SHARP INTERFACE LIMIT FOR COMPRESSIBLE NON-ISENTRIC
PHASE-FIELD MODEL

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Abstract. In this paper, the sharp interface limit for the compressible non-isentropic
Navier-Stokes/Allen-Cahn system is derived by the method of matched asymptotic ex-
pansion. We show that the leading order problem satisfies the compressible Navier-Stokes
equations with the interface being a free boundary. We discuss two cases in terms of
different phase field diffusion coefficients. One is \( M_\varepsilon = O(1) \) and \( M_\varepsilon = O(\varepsilon) \), where \( \varepsilon \)
is the interface thickness. We have observed that the velocity and the temperature of the
compressible immiscible two-phase fluids continuously through the interface. There is a
jump for the tension tensor at the interface, this jump depends on the surface tension
and the mean curvature of the interface. In particular, for the first case \( M_\varepsilon = O(1) \), no
matter how the density changes through the interface, the velocity of the interface in the
normal direction is the same as the normal velocity of the fluid along the interface. But
for the second case \( M_\varepsilon = O(\varepsilon) \), this phenomenon can’t occur where the density passes
continuously through the interface. In fact, on this part of the interface, the normal ve-
locity of the interface is determined by the mean curvature of the interface, the velocity
and the density of the compressible immiscible two-phase fluids. That’s where the phase
transition happens.

1. Introduction

Understanding the geometry and distribution of the interface is very important for
determining the immiscible two-phase flow. The treatment of such two-phase flow’s in-
terface is derived from the idea of physicist J.D. Van der Waals [13], who regarded the
interface of immiscible two-phase flow as a region with a certain thickness. Mathematical
models based on this idea are often called diffusion interface models, such as the famous
Navier-Stokes/Allen-Cahn system, which can be used to study the immiscible two-phase
flow, such as phase transformation, chemical reactions, etc., see [2]- [10] and the references
therein. In these literatures, by introducing diffusion interface instead of sharp interface,
the authors overcome the difficulties caused by the boundary condition of interface.

For compressible immiscible two-phase flow, taking any one of the volume particles
in the flow, we assume \( M_i \) the mass of the components in the representative material
volume \( V \), \( \phi_i = M_i / V \) the mass concentration, \( \rho_i = M_i / V \) the apparent mass density of the
fluid \( i \) \((i = 1, 2)\). The total density is given by \( \rho = \rho_1 + \rho_2 \) and \( \phi = \phi_1 - \phi_2 \). We call
\( \phi \) the difference of the two components for the fluid mixture. Obviously, \( \phi \) describes the
distribution of the interface. The compressible heat-conducting Navier-Stokes/Allen-Cahn
system derived by Heida-Málek-Rajagopal [8] is as following

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) &= \text{div} T, \\
\partial_t (\rho \phi) + \text{div}(\rho \phi u) &= -M \mu, \\
\rho \mu &= \rho \frac{\partial f}{\partial \phi} - \text{div}(\rho \frac{\partial f}{\partial \nabla \phi}), \\
\partial_t (\rho E) + \text{div}(\rho E u) &= \text{div}(T u + k \nabla \theta - M \epsilon \frac{\partial f}{\partial \nabla \phi}),
\end{aligned}
\]

(1.1)

where \(x \in \Omega \subset \mathbb{R}^N\), \(N\) is spatial dimension, \(t > 0\). The unknown functions \(\rho(x, t), u(x, t), \phi(x, t), \theta(x, t)\) denote the total density, the velocity, the difference of the two components for the fluid mixture, and the absolute temperature, respectively. \(\mu(x, t)\) is the chemical potential of the fluid. \(\epsilon > 0\) is the thickness of the diffuse interface. \(k > 0\) is the coefficient of heat conduction. \(M_\epsilon = M(\epsilon) > 0\) is the mobility coefficient. The Cauchy stress-tensor is represented by

\[
T = 2\nu \mathbb{D}(u) + \lambda (\text{div} u) \mathbb{I} - p \mathbb{I} - \rho \nabla \phi \otimes \frac{\partial f}{\partial \nabla \phi}.
\]

(1.2)

where \(\mathbb{D}u\) is the deformation tensor

\[
\mathbb{D}u = \frac{1}{2} (\nabla u + \nabla^\top u),
\]

(1.3)

and \(\mathbb{I}\) is the unit matrix, \(\top\) means the transpose of the matrix. \(\nu > 0, \lambda > 0\) are viscosity coefficients, satisfying

\[
\nu > 0, \quad \lambda + \frac{2}{N} \nu \geq 0.
\]

(1.4)

The total energy density \(\rho E\) is given by

\[
\rho E = \rho (e + f + \frac{1}{2} u^2),
\]

(1.5)

where \(\rho e\) is the internal energy, \(\frac{\rho u^2}{2}\) is the kinetic energy, \(f\) is the fluid-fluid interfacial free energy density, and it has the following form (refer to Heida-Málek-Rajagopal [8] and Lowengrub-Truskinovsky [12]):

\[
f(\rho, \phi, \nabla \phi) \overset{\text{def}}{=} \frac{1}{4\epsilon \rho} (1 - \phi^2)^2 + \frac{\epsilon}{2\rho} |\nabla \phi|^2.
\]

(1.6)

Here \(p = p(\rho, \theta), e = e(\rho, \theta)\) and \(f = f(\rho, \phi, \nabla \phi)\) obey the second law of thermodynamics (see Lions [11]),

\[
ds = \frac{1}{\theta} (d(e + f) + pd(\frac{1}{\rho})),
\]

(1.7)

where \(s\) is the thermodynamic entropy. Then deduced from (1.7), we have

\[
\frac{\partial s}{\partial \theta} = \frac{1}{\theta} \frac{\partial (e + f)}{\partial \theta}, \quad \frac{\partial s}{\partial \rho} = \frac{1}{\theta} \left( \frac{\partial (e + f)}{\partial \rho} - \frac{p}{\rho^2} \right),
\]

(1.8)

which implies the following compatibility equation

\[
p = \rho^2 \frac{\partial (e + f)}{\partial \rho} + \theta \frac{\partial \rho}{\partial \theta} - \frac{1}{4\epsilon} (\phi^2 - 1)^2 - \frac{\epsilon}{2} |\nabla \phi|^2.
\]

(1.9)
Therefore, we have
\[
\begin{aligned}
\partial_t \rho + \text{div} (\rho \mathbf{u}) &= 0, \\
\rho \partial_t \mathbf{u} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} - 2\nu \text{div} \mathbf{u} - \lambda \nabla \text{div} \mathbf{u} + \nabla p(\rho, \theta) &= -\epsilon \text{div} (\nabla \phi \otimes \nabla \phi), \\
\rho \partial_t \phi + \rho \mathbf{u} \nabla \phi &= -M_\epsilon \mu, \\
\rho \mu &= \frac{1}{\epsilon} (\phi^3 - \phi) - \epsilon \Delta \phi, \\
\epsilon \phi (\rho \partial_t \theta + \rho \mathbf{u} \cdot \nabla \theta) + \theta p_\theta \text{div} \mathbf{u} - k \Delta \theta &= 2\nu |\nabla \mathbf{u}|^2 + \lambda (\text{div} \mathbf{u})^2 + M_\epsilon \mu^2. \\
\end{aligned}
\]  
(1.10)

About the study of compressible immiscible two-phase flow, most of the works focused on isentropic compressible problems. Feireisl-Petzeltová-Rocca-Schimperna [7] established the global existence of finite energy weak solutions in 3-D by using the framework which was introduced by Lions [11]. Ding-Li-Lou [5] proved the global existence of the strong solutions in 1-D with large initial data. Chen-Guo [4] generalized the result of [5] to the case that the initial vacuum is allowed. Abels-Liu [1] proved the convergence of the solutions for the incompressible Stokes/Allen-Cahn system to solutions of a sharp interface model for sufficiently small times. Witterstein [15] showed that the sharp-interface limit of the isentropic phase-field model is the standard two-phase compressible Navier-Stokes equations by the method of asymptotic analysis. Wang-Wang [14], Xu-Di-Yu [16] investigated the sharp-interface limits of the incompressible phase-field model with a generalized Navier slip boundary condition. There is not much work for non-isentropic case. Kotschote [10] obtained the local existence and uniqueness result for strong solutions in 3-D for compressible non-isothermal phase-field model.

The main purpose of this paper is to derive the sharp interface limit of the non-isentropic compressible system (1.10) for two cases (C1) and (C2) by
\[
\begin{aligned}
(C1) & \quad M_\epsilon = 1, \\
(C2) & \quad M_\epsilon = \frac{1}{\epsilon}. \\
\end{aligned}
\]  
(1.11)

The physical meaning of the second case (C2) is that, the phase field mobility rate is accelerated. Before we introduce our main result, let us summarize the common symbols used in this paper. For \( \forall t > 0 \), suppose that unknown two-phase free interface is given by
\[
\Gamma(t) := \{ \mathbf{x} \in \Omega | \phi_\epsilon (\mathbf{x}, t) = 0 \}. 
\]  
(1.12)

Obviously, \( \Gamma(t) \) divides the whole domain \( \Omega \) into two separated domain \( \Omega^-(t) \) and \( \Omega^+(t) \) which represented the domains occupied by fluid 1 and fluid 2 respectively, more precisely
\[
\Omega^-(t) = \{ \mathbf{x} \in \Omega | \phi_\epsilon (\mathbf{x}, t) < 0 \}, \quad \Omega^+(t) = \{ \mathbf{x} \in \Omega | \phi_\epsilon (\mathbf{x}, t) > 0 \},
\]
and
\[
\Omega = \Omega^-(t) \cup \Gamma(t) \cup \Omega^+(t), \quad \forall t > 0.
\]
Therefore, we have the following definition
\[
\Omega^- = \{ (\mathbf{x}, t) | \mathbf{x} \in \Omega^-(t) \}, \quad \Omega^+ = \{ (\mathbf{x}, t) | \mathbf{x} \in \Omega^+(t) \}, \quad \Gamma = \{ (\mathbf{x}, t) | \mathbf{x} \in \Gamma(t) \}. 
\]  
(1.13)

We use the method of matched asymptotic expansion to determine the sharp interface limit as \( \epsilon \to 0 \). Unlike incompressible fluids, the density of compressible fluids varies. Especially for immiscible two-phase flow, the change of density may cause phase transition. To illustrate the mass density properties near the interface in more detail, we define the set \( S \subset \Gamma \) by
\[
S = \{ (\mathbf{x}, t) \in \Gamma | \rho^+ (\mathbf{x}, t) = \rho^- (\mathbf{x}, t) \},
\]  
(1.14)
which implies that the jump of the density $\rho$ can’t occur on this part of the interface $\Gamma$. Now, we give the main theorem of sharp interface limit.

**Theorem 1.1.** We assume (1.12)-(1.13). Let $(\rho, u, \phi, \mu, \theta)$ be a solution of the Navier-Stokes-Allen-Cahn system (1.10). We assume that an outer asymptotic expansion, that is

$$
\begin{align*}
\rho &= \rho_0 \pm \epsilon \rho_1 + \cdots, \quad u = u_0 + \epsilon u_1 + \cdots, \quad \phi = \phi_0 + \epsilon \phi_1 + \cdots, \\
\mu &= \epsilon^{-1} \mu_0 + \mu_1 + \cdots, \quad \theta = \theta_0 + \epsilon \theta_1 + \cdots,
\end{align*}
$$

and an inner asymptotic expansion, that is

$$
\begin{align*}
\rho &= \tilde{\rho}_0 \pm \epsilon \tilde{\rho}_1 + \cdots, \quad u = \tilde{u}_0 + \epsilon \tilde{u}_1 + \cdots, \quad \phi = \tilde{\phi}_0 + \epsilon \tilde{\phi}_1 + \cdots, \\
\tilde{\mu} &= \epsilon^{-1} \tilde{\mu}_0 + \tilde{\mu}_1 + \cdots, \quad \tilde{\theta} = \tilde{\theta}_0 + \epsilon \tilde{\theta}_1 + \cdots,
\end{align*}
$$

for $\rho, u, \phi, \mu, \theta$. We suppose that $\rho_0 > 0$ and $\tilde{\rho}_0 > 0$. Then as $\epsilon \to 0$, the system (1.10) converges to the sharp interface problem

$$
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \quad \text{in } \Omega^\pm, \\
\rho \partial_t u + \rho(u \cdot \nabla) u &= -\nu \text{div}^2 u - \lambda \nabla \text{div} u + \nabla (\rho \frac{\partial e}{\partial \rho} + \theta \frac{\partial p}{\partial \theta}) = 0, \quad \text{in } \Omega^\pm, \\
\epsilon \theta (\rho \partial_t \theta + \rho u \cdot \nabla \theta) + \theta p \rho \text{div} u - k \Delta \theta &= 2\nu \|D u\|^2 + \lambda (\text{div} u)^2, \quad \text{in } \Omega^\pm, \\
\phi &= \pm 1,
\end{align*}
$$

with the jump conditions

$$
\begin{align*}
(C1) \quad M_\epsilon &= 1 : \left\{ \begin{array}{ll}
[u]_\Gamma = 0, \quad [\theta]_\Gamma = 0, & \text{on } \Gamma, \\
V_n - u \cdot n &= 0, & \text{on } \Gamma, \\
[(2\nu D u + \lambda \text{div} u - (\rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial p}{\partial \theta}) \|u\|)n]_\Gamma &= \sigma \kappa n, & \text{on } \Gamma,
\end{array} \right.
\end{align*}
$$

$$
\begin{align*}
(C2) \quad M_\epsilon &= \frac{1}{\epsilon} : \left\{ \begin{array}{ll}
[u]_\Gamma = 0, \quad [\theta]_\Gamma = 0, & \text{on } \Gamma, \\
[\rho]_\Gamma = 0, \quad \rho^2 (V_n - u \cdot n) &= -\kappa, & \text{on } S, \\
V_n - u \cdot n &= 0 = \kappa, & \text{on } \Gamma \setminus S, \\
[(2\nu D u + \lambda \text{div} u - (\rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial p}{\partial \theta}) \|u\|)n]_\Gamma &= \sigma \kappa n, & \text{on } \Gamma,
\end{array} \right.
\end{align*}
$$

on the interface $\Gamma$, where $\Gamma$ is defined as (1.13) and $S$ defined as (1.14). $[\cdot]_\Gamma = -.+$ denotes the jump of limiting values across the interface, $\sigma$ is the constant coefficient of surface tension (see (2.82)), $\kappa$ the curvature of the interface $\Gamma$, $n$ the unit normal of the interface pointing to $\Omega^+$, $V_n$ denotes the normal velocity of the interface $\Gamma$.

**Remark 1.1.** Theorem 1.1 shows that the leading order problem satisfies the compressible Navier-Stokes equations with the interface being a free boundary. Whether the diffusion coefficient $M_\epsilon = O(1)$ or $M_\epsilon = O(\frac{1}{\epsilon})$, the velocity and the temperature of the flow continuously through the interface. There is a jump for the tension tensor at the interface, and this jump depends on the surface tension and the mean curvature of the interface. In particular, for the first case $M_\epsilon = O(1)$, no matter how the density changes through the interface, the velocity of the interface in the normal direction is the same as the normal velocity of the fluid along the interface. But for the second case $M_\epsilon = O(\frac{1}{\epsilon})$, this phenomenon can only occur at the part of the interface where the density is discontinuous, and the mean curvature for this part of the interface is zero. In other parts of the interface,
the normal velocity of the interface is determined by the velocity, density of the fluid and the curvature of the surface, and that’s where the phase transition happens.

2. Asymptotic expansion for non-isentropic compressible phase-field model

In this section we consider the asymptotic analysis for \((1.10)\) for two cases \((C1)\) and \((C2)\). The analysis process is as follows, firstly we consider the outer expansions far from the interface \(\Gamma\), and then we do the inner expansions near the interface \(\Gamma\), and finally we combine them together to obtain the sharp interface limit of the system \((1.10)\) in \(\Omega\). To make the presentation in this section clear, we use \(\rho, \mathbf{u}, \phi, \mu, \theta\) instead of \(\rho_\epsilon, \mathbf{u}_\epsilon, \phi_\epsilon, \mu_\epsilon, \theta_\epsilon\) in the system \((1.10)\), to show explicitly that these functions depend on \(\epsilon\).

2.1. Outer expansion. Far from the two-phase interface \(\Gamma\), we use the following ansatz,

\[
\begin{align*}
\rho_\epsilon^\pm &= \rho_0^\pm + \epsilon \rho_1^\pm + \epsilon^2 \rho_2^\pm + \cdots, \\
\mathbf{u}_\epsilon^\pm &= \mathbf{u}_0^\pm + \epsilon \mathbf{u}_1^\pm + \epsilon^2 \mathbf{u}_2^\pm + \cdots, \\
\phi_\epsilon^\pm &= \phi_0^\pm + \epsilon \phi_1^\pm + \epsilon^2 \phi_2^\pm + \cdots, \\
\mu_\epsilon^\pm &= \epsilon^{-1} \mu_0^\pm + \mu_1^\pm + \mu_2^\pm + \cdots, \\
\theta_\epsilon^\pm &= \theta_0^\pm + \epsilon \theta_1^\pm + \epsilon^2 \theta_2^\pm + \cdots.
\end{align*}
\]

(2.1)

Here \(g^\pm\) denotes the restriction of a function \(g\) in \(\Omega^+\) and \(\Omega^-\) respectively. Since \(\rho > 0, \theta > 0\), we presume that \(\rho_0^+ > 0, \theta > 0\).

Combined with the above analysis, let us first plug ansatz \((2.1)\) in the mass conservation equation \((1.10)_1\) and compare the coefficients of terms with the same power of \(\epsilon\), and we achieve on the lowest order

\[
O(1) : \quad \partial_t \rho_0^\pm + \text{div}(\rho_0^\pm \mathbf{u}_0^\pm) = 0.
\]

(2.2)

Similarly, plugging ansatz \((2.1)\) in the momentum conservation equation \((1.10)_2\), we obtain

\[
\begin{align*}
O(\epsilon^{-1}) : \quad &\nabla((1 - (\phi_0^\pm)^2)/4) = 0, \\
O(1) : \quad &\rho_0^\pm \partial_t \mathbf{u}_0^\pm + \rho_0^\pm \mathbf{u}_0^\pm \cdot \nabla \mathbf{u}_0^\pm - \nabla((\phi_0^\pm)^3 - \phi_0^\pm) \phi_0^\pm + \text{div} \mathbf{S}_0^\pm - \nabla p_0^\pm = 0,
\end{align*}
\]

(2.3)\(\) \(\) (2.4)

where

\[
\mathbf{S}_0^\pm = 2\nu \mathbb{I} \mathbf{D} \mathbf{u}_0^\pm + \lambda \text{div} \mathbf{u}_0^\pm \mathbb{I},
\]

(2.5)

and

\[
\rho_0^\pm = (\rho_0^\pm)^2 e_p(\rho_0^\pm, \theta_0^\pm) + \theta_0^\pm p_0(\rho_0^\pm, \theta_0^\pm).
\]

(2.6)

Next, we plug ansatz \((2.1)\) in conservation equation of phase field \((1.10)_3\) to achieve

\[
\begin{align*}
O(\epsilon^{M_\epsilon^{-2}}) : \quad &\mu_0^\pm = 0, \\
O(\epsilon^{M_\epsilon^{-1}}) : \quad &\chi_{M_\epsilon}(\rho_0^\pm \partial_t \phi_0^\pm + \rho_0^\pm \mathbf{u}_0^\pm \cdot \nabla \phi_0^\pm) = -\mu_1^\pm,
\end{align*}
\]

(2.7)\(\) \(\) (2.8)

where

\[
\chi_{M_\epsilon} = \begin{cases} 1, & \text{for } (C1) \ M_\epsilon = 1, \\
0, & \text{for } (C2) \ M_\epsilon = \frac{1}{2}.
\end{cases}
\]

(2.9)

Plugging ansatz \((2.1)\) in potential equation \((1.10)_4\) and comparing the coefficients of terms with the same power of \(\epsilon\), we obtain

\[
O(\epsilon^{-1}) : \quad \rho_0^\pm \phi_0^\pm = \phi_0^\pm ((\phi_0^\pm)^2 - 1).
\]

(2.10)
Finally, plugging ansatz (2.1) in energy equation (1.10), we obtain
\[ O(1) : e_\theta (\rho_0^\pm, \theta_0^\pm) (\rho_0^\pm \partial_\theta \theta_0^\pm + \rho_0^\pm u_0^\pm \cdot \nabla \theta_0^\pm + \theta_0^\pm p_\theta (\rho_0^\pm, \theta_0^\pm)) \text{div} u_0^\pm = -k \Delta \theta_0^\pm \]
\[ = 2\nu |\nabla u_0^\pm|^2 + \lambda (\text{div} u_0^\pm)^2 + \chi_M (|\mu_\pm^\pm|^2 + 2\mu_0^\pm \mu_0^\pm) + (1 - \chi_M) \left( 2\mu_1^\pm \mu_2^\pm + 2\mu_0^\pm \mu_0^\pm \right), \quad (2.11) \]
where \( \chi_M \) is defined in (2.9). Then from the equation (2.3), (2.7), (2.8) and (2.10), we have
\[ \phi_0^\pm = \pm 1, \quad \mu_0^\pm = 0, \quad \mu_0^\pm = 0, \quad \text{in} \Omega^\pm. \quad (2.12) \]
Combining (2.4) and (2.10), it follows that
\[ \rho_0^\pm \partial_\theta u_0^\pm + \rho_0^\pm (u_0^\pm \cdot \nabla) u_0^\pm + \nabla p_0^\pm = \text{div} \mathcal{S}_0^\pm. \quad (2.13) \]
Moreover, (2.11) and (2.12) derive that
\[ e_\theta (\rho_0^\pm, \theta_0^\pm)(\rho_0^\pm \partial_\theta \theta_0^\pm + \rho_0^\pm u_0^\pm \cdot \nabla \theta_0^\pm) + \theta_0^\pm p_\theta (\rho_0^\pm, \theta_0^\pm)) \text{div} u_0^\pm = -k \Delta \theta_0^\pm \]
\[ = 2\nu |\nabla u_0^\pm|^2 + \lambda (\text{div} u_0^\pm)^2. \quad (2.14) \]
By using (2.2), (2.12), (2.13) and (2.14), we can state the following lemma.

**Lemma 2.1.** Assume \( \rho_0^\pm > 0 \). Letting \( \epsilon \to 0 \) in ansatz (2.1), we derive from (2.2), (2.4), (2.7), (2.8), (2.10) and (2.11), in the regions outside the transition layer, to the problem
\[ \partial_t \rho^\pm + \text{div}(\rho^\pm u^\pm) = 0, \quad \text{in} \Omega^\pm, \quad (2.15) \]
\[ \rho^\pm \partial_\theta u^\pm + \rho^\pm (u^\pm \cdot \nabla) u^\pm - \text{div}(2\nu \nabla u^\pm + \lambda \text{div} u^\pm) \]
\[ + \nabla (|\rho^\pm|^2 (e_\theta (\rho^\pm, \theta^\pm) + \theta^\pm p_\theta (\rho^\pm, \theta^\pm))) = 0, \quad \text{in} \Omega^\pm, \quad (2.16) \]
\[ e_\theta (\rho^\pm, \theta^\pm)(\rho^\pm \partial_\theta \theta^\pm + \rho^\pm u^\pm \cdot \nabla \theta^\pm) + \theta^\pm p_\theta (\rho^\pm, \theta^\pm) \text{div} u^\pm = -k \Delta \theta^\pm \]
\[ = 2\nu |\nabla u^\pm|^2 + \lambda (\text{div} u^\pm)^2, \quad \text{in} \Omega^\pm, \quad (2.17) \]
and \( \phi_0^\pm = \pm 1 \) in \( \Omega^\pm \).

### 2.2. Inner expansion.
In this subsection, we propose to analysis the inner expansion near the interface \( \Gamma \). Let \( d(x, t) \) be signed distance to \( \Gamma \), which is well-defined near the interface. Then the unit normal of the interface pointing to \( \Omega^+ \) is given by \( n = \nabla d \) and the normal velocity of the interface \( \Gamma \) is given by \( V_n = -\partial_t d \). We introduce a new rescaled variable
\[ \xi = \frac{d(x, t)}{\epsilon}. \quad (2.18) \]
For any function \( g(x, t) \) (e.g. \( g = \rho, u, \phi, \mu, \theta \)), we can rewrite it as
\[ g(x, t) = \tilde{g}(x, t, \xi). \quad (2.19) \]
Then we have
\[ \nabla g = \nabla ^\xi \tilde{g} + e^{-1} \partial_\xi ^\xi \tilde{g} n, \]
\[ \Delta g = \Delta ^\xi \tilde{g} + e^{-2} (n \cdot \nabla) \partial_\xi \tilde{g} + e^{-2} \partial_\xi ^2 \tilde{g} \]
\[ = 2\nu |\nabla \tilde{g}|^2 + \lambda (\text{div} \tilde{g})^2, \quad (2.20) \]
Here we use the fact that \( \nabla \cdot n = \kappa \), the mean curvature of the interface. \( \kappa(x) \) for \( x \in \Gamma(t) \) is positive (resp. negative) if the domain \( \Omega_- \) is convex (resp. concave) near \( x \).
In the inner region, we assume that
\begin{align}
\tilde{\rho}_\epsilon &= \tilde{\rho}_0 + \epsilon \tilde{\rho}_1 + \epsilon^2 \tilde{\rho}_2 + \cdots, \\
\tilde{\mathbf{u}}_\epsilon &= \tilde{\mathbf{u}}_0 + \epsilon \tilde{\mathbf{u}}_1 + \epsilon^2 \tilde{\mathbf{u}}_2 + \cdots, \\
\tilde{\phi}_\epsilon &= \tilde{\phi}_0 + \epsilon \tilde{\phi}_1 + \epsilon^2 \tilde{\phi}_2 + \cdots, \\
\tilde{\mu}_\epsilon &= \epsilon^{-1} \tilde{\rho}_0 + \epsilon \tilde{\mu}_1 + \epsilon^2 \tilde{\mu}_2 + \cdots, \\
\tilde{\theta}_\epsilon &= \tilde{\theta}_0 + \epsilon \tilde{\theta}_1 + \epsilon^2 \tilde{\theta}_2 + \cdots.
\end{align}
(2.21)

In the next we represent the system \((1.10)\) in the new coordinates and compare the order of the \(\epsilon\) coefficients. Let us first plug ansatz \((2.21)\) in the mass conservation equation \((1.10)_1\) to infer the following equation in new coordinates
\begin{equation}
\frac{\partial \tilde{\rho}_0}{\partial t} - \epsilon^{-1} \partial_\xi (\tilde{\rho}_0 V_n) + \text{div}(\tilde{\rho}_0 \tilde{\mathbf{u}}_0) + \epsilon^{-1} \partial_\xi (\tilde{\rho}_0 \tilde{\mathbf{u}}_0) \cdot \mathbf{n} + \mathcal{O}(\epsilon) = 0.
\end{equation}
(2.22)

By comparing the coefficients of same \(\epsilon\)-order, we obtain
\begin{align}
\mathcal{O}(\epsilon^{-1}) : & \quad - \frac{\partial_\xi \tilde{\rho}_0 V_n + \partial_\xi (\tilde{\rho}_0 \tilde{\mathbf{u}}_0) \cdot \mathbf{n}}{\mathbf{n}} = 0, \\
\mathcal{O}(1) : & \quad \frac{\partial_\xi \tilde{\rho}_0 + \text{div}(\tilde{\rho}_0 \tilde{\mathbf{u}}_0) - \partial_\xi (\tilde{\rho}_0 \tilde{\mathbf{u}}_1 + \tilde{\varphi}_1 \tilde{\mathbf{u}}_0) \cdot \mathbf{n}}{\mathbf{n}} = 0.
\end{align}
(2.23)\(\quad\) (2.24)

Next, we evaluate the right-hand-side of the momentum conservation equation \((1.10)_2\). By using \((2.20)\), through differentiation, we obtain
\begin{align}
\text{div}(\nu(\nabla \mathbf{u} + \nabla^T \mathbf{u})) + \text{div}(\lambda \text{div} \mathbf{u})
&= \frac{1}{\epsilon^2} \partial_\xi ((\nu + \lambda) \partial_\xi \tilde{\mathbf{u}}_0 \cdot \mathbf{n}) + \frac{1}{\epsilon^2} \partial_\xi (\nu \partial_\xi \tilde{\mathbf{u}}_0) + \frac{1}{\epsilon} \partial_\xi (\nu (\nabla \tilde{\mathbf{u}}_0 + \nabla^T \tilde{\mathbf{u}}_0) \cdot \mathbf{n}) \\
&\quad + \frac{1}{\epsilon} \text{div}(\nu \partial_\xi \tilde{\mathbf{u}}_0 \otimes \mathbf{n}) + \frac{1}{\epsilon} \nu \partial_\xi \tilde{\mathbf{u}}_0 \kappa + \frac{1}{\epsilon} \partial_\xi (\lambda \text{div} \tilde{\mathbf{u}}_0) \cdot \mathbf{n} \\
&\quad + \frac{1}{\epsilon} \nabla (\lambda \partial_\xi \tilde{\mathbf{u}}_0 \cdot \mathbf{n}) + \frac{1}{\epsilon} ((\nu + \lambda)(\partial_\xi \tilde{\mathbf{u}}_0 \cdot \nabla) \mathbf{n} + \mathcal{O}(1),
\end{align}
(2.25)

and
\begin{align}
\nabla \rho_0 &+ \epsilon \text{div}(\nabla \phi \otimes \nabla \phi) \\
&= \frac{1}{\epsilon} \partial_\xi \rho_0 \mathbf{n} - \frac{1}{\epsilon} \nabla (\tilde{\phi}_0^3 - \tilde{\phi}_0) - \frac{1}{4\epsilon^2} \partial_\xi (\tilde{\phi}_0^2 - 1)^2 \mathbf{n} + \frac{1}{4\epsilon^2} \partial_\xi (\tilde{\phi}_0 - 1)^2 \mathbf{n} \\
&\quad + \frac{1}{\epsilon} \nabla |\partial_\xi \tilde{\phi}_0|^2 \mathbf{n} + \frac{1}{\epsilon} (\nabla \tilde{\phi}_0)^2 \kappa \mathbf{n} + \frac{1}{\epsilon} \partial_\xi \tilde{\phi}_0 \nabla \tilde{\phi}_0 + \mathcal{O}(1),
\end{align}
(2.26)

where \(p_0 = \rho_0^2 \epsilon \rho_0(\tilde{\phi}_0, \tilde{\mathbf{u}}_0) + \tilde{\varphi}_0 \varphi_0(\tilde{\phi}_0, \tilde{\mathbf{u}}_0)\). Substituting \((2.25)\) and \((2.26)\) into \((1.10)_2\), we have
\begin{align}
- \frac{1}{\epsilon} \rho_0 \partial_\xi \tilde{\mathbf{u}}_0 V_n + \frac{1}{\epsilon} \rho_0 \partial_\xi \tilde{\mathbf{u}}_0 \cdot \mathbf{n} &= \frac{1}{\epsilon^2} \partial_\xi ((\nu + \lambda) \partial_\xi \tilde{\mathbf{u}}_0 \cdot \mathbf{n}) + \frac{1}{\epsilon^2} \partial_\xi (\nu \partial_\xi \tilde{\mathbf{u}}_0) + \frac{1}{\epsilon} \partial_\xi (\nu (\nabla \tilde{\mathbf{u}}_0 + \nabla^T \tilde{\mathbf{u}}_0) \cdot \mathbf{n}) \\
&\quad + \frac{1}{\epsilon} \text{div}(\nu \partial_\xi \tilde{\mathbf{u}}_0 \otimes \mathbf{n}) + \frac{1}{\epsilon} \nu \partial_\xi \tilde{\mathbf{u}}_0 \kappa + \frac{1}{\epsilon} \partial_\xi (\lambda \text{div} \tilde{\mathbf{u}}_0) \cdot \mathbf{n} \\
&\quad + \frac{1}{\epsilon} ((\nu + \lambda)(\partial_\xi \tilde{\mathbf{u}}_0 \cdot \nabla) \mathbf{n} + \frac{1}{4\epsilon^2} \partial_\xi (\tilde{\phi}_0^2 - 1)^2 \mathbf{n} \\
&\quad - \frac{1}{2\epsilon^2} \partial_\xi |\partial_\xi \tilde{\phi}_0|^2 \mathbf{n} - \frac{1}{\epsilon} \nabla |\partial_\xi \tilde{\phi}_0|^2 \mathbf{n} - \frac{1}{\epsilon} (\nabla \tilde{\phi}_0)^2 \kappa \mathbf{n} - \frac{1}{\epsilon} \partial_\xi \tilde{\phi}_0 \nabla \tilde{\phi}_0 + \mathcal{O}(1).
\end{align}
(2.27)
By comparing the coefficients of same \(\epsilon\)-order, we achieve
\[
\mathcal{O}(\epsilon^{-2}): \quad \partial_\xi ((\nu + \lambda) \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n}) + \partial_\xi (\nu \partial_\xi \hat{\mathbf{u}}_0) + \frac{1}{4} \partial_\xi (\phi_0^2 - 1)^2 \mathbf{n} - \frac{1}{2} \partial_\xi |\partial_\xi \phi_0|^2 \mathbf{n} = 0, \tag{2.28}
\]
\[
\mathcal{O}(\epsilon^{-1}): \quad - \tilde{\rho}_0 \partial_\xi \hat{\mathbf{u}}_0 V_n + \tilde{\rho}_0 \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \hat{\mathbf{u}}_0 + \partial_\xi \rho_0 \mathbf{n} \\
= \partial_\xi ((\nu (\nabla \hat{\mathbf{u}}_0 + \nabla^T \hat{\mathbf{u}}_0)) \cdot \mathbf{n} + \text{div}(\nu \partial_\xi \hat{\mathbf{u}}_0 \otimes \mathbf{n}) + \nu \partial_\xi \hat{\mathbf{u}}_0 \kappa \\
+ \partial_\xi (\lambda \text{div} \hat{\mathbf{u}}_0) \cdot \mathbf{n} + \nabla (\lambda \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n}) + (\nu + \lambda)(\partial_\xi \hat{\mathbf{u}}_0 \cdot \nabla) \mathbf{n} \\
+ \frac{1}{4} \nabla (\phi_0^2 - 1)^2 + \partial_\xi ((\phi_0^3 - \phi_0) \vec{\phi}_1) \mathbf{n} - \mathbf{n} \cdot \nabla |\partial_\xi \phi_0|^2 \mathbf{n} - |\partial_\xi \phi_0|^2 \kappa \mathbf{n} \\
- \partial_\xi \phi_0 \nabla \phi_0 + \partial_\xi ((\nu + \lambda) \partial_\xi \hat{\mathbf{u}}_1 \cdot \mathbf{n}) + \partial_\xi (\nu \partial_\xi \hat{\mathbf{u}}_1) - \partial_\xi (\partial_\xi \phi_0 \partial_\xi \phi_1) \mathbf{n}. \tag{2.29}
\]
Similarly, equation (1.10)\_3 can be rewritten in the new coordinates as
\[
- \frac{1}{\epsilon} \tilde{\rho}_0 \partial_\xi \phi_0 V_n + \frac{1}{\epsilon} \tilde{\rho}_0 \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \phi_0 = -M_\epsilon \frac{1}{\epsilon} \tilde{\mu}_0 + \mathcal{O}(1). \tag{2.30}
\]
For (C1) \(M_\epsilon = 1\), we compare the coefficients of same \(\epsilon\)-order to get
\[
\mathcal{O}(\epsilon^{-1}): \quad - \tilde{\rho}_0 \partial_\xi \phi_0 V_n + \tilde{\rho}_0 \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \phi_0 = -\tilde{\mu}_0. \tag{2.31}
\]
For (C2) \(M_\epsilon = \frac{1}{\epsilon}\), we have
\[
\mathcal{O}(\epsilon^{-2}): \quad \tilde{\mu}_0 = 0, \tag{2.32}
\]
\[
\mathcal{O}(\epsilon^{-1}): \quad - \tilde{\rho}_0 \partial_\xi \phi_0 V_n + \tilde{\rho}_0 \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \phi_0 = -\tilde{\mu}_1. \tag{2.33}
\]
Plugging ansatz (2.21) in the potential equation (1.10)\_4, we obtain
\[
\frac{1}{\epsilon} \tilde{\rho}_0 \tilde{\mu}_0 = -2(\mathbf{n} \cdot \nabla) \partial_\xi \phi_0 - \partial_\xi \phi_0 \kappa - \frac{1}{\epsilon} \partial_\xi \phi_0 + \frac{1}{\epsilon} (\phi_0^3 - \phi_0) + \mathcal{O}(\epsilon), \tag{2.34}
\]
comparing the coefficients of same \(\epsilon\)-order, we achieve
\[
\mathcal{O}(\epsilon^{-1}): \quad \tilde{\rho}_0 \tilde{\mu}_1 + \tilde{\rho}_1 \tilde{\mu}_0 = -2(\mathbf{n} \cdot \nabla) \partial_\xi \phi_0 - \partial_\xi \phi_0 \kappa - \partial_\xi \phi_1 + (3\phi_0^2 - 1) \phi_1. \tag{2.35}
\]
Finally, plugging ansatz (2.21) in the energy equation (1.10)\_5. For (C1) \(M_\epsilon = 1\), we compare the coefficients of same \(\epsilon\)-order to get
\[
\mathcal{O}(\epsilon^{-2}): \quad k \partial_{\xi \xi} \tilde{\theta}_0 + \nu |\partial_\xi \hat{\mathbf{u}}_0|^2 + (\nu + \lambda) |\partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n}|^2 + \tilde{\mu}_0 = 0, \tag{2.36}
\]
\[
\mathcal{O}(\epsilon^{-1}): \quad e_\theta(\tilde{\rho}_0, \tilde{\theta}_0)(-\tilde{\rho}_0 \partial_\xi \tilde{\theta}_0 V_n + \tilde{\rho}_0 \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \tilde{\theta}_0) + \tilde{\theta}_0 p_0(\tilde{\rho}_0, \tilde{\theta}_0) \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n} \\
= 2k(\mathbf{n} \cdot \nabla) \partial_\xi \tilde{\theta}_0 + \kappa \partial_\xi \phi_0 \kappa + k \partial_\xi \phi_1 + 2\nu \text{div} \tilde{\mathbf{u}}_0 \otimes \mathbf{n} + 2\lambda \text{div} \tilde{\mathbf{u}}_0 \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n} \\
+ 2\nu \partial_\xi \tilde{\mathbf{u}}_0 \cdot \partial_\xi \hat{\mathbf{u}}_1 + 2(\nu + \lambda) \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \hat{\mathbf{u}}_1 \cdot \mathbf{n} + 2\tilde{\mu}_0 \tilde{\mu}_1. \tag{2.37}
\]
For (C2) \(M_\epsilon = \frac{1}{\epsilon}\), we have
\[
\mathcal{O}(\epsilon^{-3}): \quad \tilde{\mu}_0 = 0, \tag{2.38}
\]
\[
\mathcal{O}(\epsilon^{-2}): \quad k \partial_{\xi \xi} \tilde{\theta}_0 + \nu |\partial_\xi \hat{\mathbf{u}}_0|^2 + (\nu + \lambda) |\partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n}|^2 + 2\tilde{\mu}_0 \tilde{\mu}_1 = 0, \tag{2.39}
\]
\[
\mathcal{O}(\epsilon^{-1}): \quad e_\theta(\tilde{\rho}_0, \tilde{\theta}_0)(-\tilde{\rho}_0 \partial_\xi \tilde{\theta}_0 V_n + \tilde{\rho}_0 \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \tilde{\theta}_0) + \tilde{\theta}_0 p_0(\tilde{\rho}_0, \tilde{\theta}_0) \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n} \\
= 2k(\mathbf{n} \cdot \nabla) \partial_\xi \tilde{\theta}_0 + \kappa \partial_\xi \phi_0 \kappa + k \partial_\xi \phi_1 + 2\nu \text{div} \tilde{\mathbf{u}}_0 \otimes \mathbf{n} + 2\lambda \text{div} \tilde{\mathbf{u}}_0 \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n} \\
+ 2\nu \partial_\xi \tilde{\mathbf{u}}_0 \cdot \partial_\xi \hat{\mathbf{u}}_1 + 2(\nu + \lambda) \partial_\xi \hat{\mathbf{u}}_0 \cdot \mathbf{n} \partial_\xi \hat{\mathbf{u}}_1 \cdot \mathbf{n} + \tilde{\mu}_1^2 + 2\tilde{\mu}_0 \tilde{\mu}_2. \tag{2.40}
\]
Noting that the matching conditions for inner and outer expansions are as follows,

$$\lim_{\xi \to \pm \infty} \tilde{g}_0(x, \xi) = g_0^\pm(x), \quad (2.42)$$

$$\lim_{\xi \to \pm \infty} (\nabla_x \tilde{g}_0(x, \xi) + \partial_\xi \tilde{g}_1(x, \xi)n) = \nabla g_0^\pm(x). \quad (2.43)$$

**Zeroth approximation.** We consider the zeroth approximation in the inner expansion:

$$- \partial_\xi \tilde{\rho}_0 V_n + \partial_\xi (\tilde{\rho}_0 \tilde{u}_0) \cdot n = 0, \quad (2.44)$$

$$\partial_\xi ((\nu + \lambda) \partial_\xi \tilde{u}_0 \cdot n) + \partial_\xi (\nu \partial_\xi \tilde{u}_0) + \frac{1}{4} \partial_\xi (\tilde{\phi}_0^2 - 1)^2 n - \frac{1}{2} \partial_\xi |\partial_\xi \tilde{\phi}_0|^2 n = 0, \quad (2.45)$$

$$\chi_M \tilde{\rho}_0 \tilde{\phi}_0 (V_n - \tilde{u}_0 \cdot n) = \tilde{\mu}_0, \quad (2.46)$$

$$\tilde{\rho}_0 = - \partial_\xi \tilde{\phi}_0 + \tilde{\phi}_0^n - \tilde{\phi}_0, \quad (2.47)$$

$$k \partial_\xi \tilde{\theta}_0 + \nu |\partial_\xi \tilde{u}_0|^2 + (\nu + \lambda) |\partial_\xi \tilde{u}_0 \cdot n|^2 + \chi_M \tilde{\rho}_0^2 = 0, \quad (2.48)$$

with boundary conditions:

$$\tilde{\rho}_0(x, t, \pm \infty) = \rho_0^\pm(x, t), \quad (2.49)$$

$$\tilde{u}_0(x, t, \pm \infty) = u_0^\pm(x, t), \quad (2.50)$$

$$\tilde{\phi}_0(x, t, \pm \infty) = \pm 1, \quad (2.51)$$

$$\tilde{\theta}_0(x, t, \pm \infty) = \theta_0^\pm(x, t), \quad (2.52)$$

where $\chi_M$ is defined by (2.9).

**Lemma 2.2.** Let $(\tilde{\rho}_0, \tilde{u}_0, \tilde{\phi}_0, \tilde{\theta}_0)$ be a solution of (2.44)-(2.51). It holds

$$\tilde{\rho}_0(t, x, \xi) (V_n - \tilde{u}_0 \cdot n)(t, x, \xi) = \rho_0^\pm (x, t) (V_n - u_0^\pm \cdot n)(x, t), \quad (2.53)$$

$$\tilde{\phi}_0(t, x, \xi) = \tanh \left( \frac{\xi}{\sqrt{2}} \right), \quad (2.54)$$

$$\partial_\xi \tilde{u}_0 = 0, \quad \partial_\xi \tilde{\theta}_0 = 0, \quad (2.55)$$

$$\left\{ \begin{array}{l} V_n - \tilde{u}_0 \cdot n = 0, \quad \text{for (C1) } M_\epsilon = 1, \\ \partial_\xi \tilde{\rho}_0 (V_n - \tilde{u}_0 \cdot n) = 0, \quad \text{for (C2) } M_\epsilon = \frac{1}{2}. \end{array} \right. \quad (2.56)$$

**Proof.** From equation (2.44) we have

$$\partial_\xi (\tilde{\rho}_0 (V_n - \tilde{u}_0 \cdot n)) = 0, \quad (2.57)$$

and by integration we obtain (2.53). By multiplying (2.46) with $\partial_\xi \tilde{\phi}_0$, and combining with (2.47), we obtain

$$- \chi_M (V_n - \tilde{u}_0 \cdot n)|\partial_\xi \tilde{\phi}_0|^2 = \frac{1}{\rho_0^2} \left( \frac{1}{2} \partial_\xi |\partial_\xi \tilde{\phi}_0|^2 - \frac{1}{4} \partial_\xi (\tilde{\phi}_0^2 - 1)^2 \right). \quad (2.58)$$

Multiplying equation (2.45) by the norm vector $n$, we infer that

$$\partial_\xi ((\lambda + 2\nu) \partial_\xi \tilde{u}_0 \cdot n) + \frac{1}{4} \partial_\xi (\tilde{\phi}_0^2 - 1)^2 - \frac{1}{2} \partial_\xi |\partial_\xi \tilde{\phi}_0|^2 = 0. \quad (2.59)$$

By integrating equation (2.59) from $-\infty$ to $\xi$, we obtain

$$(\lambda + 2\nu) \partial_\xi \tilde{u}_0 \cdot n + \frac{1}{4} (\tilde{\phi}_0^2 - 1)^2 - \frac{1}{2} |\partial_\xi \tilde{\phi}_0|^2 = 0. \quad (2.60)$$
Multiplying equation (2.60) by $\frac{\partial}{\partial \xi}(\frac{1}{\tilde{\rho}_0})$, combining with (2.44), we infer that

$$-2(\lambda + 2\nu)\tilde{h}_0^{-4}(V_n - \tilde{u}_0 \cdot n)|\partial_\xi \tilde{h}_0|^2 = \left(\frac{1}{2} |\partial_\xi \tilde{h}_0|^2 - \frac{1}{4}(\tilde{h}_0^2 - 1)^2\right) \frac{\partial}{\partial \xi}(\frac{1}{\tilde{\rho}_0}).$$

(2.61)

Combining (2.58) with (2.61), and integrating them from $-\infty$ to $+\infty$, we obtain

$$\int_{-\infty}^{+\infty} (V_n - \tilde{u}_0 \cdot n)(\chi_M |\partial_\xi \tilde{h}_0|^2 + 2(\lambda + 2\nu)\tilde{h}_0^{-4}|\partial_\xi \tilde{h}_0|^2) \, d\xi = 0.$$  

(2.62)

Since $\tilde{\rho}_0 > 0$, $\lambda + 2\nu > 0$, and (2.9), combining (2.62) and (2.53) we obtain (2.56). Then from (2.46) and (2.58), for both (C1) and (C2), we obtain

$$\tilde{\mu}_0 = 0, \quad \frac{1}{2} |\partial_\xi \tilde{h}_0|^2 = \frac{1}{4}(\tilde{h}_0^2 - 1)^2.$$  

(2.63)

Then we have

$$\frac{\partial_\xi \tilde{h}_0}{\sqrt{2}} = \frac{1}{\tilde{\rho}_0^2}.$$  

(2.64)

Noticing $\tilde{h}_0(x, t, 0) = 0$, we obtain (2.54). Since $\lambda + 2\nu > 0$, combining (2.60) and (2.63), we have

$$\partial_\xi \tilde{h}_0 \cdot n = 0.$$  

(2.65)

Multiplying equation (2.29) by the tangential vector $\tau$, we infer that

$$\partial_\xi (\lambda \partial_\xi \tilde{h}_0) \cdot \tau = 0.$$  

(2.66)

Since $\lambda > 0$, by integration from $-\infty$ to $\xi$, we have

$$\partial_\xi \tilde{u}_0 \cdot \tau = 0.$$  

(2.67)

By integrating equation (2.48) from $-\infty$ to $\xi$, combining with (2.65) and (2.63), we obtain

$$k \partial_\xi \tilde{h}_0 = 0.$$  

(2.68)

Since $k > 0$, combining (2.65) and (2.67), we derive (2.55). The proof is completed.  

\textbf{Remark 2.1.} For (C1) $M_\xi = 1$, we have $V_n - \tilde{u}_0 \cdot n = 0$ on $\Gamma$. For (C2) $M_\xi = \frac{1}{\varepsilon}$ we introduce the set $S \subset \Gamma$ by

$$\rho^+(x, t) = \rho^-(x, t), \quad \text{on } S \times (-\infty, +\infty).$$  

(2.69)

From (2.56), it implies that

$$\rho^+(x, t) = \rho^-(x, t), \quad \text{on } \Gamma \setminus S \times (-\infty, +\infty).$$  

(2.70)

\textit{First approximation.} We derive the system from the first order asymptotic analysis by the equations (2.24), (2.29), (2.36) and (2.38). By (2.55), the equations are simplified to the following system:

$$\partial_\xi \tilde{h}_0 + \text{div}(\tilde{h}_0 \tilde{u}_0) - \partial_\xi \tilde{h}_0^2 \tilde{u}_1 - \partial_\xi \tilde{h}_0 \tilde{u}_0 \cdot n + \partial_\xi (\tilde{h}_0 \tilde{u}_0 \tilde{u}_1 \cdot n) = 0,$$  

(2.71)

$$-\partial_\xi \tilde{h}_0 n + \frac{1}{4} \nabla(\tilde{h}_0^2 - 1)^2 + \partial_\xi ((\tilde{h}_0^2 - \tilde{h}_0^2) \tilde{h}_1^2) n + \partial_\xi ((\nu + \lambda) \partial_\xi \tilde{u}_1 \cdot n) n + \partial_\xi (\nu \partial_\xi \tilde{u}_1)$$

$$= \partial_\xi (\partial_\xi \tilde{h}_0 \tilde{h}_0 \tilde{h}_1) n + n \cdot \nabla |\partial_\xi \tilde{h}_0|^2 n + |\partial_\xi \tilde{h}_0|^2 \kappa n + \partial_\xi \xi \nabla \tilde{h}_0, \quad \text{on } S \times (-\infty, +\infty).$$  

(2.72)

$$\tilde{h}_0 \tilde{u}_1 = -2(n \cdot \nabla) \partial_\xi \tilde{h}_0 - \partial_\xi \tilde{h}_0 \kappa - \partial_\xi \xi \nabla \tilde{h}_0 + (3 \tilde{h}_0^2 - 1) \tilde{h}_0,$$  

(2.73)
k\partial_{\xi\xi}\bar{\rho}_1 = 0, \quad (2.74)

and for (C2) \(M_\epsilon = \frac{1}{\epsilon}\) we have

\[ \bar{\rho}_0 \partial_{\xi} \bar{\phi}_0 (V_n - \bar{u}_0 \cdot n) = \bar{\mu}_1. \quad (2.75) \]

**Lemma 2.3.** It holds

\[ [\mathcal{S}_0 - p_0 \mathbb{I}] n]_\Gamma = \sigma \kappa n, \quad \text{on } \Gamma, \quad (2.76) \]

where \( \mathcal{S}_0^\pm = 2\nu \mathbb{D} u_0^\pm + \lambda \text{div} u_0^\pm \mathbb{I}, \ p_0^\pm = (\rho_0^\pm)^2 e_p (\rho_0^\pm, \theta_0^\pm) + \theta_0^\pm p_0 (\rho_0^\pm, \theta_0^\pm). \)

**Proof.** Since \( \bar{\phi}_0 \) is given by (2.54), then we simplify equation (2.72) to

\[
- \partial_\xi p_0 n + \partial_\xi (\lambda + \nu) \partial_\xi \bar{u}_1 + \nu (\partial_\xi \bar{u}_1 \cdot n)n = \partial_\xi (\partial_\xi \bar{\phi}_0 (2.77). \]

We integrate (2.77) from \( \xi = -\infty \) to \( \xi = +\infty \), and the first and third summand on the right-hand-side vanish. The first summand on the left-hand-side of (2.77) turns into

\[
- \int_{-\infty}^{\infty} \partial_\xi p_0 \nu d\xi = -[p(\rho_0^+, \theta_0^+) - p(\rho_0^-, \theta_0^-)] n = -[p(\rho_0, \theta_0)] n. \quad (2.78) \]

From the matching condition (2.42)-(2.43), we obtain

\[
\lim_{\xi \to \pm\infty} \partial_\xi u_1 = n \cdot \nabla u_0. \quad (2.79) \]

For the second summand on the left-hand-side of (2.77) we have

\[
\int_{-\infty}^{\infty} \partial_\xi ((\nu + \lambda) \partial_\xi \bar{u}_1 \cdot n)n + \nu \partial_\xi \bar{u}_1 d\xi = [\mathcal{S}_0] n. \quad (2.80) \]

Combining the equations above together we conclude

\[
-[p(\rho_0, \theta_0)] n + [\mathcal{S}_0] n = \int_{-\infty}^{\infty} |\partial_\xi \bar{\phi}_0|^2 \kappa n d\xi = \sigma \kappa n, \quad (2.81) \]

where

\[
\sigma = \int_{-\infty}^{\infty} |\partial_\xi \bar{\phi}_0|^2 d\xi = \frac{2}{3} \sqrt{2}. \quad (2.82) \]

then it follows (2.76). \qed

For (C1) \( M_\epsilon = 1 \), from Lemma 2.2 and Lemma 2.3, we obtain the following jump conditions on the free interface \( \Gamma \).

**Lemma 2.4.** Letting \( \epsilon \to 0 \) in ansatz (2.21) for (C1) \( M_\epsilon = 1 \), it holds

\[ |u|_\Gamma = 0, \quad [\theta]_\Gamma = 0, \quad (2.83) \]

\[ V_n - u \cdot n = 0, \quad (2.84) \]

\[ [(2\nu \mathbb{D} u + \lambda \text{div} u \mathbb{I} - (\rho^2 \frac{\partial e}{\partial p} + \theta \frac{\partial p}{\partial \theta}) \mathbb{I}) n]_\Gamma = \sigma \kappa n, \quad (2.85) \]
For \((C^2)\) \(M_\epsilon = \frac{1}{\epsilon}\), we derive the equation (2.75) to obtain the following lemma.

**Lemma 2.5.** It holds

\[
\tilde{\rho}_0(V_n - u_0 \cdot n) \int_{-\infty}^{+\infty} \tilde{\rho}_0 |\partial_\xi \tilde{\phi}_0|^2 d\xi + \sigma \kappa = 0. \tag{2.86}
\]

**Proof.** Combining (2.73) and (2.75), we have

\[
-\tilde{\rho}_0^2(V_n - u_0 \cdot n) \partial_\xi \tilde{\phi}_0 = 2(n \cdot \nabla) \partial_\xi \tilde{\phi}_0 + \partial_\xi \tilde{\phi}_0 \kappa + \partial_\xi \tilde{\phi}_1 - (3 \tilde{\phi}_0^2 - 1) \tilde{\phi}_1, \tag{2.87}
\]

multiplying (2.87) by \(\partial_\xi \tilde{\phi}_0\) and integrating from \(\xi = -\infty\) to \(\xi = +\infty\), we obtain

\[
- \int_{-\infty}^{+\infty} \tilde{\rho}_0^2(V_n - u_0 \cdot n)|\partial_\xi \tilde{\phi}_0|^2 d\xi
= \int_{-\infty}^{+\infty} |\partial_\xi \tilde{\phi}_0|^2 \kappa + \partial_\xi \tilde{\phi}_1 \partial_\xi \tilde{\phi}_0 - (3 \tilde{\phi}_0^2 - 1) \tilde{\phi}_1 \partial_\xi \tilde{\phi}_0 d\xi. \tag{2.88}
\]

From (2.35), (2.82) and (2.53), we have

\[
- \tilde{\rho}_0(V_n - u_0 \cdot n) \int_{-\infty}^{+\infty} \tilde{\rho}_0 |\partial_\xi \tilde{\phi}_0|^2 d\xi
= \int_{-\infty}^{+\infty} |\partial_\xi \tilde{\phi}_0|^2 \kappa + \int_{-\infty}^{+\infty} (\partial_\xi \tilde{\phi}_1 \partial_\xi \tilde{\phi}_0 - (3 \tilde{\phi}_0^2 - 1) \tilde{\phi}_1 \partial_\xi \tilde{\phi}_0) d\xi
= \sigma \kappa - \int_{-\infty}^{+\infty} \partial_\xi \tilde{\phi}_1 (- \partial_\xi \tilde{\phi}_0 + (\tilde{\phi}_0^3 - \tilde{\phi}_0)) d\xi = \sigma \kappa, \tag{2.89}
\]

that is (2.86).

For \((C^2)\) \(M_\epsilon = \frac{1}{\epsilon}\), from Lemma 2.2, Lemma 2.3, Lemma 2.5 and Remark 2.1, we obtain the following jump conditions on the free interface \(\Gamma\).

**Lemma 2.6.** Letting \(\epsilon \to 0\) in ansatz (2.21) for \((C^2)\) \(M_\epsilon = \frac{1}{\epsilon}\), it holds

\[
\begin{align*}
[u]_\Gamma &= 0, \quad [\theta]_\Gamma = 0, \quad [\rho]_\Gamma = 0, \\
[(2\nu D_u + \lambda \text{div} u) n + (\rho^2 \frac{\partial e}{\partial p} + \theta \frac{\partial p}{\partial \theta}) n]_\Gamma &= \sigma \kappa n, \quad \text{on } S, \\
\rho^2(V_n - u \cdot n) &= -\kappa,
\end{align*}
\]

\[
\begin{align*}
[u]_\Gamma &= 0, \quad [\theta]_\Gamma = 0, \\
V_n - u \cdot n &= 0, \\
[(2\nu D_u + \lambda \text{div} u) n - (\rho^2 \frac{\partial e}{\partial p} + \theta \frac{\partial p}{\partial \theta}) n]_\Gamma &= 0, \quad \text{on } \Gamma \setminus S. \tag{2.91}
\end{align*}
\]

**Proof of Theorem 1.1.** We analyze the limit process by the matched asymptotic expansion. By applying the results of Lemma 2.1, Lemma 2.4 and Lemma 2.6, the system (1.10) converges to the sharp interface problem (1.15) with the free interface (1.16)-(1.17).
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