Holographic flows with scalar self-interaction toward the Kasner universe

Yong-Qiang Wang\textsuperscript{1,2,3}, Yan Song\textsuperscript{1,2}, Qian Xiang\textsuperscript{1,2}, Shao-Wen Wei\textsuperscript{1,2,3}, Tao Zhu\textsuperscript{4}, Yu-Xiao Liu\textsuperscript{1,2,3}\textsuperscript{*}

\textsuperscript{1}Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China
\textsuperscript{2}Research Center of Gravitation, Lanzhou University, Lanzhou 730000, China
\textsuperscript{3}Key Laboratory for Magnetism and Magnetic of the Ministry of Education, Lanzhou University, Lanzhou 730000, China
\textsuperscript{4}Institute for Theoretical Physics and Cosmology, Zhejiang University of Technology, Hangzhou, 310023, China

Abstract

Considering a thermal state of the dual CFT with a uniform deformation by a scalar operator, we study a holographic renormalization group flow at nonzero temperature in the bulk described by the Einstein-scalar field theory with the self-interaction term $\lambda \phi^4$ in asymptotic anti-de Sitter spacetime. We show that the holographic flow with the self-interaction term could run smoothly through the event horizon of a black hole and deform the Schwarzschild singularity to a Kasner universe at late times. Furthermore, we also study the effect of the scalar self-interaction on the deformed near-singularity Kasner exponents and the relationship between entanglement velocity and Kasner singularity exponents at late times.

* liuyx@lzu.edu.cn, corresponding author
I. INTRODUCTION

According to general relativity, a gravitational singularity exists in the interior of a black hole, which contains a huge mass in an infinitely small space. Due to that the infinity of matter density and gravity could lead to infinite spacetime curvature, all laws of physics as we know break down at a singularity. Because of the existence of black hole singularity, the study of the black hole interior is very challenging. In recent years, the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] has been used to investigate the black hole interior. It has been found [4–9] that by studying the analytic behaviors of the correlation functions of conformal field theory, one could obtain the information about the singularity extracted from the corresponding correlation functions. For a dynamical spacetime geometry, the entanglement entropy is also a possible way to probe the interior of a black hole [10–15]. Especially, at late times the linear growth of the entanglement entropy could be directly related to the growth of the black hole interior measured along with a critical spatial slice [13].

Recently, the model of a massive, real scalar field minimally coupled to the four-dimensional Einstein gravity with a negative cosmological constant was investigated in [16]. Considering that the thermal state of the dual CFT field could be deformed by a relevant scalar operator sourced by the scalar field on the boundary, the authors solved numerically the equations of motion from an AdS boundary as the UV, to the timelike singularity inside the black hole as the IR, and found that the generic behavior of this family of numerical solutions near the black hole singularity could be described by a one parameter family of homogeneous, anisotropic so-called Kasner spacetime. Moreover, with the vanishing of deformation, the IR geometry has a timelike cousin of the Schwarzschild singularity and is the less generic case of the Kasner universe. Soon afterward the study of holographic flows from CFT to Kasner universe was extended to the case of charged black holes [17]. It is interesting to see that a relevant deformation of the dual CFT with a neutral scalar operator would result in the formation of a charged black hole without a Cauchy horizon. Furthermore, in [18] holographic flow inside the horizon of holographic superconductors was studied, and complex dynamical behaviors in the interior of the holographic superconductors were found.

Until now, the studies of holographic flow from the UV to the IR only focus on the case of the free scalar field. Moreover, in [18] the authors thought that in some epochs of the dynamics, holographic flows inside the horizon of holographic superconductors are likely sensitive to the scalar potential, and it is worth studying
more general scalar potentials. In this paper, we investigate a holographic renormalization group flow at nonzero temperature in the bulk described by the Einstein-scalar field theory with the self-interaction term $\lambda \phi^4$ in asymptotic AdS spacetime, and study the effect of scalar self-interaction on the near-singularity Kasner exponents. Furthermore, as a method to probe the black hole interior, entanglement velocity for the black hole with the Kasner singularity interior at late times is also investigated.

The paper is organized as follows. In Sec. II we introduce the model of the Einstein-scalar field theory with the self-interaction term $\lambda \phi^4$ in asymptotic AdS spacetime and explore the ansatz of metric and matter field. We also analyze the boundary conditions. In Sec. III we introduce the numerical method and show the numerical results for a class of holographic flows from the AdS boundary to the Kasner singularity. Moreover, the effect of the scalar self-interaction on the deformed near-singularity Kasner exponents and entanglement velocity for the black hole at late times are studied. The conclusion and discussion are given in the last section.

II. SET UP

We consider the action of a massive, real scalar field minimally coupled to the 3 + 1 dimensional Einstein gravity with a negative cosmological constant, which is written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa_4^2} (R - \Lambda) - \frac{1}{2} \left( g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + m^2 \phi^2 + \lambda \phi^4 \right) \right],$$  \hspace{1cm} (II.1)

where $\kappa_4$ is the four-dimensional gravitational constant and $\Lambda = -\frac{6}{L^2}$ is the cosmological constant with $L$ being the AdS radius. $R$ and $\phi$ are the Ricci scalar and real scalar field respectively, and the latter has a quartic self-interaction. The constants $m$ and $\lambda$ represent the mass of the scalar field and the interaction parameter, respectively. The parameter $\lambda$ controls the strength of this self-interaction, and the self-interacting scalar field reduces to the free one when $\lambda = 0$. The Einstein and scalar field equations can be derived from the above action (II.1) as

$$R_{\mu\nu} + \frac{3}{L^2} g_{\mu\nu} = \frac{\kappa_4^2}{2} \left( 2
abla_\mu \phi \nabla_\nu \phi + g_{\mu\nu}(m^2 \phi^2 + \lambda \phi^4) \right),$$  \hspace{1cm} (II.2a)

$$\nabla_\mu \nabla^\mu \phi = m^2 \phi + 2 \lambda \phi^3.$$  \hspace{1cm} (II.2b)

For simplicity, we will set the AdS radius to one and $\kappa_4^2 = 1/2$. Note that in the four-dimensional spacetime the values of mass need to satisfy the Breitenlohner-Freedman (BF) bound of $m^2 \geq -9/4$ \cite{BF}, and we will set $m^2 = -2$.

In order to obtain the plane-symmetric hairy black hole solutions, following the conventions in \cite{16} we
choose the metric ansatz in the form of
\[ ds^2 = \frac{1}{r^2} \left( -f(r)e^{-\chi(r)} dt^2 + \frac{dr^2}{f(r)} + dx^2 + dy^2 \right) , \] (II.3)

together with the scalar field
\[ \phi = \phi(r) , \] (II.4)

where the radial coordinate \( r \in (0, \infty) \). The asymmetrical AdS boundary and the singularity inside the event horizon are fixed at \( r \to 0 \) and \( r \to \infty \), respectively. The horizon radius \( r_+ \) is defined through the requirement that \( f(r_+) = 0 \). When the scalar field vanishes, there exists a solution of the planar Schwarzschild-AdS black hole with \( \chi = 0 \) and \( f = 1 - \frac{(r/r_+)^3}{3} \).

As shown in [19, 20], inside the horizon, the near-singularity behavior of the metric (II.3) has the form of the Kasner universe,
\[ ds^2 \sim -d\tau^2 + \tau^{2p_t} dt^2 + \tau^{2p_x} \left( dx^2 + dy^2 \right) , \] (II.5)

with the near-singularity behavior of the scalar field given by
\[ \phi(r) \sim -\sqrt{2} p_\phi \log \tau . \] (II.6)

Here \( \tau \) is a time function with the singularity at \( \tau = 0 \). Setting \( d\tau = (-r^2 f)^{-1/2} dr \), the metric of the plane-symmetric black hole (II.3) can take the form (II.5). The parameters \( p_\phi, p_t, p_x \) are known as the Kasner exponents, which are functions of the spatial coordinates \( t, x, y \) and must satisfy the Kasner relations \( p_t + 2p_x = 1 \) and \( p_\phi^2 + p_t^2 + 2p_x^2 = 1 \). For the Schwarzschild-AdS black hole, the Kasner-type singularity has the exponents \( p_t = -\frac{1}{3}, p_x = \frac{2}{3} \) and \( p_\phi = 0 \). The solution (II.3) is asymptotically AdS, which describes an RG flow from the AdS boundary as the UV, to the timelike Kasner singularity as the IR.

In order to obtain the numerical solution from the AdS boundary to the near-singularity inside a black hole horizon, we will choose the ingoing coordinates as follows
\[ ds^2 = \frac{1}{r^2} \left( -f(r)e^{-\chi(r)} du^2 + 2e^{-\chi(r)/2} dudr + dx^2 + dy^2 \right) . \] (II.7)

One advantage of this metric is that the apparent singularity at the event horizon where \( f(r_+) = 0 \) is only a coordinate singularity which is not physical. Substituting the ansatz of the metric (II.7) and the matter field
into the Einstein and scalar field equations (II.2), one can derive the equations of motion as follows

\[ \phi'' + \left( \frac{f'}{f} - \frac{2}{r} - \frac{\chi'}{2} \right) \phi' + \frac{2}{r^2 f} \phi - \frac{2 \lambda \phi^3}{r^2 f} = 0, \] (II.8a)

\[ \chi' - \frac{2 f'}{r} - \frac{\phi^2}{r f} - \frac{6}{r} + \frac{6 r}{r} + \frac{6 r}{r} = 0, \] (II.8b)

\[ \chi' - \frac{r}{2} (\phi')^2 = 0. \] (II.8c)

To obtain the numerical solutions of the above equations by integrating from the AdS boundary to the near-singularity region, we require that \( f = 0 \) at the event horizon \( r = r_+ \). Thus, the Hawking temperature of the black hole is

\[ T = \left| \frac{f' e^{-\chi/2}}{4 \pi} \right|_{r=r_+}. \] (II.9)

In addition, by analyzing the equations of motion (II.8) at the asymptotic boundary (\( r \to 0 \)) order by order, we obtain the asymptotic expansions as follows

\[ \phi = \phi_0 r + \langle O \rangle r^2 + \cdots, \quad \chi = \frac{\phi_0^2}{4} r^2 + \cdots, \quad f = 1 + \cdots. \] (II.10)

According to AdS/CFT dictionary, the first-order expansion coefficient \( \phi_0 \) is the source in the dual field theory on the AdS boundary, meanwhile, the second-order one \( \langle O \rangle \) is the corresponding expectation value of the operator \( O \). Moreover, solving the equations of motion (II.8) near the singularity (\( r \to \infty \)), we obtain the asymptotic behaviors in the form of

\[ \phi = 2c \log r + \cdots, \quad \chi = 2c^2 \log r + \cdots, \quad f = -f_1 r^{3+c^2} + \cdots. \] (II.11)

Here both \( c \) and \( f_1 \) are constants, and \( c = 0 \) corresponds to the case of the Schwarzschild-AdS black hole. Furthermore, by setting the coordinate \( r^{(3+c^2)} = 1/r^2 \), we can write the spacetime near the singularity in form of the Kasner university metric form (II.5) with the Kasner exponents:

\[ p_x = \frac{2}{3+c^2}, \quad p_t = \frac{c^2-1}{3+c^2}, \quad p_\phi = \frac{2 \sqrt{2} c}{3+c^2}. \] (II.12)

III. NUMERICAL RESULTS

In order to obtain the numerical solutions of the coupled equations (II.8), the integration over the whole region \( 0 \leq r \leq \infty \) should be divided into the following two steps. In the first step, we use the pseudospectral method based on the Chebyshev polynomials in the integration region \( 0 \leq r \leq r_+ \). Our iterative process is performed by using the Newton-Raphson method, and the relative error for the numerical solutions in this
process is estimated to be below $10^{-5}$. For details of the pseudospectral method, see Ref. [21]. Of course, we can also solve these equations of motion by using the numerical shooting method described in [22, 23]. In the second step, we consider the backward differentiation formula (BDF) for solving Eqs. (I.8) in the integration region $r_+ \leq r \leq \infty$, which is a family of implicit methods for the numerical integration of ordinary differential equations. The initial value for the second step is extracted from the data on the event horizon derived from the first step.

A. Holographic flows from UV to IR

After obtaining the numerical solutions of the deformed black hole with scalar hair, we study how the scalar self-interaction parameter $\lambda$ affects the deformed near-singularity Kasner exponents. As an example, we show the typical results of our numerical program for $r \frac{d\phi}{dr}$ and $r \frac{d\chi}{dr}$ as two functions of $r$ for a fixed dimensionless parameter $\phi_0/T = 13$ in Fig. 1. The blue and red lines correspond to the functions $r \frac{d\phi}{dr}$ and $r \frac{d\chi}{dr}$, respectively. The dashed and dotted curves denote different self-interaction parameters $\lambda = 0.1$ and $\lambda = 0.2$, respectively. When taking the parameter $\lambda = 0$ (solid curves), we reproduce holographic flow from AdS to a Kasner cosmology, which was first obtained in [16]. The horizontal dashed green line along the $r$ coordinator corresponds to the planar Schwarzschild-AdS black hole. The vertical dashed black line indicates the location of the event horizon. In the large $r$ limit (near singularity), the values of $r \frac{d\phi}{dr}$ and $r \frac{d\chi}{dr}$ are close to different non-zero constants, which is the universal feature with the different self-interacting parameters $\lambda$.

Moreover, we could see that the asymptotic values of the holographic flows from CFT to the Kasner universe decrease with the scalar self-interaction $\lambda$.

From the asymptotic behavior of the solutions of Eq. (II.11), we observe that the asymptotic values of the holographic flows could be related to the Kasner exponents (II.12). In Fig. 2, we show how the Kasner exponent $p_t$ varies as a function of the deformed parameter $\phi_0/T$ with different scalar self-interaction parameters $\lambda$. From top to bottom, we plot three curves with $\lambda = 0$ (black color), $\lambda = 0.1$ (red color), and $\lambda = 0.2$ (blue color), respectively. At $\phi_0/T = 0$, all of the Kasner exponents in these three curves reduce to the usual Schwarzschild-AdS singularity with the value of $p_t = -1/3$. With the increase of the deformed parameter $\phi_0/T$, the Kasner exponent $p_t$ increases firstly and then reaches a maximum point. Further increasing the value of $\phi_0/T$, the Kasner exponent begins to decrease. When the curves go into the range of large deformed parameter $\phi_0/T$, the numerical error begins to increase and a finer mesh is required. However, it is certainly possible that the curve will eventually approach the Schwarzschild singularity value of $p_t = -1/3$ at a much
FIG. 1: The holographic flow from the AdS boundary to the Kasner universe inside the event horizon. The blue and red lines correspond to the functions $r \frac{d\phi}{dr}$ and $r \frac{d\chi}{dr}$, respectively. The solid, dashed, and dotted curves denote different self-interaction parameters $\lambda = 0$, $\lambda = 0.1$ and $\lambda = 0.2$, respectively. All of curves are fixed with the deformed parameter $\phi_0/T = 13$.

larger deformed parameter $\phi_0/T$. Moreover, we could see that at a fixed deformed parameter $\phi_0/T$, the Kasner universe exponent decreases in the conditions of higher scalar self-interactions $\lambda$. As $\lambda \to \infty$, it can be

predicted that the Kasner exponent $p_t$ would return to the Schwarzschild singularity with the value of $-1/3$.

It is noteworthy that the Kasner exponents with the different self-interaction parameters $\lambda$ are close in the
range of small deformation $\phi_0/T$, the reason is that the self-interacting term $\lambda\phi^4$ is a small quantity in the range of small deformation due to its high order and thus has little effect on the deformed near-singularity Kasner exponents. While, in the range of large deformation $\phi_0/T$, the self-interacting term is big enough to affect the Kasner exponents.

**B. Probes of the Kasner exponent by entanglement entropy**

In this subsection, we would use the entanglement entropy as a probe of the black hole interior to study the Kasner singularity. It is well known that the entanglement entropy of the dual field could be obtained in holographic models as the area of the minimum surface extended into the bulk with the same AdS boundary of the quantum system [25–27]. Hartman and Maldacena in [13] showed that at late times of the dynamical system, the extremal surface eventually stops expanding on a specific critical slice inside the horizon and can not get closer to the singularity, and the entanglement entropy increases linearly with time. The linear growth of the entanglement entropy could be directly related to the growth of the black hole interior measured along this critical surface. This growth could define a so-called entanglement velocity, which is sensitive to the black hole interior, but not to the near-singularity region. In our model, at late time the entanglement velocity $v$ is written as

$$v^2 = r_+^4 \left. \frac{|f|e^{-\chi}}{r^4} \right|_{r=r_{\text{crit}}}.$$  \hspace{1cm} (III.1)

Here $r = r_{\text{crit}}$ is the radius of the critical surface, in which $-fe^{-\chi}/r^4$ has a maximum inside the horizon. The formula of the entanglement velocity in Eq. (III.1) is the same as the case of the free scalar field in [16].

In Fig. 3 we exhibit the entanglement velocity $v$ as a function of $-1/p_t$ with the scalar self-interaction parameter $\lambda = 0, 0.1, 0.2$, denoted by the black, red, and blue dotted lines, respectively. At $-1/p_t = 3$, the values of the entanglement velocity in these three curves reduce to the usual Schwarzschild-AdS singularity with the entanglement velocity $v = \sqrt{3}/2^{4/3}$. With the increase of $-1/p_t$, the entanglement velocity decreases at first and then it reaches a minimum value. Afterwards, by re-decreasing $-1/p_t$, the value of the entanglement velocity slowly goes upwards and reach the Schwarzschild-AdS singularity in the end. Moreover, we could see that the minimum value of the entanglement velocity increases when enlarging the scalar self-interaction $\lambda$. 
FIG. 3: The dependence of the entanglement velocity $v$ on $-1/p_t$. The black, red and blue curves correspond to the self-interaction parameters $\lambda = 0, \lambda = 0.1$ and $\lambda = 0.2$, respectively.

IV. CONCLUSION AND DISCUSSION

In summary, we have studied a holographic renormalization group flow at nonzero temperature in the bulk from the AdS boundary to the Kasner universe. By numerically solving the gravitational dynamics in the ingoing coordinates, we showed the holographic flow with the self-interaction term could run smoothly through the event horizon of the black hole and deform the Schwarzschild singularity to the Kasner universe, where the Kasner exponent decreases with the scalar self-interaction $\lambda$. Moreover, in the range of small deformation $\phi_0/T$, the scalar self-interaction term has a smaller effect on the deformed near-singularity Kasner exponents, and in the range of large deformation, the self-interacting term is big enough to affect the Kasner exponents. Furthermore, we used the entanglement entropy as a probe of the black hole interior to study the Kasner singularity and showed that the minimum of the entanglement velocity increases with the scalar self-interaction $\lambda$.

There are several interesting extensions of our work. The first one is to investigate how a relevant deformation of the dual CFT with a self-interacting scalar operator would result in the formation of the Kasner singularity in a charged black hole interior, and study whether the charged black hole with the scalar self-interaction has no Cauchy horizon similar as the case in [17]. The second one is to extend the study on holographic flow inside the horizon of the holographic superconductors [18] to the model with more general scalar potentials, and investigate how the dynamical behavior in the interior of the holographic superconduc-
tors performs. Finally, we are planning to study the holographic flow toward the black hole interior in the model of the excited holographic superconductor [28, 29], and investigate the configuration of the black hole with Kasner singularity in future works.

Note added: When we are submitting this paper to arXiv, we notice that there appears a paper on arXiv [30], in which the authors established a no inner-horizon theorem for black holes with charged scalar hair including the self-interaction term, and proved that in a general gravitational theory with a charged scalar field both spherical and planar black holes with scalar hair have no inner Cauchy horizon.

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