Quantum Theory of Weyl Invariant Scalar-tensor Gravity

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Abstract

We perform a manifestly covariant quantization of a Weyl invariant, i.e., a locally scale invariant, scalar-tensor gravity in the extended de Donder gauge condition (or harmonic gauge condition) for general coordinate invariance and a new scalar gauge for Weyl invariance within the framework of BRST formalism. It is shown that choral symmetry, which is a Poincaré-like $\text{IOSp}(8|8)$ supersymmetry in case of Einstein gravity, is extended to a Poincaré-like $\text{IOSp}(10|10)$ supersymmetry. We point out that there is a gravitational conformal symmetry in quantum gravity and account for how conventional conformal symmetry in a flat Minkowski space-time is related to the gravitational conformal symmetry. Moreover, we examine the mechanism of spontaneous symmetry breakdown of the choral symmetry, and show that the gravitational conformal symmetry is spontaneously broken to the Poincaré symmetry and the corresponding massless Nambu-Goldstone bosons are the graviton and the dilaton. We also prove the unitarity of the physical S-matrix on the basis of the BRST quartet mechanism.

1 Introduction

There is no question that symmetry plays the central role in both elementary-particle physics and quantum gravity. For instance, in the Yang-Mills theory it has been found that we have the non-abelian gauge symmetry and that this symmetry gives rise to physically significant effects, such as the asymptotic freedom and the quark confinement.

It is well known that there are two kinds of symmetries in nature: global symmetry and gauge symmetry. In order to understand the nature more deeply, it is necessary to understand the meaning of the both symmetries. The meaning of the global symmetry is clear in the sense that it operates physical observables in a direct manner and shows the real symmetry of a physical system. On the other hand, the meaning of the gauge symmetry is more elusive than that of the

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global symmetry since it does not operate on physical observables directly. To treat the gauge symmetry properly in quantum field theory, it is essential to fix the gauge symmetry by a suitable gauge condition and consequently physical observables are defined as BRST invariant operators. Thus, it is sometimes said that the gauge symmetry is redundancy in the mathematical description of a physical system rather than the property of the system itself.

Another important property of the symmetries is that many of the global symmetry are not exact but only approximate whereas the gauge symmetry is an exact one. For instance, there is a clear prediction of violation of baryon and lepton numbers by a quantum anomaly in the standard model. This fact is also supported by a theory of quantum gravity. In particular, when a black hole evaporates at the quantum level, the baryon and lepton numbers are not conserved whereas gauge quantum numbers such as electric and magnetic charges are precisely conserved since they are measured by the flux integrals at infinity.

Thus, if a certain global symmetry plays a critical role in physics, it must be promoted to the gauge symmetry. This statement particularly holds in constructing theories involving quantum gravity. In our previous work [1], we have presented a quantum theory of a globally scale invariant gravity with a real scalar field, which is equivalent to the well-known Brans-Dicke gravity [2], by constructing its manifestly covariant BRST formalism. Since many of studies of the Brans-Dicke gravity have been so far limited to a classical analysis, our theory has provided us with some useful information on quantum aspects of the Brans-Dicke gravity. Indeed, based on this quantum gravity we have elucidated a mechanism of how a scale invariance is spontaneously broken and consequently exactly massless “dilaton” emerges thanks to the Nambu-Goldstone theorem in quantum gravity [1,3]. Then, it is natural to generalize our formulation to a case of a locally scale invariant, or equivalently Weyl invariant, scalar-tensor gravity and ask if we can get some useful knowledge about quantum aspects of the theory.

In this article, we perform a manifestly covariant BRST quantization of a Weyl invariant scalar-tensor gravity with a real scalar field in addition to the metric tensor field, investigate the remaining global symmetries and their spontaneous symmetry breakdown, prove the unitarity of the $S$-matrix, and then elucidate that there exists a gravitational analog of conformal symmetry in our theory. Long ago, in a pioneering work by Nakanishi [4,5], on the basis of the Einstein-Hilbert action in the de Donder gauge (harmonic gauge) for general coordinate transformation (GCT), it has been shown that there remains a huge residual symmetry, which is a Poincaré-like $ISO_p(8|8)$ supersymmetry, called “choral symmetry”, including the BRST symmetry and $GL(4)$ symmetry etc. In our present formulation, adopting the extended de Donder gauge condition for the GCT and a new scalar gauge condition for the Weyl transformation, the choral symmetry is extended to a Poincaré-like $ISO_p(10|10)$ supersymmetry, which includes the scale symmetry and the gravitational special conformal symmetry. It is of interest that as in a flat Minkowski space-time, both the scale symmetry and the special conformal symmetry are spontaneously broken, and the dilation is not only a Nambu-Goldstone boson for the scale symmetry but
also its derivative provides a Nambu-Goldstone boson for the special conformal transformation.

The paper is organized as follows. In Section 2, we discuss a general gravitational theory for which there are two local symmetries, those are, the general coordinate invariance and the Weyl symmetry. It is pointed out that in such a theory, we must choose a gauge fixing condition for GCT carefully in such a way that it does not violate the Weyl symmetry, and similarly a gauge fixing condition for the Weyl transformation should be selected in order not to break the GCT. In Section 3, beginning with a Weyl invariant scalar-tensor gravity [6], we fix the GCT and the Weyl transformation by the extended de Doner gauge and the new scalar gauge conditions, and construct a gauge-fixed, BRST-invariant quantum Lagrangian. In Section 4, we calculate various equal-time (anti-)commutation relations (ETCRs) among the fundamental fields, in particular, the Nakanishi-Lautrup auxiliary field, the Faddeev-Popov (FP) ghosts. In Section 5, we derive the ETCRs involving the gravitational field. In Section 6, we prove the unitarity of the physical S-matrix by means of the BRST quartet mechanism. In Section 7, we show that there is a choral symmetry, which is an \( IOSp(10|10) \) supersymmetry, in our theory. In Section 8, we point out the existence of a gravitational conformal symmetry even in quantum gravity, and we investigate its spontaneous symmetry breaking in Section 9. The final section is devoted to discussion.

Two appendices are put for technical details. In Appendix A, a derivation of the equation for the \( b_\rho \) field is given, and in Appendix B we have accounted for the relationship between the gravitational conformal symmetry and conventional conformal symmetry.

## 2 Consistency between two BRST symmetries

We wish to perform a manifestly covariant BRST quantization of a gravitational theory which is invariant under both general coordinate transformation (GCT) and Weyl transformation, or equivalently local scale transformation. To take a more general theory into consideration, without specifying the concrete expression of the gravitational Lagrangian density, we will start with the following classical Lagrangian density:

\[
L_c = L_c(g_{\mu\nu}, \phi), \tag{2.1}
\]

which includes the metric tensor field \( g_{\mu\nu} \) and a scalar field \( \phi \) as dynamical variables.\(^2\) We assume that \( L_c \) does not involve more than first order derivatives of the metric and matter fields.

\(^1\)We follow the notation and conventions of MTW textbook [7]. Greek little letters \( \mu, \nu, \cdots \) and Latin ones \( i, j, \cdots \) are used for space-time and spatial indices, respectively; for instance, \( \mu = 0, 1, 2, 3 \) and \( i = 1, 2, 3 \). The Riemann curvature tensor and the Ricci tensor are respectively defined by \( R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\lambda\mu} \Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\lambda\nu} \Gamma^\lambda_{\sigma\mu} \) and \( R_{\mu\nu} = R^\rho_{\mu\rho\nu} \). The Minkowski metric tensor is denoted by \( \eta_{\mu\nu} \); \( \eta_{00} = -1, \eta_{11} = -\eta_{22} = -\eta_{33} = 1 \) and \( \eta_{\mu\nu} = 0 \) for \( \mu \neq \nu \).

\(^2\)It is straightforward to add the other fields such as gauge fields and spinors.
We have a physical situation in mind that we fix the general coordinate symmetry and the Weyl symmetry by suitable gauge conditions. It is a familiar fact that after introducing the gauge conditions, instead of such the two local gauge symmetries, we are left with two kinds of global symmetries named as the BRST symmetries. The BRST transformation, which is denoted as $\delta_B$, corresponding to the GCT is defined as

$$\delta_B g_{\mu\nu} = -(\nabla_\mu c_\nu + \nabla_\nu c_\mu),$$

$$\delta_B \tilde{g}^{\mu\nu} = h(\nabla^\mu c^\nu + \nabla^\nu c^\mu - g^{\mu\nu} \nabla_\rho c_\rho),$$

$$\delta_B \phi = -c^\lambda \partial_\lambda \phi, \quad \delta_B c^\rho = -c^\lambda \partial_\lambda c^\rho,$$

$$\delta_B \bar{c}_\rho = i B_\rho, \quad \delta_B B_\rho = 0,$$

(2.2)

where $c^\rho$ and $\bar{c}_\rho$ are respectively the Faddeev-Popov (FP) ghost and anti-ghost, $B_\rho$ is the Nakanishi-Lautrup (NL) field, and we have defined $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu} \equiv h g^{\mu\nu}$. For later convenience, in place of the NL field $B_\rho$ we will introduce a new NL field defined as

$$b_\rho = B_\rho - ic^\lambda \partial_\lambda \bar{c}_\rho,$$

(2.3)

and its BRST transformation reads

$$\delta_B b_\rho = -c^\lambda \partial_\lambda b_\rho.$$  

(2.4)

The other BRST transformation, which is denoted as $\bar{\delta}_B$, corresponding to the Weyl transformation is defined as

$$\bar{\delta}_B g_{\mu\nu} = 2c g_{\mu\nu}, \quad \bar{\delta}_B \tilde{g}^{\mu\nu} = 2 \tilde{g}^{\mu\nu},$$

$$\bar{\delta}_B \phi = -c_\phi, \quad \bar{\delta}_B \bar{c} = i B, \quad \bar{\delta}_B c = \bar{\delta}_B B = 0,$$

(2.5)

where $c$ and $\bar{c}$ are respectively the FP ghost and FP anti-ghost, $B$ is the NL field. Note that the two BRST transformations are nilpotent, i.e.,

$$\delta^2_B = \bar{\delta}^2_B = 0.$$  

(2.6)

To complete the two BRST transformations, we have to fix not only the GCT BRST transformation $\delta_B$ on $c, \bar{c}$ and $B$ but also the Weyl BRST transformation $\bar{\delta}_B$ on $c^\rho, \bar{c}_\rho$ and $b_\rho$. It is easy to determine the former BRST transformation since the fields $c, \bar{c}$ and $B$ are all scalar fields so their BRST transformations should take the form:

$$\delta_B B = -c^\lambda \partial_\lambda B, \quad \delta_B c = -c^\lambda \partial_\lambda c, \quad \delta_B \bar{c} = -c^\lambda \partial_\lambda \bar{c}.$$  

(2.7)

On the other hand, there is an ambiguity in fixing the latter BRST transformation, but we would like to propose a recipe for achieving this goal. The recipe is to just assume that the two BRST transformations are anti-commute with each other, that is,

$$\{\delta_B, \bar{\delta}_B\} \equiv \delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B = 0,$$  

(2.8)
which requires us to take
\[
\delta_B b_\rho = \delta_B c^\rho = \delta_B \bar{c}_\rho = 0. \tag{2.9}
\]

Now we would like to explain an important point, which is sometimes missed in the theoretical physics literature, when two BRST transformations coexist in a theory. Suppose that we fix the GCT by a gauge condition
\[
F^\alpha(g_{\mu\nu}, \phi) = 0
\]
and the Weyl transformation by a gauge condition
\[
F(g_{\mu\nu}, \phi) = 0. \tag{2.10}
\]
Then, the gauge-fixed and BRST invariant Lagrangian density is given by
\[
\mathcal{L}_q = \mathcal{L}_c + \delta_B (\bar{c}_\alpha F^\alpha) + \bar{\delta}_B (\bar{c} F), \tag{2.11}
\]
where the first term is the classical Lagrangian density (2.1). Under this situation, a natural question arises about the gauge-fixing conditions: Can we take any gauge-fixing conditions if they fix gauge symmetries anyway? If not, what gauge conditions are suitable for \( F^\alpha \) and \( F \)?

In order to answer the questions, let us take the two BRST transformations separately and check whether the quantum Lagrangian density (2.10) is really invariant under the BRST transformations up to surface terms. First, taking the Weyl BRST transformation leads to
\[
\bar{\delta}_B \mathcal{L}_q = \bar{\delta}_B \delta_B (\bar{c}_\alpha F^\alpha) = -\delta_B \bar{\delta}_B (\bar{c}_\alpha F^\alpha)
\]
\[
= -\delta_B \left[ (\bar{\delta}_B \bar{c}_\alpha) F^\alpha - \bar{c}_\alpha \bar{\delta}_B F^\alpha \right], \tag{2.12}
\]
where we have used \( \bar{\delta}_B \mathcal{L}_c = 0 \), and Eqs. (2.6) and (2.8). This equation clearly shows that the conditions
\[
\bar{\delta}_B \bar{c}_\alpha = 0, \quad \bar{\delta}_B F^\alpha = 0, \tag{2.13}
\]
are sufficient conditions such that the Lagrangian density (2.10) is invariant under the Weyl BRST transformation.

It is of interest to notice that the former condition in (2.13) leads to two remaining equations in (2.9). To see this fact, let us take the GCT BRST transformation of the former equation as follows:
\[
0 = \delta_B \delta_B \bar{c}_\alpha = -\delta_B \delta_B \bar{c}_\alpha = -i \bar{\delta}_B B_\alpha
\]
\[
= -i \left[ \bar{\delta}_B b_\alpha + i (\delta_B c^\lambda) \partial_\lambda \bar{c}_\alpha \right], \tag{2.14}
\]
which implies \( \bar{\delta}_B b_\alpha = \delta_B c^\lambda = 0 \), which coincide with the remaining two equations in (2.9).

On the other hand, the latter condition in (2.12) gives rise to important information on the gauge condition for the GCT: The gauge-fixing condition for the GCT must be invariant under the Weyl transformation. Thus, for instance, the conventional de Donder gauge condition (or harmonic gauge condition)
\[
\partial_\mu \bar{\delta}_B^{\mu\nu} = 0, \tag{2.15}
\]
is not suitable when there is the Weyl invariance\footnote{In two space-time dimensions the de Donder condition is Weyl invariant so it is allowed to use it as the gauge-fixing condition for the GCT.}.
Next, let us operate the GCT BRST transformation on $L_q$. To do that, since the Lagrangian density is in general a quantity with density, it is more convenient to write as $L_q \equiv \sqrt{-g}L'_q$ and $F = \sqrt{-g}F'$ where $F$ and $F'$ are scalars. Then, taking the GCT BRST variation leads to

$$\delta_B L_q = \delta_B (\sqrt{-g}L'_q) = \delta_B \delta_B (\sqrt{-g}cF') = -\delta_B [\sqrt{-g}(c^{\rho}c^{\rho}cF' + \sqrt{-g}(-c^{\rho}c^{\rho})cF') - \sqrt{-g}c(-c^{\rho}cF')] = \partial_\mu \delta_B (c^{\rho}cF'),$$

which means that $L_q$ is indeed invariant under the GCT BRST transformation up to a surface term. In obtaining this result, we have assumed

$$\delta_B F' = -c^{\rho}c cF',$$  

which is nothing but the requirement that the quantity $F'$ should be a scalar under the GCT. Thus, only a scalar function $F'$, or equivalently a scalar density $F$, makes sense as a gauge-fixing condition for the Weyl invariance. Of course, this scalar function must break the Weyl invariance. As suitable gauge-fixing conditions, in this paper we will choose $F^\nu = \partial_\mu (\tilde{g}^{\mu\nu} \phi^2)$ and $F = \partial_\mu (\tilde{g}^{\mu\nu} \phi \partial_\nu \phi)$.

### 3 Quantum Weyl invariant scalar-tensor gravity

In this section, as a classical Lagrangian, we will take a Weyl invariant scalar-tensor gravity whose Lagrangian is of form \[3\]

$$L_c = \sqrt{-g} \left( \frac{1}{12} \phi^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right),$$

where $\phi$ is a real scalar field with a ghost-like kinetic term, and $R$ the scalar curvature. In addition to the invariance under the general coordinate transformation (GCT), this Lagrangian is also invariant under the Weyl transformation (or the local scale transformation) defined as

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}, \quad \phi \rightarrow \phi' = \Omega^{-1}(x)\phi.$$  

Recall that in order to prove the invariance, we need to use the following transformation of the scalar curvature under \[3.2\]:

$$R \rightarrow R' = \Omega^{-2}(R - 6\Omega^{-1} \Box \Omega),$$

where $\Box \Omega \equiv h^{-1} \partial_\mu (\tilde{g}^{\mu\nu} \partial_\nu \Omega)$.

As explained in the previous section, we have to pay attention to what gauge-fixing conditions should be chosen for the GCT and the Weyl transformation in a consistent manner. For instance, taking the de Donder condition as a gauge condition for GCT is not allowed since it breaks the Weyl symmetry in four

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4For simplicity, we henceforth call a Lagrangian density a Lagrangian.
space-time dimensions. There are several interesting choices of suitable gauge conditions for the GCT, but we shall refer to only two representative examples: The first gauge condition for the GCT is a Weyl invariant version of the de Donder gauge:

$$\partial_\mu \left( (-g)^{1/4} g^{\mu \nu} \right) = 0.$$  \hspace{1cm} (3.4)

This gauge choice is invariant under the Weyl transformation (3.2) and is physically interesting in the sense that it makes use of only the metric tensor field. However, some fields such as the Nakanishi-Lautrup field become not a normal vector field but a vector field with density, which makes several formulas ugly, so we will not adopt (3.4) as a gauge condition for the GCT. The second gauge condition, which we will take in this article and call it “the extended de Donder gauge”, is given by

$$\partial_\mu (\tilde{g}^{\mu \nu} \phi^2) = 0,$$  \hspace{1cm} (3.5)

which is also invariant under the Weyl transformation (3.2).

Next, let us consider a gauge-fixing condition for the Weyl transformation. From the consistency discussed in Section 2, an appropriate gauge condition must obey the condition that it is invariant under the GCT, that is, a scalar quantity. Since there are many of scalars constructed out of the real scalar field $\phi$ and the Riemannian tensors, we might be left in the dark on this issue. However, surprisingly enough, if we impose the requirement such that the FP ghost’s Lagrangian should have a Weyl invariant metric $\tilde{g}^{\mu \nu} \phi^2$ instead of the standard metric $\tilde{g}^{\mu \nu}$, the suitable gauge condition for the GCT can be uniquely picked up. Such the gauge condition, we will call “the scalar gauge condition”, reads

$$\partial_\mu (\tilde{g}^{\mu \nu} \phi^2 \partial_\nu \phi) = 0,$$  \hspace{1cm} (3.6)

which can be alternatively written as

$$\square \phi^2 = 0.$$  \hspace{1cm} (3.7)

Incidentally, the unitary gauge $\phi = \text{constant}$ is often taken to show that the Weyl invariant scalar-tensor gravity (3.1) is equivalent to the Einstein-Hilbert term, but this gauge choice is not so interesting since there remains no conformal symmetry behind.

After taking the extended de Donder gauge condition (3.5) for the GCT and the scalar gauge condition (3.6) for the Weyl transformation, the gauge-fixed and BRST invariant quantum Lagrangian is given by

$$\mathcal{L}_q = \mathcal{L}_c + \mathcal{L}_{GF+FP} + \mathcal{L}_{GF+FP}$$
$$= \mathcal{L}_c + \imath \delta_B (\tilde{g}^{\mu \nu} \phi^2 \partial_\mu \tilde{c}_\nu) + \imath \delta_B [\bar{c} \partial_\mu (\tilde{g}^{\mu \nu} \phi \partial_\nu \phi)]$$
$$= \sqrt{-\tilde{g}} \left( \frac{1}{12} \phi^2 R + \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right) - \tilde{g}^{\mu \nu} \phi^2 (\partial_\mu b_\nu + \imath \partial_\mu \tilde{c}_\lambda \partial_\nu c^\lambda)$$
$$+ \tilde{g}^{\mu \nu} \phi \delta_\mu B \partial_\nu \phi - \imath \tilde{g}^{\mu \nu} \phi^2 \partial_\mu \bar{c} \partial_\nu c,$$  \hspace{1cm} (3.8)
where surface terms are dropped. Note that the last term, which is the FP ghost’s term for the Weyl transformation, certainly involves the Weyl invariant metric $\tilde{g}^\mu{}^\nu \phi^2$. Let us rewrite this Lagrangian concisely as

$$L_q = \sqrt{-g} \frac{1}{12} \phi^2 R - \frac{1}{2} \tilde{g}^\mu{}^\nu E_{\mu\nu},$$ \hspace{1cm} (3.9)

where we have defined

$$E_{\mu\nu} = -\frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \phi^2 (\partial_\mu b_\nu + i \partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda)$$

$$- \phi \partial_\mu B \partial_\nu \phi + i \phi^2 \partial_\mu \bar{c} \partial_\nu c + (\mu \leftrightarrow \nu).$$ \hspace{1cm} (3.10)

Moreover, it is sometimes more convenient to introduce the dilaton $\sigma(x)$ by defining

$$\phi(x) \equiv e^{\sigma(x)},$$ \hspace{1cm} (3.11)

and rewrite further into the form

$$L_q = e^{2\sigma(x)} \left( \sqrt{-g} \frac{1}{12} R - \frac{1}{2} \tilde{g}^\mu{}^\nu \tilde{E}_{\mu\nu} \right),$$ \hspace{1cm} (3.12)

where we have defined

$$\tilde{E}_{\mu\nu} = -\frac{1}{2} \partial_\mu \sigma \partial_\nu \sigma + \partial_\mu b_\nu + i \partial_\mu \bar{c}_\lambda \partial_\nu c^\lambda$$

$$- \partial_\mu B \partial_\nu \sigma + i \partial_\mu \bar{c} \partial_\nu c + (\mu \leftrightarrow \nu).$$ \hspace{1cm} (3.13)

Note that the relation between $E_{\mu\nu}$ and $\tilde{E}_{\mu\nu}$ is given by

$$E_{\mu\nu} = \phi^2 \tilde{E}_{\mu\nu} = e^{2\sigma} \tilde{E}_{\mu\nu}. \hspace{1cm} (3.14)$$

From the Lagrangian $L_q$, it is straightforward to derive the field equations by taking the variation with respect to $g_{\mu\nu}$, $\phi$ (or $\sigma$), $b_\nu$, $B$, $c^\rho$, $\bar{c}_\rho$, $c$ and $\bar{c}$ in order:

$$\frac{1}{12} \phi^2 G_{\mu\nu} - \frac{1}{12} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \phi^2 - \frac{1}{2} (E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E) = 0,$$

$$\frac{1}{6} \phi^2 R - E - 2 g^{\mu\nu} \phi \partial_\mu B \partial_\nu \phi - \phi^2 \square B = 0,$$

$$\partial_\mu (\tilde{g}^\mu{}^\nu \phi^2) = 0, \hspace{1cm} \partial_\mu (\tilde{g}^\mu{}^\nu \phi \partial_\nu \phi) = 0,$$

$$g^{\mu\nu} \partial_\mu \partial_\nu \bar{c}_\rho = g^{\mu\nu} \partial_\mu \partial_\nu c^\rho = g^{\mu\nu} \partial_\mu \partial_\nu \bar{c} = g^{\mu\nu} \partial_\mu \partial_\nu c = 0. \hspace{1cm} (3.15)$$

where we have defined the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and $E \equiv \tilde{g}^{\mu\nu} E_{\mu\nu}$. The two gauge-fixing conditions in $\text{(3.15)}$ lead to a very simple equation for the dilaton:

$$g^{\mu\nu} \partial_\mu \partial_\nu \sigma = 0. \hspace{1cm} (3.16)$$
It is worthwhile to notice that it is not the scalar field $\phi$ but the dilaton $\sigma$ that satisfies this type of equation. Furthermore, the trace part of the Einstein equation, i.e., the first field equation in (3.15) and the field equation for $\phi$ also give us the equation for $B$:

$$g^{\mu\nu}\partial_\mu\partial_\nu B = 0. \quad (3.17)$$

Finally, using the field equations obtained thus far, after some calculations, we can also derive the equation for $b^\rho$:

$$g^{\mu\nu}\partial_\mu\partial_\nu b^\rho = 0. \quad (3.18)$$

In other words, setting $X_M = \{x^\mu, b_\mu, \sigma, B, c_\mu, \bar{c}_\mu, \bar{c}, \bar{c}\}$, $X_M$ turns out to obey the very simple equation:

$$g^{\mu\nu}\partial_\mu\partial_\nu X_M = 0. \quad (3.19)$$

This fact, together with the gauge condition $\partial_\mu(\tilde{g}^{\mu\nu}\phi^2) = 0$, produces the two kinds of conserved currents:

$$\mathcal{P}^{\mu M} \equiv \tilde{g}^{\mu\nu}\phi^2\partial_\nu X_M = \tilde{g}^{\mu\nu}\phi^2(1\partial_\nu X_M)$$

$$\mathcal{M}^{\mu MN} \equiv \tilde{g}^{\mu\nu}\phi^2(X_M\partial_\nu Y^N), \quad (3.20)$$

where we have defined $X_M\partial_\mu Y^N \equiv X_M\partial_\mu Y^N - (\partial_\mu X_M)Y^N$.

## 4 Canonical quantization and equal-time commutation relations

In this section, after introducing the canonical commutation relations (CCRs), we will evaluate various equal-time commutation relations (ETCRs) among fundamental variables. To simplify various expressions, we will obey the following abbreviations adopted in the textbook of Nakanishi and Ojima [5]:

$$[A, B'] = [A(x), B(x')]|_{x^0=x'^0}, \quad \delta^3 = \delta(\vec{x} - \vec{x}')$$

$$\tilde{f} = \frac{1}{\tilde{g}^{00}} = \frac{1}{\sqrt{-gg^{00}}} = \frac{1}{\hbar\tilde{g}^{00}}, \quad (4.1)$$

where we assume that $\tilde{g}^{00}$ is invertible.

Now let us set up the canonical (anti-)commutation relations:

$$[g_{\mu\nu}, \pi^\rho_M] = i\frac{1}{2}(\delta_\rho^\mu\delta_\nu^\lambda + \delta_\rho^\lambda\delta_\nu^\mu)\delta^3, \quad [\phi, \pi^\rho_\phi] = +i\delta^3, \quad [B, \pi^\rho_B] = +i\delta^3,$$

$$\{c^\rho, \pi^\sigma_{c\lambda}\} = \{\bar{c}_\lambda, \pi^\sigma_{c}\} = +i\delta^3, \quad \{c, \pi^\rho_c\} = \{\bar{c}, \pi^\rho_{\bar{c}}\} = +i\delta^3, \quad (4.2)$$

\(^5\)The detail of the calculation is presented in Appendix A.
where the other (anti-)commutation relations vanish. Here the canonical variables are $g_{\mu\nu}, \phi, B, c^\rho, \bar{c}_\mu, c, \bar{c}$ and the corresponding canonical conjugate momenta are $\pi_{\mu\nu}^g, \pi_\phi, \pi_B, \pi_{\bar{c}}^\rho, \pi_c^\sigma, \pi_{\bar{c}}$, respectively and the $b_\mu$ field is regarded as not a canonical variable but a conjugate momentum of $\tilde{g}^{\mu\nu}$.

To remove second order derivatives of the metric involved in $R$, we perform the integration by parts once and rewrite the Lagrangian \[4.3\] as

\[
\mathcal{L}_q = -\frac{1}{12} \tilde{g}^{\mu\nu} \phi^2 (\Gamma_{\mu}^{\alpha} \Gamma_{\sigma}^{\alpha} - \Gamma_{\mu}^{\sigma} \Gamma_{\sigma}^{\alpha}) - \frac{1}{6} \phi \partial_\mu \phi (\tilde{g}^{\alpha\beta} \Gamma_{\alpha\beta}^{\mu} - \tilde{g}^{\mu\nu} \Gamma_{\nu\alpha}^{\alpha}) + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \tilde{g}^{\mu\nu} \partial_\mu B \phi \partial_\nu \phi - i \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \bar{c} \partial_\nu c + \partial_\mu \nu^\mu,
\]

where a surface term $\nu^\mu$ is defined as

\[
\nu^\mu = \frac{1}{12} \tilde{g}^2 (\tilde{g}^{\alpha\beta} \Gamma_{\alpha\beta}^{\mu} - \tilde{g}^{\mu\nu} \Gamma_{\nu\alpha}^{\alpha}) - \tilde{g}^{\mu\nu} \phi^2 b_\nu.
\]

Using this Lagrangian, the concrete expressions for canonical conjugate momenta become

\[
\pi_{g}^{\mu\nu} = \frac{\partial \mathcal{L}_q}{\partial \partial_{g}^{\mu\nu}} = -\frac{1}{24} \sqrt{-g} \phi^2 \left[ g^{0\lambda} g^{\mu\nu} g^{\sigma\tau} - g^{0\tau} g^{\mu\lambda} g^{\nu\sigma} - g^{0\sigma} g^{\mu\tau} g^{\nu\lambda} + g^{0\lambda} g^{\mu\tau} g^{\nu\sigma} \right] + g^{\sigma\tau} g^{0\mu} g^{\sigma\nu} + \frac{1}{2} \left( g^{0\mu} g^{\sigma\nu} g^{\sigma\tau} - g^{0\nu} g^{\sigma\mu} g^{\sigma\tau} \right) \partial_\lambda \partial_\sigma \phi + \frac{1}{6} \sqrt{-g} \left[ \frac{1}{2} (g^{0\mu} g^{\nu\rho} + g^{0\nu} g^{\mu\rho}) - g^{\mu\nu} g^{0\rho} \right] \phi \partial_\rho \phi - \frac{1}{2} \sqrt{-g} \left( g^{0\mu} g^{\nu\rho} + g^{0\nu} g^{\mu\rho} - g^{0\rho} g^{\mu\nu} \right) \phi^2 b_\rho,
\]

\[
\pi_\phi = \frac{\partial \mathcal{L}_q}{\partial \partial_{\phi}} = g^{0\mu} \partial_\mu \phi + 2 \tilde{g}^{0\mu} \phi b_\mu + \frac{1}{6} \phi (-\tilde{g}^{0\beta} \Gamma_{\alpha\beta}^{\mu} + \tilde{g}^{0\alpha} \Gamma_{\alpha\beta}^{\beta}) + g^{0\mu} \partial_\mu B \phi,
\]

\[
\pi_B = \frac{\partial \mathcal{L}_q}{\partial \partial_B} = g^{0\mu} \phi \partial_\mu \phi,
\]

\[
\pi_{\bar{c}}^\rho = \frac{\partial \mathcal{L}_q}{\partial \partial_{\bar{c}}^\rho} = -i \tilde{g}^{0\mu} \phi^2 \partial_\mu \bar{c}_\rho,
\]

\[
\pi_c^\sigma = \frac{\partial \mathcal{L}_q}{\partial \partial_c^\sigma} = i \tilde{g}^{0\mu} \phi^2 \partial_\mu c^\sigma,
\]

\[
\pi_{\bar{c}} = \frac{\partial \mathcal{L}_q}{\partial \partial_{\bar{c}}} = -i \tilde{g}^{0\mu} \phi^2 \partial_\mu \bar{c},
\]

\[
\pi_{\bar{c}} = \frac{\partial \mathcal{L}_q}{\partial \partial_{\bar{c}}} = i \tilde{g}^{0\mu} \phi^2 \partial_\mu \bar{c},
\]

where we have defined the time derivative such as $\dot{g}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \tau} \equiv \partial_\tau g_{\mu\nu}$, and differentiation of ghosts is taken from the right.
From now on, we would like to evaluate various nontrivial equal-time commutation relations (ETCRs) in order. Let us first work with the ETCR in Eq. (4.2):

\[
\pi^\alpha_0 \delta g \, \partial_\alpha g - \frac{1}{2} \left( \delta_\mu^\alpha \delta_\nu^\beta + \delta_\nu^\alpha \delta_\mu^\beta \right) \delta^3. \tag{4.6}
\]

The canonical conjugate momentum \( \pi^\alpha_0 \) has a structure

\[
\pi^\alpha_0 = A^\alpha + B^\alpha \delta \phi + C^\alpha b_\beta, \tag{4.7}
\]

where \( A^\alpha, B^\alpha, \) and \( C^\alpha \equiv -\frac{1}{2} \tilde{g}^0_0 g^\alpha_\beta \phi^2 \) have no \( \dot{g}_{\mu\nu} \), and \( B^\alpha \delta \phi \) does not have \( \dot{\phi} \) since \( \pi^\alpha_0 \) does not include the dynamics of the metric and the scalar fields. Then, we find that Eq. (4.6) produces

\[
\delta g_{\mu\nu}, b_\rho \right] = -i \tilde{f} \phi^{-2} \left( \delta^0_\mu g_{\rho\nu} + \delta^0_\nu g_{\rho\mu} \right) \delta^3. \tag{4.8}
\]

From this ETCR, we can easily derive ETCRs:

\[
\left[ \tilde{g}^{\mu\nu}, b_\rho \right] = i \tilde{f} \phi^{-2} \left( \tilde{g}^{0\mu} \delta_\rho^\nu + \tilde{g}^{0\nu} \delta_\rho^\mu - \tilde{g}^{\mu\nu} \delta^0_\rho \right) \delta^3. \tag{4.9}
\]

Here we have used the following fact; since a commutator works as a derivation, we can have formulae:

\[
\left[ \Phi, \delta \Phi' \right] = -\partial^0_\Phi \left[ \Phi', \Phi \right], \quad \left[ \tilde{g}^{\mu\nu}, \delta \Phi' \right] = - \left( \tilde{g}^{0\mu} \delta_\rho^\nu - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{g}^{00} \right) \left[ \Phi, \delta \Phi' \right], \tag{4.10}
\]

where \( \Phi \) is a generic field. Similarly, the ETCR, \( \left[ \pi^0_\delta, \phi' \right] = 0 \) yields

\[
\left[ \phi, \delta \phi' \right] = 0. \tag{4.11}
\]

The ETCR, \( \left[ \pi^0_\delta, B' \right] = 0 \) yields

\[
\left[ B, \delta B' \right] = 0. \tag{4.12}
\]

Moreover, the ETCRs, \( \left[ \pi_B, \delta \phi' \right] = 0 \) and \( \left[ \pi_B, B' \right] = -i \delta^3 \) respectively produce

\[
\left[ \phi, \delta \phi' \right] = 0, \quad \left[ \phi, B' \right] = -i \tilde{f} \phi^{-1} \delta^3. \tag{4.13}
\]

As for the ETCRs involving FP ghosts, let us first consider the anti-ETCRs, \( \{ \pi_{c\lambda}, c'_{\sigma} \} = \{ \pi_{\bar{c}\lambda}, \bar{c}'_{\sigma} \} = i \delta^3_\lambda \delta^3_\sigma \). These anti-ETCRs lead to the same anti-ETCR:

\[
\{ \hat{c}_\lambda, c'_{\sigma} \} = -i \hat{c}'_{\sigma}, \quad \{ \hat{c}_\lambda, c'_{\sigma} \} = -i \hat{c}'_{\sigma} \delta^3_\lambda \delta^3_\sigma, \tag{4.14}
\]

where we have used a useful identity for generic variables \( \Phi \) and \( \Psi \):

\[
[\Phi, \hat{\Psi}'] = \partial_0 [\Phi, \Psi] - [\hat{\Phi}, \Psi], \tag{4.15}
\]

\[
[\Phi, \hat{\Psi}'] = \partial_0 [\Phi, \Psi] - [\hat{\Phi}, \Psi], \tag{4.15}
\]
which holds for the anti-commutation relation as well. In a similar way, the anti-ETCRs, $\{\pi_c, c'\} = \{\pi_{\bar{c}}, \bar{c}'\} = i\delta^3$ yield

$$\{\dot{c}, c'\} = -\{\dot{\bar{c}}, \bar{c}'\} = -f \hat{\phi}^{-2}\delta^3. \tag{4.16}$$

Moreover, $[\pi^g_0, c] = [\pi^g_0, \bar{c}] = 0$ give us the ETCRs:

$$[b_\rho, c'] = [b_\rho, \bar{c}] = 0, \tag{4.17}$$

and similarly $[\pi^g_0, c] = [\pi^g_0, \bar{c}] = 0$ produce

$$[b_\rho, c] = [b_\rho, \bar{c}] = 0. \tag{4.18}$$

To calculate the ETCRs between $B$ and the FP-ghosts, it is necessary to utilize the ETCRs, $[B, \pi^\lambda_\sigma c] = [B, \pi^\lambda_\sigma \bar{c}] = [B, \pi^\rho c] = [B, \pi^\rho \bar{c}] = 0$, and consequently we have

$$[B, \dot{c}_\lambda] = [B, \dot{c}'] = [B, \dot{\bar{c}}] = [B, \dot{\bar{c}}'] = 0. \tag{4.19}$$

Furthermore, taking the Weyl BRST transformation of the third ETCR reads

$$0 = [\{i \bar{Q}_B, B\}, \dot{c}'] + [B, \{i \bar{Q}_B, \dot{c}'\}] = [B, i \dot{B'}], \tag{4.20}$$

where we have used the Weyl BRST transformation (2.5). As a result we have the ETCR:

$$[B, \dot{B'}] = 0. \tag{4.21}$$

Next, from $[\pi_B, \dot{c}_\lambda] = [\pi_B, \dot{c}'] = 0$, we find

$$[\dot{\phi}, \dot{c}_\lambda] = [\dot{\phi}, \dot{c}'] = 0. \tag{4.22}$$

Similarly, from $[\pi_B, c'] = [\pi_B, c] = 0$, we have

$$[\dot{\phi}, c'] = [\dot{\phi}, c] = 0. \tag{4.23}$$

Using the field equation for $\bar{c}_\lambda$ in (3.15), i.e., $g^{\mu
\nu} \partial_\mu \partial_\nu \bar{c}_\lambda = 0$, the ETCR, $[\phi, \bar{c}_\lambda] = 0$, the formula (4.15), and Eq. (4.22), it is easy to derive the equations:

$$[\dot{\phi}, \bar{c}_\lambda] = [\dot{\phi}, \bar{c}'] = [\dot{\phi}, c'] = 0. \tag{4.24}$$

Similar equations also hold when $\bar{c}_\lambda$ is replaced with $c''$, $c'$ or $c'$. Now, using the equations obtained above, we are ready to evaluate the type of the ETCRs, $[\dot{\Phi}, b'_\rho]$ where $\Phi$ is a generic field. First, let us focus on $[\dot{\phi}, b'_\rho]$.

---

6 We define the BRST transformation for the Weyl transformation as $\delta_B \Phi \equiv [i \bar{Q}_B, \Phi]$ where $\Phi$ is a generic field and $\{ , , \}$ denotes the graded bracket. Of course, in case of the GCT BRST transformation, it is replaced by $\delta_B \Phi \equiv [Q_B, \Phi]$. 

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To do that, we start with $\dot{\phi}, c'_\rho = 0$ in Eq. (4.22) and take its BRST variation for the GCT as follows:

$$
0 = \{iQ_B, [\dot{\phi}, c'_\rho]\}
= \{iQ_B, \dot{\phi}, c'_\rho\} + [\dot{\phi}, iQ_B, c'_\rho]\n= -\partial_\rho (c^\lambda \partial_\lambda \phi), c'_\rho) + [\dot{\phi}, i(b'_\rho + i c^\lambda \partial_\lambda c'_\rho)]\n= -\{c^\lambda, c'_\rho\} \partial_\lambda \phi + i[\dot{\phi}, b'_\rho].
$$

Using Eq. (4.14), we are able to obtain

$$
[\dot{\phi}, b'_\rho] = -i\tilde{f}\phi^{-2} \partial_\rho \phi \delta^3.
$$

It turns out that the ETCRs, $[\pi_\alpha, \pi_g^{\alpha\nu}] = [\pi_g^{\alpha\nu}, \pi_g^{\alpha\nu}] = 0$ give rise to

$$
[\dot{c}_\lambda, b'_\rho] = -i\tilde{f}\phi^{-2} \partial_\rho \tilde{c}_\lambda \delta^3, \quad [c^\rho, b'_\rho] = -i\tilde{f}\phi^{-2} \partial_\rho \phi \delta^3.
$$

Similarly, the ETCRs, $[\pi_\epsilon, \pi_g^{\alpha\nu}] = [\pi_g^{\alpha\nu}, \pi_g^{\alpha\nu}] = 0$ give us

$$
[\dot{c}, b'_\rho] = -i\tilde{f}\phi^{-2} \partial_\rho \tilde{c} \delta^3, \quad [c, b'_\rho] = -i\tilde{f}\phi^{-2} \partial_\rho \phi \delta^3.
$$

In order to evaluate $[\dot{B}, b'_\rho]$, we make use of $[\pi_\epsilon, b'_\rho] = 0$, which can be easily proved. Taking its BRST transformation for the Weyl transformation leads to the equation:

$$
[\{iQ_B, \pi_\epsilon\}, b'_\rho] = 0,
$$

where $[iQ_B, b'_\rho] = 0$ was used. We can show that $\{iQ_B, \pi_\epsilon\} = \tilde{g}^{0\mu} \phi^2 \partial_\mu B$, so using (4.9) and (4.11), we can calculate

$$
[\dot{B}, b'_\rho] = -i\tilde{f}\phi^{-2} \partial_\rho B \delta^3.
$$

Finally, the ETCR, $[g_{\mu\nu}, b'_\rho]$ (or equivalently, $[g_{\mu\nu}, \dot{b}'_\rho]$) can be obtained by using the method developed in our previous article. Only the result is written out as

$$
[g_{\mu\nu}, b'_\rho] = -i\left\{\tilde{f}\phi^{-2}(\partial_\rho g_{\mu\nu} + \delta_0^0 \tilde{g}_{\mu\nu} + \delta_0^0 \tilde{g}_{\mu\nu}) \delta^3\right.\n+ \left.\left[\delta_\mu^k - 2\delta_0^k \tilde{g}_{\mu\nu} + (\mu \leftrightarrow \nu)\right]\partial_\kappa (\tilde{f}\phi^{-2} \delta^3)\right\},
$$

or equivalently,

$$
[g_{\mu\nu}, b'_\rho] = i\left\{\tilde{f}\phi^{-2}(\partial_\rho g_{\mu\nu} - \partial_\rho (\tilde{f}\phi^{-2})(\delta_0^0 \tilde{g}_{\mu\nu} + \delta_0^0 \tilde{g}_{\mu\nu})) \delta^3\right.\n+ \left.\left[\delta_\mu^k - 2\delta_0^k \tilde{g}_{\mu\nu} + (\mu \leftrightarrow \nu)\right]\partial_\kappa (\tilde{f}\phi^{-2} \delta^3)\right\}.
$$

Following our previous calculation, we can prove

$$
[b_\mu, b'_\nu] = 0,
$$

$$
[b_\mu, b'_\nu] = i\tilde{f}\phi^{-2}(\partial_\mu b_\nu + \partial_\nu b_\mu) \delta^3.
$$
5 Equal-time commutation relations in gravitational sector

The remaining nontrivial ETCRs are related to the time derivative of the metric field, i.e., the ETCRs, \([\hat{\pi}_\phi, g'_\mu\nu]\) where \(\Phi\) is a generic field. In this section, we will evaluate such the ETCRs.

First of all, let us start with the ETCR, \([\hat{\pi}_\phi, g'_\mu\nu] = 0\). From the expression of \(\hat{\pi}_\phi\) in Eq. (4.5), this ETCR can be described as

\[
\tilde{g}^{00} [\hat{\phi}, g'_\mu\nu] + \frac{1}{6} \phi (\tilde{g}^{00} g^{0\sigma} - \tilde{g}^{0\rho} g^{0\sigma}) [\hat{g}_{\rho\sigma}, g'_\mu\nu] + \tilde{g}^{00} \phi [\hat{B}, g'_\mu\nu] = -4i \tilde{f} \phi^{-1} \sqrt{-g} \delta^0_\mu \delta^0_\nu \delta^3.
\]

(5.1)

Next, the ETCR, \([\pi_\phi, \phi'_\mu] = -i \delta^3\) produces the equation:

\[
(\tilde{g}^{00} g^{0\sigma} - \tilde{g}^{0\rho} g^{0\sigma}) [\hat{g}_{\rho\sigma}, \phi'_\mu] = 0.
\]

(5.2)

Moreover, the ETCR, \([\pi_\phi, B'] = 0\) reads

\[
(\tilde{g}^{00} g^{0\sigma} - \tilde{g}^{0\rho} g^{0\sigma}) [\hat{g}_{\rho\sigma}, B'] = 6i \phi^{-2} \delta^3.
\]

(5.3)

The extended de Donder gauge, \(\partial \mu (\tilde{g}^{0\mu} \phi^2) = 0\), can be rewritten as

\[
\mathcal{D}^{\lambda\rho\sigma} \hat{g}_{\rho\sigma} + 4 \phi^{-1} g^{\lambda\rho} \partial_{\rho} \phi = (2 g^{\lambda\rho} g^{\sigma k} - g^{\rho\sigma} g^{\lambda k}) \partial_k \hat{g}_{\rho\sigma},
\]

(5.4)

where \(\mathcal{D}^{\lambda\rho\sigma} \equiv g^{\lambda\rho} g^{\sigma\sigma} - 2 g^{\lambda\rho} g^{\sigma\sigma}\). Since the right-hand side (RHS) of Eq. (5.4) is independent of \(\hat{g}_{\mu\nu}, \phi\) or \(B\). Thus, we have three identities:

\[
\mathcal{D}^{\lambda\rho\sigma} [\hat{g}_{\rho\sigma}, g'_\mu\nu] + 4 \phi^{-1} g^{\lambda0} [\hat{\phi}, g'_\mu\nu] = 0.
\]

(5.5)

\[
\mathcal{D}^{\lambda\rho\sigma} [\hat{g}_{\rho\sigma}, \phi'_\mu] = 0.
\]

(5.6)

\[
\mathcal{D}^{\lambda\rho\sigma} [\hat{g}_{\rho\sigma}, B'] = 4i \tilde{f} \phi^{-2} g^{00} \delta^3.
\]

(5.7)

In Eqs. (5.6) and (5.7), we have used Eq. (4.13).

Putting \(\lambda = 0\) in Eq. (5.6) and using Eq. (5.2), we have

\[
\hat{g}^{\sigma\rho} [\hat{g}_{\rho\sigma}, \phi'] = g^{0\rho} g^{0\sigma} [\hat{g}_{\rho\sigma}, \phi'] = 0.
\]

(5.8)

In general, from the argument of symmetry, \([\hat{g}_{\rho\sigma}, \phi']\) must be of form:

\[
[\hat{g}_{\rho\sigma}, \phi'] = a_1 (g_{\rho\sigma} + a_2 \delta^0_\rho \delta^0_\sigma) \delta^3,
\]

(5.9)

where \(a_1, a_2\) are constants. Eq. (5.8) then requires us to take \(a_1 = a_2 = 0\). Thus, we have

\[
[\hat{g}_{\rho\sigma}, \phi'] = 0.
\]

(5.10)
Furthermore, imposing Eq. (5.13), we can determine \( c \) write down its general expression like

\[
[\hat{g}_{\mu\rho}, B'] = b_1 (\hat{g}_{\rho\sigma} + b_2 \delta^0_\rho \delta^0_\sigma) \delta^3,
\]

where \( b_1, b_2 \) are constants. From Eq. (5.13), \( b_1 \) is determined to be \( 2i \hat{f} \phi^{-2} \), and then Eq. (5.17) requires \( b_2 \) to be vanishing, so we can obtain

\[
[\hat{g}_{\rho\sigma}, B'] = 2i \hat{f} \phi^{-2} \delta_{\rho\sigma} \delta^3.
\]

Finally, we wish to evaluate \([\hat{g}_{\rho\sigma}, g'_{\mu\nu}]\), for which we need some calculations. Before doing so, let us rewrite Eq. (5.1) by means of Eqs. (5.10) and (5.12) into the form:

\[
(\hat{g}^{00} g^{\rho\sigma} - \hat{g}^{00} g^{0\sigma}) [\hat{g}_{\rho\sigma}, g'_{\mu\nu}] = -12i \hat{f} \phi^{-2} \left( g_{\mu\nu} + \frac{2}{g_{00}} \delta^0_\mu \delta^0_\nu \right) \delta^3.
\]

Similarly, Eq. (5.5) reduces to

\[
(\hat{g}^{00} g^{\rho\sigma} - 2 \hat{g}^{0\rho} g^{0\sigma}) [\hat{g}_{\rho\sigma}, g'_{\mu\nu}] = 0.
\]

We are now willing to evaluate the ETCR, \([\hat{g}_{\rho\sigma}, g'_{\mu\nu}]\). This ETCR has a symmetry under the simultaneous exchange of \((\mu\nu) \leftrightarrow (\rho\sigma)\) and primed \leftrightarrow unprimed in addition to the usual symmetry \( \mu \leftrightarrow \nu \) and \( \rho \leftrightarrow \sigma \). Then, we can write down its general expression like

\[
[\hat{g}_{\rho\sigma}, g'_{\mu\nu}] = \left\{ c_1 g_{\rho\sigma} g_{\mu\nu} + c_2 (g_{\mu\nu} g_{\sigma\tau} + g_{\rho\sigma} g_{\tau\nu}) + h \hat{f} [c_3 (\delta^0_\rho \delta^0_\sigma g_{\mu\nu} + \delta^0_\sigma \delta^0_\mu g_{\rho\nu}) + c_4 (\delta^0_\rho \delta_\mu g_{\sigma\nu} + \delta^0_\mu \delta^0_\nu g_{\rho\sigma}) + \delta^0_\sigma \delta^0_\mu g_{\rho\nu} + \delta^0_\rho \delta^0_\nu g_{\sigma\mu}) ] + (h \hat{f})^2 [c_5 \delta^0_\rho \delta^0_\sigma \delta^0_\mu \delta^0_\nu] \right\} \delta^3,
\]

where \( c_i (i = 1, \cdots, 5) \) are some coefficients. Imposing Eq. (5.14) on (5.15) leads to relations among the coefficients:

\[
c_3 = 2(c_1 + c_2), \quad c_4 = -c_2, \quad c_5 = 4(c_1 + c_2).
\]

Furthermore, imposing Eq. (5.13), we can determine \( c_2, c_3, c_4 \) and \( c_5 \) via \( c_1 \) as

\[
c_3 = -c_1 - 12i \hat{f} \phi^{-2}, \quad c_4 = -c_2 = \frac{3}{2} c_1 + 6i \hat{f} \phi^{-2}, \quad c_5 = -2c_1 - 24i \hat{f} \phi^{-2}.
\]

In order to fix the coefficient \( c_1 \), we need to calculate the ETCR, \([\hat{g}_{kl}, g'_{mn}]\) explicitly in terms of \([\pi_{g}^{kl}, g'_{mn}]\) in Eq. (4.13) and the concrete expression of \( \pi_{g}^{kl} \) in Eq. (4.1). To do that, from Eq. (4.13), let us write

\[
\pi_{g}^{kl} = \hat{A}^{kl} + \hat{B}^{klp} b_p + \hat{C}^{klmn} \hat{g}_{mn} + \hat{D}^{kl} \phi.
\]
Here $\hat{A}^{kl}$, $\hat{B}^{kl\rho}$, $\hat{C}^{klmn}$, and $\hat{D}^{kl}$ commute with $g_{mn}$, and $\hat{C}^{klmn}$ and $\hat{D}^{kl}$ are defined as

$$\hat{C}^{klmn} = \frac{1}{24} h\varphi^2 K^{klmn}, \quad \hat{D}^{kl} = \frac{1}{6} \delta(g^{00} g^{kl} - g^{0k} g^{0l}),$$

(5.19)

where the definition of $K^{klmn}$ and its property are given by

$$K^{klmn} = \left| \begin{array}{ccc} g^{00} & g^{0l} & g^{0n} \\ g^{k0} & g^{kl} & g^{kn} \\ g^{m0} & g^{ml} & g^{mn} \end{array} \right|,$$

$$K^{klmn} \frac{1}{2} (g^{00})^{-1}(g_{k0}g_{mn} - g_{m0}g_{jn} - g_{jn}g_{km}) = \frac{1}{2} (\delta^k_i \delta^l_j + \delta^l_i \delta^k_j).$$

(5.20)

From Eq. (5.18), we can calculate

$$[\dot{g}_{kl}, g'_{mn}] = \hat{C}^{-1}_{klpq} \left( [\pi^{pq}, g'_{mn}] - \hat{B}^{pq}[b, g'_{mn}] - \hat{D}^{pq}[\dot{\phi}, g'_{mn}] \right),$$

$$= -\frac{1}{2} \hat{C}^{-1}_{klpq} (\delta^p_m \delta^q_n + \delta^q_m \delta^p_n) \delta^3,$$

(5.21)

where we have used Eqs. (4.2), (4.8), and (5.10). Since we can calculate

$$\hat{C}^{-1}_{klpq} = 12 \hat{\phi}^{-2} (g_{kl}g_{pq} - g_{kp}g_{ql} - g_{kq}g_{lp}),$$

(5.22)

we can eventually arrive at the result:

$$[\dot{g}_{kl}, g'_{mn}] = -12 \hat{\phi}^{-2} (g_{kl}g_{mn} - g_{km}g_{ln} - g_{kn}g_{lm}) \delta^3.$$

(5.23)

Meanwhile, from Eq. (5.15) we have the ETCR:

$$[\dot{g}_{kl}, g'_{mn}] = \left[ c_1 g_{kl}g_{mn} + c_2 (g_{km}g_{ln} + g_{kn}g_{lm}) \right] \delta^3$$

(5.24)

Hence, comparing (5.23) with (5.24), we can obtain

$$c_1 = -12 \hat{\phi}^{-2}, \quad c_2 = 12 \hat{\phi}^{-2}.$$

(5.25)

Note that these values satisfy the relation in Eq. (5.17), $-c_2 = \frac{3}{2} c_1 + 6 i \hat{\phi}^{-2}$, which gives us a nontrivial verification of our result. In this way, we have succeeded in getting the following ETCR:

$$[g_{\rho\sigma}, g'_{\mu\nu}] = -12 \hat{\phi}^{-2} [g_{\rho\sigma}g_{\mu\nu} - g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu} + h \hat{\phi} (\delta^0_{\rho} \delta^0_{\mu} g_{\sigma\nu} + \delta^0_{\rho} \delta^0_{\nu} g_{\sigma\mu} + \delta^0_{\sigma} \delta^0_{\mu} g_{\rho\nu} + \delta^0_{\sigma} \delta^0_{\nu} g_{\rho\mu})] \delta^3.$$

(5.26)
6 Unitarity of physical S-matrix

As in the conventional BRST formalism, the physical state |phys⟩ is defined by imposing two subsidiary conditions [8]:

\[ Q_B |\text{phys}⟩ = \bar{Q}_B |\text{phys}⟩ = 0. \]  \hspace{1cm} (6.1)

It is then well known that the physical S-matrix is unitary under the assumption that all the BRST singlet states have positive norm. In this section, we would like to prove the unitarity of the physical S-matrix. Since there is a ghost-like scalar field φ as well as timelike and longitudinal components of the metric field in our formalism, this is not a trivial problem.

In analysing the unitarity, it is enough to take account of asymptotic fields of all the fundamental fields and the free part of the Lagrangian. Let us first assume the asymptotic fields as

\[ g_{μν} = \eta_{μν} + φ_{μν}, \quad φ = φ_0 + \tilde{φ}, \quad b_μ = β_μ, \quad B = β, \quad c = γ, \quad \bar{c} = \bar{γ}, \]  \hspace{1cm} (6.2)

where \( \eta_{μν} = \eta_{νμ} \) is the flat Minkowski metric with the mostly positive signature and \( φ_0 \) is a constant. In this section, the Minkowski metric is used to lower or raise the Lorentz indices. Using these asymptotic fields, the free part of the Lagrangian reads

\[ L_q = \frac{1}{12} φ_0^2 \left( \frac{1}{2} φ_{μν} \Box φ_{μν} - \frac{1}{4} φ_0 \Box φ - \frac{1}{2} φ_{μν} \partial_μ φ_ρ \partial_ν φ_ρ + \frac{1}{2} φ_{μν} \partial_μ φ \partial_ν φ_ρ \right) + \frac{1}{6} \left( -2 φ_0 \Box φ \partial_ρ φ_ρ + \frac{1}{2} φ_0 \Box φ \partial_ρ φ \partial_ρ φ_ρ \right) - \frac{1}{2} φ_0 \Box \tilde{φ} \partial_ρ φ_ρ - \frac{1}{2} φ_0 \Box \tilde{φ} \partial_ρ \bar{γ} \partial_ρ \bar{γ} \]  \hspace{1cm} (6.3)

where \( \Box \equiv \partial_μ \partial_ν + \partial_ν \partial_μ \) and \( φ \equiv φ_{μν} φ_{νμ} \). Based on this Lagrangian, it is easy to derive the linearized field equations:

\[ \frac{1}{12} \partial_0 \left( \frac{1}{2} φ_{μν} \Box φ_{μν} - \frac{1}{2} \eta_{μν} \Box φ - \partial_ρ \partial_0 (φ_{ρν}) + \frac{1}{2} \partial_0 \partial_0 φ + \frac{1}{2} φ_0 \eta_{μν} \partial_ρ \partial_0 φ_ρ \right) + \frac{1}{6} \left( -\eta_{μν} \Box + \partial_μ \partial_0 φ \right) + \partial_0 \partial_0 φ_ρ - \frac{1}{2} \eta_{μν} \partial_ρ \partial_0 φ_ρ = 0. \]  \hspace{1cm} (6.4)

\[ \frac{1}{6} \left( \partial_0 φ - \partial_0 \partial_0 φ_{μν} \right) + 2 \partial_0 β_ρ + \Box β = 0. \]  \hspace{1cm} (6.5)

\[ \partial_μ φ - \frac{1}{2} φ_0 \left( \partial_μ φ_{ρμ} - \frac{1}{2} \partial_ρ φ \right) = 0. \]  \hspace{1cm} (6.6)

\[ \Box \tilde{φ} = \Box γ_μ = \Box γ_μ = 0. \]  \hspace{1cm} (6.7)

Here we have introduced the symmetrization notation \( A_{μν} B_ν \equiv \frac{1}{2} \left( A_μ B_ν + A_ν B_μ \right) \). Now, operating \( \partial_μ \) on Eq. (6.6) and using Eq. (6.7), we can obtain

\[ \partial_μ \partial_0 φ_{μν} - \frac{1}{2} \Box φ = 0. \]  \hspace{1cm} (6.8)
Next, taking the trace of Eq. (6.4) with the help of Eqs. (6.7) and (6.8) leads to

\[ \Box \varphi + 24 \partial_\mu \beta^\mu = 0. \]  

(6.9)

Then, with the help of Eqs. (6.8) and (6.9), Eq. (6.5) can be rewritten as

\[ \Box \beta = 0. \]  

(6.10)

Moreover, acting \( \partial^\mu \) on Eq. (6.4) yields

\[ \Box \beta_{\mu} = 0. \]  

(6.11)

Finally, using various equations obtained thus far, Eq. (6.4) is reduced to the form:

\[ \Box \varphi_{\mu\nu} + 24 \partial_{(\mu} \beta_{\nu)} = 0, \]  

(6.12)

which means that the field \( \varphi_{\mu\nu} \) is a dipole field:

\[ \Box^2 \varphi_{\mu\nu} = 0. \]  

(6.13)

On the other hand, the other fields are all simple pole fields:

\[ \Box \tilde{\phi} = \Box \beta_{\mu} = \Box \beta = \Box \gamma^\mu = \Box \bar{\gamma}_\mu = \Box \gamma = \Box \bar{\gamma} = 0. \]  

(6.14)

Note that Eq. (6.14) corresponds to Eq. (3.19) in a curved space-time.

Following the standard technique, we can calculate the four-dimensional (anti-)commutation relations (4D CRs) between asymptotic fields. The point is that the simple pole fields, for instance, the Nakanishi-Lautrup field \( \beta(x) \) can be expressed in terms of the invariant delta function \( D(x) \) as

\[ \beta_{\mu}(x) = \int d^3 z D(x - z) \overset{\rightarrow}{\partial}_0 \beta_{\mu}(z), \]  

(6.15)

whereas the dipole field \( \varphi_{\mu\nu}(x) \) can be done as

\[
\varphi_{\mu\nu}(x) = \int d^3 z \left[ D(x - z) \overset{\rightarrow}{\partial}_0 \varphi_{\mu\nu}(z) + E(x - z) \overset{\rightarrow}{\partial}_0 \Box \varphi_{\mu\nu}(z) \right] \\
= \int d^3 z \left[ D(x - z) \overset{\rightarrow}{\partial}_0 \varphi_{\mu\nu}(z) - 24E(x - z) \overset{\rightarrow}{\partial}_0 \partial_{(\mu} \beta_{\nu)}(z) \right],
\]  

(6.16)

where in the last equality we have used Eq. (6.12). Here the invariant delta function \( D(x) \) for massless simple pole fields and its properties are described as

\[ D(x) = -\frac{i}{(2\pi)^3} \int d^4 k \epsilon(k_0) \delta(k^2) e^{ikx}, \quad \Box D(x) = 0, \]

\[ D(-x) = -D(x), \quad D(0, \vec{x}) = 0, \quad \partial_0 D(0, \vec{x}) = \delta^3(x), \]  

(6.17)
where \( \epsilon(k_0) \equiv \frac{\hbar \omega}{|k_0|} \). Similarly, the invariant delta function \( E(x) \) for massless dipole fields and its properties are given by

\[
E(x) = -\frac{i}{(2\pi)^3} \int d^4k \epsilon(k_0) \delta'(k^2)e^{ikx}, \quad \Box E(x) = D(x),
\]

\[
E(-x) = -E(x), \quad E(0, \vec{x}) = \partial_0 E(0, \vec{x}) = \partial_0^2 E(0, \vec{x}) = 0,
\]

\[
\partial_0^6 E(0, \vec{x}) = -\delta^3(x).
\] (6.18)

It is easy to show that the RHS of Eqs. (6.15) and (6.16) is independent of \( z^0 \). Thus, for instance, when we evaluate the four-dimensional commutation relation \( [\varphi_{\mu\nu}(x), \varphi_{\sigma\tau}(y)] \), we can put \( z^0 = y^0 \) and use the three-dimensional commutation relations among asymptotic fields. The resultant 4D CRs are summarized as

\[
[\varphi_{\mu\nu}(x), \varphi_{\sigma\tau}(y)] = 12i\phi_0^{-2}[(\eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma})D(x-y) + (\eta_{\mu\sigma}\partial_\nu + \eta_{\mu\nu}\partial_\sigma + \eta_{\nu\sigma}\partial_\mu + \eta_{\nu\tau}\partial_\sigma)E(x-y)],
\] (6.19)

\[
[\varphi_{\mu\nu}(x), \beta_{\rho}(y)] = i\phi_0^{-2}(\eta_{\mu\rho}\partial_\nu + \eta_{\mu\nu}\partial_\rho)D(x-y).\]

\[\text{It is convenient to perform the Fourier transformation of Eqs. (6.19)-(6.21).} \quad \gamma(x, \gamma(y)) = -\phi_0^{-2}\delta^3(x-y).\]

\[\text{The other 4D CRs vanish identically.} \quad \Box \]

Now we would like to discuss the issue of the unitarity of the theory in hand. To do that, it is convenient to perform the Fourier transformation of Eqs. (6.19)-(6.23). However, for the dipole field we cannot use the three-dimensional Fourier expansion to define the creation and annihilation operators. We therefore make use of the four-dimensional Fourier expansion \(^8\)

\[
\varphi_{\mu\nu}(x) = \frac{1}{(2\pi)^2} \int d^4p \theta(p_0) [\varphi_{\mu\nu}(p)e^{ixp} + \varphi_{\mu\nu}^\dagger(p)e^{-ixp}],
\] (6.25)

where \( \theta(p_0) \) is the step function. For any simple pole fields, we adopt the same Fourier expansion, for instance,

\[
\beta_{\mu}(x) = \frac{1}{(2\pi)^2} \int d^4p \theta(p_0) [\beta_{\mu}(p)e^{ixp} + \beta_{\mu}^\dagger(p)e^{-ixp}].
\] (6.26)

Incidentally, for a generic simple pole field \( \Phi \), the three-dimensional Fourier expansion is defined as

\[
\Phi(x) = \frac{1}{(2\pi)^2} \int d^3p \frac{1}{\sqrt{2|p|}} [\Phi(p)e^{ixp} + \Phi^\dagger(p)e^{-ixp}],
\] (6.27)
where the on-shell relation \( p_0 = |\vec{p}| \) must be satisfied (\( \vec{p} \) denotes the three-dimensional momentum) whereas the four-dimensional Fourier expansion reads

\[
\Phi(x) = \frac{1}{(2\pi)^2} \int d^4p \, \theta(p_0) [\Phi(p) e^{ipx} + \Phi^\dagger(p) e^{-ipx}].
\]

Thus, the annihilation operator \( \Phi(p) \) in the four-dimensional Fourier expansion has connection with the annihilation operator \( \Phi(\vec{p}) \) in the three-dimensional Fourier expansion like

\[
\Phi(p) = \theta(p_0) \delta(p^2) \sqrt{2|\vec{p}|} \Phi(\vec{p}).
\]

Based on these Fourier expansions, we can calculate the Fourier transform of Eqs. (6.29)-(6.24):

\[
\begin{align*}
&[\varphi_{\mu\nu}(p), \varphi_{\sigma\tau}^\dagger(q)] = 12\phi_0^{-2}\theta(p_0)\delta^4(p - q)[\delta(p^2)(\eta_{\mu\sigma}\eta_{\nu\tau} - \eta_{\mu\tau}\eta_{\nu\sigma}) \\
&- 3\delta(p^2)(\eta_{\mu\nu}p_{\rho}p_{\sigma} + \eta_{\nu\rho}p_{\mu}p_{\sigma} + \eta_{\mu\sigma}p_{\nu}p_{\rho} + \eta_{\nu\sigma}p_{\mu}p_{\rho})]. \\
&[\varphi_{\mu\nu}(p), \beta_{\rho}^\dagger(q)] = i\phi_0^{-2}(\eta_{\mu\rho}p_{\nu} + \eta_{\nu\rho}p_{\mu})\theta(p_0)\delta(p^2)\delta^4(p - q). \\
&[\varphi_{\mu\nu}(p), \beta^\dagger(q)] = -2\phi_0^{-1}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p - q). \\
&[\dot{\varphi}(p), \beta^\dagger(q)] = \phi_0^{-1}\theta(p_0)\delta(p^2)\delta^4(p - q). \\
&\{\gamma^\sigma(p), \gamma_{\tau}^\dagger(q)\} = i\phi_0^{-2}\delta^\sigma_\tau\theta(p_0)\delta(p^2)\delta^4(p - q). \\
&\{\gamma(p), \gamma_{\tau}^\dagger(q)\} = i\phi_0^{-2}\theta(p_0)\delta(p^2)\delta^4(p - q).
\end{align*}
\]

Next, let us turn our attention to the linearized field equations. In the Fourier transformation, Eq. (6.30) takes the form:

\[
p''\varphi_{\mu\nu} - \frac{1}{2}p_\mu\varphi = 2\phi_0^{-1}p_\mu\vec{\phi}.
\]

If we fix the degree of freedom associated with \( \vec{\phi} \), which will be discussed later, this equation gives us four independent relations in ten components of \( \varphi_{\mu\nu}(p) \), thereby reducing the independent components of \( \varphi_{\mu\nu}(p) \) to six. To deal with six independent components of \( \varphi_{\mu\nu}(p) \), it is convenient to take a specific Lorentz frame such that \( p_1 = p_2 = 0 \) and \( p_3 > 0 \), and choose the six components as follows:

\[
\begin{align*}
\varphi_1(p) &= \frac{1}{2}[\varphi_{11}(p) - \varphi_{22}(p)], \quad \varphi_2(p) = \varphi_{12}(p), \quad \omega_0(p) = -\frac{1}{2p_0}\varphi_{00}(p), \\
\omega_I(p) &= -\frac{1}{p_0}\varphi_{0I}(p), \quad \omega_3(p) = -\frac{1}{2p_3}\varphi_{33}(p),
\end{align*}
\]

where the index \( I \) takes the transverse components \( I = 1, 2 \).

In this respect, it is worthwhile to consider the GCT BRST transformation for these components. First, let us write down the GCT BRST transformation for the Fourier expansion of the asymptotic fields, which reads

\[
\begin{align*}
&\delta_B\varphi_{\mu
u}(p) = -i[p_{\mu}\gamma_{\nu}(p) + p_{\nu}\gamma_{\mu}(p)], \\
&\delta_B\gamma^\mu(p) = 0, \quad \delta_B\gamma_{\mu}(p) = i\beta_{\mu}(p), \\
&\delta_B\dot{\varphi}(p) = \delta_B\beta_{\mu}(p) = \delta_B\beta(p) = \delta_B\gamma(p) = \delta_B\gamma_{\mu}(p) = 0.
\end{align*}
\]
Using this BRST transformation, the GCT BRST transformation for the components in \((6.37)\) takes the form:

\[
\delta_B \varphi_I(p) = 0, \quad \delta_B \omega_\mu(p) = i\gamma_\mu(p), \\
\delta_B \tilde{\gamma}_\mu(p) = i\beta_\mu(p), \quad \delta_B \gamma_\mu(p) = \delta_B \beta_\mu(p) = 0,
\]

(6.39)

where \(p_1 = p_2 = 0\) was used. This BRST transformation implies that \(\varphi_I(p)\) could be the physical observable while a set of fields, \(\{\omega_\mu(p), \beta_\mu(p), \gamma_\mu(p), \tilde{\gamma}_\mu(p)\}\) might belong to the BRST quartet, which are dropped from the physical state by the Kugo-Ojima subsidiary condition, \(Q_B|\text{phys}\rangle = 0\) \[8\]. However, note that \(\beta_\mu(p), \gamma_\mu(p)\) and \(\tilde{\gamma}_\mu(p)\) are simple pole fields obeying \(p^2\beta_\mu(p) = p^2\gamma_\mu(p) = p^2\tilde{\gamma}_\mu(p) = 0\), but \(\varphi_{\mu\nu}(p)\) is a dipole field satisfying \((p^2)^2\varphi_{\mu\nu}(p) = 0\), so that a naive Kugo-Ojima’s quartet mechanism does not work.

To clarify the BRST quartet mechanism, let us calculate their 4D CRs. From Eqs. \((6.30)-(6.35)\) and the definition \((6.37)\), it is straightforward to derive the following 4D CRs:

\[
[\varphi_I(p), \varphi_J(q)] = -12\phi_0^{-2}\delta_{IJ}\theta(p_0)\delta(p^2)\delta^4(p-q), \quad (6.40)
\]

\[
[\varphi_I(p), \omega_{\mu}(q)] = [\varphi_I(p), \beta_{\mu}(q)] = [\beta_\mu(p), \beta_{\nu}(q)] = 0. \quad (6.41)
\]

\[
[\omega_{\mu}(p), \beta_{\nu}(q)] = -i\phi_0^{-2}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p-q), \quad (6.42)
\]

\[
\{\gamma_\mu(p), \tilde{\gamma}_\nu(q)\} = i\phi_0^{-2}\eta_{\mu\nu}\theta(p_0)\delta(p^2)\delta^4(p-q). \quad (6.43)
\]

In addition to them, we have a bit complicated expression for \([\omega_{\mu}(p), \omega_{\nu}(q)]\) because \(\varphi_{\mu\nu}(p)\) is a dipole field, but luckily enough this expression is not necessary for our aim \[8\]. It is known how to take out a simple pole field from a dipole field, which amounts to using an operator defined by \[8\]

\[
\mathcal{D}_p = \frac{1}{2|p|^2 p_0} \frac{\partial}{\partial p_0} + c, \quad (6.44)
\]

where \(c\) is a constant. Using this operator, we can define a simple pole field \(\hat{\varphi}_{\mu\nu}(p)\) from the dipole field \(\varphi_{\mu\nu}(p)\), which obeys \((p^2)^2\hat{\varphi}_{\mu\nu}(p) = 0\), as

\[
\hat{\varphi}_{\mu\nu}(p) = \varphi_{\mu\nu}(p) - \mathcal{D}_p p^2 \varphi_{\mu\nu}(p) = \varphi_{\mu\nu}(p) - 24i\mathcal{D}_p p_\mu \beta_\nu(p), \quad (6.45)
\]

where in the last equality we have used the Fourier transform of the linearized field equation \((6.12)\). It is then easy to verify the equation:

\[
p^2 \hat{\varphi}_{\mu\nu}(p) = 0. \quad (6.46)
\]

Then, in \((6.37)\) we replace \(\varphi_{\mu\nu}\) of \(\omega_\mu\) with \(\hat{\varphi}_{\mu\nu}\), and we redefine \(\omega_\mu\) by \(\tilde{\omega}_\mu\) as

\[
\tilde{\omega}_0(p) = -\frac{1}{2p_0} \hat{\varphi}_{00}(p), \quad \tilde{\omega}_I(p) = -\frac{1}{p_0} \hat{\varphi}_{0I}(p), \quad \tilde{\omega}_3(p) = -\frac{1}{2p_3} \hat{\varphi}_{33}(p). \quad (6.47)
\]

The key point is that with this redefinition from \(\omega_\mu\) to \(\tilde{\omega}_\mu\), the BRST transformation and the 4D CRs remain unchanged owing to \(\delta_B \beta_\mu = 0\) and \([\beta_\mu(p), \beta_{\nu}(q)] = \delta_{\mu\nu} \delta(p^2)\delta^4(p-q)\).
\[ [\varphi_I(p), \beta^I_\mu(q)] = 0, \text{ those are,} \]
\[
\delta_B \hat{\omega}_\mu(p) = i \gamma_\mu(p), \quad \hat{\omega}_\mu(p), \beta^I_\mu(q) = [\omega_\mu(p), \beta^I_\mu(q)], \\
[\varphi_I(p), \hat{\omega}_I^\dagger(q)] = [\varphi_I(p), \omega_\mu^\dagger(q)]. \quad (6.48)
\]

Now it turns out that all the fields, \( \{ \varphi_I, \hat{\omega}_\mu, \beta_\mu, \gamma_\mu, \tilde{\gamma}_\mu \} \) are simple pole fields\(^9\). Since all the fields become simple pole fields, we can obtain the standard creation and annihilation operators in the three-dimensional Fourier expansion from those in the four-dimensional one through the relation \( (6.29) \). As a result, the three-dimensional (anti-)commutation relations, which are denoted as \([\Phi(\vec{p}), \Phi^1(\vec{q})]\) with \( \Phi(\vec{p}) \equiv \{ \varphi_I(\vec{p}), \hat{\omega}_\mu(\vec{p}), \beta_\mu(\vec{p}), \gamma_\mu(\vec{p}), \tilde{\gamma}_\mu(\vec{p}) \} \), are given by\(^10\)

\[
[\Phi(\vec{p}), \Phi^1(\vec{q})] = \begin{pmatrix}
-12 \phi_0^{-2} \delta_{IJ} & [\omega_{\mu}(\vec{p}), \omega^I_{\mu}(\vec{q})] & -i \phi_0^{-2} \eta_{\mu\nu} \\
[\omega_{\mu}(\vec{p}), \omega^I_{\mu}(\vec{q})] & i \phi_0^{-2} \eta_{\mu\nu} & 0 \\
-i \phi_0^{-2} \eta_{\mu\nu} & 0 & -i \phi_0^{-2} \eta_{\mu\nu}
\end{pmatrix}
\times \delta(\vec{p} - \vec{q}). \quad (6.49)
\]

The (anti-)commutation relations \( (6.49) \) have in essence the same structure as those of the Yang-Mills theory \(^8\). Hence, we find that \( \varphi_I \) could be the physical observable while a set of fields \( \{ \hat{\omega}_\mu, \beta_\mu, \gamma_\mu, \tilde{\gamma}_\mu \} \) belongs to the BRST quartet.

Next, let us move on to another BRST transformation, which is the BRST transformation for the Weyl transformation. The Weyl BRST transformation for the asymptotic fields is of form:

\[
\delta_B \varphi_{IJ} = 2 c \eta_{IJ}, \quad \delta_B \hat{\phi} = - \phi_0 \gamma, \quad \delta_B \gamma = 0, \quad \delta_B \tilde{\gamma} = i \beta, \\
\delta_B \beta = \delta_B \beta^I_\mu = \delta_B \gamma_\mu = \delta_B \tilde{\gamma}_\mu = 0. \quad (6.50)
\]

The Weyl BRST transformation of \( \varphi_I \) is vanishing:

\[
\delta_B \varphi_I = 0, \quad (6.51)
\]

which means that together with \( \delta_B \varphi_{IJ} = 0, \varphi_{IJ} \) is truely the physical observable. The four-dimensional (anti-)commutation relations among the fields \( \{ \phi, \beta, \gamma, \tilde{\gamma} \} \) read

\[
[\hat{\phi}(p), \hat{\phi}^I(q)] = 0, \\
[\hat{\phi}(p), \beta^I_\mu(q)] = \phi_0^{-1} \theta(p_0) \delta(p^2) \delta^I(\mu - q), \\
[\gamma(p), \gamma^I_\mu(q)] = i\phi_0^{-2} \theta(p_0) \delta(p^2) \delta^I(\mu - q). \quad (6.52)
\]

As can be also seen in these 4D CRs, all the fields \( \{ \varphi_I, \hat{\phi}, \beta, \gamma, \tilde{\gamma} \} \) are massless simple pole fields. Via the relation \( (6.29) \), the three-dimensional (anti-)commutation relations \( \{ \Phi(\vec{p}), \Phi^1(\vec{q}) \} \) with \( \Phi(\vec{p}) \equiv \{ \varphi_I(\vec{p}), \hat{\phi}(\vec{p}), \beta(\vec{p}), \gamma(\vec{p}), \tilde{\gamma}(\vec{p}) \} \),

\(^9\)Without the redefinition, \( \varphi_I(p) \) is already a simple pole field as can be seen in Eq. \( (6.40) \).

\(^{10}\)The bracket \( [A, B] \) is the graded commutation relation denoting either commutator or anti-commutator, according to the Grassmann-even or odd character of \( A \) and \( B \), i.e., \( [A, B] = AB - (-)^{|A||B|} BA \).
are of form

\[
[\Phi(\vec{p}), \Phi^\dagger(\vec{q})] = \begin{pmatrix}
-12\phi_0^{-2}\delta_{IJ} & 0 & \phi_0^* \\
0 & \phi_0^{-1} & 0 \\
\phi_0^{-1} & 0 & -i\phi_0^{-2}
\end{pmatrix} \delta(\vec{p} - \vec{q}).
\]

Thus, \( \varphi_I \) is the physical observable while a set of fields, \( \{ \hat{\phi}, \beta, \gamma, \bar{\gamma} \} \) consists of the BRST quartet and is the unphysical mode by the Kugo-Ojima’s subsidiary condition [8]. Here it is worth mentioning that the ghost-like scalar field \( \phi \) belongs to the unphysical mode so together with the result obtained in the analysis of the GCT BRST cohomology, the physical S-matrix is found to be unitary.

7 Choral symmetry

As mentioned in Section 3, a set of fields (including the space-time coordinates \( x^\mu \) \( X^M \equiv \{ x^\mu, b_\mu, \sigma, B, c^\mu, \bar{c}_\mu, c, \bar{c} \} \) obeys a very simple equation:

\[
g^{\mu\nu} \partial_\mu \partial_\nu X^M = 0.
\] (7.1)

This equation holds if and only if we adopt the extended de Donder gauge and the new scalar gauge as gauge-fixing conditions for the GCT and the Weyl transformation, respectively. The existence of this simple equation suggests that there could be many of conserved currents defined in Eq. (3.20). In this section, we shall show explicitly that there exist such currents and we have a huge global symmetry called choral symmetry, which is the \( IOSp(10|10) \) in the present theory.

Let us start with the Lagrangian (3.12), which can be cast to the form:

\[
L_q = \tilde{g}^{\mu\nu} \phi^2 \left( \frac{1}{12} R_{\mu\nu} - \frac{1}{2} \tilde{E}_{\mu\nu} \right).
\] (7.2)

Here note that \( \tilde{g}^{\mu\nu} \phi^2 \) is a Weyl invariant metric and the Ricci tensor is invariant under only a global scale transformation. We can further rewrite it into the form:

\[
L_q = \tilde{g}^{\mu\nu} \phi^2 \left( \frac{1}{12} R_{\mu\nu} - \frac{1}{2} \eta_{NM} \partial_\mu X^M \partial_\nu X^N \right) = \tilde{g}^{\mu\nu} \phi^2 \left( \frac{1}{12} R_{\mu\nu} - \frac{1}{2} \partial_\mu X^M \tilde{\eta}_{MN} \partial_\nu X^N \right),
\] (7.3)

where we have introduced an \( OSp(10|10) \) metric \( \eta_{NM} = \tilde{\eta}_{MN}^T \equiv \tilde{\eta}_{MN} \) defined...
as \[ \eta_{NM} \]

Let us note that this \( OSp(10|10) \) metric \( \eta_{NM} \), which is a c-number quantity, has the symmetry property such that

\[
\eta_{MN} = (-)^{|M|+|N|} \eta_{NM} = (-)^{|M|} \eta_{NM} = (-)^{|N|} \eta_{NM},
\]

where the statistics index \(|M|\) is 0 or 1 when \( X^M \) is Grassmann-even or Grassmann-odd, respectively. This property comes from the fact that \( \eta_{MN} \) is ‘diagonal’ in the sense that its off-diagonal, Grassmann-even and Grassmann-odd, and vice versa, matrix elements vanish, i.e., \( \eta_{MN} = 0 \) when \(|M| \neq |N|\), thereby being \(|M| = |N| = |M| \cdot |N|\) in front of \( \eta_{MN} \).

Now that the quantum Lagrangian (7.3) is expressed in a manifestly \( IOSp(10|10) \) invariant form except for the Weyl invariant metric \( \tilde{g}_{\mu\nu} \phi^2 \), which will be discussed later, there could exist an \( IOSp(10|10) \) as a global symmetry in our theory. Let us show this fact first. The infinitesimal \( OSp \) rotation is defined by

\[
\delta X^M = \eta^{ML} \varepsilon_{LN} X^N \equiv \varepsilon^M_X X^N,
\]

where \( \eta^{MN} \) is the inverse matrix of \( \eta_{MN} \), and the infinitesimal parameter \( \varepsilon_{MN} \) has the following properties:

\[
\varepsilon_{MN} = (-)^{1+|M|+|N|} \varepsilon_{NM}, \quad \varepsilon_{MN} X^L = (-)^{|L|(1+|M|+|N|)} X^L \varepsilon_{MN}.
\]

Moreover, in order to find the conserved current, we assume that the infinitesimal parameter \( \varepsilon_{MN} \) depends on the space-time coordinates \( x^\mu \), i.e., \( \varepsilon_{MN} = \varepsilon_{MN}(x^\mu) \).

Assuming for a while that the metric \( \tilde{g}^{\mu\nu} \phi^2 \) and \( R_{\mu\nu} \) is invariant, the infinitesimal variation of the quantum Lagrangian (7.6) under the \( OSp \) rotation is given by

\[
\delta \mathcal{L}_q = -\tilde{g}^{\mu\nu} \phi^2 \left( \partial_\mu \varepsilon_{NM} X^M \partial_\nu X^N + \varepsilon_{NM} \partial_\mu X^M \partial_\nu X^N \right).
\]

It is easy to prove that the second term on the RHS vanishes owing to the first property in Eq. (7.7). Thus, \( \mathcal{L}_q \) is invariant under the infinitesimal \( OSp \)
rotation. The conserved current is then calculated as
\[
\delta L_q = -\tilde{g}^{\mu\nu} \phi^2 \partial_\mu \varepsilon_{NM} X^M \partial_\nu X^N
\]
\[
= -\frac{1}{2} \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \varepsilon_{NM} \left[ X^M \partial_\nu X^N \right] - \left( -1 \right)^{|M| - |N|} X^N \partial_\nu X^M
\]
\[
= -\frac{1}{2} \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \varepsilon_{NM} \left( X^M \partial_\nu X^N - \partial_\nu X^M X^N \right)
\]
\[
= -\frac{1}{2} \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \varepsilon_{NM} \left[ X^M \partial_\nu X^N \right]
\]
\[
= -\frac{1}{2} \partial_\mu \varepsilon_{NM} \mathcal{M}^{\mu MN},
\]
from which the conserved current \(\mathcal{M}^{\mu MN}\) for the \(\text{OSp}\) rotation takes the form:
\[
\mathcal{M}^{\mu MN} = \tilde{g}^{\mu\nu} \phi^2 X^M \partial_\nu X^N.
\]

In a similar way, we can derive the conserved current for the infinitesimal translation
\[
\delta X^M = \varepsilon^M,
\]
where \(\varepsilon^M\) is the infinitesimal parameter and assume that it is a local one for deriving the corresponding conserved current. Indeed, assuming again that the metric \(\tilde{g}^{\mu\nu}\) and \(R_{\mu\nu}\) are invariant under the translation, we can show that \(L_q\) is invariant under the infinitesimal translation as follows:
\[
\delta L_q = -\tilde{g}^{\mu\nu} \phi^2 \eta_{NM} \partial_\mu \varepsilon^M \partial_\nu X^N
\]
\[
= -\tilde{g}^{\mu\nu} \phi^2 \partial_\mu \varepsilon_{MN} \partial_\nu X^N
\]
\[
= -\partial_\mu \varepsilon_{MN} \mathcal{P}^{\mu MN},
\]
which implies that the conserved current \(\mathcal{P}^{\mu MN}\) for the translation reads
\[
\mathcal{P}^{\mu MN} = \tilde{g}^{\mu\nu} \phi^2 \partial_\nu X^M = \tilde{g}^{\mu\nu} \phi^2 \left( 1 \partial_\nu X^M \right).
\]

The above proofs make sense only under the assumption that the metric \(\tilde{g}^{\mu\nu}\) and \(R_{\mu\nu}\) are invariant under the \(\text{IOSp}(10|10)\). So the problem reduces to a question: Is this assumption correct? The answer is obviously not true, but the non-invariant terms can be compensated by a suitable Weyl transformation. To show this fact, let us consider only the case of the infinitesimal \(\text{OSp}\) rotation since we can treat the case of the translation in a perfectly similar manner. Under the infinitesimal \(\text{OSp}\) rotation (7.6), the dilaton \(\sigma(x)\), which is defined as \(\phi = e^\sigma\), transforms as
\[
\delta \sigma = \eta^{\alpha L} \xi_{LN} X^N = -\varepsilon_{BN} X^N,
\]
where we have used (7.14) and
\[
\begin{pmatrix}
-1 & -1 \\
-1 & 0
\end{pmatrix}^{-1} = \begin{pmatrix}
0 & -1 \\
-1 & 1
\end{pmatrix}.
\]
where recall that the matrix $\eta^{ML}$ is the inverse matrix of $\eta_{ML}$. As for the scalar field $\phi(x)$, this transformation for the dilaton can be interpreted as a Weyl transformation:

$$\phi \rightarrow \phi' = e^{\epsilon(x)}\phi,$$

(7.16)

where the infinitesimal parameter is defined as $\epsilon(x) = -\varepsilon_B N X^N$. This Weyl transformation induces the Weyl transformation for the metric tensor field at the same time:

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = e^{-2\epsilon(x)}g_{\mu\nu}.$$  

(7.17)

Let us recall that the metric $\tilde{g}^{\mu\nu} \phi^2$ is the Weyl invariant metric so that it is invariant under the Weyl transformation (7.16) and (7.17). This implies that $\tilde{g}^{\mu\nu} \phi^2$ is essentially invariant under the $OSp$ rotation if an appropriate Weyl transformation is achieved.

How about $R_{\mu\nu}$? Even if $R_{\mu\nu}$ is not invariant under the Weyl transformation in itself, this object comes from the classical Lagrangian of the Weyl invariant scalar-tensor gravity in (3.1), so together with the metric tensor and the scalar field it essentially becomes invariant under the Weyl transformation (7.16) and (7.17). Thus, in this sense, $R_{\mu\nu}$ is also invariant under the $OSp$ rotation. In any case, it is worthwhile to stress that in the present formulation, the chiral symmetry $IOSp(10|10)$ is not a symmetry of only the FP ghosts and the Nakanishi-Latrup fields but closely related to classical fields $g_{\mu\nu}$ and $\phi$ which lie in the classical Lagrangian.

An important remark is relevant to the expression of the conserved currents (7.10) and (7.13). To make the quantum Lagrangian $L_q$ be invariant under the chiral symmetry $IOSp(10|10)$, it is necessary to perform the Weyl transformation (7.16) and (7.17). Then, it is natural to ask if because of this associated Weyl transformation, the expression of the currents would be modified or not. Here a miracle happens. As shown in Refs. [10, 11], the current for the Weyl transformation identically vanishes in the Weyl invariant scalar-tensor gravity. Thus, although we make the Weyl transformation (7.16) and (7.17), the conserved currents (7.10) and (7.13) are unchanged.

From the conserved currents (7.10) and (7.13), the corresponding conserved charges become

$$M^{MN} \equiv \int d^3x \mathcal{M}^{0MN} = \int d^3x \tilde{g}^{0\nu} \phi^2 X^M \partial_\nu X^N,$$

$$P^M \equiv \int d^3x \mathcal{P}^{0M} = \int d^3x \tilde{g}^{0\nu} \phi^2 \partial_\nu X^M.$$  

(7.18)

It then turns out that using various ETCRs obtained so far, the $IOSp(10|10)$ generators $\{M^{MN}, P^M\}$ generate an $IOSp(10|10)$ algebra:

$$[P^M, P^N] = 0,$$

$$[M^{MN}, P^R] = i [P^M \tilde{\eta}^{NR} - (-)^{|M||R|} P^N \tilde{\eta}^{MR}],$$

$$[M^{MN}, P^R] = i [M^{MS} \tilde{\eta}^{NR} - (-)^{|N||R|} M^{MR} \tilde{\eta}^{NS} - (-)^{|N||R|} M^{NS} \tilde{\eta}^{MR} + (-)^{|M||R|+|N||S|} M^{NR} \tilde{\eta}^{MS}].$$  

(7.19)
As a final remark, it is worthwhile to point out that all the global symmetries existing in the present theory are expressed in terms of the generators of the choral symmetry. For instance, the BRST charges for the GCT and Weyl transformation are respectively expressed as

\[ Q_B \equiv M(b_\rho, c^\rho) = \int d^3x \tilde{g}^{0\nu} \phi^2 b_\nu c^\rho, \]

\[ \tilde{Q}_B \equiv M(B, c) = \int d^3x \tilde{g}^{0\nu} \phi^2 B \partial_\nu c. \]  

(7.20)

8 Gravitational conformal symmetry

Even if we have already fixed the Weyl symmetry by the scalar gauge condition \[ (3.6) \], we still have its linearized, residual symmetries. In order to look for the residual symmetries, it is convenient to take the extended de Donder gauge \[ (3.5) \] into consideration simultaneously.\(^\text{11}\) With the help of the extended de Donder gauge \[ (3.5) \], the scalar gauge condition \[ (3.6) \] can be rewritten as

\[ 0 = \partial_\mu (\tilde{g}^{\mu\nu} \phi \partial_\nu \phi) = \partial_\mu (\tilde{g}^{\mu\nu} \phi^2 \partial_\nu \sigma) = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu \sigma, \]  

(8.1)

where we have used the relation between the scalar field and dilaton, \( \phi = e^\sigma \). Under the Weyl transformation \[ (3.2) \] with \( \Omega(x) \equiv e^{\Lambda(x)} \), the dilaton \( \sigma \) transforms as

\[ \sigma \to \sigma' = \sigma - \log \Omega = \sigma - \Lambda, \]  

(8.2)

where we have used the Weyl transformation \[ (3.2) \] for the scalar field. Since \( \tilde{g}^{\mu\nu} \phi^2 \) is a Weyl invariant quantity, the Weyl transformation makes Eq. \[ (8.1) \] change to

\[ 0 = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu \sigma \to 0 = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu \sigma' = \tilde{g}^{\mu\nu} \phi^2 \partial_\mu \partial_\nu (\sigma - \Lambda). \]  

(8.3)

This equation shows that when we use the extended de Donder gauge, the scalar gauge condition is still invariant under the Weyl transformation as long as

\[ \tilde{g}^{\mu\nu} \partial_\mu \partial_\nu \Lambda = 0, \]  

(8.4)

is satisfied, thereby implying the existence of the residual symmetries.\(^\text{12}\) Selecting the coefficients appropriately for later convenience, the solution to Eq. \[ (8.4) \] is given by

\[ \Lambda = \lambda - 2k_\mu x^\mu, \]  

(8.5)

where \( \lambda, k_\mu \) are constants.\(^\text{12}\)

\(^{11}\) The same strategy has been adopted in different theories in Refs. \[ (12)\(13\).\)

\(^{12}\) It is shown in Appendix B that the transformations associated with the parameters \( \lambda \) and \( k_\mu \), respectively, correspond to dilatation and special conformal transformation in a flat Minkowski background.
We can also verify the invariance of the quantum Lagrangian under the residual symmetries more directly. To do that, let us assume that Λ, or equivalently, λ and \( k_\mu \), are the infinitesimal parameters. It then turns out that the quantum Lagrangian (3.12) is invariant under the residual symmetries:

\[
\begin{align*}
\delta g_{\mu\nu} &= 2(\lambda - 2k_\mu x^\mu)g_{\mu\nu}, \\
\delta \sigma &= -(\lambda - 2k_\mu x^\mu), \quad \delta b_\mu = 2k_\mu B, \\
\end{align*}
\]

where the other fields are unchanged. The generators corresponding to the transformation parameters \( \lambda \) and \( k_\mu \) are respectively constructed out of those of the choral symmetry as

\[
\begin{align*}
D_0 &\equiv -P(B) = -\int d^3x \tilde{g}^{0\nu} \phi^2 \partial_\nu B, \\
K^\mu &\equiv 2M^\mu(x, B) = 2\int d^3x \tilde{g}^{0\nu} \phi^2 x^\mu \partial_\nu B. \\
\end{align*}
\]

In addition to the generators \( D_0 \) and \( K^\mu \), one can construct the translation generator \( P_\mu \) and \( GL(4) \) generator \( G^\mu_\nu \) from those of the choral symmetry \( IOSp(10|10) \) as

\[
\begin{align*}
P_\mu &\equiv P_\mu(b) = \int d^3x \tilde{g}^{0\nu} \phi^2 \partial_\nu b_\mu, \\
G^\mu_\nu &\equiv M^\mu_\nu(x, b) - iM^\mu_\nu(c^\tau, \bar{c}_\tau) \\
&= \int d^3x \tilde{g}^{0\lambda} \phi^2 (x^\mu \partial_\lambda b_\nu - ic^\mu \partial_\lambda \bar{c}_\nu). \\
\end{align*}
\]

Now we would like to show that in our theory there is a gravitational conformal algebra which is slightly different from conformal algebra in a flat Minkowski space-time. For this aim, let us consider a set of generators, \{\( P_\mu, G^\mu_\nu, K^\mu, D_0 \}\}. From these generators, we wish to construct the generator \( D \) for a scale transformation. Recall that in conformal field theory in the four-dimensional Minkowski space-time, the dilatation generator obeys the following algebra for an local operator \( O_i(x) \) of conformal dimension \( \Delta_i \) [15, 16]:

\[
[iD, O_i(x)] = x^\mu \partial_\mu O_i(x) + \Delta_i O_i(x). \\
\]

Since the scalar field \( \phi(x) \) has conformal dimension 1, it must satisfy the equation:

\[
[iD, \phi(x)] = x^\mu \partial_\mu \phi(x) + \phi(x). \\
\]

\[13\] For clarity, we will call a global scale transformation in a flat Minkowski space-time “dilatation”. Dilatation is usually interpreted as a subgroup of the general coordinate transformation in such a way that the space-time coordinates are transformed as \( x^\mu \to \Omega x^\mu \) in the flat space-time where \( \Omega \) is a constant scale factor, whereas the global scale transformation is a rescaling of all lengths by the same \( \Omega \) by \( g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \). The two viewpoints are completely equivalent since all the lengths are defined via the line element \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu \).
To be consistent with this equation, we shall make a generator for the scale transformation. From the definitions (8.7) and (8.8), we find

\[
[iG^\mu, \phi(x)] = x^\mu \partial_\mu \phi(x), \quad [iD_0, \phi(x)] = -\phi(x).
\]  

(8.11)

The following linear combination of \(G^\mu\) and \(D_0\) does the job:

\[
D \equiv G^\mu - D_0.
\]  

(8.12)

As a consistency check, it is valuable to see how this operator \(D\) acts on the metric field whose result reads:

\[
[iD, g_{\sigma\tau}] = [iG^\mu, g_{\sigma\tau}] - [iD_0, g_{\sigma\tau}]
\]

\[
= (x^\mu \partial_\mu g_{\sigma\tau} + 2g_{\sigma\tau}) - 2g_{\sigma\tau} = x^\mu \partial_\mu g_{\sigma\tau},
\]

(8.13)

which implies that the metric field has conformal dimension 0 as desired and this result will be used later when discussing spontaneous symmetry breakdown.

Next, let us calculate an algebra among the generators \(\{P_\mu, G^\mu, K^\nu, D\}\). After some calculations, we find that the algebra closes and takes the form:

\[
[P_\mu, P_\nu] = 0, \quad [P_\mu, G^\nu_\sigma] = iP_\mu \delta^\nu_\sigma, \quad [P_\mu, K^\nu] = -2i(G^\rho - D)\delta^\nu_\rho,
\]

\[
[P_\mu, D] = iP_\mu, \quad [G^\mu_\nu, G^\rho_\sigma] = i(G^\mu_\sigma \delta^\rho_\nu - G^\rho_\nu \delta^\mu_\sigma),
\]

\[
[G^\mu_\nu, K^\rho] = iK^\mu \delta^\rho_\nu, \quad [G^\mu_\nu, D] = [K^\mu, K^\nu] = 0,
\]

\[
[K^\mu, D] = -iK^\mu, \quad [D, D] = 0.
\]

(8.14)

To extract the gravitational conformal algebra in quantum gravity, it is necessary to introduce the “Lorentz” generator, which can be contracted from the \(GL(4)\) generator as

\[
M_{\mu\nu} \equiv -\eta_{\mu\nu}G^\rho + \eta_{\nu\rho}G^\rho_\mu.
\]  

(8.15)

In terms of the generator \(M_{\mu\nu}\), the algebra (8.14) can be cast to the form:

\[
[P_\mu, P_\nu] = 0, \quad [P_\mu, M_{\rho\sigma}] = i(P_\mu \eta_{\sigma\mu} - P_{\sigma} \eta_{\mu\rho}),
\]

\[
[P_\mu, K^\nu] = -2i(G^\rho - D)\delta^\nu_\rho, \quad [P_\mu, D] = iP_\mu,
\]

\[
[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\sigma\rho} \eta_{\mu\nu} - M_{\nu\sigma} \eta_{\mu\rho} + M_{\rho\mu} \eta_{\sigma\nu} - M_{\mu\nu} \eta_{\rho\sigma}),
\]

\[
[M_{\mu\nu}, K^\rho] = i(-K_\mu \delta^\rho_\nu + K_\nu \delta^\rho_\mu), \quad [M_{\mu\nu}, D] = [K^\mu, K^\nu] = 0,
\]

\[
[K^\mu, D] = -iK^\mu, \quad [D, D] = 0.
\]

(8.16)

where we have defined \(K_\mu \equiv \eta_{\mu\nu}K^\nu\). It is of interest that the the algebra (8.16) in quantum gravity, which we call “gravitational conformal algebra”, formally resembles conformal algebra in the flat Minkowski space-time except for the expression of \([P_\mu, K^\nu]\)\footnote{In case of conformal algebra in the flat space-time, \([P_\mu, K^\nu] = -2i(\delta^\nu_\rho D + M^\rho_\nu)\).} This difference reflects from the difference of the definition of conformal dimension in both gravity and conformal field theory, for which the metric tensor field \(g_{\mu\nu}\) has 2 in gravity as seen in Eq. 3.2 while it has 0 in conformal field theory as seen in Eq. 8.13.
9 Spontaneous breakdown of symmetries

In the theory in hand, there are huge global symmetries, which are $IOSp(10|10)$ supersymmetry, so it is valuable to investigate which symmetries are spontaneously broken or survive even in quantum regime. In this section, we postulate the existence of a unique vacuum $|0\rangle$, which is normalized to be the unity:

$$\langle 0|0 \rangle = 1. \quad (9.1)$$

Furthermore, we assume that the vacuum is translation invariant:

$$P_\mu |0\rangle = 0, \quad (9.2)$$

and the vacuum expectation values (VEVs) of the metric tensor $g_{\mu\nu}$ and the scalar field $\phi$ are respectively the Minkowski metric $\eta_{\mu\nu}$ and a non-zero constant $\phi_0 \neq 0$:

$$\langle 0|g_{\mu\nu}|0 \rangle = \eta_{\mu\nu}, \quad \langle 0|\phi|0 \rangle = \phi_0. \quad (9.3)$$

By a straightforward calculation, we can obtain the following VEVs:

$$\langle 0|\hat{P}^\mu(x, b_\rho)|0 \rangle = -\delta^\mu_\rho, \quad \langle 0|\hat{P}^\mu(c_\tau, \bar{c}_\rho)|0 \rangle = i\delta^\mu_\rho, \quad \langle 0|\hat{P}_\mu(\bar{c}_\tau, c^\tau)|0 \rangle = -i\delta^\mu_\rho,$$

$$\langle 0|\hat{M}^{\mu\nu}(x, c_\tau, \bar{c}_\rho, \partial_\lambda b_\rho - \partial_\rho b_\lambda)|0 \rangle = -i\delta^{\mu\nu}_\lambda, \quad \langle 0|\hat{M}^{\mu\nu}(x, \bar{c}_\rho, \partial_\lambda c_\rho)|0 \rangle = -i\delta^{\mu\nu}_\lambda,$$

$$\langle 0|\hat{P}(\sigma, B)|0 \rangle = 1, \quad \langle 0|\hat{P}(c, \bar{c})|0 \rangle = i, \quad \langle 0|\hat{P}(\bar{c}, c)|0 \rangle = -i, \quad \langle 0|\hat{M}(\sigma, c)|0 \rangle = i\sigma_0, \quad \langle 0|\hat{M}(\sigma, \bar{c})|0 \rangle = -i\sigma_0, \quad (9.4)$$

where $\langle 0|\sigma(x)|0 \rangle \equiv \sigma_0$. Eq. (9.4) shows that the symmetries generated by the conserved charges

$$\{P^\mu(x), P^\mu(c^\tau), P_\mu(\bar{c}_\tau), M^{\mu\nu}(x, x), M^{\mu\nu}(x, c^\tau), M^{\mu\nu}(x, \bar{c}_\tau), P(\sigma), P(c), P(\bar{c}), M(\sigma, c), M(\sigma, \bar{c})\}$$

are necessarily broken spontaneously, thereby $b_\mu, c^\mu, \bar{c}_\mu, B, c$ and $\bar{c}$ acquiring massless Nambu-Goldstone modes. Note that the exact masslessness of the dilaton $\sigma$ cannot be proved in this way.

Next, on the basis of the gravitational conformal symmetry, we will show that $GL(4)$, special conformal symmetry and scale symmetry are spontaneously broken down to the Poincaré symmetry. We find that the VEV of a commutator between the $GL(4)$ generator and the metric field reads

$$\langle 0|[iG^{\mu\nu}, g_{\sigma\tau}]|0 \rangle = \delta^\mu_\rho \eta_{\nu\sigma} + \delta^\mu_\tau \eta_{\nu\sigma}. \quad (9.5)$$

Thus, the Lorentz generator, which is defined in Eq. (8.15), has the vanishing VEV:

$$\langle 0|[iM^{\mu\nu}, g_{\sigma\tau}]|0 \rangle = 0. \quad (9.6)$$
On the other hand, the symmetric part, which is defined as $\bar{M}_{\mu\nu} \equiv \eta_{\mu\rho} G^\rho_{\nu} + \eta_{\nu\rho} G^\rho_{\mu}$, has the non-vanishing VEV:

$$\langle 0 | i \bar{M}_{\mu\nu}, g_{\sigma\tau} | 0 \rangle = 2(\eta_{\mu\sigma} \eta_{\nu\tau} + \eta_{\nu\tau} \eta_{\mu\sigma}).$$  

(9.7)

Thus, the $GL(4)$ symmetry is spontaneously broken to the Lorentz symmetry where the corresponding Nambu-Goldstone boson with ten independent components is nothing but the massless graviton [17]. Here it is interesting that in a sector of the scalar field, the $GL(4)$ symmetry and of course the Lorentz symmetry as well, do not give rise to a symmetry breaking as can be seen in the commutators:

$$\langle 0 | [i G_{\mu\nu}, \phi] | 0 \rangle = \langle 0 | [i \bar{M}_{\mu\nu}, \phi] | 0 \rangle = \langle 0 | [i \bar{M}_{\mu\nu}, \phi] | 0 \rangle = 0. \quad (9.8)$$

Now we wish to clarify how the scale symmetry and special conformal symmetry are spontaneously broken and what the corresponding Nambu-Goldstone bosons are. As for the scale symmetry, it is not the gravitational field but the dilaton that gives rise to spontaneous symmetry breaking. Indeed, Eq. (8.13) gives us

$$\langle 0 | [i D, g_{\sigma\tau}] | 0 \rangle = 0. \quad (9.9)$$

On the other hand, for the dilaton, from Eq. (8.11) we have

$$\langle 0 | [i D, \sigma] | 0 \rangle = 1, \quad (9.10)$$

which elucidates the spontaneous symmetry breakdown of the scale symmetry whose Nambu-Goldstone boson is just the massless dilaton $\sigma(x)$.

Regarding the special conformal symmetry, we find

$$\langle 0 | [i K_{\mu}^\nu, \partial_{\nu} \sigma] | 0 \rangle = 2 \delta_{\mu}^\nu. \quad (9.11)$$

This equation means that the special conformal symmetry is certainly broken spontaneously and its Nambu-Goldstone boson is the derivative of the dilaton. This interpretation can be also verified from the gravitational conformal algebra. In the algebra (8.16), we have a commutator between $P_{\mu}$ and $K^\nu$:

$$[P_{\mu}, K^\nu] = -2i(G^\rho_{\rho} - D)\delta_{\mu}^\nu. \quad (9.12)$$

Let us consider the Jacobi identity:

$$[[P_{\mu}, K^\nu], \sigma] + [[K^\nu, \sigma], P_{\mu}] + [[\sigma, P_{\mu}], K^\nu] = 0. \quad (9.13)$$

Using the translational invariance of the vacuum in Eq. (9.2) and the equation

$$[P_{\mu}, \sigma] = -i \partial_{\mu} \sigma, \quad (9.14)$$

and taking the VEV of the Jacobi identity (9.13), we can obtain the VEV:

$$\langle 0 | [K^\nu, \partial_{\mu} \sigma] | 0 \rangle = -2\delta_{\mu}^\nu \langle 0 | [G^\rho_{\rho} - D, \sigma] | 0 \rangle = -2i \delta_{\mu}^\nu. \quad (9.15)$$
which coincides with Eq. (9.11) as promised. In other words, the $GL(4)$ symmetry is spontaneously broken to the Poincaré symmetry whose Nambu-Goldstone boson is the graviton, the scale symmetry and the special conformal symmetry are also spontaneously broken and the corresponding Nambu-Goldstone bosons are the dilaton and the derivative of the dilaton, respectively. It is of interest that the Nambu-Goldstone boson associated with the special conformal symmetry is not an independent field in quantum gravity as in conformal field theory [15].

10 Conclusion

In this article, we have performed a manifestly covariant quantization and constructed a quantum theory of the Weyl invariant scalar-tensor gravity within the framework of the BRST formalism. In the past, Nakanishi has made a similar quantum gravitational theory of Einstein's general relativity [4, 5], and the present work provides its natural generalization in the sense that the Weyl symmetry is treated on the same footing as the general coordinate symmetry. Since the Weyl invariant scalar-tensor gravity has been known to be equivalent to general relativity in the unitary gauge where the scalar field is gauge-fixed to a constant, it is natural to expect that our present theory shares several characteristic features with the Nakanishi’s quantum gravity. In particular, the both theories have a huge global symmetry called “choral symmetry”, but our choral symmetry $ISO_p(10|10)$ is larger than that of the Nakanishi’s theory, which is an $ISO_p(8|8)$, owing to the presence of the Weyl symmetry in our formulation. Compared with the case of general relativity, one peculiar feature of our choral symmetry is that the choral symmetry needs the Weyl symmetry in proving its invariance of the quantum Lagrangian so that it is closely related to a gravitational sector while in the case of general relativity the choral symmetry is isolated from classical Lagrangian and comes from purely the Lagrangian involving the Nakanishi-Lautrup field and the FP ghosts.

It is worth mentioning that in our quantum gravity there is a gravitational conformal algebra which is relevant to conventional conformal algebra in a flat Minkowski space-time. According to the Zumino theorem [19], the theories which are invariant under the GCT and Weyl transformation have conformal invariance in the flat Minkowski background at the classical level. The present study supports a conjecture that the Zumino theorem could be valid even in quantum gravity.

Last but not least, we should comment on a Weyl anomaly. In this respect, let us recall that in the manifestly scale invariant regularization method [20] - [25], the scale invariance is free of scale anomaly. Though a completely satisfying formalism is still missing, we believe that in the Weyl invariant regularization method, the Weyl invariance would be also kept in the operator level without the Weyl anomaly, and is spontaneously broken in considering states in the Hilbert space.

There are a lot of works to be done in future. First of all, we should make a
manifestly Weyl invariant regularization methods by introducing an additional scalar field which plays a role for the renormalization mass scale $\mu$. Secondly, we should prove a quantum Zumino theorem in case that a classical Lagrangian is an arbitrary Lagrangian which is invariant under the Weyl transformation. Thirdly, we should add the Lagrangian of conformal gravity, that is, $\mathcal{L} \sim \sqrt{-g} C_{\mu\nu\rho\sigma}^2$ with conformal tensor $C_{\mu\nu\rho\sigma}$, and investigate if the similar analysis to the present work could be done or not. Finally, it has been known that the Weyl invariant scalar-tensor gravity reduces to the Weyl transverse gravity when the longitudinal general coordinate transformation is gauge-fixed [26] - [29]. The Weyl transverse gravity possesses the Weyl symmetry, to which we could apply the present formulation and investigate various quantum aspects. We hope to return these problems in the near future.

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Appendix

A Derivation of Eq. (3.18)

In this appendix, we present a derivation of Eq. (3.18). First of all, let us notice that the scalar gauge condition (3.6) is equivalent to the equation:

$$\Box \phi^2 = 0. \quad (A.1)$$

Then, the Einstein equation in (3.15) reads

$$G_{\mu\nu} - \phi^{-2} \nabla_\mu \nabla_\nu \phi^2 - 6 \phi^{-2} (E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E) = 0. \quad (A.2)$$

With the help of Eq. (A.1), the trace part of this equation becomes

$$R = 6 \phi^{-2} E. \quad (A.3)$$

Inserting Eq. (A.3) to Eq. (A.2) leads to

$$R_{\mu\nu} = \phi^{-2} (\nabla_\mu \nabla_\nu \phi^2 + 6 E_{\mu\nu}). \quad (A.4)$$

Next, operating a covariant derivative $\nabla^\mu$ on Eq. (A.2) and using the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, we have

$$2 \nabla^\mu \phi \nabla_\mu \nabla_\nu \phi^2 - \phi R_{\nu}^{\mu} \nabla_\mu \phi^2 + 12 \nabla^\mu \phi (E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E)$$

$$- 6 \phi \nabla^\mu (E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E) = 0, \quad (A.5)$$
where Eq. (A.1) was used. Substituting Eq. (A.4) into (A.5) produces
\[
\nabla^\mu (E_{\mu\nu} - \frac{1}{2} g_{\mu\nu} E) + \phi^{-1} \nabla_\nu \phi E = 0. \quad (A.6)
\]

At this point, we make use of an identity which holds for any symmetric tensor \( S^\mu_\nu = S^\nu_\mu \): \[ (A.7) \]
\[
\nabla_\nu S^\nu_\mu = h^{-1} \partial_\alpha (h S^\nu_\mu) + \frac{1}{2} S_{\alpha\beta} \partial_\mu g^{\alpha\beta}. \]

Identifying \( S^\mu_\nu \) with \( E^\mu_\nu \), and using the relation (3.14), we can obtain
\[
g^{\rho\nu} \partial_\rho \hat{E}^\mu_\nu - \frac{1}{2} g^{\alpha\beta} \partial_\mu \hat{E}^\alpha_\beta = 0, \quad (A.8)
\]
where we used the extended de Donder gauge condition (3.5). Finally, when we calculate the LHS of Eq. (A.8) by using the definition of \( \hat{E}^\mu_\nu \) in (3.13), we can arrive at the desired equation Eq. (3.18).

### B Residual symmetry and conformal symmetry

In this Appendix, we would like to explain that the residual symmetries found in Eq. (8.5) in a curved space-time reduces to a dilatational invariance and special conformal invariance in a flat Minkowski space-time.

Before doing so, let us first recall that conformal transformation \[ (B.1) \]
\[
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2 \Lambda(x) \eta_{\mu\nu},
\]
where \( \Lambda(x) \) is the infinitesimal transformation parameter of the Weyl transformation, i.e., \( \Omega(x) \equiv e^{\Lambda(x)} \approx 1 + \Lambda(x) \).

Taking the trace of Eq. (B.1) enables us to determine \( \Lambda(x) \) to be
\[
\Lambda = \frac{1}{4} \partial^\rho \epsilon_\rho. \quad (B.2)
\]
Inserting this \( \Lambda \) to Eq. (B.1) yields
\[
\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{1}{2} \partial^\rho \epsilon_\rho \eta_{\mu\nu}, \quad (B.3)
\]
which is often called the “conformal Killing equation” in the Minkowski space-time. It is worth stressing that Eq. (B.3) implies the following fact: The flat Minkowski metric \( g_{\mu\nu} = \eta_{\mu\nu} \) is invariant in the space of the metric functions under a suitable combination of the general coordinate transformation and the Weyl transformation in such a way that
\[
\delta(\epsilon_\mu) = \delta_{\text{GCT}}(\epsilon_\mu) - \delta_W(\Lambda = \frac{1}{4} \partial^\rho \epsilon_\rho), \quad (B.4)
\]
when the vector field $\epsilon_\mu(x)$ obeys the conformal Killing equation (B.3). To put it differently, the characteristic feature of the theory under consideration is that the Lagrangian (3.1) possesses the conformal symmetry with 15 global parameters which is a subgroup of the general coordinate transformation and the Weyl transformation.

Multiplying it by $\partial^\mu \partial^\nu$, we obtain

$$\Box \partial^\mu \epsilon_\mu = 0.$$  \hspace{1cm} (B.5)

Moreover, multiplying Eq. (B.3) by $\partial_\mu \partial_\lambda$ and then symmetrizing the indices $\lambda$ and $\nu$ leads to the equation:

$$\partial_\nu \partial_\lambda \partial^\mu \epsilon_\mu = 0,$$  \hspace{1cm} (B.6)

where we have used Eqs. (B.3) and (B.5). It turns out that a general solution to Eq. (B.6) reads

$$\epsilon_\mu = a_\mu + \omega_\mu^\nu x_\nu + \lambda x_\mu + k_\mu x^2 - 2 x_\mu k_\nu x^\nu,$$  \hspace{1cm} (B.7)

where $a_\mu, \omega_\mu^\nu = -\omega_\nu^\mu, \lambda$ and $k_\mu$ are all constant parameters and they correspond to the translation, the Lorentz transformation, the dilatation and the special conformal transformation, respectively.

At this point, it is useful to verify what expression the infinitesimal parameter $\Lambda$ generated by the “conformal Killing vector” $\epsilon_\mu$ in Eq. (B.7) takes. Actually, substituting Eq. (B.7) into Eq. (B.2), we have

$$\Lambda = \lambda - 2 k_\mu x_\mu.$$  \hspace{1cm} (B.8)

This is nothing but zero-mode solutions in Eq. (S.5). This result implies that finding the residual symmetries (S.5) amounts to solving the conformal Killing equation in a flat Minkowski space-time.

To summarize, we have explicitly shown that in our quantum gravity the Weyl symmetry, together with the general coordinate invariance, generates the conformal symmetry in the flat Minkowski background. This result is a quantum-mechanical generalization of the well-known Zumino’s theorem [19] which insists that the theories invariant under both the general coordinate transformation and the Weyl transformation (or local scale transformation) possess conformal symmetry in the flat Minkowski background. Even if we used a Weyl invariant classical Lagrangian (S.1), we think that the result obtained here holds for any theories which are invariant under the GCT and the Weyl transformation if we adopt the extended de Donder gauge and the scalar gauge for these invariances.

References

[1] I. Oda, “Quantum Scale Invariant Gravity in de Donder Gauge”, Phys. Rev. D 105 (2022) 066001.
[2] C. Brans and R. H. Dicke, “Mach’s Principle and a Relativistic Theory of Gravitation”, Phys. Rev. 124 (1961) 925.

[3] I. Oda, “Scale Invariance and Dilaton Mass”, arXiv:2110.15408 [hep-th].

[4] N. Nakanishi, ”Indefinite Metric Quantum Field Theory of General Gravity”, Prog. Theor. Phys. 59 (1978) 972.

[5] N. Nakanishi and I. Ojima, ”Covariant Operator Formalism of Gauge Theories and Quantum Gravity”, World Scientific Publishing, 1990 and references therein.

[6] Y. Fujii and K. Maeda, ”The Scalar-Tensor Theory of Gravitation”, Cambridge University Press, 2003.

[7] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation”, W H Freeman and Co (Sd), 1973.

[8] T. Kugo and I. Ojima, ”Local Covariant Operator Formalism of Nonabelian Gauge Theories and Quark Confinement Problem”, Prog. Theor. Phys. Suppl. 66 (1979) 1.

[9] T. Kugo, ”Noether Currents and Maxwell-type Equations of Motion in Higher Derivative Gravity Theories”, arXiv:2107.11600 [hep-th].

[10] R. Jackiw and S-Y. Pi,”Fake Conformal Symmetry in Conformal Cosmological Models”, Phys. Rev. D 91 (2015) 067501.

[11] I. Oda, ”Fake Conformal Symmetry in Unimodular Gravity”, Phys. Rev. D 94 (2016) 044032.

[12] I. Oda, ”Restricted Weyl Symmetry”, Phys. Rev. D 102 (2020) 045008.

[13] A. Kamimura and I. Oda, ”Quadratic Gravity and Restricted Weyl Symmetry”, Mod. Phys. Lett. A 36 (2021) 2150139.

[14] I. Oda, ”Restricted Weyl Symmetry and Spontaneous Symmetry Breakdown of Conformal Symmetry”, Mod. Phys. Lett. A 36 (2021) 2150203.

[15] D. J. Gross and J. Wess, ”Scale Invariance, Conformal Invariance, and the High-Energy Behavior of Scattering Amplitudes”, Phys. Rev. D 2 (1970) 753, and references therein.

[16] Y. Nakayama, ”Scale Invariance Vs Conformal Invariance”, Phys. Rept. 569 (2015) 1, and references therein.

[17] N. Nakanishi and I. Ojima, ”Proof of the Exact Masslessness of Gravitons”, Phys. Rev. Lett. 43 (1979) 91.

[18] K. Kobayashi and T. Uematsu, ”Non-linear Realization of Superconformal Symmetry”, Nucl. Phys. B 263 (1986) 309.
[19] B. Zumino, “Effective Lagrangian and Broken Symmetries”, Lectures on Elementary Particles and Quantum Field Theory v.2, Cambridge, Brandeis Univ., pp. 437-500, 1970.

[20] F. Englert, C. Truffin and R. Gastmans, “Conformal Invariance in Quantum Gravity”, Nucl. Phys. B 117 (1976) 407.

[21] M. Shaposhnikov and D. Zenhausern, “Quantum Scale Invariance, Cosmological Constant and Hierarchy Problem”, Phys. Lett. B 671 (2009) 162.

[22] M. Shaposhnikov and F. Tkachov, “Quantum Scale-invariant Models as Effective Field Theories”, [arXiv:0905.4857 [hep-th]].

[23] A. Codello, G. D’Odorico, C. Pagani and R. Percacci, “The Renormalization Group and Weyl-invariance”, Class. Quant. Grav. 30 (2013) 115015.

[24] C. Tamarit, “Running Couplings with A Vanishing Scale Anomaly”, JHEP12 (2013) 098.

[25] D. M. Ghilencea, “Manifestly Scale-invariant Regularization and Quantum Effective Operators”, Phys. Rev. D 93 (2016) 105006.

[26] I. Oda, “Schwarzschild Solution from Weyl Transverse Gravity”, Mod. Phys. Lett. A 32 (2017) 1750022.

[27] I. Oda, “Reissner-Nordstrom Solution from Weyl Transverse Gravity”, Mod. Phys. Lett. A 31 (2016) 1650206.

[28] I. Oda, “Cosmology in Weyl Transverse Gravity”, Mod. Phys. Lett. A 31 (2016) 1650218.

[29] I. Oda, “Classical Weyl Transverse Gravity”, Eur. Phys. J. C 77 (2017) 284.