ON WARD-TAKAHASHI IDENTITIES FOR THE PARISI SPIN GLASS

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Abstract. The introduction of “small permutations” allows us to derive Ward-Takahashi identities for the spin-glass, in the Parisi limit of an infinite number of steps of replica symmetry breaking. The first identities express the emergence of a band of Goldstone modes. The next identities relate components of (the Replica Fourier Transformed) 3-point function to overlap derivatives of the 2-point function (inverse propagator). A jump in this last function is exhibited, when its two overlaps are crossing each other, in the special simpler case where one of the cross-overlaps is maximal.

This work is a tribute to the memory of Giovanni Paladin

1 Introduction and summary

The breaking of a continuous symmetry group is known to generate massless or Goldstone [1] modes in the broken symmetry phase. Infrared divergences associated with massless-modes usually complicate the renormalization process inside that phase. However the broken invariance generates Ward-Takahashi (W-T) identities, the first of which imposes masslessness. Higher order identities are then instrumental in the taming of proliferating divergences into relationships between renormalized quantities [2, 3].

For systems with quenched disorder whose broken invariance group is the permutation group of replicas, the affair is more subtle. The replica symmetry may undergo \( R \) steps of breaking [4]. For \( R = 1 \) (or \( R \) finite) there will be no Goldstone mode. For the Parisi limit [4] \( R \to \infty \), one finds bands of massless modes [5, 6, 7] and highly singular bare propagators [8]. Thus the need for help from W-T identities is even more acute in this case.

In a separate publication [9] hereafter called (I) the basic approach to a derivation of W-T identities was presented with the introduction of “small permutations” (the small parameter being e.g. \( p_r - p_{r+1} \approx 1/R \), the size difference between consecutive Parisi boxes). There, the result for the first W-T identities, that, in particular, exhibit Goldstone modes, was presented. In this work we shall follow the same approach. In the Parisi limit, invariance properties, associated with the replica permutation group, become hidden, and a new invariance, the so called reparametrization invariance, emerges. Their interrelationship, examined in (I) will not be further investigated here.

In Section 2 we recall briefly relationships leading to W-T identities in the continuum limit. Parametrization of replicas and “small permutations” are defined in Section 3. We recall the 2-point function parametrization in Section 4. The first W-T identity, yielding a continuity condition, is derived in Section 5. The W-T identity, imposing masslessness is derived in Section 6. Section 7 is devoted to a minimal discussion of the parametrization for 3-point functions. Finally sections 8–10 contain the derivation of W-T identities relating 3-point and 2-point functions. To keep the developments to a reasonable size, while still giving a detailed derivation, we have chosen the simpler case where one of the cross-overlaps is maximal (and equal to \( R + 1 \)).

2 Invariance properties and W-T relationships:

Let us consider the free energy functional \( F \{ \bar{q} \} \) where \( \bar{q} \) is the \( n(n-1)/2 \) dimensional order parameter vector \( q_{\alpha\beta} \) (with \( \alpha = 1, 2, \ldots n; q_{\alpha\beta} = q_{\beta\alpha}; q_{\alpha\alpha} = 0 \)). Let \( \bar{h} \) be a general source conjugate to \( \bar{q}, F \) being the Legendre transform of \( W \{ \bar{h} \} = \ln Z \)

\[
W \{ \bar{h} \} + F \{ \bar{q} \} = \bar{h} \cdot \bar{q} = \frac{1}{2} \sum_{\alpha,\beta} h_{\alpha\beta} q_{\alpha\beta}
\]
By construction $W$ and $F$ are invariant under a replica permutation $P$, which is also a permutation i.e. a rotation of the $n \ (n - 1)/2$ axes, changing $\bar{q}$ into $\bar{q}'$

$$\bar{q}' = P \bar{q} \tag{2.2}$$

$$P q_{\alpha \beta} = q_{P \alpha, P \beta} \tag{2.3}$$

We thus have from replica permutation invariance,

$$F \{\bar{q}'\} = F \{\bar{q}\} \tag{2.4}$$

From (2.1) we also have

$$\frac{\delta F \{\bar{q}\}}{\delta q_{\alpha \beta}} = h_{\alpha \beta} \tag{2.5}$$

and the identity

$$\frac{\delta F \{\bar{q}'\}}{\delta q_{\alpha \beta}} = h'_{\alpha \beta}. \tag{2.6}$$

This last equation states that in (2.5) it is equivalent to rotate the source $\bar{h}$ or to rotate the order parameter $\bar{q}$, $\partial F/\partial \bar{q}$ is functionally depending upon. Separating out in $P$, the identity operator

$$P \equiv 1 + \delta P \tag{2.7}$$

$$\delta P q_{\alpha \beta} = q_{P \alpha, P \beta} - q_{\alpha \beta} \tag{2.8}$$

we have

$$\frac{\partial F}{\partial \bar{q}} \{\bar{q} + \delta P \bar{q}\} = \bar{h} + \delta P \bar{h} \tag{2.9}$$

i.e. by Taylor expanding around $\bar{q}$

$$\frac{1}{2} \sum_{\mu \nu} \frac{\partial^2 F}{\partial q_{\alpha \beta} \partial q_{\mu \nu}} \{\bar{q}\} \delta P q_{\mu \nu} + ... = \delta P h_{\alpha \beta} \tag{2.10}$$

Under a “small permutation”, one can neglect higher order terms in (2.10), and recover the first W-T identity relating 2-point and 1-point functions. Applying the same procedure (2.6–9) to $\partial^2 F/\partial q_{\alpha \beta} \partial q_{\gamma \delta}$ will produce the next set of identities relating 3-point and 2-point functions.

3 “Small permutations$^1$”:

Let us parametrize a replica $\alpha$ by its address, i.e. the list of branch numbers

$$\alpha : [a_\alpha, a_1, a_2...a_R] \tag{3.1}$$

one has to follow to reach replica $\alpha$ at the bottom of the (ultrametric) tree. Here

$^1$ The first attempt at building an “infinitesimal permutation” can be found in ref.[10]. A general discussion of these transformations has been given by Goltsev[11]. Independently Parisi and Slanina[12] have introduced similar constructions in a random polymer context.
\[ a_o = 0, 1, 2, \ldots (p_o/p_1 - 1) \]
\[ a_1 = 0, 1, 2, \ldots (p_1/p_2 - 1) \]  
\[ a_r = 0, 1, 2, \ldots (p_r/p_{r+1} - 1) \]

and \( a_r \) is numbering the \( p_r/p_{r+1} \) branches descending from a node of level \( r \). Two replicas \( \alpha, \beta \) have an overlap \( r \)

\[ \alpha \cap \beta = r \]  

if

\[
\alpha = \begin{bmatrix} a_o a_1 \ldots a_{r-1} & a_r a_{r+1} \ldots a_R \\
 a_o a_1 \ldots a_{r-1} & b_r b_{r+1} \ldots b_R \end{bmatrix} \quad a_r \neq b_r
\]

When \( q_{\alpha \beta} \) takes its saddle-point value, we then have

\[ q_{\alpha \beta} = q_r \]  

We now define a “small permutation” \( P^{(r)} \) by its action upon the addresses. For example, for all replicas with a given fixed \( a_{r+1} \), say

\[ a_{r+1} = 0_{r+1} \]  

and only for those, the action of \( P^{(r)} \) will be to change \( a_r \) into \( 1 + a_r \). For all other replicas, \( P^{(r)} \) will act as the identity operator,

\[
P^{(r)} = P^{(r)} \left[ a_o, a_1 \ldots a_r, a_{r+1} \ldots a_R \right] = \left[ a_o, a_1 \ldots a_r, a_{r+1} \ldots a_R \right] \left( 1 - \delta_{a_{r+1};0_{r+1}} \right) + \left[ a_o, a_1, \ldots 1 + a_r, a_{r+1} \ldots a_R \right] \delta_{a_{r+1};0_{r+1}}
\]  

This choice of permutation will be kept throughout the paper. In words, what our chosen \( P^{(r)} \) does is the following: consider, in the ultrametric tree, a given node at level \( r \) and its \( p_r/p_{r+1} \) nodes at level \( r + 1 \) descending from it. From each one of these \( p_r/p_{r+1} \) nodes emerge \( p_{r+1}/p_{r+2} \) branches. Select the 0th one (and its descendent) and the set of replicas it leads to. We have \( p_r/p_{r+1} \) such sets. \( P^{(r)} \) circularly permutes en bloc, those \( p_r/p_{r+1} \) sets of replicas. What we have just said for one given node at level \( r \), is valid for all such \( p_o/p_r \) nodes.

Let us compute now the action of \( P^{(r)} \) upon \( q_{\alpha \beta} \), i.e.

\[
\delta P^{(r)} q_{\alpha \beta} = q_{P^{(r)};P^{(r)}} - q_{\alpha \beta}.
\]  

If \( \alpha \cap \beta = r + 1 \), i.e. for \( b_{r+1} \neq 0_{r+1} \), we have

\[
\delta P^{(r)} q_{\alpha \beta} = q \left[ \begin{array}{ccc} a_o \ldots a_{r-1} & 1 + a_r & 0_{r+1} \ldots a_R \\
 & a_r & b_{r+1} \ldots b_R \end{array} \right] - q \left[ \begin{array}{ccc} a_o \ldots a_{r-1} & a_r & 0_{r+1} \ldots a_R \\
 & a_r & b_{r+1} \ldots b_R \end{array} \right] = q_r - q_{r+1}
\]  

We could have also numbered \( a_o = 1, 2, \ldots p_o/p_1 \). One should think of \( a_o \) as taking values on a circle i.e. mod \((p_o/p_1)\).
If $\alpha \cap \beta = r$, we have, if and only if $b_{r+1} \neq 0_{r+1}$

$$
\delta P^{(r)} Q_{\alpha \beta} = \begin{bmatrix}
q_{-1} \cdots q_{-1} & q_{0} \cdots q_{0} \\
q_{1} \cdots q_{1} & q_{2} \cdots q_{2}
\end{bmatrix}
- q \begin{bmatrix}
-1 + a_{r} & 0_{r+1} \cdots a_{R} \\
a_{r} & b_{r+1} \cdots b_{R}
\end{bmatrix}
$$

$$
= q_{r+1} - q_{r}
$$

(3.10)

and likewise under $\alpha, \beta$ exchange. For all other components $q_{\alpha \beta}$, we have

$$
\delta P^{(r)} Q_{\alpha \beta} = 0
$$

(3.11)

When $R$ is large (and in particular in the Parisi limit) we obtain (except for $r = R$),

$$
q_{r} - q_{r+1} \sim p_{r} - p_{r+1} \sim O(1/R)
$$

(3.12)

allowing us to discard higher order terms in the Taylor expansion (2.9). We can then write

$$
\frac{1}{2} \sum_{\mu \nu} \frac{\partial^{2} F}{\partial q_{\alpha \beta} \partial q_{\mu \nu}} \delta P^{(r)} q_{\mu \nu} = \delta P^{(r)} h_{\alpha \beta} + O(1/R)
$$

(3.13)

$$
\frac{1}{2} \sum_{\mu \nu} \frac{\partial^{2} F}{\partial q_{\alpha \beta} \partial q_{\gamma \delta} \partial q_{\mu \nu}} \delta P^{(r)} q_{\mu \nu} = \delta P^{(r)} \frac{\delta^{2} F}{\delta q_{\alpha \beta} \delta q_{\gamma \delta}} + O(1/R)
$$

(3.14)

where, in the right side of this equation $\delta P^{(r)}$ acts upon $\alpha \beta \gamma \delta$.

We have now to implement these relationships using explicit parametrizations for the two and three-point functions, namely

$$
\frac{\partial^{2} F}{\partial q_{\alpha \beta} \partial q_{\mu \nu}} \equiv M^{\alpha \beta ; \mu \nu}
$$

(3.15)

the “mass-operator” (inverse propagator), and

$$
\frac{\partial^{3} F}{\partial q_{\alpha \beta} \partial q_{\gamma \delta} \partial q_{\mu \nu}} \equiv W^{\alpha \beta ; \gamma \delta ; \mu \nu}
$$

(3.16)

4 2-point function parametrization:

We give now a minimal analysis of the 2-point function $M^{\alpha \beta ; \gamma \delta}$, i.e. the (zero momentum) inverse propagator. Details can be found in [8] and for the Replica Fourier Transform (RFT) approach in [13].

With four replicas, $M^{\alpha \beta ; \gamma \delta}$ will depend upon three overlaps.

(i) If generically $\alpha \cap \beta \equiv \gamma \cap \delta = r$, (Replicon configurations), $M$ depends upon two cross-overlaps $u, v \geq r+1$:

$$
M^{r} u v = \max (\alpha \cap \gamma, \alpha \cap \delta) \quad v = \max (\beta \cap \gamma, \beta \cap \delta)
$$

(4.1)

The double RFT defines the Replicon kernel

$$
RM_{R,k}^{\alpha \beta} = \sum_{u=k}^{R} \sum_{v=k}^{R+1} p_{u} p_{v} \left( M^{r} u v - M^{r} u v - M^{r} u v + M^{r} u v \right)
$$

(4.2)

for $k, \ell \geq r + 1$. Hatted variables stand for RFT ones, this notation is preferred here to the one that uses a different symbol for kernels (i.e. RFT’s), so that we have the possibility of mixed (or incomplete) transforms.
The Replicon component $R M_{u;v}^{r;r}$ is in turn the inverse double transform

$$R M_{u;v}^{r;r} = \sum_{k=r+1}^{u} \sum_{\ell=r+1}^{v} \frac{1}{p_k p_\ell} \left( M_{k;\ell}^{r;r} - M_{k;\ell+1}^{r;r} - M_{k;\ell}^{r;r} + M_{k;\ell+1}^{r;r} \right).$$  (4.3)

(ii) If generically $\alpha \cap \beta = r, \gamma \cap \delta = s, r \neq s$ (Longitudinal-Anomalous configurations), $M$ depends upon one cross-overlap

$$A M_t^{r;s} \equiv M_t^{r;s} \hspace{1cm} t = \max(\alpha \cap \gamma, \alpha \cap \delta, \beta \cap \gamma, \beta \cap \delta)$$  (4.4)

and

$$M_k^{r;s} = \sum_{t=k}^{R+1} p_t^{(r,s)} (M_{t-1}^{r,s} - M_{t}^{r,s})$$  (4.5)

$$M_t^{r;s} = \sum_{k=0}^{k=t} \frac{1}{p_k} (M_{k}^{r,s} - M_{k+1}^{r,s})$$  (4.6)

with, e.g. for $r < s$,

$$p_x^{(r,s)} = \begin{cases} p_x & x \leq r < s \\ 2p_x & r < x \leq s \\ 4p_x & r < s < x \end{cases}$$  (4.7)

The description is then complete if one gives $A M$ in the Replicon configuration with $r \equiv s$, and $u, v \geq r + 1$,

$$A M_{u,v}^{r;r} = A M_u^{r;r} + A M_v^{r;r} - A M_r^{r;r}$$  (4.8)

a component thus projected out in the double RFT of (4.2). In all cases we may now think of $A M$ with a single lower index (hatted or not) and $R M$ with two lower indices (hatted or not).

5 Relating 2-point and 1-point functions: $\delta P_{\alpha\beta}^{(r)} = 0$

We explicit now eq.(3.13), where $\alpha$ and $\beta$ can be chosen at our convenience. The simplest case occurs for $\delta P_{\alpha\beta}^{(r)} = 0$, hence with a zero right hand side. For example\(^3\) if $\alpha \cap \beta = r + 1$, i.e. $a_{r+1} \neq b_{r+1}$, we have

$$\alpha \begin{bmatrix} a_r & a_{r+1} \ldots a_R \\ a_o \ldots a_{r-1} \\ a_r & b_{r+1} \ldots b_R \end{bmatrix}$$  (5.1)

and, if we then choose

$$a_{r+1}, b_{r+1} \neq 0_{r+1}$$  (5.2)

we have $\delta P^{(r)} h_{\alpha\beta} = 0$, i.e. a null right hand side.

We collect now all the non vanishing contributions coming up in the sum over $\mu, \nu$ in (3.13). The factor $\delta P^{(r)} q_{\mu\nu}$ is non zero and equal to $\pm (q_r - q_{r+1})$ when $\delta P^{(r)}$ acts upon

$$\mu \begin{bmatrix} m_o \ldots m_{r-1} \\ m_r \quad 0_{r+1} \ldots m_R \end{bmatrix} \hspace{1cm} \mu \cap \nu = r + 1$$  (5.3)

with $P^{(r)}$ as of (3.7) the choice $\alpha \cap \beta = r + 1$ leads to a slightly simpler calculation than $\alpha \cap \beta = r$.\(^3\)
or

\[
\begin{align*}
\mu & \left[ m_{\alpha} \ldots m_{\alpha-1} \quad -1 + m_r \quad 0_{r+1} \ldots m_R \right] \\
\nu & \left[ m_r \quad m'_{\beta} \quad n'_{r+1} \ldots n_R \right]
\end{align*}
\]

μ ∩ ν = r \quad (5.4)

respectively (see eq.3.9-10). Here a primed component is distinct from zero.

Let us note as a preliminary simplifying remark, that the sum over μ, ν gives a weight \((p_{r+1}/p_{r+2} - 1) \approx O(1/R)\) to all contributions where the sum can be freely carried out, ignoring the passive replicas (here \(\alpha, \beta, \alpha \cap \beta = r+1\)). As a consequence only \(\mu \nu\) configurations with \((m_\alpha = a_\alpha, m_1 = a_1 \ldots, m_r = a_r)\) need be considered.

The \(\mu, \nu\) sum, i.e. the sum over \(m'\)s and \(n'\)s, is thus reduced to two types of geometries

(i) \(\mu \cap \nu = r + 1\)

\[
\begin{align*}
\mu & \quad \left[ a_\alpha \ldots a_{r-1} \quad a_r \quad 0_{r+1} \ldots m_R \right] \\
\nu & \quad \left[ a_\alpha \ldots a_{r-1} \quad a_r \quad n'_{r+1} \ldots n_R \right]
\end{align*}
\]

(ii) \(\mu \cap \nu = r\)

\[
\begin{align*}
\mu & \quad \left[ a_\alpha \ldots a_{r-1} \quad -1 + a_r \quad 0_{r+1} \ldots m_R \right] \\
\nu & \quad \left[ a_\alpha \ldots a_{r-1} \quad a_r \quad n'_{r+1} \ldots n_R \right]
\end{align*}
\]

plus \(\mu, \nu\) exchange.

In case (i) we obtain a contribution

\[
\left[ M_{r+1}^{r+1; r+1} \left( \frac{p_{r+1}}{p_{r+2}} - 3 \right) p_{r+2}^2 + 2 \sum_{u=r+2}^{R+1} M_{u}^{r+1; r+1} (p_u - p_{u+1}) p_{r+2} \right] (q_r - q_{r+1}) \quad (5.7)
\]

Here the first term is for \(n'_{r+1} \neq a_{r+1}, b_{r+1}, 0_{r+1}\) i.e. taking \((p_{r+1}/p_{r+2} - 3)\) values, \(p_{r+2}^2\) coming from summation over free branch numbers \(m_{r+2}, m_{r+3} \ldots\) and \(n_{r+2}, n_{r+3} \ldots\). The second term arises from \(n'_{r+1} = a_{r+1}\) (or \(b_{r+1}\)) and summing over free branch numbers when \(\nu \cap \alpha = u, \ u \geq r + 2\) (or \(\nu \cap \beta = u\)). Indeed the sum is then over \(n_u \neq a_u\), and \(n_{u+1}, n_{u+2}, \ldots\) that is yielding \((p_u/p_{u+1} - 1) p_{u+1}\).

In case (ii) the contribution becomes instead

\[
\left[ M_{r+1}^{r+1; r} \left( \frac{p_{r+1}}{p_{r+2}} - 3 \right) p_{r+2}^2 + 2 \sum_{u=r+2}^{R+1} M_{u}^{r+1; r} (p_u - p_{u+1}) p_{r+2} \right] (q_{r+1} - q_r) \quad (5.8)
\]

Using now the obvious relationship,

\[
\sum_{u=r+2}^{R+1} M_{u}^{r+1; \nu} (p_u - p_{u+1}) = \sum_{u=r+2}^{R+1} p_u (M_{u}^{r+1; \nu} - M_{u-1}^{r+1; \nu}) + p_{r+2} M_{r+1}^{r+1; \nu}
\]

and the RFT definitions (4.5, 7), then, the sum of (5.7) and (5.8) becomes

\[
\left[ \frac{p_{r+2}}{4} \left( M_{r+2}^{r+1; r+1} - M_{r+2}^{r+1; r} \right) + O(1/R) \right] (q_r - q_{r+1}) = 0
\]

This first relationship expresses, in the \(R \to \infty\) limit, the continuity of the kernels (Fourier Transforms) in their overlaps.
6 Relating 2-point and 1-point functions: \( \delta P_{\alpha\beta}^{(r)} \neq 0 \)

Let us choose again \( \alpha \cap \beta = r + 1 \) but now with \( \delta P_{\alpha\beta}^{(r)} q_{\alpha\beta} = q_r - q_{r+1} \). Hence the right hand side of (3.13) is now

\[
(h_r - h_{r+1}).
\]

and we have

\[
\alpha \left[ a_0 - a_{r-1} a_r 0_{r+1} a_{R} \right] \\
\beta \left[ a_0 - a_{r-1} b_{r+1} b_{R} \right].
\]

Carrying out again the \( \mu, \nu \) sum in (3.13) we have the two cases considered above, all the others yielding zero or a contribution of order \( 1/R \) as remarked before.

(i) \( \mu \cap \nu = r + 1 \) as in (5.5). We get

\[
\sum_{u=r+2}^{R+1} M^{r+1;r+1} u \delta u \left( \frac{p_{r+1}}{p_{r+2}} - 2 \right) p_{r+2} + \sum_{u=r+2}^{R+1} \sum_{v=r+2}^{R+1} M^{r+1;r+1} v \delta u \delta v (q_r - q_{r+1})
\]

Here we use \( \delta u \equiv p_u - p_{u+1} \).

The first term in (6.3) comes from \( n'_{r+1} \neq b'_{r+1}, 0_{r+1} \) i.e. taking \( (p_{r+1}/p_{r+2} - 2) \) values, the factor \( p_{r+2} \) from summing over \( m_{r+2}, \ldots, m_{R} \), and \( \delta u \) from summing over \( n_u \neq a_u, n_{u+1}, \ldots, n_R \) when \( \gamma \cap \alpha = u, u \geq r + 2 \). The last term is from \( n'_{r+1} = b'_{r+1} \) and summing over free branch numbers \( m \) and \( n, \mu \cap \alpha = u, \gamma \cap \beta = v, u, v \geq r + 2 \) and \( \mu, \nu \) exchange.

(ii) \( \mu \cap \nu = r \) as in (5.6), we get instead

\[
M^{r+1;r} (r+1) \left( \frac{p_{r+1}}{p_{r+2}} - 2 \right) p_{r+2} + \sum_{v=r+2}^{R+1} M^{r+1;r} v \delta v p_{r+2} \]

where the first term is for \( n'_{r+1} \neq b'_{r+1}, 0_{r+1} \) and the last for \( v \cap \beta = v, v \geq r + 2 \) and the resulting sum over free branch numbers \( m \) and \( n \).

Pulling (6.1, 3.5) together, and using relationships of eq.(5.9), and

\[
\sum_{u=r+2}^{R+1} \sum_{v=r+2}^{R+1} M^{r+1;r+1}_{u,v} \delta u \delta v = M^{r+1;r+1}_{r+2,r+2} + 2p_{r+2} \sum_{u=r+2}^{R+1} M^{r+1;r+1}_{u,v} \delta u - p_{r+2}^2 M^{r+1;r+1}_{r+2,r+2}
\]

that follows from RFT definitions (4.2) and from (4.8), we obtain,

\[
M^{r+1;r+1}_{r+2,r+2} + p_{r+2} \sum_{u=r+2}^{R+1} \left[ (M^{r+1;r+1}_{u,v} - M^{r+1;r+1}_{u-1,v}) - (M^{r+1;r}_{u,v} - M^{r+1;r}_{u-1,v}) \right]
\]

\[
= (h_{r+1} - h_r) / (q_{r+1} - q_r) + O(1/R).
\]

Using definitions (4.5, 7) for RFT's, we get

\[
M^{r+1;r+1}_{r+2,r+2} + \frac{p_{r+2}}{4} \left( M^{r+1;r+1}_{r+2} - M^{r+1;r+1}_{r+2} \right) = (h_{r+1} - h_r) / (q_{r+1} - q_r) + O(1/R)
\]

i.e., with the previous W-T-like relationship of equation (5.10)

\[
M^{r+1;r+1}_{r+2,r+2} = (h_{r+1} - h_r) / (q_{r+1} - q_r) + O(1/R).
\]
Taking Parisi limit, with \( x = r/R + 1, R \to \infty \), we finally, get
\[
M_{x \to \infty}^{x \to \infty} = \hat{h}(x) / \hat{q}(x)
\]
(6.9)

This W-T identity states that one has Goldstone modes, for each overlap \( x \) for which \( \hat{h}(x) = 0 \), that is for an overlap independent magnetic field. In particular one has a band of Goldstone modes with or without the presence of a magnetic field.

7 3-point function parametrization:

A general discussion of 3-point function parametrization would require too much space. Here we give a minimal discussion and, to keep within size, we specialize to the case where one of the given cross-overlaps is maximal, namely \( \beta \equiv \delta \). Instead of investigating (3.14) we are now concerned with
\[
\frac{1}{2} \sum_{\mu \nu} W^{\alpha \beta; \beta' \gamma; \mu \nu} \delta P^{(r)} q_{\mu \nu} = \delta P^{(r)} M^{\alpha \beta; \beta' \gamma} + O(1/R).
\]
(7.1)

Further we shall concentrate upon the more interesting configuration \( \alpha \cap \beta \neq \beta \cap \gamma \) in which case, the 2-point function in the right hand side becomes
\[
M^{\alpha \beta; \beta' \gamma} = M^{r; s}_{R+1}, \quad r \neq s
\]
(7.2)
The general 3-point function (six replicas) involves five overlaps. Because of the choice \( \beta \equiv \delta \), one of the cross-overlaps is now \( \beta \cap \delta \equiv \beta \cap \beta \equiv R + 1 \). With the choice \( \alpha \cap \beta \neq \beta \cap \gamma \), \( r \neq s \), we now have only two distinct geometries of interest
(i) \( \mu \cap \nu \equiv \alpha \cap \beta = r \), with
\[
W^{\alpha \beta; \beta' \gamma; \mu \nu} = W^{r; s; r; u; v; R+1}_{u; v; R+1}, \quad u, v \geq r + 1
\]
(7.3)
where
\[
\begin{align*}
\mu &= \max(\alpha \cap \mu, \alpha \cap \nu) \\
\nu &= \max(\beta \cap \mu, \beta \cap \nu, \gamma \cap \mu, \gamma \cap \nu)
\end{align*}
\]
(7.4)
the pairs \( \alpha \beta, \mu \nu \) being in a Replicon-like geometry. The RFT writes
\[
W^{r; s; r}_{k; \ell; R+1}^{(s)}
\]
(7.5)
the \( s \) superscript in \( \hat{\ell}^{(s)} \) is to remind that for \( s > r \) the double RFT is to be calculated as in (4.2) but with weights \( p_u, p_v^{(s)} \) with
\[
p^{(s)}_v = \begin{cases} p_v & v \leq s \\ 2p_v & s < v \end{cases}
\]
(7.6)
If \( s < r \), the superscript is to be forgotten, and weights \( p_u, p_v \) used\(^4\).  
(ii) \( \mu \cap \nu \neq \alpha \cap \beta \) (and \( \mu \cap \nu \neq \beta \cap \gamma \)) generically, then
\[
W^{\alpha \beta; \beta' \gamma; \mu \nu} = A W^{r; s; q}_{t; R+1}
\]
(7.7)
\[
t = \max(\alpha \cap \mu, \alpha \cap \nu, \beta \cap \mu, \beta \cap \nu, \gamma \cap \mu, \gamma \cap \nu).
\]
(7.8)
\(^4\)To make a long story short, just like we have two geometries \( R M^{r; r}, A M^{r; s} \) for the 2-point functions, we now have four geometries \( R R W^{r; r; r}, A R W^{r; s; r}, N R W^{r; s; r}, A A W^{r; s; q} \), where \( NR \) stands for Nested Replicon. In the simpler case considered here the Replicon-like geometry hides in fact the \( AR \) and \( NR \) geometries, this distinction being transferred into the extra superscript. See more below in section 9.
Here the RFT

\[ W_{k;R+1}^{r,s|q} \]  

is to be calculated as in (4.5) with a \( p_t^{(r,s,q)} \) trivially generalizing (4.7).

\section*{8 Relating 3-point and 2-point functions: \( s < r \)}

Let us choose again \( \alpha \cap \beta = r + 1 \), and \( \beta \cap \gamma = s \), \( s \neq r, r + 1 \), which under \( P^{(r)} \) leads to \( \delta P^{(r)}_{\beta \gamma} = 0 \).
Taking besides \( \delta P^{(r)}_{\alpha \beta} = 0 \) would yield a continuity relationship analog to (5.10). Let us choose instead \( \delta P^{(r)}_{\alpha \beta} \neq 0 \) as in (6.2), which gives for the right hand side of (7.1)

\[ (M_{R+1}^{r,s} - M_{R+1}^{r+1,s}) \]  

Consider now the left hand side of (7.1) and its \( \mu, \nu \) summation. As above, the case \( \mu \cap \nu \equiv \alpha \cap \beta = r + 1 \) will give rise to the bulk contribution \( (M_{R+1}^{r+1;\gamma+1} - M_{R+1}^{r;\gamma+1}) \) in the previous case, see (6.3, 6.8) plus a remainder which combined with the contribution for \( \mu \cap \nu = r \) will construct a term of order \( 1/R \) when the continuity condition for the RFT’s ((5.10) and alike) is taken into account. This we now exhibit first when \( s < r \).

The chosen structure of the \( \alpha \beta \gamma \) tree is as

\[ \begin{array}{cccc}
\alpha & a_s \ldots a_{r-1} & a_r & 0_{r+1} \ldots a_R \\
\beta & a_o \ldots a_{s-1} & a_s \ldots a_{r-1} & a_r & b_{r+1} \ldots b_R \\
\gamma & c_s \ldots c_{r-1} & c_r & c_{r+1} \ldots c_R \\
\end{array} \]  

(8.2)

with \( c_s \neq a_s (\alpha \cap \gamma = \beta \cap \gamma = s) \) and \( b_{r+1} \neq 0_{r+1} (\alpha \cap \beta = r + 1) \).

Consider first the configurations where the pair \( \mu, \nu \) is squating the \( \alpha, \beta \) branches of the tree

(i) \( \mu \cap \nu = \alpha \cap \beta = r + 1 \) as in (5.5), or
(ii) \( \mu \cap \nu = r \neq \alpha \cap \beta \) as in (5.6)

In that case the \( \mu, \nu \) sum yields terms in strict correspondence with (6.3, 5) i.e., pulling them together, with the left hand side of (6.8):

\[ R W_{r+1;\gamma+1}^{r+1;\gamma+1} \]  

(8.3)

Note that the RFT \( R W_{r+1;\gamma+1}^{r+1;\gamma+1} \) is calculated from \( W_{u,v;R+1}^{r+1;\gamma+1} \) as in (4.2) with weights \( p_u, p_v \). But \( A W_{t;R+1}^{r+1;\gamma+1} \), with \( q = r, r + 1 \), is defined from \( A W_{t;R+1}^{r+1;\gamma+1} \) with weights \( p_t^{(r+1;\gamma+1)} \) instead of \( p_t^{(r+1,q)} \) (as of (4.5) for \( A M^{r+1,q} \)). We thus get a factor \( 1/8 \) in (8.3) (instead of \( 1/4 \) in (6.8)).

Consider now the configurations where \( \mu, \nu \) sit upon \( \gamma \) branches, that is

(i) \( \mu \cap \nu = r + 1 \), i.e. (with as in (8.2) \( c_s \neq a_s \)),

\[ \begin{array}{cccc}
\mu & a_o \ldots a_{s-1} & c_s \ldots c_{r-1} & c_r & 0_{r+1} \ldots m_R \\
\nu & c_s \ldots c_{r-1} & c_r & n_{r+1} \ldots n_R \\
\end{array} \]  

(8.4)

(ii) \( \mu \cap \nu = r \)

\[ \begin{array}{cccc}
\mu & a_o \ldots a_{s-1} & c_s \ldots c_{r-1} & -1 + c_r & 0_{r+1} \ldots m_R \\
\nu & c_s \ldots c_{r-1} & c_r & n_{r+1} \ldots n_R \\
\end{array} \]  

(8.5)

generating a contribution
\[ p_{r+2} \left( A W_{r+1}^{r+1, s;r+1} - A W_{r+1}^{r+1; s} \right) (q_r - q_{r+1}) \]  
\[ \text{Pulling together (8.1, 3, 6) we get, for } s < r \]
\[ RW_{r+1}^{r+1; s;r+1} = (M_{R+1}^{r,s} - M_{R+1}^{r+1; s}) / (q_r - q_{r+1}) + O(1/R) \]

\section{9 Relating 3-point and 2-point functions: } \( s > r + 1 \)

The starting tree is now, for \( s > r + 1 \),
\[ \alpha \left[ a_0 a_{r-1} a_r b_{r+1} b_{r+2} b_{s-1} b_s b_R \right] \]
\[ \beta \left[ a_0 a_{r-1} a_r b_{r+1} b_{r+2} \ldots \right] \]
\[ \gamma \left[ a_0 a_{r-1} a_r b_{r+1} b_{r+2} b_{s-1} c_s c_R \right] \]

where \( a \cap \beta = r + 1 \) (with \( b'_{r+1} \neq 0 \)) and \( \beta \cap \gamma = s \), with \( b_s \neq c_s \).

As above, the case \( \mu \cap \nu \equiv \alpha \cap \beta = r + 1 \) will give rise to the bulk contribution (plus a remainder). The difference is now, with \( \beta \cap \gamma = s > r + 1 \), that there is no separate squatting of the \( \gamma \) branches, as in (8.4-6). Instead we get only a contribution like (8.3)
\[ \left[ RW_{r+1}^{r+1; s;r+1} - A W_{r+1}^{r+1; s} \right] (q_r - q_{r+1}) \]

but with a difference. As noted by the \( s \) superscript on the hatted variable, the RFT of \( RW \) is calculated from \( W_{r+1; s;r+1}^{r+1} \) with \( p_u, p_v^{(s)} \) respectively as in (7.3-5). The squating of the \( \gamma \) branches by replica \( \nu \) (or \( \mu \)), as in (8.6) for \( s < r \), shows up here by the occurrence of a jump in \( p_v^{(s)} \) when \( \nu \) crosses \( s \). So that, we may alternatively separate out a \( RW \) regular, that keeps the same RFT definition (with \( p_u, p_v \) as for \( R M \) in (4.2, 3)) valid for all \( s \), and exhibit the jump by writing
\[ s < r \quad RW_{r+1; s;r+1}^{r+1} \]

\begin{equation}
\left[ RW_{r+1; s;r+1}^{r+1} - A W_{r+1; s}^{r+1} \right] (q_r - q_{r+1}) \tag{9.4}
\end{equation}

\[ s > r + 1 \quad RW_{r+1; s;r+1}^{r+1} + R W_{r+1; s;r+1}^{r+1} + R W_{r+1; s;r+1}^{r+1} \]

\[RW_{r+1; s;r+1}^{r+1} = (M_{R+1}^{r,s} - M_{R+1}^{r+1; s}) / (q_r - q_{r+1}) + O(1/R)\]

Note, also, that the difference of RFT’s as occurring in (9.2) is precisely the combination of \( O(1/R) \) that a choice \( \delta P_{\alpha \beta \gamma}^{(r)} = 0 \) would have exhibited.

We may thus write, pulling together (8.1) and (9.2-4), for the case \( s > r + 1 \)
\[ RW_{r+1; s;r+1}^{r+1} + RW_{r+1; s;r+1}^{r+1} = (M_{R+1}^{r,s} - M_{R+1}^{r+1; s}) / (q_r - q_{r+1}) + O(1/R) \]

\section{10 Continuity and jumps. W-T identities in the Parisi limit}

We comment first on continuity before returning to the above identities. Consider the RFT’s of some function \( f \) (on its cross-overlaps, the only RFT’s considered here). From its definition, we have
\[ p_u (f_u - f_{u-1}) = f_u - f_{u+1} \]

Hence, at the upper bound \( (p_{R+1} \equiv 1) \), we get
\[ 5\text{i.e. } ARW + NRW \text{ as hinted in footnote 4.} \]
shows the continuity of the RFT under change of one of its passive overlaps. Here we have a double geometry by \(s > r\) for \(p_o \equiv n\). Likewise at the lower bound \(r + 1\) of the summation domain, for \(f_u^{r_t}\) with \(u \geq r + 1\) as in a Replicon geometry, we have

\[ p_o f_o = f_0 - f_1. \]  

(10.3)

This implies a jump of \(f_u\) at \(u = R\), if \(f_{R+1}\) is non-zero. This is what is seen, e.g., on the bare propagators in the Parisi limit, between the plateau value \(x = x_1\), and the maximal value \(x = 1\).

At the other end of the summation domain, we get

\[ p_o f_o = f_0 - f_1. \]  

(10.4)

Again \(f_{r+1}^{r_t}\) is found continuous and \(f_{r+1}^{r_t}\) vanishing in the Parisi limit e.g. for the bare propagators\(^8\). Finally, in the presence of passive overlaps \(r, s\), we have

\[ \mu^{(r,s)}_{u} \left( f_{u}^{r_t} - f_{u-1}^{r_t} \right) = f_{u}^{r_t} - f_{u+1}^{r_t} \]  

(10.5)

with \(p_u^{(r,s)}\) as of (4.7). Here too the RFT (difference) is found regular, whereas \(f_{u}^{r_t} - f_{u-1}^{r_t}\) has a jump when \(s\) and \(u\) cross each other, compensating for the jump of \(p_u^{(r,s)}\). On the other hand eq.(5.10) shows the continuity of the RFT under change of one of its passive overlaps.

Let us consider now equations (8.7) and (9.5) that give relationships between the (derivative of the) mass operator \(M_{R+1}^{r_t}\) and the 3-point function RFT. Letting

\[ x = r/R + 1, \ y = s/R + 1, \ R \to \infty \]  

(10.6)

we first get from (8.7), for \(0 \leq y < x < x_1\)

\[ R W_{x,y;R+1}^{x,y} = \frac{\partial}{\partial x} M_{1}^{x,y}/\hat{q}(x), \quad y < x \]  

(10.7)

Here we have a double RFT, in a Replicon-geometry, the Parisi limit of a RFT being given in this geometry by\(^{14, 15}\)

\[ f_k = \int_k^{x_1} u \ du \ (\partial f_u / \partial u) + f_1^x - f_1^{x_1} \]  

(10.8)

In the situation (9.5), we obtain instead a left hand side with a new term \(R W_{r+1;R+1}^{s,y} \) defined only for \(s > r + 1\). With (10.6) and for \(0 \leq x < y < x_1\) we get

\[ R W_{x,y;R+1}^{s,y} + R W_{x,y;R+1}^{s,y} = \frac{\partial}{\partial x} M_{1}^{x,y}/\hat{q}(x), \quad x < y. \]  

(10.9)

The derivative of the 2-point function is thus shown to have a jump, when the overlaps \(x, y\) cross each other

\[ R W_{x,y;R+1}^{s,y} \Big|_{y = x + o} = \frac{\partial}{\partial x} M_{1}^{x,y}/\hat{q}(x) \]  

(10.10)

Note that this behavior is easily checked at the zero loop level where one keeps only the term \(wtr \ q^2/3!\) in the free energy functional. The left hand side (third derivative) is only non-vanishing when the ultrametric inequality \(x < y\) is satisfied, and yields \(w\Theta (y - x)\) as a result. The right hand side is the \(x\) derivative of \(wq \ (\min(x, y))/\hat{q}(x)\), thus verifying equations (10.7, 9).
11 In conclusion:

Let us summarize what has been accomplished. Following the approach of (I) we have first spelled out in detail the derivation of results given in (I). We have recalled how to build a “small permutation” allowing to obtain W-T relationships to order $1/R$. Then we have used it to derive an equation expressing the continuity of the RFT for the 2-point function. Secondly we have given the derivation of the identity exhibiting Goldstone modes (and massive modes when the magnetic field is overlap dependent). Finally we have derived and spelled out typical W-T identities relating 3-point functions to an overlap derivative of 2-point functions (in a case where it remains rather easily tractable).

Obviously, with enough patience, the above technique may be used to construct a complete set of W-T identities. What remains to be seen is whether this new tool will perform the job one is usually expecting from such identities. In other words, the question is now whether these W-T identities will help to control the proliferating infrared divergences that plague the computation of loop corrections in the condensed spin-glass phase. But this is another story.

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