Lensing function relation in Hadrons

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Abstract. Nontrivial model-induced relations between parton distribution functions can be helpful to interpret, at least qualitatively, the mechanisms at work in scattering processes off the nucleon. However great care is needed in the transition between qualitative and quantitative interpretation of the results. One example is given by the relations between transverse distortions in the distribution of quarks in impact-parameter space and analogous distortions in transverse-momentum space due to T-odd effects, that have been found in the context of various model calculations. We discuss the origin of such relations pointing out the very specific conditions under which they are realised. These conditions are typically verified only in relatively simple models that describe hadrons as two-body bound systems and involve a helicity-conserving coupling between the gauge boson and the spectator system.

Introduction
A three-dimensional picture of the partonic structure of the nucleon can be obtained by considering two types of parton distributions. The first ones are impact-parameter dependent parton distributions (IPDs), obtained by Fourier transforms of generalised parton distributions (GPDs), that contribute to observable asymmetries in exclusive processes involving hadrons. Secondly, there are transverse-momentum dependent parton distributions (TMDs) that give rise to observable asymmetries in semi-inclusive deep inelastic (SIDIS) processes. IPDs give access to the structure of hadrons in terms of the longitudinal component of the momentum and the transverse position of partons, whereas TMDs describe the parton distributions as function of the longitudinal and transverse components of the momentum.

At leading twist, the quark correlator which is parametrized in terms of IPDs has, formally, the same structure as the quark correlator for TMDs, with the impact parameter $b_\perp$ taking the role of the transverse momentum $k_\perp$. However $b_\perp$ and $k_\perp$ are not conjugate variables, therefore it is not possible to establish model-independent relations between IPDs and TMDs. Only in some model calculations do nontrivial relations emerge [1–3].
The most prominent case of such nontrivial relations is the translation of the T-even transverse position space asymmetry into a T-odd transverse momentum space asymmetry of the active quark, as observed in single spin asymmetries (SSAs) of SIDIS processes. This relation is realized via the factorisation in the TMDs of the final state interactions (FSIs), incorporated in a so-called “chromodynamics lensing function” [4–6].

In this contribution, we summarise the very stringent conditions for the validity of the lensing relation that have been discussed more extensively in Ref. [7], both in a model-independent way and with examples from different models.

1. Relations between GPDs and T-odd TMDs

The following light-cone correlator defines the quark GPDs

\[ F^\gamma(x, \xi, t, S) = \frac{1}{2} \int \frac{dz}{2\pi} e^{ikz} \langle p', S|\bar{\psi} \left( -\frac{z}{2} \right) \Gamma W \left( -\frac{z}{2}, \frac{z}{2} \right) \psi \left( \frac{z}{2} \right) |p, S\rangle |_{z^+ = 0, z_\perp = 0}, \]

which depends on the light-cone momentum fraction \( x = k^+/p^+ \), on the variables \( \xi = -\frac{\Delta^+}{2p^+} \) and \( t = \Delta^2 \), where \( P = (p + p')/2 \) and \( \Delta = p' - p \).

The IPD correlator can be obtained by Fourier transform of the above GPD correlator (1) at \( \xi = 0 \) from the momentum coordinates \( A \) to the impact-parameter coordinates \( b \), and reads

\[ F^\gamma(x, b \perp, S) = \frac{1}{2} \int \frac{dz}{2\pi} e^{ipz} \langle p^+, R \perp = 0 \perp, S|\bar{\psi}(z_1)\Gamma W(z_1, z_2)\psi(z_2)|p^+, R \perp = 0 \perp, S\rangle, \]

where the quark fields are evaluated at \( z_{1,2} = (0^+, \mp \frac{z}{2}, b \perp) \) and the hadron is in a state with longitudinal momentum \( p^+ \) and transverse center of momentum (the infinite-momentum frame equivalent of the center of mass position) \( R \perp = 0 \perp \). The operator \( W \), known as Wilson line or gauge link, ensures color gauge invariance of the matrix element and is defined as follows:

\[ W(a, b) = \mathcal{P}\exp \left\{ -ig_s \int_\gamma d\zeta \cdot A(\zeta) \right\}, \]

where \( g_s = \sqrt{4\pi\alpha_s} \) and \( \gamma \) is a path from \( a \) to \( b \). The explicit form of the path is determined by the physical process under consideration.

The quark TMD correlator is defined as

\[ \phi^\gamma(x, k \perp, S) = \frac{1}{2} \int \frac{dz'}{(2\pi)^2} e^{ikz'} \langle p, S|\bar{\psi} \left( -\frac{z'}{2} \right) \Gamma W \left( -\frac{z'}{2}, \frac{z'}{2} \right) \psi \left( \frac{z'}{2} \right) |p, S\rangle |_{z^+ = 0}, \]

where \( p = (p^+, p_\perp, p \perp = 0 \perp) \) and \( S \) are, respectively, the hadron-target momentum and spin, \( \psi \) is the quark field operator and \( \Gamma \) is a generic matrix in the Dirac space. The TMD correlator depends on \( x \) and on the quark transverse momentum \( k \perp \).

In this work we consider the SIDIS process to access TMDs information. In this framework, the gauge link in Eq. (3) breaks the naive time-reversal invariance of the correlator because of the transverse contribution at positive light-cone infinity [8]. As a consequence, T-odd TMDs do not vanish. At leading-twist and for spin 1/2 targets, two T-odd TMDs are present: the Sivers function \( f^\perp_{T\perp}(x, k_\perp^2) \) and the Boer-Mulders function \( h^\perp_{T\perp}(x, k_\perp^2) \). The former describes the momentum distribution of unpolarized quark in a transversely polarized target, and is obtained from the correlator (3) with \( \Gamma = \gamma^+ \). The latter function gives the momentum distribution of

\[ ^1 \text{We use light-front coordinates, with } v^\pm = 1/\sqrt{2}(v^0 \pm v^3) \text{ and } v_\perp = (v^1, v^2) \text{ for a generic four-vector } v. \]
transversely polarized quarks in an unpolarized target, and is defined from the correlator (3) with $\Gamma = i\sigma^3 + \gamma_5$. For spin-0 targets, only the contribution of the Boer-Mulders function is present.

The correlator (2) with $\Gamma = \gamma^+$ for transversely polarised spin 1/2 hadrons is parametrised in terms of the derivative of the IPD $E$. Instead, with $\Gamma = i\sigma^3 + \gamma_5$ and unpolarized target, we access the derivative of the combination $\mathcal{E}_T + 2\mathcal{H}_T$ of chiral-odd IPDs.

Once the integration over $z_\perp$ in Eq. (3) is performed, the analogy (under the exchange $k_\perp \leftrightarrow b_\perp$) between the TMD correlator and the IPD correlator becomes evident. This suggests a relation between T-odd TMDs and T-even IPDs for unpolarized (transversely) polarized quarks in a transversely polarized (unpolarized) target.

We consider the following average quark transverse momenta

$$\langle k^i_\perp(x) \rangle_{UT} = \int d^2k_\perp \langle k^i_\perp \Phi^{[i^+]}(x, k_\perp, S_\perp) \rangle, \quad \langle k^i_\perp(x) \rangle_{TU} = \int dk_\perp k^i_\perp \Phi^{[i^+\gamma_5]}(x, k_\perp),$$

where the first and second subscripts indicate the quark and hadron polarisation, respectively. We can rewrite the UT average transverse momentum as (equivalent result holds for $\langle k^i_\perp(x) \rangle_{TU}$)

$$\langle k^i_\perp(x) \rangle_{UT} = \frac{1}{2} \int \frac{dz}{2\pi} e^{ixp\cdot z} \left\langle p, S_\perp \mid \bar{v}(\frac{z}{2}) W\left(-\frac{z}{2}\frac{2}{\gamma^+}\right) I^i\left(\frac{2}{\gamma^+}\right) \mid p, S_\perp \right\rangle_{z^+ = z_\perp = 0}$$

$$= \frac{1}{2} \int d^2b_\perp \int \frac{dz}{2\pi} e^{ixp\cdot z} \left\langle p^+, R_\perp = 0, S_\perp \mid \bar{v}(z_1) W(z_1; z_2) I^i(z_2) \gamma^+ \psi(z_2) \mid p^+, R_\perp = 0, S_\perp \right\rangle .$$

The operator $I^i(z)$ encodes part of the contribution of the FSIs, and is defined as

$$I^i(z) = \frac{g_s}{2} \int dy^- W((z^+, z^+, z_\perp), (y^+, z^+, z_\perp)) G^{i+}(y^-, z^+, z_\perp) W((y^-, z^+, z_\perp), (z^+, z^+, z_\perp)) ,$$

with $G^{i+}$ being the gluon-field strength tensor. In the light-cone gauge $A^+ = 0$ with advanced boundary condition $A^-(-\infty^-) = 0$, one has $I^i(z) = \frac{2}{\gamma^+} A^i_\perp(\infty^-, z^+, z_\perp)$, and $W(z_1; z_2) = 1$, therefore in Eq. (2) the effect of the FSIs is trivial and Eq. (5) simplifies in

$$\langle k^i_\perp(x) \rangle_{UT} = \frac{g_s}{2} \int \frac{dz}{2\pi} e^{ixp\cdot z} \left\langle p, S_\perp \mid \bar{v}(\frac{z}{2}) A^i_\perp(\infty^-) \gamma^+ \psi\left(\frac{z}{2}\right) \mid p, S_\perp \right\rangle_{z^+ = z_\perp = 0} .$$

One notices from Eq. (7) that the FSIs for the average transverse momentum in the light-cone gauge $A^+ = 0$ reduce to the exchange of a single transverse gluon at light-cone infinity between the active quark and the spectator system (the same result holds regardless of the boundary conditions for $A^\perp$).

The first line of Eq. (5) can be rewritten via the insertion of two completeness relation:

$$\langle k^i_\perp(x) \rangle_{UT} = \frac{1}{2} \left\{ \langle d^2k_1 \rangle \right\} \left\{ \langle d^2k_2 \rangle \right\} \int \frac{dz}{2\pi} e^{ixp\cdot z} e^{-i\frac{z}{2}(k^i_\perp + k^j_\perp + t^i)} \sum_{n,m} \sum_{\beta\beta'} \int \frac{dq_\perp^i dq_\perp^{-i}}{(2\pi)^3 2q_\perp^i}$$

$$\times \prod_{j=1}^m \frac{dw_\perp^j}{(2\pi)^2} \left\langle p^+, p_\perp = 0, S_\perp \mid \phi(k_1) \gamma^+ \{q^i_1, q^j_\perp, i\} n, \beta \rangle \langle \{q^i_1, q^j_\perp, i\} n, \beta \mid I^i(l) \{w_\perp^i, w_\perp^{-i}, m, \beta \} \right\} \times \left\{ w_\perp^i, w_\perp^{-i}, m, \beta \right\} \psi(k_2) \{p^+, p_\perp = 0, S_\perp \} ,$$

where $\{ \ldots \}$ indicates the Lorentz invariant integration measure and $\phi \left(\frac{x}{2}\right) = \bar{v}(\frac{x}{2}) W\left(-\frac{x}{2}; \frac{2}{\gamma^+}\right)$. In Eq. (8), the index $\beta$ and $\beta'$ label the parton, color and the helicity content of the intermediate
states. The matrix elements of the lensing operator $I^i(l)$ in Eq. (9) represent the interaction between the active parton and the spectator system mediated by the Wilson gluons and correspond to the FSIs that occur in a SIDIS process. To recover in Eq. (8) the definition of the IPDs correlator and, therefore, factorize the FSIs, the operator $I^i(l)$ must satisfy the following relation

$$
\langle \{q_i^+, q_{\perp,i}\}_n, \beta' | I^i(l) | \{w_i^+, w_{\perp,i}\}_m, \beta \rangle = 2\pi L^i \left( \frac{l_i}{1 - x} \right) \delta_{n,m} \delta_{\beta,\beta'} \delta(l^+) \prod_{i=1}^n (2\pi)^d 2q_i^+ \delta(q_i^+ - w_i^+) \delta \left( q_{\perp,i} - w_{\perp,i} - x_i l_i \frac{1}{1 - x} \right), \tag{9}
$$

where $x_i$ is the light-cone momentum fraction of each constituent w.r.t. the hadron target light-cone momentum, i.e. $x_i = w_i^+ / p^+$, and should satisfy the relation $\sum_i x_i = 1 - x$. The relation (9) imposes strict conditions on the action of the FSIs that can be summarised in the following points:

1. the FSIs should connect Fock states with the same number of constituents and the same parton, helicity and color content;
2. the FSIs should transfer the total transverse momentum $l_\perp / (1 - x)$ to the whole spectator system;
3. the FSIs can not transfer momentum in the light-cone direction to the spectator system;
4. the FSIs should transfer a fraction $x_i = w_i^+ / p^+$ of the total transverse momentum to each constituent of the spectator system.

The last condition is the most stringent. It is crucial to obtain the correct transverse light-front boost that gives the non-diagonal matrix element defining the GPDs and then the transverse distortion in impact parameter space described by IPDs. In the light-cone gauge with advanced boundary conditions, one can easily deduce that the condition (iv) can be realised via a single particle coupling (e.g. perturbative coupling) between the gauge boson and the spectator system only if the latter is composed by a single constituent, i.e. the hadron target is a two-body bound system. In this case, the light-cone momentum fraction of the spectator is equal to $1 - x$ and the constraint on the transverse momentum transferred by the Wilson gluon to the spectator system follows trivially from the conservation of the total momentum of the hadron target. Otherwise, the condition (iv) imposes the need to share the transverse momentum carried by the Wilson gluon with each spectator parton in a proportion equal to the longitudinal momentum fraction $x_i$. This can not be realised in systems composed by more than two constituents by assuming an interaction vertex between the gauge boson and a single parton in the remnant.

We conclude that if and only if the above conditions are fulfilled we can write

$$
\langle k_\perp(x) \rangle_{UT} = - \int dk_\perp k_\perp \frac{e_j k_j S_j}{M} f_{T T}(x, k_\perp^2) = \int db_\perp L_i \left( \frac{b_\perp}{1 - x} \right) \mathcal{F}^{\gamma T}(x, b_\perp, S_\perp) = \int db_\perp L_i \left( \frac{b_\perp}{1 - x} \right) \frac{e_j b_j S_j}{M} \left( \mathcal{E}(x, b_\perp^2) \right)'. \tag{10}
$$

Analogously, we can analyze the average quark transverse momentum of a transversely polarized quark in an unpolarized target and establish a lensing relation between the Boer-Mulders function and the combination of IPDs $- \left( \mathcal{E}_T(x, b_\perp^2) + 2\tilde{H}_T(x, b_\perp^2) \right)'$, with $\mathcal{E}_T(x, b_\perp^2) = 0$ and $2\tilde{H}_T \to \tilde{H}_T$ for spin-zero targets.

One can verify that the conditions (i)-(iv) are verified in models that describe hadrons as a two-body system in which the remnant is either massive with spin $\leq 1/2$ or massless with
arbitrary spin, and assuming that the Wilson gluon interacts with only one particle in the spectator system. Examples of models that fall into one of the previous cases are: the quark target model [1]; for a proton target, the scalar-diquark spectator model [1, 6] and the axial-diquark models that admit only transverse polarization for the diquark [9]; for a pion target, relativistic models at the lowest order in the Fock-space expansion of the pion [10, 11]. Viceversa, the conditions are not fulfilled in a many-body model for hadrons, such as the three-quark model for the nucleon [12–16], or in two-body models in which the coupling with the Wilson gluon allows a helicity transition for the remnant, as, for example, the axial-vector diquark model with longitudinally polarised diquark.

We briefly go through two examples, one satisfying the lensing relation, and one violating it. To show the results, we use the representation of the TMDs and GPDs in terms of light-front wave functions (LFWFs). The first example is the lensing relation in the case of the pion, the second one is the three-quark model calculations for the proton.

In the case of the pion, described as a quark-antiquark bound system, the GPD and the Boer-Mulders TMD can be written as [11]

\[
\frac{\Delta^k}{2M}\bar{H}_{T,\pi}(x,0,-\Delta^2_\perp) = \frac{T^2}{2(2\pi)^3} \int dk_{\perp} G^k(x, k_{\perp}| x, k_{\perp} + (1 - x)\Delta_{\perp}),
\]

\[
k^\perp_1 h_{1,\pi}(x, k^2_{\perp}) = \frac{2\alpha_s}{(2\pi)^3} \frac{4}{3} T^2 M_\pi \int dq_{\perp} G^k(x, k_{\perp}| x, k_{\perp} - q_{\perp}),
\]

where \(q_\perp\) is the transverse momentum of the Wilson gluon and the function \(G\) is defined from the light-front wave amplitude (LFWA) overlap \(F^k\) as [7]

\[
G^k(x_1, k_{\perp,1}|| x'_1, k'_{\perp,1}) = F^k(x_1, k_{\perp,1}; 1 - x, -k_{\perp,1}|| x'_1, k'_{\perp,1}; x'_1, -k'_{\perp,1}).
\]

In Eq. (13), the arguments of the functions on the left-hand (right-hand) side of \(|\rangle\) refer to the momentum dependence of the (complex conjugate) LFWF of the pion in the initial (final) state. With the formal identification of

\[
-q_\perp = (1 - x)\Delta_{\perp},
\]

one obtains:

\[
\langle k^\prime_{\perp,1}|_{TU} = \int db_{\perp} \frac{\epsilon_{k^\prime_{\perp,1}}}{M} L^1_b(x, b_{\perp}) \left(\bar{H}_{T,\pi}(x, b_{\perp}^2)\right)',
\]

with the lensing function [6]

\[
L^i (b_{\perp}/(1 - x)) = -\frac{4}{3} \alpha_s 4\pi \int dq_{\perp} q_{\perp}^i e^{-b_{\perp}/(\alpha_s q_{\perp})} = -\frac{8}{3} \alpha_s 4\pi^2 b_{\perp}^3 (1 - x).
\]

Note that the relation in Eq. (14) corresponds exactly to condition (iv).

For the proton, described as three-quark bound system, we introduce the appropriate LFWA overlap \(G_T\) [7, 12] and obtain the following expressions for the GPD \(E\) and for the Sivers function:

\[
\frac{i\epsilon^{ij}_{\perp} \Delta^j_{\perp}}{M} S^i_{\perp} E(x, -\Delta^2_{\perp}) = \int \frac{dk_{\perp}}{2(2\pi)^3} \int_0^x dy \int \frac{dt_{\perp}}{2(2\pi)^3} G_T(x, k_{\perp}; y, t_{\perp}|| x, k_{\perp} + (1 - x)\Delta_{\perp}; y, t_{\perp} - y\Delta_{\perp}),
\]

\[
\frac{\epsilon^{ij}_{\perp} k^j_{\perp}}{M} f^i_{1T}(x, k^2_{\perp}) = -\frac{\alpha_s}{3(2\pi)^3} \int dq_{\perp} q_{\perp}^i \int_0^x dy \int dt_{\perp} G_T(x, k_{\perp}; y, t_{\perp}|| x, k_{\perp} - q_{\perp}; y, t_{\perp} + q_{\perp}).
\]
From this expression, one clearly sees that the formal identification in Eq. (14) does not apply, since \((1 - x)\) and \(y\) are independent variables. This breaks the condition (iv) and, hence, the validity of the lensing relation.

The last example, for which a more detailed discussion can be found in Ref. [7], is the axial-vector diquark model for the nucleon. In this model one can either include or dismiss the longitudinal polarisation state for the diquark. In the former case, the coupling of a vector colored particle with the Wilson gluon is such that an helicity flip for the remnant is not suppressed by a power of \(1/P^+\) and, as a consequence, condition (i) is violated. This situation is different with respect to, e.g., the case of a spin-1/2 remnant, where the helicity flip is suppressed as \(1/P^+\). If one does not include the longitudinal polarisation for the diquark, then all the four conditions are satisfied, since the Wilson gluon cannot cause an helicity flip of two unity.

To summarise, model calculations of TMDs play a crucial role for building educated Ansätze for fits of the TMDs and GPDs, and are essential towards an understanding of the non-perturbative aspects of TMDs. However, the use of model-induced relations as constraint to extract information on these distributions from data should be avoided. Instead, one should use the results of the extractions to understand how good the model-induced relations are, because this will eventually shed light on some of the non perturbative aspects of QCD.

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References
[1] Meissner S, Metz A and Goeke K 2007 Phys. Rev. D76 034002 (Preprint hep-ph/0703176)
[2] Avakian H, Efremov A V, Schweitzer P and Yuan F 2010 Phys. Rev. D81 074035 (Preprint 1001.5467)
[3] Lorcé C and Pasquini B 2011 Phys. Rev. D84 034039 (Preprint 1104.5651)
[4] Burkardt M 2004 Phys. Rev. D69 057501 (Preprint hep-ph/0311013)
[5] Burkardt M 2004 Nucl. Phys. A735 185–199 (Preprint hep-ph/0302144)
[6] Burkardt M and Hwang D S 2004 Phys. Rev. D69 074032 (Preprint hep-ph/0309072)
[7] Pasquini B, Rodini S and Bacchetta A 2019 Phys. Rev. D100 054039 (Preprint 1907.06960)
[8] Belitsky A V, Ji X and Yuan F 2003 Nucl. Phys. B656 165–198 (Preprint hep-ph/0208038)
[9] Bacchetta A, Conti F and Radici M 2008 Phys. Rev. D78 074010 (Preprint 0807.0323)
[10] Gamberg L and Schlegel M 2010 Phys. Lett. B685 95–103 (Preprint 0911.1964)
[11] Pasquini B and Schweitzer P 2014 Phys. Rev. D90 014050 (Preprint 1406.2056)
[12] Pasquini B and Yuan F 2010 Phys. Rev. D81 114013 (Preprint 1001.5398)
[13] Pasquini B, Cazzaniga S and Boffi S 2008 Phys. Rev. D78 034025 (Preprint 0806.2298)
[14] Boffi S, Pasquini B and Traini M 2003 Nucl. Phys. B649 243–262 (Preprint hep-ph/0207340)
[15] Pasquini B, Pincetti M and Boffi S 2005 Phys. Rev. D72 094029 (Preprint hep-ph/0510376)
[16] Pasquini B and Boffi S 2007 Phys. Lett. B653 23–28 (Preprint 0705.4345)