THE GENERALIZED LOCALLY CHECKABLE PROBLEM
IN BOUNDED TREEWIDTH GRAPHS

FLAVIA BONOMO-BRABERMAN AND CAROLINA LUCÍA GONZALEZ

ABSTRACT. We introduce a new problem that generalizes some previous attempts of covering locally checkable problems under the same umbrella. Optimization and decision problems such as \(k\)-dominating set, b-coloring, acyclic coloring and connected dominating set, can be seen as instances of this new problem.

We prove that this new problem can be solved, under mild conditions, in polynomial time for bounded treewidth graphs. As a consequence, we obtain polynomial-time algorithms to solve, for bounded treewidth graphs, Grundy domination and double Roman domination, among other problems for which no such algorithm was previously known. Moreover, by proving that (fixed) powers of bounded degree and bounded treewidth graphs are also bounded degree and bounded treewidth graphs, we can enlarge the family of problems that can be solved in polynomial time for these graph classes, including distance coloring problems and distance domination problems (for bounded distances).

1. Introduction

Many combinatorial optimization problems in graphs can be classified as vertex partitioning problems. The partition classes have to verify inner-properties and/or inter-properties, and there is an objective function to minimize or maximize. Some of these properties are \textit{locally checkable}, that is, the property that each vertex has to satisfy with respect to the partition involves only the vertex and its neighbors. This is the case of stable set, dominating set and \(k\)-coloring, among others.

In the spirit of generalizing this kind of problems, in [61] Telle defines the \textit{locally checkable vertex partitioning} (LCVP) problems. In [14], Bui-Xuan, Telle and Vatshelle present dynamic programming algorithms for LCVP problems that run in polynomial time on many graph classes, including interval graphs, permutation graphs and Dilworth \(k\) graphs, and in fixed-parameter single-exponential time parameterized by boolean-width. In [16], Cattaneo and Perdrix define a different generalization of LCVP problems that allows us to deal with properties of the subset that are not necessarily locally checkable, as for example being connected, and prove hardness results for LCVP problems and such generalizations.

However, none of these generalizations allows to model problems like \(k\)-domination (at least not in a straightforward way).

In this paper, we define the \textit{generalized locally checkable} (GLC) problem that successfully deals with the previously mentioned issue. In this new problem, every vertex has a list of colors that it can receive along with the cost of receiving each color. For each vertex and each coloring of its neighborhood, we have a function that \textit{checks} the neighborhood, i.e., determines if the colors of the neighbors and itself are permitted for the vertex in question. Other simple operators are also required, such as one that combines the costs and one that compares them. We include edge labels, whose values

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may be involved in the checking functions. We also consider a set of global properties
(for example, that certain sets of the partition induce a connected or an acyclic sub-
graph). In this way, problems like acyclic coloring or connected dominating set can be
modeled as instances of the GLC problem.

Let us give an example. Imagine there is a city where we need to build hospitals,
schools and police stations. Of course, the cost of building a hospital is not the same as
building a school, and the cost of building in some areas may be higher than in others,
so, for each place and each type of building permitted in that place, we have the cost
of raising such building there. Also, some streets might be easier to transit than others
(this information is stored in the edge labels). There are some rules as well, such as “a
hospital needs to be near a police station” (these rules are what the checking function
for each vertex controls, using the edge labels). As a global property, we can request
that hospitals induce a connected subgraph. Finally, we are interested in a feasible
solution of minimum cost.

A key notion to our paper is the \textit{treewidth} of a graph, which was introduced by
Robertson and Seymour \cite{56}. Graphs of treewidth at most $k$ are called \textit{partial $k$-trees}.
Some graph classes with bounded treewidth include: forests (treewidth 1); pseudo-
forests, cacti, outerplanar graphs, and series-parallel graphs (treewidth at most 2);
Halin graphs and Apollonian networks (treewidth at most 3) \cite{7, 10, 50, 56}. In addi-
tion, control flow graphs arising in the compilation of structured programs also have
bounded treewidth (at most 6) \cite{63}.

We give a polynomial-time algorithm for the GLC problem, under mild condi-
tions, for bounded treewidth graphs. Furthermore, by proving that (fixed) powers of
bounded degree and bounded treewidth graphs are also bounded degree and bounded
treewidth graphs, we can enlarge the family of problems (modeled as instances of
the GLC problem) that can be solved in polynomial time for these graph classes, in-
cluding distance coloring problems (packing chromatic number \cite{13, 29, 36, 60}, $L(p, 1)$-
coloring \cite{17, 18, 38, 39}), distance independence \cite{28}, distance domination \cite{42}, and
distance LCVP problems \cite{48}, for bounded distances. We prove that NP-complete prob-
lems can be reduced to the GLC problem in complete graphs. Thus, a generalization
of the polynomiality to bounded clique-width graphs is not possible unless P=NP.

Courcelle’s celebrated theorem (see \cite{22}) states that every graph problem definable in
Monadic Second-Order (MSO) logic can be solved in linear time for bounded treewidth
graphs. However, its main drawback is that the multiplicative constants in the running
time of the algorithm generated with an MSO-formula can be extremely large \cite{32}. In
contrast, the statement of our problem is closer to natural language, the algorithm
for bounded treewidth graphs is based on a rather simple computation of a recursive
function and its time complexity is fully detailed.

The paper is organized as follows. Section 2 contains the necessary preliminaries
and basic definitions. The central notion of the paper, the GLC problem, is formally
introduced in Section 3. In Section 4, we analyze the time complexity of such problem in
complete graphs. In Section 5 we give an algorithm to solve the GLC problem without
global properties for bounded treewidth graphs, and extend it in Section 6 with some
global properties. In Section 7 we give polynomial-time algorithms that solve, for
bounded treewidth graphs, some problems for which no such algorithm was previously
known. Section 7 is also devoted to show how to model some well known problems as
instances of the GLC problem. In Section 8 we analyze the time complexity of the GLC
problem in bounded treewidth and bounded degree graphs and prove that fixed powers
of such graphs are also bounded treewidth and bounded degree graphs. Concluding
remains and future research lines are in Section 9. We include in Appendix A the
definition of the problems mentioned in Section 7.

2. Basic definitions and preliminary results

2.1. Basic definitions on sets and operations. Let $S$ be a set. A closed binary
operation on $S$ is a function $*: S \times S \rightarrow S$. It is usual to write $*(s_1, s_2)$ as $s_1 * s_2$.
Such an operation is commutative if $s_1 * s_2 = s_2 * s_1$ for all $s_1, s_2 \in S$, and it is
associative if $(s_1 * s_2) * s_3 = s_1 * (s_2 * s_3)$ for all $s_1, s_2, s_3 \in S$. An element $e \in S$
is neutral if $e * s = s * e = s$ for all $s \in S$. An element $a \in S$ is absorbing if
$a * s = s * a = a$ for all $s \in S$. It is easy to prove that if $s \in S$ is a neutral
(resp. absorbing) element, then this element is unique. A commutative and associative
operation $*$ can be naturally extended to any nonempty finite subset of $S$, writing
$\star_{x \in X} P(x)$ when $\{P(x) : x \in X\} \subseteq S$ and $X$ is finite and nonempty, moreover, if the
operation also has a neutral element $e$ then we define $\star_{x \in \emptyset} P(x) = e$.

A binary relation $R$ on a set $S$ is a subset of the Cartesian product $S \times S$. It
is usual to write $(s_1, s_2) \in R$ as $s_1 R s_2$. We say that $R$ is reflexive if $s R s$ for all
$s \in S$, antisymmetric if $s_1 R s_2 \land s_2 R s_1 \Rightarrow s_1 = s_2$ for all $s_1, s_2 \in S$, and transitive
if $s_1 R s_2 \land s_2 R s_3 \Rightarrow s_1 R s_3$ for all $s_1, s_2, s_3 \in S$. If $R$ is reflexive, antisymmetric and
transitive, then $(S, R)$ is called a partial order (or partially ordered set). If in addition
$s_1 R s_2 \lor s_2 R s_1$ for every $s_1, s_2 \in S$, then $(S, R)$ is a total order (or totally ordered set).

Let $(S, \preceq)$ be a totally ordered set. A maximum element is an element $m \in S$ such
that $s \preceq m$ for all $s \in S$. Note that not every totally ordered set has a maximum
element, and it is easy to prove that if it does have a maximum element then this
element is unique. The minimum operation, min, is the closed binary operation on $S$
such that $\min(s_1, s_2) = s_1$ if $s_1 \preceq s_2$ and $\min(s_1, s_2) = s_2$ if $s_2 \preceq s_1$. It is easy to prove
that min is commutative and associative.

A set of natural numbers is co-finite if its complement with respect to the set of
natural numbers is finite.

We denote by $[a, b]$, with $a, b \in \mathbb{Z}$ and $a < b$, the set of all integer numbers greater
or equal $a$ and less or equal $b$, that is $\{a, a + 1, \ldots, b\}$.

Given a set $S$ and a function $f : S \rightarrow \mathbb{R}$, the weight of the function $f$ (finite or
infinite) is defined as $\sum_{s \in S} f(s)$.

Throughout this paper we will work with the set $\text{BOOL} = \{\text{True}, \text{False}\}$ of boolean
values and all the usual logical operators, such as $\neg, \land, \lor$ and $\Rightarrow$.

2.2. Basic definitions on automata. A deterministic finite-state automaton is a
five-tuple $(Q, \Sigma, \delta, q_0, F)$ that consists of

- $Q$: a finite set of states,
- $\Sigma$: a finite set of input symbols (often called the alphabet),
- $\delta$: $Q \times \Sigma \rightarrow Q$: a transition function,
- $q_0 \in Q$: an initial or start state, and
- $F \subseteq Q$: a set of final or accepting states.

We say that an automaton $M = (Q, \Sigma, \delta, q_0, F)$ accepts a string $s_1 \ldots s_n$, with $n \geq 1$,
if and only if $s_i \in \Sigma$ for all $1 \leq i \leq n$ and $\delta(... \delta(q_0, s_1) \ldots, s_n) \in F$.

For example, the automaton $M = (\{q_0, q_1\}, \{1\}, \delta, q_0, \{q_1\})$ where $\delta(q_0, 1) = q_1$ and
$\delta(q_1, 1) = q_0$, is the automata that accepts sequences of an odd number of 1s.

For more about automata theory we refer the reader to [46].
2.3. Basic definitions on graphs. Let $G$ be a finite, simple and undirected graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set, respectively, of $G$. For any $W \subseteq V(G)$, we denote by $G[W]$ the subgraph of $G$ induced by $W$. Let $N_G(v)$ (open neighborhood of $v$) be the set of neighbors of $v \in V(G)$ and let $N_G[v] = N_G(v) \cup \{v\}$ (closed neighborhood of $v$). The closed neighborhood of a set $S$ is $N_G[S] = \bigcup_{v \in S} N_G[v]$. The degree of a vertex $v$ is $d_G(v) = |N_G(v)|$. The maximum degree of a vertex in $G$ is denoted by $\Delta(G)$.

A graph class is a collection of graphs that is closed under isomorphism. Given a graph class $\mathcal{G}$, we say that $\mathcal{G}$ is of bounded degree if $\sup \{\Delta(G) \mid G \in \mathcal{G}\} < \infty$.

A graph $G$ is connected if for every pair of vertices $u, v$ in $V(G)$ there exists a path in $G$ from $u$ to $v$. A connected component of a graph is an inclusion-wise maximal connected subgraph of it. For two vertices $x, y$ in a connected graph $G$, we denote by $dist_G(x, y)$ the distance between $x$ and $y$, that is, the length (number of edges) of a shortest $x, y$-path in $G$. The $k$-th power of $G$ is the graph denoted by $G^k$ such that for all distinct vertices $x, y$ in $V(G)$, $x$ is adjacent to $y$ in $G^k$ if and only if $dist_G(x, y) \leq k$.

A complete graph is a graph whose vertices are pairwise adjacent. We denote by $K_r$ the complete graph on $r$ vertices. A clique (resp. stable set or independent set) in a graph is a set of pairwise adjacent (resp. nonadjacent) vertices. The maximum size of a clique (resp. independent set) in the graph $G$ is denoted by $\omega(G)$ (resp. $\alpha(G)$).

A graph $G$ is bipartite if $V(G)$ can be partitioned into two stable sets $V_1$ and $V_2$, and $G$ is complete bipartite if every vertex of $V_1$ is adjacent to every vertex of $V_2$. We denote by $K_{r,s}$ the complete bipartite graph with $|V_1| = r$ and $|V_2| = s$. The star $S_n$ is the complete bipartite graph $K_{1,n-1}$.

A proper $k$-coloring of a graph is a partition of its vertices into at most $k$ stable sets, each of them called color class. Equivalently, a proper $k$-coloring is an assignment of colors to vertices such that adjacent vertices receive different colors, and the number of colors used is at most $k$. The chromatic number $\chi(G)$ of a graph $G$ is the minimum $k$ that allows a proper $k$-coloring of $G$. In the more general LIST-COLORING problem, each vertex $v$ has a list $L(v)$ of available colors for it.

A pair of vertices or a pair of edges dominate each other when they are either equal or adjacent, while a vertex and an edge dominate each other when the vertex belongs to the edge. We will denote by $\gamma_{U,W}(G)$, for $U, W$ sets of elements of $G$, the minimum cardinality or weight of a subset $S$ of $U$ which dominates $W$. The parameter $\gamma_{U,V}$ is also denoted simply by $\gamma$, and the associated problem is known as MINIMUM DOMINATING SET. Parameters $\gamma_{V,E}$ and $\gamma_{E,V}$ are associated with the MINIMUM VERTEX COVER and MINIMUM EDGE COVER, respectively. In the MINIMUM $\{k\}$-DOMINATION problem, given a graph $G$ we want to find the minimum weight of a function $f: V(G) \to \{0, 1, \ldots, k\}$ such that $\sum_{u \in N_G[v]} f(u) \geq k$ for all $v \in V(G)$.

Let $\sigma$ and $\rho$ be finite or co-finite subsets of non-negative integer numbers. A subset $S$ of vertices of a graph $G$ is a sigma-rho set, or simply $(\sigma, \rho)$-set, of $G$ if for every $v$ in $S$, $|N(v) \cap S| \in \sigma$, and for every $v$ in $V(G) \setminus S$, $|N(v) \cap S| \in \rho$. The locally checkable vertex subset problems [61] consist of finding a minimum or maximum $(\sigma, \rho)$-set in an input graph $G$, possibly on vertex weighted graphs. A generalization of these problems asks for a partition of $V(G)$ into $q$ classes, with each class satisfying a certain $(\sigma, \rho)$-property, as follows. A degree constraint matrix $D_q$ is a $q \times q$ matrix with entries being finite or co-finite subsets of non-negative integer numbers. A $D_q$-partition of a graph $G$ is a partition $\{V_1, V_2, \ldots, V_q\}$ of $V(G)$ such that for $1 \leq i, j \leq q$ it holds that for every $v \in V_i$, $|N(v) \cap V_j| \in D_q[i, j]$. A locally checkable vertex partitioning problem [61] consists of deciding if $G$ has a $D_q$ partition. Optimization versions can
be defined, possibly on vertex weighted graphs. The distance-$r$ LCVP problems [48] further generalize LCVP problems by considering, for each vertex $v$, the set of vertices of a subgraph centered at $v$ with radius $r(N^r(v))$ instead of $N(v)$.

For a graph $G$ and $uv \in E(G)$, the graph obtained by subdividing $uv$ in $G$ arises from $G$ by adding a new vertex $w$, making $w$ adjacent to $u$ and $v$, and then deleting the edge $uv$. The subdivision graph $S(G)$ of $G$ is obtained by subdividing each of the edges of $G$.

The line graph of a graph $G$ is denoted by $L(G)$ and has as vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges have a common endpoint (i.e., are adjacent in $G$). The total graph of $G$, denoted by $T(G)$, is defined similarly: its vertex set is $V(G) \cup E(G)$, $V(G)$ induces $G$, $E(G)$ induces $L(G)$, and $v \in V(G)$, $uw \in E(G)$ are adjacent in $T(G)$ if and only if either $v = u$ or $v = w$.

A graph, or a subgraph of a graph, is acyclic if it does not contain a cycle of length at least three. An acyclic graph is called a forest. A tree is a connected acyclic graph. In a tree $T$, we usually call the elements in $V(T)$ nodes. A node with degree at most 1 is called a leaf and a node of degree at least 2 is called an internal node. A tree is called a rooted tree if one vertex has been designated the root, in which case the edges have a natural orientation, towards or away from the root. For a rooted tree $T$ and $u \in V(T)$, the neighbor of $u$ on the path to the root is called the parent of $u$ and a vertex $v$ is a child of $u$ if $u$ is the parent of $v$. A binary tree is a rooted tree where every internal node has at most two children.

2.4. Definitions and preliminary results on treewidth. A tree-decomposition of a graph $G$ is a family $\{X_i : i \in I\}$ of subsets of $V(G)$ (called bags), together with a tree $T$ with $V(T) = I$, satisfying the following properties:

(W1) $\bigcup_{i \in I} X_i = V(G)$.
(W2) Every edge of $G$ has both its ends in $X_i$ for some $i \in I$.
(W3) For all $v \in V(G)$, the set of nodes $\{i \in I : v \in X_i\}$ induces a subtree of $T$.

The width of the tree-decomposition is $\max\{|X_i| - 1 : i \in I\}$. The treewidth of $G$, denoted $tw(G)$, is the minimum $w \geq 0$ such that $G$ has a tree-decomposition of width less or equal $w$.

Given a graph class $\mathcal{G}$, the treewidth of $\mathcal{G}$ is $tw(\mathcal{G}) = \sup\{tw(G) \mid G \in \mathcal{G}\}$. We say that $\mathcal{G}$ is of bounded treewidth if $tw(\mathcal{G}) < \infty$.

We will often make use of the following basic properties of the treewidth, some of which can be easily deduced.

**Proposition 2.4.1.** Let $\mathcal{G}$ be a family of graphs of bounded treewidth. If $G \in \mathcal{G}$ then $|E(G)|$ is $O(|V(G)|)$.

**Proposition 2.4.2.** If $H$ is a subgraph of a graph $G$ then $tw(H) \leq tw(G)$.

**Theorem 2.4.3** ([57, pages 1 and 2]). For every graph $G$, $tw(G) \geq \omega(G) - 1$. Moreover, $tw(G) \geq \chi(G) - 1$.

**Theorem 2.4.4** ([50, page 76]). The treewidth of $S(G)$ is equal to the treewidth of $G$.

A tree-decomposition $(T, \{X_i\}_{i \in V(T)})$ is nice [50, Definition 13.1.4] if

- $T$ is a rooted binary tree;
- if a node $i$ has two children $j$ and $k$ then $X_i = X_j = X_k$; (join node)
- if a node $i$ has one child $j$, then either
  - $|X_i| = |X_j| - 1$ and $X_i \subset X_j$, or (forget node)
  - $|X_i| = |X_j| + 1$ and $X_i \supset X_j$. (introduce node)
Let $T_i$ be the subtree of $T$ rooted at node $i$. We will denote by $G_i$ the subgraph of $G$ induced by $\bigcup_{j \in V(T_i)} X_j$.

**Theorem 2.4.5** ([8, Theorem 1]). There exists an algorithm, that given an $n$-vertex graph $G$ and an integer $k$, in time $O(c^k n)$ for some $c \in \mathbb{N}$, either outputs that the treewidth of $G$ is larger than $k$, or constructs a tree-decomposition of $G$ of width at most $5k + 4$.

**Theorem 2.4.6** ([50, Lemma 13.1.3]). For constant $k$, given a tree-decomposition of a graph $G$ of width $k$ and $O(n)$ nodes, where $n$ is the number of vertices of $G$, one can find a nice tree-decomposition of $G$ of width $k$ and with at most $4n$ nodes in $O(n)$ time.

However, we will work with a slight modification of nice tree decompositions, where the bags of the root and leaves have only one vertex each.

**Definition 2.4.7.** A tree-decomposition $(T, \{X_i\}_{i \in V(T)})$ is called easy if

- $T$ is a binary tree rooted at $r$ such that $|X_r| = 1$;
- if a node $i$ has two children $j$ and $k$ then $X_i = X_j = X_k$; (join node)
- if a node $i$ has one child $j$, then either
  - $|X_i| = |X_j| - 1$ and $X_i \subset X_j$, or (forget node)
  - $|X_i| = |X_j| + 1$ and $X_i \supset X_j$; (introduce node)
- if a node $i$ has no children, then $|X_i| = 1$. (leaf node)

It is straightforward to prove that, given a nice tree-decomposition $(T, \{X_i\}_{i \in V(T)})$ of width $k$ and $O(n)$ nodes, one can construct in $O(kn)$ time an easy tree-decomposition of width $k$ and $O(kn)$ nodes.

3. The Generalized Locally Checkable Problem

Let $G$ be a simple undirected graph such that each edge $e \in E(G)$ has a label $\ell_e$. Then suppose we have the following:

- for every vertex $v \in V(G)$, a nonempty set (also called list) $L_v$ of possible colors for $v$;
- a totally ordered set $\text{Weights}$ with a maximum element $\text{ERROR}$, together with the minimum operation of the order $\preceq$ (called min) and a closed binary operation on $\text{Weights}$ (called $\oplus$) that is commutative and associative, has a neutral element (called $e_\oplus$) and an absorbing element that is equal to $\text{ERROR}$, and is such that $s_1 \preceq s_2 \Rightarrow s_1 \oplus s_3 \preceq s_2 \oplus s_3$ for all $s_1, s_2, s_3 \in \text{Weights}$;
- for every vertex $v \in V(G)$ and for every color $i \in L_v$, a weight (or cost) $w_{v,i} \in \text{Weights} - \{\text{ERROR}\}$ of assigning color $i$ to vertex $v$;
- a function $\text{check}$ that, given a vertex $v \in V(G)$ and given a color assignment $c : N_G[v] \to \bigcup_{u \in N_G[v]} L_u$ such that $c(u) \in L_u$ for all $u \in N[v]$, returns $\text{TRUE}$ if the vertex $v$ together with its neighborhood (considering the labels of the edges $uv$ with $u \in N_G(v)$) satisfies a certain condition, and $\text{FALSE}$ otherwise; and
- a set of global properties $\Pi$.

We say that an assignment of colors to vertices $c$ is valid in $V$ if $c(v) \in L_v$ for all $v \in V$, and it is proper if it is valid in $V(G)$ and $\text{check}(v, c_v)$ is true for every $v \in V(G)$, where $c_v$ is the function $c$ restricted to the domain $N_G[v]$. The weight of a color assignment $c : V \to \bigcup_{v \in V} L_v$ valid in $V$ is $w(c) = \bigoplus_{v \in V} w_{v,c(v)}$.

The main object of study in this paper is the following problem:
Generalized locally checkable (GLC) problem

Instance: A simple undirected graph $G$, $\ell_e$ for all $e \in E(G)$, $L_v$ for all $v \in V(G)$, 
(Weights, $\preceq$) together with $\oplus$, $w_{v,i}$ for all $v \in V(G)$ and all $i \in L_v$, $check$ and $\Pi$.

Question: What is the minimum weight (according to the order $\preceq$) of a proper assignment of colors to vertices that satisfies the properties in $\Pi$?

Many different optimization problems can be modeled as instances of the GLC problem. Some decision problems can be modeled as well, using weights only to determine if the answer is “yes” or “no” (if the output is ERROR then the answer is “no”, otherwise is “yes”). For the examples shown throughout this paper, we will assume that, otherwise stated, the definitions of $L_v$ are for all $v \in V(G)$, of $w_{v,i}$ for all $v \in V(G)$ and all $i \in L_v$, and of $check(v, c)$ for all $v \in V(G)$ and all color assignments $c$ valid in $N_G[v]$. Also, if the labels $\ell_e$ are not specified, we can assume they are all equal to 1.

For example, for list-coloring we can set:
- $L_v$ is part of the input;
- (Weights, $\preceq$) = ($\mathbb{R} \cup \{+\infty\}$, $\leq$) and $\oplus = +$;
- $w_{v,i} = 0$ (notice that in this case we are not interested in the minimum weight but only in the existence of such coloring);
- $check(v, c) = (c(v) \neq c(u) \forall u \in N_G(v))$; and
- $\Pi = \emptyset$.

We can also model the problem of deciding if $G$ has a $D_q$ partition as an instance of the GLC problem in the following way:
- $L_v = \lfloor 1, q \rfloor$;
- (Weights, $\preceq$) = ($\mathbb{R} \cup \{+\infty\}$, $\leq$) and $\oplus = +$;
- $w_{v,i} = 0$;
- $check(v, c) = \bigwedge_{1 \leq j \leq q} (|\{u : u \in N(v) \land c(u) = j\}| \in D_q[c(v), j])$; and
- $\Pi = \emptyset$.

Therefore, the GLC problem is indeed a generalization of LCVP problems. Furthermore, it allows us to model more problems, like $\{k\}$-domination, for which we can set:
- $L_v = \lfloor 1, k \rfloor$;
- (Weights, $\preceq$) = ($\mathbb{R} \cup \{+\infty\}$, $\leq$) and $\oplus = +$;
- $w_{v,i} = i$;
- $check(v, c) = \left( k \leq \sum_{u \in N_G[v]} c(u) \right)$; and
- $\Pi = \emptyset$.

We give more examples in Section 7.

Notice that we can model problems where the input graph is directed by considering edge labels that also carry the direction of the edge, and the $check$ function can use it to distinguish in-neighbors and out-neighbors.

4. The GLC problem in complete graphs

It is easy to see that we can polynomially reduce NP-complete problems in graphs to the GLC problem in a complete graph, even when restricting the sets of colors and edge labels to $\{0, 1\}$. Indeed, we can transform the classical domination problem in a graph $G$ to the GLC problem in a complete graph in the following way. We construct
a complete graph $G'$ such that $V(G') = V(G)$ and $\ell_{uv} = 1$ if $uv \in E(G)$ and $\ell_{uv} = 0$ otherwise. Then, for the GLC problem we set

\begin{itemize}
  \item $L_v = \{0, 1\};$
  \item $(\text{WEIGHTS}, \leq) = (\mathbb{R} \cup \{+\infty\}, \leq)$ and $\oplus = +;$
  \item $w_{u,i} = i;$
  \item $\text{check}(v, c) = \left( c(v) + \sum_{u \in N_{G'}(v)} (c(u) \cdot \ell_{uv}) \geq 1 \right);$ and
  \item $\Pi = \emptyset.$
\end{itemize}

It is clear that the minimum weight of a proper coloring in this instance equals $\gamma(G)$, and this transformation can be performed in polynomial time.

We can even find a polynomial-time reduction from the domination problem in a graph $G$ to the GLC problem in a complete graph $G'$ such that $V(G') = V(G)$ and where $\ell_e = 1$ for all $e \in E(G')$. Indeed, we can set

\begin{itemize}
  \item $L_v = \{0, N_G[v]\};$
  \item $(\text{WEIGHTS}, \leq) = (\mathbb{R} \cup \{+\infty\}, \leq)$ and $\oplus = +;$
  \item $w_{v,\emptyset} = 0$ and $w_{v,N_G[v]} = 1;$
  \item $\text{check}(v, c) = \left( v \in \bigcup_{u \in N_{G'}[v]} c(u) \right);$ and
  \item $\Pi = \emptyset.$
\end{itemize}

However, the GLC problem in a complete graph $G'$ is polynomial-time solvable when:

\begin{itemize}
  \item $\ell_e = 1$ for all $e \in E(G');$
  \item the number of all possible colors (that is, $|\bigcup_{v \in V(G')} L_v|$) is bounded by a constant;
  \item $\text{check}(v, c)$ can be computed in polynomial time and only depends on $v$, $c(v)$ and the number of neighbors of each color that $v$ has (informally, $\text{check}(v, c)$ “does not care about” the names of the neighbors of $v$); and
  \item $\Pi = \emptyset.$
\end{itemize}

Let $\text{COLORS} = \bigcup_{v \in V(G')} L_v$, $\mathcal{C} = |\text{COLORS}|$ and assume $\text{COLORS} = \{c_1, \ldots, c_{\mathcal{C}}\}$. Notice that since $G'$ is a complete graph then $N_{G'}[v] = V(G')$ for all $v \in V(G')$. Therefore, by the restrictions imposed to $\text{check}$, we can assume there exists a function $\text{check}'$ such that $\text{check}'(v, c(v), (k_1, \ldots, k_{\mathcal{C}})) = \text{check}(v, c)$, where $k_i = |\{ u \in V(G') : c(u) = c_i \}|$ for all $i \in [1, \mathcal{C}]$.

For every distribution of colors $(k_1, \ldots, k_{\mathcal{C}})$, with $k_i \in \mathbb{N}_0$ for all $i \in [1, \mathcal{C}]$ and such that $\sum_{i=1}^{\mathcal{C}} k_i = |V(G')|$, we need to verify if it can actually be achieved (that is, there exists a proper color assignment such that $k_i = |\{ u \in V(G') : c(u) = c_i \}|$ for all $i \in [1, \mathcal{C}]$), and if so, find one such proper assignment of colors to vertices of minimum weight. To this end, we construct a directed capacitated network $F$ with vertices $i$ for all $i \in \text{COLORS}$, $(v, i)$ for all $v \in V(G')$ and all $i \in L_v$, $v$ for all $v \in V(G')$, and $s$ and $t$. There is a directed edge of capacity $k_{\sigma(i)}$ and cost 0 from $s$ to $i$ for all $i \in \text{COLORS}$. There is a directed edge of capacity 1 and cost 0 from $i$ to $(v, i)$ if $k_i \geq 1$ and $\text{check}'(v, i, (k_1, \ldots, k_{\mathcal{C}}))$ is true. There is a directed edge of capacity 1 and cost $w_{v,i}$ from $(v, i)$ to $v$ for all $v \in V(G')$, $i \in L_v$. There is a directed edge of capacity 1 and cost 0 from $v$ to $t$ for all $v \in V(G')$. There are no more edges than these ones. Note that the distribution $(k_1, \ldots, k_{\mathcal{C}})$ is achievable if and only if the maximum flow in $F$ is $|V(G')|$, and in this case the proper assignment of colors to vertices of minimum weight corresponds to the maximum flow in $F$ of minimum cost.

Finally, the answer to the problem is obtained by finding the minimum proper color assignment among the ones found for all achievable distributions of colors.
Since \( C \) is bounded by a constant, the number of distributions of colors is polynomial in \(|V(G')|\) (because it is \( \left( |V(G')|^{c-1} \right) < (|V(G')| + 1)^{c-1} \)), \( \text{check}'(v, i, (k_1, \ldots, k_C)) \) can be computed in polynomial time, constructing \( F \) takes polynomial time, and the problem of finding the maximum flow of minimum cost is polynomial-time solvable, then this instance of the GLC problem on a complete graph is polynomial-time solvable.

5. The GLC Problem with \( \Pi = \emptyset \) in Bounded Treewidth Graphs

We first give a polynomial-time algorithm to solve the GLC problem with \( \Pi = \emptyset \) for bounded treewidth graphs. In Section 6 we will explain how to modify this algorithm in order to add some global properties.

Assume we are given an easy tree decomposition \((T, \{X_t\}_{t \in V(T)})\) of the input graph \( G \). We will solve the problem in a dynamic programming fashion and describe the algorithm by explaining what happens in each type of node of the tree decomposition. For every node \( t \) of \( T \), the algorithm computes a function \( \lambda_t \) that receives a tuple \((S, c, \omega, \eta, \bar{c})\) and returns an element of \( \text{WEIGHTS} \), where

- \( S \subseteq X_t \),
- \( c \) is a color assignment valid in \( X_t \),
- \( \omega: X_t \rightarrow \text{WEIGHTS} \),
- \( \eta \) is a valid partial neighborhood mapping for \( c \) (to be defined in the following subsection), and
- \( \bar{c} \) is a function that given a vertex \( v \in X_t \) returns a checking function for \((v, c(v))\) (to be defined in the following subsection).

Throughout the following subsections, we will note \( \bar{c}_v \) instead of \( \bar{c}(v) \) to make the notation less cumbersome.

5.1. Partial neighborhoods. In order to define the parameters \( \bar{c} \) and \( \eta \), we first need the following definitions and notation.

Definition 5.1.1. A partial neighborhood system for an instance of the GLC problem consists of:

- A set \( \mathcal{N}_{v,i} \), for every \( v \in V(G) \) and \( i \in L_v \), together with a closed binary operation \( \boxplus_{v,i} \) on \( \mathcal{N}_{v,i} \) that is commutative and associative and has a neutral element \( e_{v,i} \).
- A function \( \text{new}N_{v,i} \), for every \( v \in V(G) \) and \( i \in L_v \), that given \( u \in N_G(v) \) and \( j \in L_u \) returns an element of \( \mathcal{N}_{v,i} \) (possibly making use of the label of the edge \( vu \)).
- A function \( \text{check}_{v,i}: \mathcal{N}_{v,i} \rightarrow \text{BOOL} \), for every \( v \in V(G) \) and \( i \in L_v \). This function must satisfy \( \text{check}_{v,i}(v, c) \big( \bigoplus_{u \in N_G(v)} \text{new}N(u, c(u)) \big) = \text{check}(v, c) \) for every vertex \( v \in V(G) \) and every color assignment \( c \) valid in \( N_G[v] \).

Roughly speaking, a partial neighborhood system gives us tools to accumulate information from the neighbors of a vertex \( v \), for every vertex \( v \) and every color \( i \in L_v \). With \( \text{new}N_{v,i}(u, j) \) we create new information, that says how \( u \) having color \( j \) affects \( v \) when having color \( i \). The operation \( \boxplus_{v,i} \) combines two pieces of information. For a color assignment \( c \) valid in \( N_G[v] \), \( \text{check}_{v,i}(v, c) \big( \bigoplus_{u \in N_G(v)} \text{new}N(u, c(u)) \big) \) simply verifies a condition over all the information collected from the neighbors of \( v \). Finally, we require the equality \( \text{check}_{v,i}(v, c) \big( \bigoplus_{u \in N_G(v)} \text{new}N(u, c(u)) \big) = \text{check}(v, c) \) to make these tools analogous to the use of \( \text{check}(v, c) \). We refer to the elements of \( \mathcal{N}_{v,i} \) as partial neighborhoods of vertex \( v \) with color \( i \).
Remark 5.1.2. For every instance of the GLC problem there exists a partial neighborhood system. Indeed, we will show one. The idea behind the following partial neighborhood system is to store all the colors assigned to the neighbors of \( v \), where \( \perp \) represents that a neighbor has not yet been assigned a color, and \( \times \) can be thought as an error sign.

Let \( v \in V(G) \), \( i \in L_v \) and assume \( N_G(v) = \{u_1, \ldots, u_{d_G(v)}\} \). Let \( \mathcal{N}_{v,i} \) be the set of all \( d_G(v) \)-tuples \( x \) such that \( x_h \in L_{u_h} \cup \{\perp, \times\} \) for all \( h \in [1, d_G(v)] \). Let \( \bigoplus^{v,i} \) be such that

\[
(n \bigoplus^{v,i} n')_h = \begin{cases} 
  n_h & \text{if } n_h = n_h' \text{ or } n_h' = \perp \\
  n_h' & \text{if } n_h = \perp \text{ and } n_h' \neq \perp \\
  \times & \text{otherwise}
\end{cases}
\]

for all \( h \in [1, d_G(v)] \). Let \( \text{new}N_{v,i}(u_h, j) \) be the \( d_G(v) \)-tuple that has \( j \) in its position \( h \) and \( \perp \) in all its other positions. Let \( x \in \mathcal{N}_{v,i} \). If \( x = (j_1, \ldots, j_{d_G(v)}) \) with \( j_h \in L_{u_h} \) for all \( h \in [1, d_G(v)] \), let \( c \) be the color assignment in \( N_G(v) \) such that \( c(v) = i \) and \( c(u_h) = j_h \) for all \( h \in [1, d_G(v)] \), and then define \( \text{check}_{v,i}(x) = \text{check}(v, c) \). Otherwise, let \( \text{check}_{v,i}(x) = \text{FALSE} \).

Of course, finding partial neighborhood systems that have smaller sets \( \mathcal{N}_{v,i} \) is of extreme importance because it reduces the time complexity of the algorithm given in Section 5.3. For a large number of problems, we can use a “bounded sum”, that is, \( \mathcal{N}_{v,i} = [0, k_{v,i}] \) for some \( k_{v,i} \in \mathbb{N} \) and \( n \bigoplus^{v,i} n' = \min(n + n', k_{v,i}) \). For example, for the \( \{k\} \)-domination problem we can set \( \mathcal{N}_{v,i} = [0, k] \), \( \text{new}N_{v,i}(u, j) = j \), \( n \bigoplus^{v,i} n' = \min(n + n', k) \) and \( \text{check}_{v,i}(n) = (n + i \geq k) \) for all \( v \in V(G) \), \( i \in L_v \) and \( n, n' \in \mathcal{N}_{v,i} \).

Another key concept is the following. Let \( X \subseteq V(G) \) and \( c \) be a color assignment valid in \( X \). Given a partial neighborhood system, a valid partial neighborhood mapping for \( c \) is a function \( \eta \) of domain \( X \) such that \( \eta(v) \in \mathcal{N}_{v,c(v)} \) for all \( v \in X \).

Finally, given \( v \in V(G) \) and \( i \in L_v \), any function \( f : \mathcal{N}_{v,i} \rightarrow \text{BOOL} \) is called a checking function for \((v, i)\).

5.2. Notation and definitions. The following definitions and notation will be useful throughout the rest of the article.

- **Extension of a function.** Let \( f : X \rightarrow Y \) and \( x \notin X \). Then the function \( f^x \bullet : Y \cup \{x\} \rightarrow Y \cup \{y\} \) is such that \( f^x \bullet(y) = y \) and \( f^x \bullet(z) = f(z) \) for all \( z \in X \).
- **Restriction of a function.** Let \( f : X \rightarrow Y \) and \( x \in X \). Then the function \( f^{-x} : X - \{x\} \rightarrow Y \) is such that \( f^{-x}(z) = f(z) \) for all \( z \in X - \{x\} \).
- **Graph with removed edges.** Let \( H \) be a graph. Then \( H^{-S} \) is the graph such that \( V(H^{-S}) = V(H) \) and \( E(H^{-S}) = E(H) - \{uv : u, v \in S\} \).
- **Neutral weight mapping.** Let \( X \subseteq V(G) \). Then the function \( \omega_X : X \rightarrow \text{WEIGHTS} \) is such that \( \omega_X(v) = c_{\emptyset} \) for all \( v \in X \).
- **Equality checking function.** For every \( v \in V(G) \), every \( i \in L_v \) and every \( n \in \mathcal{N}_{v,i} \), let \( eq_n : \mathcal{N}_{v,i} \rightarrow \text{BOOL} \) be the function such that \( eq_n(n') = (n = n') \) for all \( n' \in \mathcal{N}_{v,i} \).
- **Function of equality checking functions.** Let \( X \subseteq V(G) \), \( c \) be a color assignment valid in \( X \) and \( \eta \) be a valid partial neighborhood mapping for \( c \). Then let \( \check{\eta} \) be the function of domain \( X \) such that \( \check{\eta} \) is \( \eta_{\eta(v)} \) for all \( v \in X \).
- **Reduction of a partial neighborhood mapping.** Let \( X \subseteq V(G) \), \( c \) be a color assignment valid in \( X \), \( \eta \) be a valid partial neighborhood mapping for \( c \) and \( v \in X \). Then the function \( \eta^{-v} \) of domain \( X - \{v\} \) is such that
then for their parents, and so on) by computing the minimum among all tuples \((S, c, \omega, \eta, \check{c})\) if \(u \in X \cap N_G(v)\) and \(\eta^v = \eta(u)\) otherwise.

- **Neutral partial neighborhood mapping.** Let \(X \subseteq V(G)\) and \(c\) be a color assignment valid in \(X\). Then the function \(\eta^c\) of domain \(X\) is such that \(\eta^c(v) = e_{v,c}(v)\) for all \(v \in X\). Observe that \(\eta^c\) is a valid partial neighborhood mapping for \(c\).

- **Partial neighborhood in a subgraph.** Let \(H\) be a subgraph of \(G\) and \(c\) be a color assignment valid in \(V(H)\). Then we define \(\text{NS}\) such that \(\text{NS}(v, c, H) = \bigoplus_{u \in N_H(v)} \text{newN}_{v,c}(u, c(u))\) for all \(v \in V(H)\). Roughly speaking, \(\text{NS}(v, c, H)\) is the information we can obtain from the neighbors of \(v\) in \(H\) and the color assignment \(c\).

5.3. **Algorithm.** Now we will merely describe the algorithm, whose proof and time complexity are given in the following subsections.

Let \(G\) be the input graph and let \((T, \{X_t\}_{t \in V(T)})\) be an easy tree decomposition of \(G\). Assume we are given a partial neighborhood system.

For every node \(t\), we define a function \(\lambda_t : D_t \rightarrow \text{WEIGHTS}\), where \(D_t\) is the set of all tuples \((S, c, \omega, \eta, \check{c})\) such that

- \(S \subseteq X_t\),
- \(c\) is a color assignment valid in \(X_t\),
- \(\omega : X_t \rightarrow \text{WEIGHTS}\) is such that \(\omega(v) \in \{e_{v,i}, w_{v,i}\}\) for all \(v \in X_t\),
- \(\eta\) is a valid partial neighborhood mapping for \(c\), and
- \(\check{c}\) is such that \(\check{c}_v \in \{\text{check}_{v,c}(\check{c})\} \cup \{eq_n : n \in N_{v,c}(v)\}\) for all \(v \in X_t\),

and its definition is as follows.

- **Leaf node.** Suppose \(t\) is a leaf node and \(X_t = \{v\}\). Then
  \[
  \lambda_t(S, c, \omega, \eta, \check{c}) = \begin{cases} 
  \omega(v) & \text{if } \check{c}_v(\eta(v)) \\ 
  \text{ERROR} & \text{otherwise}
  \end{cases}
  \]

- **Forget node.** Suppose \(t\) is a forget node, \(s\) is the child of \(t\) and \(X_s - X_t = \{v\}\). Then
  \[
  \lambda_t(S, c, \omega, \eta, \check{c}) = \min_{i \in L_v} \lambda_s(S, c, e_{v,i}, \omega \leftrightarrow w_{v,i}, \eta \leftrightarrow e_{v,i}, \check{c} \leftrightarrow \text{check}_{v,c}(\check{c})).
  \]

- **Introduce node.** Suppose \(t\) is an introduce node, \(s\) is the child of \(t\) and \(X_t - X_s = \{v\}\). Let \(n_v = \eta(v) \bigoplus_{v,c} \text{NS}(v, c, G_t^{-S}[X_t])\). Then
  \[
  \lambda_t(S, c, \omega, \eta, \check{c}) = \begin{cases} 
  \omega(v) \oplus \lambda_s(S - \{v\}, c, e_{v,i}, \omega \leftrightarrow w_{v,i}, \eta \leftrightarrow e_{v,i}, \check{c} \leftrightarrow \text{check}_{v,c}(\check{c})) & \text{if } \check{c}_v(n_v) \\ 
  \text{ERROR} & \text{otherwise}
  \end{cases}
  \]

- **Join node.** Suppose \(t\) is a join node and \(r, s\) are the children of \(t\).

  We say that a pair \((\eta_r, \eta_s)\) of valid partial neighborhood mappings for \(c\) is **good** if \(\check{c}_v(\eta(v) \bigoplus_{v,c} \text{NS}(v, c, G_t^{-S}[X_t]) \bigoplus_{v,c} \eta_r(v) \bigoplus_{v,c} \eta_s(v))\) is true for all \(v \in X_t\). Let \(W = \bigoplus_{v \in X_t} \omega(v)\). Then
  \[
  \lambda_t(S, c, \omega, \eta, \check{c}) = \min_{(\eta_r, \eta_s) \text{ is good}} \{W \oplus \lambda_r(X_r, c, \omega_{X_r}, \eta^r_r, \check{c}^r_r) \oplus \lambda_s(X_s, c, \omega_{X_s}, \eta^s_r, \check{c}^s_r)\}.
  \]

The algorithm is executed in a bottom-up fashion (that is, first for all the leaf nodes, then for their parents, and so on) by computing \(\lambda_t(S, c, \omega, \eta, \check{c}) \in \text{WEIGHTS}\) for every node \(t\) and every \((S, c, \omega, \eta, \check{c}) \in D_t\). Finally, the result is obtained by finding the minimum among all \(\lambda_r(\emptyset, c, \omega, \eta^c_r, \check{c})\) such that \(r\) is the root of \(T\), \(c\) is a color assignment valid in \(X_r\), \(\omega : X_r \rightarrow \text{WEIGHTS}\) is such that \(\omega(v) = w_{v,c}(v)\) for all \(v \in X_r\), and \(\check{c}_v = \text{check}_{v,c}(\check{c})\) for all \(v \in X_r\).
5.4. Correctness. We will prove that the above algorithm is correct.

Consider an instance of the GLC problem with a partial neighborhood system, whose input graph is $G$.

**Definition 5.4.1.** For every $X \subseteq V(G)$, every $G'$ subgraph of $G$ such that $X \subseteq V(G')$ and $N_G[V(G')-X] \subseteq V(G')$, every color assignment $c$ valid in $X$, every $\eta$ that is a valid partial neighborhood mapping for $c$, and every $\bar{c}$ such that $\bar{c}_v \in \{\text{check}_{v,c(v)}\} \cup \{eq_n : n \in N_{c,c(v)}\}$ for all $v \in X$, we say that a function $f$ is a $(X, c, \eta, \bar{c})$-coloring in $G'$ if

- $f$ is a color assignment valid in $V(G')$,
- $f(v) = c(v)$ for all $v \in X$,
- $\bar{c}_v(\eta(v) \oplus w_f(v) \text{NS}(v, f, G')) = \text{TRUE}$ for all $v \in X$, and
- $\text{check}(u, f_u) = \text{TRUE}$ for all $u \in V(G')-X$, where $f_u$ is the restriction of $f$ to $N_G[u]$.

For a $(X, c, \eta, \bar{c})$-coloring $f$ in $G'$ and a function $\omega : X \rightarrow \text{WEIGHTS}$, we define the weight under $\omega$ of $f$ as $w_\omega(f) = (\bigoplus_{v \in X} \omega(v)) \oplus \left(\bigoplus_{v \in V(G')-X} w_v(f(v))\right)$.

**Lemma 5.4.2.** Let $G$ be a graph with an easy tree decomposition $(T, \{X_t\}_{t \in V(T)})$. Let $t \in V(T)$. For every $(S, c, \omega, \eta, \bar{c}) \in D_t$ we have

$$\lambda_t(S, c, \omega, \eta, \bar{c}) = \min \{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \bar{c})\text{-coloring in } G_t^{-S}\}.$$

**Proof.** First of all, notice that if there are no $(X_t, c, \eta, \bar{c})$-colorings in $G_t^{-S}$ then we have $\min \{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \bar{c})\text{-coloring in } G_t^{-S}\} = \text{ERROR}$.

We will proceed by induction on $t$. Let $(S, c, \omega, \eta, \bar{c}) \in D_t$. The base case is when $t$ is a leaf node.

- **Leaf node.** Suppose $t$ is a leaf node and $X_t = \{v\}$. Notice that in this case $V(G_t^{-S}) = \{v\} = X_t$.

If $\bar{c}_v(\eta(v)) = \text{FALSE}$ then $\lambda_t(S, c, \omega, \eta, \bar{c}) = \text{ERROR}$. Moreover, by definition, there are no $(X_t, c, \eta, \bar{c})$-colorings in $G_t^{-S}$, leading to the desired equality.

If $\bar{c}_v(\eta(v)) = \text{TRUE}$ then, by definition, $c$ is the only possible $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$. Therefore,

$$\min \{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \bar{c})\text{-coloring in } G_t^{-S}\} = w_\omega(c) = \omega(v) = \lambda_t(S, c, \omega, \eta, \bar{c}).$$

Now assume that $t$ is not a leaf node and that the statement is true for every child $s$ of $t$ in $T$. We will analyze the three remaining cases for $t$.

- **Forget node.** Suppose $t$ is a forget node, $s$ is the child of $t$ and $X_s - X_t = \{v\}$. Notice that $v \notin S$ and also $G_t^{-S} = G_s^{-S}$.

By definition of $\lambda_t$ and the induction hypothesis we have

$$\lambda_t(S, c, \omega, \eta, \bar{c}) = \min_{i \in L_v} \{\lambda_s(S, c, \omega^{w_v \rightarrow w_{v,i}}, \eta^{v \rightarrow w_{v,i}}, \bar{c}^{v \rightarrow \text{check}_{v,i}})\}$$

$$= \min_{i \in L_v} \{\min \{w_\omega(f) : f \text{ is a } (X_s, c, \omega^{w_v \rightarrow w_{v,i}}, \eta^{v \rightarrow w_{v,i}}, \bar{c}^{v \rightarrow \text{check}_{v,i}})\text{-coloring in } G_s^{-S}\}\}$$

$$= \min \{w_\omega(f) : i \in L_v \text{ and } f \text{ is a } (X_s, c, \omega^{w_v \rightarrow w_{v,i}}, \eta^{v \rightarrow w_{v,i}}, \bar{c}^{v \rightarrow \text{check}_{v,i}})\text{-coloring in } G_s^{-S}\}.$$

Let $i \in L_v$. We claim that every $f$ that is a $(X_s, c, \omega^{w_v \rightarrow w_{v,i}}, \eta^{v \rightarrow w_{v,i}}, \bar{c}^{v \rightarrow \text{check}_{v,i}})$-coloring in $G_s^{-S}$ is also a $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$. Conversely, every $f$ that is a $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$ is also a $(X_s, c, \omega^{w_v \rightarrow w_{v,i}}, \eta^{v \rightarrow w_{v,i}}, \bar{c}^{v \rightarrow \text{check}_{v,i}})$-coloring in $G_s^{-S}$. We prove the first claim (the second one is similar) by showing each of the items of the definition of $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$ holds:
\begin{itemize}
  \item $f(w) = c(w)$ for all $w \in X_t$ (because it is true for all $w \in X_s$),
  \item $f$ is a color assignment valid in $V(G_s^{-S}) = V(G_t^{-S})$,
  \item $\text{check}(u, f) = \text{TRUE}$ for all $u \in V(G_s^{-S}) - X_s = V(G_t^{-S}) - X_t - \{v\}$ and $\text{check}(v, f) = \text{check}_{v,i}(\text{NS}(v, f, G_s^{-S}))$.
\end{itemize}

\begin{align*}
  &= \text{check}_{v,i}(\epsilon_{v,i} \nsim v, i \nsim \text{NS}(v, f, G_s^{-S})) \\
  &= \epsilon_v^{v \rightarrow \text{check}_{v,i}}(\eta(v) \nsim v, i \nsim \text{NS}(v, f, G_s^{-S})) \\
  &= \text{TRUE}
\end{align*}

(because $f(v) = i$).

\begin{itemize}
  \item $\bar{c}_w(\eta(w) \nsim w, f(w) \nsim \text{NS}(w, f, G_t^{-S})) = \text{TRUE}$ for all $w \in X_t$ (because it is true for all $w \in X_s$ and $E(G_t^{-S}) = E(G_s^{-S})$).
\end{itemize}

Clearly, $W_{\omega}^{v \rightarrow w, i(v)}(f) = w_{\omega}(f)$ for every $f$ that is a $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$.

Therefore
\begin{align*}
  \min\{W_{\omega}^{v \rightarrow w, i(v)} : i \in L_v \land f \text{ is a } (X_s, c^v, \eta^v, \bar{c}^v)\text{-coloring in } G_s^{-S}\} \\
  = \min\{W_{\omega}(f) : f \text{ is a } (X_t, c, \eta, \bar{c})\text{-coloring in } G_t^{-S}\}
\end{align*}
as we wanted.

\begin{itemize}
  \item **Introduce node.** Suppose $t$ is an introduce node, $s$ is the child of $t$ and $X_t - X_s = \{v\}$. First of all, observe that $v \notin V(G_s^{-S})$ and $N_{G_t^{-S}}[v] \subseteq X_t$ (because $(T, \{X_t\}_{t \in V(T)})$ is a tree-decomposition), and that this implies that $\text{NS}(v, c, G_t^{-S}) = \text{NS}(v, c, G_t^{-S}[X_t])$.

If $\bar{c}_v(\eta(v) \nsim v, c, G_t^{-S}[X_t]) = \text{FALSE}$ then there are no $(X_t, c, \eta, \bar{c})$-colorings in $G_t^{-S}$ (by Definition 5.4.1) and we also have $\lambda_t(S, c, \omega, \eta, \bar{c}) = \text{ERROR}$ (by definition of $\lambda_t$).

Now assume that $\bar{c}_v(\eta(v) \nsim v, c, G_t^{-S}[X_t]) = \text{TRUE}$. It is straightforward to prove that if a function $f$ is a $(X_s, c^v, \eta^v, \bar{c}^v)$-coloring in $G_s^{-S}$ then the function $f^{v \rightarrow i(v)}$ is a $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$ and $W_{\omega}(f^{v \rightarrow i(v)}) = \omega(v) \oplus W_{\omega}^{v \rightarrow i(v)}$. Furthermore, it is also straightforward to prove that for every $(X_t, c, \eta, \bar{c})$-coloring $g$ in $G_t^{-S}$, the function $g^{-v}$ is a $(X_s, c^v, \eta^v, \bar{c}^v)$-coloring in $G_s^{-S}$ and $\omega(v) \oplus W_{\omega}^{v \rightarrow g^{-v}} = W_{\omega}(g)$. Therefore
\begin{align*}
  \min\{W_{\omega}(g) : g \text{ is a } (X_t, c, \eta, \bar{c})\text{-coloring in } G_t^{-S}\} \\
  = \min\{\omega(v) \oplus W_{\omega}^{-v}(f) : f \text{ is a } (X_s, c^v, \eta^v, \bar{c}^v)\text{-coloring in } G_s^{-S}\} \\
  = \omega(v) \oplus \min\{W_{\omega}^{-v}(f) : f \text{ is a } (X_s, c^v, \eta^v, \bar{c}^v)\text{-coloring in } G_s^{-S}\} \\
  = \omega(v) \oplus \lambda_t(S - \{v\}, c^{-v}, \omega^{-v}, \eta^{-v}, \bar{c}^{-v}) \\
  = \lambda_t(S, c, \omega, \eta, \bar{c})
\end{align*}

\begin{itemize}
  \item **Join node.** Suppose $t$ is a join node and $r, s$ are the children of $t$. Recall that $X_t = X_r = X_s$.

For every color assignment $f$ valid in $V(G_t^{-S})$, denote by $f|_r$ (resp. $f|_s$) the restriction of $f$ to $V(G_r^{-X_r})$ (resp. $V(G_s^{-X_s})$). Let $W = \bigoplus_{v \in X_t} \omega(v)$. Notice that $W_{\omega}(f) = W \oplus w_{\omega}^{-X_t}(f|_r) \oplus w_{\omega}^{-X_s}(f|_s)$ for every color assignment $f$ valid in $V(G_t^{-S})$.

Suppose there exists a $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$ and let $f$ be one of them. Let $\eta_t$ and $\eta_t$ be functions of domain $X_t$ such that $\eta_t(v) = \text{NS}(v, f, G_r^{-X_r})$ and $\eta_t(v) = \text{NS}(v, f, G_s^{-X_s})$ for all $v \in X_t$.

Since $V(G_r^{-X_r}) \cap V(G_s^{-X_s}) = X_t$ then $V(G_r^{-X_r}) \cap V(G_s^{-X_s}) \cap N_{G_t^{-S}}(v) = \emptyset$.

Moreover, since $f$ is a $(X_t, c, \eta, \bar{c})$-coloring in $G_t^{-S}$ then we know that, for all $v \in X_t$,
for this instance of the GLC problem is

\[ \text{Time complexity.} \]

operations are assumed to run in constant time. In particular, we are assuming that

\[ f_r \]

check

\[ f_v \]

\[ \eta(v) \equiv \text{true}. \]

Therefore, the functions \( \eta_r \) and \( \eta_s \) form a good pair of valid partial neighborhood mappings for \( c \).

It is straightforward to prove that \( f \) is a \( (X_r, c, \eta_r, \hat{c}_r) \)-coloring in \( G_r^{-X_r} \), and that \( f |_s \) is a \( (X_s, c, \eta_s, \hat{c}_s) \)-coloring in \( G_s^{-X_s} \). Hence,

\[
\begin{align*}
    w_\omega(f) &= W \oplus w_\omega_{X_r}(f |_r) \oplus w_\omega_{X_s}(f |_s) \\
    &\geq W \oplus \lambda_t(X_r, c, \omega_{X_r}, \eta_r, \check{c}_r) \oplus \lambda_s(X_s, c, \omega_{X_s}, \eta_s, \check{c}_s) \\
    &\geq \lambda_t(S, c, \omega, \eta, \check{c}).
\end{align*}
\]

Since \( \min\{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \check{c})\text{-coloring in } G_t^{-S}\} = \text{ERROR} \) if there are no \( (X_t, c, \eta, \check{c})\text{-colorings in } G_t^{-S} \), we obtain

\[
\min\{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \check{c})\text{-coloring in } G_t^{-S}\} \geq \lambda_t(S, c, \omega, \eta, \check{c}).
\]

To conclude, we will show that the other inequality holds.

If \( \lambda_t(S, c, \omega, \eta, \check{c}) = \text{ERROR} \) then the statement trivially holds. Otherwise, by definition of \( \lambda_t \), the minimum is realized by a good pair \((\check{\eta}_r, \check{\eta}_s)\), and \( \lambda_t(X_r, c, \omega_{X_r}, \eta_r, \check{c}_r) \neq \text{ERROR} \) and \( \lambda_s(X_s, c, \omega_{X_s}, \eta_s, \check{c}_s) \neq \text{ERROR} \). Therefore there exists a \( (X_r, c, \eta_r, \check{c}_r)\)-coloring \( \hat{f}_r \) in \( G_r^{-X_r} \) and a \( (X_s, c, \eta_s, \check{c}_s)\)-coloring \( \hat{f}_s \) in \( G_s^{-X_s} \) such that \( w_{\omega_{X_r}}(\hat{f}_r) = \lambda_r(X_r, c, \omega_{X_r}, \eta_r, \check{c}_r) \) and \( w_{\omega_{X_s}}(\hat{f}_s) = \lambda_s(X_s, c, \omega_{X_s}, \eta_s, \check{c}_s) \).

Let \( f \) be a function of domain \( V(G_t^{-S}) \) such that

- \( \hat{f}(v) = c(v) \) for all \( v \in X_t \),
- \( \hat{f}(v) = \hat{f}_r(v) \) for all \( v \in V(G_r^{-X_r}) - X_r \), and
- \( \hat{f}(v) = \hat{f}_s(v) \) for all \( v \in V(G_s^{-X_s}) - X_s \).

It is straightforward to prove that \( \hat{f} \) is a \( (X_t, c, \eta, \check{c})\)-coloring in \( G_t^{-S} \), and also that \( \hat{f}_r \) (resp. \( \hat{f}_s \)) is the restriction of \( \hat{f} \) to \( V(G_r^{-X_r}) \) (resp. \( V(G_s^{-X_s}) \)). Therefore,

\[
\begin{align*}
\lambda_t(S, c, \omega, \eta, \check{c}) &= W \oplus \lambda_r(X_r, c, \omega_{X_r}, \eta_r, \check{c}_r) \oplus \lambda_s(X_s, c, \omega_{X_s}, \eta_s, \check{c}_s) \\
&= W \oplus w_{\omega_{X_r}}(\hat{f}_r) \oplus w_{\omega_{X_s}}(\hat{f}_s) \\
&= w_\omega(\hat{f}) \\
&\geq \min\{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \check{c})\text{-coloring in } G_t^{-S}\}.
\end{align*}
\]

Consequently, \( \lambda_t(S, c, \omega, \eta, \check{c}) = \min\{w_\omega(f) : f \text{ is a } (X_t, c, \eta, \check{c})\text{-coloring in } G_t^{-S}\} \).

As a result, the statement holds.

Now the following result is immediate.

**Corollary 5.4.3.** If \( r \) is the root of \( T \) then the minimum weight of a proper coloring for this instance of the GLC problem is

\[
\min\{\lambda_t(\emptyset, c, \omega, \eta_r, \check{c}_r) : (\emptyset, c, \omega, \eta_r, \check{c}_r) \in D_r, \text{ and } \omega(v) = w_{\omega_{c(v)}} \text{ and } \check{c}_v = \text{check}_{v, c(v)} \text{ for all } v \in X_r \}.
\]

### 5.5. Time complexity.

Now that we know that the algorithm is correct, we focus on its time complexity.

Let \( k = \max\{|X_t| : t \in V(T)\} \), \( N = \max\{|N_{c,i}| : v \in V(G), i \in L_v\} \) and \( C = \max\{|L_v| : v \in V(G)\} \). Let \( t_{eq}, t_{\oplus}, t_{newN}, t_{\ominus}, \text{ and } t_{min} \) be upper bounds for the executing time of all the functions \( \text{check}_{v,i} \) and \( \text{eq}_i, \oplus^i, \ominus, \text{ and } \min, \text{ respectively.} \) Other operations are assumed to run in constant time. In particular, we are assuming that
we access \( w_{v,i} \) in \( O(1) \)-time (because it either is part of the input or can be computed in \( O(1) \)).

Traversing the tree \( T \) requires \( O(|V(T)|) \) time. In \( O(k^2|V(T)| + k|E(G)|) \) time we can construct the adjacency matrices of all the graphs \( G[X_t] \) with \( t \in V(T) \) (by traversing \( T \) top-down and computing \( N_{G_t}(v) \cap X_t \) only for nodes \( t \) that are the child of a forget nodes \( s \) with \( X_t - X_s = \{ v \} \)). Also, in \( O(k) \) time we can construct each of the necessary function extensions and restrictions, and in \( O((t^\ominus + t_{newN})k) \) we can construct each of the necessary \( \eta \).

We analyze four separate cases, and the proof in each one of them is straightforward.

- Leaf node: \( O(t_\ominus) \)
- Forget node: \( O(k^2 + (k + t_{min})C) \)
- Introduce node: \( O((t_\ominus + t_{newN})k + t_\ominus + t_\oplus + k^2) \)
- Join node: \( O((t_\ominus + t_{newN})k^2 + t_\ominus k + ((t_\ominus + t_\ominus)k + t_\oplus + t_{min})N^{2k}) \)

Each one of them is computed for every possible tuple \((S, c, \omega, \eta, c)\). We know that there are no more than \( 2^k \cdot C^k \cdot 2^k \cdot N^k \cdot (1 + N)^k \) of such tuples, and that constructing each of them requires \( O(k) \) operations.

In summary, the time complexity of this algorithm is \( O((t_\ominus k + (k + t_{min})C + (t_\ominus + t_{newN})k^2 + ((t_\ominus + t_\ominus)k + t_\oplus + t_{min})N^{2k})4^kC^kN^k(1 + 1)^k|V(T)| + k|E(G)|) \).

5.6. Special instances. Since for a graph of treewidth upper bounded by a constant \( q \) we can construct an easy tree-decomposition of width \( k \leq 5q + 4 \) with \( O(|V(G)|) \) nodes in \( O(|V(G)|) \) time, the next result easily follows.

**Theorem 5.6.1.** Let \( F \) be a family of graphs of bounded treewidth. Consider a family of instances of the GLC problem with \( \Pi = \emptyset \) and a partial neighborhood system, where

- \( G \in F \),
- \( C = \max\{|L_v| : v \in V(G)\} \) and \( N = \max\{|N_{v,i}| : v \in V(G), i \in L_v\} \) are polynomial in \( |V(G)| \), and
- all the functions \( check_{v,i}, eq_n, \ominus^{v,i}, newN_{v,i}, \oplus \) and \( \min \) can be computed in polynomial time.

Then there exists an algorithm that solves these instances in polynomial time. Furthermore, if \( C \) and \( N \) are bounded by a constant, and all the mentioned functions can be computed in constant time, then the time complexity of such algorithm is \( O(|V(G)|) \).

In particular, there are some instances of the GLC problem for which we can give “good” partial neighborhood systems.

**Corollary 5.6.2.** Let \( F \) be a family of graphs of bounded treewidth. Consider a family of instances of the GLC problem with \( \Pi = \emptyset \) and where

- \( G \in F \),
- \( |\bigcup_{v \in V(G)} L_v| \) is bounded by a constant,
- \( \ominus \) and \( \min \) can be computed in polynomial time, and
- \( check(v, c) \) can be computed in polynomial time and only depends on \( v, c(v) \) and the number of neighbors of each color that \( v \) has.

Then such instances of the GLC problem can be solved in polynomial time.

**Proof.** For each instance, let \( COLORS = \bigcup_{v \in V(G)} L_v \) and define the following partial neighborhood system:

- \( N_{v,i} = [0, d_{G}(v)]^{COLORS} \),
- \( (n \ominus^{v,i} n')_j = \min(n_j + n'_j, d_{G}(v)) \) for all \( j \in COLORS \).
• new $N_{v,i}(u, j)_h = 0$ for all $h \in \text{COLORS} - \{j\}$ and new $N_{v,i}(u, j)_{j} = 1$;
• check$_{w,i}(n) = \text{check}(v, c)$ where $c$ is any color assignment valid in $N_G[v]$ such that $c(v) = i$ and $|\{u : u \in N_G(v) \land c(u) = j\}| = n_j$ for all $j \in \text{COLORS}$. Note that we can construct $c$ in polynomial time using flow algorithms.

By Theorem 5.6.1, the statement holds. \hfill \Box

6. DEALING WITH GLOBAL PROPERTIES IN BOUNDED TREewidth GRAPHS

In this section we explain how to modify the previous algorithm in order to handle some cases of $\Pi \neq \emptyset$. The general idea in all of these cases is to modify $\lambda_t$ in the original algorithm extending it with new parameters. Thus, at each node $t$ we compute $\lambda_t(S, c, \omega, \eta, \check{c}, \ldots)$.

For simplicity, in the following subsections we omit some parts of the original algorithm, writing only the necessary changes.

6.1. The size of a color class is an element of a particular set. Suppose we want the class of color $j$ to have a size that is an element of a set $\sigma \subseteq \mathbb{N}_0$.

Consider a deterministic finite-state automaton $(Q, \{1\}, \delta, q_0, F)$ that accepts a string of $n$ consecutive characters 1 if and only if $n \in \sigma$. Notice that for all finite sets $\sigma \subseteq \mathbb{N}_0$ there exists such an automaton. Indeed, let $m$ be the maximum element of $\sigma$, $Q = \{s_0, \ldots, s_{m+1}\}$, $q_0 = s_0$, $F = \{s_i : i \in \sigma\}$, and $\delta(s_i, 1) = s_{i+1}$ for all $0 \leq i \leq m$ and $\delta(s_{m+1}, 1) = s_{m+1}$. Although, for time complexity issues, when $m$ is not a constant we might be interested in another automata, with $O(1)$ number of states (for example, if $\sigma$ is the set of odd numbers in $[0, |V(G)|]$, we only need two states).

In the algorithm, at each node $t$, we add a parameter $\text{state}_j$ that stores the state of the partial size of the color class $j$, and also a parameter $\text{accept}_j$ that checks if we are in the desired state, and then proceed in the following way.

• **Leaf node:** Now we also need to check if $\text{accept}_j(\text{state}_j)$ is true.
• **Forget node:** For all $i \in L_v$, let $\text{state}_i^j = \text{state}_j$ if $i \neq j$ and $\text{state}_i^j = \delta(\text{state}_j, 1)$ otherwise. Then
  \[
  \lambda_t(\ldots, \text{state}_j, \text{accept}_j) = \min \{\lambda_s(\ldots, \text{state}_j^i, \text{accept}_j) : i \in L_v\}.
  \]
• **Introduce node:** Remains the same (with $\text{state}_j$ and $\text{accept}_j$ added to $\lambda_s$).
• **Join node:** For all $q \in Q$, let $\text{eq}_q : Q \to \text{BOOL}$ be such that $\text{eq}_q(q') = (q = q')$ for all $q' \in Q$. Then
  \[
  \lambda_t(\ldots, \text{state}_j, \text{accept}_j) = \min \{W \oplus \lambda_r(\ldots, \text{state}_j, \text{eq}_q) \oplus \lambda_s(\ldots, q, \text{accept}_j) : q \in Q \land \ldots\}.
  \]

At the root $r$ where $X_r = \{v\}$, we compute all $\lambda_r(\ldots, a, s_r)$, with $a$ such that $a(s) = (s \in F)$, and $s_r = q_0$ if $c(v) \neq j$ and $s_r = \delta(q_0, 1)$ otherwise.

Note that it is easy to generalize this idea to more classes by simply adding as many $\text{state}_j$ and $\text{accept}_j$ as needed (each of them with its own automaton), and even to a set $J$ of classes by replacing statements of the form “$i \neq j$” with “$i \notin J$”.

The time complexity now depends on the number of states and color classes to restrict. We can assume that checking if a state is an accepting one is a $O(1)$ operation and so is computing $\delta(s, 1)$. Let $R$ be the number of color classes (or sets of color classes) to restrict and let $S$ be the size of the largest set of states among all considered automata. The only changes in complexity are:

• **Leaf node:** add $R$.
• **Forget node:** add $2R$.
• **Introduce node:** add $2R$. 

• Join node: multiply by $S^R$.
• When we multiply by the number of all possible combinations of the parameters of $\Lambda_t$: add a factor $(S(S + 1))^R$.

In particular, the complexity of the algorithm remains polynomial in $|V(G)|$ if $R$ is bounded by a constant, allowing us to, for example, ask for a color class to be non-empty or to have at most one element.

### 6.2. One color class is connected.

Suppose we want the class of color $j$ to be connected.

At each node $t$, we add the parameter $comp_j : X_t \rightarrow [1,k]$ that maps vertices of color $j$ to natural numbers that represent connected components.

For the following items, let $X^j_t = \{u : u \in X_j \land c(u) = j\}$ and $N^j_t(v) = N_G(v) \cap X^j_t$.

• Leaf node: Remains the same.
• Forget node: $\lambda(t,\ldots,comp_j) = \min\{\lambda_s(\ldots,comp_j) : i \in L_v\}$ where $comp_j$ is such that:
  - if $i \neq j$ then $comp_j(u) = comp_j(u)$ for all $u \in X^j_t$, and
  - $comp_j(v) = \begin{cases} \min_{u \in N^j_t(v)}\{comp_j(u)\} & \text{if } N^j_t(v) \neq \emptyset \\ \text{any value in } [1,k] \setminus \{comp_j(u) : u \in X^j_t\} & \text{otherwise} \end{cases}$

and, for all $u \in X^j_t$, $comp_j(u) = comp_j(v)$ if there exists $z \in N^j_t(v)$ such that $comp_j(u) = comp_j(z)$, and $comp_j(u) = comp_j(u)$ otherwise.

The idea behind this is that if $v$ is a neighbor of two or more vertices of different connected components, then those connected components can be unified, and if $v$ is not a neighbor of any other vertex of color $j$ then it is in a new connected component.

• Introduce node: If $c(v) \neq j$ then it remains the same (adding $comp_j$ to $\lambda_s$).

Otherwise, we split the case related to $c_v(n_v) = \text{TRUE}$ in the following:

- If there exists $u \in X^j_s$ such that $comp_j(v) = comp_j(u)$ then $\lambda_t(\ldots,comp_j) = \ldots\lambda_s(\ldots,comp_j^{-r})$.
- If there does not exist $u \in X^j_s$ such that $comp_j(v) = comp_j(u)$, and $X^j_t \neq \emptyset$ then $\lambda_t(\ldots,comp_j) = \text{ERROR}$.
- If $X^j_s = \emptyset$ then let $M = (\{q_0,q_1\},\{1\},\delta,q_0,\{q_0\})$ be an automaton such that $\delta(q_0,1) = q_1$ and $\delta(q_1,1) = q_1$. We use $M$ to request that the class of color $j$ is empty in $G_s$, therefore $\lambda_t(\ldots,comp_j) = \ldots\lambda_s(\ldots,q_0,cq_0)$.

The idea is that if $v$ belongs to a different connected component than all the vertices of color $j$ in $X_s$, then there is no way to connect $v$ with them and we get an error. Also, if there are no vertices of color $j$ in $X_s$ then there cannot be any vertices of color $j$ in $G_s$, because $N_G(V(G) - V(G_s)) \cap V(G_s) \subseteq X_s$.

• Join node: For every $S \subseteq \{comp_j(v) : v \in X^j_t\}$ let $comp_j^S$ be a function such that, for all $v \in X^j_t$,

$$comp_j^S(v) = \begin{cases} \min(S) & \text{if } comp_j(v) \in S \\ comp_j(v) & \text{otherwise}. \end{cases}$$

Here we need to unify the different connected components, and one branch takes care of unifying a set $S$ of them while the other branch unifies a set $R$ of them, such that $S \cup R = \{comp_j(v) : v \in X^j_t\}$ and $|S \cap R| = 1$. Then $\lambda_t(\ldots,comp_j) = \min\{W \oplus \lambda_s(\ldots,comp_j^S) \oplus \lambda_s(\ldots,comp_j^R) : S \cup R = \{comp_j(v) : v \in X^j_t\} \land |S \cap R| = 1 \land \ldots\}$. At the root $r$ where $X_r = \{v\}$, the function $comp_j$ is such that $comp_j(v) = 1$.

As before, notice that it is easy to generalize this idea to more classes or sets of classes by adding as many $comp_j$ functions as needed.
Let $J$ be the number of color classes (or sets of color classes) to restrict. The only changes in complexity are:

- **Leaf node:** add $J$.
- **Forget node:** add $(C - 1 + k^2)J$.
- **Introduce node:** add $kJ$.
- **Join node:** multiply by $(k^2)J$.

When we multiply by the number of all possible combinations of the parameters of $\lambda_t$: add a factor $(k^k)^J$.

Note that because in the introduce node we require that the class of color $j$ is empty, we also need to compute $\lambda_t(\ldots, q_0, eq_0)$ for all $t \in T$, but this addition does not change the time complexity here (due to the fact that both $S$ and $R$ are bounded by a constant in this case).

### 6.3. One color class is acyclic. (For undirected graphs.)

It can be done in essentially the same way as for the connected property. The only difference is that in the introduce node we do as in the original algorithm, and in the forget node we check if $v$ is a neighbor of at least two vertices that belong to the same component (in which case there is a cycle and we raise an error).

### 7. Classical problems as instances of the GLC problem in bounded treewidth graphs

In this section we show how to model different problems as instances of the GLC problem with a partial neighborhood system. As a result, we obtain polynomial-time algorithms to solve these problems for bounded treewidth graphs or bounded treewidth and bounded degree graphs. For problems in Subsection 7.1, no such algorithms were previously known (until the date and to the best of our knowledge).

In Subsection 7.2 we study other problems that were already known to be polynomial-time solvable for the before-mentioned classes. It is worth to mention how to restate these problems as instances of the GLC problem, even when the time complexity of the proposed solution is worse than the best one known, because problems modeled this way can be easily modified or combined, adding global properties or more restrictions, or even dealing with some distance versions. On the other hand, they can inspire the statement of other problems as instances of the GLC problem.

Throughout this section, we will assume that, otherwise stated, the definitions of $N_{v,i}$ are for all $v \in V(G)$ and $i \in L_v$, of $n \boxplus^v n'$ for all $v \in V(G)$, $i \in L_v$, and $n, n' \in N_{v,i}$, of $\text{new}N_{v,i}(u, j)$ for all $v \in V(G)$, $i \in L_v$, $u \in N_G(v)$ and $j \in L_u$, and of $\text{check}_{v,i}(n)$ for all $v \in V(G)$, $i \in L_v$, and $n \in N_{v,i}$.

### 7.1. New results.

#### 7.1.1. Double Roman domination.

This problem was first defined in [6] and proved to be NP-complete for bipartite and chordal graphs in [2].

A **double Roman dominating function** on a graph $G$ is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then vertex $v$ must have at least two neighbors assigned 2 under $f$ or one neighbor $w$ with $f(w) = 3$, and if $f(v) = 1$, then vertex $v$ must have at least one neighbor $w$ with $f(w) \geq 2$. The **weight** of a double Roman dominating function $f$ is the sum $f(V(G)) = \sum_{v \in V(G)} f(v)$, and the minimum weight of a double Roman dominating function on $G$ is the **double Roman domination number** of $G$. 
We can model the Double Roman domination problem as an instance of the GLC problem with a partial neighborhood system in the following way:

1. \( L_v = \{0, 1, 2, 3\} \);
2. \( (\text{Weights}, \leq) = (\mathbb{R} \cup \{+\infty\}, \leq) \) and \( \oplus = + \);
3. \( w_{v,i} = 0 \);
4. \( \text{check}(v, c) = (c(v) = 0 \Rightarrow (\exists u, w \in N_G(v) / u \neq w \land c(u) = c(w) = 2) \lor (\exists u \in N_G(v) / c(u) = 3)) \land (c(v) = 1 \Rightarrow \exists u \in N_G(v) / c(u) \geq 2) \);
5. \( \Pi = \emptyset \);
6. \( \mathcal{N}_{v,i} = \{0, 1, 2\} \times \{0, 1\} \);
7. \((n_2, n_3) \boxplus v,i (n_2', n_3') = (\min(n_2 + n_2', 2), \min(n_3 + n_3', 1))\);
8. \( \text{new} \mathcal{N}_{v,i}(u, j) = \begin{cases} (0, 0) & \text{if } j \leq 1 \\ (1, 0) & \text{if } j = 2 \\ (0, 1) & \text{if } j = 3 \end{cases} \)
9. \( \text{check}_{v,i}(n_2, n_3) = (i = 0 \Rightarrow n_2 \geq 2 \lor n_3 \geq 1) \land (i = 1 \Rightarrow n_2 + n_3 \geq 1) \).

Notice that the partial neighborhood system simply counts (until it saturates) the number of neighbors assigned with 2 and also the ones assigned with 3. For other versions of the double Roman domination problem, we might require to also count the number of neighbors assigned with 0 or 1. It is easy to see that some of its variants, such as perfect [25], independent [53], outer independent [1] and total [59], can be modeled by making slight modifications to the previous items.

Since \( \mathcal{C} \) and \( \mathcal{N} \) are bounded by a constant, the time complexity in this case is \( O(|V(G)|) \) for a graph \( G \) in a family of graphs of bounded treewidth.

### 7.1.2. \( b \)-coloring problem

In [47], this problem was defined and proved to be NP-complete for general graphs.

Given a graph \( G \) and a positive integer \( k \), the \( b \)-coloring problem asks if there exist a proper \( k \)-coloring of the vertices of \( G \) such that every color class contains a vertex that has neighbors in all the other color classes.

We restate the problem in such a way that now we consider \( 2k \) colors: the ones greater than \( k \) represent the “\( b \)-vertices”, that are requested to have a closed neighborhood including colors \( i \) or \( i+k \) for every \( 1 \leq i \leq k \). Then we can model the \( b \)-coloring problem as an instance of the GLC problem with a partial neighborhood system in the following way:

1. \( L_v = [1, 2k] \);
2. \( (\text{Weights}, \leq) = (\mathbb{R} \cup \{+\infty\}, \leq) \) and \( \oplus = + \);
3. \( w_{v,i} = 0 \) (actually, we do not care about weights in this case);
4. \( \text{check}(v, c) = (c(v) \leq k \Rightarrow \bigwedge_{u \in N_G(v)} (c(u) \neq c(v) \land c(u) \neq c(v) + k)) \land (c(v) > k \Rightarrow \bigwedge_{u \in N_G(v)} (c(u) \neq c(v) \land c(u) \neq c(v) - k)) \land \bigwedge_{i=1}^{k} (\exists u \in N_G[v] / c(u) = i \lor c(u) = i + k) \);
5. \( (\Pi) \) for every \( i \in [1, k] \), the set of vertices with color \( i+k \) has at least 1 element;
6. \( \mathcal{N}_{v,i} = 2^{L_v} \);
7. \( n \boxplus v,i n' = n \cup n' \);
8. \( \text{new} \mathcal{N}_{v,i}(u, j) = \{j\} \);
9. \( \text{check}_{v,i}(n) = (i \leq k \Rightarrow i \notin n \land i + k \notin n) \land (i > k \Rightarrow i \notin n \land i - k \notin n) \land \bigwedge_{j=1}^{c} (j + k \neq i \Rightarrow j \in n \lor j + k \in n)). \)
For a graph $G$ in a family of graphs of bounded treewidth, the time complexity is $O(k^c c_2^k |V(G)|)$ for some constants $c_1, c_2$. Since $k$ is at most $\Delta(G) + 1$, then when $\Delta(G)$ is bounded by a constant or $O(\log |V(G)|)$ we can solve this problem in polynomial time on $|V(G)|$.

7.1.3. Minimum chromatic violation problem. This NP-hard problem was first defined in [9] as a generalization of the $k$-coloring problem.

Given a graph $G$, a set of weak edges $F \subseteq E(G)$ and a positive integer $k$, the minimum chromatic violation problem asks for a $k$-coloring of the graph $G' = (V(G), E(G) - F)$ minimizing the number of weak edges with both endpoints receiving the same color.

Let $\hat{G} = (V(G) \cup E(G), E(\hat{G}))$ be the graph obtained by subdividing the edges of $G$. Assume that there is an ordering of the vertices, and that edges are named as ordered pairs of its endpoints. Then we can set

- $L_v = [1, k]$ for all $v \in V(G)$,
- $L_{uv} = L_u \times L_v$ for all $uv \in E(G)$;
- $(\text{WEIGHTS, } \leq) = (\mathbb{R} \cup \{+\infty\}, \leq)$ and $\oplus = +$;
- $w_{v,i} = 0$ for all $v \in V(G), i \in L_v$,
- $w_{uv,(i,i)} = 1$ for all $uv \in E(G), i \in L_u \cap L_v$,
- $w_{uv,(i,j)} = 0$ for all $uv \in E(G), i \in L_u, j \in L_v - \{i\}$;
- $\text{check}(v, c) = \text{TRUE}$ for all $v \in V(G)$,
- $\text{check}(uv, c) = (c(uv) = (c(u), c(v)))$ for all $uv \in E$, and
- $\text{check}(uv, c) = (c(uv) = (c(u), c(v)) \land c(u) \neq c(v))$ for all $uv \in E(G) - F$;
- $\Pi = \emptyset$;
- $N_{v,i} = \text{BOOL}$ for all $v \in V(\hat{G}), i \in L_v$;
- $n \boxplus N^i n' = (n \land n')$ for all $v \in V(\hat{G}), i \in L_v$ and $n, n' \in N_{v,i}$;
- $\text{new}N_{v,i}(e, j) = \text{TRUE}$ for all $v \in V(G), i \in L_v, e \in N_\hat{G}(v), j \in L_e$,
- $\text{new}N_{uv,(c_u, c_v)}(u, j) = (j = c_u)$ for all $uv \in E(G), c_u \in L_u, c_v \in L_v, j \in L_u$,
- $\text{new}N_{uv,(c_u, c_v)}(v, j) = (j = c_v)$ for all $uv \in E(G), c_u \in L_u, c_v \in L_v, j \in L_v$;
- $\text{check}_{v,i}(n) = \text{TRUE}$ for all $v \in V(G), i \in L_v, n \in N_{v,i}$,
- $\text{check}_{uv,(i,j)}(n) = n$ for all $uv \in E, i \in L_u, j \in L_v, n \in N_{uv,(i,j)}$,
- $\text{check}_{uv,(i,j)}(n) = (n \land i \neq j)$ for all $uv \in E(G) - F, i \in L_u, j \in L_v, n \in N_{uv,(i,j)}$.

Basically, every edge in $G$ is colored with a pair of colors and checks if these are the colors of its endpoints. Edges in $F$ are allowed to have endpoints of the same color, while edges not in $F$ always produce an error when colored with a pair of equal colors. If an edge is colored with a pair of equal colors then its weight is 1, otherwise is 0.

In this way we have $C = k^2, N = 2$ and all the $t$'s bounded by a constant. Therefore, when $k$ is bounded by a constant the problem can be solved in $O(|V(G)|)$ time for a graph $G$ in a family of graphs of bounded treewidth.

7.1.4. Grundy domination number. This problem was introduced in [11] and proved to be NP-complete even for chordal graphs.

A sequence $S = (v_1, \ldots, v_k)$ of distinct vertices of a graph $G$ is a dominating sequence if $\{v_1, \ldots, v_k\}$ is a dominating set of $G$, and $S$ is called a legal (dominating) sequence if (in addition) $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$ for each $i$. We say that $v_i$ footprints the vertices in $N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$ and that $v_i$ is the footpointer of every $u \in N[v_i] - \bigcup_{j=1}^{i-1} N[v_j]$ (notice that every vertex in $V(G)$ has a unique footpointer). We are interested in the maximum length of a legal dominating sequence in $G$. 


Given a legal sequence $S$, every vertex $v \in V(G)$ can be associated with a pair $(p_v, f_v)$, where $p_v$ is the position of $v$ in $S$ (or $\perp$ if $v$ is not in $S$) and $f_v$ is the position in $S$ of the footprinter of $v$.

Directly from the definition of legal sequences we can deduce the following statement. A set $\{(p_v, f_v) : v \in V(G)\}$ determines a legal sequence if and only if the following conditions are satisfied: $p_v \neq p_u$ for all $u, v \in V(G)$ such that $u \neq v$ and $p_u \neq \perp$ (i.e., two vertices that appear in the sequence cannot have the same position in it), and also, for all $v \in V(G)$, we have $f_v = \min\{p_u : u \in N_G[v]\}$ (i.e., $v$ is properly footprinted) and $p_v \neq \perp \Rightarrow \exists u \in N_G[v] / f_u = p_v$ (i.e., if $v$ appears in the sequence then it footprints at least one vertex).

Let $n = |V(G)|$ and $\perp = n + 1$. We can model the Grundy domination problem as an instance of the GLC problem with a partial neighborhood system as follows:

- $L_v = \{1, \ldots, n + 1\} \times \{1, \ldots, n\}$;
- $(\text{WEIGHTS, } \preceq) = (\mathbb{R} \cup \{-\infty\}, \succeq)$ and $\oplus = +$;
- for all $f \in \{1, \ldots, n\}$, $w_{v, (\perp, f)} = 0$ and $w_{v, (p, f)} = 1$ for all $p \neq \perp$;
- $\text{check}(v, c) = (\exists u \in N_G[v] / c(v) = c(u)) \land (\not\exists u \in N_G[v] / c(v) > c(u))$
  $\land (c(v) \neq \perp \Rightarrow \exists u \in N_G[v] / c(v) = c(u))$
  $\land (c(v) \neq \perp \Rightarrow \not\exists u \in N_G[v] / c(v) = c(u));$
- $\Pi = \emptyset$;
- $\mathcal{N}_{v, (p, f)} = \{0, 1, 2\} \times \{0, 1\}$;
- $(\eta, \phi) \oplus_{v, (p, f)} (\eta', \phi') = (\min(\eta + \eta', 2), \min(\phi + \phi', 1))$
- $\text{new}\mathcal{N}_{v, (p_v, f_v)}(u, (p_u, f_u))$
  $= \begin{cases} (0, 0) & \text{if } p_u = p_v = \perp \\ (1, 0) & \text{if } p_u < p_v \land f_v = p_u \\ (0, 1) & \text{if } p_u > p_v \land f_u = p_v \\ (0, 0) & \text{if } p_u > p_v \land f_u < p_v \\ (0, 0) & \text{if } p_u > p_v \land f_u < p_v \\ (2, 0) & \text{otherwise}; \end{cases}$
- $\text{check}_{v, (p, f)}(\eta, \phi) = (f = p \land \eta = 0) \lor (\eta = 1 \land (p \neq \perp \Rightarrow \phi = 1))$.

In this instance we want to maximize the number of vertices in the sequence, hence the weights are defined as 1 for vertices that appear in the sequence and 0 otherwise. We have a function $\text{check}$ that checks if a vertex $v$ either footprint itself, or has a unique footprinter and if $v$ is in the sequence then it footprints another vertex. The partial neighborhoods of each vertex $v$ keep track of whether $v$ was properly footprinted and whether $v$ has footprinted another vertex, and the function $\text{new}\mathcal{N}_{v, (p_v, f_v)}(u, (p_u, f_u))$ tells us what happens when $v$ “sees” a neighbor $u$ considering the following cases:

- If $p_u < p_v$ and $f_v = p_u$ then $v$ found its footprinter.
- If $p_u < p_v$ and $f_v < p_u$ then nothing happens. The vertex $u$ cannot footprint $v$ because there is another vertex before $u$ that footprints $v$.
- If $p_u < p_v$ and $f_v > p_u$ then there is an error because $v$ cannot have a neighbor that appears before its footprinter in the sequence.
- If $p_u = p_v \neq \perp$ then there is an error because two vertices should not be in the same position.
- If $p_u = p_v = \perp$ then nothing happens.
- If $p_u > p_v$ and $f_u = p_v$ then $v$ found a vertex to footprint.
- If $p_u > p_v$ and $f_u < p_v$ then nothing happens. The vertex $v$ cannot footprint $u$ because there is another vertex before $v$ that footprints $u$. 

If \( p_u > p_v \) and \( f_u > p_v \) then there is an error because \( u \) cannot have a neighbor that appears before its footprinter in the sequence.

Since we apply \( \text{new}N_v(p_v, f_v)(u, (p_w, f_w)) \) to every ordered pair of neighbors \((v, u)\) and the \text{check} function checks if the vertex footprints itself, at some point we will know if every vertex was properly footprinted and has properly footprinted itself or another vertex.

Notice that for this instance we actually do not require that, for every \( p \) such that \( 1 \leq p \leq n \), the set of vertices with a color in \( \{(p, f) : 1 \leq f \leq n\} \) has at most 1 element. Indeed, suppose that after running the algorithm there are two different vertices \( u, v \) such that \( p_u = p_v \neq \perp \). Then we know that \( u \notin N_G[v] \) (because, among all the calls to \( \text{new}N_v(p_v, f_v) \), we check that there is no neighbor \( w \) of \( v \) with \( p_w = p_v \neq \perp \), and that if there exists a vertex \( w \) such that \( f_w > p_v = p_u \) then \( w \notin N_G[v] \) and \( w \notin N_G[u] \) (because we check this in \( \text{new}N_w(p_w, f_w) \), and also that if \( f_w = p_v = p_u \) and \( w \in N_G[u] \) then \( w \notin N_G[v] \) (because we check if \( w \) has two footprinters). Therefore, \( N_G[u] \cap N_G[v] = \emptyset \) such that \( f_w < p_u = p_v \) for all \( w \in F \). Now we can assign colors \((p'_w, f'_w)\) for every \( w \in V(G) \), such that:

- \( p'_w = p_w \) if \( p_w \leq p_u \) and \( w \neq v \),
- \( p'_w = p_w + 1 \) if \( p_w = p_u \) or \( w = v \),
- \( f'_w = f_w \) if \( f_w < p_u \),
- \( f'_w = f_w \) if \( f_w = p_u \) and \( w \notin N_G[v] \),
- \( f'_w = f_w + 1 \) if \( f_w = p_u \) and \( w \in N[v] \), and
- \( f'_w = f_w + 1 \) if \( p_w > p_u \).

That is, we move one position forward all the vertices that appear after \( u \) in \( S \) and increase the necessary \( f_w \). It is easy to see that this new assignment preserves the “legality” of \( u \) (i.e., if \( N_G[u] \cap \{z : z \in N_G[w] \land p_w < p_u\} \neq \emptyset \) then \( N_G[u] \cap \{z : z \in N_G[w] \land p_w < p_u\} \neq \emptyset \) and also of all the other vertices.

We can also model the Grundy total domination problem (defined in [12]) in a very similar way, by simply removing the cases where a vertex can footprint itself.

For both problems, since \( C \) is \( \mathcal{O}(|V(G)|^2) \), and \( N \) and all the \( t \)'s are bounded by a constant, the time complexity is polynomial in \(|V(G)|\) for a graph \( G \) in a family of bounded treewidth graphs.

### 7.2. Well known problems

The following problems are well known to be polynomial-time solvable in bounded treewidth graphs (or even in larger graph classes). Many of them have been studied in [4, 5, 26, 33, 62] and their definitions can be found in Appendix A.

In tables 1, 2 and 3 we show some examples of coloring, domination, independence and packing problems as instances of the GLC problem without global properties (that is, \( \Pi = \emptyset \)) and with a partial neighborhood system. Observe that for list-coloring and \( H \)-coloring we are only interested in determining whether such coloring exists or not, so we do not use the weights for optimizing the solution, instead we use them precisely for determining if a solution is valid. For \( k \)-coloring we can make use of weights to determine the smallest \( j \leq k \) for which there exists a \( j \)-coloring in \( G \) (in particular, by Theorem 2.4.3, we could set \( k = tw(G) + 1 \) to obtain the chromatic number in \( \mathcal{O}(|V(G)|) \) time). For the total version of most of these domination problems we only need to replace \( N[v] \) by \( N(v) \) and expressions like \( n + i \geq k \) by \( n \geq k \). Assuming \( k \) constant, the time complexity is \( \mathcal{O}(|V(G)|) \) in all these cases. For weighted versions of these problems, weights are part of the input.
Table 1. Coloring problems modeled as instances of the GLC problem with \( \Pi = \emptyset \) and with a partial neighborhood system.

| Problem       | \( L_v \) | \( \text{WEIGHTS, } \leq, \oplus \) | \( \text{check}(v, c) \) | \( N_{v,i} \) | \( n \iff i \) | \( \text{new}N_{v,i}(u, j) \) | \( \text{check}_{v,i}(n) \) |
|---------------|---------|----------------|----------------|----------|-------------|----------------|----------------|
| \( k \)-coloring | \([1, k]\) | \( \mathbb{R} \cup \{ +\infty \}, \leq, \text{max} \) | \( i \sum_{u \in N_G(v)} c(u) \neq c(v) \) | BOOLEAN | \( n \land n' \) | \( j \neq i \) | \( n \) |
| \( k \)-chromatic sum | \([1, k]\) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( i \sum_{u \in N_G(v)} c(u) \neq c(v) \) | BOOLEAN | \( n \land n' \) | \( j \neq i \) | \( n \) |
| List-coloring from input | | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( 0 \sum_{u \in N_G(v)} c(u) \neq c(v) \) | BOOLEAN | \( n \land n' \) | \( j \neq i \) | \( n \) |
| \( H \)-coloring | \( V(H) \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( 0 \sum_{u \in N_G(v)} c(u) \in N_H(c(v)) \) | BOOLEAN | \( n \land n' \) | \( j \in N_H(i) \) | \( n \) |

Table 2. Domination problems modeled as instances of the GLC problem with \( \Pi = \emptyset \) and with a partial neighborhood system.

| Problem       | \( L_v \) | \( \text{WEIGHTS, } \leq, \oplus \) | \( \text{check}(v, c) \) | \( N_{v,i} \) | \( n \iff i \) | \( \text{new}N_{v,i}(u, j) \) | \( \text{check}_{v,i}(n) \) |
|---------------|---------|----------------|----------------|----------|-------------|----------------|----------------|
| \( k \)-tuple domination | \( \{0, 1\} \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( i \sum_{u \in N_G(v)} c(u) \geq k \) | \([0, k]\) | \( \min(n + n', k) \) | \( j \) | \( n + i \geq k \) |
| Total \( k \)-tuple domination | \( \{0, 1\} \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( i \sum_{u \in N_G(v)} c(u) \geq k \) | \([0, k]\) | \( \min(n + n', k) \) | \( j \) | \( n \geq k \) |
| \( k \)-domination | \( \{0, 1\} \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( i c(v) = 0 \lor \sum_{u \in N_G(v)} c(u) \geq k \) | \([0, k]\) | \( \min(n + n', k) \) | \( j \) | \( i = 0 \Rightarrow n \geq k \) |
| \{\( k \)\}-domination | \( \{0, k\} \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( i \sum_{u \in N_G(v)} c(u) \geq k \) | \([0, k]\) | \( \min(n + n', k) \) | \( j \) | \( n + i \geq k \) |
| \( k \)-rainbow domination | \( 2[1, k] \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( \bigcup_{u \in N_G(v)} c(u) = k \) | \( 2[1, k] \) | \( n \lor n' \) | \( j = 2 \) | \( i = 0 \Rightarrow n \) |
| Roman domination | \( \{0, 1, 2\} \) | \( \mathbb{R} \cup \{ +\infty \}, \leq, + \) | \( i c(v) = 0 \Rightarrow \bigvee_{u \in N_G(v)} (c(u) = 2) \) | BOOLEAN | \( n \lor n' \) | \( j = 2 \) | \( i = 0 \Rightarrow n \) |

Table 3. Miscellaneous problems modeled as instances of the GLC problem with \( \Pi = \emptyset \) and with a partial neighborhood system.

| Problem       | \( L_v \) | \( \text{WEIGHTS, } \leq, \oplus \) | \( \text{check}(v, c) \) | \( N_{v,i} \) | \( n \iff i \) | \( \text{new}N_{v,i}(u, j) \) | \( \text{check}_{v,i}(n) \) |
|---------------|---------|----------------|----------------|----------|-------------|----------------|----------------|
| Independent set | \( \{0, 1\} \) | \( \mathbb{R} \cup \{-\infty\}, \geq, + \) | \( i = 1 \Rightarrow \sum_{u \in N_G(v)} c(u) = 0 \) | \( \{0, 1\} \) | \( \min(n + n', 1) \) | \( j \) | \( i = 1 \Rightarrow n = 0 \) |
| \{\( k \)\}-packing function | \( \{0, k\} \) | \( \mathbb{R} \cup \{-\infty\}, \geq, + \) | \( i \sum_{u \in N_G(v)} c(u) \leq k \) | \( \{0, k + 1\} \) | \( \min(n + n', k + 1) \) | \( j \) | \( n \leq k \) |
| \{\( k \)\}-limited packing | \( \{0, 1\} \) | \( \mathbb{R} \cup \{-\infty\}, \geq, + \) | \( i \sum_{u \in N_G(v)} c(u) \leq k \) | \( \{0, k + 1\} \) | \( \min(n + n', k + 1) \) | \( j \) | \( n \leq k \) |
7.2.1. Additive coloring. Let $\eta$ be an upper bound of the additive chromatic number. It was shown in [3] that the additive chromatic number is at most $\Delta(G)^2 - \Delta(G) + 1$.

To model the additive coloring problem as an instance of the GLC problem with a partial neighborhood system, we change the numbers assigned to the vertices with pairs of integers $(n, s)$, where $n$ represents the number assigned to the vertex and $s$ the sum of the numbers assigned to its neighbors. Then

- $L_v = [1, \eta] \times [1, \Delta(G)\eta]$;
- $(\text{Weights,} \leq) = (\mathbb{R} \cup \{+\infty\}, \leq)$ and $\oplus = \max$;
- $\mathbf{w}_{v,i} = i_1$;
- $\text{check}(v, c) = \left( \bigwedge_{u \in N_G(v)} c(u)_2 \neq c(v)_2 \right) \land \left( c(v)_2 = \sum_{u \in N_G(v)} c(u)_1 \right)$;
- $\Pi = \emptyset$;
- $N_{v,i} = [1, \Delta(G)\eta + 1]$;
- $n \boxplus^i n' = \min(n + n', \Delta(G)\eta + 1)$;
- $\text{new}N_{v,i}(u, j) = \begin{cases} j_1 & \text{if } i_2 \neq j_2 \\ \Delta(G)\eta + 1 & \text{otherwise} \end{cases}$;
- $\text{check}_{v,i}(n) = (n = i_2)$.

Then $\mathcal{C} = O(\Delta(G)\eta^2)$ and $\mathcal{N}$ is $O(\Delta(G)\eta)$, implying that there exists a polynomial-time algorithm to compute $\eta(G)$ when $G$ is in a class of graphs of bounded treewidth. Another polynomial time algorithm was obtained by R. Grappe, L. N. Grippo, and M. Valencia-Pabon (personal communication).

7.2.2. Distance domination problems. We will start by showing how to model the distance $k$-domination problem. To do this, we restate the problem in the following way: vertices receive integers from 0 to $k$ (that represent their distance to a vertex of the distance $k$-dominating set), and vertices with a number greater than 0 must satisfy the condition of having a neighbor with the preceding number assigned. Then

- $L_v = [0, k]$;
- $(\text{Weights,} \leq) = (\mathbb{R} \cup \{+\infty\}, \leq)$ and $\oplus = +$;
- $\mathbf{w}_{v,0} = 1$ and $\mathbf{w}_{v,i} = 0$ for all $i \in [1, k]$;
- $\text{check}(v, c) = \left( c(v) > 0 \Rightarrow \bigvee_{u \in N_G(v)} c(u) = c(v) - 1 \right)$;
- $\Pi = \emptyset$;
- $N_{v,i} = \text{BOOL}$;
- $n \boxplus^i n' = n \vee n'$;
- $\text{new}N_{v,i}(u, j) = (j = i - 1)$;
- $\text{check}_{v,i}(n) = (i > 0 \Rightarrow n)$.

We have $\mathcal{C} = k + 1$, $\mathcal{N} = 2$ and the $t$’s bounded by a constant. Therefore, assuming $k$ is bounded by a constant and $G$ is in a class of graphs of bounded treewidth, the time complexity of the algorithm in this case is $O(|V(G)|)$.

Notice that with a similar argument we can model a distance domination problem involving more than two sets. The idea is to make the colors indicate how far the vertices are from every other color. For example, if we have to color the graph with $\{r, g, b\}$ in such a way that every vertex is at distance at most 3 from a vertex of color $r$ and at distance at most 2 from a vertex of color $g$, following this idea to model the problem as an instance of the GLC problem, our set of colors is $\{r_{0,1}, r_{0.2}, g_{1.0}, g_{2.0}, g_{3.0}, b_{1.1}, b_{2.1}, b_{3.1}, b_{1.2}, b_{2.2}, b_{3.2}\}$.
However, when there are restrictions over the distance between vertices of the same color, the previous approach would not work. We will now explain how to model these problems when the required distance is 2.

Let us work with the semitotal domination problem. We first restate the problem in order to differentiate the two possible situations for a vertex in $D$ (that is, having a neighbor in $D$ or being at distance 2 of another vertex in $D$) and the two possible situations for a vertex not in $D$ (that is, being the nexus between two vertices in $D$ or not), as you can see Figure 1. In this way, a vertex of color $D^*$ needs at least two neighbors in $D$, a vertex of color $D_1$ needs one neighbor in $D$, a vertex of color $D_2$ needs one neighbor of color $D^*$, and a vertex of color $D_2$ needs one neighbor in $D$, and a vertex of color $D_2$ needs one neighbor of color $D^*$.

Then we can set

- $L_v = \{D_1, D_2, \overline{D}, \overline{D}^*\}$;
- $(\text{Weights, } \preceq) = (\mathbb{R} \cup \{+\infty\}, \leq)$ and $\oplus = +$;
- $w_{v,D_1} = w_{v,D_2} = 1$, and $w_{v,\overline{D}} = w_{v,\overline{D}^*} = 0$;
- $\text{check}(v,c) = (c(v) \in \{\overline{D}, D_1\} \Rightarrow \exists u \in N_G(v) / c(u) \in \{D_1, D_2\})$
  $\wedge (c(v) = D_2 \Rightarrow \exists u \in N_G(v) / c(u) = \overline{D}^*)$
  $\wedge (c(v) = \overline{D}^* \Rightarrow \exists u, w \in N_G(v) / u \neq w \wedge c(u), c(w) \in \{D_1, D_2\})$;
- $\Pi = \emptyset$;
- $N_{v,i} = \{0, 1, 2\}$;
- $n \oplus n' = \min(n + n', 2)$;
- If $i \in \{D_1, \overline{D}, \overline{D}^*\}$:
  - $\text{new}N_{v,i}(u,j) = \begin{cases} 1 & \text{if } j \in \{D_1, D_2\} \\ 0 & \text{otherwise} \end{cases}$
  - If $i = D_2$:
    - $\text{new}N_{v,i}(u,j) = \begin{cases} 1 & \text{if } j = \overline{D}^* \\ 0 & \text{otherwise} \end{cases}$
- $\text{check}_{v,i}(n) = (i \in \{\overline{D}, D_1, D_2\} \Rightarrow n \geq 1) \wedge (i = \overline{D}^* \Rightarrow n \geq 2)$.

Notice that in the restatement of semitotal domination we changed the “duties” of the vertices: the ones in charge of checking that a vertex of color $D_2$ is at distance 2 of another vertex in $D$ are now the vertices of color $\overline{D}^*$.

Combining the ideas of the two previous problems we can solve a large number of related problems, such as total distance 2-domination.

7.2.3. Problems involving edges. We consider two kind of problems: when edges do not have requirements over other edges, and when they do.
For the first class of problems, consider the graph $G' = (V(G) \cup E(G), E'(G))$ obtained by subdividing each edge and (possibly, depending on the requirements of the problem) also keeping the original edge (note that $tw(G') \leq tw(G) + 1$). We might need to duplicate the colors in order to differentiate the colors assigned to edges from the colors assigned to vertices, so that the checking functions can distinguish them. As an example, we show how to model vertex cover:

- $L_v = \{0, 1\}$ for all $v \in V(G)$ and $L_{uv} = \{0\}$ for all $uv \in E(G)$;
- $(\text{WEIGHTS}, \preceq) = (\mathbb{R} \cup \{+\infty\}, \preceq)$ and $\oplus = +$;
- $w_{v', i} = i$ for all $v' \in V(G')$, $i \in L_{v'}$;
- $check(v, c) = \text{TRUE}$ for all $v \in V(G)$, and $check(uv, c) = (c(u) + c(v) \geq 1)$ for all $uv \in E(G)$;
- $\Pi = \emptyset$;
- $N_{v', i} = \{0, 1\}$;
- $n \oplus^{v', i} n' = \min(n + n', 1)$;
- new$N_{v', i}(u', j) = j$; and
- $check_{v, i}(n) = \text{TRUE}$ for all $v \in V(G)$, $i \in L_v$, and $check_{uv, i}(n) = (n \geq 1)$ for all $uv \in E(G)$.

Edge cover is basically the same as vertex cover but interchanging vertices and edges. As regards the second class of problems involving edges, we illustrate the maximum matching problem, for which we can set:

- $L_v = \{0\}$ for all $v \in V(G)$ and $L_{uv} = \{0, 1\}$ for all $uv \in E(G)$;
- $(\text{WEIGHTS}, \succeq) = (\mathbb{R} \cup \{-\infty\}, \succeq)$ and $\oplus = +$;
- $w_{v', i} = i$ for all $v' \in V(G')$, $i \in L_{v'}$;
- $check(v, c) = \left(\sum_{u \in N_G(v)} c(u) \leq 1\right)$ for all $v \in V(G)$, and $check(uv, c) = \text{TRUE}$ for all $uv \in E(G)$;
- $\Pi = \emptyset$;
- $N_{v', i} = \{0, 1\}$;
- $n \oplus^{v', i} n' = \min(n + n', 1)$;
- new$N_{v', i}(u', j) = j$; and
- $check_{v, 0}(n) = (n \leq 1)$ for all $v \in V(G)$, and $check_{uv, 0}(n) = \text{TRUE}$ for all $uv \in E(G)$, $i \in L_{uv}$.

Notice that the idea is more similar to the one of semitotal domination, in the sense that the neighbors of the edges (the vertices) are the ones in charge of checking the requirements of the edges (in the case of matching, that not two chosen edges share an endpoint).

7.2.4. LCVP problems. Let $m(S)$ be the maximum of $S$ if $S$ is finite or the maximum of $\overline{S}$ if $S$ is co-finite.

We have already seen (in Section 3) how to model the problem of deciding if $G$ has a $D_q$ partition as an instance of the GLC problem, so now we only need to extend it with a partial neighborhood system:

- $N_{v', i}$ is the Cartesian product of the sets $[1, m(D_q[i, j])]$ for all $1 \leq j \leq q$;
- $n \oplus^{v', i} n'$ is such that $(n \oplus^{v', i} n')_j = \min(n_j + n'_j, m(D_q[i, j]))$ for all $1 \leq j \leq q$;
- new$N_{v', i}(u, j)$ is the tuple such that its $j$th entry is 1 and all its other entries are 0; and
- $check_{v, i}(n) = \land_{1 \leq j \leq q} (n_j \in D_q[i, j])$. 
8. THE GLC PROBLEM IN BOUNDED TREEWIDTH AND BOUNDED DEGREE GRAPHS

Recall the partial neighborhood system defined in Remark 5.1.2, for which $N \leq (c + 2)^{\delta(G)}$. Hence, the next result easily follows.

**Theorem 8.0.1.** Let $F$ be a family of graphs of bounded treewidth and bounded degree, and let $G \in F$. Then there exists a polynomial-time algorithm that solves the GLC problem with $C$ polynomial in $|V(G)|$, $\Pi = \emptyset$, and all functions check, min and $\oplus$ computable in polynomial time.

Furthermore, the next lemma shows that fixed powers of bounded treewidth and bounded degree graphs are also bounded treewidth and bounded degree graphs, therefore extending the results of the previous sections to more problems in these graph classes.

**Lemma 8.0.2.** Let $G$ be a graph and $p \geq 2$. Then

$$\Delta(G) \leq \Delta(G^p) \leq \Delta(G)^p$$

and

$$\max(\text{tw}(G), \Delta(G)) \leq \text{tw}(G^p) \leq (\text{tw}(G) + 1)(\Delta(G) + 1)^{\lceil \frac{p}{2} \rceil} - 1.$$

**Proof.** The inequality $\Delta(G) \leq \Delta(G^p) \leq \Delta(G)^p$ follows easily from the definition of power of a graph. Let $v$ be a vertex of $G$ of maximum degree. In $G^p$, the graph induced by $N_G[v]$ is a clique of size $\Delta(G) + 1$ and, by Theorem 2.4.3, we get that $\text{tw}(G^p) \geq \Delta(G)$. Since $G$ is a subgraph of $G^p$ and by Proposition 2.4.2, we have $\text{tw}(G^p) \geq \text{tw}(G)$.

Now assume $(T, \{X_t\}_{t \in V(T)})$ is a tree decomposition of $G$. For every $t \in V(T)$, let $Y_t$ be the set of vertices that are at distance less than or equal to $\lceil \frac{p}{2} \rceil$ from a vertex of $X_t$. We will prove that $(T, \{Y_t\}_{t \in V(T)})$, is a tree decomposition of $G^p$.

Clearly, $\bigcup_{t \in V(T)} Y_t = V(G) = V(G^p)$, so property (W1) holds.

Let $u, v$ be two vertices that are neighbors in $G^p$. If they are also neighbors in $G$, then there exists a bag $X_t$ that contains both of the vertices and since $X_t \subseteq Y_t$, we get that property (W2) holds in this case. If not, there exists a vertex $w$ that is at distance at most $\lceil \frac{p}{2} \rceil$ from both $u$ and $v$ in $G$. Therefore, since there exists a bag $X_t$ that contains $w$, this implies that $w \in Y_t$ (because $X_t \subseteq Y_t$) and $u, v \in Y_t$ (because $u, v$ are at distance at most $\lceil \frac{p}{2} \rceil$ from $w \in X_t$). Consequently, property (W2) also holds in this remaining case.

Now we will prove that (W3) holds. For all $u \in V(G)$, let $T_u^X = T[t \in V(T) : u \in X_t]$ and $T_u^Y = T[t \in V(T) : u \in Y_t]$. Applying property (W3) to $(T, \{X_t\}_{t \in V(T)})$, we obtain that $T_u^X$ is a subtree of $T$ for every $u \in V(G)$. Let $v \in V(G)$. We will prove that $T_v^Y$ is connected. By definition of the bags $Y_t$, we know that $v \in Y_t$ if and only if $t \in V(T_u^X)$ or $t \in V(T_u^X)$ for some $u$ such that $d(v, u) \leq \lceil \frac{p}{2} \rceil$. Let $t \in V(T_v^X)$. Notice that in order to prove that $T_v^Y$ is connected it is sufficient to prove that there exists a path in $T$ between $t$ and every $s \in V(T_v^X) - V(T_v^X)$. Let $s \in V(T_v^X) - V(T_v^X)$ and let $v_s \in Y_s$ be such that $d(v, v_s) \leq \lceil \frac{p}{2} \rceil$. Since $T_u^X$ is a subtree of $T$ for every $u \in V(G)$, and $V(T_u^X) \cap V(T_u^X) \neq \emptyset$ for all $uv \in E(G)$ (because of property (W2) applied to $(T, \{X_t\}_{t \in V(T)})$ and the edge $uv$), and there exists a path in $G$ between $v$ and $v_s$, we get that there exists a path in $T$ between $t$ and $s$. Therefore (W3) holds for $(T, \{Y_t\}_{t \in V(T)})$.

Since every bag $Y_t$ has at most $(\text{tw}(G) + 1)(\Delta(G) + 1)^{\lceil \frac{p}{2} \rceil}$ vertices, we obtain $\text{tw}(G^p) \leq (\text{tw}(G) + 1)(\Delta(G) + 1)^{\lceil \frac{p}{2} \rceil} - 1$. \qed
The next result follows directly from the previous lemma, the results in Section 5 and the ideas in Section 7.

**Corollary 8.0.3.** Let $\mathcal{F}$ be a family of graphs of bounded treewidth and bounded degree. Given $G \in \mathcal{F}$ with a tree-decomposition of width $k$, and a fixed number $p$, label the edges of $G^p$ with the distance of their endpoints in $G$. Then the algorithm of Section 5 can be instantiated to solve, in polynomial time, distance coloring problems [13, 17, 18, 29, 36, 38, 39, 60], distance independence [28], distance domination problems [42], and distance LCVP problems [48], for bounded distances, for the input graph $G$.

A similar result has been obtained by Jaffke, Kwon, Strømme and Telle for the distance versions of the LCVP problems in bounded MIM-width graphs [48].

9. **Conclusions and further work**

We introduced the *generalized locally checkable problem* which encompasses many problems with locally checkable restrictions, and showed that it generalizes some previous attempts of doing so. While in these former frameworks the restrictions have to be expressed using logic formulas or matrices, in the generalized locally checkable problem they are expressed as *functions*. This feature brings our problem statement closer to natural language, making it easier to employ.

We introduced the concept of *partial neighborhood system*, which is very natural despite its technical definition. A partial neighborhood system allows us to restructure the function $\text{check}$ in simpler terms, and provides us with tools to collect and compress the information of the neighbors of a vertex.

In Theorem 5.6.1 we proved that, for bounded treewidth graphs, there exists a polynomial-time algorithm that solves the generalized locally checkable problem with a given partial neighborhood system (under appropriate conditions, such as having the number of colors and partial neighborhoods polynomial in $|V(G)|$, and having functions $\text{check}_{v,i}$ computable in polynomial time). Moreover, in Corollary 5.6.2 we prove that, for some instances, we do not need to describe the partial neighborhood system in order to obtain a polynomial-time algorithm. The conditions required are in fact very natural and many well known locally checkable problems satisfy them (for example, $k$-coloring, $\{k\}$-domination and $\{k\}$-packing function, for constant $k$). Nonetheless, better choices of the partial neighborhood systems can lead to algorithms with a better time complexity (linear in most cases, as we show in Section 7).

We also showed how to solve instances including some global properties (Section 6). As a result, we can solve problems like acyclic coloring and connected dominating set in polynomial-time for bounded treewidth graphs.

By modeling double Roman domination (and some of its variants), minimum chromatic violation, and Grundy (total) domination as instances of the GLC problem, we proved that these problems are polynomial time solvable in bounded treewidth graphs (Section 7.1). In addition, we proved that $b$-coloring is polynomial time solvable in bounded treewidth and bounded degree graphs.

Finally, we proved that for any instance of the GLC problem where $G$ is in a class of graphs of bounded treewidth and bounded degree, the number of colors is polynomial in $V(G)$, $\Pi = \emptyset$, and all functions $\text{check}$, $\min$ and $\oplus$ are computable in polynomial time, there exists a polynomial-time algorithm that solves it (Theorem 8.0.1). Furthermore, by proving that (fixed) powers of bounded degree and bounded treewidth graphs are also bounded degree and bounded treewidth graphs (Lemma 8.0.2), we were able to enlarge the family of problems that can be solved in polynomial time for these graph
classes, including, for example, distance coloring problems and distance domination problems for bounded distances (Corollary 8.0.3).

Possible future lines of research include developing polynomial-time algorithms for other graph classes (and analyzing what restrictions are necessary for the other parameters of the problem), providing classes of global properties for which we can extend the algorithm in Section 5, studying NP-complete instances, and modeling more problems as instances of the GLC problem.

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REFERENCES

[1] H. Abdollahzadeh Ahangar, M. Chellali, and S. Sheikholeslami. Outer independent double Roman domination. *Applied Mathematics and Computation*, 364:124617, 2020.

[2] H. A. Ahangar, M. Chellali, and S. M. Sheikholeslami. On the double Roman domination in graphs. *Discrete Applied Mathematics*, 232:1–7, 2017.

[3] S. Akbari, M. Ghanbari, R. Manaviyat, and S. Zare. On the lucky choice number of graphs. *Graphs and Combinatorics*, 29(2):157–163, Mar. 2013.

[4] G. Argiroffo, V. Leoni, and P. Torres. \( k \)-domination for chordal graphs and related graph classes. *Electronic Notes in Discrete Mathematics*, 44:219–224, 2013.

[5] S. Arnborg, J. Lagergren, and D. Seese. Easy problems for tree-decomposable graphs. *Journal of Algorithms*, 12(2):308–340, 1991.

[6] R. A. Beeler, T. W. Haynes, and S. T. Hedetniemi. Double Roman domination. *Discrete Applied Mathematics*, 211:23–29, 2016.

[7] H. L. Bodlaender. A partial \( k \)-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1):1–45, 1998.

[8] H. L. Bodlaender, P. G. Drange, M. S. Dregi, F. V. Fomin, D. Lokshtanov, and M. Pilipczuk. An \( O(ckn) \) 5-approximation algorithm for treewidth. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 499–508, 2013.

[9] M. Braga, D. Delle Donne, M. Escalante, J. Marenco, M. Ugarte, and M. Varaldo. The minimum chromatic violation problem: a polyhedral study. *Electronic Notes in Discrete Mathematics*, 62:309–314, 2017. LAGOS’17 – IX Latin and American Algorithms, Graphs and Optimization.

[10] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey*, volume 3 of *SIAM Monographs on Discrete Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[11] B. Brešar, T. Gologranc, M. Milanič, D. F. Rall, and R. Rizzi. Dominating sequences in graphs. *Discrete Mathematics*, 336:22–36, 2014.

[12] B. Brešar, M. A. Henning, and D. F. Rall. Total dominating sequences in graphs. *Discrete Mathematics*, 339(6):1665–1676, 2016.

[13] B. Brešar, S. Klavžar, and D. F. Rall. On the packing chromatic number of Cartesian products, hexagonal lattice, and trees. *Discrete Applied Mathematics*, 155(17):2303–2311, 2007.

[14] B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theoretical Computer Science*, 511:66–76, 2013.

[15] T. Calamoneri. The \( L(h,k) \)-Labelling Problem: A Survey and Annotated Bibliography. *The Computer Journal*, 49(5):585–608, 05 2006.

[16] D. Cattaneo and S. Perdrix. The parameterized complexity of domination-type problems and application to linear codes. In T. V. Gopal, M. Agrawal, A. Li, and S. B. Cooper, editors, *Theory and Applications of Models of Computation - 11th Annual Conference, TAMC 2014, Chennai, India, April 11-13, 2014. Proceedings*, volume 8402 of *Lecture Notes in Computer Science*, pages 86–103. Springer, 2014.

[17] G. J. Chang and D. Kuo. The \( L(2,1) \)-labeling problem on graphs. *SIAM Journal on Discrete Mathematics*, 9(2):309–316, 1996.
[18] G. J. Chang and C. Lu. Distance-two labelings of graphs. *European Journal of Combinatorics*, 24(1):53–58, 2003.
[19] G. J. Chang, J. Wu, and X. Zhu. Rainbow domination on trees. *Discrete Applied Mathematics*, 158(1):8–12, 2010.
[20] E. J. Cockayne, R. M. Dawes, and S. T. Hedetniemi. Total domination in graphs. *Networks*, 10(3):211–219, 1980.
[21] E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi, and S. T. Hedetniemi. Roman domination in graphs. *Discrete Mathematics*, 278(1):11–22, 2004.
[22] B. Courcelle and M. Mosbah. Monadic second-order evaluations on tree-decomposable graphs. *Theoretical Computer Science*, 109(1):1078–1081, 2009.
[23] S. Czerwiński, J. Grytczuk, and W. Żelazny. Lucky labelings of graphs. *Information Processing Letters*, 109(18):1078–1081, 2009.
[24] J. Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17:449–467, 1965.
[25] A. T. Egunjobi and T. W. Haynes. Perfect double Roman domination of trees. *Discrete Applied Mathematics*, 2020.
[26] E. Eiben, R. Ganian, and J. Lauri. On the complexity of rainbow coloring problems. *Discrete Applied Mathematics*, 246:38–48, 2018. The Combinatorics of Graphs and Strings.
[27] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979.
[28] H. Eto, F. Guo, and E. Miyano. Distance-d independent set problems for bipartite and chordal graphs. *J. Comb. Optim.*, 27(1):88–99, Jan. 2014.
[29] J. Fiala and P. A. Golovach. Complexity of the packing coloring problem for trees. *Discrete Applied Mathematics*, 158(7):771–778, 2010. Third Workshop on Graph Classes, Optimization, and Width Parameters Eugene, Oregon, USA, October 2007.
[30] J. F. Fink and M. S. Jacobson. n-Domination in Graphs, pages 283–300. John Wiley & Sons, Inc., USA, 1985.
[31] P. Firby and J. Haviland. Independence and average distance in graphs. *Discrete Applied Mathematics*, 75(1):27–37, 1997.
[32] M. Frick and M. Grohe. The complexity of first-order and monadic second-order logic revisited. *Annals of Pure and Applied Logic*, 130(1):3–31, 2004. Papers presented at the 2002 IEEE Symposium on Logic in Computer Science (LICS).
[33] E. Galby, A. Munaro, and B. Ries. Semitotal domination: New hardness results and a polynomial-time algorithm for graphs of bounded mim-width. *Theoretical Computer Science*, 814:28–48, 2020.
[34] R. Gallant, G. Gunther, B. Hartnell, and D. Rall. Limited packings in graphs. *Discrete Applied Mathematics*, 158(12):1357–1364, 2010.
[35] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., New York, NY, USA, 1979.
[36] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris, and D. F. Rall. Broadcast chromatic numbers of graphs. *Ars Combinatoria*, 86:33–49, 2008.
[37] W. Goddard, M. A. Henning, and C. A. McPillan. Semitotal domination in graphs. *Utilitas Mathematica*, 94:67–81, 2014.
[38] D. Gonçalves. On the L(p,1)-labelling of graphs. In S. Felsner, editor, *2005 European Conference on Combinatorics, Graph Theory and Applications (EuroComb ’05)*, volume DMTCS Proceedings vol. AE, European Conference on Combinatorics, Graph Theory and Applications (EuroComb ’05) of *DMTCS Proceedings*, pages 81–86, Berlin, Germany, 2005. Discrete Mathematics and Theoretical Computer Science.
[39] J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM Journal on Discrete Mathematics*, 5(4):586–595, 1992.
[40] B. Grünbaum. Acyclic colorings of planar graphs. *Israel Journal of Mathematics*, 14(4):390–408, 1973.
[41] F. Harary and T. W. Haynes. Double domination in graphs. *Ars Combinatoria*, 55:201–213, 04 2000.
[42] T. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of Domination in Graphs*. Boca Raton: CRC Press, 1998.
[43] J. He and H. Liang. Complexity of total \( k \)-domination and related problems. In M. Atallah, X.-Y. Li, and B. Zhu, editors, *Frontiers in Algorithmics and Algorithmic Aspects in Information and Management*, pages 147–155, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[44] M. A. Henning. Distance domination in graphs. In T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, editors, *Domination in Graphs: Advanced Topics*, chapter 12. Marcel Dekker, Inc., 1997.

[45] M. A. Henning and A. P. Kazemi. \( k \)-tuple total domination in graphs. *Discrete Applied Mathematics*, 158(9):1006–1011, 2010.

[46] J. E. Hopcroft and J. D. Ullman. *Introduction To Automata Theory, Languages, And Computation*. Addison-Wesley Longman Publishing Co., Inc., USA, 1st edition, 1990.

[47] R. W. Irving and D. F. Manlove. The \( b \)-chromatic number of a graph. *Discrete Applied Mathematics*, 91(1):127–141, 1999.

[48] L. Jaffke, Joung Kwon, T. J. F. Strekke, and J. A. Telle. Generalized Distance Domination Problems and Their Complexity on Graphs of Bounded \( mim \)-width. In C. Paul and M. Pilipczuk, editors, *13th International Symposium on Parameterized and Exact Computation (IPEC 2018)*, volume 115 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 6:1–6:14, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[49] R. M. Karp. *Reducibility among Combinatorial Problems*, pages 85–103. Springer US, Boston, MA, 1972.

[50] T. Kloks. *Treewidth*, volume 842 of *Lecture Notes in Computer Science*. Springer-Verlag Berlin Heidelberg, 1 edition, 1994.

[51] E. Kubicka and A. J. Schwenk. An introduction to chromatic sums. In *Proceedings of the 17th Conference on ACM Annual Computer Science Conference*, CSC ’89, pages 39–45, New York, NY, USA, 1989. Association for Computing Machinery.

[52] V. A. Leoni and E. G. Hinrichsen. \( k \)-packing functions of graphs. In P. Fouilhoux, L. E. N. Gouveia, A. R. Mahjoub, and V. T. Paschos, editors, *Combinatorial Optimization*, pages 325–335, Cham, 2014. Springer International Publishing.

[53] H. Maimani, M. Momeni, S. Nazari Moghaddam, F. Rahimi Mahid, and S. Sheikholeslami. Independent double Roman domination in graphs. *Bulletin of the Iranian Mathematical Society*, 46:543–555, 2020.

[54] A. Meir and J. W. Moon. Relations between packing and covering numbers of a tree. *Pacific Journal of Mathematics*, 61(1):225–233, 1975.

[55] Ø. Ore. *Theory of graphs*, volume XXXVIII of *Colloquium Publications*. American Mathematical Society, Providence, 3rd edition, 1962.

[56] N. Robertson and P. Seymour. Graph minors. III. Planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984.

[57] N. Robertson and P. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *Journal of Algorithms*, 7(3):309–322, 1986.

[58] E. Sampathkumar and H. B. Walikar. The connected domination number of a graph. *Journal of Mathematical and Physical Sciences*, 13:607–613, 1979.

[59] Z. Shao, J. Amjadi, S. M. Sheikholeslami, and M. Valinavaz. On the total double Roman domination. *IEEE Access*, 7:52035–52041, 2019.

[60] C. Sloper. An eccentric coloring of trees. *Australasian Journal of Combinatorics*, 29:309–321, 2004.

[61] J. A. Telle. *Vertex Partitioning Problems: Characterization, Complexity and Algorithms on Partial k-Trees*. PhD thesis, University of Oregon, 1994.

[62] J. A. Telle and A. Proskurowski. Algorithms for vertex partitioning problems on partial \( k \)-trees. *SIAM Journal on Discrete Mathematics*, 10(4):529–550, 1997.

[63] M. Thorup. All structured programs have small tree width and good register allocation. *Information and Computation*, 142(2):159–181, 1998.

[64] V. G. Vizing. Vertex colorings with given colors. *Diskret. Analiz*, 29:3–10, 1976.

[65] M. Yannakakis and F. Gavril. Edge dominating sets in graphs. *SIAM Journal on Applied Mathematics*, 38(3):364–372, 1980.
APPENDIX A. PROBLEMS DEFINITIONS

We define here the decision versions of the problems mentioned along the paper. Other similar problems can also be modeled as instances of the GLC problem.

A.1. Domination problems.

**DOMINATING SET [55]**

Instance: A (weighted) graph $G$ and a positive integer $k$.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most $k$ and such that $|N[v] \cap S| \geq 1$ for every $v \in V(G)$?

**TOTAL DOMINATION [20]**

Instance: A (weighted) graph $G$ and a positive integer $k$.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most $k$ and such that $|N(v) \cap S| \geq 1$ for every $v \in V(G)$?

**k-TUPLE DOMINATION [41]**

Instance: A (weighted) graph $G$, a positive integer $k$ and a positive integer $\ell$.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most $\ell$ and such that $|N[v] \cap S| \geq k$ for every $v \in V(G)$?

**TOTAL k-TUPLE DOMINATION [45]**

Instance: A (weighted) graph $G$, a positive integer $k$ and a positive integer $\ell$.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most $\ell$ and such that $|N(v) \cap S| \geq k$ for every $v \in V(G)$?

**k-DOMINATION [30]**

Instance: A (weighted) graph $G$, a positive integer $k$ and a positive integer $\ell$.

Question: Does there exist $S \subseteq V(G)$ of size (weight) at most $\ell$ and such that $|N(v) \cap S| \geq k$ for every $v \in V(G) \setminus S$?

**\{k\}-DOMINATION [43]**

Instance: A graph $G$, a positive integer $k$ and a positive integer $\ell$.

Question: Does there exist a function $f : V(G) \to \{0, 1, \ldots, k\}$ of weight at most $\ell$ and such that $f(N[v]) \geq k$ for every $v \in V(G)$?

**k-RAINBOW DOMINATION [19]**

Instance: A graph $G$, a positive integer $k$ and a positive integer $\ell$.

Question: Does there exist a function $f : V(G) \to 2^{\{1, \ldots, k\}}$ of weight $(\sum_{v \in V(G)} |f(v)| \leq k)$ at most $\ell$ and such that for every vertex $v \in V(G)$ for which $f(v) = \emptyset$ we have $\bigcup_{u \in NG[v]} f(u) = \{1, \ldots, k\}$?

**SEMITOTAL DOMINATING SET [37]**

Instance: A (weighted) graph $G$ with no isolated vertex and a positive integer $k$.

Question: Does there exist a dominating set $D \subseteq V(G)$ of size (weight) at most $k$ and such that every vertex in $D$ is within distance two of another vertex in $D$?

**DISTANCE k-DOMINATION (also called k-COVERING) [44, 54]**

Instance: A (weighted) graph $G$, a positive integer $k$ and a positive integer $\ell$.

Question: Does there exist a set $S \subseteq V(G)$ of size (weight) at most $\ell$ and such that every vertex in $G$ is within distance $k$ from a vertex in $S$?

**CONNECTED DOMINATING SET [58]**

Instance: A (weighted) graph $G$ and a positive integer $k$.

Question: Does there exist a dominating set of $G$ of size (weight) at most $k$ that induces a connected subgraph of $G$?
Roman domination [21]

**Instance:** A graph $G$ and a positive integer $k$.

**Question:** Does there exist a function $f : V(G) \rightarrow \{0, 1, 2\}$ of weight at most $k$ and such that every vertex $u \in V(G)$ for which $f(u) = 0$ is adjacent to at least one vertex $v \in V(G)$ for which $f(v) = 2$?

Grundy total domination [12]

**Instance:** A graph $G$ and a positive integer $k$.

**Question:** Does there exist a sequence $(v_1, \ldots, v_\ell)$ of distinct vertices of $G$ such that $\ell \geq k$, $\{v_1, \ldots, v_\ell\}$ is a dominating set of $G$ and $N(v_i) - \bigcup_{j=1}^{i-1} N(v_j) \neq \emptyset$ for each $i$?

A.2. Coloring problems.

$k$-COLORING [35]

**Instance:** A graph $G$ and a positive integer $k$.

**Question:** Does there exist a function $c : V(G) \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$?

LIST-COLORING [27, 64]

**Instance:** A graph $G$ and a set $L(v)$ of colors for each vertex $v \in V(G)$.

**Question:** Does there exist a proper coloring $c$ such that $c(v) \in L_v$ for all $v \in V(G)$?

$k$-CHROMATIC SUM [51]

**Instance:** A graph $G$ and a positive integer $k$.

**Question:** Is there a proper coloring $c$ of the graph $G$ such that $\sum_{v \in V} c(v) \leq k$?

ADDITIVE COLORING (also called LUCKY LABELING) [23]

**Instance:** A graph $G$ and a positive integer $k$.

**Question:** Does there exist a function $f : V(G) \rightarrow \{1, \ldots, k\}$ such that for every two adjacent vertices $u, v$ the sums of numbers assigned to their neighbors are different (that is, $\sum_{w \in N(u)} f(w) \neq \sum_{z \in N(v)} f(z)$)?

ACYCLIC COLORING [40]

**Instance:** A graph $G$ and a positive integer $k$.

**Question:** Does there exist a $k$-coloring of $G$ such that of every subgraph of $G$ spanned by vertices of two of the colors is acyclic (in other words, is a forest)?

$L(h, k)$-LABELING [15]

**Instance:** A graph $G$ and positive integers $h$, $k$ and $s$.

**Question:** Does there exist a labeling of its vertices by integers between 0 and $s$ such that adjacent vertices of $G$ are labeled using colors at least $h$ apart, and vertices having a common neighbor are labeled using colors at least $k$ apart?

A.3. Independence problems.

INDEPENDENT SET [35]

**Instance:** A (weighted) graph $G$ and a positive integer $k$.

**Question:** Does there exist an independent set of $G$ of size (weight) at least $k$?

$k$-INDEPENDENT SET (also called DISTANCE $d$-INDEPENDENT SET) [28, 31]

**Instance:** A graph $G$, a positive integer $k$ and a positive integer $s$.

**Question:** Does there exist $X \subseteq V(G)$ of size at least $s$ such that the distance between every two vertices of $X$ is at least $k + 1$?
A.4. Packing problems.

{k}-packing function [52]
Instance: A graph $G$, a positive integer $k$ and a positive integer $\ell$.
Question: Does there exist a function $f : V(G) \rightarrow \{0, 1, \ldots, k\}$ of weight at least $\ell$ and such that $f(N[v]) \leq k$ for every $v \in V(G)$?

{k}-limited packing [34]
Instance: A graph $G$, a positive integer $k$ and a positive integer $\ell$.
Question: Does there exist a function $f : V(G) \rightarrow \{0, 1\}$ of weight at least $\ell$ and such that $f(N[v]) \leq k$ for every $v \in V(G)$?

Packing chromatic number [13]
Instance: A graph $G$ and a positive integer $k$.
Question: Can $G$ be partitioned into disjoint classes $X_1, \ldots, X_k$ where vertices in $X_i$ have pairwise distance greater than $i$?

A.5. Problems involving edges.

Matching [24]
Instance: A(n) (edge weighted) graph $G$ and a positive integer $k$.
Question: Does there exist a set $M \subseteq E(G)$ of pairwise non-adjacent edges of size (weight) at least $k$?

Edge domination [65]
Instance: A(n) (edge weighted) graph $G$ and a positive integer $k$.
Question: Does there exist $F \subseteq E(G)$ of size (weight) at most $k$ and such that every edge in $E(G)$ shares an endpoint with at least one edge in $F$?

Vertex cover [49]
Instance: A (weighted) graph $G$ and a positive integer $k$.
Question: Does there exist $S \subseteq V(G)$ of size (weight) at most $k$ and such that every edge in $E(G)$ has at least one endpoint in $S$?

Edge cover [35]
Instance: A(n) (edge weighted) graph $G$ and a positive integer $k$.
Question: Does there exist $F \subseteq E(G)$ of size (weight) at most $k$ and such that every vertex in $V(G)$ belongs to at least one edge in $F$?

Universidad de Buenos Aires. Facultad de Ciencias Exactas y Naturales. Departamento de Computación. Buenos Aires, Argentina. / CONICET-Universidad de Buenos Aires. Instituto de Investigación en Ciencias de la Computación (ICC). Buenos Aires, Argentina.

E-mail address: fbonomo@dc.uba.ar

CONICET-Universidad de Buenos Aires. Instituto de Investigación en Ciencias de la Computación (ICC). Buenos Aires, Argentina.

E-mail address: cgonzalez@dc.uba.ar