sets \( A \) and \( B \), then for every \( a, b \in \mathcal{R} \)

\[
\int_{A} (af + bg) = a \int_{A} f + b \int_{A} g \quad \text{and} \quad \int_{A \cup B} f = \int_{A} f + \int_{B} f.
\]

The only assertion that needs to be verified for unbounded measurable functions is Theorem 3.7 of [26]. The next proposition ensures that it is still true for measurable unbounded functions.

**Proposition 3.22.** Let \( A \subset \mathcal{R} \) be measurable and let \( f, g : A \to \mathcal{R} \) satisfy \( f = g \) a.e. on \( A \). Then \( f \) is measurable on \( A \) if and only if \( g \) is measurable on \( A \); moreover

1. \( \int_{A} f \) is defined if and only if \( \int_{A} g \) is defined;
2. if \( \int_{A} f \) is defined, then \( \int_{A} f = \int_{A} g \) (including the case \( \int_{A} f = \int_{A} g = \pm \infty \)).

**Proof.** If \( f \) and \( g \) are bounded, then the desired result is a consequence of Theorem 3.7 of [26].

We only need to address the case when \( f \) and/or \( g \) are not bounded. As in the proof of Theorem 3.7 of [26], let \( B = \{ x \in A : f(x) = g(x) \} \) and \( S = A \setminus B \). Then \( B \) and \( S \) are measurable, \( m(B) = m(A) \) and \( m(S) = 0 \). Since \( f = g \) on \( B \), it is clear that \( \int_{B} f \) is defined if and only if \( \int_{B} g \) is defined and that \( \int_{B} f = \int_{B} g \).

In light of Proposition 4.13 of [29], it is sufficient to prove that for every measurable function \( F : A \to \mathcal{R} \) such that \( \int_{A} f \) is defined, then \( \int_{S} F = 0 \) for every \( S \subset A \) with \( m(S) = 0 \). However, this is a consequence of the fact that, if \( m(S) = 0 \), then the family \( \mathcal{F}(f, S) \) introduced in Definition 3.20 contains only the empty set. \( \square \)

Even if measurable functions are not bounded, continuous measurable functions over intervals are bounded almost everywhere.

**Lemma 3.23.** Let \( f : I(a, b) \to \mathcal{R} \) be measurable and continuous. Then \( f \) is bounded almost everywhere in \( I(a, b) \).

**Proof.** Let \( \varepsilon \in \mathcal{R} \), \( \varepsilon > 0 \) and let \( \{I(a_{n}, b_{n})\}_{n \in \mathbb{N}} \) be a sequence of mutually disjoint intervals satisfying conditions (1)–(4) of Definition 3.12. By Lemma 4.7 of [4], \( f \) is bounded on each interval \( I(a_{n}, b_{n}) \). Moreover, by Proposition 3.11, for every \( n \in \mathbb{N} \) \( \lim_{x \to a_{n}^{+}} f(x) \in \mathcal{R} \) and \( \lim_{x \to b_{n}^{-}} f(x) \in \mathcal{R} \), i.e., both limits exist and are finite. By continuity of \( f \), whenever \( a_{m} = b_{n} \) for some \( m, n \in \mathbb{N} \), then \( \lim_{x \to a_{n}^{+}} f(x) = \lim_{x \to b_{n}^{-}} f(x) \in \mathcal{R} \). As a consequence, \( f \) is bounded on \( \bigcup_{n \in \mathbb{N}} I(a_{n}, b_{n}) \). By the arbitrariness of \( \varepsilon \), \( f \) is bounded almost everywhere on \( I(a, b) \). \( \square \)

Thanks to the previous result, we can show that the antiderivative of a continuous, measurable function \( f \) can be defined from the integral of \( f \).

**Proposition 3.24.** Let \( f : I(a, b) \to \mathcal{R} \) be measurable and continuous. Then the function \( F : I(a, b) \to \mathcal{R} \) defined by \( F(x) = \int_{I(a, x)} f \) is well-defined and measurable. Moreover, \( F'(x) = f(x) \) for almost every \( x \in I(a, b) \).

**Proof.** We begin our proof by showing that \( F(x) \) is defined for every \( x \in I(a, b) \). By Proposition 3.8 of [29], \( f \) is measurable on each set of the form \( I(a, x) \) for \( x \in I(a, b) \). Since \( f \) is continuous, by Lemma 3.23 it is bounded almost everywhere in \( I(a, b) \). Thus there exists a bounded measurable function \( g \) over \( I(a, b) \) such that \( g(x) = f(x) \) for almost every \( x \in I(a, b) \). Corollary 4.6 of [29] entails that \( \int_{I(a, x)} g \) is defined for every \( x \in I(a, b) \). By Proposition 3.22, we conclude that \( F(x) = \int_{I(a, x)} f \) is also defined and not infinite for every \( x \in I(a, b) \).
We will now prove that $F$ is measurable on $I(a, b)$. Let $\varepsilon \in \mathcal{R}$, $\varepsilon > 0$ and let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of mutually disjoint intervals satisfying conditions (1)–(4) of Definition 3.12 for $f$. Then for every $n \in \mathbb{N}$ the restriction of $f$ to $I_n$ is analytic. As a consequence, for every $n \in \mathbb{N}$ there exists an antiderivative $F_n$ of $f$ such that for every $c, d \in I_n$, $c < d$, $\int_{[c,d]} f = F_n(d) - F_n(c)$. Since $F_n = \int_{[a,b]} f - \int_{[a,c]} f$, $F - F_n$ is constant over $I_n$, so that $F$ is a difference of simple functions. By Lemma 3.13, $F$ is also simple on each of the intervals $I_n$. We conclude that $F$ satisfies Definition 3.12, hence it is measurable.

In order to prove that $F'(x) = f$ for almost every $x \in I(a, b)$, consider $\varepsilon$ and $\{I_n\}_{n \in \mathbb{N}}$ as in the previous part of the proof. Let also $x \in I_n$ such that there exist $h_x > 0$ that satisfies the inclusion $[x, x + h] \subset I_n$. As a consequence, if $0 < h < h_x$,

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_{[x, x + h]} f.$$

Let $m_h = \min_{y \in [x, x + h]} \{f(y)\}$ and $M_h = \max_{y \in [x, x + h]} \{f(y)\}$. Notice that $m_h$ and $M_h$ exist by Theorem 3.3 of [24]. Moreover, continuity of $f$ entails that $s\text{-lim}_{h \to 0^+} m_h = s\text{-lim}_{h \to 0^+} M_h = f(x)$. However,

$$m_h \leq \frac{1}{h} \int_{[x, x + h]} f \leq M_h,$$

so that, by the squeeze theorem for strong convergence,

$$s\text{-lim}_{h \to 0^+} \frac{F(x + h) - F(x)}{h} = s\text{-lim}_{h \to 0^+} \frac{1}{h} \int_{[x, x + h]} f = f(x).$$

A similar argument applies to the case when there exist $h_x > 0$ that satisfies the inclusion $[x - h, x] \subset I_n$. Thus $F'(x) = f(x)$ for every $x \in \bigcup_{n \in \mathbb{N}} I_n$. By the arbitrariness of $x$, we conclude that $F'(x) = f(x)$ almost everywhere in $I(a, b)$. \qed

We have recalled that Shamseeddine and Berz proved that bounded measurable functions always have a well-defined integral. However, there are some unbounded measurable functions whose integral is infinite or undefined.

**Example 3.25.** Let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of mutually disjoint intervals satisfying

- $I_n \neq \emptyset$ for all $n \in \mathbb{N}$;
- $l(I_n) < d^{-n}$ for all $n \in \mathbb{N}$.

Let $A = \bigcup_{n \in \mathbb{N}} I_n$: by our choice of $\{I_n\}_{n \in \mathbb{N}}$, $A$ is measurable.

Define $f : A \to \mathcal{R}$ by $f(x) = \frac{1}{l(I_n)}$ whenever $x \in I_n$. Then $f$ is measurable by definition, but

$$\int_A f(x) = \sum_{n \in \mathbb{N}} \int_{I_n} f(x) = \sum_{n \in \mathbb{N}} \frac{l(I_n)}{l(I_n)^2} = +\infty.$$

Now define $g : A \to \mathcal{R}$ by $g(x) = \frac{1}{l(I_n)}$ whenever $x \in I_n$. Then $g$ is also measurable, but

$$\int_A g(x) = \sum_{n \in \mathbb{N}} \int_{I_n} g(x) = \sum_{n \in \mathbb{N}} \frac{l(I_n)}{l(I_n)}$$

and the last series of the above equality is undefined, since it neither converges strongly nor diverges.

The existence of measurable functions with infinite or undefined integral is one of the motivations for the introduction of the $L^p$ spaces over the Levi-Civita field.
4. $L^p$ spaces on the Levi-Civita field

We have seen in the previous section that, according to Definitions 3.12 and 3.20, some measurable functions do not have a well-defined integral or have an infinite integral. This feature suggests that it would be meaningful to introduce spaces of functions whose $p$-th power is measurable and has a finite integral. In order to do so, we need to be able to identify functions that are equal almost everywhere, as in the case of real Lebesgue measurable functions. This identification can be successfully introduced as a consequence of Proposition 3.22 and of the nontrivial fact that equality almost everywhere is an equivalence relation.

**Proposition 4.1.** Let $A \subset \mathcal{R}$ be a measurable set. The relation $\cong$ over $\mathcal{M}(A)$ defined by $f \cong g$ if $f = g$ a.e. is an equivalence relation.

*Proof.* Clearly, $\cong$ is reflexive and symmetric. We only need to prove that it is transitive. Let $f_1 \cong f_2$ and $f_2 \cong f_3$ in $\mathcal{M}(A)$. Define $A_1 = \{x \in A : f_1(x) = f_2(x)\}$, $A_2 = \{x \in A : f_2(x) = f_3(x)\}$ and $A_3 = \{x \in A : f_1(x) = f_3(x)\}$. Then $A_1 \cap A_2 \subseteq A_3 \subseteq A$. By Lemma 3.8, $A_1 \cap A_2$ is measurable and $m(A_1 \cap A_2) = m(A)$. Lemma 3.4 ensures then that $A_3$ is measurable and $m(A_3) = m(A)$, so that $f_1 = f_3$ a.e. on $A$. \hfill $\Box$

From now on, we will identify a measurable function $f : A \rightarrow \mathcal{R}$ with the equivalence class

$$[f] = \{g : A \rightarrow \mathcal{R} : f(x) = g(x) \text{ for almost every } x \in A\}$$

and, with an abuse of notation, we will often write $f$ instead of $[f]$.

As a consequence of the above identification, it is possible to refine the notion of measurable function with the introduction of $L^p$ spaces.

**Definition 4.2.** Let $A \subset \mathcal{R}$ be a measurable set.

If $1 \leq p < \infty$, we define

$$L^p(A) = \left\{ [f] : f \in \mathcal{M}(A), f^p \in \mathcal{M}(A), \int_A |f|^p \text{ is defined and } \int_A |f|^p < +\infty \right\}.$$

If $f \in L^p(A)$, then we define $\|f\|_p = (\int_A |f|^p)^{1/p}$.

For $p = \infty$, we define a function $\|\|_{\infty} : \mathcal{M}(A) \rightarrow \mathcal{R} \cup \{\text{undefined}\}$ by posing

$$\|f\|_{\infty} = \inf_{c \in \mathcal{R}} \{ |f(x)| \leq c \text{ for almost every } x \in A\}$$

and we define $L^\infty$ as the set of measurable functions with a well-defined $\infty$-norm:

$$L^\infty(A) = \{ [f] : f \in \mathcal{M}(A) \text{ and } \|f\|_{\infty} \neq \text{undefined} \}.$$

Notice that the functions $\|\|_p$ are well-defined thanks to Proposition 3.22 and to Proposition 4.1.

**Remark 4.3.** If a function $f$ belongs to $L^\infty$, then it is essentially bounded, but the converse is in general false. In fact, it is not possible to define the $L^\infty$ space and the function $\|\|_{\infty}$ by defining $L^\infty(A) = \{ [f] : f \in \mathcal{M}(A) \text{ and } f \text{ is essentially bounded} \}$ and then posing

$$\|f\|_{\infty} = \inf_{c \in \mathcal{R}} \{ |f(x)| \leq c \text{ for almost every } x \in A\}$$

for all $f \in L^\infty(A)$. Indeed, with such a definition $\|f\|_{\infty}$ would not be defined for all functions in $L^\infty$. 

A counterexample is constructed as follows: let \( I_n = [n - d^n, n + d^n] \). Then \( A = \bigcup_{n \in \mathbb{N}} I_n \) is measurable, and the function \( f : A \to \mathbb{R} \) defined by \( f_{I_n} = n \) is measurable and essentially bounded, since \( |f(x)| < d^{-1} \) for all \( x \in A \). However, \( \|f\|_\infty \) is not well-defined, since
\[
\{ c \in \mathbb{R} : f(x) < c \text{ for almost every } x \in A \} = \{ c \in \mathbb{R} : f_{I_n} < c \text{ for all } n \in \mathbb{N} \},
\]
and it is well-known that the latter set is bounded from below but has no infimum.

**Remark 4.4.** The distinction between \( L^p \) functions and measurable ones is significant, i.e. \( L^p(A) \neq \mathcal{M}(A) \). For instance, let \( \{I_n\}_{n \in \mathbb{N}} \) and \( g \) be defined as in Example 3.25. We have already argued that \( g \) is measurable, but \( g \notin L^p(A) \) for any \( p \geq 1 \), since \( g \) is not bounded and
\[
\int_A |g(x)|^p = \sum_{n \in \mathbb{N}} \int_{I_n} |g(x)|^p = \sum_{n \in \mathbb{N}} \frac{1}{|I_n|^p},
\]
and the latter sum is undefined if \( p = 1 \) and diverges \( +\infty \) if \( p > 1 \). Similarly, if \( a > 1 \), \( g^{\frac{1}{a}} \in L^p(A) \) for all \( 1 \leq p < a \).

Many properties of the \( L^p \) spaces are a consequence of the following Hölder’s inequality, that is similar to the one valid for real \( L^p \) spaces.

**Lemma 4.5** (Hölder’s inequality). Suppose that \( A \subset \mathbb{R} \) is measurable. For every \( 1 \leq p \leq q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) (allowing also for \( p = 1 \) and \( q = \infty \)), for every \( f \in L^p(A) \) and \( g \in L^q(A) \), then \( fg \in L^1(A) \) and
\[
\|fg\|_1 \leq \|f\|_p \|g\|_q.
\]

**Proof.** If \( fg(x) = 0 \) for almost every \( x \in A \), then the proof is trivial. Suppose then that \( fg(x) \neq 0 \) on a measurable set of positive measure.

If \( p = 1 \) and \( q = \infty \), then the result is a consequence of Corollary 4.12 of [29]. Otherwise, notice that the Young’s inequality \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \) is true for every \( a, b \in \mathbb{R} \). As a consequence, the usual proof for the Hölder inequality still applies to this setting: since \( \|f\|_p \neq 0 \) and \( \|g\|_q \neq 0 \) by hypothesis, then for every \( x \in A \)
\[
\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p \|f\|_p} + \frac{|g(x)|^q}{q \|g\|_q}.
\]
Corollary 4.11 of [29] entails the inequality
\[
\frac{1}{\|f\|_p \|g\|_q} \int_A |f(x)g(x)| \leq \frac{1}{\|f\|_p} \int_A |f(x)|^p + \frac{1}{\|g\|_q} \int_A |g(x)|^q = 1,
\]
that is equivalent to the Hölder’s inequality.

**Corollary 4.6.** For all measurable sets \( A \subset \mathbb{R} \) and for all \( 1 \leq p \leq \infty \), the maps \( \|\cdot\|_p : L^p(A) \to \mathbb{R} \) are norms.

**Proof.** We need to prove that, for all \( f, g \in L^p \) and for all \( a \in \mathbb{R} \),
\begin{enumerate}
  \item \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \);
  \item \( \|af\|_p = |a| \|f\|_p \);
  \item \( \|f\|_p = 0 \iff f = 0 \).
\end{enumerate}

Thanks to linearity of the integral on the Levi-Civita field, established in Proposition 4.3 of [29], and thanks to the Hölder’s inequality established in Lemma 4.5, the proof of property 1 is analogous to the proof that the norms of the usual \( L^p \) spaces are subadditive.

Property 2 can be obtained from the definition of integral and by Proposition 4.3 of [29].
Property 3 is a consequence of the identification of functions a.e. equal and of Theorem 3.7 of [26].

A result by Shamseddine and Berz entails an inclusion between the $\mathcal{L}^p$ spaces.

**Lemma 4.7.** For all measurable sets $A \subset \mathcal{R}$, if $f$ is bounded then $f \in \mathcal{L}^p(A)$ for all $1 \leq p$. As a consequence, $\mathcal{L}^\infty(A) \subseteq \mathcal{L}^p(A)$ for all $1 \leq p < \infty$.

**Proof.** Recall that if $A$ is measurable, then $m(A) \in \mathcal{R}$ (i.e. the measure of $A$ is not infinite). It is easy to check that if $f$ is measurable, then $|f|$ and $|f|^p$ are also measurable for all $1 \leq p < \infty$.

By Corollary 4.12 of [29], for all bounded $f$,

$$\int_A |f|^p \leq M^p \cdot m(A),$$

where $M$ is a bound for $|f|$, so that $f \in \mathcal{L}^p(A)$ for all $1 \leq p < \infty$.

If $f \in \mathcal{L}^\infty(A)$, we can choose $M = \|f\|_\infty$. □

**Remark 4.8.** Lemma 4.7 entails that, under the hypothesis that measurable functions are bounded, then the theory of $\mathcal{L}^p$ spaces over the Levi-Civita field becomes trivial. In fact, all bounded measurable functions have a well-defined integral. Moreover, since the $p$-th power of a bounded measurable function is still bounded and measurable, we would have $\mathcal{M}(A) = \mathcal{L}^p(A)$ for all $1 \leq p < \infty$.

It is possible to prove that the usual inclusions between the real $L^p$ spaces are valid also for the $\mathcal{L}^p$ spaces.

**Lemma 4.9.** If $A \subset \mathcal{R}$ is measurable and if $1 \leq p \leq q \leq \infty$, then $\mathcal{L}^q(A) \subseteq \mathcal{L}^p(A)$.

**Proof.** If $q = \infty$, then $f \in \mathcal{L}^p(A)$ by Lemma 4.7. Suppose then that $q < \infty$.

Thanks to the Hölder’s inequality, the proof is the same as in the case of the Lebesgue measure: if $f \in \mathcal{L}^q(A)$, then

$$\int_A |f|^p \leq \left(\int_A |f|^{pq/p}\right)^{p/q} \left(\int_A 1\right)^{1-p/q} = \|f\|_q^p \cdot m(A)^{1-p/q}.$$

The hypothesis $f \in L^q(A)$ and the fact that $m(A) \in \mathcal{R}$ for any measurable set $A$ entail that $m(A)^{1-p/q} \in \mathcal{R}$, so that indeed $f \in \mathcal{L}^p(A)$. □

We conclude with some coherence results for the integral on the Levi-Civita field and the Lebesgue integral.

**Lemma 4.10.** If $a, b \in \mathcal{R}$ and $f \in C^0([a, b])$, then $\mathcal{T}_\infty \in \mathcal{L}^1([a, b]_{\mathcal{R}})$ and

$$\int_{[a, b]} \mathcal{T}_\infty = \int_{[a, b]} f(x)dx.$$

Moreover, if $I(c, d) \subseteq [a, b]_{\mathcal{R}}$, then

$$\int_{(c, d)} \mathcal{T}_\infty \approx \int_{[c, d]} f(x)dx.$$

**Proof.** If $f \in C^0([a, b])$, then $\mathcal{T}_\infty$ is simple over $[a, b]_{\mathcal{R}}$. Let $F$ be an antiderivative of $f$: then, by Theorem 3.14 of [27], $F_\infty$ is an antiderivative of $\mathcal{T}_\infty$. By definition 3.20,

$$\int_{[a, b]} \mathcal{T}_\infty = F_\infty(b) - F_\infty(a),$$

where $F_\infty(b) - F_\infty(a)$ is a number.
while from real analysis it is well-known that
\[
\int_{[a,b]} f(x) dx = F(b) - F(a).
\]
Since \(a, b \in \mathbb{R}\), the desired equality is a consequence of the property \(T_\infty(r) = F(r)\) for all \(r \in \mathbb{R}\).

As for the second result, by definition of \(T_\infty\) we have \(T_\infty(c) \approx F(c[0])\) and \(F(d) \approx T_\infty(d[0])\); these hypotheses are sufficient to entail the desired result. \(\square\)

From the previous lemma we deduce that the extension of a real analytic function in \(L^p\) belongs to the space \(\mathcal{L}^p\) also in the Levi-Civita field.

**Corollary 4.11.** If \(a, b \in \mathbb{R}\) and \(f \in C^\infty([a, b], \mathbb{R})\), then \(\mathcal{T}_\infty \in \mathcal{L}^p(A)\) for all measurable \(A \subseteq [a, b]_{\mathcal{A}}\) and for all \(1 \leq p \leq \infty\). Moreover, \(\|f\|_p = \|\mathcal{T}_\infty\|_p\), where the former is the norm in the space \(L^p([a, b])\) and the latter is the norm in \(\mathcal{L}^p([a, b]_{\mathcal{A}})\).

In the previous results, it is essential that the intervals under consideration are closed. In fact, as a consequence of Proposition 3.18, functions of the form \(x^{-\frac{1}{a}}\) with \(a > 1\) are not in \(L^p(0, 1)\) for any \(p \geq 1\), even if their restriction to \(\mathbb{R}\) is in \(C^\infty(0, 1) \cap L^p(0, 1)\) for \(1 \leq p < a\).

### 4.1. Convergence in the \(\mathcal{L}^p\) spaces

In the \(\mathcal{L}^p\) spaces it is possible to consider sequences of functions that converge with respect to the \(\mathcal{L}^p\) norm. Strong and weak convergence on the Levi-Civita field yield two corresponding notions of convergence in norm.

**Definition 4.12.** Let \(A \subseteq \mathcal{A}\) be a measurable set and let \(\{f_n\}_{n \in \mathbb{N}}\) be a sequence of functions with \(f_n \in \mathcal{L}^p(A)\) for all \(n \in \mathbb{N}\).

The sequence \(\{f_n\}_{n \in \mathbb{N}}\) is strongly Cauchy in \(\mathcal{L}^p(A)\) iff
\[
\forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : \forall m, n > n_\epsilon \|f_n - f_m\|_p < \epsilon.
\]
If there exists \(f \in \mathcal{L}^p(A)\) such that \(\lim_{n \to \infty} \|f_n - f\|_p = 0\), then we will say that \(\{f_n\}_{n \in \mathbb{N}}\) converges strongly to \(f\) with respect to the \(\mathcal{L}^p\) norm.

The sequence \(\{f_n\}_{n \in \mathbb{N}}\) is weakly Cauchy in \(\mathcal{L}^p(A)\) iff
\[
\forall \epsilon \in \mathbb{R}, \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : \forall m, n > n_\epsilon \max_{q \in \mathbb{Q}, q < \epsilon^{-1}} \|f_n - f_m\|_p \leq \epsilon.
\]
If there exists \(f \in \mathcal{L}^p(A)\) such that \(w\text{-}\lim_{n \to \infty} \|f_n - f\|_p = 0\), then we will say that \(\{f_n\}_{n \in \mathbb{N}}\) converges weakly to \(f\) with respect to the \(\mathcal{L}^p\) norm.

As a consequence of the fact that every sequence that converges strongly also converges weakly, if a sequence is strongly Cauchy then it is also weakly Cauchy.

In the next sections we will show that the \(\mathcal{L}^p\) spaces are not complete with respect to both notions of convergence.

### 4.2. Completeness of the \(\mathcal{L}^p\) spaces with respect to strong convergence

Some results addressing completeness of the space \(\mathcal{L}^\omega\) with respect to strong convergence have been obtained in [26, 29].

**Lemma 4.13.** If \(A \subseteq \mathcal{A}\) is measurable and \(m(A) > 0\), then \(\mathcal{L}^\omega(A)\) is not complete.

**Proof.** For a proof that \(\mathcal{L}^\omega([0, 1])\) is not complete, see example 3.10 of [29]. From this example, one can obtain a proof that \(\mathcal{L}^\omega(A)\) is not complete for every set \(A \subseteq \mathcal{A}\) by rescaling the argument of example 3.10 to an interval \(I(a, b) \subseteq A\) with \(a \neq b\). \(\square\)

From this result, one immediately obtains that the \(\mathcal{L}^p\) spaces are not complete with respect to strong convergence.
Proposition 4.14. For every measurable $A \subset \mathcal{B}$ with $m(A) > 0$ and for every $1 \leq p \leq \infty$, the spaces $L^p(A)$ are not complete.

Proof. By Lemma 4.13, $L^\infty(A)$ is not complete. Let $\{f_n\}_{n \in \mathbb{N}}$ be a strongly Cauchy sequence in $L^\infty(A)$ that does not converge in $L^\infty(A)$, and let $f$ be its uniform limit. Recall that $f$ is not measurable.

Lemma 4.9 ensures that $L^\infty(A) \subseteq L^p(A)$; thus by the Hölder’s inequality

$$\|f_n - f_m\|_p^p = \int |f_n - f_m|^p \leq \|f_n - f_m\|_\infty \cdot m(A) = \|f_n - f_m\|_\infty \cdot m(A).$$

Since $\{f_n\}_{n \in \mathbb{N}}$ is strongly Cauchy in $L^\infty(A)$, from the previous inequality and from the squeeze theorem for the strong convergence we obtain that $\{f_n\}_{n \in \mathbb{N}}$ is strongly Cauchy also in $L^p(A)$ for every $1 \leq p \leq \infty$. However, since $f \notin \mathcal{M}(A)$, $\{f_n\}_{n \in \mathbb{N}}$ is strongly Cauchy but does not converge in $L^p(A)$ for any $1 \leq p \leq \infty$. We conclude that the spaces $L^p(A)$ are not complete.

Despite these negative results, strong convergence of a sequence $\{f_n\}_{n \in \mathbb{N}}$ to a function $f$ in $L^\infty$ implies strong convergence of $\{f_n\}_{n \in \mathbb{N}}$ to $f$ in $L^1$.

Theorem 4.15. If $A \subset \mathcal{B}$ is measurable and if $\{f_n\}_{n \in \mathbb{N}}$ strongly converges to $f$ in $L^\infty(A)$, then $f \in L^1(A)$ and $\lim_{n \to \infty} \int_A f_n = \int_A f$.

Proof. See Theorem 3.9 of [26].

This result can be extended to show that strong convergence of a sequence $\{f_n\}_{n \in \mathbb{N}}$ to a function $f$ in $L^\infty$ implies strong convergence of $\{f_n\}_{n \in \mathbb{N}}$ to $f$ in $L^p$.

Proposition 4.16. If $A \subset \mathcal{B}$ is measurable and if $\{f_n\}_{n \in \mathbb{N}}$ strongly converges to $f$ in $L^\infty(A)$, then $f \in L^p(A)$ and $\lim_{n \to \infty} \|f_n - f\|_p = 0$ for every $1 \leq p \leq \infty$.

Proof. We proceed as in the proof of Proposition 4.14. The fact that $f \in L^p(A)$ is a consequence of Lemma 4.9. By the Hölder’s inequality,

$$\|f_n - f_m\|_p \leq \|f_n - f\|_\infty \cdot m(A).$$

Since $\lim_{n \to \infty} \|f_n - f\|_\infty = 0$, we have also $\lim_{n \to \infty} \|f_n - f\|_p^p = 0$. By the squeeze theorem for the strong convergence in $\mathcal{B}$, we obtain the desired conclusion.

4.3. Completeness with respect to weak convergence. There are many arguments against completeness of the $L^p$ spaces with respect to weak convergence. At a fundamental level, the field $\mathcal{B}$ itself is not complete with respect to weak convergence: thus if $\{a_n\}_{n \in \mathbb{N}}$ is a weakly Cauchy sequence that does not converge in $\mathcal{B}$ and if $A$ is a measurable subset of $\mathcal{B}$, then the sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ defined by $f_n(x) = a_n$ for all $x \in A$ satisfies

- $f_n \in L^p(A)$ for every $1 \leq p < \infty$;
- $\{f_n\}_{n \in \mathbb{N}}$ is weakly Cauchy;
- $\{f_n\}_{n \in \mathbb{N}}$ does not converge in $L^p(A)$, otherwise $\lim_{n \to \infty} a_n \in \mathcal{B}$, against the hypothesis.

Moreover, we have seen in Section 4.2 that the $L^p$ spaces are not even complete with respect to strong convergence. Since strongly Cauchy sequences are also weakly Cauchy, then from Proposition 4.14 one immediately obtains that the $L^p$ spaces are not complete also with respect to weak convergence.
4.4. A completion of the \( L^p \) spaces with respect to strong convergence. In Proposition 4.14 we proved that the \( L^p \) spaces are not complete with respect to the strong or the weak convergence. In this section we will introduce the completions of the \( L^p \) spaces with respect to strong convergence. The resulting spaces will be complete with respect to strong convergence, and the completion of \( L^2 \) will also have a well-defined inner product, making it a Hilbert space.

The completion of the \( L^p \) spaces with respect to strong convergence is obtained by considering all sequences of \( L^p \) functions that are strongly Cauchy with respect to the \( p \)-norm, and by identifying those sequences whose difference strongly converges to zero.

**Definition 4.17.** Let \( A \) be a measurable set. Define

\[
B^p_s(A) = \{ \{ f_n \}_{n \in \mathbb{N}} : f_n \in L^p(A) \text{ for all } n \in \mathbb{N} \text{ and } \{ f_n \}_{n \in \mathbb{N}} \text{ is strongly Cauchy} \},
\]

and define the relation \( \cong_s \) over \( B^p_s \times B^p_s \) by posing

\[
\{ f_n \}_{n \in \mathbb{N}} \cong_s \{ g_n \}_{n \in \mathbb{N}} \text{ iff } s-lim_{n \to \infty} \| f_n - g_n \|_p = 0.
\]

In order to consider the quotient with respect to the relation \( \cong_s \), it is necessary to ensure that \( \cong_s \) is an equivalence relation.

**Lemma 4.18.** For every measurable \( A \subset \mathcal{R} \), the relation \( \cong_s \) is an equivalence relation over \( B^p_s(A) \).

**Proof.** From the definition it is immediate to prove that \( \cong_s \) is symmetric and reflexive. In order to prove that it is transitive, suppose that \( \{ f_n \}_{n \in \mathbb{N}} \cong_s \{ g_n \}_{n \in \mathbb{N}} \) and that \( \{ g_n \}_{n \in \mathbb{N}} \cong_s \{ h_n \}_{n \in \mathbb{N}} \). Since \( \| f_n - z_n \|_p = \| f_n - g_n + g_n - z_n \|_p \leq \| f_n - g_n \|_p + \| g_n - z_n \|_p \), by the squeeze theorem for strong convergence we conclude that \( \cong_s \) is also transitive. \( \square \)

Thanks to the above Lemma, we can define the completion of the \( L^p \) spaces with respect to strong convergence.

**Definition 4.19.** We define \( \mathcal{L}^p_s(A) = B^p_s(A) / \cong_s \). If \( \{ f_n \}_{n \in \mathbb{N}} \in B^p_s(A) \), we will denote its equivalence class in \( \mathcal{L}^p_s(A) \) by \( \langle f_n \rangle \).

These completions are vector spaces over \( \mathcal{R} \).

**Lemma 4.20.** For every measurable \( A \subset \mathcal{R} \), \( \mathcal{L}^p_s(A) \) is a vector space over \( \mathcal{R} \).

**Proof.** Corollary 4.6 entails that \( \mathcal{L}^p_s(A) \) is closed under linear combinations: if \( \{ f_n \}_{n \in \mathbb{N}} \) and \( \{ g_n \}_{n \in \mathbb{N}} \) are sequences in \( B^p_s(A) \) and if \( \alpha, \beta \in \mathcal{R} \), then the linear combination \( \{ \alpha f_n + \beta g_n \}_{n \in \mathbb{N}} \) satisfies the inequality

\[
\| \alpha f_n + \beta g_n \|_p \leq |\alpha| \| f_n \|_p + |\beta| \| g_n \|_p.
\]

As a consequence,

\[
\| \alpha f_n + \beta g_n - (\alpha f_n + \beta g_m) \|_p \leq |\alpha| \| f_n - f_m \|_p + |\beta| \| g_n - g_m \|_p.
\]

Since \( \{ f_n \}_{n \in \mathbb{N}} \) and \( \{ g_n \}_{n \in \mathbb{N}} \in B^p_s(A) \), we deduce that also \( \{ \alpha f_n + \beta g_n \}_{n \in \mathbb{N}} \in B^p_s(A) \), as desired. \( \square \)

It is possible to introduce a norm on each of the \( \mathcal{L}^p_s \) spaces: the \( \mathcal{L}^p_s \) norm of \( \langle f_n \rangle \) is defined as the strong limit of the \( L^p \) norms of \( f_n \). At first we will prove that this function does not depend on the representative of \( \langle f_n \rangle \), and then that it is indeed a norm over \( \mathcal{L}^p_s \).

**Lemma 4.21.** The function \( \| \|_p : \mathcal{L}^p_s(A) \to \mathcal{R} \) defined by \( \| \langle f_n \rangle \|_p = s-lim_{n \to \infty} \| f_n \|_p \) is well-defined, i.e. it does not depend on the representative of the equivalence class \( \langle f_n \rangle \).
Proof. Suppose that \( \langle f_n \rangle = \langle g_n \rangle \) in \( L^p(A) \). Then \( s\text{-lim}_{n \to \infty} \| f_n - g_n \|_p = 0 \), so that
\[
\lim_{n \to \infty} \| f_n \|_p = \lim_{n \to \infty} \| f_n - g_n + g_n \|_p \leq \lim_{n \to \infty} \| f_n - g_n \|_p + \lim_{n \to \infty} \| g_n \|_p = \lim_{n \to \infty} \| g_n \|_p.
\]
By reversing the roles of \( \{f_n\}_{n \in \mathbb{N}} \) and \( \{g_n\}_{n \in \mathbb{N}} \), it is possible to obtain the other inequality \( s\text{-lim}_{n \to \infty} \| g_n \|_p \leq \lim_{n \to \infty} \| f_n \|_p \). These inequalities entail also \( s\text{-lim}_{n \to \infty} \| f_n \|_p = s\text{-lim}_{n \to \infty} \| g_n \|_p \). As a consequence, the function \( \| \cdot \|_p \) does not depend upon the representative of \( \langle f_n \rangle \).
\(\Box\)

Lemma 4.22. For every measurable \( A \subset \mathcal{R} \) and for every \( 1 \leq p \leq \infty \), the function \( \| \cdot \|_p : L_s^p(A) \to \mathcal{R} \) is a norm over \( L_s^p(A) \).

Proof. We need to prove that, for all \( \langle f_n \rangle, \langle g_n \rangle \in L^p \) and for all \( a \in \mathcal{R} \),

1. \( \| \langle f_n \rangle + \langle g_n \rangle \|_p \leq \| \langle f_n \rangle \|_p + \| \langle g_n \rangle \|_p \);
2. \( \| a \langle f_n \rangle \|_p = |a| \| \langle f_n \rangle \|_p \);
3. \( \| \langle f_n \rangle \|_p = 0 \) iff \( \langle f_n \rangle = 0 \).

The first property is a consequence Corollary 4.6 and of the fact that inequalities are preserved under strong limits.

For the proof of the second property, notice that
\[
\| \langle a f_n \rangle \|_p = s\text{-lim}_{n \to \infty} \| a f_n \|_p = s\text{-lim}_{n \to \infty} |a| \| f_n \|_p = |a| \lim_{n \to \infty} \| f_n \|_p = |a| \| \langle f_n \rangle \|_p,
\]
so that \( \| \langle a f_n \rangle \|_p = |a| \| \langle f_n \rangle \|_p \).

Finally recall that, by definition, if \( \| \langle f_n \rangle \| = 0 \), then \( s\text{-lim}_{n \to \infty} \| f_n \|_p = 0 \), so \( \{f_n\}_{n \in \mathbb{N}} \equiv_s \{0\}_{n \in \mathbb{N}} \), that is \( \langle f_n \rangle = 0 \).
\(\Box\)

It turns out that the spaces \( L_s^p \) are complete with respect to the norm defined above.

Proposition 4.23. For every measurable \( A \subset \mathcal{R} \), \( L_s^p(A) \) is complete with respect to the norm \( \| \cdot \|_p \). As a consequence, it is a Banach space over \( \mathcal{R} \).

Proof. Taking into account Lemmas 4.20 and 4.22, we only need to prove that \( L_s^p(A) \) is complete with respect to the \( L_s^p \) norm. The proof is analogous to the real case, and hinges upon completeness of \( \mathcal{R} \) with respect to strong convergence. Let \( \{\phi_k\}_{k \in \mathbb{N}} \) be a sequence in \( L_s^p(A) \) such that
\[
\forall \varepsilon \in \mathcal{R}, \varepsilon > 0 \exists k_\varepsilon \in \mathbb{N} : \forall h, k > k_\varepsilon \| \phi_h - \phi_k \|_p < \varepsilon.
\]

By definition of \( L_s^p(A) \), for every \( k \in \mathbb{N} \) there exists \( g_k \in L^p(A) \) such that
\[
\| g_k - \phi_k \|_p \leq d^{-k}.
\]

We claim that \( \{g_n\}_{n \in \mathbb{N}} \) is strongly Cauchy in \( L^p(A) \) and that \( s\text{-lim}_{k \to \infty} \| \phi_k - \langle g_n \rangle \|_p = 0 \).

Let us show that \( \{g_n\}_{n \in \mathbb{N}} \) is strongly Cauchy. Notice that, for every \( h, k \in \mathbb{N} \),
\[
\| g_k - g_h \|_p \leq \| g_k - \phi_k \|_p + \| \phi_k - \phi_h \|_p + \| \phi_h - g_h \|_p.
\]
Thanks to equation (4.2), we have also \( \| g_k - \phi_k \|_p < d^{-k} \) and \( \| \phi_h - g_h \|_p < d^{-h} \). Let \( \varepsilon \in \mathcal{R}, \varepsilon > 0 \). Then if \( \max \{h, k\} > -\lambda(\varepsilon) \), \( \| g_k - \phi_k \|_p + \| \phi_h - g_h \|_p < d^{-k} + d^{-h} < \varepsilon \). Similarly, if \( h, k > k_\varepsilon \), by equation (4.1) we have also \( \| f_n - f_n \|_p < \varepsilon \). Thus \( \| g_k - g_h \|_p < 2\varepsilon \), and \( \{g_k\}_{k \in \mathbb{N}} \) is strongly Cauchy in \( L_s^p(A) \).

We now want to prove that \( \{\phi_k\}_{k \in \mathbb{N}} \) converges to \( \langle g_n \rangle \) in the \( L_s^p \) norm. Observe that for every \( k \in \mathbb{N} \)
\[
\| \phi_k - \langle g_n \rangle \|_p \leq \| \phi_k - g_k \|_p + \| g_k - \langle g_n \rangle \|_p.
\]
By equation (4.2), \( \| \phi_k - g_k \|_p < d^{-k} \) and, by definition of \( \{ g_k \}_{k \in \mathbb{N}} \), \( \text{s-lim}_{k \to \infty} \| g_k - \langle g_n \rangle \|_p = 0 \). Thus \( \text{s-lim}_{k \to \infty} \| \phi_k - \langle g_n \rangle \|_p = 0 \), and the proof is concluded. \( \square \)

It is also possible to define a duality pairing on the spaces \( L^p_l \).

**Definition 4.24.** For every measurable \( A \subset \mathcal{B} \) and for every \( 1 \leq p \leq q \leq 1 \) (allowing also for \( p = 1 \) and \( q = \infty \)), define an operator \( f_A : L^p_l(A) \times L^q(A) \to \mathcal{B} \) by posing
\[
\int_A \langle f_n \rangle \langle g_n \rangle = \text{s-lim} \int_A f_n g_n.
\]

The operator is well-defined, i.e. it is independent on the choice of the representatives for \( \langle f_n \rangle \) and for \( \langle g_n \rangle \).

**Lemma 4.25.** For every measurable \( A \subset \mathcal{B} \), for every \( 1 \leq p \leq q \leq \infty \) with \( 1/p + 1/q = 1 \) (allowing also for \( p = 1 \) and \( q = \infty \)), for every \( \langle f_n \rangle \in L^p_l(A) \) and for every \( \langle g_n \rangle \in L^q_l(A) \), if \( \langle f_n \rangle = \langle F_n \rangle \) and if \( \langle g_n \rangle = \langle G_n \rangle \), then
\[
\int_A \langle f_n \rangle \langle g_n \rangle = \int_A \langle F_n \rangle \langle G_n \rangle.
\]

**Proof.** Let us calculate
\[
\int_A \langle f_n \rangle \langle g_n \rangle - \int_A \langle F_n \rangle \langle G_n \rangle = \text{s-lim} \int_A f_n g_n - \text{s-lim} \int_A F_n G_n.
\]
Notice that
\[
(4.3) \quad \int_A F_n G_n = \int_A (F_n - f_n) G_n + \int_A f_n G_n
\]
By the Hölder’s inequality,
\[
\int_A |(F_n - f_n) G_n| \leq \| F_n - f_n \|_p \| G_n \|_q
\]
and, since \( \text{s-lim}_{k \to \infty} \| F_n - f_n \|_p = 0 \), then by the squeeze theorem for strong convergence we also have \( \text{s-lim}_{k \to \infty} \int_A (F_n - f_n) G_n = 0 \). Since \( \int_A (F_n - f_n) G_n \leq \int_A |(F_n - f_n) G_n| \), we obtain also \( \text{s-lim}_{k \to \infty} \int_A (F_n - f_n) G_n = 0 \). From equation (4.3), we conclude that \( \int_A F_n G_n = \int_A f_n G_n \).

With a similar argument, it is possible to deduce also \( \int_A f_n G_n = \int_A f_n g_n \), as desired. \( \square \)

Thanks to the properties of the integral over \( \mathcal{B} \) and of the strong convergence, it is possible to prove that \( f \) is an inner product over \( L^2_{\mathcal{B}} \).

**Lemma 4.26.** For every measurable \( A \subset \mathcal{B} \), \( f \) is a scalar product over \( L^2_{\mathcal{B}}(A) \).

**Proof.** Symmetry and bilinearity can be obtained from the definition of \( f \). In order to prove that \( f(\langle f_n \rangle) > 0 \) whenever \( \langle f_n \rangle \neq 0 \), recall that \( f(\langle f_n \rangle) = \| \langle f_n \rangle \|_2^2 \) and that, as a consequence of Lemma 4.22, if \( \langle f_n \rangle \neq 0 \) then \( \| \langle f_n \rangle \|_2 > 0 \). \( \square \)

As a consequence of the completeness of \( L^2_{\mathcal{B}} \) and of the existence of the scalar product \( f \), we conclude that \( L^2_{\mathcal{B}} \) is a Hilbert space.

**Proposition 4.27.** For every measurable \( A \subset \mathcal{B} \), the space \( L^2(A) \) is a Hilbert space.

**Proof.** In Proposition 4.23 we have proved that \( L^2(A) \) with the scalar product \( f \) is a Banach space. Since \( \int_A \) is a scalar product over \( L^2_{\mathcal{B}}(A) \), as shown in Lemma 4.26, \( L^2_{\mathcal{B}}(A) \) is a Hilbert space. \( \square \)
The previous results show that the \( \mathcal{L}_p^p \) spaces share many good properties with the real \( L^p \) spaces based upon the Lebesgue measure. Despite these positive features, however, they are still not rich enough to represent real functions that are not locally analytic. More precisely, we will prove that for every non-negative real measurable function \( f \) over the interval \([a, b]\), if \( f \) is not locally analytic almost everywhere, then there exist no \( \phi \in \mathcal{L}_p^p([a, b]_{\mathbb{R}}) \) such that

\[
(4.4) \quad \int_I |f(x)|^p \, dx \approx \int_{I_{\phi}} |\phi|^p \quad \text{for every interval } I \subseteq [a, b].
\]

**Proposition 4.28.** If \( f \in L^p([a, b]) \) is a non-negative function that is not locally analytic at any point in \([a, b]\), then for every \( \langle g_n \rangle \in \mathcal{L}_p^p([a, b]_{\mathbb{R}}) \) there exists an interval \( I(c, d) \) with \( c, d \in \mathbb{R} \) and \( c \neq d \) such that

\[
\left\| \langle g_n \cdot \chi_{I(c, d)} \rangle \right\|_p \approx \left\| f(x) \cdot \chi_{I(c, d)} \right\|_p.
\]

**Proof.** By Proposition 4.23, \( \mathcal{L}_p^p([a, b]_{\mathbb{R}}) \) is dense in \( \mathcal{L}_p^p([a, b]_{\mathbb{R}}) \). As a consequence, it is sufficient to prove that for every \( g \in \mathcal{L}_p^p([a, b]_{\mathbb{R}}) \) there exists an interval \( I(c, d) \) with \( c, d \in \mathbb{R} \) and \( c \neq d \) such that

\[
\left\| g \cdot \chi_{I(c, d)} \right\|_p \approx \left\| f(x) \cdot \chi_{I(c, d)} \right\|_p.
\]

If \( \lambda(g(x)) < 0 \) for every \( x \in [a, b]_{\mathbb{R}} \), then the desired inequality is trivial, since in this case \( \lambda \left( \left\| g \cdot \chi \right\|_p \right) < 0 \). Suppose then that there exists \( I \subseteq [a, b]_{\mathbb{R}} \) with \( l(I) \neq 0 \) such that \( g \) is analytic over \( I \) and \( \lambda(g(x)) \geq 0 \) for every \( x \in I \). Then it is possible to define a real function \( G : I \cap \mathbb{R} \to \mathbb{R} \) by posing \( G(r) = g(r)[0] \) for every \( r \in I \cap \mathbb{R} \). Moreover, by our choice of \( I \), \( G \) is analytic in \( I \cap \mathbb{R} \). As a consequence, by Lemma 4.10, \( \left\| G \cdot \chi_{I_{\mathbb{R}}} \right\|_p \approx \left\| g \cdot \chi_{I_{\mathbb{R}}} \right\|_p \) for every interval \( J \subseteq I \). However, since \( f \) is not locally analytic at every point of its domain, there exist \( c, d \in I \) such that

\[
\left\| G \cdot \chi_{I(c, d)} \right\|_p \neq \left\| f \cdot \chi_{I(c, d)} \right\|_p
\]

for every \( 1 \leq p \leq \infty \). This is sufficient to conclude the proof. \( \square \)

**Remark 4.29.** We have chosen to formalize the notion of representation of a real function by an element of \( \mathcal{L}_p^p \) with condition (4.4) since for any pair of non-negative functions \( f, g \in L^p([a, b]) \) the property

\[
\int_I |f(x)|^p \, dx = \int_I |g(x)|^p \quad \text{for every interval } I \subseteq [a, b]
\]

is equivalent to \( f = g \) a.e. in \([a, b]\).

5. Representing real functions and distributions

We have seen in Proposition 3.18 that both the space of measurable functions and the \( \mathcal{L}_p \) spaces over the Levi-Civita field are not expressive enough to represent real continuous functions. Similarly, Proposition 4.28 entails that it is not possible to represent real functions that are not locally analytic as strongly converging sequences of \( \mathcal{L}_p \) functions. However, it is possible to represent a large class of real functions as weakly Cauchy sequences of measurable functions. This, in turn, will enable the representation of real distributions with measurable functions on the Levi-Civita field.
5.1. **Representing real functions as weakly Cauchy sequences of \( \mathcal{L}^p \) functions.** Even if it is not possible to represent real continuous functions with elements of \( \mathcal{L}^p \), weakly Cauchy sequences of \( \mathcal{L}^p \) functions are expressive enough to represent not only real continuous functions, but every real function whose \( p \)-th power is summable.

**Theorem 5.1.** Let \( I \subset \mathbb{R} \) be an interval. For every real \( 1 \leq p \leq \infty \) and for every \( f \in L^p(I) \), there exists a weakly Cauchy sequence \( \{ \overline{s}_n \}_{n \in \mathbb{N}} \) of measurable functions in \( \mathcal{M}(I, \mathbb{R}) \) such that

1. \( \text{w-lim}_{n \to \infty} \| \overline{s}_n \|_p = \| f \|_p \),
2. for every interval \( J \subseteq I \), \( \text{w-lim}_{n \to \infty} \| \overline{s}_n \cdot \chi_J \|_p = \| f \cdot \chi_J \|_p \), and
3. for a.e. \( x \in I \) there exists a subsequence \( \{ \overline{s}_{n_k} \}_{k \in \mathbb{N}} \) such that \( \text{w-lim}_{k \to \infty} \overline{s}_{n_k}(x) = f(x) \).

**Proof.** Let \( f \in L^p(I) \) and let \( \{ s_n \}_{n \in \mathbb{N}} \) be a sequence of simple functions satisfying \( \lim_{n \to \infty} \| s_n - f \|_p = 0 \). Since \( s_n \) is simple for every \( n \in \mathbb{N} \), there exists \( k_n \) such that \( s_n = \sum_{i=1}^{k_n} s_{n,i} \), with \( s_{n,i} \) constant over an interval \( I_{n,i} \subseteq I \). Then, if we let \( \overline{s}_{n,i}(x) \) be the canonical extension of \( s_{n,i} \) to \( (I_{n,i})_{\mathbb{R}} \), we can define \( \overline{s}_n = \sum_{i=1}^{k_n} \overline{s}_{n,i} \). The functions \( \overline{s}_{n,i} \) are measurable by Lemma 4.10, so that \( \overline{s}_n \) is also measurable. Moreover, the same Lemma entails that

\[
\| \overline{s}_n \|_p = \left\| \sum_{i=1}^{k_n} s_{n,i} \right\|_p = \left\| \sum_{i=1}^{k_n} s_{n,i} \right\|_p = \| s_n \|.
\]

This equality, the hypothesis that \( \lim_{n \to \infty} \| s_n - f \|_p = 0 \) and Corollary 2.14 entail that

- \( \{ \overline{s}_n \}_{n \in \mathbb{N}} \) is weakly Cauchy in \( \mathcal{L}^p(I, \mathbb{R}) \)
- \( \text{w-lim}_{n \to \infty} \| \{ \overline{s}_n \}_{n \in \mathbb{N}} \|_p = \| f \|_p \).

This is sufficient to prove (1). The proof of equality (2) can be obtained by localizing the above proof to any interval \( J \subseteq I \).

Property (3) can be obtained from Corollary 2.14 and from the hypothesis that \( s_n \) converges to \( f \) in the \( L^p \) norm. The latter entails that for a.e. \( x \in I \) there exists a subsequence \( \{ s_{n_k} \}_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} s_{n_k}(x) = f(x) \). Thus by Corollary 2.14 we also have \( \text{w-lim}_{k \to \infty} \overline{s}_{n_k}(x) = f(x) \).

As a consequence of the above theorem, we can represent \( L^p \) functions as weakly Cauchy sequences of measurable functions.

**Remark 5.2.** Theorem 5.1 might suggest that it could be convenient to define not only the spaces \( \mathcal{L}^p \), but also some \( \mathcal{L}^p \) spaces obtained by substituting strong limits with weak limits in Definitions 4.17 and 4.19. However, this route presents many obstacles. The principle is that weak convergence does not satisfy a squeeze theorem, so many results of Section 4.4 turn out to be false for the \( \mathcal{L}^p \) spaces. For instance, if \( \{ f_n \}_{n \in \mathbb{N}} \) is a weakly Cauchy sequence such that \( \text{w-lim}_{n \to \infty} \| f_n \|_p \) is finite, one might define \( \| \{ f_n \} \|_p = \text{w-lim}_{n \to \infty} \| f_n \|_p \). However, with this definition there would be sequences \( \{ f_n \}_{n \in \mathbb{N}} \) satisfying \( \| f_n \|_p > 0 \) for all \( n \in \mathbb{N} \), but \( \| \{ f_n \} \|_p < 0 \) (one of such sequences can be obtained from Example 2.16). Consequently, the map \( \| \cdot \|_p \) would not be a norm or even a seminorm.

Moreover, the value \( \| \{ f_n \} \|_p \) would not be independent from the representative \( \{ f_n \}_{n \in \mathbb{N}} \). This happens because there are sequences of measurable functions \( \{ f_n \}_{n \in \mathbb{N}} \) and \( \{ g_n \}_{n \in \mathbb{N}} \) such that \( \text{w-lim}_{n \to \infty} \| f_n - g_n \|_p = 0 \), but \( \text{w-lim}_{n \to \infty} \| f_n \|_p \neq \text{w-lim}_{n \to \infty} \| g_n \|_p \). A possible workaround could be that of defining a function \( \| \{ f_n \} \|_p = \lim_{n \to \infty} \| f_n \|_p \) and in identifying two sequences \( \{ f_n \}_{n \in \mathbb{N}} \) and \( \{ g_n \}_{n \in \mathbb{N}} \) whenever \( \lim_{n \to \infty} \| f_n - g_n \|_p = 0 \).
However, such a definition would entail the identification of all functions whose values differ up to an infinitesimal.

We remark that some classes of measurable functions could have a more meaningful representation than the one obtained in the proof of Theorem 5.1. The strongest result can be obtained for real continuous functions defined over compact intervals.

**Proposition 5.3.** For every \( f \in C^0([a,b]) \) and for every \( n \in \mathbb{N} \), there exists a sequence of measurable functions \( \{ \psi_n \}_{n \in \mathbb{N}} \) that satisfies the following properties for every \( 1 \leq p \leq \infty \):

1. \( \{ \psi_n \}_{n \in \mathbb{N}} \) is weakly Cauchy in \( L^p([a,b],\mathbb{R}) \);
2. \( \operatorname{w-lim}_{n \to \infty} \| \psi_n \|_p = \| f \|_p \);
3. \( \operatorname{w-lim}_{n \to \infty} \psi_n(x) = f(x) \) for every \( x \in [a,b] \).

**Proof.** Let \( [a,b] \subset \mathbb{R} \) and let \( \mathbb{P}_n([a,b]) \) the space of real polynomials of degree at most \( n \) defined over \( [a,b] \):

\[
\mathbb{P}_n([a,b]) = \{ p : [a,b] \to \mathbb{R} : \exists a_0, \ldots, a_n \in \mathbb{R} \text{ such that } p(x) = \sum_{i=0}^{n} a_i x^i \}.
\]

Recall that for every \( f \in C^0([a,b]) \) and for every \( n \in \mathbb{N} \), there exists a unique \( p^*_n \in \mathbb{P}_n \) such that

\[
\| f - p^*_n \|_\infty \leq \| f - p \|_\infty \quad \text{for all } p \in \mathbb{P}_n
\]

and

\[
\lim_{n \to \infty} \| p^*_n \|_p = \| f \|_p
\]

for every \( 1 \leq p \leq \infty \). For a proof of these statements, we refer for instance to [12].

For every \( n \in \mathbb{N} \) define \( \psi_n \), as the canonical extension of the polynomial \( p^*_n \) to \( [a,b] \).

Properties (1) and (2) can be obtained from equation (5.3), from Lemma 4.10 and from Corollary 2.14: since

\[
\int_{[a,b]} |\psi_n(x)|^p = \int_{[a,b]} |p^*_n(x)|^p dx \quad \text{and} \quad \lim_{n \to \infty} \int_{[a,b]} |p^*_n(x)|^p dx = \int_{[a,b]} |f(x)|^p dx,
\]

then \( \operatorname{w-lim}_{n \to \infty} \| \psi_n \|_p = \| f \|_p \). We deduce that the sequence \( \{ \psi_n \}_{n \in \mathbb{N}} \) is weakly Cauchy in \( L^p([a,b],\mathbb{R}) \) and that \( s\lim_{n \to \infty} \| \psi_n \|_p = \| f \|_p \).

Property (3) is a consequence of the equality \( \psi_n(x) = p^*_n(x) \) for all \( x \in [a,b] \), of uniform convergence of \( p^*_n \) to \( f \) and of Corollary 2.14. \( \square \)

For reciprocals of simple functions that diverge at a point, like the ones discussed in Proposition 3.18, it might be more convenient to work with a different approximation, as shown in the next example.

**Example 5.4.** Let \( a > 1 \) and consider the function \( f(x) = x^{-\frac{a}{2}} \) for \( x \in (0,1]_{\mathbb{R}} \). Recall that, by Proposition 3.18, \( f(x) \) is not measurable over \((0,1]_{\mathbb{R}}\).

Define the functions \( f_n = f|_{[n^{-1},1]} \) for every \( n \in \mathbb{N} \). By Lemma 4.10,

\[
\int_{[n^{-1},1]} |f_n|^p = \int_{n^{-1}}^{1} x^{-\frac{a}{2}} dx.
\]

It is well-known that the real sequence \( \int_{n^{-1}}^{1} x^{-\frac{a}{2}} dx \) converges for every \( 1 \leq p < a \): consequently, under these conditions over \( p \), we have

- \( \{ f_n \}_p \) is weakly Cauchy in \( L^p((0,1]_{\mathbb{R}}) \).
\[ \text{w-lim}_{n \to \infty} \|f_n\|_p = \|f\|_p. \]

In addition, we have \( \text{w-lim}_{n \to \infty} f_n(x) = f(x) \) for every \( x \in (0, 1] \), \( x \notin M_n \) (i.e. for every nearstandard \( x \in (0, 1] \)).

5.2. **Dirac-like measurable functions.** The Dirac distribution centred at \( r \in \mathbb{R} \) is a continuous linear functional defined over \( C^0(\mathbb{R}) \) by the formula
\[ \langle D_r, \varphi \rangle = \varphi(r) \]
for every \( \varphi \in C^0(\mathbb{R}) \) (where \( \langle \cdot, \cdot \rangle \) denotes the duality between a distribution and a test function).

Measurable functions that represent the Dirac distribution have already been studied by Shamseddine and Flynn [16]. The authors defined some specific representatives of the Dirac distributions and applied them for the description of physical phenomena. For instance, they discussed the measurable solutions to the Poisson equation \( y''(t) = f(t) \) and determined the Green function of a damped driven harmonic oscillator.

In this section we will define a family of measurable functions over the Levi-Civita field that represent the Dirac distribution, and study some of their theoretical properties. We will also discuss the product between the Dirac and the Heaviside distribution, a well-known problem of the nonlinear theory of distributions (for a discussion of this problem and for a solution in an algebra of generalized functions, we refer to [9]). This problem cannot be treated in the space of real distributions; instead it requires algebras of generalized functions, such as Colombeau algebras [13], asymptotic functions [22], ultrafunctions [6] and grid functions [9]. Notice that these algebras have non-Archimedean rings of scalars and, if one exclude Colombeau algebras, the ring of scalar of the other algebras of generalized functions are fields obtained with techniques of Robinson’s nonstandard analysis. The non-Archimedean rings of scalars underlying these spaces of generalized functions suggests that nonlinear theories of distributions require a non-Archimedean setting. Inspired by this observation, we aim at proving that that the results concerning the product between the Dirac and the Heaviside distribution obtained with these algebras of generalized functions can also be obtained with measurable functions on on the Levi-Civita field.

The Dirac measurable functions discussed by Shamseddine and Flynn are non-negative measurable functions, their support is included in an infinitesimal neighbourhood of a point and that their integral equals 1. We will say that a measurable function with these properties is a Dirac-like function.

**Definition 5.5.** A measurable function \( \delta_r \in \mathcal{L}^1(\mathbb{A}) \) is Dirac-like at \( r \in \mathbb{A} \) iff

1. \( \delta_r(x) \geq 0 \) for all \( x \in A \);
2. there exists \( h \in M_n, h > 0 \) such that \( \text{supp} \delta_r \subseteq [r - h, r + h] \subset A \);
3. \( \|\delta_r\|_1 = 1 \).

By using the representation of real continuous functions of Proposition 5.3, we will now argue that every Dirac-like function represents the real Dirac distribution.

**Proposition 5.6.** Let \( r \in [a, b], r \neq a \) and \( r \neq b \). For all Dirac-like measurable functions \( \delta_r \in \mathcal{L}^1([a, b]_\mathbb{A}) \) and for all \( f \in C^0([a, b]) \), if \( \mathcal{P}_n \) is defined as in Proposition 5.3, then
\[ \lim_{n \to \infty} \gamma \left( \int_{[a, b]} \delta_r \cdot \mathcal{P}_n \right) = \lim_{n \to \infty} \gamma \left( \int_{[a, b]} \delta_r \cdot \mathcal{P}_n \right) [0] = f(r). \]

**Proof.** If \( r \notin \text{supp} f \), then by uniform convergence of \( p'_n \) to \( f \) in \([a, b]\), we have also \( \text{w-lim}_{n \to \infty} \mathcal{P}_n = 0 \) for almost every \( x \sim r, x \in [a, b]_\mathbb{A} \). As a consequence, the desired assertion is true.
Suppose then that \( r \in \text{supp} f \) and let \( h \in M_n \) such that \( \text{supp} \delta \subseteq [r - h, r + h)_0 \). Since \( \mathfrak{p}_n \) is analytic over \([a, b]_0\), for all \( n \in \mathbb{N} \) the numbers
\[
m_n = \min_{x \in [r - h, r + h)_0} \mathfrak{p}_n(x) \quad \text{and} \quad M_n = \max_{x \in [r - h, r + h)_0} \mathfrak{p}_n(x)
\]
are well-defined [24]. As a consequence of these inequalities and thanks to Corollary 4.6 of [29],
\[
(5.4) \quad m_n = m_n \int_{[r-h,r+h)_0} \delta_n \leq \int_{[r-h,r+h)_0} \delta_n \mathfrak{p}_n \leq M_n \int_{[r-h,r+h)_0} \delta_n = M_n
\]
for all \( n \in \mathbb{N} \). Thus it is sufficient to prove that \( \lim_{n \to \infty} M_n[0] = \lim_{n \to \infty} m_n[0] = f(r) \). The desired result is a consequence of the equalities \( M_n[0] = m_n[0] = \mathfrak{p}_n(r) = p'_n(r) \) and of Proposition 5.3. \( \square \)

5.3. **Measurable representatives of the Heaviside distribution.** In principle, the Heaviside distribution could be represented by any non decreasing measurable function \( H \) such that \( H(x) = 0 \) whenever \( x < 0 \), \( x \notin M_0 \) and \( H(x) = 1 \) whenever \( 0 < x \), \( x \notin M_0 \). However, it will be more convenient to define it as the antiderivative of a continuous Dirac-like distribution centred at 0.

**Definition 5.7.** Let \( \delta_h \) be a continuous Dirac-like measurable function. Define \( H : \mathbb{R} \to \mathbb{R} \) as
\[
H(x) = \int_{[-1,1]} \delta_h.
\]

We remark that every continuous Dirac-like measurable function yields a corresponding function \( H \). With a slight abuse of notation, we will not explicitly denote the dependence of \( H \) upon \( \delta_h \).

The functions obtained with Definition 5.7 represent the Heaviside distribution.

**Proposition 5.8.** For all \( f \in C^0([a, b]) \), if \( \mathfrak{p}_n \) is defined as in Proposition 5.3, then
\[
\lim_{n \to \infty} \left( \int_{[a, b]} H \cdot \mathfrak{p}_n \right) = \lim_{n \to \infty} \left( \int_{[a, b]} H \cdot \mathfrak{p}_n \right) [0] = \int_0^\infty f(x) \, dx.
\]

**Proof.** If \( f \in C^0([a, b]) \) and if \( H_\mathbb{R} \) is the restriction of \( H \) to \( \mathbb{R} \), then \( H_\mathbb{R} \cdot f \) is equal to the restriction of \( f \) to \([a, b] \cap (0, \infty) \).

Let \( h \in M_n \), \( h > 0 \) satisfy \( \text{supp}(\delta_h) \subseteq [-h, h)_0 \). Then
\[
\int_{[a, b]} H \cdot \mathfrak{p}_n = \int_{[a, h]} H \cdot \mathfrak{p}_n + \int_{[h, b]} \mathfrak{p}_n.
\]
Since \( h \in M_0 \) and since both \( h \) and \( \mathfrak{p}_n \) are bounded by some real number \( r \in \mathbb{R} \),
\[
\left| \int_{[a, b]} H \cdot \mathfrak{p}_n \right| \leq h \cdot r \approx 0.
\]
Moreover, \( H(x) \cdot \mathfrak{p}_n(x) = p'_n(x) \) for every \( x \in [a, b]_0 \cap (h, \infty) \). Thus, by Lemma 4.10,
\[
\left( \int_{[h, b]} H \cdot \mathfrak{p}_n \right) = \int_{[0, b]} p'_n \, dx.
\]
The desired equality can then be obtained from uniform convergence of \( \{ p'_n \}_{n \in \mathbb{N}} \) to \( f \) over \([a, b] \cap (0, \infty) \) with an argument similar to the one used in the proof of Theorem 5.1. \( \square \)
One of the many advantages of working with representatives of the Heaviside distribution defined over non-Archimedean domains is that they allow for some calculations that are not possible with the real distributions. For instance, in the description of shock waves Colombeau works with a representative of the Heaviside distribution and of the Dirac distribution that satisfy the equality

\begin{equation}
\int_{\mathbb{R}} (H^m - H^n) \cdot \delta_0 = \frac{1}{m+1} - \frac{1}{n+1}.
\end{equation}

As recalled in [9], this calculation is not justified in the theory of distributions. A first obstacle is that, in the real case, the powers \(H^n\) of the Heaviside distribution are all equal to the original function \(H\). However, this does not happen for the measurable representatives introduced in Definition 5.7.

**Lemma 5.9.** For every \(n \in \mathbb{N}\), and for every \(f \in C^0([a, b])\),

\[
\lim_{n \to \infty} \left( \int_{[a, b]} H^n \cdot \mathcal{P}_n \right) = \lim_{n \to \infty} \left( \int_{[a, b]} H^n \cdot \mathcal{P}_n \right) [0] = \int_0^{+\infty} f(x) \, dx.
\]

However, \(H^n = H^m\) if and only if \(n = m\).

**Proof.** Let \(h \in M_0\) such that \(\text{supp} \delta_0 \subseteq [-h, h]_{\mathcal{R}}\). Then for every \(m, n \in \mathbb{N}\)

1. if \(x \leq -h\), then \(H^n(x) = H^m(x) = 0\).
2. if \(x \geq h\) then \(H^n(x) = H^m(x) = 1\).
3. there exists \(x \in [-h, h]\) such that \(0 < H(x) < 1\).

By point (3), we immediately conclude that \(H^n = H^m\) if and only if \(n = m\).

Thanks to properties (1) and (2), if \(b < -h\) or \(a > h\) the equality \(\lim_{n \to \infty} \left( \int_{[a, b]} H^n \cdot \mathcal{P}_n \right) [0] = \int_0^{+\infty} f(x) \, dx\) is trivially true.

Suppose then that \(a \leq h \leq b\). In this case, we can write

\[
\int_{[a, b]} H^n \cdot \mathcal{P}_n = \int_{[a, h]} H^n \cdot \mathcal{P}_n + \int_{[h, b]} \mathcal{P}_n.
\]

Since \(h \in M_0\), by Lemma 4.10 and by equation (5.3) we have

\[
\lim_{n \to \infty} \left( \int_{[a, b]} H^n \cdot \mathcal{P}_n \right) = \lim_{n \to \infty} \left( \int_{[a, h]} H^n \cdot \mathcal{P}_n \right) + \int_{[h, b]} \mathcal{P}_n [0] = \int_0^{+\infty} f(x) \, dx.
\]

Similarly, since \(f\) is bounded and since \(0 \leq H^n(x) \leq 1\) for every \(n \in \mathbb{N}\) and for every \(x \in \mathcal{R}\),

\[
\int_{[a, b]} H^n \cdot \mathcal{P}_n \approx 0.
\]

We conclude

\[
\lim_{n \to \infty} \left( \int_{[a, b]} H^n \cdot \mathcal{P}_n \right) [0] = \lim_{n \to \infty} \left( \int_{[a, b]} H^n \cdot \mathcal{P}_n \right) [0] + \lim_{n \to \infty} \left( \int_{[a, h]} H^n \cdot \mathcal{P}_n \right) [0] = \int_0^{+\infty} f(x) \, dx.
\]

The case where \(a \leq -h \leq b\) can be treated in a similar way.

The second obstacle for the evaluation of the expression (5.5) is that the product between the Heaviside distribution and the Dirac distribution is not defined. However, the measurable representatives of the Dirac and of the Heaviside distribution allow to evaluate the product \(H \cdot \delta_0\), similarly to what happens in Colombeau algebras [13] and in spaces of generalized functions of nonstandard analysis [9, 22].
**Proposition 5.10.** For all $f \in C^0([a,b])$ and for all $1 \leq p \leq \infty$, if $\mathcal{P}_n$ is defined as in Proposition 5.3, then

$$
\lim_{n \to \infty} \mathcal{P}_n \left( \int_{[a,b]} (H \cdot \delta_0) \cdot \mathcal{P}_n \right) = \lim_{n \to \infty} \mathcal{P}_n \left( \int_{[a,b]} (H \cdot \delta_0) \cdot \mathcal{P}_n \right) [0] = \frac{1}{2} f(0)
$$

and

$$
\int_{[a,b]} (H^m - H^n) \cdot \delta_0 = \frac{1}{m+1} - \frac{1}{n+1}.
$$

**Proof.** Recall that we have supposed that $\delta_0$ is continuous: thanks to Proposition 3.24, $H \in C^1(\mathcal{R})$ and $H'(x) = \delta_0(x)$ for a.e. $x \in \mathcal{R}$. Recall also that the well-known product formula for the derivative is still true for differentiable functions defined on the Levi-Civita field [2]: consequently, we have $(H^{m+1})' = (m+1)H^m \cdot \delta_0$ for all $m \in \mathbb{N}$. Taking $m = 1$, we deduce that $\int_{[a,b]} (H \cdot \delta_0) \cdot \mathcal{P}_n = \frac{1}{2} \int_{[a,b]} (H^2)' \cdot \mathcal{P}_n$. Now notice that

1. $(H^2)'(x) \geq 0$ for every $x \in \mathcal{R}$;
2. there exists $h \in M_n, h > 0$ such that $\text{supp}(H^2)' \subseteq [-h,h]$;
3. by definition of the integral, $\int_{-h,h} (H^2)' = H^2(h) - H^2(-h)$.

We have already observed in the proof of Lemma 5.9 that $H(h) = 1$ and $H(-h) = 0$, so that also $H^2(h) = 1$ and $H^2(-h) = 0$. Thus $(H^2)'$ is a Delta-like measurable function in the sense of Definition 5.5. Proposition 5.6 ensures that

$$
\lim_{n \to \infty} \left( \frac{1}{2} \int_{[a,b]} (H^2)' \cdot \mathcal{P}_n \right) = \lim_{n \to \infty} \left( \frac{1}{2} \int_{[a,b]} (H^2)' \cdot \mathcal{P}_n \right) [0] = \frac{1}{2} f(0),
$$

as desired.

In order to prove the second equality, let $h \in M_n$ such that $\text{supp} \delta_0 \subseteq [-h,h]$. Since $(H^{m+1})' = (m+1)H^m \cdot \delta_0$,

$$
\int_{[a,b]} (H^m - H^n) \cdot \delta_0 = \int_{[-h,h]} \left( \frac{H^{m+1}}{m+1} - \frac{H^{n+1}}{n+1} \right) \cdot \delta_0 = \left( \frac{H^{m+1}(h)}{m+1} - \frac{H^{n+1}(h)}{n+1} \right) - \left( \frac{H^{m+1}(-h)}{m+1} - \frac{H^{n+1}(-h)}{n+1} \right).
$$

Since $H^m(-h) = H^n(-h) = 0$ and $H^m(h) = H^n(h) = 1$ for all $m,n \in \mathbb{N}$, we obtain the desired result. \hfill \square

The previous Proposition shows that, whenever $\delta_0$ is a Dirac-like continuous and measurable function, the product $H \cdot \delta_0$ is well-defined and is equal to $\frac{1}{2} \delta_0$. This result agrees with similar calculations obtained with other generalized functions. Moreover, the evaluation of the integral (5.5) is justified also in the Levi-Civita field. For a more detailed discussion on these topics from the nonlinear theory of distributions, we refer to [9].

### 5.4. Derivatives of the Dirac distribution

In this section, we will show how the derivatives of Dirac-like measurable functions represent the distributional derivatives of the Dirac distributions. In order to do so, we need to establish some results on the representation of the derivative of a real continuous function by means of sequences of measurable functions in the Levi-Civita field.

Recall that, if $F \in C^1([a,b])$ and if $\{P_n\}_{n \in \mathbb{N}}$ uniformly converges to $F$ over $[a,b]$, then the sequence $\{P'_n\}_{n \in \mathbb{N}}$ might not converge to $F'$ over $[a,b]$. This property suggests a subtler approach to the representation of the derivative of a differentiable function.
If \( f \in C^0([a,b]) \), denote by \( F \in C^1([a,b]) \) the function defined by \( F(x) = \int_a^x f(t) \, dt \).

It is well-known that, if \( P_n \) are defined as in the proof of Proposition 5.3 and if \( P_n(x) = \int_a^x p_n(t) \, dt \), then the sequence \( \{P_n\}_{n \in \mathbb{N}} \) uniformly converges to \( F \) over \([a,b]\). Let \( \overline{P}_n \) be the canonical extension of \( P_n \) over \([a,b]\). Then, as a consequence of Theorem 5.1 and of Proposition 5.3, the sequence \( \{\overline{P}_n\}_{n \in \mathbb{N}} \) satisfies the conditions

1. \( \{\overline{P}_n\}_{n \in \mathbb{N}} \) is weakly Cauchy in \( \mathcal{L}^p([a,b]) \) for every \( 1 \leq p \leq \infty \);  
2. \( \text{w-lim}_{n \to \infty} \|\overline{P}_n\|_p = \|F\|_p \);  
3. \( \text{w-lim}_{n \to \infty} \overline{P}_n(x) = F(x) \) for every \( x \in [a,b] \).

From this representation of \( F \) we can obtain a measurable representation of the derivative of the Dirac distribution.

**Proposition 5.11.** Let \( r \in [a,b] \), \( r \neq a \) and \( r \neq b \) and let \( \delta_r \) be a Dirac-like measurable function of class \( C^1 \). Then for every \( f \in C^0([a,b]) \),

\[
\lim_{n \to \infty} \left( \int_{[a,b]} \delta_r' \cdot \overline{P}_n \right) = \lim_{n \to \infty} \left( \int_{[a,b]} \delta_r' \cdot P_n \right) [0] = -f(r).
\]

**Proof.** By the product formula for the derivative and by definition of \( \overline{P}_n \), we have

\[
(\delta_r \cdot \overline{P}_n)' = \delta_r \cdot \overline{P}_n + \delta_r' \cdot \overline{P}_n.
\]

Thus

\[
\int_{[a,b]} (\delta_r \cdot \overline{P}_n)' = \int_{[a,b]} \delta_r' \cdot \overline{P}_n + \int_{[a,b]} \delta_r \cdot \overline{P}_n.
\]

The hypothesis that \( r \neq a \) and \( r \neq b \) entail

\[
\int_{[a,b]} (\delta_r \cdot \overline{P}_n)' = \delta_r(b) \cdot \overline{P}_n(b) - \delta_r(a) \cdot \overline{P}_n(a) = 0,
\]

so that

\[
\int_{[a,b]} \delta_r' \cdot \overline{P}_n = -\int_{[a,b]} \delta_r \cdot \overline{P}_n.
\]

By Proposition 5.6 we obtain the desired equality. \( \square \)

By iterating the previous result, one can obtain a general representation of the \( k \)-th derivative of the Dirac distribution.

**Corollary 5.12.** Let \( r \in [a,b] \), \( r \neq a \) and \( r \neq b \) and let \( \delta_r \) be a Dirac-like measurable function of class \( C^k \). Let \( F \in C^k([a,b]) \) and define \( f = F^{(k)} \). If the sequence of measurable functions \( \{\overline{P}_n\}_{n \in \mathbb{N}} \) represents \( f \) in the sense of Proposition 5.3, and if \( \{\overline{P}_n\}_{n \in \mathbb{N}} \) is a sequence of simple functions that satisfy \( \overline{P}_n^{(k)} = P_n \) for every \( n \in \mathbb{N} \), then

\[
\lim_{n \to \infty} \delta_r^{(k)} \cdot \overline{P}_n = \lim_{n \to \infty} \delta_r^{(k)} \cdot P_n [0] = (-1)^k f(r).
\]

6. **Final remarks**

In Section 5 we have shown that it is possible to represent a class of real measurable functions with sequences of measurable functions in the Levi-Civita field, and to represent some real distributions with measurable functions on the Levi-Civita field. This approach has allowed us to obtain some classic results from the nonlinear theory of distributions by working with measurable functions on the Levi-Civita field.

Despite these positive results, we believe that there are some obstacles towards representing every real distribution with measurable functions over \( \mathcal{R} \). Recall the following representation theorem for real distributions: if \( T \) is a distribution with compact support,
then there exists $f \in C^0(\mathbb{R})$ and $k \in \mathbb{N}$ such that $T = f^{(k)}$, where the derivative is assumed in the sense of distributions [31]. Thanks to this equality and from the approach used in Section 5.4 to obtain Corollary 5.12, if $f$ is locally analytic then it might be possible to represent the distribution $T$ with a measurable function defined on the Levi-Civita field (but recall that if $f$ is locally analytic, then $T$ can be identified with a locally analytic function). However, Proposition 3.16 and Proposition 4.28 entail that it is not possible to represent every real distribution with compact support in this way. In particular, if $T = f^{(k)}$ for some $f \in C^0$ that is not locally analytic at almost every point of its domain, then $T$ does not admit a representation as a measurable function defined on the Levi-Civita field.

If one is interested in representing distributions of arbitrary support and whose order is not finite, i.e. such that $T = \sum_{n \in \mathbb{N}} f^{(k_n)}_n$ with $f_n \in C^0(\mathbb{R})$ for every $n \in \mathbb{N}$ and with $\lim_{n \to \infty} k_n = +\infty$ (for a more precise definition of order of a distribution see Section 6.2 of [31]), then it might not even possible to use the approach proposed in Section 5.4.

Theorem 5.1 and the above examples might suggest that it would be more convenient to use the weak limit instead of the strong limit in the definition of measurable sets and functions. However, this idea needs to be worked out with great care. As we have seen in Remark 5.2, it is not possible to define a notion of norm, and consequently also a notion of integral, by using the weak limit instead of the strong limit in the definition of the $\| \cdot \|_p$ norm (see Lemma 4.21). In addition, since the weak limit does not satisfy a squeeze theorem, it might happen that $f_n(x) > g_n(x)$ for all $n \in \mathbb{N}$, but that $\text{w-lim}_{n \to \infty} \|f_n\|_p < \text{w-lim}_{n \to \infty} \|g_n\|_p$. As a consequence, results such as Proposition 4.4 and Corollary 4.5 of [29] would not be true.

A measure defined by replacing the strong limit with the weak limit in Definition 3.2 would suffer from similar drawbacks. For instance, this “measure” would not be monotone, since it would be possible to prove that the monad $\mu (r)$ of a point $r \in \mathbb{R}$ is measurable with measure $0$ (since it can be obtained as the weak limit of the measure of the sets $(-n^{-1}, n^{-1})$). As a consequence, for every $h \in M_\mu$, $h > 0$ we would have the inclusion $(r - h, r + h) \subseteq \mu (r)$ together with the opposite inequality $0 = m(\mu (r)) < \mu ((r - h, r + h) \subseteq \mu (r)) = 2h$.

In order to overcome these obstacles and to exploit the strength of Theorem 5.1, we believe that it could be possible to define a uniform real-valued measure over the Levi-Civita field by adapting the Löeb measure construction of nonstandard analysis [15].

Notice however that allowing a too large class of functions to be measurable would lead to further difficulties. For instance, if the measure allows for the existence of non-constant measurable functions with null derivative, then the Mean Value Theorem for the integral would fail, and it would not be possible to represent the distributional derivative with the pointwise derivative on the Levi-Civita field.

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**References**

[1] M. Berz, *Analysis on a Nonarchimedean Extension of the Real Numbers*. Lecture Notes, 1992.

[2] M. Berz, *Calculus and numerics on Levi-Civita fields*. In M. Berz, C. Bischof, G. Corliss, and A. Griewank, editors, *Computational Differentiation: Techniques, Applications, and Tools*, pages 19–35, Philadelphia, 1996. SIAM.

[3] M. Berz, K. Shamseddine, *Analysis on the Levi-Civita field, a brief overview*, Contemporary Mathematics, 508, 2010, pp. 215-237.
[4] M. Berz, K. Shamseddine, Analytical properties of power series on Levi-Civita fields, Annales Mathématiques Blaise Pascal, Volume 12 n 2, 2005, pp. 309-329.
[5] M. Berz, K. Shamseddine, The differential algebraic structure of the Levi-Civita field and applications, Int. J. Appl. Math., Volume 3 (2000), pp. 449-465.
[6] V. Benci, Ultrafunctions and generalized solutions, Advanced Nonlinear Studies 13.2 (2013): 461-486.
[7] V. Benci and L. Luperi Baglini, A non-archimedean algebra and the Schwarz impossibility theorem, Monatshefte für Mathematik 176.4 (2015): 503-520.
[8] E. Bottazzi, A transfer principle for the continuation of real functions to the Levi-Civita field, p-Adic Numbers, Ultrametric Analysis, and Applications, issue 3, vol 10 (2018).
[9] E. Bottazzi, Grid functions of nonstandard analysis in the theory of distributions and in partial differential equations, Advances in Mathematics 345 (2019): 429-482.
[10] E. Bottazzi, A grid function formulation of a class of ill-posed parabolic equations (2017), submitted. See https://arxiv.org/abs/1704.00472.
[11] E. Bottazzi, M. G. Katz, Infinite lotteries, spinners, and the applicability of hyperreals, submitted.
[12] J. G. Burkill, Lectures On Approximation By Polynomials, Tata Institute of Fundamental Research (1959), ISBN/ASIN: B0007J A0CC
[13] J.F. Colombeau, Nonlinear generalized functions: their origin, some developments and recent advances, São Paulo J. Math. Sci. (ISSN2316-9028) 7(2) (2013) 201–239.
[14] Costin, O., Ehrlich, P., Friedman, H. M. Integration on the surreals: a conjecture of Conway, Kruskal and Norton, preprint (2015). See https://arxiv.org/abs/1505.02478.
[15] N. J. Cutland, Loeb measure theory, in Loeb Measures in Practice: Recent Advances, Springer Berlin Heidelberg (2000): 1–28.
[16] D. Flynnn, K. Shamseddine, On Integrable Delta Functions on the Levi-Civita Field, p-Adic Numbers, Ultrametric Analysis and Applications, 2018, 10.1: 32-56.
[17] B. Crowell, M. Khafateh, Inf, http://www.lightandmatter.com/calc/inf/.
[18] T. Levi-Civita, Sugli infiniti ed infinitesimi attuali quali elementi analitici, Atti Ist. Veneto di Sc., Lett. ed Art., 7a (4) (1892), p. 1765.
[19] T. Levi-Civita, Sui numeri transfiniti, Rend. Acc. Lincei, 5a (7) (1898), pp. 91-113.
[20] A. R. Mészáros, K. Shamseddine, On the solutions of linear ordinary differential equations and Bessel-type special functions on the Levi-Civita field, Journal of Contemporary Mathematical Analysis 50, 53–62, doi=10.3103/S1068362315020016
[21] H. M. Moreno, Non-measurable sets in the Levi-Civita field, in Advances in Ultrametric Analysis: 12th International Conference on P-adic Functional Analysis, July 2-6, 2012, University of Manitoba, Winnipeg, Manitoba, Canada. American Mathematical Soc., 2013.
[22] M. Oberguggenberger, T. Todorov, An embedding of Schwartz distributions in the algebra of asymptotic functions, Int. J. Math. Math. Sci. 21 (1998) 417–428.
[23] A. Robinson, Non-standard analysis, Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math. 23 (1961), 432–440
[24] K. Shamseddine, Absolute and relative extrema, the mean value theorem and the inverse function theorem for analytic functions on a Levi-Civita field, Contemp. Math 551 2011: 257-268.
[25] K. Shamseddine, New Elements of Analysis on the Levi-Civita Field, PhD thesis, Michigan State University, East Lansing, Michigan, USA, 1999. Also Michigan State University report MSUCL-1147.
[26] K. Shamseddine, *New results on integration on the Levi-Civita field*, Indagationes Mathematicae, 24(1) 2013, pp.199-211.

[27] K. Shamseddine, M. Berz, *Analytical properties of power series on Levi-Civita fields*, Ann. Math. Blaise Pascal, 12(2) 2005, pp.309-329.

[28] K. Shamseddine, M. Berz, *Convergence on the Levi-Civita field and study of power series*, Proc. Sixth International Conference on Nonarchimedean Analysis, pages 283–299, New York, NY, 2000. Marcel Dekker.

[29] K. Shamseddine, M. Berz, *Measure theory and integration on the Levi-Civita field*, Contemporary Mathematics, 319, 2003, pp.369-388.

[30] K. Shamseddine, D. Flynn, *Measure theory and Lebesgue-like integration in two and three dimensions over the Levi-Civita field*, Contemporary Mathematics, 665, 2016, pp. 289-325.

[31] R. Strichartz, *A guide to distribution theory and Fourier Transforms*, CRC Press (1994).

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