Threshold and Revocation Encryptions via Threshold Trapdoor Function

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Abstract. We introduce a cryptographic primitive named threshold trapdoor function (TTDF), from which we give generic constructions of threshold and revocation encryptions under adaptive corruption model. Then, we show TTDF can be instantiated under the decisional Diffie-Hellman (DDH) assumption and the learning with errors (LWE) assumption. By combining the instantiations of TTDF with the generic constructions, we obtain threshold and revocation encryptions which compare favorably over existing schemes. The experimental results show that our proposed schemes are practical.

Keywords: Threshold trapdoor function, Threshold encryption, Revocation encryption, Generic construction, Adaptive corruptions, Lattice

1 Introduction

Threshold public-key encryption. TPKE \cite{1,2,3,4} can distribute the decryption power among many servers in order to ensure threshold servers can decrypt ciphertexts, while any probabilistic polynomial-time (PPT) adversary corrupting less than threshold servers is unable to obtain the message. TPKE itself provides useful functionalities, and it is also an important building block for other cryptographic primitives, such as mix-net (anonymous channel) \cite{5}, public key encryption with non-interactive opening \cite{6,7}.

Designing generic construction of TPKE has proved to be a highly non-trivial task. Dodis and Katz \cite{8} gave a generic construction of TPKE from multiple encryption technique. Wee \cite{9} introduced a new primitive called threshold extractable hash proofs and presented a generic construction of TPKE from it. However, both of above constructions are only secure under the static corruption model where the adversary fixes the servers that will be corrupted before the scheme is set up. Following the work of Wee \cite{9}, Libert and Yung \cite{10} introduced a primitive named all-but-one perfectly sound threshold hash proof systems, from which they gave a generic construction of TPKE under adaptive corruption model where the adversary can corrupt servers at any time. The results are important since it is known that the adaptive adversary is strictly stronger than the static one \cite{11,12}. But they only showed concrete instantiations under number-theoretic assumptions in bilinear groups which are vulnerable to quantum attacks. Recently, lattices have been recognized as a viable foundation for quantum-resistant cryptography. Bendlin and Damgård \cite{13} gave the first lattice-based TPKE based on a variant of Regev’s scheme \cite{14}. Xie et al. \cite{15} designed the first chosen-ciphertext secure (IND-CCA) TPKE under the LWE assumption. However, both of above TPKEs are only statically secure, and the size of the public key and the ciphertext is at least linear in the number of servers. Bendlin et al. \cite{16} converted Identity Based Encryption (IBE) \cite{17} into threshold one, which can be transformed into a TPKE via the generic transformation in \cite{18}. However, in an offline phase, their scheme needs the parties to perform lots of interactive precomputation. In summary, the state-of-the-art TPKE is not entirely satisfactory. On one hand, existing generic constructions of TPKE are designed in the limited static corruption model which fails to capture realistic attacks. On the other hand, most existing TPKE schemes are based on number-theoretic assumptions which are insecure against quantum attacks.

Revocation public-key encryption. RPKE \cite{19,20} enables a sender to broadcast ciphertexts and all but some revoked users can do the decryption. It is a special kind of broadcast encryption \cite{21} which
enables a sender to encrypt messages and transmit ciphertexts to users on a broadcast channel in order to the chosen users can decrypt ciphertexts. RPKE has numerous applications, including pay-TV systems, streaming audio/video and many others.

Naor and Pinkas [19] considered the following scenario: a group controller (GC) controls the decryption capabilities of users. If a subgroup of users is disallowed to do the decryption, the GC needs to generate a new key which should be known to other users and be used to encrypt in the further group communication. Then they constructed a RPKE scheme under the DDH assumption. Unlike the scenario of [19], Dodis and Fazio [20] designed a RPKE in which every user who knows the revoked identities can encrypt messages and every non-revoked user can decrypt ciphertexts. Then, they constructed IND-CCA RPKE under the DDH assumption. Wee [9] presented a generic construction of RPKE in static corruption model and instantiated the construction under the DDH assumption and factoring assumption respectively. However, all of aforementioned schemes are designed under the number-theoretic assumptions which are insecure against quantum attack.

1.1 Motivations

A central goal in cryptography is to construct cryptosystems in strong security models which can resist lots of possible attacks. Another goal is to build cryptosystems under intractability assumptions which are as general as possible; in this way, we can replace the underlying assumption, if some assumption is vulnerable to a new attack or if another yields better performance. Therefore, generic constructions of TPKE and RPKE in stronger adaptive corruption model are advantageous. Meanwhile, with the development of quantum computer, designing the quantum-resistant TPKE and RPKE is also necessary. Last but not least, constructing cryptosystems based on the same cryptographic primitive brings additional advantages such as reducing the footprint of cryptographic code and easily embedding into systems.

Motivated by above discussions, we ask the following challenging questions:

*Can we construct TPKE and RPKE under adaptive corruption model from one cryptographic primitive? Can we instantiate this primitive based on quantum-resistant assumptions?*

1.2 Our Contributions

We introduce a cryptographic primitive named TTDF, and derive generic constructions of TPKE and RPKE under adaptive corruption model from it. Along the way to instantiate TTDF, we propose a notion called threshold lossy trapdoor function (TLTDF) and prove that TTDF is implied by TLTDF, while the latter can be instantiated based on the DDH assumption and the LWE assumption. Moreover, we show a relaxation of TTD called threshold trapdoor relation (TTDR), which enables the same applications of TPKE and RPKE, and admits more efficient instantiation based on the DDH assumption. An overview of the constructions of this work is given in Figure 1.

![Fig. 1. Overview of the results in this work.](image-url)
Threshold Trapdoor Function. Informally, $(t, n)$-TTDF is a threshold version of trapdoor function. It is parameterized by the threshold value $t$ and the number of identities $n$. $(t, n)$-TTDF splits the master trapdoor into $n$ shared trapdoors which can be transmitted to $n$ users securely. Every user who holds shared trapdoor can compute a piece of inversion share. Then, by collecting more than $t$ inversion shares, the combiner can recover the preimage. Especially, it can even compute any other inversion shares (threshold shares mean all shares). We formalize security notion for TTDF, namely threshold one-wayness, which requires that the function is hard to invert, even if the adversary can adaptively obtain less than $t$ shared trapdoors.

TPKE from TTDF. $(t, n)$-TTDF gives rise to a simple construction of $(t, n)$-TPKE. The public key consists of an injective trapdoor function index $ek$, and the master secret key consists of the master trapdoor $mtd$. The sharing algorithm splits the master secret key into $n$ shared secret keys. Given a message $m$, the encryption algorithm chooses a random input $x$ and outputs the ciphertext $(F(ek, x), m \oplus \text{hc}(x))$, where $\text{hc}$ is a hardcore function. The decryption algorithm uses the shared secret key to compute a decryption share. The combining algorithm retrieves $x$ upon receiving at least $t$ decryption shares, and then extracts the message $m$. Moreover, threshold one-wayness prevents any PPT adversary who can adaptively obtain less than $t$ shared secret keys from decrypting ciphertext.

RPKE from TTDF. $(t, n)$-TTDF also gives rise to a simple construction of $(t - 1, n)$-RPKE. The public key consists of an injective trapdoor function index $ek$, and the master secret key consists of the master trapdoor $mtd$. The sharing algorithm splits the master secret key into $n$ shared secret keys. To encrypt a session key $s$, the encryption algorithm chooses a random input $x$ and computes $c_i = F(ek, x)$, $\delta_i = F^{-1}(sk_{i_1}, c_1), j = 1, \cdots, t - 1$, where $(sk_{i_1}, \cdots, sk_{i_n})$ are revoked secret keys, then outputs the ciphertext $c = (c_1, c_2 = \text{hc}(x) \oplus s, (\delta_i_1, \cdots, \delta_i_{t-1}))$. The decryption algorithm takes in any non-revoked secret key and computes $\delta_i = F^{-1}(sk_{i_1}, c_1), j \neq 1, \cdots, t - 1$ to retrieve $x$, and then extracts the session key $s$. Moreover, threshold one-wayness ensures that no PPT adversary can decrypt ciphertext without the non-revoked secret key.

Instantiation. Along the way to instantiate TTDF, we introduce the notion of TLTDF, which is a threshold version of the lossy trapdoor function (LTDF) [22]. Informally, the LTDF has two modes. In the injective mode, it is an injective trapdoor function. In the lossy mode, it statistically loses a significant amount of information about its input. The two modes are computationally indistinguishable. However, in both modes of TLTDF, the master trapdoor can be split into many shared trapdoors and every shared trapdoor can be used to compute an inversion share. Especially, in the injective mode any threshold inversion shares can be used to recover preimage. Moreover, any PPT adversary cannot distinguish both modes, even if the adversary can adaptively obtain less than threshold shared trapdoors.

The proof that TTDF is implied by TLTDF, while the latter can be instantiated under the DDH assumption and the LWE assumption respectively. DDH-based TLTDF is easy to design, while building LWE-based TLTDF is a non-trivial task. Intuitively, we transform the inversion algorithm of LTDF into threshold version by using $(t, n)$-threshold secret sharing scheme [24]. Every user gets a shared trapdoor $td_i, i \in [n]$, and computes the inversion share $(a, td_i) + e_i$. Then the combiner obtains $t$ inversion shares to compute the Lagrangian coefficients $L_i$ for any identity set of size $t$ and recombinates the $(a, td)$ by computing

$$L_1((a, td_1) + e_1) + \cdots + L_t((a, td_t) + e_t) = \langle a, \sum_{i=1}^t L_i \cdot td_i \rangle + \sum_{i=1}^t L_i \cdot e_i = \langle a, td \rangle + \sum_{i=1}^t L_i \cdot e_i$$

Unfortunately, choosing identities in a large identity space causes the norm of errors out of control and prevents correct inversion. To resolve this problem, we take advantage of the technique of "clearing out the denominator" [24,25,26]. Note that since the Lagrangian coefficients are rational numbers and the identity is chosen in $[n]$, we can scale them to be integers by computing $(n!)^2 L_i$. By instantiating appropriate parameters, we prove that the quantity of errors preserves bounded, which does not affect the correctness of inversion.

Optimization. We show a relaxation of TTDF called TTDR. Informally, TTDR replaces the evaluation algorithm of TTDF with a relation sampling algorithm which can generate a random input with its image.
of a function, while the function need not be efficiently computable. We also formalize security notion for TTDR, namely threshold one-wayness, which requires that the function is hard to invert, even if the adversary can adaptively obtain less than threshold shared trapdoors.

Similar to instantiating TTF from TLTDF, we instantiate TTDR by introducing the notion of threshold lossy trapdoor relation (TLTDR), which is a threshold version of lossy trapdoor relation (LTD)\(^{27}\). We prove TTDR is naturally implied by TLTDR. Moreover, we instantiate TLTDR based on the DDH assumption to obtain an instantiation of TTDR, which is more efficient than TTDF.

2 Preliminaries

2.1 Notations

We denote the natural numbers by \(\mathbb{N}\), the integers by \(\mathbb{Z}\), the real numbers by \(\mathbb{R}\). We use lower-case bold letters and upper-case bold letters to denote vectors and matrices (e.g. \(\mathbf{x}\) and \(\mathbf{X}\)). Let \(x^T\) and \(X^T\) denote transpositions of vector \(x\) and matrix \(X\). For \(n \in \mathbb{N}\), \(1^n\) denotes the string of \(n\) ones, and \([n]\) denotes the set \(\{1, \cdots, n\}\). We use standard asymptotic \((O, o, \Omega, \omega)\) notation to denote the growth of positive functions. We denote a negligible function by \(\text{negl}(\lambda)\), which is an \(f(\lambda)\) such that \(f(\lambda) = o(\lambda^{-c})\) for every fixed constant \(c\), and we let \(\text{poly}(\lambda)\) denote an unspecified function \(f(\lambda) = O(\lambda^c)\) for some constant \(c\). If \(S\) is a set then \(s \leftarrow S\) denotes the operation of sampling an element \(s\) of \(S\) uniformly at random.

Let \(X\) and \(Y\) be two random variables over some countable set \(S\). The statistical distance between \(X\) and \(Y\) is defined as
\[
\Delta(X, Y) = \frac{1}{2} \sum_{s \in S} |\Pr[X = s] - \Pr[Y = s]|.
\]

2.2 Assumptions

DDH Assumption. The generation algorithm \(\text{Gen}\) takes in a security parameter \(1^\lambda\) and outputs \((p, G, g)\), where \(p\) is a prime, \(G\) is a cyclic group of order \(p\) and \(g\) is a generator of \(G\).

The DDH assumption \(^{28}\) is that the ensemble \(\{(\mathbb{G}, g^a, g^b, g^{ab})\}_{\lambda \in \mathbb{N}}\) and \(\{(\mathbb{G}, g^a, g^b, g^c)\}_{\lambda \in \mathbb{N}}\) are computationally indistinguishable, where \((p, G, g) \leftarrow \text{Gen}(1^\lambda)\), and \(a, b, c \leftarrow \mathbb{Z}_p\).

LWE Assumption. Let \(d\) be the dimension of lattice, an integer \(q = \text{poly}(d)\) and all operations be performed in \(\mathbb{Z}_q\). For an error distribution \(\chi : \mathbb{Z}_q \rightarrow \mathbb{R}^+\), an integer dimension \(d \in \mathbb{Z}^+\) and a vector \(z \in \mathbb{Z}_q^d\), \(A_{z, \chi}\) is the distribution on \(\mathbb{Z}_q^d \times \mathbb{Z}_q\) of the variable \((a, (a, z) + c)\) where \(a \leftarrow \mathbb{Z}_q^d\) and \(c \leftarrow \chi\).

The LWE assumption is that independent samples from the LWE distribution \(A_{z, \chi}\) for some secret \(z \in \mathbb{Z}_q^d\), and independent samples from the uniform distribution on \(\mathbb{Z}_q^d \times \mathbb{Z}_q\) are computationally indistinguishable. For normal error distributions, the LWE problem is as hard as the worst-case lattice problem \(^{14}\).

2.3 Randomness Extraction

We use the notion of average min-entropy \(^{29}\), that captures the remaining unpredictability of \(X\) conditioned on the value of \(Y\):
\[
\tilde{H}_\infty(X|Y) = -\log(\mathbb{E}_{y \leftarrow Y} [2^{-\tilde{H}_\infty(X|Y=y)}]).
\]

We review the following useful lemmas from \(^{29}\).

Lemma 1. If \(Y\) takes at most \(2^r\) values and \(Z\) is any random variable, then \(\tilde{H}_\infty(X|(Y, Z)) \geq \tilde{H}_\infty(X|Z) - r\).

Lemma 2. Let \(X, Y\) be random variables such that \(X \in \{0, 1\}^l\) and \(\tilde{H}_\infty(X|Y) \geq k\). Let \(H\) be a family of pairwise independent hash functions from \(\{0, 1\}^l\) to \(\{0, 1\}^l\). Then for \(h \leftarrow H\), we have
\[
\Delta((Y, h(X)), (Y, h(U))) \leq \epsilon
\]
as long as \(l' \leq k - 2\lg(1/\epsilon)\).

\(^{2}\) We give a refined definition of LTDR in Section 8 which is more simple and intuitive than the one introduced in \(^{27}\).
2.4 Threshold Secret Sharing

We now recall the threshold secret sharing scheme [23]. Let $\mathbb{F}$ be a finite field, $|\mathbb{F}| > n$. Let $id_i \in \mathbb{F}, i = 1, \ldots, n$ be distinct, nonzero elements that are fixed and publicly known. The scheme works as follows:

- **Share($s, id_i$) → $s_i$:** On input a secret $s \in \mathbb{F}$, and any identity $id_i, i \in [n]$. It chooses $a_1, \ldots, a_t - 1 \in \mathbb{F}$, and defines the polynomial $p(x) = s + \sum_{i=1}^{t-1} a_i x^i$. This is a uniform degree-$(t-1)$ polynomial with constant term $s$. The share of user $id_i$ is $s_i = p(id_i) \in \mathbb{F}$.

- **Combine($\langle id_{i_1}, s_{i_1} \rangle, \ldots, \langle id_{i_t}, s_{i_t} \rangle$) → $s$:** On input any $t$ identities $id_{i_j}, j = 1, \ldots, t$, and associated shares $s_{i_j}, j = 1, \ldots, t$. Using polynomial interpolation, it computes the unique degree-$(t-1)$ polynomial $p'$ for which $p'(id_{i_j}) = s_{i_j}, j = 1, \ldots, t$. The combining algorithm outputs the secret $s = p'(0)$.

**Correctness.** It is clear that the combining algorithm works since the secret $p(0) = s$ can be constructed from any $t$ shares.

By the Lagrange interpolation formula, given any $t$ points $(id_{i_j}, p(id_{i_j}))$, $j = 1, \ldots, t$,

$$p(x) = \sum_{j=1}^{t} p(id_{i_j}) \prod_{l=1, l \neq j}^{t} \frac{x - id_{i_l}}{id_{i_j} - id_{i_l}},$$

we can compute any other points (threshold points mean all points) $(id_{i_v}, p(id_{i_v})), v \neq 1, \ldots, t, id_{i_v} \in \mathbb{F}$, where the secret is a special point $(0, s = p(0))$.

**Security.** The sharing algorithm Share has perfect privacy, that is, any $t-1$ users learn nothing of secret $s$ from their shares. For any $t-1$ users corresponding to identities $id_{i_j}, j = 1, \ldots, t-1$ and for any secret $s$ (namely, $p(0)$), the distributions of $t-1$ shares of $s$ are perfectly indistinguishable from $t-1$ independently uniform distributions.

In this paper, when building TLTDF from the LWE assumption, we take advantage of the technique of “clearing out the denominator” [24][25][26] and the fact that the term $(nl)^2 \cdot L_j$ is an integer, where $L_j, j = 1, \ldots, t$ are the Lagrangian coefficients.

**Lemma 3. ([27], Lemma 2.2).** For any $t$ identities $id_{i_j} = i, i_j \in [n], j = 1, \ldots, t$, the product $(nl)^2 \cdot L_j$ is an integer, and $|nl|^2 \cdot L_j | \leq (nl)^3$.

2.5 Threshold Encryption

We now recall the definition of TPKE from [9]. A $(t,n)$-TPKE consists of four algorithms as follows:

- **Gen$(1^\lambda)$ → $(pk, sk)$:** On input the security parameter $1^\lambda$, the key generation algorithm outputs a public key $pk$ and a secret key $sk = (sk_1, \ldots, sk_n)$.

- **Enc$(pk, m)$ → $c$:** On input the public key $pk$ and a message $m$, the encryption algorithm outputs a ciphertext $c$.

- **Dec$(sk_i, c)$ → $\delta_i$:** On input a shared secret key $sk_i, i \in [n]$ and the ciphertext $c$, the decryption algorithm outputs a decryption share $\delta_i$.

- **Combine$(\delta_{i_1}, \ldots, \delta_{i_t}, c) \rightarrow m$:** On input any $t$ decryption shares $\delta_{i_j}, j = 1, \ldots, t$ and the ciphertext $c$, the combining algorithm outputs the message $m$.

**Correctness.** For any message $m$, $c \leftarrow \text{Enc}(pk, m)$, and any $t$ decryption shares $\delta_{i_1}, \ldots, \delta_{i_n}$, we have $\text{Combine}(\delta_{i_1}, \ldots, \delta_{i_t}, c) = m$.

**Security.** Let $\mathcal{A}$ be a PPT adversary against IND-CPA security of TPKE scheme with adaptive corruption. Its advantage function is defined as

$$\text{Adv}^{\text{ind-cpa}}_{\text{TPKE}, \mathcal{A}}(\lambda) = \left| \Pr \left[ \begin{array}{l} \text{(pk, sk)} \leftarrow \text{Gen}(1^\lambda); \\
(id_{i_1}, \ldots, id_{i_{t-1}}) \leftarrow \mathcal{A}(1^\lambda); \\
b = b' : (m_1, m_1) \leftarrow \mathcal{A}(pk, sk_{i_1}, \ldots, sk_{i_{t-1}}); \\
b \leftarrow \{0, 1\}, c^* \leftarrow \text{Enc}(pk, m_b); \\
b' \leftarrow \mathcal{A}^{\text{Dec}}(\text{Enc}(pk))(pk, sk_{i_1}, \ldots, sk_{i_{t-1}}, c^*) \end{array} \right] - \frac{1}{2} \right|$$
Here, $\text{Dec}(\cdot, \text{Enc}(pk))$ denotes an oracle that given an input of any identity $id$, computes a fresh ciphertext $c$ using Enc$(pk)$ and returns a decryption share $\text{Dec}(sk_{id}, c)$. This captures that the adversary may obtain decryption shares of fresh encryptions of known messages. The $(t, n)$-TPKE scheme is IND-CPA secure, if for all PPT adversary the advantage function is negligible.

2.6 Revocation Encryption

We recall the definition of RPKE from [13]. A $(r, n)$-RPKE consists of four algorithms as follows:

- $\text{Gen}(1^\lambda, t) \rightarrow (pk, msk)$: On input the security parameter $1^\lambda$, and the revocation threshold $r$, the key generation algorithm outputs a public key $pk$ and a master secret key $msk$.
- $\text{Reg}(msk, id_i) \rightarrow sk_i$: On input the master secret key $msk$ and a new identity $id$ associated with the user, the registration algorithm outputs the shared secret key $sk_i$.
- $\text{Enc}(pk, S, s) \rightarrow c$: On input the public key $pk$, a set $S$ of revoked users (with $|S| \leq r$) and a session key $s$, the encryption algorithm outputs a ciphertext $c$.
- $\text{Dec}(sk_i, c) \rightarrow s$: On input a shared secret key $sk_i$ of user $id_i$ and the ciphertext $c$, the decryption algorithm outputs the session key $s$, if $id_i$ is a legitimate user when $c$ is constructed.

**Correctness.** For any $id_i, i \in [n]$, $(pk, msk) \leftarrow \text{Gen}(1^\lambda)$, any $s$, and any set $S, c \leftarrow \text{Enc}(pk, S, s)$, we require that for any non-revoked secret key $sk_i$, $s = \text{Dec}(sk_i, c)$.

**Security.** Let $\mathcal{A}$ be a PPT adversary against IND-CPA security of RPKE scheme with adaptive corruption. Its advantage function is defined as

$$
\text{Adv}^{\text{Ind-cpa}, \text{RPKE}, \mathcal{A}}_\lambda = \Pr \left[ b = b' \middle| \begin{array}{l}
(pk, msk) \leftarrow \text{Gen}(1^\lambda);
(id_{i_1}, \ldots, id_{i_r}) \leftarrow \mathcal{A}(1^\lambda);
(sk_{i_1}, \ldots, sk_{i_r}) \leftarrow \text{Reg}(msk, id_{i_1}, \ldots, id_{i_r});
b \leftarrow \{0, 1\}, c^* \leftarrow \text{Enc}(pk, S, s_0);
b' \leftarrow \mathcal{A}(pk, sk_{i_1}, \ldots, sk_{i_r}, c^*)
\end{array} \right] \frac{1}{2}
$$

If for all PPT adversary the advantage function is negligible, the $(r, n)$-RPKE scheme is IND-CPA secure.

3 Threshold Trapdoor Function

We give the definition and the security of TTDF as follows.

**Definition 1.** A collection of $(t, n)$-TTDFs is a tuple of polynomial-time algorithms defined as follows:

- $\text{Gen}(1^\lambda) \rightarrow (ek, mtd)$: On input the security parameter $1^\lambda$, the generation algorithm outputs a function index $ek$ and a master trapdoor $mtd$.
- $\text{Share}(mtd, id_i) \rightarrow t_{i_1}$: On input the master trapdoor $mtd$ and any identity $id_i, i \in [n]$, the sharing algorithm outputs the shared trapdoor $t_{i_1}, i \in [n]$.
- $F(ek, x) \rightarrow y$: On input the function index $ek$ and $x \in \mathbb{Z}_2^n$, the evaluation algorithm outputs $y$.
- $F^{-1}(t_{i_1}, y) \rightarrow \delta_i$: On input any shared trapdoor $t_{i_1}$, and the value $y$, the partial inversion algorithm outputs the inversion share $\delta_i$.
- $\text{Combine}^F(ek, x, \delta_{i_1}, \ldots, \delta_{i_{t-1}}, id_{i_t}) \rightarrow \delta_{i_t}$: On input $ek, x \in \mathbb{Z}_2^n$ and any $t - 1$ inversion shares $\delta_{i_1}, \ldots, \delta_{i_{t-1}}$ of the image of $x$, and identity $id_{i_t}$, the combining inversion algorithm outputs the inversion share $\delta_{i_t}$ of identity $id_{i_t}$.
- $\text{Combine}(\delta_{i_1}, \ldots, \delta_t, y) \rightarrow x$: On input any $t$ inversion shares $\delta_{i_j}, j = 1, \ldots, t$ and the value $y$, the combining algorithm outputs $x$.

Note that the generation algorithm is a probabilistic algorithm, while the rest five algorithms are deterministic algorithms, and we require that in the partial inversion algorithm and the combining algorithm, if a value $y$ is not in the image, the behavior of the algorithms are unspecified.

\footnote{The set $S$ contains the identities and shared secret keys of revoked users.}
Correctness. For any \( id_i, (ek, mtd) \leftarrow \text{Gen}(1^\lambda), \, td_i \leftarrow \text{Share}(mtd, id_i), \, i \in [n], \, x \leftarrow \{0, 1\}^t, \, y = F(ek, x) \), we require that for any \( t \) shared trapdoors \( td_{i_1}, \ldots, td_{i_t} \), we have
\[
x = \text{Combine}(F^{-1}(td_{i_1}), \ldots, F^{-1}(td_{i_t}), y).
\]

Security. Let \( \mathcal{A} \) be a PPT adversary against \((t, n)-\text{TTDF}\) and define its advantage function as
\[
\text{Adv}_{\text{TTDF}, \mathcal{A}}^\text{low}(\lambda) = \Pr \left[ x = x' : (ek, mtd) \leftarrow \text{Gen}(1^\lambda); \right.
\]
\[
\left. (id_{i_1}, \ldots, id_{i_{t-1}}) \leftarrow \mathcal{A}(1^\lambda); \quad t, n \right. \n\]
\[
\left. \left. x \leftarrow \{0, 1\}^t, \, y = F(ek, x); \quad x' \leftarrow \mathcal{A}(ek, td_{i_1}, \ldots, td_{i_{t-1}}, y) \right) \right] \right]
\]

If for any PPT adversary the advantage function is negligible, \((t, n)-\text{TTDF}\) is threshold one-way.

3.1 Connection to Function Sharing

De Santis et al. [2] introduced the notion of function sharing (FS) parameterized by the threshold value \( t \) and the number of identities \( n \). \((t, n)\)-FS can split the master trapdoor into \( n \) shared trapdoors, where \( n \) is a fixed polynomial of the security parameter. The function is easy to invert when given threshold (at least \( t \) out of \( n \)) shared trapdoors, while any PPT adversary cannot invert the function even if it obtains any \( t-1 \) shared trapdoors and a history tape \( H \) with partial inversion shares of polynomial many random images. Then they constructed threshold cryptosystems based on FS and instantiated it under the RSA assumption. However, the number of identities of their FS and TPKE is limited in a fixed polynomial of security parameter.

In this paper, we propose the notion of TTDF that differs from FS of the number of identities. In TTDF, the generation algorithm and the sharing algorithm are independent of the number of identities, and the total number of identities could be an exponential number. Therefore, \((t, n)-\text{TTDF}\) implies \((t, n)\)-FS. \((t, n)-\text{TTDF}\) has an additional combining inversion algorithm that given the function index \( ek \), any preimage \( x \) and any \( t-1 \) inversion shares of the image of \( x \), can compute the inversion share of any other identity. Therefore, \((t, n)-\text{TTDF}\) can be used to construct the TPKE scheme [9] which supports ad-hoc groups (i.e., exponential number of identities and the generation algorithm is independent of the total number of identities), the reason is that the reduction algorithm who holds any \( t-1 \) shared trapdoors can answer the oracle \( \text{Dec} (\cdot, \text{Enc} (pk)) \) of all identities.

4 Threshold Encryption from TTDF

Let \((\text{Gen}, \text{Share}, F, F^{-1}, \text{Combine}F^{-1}, \text{Combine})\) be a \((t, n)\)-TTDF and \( hc(\cdot) \) be a hardcore function. We construct a TPKE as follows:

- \( \text{Gen}(1^\lambda) \rightarrow (pk, msk) \): On input the security parameter \( 1^\lambda \), the generation algorithm runs \((ek, mtd) \leftarrow \text{TTDF}.\text{Gen}(1^\lambda)\) and outputs a public key \( pk = ek \) and a master secret key \( msk = mtd \).
- \( \text{Share}(msk, id_i) \rightarrow sk_i \): On input the master secret key \( msk \) and any identity \( id_i, i \in [n] \), the sharing algorithm runs \( td_i \leftarrow \text{TTDF}.\text{Share}(msk, id_i) \) and outputs the shared secret key \( sk_i = td_i, i \in [n] \).
- \( \text{Enc}(pk, m) \rightarrow c \): On input the public key \( pk \) and a message \( m \), the encryption algorithm chooses \( x \leftarrow \{0, 1\}^t \), computes \( c_1 = \text{TTDF}.F(pk, x), \, c_2 = hc(x) \oplus m \), and outputs the ciphertext \( c = (c_1, c_2) \).
- \( \text{Dec}(sk_i, c) \rightarrow \delta_i \): On input a secret key \( sk_i \) and a ciphertext \( c \), the decryption algorithm computes \( \delta_i = \text{TTDF}.F^{-1}(sk_i, c_1) \), and outputs a decryption share \( \delta_i \).
- \( \text{Combine}(\delta_{i_1}, \cdots, \delta_{i_t}, c) \rightarrow m \): On input any \( t \) decryption shares \( \delta_{i_j}, j = 1, \cdots, t \) and the ciphertext \( c = (c_1, c_2) \), the combining algorithm computes \( x = \text{TTDF}.\text{Combine}(\delta_{i_1}, \cdots, \delta_{i_t}, c_1), m = hc(x) \oplus c_2 \). It outputs the message \( m \).

**Theorem 1.** If the TTDF is threshold one-way, then the TPKE is IND-CPA secure.

**Proof.** We define two hybrid experiments \( \text{Game}_1, \text{Game}_2 \).
Game 1: The game is identical to the IND-CPA experiment. At the beginning, the challenger runs Gen to obtain pk and msk. The challenger sends pk to A. A chooses any t − 1 identities id₁, j = 1, · · · , t − 1 to corrupt. Then the challenger runs the sharing algorithm Share to obtain sk₁, j = 1, · · · , t − 1 and sends them to A. A can choose any message m′ and any identity id′ to query the oracle Dec(·, Enc(pk)) many times, and obtain the decryption share δ′ ← Dec(sk′, Enc(pk, m′)), where sk′ is shared secret key of identity id′. Upon receiving the messages m₀, m₁ from A, the challenger chooses b ∈ {0, 1} at random and returns c∗ = Enc(pk, m₀) to A. A is still able to have access to the oracle Dec(·, Enc(pk)). At the end of the game, A outputs b′ ∈ {0, 1} as the guess of b. If b′ = b, A wins this game, otherwise fails.

Game 2: The game is identical to Game 1, except that when the challenger generates the challenge ciphertext c∗ = (c₁, c₂), it replaces c₂ = m₀ ⊕ hc(x) with c₂ = m₀ ⊕ r.

For i ∈ {1, 2}, let Pr[A^{Game i} = b] be the probability that A outputs the bit b when executed in Game i. We claim that if there is an adversary A against the TPKE such that Pr[A^{Game 1} = b] − Pr[A^{Game 2} = b] is non-negligible, we can construct a distinguisher D against the hardcore function. On input (ek, y, r), where ek is a function index, y = F(ek, x) with x ← {0, 1}t and r is either hc(x) or a random string, D works as follows:

1. D runs A on input pk = ek and gets identities id₁, · · · , idt−1 output by A.
2. D chooses these identities to corrupt, and obtains associated shared trapdoors td₁, · · · , tdt−1, then returns these shared trapdoors to A. A can choose any message m′ and any identity id′ to query the oracle Dec(·, Enc(pk)). D chooses x′ in domain at random and computes c₁ = TTDF.F(pk, x'), c₂ = hc(x') ⊕ m', δ₁ = TTDF.F⁻¹(td₁, c₁), · · · , δt−1 = TTDF.F⁻¹(tdₜ₋₁, c₁), δ = TTDF.CombineF⁻¹(x', c₁, δ₁, · · · , δt−1, id'). Then D returns δ' to A. Upon receiving two messages m₀, m₁ from A, D chooses b ∈ {0, 1} at random, let c₁ = y, c₂ = m₀ ⊕ r, and returns c = (c₁, c₂) to A. A is still able to have access to the oracle Dec(·, Enc(pk)). D can also simulate the oracle. At last D outputs what A outputs.
3. if b = b', D returns “1” to denote r is the output of the hardcore function, otherwise returns “0” to denote r is a random string.

The distinguisher D can give a perfect simulation of either Game 1 or Game 2. The advantage of D is non-negligible, which is a contradiction of the threshold one-wayness. Therefore, | Pr[A^{Game 1} = b] − Pr[A^{Game 2} = b]| ≤ negl(λ).

Finally, in Game 2 the output of hardcore function has been replaced with a random string, so Pr[A^{Game 2} = b] = 1/2. We have:

Pr[A^{Game 1} = b] ≤ | Pr[A^{Game 1} = b] − Pr[A^{Game 2} = b]| + Pr[A^{Game 2} = b] ≤ 1/2 + negl(λ)

Therefore, the TPKE is IND-CPA secure.

5 Revocation Encryption from TTDF

Let (Gen, Share, F, F⁻¹, CombineF⁻¹, Combine) be a (t, n)-TTDF and hc(·) be a hardcore function. We construct a (t − 1, n)-RPKE as follows:

- Gen(1^λ) → (pk, msk): On input the security parameter 1^λ, the generation algorithm runs (ek, mtd) ← TTDF.Gen(1^λ) and outputs a public key pk = ek and a master secret key msk = mtd.
- Reg(msk, id₁) → sk₁: On input the master secret key msk and any identity id₁, i ∈ [n], the registration algorithm runs td₁ ← TTDF.Share(mtd, id₁) and outputs the shared secret key sk₁ = td₁, i ∈ [n].
- Enc(pk, sk₁, · · · , skt−₁, s) → c: On inputs the public key pk, a set of t − 1 revoked secret keys sk₁, j = 1, · · · , t − 1 and a session key s, the encryption algorithm chooses x ← {0, 1}t, computes c₁ = TTDF.F(pk, x), c₂ = hc(x) ⊕ s and δ₁ = TTDF.F⁻¹(sk₁, c₁), j = 1, · · · , t − 1. It outputs the ciphertext c = (c₁, c₂, δ₁, · · · , δt−₁).
• \( \text{Dec}(sk_{i_j}, c) \rightarrow s \): On inputs a secret key \( sk_{i_j}, j \neq 1, \ldots, t - 1 \) and a ciphertext \( c \), the decryption algorithm computes \( \delta_{i_j} = \text{TTDF}^{-1}(sk_{i_j}, c_1), j \neq 1, \ldots, t - 1, x = \text{TTDF} \). Combine \((\delta_{i_1}, \ldots, \delta_{i_{t-1}}, \delta_{i_j}, c_1) \) and \( s = \text{hc}(x) \oplus c_2 \). It outputs the session key \( s \).

**Theorem 2.** If the TTDF is threshold one-way, then the RPKE is IND-CPA secure.

**Proof.** We define two hybrid experiments \( \text{Game}_1, \text{Game}_2 \).

- **Game1:** The game is identical to the IND-CPA experiment. At the beginning, the challenger runs \((pk, msk) \leftarrow \text{Gen}(1^\lambda) \) and gives the \( pk \) to the adversary \( A \). \( A \) can choose any \( t - 1 \) identities \( id_{i_j}, j = 1, \ldots, t - 1 \) to corrupt. The challenger runs the registration algorithm to generate \( sk_{i_j}, j = 1, \ldots, t - 1 \) and gives them to \( A \). Upon receiving two session keys \( s_0, s_1 \) from \( A \), the challenger chooses \( b \in \{0, 1\} \) at random and returns \( c^* = \text{Enc}(pk, sk_{i_1}, \ldots, sk_{i_{t-1}}, s_b) \) to \( A \). At the end of the game, \( A \) outputs \( b' \in \{0, 1\} \) as the guess of \( b \). If \( b' = b \), \( A \) wins this game, otherwise fails.

- **Game2:** The game is identical to \( \text{Game}_1 \), except when the challenger generates the challenge ciphertext \( c^* = (c_1, c_2, \delta_{i_1}, \ldots, \delta_{i_{t-1}}) \), it replaces \( c_2 = s_0 \oplus \text{hc}(x) \) with \( c_2 = s_0 \oplus r \), where \( r \) is a random string.

For \( i \in \{1, 2\} \), let \( \Pr[A|\text{Game}_i = b] \) be the probability that \( A \) outputs the bit \( b \) when executed in \( \text{Game}_i \). We claim that if there is an adversary \( A \) such that \( \Pr[A|\text{Game}_1 = b] - \Pr[A|\text{Game}_2 = b] \) is non-negligible, we can construct a distinguisher \( D \) against the hardcore function. On input \((ek, y, r)\), where \( ek \) is a function index, \( y = F(ek, x) \) with \( x \leftarrow \{0, 1\}^t \) and \( r \) is either \( \text{hc}(x) \) or a random string, \( D \) works as follows:

1. \( D \) runs \( A \) on input \( pk = ek \) and gets identities output by \( A \).
2. \( D \) chooses these identities to corrupt and obtains associated trapdoors, then runs \( A \) on input these associated trapdoors and obtains two session keys \( s_0, s_1 \). \( D \) chooses \( b \in \{0, 1\} \) at random, let \( c_1 = y, c_2 = s_0 \oplus r \), computes \( \delta_{i_j} = F^{-1}(sk_{i_j}, c_1), j = 1, \ldots, t - 1 \), returns \( c = (c_1, c_2, \delta_{i_1}, \ldots, \delta_{i_{t-1}}) \) to \( A \) and gets a bit \( b' \) output by \( A \).
3. if \( b = b' \), \( D \) returns “1” to denote \( r \) is the output of the hardcore function, otherwise returns “0” to denote \( r \) is a random string.

The distinguisher \( D \) can give a perfect simulation of either \( \text{Game}_1 \) or \( \text{Game}_2 \). The advantage of \( D \) is non-negligible, which is a contradiction of the threshold one-wayness. Therefore, \( |\Pr[A|\text{Game}_1 = b] - \Pr[A|\text{Game}_2 = b]| \leq \text{negl}(\lambda) \).

Finally, in \( \text{Game}_2 \) the output of hardcore function has been replaced with a random string, so \( \Pr[A|\text{Game}_2 = b] = 1/2 \). We have:

\[
\Pr[A|\text{Game}_1 = b] \leq |\Pr[A|\text{Game}_1 = b] - \Pr[A|\text{Game}_2 = b]| + \Pr[A|\text{Game}_2 = b] \leq \frac{1}{2} + \text{negl}(\lambda)
\]

Therefore, the RPKE is IND-CPA secure.

## 6 Threshold Lossy Trapdoor Function

Let \( l(\lambda) = \text{poly}(\lambda) \) denote the input length of the function, \( k(\lambda) \leq l(\lambda) \) and \( r(\lambda) = l(\lambda) - k(\lambda) \) denote the lossiness and the residual leakage. We often omit the dependence on \( \lambda \).

**Definition 2.** A collection of \((t, n, l, k)\)-TLTDFs is a tuple of polynomial-time algorithms defined as follows. For notational convenience, define the sampling algorithm \( \text{Samp}_{\text{inj}}^*() := \text{Samp}(\cdot, 1) \) samples injective mode and \( \text{Samp}_{\text{loss}}^*() := \text{Samp}(\cdot, 0) \) samples lossy mode.

- \( \text{Samp}_{\text{inj}}^*(1^\lambda) \rightarrow (ek, mtd) \): On input the security parameter \( 1^\lambda \), the sampling algorithm outputs a function index \( ek \) and a master trapdoor \( mtd \).
- \( \text{Samp}_{\text{loss}}^*(1^\lambda) \rightarrow (ek, mtd) \): On input the security parameter \( 1^\lambda \), the sampling algorithm outputs a function index \( ek \) and a master trapdoor \( mtd \).
- \( \text{Share}(mtd, id_i) \rightarrow td_i \): On input the master trapdoor \( mtd \) and any identity \( id_i, i \in [n] \), in both modes the sharing algorithm outputs the shared trapdoor \( td_i, i \in [n] \).
\[ F(ek, x) \rightarrow y: \text{On input the function index } ek \text{ and } x \in \{0, 1\}^t, \text{ in both modes the evaluation algorithm output } y, \text{ but in the lossy mode the image has size at most } 2^r = 2^{t-k}. \]

\[ F^{-1}(td_i, y) \rightarrow \delta_i: \text{On input any shared trapdoor } td_i, \ i \in [n] \text{ and the value } y, \text{ the partial inversion algorithm outputs an inversion share } \delta_i. \]

\[ \text{Combine}^{-1}(ek, x, \delta_{i_1}, \cdots, \delta_{i_{t-1}}, id_{i_t}) \rightarrow \delta_{i_t}: \text{On input } ek, x \in \{0, 1\}^t, \text{ any } t-1 \text{ inversion shares } \delta_{i_1}, \cdots, \delta_{i_{t-1}} \text{ of the image of } x, \text{ and identity } id_{i_t}, \text{ the combining inversion algorithm outputs the inversion share } \delta_{i_t} \text{ of identity } id_{i_t}. \]

\[ \text{Combine}(\delta_{i_1}, \cdots, \delta_{i_t}, y) \rightarrow x: \text{On input any } t \text{ inversion shares } \delta_{i_j}, j = 1, \cdots, t \text{ and the value } y, \text{ in injective mode the combining algorithm outputs } x. \]

Note that the sampling algorithms in both modes are probabilistic algorithms, while the rest five algorithms are deterministic algorithms, and we require that the shared trapdoors in both modes have the same space, and the behavior of the partial inversion algorithm and the combining algorithm is unspecified, if a value y is not in the image.

**Security.** Let \( \mathcal{A} \) be a PPT adversary against TLTDF and define its advantage function as

\[ \text{Adv}^{\text{ind}}_{\text{TLTDF}, \mathcal{A}}(\lambda) = \Pr \left[ \begin{array}{c}
    b \leftarrow \{0, 1\}; \\
    (ek, mtd) \leftarrow \text{Samp}(1^\lambda, b); \\
    (id_{i_1}, \cdots, id_{i_{t-1}}) \leftarrow \mathcal{A}(1^\lambda); \\
    td_i \leftarrow \text{Share}(mtd, id_i), i \in [n]; \\
    b' \leftarrow \mathcal{A}(ek, td_{i_1}, \cdots, td_{i_{t-1}})
\end{array} \right] - \frac{1}{2} \]

A TLTDF is said to produce indistinguishable function indexes if \( \text{Adv}^{\text{ind}}_{\text{TLTDF}, \mathcal{A}}(\lambda) \) is negligible for all adversary.

**Theorem 3.** If the sharing algorithm holds perfect privacy and the injective and lossy modes of LTDF are indistinguishable, then the TLTDF described above is also hard to distinguish injective from lossy.

**Proof.** We define four hybrid experiments Game1, Game2, Game3, Game4.

- **Game1:** The challenger runs \((ek, mtd) \leftarrow \text{Samp}_{\text{inj}}(1^\lambda)\), and gives \(ek\) to \(\mathcal{A}\). \(\mathcal{A}\) chooses any \(t-1\) identities \(id_{i_j}\) to corrupt. Then the challenger runs the sharing algorithm to generate \(t-1\) associated trapdoor \(td_{i_j}, j = 1, \cdots, t-1\). The challenger gives the \(t-1\) shared trapdoors to \(\mathcal{A}\).

- **Game2:** The game is identical to Game1, except that the challenger generates the corrupted trapdoors by choosing \(r_i, i = 1, \cdots, t-1\) at random in the shared trapdoor space and then gives them to \(\mathcal{A}\).

- **Game3:** The game is identical to Game2, except that the challenger runs \((ek, mtd) \leftarrow \text{Samp}_{\text{loss}}(1^\lambda)\) instead of running \((ek, mtd) \leftarrow \text{Samp}_{\text{inj}}(1^\lambda)\).

- **Game4:** The game is identical to Game3, except that the challenger runs the sharing algorithm to generate \(t-1\) shared trapdoor \(td_{i_j}, j = 1, \cdots, t-1\) and gives the \(t-1\) shared trapdoors to \(\mathcal{A}\).

The adversary’s view is perfectly indistinguishable in Game1 and Game2 with the replacement of the shared trapdoors, since the sharing algorithm has the perfect privacy. Similarly, the adversary’s view is perfectly indistinguishable in Game3 and Game4. The only difference between Game2 and Game3 is the sampling algorithm. So the adversary’s view is computationally indistinguishable in Game2 and Game3, the fact follows that the injective and lossy modes of LTDF are indistinguishable [22]. Therefore, the adversary’s view is computationally indistinguishable in Game1 and Game4 and the TLTDF described above is hard to distinguish injective from lossy.

**Theorem 4.** Let \(\text{TLTDF} = (\text{Samp}_{\text{inj}}, \text{Samp}_{\text{loss}} \text{ Share}, F, F^{-1}, \text{Combine}^{-1}, \text{Combine})\) give a collection of \((t, n, l, k)\)-TLTDFs with \(k = \omega(\log \lambda)\). Then \(\text{TTDF} = (\text{Samp}_{\text{inj}}, \text{Share}, F, F^{-1}, \text{Combine}^{-1}, \text{Combine})\) give a collection of \((t, n)\)-TTDFs.

**Proof.** By definition, for any \(id_i, i \in [n], (ek, mtd) \leftarrow \text{Samp}_{\text{inj}}(1^\lambda), td_i \leftarrow \text{Share}(mtd, id_i), x \leftarrow \{0, 1\}^t, y = F(ek, x), \) for any \(t-1\) inversion shares \(\delta_{i_1} = F^{-1}(td_{i_1}, y), \cdots, \delta_{i_{t-1}} = F^{-1}(td_{i_{t-1}}, y)\) and identity \(id_{i_t}\), we have \(\delta_{i_t} = F^{-1}(td_{i_t}, y) = \text{Combine}^{-1}(x, y, \delta_{i_1}, \cdots, \delta_{i_{t-1}}, id_{i_t})\), and for any \(t\) shared trapdoors...
Since $k$ injective mode from lossy mode, a contradiction of the hypothesis. First, by the assumption on bounded algorithm index $td_i$, we have that $D$ outputs “1” if $x' = x$ with non-negligible probability and $D$ outputs “0” if $x'$ is generated in the injective mode and generate a ciphertext $ek, td_i, \cdots, td_{i-1}, F(ek, x)$, Because $F(ek, x)$ takes at most $2^{l-k}$ values, $ek$ and $x$ are independent. By (22, Lemma 2.1)

$$\tilde{H}_\infty(x|ek, td_i, \cdots, td_{i-1}, F(ek, x)) \geq \tilde{H}_\infty(x|ek, td_i, \cdots, td_{i-1}) - (l - k) = l - (l - k) = k$$

where by the perfect privacy of the sharing algorithm, $td_i, j = 1, \cdots, t - 1$ look like random numbers. Since $k = \omega(\log \lambda)$, the probability that $A$ outputs $x'$ and $D$ outputs “0” is negl$(\lambda)$. $D$ distinguishes injective mode from lossy mode, a contradiction of the hypothesis.

Remarks. In our applications of TPKE and RPKE, we use the pairwise independent hash function $[30]$ as a hardcore function. Let $H : \{0, 1\}^l \rightarrow \{0, 1\}^{l'}$ be a family of pairwise independent hash functions, where $l' \leq k - 2 \log(1/\epsilon)$ for some negligible $\epsilon = \text{negl}(\lambda)$, and we choose $hc \leftarrow H$. Following the Theorem 4, $\tilde{H}_\infty(x|ek, td_i, \cdots, td_{i-1}, F(ek, x)) \geq k$. By the hypothesis that $l' \leq k - 2 \log(1/\epsilon)$ and Lemma 2, we have that $hc(x)$ is $\epsilon$-close to uniform.

7 Instantiations of TLTDF

In this section, we give instantiations of TLTDF based on the DDH assumption and the LWE assumption.

7.1 Instantiation of TLTDF Based on the DDH Assumption

By using the ElGamal-like encryption primitive in [22], we generate a ciphertext $C_1$ by encrypting the identity matrix $I$ in the injective mode and generate a ciphertext $C_0$ by encrypting the all-zeros matrix $0$ in the lossy mode.

Lemma 4. (22, Lemma 5.1). The matrix encryption scheme produces indistinguishable ciphertexts under the DDH assumption.

Construction. We now describe a DDH-based TLTDF as follows. The identity space is given by $\mathbb{Z}_p \setminus \{0\}$.

• $Samp_{Inj}$: On input $1^l$, it chooses $(p, G, g) \leftarrow \text{Gen}(1^\lambda)$, samples $r_1, s_i, b_{ij} \leftarrow \mathbb{Z}_p, i = 1, \cdots, l, j = 1, \cdots, t - 1$ and computes

$$C_1 = \begin{pmatrix} g^{r_1} & g^{r_1 s_1} & \cdots & g^{r_1 s_l} \\ g^{r_2} & g^{r_2 s_1} & \cdots & g^{r_2 s_l} \\ \vdots & \vdots & \ddots & \vdots \\ g^{r_l} & g^{r_l s_1} & \cdots & g^{r_l s_l} \end{pmatrix}$$

The function index is $ek = C_1$ and the master trapdoor is $mtd = ((s_i), D = (b_{ij}))$. 

$$td_i, \cdots, td_{i-1}, \text{we have } x = \text{Combine}(F^{-1}(td_i, y), \cdots, F^{-1}(td_{i-1}, y), y)$$

Therefore, the correctness condition holds. We prove that the function also holds the threshold one-wayness:

Suppose $A$ is a PPT inverter, if $A$ can break the threshold one-wayness with non-negligible probability, we can build an adaptive distinguisher $D$ between injective modes and lossy ones. $D$ is given a function index $ek$ as input. Its goal is to distinguish $ek$ is generated in the injective or lossy mode. $D$ works as follows:

1. $D$ runs inverter $A$ on input the function index $ek$ and gets identities output by $A$.
2. $D$ chooses these identities to corrupt and obtains associated trapdoors, then $D$ chooses $x \leftarrow \{0, 1\}^l$, computes $y = F(ek, x)$, gives the value $y$ and the associated trapdoors to $A$, and then obtains the value $x'$ output by $A$.
3. if $x' = x$, $D$ returns “1” to denote $ek$ is generated in the injective mode, otherwise returns “0” to denote $ek$ is generated in the lossy mode.

First, by the assumption on $A$, if $ek$ is generated by $\text{Samp}_{Inj}(1^\lambda)$, we have $x' = x$ with non-negligible probability and $D$ outputs “1”. Suppose $ek$ is generated by $\text{Samp}_{Loss}(1^\lambda)$. The probability that even an unbounded algorithm $A$ predicts $x$ is given by the average min-entropy of $x$ conditioned on $(ek, td_i, \cdots, td_{i-1}, F(ek, \cdot))$, Because $F(ek, \cdot)$ takes at most $2^{l-k}$ values, $ek$ and $x$ are independent. By (22, Lemma 2.1)
The function index is $ek = C_0$ and the master trapdoor is $mtd = ((s_i), D = (b_{ij}))$.
• Share: On input the master trapdoor $mtd$ and any identity $id_i, i = 1, \cdots, n$, it sets
  \[
  \frac{f_1(x)}{f_2(x)} = \begin{pmatrix}
  f_1(x) & f_2(x) & \cdots & f_{t-1}(x)
  \end{pmatrix}
  \begin{pmatrix}
  f_1(x) \ b_1 \ b_2 \ \cdots \ b_{t-1} \\
  f_2(x) \ b_2 \ b_3 \ \cdots \ b_{t-1} \\
  \vdots \ \vdots \ \cdots \ \vdots \\
  f_{t-1}(x) \ b_1 \ b_2 \ \cdots \ b_{t-2}
  \end{pmatrix}
  \begin{pmatrix}
  1 \\
  x \\
  \vdots \\
  x^{t-2}
  \end{pmatrix}
  \]
  and outputs $td_i^T = (f_1(id_i), f_2(id_i), \cdots, f_{t-1}(id_i))$.

• $F$: On input a function index $ek = (c_{ij})_{i \times (t+1)}$ and $x \in \{0,1\}^t$, $x = (x_1, \cdots, x_t)$, it outputs $y = (y_1, \cdots, y_{t+1})$.

• $F^{-1}$: On input any shared trapdoor $td_i$ and the value $y_i$. It outputs $\delta_T = (\sum_{r=0}^{t-1} L_r f_1(id_i), \sum_{r=0}^{t-1} L_r f_2(id_i), \cdots, \sum_{r=0}^{t-1} L_r f_{t-1}(id_i))$.

Because of $f_j(id_i) = \sum_{r=0}^{t-1} L_r f_j(id_i), j = 1, \cdots, t$, where $L_v, v = 0, 1, \cdots, t-1$ are the Lagrangian coefficients which may be efficiently computed given $(id_i = 0, id_i, \cdots, id_{i-1})$, it computes $y_i = F(ek, x)$, and $y_i^{(id_i)} = y_i \in \{0,1\}$, $i = 1, \cdots, l$.

• Combine: On input any $t$ inversion shares $\delta_i, j = 1, \cdots, t$ and the value $y$. Because of $f_j(0) = \sum_{v=0}^{t-1} L_v \cdot f_j(id_i), j = 1, \cdots, t$, where $L_v, v = 1, \cdots, t$ are the Lagrangian coefficients which may be efficiently computed given $(id_i, \cdots, id_{i-t})$, it computes

\[
\begin{pmatrix}
  y_1^* \\
  y_2^* \\
  \vdots \\
  y_t^*
  \end{pmatrix} = \begin{pmatrix}
  \prod_{v=0}^{t-1} y_1^{f_1(id_i)} \\
  \prod_{v=0}^{t-1} y_2^{f_2(id_i)} \\
  \vdots \\
  \prod_{v=0}^{t-1} y_t^{f_t(id_i)}
  \end{pmatrix}
  \begin{pmatrix}
  \sum_{v=0}^{t-1} L_v f_1(id_i) \\
  \sum_{v=0}^{t-1} L_v f_2(id_i) \\
  \vdots \\
  \sum_{v=0}^{t-1} L_v f_{t-1}(id_i)
  \end{pmatrix}
  \begin{pmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_t
  \end{pmatrix}
\]

and outputs $x = (x_1, \cdots, x_t)$, where $x_i = y_{i+1}/y_i^*, i = 1, \cdots, l$.

**Lemma 5.** The algorithms give a collection of $(t, n, l, l - \log p)$-TLTDFs under the DDH assumption.

**Proof.** The $(t, n)$-threshold secret sharing scheme holds the perfect privacy. Both modes of LTDF are computationally indistinguishable. Therefore, we can show the indistinguishability between injective and lossy mode of TLTDF.

We transform the inversion algorithm into threshold version which does not change the lossy mode. In the lossy mode, the number of possible function outputs is at most $p$, the residual leakage $r \leq \log p$, and the lossiness is $k = n - r \geq l - \log p$.
7.2 Instantiation of TLTDF Based on the LWE Assumption

We recall a variant of LWE-based symmetric key cryptosystem [22] which has a small message space. Let \( T = \mathbb{R}/\mathbb{Z} \), \( \eta \in \mathbb{N} \). For every message \( m \in \mathbb{Z}_p \), we define the “offset” \( c_m = m/p \in T \). The secret key is \( z \leftarrow \mathbb{Z}_q^d \). To encrypt \( m \in \mathbb{Z}_p \), we choose \( a \leftarrow \mathbb{Z}_q^d \) and an error term \( e \leftarrow \chi \). The ciphertext is

\[
E_z(m; u; a, e) = (a, (a, z) + qc_m + u + e) \in \mathbb{Z}_q^d \times \mathbb{Z}_q
\]

where the rounding error \( u = [qc_m] - qc_m \in \{-1/2, 1/2\} \). For a ciphertext \( c = (a, c') \), the decryption algorithm computes \( t = \eta(c' - \langle a, z \rangle)/q \) and outputs \( m \in \mathbb{Z}_p \), such that \( t - \eta c_m \) is closest to 0. Note that for any ciphertext, as long as the absolute total error \(|pe + \eta u| \leq \eta q/2p\), the decryption is correct.

We use “matrix encryption” mechanism in [22] to generate the ciphertext

\[
C = E_Z(M, U; A, E)
\]

where \( M = (m_{i,j}) \in \mathbb{Z}_p^{h \times w} \) is a message matrix, \( U = (u_{i,j}) \) is a matrix of rounding errors, \( E = (e_{i,j}) \in \mathbb{Z}_q^{h \times w} \) is error matrix, \( e_{i,j} \leftarrow \chi \), choose independent \( z_i \leftarrow \mathbb{Z}_q \), \( Z = (z_1, \cdots, z_w) \), for each row \( i \in [h] \) of the random matrix \( A \in \mathbb{Z}_q^{h \times d} \), choose independent \( a_i \leftarrow \mathbb{Z}_q^d \).

In the injective mode, the message matrix \( M \) is a matrix \( B \), which is the tensor product \( I \otimes b \), where \( I \in \mathbb{Z}_p^{w \times w} \) is the identity and \( b = (1, 2, \cdots, 2^{l-1})^T \in \mathbb{Z}_p^l \), \( l = \lceil \log p \rceil \), \( w = h/l \). In the lossy mode, the message matrix \( M \) is all-zeros matrix 0.

Lemma 6. ([22], Lemma 6.2). For \( h, w = \text{poly}(d) \), the matrix encryption scheme produces indistinguishable ciphertexts under the assumption that LWE_{q, \chi} is hard.

Construction. We describe a LWE-based TLTDF as follows. By using the technique of clearing out the denominator to bound the quantity of errors, we require that the identity space \( ID = [n] \), \( n \in \mathbb{N} \) and set \( \eta = (n!)^3 \).

- \textbf{Samp}_{\text{ inj}}: On input 1\(^d\), it generates

\[
C = E_Z(B, U; A, E)
\]

and outputs the function index \( C \) and the master trapdoor \( mtd = (z_i, D_i) \), where \( z_i = (z_i^{(1)}, D_i = (b_{ij}^{(1)})), z_j^{(1)}, b_{jk}^{(1)} \leftarrow \mathbb{Z}_q, i = 1, \cdots, w, j = 1, \cdots, d, k = 1, \cdots, t - 1. \)

- \textbf{Samp}_{\text{ loss}}: On input 1\(^d\), it generates

\[
C = E_Z(0, U; A, E)
\]

and outputs the function index \( C \) and the master trapdoor \( mtd = (z_i, D_i) \), where \( z_i = (z_j^{(1)}), D_i = (b_{ij}^{(1)}), z_j^{(1)}, b_{jk}^{(1)} \leftarrow \mathbb{Z}_q, i = 1, \cdots, w, j = 1, \cdots, d, k = 1, \cdots, t - 1. \)

- \textbf{Share}: On input the master trapdoor \( mtd \) and any identity \( id_{i_v} = i_v \in [n] \), it sets

\[
\begin{pmatrix}
 f_1^{(1)}(x) \\
 \vdots \\
 f_d^{(1)}(x)
\end{pmatrix} =
\begin{pmatrix}
 z_1^{(1)} b_1^{(1)} \cdots b_1^{(t-1)} \\
 \vdots \\
 z_d^{(1)} b_d^{(1)} \cdots b_d^{(t-1)}
\end{pmatrix}
\begin{pmatrix}
 1 \\
 \vdots \\
 x^{t-1}
\end{pmatrix}
\]

and outputs

\[
td_{i_v} =
\begin{pmatrix}
 f_1^{(1)}(i_v) f_1^{(2)}(i_v) \cdots f_1^{(w)}(i_v) \\
 \vdots \\
 f_d^{(1)}(i_v) f_d^{(2)}(i_v) \cdots f_d^{(w)}(i_v)
\end{pmatrix}
\]

- \textbf{F}: On input the function index \( C \) and \( x \in \{0, 1\}^h \), it outputs the vector \( a = xA \) and \( y = xC \).
By definition, $\alpha$ and standard deviation $\phi$:

\[ \alpha \cdot e \cdot \cdot \cdot \phi \]

Combine

1. $t d_{i_v}$ computed given any $t d_{i_v}$, $v = 1, \ldots, t - 1$ and identity $i d_{i_v}$. Because of $f^{(i)}(i_v) = \sum_{v = 0}^{t - 1} L_v \cdot f^{(i)}(i_v), j = 1, \ldots, d, i = 1, \ldots, w$, where $L_v$, $v = 0, 1, \ldots, t - 1$ are Lagrangian coefficients which may be efficiently computed given $i_0 = 0, i_1, \ldots, i_t - 1$, it computes the image $y = (y_1, \ldots, y_w)$ of $x$ and $xB = (m_1, \ldots, m_w)$. For every $i = 1, \ldots, w$, $\delta^{(i)}(0) = (a, z_i) + e_i = y_i - [q c_m] = (a, z_i) + (xE)$, it outputs the inversion share $\delta_{i_v} = \sum_{v = 0}^{t - 1} L_v \delta_{i_v}$.

2. Combine: On input any $t$ inversion shares $\delta_{i_v}, \ldots, \delta_{i_v}$. Because of $f^{(i)}(0) = \sum_{v = 1}^{t} L_v \cdot f^{(i)}(i_v), j = 1, \ldots, d, i = 1, \ldots, w$, where $L_v$, $v = 1, \ldots, t$ are Lagrangian coefficients which can be efficiently computed given any $t$ identities $i_1, \ldots, i_t$, it computes

\[ L_v \delta^{(i)}_{i_v} = \left\langle a, L_v \left( \begin{array}{c} f^{(i)}_1(i_v) \\ \vdots \\ f^{(i)}_d(i_v) \end{array} \right) \right\rangle + L_v e^{(i)}_{i_v}, \]

where $v = 1, \ldots, t$.

\[ y_i = \sum_{v = 1}^{t} L_v \delta^{(i)}_{i_v} = \left\langle a, \left( \sum_{v = 1}^{t} L_v f^{(i)}_1(i_v) + \cdots + L_v f^{(i)}_d(i_v) \right) \right\rangle + \sum_{v = 1}^{t} L_v e^{(i)}_{i_v} = (a, z_i) + \sum_{v = 1}^{t} L_v e^{(i)}_{i_v} \]

where $i = 1, \ldots, w$ and gets $y' = (y'_1, \ldots, y'_w)$, then it computes $y'' = \eta(y_i - y'_i)/q, i = 1, \ldots, w$ and obtains $m_i \in Z_q$ such that $y'' - \eta c_m_i$ is closest to 0. Finally, it outputs $x \in \{0, 1\}^h$, so that $xB = (m_1, \ldots, m_w)$.

We show correctness and lossy properties of our TLTDF as follows.

We recall some probability distributions in [22]. For $\alpha \in \mathbb{R}^+$, let $\psi_\alpha$ be a normal variable with mean 0 and standard deviation $\frac{\sqrt{2 \pi} \alpha}{\sqrt{2 \pi} \alpha}$ on $\mathbb{T}$. For any probability $\phi: \mathbb{T} \to \mathbb{R}^+$ and $q \in \mathbb{Z}^+$, let its discretization $\hat{\phi}: \mathbb{Z}_q \to \mathbb{R}^+$ be the discrete distribution over $\mathbb{Z}_q$ of the random variable $[q \cdot X_\phi] \mod q$, where $X_\phi$ is the distribution $\phi$.

**Lemma 7.** Let $q \geq 4p(h + \gamma), \alpha \leq 1/(16p(h + g))$ for $g \geq \gamma^2$, $\gamma = \sum_{i=1}^{t} \eta L_v$, where $L_v, v \in [t]$ is the Lagrangian coefficient and let $E = (e_{i,j}) \in \mathbb{Z}_q^{h \times w}$ be an error matrix generated by choosing independent error terms $e_{i,j} \leftarrow \chi = \psi_\alpha$ and $e_{i,v} \leftarrow \chi = \psi_\alpha, v \in [t]$. Every entry of $xE + \sum_{v=1}^{t} \eta L_v e_{i,v}$ has absolute value less than $q/4p$ for all $x \in \{0, 1\}^h$, except with probability at most $w \cdot 2^{-g}$ over the choice of $E$ and $e_{i,v}$.

**Proof.** By definition, $e_i = [qs_i] \mod q, e_{i,v} = [qs_{i,v}] \mod q$ where $s_i, s_{i,v}$ are independent normal variables with mean 0 and variance $\alpha^2$ for each $i \in [h], v \in [t]$. Let $s' = (x, e) + \sum_{v=1}^{t} \eta L_v e_{i,v}$, where $e = (e_1, \cdots, e_h)^T$. Then $s'$ is at most $(h + \gamma)/2 \leq q/8p$ away from $q(\langle x, s \rangle + \sum_{v=1}^{t} \eta L_v s_{i,v}) \mod 1).$
Since the $s_i, s_{i'}$ are independent, $(x, s) + \sum_{v=1}^{t} \eta L_v s_v$ is distributed as a normal variable with mean 0 and variance at most $(h + \gamma^2)\alpha^2 \leq (h + g)\alpha^2$, where $\gamma^2 > \sum_{v=1}^{t} (\eta L_v)^2$, hence a standard deviation of at most $(\sqrt{h + g})\alpha$. Then by the tail inequality on normal variables and the hypothesis on $\alpha$,

$$\Pr[\|x, s\| + \sum_{v=1}^{t} \eta L_v s_v \mid \|x, s\| + \sum_{v=1}^{t} \eta L_v s_v \leq 2 \sqrt{h + g} \alpha \leq \frac{\exp(-2(h + g))}{2\sqrt{h + g}} < 2^{-(h + g)}.$$ 

We show that for any fixed $x \in \{0, 1\}^h$, $\Pr[|s'| \geq 4q/p] \leq 2^{-(h + g)}$. Taking a union bound over all $x \in \{0, 1\}^h$, we can conclude that $|s'| < 4q/p$ for all $x \in \{0, 1\}^h$ except with probability at most $2^{-g}$.

Therefore, for each column $e$ of $E$ and $e_{i_v}, v \in [t]$; $|s'| < 4q/p$, for all $x$ except with probability at most $2^{-g}$ over the choice of $e$ and $e_{i_v}, v \in [t]$. The lemma follows by a union bound over all $w$ columns of $E$.

**Parameters.** Instantiate the parameters: let $p = h^{c_1}$ for constant $c_1 > 0$, $h = d^3$ for constant $c_3 > 1$, $\gamma = \sum_{v=1}^{t} \eta L_v$, where $L_v$ is the Lagrangian coefficient, $e_{i_v} \leftarrow \chi$ for $v \in [t]$, let $\chi = \psi_\alpha$ where $\alpha \leq 1/(32ph)$ and let $q \leq 2^{h/\alpha}$, $O(ph^{c_2})$ for constant $c_2 > 1$.

Note that for $A \in \mathbb{Z}_q^{2h \times d}$, the size of the function index is $hd \log q = d^{c_3 + 1} \log q = \Omega(d^2 \log d)$ and for $(xA, xC) \in \mathbb{Z}_q^d \times \mathbb{Z}_q^d$, the size of the image is $(h + w) \log q = (d^{c_3} + d^3/\log p) \log q = \Omega(d \log d)$.

**Correctness.** We now show correctness of the above TLTDF by proving the following theorem.

**Theorem 5.** The TLTDF with above parameters instantiated satisfies the correctness.

**Proof.** The combining algorithm computes $y' = (y'_1, \ldots, y'_w)$ as follows:

$$y'_i = (a, z_i) + \sum_{v=1}^{t} L_v e^{(i)}_{i_v} = (a, z_i) + \sum_{v=1}^{t} L_v e^{(i)}_{i_v}.$$ 

We have

$$y''_i = \frac{|\eta(y_i - y'_i)|}{q} = \frac{|\eta e_{m, q + \eta(xU)_i + \eta(xE)_i} - \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v}|}{q}.$$ 

Let $g = h \geq \gamma^2$ in above Lemma 7, the absolute total error

$$\|\mathbf{xU}i + (\mathbf{xE})i - \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v} \| \leq \|\mathbf{xU}_i\| + \|((\mathbf{xE})_i - \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v})\| \leq \frac{q}{8p} + \frac{q}{4p} < \frac{q}{2p}.$$ 

Therefore, we have

$$|\eta((\mathbf{xU})_i) + \eta((\mathbf{xE})_i) - \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v})| \leq |\eta((\mathbf{xU})_i) + (\eta((\mathbf{xE})_i) + \eta) \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v})| \leq \frac{\eta q}{2p}$$

the inversion is correct.

**Theorem 6.** The TLTDF with above parameters produces indistinguishable function indexes under the LWE $\eta_{q, \chi}$ assumption. Moreover, the algorithms give a collection of $(t, n, h, k)$-TLTDFs under the LWE $\eta_{q, \chi}$ assumption is hard. The residual leakage $\tau = h - k$ is

$$\tau \leq \left(\frac{c_2}{c_1} + o(1)\right) \cdot h.$$ 

**Proof.** The $(t, n)$-threshold secret sharing scheme holds the perfect privacy and the injective and lossy modes of LTDF are indistinguishable [22]. Therefore, we can show the indistinguishability between injective and lossy mode of TLTDF.

We transform the inversion algorithm into threshold version which does not change the lossy mode. In the lossy mode, as in the correctness argument, $\|\mathbf{xU}_i\| + \|((\mathbf{xE})_i + \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v})\| < \frac{\eta q}{2p}$. Therefore, for $i \in [w]$, the function output $y_i = (xA, z_i) + \|\mathbf{xU}_i\| + 0 + \|((\mathbf{xE})_i + \sum_{v=1}^{t} \eta L_v e^{(i)}_{i_v})\|$ can take at most $q/p$ possible values. Then the number of possible function outputs is at most $q^d (q/p)^w$. The proof follows (22, Theorem 6.4), we omit the details.
8 Threshold Trapdoor Relation

We show a relaxation of TTDF called TTDR and prove that TTDR maintains the same application of TPKE and RPKE.

**Definition 3.** A collection of $(t,n)$-TTDRs is a tuple of polynomial-time algorithms as follows:

- $\text{Gen}(1^\lambda) \rightarrow (ek,mtd)$: On input the security parameter $1^\lambda$, the generation algorithm outputs a function index $ek$ and a master trapdoor $mtd$.
- $\text{Share}(mtd, id_i) \rightarrow td_i$: On input the master trapdoor $mtd$ and any identity $id_i$, $i \in [n]$, the sharing algorithm outputs the shared trapdoor $td_i$, $i \in [n]$.
- $\text{Samp}(ek) \rightarrow (x,y)$: On input the function index $ek$, the relation sampling algorithm samples a relation $(x,y = F(ek,x))$.
- $F^{-1}(td_i,y) \rightarrow \delta_i$: On input any shared trapdoor $td_i$ and the value $y$, the partial inversion algorithm outputs the inversion share $\delta_i$.
- $\text{Combine}^{-1}(x,y,\delta_{i_1}, \ldots, \delta_{i_{t-1}}, id_i) \rightarrow \delta_i$: On input $x \in \{0,1\}^l$, its image $y = F(ek,x)$, any $t-1$ inversion shares $\delta_{i_1}, \ldots, \delta_{i_{t-1}}$, and identity $id_i$, the combining inversion algorithm outputs the inversion share $\delta_i$ of identity $id_i$.
- $\text{Combine}(\delta_i, \ldots, \delta_i, y) \rightarrow x$: On input any $t$ inversion shares $\delta_i,j = 1, \ldots, t$ and the value $y$, the combining algorithm outputs $x$.

Note that the generation algorithm and the relation sampling algorithm are probabilistic algorithms, while the rest four algorithms are deterministic algorithms, and we require that in the partial inversion algorithm, the combining inversion algorithm and the combining algorithm, the behavior of the algorithms is unspecified, if the $y$ is not in the image.

**Correctness.** For any $id_i, i \in [n]$, $(ek,mtd) \leftarrow \text{Gen}(1^\lambda)$, $td_i \leftarrow \text{Share}(mtd, id_i)$, any relation $(x,y = F(ek,x))$, we require that for any $t$ shared trapdoors $td_{i_1}, \ldots, td_{i_t}$, we have

$$x = \text{Combine}(F^{-1}(td_{i_1},y), \ldots, F^{-1}(td_{i_t},y), y).$$

**Security.** Let $A$ be a PPT adversary and define its advantage function as

$$\text{Adv}_{\text{TTDR},A}^{\text{low}}(\lambda) = \Pr \left[ \begin{array}{l} (ek,mtd) \leftarrow \text{Gen}(1^\lambda); \\
(id_{i_1}, \ldots, id_{i_{t-1}}) \leftarrow A(1^\lambda); \\
(x,y) \leftarrow \text{Samp}(ek); \\
x' \leftarrow A(ek,td_{i_1}, \ldots, td_{i_{t-1}}, y) \end{array} \right| x = x' : td_i \leftarrow \text{Share}(mtd, id_i), i \in [n];$$

A $(t,n)$-TTDR is threshold one-way if for any PPT adversary the advantage function is negligible.

Following the construction of TPKE and RPKE from TTDF, we can show generic constructions of TPKE and RPKE from TTDR by running the relation sampling algorithm of TTDR instead of the evaluation algorithm of TTDF in the encryption algorithm. The threshold one-wayness ensures the TPKE and RPKE are IND-CPA secure.

**Threshold Lossy Trapdoor Relation.** Following the definitions of TTDR and TLTF, by relaxing the evaluation algorithm of TLTF into relation sampling algorithm, we present the definition of TLTD and show that TLTF also produces indistinguishable function indexes. Similarly, we can prove TLTD implies TTDR.

We propose a refined definition of the relation by omitting the public computable injective map in LTDR [27]. Informally, the function index $ek$ is a composite function description which consists of the inverse map of the public computable injective map. The relation sampling algorithm outputs a relation $(x,y = F(ek,x))$. The inversion algorithm takes in the trapdoor and the image $y = F(ek,x)$, outputs $x$.

**Instantiations of TLTD.** Following the instantiation of TLTF under the DDH assumption and the instantiation of LTDR [27], we give an efficient instantiation under the DDH assumption by relaxing
evaluation algorithm into relation sampling algorithm. We constructs the TLTDR by using $2 \times 3$ matrix encryption. The function indexes in the injective mode and in the lossy mode of TLTDR are

$$
C_1 = \left( \begin{array}{ccc}
g^{r_1} & g^{r_1 s_1} & g^{r_1 s_2} \\
g^{r_2} & g^{r_2 s_1} & g^{r_2 s_2} \
g^{r_3} & g^{r_3 s_1} & g^{r_3 s_2} 
\end{array} \right),
C_0 = \left( \begin{array}{ccc}
g^{r_1} & g^{r_1 s_1} & g^{r_1 s_2} \\
g^{r_2} & g^{r_2 s_1} & g^{r_2 s_2} \
g^{r_3} & g^{r_3 s_1} & g^{r_3 s_2} 
\end{array} \right)
$$

For $C = (c_{ij})_{2 \times 3}$ and $(x_1, x_2) \in \mathbb{Z}_p^2$, the relation sampling algorithm outputs a relation $(x = (g^{x_1}, g^{x_2}))$, $F(ek, x) = (c_{11}^{-1}c_{12}^2, c_{12}^{-1}c_{22}^2, c_{13}^{-1}c_{23}^{-1})$. The combining algorithm computes $x$ by taking as input any $t$ inversion shares and the image $F(ek, x)$. It is not hard to show a collection of $(t, n, 2 \log p, \log p)$-TLTDRs under the DDH assumption.

9 Performance Evaluations

9.1 Theoretical Analysis

Communication Cost and Efficiency Comparison. Table 1 compares the communication costs and computational costs of our lattice-based TPKE schemes with that in [13, 15, 16]. For lattice-based TPKE, the communication cost of our scheme is less than [13], in which they need to use a large modulus which causes larger ciphertexts. Compared with [15], they split the message into many pieces and encrypt every piece by a different tag-based encryption, that cause the size of the public key and the ciphertext is at least linear in the number of users, while our scheme splits the master secret key and encrypt every piece by a different tag-based encryption, that cause the size of the public key and modulus which causes larger ciphertexts. Compared with [15], they split the message into many pieces and a modular exponentiation respectively. We construct the scheme of Ours1 and Ours2 from TTDF where $l$ denotes input length of function and the scheme of Ours3 from TTDR.

Table 1. Comparisons Among TPKE Schemes

| Scheme | pk size | ciphertext size | Enc | Dec | assumption | adaptive corruption | generic construction | IND-CCA |
|--------|---------|-----------------|-----|-----|------------|---------------------|---------------------|---------|
| BDH    | $d^3$   | $d^3$           | 1Mvp| 1Mvp| LWE        | ×                   | ×                   | ×       |
| XXXZ11 | $2d^3 \log d$ | $2d^2 \log d$ | 1SS+10TS+1TBE | 1InvLTDF | LWE | × | √ | × |
| BKPP13 | $d^2 \log d$ | $d\log d$ | 1OTS+2Mvp | 1InvPGLF | LWE | √ | × | √ |
| Ours1  | $d^2 \log d$ | $d\log d$ | 1Mvp | 1Mvp | LWE | √ | × | × |
| Ours2  | $\ell(l[l])$ | $l[l]$ | $\ell$Exp | 1Exp | DDH | √ | × | × |
| Ours3  | $6[l]$ | $3[l]$ | 6Exp | 1Exp | DDH | √ | × | × |

\* $n$ and $d$ denotes the number of users and the dimension of lattice respectively. Mvp, SS, OTS, TBE, InvLTDF, InvPGLF and Exp denote the cost of a matrix-vector product, a secret sharing, a one-time signature, a tag-based encryption, inverting an image of LTDF, sampling a preimage of preimage sampleable function and a modular exponentiation respectively. We construct the scheme of Ours1 and Ours2 from TTDF where $l$ denotes input length of function and the scheme of Ours3 from TTDR.

Table 2 compares the communication costs of our RPKE schemes with that in [19, 20, 9]. The size of the public key of our DDH-based RPKE is a $2 \times 3$ matrix which is less than [20] and [9] in which the size of the public key is at least linear with the revocation threshold value.
### 9.2 Experimental Analysis

In order to evaluate the practical performance of our schemes, we implement the TTDF in Section 3, TPKE in Section 4 and RPKE in Section 5 based on the NTL library. The program is executed on an Intel Core i7-2600 CPU 3.4GHz and 4GB RAM running Linux Deepin 15.4.1 64-bit system.

**Experiment Setting and Computation Time.** As depicted in Table 3 and Table 4, we set the security parameter $\lambda = 128, 256, 512$ respectively, and the dimension of lattice $d = 512, 768, 1024$ respectively and compute other parameter by the Section 7, where $h = d^{\alpha}, c_3 > 1, p = h^{c_3}, c_1 > 0, l = \log p_1, w = m = h/l, \alpha \leq 1/(2\pi h)$ and $q > 2\sqrt{\alpha/d}$. What’s more, we set the number of users is $n = 4$ and the threshold value is $t = 3$ in TPKE, and the number of revoked users is $r = 2$ in RPKE.

#### Table 3. Experiment Setting and Computation Time of DDH-Based TPKE and RPKE

| Parameter | TPKE Time (ms) | RPKE Time (ms) |
|-----------|----------------|----------------|
| $\lambda$ | KeyGen Encrypt Decrypt Combine | KeyGen Encrypt Decrypt |
| 128 | 2.184 0.266 0.131 0.309 | 4.978 0.396 0.467 |
| 256 | 9.499 0.517 0.312 0.766 | 11.59 0.839 0.770 |
| 512 | 68.08 1.414 0.749 1.348 | 34.64 1.390 1.786 |

$^d$ $\lambda$, $n$, and $r$ indicate the security parameter, the number of users, the threshold value, and the number of revoked users, respectively.

#### Table 4. Experiment Setting and Computation Time of LWE-Based TPKE and RPKE

| Parameter | TPKE Time (s) | RPKE Time (s) |
|-----------|----------------|----------------|
| $d$ | KeyGen Encrypt Decrypt Combine | KeyGen Encrypt Decrypt |
| $p$ | 2.178 0.016 0.069 2.218 | 2.259 0.092 2.446 |
| $w$ | 4.161 0.164 0.014 8.949 | 4.215 0.305 9.032 |
| $t$ | 7.724 0.293 0.021 17.554 | 7.367 0.382 17.962 |

$^d$ $d$, $h$, $p$, $w$, $n$, $t$, and $r$ indicate the dimension of lattice, the number of rows of matrix $A$ of the public key, the size of the message space $\mathbb{Z}_p$, the number of columns of matrix $Z$ of the master secret key, the number of users, the threshold value, and the number of revoked users, respectively.

As depicted in Table 3 and Table 4, we show the average running times of all algorithms in our TPKE and RPKE schemes. For different security levels, we set the security parameter $\lambda = 128, 256, 512$ respectively. The average running times of all algorithms in both DDH-based TPKE and RPKE are the level of milliseconds. Therefore, our schemes are efficient and practical. Meanwhile, we set the dimension of $d$ and $d$ denote the threshold value and the dimension of lattice, respectively. We construct the scheme of Ours_1 and Ours_2 from TTDR where $l$ denotes input length of function and the scheme of Ours_3 from TTDR.

We use the technique of “clearing out the denominator” to preserve correct decryption, but the number of errors still increases $(nl)^2$. Therefore, we set the number of users $n = 4$ enable to run the LWE-based TPKE and RPKE practically.
of lattice $d = 512, 768, 1024$ respectively. In LWE-based TPKE scheme, the average running times of the encryption algorithm and the decryption algorithm are 0.076s, 0.167s, 0.293s, and 0.005s, 0.012s, 0.021s. In LWE-based RPKE scheme, the average running times of the encryption algorithm are 0.092s, 0.209s, 0.382s. From these outcomes, we note that the encryption algorithm and the decryption algorithm of our TPKE schemes and the encryption algorithm of our RPKE schemes are efficient.

Acknowledgements: This work is supported by the National Natural Science Foundation of China (No. 61772522) and Youth Innovation Promotion Association CAS.

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