Specific examples of the generalized Raychaudhuri Equations for the evolution of deformations along families of \( D \) dimensional surfaces embedded in a background \( N \) dimensional spacetime are discussed. These include string worldsheets embedded in four dimensional spacetimes and two dimensional timelike hypersurfaces in a three dimensional curved background. The issue of focussing of families of surfaces is introduced and analysed in some detail.

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I. INTRODUCTION

The Raychaudhuri equations for null or timelike geodesic congruences [1] provide us with a clear picture of the evolution of deformations along these specific families of curves. Together with the Einstein equations and an assumption about the nature of matter (in terms of an Energy Condition [2]) one arrives at the Focussing theorem which states that timelike/null geodesic congruences tend to get focussed within a finite value of the affine parameter used in defining these curves [2,3]. The Focussing theorem along with some other global arguments lead to the famous Singularity Theorems of GR [2].

In recent times, one has seen the emergence of the string or membrane viewpoint as the alternative for the point particle. The motivation for this is to arrive at a viable theory of quantum gravity as well as a unification of all forces. At Planck length scales the stringy nature of the point particle is believed to remove the problem of renormalizability of gravity and solve the singularity problem of GR.

If one accepts the string or membrane viewpoint then one should be able to write down the corresponding generalized Raychaudhuri equations for timelike/null worldsheet congruences and arrive at similar focussing and singularity theorems in Classical String theory. Very recently, Capovilla and Guven [4] have written down the generalized Raychaudhuri equations for timelike worldsheet congruences. In this paper, we construct explicit examples of these rather complicated set of equations by specializing to certain simple extremal families of surfaces. Our principal aim is to extract some information regarding focussing of families of surfaces in a way similar to the results for geodesic congruences in GR.

Sec. II of the paper contains a brief review of the generalized Raychaudhuri equations \textit{a la} Capovilla and Guven.

Sec. III contains the case of extremal string worldsheets in flat and curved backgrounds. After a general treatment of the generalized Raychaudhuri equations in string theory we move on to specific cases. Using the well–known string configurations in Rindler spacetime we analyse the resulting Raychaudhuri equation. For De Sitter spacetime we are able to make some general comments primarily because of certain specific properties of the spacetime itself.

In the fourth section of the paper we shall deal with the case of hypersurfaces(i.e. \( N – 1 \) dimensional surfaces embedded in a \( N \) dimensional background). We begin by writing down the full set of generalized Raychaudhuri equations for these objects and then move on to a discussion of the special case of an extremal timelike hypersurfaces (two dimensional) in a curved Lorentzian background (a wormhole metric in \( 2 + 1 \) dimensions). The final section of the paper contains a summary and remarks on future directions.

In the Appendix to the paper we present the pedagogical example of a catenoidal membrane embedded in a three dimensional Euclidean background. This example, it is hoped, will serve as an useful exercise while learning about these equations. It may also turn out to be a relevant calculation in the very active area of biological(amphiphilic) membranes.

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II. FORMALISM

This section reviews the recent work of Capovilla and Guven[4] which deals with a generalisation of the Raychaudhuri Equations for $D$ dimensional surfaces embedded in an $N$ dimensional background.

We define a $D$ dimensional surface in an $N$ dimensional background through the embedding $x^\mu = X^\mu(\xi^a)$ where $\xi^a$ are the coordinates on the surface and $x^\mu$ are the ones in the background. Furthermore, with the help of an orthonormal basis $(E_\mu^a, n_\mu^a)$ consisting of $D$ tangents and $N - D$ normals we can write down the Gauss–Weingarten equations using the usual definitions of extrinsic curvature, twist potential and the worldsheet Ricci rotation coefficients.

In order to analyse deformations normal to the worldsheet we need to consider the normal gradients of the spacetime basis. The corresponding analogs of the Gauss–Weingarten equations are:

\begin{align}
D_i E_a &= J_{aij} n^j + S_{abi} E^b \\
D_i n_j &= -J_{aij} E^a + \gamma^k_{ij} n_k
\end{align}

where $D_i \equiv n^\mu D_\mu$ ($D_\mu$ being the usual spacetime covariant derivative). The quantities $J_{aij}^i$, $S_{abi}$ and $\gamma^k_{ij}$ are defined as:

\begin{align}
S_{ab} &= g_{\mu\nu} n^a (D_\alpha E^\mu_a) E^\nu_b \\
\gamma^k_{ij} &= g_{\mu\nu} n^a (D_\alpha n^\mu_a) n^\nu_k \\
J_{aij}^i &= g_{\mu\nu} n^{a\nu} (D_\alpha E^\mu_a) n^{j\nu}
\end{align}

The full set of equations governing the evolution of deformations can now be obtained by taking the worldsheet gradient of $J_{aij}$

\begin{equation}
\tilde{\nabla}_b J_{aij} = -\tilde{\nabla}^i K_{ab}^j - J_{bk}^i J_{kj}^j - K_{bi}^j K_{aj}^i - g(R(E_b, n^i) E_a, n^j)
\end{equation}

where the extrinsic curvature tensor components are $K_{ab}^j = -g_{\mu\nu} E^\alpha_a (D_\mu E^\nu_b) n^{j\nu}$

On tracing over worldsheet indices we get

\begin{equation}
\tilde{\nabla}_a J_{aij} = -J_{ak}^i J_{kj}^a - K_{ai}^j K_{aj}^i - g(R(E_a, n^i) E_a, n^j)
\end{equation}

where we have used the equation for extremal membranes (i.e. $K^i = 0$)

The antisymmetric part of (6) is given as:

\begin{equation}
\tilde{\nabla}_b J_{aij} - \tilde{\nabla}_a J_{bij} = G_{ab}^i
\end{equation}

where $g(R(Y_1, Y_2) Y_3, Y_4) = R_{(\alpha \beta \mu \nu} Y_{\gamma}^\alpha Y_{\beta}^\beta Y_{\alpha}^\mu Y_{\nu}^\gamma$ and

\begin{equation}
G_{ab}^i = -J_{bk}^i J_{kj}^b - K_{bi}^j K_{aj}^i - g(R(E_b, n^i) E_a, n^j) - (a \rightarrow b)
\end{equation}

One can further split $J_{aij}$ into its symmetric traceless, trace and antisymmetric parts ($J_{aij}^i = \Sigma_{aij}^i + \Lambda_{aij}^i + \sum_a \delta_{i}^a \theta_{aij}$) and obtain the evolution equations for each of these quantities. The one we shall be concerned with mostly is given as

\begin{equation}
\Delta \gamma + \frac{1}{2} \partial_a \gamma \partial^a \gamma + (M^2)_{aij} = 0
\end{equation}

with the quantity $(M^2)_{aij}$ is given as:

\begin{equation}
(M^2)_{aij} = K_{ab}^i K_{aj}^b + R_{\mu\nu\rho\sigma} E_{a}^{\mu} n^{\nu} E_{b}^{\rho} n^{\sigma}
\end{equation}

$\nabla_a$ is the worldsheet covariant derivative ($\Delta = \nabla^a \nabla_a$) and $\partial_a \gamma = \theta_a$. Notice that we have set $\Sigma_{aij}^i$ and $\Lambda_{aij}^i$ equal to zero. This is possible only if the symmetric traceless part of $(M^2)_{aij}$ is zero. One can check this by looking at the full set of generalized Raychaudhuri equations involving $\Sigma_{aij}^i$, $\Lambda_{aij}^i$ and $\theta_a$ [4]. For geodesic curves the usual Raychaudhuri equations can be obtained by noting that $K_{b0}^i = 0$, the $J_{aij}$ are related to their spacetime counterparts $J_{\mu\nu\alpha}$ through the equation $J_{\mu\nu\alpha} = n_{\mu}^a n_{\nu}^b J_{aij}$, and the $\theta$ is defined by contracting with the projection tensor $h_{\mu\nu}$.

The $\theta_a$ or $\gamma$ basically tell us how the spacetime basis vectors change along the normal directions as we move along the surface. If $\theta_a$ diverges somewhere, it induces a divergence in $J_{aij}$, which, in turn means that the gradients of the spacetime basis along the normals have a discontinuity. Thus the family of worldsheets meet along a curve and a cusp/kink is formed. This, we claim, is a focussing effect for extremal surfaces analogous to geodesic focussing in GR where families of geodesics focus at apoint if certain specific conditions on the matter stress energy are obeyed.

We now move on to the discussion of the special cases.
III. STRING WORLDSHEETS

Two dimensional timelike surfaces embedded in a four dimensional background are the objects of discussion in this section. We begin by writing down the generalised Raychaudhuri equation for the case in which $\Sigma_{ij}^\mu$ and $A_{ij}^\mu$ are set to zero (i.e. implicitly assuming that $(M^2)^{ij}$ does not have a nonzero symmetric traceless part). Thus we have

$$-\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma^2} + \Omega^2(\sigma, \tau)(M^2)_{ij}^2(\sigma, \tau)F = 0$$  \hspace{1cm} (12)

where $\Omega^2$ is the conformal factor of the induced metric written in isothermal coordinates. Notice that the above equation is a second–order, linear, hyperbolic partial differential equation. On the contrary, the Raychaudhuri equation for curves is a linear, second order, ordinary differential equation. The easiest way to analyse the solutions of this equation with respect to focussing is to assume separability of the quantity $\Omega^2(M^2)$.

Thus, focussing will entirely depend on the sign of the quantity $\Omega^2(M^2)$. Then, we have

$$\Omega^2(M^2)_{ij}^2 = M_1^2(\tau) + M_2^2(\sigma)$$ \hspace{1cm} (13)

$$F(\tau, \sigma) = F_1(\tau) \times F_2(\sigma)$$ \hspace{1cm} (14)

With these we can now split the partial differential equation into two ordinary differential equations given by

$$\frac{d^2 F_1}{d\tau^2} + (\omega^2 - M_2^2(\tau))F_1 = 0 \hspace{1cm} (15)$$

$$\frac{d^2 F_2}{d\sigma^2} + (\omega^2 + M_2^2(\sigma))F_2 = 0 \hspace{1cm} (16)$$

Since the expansions along the $\tau$ and $\sigma$ directions can be written as $\theta_{\tau} = \frac{\dot{F}_1}{F_1}$ and $\theta_\sigma = \frac{\dot{F}_2}{F_2}$ we can analyse focussing effects by locating the zeros of $F_1$ and $F_2$ much in the same way as for geodesic curves [5]. The well–known theorems on the existence of zeros of ordinary differential equations as discussed in [5] make our job much simpler. The theorems essentially state that the solutions of equations of the type $\frac{d^2 \lambda}{d\tau^2} + H(\tau)\lambda = 0$ will have at least one zero if $H(\tau)$ is positive definite. Thus for our case here, we can conclude that, focussing along the $\tau$ and $\sigma$ directions will take place only if

$$\omega^2 \geq max[M_1^2(\tau)] \hspace{1cm} ; \hspace{1cm} \omega^2 \geq max[-M_2^2(\sigma)]$$ \hspace{1cm} (17)

For stationary strings, one notes that $(M^2)^{ij}$ will not have any dependence on $\tau$. Thus we can set $M_1^2 = 0$ equal to zero. Thus, focussing will entirely depend on the sign of the quantity $M_2^2$. We can write $M_2^2$ alternatively as follows. Consider the Gauss–Codazzi integrability condition:

$$R_{\mu \nu \alpha \beta}E^\mu_aE^\nu_bE^\alpha_cE^\beta_d = R_{abcd} - K_{aci}K^{aci} + K_{bdi}K^{bdi}$$ \hspace{1cm} (18)

Trace the above expression on both sides with $\eta^{ac} \eta^{bd}$ and rearrange terms to obtain:

$$K_{abi}K^{abi} = -2R + K^{ai}K_i + R_{\mu \nu \alpha \beta}E^\mu_aE^\alpha_cE^\nu_bE^\beta_d$$ \hspace{1cm} (19)

Thereafter, use this expression and the fact that $n^\mu n^\nu = g^{\mu \nu} - E^\mu_aE^\nu_a$ in the original expression for $(M^2)_{ij}$ (see Eqn.(11)) to get (for extremal configurations with $K^i = 0$)

$$M_2^2 = -2R + R_{\mu \nu}E^\mu_aE^\nu_a$$ \hspace{1cm} (20)

One can notice the following features from the above expression:

(i) If the background spacetime is a vacuum solution of the Einstein equations then the positivity of $M_2^2$ is guaranteed iff $2R \leq 0$. Thus all string configurations in vacuum spacetimes which have negative Ricci curvature everywhere will necessarily imply focussing. This includes the well known string solutions in Schwarzschild and Kerr backgrounds.

(ii) If the background spacetime is a solution of the Einstein equations then we can replace the second and third terms in the expressions for $M_2^2$ by the corresponding ones involving the Energy Momentum tensor $T_{\mu \nu}$ and its trace. Thus we have

$$M_2^2 = \left(-\frac{1}{2}g^{\mu \nu}2R + T_{\mu \nu} - \frac{1}{2}Tg_{\mu \nu}\right)E^\mu_aE^\nu_a$$ \hspace{1cm} (21)

Notice that if we split the quantity $E^\mu_aE^\nu_a$ into two terms such as $E^\mu_aE^{\nu \gamma}$ and $E^\mu_aE^{\nu \sigma}$ then we have:
\[ M_2^2 = -2 R + \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) E^\mu E^\nu + \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) E^\mu E^\nu \]  

(22)

The second term in the above equation is the L. H. S. of the Strong Energy Condition (SEC). Apart from this we have two other terms which are entirely dependent on the fact that we are dealing with extended objects. The positivity of the whole quantity can therefore be thought of as an Energy Condition for the case of strings. Thus even if the background spacetime satisfies the SEC, focussing of string world-sheets is not guaranteed–worldsheet curvature and the projection of the combination \( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \) along the \( \sigma \) direction have an important role to play in deciding focussing/defocussing.

Let us now try to understand the consequences of the above equations for certain specific flat and curved backgrounds for which the string solutions are known.

**A. Rindler Spacetime**

The metric for four dimensional Rindler spacetime is given as

\[ ds^2 = -a^2 x^2 dt^2 + dx^2 + dy^2 + dz^2 \]  

(23)

We recall from [6] the a string solution in a Rindler spacetime:

\[ t = \tau ; \quad x = ba \cosh a\sigma_c ; \quad y = ba^2 \sigma_c ; \quad z = z_0 \quad (constant) \]  

(24)

where \( d\sigma_c = \frac{d\sigma}{a \tau} \) and \( b \) is an integration constant. The orthonormal set of tangents and normals to the worldsheet can be chosen to be as follows:

\[ E^\mu_\tau \equiv \left( \frac{1}{a x}, 0, 0, 0 \right) \quad ; \quad E^\mu_\sigma \equiv (0, \tanh a\sigma_c, \sech a\sigma_c, 0) \]  

(25)

\[ n^\mu_1 \equiv (0, 0, 0, 1) \quad ; \quad n^\mu_2 \equiv (0, \sech a\sigma_c, -\tanh a\sigma_c, 0) \]  

(26)

In the worldsheet coordinates \( \tau, \sigma_c \) the induced metric is flat and the components of the extrinsic curvature tensor turn out to be

\[ K^1_{\ ab} = 0 ; \quad K^2_{\tau\tau} = -K^2_{\sigma_c\sigma_c} = \frac{1}{ba \cosh^2 a\sigma_c} ; \quad K^2_{\sigma\tau} = 0 \]  

(27)

The quantity \( (M^2)_i^i \) which is dependent only on the extrinsic curvature of the worldsheet (the background spacetime being flat) turns out to be

\[ (M^2)_i^i = \frac{2}{b^2 a^2 \cosh^4 a\sigma_c} \]  

(28)

Therefore the generalized Raychaudhuri equation turns out to be

\[ -\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma_c^2} + \frac{2a^2}{\cosh^2 a\sigma_c} F = 0 \]  

(29)

Separating variables \( F = T(\tau)\Sigma(\sigma) \) one gets the harmonic oscillator equation for \( T \) and the Poschl Teller equation for positive eigenvalues [7] for \( \Sigma \) which is given as:

\[ \frac{d^2 \Sigma}{d\sigma^2} + \left( \omega^2 + \frac{2a^2}{\cosh^2 a\sigma_c} \right) \Sigma = 0 \]  

(30)

From the results of Tipler [5] on the zeros of differential equations one can conclude that focussing will occur \( (H(\sigma) > 0 \text{ always}) \)
B. De Sitter Spacetime

The metric for De Sitter spacetime is given as:

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  

(31)

with \( f(r) = 1 - H^2 r^2 \).

Since De Sitter spacetime is an Einstein space we have a clear advantage. The Riemann, Ricci tensors can be written as

\[ R_{\mu\nu\alpha\beta} = H^2 (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) \quad ; \quad R_{\mu\nu} = 3H^2 g_{\mu\nu} \]  

(32)

Therefore a little bit of calculation will reveal that the Raychaudhuri equation for all string configurations in DeSitter space can be written as:

\[ -\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma^2} + \Omega^2 (-2R + 6H^2) F = 0 \]  

(33)

Focussing in this case thus depends only on the Ricci scalar and the conformal factor \( \Omega^2 \) of the worldsheet metric. One can write down the solutions of the above equation for the various string configurations in De Sitter space mentioned in [6]. This is simple enough and we shall refrain from writing them out explicitly.

Some other examples have been constructed in [8]. These involve the case of a 3 + 1 dimensional Lorentzian wormhole geometry as the background.

IV. HYPERSURFACES

We now move on to the special case of timelike hypersurfaces. Here we have \( D \) quantities \( J_a \) but only one normal defined at each point on the surface. The Eqn. (8) turns out to be:

\[ \partial_b J_a - \partial_a J_b = 0 \]  

(34)

Therefore one can write \( J_a = \partial_a \gamma \) and the traced equation (7) becomes,

\[ \Delta \gamma + (\partial_a \gamma)(\partial^a \gamma) + M^2 = 0 \]  

(35)

with

\[ M^2 = K_{ab}K^{ab} + R_{\sigma\alpha}n^\sigma n^\alpha \]

\[ = -2R + R_{\mu\nu}E^{\mu a}E^{\nu a} = -2R + 3R - R_{\mu\nu}n^\mu n^\nu \]  

(36)

where we have used \( n^\mu n^\nu = g^{\mu\nu} - E_a^{\mu}E^a_{\nu} \) and the Gauss–Codazzi integrability condition.

If we assume that the background spacetime satisfies the Einstein equation then we have:

\[ M^2 = -\left( 2R_{\mu\nu} + T_{\mu\nu} + T g_{\mu\nu} \right) n^\mu n^\nu \]  

(37)

Thus, for stationary two dimensional hypersurfaces (strings in 3D backgrounds) we have the same conclusions as obtained in the previous section. For a two–dimensional hypersurface in three–dimensional flat background the task is even simpler. \( M^2 \) can be shown to be equal to the negative of the Ricci scalar of the membrane’s induced metric and \( 2R \leq 0 \) guarantees focussing. We will discuss in detail the case of a 2D hypersurface(catenoid) embedded in a 3D flat, Euclidean background in the Appendix to this paper.

Let us now turn to a specific case where the equations are exactly solvable.
A. Hypersurfaces in a 2 + 1 Curved Background

Our background spacetime here is curved, Lorentzian background and 2 + 1 dimensional. The metric we choose is that of a Lorentzian wormhole in 2 + 1 dimensions given as:

\[ ds^2 = -dt^2 + dl^2 + (l_0^2 + l^2) \, d\theta^2 \] (38)

A string configuration in this background can be easily found by solving the geodesic equations in the 2D spacelike hypersurface [8]. This turns out to be

\[ t = \tau ; \quad l = \sigma ; \quad \theta = \theta_0 \] (39)

The tangents and normal vectors are simple enough:

\[ E^\mu_\tau \equiv (1, 0, 0) ; \quad E^\mu_\sigma \equiv (0, 1, 0) ; \quad n^\mu = \left(0, 0, 1, \frac{1}{l_0^2 + l^2}\right) \] (40)

The extrinsic curvature tensor components are all zero as the induced metric is flat. Using the Riemann tensor components (which can be evaluated simply using the standard formula) we can write down the generalised Raychaudhuri equation. This turns out to be:

\[ -\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma^2} + \left(-\frac{l_0^2}{(l_0^2 + \sigma^2)^2}\right) F = 0 \] (41)

A separation of variables \( F = T(\tau)\Sigma(\sigma) \) will result in two equations—one of which is the usual Harmonic Oscillator and the other given by:

\[ \frac{d^2 \Sigma}{d\sigma^2} + \left(\omega^2 - \frac{l_0^2}{(l_0^2 + \sigma^2)^2}\right) \Sigma = 0 \] (42)

The above equation can be recast into the one for Radial Oblate Spheroidal Functions by a simple change of variables \( \Sigma' = \sqrt{l_0^2 + \sigma^2}\Sigma \).

\[ (1 + \xi^2) \frac{d^2 \Sigma'}{d\xi^2} + 2\xi \frac{d \Sigma'}{d\xi} + \left(\omega^2 l_0^2(1 + \xi^2)\right) \Sigma' = 0 \] (43)

where \( \xi = \frac{\sigma}{l_0} \).

The general equation for Radial Oblate Spheroidal Functions is given as:

\[ (1 + \xi^2) \frac{d^2 V_{mn}}{d\xi^2} + 2\xi \frac{d V_{mn}}{d\xi} + \left(-\lambda_{mn} + k^2 \xi^2 - \frac{m^2}{1 + \xi^2}\right) V_{mn} = 0 \] (44)

Assuming \( m = 0 \) and \( \lambda_{mn} = -k^2 = -\omega^2 l_0^2 \) we get the equation for our case. The solutions are finite at infinity and behave like simple sine/cosine waves in the variable \( \sigma \). Consulting the tables in [9] we conclude that only for \( n = 0, 1 \) we can have \( \lambda_{mn} \) to be negative. In general, the scattering problem for the Schroedinger–like equation has been analysed numerically in [10].

As regards focussing, one can say from the differential equations and the theorems stated in [5] that the function \( \Sigma' \) will always have zeros if \( \omega^2 \geq \frac{1}{l_0} \). Even from the series representations (see [9]) of the Radial Oblate Spheroidal Functions we can exactly locate the zeros and obtain explicitly the focal curves. However, we shall not attempt such a task here.

V. CONCLUSIONS

The basic aim of this paper has been to obtain explicit examples of the generalised Raychaudhuri equations derived in [4]. To this end we have discussed two specific cases—that of string worldsheets and hypersurfaces. In the latter case the full set of equations simplify considerably. We have solved them for the case of an extremal 2D timelike membrane in a curved 2 + 1 dimensional Lorentzian wormhole background.
In the case of strings we have been able to reduce the generalised Raychaudhuri equation to a form reminiscent of the one for geodesic congruences by assuming that the background spacetime satisfies the Einstein equations with a specific form of matter. Any assumption of an Energy Condition for the matter generating the background spacetime does not seem to lead to focussing effects. The presence of the worldsheet Ricci scalar and the spacetime Ricci tensor together control focussing effects. For backgrounds obeying $R_{\mu\nu} = 0$ it is necessary to have worldsheets of negative worldsheet curvature in order to ensure focussing for stationary strings. If the backgrounds obey the Einstein field equations with a specific energy–momentum tensor then we have a specific condition which ensures focussing.

However, it is not completely clear what role the extrinsic curvature term in $(M^2)^i_j$ term plays as regards worldsheet focussing in the general case (i.e. with $\Omega^2(M^2)^i_j$ not separable). This requires a more extensive analysis of the general features of the partial differential equation. Moreover, if one wishes to conclude about the existence/nonexistence of spacetime singularities one has to frame the notions of worldsheet completeness in analogy with geodesic completeness and relate the idea of worldsheet incompleteness with the presence of a singularity in the background spacetime.

This, indeed is a fairly difficult problem (in fact it doesn’t even have a clear formulation). However, a solution of such a problem would actually tell us whether a string description as an alternative to the point particle can actually lead to the presence/absence of spacetime singularities at the classical level. From the fact that focussing is present in almost all the cases under discussion here we cannot conclusively say anything about spacetime singularities. The only progress we have been able to make in this paper is to introduce the idea and the conditions under which focussing can take place for families of string worldsheets or hypersurfaces.

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APPENDIX A: A PEDAGOGICAL EXAMPLE – THE CATENOIDAL MEMBRANE

This appendix contains a pedagogical example of the generalised Raychaudhuri equations which may help the reader in understanding the equations better. Moreover, there do exist a whole class of physically relevant systems which can be modelled using the theory of 2D surfaces embedded in a Euclidean background [11]. It is hoped that the example below may serve to be useful in that context also.

The background metric here is flat and Euclidean. It is given as:

$$ds^2 = dx^2 + dy^2 + dz^2$$  \hspace{1cm} (A1)

The embedding of a two dimensional surface in this three dimensional background is specified by three functions $x(u, v), y(u, v)$ and $z(u, v)$ where $u$ and $v$ are the coordinates on the surface (worldsheet).

For the catenoid we have:

$$x(u, v) = b_0 \sinh u \cos v; \quad y(u, v) = b_0 \sinh u \sin v; \quad z(u, v) = b_0 v$$  \hspace{1cm} (A2)

The induced metric on the surface $\gamma_{ab}$ has only nonzero diagonal elements:

$$\gamma_{uu} = \gamma_{vv} = b_0^2 \cosh^2 u; \quad \gamma_{uv} = \gamma_{vu} = 0$$  \hspace{1cm} (A3)

We now choose an orthonormal basis for which $g_{\mu\nu} E^\mu_a E^\nu_b = \delta_{ab}$ which is convenient for calculating the extrinsic curvature The tangent vectors and the normal are given as:

$$E^\mu_u \equiv (\cos v, \sin v, 0)$$

$$\equiv \left( \frac{x}{r}, \frac{y}{r}, 0 \right)$$  \hspace{1cm} (A4)

$$E^\mu_v \equiv (-\tanh u \sin v, \tanh u \cos v, \sech u)$$

$$\equiv \left( -\frac{y}{r_1}, \frac{x}{r_1}, \frac{b_0}{r_1} \right)$$  \hspace{1cm} (A5)

$$n^\mu \equiv \frac{\sin v}{\cosh u} (\frac{\cos v}{\cosh u}, -\tanh u)$$

$$\equiv \left( -\frac{yb_0}{rr_1}, \frac{xb_0}{rr_1}, \frac{r}{r_1} \right)$$  \hspace{1cm} (A6)
where \( r = \sqrt{x^2 + y^2} \) and \( r_1 = \sqrt{x^2 + y^2 + b_0^2} \).

The extrinsic curvature tensor components are defined as \( K_{ab} = -g_{\mu\nu}E_a^\mu(D_\nu E_b^\nu)n^\nu \). For the embedding under consideration here one gets:

\[
K_{uu} = K_{vv} = 0 ; \quad K_{uv} = K_{vu} = -\frac{1}{b_0 \cosh^2 u}
\]

With the above expressions we can now straightaway write down the Raychaudhuri equations. In this case we have only one of these. Using \( \gamma = \ln F \) and

\[
\Delta \equiv \frac{1}{b_0 \cosh^2 u} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right)
\]

we get:

\[
\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} + \frac{2}{\cosh^2 u} F = 0
\]

We can now separate variables using \( F(u, v) = U(u)V(v) \) and obtain two independent equations for \( U(u) \) and \( V(v) \). These are:

\[
\frac{d^2 V}{dv^2} + \omega^2 V = 0
\]

\[
\frac{d^2 U}{du^2} + \left( -\omega^2 + \frac{2}{\cosh^2 u} \right) U = 0
\]

The second of these is the well–known Poschl–Teller equation which occurs in quantum mechanics [7]. We now write down the solutions to these equations and obtain the expressions for the expansions \( J_u \) and \( J_v \). These expansions will tell us about the nature of the deformations of the catenoidal membrane.

The solutions to the equation for \( V(v) \) (for \( \omega \neq 0 \)) are simple:

\[
V(v) = \sin \omega v \quad \text{or} \quad \cos \omega v
\]

Now since \( v \) is an angle coordinate we must have \( V(v + 2\pi) = V(v) \) which implies \( \omega n = n \) where \( n \) is integral \((n \neq 0)\). For \( n = 0 \) one can easily check that \( V(v) = \text{constant} \) is the only possible solution (this is once again because \( v \) is an angle coordinate). We shall now have to use this input in the equation for \( U \) in order to obtain the relevant solutions.

Before we do that let us evaluate \( J_v \). From the relation \( F = UV \) and \( \gamma = \ln F \) we get \( \gamma = \ln U + \ln V \). Thus we have (from \( J_a = \partial_a \gamma \)):

\[
J_u = \frac{U'}{U} ; \quad J_v = \frac{V'}{V}
\]

Thus depending on which of the solutions \( V(v) \) we choose, we get a separate expression for \( J_v \).

\[
V(v) = \sin nv \quad ; \quad J_v = n \cot nv
\]

\[
V(v) = \cos nv \quad ; \quad J_v = -n \tan nv
\]

Thus focussing in the angular direction can occur at \( v = \pi, 2\pi(0) \) (sin solution) or \( v = \frac{\pi}{2}, \frac{3\pi}{2} \) (cos solution). At these points the families of \( u = \text{constant} \) closed curves meet.

Now we move over to the solutions of the \( U \) equation which are of course much more nontrivial. First let us rewrite the \( U \) equation in a different form by introducing the variable \( J_a = \frac{U'}{U} \). This results in a first order equations of the Riccati type and is useful in discussing focussing.

\[
\frac{dJ_u}{du} + J_u^2 = -\left( -n^2 + \frac{2}{\cosh^2 u} \right)
\]
This equation is similar to the Raychaudhuri equation in GR. Thus, if the R.H.S is negative then we can get focussing. For \( n = 0 \) there seems to be no problem with focussing whereas for \( n = 1 \) focussing is possible only within a finite region of \( u \) (as the R.H.S. is negative only in that domain. For all \( n \geq 2 \) one does not get any focussing (the R.H.S is positive for all \( u \)).

The solutions to the Poschl–Teller equations can be obtained in terms of Hypergeometric Functions \([7]\). Since, in this case we are interested in the Poschl–Teller equation only as a differential equation and not as a potential problem in quantum mechanics we shall be concerned with solutions which are finite everywhere as well as those which diverge at specific values of the independent variable.

We now list the various solutions for different \( n \) and the corresponding \( J_u \).

\( n = 0 \) : First Solution

\[
U_1(u) = F[-1, 2, 1, \frac{1}{2}(1 - \tanh u)] = P_1(\tanh u) = \tanh u
\]  

\[
J_u^{(1)} = \frac{1}{\sinh u \cosh u}
\]  

\( n = 0 \) : Second Solution

\[
U_2(u) = u \tanh u - 1
\]  

\[
J_u^{(2)} = \frac{2}{\sinh 2u} + \frac{\coth^2 u}{u - \coth u}
\]

\( n = 1 \) : First Solution

\[
U_1(u) = \text{sechu} F[0, 3, 2, \tanh u] = \text{sechu}
\]  

\[
J_u^{(1)} = -\tanh u
\]

\( n = 1 \) : Second Solution

\[
U_2(u) = \frac{u + \sinh u \cosh u}{\cosh u}
\]  

\[
J_u^{(2)} = -\tanh u + \frac{1 + \cosh 2u}{u + \sinh u \cosh u}
\]

It is worthwhile to point out that the second solutions have been obtained in both cases by solving the nonlinear first order equation involving \( J_u \) (Riccati equation). We have used the well known fact that if one solution of a Riccati equation is known, one can derive a second solution by writing it as a sum of the known one and an unknown function. (The differential equation in the unknown function reduces to a Bernoulli equation). Thus obtaining \( J_u \) we integrate to get \( U(u) \).

What do the \( J_u \) obtained above imply? For \( n = 0 \) both the solutions have the property that at \( u = 0 \) \( J_u \rightarrow -\infty \) from below (i.e. \( u \) negative) and \( J_u \rightarrow \infty \) from above. Thus \( u = 0 \) is a focal curve (a circle in this case). Additionally, the second solution has the intriguing feature that at two symmetrically placed points (which are solutions to the transcendental equation \( u = \coth u \)), \( J_u \) has exactly similar behaviour. These are focal curves too. In the \( n = 1 \) case however one of the solutions (the first one) does not lead to any focussing at all– we get an almost parallel family of surfaces. The other solution for \( n = 1 \) indicates focussing only at \( u = 0 \) and nowhere else. If one is inclined to consider a membrane of finite extent (such as a soap film formed between coaxial rings placed a certain distance apart \([12]\)) then one needs to have \( J_u \) diverging at a finite value of \( u \). This seems possible only for the second solution for \( n = 0 \).

Another extremal two–dimensional surface (embedded in three dimensional Euclidean space) is the helicoid. Interestingly there exists a local isometry of the helicoid into the catenoid. If \( u_1 \) and \( v_1 \) are the coordinates on the helicoid this is given as :

\[
v_1 = v \quad u_1 = \sinh u
\]
Thus, one can essentially use the same pair of differential equations for $u$ and $v$ given above for the catenoid. However there is one striking difference. For the helicoid $v_1$ is no longer an angle variable (the ranges of $u_1$ and $v_1$ are $0 < u_1 < \infty$ and $-\infty < v_1 < \infty$. Therefore, we do not have a restriction on the allowed values on $\omega^2$ arising solely from the equation for $v_1$. The equation for the $u_1$ variable however yields a restriction $\omega^2 \leq \frac{3}{2}$ if one is interested in focusing effects.

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