A variational principle for the system of P.D.E. of porous metal bearings

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Abstract Porous metal bearings are widely used in small and micro devices. To compute the pressure one has to solve the Reynolds equation coupled with the Laplace equation. We show that it is possible to give to the relevant boundary value problem a variational formulation. We show that the pressure of the film in a porous bearing is less than that of the corresponding non-porous bearing.

Keywords Lubrication theory · Reynolds’s equation · Variational principle

Mathematics Subject Classification 49S99 · 35Q35

1 Introduction

The presence in all sort of machines of rotating parts requires an efficient way to keep separate the sliding surfaces with as little frictional resistance as possible. Reynolds [9] derived in 1886 the equation

\[ \nabla \cdot \left( \frac{h^3}{\mu} \nabla p \right) = 6U \frac{dh}{dx}, \quad h(x) \geq h_0 > 0 \]  

starting from heuristic approximations of the Navier–Stockes system and exploiting the special shape ”film-like” of the domain relevant in lubrication. In (1) \( p \) is the pressure in the lubricating fluid, \( \mu \) the viscosity, \( U \) the velocity of the moving surface and \( h \) describes the small gap between the sliding surfaces. Once \( p \) is known the theory of lubrication gives the approximate components of the velocity. There exists a wide variety in the shape of the bearing surfaces and in the way in which the lubricant is injected in the device (see [2, 8]). In this paper we consider the case of the self-lubricating bearing in which a porous matrix impregnated with the lubricant is the source of the lubricating fluid. Let us consider first the mathematical model in three-dimensional cartesian coordinates.

Let \( \Omega = \{(x, y, z), 0 < x < M, 0 < y < L, 0 < z < R \} \) denote the porous matrix. The domain \( \{(x, y, z), 0 < x < M, 0 < y < L, R < z < R + h(x)\} \) is the region separating the bearing surfaces and occupied by the lubricating fluid which the theory of lubrication treats as two-dimensional. The Reynolds equation holds on \( C_2 = \{0 < x < M, 0 < y < L, z = R \} \) and reads, [2, p. 548],

\[ \frac{\partial}{\partial x} \left( \frac{h^3}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{h^3}{\mu} \frac{\partial p}{\partial y} \right) = 6U \frac{\partial h}{\partial x} + 12V_0, \]  

(2)

\( V_0 \) is the rate of flow per unit area across \( C_2 \). Let \( v \) denote the mean velocity of fluid in the porous matrix and \( \rho(x, y, z) \) the corresponding pressure. By Darcy’s law we have in \( \Omega \)
\[ \mathbf{v} = -\frac{\Phi}{\mu} \nabla p. \]  
\hspace{1cm} (3)

Since \( \nabla \cdot \mathbf{v} = 0 \) we obtain
\[ \Delta p = 0 \quad \text{in} \quad \Omega. \]  
\hspace{1cm} (4)

On the other hand, the rate of flow per unit area coming out (or entering) from the matrix is equal to the rate of flow entering (or coming out) from the lubricant fluid. Hence we must have on \( C_2 \)
\[ \frac{\partial p}{\partial z} = \frac{\partial \bar{p}}{\partial z}. \]  
\hspace{1cm} (5)

Since \( V_0 = \frac{\Phi}{\mu} \frac{\partial \bar{p}}{\partial z} \) we obtain, from (2) (see [2], p. 548),
\[ \frac{\partial}{\partial x} \left( h^3 \frac{\partial \bar{p}}{\partial x} \right) + \frac{\partial}{\partial y} \left( h^3 \frac{\partial \bar{p}}{\partial y} \right) = 6U \frac{\partial h}{\partial x} + 12\Phi \frac{\partial \bar{p}}{\partial z} \quad \text{on} \quad C_2. \]  
\hspace{1cm} (6)

Thus, to determine the pressure in both the porous matrix and in the lubricating film, we must solve the Eq. (4) with the Eq. (6) playing the role of a boundary condition on \( C_2 \) and additional boundary conditions (Dirichlet or Neumann) on the remaining part of the boundary of \( \Omega \). This situation is unusual from the mathematical point of view because we must satisfy on part of the boundary an elliptic equation of the same order of the operator (in this case the laplacian) valid in \( \Omega \). It seems that this kind of problem have been rarely studied in the vast literature on elliptic equations of the second order.

2 The case of full cylindrical porous bearings

In this Section we treat the self-lubricating cylindrical porous bearing of finite length \( L \) with the cross-section of Fig. 1.

We use the notations of [2]. Thus we use cylindrical coordinates \( (\rho, \theta, y) \), \( h(\theta) = c(1 + \epsilon \cos(\theta)) \) is the oil film thickness with \( c \) the radial clearance in the journal bearing and \( 0 < \epsilon < 1 \). \( \mu \) denotes the viscosity, \( U \) the velocity of the shaft and \( \Phi \) the permeability of the porous matrix.

Let
\[ \Omega = \{ (\rho, \theta, y); \ R_1 < \rho < R_2, \ 0 < \theta \leq 2\pi, \ 0 < y < L \} \]
denote the porous matrix, let

\[ C_1 = \{ (\rho, \theta, y); \ \rho = R_1, \ 0 < \theta \leq 2\pi, \ 0 < y < L \} \]
be the interior surface of the matrix and let
\[ C_2 = \{ (\rho, \theta, y); \ \rho = R_2, \ 0 < \theta \leq 2\pi, \ 0 < y < L \} \]
be the exterior part of the matrix, where the Reynolds equation holds. Finally let
\[ A_1 = \{ (\rho, \theta, y); \ R_1 < \rho < R_2, \ 0 < \theta \leq 2\pi, \ y = 0 \} \]
\[ A_2 = \{ (\rho, \theta, y); \ R_1 < \rho < R_2, \ 0 < \theta \leq 2\pi, \ y = L \} \]
be the two end surfaces of the computational domain. Define (see [2], p. 549)
\[ k_1 = \frac{c^2}{12\Phi R_1^2}, \quad k_2 = \frac{c^2}{12\Phi R_2^2}, \quad k_3 = \frac{6U\mu}{12\Phi R_2^2}, \]
k_1, k_2 and k_3 are all positive constant. The permeability \( \Phi \) of the porous matrix and the viscosity \( \mu \) are taken as constants. By the consideration made in the Introduction the pressure \( p(\rho, \theta, y) \) in the lubricating film and in the matrix is determined by the following boundary value problem: to find \( p(\rho, \theta, y) \) such that
\[ \Delta p = 0 \quad \text{in} \quad \Omega \]  
\hspace{1cm} (7)
\[ k_1(h^3 p_0)_\rho + k_2(h^3 p_y)_y = p_\rho + k_3 h' \quad \text{on} \quad C_2 \]  
\hspace{1cm} (8)
\[ p = 0 \quad \text{on} \quad C_1, \quad p = 0 \quad \text{on} \quad A_1 \cup A_2 \]  
\hspace{1cm} (9)
\[ p(\rho, \theta, y) = p(\rho, \theta + 2\pi, y). \]  
\hspace{1cm} (10)
As before, the interaction between the pressure in the porous matrix and the pressure in the lubricating film is modeled by Eq. (8). A crucial phenomenon in real bearings is the fact that the lubricant film cannot sustain pressure below the atmospheric pressure leading to the phenomenon of rupture of the film and cavitation. The fluid film breaks down in a "a priori" unknown region. To take this fact into account we must add to problem (7)–(10) the unilateral condition

\[ p \geq 0 \quad \text{on} \quad C_2 \]

with Eq. (8) holding only where \( p > 0 \). Two approaches are possible: the so-called "half-Sommerfeld" condition in which the cavitated solution is simply obtained by setting the solution of the bilateral problem equal to zero in all points in which it becomes negative, or the more precise approach where the problem is restated as a free boundary problem. This makes possible to use the Lax–Milgram lemma and the Stampacchia theorem to prove that both problems have one and only one solution.

The special geometry of the long bearing is considered in Sect. 4. In this case the pressure will depend only from two variables and the Reynolds equation becomes one-dimensional. Using the Fourier’s method the equations are uncoupled and we solve separately a non standard two-point problem for the Reynolds equation and a Dirichlet’s problem in the porous matrix. In Sect. 5 the two-point problem is solved in the special case of small eccentricity ratio. In this way we obtain a simple expression for the pressure in the film, and the effect of the porous matrix reduces to a single numerical parameter. Moreover, if we adopt the half-Sommerfeld condition we find that the pressure in the film is less than the pressure in the corresponding non-porous bearing. Finally in Sect. 6 some open problems concerning the regularity of the solutions and the shape of the region of cavitation are proposed.

### 3 Variational formulation

Define the functional

\[
J(p) = \int_\Omega |\nabla p|^2 dV + \int_{C_2} \left[ k_1 p_0^2 + k_2 p_1^2 + k_3 h'p \right] dS.
\]

(13)

Let us take, for the moment, as class of admissible functions for \( J(p) \)

\[
\mathcal{H} = \{ p(\rho, \theta, y) \in C^2(\bar{\Omega}), \quad p = 0 \quad \text{on} \quad C_1, \quad p = 0 \quad \text{on} \quad A_1 \cup A_2, \quad p(\rho, \theta, y) = p(\rho, \theta + 2\pi, y) \}.
\]

Note that \( p \) is not prescribed on \( C_2 \). Let \( p \) be a function which makes \( J(p) \) stationary in \( \mathcal{H} \) and let \( v \) be an arbitrary function of \( \mathcal{H} \) vanishing on the boundary of \( \Omega \). If \( x \in \mathbb{R}^1 \) we have \( p + xv \in \mathcal{H} \). Define \( g_1(x) = J(p + xv) \). Using the condition \( g'_1(0) = 0 \) we find

\[
\int_\Omega \nabla p \cdot \nabla v dV + \int_{C_2} \left[ h^3 \left( k_1 p_0 v_0 + k_2 p_1 v_1 \right) + k_3 h'v \right] dS = 0.
\]

(14)

Integrating by parts in (14) and taking into account that \( v \) vanishes on the boundary of \( \Omega \), we find

\[
\int_\Omega v \Delta p \, dV = 0.
\]

(15)

Since \( v \) is arbitrary we conclude that (7) holds i.e.

\[
\Delta p = 0 \quad \text{in} \quad \bar{\Omega}.
\]

(16)

Let us take now a different variation. More precisely let \( w \in \mathcal{H} \), thus \( w \) is arbitrary on \( C_2 \). Consider \( g_2(x) = J(p + xw) \). Computing \( g'_2(0) = 0 \) we find

1. Taken here equal to zero in a suitable scale.
2. The conditions (12) are valid in the non-porous case. They are justified by the Lewy–Stampacchia theorem (see [6], and [7, p. 223]). For, the solution of the variational inequality which gives the unilateral solution is globally of class \( C^{1,2} \). Hence, on the side of the free boundary where \( p = 0 \), we have \( p_0 = 0, p_1 = 0 \). This implies, by continuity, (12). The conditions valid on the free boundary in the porous case, are open questions, see Sect. 6.
\[
\int_\Omega \nabla p \cdot \nabla wdV + \int_{\Omega_C} [h^3 (k_1 p_0 w_0 + k_2 p_y w_y) + k_3 h' w] dS = 0,
\]
(17)
i.e. (14) with \(w\) in place of \(v\). This time, however, integrating by parts in the first integral of (17) we have, taking into account (16),
\[
\int_\Omega \nabla p \cdot \nabla wdV = \int_{\Omega_C} \frac{dp}{dn} wdS = \int_{\Omega_C} p_0 wdS.
\]
(18)
Let us now integrate by parts in the second integral of (17). Recall that on \(C_2\) we have
\[
p(R_2, 0, 0) = p(R_2, 0, L) = 0, \quad p(R_2, 0, y) = p(R_2, 2\pi, y).
\]
(19)
By (19) we have
\[
\int_{C_2} (k_1 h^3 p_0 w_0 + k_2 h^3 p_y w_y + k_3 h' w) dS \\
= \int_{C_2} [-k_1 (h^3 p_0)_0 w - k_2 (h^3 p_y)_y w + k_3 h' w] dS.
\]
(20)
Since \(w\) is arbitrary on \(C_2\), adding (20) and (18) we obtain, in view of (17),
\[
k_1 (h^3 p_0)_0 + k_2 (h^3 p_y)_y = p_0 + k^3 h',
\]
i.e. (8). We use the established variational principle to prove that problem (7)–(10) has one and only one solution. To this end, we define a new space of admissible functions more mathematically convenient. Let \(\hat{H}((\Omega)\) be the set of functions \(v\) of class \(C^\infty(\Omega)\) periodic with period \(2\pi\) with respect to \(\theta\) which take arbitrary values on \(C_2\) and vanish in a neighborhood of \(C_1 \cup A_1 \cup A_2\). We define in \(\hat{H}((\Omega)\) the norm
\[
\|v\| = \left[ \int_\Omega |\nabla v|^2 dV + \int_{\Omega_C} \left( k_1 v_0^2 + k_2 v_y^2 \right) dS \right]^{1/2}.
\]
(22)
Let \(H_{00}^1(\Omega)\) be the completion of \(\hat{H}\) with respect to the norm (22). To prove existence and uniqueness of solutions of problem (7)–(10) we use the Lax–Milgram lemma (see [10, p. 383] and [1]) which, for the sake of completeness, we quote below.

**Lemma 1** Let \(H\) be an Hilbert space and \(a(u, v)\) a bilinear form defined in \(H \times H\). Assume that there exists a constant \(C > 0\) such that
\[
|a(u, v)| \leq C \|u\| \|v\| \quad \text{for all} \quad u, v \in H
\]
(23)
and a constant \(K > 0\) such that
\[
|a(v, v)| \geq K \|v\|^2 \quad \text{for all} \quad v \in H.
\]
(24)
Then, if \(f \in H'\), there exists a unique solution of the problem
\[
u \in H, \quad a(u, v) = \langle f, v \rangle \quad \text{for all} \quad v \in H.
\]
(25)
Moreover, if \(a(u, v)\) is symmetric, \(u\) is characterized by the minimum property
\[
\frac{1}{2} a(u, u) - \langle f, u \rangle = \min_v \left\{ \frac{1}{2} a(v, v) - \langle f, v \rangle \right\}.
\]
(26)
In our context Lemma 1 is used as follows: \(H\) will be \(H_{00}^1(\Omega)\), and
\[
a(u, v) = \int_\Omega \nabla u \cdot \nabla v dV + \int_{\Omega_C} h^3 (k_1 u_0 v_0 + k_2 u_y v_y) dS,
\]
(27)
finally
\[
\langle f, v \rangle = \int_{\Omega_C} k_3 h' v dS.
\]
(28)
Recalling the definition (22) of the norm in \(H_{00}^1(\Omega)\) and using the Cauchy–Schwartz inequality we find that (23) is satisfied. Since \(h(\theta) = c(1 + \epsilon \cos(\theta)) \geq c(1 - \epsilon) > 0\), also (24) is satisfied. In the case at hand, (25) is the weak formulation of problem (7)–(10), thus this problem has one and only one weak solution. If the weak solution is sufficiently regular to permit the integration by parts leading to (7) and (8) we obtain a classical solution. In the same vein, i.e. in term of a weak formulation, we can treat the problem with cavitation. To this end use will be made of a generalization of the Lax–Milgram lemma, known as the Stampacchia theorem, see [6], p. 24, which we quote below.

**Theorem 2** Let \(H\), \(a(u, v)\) and \(f\) be as in Lemma 1 and \(K\) be a convex closed subset of \(H\). Then there exists a unique \(u\) such that
$u \in \mathcal{K}, \quad a(u, v - u) \geq \langle f, v - u \rangle \quad \text{for all } v \in \mathcal{K}.$  
\begin{equation}
(29)
\end{equation}

Moreover, if $a(u, v)$ is symmetric $u$ is characterized by the minimum property
\begin{equation}
\frac{1}{2} a(u, u) - \langle f, u \rangle = \min_{v \in \mathcal{K}} \left\{ \frac{1}{2} a(v, v) - \langle f, v \rangle \right\}.
\end{equation}
\begin{equation}
(30)
\end{equation}

We apply this theorem with $\mathcal{K} = \{ v \in H^1_{00}(\Omega), \ v \geq 0 \text{ on } C_2 \}$. The space $H$, the bilinear form $a(u, v)$ and $f$ shall remain as prescribed in (27) and (28). We conclude that problem (7)–(12) has a unique weak solution. For methods related to Lemma 1 we refer to the book [10] chapter 6.

### 4 The porous long bearing

We treat in this Section the case of the porous long bearing. This leads to a one-dimensional Reynolds equation with the pressure depending only on $x$ and $y$. Let $D = \{(x, y); \ 0 < x < \pi, \ 0 < y < a\}$ represents the porous matrix. If cavitation is not taken into account, we arrive, for the determination of the pressure, to the following non standard boundary value problem: to find $p(x, y)$ such that
\begin{equation}
p_{xx} + p_{yy} = 0 \quad \text{in } D
\end{equation}
\begin{equation}
p(0, y) = 0, \quad p(\pi, y) = 0 \quad \text{for } 0 < y < a
\end{equation}
\begin{equation}
p(x, 0) = 0, \quad \text{for } 0 < x < \pi
\end{equation}
\begin{equation}
(33)
\end{equation}
\begin{equation}
- (h^3(x)p_x(x, a))_x + p_x(x, a) = -h'(x) \quad \text{for } 0 < x < \pi,
\end{equation}
\begin{equation}
(34)
\end{equation}

where $h(x) = 1 + \epsilon \cos(2x), \ 0 < \epsilon < 1$. We have set, to simplify notations, $k_1 = 1, \ k_3 = 1$ and $c = 1$.\footnote{These constant can easily reintroduced with minor notational changes.} To prove that problem (4.1)–(34) has one and only one solution we could follow the variational method of Sect. 3 defining the functional
\begin{equation}
J(p) = \int_D |\nabla p|^2 \, dx + \int_0^\pi [h^3 p_x^2(x, a) + 2h'p(x, a)] \, dx
\end{equation}
and the corresponding class of admissible function. However, we prefer to use a fixed point argument to get existence and uniqueness since the calculations involved will be needed in Sect. 5. We first consider the following standard Dirichlet’s problem. Given $P(x) \in H^1_0([0, \pi])$, find $p(x, y)$ satisfying
\begin{equation}
p_{xx} + p_{yy} = 0 \quad \text{in } D
\end{equation}
\begin{equation}
p(0, y) = 0, \quad p(\pi, y) = 0 \quad \text{for } 0 < y < a
\end{equation}
\begin{equation}
p(x, 0) = 0, \quad p(x, a) = P(x) \quad \text{for } 0 < x < \pi.
\end{equation}
\begin{equation}
(37)
\end{equation}
The problem (35)–(37) has one and only one solution (see [11, p. 95]) given by
\begin{equation}
p(x, y) = \sum_{n=1}^\infty \frac{\sinh(ny) \sin(nx)}{\sinh(na)} b_n, \quad (38)
\end{equation}

where
\begin{equation}
b_n = \frac{2}{\pi} \int_0^\pi \sin(n \zeta) P(\zeta) \, d\zeta.
\end{equation}
\begin{equation}
(39)
\end{equation}

Note that
\begin{equation}
\frac{\partial p}{\partial y}(x, a) = \sum_{n=1}^\infty n \coth(na) b_n \sin(nx).
\end{equation}
\begin{equation}
(40)
\end{equation}

The problem (35)–(37) defines the operator
\begin{equation}
p = T(P), \quad T : H^1_0(0, \pi) \to H^1(D).
\end{equation}
\begin{equation}
(41)
\end{equation}

If we can find a solution $P(x) \in H^1_0(0, \pi)$ of the following integro-differential two point problem
\begin{equation}
- \left(h^3(x)p'(x)\right)' + \sum_{n=1}^\infty n \coth(na) b_n \sin(nx) = -h'(x)
\end{equation}
\begin{equation}
(42)
\end{equation}
\begin{equation}
P(0) = 0, \quad P(\pi) = 0, \quad (43)
\end{equation}

where
\begin{equation}
b_n = \frac{2}{\pi} \int_0^\pi \sin(n \zeta) P(\zeta) \, d\zeta
\end{equation}
\begin{equation}
(44)
\end{equation}

the sought for solution of problem (32)–(34) is given by $p(x, y) = T(P(x))$. We have
Theorem 3  There exists one and only one solution in $H^1_0(0, \pi)$ to the integro-differential problem (42), (43).

Proof  Let

$$P(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \in H^1_0(0, \pi),$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} P(\xi) \sin(n\xi) d\xi$$

$$V(x) = \sum_{n=1}^{\infty} c_n \sin(nx) \in H^1_0(0, \pi),$$

$$c_n = \frac{2}{\pi} \int_0^{\pi} V(\xi) \sin(n\xi) d\xi.$$  (44)

We have

$$\int_0^{\pi} P'(x)V'(x) dx = \sum_{n=1}^{\infty} n^2 b_n c_n$$  (45)

and

$$\|V\|_{H^1_0(0, \pi)} = \left( \sum_{n=1}^{\infty} n^2 c_n^2 \right)^{1/2}, \quad \|V\|_{L^2(0, \pi)} = \left( \sum_{n=1}^{\infty} c_n^2 \right)^{1/2}.$$  (46)

Let us multiply (42) by $V(x) \in H^1_0(0, \pi)$ and integrate by parts over $(0, \pi)$. We obtain, for all $V(x) \in H^1_0(0, \pi)$,

$$\int_0^{\pi} h^3(x)P'(x)V'(x) dx + \frac{\pi}{2} \sum_{n=1}^{\infty} n \coth(na)b_n c_n$$

$$= - \int_0^{\pi} h'(x)V(x) dx,$$  (47)

where we have used the relation\(^4\)

$$\int_0^{\pi} \left[ \sum_{n=1}^{\infty} n \coth(na)b_n \sin(nx) \right] \left[ \sum_{k=1}^{\infty} c_k \sin(kx) \right] dx$$

$$= \frac{\pi}{2} \sum_{n=1}^{\infty} n \coth(na)b_n c_n.$$

Therefore, we can restate problem (42), (43) in weak form as follows: to find $P(x) \in H^1_0(0, \pi)$ such that (47) holds. We use the Lax–Milgram lemma to prove that (47) has one and only one solution. Define in $H^1_0(0, \pi)$ the bilinear form

$$a(P, V) = \int_0^{\pi} h^3(x)P'(x)V'(x) dx + \frac{\pi}{2} \sum_{n=1}^{\infty} n \coth(na)b_n c_n.$$  (48)

Since $\coth(na) \leq \coth(a)$ for $n \geq 1$, we have

$$\| \sum_{n=1}^{\infty} n \coth(na)b_n c_n \| \leq \coth(a) \left( \sum_{n=1}^{\infty} n^2 b_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} c_n^2 \right)^{1/2} \leq \coth(a) \|P\|_{H^1_0(0, \pi)} \|V\|_{H^1_0(0, \pi)}.$$

Thus $a(P, V)$ is bounded and also coercive. In fact, since $h^3 \geq 1 - \epsilon^3$ we have

$$a(P, P) = \int_0^{\pi} h^3(x)P'(x)^2 dx + \sum_{n=1}^{\infty} n \coth(na)b_n^2$$

$$\geq (1 - \epsilon^3) \|P\|_{H^1_0(0, \pi)}^2.$$  (49)

Since $\int_0^{\pi} h'(x)V(x) dx$ defines a linear functional on $H^1_0(0, \pi)$ we conclude that problem (47) has one and only one solution. \(\square\)

Invoking Lemma 1 we obtain the unique solution of problem (31)–(34) which is given by $p(x, y) = T(P(x))$.

5 Dependence of the pressure from the parameter $a$ in the case of the infinitely long bearing of small eccentricity ratio

The case of the infinitely long bearing of small eccentricity ratio [2] simplifies considerably the situation of the previous Section. The two-point problem (42), (43) becomes now

$$-P''(x) + \sum_{n=1}^{\infty} n \coth(na)b_n \sin(nx) = \sin(2x),$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} P(\xi) \sin(n\xi) d\xi,$$  (49)

$$P(0) = 0, \quad P(\pi) = 0.$$  (50)

The problem (49)–(50) has an explicit and particularly simple solution. Substituting

\(^4\) To verify this relation, recall that $\int_0^{\pi} \sin(mx) \sin(nx) dx$ is equal to $\frac{\pi}{2}$ if $m = n$ and equal to 0 if $m \neq n$.  

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\[ P(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \]  
(51)

In (49) we have
\[ \sum_{n=1}^{\infty} n^2 b_n \sin(nx) + \sum_{n=1}^{\infty} n \coth(na) b_n \sin(nx) = \sin(2x). \]  
(52)

The \( b_n \)'s entering in (52) are easily computed. For, if we multiply (52) by \( \sin(nx) \), \( n = 1, 2, \ldots \) and integrate over \([0, \pi]\) with respect to \( x \) we find \( b_n = 0 \) if \( n \neq 2 \) and
\[ b_2 = \frac{1}{2 [2 + \coth(2a)]}. \]  
(53)

Whence from (51) we obtain
\[ P(x) = \frac{\sin(2x)}{2 [2 + \coth(2a)]} \]  
(54)

with the pressure in \( \mathcal{D} \) given, in view of (38), by
\[ p(x, y) = \frac{\sinh(2y) \sin(2x)}{2 \sinh(2a) [2 + \coth(2a)]}. \]  
(55)

We contrast (55) with the corresponding solution of the non-porous case. This does not depend on \( y \), and solves the problem
\[- p''_0(x) = \sin(2x), \quad p_0(0) = 0, \quad p_0(\pi) = 0 \]  
(56)

whose solution is
\[ p_0(x) = \frac{\sin(2x)}{4}. \]  
(57)

Hence we have
\[ |p_0(x)| > |p(x, a)|. \]  
(58)

If we assume the "half-Sommerfeld" condition to take into account of cavitation, we have from (5.6)
\[ p(x, a) = \begin{cases} \frac{\sin(2x)}{2 [2 + \coth(2a)]} & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \leq \pi. \end{cases} \]  
(59)

From (59) we can draw the conclusion that, at least in the present simplified model, the pressure in porous bearings is always less than that in the corresponding non-porous bearings and that the pressure increases as the porous matrix increases in size.

6 Conclusion

Consider the problems without cavitation. Using Lemma 1 we proved that there is one and only one weak solution of problem (7)–(10). Can we prove that this weak solution is also a classical solution, in other words that the weak solutions has all the derivatives needed to satisfy the equations (7) and (8)? One would expect even more, i.e. that the solution is of class \( C^\infty \).

Completely different is the situation for the problem in which cavitation is taken into account. In this case there is certainly a threshold of regularity. Probably one cannot go beyond the \( C^{0, \alpha} \)-regularity as found in a special case in [5]. Another question worthy of consideration is the shape of the region of cavitation in the porous case, (see [3] for the non porous case).

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Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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