The partition dimension of corona product graphs

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Abstract

Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph $G$, the metric representation of a vertex $v$ of $G$ with respect to $S$ is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$, where $d(v, v_i)$, $i \in \{1, ..., k\}$ denotes the distance between $v$ and $v_i$. $S$ is a resolving set of $G$ if for every pair of vertices $u, v$ of $G$, $r(u|S) \neq r(v|S)$. The metric dimension $\dim(G)$ of $G$ is the minimum cardinality of any resolving set of $G$. Given an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of vertices of a connected graph $G$, the partition representation of a vertex $v$ of $G$, with respect to the partition $\Pi$ is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, $1 \leq i \leq t$, represents the distance between the vertex $v$ and the set $P_i$, that is $d(v, P_i) = \min_{u \in P_i} d(v, u)$. $\Pi$ is a resolving partition for $G$ if for every pair of vertices $u, v$ of $G$, $r(u|\Pi) \neq r(v|\Pi)$. The partition dimension $pd(G)$ of $G$ is the minimum number of sets in any resolving partition for $G$. Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$ respectively. The corona product $G \odot H$ is defined as the graph obtained from $G$ and $H$ by
taking one copy of $G$ and $n_1$ copies of $H$ and then joining by an edge, all the vertices from the $i$th-copy of $H$ with the $i$th-vertex of $G$. Here we study the relationship between $pd(G \circ H)$ and several parameters of the graphs $G \circ H$, $G$ and $H$, including $\dim(G \circ H)$, $pd(G)$ and $pd(H)$.

**Keywords:** Resolving sets, resolving partition, metric dimension, partition dimension, corona graph.

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## 1 Introduction

The concepts of resolvability and location in graphs were described independently by Harary and Melter [10] and Slater [19], to define the same structure in a graph. After these papers were published several authors developed diverse theoretical works about this topic [3, 4, 5, 6, 7, 8, 9, 16, 18, 20]. Slater described the usefulness of these ideas into long range aids to navigation [19]. Also, these concepts have some applications in chemistry for representing chemical compounds [14, 15] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [17]. Other applications of this concept to navigation of robots in networks and other areas appear in [6, 12, 16]. Some variations on resolvability or location have been appearing in the literature, like those about conditional resolvability [18], locating domination [11], resolving domination [1] and resolving partitions [5, 8, 9]. In this work we are interested into study the relationship between $pd(G \circ H)$ and several parameters of the graphs $G \circ H$, $G$ and $H$, including $\dim(G \circ H)$, $pd(G)$ and $pd(H)$.

We begin by giving some basic concepts and notations. Let $G = (V, E)$ be a simple graph. Let $u, v \in V$ be two different vertices in $G$, the distance $d_G(u, v)$ between two vertices $u$ and $v$ of $G$ is the length of a shortest path between $u$ and $v$. If there is no ambiguity, we will use the notation $d(u, v)$ instead of $d_G(u, v)$. The diameter of $G$ is defined as $D(G) = \max_{u, v \in V}\{d(u, v)\}$. Given $u, v \in V$, $u \sim v$ means that $u$ and $v$ are adjacent vertices. Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of a connected graph $G$, the metric representation of a vertex $v \in V$ with respect to $S$ is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. We say that $S$ is a resolving set for $G$ if for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The metric dimension
of $G$ is the minimum cardinality of any resolving set for $G$, and it is denoted by $\dim(G)$.

Given an ordered partition $\Pi = \{P_1, P_2, ..., P_t\}$ of vertices of a connected graph $G$, the partition representation of a vertex $v \in V$ with respect to the partition $\Pi$ is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), ..., d(v, P_t))$, where $d(v, P_i)$, $1 \leq i \leq t$, represents the distance between the vertex $v$ and the set $P_i$, that is $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$. We say that $\Pi$ is a resolving partition of $G$ if for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \neq r(v|\Pi)$. The partition dimension of $G$ is the minimum number of sets in any resolving partition for $G$ and it is denoted by $pd(G)$. The partition dimension of graphs is studied in [5, 8, 18, 20, 21].

Let $G$ and $H$ be two graphs of order $n_1$ and $n_2$, respectively. The corona product $G \circ H$ is defined as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n_1$ copies of $H$ and joining by an edge each vertex from the $i^{th}$-copy of $H$ with the $i^{th}$-vertex of $G$. We will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of $G$ and by $H_i = (V_i, E_i)$ the copy of $H$ such that $v_i \sim v$ for every $v \in V_i$.

2 Majorizing $pd(G \circ H)$

It was shown in [8] that for any nontrivial connected graph $G$ we have $pd(G) \leq dim(G) + 1$. Thus,

$$pd(G \circ H) \leq dim(G \circ H) + 1.$$  \hfill (1)

In order to give another interesting relationship between $pd(G \circ H)$ and $dim(G \circ H)$ that allow us to derive tight bounds on $pd(G \circ H)$, we present the following lemma.

Lemma 1. \textbf{[22]} Let $G = (V, E)$ be a connected graph of order $n \geq 2$ and let $H$ be a graph of order at least two. Let $H_i = (V_i, E_i)$ be the subgraph of $G \circ H$ corresponding to the $i^{th}$ copy of $H$.

(i) If $u, v \in V_i$, then $d_{G \circ H}(u, x) = d_{G \circ H}(v, x)$ for every vertex $x$ of $G \circ H$ not belonging to $V_i$.

(ii) If $S$ is a resolving set for $G \circ H$, then $V_i \cap S \neq \emptyset$ for every $i \in \{1, ..., n\}$.

(iii) If $S$ is a resolving set for $G \circ H$ of minimum cardinality, then $V \cap S = \emptyset$. 

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Theorem 2. Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2$. Then

$$pd(G \odot H) \leq \frac{1}{n_1}dim(G \odot H) + pd(G) + 1.$$ 

Proof. Let $S$ be a resolving set for $G \odot H$ of minimum cardinality. By Lemma 1 (ii) and (iii) we conclude that $S = \bigcup_{i=1}^{n_1} S_i$, where $\emptyset \neq S_i \subset V_i$. We note that $|S_i| = \frac{|S_i|}{n_1} = \frac{1}{n_1}dim(G \odot H)$ for every $i \in \{1, ..., n_1\}$. In order to build a resolving partition for $G \odot H$, we need to introduce some additional notation. Let $\Pi(G) = \{W_1, W_2, ..., W_{pd(G)}\}$ be a resolving partition for $G$, let $A = \bigcup_{i=1}^{n_1} (V_i - S_i)$, let $S_i = \{v_{i1}, v_{i2}, ..., v_{it}\}$, and let $B_j = \bigcup_{i=1}^{n_1} \{v_{ij}\}$, $j = 1, ..., t$. Let us prove that $\Pi = \{A, B_1, ..., B_t, W_1, ..., W_{pd(G)}\}$ is a resolving partition for $G \odot H$. Let $x, y$ be two different vertices of $G \odot H$. We have the following cases.

Case 1. $x, y \in V_i$. If $x \in S_i$ or $y \in S_i$ then $x$ and $y$ belong to different sets of $\Pi$, so $r(x|\Pi) \neq r(y|\Pi)$. We suppose $x, y \in V_i - S_i$. Since $S$ is a resolving set for $G \odot H$, we have $r(x|S) \neq r(y|S)$. By Lemma 1 (i), $d_{G \odot H}(x, u) = d_{G \odot H}(y, u)$ for every vertex $u$ of $G \odot H$ not belonging to $V_i$. So, there exists $v \in S_i$ such that $d_{G \odot H}(x, v) \neq d_{G \odot H}(y, v)$. Thus, either $(v \sim x$ and $v \not\sim y)$ or $(v \not\sim x$ and $v \sim y)$. In the first case we have $d_{G \odot H}(x, v) = d_{H_i}(x, v) = 1$ and $d_{G \odot H}(y, v) = 2 \leq d_{H_i}(y, v)$. The case $v \not\sim x$ and $v \sim y$ is analogous. Therefore, for every $x, y \in V_i$ there exists $v_u \in S_i$ such that $d_{G \odot H}(x, B_i) = d_{G \odot H}(x, B_i) \neq d_{G \odot H}(y, B_i) = d_{G \odot H}(y, B_i)$.

Case 2. $x \in V_i$ and $y \in V_j$, $j \neq i$. There exists $W_k \in \Pi(G)$ such that $d_G(v_i, W_k) \neq d_G(v_j, W_k)$. Thus, $d_{G \odot H}(x, W_k) = 1 + d_G(v_i, W_k) \neq d_G(v_j, W_k) + 1 = d_{G \odot H}(y, W_k)$.

Case 3. $x, y \in V$. There exists $W_k \in \Pi(G)$ such that $d_G(x, W_k) \neq d_G(y, W_k)$. Thus, $d_{G \odot H}(x, W_k) \neq d_{G \odot H}(y, W_k)$.

Case 4. $x \in V$ and $y \not\in V$. In this case $x$ and $y$ belong to different sets of $\Pi$, so $r(x|\Pi) \neq r(y|\Pi)$.

Therefore, $\Pi$ is a resolving partition for $G \odot H$. 

We denote by $K_n$ and $P_n$ the complete graph and the path graph of order $n$, respectively. The following proposition allows us to conclude that for every connected graphs $G$ and $H$ of order greater than or equal to two such that $G \odot H \not\cong K_{n_1} \odot P_2$ and $G \odot H \not\cong K_{n_1} \odot P_3$, the equation in Theorem 2 is never worse than equation (1).
Proposition 3. Let $G$ and $H$ be two connected graph of order greater than or equal to two. Let $n_1$ denote the order of $G$. If $G \circ H \not\cong K_{n_1} \circ P_2$ and $G \circ H \not\cong K_{n_1} \circ P_3$, then

$$\dim(G \circ H) \geq \frac{n_1}{n_1 - 1}pd(G).$$

Proof. It was shown in [22] that

$$\dim(G \circ H) \geq n_1 \dim(H). \tag{2}$$

So we differentiate two cases. Case 1: $\dim(H) \geq 2$. Since $n_1 \geq 2$, we have $2n_1(n_1 - 1) \geq n_1^2$. Thus,

$$\dim(H)n_1(n_1 - 1) \geq 2n_1(n_1 - 1) \geq n_1^2 \geq n_1 pd(G).$$

Hence, by equation (2) we obtain $\dim(G \circ H)(n_1 - 1) \geq n_1pd(G)$.

Case 2: $\dim(H) = 1$. It was shown in [6] that a connected graph $H$ has dimension 1 if and only if $H$ is a path graph. So we have $H \cong P_{n_2}$. Now we consider two subcases.

Subcase 2.1: $G \not\cong K_{n_1}$ and $n_2 \geq 2$. Then by equation (2) we obtain

$$(n_1 - 1)\dim(G \circ P_{n_2}) \geq n_1(n_1 - 1) \geq n_1 pd(G)$$

and, as a consequence, $\dim(G \circ H) \geq \frac{n_1}{n_1 - 1}pd(G)$.

Subcase 2.2: $G \cong K_{n_1}$ and $n_2 \geq 4$. Let $S$ be a resolving set for $K_{n_1} \circ P_{n_2}$ of minimum cardinality. As above we denote by $\{v_1, ..., v_{n_1}\}$ the set of vertices of $K_{n_1}$ and by $H_i = (V_i, E_i)$, $i \in \{1, ..., n_1\}$ the corresponding copies of $P_{n_2}$ in $K_{n_1} \circ P_{n_2}$. By Lemma 1 (ii) we know that $V_i \cap S \neq \emptyset$, for every $i \in \{1, ..., n_1\}$. We suppose $V_i \cap S = \{x_i\}$. In this case, since $n_2 \geq 4$ and $H_i \cong P_{n_2}$, there exist $a, b \in V_i$ such that either $d_{K_{n_1} \circ P_{n_2}}(a, x_i) = d_{K_{n_1} \circ P_{n_2}}(b, x_i) = 1$ or $d_{K_{n_1} \circ P_{n_2}}(a, x_i) = d_{K_{n_1} \circ P_{n_2}}(b, x_i) = 2$. Thus, By Lemma 1 (i) we conclude that $r(a|S) = r(b|S)$, a contradiction. Hence, $|V_i \cap S| \geq 2$ and, as a consequence, $\dim(K_{n_1} \circ P_{n_2}) \geq 2n_1$. Then

$$\dim(K_{n_1} \circ P_{n_2})(n_1 - 1) \geq 2n_1(n_1 - 1) \geq n_1^2 = n_1pd(K_{n_1}).$$

Therefore, the result follows. □
In [22] we showed that for every connected graph $G$ of order $n_1 \geq 2$ and every graph $H$ of order $n_2 \geq 2$,
\[
\dim(G \odot H) \leq \begin{cases} 
  n_1(n_2 - \alpha - 1) & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1, \\
  n_1(n_2 - \alpha) & \text{for } \alpha \geq 1 \text{ and } \beta = 0, \\
  n_1(n_2 - 1) & \text{for } \alpha = 0,
\end{cases}
\]
where $\alpha$ denotes the number of connected components of $H$ and $\beta$ denotes the number of isolated vertices of $H$.

By using the above bound on $\dim(G \odot H)$ we obtain the following direct consequence of Theorem 2.

**Corollary 4.** Let $G$ be a connected graph of order $n_1 \geq 2$ and let $H$ be a graph of order $n_2 \geq 2$. Let $\alpha$ be the number of connected components of $H$ of order greater than one and let $\beta$ be the number of isolated vertices of $H$. Then
\[
\text{pd}(G \odot H) \leq \begin{cases} 
  \text{pd}(G) + n_2 - \alpha & \text{for } \alpha \geq 1 \text{ and } \beta \geq 1, \\
  \text{pd}(G) + n_2 - \alpha + 1 & \text{for } \alpha \geq 1 \text{ and } \beta = 0, \\
  \text{pd}(G) + n_2 & \text{for } \alpha = 0.
\end{cases}
\]

The reader is referred to [22] for several upper bounds on $\dim(G \odot H)$ which lead to bounds on $\text{pd}(G \odot H)$.

**Theorem 5.** Let $G$ and $H$ be two connected graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. If $D(H) \leq 2$, then
\[
\text{pd}(G \odot H) \leq \text{pd}(G) + \text{pd}(H).
\]

**Proof.** Let $P = \{A_1, A_2, ..., A_k\}$ be a resolving partition in $G$ and let $Q_i = \{B_{i1}, B_{i2}, ..., B_{it}\}$ be a resolving partition in the corresponding copy $H_i$ of $H$. Let $B_j = \bigcup_{i=1}^{\mu} B_{ij}$, $j \in \{1, ..., t\}$. We will show that
\[
\Pi = \{A_1, A_2, ..., A_k, B_1, B_2, ..., B_t\}
\]
is a resolving partition for $G \odot H$. Let $x, y$ be two different vertices of $G \odot H$. If $x, y \in A_i$, then there exists $A_j \in P \subset \Pi$, $j \neq i$, such that $d(x, A_j) \neq d(y, A_j)$. On the other hand, if $x, y \in B_j$, then we have the following cases.
Case 1: \(x, y \in B_{ij}\). Hence, there exists \(B_{ik} \in Q_i, k \neq j\), such that 
\(d_{H_1}(x, B_{ik}) \neq d_{H_1}(y, B_{ik})\). Since \(D(H) \leq 2\), for every \(u \in B_{ij}\) we have 
\(d_{H_1}(u, B_{ik}) = d_{G \odot H}(u, B_k)\) and \(d_{H_1}(u, B_{ik}) = d_{G \odot H}(u, B_k)\). So, we obtain 
\(d_{G \odot H}(x, B_k) = d_{H_1}(x, B_{ik}) \neq d_{H_1}(y, B_{ik}) = d_{G \odot H}(y, B_k)\).

Case 2: \(x \in B_{ij}\) and \(y \in B_{kj}, k \neq i\). If \(v_i, v_k \in A_i\), then there exists \(A_q \in P \subset \Pi\) such that 
\(d_G(v_i, A_q) \neq d_G(v_k, A_q)\). Hence, we have 
\(d_{G \odot H}(x, A_q) = 1 + d_G(v_i, A_q) = 1 + d_G(v_k, A_q) = d_{G \odot H}(y, A_q)\).

Corollary 6. Let \(G\) and \(H\) be two connected graphs of order \(n_1 \geq 2\) and \(n_2 \geq 2\), respectively. If \(D(H) \leq 2\), then 
\[pd(G \odot H) \leq \dim(G) + \dim(H) + 2.\]

In the next section we will show that all the above inequalities are tight.

3 Minorizing \(pd(G \odot H)\)

Theorem 7. Let \(G\) and \(H\) be two connected graphs. Let \(\Pi\) be a resolving partition of \(G \odot H\) of minimum cardinality. Let \(H_i = (V_i, E_i)\) be the subgraph of \(G \odot H\) corresponding to the \(i^{th}\)-copy of \(H\), and let \(\Pi_i\) be the set composed by all non-empty sets of the form \(S \cap V_i\), where \(S \in \Pi\). Then \(\Pi_i\) is a resolving partition for \(H_i\).

Proof. If \(\Pi_i\) is composed by sets of cardinality one, then the result immediately follows. Now, let \(x, y\) be two different vertices of \(H_i\) belonging to the same set of \(\Pi\). We know that there exists \(S \in \Pi\) such that 
\(d_{G \odot H}(x, S) \neq d_{G \odot H}(y, S)\). By Lemma 1 (i) we have that for every vertex \(v\) of \(G \odot H\) not belonging to \(V_i\), it follows that 
\(d_{G \odot H}(x, v) = d_{G \odot H}(y, v)\). Hence we conclude \(S' = S \cap V_i \neq \emptyset\) and we can assume, without loss of generality, that 
\(d_{G \odot H}(x, S) = 1\) and \(d_{G \odot H}(y, S) = 2\). As a result, \(S' \in \Pi_i\) and 
\(d_{H_i}(x, S') = d_{G \odot H}(x, S) = 1 < 2 = d_{G \odot H}(y, S) \leq d_{H_i}(y, S')\). Therefore, the result follows.

Corollary 8. For any connected graphs \(G\) and \(H\),
\[pd(G \odot H) \geq pd(H).\]
It is easy to check that for the star graph \( K_{1,n} \), \( n \geq 2 \), it follows \( pd(K_{1,n}) = n \). So the following result shows that the above inequality is tight.

**Proposition 9.** Let \( G \) denote a connected graph of order \( n_1 \) and let \( n \) be an integer. If \( n \geq 2n_1 \geq 4 \) or \( n > 2n_1 = 2 \), then

\[
pd(G \circ K_{1,n}) = n.
\]

**Proof.** Let us suppose \( n \geq 2n_1 \geq 4 \). For each \( v_i \in V \), let \( \{a_i, u_{i1}, u_{i2}, \ldots, u_{in}\} \) be the set of vertices of the \( i^{th} \) copy of \( K_{1,n} \) in \( G \circ K_{1,n} \), where \( a_i \) is the vertex of degree \( n \).

We will show that \( \Pi = \{S_1, S_2, \ldots, S_n\} \) is a resolving partition for \( G \circ K_{1,n} \), where

\[
S_1 = \{a_1, u_{11}, u_{21}, \ldots, u_{n1}\},
S_2 = \{v_1, u_{12}, u_{22}, \ldots, u_{n2}\},
S_3 = \{a_2, u_{13}, u_{23}, \ldots, u_{n3}\},
S_4 = \{v_2, u_{14}, u_{24}, \ldots, u_{n4}\},
\]

\[
S_{2n_1} = \{v_{n1}, u_{1(2n_1)}, u_{2(2n_1)}, \ldots, u_{n1(2n_1)}\},
S_{2n_1+1} = \{u_{1(2n_1+1)}, u_{2(2n_1+1)}, \ldots, u_{n1(2n_1+1)}\},
\]

\[
S_n = \{u_{1n}, u_{2n}, \ldots, u_{nn}\}.
\]

Let \( x, y \) be two different vertices of \( G \circ K_{1,n} \). We differentiate three cases. Case 1: \( x = u_{ii} \) and \( y = u_{jl} \), \( i \neq j \). If \( l \neq 2i - 1 \), then

\[
d(u_{ii}, S_{2i-1}) = d(u_{ii}, a_i) = 1 < 2 = d(u_{jl}, u_{j(2i-1)}) = d(u_{jl}, S_{2i-1}).
\]

If \( l = 2i - 1 \), then

\[
d(u_{jl}, S_{2j-1}) = d(u_{jl}, a_j) = 1 < 2 = d(u_{il}, u_{i(2j-1)}) = d(u_{il}, S_{2j-1}).
\]

Case 2: \( x = v_i \) and \( y = u_{j(2i)} \). If \( j = i \), then

\[
d(v_i, S_i) = d(v_i, u_{ii}) = 1 < 2 = d(u_{i(2i)}, u_{ii}) = d(u_{i(2i)}, S_i).
\]

If \( j \neq i \), then

\[
d(v_i, S_i) = d(v_i, u_{ii}) = 1 < 2 = d(u_{j(2i)}, u_{ji}) = d(u_{j(2i)}, S_i).
\]
Case 3: $x = a_i$ and $y = u_{j(2i-1)}$. If $j = i$, then
\[ d(a_i, S_i) = d(a_i, u_{ii}) = 1 < 2 = d(u_{i(2i-1)}, u_{ii}) = d(u_{i(2i-1)}, S_i). \]

If $j \neq i$, then
\[ d(a_i, S_i) = d(a_i, u_{ii}) = 1 < 2 = d(u_{j(2i-1)}, u_{ji}) = d(u_{j(2i-1)}, S_i). \]

Therefore, we conclude that \( \Pi \) is a resolving partition for \( G \odot K_{1,n} \).

For \( n_1 = 1 \) and \( n \geq 3 \) we denote by \( v \) the vertex of \( G \), by \( a \) the vertex of \( K_{1,n} \) of degree \( n \), and by \( \{u_1, u_2, ..., u_n\} \) the set of leaves of \( K_{1,n} \). Thus, from \( d(v, u_3) = 1 < 2 = d(u_2, u_3) \) and \( d(a, u_3) = 1 < 2 = d(u_1, u_3) \), we conclude that \( \Pi = \{S_1, S_2, ..., S_n\} \) is a resolving partition for \( G \odot K_{1,n} \), where \( S_1 = \{a, u_1\}, S_2 = \{v, u_2\}, S_3 = \{u_3\}, ..., S_n = \{u_n\} \).

\[ \square \]

**Lemma 10.** Let \( G \) be a connected graph. If \( \Pi \) is a resolving partition for \( G \odot K_n \) of cardinality \( n+1 \), then for every vertex \( v \) of \( G \odot K_n \) and every \( A \in \Pi \), it follows \( d(v, A) \leq 3 \).

**Proof.** Let \( v_i, v_j \) be two adjacent vertices of \( G \) and let \( H_l = (V_l, E_l) \) \((l \in \{i, j\})\) be the copy of \( K_n \) in \( G \odot K_n \) such that \( v_l \) is adjacent to every vertex of \( H_l \). If there exists a vertex \( v \) of the subgraph of \( G \odot K_n \) induced by \( V_i \cup V_j \cup \{v_i, v_j\} \) such that \( d(v, A) > 3 \), for some \( A \in \Pi \), then, since different vertices of \( V_i \) (respectively, \( V_j \)) belong to different sets of \( \Pi \), there exist \( B, C \in \Pi \), \( u_i \in V_i \) and \( u_j \in V_j \) such that \( d(v, A) \leq 3 \), for every \( A \in \Pi \). \[ \square \]

Given a graph \( H \) which contains a connected component isomorphic to a complete graph, we denote by \( c(H) \) the maximum cardinality of any connected component of \( H \) which is isomorphic to a complete graph.

**Theorem 11.** Let \( G \) be a connected graph of order \( n \). Then for any graph \( H \) such that \( n > 2c(H) + 1 \geq 5 \),
\[ pd(G \odot H) \geq c(H) + 2. \]
Proof. We denote by $S_i$ a connected component of $H_i$ isomorphic to $K_{c(H)}$, $i \in \{1, ..., n\}$. Since different vertices of $S_i$ belong to different sets of any resolving partition for $G \odot H$, we conclude $pd(G \odot H) \geq c(H)$. If $pd(G \odot H) = c(H)$, then there exist two vertices $a, b \in S_i \cup \{v_i\}$ such that they belong to the same set of any resolving partition for $G \odot H$. Thus, $a$ and $b$ have the same partition representation, which is a contradiction. So, $pd(G \odot H) \geq c(H) + 1$.

Now, let us suppose $pd(G \odot H) = c(H) + 1$ and let $\Pi(G \odot H) = \{A_1, A_2, ..., A_{c(H)+1}\}$ be a resolving partition for $G \odot H$. Now, let $S = \bigcup_{i=1}^{n} (S_i \cup \{v_i\})$ and let $u \in S$. Suppose $u \in A_j, j \in \{1, ..., c(H) + 1\}$.

So, we have that the partition representation of $u$ is given by

$$r(u|\Pi) = (1, 1, 1, 0, 1, ..., 1, t, 1, ..., 1),$$

where $i, j \in \{1, ..., c(H) + 1\}, i \neq j$, and, by Lemma 10, $t \in \{1, 2, 3\}$. Since for every different vertices $a, b \in S$, $r(a|\Pi) \neq r(b|\Pi)$, the maximum number of possible different partition representations for vertices of $S$ is given by $(c(H)+1)(2c(H)+1)$, i.e., for $t = 1$ there are at most $c(H)+1$ different vectors and for $t \in \{2, 3\}$ there are at most $2(c(H)+1)c(H)$. Hence, $n(c(H)+1) = |S| \leq (2c(H)+1)(c(H)+1)$ and, as a consequence, $n \leq 2c(H)+1$. Therefore, if $n > 2c(H)+1$, then $pd(G \odot H) \geq c(H) + 2$. \qed

Corollary 12. Let $G$ be a graph of order $n_1$ and let $n_2 \geq 2$ be an integer. If $n_1 > 2n_2 + 1$, then

$$pd(G \odot K_{n_2}) \geq n_2 + 2.$$
**Theorem 14.** Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be any graph. If $n > \beta(H) \geq 2$, then

$$pd(G \odot H) \geq \beta(H) + 1.$$ 

**Proof.** We will proceed similarly to the proof of Theorem 11. Let $S_i$ denote the set of isolated vertices of $H_i$, $i \in \{1, ..., n\}$.

Since different vertices of $S_i$ belong to different sets of any resolving partition for $G \odot H$, we have $pd(G \odot H) \geq \beta(H)$. Let us suppose $pd(G \odot H) = \beta(H)$ and let $\Pi(G \odot H) = \{A_1, A_2, ..., A_{\beta(H)}\}$ be a resolving partition for $G \odot H$. Now, let $S = \bigcup_{i=1}^{n}(S_i \cup \{v_i\})$ and let $u \in S$. If $u \in A_j \cap S_j$, $j \in \{1, ..., n\}$, then the partition representation of $u$ is given by

$$r(u|\Pi) = (2, 2, ..., 2, 0, 2, ..., 2, t, 2, ..., 2),$$

with $i, j \in \{1, ..., \beta(H)\}, i \neq j$ and $t \in \{1, 2\}$. On the other side, if $u \in A_j \cap V$, then

$$r(u|\Pi) = (1, 1, ..., 1, 0, 1, ..., 1),$$

with $j \in \{1, ..., \beta(H)\}$. Thus, the maximum number of possible different partition representations for vertices of $S$ is given by $(\beta(H) + 1)\beta(H)$. Hence, $n(\beta(H) + 1) = |S| \leq \beta(H)(\beta(H) + 1)$. Thus, $n \leq \beta(H)$. Therefore, if $n > \beta(H)$, then $pd(G \odot H) \geq \beta(H) + 1$. 

**Corollary 15.** Let $G$ be a graph of order $n_1$ and let $n_2 \geq 2$ be an integer. If $n_1 > n_2$, then

$$pd(G \odot N_{n_2}) \geq n_2 + 1.$$ 

**Proposition 16.** If $n_1 \geq n_2 \geq 2$, then

$$pd(P_{n_1} \odot N_{n_2}) = n_2 + 1.$$ 

**Proof.** Let $V = \{v_1, ..., v_n\}$ be the set of vertices of $P_{n_1}$ and, for each $v_i \in V$, let $V_i = \{u_{i1}, ..., u_{in_2}\}$ be the set of vertices of the $i^{th}$ copy of $N_{n_2}$ in $P_{n_1} \odot N_{n_2}$. Let $\Pi = \{A_1, ..., A_{n_2 + 1}\}$, where $A_1 = \{v_1, u_{11}\}$, $A_2 = \{v_i, u_{i1} : i \in \{2, ..., n_1\}\}$ and $A_j = \{u_{ij-1} : i \in \{1, ..., n_1\}\}$ for $j \in \{3, ..., n_2 + 1\}$. Note that $d_{P_{n_1} \odot N_{n_2}}(v_1, A_2) \neq d_{P_{n_1} \odot N_{n_2}}(u_{11}, A_2)$. Moreover, for two different vertices $x, y \in A_j, j \in \{3, ..., n_2 + 1\}$, we have $d_{P_{n_1} \odot N_{n_2}}(x, A_1) \neq d_{P_{n_1} \odot N_{n_2}}(y, A_1)$. Now on we suppose $x, y \in A_2$. If $x, y \in V$ or $x, y \in V_i$, for some $i$, then
Finally, if \( x \in V \) and \( y \notin V \), then \( d_{P_{n_1} \odot N_{n_2}}(x, A_3) \neq d_{P_{n_1} \odot N_{n_2}}(y, A_3) \). Therefore, \( \Pi \) is a resolving partition for \( P_{n_1} \odot N_{n_2} \) and, as a consequence, \( pd(P_{n_1} \odot N_{n_2}) \leq n_2 + 1 \). By corollary 15 we conclude the proof.

\[ \square \]

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