Concerning Classical Forces, Energies, and Potentials for
Accelerated Point Charges

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Abstract

Although the expressions for energy densities involving electric and magnetic fields are exactly analogous, the connections to forces and electromagnetic potentials are vastly different. For electrostatic situations, the changes in the electric energy can be related directly to electric forces and to the electrostatic potential. The situation involving magnetic forces and energy changes involves two fundamentally different situations. For charged particles moving with constant velocities, the changes in both electric and magnetic field energies are provided by the external forces that keep the particles’ velocities constant; there are no Faraday acceleration electric fields in this situation. However, for particles that change speed, the changes in magnetic energy density are related to acceleration-dependent Faraday electric fields. Current undergraduate and graduate textbooks deal only with highly symmetric situations where the back Faraday electric fields are easily calculated from the time-changing magnetic flux. However, in situations that lack high symmetry, such as the magnetic Aharonov-Bohm situation, the back (Faraday) acceleration electric fields of point charges seem unfamiliar. In this article, we present a simple unsymmetric example and analyze it using the Darwin Lagrangian. In all cases involving changing velocities of the current carriers, it is the work done by the back (Faraday) acceleration electric fields that balances the magnetic energy changes.
I. INTRODUCTION

The expression $E^2/(8\pi)$ for the energy density associated with electric fields is exactly analogous to the energy density $B^2/(8\pi)$ for magnetic fields. However, the connections of the energy expressions with the forces exerted by the fields are vastly different between the electric and magnetic cases. The work done by electrostatic forces leads directly to changes in electric field energy. On the other hand, magnetic fields do no work. Accordingly, the association between changes in magnetic energy and work are more subtle. In the quasistatic regime, electric field energies depend upon only the relative positions of charges, whereas magnetic field energies depend upon both the relative positions of charges and also the velocities of the charges. Current textbooks of electromagnetism discuss quasistatic magnetic energy changes for only two situations: 1) charges that move with constant speed, and 2) charges which change their speed but are in highly symmetric configurations. This limited perspective leaves out magnetic energy changes for situations that lack high symmetry. Here we present a simple point-charge example that lacks high symmetry and illustrates the connections between forces, energies, and potentials in quasistatic classical electrodynamics when radiation fields are excluded.

Although electrostatics and magnetostatics can be treated accurately using continuous charge densities and currents, classical electrodynamics is a relativistic theory and requires the use of point charges. Accordingly, we will use point charges in our analysis of quasistatic energy changes. In order to simplify the situation, we turn to the Darwin Lagrangian which treats classical electrodynamics accurately through order $1/c^2$. Use of this approximation simplifies the analysis by eliminating radiation fields and retarded times.

Reference to an analogy may clarify the purpose of the present article. Magnetostatics, like electrostatics, involves only two of Maxwell’s equations for the field, $\nabla \cdot B = 0$ and $\nabla \times B = (4\pi/c)J$. However, if one considers only situations of high symmetry, then one can apply only Ampere’s law and ignore the divergence requirement for $B$. Thus if a current density $J(r)$ has axial symmetry, then Ampere’s law plus symmetry is sufficient to determine the magnetic field $B$, without any need for the divergence equation. However, the highly-symmetric situations give a limited and false impression of the theory. When the situation lacks high symmetry, then the Biot-Savart law must be used. The Biot-Savart law is a particular solution requiring both Maxwell equations for $B$, and may be complemented by
a solution of the homogeneous equations. Similarly, highly-symmetric situations involving axial symmetry give the impression that Faraday’s law $\nabla \times \mathbf{E} = -(1/c) \partial \mathbf{B}/\partial t$ can be used without the need for the rest of Maxwell’s equations because the confounding terms are suppressed by the symmetry. However, the symmetric situations give only a limited understanding of slowly-varying electromagnetic fields.

This understanding limited to symmetric situations makes physicists unprepared to analyze situations which lack high symmetry, such as the simple example in the present article or such as the Aharonov-Bohm situation. Although the azimuthally-symmetric vector potential associated with a circular solenoid is often treated in junior-level courses in classical electromagnetism,[1] the experimentally-realized situation of the magnetic Aharonov-Bohm effect (where electrons pass on both sides of a long solenoid) involves time-dependent interactions whose classical electromagnetic aspects are not azimuthally symmetric. It is often claimed in electromagnetism texts[2] and repeatedly in quantum texts[3] that there is no classical electromagnetic interaction of the solenoid back on the passing electrons. This claim is made despite the magnetic energy changes associated with the overlap of the magnetic field of a passing electron with the magnetic field of the solenoid, which changes depend on the side of the solenoid on which the electron passes. However, such no-interaction claims do not take into account the Faraday acceleration fields of the charges in the solenoid which are related to the magnetic energy changes. Faraday acceleration fields associated with magnetic energy changes are treated in the present article, but discussion of the Aharonov-Bohm situation is made elsewhere.[4]

II. THE DARWIN-LAGRANGIAN APPROXIMATION

1. The Darwin Lagrangian

The classical electrodynamics of charged particles at arbitrary changing velocities is enormously complicated, particularly because electromagnetic fields depend upon sources at retarded times. In order to simplify the situation, we will consider the behavior of just two point charges that move at speeds small compared to the speed $c$ of light in vacuum. This situation corresponds to that described by the Darwin Lagrangian[5][6]. This approximation represents an enormous simplification; it excludes radiation and involves no retarded times.
The Darwin Lagrangian for two point charges $e$ and $q$ is given by:

$$L = -m_e c^2 \sqrt{1 - v_e^2/c^2} - m_q c^2 \sqrt{1 - v_q^2/c^2} - \frac{eq}{|r_e - r_q|} + \frac{eq}{2c^2} \left[ \mathbf{v}_e \cdot \mathbf{v}_q + \frac{[\mathbf{v}_e \cdot (r_e - r_q)] [\mathbf{v}_q \cdot (r_e - r_q)]}{|r_e - r_q|^3} \right]$$

$$= -m_e c^2 \sqrt{1 - v_e^2/c^2} - m_q c^2 \sqrt{1 - v_q^2/c^2} - \frac{1}{2} [e\Phi_q(r_e, t) + q\Phi_e(r_q, t)]$$

$$+ \frac{1}{2} \left[ \frac{e}{c} \mathbf{v}_e \cdot [\mathbf{A}_q(r_e, t)] + q \frac{\mathbf{v}_q}{c} \cdot [\mathbf{A}_q(r_q, t)] \right]$$

where it is understood that the square roots are to be expanded in powers of $1/c^2$ through order $1/c^4$. Here we have not expanded the square roots because maintaining the exact mechanical forms leads to the familiar expressions for the particle mechanical energies on calculations from the Lagrangian.

A. Potentials and Fields in the Darwin-Lagrangian Approximation

For a point charge $q$ located at $r_q(t)$ and moving with small velocity $v_q(t)$ and acceleration $a_q(t)$, through order $1/c^2$, the system Lagrangian in Eq. (1) can be described as involving a scalar potential

$$\Phi_q(r, t) = \frac{q}{|r - r_q(t)|},$$

a vector potential

$$\mathbf{A}_q(r, t) = \frac{q}{2c} \left[ \frac{\mathbf{v}_q}{|r - r_q|} + \frac{[\mathbf{v}_q \cdot (r - r_q)] (r - r_q)}{|r - r_q|^3} \right],$$

an electric field

$$\mathbf{E}_q(r, t) = -\nabla \Phi_q(r, t) - \frac{1}{c} \frac{\partial \mathbf{A}_q(r, t)}{\partial t}$$

$$= q \frac{r - r_q}{|r - r_q|^3} \left[ 1 + \frac{v_q^2}{2c^2} - \frac{3}{2} \left( \frac{\mathbf{v}_q \cdot (r - r_q)}{c |r - r_q|} \right)^2 \right]$$

$$- \frac{q}{2c^2} \left[ \frac{a_q}{|r - r_q|} + \frac{a_q \cdot (r - r_q)}{|r - r_q|^3} \right],$$

and a magnetic field

$$\mathbf{B}_q(r, t) = \nabla \times \mathbf{A}_q(r, t) = q \frac{\mathbf{v}_q(t)}{c} \times \frac{r - r_q(t)}{|r - r_q(t)|^3}.$$
B. Comments on the Electromagnetic Fields in the Darwin Approximation

We notice immediately that the scalar potential in Eq. (2) is the same as that for a point charge in electrostatics, except that the position \( r_q(t) \) is now time dependent. Just how different the theory is from electrostatics can be seen in the expression (4) for the electric field which now involves velocities and even accelerations of the charged particle. Similarly, the magnetic field in Eq. (5) is that which one might be tempted to use for a point charge from the Biot-Savart law in magnetostatics using \( J_q(r,t) = qv_q(t) \delta^3(r-r_q(t)) \). However, these expressions are now only approximations since we are not dealing with electrostatics or magnetostatics. The scalar potential appears in the connection with the electric field \( E_q(r,t) \) in Eq. (4), but not in connection with the magnetic field \( B_q(r,t) \) in Eq. (5). The scalar potential \( \Phi_q(r,t) \) depends on the relative positions \( |r-r_q| \) of the charge and the field point. On the other hand, the vector potential \( A_q(r,t) \) is connected to both \( E_q(r,t) \) and \( B_q(r,t) \). The vector potential \( A_q(r,t) \) depends not only on the relative displacement \( (r-r_q) \) between the charge and the field point, but also on the velocity \( v_q \) of the charge \( q \). The magnetic field \( B_q(r,t) \) depends upon spatial derivatives of \( A_q(r,t) \). In contrast, the electric field \( E_q(r,t) \) in Eq. (4) depends upon the time rate of change of the vector potential \( A_q(r,t) \), which may involve changes in both the charge’s position \( r_q \) and/or the charge’s velocity \( v_q \). The terms in Eq. (4) for \( E_q(r,t) \) involving \( v_q^2 \) arise from position-dependent changes in \( A_q(r,t) \) with time. The terms involving acceleration \( a_q \) arise from time-dependent changes associated with velocity. The acceleration-dependent terms give the Faraday acceleration fields of the charged particle. The acceleration-dependent terms for the electric field in Eq. (4) appear in an electromagnetism textbook published in 1940, but do not seem to appear in more recent textbooks.

C. Equations of Motion from the Darwin Lagrangian

The Euler-Lagrange equations of motion for the charges \( e \) and \( q \) can be obtained in terms of the canonical momentum of a particle. However, these equations can be rewritten in the conventional form for Newton’s second law as \( dP_{\text{mechanical}}/dt = qE + q(v/c) \times B \) where \( E = -\nabla \Phi - (1/c) \partial A/\partial t \) and \( B = \nabla \times A \).
D. The Total Energy

The associated system total energy follows directly from the Lagrangian in Eq. (1), giving a sum of mechanical, electric, and magnetic energies,

\[ U^{(total)} = U^{(m)} + U^{(E)} + U^{(B)}, \]

where the mechanical energy is

\[ U^{(m)} = \frac{m_e c^2}{\sqrt{1 - v_e^2/c^2}} + \frac{m_q c^2}{\sqrt{1 - v_q^2/c^2}}, \]

the electric field energy is

\[ U^{(E)} = \frac{1}{2} \left[ e \Phi_q (r_e, t) + q \Phi_e (r_q, t) \right] = \frac{e q}{|r_e - r_q|}, \]

and the magnetic field energy is

\[ U^{(B)} = \frac{1}{2} \left[ e \frac{v_e}{c} \cdot [A_q (r_e, t)] + q \frac{v_q}{c} \cdot [A_e (r_q, t)] \right] \]
\[ = \frac{e q}{2c^2} \left[ \frac{v_e \cdot v_q}{|r_e - r_q|^2} + \frac{[v_e \cdot (r_e - r_q)] [v_q \cdot (r_e - r_q)]}{|r_e - r_q|^3} \right]. \]

III. FIELD-POTENTIAL-ENERGY CONNECTIONS FOR ELECTRIC AND MAGNETIC FIELDS

A. Electric Energy Changes

In the Darwin approximation, the changes in electric field energy are directly analogous to those familiar from electrostatics. The electric energy in Eq. (8) involves only the scalar potentials which take the same form as in electrostatics. The electric forces associated with the electric potential are directly associated with energy in the electric field. The electric energy involves only the separations \(|r_e - r_q|\) between the charged particles.

B. Magnetic Energy Changes

In contrast to electric field energy changes which involve only the separations between the charges, magnetic energy changes involve both changes in the separations between the
charges and also changes in the velocities of the charges, as indicated in Eq. (9). It seems convenient to separate magnetic energy changes into two separate situations—those involving charged particles moving with constant velocity and those involving accelerating charges.

1. Charged Particles Moving with Constant Velocity

The situation of charges moving with constant velocities turns out to be uncomplicated, and the connections between forces and energy changes are exactly soluble. It can be shown[8] that for point charges moving with constant velocities, the external forces that hold the particles at constant velocities also provide exactly the changing energies in the electric and magnetic fields for any particle velocities less than the speed of light in vacuum. There are no kinetic energy changes and no Faraday acceleration electric fields.

A charged particle at rest in an inertial frame does not cause any emf around any closed curve in that space. However, a charge particle moving with constant velocity will indeed cause an emf around a general closed curve in the inertial frame. Also, the charge moving with constant velocity has an associated magnetic field, and the emf caused by the moving charge agrees with the changing magnetic flux through the closed curve. Thus, depending upon the speed of the charge in an inertial frame, a charge q may or may not cause an emf. Indeed, if we are dealing with only two charges moving with constant velocities, we can always go to the rest frame of one of the charges and so have changes in only electric field energy and no magnetic field energy changes at all.

Situations involving motional emfs are often treated as though the relevant charge carriers were moving with constant velocity, and so can be understood in terms of energy changes being balanced by the work done by external forces[9].

2. Accelerating Charges

The particularly-troubling magnetic energy situations involve groups of charges where particles change their speeds. These cases generally involve Faraday acceleration electric forces, and the energy changes may involve particle kinetic energy changes and/or changes in electromagnetic field energy and/or work done by external forces. We will discuss magnetic energy changes in connection with two different examples, one involving a highly symmetric
situation and one involving an unsymmetric situation.

IV. SITUATIONS INVOLVING A LONG SOLENOID

A. Azimuthally Symmetric Solenoid with Increasing Currents

The most familiar situation of a back (Faraday) acceleration electric field occurs for the highly-symmetric situation of a circular solenoid. When the currents in the solenoid are constant, the charge carriers are accelerating towards the central axis since they are moving in a circle. A single charge moving in a circle does not have just electrostatic and magnetostatic fields, but rather has higher terms in \(1/c\), including radiation terms. However, the situation changes for a many-particle current which can be approximated as a steady current. Even though the charges are still accelerating when the current in a circuit is constant, all the higher-order contributions to the fields cancel, leaving only the electrostatic and magnetostatic contributions\(^{[10]}\). For a circular solenoid, the vector potential \(A\) inside may be chosen as symmetric in the azimuthal \(\hat{\phi}\)-direction with \(A = \hat{\phi}Br/2 = \hat{\phi}2\pi nevr/c\), and outside the solenoid \(A = \hat{\phi}2\pi nevR^2/(cr)\), where \(n\) is the number of current carriers per unit area, \(e\) and \(v\) are the charge and speed of each current carrier, and \(R\) is the radius of the long solenoid. This corresponds to the Coulomb gauge for \(A\).

When the currents in a solenoid are changing in an azimuthally symmetric pattern, the charge carriers in the winding of the solenoid have a tangential acceleration associated with the same change in speed \(v\) for each charge. If there were no change in magnetic field energy, the agent causing the changing speed of the charge carriers would need to provide power so as to change only the kinetic energy of the current carriers. This change in kinetic energy of the current carriers is so small as to escape any mention in textbooks of classical electromagnetism. It is the change in magnetic field energy that is overwhelmingly dominant. The agent changing the speed of the current carriers must provide the power against the back (Faraday) acceleration electric fields corresponding to the acceleration terms in Eq. (4). We emphasize: The magnetic energy balance for quasistatic systems requires the existence of Faraday electric forces associated with the accelerations of the charged particles.

The back electric fields associated with the changing speeds of the charge carriers can be described in terms of the changing vector potential. Thus, for a long solenoid with
slowly increasing currents, the changes of speed of all the current carriers give a changing vector potential with non-zero $\partial A / \partial t$ and so a back emf from the non-vanishing (Faraday) acceleration electric field appearing from $E = -\nabla \Phi - (1/c) \partial A / \partial t$. Although the speed $v$ of all the particles changes (producing a changing magnetic field and magnetic energy), the functional spatial dependence of the magnetic field $B$ on the vector potential $A$ involves only spatial separations and does not change, $B = \nabla \times A$. Inside the solenoid, an increasing vector potential is associated with increasing speed $v$ and an increasing magnetic field, since inside, $A$ depends linearly on the distance $r$ from the central axis $A = \hat{\phi} Br/2 = \hat{\phi} 2\pi nevr/c$ and $B = \nabla \times A \neq 0$. However, outside the solenoid, even though the magnitude of the vector potential is increasing in $A = \hat{\phi} 2\pi nevR^2/(cr)$ due the increase in $v$, the $1/r$ fall-off of the vector potential with distance from the central axis still gives $B = \nabla \times A = 0$. To the surprise of many students, there is an acceleration-dependent electric field outside the solenoid from $\partial A / \partial t$, but still no magnetic field outside.

In our elementary electromagnetism classes where our examples and problems are highly symmetric, it is easy to relate the back emf of the solenoid to Faraday’s law involving a changing magnetic field,

$$\text{emf} = \oint d\mathbf{r} \cdot \mathbf{E} = -\frac{1}{c} \frac{d}{dt} \left( \int d\mathbf{S} \cdot \mathbf{B} \right), \quad (10)$$

where $\mathbf{S}$ is the area, and to take advantage of the high symmetry to calculated $\mathbf{E}$. The textbooks emphasize that changing currents in a solenoid lead to changing magnetic fields and therefore to Faraday electric fields. They do not emphasize that this back emf is due directly to the electric fields produced by the accelerating current carriers. There is no need to go through a changing magnetic field to obtain the emf. The same current carriers that produce the Faraday acceleration electric field also produce the magnetic field of the solenoid in the first place[12].

**B. Unsymmetric Situations**

The Darwin Lagrangian is rarely mentioned in junior-level electromagnetism texts. Although the full Lienard-Wiechert fields involving retarded times are often given, the quasistatic approximation through $1/c^2$ for the electric fields in Eq. (4) goes unmentioned even in graduate-level texts. Thus many physicists are unfamiliar with the acceleration-
dependent terms for the electric field of a point charge that are given in Eq. [1], and so they are unable to understand the classical electromagnetic aspects of an unsymmetric situation such as that proposed by Aharonov-Bohm where electrons pass a long solenoid.

When an electron passes alongside a long solenoid, the magnetic field of the passing charge will overlap with the magnetic field of the solenoid giving a change in the total magnetic energy, a relativistic $1/c^2$ effect. Depending upon which side the electron passes, the change in total magnetic field energy can be positive or negative. Naturally, one asks, “What provides the energy which balances the magnetic energy change?” If the solenoid is a superconductor, there is no external source of energy. In other cases, there is a battery providing current to the solenoid. In this latter case, one sometimes hears the suggestion that the battery provides the necessary energy changes. However, this side-dependent magnetic energy change does *not* correspond to a highly symmetric situation. Indeed, a relativistic $1/c^2$ effect is involved, and the forces on the charged particles of the solenoid due to the fields of a passing charge will be in *different directions in different locations*, some solenoid charges increasing their speeds and some decreasing their speeds. For situations that are not highly symmetric, we cannot easily use the familiar two-step procedure through the magnetic field to obtain the induced electric field.

In the past, it has been suggested repeatedly [11] that the full magnetic energy change associated with a passing electron was balanced by the kinetic energy change of the passing electron itself. Such a kinetic energy change for a passing charge indeed accounts for the magnitude of the experimentally observed Aharonov-Bohm phase shift in terms of a classical electromagnetic lag effect rather than the currently prevailing idea of a “purely quantum topological effect” of magnetic flux leading to the Aharonov-Bohm phase shift. In any case, the classical electromagnetic situation of charged particles passing a long solenoid lacks high symmetry, and therefore we cannot use symmetry and the integral form of Faraday’s law to calculate the force on the passing electrons. One must turn to a more fundamental analysis, such as one through the Darwin Lagrangian.
V. EXAMPLE OF TWO POINT CHARGES MOVING SIDE-BY-SIDE

A. Side-by-Side Charges

For the simplest possible illustration of magnetic energy changes in a situation lacking high symmetry, we consider two point charges $e$ and $q$ moving side-by-side at separation $l = |\mathbf{r}_e - \mathbf{r}_q|$ along frictionless rods parallel to the $z$-axis with velocities $\mathbf{v}_e$ and $\mathbf{v}_q$. This system will have both electric and magnetic fields and associated energies. For motion with constant velocity, the electric and magnetic forces of one charge on the other are perpendicular to the frictionless rods. This same system, where the forces of constraint introduce neither energy nor net linear momentum, was used previously\[13\] as an illustration of Einstein’s mass-energy relation $E = mc^2$.

In Darwin’s $1/c^2$ approximation for the energy (given in Eqs. (6)-(9)), the electric field energy is still $U^{(E)}(\mathbf{r}_e, \mathbf{r}_q) = eq/|\mathbf{r}_e - \mathbf{r}_q|$ in Eq. (8), whereas the magnetic field energy is given in Eq. (9). For our simple side-by-side situation, where the separation $l = |\mathbf{r}_e - \mathbf{r}_q|$ of the charges is perpendicular to the direction of motion $\mathbf{v}$, we have the magnetic field energy

$$U^{(B)}(\mathbf{r}_e, \mathbf{r}_q, \mathbf{v}) = \frac{1}{2} \left[ e \frac{v}{c} \left( \frac{q}{2c} \frac{v}{l} \right) + q \frac{v}{c} \left( \frac{e}{2c} \frac{v}{l} \right) \right] = \frac{eqv^2}{2c^2l}. \quad (11)$$

B. Accelerating the System of Two Point Charges

Suppose that external forces $\mathbf{F}_{ext}^{(e)}$ and $\mathbf{F}_{ext}^{(q)}$ parallel to the $z$-axis are applied to the two charges so as to give the particles a small common acceleration $\mathbf{a} = \hat{z}a$ for a short time $\tau$, thus changing the particles’ common speed by $\Delta v = a\tau$. The kinetic energy $mc^2(\gamma - 1)$ of each particle changes due to the change in speed. The field energy in the electric field does not change according to the Darwin approximation in Eq. (8). However, the energy in the magnetic field changes from that given in Eq. (11) over to that with $v$ replaced by $v + \Delta v$, so that the change in magnetic energy is

$$\Delta U^{(B)} = \frac{eq(v + \Delta v)^2}{2c^2l} - \frac{eqv^2}{2c^2l} \approx \frac{eq\Delta v}{c^2l} \quad (12)$$

During the acceleration of the parallel-moving charges $e$ and $q$, the vector potential $\mathbf{A} = \mathbf{A}_e + \mathbf{A}_q$ changes as the speed changes, leading to changes in the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and in the magnetic field energy. However, the acceleration of the charges
and the change in the vector potential $A$ also leads to a change in the electric field through $E = -\nabla \Phi - (1/c) \partial A/\partial t$. Thus, the electric field has a new acceleration-related component parallel to the velocity $v$ of the particles. Therefore, from Eq. (10), each charge exerts an additional (acceleration-dependent) electric force

$$eE_{\text{acc} - q} = qE_{\text{acc} - e} = -\frac{eqa}{2c^2 |r_e - r_q|} = -\frac{eqa}{2c^2 l},$$

(13)
a retarding force, on the other charge. But now Newton’s second law for the mechanical momentum change in the $z$-direction for each particle involves not just one external force on each particle, but rather two forces, both the external force and the electric force of one charge on the other,

$$\frac{dp_e}{dt} = F_{\text{ext}}^{(e)} - \frac{eqa}{2c^2 l} \quad \text{and} \quad \frac{dp_q}{dt} = F_{\text{ext}}^{(q)} - \frac{qea}{2c^2 l}.$$ 

(14)

Therefore, the external forces (which provided the acceleration $a$ of the charges) have to be larger so as to overcome these retarding forces. Rewriting Newton’s second-law equations in (14) by moving the electric forces to the opposite sides of the equal signs, we require

$$F_{\text{ext}}^{(e)} = \frac{dp_e}{dt} + \frac{eqa}{2c^2 l} \quad \text{and} \quad F_{\text{ext}}^{(q)} = \frac{dp_q}{dt} + \frac{qea}{2c^2 l}.$$ 

(15)

The change in the energy of the system due to the external forces is

$$\Delta U^{(\text{total})} = \int_0^\tau dt \left( F_{\text{ext}}^{(e)} + F_{\text{ext}}^{(q)} \right) \cdot v = \int_0^\tau dt \left( \frac{dp_e}{dt} + \frac{dp_q}{dt} + \frac{eqa}{c^2 l} \right) \cdot v$$

$$\approx \Delta U^{(me)} + \Delta U^{(mq)} + \frac{eqv \cdot a \tau}{c^2 l} = \Delta U^{(me)} + \Delta U^{(mq)} + \frac{qv \Delta v}{c^2 l},$$

(16)

where we have used $\Delta v = a \tau$ and have kept only first-order terms. Thus the external forces indeed account for the energy change, both the mechanical energy change $\Delta U^{(me)} + \Delta U^{(mq)}$ (from the time-integrations over the changes in momentum), and also the magnetic energy change (from the time-integration over the Faraday acceleration electric fields) corresponding to Eq. (12).

In this example, the external forces had to be larger because of the changing vector potential $A = A_q + A_e$, which gave a new acceleration-dependent term in the expression $E = -\nabla \Phi - (1/c) \partial A/\partial t$. This is a common situation for charges that are changing speed. The magnetic energy balance for quasistatic systems requires the existence of Faraday electric forces associated with the accelerations of the charged particles.
Thus for unsymmetrical situations (such as the simple example here of charges accelerating side-by-side, or indeed of charges passing outside a solenoid in the Aharonov-Bohm situation), we can not use high symmetry together with Faraday’s law involving changing magnetic flux, but rather we must fall back on more fundamental considerations; we need to connect changes in magnetic field energy directly with particle accelerations and with the back (Faraday) acceleration terms in Eq. (4) for the electric field of a point charge. Indeed, for groups of closely-spaced charges, the inertial effects of the back (Faraday) acceleration field become important, and failure to recognize these fields in unsymmetric cases can lead to paradoxes.

This example involving two point charges illustrates some of the contrasts involving forces, energies, and potentials for the electric and magnetic fields. In the electric case, the work associate with the electric field appears directly as energy stored in the electric fields, as is familiar in electrostatics. In the magnetic case, external or internal forces can change the speeds of the particles that are producing the magnetic fields. The change in the speeds of the charged particles leads to changes in magnetic energy. However, the change of charged particles’ speeds also causes back (Faraday) acceleration electric fields which make it harder for the external or internal forces to accelerate the charges. It is the additional work against the back (Faraday) acceleration electric fields that balances the change in magnetic field energy.

VI. CLOSING SUMMARY

The Darwin energy expressions in Eqs. (6)-(9) give the energy of two low-speed interacting charges. The electric field energy is the familiar expression from electrostatics. The magnetic field energy involves the separations and velocities of the charged particles. The magnetic field energy can change while keeping the particle velocities $v_e$ and $v_q$ constant if the relative displacement $(r_e - r_q)$ between the charges changes. In this case of charges moving with constant velocities, the change in magnetic field energy is provided by the external forces which keep the charges moving with constant velocity. There are no Faraday acceleration electric fields for charges moving with constant velocities. On the other hand, the magnetic field energy can also change due to accelerations $a_e$ and $a_q$ of the charges. In this case, the change in magnetic field energy is associated with the Faraday acceleration
terms in the electric field given in Eq. (4).

Particularly for situations where the speed of current carriers varies, the vector potential $A(\mathbf{r},t)$ serves as an intermediate connection between magnetic fields and electric fields through $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla \Phi - (1/c) \partial \mathbf{A}/\partial t$. Any external or internal agent that tries to accelerate the current carriers for a magnetic field produces a time-changing vector potential, which creates an electric field, that (fitting with Lenz’s law) tries to balance the energy change against work done by the external or internal agent. The change in the speed of the current carriers may produce a change in the magnetic field energy, and this change in magnetic field energy is consistent with energy conservation because the external or internal agent causing the change had to do additional work against the back Faraday electric fields of the accelerating charges.

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[1] D. J. Griffiths, *Introduction to Electrodynamics* 4th edn (Pearson, New York 2013), pp. 247-248.

[2] A. Shadowitz, *The Electromagnetic Field* (Dover, New York, 1988), pp. 197, 208-209, 517-522.

A. Garg, *Classical Electromagnetism in a Nutshell* (Princeton U. Press, Princeton, NJ 08450, 2012), pp. 107-108.

[3] See for example, D. J. Griffiths, *Introduction to Quantum Mechanics* 2nd ed. (Pearson Prentice Hall, Upper Saddle River, NJ 2005), pp. 384-391 or L. E. Balentine, *Quantum Mechanics* (Prentice Hall, Englewood Cliffs, New Jersey 07632, 1990), pp. 220-223.

[4] T. H. Boyer, “The Aharonov-Bohm phase shift as a Relativity Paradox,” and “A Classical Electromagnetic Basis for the Aharonov-Bohm Phase Shift,” submitted for publication.

[5] C. G. Darwin, “The Dynamical Motions of Charged Particles,” Phil. Mag. 39, 537-551 (1920).
[6] J. D. Jackson, *Classical Electrodynamics* 2nd edn (John Wiley & Sons, New York, 1975), pp. 593-595, and p. 616, Problem 12.12.

[7] L. Page and N. I. Adams, *Electrodynamics* (Van Nostrand, New York, 1940), p. 175. See also, L. Page and N. I. Adams, “Action and reaction between moving charges,” Am. J. Phys. 13, 141–147 (1945).

[8] T. H. Boyer, “Energy and Momentum in Electromagnetic Field for Charged Particles Moving with Constant Velocities,” Am. J. Phys. 39, 257-270 (1971).

[9] See ref. 1, pp. 373-378.

[10] See reference 6, p. 697, Problems 14.12 and 14.13.

[11] T. H. Boyer, “Classical electromagnetic deflections and lag effects associated with quantum interference pattern shifts: considerations related to the Aharonov-Bohm effect,” Phys. Rev. D 8, 1679-1693 (1973); “The Aharonov-Bohm effect as a classical electromagnetic lag effect: an electrostatic analogue and possible experimental test,” Il Nuovo Cimento 100, 685-701 (1987); “Does the Aharonov-Bohm effect exist?” Found. Phys. 30, 893-905 (2000); “Classical electromagnetism and the Aharonov-Bohm phase shift,” Found. Phys. 30, 907-932 (2000); “Darwin-Lagrangian analysis for the interaction of a point charge and a magnet: Considerations related to the controversy regarding the Aharonov-Bohm and Aharonov-Casher phase shifts,” J. Phys. A: Math. Gen. 39, 3455-3477 (2006).

[12] T. H. Boyer, “Faraday induction and the current carriers in a circuit,” Am. J. Phys. 83, 263-271 (2015).

[13] T. H. Boyer, “Electrostatic Potential Energy Leading to an Inertial Mass Change for a System of Two Point Charges,” Am. J. Phys. 46, 383-385 (1978).