STRING THEORY, SCALE RELATIVITY AND
THE GENERALIZED UNCERTAINTY PRINCIPLE

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ABSTRACT

Extensions (modifications) of the Heisenberg Uncertainty principle are derived within
the framework of the theory of Special Scale-Relativity proposed by Nottale. In particular,
generalizations of the Stringy Uncertainty Principle are obtained where the size of the
strings is bounded by the Planck scale and the size of the Universe. Based on the fractal
structures inherent with two dimensional Quantum Gravity, which has attracted consider-
able interest recently, we conjecture that the underlying fundamental principle behind
String theory should be based on an extension of the Scale Relativity principle where both
dynamics as well as scales are incorporated in the same footing.

I. INTRODUCTION

In recent years considerable attention has been given to the generalizations of Heisen-
berg’s Uncertainty principle and to formulations of nonlinear Quantum Mechanics [1,2,3].
It was argued [3] that an extension of quantum mechanics might be required in order to
accomodate gravity. Although the current work on string theory suggests that no modifi-
cations of Quantum Mechanics might be necessary to include gravity, evidence of Planck
scale physics suggests otherwise. The poor understanding of physics at very short distances
indicates that the small scale structure of space-time might not be adequately described
by classical continuum geometry. One new theory consistent with the Standard Model of
particle physics is Connes [4] non-commutative geometry. Roughly it is based on the non-
commutative algebra of functions defined on a manifold whose ‘points’ have been smeared
( into operators) according to the spacetime foam picture at tiny distances.

In the past two years there has been a considerable advance in the non-perturbative
behaviour of string theory where the existence of duality symmetries in the theory indicate
that strings do not distinguish small spacetime scales from large ones. This is the so-called
T duality symmetry. This in turn calls for a modification of Heisenberg’s Uncertainty
principle where beyond Planck scale energies the size of the string grows with momenta
instead of the opposite. For a pedagogical introduction on the new description of spacetime
as a result of string theory see Witten [28].

Sometime ago [5,6] argued that some sort of enlarged equivalence principle is operating
in string theory in which dynamics is not only independent of coordinate transformations
but also of structures occurring at distances shorter than the fundamental string length,
$\lambda_s = \sqrt{2\alpha\hbar}$, in units $c = 1$; i.e. distances smaller than $\lambda_s$ are not relevant in string
theory. Many arguments have been given to support this fact: high energy scattering at
fixed angle [7]; renormalization group theory analysis based on a discretized version of
Polyakov’s generating functional [8,9]; high temperature behaviour of the free energy [10]; the above-mentioned duality of strings and other higher dimensional extended objects: p-branes or extendons [11]; particle size-growth with momentum related to information spreading near black hole horizons [12].

This all suggests that below the Planck length the very concept of spacetime changes meaning and that the Heisenberg Uncertainty principle needs to be modified. The main purpose of this letter is to write down generalizations of the Heisenberg’s Uncertainty relations.

One formula which is a faithful interpolation of the results above is:

\[
\frac{\Delta X^\mu}{\lambda_s} \geq \left( \frac{p_s}{\Delta p^\mu} \right) + \left( \frac{\Delta p^\mu}{p_s} \right), \quad p_s \lambda_s = \hbar.
\]

Eq-(1) is valid in flat backgrounds and holds for each spacetime component.

In this letter we will generalize eq-(1) and write down a more accurate equation that encodes the scale relativity principle as well as incorporating the duality principle in string theory. Despite many efforts, the fundamental principles underlying string theory are still unknown, in particular the search for higher symmetries. The second purpose of this letter is to conjecture that the special theory of scale relativity recently proposed by Nottale [13,14] must play a fundamental role in string theory, specially in regards to the fact that this theory demonstrates that there is a universal, absolute and impassable scale in Nature, which is invariant under dilatations. This lower limit is the Planck scale.

It was emphasized by Nottale in his book that a full motion plus scale relativity including all spacetime components, angles and rotations remains to be constructed. In particular the general theory of scale relativity. Our aim is to show that string theory provides an important step in that direction and vice versa: the scale relativity principle must be operating in string theory. The cosmological implications of such principle allowed Nottale to provide a very simple and elegant proposal for the resolution of the cosmological constant problem [14]. In particular, the fundamental scales in Nature are determined by constraints which are set at both the small and large scales. This is in perfect accordance with the duality principle in string theory. Applying the scale relativity principle to the Universe one arrives at the conclusion that there must exist an absolute, impassable, upper scale in Nature which is invariant under dilatations (invariant under the expansion of the Universe) which would hold all the properties of infinity. This upper scale, \(L\), defines the "radius" of the Universe and, when it is seen at its own resolution, it becomes invariant under dilatations [14]. We must emphasize that scale-relativity does not force the universe to be closed; an open expanding universe can have an upper characteristic scale (invariant under dilatations) \(L\) which holds the property of infinity.

The term in (1): \(\Delta X^\mu \sim \Delta p^\mu\) is a long distance effect due to the emergence of classical gravity effects from nonperturbative quantum string theory dynamics [5,6,7,8,9]. These authors showed that string amplitudes at or above the Planck energy resummed over all orders of perturbation theory around the flat metric and turned out to yield classical gravity effects in appropriate kinematic regions. The \(S\) matrix above Planck energies is dominated by graviton exchange at large impact parameters. G’t Hooft [15] has shown that high energy particle scattering is dominated by graviton exchange. At energies higher than...
black hole production sets in accompanied by coherent emission of real gravitons. In contrast the regime where \( \Delta X^\mu \sim (1/\Delta p^\mu) \) is characterized by ordinary gauge interactions based on standard Quantum Mechanics. So there appears to be two physics regimes which are manifestations of the same quantum string dynamics. One where energies are much larger than the Planck energy and other where the energies are much smaller.

Within the context of scale relativity we propose that the \( \Delta X^\mu \sim \Delta p^\mu \) behaviour originates from the existence of the upper scale in Nature; i.e. it is also a long distance effect pertaining to classical gravity interactions at cosmological scales; whereas the \( \Delta X^\mu \sim \frac{1}{\Delta p^\mu} \) terms are the standard Quantum Mechanics results originating from the fractal structure of spacetime at microphysical scales [13,14,29]. It is in this fashion how string theory and scale relativity merge. At first sight it seems surprising how the upper scale, \( L \), can have a connection to Planck scale physics. This was the basis of Nottale’s proposal to the resolution of the cosmological constant problem. The relationship to string theory is realized via the target space \((T)\) duality symmetry: \( R \leftrightarrow \alpha'/R \); i.e string theory does not distinguish one spacetime (large radius) from the other (small radius). Fractal structures also occur at cosmological scales and for this reason Nottale introduced the upper scale in order to accommodatethe scale relativity principle to the whole universe as well.

The problem of time measurement in quantum gravity has been discussed by [30] and a minimum time interval of the order of the Planck time was found. A review of the series of different physical arguments leading to the Planck scale as the minimum scale in Nature was given by Garay [27]. For a discussion of quantum states for non-perturbative quantum gravity which exhibit a discrete structure at the Planck scale see Ashtekar et al [27].

The authors in [16,17,18,19] have emphasized that the fractal structure of spacetime is a crucial question in any theory of Quantum Gravity. Quantum Mechanics requires a functional average over all possible equivalence classes of metrics and it is in this way how the effective dimension of spacetime is measured. It has been shown [13] that quantum mechanics "arises" from the fractal nature of particle trajectories. In particular, the fractal dimension of space as well as time turned out to be equal to 2 for a point particle in the relativistic domain. Hence, the total fractal space-time dimension is \( 2 + 2 = 4 \). The authors [16] provided ample evidence that the Hausdorff dimension is \( D_H = 4 \) for two-dimensional quantum gravity coupled to matter with a central charge \( c < 1 \) whereas the spectral dimension was \( D_S = 2 \). The fact that the \( D_H, D_S \) were constant for any value of \(-\infty < c < 1\) seems to indicate that both dimensions are intrinsic properties of two-dim quantum gravity independent of the coupling to matter and hence an intrinsic property of the geometry of spacetime. The essence of the observations of [13] was based on the fact that the relevant paths in Feynman’s path integral description of Quantum Mechanics are the fractal trajectories; i.e; continuous but nondifferentiable at any point.

It is the purpose of this letter to bridge both string theory and scale relativity within the framework of Heisenberg’s Uncertainty principle and derive a more general expression than the one in (1) and the one presented in [5-9]. The expression below Eq-(15) is very relevant if one were to include the fractal nature of two-dimensional quantum gravity in string theory; i.e. a fractal two dimensional surface moving in a fractal target spacetime.

The importance of noncontinuous maps in string theory has been discussed in [24]. The space of string configurations in string theory required both continuous and noncontinuous
square integrable maps in order to reproduce the results from the dual models. The size and shape of strings in their ground state in the lightcone gauge was investigated in [25]. It was found that in two-dimensions the extrinsic curvature was divergent. A regularization scheme was used where the string was kept continuous. As the dimensionality of spacetime increased the string became smoother and had divergent average size. This is unphysical since their size cannot exceed the size of the Universe. It is for this reason that the upper scale in nature must also appear in eq-(15). The average curvature diverges in $D = 4$ due to kinks and cusps on a string. It is important to study these properties further.

It has been suggested by many authors that if one wishes to study Planckian physics one must abandon the Archimedean axiom where any given large segment on a straight line can be surpassed by successive addition of small segments along the same line. A non-Archimedean geometry has been proposed using $p-adic$ analysis. For a review of $p-adic$ strings see [26].

Having presented the reasons why scale relativity is a relevant issue in string theory we shall derive the scale-relativity extension of the stringy-uncertainty principle which is a more general uncertainty relation than the ones considered so far.

II. The Generalized Uncertainty Principle

Our goal now is to write down a more precise expression for the enlarged uncertainty principle (1) based on the scale relativity principle. The expression below agrees qualitatively with the results in [20] where for energies not too large the size of the interaction region increases linearly with energy while at higher energies it remains constant. ( see Diagram ).

According to scale relativity all physical quantities depend on the resolution at which they are observed. At scales of the order $r$ the generalized de Broglie-Compton relation is:

$$\ln\left(\frac{m}{m_o}\right) = \frac{\ln\left(\frac{\lambda_o}{r}\right)}{\sqrt{1 - \frac{\ln^2\left(\frac{\lambda_o}{r}\right)}{\ln^2\left(\frac{\lambda_o}{\Lambda}\right)}}} = \ln\left(\frac{\lambda_o}{r}\right)^{\delta}. \quad (2)$$

The quantities $m_o, \lambda_o$ are the reference mass and length scales with respect to which we measure the other scales. If one chose these scales to be the electron’s mass and its de Broglie wavelength this would be the signal of the quantum mechanical-classical physics transition of the electron. Scale relativity proposes that in general $\lambda_o \sim \left(\frac{h}{m_o c}\right)$ instead of being equal and the lowest scale is $\Lambda$, the Planck length $\Lambda = \sqrt{\frac{hG}{c^3}} = \left(\frac{h}{m_p c}\right)$. The square root factor in the second term of the r.h.s of (2) is a Lorentzian-like gamma factor of the same type which appears in special relativity: $m = m_o \left[1 - v^2/c^2\right]^{-1/2}$. Now it plays in (2) the role of a scale-dependent anomalous dimension $\delta = D_F - D_T$; where $D_F$ is the fractal spatial dimension of a fractal curve trajectory and $D_T$ is the topological dimension. At the quantum-classical transition : $r = \lambda_0$ one has $D_F = 2; D_T = 1$ and $\delta$ becomes equal to 1 whereas for other values of $r < \lambda_o; \delta > 1$. Similar considerations apply to the time coordinate [13,14].

We can see that if one sets the minimum length to zero, eq-(2) yields the ordinary Compton-de Broglie relation since for $\delta = 1$ :
\[ p\lambda = p_o\lambda_o = \hbar \]  
Equation (3)

Setting \( p \sim \Delta p \) and \( \delta \sim \Delta x \) in (3) gives the original Heisenberg’s uncertainty relation:

\[ \Delta p\Delta x \sim \hbar \]  
Equation (4)

Whereas from (2) one learns that a plausible modification of the Heisenberg’s uncertainty relation is:

\[ \ln\left(\frac{\Delta p}{p_o}\right) = \ln\left(\frac{\Delta x}{\lambda_o}\right) \sqrt{1 - \frac{\ln^2(\lambda_o/\Delta x)}{\ln^2(\lambda_o/\Lambda)}}. \]  
Equation (5)

Similar considerations apply to the other spacetime coordinates and momenta.

Now we shall concentrate on the large scale behaviour of spacetime. The absolute, impassable, large scale in nature [14] is relevant to fractal structures in cosmology. In particular the cosmological generalized-Schwarzschild mass relation is:

\[ \ln\left(\frac{M}{m}\right) = \ln\left(\frac{R}{\lambda_g}\right) \sqrt{1 - \frac{\ln^2(R/\lambda_g)}{\ln^2(L/\lambda_g)}}. \]  
Equation (6)

where \( m_g, \lambda_g \) are the mass and length scales which signal the classical physics (scale independence) cosmological domain transition (where fractal structures over large scales in the Universe become important); the quantity : \( \lambda_g \sim (Gm_g/c^2) \) where \( G \) is Newton’s constant and \( L \) is the upper, impassable, absolute scale we referred to earlier which is invariant under dilatations. The value of \( L \) was set in [14] to be \( L \sim 10^{61}\Lambda \). To simplify matters we can set \( \hbar = c = 1 \). We also notice that eqs-(2,6) are valid in any logarithm base, we opt to use the natural base.

When the upper scale \( L \to \infty \), eq-(6) becomes the original Schwarzschild mass relation (up to a factor of two):

\[ M \sim \frac{Rc^2}{G} \]  
Equation (7)

Setting \( M \sim \Delta p \) and \( R \sim \Delta x \) in (7) one recovers the behaviour of the second term in the r.h.s of (1) signaling the graviton exchange dominance in high energy string scattering. At higher energies than the Planck energy the size of the string increases instead of decreasing. The energy imparted to the string is used to break the strings into pieces. This picture is the one proposed by [12] consistent with the Bekenstein-Hawking bound for the entropy of a black hole in terms of the horizon’s area: \( S = A/4G \). This is the statement that one cannot have more than one bit of information per unit area in Planck units.

Inverting the relations given by eqs-(2,6) yields:

\[ \ln\left(\frac{\lambda_o}{r}\right) = \frac{\ln\left(\frac{m}{m_o}\right)}{\sqrt{1 + \frac{\ln^2(m/m_o)}{\ln^2(\lambda_o/\Lambda)}}}, \]  
Equation (8)
and:

\[
\ln(R/\lambda_g) = \frac{\ln(M/m_g)}{\sqrt{1 + \frac{\ln^2(M/m_g)}{\ln^2(L/\lambda_g)}}}.
\]  

(9)

We learnt from (1) that the minimum value of \(\Delta X\) is \(\sim \lambda_s\). Let's suppose that one wishes the two curves given by eqs-(8,9) to intersect at the point \((\lambda_p, m_p)\) with \(m_p \equiv (1/\Lambda)\), where \(m_o \sim (1/\lambda_o)\) and \(\lambda_p \neq \Lambda!!\). The latter occurs because in scale relativity the high energy length and mass scales decouple: energy tends to infinity when distances reach the Planck scale.

The intersection of the two graphs described by (8,9) requires to extend the domain of validity of the scaling exponents appearing in eqs-(2,6) and the remaining equations that follow. Therefore we must have the following expressions for the scaling exponents in the extended domains:

\[
\delta_{Compton} = \frac{1}{\sqrt{1 - \frac{\ln^2(\lambda_o/r)}{\ln^2(\lambda_o/L)}}}, \quad r > \lambda_o.
\]  

(10)

\[
\delta_{Schwarzschild} = \frac{1}{\sqrt{1 - \frac{\ln^2(\lambda_g/r)}{\ln^2(\lambda_g/\Lambda)}}}, \quad r < \lambda_g.
\]  

(11)

The reason this is necessary is to ensure that the scaling exponents are always real. In ordinary relativity the velocity cannot exceed the speed of light and the Lorentz dilation factor for this reason is never complex.

With this in mind the intersection of the two graphs described by (8,9) in the region \(\Lambda < \lambda_p < r < \lambda_o\) requires to have:

\[
\ln(\lambda_o/\lambda_p) = \frac{\ln(m_p/m_o)}{\sqrt{1 + \frac{\ln^2(m_p/m_o)}{\ln^2(\lambda_o/\Lambda)}}}.
\]  

(12)

one learns from (12) that \(\Lambda < \lambda_p < \lambda_o\). The other relation is

\[
\ln(\lambda_p/\lambda_g) = \frac{\ln(m_p/m_g)}{\sqrt{1 + \frac{\ln^2(m_p/m_g)}{\ln^2(\lambda_o/\Lambda)}}}.
\]  

(13)

Notice the factor of \(\Lambda\) appearing in the scaling exponent of (13) versus the \(L\) appearing in eq-(9). Eliminating \(\lambda_p\) from (10,11) gives an algebraic relationship between the \(\lambda_o\) and the \(\lambda_g\) scales based on the input values of \(\Lambda, L\). The value of \(m_g\) is \(\sim (\lambda_g/\Lambda^2)\) and of \(m_o \sim 1/\lambda_o\).

We have learnt from scale relativity that the Planck mass \(m_p\) does not correspond any more to the Planck length \(\Lambda\). The Planck length now corresponds to an infinite mass. In motion special relativity it is required an infinite energy to accelerate a particle of non-zero rest mass to the speed of light. Not surprisingly, when we reach the resolution of the Planck’s length an infinite amount of energy is required.
In order to solve for eqs-(12,13) let us assume that $\lambda_s \sim \lambda_p$. The string scale $\lambda_s$ is model dependent \[8,9\]. In the heterotic string $\Lambda = (\alpha_{GUT}/4)\lambda_s \sim 10^{-2}\lambda_s$. Now let us compute the fundamental lengths $\lambda_o, \lambda_g$ starting from eqs-(12,13). An estimate is obtained by setting $\ln(\lambda_o/\Lambda) \sim \ln(m_p/m_o)$. Eq-(12) becomes in this approximation:

$$\lambda_p \sim \lambda_o (\Lambda/\lambda_o)^{1/2}.$$ \quad (14)

Therefore, if we were to impose $\lambda_s \sim \lambda_p$, the length scale $\lambda_o$ is of the order $\lambda_o \sim 10^7 \Lambda \sim 10^{-26}$ cm. In any case we have that $\Lambda < \lambda_s < \lambda_o$. Having fixed $\lambda_o$, eq-(13) generates the other scale $\lambda_g$. For this particular choice for $\lambda_p$ one gets that $\lambda_g \sim \lambda_o \sim 10^7 \Lambda$, also which would not be very physical. Clearly our choice of $\lambda_p, m_p$ were not that appropriate.

The correct choice of intersection point due to the fact that the scaling exponents are not equal in eqs-(12,13) is to choose instead $\lambda_s$ and $m_s = \frac{\sigma_s}{\Lambda} \neq \frac{\lambda_s}{\lambda_o}$ as the true intersection point. The value of $\sigma_s < 1$ so the intersection point lies below the $m_p$ value. Now with the value of $\lambda_s \sim 10^2 \Lambda$ the value of $m_s$ is constrained to obey $m_p > m_s > \frac{10^{-2}}{\Lambda}$. Therefore, now one would obtain more reasonable values for $\lambda_o, \lambda_g$; (see diagram).

In any case one has a free parameter to vary which we can take to be $\lambda_s$ assuming the upper scale $L$ is fixed in terms of $\Lambda$. Tuning the value of $\lambda_s$ furnishes a range of values for the transition scales $\lambda_o, \lambda_g$ for $L = 10^{60} \Lambda$. Other choices for $L$ leaves us even more freedom to work with.

The diagram referred as figure 1, after the extension of the scaling exponents has been performed in the whole regions of $r$, shows the three ”quantum”, ”classical”, ”cosmological” domains from the smallest scale, $\Lambda$ to the largest one $L$. A similar diagram can be found in Nottale’s book \[14 ,ch.7\]. The relevance of this diagram is that the region comprised within the two curves representing the generalized Schwarzschild and Compton-de Broglie formulae (2,3) is similar in shape to the allowed region obtained from the Stringy Uncertainty Principle (1). This is not to say that both regions are identical !.

Now we are in a position to write down the scale relativity extension/modification of the Stringy-Uncertainty Principle. Using eqs-(8,9), replacing masses for momentum quantities , $\Delta p$, and lengths for $\Delta X$ one can infer that :

$$\frac{\Delta X^\mu}{\lambda_o} \geq \frac{1}{2} (\frac{p_o}{\Delta p^\mu})^\alpha + \frac{1}{2} \frac{\lambda_g}{\lambda_o} (\frac{m_o}{m_g})^\beta (\frac{\Delta p^\mu}{p_o})^\beta.$$ \quad (15)

This expression is valid for each spacetime component separately. The momentum dependent coefficients $\alpha, \beta$ are :

$$\alpha = \frac{1}{\sqrt{1 + \frac{\ln^2(\Delta p^\mu/p_o)}{\ln^2(\lambda_s/\lambda)}}}, \quad r < \lambda_o$$ \quad (16)

$$\beta = \frac{1}{\sqrt{1 + \frac{\ln^2(\Delta p^\mu/m_o)}{\ln^2(L/\lambda_g)}}, \quad r > \lambda_g$$ \quad (17)

and :
\[ \alpha = \frac{1}{\sqrt{1 + \frac{\ln^2(\Delta p^\mu / p_o)}{\ln^2(\lambda_o/L)}}}, \quad r > \lambda_o \] (18)

\[ \beta = \frac{1}{\sqrt{1 + \frac{\ln^2(\Delta p^\mu / m_g)}{\ln^2(\lambda / \lambda_g)}}}, \quad r < \lambda_g \] (19)

with the proviso that \( \lambda_o, \lambda_g \) are expressed in terms of \( \Lambda, L \) as explained earlier.

Let's analyse eq-(15) in detail. Firstly, whenever \( \Delta X = \Lambda \) or \( L \) the \( \Delta p = \infty \) or 0. The values of \( \Delta X \) are confined to the Planck scale and the upper impassible scale whereas there is no ultraviolet nor infrared cutoff in the momenta. The limiting behaviour of (15) is :

(i). When \( \Delta p^\mu \) is very small the first term in the r.h.s of (15) dominates and one has the standard Heisenberg’s relation for \( \alpha \to 1 \) when \( \Lambda \to 0 \).

(ii). When \( \Delta p^\mu \) is very large the second term of the r.h.s of (15) dominates and one has

\[ \frac{\Delta X^\mu}{\lambda_g} \geq \frac{(\Delta p^\mu)}{m_g c} \beta. \] (20)

in the \( L \to \infty; \beta \to 1 \) and one recovers the regime where graviton exchange dominates over other interactions and the uncertainty grows with momentum : \( \Delta X \sim \Delta p \).

(iii). The stringy enlarged uncertainty principle is recovered for the following behaviour in the middle region of the diagram \( \lambda_o < r < \lambda_g \):

\[ \alpha, \beta \sim 1. \quad \lambda_o = a\lambda_s, \quad p_o = a^{-1} p_s, \quad \lambda_o p_o = \lambda_s p_s = 1; \quad \lambda_o \sim \frac{1}{m_o}, \quad m_g \sim \frac{\lambda_g}{\Lambda^2}; \] (21)

so that (15) becomes in that region :

\[ \frac{\Delta X^\mu}{\lambda_s} \geq \frac{1}{2} \left( \frac{p_s}{\Delta p^\mu} \right) + \frac{1}{2} \left( \frac{\Lambda}{\lambda_s} \right)^2 \left( \frac{\Delta p^\mu}{p_s} \right). \] (22)

with \( p_s \lambda_s = 1 \). The factors of \( a \) explicitly cancel.

One recognizes that the ”Galilean” limit ( analogous of \( c \to \infty \)) is attained when lowest scale tends to zero and the upper scale tends to infinity. In eq-(22) we can see that the ratio \( (\Lambda/\lambda_s)^2 \) appearing in the second term of the r.h.s of (22) tends to one when the string scale \( \lambda_s \to \Lambda \). Therefore, eq-(22) reduces to eq-(1) in that limit. The results of [5-9] confirm that the string is not sensitive to scales smaller than \( \lambda_s \); i.e scales of the order of the Planck length. Therefore our results based on scale relativity are compatible with the results obtained from string theory.

(iv) A time-energy uncertainty relation which involves the ratio of two scales has been given by Itzhaki [30] by assuming that the gravitational field fluctuations due to the mass-energy of a clock ( whose metric is the Schwarzschild type ) contributes to a time uncertainty of the order :

\[ \Delta t \geq \frac{\hbar}{\Delta E} + \frac{2G \log(x/x_c) \Delta E}{c^5}. \] (23)
with $x_c$ being the shortest distance for which general relativity is a good approximation to quantum gravity (it can be taken as the size of the clock) and $x$ is the distance from the clock to the observer. Once again, we can see that eq-(19) has the same energy dependence as the second term of eq-(18) and of eq-(13) for $\alpha = 1; \beta = 1$.

To the author one important lesson gained from these results is not only the role that scale relativity will play in underlying the fundamental principles to formulate string theory but in the need to modify Quantum Mechanics in order to accommodate gravity. Some time ago we were able to show how Nonlinear Quantum mechanics admitted a geometrical interpretation based on Weyl geometry [3]. In particular, the nonlinear corrections to the Hydrogen atom ground sate energy was $\sim 10^{-35}$ ev. For an earlier geometrical origins of Quantum Mechanics based on Weyl geometry see [22]. Later, Nottale extended the notion of scale covariance within the fractal nature of spacetime microphysics to account for Quantum effects. An earlier treatment of fractal trajectories and quantum mechanics was given by G.N. Ord [29]. Since the uncertainty principle has been modified it is not surprising that one might have to modify accordingly the standard Quantum Mechanics in order to accommodate these results: in particular the standard commutation relations.

For a recent proof that $[p, q] \neq i\hbar$ see [23].

Also relevant about the minimum length in Nature are the arguments of [8,9] pertaining to the nonexistence of black holes of sizes $2M$ smaller than $\lambda_s$ and the subsequent fact that Hawking radiation must stop at some point when the Hawking temperature reaches the Hagedorn temperature $T = (1/8\pi M) = T_H$. It was suggested that at this point conformal invariance is broken and no consistent string propagation is possible.

A more detailed study of fractal surfaces within the context of string theory is warranted as well as the role that scale relativity will play in a theory of Quantum Gravity. An extension of scale relativity where both dynamics as well as scales are on the same footing should reveal very relevant information about the underlying principle behind string theory as suggested in [5]. Whether or not Quantum Mechanics remains intact still remains as a challenging question.

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