QUANTUM DYNAMICS OF ELLIPTIC CURVES

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Abstract. We calculate the $K$-theory of a crossed product $C^*$-algebra $\mathcal{A}_{RM} \rtimes \mathcal{E}(K)$, where $\mathcal{A}_{RM}$ is the noncommutative torus with real multiplication and $\mathcal{E}(K)$ is an elliptic curve over the number field $K$. We use this result to evaluate the rank and the Shafarevich-Tate group of $\mathcal{E}(K)$.

1. Introduction

The noncommutative torus $\mathcal{A}_\theta$ is a $C^*$-algebra on the generators $u$ and $v$ satisfying relation $vu = e^{2\pi i \theta} uv$ for a real constant $\theta$. The algebra $\mathcal{A}_\theta$ is said to have real multiplication (RM), if $\theta$ is an irrational quadratic number. We shall denote such an algebra by $\mathcal{A}_{RM}$.

Let $K$ be a number field and let $\mathcal{E}(K)$ be an elliptic curve over $K$. Here we consider a functor $F$ between elliptic curves $\mathcal{E}(K)$ and the $C^*$-algebras $\mathcal{A}_{RM}$, see [10, Section 1.3] for the details. Such a functor maps $K$-isomorphic elliptic curves $\mathcal{E}(K)$ and $\mathcal{E}'(K)$ to isomorphic $C^*$-algebras $\mathcal{A}_{RM}$ and $\mathcal{A}'_{RM}$, respectively. It is useful to think of the $\mathcal{A}_{RM}$ as a non-commutative analog of the coordinate ring of $\mathcal{E}(K)$.

Recall that $\mathcal{E}(K)$ is an algebraic group over $K$; such a group is compact and thus abelian. The Mordell-Weil Theorem says that $\mathcal{E}(K) \cong \mathbb{Z}^r \oplus \mathcal{E}_{tors}(K)$, where $r = rk \mathcal{E}(K) \geq 0$ is the rank of $\mathcal{E}(K)$ and $\mathcal{E}_{tors}(K)$ is a finite abelian group. The group operation $\mathcal{E}(K) \times \mathcal{E}(K) \to \mathcal{E}(K)$ defines an action of the group $\mathcal{E}(K)$ by the $K$-automorphisms of $\mathcal{E}(K)$. Recall that each $K$-automorphism of $\mathcal{E}(K)$ gives rise to an automorphism of $\mathcal{A}_{RM}$. Thus one gets an action of $\mathcal{E}(K)$ on $\mathcal{A}_{RM}$ by automorphisms of the algebra $\mathcal{A}_{RM}$. The object of our study is a crossed product $C^*$-algebra coming from such an action, i.e. the $C^*$-algebra:

$$\mathcal{A}_{RM} \rtimes \mathcal{E}(K).$$ (1.1)

The crossed product (1.1) is often identified with a dynamical system on the noncommutative topological space $\mathcal{A}_{RM}$. In other words, we deal with a "quantum" dynamical system; hence the title of our note. We refer the reader to [Pedersen 1979] [12] and [Blackadar 1986] [1, Chapter V] for an excellent introduction to the quantum dynamics.

The aim of our note is the $K$-theory of the crossed product (1.1); we refer the reader to theorem 1.1. We apply this result to evaluate the rank and the Shafarevich-Tate group of elliptic curve $\mathcal{E}(K)$; see corollaries 1.2 and 1.3, respectively. To formalize our results, let us recall the following facts.

2010 Mathematics Subject Classification. Primary 11G05; Secondary 46L85.
Key words and phrases. elliptic curves, Shafarevich-Tate group, real multiplication.
Denote by \( \tau \) the canonical tracial state on the crossed product \( \mathcal{A}_R \times \mathcal{E}(K) \); existence of \( \tau \) follows from [Phillips 2005] [13, Theorem 3.4]. It is well known, that \( K_0(\mathcal{A}_R) \cong \mathbb{Z}^2 \). Moreover, \( \tau \) defines an embedding \( K_0(\mathcal{A}_R) \hookrightarrow \mathbb{R} \) given by the formula \( \tau(K_0(\mathcal{A}_R)) = \mathbb{Z} + \theta \mathbb{Z} \subset \mathbb{R} \) [Blackadar 1986] [1, Exercise 10.11.6]. Following Yu. I. Manin, we shall call \( \mathbb{Z} + \theta \mathbb{Z} \) a “pseudo-lattice”. By \( \Lambda \) we understand the ring of endomorphisms of the pseudo-lattice \( \tau(K_0(\mathcal{A}_R)) \). Since \( \theta \) is an irrational quadratic number, the ring \( \Lambda \) is an order in the real quadratic field \( k = \mathbb{Q}(\theta) \). Thus \( \Lambda \cong \mathbb{Z} + f \mathcal{O}_k \), where \( \mathcal{O}_k \) is the ring of integers of \( k \) and \( f \geq 1 \) is a conductor of the order. We shall write \( Cl (\Lambda) \) to denote the class group of the ring \( \Lambda \). By \( h_{\Lambda} = |Cl (\Lambda)| \) we understand the class number of \( \Lambda \). Denote by \( K_{ab} \) the maximal abelian extension of the field \( k \) modulo conductor \( f \). If \( f = 1 \), then the extension \( K_{ab} \) is unramified, i.e. the Hilbert class field of \( k \). It is known, that \( Gal (K_{ab}|k) \cong Cl (\Lambda) \), where \( Gal (K_{ab}|k) \) is the Galois group of the extension \( k \subseteq K_{ab} \). Let \( \{ \alpha_i \mid 1 \leq i \leq h_{\Lambda} \} \) be generators of the field \( K_{ab} \), such that \( \alpha_i \) are conjugate algebraic numbers. Consider a normalization of \( \alpha_i \) given by the formula \( \lambda_i = \alpha_i a_{h_\Lambda}^{-1} \mid 1 \leq i \leq h_{\Lambda} - 1 \}. Our main result can be formulated as follows.

**Theorem 1.1.** The \( K \)-theory of the crossed product \( C^*-\text{algebra} \) (1.1) is described by the following formulas:

\[
\begin{align*}
K_0(\mathcal{A}_R \times \mathcal{E}(K)) & \cong \mathbb{Z}^{h_{\Lambda}+1}, \\
\tau(K_0(\mathcal{A}_R \times \mathcal{E}(K))) & = \mathbb{Z} + \theta \mathbb{Z} + \lambda_1 \mathbb{Z} + \cdots + \lambda_{h_{\Lambda}-1} \mathbb{Z}.
\end{align*}
\]

Denote by \( rk \mathcal{E}(K) \) the rank of elliptic curve \( \mathcal{E}(K) \). By \( III(\mathcal{E}(K)) \) we understand the Shafarevich-Tate group of \( \mathcal{E}(K) \). Theorem 1.1 implies the following formulas.

**Corollary 1.2.** \( rk \mathcal{E}(K) = h_{\Lambda} - 1 \).

**Corollary 1.3.** \( III(\mathcal{E}(K)) \cong Cl (\Lambda) \oplus Cl (\Lambda) \).

**Remark 1.4.** For the sake of simplicity, we treat the case of elliptic curves only. However, the results of 1.1 – 1.3 can be extended to any abelian variety over a number field \( K \).

**Remark 1.5.** It follows from 1.2 and 1.3, that

\[
|III(\mathcal{E}(K))| = (1 + rk \mathcal{E}(K))^2. \tag{1.3}
\]

It is hard to verify (1.3) directly, since the group \( III(\mathcal{E}(K)) \) is unknown for a single \( \mathcal{E}(K) \) [Tate 1974] [16, p.193]. Indirectly, one can predict “analytic” values of \( rk \mathcal{E}(K) \) and \( |III(\mathcal{E}(K))| \) assuming the BSD Conjecture [Swinnerton-Dyer 1967] [15]. While many of such values satisfy (1.3), the other are not [Cremona et al. 2017] [4]. We do not know an exact relation between the analytic values and those described by formula (1.3).

The article is organized as follows. The preliminary facts are introduced in Section 2. The proof of theorem 1.1, corollary 1.2 and 1.3 can be found in Section 3.

2. Preliminaries

We briefly review elementary facts about \( C^*-\text{algebras} \), class field theory for quadratic fields and the Shafarevich-Tate groups. We refer the reader to [Pedersen 1979] [12], [Neukirch 1999] [9] and [Silverman 1985] [14] for a detailed account.
2.1. $C^*$-dynamical systems.

2.1.1. $C^*$-algebras. A $C^*$-algebra $\mathcal{A}$ is an algebra over $\mathbb{C}$ with a norm $a \mapsto ||a||$ and an involution $a \mapsto a^*$ such that it is complete with respect to the norm and $||ab|| \leq ||a|| \cdot ||b||$ and $||a^*a|| = ||a||^2$ for all $a, b \in \mathcal{A}$. Any commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$; otherwise, $\mathcal{A}$ can be thought of as a noncommutative topological space.

2.1.2. $K$-theory of $C^*$-algebras. For a unital $C^*$-algebra $\mathcal{A}$, let $V(\mathcal{A})$ be the union over $n$ of projections in the $n \times n$ matrix $C^*$-algebra with entries in $\mathcal{A}$; projections $p, q \in V(\mathcal{A})$ are equivalent if there exists a partial isometry $u$ such that $p = u^*u$ and $q = uu^*$. The equivalence class of projection $p$ is denoted by $[p]$; the equivalence classes of orthogonal projections can be made to a semigroup by putting $[p] + [q] = [p + q]$. The Grothendieck completion of this semigroup to an abelian group is called the $K_0$-group of the algebra $\mathcal{A}$. The functor $\mathcal{A} \to K_0(\mathcal{A})$ maps the category of unital $C^*$-algebras into the category of abelian groups, so that projections in the algebra $\mathcal{A}$ correspond to a positive cone $K_0^+ \subset K_0(\mathcal{A})$ and the unit element $1 \in \mathcal{A}$ corresponds to an order unit $u \in K_0(\mathcal{A})$. The ordered abelian group $(K_0, K_0^+, u)$ with an order unit is called a dimension group; an order-isomorphism class of the latter we denote by $(G, G^+)$. 

2.1.3. Crossed products. Let $\mathcal{A}$ be a $C^*$-algebra and $G$ a locally compact group. We shall consider a continuous homomorphism $\alpha$ from $G$ to the group $Aut \mathcal{A}$ of $*$-automorphisms of $\mathcal{A}$ endowed with the topology of pointwise norm-convergence. Roughly speaking, the idea of the crossed product construction is to embed $\mathcal{A}$ into a larger $C^*$-algebra in which the automorphism becomes the inner automorphism. A covariant representation of the triple $(\mathcal{A}, G, \alpha)$ is a pair of representations $(\pi, \rho)$ of $\mathcal{A}$ and $G$ on the same Hilbert space $\mathcal{H}$, such that $\rho(g)\pi(a)\rho(g)^* = \pi(\alpha_g(a))$ for all $a \in \mathcal{A}$ and $g \in G$. Each covariant representation of $(\mathcal{A}, G, \alpha)$ gives rise to a convolution algebra $C(G, \mathcal{A})$ of continuous functions from $G$ to $\mathcal{A}$; the completion of $C(G, \mathcal{A})$ in the norm topology is a $C^*$-algebra $\mathcal{A} \rtimes_\alpha G$ called a crossed product of $\mathcal{A}$ by $G$. If $\alpha$ is a single automorphism of $\mathcal{A}$, one gets an action of $\mathbb{Z}$ on $\mathcal{A}$; the crossed product in this case is called simply the crossed product of $\mathcal{A}$ by $\alpha$.

2.1.4. AF-algebras. An AF-algebra (Approximately Finite $C^*$-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional $C^*$-algebras $M_n$, where $M_n$ is the $C^*$-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \oplus \cdots \oplus M_{n_k}$. The ascending sequence mentioned above can be written as $M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots$, where $M_i$ are the finite dimensional $C^*$-algebras and $\varphi_i$ the homomorphisms between such algebras. If $\mathring{A}$ is an AF-algebra, then its dimension group $(K_0(\mathring{A}), K_0^+(\mathring{A}), u)$ is a complete isomorphism invariant of algebra $\mathring{A}$. The order-isomorphism class $(K_0(\mathring{A}), K_0^+(\mathring{A}))$ is an invariant of the Morita equivalence of algebra $\mathring{A}$, i.e. an isomorphism class in the category of finitely generated projective modules over $\mathring{A}$.

2.2. Abelian extensions of quadratic fields. Let $D$ be a square-free integer and let $k = \mathbb{Q}(\sqrt{D})$ be a quadratic number field, i.e. an extension of degree two of the field of rationals. Denote by $O_k$ the ring of integers of $k$ and by $\Lambda$ an order in
O_k, i.e. a subring of the ring \( O_k \) containing 1. The order \( \Lambda \) can be written in the form \( \Lambda = \mathbb{Z} + fO_k \), where the integer \( f \geq 1 \) is a conductor of \( \Lambda \). If \( f = 1 \), then \( \Lambda \cong O_k \) is the maximal order.

Denote by \( \text{Cl} (\Lambda) \) the ideal class group and by \( h_\Lambda = |\text{Cl} (\Lambda)| \) the class number of the ring \( \Lambda \). If \( \Lambda \cong O_k \), then \( h_\Lambda \) coincides with the class number \( h \) of the field \( k \). The integer \( h \leq h_\Lambda \) is always a divisor of \( h_\Lambda \) given by the formula:

\[
h_\Lambda = f e_f \prod_{p|f} \left(1 - \left(\frac{D}{p}\right) \frac{1}{p}\right),
\]

where \( e_f \) is the index of the group of units of \( \Lambda \) in the group of units of \( O_k \), \( p \) is a prime number and \( \left(\frac{D}{p}\right) \) is the Legendre symbol.

Let \( K_{ab} \) be the maximal abelian extension of the field \( k \) modulo conductor \( f \geq 1 \).

The class field theory says that

\[
\text{Gal} (K_{ab}|k) \cong \text{Cl} (\Lambda),
\]

where \( \text{Gal} (K_{ab}|k) \) is the Galois group of the extension \( (K_{ab}|k) \). The \( K_{ab} \) is the Hilbert class field (i.e. a maximal unramified abelian extension) of \( k \) if and only if \( f = 1 \).

For \( D < 0 \) an explicit construction of generators of the field \( K_{ab} \) is realized by elliptic curves with complex multiplication, see e.g. [Neukirch 1999] [9, Theorem 6.10]. For \( D > 0 \) an explicit construction of generators of the field \( K_{ab} \) is realized by noncommutative tori with real multiplication [10, Theorem 6.4.1].

2.3. Shafarevich-Tate group of elliptic curve. The Shafarevich-Tate group \( \text{III}(\mathcal{E}(K)) \) is a measure of failure of the Hasse principle for the elliptic curve \( \mathcal{E}(K) \). Recall that if \( \mathcal{E}(K) \) has a \( K \)-rational point, then it has also a \( K_v \)-point for every completion \( K_v \) of the number field \( K \). A converse of this statement is called the Hasse principle. In general, the Hasse principle fails for the elliptic curve \( \mathcal{E}(K) \).

Denote by \( H^1(K, \mathcal{E}) \) the first Galois cohomology group of \( \mathcal{E}(K) \) [Silverman 1985] [14, Appendix B]. There exists a natural homomorphism

\[
\omega : H^1(K, \mathcal{E}) \to \prod_v H^1(K_v, \mathcal{E}),
\]

where \( H^1(K_v, \mathcal{E}) \) is the first Galois cohomology over the field \( K_v \). The Shafarevich-Tate group of an elliptic curve \( \mathcal{E}(K) \) is

\[
\text{III}(\mathcal{E}(K)) := \text{Ker } \omega.
\]

The group \( \text{III}(\mathcal{E}(K)) \) is trivial if and only if elliptic curve \( \mathcal{E}(K) \) satisfies the Hasse principle.

Remark 2.1. The Shafarevich-Tate group \( \text{III}(A(K)) \) of an abelian variety \( A(K) \) over the number field \( K \) is defined similarly and has the same properties as \( \text{III}(\mathcal{E}(K)) \).

3. Proofs

3.1. Proof of theorem 1.1. For the sake of clarity, let us outline the main ideas. Our proof is based on a “rigidity principle” for extensions of the pseudo-lattice \( \mathbb{Z} + \theta \mathbb{Z} \) corresponding to the algebra \( \mathcal{A}_{RM} \). Such a rigidity follows from the class field theory for the real quadratic field \( k = \mathbb{Q}(\theta) \). Namely, the canonical embedding
\( \mathcal{A}_{RM} \hookrightarrow \mathcal{A}_{RM} \times \mathcal{E}(K) \) implies an inclusion \( K_0(\mathcal{A}_{RM}) \subseteq K_0(\mathcal{A}_{RM} \times \mathcal{E}(K)) \). Using the canonical tracial state \( \tau \) on \( \mathcal{A}_{RM} \times \mathcal{E}(K) \), one gets an inclusion:

\[
Z + \theta Z \subseteq \lambda_1 Z + \cdots + \lambda_m Z, \tag{3.1}
\]

where \( \lambda_i \) are generators of the pseudo-lattice \( \tau(K_0(\mathcal{A}_{RM} \times \mathcal{E}(K))) \). It is easy to see, that each \( \lambda_i \in \mathbb{R} \) is an integer algebraic number. But the crossed product (3.1) depends solely on the algebra \( \mathcal{A}_{RM} \), see formula (3.5). Therefore the extension (3.1) satisfies a “rigidity principle”. In other words, the arithmetic of the number field \( k(\lambda_i) \) must be controlled by the arithmetic of the field \( k \). It is well known, that this happens if and only if \( k(\lambda_i) \cong \mathcal{K}_{ab} \), where \( \mathcal{K}_{ab} \) is the maximal abelian extension of the field \( k \) modulo conductor \( f \geq 1 \). Thus \( m = h_\Lambda \), where \( h_\Lambda \) is the class number of the order \( \Lambda \subseteq O_K \); we refer the reader to (2.1) for an explicit formula. We pass to a detailed argument by splitting the proof in a series of lemmas and corollaries.

**Lemma 3.1.** The real numbers \( \lambda_i \) in formula (3.1) are algebraic integers.

**Proof.** Recall that the endomorphism ring \( \Lambda \) of the pseudo-lattice \( Z + \theta Z \) is an order \( Z + fO_k \) in the number field \( k \). In particular, since \( f \neq 0 \) we conclude that \( \Lambda \) is a non-trivial ring, i.e. \( \Lambda \not\subseteq Z \).

On the other hand, the inclusion (3.1) implies that:

\[
\Lambda \subseteq \text{End} (\lambda_1 Z + \cdots + \lambda_m Z), \tag{3.2}
\]

where \( \text{End} (\lambda_1 Z + \cdots + \lambda_m Z) \) is the endomorphism ring of the pseudo-lattice \( \lambda_1 Z + \cdots + \lambda_m Z \subset \mathbb{R} \). We conclude from (3.2) that the ring \( \text{End} (\lambda_1 Z + \cdots + \lambda_m Z) \) is non-trivial, i.e. bigger than the ring \( Z \).

Recall that the endomorphisms of pseudo-lattice \( \lambda_1 Z + \cdots + \lambda_m Z \subset \mathbb{R} \) coincide with multiplication by the real numbers. In other words, the ring \( \text{End} (\lambda_1 Z + \cdots + \lambda_m Z) \) is the coefficient ring of a \( Z \)-module \( \lambda_1 Z + \cdots + \lambda_m Z \subset \mathbb{R} \) [Borevich & Shafarevich 1966] [3, p. 87]. Up to a multiple, any such ring must be an order in a real number field \( \mathcal{K} \). Thus we have a field extension \( \mathcal{K} | k \) and the following inclusions:

\[
\Lambda \subseteq O_k \subseteq O_{\mathcal{K}}, \tag{3.3}
\]

where \( O_{\mathcal{K}} \) is the ring of integers of the field \( \mathcal{K} \).

On the other hand, it is known that the full \( Z \)-module \( \lambda_1 Z + \cdots + \lambda_m Z \) is contained in its coefficient ring \( O_{\mathcal{K}} \) [Borevich & Shafarevich 1966] [3, Lemma 1, p. 88]. In particular, each \( \lambda_i \) is an algebraic integer. Lemma 3.1 is proved. \( \square \)

**Remark 3.2.** It is useful to scale the RHS of inclusion (3.1) dividing it by the real number \( \lambda_m \neq 1 \). Such a normalization is always possible, since the embedding \( \tau: K_0(\mathcal{A}_{RM} \times \mathcal{E}(K)) \rightarrow \mathbb{R} \) is defined up to a scalar multiple. Thus we can rewrite inclusion (3.1) in the form:

\[
\tau(K_0(\mathcal{A}_{RM} \times \mathcal{E}(K))) = Z + \theta Z + \lambda_1 Z + \cdots + \lambda_{m-1} Z. \tag{3.4}
\]

**Lemma 3.3.** The number field \( k(\lambda_1, \ldots, \lambda_m) \) is the maximal abelian extension of the field \( k \) modulo conductor \( f \geq 1 \).

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\(^1\)The existence of \( \tau \) follows from [Phillips 2005] [13, Theorem 3.4].
Proof. Let \( \mathcal{E}(K) \) be an elliptic curve over the number field \( K \) and let \( \mathcal{A}_{RM} = F(\mathcal{E}(K)) \) be the corresponding noncommutative torus with real multiplication [10, Section 1.3]. The functor \( F \) is faithful on the category of \( K \)-rational elliptic curves and therefore \( F \) has a correctly defined inverse \( F^{-1} \). Thus \( \mathcal{E}(K) = F^{-1}(\mathcal{A}_{RM}) \) and one can write the crossed product (1.1) in the form:

\[
\mathcal{A}_{RM} \rtimes F^{-1}(\mathcal{A}_{RM}).
\]

(3.5)

Consider an endomorphism ring, \( M \), of the pseudo-lattice \( \lambda_1 \mathbb{Z} + \cdots + \lambda_m \mathbb{Z} \subset \mathbb{R} \). Denote by \( K \equiv M \otimes \mathbb{Q} \) a number field, such that \( M \) is an order in the ring of integers \( \mathcal{O}_K \) of the field \( K \). In view of the inclusion (3.2), one gets an inclusion \( \Lambda \subset M \), where \( \Lambda \) is an order in the ring \( \mathcal{O}_K \). Since \( \Lambda \subset \mathcal{O}_K \) and \( M \subset \mathcal{O}_K \), we get the inclusions:

\[
\mathcal{O}_K \subset \mathcal{O}_K \text{ and } k \subset K.
\]

(3.6)

On the other hand, it follows from the formula (3.5) that the crossed product \( \mathcal{A}_{RM} \rtimes \mathcal{E}(K) \) depends only on the inner structure of algebra \( \mathcal{A}_{RM} \). The same is true for the inclusions of groups \( K_0(\mathcal{A}_{RM}) \subset K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K)) \), the inclusion of pseudo-lattices \( \mathcal{E}(K) \subset \mathcal{E}(K) \), the inclusion of rings \( \text{End} \mathcal{A}_{RM} \subset \mathcal{O}_K \) and the inclusion of \( \text{End} \mathcal{E}(K) \subset \text{End} \mathcal{E}(K) \). In particular, the last inclusion says that arithmetic of the number field \( K \) in formula (3.6) is controlled by the arithmetic of the field \( k \). In other words, there exists an isomorphism:

\[
\text{Gal} (K|k) \cong \text{Cl} (\Lambda),
\]

(3.7)

where \( \text{Gal} (K|k) \) is the Galois group of the extension \( k \subset K \). Therefore \( \Lambda \) is the maximal abelian extension of the field \( k \) modulo conductor \( f \geq 1 \), see Section 2.2.

Let us show that \( \Lambda \) is an extension of the real quadratic field \( k \) by the values \( \{\lambda_i\}_{i=1}^m \). Indeed, since the coefficient ring of the full \( \mathbb{Z} \)-module \( \lambda_1 \mathbb{Z} + \cdots + \lambda_m \mathbb{Z} \) is isomorphic to the order \( M \subset \mathcal{O}_K \), we conclude that \( \lambda_i \in \mathcal{O}_K \) [Borevich & Shafarevich 1960] [3, Lemma 1, p. 88]. Moreover, by a change of basis in the \( \mathbb{Z} \)-module, we can always arrange the generators \( \{\lambda_i\}_{i=1}^m \) to be algebraically conjugate numbers of the field extension \( K|k \). In particular, one gets \( \Lambda = k(\lambda_1, \ldots, \lambda_m) \). Lemma 3.3 follows.

Corollary 3.4. The cardinality of the set of generators \( \lambda_i = m = h_\Lambda \).

Proof. Indeed, since \( \{\lambda_i\}_{i=1}^m \) are algebraically conjugate numbers of the field extension \( K|k \), we conclude that \( m = |\text{Gal} (K|k)| \). But \( \text{Gal} (K|k) \cong \text{Cl} (\Lambda) \) and therefore \( m = |\text{Cl} (\Lambda)| = h_\Lambda \). Corollary 3.4 follows.

Remark 3.5. To prove our results, we do not need an explicit formula for the values of generators \( \lambda_i \) in terms of \( \theta \in k \); however, we refer an interested reader to [10, Theorem 6.4.1] for such a formula.

Corollary 3.6. \( \tau(K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K))) = \mathbb{Z} + \theta \mathbb{Z} + \lambda_1 \mathbb{Z} + \cdots + \lambda_{h_\Lambda - 1} \mathbb{Z} \).

Proof. The formula follows from remark 3.2 and corollary 3.4.

Corollary 3.7. \( K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K)) \cong \mathbb{Z}^{h_\Lambda + 1} \).

Proof. Indeed, the rank of the abelian group \( K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K)) \) is equal to the number of generators of the pseudo-lattice \( \tau(K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K))) \subset \mathbb{R} \). It follows from corollary 3.6, that such a number is equal to \( h_\Lambda + 1 \). Corollary 3.7 follows.

Theorem 1.1 follows from the corollaries 3.6 and 3.7.
3.2. Proof of corollary 1.2. Let $\mathcal{E}(K)$ be an elliptic curve over the number field $K$. The Mordell-Weil Theorem says that $\mathcal{E}(K) \cong \mathbb{Z}^r \oplus \mathcal{E}_{\text{tors}}(K)$, where $r = rK \mathcal{E}(K)$ and $\mathcal{E}_{\text{tors}}(K)$ is a finite abelian group.

Consider again the pseudo-lattice $\tau(K_0(\mathcal{A}_{RM} \times \mathcal{E}(K))) \subset \mathbb{R}$ and substitute $\mathcal{E}(K) \cong \mathbb{Z}^r \oplus \mathcal{E}_{\text{tors}}(K)$:

$$
\tau[K_0(\mathcal{A}_{RM} \times \mathcal{E}(K))] = \tau[K_0(\mathcal{A}_{RM} \times (\mathbb{Z}^r \oplus \mathcal{E}_{\text{tors}}(K)))] = \tau[K_0(\mathcal{A}_{RM} \times \mathbb{Z}^r)] + \tau[K_0(\mathcal{A}_{RM} \times \mathcal{E}_{\text{tors}}(K))] = (3.8)
$$

In the last line of (3.8) we have the following two terms:

(i) $\tau[K_0(\mathcal{A}_{RM} \times \mathcal{E}_{\text{tors}}(K))] = \frac{1}{k}(\mathbb{Z} + \theta \mathbb{Z})$, where $k \geq 2$ is an integer depending on the order of finite group $\mathcal{E}_{\text{tors}}$: we refer the reader to [Echterhoff, Lück, Phillips & Walters 2010] [5, Theorem 0.1] for the proof of this fact.

(ii) $\tau[K_0(\mathcal{A}_{RM} \times \mathbb{Z}^r)] = \mathbb{Z} + \theta \mathbb{Z} + \lambda_1 \mathbb{Z} + \cdots + \lambda_r \mathbb{Z}$. For $r = 0$ this formula follows from [Echterhoff, Lück, Phillips & Walters 2010] [5, Theorem 0.1] after one rescales pseudo-lattice $\frac{1}{k}(\mathbb{Z} + \theta \mathbb{Z}) \subset \mathbb{R}$ to the pseudo-lattice $\mathbb{Z} + \theta \mathbb{Z} \subset \mathbb{R}$, see remark 3.2. For $r = 1$ the formula follows from [Farsi & Watling 1994] [6, Proposition 19]. For $r \geq 2$ the formula is proved by an induction. Namely, it is verified directly that the case $i + 1$ adds an extra generator $\lambda_{i+1}$ of the pseudo-lattice $\mathbb{Z} + \theta \mathbb{Z} + \lambda_1 \mathbb{Z} + \cdots + \lambda_r \mathbb{Z}$ corresponding to the case $i$.

It follows from (i) and (ii) that after a scaling, one gets the following inclusion of the pseudo-lattices:

$$
\tau[K_0(\mathcal{A}_{RM} \times \mathcal{E}_{\text{tors}}(K))] \subseteq \tau[K_0(\mathcal{A}_{RM} \times \mathbb{Z}^r)].
$$

From (3.8) and (3.9) we get the following equality:

$$
\tau[K_0(\mathcal{A}_{RM} \times \mathcal{E}(K))] = \tau[K_0(\mathcal{A}_{RM} \times \mathbb{Z}^r)].
$$

Using formula (1.2) and calculations of item (ii), one obtains from (3.10) the following equation:

$$
\mathbb{Z} + \theta \mathbb{Z} + \lambda_1 \mathbb{Z} + \cdots + \lambda_{h_A - 1} \mathbb{Z} = \mathbb{Z} + \theta \mathbb{Z} + \lambda_1 \mathbb{Z} + \cdots + \lambda_r \mathbb{Z},
$$

(3.11)

It is easy to see, that equation (3.11) is solvable if and only if $r = h_A - 1$. In other words, $r := rK \mathcal{E}(K) = h_A - 1$. Corollary 1.2 follows.

3.3. Proof of corollary 1.3. Let us make general remarks and outline the main ideas of the proof.

A relation between quadratic number fields and ranks of elliptic curves has been known for a while [Goldfeld 1976] [7]. In fact, the famous Birch and Swinnerton-Dyer Conjecture uses the relation to compare (special values of) the Dirichlet $L$-functions of a number field with the Hasse-Weil $L$-function of an elliptic curve [Swinnerton-Dyer 1967] [15]. Let us mention a recent generalization of this idea by [Bloch & Kato 1990] [2].

Our idea is to show that there exists a natural correspondence between the arithmetic of ideals of the real quadratic fields and the Hasse principle for elliptic curves. Namely, denote by $\mathcal{A}_{RM}^{(1)}, \ldots, \mathcal{A}_{RM}^{(h_A)}$ the companion noncommutative tori corresponding to the $\mathcal{A}_{RM}$ [10, p.163]: simply speaking, these are pairwise non-isomorphic algebras $\mathcal{A}_{RM}^{(i)}$, such that $\text{End} \ K_0(\mathcal{A}_{RM}^{(i)}) \cong \Lambda$ for all $1 \leq i \leq h_A$. 
Since the companion algebras $\mathcal{A}^{(i)}_{RM}$ have the same endomorphisms, so will be their “quantum dynamics”, i.e. the crossed product $\mathcal{A}^{(i)}_{RM} \rtimes \mathcal{E}(K)$, see lemma 3.8.

On the other hand, we establish a natural isomorphism between the abelian groups $K_0(\mathcal{A}_{RM}) \cong H^1(K, \mathcal{E})$ and $K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K)) \cong \prod_v H^1(K_v, \mathcal{E})$, see lemma 3.9. In view of formula (2.3), this means that the preimage of each cocycle in $\prod_v H^1(K_v, \mathcal{E})$ under the homomorphism $\omega$ consists of the $h_\Lambda \geq 1$ distinct cocycles of the $H^1(K, \mathcal{E})$. In other words, we get an inclusion $Cl(\Lambda) \subset \mathbb{I}(\mathcal{E}(K))$.

A precise formula is derived from Atiyah’s pairing between the $K$-theory and the $K$-homology of $C^*$-algebras, see e.g. [10, Section 10.2]. Namely, it is known that the $K^0(\mathcal{A}_0) \cong K_0(\mathcal{A}_0)$, where $K^0(\mathcal{A}_0)$ is the zero $K$-homology group of the non-commutative torus $\mathcal{A}_0$ [Hadfield 2004] [8, Proposition 4]. Repeating the argument for the group $K^0(\mathcal{A}_{RM})$, we get another subgroup $Cl(\Lambda) \subset \mathbb{I}(\mathcal{E}(K))$. In view of the Atiyah pairing, one gets $\mathbb{I}(\mathcal{E}(K)) \cong Cl(\Lambda) \oplus Cl(\Lambda)$. We pass to a detailed argument by splitting the proof in a series of lemmas.

**Lemma 3.8.** The $\mathcal{A}^{(i)}_{RM}$ and $\mathcal{A}^{(j)}_{RM}$ are companion noncommutative tori, if and only if, the crossed products $\mathcal{A}^{(i)}_{RM} \rtimes \mathcal{E}(K) \cong \mathcal{A}^{(j)}_{RM} \rtimes \mathcal{E}(K)$ are Morita equivalent $C^*$-algebras.

**Proof.** (i) Let $\mathcal{A}^{(i)}_{RM}$ and $\mathcal{A}^{(j)}_{RM}$ be companion noncommutative tori. In this case we have:

$$End\ K_0(\mathcal{A}^{(i)}_{RM}) \cong End\ K_0(\mathcal{A}^{(j)}_{RM}) \cong \Lambda. \quad (3.12)$$

Denote by

$$\tilde{\text{End}}\ (\mathcal{A}^{(i)}_{RM}) \cong \tilde{\text{End}}\ (\mathcal{A}^{(j)}_{RM}) \quad (3.13)$$

a pull back of the isomorphism (3.12) to the category of noncommutative tori.

Recall that the crossed product $\mathcal{A}^{(i)}_{RM} \rtimes \mathcal{E}(K)$ is an extension of the algebra $\mathcal{A}^{(i)}_{RM}$ by the elements $v \in \mathcal{A}^{(i)}_{RM} \rtimes \mathcal{E}(K)$, such that each $\phi \in \tilde{\text{End}}\ (\mathcal{A}^{(i)}_{RM})$ becomes an inner endomorphism, i.e. $\phi(u) = v^{-1}uv$ for every $u \in \mathcal{A}^{(i)}_{RM}$, see Section 2.1. From (3.13) we conclude that the algebras $\mathcal{A}^{(i)}_{RM}$ and $\mathcal{A}^{(j)}_{RM}$ must have $*$-isomorphic extensions. In other words, the corresponding crossed products $\mathcal{A}^{(i)}_{RM} \rtimes \mathcal{E}(K)$ and $\mathcal{A}^{(j)}_{RM} \rtimes \mathcal{E}(K)$ are isomorphic up to an adjustment of generators of the extension, i.e. the Morita equivalence. The ‘only if’ part of lemma 3.8 is proved.

(ii) Let $\mathcal{A}^{(i)}_{RM} \rtimes \mathcal{E}(K) \cong \mathcal{A}^{(j)}_{RM} \rtimes \mathcal{E}(K)$ be the Morita equivalent crossed products. Let us show that $\mathcal{A}^{(i)}_{RM}$ and $\mathcal{A}^{(j)}_{RM}$ are companion noncommutative tori. Indeed, our assumption implies immediately an isomorphism $\tilde{\text{End}}\ (\mathcal{A}^{(i)}_{RM}) \cong \tilde{\text{End}}\ (\mathcal{A}^{(j)}_{RM})$. We apply the $K_0$-functor and we get an isomorphism $End\ K_0(\mathcal{A}^{(i)}_{RM}) \cong End\ K_0(\mathcal{A}^{(j)}_{RM}) \cong \Lambda$. In other words, the $\mathcal{A}^{(i)}_{RM}$ and $\mathcal{A}^{(j)}_{RM}$ are companion algebras. The ‘if’ part of lemma 3.8 is proved. \hfill \Box

**Lemma 3.9.** Let $H^1(K, \mathcal{E})$ and $H^1(K_v, \mathcal{E})$ be the first Galois cohomology over the field $K$ and over the completion $K_v$ of $K$, respectively. There exists a natural isomorphism between the following groups:

$$\begin{aligned}
H^1(K, \mathcal{E}) &\cong K_0(\mathcal{A}_{RM}), \\
\prod_v H^1(K_v, \mathcal{E}) &\cong K_0(\mathcal{A}_{RM} \rtimes \mathcal{E}(K)).
\end{aligned} \quad (3.14)$$
Proof. (i) Let us show that $H^1(K, \mathcal{E}) \cong K_0(\mathcal{A}_R M)$. Indeed, such an isomorphism is a special case of [11, Theorem 1.1] saying that $H^1(\text{Gal}(C|K), \text{Aut}_C^{ab}(V)) \cong (K_0(\mathcal{A}_R) \times K_0^+(\mathcal{A}_R))$. For that, one has to restrict to the case $\mathcal{V} = \mathcal{E}(K)$ and notice that $\mathcal{A}_R = \mathcal{A}_R M$. On the other hand, since $\mathcal{E}(K)$ is an algebraic group, one gets $\text{Aut}_C^{ab}(\mathcal{E}(K)) \cong \mathcal{E}(K)$. The rest of the formula follows from the definition of the group $H^1(K, \mathcal{E})$.

(ii) Let us prove that $\prod_v H^1(K_v, \mathcal{E}) \cong K_0(\mathcal{A}_R M \times \mathcal{E}(K))$. An idea of the proof is to construct an AF-algebra, $\Lambda$, connected to the profinite group $\prod_v H^1(K_v, \mathcal{E})$; we refer the reader to Section 2.1.4 or [Blackadar 1986] [1, Chapter 7] for the definition of an AF-algebra. Next we show that the crossed product $\mathcal{A}_R M \times \mathcal{E}(K)$ embeds into $\Lambda$, so that $K_0(\mathcal{A}_R M \times \mathcal{E}(K)) \cong K_0(\Lambda)$. The rest of the proof will follow from the properties of the AF-algebra $\Lambda$. We pass to a detailed argument.

Recall that $\prod_v H^1(K_v, \mathcal{E})$ is a profinite group, i.e.

$$
\prod_v H^1(K_v, \mathcal{E}) \cong \varprojlim G_k,
$$

(3.15)

where $G_k = \prod_{v=1}^k H^1(K_v, \mathcal{E})$ is a finite group. Consider a group algebra

$$
\mathbb{C}[G_k] \cong M_{n_1}^{(k)}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}^{(k)}(\mathbb{C})
$$

(3.16)

corresponding to $G_k$. Notice that the $\mathbb{C}[G_k]$ is a finite-dimensional $C^*$-algebra. The inverse limit (3.15) defines an ascending sequence of the finite-dimensional $C^*$-algebras:

$$
\mathcal{A} := \varprojlim M_{n_1}^{(k)}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}^{(k)}(\mathbb{C}).
$$

(3.17)

In other words, the limit $\mathcal{A}$ is an AF-algebra, such that $K_0(\mathcal{A}) \cong \prod_v H^1(K_v, \mathcal{E})$.

To prove that $K_0(\mathcal{A}_R M \times \mathcal{E}(K)) \cong K_0(\mathcal{A})$, we shall use the “rigidity principle” described in Section 3.1. Namely, the extension $H^1(K, \mathcal{E}) \subset \prod_v H^1(K_v, \mathcal{E})$ is defined solely by the group $H^1(K, \mathcal{E})$ [Silverman 1985] [14, Appendix B]. Since $H^1(K, \mathcal{E}) \cong K_0(\mathcal{A}_R M)$ and $\prod_v H^1(K_v, \mathcal{E}) \cong K_0(\mathcal{A})$, we conclude that the extension $K_0(\mathcal{A}_R M) \subset K_0(\mathcal{A})$ is defined by the group $K_0(\mathcal{A}_R M)$ alone. But the extension $K_0(\mathcal{A}_R M) \subset K_0(\mathcal{A}_R M \times \mathcal{E}(K))$ is the only extension with such a property. Thus $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_R M \times \mathcal{E}(K))$ and the crossed product $\mathcal{A}_R M \times \mathcal{E}(K)$ embeds into the AF-algebra $\mathcal{A}$.

To finish the proof of lemma 3.9, we recall that $K_0(\mathcal{A}) \cong \prod_v H^1(K_v, \mathcal{E})$ and therefore $\prod_v H^1(K_v, \mathcal{E}) \cong K_0(\mathcal{A}_R M \times \mathcal{E}(K))$. $\blacksquare$

Lemma 3.10. $\text{T}(\mathcal{E}(K)) \cong Cl(\Lambda) \oplus Cl(\Lambda)$.

Proof. Let $\{\mathcal{A}_R^{(i)}\}_{i=1}^{h_{\Lambda}}$ be the companion noncommutative tori of the $\mathcal{A}_R M$. Consider a group homomorphism

$$
h : K_0(\mathcal{A}_R M) \rightarrow K_0(\mathcal{A}_R M \times \mathcal{E}(K)),
$$

(3.18)

induced by the natural embedding $\mathcal{A}_R M \hookrightarrow \mathcal{A}_R M \times \mathcal{E}(K)$. It follows from lemma 3.8, that

$$
h(K_0(\mathcal{A}_R^{(i)})) = \mathbb{Z} + \theta \mathbb{Z} \quad \text{for all} \quad 1 \leq i \leq h_{\Lambda}.
$$

(3.19)

In other words, one gets $\text{Ker} h \cong Cl(\Lambda)$, where $Cl(\Lambda)$ is the class group of the order $\Lambda$ in the real quadratic field $\mathbb{Q}(\theta)$.  


But $K_0(\mathcal{A}_{\text{RM}}) \cong H^1(K, \mathcal{E})$ and $K_0(\mathcal{A}_0 \otimes \mathcal{E}(K)) \cong \prod_n H^1(K_n, \mathcal{E})$, see lemma 3.9. Therefore, in view of the formulas (2.3) and (2.4), the abelian group $\text{Cl}(\Lambda)$ is an obstacle to the Hasse principle for the elliptic curve $\mathcal{E}(K)$. In other words, $\text{Cl}(\Lambda) \subseteq \text{III}(\mathcal{E}(K))$.

To calculate an exact relation between the groups $\text{Cl}(\Lambda)$ and $\text{III}(\mathcal{E}(K))$, recall that the $K$-homology is the dual theory to the $K$-theory, see e.g. [Blackadar 1986] [1, Section 16.3]. Roughly speaking, cocycles in $K$-theory are represented by vector bundles. Atiyah proposed using elliptic operators to represent the $K$-homology cycles. An elliptic operator can be twisted by a vector bundle, and the Fredholm index of the twisted operator defines a pairing between the $K$-homology and the $K$-theory with values in $\mathbb{Z}$.

In particular, it is known that for the algebra $\mathcal{A}_\theta$, it holds $K^0(\mathcal{A}_\theta) \cong K_0(\mathcal{A}_\theta)$, where $K^0(\mathcal{A}_\theta)$ is the zero $K$-homology group of $\mathcal{A}_\theta$ [Hadfield 2004] [8, Proposition 4]. Repeating the argument for the group $K^0(\mathcal{A}_{\text{RM}})$, one can prove an analog of theorem 1.1 for such a group. In other words, we get another subgroup $\text{Cl}(\Lambda) \subseteq \text{III}(\mathcal{E}(K))$. Since there are no other duals to the $K$-theory of $C^*$-algebras, we conclude from the Atiyah pairing, that $\text{III}(\mathcal{E}(K)) \cong \text{Cl}(\Lambda) \oplus \text{Cl}(\Lambda)$. Lemma 3.10 is proved.

Corollary 1.3 follows from lemma 3.10.

Remark 3.11. The reader can observe, that construction of a generator of $\mathcal{E}(K)$ is similar to construction of an “ideal number” (i.e. a principal ideal) of the number field $k$. Namely, it is well known that not every ideal of the ring $\Lambda \subseteq O_K$ principal; an obstruction is a non-trivial group $\text{Cl}(\Lambda)$. However, this can be repaired in a bigger field $K = K_{ab}$; there exists a finite extension $k \subseteq K$, such that every ideal of $\Lambda$ is principal in the ring $O_K$. Likewise, one cannot construct a generator of $\mathcal{E}(K)$ by a finite descent in general; an obstruction is a non-trivial group $\text{III}(\mathcal{E}(K))$. However, in an extension $\mathcal{A}_{\text{RM}} \otimes \mathcal{E}(K)$ of the coordinate ring $\mathcal{A}_{\text{RM}}$ of $\mathcal{E}(K)$, the descent will be always finite and give a generator of the $\mathcal{E}(K)$. Such an analogy explains the formula $\text{III}(\mathcal{E}(K)) \cong \text{Cl}(\Lambda) \oplus \text{Cl}(\Lambda)$ on an intuitive level. Notice also that the $\mathcal{A}_{\text{RM}} \otimes \mathcal{E}(K)$ is the coordinate ring of an abelian variety $A(K)$, which is related to the Euler variety $V_E$ coming from the continued fraction of $\theta$ [10, Section 6.2.1].

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