Abstract

Abelian orbifolds of $\mathbb{C}^3$ are known to be encoded by hexagonal brane tilings. To date it is not known how to count all such orbifolds. We fill this gap by employing number theoretic techniques from crystallography, and by making use of Polya’s Enumeration Theorem. The results turn out to be beautifully encoded in terms of partition functions and Dirichlet series. The same methods apply to counting orbifolds of any toric non-compact Calabi-Yau singularity. As additional examples, we count the orbifolds of the conifold, of the $L^\text{\eta}_{\text{ba}}$ theories, and of $\mathbb{C}^4$. 
1 Introduction

Brane tilings [1, 2] have met with a lot of interest in the past few years. Each brane tiling gives rise to a quiver gauge theory, which can describe either the theory living on D3 branes probing a toric Calabi–Yau–three singularity, or the theory living on M2 branes probing a toric Calabi–Yau–four singularity [3]. Faces, edges and nodes of the brane tiling – a periodic bipartite tiling of the plane – correspond respectively to gauge groups, chiral bifundamental fields and interaction terms in the superpotential. A periodic quiver can be constructed from the brane tiling by substituting nodes by faces and edges by arrows. The faces of the periodic quiver thus represent terms in the superpotential with a (negative) positive sign for (anti–) clockwise orientation and are in fact extensions of the usual notion of a quiver which does not have a “built–in” superpotential.

Recently, the question of enumerating all possible brane tilings has been raised [4]. A classification of all brane tilings with up to and including $N_T = 6$ terms in the superpotential was given in this paper, and the results for all brane tilings with $N_T = 8$ terms were computed but not published. These results were derived using a computer code that reaches its limits for $N_T = 10$ terms. Thus a need for a better algorithm or a different approach arises. One possible approach is to count the number of tilings for a fixed number of terms $N_T$ and collect the answer into a generating function. This turns out to be a difficult task and to date an answer is still unknown. We can therefore simplify the problem further and attempt to count the number of “sub-tilings”. For example, we can ask how many tilings with $n$ hexagons in the fundamental domain there are. Or we can ask how many inequivalent
tilings with $2n$ squares one can construct. Such questions turn out to be relatively easy to solve and are the subject of the present work.

It is known that all orbifolds of a given geometry correspond to a repetition of the fundamental domain [1]. For example, the number of inequivalent hexagonal tilings with $n$ tiles equals the number of inequivalent orbifolds of $C^3$ with an Abelian group of order $n$. Similarly, the number of tilings with $2n$ squares is the number of inequivalent orbifolds of the conifold by an Abelian group of order $n$, etc. Note that in the case of compact Calabi–Yau manifolds, the problem is different because of the gluing conditions for the patches that result in a finite number of admissible orbifolds.

It is useful to map the problem of counting brane tilings to the problem of counting sublattices. Take for example the problem of counting the Abelian orbifolds of $C^3$. Instead of counting the lattices obtained by the repetition of $n$ tiles, we can take the standard bipartite hexagonal lattice and count its sublattices of index $n$. Similarly, the problem of counting Abelian orbifolds of the conifold is equivalent to counting a certain type of square sublattices. We are thus led to the subject of enumeration of sublattices which has been studied by the crystallography community in great detail (see e.g. [5, 6, 7, 8] and references therein). Fortunately, we are able to take results from this field and apply them to the questions of interest in this paper.

In the following we will outline some methods for counting the sublattices of a given lattice. While it is possible to enumerate by hand the first few orbifolds, this quickly becomes cumbersome. A general understanding of how the number of sublattices $f(n)$ with index (i.e. size of the fundamental cell) $n$ behaves and grows is therefore desirable. It turns out that this number decomposes into the symmetries of a given lattice. Our main results are closed formulae for the number of Abelian orbifolds of $C^3$, of the conifold, of $L^{aba}$ theories and of $C^4$, which are furthermore generalizable to any toric non–compact Calabi–Yau. In all the cases corresponding to Calabi–Yau–three geometries we find that for large $n$ the dominant contribution is

$$f(n) \sim \frac{\sigma(n)}{|G|}, \quad \text{for } n \gg 1,$$

where $G$ is the symmetry group of the fundamental cell of the brane tiling, and $\sigma$ is the sum–of–divisors function.

The plan of this note is as follows. In Section 2 we outline the problem we are studying. In the following sections, we introduce the knowledge necessary to count all the sublattices of a given lattice and show how to apply it to the case of the hexagonal lattice (Abelian orbifolds of $C^3$). In Section 3 we discuss the cycle index which captures the symmetries of a given lattice. The Hermite normal form is introduced in Section 4. In Section 5 the concepts of the Dirichlet convolution and the Dirichlet series are introduced. In Section 6 we discuss the examples of the square lattice (Abelian orbifolds of the conifold), the $L^{aba}$ theories, and the tetrahedral lattice (Abelian orbifolds of $C^4$) following the same steps as detailed in the previous sections for the case of the hexagonal lattice. In Appendix A we briefly discuss the general lattice in $d$ dimensions.
identify the symmetries of each lattice, which is the topic of the next section. This is going to be the language used in the next sections. The first step is to state above, more details can be found in \[9\].

To get a better understanding of our counting problem, let us give a brief description of some details. A more exhaustive explanation will appear in a forthcoming publication \[9\] which will describe several computer codes that were used in order to obtain some of the results used in this note.

We start by looking at orbifolds of $\mathbb{C}^3$. For simplicity, let us focus on $\mathbb{Z}_n$ orbifolds. More general Abelian orbifolds such as $\mathbb{Z}_n \times \mathbb{Z}_m$ can be treated in a similar fashion. Let us denote the coordinates of $\mathbb{C}^3$ by $\{z_1, z_2, z_3\}$, and the orbifold action by $(a_1, a_2, a_3)$ such that $\{z_1, z_2, z_3\} \sim \{\omega^m z_1, \omega^{m^2} z_2, \omega^{m^3} z_3\}$, with $\omega^n = 1$ and $a_1 + a_2 + a_3 = 0 \mod n$. In this notation, the problem is to find all triples $(a_1, a_2, a_3)$ that give inequivalent orbifolds of $\mathbb{C}^3$. The first few cases are as follows. For $n = 1$, the orbifold group is trivial and there is only one case, $\mathbb{C}^3$. For $n = 2$, there is again one case, which is commonly denoted in the literature as $\mathbb{C}^2 / \mathbb{Z}_2 \times \mathbb{C}$. For $n = 3$, there are two cases, $\mathbb{C}^2 / \mathbb{Z}_3 \times \mathbb{C}$ with orbifold action $(1, 2, 0)$ and $\mathbb{C}^3 / \mathbb{Z}_3$ with orbifold action $(1, 1, 1)$. For $n = 4$, there are three cases, $\mathbb{C}^2 / \mathbb{Z}_4 \times \mathbb{C}$, $\mathbb{C}^3 / \mathbb{Z}_4$ with orbifold action $(1, 1, 2)$ and $\mathbb{C}^3 / \mathbb{Z}_2 \times \mathbb{Z}_2$, etc. The brane tilings for the first examples can be found in Table 1, a count of the first 16 cases can be found in the last row, $f^{\Delta}$, of Table 2.

An alternative way of formulating the problem is by looking at the toric diagrams of these orbifolds. Since the toric diagram of $\mathbb{C}^3$ is a triangle of unit area, the toric diagram of an orbifold of $\mathbb{C}^3$ by an Abelian group of order $n$ is again a triangle but with an area which is $n$ times larger. The problem of counting all inequivalent orbifolds of $\mathbb{C}^3$ is therefore equivalent to the problem of finding all triangles with vertices on integral points and area $n$. Of course, since these are toric diagrams, two triangles which are related by a $GL(2, \mathbb{Z})$ transformation are equivalent. This provides another approach to counting orbifolds. A systematic method of finding such triangles is algorithmically different from that of finding inequivalent orbifold actions and therefore provides an alternative approach to the counting problem. As stated above, more details can be found in \[9\].

Yet a third approach is to think of the brane tiling as forming a bipartite hexagonal lattice and the problem of finding inequivalent toric diagrams is mapped to the problem of finding its sublattices. This is going to be the language used in the next sections. The first step is to identify the symmetries of each lattice, which is the topic of the next section.

There is however a subtle point that needs to be emphasized. The brane tiling encodes

| $n$ | 1 | 2 | 3 | 3 |
|-----|---|---|---|---|
| brane tiling | ![brane tiling 1](image1.png) | ![brane tiling 2](image2.png) | ![brane tiling 3](image3.png) | ![brane tiling 4](image4.png) |
| geometry | $\mathbb{C}^3$ | $\mathbb{C}^2 / \mathbb{Z}_2 \times \mathbb{C}$ | $\mathbb{C}^2 / \mathbb{Z}_3 \times \mathbb{C}$ | $\mathbb{C}^3 / \mathbb{Z}_3$ |

Table 1: Brane tilings for the first orbifolds of $\mathbb{C}^3$.  

2 What we are counting and how

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There is however a subtle point that needs to be emphasized. The brane tiling encodes
the same information as the quiver diagram plus the superpotential, and in certain cases, one toric geometry admits several possible quiver gauge theories. In this case we speak of different toric phases. The lattices of the brane tilings pertaining to the different toric phases are different, but they preserve the same symmetries, and for the enumeration of toric geometries, the symmetries are what matters. Even though our construction works starting from any lattice (brane tiling), if we want to count all the resulting quiver gauge theories, we need to keep track of the different toric phases that can appear in the process of orbifolding a given geometry. Consider for example the case of the conifold (see Section 6.1). The brane tiling is a bipartite square lattice. Dividing by $\mathbb{Z}_2$, one obtains $\mathbb{F}_0$. This variety has two different toric phases, one corresponds to a sublattice of the square lattice, the other one is represented by a square–octagon lattice. While the quiver gauge theory of the former lattice is captured by counting the orbifolds of the conifold, the gauge theories stemming from the latter have to be considered separately. The final consequence is that even if we use the lattice of the brane tiling (which corresponds directly to the gauge theory), our counting covers the geometries but in general not all the gauge theories or toric phases that can arise from the orbifolds of a given geometry.

3 Symmetries and the cycle index

In the following, we need a way to capture the symmetries of a given lattice. Let us label the vertices of the fundamental cell by the numbers $\{1, \ldots, m\}$. We now want to describe the group of permutations $G$ of the set $X = \{1, \ldots, m\}$ which result in the same fundamental cell. The cycle index encodes this information [10]. For the admissible symmetries, the fact that we are considering bipartite lattices plays an important role, since the preservation of the coloring of the vertices results in an additional constraint.

In our case, it is most convenient to express the permutations in cycle notation. The cycles of $g \in G$ are the orbits of the elements $\varepsilon \in X$ under $g$. For each group element $g$ we start with $\varepsilon_1 \in X$ and write down its orbit in parentheses, $(\varepsilon_1 \, g(\varepsilon_1) \, g^2(\varepsilon_1) \, \ldots \, g^{k-1}(\varepsilon_1))$, where $g^k(\varepsilon_1) = \varepsilon_1$. We continue to do the same with the next element that has not yet appeared in an orbit until we have exhausted all the elements of $X$. Each $g \in G$ can thus be expressed in terms of $\alpha_k$ disjoint cycles of length $k$; cycles of length one correspond to elements that are fixed under $g$. The type of $g$ is given by the partition of $m \, [1^{a_1} 2^{a_2} \ldots l^{a_l}]$, where $m = a_1 + 2a_2 + \ldots + la_l$. The partition is represented by the expression

$$\zeta_g(x_1, \ldots, x_l) = x_1^{a_1} x_2^{a_2} \ldots x_l^{a_l}.$$  \hspace{1cm} (3.1)

**Definition.** The cycle index of $G$ is obtained by summing the $\zeta_g$ over all elements $g \in G$ and dividing by the number of elements $|G|$:

$$Z_G(x_1, \ldots, x_l) = \frac{1}{|G|} \sum_{g \in G} \zeta_g(x_1, \ldots, x_l) = \frac{1}{|G|} \sum_{\alpha} c(\alpha_1, \ldots, \alpha_l) \, x_1^{a_1} \ldots x_l^{a_l},$$  \hspace{1cm} (3.2)

where $c(\alpha_1, \ldots, \alpha_l)$ is the number of permutations of type $[1^{a_1} 2^{a_2} \ldots l^{a_l}]$, and the sum runs over all
Table 2: Cycle index for the symmetric group $S_3$. $Z_{S_3} = \frac{1}{6} \left( x_1^3 + 3x_1x_2 + 2x_3 \right)$.

Some cases of interest are:

1. The **cyclic group** $C_m$ is the group of symmetries associated to a circular object where reflections are excluded:

   $$Z(C_m) = \frac{1}{m} \sum_{d \mid m} \varphi(d) x_d^{m/d},$$

   where $\varphi(d)$ is the totient function.

2. The **dihedral group** $D_m$ is the group of symmetries associated to a circular object where reflections are allowed:

   $$Z(D_m) = \frac{1}{2} Z(C_m) + \begin{cases} 
   \frac{1}{2} x_1x_2^{(m-1)/2}, & \text{if } m \text{ is odd}, \\
   \frac{1}{4} \left( x_1^2x_2^{(m-2)/2} + x_2^{m/2} \right), & \text{if } m \text{ is even}.
   \end{cases}$$

3. The **symmetric group** $S_m$ is the group of all permutations of $n$ symbols:

   $$Z(S_m) = \sum_{a_1+2a_2+\cdots+k_k=m} \frac{1}{k_1!k_2!\cdots k_k!} \prod_{k=1}^m x_k^{a_k}.$$ 

   A convenient recursion formula is given by

   $$Z(S_m) = \frac{1}{m} \sum_{k=1}^m x_k Z(S_{m-k}), \quad Z(S_0) = 1.$$  

The link between the cycle index and the number of sublattices of the lattice $L$ is provided by Burnside’s lemma.
Lemma. Let $G$ be a group of permutations of the set $X$. The number $N(G)$ of orbits of $G$ is given by the average over $G$ of the sizes of the fixed sets:

$$N(G) = \frac{1}{|G|} \sum_{g \in G} |F_g| ; \quad F_g = \{ x \in X \mid g(x) = x \} . \quad (3.7)$$

The number $f^L(n)$ of sublattices of index $n$ can be understood as the number of orbits of the symmetry group $G$ when acting on the set $X_n$ of sublattices of index $n$. According to the lemma, this can be written as the average of the number of elements in $X_n$ that are left invariant by the action of $g \in G$:

$$f^L(G) = \frac{1}{|G|} \sum_{g \in G} f^L_g(n) ; \quad f^L_g(n) = |\{ x \in X_n \mid g(x) = x \}| . \quad (3.8)$$

Using the cycle decomposition introduced above we can rewrite this expression as a sum over the types of the elements $g$, indexed by partitions $\alpha$:

$$f^L(n) = \frac{1}{|G|} \sum_{\alpha} c(\alpha) f^L_{\alpha}(n) . \quad (3.9)$$

By comparison with Equation (3.2) we see that we obtain a subsequence for each monomial in the cycle index $Z_G(x)$.

Every group contains the identity, which is represented by the partition $[1^m]$. The corresponding number of invariant sublattices $f^L_{[1^m]}(n)$ only depends on the dimension of the lattice $d = \dim[L]$ and is given by the formula in Equation (A.10):

$$f^L_{[1^m]}(n) = \sum_{k_0, \ldots, k_{d-1}=1 \atop k_0 k_1 \cdots k_{d-1} = n} k_1 k_2^2 \cdots k_{d-1}^{d-1} . \quad (3.10)$$

Example ($\triangle$). Consider the bipartite hexagonal lattice corresponding to the geometry of $C^3$. Because of the bipartiteness, the symmetry group is not $D_6$ as expected for a hexagon, but $S_3$, which we will also denote by a triangle. Table 2 shows the cycle decomposition for the symmetric group $S_3$:

$$Z_{S_3} = \frac{1}{6} \left( x_1^3 + 3 x_1 x_2 + 2 x_3 \right) . \quad (3.11)$$

Table 3 gives the number of sublattices of index $n$ for the bipartite hexagonal lattice for each of the monomials appearing in the expression above.

$$f^\triangle = \frac{1}{6} \left( f^\triangle_{x_1^3} + 3 f^\triangle_{x_1 x_2} + 2 f^\triangle_{x_3} \right) \quad (3.12)$$

The numbers $f^\triangle(n)$, $n \leq 500$ are represented in Figure 1, where the prime numbers are emphasized.
Table 3: Number of sublattices of index $n$ for the hexagonal lattice, classified by the cycles of the symmetric group $S_3$. According to the cycle index decomposition, $f^\Delta = \frac{1}{6} \left( f_{x_1^3}^\Delta + 3 f_{x_1 x_2}^\Delta + 2 f_{x_3}^\Delta \right)$.

| $n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $f_{x_1^3}^\Delta$ | 1  | 3  | 4  | 7  | 6  | 12 | 8  | 15 | 13 | 18 | 12 | 28 | 14 | 24 | 24 | 31 |
| $f_{x_1 x_2}^\Delta$ | 1  | 1  | 2  | 3  | 2  | 5  | 3  | 2  | 6  | 2  | 4  | 7  | 1  | 0  | 0  | 1  |
| $f_{x_3}^\Delta$     | 1  | 0  | 1  | 1  | 0  | 0  | 1  | 0  | 0  | 1  | 2  | 0  | 0  | 1  |    |    |
| $f^\Delta$          | 1  | 1  | 2  | 3  | 2  | 3  | 5  | 4  | 4  | 3  | 8  | 4  | 5  | 6  | 9  |    |

Figure 1: Scatter plot of the sequence $f^\Delta$ for a hexagonal lattice. Prime numbers are emphasized in red. The two lines correspond to $n/6$ and $e^n \log \log n/6$. 


4 Hermite normal form

Consider a lattice $L_d$ generated by the $d$ vectors $\langle y_1, \ldots, y_d \rangle$. Any sublattice $L'$ of $L_d$ is generated by $d$ vectors $\langle x_1, \ldots, x_d \rangle$ which can be written as

$$
\begin{align*}
  x_1 &= a_{11} y_1 \\
  x_2 &= a_{21} y_1 + a_{22} y_2 \\
  \vdots \\
  x_d &= a_{d1} y_1 + a_{d2} y_2 + \cdots + a_{dd} y_d,
\end{align*}
$$

where the integer coefficients $a_{ij}$ satisfy the conditions

$$
0 \leq a_{ij} < a_{ii} \quad \forall j < i. \tag{4.2}
$$

The above construction is the so-called Hermite normal form. The index $n$ of the lattice is given by the product

$$
n = \prod_{i=1}^{d} a_{ii}. \tag{4.3}
$$

Expressing the sublattices via Equation (4.1), the problem of counting the sublattices of a generic lattice turns into the problem of counting the number of matrices $a_{ij}$ that satisfy the conditions Equation (4.2) and Equation (4.3).

Consider a two–dimensional lattice $L_2$. The condition $a_{11}a_{22} = n$ can be satisfied by the choice $a_{22} = m$ and $a_{11} = n/m$, where $m$ is a divisor of $n$. If we want to count the number of sublattices invariant under the symmetry $x^\alpha$, we need to enumerate the possible values of $a_{21}$. The constraint $a_{21} < a_{22}$ introduces a dependence of the number of possible values of $a_{21}$, $\# \{ a_{21} \} = g_{x^\alpha}^{L_2}(a_{22})$, on $a_{22}$. The total number of sublattices $f_{x^\alpha}^{L_2}(n)$ is thus given by summing $g_{x^\alpha}^{L_2}(m)$ over all the divisors of $n$:

$$
f_{x^\alpha}^{L_2}(n) = \sum_{m|n} g_{x^\alpha}^{L_2}(m). \tag{4.4}
$$

Repeating the same construction for a three–dimensional lattice $L_3$, we find that

$$
f_{x^\alpha}^{L_3}(n) = \sum_{m_1|n} \sum_{m_2|m_1} h_{x^\alpha}^{L_3}(\frac{m_1}{m_2}) g_{x^\alpha}^{L_2}(m_2). \tag{4.5}
$$

A similar decomposition into $d – 1$ nested sums appears for any lattice of dimension $d$. 

8
5 Generating functions

5.1 The Dirichlet convolution

Looking at Figure 1, one realizes that prime numbers play a special role. This is one of the clues that point to the fact that the sequences corresponding to the monomials of the cycle index have the property of being multiplicative. A possible explanation for this may be linked to the observation that the orbifold group \( \mathbb{Z}_{pq} \) is isomorphic to the group \( \mathbb{Z}_p \times \mathbb{Z}_q \) for \( p, q \) primes.

**Definition.** A sequence \( f \) is multiplicative if

\[
f(nm) = f(n)f(m), \quad \text{when} \ (n, m) = 1, \quad (5.1)
\]

where \((n, m)\) denotes the greatest common divisor between \( n \) and \( m \).

It follows in particular that \( f \) is completely determined by its values for primes and their powers, since for any \( n \) we can use the factorization \( n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \), and

\[
f(n) = f(p_1^{a_1})f(p_2^{a_2}) \ldots f(p_r^{a_r}). \quad (5.2)
\]

Multiplicative sequences form a group under the Dirichlet convolution. For our examples, it is convenient to use this property and decompose each of the sequences into products of other sequences that are easier to deal with. First, we need the following

**Definition.** The Dirichlet convolution of two sequences \( f \) and \( g \) is the sequence \( h \) defined by

\[
f(n) = (g * h)(n) = \sum_{m|n} g(m)h\left(\frac{n}{m}\right), \quad (5.3)
\]

where the notation \( m|n \) means that the sum runs over all the divisors \( m \) of \( n \).

One can prove that this convolution is commutative, \( f * g = g * f \), and associative, \( f * (g * h) = (f * g) * h \), and that the sequence \( \text{Id} \) defined by

\[
\text{Id}(n) = \{ 1, 0, 0, \ldots \} \quad (5.4)
\]

is the identity, \( f * \text{Id} = f \). To each sequence \( f \) one can associate its inverse \( f^{-1} \) satisfying

\[
f * f^{-1} = f^{-1} * f = \text{Id}. \quad (5.5)
\]

The inverse can be evaluated recursively via

\[
f^{-1}(n) = \frac{1}{f(1)} \sum_{\substack{d|n \\& \ \text{d < n}}} f\left(\frac{n}{d}\right)f^{-1}(d). \quad (5.6)
\]

We have observed above that in two dimensions, the number of invariant sublattices can
be put in the form of Equation (4.4),

\[ f_{\Delta}^{\triangle}(n) = \sum_{m|n} g_{\Delta}^{\triangle}(m). \quad (5.7) \]

This is equivalent to the statement that the sequence \( f_{\Delta}^{\triangle} \) is the convolution of \( g_{\Delta}^{\triangle} \) with the unit \( u \) defined by

\[ u(n) = \{1, 1, 1, \ldots\} . \quad (5.8) \]

Its inverse is the Möbius function defined by

\[ \mu(n) = \begin{cases} 1 & \text{if } n \text{ is square–free with an even number of distinct prime factors,} \\ -1 & \text{if } n \text{ is square–free with an odd number of distinct prime factors,} \\ 0 & \text{otherwise.} \end{cases} \quad (5.9) \]

It follows that if \( f = g * u \), then \( g = \mu * f \).

**Example (\( \triangle \)).** In the case of the bipartite hexagonal lattice we find:

1. The sequence \( f_{\Delta}^{\triangle} = \{1, 3, 4, 7, 6, 12, 8, 15, \ldots\} \) corresponds to the identity permutation \( \chi_3^{\Delta} \) and it is given by Equation (3.10) with \( d = 2 \). It can also be written as the convolution

\[ f_{\Delta}^{\triangle} = u * N, \quad (5.10) \]

where

\[ N(n) = \{1, 2, 3, \ldots\} . \quad (5.11) \]

2. The sequence \( f_{\Delta}^{\triangle} = \{1, 1, 2, 3, 2, 2, 2, 5, \ldots\} \) can be written as the convolution of a periodic sequence of period 4 and the unit:

\[ f_{\Delta}^{\triangle} = \{1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots\} * u . \quad (5.12) \]

\( g_{\Delta}^{\triangle} \) is in turn the convolution of a finite sequence and \( u \):

\[ f_{\Delta}^{\triangle} = \{1, 0, 1, 2, 1, 0, 1, 2, 1, \ldots\} * u = \{1, -1, 0, 2\} * u * u . \quad (5.13) \]

3. The last sequence \( f_{\Delta}^{\triangle} = \{1, 0, 1, 0, 0, 2, 0, \ldots\} \) also has the form of the convolution of the unity with a periodic sequence of period 3:

\[ f_{\Delta}^{\triangle} = \{1, -1, 0, 1, -1, 0, 1, -1, 0, \ldots\} * u . \quad (5.14) \]

The periodic sequence is the (non–principal) Dirichlet character of modulus three (see [11]):

\[ g_{\Delta}^{\triangle} = \chi_3^{\triangle}(n) = \{1, -1, 0, 1, -1, 0, \ldots\} . \quad (5.15) \]
Putting all together we find that the sequence $f^\Delta$ can be written as

$$f^\Delta = \frac{1}{6} \left( f_{x_1}^\Delta + 3 f_{x_1x_2}^\Delta + 2 f_{x_2}^\Delta \right) = \frac{1}{6} \left( N + 3 \{1, 0, -1, 2\} * u + 2 \chi_{3,2} * u \right). \tag{5.16}$$

### 5.2 Dirichlet series and power series

The information contained in a sequence $f$ can be usefully encoded into a generating function (more commonly used as a partition function in the physics literature). In the following, we will use two types of generating functions:

1. the formal power series (partition function)

   $$F(t) = \sum_{n=1}^{\infty} f(n)t^n; \tag{5.17}$$

2. the Dirichlet series

   $$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{5.18}$$

The corresponding inverse transformations are given by

$$f(n) = \frac{1}{2\pi i} \oint F(t) \frac{dt}{tn + 1}, \tag{5.19}$$

$$f(n) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} F(s)n^s|_{s=\sigma+i\tau} \, d\tau. \tag{5.20}$$

Dirichlet series are appropriate in the case of multiplicative sequences. In particular, if $f$ is multiplicative, the series can be expanded in terms of an infinite product over the primes, the Euler product:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots \right). \tag{5.21}$$

This is consistent with the observation that a multiplicative sequence is determined by the values taken for powers of prime numbers.

Let us now consider the sequences that appeared in the example above:

1. For the identity $\text{Id}$:

   $$\text{Id}(s) = 1, \quad \text{Id}(t) = t; \tag{5.22}$$

2. For the unit $u$ we obtain Riemann’s zeta function:

   $$u(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s), \quad u(t) = \frac{1}{1-t} - 1; \tag{5.23}$$
3. For \(N:\)
\[
N(s) = \sum_{n=1}^{\infty} \frac{n}{n^s} = \zeta(s-1), \quad N(t) = \frac{1 + t^3}{(1-t)(1-t^2)} - 1; \quad (5.24)
\]

4. For the finite sequence \(\{1, -1, 0, 2\}:\)
\[
\{1, -1, 0, 2\} (s) = 1 - \frac{1}{2^s} + \frac{2}{4^s}, \quad \{1, -1, 0, 2\} (t) = t - t^2 + 2t^4; \quad (5.25)
\]

5. For the Dirichlet character \(\chi_{3,2},\) the corresponding Dirichlet series is the so-called \(L-\)function
\[
\chi_{3,2}(s) = \sum_{n=1}^{\infty} \frac{\chi_{3,2}(n)}{n^s} = L(s, \chi_{3,2}), \quad \chi_{3,2}(t) = \frac{(1 + t)(1 - t^2)}{1 - t^3} - 1. \quad (5.26)
\]

Both types of generating functions have a simple behavior under Dirichlet convolution. Let \(f, g\) and \(h\) be such that
\[
f = g \ast h. \quad (5.27)
\]
The power series for \(h\) reads:
\[
F(t) = \sum_{n=1}^{\infty} f(n)n^n = \sum_{n=1}^{\infty} \sum_{m|n} g(m) h(\frac{n}{m}) t^n = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} g(m) h(k) t^{mk}. \quad (5.28)
\]
This can be expressed in two ways, using the generating function for \(g\) or for \(h:\)
\[
F(t) = \sum_{m=1}^{\infty} g(m) H(t^m) = \sum_{k=1}^{\infty} h(k) G(t^k). \quad (5.29)
\]
In particular, since all our sequences can be written as sums over divisors (or equivalently as Dirichlet convolutions with the unit), we will always write
\[
F(t) = \sum_{k=1}^{\infty} G(t^k). \quad (5.30)
\]
It is also possible to write the power series for the inverse of the Dirichlet convolution as follows. Let
\[
f(t) = \sum_{k,m=1}^{\infty} g(m) h(k) t^{mk}, \quad (5.31)
\]
then
\[
H(t) = \sum_{k=1}^{\infty} h(k) t^k = \sum_{m=1}^{\infty} \mu(k) g(k) F(t^k), \quad (5.32)
\]
where \(\mu\) is the Möbius function.
More directly, the Dirichlet series is decomposed as
\[
F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=1}^{\infty} \sum_{m|n} \frac{g(m) h\left(\frac{n}{m}\right)}{n^s} = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{g(m) h(k)}{m^s k^s} = G(s) H(s). \tag{5.33}
\]

Note that the Dirichlet series corresponding to a sequence can be also understood as the Laplace transform of a discrete measure. This explains why it exchanges convolution and pointwise products.

Both generating functions can be seen as linear transformations, hence the decomposition in Equation (3.9) still holds.

**Example (△).** Let us now apply these formulas to the terms in \( f^\triangle \):

1. The sequence \( f^\triangle_{x_1^3} \) is decomposed as \( f^\triangle_{x_1^3} = u * N \), hence the corresponding power series is generated by

   \[
   G^\triangle_{x_1^3}(t) = \sum_{k=1}^{\infty} k t^k = \frac{1 + t^3}{(1 - t)(1 - t^2)} - 1, \tag{5.34}
   \]

   and the Dirichlet series reads

   \[
   F^\triangle_{x_1^3}(s) = \zeta(s) \zeta(s - 1). \tag{5.35}
   \]

2. The sequence \( f^\triangle_{x_1 x_2} = \{1, -1, 0, 2\} * u * u \) gives

   \[
   G^\triangle_{x_1 x_2}(t) = \frac{1 + t^3}{(1 - t)(1 + t^2)} - 1, \tag{5.36}
   \]

   \[
   F^\triangle_{x_1 x_2}(s) = \left(1 - 2^{-s} + 2^{1-2s}\right) \zeta(s)^2. \tag{5.37}
   \]

3. The sequence \( f^\triangle_{x_3} = \chi_{3,2} * u \) gives

   \[
   G^\triangle_{x_3}(t) = \frac{(1 + t)(1 - t^2)}{1 - t^3} - 1, \tag{5.38}
   \]

   \[
   F^\triangle_{x_3}(s) = L(s, \chi_{3,2}) \zeta(s). \tag{5.39}
   \]

Collecting all the terms we find:

1. For the power series:

   \[
   G^\triangle(t) = \frac{1}{6} G^\triangle_{x_1}(t) + \frac{1}{2} G^\triangle_{x_1 x_2}(t) + \frac{1}{3} G^\triangle_{x_3}(t) = \frac{1}{(1 - t)(1 + t^2)(1 - t^3)} - 1, \tag{5.40}
   \]

   whence

   \[
   F^\triangle(t) = \sum_{m=1}^{\infty} \left[\frac{1}{(1 - t^m)(1 + t^{2m})(1 - t^{3m})} - 1\right] = \sum_{m=1}^{\infty} \sum_{n_1, n_2, n_3 = 0}^{\infty} (-1)^{n_2} t^{m(n_1 + 2n_2 + 3n_3)} \tag{5.41}
   \]

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2. For the Dirichlet series:

\[
F^\triangle(s) = \zeta(s) + 3 \left(1 - 2^{-s} + 2^{1-2s}\right) \zeta(s) + 2L(s, \chi_{3,2}) - 1.
\] (5.42)

The generating functions for the hexagonal lattice are summarized in Table 4.

### 5.3 Asymptotic behavior

The asymptotic behavior of a sequence can be derived by looking at the corresponding Dirichlet series.

**Theorem.** Let \( F(s) \) be a Dirichlet series with non-negative coefficients that converges for \( \Re(s) > \alpha > 0 \), and suppose that \( F(s) \) is holomorphic in all points of the line \( \Re(s) = \alpha \), except for \( s = \alpha \). If for \( s \to \alpha^+ \), the Dirichlet series behaves as

\[
F(s) \sim A(s) + \frac{B(s)}{(s - \alpha)^{-m+1}},
\] (5.43)

where \( m \in \mathbb{N} \), and both \( A(s) \) and \( B(s) \) are holomorphic in \( s = \alpha \), then the partial sum of the coefficients is asymptotic to:

\[
\sum_{n=1}^{N} a_n \sim \frac{B(\alpha)}{\alpha^m} N^\alpha \log^m N. 
\] (5.44)

In order to apply this theorem, we can make use of the following facts:

1. The Riemann zeta function \( \zeta(s) \) is analytic everywhere, except for a simple pole at \( s = 1 \) with residue 1;
2. The L–function \( L(s, \chi) \) is analytic everywhere, except for a simple pole at \( s = 1 \) if \( \chi \) is a principal character.

Another useful fact is Robin’s inequality for the \( \sigma \) function (sum of the divisors):

\[
\sigma(n) < e^\gamma n \log \log n, \quad n \text{ large},
\] (5.45)
where \( \gamma \) is Euler’s constant. This is true for large \( n \), where large means \( n \geq 5041 \), and if and only if Riemann’s hypothesis is true \([12]\).

**Example (\( \triangle \)).** The rightmost pole of the Dirichlet series \( F^\triangle (s) \) in Equation (5.42) is found for \( s = 2 \), has order 1 and its residue is \( \zeta(2)/6 \). Using the above theorem we conclude that the partial sum of the terms in the sequence \( f^\triangle \) behaves asymptotically as

\[
\sum_{n=1}^{N} f^\triangle(n) \sim \frac{\zeta(2)}{12} N^2 = \frac{\pi^2}{72} N^2,
\]

and the sequence itself grows asymptotically as \( f^\triangle(N) = \mathcal{O}(N) \).

For large \( n \), the leading term is \( \zeta(s)\zeta(s - 1)/6 \), hence

\[
f^\triangle (n) < e^{\gamma n \log \log n} \frac{\log \log n}{6}, \quad n \text{ large}.
\]

### 6 Examples

#### 6.1 Orbifolds of the conifold

Another simple geometry that lends itself to the counting of its orbifolds is the *conifold*. In terms of the dimer model description, it corresponds to the bipartite square lattice with a black and a white vertex in its unit cell. The brane tilings for the first orbifolds can be found in Table 5. Note that as mentioned earlier, we are here enumerating all the toric geometries stemming from orbifolds of the conifold and *not* all possible quiver gauge theories, since we are not taking into account the multiple toric phases.

The brane tiling corresponding to the conifold is a bipartite square lattice with two faces in the unit cell. Its symmetry is given by Klein’s Vierergruppe \( V = \mathbb{Z}_2 \times \mathbb{Z}_2 \), with cycle index

\[
Z_V = \frac{1}{4} \left( x_1^4 + 2 x_1^2 x_2 + x_2^2 \right).
\]

Accordingly, we can decompose \( f^\square \) as in Table 6. The numbers \( f^\square(n), n \leq 500 \) are represented in Figure 2.
Table 6: Number of sublattices of index $n$ for the square lattice, classified by the cycles of the Vierergruppe $V = \mathbb{Z}_2 \times \mathbb{Z}_2$.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $f_{\mathbb{Z}_4}$ | 1 | 3 | 4 | 7 | 6 | 12 | 8 | 15 | 13 | 18 | 12 | 28 | 14 | 24 | 24 | 31 |
| $f_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ | 1 | 1 | 1 | 2 | 2 | 1 | 1 | 3 | 2 | 2 | 1 | 3 | 2 | 1 | 2 | 4 |
| $f_{\mathbb{Z}_2^2}$ | 1 | 3 | 2 | 5 | 2 | 6 | 2 | 7 | 3 | 6 | 2 | 10 | 2 | 6 | 4 | 9 |
| $f$ | 1 | 2 | 2 | 4 | 3 | 5 | 3 | 7 | 4 | 11 | 5 | 8 | 8 | 12 |

Figure 2: Scatter plot of the sequence $f^{\square}$ for a square lattice. Prime numbers are given in red. The two lines correspond to $n/4$ and $e^\gamma n \log \log n/4$. 

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Table 7: Generating functions for the Abelian orbifolds of the conifold organized by the symmetries of the brane tiling.

The analysis of the sequences corresponding to the terms in $Z_V$ gives the following results.

1. For the monomial $x_1^4$ we obtain the by now usual term
   \[ f_{x_1^4} = u \ast N. \]  
   (6.2)
   This is the same as for the example of the hexagon since it depends only on the dimension of the lattice.

2. The monomial $x_1^2 x_2$ receives two contributions (this is different from the other cases we have encountered):
   \[ f_{x_1^2 x_2} = \frac{1}{2} u \ast (\chi_{4,2} + \{ 1, -1, 0, 2 \} \ast u). \]  
   (6.3)
   The corresponding power series is given by
   \[ G_{x_1^2 x_2}(t) = \frac{1}{(1-t)(1+t^2)} - 1. \]  
   (6.4)

3. The monomial $x_2^2$ corresponds to
   \[ f_{x_2^2} = \{ 1, 1 \} \ast u \ast u. \]  
   (6.5)
   and the corresponding power series is
   \[ G_{x_2^2}(t) = \frac{1-t^3}{(1-t)(1-t^2)} - 1. \]  
   (6.6)

The generating functions in this decomposition are collected in Table 7.

Collecting all the terms and summing them according to the coefficients of the cycle index, we find

1. For the power series:
   \[ G(t) = \frac{1}{(1-t)(1-t^4)} - 1, \]  
   (6.7)
Table 8: Number of sublattices of index \( n \) for the \( \text{L}_{aba} \) theories (\( \mathbb{Z}_2 \) symmetry).

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| \( f_{x_1}^{\Box} \) | 1 | 3 | 4 | 7 | 6 | 12 | 8 | 15 | 13 | 18 | 12 | 28 | 14 | 24 | 24 | 31 |
| \( f_{x_2}^{\Box} \) | 1 | 1 | 2 | 3 | 2 | 2 | 2 | 5 | 3 | 2 | 2 | 4 | 7 |    |    |    |    |
| \( f^{\Box} \)   | 1 | 2 | 3 | 5 | 4 | 7 | 5 | 10 | 8 | 10 | 7 | 17 | 8 | 13 | 14 | 19 |

whence

\[
F^{\Box}(t) = \sum_{m=1}^{\infty} \left[ \frac{1}{(1-t^m)(1-t^{4m})} - 1 \right] = \sum_{m=1}^{\infty} \sum_{n_1,n_2=0 \atop n \neq (0,0)} t^{m(n_1+4n_2)}. \tag{6.8}
\]

2. For the Dirichlet series:

\[
F^{\Box}(s) = \frac{1}{4} \zeta(s) \left[ \zeta(s-1) + 2 \left( 1 + 2^{-2s} \right) \zeta(s) + L(s, \chi_{4,2}) \right]. \tag{6.9}
\]

The asymptotic behavior of the partial sum of the terms in the sequence \( f^{\Box} \) is given by

\[
\sum_{n=1}^{N} f^{\Box}(n) \sim \frac{1}{8} \frac{\zeta(2)}{N^2} = \frac{\pi^2}{48} N^2, \tag{6.10}
\]

and the sequence itself grows asymptotically as \( f^{\Box}(N) = O(N) \). By Robin’s inequality, the sequence \( f^{\Box} \) is bounded for large \( n \) by

\[
f^{\Box}(n) < e^\gamma n \log \log n, \quad n \text{ large}. \tag{6.11}
\]

### 6.2 Orbifolds of the \( \text{L}_{aba} \) theories

In the case of the \( \text{L}_{aba} \) theories with \( a \neq b \), the brane tiling lattice has \( \mathbb{Z}_2 \) symmetry, and the cycle index is given by

\[
Z_{\mathbb{Z}_2} = \frac{1}{2} (x_1^2 + x_2). \tag{6.12}
\]

The number of orbifolds \( f^{\Box}(n) \) (the first terms are collected in Table 8) can be decomposed into two contributions:

1. the usual term corresponding to the identity,

\[
f_{x_1}^{\Box} = u \ast N; \tag{6.13}
\]

2. the term corresponding to the reflection \( x_2 \):

\[
f_{x_2}^{\Box} = \{ 1, -1, 0, 2 \} \ast u \ast u \tag{6.14}
\]
The power series reads:
\[
F^\triangle(t) = \sum_{m=1}^{\infty} \frac{1 + t^{3m}}{(1 - t^m)(1 - t^{4m})} = \sum_{m=1}^{\infty} \sum_{n_1, n_2 = 0}^{\infty, \neq (0,0)} t^{m(n_1 + 4n_2)} (1 + t^{3m}).
\] (6.15)

The Dirichlet series is
\[
F^\triangle(s) = \frac{1}{2} \zeta(s) \left[ \zeta(s - 1) + \left( 1 - 2^{-s} + 2^{1-2s} \right) \zeta(s) \right].
\] (6.16)

From here we can read the asymptotic behavior of the partial sum of the terms in the sequence \(F^\triangle\):
\[
\sum_{n=1}^{N} f^\triangle(n) \sim \frac{\zeta(2)}{4} N^2 = \frac{\pi^2}{24} N^2.
\] (6.17)

By Robin’s inequality, the sequence \(F^\triangle\) is bounded for large \(n\) by
\[
f^\triangle(n) < \frac{e^{\gamma n} \log \log n}{2}, \quad n \text{ large.}
\] (6.18)

If \(a = b\), the brane tiling acquires an extra symmetry and the relevant group is \(\mathbb{Z}_2 \times \mathbb{Z}_2\). Since this is the same symmetry as for the brane tiling of the conifold that has been described in the previous section, the formulae in Equation (6.8) and Equation (6.9) apply also to this case.

### 6.3 Orbifolds of \(\mathbb{C}^4\)

For the last example we consider Abelian orbifolds of \(\mathbb{C}^4\). The three–dimensional counterpart of the brane tiling was described in \[13\]. The bipartite lattice has \(S^4\) symmetry, like a tetrahedral lattice (see Figure 3). For this reason we will denote the counting function \(f_{\phi}(n)\) by a tetrahedron, just as before we used a triangle for the \(S_3\) symmetry of the orbifolds of \(\mathbb{C}^3\).

The cycle index for \(S_4\) is
\[
Z_{S_4} = \frac{1}{24} \left( x_1^4 + 6 x_1^2 x_2 + 3 x_2^2 + 8 x_1 x_3 + 6 x_4^2 \right).
\] (6.19)

The first terms of the subsequences \(f_{\phi_i}(n)\) are collected in Table 9, and the first 500 numbers of \(f_{\phi}(n)\) are represented in Figure 4.

Following equation (4.5), we know that each of the subsequences can be written in the form of a double Dirichlet convolution where one of the factors is the unity \(u\):

1. The sequence \(f_{\phi_1}^\phi\) is the one corresponding to the identity. This means that it is given by Equation (3.10) for \(d = 3\). Equivalently, it can be written as the convolution
\[
f_{\phi_1}^\phi = u * N * N^2.
\] (6.20)
Figure 3: The tetrahedral lattice. Points are at the vertices and the center of a tetrahedron. Four such tetrahedra connected at their vertices fit into a cube.

Figure 4: Scatter plot of the number of sublattices of index $n$ for a tetrahedral lattice. Prime numbers are given in red. The line corresponds to $n^2 / 24$.

| $f_{x, y, z}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $f_{x_1}$ | 1 | 7 | 13 | 35 | 31 | 91 | 57 | 155 | 130 | 217 | 133 | 455 | 183 | 399 | 403 | 651 |
| $f_{x_2}$ | 1 | 3 | 5 | 11 | 7 | 15 | 9 | 31 | 18 | 21 | 13 | 55 | 15 | 27 | 35 | 75 |
| $f_{x_3}$ | 1 | 3 | 5 | 11 | 7 | 15 | 9 | 31 | 18 | 21 | 13 | 55 | 15 | 27 | 35 | 75 |
| $f_{x_4}$ | 1 | 1 | 1 | 2 | 1 | 1 | 3 | 2 | 4 | 1 | 1 | 2 | 3 | 3 | 1 | 3 |
| $f_{x_5}$ | 1 | 1 | 1 | 3 | 3 | 1 | 1 | 5 | 2 | 3 | 1 | 3 | 3 | 1 | 3 | 7 |

Table 9: Number of sublattices of index $n$ for the tetrahedral lattice, classified by the cycles of the symmetric group $S_4$. 

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2. The sequence $f_{\mathfrak{x}_1 \mathfrak{x}_2}^\Phi$ can be written as
\[ f_{\mathfrak{x}_1 \mathfrak{x}_2}^\Phi = \{1, -1, 0, 4\} \ast u \ast u \ast N . \quad (6.21) \]

3. The sequence $f_{\mathfrak{x}_2}^\Phi$ coincides with $f_{\mathfrak{x}_1 \mathfrak{x}_2}^\Phi$.

4. The sequence $f_{\mathfrak{x}_1 \mathfrak{x}_3}^\Phi$ can be written as a convolution with the non–principal Dirichlet character of modulus three that has already appeared before:
\[ f_{\mathfrak{x}_1 \mathfrak{x}_3}^\Phi = \{1, 0, -1, 0, 0, 0, 3\} \ast u \ast \chi_{3,2} . \quad (6.22) \]

5. The sequence $f_{\mathfrak{x}_4}^\Phi$ is the convolution of the non–principal character of modulus four:
\[ f_{\mathfrak{x}_4}^\Phi = \{1, -1, 0, 2\} \ast u \ast \chi_{4,2} . \quad (6.23) \]

We already have the Dirichlet series corresponding to each of these terms and we can collect them by using Burnside’s lemma. The final result is:
\[ F^\Phi(s) = \frac{1}{24} \left( F_{\mathfrak{x}_1}^\Phi(s) \ast 6 F_{\mathfrak{x}_2}^\Phi(s) + 3 F_{\mathfrak{x}_3}^\Phi(s) + 8 F_{\mathfrak{x}_4}^\Phi(s) + 6 F_x^\Phi(s) \right) = \]
\[ = \frac{\zeta(s)\zeta(s-1)}{24} (\zeta(s-2) + 9 (1 - 2^{-s} + 2^{1-2s}) \zeta(s)) + \]
\[ + \frac{\zeta(s)^2}{24} \left( 8 \left( 1 - 3^{-s} + 3^{1-2s} \right) L(\chi_{3,2}, s) + 6 \left( 1 - 2^{-s} + 2^{1-2s} \right) L(\chi_{4,2}, s) \right) . \quad (6.24) \]

The rightmost pole is at $s = 3$ and has order 1. This means that the partial sum of the terms in the sequence behaves like
\[ \sum_{n=1}^{N} f_{\mathfrak{x}_1}^\Phi(n) \sim \frac{\zeta(3) \zeta(3)}{3 \times 24} N^3 \sim 0.0274 N^3 . \quad (6.25) \]

It is also possible to write the power series corresponding to each term as follows:
\[ F_{\mathfrak{x}_1}^\Phi(t) = \sum_{m=1}^{\infty} G_{\mathfrak{x}_1}^\Phi(t^m) , \quad (6.26) \]
Table 10: Generating functions for the Abelian orbifolds of $\mathbb{C}^4$, organized by the symmetries of the brane tiling.

where

\[
G_{x_1^4}^\Phi(t) = \sum_{n,m=1}^{\infty} nm^2 t^{mn},
\]

(6.27a)

\[
G_{x_1^2 x_2}^\Phi(t) = G_{x_2^2}^\Phi(t) = \sum_{n,m=1}^{\infty} m \left( t^{mn} - t^{2mn} + 4t^{4mn} \right),
\]

(6.27b)

\[
G_{x_1 x_3}^\Phi(t) = \frac{1}{2} \left[ \sum_{n,m=-\infty}^{\infty} t^{n^2+4m^2} - 1 \right],
\]

(6.27c)

\[
G_{x_4}^\Phi(t) = \frac{1}{2} \left[ \sum_{n,m=-\infty}^{\infty} t^{n^2+mn+7m^2} - 1 \right].
\]

(6.27d)

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A Generic lattice in \(d\) dimensions

In this appendix we derive a general expression for the number \(f^{L_d}(n)\) of sublattices of index \(n\) for a generic lattice in \(d\) dimensions. Using the construction in Section 4, we want to count matrices of the form

\[
\begin{pmatrix}
    a_{11} & a_{22} & \cdots & \\
    a_{21} & a_{22} & & \\
    \ldots & \ldots & \ddots & \\
    a_{d1} & a_{d2} & \cdots & a_{dd}
\end{pmatrix}
\]  
(A.1)

where \(a_{11}a_{22}\cdots a_{dd} = n\), and \(0 \leq a_{ij} < a_{ii}, \forall j < i\).

- For \(d = 1\), the only condition is \(a_{11} = n\), so there is exactly one sublattice for each choice of the index:
  \[
f^{L_1}(n) = 1.\]
  (A.2)

- For \(d = 2\) we want to count the triangular matrices
  \[
  \begin{pmatrix}
    a_{11} & 0 \\
    a_{21} & a_{22}
  \end{pmatrix}
  \]
  (A.3)

  such that \(a_{11}a_{22} = n\) and \(a_{21} = 0,1,\ldots,a_{22} - 1\). It is immediate to see that there are \(m\) such matrices for each choice of a divisor \(m\) of \(n\):
  \[
f^{L_2}(n) = \sum_{m|n} m = \sum_{m|n} mf^{L_1}(m).\]
  (A.4)

- In \(d = 3\) we count the matrices
  \[
  \begin{pmatrix}
    a_{11} & 0 & 0 \\
    a_{21} & a_{22} & 0 \\
    a_{31} & a_{32} & a_{33}
  \end{pmatrix}
  \]
  (A.5)

  where \(a_{11}a_{22}a_{33} = n, a_{21} = 0,1,\ldots,a_{22} - 1\) and both \(a_{31}\) and \(a_{32}\) are in \(\{0,1,\ldots,a_{33} - 1\}\). Let us write
  \[
a_{11} = \frac{n}{m_1}, \quad a_{22} = \frac{m_1}{m_2}, \quad a_{33} = m_2,
  \]
  (A.6)

  then we have \(a_{22}a_{33}^2 = m_1/m_2 \times m_2^2 = m_1m_2\) such matrices for any divisor \(m_1\) of \(n\) and any divisor \(m_2\) of \(m_1\). Hence
  \[
f^{L_3}(n) = \sum_{m_1|m_2|m_2|m_1} m_1m_2 = \sum_{m|m} mf^{L_2}(m).\]
  (A.7)

- For general \(d\), one has the condition \(a_{11}a_{22}\cdots a_{dd} = n\) and a total of \(a_{22}a_{33}^2\cdots a_{dd}^{d-1}\)
matrices. Writing the coefficients as
\[ a_{11} = \frac{n}{m_1}, \quad a_{22} = \frac{m_1}{m_2}, \quad \ldots \quad a_{(d-1)(d-1)} = \frac{m_{d-2}}{m_{d-1}}, \quad a_{dd} = m_{d-1}, \] (A.8)
we obtain
\[ f^{L_d}(n) = \sum_{m_1|m} \sum_{m_2|m_1} \cdots \sum_{m_{d-1}|m_{d-2}} m_1 m_2 \cdots m_{d-1} = \sum_{m|n} m f^{L_{d-1}}(m). \] (A.9)

The generating functions for these sequences are written as follows. From the observation that there are \(a_{22}a_{33}^2 \cdots a_{dd}^{d-1}\) matrices of index \(n = a_{11}a_{22} \cdots a_{dd}\) we can obtain directly the power series
\[ F^{L_d}(t) = \sum_{n=1}^{\infty} f^{L_d}(n) t^n = \sum_{k_1,k_2,\ldots,k_d=1}^{\infty} k_2 k_3^2 \cdots k_d^{d-1} k_1 k_2 \cdots k_d. \] (A.10)
The Dirichlet series can be written by using the recursion relation
\[ f^{L_d}(n) = \sum_{m|n} m f^{L_{d-1}}(m). \] (A.11)
Rewriting it as
\[ \frac{f^{L_d}(n)}{n} = \sum_{m|n} \frac{1}{n/m} f^{L_{d-1}}(m), \] (A.12)
we see that the sequence \([f^{L_d}(n)/n]\) is the Dirichlet convolution of the sequences \([1/n]\) and \(f^{L_{d-1}}\):
\[ \left[ \frac{f^{L_d}(n)}{n} \right] = \left[ \frac{1}{n} \right] * \left[ f^{L_{d-1}}(n) \right]. \] (A.13)
It follows that
\[ F^{L_d}(s+1) = \zeta(s+1) F^{L_{d-1}}(s). \] (A.14)
This recursion relation can be solved starting from
\[ F^{L_1}(s) = \zeta(s), \] (A.15)
and we find
\[ F^{L_d}(s) = \zeta(s) \zeta(s-1) \cdots \zeta(s-d+1), \] (A.16)
or in the form of a Dirichlet convolution
\[ f^{L_d} = u * N * N^2 * \cdots * N^{d-1}, \] (A.17)
where \(N^k(n) = \{ 1^k, 2^k, 3^k, \ldots \} \).