Abstract

In this paper, we consider the minimal entropy of qubit states transmitted through two uses of a noisy quantum channel, which is modeled by the action of a completely positive trace-preserving (or stochastic) map. We provide strong support for the conjecture that this minimal entropy is additive, namely that the minimum entropy can be achieved when product states are transmitted. Explicitly, we prove that for a tensor product of two unital stochastic maps on qubit states, using an entanglement that involves only states which emerge with minimal entropy cannot decrease the entropy below the minimum achievable using product states. We give a separate argument, based on the geometry of the image of the set of density matrices under stochastic maps, which suggests that the minimal entropy conjecture holds for non-unital as well as for unital maps. We also show that the maximal norm of the output states is multiplicative for most product maps on n-qubit states, including all those for which at least one map is unital.
For the class of unital channels on $\mathbb{C}^2$, we show that additivity of minimal entropy implies that the Holevo capacity of the channel is additive over two inputs, achievable with orthogonal states, and equal to the Shannon capacity. This implies that superadditivity of the capacity is possible only for non-unital channels.

Key words: Entangled state; minimal entropy; qubit; stochastic map.

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1 Introduction

1.1 Entropy and unital stochastic maps

When a pure state, represented by a density matrix \( \rho \), is transmitted along a noisy channel, it is mapped into a mixed state \( \Phi(\rho) \). The entropy of the initial pure state is necessarily zero, i.e., \( S(\rho) \equiv -\Tr \rho \log \rho = 0 \) since \( \rho^2 = \rho \) and so the only eigenvalues of \( \rho \) are 0 and 1. However, the entropy \( S[\Phi(\rho)] \) of the mixed state which emerges need not be zero. One seeks states \( \rho \) which minimize the effect of the noise in the sense of minimizing the entropy \( S[\Phi(\rho)] \) of the state that emerges from the channel. There are a number of reasons for studying such states, most notably the connection between minimizing entropy and maximizing channel capacity, which will be discussed in Section 2.3. However, in this paper we focus attention on the entropy.

The noise, which results from interactions between the states in a Hilbert space \( \mathcal{H} \) and the environment, is represented by the action of a completely positive, trace-preserving map \( \Phi \) on the trace class operators in \( \mathcal{B}(\mathcal{H}) \). We use the term stochastic to describe such maps. (Following a similar use by Alberti and Uhlmann [1], this terminology was used by Petz, [13] and reflects the fact that \( \Phi \) is the non-commutative analogue of the action of a column stochastic matrix on a probability vector.) We restrict attention to two-level quantum systems in which case \( \mathcal{H} = \mathbb{C}^2 \) or tensor products of \( \mathbb{C}^2 \). A stochastic map \( \Phi \) acting on states in \( \mathbb{C}^2 \), can be naturally extended to tensor products, e.g., \( \Phi \otimes \Phi \) acting on states on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) etc., and leads to questions about the additivity of the minimal entropy and capacity of product channels.

In particular, Shor has conjectured that the minimal entropy is additive. We were led independently to this conjecture because it would imply additivity of channel capacity for unital stochastic maps, i.e., maps which take the identity operator to itself so that \( \Phi(I) = I \). Although we have a convincing argument that our results for entropy, unlike those for channel capacity, extend to non-unital maps, we focus most of our attention on unital maps. In the last section we briefly consider non-unital maps.

Recall that every completely positive map \( \Phi \) can be represented (non-uniquely) in the Kraus form

\[
\Phi(\rho) = \sum_k A_k^\dagger \rho A_k. \tag{1}
\]

Every map \( \Phi \) representing noisy evolution in a quantum channel must preserve the trace of \( \rho \), since \( \Phi(\rho) \) is also a state. In terms of the Kraus operators, the condition that \( \Phi \) be stochastic, that is completely positive and trace preserving, is

\[
\Tr \Phi(\rho) = \Tr \rho \quad \forall \rho \quad \Leftrightarrow \quad \sum_{k=1}^n A_k A_k^\dagger = I. \tag{2}
\]
The map $\Phi$ is \textit{unital} if $\Phi(I) = I$, that is if $\Phi$ maps the identity operator to itself. In terms of the Kraus operators, the condition for $\Phi$ to be completely positive and unital is

$$\Phi(I) = I \iff \sum_{k=1}^{n} A_k^\dagger A_k = I. \quad (3)$$

A sufficient condition that a stochastic map be unital is that the Kraus operators are self-adjoint, i.e., $A_k = A_k^\dagger \forall k$. This condition is not necessary; for example, a double-stochastic matrix which is not symmetric corresponds to a unital stochastic map which is not self-adjoint. Henceforth in this paper, “unital map” will mean “unital stochastic map” unless otherwise stated.

It is worth noting that the Kraus operators are self-adjoint if and only if $\Phi$ is self-adjoint with respect to the Hilbert-Schmidt inner product $\langle P, Q \rangle = \text{Tr} P^\dagger Q$. The dual or adjoint map $\hat{\Phi}$ is then defined by the condition $\text{Tr} \left( \hat{\Phi}(P) Q \right) = \text{Tr} P^\dagger \Phi(Q)$. It is easy to see that if $\Phi$ is given by (I) then $\hat{\Phi}(\rho) = \sum_k A_k \rho A_k^\dagger$. In addition, since the dual of any trace-preserving map satisfies $\hat{\Phi}(I) = I$, any stochastic map (considered as a linear map on the space of Hilbert-Schmidt operators) has eigenvalue 1, and hence a fixed point $P$ such that $\Phi(P) = P$.

For unital maps, the identity is an eigenvector whose orthogonal complement is the set of operators with trace zero. Hence, a unital map also defines a linear map on the traceless part of a density matrix. By contrast, a non-unital map is only affine on the set of traceless matrices. This distinction is easily seen for the $\mathbb{C}^2$ case when the Bloch sphere representation is used as described in section 2.1. Appendix C contains a list of examples of unital and non-unital maps.

Recall that the entropy of a density matrix can be written in terms of its eigenvalues $\lambda_k$, namely $S(\rho) = -\sum_k \lambda_k \log \lambda_k$. The minimal entropy $S(\rho) = 0$ occurs if and only if one eigenvalue of $\rho$ is 1 and all others 0; the maximal entropy (in $d$ dimensions) of $S(\rho) = \log d$ occurs if and only if all eigenvalues are $1/d$ so that $\rho = \frac{1}{d} I$. Thus, if $S(\rho) \approx 0$, one must have one eigenvalue close to 1 and the others near 0. Hence, states with small entropy are those for which $\|\rho\| \approx 1$. Thus, in seeking pure states $\rho$ which have minimal entropy $S[\Phi(\rho)]$ after emerging from a noisy channel, we are led to seek states for which $\|\Phi(\rho)\|$ is maximal. In section 6 we give a precise definition of the maximal norm and show that it is multiplicative when (at least) one channel is a unital map on $\mathbb{C}^2$.

Our results suggest that multiplicativity of the maximal norm may hold for general channels; in fact, we can extend our result to some non-unital channels (see Remarks at the end of Section 3). In any case, our results provide strong support for the additivity of minimal entropy for unital channels.

Another property of unital channels is that the entropy of a state is non-decreasing under the action of a unital stochastic map. This follows easily from
the fact that the relative entropy

\[ H(P, Q) = \text{Tr} P [\log P - \log Q] \] (4)

decreases under stochastic maps, i.e.,

\[ H[\Phi(P), \Phi(Q)] \leq H(P, Q) \] (5)

Since \( S(\rho) = -H(\rho, \frac{1}{d} I) + \log d \) if \( \Phi \) is unital, it follows from (5) that \( S[\Phi(\rho)] \geq S(\rho) \). For a non-unital map, the entropy of a pure state cannot decrease; however, one can have mixed states for which the entropy actually decreases.

### 1.2 Channel Capacity

We now discuss the information capacity of a noisy quantum channel \([12], [13]\) used for what is sometimes called “classical” communication, i.e., communications in which signals are sent using quantum particles but without additional or prior entanglement between sender and receiver. In the simplest case where no entanglement is used in either the transmission or the measurement, each letter \( i \) from the source alphabet is represented by a pure state which we represent by its density matrix \( \rho_i \) on a quantum Hilbert space. During transmission the channel transforms this state into \( \tilde{\rho}_i \equiv \Phi(\rho_i) \), where \( \Phi \) implements the noisy interaction between states and the environment. The map \( \Phi \) is a completely positive, trace-preserving map on the set of states. The resulting state \( \tilde{\rho}_i \) is measured, and the outcome determines a letter from the output alphabet. In the general case this measurement is effected by a positive operator-valued measurement (POVM) – namely, there is a positive operator \( E_j \) assigned to each letter \( j \) of the output alphabet, which together satisfy the constraint \( \sum_j E_j = I \). When the measurement is performed on a state \( \rho \), the result will be \( j \) with probability \( \text{Tr}(\rho E_j) \).

Several definitions of channel capacity have been proposed, corresponding to whether or not entangled states are used for transmission, and whether or not entangled measurements are made by the receiver. Bennett and Shor [5] identify four possibilities, which we denote \( C_{PP}, C_{PE}, C_{EP} \) and \( C_{EE} \) where the subscripts \( P \) and \( E \) refer to product and entangled processes respectively.

In the process described above, with no entanglement at either end, the channel is equivalent to a classical noisy channel with transition probabilities \( \{p_{ij} = \text{Tr}(\tilde{\rho}_i E_j)\} \). Therefore its maximum rate of information transmission is given by

\[ C_{PP}(\Phi) = C_{\text{Shan}}(\Phi) = \sup_{\pi, \rho, E} \sum_i \sum_j \pi_i p_{ij} \log \left( \frac{p_{ij}}{\sum_k \pi_k p_{kj}} \right), \] (6)

where we take the sup over all probability distributions \( \{\pi_i\} \) for the input alphabet, as well as all choices of input states and measurements. We call this the Shannon capacity of the channel, since it bears closest resemblance to the classical case.
It is reasonable to expect that by transmitting entangled states and by using entangled measurements it may be possible to exceed the Shannon capacity for a noisy channel. The Holevo-Schumacher-Westmoreland Theorem \([12], [22]\) provides a closed form expression for the capacity in the case where product states are transmitted at the input and entangled measurements of arbitrary length are allowed at the output:

\[
C_{PE} = C_{\text{Holv}}(\Phi).
\] (7)

Here \(C_{\text{Holv}}(\Phi)\) is the **Holevo capacity** of the channel:

\[
C_{\text{Holv}}(\Phi) = \sup_{\pi, \rho} \left( S(\tilde{\rho}) - \sum_i \pi_i S(\tilde{\rho}_i) \right),
\] (8)

where \(\rho = \sum_i \pi_i \rho_i\) and \(\tilde{\rho} = \Phi(\rho)\). The well-known Holevo bound states that

\[
C_{\text{Shan}}(\Phi) \leq C_{\text{Holv}}(\Phi).
\] (9)

Holevo \([13, 14]\) provided examples of channels in which this inequality is strict, i.e., \(C_{\text{Shan}}(\Phi) < C_{\text{Holv}}(\Phi)\). Furthermore, it has been shown \([11, 18]\) that a necessary and sufficient condition for strict inequality is that the output states \(\{\tilde{\rho}_i\}\) do not commute.

One important open question is whether or not the Holevo capacity can be exceeded when entangled states are used at the input, that is whether \(C_{EE}\) exceeds \(C_{PE}\). This would be equivalent to the superadditivity of the Holevo capacity. In this paper we address this question in the case of messages which are entangled over two inputs only. This is equivalent to the question whether \(C_{\text{Hol}}(\Phi \otimes \Phi)\) exceeds \(2C_{\text{Hol}}(\Phi)\). Holevo \([13]\) has shown that \(C_{\text{Shan}}(\Phi \otimes \Phi) > 2C_{\text{Shan}}(\Phi)\) for the quantum binary channel, but to the best of the authors’ knowledge there is no known example of a superadditive channel for the Holevo capacity. Bruss et al \([7]\) showed that \(C_{\text{Hol}}(\Phi \otimes \Phi) = 2C_{\text{Hol}}(\Phi)\) for the depolarising channel, which is an example of a unital channel. As we will show, our results strongly suggest that if the Holevo capacity is superadditive then the channel must be non-unital.

### 1.3 Summary of Results

We prove several theorems about the minimal entropy and the maximal norm for states of the form \((\Phi \otimes \Omega)(\rho_{12})\), where \(\Phi\) and \(\Omega\) are unital stochastic maps on \(\mathbb{C}^{2 \times 2}\) and \(\rho_{12}\) is an entangled state. In addition, we explain how these results provide evidence for the conjecture that **minimal entropy is additive** for all stochastic maps on \(\mathbb{C}^{2 \times 2}\). We also show that this conjecture has important implications for the capacity of unital quantum channels. In particular, we show that the conjecture implies that if \(\Phi\) is unital, then the Holevo capacity is additive over two inputs, that is \(C_{\text{Holv}}(\Phi \otimes \Phi) = 2C_{\text{Holv}}(\Phi)\).
Our first theorem concerns the maximal value of $\|\Phi(\rho)\|$ as $\rho$ varies over states on $\mathbb{C}^2$. We will consider the general possibility of two stochastic maps $\Phi$ and $\Omega$ on $\mathbb{C}^{2\times 2}$ and denote their maximal values by $M_\Phi$ and $M_\Omega$ respectively. In Theorem 6 we prove, under mild conditions on one of these maps, that the maximal value of $\|(\Phi \otimes \Omega)(\rho_{12})\|$ is $M_\Phi M_\Omega$, as $\rho_{12}$ varies over states on $\mathbb{C}^{2\times 2}$. That is, the norm of $(\Phi \otimes \Omega)(\rho_{12})$ achieves its maximal value on product states $\rho_{12} = \rho_1 \otimes \rho_2$, rather than on entangled states.

In Theorem 12, we prove a similar, though slightly weaker, result for the minimal entropy of $(\Phi \otimes \Omega)(\rho_{12})$. Namely, we restrict $\rho_{12}$ to the family of entangled states whose reduced density matrices $\rho_1$ and $\rho_2$ are such that $\Phi(\rho_1)$ and $\Omega(\rho_2)$ have minimal entropy. Then we prove that the minimal entropy of $(\Phi \otimes \Omega)(\rho_{12})$, as $\rho_{12}$ varies over this family, is the sum of the minimal entropies of $\Phi(\rho_1)$ and $\Omega(\rho_2)$. That is, the entropy also achieves its minimal value on product states.

It seems extremely unlikely that we could further decrease the entropy of $(\Phi \otimes \Omega)(\rho_{12})$ by using an entangled state whose reduced density matrices do not have minimal entropy themselves. Therefore we believe that the conclusion of Theorem 12 also holds as $\rho_{12}$ varies over all entangled states on $\mathbb{C}^{2\times 2}$. In fact, similar arguments and numerical evidence support an even stronger conclusion, namely that minimal entropy is additive for all stochastic maps on $\mathbb{C}^{2\times 2}$. This is the content of the conjecture below.

**Conjecture 1** If $\Phi$ and $\Omega$ are stochastic maps on $\mathbb{C}^{2\times 2}$, then

$$\inf_{\rho \text{ pure}} S[(\Phi \otimes \Omega)(\rho)] = \inf_{\rho \text{ pure}} S[\Phi(\rho)] + \inf_{\rho \text{ pure}} S[\Omega(\rho)].$$

We will discuss the evidence for this conjecture in detail in Section 4 and Section 5.2. We note that P. Shor \cite{23} earlier made a similar conjecture, and together with J. Smolin obtained numerical evidence which supports it.

The unital case is particularly important because it yields the following results as immediate corollaries.

**Corollary 2** If $\Phi$ is unital, then the Holevo capacity is additive, i.e.,

$$C_{\text{Holv}}(\Phi \otimes \Phi) = 2C_{\text{Holv}}(\Phi).$$

**Corollary 3** If $\Phi$ is unital, then the Holevo capacity can be achieved with orthogonal states.

**Corollary 4** If $\Phi$ is unital, then $C_{\text{Holv}}(\Phi) = C_{\text{Shan}}(\Phi)$ and $C_{\text{Holv}}(\Phi \otimes \Phi) = C_{\text{Shan}}(\Phi \otimes \Phi)$. 

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Corollary 5 If $\Phi$ is unital, then the Shannon capacity is also additive, i.e., $C_{\text{Shan}}(\Phi \otimes \Phi) = 2C_{\text{Shan}}(\Phi)$.

In Section 2.3 we will explain in detail how these Corollaries follow if Conjecture 1 holds for unital maps.

This paper is organized as follows. In Section 2 we introduce the notation we will use for the Stokes parametrization for representing both states and maps in a basis consisting of the Identity and Pauli matrices, and show how the various conditions of unital, trace-preserving, and complete positivity can be expressed in this representation. With this background, we conclude Section 2 by presenting the arguments leading to the Corollaries above.

In Section 3 we prove the multiplicativity of the maximum value of $\|\Phi(\rho)\|$. Section 4 contains the heart of the paper in which we give the details of the proof of our theorem about additivity of minimal entropy. In Section 5 we discuss some of the features of non-unital maps using a special subclass and then present the evidence for additivity of minimal entropy in general. Finally, in Section 6 we summarize our results and discuss their implications for channel design.

We also include three Appendices. Appendix A gives some important background on singular value decompositions and the details needed for the diagonal representation introduced in Section 2.1. Appendix B gives the details needed to verify the complete positivity conditions of Section 2.2. In Appendix C we provide a number of examples of different types of channels and show how some familiar examples appear in the representation and notation we use.

2 Preliminaries

2.1 Stokes parametrization and Bloch sphere

Recall that the identity and Pauli matrices form a basis for $\mathbb{C}^{2 \times 2}$ so that any $2 \times 2$ matrix $C$ can be written as $w_0 I + w \cdot \sigma$ where $\sigma$ denotes the vector of Pauli matrices and $w \in \mathbb{C}^3$. Then for $C = w_0 I + w \cdot \sigma$

a) $C$ is self-adjoint $\iff (w_0, w)$ is real, i.e., $w_0 \in \mathbb{R}$ and $w \in \mathbb{R}^3$;

b) $\text{Tr}C = 1 \iff w_0 = \frac{1}{2}$; and

c) $C > 0 \iff |w| \leq w_0$.

Thus, $\{I, \sigma\}$ also form a basis for the real vector space of self-adjoint matrices in $\mathbb{C}^{2 \times 2}$ and every density matrix can be written in this basis as $\rho = \frac{1}{2}[I + w \cdot \sigma]$ with $w \in \mathbb{R}^3$ and $|w| \leq 1$. Furthermore

d) $\rho$ is a one-dimensional projection (or pure state) $\iff |w| = 1$. 

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Every linear map $\Phi : \mathbb{C}^{2 \times 2} \to \mathbb{C}^{2 \times 2}$ can be represented in this basis by a unique $4 \times 4$ matrix $T$, and $\Phi$ is trace-preserving if and only if the first row satisfies $t_{1k} = \delta_{1k}$, i.e., $T = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}$ where $T$ is a $3 \times 3$ matrix (and $0$ and $t$ are row and column vectors respectively) so that

$$
\Phi(w_0 I + w \cdot \sigma) = w_0 I + (t + T w) \cdot \sigma.
$$

(11)

The matrix $T$ is self-adjoint if and only if $\Phi$ is self-adjoint as an operator on $\mathbb{C}^{2 \times 2}$ with respect to the Hilbert-Schmidt inner product. We are interested in those $\Phi$ which map a subspace of self-adjoint matrices into itself, which holds if and only if $T$ is real.

The map $\Phi$ is unital if and only if $t = 0$. Thus, any unital stochastic map $\Phi$ acting on density matrices on $\mathbb{C}^2$ can be written in the form

$$
\Phi\left(\frac{1}{2}[I + w \cdot \sigma]\right) = \frac{1}{2}[I + (T w) \cdot \sigma],
$$

where $T$ is a real $3 \times 3$ matrix. Using the singular value decomposition (see Appendix A), we can write

$$
T = RS
$$

(13)

where $R$ is a rotation and $S$ is self-adjoint. Define the map $\Phi_S$ by

$$
\Phi_S\left(\frac{1}{2}[I + w \cdot \sigma]\right) = \frac{1}{2}[I + (S w) \cdot \sigma]
$$

(14)

As explained in Appendix A, the rotation $R$ defines a unitary operator $U$ such that for any state $\rho$,

$$
\Phi(\rho) = U \left[ \Phi_S(\rho) \right] U^\dagger
$$

(15)

In this paper we are interested only in the critical values of certain functions of the spectrum of $\Phi(\rho)$, as $\rho$ varies over the space of states – the maximum value of the norm, the minimum value of the entropy. Since a unitary transformation leaves the spectrum unchanged, these are the same for $\Phi$ and $\Phi_S$. Also, since $S$ is self-adjoint it can be diagonalized by a change of basis. Hence without loss of generality we need henceforth consider only unital stochastic maps whose matrix $T$ defined in (11) is diagonal, with eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$. As a shorthand, we will denote this diagonal map by $\Phi[\lambda_1, \lambda_2, \lambda_3]$. The image of the set of pure state density matrices $\rho = \frac{1}{2}[I + w \cdot \sigma]$ (with $|w| = 1$) under the action of $\Phi[\lambda_1, \lambda_2, \lambda_3]$ is the ellipsoid

$$
\left(\frac{x_1}{\lambda_1}\right)^2 + \left(\frac{x_2}{\lambda_2}\right)^2 + \left(\frac{x_3}{\lambda_3}\right)^2 = 1,
$$

(16)
and the image under the action of $\Phi$ is obtained by a further rotation of the ellipsoid, corresponding to the operator $U$ in (13).

Similar reasoning applies when $\Phi$ is non-unital. Using (75) and (79) from Appendix A, the map $\Phi$ can be written in the form $\Phi(\rho) = U\Phi_D(V\rho V^\dagger)U^\dagger$ where $U, V$ are unitary, $D$ is diagonal and $\Phi_D$ is represented by the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1' & \lambda_1 & 0 & 0 \\ t_2' & 0 & \lambda_1 & 0 \\ t_3' & 0 & 0 & \lambda_3 \end{pmatrix}$$

(17)

The vector $t' = (t_1', t_2', t_3')$ is equal to $R_2R_1^T t$ in the notation of (75). In this case, the image of the set of pure state density matrices $\rho = \frac{1}{2}[I + w \cdot \sigma]$ (with $|w| = 1$) under the action of $\Phi_D$ is the translated ellipsoid

$$\left(\frac{x_1 - t_1'}{\lambda_1}\right)^2 + \left(\frac{x_2 - t_2'}{\lambda_2}\right)^2 + \left(\frac{x_3 - t_3'}{\lambda_3}\right)^2 = 1,$$

(18)

and again the image under $\Phi$ is a rotation of this.

It will be useful to write out explicitly the action of the diagonal unital map $\Phi[\lambda_1, \lambda_2, \lambda_3]$ on a density matrix in the form

$$\Phi[\lambda_1, \lambda_2, \lambda_3](\rho) = \Phi[\lambda_1, \lambda_2, \lambda_3]\left(\begin{array}{cc} a & b \\ b^\dagger & c \end{array}\right)$$

(19)

$$= \frac{1}{2}\begin{pmatrix} (a + c) + \lambda_3(a - c) & \lambda_1(b + b^\dagger) + \lambda_2(b - b^\dagger) \\ \lambda_1(b + b^\dagger) - \lambda_2(b - b^\dagger) & (a + c) - \lambda_3(a - c) \end{pmatrix}.$$ 

### 2.2 Complete Positivity Conditions

The requirement that $\Phi$ be stochastic imposes a number of constraints on the matrix $T$. We describe these in Appendix B in which we give explicit formulas for the matrix elements of $T$ in terms of the Stokes parameterization of the operators $A_k$. These formulas in turn imply constraints on the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ described in the previous section.

Let $T_{jk}$ denote the elements of $T$ using the convention that $j, k \in 0 \ldots 3$. Then the point with coordinates $(T_{11}, T_{22}, T_{33})$ must lie inside a tetrahedron with corners at $(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)$. These conditions are equivalent to four linear inequalities which can be written compactly as

$$|T_{11} \pm T_{22}| \leq |1 \pm T_{33}|.$$ 

(20)

(Note that we always have $T_{00} = 1$.)
In the special case where \( \Phi \) is unital, (20) implies that the eigenvalues (which are necessarily real) satisfy
\[
|\lambda_1 \pm \lambda_2| \leq |1 \pm \lambda_3|.
\]
(21)

In fact, for unital \( \Phi \) the condition (21) is a necessary and sufficient condition for the numbers \((\lambda_1, \lambda_2, \lambda_3)\) to arise as eigenvalues of the self-adjoint part of a unital stochastic map.

These conditions were discussed earlier by Algoet and Fujiwara [2]. In addition they gave conditions for complete positivity of some non-unital maps. In particular, for the special case of (17) with the form
\[
\Phi\left(\frac{1}{2}[I + w \cdot \sigma]\right) = \frac{1}{2}[I + w_1 \lambda_1 \sigma_1 + (t + w_3 \lambda_3)\sigma_3],
\]
(22)
they showed that the necessary and sufficient condition for complete positivity is
\[
\lambda_1^2 + t^2 \leq (1 - |\lambda_3|)^2
\]
(23)

### 2.3 Relation between capacity and minimal entropy for unital maps

We now assume wlog that \( \Phi \) is self-adjoint and written in the diagonal form \( \Phi[\lambda_1, \lambda_2, \lambda_3] \). Let \( \mu = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) \), and let \( w_\mu \) be a unit vector satisfying \( T w_\mu = \pm \mu w_\mu \). Then it is easy to show that
\[
\inf_{\rho} S[\Phi(\rho)] = S\left(\Phi(\frac{1}{2}[I + w_\mu \cdot \sigma])\right) = h(\mu)
\]
(24)
where
\[
h(\mu) = -\frac{1}{2}(1 + \mu) \ln \frac{1}{2}(1 + \mu) - \frac{1}{2}(1 - \mu) \ln \frac{1}{2}(1 - \mu).
\]
(25)

Consider now the question of computing the Holevo capacity \( C_{\text{Hol}}(\Phi) \) defined in [8]. The choice \( \rho_1 = \frac{1}{2}[I + w_\mu \cdot \sigma] \) and \( \rho_2 = \frac{1}{2}[I - w_\mu \cdot \sigma] \), and \( \pi_1 = \pi_2 = \frac{1}{2} \), both maximizes the first term \( S(\rho) = S(\frac{1}{2}I) = \ln 2 \) and minimizes the second term \( \sum \pi_i S(\rho_i) = h(\mu) \). Hence it also maximizes their difference, which gives \( C_{\text{Hol}}(\Phi) = \ln 2 - h(\mu) \).

In Section 1.3 we stated several corollaries of Conjecture [4]. Here we will show how these corollaries follow from the assumption that minimal entropy is additive for unital maps.

So suppose that Conjecture [4] holds for unital maps, that is suppose that the minimum value of \( S(\Phi \otimes \Phi(\rho_{12})) \) over states \( \rho_{12} \) on \( C^{2 \times 2} \) is \( 2h(\mu) \). Then there are four product states, namely \( \rho_i = \frac{1}{2}[I \pm w_\mu \cdot \sigma] \otimes \frac{1}{2}[I \pm w_\mu \cdot \sigma] \), such that \( S(\Phi \otimes \Phi(\rho_i)) \) achieves this minimum value for each \( i \). If we take \( \pi_i = 1/4 \) for each \( i \), then
\[ \rho = \frac{1}{4} I \otimes I \quad \text{and} \quad S(\Phi \otimes \Phi(\rho)) = \ln 4 \] achieves its maximum possible value. Hence with these choices, we can separately maximize each term on the right side of (8) and therefore maximize the Holevo capacity. Therefore, for unital maps the equality \( C_{\text{Hol}}(\Phi \otimes \Phi) = 2C_{\text{Hol}}(\Phi) \) is implied by the minimal entropy conjecture. This demonstrates Corollary 2. Furthermore, the two minimal entropy states \( \frac{1}{2}[I \pm w_{\mu} \cdot \sigma] \) are orthogonal. Hence both \( C_{\text{Hol}}(\Phi) \) and \( C_{\text{Hol}}(\Phi \otimes \Phi) \) are achieved with orthogonal states, and this establishes Corollary 3. Also, a simple calculation shows that the expression inside the sup in the definition of \( C_{\text{Shan}}(\Phi) \) in (8) equals \( C_{\text{Hol}}(\Phi) \) when we choose the input states to be \( \rho_i = \frac{1}{2}[I \pm w_{\mu} \cdot \sigma] \) with \( \pi_i = \frac{1}{2} \), and the POVM to be \( E_i = \frac{1}{2}[I \pm w_{\mu} \cdot \sigma] \). This shows the first statement of Corollary 4, and the second statement follows immediately. Then Corollary 5 is a direct consequence.

The essential observation in this argument is that we can find a partition of unity in terms of a set of orthogonal input states which are mapped into a level set of minimal entropy. For such inputs, uniform averaging yields the state \( \rho = I \) whose output \( \Phi(\rho) = I \) has maximal entropy. Hence both terms in the Holevo capacity are simultaneously maximised. On \( C^2 \) orthogonal inputs have the form \( \frac{1}{2}[I \pm w \cdot \sigma] \), and the corresponding output states have the same entropy if and only if \( \Phi \) is unital. In that case, the products of these states form a set of orthogonal inputs on \( C^4 \) which map onto a level set of entropy under the product map \( \Phi \otimes \Phi \). If the minimal entropy is additive, one such set of product states will be mapped onto a set of minimal entropy. For non-unital maps on \( C^2 \), and more general (non-product) maps on \( C^3 \) or \( C^4 \), this need not hold. (Fuchs and Shor \[23\] have found an explicit example of a map on \( C^3 \) which does not have such a set of orthogonal inputs.) Hence the above argument is quite special and does not provide a direct link between the additivity of minimal entropy and additivity of the Holevo capacity.

### 3 Upper Bound on Norm

For any linear map \( \Omega \) define

\[ M_{\Omega} \equiv \sup_{\rho \in \text{DenMat}} \| \Omega(\rho) \| = \sup_{Q > 0} \frac{\| \Omega(Q) \|}{\text{Tr}Q} \tag{26} \]

so that for any \( \rho > 0, \| \Omega(\rho) \| \leq M_{\Omega} \text{Tr} \rho \). Since the matrix norm \( \| \cdot \| \) used in (26) is convex, it suffices to consider the supremum over pure states or, equivalently one-dimensional projections \( \rho = |\psi\rangle\langle\psi| \). Then \( M_{\Omega} \) can be rewritten using the representation (1)

\[ M_{\Omega} = \sup_{\psi, \chi} \sum_k |\langle \chi, A_k \psi \rangle|^2 \tag{27} \]

where the supremum is taken over those vectors satisfying \( |\psi| = |\chi| = 1 \).
We restrict attention now to unital maps. As discussed in Section 2.1, wlog we assume that $\Phi$ is diagonal of the form $\Phi[\lambda_1, \lambda_2, \lambda_3]$. Then it follows from the discussion in section 2.1 that

$$M_\Phi = \frac{1}{2}(1 + \max_k |\lambda_k|). \quad (28)$$

In this section we will show that for unital maps on $\mathbb{C}^{2\times2}$, the norm $M_\Phi$ is multiplicative, i.e., $M_{\Phi \otimes \Omega} = M_\Phi M_\Omega$. In fact, we will show a slightly stronger result.

**Theorem 6** Let $\Omega$ be any 2-positive map on $\mathbb{C}^{n \times n}$ and let $\Phi$ be a unital stochastic map on $\mathbb{C}^{2\times2}$. Then $M_{\Phi \otimes \Omega} = M_\Phi M_\Omega$.

Notice that Theorem 6 implies that $\| (\Phi \otimes \Omega)(\rho) \|$ is maximised on product states of the form $\rho = \rho_1 \otimes \rho_2$, where $\|\Phi(\rho_1)\| = M_\Phi$ and $\|\Omega(\rho_2)\| = M_\Omega$.

Our proof will need the following well-known result. (See, e.g., [15].)

**Lemma 7** Let $S = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$ be a matrix in block form with $A, C > 0$. Then $S$ is (strictly) positive definite (i.e. $S > 0$) if and only if $A > BC^{-1}B^\dagger$ if and only if $C > B^\dagger A^{-1}B$.

As immediate corollaries we find

**Corollary 8** Let $S = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$ be a matrix in block form with $A, C$ positive semi-definite. Then $S$ is positive semi-definite if and only if for all $u > 0$ one of the following two equivalent conditions holds

$$A + uI > B(C + uI)^{-1}B^\dagger$$
$$C + uI > B^\dagger(A + uI)^{-1}B.$$

**Corollary 9** If $S = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \geq 0$, then

$$\|B\|^2 = \|BB^\dagger\| \leq \|A\| \|C\|. \quad (29)$$

To prove this note that

$$\langle v, BB^\dagger v \rangle \leq \|C + uI\| \langle v, B(C + uI)^{-1}B^\dagger v \rangle$$
$$\quad < \|C + uI\| \langle v, (A + uI)v \rangle \leq \|C + uI\| \|A + uI\| \|v\|^2.$$

Choosing $v$ an eigenvector of $BB^\dagger$ and letting $u \to 0$ proves (29).
Returning to the proof of Theorem 6, let \( \rho = \begin{pmatrix} \rho_1 & \gamma \\ \gamma^\dagger & \rho_2 \end{pmatrix} \) be a density matrix on \( \mathbf{C}^2 \otimes \mathbf{C}^n \) written in block form with \( \rho_1, \rho_2, \gamma \) each \( n \times n \) matrices. First observe that the two-positivity of \( \Omega \) implies that

\[
\left( I \otimes \Omega \right)(\rho) = \begin{pmatrix} \Omega(\rho_1) & \Omega(\gamma) \\ \Omega(\gamma)^\dagger & \Omega(\rho_2) \end{pmatrix} \geq 0
\] (30)

Hence, it follows from (29) that

\[
\| \Omega(\gamma) \|^2 = \| \Omega(\gamma) \Omega(\gamma)^\dagger \| \leq \| \Omega(\rho_1) \| \| \Omega(\rho_2) \|
\] \leq \text{Tr} \rho_1 \text{Tr} \rho_2 \frac{2}{\omega_1} \quad (31)

Now use the form (19) and the linearity of \( \Omega \) to write

\[
\left( \Phi \otimes \Omega \right)(\rho) = \begin{pmatrix} P & L \\ L^\dagger & Q \end{pmatrix}
\] (32)

\[
= \frac{1}{2} \begin{pmatrix} \Omega[\rho_1 + \rho_2 + \lambda_3(\rho_1 - \rho_2)] & (\lambda_1 + \lambda_2) \Omega(\gamma) + (\lambda_1 - \lambda_2) \Omega(\gamma)^\dagger \\ (\lambda_1 + \lambda_2) \Omega(\gamma)^\dagger + (\lambda_1 - \lambda_2) \Omega(\gamma) & \Omega[\rho_1 + \rho_2 - \lambda_3(\rho_1 - \rho_2)] \end{pmatrix}
\]

Note that the complete positivity of \( \Phi \) implies that \( \rho_1 + \rho_2 + \lambda_3(\rho_1 - \rho_2) > 0 \). Thus if \( x = \text{Tr} \rho_1 \)

\[
\| P \| \leq M_\Omega \frac{1}{2} \text{Tr}[\rho_1 + \rho_2 + \lambda_3(\rho_1 - \rho_2)]
\]

\[
= M_\Omega \left[ \frac{1}{2} + \lambda_3(x - \frac{1}{2}) \right], \quad \text{and}
\]

\[
\| Q \| \leq M_\Omega \left[ \frac{1}{2} - \lambda_3(x - \frac{1}{2}) \right].
\] (33) (34)

Now we can assume wlog that \( \lambda_3 = \max_k |\lambda_k| \) so that \( M_\Phi = \frac{1}{2}(1 + \lambda_3) \). Then to prove Theorem 6, it suffices to show that

\[
z > \frac{1}{2}(1 + \lambda_3)M_\Omega \Rightarrow zI - \left( \Phi \otimes \Omega \right)(\rho) > 0.
\] (35)

Note that

\[
\| LL^\dagger \| \leq (z - \| P \|)(z - \| Q \|)
\] (36)

and the general property \( P \leq \| P \| \) imply

\[
L(zI - P)^{-1}L^\dagger \leq L(z - \| P \|)^{-1}L^\dagger \leq LL^\dagger \| z - \| P \| \|^{-1}
\]

\[
\leq (z - \| Q \|) \leq zI - Q.
\]

Therefore, by Lemma 4, to verify (33), it suffices to show (36). But it follows from (31) that

\[
\| LL^\dagger \| = \frac{1}{2} \left[ (\lambda_1 + \lambda_2) \Omega(\gamma) + (\lambda_1 - \lambda_2) \Omega(\gamma)^\dagger \right] \left[ (\lambda_1 + \lambda_2) \Omega(\gamma)^\dagger + (\lambda_1 - \lambda_2) \Omega(\gamma) \right]
\]

\[
\leq \frac{1}{2} \left[ (\lambda_1 + \lambda_2)^2 + 2|\lambda_1 + \lambda_2| |\lambda_1 - \lambda_2| + (\lambda_1 - \lambda_2)^2 \right] \| \Omega(\gamma) \|^2
\]

\[
\leq \lambda_1^2 x(1 - x) M_\Omega^2.
\]
where we have used \( \|\Omega(\gamma)\|^2 = \|\Omega(\gamma)^\dagger\|^2 = \|\Omega(\gamma)\Omega(\gamma)^\dagger\| \). However, (33) and (34) also imply

\[
(zI - \|P\|)(zI - \|Q\|) \geq \left( \lambda_3(1 - x)M_{\Omega} \right) \left( x\lambda_3M_{\Omega} \right) = x(1 - x)\lambda_3^2M_{\Omega}^2.
\]

Since we have assumed \( \lambda_3^2 > \lambda_1^2 \), these inequalities imply (36).

**Remark:** At this point the only use we made of the unital character of \( \Phi \) was

(a) to give a specific formula for \( M_{\Phi} \)

(b) to use the special form (19) of representing \( \Phi \).

It is possible to generalize these formulas to some non-unital stochastic maps. Any stochastic map \( \Phi \) can be written in the form (17), where \( T \) is the \( 4 \times 4 \) matrix which represents its action on \((I, \sigma_1, \sigma_2, \sigma_3)\). Suppose \( \lambda_1 \) is the smallest diagonal entry of \( T \). If \( t_1 = 0 \), then the above method can be extended in a straightforward way to deduce that \( \Phi \) also satisfies the conclusion of Theorem 6, for any values of \( t_2, t_3 \) allowed by complete positivity. That is, as long as we do not translate the ellipsoid in the direction of its shortest major axis, the conclusion still holds. This is consistent with the conclusions of Section 5, where we argue that this is the hardest case to analyse. The difficulty occurs when the two other major axes have equal lengths, so that the ellipsoid is a ‘flying saucer’. This produces a circle of states of maximal norm and minimal entropy in the ellipsoid. It is necessary to show that no entanglement of these minimal entropy states can increase the norm above the product bound, or can lower the entropy below the product sum. We discuss this situation further in (5.2).

### 4 Minimal Entropy Analysis

#### 4.1 Reduction via Convexity

**Lemma 10** Let \( \Phi, \Omega \) be unital stochastic maps with \( M_{\Phi} \) and \( M_{\Omega} \) equal to \( \mu \) and \( \nu \), respectively. Then

\[
\inf_{\rho \text{ pure}} S(\Phi \otimes \Omega)(\rho) \geq \inf_{\rho \text{ pure}} S\left( \Phi[\mu, u, \mu] \otimes \Omega[\nu, v, \nu] \right)(\rho) \tag{37}
\]

where \( |u| \leq \mu \) and \( |v| \leq \nu \).

**Proof of Lemma:** By the results of sections 2.2 and 2.3, we can assume wlog that \( \Phi \) and \( \Omega \) are self-adjoint and diagonal, with eigenvalues \((\lambda_1, \lambda_2, \lambda_3)\) and \((\omega_1, \omega_2, \omega_3)\) respectively, where \( \lambda_3 = \mu > 0 \) and \( \omega_3 = \nu > 0 \). We first consider the case \( \mu, \nu > 1/3 \). It follows from (21) that the eigenvalues \( \lambda_1, \lambda_2 \) lie in a convex set with extreme points

\[
(\mu, \mu), (\mu, 2\mu - 1), (2\mu - 1, \mu), (-\mu, -\mu), (-\mu, 1 - 2\mu), (1 - 2\mu, -\mu)
\]
If we let $\Phi_1 \equiv \Phi[\mu, \mu, \mu]$, $\Phi_2 \equiv \Phi[\mu, 2\mu - 1, \mu]$ etc. so that $\Phi_j$ $(j = 1 \ldots 6)$ denote the stochastic maps corresponding to these six points, we can write $\Phi = \sum_{j=1}^{6} a_j \Phi_j$ as a convex combination of these six maps and similarly for $\Omega = \sum_{k=1}^{6} b_k \Omega_k$. Then, since the entropy is concave we find

$$S(\Phi \otimes \Omega)(\rho) = S\left(\sum_{j=1}^{6} a_j \Phi_j \otimes \sum_{k=1}^{6} b_k \Omega_k\right)(\rho)$$

$$= S\left(\sum_{j=1}^{6} \sum_{k=1}^{6} a_j b_k \Phi_j \otimes \Omega_k\right)(\rho)$$

$$\geq \sum_{j=1}^{6} \sum_{k=1}^{6} a_j b_k S(\Phi_j \otimes \Omega_k)(\rho)$$

$$\geq \min\{S(\Phi_j \otimes \Omega_k)(\rho) : j = 1 \ldots 6, k = 1 \ldots 6\} \quad (38)$$

But now we note that $\Phi_4 = \Upsilon_3 \circ \Phi_1$ and $\Phi_5 = \Upsilon_3 \circ \Phi_2$, where $\Upsilon_3(\rho) = \sigma_z \rho \sigma_z$. Hence, e.g.,

$$(\Phi_5 \otimes \Omega_4)(\rho) = (\sigma_z \otimes \sigma_z)\left[(\Phi_2 \otimes \Omega_1)(\rho)\right](\sigma_z \otimes \sigma_z)$$

so that

$$S\left(\Phi_5 \otimes \Omega_4\right)(\rho) = S\left(\Phi_2 \otimes \Omega_1\right)(\rho).$$

and similarly for $S\left(\Phi_1 \otimes \Omega_4\right)(\rho) = S\left(\Phi_1 \otimes \Omega_1\right)(\rho)$ etc. Hence we can replace (38) by

$$S(\Phi \otimes \Omega)(\rho) \geq \min\{S(\Phi_j \otimes \Omega_k)(\rho) : j, k = 1, 2, 3\}. \quad (39)$$

Since we also have $\inf_{\rho \text{ pure}} S(\Phi[\mu, u, \mu] \otimes \Omega(\rho)) = \inf_{\rho \text{ pure}} S(\Phi[u, u, \mu] \otimes \Omega(\rho))$ for any $\Omega$, and since $\Phi_j$, $j = 1, 2, 3$ and $\Omega_k$, $k = 1, 2, 3$ have the form given in the lemma, the result follows.

For $\mu < 1/3$ we proceed similarly, but with the convex set given by the rectangle with corners $(\pm\mu, \pm\mu)$.

### 4.2 Special Form of Pure State

As shown in Section 4.1, to show additivity of minimal entropy for unital stochastic maps it is sufficient to consider self-adjoint maps of the special form $\Phi[\mu, u, \mu]$ and $\Omega[\nu, v, \nu]$. In this section we prove additivity for these maps over a special class of entangled states.
Theorem 11 Let $\Phi[\mu, u, \mu]$ and $\Omega[\nu, v, \nu]$ be diagonal stochastic maps, satisfying $\mu \geq |u|$ and $\nu \geq |v|$, so that $\mu$ and $\nu$ are the largest eigenvalues of $\Phi$ and $\Omega$ respectively. Let $|\psi\rangle$ be a pure state of the form $|\psi\rangle = a|00\rangle + e^{i\theta}d|11\rangle$. Then

$$S(\Phi \otimes \Omega)(|\psi\rangle\langle\psi|) \geq h(\mu) + h(\nu) = \inf_{\rho} S[\Phi(\rho)] + \inf_{\gamma} S[\Omega(\gamma)].$$

We prove Theorem 11 in the next section. Here we derive some intermediate results which will be used in the proof. For generality we consider diagonal maps $\Phi[\lambda_1, \lambda_2, \lambda_3]$ and $\Omega[\omega_1, \omega_2, \omega_3]$, and for definiteness we also assume that $|\lambda_3|$ and $|\omega_3|$ are their largest singular values.

We find the density matrix for a pure state of the form $|\psi\rangle = a|00\rangle + e^{i\theta}d|11\rangle$ with $a, d$ real and $a^2 + d^2 = 1$. Then if $\rho = |\psi\rangle\langle\psi|$}

$$\rho = \left(\begin{array}{cccc} a^2 & 0 & 0 & ade^{i\theta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ ade^{-i\theta} & 0 & 0 & d^2 \end{array}\right) = \left(\begin{array}{cccc} \alpha & 0 & 0 & e^{i\theta}\sqrt{t}/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\theta}\sqrt{t}/2 & 0 & 0 & 1 - \alpha \end{array}\right),$$

where $\alpha = a^2$ and $t = 4\alpha(1 - \alpha)$, so that $t \in [0, 1]$. Applying the stochastic maps gives

$$[\Phi \otimes \Omega](\rho) = \frac{1}{t}\alpha \left[I \otimes I + \lambda_3 \omega_3 \sigma_z \sigma_z + \omega_3 I \otimes \sigma_z + \sigma_z \otimes I\right]$$

$$+ \frac{1}{t}(1 - \alpha) \left[I \otimes I + \lambda_3 \omega_3 \sigma_z \sigma_z - I \otimes \sigma_z - \sigma_z \otimes I\right]$$

$$+ \frac{1}{t}\sqrt{t} \left[\cos \theta \left(\sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y\right) - \sin \theta \left(\sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x\right)\right].$$

Notice that because $\Phi$ and $\Omega$ are diagonal, the result remains a linear combination of terms of the form $\sigma_k \otimes \sigma_k \quad (k = 0 \ldots 3)$ and $I \otimes \sigma_z$ and $\sigma_z \otimes I$; no cross terms of the form $\sigma_x \otimes \sigma_z$ etc arise. Therefore, the only non-zero terms in $[\Phi \otimes \Omega](\rho)$ lie along the diagonal or skew diagonal. Thus $[\Phi \otimes \Omega](\rho)$ can be written in the form

$$\begin{pmatrix} X & 0 & 0 & X \\ 0 & X & X & 0 \\ 0 & X & X & 0 \\ X & 0 & 0 & X \end{pmatrix}$$

where $X$ denotes a non-zero matrix element. Thus, $[\Phi \otimes \Omega](\rho)$ is equivalent to a block diagonal matrix.
A straightforward computation shows that these blocks, which we refer to as “outer” and “inner” can be written respectively as

\[ \frac{1}{4} \left( 1 + \lambda_3 \omega_3 + \sqrt{1 - t}(\lambda_3 + \omega_3) \right) \]

\( \frac{1}{2} \sqrt{t} e^{i\theta}(\lambda_4 \omega_+ - e^{-i\theta} \lambda_4 \omega_-) \frac{1}{4} \left( 1 + \lambda_3 \omega_3 - \sqrt{1 - t}(\lambda_3 + \omega_3) \right) \tag{44} \]

and

\[ \frac{1}{4} \left( 1 - \lambda_3 \omega_3 + \sqrt{1 - t}(\lambda_3 - \omega_3) \right) \]

\( \frac{1}{2} \sqrt{t} e^{i\theta}(\lambda_4 \omega_+ + e^{-i\theta} \lambda_4 \omega_-) \frac{1}{4} \left( 1 - \lambda_3 \omega_3 - \sqrt{1 - t}(\lambda_3 - \omega_3) \right) \) \tag{45}.

where \( \lambda_\pm = \lambda_1 \pm \lambda_2 \) and similarly for \( \omega_\pm \).

The first has eigenvalues

\[ \frac{1}{4} \left[ 1 + \lambda_3 \omega_3 \right] \pm \frac{1}{4} \left[ (1 - t)(\lambda_3 + \omega_3)^2 + \frac{1}{2} t (\lambda_1^2 \omega_+^2 + \lambda_2^2 \omega_-^2 + 2 \cos(2\theta) \gamma) \right]^{1/2} \] \tag{46}

while the second has eigenvalues

\[ \frac{1}{4} \left[ 1 - \lambda_3 \omega_3 \right] \pm \frac{1}{4} \left[ (1 - t)(\lambda_3 - \omega_3)^2 + \frac{1}{2} t (\lambda_1^2 \omega_+^2 + \lambda_2^2 \omega_-^2 + 2 \cos(2\theta) \gamma) \right]^{1/2} \] \tag{47}

where \( \gamma = \lambda_+ \lambda_- \omega_+ \omega_- \).

Minimum entropy occurs when both pairs of eigenvalues are spread out as far as possible. This happens either at \( \theta = 0 \) (if \( \gamma \geq 0 \)) or at \( \theta = \pi/2 \) (if \( \gamma \leq 0 \)). We will be interested in the case \( \lambda_1 \geq |\lambda_2| \) and \( \omega_1 \geq |\omega_2| \), which means that \( \gamma \geq 0 \), so we assume that \( \theta = 0 \) henceforth.

We need more compact notation for the eigenvalues. Define

\[ f(t) = \left( (\lambda_3 + \omega_3)^2 - t[(\lambda_3 + \omega_3)^2 - (\lambda_1 \omega_1 + \lambda_2 \omega_2)^2] \right)^{1/2} \] \tag{48}

and

\[ g(t) = \left( (\lambda_3 - \omega_3)^2 - t[(\lambda_3 - \omega_3)^2 - (\lambda_1 \omega_1 - \lambda_2 \omega_2)^2] \right)^{1/2} \] \tag{49}

Also define

\[ A = 1 + \lambda_3 \omega_3, \quad B = 1 - \lambda_3 \omega_3 \tag{50} \]

Then the four eigenvalues of \([\Phi \otimes \Omega](\rho)\) at \( \theta = 0 \) are simply

\[ \frac{1}{4}[A \pm f(t)], \quad \frac{1}{4}[B \pm g(t)] \] \tag{51}

The state \( |\psi\rangle \) is unentangled when \( t = 0 \) and maximally entangled for \( t = 1 \). We want to show that the entropy is minimized at \( t = 0 \). We will let \( S(t) \) denote
the entropy of \([\Phi \otimes \Omega](\rho)\) considered as a function of \(t\). To analyze its behavior, it is convenient to use the function

\[
\eta(\alpha, x) = -(\alpha + x) \log(\alpha + x) - (\alpha - x) \log(\alpha - x).
\]  
(52)

It follows that

\[
S(t) = \frac{1}{2} \eta[A, f(t)] + \frac{1}{2} \eta[B, g(t)] + \log 4,
\]
(53)

where we have used the fact that \(\frac{1}{2}(A + B) = 1\). To find the minimum of \(S(t)\) it suffices to analyze the behavior of \(\eta[A, f(t)]\) and \(\eta[B, g(t)]\). First observe that

\[
\frac{d}{dt} \eta[A, f(t)] = f'(t) \log \frac{A - f(t)}{A + f(t)}
\]
(54)

and

\[
\frac{d^2}{dt^2} \eta[A, f(t)] = f''(t) \log \frac{A - f(t)}{A + f(t)} - \frac{|f'(t)|^2}{A - |f(t)|^2} \left( \frac{2A}{A - |f(t)|^2} \right)
\]
\[
\leq \frac{2A}{A - |f(t)|^2} \left( f''(t) f(t) - |f'(t)|^2 \right)
\]
(55)

if \(f''(t) \leq 0\) and \(0 \leq f(t) \leq A\). This follows from the elementary inequality

\[
\log \left(\frac{1+x}{1-x}\right) \geq \frac{2x}{1-x^2}\quad \text{ (which holds for } x \in [0, 1]\text{)}
\]

applied to \(x = f(t)/A\). Now \(f(t)\) is a function of the form \(\sqrt{a - bt}\) for which one easily checks that \(f''(t) < 0\) and \(|f''(t)| f(t) - |f'(t)|^2 = 0\). Therefore, it follows immediately from (55) that \(\frac{d^2}{dt^2} \eta[A, f(t)] \leq 0\). Since \(g(t)\) also has the form \(\sqrt{a - bt}\), a similar argument holds for \(\eta[B, g(t)]\). Hence \(S''(t) < 0\) from which we conclude that \(S(t)\) is a concave function on \([0, 1]\), and therefore attains its minimum at either \(t = 0\) or \(t = 1\).

Hence to prove that \(S\) attains its minimal value at \(t = 0\), it is necessary and sufficient to show that \(S(1) \geq S(0)\). In essence, we have shown that for a state of the form given in Theorem 1 the minimal entropy is attained for either a maximally entangled state or a simple product state.

If we again think of \(f(t)\) in the form \(\sqrt{a - bt}\), then (18) (together with our assumption that \(\lambda_3\) and \(\omega_3\) are the maximal singular values) implies that \(b > 0\). Combined with (54) this implies that \(\eta[A, f(t)]\) is increasing. However, this need not be true for \(g(t)\). For example, when \(\lambda_3 = \omega_3\) and \(t = 1\), \(g'(1) = \frac{1}{2} |\lambda_1 \omega_1 - \lambda_2 \omega_2|\) which implies that \(g(t)\) is increasing and \(\eta[B, g(t)]\) decreasing. Thus, the general situation is that the entropy \(S(t)\) is a linear combination of two concave functions corresponding to the contributions from the “outer” and “inner” eigenvalues respectively. The former is always increasing, while the latter can be decreasing as \(t\) goes from 0 to 1. To illustrate this, Figure 1 shows how the eigenvalues of the product state move in the case where \(\lambda_3 = \omega_3 = \mu\). The “inner” eigenvalues are both \(\frac{1}{4}(1 - \mu^2)\), and they move apart as \(t\) increases away from 0, which lowers their
contribution to the entropy. The “outer” eigenvalues \( \frac{1}{4}(1 \pm \mu)^2 \) move together as \( t \) increases, which raises their contribution to the entropy.

To examine the difference \( S(1) - S(0) \), we note that

\[
S(1) = \log 4 + \frac{1}{4} \eta(1 + \lambda_3 \omega_3, |\lambda_1 \omega_1 + \lambda_2 \omega_2|) + \frac{1}{4} \eta(1 - \lambda_3 \omega_3, |\lambda_1 \omega_1 - \lambda_2 \omega_2|) \tag{56}
\]

\[
S(0) = \log 4 + \frac{1}{4} \eta(1 + \lambda_3 \omega_3, \lambda_3 + \omega_3) + \frac{1}{4} \eta(1 - \lambda_3 \omega_3, |\lambda_3 - \omega_3|) \tag{57}
\]

so that

\[
4[S(1) - S(0)] = \eta(1 + \lambda_3 \omega_3, |\lambda_1 \omega_1 + \lambda_2 \omega_2|) - \eta(1 + \lambda_3 \omega_3, \lambda_3 + \omega_3) + \eta(1 - \lambda_3 \omega_3, |\lambda_1 \omega_1 - \lambda_2 \omega_2|) - \eta(1 - \lambda_3 \omega_3, |\lambda_3 - \omega_3|) \tag{58}
\]

Since \( \frac{d}{dx} \eta(\alpha, x) = \log \frac{\alpha - x}{\alpha + x} < 0 \) if \( x > 0 \), \( \eta(\alpha, x) \) is decreasing in \( x \). By our assumptions, \( \lambda_3 \geq |\lambda_1| \geq |\lambda_1 \omega_1| \) and \( \omega_3 \geq |\omega_2| \geq |\lambda_2 \omega_2| \) so that

\[
\lambda_3 + \omega_3 > |\lambda_1 \omega_1 + \lambda_2 \omega_2|,
\]

and hence the difference of the the first two terms in (58) (which corresponds to the change in entropy from the “outer” eigenvalues) is always positive. The change from the inner eigenvalues need not be positive however; indeed, when \( \lambda_3 = \omega_3 \) it must be negative. Thus we need to show that the contribution from the inner eigenvalues cannot dominate.

We gain some intuition from an elementary analysis of \( \eta(\frac{1}{2}, x) \). This function is largest near \( x = 0 \), where it is flat, but has its largest derivative near \( x = \pm 1 \). Hence one expects the change from the larger “outer” eigenvalues to dominate. Explicit analysis of the extreme points in the next section verifies this.

For the proof of Theorem 11 we will restrict to the values \( \lambda_1 = \lambda_3 = \mu \) and \( \lambda_2 = u \) with \( |u| \leq \mu \), and \( \omega_1 = \omega_3 = \nu \) and \( \omega_2 = v \) with \( |v| \leq \nu \). In this case (58) becomes

\[
4[S(1) - S(0)] = \eta(1 + \mu \nu, \mu \nu + uv) - \eta(1 + \mu \nu, \mu + \nu) + \eta(1 - \mu \nu, \mu \nu - uv) - \eta(1 - \mu \nu, \mu - \nu). \tag{59}
\]

### 4.3 Analysis of Extreme Points

In this section we will complete the proof of Theorem 11. By the argument in section 4.1, it suffices to consider either \( u = \pm \mu \) with \( \mu \in [0, \frac{1}{3}] \), or \( u = \mu \) with \( \mu \in [\frac{3}{5}, 1] \), or \( u = 2\mu - 1 \) with \( \mu \in [\frac{1}{2}, 1] \). Similarly for \( \Omega \): either \( v = \pm \nu \) with \( \nu \in [0, \frac{1}{2}] \), or \( v = \nu \) with \( \nu \in [\frac{1}{3}, 1] \), or \( v = 2\nu - 1 \) with \( \nu \in [\frac{1}{4}, 1] \). So we wish to prove the positivity of \( S(1) - S(0) \) for these values of the parameters.

The simplest case occurs when \( u = \mu, v = \nu \). In this case (59) becomes simply

\[
4[S(1) - S(0)] = \eta(1 + \mu \nu, 2\mu \nu) - \eta(1 + \mu \nu, \mu + \nu) + \eta(1 - \mu \nu, 0) - \eta(1 - \mu \nu, \mu - \nu). \tag{60}
\]
Since \( \eta(\alpha, x) \) is decreasing in \(|x|\), the first term dominates the second, and the third term dominates the fourth, hence \( S(1) - S(0) > 0 \) in this case.

The remaining cases are handled numerically. It is useful to first consider a special case, namely \( \mu = \nu \). This arises when the convexity argument is applied to the product channel \( \Phi \otimes \Phi \), which is the situation of most interest to us. Then (60) yields

\[
4[S(1) - S(0)] = -(1 + 2\mu^2 + \nu^2) \log(1 + 2\mu^2 + \nu^2) \\
-(1 - 2\mu^2 + \nu^2) \log(1 - 2\mu^2 + \nu^2) - 2(1 - \nu) \log(1 - \nu) \\
+4(1 + \mu) \log(1 + \mu) + 4(1 - \mu) \log(1 - \mu). \tag{61}
\]

Graphing verifies that (61) is positive for the two extreme values \( \nu = \pm \mu^2 \) in the range \( 0 \leq \mu \leq \frac{1}{3} \) (see Figure 2), and for the three extreme values \( \nu = \mu(2\mu - 1) \) and \( \nu = (2\mu - 1)^2 \) in the range \( \frac{1}{3} \leq \mu \leq 1 \) (see Figure 3). The graphs show that (61) is smallest when \( \mu \approx 0, 1 \), so we analyze these regions more carefully. In the first case when \( 0 \leq \mu \leq \frac{1}{3} \) and \( \nu = -\mu^2 \), we expand around \( \mu = 0 \). This gives

\[
4[S(1) - S(0)] \approx 4\mu^2
\]

In the second case when \( \frac{1}{3} \leq \mu \leq 1 \) and \( \nu = \mu(2\mu - 1) \), write \( x = 1 - \mu \), and expand in \( x \):

\[
4S(1) - 4S(0) \approx 3x \log \frac{1}{x} + x \left[ 7(1 + \log 4) - 6 \log 3 - 4(1 + \log 2) \right] \\
\approx 3x \log \frac{1}{x} + 3.34x. \tag{62}
\]

This is manifestly positive for \( x \) small. Similarly in the case \( \nu = (2\mu - 1)^2 \) we have

\[
4S(1) - 4S(0) \approx 4x \log \frac{1}{x} + 4x(1 - \log 2) \\
\approx 4x \log \frac{1}{x} + 1.23x \tag{63}
\]

The general case \( \mu \neq \nu \) is handled similarly. We have

\[
4[S(1) - S(0)] = -(1 + 2\mu \nu + \nu^2) \log(1 + 2\mu \nu + \nu^2) \\
-(1 - 2\mu \nu + \nu^2) \log(1 - 2\mu \nu + \nu^2) - 2(1 - \nu) \log(1 - \nu) \\
+2(1 + \mu) \log(1 + \mu) + 2(1 - \mu) \log(1 - \mu) \\
+2(1 + \nu) \log(1 + \nu) + 2(1 - \nu) \log(1 - \nu). \tag{64}
\]

By symmetry it suffices to assume that \( \nu \leq \mu \). For \( 0 \leq \mu \leq \frac{1}{3} \) and \( 0 \leq \nu \leq \mu \) we have two extreme values \( \nu = \pm \mu \nu \). Graphing (64) shows that it is positive in
both of these cases. Again the smallest values occur near \( \mu = \nu = 0 \), so we expand (64) around this point. For both values \( uv = \pm \mu \nu \) this gives
\[
4[S(1) - S(0)] \simeq 2(\mu^2 + \nu^2).
\]
For \( \frac{1}{3} \leq \mu \leq 1 \) and \( 0 \leq \nu \leq \frac{1}{3} \), there are four extreme values \( uv = \mu \nu, uv = -\mu \nu, uv = (2\mu - 1)\nu \) and \( uv = -(2\mu - 1)\nu \). The graph of (64) is positive in all cases, with smallest values around \( \mu = \frac{1}{3} \). For \( \frac{3}{4} \leq \mu \leq 1 \) and \( 0 \leq \nu \leq \frac{1}{3} \), there are also four extreme values, \( uv = \mu \nu, uv = \mu(2\nu - 1), uv = (2\mu - 1)\nu \) and \( uv = (2\mu - 1)(2\nu - 1) \) (see Figure 4 for the last of these). The graphs of (64) are positive in each case, and the smallest values occur near \( \mu = \nu = 1 \). This region can be analyzed more carefully by expanding the functions to leading order in \( \mu = 1 - \mu \) and \( \nu = 1 - \nu \).

For example, when \( uv = (2\mu - 1)(2\nu - 1) \) the expansion of (64) yields
\[
4S(1) - 4S(0) \simeq 2[x \log x + y \log y - 2(x + y) \log(x + y)]
+ x(2 + 2 \log 2) + y(2 + 2 \log 2) \tag{65}
\]
Using convexity of the function \( x \log x \), we can bound (65) from below by \( 2(x + y) \), which demonstrates positivity for \( x, y \) small. Similar results are obtained for the other cases.

### 4.4 Mixing Discussion

In this section we extend Theorem 11 to pure states \( |\psi\rangle \) formed from any entanglement of states of minimal entropy. After a precise statement of this and proof of this extension, we discuss its interpretation and the evidence for more general validity of Conjecture 1.

Let \( \Phi \) be a unital stochastic map. As explained in Appendix A, we can write \( \Phi = U\Phi_S U^\dagger \) where \( U \) is a unitary operator and \( \Phi_S \) is self-adjoint. Let \( \mu = ||S|| \).

Define
\[
\mathcal{L}(\Phi) = \{ \rho = \frac{1}{2}(I + N) \mid \Phi_S(N) = \pm \mu N \}. \tag{66}
\]
In words, \( \mathcal{L}(\Phi) \) is the collection of density matrices which lie in the direction of the largest eigenvalue of \( \Phi_S \). If this largest eigenvalue is non-degenerate, then \( \mathcal{L}(\Phi) \) is a line segment between antipodal points on the Bloch sphere. In case of degeneracy it may be a disk, or even the entire Bloch sphere.

If \( \rho_{12} \) is a density matrix on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), we denote by \( \rho_1 = T_2(\rho_{12}) \) and \( \rho_2 = T_1(\rho_{12}) \) the reduced density matrices on \( \mathbb{C}^2 \) obtained by taking the indicated partial traces.

**Theorem 12** Let \( \Phi \) and \( \Omega \) be unital stochastic maps. Let \( \rho_{12} \) be a density matrix on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), such that \( \rho_1 \) lies in \( \mathcal{L}(\Phi) \), and \( \rho_2 \) lies in \( \mathcal{L}(\Omega) \). Then
\[
S(\Phi \otimes \Omega)(\rho_{12}) \geq \inf_{\rho} S[\Phi(\rho)] + \inf_{\gamma} S[\Omega(\gamma)]. \tag{67}
\]
**Proof:** We assume wlog that $\Phi$ and $\Omega$ are diagonal maps in the form $\Phi[\lambda_1, \lambda_2, \lambda_3]$ and $\Omega[\omega_1, \omega_2, \omega_3]$. Furthermore we can arrange that $\lambda_3 = \mu$ is the largest eigenvalue, so that $\frac{1}{2}[I \pm \sigma_3]$ lies in $\mathcal{L}(\Phi)$. Similarly we can arrange that $\omega_3 = \nu$ is the largest eigenvalue, so that $\frac{1}{2}[I \pm \sigma_3]$ also lies in $\mathcal{L}(\Omega)$.

Wlog we can assume that $\rho_{12} = |\psi\rangle\langle\psi|$ is a pure state. In the bases which diagonalize $\Phi$ and $\Omega$, we have

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle \quad (68)$$

Define the matrix $A$ to be

$$A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \quad (69)$$

Then as shown in Appendix A, the reduced density matrices are

$$\rho_1 = A A^\dagger, \quad \rho_2 = (A^\dagger A)^T \quad (70)$$

We obtain the “Schmidt decomposition” by applying the singular value decomposition to $A$. The result is a new basis in which $|\psi\rangle$ has the diagonal form assumed in Theorem 1. This decomposition is obtained by finding unitary operators $U_1, U_2$ which diagonalize $AA^\dagger$ and $A^\dagger A$ respectively, so that $U_1 A U_2^\dagger$ is also diagonal. By assumption $\rho_1$ lies in $\mathcal{L}(\Phi)$, and hence so does $AA^\dagger$. If $\mu$ is non-degenerate, then $\mathcal{L}(\Phi)$ is the line segment consisting of the diagonal density matrices. Therefore $AA^\dagger$ is also diagonal, so $U_1$ is equal to the identity, up to a phase. If $\nu$ is also non-degenerate then $U_2$ is also proportional to the identity, and hence $\rho_{12}$ is already in diagonal form. The result follows immediately by applying Theorem 1.

In general either or both of $\mu$ and $\nu$ may be degenerate. For example, if $\mu$ is 2-fold degenerate, then $\mathcal{L}(\Phi)$ is a disk containing the $z$-axis in the Bloch sphere. So $AA^\dagger$ lies in this disk, and hence it is diagonalized by a rotation of the Bloch sphere which preserves this disk. By definition, such a rotation commutes with the action of $\Phi$, since the plane which contains this disk is an eigenspace of $\Phi$. Hence the unitary operator $U_1$ commutes with $\Phi$. Similarly if $\mu$ is 3-fold degenerate, then every unitary operator commutes with $\Phi$. To apply the argument to $\Omega$, note that by assumption $\rho_2$ lies in $\mathcal{L}(\Omega)$. If $\nu$ is non-degenerate, or is 3-fold degenerate, the same argument applies. If $\nu$ is 2-fold degenerate, then $\mathcal{L}(\Omega)$ is a disk. The transpose operation on the Bloch sphere is the reflection in the $xz$-plane, and this does not in general preserve a disk containing the $z$-axis. However we have assumed that $\Omega$ is diagonal, and hence $\mathcal{L}(\Omega)$ either lies in the $xz$-plane or the $yz$-plane. In both cases the transpose leaves $\mathcal{L}(\Omega)$ invariant, and hence the same argument can be applied to deduce that $U_2$ also commutes with $\Omega$. It follows that $(\Phi \otimes \Omega)(|\psi\rangle\langle\psi|)$ is unitarily equivalent to $(\Phi \otimes \Omega)(|\psi'\rangle\langle\psi'|)$ where $|\psi'\rangle$ has the form assumed in Theorem 1, and hence the result follows.

**Remarks:**
1. The set $\mathcal{L}(\Phi)$ contains the states of minimal entropy for $\Phi$. Theorem 12 shows that by entangling the minimal entropy states of the individual channels we cannot decrease the entropy of the product channel. Since it seems unlikely that entangling states of higher entropy will improve the situation, we present this as strong evidence for our conjecture.

2. We illustrate Theorem 12 in the case where $\Phi[\lambda_1, \lambda_2, \lambda_3]$ is self-adjoint and diagonal, with $\lambda_1 = \lambda_3$. If the $|\psi\rangle$ is real, i.e., the matrix $A = a_{jk}$ is real, then the $2 \times 2$ unitary matrices $U_1$ and $U_2$ which diagonalize $AA^\dagger$ and $A^\dagger A$ can be chosen real and orthogonal, in which case we emphasize this by writing them as $O_1$ and $O_2$. Now suppose that our original orthogonal basis $|0\rangle, |1\rangle$ on $\mathbb{C}^2$ corresponds to the eigenvectors of $\sigma_z$ so that the corresponding pure state projections are $\frac{1}{2}[I + w \cdot \sigma]$, with $w = (0, 0, 1)$ corresponding to the “North pole” of a sphere. Each unitary $2 \times 2$ matrix $U$ can be associated with a real orthogonal $3 \times 3$ matrix, and the effect of $U$ on the basis vectors corresponds to a rotation on the sphere or the action of a real orthogonal $3 \times 3$ matrix on $w$. When the original $2 \times 2$ matrix is real orthogonal, the corresponding rotation on the sphere reduces to a rotation in the $xz$-plane.

If we now write the unital stochastic map $\Phi$ in the form
\[
\Phi : \rho = \frac{1}{2}[I + w \cdot \sigma] \rightarrow \Phi(\rho) = \frac{1}{2}[I + T w \cdot \sigma],
\]
then the change of basis is equivalent to replacing $T$ by $\hat{O} T \hat{O}^{-1}$ where $\hat{O}$ denotes the $3 \times 3$ orthogonal matrix associated with $O$. Thus, for example, when $O = \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix}$, $\hat{O} = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$. If $T$ is diagonal with eigenvalues $\lambda_x = \lambda_z$, then $\hat{O} T \hat{O}^{-1} = T$. Hence every state $|\psi\rangle$ with real coefficients can be diagonalized, and the matrix $T$ is unchanged.

3. Suppose that $\Phi$ has one very large singular value and two small ones. Then the unit sphere corresponding to the set of density matrices is mapped into an ellipsoid shaped like a football, and the states of minimal entropy lie at the ends of the football (see Figure 5). We can interpret Theorem 11 as saying that entangling these minimal entropy states will not decrease the entropy below the sum of the minimum entropies.

Now suppose that we always keep two eigenvalues equal, but vary the parameters so that the ends of the football move in until it becomes a sphere and then a “flying saucer” (see Figure 6). The ends of the football have moved in to states corresponding to maximal entropy. The minimal entropy states now form a great circle. As we explained above, our special form for $\psi$ allows a general entanglement of states corresponding to these great circles of
minimal entropy. Yet even this more general entanglement does not decrease
the entropy below that of product states.

4. The discussion above shows that, at least in the case of unital maps, if Con-
jecture 1 does not hold, then the entanglements which use states of higher
entropy would achieve a lower entropy on the product space than entangle-
ments of states of minimal entropy. In addition, such entanglements would
have to lower the entropy without increasing the largest eigenvalue of \( \Phi(\rho_{12}) \)
beyond the product value given in Theorem 1. We do not find this plausible.

5 Non-unital Maps

In this section we give some heuristic evidence to support Conjecture 1. Before
doing so, we illustrate the differences between unital and non-unital maps by dis-
covering some of the properties of a special class of maps on \( \mathbb{C}^{2 \times 2} \).

A non-unital map is one for which \( \Phi(I) \neq I \). This means that it takes a ran-
domly distributed alphabet to a non-random distribution. One would intuitively
expect that the maximum capacity would then be achieved for alphabets which
are not evenly distributed. Although this is true classically, it need not be true for
quantum stochastic maps as shown by the example below.

It follows from equation (11) that a unital stochastic map defines a linear map
on the subspace of matrices with trace zero. However, a non-unital map is affine
when restricted to this subspace.

5.1 Special subclass

We now consider non-unital maps which correspond, in the notation of Section 2.1
to the matrix

\[
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
t & 0 & 0 & \lambda_3
\end{pmatrix}
\]

(71)

with \( t \neq 0 \). This is easily seen to yield the map

\[
\Phi(\frac{1}{2}[I + w \cdot \sigma]) = \frac{1}{2} [I + w_1 \lambda_1 \sigma_1 + (t + w_3 \lambda_3) \sigma_3].
\]

(72)

If \( \lambda_1 = \frac{1}{\sqrt{3}} \) and \( t = \lambda_3 = \frac{1}{3} \), this is equivalent to the “splaying” channel introduced
by Fuchs 14 to demonstrate that there exist stochastic maps for which the Holevo
capacity is achieved only by non-orthogonal states. The case \( \lambda_3 = 0 \) was con-
sidered in 17 in a different context. The case \( \lambda_1 = 0 \) is essentially classical, i.e., if
\( \rho \) is restricted to the subset of states of the form \( \frac{1}{2}[I \pm w\sigma_3] \), the action of \( \phi \) is equivalent to the action of the column stochastic matrix \( \frac{1}{2} \begin{pmatrix} 1 + t + \lambda_3 & 1 - t - \lambda_3 \\ 1 - t - \lambda_3 & 1 + t - \lambda_3 \end{pmatrix} \) on the probability vector \( \frac{1}{2} \begin{pmatrix} 1 + w \\ 1 - w \end{pmatrix} \).

Since equality in (22) holds for Fuchs example, it can be regarded as an extreme case. Because \( \lambda_2 = 0 \) for this class of maps, they map the unit sphere of density matrices into the ellipse in the \( x-z \) plane satisfying the equation

\[
\frac{w_1^2}{\lambda_1^2} + \frac{(w_3 - t)^2}{\lambda_3^2} = 1
\]  

(73)

In the special cases, \( \lambda_1 = 0 \) and \( \lambda_3 = 0 \), these ellipses become vertical and horizontal line segments respectively.

In the classical case (\( \lambda_1 = 0 \)) it is not hard to show that the maximal capacity is never achieved for \( \pi = \frac{1}{2} \). One has only two pure states \( \rho_{\pm} = \frac{1}{2}[I \pm \sigma_3] \) (which are orthogonal) for which \( S[\Phi_{\pm}(\rho)] \) is not identical.

By contrast, for the non-unital quantum case with \( \lambda_1 > \lambda_3 \), it appears, in general, that maximal capacity is achieved at \( \pi = \frac{1}{2} \) and with non-orthogonal states. Moreover, these non-orthogonal states need not correspond to the minimal entropy states. Some insight into these observations can be obtained by looking at the ellipse (73) in Figure 7 corresponding to Fuchs channel. (Fuchs [10] showed explicitly that non-orthogonal states are required to achieve maximum capacity, and that the maximal capacity achievable with orthogonal states occurs for \( \pi = \frac{1}{2} \).)

The endpoints of the ellipse (denoted \( A_{\pm} \)) correspond to \( \Phi(\frac{1}{2}[I \pm \sigma_1]) \) and have entropy \( h[\frac{1}{2}(1 + 2/3)] \) while the minimal entropy states (denoted \( C_{\pm} \)) correspond to the images of \( \Phi \left( \frac{1}{2}\left[I + \left( \pm \frac{\sqrt{2}}{2}, 0, \frac{1}{2}\right)\sigma_3\right]\right) \) and have entropy \( h[\frac{1}{2}(1 + 1/\sqrt{2})] \). Note that this is the point at which the ellipse meets the circle \( x^2 + z^2 = \frac{1}{2} \), which is a level set for the entropy on the Bloch sphere.

The states \( \frac{1}{2}[I \pm \sigma_1] \) are the only pair of orthogonal states with identical entropy. If one tries to move from \( A_+ \) toward \( C_+ \) to lower the entropy from one of a pair of orthogonal states, the other orthogonal state must move along the ellipse away from \( A_- \), down and closer to the origin and hence has a higher entropy than \( A_- \). Explicit computation shows that the entropy price paid by moving away from \( A_- \) is greater than that gained by moving \( A_+ \) toward \( C_+ \). For any pair of states \( \rho_i \) \( (i = 1, 2) \), the state \( \tilde{\rho} = \pi \tilde{\rho}_1 + (1 - \pi)\tilde{\rho}_2 \) which occurs in (3) lies along the line segment between \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \). For the points \( A_\pm \) it is easy to see that the maximum capacity will occur for \( \pi = \frac{1}{2} \). If one does not require orthogonal states, then simultaneously moving both \( A_\pm \) toward \( C_\pm \) to decrease the entropy seems advantageous. However, this will also decrease the entropy of the convex
combination $\tilde{\rho}$ which *increases* the capacity. Hence, maximal capacity does not occur at the minimal entropy states but at states on the ellipse which lie between $C_\pm$ and $A_\pm$. (As long as the move is symmetric, maximal capacity will occur at $\pi = \frac{1}{2}$. Symmetry suggests this is optimal, but that does not seem to have been proved explicitly.)

One expects similar behavior for any map $\Phi$ of the form (71) for which $\lambda_1 > \lambda_3$. When $\lambda_1 < \lambda_3$ the major axis of the ellipse lies along the x-axis, there is only one state of minimal entropy at the “top” of the ellipse, and one expects the channel to behave more and more like a classical channel as the ratio $\lambda_1/\lambda_3$ increases.

When $\lambda_3 = 0$, the ellipse becomes a horizontal line so that the minimal entropy states and the endpoints of the ellipse coincide. Hence, the maximal capacity is again achieved for orthogonal states. In the limiting case $t^2 + \lambda_3^2 = 1$, the endpoints of the ellipse lie on the unit circle of pure states and hence, have entropy zero. However, the capacity does *not* achieve its maximum value of $\log 2$ but instead has the value $h(t)$.

As was noted earlier, $\Phi$ always has a fixed point. For channels of the form (71), this fixed point is at $(0, 0, \frac{1}{\sqrt{1 - \lambda_3}})$. However, we have been unable to attach any significance to the fixed point. (For Fuchs channel, the fixed point is at $\frac{1}{2}[C_+ + C_-]$; however, this seems coincidental.)

At first glance, it might seem that the price paid for the versatility of non-unital channels is too great. If we fix $\lambda_2$ and $\lambda_3$, then when $t \neq 0$ the requirement (28) implies that $\lambda_1$ must be smaller (i.e., “noisier”) than for the corresponding unital channel. For example, using Fuchs values $t = \lambda_3 = \frac{1}{3}$, $\lambda_2 = 0$, one finds that $\lambda_1 = \frac{1}{\sqrt{3}} \approx 0.577$ is optimal and corresponds to the least “noisy” direction. However, if $t = 0$, one could increase this to $\lambda_1 = 2/3 \approx 0.667$ with corresponding decrease in noise so that the minimal entropy (which comes from the states $\frac{1}{2}[I \pm \sigma_3]$) is $h(0.667)$. For the non-unital channels these same states would yield an entropy of only $h(0.577)$. However, the states $\frac{1}{2}[I + (\pm \frac{\sqrt{3}}{2}, 0, \frac{1}{2}) \cdot \sigma]$ will emerge with entropy $h(0.707)$. The decrease in eigenvalues of the part of $\Phi$ corresponding to the restriction to matrices of trace zero, is overcome by the contribution to the emerging state of the non-unital part of the map. We expect that this behavior is generic for non-unital maps.

### 5.2 Minimal entropy

Recall the discussion in Section 2.1. For unital maps, the generic situation is that there are two states of minimal entropy corresponding to the endpoints of the major axis. In the generic situation for a non-unital map, unless $t$ is perpendicular to the major axis, the ellipsoid in (18) will have *only one* state of minimal entropy. Hence *any* entanglement will require mixing with a state of higher entropy. In the case of a map (such as the Fuchs map (72) with $|\lambda_3| > |\lambda_1|$) for which the translation $t$ is
perpendicular to the (non-degenerate) major axis of the ellipsoid, there will be two states of minimal entropy. However, because the translated ellipsoid is not centered at the origin, these two states will not be the images of two orthogonal states, but rather the images of two non-orthogonal pure states corresponding to vectors $|\psi_1\rangle$ and $|\psi_2\rangle$. This suggests that the best candidate for a minimal entropy entangled state should have the form $\Psi = \sum_{jk} a_{jk} |\psi_j \otimes \psi_k\rangle$. However, after changing to orthogonal bases and using the SVD decomposition (83), this can be rewritten in the form $\Psi = a |\chi_1 \otimes \chi_3\rangle + d |\chi_2 \otimes \chi_4\rangle$ where $\langle \chi_1, \chi_2 \rangle = \langle \chi_3, \chi_4 \rangle = 0$ and, at most, only one state in each of the pairs $\chi_1, \chi_2$ and $\chi_3, \chi_4$ can equal either $\psi_1$ or $\psi_2$. Hence any entanglement must include states which are mapped into states of higher entropy under the action of $\Phi_S$. Thus one expects states of minimal entropy under the action of $\Phi_S \otimes \Phi_S$ to have the product form $|\psi_j \otimes \psi_k\rangle$.

In the case of two-fold degeneracy (of the form $|\lambda_j\rangle = |\lambda_k\rangle$), a shift orthogonal to both major axes will yield a circle of states of minimal entropy; however, that circle will not correspond to a “great circle” but rather a circle of constant latitude. Such a circle never includes the image of two orthogonal states and hence the argument above still holds. In the case of three-fold degeneracy, the image will be a sphere and any shift will again yield only a single state of minimal entropy.

Since the maps $\Phi$ and $\Phi_S$ differ only by rotations of the Bloch sphere, the same conclusions hold for $\Phi$. Moreover, our analysis suggests that for any pair of maps $\Phi$ and $\Omega$, the states which yield minimum entropy under $\Phi \otimes \Omega$ will be simple product states, regardless of whether one or both are non-unital.

6 Conclusion

Our main result in this paper is that channel capacity is additive for unital channels, where unital means that the channel maps the totally random input state (whose density matrix is proportional to the identity) into itself (for example, every self-adjoint channel is unital). Specifically, we present strong evidence that both the Shannon capacity (no entanglement for input states or output measurements) and the Holevo capacity (inputs unentangled, but output measurements may be entangled) are additive over two uses of a two-dimensional unital channel. This is the first result that establishes additivity of channel capacity for a broad class of quantum channels.

We show that the problem reduces to finding the states of minimal entropy for two uses of the channel. If these minimal entropy states are product states, then this implies additivity of capacity. We then prove that one of two possibilities occurs: either these minimal entropy states are product states, or else they are entangled states whose reduced density matrices have greater than minimal entropy. We argue that the latter case is very unlikely (numerical experiments confirm this), and so conclude that the former is true. As further supporting evidence, we prove
that the maximal norm of states which emerge from the channel is multiplicative over two channel uses.

Our results rely heavily on the Stokes parametrization and the properties of the image of the Bloch sphere under stochastic maps. In order to extend them to higher dimensions, we would need an effective parameterization of the subspace of matrices of trace zero in higher dimensions. One can always write a density matrix in $\mathbb{C}^{d \times d}$ as $\rho = \frac{1}{d} [I + N]$ where $\text{Tr} N = 0$. However, for $d = 4$, we do not know what the analog of the Bloch sphere looks like. We know only that its boundary corresponds to those $N$ with eigenvalues $+3, -1, -1, -1$ which is not the analogue of the surface of a sphere. Without knowing the geometry of this region, we can hardly hope to answer the important question of how it transforms under maps of the form $\Phi \otimes \Phi$. Thus, we have been forced to use indirect methods to reach conclusions about the states of minimal entropy emerging from $\Phi \otimes \Phi$.

Our results have implications for the design of communication channels. Ideally, one wants to eliminate all noise. However, this will not be practical and one wants to know how best to allocate resources.

In the case of unital channels, minimal entropy and maximal capacity are achieved if signal codes are chosen to correspond to the least noisy “direction” or polarization. (Here, we use “direction” in the sense of the Bloch sphere or maximum $\lambda_k$ in our notation. This is unrelated to direction of signal transmission.) Hence, if maximizing capacity is the primary goal, then it would seem sufficient to minimize noise in only one direction. Even if the orthogonal directions are extremely noisy, signals sent using optimal codes will not be affected. However, in this case “classical communication” becomes truly classical. If codes are restricted to one direction, then one is back in the classical situation with one choice for encoding 0 and 1. One has effectively lost the versatility of rotating the code basis as a tool for such purposes as signal encryption.

Non-unital channels have far more versatility, some aspects of which were discussed briefly in section 5.1. Much more work needs to be done analyzing the properties of non-unital channels. Thus far, most authors have looked for examples of particular maps which illustrate particular facets of stochastic maps (such as Fuchs [10] example demonstrating the possibility of maximizing capacity with non-orthogonal states). Our approach has been to try to find parameters which characterize subclasses of stochastic maps with certain properties. As summarized in Appendix C, most of the known examples of noisy channels can easily be shown to belong to one of the groups discussed in this paper. A complete analysis of non-unital maps would seem to require an extension of conditions of the type (22) to general maps of the form (17) with $t_k, \lambda_k \neq 0$.
A  Singular Value, Polar and Schmidt Decompositions

In this paper we make repeated, and sometimes subtle, use of the singular value decomposition (SVD) of matrices. In view of this, and of some confusion in the literature about the connection between the SVD and what is often referred to as the “Schmidt” decomposition, we provide a brief summary and review of these closely related decompositions and their connection to the better known polar decomposition.

We begin with the polar decomposition which can be extended to bounded, and even some unbounded, operators \[20\] on a Hilbert space.

**Theorem 13** (Polar Decomposition) Any \(m \times n\) matrix \(A\) can be written in the form \(A = U |A|\) where the \(n \times n\) matrix \(|A| = \sqrt{A^\dagger A}\) is positive semi-definite and the \(m \times n\) matrix \(U\) is a partial isometry. The term partial isometry means that \(U^\dagger U\) (or, equivalently, \(UU^\dagger\)) is a projection. In general, \(U\) need not be unique but can be uniquely determined by the condition \(\ker U = \ker A\). If \(A\) is a square \(n \times n\) matrix, then \(U\) can instead be chosen (non-uniquely) to be unitary. Since \(|A|\) is self-adjoint, it can be written as \(|A| = VDV^\dagger\) where \(D\) is a diagonal matrix with non-negative entries and \(V\) is unitary. Inserting this in Theorem 13 with \(U\) chosen to be unitary yields the SVD since \(W = UV\) is also unitary.

**Theorem 14** (Singular Value Decomposition) Any \(n \times n\) matrix \(A\) can be written in the form \(A = WDV^\dagger\) with \(V, W\) unitary and \(D\) a positive semi-definite diagonal matrix.

The non-zero elements of \(D\) are called the singular values of \(A\). They are easily seen to be the eigenvalues of \(|A|\) and, hence, their squares yield the non-zero eigenvalues of \(A^\dagger A\). As an immediate corollary, one finds that \(A^\dagger A\) and \(AA^\dagger\) are unitarily equivalent and that \(V\) and \(W\) are, respectively, the unitary transformations that diagonalize \(A^\dagger A\) and \(AA^\dagger\). These results can be extended to non-square matrices if the requirement that \(V, W\) be unitary is relaxed to partial isometry.

Using the notation of Section 2.1, we can apply the SVD to the \(3 \times 3\) matrix \(T\) which corresponds to the restriction of the stochastic map \(\Phi\) to the subspace of matrices with trace zero. Because \(T\) is real, the matrices \(V, W\) can be chosen to be real orthogonal so that we can write

\[
T = O_1 DO_2^T
\]

where \(O_1, O_2\) are orthogonal and the superscript \(T\) denotes transpose. Now, every \(3 \times 3\) orthogonal matrix is either a rotation, or the product of a rotation with the inversion \(-I\). Hence we can rewrite (74) as

\[
T = R_1(\pm D)R_2^T = (R_1R_2^T)R_2(\pm D)R_2^T
\]

(75)
where \( R_1 \) and \( R_2 \) are rotations, and we conclude that \( T \) can be written as

\[
T = RS
\]  

where \( S \) is self-adjoint and \( R \) is a rotation. If \( \Phi \) is unital, define the map \( \Phi_S \) by

\[
\Phi_S(w_0I + w \cdot \sigma) = w_0I + S w \cdot \sigma
\]  

Since every rotation is implemented by a unitary transformation on \( \mathbb{C}^2 \), there is a unitary operator \( U \) such that for any state \( \rho \),

\[
\Phi(\rho) = U\Phi_S(\rho)U^\dagger
\]  

For non-unital maps, a similar argument can be used to show that any stochastic map has the form (11) where the restriction of \( \Phi \) to the matrices with trace zero has the form (76) and \( t \rightarrow R^T t \), i.e. \( \Phi(\rho) = U\Phi_S(\rho)U^\dagger \) where

\[
\Phi_S(w_0I + w \cdot \sigma) = w_0I + (R^T t + S w) \cdot \sigma
\]

and \( R \) is the rotation on \( \mathbb{R}^3 \) corresponding to \( U \).

By construction, either \( S \) or \(-S\) is positive semi-definite. However, if \( \Upsilon_k(\rho) = \sigma_k \rho \sigma_k \), then composing \( \Upsilon_k \) with the diagonal map \( \Phi_D(w_0I + w \cdot \sigma) = w_0I + D w \cdot \sigma \) merely changes the signs of two of the diagonal elements of \( D \). Since \( U \sigma_k \) is also a unitary map, we can drop the restriction that \( S \) be semi-definite by modifying \( U \) if necessary. This is useful because, as we will see in the next section, the general conditions on the eigenvalues of a matrix \( S \) corresponding to a self-adjoint map \( \Phi \) include the possibility of negative and positive eigenvalues. We can summarize this discussion in the following

**Theorem 15**

Any stochastic map \( \Phi \) on \( \mathbb{C}^{2 \times 2} \) can be written in the form \( \Phi(\rho) = U\Phi_S(\rho)U^\dagger \) where \( U \) is unitary and \( \Phi_S \) is a stochastic map whose restriction to matrices with trace zero is self-adjoint.

It may be worth noting that we can apply the polar decomposition theorem directly to a completely positive map \( \Phi \). If we use \( \tilde{\Phi} \) to denote the adjoint with respect to the Hilbert-Schmidt inner product, then Theorem 13 implies that we can write

\[
\Phi = \Upsilon \circ |\Phi|
\]

where \( |\Phi| = \sqrt{\tilde{\Phi} \circ \Phi} \) and \( \Upsilon \) is a partial isometry. If \( \Phi \) takes an algebra (e.g., \( \mathbb{C}^{n \times n} \)) to itself, then \( \Upsilon \) can be chosen to be an isometry, i.e., \( \tilde{\Upsilon} \Upsilon = \Upsilon \tilde{\Upsilon} = I \). However, neither \( |\Phi| \) nor \( \Upsilon \) need be stochastic in general (even though their composition is). On the contrary, if \( \Phi_S \) has an odd number of negative eigenvalues, then
\[ \Phi_S = \Gamma \circ |\Phi| \] where \( \Gamma \) changes the sign of an odd number of eigenvalues and, hence, is not a completely positive map. (See Appendix C for further discussion and explicit examples.)

What we have shown in the argument above is that for unital maps on \( \mathbb{C}^{2 \times 2} \), the isometry \( \Upsilon \) can always be implemented by a unitary transformation, possibly composed with a map \( \Gamma \) that takes \( [w_0 I + w \cdot \sigma] \rightarrow [w_0 I - w \cdot \sigma] \), i.e., there is a unitary matrix \( U \) such that \( \Upsilon \) has the form \( \Upsilon \pm \left( w_0 I + w \cdot \sigma \right) = U[w_0 I \pm w \cdot \sigma]U^\dagger \) where one and only one sign holds. If \( \Phi \) is a non-unital stochastic map, then \( |\Phi| \) will not even be trace-preserving and the isometry \( \Upsilon \) will correspond to a change of basis that mixes the identity \( I \) with the three \( \sigma \) matrices (in contrast to the unital case in which the change of basis affects only the subspace of traceless matrices spanned by the three Pauli matrices). Hence for non-unital maps, the full polar decomposition (80) of a stochastic map \( \Phi \) may be less useful than the polar decomposition on the restriction to matrices of trace zero. For unital maps on \( \mathbb{C}^{n \times n} \) with \( n \geq 3 \) it would be interesting to know how much \( \Upsilon \) can differ from a map of the form \( \Upsilon(\rho) = U\rho U^\dagger \) where \( U \) is an \( n \times n \) unitary matrix.

In order to see the connection between the SVD and the so-called “Schmidt decomposition”, consider a wave function or vector of the form

\[ \Psi = \sum_{jk} a_{jk} \psi_j \otimes \chi_k \] (81)

with \( \{\psi_j\} \) and \( \{\chi_k\} \) orthonormal. It is not hard to see that there is an isomorphism between such vectors and operators of the form

\[ K_{\Psi} = \sum_{jk} a_{jk} |\psi_j \rangle \langle \chi_k| \] (82)

and that \( K_{\Psi} \) is a Hilbert-Schmidt operator if and only if \( \Psi \) is square-integrable (in the case of wave functions). Moreover, if \( \rho_{12} = |\Psi \rangle \langle \Psi| \), then

\[ \rho_1 \equiv T_2(\rho_{12}) = K_{\Psi} K_{\Psi}^\dagger \]
\[ \rho_2 \equiv T_1(\rho_{12}) = (K_{\Psi}^\dagger K_{\Psi})^T \]

where \( \rho_1 \) and \( \rho_2 \) are the reduced density matrices obtained by taking the indicated partial traces \( T_2 \) and \( T_1 \). The “Schmidt decomposition” is an immediate consequence of the application of the SVD to the operator \( K_{\Psi} \) given by (82) which implies the following result.

**Theorem 16 (Schmidt)** Any wave function of the form (81) can be rewritten as

\[ \Psi = \sum_k \mu_k \tilde{\psi}_k \otimes \tilde{\chi}_k \] (83)

where \( \mu_k \) are the singular values of the matrix \( A \), the bases \( \{\tilde{\psi}_k\} \) and \( \{\tilde{\chi}_k\} \) are orthonormal and related by \( \mu_k \tilde{\psi}_k = K_{\Psi} \tilde{\chi}_k \) with \( K_{\Psi} \) given by (82).
It follows immediately that the reduced density matrices $\rho_1$ and $\rho_2$ have the same non-zero eigenvalues $\{\mu_k^2\}$ and $\{\tilde{\psi}_k\}$ and $\{\tilde{\chi}_k\}$ are the eigenvectors of $\rho_1$ and $\rho_2$ respectively.

There is an interesting history associated with both the SVD and Schmidt decompositions, as well as the attachment of Schmidt’s name to (83) in the physics literature. In Chapter 3 of [16], Horn and Johnson give a detailed account of the history of the SVD which goes back to Beltrami and Jordan who independently obtained the SVD for real $n \times n$ matrices in the 1870’s. In 1902 Autonne obtained the SVD for general nonsingular complex $n \times n$ matrices and later made explicit the straightforward generalization to singular matrices in a long paper in 1915 which seems to have been subsequently overlooked by many researchers.

Independently, Schmidt obtained analogous results in 1907 for operators associated with integral kernels. When quantum chemists became interested in density matrices in the 1960’s, Carlson and Keller [8] rediscovered some of his results. However, John Coleman [9] soon pointed out the connection with Schmidt’s much earlier work. Coleman’s observation probably made physicists and chemists aware of that work although he did not use the term “Schmidt decomposition” which has recently become popular in the quantum computing literature. Physical chemists were initially interested in the reduced density matrices which arise from a multi-particle wave function of the form $\Psi(w_1 \ldots w_n)$ where $w_i$ denotes the space and spin coordinates associated with the i-th particle. It was natural to decompose these coordinates into two subsets $x = w_1 \ldots w_p$ and $y = w_{p+1} \ldots w_n$ so that the density matrices $\rho_1$ and $\rho_2$ actually correspond to $p$-th and $(N-p)$-th order reduced density matrices of $\Psi$. It is noteworthy that if the original wave function $\Psi$ has some symmetry, then the functions $\{\tilde{\psi}_k\}$ and $\{\tilde{\chi}_k\}$ can always be chosen so that (83) has the same symmetry. In particular, if $\Psi$ is antisymmetric, (as required by the Pauli exclusion principle for fermions) then these bases can be chosen so that (83) is also antisymmetric; i.e., there is no need to apply an additional antisymmetrizer.

In the case of multi-particle wave functions $\Psi(x, y)$ is readily interpreted as the kernel of an integral operator (which corresponds to $K_\Psi$ defined above) acting on an infinite dimensional Hilbert space. Hence, it is quite natural to attribute the results to Schmidt when used in this context. However, in quantum computation, one only considers the “spin” part of the wave function. Since this can always be represented by a finite dimensional matrix, the “diagonalization” of $a_{jk}$ in (83) can be obtained directly from the SVD; there is no need to detour into infinite dimensional Hilbert spaces to use results for integral operators. Moreover, since the matrix form of the SVD preceded Schmidt’s work, it seems natural to use the term SVD decomposition.
B  Matrix Representation of Stochastic Maps

To prove the eigenvalue conditions (21), we rewrite the Kraus operators defined by (1) in the form

$$A_k = v_{k0}I + v_k \cdot \sigma$$  \hspace{1cm} (84)

where \((v_{k0}, v_k)\) is a vector in \(\mathbb{C}^4\). We will let \(V\) denote the \(n \times 4\) matrix with elements \(v_{kj}\), \(V_j\) its columns as vectors in \(\mathbb{C}^n\) and \(V = (V_1, V_2, V_3)\). Using the relation

\[(aI + u \cdot \sigma)(bI + w \cdot \sigma) = (ab + u \cdot w)I + (aw + bu + iu \times w) \cdot \sigma\]  \hspace{1cm} (85)

one finds

$$A_k A_k^\dagger = \sum_{j=0}^3 |v_{kj}|^2 I + (v_{k0} \overline{v}_k + \overline{v}_{k0} v_k + i v_k \times \overline{v}_k) \cdot \sigma$$  \hspace{1cm} (86)

$$A_k^\dagger A_k = \sum_{j=0}^3 |v_{kj}|^2 I + (v_{k0} \overline{v}_k + \overline{v}_{k0} v_k - i v_k \times \overline{v}_k) \cdot \sigma$$  \hspace{1cm} (87)

so that

$$\sum_k A_k A_k^\dagger = (\sum_{j=0}^3 |V_j|^2)I + 2 \left( \Re\langle V_0, V \rangle + \sum_k \Re v_k \times \Im v_k \right) \cdot \sigma$$  \hspace{1cm} (88)

$$\sum_k A_k^\dagger A_k = (\sum_{j=0}^3 |V_j|^2)I + 2 \left( \Re\langle V_0, V \rangle - \sum_k \Re v_k \times \Im v_k \right) \cdot \sigma$$  \hspace{1cm} (89)

where \(\langle ., . \rangle\) denotes the standard inner product in \(\mathbb{C}^n\) and \(V = (V_1, V_2, V_3)\). Hence if \(\Phi\) is either unital or trace-preserving, then

$$\sum_{j=0}^3 |V_j|^2 = \text{Tr} V^\dagger V = 1.$$  \hspace{1cm} (90)

In addition, it follows from (2) and (3) that

$$\text{Tr} \Phi(\rho) = \text{Tr} \rho \Rightarrow \Re\langle V_0, V \rangle + \sum_k \Re v_k \times \Im v_k = 0$$  \hspace{1cm} (91)

$$\Phi(I) = I \Rightarrow \Re\langle V_0, V \rangle - \sum_k \Re v_k \times \Im v_k = 0.$$  \hspace{1cm} (92)

This implies that \(\Phi\) is both unital and trace-preserving if and only if, in addition to (88),

$$\Re\langle V_0, V \rangle = 0$$  \hspace{1cm} (93)

$$\sum_k \Re v_k \times \Im v_k = 0.$$  \hspace{1cm} (94)
where (92) can be rewritten as
\[ \mathfrak{I} \langle V_j, V_k \rangle = 0, \quad j \neq k \in 1, 2, 3 \] (93)

By defining \( \hat{V} \) to be the matrix obtained by replacing the first column of \( V \) by \( \hat{V}_0 = iV_0 \), one can rewrite conditions (91) and (92) as the single requirement that \( \mathfrak{I} (\hat{V}^\dagger \hat{V}) \) is diagonal or, equivalently, \( \mathfrak{I} \langle \hat{V}_j, \hat{V}_k \rangle = 0, \quad j \neq k \in 0 \ldots 3. \)

We will now derive the general form of the real 3 \times 3 matrix \( T \) defined by (12) so that
\[ T_{jk} = \frac{1}{2} \text{Tr}(\sigma_j \Phi(\sigma_k)) \] (94)

In general \( T \) is not symmetric; in fact, it is symmetric if and only if every operator \( A_k \) is self-adjoint. One finds after straightforward calculation that
\[ T_{jj} = \langle V_0, V_0 \rangle + \langle V_j, V_j \rangle - \sum_{i \neq j} \langle V_i, V_i \rangle \] (95)
\[ T_{ij} = 2 \Re \langle V_i, V_j \rangle \mp 2 \Im \langle \hat{V}_0, \hat{V}_k \rangle \] (96)

where the \(-\) holds in (96) if \( \{i, j, k\} \) is an even or cyclic permutation of \( \{1, 2, 3\} \) and the \(+\) sign if it is an odd permutation. Thus, for example,
\[ T_{11} = \langle V_0, V_0 \rangle + \langle V_1, V_1 \rangle - \langle V_2, V_2 \rangle - \langle V_3, V_3 \rangle \]
\[ T_{12} = 2 \Re \langle V_1, V_2 \rangle - 2 \Im \langle \hat{V}_0, \hat{V}_3 \rangle \]

In the special case where \( T \) is diagonal its eigenvalues can easily be obtained from (93). Let \( (\lambda_1, \lambda_2, \lambda_3) \) be the eigenvalues, and define \( q_j = \langle V_j, V_j \rangle \quad j = 0 \ldots 3. \) Then (93) becomes
\[ \lambda_1 = q_0 + q_1 - q_2 - q_3 \]
\[ \lambda_2 = q_0 - q_1 + q_2 - q_3 \]
\[ \lambda_3 = q_0 - q_1 - q_2 + q_3 \] (97)

Together with the condition (88) which can be written as
\[ 1 = q_0 + q_1 + q_2 + q_3 \] (98)

this implies that the point with coordinates \( (\lambda_1, \lambda_2, \lambda_3) \) must lie inside the tetrahedron with corners at \( (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1) \). Furthermore by taking \( n \geq 4 \) and choosing the vectors \( V_0, V_1, V_2, V_3 \) to be orthogonal we see that every point in this tetrahedron defines a triplet of eigenvalues which can arise
from a unital stochastic operator. These conditions are equivalent to four linear inequalities which must be satisfied by the eigenvalues, namely

\[
\begin{align*}
\lambda_1 + \lambda_2 & \leq 1 + \lambda_3 \\
\lambda_1 - \lambda_2 & \leq 1 - \lambda_3 \\
-\lambda_1 + \lambda_2 & \leq 1 - \lambda_3 \\
-\lambda_1 - \lambda_2 & \leq 1 + \lambda_3
\end{align*}
\]

which is equivalent to the more compact (21).

These conditions were obtained earlier by Algoet and Fujiwara [2].

The expression in (95) and (96) also hold for non-unital Φ. To calculate the non-zero elements in the first column extend (94) to \( j = 0 \) and observe that the trace-preserving condition (89) implies \( \Re \langle V_0, V_j \rangle = -\sum_k \Re v_k \times \Im v_k \). Using this in (87) yields

\[
t_j = T_{j0} = \frac{1}{2} \text{Tr} \sigma_j \Phi(I) = 4 \Re \langle V_0, V_j \rangle
\]

(99)

C Examples

We now give some examples of unital and non-unital maps which illustrate some of the features discussed earlier and show the correspondence between our parameterizations and some well-known examples which are usually described by their Kraus operators. Following the notation of Section 2.1, we will let \( \Phi[\lambda_1, \lambda_2, \lambda_3] \) denote a diagonal unital map.

Before doing so we note that the classic example of a linear, positivity preserving map which is not completely positive is the transpose, which corresponds to \( \Phi[1, -1, 1] \). We also note that if \( \Upsilon_k \) denotes the maps \( \Upsilon_k(\rho) = \sigma_k \rho \sigma_k \), the composition \( \Upsilon_k \circ \Phi[\lambda_1, \lambda_2, \lambda_3] \) changes the sign of the two eigenvalues whose subscript is not \( k \). (For example \( \Upsilon_2 \circ \Phi[\lambda_1, \lambda_2, \lambda_3] = \Phi[-\lambda_1, \lambda_2, -\lambda_3] \).) Thus the map \( \Phi[-1, -1, -1] \) which takes \( \frac{1}{2}[I + w \cdot \sigma] \rightarrow \frac{1}{2}[I - w \cdot \sigma] \) is also not completely positive.

Examples of Unital Channels

- Depolarizing channel: \( \Phi[1 - \frac{4x}{3}, 1 - \frac{4x}{3}, 1 - \frac{4x}{3}] \)

\[
A_0 = \sqrt{1 - x} I, \quad A_k = \sqrt{x/3} \sigma_k \quad (k = 1, 2, 3)
\]

- BFS [4] two-Pauli channel: \( \Phi[x, x, 2x - 1] \)

\[
A_0 = \sqrt{x} I, \quad A_k = \sqrt{\frac{1}{2}(1 - x)} \sigma_k \quad (k = 1, 2)
\]
• Phase-damping channel: \( \Phi[1 - x, 1 - x, 1] \)

\[
A_0 = \sqrt{1 - x} I, \quad A_\pm = \sqrt{x} \frac{1}{2} [I \pm \sigma_z]
\]

• Rotation: A single rotation is the simplest example of a unital, non-self-adjoint map. \( A_1 = U \) where \( U \) is unitary and \( \det(U) = +1 \).

A convex combination of such rotations using Kraus operators \( A_k = \tau_k U_k \) where \( \sum_k |\tau_k|^2 = 1 \) yields a more general example of a unital map which is not self-adjoint, namely, \( \Phi(\rho) = \sum_k |\tau_k|^2 U_k^\dagger \rho U_k \). Then the associated 3 \times 3 matrix \( T = \sum_k |\tau_k|^2 R_k \) where \( R_k \) is the rotation corresponding to \( U_k \).

The BFS two-Pauli channel was introduced by Bennett, Fuchs, and Smolin \[4\] to demonstrate that entangled states could reduce the probability of error. It is worth noting that this map corresponds to the extreme points studied in detail in Section \[4\]. Hence, it is particularly noteworthy that our analysis provides particularly strong evidence that entanglements do not decrease the entropy for these maps. It is also worth noting that when \( x = \frac{1}{3} \) this map becomes \( \Phi[\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}] \).

Thus, although \( \Phi[1, 1, -1] \) and \( \Phi[-1, -1, -1] \) are not completely positive, both \( \Phi[\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}] \) and \( \Phi[-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}] \) are completely positive.

**Examples of Non-Unital Channels**

• Amplitude-damping channel

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1-t} \\ 0 & 0 & \sqrt{1-t} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{t} \\ 0 & 0 \end{pmatrix}
\]

\( I \rightarrow \Phi(I) = \begin{pmatrix} 1 + t & 0 \\ 0 & 1 - t \end{pmatrix} = I + t \sigma_z \)

\[
T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-t} & 0 & 0 \\ 0 & 0 & \sqrt{1-t} & 0 \\ t & 0 & 0 & 1 - t \end{pmatrix}
\]

Equality in \((\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2 - t^2\)

• Fuchs channel and related examples discussed in Section \[5.1\]
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Figure Captions

Figure 1. The four eigenvalues of the product state of minimal entropy, with their direction of movement shown as $u$ increases away from 0.

Figure 2. The entropy difference $4(S(1) - S(0))$ for the two channels $\Phi[\mu, \mu, \mu] \otimes \Phi[\mu, \mu, \mu]$ (upper curve) and $\Phi[\mu, \mu, \mu] \otimes \Phi[\mu, -\mu, \mu]$ (lower dashed curve), in the range $0 \leq \mu \leq \frac{1}{3}$. Note that the curves coincide over most the interval.

Figure 3. The entropy difference $4(S(1) - S(0))$ for the three channels $\Phi[\mu, \mu, \mu] \otimes \Phi[\mu, \mu, \mu]$ (dots), $\Phi[\mu, \mu, \mu] \otimes \Phi[\mu, 2\mu - 1, \mu]$ (dashes), and $\Phi[\mu, 2\mu - 1, \mu] \otimes \Phi[\mu, 2\mu - 1, \mu]$ (full) in the range $\frac{1}{3} \leq \mu \leq 1$.

Figure 4. The entropy difference $4(S(1) - S(0))$ for the channel $\Phi[\mu, 2\mu - 1, \mu] \otimes \Omega[\nu, 2\nu - 1, \nu]$ in the range $\frac{1}{3} \leq \mu, \nu \leq 1$.

Figure 5. The Bloch sphere and its image (the ellipsoid) under a unital map with one large and two small singular values. The endpoints of the ellipsoid are shown separately – these are the unique states of minimal entropy.

Figure 6. The Bloch sphere and its image under a unital map with two large and one small singular values. The “waistband” of the ellipsoid is shown separately; this entire circle consists of minimal entropy states.

Figure 7. The Bloch sphere (the circle of radius 1) and its image (the translated ellipse) under the Fuchs map, together with the circle of radius $1/\sqrt{2}$. The endpoints of the ellipse are marked $A \pm$, and the points of minimal entropy are marked $C \pm$. 

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Figure 1: The four eigenvalues of the product state of minimal entropy, with their
direction of movement shown as \( u \) increases away from 0.

\[ \begin{align*}
1 & \sim \frac{1}{4}(1 + |\mu|)^2 \\
\frac{1}{4}(1 - |\mu|^2) & \sim \frac{1}{4}(1 - |\mu|)^2 \\
0 & \end{align*} \]

Figure 2: The entropy difference \( 4[S(1) - S(0)] \) for the two channels
\( \Phi[\mu, \mu, \mu] \otimes \Phi[\mu, \mu, \mu] \) (upper curve) and \( \Phi[\mu, \mu, \mu] \otimes \Phi[\mu, -\mu, \mu] \) (lower dashed
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