Designing the unambiguous discriminator from the one-photon interferometer

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The quantum states filtering, whose general theorem was given by Bergou et al. (Phys.Rev.A 71, 042314(2005)), should find its important applications in present scheme, where we are trying to show that the problem of quantum states unambiguous discrimination may be solved by applying the argument of filtering. Let’s use the quantum filtering, as an example, to show the basic idea of present scheme. Suppose there are $N$ linearly independent states, if we are able to find a $(N+1)$-dimensional unitary transformation, $R(\omega)$ (with $\omega$ is an adjustable variable), which will be performed on each $|\Psi\rangle$ in the way like: $R(\omega)|\Psi_1\rangle = d_{11}(\omega)|D_1\rangle + \sum_{i=2}^{N} d_{i1}(\omega)|D_i\rangle > + f_k(\omega)|F\rangle >$, then, according to the definition of the operators for filtering, there should be: $E_1(\omega) = R^{-1}(\omega)|D_1\rangle >> D_1|R(\omega)|$, $E_2(\omega) = \sum_{k=2}^{N} R^{-1}(\omega)|D_k\rangle >> D_k|R(\omega)|$ and $E_0(\omega) = R^{-1}(\omega)|F\rangle << F|R(\omega)|$. With this this $\{E_n(\omega)\}$ in hands, we could find the optimal operators which lets the function $F(\omega) = \sum_{j=1}^{N} \eta_j < \Psi_1|E_0(\omega)|\Psi_j\rangle$, with $\eta_j$ to be the a priori probability of $|\Psi_j\rangle$, have its minimum value. For the system with $N=3$, there are three types of operators: (a) $E_1(\omega)|\Psi_2\rangle = \sum_{i=2}^{N} d_{i2}(\omega)|D_i\rangle > + f_k(\omega)|F\rangle >$, (b) $E_2(\omega)|\Psi_1\rangle = \sum_{i=2}^{N} d_{i2}(\omega)|D_i\rangle > + f_k(\omega)|F\rangle >$, (c) $E_3(\omega)|\Psi_3\rangle = \sum_{i=1}^{N} d_{i3}(\omega)|D_i\rangle > + f_k(\omega)|F\rangle >$, for $i \neq j$ for $i=1,2,3$ and $E_0$ for failure; and (c) $E_1(\omega)|\Psi_2\rangle >> E_2(\omega)|\Psi_3\rangle > 0$, $E_2(\omega)|\Psi_1\rangle >> E_1(\omega)|\Psi_3\rangle > 0$, while $E_0 = I - \sum_{k=1}^{2} E_k$. We shall show that all these three types of operators, which may be performed on a $N=3$ systems, can be get by applying argument of filtering: the case $a$ is in fact the filtering with $N=3$, case $b$ can be viewed as successive filtering and the case $c$ can also be solved by an argument of filtering in subspace. It can be shown that each case, which belongs to the above three, can be solved by reducing it to the problem of filtering. An important case of $N=4$ system, has also been discussed.

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I. INTRODUCTION

As a very recent development, the possibility of unambiguous discrimination between unknown quantum states can be potentially useful for many applications in quantum computing and quantum communications. The problem of unambiguous discriminating pure states, which are successfully identified with nonunit probability but witout error, was originally formulated and analyzed by Ivanovic, Dicks and Peres [1-3] in 1987. Later, Jeager and Shimony solved the question of unambiguous discrimination of two known pure states with arbitrary a priori probability. Shortly after this result, Chefu and Barnett proved that only linearly independent pure states can be unambiguously discriminated [5]. The problem of discrimination among three nonorthogonal states was first considered by Peres and Terno [6], and the same question has also been discussed by Duan and Guo[7] and Sun et al. [8]. Chefu and Barnett also provided the optimal failure probability and its corresponding optimal measurement for a $n$ symmetric states [9], and an experimental set for discriminating four linearly independent nonorthogonal symmetric states was given by Jimenez et al. [10]. A new strategy for optimal unambiguous discrimination of quantum states was also offered by Jafariadhe et al. [11].

Unambiguous discrimination involving mixed state or a set of pure states, became an object of research recently. Several necessary and sufficient conditions for the optimum measurement have been given by Zhang et al. [12] and Eldar et al. [13]. Reduction theorems, which can simplify the discrimination theorem, have been developed by Raynal et al. [14-15]. Low bounds for the failure and the conditions for saturating the bounds, have also been studied [16-20]. There are only a few special cases have analytical solution for the quantum measurement, for example, the quantum state filtering [21-23], two mixtures with orthogonal or one-dimensional kernels [14-15], two mixtures in The Jordan basis [24] and other cases [26-30].

In present work, we shall present a new scheme to solve the problem of quantum state unambiguous discrimination. Let’s use the quantum state filtering originated from [21-23], as an example, to show the basic idea of present scheme. Suppose there are $N$ linearly independent states, the task of the quantum state filtering can be viewed as to find a set of operators $\{E_m\}$, whose elements are defined by: $E_2(\omega)|\Psi_1\rangle > 0$, $E_1(\omega)|\Psi_k\rangle > 0$ with $2 \leq k \leq N$, and $E_0$ corresponds to fail. If we are able to find a $(N+1)$-dimensional unitary transformation, $R(\omega)$ with $\omega$ is an adjustable variable, which will be performed on each $|\Psi\rangle$ in the way like: $R(\omega)|\Psi_1\rangle = d_{11}(\omega)|D_1\rangle + \sum_{k=2}^{N} d_{k1}(\omega)|D_k\rangle > + f_k(\omega)|F\rangle >$, according to the definition of the operators, there should be: $E_1(\omega) = R^{-1}(\omega)|D_1\rangle >> D_1|R(\omega)|$, $E_2(\omega) = \sum_{k=2}^{N} R^{-1}(\omega)|D_k\rangle >> D_k|R(\omega)|$ and $E_0(\omega) = R^{-1}(\omega)|F\rangle << F|R(\omega)|$. With this $\{E_m(\omega)\}$ in hands, we could find the optimal operators which lets the function $F(\omega) = \sum_{j=1}^{N} \eta_j < \Psi_1|E_0(\omega)|\Psi_j\rangle$, with $\eta_j$ to be the a priori probability of $|\Psi_j\rangle$, have it’s minimal value.

For the system with $N=3$, there are three types of operations: (a) $E_1(\omega)|\Psi_2\rangle >> E_2(\omega)|\Psi_3\rangle > 0$, $E_2(\omega)|\Psi_1\rangle >> E_1(\omega)|\Psi_3\rangle > 0$ and $E_0$
corresponds to fail; (b) $E^b_i |\Psi_j > = 0$ if $i \neq j$ for $i=1,2,3$, and $E^b_0$ for failure; and (c) $E^c_1 |\Psi_2 > = E^c_2 |\Psi_3 > = 0$, $E^c_1 |\Psi_1 > = E^c_2 |\Psi_2 > = 0$, while $E^c_0 = I - \sum_{k=1}^2 E_k$. We shall show that all these three types of operators, which may be performed on a $N=3$ systems, can be get by applying argument of filtering: the case a is in fact the filtering with $N=3$, case b can be viewed as successive filtering and the case c can also be solved by an argument of filtering in subspace. It looks as if each case, which belongs to the above three, can be solved by reducing it to the problem of filtering. An important case of $N=4$ system, has also been discussed.

Our present paper is organized as follows. Section II is a preliminary section in which we introduce the so-called double-triangle representation. In section III, we shall give a different way of solving the question of quantum states filtering. A concept of filtering in subspace will be introduced in Sec.IV. Two examples, discriminating three states filtering. A concept of filtering in subspace. It looks as if each case, which can also be solved by an argument of filtering: the case a is in fact the filtering with $N=3$, case b can be viewed as successive filtering and the case c can also be solved by reducing it to the problem of filtering. An important case of $N=4$ system, has also been discussed.

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\( i = 1, 2, \ldots, N \), and \( j = 1, 2, \ldots, M \), for it’s “in-space” while \(|D_k\rangle, |F_l\rangle\) for it’s “out-space” with \( 1 \leq k \leq K \), 
\( 1 \leq l \leq L \), and \( N+M = K+L \). \( H^e \) is the Hilbert space 
where the states are defined:

\[
|\Psi_i\rangle = \sum_{j=1}^{N} c_{ij}|e_j\rangle,
\]

(2.5)

\( H^A \) is the subspace for ancillas, \( U(N+M) \) will couple this 
two subspace together. Let’s use \( \omega \) to denote the 
adjustable parameter(s) in the unitary transformation, we 
can define \(|\psi\rangle_{out} = U(\omega)|\Psi\rangle > \) and express it in the 
“out-space”

\[
|\psi_\omega\rangle_{out} = \sum_{k=1}^{K} d_{jk}(\omega)|D_k\rangle + \sum_{l=1}^{L} f_{jl}(\omega)|F_l\rangle >, 
\]

(2.6)

with the normalization constraint \( \sum_{k=1}^{K} |d_{jk}|^2 + \sum_{l=1}^{L} |f_{jl}|^2 = 1 \). If we want to unambiguously discrimi-
nate all \( \Psi_j \) in \( G \), we should find the general \( U(\omega) \) which 
gives

\[
|\psi_\omega\rangle_{out} = d_{jj}(\omega)|D_j\rangle + \sum_{l=1}^{L} f_{jl}(\omega)|F_l\rangle >, 
\]

(2.7)

then, \( |D_j\rangle \gg |D_l\rangle \) and \( \sum_{l=1}^{L} |F_l\rangle \gg |F_l\rangle \) should be the 
projective operator corresponds to \( E_j \) and \( E_0 \), re-
spectively. With the \( U(\omega) \), we could write , for example, 
\( |D_j\rangle \) in the “in-space” as \( |D_j\rangle \rightarrow U^{-1}|D_j\rangle = 
|D_j\rangle ^e + |D_j\rangle ^A \) with \( |D_j\rangle ^e \) and \( |D_j\rangle ^A \) are two non-
normalized vectors which lies in \( H^e \) and \( H^A \), respectively.

There should be

\[
E_j(\omega) = |D_j\rangle ^e < |D_j\rangle, \quad E_0(\omega) = \sum_{l=1}^{L} |F_l\rangle ^e < |F_l\rangle, 
(2.8)
\]

and \( E_0(\omega) + \sum_{j=1}^{N} E_j(\omega) = \Gamma^r \). With these operators 
in hands, we could get both the optimal operators and the 
maximum values for discriminating \(|\Psi_j\rangle > \). When the 
projective operators are expressed in the “in-space”, 
there are written formally in terms of \(|e_i\rangle \gg |e_j\rangle \). If we 
could define \(|\{e_j\rangle \rangle \} \) by \(|\{\Psi_j\rangle > \) at the beginning, 
then we shall be able to complete the task of expressing \( E_m \) 
in terms of \(|\Psi_j^+ > \rangle < \Psi_j^+ |. \) The argument above can also 
be generalized to other cases with different operators on \( G \).

In present works, we always adopt the proposals 
originated from the works by Sun et al. [8]: (a) any pure 
state can be realized by a single-photon state and (b), 
according to Heck’s theorem [34], any unitary transfor-
mation matrix can also be realized by an optical network 
consisting of beam-splitters, phase-shifters, etc., all these 
optical elements should construct an one-photon interfer-
ometer (OPI). The device in Fig.2 is a typical four-port 
beam splitter which is used to realize a two-dimensional 
unitary transformation \( U_2(\omega) \):

\[
\begin{pmatrix}
-\sin \omega & \cos \omega \\
\cos \omega & -\sin \omega
\end{pmatrix}
\begin{pmatrix}
|\omega, r\rangle \\
|\omega, d\rangle
\end{pmatrix} = 
\begin{pmatrix}
|\omega, l\rangle \\
|\omega, u\rangle
\end{pmatrix}.
\]

(2.9)

A property of this beam-splitter, which is frequently ap-
plied in present works, should be noted: suppose there is an input

\[
|\phi\rangle = c_{w,l}|\omega, l\rangle + c_{w,u}|\omega, u\rangle, 
(2.10)
\]

where \( c_{w,l} \) and \( c_{w,u} \) are real parameters for simplicity, 
after performing the \( U_2(\omega) \), the output should be

\[
|\phi'\rangle = U_2(\omega)|\phi\rangle = c_{w,r}|\omega, r\rangle + c_{w,d}|\omega, d\rangle 
(2.11)
\]

with the coefficients satisfy:

\[
c_{w,r} = -c_{w,l} \sin \omega + c_{w,u} \cos \omega, 
(2.12)
\]

\[
c_{w,d} = c_{w,l} \cos \omega + c_{w,u} \sin \omega, 
(2.13)
\]

if we choose

\[
\sin \omega = \frac{c_{w,u}}{c_{w,l}^2 + c_{w,u}^2}, \quad \cos \omega = \frac{c_{w,r}}{c_{w,l}^2 + c_{w,u}^2}, 
(2.14)
\]

then \(|\phi'\rangle = (c_{w,l}^2 + c_{w,u}^2)|\omega, d\rangle > \) while it’s output along 
\(|\omega, r\rangle > \) is zero.

B. which parameters are known?

In present work, we shall deal with the case that all the 
states in \( G \) are linearly independent and their overlaps 
are also known.

\textit{Definition 2.1}: a \( N \)-dimensional matrix \( O(N) \) is de-

fined by it’s matrix elements

\[
a_{ij} = <\Psi_i|\Psi_j>, 
(2.15)
\]

with constraint that \(|O_{ij}| < 1 \) holds for \( i \neq j \).

Certainly, \( O(N) \) is Hermitian. Using \( O^*(N) \) and \( \bar{O}(N) \) 
for it’s conjugate matrix and transposed matrix, respec-
tively, there should be \( O_{ij}^* = O_{ij} = a_{ji} \).
Definition 2.2: A(N) is used to denote the adjoint matrix of O(N), \(A_{ij} = a_{ij}\), the inverse of O should be

\[
O^{-1}(N) = \frac{A(N)}{\det(O)},
\]

(2.16)

where \(\det(O)\) denotes the determinant of O(N).

Definition 2.3: \(t_j\) is used to denote

\[
t_j = <\psi_j^\perp|\Psi >.
\]

(2.17)

From the definition of the reciprocal states, if \(|\Psi j >\) is a reciprocal state of \(|\Psi j >\), then \(\exp\{i\phi_j\}|\Psi j >\) is also a reciprocal state of \(|\Psi j >\). We can always let \(t_j = <\psi_j^\perp|\Psi j > = <\psi_j|\Psi j^\perp >\) by choosing a suitable set of \(\{\phi_j\}\). Defining

\[
|\Psi j > = \frac{1}{\sqrt{1 - t_j^2}}(|\psi_j > - t_j|\Psi j^\perp >),
\]

(2.18)

one may verify that \(|\Psi j >\) and \(|\Psi j^\perp >\) form an orthonormal basis for \(|\Psi j >\) and

\[
|\Psi j > = \sqrt{1 - t_j^2}|\Psi j^\perp > + t_j|\Psi j >, \quad (2.19)
\]

it should be emphasized here that, either \(|\Psi j >\) or \(|\Psi j^\perp >\), is defined from all the states in G:

Theorem 2.1: \( <\psi_j^\perp|\Psi k > = t_j \delta_{jk}.\)

(2.20)

With the \(t_j\) and O(N) defined above, we may introduced another transformation matrix:

Theorem 2.2: denoting \((\delta_{ij}) = I\), and defining

\[
R(N) = O(N)(\frac{1}{t_j} \delta_{ij}),
\]

(2.21)

there should be

\[
R(N)|\Psi > = |\Psi >. \quad (2.22)
\]

Proof: formally, we can write \(|\Psi j >\) as a linear combination of \(|\Psi k >\) in the way like \(|\Psi j > = \sum_{k=1}^{N} h_{jk} |\Psi k >\), there should be \(o_{ij} = <\psi_j|\Psi j > = \sum_{k=1}^{N} h_{jk} <\psi_i|\Psi k > = h_{ji} t_i\), which gives \(|\Psi j > = \sum_{k=1}^{N} \frac{o_{jk}}{t_j} t_i|\Psi k >.\)

Let \(N = 3\), as an example, we have

\[
\begin{align*}
|\Psi j > & = \frac{o_{21} |\Psi j >}{|\Psi 1 >} + \frac{o_{22} |\Psi j >}{|\Psi 2 >} + \frac{o_{23} |\Psi j >}{|\Psi 3 >} \quad = |\Psi 1 >, \\
|\Psi 2 > & = \frac{o_{12} |\Psi j >}{|\Psi 1 >} + \frac{o_{22} |\Psi j >}{|\Psi 2 >} + \frac{o_{32} |\Psi j >}{|\Psi 3 >} \quad = |\Psi 2 >, \\
|\Psi 3 > & = \frac{o_{13} |\Psi j >}{|\Psi 1 >} + \frac{o_{23} |\Psi j >}{|\Psi 2 >} + \frac{o_{33} |\Psi j >}{|\Psi 3 >} \quad = |\Psi 3 >. \\
\end{align*}
\]

(2.23)

It is possible to express \(|\Psi j >\) in terms of \(|\Psi >\) through introducing the inverse of R(N)

\[
R^{-1}(N) = (t_j \delta_{ij}) \frac{A(N)}{\det(O)},
\]

(2.24)

from Eq.(2.16) and Eq.(2.21) while the relation, \(\det(O) = \det(\tilde{O})\), has been used [33]. Naturally,

\[
R^{-1}(N)|\Psi > = |\Psi j >. \quad (2.25)
\]

Both R(N) and \(R^{-1}\), which are known from O(N), can be used to derive the value of \(t_j\). Let’s use \(N=3\), as an example, to give the derivation. From Eq.(2.25), we have

\[
\frac{t_1}{\det(O(3))}(a_{11}|\Psi 1 > + a_{21}|\Psi 2 > + a_{31}|\Psi 3 >) = |\Psi 1 >, \\
\frac{t_2}{\det(O(3))}(a_{12}|\Psi 1 > + a_{22}|\Psi 2 > + a_{32}|\Psi 3 >) = |\Psi 2 >, \\
\frac{t_3}{\det(O(3))}(a_{13}|\Psi 1 > + a_{23}|\Psi 2 > + a_{33}|\Psi 3 >) = |\Psi 3 >,
\]

(2.26)

times <\psi_j^\perp | on both sides of each equation, there are \(t_j^2 a_{jj} / \det(O(3))\) = 1 for \(j = 1, 2, 3\). This calculation can be generalized to

Theorem 2.3: \(t_j = \sqrt{\frac{\det(O(N))}{a_{jj}}}\).

(2.27)

Now, we have shown how to get \(|\Psi j >\) from G, and their overlaps can be expressed thorough

Theorem 2.4: defining the matrix \(O^-(N)\) by

\[
O^-(N) = a_{ij} = <\psi_i^\perp|\Psi j >, \quad (2.28)
\]

there should be

\[
o_{ij} = \frac{a_{ij}}{\sqrt{a_{ii} a_{jj}}} \quad (2.29)
\]

Proof: we could suppose \(|\Psi j >\) is known at first while \(|\Psi >\) can be viewed as it’s "reciprocal" state, and there should be

\[
\tilde{O}^{-\frac{1}{t_j}}(\frac{1}{t_j} \delta_{ij})|\Psi > = |\Psi j >. \quad (2.30)
\]

by following the argument for the case where \(|\Psi >\) is known at first. Comparing it with Eq.(2.25), we find

\[
O^-(t_j \delta_{i'j'}) = \frac{A}{\det(O)}(\frac{1}{t_j} \delta_{ij}), \quad (2.31)
\]

it can be written in the form of Eq.(2.29) by using Eq.(2.27). Some \(O^\perp\) shall be given in the appendix.

C. the double-triangle representation

A complete set of reciprocal states exists if, and only if, the state \(|\psi_j >\) are linearly independent while the reciprocal states are also linearly independent, this fact will
be used in deriving a set of normalized basis set \( \{|e_j>\} \). Letting

\[
|\Psi_1^+> = |e_1>, \quad |\Psi_2^+> = a_{12}^1|e_1> + \sqrt{1 - |a_{12}^1|^2}|e_2>,
\]

the coefficients of \( |\Psi_2^+> \) are decided by the two requirements (a) it’s overlap with \( |\Psi_1^+> \) keeps unchanged and (b) the state should be normalized. These requirements may also used in deriving the coefficients of \( |\Psi_3> \): suppose

\[
|\Psi_3^+> = \sum_{k=1}^{3} c_{3k}^1|e_k>, \quad (2.33)
\]

from the requirements (a) and (b), there are three equations

\[
<\Psi_1^+|\Psi_3^+> = c_{31}^1, \quad <\Psi_2^+|\Psi_3^+> = 1, \quad (2.34)
\]

\[
<\Psi_2^+|\Psi_3^+> = a_{21}^1 a_{13}^1 + \sqrt{1 - |a_{12}^1|^2} c_{32}^1, \quad (2.35)
\]

their solutions should be

\[
c_{31}^1 = a_{13}^1, \quad c_{32}^1 = \frac{a_{23}^1 - a_{21}^1 a_{13}^1}{\sqrt{1 - |a_{12}^1|^2}}, \quad c_{33}^1 = \sqrt{1 - |c_{31}^1|^2 - |c_{32}^1|^2}. \quad (2.36)
\]

In principle, this process can be continued until we get all the coefficients, \( c_{ij}^1 \), used as the matrix elements for the matrix \( C^\perp(N) \). Introducing another \( N \)-dimensional Matrix \( E \), which is defined by \( E = (C^\perp)^{-1} \), with it’s matrix elements denoted by \( e_{ij} \), we can define the basis, \( \{|e_j>\} \), in the way like

\[
|e_1> = |\Psi_1^+>, \\
|e_2> = \frac{|\Psi_2^+> - a_{12}^1 |\Psi_1^+>}{\sqrt{1 - |a_{12}^1|^2}}, \\
|e_j> = \sum_{i=1}^{j} e_{ji} |\Psi_i^+>.
\]

After introducing this basis, every input state can be expressed in it by defining

\[
e_{ij} = <e_j|\Psi_i> . \quad (2.38)
\]

One may verify that there should be \( e_{ij} = 0 \) if \( i > j \) according to theorem 2.1, this makes

\[
|e_N> = |\Psi_N> \quad (2.39)
\]

according to a simple reasoning, and the expression, \( |e_N> = \sum_{i=1}^{N} e_{ji} |\Psi_i^+> \), is an equivalent form of it. The matrix \( C \) with \( c_{ij} \) is a upper-triangle matrix while \( C^\perp \) is a lower-triangle matrix, for examples,

\[
\begin{pmatrix}
c_{11}^1 & 0 & 0 \\
c_{21}^1 & c_{22}^1 & 0 \\
c_{31}^1 & c_{32}^1 & c_{33}^1
\end{pmatrix}
\begin{pmatrix}
|e_1> \\
|e_2> \\
|e_3>
\end{pmatrix}
= \begin{pmatrix}
|\Psi_1^> \\
|\Psi_2^> \\
|\Psi_3^>
\end{pmatrix}, \quad (2.40)
\]

\[
\begin{pmatrix}
c_{11} & c_{12} & c_{13} \\
0 & c_{22} & c_{23} \\
0 & 0 & c_{33}
\end{pmatrix}
\begin{pmatrix}
|e_1> \\
|e_2> \\
|e_3>
\end{pmatrix}
= \begin{pmatrix}
|\Psi_1> \\
|\Psi_2> \\
|\Psi_3>
\end{pmatrix}, \quad (2.41)
\]

this is the reason why we call \( \{|e_j>\} \) the double-triangle representation (DTR). In the argument below, we always suppose that the states, either \( \{|\Psi_i>\} \) or \( \{|\Psi_j^+>\} \), have been expressed in the DTR. At the end of this section, we would like to emphasis again: if \( O(N) \) is known, then \( t_j, O^\perp(N), \{|\Psi_j^+>\} \}, \{|e_i>\}, C \) and \( C^\perp \) are also given at the same time.

III. THE QUANTUM STATE FILTERING

A. the POVMs for the filtering

The quantum state filtering, which was termed in [21-23], is a special case of telling whether a state \( |\Psi_i> \) belongs to, \( G_1 \equiv \{|\Psi_1>\} \) or \( G_2 \equiv \{|\Psi_k>\} \) for \( k \geq 2 \), with a non-zero probability of failure. The derivation of the optimal measurement strategy, in terms of \( \{E_i, E_0\} \) for \( i=1, 2 \), to distinguish \( |\Psi_1> \) from \( G_2 \) has been given and it is shown that this problem is equivalent to the discrimination of a pure state and an arbitrary mixed states. The quantum state filtering, as we shall shown, plays important roles in present works: (1) it’s an excellent example to show how our scheme works while (2)
the filtering in a successive way will be used to complete other operations on \( G \). The POVMs \( \{E_n\} \) for filtering are defined by \( E_1|\Psi_k\rangle > 0 \) for \( k \geq 2 \), \( E_2|\Psi_1\rangle = 0 \) and \( E_0 \) for failure, our task is to find an general unitary transformation \( U(\omega) \) which transfers each state \( |\Psi_j\rangle > \) in the “out-space” in way like:

\[
|\Psi_1\rangle >_{\text{out}} = d_{11}|D_1\rangle > + f_1|F\rangle >,  \tag{3.1}
\]

\[
|\Psi_k\rangle >_{\text{out}} = \sum_{j=2}^{N} d_{kj}|D_j\rangle > + f_k|F\rangle >.  \tag{3.2}
\]

This \( U(\omega) \) can be realized by the OPI in Fig.3. Keeping in mind that \( |\Psi_k\rangle > \) has no input along the rail \( e_1 \), the input state should be \( |\Psi_1\rangle > \) when the detector \( D_1 \) has been triggered. By applying Eqs.(2.9-14), we are always possible to prevent the signals of \( |\Psi_1\rangle > \) from appearing in the detector \( D_k \). Usually, a complex parameter, say, \( c_{ij} \) may be expressed as

\[
c_{ij} = |c_{ij}| \exp\{i\phi_{ij}\},  \tag{3.3}
\]

with \( \exp\{i\phi_{ij}\} = c_{ij}/|c_{ij}| \). In Fig.3, a phase-shift, \( \Phi_{ij} \), is placed in front of each beam-splitter denoted by \( \omega_{ij} \), we always choose the phase-shift

\[
\Phi_{ij} = -\phi_{ij},  \tag{3.4}
\]

while each beam-splitter takes the value

\[
\sin \omega_{k1} = \frac{\sqrt{\sum_{j=1}^{k-1} |c_{ij}|^2 - c_{11}^2 \sin^2 \omega_{ij}}}{\sqrt{\sum_{j=1}^{k} |c_{ij}|^2 - c_{11}^2 \sin^2 \omega_{ij}}},  \tag{3.5}
\]

\[
\cos \omega_{k1} = \frac{\sqrt{\sum_{j=1}^{k} |c_{ij}|^2 - c_{11}^2 \sin^2 \omega_{ij}}}{c_{1k}},  \tag{3.6}
\]

with \( \omega_{11} \equiv \omega \) and \( c_{ij} = < e_i |\Psi_1\rangle > \), for examples,

\[
c_{11} = t_1, \quad c_{12} = -\frac{\delta_{12}t_1}{\sqrt{1 - |\delta_{12}|^2}},  \tag{3.7}
\]

\[
c_{21} = 0, \quad c_{22} = \frac{t_2}{\sqrt{1 - |\delta_{12}|^2}}.  \tag{3.8}
\]

In Fig.3, we could read

\[
|D_1\rangle >_{\text{in}} = \exp\{-i\Phi_{11}\} U_{-1}^{-1}(\omega_{11})|D_1\rangle >,  \tag{3.9}
\]

and get

\[
|D_1\rangle >_{\text{in}} = -\sin \omega |e_1\rangle > + \sin \omega |v\rangle >,  \tag{3.10}
\]

where \( \Phi_{11} = 0 \) and Eq.(2.9) have been used. Through a similar argument, we could arrive at

\[
|D_2\rangle >_{\text{in}} = |D_2\rangle > + \cos \omega \sin \omega |v\rangle >,  \tag{3.11}
\]

\[
|D_2\rangle > = -\sin \omega \sin \omega e^{-i\Phi_{21}} |e_2\rangle > + \cos \omega \sin \omega |e_1\rangle >,  \tag{3.11}
\]

it can be proved that

\[
|D_2\rangle > = -\frac{\cos \omega \sin \omega e^{i\Phi_{12}}}{\sqrt{\cos^2 \omega + \sin^2 \omega |\omega_{12}|^2}} |\psi_{1/2}\rangle >.  \tag{3.12}
\]

In fact, we may use the relation, \( E_2(\omega) = |1\rangle < - E_1(\omega) - E_0(\omega) \), instead of giving all \( |D_k\rangle > \) in detail. Using Eq.(3.1) and Eq.(3.11), we may get

\[
|D_1\rangle > = -\sin \omega |\psi_{1/2}\rangle >,  \tag{3.13}
\]

\[
|F\rangle > = \frac{(1 - |D_1\rangle > \langle D_1|)^e |\Psi_1\rangle >}{\sqrt{\langle \Psi_1| (1 - |D_1\rangle > \langle D_1|) |\Psi_1\rangle >}},
\]

and the POVMs of filtering should be

\[
E_1(\omega) = |D_1\rangle \langle D_1|, \quad E_0(\omega) = |F\rangle \langle F|,  \tag{3.14}
\]

\[
E_2(\omega) = \sum_{k=2}^{N} |D_k\rangle \langle D_k| = |\Gamma\rangle - E_1(\omega) - E_0(\omega).
\]

If the POVMs \( \{E_n\} \) were known, then the calculation of the optimal value of filtering should be easily completed. Suppose \( \eta_j \) is the a priori probability of \( |\psi_j\rangle > \), we denote \( F_1/N(-1)(\omega) \) and \( F_1/N(1)(\omega) \) the average value of success and failure of filtering, respectively,

\[
P_1/N(-1)(\omega) = \sum_{i=1}^{2} \text{Tr}(E_i \rho_i),  \tag{3.15}
\]

\[
P_1/N(1)(\omega) + F_1/N(1)(\omega) = \sum_{j=1}^{N} \eta_j,  \tag{3.16}
\]

with \( \rho_1 = \eta_1 |\psi_1\rangle < \psi_1|, \rho_2 = \sum_{k=2}^{N} \eta_k |\psi_k\rangle < \psi_k| \) and \( \rho = \rho_1 + \rho_2 \). A simple calculation shows that

\[
F_1/N(1)(\omega) = \eta_1 |f_1| = \sum_{k=2}^{N} \eta_k |f_k|^2,  \tag{3.17}
\]

with \( f_j = < F |\psi_j > \) and

\[
f_1 = \sqrt{1 - t_1^2 \sin^2 \omega}, \quad f_k = \frac{\omega_{1k}}{\sqrt{1 - t_1^2 \sin^2 \omega}}.  \tag{3.18}
\]

The optimal value of \( F_1/N(-1)(\omega) \) and \( F_1/N(-1)(\omega) \) is defined to be minimum value of \( F_1/N(-1)(\omega) \) in the domain of \( 0 \leq \omega < 2\pi \). From Eqs.(3.17-18), there is

\[
\frac{dF_1/N(1)(\omega)}{d\omega} = t_1^2 \sin 2\omega (\eta_1 - \sum_{k=2}^{N} \eta_k |\omega_{1k}|^2),  \tag{3.19}
\]

and \( F_1/N(1)(\omega) \) happens at \( dF_1/N(1)(\omega)/d\omega = 0 \). Now, we are able to give the optimal values of filtering:

(a) if \( \sqrt{\sum_{k=2}^{N} \frac{\eta_k}{\eta_1} |\omega_{1k}|^2} > 1 \), by letting \( \sin^2 \omega opt = 0 \), we have

\[
F_1/N(1)(\omega) = \eta_1 + \sum_{k=1}^{N} |\omega_{1k}|^2,  \tag{3.20}
\]
is the most interesting case in the field of quantum states

Besides all this optimal results, we could also get the optimal POVMs \( \{F_{m}^{\text{opt}}\} \).

\[ F_{1/(N-1)}^{\text{opt}}(\omega) = \eta_{1}(1-\eta_{1}^{2}) + \frac{\sum_{k=2}^{N} \eta_{k}|\alpha_{1k}|^{2}}{1-\eta_{1}^{2}} \]  

while \( \sin^{2}\omega_{\text{opt}} = 1 \). Substituting \( \omega_{\text{opt}} \) for \( \omega \) in Eqs.(3.13-14), we could also get the optimal POVMs \( \{F_{m}^{\text{opt}}\} \).

\[ F_{1/(N-1)}^{\text{opt}}(\omega) = \eta_{1}(1-\eta_{1}^{2}) + \frac{\sum_{k=2}^{N} \eta_{k}|\alpha_{1k}|^{2}}{1-\eta_{1}^{2}} \]

while \( \sin^{2}\omega_{\text{opt}} = 1 \). Substituting \( \omega_{\text{opt}} \) for \( \omega \) in Eqs.(3.13-14), we could also get the optimal POVMs \( \{F_{m}^{\text{opt}}\} \).

**B. an example: filtering for N=3**

The filtering of N=3 is a case with a fully analytical solution and an optical implementation of the optimal strategy [23], we shall show, via a simple optical setting, how to recover all the optimal values in [23]. For N=3, there is

\[ 1 - \eta_{1}^{2} = \frac{|\alpha_{1}|^{2} + |\alpha_{2}|^{2} + |\alpha_{3}|^{2} - \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{32}}{1 - |\alpha_{3}|^{2}} \]  

Defining

\[ A = \eta_{2}|\alpha_{12}|^{2} + \eta_{3}|\alpha_{13}|^{2} \]

we can write the general results of filtering to the N=3: (a) for \( A > \eta_{1} \), there is

\[ F_{1/2}^{\text{opt}} = \eta_{1} + A \]

(b) for \( \eta_{1}(1 - \eta_{1}^{2}) \leq A \leq \eta_{1} \), there should be

\[ F_{1/2}^{\text{opt}} = 2\sqrt{\eta_{1}A} \]

and (c) else, \( A < \eta_{1}(1 - \eta_{1}^{2}) \),

\[ F_{1/2}^{\text{opt}} = \eta_{1}(1 - \eta_{1}^{2}) + \frac{A}{1 - \eta_{1}^{2}} \]

Besides all this optimal results, we could also get the optimal POVMs for filtering with N=3.

**C. filtering with N=2: discriminating two pure states**

The problem, how to discriminate \( |\Psi_{1}\rangle \) from \( |\Psi_{2}\rangle \), is the most interesting case in the field of quantum states

unambiguously discrimination. Here, it can be solved as a special case of filtering with N=2. The present solution is complete in the sense that: not only the optimal values but also the optimal POVMs should be given at the same time while the POVMs have the forms required by Eqs.(2.2-4). The OPI in Fig.4 is for the filtering with N=2 and it takes \( |\Psi_{1}\rangle|\Psi_{2}\rangle = |\Psi_{2}\rangle|\Psi_{1}\rangle \) for simplicity. From the DTR for N=2, the basis vectors should be

\[ |e_{1}\rangle = |\Psi_{1}\rangle >, \quad |e_{2}\rangle = |\Psi_{2}\rangle > \]

and the states have the forms

\[ |\Psi_{1}\rangle = \sqrt{1 - |\alpha_{12}|^{2}}|e_{1}\rangle + \alpha_{21}|e_{2}\rangle, \quad |\Psi_{2}\rangle = |e_{2}\rangle >. \]

With known parameters for N=2, which have been given in the Appendix, we have

\[ |\psi_{1}\rangle > = \frac{|\Psi_{1}\rangle - \alpha_{21}|\Psi_{2}\rangle >}{\sqrt{1 - |\alpha_{12}|^{2}}}, \quad |\psi_{2}\rangle > = \frac{|\Psi_{2}\rangle - \alpha_{21}|\Psi_{1}\rangle >}{\sqrt{1 - |\alpha_{12}|^{2}}} \]

by applying Eq.(2.25). The POVMs for discriminating two linearly independent states, \( |\Psi_{1}\rangle > \) and \( |\Psi_{1}\rangle > \), should be

\[ E_{1} = \sin^{2}\omega|\psi_{1}\rangle > < \psi_{1}\rangle |, \quad E_{2} = \cos^{2}\omega|\psi_{1}\rangle > < \psi_{1}\rangle |, \quad E_{0} = (|\Psi_{1}\rangle > - t_{1}^{2}\sin^{2}\omega|\psi_{1}\rangle >)(|\Psi_{1}\rangle > - t_{1}^{2}\sin^{2}\omega < \psi_{1}\rangle |) \]

while \( t_{1} = \sqrt{1 - |\alpha_{12}|^{2}} \), and

\[ F_{1/1}(\omega) = \eta_{1}(1 - t_{1}^{2}\sin^{2}\omega) + \frac{\eta_{2}|\alpha_{12}|^{2}}{1 - t_{1}^{2}\sin^{2}\omega}. \]
Now, we could give the optimal values and the optimal POVMs at the same time: (1) for $\sqrt{\frac{2}{\eta_1}} > \frac{1}{|\omega_{12}|}$, let $\sin^2 \omega_{\text{opt}} = 0$, there should be

$$E_{1}^{\text{opt}} = 0, \quad E_{2}^{\text{opt}} = |\Psi_{2}^+ > < \Psi_{2}^+|,$$

$$E_{0}^{\text{opt}} = |\Psi_{1} > < \Psi_{1}|,$$

$$F_{1/1}^{\text{opt}}(\omega) = \eta_{1} + \eta_{2}|\omega_{12}|^2,$$

$$F_{1/1}^{\text{opt}}(\omega) = \eta_{2}(1 - |\omega_{12}|^2),$$

(3.34)

(2) if $|\omega_{12}| \leq \sqrt{\frac{2}{\eta_1}} \leq \frac{1}{|\omega_{12}|}$, by letting

$$\sin^2 \omega_{\text{opt}} = \frac{1 - \sqrt{\frac{2}{\eta_1}}|\omega_{12}|}{1 - |\omega_{12}|^2},$$

(3.35)

we shall get the optimal POVMs

$$E_{1}^{\text{opt}} = \frac{1 - \sqrt{\frac{2}{\eta_1}}|\omega_{12}|}{1 - |\omega_{12}|^2}|\Psi_{1}^+ > < \Psi_{1}^+|,$$

$$E_{2}^{\text{opt}} = \frac{1 - \sqrt{\frac{2}{\eta_1}}|\omega_{12}|}{1 - |\omega_{12}|^2}|\Psi_{2}^+ > < \Psi_{2}^+|,$$

$$E_{0}^{\text{opt}} = I - \sum_{j=1}^{2} E_{j}^{\text{opt}},$$

(3.36)

which give the optimal results

$$F_{1/1}^{\text{opt}}(\omega) = 2\sqrt{\eta_1}\eta_2|\omega_{12}|,$$

$$F_{1/1}^{\text{opt}}(\omega) = \sum_{j=1}^{2} \eta_j - 2\sqrt{\eta_1}\eta_2|\omega_{12}|,$$

(3.37)

(3.38)

and (3) when $\sqrt{\frac{2}{\eta_1}} < |\omega_{12}|$, through choosing $\sin^2 \omega_{\text{opt}} = 1$, we arrive at

$$E_{1}^{\text{opt}} = |\Psi_{1}^+ > < \Psi_{1}^+|, \quad E_{0}^{\text{opt}} = |\Psi_{2} > < \Psi_{2}|,$$

while $E_{2}^{\text{opt}} = 0$, all these operators leads to

$$F_{1/1}^{\text{opt}}(\omega) = \eta_{1}|\omega_{12}|^2 + \eta_2,$$

$$F_{1/1}^{\text{opt}}(\omega) = \eta_2(1 - |\omega_{12}|^2).$$

(3.39)

Compare with other methods of solving the same question, the present scheme states that the general POVMs may be given before the decision of the optimal values of success and failure.

IV. FILTERING IN SUBSPACE

A. Filtering with the background

Suppose $G_1 \equiv \{ |\Psi_k > \}$ and $G_2 \equiv \{ |\Psi_l > \}$, a new operation of G can be specified by the definition of

the POVMs as $E_1 |\Psi_2 > = 0$ if $|\Psi_2 >$ belongs to $G_2$, $E_2 |\Psi_2 > = 0$ if $|\Psi_2 >$ belongs to $G_1$ and $E_0$ corresponds to failure. If a state, say, $|\Psi_2 >$, is shared by both $G_1$ and $G_2$, then $E_1 |\Psi_2 > = E_2 |\Psi_2 > = 0$ according to the definition of the POVMs $\{ E_m \}$. We call this case the name of discriminating with the background. In this section, we shall consider a simple case of discriminating $|\Psi_1 >$ from $|\Psi_2 >$ with $|\Psi_3 >$ as the background. In the DTR, this operation on G can also be viewed as a filtering in a two-dimensional subspace.

For $N=3$, the the basis vectors in the DTR should be:

$$|e_1 > = |\Psi_{1}^+ >,$$

$$|e_2 > = \frac{|\Psi_{2}^+ > - |\omega_{12}| |\Psi_{3}^+ >}{\sqrt{1 - |\omega_{12}|^2}},$$

$$|e_3 > = |\Psi_{3} >,$$

(4.1)

with

$$|\omega_{12}|^2 = \frac{a_{13}a_{32} - a_{12}^2}{\sqrt{(1 - |a_{23}|^2)(1 - |a_{13}|^2)}},$$

(4.2)

while the matrix $C(3)$ takes the form

$$\begin{pmatrix}
  t_1 & a_{13} - a_{12}a_{32} & a_{31} \\
  0 & \sqrt{1 - |a_{23}|^2} & a_{32} \\
  0 & 0 & 1
\end{pmatrix}$$

(4.3)

with $t_1$, which holds for $N=3$, is

$$t_1 = \sqrt{\frac{\det(O(3))}{1 - |a_{23}|^2}},$$

(4.4)

The $U(\omega)$ in Fig.5 is required to transform each $|\Psi_j >$
to $|\Psi_j >_{\text{out}}$ as
\[
|\Psi_1 >_{\text{out}} = d_1|D_1 > + f_{11}|F_1 > + f_{12}|F_2 >, \\
|\Psi_2 >_{\text{out}} = d_2|D_2 > + f_{21}|F_1 > + f_{22}|F_2 >, \\
|\Psi_3 >_{\text{out}} = |F_2 >,
\]
and this goal can be reached, as we shall shown later, by applying the argument of filtering. The $U^{-1}(\omega)$ shall give
\[
|D_1 >^e = -\sin \omega |\Psi_1 >^e, \\
|F_2 >^e = |\Psi_3 >, \\
|F_1 >^e = \frac{(|e > - |F_2 >^e < F_2 | - |D_1 >^e < D_1 |)|\Psi_1 >}{\sqrt{1 - (|f_{12}|^2 + |d_1|^2})},
\]
with $|D_2 >^e$ has given by Eq.(3.12). The POVMs are defined by
\[
E_1(\omega) = |D_1 >^e < D_1 |, \\
E_2(\omega) = |D_2 >^e < D_2 |, \\
E_0(\omega) = |F_1 >^e < F_1 | + |F_1 >^e < F_1 |.
\]
Defining
\[
F_1(\omega) = \sum_{j=1}^{2} \eta_j |f_{2j}|^2, \\
F_2(\omega) = \sum_{i=1}^{3} \eta_i |f_{2i}|^2,
\]
with $f_{ij} = |F_j |\Psi_i >$, the average value of failure should be
\[
F(\omega) = F_1(\omega) + F_2(\omega).
\]
A simple calculation shows
\[
F_2(\omega) = \sum_{j=1}^{2} \eta_j |o_{3j}|^2 + \eta_3,
\]
it is still in a special form of filtering with N=3, see Eq.(3.26). The $F_1(\omega)$ is left to be decided by the general results of filtering with N=2.

Formally, $|\Psi_1 >$ and $|\Psi_1 >$ can written by
\[
|\Psi_1 > = \sqrt{1 - |c_{13}|^2} |\Psi'_1 > + c_{13} |e_3 >, \\
|\Psi_2 > = \sqrt{1 - |c_{23}|^2} |\Psi'_2 > + c_{23} |e_3 >,
\]
with $|\Psi'_1 >$ and $|\Psi'_2 >$ are two normalized states defined in the subspace specified by $\{|e_j >\}$ for $j=1,2$:
\[
|\Psi'_1 > = \frac{c_{11}|e_1 > + c_{12}|e_2 >}{\sqrt{1 - |c_{13}|^2}}, \\
|\Psi'_2 > = |e_2 >,
\]
and their overlap should be
\[
o'_{12} =< \Psi'_1 |\Psi'_2 > = \frac{|o_{12} - o_{13}o_{22}|}{\sqrt{(1 - |o_{31}|^2)(1 - |o_{32}|^2)}}.
\]
Now, in the two-dimensional subspace with $\{|e_1 >, |e_2 >\}$, our task is to discriminate $|\Psi_1 >$ from $|\Psi_2 >$ with
\[
\eta'_j = \eta_j (1 - |o_{3j}|^2),
\]
for $j=1,2$, to be their a prior probability, respectively. According to our discussion about filtering, we have
\[
F_1(\omega) = \eta'_1 (1 - (1 - |o'_{12}|^2) \sin^2 \omega) + \frac{\eta'_2 |o'_{12}|^2}{1 - (1 - |o'_{12}|^2) \sin^2 \omega},
\]
which is equivalent with the one given by Eq.(4.9). This is the reason why the present case is viewed as a process of filtering with N=2, certainly, in the subspace without the background. It’s optimal results have nearly the same forms for filtering with N=2: (1) for $\sqrt{\frac{d_1}{H_1}} > \frac{1}{|o'_{12}|}$, let $\sin^2 \omega_{\text{opt}} = 0$, there should be
\[
F_{\text{opt}}(\omega) = F_2(\omega) + \eta'_1 + \eta'_2 |o'_{12}|^2,
\]
\[
= \eta'_1 (1 - |o'_{12}|^2),
\]
(2) if $|o'_{12}| \leq \sqrt{\frac{d_1}{H_1}} \leq \frac{1}{|o'_{12}|}$, by letting
\[
\sin^2 \omega_{\text{opt}} = \frac{1 - \sqrt{\frac{d_1}{H_1}} |o'_{12}|}{1 - |o'_{12}|^2},
\]
we shall get the optimal results
\[
F_{\text{opt}}(\omega) = F_2(\omega) + 2 \sqrt{\frac{d_1}{H_1}} |o'_{12}|,
\]
\[
= \eta'_1 (1 - |o'_{12}|^2),
\]
and (3) when $\sqrt{\frac{d_1}{H_1}} < |o'_{12}|$, through choosing $\sin^2 \omega_{\text{opt}} = 1$, we arrive at
\[
F_{\text{opt}}(\omega) = F_2(\omega) + \eta'_1 |o'_{12}|^2 + \eta_2,
\]
\[
= \eta'_1 (1 - |o'_{12}|^2).
\]
It should be noted that $F_2(\omega)$, which has been given in Eq.(4.11), is in fact a constant. The present argument, which is suitable for discriminating $G_1 \equiv \{|\psi_1 >, |\Psi_3 >\}$ and $G_2 \equiv \{|\psi_2 >, |\Psi_3 >\}$, can be generalized to the discriminating two general mixtures sharing part of states in comm.

B. discriminating two mixtures in Jordan basis

Suppose there are two mixtures,
\[
\rho_1 = \sum_{i=1}^{K} p_i |\Psi_i > < \Psi_i |,
\]
\[
\rho_2 = \sum_{j=K+1}^{2K} p_j |\Psi_j > < \Psi_j |,
\]
POVMs, \( \{ E^i_m \} \), to complete this task should be

\[
E^1_i(\omega_i) = \sin^2 \omega_i |\Psi_i^+\rangle < |\Psi_i^-\rangle, \\
E^2_i(\omega_i) = \cos^2 \omega_i \frac{\cos^2 \omega_i + \sin^2 \omega_i \cos^2 \theta_i}{\cos^2 \omega_i + \sin^2 \omega_i \cos^2 \theta_i} |\Psi_{K+i}^-\rangle < |\Psi_{K+i}^+\rangle, \\
E^0_i(\omega_i) = I - \sum_{k=1}^{2} E^k_i(\omega_i),
\]

(4.25)

the average value of the failure in \( H^i \) should be

\[
F^i(\omega_i) = \eta_i (1 - \sin^2 \theta_i \sin^2 \omega_i) + \frac{\eta_{K+i} \cos^2 \theta_i}{1 - \sin^2 \theta_i \sin^2 \omega_i},
\]

(4.26)

it’s optical values are given by the theorem of filtering with \( N=2 \). Finally, we can define the POVMs \( \{ E_m \} \) by

\[
E_m = \sum_{i=1}^{K} E^i_m,
\]

(4.27)

for \( m=0, 1, 2 \). The average value of fail can be expressed by

\[
F(\omega) = \sum_{i=1}^{K} F^i(\omega_i),
\]

(4.28)

while it’s optimal value

\[
F^{opt}(\omega) = \sum_{i=1}^{K} F^i(\omega_i = \omega_i^{opt}),
\]

(4.29)

where \( \omega_i^{opt} \) should depend on the actual value of the parameters, \( \eta_i, \eta_{K+i} \) and \( \cos \theta_i \), here, this requirement has also been pointed by the recent work [20].

V. THE SUCCESSIVE FILTERING FOR DISCRIMINATION OF PURE STATES

A. the optical realization of \( U(\omega) \)

In present section, we shall show the POVMs, which are defined for discriminating of pure states, can be realized in an enlarged Hilbert space by applying the successive filtering. The OPI in Fig.7 is designed to discriminate three linearly independent states, \( |\Psi_i\rangle \) for \( i=1, 2, 3 \), here. The \( U(\omega) \) realized by this OPI can be written as

\[
U(\omega) = R(2)R(1),
\]

(5.1)

with \( R(1) \) denotes the unitary transformation done by the beam-splitters and phase-shifters, \( \omega_{j1} \) and \( \Phi_{j1} \) for \( j=1, 2, 3 \), on the left part of Fig.7, while \( R(2) \) denotes the unitary transformation realizes by, \( \omega_{k1} \) and \( \Phi_{k1} \) for \( k=2, 3 \), the beam-splitters and phase-shifters on the right.
part. At first, the R(1) is defined to filter $|\Psi_1>$ from the states $|\Psi_2>$ and $|\Psi_3>$, here.

$$R(1)|\Psi_1> = d_{11}|D_1 > + f_{11}|F_1 >,$$

$$R(1)|\Psi_k > = \sqrt{1-|f_{k1}|^2}|\Psi'_k > + f_{k1}|F_1 >,$$

for k=2, 3, while $|\Psi'_k >$ lies in a two-dimensional subspace $H'$, which is specified by it’s basis as $\{|\omega_{k1}, r >\}$ for k=2, 3, and

$$|\Psi'_2 > = \sqrt{1-|\alpha'_2|^2}|\omega_{21}, r > + \alpha'_2|\Psi'_3 >,$$

$$|\Psi'_3 > = e^{i\phi'_3}|\omega_{31}, r >,$$

where $\alpha'_2 = <\Psi'_2|\Psi'_3 >$, using Eq.(5.2), we get

$$\alpha'_2 = \frac{\alpha_{23} - f_{23}^* f_{31}}{\sqrt{(1-|f_{21}|^2)(1-|f_{31}|^2)}}.$$

In this run of filtering, the average value of fail should be

$$F_1(\omega) = \sum_{j=1}^{3} \eta_j |f_{j1}|^2,$$

with $f_{j1} = <F_1|\Psi_j >$. After the first turn of filtering, we are left with two states $|\Psi'_2 >$ and $|\Psi'_3 >$ in $H'$, with their a priori probabilities to be

$$\eta'_k = \eta_k (1-|f_{k1}|^2),$$

for k=2, 3, respectively. R(2) is designed to filter $|\Psi'_2 >$ from $|\Psi'_3 >$ in the way like:

$$R(2)|\Psi'_2 > = \frac{1}{\sqrt{1-|f_{21}|^2}}(d_{22}|D_2 > + f_{22}|F_2 >),$$

$$R(2)|\Psi'_3 > = \frac{1}{\sqrt{1-|f_{31}|^2}}(d_{33}|D_3 > + f_{32}|F_2 >),$$

and in this run of filtering, the average value of the failure should be

$$F_2(\omega) = \eta'_2 \left(1 - (1-|\alpha'_2|^2)^2 \right) \sin^2 \omega_2 + \eta'_3 |\alpha'_3|^2 \frac{2}{1 - (1-|\alpha'_2|^2)^2 \sin^2 \omega_2}.$$

with $\omega_{22} = \omega_2$. If

$$\eta_2 = \eta_3 = \eta, \quad |f_{21}| = |f_{31}|,$$

then we shall get

$$F_{2>opt} = 2\eta|\alpha_{23} - f_{23} f_{31}|,$$

by letting

$$\sin^2 \omega_{2>opt} = \frac{1 - |\alpha'_2|^2}{1 - |\alpha'_2|^2\sin^2 \omega_2}.$$

B. the POVMs for discriminating three pure states

By performing the R(2) after R(1), the state $|\Psi_j >$ are transformed into:

$$|\Psi_1 >^{\text{out}} = d_{11}|D_1 > + f_{11}|F_1 >,$$

$$|\Psi_2 >^{\text{out}} = d_{22}|D_2 > + f_{21}|F_1 > + f_{22}|F_2 >,$$

$$|\Psi_3 >^{\text{out}} = d_{33}|D_3 > + f_{31}|F_1 > + f_{32}|F_2 >.$$

With $U^{-1}(\omega) = R^{-1}(1)R^{-1}(2)$, we can arrive at

$$|D_1 >^e = -\sin \omega_1 |\Psi_1 >$$

$$|D_2 >^e = \frac{\cos \omega_1 \sin \omega_2}{\sqrt{\cos \omega_1^2 + \sin^2 \omega_1 |\alpha_{2}^2|^2}}|\Psi_2 >^e,$$

$$|F_1 >^e = \frac{|\Psi_1 > - t_{11} \sin^2 \omega |\Psi_1 >}{\sqrt{1 - t_{11}^2 \sin^2 \omega}},$$

$$|F_2 >^e = \frac{|(\mathbf{I} - D_2 >^e < D_2 > | - |F_1 >^e < F_1 >)||\Psi_2 >}{\sqrt{1 - |d_{21}|^2 - |f_{21}|^2}},$$

with $d_{ij} = <D_j|\Psi_i >, f_{ij} = <F_j|\Psi_i >$ and $\omega_1 = \omega_{11}$. For discriminating three pure states, the POVMs are defined by

$$E_1(\omega) = |D_1 >^e < D_1 |,$$

$$E_2(\omega) = |D_2 >^e < D_2 |,$$

$$E_0(\omega) = \sum_{k=1}^{2} |F_k >^e < F_k |,$$

$$E_3(\omega) = \mathbf{I} - \sum_{l=1}^{2} E_l - E_0,$$

the average value of failure is defined by

$$F(\omega) = \sum_{j=1}^{3} \eta_j |<\Psi_j|F_1 >|^2 + \sum_{l=1}^{2} \eta_l |<\Psi_l|F_2 >|^2.$$
Using Eq.(5.10), there should be

\[ F(\omega) = F_1(\omega) + F_2(\omega), \tag{5.16} \]

the \( F_2(\omega) \), which has been given in Eq.(5.8), is in the form of filtering with \( N=2 \). With calculations that

\[ f_{11} = \sqrt{1 - t^2 \sin \omega_1}, \]
\[ f_{k1} = = \frac{\omega_{1k}}{1 - t^2 \sin \omega_1}, \tag{5.17} \]

for \( k=1,2 \), \( F_1(\omega) \) should be

\[ F_1(\omega) = \eta_1 (1 - t^2 \sin \omega_1) + \frac{\sum_{k=2}^3 \eta_k |a_{1k}|^2}{1 - t^2 \sin \omega_1}, \tag{5.18} \]

certainly, it is also in a typical form of filtering with \( N=3 \).

C. the analytic optimal results for a special case

Usually, it is difficult for us to give an analytic solution for the optimal values of the \( F(\omega) \) in Eq.(5.16), while the following case, which has been discussed in [8], is an exception. Considering the case, where \( \langle \Psi_{1}\rangle \langle \Psi_{2} \rangle = \langle \Psi_{1} | \Psi_{2} \rangle = s_1 \) and \( \langle \Psi_{2} | \Psi_{3} \rangle = s_2 \) under the conditions that \( s_1^2 < s_2 \), we find that

\[ t_1 = \frac{\det(O(3))}{\sqrt{1 - s_2^2}}, \]
\[ \det(O(3)) = 1 - 2s_1^2 - s_2^2 + 2s_1^2 s_2^2, \tag{5.19} \]

according to the results given in the appendix. Suppose \( \eta_j = \frac{1}{j} \) for \( j=1,2,3 \), we could find

\[ f_{21} = f_{31} = \frac{s_1}{\sqrt{1 - t_1^2 \sin \omega_1}}. \tag{5.20} \]

Using Eq.(5.10), there should be

\[ F_2^{opt}(\omega_1) = \frac{2}{3} s_2 - \frac{s_1^2}{1 - t_1^2 \sin^2 \omega_1}, \tag{5.21} \]

the optimal value is defined as the minimum value of the function

\[ F(\omega) = F_1(\omega_1) + F_2^{opt}(\omega_1), \tag{5.22} \]

and it should depend on the actual situations about the \( \omega_{ij} \): (1) if \( 2s_1 \leq s_2^2 \), by letting

\[ \sin^2 \omega_{ij}^{opt} = \frac{1 - 2s_1}{1 - \frac{2s_1^2}{1 + s_2^2}}, \tag{5.23} \]

we shall get the optimal result

\[ F^{opt} = \frac{2}{3} (2s_1 - s_2). \tag{5.24} \]

and (2), is \( s_1 \leq 2s_2 \), the optimal value

\[ F^{opt} = \frac{1}{3} s_1^2 + 2s_2, \tag{5.25} \]

with the choice of

\[ \sin^2 \omega_{ij}^{opt} = \frac{1 - s_1^2}{1 - \frac{2s_1^2}{1 + s_2^2}}, \tag{5.26} \]

substituting \( \omega_{ij}^{opt} \) for \( \omega_1 \) in Eq.(5.11), which is \( \omega_1 \)-dependent, we can get the actual optimal setting for \( \omega_{ij}^{opt} \). One check that: the optimal values for \( F(\omega) \), which have been given in Eqs.(5.24-25), are consistent with the optimal results in [8].

VI. THE SUCCESSIVE FILTERING FOR DISCRIMINATION OF TWO MIXTURES

A. the optical realization in the enlarged space

Suppose there are two mixtures

\[ \rho_1 = \sum_{i=1}^2 P_i |\Psi_i \rangle \langle \Psi_i|, \quad \rho_2 = \sum_{j=3}^4 P_j |\Psi_j \rangle \langle \Psi_j|, \tag{6.1} \]

with \( \sum_{i=1}^2 P_i = \sum_{j=3}^4 P_j = 1 \), each \( \rho_i \) with it’s a priori probability to be \( \lambda_i \), and \( \sum_{i=1}^2 \lambda_i = 1 \). Letting \( \eta_i = \lambda_1 P_i \) for \( i=1,2 \), and \( \eta_j = \lambda_2 P_j \) for \( j=3,4 \), the above question can also be viewed as an operation on \( G \equiv \{ |\Psi_k \rangle \} \) with \( \eta_k \) as its probability for \( k=1,2,3,4 \), we are required to tell if a state \( |\Psi_j \rangle \in G \) belongs to \( S_1 \equiv \{ |\Psi_1 \rangle, |\Psi_2 \rangle \} \) or \( S_2 \equiv \{ |\Psi_3 \rangle, |\Psi_4 \rangle \} \) while there is a non-vanishing probability for failure. In terms of POVMs, \( E_1 |\Psi_j \rangle \rangle = E_2 |\Psi_j \rangle \rangle = 0 \) holds for \( |\Psi_j \rangle \rangle \in S_2 \) and \( |\Psi_j \rangle \rangle \in S_1 \), respectively. Certainly, there is \( E_0 = \Gamma - \sum_{k=1}^2 E_k \). The OPI in Fig.8 is designed to realize the unitary transformation in the way like

\[ U(\omega) = R(2)R(1), \tag{6.2} \]

where \( R(1) \) is the unitary transformation for filtering \( |\Psi_2 \rangle \) from the rest of the states in \( G \), \( R(2) \) is used to filter \( |\Psi_4 \rangle \) from \( |\Psi_3 \rangle \rangle > \) and \( |\Psi_4 \rangle \rangle > \), whose definitions shall be given later. It should be noted that, when the detector \( D_1 \) fired, we can not tell whether this signal is from \( |\Psi_1 \rangle \rangle \) or \( |\Psi_2 \rangle \rangle \) since the fact that these two states may have non-zero coefficients, \( c_{12} \) and \( c_{22} \), along the rail \( c_2 \), respectively. It is certain that this signal cannot come from the states, \( |\Psi_3 \rangle \rangle > \) and \( |\Psi_4 \rangle \rangle > \), according to our discussion of DTR. One may compare the present \( R(1)|\Psi_j \rangle \rangle > \), with the one in discriminating three pure states,

\[ R(1)|\Psi_2 \rangle \rangle = d_{21}|D_1 \rangle \rangle > + f_{21}|F_1 \rangle \rangle >, \]
\[ R(1)|\Psi_1 \rangle \rangle = \sqrt{1 - |f_{11}|^2 - |d_{11}|^2}|\Psi_1 \rangle \rangle > + d_{11}|D_1 \rangle \rangle > + f_{11}|F_1 \rangle \rangle >, \tag{6.3} \]
\[ R(1)|\Psi_k \rangle \rangle = \sqrt{1 - |f_{k1}|^2}|\Psi_k \rangle \rangle > + f_{k1}|F_1 \rangle \rangle >, \]
for k=3,4. Defining a Hilbert space $H'$ with its basis as \{$(\omega_{21}, r >, |\omega_{31}, r >, |e_{1} >)$\}, there are three states:

\[
|\Psi'_{1} > = \frac{c_{11}}{\sqrt{1 - |f_{11}|^2 - |d_{11}|^2}}|e_{1} > + o'_{34}a'_{41} \sqrt{1 - |o'_{41}|^2}|\omega_{21}, r > + o'_{43}|\Psi'_{4} >
\]
\[
|\Psi'_{3} > = \sqrt{1 - |o'_{41}|^2}|\omega_{21}, r > + o'_{43}|\Psi'_{4} >, \quad \text{(6.4)}
\]
\[
|\Psi'_{4} > = e^{i\phi}r'|\omega_{31}, r >,
\]
with $o'_{ij} = <\Psi'_{i}|\Psi'_{j} >$, these overlaps can be derived from Eq.(6.3) by the requirement that the $o_{ij}$ keeps unchanged when the unitary transformation is performed on the input,

\[
o'_{1k} = \frac{o_{1k} - f_{11}f_{k1}}{\sqrt{(1 - |f_{11}|^2 - |d_{11}|^2)(1 - |f_{k1}|^2)}}, \quad \text{(6.5)}
\]
for k=3,4, while

\[
o'_{34} = \frac{o_{34} - f_{31}f_{41}}{\sqrt{(1 - |f_{31}|^2)(1 - |f_{41}|^2)}}, \quad \text{(6.6)}
\]

In principle, we could realize $R(2)$ as the filtering for $N=3$,

\[
R(2)||\Psi'_{1} > = \frac{d_{12}|D_{1} > + f_{12}|F_{2} >}{\sqrt{(1 - |f_{11}|^2 - |d_{11}|^2)}}
\]
\[
R(2)||\Psi'_{k} > = \frac{\sum_{i=3}^{4}d_{ki}|D_{i} > + f_{k2}|F_{2} >}{\sqrt{1 - |f_{k1}|^2}}. \quad \text{(6.7)}
\]

In the first run of filtering, the average value of failure is defined by

\[
F_{1}(\omega) = \sum_{j=1}^{N=4} \eta_{j}|f_{j1}|^2, \quad \text{(6.8)}
\]

while the one for the second run of filtering is

\[
F_{2}(\omega) = \eta'_{1}(1 - c'_{11}\sin^2 \omega_{2}) + \sum_{k=3}^{4} \eta'_{k}|o'_{1k}|^2, \quad \text{(6.9)}
\]

in a standard form of filtering with $N=3$, and their a priori probability is

\[
\eta'_{1} = \eta_{1}(1 - |f_{11}|^2 - |d_{11}|^2), \quad \eta'_{k} = \eta_{k}(1 - |f_{k1}|^2), \quad \text{(6.10)}
\]

for k=3, 4, respectively. The coefficient, $c'_{11}$, may be read from Eq.(6.4) as

\[
c'_{11} = \frac{c_{11}}{\sqrt{(1 - |f_{11}|^2 - |d_{11}|^2)}}, \quad \text{(6.11)}
\]

**FIG. 8:** The OPI for discriminating two mixtures

**B. the POVMs realized by the OPI in Fig.8**

As we have shown, it’s possible for us to get a $U(\omega)$ which transforms each $|\Psi_{j} >$ in the out-space:

\[
|\Psi_{2} >^\text{out} = d_{21}|D_{1} > + f_{21}|F_{1} >,
\]
\[
|\Psi_{1} >^\text{out} = \sum_{j=1}^{2} d_{1j}|D_{j} > + f_{11}|F_{11} >, \quad \text{(6.12)}
\]
\[
|\Psi_{k} >^\text{out} = \sum_{i=1}^{4} d_{ki}|D_{i} > + f_{ki}|F_{i} >.
\]

Using these expressions and the inverse $R^{-1}(1)R^{-1}(2)$, we can get

\[
|D_{1} >^\epsilon = -\sin \omega_{1}|e_{2} >,
\]
\[
|F_{1} >^\epsilon = \frac{(\Gamma - |D_{1} >^\epsilon < D_{1} |)|\Psi_{2} >}{\sqrt{1 - |d_{21}|^2}},
\]
\[
|D_{2} >^\epsilon = -\sin \omega_{2}|e_{1} >,
\]
\[
|F_{2} >^\epsilon = \frac{(\Gamma - \sum_{i=1}^{2} |D_{i} >^\epsilon < D_{i} | - |F_{1} >^\epsilon < F_{1} |)|\Psi_{1} >}{\sqrt{1 - |f_{11}|^2 - \sum_{i=1}^{2} |D_{i} |^2}}, \quad \text{(6.13)}
\]

with $d_{ij} = <D_{j}||\Psi_{i} >$ and $f_{ij} = <F_{j}||\Psi_{i} >$, the POVMs are defined by

\[
E_{1} = \sum_{i=1}^{2} |D_{i} >^\epsilon < D_{i} |,
\]
\[
E_{0} = \sum_{i=1}^{2} |F_{i} >^\epsilon < F_{i} |, \quad \text{(6.14)}
\]
\[
E_{2} = \sum_{j=3}^{4} |D_{j} >^\epsilon < D_{j} | = \Gamma - \sum_{m=0}^{1} E_{m}.
\]
With these operators in hands, we could define the average value of fail

\[ F(\omega) = \sum_{j=1}^{N=4} \eta_j < \Psi_j | E_0 | \Psi_j >, \quad (6.15) \]

and one may check that

\[ F(\omega) = F_1(\omega) + F_2(\omega). \quad (6.16) \]

The coefficients, which are needed in the calculation of \( F(\omega) \), are list here

\[
\begin{align*}
    f_{11} &= \frac{c_{12} c_{22} \sin^2 \omega_1}{\sqrt{1 - c_{22}^2 \sin^2 \omega_1}}, \\
    f_{21} &= \sqrt{1 - c_{22}^2 \sin^2 \omega_1}, \\
    f_{k1} &= \frac{c_{12} \sin \omega_1}{\sqrt{1 - c_{22}^2 \sin^2 \omega_1}}, \\
    d_{11} &= -c_{12} \sin \omega_1, \\
    d_{12} &= -c_{11} \sin \omega_2, \\
    f_{12} &= \sqrt{1 - \sum_{i=1}^{2} |d_{i1}|^2 - |f_{11}|^2}, \\
    f_{k2} &= \frac{c_{12}^* f_{k1} - \Delta_{k1}}{\sqrt{1 - \sum_{i=1}^{2} |d_{i1}|^2 - |f_{11}|^2}},
\end{align*}
\]

with \( k=3, 4 \).

C. an application of the POVMs

In a recent work, Raynal et al. considered the question, which came from the implementation of the BB84 by using the four quantum optical coherent states \(|\pm \alpha\rangle, |\pm i\alpha\rangle\) [35], of how to discriminate the following two mixtures:

\[
\begin{align*}
    \rho_1 &= \frac{1}{2} (|\alpha > < \alpha| + - \alpha > < -\alpha|), \\
    \rho_2 &= \frac{1}{2} (|i\alpha > < i\alpha| + - i\alpha > < -i\alpha|), \quad (6.18)
\end{align*}
\]

and the authors expressed the optimal failure probability in terms of the mean photon number:

\[ F_{\text{opt}} = e^{-\mu (|\cos \mu| + |\sin \mu|)}, \quad (6.19) \]

with \( \mu \equiv |\alpha|^2 \).

Here, we shall reconsider this problem with the POVMs in Eqs.(13-14). Writing the all the known parameters in terms of \( \alpha \), we designed a program to get the optimal value of \( F(\omega) \) in Eq.(6.15) by scanning in the parameters space \( 0 \leq \omega_1, \omega_2 \leq 2\pi \). The final result of our calculation and the analytic solution in Eq.(6.19) are both presented in Fig.9, while the optimal values of the \( \omega_1 \) and \( \omega_2 \) are given in Fig.10. Although in some regions of Fig.9, small discrepancy still exists, our numerical calculations are consistent with the analytical solutions well in most parts of the parameter space, this fact shall give great supports to our present proposal.

VII. DISCUSSION

In the present paper, we always adopt a naive understanding of the mixture: suppose a mixtures is denoted by \( \rho \), for examples, \( \rho = \sum_{j=1}^{N} \eta_j | \Psi_j > < \Psi_j | \), in each run of the experiment, the input for our OPI is still a pure state belonging to the set \( \{ |\Psi_j > \} \).

For the system of \( N=2 \) and \( N=3 \), we have given a series of derivations to show why these cases can be solved by applying the argument of filtering, certainly, within the DTR. The case discussed in Sec.VI, is an important case of \( N=4 \). There are still others types of operations for \( N=4 \) system: (a) filtering \( |\Psi_1 > \) from \( |\Psi_2 > \) and \( |\Psi_3 > \) with \( |\Psi_4 > \) as the background; (b) filtering \( |\Psi_1 > \) from \( |\Psi_2 > \) with \( |\Psi_3 > \) and \( |\Psi_4 > \) as the background; (c) discriminating three pure states with \( |\Psi_4 > \) as the background.
A important profit of our scheme should be mentioned here: the POVMs for each case shall be able to, although maybe not in an optimal way, complete the task of discriminating when the a priori probability of each state is not completely decided.

In end of this paper, we would like to emphasize that: first, for a given case, if one could prepare the input in the one-photon state, then our OPI can be directly used for the optical experimental realization. Then, although the POVMs are from the one-photon picture, yet they are general and state-type independent. Finally, a proposal, rather than a complete proof, has been given here in order to find a solution to the problem of the quantum state unambiguous discrimination. It’s still a open question that: whether the task of quantum state unambiguous discriminating, either of pure states or of mixtures, can be solved by reducing it to the problem of quantum state filtering?

FIG. 11: The OPIs for the following cases: (a) filtering $|\Psi_1\rangle$ from $|\Psi_2\rangle$ and $|\Psi_3\rangle$ with $|\Psi_4\rangle$ as the background; (b) filtering $|\Psi_1\rangle$ from $|\Psi_2\rangle$ with $|\Psi_3\rangle$ and $|\Psi_4\rangle$ as the background; (c) discriminating three pure states with $|\Psi_4\rangle$ as the background; (d) discriminating four pure states and (e) discriminating $|\Psi_1\rangle$, $|\Psi_2\rangle$ and $\{|\Psi_3\rangle, |\Psi_4\rangle\}$.

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**APPENDIX: SOME KNOWN MATRICES FOR THE LOW-DIMENSIONAL CASES.**

For $N=2$,

$$O(2) = \begin{pmatrix} 1 & o_{12} \\ o_{21} & 1 \end{pmatrix}, \quad (1.1)$$

$$O^{-1}(2) = \begin{pmatrix} 1 & -o_{12} \\ -o_{21} & 1 \end{pmatrix}, \quad (1.2)$$

with

$$\det(O^{-1}(2)) = 1 - |o_{12}|^2, \quad t_1 = t_2 = \sqrt{1 - |o_{12}|^2} \quad (1.3)$$

For $N=3$,

$$O(3) = \begin{pmatrix} 1 & o_{12} & o_{13} \\ o_{21} & 1 & o_{23} \\ o_{31} & o_{32} & 1 \end{pmatrix}, \quad (1.4)$$

with $A(3)$, the adjoint of $O(3)$, to be

$$\begin{pmatrix} 1 - |o_{23}|^2 & o_{13}o_{32} - o_{12} & o_{12}o_{23} - o_{13} \\ o_{23}o_{31} - o_{21} & 1 - |o_{13}|^2 & o_{21}o_{13} - o_{23} \\ o_{32}o_{21} - o_{31} & o_{31}o_{12} - o_{32} & 1 - |o_{12}|^2 \end{pmatrix} \quad (1.5)$$
The inverse $O^{-1}(3)$ could be given by $O^{-1}(3) = \tilde{A}(3)/\det(O)$ with

$$\det(O(3)) = 1 - |o_{12}|^2 - |o_{13}|^2 - |o_{23}|^2 + o_{12}o_{23}o_{31} + o_{21}o_{13}o_{22}$$  \hspace{1cm} (1.6)

which has applications in

$$t_1 = \sqrt{\frac{\det(O)}{1 - |o_{23}|^2}}, t_2 = \sqrt{\frac{\det(O)}{1 - |o_{13}|^2}}, t_3 = \sqrt{\frac{\det(O)}{1 - |o_{21}|^2}}. \hspace{1cm} (1.7)$$

The overlap matrix, $O^\perp(3)$, is known with the form:

$$O^\perp(3) = \begin{pmatrix}
\frac{1}{(1 - |o_{23}|^2)(1 - |o_{12}|^2)} & \frac{o_{13}o_{23} - o_{12}}{(1 - |o_{23}|^2)(1 - |o_{13}|^2)} & \frac{o_{12}o_{23} - o_{13}}{(1 - |o_{21}|^2)(1 - |o_{12}|^2)} \\
\frac{o_{23}o_{13} - o_{21}}{(1 - |o_{21}|^2)(1 - |o_{13}|^2)} & \frac{1}{(1 - |o_{21}|^2)(1 - |o_{13}|^2)} & \frac{o_{12}o_{13} - o_{23}}{(1 - |o_{23}|^2)(1 - |o_{12}|^2)} \\
\frac{o_{12}o_{13} - o_{23}}{(1 - |o_{23}|^2)(1 - |o_{12}|^2)} & \frac{o_{23}o_{12} - o_{13}}{(1 - |o_{13}|^2)(1 - |o_{23}|^2)} & \frac{1}{(1 - |o_{13}|^2)(1 - |o_{23}|^2)}
\end{pmatrix}. \hspace{1cm} (1.8)$$

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