Time-dependent correlation functions in open quadratic fermionic systems

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Abstract. We formulate and discuss the explicit computation of dynamic correlation functions in open quadratic fermionic systems which are driven and dissipated by Lindblad jump processes that are linear in canonical fermionic operators. Dynamic correlators are interpreted in terms of a local quantum quench where the pre-quench state is a non-equilibrium steady state, i.e. a fixed point of the Liouvillian. As an example, we study the $XY$ spin $1/2$ chain and quadratic Majorana chains with boundary Lindblad driving, whose dynamics exhibit asymmetric (skewed) light cone behaviour. We also numerically treat the two-dimensional $XY$ model and the $XY$ spin chain with additional Dzyaloshinskii–Moriya interactions. The latter exhibits a new non-thermal phase transition which can be understood in terms of bifurcations of the quasi-particle dispersion relation.

Finally, considering in some detail the periodic quadratic fermionic ring with dissipation at a single (arbitrary) site, we present analytical expressions for first order corrections (in the strength of dissipation) to the spectrum and non-equilibrium steady state (NESS) correlation functions.

Keywords: correlation functions, quantum dissipative systems, quantum criticality, quantum quenches
1. Introduction

The motivation for studying open quantum systems, where a finite but possibly large quantum system interacts with an infinite environment, is twofold; on the one hand it is a way of attacking the fundamental questions regarding quantum mechanics, such as the quantum measurement problem, the decoherence and the dynamics far from equilibrium. On the other hand, it is very relevant for experimental quantum physics, which has advanced significantly over the last few years and is capable of demonstrating
a good control over the many body states and their dynamics (e.g. in ultracold gases, trapped ions, etc).

Decoherence is one of the main players in second quantum revolution [1], which will develop new quantum technologies. It plays an important role in quantum computing, state preparation and quantum memories. Furthermore, studying the non-equilibrium steady states (NESS) of open systems gives us new interesting phases of matter separated by non-equilibrium phase transitions [2–5].

The dynamics far from equilibrium is still poorly understood, even in the context of integrable systems. In recent years, there has been much progress in understanding dynamics following an instantaneous local or global quantum quench [6–10] in closed systems. In the context of open systems, even though some progress in understanding the structure of non-equilibrium steady states of boundary driven integrable systems has been achieved [11], relaxation (Liouvillian) dynamics is still essentially unexplored.

In this paper, we study the simplest of such problems. Specifically, we consider a protocol in which we start in a non-equilibrium steady state, perform a local quench and then study the relaxation dynamics in terms of local observables. In this way, we define Liouvillian dynamic correlation functions and set up a procedure to calculate them for quasi-free fermionic many-body systems.

First we discuss a general quadratic one-dimensional fermionic model that is governed by the Lindblad master equation with general linear Lindblad operators. We investigate its spectrum and the static and dynamic correlation functions (section 2). In the second part, we focus on a specific problem given by a weakly coupled fermionic ring with superconducting pairing terms (section 3), which can be applied to study a boundary driven XY spin chain model as well. In section 4, we provide some interesting numerical results for static and dynamic NESS correlation functions of the one-dimensional and two-dimensional quadratic fermionic models. In particular, in the boundary driven XY chain, we observe skewed (asymmetric) light-cone behaviour, with the real part of two-point correlations with points at late times to the left (right) well approximated by the thermal correlators of the right (left) bath. Studying a boundary driven XY spin chain with the presence of Dzialoshinsky–Moriya (DM) interaction terms, we observe a rich non-equilibrium phase diagram with additional critical lines which correspond to bifurcations of a quasi-particle dispersion relation. In section 5, we then focus on the perturbative/asymptotic analysis of the Liouvillian spectrum and static NESS correlator of the fermionic ring in the presence of weak dissipation/driving. Most interestingly, with dissipative driving at just one site, we encounter a unique steady state, where dissipation creates entanglement/quantum correlations between opposite pairs of fermions/spins over large distances.

2. Statement of the Fermionic problem

The evolution of an open system’s density matrix $\rho(t)$ is given by the Lindblad master equation [12], which is the most general Markovian, trace preserving and completely positive non-unitary map:

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\[ \frac{d\rho}{dt} = \hat{\mathcal{L}}\rho := -i[H, \rho] + \sum_{\mu=1}^{M} \left( 2L_\mu \rho L_\mu^\dagger - \{ L_\mu^\dagger L_\mu, \rho \} \right). \]  

We focus on quadratic fermionic Hamiltonians \( H \) for systems of length \( n \), written in terms of \( 2n \) Majorana fermions \( w_j \) (we use a convention \( w_j^2 = 1 \)) combined together with a \( 2n \times 2n \) anti-symmetric matrix \( \mathbf{H} \) and write the Hamiltonian as

\[ H = \sum_{j,k=1}^{2n} w_j \mathbf{H}_{jk} w_k. \]

The linear Lindblad operators are written with the help of \( 2n \)-component vectors \( l_\mu \) as

\[ L_\mu = \sum_j l_{\mu,j} w_j. \]

### 2.1. Third quantization

A convenient general procedure for treating such problems is the so-called third quantization [13–16] that is based on the quantization in the Fock space of operators. Here we quickly review it and fix the notation. See also [17] for an alternative but equivalent super-fermion formulation. However, an idea to consider Hilbert spaces of observables has appeared in several other places in the literature, see e.g. [18–20].

We are interested in the dynamics of a density matrix \( \rho(t) \) which is a non-negative unit trace operator. Vectorizing the space of operators, the latter are now expressed as vectors in a \( 4^n \)-dimensional Fock space \( \mathcal{K} \) of operators (Liouville–Fock space) and the superoperators become operators over \( \mathcal{K} \). We select an orthonormal canonical basis for \( \mathcal{K} \) as:

\[ P_{\alpha_1,\ldots,\alpha_{2n}} := 2^{-n/2} w_{\alpha_1} w_{\alpha_2} \cdots w_{\alpha_{2n}}, \quad \alpha_j \in \{0,1\}, \]

and introduce the creation and annihilation linear superoperators (superoperators are denoted by a hat \( \hat{\cdot} \) to distinguish them from conventional operators) \( \hat{c}_j, \hat{c}_j^\dagger \) over \( \mathcal{K} \): 

\[ \hat{c}_j |P_{\alpha\beta}\rangle = \delta_{\alpha,0} |w_j P_{\beta}\rangle \text{ and } \hat{c}_j |P_{\alpha\beta}\rangle = \delta_{\alpha,1} |w_j P_{\beta}\rangle \]

which obey the canonical anticommutation relations (CAR). They are combined with the \( 4n \) Majorana fermionic superoperators over \( \mathcal{K} \):

\[ \hat{a}_{1,j} := \frac{1}{\sqrt{2}} (\hat{c}_j + \hat{c}_j^\dagger), \quad \hat{a}_{2,j} := \frac{i}{\sqrt{2}} (\hat{c}_j - \hat{c}_j^\dagger), \]

obeying CAR \( \{ \hat{a}_{\nu,j}, \hat{a}_{\nu',j'} \} = \delta_{\nu,\nu'} \delta_{j,j'} \), with which we can express the Liouvillian map \( \hat{\mathcal{L}} \) simply as:

\[ \frac{d}{dt} |\rho(t)\rangle = \hat{\mathcal{L}} |\rho(t)\rangle, \quad \hat{\mathcal{L}} = \hat{\mathbf{A}} \hat{\rho} - A_0 \mathbf{1}_n, \]

where we have used the notation \( \hat{\rho} = (\hat{a}_{1,1}, \hat{a}_{1,2}, \ldots, \hat{a}_{1,n}, \hat{a}_{2,1}, \ldots, \hat{a}_{2,n}) \).

The constant \( A_0 \) is unimportant [13], and the \( 4n \times 4n \) matrix \( \mathbf{A} \) can be expressed in terms of the Hamiltonian matrix \( \mathbf{H} \) and the positive semidefinite Hermitian Lindblad bath matrix \( \mathbf{M} = \sum_{\mu} l_{\mu} \otimes l_{\mu}^\dagger \). The matrices \( \mathbf{M}_r \) and \( \mathbf{M}_i \) are its real and imaginary parts:

\[ \mathbf{A} = \begin{pmatrix} \mathbf{X} & 2i\mathbf{M} \\ -2i\mathbf{M}^T & \mathbf{X} - 4i\mathbf{M}_i \end{pmatrix}, \quad \mathbf{X} = -2i\mathbf{H} + 2\mathbf{M}_r. \]
The $\mathbf{M}$ and $\mathbf{X}$ are $2n \times 2n$ matrices, the real and imaginary parts of $\mathbf{M}$ are symmetric, anti-symmetric under transposition, respectively, i.e. $\mathbf{M}_r^T = \mathbf{M}_r$, $\mathbf{M}_i^T = -\mathbf{M}_i$, and $\mathbf{X}$ is non-symmetric but real. The matrix $\mathbf{A}$ is unitarily equivalent to the block triangular matrix $\hat{\mathbf{A}} = \mathbf{U} \mathbf{A} \mathbf{U}^\dagger$,

$$
\hat{\mathbf{A}} = \begin{pmatrix} -\mathbf{X}^T & 4i\mathbf{M}_i \\ 0 & \mathbf{X} \end{pmatrix}, \quad \mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \otimes \mathbf{1}_{2n},
$$

therefore its spectrum is determined by the eigenvalues $\{\Lambda_j\}_{j=1}^{2n}$ of the $2n \times 2n$ non-Hermitian matrix $\mathbf{X}$. The imaginary part of these eigenvalues is mostly determined by the Hamiltonian, whereas the real part is positive semidefinite and is a consequence of the dissipation encoded in $\mathbf{M}_r$. The spectrum of the Liouvillian operator $\hat{\mathcal{L}}$ consists of $4^n$ elements $\{\sum_{2\in\{0,1\}^\otimes 2n} \nu_k \Lambda_k\}$ and does not depend on $\mathbf{M}_i$.

We could proceed by constructing a ‘non-unitary Bogoliubov transformation’ diagonalising $\mathbf{A}$ and express the Liouvillian operator in terms of the normal master modes $\hat{\mathbf{L}} = -2\sum_{j=1}^{2n} \Lambda_j \hat{b}_j \hat{b}_j$, where $\Lambda_j$ are the eigenvalues of $\mathbf{X}$, and CAR obeying operators $\hat{b}_j$ and $\hat{b}_j^\dagger$ are the linear combinations of the Majorana operators $\hat{a}$ [13, 14, 21]. But this is not necessary, as will become clear in the next subsection.

### 2.2. Correlation functions

The equal-time two-point correlation functions in NESS can be calculated using the normal master modes, or directly by writing the dynamics of the two-point correlations and calculating the fixed point [2, 14, 22]. Since the Hamiltonian is quadratic and the coupling Lindblad operators $L_\mu$ are linear in fermionic operators, the NESS is a Gaussian state and the higher order correlation functions can be calculated from the two-point correlation functions using Wick contractions.

The two-point correlation functions are given in terms of a covariance (or correlation) matrix $\mathbf{C}$ [2, 14, 22]:

$$
\mathbf{C}_{jk}(t) = \text{Tr} w_j w_k \rho(t) - \delta_{j,k} = 2\langle 1| \hat{a}_{1,j} \hat{a}_{1,k} |\rho(t)\rangle - \delta_{j,k}.
$$

To express the dynamics, we focus on the time derivative of the two-point correlation functions. We adopt the super-Heisenberg picture and write the time derivative in terms of the time-dependent Majorana superoperators $\hat{a}_{\mu,z}(t) = e^{-\hat{\mathcal{L}}t} \hat{a}_{\mu,z} e^{\hat{\mathcal{L}}t}$

$$
\frac{d\mathbf{C}_{jk}(t)}{dt} = \langle \frac{dw_j(t)}{dt} w_k(t) \rangle_{\rho(0)} + \langle w_j(t) \frac{dw_k(t)}{dt} \rangle_{\rho(0)}
$$

$$
= 2\langle 1| \frac{d\hat{a}_{1,j}(t)}{dt} \hat{a}_{1,k}(t) + \hat{a}_{1,j}(t) \frac{d\hat{a}_{1,k}(t)}{dt} |\rho(0)\rangle.
$$

The time derivative of the Majorana superoperators is:

$$
\frac{d\hat{a}_{1,z}(t)}{dt} = [\hat{a}_{1,z}(t), \hat{\mathcal{L}}] = 2\sum_s \left( \mathbf{A}_{(1,z),(s)} \hat{a}_{1,s} + \mathbf{A}_{(1,z),(2,s)} \hat{a}_{2,s} \right).
$$

After simple manipulations we obtain the continuous Lyapunov equation that governs the time evolution of the equal-time correlation functions:

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\(-\frac{1}{2} \frac{dC(t)}{dt} = X^T C(t) + C(t)X - 4iM_i. \tag{11}\)

We have used \(\text{Tr}\rho(0) = 1\), but note that \(\rho(0)\) is self-adjoint. The time evolution of the correlation functions is closed and Markovian, i.e. it does not depend on the whole history of \(\rho(t')\) but only on the instantaneous covariance matrix \(C(t)\).

### 2.2.1. Correlations in a NESS.

The two-point correlations in a NESS \(C(\infty) = C^\infty\) are determined by the condition \(\frac{dC^\infty}{dt} = 0\). The NESS is unique if the eigenvalues of \(X\) have a strictly positive real part \([13]\). Moreover, there are more general theorems stating the necessary conditions for the uniqueness of the NESS \([23]\), or non-uniqueness in the presence of strong symmetry \([24]\). We can calculate \(C^\infty\) using the spectral decomposition of the matrix \(X\) or non-uniqueness \([23]\): we can calculate \(C^\infty\) using the spectral decomposition of the matrix \(X\) or non-uniqueness \([23]\).

For the analytical solution it is useful to use the spectral decomposition of the matrix \(X\): \(XP = \Lambda X\), \(\Lambda = \text{diag}(\ldots, \Lambda_j, \ldots)\). The correlations in the NESS can be expressed as:

\[
\{\Lambda, \hat{P}^T C^\infty \hat{P}\} = 4i \hat{P}^T M_i \hat{P}, \quad (\hat{P}^T C^\infty \hat{P})_{kl} = 4i \frac{(\hat{P}^T M_i \hat{P})_{kl}}{\Lambda_k + \Lambda_l}. \tag{12}\]

Having computed \(\hat{P}^T C^\infty \hat{P}\) from equation (12), we can express the correlations in the NESS by \(C^\infty = \hat{P}^{-T}(\hat{P}^T C^\infty \hat{P}) \hat{P}^{-1}\). In the right part of equation (12), we see that the driving encoded in \(M_i\) sets the correlations in the NESS. Note that \(M_i\) does not appear anywhere else in our derivation.

### 2.2.2. Dynamics of the correlations.

The main goal of this paper is to study the dynamic response of non-equilibrium open systems. Writing the correlation matrix as \(C(t) = C^\infty + (C(t) - C^\infty)\) and inserting it in equation (11), we see that the general solution for the correlation matrix is:

\[
C(t) = e^{-2tx^T} (C(0) - C^\infty) e^{-2tx} + C^\infty, \quad C(0)_{j,m} = \text{Tr} w_j w_m \rho(0) - \delta_{j,m}. \tag{13}\]

Using the diagonalization \(X \hat{P} = \hat{P} \Lambda\) and definition \(G(t) = \hat{P}^T (C(t) - C^\infty) \hat{P}\), the simple evolution of the correlations in the new basis has the form:

\[
G(t)_{j,m} = e^{-2t(\Lambda_j + \Lambda_m)} G(0)_{j,m}. \tag{14}\]

Equations (13) and (14) are very general and give us important information about the dynamics in quadratic open quantum systems. The time-dependent correlation functions are thus given in terms of the simple propagation of the initial condition.

The positive real part of the spectrum of matrix \(X\), which is mostly determined by dissipation encoded in \(M_i\), determines the decay of the correlations in the NESS. The smallest real part of \(\Lambda_j\), i.e. the Liouvillian gap, determines the longest possible time scale of the decay. The imaginary part of the spectrum is mostly determined by the Hamiltonian \(H\) and produces oscillations and a light cone-like effect which will be discussed later. The driving encoded in \(M_i\) influences the correlations in the NESS \(C^\infty\).

In order to investigate the dynamics, we would like to calculate the time-dependent correlation functions for specific simple initial states. To avoid breaking the super-parity \([2]\), we are interested in the four-point time-dependent correlation functions of the form
which captures, for example, the time-dependent density–density correlations and spin–spin correlator \( \langle \sigma^z_k(t)\sigma^z_l(0) \rangle \) for the spin systems after performing the Jordan–Wigner transformation. It is especially interesting and simple to look at the case when \( \rho = \rho_{NESS} \) is the NESS density matrix. In the derivation of the map \( \mathbf{C}(0) \rightarrow \mathbf{C}(t) \), equations (13) and (14), we have never required \( \rho(0) \) to be a valid density matrix (i.e. a non-negative Hermitian operator). In fact, we can use any element of the Fock space \( \mathcal{K}^+ \) with positive super-parity (see [13] for definitions). In order to calculate the four-point correlation function (15), we take the initial condition \( \rho(0) = \frac{w_kw_l\rho_{NESS}}{\text{Tr}(w_kw_l\rho_{NESS})} \) for some fixed indices \( k, l \), so the initial value correlation matrix reads:

\[
\mathbf{C}(0)_{j,m} + \delta_{j,m} = \text{Tr}w_jw_m\rho(0) = \frac{\langle w_jw_mw_kw_l \rangle}{\langle w_kw_l \rangle} = \frac{\langle w_jw_m \rangle + (\langle w_jw_l \rangle\langle w_mw_k \rangle - \langle w_jw_k \rangle\langle w_mw_l \rangle)}{\langle w_kw_l \rangle},
\]

where we have used a short hand notation for expectation values with respect to the NESS, \( \langle A \rangle \equiv \text{Tr}(A\rho_{NESS}) \). The solution to this problem can also be seen as a Green’s function of the time evolution, and can be used to study any initial condition. Another important initial condition is the case of a thermal initial state, where the initial condition is given by equation (16), where we take expectation values with respect to \( \rho = Z^{-1}\exp(-\beta H) \).

### 2.3. Local open quantum quench

Recently, there has been much interest in studying global quantum quenches. Naturally, this also leads to interest in local quantum quenches, where one starts with a ground state of a Hamiltonian that is only locally different from the post-quench Hamiltonian (see, e.g., [9, 10]). Here we look at the local quench from a different, open system’s perspective and define it via the following protocol:

1. Wait for the open quantum system to relax into the steady state (NESS).
2. Perform a local quench by measuring a local observable (e.g. \( n_k = c_k^\dagger c_k \)) at some fixed site \( k \).
3. Use the resulting density matrix as the initial condition.
4. Measure the correlations in the state at later times.

Let us look at an example where we measure a particle number \( n_k \) at site \( k \) at time \( t = 0 \) in the NESS, and continue with the experiment provided we measured \( n_k = 1 \). The initial condition is then given by the density matrix

\[
\rho(0) = \frac{P\rho_{NESS}P}{\text{Tr}P\rho_{NESS}P}, \quad P = c_k^\dagger c_k = \frac{1 - i\hbar w_{2k-1}w_{2k}}{2},
\]

where \( P \) is a projector on the subspace with a particle at site \( k \), and \( c_k \) are conventional fermionic operators (not to be confused with the superoperators \( \hat{c}_k \)). Explicitly, \( \rho(0) \) and the initial condition \( \mathbf{C}(0) \) for the later time correlation functions are given by:
\[ \rho(0) = \frac{1}{2} \rho - \frac{w_{2k-1} w_{2k} \rho w_{2k-1} w_{2k} - i w_{2k-1} w_{2k} \rho - i \rho w_{2k-1} w_{2k}}{1 - i \langle w_{2k-1} w_{2k} \rangle}, \]  
(18)

\[ C(0)_{j,m} = \frac{\langle w_j w_m w_{2k-1} w_{2k} \rangle}{1 + \langle w_{2k-1} w_{2k} \rangle}, \quad C(0)_{j,j} = 0. \]  
(19)

The initial condition can be considered to be a special case of (16) up to the normalization, so later we will focus on calculating the time-dependent correlation functions resulting from local quantum quenches.

### 3. Application to a one-dimensional quadratic Fermionic chain

In the second part of this paper, we will focus on the quadratic fermionic Hamiltonian with superconductive pairing

\[ H = \sum_m (c_m^{\dagger} c_{m+1} + \gamma c_m^{\dagger} c_m^{\dagger} + \text{h.c.}) + h \left( 2 c_m^{\dagger} c_m - 1 \right) \]

\[ = -\frac{i}{2} \sum_m \left( 1 + \gamma w_{2m} w_{2m+1} - \frac{1 - \gamma}{2} w_{2m-1} w_{2m+2} + h w_{2m-1} w_{2m} \right), \]  
(20)

which corresponds to the Heisenberg XY spin chain after the Jordan–Wigner transformation for an open-boundary spin chain. This problem is both rich and treatable, which can be seen from the huge number of works treating it. It is also well known as the Majorana chain. The main focus will be on the periodic boundary conditions, but we will also numerically treat the open (free) boundary conditions (relevant for the spin problem).

Coupling with the environment is parametrized by the following bath (Lindblad) operators $L_{\mu}$, which create/destroy fermionic particles at site $j$:

\[ L_{\mu}^j = \sqrt{\Gamma_{\mu}^j} c_j^{\dagger}, \quad L_{-}^j = \sqrt{\Gamma_{-}^j} c_j, \quad L_{+}^j = \frac{1}{2} \sqrt{\Gamma_{+}^j} (w_{2j-1} + iw_{2j}), \]  
(21)

where $\Gamma_{\mu}^j$ are positive coupling constants. In the XY model, they are related to the bath temperatures/magnetizations of non-interacting spins as $\Gamma_{\mu}/\Gamma_{\mu} = e^{-2h\beta}$ [13], where $h$ is the transverse magnetic field in the spin formulation of the model. For ease of comparison with the thermal closed system, we parametrize these constants in a slightly different way in terms of new parameters $S_j$, $\beta_j$:

\[ \Gamma_{\mu}^j = \frac{S_j e^{\pm h\beta_j}}{2 \cosh h \beta_j}, \quad \Gamma_{+}^j + \Gamma_{-}^j = S_j, \quad \Gamma_{+}^j - \Gamma_{-}^j = S_j \tanh h \beta_j. \]  
(22)

As we have seen before, all physical information regarding our system is contained in the matrices $X$ and $M_i$. To write them explicitly, it is convenient to reorder the sequence of Majoranas as \{w_{1}, w_{3}, \ldots, w_{2n-1}, w_{2}, w_{4}, \ldots, w_{2n} \} \{w_{1}(1,1), \ldots, w_{1}(n,1), w_{1}(1,2), \ldots, w_{1}(n,2) \} so the reordered matrix $X = -2iH + 2M_i$ reads

---

1 The Jordan–Wigner transformation complicates the expressions for the spin systems, where we can couple to the bath only the first and the last spin site in order to maintain the quadratic structure of the problem.
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\[ X = \begin{pmatrix} D & T \\ -T^T & D \end{pmatrix}, \quad D = \frac{1}{4} \text{diag}(S_1, S_2, \ldots, S_n), \]  

(23)

\[
T = \begin{pmatrix}
-\hbar & \frac{1-\gamma}{2} & 0 & \ldots & 0 & \frac{1+\gamma}{2} \\
\frac{1+\gamma}{2} & -\hbar & \frac{1-\gamma}{2} & 0 & \ldots & 0 \\
0 & \frac{1+\gamma}{2} & -\hbar & \frac{1-\gamma}{2} & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & \frac{1+\gamma}{2} & -\hbar & \frac{1-\gamma}{2} \\
\frac{1-\gamma}{2} & 0 & \ldots & 0 & \frac{1+\gamma}{2} & -\hbar
\end{pmatrix}.
\]  

(24)

The matrix \( D \) encodes the dissipation, and the circulant \( n \times n \) matrix \( T \) (tri-diagonal Toeplitz matrix for open boundary conditions) encodes the Hamiltonian. The matrix expressing the driving is written as

\[
M_i = \begin{pmatrix} 0 & \tilde{M}_i \\ -\tilde{M}_i & 0 \end{pmatrix}, \quad \tilde{M}_i = \text{diag}(\ldots, S_j \tanh \beta j h, \ldots).
\]  

(25)

In section 5 we will treat the problem perturbatively for small \( S_j \).

3.1. Application to the XY spin chain

The XY spin chain has attracted much interest in the past (see [26–28, 44]) since it is a prime example of a solvable system. It was recently reviewed in the quantum information community, where it is a popular example for understanding the entanglement generation [29–32], the quantum speed limit [33, 34] and other quantum information concepts. It was also a prime example for open systems treated with the method of third quantization (see, e.g., [13, 21, 29]); the dynamic response was computed in [35]. The XX chain was solved directly [36, 37] and its transport properties were studied in [38]. The Hamiltonian (20) is the Jordan–Wigner transformation of the XY spin chain

\[
H = \sum_{m=1}^{n-1} \left( \frac{1 + \gamma}{2} \sigma^x_m \sigma^x_{m+1} + \frac{1 - \gamma}{2} \sigma^y_m \sigma^y_{m+1} \right) + \sum_{m=1}^{n} \hbar \sigma^z_m,
\]  

(26)

with the Jordan–Wigner transformation given by:

\[
\sigma^x_m = (-i)^{m-1} \prod_{j=1}^{2m-1} w_j, \quad \sigma^y_m = (-i)^{m-1} \prod_{j=1}^{2m-2} w_j w_{2m}.
\]  

(27)

Because of the non-locality of the Jordan–Wigner transformation, only linear driving on the first and the last spin can be included:

\[
L_1 = \frac{1}{2} \sqrt{\Gamma_1} (w_1 \mp iw_2), \quad L^n = \frac{1}{2} \sqrt{\Gamma^n} (w_{2n-1} \mp iw_{2n}) W.
\]  

(28)

At the last site we encounter the parity operator \( W = -(-i)^{n} w_1 w_2 \ldots w_{2n} \), which is unimportant since \( L_1, L_n \) enter quadratically into the Lindblad equation, \( WW^\dagger = 1 \), and \( W \) is a Casimir operator [13].
We can use the above construction to treat the spin chain with open boundaries. Even though the $XY$ spin ring can be exactly solved in a closed system context, this can no longer be done once we consider it as an open quantum system. Imposing periodic boundary conditions (PBC) $\sigma_\alpha^\mu = \sigma_{\alpha+1}^\mu$ results in $(-W)w_{2n+1} = w_1$, therefore the solution of the closed problem can be obtained from independent solutions to the two fermionic problems: the odd (even) parity sector is the solution for the periodic (anti-periodic) fermionic problem \cite{27, 39}. The $L_\mu \rho L_\mu^\dagger$ term mixes the two parity sectors, and the driven spin problem with the periodic boundary condition becomes unsolvable in this manner.

3.1.1. Correlations in the open $XY$ spin chain. The correlations containing $\sigma^z$ are easily expressible in terms of the fermionic problem using $\sigma^z_m = -i w_{2m-1} w_{2m}$, for example, the $S^z S^z$ time-dependent correlation functions read

$$\langle S^z_k(t) S^z_j(0) \rangle = \frac{1}{4} \langle \sigma^z_k(t) \sigma^z_j(0) \rangle = -\frac{1}{4} \langle (w_{2k-1} w_{2k})(t)(w_{2j-1} w_{2j})(0) \rangle.$$  \hspace{1cm} (29)

In comparison, the thermal correlation functions of the closed system in the thermodynamic limit are given by the following fermionic correlations \cite{27, 28}:

$$\langle w_{2l-1} w_{2k} \rangle = \frac{-i}{2\pi} \int_{-\pi}^{\pi} \frac{(\cos\phi - h) + i\gamma \sin\phi}{\Lambda(\phi)} \tanh \left( \frac{\beta h \Lambda(\phi)}{2} \right) e^{i(k-l)\phi} d\phi,$$

$$\Lambda(\phi) = \sqrt{(\cos\phi - h)^2 + \gamma^2 \sin^2 \phi},$$  \hspace{1cm} (30)

where $\Lambda(\phi)$ is the dispersion in the thermodynamic limit and furthermore $\langle w_{2l} w_{2k} \rangle = \langle w_{2l-1} w_{2k-1} \rangle = \delta_{l,k}$.

Using the general framework presented in the previous section, we can numerically calculate the equal and non-equal time correlation functions for systems larger than 1000 sites or look at the response to the local quench. The numerical results will be presented in the following section.

4. Numerical results

4.1. Correlations in the NESS

As was argued in \cite{21, 40}, the open $XY$ spin chain exhibits a non-equilibrium phase transition at $h_c = |1 - \gamma^2|$. The two-point correlations are shown in figure 1, and they are distinct in different phases. The long-range phase $h < h_c$ coincides with the appearance of the double minima in the dispersion relation, as is indicated in figure 2. For more details see \cite{21, 40}.

4.2. Time-dependent correlations

In figures 3 and 4, we show the time-dependent correlation function (also interpreted as a local quench) of the open $XY$ model, which shows light cone behaviour. The response to the left and the right has the same velocity, but the response is asymmetric for the asymmetric driving, as can be clearly seen in figure 4.
Figure 1. The size of the equal-time two-point correlation function in the NESS (elements of the matrix $C^\infty$) for $h < h_c$ (left) and $h > h_c$ (right) of the open XY spin chain consisting of 50 spins with $\gamma = 0.5$, $S_1 = 0.3$, $S_0 = 0.1$, $\beta_1 = 0.1$ and $\beta_n = 5$. We have reordered the sequence of Majoranas as $\{w_1, w_3, \ldots, w_{2n-1}, w_2, w_4, \ldots, w_{2n}\}$.

Figure 2. The free dispersion of the XY model $\Lambda(\phi)$, as defined in equation (30), changes the number of local minima when we cross the non-equilibrium phase transition at $h_c = |1 - \gamma^2|$.

Figure 3. A typical asymmetric light cone response of the real part of the connected time-dependent correlation function in the open XY spin chain for an asymmetric driving. Each new line is $\Delta t = 1$ later and is lifted for better visibility. In the left (right) figure $h = 0.6$ ($h = 0.9$), so we are in different non-equilibrium phases ($h_c = 0.75$) and the central region behaves differently. The parameters are $\gamma = 0.5$, $n = 100$, $S_1 = S_n = 1$, $\beta_1 = 0.5$ and $\beta_n = 2$. 

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4.3. Open XY model with DM interactions

An interesting solvable extension to the XY model is the addition of DM interactions that break the space reflection symmetry.

We focus on the special form \( \vec{D} = D \hat{e}_z \):

\[
H_{DM} = D \sum_m \hat{e}_z (\vec{\sigma}_m \times \vec{\sigma}_{m+1}) = D \sum_m (\sigma^x_m \sigma^y_{m+1} - \sigma^y_m \sigma^x_{m+1})
= -iD \sum_{l}^{2n} w_l w_{l+2}. \tag{31}
\]

The model is exactly solvable in the thermodynamic limit of a closed system, where the dispersion is given by [41]:

\[
\Lambda(\phi) = \left| 2D\sin \phi + \sqrt{(h - \cos \phi)^2 + \gamma^2 \sin^2 \phi} \right| \tag{32}
\]

In figure 5 we show the static (equal-time) correlations in the steady state, which were used as the indicator of the non-equilibrium phase transition in the open XY model [21]. We notice a novel non-thermal phase transition between the two steady

Figure 4. The real and imaginary part of the connected time-dependent correlation functions of the open XY spin chain for the baths at different temperatures. We see that \( \langle S^z_{r+n/2}(t)S^z_{n/2} \rangle \) labeled \(+r\) and \( \langle S^z_{r+n/2}(t)S^z_{n/2} \rangle \) labeled \(-r\) are different. The real part of the correlation to the right \(+r\) \((-r\)) is very similar to the thermal response of the system with \( \beta_l (\beta_r) \), corresponding to the temperature of the left bath \( \beta_l = 0.1 \)(right bath \( \beta_r = 5 \)). We can imagine the bath sending thermal quasiparticles to the right (left). Other parameters are \( r = 10, n = 64, \gamma = 0.5, h = 0.9, S_1 = 0.3 \) and \( S_n = 1 \).
state phases with different long-range correlations, similar to the known transition from the XY model without a DM term, that is still present. Interestingly, the new phase transition again coincides with the appearance of double minima in the dispersion relation which is plotted schematically on the right. When we go from point A to B, the second minimum of the dispersion relation disappears at the phase boundary. When we go from B to C, the left minimum hits the zero-level and bifurcates to a pair of Dirac points. D is a special point, and the boundary line continuing through it is the transition of the B → C type. The parameters are $n = 100$, $\gamma = 0.5$, $S_1 = S_n = 1$, $\beta_1 = \beta_n = 2$.

Figure 5. The phase diagram of the open XY model with DM interactions at fixed anisotropy $\gamma = 0.5$. The average correlations $C_{\text{long}} = \frac{1}{4} \sum_{i=3}^{n/2} \left( \langle S_{n/2}^{z} S_{i}^{z} \rangle - \langle S_{n/2}^{z} \rangle \langle S_{i}^{z} \rangle \right)$ between the middle site and the last quarter of the spin chain with 100 spins are plotted. The darker region with residual long-range correlations coincides with the appearance of double minima in the dispersion relation which is plotted schematically on the right. When we go from point A to B, the second minimum of the dispersion relation disappears at the phase boundary. When we go from B to C, the left minimum hits the zero-level and bifurcates to a pair of Dirac points. D is a special point, and the boundary line continuing through it is the transition of the B → C type. The parameters are $n = 100$, $\gamma = 0.5$, $S_1 = S_n = 1$, $\beta_1 = \beta_n = 2$.

4.4. Two-dimensional model

Next, we present the time-dependent correlation function (figure 7) for a two-dimensional problem given by the Hamiltonian

$$H = \sum_{(j,k)} (c_j^\dagger c_k + \gamma c_j^\dagger c_k^\dagger + \text{h.c.}) + 2\hbar \sum_j c_j^\dagger c_j,$$

(33)

with $(j,k)$ denoting the nearest neighbours on a square $n \times n$ lattice $\{1,2,\ldots n\} \times \{1,2,\ldots n\}$ with open boundary conditions. The system is driven at the top-left $(1,1)$ and the bottom-right $(n,n)$ corners with the Lindblad operators given by equation (21). We looked at the connected part of the density–density correlations at different times.
Time-dependent correlation functions in open quadratic fermionic systems

\[ C(t)_{(x,y)} = \frac{\langle (w_{2r-1}w_{2r})(t)(w_{2s-1}w_{2s})(0) \rangle - \langle (w_{2r-1}w_{2r})(0)(w_{2s-1}w_{2s})(0) \rangle}{\langle w_{2s-1}w_{2s} \rangle}, \]

where the sequential indices of the Majoranas are determined by \( r = x + n(y - 1) \) and \( s = \frac{n}{2} + n \frac{n^2}{2} - 1 \).

We notice that the spread of the correlations is non-uniform, since they spread faster in the driving direction.

### 5. Analytic solutions for the weakly coupled Fermionic ring

Analytical results provide us with important new information and insights. In this section we will consider a one-dimensional systems of \( n \) sites with periodic boundary conditions governed by the fermionic Hamiltonian \((20)\). Each site is coupled to a bath with a linear term, as described by equations \((21)\) and \((22)\). The coupling at site \( j \) is parametrized by strength \( S_j \) and the effective inverse temperature \( \beta_j \).

When there is no dissipation, the fermionic Hamiltonian \((20)\) is connected to the XY spin problem. There is no known analytical solution for the non-driven (closed) XY spin chain with open boundary conditions (except for \( h = 0 \) \([42, 43]\)), so we have treated the driven problem with open boundary conditions only numerically in \( 4 \). Note that obtaining the spectrum analytically is possible for the boundary driven XX spin chain (\( \gamma = 0 \)) with special values of driving parameters \( S_1 \) and \( S_n \), which has been performed by Guo and Poletti \([16]\).

Our approach in this section is to use the analytical solution for the XY problem in the closed quantum system \([26–28]\) to analytically diagonalise the matrix \( X \), equation \((5)\), and then include the coupling with baths as a small perturbation, therefore solving the weakly coupled fermionic problem. Since the driving couples different parity sectors, our result does not apply to the driven XY spin ring. For simplicity we will assume that the number of sites \( n \) is even.

#### 5.1. Spectrum of the matrix X

Firstly, we will diagonalize the \( X \) in the limit of the small \( S_j \). This will give us an analytical insight in the dynamics of the weakly coupled open system. As we have

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mentioned before, the matrix $X$ does not depend on the inverse temperatures $\beta_j$ and is given explicitly in equations (23) and (24).

In the limit of the small couplings $S_j$ we can treat the coupling to the baths as a perturbation and calculate the real corrections to the imaginary eigenvalues, which govern the decay to the steady state. A closed system corresponding to $X_0$ with all $S_j = 0$ and thus $D = 0$ can be diagonalised as [28]:

$$PX_0P^\dagger = \begin{pmatrix} i\Lambda^0 & 0 \\ 0 & -i\Lambda^0 \end{pmatrix}, \quad P = \begin{pmatrix} \Phi & i\Psi \\ \Phi & -i\Psi \end{pmatrix}, \quad PP^\dagger = 1_{2n},$$

(35)

$\Psi\Psi^T = \Phi\Phi^T = 1_n$ and $\Phi T = \Lambda^0\Psi$. The eigenvalues and eigenvectors are:

\[\text{Figure 7.}\] The dynamic response of a two-dimensional model given by the Hamiltonian (33) and the dissipation in the top-left and bottom-right corners. We plot the connected time-dependent correlation between the $(x, y)$ at time $t$ and $(\frac{n}{2}, \frac{n}{2})$ at time 0 given by the $C(t)^c$ defined in equation (34). We notice a non-uniform $2 + 1$-dimensional light cone, since the correlations spread faster in the driving direction. The bottom two pictures show the real and imaginary parts of the dynamic response at points $(\frac{n}{2}, \frac{n}{2}) + r$ for different vectors $r$, which show different responses. Parameters are $n = 20$, $\gamma = 0.5$, $S_{(1,1)} = 0.3$, $S_{(n,n)} = 0.1$, $\beta_{(1,1)} = 0.1$ and $\beta_{(n,n)} = 5$. 

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The parameter $\lambda_k$ captures the nonzero anisotropy $\gamma$, with a function $\arctan$ defined such that $0 < \lambda_k \leq \pi$. The quantization condition gives $\phi_k = k\frac{2\pi}{n}$ for the periodic boundary conditions of the fermionic problem ($\phi_k = (k - \frac{1}{2})\frac{2\pi}{n}$ for the anti-periodic boundary conditions); $k = 1, \ldots, n$.

We will now compute the first order corrections for the weakly coupled $XY$ ring using first order perturbation theory. The eigenvectors in non-degenerate subspaces remain the same, so an attempt to diagonalise $X$ with unperturbed eigenvectors results in:

$$
\mathbf{PXP}^\dagger = \left( \begin{array}{cc} i\Lambda^0 + \frac{\Phi\Phi^T + \Psi\Psi^T}{2} & \frac{\Phi\Phi^T - \Psi\Psi^T}{2} \\ \frac{\Phi\Phi^T - \Psi\Psi^T}{2} & -i\Lambda^0 + \frac{\Phi\Phi^T + \Psi\Psi^T}{2} \end{array} \right).
$$

In order to correctly calculate the first order corrections, we need to take into account the fact that the spectrum of the closed system is degenerate: $\Lambda_k^0 = \Lambda_{n-k}^0$ (for a PBC $\Lambda_k^0 = \Lambda_{n-k+1}^0)$. Let us denote $\mathbf{K} = \frac{\Phi\Phi^T + \Psi\Psi^T}{2}$ which can be calculated as

$$
\mathbf{K}_{k,l} = \sum_{j=1}^{n} \frac{S_j}{8} \left( \Psi_{k,j} \Psi_{l,j} + \Phi_{k,j} \Phi_{l,j} \right) = \sum_{j=1}^{n} \frac{S_j}{4n} \left( \cos(j(\phi_k - \phi_l))\cos(\lambda_k - \lambda_l) + \sin(j(\phi_k + \phi_l))\cos(\lambda_k + \lambda_l) \right).
$$

Elements of the degenerate subspaces can be further simplified using $\lambda_{n-k} = -\lambda_k + \pi$:

$$
\mathbf{K}_{k,k} = \frac{1}{4n} \sum_{j=1}^{n} S_j (1 + \sin 2j \phi_k \cos 2\lambda_k) = \frac{1}{4n} \left( \sum_{j=1}^{n} S_j + (\vec{S}_k)_y \cos 2\lambda_k \right),
$$

$$
\mathbf{K}_{n-k,k} = \mathbf{K}_{k,n-k} = -\frac{\cos 2\lambda_k}{4n} \sum_{j=1}^{n} S_j \cos 2j \phi_k = -\frac{\cos 2\lambda_k}{4n} (\vec{S}_k)_x,
$$

$$
\vec{S}_k = \left( \sum_{j=1}^{n} S_j \cos 2j \phi_k \right) \left( \sum_{j=1}^{n} S_j \sin 2j \phi_k \right) = \left( \frac{\vec{S}_k}{2n} \right).
$$

In the last equalities, we introduced a convenient vector notation, which we use to express the first order real corrections to eigenvalues and eigenvectors of the degenerate subspaces as:

$$
\Lambda_{\pm,k}^1 = \sum_{j=1}^{n} \frac{S_j}{2n} \pm \frac{|\vec{S}_k|}{2n} \cos 2\lambda_k, \quad v_{\pm,k} = \frac{1}{\sqrt{|v_{\pm,k}|}} \left( \frac{(\vec{S}_k)_y \pm |\vec{S}_k|}{-|\vec{S}_k|} \right).
$$
Therefore the correct eigenvectors that diagonalise $X_0$ and the correction $K$ in the degenerate subspaces are:

$$
\Phi_{k,l}^\pm = v_{\pm,k,1}\Phi_{k,l} + v_{\pm,k,2}\Phi_{n-k,l}, \quad \Psi_{k,l}^\pm = v_{\pm,k,1}\Psi_{k,l} + v_{\pm,k,2}\Psi_{n-k,l}.
$$

(43)

The new basis is labeled by the index pair $\{\pm, k\}; k = 1, \ldots, n$ instead of one index $k = 1, \ldots, n$.

5.1.1. Special cases. We will take a closer look at three special cases:

(a) $S_n = S \neq 0$

(b) $S_n = S_1 = S \neq 0$

(c) $S_j = S$ for every $j$ with all other (unspecified) coupling constants $S_j$ equal to zero.

In case (a), $\Lambda_{\pm,k}^1 = \frac{S}{n}(1 \pm \cos 2\lambda_k)$ so $\Lambda_{\pm,k}^1 = \frac{S}{n}\cos^2 \lambda_k$, $\Lambda_{\pm,k}^1 = \frac{S}{n}\sin^2 \lambda_k$ and eigenvectors are $(1,1)^T$ and $(1,-1)^T$ for $k = 1, \ldots, n/2$.

On the other hand, in case (b) we have:

$$\Lambda_{\pm,k}^1 = \frac{S}{n}(1 \mp \cos 2\lambda_k \cos \phi_k), \quad \bar{S} = 2\cos \phi_k \left(\frac{\cos \phi_k}{\sin \phi_k}\right), \hspace{1cm} (44)$$

$$v_{\pm,k} = \frac{1}{\sqrt{2(1 \pm \sin \phi_k)}} \left(\sin \phi_k \pm 1, -\cos \phi_k\right). \hspace{1cm} (45)$$

Case (c) results in $\bar{S} = 0$ and $\Lambda^1 = \frac{S}{2}$ with no dependence on $n$. The results are illustrated in figure 8.

5.2. Static correlation functions in the NESS

As seen in section 2.2 (and noting in our case $\hat{P} = P^\dagger$), the static correlation functions are given by the matrix $C$ (we will skip the superscript $\infty$), which is determined by:

$$(\mathbf{P C P}^\dagger)_{jk} = 4i \frac{(\mathbf{P M}_{i} P^\dagger)_{jk}}{\Lambda_j + \Lambda_k^*}, \quad \mathbf{P} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi^+ & i\Psi^+ \\ \Phi^- & i\Psi^- \\ \Phi^+ & -i\Psi^+ \\ \Phi^- & -i\Psi^- \end{pmatrix}, \hspace{1cm} (46)$$

with $\Phi^\pm, \Psi^\pm$ defined in (43). Notice that the static correlations in the NESS depend on the driving encoded in the matrix $M_i$ by the inverse temperatures $\beta_j$ at sites $j$, which we have written explicitly in equation (25). To calculate the correlation functions we need to compute the relevant elements of $(\mathbf{P M}_{i} P^\dagger)$:

$$\mathbf{P M}_{i} P^\dagger = \frac{1}{2} \begin{pmatrix} -i(\Phi \tilde{M}_i \Psi^\dagger + \Psi \tilde{M}_i \Phi^\dagger) & i(\Phi \tilde{M}_i \Psi^\dagger - \Psi \tilde{M}_i \Phi^\dagger) \\ -i(\Phi \tilde{M}_i \Psi^\dagger - \Psi \tilde{M}_i \Phi^\dagger) & i(\Phi \tilde{M}_i \Psi^\dagger + \Psi \tilde{M}_i \Phi^\dagger) \end{pmatrix}, \hspace{1cm} (47)$$

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where we skipped indices ± for better visibility and wrote \( \Phi = \left( \Phi^+ \Phi^- \right) \). The largest contribution for \( \text{PCP}^1 \) comes from \( \Lambda_j + \Lambda_k^* \approx 0 \). Up to the first order correction in \( S \) it means that \( i(\Lambda_j^0 - \Lambda_k^0) + \Lambda_j^1 + \Lambda_k^1 \approx S \) and we obtain the largest correction in the degenerate subspaces. Using the explicit expressions

\[
\Phi_{kj}^+ = \frac{1}{\sqrt{n|v_{\pm,k}|}} \left\{ (v_{\pm,k,1} + v_{\pm,k,2}) (\cos\lambda_k \sin j\phi_k - \sin\lambda_k \cos j\phi_k) \\
+ (v_{\pm,k,1} - v_{\pm,k,2}) (\cos\lambda_k \cos j\phi_k + \sin\lambda_k \sin j\phi_k) \right\},
\]

and the same expression for \( \Psi \) with \(-\lambda_k\) instead of \(\lambda_k\) we derive:

\[
\Phi_{kj}^+ \Psi_{kj}^+ = \frac{1}{n} \left\{ \cos 2\lambda_k \pm \frac{S_x}{|S|} \cos 2 j\phi_k \pm \frac{S_y}{|S|} \sin 2 j\phi_k \right\},
\]

\[
\Phi_{kj}^+ \Psi_{kj}^- = \frac{1}{n} \left\{ \mp \frac{S_x}{|S|} \sin 2\lambda_k - \frac{S_y}{|S|} \sin 2 j\phi_k \right\},
\]

which is all that we need since \( \tilde{M}_i \) is a diagonal matrix and we are interested in the first non-trivial order in \( S \).

Figure 8. *Top*: numerically calculated first order corrections \( \Lambda^1 \) resized by a factor \( \frac{n}{S} \) and ordered by size for different magnitudes of the coupling \( S \) in case (b). The shape stays mostly the same, with the exception of the largest correction, which becomes much larger for larger values of \( S \). In the inset we show the logarithm of the error of analytical values given by (44) for a small coupling \( S \) (coupling at sites 1 and \( n, h = .8, \gamma = 0.5, \ n = 50 \). *Bottom*: the analytical corrections in cases (a) (rescaled by \( \frac{n}{2S} \)), (b) (rescaled by \( \frac{n}{S} \)) and (c) (rescaled only by \( S \)).
5.2.1. Static correlation functions: case (a). In case (a), dissipation at a single site equation (46) results in the diagonal elements:

\[(PCP^\dagger)_{k,k}^{\pm,\pm} = \pm 4 \tanh \beta,\]

with other elements equal to 0 and \(k = 1, \ldots, n/2\) (for the second half of the diagonal we get an additional minus sign, as seen in equation (47)). Interestingly, other elements are exactly zero and the result is robust against stronger coupling (the numerics and analytics suggest that the result is correct and exact for any coupling). The matrix \(C\) encoding the correlations in the NESS is simple:

\[C_{(k,1),(l,2)} = -i \tanh \beta (\delta_{k,n-l} - \delta_{k,n} \delta_{l,n}),\]

and all other nonzero elements are determined by the asymmetry of the matrix \(C\) (\(C^T = -C\)). We denoted by 1 (2) odd (even) Majorana subspaces (\(w_{(j,1)} = w_{2j-1}\) and \(w_{(j,2)} = w_{2j}\)). This gives very interesting long-range correlations between even–odd Majoranas that are the same distance from site \(n\) in opposite directions, as shown in figure 9, and do not depend on the bath coupling strength \(S\).

5.2.2. Static correlation functions: case (b). The first order results for the special case (b) \((S_1 = S_n = S)\) are given by:

\[(PCP^\dagger)_{k,k}^{+,+} = \frac{(\tanh \beta_1 + \tanh \beta_n)(\cos 2\lambda_k + \cos \phi_k)}{2(1 + \cos 2\lambda_k \cos \phi_k)},\]

\[(PCP^\dagger)_{k,k}^{-,-} = \frac{(\tanh \beta_1 + \tanh \beta_n)(\cos 2\lambda_k - \cos \phi_k)}{2(1 - \cos 2\lambda_k \cos \phi_k)},\]

\[(PCP^\dagger)_{k,k}^{+,\mp} = \frac{(\tanh \beta_1 - \tanh \beta_n)\sin \phi_k \cos \phi_k}{2 |\cos \phi_k|}.\]

Figure 9. Left: a graphic showing the interesting correlations in case (a), which are between even–odd Majoranas and the same in size. The arrow shows the driven site. Right: the second order corrections in case (b) for \(h < h_c\) are enhanced along the arcs which indicate long distance correlations in the NESS (a further explanation is given in the final paragraph of section 5.2.2).
In this case the resulting expression is more complicated:

\[
C_{(k,1), (l,2)} = -\frac{\tanh h \beta_k - \tanh h \beta_l}{4} \left( \delta_{k+l,2+n} - \delta_{k+l,n} \right) + \frac{\tanh h \beta_k + \tanh h \beta_l}{2n} \sum_{a=1}^{n/2} \frac{1}{(\cos \phi_a - h)^2 + \gamma^2} \left( 2(\cos \phi_a - h)^2 \left[ \cos(k-l)\phi_a + \frac{\gamma \sin \phi_a \sin(k-l)\phi_a}{\cos \phi_a - h} \right] + \gamma^2 (\cos(k+l)\phi_a + \cos(k+l+2)\phi_a) \right).
\] (55)

Again there are interesting correlations between sites \( j \) and \( n - j \) (and \( n - j + 2 \)), similar to case (a). This is the first order result, and the higher orders are non-trivial. In fact it misses an important feature: a change in the long distance correlation behaviour when we cross the non-equilibrium phase transition. This is a \( O(S) \) effect in the weakly coupled regime (it does not depend on \( S \) for larger couplings or the open chain). It is a consequence of the form of the dispersion for the elements \((\text{PCP}^1)_{k,l}\) which cause the correlations at larger distances. They are shown in figure 9 on the right and are arc shaped.

5.3. Validity of first order perturbation theory

Here we will take a closer look at the validity of perturbation theory, with a focus on the large \( n \) (thermodynamic) limit. This limit can be problematic because the unperturbed energy level spacing gets smaller and smaller, and from some point on, we may need to treat the neighbouring energy levels as almost degenerate and use degenerate perturbation theory.

In general we can say that first order perturbation theory will work if the first order correction for the eigenvectors are small:

\[
(\Psi_k^+) = \sum_{l \neq k} \frac{K^\pm_{kl}}{\Lambda^0_k - \Lambda^0_l} (\Psi^0_l)^0, \quad K^\sigma = \frac{\Phi^{\sigma} \mathcal{D}(\Phi^{\nu})^T + \Phi^{\nu} \mathcal{D}(\Phi^{\nu})^T}{2}, \quad \sigma, \nu \in \{+, -\}.
\] (56)

The energy difference is the smallest for \( k, l \) close to 0, \( \frac{n}{2}, n \). There we can expand the dispersion \( \Lambda^0_k = a_2 \phi_k^2 + a_4 \phi_k^4 \) with \( a_2 = \frac{h(1-\gamma^2)}{2|1+\gamma h|} \), where the first sign holds close to 0 and \( n \) and the second one close to \( \frac{n}{2} \). At the non-equilibrium phase transition \( h_c = |1 - \gamma^2| \), when for \( k \) close to 0 or \( \frac{n}{2}, a_2 = 0 \), the next order has a prefactor \( a_4 = \frac{3}{2} \left( \frac{1}{\gamma^2} - 1 \right) \).

We can bound \( |K^\pm_{kl}| < \sum_j S_j/n \), where \( \sigma \in \{+, -\} \). This gives us the bound \( n_b \) below in which perturbation theory is non-problematic:

\[
\left| \frac{K^\pm_{kl}}{\Lambda^0_k - \Lambda^0_l} \right| < \sum_j S_j/n \left| a_2 \left( \frac{2\pi}{n} \right)^2 + a_4 \left( \frac{2\pi}{n} \right)^4 \right|, \quad n_b = \frac{a_2 (2\pi)^2}{\sum_j S_j}, \quad (n_b)_c = \sqrt{a_4 (2\pi)^4 \sum_j S_j},
\] (57)

where the last bound \( (n_b)_c \) is valid at the criticality when \( a_2 = 0 \). For larger \( n \) we may need to use the degenerate perturbative theory at least for some energy levels and treat the non-exact degeneracy as a perturbation.

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5.4. Spectral gap

The spectral gap is an important feature of the system since it sets the longest relaxation time. Looking at the smallest correction, we can calculate the spectral gap of the Liouvillian \( \Delta = 2 \min(\text{Re } \Lambda) = 2 \min \Lambda \), which we can express explicitly for the special cases (a) and (b).

The smallest gap appears at \( \phi_k = \frac{2\pi}{n}, \pi - \frac{2\pi}{n} \quad (k = 1, \frac{n}{2} - 1) \), where at large \( n \) holds that:

\[
\lambda_1 \approx \frac{\gamma}{1 - \frac{\pi}{n}}, \quad \lambda_{2+1} \approx -\frac{\gamma}{1 + \frac{\pi}{n}},
\]

where subscripts \( a \) and \( b \) refer to cases (a) and (b) respectively. The case with \( 1 + h \) results in lower energy. Notice the \( n^{-3} \) dependence, which is the same as for the open \( XY \) chain [13] (which has the same fermionic Hamiltonian up to a periodic boundary term).

This result is valid as long as we are away from the critical \( h_c \), where the unperturbed dispersion is quartic. It holds for every large \( n \), not just \( n < n_b \), because, in case (a):

\[
K_{kl}^- = \frac{S}{2n} \sin \lambda_k \sin \lambda_l, \quad K_{kl}^+ = \frac{S}{2n} \cos \lambda_k \cos \lambda_l, \quad K_{kl}^{\pm} = 0.
\]

The smallest eigenvalues’ correction eigenvectors can couple only to \( K^{-\sigma} \) and the correction to eigenvectors is \( \frac{S}{a_2(\frac{n}{a_2})^2} \approx \frac{S}{a_2 n} \).

In case (b), the expressions become more complicated, so we state here only the dependencies on \( n \) for \( k, l \) close to 0

\[
K_{kl}^- \approx \frac{S}{n^3}, \quad K_{kl}^+ \approx \frac{S}{n}, \quad K_{kl}^{\pm} \approx \frac{S}{n^2}.
\]

Again the smallest eigenvalues’ correction eigenvectors can couple only to \( K^{-\sigma} \) and the largest correction is \( \frac{S}{a_2(\frac{n}{a_2})^2} \approx \frac{S}{a_2} \), so the expression for the NESS gap stays valid even for \( n > n_b \), providing we are not too close to the criticality where \( a_2 \to 0 \).

5.4.1. Spectral gap at the non-equilibrium phase transition. At the non-equilibrium phase transition \( a_2 = 0 \) the energy levels are much closer together, therefore we can use the above arguments only up to the bound in the critical case \( (n_b)_c \). After that we would need to treat more and more energy levels as degenerate and look at the almost degeneracy as a perturbation if we wished to continue with the perturbation theory approach.

The numerical results reveal an interesting behaviour that is summarized in figure 10. The gap size scaling in \( n \) changes from \( Sn^{-3} \) to \( n^{-7}/S \) in case (a). It is interesting that the gap closes faster for larger couplings \( S \), which is counterintuitive.

In case (b), we make a transition from \( n^{-3} \) to \( n^{-7} \) and for very large \( n \) to \( n^{-5} \); the latter is the same as in the open \( XY \) spin chain [13].
We have discussed open fermionic systems with a quadratic Hamiltonian and a linear dissipation governed by the Lindblad equation and have written a simple equation governing the dynamics of the system after a local quench. The driving only influences the correlations in the NESS, whereas the dissipation governs the decay to the NESS, and the Hamiltonian gives the oscillatory response.

Looking at the response to a local quench in the open XY spin chain driven at the edges, which can be mapped to our problem, reveals a light cone behaviour, which is asymmetric for the driving at different effective temperatures. The dynamic response of the two-dimensional problem behaves similarly. Another interesting example is the XY spin chain with DM interactions, where we have noticed a new type of non-thermal phase transition with the long-range correlated NESS which is due to the appearance of a pair of Dirac points in the dispersion relation.

In the second part of the paper, we have presented an analytic solution for a weakly-coupled fermionic ring with nearest-neighbour hopping, and superconductive and chemical potential terms in the Hamiltonian. The explicit real corrections to the imaginary eigenvalues and the correlations in the NESS are calculated based on the solution of the closed XY spin ring, which is perturbed by a weak coupling.
The special case of dissipative driving at just a single site gives a very simple and highly non-local NESS, where each pair of opposite sites is strongly correlated (entangled). We mention the non-equilibrium phase transition which gives long-range correlations in the NESS (that was reported for an open spin XY model [21]) by looking at the spectral gap and the two-point correlation functions in the NESS. Interestingly, in order to observe the effect with long-range correlations, we need driving at more than one site.

Many interesting open questions still remain. One can easily numerically treat general quadratic driven systems, so one could look at the far-from-equilibrium dynamics in many other systems, for example in disordered systems, three-dimensional systems or long-range coupled systems. This framework can also be a starting point for the perturbation theory of non-quadratic (weakly interacting) systems.

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