Some consequences of the Einstein covariance principle in electrodynamics of media

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Abstract

We show that the Einstein covariance principle provides an opportunity to generate infinitely many solutions of given covariant equation from a known one. With use of this statement we derive exact expressions for charge and current densities in a medium in an arbitrary external field. We obtain that in a homogeneous nondispersive medium the field of a point particle differs at short-distances from the well known expression. We also demonstrate that in a linear medium second harmonic generation and optical detection become possible in the field of radially polarized radiation.

Key words General relativity, covariance principle, charge and current densities.

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1 Introduction

The requirement of general covariance of the equations describing different processes in the nature is one of the corner-stones of Einstein general relativity [1] and has enormous significance in modern theoretical physics. When combined with the equivalence principle it reduces the gravitation to the metric properties of space-time and shows interconnection between the geometry of space-time continuum and the material processes.

In this article we make an attempt to discover another powerful aspect of the Einstein covariance principle. Namely, we are going to show that, the covariance principle provides new methods (algorithms) for solution of different problems of theoretical and mathematical physics.
In Sec. 2 we show that with use of the covariance principle it is possible to generate infinitely many solutions of given covariant equation from a known one. The new, generated, solutions correspond to physical situations differing (in boundary conditions, external fields) from those in the known solution. For instance, having a free solution of the given covariant equation, we can generate solutions of the same equation in the external field.

We succeeded in Sec. 3 to perform this procedure for the continuity equation for an ensemble of charged point particles and in Sec. 4 for a continuous medium using the Euler transformation i.e. with the help of Euler transformation we can pick up the solutions of the continuity equation in the external field from the infinite set of the new solutions of continuity equation.

As a result, in Sec. 4 we obtain exact expressions for the charge and current densities in a medium in the presence of an arbitrary external field. These formulae contain the external field in implicit way via Euler displacement field, which provides an opportunity to define polarization of the medium in natural way independent of the kind of the medium. Then, we show that in the case of homogeneous isotropic media it is possible to unambiguously determine the polarization and magnetization vectors in terms of the medium polarization.

In the Secs. 5 and 6 we discuss some physical consequences of the results of Sec. 4 considering two "textbook" problems of the linear electrodynamics.

In Sec. 5 we solve the problem of the field of a charged point particle in a homogeneous dielectric medium and find out that in this case the electric field is determined by a cubic algebraic equation. Analyzing solutions of this equation we detain a deviation from the well known result for small distances from the point charge.

In Sec. 6 we consider propagation of radially polarized radiation in a homogeneous linear dielectric medium. Our calculations show that in this case (linear medium) it is possible to observe nonlinear phenomena like optical detection and second harmonic generation.

Sec. 7 concludes the paper with some remarks.

2 The main statement

The Einstein covariance principle [1] claims that any physical law must have a covariant form i.e. if physical quantities $A, B, ...$ according to some physical law, are related by an equation

$$F \left( A(X), \hat{L} [B(X)], ... \right) = 0,$$  \hspace{1cm} (1.a)
in coordinates $X$ ( $\hat{L}$ is an operator, differential, integral, etc.), the functional relation should be the same in any other coordinates $X'$ i.e.

$$F \left( A'(X'), \hat{L}' [B'(X')] , ... \right) = 0. \quad (1.b)$$

But, each physical quantity has certain transformation property (scalar, vector, tensor etc.), i.e., for a given coordinate transformation

$$X'^i = W^i (X), \ (i = 0, 1, 2, 3) \quad (2)$$

these quantities are transformed as

$$A(X) = \hat{\Lambda}(X) A'(W(X)), \ B(X) = \hat{\Lambda}(X) B'(W(X)), \ ... \quad (3)$$

where $\hat{\Lambda} = 1$ for scalar quantities, for the vectors $\hat{\Lambda}$ is a matrix (see.(8)), for higher rank tensors $\hat{\Lambda}$ is a direct product of matrices. Insertion of (3) into (1.a) gives the equation

$$F \left( \hat{\Lambda}(X) A'(W(X)), \hat{L}' \left[ \hat{\Lambda}(X) B'(W(X)) \right], ... \right) = 0, \quad (4)$$

which, by comparing with (1.b), leads to the following statement:

*If $A(X), B(X), ...$ satisfy the covariant equation

$$F \left( A(X), \hat{L} [B(X)], ... \right) = 0, \quad (5.a)$$

then

$$\hat{\Lambda}(X) A(W(X)), \hat{\Lambda}(X) B(W(X)), \ ... \quad (5.b)$$

satisfy the same equation for any $W^i (X) \ (i = 0, 1, 2, 3)$.  

Thus, Einstein covariance principle provides an opportunity to generate infinite many new solutions for a given covariant equation from a known one by means of transformations within the frames of general relativity. The kind of covariant equation is not important for this statement: the equation may be linear or non-linear, differential or integro-differential, etc..

3 Continuity equation

As an application of the statement above let us consider the covariant continuity equation in electrodynamics

$$\frac{1}{\sqrt{-g(X)}} \partial_i \left( \sqrt{-g(X)} j^i (X) \right) = 0 \quad (6)$$
where \(-g(X)\) is the determinant of metric tensor (we use notations of the well known Landau-Lifshitz book [2]). It follows from (5) that, for any transformation \(W^i(X)\),
\[
\hat{\Lambda}\sqrt{-g} \equiv \|\Lambda(X)\|\sqrt{-g(W(X))}, \quad \hat{\Lambda}j^i \equiv \tilde{\Lambda}^i_n (X)j^n(W(X)) ,
\]
(7) satisfies the Eq.(6). Here we took into account the four-vector character of the current and the transformation law for the determinant of metric tensor:
\[
\Lambda^i_j(X) \equiv \partial_j W^i(X), \quad \tilde{\Lambda}^i_n \Lambda^n_j = \delta^i_j.
\]
(8)

Writing (6), (7) in the Cartesian coordinates, we now claim that if \(j^i_0(X)\) is a solution of
\[
\partial_i j^i_0 = 0, \quad (9)
\]
then
\[
\hat{j}^i(X) \equiv \|\Lambda(X)\|\tilde{\Lambda}^i_n (X)j^n_0(W(X))\]
(10) satisfies the same equation (9) for any transformations \(W^i(X)\).

The expression (10) has very important consequences in the electrodynamics. It provides a possibility to express charge and current densities in the arbitrary external field via charge and current densities of the undisturbed system.

This suggestion may be proved for an ensemble of charged point particles without using the statement (5). Indeed, let us consider an ensemble of particles having charge \(e_a\) and trajectories \(r^0_a(t)\) \((a = 1, 2, \ldots)\) in the absence of the external field. In this case the current four-vector at the point \(X' \equiv (ct', r')\) is given by [2]
\[
j^i_0(X') = c \sum_a e_a \delta (r' - r^0_a(t')) \frac{dX'^i}{dX^0} . \quad (11.a)
\]

In the presence of an external field the trajectories of particles are changed \(r^0_a(t) \rightarrow r_a(t)\) and the current at the point \(X \equiv (ct, r)\) becomes
\[
j^i (X) = c \sum_a e_a \delta (r - r_a(t)) \frac{dX^i}{dX^0} . \quad (11.b)
\]

Now, if we perform the Euler transformation (see Appendix)
\[
X^n = X^i - U^i (X), \quad U^i = (0, \mathbf{u}(r,t))
\]
(12) in (11.a) and use the well-known formula
\[
\prod_{i=1}^n \delta (x_i - \alpha_i) = \frac{1}{|J|} \prod_{i=1}^n \delta (\xi_i - \beta_i), \quad J \equiv \frac{\partial (x_1,...,x_n)}{\partial (\xi_1,...,\xi_n)},
\]
we arrive at the expression

\[ j^i(X) \equiv \parallel \Lambda (X) \parallel \hat{A}^i_n (X) j^a_0 (X - U), \]  

which coincides with \((11.b)\) under the condition

\[ u(r_a(t), t) = r_a(t) - r^0_a(t). \]  

This means that in an arbitrary external field the current four-vector \(j^i(X)\) is expressed linearly in terms of the undisturbed current four-current \(j^a_0\) at the point \(X - U\).

As we see, \(u(r,t)\) is a field which equals, on the trajectories of particles, to the displacement of these trajectories caused by external forces and, hence, it has the similar meaning as in the theory of elasticity [3].

On the other hand, the expression \((13)\) is a special case of \((10)\) when \(W^i(X)\) is the Euler transformation \((12)\) with the additional condition \((14)\). Hence, we can say that, by using the covariance principle, we are able to connect current and charge densities of disturbed and undisturbed system of an ensemble of point particles in any external field with the help of the Euler transformation. This suggestion is rather general. In Sec.4 we will come to the same conclusion for the four-current in a medium without referring to point structure of the particles in medium.

## 4 Charge and current densities in a medium

The expression \((10)\) for the current provides an opportunity to reformulate unambiguously the electrodynamics of continuous media independently of a specific type of the medium. Realistic media are complex systems with a number of subsystems (free electrons, electrons of different atoms in the same atomic state, nuclei, etc.). In the macroscopic electrodynamics, after Lorentz averaging, a medium is considered as continuous substance consisting of different continuous subsystems. Deformations in media caused by external forces appear in this approach as a consequence of modifications in particle trajectories caused by the interaction of those with the external fields. This means that we can use the methods of the description of the theory of elasticity [3] in the electrodynamics of continuous media. So, the problem of determination of the four-current of a certain subsystem in an external field reduces to the problem of determination of the four-current in the deformed subsystem of the medium. The coordinates of deformed and undeformed subsystems are connected by Euler transformation in terms of the displacement field \(u(r,t)\) [3]

\[ W^i(X) = (ct, r - u(r,t)) \]  

\[ 15 \]
and, hence, the expression (13) which is (10) written for Euler transformation (15) gives the four-current $j^i$ of the subsystem in the external field as a linear combination of the same quantity $j^i_0$ for the undisturbed subsystem. By using (A.5) of Appendix we obtain the three-dimensional form of the current (13):

$$\rho (r,t) = \|\Lambda\| \rho_0 (r - u, t), \quad (16.a)$$

$$j^\alpha (r,t) = \|\Lambda\| \left( S^{-1}_{\alpha\beta} [ (\partial_\tau u_\beta) \rho_0 (r - u, t) + j^\beta_0 (r - u, t) ] \right), \quad (16.b)$$

where $\|\Lambda\|$, $S^{-1}_{\alpha\beta}$ are given by expression (A.4), (A.6) of Appendix. The expression (16) is exact for any displacement field. Now, summing the expressions (16) over all subsystems (each with its own displacement fields) we arrive at the exact formulas for the charge and current densities of the medium in any external field.

Consider a medium consisting of a subsystem of identical particles and a background which ensures neutrality of the medium. Let the particle distribution in equilibrium be homogeneous and isotropic, then for averaged charge and current densities we have $\bar{\rho}_0 = \text{const}$, $\bar{j}_0 \equiv 0$. Defining the polarization of the medium as $\mathbf{P}(r,t) = \rho_0 \mathbf{u}(r,t)$, which is induced dipole moment density of the particles, and taking into account the neutrality condition, we obtain from (16), (A.4), (A.6) the following expressions for the charge and current densities of the medium in the external field

$$\bar{\rho} (r,t) = -\nabla \cdot \mathbf{\Pi}, \quad (17.a)$$

$$\mathbf{\bar{j}} (r,t) = \partial_t \mathbf{\Pi} + c [\nabla \times \mathbf{M}]. \quad (17.b)$$

Here

$$\Pi_\alpha \equiv P_\alpha + \frac{1}{2\rho_0} [ P_\nu \partial_\nu P_\alpha - P_\alpha \partial_\nu P_\nu ] + \frac{1}{6\rho_0^2} \epsilon_{\alpha\mu\nu} \epsilon_{\beta\lambda\sigma} (\partial_\mu P_\lambda) (\partial_\nu P_\sigma) P_\beta, \quad (18.a)$$

$$M_\alpha \equiv \frac{1}{2c\rho_0} \epsilon_{\alpha\lambda\nu} P_\lambda (\partial_\tau P_\nu) + \frac{1}{3c\rho_0^2} \epsilon_{\nu\sigma\lambda} (\partial_\alpha P_\lambda) (\partial_\nu P_\sigma) P_\tau \quad (18.b)$$

are the electric and magnetic polarization vectors of the medium.

In what follows we will restrict ourselves to consideration of linear regime of the interaction of external electromagnetic field with isotropic medium, i.e., we will consider weak electromagnetic fields in an isotropic
medium where \( P = \frac{\varepsilon - 1}{4\pi} E \). This is a restriction on the value of external electromagnetic field, but it could have large spatial derivatives which we cannot neglect in this approximation. Hence, even in linear regime of the interaction expressions (18) remain nonlinear.

5 The field of a point charge in a medium

Consider the electrostatic field of the point particle with charge \( q \) located at the origin of the coordinates in a medium. Because of spherical symmetry and stationarity of the problem, for the only nonzero component of (18) we have

\[
\Pi_r = P_r - \frac{1}{\rho_0} \frac{P_r^2}{r} + \frac{1}{3\rho_0^2} \frac{P_r^3}{r^2}.
\]

Then the solution to the Maxwell equation for the static field

\[
\nabla \cdot (E + 4\pi \Pi) = 4\pi q \delta (r)
\]

in linear nondispersive medium is given by the cubic algebraic equation

\[
E - \frac{(\varepsilon - 1)^2 E^2}{\varepsilon (4\pi \rho_0)^2 r} + \frac{(\varepsilon - 1)^3 E^3}{\varepsilon (4\pi \rho_0)^2 3r^2} = \frac{q}{\varepsilon r^2}.
\]

(19)

Here \( \varepsilon \) is the static dielectric constant of the medium. For large distances from the particle, where \(|u_r| \ll r\), we may ignore nonlinear terms in (19) and get the well known result:

\[
E = \frac{q}{\varepsilon r^2}.
\]

But, for small distances, when \(|u_r| \sim r\), nonlinear corrections in (19) are important. So, for an elementary charge \(|q| = |e|\) embedded in the electron gas with the concentration \( \sim 10^{22} cm^{-3} \) in solids with \( \varepsilon = 10\), the corrections are \( \sim 1 - 10\% \) at the distances of a few atomic radii. These corrections can be important in the problems of an exciton, solvated electrons and ions, the electronic centers in dye crystals, etc.

As we see, screening depends on relative sign of \( \rho_0 \) and \( q \). For the same signs of \( \rho_0 \) and \( q \) the effective charge is larger than \( q/\varepsilon \), and for different signs it is smaller. This fact has a simple geometrical explanation. According to the Gauss theorem, the field of the point particle at the point \( r \) is proportional to the charge within the sphere of the radius \( r \). In case of the same signs of \( \rho_0 \) and \( q \) medium particles between the spheres \( r - |u_r| \) and \( r \) leave the sphere \( r \). For the case of the different signs of \( \rho_0 \) and \( q \) medium particles come into the sphere \( r \) from between those of radii \( r \) and \( r + |u_r| \). But in these two cases the magnitude of outgoing and incoming charges are not equal: in the second case the incoming is greater, and we have the asymmetry mentioned above.
6 Nonlinear effects in a linear medium

Let us consider now a problem with cylindrical symmetry, i.e., propagation of a radially polarized radiation \( E_0 (r,z,t) = \hat{r} r E(r) \exp i (k_0 z - \omega_0 t) \) (for instance \( TEM_{01} \) radially polarized mode [4]) in a linear, isotropic medium. For the nonzero component of (18), i.e., radially polarized polarization \( P = \hat{r} P_r (r,z,t) \) \((r^2 = x^2 + y^2)\) we have

\[
\Pi_r = P_r - \frac{1}{2 \rho_0} \frac{P_r^2}{r}. \quad (20)
\]

Then (17) and (20) give rise to:

a. Static charge with density

\[
\rho (r) = 2 \chi^{(2)} \partial_r |E(r)|^2 , \quad (21.a)
\]

b. Charge and current densities at the second harmonic frequency

\[
\rho (r,z,t) = \chi^{(2)} \partial_r E_0^2 (r,z,t) , \quad j_r (r,z,t) = -\chi^{(2)} \partial_t E_0^2 (r,z,t) , \quad (21.b)
\]

the coefficient of nonlinearity in (20) is

\[
\chi^{(2)} = \left[ \varepsilon (\omega_0) - \frac{1}{4 \pi} \right]^2 \cdot \frac{1}{2 \rho_0} . \quad (22)
\]

So, in a linear medium it is possible to obtain optical detection and second harmonic generation. Let us estimate coefficient of the quadratic nonlinearity \( \chi^{(2)} \) for the medium with \( \varepsilon = 3 \) and electron concentration \( 10^{20} \div 10^{21} \text{cm}^{-3} \). For the width of radiation beam \( r \sim 0.1 \text{cm} \) we have

\[
\chi^{(2)} \sim 10^{-11} \div 10^{-12} \text{CGSE}.
\]

In case of focusing this coefficient may reach \( \sim 10^{-8} \div 10^{-9} \text{CGSE} \) because of strong inhomogeneity in the focus of the beam; this is large enough for observation.

7 Concluding remarks

We have shown how to generate with the help of Einstein covariance principle infinite many new solutions of a given covariant equation having a known one. By using Euler transformation we have separated a solution to the generated set of continuity equation which is a solution to the same equation in an arbitrary external field. In this way we got an exact expression for the charge and current densities for a medium.
in the external field. This new approach gives us an opportunity to determine the polarization of a homogeneous medium in a very general manner, i.e., independently of the model of the medium. In addition, it is possible, in this case, to unambiguously determine the electric polarization and magnetization vectors in terms of the medium polarization and its derivatives. Further on we considered applications of this expression. Solving the problem of charged point particle in a homogeneous dielectric medium we arrived at a cubic algebraic equation for the electric field, whose solution shows deviation of the electric field from the classical result at small distances from the point charge. We predicted second harmonic generation and optical detection phenomena in a linear medium.

It would be interesting to apply this approach to other equations in mathematical physics. These type of investigations are in progress and will be published elsewhere.

We note finally that the boundary-value problem of the electrodynamics of an expanding-contracting sphere has been for the first time solved in [5] with use of the covariance principle, but the authors did not realize, at that time, the universality and importance of that approach.

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8 Appendix: Euler transformation

We use in the paper the Euler transformation which is very well known in the theory of elasticity:

\[ r' = r - \mathbf{u}(r,t) \] (A.1)

where \( \mathbf{u}(r,t) \) is the displacement field of the medium.

For the transformation matrix \( \Lambda^j_i(X) \equiv \frac{\partial X}{\partial X'} \) and its determinant we get

\[ \Lambda^0_0 = 1, \quad \Lambda^0_\alpha = 0, \quad \Lambda^\alpha_0 = -\frac{1}{c} \partial_t u_\alpha, \quad \Lambda^\alpha_\beta = \delta_{\alpha\beta} - \partial_\beta u_\alpha \equiv S_{\alpha\beta}, \] (A.3)

\[ \|\Lambda\| = 1 - \partial_\lambda \left\{ u_\lambda + \frac{1}{2} \left[ u_\nu \partial_\nu u_\lambda - u_\lambda \partial_\nu u_\nu \right] + \frac{1}{6} e_{\lambda\mu\nu} e_{\beta\rho\sigma} (\partial_\mu u_\rho)(\partial_\nu u_\sigma) u_\beta \right\} . \] (A.4)
Here \( u_\alpha (\alpha = 1, 2, 3) \) are components of \( u \). \( e_{\alpha \beta \gamma}, \delta_{\alpha \beta} \) are three dimensional Levi-Civita and Kronecker symbols, respectively.

For the reciprocal matrix \( \tilde{\Lambda} \) of \( \Lambda \left( \Lambda_i^m \tilde{\Lambda}^m_j = \Lambda_i^m \Lambda_j^m = \delta_i^j \right) \) we find from (A.3):

\[
\tilde{\Lambda}_0^0 = 1, \quad \tilde{\Lambda}_0^\alpha = 0, \quad \tilde{\Lambda}_\alpha^0 = \frac{1}{c} S^{-1}_{\alpha \beta} \partial_t u_\beta, \quad \tilde{\Lambda}_\alpha^\beta = S^{-1}_{\alpha \beta},
\]

(A.5)

where \( S^{-1} \) is the reciprocal matrix of \( S \):

\[
||\Lambda|| S^{-1}_{\alpha \beta} = \delta_{\alpha \beta} - \partial_\nu \left[ u_\nu \delta_{\alpha \beta} - u_\alpha \delta_{\nu \beta} - \frac{1}{2} u_\lambda e_{\lambda \beta \sigma} e_{\nu \alpha \mu} \partial_\mu u_\sigma \right].
\]

(A.6)

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