ON THE HASSE PRINCIPLE FOR THE CHOW GROUPS OF ZERO-CYCLES ON QUADRIC FIBRATIONS

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ABSTRACT. We give a sufficient condition for the injectivity of the global-to-local map of the relative Chow group of zero-cycles on a quadric fibration of dimension \( \leq 3 \) defined over a number field.

1. Introduction

Let \( k \) be a number field and \( \Omega \) the set of its places. For any variety \( X \) over \( k \), \( \text{CH}_0(X) \) denotes the Chow group of zero-cycles on \( X \) modulo rational equivalence. Then we have the global-to-local map

\[
\chi(X) \longrightarrow \prod_{v \in \Omega} \chi(X \otimes_k k_v),
\]

where for each place \( v \in \Omega \), \( k_v \) denotes the completion of \( k \) at \( v \). If there exists a proper morphism \( X \to C \) from \( X \) to another variety \( C \), we also have the relative version of the global-to-local map

\[
\Phi : \chi(X/C) \longrightarrow \prod_{v \in \Omega} \chi(X \otimes_k k_v/C \otimes_k k_v),
\]

where \( \chi(X/C) \) denotes the kernel of the push-forward map \( \chi(X) \to \chi(C) \). In this paper, we study the injectivity of \( \Phi \) for quadric fibrations over curves.

First, let us recall some known results for a surface \( X \). In the case where \( X \) is a conic bundle surface over the projective line \( \mathbb{P}^1_k \), Salberger proved that the kernel of \( \Phi \) is controlled by the Tate-Shafarevich group of the Néron-Severi torus of \( X \) (see [7] for the details). In the case where \( X \) is a quadric fibration of dimension \( \geq 4 \), few results are known. Parimala and Suresh proved that if \( X \to C \) is a quadratic fibration over a smooth projective curve \( C \) whose generic fiber is defined by a Pfister neighbour of rank \( \geq 5 \), then the global-to-local map restricted to real places

\[
\Phi_{\text{real}} : \chi(X/C) \longrightarrow \bigoplus_{v: \text{real place}} \chi(X \otimes_k k_v/C \otimes_k k_v)
\]

is injective [6]. By using this injectivity, they deduced a finiteness result of the torsion subgroup of the Chow group \( \chi(X) \) of zero-cycles on \( X \).

When \( X \to C \) is a quadric fibration of dim \( \leq 3 \), not only that the map \( \Phi_{\text{real}} \) is not injective in general, but also the map \( \Phi \) is not injective [8]. However, the map \( \Phi \) can be injective. If the generic fiber of \( X \to C \) is defined by a quadratic form over a base field \( k \), the map \( \Phi \) is injective (Theorem 3.1). The above condition does not imply the injectivity of \( \Phi_{\text{real}} \), and we give an example of this (Proposition 3.3).
Note that we don’t assume that quadric fibrations are admissible (for the definition of admissibility, see [6]).

**Notation and conventions.** In section 2, \( k \) denotes a field of characteristic different from 2. In section 3, \( k \) denotes a number field (i.e. a finite extension field of \( \mathbb{Q} \)). For a variety \( X \), \(|X|\) denotes the set of closed points on \( X \). We denote by \( \text{CH}_i(X) \) the Chow group of cycles of dimension \( i \) on \( X \) modulo rational equivalence [3]. For a geometrically integral variety \( X \) over \( k \), \( k(X) \) denotes the function field of \( X \). For any extension \( L/k \) of fields, \( L(X) \) denotes the function field of \( X \otimes_k L \). If \( x \) is a point in \( X \), \( k(x) \) denotes the residue field at \( x \).

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2. Definition of the map \( \delta \)

By a **quadratic space** over \( k \), we mean a nonsingular quadratic form over \( k \). We denote by \( W(k) \) the Witt group of quadratic spaces over \( k \) and by \( Ik \) the fundamental ideal of \( W(k) \) consisting of classes of even rank quadratic spaces. We represent quadratic spaces over \( k \) by diagonal matrices \( \langle a_1, \ldots, a_n \rangle(a_i \in k^*) \) with respect to the choice of an orthogonal basis. By an \( n \)-fold **Pfister form** over \( k \), we mean a quadratic space of the type \( \langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle \). The set of nonzero values of a Pfister form is a subgroup of the multiplicative group \( k^* \) of \( k \) [3, Theorem 1.8, p. 319]. By a **Pfister neighbor** of an \( n \)-fold Pfister form \( q \), we mean a quadratic space of rank at least \( 2^{n-1} + 1 \) which is a subform of \( q \) [4, Example 4.1].

For any quadratic space \( q \) over \( k \), let \( N_q(k) \) be the subgroup of \( k^* \) generated by norms from finite extensions \( E \) of \( k \) such that \( q \) is isotropic over \( E \). If \( q \) is isotropic, then clearly \( N_q(k) = k^* \). For any \( a \in k^* \), \( q \) is isotropic if and only if \( \langle a \rangle \otimes q \) is isotropic. Therefore \( N_q(k) = N_{\langle a \rangle \otimes q}(k) \). By Knebusch’s norm principle, \( N_q(k) \) is generated by elements of the form \( xy \), with \( x, y \in k^* \) which are values of \( q \) over \( k \) [2, Lemma 2.2]. In particular, if a quadratic form \( q \) is of the form \( q = \langle 1, a \rangle \otimes \langle 1, b \rangle \), then \( N_q(k) \) is equal to the group \( \text{Nrd}_{D/k}(D^*) \) of reduced norms of the quaternion algebra \( D = (-a, -b)_k \). Suppose that \( q' \) is a Pfister neighbor of a Pfister form \( q \). Then, for any extension \( E/k \), \( q' \) is isotropic over \( E \) if and only if \( q \) is isotropic over \( E \) [4, Example 4.1]. So we have \( N_q(k) = N_{q'}(k) \).

The following lemma is elementary and well-known, but the proof does not seem to be written explicitly in the literature.

**Lemma 2.1.** Let \( q \) be a Pfister form over \( k \). Then

\[
N_q(k) = \{ x \in k^* \mid q \otimes \langle 1, -x \rangle \text{ is isotropic} \}.
\]

In particular, \( x \) belongs to \( N_q(k) \) if and only if \( q \otimes \langle 1, -x \rangle = 0 \) in \( W(k) \).

**Proof.** Since \( q \) is a Pfister form, \( N_q(k) \) is the set of non-zero values of \( q \). If \( q \) is isotropic, the assertion is clear. We suppose that \( q \) is anisotropic. If \( q \otimes \langle 1, -x \rangle \) is isotropic, then there exist two vectors \( v_1, v_2 \) in the underlying vector space of \( q \), not both zero, such that \( q(v_1) - xq(v_2) = 0 \). Therefore

\[
x = q(v_1)/q(v_2) \in N_q(k).
\]
The other implication follows from the fact that a Pfister form represents 1.

The last assertion results from the basic fact on Pfister forms [5, Theorem 1.7, p. 319]. □

**Definition 2.2.** Let $C$ be a smooth projective geometrically integral curve over $k$. A *quadric fibration* $(X, \pi)$ over $C$ is a geometrically integral variety $X$ over $k$, together with a proper flat $k$-morphism $\pi : X \to C$ such that each point $P$ of $C$ has an affine neighborhood $\text{Spec} A(P)$, with $X \times_C \text{Spec} A(P)$ isomorphic to a quadric in $\mathbb{P}^n_{A(P)}$ and such that the generic fiber of $\pi$ is a smooth quadric.

Given a quadratic space $q$ over the function field $k(C)$ of $C$ of rank $\geq 3$, we can easily construct a quadric fibration $\pi : X \to C$, whose generic fiber is given by the quadratic space $q$. It is not unique, but two quadric fibrations having the same generic fiber are birational over $C$.

Let $\pi : X \to C$ be a quadric fibration with the generic fiber given by a quadratic space $q$, and $\text{CH}_0(X/C)$ denote the kernel of the map

$$
\pi_* : \text{CH}_0(X) \to \text{CH}_0(C).
$$

We have the following commutative diagram with exact rows (see [2])

$$
\begin{array}{ccccccccc}
\bigoplus_{x \in |X_\eta|} k(x)^* & \longrightarrow & \bigoplus_{P \in |C|} \text{CH}_0(X_P) & \longrightarrow & \text{CH}_0(X) & \longrightarrow & 0 \\
\downarrow \oplus_{N_{k(x)/k(C)}} & & \downarrow \oplus \deg_{X_P/k(P)} & & \pi_* & & \\
0 & \longrightarrow & k(C)^*/k^* & \longrightarrow & \bigoplus_{P \in |C|} \mathbb{Z} & \longrightarrow & \text{CH}_0(C) & \longrightarrow & 0,
\end{array}
$$

where $X_\eta$ is the generic fiber of $\pi : X \to C$ and $\deg_{X_P/k(P)} : \text{CH}_0(X_P) \to \mathbb{Z}$ is the degree map. Since the map $\bigoplus_{x \in |X_\eta|} k(x)^* \to k(C)^*$ is induced by norms, the image is precisely $N_\eta(k(C))$. By the snake lemma and the fact that $A_0(X_P) = 0$ ([2]), we have an exact sequence

$$
0 \longrightarrow \text{CH}_0(X/C) \xrightarrow{\delta} k(C)^*/k^*N_\eta(k(C)) \longrightarrow \bigoplus_{P \in |C^{(1)}|} \mathbb{Z}/\deg_{X_P/k(P)}(\text{CH}_0(X_P)).
$$

**Remark 2.3.** We denote by $k(C)^*_{\text{ad}}(q)$ the subgroup of $k(C)^*$ consisting of functions, which, at each closed point $P \in C$, can be written as a product of a unit at $P$ and an element of $N_\eta(k(C))$. Colliot-Thélène and Skorobogatov [2] proved that the above homomorphism $\delta$ defines an isomorphism

$$
\delta : \text{CH}_0(X/C) \xrightarrow{\sim} k(C)^*_{\text{ad}}(q)/k^*N_\eta(k(C))
$$

for an *admissible* quadric fibration $\pi : X \to C$. However, we don’t have to assume the admissibility for our main results.

3. **Injectivity of the global-to-local map**

Let $k$ be a number field and $\Omega$ be the set of places of $k$. For any $v \in \Omega$, $k_v$ denotes the completion of $k$ at $v$.

In [6, Theorem 5.4], Parimala and Suresh proved that if $X \to C$ is an admissible quadric fibration whose generic fiber is given by a Pfister neighbor over $k(C)$ of
rank \geq 5, then the map

\[ \Phi_{\text{real}} : \text{CH}_0(X/C) \to \bigoplus_{v \in \Omega} \text{CH}_0(X \otimes_k k_v/C \otimes_k k_v) \]

is injective, where \( v \) runs over all real places of \( k \). In the case where the rank of the quadratic form defining the generic fiber is less than 5, we give a sufficient condition for the injectivity of the global-to-local map \( \Phi \).

**Theorem 3.1.** Let \( \pi : X \to C \) be a quadric fibration over a number field \( k \). Assume that \( \dim X = 2 \) or 3, and the generic fiber of \( \pi \) is isomorphic to a quadric defined over \( k \) (i.e. there exists a quadric \( Q \subset \mathbb{P}^N_k \) such that the generic fiber is isomorphic to \( Q \otimes_k k(C) \) over \( k(C) \)). Then, the natural map

\[ \Phi : \text{CH}_0(X/C) \to \bigoplus_{v \in \Omega} \text{CH}_0(X \otimes_k k_v/C \otimes_k k_v) \]

is injective.

**Proof.** Let \( q \) be a quadratic form defining the generic fiber of the quadric fibration \( \pi : X \to C \). In order to prove the theorem, it is sufficient to show that the natural map

\[ k(C)^* / k^* N_q(k(C)) \to \prod_{v \in \Omega} k_v(C)^* / k_v^* N_q(k_v(C)) \]

is injective.

We may assume that \( q = (1, a, b, abd) \), \( a, b, d \in k^* \). Put \( L := k(\sqrt{d}) \). Note that \( q \) is isometric to \( (1, a) \otimes (1, b) \) over \( L(C) \). Let \( f \in k(C)^* \) such that \( f \in k_v^* N_q(k_v(C)) \) for all places \( v \) of \( k \). For any real place \( w \) of \( L \), denote by \( w' \) the restriction of \( w \) to \( k \). Since \( f \in k_w^* N_q(k_w(C)) \), there exists \( \mu_w' \in k_w^* \) such that \( \mu_w' f \in N_q(k_w(C)) \). We can choose \( \mu \in k^* \) such that the sign of \( \mu \) is the same as that of \( \mu_w' \) for each real place \( w \) of \( L \). Thus we have \( \mu f \in N_q(k_w(C)) \subset N_q(L_w(C)) \). Therefore \( q \otimes (1, -\mu f) \) is hyperbolic over \( L_w(C) \) for each real place \( w \) of \( L \). For a complex place \( w \) of \( L \), it is clear that \( q \otimes (1, -\mu f) \) is hyperbolic over \( L_w(C) \). Further, for a finite place \( w \) of \( L \), we have \( f \in k_w^* N_q(k_w(C)) \), where \( v \) is the place of \( k \) below \( w \). Since \( k_w^* \subset N_q(k_w(C)) \),

\[ \mu f \in k_w^* N_q(k_w(C)) = N_q(k_w(C)) \subset N_q(L_w(C)). \]

Therefore \( q \otimes (1, -\mu f) \) is hyperbolic over \( L_w(C) \) for all places \( w \) of \( L \). By [1] Theorem 4], the natural map

\[ I^3 L(C) \to \prod w I^3 L_w(C) \]

is injective, where \( w \) runs over all places of \( L \). Hence \( q \otimes (1, -\mu f) \) is hyperbolic over \( L(C) \). By [2] Proposition 2.3], we have

\[ \mu f \in N_q(L(C)) \cap k(C)^* = N_q(k(C)). \]

This proves the required injectivity.

The image of the global-to-local map \( \Phi \) lies in the direct sum \( \bigoplus_v \text{CH}_0(X \otimes_k k_v/C \otimes_k k_v) \). Indeed, a quadratic form of rank 4 defined over a number field \( k \) is isotropic over \( k_v \) for all but finitely many places \( v \) of \( k \). Therefore for all but finitely many \( v \), \( \text{CH}_0(X \otimes_k k_v/C \otimes_k k_v) = 0 \). \( \square \)
Remark 3.2. Without the assumption that the generic fiber is defined over \( k \), the natural map
\[
\Phi : \text{CH}_0(X/C) \longrightarrow \prod_{v \in \Omega} \text{CH}_0(X \otimes_k k_v/C \otimes_k k_v)
\]
is not injective \cite{8}.

Finally, we consider the restricted global-to-local map \( \Phi_{\text{real}} \). Parimala and Suresh’s result \cite[Theorem 5.4]{6}, which is for quadratic forms of rank at least 5, does not hold for forms of smaller rank. We give the following example, which is a variation of \cite[Proposition 6.1]{6}.

Proposition 3.3. Let \( C \) be the elliptic curve over \( \mathbb{Q} \) defined by
\[
y^2 = -x(x+2)(x+3).
\]
Assume that the generic fiber of a quadric fibration \( \pi : X \to C \) is isomorphic to the quadric defined by the quadratic form \( q = \langle 1, -2, 3, -6 \rangle \). Then the natural map
\[
\Phi_{\text{real}} : \text{CH}_0(X/C) \longrightarrow \text{CH}_0(X \otimes_{\mathbb{Q}} \mathbb{R}/C \otimes_{\mathbb{Q}} \mathbb{R})
\]
is not injective.

Proof. Since \( q \) is isotropic over \( \mathbb{R} \), \( N_q(\mathbb{R}(C)) = \mathbb{R}(C)_\ast \). So we have \( \text{CH}_0(X \otimes_{\mathbb{Q}} \mathbb{R}/C \otimes_{\mathbb{Q}} \mathbb{R}) = 0 \).

Since \( \text{div}_{C}(x) = 2D \), for some divisor \( D \) on \( C \), \( x \in \mathbb{Q}(C)_\ast/\mathbb{Q}_\ast N_q(\mathbb{Q}(C)) \) is contained in \( \text{Im} \delta \). On the other hand, \( q \) is isometric to \( \langle 1, 1, 3, 3 \rangle \) over \( \mathbb{Q}_3 \). Thus \( x \notin \mathbb{Q}_3 N_q(\mathbb{Q}_3(C)) \) by \cite[Proposition 6.1]{6}. Therefore we have \( \text{CH}_0(X/C) \neq 0 \). \( \square \)

Remark 3.4. Note that in the above case the map
\[
\Phi : \text{CH}_0(X/C) \longrightarrow \bigoplus_{v \in \Omega} \text{CH}_0(X \otimes_k k_v/C \otimes_k k_v)
\]
is injective by Theorem \cite[3.1]{3}. The summands of the right hand side vanish except for \( \text{CH}_0(X \otimes_{\mathbb{Q}} \mathbb{Q}_2/C \otimes_{\mathbb{Q}} \mathbb{Q}_2) \) and \( \text{CH}_0(X \otimes_{\mathbb{Q}} \mathbb{Q}_3/C \otimes_{\mathbb{Q}} \mathbb{Q}_3) \), since the quadratic form \( \langle 1, -2, 3, -6 \rangle \) is isotropic over \( \mathbb{R} \) and over \( \mathbb{Q}_p \) for all primes \( p \) except 2 and 3.

References

[1] J. Kr. Arason, R. Elman, and B. Jacob, Fields of cohomological 2-dimension three, Math. Ann. 274 (1986), no. 4, 649–657.
[2] J.-L. Colliot-Thélène and A. N. Skorobogatov, Groupe de Chow des zéros-cycles sur les fibrés en quadriques, K-Theory 7 (1993), no. 5, 477–500.
[3] W. Fulton, Intersection theory, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.
[4] M. Knebusch, Generic splitting of quadratic forms. I, Proc. London Math. Soc. (3) 33 (1976), no. 1, 65–93.
[5] T. Y. Lam, Introduction to quadratic forms over fields, Graduate Studies in Mathematics, vol. 67, American Mathematical Society, Providence, RI, 2005.
[6] R. Parimala and V. Suresh, Zero-cycles on quadric fibrations: finiteness theorems and the cycle map, Invent. Math. 122 (1995), no. 1, 83–117.
[7] P. Salberger, Zero-cycles on rational surfaces over number fields, Invent. Math. 91 (1988), no. 3, 505–524.
[8] V. Suresh, Zero cycles on conic fibrations and a conjecture of Bloch, K-Theory 10 (1996), no. 6, 597–610.

[9] R. G. Swan, Zero cycles on quadric hypersurfaces, Proc. Amer. Math. Soc. 107 (1989), no. 1, 43–46.

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