Equivalences among parabolicity, comparison principle and capacity on complete Riemannian manifolds

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Abstract

In this work we establish new equivalences for the concept of $p$-parabolic Riemannian manifolds. We define a concept of comparison principle for elliptic PDE’s on exterior domains of a complete Riemannian manifold $M$ and prove that $M$ is $p$-parabolic if and only if this comparison principle holds for the $p$-Laplace equation. We show also that the $p$-parabolicity of $M$ implies the validity of this principle for more general elliptic PDS’s and, in some cases, these results can be extended for non $p$-parabolic manifolds or unbounded solutions, provided that some growth of these solutions are assumed.

1 Introduction

We recall that a complete Riemannian manifold $M$ is $p$–parabolic, $p > 1$, if $M$ does not admit a Green function that is, a positive real function $G(x,y)$ defined for $x, y \in M$ with $x \neq y$ such that $G(\cdot,y)$ is of $C^1$ class in $M \setminus \{y\}$ for any fixed $y \in M$, and

\[- \text{div} \left( \| \nabla G(\cdot,y) \|^{p-2} \nabla G(\cdot,y) \right) = \delta_y, \quad y \in M.\]

This equality is in the sense of distributions that is,

\[\int_M \left\langle \| \nabla G(\cdot,y) \|^{p-2} \nabla G(\cdot,y), \nabla \varphi \right\rangle = \varphi(y)\]

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for all $\varphi \in C^1_c(M)$.

A well known subject of research is to find characterizations for the $p-$parabolicity of complete Riemannian manifolds. Gathering together results which have been proved in the last decades, see Grigor’yan [5] and Holopainen [7, 8, 9, 10] for instance, we have (the notions used in the theorem are all defined below):

**Theorem 1.1.** Let $M$ be a complete Riemannian manifold. Then the following alternatives are equivalent:

(a) $M$ is $p-$parabolic;
(b) the $p-$capacity of any compact subset of $M$ is zero;
(c) the $p-$capacity of some precompact open subset of $M$ is zero;
(d) any bounded from below supersolution for the $p-$Laplacian operator is constant.

The case $p = 2$ is more classical, with a longer history, and there are other equivalences (see [5, 6]).

For a domain $\Omega \subset M$, we define the $p$-capacity of a set $E \subset \Omega$ with respect to $\Omega$ by

$$\text{cap}_p(E; \Omega) := \inf_{v \in \mathcal{F}_{E,\Omega}} \int_{\Omega} |\nabla v|^p dx,$$

where the infimum is taken over all the functions $v$ that belongs to

$$\mathcal{F}_{E,\Omega} = \{v \in C^\infty_c(M) : v = 1 \text{ in } E \text{ and } \text{supp}(v) \subset \Omega\}. \tag{2}$$

Observe that the $p-$capacity of a bounded subset $E \subset M$ with respect to $M$ can be obtained by taking a sequence of geodesic balls $B(o, R_k)$ centered at a given point $o \in M$, with $R_k \to +\infty$, and taking the limit

$$\text{Cap}_p(E) := \lim_{k \to +\infty} \text{cap}_p(E; B(o, R_k)). \tag{3}$$

It is not difficult to prove that the limit always exists, that does depend not on the sequence of balls and that $\text{Cap}_p(E) = \text{cap}_p(E; M)$.

Now we observe that as a result of the equivalence between (b) and (c) of Theorem [1.1] (see [7, 8, 9, 10]), we obtain the following result which is an important tool to the study of $p$-parabolicity and comparison principle:

**Corollary 1.2** (Dichotomy). Let $M$ be a complete noncompact Riemannian manifold and $p > 1$. If $\text{Cap}_p(B_0) > 0$ for some ball $B_0 \subset M$, then 

$$\text{Cap}_p(B) > 0$$

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for any ball $B \subset M$. More generally, if $\text{Cap}_p(E) > 0$ for some bounded set $E$, then $\text{Cap}_p(F) > 0$ for any set $F$ that contains some interior point.

Recall that the $p$–Laplacian operator $\Delta_p$ is defined by

$$\Delta_p u = \text{div} \left( \|\nabla u\|^{p-2} \nabla u \right), \quad u \in C^1(M).$$

A function $w \in C^1(M)$ is a supersolution of $\Delta_p$ if $\Delta_p w \leq 0$ in the weak sense.

In the main result of our paper we add two other equivalences in Theorem 1.1. The main one relates $p$–parabolicity with another very classical notion, the comparison principle, which is likely the most fundamental tool on PDE (see [15], Abstract). In the second one we use the $p$–capacity to show in an explicit way that the $p$–parabolicity of a Riemannian manifold depends on its behaviour at infinity.

We first define that $M$ satisfies the comparison principle for the $p$–Laplace operator (or the $p$–comparison principle for short) if, given any exterior domain $\Omega$ of $M$ ($\Omega = M \setminus K$, where $K$ is a compact subset), if $v, w \in C^0(\Omega)$ are bounded sub and super solutions for $\Delta_p$ then, whenever $v|_{\partial \Omega} \leq w|_{\partial \Omega}$, it follows that $v \leq w$ in $\Omega$. (In this case, we also say that the comparison principle for exterior domains in $M$ holds for the $p$-Laplace operator.) We prove:

**Theorem 1.3.** Let $M$ be a complete Riemannian manifold. Then the following alternatives are equivalent:

(a) $M$ is $p$–parabolic

(b) $M$ satisfies the comparison principle for $\Delta_p$

(c) Given $o \in M$ there are sequences $R^k_1, R^k_2$ satisfying $R^k_1 < R^k_2$ and

$$\lim_{k \to \infty} R^k_1 = \lim_{k \to \infty} R^k_2 = \infty$$

such that

$$\sup_k \text{cap}_p(B(o, R^k_1); B(o, R^k_2)) < +\infty$$

The existence of many different conditions implying $p$–parabolicity is well known and hence, where the $p$–comparison principle holds (see [12], [13], [18], [4], [8], [13], [17], [10] and also references therein). Among them, there are several works giving conditions in terms of the volume growth of geodesic balls. We prove here the following result:
Theorem 1.4. Let $M$ be a complete noncompact Riemannian manifold and $p > 1$. Suppose that, for a fixed point $o \in M$, there exist two increasing sequences $r_k$ and $s_k$ such that $s_k > r_k \to +\infty$ and

$$\sup_k \left( \frac{2}{s_k - r_k} \right)^p \text{Vol}(B(o, s_k) \setminus B(o, r_k)) < +\infty.$$  

Then $M$ is $p$–parabolic.

We note that condition (4) is satisfied if there exists some sequence $r_k \to +\infty$ such that

$$\text{Vol}(B(o, r_k)) \leq C r_k^q$$

for any $k$, where $q \leq p$ and $C > 0$ is a constant that depends only on $M$.

2 A general comparison result

In this section, we prove that a comparison result holds for a general class of equations which contains the $p$–Laplacian provided that the $p$–capacity of a sequence of annuli in the manifold has some decay. This section is of independent interest, but it will be important for the next one.

We will study a class of second order elliptic PDE’s of the form

$$\text{div} \left( \frac{A(|\nabla u|)}{|\nabla u|} \nabla u \right) = 0 \quad \text{in} \quad M \setminus K,$$  

where $K \subset M$ is a compact set and $A \in C[0, +\infty) \cap C^1(0, +\infty)$ satisfies, for some $\alpha \in (0, 1]$ and $\beta \geq 1$, the following conditions:

- $A(0) = 0$ and $A(s) > 0$ for $s > 0$;  

- $sA'(s) \geq \alpha A(s)$ for $s \geq 0$;  

- $sA'(s) \leq \beta A(s)$ for $s \geq 0$;  

There exist $D_2$ and $D_1$ positive such that

- $A(s) \geq D_1 s^{p-1}$ for any $s \geq 0$;  

- $A(s) \leq D_2 s^{p-1}$ for any $s \geq 0$.

Related to the operator in (5), define $A : \mathbb{R}^n \to \mathbb{R}^n$ by

$$A(0) = 0 \quad \text{and} \quad A(v) = \frac{A(|v|)}{|v|} v \quad \text{for} \quad v \neq 0.$$
Hence equation (5) is equivalent to
\[ \text{div} \mathbf{A}(\nabla u) = 0 \quad \text{in} \quad M \setminus K. \]

Remind that \( u \in C^1(M \setminus K) \) is said to be a weak solution to this equation if
\[ \int_{M \setminus K} \nabla \varphi \cdot \mathbf{A}(\nabla u) \, dx = 0 \]
for any \( \varphi \in C^1_c(M \setminus K) \). By an approximating argument, this holds for any \( \varphi \in W^{1,p}_0(M \setminus K) \).

Observe that conditions (7) and (9) imply that
\[ \sum_{i,j=1}^n \frac{\partial A_j(v)}{\partial v_i} \xi_i \xi_j \geq D_1 \alpha |v|^{p-2} |\xi|^2 \]  
(11)

and, from (8) and (10), we have that
\[ \sum_{i,j=1}^n \left| \frac{\partial A_j(v)}{\partial v_i} \right| \leq n^2 D_2 (\beta + 2) |v|^{p-2} \]  
(12)

for any \( \xi \in \mathbb{R}^n \) and \( v \in \mathbb{R}^n \setminus \{0\} \), where \( A_j \) is the \( j \)th coordinate functions of \( A \). Moreover, \( A \in C(\mathbb{R}^n; \mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n) \) and \( A(0) = 0 \). Hence, we have the following result, proved in [2] (Lemma 2.1):

\textbf{Lemma 2.1.} There exist positive constants \( c_1 = c_1(n, p, \alpha) \) and \( c_2 = c_2(n, p, \beta) \) such that
\[ (\mathbf{A}(v_2) - \mathbf{A}(v_1)) \cdot (v_2 - v_1) \geq c_1 |v_2| + |v_1| |v_2 - v_1|^2 \]  
(13)
\[ |\mathbf{A}(v_2) - \mathbf{A}(v_1)| \leq c_2 |v_2| + |v_1| |v_2 - v_1| \]  
(14)

for \( v_1, v_2 \in \mathbb{R}^n \) that satisfy \( |v_1| + |v_2| \neq 0 \). If \( p \geq 2 \), then
\[ (\mathbf{A}(v_2) - \mathbf{A}(v_1)) \cdot (v_2 - v_1) \geq c_1 |v_2 - v_1|^p \]  
(15)

for any \( v_1, v_2 \in \mathbb{R}^n \). If \( 1 < p \leq 2 \), then
\[ |\mathbf{A}(v_2) - \mathbf{A}(v_1)| \leq c_2 |v_2 - v_1|^{p-1} \]  
(16)

for \( v_1, v_2 \in \mathbb{R}^n \).
We can also conclude directly, from (10) and (9), that

\[ |A(v)| \leq D_2 |v|^{p-1} \quad \text{and} \quad |A(v)| \geq D_1 |v|^{p-1} \quad \text{for} \; v \in \mathbb{R}^n. \]  

(17)

We need the following result:

**Lemma 2.2.** Let \( K \) be a compact set of \( M \) and \( o \in M \) be a fixed point. Suppose that \( U \) and \( V \) are bounded domains of \( M \) that satisfy \( K \subset V \subset U \). For \( R_2 > R_1 > 0 \) such that the ball \( B(o, R_1) \) contains \( \overline{U} \), there exists a function \( \eta = \eta_{R_1, R_2} \in W^{1,p}_0(M \setminus K) \) such that

- \( 0 \leq \eta \leq 1 \);
- \( \eta(x) = 1 \) for \( x \in B(o, R_1) \setminus U \);
- \( \eta(x) = 0 \) if \( x \in V \) or \( d(x, o) \geq R_2 \);
- \( |D\eta(x)| \leq m \) for \( x \in U \setminus V \), where \( m \) is a constant that depends on \( U , V \) and \( M \), but not on \( R_1 \) and \( R_2 \);
- \( \Delta_p \eta(x) = 0 \) if \( R_1 < d(x, o) < R_2 \) in the weak sense.

**Proof.** First minimize the functional

\[ J(v) := \int_{B(o, R_2)} |\nabla v|^p dx \]

in the convex set \( \{ v \in W^{1,p}_0(B(o, R_2)) : v = 1 \text{ in } B(o, R_1) \} \). From the classical theory, there exists a unique minimizer \( v_0 \) in this convex set such that \( v_0 = 1 \text{ in } B(o, R_1) \) and \( \Delta_p v_0 = 0 \text{ in } B(o, R_2) \setminus B(o, R_1) \) in the weak sense. Now, for some neighborhood \( V_\epsilon \) of \( V \) such that \( V \subset V_\epsilon \subset U \), let \( v_1 \in C^\infty(U) \) satisfying \( 0 \leq v_1 \leq 1 \text{ in } U \), \( v_1 = 0 \text{ in } V \) and \( v_1 = 1 \text{ in } U \setminus V_\epsilon \). Defining \( \eta \) by \( \eta := v_1 \text{ in } U \), \( \eta := v_0 \text{ in } B(o, R_2) \setminus U \) and \( \eta := 0 \text{ in } M \setminus B(o, R_2) \), it follows the result. \( \square \)

Now we fix some notation before stating the main result. For a fixed point \( o \in M \), we denote the capacity of \( B(o, R_1) \) with respect to \( B(o, R_2) \), for \( R_2 > R_1 \), by

\[ \text{cap}_p (R_1, R_2, o) := \text{cap}_p (B(o, R_1); B(o, R_2)). \]

If it is clear that the center of the balls is \( o \), we simply denote by \( \text{cap}_p (R_1, R_2) \).

The annulus \( B(o, R_2) \setminus \overline{B(o, R_1)} \), centered at \( o \), is denoted by

\[ \mathcal{A}_{R_1, R_2} = \mathcal{A}_{R_1, R_2, o} := B(o, R_2) \setminus \overline{B(o, R_1)} \quad \text{for} \; R_2 > R_1 \]
and the oscillation of a function $v$ in this annulus is defined by

\[
\text{osc}_{\mathcal{A}_{R_1,R_2}} v := \sup_{\mathcal{A}_{R_1,R_2}} v - \inf_{\mathcal{A}_{R_1,R_2}} v.
\] (18)

The next proposition is a variant of the result that the derivatives of global solutions are in $L^p$ if the manifold is $p$-parabolic.

**Proposition 2.3.** Let $u \in C^1(M \setminus K)$ be a weak solution of (5) in $M \setminus K$. Suppose that $A$ satisfies (6) - (10) and $U \subset M$ is an open bounded set such that $K \subset U$.

(a) If $u$ is bounded and

\[
\sup_k \left( \text{osc}_{\mathcal{A}_{R_1^k,R_2^k}} u \right)^p \cdot \text{cap}_p (R_1^k, R_2^k) < +\infty,
\]

where $B(o, R_1^k)$ and $B(o, R_2^k)$ are increasing sequences of balls that contain $U$ and such that $R_2^k > R_1^k \to \infty$, then $|Du| \in L^p(M \setminus U)$ and there exist positive constants $C_1$ and $C_2$, such that

\[
\|Du\|_{L^p(M \setminus U)} \leq C_2 + C_1 \limsup_{k \to \infty} \left( \text{osc}_{\mathcal{A}_{R_1^k,R_2^k}} u \right)^p \cdot \text{cap}_p (R_1^k, R_2^k).
\]

(b) In the case that $u$ is not necessarily bounded, if we assume

\[
\sup_k \left( \max_{\mathcal{A}_{R_1^k,R_2^k}} |u| \right)^p \cdot \text{cap}_p (R_1^k, R_2^k) < +\infty,
\]

then $|Du| \in L^p(M \setminus U)$ and

\[
\|Du\|_{L^p(M \setminus U)} \leq C_2 + C_1 \limsup_{k \to \infty} \left( \max_{\mathcal{A}_{R_1^k,R_2^k}} |u| \right)^p \cdot \text{cap}_p (R_1^k, R_2^k).
\]

(In both cases, $C_1$ depends on $p$, $c_1$, $D_2$ and $C_2$ depends on $p$, $c_1$, $D_2$, $K$, $U$, and $u$.)

**Proof.** (a) Let $V$ be an open set and $R_0 > 0$ s.t. $K \subset V \subset\subset U \subset\subset B(o, R_0)$. For $R_1 > R_0$ and $R_2 > R_0$, consider the function $\eta = \eta_{R_1,R_2}$ as in Lemma 2.2 associated to $V$, $U$, $R_1$ and $R_2$. Note that the function

\[
\varphi := \eta^p(u - I), \quad \text{where} \quad I = \inf_{\mathcal{A}_{R_1,R_2}} u,
\]
belongs to $W^{1,p}_{0}(M \setminus K)$. Hence, using that $u$ is a weak solution of (5), we obtain
\[ \int_F (u - I) \nabla \eta^p \cdot A(\nabla u) \, dx + \int_F \eta^p \nabla u \cdot A(\nabla u) \, dx = 0, \]
where $F \subset B(o, R_2) \setminus V$ is the compact support of $\eta$. Hence, (13) and (17) imply that
\[ c_1 \int_F \eta^p \lvert \nabla u \rvert^p \, dx \leq \int_F \eta^p \nabla u \cdot A(\nabla u) \, dx \]
\[ = - \int_F (u - I) \nabla \eta^p \cdot A(\nabla u) \, dx \]
\[ \leq \int_F p\lvert u - I \rvert \eta^{p-1} \lvert \nabla \eta \rvert \lvert A(\nabla u) \rvert \, dx \]
\[ \leq p D_2 \int_F \lvert u - I \rvert \eta^{p-1} \lvert \nabla \eta \rvert \lvert \nabla u \rvert^{p-1} \, dx. \]

From Hölder inequality,
\[ c_1 \int_F \eta^p \lvert \nabla u \rvert^p \, dx \leq p D_2 \left( \int_F \eta^p \lvert \nabla u \rvert^p \, dx \right)^{\frac{p-1}{p}} \left( \int_F \lvert u - I \rvert \eta^{p-1} \lvert \nabla \eta \rvert \lvert A(\nabla u) \rvert \, dx \right)^{\frac{1}{p}}, \tag{19} \]
that is,
\[ \int_F \eta^p \lvert \nabla u \rvert^p \, dx \leq \left( \frac{p D_2}{c_1} \right)^{\frac{p}{p-1}} \int_F \lvert u - I \rvert \eta^{p-1} \lvert \nabla u \rvert \, dx. \tag{20} \]

Using the hypotheses on the derivative of $\eta$, we obtain
\[ \int_F \lvert u - I \rvert \eta^{p-1} \lvert \nabla \eta \rvert \lvert A(\nabla u) \rvert \, dx \leq 2^p S_0^p \, m^p \text{Vol}(U \setminus V) + \left( \frac{\text{osc}_{\mathcal{A}_{R_1,R_2}} u}{\mathcal{A}_{R_1,R_2}} \right)^p \int_{\mathcal{A}_{R_1,R_2}} \lvert \nabla \eta \rvert^p \, dx, \tag{21} \]
where $S_0 = \sup_{M \setminus V} \lvert u \rvert$. Since $\eta$ is $p$-harmonic in $\mathcal{A}_{R_1,R_2}$, $\eta = 1$ on $\partial B(o, R_1)$ and $\eta = 0$ on $\partial B(o, R_2)$, then
\[ \int_{\mathcal{A}_{R_1,R_2}} \lvert \nabla \eta \rvert^p \, dx = \inf_{v \in \mathcal{F}_{B(o, R_1), B(o, R_2)}} \int_M \lvert \nabla v \rvert^p \, dx = \text{cap}_p (R_1, R_2). \]
Hence, from (20) and (21), we conclude that
\[ \int_F \eta^p |\nabla u|^p \, dx \leq C_1 \, 2^p \, S_0^p \, m^p \, \text{Vol}(U \setminus V) + C_1 \left( \text{osc}_{A_{R_1,R_2}} u \right)^p \text{cap}_p (R_1, R_2), \]
where \( C_1 = (p \, D_2/c_1)^p \). Since \( B(o, R_1) \setminus U \subset F \) and \( \eta = 1 \) in \( B(o, R_1) \setminus U \),
\[ \int_{B(o,R_1) \setminus U} |\nabla u|^p \, dx \leq C_2 + C_1 \left( \text{osc}_{A_{R_1,R_2}} u \right)^p \text{cap}_p (R_1, R_2), \]
where \( C_2 = C_1 \, 2^p \, S_0^p \, m^p \, \text{Vol}(U \setminus V) \). In particular, this holds for \( R_k^1 \) and \( R_k^2 \).

Then, according to the hypotheses, the right-hand side is bounded, proving (a).

(b) Taking the test function \( \varphi := \eta^p u \) and applying the same argument as in (a), we conclude (b). The main difference in relation to (a) is that we have to replace \( S_0 \) by \( \sup_{U \setminus V} |u| \) and \( \text{osc}_{A_{R_1,R_2}} u \) by \( \max_{A_{R_k^1,R_k^2}} |u| \).

The idea of the proof of the next theorem is based on the proof of Theorem 2 of [1].

**Theorem 2.4.** Let \( u, v \in C(M \setminus K) \cap C^1(M \setminus K) \) be weak solutions of (5) in \( M \setminus K \), where \( A \) satisfies (6) - (10) and \( p > 1 \). Assume also that
\[ \sup_k \left( \max_{A_{R_k^1,R_k^2}} |u| \right)^p \text{cap}_p (R_1^k, R_2^k) \quad \text{and} \quad \sup_k \left( \max_{A_{R_k^1,R_k^2}} |v| \right)^p \text{cap}_p (R_1^k, R_2^k) \quad (22) \]
are finite for some sequences \((R_1^k)\) and \((R_2^k)\) such that \( R_2^k > R_1^k \to \infty \). If \( u \leq v \) on \( \partial K \), then
\[ u \leq v \quad \text{in} \quad M \setminus K. \]

**Proof.** Let \((\varepsilon_k)\) be a sequence of positive numbers such that \( \varepsilon_k \downarrow 0 \) and
\[ \lim_{k \to \infty} \varepsilon_k^p \text{cap}_p (R_1^k, R_2^k) = 0. \quad (23) \]
Observe that \( v - u + \varepsilon_k \geq \varepsilon_k \) on \( \partial K \) for any \( k \). Then, using that \( v - u + \varepsilon_k \) is continuous, we conclude that there exists a bounded open set \( U = U_k \supset K \) such that \( v - u + \varepsilon_k > 0 \) in \( U \setminus K \). We can suppose that \( U_k \) is a decreasing sequence and \( \bigcap_{k=1}^\infty U_k = K \). As in the previous proposition, let \( V = V_k \) be
an open set and \( R_0 \geq 1 \) such that \( K \subset V_k \subset U_k \subset U_1 \subset B(o, R_0) \). For \( R_2^k > R_1^k > R_0 \), let \( \eta = \eta_{R_1^k, R_2^k} \) as described in Lemma 2.2. Observe that

\[
(v - u + \varepsilon_k)^- = \max \{- (v - u + \varepsilon_k), 0\} = 0 \quad \text{in} \quad U \setminus K.
\]

Then \( \varphi := \eta^p (v - u + \varepsilon_k)^- \) has a compact support

\[
F_k \subset B(o, R_2^k) \setminus U_k \subset M \setminus K.
\]

Hence, using that \( v - u + \varepsilon_k \in C^1(M \setminus K) \), we have that \( \varphi \in W_0^{1,p}(M \setminus K) \). Since \( u \) and \( v \) are weak solutions of (5) and \( \varphi \in W_0^{1,p}(M \setminus K) \), we get

\[
\int_{M \setminus K} \nabla \varphi \cdot [A(\nabla v) - A(\nabla u)] \, dM = 0.
\] (24)

Observe that \( F_k \subset \{ x \in M \setminus K : v(x) - u(x) + \varepsilon_k \leq 0 \} \) and, therefore,

\[
\nabla \varphi = \eta^p \chi_{F_k} \nabla (v - u) + p \eta^p(v - u + \varepsilon_k)^- \nabla \eta \quad \text{a.e. in} \quad M \setminus K,
\]

where \( \chi_{F_k} \) is the characteristic function of the set \( F_k \). Hence, it follows that

\[
\int_{F_k} \eta^p \nabla (v - u) \cdot [A(\nabla v) - A(\nabla u)] \, dM =
\]

\[
\int_{A_{R_1^k, R_2^k}} p \eta^p(v - u + \varepsilon_k)^- \nabla \eta \cdot [A(\nabla v) - A(\nabla u)] \, dM.
\] (25)

Replacing \( F_k \) by \( G_k := F_k \cap \{ x \in M \setminus K : |\nabla u(x)| + |\nabla v(x)| \neq 0 \} \), the left-hand side of (25) does not change. Then, from (13) and (17), we have

\[
c_1 \int_{G_k} \eta^p (|\nabla v| + |\nabla u|)^{p-2} \nabla (v - u)^2 \, dM \leq
\]

\[
\leq \int_{A_{R_1^k, R_2^k}} p \eta^p(v - u + \varepsilon_k)^- \nabla \eta \cdot [A(\nabla v) - A(\nabla u)] \, dM.
\] (26)

\[
\leq D_{2p} \int_{A_{R_1^k, R_2^k}} \eta^p \nabla \eta \cdot (|\nabla v|^{p-1} + |\nabla u|^{p-1}) \, dM,
\] (27)
where
\[ S_k = \max_{A_{R_k^1, R_k^2}} (v - u + \varepsilon_k)^- \leq \max_{A_{R_k^1, R_k^2}} |u| + \max_{A_{R_k^1, R_k^2}} |v| + \varepsilon_k < \infty. \]

Using Hölder’s inequality, we have
\[
c_1 \int_{G_k} \eta^p \left( |\nabla v| + |\nabla u| \right)^{p-2} |\nabla (v - u)|^2 \, dM \\
\leq D_2 p \int_{A_{R_k^1, R_k^2}} S_k |\nabla \eta| \eta^{p-1} 2 \max \{ |\nabla u|, |\nabla v| \}^{p-1} \, dM \\
\leq 2D_2 p \left( \int_{A_{R_k^1, R_k^2}} S_k^p |\nabla \eta|^p \, dM \right)^{\frac{1}{p}} \left( \int_{A_{R_k^1, R_k^2}} \eta^p \left( |\nabla u|^p + |\nabla v|^p \right) \, dM \right)^{\frac{p-1}{p}}.
\]

Therefore, since $|\eta| \leq 1$ and $p > 1$,
\[
\int_{G_k} \eta^p \left( |\nabla v| + |\nabla u| \right)^{p-2} |\nabla (v - u)|^2 \, dM \\
\leq \frac{2D_2 p}{c_1} \left( \int_{A_{R_k^1, R_k^2}} S_k^p |\nabla \eta|^p \, dM \right)^{\frac{1}{p}} \times \left( \| \nabla u \|_{L^p(A_{R_k^1, R_k^2})}^{p-1} + \| \nabla v \|_{L^p(A_{R_k^1, R_k^2})}^{p-1} \right).
\]

Observe now that
\[
\int_{A_{R_k^1, R_k^2}} S_k^p |\nabla \eta|^p \, dM \\
\leq 3^p \max_{A_{R_k^1, R_k^2}} (v - u + \varepsilon_k)^- \int_{A_{R_k^1, R_k^2}} |\nabla \eta|^p \, dM \\
\leq 3^p \left[ \max_{A_{R_k^1, R_k^2}} |v|^p + \max_{A_{R_k^1, R_k^2}} |u|^p + \varepsilon_k^p \right] \text{cap}_p (R_1^k, R_2^k).
\]

for all $k$. From this and (28), we conclude that
\[
\int_{G_k} \eta^p \left( |\nabla v| + |\nabla u| \right)^{p-2} |\nabla (v - u)|^2 \, dM \\
\leq \frac{2D_2 p}{c_1} 3^p \left[ \max_{A_{R_k^1, R_k^2}} |v|^p + \right. \\
\left. \max_{A_{R_k^1, R_k^2}} |u|^p + \varepsilon_k^p \right] \text{cap}_p (R_1^k, R_2^k) \left( \| \nabla u \|_{L^p(A_{R_k^1, R_k^2})}^{p-1} + \| \nabla v \|_{L^p(A_{R_k^1, R_k^2})}^{p-1} \right).
\]
Note that the last two terms in this inequality converges to zero, according to Proposition 2.3. Moreover,

$$\max_{\mathcal{A}} |v|^p \text{cap}_p (R_1^k, R_2^k) \text{ and } \max_{\mathcal{A}} |u|^p \text{cap}_p (R_1^k, R_2^k)$$

are bounded from hypothesis and \( \varepsilon_k^p \text{cap}_p (R_1^k, R_2^k) \) is bounded from (23). Hence, the right-hand side of (30) converges to 0 as \( k \to \infty \). On the other hand,

$$\int_{G_k} \eta_R^p \left( |\nabla v| + |\nabla u| \right) \left( |v - u| \right)^{p-2} \left( |\nabla (v - u)| \right) dM \to \int_{G_0} \left( |\nabla v| + |\nabla u| \right) \left( |v - u| \right)^{p-2} \left( |\nabla (v - u)| \right) dM$$

as \( k \to \infty \), where

$$G_0 = \{ x \in M \setminus K : v(x) < u(x) \} \cap \{ x \in M \setminus K : |\nabla u(x)| + |\nabla v(x)| \neq 0 \}$$

since \( \eta_{R_1^k, R_2^k} \to 1 \) in \( M \setminus K \) as \( k \to \infty \) and \( G_k \) is an increasing sequence (\( F_k \) is an increasing sequence) of sets such that \( \bigcup G_k = G_0 \). Then

$$\int_{G_0} \left( |\nabla v| + |\nabla u| \right) \left( |v - u| \right)^{p-2} \left( |\nabla (v - u)| \right) dM = 0. \quad (31)$$

Therefore, \( \chi_{G_0} \nabla (v - u) = 0 \). Since

$$\nabla (v - u)^- = \chi_{\{x \in M \setminus K : v(x) < u(x)\}} \nabla (u - v) = \chi_{G_0} \nabla (u - v) \text{ a.e. in } M \setminus K,$$

it follows that \( \nabla (v - u)^- = 0 \) a.e. in \( M \setminus K \). Using this and that \( (v - u)^- = 0 \) on \( \partial K \), we conclude that \( (v - u)^- = 0 \) in \( M \setminus K \). Therefore,

$$u(x) \leq v(x) \text{ for } \ x \in M \setminus K,$$

proving the result.

In particular, if \( u \) and \( v \) are bounded, the control of \( \text{cap}_p (R_1^k, R_2^k) \) guarantees the comparison principle for exterior domains:

**Corollary 2.5.** Let \( M \) be a complete noncompact Riemannian manifold. Let \( A \) be a function that satisfies (6) - (10) and \( p > 1 \). If

$$\sup_k \text{cap}_p (R_1^k, R_2^k) < +\infty,$$

then the comparison principle holds for the exterior problem (5).
Remark 2.6. For \( p = 2 \), Theorem 2.4 holds even if the terms in (22) are not necessarily bounded, provided that

\[
\max_{A_{R_k^1, R_k^2}} |v - u|^p \cap p(R_1^k, R_2^k) \to 0 \quad \text{as} \quad k \to +\infty. \tag{32}
\]

Indeed, according to (25), (15), (16) and the definition of \( F_k \), for \( p = 2 \),

\[
c_1 \int_{F_k} \eta^2 |\nabla (v - u)|^2 dM \leq 2 c_2 \int_{A_{R_k^1, R_k^2} \cap F_k} \eta S_k |\nabla \eta| |\nabla v - \nabla u| dM \leq \left( \int_{F_k} \eta^2 |\nabla (v - u)|^2 dM \right)^{\frac{1}{2}} \left( \int_{A_{R_k^1, R_k^2}} S_k^2 |\nabla \eta|^2 dM \right)^{\frac{1}{2}}.
\]

Therefore

\[
\int_{F_k} \eta^2 |\nabla (v - u)|^2 dM \leq \left( \frac{2 c_2}{c_1} \right)^2 \int_{A_{R_k^1, R_k^2}} S_k^2 |\nabla \eta|^2 dM.
\]

Doing the same computation as in (29), we have

\[
\int_{A_{R_k^1, R_k^2}} S_k^2 |\nabla \eta|^2 dM \leq 2^2 \left[ \sup_{A_{R_k^1, R_k^2}} |v - u|^p + \varepsilon_k^p \right] \text{cap}_p (R_1^k, R_2^k)
\]

and, therefore,

\[
\int_{F_k} \eta^2 |\nabla (v - u)|^2 dM \leq \left( \frac{2 c_2}{c_1} \right)^2 4 \left[ \sup_{A_{R_k^1, R_k^2}} |v - u|^p + \varepsilon_k^p \right] \text{cap}_p (R_1^k, R_2^k).
\]

Hence, using (32) and (23), we have that

\[
\int_{F_k} \eta_{R_k^1, R_k^2}^2 |\nabla (v - u)|^2 dM \to 0.
\]

Then, as in the theorem, relation (31) holds and following the same argument as before, we conclude that \( u \leq v \).
3 Equivalence between the $p$–parabolicity and the $p$–comparison principle

In this section we prove that the $p$-comparison principle, as defined previously, holds for the exterior problem (5) if and only if $M$ is $p$-parabolic.

First note that

$$\operatorname{Cap}_p(E) \leq \operatorname{cap}_p(E; \Omega) \quad \text{and} \quad \operatorname{Cap}_p(E) \leq \operatorname{Cap}_p(F)$$

(33)

for any $\Omega \subset M$ and $E \subset F \subset M$, since $\mathcal{F}_{E,\Omega} \subset \mathcal{F}_{E,M}$ and $\mathcal{F}_{E,M} \supset \mathcal{F}_{F,M}$ in this case.

We also need the next result, that is an extension of the one established in Corollary 4.6 of [5] to the case $p = 2$.

**Lemma 3.1.** If $U \subset M$ is a bounded domain with a $C^2$ boundary and $\operatorname{Cap}_p(U) > 0$, then there exists some non-constant $p$-harmonic function $u$ such that $u = 1$ on $\partial U$ and $0 < u < 1$ in $M \setminus U$.

**Proof.** Let $(W_k)$ be an increasing sequence of bounded domains with $C^2$ boundary such that

$$\bigcup_{k=1}^{\infty} W_k = M \quad \text{and} \quad U \subset \subset W_k \subset \subset W_{k+1} \quad \text{for any } k \in \mathbb{N}.$$  

(If there exists an increasing sequence of balls $B(o, R_k)$ with $C^2$ boundary, where $R_k \to +\infty$, we can take $W_k = B(o, R_k)$.) From the theory for PDE, there exists a function $u_k \in \mathcal{F}_{U,W_k}$ such that

$$\int_M |\nabla u_k|^p \, dx = \inf_{v \in \mathcal{F}_{U,W_k}} \int_M |\nabla v|^p \, dx = \operatorname{cap}_p(U; W_k).$$

(34)

Moreover, $u_k$ is the $p$-harmonic function in the $A_k = A_{U,W_k} := W_k \setminus U$ such that

$$\begin{cases}
    u_k = 1 \text{ on } \partial U & \text{and} \\
    u_k = 0 \text{ on } \partial W_k
\end{cases}$$

in the trace sense.

Due to smoothness of $\partial U$ and $\partial W_k$, the theory of regularity implies that $u_k \in C^1(\overline{A_k})$ (for instance, see [11]). From the strong maximum principle
0 < u_k < 1 in A_k. Hence u_k > 0 on \( \partial W_{k-1} \) for any \( k \geq 2 \), since \( \partial W_{k-1} \subset A_k = W_k \setminus \overline{U} \). Therefore, using the comparison principle, we conclude that

\[ u_{k-1} < u_k \quad \text{in} \quad A_{k-1}. \]  

(35)

Observe also that \( u_k = 1 \) in \( \overline{U} \) and \( u_k = 0 \) in \( M \setminus W_k \), since \( u_k \in \mathcal{F}_{U,W_k} \). (This implies that \( u_k \) is continuous in \( M \) due to the fact that \( u_k \in C^1(A_k) \).) Then, from (35), we have that

\[ u_{k-1} \leq u_k \quad \text{in} \quad M \quad \text{for} \quad k \in \{2, 3, \ldots \}. \]

Hence, using that \( 0 \leq u_k \leq 1 \) for any \( k \), we have that \( u_k \) converges to some function \( u \) defined in \( M \) satisfying \( 0 \leq u \leq 1 \). In particular, \( u = 1 \) in \( \overline{U} \).

Note also that \( u \geq u_k > 0 \) in \( W_k \) for any \( k \). Thus, \( u > 0 \) in \( M \).

**• Statement 1:** \( u \) is \( p \)-harmonic. Observe that for any bounded domain \( V \subset M \setminus \overline{U} \), we have that \( V \subset A_k \) for large \( k \). Then, starting from some large \( k \), \( (u_k) \) is a uniformly bounded sequence of \( p \)-harmonic functions in \( V \). Hence, according to Theorem 1.1 of [19], there exists some constant \( C > 0 \) that depends on \( V, M, n \) and \( p \) such that \( |\nabla u_k| \leq C \) in \( V \). Thus, \( u_k \) is uniformly bounded in \( W^{1,p}(V) \) and, therefore, up to a subsequence, we have that

\[ u_k \to v_0 \quad \text{in} \quad W^{1,p}(V) \quad \text{and} \quad u_k \to v_0 \quad \text{in} \quad L^p(V) \]

for some \( v_0 \in W^{1,p}(V) \), due to the reflexivity of \( W^{1,p}(V) \) and the Rellich-Kondrachov theorem. Indeed, \( v_0 = u \) since \( u_k \to u \) pointwise. Hence, using that \( u_k \) is \( p \)-harmonic and the same argument as in [3] (see Theorem 3 of page 495), we conclude that

\[ 0 = \lim_{k \to \infty} \int_V |\nabla u_k|^{p-2} \nabla u_k \nabla \varphi \, dx = \int_V |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx, \]

for any \( \varphi \in C_c^\infty(V) \), that is, \( u \) is \( p \)-harmonic in \( V \), proving the statement.

**• Statement 2:** \( u \) is non-constant and \( 0 < u < 1 \) in \( M \setminus \overline{U} \). Since we already proved that \( 0 < u \leq 1 \) in \( M \setminus \overline{U} \), from the maximum principle, we just need to show that \( u \) is non-constant. Suppose that \( u \) is constant. Let \( B^* = B(o,R_0) \) be an open ball such that \( \overline{U} \subset B^* \) and \( \rho \in C^\infty(M) \) be a function that satisfies \( 0 \leq \rho \leq 1 \), \( \rho = 0 \) in some neighborhood of \( \overline{U} \) and \( \rho = 1 \)

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in $M \setminus \overline{B^*}$. (We can suppose w.l.o.g. that $\overline{B^*} \subset W_k$ for any $k$.) Since $u_k$ is $\rho$-harmonic in $A_k$ and $\varphi = \rho^p u_k \in W_0^{1,p}(A_k)$, we have

$$- \int_{A_k} p u_k \rho^{-1} |\nabla u_k|^{p-2} \nabla u_k \nabla \rho \, dx = \int_{A_k} \rho^p |\nabla u_k|^p \, dx \geq \int_{M \setminus \overline{B^*}} |\nabla u_k|^p \, dx.$$  

Hence, using (34) and (33), we conclude that

$$- \int_{A_k} p u_k \rho^{-1} |\nabla u_k|^{p-2} \nabla u_k \nabla \rho \, dx + \int_{\overline{B^*}} |\nabla u_k|^p \, dx \geq \int_{M \setminus \overline{B^*}} |\nabla u_k|^p \, dx + \int_{\overline{B^*}} |\nabla u_k|^p \, dx = \text{cap}_p (U; W_k) \geq \text{Cap}_p (U) > 0.$$  

Then, from the fact that $\nabla \rho = 0$ in $M \setminus \overline{B^*}$ and $\nabla u_k = 0$ in $U$, we have

$$- \int_{\overline{B^*} \setminus U} p u_k \rho^{-1} |\nabla u_k|^{p-2} \nabla u_k \nabla \rho \, dx + \int_{\overline{B^*} \setminus U} |\nabla u_k|^p \, dx \geq \text{Cap}_p (U). \quad (36)$$

Now the idea is to show that the left-hand side goes to zero as $k \to +\infty$, leading a contradiction. For that, observe that since $u_1$ is $C^1(A_1)$, $u_1 = 1$ on $\partial U$, $\partial U$ is $C^2$ and $0 < u_1 < 1$ in $A_1$, there exists $c_1 > 0$ such that

$$1 - c_1 \text{dist}(x, \partial U) \leq u_1(x) < 1 \quad \text{for} \quad x \in A_1.$$  

Using that $0 < u_1 \leq u_k < 1$ in $A_1$ for any $k$, we have

$$0 < 1 - u_k(x) \leq \min \{1, c_1 \text{dist}(x, \partial U)\} \quad \text{for} \quad x \in A_1. \quad (37)$$

Furthermore, the inclusion $\overline{B^*} \subset W_1$ implies that there exists $r_0 > 0$ such that $B(x, r_0) \subset W_1$ for any $x \in \overline{B^*}$. Therefore, for any $x \in \overline{B^*} \setminus U$ it follows that

$$B(x, r) \subset W_1 \setminus U = A_1 \quad \text{if} \quad r \leq r_1 := \min \{r_0, \text{dist}(x, \partial U)\}.$$  

Hence, using that $1 - u_k$ is positive and $p$-harmonic in $A_1$, Theorem 1.1 of [19] and (37), we have

$$|\nabla u_k(x)| \leq \tilde{C} \frac{(1 - u_k(x))}{r_1} \leq \tilde{C} \frac{\min \{1, c_1 \text{dist}(x, \partial U)\}}{r_1} \leq \tilde{C} \max \{c_1, 1/r_0\},$$
for \( x \in \overline{B^* \setminus U} \), where \( \tilde{C} > 0 \) is a constant that depends on \( n, p, M \) and \( A_1 \). Therefore, the sequence \( |\nabla u_k| \) is uniformly bounded in \( \overline{B^* \setminus U} \). Hence, if we prove that \( |\nabla u_k| \) converges to zero pointwise in \( \overline{B^* \setminus U} \), then the bounded convergence theorem implies that the left-hand side of (30) converges to zero generating a contradiction. For that, observe that \( u = 1 \) in \( M \), since we are assuming that \( u \) is a constant and \( u = 1 \) in \( U \). From the fact that \( u_k \to u \), we conclude that \( 1 - u_k \to 0 \). Then, using that \( B(x, r_1) \subset A_1 \) for any \( x \in \overline{B^* \setminus U} \) and

\[
|\nabla u_k(x)| \leq \tilde{C} \frac{(1 - u_k(x))}{r_1}
\]
as before, it follows that \( |\nabla u_k(x)| \to 0 \) as \( k \to +\infty \) for \( x \in \overline{B^* \setminus U} \). Therefore, from the bounded convergence theorem we have that left-hand side of (30) converges to zero, contradicting \( \text{Cap}_p(U) > 0 \).

The following result is a consequence of Corollary 1.2.

**Lemma 3.2.** Let \( M \) be a complete noncompact Riemannian manifold, \( p > 1 \) and \( B_0 = B(o, R_0) \) some open ball in \( M \). Then \( \text{Cap}_p(B_0) = 0 \) if and only if there exist two sequences \( R^k_1 \) and \( R^k_2 \) such that \( R^k_2 > R^k_1 \to +\infty \) and \( \text{cap}_p(R^k_1, R^k_2, o) \to 0 \).

**Proof.** Suppose that \( \text{Cap}_p(B_0) = 0 \). Let \( (R^k_1) \) be an increasing sequence such that \( R^k_1 \to +\infty \). Then, from Corollary 1.2, \( \text{Cap}_p(B(o, R^k_1)) = 0 \) for any \( k \). Moreover, for a fixed \( k \) and any sequence \( (R_j) \) satisfying \( R_j \to +\infty \), (33) implies that

\[
0 = \text{Cap}_p(B(o, R^k_1)) = \lim_{j \to +\infty} \text{cap}_p(B(o, R^k_j), B(o, R_j))
\]

Hence, there exists some \( R^k_2 > R^k_1 \) such that

\[
\text{cap}_p(B(o, R^k_1), B(o, R^k_2)) < 1/k,
\]

proving that \( \text{cap}_p(R^k_1, R^k_2, o) \to 0 \). Reciprocally, assume that \( \text{cap}_p(R^k_1, R^k_2, o) \to 0 \), where \( R^k_2 > R^k_1 \to +\infty \). From (33), for \( R^k_1 > R^k_1 \), we have

\[
\text{Cap}_p(B(o, R^k_1)) \leq \text{Cap}_p(B(o, R^k_1)) \leq \text{cap}_p(R^k_1, R^k_2, o) \to 0.
\]

Therefore, Corollary 1.2 implies that \( \text{Cap}_p(B_0) = 0 \). \( \square \)
Combining this lemma with the results of the previous section, we obtain a comparison principle for exterior problems where the operator involved is more general than the $p$-laplacian operator. Since the comparison principle is an important issue in the PDE theory, the following result that holds for a larger class of operators might be interesting by itself.

**Theorem 3.3.** Let $M$ be a complete noncompact Riemannian manifold, $A$ be a function that satisfies (6) - (10) and $p > 1$. If $M$ is $p$-parabolic, then the comparison principle holds for the exterior problem (5).

**Proof.** Let $u, v \in C(M \setminus K) \cap C^1(M \setminus K)$ be bounded weak solutions of (5) such that $u \leq v$ on $\partial K$, where $K \subset M$ is a compact set. Consider a ball $B_0 = B(o, R_0)$ such that $K \subset B_0$. Then $\text{Cap}_p(B_0) = 0$, since $M$ is $p$-parabolic. Therefore, Lemma 3.2 implies that there exist two sequences $R_1^k$ and $R_2^k$ such that $R_2^k > R_1^k \to +\infty$ and $\text{cap}_p(R_1^k, R_2^k, o) \to 0$. Hence, using that $u$ and $v$ are bounded, we have that

$$\max_{\partial R_1^k, R_2^k} |u|^p \text{ cap}_p(R_1^k, R_2^k) \quad \text{and} \quad \max_{\partial R_1^k, R_2^k} |v|^p \text{ cap}_p(R_1^k, R_2^k)$$

are bounded. Then, from Theorem 2.4 we conclude that $u \leq v$ in $M \setminus K$. \qed

Now we obtain the equivalence between (a) and (b) of Theorem 1.3 in the following result:

**Theorem 3.4.** Let $M$ be a complete noncompact Riemannian manifold and $p > 1$. The following are equivalents

(i) $M$ is $p$-parabolic;

(ii) the comparison principle holds for the exterior problem

$$\Delta_p v = 0 \quad \text{in} \quad M \setminus K,$$

(iii) the comparison principle holds for the exterior problem (38) for some compact set $K_0 = \overline{U_0}$, where $U_0 \neq \emptyset$ is some bounded open set with $C^2$ boundary.

**Proof.** (iii) $\Rightarrow$ (i) : Suppose that $M$ is not $p$-parabolic. Then, $\text{Cap}_p(E) > 0$ for some compact set $E$. Since, we are assuming (iii), the comparison principle holds for some compact $K_0 = \overline{U_0}$, where $U_0 \neq \emptyset$ is open. Then, Corollary 1.2 implies that $\text{Cap}_p(K_0) > 0$. Remind also that $\partial U_0$ is $C^2$. 18
Therefore, from Lemma 3.1, there exists a $p$-harmonic function $w$ such that $w = 1$ on $\partial K_0$ and $0 < w < 1$ in $M \setminus K_0$. Let $u$ be defined by $u = 1$ in $M$. Thus $u$ and $w$ are bounded $p$-harmonic functions, $u \leq w$ on $\partial K_0$, but $w < 1 = u$ in $M \setminus K_0$. That is, the comparison principle does not hold for $K_0 = U_0$, contradicting the hypothesis. Hence $M$ is $p$-parabolic.

$(i) \Rightarrow (ii)$: Assuming that $M$ is $p$-parabolic, the comparison principle for the exterior domains with the $p$-laplacian operator is a consequence of Theorem 3.3.

$(ii) \Rightarrow (iii)$: Trivial.

From the previous results we obtain the following property about the $p$-capacity, that corresponds the equivalence between (a) and (c) of Theorem 1.3:

**Corollary 3.5.** Let $M$ be a complete noncompact Riemannian manifold and $p > 1$. There exist two sequences $R_1^k$ and $R_2^k$ such that $R_2^k > R_1^k \to +\infty$ and

$$\text{cap}_p (R_1^k, R_2^k, o) \to 0$$

if and only if there exist two sequences $\tilde{R}_1^k$ and $\tilde{R}_2^k$ such that $\tilde{R}_2^k > \tilde{R}_1^k \to +\infty$ and

$$\sup_k \text{cap}_p (\tilde{R}_1^k, \tilde{R}_2^k, o) < +\infty.$$

Moreover, $M$ is $p$-parabolic if and only if some of these two conditions holds.

**Proof.** If $M$ is $p$-parabolic, then Lemma 3.2 guarantees that there exist two sequences $R_1^k$ and $R_2^k$ such that $R_2^k > R_1^k \to +\infty$ and $\text{cap}_p (R_1^k, R_2^k, o) \to 0$. Therefore, it is trivial that $\sup_k \text{cap}_p (R_1^k, R_2^k, o) < +\infty$.

Hence, if $M$ is $p$-parabolic then the first condition holds which implies the second one.

Now suppose that

$$\sup_k \text{cap}_p (\tilde{R}_1^k, \tilde{R}_2^k, o) < +\infty.$$

Hence, from Corollary 2.5, the comparison principle holds for the exterior problem (38). Then, Theorem 3.4 implies that $M$ is $p$-parabolic, concluding the proof. □
4 Comparison Principle under volume growth conditions

In [10], Holopainen proves that $M$ is $p$-parabolic assuming the following condition on the volume growth of geodesic balls:

$$\int_{-\infty}^{+\infty} \left( \frac{R}{V(R)} \right)^{\frac{1}{p-1}} dR = +\infty \quad \text{or} \quad \int_{-\infty}^{+\infty} \left( \frac{1}{V'(R)} \right)^{\frac{1}{p-1}} dR = +\infty, \quad (39)$$

for $p > 1$, where $V(R) = \text{Vol}(B(o,R))$. Therefore, according to Theorem 1.3, the $p$-comparison principle for exterior domains in $M$ holds if those conditions are satisfied.

In this section we follow a different kind of assumptions, using Theorem 2.4 to obtain comparison principles. One advantage is that we can prove results even for solutions that are not bounded a priori and for manifolds that are not $p$-parabolic, provided there is some relation between the growth of the solution and the volume of the geodesic balls.

We have to observe the following:

Remark 4.1. For $R_2 > R_1 > 0$ and $o \in M$, there exists a function $w_0 \in \mathcal{F}_{B(o,R_1),B(o,R_2)}$ such that $|\nabla w_0(x)| \leq 2/(R_2 - R_1)$ for any $x \in A_{R_1,R_2} = B(o,R_2) \setminus B(o,R_1)$. For instance, take $w_0$ as a mollification of

$$w_1(x) = \begin{cases} 1 & \text{if } \text{dist}(x,o) \leq R_1 \\ \frac{R_2 - \text{dist}(x,o)}{R_2 - R_1} & \text{if } R_1 < \text{dist}(x,o) < R_2 \\ 0 & \text{if } \text{dist}(x,o) \geq R_2. \end{cases}$$

Then

$$\text{cap}_p (R_1, R_2, o) \leq \int_{A_{R_1,R_2}} |\nabla w_0|^p dx \leq \left( \frac{2}{R_2 - R_1} \right)^p \text{Vol}(A_{R_1,R_2}). \quad (40)$$

As a consequence of Remark 4.1 and Theorem 2.4, we have the following result that can be applied for possibly unbounded solutions or non $p$-parabolic manifolds:
Theorem 4.2. Let $u, v \in C(M\setminus K) \cap C^1(M\setminus K)$ be weak solutions of (5) in $M\setminus K$, where $A$ satisfies (6) - (10) and $p > 1$. Assume also that $\max_{A_{R^k_1, R^k_2}} |u|_p \frac{\text{Vol}(A_{R^k_1, R^k_2})}{(R_2^k - R_1^k)^p}$ and $\max_{A_{R^k_1, R^k_2}} |v|_p \frac{\text{Vol}(A_{R^k_1, R^k_2})}{(R_2^k - R_1^k)^p}$ are bounded sequences for some $(R^k_1)$ and $(R^k_2)$ such that $R^k_2 > R^k_1 \to \infty$.

If $u \leq v$ on $\partial K$, then $u \leq v$ in $M\setminus K$.

Corollary 4.3. Assume the same hypotheses as in the previous theorem. Then the solutions $u$ and $v$ are bounded.

Proof. First Case: The sequence of quotients

$$Q_k := \frac{\text{Vol}(A_{R^k_1, R^k_2})}{(R_2^k - R_1^k)^p}$$

is also bounded for the same $(R^k_1)$ and $(R^k_2)$ given by Theorem 4.2. Observe that the constant function $w = \max_{\partial K} v$ is a solution (5). Since the sequence in (42) is bounded, then the sequence of (41) with $u$ replaced by $w$ is also bounded. Then we can apply the previous theorem to $w$ and $v$. Hence $v \leq w = \max_{\partial K} v < +\infty$. Similarly, $v$ is bounded from below by $\min_{\partial K} v$. The argument for $u$ is the same.

Second case: The sequence of ratios $Q_k$ is not bounded. Then there exists some subsequence $Q_{kj}$ such that $Q_{kj} \to +\infty$. Hence, from the boundedness of the sequences given in (41), we have

$$\max_{A_{R^k_1, R^k_2}} |u|_p \to 0.$$ 

In particular, this sequence is bounded by some constant $C > 0$. Therefore, $|u| \leq C^{1/p}$ on the spheres $\partial B(o, R_{kj}^k)$ for any $j$. Since $u$ satisfies the maximum principle, we get

$$|u| \leq D := \max_{\partial K} \{C^{1/p}, \max_{\partial K} |u|\} < +\infty \quad \text{in} \quad B(o, R_{kj}^k) \setminus K \quad \text{for any} \ j.$$ 

Using that $R_{kj}^k \to +\infty$, we conclude that $|u| \leq D$ in $M\setminus K$. The same holds for $v$. \hfill \square
**Corollary 4.4.** Assume the same hypotheses as in Theorem 4.2. If \( u = v \) on \( \partial K \), then \( u = v \) in \( M \setminus K \). Moreover \( u \) is bounded.

Observe that condition (41) holds, for example, if there exist \( C > 0, q > 0 \) and some sequence \( R_k^2 \to +\infty \) such that

\[
\max_{B(o,R_k^2)} |u| \leq C(R_k^2)^{(p-q)/p}, \quad \max_{B(o,R_k^2)} |v| \leq C(R_k^2)^{(p-q)/p}
\]

and

\[
\text{Vol}(B(o,R_k^2)) \leq C(R_k^2)^q.
\]

In particular, condition (41) is satisfied if we assume that

\[
\max_{B(o,R)} |u|, \max_{B(o,R)} |v| \leq CR^{(p-q)/p} \quad \text{and} \quad \text{Vol}(B(o,R)) \leq CR^q, \quad (43)
\]

for any \( R \geq R_0 \), where \( R_0 > 0 \) is any fixed positive. Hence we have the following result:

**Corollary 4.5.** Let \( u, v \in C(M \setminus K) \cap C^1(M \setminus K) \) be weak solutions of (5) in \( M \setminus K \), where \( A \) satisfies (6) - (10) and \( p > 1 \). If (43) holds and \( u \leq v \) on \( \partial K \), then \( u \leq v \) in \( M \setminus K \). Moreover \( u \) and \( v \) are bounded.

From this corollary, for \( p > q \), we do not need to assume that the solutions are bounded to have a comparison principle. It is sufficient that they satisfy (43). Anyway we conclude that they are bounded from Corollary 4.3. On the other hand, if \( p < q \), we cannot guarantee that \( M \) is \( p \)-parabolic, since the quotient given by (42) may diverge to infinity. Still, we have some comparison principle provided we assume that \( u \) and \( v \) go to zero at infinity with some speed.

**Remark 4.6.** If the sequence of \( Q_k \), defined in (42), is bounded, then the sequence of capacities \( \text{cap}_p(R_1^k, R_2^k, o) \) is also bounded, according to (40). Then, from (c) of Theorem 1.3, we conclude that \( M \) is \( p \)-parabolic. Therefore, from Theorem 3.3, it holds the comparison principle for bounded solutions.

This implies the uniqueness of bounded solution for a Dirichlet problem in exterior domains. Moreover, from the fact that \( v \equiv \text{const.} \) is a bounded solution of (5), we have the following extension of Liouville’s result to exterior domains:
**Corollary 4.7.** Assume the same hypotheses as in Theorem 4.2 about $M$, $K$, $p$ and $A$. Suppose that the quotient $Q_k$ of (42) is bounded (or simply, $M$ is $p$-parabolic). If $u$ is a bounded weak solution of (5) and $u$ is constant on $\partial K$, then $u$ is constant. More generally, given a continuous function $\phi$ on $\partial K$, there exists at most one bounded solution of (5) such that $u = \phi$ on $\partial K$.

### 5 Comparison principle for rotationally symmetric manifolds

According to Milnor’s lemma in \[14\], a complete rotationally symmetric 2-dimensional Riemannian manifold with the metric $ds^2 = dr^2 + f^2(r)d\theta$ is parabolic if and only if

$$\int_a^\infty \frac{1}{f(r)} dr = +\infty$$

for some $a > 0$. For higher dimension and $p > 1$, a complete rotationally symmetric $n$-dimensional Riemannian manifold $M = \mathbb{R}^+ \times S^{n-1}$ with respect to a point $o \in M$ endowed with the metric

$$ds^2 = dr^2 + f^2(r)d\omega^2,$$

where $r = \text{dist}(x, o)$, $d\omega^2$ is the standard metric of $S^{n-1}$ and $f$ is a $C^1$ positive function in $(0, +\infty)$, is $p$-parabolic if and only if

$$\int_a^\infty f^{-\frac{n-1}{p-1}}(r) dr = +\infty \quad \text{for some } a > 0. \quad (44)$$

Indeed, that the divergence of this integral implies the $p$-parabolicity is a consequence of the second integral condition in (39) (or Ilkka’s condition) and the fact that the volume of the ball $B(o, r)$ is $V(r) = n\omega_n \int_0^r (f(s))^{n-1} ds$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. The converse is true since the convergence of this integral implies the existence of a nonconstant bounded $p$-superharmonic function: for instance, consider the function $\eta_{a, +\infty}$ defined by $\eta_{a, +\infty}(x) = 1$ if $\text{dist}(x, o) \leq a$ and

$$\eta_{a, +\infty}(x) = \frac{\int_{\text{dist}(x, o)}^{+\infty} f^{-\frac{n-1}{p-1}}(s) ds}{\int_a^{+\infty} f^{-\frac{n-1}{p-1}}(s) ds} \quad \text{if } \text{dist}(x, o) > a,$$
that is $p$-harmonic outside the ball $B(o,a)$ as we will see later and, therefore, $p$-superharmonic in $M$. Hence, from Theorems 3.4 and 3.3, we conclude respectively the following results:

**Corollary 5.1.** Let $M = \mathbb{R}^+ \times S^{n-1}$ be a complete rotationally symmetric $n$-dimensional Riemannian manifold with respect to a point $o \in M$ endowed with the metric
\[ d s^2 = d r^2 + f^2(r) d \omega^2. \]
Then the comparison principle holds for the exterior problem $\Delta_p v = 0$ in $M \setminus K$, where $K$ is any compact set, if and only if condition (44) holds.

**Corollary 5.2.** Let $(M, ds^2)$ be as in the previous corollary and assume $A$ satisfies (6) - (10) for $p > 1$. Suppose also that $u, v \in C(M \setminus K) \cap C^1(M \setminus K)$ are bounded weak solutions of (5) in $M \setminus K$, where $K$ is a compact set of $M$, and that condition (44) is satisfied. If $u \leq v$ on $\partial K$, then
\[ u \leq v \quad \text{in} \quad M \setminus K. \]
In particular, if $u$ is constant on $\partial K$, then $u$ is constant in $M \setminus K$.

Now we show some comparison result that holds for non bounded solutions or hyperbolic manifolds. Taking the point $o$ as the reference, let $\eta = \eta_{R_1,R_2}$ be the function defined in Lemma 2.2 for $R_2 > R_1 > 0$. Since $\eta$ is $p$-harmonic in $\mathcal{A}_{R_1,R_2}$, it is radially symmetric due to the symmetry of $M$ with respect to $o$. Then $\eta$ is a function of $r$ in $\mathcal{A}_{R_1,R_2}$ and satisfies the equation
\[ \frac{\Delta_p \eta}{|\nabla \eta|^{p-2}} = (p-1)\eta''(r) + (n-1)\frac{f'(r)}{f(r)} \eta'(r) = 0 \quad \text{for} \quad r \in (R_1, R_2), \]
where the prime denotes the derivative with respect to $r$. Moreover, $\eta(R_1) = 1$ and $\eta(R_2) = 0$. Hence
\[ \eta(r) = \frac{\int_{R_1}^{R_2} f^{\frac{1-n}{p-1}}(s) \, ds}{\int_{R_1}^{R_2} f^{\frac{1-n}{p-1}}(s) \, ds}. \]
Then, using that the element of volume is $dx = n\omega_n f^{n-1}(r) dr$, we have
\[ \int_{\mathcal{A}_{R_1,R_2}} |\nabla \eta|^p \, dx = n\omega_n \int_{R_1}^{R_2} (\eta')^p f^{n-1}(r) \, dr = \frac{n\omega_n}{\left(\int_{R_1}^{R_2} f^{\frac{1-n}{p-1}}(s) \, ds\right)^{p-1}}. \]
Observe that the capacity \( \text{cap}_p (R_1, R_2) := \text{cap}_p (B(o, R_1); B(o, R_1)) \) is attained at \( \eta_{R_1, R_2} \), since it is harmonic and, therefore, minimizes the Dirichlet integral over \( A_{R_1, R_2} \) in the set

\[
\{ v \in C^0(A_{R_1, R_2}) \cap C^1(A_{R_1, R_2}) : v = 1 \text{ on } \partial B(o, R_1) \text{ and } v = 0 \text{ on } \partial B(o, R_2) \}.
\]

Therefore, from the last equation,

\[
\left( \max_{A_{R_1, R_2}} |u| \right)^p \text{cap}_p (R_1, R_2) = n\omega_n \left( \max_{A_{R_1, R_2}} |u| \right)^p \left( \int_{R_1} f^{-\frac{n-1}{p-1}} (s) \, ds \right)^{-(p-1)}
\]

From this expression, Theorem 2.4 and Remark 2.6, we have the following:

**Corollary 5.3.** Assume the same hypotheses as in the previous corollary about \( M, K \) and \( A \). Let \( u, v \in C(M \setminus K) \cap C^1(M \setminus K) \) be weak solutions of (5) in \( M \setminus K \). Suppose also that

\[
\max_{A_{R_1, R_2}} |u|, \quad \max_{A_{R_1, R_2}} |v|
\]

are bounded,\(^{(45)}\)

where \( R^k_1 \) and \( R^k_2 \) are sequences such that \( R^k_2 > R^k_1 \to +\infty \). For \( p = 2 \), this condition can be replaced by

\[
\max_{A_{R_1, R_2}} |v - u|^2 \to 0.
\]

If \( u \leq v \) on \( \partial K \), then

\[
u \leq v \quad \text{in} \quad M \setminus K.
\]

As an application we present a result where the functions \( u \) and \( v \) can go to infinity, provided that their growth are bounded by some specific function.

**Corollary 5.4.** Let \( (M, ds^2) \) be as in the previous corollary, where \( 0 < f(r) \leq E_1r \) for some \( E_1 > 0 \) and any large \( r \). Suppose that \( A \) satisfies (6)
- (10) with \( p = n \). Let \( K \subset M \) be a compact set and \( u, v \in C(M \setminus K) \cap C^1(M \setminus K) \) be weak solutions of (5) in \( M \setminus K \) such that

\[
|u(r)| \leq E_2 \left( \ln r \right)^{\frac{n-2}{n}} \quad \text{and} \quad |v(r)| \leq E_2 \left( \ln r \right)^{\frac{n-2}{n}}
\]

(47)

for some \( E_2 > 0 \) and any large \( r \). If \( u \leq v \) on \( \partial K \), then

\[
u \leq v \quad \text{in} \quad M \setminus K.
\]

**Proof.** By hypothesis, there exists \( R_0 > 0 \) such that \( f(r) \leq E_1 r \) for \( r \geq R_0 \). Then,

\[
\int_{R_1^k}^{R_2^k} \frac{1}{f(s)} \, ds \geq \int_{R_1^k}^{R_2^k} \frac{1}{E_1 s} \, ds = \frac{1}{E_1} \ln \left( \frac{R_2^k}{R_1^k} \right) \quad \text{for} \quad R_2^k > R_1^k > R_0.
\]

Let \((R_k)\) be a sequence that goes to infinity. Take

\[
R_1^k = R_k \quad \text{and} \quad R_2^k = (R_1^k)^2 = (R_k)^2.
\]

Hence, using the last inequality, we have

\[
\int_{R_1^k}^{R_2^k} \frac{1}{f(s)} \, ds \geq \frac{1}{E_1} \ln R_k.
\]

Therefore, using the growth hypothesis about \( u \) and \( v \), we have

\[
\max_{\lambda R_1^k, R_2^k} |u|^{n^2} \leq \frac{E_1^{n-1} E_2^n \left( \ln R_2^k \right)^{n-1}}{\left( \ln R_k \right)^{n-1}} = (2E_1)^{n-1} E_2^n
\]

and a similar estimate for \( v \). Then, from Corollary 5.3 we conclude the result. \( \square \)

**Remark 5.5.** This corollary holds, for instance, for any complete noncompact rotationally symmetric Riemannian manifold such that the sectional curvature is nonnegative, since \( f(r) \leq r \) in this case.

It can be applied also for some Hadamard manifolds provided the curvature goes to zero sufficiently fast.
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