ON THE FAMILIES OF HYPERPLANE SECTIONS OF SOME SMOOTH PROJECTIVE VARIETIES

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Abstract. In this note, we give two applications of [5, Theorem 3.1]. We first study the free family $\mathcal{K}$ of hyperplane sections of the smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 3$. We prove that $X$ is determined by the free family $\mathcal{K}$ if $\dim(X) \geq 4$. As an application, we deduce that for $n \geq 4$, the hyperplane section of $X$ varies maximally in the moduli space of the smooth hypersurface of degree $d \geq 3$ in $\mathbb{P}^n$. We then study the free family of hyperplane sections of the smooth projective surface $X$ with Kodaira dimension $\kappa(X) \geq 0$. We prove that $X$ is determined by this free family.

1. Introduction

We work over complex number field $\mathbb{C}$. Unless otherwise stated, we work in the complex-analytic setting.

Let $X$ be a complex manifold and $Y \subset X$ be a compact complex submanifold. For a non-negative integer $l$, we use $(Y/X)_l$ to denote the $l$-th infinitesimal neighborhood of $Y$ in $X$. Denote by Douady($X$) the Douady space of $X$. We refer the reader to [5, Section 1 and Section 2] for the background and definitions. In [5, Question 1.5], a family version of the question on holomorphic embeddings posed by Nirenberg and Spencer is formulated. One of Hwang’s results related to this question is the following theorem (see [5, Theorem 3.1]).

Theorem 1.1. Let $\mathcal{K} \subset$ Douady($X$) be a free family in a complex manifold $X$, a member $A \subset X$ of which satisfies $H^0(A, T_A) = 0$. Then for any free family $\tilde{\mathcal{K}} \subset$ Douady($\tilde{X}$) in a complex manifold $\tilde{X}$, if $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are iso-equivalent up to order 1, then they are germ-equivalent.

Notice that there are a lot of smooth projective varieties $A$ with $H^0(A, T_A) = 0$. For example, it is well known that we have $H^0(A, T_A) = 0$ if $A$ is a smooth projective variety of general type. Thus Theorem 1.1 can be applied to a wide class of submanifolds. Some applications of Theorem 1.1 have been given in [5] (see for instance [5, Theorem 1.8]).

The main aim of this note is to give more applications of Theorem 1.1. The first result of this note is the following theorem.

Theorem 1.2. Let $X \subset \mathbb{P}^{n+1}$ and $\tilde{X} \subset \mathbb{P}^{n+1}$ be two smooth hypersurfaces of degree $d \geq 3$. Suppose that the free families $\mathcal{K}$ and $\tilde{\mathcal{K}}$ of hyperplane sections of $X \subset \mathbb{P}^{n+1}$ and $\tilde{X} \subset \mathbb{P}^{n+1}$ are iso-equivalent up to order 0. Assume that $n \geq 4$. Then $X$ and $\tilde{X}$ are isomorphic by a projective transformation of $\mathbb{P}^{n+1}$.

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Let $n \geq 2$ be a positive integer. Suppose that $X \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d \geq 3$. Denote by $M_{d,n-1}$ the moduli space of smooth hypersurfaces of degree $d$ in $\mathbb{P}^n$. Let $U_0 \subset |\mathcal{O}_{\mathbb{P}^{n+1}}(1)|$ be the Zariski open subset which parametrizes the smooth hyperplane section of $X$. We have the natural morphism:

$$\mu : U_0 \rightarrow M_{d,n-1}$$

$$H \mapsto [X \cap H]$$

One may ask the following interesting question:

**Question 1.3.** Is it possible to determine $\dim \mu(U_0)$? When is $\mu$ a generically finite morphism onto its image?

In [1], Beauville proved that we have $\dim \mu(U_0) \geq 1$. In [2], Cheng proved that $\mu$ is generically finite onto its image if $d > n > 1$ and $(n, d) \neq (2, 3), (3, 4)$.

As an application of Theorem 1.2, we have the following positive result on Question 1.3.

**Corollary 1.4.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 3$. Denote by $M_{d,n-1}$ the moduli space of smooth hypersurfaces of degree $d$ in $\mathbb{P}^n$. Let $U_0 \subset |\mathcal{O}_{\mathbb{P}^{n+1}}(1)|$ be the Zariski open subset which parametrizes the smooth hyperplane section of $X$. Let $\mu : U_0 \rightarrow M_{d,n-1}$ be the natural morphism which sends the hyperplane $H$ to the corresponding hyperplane section $[H \cap X]$. Suppose that $n \geq 4$. Then $\mu$ is a generically finite morphism onto its image.

Combining Corollary 1.4 with [2, Theorem 0.2], one has the following theorem.

**Theorem 1.5.** Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d \geq 3$. Keep the notation as in Corollary 1.4. Suppose that $n \geq 2$ and $(n, d) \neq (2, 3), (3, 3), (3, 4)$. Then $\mu : U_0 \rightarrow M_{d,n-1}$ is a generically finite morphism onto its image.

Our next result is the following theorem.

**Theorem 1.6.** Let $X \subset \mathbb{P}^N$ and $\tilde{X} \subset \mathbb{P}^N$ be two smooth surfaces with Kodaira dimensions $\kappa(X) \geq 0$ and $\kappa(\tilde{X}) \geq 0$. Suppose that the free families $\mathcal{K}$ and $\tilde{\mathcal{K}}$ of hyperplane sections of $X \subset \mathbb{P}^N$ and $\tilde{X} \subset \mathbb{P}^N$ are iso-equivalent up to order $0$. Let $A \subset X$ be a general hyperplane section. Denote by $F_A : A \rightarrow \tilde{A}$ the isomorphism induced by the iso-equivalence, where $A \subset X$ is a general hyperplane section. Suppose furthermore that $F_A^*\mathcal{O}_{\tilde{A}}(1) \cong \mathcal{O}_A(1)$, where $\mathcal{O}_A(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_A$ and $\mathcal{O}_{\tilde{A}}(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_{\tilde{A}}$. Then $X$ and $\tilde{X}$ are isomorphic by a projective transformation of $\mathbb{P}^N$.

**Remark 1.7.** The author would like to thank Professor Jun-Muk Hwang for informing him that Theorem 1.6 was asked by Professor Ciro Ciliberto.

**Remark 1.8.** Notice that when $K_X \cong \mathcal{O}_X$ and $K_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$, we have $\mathcal{O}_A(1) \cong K_A$ and $\mathcal{O}_{\tilde{A}}(1) \cong K_{\tilde{A}}$. Thus the assumption $F_A^*\mathcal{O}_A(1) \cong \mathcal{O}_{\tilde{A}}$ in Theorem 1.6 holds. In particular, this implies that Theorem 1.6 can be applied to study the family of hyperplane sections of $K3$ surfaces and Abelian surfaces.
The family of hyperplane sections of $K3$ surfaces was studied in [5, Theorem 1.8].

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2. **Proof of Theorem 1.2 and Corollary 1.4**

In this section, we will prove Theorem 1.2 and Corollary 1.4. We refer the reader to [7, Section 6.2] for details of Jacobian rings of smooth hypersurfaces in projective spaces.

2.1. **Proof of Theorem 1.2.** First, we prove the following lemma.

**Lemma 2.1.** Let $A \subset X$ be a smooth hyperplane section of the smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 3$. Suppose that $n \geq 4$. Then the Kodaira-Spencer map $H^0(A, N_{A/X}) \rightarrow H^1(A, T_A)$ of hyperplane sections determines the extension class of

$$(*) \quad 0 \rightarrow T_A \rightarrow T_X|_A \rightarrow N_{A/X} \rightarrow 0$$

up to non-zero scalar multiplications.

**Proof.** Let $H \subset \mathbb{P}^{n+1}$ be the hyperplane such that $A = X \cap H$. Denote by $x_0, x_1, \ldots, x_n$ the homogeneous coordinates on $H \cong \mathbb{P}^n$ and by $f$ the defining equation of $A \subset H$. Let $R_f = S/J_f$ be the Jacobian ring of $A$, where $S = \mathbb{C}[x_0, x_1, \ldots, x_n]$. For an integer $i$, denote by $R^i_f \subseteq R_f$ the subspace consists of elements in $R_f$ of degree $i$. Set $N = (d-1)(n+1) - n - 1$.

Denote by $K_A$ the canonical bundle of $A$. Let $e \in H^1(A, T_A \otimes N_A^\vee/X)$ the extension class of $(*)$. Denote by

$$e^* \in \text{Hom}(H^{n-2}(A, K_A \otimes \Omega_A \otimes N_A^\vee/X), H^{n-1}(A, K_A))$$

the Serre dual of $e$. The Kodaira-Spencer map is the boundary homomorphism

$$\partial : H^0(A, N_{A/X}) \rightarrow H^1(A, T_A)$$

which is induced by taking cup product with $e$. Let

$$\partial^* : H^{n-2}(A, K_A \otimes \Omega_A) \rightarrow H^{n-1}(A, K_A \otimes N_A^\vee/X)$$

be the dual of $\partial$. Let

$$\alpha : H^0(A, N_{A/X}) \otimes H^{n-2}(A, K_A \otimes \Omega_A) \rightarrow H^{n-2}(A, K_A \otimes \Omega_A \otimes N_{A/X})$$

be the natural cup product. Denote by

$$\beta : H^0(A, N_{A/X}) \otimes H^{n-1}(A, K_A \otimes N_A^\vee/X) \rightarrow H^{n-1}(A, K_A)$$

the natural cup product. Then the following diagram is commutative up to sign,

$$\begin{array}{ccc}
H^0(N_{A/X}) \otimes H^{n-2}(K_A \otimes \Omega_A) & \xrightarrow{\text{id} \otimes \partial^*} & H^0(N_{A/X}) \otimes H^{n-1}(K_A \otimes N_A^\vee/X) \\
\alpha \downarrow & & \downarrow \beta \\
H^{n-2}(K_A \otimes \Omega_A \otimes N_{A/X}) & \xrightarrow{e^*} & H^{n-1}(A, K_A)
\end{array}$$
Claim. \(\alpha\) is surjective.

Suppose that the Claim holds. Notice that we have \(H^{n-1}(A, K_A) \cong \mathbb{C}\). Then up to nonzero scalar multiplications, the Kodaira-Spencer homomorphism \(\partial\) determines the extension class \(e\) by

\[
\alpha^{-1}(\text{Ker}(\alpha^*)) = \text{Ker}(\beta \circ (\text{Id} \otimes \partial^*)).
\]

To prove the Claim, we first notice that we have \(N_{A/X} \cong \mathcal{O}_A(1)\), where \(\mathcal{O}_A(1) = \mathcal{O}_H(1)|_A\). By Serre duality theorem, the surjectivity of \(\alpha\) is equivalent to the injectivity of

\[
\alpha^* : H^1(A, T_A \otimes \mathcal{O}_A(-1)) \to \text{Hom}(H^0(A, \mathcal{O}_A(1)), H^1(A, T_A)).
\]

Notice that \(\alpha^*\) is induced by the natural cup product

\[
m : H^1(A, T_A \otimes \mathcal{O}_A(-1)) \otimes H^0(A, \mathcal{O}_A(1)) \to H^1(A, T_A).
\]

Consider the normal exact sequence corresponds to \(A \subset H \cong \mathbb{P}^n\):

\[
(\ast\ast) \ 0 \to T_A \to T_{\mathbb{P}^n}|_A \to N_{A/\mathbb{P}^n} \to 0,
\]

where \(N_{A/\mathbb{P}^n} \cong \mathcal{O}_A(d)\). Let \(\lambda \in H^1(A, T_A \otimes \mathcal{O}_A(-1))\) be the extension class of \((\ast\ast)\). Since \(n \geq 4\), by [7, Lemma 6.15], taking cup product with \(\lambda\) induces an isomorphism:

\[
\rho : R^d_{\partial} \to H^1(A, T_A)
\]

By the same argument as in the proof of [7, Lemma 6.15] and our assumption \(n \geq 4\), taking cup product with \(\lambda\) induces an isomorphism:

\[
\rho_{-1} : R^{d-1}_{\partial} \to H^1(A, T_A \otimes \mathcal{O}_A(-1)).
\]

Notice that we have the natural isomorphism \(\theta : R^1_{\partial} \to H^0(A, \mathcal{O}_A(1))\). Denote by \(m_{1,d-1} : R^1_{\partial} \otimes R^{d-1}_{\partial} \to R^d_{\partial}\) the natural multiplication induced by multiplication of polynomial ring. In particular, we have the following commutative diagram.

\[
\begin{array}{c}
R_{\partial}^1 \otimes R_{\partial}^{d-1} \\
\downarrow \theta \otimes \rho_{-1} \\
H^0(\mathcal{O}_A(1)) \otimes H^1(T_A \otimes \mathcal{O}_A(-1)) \\
\downarrow m \\
H^1(A, T_A)
\end{array}
\]

Since \(\theta\), \(\rho_{-1}\) and \(\rho\) are isomorphisms, we conclude that the injectivity of \(\alpha^*\) is equivalent to the injectivity of

\[
\mu : R^{d-1}_{\partial} \to \text{Hom}(R^1_{\partial}, R^d_{\partial}),
\]

where \(\mu\) is induced by \(m_{1,d-1}\). Notice that we have

\[
N = (d - 2)(n + 1)
\geq d + n - 2
\geq d + 2,
\]

where the first inequality follows by \(d \geq 3\) and the last inequality follows by \(n \geq 4\). By [7, Corollary 6.20 (ii)] and \(N \geq d + 2\), \(\mu\) is injective. Thus \(\alpha^*\) is injective. We conclude that \(\alpha\) is surjective. The Claim is proved. \(\Box\)
Proposition 2.2. The free families $\mathcal{K}$ and $\tilde{\mathcal{K}}$ in Theorem 1.2 are iso-
equivalent up to order 1.

Proof. After shrinking $\mathcal{K}$ and $\tilde{\mathcal{K}}$, we may assume that there is a biholomor-
phic map $f : \mathcal{K} \to \tilde{\mathcal{K}}$ such that for each $[A] \in \mathcal{K}$, $[\tilde{A}] = f([A]) \in \tilde{\mathcal{K}}$, we have $A \cong \tilde{A}$.

Since $A$ and $\tilde{A}$ are hypersurfaces of $\mathbb{P}^n$ of degree $d \geq 3$ and $\text{dim}(A) = \text{dim}(\tilde{A}) \geq 3$, the biholomorphic map $A \cong \tilde{A}$ is induced by a projective transformation of $\mathbb{P}^n$. Thus the biholomorphic map induces the isomorphisms

$$T_A \cong T_{\tilde{A}}, \quad O_A(1) \cong O_{\tilde{A}}(1).$$

It is well known (see the remark after Proposition 1.7 in [3]) that the first infinitesimal neighborhood of a submanifold $A \subset X$ is determined by the extension class of

$$0 \to T_A \to T_X|_A \to N_{A/X} \to 0.$$

Notice that we have the natural isomorphisms:

$$N_{A/X} \cong O_A(1), \quad N_{\tilde{A}/\tilde{X}} \cong O_{\tilde{A}}(1).$$

By Lemma 2.1 and our assumption, we deduce that $T_X|_A$ and $T_{\tilde{X}}|_{\tilde{A}}$ are isomorphic (up to a nonzero scalar multiplication) as extensions of $O_A(1) \cong O_{\tilde{A}}(1)$ by $T_A \cong T_{\tilde{A}}$. Thus we have $(A/X)_1 \cong (\tilde{A}/\tilde{X})_1$.

Proof of Theorem 1.2. By our assumption, we always have

$$H^0(A, T_A) = H^0(\tilde{A}, T_{\tilde{A}}) = 0.$$

By Proposition 2.2 and [5, Theorem 3.1], we deduce that $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are germ-equivalent. Then there exist Euclidean open neighborhoods of some hyperplane sections $A \subset U \subset X$ and $\tilde{A} \subset \tilde{U} \subset \tilde{X}$, a biholomorphic map $\Phi : U \to \tilde{U}$ such that $\Phi(A) = \tilde{A}$. By [4, Corollary V.2.3], $\Phi$ can be extended to a birational map $\Psi : X \dasharrow \tilde{X}$. Since $A \subset X$ and $\tilde{A} \subset \tilde{X}$ are hyperplane sections of smooth projective varieties, $\Psi$ is an isomorphism by Zariski main theorem. Since $X \subset \mathbb{P}^{n+1}$ and $\tilde{X} \subset \mathbb{P}^{n+1}$ are smooth hypersurfaces of degree $d \geq 3$ and $\text{dim}(X) = \text{dim}(\tilde{X}) \geq 4$, the isomorphism $\Psi$ is induced by a projective transform of $\mathbb{P}^{n+1}$.

2.2. Proof of Corollary 1.4. Let $\rho : \mathcal{Y}_{U_0} \to U_0$ be the universal family of smooth $(n - 1)$-folds of degree $d$ obtained as hyperplane sections of $X$. For $t \in U_0$, denote by $Y_t$ the corresponding smooth hyperplane section of $X \subset \mathbb{P}^{n+1}$. For any open subset $V \subset U_0$, write

$$\mathcal{Y}_V = \mathcal{Y}_{U_0} \times_{U_0} V.$$

Set $\rho_V = \rho|_{\mathcal{Y}_V} : \mathcal{Y}_V \to V$.

Suppose that $\mu$ is not generically finite onto its image. Then a general fiber $F$ of $\mu : U_0 \to \mu(U_0)$ is of dimension $k \geq 1$. Let $x$ and $y$ be two general points of $F$. Then there are two very small Euclidean open subsets $x \in U_x \subset U_0$, $y \in U_y \subset U_0$ and an isomorphism $f_{x,y} : U_x \to U_y$ satisfying $\mu|_{U_x} \circ f_{x,y} = \mu|_{U_y}$. Notice that we have $H^0(Y_t, T_{Y_t}) = 0$ for any $t \in U_0$. We can choose two Euclidean open subsets $V_x \subset U_x$ and $V_y \subset U_y$ such that:
(1) $f_{x,y}$ induces an isomorphism between $V_x$ and $V_y$. By abuse of notation, we still use $f_{x,y}$ to denote this isomorphism.

(2) there is an isomorphism $F_{x,y} : \mathcal{Y}_V_x \to \mathcal{Y}_V_y$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{Y}_V_x & \xrightarrow{F_{x,y}} & \mathcal{Y}_V_y \\
\rho_{V_x} & & \rho_{V_y} \\
V_x & \xrightarrow{f_{x,y}} & V_y
\end{array}
$$

In particular, $F_{x,y}$ gives an iso-equivalence up to order 0 between two free families $\mathcal{Y}_V_x$ and $\mathcal{Y}_V_y$. By Theorem 1.2, there is an isomorphism $\Phi_{x,y} : X \to X$ such that $\Phi_{x,y}^*(Y_{f_{x,y}(t)}) = Y_t$ for any $t \in V_x$.

For a fixed $x$, we can choose infinitely many $y_i$’s such that $U_{y_i} \cap U_{y_j} = \emptyset$ for any $i \neq j$. By the above arguments, we can find infinitely many automorphisms $\Phi_i = \Phi_{x,y_i}$ such that $\Phi_i^*(Y_{f_{x,y_i}(t)}) = Y_t$ for any $t \in V_x$. Here $V_x$ depends on $i$. By the choise of $U_{y_i}$, we conclude that $\Phi_i \neq \Phi_j$ if $i \neq j$. So the automorphism group of $X$ is an infinite group, which is a contradiction. The proof is completed.

3. PROOF OF THEOREM 1.6

Similarly as in the proof of Theorem 1.2, the key step is to prove the following lemma.

**Lemma 3.1.** Let $A \subset X$ be a smooth hyperplane section of the smooth surface $X \subset \mathbb{P}^N$ with $\kappa(X) \geq 0$. Then the Kodaira-Spencer map $H^0(A, N_{A/X}) \to H^1(A, T_A)$ of hyperplane sections determines the extension class of

$$(*) \quad 0 \to T_A \to T_{X|A} \to N_{A/X} \to 0$$

up to non-zero scalar multiplications.

**Proof.** Denote by $K_A$ the canonical bundle of $A$. Let $e \in H^1(A, T_A \otimes N_{A/X}^\vee)$ the extension class of $(*)$. Denote by $e^* \in \text{Hom}(H^0(A, K_A \otimes 2 \otimes N_{A/X}), H^1(A, K_A))$ the Serre dual of $e$. The Kodaira-Spencer map is the boundary homomorphism

$$\partial : H^0(A, N_{A/X}) \to H^1(A, T_A)$$

which is induced by taking cup product with $e$. Let

$$\partial^* : H^0(A, K_A \otimes 2) \to H^1(A, K_A \otimes N_{A/X}^\vee)$$

be the dual of $\partial$. Let

$$\alpha : H^0(A, N_{A/X}) \otimes H^0(A, K_A \otimes 2) \to H^0(A, K_A \otimes 2 \otimes N_{A/X})$$

be the natural multiplication product. Denote by

$$\beta : H^0(A, N_{A/X}) \otimes H^1(A, K_A \otimes N_{A/X}^\vee) \to H^1(A, K_A)$$
the natural cup product. Then the following diagram is commutative up to sign,

$$
\begin{array}{ccc}
H^0(N_{A/X}) \otimes H^0(K_A^{\otimes 2}) & \xrightarrow{\text{Id} \otimes \partial^*} & H^0(N_{A/X}) \otimes H^1(K_A \otimes N_{A/X}') \\
\alpha & & \beta \\
H^0(K_A^{\otimes 2} \otimes N_{A/X}) & \xrightarrow{e^*} & H^1(A, K_A)
\end{array}
$$

**Claim.** $\alpha$ is surjective.

Suppose that the **Claim** holds. Notice that we have $H^1(A, K_A) \cong \mathbb{C}$. Then up to nonzero scalar multiplications, the Kodaira-Spencer homomorphism $\partial$ determines the extension class $e$ by

$$
\alpha^{-1}(\text{Ker}(e^*)) = \text{Ker}(\beta \circ (\text{Id} \otimes \partial^*)).
$$

To prove the **Claim**, we first notice that we have $N_{A/X} = O_A(A) \cong O_A(1)$, where $O_A(1) = O_F^N(1)|_A$. By adjunction formula on $X$, we have

$$
\deg(K_A) = ((K_X + A) \cdot A) > 0,
$$

where the last inequality follows by $\deg(O_A(A)) = \deg(O_A(1)) > 0$ and $\kappa(X) \geq 0$. So $A$ is a smooth curve of genus $g \geq 2$. Since $O_A(1)$ is very ample, we have $h^0(A, O_A(1)) \geq 3$ and $\deg(O_A(1)) \geq 3$. We conclude that $K_A \otimes O_A(1)$ is also very ample. By [6, Proposition 3.1 (1) (b)], the multiplication maps

$$
H^0(A, K_A) \otimes H^0(A, N_{A/X}) \rightarrow H^0(A, K_A \otimes N_{A/X})
$$

and

$$
H^0(A, K_A) \otimes H^0(A, K_A \otimes N_{A/X}) \rightarrow H^0(A, K_A^{\otimes 2} \otimes N_{A/X})
$$

are surjective. Thus the multiplication map $\alpha$ is surjective. We finish the proof of **Claim.**

**Proof of Theorem 1.6.** By the same argument as in the proof of Theorem 1.2, we can complete the proof of Theorem 1.6. Since the arguments are same, we omit the details.

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