GROWTH OF BILINEAR MAPS III: DECIDABILITY

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Abstract. The following notion of growth rate can be seen as a generalization of joint spectral radius: Given a bilinear map $\star : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with nonnegative coefficients and a nonnegative vector $s \in \mathbb{R}^d$, denote by $g(n)$ the largest possible entry of a vector obtained by combining $n$ instances of $s$ using $n - 1$ applications of $\star$. Let $\lambda$ denote the growth rate $\limsup_{n \to \infty} \sqrt[n]{g(n)}$. Rosenfeld showed that the problem of checking $\lambda \leq 1$ is undecidable by reducing the problem of joint spectral radius.

In this article, we provide a simpler reduction using the observation that matrix multiplication is actually a bilinear map. Suppose there is no restriction on the signs, an application of this reduction is that the problem of checking if the system can produce a zero vector is undecidable by reducing the problem of checking the mortality of a pair of matrices. This answers a question asked by Rosenfeld. Another application is that the problem does not become harder when we introduce more bilinear maps or more starting vectors, which was remarked by Rosenfeld.

It is known that if the vector $s$ is strictly positive, then the limit superior $\lambda$ is actually a limit. However, we show that when $s$ is only nonnegative, the problem of checking the validity of the limit is undecidable. This also answers a question asked by Rosenfeld.

We provide a formula for the growth rate $\lambda$. A condition is given so that the limit is always ensured. This actually gives a simpler proof for the limit $\lambda$ when $s > 0$. An important corollary of the formula is the computability of the growth rate, which answers another question by Rosenfeld. Another corollary is that the problem of checking $\lambda \leq 1$ is reducible to the problem of joint spectral radius, via the halting problem, i.e. the two problems are Turing equivalent. Also, we relate the finiteness property of a set of matrices to a special structure called “linear pattern” for the problem of bilinear system.

1. Introduction

Given a bilinear map $\star : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with nonnegative coefficients and a positive vector $s \in \mathbb{R}^d$, denote by $g(n)$ the largest possible entry of a vector obtained by combining $n$ instances of $s$ using $n - 1$ applications of $\star$. For example, all the combinations of 4 instances of $s$ are

$$s \ast (s \ast (s \ast s)), s \ast ((s \ast s) \ast s), (s \ast s) \ast (s \ast s), (s \ast (s \ast s)) \ast s, ((s \ast s) \ast s) \ast s.$$  

It was shown in [1] that the following growth rate is valid

$$\lambda = \lim_{n \to \infty} \sqrt[n]{g(n)}.$$  

When the entries of $s$ are not necessarily positive but only nonnegative, the limit $\lambda$ may be no longer valid. However, relaxing the requirements on the signs in this way is often asked in applications. Therefore, Rosenfeld in [2] extends the notion of the growth rate $\lambda$ for the case $s$ is nonnegative by defining

$$\lambda = \limsup_{n \to \infty} \sqrt[n]{g(n)},$$  

which is called the growth rate of the bilinear system $\ast, s$.

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Let us call the original setting the positive setting and the latter setting the nonnegative setting (with respect to the sign of $s$).

The study of this problem was first started by Rote in [3] with the maximum number of minimal dominating sets in a tree of $n$ leaves as an example. Later on, a richer set of applications to the maximum number of different types of dominating sets, perfect codes, different types of matchings, and maximal irredundant sets in a tree was given by Rosenfeld in [4].

For application purpose, estimating $\lambda$ is a natural problem. In [5] the limit in the positive setting can be approximated to an arbitrary precision. In [4] the growth rate in the nonnegative setting was shown to be upper semi-computable, i.e. we can generate a sequence of upper bounds converging to $\lambda$. In this article, we show that the growth rate is computable. However, it still remains the problem of checking if $\lambda \leq 1$. In [2] Rosenfeld shows that checking $\lambda \leq 1$ is undecidable for the nonnegative setting by reducing the problem of checking $\rho \leq 1$ for the joint spectral radius $\rho$. The notion of joint spectral radius was first introduced in [6] and the growth of bilinear maps can be seen as a generalization.

In this paper, we provide another proof with a simpler reduction using the observation that matrix multiplication is also a bilinear map, as in Theorem 1. The reduction is so natural, and the products of the matrices can be found in an embedded form in the resulting vectors. Note that it is still left open whether the problem of checking $\lambda \leq 1$ in the positive setting is undecidable. We attempt to prove its undecidability under the assumption that it is undecidable to check $\rho \leq 1$ for the joint spectral radius $\rho$ of a pair positive matrices, as in Section 7.

The undecidability of the problem for joint spectral radius is actually proved by

$$HP \leq PFAE \leq JSR,$$

where $HP$ denotes the halting problem, $PFAE$ denotes the problem of probabilistic finite automaton emptiness and $JSR$ denotes the problem of joint spectral radius. (We denote $A \leq B$ if Problem $A$ can be reduced to Problem $B$.) To be more precise, we mean by $JSR$ the problem of checking $\rho \leq 1$. Note that all these problem are Turing equivalent (i.e. each is reducible to any other) since we a reduction from $JSR$ to $HP$ by the joint spectral radius theorem, which states that for a finite set $\Sigma$ of matrices we have

$$\rho(\Sigma) = \sup_{n} \max_{A_1, \ldots, A_n \in \Sigma} \sqrt[n]{\rho(A_1 \ldots A_n)},$$

where $\rho$ denotes both the joint spectral radius of a set of matrices and the ordinary spectral radius of a matrix, depending on the argument. Indeed, we just run the program that looks for a sequence of matrices whose product has the spectral radius greater than 1. The program does not stop if and only if $\rho(\Sigma) \leq 1$. (Note that the problem for spectral radius ($SR$) is decidable.)

On the other hand, a formula of $\lambda$ in Section 4 allows a reduction from checking $\lambda \leq 1$ to the halting problem. We show the formula as follows in a form that looks similar to the joint spectral radius theorem:

$$\lambda = \sup_{n} \max_{\text{linear pattern } P \mid |P| = n} \sqrt[n]{\rho(M(P))}.$$

We do not explain the terms in detail, which is done in Section 4, but we may say roughly that a linear pattern is a sequence $x_n$ for $n = 0, 1, 2, \ldots$ so that $x_0 = s$ and $x_n$ for $n \geq 1$ is a combination of some instances of $s$ and only one instance of $x_{n-1}$. The notation $|P|$ denotes the number of instances of $s$ and the matrix $M = M(P)$ represents the linear
relation \( x_n = Mx_{n-1} \) for every \( n \geq 1 \). The reduction from checking \( \lambda \leq 1 \) to the halting problem is similar to the one for the problem of checking \( \rho(\Sigma) \leq 1 \).

It means we have established the relation of these problems to the problem of the growth rate of a bilinear system (GRBS). An interesting point is that using reductions of the same kind as the one for JSR \( \leq \) GRBS we can show that the problem for the growth rate does not become harder when multiple operators and multiple starting vectors are allowed. This was first remarked by Rosenfeld in [2]. Let us call it the joint growth rate of a bilinear system (JGRBS). In total, we have

\[ SR < HP = PFAE = JSR = GRBS = JGRBS, \]

where \( A < B \) means \( A \leq B \) but we do not have \( B \leq A \), and \( A = B \) means \( A \leq B \) and \( B \leq A \), that is each of \( A, B \) is reducible to the other, i.e. they are Turing equivalent.

Note that we still do not yet have a natural reduction from GRBS to JSR as the one for JSR \( \leq \) GRBS. Such a reduction is very desirable with some consequences, as discussed in Section 5.

In [2] Rosenfeld asks the following question: Suppose the coefficients of \( * \) and the entries of \( s \) have no condition on the signs (they can even be complex), then is the problem of checking if the system can produce a zero vector decidable? A negative answer is given in Theorem 2. It uses almost the same construction as the one for JSR \( \leq \) GRBS but reduces the problem of checking the mortality of a pair of matrices instead.

Since the reduction for JSR \( \leq \) GRBS is quite natural, we can relate the finiteness property [7] for the joint spectral radius to a result on whether the rate of a linear pattern can attain the growth rate. A set \( \Sigma \) of matrices is said to have the finiteness property if the supremum in Equation (1) for the joint spectral radius theorem is attainable, that is there exist \( A_1, \ldots, A_n \in \Sigma \) so that \( \sqrt[n]{\rho(A_1 \ldots A_n)} = \rho(\Sigma) \). Meanwhile, the rate of a linear pattern \( P \) is \( \sqrt{\rho(M(P))} \) and the supremum in Equation (2) is not always attainable, that is there exists a system where no linear pattern has the same rate as the growth rate (e.g. see [1]). The relation is presented in Section 5.

Checking if \( \lambda \) is actually a limit is also interesting problem, whose decidability was asked by Rosenfeld in a correspondence. Theorem 3 shows that it is undecidable by reducing the problem of checking \( \lambda \leq 1 \). During the course, there is a transform of \((*, s)\) to a new system with the corresponding function \( g'(n) \) so that for every \( m \geq 1 \) we have \( g'(2m) = g(m) \) and \( g'(2m + 1) = 0 \).

As an attempt to study the nonnegative setting, we extend the formula of \( \lambda \) in [5] to the nonnegative setting in Section 4. Using the formula, we give a condition so that the limit is always ensured. This actually serves as a proof of the limit \( \lambda \) in the positive setting, which is quite simpler than the proof in [1]. Another corollary is a transform so that the new system still has the same growth rate as the original one but with the valid limit. In fact, the computability of the growth rate in the nonnegative setting is derived from the formula, as in Theorem 7.

2. Reductions

The common construction. We present a construction that will be used in the proofs of both Theorems 1 and 2.

As a matrix multiplication is also a bilinear map in \( \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \), we use some consistent embedding of a matrix \( A \) to a vector \( v \) in the space \( \mathbb{R}^{d^2} \). We denote the embedding by two functions \( \Gamma, \tilde{\Gamma} \) so that

\[ A = \Gamma(v), \quad v = \tilde{\Gamma}(A). \]

We also denote by \( v_{[i,j]} \) the subvector of \( v \) containing the \( k \)-th entries for \( i \leq k \leq j \).
Given a pair of matrices $A, B$ in $\mathbb{R}^d$. Consider the space $\mathbb{R}^{3d^2+2}$ and denote by $R_A, R_B, R_C$ the ranges $[1, d^2], [d^2+1, 2d^2], [2d^2+1, 3d^2]$, respectively, and $i = 3d^2+1, j = 3d^2+2$. Let the system $(*, s)$ with $*: \mathbb{R}^{3d^2+2} \times \mathbb{R}^{3d^2+2} \to \mathbb{R}^{3d^2+2}$ and $s \in \mathbb{R}^{3d^2+2}$ be so that

$$s_{R_A} = \tilde{\Gamma}(A), \quad s_{R_B} = \tilde{\Gamma}(B), \quad s_{R_C} = 0,$$

and for any two vectors $x, y$,

$$(x * y)_{R_A} = (x * y)_{R_B} = 0,$$

$$(x * y)_{R_C} = \tilde{\Gamma}(\Gamma(x_{R_C}\Gamma(y_{R_C})) + x_jy_{R_A} + x_{R_B}y_j,$$

$$(x * y)_i = 0, \quad (x * y)_j = x_iy_j.$$

Let us make some analysis for the vector $v$ obtained from a combination of $n$ instances of $s$. Obviously, $v_i$ is nonzero for only $n = 1$, and $v_j$ is nonzero for only $n = 2$. Also, $v_{R_A}, v_{R_B}$ are zero whenever $n \geq 2$. It remains to consider the dimensions in $R_C$.

**Proposition 1.** The vector $\tilde{v} = v_{R_C}$ is zero whenever $n$ is not divisible by 3. When $n = 3m$, if $\tilde{v}$ is nonzero then the matrix form $\Gamma(\tilde{v})$ is a product of $m$ matrices from $\{A, B\}$. On the other hand, for any $M_1, \ldots, M_n \in \{A, B\}$, we have a combination for $n = 3m$ so that $\Gamma(\tilde{v}) = M_1 \ldots M_m$. Moreover, if every subcombination (contained in a matching pair of brackets) has either less than 3 instances of $s$ or a multiple of 3 instances of $s$, then $\tilde{v}$ is a product of $m$ matrices from $\{A, B\}$ for $n = 3m$.

**Proof.** At first, we make an observation: If the sum $x_jy_{R_A} + x_{R_B}y_j$ in the expression of $(x * y)_{R_C}$ is not zero, then $n = 3$ due to the analysis of $v_{R_A}, v_{R_B}$ and $v_i, v_j$. We have $\tilde{v} = \Gamma(A)$ if the combination is $(s * s) * s$, and $\tilde{v} = \Gamma(B)$ for $s * (s * s)$.

We prove the proposition by induction. It trivially holds for any $n \leq 3$. We show that it holds for any $n > 3$ provided that it holds for smaller numbers.

Let the combination be $X * Y$, where $X, Y$ are the combinations of $n_1, n_2$, respectively, instances of $s$, with $n = n_1 + n_2$.

As $n > 3$, the summands $X_jY_{R_A}, X_{R_B}Y_j$ in the expression of $(X * Y)_{R_C}$ are zero, hence we can safely ignore these summands, that is

$$\tilde{v} = (X * Y)_{R_C} = \tilde{\Gamma}(\Gamma(X_{R_C})\Gamma(Y_{R_C})).$$

Suppose $n$ is not divisible by 3, then one of $n_1, n_2$ is not divisible by 3. By induction hypothesis, either $X_{R_C}$ or $Y_{R_C}$ is zero. It follows that $\tilde{v} = 0$.

Suppose $n = 3m$ and $\tilde{v}$ is not zero, then $n_1, n_2$ are also divisible by 3, say $n_1 = 3m_1$, $n_2 = 3m_2$. By induction hypothesis, both $\Gamma(X_{R_C})$ and $\Gamma(Y_{R_C})$ are products of $m_1, m_2$, respectively, matrices from $\{A, B\}$. Hence $\Gamma(\tilde{v}) = \Gamma(X_{R_C})\Gamma(Y_{R_C})$ is also a product of $m_1 + m_2 = m$ matrices from $\{A, B\}$.

On the other hand, for any $M_1, \ldots, M_m \in \{A, B\}$, by induction hypothesis let $X$ be the combination of $3(m - 1)$ instances of $s$ so that $\Gamma(X_{R_C}) = M_1 \ldots M_{m-1}$ and $Y$ be the combination of $3$ instances of $s$ so that $\Gamma(Y_{R_C}) = M_m$. It follows that $\Gamma(\tilde{v}) = \Gamma(X_{R_C})\Gamma(Y_{R_C}) = M_1 \ldots M_m$.

The conclusion relating to the number of instances of $s$ in a subcombination is argued in a similar way.

The proof finishes by induction. $\square$

**Joint spectral radius.** Before proving some problems are undecidable, we remind the readers the notion of the joint spectral radius.
Given a set of matrices $\Sigma$ in $\mathbb{R}^d$, the joint spectral radius $\rho(\Sigma)$ of $\Sigma$ is defined to be the limit

$$\rho(\Sigma) = \lim_{n \to \infty} \sqrt[n]{\max_{M_1, \ldots, M_n \in \Sigma} \|M_1 \cdots M_n\|}.$$ 

The limit is introduced and proved in [6] and it is independent on the norm. For convenience, we let the norm be the maximum norm, i.e. the largest absolute value of an entry in the matrix.

**Checking $\lambda \leq 1$ is undecidable.** Using the construction, we can establish the reductions in the following theorems.

**Theorem 1.** The problem of checking if $\lambda \leq 1$ for the nonnegative setting is undecidable.

**Proof.** Consider the problem of checking if the joint spectral radius $\rho\{A, B\}$ of a pair of nonnegative matrices $A, B$ in $\mathbb{R}^d$ is at most 1, which is known to be undecidable in [8]. We reduce this problem to the problem of checking if $\lambda \leq 1$ for the system $(\ast, s)$ that has been constructed.

By Proposition 1, we obtain

$$g(3m) = \max_{M_1, \ldots, M_m \in \{A, B\}} \|M_1 \cdots M_m\|.$$ 

Also, for $n > 2$ and $n$ not divisible by 3, we have

$$g(n) = 0.$$ 

Therefore,

$$\lambda = \sqrt[3]{\rho\{A, B\}}.$$ 

It means that we have reduced the problem of the joint spectral radius to the problem of the growth rate. The conclusion on the undecidability follows. \[ \Box \]

The variant of checking $\lambda = 1$ is also undecidable due to the undecidability of the corresponding problem of checking $\rho = 1$ for the joint spectral radius. In fact, we can reduce the problem of checking $\lambda \leq 1$ to the problem of checking $\lambda = 1$ by adding an extra dimension that is always 1. However, the question for $\rho \geq 1$ still remains open (see [8], Section 2.2.3) for a discussion) as we restated as follows.

**Conjecture 1 ([8]).** It is undecidable to check if $\rho \geq 1$ for the joint spectral radius $\rho$.

This has applications in the stability of dynamical systems. If the conjecture holds, then the problem checking $\lambda \geq 1$ is also undecidable. Note that the problem of comparing $\rho$ with 1 for a pair of matrices and a set of matrices are equivalent, see [8].

**Checking the mortality is undecidable.** Other properties of a pair of matrices can be also reduced to the corresponding ones of a bilinear system. The following theorem is an example.

**Theorem 2.** When there is no condition on the signs of the coefficients and the entries, the problem of checking if the system can produce a zero vector is undecidable.

**Proof.** We reduce to this problem the problem of checking if a pair of matrices $A, B$ is mortal, that is checking if there exists a sequence of matrices $M_1, \ldots, M_m$ drawn from $\{A, B\}$ for some $m$ so that $M_1 \ldots M_m$ is a zero matrix. The problem of mortality for a pair of matrices is known to be undecidable in [10]. We use the same system $(\ast, s)$ that has been constructed. However, we add three extra dimensions $3d^2 + 3, 3d^2 + 4, 3d^2 + 5$ with $s_{3d^2+3} = 1, s_{3d^2+4} = 0, s_{3d^2+5} = 0$ and

$$(x \ast y)_{3d^2+3} = x_{3d^2+4}y_{3d^2+4}, \quad (x \ast y)_{3d^2+4} = x_{3d^2+3}y_{3d^2+3},$$

$$(x \ast y)_{3d^2+5} = x_{3d^2+3}y_{3d^2+4} + x_{3d^2+4}y_{3d^2+3} + x_{3d^2+5}y_{3d^2+5} - x_i y_j - x_j y_i.$$
where we still denote \( i = 3d^2 + 1, j = 3d^2 + 2 \). (Note that we are allowed to use negative coefficients here.)

It is not hard to see that for any vector \( v \) obtained from a combination of \( n \) instances of \( s \), the entry \( v_{3d^2+3} \) is nonzero if and only if \( n = 3m+1 \), the entry \( v_{3d^2+4} \) is nonzero if and only if \( n = 3m+2 \).

**Claim**: For \( n \) divisible by 3, if the entry \( v_{3d^2+5} \) is zero then every subcombination has either less than 3 instances of \( s \) or a multiple of 3 instances of \( s \).

The claim trivially holds for \( n = 3 \), since both combinations of 3 instances of \( s \) give \( v_{3d^2+5} = 0 \). Note that \( v_i \neq 0 \) for only \( n = 1 \) and \( v_j \neq 0 \) for only \( n = 2 \), hence for \( n \neq 3 \), both \( x_i y_j \) and \( x_j y_i \) are zero, that is we can ignore them whenever \( n \neq 3 \). This also means \( v_{3d^2+5} \) is always nonnegative despite some negative coefficients.

We prove by induction for \( n = 3m \) with \( m > 1 \) provided that the claim holds for smaller number. If the combination is \( X \ast Y \), then the numbers of instances of \( s \) in both \( X, Y \) are divisible by 3, otherwise \( X_{3d^2+3}Y_{3d^2+4} + X_{3d^2+4}Y_{3d^2+3} \) is positive while \( X_{3d^2+3}Y_{3d^2+5} \) is nonnegative. As we need \( X_{3d^2+5}Y_{3d^2+5} = 0 \), by induction hypothesis every subcombination also needs to follow the requirement. The claim is verified.

Therefore, if the resulting vector \( v \) is zero, then the combination must follow the requirement for the subcombinations. We can obtain such a zero vector if and only if \( \{A, B\} \) is mortal by Proposition 1. The conclusion follows.

3. **Checking if the limit holds is undecidable**

**Proposition 2.** For every \((\ast, s)\) there exists \((\ast', s')\) so that for every \( m \geq 1 \) we have \( g'(2m + 1) = 0 \) and \( g'(2m) = g(m) \).

**Proof.** Suppose the space of \((\ast, s)\) is \( \mathbb{R}^d \) and the coefficients of \( \ast \) are \( c_{i,j}^{(k)} \), that is \((\ast y)_k = \sum_{i,j} c_{i,j}^{(k)} x_i y_j \) for any vectors \( x, y \) and index \( k \). Let \( \ast' : \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) and \( s \in \mathbb{R}^{d+1} \) be so that

\[
\begin{align*}
    s'_{[1,d]} &= 0, \quad s'_{d+1} = 1,
    \\
    (x \ast' y)_k &= s_k x_{d+1} y_{d+1} + \sum_{i,j \in [1,d]} c_{i,j}^{(k)} x_i y_j,
    \\
    (x \ast' y)_{d+1} &= 0.
\end{align*}
\]

For every vector \( v \) obtained from a combination of \( n \) instances of \( s' \) (using operator \( \ast' \)), the last dimension is obvious. The vector \( v_{d+1} = 1 \) if \( n = 1 \) and \( v_{d+1} = 0 \) otherwise. The verification that \((\ast', s')\) satisfies the requirements can be reduced to proving the following claim.

**Claim**: Denote \( \bar{w} = w_{[1,d]} \) for any vector \( w \). The vector \( \bar{v} \) is zero whenever \( n \) is odd. When \( n = 2m \), if \( \bar{v} \) is not zero then \( \bar{v} \) is a vector obtained from a combination of \( m \) instances of \( s \) (using operator \( \ast \)). On the other hand, for every vector \( u \) obtained from a combination of \( m \) instances of \( s \) we have a combination of \( 2m \) instances of \( s' \) so that \( \bar{v} = u \).

Let \( X \ast Y \) be the combination for \( v \), where \( X, Y \) are combinations of \( n_1, n_2 \), respectively instances of \( s \) with \( n = n_1 + n_2 \).

At first, we should note that if the summand \( s_k x_{d+1} y_{d+1} \) in the expression of \((x \ast' y)_k\) is nonzero, then \( n = 2 \). Therefore, when \( n > 2 \) we can safely remove it from the expression, that is

\[
\bar{v} = X \ast Y.
\]
It is not so hard to prove the claim by induction. The claim trivially holds for $n \leq 2$. We prove it for $n > 2$ provided that it holds for smaller numbers.

If $n$ is odd then either $n_1$ or $n_2$ is odd, hence either $X = 0$ or $Y = 0$. It follows that $\bar{v} = 0$.

When $n = 2m$, if $\bar{v}$ is not zero, then both $n_1, n_2$ are even. It follows from induction hypothesis that $X, Y$ are vectors obtained from combinations of $m_1, m_2$, respectively instances of $s$, where $m_1 + m_2 = m$. It means $\bar{v} = X \ast Y$ is a vector obtained from combining $m$ instances of $s$.

The remaining conclusion on the construction of a combination of $2m$ instances of $s'$ is argued similarly.

We finish proving the claim, hence also get the proposition proved. \hfill \Box

**Theorem 3.** Checking the validity of the limit of $\sqrt[3]{g(n)}$ is undecidable.

**Proof.** We will reduce the problem of checking if $\lambda = \limsup_{n \to \infty} \sqrt[3]{g(n)} \leq 1$ for a system $(\ast, s)$ to the problem of checking the validity of the limit.

By Proposition 2, there exists a system $(s', s')$ so that for every $m \geq 1$ we have $g'(2m + 1) = 0$ and $g'(2m) = g(m)$. Let the space of $(s', s')$ be $\mathbb{R}^d$, we construct $s'' : \mathbb{R}^{d_1+1} \times \mathbb{R}^{d_1+1} \to \mathbb{R}^{d_1+1}$ and $s'' \in \mathbb{R}^{d_1+1}$ so that the system $(s', s')$ is brought into the first $d'$ dimensions of the new system and

$$s''_{d' + 1} = 1, \quad (x \ast y)_{d'+1} = x_{d'+1}y_{d'+1}. $$

The last dimension is obviously always 1. So we are ensured that $\lambda''$ of the new system is at least 1. It means the following are now equivalent: (i) the limit of $\sqrt[3]{g'(n)}$ exists for $(s'', s'')$ and (ii) the growth rate $\lambda = (\lambda')^2$ is at most 1. The reduction is finished, and the conclusion on undecidability follows. \hfill \Box

4. Growth rate in nonnegative setting

**Some definitions.** Before presenting a formula of the growth rate, we present some definitions that can be found in [1] and [5]. The definitions are self-contained, but the readers are advised to check the original source for more intuitions and explanations. In fact, the proof of the formula is a simplified and adapted version of the argument in [5] for the nonnegative setting.

Beside $g(n)$, we also denote by $g_i(n)$ the largest possible $i$-th entry of any vector obtained from a combination of $n$ instances of $s$.

We make an assumption that for every $i$ there exists some $n$ so that $g_i(n) > 0$, otherwise we can safely eliminate such a degenerate dimension $i$. How to check for some $i$ if $g_i(n) = 0$ for every $n$ is left as an exercise for the readers. Note that without the assumption, some later results may not hold in their current forms.

A *composition tree* is a rooted binary tree where each vertex is assigned a vector in the following way. We assign the same vector $s$ to all leaves, and assign to each non-leaf vertex the value $x \ast y$ where $x, y$ are respectively the vectors of the left and right children. The vector obtained at the root is called the *vector associated with* the composition tree. We often call a composition tree a tree for short. It can be seen that there is a one-to-one correspondence between a tree of $n$ leaves and a combination of $n$ instances of $s$.

If every leaf is assigned the same vector $s$ but a specially marked leaf is assigned a vector variable $u$, then the vector $v$ at the root depends linearly on $u$ by a matrix $M = M(P)$, that is $v = Mu$. If the tree is $T$ and the leaf is $\ell$, we say such a setting is a *linear pattern* $P = (T, \ell)$. We call $M$ the *matrix associated with* $P$.
A composition $P_1 \oplus P_2$ of two linear patterns $P_1 = (T_1, \ell_1), P_2 = (T_2, \ell_2)$ is the pattern $(T, \ell)$ so that $T$ is obtained from $T_1$ by replacing $\ell_1$ by $T_2$, and $\ell = \ell_2$. If $M_1, M_2$ are the matrices associated to $P_1, P_2$, then $M_1 M_2$ is the matrix associated to $P_1 \oplus P_2$.

For $m \geq 1$ we denote $P^m = P \oplus \cdots \oplus P$ where there are $m$ instances of $P$. If $M$ is associated to $P$ then obviously $M^m$ is associated to $P^m$.

The number of leaves $|P|$ of a pattern $P = (T, \ell)$ is defined to be the number of leaves excluding the marked leaf (i.e. one less than the number of leaves in $T$). One can see that $|P \oplus Q| = |P| + |Q|$.

For convenience, we also denote by $P \oplus T'$ the tree obtained from the tree of the pattern $P$ by replacing the marked leaf by the tree $T'$. Let $u$ be the vector associated with $T'$, the vector $v$ associated with $P \oplus T'$ is $M u$ for $M = M(P)$. Let $T'$ be a tree of a bounded number of leaves with $u_j > 0$, then $M_{i,j} \leq \text{const} M_{i,j} u_j \leq \text{const} v_i \leq \text{const} g_i(|P| + O(1))$.

Let $*$ be represented by the coefficients $c_{i,j}^{(k)}$ so that for any vectors $x, y$ and an index $k$,

$$(x * y)_k = \sum_{i,j} c_{i,j}^{(k)} x_i y_j.$$ 

The dependency graph is the directed graph where the dimensions are the vertices and there is an edge from $k$ to $i$ if and only if either $c_{i,j}^{(k)} \neq 0$ or $c_{j,i}^{(k)} \neq 0$ (loops are allowed). The dependency graph can be partitioned into strongly connected components, for which we call components for short. These components define a partial order so that for two different components $C', C$, we say $C' < C$ if there is a path from $i$ to $j$ for $i \in C$ and $j \in C'$.

If there is a path from $i$ to $j$, then there is a linear pattern $P_{i \to j}$ of a bounded number of leaves so that $M(P_{i \to j})_{i,j} > 0$. It can be seen from the fact that if there is an edge $ki$ then $M(P)_{k,i} > 0$ for $P = (T, \ell)$ where $\ell$ is the left (or right) branch if $c_{i,j}^{(k)} \neq 0$ (or $c_{j,i}^{(k)} \neq 0$), and the right (or left) branch has a bounded number of leaves whose associated vector has a positive $j$-th entry. The remaining is done by compositions if the distance from $i$ to $j$ is greater than 1.

The formula. Now we have enough material for proving the following formula of the growth rate.

**Theorem 4.** The growth rate can be expressed as a supremum by

$$\lambda = \limsup_{n \to \infty} \sqrt[n]{g(n)} = \sup_{\text{linear pattern } P} \max_i |P|^{\sqrt[n]{M(P)_{i,i}}}.$$  

**Proof.** Let $\theta$ denote the supremum in the theorem. It is obvious that $\lambda \geq \theta$. Indeed, for any $P$ and $i$, consider the sequence $n = q|P| + r$ for $q = 1, 2, \ldots$, where $r$ satisfies $g_i(r) > 0$ by a tree $T_0$. For such $n$, consider the tree $P^q \oplus T_0$, the associated vector has the $i$-th entry at least $\text{const} (M(P)_{i,i})^q$. As $r$ is bounded, the lower bound of $\lambda$ follows.

It remains to prove the other direction $\lambda \leq \theta$ by the fact that for every $i$ there exists some $r$ so that

$$g_i(n) \leq \text{const} n^{O((\log n)^r)} \theta^n. \tag{3}$$

At first, we make an observation: If $i, j$ are in the same connected component, then for every linear pattern $P$,

$$M(P)_{i,j} \leq \text{const} \theta^{|P|}. \tag{4}$$

\[1\]In \[3\] the bound is even shown to be $\text{const} n^r \theta^n$, but we would keep the approach simpler for the purpose of proving the theorem only.
Indeed, let $P_{j 	o i}$ be the pattern of a bounded number of leaves so that $M(P_{j 	o i})_{j,i} > 0$, we have $M(P + P_{j 	o i})_{i,i} \geq M(P)_{i,j} M(P_{j 	o i})_{j,i} \geq \text{const} \, (P)_{i,j}$. Meanwhile, $M(P + P_{j 	o i})_{i,i} \leq \theta^{P + P_{j 	o i}} \leq \text{const} \, \theta^{|P|}$. The observation is clarified.

When the component is not connected (containing a single vertex without loops), the observation is trivial.

We prove (3) by induction on the components. The observation in (3) can be seen to be the base case actually, since for any $i$ in a minimal component let $P$ be any pattern with the tree associated to $g_i(n)$ we have $g_i(n) = \sum_j M(P)_{i,j}s_j \leq \text{const} \, M(P)_{i,j}$ for some $j$ (note that $j$ is in the same component). Suppose (3) holds for any vertex in a component lower than the component of $i$ with the degree $r'$, we prove that it also holds for $i$ with some degree $r$.

Let $T$ be the tree associated with $g_i(n)$. Pick a subtree $T_0$ of $m$ leaves so that $n/3 \leq m \leq 2n/3$. Let the pattern $P'$ be so that we have the decomposition $T = P' \oplus T_0$. Let $M'$ be the matrix associated with $P'$ and $u$ be the vector associated with $T_0$, we have

$$g_i(n) = \sum_j M'_{i,j} u_j \leq \text{const} \, M'_{i,j} u_j \leq \text{const} \, M'_{i,j} g_j(m)$$

for some $j$.

If $j$ is in the same component as $i$, then we have $M'_{i,j} \leq \text{const} \, \theta^{n-m}$. Therefore,

$$g_i(n) \leq \text{const} \, \theta^{n-m} g_j(m).$$

If $j$ is not in the component of $i$, then $g_j(m) \leq \text{const} \, m^{O((\log m)^r')} \theta^m$ by induction hypothesis. Since $M'_{i,j} \leq \text{const} \, g_i(|P'| + O(1))$, we have

$$g_i(n) \leq \text{const} \, g_i(n - m + O(1)) m^{O((\log m)^{r'})} \theta^m.$$

In either case we have reduced $n$ to at most a fraction of $n$ and $g_i$ to $g_k$ with $k$ still in the same component of $i$, in $g_i(n)$ to $g_j(m)$ for the first case, and in $g_i(n)$ to $g_i(n - m + O(1))$ for the second case. Repeating the process recursively to either $g_j(m)$ or $g_i(n - m + O(1))$ an $O(\log n)$ number of times until the argument is small enough, we obtain

$$g_i(n) \leq \text{const} \, K^{O((\log n)^r')} (n^{O((\log n)^{r'})})^{O(\log n)} \theta^{n + O(\log n)} \leq \text{const} \, n^{O((\log n)^{r'})} \theta^n$$

where $r = r' + 1$ and $K$ is some constant. (Note that $a^{\log b} = b^{\log a}$.)

The proof finishes by induction.

\hfill \Box

**A condition for the limit to hold.** We provide a condition in the nonnegative setting so that $\lambda$ is a limit.

**Theorem 5.** Suppose there exists some $n_0$ so that for every $n \geq n_0$ and for every $i$ we have $g_i(n) > 0$, then $\lambda$ is actually a limit.

**Proof.** Let $\theta$ denote the supremum in Theorem 4 it suffices to prove that

$$\lim_{n \to \infty} \sqrt[n]{g(n)} \geq \theta,$$

which can be reduced to showing that for any pattern $P$ and any index $i$, we have

$$\lim_{n \to \infty} \sqrt[n]{g(n)} \geq \sqrt[n]{M(P)_{i,i}}.$$

Indeed, for every $n$ large enough, let $n = q|P| + r$ so that $n_0 \leq r < n_0 + |P|$. Consider the tree $P^n \oplus T_0$ where $T_0$ is the tree associated with $g_i(r)$, the associated vector has the $i$-th entry at least a constant times $(M(P)_{i,i})^r$. Since $r$ is bounded, the conclusion follows. \hfill \Box
We have actually provided another proof that the limit is always valid in the positive setting. This proof is actually simpler than both other versions in [1] and [5].

**Corollary 1.** The growth rate $\lambda$ is always a limit in the positive setting.

5. **Applications of the formula**

**A formula for the spectral radius.** The formula in Theorem 4 turns out to give a formula for the spectral radius of a nonnegative matrix $A$.

**Theorem 6.** For every nonnegative matrix $A$, the spectral radius $\rho(A)$ can be written as

$$\rho(A) = \sup_n \max_i \sqrt[n]{(A^n)_{i,i}}.$$ 

**Proof.** Suppose $A$ is a $d \times d$ matrix, consider a consistent embedding of a $d \times d$ matrix $B$ to a vector $v$ in $\mathbb{R}^{d^2}$ by the functions $\Gamma, \tilde{\Gamma}$ so that

$$B = \Gamma(v), \quad v = \tilde{\Gamma}(B).$$

Let the system $(\ast, s)$ in the space $\mathbb{R}^{d^2}$ be so that $s = \tilde{\Gamma}(A)$ and

$$u \ast v = \tilde{\Gamma}(\Gamma(u)\Gamma(v)).$$

One can see that every combination of $n$ instances of $s$ gives $\tilde{\Gamma}(A^n)$. Therefore,

$$\lambda = \rho(A).$$

On the other hand, if $P$ is a linear pattern with $|P| = m$, then the relation between the vector at the root $v$ and the vector at the marked leaf $u$ is

$$\Gamma(v) = A^m\Gamma(u).$$

Let $M$ be the $d^2 \times d^2$ matrix so that $v = Mu$, it is not hard to see that the diagonal $M_{j,j}$ are all zero except for those $j$ that correspond to the entries $(i, i)$ for $i = 1, \ldots, d$, that is $M_{j,j} = (A^m)_{i,i}$.

It follows that

$$\rho(A) = \lambda = \sup_{|P| = n} \max_{P} \max_i \sqrt[n]{M(P)_{i,i}} = \sup_n \max_i \sqrt[n]{(A^n)_{i,i}}.$$ 

□

**Remark 1.** One can also obtain a similar formula for the joint spectral radius using this method with the construction in Section 2. However, it would be more complicated to argue.

**Finiteness property and linear patterns.** The growth rate can be written a bit differently as

$$\lambda = \sup_{P} \sqrt[|P|]{\sup_n \max_i \sqrt[n]{M(P)_{i,i}}}.$$ 

since the linear pattern $P^n$ has the associated matrix $M(P)^n$ and has $|P^n| = n|P|.

By the formula of the spectral radius in Theorem 6 we have

$$\lambda = \sup_{P} \sqrt[|P|]{\rho(M(P))}.$$ 

We call $\bar{\lambda}_P = \sqrt[|P|]{\rho(M(P))}$ the rate of the pattern $P$. The formula for the new notation is

$$\lambda = \sup_{P} \bar{\lambda}_P.$$ 

(5)
The rate of a linear pattern is the original motivation for the proof of the limit in the positive setting in [1]. Although it is not technically more important than Theorem 4, its meaning is worth mentioning: Consider the sequence of the trees of $P^1, P^2, \ldots$, the vectors $v^{(1)}, v^{(2)}, \ldots$ associated with these trees are $M_1, M_2, \ldots$ for $M = M(P)$. As $s > 0$, the growth $\lambda_P = \lim_{n \to \infty} \sqrt[n]{\|v^{(n)}\|}$ of the norms $\|v^{(n)}\|$ is the spectral radius of $M$. However, a lower bound on the growth rate should be $\rho(M)$ after being normalized, by taking the $|P|$-th root, as the number of leaves in $P_t$ grows by $|P|$ in each step, that is $\lambda \geq \bar{\lambda}_P = \sqrt[|P|]{\rho(M)}$. The proof in [1] manages to show that this is also the upper bound.

Representing the growth rate in terms of the rates of linear patterns gives some new insight. While the supremum is almost never attained in the form of Theorem 4 (as rare as in the case of Theorem 3), it is quite common that some linear pattern attains the growth rate in the form of (5), that is $\lambda = \bar{\lambda}_P$ for some $P$. For example, the system in [1] Theorem 3, where $s = (1, 1)$ and $x \ast y = (x_1 y_2 + x_2 y_1, x_1 y_2)$, takes the golden ratio as the growth rate, and the growth rate is attained by a linear pattern where the tree has two leaves with the marked leaf on the left. The readers can also check [3] for a more complicated example.

A natural question would be:

When is the growth rate actually the rate of a linear pattern?

We relate this question to the finiteness property of a set of matrices.

Given a pair of matrices $A, B$ and the associated bilinear system that is constructed as in Section 2, we have

$$\lambda = \sqrt[n]{\rho(A, B)}.$$ 

Suppose the pair $A, B$ has the finiteness property, that is there exists a sequence $M_1, \ldots, M_m$ where each matrix is in $\{A, B\}$ so that $\sqrt[n]{\rho(M_1 \ldots M_m)} = \rho(A, B)$. We can then build a pattern $P = (T, \ell)$ so that $\bar{\lambda}_P = \sqrt[|P|]{\rho(A, B)}$. Indeed, if $T'$ is the tree of $3m$ leaves that is associated to $M_1 \ldots M_m$ (as in Proposition 1), we can let $T$ be the tree of $3m + 1$ leaves where one branch is $T''$ and the other branch is the marked leaf $\ell$. The readers can check that $\bar{\lambda}_P = \sqrt[|P|]{\rho(A, B)}$.

On the other hand, suppose the pair $A, B$ does not have the finiteness property, e.g. the class of pairs in [11], or an explicit instance in [12]. In this case, there is no linear pattern where $\bar{\lambda}_P = \sqrt[|P|]{\rho(A, B)}$, since otherwise, by considering the sequence of $P^n$ for $t = 1, 2, \ldots$, we would have a periodic sequence of products of matrices whose norms follow the rate $\rho(A, B)$ (with respect to the number of matrices).

In fact, the readers can find in [1] Theorem 2 a simple example in the positive setting where no linear pattern has the same rate as the growth rate. The example is not related to the joint spectral radius and involves only binary entries and coefficients with $s = (1, 1)$ and $x \ast y = (x_1 y_1 + x_2 y_2, x_1 y_2)$. On the other hand, the algebraic nature of the entries in the example of [12] is quite complicated. It seems that $JSR \leq GRBS$ suggests some phenomenon of $GRBS$ may be easier to construct than a similar one of $JSR$. Nevertheless, the finiteness conjecture is still open for the case of rational (and equivalently binary) matrices, see [13]. Note that if we have a reduction from $GRBS$ to $JSR$ that is as natural as the one in Section 2 and keeps the resulting vectors in some form in the resulting matrices, then we can obtain a set of binary matrices without finiteness property.

**Computability of the growth rate.** Before proving the computability of the growth rate in Theorem 7 below, we give an extension of Fekete’s lemma.

**Lemma 1** (An extension of Fekete’s lemma for nonnegative sequences). *Given a nonnegative sequence $a_n$ for $n = 1, 2, \ldots$, if the sequence is supermultiplicative (i.e. $a_{m+n} \geq$
is the pattern for that this sequence is supermultiplicative. Indeed, suppose we have
\[ P \]
semi-computable. It remains to show that it is lower semi-computable, by showing a
\[ \lambda \]
was shown in [4] that the growth rate \( \lambda \) in the nonnegative setting is computable.

**Theorem 7.** The growth rate in the nonnegative setting is computable.

**Proof.** It was shown in [1] that the growth rate \( \lambda \) in the nonnegative setting is upper semi-computable. It remains to show that it is lower semi-computable, by showing a sequence of lower bounds converging to \( \lambda \). As
\[ \lambda = \sup_{\text{linear pattern } P} \max_i \sqrt[n]{M(P)_{i,i}}, \]
we have
\[ \lambda = \max_i \sup_n \sup_{\text{linear pattern } P} \sqrt[n]{M(P)_{i,i}}. \]

For each \( i \), consider the sequence \( a_n^{(i)} = \sup_{|P|=n} M(P)_{i,i} \) for \( n = 1, 2, \ldots \), and we can see that this sequence is supermultiplicative. Indeed, suppose \( P \) is the pattern for \( a_n^{(i)} \) and \( Q \) is the pattern for \( a_n^{(i)} \). Since the pattern \( P \oplus Q \) has \( |P \oplus Q| = |P| + |Q| = m + n \), we have
\[ a_{m+n}^{(i)} \geq M(P \oplus Q)_{i,i} = (M(P)M(Q))_{i,i} \geq M(P)_{i,i}M(Q)_{i,i} = a_m^{(i)}a_n^{(i)}. \]

By the extension of Fekete’s lemma for nonnegative sequence, the subsequence \( \{ \sqrt[n]{a_n^{(i)}} : a_n^{(i)} > 0 \} \), if nonempty, converges to \( \sup_n \sqrt[n]{a_n^{(i)}} \). Consider the sequence \( x_n = \max_i a_n^{(i)} \), either the sequence is all zero or
\[ \lim_{n \to \infty} \{ \sqrt[n]{x_n} : x_n > 0 \} = \max_i \lim_{n \to \infty} \{ \sqrt[n]{a_n^{(i)}} : a_n^{(i)} > 0 \} = \max_i \sup_n \sqrt[n]{a_n^{(i)}} = \lambda. \]
The conclusion follows from the fact that each \( \sqrt[n]{x_n} \) is a lower bound for \( \lambda \).

**Theorem 7.** The growth rate in the nonnegative setting is computable.

**Proof.** It was shown in [1] that the growth rate \( \lambda \) in the nonnegative setting is upper semi-computable. It remains to show that it is lower semi-computable, by showing a sequence of lower bounds converging to \( \lambda \). As
\[ \lambda = \sup_{\text{linear pattern } P} \max_i \sqrt[n]{M(P)_{i,i}}, \]
we have
\[ \lambda = \max_i \sup_n \sup_{\text{linear pattern } P} \sqrt[n]{M(P)_{i,i}}. \]

For each \( i \), consider the sequence \( a_n^{(i)} = \sup_{|P|=n} M(P)_{i,i} \) for \( n = 1, 2, \ldots \), and we can see that this sequence is supermultiplicative. Indeed, suppose \( P \) is the pattern for \( a_n^{(i)} \) and \( Q \) is the pattern for \( a_n^{(i)} \). Since the pattern \( P \oplus Q \) has \( |P \oplus Q| = |P| + |Q| = m + n \), we have
\[ a_{m+n}^{(i)} \geq M(P \oplus Q)_{i,i} = (M(P)M(Q))_{i,i} \geq M(P)_{i,i}M(Q)_{i,i} = a_m^{(i)}a_n^{(i)}. \]

By the extension of Fekete’s lemma for nonnegative sequence, the subsequence \( \{ \sqrt[n]{a_n^{(i)}} : a_n^{(i)} > 0 \} \), if nonempty, converges to \( \sup_n \sqrt[n]{a_n^{(i)}} \). Consider the sequence \( x_n = \max_i a_n^{(i)} \), either the sequence is all zero or
\[ \lim_{n \to \infty} \{ \sqrt[n]{x_n} : x_n > 0 \} = \max_i \lim_{n \to \infty} \{ \sqrt[n]{a_n^{(i)}} : a_n^{(i)} > 0 \} = \max_i \sup_n \sqrt[n]{a_n^{(i)}} = \lambda. \]
The conclusion follows from the fact that each \( \sqrt[n]{x_n} \) is a lower bound for \( \lambda \).

**Transform to make the limit valid.** Beside the transform in Proposition [2] we also present the following transform, as an application of Theorem [4]. While the former transform makes the limit not valid, the latter ensures the opposite.

**Proposition 3.** For every \( (\ast, s) \) there exists \( (\ast', s') \) so that \( (\ast', s') \) has the same growth rate as \( (\ast, s) \) and the limit of \( \sqrt[n]{g'(n)} \) is ensured, where \( g'(n) \) is the function for \( (\ast', s') \).
Proof. Consider $*: \mathbb{R}^{d+2} \times \mathbb{R}^{d+2} \to \mathbb{R}^{d+2}$ so that the coefficients of $*$ and the entries of $s$ are brought to the first $d$ dimensions of $(s', s')$. We let $s'_{d+1} = s'_{d+2} = \alpha$ (in fact, the value of $s'_{d+1}$ does not matter), where $0 < \alpha \leq \lambda$, say we can take any lower bound of $\lambda$, e.g. by Theorem 4. (Note that we assume $\lambda > 0$, otherwise it is trivial.) The operator $s'$ is defined so that

$$(x \ast y)_{d+1} = \sum_{i=1}^{d} x_i y_{d+2},$$

and

$$(x \ast y)_{d+2} = x_{d+2} y_{d+2}.$$

The $(d + 2)$-th entry of any resulting vector is obviously a power of $\alpha$ to the number of instances of $s$. It follows that for any index $i$ and a bounded $\delta$, we have

$$g'_d(n + \delta) \geq \text{const} g_i(n),$$

by considering the composition tree where the left branch is associated with $g_i(n)$ and the right branch is any tree of $\delta$ leaves.

This means that for a bounded $\delta$, we have

$$g'(n + \delta) \geq g'_d(n + \delta) \geq \max_i \text{const} g_i(n) = \text{const} g(n).$$

However, it is not hard to see (exercise) that

$$\limsup_{n \to \infty} \sqrt[n]{g'(n)} \leq \limsup_{n \to \infty} \sqrt[n]{g(n)}.$$

For any linear pattern $P$ with the associated matrix $M$ and any index $i$, we prove that

$$(6) \quad \liminf_{n \to \infty} \sqrt[n]{g'(n)} \geq \sqrt[n]{|P| M_{i,i}}.$$

Indeed, for any $n$ large enough, let $n'$ be a multiple of the number of leaves in $P$, say $n' = q|P|$, so that $n - n'$ is bounded but not too small. Consider the pattern $P^n$ with the associated matrix $M^n$. We have $(M^n)_{i,i} \geq (M_{i,i})^q$. Let $T_0$ be the tree associated to $g_i(n_0) > 0$ for a bounded $n_0$, the $i$-th entry of the vector associated to $P^n \oplus T_0$ is at least a constant times $(M_{i,i})^q$.

We choose $n'$ so that let $r = n - (n' + n_0) > 0$. As $r$ is bounded, we have

$$g'(n) \geq \text{const} g(n - r) \geq \text{const} (M_{i,i})^q.$$

As $n - n'$ is bounded, we have proved the inequality (6). It follows that

$$\liminf_{n \to \infty} \sqrt[n]{g'(n)} \geq \sup_{\text{linear pattern } P} \max_{i} |P| \sqrt[n]{M(P)_{i,i}}.$$

By Theorem 4 we have

$$\liminf_{n \to \infty} \sqrt[n]{g'(n)} \geq \lambda.$$

In total, we have the limit

$$\lim_{n \to \infty} \sqrt[n]{g'(n)} = \liminf_{n \to \infty} \sqrt[n]{g'(n)} = \limsup_{n \to \infty} \sqrt[n]{g'(n)} = \lambda.$$

□

Assume Conjecture 1 holds, that is checking $\rho \geq 1$ and checking $\lambda \geq 1$ are undecidable, we give another approach to the undecidability of the problem of checking if the limit $\lambda$ holds, as an application of Proposition 3.
Given a system \((s, s)\), let the system \((s', s')\) obtained from Proposition 3 be in the space \(\mathbb{R}^{d'}\). Consider \(s'' : \mathbb{R}^{d'+2} \times \mathbb{R}^{d'+2} \to \mathbb{R}^{d'+2}\) and \(s'' \in \mathbb{R}^{d'+2}\) where the first \(d'\) dimensions are deduced from \((s', s')\). We let \(s''_{d'+1} = 1, s''_{d'+2} = 0, and (x \ast'' y)_{d'+1} = x_{d'+2}y_{d'+2}, (x \ast'' y)_{d'+2} = x_{d'+1}y_{d'+1}.

We can see that the last 2 dimensions are independent of the remaining dimensions, and \(\max\{g'''_{d'+1}(n), g'''_{d'+2}(n)\} = 0\) if \(3 \mid n\) and it is 1 otherwise, where \(g''\) is the function for \((s'', s'')\).

The following are now equivalent: (i) the limit of \(\sqrt[3]{g''(n)}\) is valid, and (ii) \(\lambda \geq 1\) if the problem of checking \(\lambda \geq 1\) is undecidable. Therefore, the latter is undecidable, under the assumption on the undecidability of \(\lambda \geq 1\).

6. MULTIPLE OPERATORS AND MULTIPLE STARTING VECTORS

Rosenfeld in [2] made a remark that the problem of the bilinear system is not harder when we allow multiple operators and multiple starting vectors. We give reductions that are similar to those in Section 2.

The construction in Section 2 is well suited for reducing the problem for \((s, s, s')\) to the original problem. By the problem for \((s, s, s')\), we mean the problem where we can choose either \(s\) or \(s'\) in the place of each \(s\) instead of fixing the vector \(s\). The two vectors \(s, s'\) play the roles of \(A, B\) in the construction. We rewrite it formally without repeating the verification.

For a bilinear map \(* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d\) and two vectors \(s, s' \in \mathbb{R}^d\), consider the space \(\mathbb{R}^{3d+2}\) and denote by \(R_A, R_B, R_C\) the ranges \([1, d], [d+1, 2d], [2d+1, 3d]\), respectively, and \(i = 3d+1, j = 3d+2\). Let the system \((\bullet, u)\) with \(\bullet : \mathbb{R}^{3d+2} \times \mathbb{R}^{3d+2} \to \mathbb{R}^{3d+2}\) and \(u \in \mathbb{R}^{3d+2}\) be so that

\[
\begin{align*}
    u_{R_A} &= s, & u_{R_B} &= s', & u_{R_C} &= 0, \\
    u_i &= 1, & u_j &= 0,
\end{align*}
\]

and for any two vectors \(x, y,\)

\[
\begin{align*}
    (x \bullet y)_{R_A} &= (x \bullet y)_{R_B} = 0, \\
    (x \bullet y)_{R_C} &= x_{R_C} \ast y_{R_C} + x_yR_A + x_{R_B}y_j, \\
    (x \bullet y)_i &= 0, & (x \bullet y)_j &= x_iy_i.
\end{align*}
\]

By the same analysis, we obtain that the growth rate of \((\bullet, u)\) is the cube root of the growth rate of \((s, s, s')\), like in Theorem 1.

Using the idea of the previous construction, we can reduce the problem for \((s', s', s)\) to the original problem. By the problem for \((s', s', s)\) we mean the problem where we can choose either \(s'\) or \(s'\) in the place of each instance of \(*\) instead of fixing \(*\).

For two bilinear maps \(*, *' : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d\) and a vector \(s \in \mathbb{R}^d\), consider the space \(\mathbb{R}^{3d+2}\) and denote by \(R_A, R_B, R_C\) and \(i, j\) as in the first reduction. Let the system \((\bullet, u)\) with \(\bullet : \mathbb{R}^{3d+2} \times \mathbb{R}^{3d+2} \to \mathbb{R}^{3d+2}\) and \(u \in \mathbb{R}^{3d+2}\) be so that

\[
\begin{align*}
    u_{R_A} &= s, & u_{R_B} &= s, & u_{R_C} &= 0, \\
    u_i &= 1, & u_j &= 0,
\end{align*}
\]
and for any two vectors $x, y$,

$$(x \bullet y)_{R_A} = x_{R_C} \ast y_{R_C}, \quad (x \bullet y)_{R_B} = x_{R_C} \ast' y_{R_C},$$

$$(x \ast y)_{R_C} = x_j y_{R_A} + x_{R_B} y_j,$$

$$(x \ast y)_i = 0, \quad (x \ast y)_j = x_i y_i.$$

It is not hard to see that for any vector $v$ obtained from combining $n$ instances of $u$ using $\bullet$, if $v_{R_C} \neq 0$ then $n = 5k + 3$ for some $k$. Also, if $v_{R_A}$ or $v_{R_B}$ is not a zero vector, then $n = 5k + 1$ for some $k$. The growth rate of $(\bullet, u)$ is the fifth root of the growth rate of $(\ast, \ast', s)$. The verification is similar to that in Proposition 1 and we leave it to the readers.

A construction for a higher number of starting vectors or a higher number of bilinear operators, or both, can be established similarly by introducing more dimensions. We leave it to the readers as an exercise since the details would be tedious.

In conclusion, introducing more vectors and more operators does not make the problem any harder.

7. Undecidability of checking $\lambda \leq 1$ in the positive setting under an assumption

As we can reduce the problem checking $\rho \leq 1$ for the joint spectral radius $\rho$ to the problem checking $\lambda \leq 1$ for the growth of bilinear maps in the nonnegative setting, one may wonder if there is a similar reduction for the positive setting, where all the entries of $s$ have to be positive. In this section, we give such a reduction under the assumption that the following conjecture holds.

Conjecture 2. It is undecidable to check $\rho \leq 1$ for the joint spectral radius $\rho$ of a pair of positive matrices.

Provided the conjecture holds, we have a reduction that is almost the same as the one in Section 2 but with a slightly more complicated argument. For a pair of positive matrices $A, B$, we reuse the notations there and keep the operator $\ast$, but set the starting vector $s$ by

$$s_{R_A} = \tilde{\Gamma}(A - X), \quad s_{R_B} = \tilde{\Gamma}(B - Y), \quad s_{R_C} = \epsilon,$$

$$s_i = 1, \quad s_j = \epsilon,$$

where the value $\epsilon > 0$ is small enough and $X, Y$ are two matrices that will be given later. (The notation $\epsilon$ may denote a number or a vector, depending on the context.)

By the description of the operator $\ast$, one needs to take care of the range $R_C$ only. Let us analyze some beginning value. At first,

$$\Gamma(((s \ast s)_{R_C}) = \Gamma(\epsilon)^2 + \epsilon(A - X) + \epsilon(B - Y),$$

$$\Gamma(((s \ast s) \ast s)_{R_C}) = (\Gamma(\epsilon)^2 + \epsilon(A - X) + \epsilon(B - Y))\Gamma(\epsilon) + (A - X),$$

$$\Gamma(((s \ast (s \ast s))_{R_C}) = \Gamma(\epsilon)(\Gamma(\epsilon)^2 + \epsilon(A - X) + \epsilon(B - Y)) + (B - Y).$$

We need $X, Y$ be so that

$$\Gamma(((s \ast s) \ast s)_{R_C}) = A,$$

$$\Gamma((s \ast (s \ast s))_{R_C}) = B,$$

$$X < A, \quad Y < B.$$
The first two requirements are equivalent to
\[ X + (X + Y) \Gamma(\epsilon^2) = \Gamma(\epsilon)^3 + A \Gamma(\epsilon^2) + B \Gamma(\epsilon^2), \]
\[ Y + \Gamma(\epsilon^2)(X + Y) = \Gamma(\epsilon)^3 + \Gamma(\epsilon^2)A + \Gamma(\epsilon^2)B. \]

Such \( X, Y \) always exist. (As the solution would be quite tedious, we leave it as an exercise for the readers.) It follows from the smallness of \( \epsilon \) that \( X, Y \) are also small. It means the requirements \( X < A \) and \( Y < B \) are guaranteed.

Denote \( M_1 = \Gamma(s_{RC}) = \Gamma(\epsilon) \) and \( M_2 = \Gamma((s * s)_{RC}) \), we have both \( M_1 < \Gamma(\epsilon') \) and \( M_2 < \Gamma(\epsilon') \) for some \( \epsilon' \) that depends on \( \epsilon \). The value \( \epsilon' \) can be made arbitrarily small by reducing \( \epsilon \).

We make the following observation, whose verification is similar to that of Proposition 1 and left to the readers.

**Proposition 4.** The matrix form \( \Gamma(v_{RC}) \) for any vector \( v \) obtained by combining \( n \) instances of \( s \) is the product of some matrices from \( \{ A, B, M_1, M_2 \} \). In particular, if \( m_A, m_B, m_1, m_2 \) are respectively the numbers of instances of \( A, B, M_1, M_2 \), then \( m_1 + 2m_2 + 3(m_A + m_B) = n \). On the other hand, for any product of \( m \) matrices from \( \{ A, B \} \), we have a combination for \( n = 3m \) so that \( \Gamma(v_{RC}) \) is the product.

Since \( \epsilon' \) can be made arbitrarily small, the number \( m_1, m_2 \) should be made minimal. It follows that \( \lambda = \sqrt[3]{\rho(A, B)} \) like in Theorem 1. Therefore, the problem of checking \( \lambda \leq 1 \) is undecidable in the positive setting under the assumption that Conjecture 2 holds.

**References**

[1] Vuong Bui. Growth of bilinear maps. *Linear Algebra and its Applications*, 624:198–213, 2021.
[2] Matthieu Rosenfeld. It is undecidable whether the growth rate of a given bilinear system is 1. *arXiv preprint arXiv:2201.07630*, 2022.
[3] Günter Rote. The maximum number of minimal dominating sets in a tree. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1201–1214. SIAM, 2019.
[4] Matthieu Rosenfeld. The growth rate over trees of any family of sets defined by a monadic second order formula is semi-computable. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 776–795. SIAM, 2021.
[5] Vuong Bui. Growth of bilinear maps II: Bounds and orders. *arXiv preprint arXiv:2110.15060*, 2021.
[6] Gian–Carlo Rota and W. Gilbert Strang. A note on the joint spectral radius. *Indagationes Mathematicae (Proceedings)*, 63:379–381, 1960.
[7] Jeffrey C Lagarias and Yang Wang. The finiteness conjecture for the generalized spectral radius of a set of matrices. *Linear Algebra and its Applications*, 214:17–42, 1995.
[8] Vincent D Blondel and John N Tsitsiklis. The boundedness of all products of a pair of matrices is undecidable. *Systems & Control Letters*, 41(2):135–140, 2000.
[9] Raphaël Jungers. *The joint spectral radius: theory and applications*, volume 385. Springer Science & Business Media, 2009.
[10] Vincent D Blondel and John N Tsitsiklis. When is a pair of matrices mortal? *Information Processing Letters*, 63(5):283–286, 1997.
[11] Vincent D Blondel, Jacques Theys, and Alexander A Vladimirov. An elementary counterexample to the finiteness conjecture. *SIAM Journal on Matrix Analysis and Applications*, 24(4):963–970, 2003.
[12] Kevin G Hare, Ian D Morris, Nikita Sidorov, and Jacques Theys. An explicit counterexample to the lagarias–wang finiteness conjecture. *Advances in Mathematics*, 226(6):4667–4701, 2011.
[13] Raphaëll M Jungers and Vincent D Blondel. On the finiteness property for rational matrices. *Linear Algebra and its Applications*, 428(10):2283–2295, 2008.