Dense Coding with Linear Optics without Bell State Measurements

P. Lougovski†, * and D. B. Uskov†, ‡

†Quantum Information Science Group, Oak Ridge National Laboratory, Oak Ridge, TN 37831
‡Department of Mathematics and Natural Sciences, Brescia University, Owensboro, KY 42301

It is well known that the efficiency of linear optical implementations of the dense coding is limited by one’s ability to discriminate between the four optically encoded Bell states. The best experimental demonstration up to date reports the transmission of \( \approx 1.63 \) bits of information per single optical qubit which is less than the theoretical bound of 2.0 bits for a generic qubit. We show that besides the Bell states there is a class of bipartite two-photon entangled states that can also facilitate dense coding. However, in contrast to the Bell states, they can be deterministically discriminated by means of linear optics and coincidence photo detection without using any auxiliary entanglement resources. We discuss how the proposed dense coding scheme can be generalized to the case of two-photon \( N \)-mode entangled states for \( N = 6, 8 \).

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I. INTRODUCTION

Dense coding is a quantum communication technique that allows one to send two bits of classical information per single qubit transmitted over a quantum channel. This becomes possible when a sender and a receiver share a maximally entangled two-qubit state (Bell state). The original protocol [1] – closely followed in all optical qubit implementations [2–5] – was designed for generic qubits and has two main requirements. First, the sender must be able to generate all four Bell states from any given Bell state via a local (single-qubit) operations only. Second, the receiver must be able to perform an unambiguous Bell state discrimination (typically using a two-qubit CNOT gate). For optical qubits – most suitable for a long distance communication – the first requirement can be readily fulfilled by combining spontaneous parametric down conversion entangled photon sources with linear optical devices such as beam splitters and phase shifters. However, the second requirement of the original protocol is very stringent. Unfortunately, one cannot deterministically distinguish all four Bell states only by means of linear optical devices and coincidence measurements [6]. Either hyper entanglement [3–5, 7] or additional entangled ancillae are needed [8, 9] making all-optical implementations challenging. As a result, the best information transmission rate of \( \approx 1.63 \) bits per single qubit has been achieved so far by using hyper-entangled photons [5].

In this paper we propose a different implementation of the linear optical dense coding protocol that neither relies on Bell state discrimination nor requires any additional physical resources (such as entangled ancillae or hyper-entangled photons) and only utilizes photon coincidence measurements that can unambiguously resolve four possible local operations performed by the sender. We avoid using Bell states as a dense coding resource by eliminating the notion of qubit from the problem. Instead, we identify bipartite two-photon entangled states that can be transformed into each other by means of linear optical operations on “local” mode(s) of one of the parties and, at the same time, can be deterministically discriminated by a photon coincidence measurement. This is done by recasting dense coding as a maximization problem where the optimization objective is the mutual information between a sender and a receiver. We find that multiple solutions exist that allow the sender to communicate 2 bits of information by sending one photon on average. Moreover, as we show below, our scheme can be extended to the case of multiple shared modes and two photons.

II. DENSE CODING FROM THE INFORMATION-THEORETICAL PERSPECTIVE

When implementing an abstract two-qubit system using single photons two so-called “dual rail” schemes are prevalent. The first is the polarization encoding where the logical zero and one states of each qubit are realized as the horizontally \( |H\rangle \) and vertically \( |V\rangle \) polarized single photon in a given spatial mode and the second is the spatial mode encoding where a single photon placed in either one of two spatial modes i.e. \( |0, 1\rangle \) and \( |1, 0\rangle \) represents the logical zero and one states. The schemes can be mapped onto each other by setting \( |H\rangle \equiv |0, 1\rangle, |V\rangle \equiv |1, 0\rangle \). An arbitrary local (single-qubit) operation can be performed by means of linear optical elements such as a polarization rotator or a beam splitter in the case of dual rail encoding. This class of operations will map the two-qubit computational space \( \mathbb{C}^4 = \text{span}\{ |0, 1, 0, 1\rangle, |0, 1, 1, 0\rangle, |1, 0, 0, 1\rangle, |1, 0, 1, 0\rangle \} \) onto itself. However, when implementing a non-local (two-qubit) operation with linear optics an input state from \( \mathbb{C}^4 \) may end up in a larger Hilbert space \( \mathbb{C}^{10} \). For example, consider the action of a 50/50 beam splitter between

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*Electronic address: lougovskip@ornl.gov
†Electronic address: dmitry.uskov@brescia.edu
Bob can unambiguously detect which state whether such states. In this case, analogously to the original dense measurement Bob tries to guess the index $i$ is when Alice can prepare four orthogonal states Alice has sent.

As a result, Alice and Bob share the state resulting state $|\psi\rangle$ distributed by the source. Alice performs a "local" mode transformation $U_i$ on her two modes and sends them to Bob who performs a coincidence detection on all four modes and estimates index $i$.

FIG. 1: (Color on-line) Dense coding scheme illustration. Alice and Bob share a two-photon four-mode state $|\Psi_{in}\rangle$ distributed by the source. Alice performs a "local" mode transformation $U_i$ on her two modes and sends them to Bob who performs a coincidence detection on all four modes and estimates index $i$.

we still would like to know what is the largest amount of information that Alice and Bob can share in this setting (is it $\geq 1.63$ bits?). To answer these questions we need to formulate the problem of dense coding in terms of communication channel capacity determination.

From the information-theoretic perspective Alice and Bob form a two-mode communication system in which Alice encodes each letter of her message using a two-bit alphabet $X = \{01, 10\}$ with probability $p(\psi_i) = p(U_i)$. The receiver, Bob, detects the message sent by Alice as a collection of random signals from the set $Y = \{01, 10\}$ with the conditional probability $p(Y|X) = p(\psi_i|\phi_j) = \langle \phi_j|\psi_i\rangle^2, i = 1, \ldots, 10; j = 1, \ldots, 4$ and applies a decoding rule to estimate the original message. Note that $p(Y|X)$ is a function of the initial state $|\Psi_{in}\rangle$. Alice’s unitary operations $U_i, i = 1, \ldots, 4$ and Bob’s detection setup $U_{Bob}$. In this context, finding the highest rate (in bits) at which information can be sent from Alice to Bob is equivalent to determining information channel capacity of the Alice-Bob system. The information channel capacity $C$ of Alice-Bob channel is defined as \cite{10},

$$C = \max I(\psi; \phi),$$

where the maximization is performed over all possible input distributions $p(\psi_i)$, states $|\Psi_{in}\rangle$, unitary mode transformations $U_i, U_{Bob}$; and $I$ is the mutual information,

$$I(\psi; \phi) = \sum_{j=1}^{10} \sum_{k=1}^{4} p(\psi_j, \phi_k) \log \frac{p(\psi_j|\phi_k)}{p(\psi_j)}.$$ (2)

Here $p(\psi_j, \phi_k) = p(\psi_j|\phi_k)p(\phi_k)$ denotes the joint probability of Alice preparing the state $|\psi_j\rangle$ and Bob detecting the state $|\phi_k\rangle$. The marginal probability $p(\phi_k)$ is defined as $p(\phi_k) = \sum_{j=1}^{10} p(\psi_j, \phi_k)$.

By definition, the mutual information $I(\psi; \phi)$ is a concave function of $p(\psi_i)$ (3 independent real-valued parameters) and a non-concave function of the unitary mode transformation matrices $U_i \in U(2)$, $i = 1, \ldots, 4$ (16 real-valued parameters), and $U_{Bob} \in SU(4)$ (15 independent real-valued parameters) and the input state $|\Psi_{in}\rangle$ that we parametrize using ten complex parameters $c_k$ (18 independent real-valued parameters \cite{11}) as $|\Psi_{in}\rangle = \sum_{k=1}^{10} c_k|\phi_k\rangle$. We can always set one of the matrices $U_i$ to be the identity $I$ matrix which will leave us with only three independent $2 \times 2$ unitary matrices. Therefore, the total number of real-valued optimization parameters in Eq.(1) is 48.

Since Alice’s alphabet only contains four letters, the global maximum of the channel capacity $C$ over all possible physical setup parameters cannot in principle exceed $\log_2(4) = 2$ bits. This follows form the definition of the mutual information in Eq.(2). We observe that for any two random variables $X$ and $Y$: $I(X; Y) = H(X) - H(X|Y)$, where $H$ denotes Shannon entropy \cite{10}. Since $H \geq 0$, the maximum of $I(X; Y)$ is achieved when

modes 2 and 3 on the state $|\psi\rangle = |0, 1\rangle \otimes |1, 0\rangle \in \mathbb{C}^4$. The resulting state $|\psi\rangle = \frac{1}{\sqrt{2}} (|0, 0, 2, 0\rangle - |2, 0, 0, 0\rangle)$ actually lies outside of $\mathbb{C}^4$ in a larger Hilbert space $\mathbb{C}^{10}$. In fact, this becomes a problem when one tries to implement a quantum computer using only linear optical transformations. But for quantum communication problems it may be more advantageous to operate in the full (two photons in four modes) Hilbert space $\mathbb{C}^{10}$.

Indeed, consider the following modified dense coding protocol (see Fig. 1). The source generates a special initial two-photon four-mode state $|\Psi_{in}\rangle \in \mathbb{C}^{10}$ and sends half of the modes to Alice and the other half to Bob. For the sake of concreteness we assume that Alice gets modes labeled 1 and 2 (1 through $N_A$ if $N$ modes are shared). As a result, Alice and Bob share the state $|\Psi_{in}\rangle$. Upon receiving her modes, Alice transforms them using one of the four predetermined two-mode unitary transformations $U_i, i = 1, \ldots, 4$. She chooses which unitary operation $U_i$ to apply according to a probability distribution $p(U_i)$ and does not disclose her choice to Bob. Next, Alice sends her modes to Bob who now has one of four possible two-photon four-mode states $|\psi_i\rangle$ with probability $p(|\psi_i\rangle) = p(U_i)$. Bob wants to learn which unitary transformation $U_i$ was performed by Alice i.e. which state $|\psi_i\rangle$ he has at hand. To do that he sends all four modes through a detection setup that performs a four-mode unitary transformation $U_{Bob}$ and measures the projection of the output state onto the two-photon four-mode Fock basis states $\{|\phi_1\rangle = |2000\rangle, |\phi_2\rangle = |1100\rangle, \ldots, |\phi_9\rangle = |0011\rangle, |\phi_{10}\rangle = |0002\rangle\}$. Using the outcome of the measurement Bob tries to guess the index $i$ of the state that Alice has sent.

Naturally, the best case scenario in terms of information transmission (assuming a noiseless quantum channel) is when Alice can prepare four orthogonal states $|\psi_i\rangle$ and Bob can unambiguously detect which state $|\psi_i\rangle$ Alice has sent him just by means of photon coincidence measurements. In this case, analogously to the original dense coding proposal \cite{1}, Alice and Bob can communicate two bits of information by sending photons in just two modes instead of four. However, it remains an open question whether such states $|\psi_i\rangle$ exist. If the answer is no, then
$H(\mathcal{X})$ is maximal (= $\log_2(|\mathcal{X}|)$) and $H(\mathcal{X}|\mathcal{Y})$ is minimal (=0) i.e. max $I(\mathcal{X};\mathcal{Y}) = \log_2(|\mathcal{X}|)$. However, it is not clear if this bound is physically attainable. Also, due to the non-concave nature of the optimization objective function many local maximums may exist. Of course, when optimizing $I(\psi; \phi)$ numerically we are interested in finding a supremum of all local maximums and hope that it is 2 bits. Note that because $I$ is concave in parameters $p(\psi_i)$, if the global (2 bits) maximum is attained, using the preceding argument one can immediately show that the only possible values of $p(\psi_i) = \frac{1}{2}$.

Therefore, we can further reduce the number of real optimization parameters to 45 by setting $p(\psi_i) = \frac{1}{4}$.

III. OPTIMIZATION RESULTS FOR $N = 4$ MODES $n = 2$ PHOTONS STATES

First, to test our approach, we solved the optimization problem in Eq.(1) using a fixed state $|\Psi_{in}\rangle$ provided by the source. We set $|\Psi_{in}\rangle$ to be equal to one of the Bell states (it does not matter which Bell state is selected, optimization works equally well for all of them) and found by numerical optimization that in this case $C = \log_2 3$. Moreover, Alice’s mode transformation matrices that correspond to this solution are the same as the ones originally proposed by Bennett and Wiesner [1]. It means that by setting the initial state to a Bell state the conventional Bell state-based dense coding protocol [1, 2] is recovered.

Next, we have discovered, by using gradient-based optimization methods, that the global maximum ($C = 2$ bits) is indeed achievable for dense coding schemes in $C^{10}$. The structure of globally optimal solutions encountered in our numerical search can be parametrized as follows. All globally optimal input states $|\Psi_{in}\rangle$ prepared by the source are, up to a swap of any two modes, equivalent to the state,

$$|\Psi_{in}\rangle = \frac{1}{2}(|1, 1, 0, 0\rangle + |0, 1, 1, 0\rangle + |1, 0, 0, 1\rangle$$
$$+ |0, 0, 1, 1\rangle).$$

For example, the following input state

$$|\tilde{\Psi}\rangle = \frac{1}{2}(|1, 0, 1, 0\rangle + |0, 1, 1, 0\rangle + |1, 0, 0, 1\rangle$$
$$+ |0, 1, 0, 1\rangle),$$

obtained from $|\Psi_{in}\rangle$ by swapping modes 2 and 3 also leads to the globally optimal solution with $C = 2$ bits.

Moreover, $|\Psi_{in}\rangle$ in Eq.(3) also defines a class of globally optimal input states that are equivalent to $|\Psi_{in}\rangle$ up to a four-mode unitary transformation:

$$U_t = \begin{bmatrix} U_A & 0 \\ 0 & U_B \end{bmatrix},$$

where $U_{A,B}$ are arbitrary unitary matrices $\in U(2)$,

$$U_A = \begin{bmatrix} e^{i\phi_1} \cos \theta_1 & -e^{i\phi_2} \sin \theta_1 \\ e^{i\phi_2} \sin \theta_1 & e^{i(\phi_1 + \phi_2)} \cos \theta_1 \end{bmatrix},$$

$$U_B = \begin{bmatrix} e^{i\phi_3} \cos \theta_2 & -e^{i\phi_4} \sin \theta_2 \\ e^{i\phi_4} \sin \theta_2 & e^{i(\phi_3 + \phi_4)} \cos \theta_2 \end{bmatrix},$$

and parameters $\theta_1, \phi_1, \ldots, \phi_6$ are arbitrary angles $\in [0, 2\pi]$.

Given matrices $U_A$ and $U_B$, Alice’s globally optimal mode transformation matrices (acting on modes 1 and 2) can be decomposed as $U_1 = U_A^{-1} U_C$, $U_2 = -U_A^{-1} \sigma_z U_C$, $U_3 = -U_2$, $U_4 = U_2 \cdot U_3$, where $U_A$ is defined in Eq.(6), $U_C$ is an arbitrary $2 \times 2$ unitary matrix $\in U(2)$ with a similar parametrization and $\sigma_z$ denoted Pauli sigma $Z$ matrix.

Lastly, Bob’s four-mode transformation matrix $U_{Bob}$ can be represented as follows,

$$U_{Bob} = \frac{1}{\sqrt{2}} \begin{bmatrix} U_C^{-1} & 0 \\ 0 & U_B^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Implementing this mode transformation matrix in an experiment, Bob will detect four distinct coincidence patterns: a coincidence between detectors in modes 1 and 2 correspond to Alice’s choice $U_1$, modes 2 and 3 correspond to $U_2$, modes 3 and 4 correspond to $U_3$, modes 1 and 4 correspond to $U_4$. We remark that this outcome mapping is not unique. Other choices of coincidence assignment are possible and can be realized by additional four-mode unitary rotation on Bob’s end i.e. $U_{Bob} \rightarrow U_{Bob} \cdot U_{swap}$.

We emphasize that $U_t$, $U_1, \ldots, U_4$ and $U_{Bob}$ define globally optimal unitary mode transformation. The result of their action on the input state $|\Psi_{in}\rangle$ can be determined in the following fashion. Let us denote input mode photon creation operators as $a_k^\dagger$, $k = 1, \ldots, 4$, then,

$$|\Psi_{in}\rangle = \frac{1}{2}(a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger + a_1^\dagger a_4^\dagger + a_3^\dagger a_4^\dagger)|0\rangle,$$

where $|0\rangle$ is a four-mode vacuum state. Consider first the unitary mode operation $U_t$ defined earlier. It acts...
Bell-state based protocol [2] our scheme deterministically channel capacity converts the states in the case of noiseless channel and perfect detectors this concerns and sends them for the coincidence detection. In

\[ a_k^+ \rightarrow \sum_{j=1}^{4} (U_k)_{kj} a_j^+ \]

where \((U_k)_{kj}\) are the matrix elements of \(U_k\). As a result, the input state \(|\Psi_{in}\rangle\) in Eq.(9) is transformed to

\[
|\Psi_t\rangle = \frac{1}{2} \left( \sum_{i,j=1}^{4} [(U_1)_{i1}(U_2)_{j2} + (U_3)_{i3}(U_4)_{j4} + (U_2)_{i2}(U_1)_{j1} + (U_4)_{i4}(U_3)_{j3}] a_i^+ a_j^+ \right) |0\rangle.
\]

(10)

Similarly, unitary mode transformations \(U_1, \cdots, U_4\) and \(U_{Bob}\) can now be applied to the state \(|\Psi_t\rangle\) in Eq.(10) in sequence, resulting in the desired output state at Bob's detectors.

The structure of the globally optimal solution is most transparent when \(U_A = U_B = U_C = 1\). In this case the physical setup that implements our dense coding protocol is illustrated in Fig.(2). Surprisingly, its structure is equivalent to a double Mach-Zehnder interferometer. The source prepares the state \(|\Psi_{in}\rangle\) in Eq.(3) by placing the product state containing two single photons in modes 1 and 2 i.e. \(|1, 1, 0, 0\rangle\) onto two 50/50 beam splitters coupling modes (1,3) and (2,4). To transform her two modes, Alice needs only 180° phase shifters. Note that the transformation \(U_3\) applies a phase shift to both Alice's modes with respect to Bob's modes. Bob combines modes (1,3) and (2,4) on two 50/50 beam splitters and sends them for the coincidence detection. In the case of noiseless channel and perfect detectors this converts the states \(|\psi_1\rangle \rightarrow |1, 1, 0, 0\rangle, |\psi_2\rangle \rightarrow |0, 1, 1, 0\rangle, |\psi_3\rangle \rightarrow |0, 0, 1, 1\rangle, |\psi_4\rangle \rightarrow |1, 0, 0, 1\rangle\) which results in the channel capacity \(C = 2\) bits.

In contrast to the linear optical implementations of the Bell-state based protocol [2] our scheme deterministically discriminates between the four possible orthogonal states generated by Alice enabling the transmission of two bits of classical information. However, unlike the Bell-state based protocol, where Alice always receives one photon from the source, in our protocol the exact number of photons Alice receives is undefined but equals one on average. The latter statement is not surprising since the proposed protocol does not make use of qubits and encourages the utilization of quantum states \(\psi \in \mathbb{C}^{10}\). In fact, it poses the following question: is it still possible to communicate two bits of classical information by sending less than one photon on average? To answer this question we need to determine the channel capacity dependence on the average communicated photon number. This can be readily done by modifying our approach slightly as to incorporate the average photon number constraint into the optimization task in Eq.(1). Note that, since Alice only uses passive optical elements, the average number of photons she sends to Bob is actually controlled by the source i.e.

\[
\langle N_{Alice} \rangle = \langle \Psi_{in} | a_1^+ a_1 + a_2^+ a_2 | \Psi_{in} \rangle,
\]

(11)

where \(a_{1,2}\) denote photon annihilation operators for Alice's modes 1,2. Thus, the new optimization problem at hand is,

\[
\text{maximize } I(\psi;\phi) \quad \text{s.t. } \langle N_{Alice} \rangle = n,
\]

(12)

where \(n\) is a constant \(\in [0, 1]\). We used a gradient-based solver to optimize Eq.(12) numerically for various values of the average photon number \(n\). For each fixed value...
of $n$ we ran 100 independent optimizations using random starting points. We then selected the largest value of \( \max f(\psi; \phi) \) over 100 runs and depicted it as a function of $n$ in Fig.(3). There, we notice that if Alice sends to Bob less then one photon on average then their channel capacity falls below two bits. However, they can achieve channel capacity $C \approx 1.63$ bits (the best value demonstrated to date [5]) by communicating $\approx 0.68 < 1$ photons on average.

In principle, Alice may want to use larger alphabets than just the four symbol one. After all, we are operating with the states in \( C^{10} \) and naturally the question arises whether a physical setup exists that attains the channel capacity $C = \log_2 M$ for some integer $M \in [5, 10]$. To answer this question we modified our scheme by allowing Alice to perform $M > 4$ unitary transformations on her two modes. At the same time we still require Bob to measure in the Fock basis \( \{|\phi_j\rangle\}, j = 1, 10 \). We numerically optimized the channel capacity in Eq.(1) for the cases of $M = 5, \cdots , 10$ and normalized the respective values to the maximal theoretically attainable channel capacity \( C = \log_2 M \). The results are plotted in Fig.(4). We notice that the maximal theoretical channel capacity is only achievable for the case of the two-bit alphabet \( (M=4) \). When Alice is trying to use $M > 4$ symbols in her alphabet the normalized channel capacity decreases. This is because Bob is unable to deterministically discriminate between the states \( |\psi_i\rangle, i = 1, \cdots , M; M > 4 \) by using projective measurements in Fock basis.

**IV. OPTIMIZATION RESULTS FOR $N = 6, 8$ MODES $n = 2$ PHOTONS STATES**

In principle, the approach described in Sec. II can be used for linear optical circuits with arbitrary number of modes $N$ and photons $n$. Here, we study two simple extensions of our method with two photons ($n = 2$) distributed over $N = 6$ and $N = 8$ modes. We assume that in both cases Alice and Bob receive $N/2$ modes from the source. Our goal is again to solve numerically the channel capacity problem posed in Eq.(1).

For the $n = 2$, $N = 6$ case the dimensionality of the correspondent Hilbert space is $\dim C = \frac{(n+N-1)!}{n!(N-1)!} = 21$ which naturally leads to the question: is it possible to design a dense coding scheme that provides channel capacity of $\log_2 21 \approx 4.39$ bits by sending just three modes from Alice to Bob? The necessary condition for that is Alice must be able to prepare 21 orthogonal states from an input state $|\Psi_{in}\rangle$ by means of “local” three-mode unitary transformations. However, we discovered numerically that Alice can at best prepare 12 orthogonal states using three-mode unitary transformations. This means that the channel capacity cannot possibly exceed $\log_2 12 \approx 3.58$ bits. Next, we numerically solved the optimization task in in Eq.(1) for the case of $M = 12$ local $U(3)$ operations performed by Alice. Obtained solutions suggest $C = 3.0$ bits which implies that even if Alice can locally prepare 12 orthogonal states, Bob cannot discriminate then deterministically by means of linear optics and coincidence detection. Indeed, we discovered that Bob can only perform a non-ambiguous detection of 8 orthogonal states encoded by Alice. Therefore, practical channel capacity in the case of $n = 2$ photons in $N = 6$ modes is limited to 3 bits.

Similar analysis for the case of $n = 2$ photons in $N = 8$ modes revealed that the practical channel capacity of four-mode communication is limited to $\log_2 12 \approx 3.58$ bits.

**V. SUMMARY**

We discussed the problem of the communication channel capacity for a linear optical circuit with $N$ modes populated by $n$ photons. We discovered that in the case of $N = 4$, $n = 2$ there is a class of bipartite two-photon entangled states that supports a dense coding protocol with the attainable channel capacity of 2 bits. We studied numerically 6 and 8 mode extensions of this protocol and provided estimates for the channel capacity in those cases.

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only 9 of the coefficients $c_k$ are independent because of the normalization constraint $\sum_{k=1}^{10} |c_k|^2 = 1$ and one of the phases can be set to 0.