The Holevo-Schumacher-Westmoreland Channel Capacity for a Class of Qudit Unital Channels

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Abstract

Using the unique nature of the average output state of an optimal signalling ensemble, we prove that for a special class of qudit unital channels, the HSW channel capacity is \( C = \log_2(d) - \min_\rho S(\mathcal{E}(\rho)) \), where \( d \) is the dimension of the qudit. The result is extended to products of the same class of unital qudit channels. Thus, the connection between the minimum von Neumann entropy at the channel output and the transmission rate for classical information over quantum channels extends beyond the qubit domain.

1 Introduction

The Holevo-Schumacher-Westmoreland theorem tells us the asymptotic rate at which classical information can be transmitted over a quantum channel \( \mathcal{E} \) per channel use is given by the maximum output Holevo quantity \( \chi \) across all possible signalling ensembles.

\[
C = \max_{\{p_i, \rho_i\}} \chi(\{p_i, \rho_i = \mathcal{E}(\rho_i)\})
\]

Here \( \chi \) is the Holevo quantity of an ensemble \( \{p_i, \rho_i\} \), defined as

\[
\chi = S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i)
\]

where \( S \) is the von Neumann entropy.\(^1\) We call \( C \) the Holevo-Schumacher-Westmoreland (HSW) channel capacity.

We call any input ensemble \( \{p_i, \rho_i\} \) that achieves \( C \) an optimal ensemble. There may be several different optimal input ensembles which achieve the optimum HSW channel capacity \( C \). However, it was shown in\(^1\) that the average channel output state of an optimal ensemble is a unique state for all optimal ensembles for that channel. That is,

\(^1\)We shall use \( \rho \) to denote a density operator at the channel input, and \( \tilde{\rho} \) as the corresponding channel output density operator.
given a set of optimal input ensembles \( \{ p_i^{(1)} , \rho_i^{(1)} \} , \{ p_i^{(2)} , \rho_i^{(2)} \} , \cdots , \{ p_i^{(N)} , \rho_i^{(N)} \} \), all achieving \( C \), we define \( \tilde{\Phi}^{(k)} = \mathcal{E} \left( \sum_i p_i^{(k)} \rho_i^{(k)} \right) \). Then it has been shown we must have \( \tilde{\Phi}^{(1)} = \tilde{\Phi}^{(2)} = \cdots = \tilde{\Phi}^{(N)} \).

The main idea of this paper is the unique nature of the output ensemble average state of an optimal signalling ensemble for a quantum channel \( \mathcal{E} \) tells us a lot about \( C \) for that channel.

2 Background Material

2.1 Invariance of \( S \) and \( \chi \) under unitary operators

Consider any ensemble \( \{ p_i, \rho_i \} \). Acting on each \( \rho_i \) with the same unitary operator \( U \) yields a set of valid quantum states \( U \rho_i U^\dagger \) and the ensemble \( \{ p_i, U \rho_i U^\dagger \} \). Furthermore, each \( \rho_i \) has the same eigenvalues as the corresponding \( U \rho_i U^\dagger \). Since von Neumann entropy depends only on a density operators eigenvalues, we conclude \( S(\rho_i) = S(U \rho_i U^\dagger) \). Furthermore, this implies the Holevo quantity \( \chi \) of the ensembles \( \{ p_i, \rho_i \} \) and \( \{ p_i, U \rho_i U^\dagger \} \) is equal, since

\[
\chi \left( \{ p_i, U \rho_i U^\dagger \} \right) = S \left( \sum_i p_i U \rho_i U^\dagger \right) - \sum_i p_i S \left( U \rho_i U^\dagger \right) \tag{I}
\]

\[
= S \left( U \left( \sum_i p_i \rho_i \right) U^\dagger \right) - \sum_i p_i S \left( U \rho_i U^\dagger \right) 
\]

\[
= S \left( \sum_i p_i \rho_i \right) - \sum_i p_i S \left( \rho_i \right) = \chi \left( \{ p_i, \rho_i \} \right) .
\]

3 HSW Channel Capacity for single qubit unital channels

As an example of the approach we shall be taking, we derive the HSW channel capacity for single qubit unital channels. This result was previously derived in [4] by a different technique.

We describe a single qubit density operator using the Bloch sphere representation.

\[
\rho = \frac{1}{2} \left( \mathcal{I} + \vec{W}_\rho \cdot \vec{\sigma} \right)
\]

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The symbol $\vec{\sigma}$ is the vector of 2 by 2 Pauli matrices
\[
\vec{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} \quad \text{where} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

The Bloch vector $\vec{\rho}$ is a real, three dimensional vector which has magnitude equal to one when representing a pure state density matrix, and magnitude less than one for a mixed (non-pure) density matrix.

It was shown in [2] that the action of a single qubit unital channel $E$ on an input state $\rho$ could be represented as $\tilde{\rho} = E(\rho)$, where $\rho$ has Bloch vector $\begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$ and $\tilde{\rho}$ has Bloch vector $\begin{bmatrix} \lambda_x w_x \\ \lambda_y w_y \\ \lambda_z w_z \end{bmatrix}$. Here the $\lambda_k \in [-1, 1]$. Using the unique nature of the average output state of an optimal signalling ensemble, we shall show the HSW channel capacity $C$ is
\[
C = 1 - \max_\rho S(E(\rho)).
\]

### 3.1 Achievability of Output Ensembles

We say an ensemble $\{q_j, \phi_j\}$ at the channel output is achievable if there exists an input ensemble $\{q_j, \varphi_j\}$ such that the $\{\varphi_j\}$ are all valid density operators and $E(\varphi_j) = \phi_j \ \forall j$. Let us recall some properties of the Pauli matrices $\{\sigma_k\}$. The $\{\sigma_k\}$ obey the relations $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$ and $\sigma_i \sigma_j = I_2$ for $i = j$. Thus, we find $\sigma_i \sigma_j \sigma_i = -\sigma_j$ for $i \neq j$ and $\sigma_i \sigma_j \sigma_i = \sigma_i$ for $i = j$. The $\sigma_k$ are Hermitian, so $\sigma_k^2 = I_2$ implies the $\sigma_k$ are unitary, yielding $\sigma_k^* = \sigma_k$.

Let $\tilde{\rho}$ be an optimal input ensemble with corresponding output ensemble $\{p_i, E(\rho_i) = \tilde{\rho}_i\}$. Apply a Pauli operator $\sigma_k$ to all the density matrices in $\{p_i, E(\rho_i) = \tilde{\rho}_i\}$, yielding an ensemble $\{p_i, \sigma_k \tilde{\rho}_i \sigma_k^\dagger\}$. We know the density operators $\{\sigma_k \tilde{\rho}_i \sigma_k^\dagger\}$ are valid density operators because $\sigma_k$ is a unitary operator, and hence acting with $\sigma_k$ implements a change of basis at the channel output. The question we are interested in is whether the output ensemble $\{p_i, \sigma_k \tilde{\rho}_i \sigma_k^\dagger\}$ is achievable. To answer this, we know for each $\tilde{\rho}_i$, there is a valid input $\rho_i$ such that $E(\rho_i) = \tilde{\rho}_i$. Consider the following.

\[
\sigma_k \tilde{\rho}_i \sigma_k^\dagger = \sigma_k E(\rho_i) \sigma_k^\dagger = \sigma_k E \left( \frac{1}{2} \left( I_2 + \omega_x \sigma_x + \omega_y \sigma_y + \omega_z \sigma_z \right) \right) \sigma_k^\dagger
\]

\[
= \sigma_k \left( \frac{1}{2} \left( I_2 + \lambda_x \omega_x \sigma_x + \lambda_y \omega_y \sigma_y + \lambda_z \omega_z \sigma_z \right) \right) \sigma_k^\dagger
\]

\footnote{We write $I_d$ for the $d$ by $d$ identity matrix.}
Define \( \delta_{k,l} = 0 \) if \( k = l \), and 1 if \( k \neq l \). Note that \( \sigma_k \sigma_l \sigma_k = ( -1 )^{\delta_{k,l}} \sigma_l \). If \( \varphi_i \) has the Bloch vector 
\[
\begin{pmatrix}
-1^{\delta_{k,x}} \omega_x \\
-1^{\delta_{k,y}} \omega_y \\
-1^{\delta_{k,z}} \omega_z
\end{pmatrix},
\]
then the channel output of \( \varphi_i \) is
\[
\mathcal{E}(\varphi_i) = \frac{1}{2} \left( I_2 + ( -1 )^{\delta_{k,x}} \lambda_x \omega_x \sigma_x + ( -1 )^{\delta_{k,y}} \lambda_y \omega_y \sigma_y + ( -1 )^{\delta_{k,z}} \lambda_z \omega_z \sigma_z \right)
\]
\[
= \frac{1}{2} \left( I_2 + \lambda_x \omega_x \sigma_{kx} \sigma_k^\dagger + \lambda_y \omega_y \sigma_{ky} \sigma_k^\dagger + \lambda_z \omega_z \sigma_{kz} \sigma_k^\dagger \right) = \sigma_k \mathcal{E}(\varphi_i) \sigma_k^\dagger = \sigma_k \tilde{\rho}_i \sigma_k^\dagger.
\]

If we can show the \( \varphi_i \) are valid density operators, then we have shown that the output ensemble \( \{ p_i, \sigma_k \tilde{\rho}_i \sigma_k^\dagger \} \) is achievable. In order for \( \varphi_i \) to be a valid density operator, we must have the corresponding Bloch vector composed of three real entries, and the magnitude of the Bloch vector less than or equal to one. Since the \( \rho_i \) are valid density operators, the three \( \omega_k \) are real, and obey \( \omega_x^2 + \omega_y^2 + \omega_z^2 \leq 1 \). Now \( ( -1 )^{\delta_{k,l}} \) for \( k, l = \{ x, y, z \} \) is real and equal in magnitude to one. The magnitude of the Bloch vector for \( \varphi_i \) is 
\[
( -1 )^{\delta_{k,x}} \omega_x^2 + ( -1 )^{\delta_{k,y}} \omega_y^2 + ( -1 )^{\delta_{k,z}} \omega_z^2 \leq 1,
\]
where the last inequality follows from our knowledge that the \( \rho_i \) are valid density operators. Thus the \( \varphi_i \) are valid density operators. We conclude that if there exists an optimal input ensemble \( \rho \), with corresponding output ensemble \( \{ p_i, \mathcal{E}(\rho_i) = \tilde{\rho}_i \} \), then the ensemble \( \{ p_i, \sigma_k \tilde{\rho}_i \sigma_k^\dagger \} \) is achievable, with corresponding input ensemble \( \{ p_i, \varphi_i \} \). Furthermore, the input ensemble \( \{ p_i, \varphi_i \} \) is optimal, since \( \sigma_k \) is a unitary operator, and we showed in equation (1) that a unitary operator acting on an ensemble does not change the Holevo quantity of that ensemble. Since \( \{ p_i, \mathcal{E}(\rho_i) = \tilde{\rho}_i \} \) attained the maximal Holevo quantity \( \mathcal{C} \) at the channel output, the output ensemble \( \{ p_i, \sigma_k \tilde{\rho}_i \sigma_k^\dagger \} \) also has a Holevo value of \( \mathcal{C} \). Thus \( \{ p_i, \varphi_i \} \) is an optimal input ensemble.

To summarize, we first chose a basis of operators \( E_i \), in this case the identity \( I_2 \) and the three Pauli operators \( \{ \sigma_x, \sigma_y, \sigma_z \} \), in which to expand the density matrix \( \rho = \sum_i \alpha_i E_i \). Next, we found a set of unitary operators \( U_k \), in this case again the Pauli operators \( \sigma_k \), such that the \( U_k \) act on the \( E_i \) resulting in a multiplicative phase factor: \( U_k E_i U_k^\dagger = \kappa_{(k,i)} E_i \), where \( \kappa_{(k,i)} \) is a complex quantity. The unital nature of the qubit channel \( \mathcal{E} \) tells us that \( \mathcal{E}(E_i) = \lambda_i E_i \) \( \forall i \) in the operator basis \( \{ E_i \} \). This leads to the commutation of the channel \( \mathcal{E} \) with the set of unitaries \( \{ U_k \} = \{ \pm I_2, \pm \sigma_x, \pm \sigma_y, \pm \sigma_z \} \).

\[
U_k \mathcal{E}(E_i) U_k^\dagger = U_k \lambda_i E_i U_k^\dagger = \lambda_i U_k E_i U_k^\dagger = \lambda_i \kappa_{(k,i)} E_i = \kappa_{(k,i)} \mathcal{E}(E_i)
\]

( By linearity of quantum channels ) \( \mathcal{E}(\kappa_{(k,i)} E_i) = \mathcal{E} \left( U_k E_i U_k^\dagger \right) \).

Since we have an expansion of \( \rho \) in terms of the \( E_i \), using the linearity of quantum channels,
we conclude that
\[
U_k \mathcal{E}(\rho) U_k^\dagger = U_k \mathcal{E} \left( \frac{1}{2} \sum_i \alpha_i E_i \right) U_k^\dagger = U_k \left( \frac{1}{2} \sum_i \alpha_i \mathcal{E}(E_i) \right) U_k^\dagger = \frac{1}{2} \sum_i \alpha_i U_k \mathcal{E}(E_i) U_k^\dagger
\]

Equation (II) allows us to conclude that any optimal ensemble \( \rho \) yields an output ensemble \( \sigma \) and average output state \( \rho \) of an optimal input ensemble will be a critical tool in extending our unital qubit channel analysis to the determination of the Holevo-Schumacher-Westmoreland channel capacity \( \mathcal{C} \) for a special class of qudit unital channels.

### 3.2 Using Symmetry Properties of Optimal Ensembles

Consider a unital qubit channel with an optimal input ensemble \( \{p_i, \rho_i\} \), \(^3\) average input state \( \Phi = \sum_i p_i \rho_i \) and average output state \( \bar{\Phi} = \mathcal{E}(\Phi) \). Let \( \Phi \) have Bloch vector \( \bar{V} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \) and \( \bar{\Phi} \) have Bloch vector \( \bar{V}' = \begin{bmatrix} \tilde{v}_x \\ \tilde{v}_y \\ \tilde{v}_z \end{bmatrix} = \begin{bmatrix} \lambda_x v_x \\ \lambda_y v_y \\ \lambda_z v_z \end{bmatrix} \). Choose one of the three \( \{\sigma_k\} \) and apply this \( \sigma_k \) to the output states \( \bar{\rho_i} \) to obtain a new output ensemble \( \{p_i, \sigma_k \bar{\rho_i} \sigma_k^\dagger\} = \{p_i, \bar{\rho}'_i\} \). We know from our work above that the output ensemble \( \{p_i, \sigma_k \bar{\rho_i} \sigma_k^\dagger\} \) is achievable and optimal. The action of \( \sigma_k \) on the output ensemble \( \{p_i, \mathcal{E}(\rho_i) = \bar{\rho}_i\} \) generates a corresponding transformation of the average output state of the optimal ensemble \( \bar{\Phi} \),

\[
\sum_i p_i \sigma_k \bar{\rho_i} \sigma_k^\dagger = \sigma_k \left( \sum_i p_i \bar{\rho_i} \right) \sigma_k^\dagger = \sigma_k \bar{\Phi} \sigma_k^\dagger = \bar{\Phi}'.
\]

By the invariance property shown in [1], we have \( \bar{\Phi}' \equiv \bar{\Phi} \). Now \( \bar{\Phi} \) has Bloch vector \( \bar{V} = \begin{bmatrix} \bar{v}_x \\ \bar{v}_y \\ \bar{v}_z \end{bmatrix} = \begin{bmatrix} \lambda_x v_x \\ \lambda_y v_y \\ \lambda_z v_z \end{bmatrix} \) and \( \bar{\Phi}' \) has Bloch vector \( \bar{V}' = \begin{bmatrix} (-1)^{\delta_{k,x}} \bar{v}_x \\ (-1)^{\delta_{k,y}} \bar{v}_y \\ (-1)^{\delta_{k,z}} \bar{v}_z \end{bmatrix} \). For \( k = \{x, y, z\} \), \( \bar{\Phi} \equiv \bar{\Phi}' \) implies

\[
\bar{v}_x = (-1)^{\delta_{k,x}} \bar{v}_x \quad \text{and} \quad \bar{v}_y = (-1)^{\delta_{k,y}} \bar{v}_y \quad \text{and} \quad \bar{v}_z = (-1)^{\delta_{k,z}} \bar{v}_z.
\]

The only way all three relationships in equation (III) can be true \( \forall k = \{x, y, z\} \) is if \( \bar{v}_x = \bar{v}_y = \bar{v}_z = 0 \). The fact \( \bar{\Phi} \) has Bloch vector \( \bar{V} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) leads to the conclusion that

\(^3\)That such an ensemble exists was shown in [3].
\[ \tilde{\Phi} = \frac{1}{2} \left( I_2 + \vec{V} \cdot \vec{\sigma} \right) = \frac{1}{2} I_2 \text{ for all optimal ensembles.} \]

A second way to see that \( \tilde{\Phi} \equiv \frac{1}{2} I_2 \) is via Schur’s lemma \([4]\). Consider the group \( \mathcal{H} \) composed of the eight operations \( \{ \pm I_2, \pm \sigma_x, \pm \sigma_y, \pm \sigma_z \} \). A necessary and sufficient condition for a finite group \( \mathcal{G} \) to be irreducible is if the relation \( \frac{1}{||\mathcal{G}||} \sum_{g \in \mathcal{G}} \left| Trace[g] \right|^2 = 1 \) is true \([4]\). Here \( ||\mathcal{G}|| \) is the order of the group \( \mathcal{G} \). Noting our group \( \mathcal{H} \) above is finite, and computing the sum with our group \( \mathcal{H} \), we find \( \mathcal{H} \) is irreducible.

Schur’s lemma states that if a group \( \mathcal{G} \) is irreducible and has a \( d \)-dimensional representation \( \Gamma(\mathcal{G}) \) in which each representation element \( \Gamma(g) \) commutes with a \( d \times d \) matrix \( M \forall g \in \mathcal{G} \), then \( M \) is proportional to \( I_d \) \([4]\). The fact that we found \( \sigma_k \tilde{\Phi} \sigma_k^\dagger = \tilde{\Phi} \ \forall k \in \{x, y, z\} \), together with the same trivial result for \( I_2 \), implies that all elements of \( \mathcal{H} \) commute with \( \tilde{\Phi} \) and thus \( \tilde{\Phi} \propto I_2 \). The trace condition \( Trace(\tilde{\Phi}) = 1 \) leads us to conclude \( \tilde{\Phi} = \frac{1}{2} I_2 \).

Having determined \( \tilde{\Phi} \), we can now rewrite the Holevo-Schumacher-Westmoreland channel capacity \( C \) as \( C = \log_2(2) - \sum_i p_i S(\mathcal{E}(\rho_i)). \) To further simplify this result, we use two results from \([3]\). In their paper, Schumacher and Westmoreland worked with the relative entropy function, \( D[|\rho\rangle\langle\rho|] \) defined as \( Trace[|\rho\rangle\langle\rho| - \rho \log_2(\phi)] \). Using \( D \), they proved the following two results.

I) The equal distance property of optimal ensembles.
For any optimal ensemble \( \sigma \), we have
\[ D[\mathcal{E}(\rho_i)\|\mathcal{E}(\Phi)] = C \ \forall i. \] (IV)

II) The sufficiency of the maximal distance property.
For any optimal ensemble \( \sigma \) with average input state \( \Phi = \sum_i p_i \rho_i \), we have
\[ D[\mathcal{E}(\phi)\|\mathcal{E}(\Phi)] \leq C \text{ for any input density matrix } \phi. \] (V)

In both I) and II), \( \Phi = \sum_i p_i \rho_i \) and \( C \) is the Holevo-Schumacher-Westmoreland channel capacity. For the case of qubit unital channels, we have found that every optimal ensemble \( \sigma \) must obey \( \mathcal{E}(\sum_i p_i \rho_i) = \frac{1}{2} I_2 \). Looking at the relative entropy formula, we see that
\[ D[\mathcal{E}(\phi)\|\frac{1}{2} I_d] = \log_2(d) - S(\mathcal{E}(\phi)), \] where \( S \) is the von Neumann entropy and \( \phi \) is any input density matrix. (For more details on this derivation, please see the appendices in \([1]\).) Using the fact that for qubit unital channels we have found, for all optimal ensembles \( \sigma \), that \( \mathcal{E}(\sum_i p_i \rho_i) = \frac{1}{2} I_2 \), the above two Schumacher and Westmoreland results become, in the qubit unital channel case,
\[ I') \quad 1 - S(\mathcal{E}(\rho_i)) = C \quad \forall i \quad \text{implying} \quad S(\mathcal{E}(\rho_i)) = S(\mathcal{E}(\rho_j)) \quad \forall i, j. \quad (VI) \]

\[ II') \quad 1 - S(\mathcal{E}(\phi)) \leq C \quad \forall \text{input density matrices } \phi. \quad (VII) \]

We know that II') is achieved with equality when \( \phi \) is any of the \( \rho_i \) in the optimal ensemble \( \omega \). Thus I') and II') taken together yield \( 1 - S(\mathcal{E}(\phi)) \leq 1 - S(\mathcal{E}(\rho_i)) \) or \( S(\mathcal{E}(\phi)) \geq S(\mathcal{E}(\rho_i)) \), which, since \( \phi \) can be any input density matrix, implies \( S(\mathcal{E}(\rho_i)) = \min_\phi S(\mathcal{E}(\phi)) \).

Plugging this result into I') yields our final result for the Holevo-Schumacher-Westmoreland channel capacity for qubit unital channels.

\[ C = 1 - \min_\phi S(\mathcal{E}(\phi)). \]

For qubit unital channels, the minimum channel output von Neumann entropy determines the Holevo-Schumacher-Westmoreland channel capacity \( C \).

### 3.3 Ensemble Achievability

The achievability of a transformed output ensemble is a concept worth emphasizing. In our discussion of unital qubit channels, the reason why we could conclude the average output state of an optimal ensemble commuted with all eight members of our group \( \mathcal{H} = \{\pm I_2, \pm \sigma_x, \pm \sigma_y, \pm \sigma_z\} \) was because, given an optimal ensemble \( \omega \), each of the eight output ensembles \( \{p_i, h\tilde{\rho}_i h^{-1}\} \), where \( h \in \mathcal{H} \), was achievable. The existence of an optimal input ensemble \( \{p_i, \phi_i\} \) which maps via the quantum channel \( \mathcal{E} \) to \( \{p_i, h\tilde{\rho}_i h^{-1}\} \) is what allowed us to conclude the relationship \( h\hat{\Phi}h^{-1} = \hat{\Phi} \) was valid, and apply Schur’s lemma.

For a generic group \( \mathcal{M} \) acting on the channel output of an optimal ensemble \( \omega \), there will typically be \( m_0 \in \mathcal{M} \) such that \( \{p_i, m_0\tilde{\rho}_i m_0^{-1}\} \) are not achievable ensembles. In these cases, we cannot conclude \( m_0\hat{\Phi}m_0^{-1} = \hat{\Phi} \) holds, where \( \hat{\Phi} \) is the average output state of an optimal ensemble. Yet it was the fact that \( m_0\hat{\Phi}m_0^{-1} = \hat{\Phi} \) holds \( \forall m \in \mathcal{M} \) that led us to apply Schur’s Lemma and conclude \( \hat{\Phi} \propto I_2 \). The lack of achievability for one or more of the transformed output ensembles \( \{p_i, m\tilde{\rho}_i m^{-1}\} \) prevents us from appealing to Schur’s Lemma. An example of the limitations to determining HSW channel capacity which results from output ensemble non-achievability arises in the case of non-unital qubit channels.

### 3.4 A Non-Unital Qubit Channel Example

Our technique fails for non-unital qubit channels. The reason why is the lack of achievability of output ensembles generated by members of the Pauli group acting on an output optimal
ensemble. For example, consider the non-unital linear qubit channel specified in the Ruskai-King-Swarez-Werner notation as \(\{t_x = t_y = 0, t_z = 0.2, \lambda_x = \lambda_y = 0, \lambda_z = 0.4\}\). This channel maps an input Bloch vector \(\vec{W}\) to an output Bloch vector \(\vec{W}'\) as:

\[
\vec{W} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \\ t_z + \lambda_z w_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.2 + 0.4w_z \end{bmatrix} = \vec{W}'.
\]

By inspection, an optimal input ensemble is \(\sigma\) with \(\rho_{1.2} = \frac{1}{2}(I_2 \pm \sigma_z)\), and corresponding output density matrices \(\rho_1 = \frac{1}{2}(I_2 - 0.2\sigma_z)\) and \(\rho_2 = \frac{1}{2}(I_2 + 0.6\sigma_z)\). Numerical analysis for this channel indicates the optimum output average state is \(\Phi \approx \frac{1}{2}(I_2 + 0.2125\sigma_z)\). Since \(\Phi \neq \frac{1}{2}I_2\), we anticipate we will not be able to meet the conditions for the application of Schur’s lemma.

Consider applying the unitary operator \(\sigma_x\) to the output optimal ensemble \(\{p_i, \mathcal{E}(p_i) = \rho_i\}\) determined in the previous paragraph. We obtain

\[
\sigma_z \rho_1 \sigma_z = \sigma_z \left(\frac{1}{2}(I_2 - 0.2\sigma_z)\right) \sigma_z^\dagger = \rho_1 \text{ and } \sigma_z \rho_2 \sigma_z = \sigma_z \left(\frac{1}{2}(I_2 + 0.6\sigma_z)\right) \sigma_z^\dagger = \rho_2.
\]

Thus the output ensemble \(\{p_i, \sigma_x \mathcal{E}(p_i) \sigma_x^\dagger = \sigma_z \rho_1 \sigma_z^\dagger\}\) is identical to the output ensemble \(\{p_i, \mathcal{E}(p_i) = \rho_i\}\), both being generated by the input ensemble \(\sigma\). Thus the output ensemble \(\{p_i, \sigma_x \mathcal{E}(p_i) \sigma_x^\dagger = \sigma_z \rho_1 \sigma_z^\dagger\}\) is an achievable output ensemble.

The application of \(\sigma_x\) or \(\sigma_y\) to \(\{p_i, \mathcal{E}(p_i) = \rho_i\}\) however does not yield an achievable ensemble. To see why, consider applying \(\sigma_x\) to \(\rho_2 = \frac{1}{2}(I_2 + 0.6\sigma_z)\), which since \(\sigma_x \sigma_z \sigma_x^\dagger = -\sigma_z\), yields the output density operator \(\rho_2' = \frac{1}{2}(I_2 + 0.6\sigma_z)\). The corresponding input density operator would have Bloch vector \(\vec{W}' = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}\), which is not a valid qubit density operator, since \(\|\vec{W}'\| > 1\). Since the output state \(\sigma_z \rho_2 \sigma_z^\dagger\) can never be mapped to by a valid input qubit density operator, we cannot assume the relation \(\sigma_z \Phi \sigma_z^\dagger = \Phi\) holds. Thus, we do not have the necessary Schur commutation requirement that \(g\Phi = \Phi g\) for all members \(g\) of the Pauli group \(\{\pm I_2, \pm \sigma_x, \pm \sigma_y, \pm \sigma_z\}\), and hence cannot conclude \(\Phi = \frac{1}{2}I_2\), as we anticipated.

As we shall develop in more detail below, working with qudits, if we can find a group \(\mathcal{G}\) such that we are assured all elements \(g \in \mathcal{G}\) are unitary and acting on the output states of an optimal ensemble \(\{\rho_i\}\) yield achievable ensembles \(\forall g \in \mathcal{G}\), then we will be able to conclude the average output state of any optimal ensemble is \(\Phi = \frac{1}{2}I_d\). From this conclusion, we can use the Schumacher-Westmoreland relative entropy results from equations [V],[VI], and [VII] to conclude the states in any input optimal ensemble must be a subset of those
input states which yield the minimum output von Neumann entropy. This in turn leads us to a HSW channel capacity $C$ of

$$C = \log_2(d) - \min_\phi S(\mathcal{E}(\phi))$$

for those qudit channels to which we can successfully apply Schur’s lemma. We now proceed to determine the subset of qudit channels which meet the Schur’s lemma requirements.

### 4 Qudit Channels

The HSW channel capacity result for unital qubit channels was previously proven in [2] by a method which did not generalize to the general qudit case (i.e. for qudit dimension $d > 2$). The technique discussed in this paper does generalize to a special subclass of unital qudit channels. Before describing that generalization, we present some background material on qudits and qudit channels.

#### 4.1 Qudits

A qudit is a system with $d$ orthogonal pure states $|j\rangle$, $j = 0, 1, 2, \cdots, d-1$. The generalization of the qubit Pauli operators $\sigma_x$ and $\sigma_z$ are the two operators $\hat{X}$ and $\hat{Z}$, whose action on the states $|j\rangle$ is $\hat{X}|j\rangle = |j+1 \text{ (mod } d)\rangle$ and $\hat{Z}|j\rangle = \Omega^j |j\rangle$. Here $\Omega = e^{2\pi i/d}$. The extension of the qubit Bloch representation for a density matrix $\rho$ to qudits is shown in appendix A to be

$$\rho = \frac{1}{d} \sum_{a,b\in\{0,1,2,\cdots,d-1\}} \alpha_{a,b} \hat{X}^a \hat{Z}^b.$$ 

The $\alpha_{a,b}$ are complex quantities. Define $E_{a,b} = \hat{X}^a \hat{Z}^b$. Note that $E_{0,0} = I_d$. In appendix A it is shown $\text{Trace}(E_{a,b}) = d \delta_{a,0} \delta_{b,0}$, where $\delta$ is the Kronecker delta function. The trace condition $\text{Trace}(\rho) = 1$ allows us to conclude $\alpha_{0,0} = 1$. Let $\Upsilon$ denote the set of $d^2 - 1$ elements $a, b \in \{0, 1, 2, \cdots, d-1\}$ with the exception that $a$ and $b$ cannot both be zero. Then we can write the qudit density matrix $\rho$ as $\rho = \frac{1}{d} \left( I_d + \sum_{(a,b)\in\Upsilon} \alpha_{a,b} E_{a,b} \right)$. A qudit quantum channel $\mathcal{E}$ is a linear map. One can write such a map as a $d^2$ by $d^2$ complex matrix $M$ taking the $d^2$ vector of coefficients $\alpha_{a,b}$ of $\rho$ to the $d^2$ set of coefficients $\tilde{\alpha}_{a,b}$ of $\tilde{\rho} = \mathcal{E}(\rho)$.

If the qudit quantum channel $\mathcal{E}$ is unital, meaning $\mathcal{E}(I_d) = I_d$, then the first row and column of $M$ are a one followed by $d^2 - 1$ zeros. Hence we can represent a qudit unital channel by a matrix $N$ of $d^2 - 1$ by $d^2 - 1$ complex entries mapping the vector of $d^2 - 1$

---

4 Our qudit matrix development in which we write $\mathcal{E}$ as a $d^2$ by $d^2$ matrix closely follows work done in [8] for the unital qubit channel case.
coefficients $\alpha_{(a,b)}$, with $(a, b) \in \mathcal{Y}$, representing $\rho$ to the vector of $d^2 - 1$ coefficients $\tilde{\alpha}_{(a,b)}$, with $(a, b) \in \mathcal{Y}$, representing $\tilde{\rho} = \mathcal{E}(\rho)$. The specific class of qudit channels we shall be interested in are those completely positive unital quantum channels for which $N$ is diagonal. This class of channels is nonempty. For example, consider the channel corresponding to all unitary operators, which maps any input density matrix to itself. This channel has all ones on the diagonal of the matrix $N$.

The approach we take to determine the HSW channel capacity for this special class of diagonal unital qudit channels closely follows our unital qubit channel derivation above. Note the operators $E_{a,b}$ are unitary. Using the commutation relation shown in appendix A, \[ \hat{Z} \hat{X} = \Omega \hat{Z} \hat{X}, \] where $\Omega = e^{\frac{2\pi i}{d}}$, we have

\[ E_{a,h} E_{a,b} E_{a,h}^\dagger = \hat{X}^g \hat{Z}^b \hat{X}^a \hat{Z}^{-h} \hat{X}^{-g} = \Omega^{ah} \hat{X}^g \hat{X}^a \hat{Z}^{-h} \hat{Z}^b \hat{X}^{-g} \quad \text{(VIII)} \]

\[ = \Omega^{ah} \hat{X}^g \hat{X}^a \hat{Z}^{-h} \hat{X}^b = \Omega^{ah} \Omega^{-bg} \hat{X}^g \hat{X}^a \hat{Z}^{-h} \hat{Z}^b \]

\[ = \Omega^{ah} \Omega^{-bg} \hat{X}^a \hat{Z}^b = \Omega^{ah-bg} E_{a,b}. \]

Define $F_{a,b,c} = \Omega^c E_{a,b}$, where $a, b, c \in \{0, 1, 2, \cdots, d-1\}$. Since $\Omega^c$ and the $E_{a,b}$ are unitary operators, $F_{a,b,c}$ is a unitary operator. The action of the $F_{a,b,c}$ on a diagonal unital qudit channel output density operator $\tilde{\rho}$ is

\[ F_{a,b,c} \tilde{\rho} F_{a,b,c}^\dagger = E_{a,b} \tilde{\rho} E_{a,b}^\dagger = E_{a,b} \mathcal{E}(\rho) E_{a,b} \]

\[ = E_{a,b} \frac{1}{d} \left( I_d + \sum_{(q,r) \in \mathcal{Y}} \lambda_{q,r} \alpha_{q,r} \mathcal{E}(q,r) E_{a,b} \right) E_{a,b}^\dagger = \frac{1}{d} \left( I_d + \sum_{(q,r) \in \mathcal{Y}} \lambda_{q,r} \alpha_{q,r} \mathcal{E}(q,r) E_{a,b} \right) \]

\[ = \frac{1}{d} \left( I_d + \sum_{(q,r) \in \mathcal{Y}} \lambda_{q,r} \alpha_{q,r} \mathcal{E}(q,r) E_{a,b} \right) = \mathcal{E} \left( \frac{1}{d} \left( I_d + \sum_{(q,r) \in \mathcal{Y}} \alpha_{q,r} \mathcal{E}(q,r) E_{a,b} \right) \right) \]

\[ = \mathcal{E} \left( E_{a,b} \frac{1}{d} \left( I_d + \sum_{(q,r) \in \mathcal{Y}} \alpha_{q,r} \mathcal{E}(q,r) E_{a,b} \right) E_{a,b}^\dagger \right) = \mathcal{E} \left( E_{a,b} \tilde{\rho} E_{a,b} \right) = \mathcal{E} \left( F_{a,b,c} \tilde{\rho} F_{a,b,c}^\dagger \right). \]

Since the $F_{a,b,c}$ are unitary operators, we conclude that given any optimal input ensemble $\mathcal{E}$, the output ensemble $\Theta_{a,b,c}$ obtained by applying $F_{a,b,c}$ to $\{p_1, \mathcal{E}(\rho_1) = \tilde{\rho_1}\}$ is achievable and $\Theta_{a,b,c}$ has the optimal input ensemble $\{p_1, \phi_i = F_{a,b,c} \rho_1 F_{a,b,c}^\dagger\}$. Each of the $\phi_i$ is a valid input density operator due to the fact that $F_{a,b,c}$ is a unitary operator, and is implementing a change of basis on $\rho_i$.

The set of operators $\{F_{a,b,c}\}$ forms a group of order $d^3$ which we shall call $\mathcal{Q}$. Recall our theorem for proving a finite group is reducible. The group $\mathcal{Q}$ is reducible since $\text{Trace} \left[ F_{a,b,c} \right]$ equals zero when either $a$ and $b$ are non-zero, and $\text{Trace} \left[ F_{a,b,c} \right]$ equals
5.1 The mapping \( \{a_1, a_2, a_3, \cdots, a_{N-1}, a_N\} \Leftrightarrow \{a\} \). The set of possible coefficients \( \{a_1, a_2, a_3, \cdots, a_{N-1}, a_N\} \) and the set of possible \( \{a^{\otimes}\} \) both have \( d \) elements, where \( d = \prod_{k=1}^{N} d_k \). Here the \( \{a_k\} \in \{0, 1, 2, \cdots, d_k - 1\} \) and \( \{a\} \in \{0, 1, 2, \cdots, d\} \). There are many bijective mappings between these two sets, and it is useful to have one particular map in mind as we proceed. The one we shall use is presented in the table below.
Thus we conclude

Another property of the $E$ and so on. Below, we associate an $E_{a,b}$ with the tensor product $\{E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)}\}$ by using this mapping twice, once for the association $\{a_1,a_2,a_3,\cdots,a_{N-1},a_N\} \iff \{a^\otimes\}$ and again for $\{b_1,b_2,b_3,\cdots,b_{N-1},b_N\} \iff \{b^\otimes\}$.

### 5.2 Orthonormality of the $\{E_{a,b}\}$

The operators $E_{a,b}$ form, with respect to the Hilbert-Schmidt norm, a set of $d^2$ orthogonal operators. The orthogonality of the $\{E_{a,b}\}$ is inherited from the orthogonality of the operators $\{E_{a,b}\}$, which is shown in appendix A, equation (XIV). Using properties of tensors from \([4]\), we have

$$\begin{align*}
\langle E_{a,b}^\otimes, E_{g,h}^\otimes \rangle &= \text{Trace} \left[ E_{a,b}^\otimes E_{g,h}^\otimes \right] \\
&= \text{Trace} \left[ \left( E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)} \right) \right. \times \left. \left( E_{g_1,h_1}^{(1)} \otimes E_{g_2,h_2}^{(2)} \otimes \cdots \otimes E_{g_N,h_N}^{(N)} \right) \right] \\
&= \text{Trace} \left[ \left( E_{a_1,b_1}^{(1)} \otimes E_{g_1,h_1}^{(1)} \right) \times \left( E_{a_2,b_2}^{(2)} \otimes E_{g_2,h_2}^{(2)} \right) \times \cdots \times \left( E_{a_N,b_N}^{(N)} \otimes E_{g_N,h_N}^{(N)} \right) \right] \\
&= \text{Trace} \left[ E_{a_1,b_1}^{(1)} E_{g_1,h_1}^{(1)} \right] \times \text{Trace} \left[ E_{a_2,b_2}^{(2)} E_{g_2,h_2}^{(2)} \right] \times \cdots \times \text{Trace} \left[ E_{a_N,b_N}^{(N)} E_{g_N,h_N}^{(N)} \right] \\
&= (d_1 \delta_{a_1,g_1} \delta_{b_1,h_1}) (d_2 \delta_{a_2,g_2} \delta_{b_2,h_2}) \cdots (d_N \delta_{a_N,g_N} \delta_{b_N,h_N}) = d \delta_{a,g} \delta_{b,h},
\end{align*}$$

where we used our map between the sets $\{a^{(k)},b^{(k)}\} \rightarrow \{a^\otimes,b^\otimes\}$, and the fact $d = \prod_{k=1}^{N} d_{k}$. Thus we conclude $\langle E_{a,b}^\otimes, E_{g,h}^\otimes \rangle = \text{Trace} \left[ E_{a,b}^\otimes E_{g,h}^\otimes \right] = \delta_{a,g} \delta_{b,h}$. The orthogonality of the $\{E_{a,b}\}$ means we can expand $\rho^\otimes$ in terms of the $\{E_{a,b}\}$, yielding $\rho^\otimes = \frac{1}{d} \sum_{a,b \in \{0,1,\cdots,d-1\}} \alpha_{a,b} E_{a,b}^\otimes$.

Another property of the $E_{a,b}$ we shall need is the result of $E_{g,h}^\otimes E_{a,b}^\otimes E_{g,h}^\dagger$. Using equation (VIII), and the tensor nature of $E_{a,b}$, we have $E_{g,h}^\otimes E_{a,b}^\otimes E_{g,h}^\dagger = \left( E_{g_1,h_1}^{(1)} \otimes E_{g_2,h_2}^{(2)} \otimes \cdots \otimes E_{g_N,h_N}^{(N)} \right) \left( E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)} \right) \left( E_{g_1,h_1}^{(1)} \otimes E_{g_2,h_2}^{(2)} \otimes \cdots \otimes E_{g_N,h_N}^{(N)} \right)^\dagger$.
where \( \omega_k = e^{\frac{2\pi i}{N}} \), \( \Omega = e^{\frac{2\pi i}{d}} \), and \( c = \sum_{k=1}^{k=N} (a_kh_k - b_kg_k) \frac{4}{d_k} \).

5.3 The channel \( \mathcal{E}^\otimes \) is unital and diagonal in the \( E_{a,b}^\otimes \) basis.

The channel \( \mathcal{E}^\otimes \) is diagonal in the \( E_{a,b}^\otimes \) basis. To see this, note that

\[
\mathcal{E}^\otimes \left(E_{a,b}^\otimes\right) = \mathcal{E}^\otimes \left(E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)}\right) = \mathcal{E}^{(1)} \otimes \mathcal{E}^{(2)} \otimes \cdots \otimes \mathcal{E}^{(N)} \left(E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)}\right)
\]

By using our bijective map \( \{a^{(k)},b^{(k)}\} \leftrightarrow \{a^\otimes,b^\otimes\} \) to move back and forth between the operator basis set \( \{E_{a,b}^\otimes\} \) and the operator basis set \( \{E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)}\} \). Thus the tensor product of diagonal qudit channels yields a diagonal qudit channel.

Next note that \( E_{0,0}^\otimes = E_{0,0}^{(1)} \otimes E_{0,0}^{(2)} \otimes \cdots \otimes E_{0,0}^{(N)} = I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes I_{d_N} = I_d \). Taking a special case of the result in equation (XII), we obtain

\[
\mathcal{E}^\otimes (I_d) = \mathcal{E}^\otimes \left(E_{0,0}^\otimes\right) = \mathcal{E}^{(1)} \left(E_{0,0}^{(1)}\right) \otimes \mathcal{E}^{(2)} \left(E_{0,0}^{(2)}\right) \otimes \cdots \otimes \mathcal{E}^{(N)} \left(E_{0,0}^{(N)}\right)
\]

We conclude that \( \mathcal{E}^\otimes (I_d) = I_d \), and the channel \( \mathcal{E}^\otimes \) is unital. Thus the tensor product of diagonal, unital qudit channels yields a diagonal unital qudit channel.
As an example, consider the product of two qubit (diagonal) unital channels, $\mathcal{E}^{(1)}$ with diagonal parameters $\{\lambda_1, \lambda_2, \lambda_3\}$, and $\mathcal{E}^{(2)}$ with diagonal parameters $\{\xi_1, \xi_2, \xi_3\}$. The product channel $\mathcal{E}^{\otimes} = \mathcal{E}^{(1)} \otimes \mathcal{E}^{(2)}$ is a diagonal, unital channel, taking an input vector of $(d_1 d_2)^2 - 1 = 4^2 - 1 = 15$ input density matrix coefficients $\alpha_{a,b}$ to the output density matrix coefficients $\tilde{\alpha}_{a,b}$, as shown below.

\[
\begin{aligned}
\{ & \text{basis element } I_2 \otimes \sigma_x \} \\
\{ & \text{basis element } I_2 \otimes \sigma_y \} \\
\{ & \text{basis element } I_2 \otimes \sigma_z \} \\
\{ & \text{basis element } \sigma_x \otimes I_2 \} \\
\{ & \text{basis element } \sigma_y \otimes I_2 \} \\
\{ & \text{basis element } \sigma_z \otimes I_2 \} \\
\{ & \text{basis element } \sigma_x \otimes \sigma_x \} \\
\{ & \text{basis element } \sigma_y \otimes \sigma_y \} \\
\{ & \text{basis element } \sigma_z \otimes \sigma_z \} \\
\{ & \text{basis element } \sigma_x \otimes \sigma_y \} \\
\{ & \text{basis element } \sigma_x \otimes \sigma_z \} \\
\{ & \text{basis element } \sigma_y \otimes \sigma_x \} \\
\{ & \text{basis element } \sigma_y \otimes \sigma_y \} \\
\{ & \text{basis element } \sigma_y \otimes \sigma_z \} \\
\{ & \text{basis element } \sigma_z \otimes \sigma_x \} \\
\{ & \text{basis element } \sigma_z \otimes \sigma_y \} \\
\{ & \text{basis element } \sigma_z \otimes \sigma_z \} \\
& \end{aligned}
\]
\[
\begin{bmatrix}
\alpha_{0,1} \\
\alpha_{0,2} \\
\alpha_{0,3} \\
\alpha_{1,0} \\
\alpha_{2,0} \\
\alpha_{3,0} \\
\alpha_{1,1} \\
\alpha_{1,2} \\
\alpha_{1,3} \\
\alpha_{2,1} \\
\alpha_{2,2} \\
\alpha_{2,3} \\
\alpha_{3,1} \\
\alpha_{3,2} \\
\alpha_{3,3} \\
\end{bmatrix} \xrightarrow{\mathcal{E}}
\begin{bmatrix}
\tilde{\alpha}_{0,1} = \xi_1 \alpha_{0,1} \\
\tilde{\alpha}_{0,2} = \xi_2 \alpha_{0,2} \\
\tilde{\alpha}_{0,3} = \xi_3 \alpha_{0,3} \\
\tilde{\alpha}_{1,0} = \lambda_1 \alpha_{1,0} \\
\tilde{\alpha}_{2,0} = \lambda_2 \alpha_{2,0} \\
\tilde{\alpha}_{3,0} = \lambda_3 \alpha_{3,0} \\
\tilde{\alpha}_{1,1} = \lambda_1 \xi_1 \alpha_{1,1} \\
\tilde{\alpha}_{1,2} = \lambda_1 \xi_2 \alpha_{1,2} \\
\tilde{\alpha}_{1,3} = \lambda_1 \xi_3 \alpha_{1,3} \\
\tilde{\alpha}_{2,1} = \lambda_2 \xi_1 \alpha_{2,1} \\
\tilde{\alpha}_{2,2} = \lambda_2 \xi_2 \alpha_{2,2} \\
\tilde{\alpha}_{2,3} = \lambda_2 \xi_3 \alpha_{2,3} \\
\tilde{\alpha}_{3,1} = \lambda_3 \xi_1 \alpha_{3,1} \\
\tilde{\alpha}_{3,2} = \lambda_3 \xi_2 \alpha_{3,2} \\
\tilde{\alpha}_{3,3} = \lambda_3 \xi_3 \alpha_{3,3} \\
\end{bmatrix}
\]

5.4 The average output state of an optimal ensemble $\tilde{\Phi}$ is $\propto I_d$ for $\mathcal{E}^{\otimes}$.

Define the set of $d^3$ operators $\{F_{a,b,c}^{\otimes}\}$ as $F_{a,b,c}^{\otimes} = e^{\frac{2\pi i}{d^3}} E_{a,b}^{\otimes}$. Using our bijective map between the sets $\{a_1, a_2, a_3, \ldots, a_{N-1}, a_N\}$ and $\{a\}$, we expand $F_{a,b,c}^{\otimes}$ in terms of a phase $e^{\frac{2\pi i}{d^3}}$ and the matrix elements of the individual operators $\{E_{a,b}^{(k)}\}$. Our expression for $F_{a,b,c}^{\otimes}$ becomes

$$F_{a,b,c}^{\otimes} = e^{\frac{2\pi i a}{d}} E_{a,b}^{\otimes} = e^{\frac{2\pi i}{d}} E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)}.$$ 

The set of operators $\{F_{a,b,c}^{\otimes}\}$ are the product of a phase $e^{\frac{2\pi i}{d}}$ and the tensor products of the individual operators $\{E_{a,b}^{(k)}\}$. The $\{F_{a,b,c}^{\otimes}\}$ are unitary operators, inheriting this behavior from the unitary nature of the phase factor and the unitary nature of the subsystem operators $\{E_{a,b}^{(k)}\}$. To see this, note

$$F_{a,b,c}^{\otimes} F_{a,b,c}^{\otimes \dagger} = \left( e^{\frac{2\pi i}{d}} E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)} \right) \left( e^{\frac{2\pi i}{d}} E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)} \right)$$

$$= e^{\frac{2\pi i}{d}} e^{\frac{2\pi i}{d}} \left( E_{a_1,b_1}^{(1)\dagger} \otimes E_{a_2,b_2}^{(2)\dagger} \otimes \cdots \otimes E_{a_N,b_N}^{(N)\dagger} \right) \left( E_{a_1,b_1}^{(1)} \otimes E_{a_2,b_2}^{(2)} \otimes \cdots \otimes E_{a_N,b_N}^{(N)} \right)$$

$$= 1 \left( E_{a_1,b_1}^{(1)\dagger} E_{a_1,b_1}^{(1)} \right) \otimes \left( E_{a_2,b_2}^{(2)\dagger} E_{a_2,b_2}^{(2)} \right) \otimes \cdots \otimes \left( E_{a_N,b_N}^{(N)\dagger} E_{a_N,b_N}^{(N)} \right)$$

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\[ I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes I_{d_N} = I_d, \]

where we used the unitary nature of the individual \( E^{(k)}_{a_k, b_k} \) to say \( E^{(k)}_{a_k, b_k} E^{(k)}_{a_k, b_k} = I_{d_k} \).

The \( \{F^{\otimes}_{a,b,c}\} \) form an irreducible group which we shall call \( Q \). To see why \( Q \) is irreducible, recall our relation for irreducibility from [4] discussed above. A necessary and sufficient condition for a finite group \( G \) to be irreducible is if the relation

\[ \frac{1}{|G|} \sum_{g \in G} \left| \text{Trace}[g] \right|^2 = 1 \]

is true\[4]. Here \( |G| \) is the order of the group \( G \). Let the group \( Q \) be the set \( \{F^{\otimes}_{a,b,c}\} \), where \( a, b, c \in \{0, 1, 2, \ldots, d - 1\} \). \( Q \) is of order \( d^3 \) and hence finite. Previously, we noted that \( E_{0,0} = I_d \) and \( \text{Trace}\left[ E^{\otimes}_{a,b}, E^{\otimes}_{g,h} \right] = \delta_{a,g} \delta_{b,h} \). Thus \( \text{Trace}\left[ E^{\otimes}_{a,b} \right] = d \delta_{a,0} \delta_{b,0} \). Computing the Trace sum yields

\[ \frac{1}{|Q|} \sum_{q \in Q} \left| \text{Trace}[q] \right|^2 = \frac{1}{d^3} \sum_{c \in \{0,1,2,\ldots,d-1\}} \sum_{b \in \{0,1,2,\ldots,d-1\}} \sum_{a \in \{0,1,2,\ldots,d-1\}} \left| \text{Trace}[E^{\otimes}_{a,b,c}] \right|^2 \]

\[ = \frac{1}{d^3} \sum_{c \in \{0,1,2,\ldots,d-1\}} \sum_{b \in \{0,1,2,\ldots,d-1\}} \sum_{a \in \{0,1,2,\ldots,d-1\}} \left| \text{Trace}[e^{2\pi i c a} E^{\otimes}_{a,b}] \right|^2 \]

\[ = \frac{1}{d^3} \sum_{c \in \{0,1,2,\ldots,d-1\}} \sum_{b \in \{0,1,2,\ldots,d-1\}} \sum_{a \in \{0,1,2,\ldots,d-1\}} \left| e^{2\pi i c a} \text{Trace}[E^{\otimes}_{a,b}] \right|^2 \]

\[ = \frac{1}{d^3} \sum_{c \in \{0,1,2,\ldots,d-1\}} \sum_{b \in \{0,1,2,\ldots,d-1\}} \sum_{a \in \{0,1,2,\ldots,d-1\}} \left| \text{Trace}[E^{\otimes}_{a,b}] \right|^2 \]

\[ = \frac{1}{d^3} d \sum_{b \in \{0,1,2,\ldots,d-1\}} \sum_{a \in \{0,1,2,\ldots,d-1\}} \left| d \delta_{a,0} \delta_{b,0} \right|^2 = 1. \]

Thus we find the group \( Q \) is irreducible.

The fact that the channel \( \mathcal{E}^{\otimes} \) is diagonal in the operator basis \( \{E^{\otimes}_{a,b}\} \), coupled with the equation (XI) result that \( E^{\otimes}_{g,h} E^{\otimes}_{a,b} E^{\otimes\dagger}_{g,h} = \Omega' E^{\otimes}_{a,b} \), and the equation (XII) result that \( \mathcal{E}\left(E^{\otimes}_{a,b}\right) = \Lambda_{a,b} E^{\otimes}_{a,b} \), allows us to conclude the operators \( \{F^{\otimes}_{a,b,c}\} \) and the channel \( \mathcal{E}^{\otimes} \) commute.

\[ F^{\otimes}_{g,h,j} \mathcal{E}(\rho) F^{\otimes\dagger}_{g,h,j} = E^{\otimes}_{g,h} \mathcal{E}(\rho) E^{\otimes\dagger}_{g,h} = E^{\otimes}_{g,h} \left( \frac{1}{d} \sum_{a,b} \alpha_{a,b} \Lambda_{a,b} E^{\otimes}_{a,b} \right) E^{\otimes\dagger}_{g,h} \]

\[ = \frac{1}{d} \sum_{a,b} \alpha_{a,b} \Lambda_{a,b} \left( E^{\otimes}_{g,h} \mathcal{E}(\rho) E^{\otimes\dagger}_{g,h} \right) = \mathcal{E}\left[E^{\otimes}_{g,h,j} \rho F^{\otimes\dagger}_{g,h,j}\right]. \]

Note that the product channel analysis in equation (XIII) is essentially the same derivation as was done in equation (X) for qudits in the \( \hat{X}^a \hat{Z}^b \) operator basis.

This is the key criterion for ensemble achievability. Since the \( \{F^{\otimes}_{a,b,c}\} \) are unitary, \( F^{\otimes}_{g,h,j} \rho F^{\otimes\dagger}_{g,h,j} \) is a valid density operator. Applying any member of \( \{F^{\otimes}_{a,b,c}\} \) to an output optimal ensemble
\{p_i, \tilde{\rho}_i^\otimes\} \text{ yields an achievable ensemble. Since the group } \{F_{a,b,c}^\otimes\} \text{ is irreducible, we can apply Schur's lemma and conclude the average output state } \tilde{\Phi}^\otimes \text{ for an optimal ensemble for the product channel } \mathcal{E}^\otimes \text{ must equal } \frac{1}{d^2} I_d.

The remainder of our analysis for diagonal unital qudit channels uses the Schumacher and Westmoreland results summarized in equations (IV), (V), (VI) and (VII) in the manner seen previously, and directly carries over to the product channel case. Thus we conclude for the product channel \(\mathcal{E}^\otimes\), the HSW channel capacity is

\[
\mathcal{C} = \log_2(d) - \min_\rho S(\mathcal{E}^\otimes(\rho)) = \sum_{k=1}^{N} \log_2(d_k) - \min_\rho S(\mathcal{E}^\otimes(\rho)).
\]

### 6 Discussion and Conclusions

The HSW channel capacity for single qubit unital channels was originally derived in [3] as

\[
\mathcal{C} = 1 - \min_\rho S(\mathcal{E}(\rho)).
\]

This result was extended in [3] to the tensor product of single qubit unital channels. For qubits, it was shown in [3] that there always exists a special basis in which a qubit unital channel can be written in diagonal form. A key step in their proof was a homomorphism between \(SU(d)\) and \(SO(d^2 - 1)\). Such a homomorphism would be necessary for the method of proof in [3] to carry through to the general qudit case for \(d > 2\). However this homomorphism only occurs for \(d = 2\). Our method for deriving the HSW channel capacity depends on the qudit unital channel being diagonal, so our method only allows us to conclude that

\[
\mathcal{C} = \log_2(d) - \min_\rho S(\mathcal{E}(\rho))
\]

holds for diagonal unital channels. However, our proof was handcrafted in two key respects. The first was the choice of a fixed operator basis, the Generalized Pauli basis, in which the density matrix expansions were made. There exists the possibility that, given a specific channel, a custom operator basis could be constructed in which the channel \(\mathcal{E}\) would be diagonal. This in essence is how the proof showing any unital qubit channel is diagonal in some operator basis, was done in [3].

The second assumption was the explicit manner by which we showed ensemble achievability. To summarize, we showed an output ensemble was achievable by

1) restricting our attention to a preordained unitary operator basis consisting of elements \(g \in \mathcal{G}\)

and

2) considering only diagonal channels in the basis \(\mathcal{G}\).
The result was an algorithm by which we were able to determine, given an optimal ensemble $\omega$, if the output ensemble $\left\{ p_i, g\tilde{\rho}_i g^{-1} \right\}$ was achievable for $g \in G$.

The possibility remains that, given a channel $E$, we could use a technique other than that developed in this paper to assure ensemble achievability across all elements of a group $G$ acting on the channel outputs of an optimal input ensemble. Again, this is essentially what occurs in the unital qubit channel scenario analyzed in [2].

As a result, we feel we have “overconstrained” the requirements for our proofs. We conjecture the relation

$$C = \log_2(d) - \min_{\rho} S(\mathcal{E}(\rho))$$

holds for all unital qudit channels, rather than just those unital channels which are diagonal in the Generalized Pauli basis.

As our final remark, the diagonal unital qudit channel capacity result extends the connection between the minimum von Neumann entropy at the channel output and the HSW channel capacity, which had previously been established in the qubit case, to a non-empty set of channels in any dimension. This implies a more universal connection between the minimum von Neumann entropy at the channel output and the classical information capacity for that quantum channel than had previously been shown.

Furthermore, recall that the Holevo quantity $\chi$ utilizes von Neumann entropy to obtain a relation for the distinguishability of quantum states. Hence it is reassuring that von Neumann entropy appears explicitly in our qudit channel capacity result. This is an indicator of consistency that reaffirms the fundamental role von Neumann entropy appears to play in Quantum Information Science.

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A The Generalized Pauli Group

The generalized Pauli operators $\hat{X}$ and $\hat{Z}$ are used in our qudit analysis. This section describes some of the properties of these operators. Their definitions are

$$\hat{X}|j\rangle = |j + 1 \text{(mod } d)\rangle \quad \text{and} \quad \hat{Z}|j\rangle = \Omega^j|j\rangle.$$ 

The quantity $\Omega = e^{\frac{2\pi i}{d}}$. Note that $\hat{X}^d = \hat{Z}^d = I_d$. The commutation relation of $\hat{X}$ and $\hat{Z}$ follows directly, yielding $\hat{Z}\hat{X} = \Omega\hat{X}\hat{Z}$. Using the fact that $\langle j + 1 | \hat{X} | j \rangle = 1$, taking the Hermitian conjugate of both sides yields $\langle j | \hat{X}^\dagger | j + 1 \rangle = 1$, allowing us to conclude $\hat{X}^\dagger |j\rangle = |j - 1 \text{(mod } d)\rangle$. This in turn implies $\hat{X}$ is unitary, since $\hat{X}\hat{X}^\dagger = \hat{X}^\dagger\hat{X} = I_d$. Similarly $\hat{Z}^\dagger |j\rangle = \Omega^{-j} |j\rangle$, from which it follows that $\hat{Z}$ is a unitary operator.

In our application of Schur’s Lemma, we use the operator set of $E_{a,b} = \hat{X}^a\hat{Z}^b$, where \{a, b\} = 0, 1, 2, \ldots, d - 1. We shall also use the operators $F_{a,b,c} = \Omega^c \hat{X}^a\hat{Z}^b$, where \{a, b, c\} = 0, 1, 2, \ldots, d - 1. The operators $E_{a,b}$ and $F_{a,b,c}$ are unitary, since the composition of unitary operators is unitary. Note that $E_{a,b}^\dagger = \hat{Z}^{-b}\hat{X}^{-a}$ and $F_{a,b,c}^\dagger = \Omega^{-c} E_{a,b}^\dagger$.

We now show that any qudit density operator $\rho$ can be expanded as

$$\rho = \frac{1}{d} \sum_{a,b \in \{0, 1, 2, \ldots, d-1\}} \alpha_{a,b} \hat{X}^a \hat{Z}^b = \frac{1}{d} \sum_{a,b \in \{0, 1, 2, \ldots, d-1\}} \alpha_{a,b} E_{a,b},$$

where the $\alpha_{a,b}$ are complex quantities. We shall work in the Hilbert-Schmidt operator norm, which for qudit operators $A$ and $B$ is defined as $\langle A, B \rangle = \text{Trace}[A^\dagger B]$. Define the rescaled operators $Q_{a,b} = \frac{E_{a,b}}{\sqrt{d}} = \frac{\hat{X}^a\hat{Z}^b}{\sqrt{d}}$. The operators $Q_{a,b}$ are a set of $d^2$ orthonormal operators in the Hilbert-Schmidt inner product, as shown below.

$$\langle Q_{a,b}, Q_{q,r} \rangle = \frac{1}{d} \langle E_{a,b}, E_{q,r} \rangle = \frac{1}{d} \text{Trace}[E_{a,b}^\dagger E_{q,r}] = \frac{1}{d} \text{Trace}[\hat{Z}^{-b}\hat{X}^{-a}\hat{X}^q\hat{Z}^r] \quad \text{(XIV)}$$

(By the cyclic nature of trace)

$$= \frac{1}{d} \sum_{j=0}^{d-1} \Omega^{(r-b)j} \langle j | \hat{X}^{q-a} \hat{Z}^{r-b} | j \rangle = \frac{1}{d} \sum_{j=0}^{d-1} \Omega^{(r-b)j} \langle j | j + q - a \text{ (mod } d) \rangle$$

$$= \frac{1}{d} \delta_{a,q} \sum_{j=0}^{d-1} \Omega^{(r-b)j} = \frac{1}{d} \delta_{a,q} \delta_{b,r} = \delta_{a,q} \delta_{b,r}. $$

Here $\delta_{a,b}$ is the Kronecker delta function. Recall any qudit density operator $\rho$ can be written as

$$\rho = \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \beta_{a,b} |a\rangle \langle b|,$$
where the $\beta_{a,b}$ are complex quantities. We shall show that $|a\rangle\langle b|$ may be written as $|a\rangle\langle b| = \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \zeta_{r,s} Q_{r,s}$, where the $\zeta_{r,s}$ are complex quantities. Rescaling the $\zeta_{r,s}$, we will conclude that $\rho$ may be written as

$$\rho = \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \alpha_{a,b} E_{a,b}. $$

To begin, write $Q_{r,s}$ as

$$Q_{r,s} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \Omega^{js} |j + r\rangle\langle j|. $$

Define $\zeta_{r,s}$ as

$$\zeta_{a,b} = \text{Trace} \left[ Q_{r,s}^\dagger |a\rangle\langle b| \right] = \frac{1}{\sqrt{d}} \text{Trace} \left[ \sum_{j=0}^{d-1} \Omega^{-js} |j + r\rangle\langle j + r| \langle b| \right] = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \Omega^{-js} \delta_{b,i} \delta_{j,r} \delta_{i,j} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \Omega^{-js} \delta_{j,b} \delta_{j+r,a} = \frac{1}{\sqrt{d}} \Omega^{-bs} \delta_{a,b+r},$$

where $\delta$ is the Kronecker delta function.

Consider the operator $L = |a\rangle\langle b|$, and the corresponding complex coefficients $\xi_{r,s} = \langle Q_{r,s}, L \rangle = \text{Trace} \left[ Q_{r,s}^\dagger |a\rangle\langle b| \right]$. We would like to expand $L$ as $L = \sum_{r,s} \langle Q_{r,s}, L \rangle Q_{r,s} = \sum_{r,s} \xi_{r,s} Q_{r,s}$. Note that $||L|| = \sqrt{\langle L, L \rangle} = 1$. Using the result of equation (XV), we can conclude that

$$\sum_{r} \sum_{s} |\xi_{r,s}|^2 = \sum_{r} \sum_{s} \left| \frac{1}{\sqrt{d}} \Omega^{-bs} \delta_{a,b+r} \right|^2 = \frac{1}{d} \sum_{r} \sum_{s} \left| \delta_{a,b+r} \right|^2 = \frac{1}{d} \sum_{r} \sum_{b} |\delta_{a,b+r}|^2 = 1. $$

Thus $\sum_{r} \sum_{s} |\xi_{r,s}|^2 = 1 = ||L||^2$. This fact, for arbitrary $a$ and $b$ in $|a\rangle\langle b|$ allows us to conclude the $Q_{r,s}$ form a complete, orthonormal basis for the $L$'s, and we can expand $L$ in terms of the $Q_{r,s} \forall a,b$. Thus the expansion $|a\rangle\langle b| = \sum_{r,s} \langle Q_{r,s}, L \rangle Q_{r,s}$ holds $\forall a, b$. This leads to an expansion for the qudit density operator $\rho$.

$$\rho = \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \beta_{a,b} |a\rangle\langle b| = \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \beta_{a,b} \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \langle Q_{r,s}, (|a\rangle\langle b|) \rangle Q_{r,s} = \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \beta_{a,b} \langle Q_{r,s}, (|a\rangle\langle b|) \rangle Q_{r,s} = \left( \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \beta_{a,b} |a\rangle\langle b| \right) \left( \sum_{r=0}^{d-1} \sum_{s=0}^{d-1} Q_{r,s} \right).$$
implies
These constraints arise from the Hermitian nature of $\rho$ in independent degrees of freedom. Hence there are constraint relations between the set of coefficients $\alpha_{a,b}$. The linearity of the inner product in the second argument was used to move the sum over the indices $a$ and $b$ inside the inner product.

To obtain the final form of the expansion for the qudit operator $\rho$ we shall use, note that $E_{0,0} = I_d$. Our result above, $\langle E_{a,b}, E_{q,r} \rangle = \text{Trace}[E_{a,b}^\dagger E_{q,r}] = d \delta_{a,q} \delta_{b,r}$, tells us that $\text{Trace}(E_{a,b}) = d \delta_{a,0} \delta_{b,0}$. Thus of the $d^2$ possible $E_{a,b}$, only $E_{0,0}$ has nonzero $\text{Trace}$. The trace condition $\text{Trace}(\rho) = 1$ allows us to conclude $\alpha_{0,0} = 1$. Using this, let $\Upsilon$ denote the set of $d^2 - 1$ elements $a,b \in \{0,1,2,\cdots,d-1\}$ with the exception that $a$ and $b$ cannot both be zero. Then we may write the qudit density matrix $\rho$ as $\rho = \frac{1}{d} \left( I_d + \sum_{(a,b) \in \Upsilon} \alpha_{a,b} E_{a,b} \right)$ with $\alpha_{a,b} = \langle E_{a,b}, \rho \rangle = \text{Trace}[E_{a,b}^\dagger \rho]$.

In the expansion of $\rho$ above, there are $2d^2 - 2$ real, independent degrees of freedom in the set of coefficients $\alpha_{a,b}$. However, in the density operator $\rho$, there are only $d^2 - 1$ real, independent degrees of freedom. Hence there are constraint relations between the $\alpha_{a,b}$.

These constraints arise from the Hermitian nature of $\rho$. Note that $E_{a,b}^\dagger = \hat{Z}^{a}\hat{X}^{b} \Omega_{d-b}^{a-d} \hat{X}^{d-a} \hat{Z}^{d-b} \Omega_{d-b}^{d-a} E_{d-a,d-b}$. Consideration of $\rho^\dagger = \rho$ then implies

$$\frac{1}{d} \left( I_d + \sum_{(a,b) \in \Upsilon} \alpha_{a,b} E_{a,b} \right) = \frac{1}{d} \left( I_d + \sum_{(a,b) \in \Upsilon} \alpha_{a,b}^* E_{a,b}^\dagger \right) = \frac{1}{d} \left( I_d + \sum_{(a,b) \in \Upsilon} \alpha_{a,b}^* \Omega_{d-a}^{d-b} E_{d-a,d-b} \right)$$

or $\alpha_{d-a,d-b} = \alpha_{a,b}^* \Omega_{d-a}^{d-b}$. Here $\ast$ indicates complex conjugation, and index arithmetic is modulo $d$.

For example, for qubits, $d = 2$, and $\Omega = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1$. Applying the constraint equation above leads to $\alpha_{0,1}^* \Omega^{(2-0)(2-1)} = \alpha_{2-0,2-1}^* = \alpha_{0,1} = \alpha_{0,1}$, implying the coefficient of $E_{0,1} = \hat{Z}$ must be real. Similarly, $\alpha_{1,0}^* \Omega^{(2-1)(2-0)} = \alpha_{2-1,2-0}^* = \alpha_{2-1,2-0}$ or $\alpha_{1,0}^* = \alpha_{1,0}$, implying the coefficient of $E_{1,0} = \hat{X}$ must be real. Lastly, $\alpha_{1,1}^* \Omega^{(2-2)(2-1)} = \alpha_{2-2,2-1}^* = -\alpha_{1,1}^*$, implying the coefficient of $E_{1,1} = \hat{X}\hat{Z}$ must be pure imaginary. Note that $\hat{X} = \sigma_x$, $\hat{Z} = \sigma_z$, and $\hat{X}\hat{Z} = -i\sigma_y$. Hence we have reproduced the Bloch Sphere representation for qubits, $\rho = \frac{1}{2} \left( I_2 + \alpha_{0,1} \hat{X} + \alpha_{1,1} \hat{X} \hat{Z} + \alpha_{0,1} \hat{Z} \right) = \frac{1}{2} \left( I_2 + w_x \sigma_x + i w_y ( -i \sigma_y ) + w_z \sigma_z \right)$, with the $w_k$ real. For qudits, we end up with $3 = d^2 - 1$ real independent parameters, and not $2d^2 - 2 = 6$. The constraint equations for the $\alpha_{a,b}$ eliminated three real degrees of freedom. In general, the constraint equations will eliminate $d^2 - 1$ real extra degrees of freedom, leaving $d^2 - 1$ actual real parameters.
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