Unified Construction of Normal Bimagic Squares of Doubly Even Orders Based on Quasi Bimagic Pairs

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Abstract. A general magic square of order \( n \) is an \( n \times n \) matrix consisting of integers in such a way that the sum of all elements in each row, each column, main diagonal and back diagonal is the same number called the magic sum of this matrix. A general magic square of order \( 2n \) is normal if its entries are \( 4n^2 \) consecutive odd integers \( 1-4n^2, 3-4n^2, \ldots, 4n^2-3, 4n^2-1 \). A normal bimagic square of order \( 4n \) is a normal magic square such that the sum of squares of all elements in each row, each column, main diagonal and back diagonal is the same number. Using the reflection matrix \( R \) and a quasi bimagic pair \( (A,B) \) where \( A \) and \( B \) are two special \( 2n \times 2n \) matrices consisting of odd integers, we give a unified and very simple construction of normal bimagic square \( H \) of order \( 4n \) for all \( n \geq 2 \): \( H = \ldots \). We construct a quasi bimagic pair by means of orthogonal diagonal latin squares for \( n \neq 3 \) and by means of the computer seeking for \( n = 3 \). AMS subject classifications: 05C50, 15B35, 05C05, 15A03, 15A18.

1. Introduction
A general magic square of order \( n \) is an \( n \times n \) matrix consisting of integers in such a way that the sum of all elements in each row, each column, main diagonal and back diagonal is the same number called the magic sum of this matrix. A general magic square of order \( n \) is normal, denoted by \( MS(n) \), if its entries are \( n^2 \) consecutive odd integers (consecutive integers) and its magic sum is zero. Usually, the \( (i,j) \) entry of a matrix \( M \) is denoted by \( m_{ij} \); rows are enumerated from the top, beginning with 0 and ending with \( n-1 \); columns are enumerated in the same way from left to right. For the constructions of normal magic squares, we refer the interested readers to references [2−5, 7, 9]. As examples, we give Dürer magic square \( M_4 \) and its variant \( M_4' \) consisting of the consecutive odd integers \(-15, -13, \ldots, 13, 15\) as follows:

\[
M_4 = \begin{bmatrix}
16 & 3 & 2 & 13 \\
5 & 10 & 11 & 8 \\
9 & 6 & 7 & 12 \\
4 & 15 & 14 & 1
\end{bmatrix}, \quad M_4' = \frac{1}{4} \begin{bmatrix}
15 & -11 & -13 & 9 \\
-7 & 3 & 5 & -1 \\
1 & -5 & -7 & 3 \\
9 & 13 & 11 & -15
\end{bmatrix} + \begin{bmatrix}
3 & -3 & -3 & 3 \\
-1 & 1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
-3 & 3 & 3 & -3
\end{bmatrix}.
\]

Let \( J \) denote the unity matrix with all elements 1, and \( R \) denote the reflection matrix (or counteridentity matrix) with 1’s on the back diagonal and all other elements 0. Clearly, \( M_4 = 2M_4 - 17J \) and \( M_4' = \frac{1}{2}(M_4 + 17J) \). Let \( A = \begin{smallbmatrix}
3 & -3 \\
-1 & 3
\end{smallbmatrix} \), \( B = \begin{smallbmatrix}
3 & 1 \\
1 & 3
\end{smallbmatrix} \), then
Let $n \in \mathbb{N}$ be the absolute value of $\left[1,\frac{n}{2}\right]$ and it was shown by Lucas [11] that there is no MS(3,2) and no MS(4,2). Quite recently, Chen and Li [5] introduced a magic pair of orthogonal general bimagic squares to provide a new construction of bimagic squares of doubly even orders is the following.

**Theorem 1.1.** ([3, 5, 11]) There exists an MS(4n,2) for all $n \geq 2$, there is no MS(4, 2).

For $n \neq 1, 3$, using idempotent self-orthogonal latin squares and magic rectangles (see [5] for the definition), Chen and Li [5] obtained a magic pair of orthogonal general MS(2n, 2) by doing some column row permutations. Combining the existence of MS(12, 2) [3] and the non-existence of MS(4, 2) [11], they proven Theorem 1.1. Obviously, since there is no a pair of latin squares of order 6, the above proof is not unified.

A zero sum row (column) general magic square of order $n$ is an $n \times n$ matrix consisting of integers in such a way that the sum of all elements in each row (column) is zero. Clearly, using the special $2 \times 2$ zero sum row and column general magic squares, we can construct some magic squares of order 4.

The rest of this paper is arranged as follows. In Section 2, we will take advantage of quasi bimagic pairs and the above method constructed $M_{\beta}$ and described as (1.1) to provide a construction of MS(4n, 2). In Section 3, by means of the construction of quasi bimagic pairs to complete a unified proof of Theorem 1.1 and provide two examples.

### 2. Construction of an MS(4n, 2)

Let $n$ be a positive integer and $In = \{0, 1, \ldots, n-1\}$. Let $[1-4n, 4n-1]^{2} = \{1-4n+2x : x \in I_{4n}\}, [1, 4n-1]^{2} = \{1+2x : x \in I_{4n}\}$ and $S_{n} = \Sigma x \in I_{2n} (2x+1)^{2}$. Given an integer $x$ and a matrix $A = (a_{i,j})$, let $|x|$ be the absolute value of $x$ and $|A| = (|a_{i,j}|$).

A $2n \times 2n$ matrix $A$ with entries in the set $[1-4n, 4n-1]$ is an absolute balanced square if each element of $[1, 4n-1]^{2}$ appears $2n$ times in $|A|$. Two absolute balanced squares $A$ and $B$ of order $2n$ are absolute orthogonal if $f(|a_{i,j}|, |b_{i,j}|) = i, j \in I_{2n}, f = [1-4n, 4n-1]^{2} \times [1, 4n-1]^{2}$.

A row (column) quasi bimagic square of order $2n$, denoted by RQMS(2n, 2) (CQMS(2n, 2)), is an absolute balanced square and a zero sum row (column) general magic square of order $2n$ having the property that the sum of all elements in each row, each column and main diagonal is $S_{n}$. We call $(A,B)$ a quasi bimagic pair, denoted QBMP(2n), if the sum of all elements in main diagonal of $A \ast B$ is zero, $A$ is a RQMS(2n, 2), $B$ is a CQMS(2n, 2), and $A$ and $B$ are absolute orthogonal.

**Construction 2.1.** Let $(A,B)$ be a QBMP(2n) with $n \geq 2$. Let $C = (\frac{A}{A \ast B})$, $D = (\frac{D}{D \ast B})$, and $H = 4nC + D$. Then $H$ is an MS(4n, 2).

**Proof.** Write $A = (a_{i,j}), B = (b_{i,j}), C = (c_{i,j}), D = (d_{i,j}), H = (h_{i,j}).$ For $k \in I_{4n}$ let $k^{*} = 4n - 1 - k$.

By the definition, we have

$$
\begin{align*}
&c_{i,j}^{*} = a_{i,j}, d_{i,j}^{*} = a_{i,j},
&c_{i,j}^{*} = a_{i,j}, d_{i,j}^{*} = a_{i,j},
&h_{i,j}^{*} = b_{i,j}, -b_{i,j},
&h_{i,j}^{*} = b_{i,j}, -b_{i,j},
\end{align*}
$$

(1) First, we prove $H$ is normal. Let $\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \in \{1, -1\}$ and $i, j, k, l \in I_{2n}$, then the entries of $H$ have the following form: $4n \alpha_{1}a_{i,j} + \beta_{1}b_{i,j} = 4n \alpha_{2}a_{k,l} + \beta_{2}b_{k,l}$. Suppose that $i, j, k, l \not\equiv (k, l)$.

We have
\[ \alpha_i a_{ij} = \alpha_j a_{ki} , \ \ \beta_i b_{ij} = \beta_j b_{kl} , \ (i,j) \neq (k,l) . \]

We obtain

\[ | a_{ij} | = | a_{ki} | , \ | b_{ij} | = | b_{kl} | , \ (i,j) \neq (k,l). \]

Since \( A \) and \( B \) are absolute orthogonal, we get \((i, j) = (k, l)\). This contradiction indicates the entries of \( H \) are distinct. Since \( a_{ij} , b_{ij} \in [1 - 4n; 4n - 1] \), we have

\[ 1 - (4n)^2 = 4n(1 - 4n) + (1 - 4n) \leq 4n \alpha_i a_{ij} + \beta_j b_{ij} \leq 4n(4n - 1) + (4n - 1) = (4n)^2 - 1 , \]

which indicates \( \{ h_{u,v} : u, v \in I_{4n} \} = [1 - (4n)^2; (4n)^2 - 1] \). Hence \( H \) is normal.

(2) Next, we show that \( H \) is a normal magic square. Since \( A \) and \( B \) are zero sum row and column general magic squares respectively, we see that \( C \) and \( D \) are general magic squares, hence \( H \) is a normal magic square.

(3) Finally, we show that \( H^2 \) is a general magic square. By the definition of \( A \) and \( B \), we have

\[ \sum_{k=0}^{2n-1} a_{ik}^2 = \sum_{k=0}^{2n-1} a_{kj}^2 = S_n , \quad i,j \in I_{2n} , \]

\[ \sum_{k=0}^{2n-1} b_{ik}^2 = \sum_{k=0}^{2n-1} b_{kj}^2 = S_n , \quad i,j \in I_{2n} , \]

\[ \sum_{k=0}^{2n-1} a_{ik}b_{kj} = 0 \quad (2.1) \]

(3.1) Rows. For \( i \in I_{2n} \), by (2.1) we obtain

\[ \sum_{u=0}^{4n-1} h_{u,i}^2 = \sum_{j=0}^{2n-1} ((4na_{i,j} + b_{i,j})^2 + (4na_{i,j} - b_{i,j})^2) = \sum_{j=0}^{2n-1} ((16n^2 a_{i,j}^2 + 8na_{i,j}b_{i,j} + b_{i,j}^2) + (16n^2 a_{i,j}^2 - 8na_{i,j}b_{i,j} + b_{i,j}^2)) = \sum_{j=0}^{2n-1} (32n^2 a_{i,j}^2 + 2b_{i,j}^2) = 32n^2 \sum_{j=0}^{2n-1} a_{i,j}^2 + 2 \sum_{j=0}^{2n-1} b_{i,j}^2 = 2(16n^2 + 1)S_n . \]

Similarly, we have \( \sum_{u=0}^{4n-1} h_{v,i}^2 \), \( v = 2(16n^2 + 1)S_n \).

(3.2) Columns. Similarly, for \( j \in I_{2n} \), by (2.1) we get

\[ \sum_{u=0}^{4n-1} h_{u,j}^2 = \sum_{i=0}^{2n-1} ((4na_{i,j} + b_{i,j})^2 + (4na_{i,j} - b_{i,j})^2) = 32n^2 \sum_{i=0}^{2n-1} a_{i,j}^2 + 16n \sum_{i=0}^{2n-1} a_{i,j}b_{i,j} + 2 \sum_{i=0}^{2n-1} b_{i,j}^2 = 2(16n^2 + 1)S_n . \]

And

\[ \sum_{u=0}^{4n-1} h_{u,n}^2 = \sum_{i=0}^{2n-1} ((4na_{i,t} - b_{i,t})^2 + (4na_{i,t} + b_{i,t})^2) = 32n^2 \sum_{i=0}^{2n-1} a_{i,t}^2 - 16n \sum_{i=0}^{2n-1} a_{i,t}b_{i,t} + 2 \sum_{i=0}^{2n-1} b_{i,t}^2 = 2(16n^2 + 1)S_n . \]

The above results show that the conclusion holds.

3. Unified Proof of Theorem 2.1

By Construction 2.1, to obtain an MS(4n,2), it suffices to find a quasi bimagic pair QBMP(2n). In this section, for convenience we shall construct a QBMP(2n) by means of orthogonal diagonal latin squares for \( n \neq 3 \) and a QBMP(6) by means of the computer seeking.

A diagonal latin square of order \( n \) over an \( n \)-set \( T \), denoted by DLS(n), is an \( n \times n \) array such that the
set of elements in each row, each column, main diagonal and back diagonal is the set $T$. Two $n \times n$

diagonal latin square $A$ over an $n$-set $T_1$ and $B$ over an $n$-set $T_2$ are orthogonal if $(a_{ij} , b_{ij}) : i, j \in I_n$ $= T_1 \times T_2$. The following can be found in [1].

Lemma 3.1. There exists a pair of orthogonal DLS($n$) if and only if $n \neq 2, 3, 6$.

Let $W$ be a $2n$-subset of $[1-4n, 4n-1]$. The $2n$-subset $W$ is a bimagic subset if $\sum_{y \in W} y^2 = 0$, $\sum_{x \in W} x = S_n$ and $W \cup \{-x : x \in W\} = [1-4n, 4n-1]$.

Lemma 3.2. There exists a bimagic subset of $[1-4n, 4n-1]$, where $n$ is an integer not less than 2.

Proof. Let

$$W = \begin{cases}
\bigcup_{u=1}^{n} \{8x + 1, -8x - 3, -8x - 5, 8x + 7\}, & n = 2u, \quad u \geq 1, \\
\{1, 3, 5, -7, 9, -11\}, & n = 2u + 1, \quad u = 1, \\
\{1, 3, 5, -7, 9, -11\}\{1 + 2x, 3 - 8x, 13 - 8x, 15 - 8x, 17, 8x + 19\}, & n = 2u + 1, u \geq 2.
\end{cases}$$

Clearly, $W \cup \{-x : x \in W\} = [1-4n, 4n-1]$ and $\sum_{y \in W} y^2 = 0$. Since $\{y^2 : y \in W\} = \{(2x + 1)^2 : x \in I_{2n}\}$, we have $\sum_{y \in W} y^2 = S_n$.

Lemma 3.3. There exists a quasi bimagic pair QBMP(2n) for all positive integers $n$ with $n \geq 2$.

Proof. By Lemma 3.2, there exists a bimagic subset $W$ of $[1-4n, 4n-1]$. By Lemma 3.1, for $n \geq 2$ and $n \neq 3$ there exists a pair of orthogonal diagonal latin squares over $W$ of order $2n$, $A$ and $F$. Without loss of generality, we can suppose that $f_{2n-1-i, 2n-1-i} = a_{i,i}$ for $i \in I_{2n}$. Define $B = (b_{i,j})(i, j \in I_{2n})$ by

$$b_{i,j} = \begin{cases}
f_{i,j}, & 0 \leq j \leq n - 1, \\
-f_{i,j}, & n \leq j \leq 2n - 1.
\end{cases}$$

By the definition, the matrices $A, A^* \text{ and } B^*$ are all general magic squares with magic sum 0, $S_n$ and $S_n$ respectively. Since $F$ is a general magic square with magic sum 0, so is $B$. Since $A$ and $F$ is orthogonal, $A$ and $B$ is absolute orthogonal. Now we show that the sum of all elements in main diagonal of $A^* \text{ and } B^*$ is zero. Note that $a_{2n-1-i, 2n-1-i} = f_{i,i}^2$ and $f_{2n-1-i, 2n-1-i} = a_{i,i}$, we have

$$\sum_{i=0}^{2n-1} a_{i,i} b_{i,i} = \sum_{i=0}^{n-1} (a_{i,i} - a_{2n-1-i, 2n-1-i}) = \sum_{i=0}^{n-1} (a_{i,i} - f_{i,i}^2) = 0.$$

The above shows that $(A, B)$ is a quasi bimagic pair. It follows that the conclusion holds for all $n \geq 2$ and $n \neq 3$. When $n = 3$, using the computer seeking, we can get

$$A = \begin{pmatrix}
-11 & 1 & 2 & 3 & -7 & 5 \\
1 & 9 & -7 & -11 & 3 & 3 \\
9 & 1 & -7 & 5 & -11 & 3 \\
-7 & 9 & 3 & 5 & 1 & -11 \\
5 & 1 & -7 & 9 & -3 & -11 \\
3 & 11 & -7 & 9 & 5 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 9 & 1 & 11 & -1 & -9 \\
5 & 3 & -11 & -9 & -5 & -5 \\
9 & -7 & 5 & -3 & 11 & -1 \\
-7 & -11 & 7 & -3 & -3 & -3 \\
-11 & 1 & 3 & -5 & -9 & 7 \\
3 & 5 & 9 & -1 & 7 & 11
\end{pmatrix}. \quad (3.1)$$

It is readily checked that $(A, B)$ is a quasi bimagic pair QBMP(6), which shows that the conclusion holds for $n = 3$. This complete the proof.

We are now in a position to give the unified proof of Theorem 1.1. For $n = 1$, by [11], there is no MS(4, 2). For $n \geq 2$, by Lemma 3.3 there exists a quasi bimagic pair QBMP(2n) and by Construction 2.1, there is an MS(4n, 2).

To illustrate the above constructions, we provide two examples below.

Example 3.4. By Construction 2.1 and $A$ and $B$ defined by (3.1), we can construct $C$ and $D$ as follows:
One can readily check that $H$ is an MS(12,2).

Example 3.5. Let $W = \{1, -3, -5, 7\}$, then $W$ is a bimagic subset of $[-7, 7]$. Let

\[
A = \begin{pmatrix}
1 & -3 & -5 & 7 \\
7 & -5 & -3 & 1 \\
-3 & 1 & 7 & -5 \\
-5 & 7 & -1 & -3
\end{pmatrix}, \quad
F = \begin{pmatrix}
-3 & -5 & 1 & 7 \\
1 & 7 & -3 & -5 \\
7 & 1 & -5 & -3 \\
-5 & -3 & 7 & 1
\end{pmatrix}, \quad
B = \begin{pmatrix}
-3 & -5 & 1 & -7 \\
1 & 7 & 3 & 5 \\
7 & 1 & 5 & 3 \\
-5 & -3 & 7 & 1
\end{pmatrix}.
\]
Let

\[
C = \begin{pmatrix}
A & AR \\
-RA & -RAR
\end{pmatrix} =
\begin{pmatrix}
1 & -3 & -5 & 7 & 7 & -5 & -3 & 1 \\
7 & -5 & -3 & 1 & 1 & -3 & -5 & 7 \\
-3 & 1 & 7 & -5 & -5 & 7 & 1 & -3 \\
-5 & 7 & 1 & -3 & -3 & 1 & 7 & -5 \\
5 & -7 & -1 & 3 & 3 & -1 & -7 & 5 \\
3 & -1 & -7 & 5 & 5 & -7 & 1 & 3 \\
-7 & 5 & 3 & -1 & -1 & 3 & 5 & -7 \\
-1 & 3 & 5 & -7 & -7 & 5 & 3 & -1
\end{pmatrix}
\]

and

\[
D = \begin{pmatrix}
B & -BR \\
RB & -BRR
\end{pmatrix} =
\begin{pmatrix}
-3 & -5 & -1 & -7 & 7 & 1 & 5 & 3 \\
1 & 7 & 3 & 5 & -5 & -3 & -7 & -1 \\
7 & 1 & 5 & 3 & -3 & -5 & -1 & -7 \\
-5 & -3 & -7 & -1 & 1 & 7 & 3 & 5 \\
-5 & -3 & -7 & -1 & 1 & 7 & 3 & 5 \\
7 & 1 & 5 & 3 & -3 & -5 & -1 & -7 \\
1 & -3 & 5 & -5 & -1 & -7 & 7 & 1 & 5 & 3
\end{pmatrix}
\]

Let

\[
H = 8C + D =
\begin{pmatrix}
5 & -29 & -41 & 49 & 63 & -39 & -19 & 11 \\
-57 & -33 & -21 & 13 & 3 & -27 & -47 & 55 \\
-17 & 9 & 61 & -37 & -43 & 51 & 7 & 31 \\
-45 & 53 & 1 & -25 & -23 & -15 & 59 & -35 \\
35 & -59 & -15 & 23 & 05 & -1 & -53 & 45 \\
31 & -7 & -51 & 43 & 37 & -61 & -9 & 17 \\
-55 & 47 & 27 & -3 & -13 & 21 & 33 & -57 \\
-11 & 19 & 39 & -63 & -49 & 41 & 29 & -5
\end{pmatrix}
\]

One can readily check that H is an MS(8,2).

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5. References
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