An analogue of cyclotomic units for products of elliptic curves

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Abstract

We construct certain elements in the integral motivic cohomology group $H^3_{\text{Mot}}(E \times E', \mathbb{Q}(2))_{\mathbb{Z}}$, where $E$ and $E'$ are elliptic curves over $\mathbb{Q}$. When $E$ is not isogenous to $E'$ these elements are analogous to ‘cyclotomic units’ in real quadratic fields as they come from modular parametrisations of the elliptic curves. We then find an analogue of the class number formula for real quadratic fields. Finally we use the Beilinson conjectures for $E \times E'$ to deduce them for products of $n$ elliptic curves. A certain amount of this paper is expository in nature.

1 Introduction

1.1 Beilinson’s Conjecture

Let $X$ be a variety over a number field $F$. The well known Tate Conjecture relates the order of the pole at a certain point of a cohomological $L$-function of $X$ as a variety over $K$, a number field containing $F$, to the rank of a group of cycles modulo homological equivalence defined over $K$. However, Tate did not give any interpretation of the residue at that point.

Beilinson [Be] made the following generalisation of the Tate conjecture (For simplicity we will state it over $\mathbb{Q}$, as that is the case we will be concerned with):

**Conjecture 1.1 (Beilinson).** Let $X$ be a smooth projective variety defined over $\mathbb{Q}$. Let $i$ be an even integer and $m = \frac{i}{2}$. Let $L(X, s)$ denote the $L$-function defined by $H^i(X)$. Let $B^m(X)_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space generated by the codimensional $m$ cycles modulo homological equivalence. Then:
\[ (i) \tilde{r}_D(B^m(X) \oplus H^{i+1}_M(X, \mathbb{Q}(m+1))_\mathbb{Z}) \text{ induces a } \mathbb{Q} \text{ structure on } H^{i+1}_D(X/\mathbb{R}, \mathbb{R}(m+1)) \]
\[ (ii) \text{ord}_{s=m}L(X, s) = \dim_{\mathbb{Q}}H^{i+1}_M(X, \mathbb{Q}(m+1))_\mathbb{Z} \]
\[ (iii) \text{ord}_{s=m+1}L(X, s) = -\dim_{\mathbb{Q}}(B^m(X)_{\mathbb{Q}}) \text{ (Tate)} \]
\[ (iv) L^*(X, s)_{s=m} \sim_{\mathbb{Q}^*} c_X(m) \]

Here \( \tilde{r}_D \) is a certain ‘thickened’ regulator map which generalises the usual regulator map for number fields, \( H^{i+1}_M(X, \mathbb{Q}(m+1))_\mathbb{Z} \) is the ‘integral’ motivic cohomology, which is the motivic cohomology of a regular proper model if it exists, otherwise there is an unconditional definition due to Scholl [Sc], and \( H^{i+1}_D(X/\mathbb{R}, \mathbb{R}(m+1)) \) is the ‘Real’ Deligne cohomology which is a real vector space of dimension \( \text{ord}_{s=m}L(X, s) - \text{ord}_{s=m+1}L(X, s) \) over \( \mathbb{R} \). \( L^*(X, s)_{s=m} \) is the first non-zero term in the Taylor expansion and \( c_X(m) \) is an element of \( \mathbb{R}^*/\mathbb{Q}^* \) related to the covolume of the image of \( \tilde{r}_D \).

Beilinson proved this when \( X \) is a product of modular curves and \( m = 1 \). From that it follows for the product of two elliptic curves over \( \mathbb{Q} \).

The purpose of this note is to directly prove this for the product of two elliptic curves over \( \mathbb{Q} \). In this situation, if \( E \) and \( E' \) are isogenous, the conjecture asserts that the motivic cohomology is 0 dimensional and hence the value of \( L(X, 1) \) should be a non-zero rational multiple of the period \( c_X(1) \). Further, if \( E \) and \( E' \) are not isogenous, the conjecture asserts that the motivic cohomology is 1 dimensional, and hence the value of the first derivative \( L'(X, 1) \) should be the period \( c_X(1) \) up to a non-zero rational number. Here this period is essentially the regulator of some element of the motivic cohomology. In this case we show a little more by finding an explicit element of \( H^{i+1}_M(X, \mathbb{Q}(2))_\mathbb{Z} \) whose regulator is equal to the value of the first derivative of the \( L \)-function at \( s = 1 \). This element comes from a modular parameterization of the elliptic curves and is an analogue of a cyclotomic unit.

We finally end up with an analogue of the class number formula for real quadratic fields:

**Theorem 1.2.** Let \( E, E' \) be non-isogenous elliptic curves over \( \mathbb{Q} \) corresponding modular forms \( f \) and \( g \) of level \( N_1 \) and \( N_2 \) respectively. Let

\[
\log_q(t) = \log|qt| + \sum_{n=1}^{\infty} \log|1 - q^n t|.
\]
where \( q = e^{2\pi i z} \). Let \( \xi \) be a primitive \( N \)th root of unity where \( N = \text{l.c.m.}(N_1, N_2) \).

Then

\[
L'(H^2(E \times E^\prime), 1) = \frac{-H(0)}{8} \sum_{k \mod N \atop (k,N)=1} \frac{1}{2\pi i} \int_{X_0(N)} \log_q(\xi^k) f(q) g(q) \frac{dq \bar{dq}}{q \bar{q}}
\]

where \( H(0) \) is a certain rational number corresponding to the terms in the \( L \)-function for primes dividing \( N \).

Stark made conjectures relating the exact values of \( L \)-functions of Number fields to regulators of units in some auxiliary number fields, and from that point of view, this can be regarded as a special case of a generalization of those conjectures.

The proof follows by looking at Ogg's original proof of the Tate conjecture for products of two elliptic curves more carefully and using Kronecker's first limit formula.

In the second part we show that the conjecture for the product of two elliptic curves implies the conjecture for \( H^{2n-1}(\prod E_i, \mathbb{Q}(n))_{\mathbb{Z}} \), the \( n \)-fold product of elliptic curves. It appears that one does not get any elements excepting those induced from lower products.

### 1.2 Analogies with quadratic extensions of \( \mathbb{Q} \)

There is a suggestive analogy of this situation with that of quadratic extensions of \( \mathbb{Q} \), which is a special case of \( m = 0 \). Consider the group

\[
\Sigma_m := \{ \text{Ker} : H^2_{\mathcal{M}}(X, \mathbb{Q}(m+1))_{\mathbb{Z}} \rightarrow H^2_{\mathcal{M}}(X, \mathbb{Q}(m+1)) \}
\]

namely the group of codimensional \( m \) cycles supported on special fibres. Conjecturally \( \Sigma_m \) is finite. When \( m = 0 \) and \( X = \text{Spec}(K) \), where \( K \) is a number field, \( \Sigma_0 \) is the class group, which is well known to be finite.

The class number formula gives an expression for the class number \( h_K \) in terms of the Dirichlet \( L \)-functions associated to \( K \). Let \( K^+ \) denote the maximal real subfield and \( h^+_K \) be the class number of \( K^+ \). Let \( h^*_K = h_K/h^+_K \) (so \( h^+ \) is the class number if \( K \) is real and \( h^* \) is the class number if \( K \) is imaginary quadratic.) Then the class number formula give significantly different expressions for \( h^* \) and \( h^+ \).

#### 1.2.1 Imaginary Quadratic Fields

If \( K \) is imaginary (so \( h_K = h^*_K \)) for each rational prime \( p \) there is a certain element \( g(p) \) of \( K^* \) called the ‘Gauss sum’, coming from a cyclotomic field.
containing $K$, which has the property that its ideal factorization involves only the primes lying over $p$ and does not depend on $p$. Namely

$$g(p) = \prod_{\mathfrak{p} | p} \mathfrak{p}^\theta$$

where $\theta$ is a certain element of the group ring of the integral Galois group, the Stickleberger element, which does not depend on $p$. The index of the ideal generated by the Stickleberger element is the class number. Hence the special element, the Gauss sum, gives rise to annihilators of the class group and is related to the value of the $L$-function, though in a roundabout manner.

Mildenhall studied the group $\Sigma_1$ when $X$ is the self product of an elliptic curve over $\mathbb{Q}$. He showed that it is torsion by constructing annihilators coming from certain special elements of $H^3_M(Y, \mathbb{Q}(2))$ where $Y$ is the self product of a modular parametrisation of the elliptic curve. These elements are analogues of Gauss sums as they too ramify at precisely one place. However, the relation with the $L$-function is not clear. Flach studied a Selmer group associated to the symmetric square of an elliptic curve which is conjecturally the same as $\Sigma_1$ and did find some relation between the $L$-value and the order of this group, though it is still not known whether this group is finite.

### 1.2.2 Real Quadratic Fields

Similarly, if $K$ is real (so $h^+ = h$) and $\chi$ is its quadratic character one has

$$L'(0, \chi) = h^+ \log |\epsilon|$$

where $\epsilon$ is the fundamental unit. On the other hand, one also has the formula

$$L'(0, \chi) = \log \prod_{\substack{k \mod N \\ (k,N)=1}} |1 - \xi^k|^{-\frac{1}{2}\chi(k)} = \sum_{\substack{k \mod N \\ (k,N)=1}} \frac{-\chi(k)}{2} \log |1 - \xi^k| \quad (1.1)$$

where $N$ is the conductor of $\chi$ and $\xi = e^{2\pi i N}$. This shows that the exact value of $L'(0, \chi)$ is the regulator of a naturally constructed unit coming from a cyclotomic field containing $K$. Further, the index of the subgroup of the units group generated by the cyclotomic units is the class number. This fact is a lot harder to prove directly without using the analytic class number formula and was only done about ten years ago by Thaine.

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Our result can be viewed as an analogue of the second statement as we compute the exact value of the $L$-function of $E \times E'$ in terms of the regulator of a special element coming from modular parametrisations. However, in this case a lot less is known about $\Sigma_1$, it is not known even whether it is torsion and at present it is not clear whether one can apply Thaine’s method to construct annihilators. As far as we are aware, there is no construction of elements of $H^1_{\text{BM}}(E \times E', \mathbb{Q}(2))_{\mathbb{Z}}$ without using the modularity except over a local field by Spiess. Such a construction could suggest how to find an analogue of the ‘fundamental unit’.

Curiously it appears that for the imaginary quadratic and isogenous cases, it is easy to use the special element to construct annihilators but hard to relate to $L$-values, while in the real quadratic and non-isogenous cases, it is easy to relate the special elements to $L$-values, but hard to construct annihilators.

Sinnott has a uniform theory of cyclotomic units and Stickelberger elements and it is conceivable that something along those lines would generalise.

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2 The Rankin-Selberg Method

2.1 Preliminaries

Let $E$ and $E'$ be the two elliptic curves over $\mathbb{Q}$. Let $\omega_E$ and $\omega_{E'}$ be Neron differentials corresponding to the global minimal Weierstrass models. These are defined up to $\pm 1$. Let $f$ and $g$ be the modular forms of weight 2 of levels $N_1$ and $N_2$ corresponding to $E$ and $E'$ respectively. Let $N = \text{l.c.m}(N_1, N_2)$. We will think of $f$ and $g$ as modular forms for $\Gamma_0(N)$. Let $\phi$ and $\phi'$ be the modular parametrisations from $X_0(N)$ to $E$ and $E'$ respectively. Define $c(\phi)$ and $c(\phi')$ in $\mathbb{Q}^*$ by

$$\phi^*(\omega_E) = c(\phi)2\pi if(z)dz \quad \text{and} \quad \phi'^*(\omega_{E'}) = c(\phi')2\pi ig(z)dz$$

where by $i$ we denote a choice of a $\sqrt{-1}$ that we make once and for all. It turns out that $c(\phi)$ and $c(\phi')$ are actually in $\mathbb{Z}\backslash\{0\}$. 

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Let
\[ f(z) = \sum_{n=1}^{\infty} a_n q^n \] and \[ g(z) = \sum_{n=1}^{\infty} b_n q^n \]
be the Fourier expansions at \( \infty \) of \( f \) and \( g \), where \( q = e^{2\pi iz} \). These modular forms are *normalised* in the sense that they are eigenfunctions for all the Hecke operators for \( p \nmid N \) and for the Fricke involution, and \( a_1 = b_1 = 1 \).

Let \( X_0(N) \) denote the compactification of the fundamental domain for \( \Gamma_0(N) \). Let
\[ \delta(f,g) = \int_{X_0(N)} \overline{f(z)g(z)} dzd\bar{z} \]
where \( z = x + iy \). Define the **Petersson Inner product** by
\[ (f,g) = \frac{1}{[\Gamma : \Gamma_0(N)]} \int_{X_0(N)} \delta(f,g) \]

We will use the following two theorems of Ogg \[ Og \].

**Theorem 2.1 (Ogg).** If \( f \) and \( g \) are normalised of levels \( N_1 \) and \( N_2 \) respectively and \( (f,g) \neq 0 \) then \( f = g \) (and \( N_1 = N_2 \)).

**Theorem 2.2 (Ogg).** If \( f = \sum_{n=1}^{\infty} a_n q^n \) is a normalised cusp form of square-free level \( N \) and \( p | N \) then
\[ a_p = \pm 1 \]

Let \( L(H^2(E \times E', s)) \) be the \( L \)-function of the product of the two elliptic curves. Then one has
\[ L(H^2(E \times E'), s) = \zeta(s - 1)^2 H(s - 1) L_{f,g}(s - 1) \]
where \( \zeta(s) \) is the Riemann Zeta function, \( \zeta_N(s) \) is the same function with the primes dividing \( N \) removed, \( H(s) \) is a polynomial in \( p^{-s} \) coming from the primes dividing \( N \) and
\[ L_{f,g}(s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_n b_n}{n^s} \]
2.2 Rankin-Selberg Convolution

We use the Rankin-Selberg convolution to get an integral representation of the $L$-function.

Let $f, g$ and $z$ be as above. Then

$$\int_{\Re s > \frac{1}{2}} f(z)g(z)dx = \sum_{n=1}^{\infty} a_n b_n e^{-4\pi n y}$$

and so, integrating over the region $y > 0$ as well,

$$\int_{|x| \leq \frac{1}{2}} f(z)g(z)y^{s-1}dx = (4\pi)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{-s}}$$

Replacing $s$ by $s + 1$ we have

$$(4\pi)^{-s-1} \Gamma(s + 1) \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{-(s+1)}} = \int_{|x| \leq \frac{1}{2}} y^s \delta(f, g)$$

(2.2)

The region $|x| \leq \frac{1}{2}$ is the fundamental domain for the stabilizer of the cusp $\infty$

$$\Gamma_\infty = \Gamma_0(N)_\infty = \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} | n \in \mathbb{Z} \}$$

so one has

$$\int_{|x| \leq \frac{1}{2}} y^s \delta(f, g) = \int_{h/\Gamma_\infty} y^s \delta(f, g) = \int_{X_0(N)} E^N_\infty(z, s) \delta(f, g)$$

where $E^N_\infty(z, s)$ is the Eisenstein Series

$$E^N_\infty(z, s) = \sum_{\gamma \in \Gamma_0(N)/\Gamma_\infty} (Im(\gamma z))^s = 1 + \sum_{m>0} \sum_{(mN, n)=1} \frac{y^s}{|mNz + n|^{2s}}$$

Let

$$\zeta_N(s) = \sum_{n>0} \sum_{(n, N)=1} n^{-s} = \prod_{p \mid N} (1 - p^{-s})^{-1}$$

One then has

$$2\zeta_N(2s) E^N_\infty(z, s) = \sum_{d \mid N} \mu(d) d^s E_{\infty}(\frac{Nz}{d}, s)$$
where \( \mu(d) \) is the Möbius function and
\[
E_\infty(z, s) = \sum_{m,n} y^s \frac{1}{|mz+n|^2s}
\]
Let
\[
L_{f,g}(s) = \zeta(N(2s)) \sum_{n=1}^{\infty} a_n b_n n^{-(s+1)}
\]
Substituting this into the equation 2.2 we get
\[
2(4\pi)^{-s-1}\Gamma(s+1)L_{f,g}(s) = \sum_{d|N} \frac{\mu(d)}{d^{2s}} \int_{\mathcal{X}_0(N)} \delta(f,g)E_\infty(\frac{Nz}{d}, s) \quad (2.3)
\]

2.2.1 The Epstein-Zeta Function

The function
\[
E_\infty(z, s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \text{Im}(\gamma z)^s = \sum_{m,n} y^s \frac{1}{|mz+n|^2s}
\]
appears in many different guises and is sometimes known as the Epstein Zeta Function or an Eisenstein-Kronecker-Lerch Series. It converges for \( \text{Re}(s) > 1 \) and has a meromorphic continuation to the entire complex plane. Further, the function
\[
E^*_\infty(z, s) = \left(\frac{1}{\pi}\right)^s \Gamma(s)E_\infty(z, s)
\]
satisfies the functional equation
\[
E^*_\infty(z, s) = E^*_\infty(z, 1-s)
\]
and has a simple pole with residue 1 at \( s = 1 \) independent of \( z \). A good reference for all these facts is Lang’s Elliptic Functions [La].

2.2.2 An Integral Representation

Multiplying by \( (\frac{N}{\pi})^2 \Gamma(s) \) and using the function \( E^*_\infty(z, s) \) in equation 2.3 above gives
\[
\Phi(s) := (\frac{2\pi}{\sqrt{N}})^{-2s}\Gamma(s)\Gamma(s+1)L_{f,g}(s) = 2\pi \sum_{d|N} \frac{\mu(d)}{d^{2s}} \int_{\mathcal{X}_0(N)} \delta(f,g)E^*_\infty(\frac{Nz}{d}, s)
\]
Since the residue of \( E^*_\infty(z, s) \) at \( s = 1 \) is 1 independent of \( z \) and it is a simple pole one sees that the residue of \( \Phi(s) \) at \( s = 1 \) is a constant times \((f,g)\). From that one has
Theorem 2.3 (Rankin).

\[ L_{f,g}(s) = \zeta_N(2s) \sum_{n=1}^{\infty} a_n b_n n^{-(s+1)} \]

is entire if \((f,g) = 0\) and is entire except for a simple pole at \(s = 1\) if \((f,g) \neq 0\). In this case the residue is a rational number times \((f,g)\).

3 The Tate Conjecture for \(E \times E'\)

The Tate conjecture amounts to the following two statements:

- \(L(H^2(E \times E'), s)\) has a pole of order 3 at \(s = 2\) when \(E\) and \(E'\) are isogenous.
- \(L(H^2(E \times E'), s)\) has a pole of order 2 at \(s = 2\) when \(E\) and \(E'\) are not isogenous.

Note that it does not matter if \(E\) has complex multiplication as we are looking at the cycles defined over \(\mathbb{Q}\) and the extra cycle is only defined over the field of CM.

Since \(\zeta(s)\) has a simple pole at \(s = 1\), using 2.1 the conjecture reduces to the statements that

- \(L_{f,g}(s)\) has a simple pole at \(s = 1\) if \(E\) and \(E'\) are isogenous.
- \(L_{f,g}(s)\) is holomorphic and non-vanishing at \(s = 1\) if \(E\) and \(E'\) are not isogenous.

From Rankin’s theorem one has that \(L_{f,g}(s)\) has a simple pole at \(s = 1\) when \(E\) and \(E'\) are isogenous as \((f,g) \neq 0\). When \(E\) and \(E'\) are not isogenous, \((f,g) = 0\) hence there is no pole at \(s = 1\).

To complete the proof of the Tate conjecture, we use the following theorem of Ogg’s [Og]

Theorem 3.1 (Ogg). \(L_{f,g}(1) \neq 0\) if \((f,g) = 0\).

The proof of this theorem is by using the Euler product for \(L_{f,g}\) to construct a Dirichlet series with positive real coefficients which does not have a pole contradicting the fact that such a Dirichlet series has a pole on the real point of the critical line. Details can be found in Ogg’s paper [Og].
4 Beilinson’s Conjecture for $E \times E'$

To verify Beilinson’s conjecture, we first have to get some understanding of what the integral motivic cohomology groups are. In the following sections, we describe the group $H^3_M(X, \mathbb{Q}(2))$, where $X$ is a surface defined over $\mathbb{Q}$. The integral motivic cohomology group $H^3_M(X, \mathbb{Q}(2))_\mathbb{Z}$ is a certain subgroup of this group, first described conditionally by Beilinson, though more recently unconditionally by Scholl [Sc].

4.1 Elements of $H^3_M(X, \mathbb{Q}(2))$

Let $X$ be a surface defined over $\mathbb{Q}$. The group $H^3_M(X, \mathbb{Q}(2))$ has several different descriptions: First, in terms of a graded piece of $K_1(X)$, second, as the higher Chow group $CH^2(X, 1)$ and finally, as the $K$-cohomology group $H^1(X, K_2)$. From the third description and the Gersten-Quillen resolution, an element of the group is represented by a formal sum

$$\sum (C, f)$$

where $C$ are curves on $X$ and $f$ are functions on these curves subject to the cocycle condition

$$\sum \text{div}(f) = 0$$

This is a generalization of the fact that elements of $F^*$ are elements of $K_1$ of a number field $F$.

4.1.1 Construction of the elements on products of curves

We construct some elements of the group $H^3_M(X, \mathbb{Q}(2))$ when $X$ is the self product of a curve $C$. We use a construction of Bloch’s which was generalized by Beilinson [Be].

Let $C$ be a curve which contains a set $S$ such that any divisor of degree 0 supported on $S$ is torsion in the jacobian of $C$. Let $Y = C \setminus S$. The condition above can be stated as the statement that the exact sequence

$$0 \to H^1_M(C, \mathbb{Q}(1)) \to H^1_M(Y, \mathbb{Q}(1)) \xrightarrow{\partial} (H^0_M(S, \mathbb{Q}(0)))_0 \to 0 \quad (4.1)$$

splits as a sequence of motivic cohomology groups as the class in the Chow group of a divisor of degree 0 can be described in terms of such an extension. The groups $H^1_M(C, \mathbb{Q}(1))$ and $H^1_M(Y, \mathbb{Q}(1))$ are simply $O_C^* \otimes \mathbb{Q}$ and $O_Y^* \otimes \mathbb{Q}$ respectively and

$$(H^0_M(S, \mathbb{Q}(0)))_0 = (z_1, z_2, ...,) \in \bigoplus_{s \in S} \mathbb{Z}$$

such that $\sum z_i = 0$. 

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Splitting is the statement that for a divisor $D$ of degree 0 supported on $S$, there is a canonical choice of a function $\epsilon(D)$ whose divisor is $D$.

One then has the following Lemma:

**Lemma 4.1.** The sequence

$$0 \to H^3_M(C^2, \mathbb{Q}(2)) \to H^3_M(C^2 \setminus S^2, \mathbb{Q}(2)) \xrightarrow{\partial} (H^0_M(S^2, \mathbb{Q}(0)))_0 \to 0 \quad (4.2)$$

splits as a sequence of motivic cohomology groups.

**Proof.** The idea is to use the splitting in the first case, equation (4.1), to split it in this case. We need to produce a canonical element of $H^3_M(C^2 \setminus S^2, \mathbb{Q}(2))$ for any 0-cycle of degree 0 supported on $S^2$. It suffices to do this for cycles of the form $D = (P_1, P_2) - (Q_1, Q_2)$. We use a trick familiar to all first year students of calculus.

$$(P_1, P_2) - (Q_1, Q_2) = (P_1, P_2) - (Q_1, P_2) + (Q_1, P_2) - (Q_1, Q_2)$$

For the pair $(P_1, P_2) - (Q_1, P_2)$ we take the element $(Y \times P_2, \epsilon(P_1 - Q_1))$ and similarly, for the pair $(Q_1, P_2) - (Q_1, Q_2)$ we take the element $(Q_1 \times Y, \epsilon(P_2 - Q_2))$. The sum of these two elements give the canonical lift of $D$. A different way of splitting $D$ gives the same element as two such liftings differ by something coming from the tame symbol, which is a coboundary. Since any divisor of degree 0 can be written as a sum of such $D$, this gives a splitting.

$$\square$$

For a divisor $D$ of degree 0 on $S^2$, let $\epsilon^2(D)$ denote the lifting. Now suppose one has a map $\Psi : X \to X^2$ such that

- $\Psi(Y) \subset X^2 \setminus S^2$
- $\Psi(S) \subset S^2$

One then has an induced pushforward,

$$\Psi_* : H^1_M(Y, \mathbb{Q}(1)) \to H^3_M(X^2 \setminus S^2, \mathbb{Q}(2))$$

so if $D$ is a divisor of degree 0 on $S$, one gets an element

$$\Psi_*(\epsilon(D)) \in H^3_M(X^2 \setminus S^2, \mathbb{Q}(2))$$

The simplest example of such a map $\Psi$ is the diagonal embedding.
Now if $D$ is a divisor of degree 0 on $S$ one gets two elements, $\epsilon^2(\Psi(D))$ and $\Psi_*(\epsilon(D))$ in $H^3_{\overline{\mathcal{M}}}(X^2\backslash S^2, \mathbb{Q}(2))$. The element $\chi(D) = \epsilon^2(\Psi(D)) - \Psi_*(\epsilon(D))$ satisfies the condition
\[ \partial(\epsilon^2(\Psi(D)) - \Psi_*(\epsilon(D))) = \Psi(\epsilon(D)) - \Psi(D) = 0 \]
Hence it lifts to give an element of $H^3_{\overline{\mathcal{M}}}(X^2, \mathbb{Q}(2))$.

4.1.2 Products of modular elliptic curves

Returning to our case, let $N = \text{l.c.m}(N_1, N_2)$ where $N_1$ and $N_2$ are the conductors of $E$ and $E'$ respectively, and let $X_0(N)$ be the modular curve of level $N$. One then has modular parametrisations,
\[ \phi : X_0(N) \rightarrow E \quad \text{and} \quad \phi' : X_0(N) \rightarrow E' \]
We take $C = X_0(N)$. By the Manin-Drinfel’d theorem, any divisor of degree 0 supported on the set of cusps is torsion, so we can take this set as the set $S$. Applying the above lemma, we can construct elements of $H^3_{\overline{\mathcal{M}}}(X_0(N)^2, \mathbb{Q}(2))$ and using the modular parametrisations, we can push these elements down to $H^3_{\overline{\mathcal{M}}}(E \times E', \mathbb{Q}(2))$.

4.1.3 Remarks on Integrality

Beilinson’s conjecture is about the integral motivic cohomology. This is a subspace of the motivic cohomology which was originally defined to be the image of the motivic cohomology of a regular proper model, if it exists. Scholl [Sc] gave an unconditional definition of this subspace, denoted by $H^3_{\overline{\mathcal{M}}}(X, \mathbb{Q}(2))_{\mathbb{Z}}$, using De Jong’s theory of alterations. This is the analogue of $\mathcal{O}^*_K$ as opposed to $K^*$ where $K$ is a number field.

In general the elements we construct using the above method do not lie in the integral motivic cohomology. However, Scholl [Sc] showed that the projection of the elements of $H^3_{\overline{\mathcal{M}}}(X_0(N) \times X_0(N), \mathbb{Q}(2))$ onto the $H^3_{\overline{\mathcal{M}}}(E \times E', \mathbb{Q}(2))$ lie in $H^3_{\overline{\mathcal{M}}}(E \times E', \mathbb{Q}(2))_{\mathbb{Z}}$ if $E$ and $E'$ are not isogenous. Further Harris and Scholl [Ha-Sc] show that this subspace of $H^3_{\overline{\mathcal{M}}}(E \times E', \mathbb{Q}(2))_{\mathbb{Z}}$ is zero dimensional if $E$ and $E'$ are isogenous and at most one dimensional if $E$ and $E'$ are not isogenous. It is not known if $H^3_{\overline{\mathcal{M}}}(E \times E', \mathbb{Q}(2))_{\mathbb{Z}}$ is even finitely generated.

To show that it is at least one, we will use the fact that if the regulator of an element is non-zero, then the element cannot be zero. This regulator turns out to be the value $L'_{f,g}(0)$. From Theorem 3.1 and the functional
equation, we know that this is non-zero and we can conclude that part of the Beilinson conjecture. Towards that end, in the next section we will describe this regulator.

If one does not require integrality, Flach [Fl] and Mildenhall [Mi] have shown independently that the group $H^3_M(E \times E', \mathbb{Q}(2))$ is infinitely generated.

4.2 Regulator Maps and the Real Deligne Cohomology

Let $X$ be a smooth projective variety over $\mathbb{Q}$, $i$ an even integer and $m = i/2$. In this section we describe what the Real Deligne cohomology groups are and explain what the constants $c_X(m)$ are. Details can be found in the articles of Esnault-Viehweg and Schneider in [SSR].

The regulator map $\tilde{r}_D$ has two components, $\tilde{r}_D = r_D \oplus z_D$, where

- $z_D$ is a cycle class map induced by the usual cycle class map to De Rham cohomology,
  
  $$z_D : B^m(X)_\mathbb{Q} \longrightarrow H^{i+1}_D(X/\mathbb{R}, \mathbb{R}(m+1))$$

- $r_D$ is a higher cycle class map, generalizing Dirichlet’s regulator map for units,
  
  $$r_D : H^{i+1}_M(X, \mathbb{Q}(m+1))_{\mathbb{Z}} \longrightarrow H^{i+1}_D(X/\mathbb{R}, \mathbb{R}(m+1))$$

Here $H^{i+1}_D(X/\mathbb{R}, \mathbb{R}(m+1))$ is the Real Deligne cohomology described below.

4.2.1 Real Deligne Cohomology

The Real Deligne cohomology is a real vector space generalising the vector space $\mathbb{R}^{r_1+r_2}$, which appears as the target space for Dirichlet’s regulator map. While the precise definition of the Real Deligne cohomology is a little involved, there are two key properties:

- (i) There is an exact sequence,
  
  $$0 \rightarrow F^{m+1}H^i_{DR}(X/\mathbb{R}) \rightarrow H^i_B(X(\mathbb{C}), \mathbb{R}(m))(-1)^m \rightarrow$$
  
  $$\rightarrow H^{i+1}_D(X/\mathbb{R}, \mathbb{R}(m+1)) \rightarrow 0$$

  (4.3)
where the $-1$ indicates that it is a $-1$ eigenspace for the involution induced by complex conjugation on the complex manifold $X(\mathbb{C})$, $H_B$ is the Betti (singular) cohomology and $H_{DR}$ is the algebraic De Rham cohomology.

- (ii) The dimension is related to the order of vanishing of $L$-functions:

$$
\dim_{\mathbb{R}} H_{DR}^{i+1}(X_{/\mathbb{R}}, \mathbb{R}(m+1)) = \text{ord}_{s=m} L(H^i(X, s)) - \text{ord}_{s=m+1} L(H^i(X, s))
$$

### 4.2.2 The period $c_X(m)$

From 4.3 there is an isomorphism of one dimensional vector spaces

$$
det(F^{m+1}H_{DR}^i(X_{/\mathbb{R}})) \otimes det(H_{DR}^{i+1}(X_{/\mathbb{R}}, \mathbb{R}(m+1))) \simeq det(H_B^i(X(\mathbb{C}), \mathbb{R}(m))^{(-1)^m}
$$

As $X$ is defined over $\mathbb{Q}$, $F^{m+1}H_{DR}^i(X_{/\mathbb{R}})$ has a rational structure coming from the algebraic De Rham cohomology. $H_B^i(X(\mathbb{C}), \mathbb{R}(m))^{(-1)^m}$ has an obvious rational structure. Part (i) of the Beilinson conjecture asserts that $\text{Im}(\tilde{r}_D)$ gives a rational structure on the Deligne cohomology $H_{DR}^{i+1}(X_{/\mathbb{R}}, \mathbb{R}(m+1))$. So all the vector spaces involved have rational structures, at least conjecturally. $c_X(m)$ is the determinant of the isomorphism above computed with respect to these $\mathbb{Q}$-structures. It is an element of $\mathbb{R}^*/\mathbb{Q}^*$.

### 4.2.3 Explicit formulae for the Regulator map

The regulator map has the following explicit description as a current on $(m, m)$ forms:

- On $B^m(X)_\mathbb{Q}$ it is given by a current of integration. If $Z$ is an element of $B^m$ and $\omega$ is a $(\dim(X) - m, \dim(X) - m)$ form in $H_{DR}^i(E \times E')$ then

$$
(z_D(Z), \omega) := (\frac{1}{2\pi i})^m \int_Z \omega \tag{4.4}
$$

- On the motivic cohomology side. If $\sum(C, f)$ is an element of $H_{\mathcal{M}}^{i+1}(E \times E', \mathbb{Q}(m+1))$ and $\omega$ is a $(\dim(X) - m, \dim(X) - m)$ form then

$$
(r_D(\sum(C, f)), \omega) := (\frac{1}{2\pi i})^m \sum \int_C \log |f| \omega \tag{4.5}
$$
This regulator map is conjecturally injective. However, it is clear that if 
\( r_D(\sum (C, f), \omega) \neq 0 \) for some \( \omega \), then the element is non-trivial in \( H^{i+1}_M(E \times E', \mathbb{Q}(m)) \).

We are interested in the case \( X = E \times E' \) and \( i = 2 \) so \( m = 1 \). In this case the corresponding Deligne cohomology is 3 dimensional. However, it further breaks down according to the motivic decomposition described in the next section.

5 Calculation of \( L \)-values

5.1 Motivic Decomposition

To compute the \( L \)-values we first observe that we can use the Künneth formula to get a decomposition of the motive \( H^2(E \times E') \).

The motive \( H^2(E \times E') \) splits up into 4 or 3 submotives depending on whether \( E \) is isogenous or not to \( E' \). From the Künneth formula one has

\[
H^2(E \times E') = H^2(E) \oplus H^2(E') \oplus H^1(E) \otimes H^1(E')
\]  

(5.1)

If \( E \simeq E' \) then the motive \( H^1(E) \otimes H^1(E') \) further splits up into

\[
H^1(E) \otimes H^1(E') = \Lambda^2 H^1(E) \oplus \text{Sym}^2 H^1(E)
\]

(5.2)

and \( \Lambda^2 H^1(E) \simeq H^2(E) \).

As this decomposition is at the level of motives, all the corresponding objects such as \( L \)-functions and the constants \( c_X(1) \) also decompose and we can reduce the problem to verifying each case individually.

The case of real interest to us is that of \( L(H^1(E) \otimes H^1(E'), s) \) when \( E \) and \( E' \) are not isogenous, as the other cases, namely \( L(H^2(E)) \), \( L(H^2(E')) \) and \( L(\text{Sym}^2(H^1(E))) \) either reduce to the cases of fields, as in the first two cases, or have been treated in detail elsewhere [Fl], as in the third. However, for completeness we will describe them.

5.1.1 The motives \( H^2(E) \) and \( H^2(E') \)

In either case here the Deligne cohomology is 1 dimensional as \( H^2(E) \) is 1 dimensional, complex conjugation acts by \(-1\) and \( F^2 H^2_{DR} = 0 \). From the exact sequence [4.3] we have \( c_{H^2(E)}(1) \) is given by

\[
\frac{1}{2\pi i} \int_E \alpha
\]
where $\alpha$ is a rational De Rham cohomology class. Such a form is obtained by $\omega_E \wedge \eta_E$ where $\omega_E$ is the canonical differential and $\eta_E$ is the form $\frac{XdX}{Y}$. The Legendre relation shows

$$\frac{1}{2\pi i} \int_E \omega_E \wedge \eta_E = 1$$

so it is rational.

The $L$-function is $L(H^2(E), 1) = \zeta(s - 1)$ and so

$$L(H^2(E), 1) = \zeta(0) = \frac{1}{2} \sim c_{H^2(E)}(1)$$

### 5.1.2 Functional Equation

For the other two cases, we need the functional equation of the $L$-function $L_{f,g}(s)$ to compute the value at $s = 1$. Here we make the further assumption that $N_1$ and $N_2$ and hence $N = \text{l.c.m.}(N_1, N_2)$ are square-free. Then one has

**Theorem 5.1 (Ogg).** Let $f$ and $g$ be normalized cusp forms of square-free level $N_1, N_2$; Let $N = \text{l.c.m.}(N_1, N_2)$ and $M = \text{g.c.d.}(N_1, N_2)$. For $p | M$, let

$$c_p = a_pb_p = \pm 1$$

Recall

$$\Phi(s) := (\frac{2\pi}{\sqrt{N}})^{-2s}\Gamma(s)\Gamma(s + 1)L_{f,g}(s)$$

and set

$$\Phi^+(s) = \Phi(s)A(s)$$

where

$$A(s) = \prod_{p|M}(1 - c_pp^{-s})^{-1}$$

Then

$$\Phi^+(s) = \Phi^+(1 - s)$$

A generalization of this theorem for arbitrary $N$ was proved by [Li]. The $L$-function of $H^1(E) \otimes H^1(E')$ is

$$L(H^1(E) \otimes H^1(E'), s) = H(s - 1)L_{f,g}(s - 1)$$

where $H(s)$ is a term depending on primes dividing $N$. In general it can be quite complicated, but the following ad hoc definition, due to Ogg [Og], seems to make the formulae cleaner.
If \( p \mid M \), then the factor is
\[
\frac{1}{(1 - c_pp^{-s})(1 - c_pp^{-(s+1)})}
\]
while if \( p \mid N_1 \), \( p \nmid N_2 \), it is
\[
\frac{1}{(1 - a_pb_pp^{-(s+1)} + p^{-1-2s})}
\]
at least when \( E \) has multiplicative reduction at \( p \).

5.2 Assume \( E \) isogenous to \( E' \)

In this case, the motivic decomposition shows that we have to understand the case of \( \text{Sym}^2(E) \). This is a critical motive in the sense of Deligne [De] as the Deligne cohomology vanishes. The period turns out to be [De]

\[
c_{\text{Sym}^2(E)}(1) = \frac{1}{2\pi i} \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_E}
\]

(5.5)

where \( \omega_E \) is the canonical differential.

Here \( N_1 = N_2 = N \) and \( f = g \) so \( c_p = a_p^2 = 1 \) for all \( p \mid N \). Let \( m \) be the number of primes dividing \( N \). From Rankin’s theorem, \( L_{f,g}(s) = L_{f,f}(s) \) has a pole at \( s = 1 \). From the motivic decomposition, we have

\[
L(H^1(E) \otimes H^1(E), s) = L(H^2(E), s)L(\text{Sym}^2(E), s)
\]

so from 5.4 above, we have

\[
L(\text{Sym}^2(E), s) = \frac{H(s-1)L_{f,f}(s-1)}{\zeta(s-1)}
\]

(5.6)

From the functional equation we get

\[
\Phi^+(0) = \Gamma(0)\Gamma(1)L_{f,f}(0)A(0) = A(0)\Phi(0)
\]

and \( A \) has a pole of order \( m \) at 0.

From the integral expression for \( \Phi(s) \) we have,

\[
\Phi(0) = \lim_{s \to 0} 2\pi \sum_{d \mid N} \frac{\mu(d)}{d^s} \int_{X_0(N)} \delta(f, f) E_\infty^*(\frac{Nz}{d}, s)
\]

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Since $E^*_\infty(\frac{Nz}{d},0)$ has a simple pole with residue 1 at $s = 0$ and

$$
\sum_{d|N} \mu(d) = \prod_{p|N} (1 - 1) = \frac{1}{\mathcal{A}(0)}
$$

one has

$$
L_{f,f}(0) = \frac{\Phi(0)}{\Gamma(0)} = 2\pi \lim_{s \to 0} \frac{1}{s} \int_{X_0(N)} \delta(f,f) E^*_\infty(\frac{Nz}{d},s) = 2\pi (f,f)[\Gamma : \Gamma_0(N)]
$$

(5.7)

From the relation between the canonical differential and the modular form one has

$$
(f,f) = \frac{\text{ideg}(\phi)}{8\pi^2 c(\phi)^2} \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_{E'}}
$$

Using this in the formula (5.7) and the expression (5.5) one gets

$$
L_{f,f}(0) = \frac{-\text{deg}(\phi)\mathcal{H}(0)[\Gamma : \Gamma_0(N)]}{2c(\phi)^2} \frac{1}{2\pi i} \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_{E'}} \sim \mathbb{Q}^* c_{\text{Sym}^2(E)}(1)
$$

(5.8)

Finally we have, from (5.4) and (5.5),

$$
L(\text{Sym}^2 E,1) = \frac{-\text{deg}(\phi)\mathcal{H}(0)[\Gamma : \Gamma_0(N)]}{c(\phi)^2} \frac{1}{2\pi i} \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_{E'}} \sim \mathbb{Q}^* c_{\text{Sym}^2(E)}(1)
$$

(5.9)

which is precisely what the conjecture predicts.

### 5.3 Assume $E$ is not isogenous to $E'$

In this case the $L$-function is

$$
L(H^1(E) \otimes H^1(E'),s) = H(s - 1)L_{f,g}(s - 1)
$$

(5.10)

The Deligne cohomology is one dimensional and it turns out that the period $c_{H^1(E)}(1)$ is the regulator of an element of the motivic cohomology. So we have to show that the value of $L'(H^1(E) \otimes H^1(E'),1)$ is rational up to the regulator of such an element. From (3.1) we know this value is non-zero, so that will also show that the element is non-zero.

We assume for simplicity that $N_1$ and $N_2$ are coprime so $\Phi^+(s) = \Phi(s)$ and hence $\Phi(s) = \Phi(1 - s)$. Since

$$
\Phi(0) = \Gamma(0)\Gamma(1)L_{f,g}(0)
$$

and $\Gamma(0)$ has a simple pole and $L_{f,g}(0) = 0$, we have

$$
L'_{f,g}(0) = \Phi(0)
$$

(5.11)
5.3.1 The Kronecker Limit Formula

To compute $\Phi(0)$ we need **Kronecker’s First Limit Formula**. This allows us to compute the constant term in the Laurent series expansion of $E_\infty(z, s)$.

**Theorem 5.2 (Kronecker).** Let

$$E_\infty(z, s) = \sum_{m,n} \frac{y^s}{|mz+n|^{2s}}$$

and let

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

where $q = e^{2\pi iz}$. Let $\gamma$ be Euler’s constant. Then

$$E_\infty(z, s) = \frac{\pi}{s-1} - \pi \log y + 2\pi(\gamma - \log 2) - 4\pi \log |\eta(z)| + O(s-1)$$  \hspace{1cm} (5.12)

**Proof.** The proof of this theorem can be found in Lang [La].

Recall

$$E_\infty^*(z, s) = \frac{1}{\pi^s \Gamma(s)} E_\infty(z, s)$$

From the functional equation, we have

$$E_\infty^*(\frac{Nz}{d}, s) = E_\infty^*(\frac{Nz}{d}, 1-s)$$

Combining this with the limit formula \(5.12\), we have

$$\lim_{s\to 0} E_\infty^*(\frac{Nz}{d}, s) = \lim_{s\to 0} E_\infty^*(\frac{Nz}{d}, 1-s)$$

$$= \frac{1}{s} - \log \frac{Ny}{d} + 2(\gamma - \log 2) - 4 \log |\eta(\frac{Nz}{d})| + O(-s)$$

From this it follows that

$$\lim_{s\to 0} \sum_{d|N} \frac{\mu(d)}{d^s} E_\infty^*(\frac{Nz}{d}, s) = \lim_{s\to 0} \sum_{d|N} \frac{\mu(d)}{d^s} E_\infty^*(\frac{Nz}{d}, 1-s)$$

$$= \sum_{d|N} -4 \log |\eta(\frac{Nz}{d})|$$
As all the other terms vanish from the facts that, for $N > 1$,

$$\sum_{d|N} \mu(d) = 0 \text{ and } \prod_{d|N} \left(\frac{N}{d}\right)^{\mu(d)} = 1$$

Recall,

$$\Phi(s) = 2\pi \sum_{d|N} \frac{\mu(d)}{d^s} \int_{X_0(N)} E_\infty' \left(\frac{Nz}{d}, s\right) \delta(f, g)$$

Using the above result we have

$$\Phi(0) = 2\pi \int_{X_0(N)} \sum_{d|N} -4 \log |\eta(\frac{Nz}{d})| \delta(f, g) \quad (5.13)$$

Let $\Delta(z)$ be the usual cusp form of weight 12 for $SL_2(\mathbb{Z})$. We have $\Delta(z) = \eta(z)^{24}$. Define

$$\Delta_N(z) := \prod_{d|N} \Delta' \left(\frac{Nz}{d}\right)^{\mu(d)}$$

This is a modular unit as $\sum_{d|N} \mu(d) = 0$ and its divisor is supported on the cusps.

Thus, the Eisenstein series tends to $\frac{1}{24} \log |\Delta_N(z)|$ and one has

$$\Phi(0) = \frac{-\pi}{3} \int_{X_0(N)} \log |\Delta_N(z)| \delta(f, g) \quad (5.14)$$

5.3.2 A natural element of $H^3_M(X_0(N) \times X_0(N), \mathbb{Q}(2))$

Let $D$ be the divisor of $\Delta_N$. Then $\epsilon(D) = \Delta_N$ and one can consider the element $\chi(D)$ in $H^3_M(X_0(N) \times X_0(N), \mathbb{Q}(2))$ constructed in section 5.1. There is a map

$$\psi : X_0(N) \times X_0(N) \to E \times E'$$

induced by the modular parametrizations $\phi$ and $\phi'$ and one has the element $(\psi_*(\chi(D)))$ in $H^3_M(E \times E', \mathbb{Q}(2))$.

Let $\omega_E$ and $\omega_{E'}$ be the canonical differentials on $E$ and $E'$ respectively. Then one has $\phi^*(\omega_E) = 2\pi i c(\phi) f(z) dz$ and $\phi'^*(\omega_{E'}) = 2\pi i c(\phi') g(z) dz$. So from that,

$$\delta(f, g) = \frac{\deg(\psi)}{8\pi^2 i c(\phi)c(\phi')} \omega_E \wedge \overline{\omega_{E'}}$$

where $\deg(\psi)$ denotes the degree of the map $\psi|_{\text{diagonal}}$. 

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Therefore one has

\[
L'_{f,g}(0) = \frac{\pi}{3} \int_{X_0(N)} \log |\Delta_N(z)| \delta(f,g) \quad (5.15)
\]

\[
= \frac{2\pi^2 i}{3} \frac{1}{2\pi i} \int_{X_0(N)} \log |\Delta_N(z)| \delta(f,g) \quad (5.16)
\]

\[
= \frac{2\pi^2 i}{3} \frac{1}{2\pi i} \int_{\psi_*(X_0(N))} \log |\psi_*(\Delta_N(z))| \frac{\deg(\psi)}{8\pi^2 i c(\phi)c(\phi')} \omega_E \wedge \overline{\omega_{E'}} \quad (5.17)
\]

Using the explicit description of the regulator map \[4.3\] we have

\[
L'_{f,g}(0) = -\frac{\deg(\psi)}{12\phi(c(\phi') (r_D(\psi_*(\chi(D))), \omega_E \wedge \overline{\omega_{E'}}) \quad (5.18)
\]

Since \(L'(H^1(E) \otimes H^1(E'), 1)) = H(0)L'_{f,g}(0)\) we have

\[
L'(H^1(E) \otimes H^1(E'), 1)) = -\frac{\deg(\psi)H(0)}{12\phi(c(\phi')} (r_D(\psi_*(\chi(D))), \omega_E \wedge \overline{\omega_{E'}}
\sim \mathcal{Q} c_{H^1(E) \otimes H^1(E')}^{1)}
\]

### 5.4 A ‘class number formula’

To add credence to our claim that the element we have is an analogue of a cyclotomic unit, we show there is a ‘class number formula’ analogous to the expression \[1.1\] for \(\zeta_K'\).

We have the following curious product formula for our function \(\Delta_N(z)\) which can be found in Asai \[\text{As}\].

\[
\Delta_N(z) = \sum_{d|N} \Delta(Nz/d)^{\mu(d)} = q^{\phi(N)} \prod_{n=1}^{\infty} \Phi_N(q^n)^{24}
\]

where \(\phi(N)\) is Euler’s totient function, \(\Phi_N(X)\) is the \(N^{th}\) cyclotomic polynomial and \(q = e^{2\pi iz}\). This follows from the Möbius inversion formula applied to the situation

\[
\sum_{d|N} \log \Phi_d(X) = \log(1 - X^N)
\]

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The inversion formula implies
\[
\log(\Phi_N(X)) = \sum_{d|N} \mu(d) \log(1 - X^{\frac{d}{N}})
\]
We also have
\[
\Phi_N(X) = \prod_{k \mod N \atop (k,N) = 1} (1 - \xi^k X)
\]
where \( \xi = e^{\frac{2\pi i}{N}} \).

Define the \( q \)-logarithm for \( q = e^{2\pi i z} \) as follows:
\[
\log_q(t) = \frac{1}{24} \log|qt| + \sum_{n=1}^{\infty} \log|1 - q^n t|
\]

Combining this with the formula for \( L'(H^2(E \times E'), 1) \), we get

**Theorem 5.3 (An ‘elliptic class number formula’).** Let \( E, E', f, g \) be as before. We have
\[
L'(H^1(E) \otimes H^1(E'), 1) = \frac{-H(0)}{2} \sum_{k \mod N \atop (k,N) = 1} \frac{1}{2\pi i} \int_{X_0(N)} \log_q(\xi^k f(q) g(q)) \frac{dq \overline{dq}}{q \overline{q}}
\]

**Proof.** From [5.14] and [5.15] we have
\[
L'_{f,g}(0)) = -4\pi^2 i \frac{1}{2\pi i} \int_{X_0(N)} \phi(N) \log|\delta(f, g)|
\]

\[
-4\pi^2 i \frac{1}{2\pi i} \int_{X_0(N)} \sum_{n=1}^{\infty} \sum_{k \mod N \atop (k,N) = 1} \log|1 - \xi^k q^n| \delta(f, g)
\]

So it follows from the definition of \( \log_q(t) \). Note the similarity to [1.1]. \( \square \)

**6 Elements of \( H^{2n-1}_{\mathcal{M}}(E_1 \times E_2 \times \ldots \times E_n, \mathbb{Q}(n)) \)\)**

We can generalise this construction to prove the Beilinson and Tate conjectures for codimension \( n - 1 \) cycles on products of \( n \) modular elliptic curves. We will work it out in detail for \( H^5_{\mathcal{M}}(E_1 \times E_2 \times E_3, \mathbb{Q}(3)) \) of the product of 3
elliptic curves and remark how it generalises. It appears that all the cycles come from exterior products.

Let $E_f, E_g$ and $E_h$ be 3 modular elliptic curves corresponding to the normalised eigenforms $f, g$ and $h$. Let $X = E_f \times E_g \times E_h$. From the description of the real Deligne cohomology, one can see that $H^5_D(X/\mathbb{R}, \mathbb{R}(3))$ is a 6 dimensional.

We have to consider the $L$-function of $H^4$ at $s = 2$. From the Kunneth formula one can see that

$$L(H^4(X), s) = \zeta(s - 2)^3 L_{f,g}(s - 2)L_{f,h}(s - 2)L_{g,h}(s - 2)$$

where $L_{f,g}(s)$ corresponds to the Rankin Selberg convolution of $f$ and $g$.

There are three cases that we have to consider, namely when all, two or none of the three curves are isogenous.

### 6.1 All the elliptic curves are isogenous

In this case there are six elements of the Neron-Severi itself, namely

$$(x, e, e), (e, x, e), (e, e, x)$$

$$(x, x, e), (x, e, x), (e, x, x)$$

There is one more ‘obvious’ element, namely $(x, x, x)$, but there is a homology relation between these 7 elements, giving rise to the ‘modified diagonal cycle’.

From the calculations in the previous section, one has that $L_{f,g}(s - 2), L_{g,h}(s - 2)$ and $L_{f,h}(s - 2)$ all have simple poles are $s = 3$ so that shows that the L function has a pole of order 6 as expected. From the functional equation one sees that $L(H^4(X), s)$ is a nonzero rational number at $s = 2$.

### 6.2 Two are isogenous

Assume only $E_f$ and $E_g$ are isogenous. In this case the conjecture predicts that the rank of the motivic cohomology is 2 while the rank of the Neron-Severi is 4. One has an exterior product map

$$H^3_M(E_f \times E_h, \mathbb{Q}(2)) \otimes H^2_M(E_g, \mathbb{Q}(1)) \rightarrow H^5_M(X, \mathbb{Q}(3))$$

From the earlier section, since $E_f$ and $E_h$ are not isogenous, one has an element of $H^3(E_f \times E_h, \mathbb{Q}(2))$ coming from the modular parametrisation, and one has the rational point on $CH^1(E_g)$. This gives an element of
$H^3_{\mathcal{M}}(X, \mathbb{Q}(3))$. Similarly, using the other pair of non-isogenous elements, one gets the other element. These are non-trivial as from the expression for the $L$-function as a product, the functions $L_{f,h}(s-1)$ and $L_{g,h}(s-1)$ have simple zeroes at $s = 2$ and the value of $L''(H^4(X), s)_{s=2}$ is the product of the regulators, which is the determinant of the regulator matrix.

6.3 None are isogenous

In this case the conjectures predict that there are three independent elements of the motivic cohomology and three independent elements of the Neron Severi.

We use the same argument as above, namely the exterior product and the construction in the previous section to construct the three elements.

Remark 6.1. The same argument can also be used to prove this conjecture in the case of products of modular curves as once again all the interesting elements come from $H^3_{\mathcal{M}}(X_1 \times X_2, \mathbb{Q}(2))$.

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