Anholonomic Transformations of Mechanical Action Principle

P. Fiziev *

International Centre for Theoretical Physics, Trieste, Italy

and

Department of Theoretical Physics, Faculty of Physics, Sofia University,
Boulevard 5 James Boucher, Sofia 1126,
Bulgaria

H. Kleinert

Institut für Theoretische Physik, Freie Universität Berlin
Arnimallee 14, D - 14195 Berlin

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Abstract

We exhibit the transformation properties of the mechanical action principle under anholonomic transformations. Using the fact that spaces with torsion can be produced by anholonomic transformations we derive the correct action principle in these spaces which is quite different from the conventional action principle.

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1 Introduction

It is well known that the action formalism of classical mechanics is not invariant under anholonomic transformations. When transforming the equations of motion anholonomically to new coordinates, the result does not agree with the naively derived equations of motion of the anholonomically transformed action [1]. The simplest example is the free motion of a particle with unit mass in a three-dimensional euclidean space \( M^3 \{x\} = \mathbb{R}^3 \ni x = \{x^i\}_{i=1,2,3} \).

For an orbit \( x(t) \) with velocity \( \dot{x}(t) \), the mechanical action of the particle is

\[
A[x(t)] := \int_{t_1}^{t_2} \frac{1}{2} |\dot{x}(t)|^2 \, dt
\]

and the Hamilton action principle:

\[
\delta A[x(t)] = 0 \tag{1}
\]

leads to the dynamical equations

\[
\ddot{x} = 0 \tag{2}
\]

defined by a straight line with uniform velocity.

Let us perform the following anholonomic transformation [1]–[5]:

\[
\dot{x}^i(t) = e^i_{\mu}(q(t)) \dot{q}^\mu(t) \tag{3}
\]

from the Cartesian to some new coordinates \( q = \{q^\mu\}_{\mu=1,2,3} \), where \( e^i_{\mu}(q) \) are elements of some nonsingular \( 3 \times 3 \) matrix \( e(q) = \{e^i_{\mu}(q)\} \) with \( \det[e(q)] \neq 0 \). By assumption, they are defined at each point of the space \( M^3 \{q\} \ni q \) and satisfy the anholonomy condition

\[
\partial_{[\nu} e^i_{\mu]}(q) \neq 0. \tag{4}
\]

When inserted into the dynamical equations (2) the transformation (3) gives

\[
0 = \ddot{x}^i(t) = \frac{d}{dt} \left( e^i_{\mu}(q(t)) \dot{q}^\mu(t) \right) = e^i_{\mu}(q(t)) \ddot{q}^\mu(t) + \partial_{\nu} e^i_{\mu}(q(t)) \dot{q}^\nu(t) \dot{q}^{\mu}(t),
\]

or, after multiplying by the inverse matrix \( e^\alpha_{\iota}(q(t)) \):

\[
\ddot{q}^\lambda + \Gamma^\lambda_{\mu\nu} \dot{q}^\mu \dot{q}^\nu = 0. \tag{5}
\]

These dynamical equations show that in the space \( M^3 \{q; \Gamma\} \) the trajectories of the particle are autoparallels. Here \( \Gamma^\alpha_{\iota\nu}(q) := e^\alpha_{\iota} \partial_{\nu} e^i_{\iota} \) are the coefficients...
of the affine flat connection with zero Cartan curvature and nonzero torsion
\[ S_{\mu \nu}^\lambda(q) := \Gamma_{\mu \nu}^\lambda(q) \neq 0 \] (because of the anholonomic condition (3)). A different result is obtained when transforming the action nonholonomically. Under the transformation (3) it goes over into
\[ A^\lambda(q) = \int_{t_1}^{t_2} \frac{1}{2} g_{\mu \nu}(q) \dot{q}^\mu \dot{q}^\nu \, dt, \]
where \( g_{\mu \nu}(q) := \sum_i e^i_{\mu}(q) e^i_{\nu}(q) \) is the metric tensor in \( M^3\{q\} \) induced by the euclidean metric in \( M^3\{x\} \) by the anholonomic transformation (3). Applying variational principle in the \( M^3\{q; g\} \) space,
\[ \delta A[q(t)] = 0, \quad (6) \]
produces an equation of motion:
\[ \ddot{q}^\lambda + \bar{\Gamma}^\lambda_{\mu \nu} \dot{q}^\mu \dot{q}^\nu = 0 \quad (7) \]
where \( \bar{\Gamma}^\lambda_{\mu \nu} := g^{\lambda \kappa} \Gamma^\kappa_{\mu \nu}, \quad \bar{\Gamma}^\lambda_{\mu \nu} := \frac{1}{2} (\partial_{\mu} g_{\nu \lambda} + \partial_{\nu} g_{\mu \lambda} - \partial_{\lambda} g_{\mu \nu}) \) are the coefficients of the symmetric Levi-Cevita connection with zero torsion \( \bar{S}^\lambda_{\mu \nu}(q) := \bar{\Gamma}^\lambda_{\mu \nu}(q) \equiv 0 \). The equations of motion (7) imply that the trajectories of the particle are geodesics in the Riemannian space \( M^3\{q; g\} \). For anholonomic coordinate transformations these results contradict each other because of (3). It is well-known from many physical examples, that the correct equation of motion in the space \( M^3\{q; \Gamma, g\} \) are the equations (6) — the true particle trajectories are autoparallel. Hence, something is wrong with the variational principle (3) in the space \( M^3\{q; \Gamma, g\} \) naturally endowed by a Riemannian metric \( g \), and by a nonmetric affine connection \( \Gamma \).

The purpose of this article is to show how to resolve this conflict by an appropriate correct modification of the action principle for anholonomic coordinates. The result shows that the right action principle must be based on the affine geometry given by the connection \( \Gamma \), not on the Riemannian geometry given by the metric \( g \). It has implications on the classical mechanics in spaces with torsion calling for a revision of many previous publications on the subject (See, for example, [6], [7], and the references herein).

2 Properties of the two tangent mappings \( e_{q \to x} \) and \( e_{x \to q} \)

Consider the two \( n \)-dimensional manifolds \( M^n\{q\} \) and \( M^n\{x\} \) with some local coordinates \( q = \{q^\mu\}_{\mu=1,...,n} \) and \( x = \{x^i\}_{i=1,...,n} \). We shall call the space
\[ M^n\{q\} \] a holonomic space, and the space \( M^n\{x\} \) an anholonomic space.

A reference system on \( M^n\{q\} \) is defined by a local frame consisting of \( n \) linearly independent basis vector fields \( e^i(q) \) on \( M^n\{q\} \). They are specified by their components \( e^i_\mu(q) \). These are supposed to form a nonsingular \( n \times n \) matrix \( e(q) = \| e^i_\mu(q) \| \) with \( \det(e(q)) \neq 0 \). Thus, there exists a set of conjugate basic vector fields the components of which form the inverse matrix \( e^{-1}(q) = \| e^\mu_i(q) \| \). The matrix elements satisfy \( e^i_\mu e^\mu_j = \delta^i_j \), \( e^\mu_i e^\nu_\mu = \delta^\nu_i \).

We now define the tangent map:

\[
e_{q \to x} : T_q M^n\{q\} \to T_x M^n\{x\} :\]

by the relations:

\[
\begin{align*}
  dx^i &= e^i_\mu(q) dq^\mu, \\
  \partial_i &= e^\mu_i(q) \partial_\mu,
\end{align*}
\]

and the inverse map

\[
e_{q \to x} : T_x M^n\{x\} \to T_q M^n\{q\} :\]

by the inverse relations:

\[
\begin{align*}
  dq^\mu &= e^\mu_i(q) dx^i, \\
  \partial_\mu &= e^i_\mu(q) \partial_i.
\end{align*}
\]

Here \( T_x M^n\{x\} \) and \( T_q M^n\{q\} \) are the linear tangent spaces above the points \( x \in M^n\{x\} \) and \( q \in M^n\{q\} \), respectively. They are linearly transformed by the matrices \( e(q) \) and \( e^{-1}(q) \), which depend only on the points \( q \in M^n\{q\} \). As a consequence of this asymmetry there are important differences between the basic properties of the two mappings.

In the literature \cite{1,2,3,4}, the mappings \( e_{q \to x} \) and \( e_{q \to x} \) have appeared in various forms. The mapping \( e_{q \to x} \) is called an anholonomic (noncoordinate) transformation if the one-forms \( dx^i \) are not exact: \( d(dx^i) = \Omega_{ij}^k dx^i \wedge dx^j \neq 0 \), \( [\partial_i, \partial_j] = -2\Omega_{ij}^k \partial_k \neq 0 \). The tensor \( \Omega_{ij}^k(q) = e^\mu_\nu e^\nu_j \partial_\mu e^k_\nu = -e^k_\lambda \partial_\mu e^\lambda_j \) is the so-called object of anholonomy. For \( \Omega_{ij}^k \equiv 0 \), the coordinate transformation becomes holonomic. In this case one can find coordinate functions \( x^i(q) \) on \( M^n\{q\} \) so that \( e^i_\mu(q) = \partial_\mu x^i(q) \). For \( \Omega_{ij}^k \neq 0 \), no such functions exist, and the \( M^n\{q\} \)-space coordinates \( \{q^\mu\} \) can not be treated as a true coordinates in the space \( M^n\{x\} \). Nevertheless, they are convenient for solving dynamical equations in the last space.
In the case of the mapping \( e_{q \rightarrow x} \), the basics vector fields with components \( e^{n}_{i}(q) \) define an affine geometry on \( M^{n}\{q\} \), and it becomes an affine space \( M^{n}\{q; \Gamma\} \). By definition, these vector fields are covariantly constant, i.e., \( \nabla_{\mu}e^{\lambda}_{i} = \partial_{\mu}e^{\lambda}_{i} + \Gamma_{\mu\nu}^{\lambda}e^{\nu}_{i} = 0 \). This determines the coefficients of the corresponding affine connection to be \( \Gamma_{\mu\nu}^{\lambda} = e^{\lambda}_{k}\partial_{\mu}e^{k}_{\nu} \). This is an affine connection with torsion tensor \( S_{\mu\nu}^{\lambda}(q) := \Gamma_{\mu\nu}^{\lambda}(q) \). The torsion tensor is nonzero in the anholonomic case where \( d(dx^{i}) = S_{jk}^{i}dx^{j} \wedge dx^{k}, [\partial_{i}, \partial_{j}] = -2S_{ij}^{k}\partial_{k} \).

The affine geometry on \( M^{n}\{q; \Gamma\} \) induces an affine geometry on the space \( M^{n}\{x; \Gamma\} \). The connection coefficients are related by \( \Gamma_{ij}^{k} = e^{\nu}_{i}e^{\mu}_{j}e^{k}_{\lambda}\Gamma_{\mu\nu}^{\lambda} + e^{k}_{\mu}\partial_{i}e^{\mu}_{j} \). The right-hand side vanishes showing that the space \( M^{n}\{x; \Gamma\} \) is an affine euclidean space.

In the anholonomic case, the two mappings \( e_{q \rightarrow x} \) and \( e_{q \leftarrow x} \) carry only the tangent spaces of the manifolds \( M^{n}\{q; \Gamma\} \) and \( M^{n}\{x; \Gamma\} \) into each other. They do not specify a correspondence between the points \( q \) and \( x \) themselves. Nevertheless, the maps \( e_{q \rightarrow x} \) and \( e_{q \leftarrow x} \) can be extended to maps of some special classes of paths on the manifolds \( M^{n}\{q; \Gamma\} \) and \( M^{n}\{x; \Gamma\} \). Consider the set of continuous, twice differentiable paths on the manifolds \( M^{n}\{q; \Gamma\} \) and \( M^{n}\{x; \Gamma\} \). Let \( C_{q} \) and \( C_{x} \) be the corresponding closed paths (cycles).

The tangent maps \( e_{q \rightarrow x} \) and \( e_{q \leftarrow x} \) imply the velocity maps:

\[
\dot{x}^{i}(t) = e^{i}_{\mu}(q(t))\dot{q}^{\mu}(t), \quad \dot{q}^{\mu}(t) = e^{\mu}_{i}(q(t))\dot{x}^{i}(t).
\]

If in addition a correspondence between only two points \( q_{1} \in M^{n}\{q; \Gamma\} \) and \( x_{1} \in M^{n}\{x; \Gamma\} \) is specified, say \( q_{1} \equiv x_{1} \), then the maps \( e_{q \rightarrow x} \) and \( e_{q \leftarrow x} \) can be extended to the unique maps of the paths \( \gamma_{q_{1}}(t) \) and \( \gamma_{x_{1}}(t) \), starting at the points \( q_{1} \) and \( x_{1} \), respectively. The extensions are

\[
\{q^{\mu}(t); q(t_{1}) = q_{1}\} \rightarrow \bigg\{ x^{i}(t) = x_{1}^{i} + \int_{t_{1}}^{t} e^{i}_{\mu}(q(t))\dot{q}^{\mu}(t)dt; \ x(t_{1}) = x_{1} \bigg\}, \quad (10)
\]

\[
\{x^{i}(t); x(t_{1}) = x_{1}\} \rightarrow \bigg\{ q^{\mu}(t) = q_{1}^{\mu} + \int_{t_{1}}^{t} e^{\mu}_{i}(q(t))\dot{x}^{i}(t)dt; \ q(t_{1}) = q_{1} \bigg\}. \quad (11)
\]

Note the asymmetry between the two maps: In order to find \( \gamma_{x_{1}}(t) = e_{q \rightarrow x}(\gamma_{q_{1}}(t)) \) explicitly, one has to evaluate the integral \( (10) \). In contrast, specifying \( \gamma_{q_{1}}(t) = e_{q \leftarrow x}(\gamma_{x_{1}}(t)) \) requires solving the integral equation \( (11) \).

Another important property of the anholonomic maps \( e_{q \rightarrow x} \) and \( e_{q \leftarrow x} \) is that these do not map the cycles in one space into the cycles in the other
space: \( e_{q\rightarrow x}(C_{qq_1}) \neq C_{xx_1} \) and \( e_{q\rightarrow x}(C_{xx_1}) \neq C_{qq_1} \). There exists, in general, a nonzero Burgers vector:

\[
b^i[C_q] := \oint_{C_q} e^i_\mu dq^\mu \neq 0, \quad \text{if } \Omega_{ij}^k \neq 0,
\]

and

\[
b^\mu[C_x] := \oint_{C_x} e^\mu_i dx^i \neq 0, \quad \text{if } S_{\mu\nu}^\lambda \neq 0.
\]

### 3 The Variations of the Paths in the Space \( \mathcal{M}^n\{q; \Gamma\} \)

We shall now consider variations of the paths with fixed ends in the holonomic space \( \mathcal{M}^n\{q; \Gamma\} \) [See Fig. 1]. This was first done by Poincare [8] for group spaces \( \mathcal{M}^n\{q; \Gamma\} \). Some generalization may be found in [9] and [10]. Here we give a detailed geometrical treatment of this subject.

Let \( \gamma_q, \bar{\gamma}_q \in \mathcal{M}^n\{q; \Gamma\} \) be two paths with common ends [See Fig. 1]. According to the standard definitions, we consider two-parametric functions \( q^\mu(t, \epsilon) \in C^2 \) for which:

\[
q^\mu(t, 0) = q^\mu(t), \quad q^\mu(t, 1) = \bar{q}^\mu(t), \quad q^\mu(t_1, 2, \epsilon) = q^\mu(t_1, 2).
\]

Then the infinitesimal increment along the path is \( dq^\mu := \partial_t q^\mu(t, \epsilon) d\epsilon \), and the variation of the path is \( \delta q^\mu := \partial_t q^\mu(t, \epsilon) \delta \epsilon \), with fixed ends condition: \( \delta q^\mu|_{t_1, 2} = 0 \). We call these variations “\( \delta_q \)-variations”, or more explicitly “\( \mathcal{M}^n\{q; \Gamma\}\)-space-variations”. The above definition leads to the obvious commutation relation

\[
\delta_q(dq^\mu) - d(\delta_q q^\mu) = 0. \tag{12}
\]

The mapping \( e_{q\rightarrow x} \) brings these paths and their variations from the space \( \mathcal{M}^n\{q; \Gamma\} \) to the space \( \mathcal{M}^n\{x; \Gamma\} \). The space \( \mathcal{M}^n\{x; \Gamma\} \) contains the image paths before and after the variation \( \gamma_x = e_{q\rightarrow x}(\gamma_q), \quad \bar{\gamma}_x = e_{q\rightarrow x}(\bar{\gamma}_q) \). We now distinguish the following variations in \( \mathcal{M}^n\{x; \Gamma\} \) [see Fig. 1]: the total \( \delta_q \)-variation of the coordinate \( x^k \): \( \delta_q x^k(t) := \delta_q x^k(t_1) + \int_{t_1}^t e^k_\mu dq^\mu \), the “holonomic variation” of the coordinates \( x^k : \delta x^k(t) := e^k_\mu \delta q^\mu \), and the variations \( \delta_q(dx^k) := \delta_q(e^k_\mu dq^\mu) \). These have the following basic properties:

1) \( \delta x^k(t_1, 2) = 0 \),
i.e., the fixed end condition for the holonomic $\delta_q$-variations in the space $\mathcal{M}^n\{x; \Gamma\}$ (note that the total $\delta_q$-variation does not possess this property).

2) $\delta_q x^k(t) = \delta x^k(t) + \Delta^k_q(t)$, where

$$\Delta^k_q(t) := 2 \int_{t_1}^t \partial_{[\mu} e^{k}_{\nu]} dq^\mu dq^\nu = 2 \int_{t_1}^t \Omega_{ij}^k dx^i dx^j \quad (13)$$

is the anholonomic deviation. The function $\Delta^k_q(t)$ describes the time-evolution of the effect of the anholonomy: initially, $\Delta^k_q(t_1) = 0$. The final value $\Delta^k_q(t_2) = b^k$ is equal to the Burgers vector. Then we derive the
equation:
\[ \delta q(dx^k) - d(\delta x^k) = 2\Omega_{ij}^k dx^i dx^j. \]  
(14)

This is Poincare’s relation.

3) By combining the relation 2) and 3), we find
\[ \delta q(dx^k) - d(\delta x^k) = 0 \]  
(15)

Under \( e_q \rightarrow x \) mapping the mechanical action \( A[\gamma_q] = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \) of mechanical system in the space \( \mathcal{M}^n \{ q; \Gamma \} \), is mapped into \( \mathcal{M}^n \{ x; \Gamma \} \) - action \( A[\gamma_x] \) as follows:

\[ A[\gamma_q] \rightarrow A[\gamma_x] = \int_{t_1}^{t_2} L(q, e^{-1} \dot{x}, t) dt = \int_{t_1}^{t_2} \Lambda(q, \dot{x}, t) dt. \]

There exists an associated integral equation \((\Pi)\) for the orbits \( q(t) = q[\gamma_x] \).

A more general form of the Lagrangian \( \Lambda \) on the space \( \mathcal{M}^n \{ q; \Gamma \} \) is: \( \Lambda = \Lambda(q; x, \dot{x}, t) \), where \( q(t) = q[\gamma_x] \) is the corresponding functional of the path \( \gamma_x \). Then the Poincare relation \((\Pi)\) and the definition for the anholonomic deviation \( \Delta^k_x(t) \) give the following expression for the variational derivative

\[ \frac{\delta A[\gamma_x]}{\delta q} (q(t)) = \partial_i \Lambda + \partial_i \Lambda - \frac{d}{dt} \left( \partial_{\dot{x}i} \Lambda \right) + 2\Omega_{ij}^k \dot{x}^j \left( \partial_{\dot{x}k} \Lambda + \int_{t_1}^{t_2} \partial_k \Lambda d\tau \right) \]  
(16)

where \( \partial_i = e^{\mu i} \partial_\mu \). Note the additional force-term proportional to the object of the anholonomy \( \Omega_{ij}^k \).

The variational principle \( \delta q A[\gamma_x] = 0 \) implies therefore the following generalized Poincare equations \((\Pi)\):

\[ \frac{\delta A[\gamma_x]}{\delta q} (q(t)) = 0; \quad q^n(t) = q^n_1 + \int_{t_1}^{t} e^{\mu_i(q(t))} \dot{x}^i(t) dt \]  
(17)

in the anholonomic space \( \mathcal{M}^n \{ x; \Gamma \} \). We have to stress that:

1. In general, \((\Pi)\) is a system of integro-differential equations, instead of systems of ordinary differential equations.

2. The nonlocal term which is proportional to \( \int_{t_1}^{t_2} \partial_k \Lambda d\tau \) violates causality. It may be shown that if, and only if, the Lagrangian \( \Lambda \) originates from some \( \mathcal{M}^n \{ q; \Gamma \} \)-space Lagrangian \( L(q, \dot{q}, t) \), then \( \partial_k \Lambda \equiv 0 \) and no causality problem appears.
3. Poincare himself considered only the special case when:
i) $\bar{\partial}_i$ are independent left invariant vector fields on a Lie group space. In this case, $-2\Omega_{ij}^k = C_{ij}^k = \text{const}$ are just structure constants of the Lie-group.
ii) $L = L_0$, where $L_0$ is invariant under corresponding group transformations, and hence, $\partial_k\Lambda_0 = 0$.

Under these two conditions, the (17) reduces to a closed system of ordinary differential equations:

$$\frac{d}{dt}(\partial_x\Lambda_0) + C_{ij}^k\dot{x}^j\partial_x\Lambda_0 = 0.$$ 

If instead of ii) we have $L = L_0 - U(q)$, then there exists causal system of integro-differential equations:

$$\begin{cases} 
\frac{d}{dt}(\partial_x\Lambda_0) + C_{ij}^k\dot{x}^j\partial_x\Lambda_0 = -\partial_iU(q), \\
q^\mu(t) = q_1^\mu + \int_{t_1}^t e_i^\mu(q(t))\dot{x}^i dt.
\end{cases}$$

which are called Poincare equations [10]–[12].

The most popular example where the Eqs. (18) apply are the Euler equations for rigid body rotations in the body-fixed reference system [10]–[12], mentioned already by Poincare himself [8]. Let us derive these equations and extend them to a description of the full rigid body dynamics (including translations) in the body system [13].

In this case $\mathcal{M}^{(6)}\{q\} = SO(3) \times \mathcal{R}^3$, $q = \{\varphi^\mu, x^\mu\}$, where $\varphi^\mu$ are the Euler angles, and $x^\mu$ are center-of-mass-coordinates in the stationary system; $\mu, i = 1, 2, 3$; the group constants are $C_{ijk} = -2\Omega_{ijk} = \epsilon_{ijk}$, is the Levi-Cevita antisymmetric symbol. The Lagrangian in the space $\mathcal{M}^{(6)}\{q\}$ reads

$L = \frac{1}{2}I_{\mu\nu}(\varphi)\dot{\varphi}^\mu\dot{\varphi}^\nu + \frac{1}{2}M\delta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ where $I_{\mu\nu}(\varphi) = \sum_i I_i e_i^\mu(\varphi)e_i^\nu(\varphi)$ are the components of the bodies inertial tensor, and $I_i = \text{const}$ are the principal inertia momenta. The anholonomic space is $\mathcal{M}^{(6)}\{x\} = \mathcal{M}^{(6)}\{\Phi^i, X^i\}$, where $d\Phi^i = e^i_\alpha(\varphi)d\varphi^\alpha = \Omega^i dt$ ( $\Omega^i$ are the components of the angular velocity in the body system), and $dX^i = e^i_\alpha(\varphi)dx^\alpha = V^i dt$ ( $V^i$ are the components of the center-of-mass velocity in the same system). Then, in the space $\mathcal{M}^{(6)}\{\Phi^i, X^i\}$, the Lagrangian reads

$$\Lambda(\dot{\Phi}, \dot{X}) = \frac{1}{2} \sum_i I_i \dot{\Phi}^i \dot{\Phi}^i + \frac{1}{2} M \sum_i \dot{X}^i \dot{X}^i.$$
Using the vectors of angular momentum $K = IΩ$, and momentum $P = MV$, we can write down the variational derivatives in the form:

$$\frac{δA}{δqΦ} = \frac{dK}{dt} + Ω × K, \quad \frac{δA}{δqX} = \frac{dP}{dt} + Ω × P.$$ 

Hence, the variational principle $δqA[γx] = 0$ gives the correct equations of motion. They contain gyroscopic terms proportional to $Ω : Ω × K$ and $Ω × P$. The usual variational principle $δxA[γx] = 0$ in the anholonomic space $M\{Φ^i, X^i\}$ would have produced wrong equations of motion without gyroscopic terms.

4 The Variations of the Paths in the Space $M^m\{x; Γ\}$

Consider now the variation of the paths with fixed ends in the anholonomic space $M^m\{x; Γ\}$ [See Fig. 2]. This consideration is similar to the one above, but it has some specific features. Let $γ_x, ˘γ_x \in M^m\{x\}$ are two paths with common ends [see Fig. 2]. We consider two-parametric functions $x^i(t, ε) \in C^2$ for which: $x^i(t, 0) = x^i(t), x^i(t, 1) = ˘x^i(t)$, and $x^i(t_{1,2}, ε) = x^i(t_{1,2})$. Then the infinitesimal increment along the path is $dx^i := \partial_t x^i(t, ε)dt$, and the variation of the path is $δx^i := \partial_ε x^i(t, ε)δε$, with fixed ends: $δx^i|_{t_{1,2}} = 0$. We call these variations “δ$_x$-variations”, or more explicitly “$M^m\{x; Γ\}$-space-variations”. The above definitions lead to the obvious commutation relation

$$δ_x(dx^i) - d(δ_x x^i) = 0. \quad (19)$$

The $e_{q¬x}$ mapping maps these paths and their variations from the space $M^n\{x; Γ\}$ to the space $M^n\{q; Γ\}$. According to our definitions, they go over into $M^n\{q; Γ\}$-paths $γ_q = e_{q¬x}(γ_x), ˘γ_q = e_{q¬x}(˘γ_x)$ [see Fig. 2], and we have got the “total δ$_x$-variation” of the coordinates $q^μ : δ_x q^μ(t) := δ_x [q^μ(t) + \int_{t_1}^t e^μ_i(q)dx^i]$ (this is an indirect definition accomplished by integral equation [11]), the “holonomic variation” of the coordinates $q^μ : δ_x q^μ(t) := e^μ_i(q)δx^i$, and the variations $δ_x(dq^μ) := δ_x [e^μ_i(q)dx^i]$. These have the following basic properties:
1) $\delta_x q^\mu(t_{1,2}) = 0,$
i.e. the fixed end condition for the holonomic $\delta_x$-variations in the space $\mathcal{M}^n\{q, \Gamma\}$ (note that the total $\delta_x$-variations do not possess this property).

2) $\delta_x q^\mu(t) = \delta_x q^\mu(t) + \Delta_x^\mu(t),$ where

$$\Delta_x^\mu(t) := \int_{t_1}^t \!\! d\tau \Gamma_{\alpha\beta}^\mu \left( dq^\alpha \delta_x q^\beta - \delta_x q^\alpha dq^\beta \right)$$

is the anholonomic deviation. The function $\Delta_x^\mu(t)$ describes the time-evolution of the effect of the anholonomy: $\Delta_x^\mu(t_1) = 0$. The final value $\Delta_x^\mu(t_2) = b^\mu$ is the Burgers vector.

3) $\delta_x (dq^\mu) - d(\delta_x q^\mu) = \Gamma_{\alpha\beta}^\mu \left( dq^\alpha \delta_x q^\beta - \delta_x q^\alpha dq^\beta \right) .

This relation is new, replacing the Poincare relation (14). It may be rewritten in a more elegant form:

$$\delta_x A(dq^\mu) - d_A(\delta_x q^\mu) = 0. \quad (20)$$

Here $\delta_x A(dq^\mu) := \delta_x (dq^\mu) + \Gamma_{\alpha\beta}^\mu \delta_x q^\alpha dq^\beta$ is the “absolute variation” and $d_A(\delta_x q^\mu) := d(\delta_x q^\mu) + \Gamma_{\alpha\beta}^\mu dq^\alpha \delta_x q^\beta$ is the corresponding “absolute differential”. By combining the last two equations we find

$$\delta_x (dq^\mu) - d(\delta_x q^\mu) = 0. \quad (21)$$

For anholonomic deviation $\Delta_x^\mu(t)$, we introduce the vector notations $\Delta_x(t) = \{\Delta_x^\mu(t)\}_{\mu=1,...,n}$ and derive an ordinary differential equation. Written in a matrix form and imposing the proper initial condition, it reads:

$$\begin{cases}
\dot{\Delta}_x(t) = -G(t)\Delta_x(t) + S(t)\delta_x q(t), \\
\Delta_x(t_1) = 0.
\end{cases} \quad (22)$$

Here $G(t) = \{G^\nu_{\mu}(t)\} := \{\Gamma_{\mu\nu}(q(t))\dot{q}^\nu(t)\}$ and $S(t) = \{S^\nu_{\mu}(t)\} := \{S_{\lambda\mu}(q(t))\dot{q}^\lambda(t)\}$ are time dependent $n \times n$-matrices. The solution of the initial value problem is:

$$\Delta_x(t) = \int_{t_1}^t \!\! d\tau \! U(t, \tau) S(\tau) \delta_x q(\tau), \quad (23)$$

where

$$U(t, \tau) = T - \exp \left( \int_{\tau}^t \!\! G(\tau') d\tau' \right). \quad (24)$$
For a mechanical system in the space $\mathcal{M}^n\{x; \Gamma\}$ with action $A[\gamma_x] = \int_{t_1}^{t_2} \Lambda(x, \dot{x}, t) dt$, the action is mapped under $e_{q\leftarrow x}$ mapping as follows:

$$A[\gamma_x] \rightarrow A[\gamma_q] = \int_{t_1}^{t_2} \Lambda(x, \dot{q}, t) dt = \int_{t_1}^{t_2} L(x, q, \dot{q}, t) dt. \quad (25)$$

The correspondence between the orbits is given by the associate integral (10) for $x(t) = x[\gamma_q]$.

A more general form of the Lagrangian $L$ on the space $\mathcal{M}^n\{q; \Gamma\}$ is $L = L(x, q, \dot{q}, t)$, where $x(t) = x[\gamma_q]$ is the corresponding functional of the path $\gamma_q$. Then the relation (20) and the definition for the anholonomic deviation $\Delta x(t)$ give the following expression for the variational derivative

$$\delta A[\gamma_q] = \partial_{x} q^\mu (t) = \partial_\mu L + \partial_\mu L - \frac{d}{dt} (\partial_{\dot{q}^\mu} L) + 2 S_{\nu \mu \lambda} \dot{q}^\nu \partial_{\dot{q}^\nu} L U_\lambda^\sigma d\tau \quad (26)$$

where $\partial_\mu = e^i_\mu \partial_i$. Observe the additional force-term proportional to the torsion $S_{\nu \mu \lambda}$.

Hence, the variational principle $\delta x A[\gamma_q] = 0$ implies the following integrodifferential dynamical equations:

$$\begin{align*}
\frac{\delta A[\gamma_q]}{\delta x q^\mu (t)} &= \partial_\mu L + \partial_\mu L - \frac{d}{dt} (\partial_{\dot{q}^\mu} L) + 2 S_{\nu \mu \lambda} \dot{q}^\nu \partial_{\dot{q}^\nu} L U_\lambda^\sigma d\tau \\
x^i(t) &= x^i_1 + \int_{t_1}^{t} e^i_\mu(q(t)) \dot{q}^\mu(t) dt \quad (27)
\end{align*}$$

in the holonomic space $\mathcal{M}^n\{q; \Gamma\}$.

The following points have to be emphasized.

1. The nonlocal term which is proportional to the term

$$\int_{t_1}^{t_2} \left( \partial_\sigma L - \Gamma_{\sigma \alpha \beta} \dot{q}^\alpha \partial_{\dot{q}^\beta} L \right) U_\lambda^\sigma d\tau \quad (28)$$

violates the causality.

2. It may be shown that if, and only if, the Lagrangian $L$ originates from some $\mathcal{M}^n\{x; \Gamma\}$-space Lagrangian $\Lambda(x, \dot{x}, t)$ then $\partial_\mu L - \Gamma_{\sigma \alpha \beta} \dot{q}^\alpha \partial_{\dot{q}^\beta} L \equiv 0$, so that no causality problem appears. In this case, we have the following system of nonlocal, but causal integrodifferential equation as dynamical equations:

$$\begin{align*}
\left\{ \begin{array}{l}
\partial_\mu L + \partial_\mu L - \frac{d}{dt} (\partial_{\dot{q}^\mu} L) + 2 S_{\nu \mu \lambda} \dot{q}^\nu \partial_{\dot{q}^\nu} L = 0 \\
x^i(t) = x^i_1 + \int_{t_1}^{t} e^i_\mu(q(t)) \dot{q}^\mu(t) dt
\end{array} \right. \quad (29)
\end{align*}$$
3. If the \( \mathbf{x} \)-space Lagrangian has the form \( \Lambda(\mathbf{x}, \dot{\mathbf{x}}, t) = \Lambda(\dot{\mathbf{x}}, t) \), then there exists a complete local system of ordinary differential equation as a dynamical equations:

\[
\partial_\mu L - \frac{d}{dt} (\partial_{q^\mu} L) + 2S_{\nu\mu} \dot{q}^\nu \partial_{q^\lambda} L = 0.
\] (30)

In these equations, an additional "torsion force"

\[
\mathcal{F}_\mu = 2S_{\nu\mu} \dot{q}^\nu \partial_{q^\lambda} L = 0
\] (31)

appears. It is easy to show, that this force preserves the mechanical energy but it is a nonpotential force even in the generalized sense: the force cannot be written as a variational derivative of some suitable functional: \( \mathcal{F}_\mu \neq \delta(\int V(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}, \ldots; t)dt)/\delta q^\mu(t) \).

The dynamical equation in presence of torsion, generated by anholonomic transformation, may be rewritten in a form:

\[
\frac{\delta A[\gamma_\mathbf{q}]}{\delta_x q^\mu(t)} = \frac{\delta A[\gamma_\mathbf{q}]}{\delta q^\mu(t)} + \mathcal{F}_\mu = 0.
\] (32)

The correct variational principle \( \delta_x A[\gamma_\mathbf{q}] = 0 \) can be replaced by a modified D’Alembert’s principle:

\[
\delta_q A[\gamma_\mathbf{q}] + \int_{t_1}^{t_2} \mathcal{F}_\mu \delta q^\mu = 0.
\] (33)

In this form, it involves only \( \mathcal{M}^n\{\mathbf{q}; \Gamma\} \)-space variables.

**Examples:**

1. Consider a free particle in an affine flat space \( \mathcal{M}^n\{\mathbf{x}; \Gamma\} \) with nonzero torsion [12]. We can think this space as produced by some tangent mapping \( e_{q-x} \) from the euclidean space \( \mathcal{M}^n\{\mathbf{x}; \Gamma\} \). Then under this mapping \( \Lambda(\dot{\mathbf{x}}) = \frac{1}{2}m \sum_i (\dot{x}_i)^2 \rightarrow L = \frac{1}{2}mg_{\mu\nu}(\mathbf{q}) \dot{q}^\mu \dot{q}^\nu, \quad g_{\mu\nu}(\mathbf{q}) = \sum_i e^i_\mu(\mathbf{q}) e^i_\nu(\mathbf{q}), \) and the variational principle \( \delta_x A[\gamma_\mathbf{q}] = 0 \) leads to the correct equation of motion (3):

\[
g_{\mu\nu} \left( \ddot{q}^\nu + \Gamma_{\lambda\sigma}^\nu \dot{q}^\lambda \dot{q}^\sigma \right) = g_{\mu\nu} \left( \ddot{q}^\nu + \Gamma_{\lambda\sigma}^\nu \dot{q}^\lambda \dot{q}^\sigma \right) + \mathcal{F}_\mu = 0.
\]

The torsion force reaches the conflict described in the introduction.

2. The Kustaanheimo-Stiefel transformation in celestial mechanics:

Consider the Kepler problem in the celestial mechanics. Here \( \mathbf{x} \in \mathcal{R}^3, r = |\mathbf{x}|, \) and the Lagrangian reads: \( \Lambda(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}m|\dot{\mathbf{x}}|^2 + \frac{\alpha}{r} (\alpha = \text{const}) \). The well-known dynamical equations in \( \mathcal{R}^3\{\mathbf{x}\} \) space are \( m\ddot{\mathbf{x}} + \alpha \mathbf{x}/r^3 = 0 \). Let us
perform the Kustaanheimo-Stiefel transformation \[1\], \[13\], which transforms the Kepler problem to the harmonic oscillator problem in \(\mathbb{R}^4\), setting:

\[
\begin{align*}
x_1 &= 2(u_1u^3 + u_2u^4), \\
x_2 &= 2(u_1u^4 + u_2u^3), \\
x_3 &= (u_1)^2 + (u_2)^2 - (u_3)^2 - (u_4)^2, \\
\end{align*}
\]

for \(\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \in \mathbb{R}^4\).

Note that a variable \(x_4\) does not exist; this is not a coordinate transformation.

An anholonomic transformation may be defined by \(d\mathbf{x}^i = e^i_\mu(\mathbf{u})du^\mu\), \(i = 1, 2, 3, 4\); using the matrix:

\[
e(\mathbf{u}) = \|e_\mu^i(\mathbf{u})\| = \begin{vmatrix} u^3 & u^4 & u^1 & u^2 \\ u^4 & -u^3 & -u^2 & u^1 \\ u^1 & u^2 & -u^3 & -u^4 \\ u^2 & -u^1 & u^4 & -u^3 \end{vmatrix}.
\]

It is easy to check that \(d(dx^i) = 0\) for \(i = 1, 2, 3\); but \(d(dx^4) \neq 0\). Hence we have indeed anholonomic transformation of the type \(e_{u\leftarrow x}\). The fourth (anholonomic) coordinate \(x_4\) must be inserted into the original Lagrangian \(\Lambda(x, \dot{x})\), replacing it by a new one: \(\bar{\Lambda}(\mathbf{x}, \dot{\mathbf{x}}) = \Lambda(x, \dot{x}) + \frac{1}{2}m(\dot{x}^4)^2\). The new four-dimensional problem in \(\mathbb{R}^4\{\mathbf{x}\}\) will be equivalent to the old one if one enforces the constraint \(x_4 = \text{const}\). The \(e_{u\leftarrow x}\) mapping maps the Lagrangian \(\bar{\Lambda}(\mathbf{x}, \dot{\mathbf{x}})\) into the space \(\mathbb{R}^4\{\mathbf{u}\}\), where it reads: \(L(\mathbf{u}, \dot{\mathbf{u}}) = 2mu^2\dot{u}^2 + \alpha/\mathbf{u}^2\).

One can produce the correct dynamical equations in the space \(\mathbb{R}^4\{\mathbf{u}\}\) using our modification of the variational principle: Here \(\delta_{\mathbf{u}}A[\mathbf{u}(t)] = 0\) implies the equation of motion \(\delta A/\delta u^\mu + F_\mu = 0\) containing a torsion force \(F_\mu = m\dot{x}^4\dot{S}_{1\mu2}\) (in this problem the only nonzero components of the torsion are \(\{S_{1\mu2}\} = \{S_{3\mu4}\} = 4\{u^2, -u^1, u^4, -u^3\}\)). The application of the naive action principle in the space \(\mathbb{R}^4\{\mathbf{u}\}\) would produce a wrong equation of motion, lacking the torsion force.

5 Concluding remarks

It is obvious, that the variational principles described in this article have many other applications, for example in the \(n\)-body problem of celestial mechanics, in rotating systems (analogously to the Euler equations for a rigid
body), in the field theory of gravitation with torsion \[6\], \[7\], and in many other physical systems \[13\] – \[18\], where anholonomic coordinates are a convenient and a useful tool.

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