Azimuthal asymmetry of transversely polarized \( \Lambda \) hyperon production in \( e^+e^- \rightarrow \Lambda^\mp \pi X \) within TMD factorization

Hui Li,\(^1\) Xiaoyu Wang,\(^2,^*\) Yongliang Yang,\(^3\) and Zhun Lu\(^{1,†} \)

\(^1\)School of Physics, Southeast University, Nanjing 211189, China
\(^2\)School of Physics and Microelectronics, Zhengzhou University, Zhengzhou, Henan 450001, China
\(^3\)College of Physics, Qingdao University, Qingdao 266071, China

We investigate the single transverse-spin asymmetry with a \( \sin(\phi - \phi_S) \) modulation in the transversely polarized \( \Lambda \) hyperon production process \( e^+e^- \rightarrow \Lambda^\mp \pi X \) within the framework of the transverse momentum dependent (TMD) factorization. The asymmetry is contributed by the convolution of the transversely polarized fragmentation function \( D_{1T} \) and the unpolarized fragmentation function \( D_1 \). We adopt the spectator diquark model result for \( D_{1T} \) to numerically estimate the \( \sin(\phi - \phi_S) \) asymmetry in \( e^+e^- \rightarrow \Lambda^\mp \pi X \) process at the kinematical region of Belle and BaBar Collaboration. We also apply the recent parameterized result for \( D_{1T} \) to perform the calculation as a comparison. To implement the TMD evolution formalism of the fragmentation functions, we use the nonperturbative Sudakov form factor associated with the fragmentation functions of the \( \Lambda \) and the pion. It is found that our prediction on the \( \sin(\phi - \phi_S) \) asymmetry as functions of \( P_{hT} \), \( z_1 \) and \( z_2 \) is sizable and could be measured at Belle and BaBar.

I. INTRODUCTION

Understanding the origin of the single-spin asymmetry in semi-inclusive process involving hadrons is one of the main goals in QCD and spin physics. Particularly, the production of a polarized \( \Lambda \) hyperon from unpolarized \( pp \) collisions has been observed \([1, 2]\) and it formed a long-standing challenge \([3, 4]\) in high energy physics. It is suggested \([5]\) that a polarized fragmentation function \([6]\), denoted by \( D_{1T}^\pm \), can account for the polarization of the \( \Lambda \) production. As a time-reversal-odd and transverse momentum dependent (TMD) fragmentation function, \( D_{1T}^\pm \) describes the fragmentation of an unpolarized quark to a transversely polarized hadron, and it is usually viewed as the analog of the Sivers function which gives the azimuthal asymmetry in the distribution of unpolarized quarks inside a transversely polarized nucleon. Furthermore, \( D_{1T}^\pm \) may play an important role in the spontaneous polarization, such as: \( q \rightarrow \Lambda^\mp X \) \([7]\). Thus, the study on the production of polarized \( \Lambda \) could also provide the information on the spin structure of the hyperon. This is intriguing since the \( \Lambda \) hyperon can not serve as a target in high energy scattering processes.

As \( D_{1T}^\pm \) is a chiral-even function, it can be accessed directly without any unknown, chiral-odd, counterpart. However, the single inclusive \( e^+e^- \) annihilation (SIA) experiment performed by OPAL at LEP has not observed significant signal on the transverse polarization of the \( \Lambda \) hyperon \([8]\). As an alternative to SIA, the processes \( e^+e^- \rightarrow \Lambda^\mp + h + X \) \([9–11]\) and semi-inclusive deep inelastic scattering (SIDIS) \( \ell p \rightarrow \ell' + \Lambda^\mp + X \) have been suggested \([9]\) to study the \( \Lambda \) polarization, where \( D_{1T}^\pm \) contribute to the spin asymmetries. Those measurement could provide a further understanding of the origin of the sizable transverse polarisation of hyperons observed in different processes \([2, 5, 6, 12–15]\). Recently, a nonzero transverse polarization of \( \Lambda \) production in SIA and semi-inclusive \( e^+e^- \rightarrow \Lambda(\bar{\Lambda}) + K^\mp (\pi^\mp) + X \) process was measured by the Belle Collaboration \([11]\), making the extraction \([16, 17]\) of the polarized fragmentation function of the \( \Lambda \) possible. It is worth pointing out that, since no hadrons exist in the initial state, electron-positron annihilation is a clean process to access TMD fragmentation functions. On the other hand, model calculations may also provide an approach to acquire knowledge of this quantity. A calculation of \( D_{1T}^\pm \) for light flavors based on a spectator-diquark model has been performed in Ref. \([18]\) and was used to make predictions on the physical observable.

The main purpose of this work is to apply the TMD factorization \( (P_{hT}/z_h \ll Q) \) \([19–23]\) to estimate the spin asymmetry in reaction \( e^+e^- \rightarrow \Lambda^\mp + \pi + X \). In this process, the convolution of the polarized fragmentation function of the \( \Lambda \) hyperon and the pion unpolarized fragmentation function can give rise to a \( \sin(\phi - \phi_S) \) asymmetry with \( \phi_S \) the azimuthal angle of \( \Lambda \) hyperon transverse spin. In the last decades, TMD factorization has been applied in various high energy processes \([23–33]\). Particularly, we take into account the TMD evolution for both \( D_{1T}^\pm \) and the unpolarized fragmentation function \( D_1 \). In the TMD formalism, the differential cross section in the region \( P_{hT}/z_h \ll Q \) can be expressed as the convolution of the hard scattering factors and the well-defined TMD distributions or fragmentation functions. The TMD formalism embeds evolution information of those functions, of which the energy evolution (or

\(^*\)Electronic address: xiaoyuwang@zzu.edu.cn
\(^†\)Electronic address: zhunlu@seu.edu.cn
the scale dependence) are governed by the so-called Collins-Soper equation [19, 20, 23, 34]. The solution of the evolution equation indicates that the changes of TMDs from a initial scale to another scale may be determined by an exponential form of the Sudakov-like form factor [20, 23, 26, 35], which can be separated to the perturbative part and nonperturbative part. The former one is perturbatively calculable, while the later one can not be calculated directly and is usually obtained by phenomenologial extraction from experimental data. In Refs. [26, 30, 32, 36] the authors extracted the nonperturbative Sudakov form factor corresponding to the unpolarized fragmentation function. In this work, we adopt two different parameterizations on the nonperturbative part for the fragmentation functions [30, 32].

The remaining content of the paper is organized as follows. In Sec. II, we present the formalism of the \(\sin(\phi - \phi_S)\) asymmetry contributed by the convolution of \(D_{1T}^\perp\) and the unpolarized fragmentation function \(D_1\) in process \(e^+e^- \rightarrow \Lambda^\mp + \pi + X\) process within TMD factorization. In Sec. III, we investigate the evolution effect for the unpolarized and transversely polarized fragmentation functions at leading order and present our choice on the nonperturbative Sudakov form factors associated with the fragmentation functions in details. In Sec. IV, we numerically estimate the \(\sin(\phi - \phi_S)\) asymmetry at the energy \(\sqrt{s} = 10.52\) GeV which is accessible at Belle and BaBar. We also compare the results calculated from different choices of the nonperturbative ingredients associated with the TMD evolution as well as different sets of \(D_{1T}^\perp\). Finally, We summarize the paper in Sec. V.

II. \(\sin(\phi - \phi_S)\) ASYMMETRY IN \(e^+e^- \rightarrow \Lambda^\mp + \pi + X\) PROCESS

In this section, we will present the detailed framework of the \(\sin(\phi - \phi_S)\) asymmetry in \(e^+e^-\) annihilation process, in which a transversely polarized \(\Lambda\) hyperon and a pion meson are produced in the final state:

\[
e^+(\ell) + e^-(\ell') \rightarrow \gamma^*(q) + X \rightarrow q(k) + \bar{q}(p) + X \rightarrow \Lambda^\uparrow(K) + \pi^0(P) + X.
\]

Here, the electron with momentum \(\ell\) and the positron with momentum \(\ell'\) annihilate into a virtual photon with momentum \(q = \ell + \ell'\) decaying to a quark-antiquark pair, which eventually fragment into the transversely polarized \(\Lambda\) hyperon and the pion meson. Here, \(X\) stands for additional undetected final state particles, \(\uparrow\) denotes the transverse polarization of the \(\Lambda\) hyperon. We should note that the virtual photon is timelike, which means \(q^2 = Q^2 > 0\). \(K\) and \(q\) are the four-momenta of the quark and antiquark, respectively, while \(K\) and \(P\) are the four-momenta of the final-state \(\Lambda\) and pion. The center of mass energy for the process can be written as \(s = (\ell + \ell')^2 = Q^2\). The invariants \(z_1\) and \(z_2\) are defined as \(z_1 = \frac{2Kq}{Q^2}\) and \(z_2 = \frac{2Pq}{Q^2}\), which can further be identified as the momentum fractions in fragmentation function of the \(\Lambda\) and the \(\pi\) meson, respectively.

In the ideal case, the transversely polarized \(\Lambda\) and the pion meson should be produced completely back-to-back. However, the radiation of the gluon in the fragmentation process and the transverse momentum dependence make the hadrons deviate from the ideal back-to-back state. The TMD factorization can be used to describe the imbalance from the back-to-back state as well as calculate the differential cross section. There are two experimental methods to define the reference frame in the \(e^+e^-\) annihilation process in the literature [24, 37–39]. In this work, we adopt the second-hadron momentum frame, which means that the momentum of second hadron-pion meson is defined as \(z\) axis, the \(xz\) plane is determined by the lepton and the pion meson momentum directions, the hadron plane is determined by \(z\) axis and the momentum direction of the \(\Lambda\) hadron. Hence \(\phi\) is defined as the azimuthal angle of the hadron plane relative to the lepton plane, while \(\phi_S\) represents the azimuthal angle of the \(\Lambda\) hyperon polarization vector \(S_\Lambda\) in the lepton frame. \(q_T\) is the transverse momentum of the virtual photon, with \(k_T\) and \(p_T\) the transverse momenta of the two fragmenting quarks, which is related to the transverse momenta of the final hadrons through \(K_\perp = -z_1k_T\) and \(P_\perp = -z_2p_T\). Finally, \(P_{h\perp} = -z_1q_T\) is the transverse momentum of \(\Lambda\) hadron in hadron frame.

The \(\sin(\phi - \phi_S)\) asymmetry shows up at leading twist in the differential cross section according to the TMD factorization [9, 24]:

\[
d\sigma(e^+e^- \rightarrow h_1h_2X) = \frac{3\alpha^2}{Q^2} z_1z_2A(y) \left\{ F[D_1D_1] - A(y)\right\} \frac{\sin(\phi - \phi_S)\mathcal{F}[\hat{h} \cdot k_T\hat{D}_{1T}D_{1T}]}{M_\Lambda}\right\} \right\}.
\]

where \(M_\Lambda\) is the mass of the \(\Lambda\) hyperon and the unit vector \(\hat{h}\) is defined as \(\hat{h} = \frac{P_T}{|P_T|} = \frac{q_T}{q_T} [25, 40]\). The single transverse-spin asymmetry with a \(\sin(\phi - \phi_S)\) modulation can be written as

\[
A_{UT}^{\sin(\phi - \phi_S)} = \frac{\mathcal{F}[\hat{h} \cdot k_T\hat{D}_{1T}D_{1T}]}{F[D_1D_1]}.
\]

where the notation \(\mathcal{F}\) represents the convolution

\[
\mathcal{F} = \sum_q e_q^2 \int d^2k_Td^2p_T\delta^2(q_T - k_T)\omega(p_T, k_T)D_A/(q(z_1, z_2, k_T)\hat{D}_{1\perp}D_{1\perp}(z_2, z_2, p_T)\),
\]
with $\omega(p_T, k_T)$ an arbitrary function of $p_T$ and $k_T$. Since it is convenient to deal with the TMD evolution effect in the $b$ space that is conjugate to the transverse momentum space, one can perform the Fourier transformation for the delta function

$$\delta^2(q_T - p_T - k_T) = \frac{1}{(2\pi)^2} \int d^2b_\perp e^{-ib_\perp \cdot (q_T - p_T - k_T)}$$

(5)

to obtain the denominator in Eq. (3) as

$$\mathcal{F}[D_1 D_1] = \sum_q e_q^2 \int d^2k_T d^2p_T \delta^2(q_T - p_T - k_T) D_1^{\Lambda/q}(z_1, z_1^2 k_T^2; Q) \tilde{D}_1^{\pi/q}(z_2, z_2^2 p_T; Q)$$

$$= \frac{1}{z_1^2 z_2^2} \sum_q e_q^2 \int d^2k_\perp d^2P_\perp \delta^2(-P_{h,1}/z_1 + P_\perp/z_1 + K_\perp/z_2) D_1^{\Lambda/q}(z_1, K_\perp^2; Q) \tilde{D}_1^{\pi/q}(z_2, P_\perp^2; Q)$$

$$= \frac{1}{z_1^2 z_2^2} \sum_q e_q^2 \int \frac{d^2b}{(2\pi)^2} e^{ib P_{h,1}/z_1} \tilde{D}_1^{\Lambda/q}(z_1, b; Q) \tilde{D}_1^{\pi/q}(z_2, b; Q)$$

$$= \frac{1}{z_1^2 z_2^2} \sum_q e_q^2 \int_0^{\infty} \frac{dbb}{(2\pi)^2} J_0(\frac{b}{z_1}) \tilde{D}_1^{\Lambda/q}(z_1, b; Q) \tilde{D}_1^{\pi/q}(z_2, b; Q),$$

(6)

where the definition of the unpolarized fragmentation function in $b$ space is adopted as (hereafter the tilde terms represent the ones in $b$ space)

$$\tilde{D}_1^{\Lambda/q}(z, b; Q) = \int d^2P_\perp e^{-iP_{\perp,b}} D_1^{\Lambda/q}(z, P_\perp^2; Q),$$

(7)

with $P_\perp$ the transverse component of the corresponding hadron with respect to the fragmenting quark momentum. The numerator can be analyzed similarly:

$$\mathcal{F}[\hat{h} \cdot k_T \frac{D_1^{\perp T}}{M_\Lambda}]$$

$$= \frac{1}{z_1^2 z_2^2} \sum_q e_q^2 \int d^2k_\perp d^2P_\perp \delta^2(-P_{h,1}/z_1 + P_\perp/z_1 + K_\perp/z_2) (\frac{\hat{h} \cdot K_\perp}{z_1 M_\Lambda}) D_1^{\perp T/q}(z_1, K_\perp^2; Q) \tilde{D}_1^{\pi/q}(z_2, P_\perp^2; Q)$$

$$= \frac{1}{z_1^3 z_2^3} \sum_q e_q^2 \int \frac{d^2b}{(2\pi)^2} e^{ib P_{h,1}/z_1} \hat{h}_\alpha \tilde{D}_1^{\perp T/q(\alpha)}(z_1, b; Q) \tilde{D}_1^{\pi/q}(z_2, b; Q),$$

(8)

with the polarized fragmentation function of $\Lambda$ hyperon defined as

$$\tilde{D}_1^{\perp T/q(\alpha)}(z, b; Q) = \int d^2P_\perp e^{-iP_{\perp,b}} \frac{P_\perp^2}{M_\Lambda} D_1^{\perp T/q}(z, P_\perp^2; Q)$$

(9)

The energy dependence of the fragmentation functions here will be discussed in details in the following section.

III. THE TMD EVOLUTION OF FRAGMENTATION FUNCTIONS

In this section, we set up the formalism of the TMD evolution for the fragmentation functions. Generally, the TMD evolution is performed in the $b$ space since the cross section can be written as the production instead the complicated convolution of the related fragmentation functions. There are two energy dependencies in the TMD fragmentation functions $\tilde{D}(z, b; \mu, \zeta_D)$, one is $\mu$, which denotes the renormalization scale related to the corresponding collinear fragmentation functions, and the other one $\zeta_D$ is the scale related to the cutoff in the definition of the TMD operators to regularize the singularity [19, 20, 23, 26, 28, 36]. Hereafter, we set $\mu = \sqrt{\zeta_D} = Q$ for simplicity, thus the TMD fragmentation functions can be written as $\tilde{D}(z, b; Q)$. Our main focus is $D(z, P_\perp; Q)$ [23, 41] in which $z$ is the collinear momentum fraction and $P_\perp$ is transverse component of the momentum. Since it is simpler to solve the evolution formalism in the coordinate space (conjugate to momentum space $P_\perp$), we define the Fourier transform of $D(z, P_\perp; Q)$ in the two-dimensional impact space (referred to as b-space below) as $|b_\perp|$

$$\tilde{D}(z, b; Q) = \int d^2P_\perp e^{-iP_{\perp,b}} D(z, P_\perp; Q).$$

(10)
The two energy scale dependencies $\mu$ and $\zeta_D$ can be encoded in the TMD evolution equations as the renormalization group equation and the Collins-Soper equation, respectively. After solving the evolution equations, the solutions are identical to each other in all TMD factorization schemes, and the evolution effects are included in the exponential form factors [19–23, 42] as

$$\tilde{D}(z, b; Q) = D \times e^{-S(Q,b)} \times \tilde{D}(z, b, \mu_i).$$

(11)

Where $D$ is the hard scattering factor, $S(Q, b)$ is the Sudakov form factor. Eq. (11) demonstrates that the TMD fragmentations $D$ at an arbitrary scale $Q$ can be evolved from an initial scale $\mu_i$ through the evolution encoded by the exponential form $\exp(-S(Q,b))$. The function $\tilde{D}(z, b; Q)$ can represent any TMD functions in the $b$-space. The relevant ones in this work are the unpolarized fragmentation function, and the $P_\perp$-weighted transversely polarized fragmentation function of the $\Lambda$ hyperon, which has been defined in Eq. (7) and Eq. (9) through the Fourier transformation for $D_1^{\pi/q}(z, P_+^2; Q)$, $D_1^{\Lambda/q}(z, P_+^2; Q)$, and $D_{1T}^{\Lambda/q}(z, P_+^2; Q)$.

In the small $b$ region $1/Q \ll b \ll 1/\Lambda$, the $b$-dependence is perturbative, while it turns to be non-perturbative in the large $b$ region. In order to combine the information between the two region, a matching procedure should be adopted. In the original Collins-Soper-Sterman (CSS) approach [40, 43–45], a parameter $b_{\text{max}}$ is introduced as the boundary between the two different regions to allow a smooth transition of $b$ from perturbative region to nonperturbative region as well as to avoid hitting on the Landau pole. The typical value of $b_{\text{max}}$ is chosen around $1 \GeV^{-1}$ to guarantee that $b_s$ is always in the perturbative region. A $b$-dependent function $b_s(b)$ may be also introduced to have the property $b_s \approx b$ at small $b$ value and $b_s \approx b_{\text{max}}$ at large $b$ value. In the original CSS approach it has the following form [20, 31]

$$b_s = b/\sqrt{1 + b^2/b_{\text{max}}^2} , \quad b_{\text{max}} < 1/\Lambda_{\text{QCD}}$$

(12)

There are also several different choices on the form of $b_s(b)$ in literature [32, 46].

The Sudakov form factor can be separated into two parts: the perturbative part $S_P$ and the non-perturbative part $S_{NP}$. It is important to keep in mind that $\tilde{D}_1^{\pi/q}(z, Q, b_{\perp}^2)$, $\tilde{D}_1^{\Lambda/q}(z, Q, b_{\perp}^2)$ and $D_{1T}^{\Lambda/q}(z, Q, b_{\perp}^2)$ follow exactly the same QCD evolution in the perturbative region. The universal part $S_P(Q, b)$, which is the same for different kinds of fragmentation functions (namely, $S_P$ is also spin-independent), has been studied in details in literature [28, 30, 36, 47, 48]:

$$S_P(Q, b) = \int_{\mu_i^2}^{Q^2} d\mu^2 \int d\mu^2 \left[A(\alpha_s(\mu)) \ln \frac{Q^2}{\mu^2} + B(\alpha_s(\mu))\right],$$

(13)

In addition, the coefficients $A$ and $B$ in Eq. (13) can be expanded as the series of $\alpha_s/\pi$:

$$A = \sum_{n=1}^{\infty} A^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n, \quad B = \sum_{n=1}^{\infty} B^{(n)} \left(\frac{\alpha_s}{\pi}\right)^n.$$

(14)

(15)

In this work, we adopt $A^{(n)}$ up to $A^{(2)}$ and $B^{(n)}$ up to $B^{(1)}$ in the accuracy of next-to-leading-logarithmic (NLL) order [20, 26, 28, 43, 45, 47]:

$$A^{(1)} = C_F,$$

$$A^{(2)} = \frac{C_F}{2} \left[ C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9} T_R n_f \right],$$

(16)

(17)

$$B^{(1)} = -\frac{3}{2} C_F.$$

(18)

On the other hand, the non-perturbative part $S_{NP}$ can not be calculated from perturbative QCD, it is usually extracted from experimental data. There are several different approaches to parameterize $S_{NP}$, we will discuss two of them in details. The first method is the widely used non-perturbative Sudakov form factor $S_{NP}$ [45, 49–51]:

$$S_{NP}^g(b, Q) = b^2 g_1^g + g_2^g \ln \frac{Q}{Q_0}. $$

(19)

Here, $g_1(b)$ are the functions of the impact parameter $b$. Particularly, $g_2(b)$ contains the information on the large $b$ behavior of the evolution kernel $K$, while $g_1^g$ contains the information about the intrinsic nonperturbative transverse
motion of bound partons, i.e., it depends on the type of the hadron and quark flavor. It might also depend on the momentum fraction of the partons \( z \) [52]. It is also worth pointing out that \( g_2(b) \) is universal for different types of TMDs and does not depend on the particular process, which is one of the important predictions of TMD factorization [23, 26, 30, 31]. Here, \( g_1^H \) was assumed a Gaussian form as:

\[
g_1^H = \frac{\langle p_T^2 \rangle_{Q_0}}{4z^2}, \tag{20}
\]

where \( \langle p_T^2 \rangle_{Q_0} \) is the relevant averaged intrinsic transverse momenta squared for TMD fragmentation functions at the momentum scale \( Q_0 \). Ref. [30] shows that the Sudakov factor with the following parameters (and \( Q_0 = \sqrt{2.4} \text{ GeV} \)) leads to a reasonably good description of all experimental data on SIDIS, DY lepton pair and W/Z boson production,

\[
\langle p_T^2 \rangle_{Q_0} = 0.19 \text{ GeV}^2, \quad g_2 = 0.16 \text{ GeV}^2, \quad b_{\text{max}} = 1.5 \text{ GeV}^{-1}. \tag{21}
\]

Therefore, \( S_{\text{NP}} \) associated with a unpolarized TMDFFs can be expressed as (G. Echevarria-Idilbi-Kang-Vitev (GIKV) parametrization)

\[
S_{\text{NP}}^D(b, Q) = \nu^2(g_1^H + g_2 \ln \frac{Q}{Q_0}). \tag{22}
\]

Since the information of the nonperturbative Sudakov form factor associated with the polarized fragmentation function of the \( \Lambda \) hyperon still remains unknown, we assume it to be the same as \( S_{\text{NP}}^D \), i.e., \( S_{\text{NP}}^{H,A/Q} = S_{\text{NP}}^D \).

In the perturbative region \( 1/Q \ll b \ll 1/\Lambda \), the TMD fragmentation function can be expressed as the convolution of the perturbatively calculable coefficients and the corresponding collinear counterparts for the TMDs at the fixed energy scale \( \mu_b \), which is a dynamic scale related to \( b \), through \( \mu_b = c/b_s \), with \( \gamma_E \approx 0.577 \) the Euler’s constant [19],

\[
\bar{D}(z, b; \mu_b) = \sum_i \int_1^z d\xi \frac{d}{\xi} C_{q \rightarrow i}(z/\xi, b; \mu) D_{i/H}(\xi, \mu). \tag{23}
\]

where \( C_{q \rightarrow i}(z/\xi, b; \mu) = \sum_{\alpha=0}^{\infty} C_{\xi-i}^{(\alpha)}(\alpha_s/\pi)^n \) is the perturbatively calculable coefficient function with \( \sum_i \) summing over the quark and antiquark flavors. Here, we will adopt the leading order (LO) result, i.e. \( C_{q \rightarrow i}^{(0)} = \delta_{iq} \delta(1 - z) \). In other words,

\[
\bar{D}_1^{\pi/q}(z, b; \mu_b) = D_1^{\pi/q}(z, \mu_b), \tag{24}
\]

\[
\bar{D}_1^{A/q}(z, b; \mu_b) = D_1^{A/q}(z, \mu_b), \tag{25}
\]

\[
\bar{D}_{1T}^{H,A/Q}(z, b; \mu_b) = (\frac{ib}{2}) \bar{D}_{1T}^{(3)}(z, z, \mu_b), \tag{26}
\]

where \( D_1^{\pi/q}(z, \mu_b) \) and \( D_1^{A/q}(z, \mu_b) \) are the collinear unpolarized fragmentation functions for pion meson and \( \Lambda \) hyperon, while \( \bar{D}_{1T}^{(3)}(z, z, \mu_b) \) is a twist-3 fragmentation function of quark flavor \( q \) to \( \Lambda \) hyperon, which has the following relation with \( D_{1T}^{h/q} \) and \( D_{1T}^{(1)} \) [53]:

\[
\bar{D}_{1T}^{(3)}(z, z, \mu_b) = \int d^2 P_\perp \frac{\langle p_T^2 \rangle_{M_h}}{M_h} D_{1T}^{h/q}(z, P_\perp) = 2M_{\Lambda}D_{1T}^{(1)}, \tag{27}
\]

where \( D_{1T}^{(1)} \) is the first transverse moment of \( D_{1T}^{(1)} \).

It is straightforward to rewrite the scale-dependent TMD fragmentation functions of the pion meson and the \( \Lambda \) hyperon in \( b \) space

\[
\bar{D}_{b/q}(z, b; Q) = e^{-\frac{4}{3} S_p(Q, b_s) - S_{b/q}^{H,A/Q}(Q, b)} D_{b/q}(z, \mu_b), \tag{28}
\]

The factor of \( \frac{1}{3} \) in front of \( S_p \) comes from the fact that \( S_p \) of quarks and antiquarks satisfies the relation [54]

\[
S_p^q(Q, b_s) = S_p^q(Q, b_s) = S_p(Q, b_s)/2. \tag{29}
\]
Thus, the fragmentation function in the transverse momentum space can be obtained by performing the Fourier transform of the fragmentation functions, which is parametrized as

$$D_1^{\phi}(z, b; Q) = e^{-\frac{1}{2} S_P(Q, b_s)} S_{NP}^{\phi}(Q, b) D_1^{\phi}(z, \mu_b),$$

$$D_1^{\Lambda/q}(z, b; Q) = e^{-\frac{1}{2} S_P(Q, b_s)} S_{NP}^{\Lambda/q}(Q, b) D_1^{\Lambda/q}(z, \mu_b),$$

$$\tilde{D}_{1T}^{\lambda/q}(z, b; Q) = (\frac{db}{2}) e^{-\frac{1}{2} S_P(Q, b_s)} S_{NP}^{\lambda/q}(Q, b) \tilde{D}_{1T}^{\lambda/q}(z, z, \mu_b).$$

(30)

Thus, the fragmentation function in the transverse momentum space can be obtained by performing the Fourier transformation

$$D_1^{\phi}(z, P_\perp; Q) = \int_0^\infty \frac{db}{2\pi} J_0(P_\perp b/z) e^{-\frac{1}{2} S_P(Q, b_s)} S_{NP}^{\phi}(Q, b) D_1^{\phi}(z, \mu_b),$$

$$D_1^{\Lambda/q}(z, P_\perp; Q) = \int_0^\infty \frac{db}{2\pi} J_0(P_\perp b/z) e^{-\frac{1}{2} S_P(Q, b_s)} S_{NP}^{\Lambda/q}(Q, b) D_1^{\Lambda/q}(z, \mu_b),$$

$$\frac{P_\perp}{M_\Lambda} \tilde{D}_{1T}^{\lambda/q}(z, P_\perp; Q) = \int_0^\infty \frac{dbb^2}{2\pi} J_1(P_\perp b/z) e^{-\frac{1}{2} S_P(Q, b_s)} S_{NP}^{\lambda/q}(Q, b) \tilde{D}_{1T}^{\lambda/q}(z, z, \mu_b).$$

(31)

(32)

(33)

where $J_i$ ($i=0,1$) are the Bessel function, and $P_\perp = |P_\perp|$.

Besides the above GIKV parametrization, some alternative forms have been also proposed [23, 26, 30, 36, 47, 48] recently. Particularly, we adopt the evolution formalism in Ref. [32] to be the second method (Bacchetta-Delcarro-Pisano-Radici-Signori (BDPRS) parametrization). In Ref. [32], the fragmentation function from the initial energy scale to the final energy scale has the following form

$$D_1^{a-h}(z, b^2; Q^2) = D_1^{a-h}(z; \mu_b^2) e^{-S(P, Q^2)} e^{\frac{1}{2} g_K(b) ln(Q^2/Q_0^2)} D_1^{a-h}(z, b^2),$$

(34)

where $g_K = -g_2 b^2/2$, following the choice in Refs. [45, 49, 55], and $D_1^{a-h}(z, b^2)$ is the intrinsic nonperturbative part of the fragmentation functions, which is parametrized as

$$\tilde{D}_{1NP}^{a-h}(z, b^2) = g_3 e^{-g_3 \frac{b^2}{4\xi^2}} + (\frac{\lambda}{\xi^2}) g_4^2 (1 - g_4 \frac{b^2}{4\xi^2}) e^{-g_4 \frac{b^2}{4\xi^2}},$$

(35)

with

$$g_{3,4}(z) = N_{3,4} (z^{\beta + \delta} (1 - z)^{\gamma} (\tilde{z}^{\beta + \delta} (1 - \tilde{z})^{\gamma},$$

(36)

where $\beta, \gamma, \delta$ and $N_{3,4} = g_{3,4}(\tilde{z})$ with $\tilde{z} = 0.5$ are free parameters fitted to the available data from SIDIS, Drell-Yan, and Z boson production processes yielding $\beta = 1.65, \gamma = 2.28, \delta = 0.14, \lambda_P = 5.50$ GeV$^{-2}, g_2 = 0.13, N_3 = 0.21$ GeV$^2, N_4 = 0.03$ GeV$^2$. Furthermore, in Ref. [32], the new $b_*$ prescription different from Eq. (12) was also proposed as

$$b_* = b_{\max}(1 - e^{-b^4/b_{\max}^4})^{1/4}$$

(37)

where $b_{\max}$ is again the boundary of the nonperturbative and perturbative $b$-space region with fixed value of $b_{\max} = 2e^{-\gamma_e}$ GeV$^{-1} \approx 1.123$ GeV$^{-1}$. Besides, the authors in Ref. [32] also chose to saturate $b_*$ at the minimum value $b_{\min} \approx 2e^{-\gamma_e}/Q$.

In this work, we will adopt the both the GIKV evolution formalism and the BDPRS evolution formalism to calculate the $\sin(\phi - \phi_S)$ asymmetry in order to investigate the impact of the different evolution formalism on the asymmetry.

IV. NUMERICAL CALCULATION

Using the framework set up above, in this section, we numerically estimate the $\sin(\phi - \phi_S)$ azimuthal asymmetry in the process $e^+ e^- \rightarrow \Lambda^+ + \pi + X$ at the energy scale of Belle experiments. In order to obtain the numerical results of the asymmetry, one needs to utilize the corresponding collinear parts of the TMD fragmentation functions as
the inputs of the TMD evolution effects. For the unpolarized collinear fragmentation function $D_1(z)$, we adopt the leading order DSS parametrization [56]. The other important input is the twist-3 collinear correlation function $D_{1T}^{(-)}$ corresponding to the $D_{1T}^+$ for the $\Lambda$ hyperon, for which we adopt two different choices for comparison.

The first choice is the spectator diquark model result including both the scalar diquark and axial-vector diquark spectators from Ref. [18] at the initial scale $\mu_0^2 = 0.23 \text{ GeV}^2$. Assuming the SU(6) spin-flavor symmetry, the fragmentation functions of the $\Lambda$ hyperon for light flavors satisfy the relations between different quark flavors and diquark types

\[
D_{1T}^+ = D_{1T}^- = \frac{1}{4} D_{1T}^{(+)} + \frac{3}{4} D_{1T}^{(+)} , \quad D_{1T}^{(s)} = D_{1T}^{(-)} ,
\]  

(38)

where $u$, $d$, and $s$ denote the up, down, and strange quarks, respectively. $D_{1T}^{(+)}$ and $D_{1T}^{(-)}$ represent the contribution from the axial-vector diquark and scalar diquark, and have the form

\[
D_{1T}^{(+)}(z, k_T^2) = \frac{\alpha_s g_2^2 C_F}{(2\pi)^3} \frac{e^{-\frac{k_T^2}{2k^2}}}{z^2(1-z)(k^2 - m_q^2)}
\]

\times (D_{1T}^{(+)}(z, k_T^2) + D_{1T}^{(+)}(z, k_T^2) + D_{1T}^{(+)}(z, k_T^2) + D_{1T}^{(+)}(z, k_T^2)),
\]

(39)

\[
D_{1T}^{(-)}(z, k_T^2) = \frac{\alpha_s g_2^2 C_F}{(2\pi)^3} \frac{e^{-\frac{k_T^2}{2k^2}}}{z^2(1-z)(k^2 - m_q^2)}
\]

\times (D_{1T}^{(-)}(z, k_T^2) + D_{1T}^{(-)}(z, k_T^2) + D_{1T}^{(-)}(z, k_T^2) + D_{1T}^{(-)}(z, k_T^2)),
\]

(40)

where $k^2$ can be written as $k^2 = \frac{z}{1-z} k_1^2 + \frac{m_q^2}{1-z} + \frac{M_A^2}{z}$, with $m_q, m_D, M_A$ the masses of the parent quark, the spectator diquark and fragmenting $\Lambda$ hyperon, respectively. The $\Lambda^2$ has the general form $\Lambda^2 = \Lambda^2(z^2(1-z)^3)$. At one loop level, there are four diagrams that can generate imaginary phases to the nonzero contribution of the fragmentation. For the scalar case, the four nonzero contributions are

\[
D_{1T}^{(+)}(z, k_T^2) = m_q M_A \frac{k^2 - m_q^2}{(k^2 - m_q^2)^2} I_1 , \quad D_{1T}^{(+)}(z, k_T^2) = M_A (2I_2 - A) + M_\Lambda (2I_2 - B - 2A) , \quad D_{1T}^{(+)}(z, k_T^2) = 0 ,
\]

(41)

\[
D_{1T}^{(+)}(z, k_T^2) = M_A z \left( 2(1-z)(m_q C P_A - M_\Lambda D P_A^-) + z(m_q A - M_\Lambda B) \right),
\]

while for the axial-vector case, the four contributions are

\[
D_{1T}^{(+)}(z, k_T^2) = \frac{m_q M_A}{k^2 - m_q^2} (1 - \frac{m_q^2}{3k^2}) I_1 , \quad D_{1T}^{(+)}(z, k_T^2) = \frac{1}{3} \left( 2M_\Lambda (m_q (I_2 - A) + M_\Lambda (A - I_2 - B)) + k . P_\Lambda (4I_2 - 6A) - A k . P_\Lambda - BM_\Lambda^2 \right.
\]

\left. + \frac{3}{2} \left( \frac{k^2 - m_q^2}{2k^2} I_1 + (k^2 - m_q^2) A \right) \right\},
\]

(42)

\[
D_{1T}^{(+)}(z, k_T^2) = 0 , \quad D_{1T}^{(a)}(z, k_T^2) = \frac{1}{3 M_\Lambda} \left( [M_\Lambda ((k^2 - m_q^2) C P_A + 2M_\Lambda D P_A^- - 2m_q M_\Lambda C P_A^-) + 2k . P_\Lambda (m_q C P_A^- - M_\Lambda D P_A^-) + z \frac{m_q^2}{2} I_1 + \frac{k^2 - m_q^2}{2} (M_\Lambda D P_A^- - m_q C P_A^-)] + M_\Lambda (m_q M_\Lambda A + M_\Lambda^2 B + 2k . P_\Lambda A) - \frac{2M_\Lambda}{z} (m_q M_\Lambda C P_A^- + k . P_\Lambda C P_A^-) \right) .
\]
FIG. 1: The $\sin(\phi - \phi_q)$ asymmetry in $e^+ e^- \to \Lambda^+ + \pi + X$ process calculated from the polarized fragmentation function of $\Lambda$ in Ref. [18]. The solid lines show the result from the BDPRRS parametrization [32] [Eqs. (34) and Eq. (35)] for the nonperturbative form factor. The dashed lines correspond to the result calculated from the GIKV parametrization [30] [Eqs. (34) and Eq. (35)] for the nonperturbative Sudakov form factor.

Here, the expressions for the functions of $A, B, C, D, I_i$ have been given in Ref. [18]. Besides, the values of the model parameters are obtained by fitting the model result of the unpolarized fragmentation function $D^{\Lambda}_1$ to the DSV parametrization [57] at the model scale $Q_0^2 = 0.23$ GeV as

$$m_D = 0.745 \text{ GeV}, \quad \lambda = 5.967 \text{ GeV}, \quad g_s = 1.982 \quad m_q = 0.36 \text{ GeV}, \quad M_\Lambda = 1.116 \text{ GeV}, \quad \alpha = 0.5, \quad \beta = 0, \quad (43)$$

where the values of the last four parameters are fixed, and the coupling constant $\alpha_s$ is chosen as 0.817 at model scale. Then the corresponding collinear twist-3 fragmentation function of quark flavor $q$ to $\Lambda$ hyperon $D^{\Lambda/3}_1(z, z, \mu_b)$ at the model scale can be obtained by using Eq. (27). For consistency, we apply the unpolarized fragmentation function of the $\Lambda$ hyperon $D^{\Lambda/3}_1(z)$ using the same model. Furthermore, for the energy dependence of the collinear counterparts for the fragmentation functions, we apply the QCDNUM evolution package [58] to evolve the unpolarized fragmentation function $D^{\Lambda/3}_1$ from the model scale $\mu_0$ to another scale. As for $D^{+}_1$, we adopt the diagonal piece of the DGLAP evolution kernel corresponding to twist-3 collinear correlation function $D^{\Lambda/3}_1(z, z, \mu_b)$ in Ref. [59], which has the same form as that for the unpolarized fragmentation function.

The second choice is the parametrization of $\Lambda$ PFF in Ref. [16] extracted from the phenomenological analysis of the experimental data on the transverse polarization of $\Lambda$ production, for the case of inclusive (plus a jet) and associated production with a light charged hadron. Adopting a Gaussian ansatz for the TMD fragmentation functions, the transversely polarized fragmentation function is parameterized in Ref. [16] as

$$\Delta D^{\Lambda\perp/3}_q = N_q e^{a_q(1 - z)} e^{b_q z} D^{\Lambda\perp/3}_q(z), \quad (44)$$
with $M_{\text{pol}}$ and $\langle K^2_{\text{pol}} \rangle_{\text{pol}}$ satisfying the relation

$$\langle K^2_{\text{pol}} \rangle_{\text{pol}} = \frac{M^2_{\text{pol}}}{M^2_{\text{pol}} + \langle K^2_{\perp} \rangle_{\text{pol}}},$$

(45)

where the unpolarised Gaussian width $\langle K^2_{\text{pol}} \rangle = 0.2 \text{ GeV}^2$ was extracted in Ref. [60] both for light and heavy hadrons. $D_{\Lambda/q}(z)$ is the unpolarized collinear fragmentation function for $\Lambda$ for which we adopt the AKK08 set for $\Lambda$ unpolarised fragmentation function $D_{\Lambda/q}(z)$ [61]: it holds for the quark fragmentation into $\Lambda + \bar{\Lambda}$.  The first-$K_{\perp}$ moment was also obtained as

$$D_{1T}^{\Lambda}(z) = \sqrt{\frac{e}{2 z M_{\Lambda}} \frac{\langle K^2_{\perp} \rangle}{\langle K^2_{\perp} \rangle_{\text{pol}}}} \Delta D_{\Lambda/q}(z).$$

(46)

So far, concerning the fragmentation function for $\Lambda$, all available parametrization are performed for the combination $\Lambda + \bar{\Lambda}$, we adopt the following expression of $D_{\Lambda/q}(z)$ given in Ref. [61] to separate the fragmentation function,

$$D_{\Lambda/q}(z_p) = D_q^{\Lambda + \bar{\Lambda}}(z_p) \frac{1}{1 + (1 - z_p)^s},$$

(47)

where the power $s = 1$ and the scaling variable $z_p$ is related to $z_1$ by $z_p \simeq z_1 [1 - M_{\perp}^2/(z^2 Q^2)]$. The above setting is different from that in Ref. [18], which assumes $SU(3)$ symmetry and take $D_{\Lambda/u} = D_{\Lambda/d} = D_{\Lambda/s}$. The corresponding collinear twist-3 fragmentation function of quark flavor $q$ to $\Lambda$ hyperon $D_{1T}^{\Lambda}(z, z, \mu_b)$ can also be obtained from Eq. (27). The parameters in Eq. (44) are adopted from Ref. [16]:

$$N_u = 0.47, \quad N_d = -0.32, \quad N_s = -0.57, \quad a_u = 0, \quad a_d = 0, \quad a_s = 2.30, \quad b_u = 3.50, \quad b_d = 0, \quad b_s = 0, \quad \langle K^2_{\perp} \rangle_{\text{pol}} = 0.1 \text{ GeV}^2.$$

(48)

The Belle data were measured in $(z_1, z_2)$ bins with boundaries at $z_{h_1} = 0.2, 0.3, 0.4, 0.5, 0.9$, where $i = 1, 2$ [11] stands for the final state $\Lambda$ and pion meson. Therefore, in Fig. 1, we plot the numerical results of the $\sin(\phi - \phi_S)$ azimuthal asymmetry in the process $e^+e^- \rightarrow \Lambda^+\pi^+X$ at $Q = 10.52$ GeV as functions of $z_1$ (upper panels), $z_2$ (central panels) and $P_{hT}$ (lower panels) for different $z_i$ bins ($i=1,2$), respectively. In this result we adopt the first choice on the Lambda fragmentation functions, namely, the spectator model results in Ref. [18]. Since the positivity bound of $D_{1T}^{\Lambda}$ in the model calculation may be violated at large $z$ region ($z > 0.75$) [18], the $z_i$ ($i = 1, 2$) bins are shown in Fig. 1 up to 0.5, i.e. the analysis is performed in $(z_1, z_2)$ bins with boundaries at $z_i = 0.2, 0.3, 0.4, 0.5$. To make the TMD factorization valid in the kinematic region $P_{h\perp}/z_1 \ll Q$, the integration over the transverse momentum $P_{h\perp}$ is performed in the region of $0 < P_{h\perp} < 0.7$ GeV. The solid lines show the asymmetry calculated from the BDPRS evolution formalism [32] and the $b_s$ prescription in Eq. (37), while the dashed lines show the asymmetry using the GIKV parametrization [30] for the nonperturbative Sudakov form factor as a comparison, with the $b_s$ prescription in Eq. (12). As shown in Fig. 1, in all the cases $\sin(\phi - \phi_S)$ azimuthal asymmetry is positive and is around several percents. Our estimates also show that the size of the asymmetry increases with increasing $z_1$, $z_2$, and the results from the GIKV parameterization are close to those from the BDPR3 parameterization. We can conclude that the asymmetry from different methods to deal with the non-perturbative evolution lead to the similar results.

In Fig. 2, we compare the results calculated from the two different choices of the $\Lambda$ PFF to study their impact on the $\sin(\phi - \phi_S)$ asymmetry. The solid lines correspond to results from the spectator model results (which is also the solid lines in Fig. 1), while the dashed lines show the asymmetry from the parametrization on $D_{1T}^{\Lambda}$ of $\Lambda$ in Ref. [16]. In both calculations the non-perturbative part of the evolution are adopted as BDPRS parametrization. We can observe from Figs. 2 that the asymmetries from different sets of $D_{1T}^{\Lambda}$ are different in the magnitude and sign. It is found that in the small $z_1$ region, the result calculated from the parametrization of $D_{1T}^{\Lambda}$ is sizable, while in the region $z_1 > 0.3$, the asymmetry is negligible. Particularly, in the region $0.2 < z_1 < 0.3$ the results from two sets of $D_{1T}^{\Lambda}$ have the similar size but opposite sign, that is, the result from the parametrization of $D_{1T}^{\Lambda}$ is negative, and the size is around 0.2 at most.

V. CONCLUSION

The production of the transversely polarized $\Lambda$ hyperon can serve as a useful tool to study its spin structure and further information on the non-perturbative hadronization mechanism. In this work, we have applied the TMD factorization approach to study the $\sin(\phi - \phi_S)$ azimuthal asymmetry in $e^+e^- \rightarrow \Lambda^+\pi^+X$ process that is accessible
FIG. 2: Comparison of the $\sin(\phi - \phi_S)$ asymmetry calculated from the model result [18] for $D_{1T}^-$ (solid lines) and the one calculated from the parameterized result [16] for $D_{1T}^-$ (dashed lines).

The asymmetry arises from the convolution of the T-odd polarized fragmentation function $D_{1T}$ which depicts the fragmentation of an unpolarized quark into a transversely $\Lambda$ hyperon, and the unpolarized fragmentation function of pion. We have taken into account the TMD evolution effects of the TMD fragmentations of the pion and $\Lambda$ hyperon by including the Sudakov form factor. The Sudakov form factor can be separated into perturbative part and non-perturbative part, for the former one, we adopted the results from the perturbative QCD at NLL accuracy, while for the latter one, we considered two different non-perturbative TMD evolution formalism for comparison. As the nonperturbative Sudakov form factor associated with the $\Lambda$ PFF is still unknown, we assume that it is the same as that of the unpolarized fragmentation function. The hard coefficients associated with the corresponding collinear functions in the TMD evolution formalism are kept in the leading-order accuracy. For the fragmentation function of $\Lambda$ hyperon, we have chosen the results from two different methods, one is from the diquark spectator model, the other one is the parametrization extracting from the Belle $e^+e^-$ data. We have found that different choices of nonperturbative Sudakov form factors in the TMD evolution formalism have similar results for $\sin(\phi - \phi_S)$ azimuthal asymmetry in process $e^+e^- \rightarrow \Lambda^+ + \pi^- + X$ at $\sqrt{s}=10.52$ GeV, while the asymmetry from difference choices of the $\Lambda$ fragmentation function has different size and sign, and also different $z$-dependence and $P_T$-dependence. Our analysis demonstrated that, within the framework of TMD evolution, $\sin(\phi - \phi_S)$ asymmetry is sizable and can be measured at Belle and BaBar. Furthermore, the measurement of the $e^+e^- \rightarrow \Lambda^+ + \pi^- + X$ process can be used to discriminate different sets of $D_{1T}^-$. More precise data from the $e^+e^-$ annihilation process will rigorously constrain the polarized fragmentation function of the $\Lambda$ hyperon.
Acknowledgements

This work is partially supported by the NSFC (China) grants 11575043, 11905187, 11847217 and 11120101004. X. Wang is supported by the China Postdoctoral Science Foundation under Grant No. 2018M640680 and the Academic Improvement Project of Zhengzhou University.

[1] A. Lesnik et al., Phys. Rev. Lett. 35, 770 (1975).
[2] G. Bunce et al., Phys. Rev. Lett. 36, 1113 (1976).
[3] G. L. Kane, J. Pumplin and W. Repko, Phys. Rev. Lett. 41, 1689 (1978).
[4] W. G. D. Dharmaratna and G. R. Goldstein, Phys. Rev. D 53, 1073 (1996).
[5] M. Anselmino, D. Boer, U. D’Alesio and F. Murgia, Phys. Rev. D 63, 054029 (2001).
[6] P. J. Mulders and R. D. Tangerman, Nucl. Phys. B 461, 197 (1996) Erratum: [Nucl. Phys. B 484, 538 (1997)].
[7] D. Boer, arXiv:0907.1610 [hep-ph].
[8] K. Ackerstaff et al. [OPAL Collaboration], Eur. Phys. J. C 2, 49 (1998) [hep-ex/9708027].
[9] D. Boer, R. Jakob and P. J. Mulders, Nucl. Phys. B 504, 345 (1997).
[10] S. Y. Wei, K. b. Chen, Y. k. Song and Z. t. Liang, Phys. Rev. D 91, no. 3, 034015 (2015).
[11] Y. Guan et al. [Belle Collaboration], Phys. Rev. Lett. 122, no. 4, 042001 (2019).
[12] K. J. Keller et al., Phys. Rev. Lett. 41, 607 (1978) Erratum: [Phys. Rev. Lett. 45, 1043 (1980)].
[13] M. Anselmino, D. Boer, U. D’Alesio and F. Murgia, Phys. Rev. D 65, 114014 (2002).
[14] Y. Kolke, A. Metz, D. Pitonyak, K. Yabe and S. Yoshida, Phys. Rev. D 95, no. 11, 114013 (2017).
[15] L. Gamborg, Z. B. Kang, D. Pitonyak, M. Schlegel and S. Yoshida, JHEP 1901, 111 (2019).
[16] U. D’Alesio, F. Murgia and M. Zaccheddu, arXiv:2003.01128 [hep-ph].
[17] D. Callos, Z. B. Kang and J. Terry, arXiv:2003.04828 [hep-ph].
[18] S. M. Aybat, T. C. Rogers, Phys. Rev. D 83, 114042 (2011).
[19] J. C. Collins and T. C. Rogers, Phys. Rev. D 87, 034018 (2013).
[20] M. G. Echevarria, A. Idilbi, A. Schäfer and I. Scimemi, Eur. Phys. J. C 80, 23 (2019).
[21] J. Collins, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 32, 1 (2011).
[22] D. Boer, Nucl. Phys. B 806, 23 (2009).
[23] S. Arnold, A. Metz and M. Schlegel, Phys. Rev. D 79, 034005 (2009).
[24] M. G. Echevarria, A. Idilbi, A. Schäfer and I. Scimemi, Eur. Phys. J. C 73, 2636 (2013).
[25] D. Pitonyak, M. Schlegel and A. Metz, Phys. Rev. D 89, 054032 (2014).
[26] M. G. Echevarria, A. Idilbi, Z. B. Kang and I. Vitev, Phys. Rev. D 89, 074013 (2014).
[27] Z. B. Kang, A. Prokudin, P. Sun and F. Yuan, Phys. Rev. D 93, 014009 (2016).
[28] A. Bacchetta, F. Delcarro, C. Pisano, M. Radici and A. Signori, JHEP 1706, 081 (2017) Erratum: [JHEP 1906, 051 (2019)].
[52] P. Sun, J. Isaacson, C.-P. Yuan and F. Yuan, arXiv:1406.3073 [hep-ph].
[53] F. Yuan and J. Zhou, Phys. Rev. Lett. 103, 052001 (2009).
[54] A. Prokudin, P. Sun and F. Yuan, Phys. Lett. B 750, 533 (2015).
[55] P. M. Nadolsky, D. R. Stump and C. P. Yuan, Phys. Rev. D 61, 014003 (2000) Erratum: [Phys. Rev. D 64, 059903 (2001)].
[56] D. de Florian, R. Sassot and M. Stratmann, Phys. Rev. D 75, 114010 (2007).
[57] D. de Florian, M. Stratmann and W. Vogelsang, Phys. Rev. D 57, 5811 (1998).
[58] M. Botje, Comput. Phys. Commun. 182, 490 (2011).
[59] Z. B. Kang, Phys. Rev. D 83, 036006 (2011).
[60] M. Anselmino et al., Phys. Rev. D 71, 074006 (2005).
[61] S. Albino, B. A. Kniehl and G. Kramer, Nucl. Phys. B 803, 42 (2008).