Quasi-particle dephasing time in disordered d-wave superconductors

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Abstract

We evaluate the low-temperature cutoff for quantum interference $1/\tau_\varphi$ induced in a $d$-wave superconductor by the diffusion enhanced quasiparticle interactions in the presence of disorder. We carry out our analysis in the framework of the non-linear $\sigma$-model which allows a direct calculation of $1/\tau_\varphi$, as the mass of the transverse modes of the theory. Only the triplet amplitude in the particle-hole channel and the Cooper amplitude with is pairing symmetry contribute to $1/\tau_\varphi$. We discuss the possible relevance of our results to the present disagreement between thermal transport data in cuprates and the localization theory for $d$-wave quasiparticles.

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I. INTRODUCTION

During the last ten-fifteen years overwhelming experimental evidences for a $d-$wave pairing symmetry in high-$T_c$ superconducting cuprates have been collected\(^{1}\). Such an unconventional symmetry plays a crucial role in determining the physics of the superconducting phase. In particular the presence of four nodes in the $d-$wave superconducting gap $\Delta_k = \Delta (\cos k_x - \cos k_y)/2$ and, consequently, the existence of gapless quasiparticles (QP) excitations down to zero energy strongly affects the low temperature transport of these systems. The energy spectrum of the QP in the proximity of the $d-$wave gap four nodes has a Dirac cone shape

$$E_k = \sqrt{\xi_k^2 + \Delta_k^2} \simeq \sqrt{v_F^2 k_x^2 + v_F^2 k_y^2}$$

where $\xi_k$ is the energy dispersion of the non interacting electrons, while $v_F$ and $v_\Delta$ represent respectively the Fermi velocity and the slope of the superconducting gap $\Delta_k$ at the nodes. A weak amount of disorder originates\(^2\) a finite quasiparticle elastic lifetime $\tau_0$ and a finite density of states at the Fermi level $N_0 = 1/(\pi^2 v_F v_\Delta \tau_0) \ln(p_0/\tau_0)$, $p_0$ being a cutoff of order $\Delta$. In the low temperature and frequency regime the conductivities of the system (both the electrical $\sigma$ and the thermal $\kappa$) do not depend on the amount of disorder but only on the bare QP spectrum parameters $v_F$ and $v_\Delta$. This leads to the so-called universality of the low energy values of the conductivities in $d-$wave superconductors. Indeed, by allowing the impurity-scattering between all the four nodes of the $d-$wave spectrum and neglecting the vertex corrections, the following “universal” expressions are found\(^2\)

$$\lim_{\Omega,T \to 0} \sigma(\Omega,T) = \frac{e^2 v_F}{\pi^2 v_\Delta}, \quad \lim_{\Omega,T \to 0} \kappa(\Omega,T) = \frac{k_B^2 v_F^2 + v_\Delta^2}{3 v_F v_\Delta}.$$  

(2)

The inclusion of vertex corrections modifies $\sigma$ by a factor depending on disorder and Fermi-Liquid parameters, while it leaves $\kappa$ totally unchanged\(^2\).

However, in cuprate superconductors this generic result could be strongly affected by localization corrections because these materials are quasi-bidimensional and quantum interference effects are usually relevant in low-dimensional systems\(^3\).

¿From a theoretical point of view the evaluation of the localization corrections to the conductivities in a $d-$wave superconductor is a really complex topic. Indeed, when considering a $d-$wave BCS Hamiltonian in the presence of disorder and neglecting interactions...
among quasiparticles, the theory for the quantum interference predicts a multitude of different regimes and crossovers depending on the specific symmetries of the underlying Fermi surface and of the disorder. Anyway, in the generic case of non-magnetic disorder connecting all nodal points the theoretical calculations indicate the occurrence of an Anderson localization of the $d$–wave QP excitations. The theory also predicts precursor effects, the so-called weak localization corrections, which are logarithmic in temperature.

The analysis of experimental findings for the thermal transport in the cuprates does not appear to fit the above theoretical scenario. Indeed when comparing these predictions with the experimental measurements a convincing agreement is found neither with the “universal behavior” nor with the complete localization. It is worth stressing that thermal conductivity is conveniently used in order to test the theoretical results, because, differently from charge conductivity, it is not affected by the vertex corrections allowing for a direct comparison between the expression of $\kappa(T)$ in Eq. (2) and the experimental data.

Thermal conductivity measurements have been performed on various cuprates, down to temperatures of the order of 100 mK. In particular the residual thermal electronic conductivity $\kappa_{\text{res}}/T$ (= $\lim_{T \to 0} \kappa/T$), extrapolated from measurements carried out on optimally doped $YBa_2Cu_3O_{6+x}$ ($YBCO(123)$) and $Bi_2Sr_2CaCu_2O_8$ ($BSCCO$) seems to be in agreement with the universal result (Eq. (2) with $v_F$ and $v_\Delta$ extracted from ARPES measurements). The agreement is particularly good for $BSCCO$, leaving no room for sizeable weak localization corrections. The situation is partially different for $YBCO(123)$, where the inclusion of sizeable weak localization corrections would provide not un-plausible larger ratios $v_F/v_\Delta$ than the estimate of Ref. 12. However (or moreover) thermal magnetoconductance measurements do not give any indications of the presence of detectable weak localization corrections which should appear as a positive $H^2$ crossing to a $\ln H$ contribution at very small fields. This result is quite generic for all materials at different dopings. On the other hand, measurements performed on $YBa_2Cu_4O_8$ ($YBCO(124)$) underdoped and $Pr_{2-x}Ce_xCuO_4$ ($PCCO$) optimally doped provide a residual conductivity $\kappa_{\text{res}}/T$ much smaller than the universal value. Such a result, compatible with $\kappa_{\text{res}}/T = 0$ could be ascribed to the localization of the quasi-particles responsible of the low temperature energy transport. Recently measurements of $\kappa_{\text{res}}/T$ at various doping became available in $La_{2-x}Sr_xCuO_4$ (LSCO, Ref. 16) and in $YBCO(123)$ and made the theoretical scenario even more involved. In particular a significant doping dependence of $\kappa_{\text{res}}/T$ has been
observed in these cuprates. Such a dependence is hardly compatible with the universal behavior, particularly in the LSCO compounds, since it would imply a strong variation on doping of \( v_F \) and \( v_\Delta \) in partial contrast with ARPES measurements\(^{13}\) and the temperature dependence of the penetration depth. Moreover \( \kappa_{\text{res}}/T \) turns out to be finite and no localization is found for all dopings in the superconducting phase.

Concerning the absence of weak localization contributions a main issue is the estimate of the temperature below which quantum interference effects start to be visible before localization occurs. In the metallic phase this issue involves the determination of the dephasing time \( \tau_\varphi \), i.e. the time scale above which coherence is destroyed by inelastic processes. The energy scale \( 1/\tau_\varphi \) represents a cutoff for quantum interference and plays a crucial role in determining quantitatively the magnitude of the localization effects at low temperature\(^3\).

In principle, the situation becomes more involved when considering a \( d^-\) wave superconductor. Due to the extra symmetry of the Cooper pairing, quantum interference determines a relevant correction to the density of states which is logarithmic in energy since it derive from integration on diffusive modes with energy as a cutoff\(^4,18\). This fact reflects in the conductivity with the onset of an other source of logarithmic corrections coming from the thermal average of the density of states contribution. These corrections no longer depend logarithmically on the dephasing time \( (\propto -ln(\tau_\varphi/\tau_0)) \), but directly on \( T \) similarly to the interaction contributions\(^2\) \( (\propto -ln(1/T\tau_0)) \). As a consequence one can naturally individuate two contributions to the conductivity corrections, coming out respectively from the modes damped by \( \tau_\varphi^{-1} \) (as in the normal metal) and from those damped by the energy

\[
\frac{\delta \sigma_s}{\sigma_s} = \left( \frac{\delta \sigma_s}{\sigma_s} \right)_{\tau_\varphi^{-1}} + \left( \frac{\delta \sigma_s}{\sigma_s} \right)_\epsilon
\]

that can be viewed, roughly speaking, as the renormalization of the diffusion coefficient and of the density of states \( (\delta \sigma_s \sim \delta D N_0 + D\delta N_0) \). Indeed half of the contribution to the Renormalization Group (RG) equation\(^4,9\) for \( \sigma_s \) comes from the first term and half from the second term.

However it can be easily shown that in the range of temperature from \( 100mK \) to \( 1K \), to which we are interested in, the second kind of corrections can be neglected\(^{19}\), because of the rather small value of \( 1/T\tau_0 \) which is usually much less than \( \tau_\varphi/\tau_0 \). We can therefore estimate the localization correction to the spin conductivity \( \sigma_s \) of a \( d^-\) wave superconductor, (and consequently to the thermal conductivity which is related to \( \sigma_s \) by a generalized Wiedmann-
Franz law\textsuperscript{2}, considering only the expression for \((\delta \sigma_s/\sigma_s)_{\tau \phi^{-1}}\) given by
\[\left(\frac{\delta \sigma_s}{\sigma_s}\right)_{\tau \phi^{-1}} = \frac{\delta \kappa_{\text{res}}}{\kappa_{\text{res}}} = -\frac{t}{2} \ln \frac{\tau \phi}{\tau_0} \tag{3}\]
where \(t = 1/2\pi^2 \sigma_s\) and \(\sigma_s = (v_F^2 + v_\Delta^2)/(\pi^2 v_F v_\Delta)\).

Note that the right term of the Eq. \textsuperscript{3} differs for a factor 1/2 from the analogous RG expression\textsuperscript{4} since, according to the above discussion, only in half of RG contribution the relevant time region for the interference ranges from \(\tau_0\) to \(\tau \phi\), while in the other half it ranges from \(\tau_0\) to \(1/\epsilon \sim T\).

In normal metals the most effective mechanism of dephasing is provided by the interactions among electrons\textsuperscript{20}. Indeed interactions in presence of disorder have a twofold effect: on one side they generate corrections which can compete with those due to pure quantum interference, on the other they provide an intrinsic dephasing time \(\tau \phi\) which limits the quantum interference processes for \(T \neq 0\).

A systematic evaluation of interaction corrections to the conductivities in the d-wave superconducting phase was carried out in Refs.\textsuperscript{8,9} (see also \textsuperscript{18}). Since these corrections are proportional to the disorder induced density of states \(N_0\), they should be negligible for clean enough systems and for temperatures of the order of 100 mK, at which the thermal transport measurements are presently available\textsuperscript{8}.

The aim of this work is to extend the analysis of Ref.\textsuperscript{8} and to derive an explicit expression for the dephasing time \(\tau \phi\) induced in the superconducting phase by the interactions among the d-wave quasiparticles. In particular we will determine \(1/\tau \phi\) as the mass of the diffuson and the Cooperon (i.e., the two particle propagators in the particle-hole and particle-particle channels) in analogy with Refs.\textsuperscript{21,22,23} where \(1/\tau \phi\) was evaluated in the metallic phase. Here we will perform a one-loop analysis of the interaction terms, within the non-linear \(\sigma\)-model which is described in detail elsewhere\textsuperscript{8,9}. We find that the most relevant contribution to \(1/\tau \phi\) comes from the triplet channel. Indeed the singlet channel contribution is ruled out by the lack of particle conservation in the superconducting state\textsuperscript{8,18}. On the other hand the residual interaction in the Cooper channel, which would lead to a \(d + is\) instability when it is attractive, provides a subleading contribution to \(1/\tau \phi\) when it is repulsive, as we shall assume here. As a final outcome the following estimate for ratio \(\tau \phi/\tau_0\), entering in Eq. \textsuperscript{3} will be derived
\[\frac{\tau \phi}{\tau_0} \propto \frac{\varepsilon_F}{T} (\Delta/\tau_0). \tag{4}\]
where $\varepsilon_F$ is the Fermi energy and the proportionality factor is of order one. In agreement with the results for the metallic phase $\tau_\varphi$ is proportional to the inverse of temperature $1/T$. Moreover the linear dependence of the ratio $\tau_\varphi/\tau_0$ on the elastic scattering time suggests that quantum interference corrections at finite temperature would be larger in cleaner systems, differently from what one could expect intuitively.

By inserting in Eq. (4) the experimental determinations of $v_F$, $v_\Delta$, $\Delta$ and $\epsilon_F$, we will estimate the value $\tau_\varphi$ for different cuprates, in order to predict from Eq. (3) the temperature below which localization corrections should become relevant. In particular in $YBCO(123)$ and $BSCCO$, optimally doped, we will find that the corrections to the universal results should become relevant, of order of $20 \div 30$ per cent, for temperatures of the order of $0.1 \div 1$ K, in disagreement with the experimental results. Actually, it is not clear why localization of the quasiparticles does not occur in $YBCO(123)$, while an explanation can be found for the bismuth cuprate $BSCCO$, as we will explicitly point out in Sec. IV. Indeed a rough estimate of the transport times, from electrical conductivity at higher temperature, suggests the existence in this last compound of other mechanisms for decoherence and dephasing, more efficient than the mechanism of interaction in a disordered environment we have considered here.

Finally, in sec. V we will present the concluding remarks on the problems posed by our findings.

II. FIELD THEORY FOR DISORDER IN THE SUPERCONDUCTING PHASE

In this section we summarize the field theory approach and introduce the non-linear $\sigma$-model (NL$\sigma$M), that will be used in the following to evaluate the dephasing time $1/\tau_\varphi$ in the $d$-wave superconducting phase. It is worth noting that in the superconducting phase the evaluation of the dephasing time using the NL$\sigma$M is simpler and more direct than using the standard perturbation theory, because of the presence of both the normal and the anomalous Green functions with specific $k$ dependences.

For the sake of clarity, we start discussing in this section the NL$\sigma$M in the absence of quasiparticle interactions, which will be included in a second step. In order to describe the properties of the disordered system within a path integral formulation, we introduce the
spinorial representation

\[
\Psi(r) = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{c}(r) \\ i\sigma_y c(r) \end{pmatrix}, \quad \bar{\Psi}(r) = [C\Psi]^T.
\]

\(\Psi(r)\) is an Nambu spinor, while \(\bar{c}(r)\) and \(c(r)\) are Grassmann variables with components \(c(r)_{\sigma,p,a}\) and \(\bar{c}(r)_{\sigma,p,a}\), where \(\sigma\) refers to the spin, and \(p = \pm\) is the index of positive (+\(\Omega\)) and negative (−\(\Omega\)) frequency components. Later \(p\) will be extended to label the positive and negative Matsubara frequencies \(\omega_m = \pi T m\), where \(m\) is an odd integer. \(a = 1...n\) is the replica index introduced by the replica trick used to average the action over disorder, and \(C\) is the charge conjugation matrix

\[
C = i\sigma_y \tau_1.
\]

Here and thereafter, the Pauli matrices \(\sigma_i(i = x, y, z)\) will act on the spin components, \(s_i(i = 1, 2, 3)\) on the frequency components, and \(\tau_i(i = 1, 2, 3)\) on the components of the Nambu spinors \(\bar{c}_i\) and \(c_i\).

In this representation the BCS action, describing the system in the superconducting phase in the absence of disorder, reads

\[
S_0 = \sum_k \bar{\Psi}_k \left( \xi_k + i\Delta_k \tau_2 s_1 - i\Omega s_3 \right) \Psi_k
\]

where the term \(-i\Omega s_3\) has been introduced in order to define retarded and advanced Green functions, and the presence of the Pauli matrix \(s_1\) in the second term is due to the fact that the Cooper pairing couples states with opposite energies.

As in the standard theory of localization in the normal phase\(^{25}\), disorder is introduced via an impurity scattering potential \(V(r)\) with local Gaussian distribution and zero mean value (the overline indicates the impurity spatial average)

\[
\overline{V(r)} = 0, \quad \overline{V(r)V(r')} = u^2\delta(r - r')
\]

In the following we will write \(u^2 = 1/(2\pi N_0\tau_0)\) as set by the saddle point solution (see below).

The average over disorder distribution is performed by using the replica trick method. A
four field term is generated which couples fields with different replica indices

\[ S_{imp} = -\frac{1}{4\pi N_0 \tau_0} \int dy (\bar{\Psi}(y)\Psi(y))^2 \]  

(8)

This contribution is then decoupled with a standard Hubbard-Stratonovich transformation. Consequently we introduce an hermitian bosonic field \( Q \), \( Q \) being a matrix in the \((\sigma, \tau, p, a)\)-space. The decoupled action has the following expression

\[ S_{imp} = \frac{1}{2\tau_0} \left\{ \int dx \left[ \frac{\pi}{4} N_0 \text{Tr} Q^2 - i(\bar{\Psi}, Q\Psi) \right] \right\} \]

where \((\bar{\Psi}, Q\Psi)\) represents a scalar product.

The NL\( \sigma \)M can now be derived, after an integration over the Grassman variables, by expanding the effective action around the saddle point \( Q_{sp} = 1/(2\tau_0)s_3 \) and taking into account only the low energy transverse fluctuations. As a consequence, the fluctuating fields \( Q \) satisfy the conditions \( Q^2 = 1 \), \( \text{Tr}Q = 0 \) and can be written in the form

\[ Q = U^{-1}Q_{sp}U \]

(9)

where \( U \) is a unitary matrix with suitable symmetries. We also rescale \( Q \to 1/2\tau_0Q \) in order to express the action in term of an adimensional field.

The outcome of this procedure is the following NL\( \sigma \)M

\[ S_{sc}[Q] = \frac{\pi}{16} \left\{ \sigma_s \int dx (\text{Tr}(\nabla Q))^2 \right. \]

\[ \left. - 8 N_0 \Omega \int dx \text{Tr}(s_3Q) \right\} \]

(10)

where

\[ \sigma_s = \frac{(2\tau_0)^{-2}}{\pi V} \sum_k \frac{\bar{\nabla}\xi_k \bar{\nabla}\xi_k + \bar{\nabla}\Delta_k \bar{\nabla}\Delta_k}{(E_k^2 + (2\tau_0)^{-2})^2} \approx \frac{1}{\pi^2} \frac{v_F^2 + v_\Delta^2}{v_F v_\Delta} \]

(11)

is the quasi-particle conductance in the Born approximation which can be interpreted as the spin conductance of the system\(^8\). The appearance of \( \sigma_s \) (instead of the charge conductivity, as in the metallic case) in front of the term \( \text{Tr}(\nabla Q)^2 \) in Eq. (10) individuates the inverse spin conductivity as the natural expansion parameter of the theory for the superconducting phase.

The expressions for the diffuson and the Cooperon (which are the massless modes of the theory showing diffusive poles) can be calculated by expanding the NL\( \sigma \)M action \( S_{sc} \) in
terms of independent fields. The Gaussian terms will define the bare diffusion and Cooperon propagators, while higher order terms will provide the corrections due to quantum interference. To this end it is useful writing the unitary transformation $U$, introduced in Eq. (9), in an exponential form $U = e^{W/2}$. The symmetry properties of $W$ are the following: (i) anti-hermiticity $W = -W^\dagger$, (ii) invariance for charge conjugation $CW^TC^T = W^\dagger$, (iii) non commutation with $Q_{sp}$ (i.e. with $s_3$). Because of this last property, $W$ has the energy structure $W \propto \alpha s_1 + \beta s_2$ and $Q$ can be written as

$$Q = Q_{sp} e^W$$

To the above properties of $W$, one has to add a further constraint when considering the superconducting phase. Indeed, the presence in Eq. (7) of the Cooper pairing term, which is proportional to $\tau_2s_1$ and determines the non conservation of the electric charge, requires the invariance condition $[U, \tau_2s_1] = 0$. This leads to

$$[W, \tau_2s_1] = 0$$

thus reducing the number of massless modes in the superconductor. Indeed, in this case, the $W$ field will be forced to be proportional in the energy space either to $s_1$ or to $s_2$. This symmetry reduction plays a crucial role in the evaluation of the dephasing time, since it implies the vanishing of the singlet term of the electron-electron interaction in the superconducting system, as explicitly discussed below.

The final outcome for $W$ fields both in metallic and in superconducting phase is summarized in the table of the appendix taken from Ref. 9. The table states whether the various components of the $W$-fields are real or imaginary and symmetric or antisymmetric.

By inserting Eq. (12) in Eq. (10), in which $\Omega s_3 \rightarrow \lambda_m \omega_m$ and the traces are extended to many Matsubara frequencies as required by analysis of interacting case, and by taking into account the symmetries of the $W$’s the two-particle propagators read

$$\langle W_{ab}^{\sigma_0,\nu} \sigma_{\sigma T}^{il} W_{cd}^{\sigma_0,\nu} \sigma_{\sigma T}^{il'} \rangle = (\pm) \frac{(1-\lambda_n \lambda_m)}{2} \delta(q + q') D_{nm}(q)$$

$$\times \left( \delta^{ac}_{nr} \delta^{bd}_{mq} \pm \delta^{ad}_{mq} \delta^{bc}_{mr} \mp [\delta^{ac}_{n-r} \delta^{bd}_{m-q} \pm \delta^{ad}_{n-q} \delta^{bc}_{m-r}] \right)$$

where $S, T$ refers to the singlet-triplet components (i.e. $W = W_S \sigma_0 + i W_T \cdot \sigma$), $i = 0, 1, 2, 3$ is the index of the $\tau$ matrices representing the particle-hole component (i.e., $W_{S/T} = W_{S/T}^0 \tau_0 +$
$i \tilde{W}_{S/T} \cdot \tau$), $abcd$ and $nmrq$ are replica and energy indexes respectively, and $\lambda_n = \text{sign}(\omega_n)$. On the right side ($\pm$) applies depending whether the W-components are real or imaginary, and $[\pm]$ whether they are symmetric or antisymmetric. Finally $D_{nm}$ is the (Gaussian) propagator

$$D_{nm}(q) = \frac{1}{4\pi N_0} \frac{1}{Dq^2 + |\omega_n - \omega_m|}$$

(15)

with the diffusion coefficient $D = \sigma_s/(2N_0)$.

The coefficients in $D_{nm}$ will be corrected both by higher order quantum interference effects and by the electron-electron interactions. Quite generally, both these effects determine a renormalization of the diffusion coefficient. In particular, interaction effects result into a self-energy $\Sigma(q, \Omega)$ which gives rise both to a renormalization of the diffusion coefficient, $D \rightarrow \tilde{D}$, and to a renormalization of the frequencies by a factor $Z$. More interestingly the interactions determine also the appearing in $D(q, \Omega)$ of a mass term (a non vanishing $\Sigma(q = 0, \Omega = 0)$), which represents the inverse dephasing time $\tau_{\varphi}^{-1}$. When all corrections are included the renormalized propagator will then read

$$D_{n,m}(q) = \frac{1}{4\pi N_0} \frac{1}{\tilde{D}q^2 + Z |\omega_n - \omega_m| + \frac{1}{\tau_{\varphi}}}$$

(16)

where $\tilde{D}$ and $Z$ indicate the renormalizations evaluated in Refs. 8,9 while $\tau_{\varphi}^{-1}$ will be explicitly computed in the following section.

III. EFFECTS OF THE INTERACTION: CALCULATION OF THE DEPHASING TIME

In the metallic phase the relevant interactions in the presence of disorder are the zero harmonic singlet and triplet amplitudes in the particle-hole channel $\Gamma_s$ and $\Gamma_t$ and the singlet $s$-wave amplitude in the Cooper particle-particle channel. In the $d$-wave superconducting phase the relevant residual Cooper interaction comes from the $is$-wave channel. In the following we will not consider this Cooper term which has been already analyzed in Ref. 18.

Indeed in the case of an additional repulsive $is$-wave Cooper interaction, it can be shown with straightforward but lengthy calculation that its contribution to $1/\tau_{\varphi}$ is subleading with
respect to the triplet particle-hole contribution (Eq. (27) below). This is because a repulsive coupling in the Cooper channel scales to zero and no relevant corrections survive from the particle-particle interaction. The situation would be completely different in the case of an attraction in the $is$-wave Cooper channel, leading to a $d + is$ instability, but the analysis of this regime, considered in Ref. 18, is beyond the scope of this paper.

Starting from $\Gamma_s$ and $\Gamma_t$, and using the same formal steps for the metallic phase\textsuperscript{26}, the following additional contribution to $S_{sc}$ is obtained\textsuperscript{9}

$$S_{int}[Q] = -T \frac{\pi^2 N_0}{8} \sum_{n,m,\omega,a,l=0,3} \int dx$$

$$\times \left\{ \Gamma_s \ Tr(Q_{n,n+\omega,\tau}^{aa} \sigma_0)Tr(Q_{m+\omega,m,\tau}^{aa} \sigma_0)$$

$$+ \Gamma_t Tr(Q_{n,n+\omega,\tau}^{aa} \sigma)Tr(Q_{m+\omega,m,\tau}^{aa} \sigma) \right\}$$

Notice that the $Q$ matrices are diagonal in the replica space, since the interactions are present at each fixed disorder configuration.

At this point we can make explicit the vanishing of the singlet contribution in Eq.(17) due to the additional constraint (13) related to the absence of particle conservation in the superconducting phase. Indeed, as discussed in the previous section, the presence of the superconducting gap causes a reduction of diffusive degrees of freedom with respect to the normal case. More precisely, the charge-conjugation invariance, which stands both in normal and in superconducting systems, implies

$$C^i Q^i C = Q \Rightarrow Q_{Si,nm}^{ab} = (-1)^i Q_{Si,mm}^{ba}$$

$$Q_{Ti,nm}^{ab} = (-1)^{i+1} Q_{Ti,mm}^{ba}$$

(18)

expliciting the charge-conjugation invariance for the singlet ($S$) and the triplet ($T$) particle-hole channels ($i = 0, 3$), while in the superconducting case the following constraint also holds

$$\tau_2 s_1 Q \tau_2 s_1 = -Q \Rightarrow Q_{Si,nm}^{ab} = (-1)^{i+1} Q_{Si,mm}^{ba}$$

$$Q_{Ti,nm}^{ab} = (-1)^{i+1} Q_{Ti,mm}^{ba}$$

(19)

which follows from Eq.(13).
In general, if we consider the following transformation for the $Q$’s

$$Q_{nm} \rightarrow Q_{-m-n}$$  \hspace{1cm} (20)

which is achieved by both permuting and changing the sign of the energy indices, we can decompose every $Q$ field in terms of its symmetric ($\text{sym}$) and anti-symmetric ($\text{ant}$) components under the transformation (20). Such a decomposition is useful, since in the superconducting case the singlet and the triplet components of the $Q^{aa}$-fields have definite symmetry properties under the transformation (20). In particular, from Eq. (18) and (19) one finds that

$$Q^{aa}_{S,\text{sym}} = 0; \quad Q^{aa}_{T,\text{ant}} = 0 \hspace{1cm} (21)$$

On the other hand, because of energy conservation, Eq. (17) can be written only in terms of the symmetric components, by rearranging the energy indices in the energy sum. As a result, only the triplet survives in Eq. (17).

We are now in position to evaluate the dephasing time, i.e. the mass of the propagators which is generated by the electron-electron interaction. In the metallic phase this mass term comes from a specific one loop self-energy diagram containing a finite contribution from a branch cut. We get the same result here. To this end, after expanding the $Q$-field appearing in $S_{\text{int}}[Q]$ in term of the $W$-fields, we consider the four-$W$-field term

$$S_{\text{int}}(W) = -T \frac{\pi^2N_0}{32} \sum_{n_1,m_1} \sum_{a,b,c} \sum_{l=0,3} \sum_{k,k',q} \Gamma_l \times \text{Tr} \left( \lambda_{n_1} W^{ab}_{n_1m_1}(k) W^{ba}_{m_1n_2}(-k+q) \gamma_l \bar{\sigma} \right) \times \text{Tr} \left( \lambda_{n_3} W^{ac}_{n_3m_2}(k') W^{ca}_{m_2n_4}(-k'-q) \gamma_l \bar{\sigma} \right) \delta_{n_1-n_2,n_4-n_3}$$ \hspace{1cm} (22)

By performing the contractions of the $W$-fields as it is shown in Fig. 1 and using the Gaussian propagator defined in Eq. (14), we obtain the following contribution to the self-energy

$$\Sigma_{n,m} = 3\pi T (1 - \lambda_m \lambda_n) \sum_{n_1} \Gamma_l (1 - \lambda_{n_1} \lambda_m) D_{n_1m}$$ \hspace{1cm} (23)
where we relabeled $m_1 = m$, $n_2 = n$. This contribution is the counterpart in the superconducting phase of the contribution coming from diagram (h) in Ref.22. To be specific, in the following we take $\omega_m > 0$ and $\omega_n < 0$

\[ \omega_m - \omega_{n1} \]

![Diagram](image)

**FIG. 1**: (a) Four field term coming from the perturbative expansion of $S_{int}$ in $W$ (Eq. 22); (b) Diagrammatic representation of the self-energy term of Eq. (23)

As in the metallic phase, the dynamic structure of the interaction must be included in Eq.(24). Indeed the one-loop self-energy is first order in $t$ (the disorder parameter) but its calculation can be carried out to infinite order in $\Gamma_t$. This is achieved simply by replacing the static amplitude $\Gamma_t$ in Eq.(23) with the dynamic amplitude $\Gamma_t(\omega)$, given by

\[ \Gamma_t(q, \omega) = \Gamma_t \frac{Dq^2 + |\omega|}{Dq^2 + (1 - 2\Gamma_t)|\omega|} \]  

(24)

where $\omega = \omega_n - \omega_{n1}$.

The self-energy $\Sigma$, therefore, becomes

\[ \Sigma(q = 0, \Omega) = 3 \frac{T}{N_0} \sum_{\omega_n < \omega < \omega_n + \Omega} \int \frac{dp}{(2\pi)^d} \frac{\Gamma_t(p, \omega)}{Dp^2 + |\omega + \Omega|} \]  

(25)

with $\Omega = \omega_m - \omega_n > 0$.

In writing the energy limits in Eq.(25) we have considered the partial cancellation discussed in Ref.22 leading to the upper cutoff $\omega < \omega_n + \Omega$, so that finally the frequency sum has to be performed in the interval $\omega_n < \omega < \omega_n + \Omega$. Apparently this contribution is proportional to $\Omega$ and should be taken as a contribution to $Z$. However in carrying out the sum by analytically continuing on the real frequencies $\omega \to -i\omega$ a branch cut at $\omega = 0$ appears. Along the branch cut $\Gamma_t(p, \omega)$ and $D(p, \omega + \Omega)$ have different analytic continuations. Because of this, the sum does not vanish in the limit $\Omega = 0$ and, in addition to the contribution to $Z$, the following mass term is also obtained.

13
\[
\frac{1}{\tau_\phi} = \frac{2}{\pi N_0} \int_{-\infty}^{\infty} \frac{d\omega}{\sinh \beta \omega} \int d^d p \frac{3 \text{Im} \{\Gamma_t(\omega, p)\} R_R(\omega, p)}{(2\pi)^d}.
\]

(26)

Notice that the integral in Eq. (26) is infrared divergent and takes contribution from momentum-energy scale \(0 < Dq^2, \omega < T\). As pointed out by Fukuyama\(^{21}\), \(1/\tau_\phi\) should enter selfconsistently the diffusion propagator in the right side of Eq. (26). In practice, following Fukuyama we evaluate the integral with a low frequency cutoff of the order of \(1/\tau_\phi\). Eventually we get:

\[
\frac{1}{\tau_\phi} = \frac{2\pi tT}{3} \left[ \frac{3}{(1 - \Gamma_t)} \right] \ln \frac{1}{\delta} \left( \frac{1}{1 - \Gamma_t} \right)
\]

(27)

where \(\delta = 3 t \Gamma_t^2 / (1 - \Gamma_t)\) represents the cut-off coming from the self-consistency within logarithmic accuracy.

This result indicates the existence of a direct proportionality between the energy scale \(1/\tau_\phi\) and the temperature, with a coefficient of proportionality that contains the square of the scattering amplitude \(\Gamma_t\). Comparing Eq. (27) with the results for normal metals\(^{22}\), we find that, despite the more involved Nambu formalism and the reduction of symmetry introduced by superconductivity, the triplet contribution to \(1/\tau_\phi\) in the \(d\)-wave superconducting phase has the same expression as in the normal phase. Conversely, as discussed above, one finds the vanishing of the singlet contribution, which is usually the most relevant contribution in the normal phase, due to the symmetries of the \(W\) fields in the superconducting phase. Notice also that usually other inelastic processes (i.e. electron-phonon or electron-electron interactions within the standard “clean” Fermi-liquid picture) lead to \(1/\tau_\phi \propto T^p\) with \(p > 1\) so that they should be subleading at low temperature with respect to (27).

IV. SIZE OF THE LOCALIZATION CORRECTIONS AND COMPARISON WITH EXPERIMENTS IN CUPRATES

In this section we use the expression (27) to estimate the ratio \(\tau_\phi/\tau_0\) in the superconducting phase of the cuprates at low temperature. To this end we express the parameters appearing in Eq. (27) in term of experimentally accessible quantities, within few reasonable
assumptions. The parameter $\Gamma_t$ can be expressed as

$$\Gamma_t = N_0^s V_t$$  \hspace{1cm} (28)$$

where $V_t$ is the dimensional static scattering amplitude and for clarity we added the index $s$ to the density of states of the superconductor, $N_0^s$. An estimation of $V_t$ is then obtained considering that its value should be essentially unaffected by the superconducting transition and that, assuming a weak- or intermediate-coupling for the metallic phase, one has $N_0^m V_t \sim 1$, i.e., $V_t \sim 1/N_0^m$, where $N_0^m$ is the metallic density of states. As a consequence, being $N_0^s \sim 1/(\pi^2 v_F v_\Delta \tau_0) \ll N_0^m \sim k_F/(2\pi v_F)$, the parameter $\Gamma_t$ will read

$$|\Gamma_t| \sim \frac{N_0^s}{N_0^m} \sim \frac{2}{\pi k_F v_\Delta \tau_0} \sim \frac{2}{\pi \Delta \tau_0} \ll 1$$  \hspace{1cm} (29)$$

where $\Delta \sim k_F v_\Delta$, being $k_F$ the Fermi momentum.

Inserting Eq. (29) and $t = v_F v_\Delta/2(v_F^2 + v_\Delta^2) \simeq v_\Delta/(2v_F)$ in Eq. (27) one gets

$$\frac{\tau_\phi}{\tau_0} \sim \frac{\pi k_F v_F}{3T} (k_F v_\Delta \tau_0) \sim \frac{\pi \varepsilon_F}{3T} (\Delta \tau_0).$$  \hspace{1cm} (30)$$

where we have dropped all the logarithmic corrections and $\varepsilon_F \sim k_F v_F$. Notice that Eq. (29) should be considered as an upper bound for $\Gamma_t$ and consequently Eq. (30) represents a lower bound for $\tau_\phi/\tau_0$. By comparing with the normal phase, where the largest contribution to $1/\tau_\phi$ is given by the singlet interaction\textsuperscript{20} and $1/\tau_\phi \sim \pi t T \sim T/(k_F v_F \tau_0)$, we find that $\tau_\phi/\tau_0$ is enhanced in the superconducting phase by a factor $\Delta \tau_0$, which can be quite large if disorder is weak.

At optimal doping the values for $k_F$, $v_F$ and $v_\Delta$ can be extracted by the ARPES data, while the elastic scattering time $\tau_0$ can be deduced from optical conductivity measurements. One gets $k_F \simeq 0.8A^{-1}$, $v_F \simeq 1.6eVA$, $v_F/v_\Delta \simeq 19$ (BSCCO) and $v_F/v_\Delta \simeq 14$ (YBCO(123))\textsuperscript{12,14,28}. The evaluation of $\tau_0$ is the most problematic one. A rough estimate of $1/\tau_0$ is obtained by extrapolating to lower temperatures $1/\tau(T)$ derived from microwave measurements of the charge conductivity $\sigma(T)$ (in the temperature range of $5 \div 30$ K) and taking $1/\tau_0 = \lim_{T\to0}1/\tau(T)$. This leads to $1/\tau_0 \simeq 0.4K$ (YBCO(123)) and $1/\tau_0 \simeq 8K$ (BSCCO)\textsuperscript{29,30,31}.

From these values we obtain the following estimates for the dephasing time in optimally
doped YBCO(123) and BSCCO

\[
\frac{\tau_{\phi}}{\tau_0} \approx 4 \cdot 10^7 \cdot \frac{1}{T} \quad \text{YBCO(123)}
\]
\[
\frac{\tau_{\phi}}{\tau_0} \approx 10^6 \cdot \frac{1}{T} \quad \text{BSCCO}
\]  

(31)

We can turn now to estimate the localization corrections to the transport coefficients according to Eq. (3):

\[
\frac{\delta \sigma_s}{\sigma_s} = \frac{\delta \kappa_{res}}{\kappa_{res}} \approx -\frac{1}{4} \frac{v_\Delta}{v_F} \ln \left[ \frac{\pi k_F v_F}{3} \frac{v_\Delta}{k_F v_\Delta \tau_0} \right]
\]  

(32)

As stated before, the log factor contains the elastic scattering time. As a consequence, the localization corrections tend -rather surprisingly- to increase with decreasing disorder. The outcome of our calculation is that for both materials we would predict that localization effects should be visible in the temperature range we are interested in, i.e., hundreds of mK. In particular for a temperature of 1 K we get for \( \delta \sigma_s/\sigma_s = \delta \kappa_{res}/\kappa_{res} \) the values \(-0.32\) and \(-0.18\) in the case of YBCO(123) and BSCCO. At 100 mK we get \(-0.36\) and \(-0.23\) respectively. More than these specific values we want to stress that our calculation (which possibly underestimates the effect) gives a relative variation of \( \sigma_s \) and \( \kappa_{res} \) of about (or even more than) 20 ÷ 30 per cent for temperatures lower than 1K. The point is that the \( t = v_\Delta/(2v_F) \) is small (but not dramatically small) while \( \tau_{\phi}/\tau_0 \) is large enough to provide a sizeable correction. For YBCO(124) we expect similar values than YBCO(123) or even larger since these compounds are usually quite clean.

These results are quite puzzling because, as we mentioned in the Introduction, no relevant localization effects are found in this temperature regime but for the case of YBCO(124) and PCCO.

The simplest possible explanation of such a discrepancy could be related to the existence in the real materials of other possible sources of inelastic scattering, beyond the diffusion enhanced electron-electron interaction (in particular scattering with some kind of quantum critical fluctuations), and in principle this would be responsible of a further contribution to the dephasing effects.

We can try to get the effective dephasing time from the temperature dependence of \( \tau(T) \) obtained in Ref. 30 and 31, as we did for \( \tau_0 \). By assuming that the temperature dependence comes from inelastic processes and writing \( 1/\tau(T) = 1/\tau_0 + 1/\tau_{in}(T) \), we take as a rough
estimate for the dephasing time $\tau^{exp}_{\phi} \simeq \tau_{in}(T)$, even though the two times are conceptually different. This procedure gives

$$\left( \frac{\tau_{\phi}}{\tau_0} \right)_{exp} \simeq 2 \cdot 10^6 \cdot \left( \frac{1}{T} \right)^{4.2} \quad \text{YBCO(123)}$$

$$\left( \frac{\tau_{\phi}}{\tau_0} \right)_{exp} \simeq 10 \cdot \frac{1}{T} \quad \text{BSCCO}$$

(33)

The comparison of Eqs. (33) and (31) does not reveal any inconsistency in taking the diffusion enhanced electron-electron interaction as the predominant source of dephasing at low temperature in YBCO(123). In particular in the temperature range around 1K our previous estimate of $\tau_{\phi}$ and the value of $\tau^{exp}_{\phi}$ are of the same order of magnitude but display a different temperature dependence with the “experimental” estimate growing as $1/T^4$ in the low temperature regime. As a consequence the scattering due to the diffusion enhanced el-el interaction is dominant below $T \simeq 0.5$ K and our previous theoretical estimate of Eq. (31) has not to be corrected. On the contrary, in BSCCO the “experimental” $\tau_{\phi}$ displays the same $1/T$ behavior of the theoretical prediction, but with a much smaller coefficient, suggesting the presence of some other mechanism of inelastic scattering more effective than the diffusion enhanced electron-electron interaction. In this case it becomes more appropriate to use the “experimental” estimation of $\tau_{\phi}$ to evaluate the value of $\delta \kappa_{res}/\kappa_{res}$, the result being $\delta \kappa_{res}/\kappa_{res} \simeq -0.03$ (at 1K) and -0.05 (at 100mK). These values can be taken as an indication of negligible weak localization corrections in BSCCO because of strong decoherence effects.

According to the above discussion and contrary to the experimental finding, significative localization effects would be expected in YBCO(123) as well as in YBCO(124), while the presence of a different source of strong inelastic scattering in BSCCO would explain the irrelevance of the localization corrections even at the lowest temperatures experimentally accessed.

V. CONCLUSIONS

In this paper we have dealt with the problem of the localization in high-$T_c$ superconductors. Their quasi 2-dimensional lattice structure would suggest that localization effects should become relevant at least in the low-temperature regime. However, as discussed in the
Introduction, the low-temperature transport measurements in the cuprates depend strongly on the different materials and point out to the inadequacy of the prediction of a complete localization with sizeable weak localization precursor effects. Of course the electronic interactions could play in principle a relevant role in the explanation of the real material properties.

A main effect of the interactions, which are a source of inelastic scattering, is to introduce a low-energy scale, the inverse of the dephasing time, which limits the quantum interference processes. Interactions also generate completely new corrections to transport, which could in principle compete with the localization effects. However they are expected to be negligible in the temperature-range of interest if disorder is weak.

We have derived the expression for the dephasing time $\tau_\phi$ in $d$-wave superconductors within a non-linear $\sigma$-model formulation of the disorder problem in the presence of interactions. Our result differs from the result for the metallic phase because of the vanishing of the singlet contribution, due to the symmetries of the superconducting phase. The main contribution to $1/\tau_\phi$ comes from the interaction in the triplet channel, while a repulsive residual interaction in the Cooper channel only produces a subleading term.

The scale $1/\tau_\phi$ allows to estimate the size of the localization corrections to the heat (and spin) transport. Indeed we have estimated the localization corrections that are theoretically expected in the various materials. We have also considered the possibility that other sources of inelastic scattering could provide smaller dephasing times in real materials, and attempted to provide rough experimental estimates of $1/\tau_\phi$ to make a direct comparison with the data for thermal transport. The outcome of this comparison is that the estimated dephasing time is not small enough to reconcile the theoretical and the experimental findings in the cuprates. This is particularly evident from the failure of the theoretical predictions for the localization corrections in the case of YBCO(123). Moreover, as pointed out in the Introduction, the scenario of the low-temperature transport in cuprates was made even more involved by very recent experiments in LSCO and in YBCO(123), which claim for a strong doping-dependence of the thermal conductivity.

The permanence of inconsistencies with the prediction of the localization theory, even when the electron-electron interactions and dephasing are taken into account, provides a clue of the relevance of some other effects which have been neglected in our analysis. For instance our theoretical predictions could be changed substantially by the proximity to
a $d + is$ instability or to a Quantum Critical Point (which would enhance considerably the estimate of the energy scale $1/\tau_\varphi$), by proximity to a nesting condition, by small $q$ scattering and domain wall scattering, by the removal of the hypothesis of weak-coupling interactions, which could become questionable especially for the underdoped cuprates and large disorder. All the above effects will lower the temperature scale below which localization become sizeable.

Finally we would like briefly to mention the relevant problem of magnetic impurities in the localization process. Since the scattering with magnetic impurities determines a vanishing of the localization corrections, a measurement of the low temperature transport coefficients in presence of magnetic impurities would be of a great relevance for understanding the role of the Anderson localization in the low-temperature transport of the cuprates, especially in the case of YBCO(124), where localization could be at work in providing a vanishing thermal transport. In YBCO(124) it would be also useful a systematic and careful analysis of magnetoconductance and thermal transport at small fields to compare with the findings in the cuprates which do not show localization. For these latter systems it would instead be important to assess the absence of magnetic impurities and local moments by magnetic measurements.

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APPENDIX

We report here explicitly the symmetries of the transverse massless modes $W$: 
\[ W_{ab}^{\text{sc},nm} = W_{ab}^{\text{sc}*} = -W_{ba}^{\text{sc},mn} = W_{ab}^{\text{sc},-n-m} \]
\[ W_{ab}^{\text{sc},nm} = -W_{ab}^{\text{sc}*} = -W_{ba}^{\text{sc},mn} = W_{ab}^{\text{sc},-n-m} \]
\[ W_{ab}^{\text{sc},nm} = W_{ab}^{\text{sc}*} = W_{ba}^{\text{sc},mn} = -W_{ab}^{\text{sc},-n-m} \]
\[ W_{ab}^{\text{sc},nm} = -W_{ab}^{\text{sc}*} = W_{ba}^{\text{sc},mn} = -W_{ab}^{\text{sc},-n-m} \]
\[ \vec{W}_{ab}^{\text{T},0,\text{nm}} = \vec{W}_{ab}^{\text{T}*} = \vec{W}_{ba}^{\text{T},0,mn} = \vec{W}_{ab}^{\text{T},0,-n-m} \]
\[ \vec{W}_{ab}^{\text{T},1,\text{nm}} = -\vec{W}_{ab}^{\text{T}*} = \vec{W}_{ba}^{\text{T},1,mn} = -\vec{W}_{ab}^{\text{T},1,-n-m} \]
\[ \vec{W}_{ab}^{\text{T},2,\text{nm}} = -\vec{W}_{ab}^{\text{T}*} = \vec{W}_{ba}^{\text{T},2,mn} = -\vec{W}_{ab}^{\text{T},2,-n-m} \]
\[ \vec{W}_{ab}^{\text{T},3,\text{nm}} = \vec{W}_{ab}^{\text{T}*} = \vec{W}_{ba}^{\text{T},3,mn} = -\vec{W}_{ab}^{\text{T},3,-n-m} \]

where \( n \) and \( m \) are odd integers which label the Matsubara frequencies \( \omega_n = T\pi n \) and \( \omega_m = T\pi m \), which have opposite sign (\( \omega_n \omega_m < 0 \)). Notice that because of Eq. \[13\] in the superconducting phase \( W_{nm} \) and \( W_{-n-m} \) are not independent.

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The expression of this kind of corrections would read $1/2 t \ln(\tau_0 T)$, so considering that for the materials examined here $t$ is always between $1/20 \div 1/40$ and $1/\tau_0$ lower than 8 K, the size of such conductivity corrections is at most of the order of few percent. Notice that the factor $1/2t$ is the same factor appearing in the energy dependent correction to the density of states $\delta N_0/N_0 = 1/2t ln(\tau_0 \epsilon)$.

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It has to be remarked that our results have been obtained in the low temperature limit and that the validity of our theoretical description is limited to $T \leq 1/\tau_0$.

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