NONLINEAR FRACTIONAL DAMPED WAVE EQUATION ON COMPACT LIE GROUPS

APARAJITA DASGUPTA, VISHVESH KUMAR, AND SHYAM SWARUP MONDAL

Abstract. In this paper, we deal with the initial value fractional damped wave equation on $G$, a compact Lie group, with power-type nonlinearity. The aim of this manuscript is twofold. First, using the Fourier analysis on compact Lie groups, we prove a local in-time existence result in the energy space for the fractional damped wave equation on $G$. Moreover, a finite time blow-up result is established under certain conditions on the initial data. In the next part of the paper, we consider fractional wave equation with lower order terms, that is, damping and mass with the same power type nonlinearity on compact Lie groups, and prove the global in-time existence of small data solutions in the energy evolution space.

Contents

1. Introduction
1.1. Main results
2. Preliminaries: Analysis on compact Lie groups
2.1. Notations
2.2. Representation theory on compact Lie groups
2.3. Fourier analysis on compact Lie groups
3. A local existence result
3.1. Fourier multiplier expressions and $L^2(G) - L^2(G)$ estimates
3.2. Local in time existence
3.3. Blow-up result
4. A global existence result
4.1. Fourier multiplier expressions and $L^2(G) - L^2(G)$ estimates
4.2. Global in time existence
5. Final remarks
6. Data availability statement
References

1. INTRODUCTION

The study of partial differential equations is indeed one of the fundamental tools for understanding and modeling natural and real-world phenomena. Fractional differential operators are nonlocal operators that are considered as a generalization of classical differential operators of arbitrary non-integer orders. For the last few decades, the study of partial
differential equations involving nonlocal operators have gained a considerable amount of interest and have become one of the essential topics in mathematics and its applications. Many physical phenomena in engineering, quantum field theory, astrophysics, biology, materials, control theory, and other sciences can be successfully described by models utilizing mathematical tools from fractional calculus [37, 42, 26, 30, 17]. In particular, the fractional Laplacian is represented as the infinitesimal generator of stable radially symmetric Lévy processes [3]. For other exciting models related to fractional differential equations, we refer to the reader [13, 18, 22, 25] to mention only a few of many recent publications.

In recent years, due to the nonlocal nature of the fractional derivatives, considerable attention has been devoted to various models involving fractional Laplacian and nonlocal operators by several researchers. There is a vast literature available involving the fractional Laplacian on the Euclidean framework, which is difficult to mention; we refer to important papers [4, 6, 7, 8, 11, 25, 21, 38] and the references therein. Here we would like to point out that the fractional Laplacian operator \((-\Delta)^\alpha\) can be reduced to the classical Laplace operator \(-\Delta\) as \(\alpha \to 1\). We refer to [25] for more details. In particular, many interesting results in some classical elliptic problems have been extended in the fractional Laplacian setting, see [10].

For the classical semilinear damped wave equation in \(\mathbb{R}^n\), the global existence or a blow-up result depending on the critical exponent has been studied in [19, 23, 43, 41]. We refer to the excellent book [14] for global in-time small data solutions for the semilinear damped wave equation on the Euclidean framework.

The study of the semilinear damped wave equation has also been extended in the non-Euclidean framework. Several papers have studied linear PDE in non-Euclidean structures in the last decades. For example, the semilinear wave equation with or without damping has been investigated for the Heisenberg group [24, 31]. In the case of graded groups, we refer to the recent works [32, 36, 40]. Concerning the damped wave equation on compact Lie groups, we refer to [27, 29, 28, 16, 5]. Particularly, the author in [27] studied semilinear damped wave equation with power type nonlinearity \(|u|^p\) on compact Lie groups and proved a local in-time existence result in the energy space via Fourier analysis on compact Lie groups. He also derived a blow-up result for the semilinear Cauchy problem for any \(p > 1\). Also, considering the semilinear wave equation with damping and mass with power nonlinearity \(|u|^p\) on compact Lie groups and without any lower bounds for \(p > 1\), the author proved the global in time existence of small data solutions in the evolution energy space in [29]. For the study of semilinear wave equation of general compact manifolds, we refer to the seminal works [9, 20] where the global in-time solution were investigated by establishing famous Strichartz type estimates. Recently, the wave equation were also explored in the noncompact manifolds setup, see [2, 44, 45, 39] and reference therein.

Then, an interesting and viable problem is to study the fractional wave equation (1.1) and (1.2) of order \(\alpha\) with \(0 < \alpha < 1\), with power-type nonlinearity. In [1], the authors have investigated the nonexistence of global weak solutions to the nonlinear fractional wave equation with power type nonlinearity on the Heisenberg group. In the setting for compact Lie groups, we have recently started a systematic study of the nonlinear fractional wave equation on compact Lie groups. This work is a continuation of our previous work [12]. To state our problem, let \(G\) be a compact Lie group with normalized Haar measure \(dx\) and let \(L\) be the Laplace-Beltrami operator on \(G\) (which also coincides with the Casimir element of the universal enveloping algebra of Lie algebra of \(G\)). For \(0 < \alpha < 1\), we consider the following two Cauchy problems for the fractional wave equation with power type nonlinearity, namely, with damping term,

\[
\begin{align*}
\partial^2_t u + (-L)^\alpha u + \partial_t u &= |u|^p, \quad x \in G, t > 0, \\
\sigma(0,x) &= \varepsilon u_0(x), \quad x \in G, \\
\sigma_t(x,0) &= \varepsilon u_1(x), \quad x \in G,
\end{align*}
\] (1.1)
Then, bounded for compact Lie group we have the following local existence result.

\[ \frac{\partial^2 u + (-\mathcal{L})^\alpha u + b \partial_t u + m^2 u = |u|^p}{u(0, x) = u_0(x), \quad x \in G,} \quad x \in G, \]

where \( p > 1, b, m^2 \) are positive constants and \( \varepsilon \) is a positive constant describing the smallness of the Cauchy data. Here for the moment, we assume that \( u_0 \) and \( u_1 \) are taken from the energy space \( H^2_0(G) \) (see (1.3) for the definition) and \( L^2(G) \), respectively.

This paper investigates a finite time blow-up result for solutions to the fractional damped wave equation involving the Laplace-Beltrami operator on compact Lie groups under a suitable sign assumption for the initial data. Moreover, we show that the presence of a positive damping term and a positive mass term in the Cauchy problem completely reverses the scenario, i.e., we prove the global existence of small data solutions for the fractional wave equation with damping and mass. More precisely, using the Gagliardo-Nirenberg type inequality (in order to handle power nonlinearity in \( L^2(G) \)) and Fourier analysis on compact Lie groups, we prove the local well-posedness of the Cauchy problem (1.1) in the energy evolution space \( C([0, T], H^2_0(G)) \cap C^1([0, T], L^2(G)) \) and the global in time existence of small data solutions for the Cauchy problem (1.2).

1.1. Main results. We denote \( L^q(G) \), \( 1 \leq q < \infty \), the space of \( q \)-integrable functions on the compact Lie group \( G \) concerning the normalized Haar measure \( dx \) on \( G \) and essentially bounded for \( q = \infty \) throughout the paper. For \( s > 0 \) and \( q \in (1, \infty) \), the fractional Sobolev space \( H^{s,q}_\alpha(G) \) of order \( \alpha \) is defined as

\[ H^{s,q}_\alpha(G) = \left\{ f \in L^q(G) : (\mathcal{L})^{s/2} f \in L^q(G) \right\}, \tag{1.3} \]

endowed with the norm \( \| f \|_{H^{s,q}_\alpha(G)} = \| f \|_{L^q(G)} + \| (\mathcal{L})^{s/2} f \|_{L^q(G)} \). We simply denote the Hilbert space \( H^{2,2}_\alpha(G) \) by \( H^2_\alpha(G) \).

By employing noncommutative Fourier analysis for compact Lie groups, our first result concerning \( L^2 \)-decay estimates for the solution of the linear version of the Cauchy problem (1.1) (when \( f = 0 \)) is stated in the following proposition.

\textbf{Proposition 1.1.} Let \( 0 < \alpha < 1 \). Suppose that \( (u_0, u_1) \in H^2_\alpha(G) \times L^2(G) \) and \( u \in C([0, \infty), H^2_\alpha(G)) \cap C^1([0, \infty), L^2(G)) \) be the solution to the homogeneous Cauchy problem

\[ \begin{align*}
\frac{\partial^2 u + (-\mathcal{L})^\alpha u + \partial_t u}{u(0, x) = u_0(x),} \quad x \in G, \quad t > 0, \\
\partial_t u(x, 0) = u_1(x), \quad x \in G.
\end{align*} \tag{1.4} \]

Then, \( u \) satisfies the following \( L^2(G) \) – \( L^2(G) \) estimates

\[ \| u(t, \cdot) \|_{L^2(G)} \lesssim \left( \| u_0 \|_{L^2(G)}^2 + t \| u_1 \|_{L^2(G)}^2 \right), \tag{1.5} \]

\[ \left\| (-\mathcal{L})^{\alpha/2} u(t, \cdot) \right\|_{L^2(G)} \lesssim (1 + t)^{-\frac{\alpha}{2}} \left( \| u_0 \|_{H^2_\alpha(G)}^2 + \| u_1 \|_{L^2(G)}^2 \right), \]

\[ \| \partial_t u(t, \cdot) \|_{L^2(G)} \lesssim (1 + t)^{-1} \left( \| u_0 \|_{H^2_\alpha(G)}^2 + \| u_1 \|_{L^2(G)}^2 \right). \]

for any \( t \geq 0 \).

Next we prove the local well-posedness of the Cauchy problem (1.1) in the energy evolution space \( C([0, T], H^2_\alpha(G)) \cap C^1([0, T], L^2(G)) \). In this case, a Gagliardo-Nirenberg type inequality (proved in [35]) will be used to estimate the power nonlinearity in \( L^2(G) \). Indeed, we have the following local existence result.
Theorem 1.2. Let $0 < \alpha < 1$ and let $G$ be a compact connected Lie group with the topological dimension $n$. Assume that $n \geq 2[\alpha]+2$. Suppose that $(u_0, u_1) \in H^G_\infty(G) \times L^2(G)$ and $p > 1$ such that $p \leq \frac{n}{n-2\alpha}$. Then there exists $T = T(\varepsilon) > 0$ such that the Cauchy problem (1.1) admits a uniquely determined mild solution

$$u \in C([0,T], H^G_\infty(G)) \cap C^1([0,T], L^2(G)).$$

Remark 1.3. Note that the restriction $p \leq \frac{n}{n-2\alpha}$ and $n \geq 2[\alpha]+2$ in the above theorem is necessary in order to apply Gagliardo-Nirenberg type inequality.

Our next result is about the non-existence of global in-time solutions to (1.1) for any $p > 1$ regardless of the size of initial data. Before stating the blow-up result, we first introduce a suitable notion of energy solutions for the Cauchy problem (1.1).

Definition 1.4. Let $0 < \alpha < 1$ and $(u_0, u_1) \in H^G_\infty(G) \times L^2(G)$. For any $T > 0$, we say that

$$u \in C \left( [0,T), H^G_\infty(G) \right) \cap C^1 \left( [0,T), L^2(G) \right) \cap L^p_{\text{loc}} \left( [0,T) \times G \right)$$

is an energy solution on $[0,T]$ to (1.1) if $u$ satisfies the following integral relation:

$$\int_G \partial_t u(t,x) \phi(t,x) dx - \int_G u(t,x) \partial_s \phi(t,x) dx + \int_G u(t,x) \phi(t,x) dx \ni$$

+ $\varepsilon \int_G u_0(x) \partial_s \phi(0,x) dx - \varepsilon \int_G u_1(x) \phi(0,x) dx - \varepsilon \int_G u_0(x) \phi(0,x) dx$

$$+ \int_0^t \int_G u(s,x) \left( \partial_s^2 \phi(s,x) + (-\mathcal{L})^\alpha \phi(s,x) + \partial_t \phi(s,x) \right) dx ds = \int_0^t \int_G |u(s,x)|^p \phi(s,x) dx ds$$

for any $\phi \in C^\infty_0([0,T) \times G)$ and any $t \in (0,T)$.

Theorem 1.5. Let $0 < \alpha < 1$, $p > 1$, and let $(u_0, u_1) \in H^G_\infty(G) \times L^2(G)$ be nonnegative and nontrivial functions. Suppose

$$u \in C \left( [0,T), H^G_\infty(G) \right) \cap C^1 \left( [0,T), L^2(G) \right) \cap L^p_{\text{loc}} \left( [0,T) \times G \right)$$

be an energy solution to the Cauchy problem (1.1) with lifespan $T = T(\varepsilon)$. Then there exists a constant $c_0 = c_0(u_0, u_1, p) > 0$ such that for any $\varepsilon \in (0, c_0)$, the energy solution $u$ blows up in finite time. Furthermore, the lifespan $T$ satisfies the following estimates

$$T(\varepsilon) \leq C\varepsilon^{-1-p}.$$  

(1.7)

Remark 1.6. (i) Here we note that the fractional Laplace-Beltrami operator $(-\mathcal{L})^\alpha$ gives the classical Laplace-Beltrami operator $-\mathcal{L}$ as $\alpha \to 1$ and all our results coincides with the results proved for the Cauchy problem for the semilinear damped wave equation on compact Lie groups in [27].

(ii) From Theorem 1.5 one can see that the sharp lifespan estimates for local in-time solutions to (1.1) is independent of $\alpha, 0 < \alpha < 1$. Thus, for any $0 < \alpha < 1$, the lifespan estimates for solutions to the Cauchy problem for the fractional wave equation (1.1) will be the same as the sharp lifespan estimates for the semilinear wave equation on compact Lie group $G$ proved in [27].

In the next part of the paper, we study the global existence of small data solutions for the nonlinear fractional wave equation with damping and mass and involving power type nonlinearity. More precisely, we consider the Cauchy problem (1.2), i.e.,

$$\begin{cases}
\partial_t^2 u + (-\mathcal{L})^\alpha u + bu + m^2 u = |u|^p, & x \in G, t > 0, \\
u(0,x) = u_0(x), & x \in G, \\
\partial_t u(x,0) = u_1(x), & x \in G,
\end{cases}$$

where $p > 1$, $b$, $m^2$ are positive constants, $u_0(x)$ and $u_1(x)$ are two given functions on $G$.  

4
First, we prove the following \( L^2 \)-decay estimates with exponential decay rates related to the time variable for the solution of the homogeneous Cauchy problem (1.2) (when \( f = 0 \)).

**Proposition 1.7.** Let \( 0 < \alpha < 1 \). Suppose that \((u_0, u_1) \in H^\alpha_2(G) \times L^2(G) \) and \( u \in \mathcal{C}([0, \infty), H^\alpha_2(G)) \cap \mathcal{C}^1([0, \infty), L^2(G)) \) be the solution to the homogeneous Cauchy problem

\[
\begin{align*}
\partial_t^\alpha u + (\mathcal{L})^\alpha u + b \partial_t u + m^2 u &= 0, \quad x \in G, \quad t > 0, \\
\partial_t u(x, 0) &= u_1(x), \quad x \in G.
\end{align*}
\]

Then, \( u \) satisfies the following \( L^2(G) - L^2(G) \) estimates

\[
\|u(t, \cdot)\|_{L^2(G)} \lesssim CA_{b,m^2}(t)(\|u_0\|_{L^2(G)} + t\|u_1\|_{L^2(G)}),
\]

\[
\left\|(-\mathcal{L})^{\alpha/2}u(t, \cdot)\right\|_{L^2(G)} \lesssim CA_{b,m^2}(t)(\|u_0\|_{H^\alpha_2(G)}^2 + \|u_1\|_{L^2(G)}^2),
\]

\[
\left\|\partial_t u(t, \cdot)\right\|_{L^2(G)} \lesssim CA_{b,m^2}(t)(\|u_0\|_{H^\alpha_2(G)}^2 + \|u_1\|_{L^2(G)}^2).
\]

for any \( t \geq 0 \), where \( C \) is a positive multiplicative constant and the decay function \( A_{b,m^2}(t) \) is given by

\[
A_{b,m^2}(t) = \begin{cases}
    e^{-\frac{b}{2}t} & \text{if } b^2 < 4m^2, \\
    (t + 1)e^{-\frac{b}{2}t} & \text{if } b^2 = 4m^2, \\
    e^{-\frac{b}{2} + \sqrt{\frac{b^2}{4} - m^2}}t & \text{if } b^2 > 4m^2.
\end{cases}
\]

Using these above \( L^2 \)-decay estimates, we will prove the global existence of small data solutions to the nonlinear fractional Cauchy problem (1.2) in the energy evolution space \( \mathcal{C}([0, \infty), H^\alpha_2(G)) \cap \mathcal{C}^1([0, \infty), L^2(G)) \). In this case, a Gagliardo-Nirenberg type inequality (proved in [35]) will be used to estimate the power nonlinearity in \( L^2(G) \). The following result is about the global existence of the mild solution of the Cauchy problem (1.2). For the definition of the mild solution, see subsection 3.2.

**Theorem 1.8.** Let \( 0 < \alpha < 1 \) and let \( G \) be a compact connected Lie group with the topological dimension \( n \). Assume that \( n \geq 2[\alpha] + 2 \). Suppose that \((u_0, u_1) \in H^\alpha_2(G) \times L^2(G) \) and \( p > 1 \) such that \( p \leq \frac{n}{n-2\alpha} \). Then there exists \( \varepsilon_0 > 0 \) such that for any \( \|u_0, u_1\|_{H^\alpha_2(G) \times L^2(G)} \leq \varepsilon_0 \), the Cauchy problem (1.2) admits a uniquely determined mild solution

\[
u \in \mathcal{C}([0, \infty), H^\alpha_2(G)) \cap \mathcal{C}^1([0, \infty), L^2(G)).
\]

**Remark 1.9.** Here we note that the fractional Laplace-Beltrami operator \((-\mathcal{L})^\alpha\) can be reduced to the classical Laplace-Beltrami operator \(-\mathcal{L}\) as \( \alpha \to 1 \) and Proposition 1.7 and Theorem 1.8 coincides with the results proved for the Cauchy problem for the fractional wave equation with damping and mass on compact Lie groups in [29].

**Remark 1.10.** We note that in the statement of Theorem 1.8, the restriction on the upper bound for the exponent \( p \) which is \( p \leq \frac{n}{n-2\alpha} \) is necessary in order to apply Gagliardo-Nirenberg type inequality (3.24) in (3.26). Also, the other restriction \( n \geq 2[\alpha] + 2 \) is made to fulfill the assumptions for the employment of such inequality.

Before studying the nonhomogeneous Cauchy problem (1.1) and (1.2) we first deal with the corresponding homogeneous problem, i.e., when \( f = 0 \). Particularly, using the group Fourier transform with respect to the spatial variable, we determine \( L^2 - L^2 \) estimates for the solution of the homogeneous fractional damped wave equation on the compact Lie group \( G \). Once we have these estimates, applying a Gagliardo-Nirenberg type inequality on compact Lie groups [27, 29, 28] (see also [35] for Gagliardo-Nirenberg type inequality on a more general frame of connected Lie groups), we prove the local well-posedness result for (1.1) and the global in time solution for (1.2).
Apart from the introduction, this paper is organized as follows. In Section 2, we recall the Fourier analysis on compact Lie groups which will be used frequently throughout the paper for our approach. In Section 3, first, we show an appropriate decomposition of the propagators for the nonlinear equation in the Fourier space. Further, by recalling the notion of mild solutions in our framework, we prove Theorem 1.2, the local existence result, by deriving some $L^2 - L^2$ estimates for the solution of the homogeneous fractional wave equation on the compact Lie group $G$. Moreover, under certain conditions on the initial data, a finite time blow-up result is established. In Section 4, we prove Theorem 1.8, the global existence for the mild solution, by deriving some $L^2 - L^2$ estimates for the solution of the homogeneous fractional wave equation with damping and mass (1.2) on the compact Lie group $G$.

2. Preliminaries: Analysis on compact Lie groups

In this section, we recall some basics of Fourier analysis on compact Lie groups to make the manuscript self-contained. A complete account of the representation theory of the compact Lie groups can be found in [16, 34, 33]. However, we mainly adopt the notation and terminology given in [33].

2.1. Notations. Throughout the article, we use the following notations:

- $f \lesssim g$: There exists a positive constant $C$ (whose value may change from line to line in this manuscript) such that $f \leq Cg$.
- $G$: Compact Lie group.
- $dx$: The normalized Haar measure on the compact group $G$.
- $\mathcal{L}$: The Laplace-Beltrami operator on $G$.
- $\mathbb{C}^{d \times d}$: The set of matrices with complex entries of order $d$.
- $\text{Tr}(A) = \sum_{j=1}^{d} a_{jj}$: The trace of the matrix $A = (a_{ij})_{1 \leq i,j \leq d} \in \mathbb{C}^{d \times d}$.
- $I_d \in \mathbb{C}^{d \times d}$: The identity matrix of order $d$.

2.2. Representation theory on compact Lie groups. Let us first recall the definition of a representation of a compact group $G$. A unitary representation of $G$ is a pair $(\xi, \mathcal{H})$ such that the map $\xi : G \rightarrow U(\mathcal{H})$, where $U(\mathcal{H})$ denotes the set of unitary operators on complex Hilbert space $\mathcal{H}$, such that it satisfies following properties:

- The map $\xi$ is a group homomorphism, that is, $\xi(xy) = \xi(x)\xi(y)$.
- The mapping $\xi : G \rightarrow U(\mathcal{H})$ is continuous with respect to strong operator topology (SOT) on $U(\mathcal{H})$, that is, the map $g \mapsto \xi(g)v$ is continuous for every $v \in \mathcal{H}$.

The Hilbert space $\mathcal{H}$ is called the representation space. To avoid any confusion, we represent a representation $(\xi, \mathcal{H})$ of $G$ by $\xi$. Two unitary representations $\xi, \eta$ of $G$ are called equivalent if there exists an unitary operator, namely intertwiner, $T$ such that $T\xi(x) = \eta(x)T$ for any $x \in G$. An intertwiner is an irreplaceable tool in the theory of representation of compact groups and is helpful in the classification of representation. A (linear) subspace $V \subset \mathcal{H}$ is said to be invariant under the unitary representation $\xi$ of $G$ if $\xi(x)V \subset V$, for any $x \in G$. An irreducible unitary representation $\xi$ of $G$ is a representation such that the only closed and $\xi$-invariant subspaces of $\mathcal{H}$ are trivial once, that is, $\{0\}$ and the full space $\mathcal{H}$.

The set of all equivalence classes $[\xi]$ of continuous irreducible unitary representations of $G$ is denoted by $\hat{G}$ and called the unitary dual of $G$. Since $G$ is compact, $\hat{G}$ is a discrete set. It is known that an irreducible unitary representation $\xi$ of $G$ is finite-dimensional, i.e., the Hilbert space $\mathcal{H}$ is finite-dimensional, say, $d_{\xi}$. Therefore, if we choose a basis $\mathcal{B} := \{e_1, e_2, \ldots, e_{d_{\xi}}\}$ for the representation space $\mathcal{H}$ of $\xi$, we can identify $\mathcal{H}$ as $\mathbb{C}^{d_{\xi}}$ and consequently, we can view $\xi$ as a matrix-valued function $\xi : G \rightarrow U(\mathbb{C}^{d_{\xi} \times d_{\xi}})$, where $U(\mathbb{C}^{d_{\xi} \times d_{\xi}})$ denotes the space of all unitary matrices. The matrix coefficients $\xi_{ij}$ of the representation
ξ with respect to \( \mathfrak{B} \) are given by \( \xi_{ij}(x) := \langle \xi(x)e_j, e_i \rangle \), for all \( i, j \in \{1, 2, \ldots, d_\xi\} \). It follows from the Peter-Weyl theorem that the set

\[
\left\{ \sqrt{d_\xi} \xi_{ij} : 1 \leq i, j \leq d_\xi, [\xi] \in \hat{G} \right\}
\]

forms an orthonormal basis of \( L^2(G) \).

2.3. **Fourier analysis on compact Lie groups.** Let \( G \) be a compact Lie group. The group Fourier transform of \( f \in L^1(G) \) at \( \xi \in \hat{G} \), denoted by \( \hat{f}(\xi) \), is defined by

\[
\hat{f}(\xi) := \int_G f(x)\xi(x)^* dx,
\]

where \( dx \) is the normalized Haar measure on \( G \). It is apparent from the definition that \( \hat{f}(\xi) \) is matrix-valued and therefore, this definition can be interpreted in weak sense, i.e., for \( u, v \in \mathcal{H} \),

\[
(\hat{f}(\xi)u, v) := \int_G f(x)\langle \xi(x)^*u, v \rangle dx.
\]

It follows from the Peter-Weyl theorem that, for every \( f \in L^2(G) \), we have the following Fourier series representation:

\[
f(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(x)\hat{f}(\xi)).
\]

The Plancherel identity for the group Fourier transform on \( G \) takes the following form

\[
\|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \hat{G}} d_\xi \|\hat{f}(\xi)\|_{HS}^2 \right)^{1/2} = \|\hat{f}\|_{L^2(\hat{G})}, \tag{2.1}
\]

where \( \| \cdot \|_{HS} \) denotes the Hilbert-Schmidt norm of a matrix \( A := (a_{ij}) \in \mathbb{C}^{d_\xi \times d_\xi} \) defined as

\[
\|A\|_{HS}^2 = \text{Tr}(AA^*) = \sum_{i,j=1}^{d_\xi} |a_{ij}|^2.
\]

We would like to emphasize here that the Plancherel identity is one of the crucial tools to establish \( L^2 \)-estimates of the solution to PDEs.

Let \( \mathcal{L} \) be the Laplace-Beltrami operator on \( G \). It is important to understand the action of the group Fourier transform on the Laplace–Beltrami operator \( \mathcal{L} \) for developing the machinery of the proofs. For \( [\xi] \in \hat{G} \), the matrix elements \( \xi_{ij} \), are the eigenfunctions of \( \mathcal{L} \) with the same eigenvalue \(-\lambda_\xi^2\). In other words, we have, for any \( x \in G \),

\[
-\mathcal{L}\xi_{ij}(x) = \lambda_\xi^2 \xi_{ij}(x), \quad \text{for all } i, j \in \{1, \ldots, d_\xi\}.
\]

The symbol \( \sigma_\mathcal{L} \) of the Laplace-Beltrami operator \( \mathcal{L} \) on \( G \) is given by

\[
\sigma_\mathcal{L}(\xi) = -\lambda_\xi^2 I_{d_\xi}, \tag{2.2}
\]

for any \( [\xi] \in \hat{G} \) and therefore, the following holds:

\[
\widehat{\mathcal{L}f}(\xi) = \sigma_\mathcal{L}(\xi)\hat{f}(\xi) = -\lambda_\xi^2 \hat{f}(\xi)
\]

for any \( [\xi] \in \hat{G} \).

For \( s > 0 \), the Sobolev space \( H^s_\mathcal{L}(G) \) of order \( s \) is defined as follows:

\[
H^s_\mathcal{L}(G) := \left\{ u \in L^2(G) : \|u\|_{H^s_\mathcal{L}(G)} < +\infty \right\},
\]

where \( \|u\|_{H^s_\mathcal{L}(G)} = \|u\|_{L^2(G)} + \|(-\mathcal{L})^{s/2}u\|_{L^2(G)} \) and \((-\mathcal{L})^{s/2}\) is defined in terms of the group Fourier transform by the following formula

\[
(-\mathcal{L})^{s/2} f := F^{-1} \left( \lambda_\xi^{s/2} (F f) \right), \quad \text{for all } [\xi] \in \hat{G}.
\]
Further, using Plancherel identity, for any $s > 0$, we have that
\[ \|(-L)^{s/2}f\|_{L^2(G)}^2 = \sum_{|\xi| \in \hat{G}} d_\xi \lambda_\xi^2 \|\hat{f}(\xi)\|_{HS}^2. \]

3. A local existence result

In this section, we study the local well-posedness of the Cauchy problem (4.1), i.e.,
\[
\begin{aligned}
\partial_t^2 u + (-L)^s u + \partial_t u &= |u|^p, \quad x \in G, t > 0, \\
u(0, x) &= \varepsilon u_0(x), \quad x \in G, \\
\partial_t u(x, 0) &= \varepsilon u_1(x), \quad x \in G,
\end{aligned}
\]

where $u_0(x)$ and $u_1(x)$ are two given functions on $G$ and $\varepsilon$ is a positive constant describing the smallness of the Cauchy data.

3.1. Fourier multiplier expressions and $L^2(G) - L^2(G)$ estimates. In this subsection, we derive $L^2(G) - L^2(G)$ estimates for the solutions to the homogeneous problem (1.4). We employ the group Fourier transform on the compact group $G$ with respect to the space variable $x$ together with the Plancherel identity in order to estimate $L^2$-norms of $u(t, \cdot)$, $(-L)^{s/2} u(t, \cdot)$, and $\partial_t u(t, \cdot)$.

Let $u$ be a solution to (1.4). Let $\hat{u}(t, \xi) = (\hat{u}(t, \xi)_{kl})_{1 \leq k, l \leq d_\xi} \in \mathbb{C}^{d_\xi \times d_\xi}$, $[\xi] \in \hat{G}$ denote the Fourier transform of $u$ with respect to the $x$ variable. Invoking the group Fourier transform with respect to $x$ on (1.4), we deduce that $\hat{u}(t, \xi)$ is a solution to the following Cauchy problem for the system of ODE’s (with the size of the system that depends on the representation $\xi$)
\[
\begin{aligned}
\partial_t^2 \hat{u}(t, \xi) + (-\sigma_\xi(\xi))^{s/2} \hat{u}(t, \xi) + \partial_t \hat{u}(t, \xi) &= 0, \quad [\xi] \in \hat{G}, \quad t > 0, \\
\hat{u}(0, \xi) &= \hat{u}_0(\xi), \quad [\xi] \in \hat{G}, \\
\partial_t \hat{u}(0, \xi) &= \hat{u}_1(\xi), \quad [\xi] \in \hat{G},
\end{aligned}
\] (3.1)

where $\sigma_\xi$ is the symbol of the operator operator $L$. Using the identity (2.2), the system (3.1) can be written in the form of $d_\xi^2$ independent ODE’s, namely,
\[
\begin{aligned}
\partial_t^2 \hat{u}(t, \xi)_{kl} + \partial_t \hat{u}(t, \xi)_{kl} + \lambda_\xi^{2\alpha} \hat{u}(t, \xi)_{kl} &= 0, \quad [\xi] \in \hat{G}, \quad t > 0, \\
\hat{u}(0, \xi)_{kl} &= \hat{u}_0(\xi)_{kl}, \quad [\xi] \in \hat{G}, \\
\partial_t \hat{u}(0, \xi)_{kl} &= \hat{u}_1(\xi)_{kl}, \quad [\xi] \in \hat{G},
\end{aligned}
\] (3.2)

for all $k, l \in \{1, 2, \ldots, d_\xi\}$. Then, the characteristic equation of (3.2) is given by
\[ \lambda^2 + \lambda + \lambda_\xi^{2\alpha} = 0, \]
and consequently the characteristic roots are $\lambda = -\frac{1}{2} \pm \sqrt{1 - 4\lambda_\xi^{2\alpha}}$. Thus the solution to the homogeneous problem (3.2) is given by
\[
\hat{u}(t, \xi)_{kl} = e^{-\frac{t}{2}} A_0(t, \xi) \hat{u}_0(\xi)_{kl} + e^{-\frac{t}{2}} A_1(t, \xi) \left( \hat{u}_1(\xi)_{kl} + \frac{1}{2} \hat{u}_0(\xi)_{kl} \right)
\]
\[
= e^{-\frac{t}{2}} \left[ A_0(t, \xi) + \frac{A_1(t, \xi)}{2} \right] \hat{u}_0(\xi)_{kl} + e^{-\frac{t}{2}} A_1(t, \xi) \hat{u}_1(\xi)_{kl},
\]
\[ A_0(t, \xi) = \begin{cases} 
\cosh \left( \frac{1}{2} \sqrt{1 - 4\lambda_\xi^{2\alpha}} t \right) & \text{if } 4\lambda_\xi^{2\alpha} < 1, \\
1 & \text{if } 4\lambda_\xi^{2\alpha} = 1, \\
\cos \left( \frac{1}{2} \sqrt{4\lambda_\xi^{2\alpha} - 1} t \right) & \text{if } 4\lambda_\xi^{2\alpha} > 1,
\end{cases}
\] (3.4)
and

\[
A_1(t, \xi) = \begin{cases} 
2 \sinh \left( \frac{\sqrt{1 - 4\lambda_\xi^2} t}{\sqrt{1 - 4\lambda_\xi^2}} \right) & \text{if } 4\lambda_\xi^2 < 1, \\
t & \text{if } 4\lambda_\xi^2 = 1, \\
\sin \left( \frac{\sqrt{4\lambda_\xi^2 - 1} t}{\sqrt{4\lambda_\xi^2 - 1}} \right) & \text{if } 4\lambda_\xi^2 > 1.
\end{cases}
\]

(3.5)

We notice that \( A_0(t, \xi) = \partial_t A_1(t, \xi) \) for any \( [\xi] \in \hat{G} \) and

\[
\partial_t \hat{u}(t, \xi)_{kl} = -e^{-\frac{t}{2}} A_1(t, \xi) \lambda_\xi^{2\alpha} \hat{u}_0(\xi)_{kl} + e^{-\frac{t}{2}} \left[ A_0(t, \xi) - \frac{1}{2} A_1(t, \xi) \right] \hat{u}_1(\xi)_{kl}.
\]

(3.6)

To simplify the presentation, we introduce the following partition of the unitary dual \( \hat{G} \) as:

\[
\mathcal{R}_1 = \{ [\xi] \in \hat{G} : 0 \leq \lambda_\xi^{2\alpha} < \frac{1}{16} \},
\]

\[
\mathcal{R}_2 = \{ [\xi] \in \hat{G} : \lambda_\xi^{2\alpha} \geq \frac{1}{16} \}.
\]

Note that the choice of \( \frac{1}{16} \) as a threshold in the previous definitions is irrelevant since our goal is to separate 0 (which is an eigenvalue for the continuous irreducible unitary representation \( 1 : x \in G \to 1 \in \mathbb{C} \)) from the other eigenvalues. Now we estimate \( L^2 \)-norms of \( u(t, \cdot), (-L)^\frac{\alpha}{2} u(t, \cdot), \) and \( \partial_t u(t, \cdot) \).

**Estimate on \( \mathcal{R}_1 \):** In this case, \(|A_0(t, \xi)| \leq \cosh \frac{t}{2}\) and \(|A_1(t, \xi)| \leq \sin \frac{t}{2}\). Therefore from (3.3), we have

\[
|\hat{u}(t, \xi)_{kl}| \lesssim |\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}|.
\]

(3.7)

Again for \([\xi] \in \mathcal{R}_1\), we have

\[
A_0(t, \xi) + \frac{A_1(t, \xi)}{2} = \frac{e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t + e^{-\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t}{2} + \frac{e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t - e^{-\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t}{2\sqrt{1 - 4\lambda_\xi^2}}
\]

\[
= \left( \frac{1}{2} + \frac{1}{4\sqrt{1 - 4\lambda_\xi^2}} \right) e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t + \left( \frac{1}{2} - \frac{1}{2\sqrt{1 - 4\lambda_\xi^2}} \right) e^{-\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t
\]

\[
\approx \left( \frac{1}{2} + \frac{1}{4\sqrt{1 - 4\lambda_\xi^2}} \right) e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t - \frac{\lambda_\xi^{2\alpha}}{\sqrt{1 - 4\lambda_\xi^2}} e^{-\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t.
\]

Thus, from (3.4) we deduce that

\[
\hat{u}_0(t, \xi)_{kl} \approx e^{-\frac{t}{2}} \left[ \left( \frac{1}{2} + \frac{1}{4\sqrt{1 - 4\lambda_\xi^2}} \right) e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t - \lambda_\xi^{2\alpha} e^{-\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t \right] \hat{u}_0(\xi)_{kl}
\]

\[
+ e^{-\frac{t}{2}} \left[ \frac{e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t - e^{-\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t}{2\sqrt{1 - 4\lambda_\xi^2}} \right] \hat{u}_1(\xi)_{kl},
\]

and therefore,

\[
|\hat{u}(t, \xi)_{kl}| \lesssim e^{-\frac{t}{2}} \left[ e^{\frac{t}{2}} \sqrt{1 - 4\lambda_\xi^2} t (|\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}|) \right]
\]
Thus from
\[ e^{-\frac{t}{2} \sqrt{ 1 - 4 \lambda_2^2 \xi^2 } t} \left( \lambda_2^2 \xi^2 \left| \tilde{u}_0(\xi)_{k\ell} \right| + \frac{1}{2} \left| \tilde{u}_1(\xi)_{k\ell} \right| \right) \]

\[ \leq e^{-\frac{t}{2} \left( \left| \tilde{u}_0(\xi)_{k\ell} \right| + \left| \tilde{u}_1(\xi)_{k\ell} \right| \right)} \left[ e^{\frac{t}{2} \sqrt{ 1 - 4 \lambda_2^2 \xi^2 } t} + e^{-\frac{t}{2} \sqrt{ 1 - 4 \lambda_2^2 \xi^2 } t} \right] \]

\[ \leq e^{-\frac{t}{2} + \frac{1}{2} \sqrt{ 1 - 4 \lambda_2^2 \xi^2 } t} \left( \left| \tilde{u}_0(\xi)_{k\ell} \right| + \left| \tilde{u}_1(\xi)_{k\ell} \right| \right) \left[ 1 + e^{-\frac{t}{2} \sqrt{ 1 - 4 \lambda_2^2 \xi^2 } t} \right] \]

\[ \approx e^{-\frac{t}{2} + \frac{1}{2} \left( 1 - 2 \lambda_2^2 \xi^2 \right) t} \left( \left| \tilde{u}_0(\xi)_{k\ell} \right| + \left| \tilde{u}_1(\xi)_{k\ell} \right| \right) \]

\[ \leq e^{-\lambda_2^2 t \left( \left| \tilde{u}_0(\xi)_{k\ell} \right| + \left| \tilde{u}_1(\xi)_{k\ell} \right| \right)} . \]

This implies using AM-GM inequality that
\[ \lambda_2^2 \xi^2 |\hat{u}(t, \xi)_{k\ell}|^2 \leq \lambda_2^2 e^{-2\lambda_2^2 t} \left( \left| \tilde{u}_0(\xi)_{k\ell} \right|^2 + \left| \tilde{u}_1(\xi)_{k\ell} \right|^2 \right) \leq (1 + t)^{-1} \left( \left| \tilde{u}_0(\xi)_{k\ell} \right|^2 + \left| \tilde{u}_1(\xi)_{k\ell} \right|^2 \right). \]

We note that, for $[\xi] \in \mathcal{R}_1$, we have
\[
A_0(t, \xi) - \frac{1}{2} A_1(t, \xi) = \frac{e^{-\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t} - e^{-\frac{t}{2} \left( 1 - 2 \lambda_2^2 \xi^2 \right) t}}{2} \left[ e^{\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t} - e^{-\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t} \right]
\]

\[
= \left( \frac{1}{2} - \frac{1}{2 \sqrt{1 - 4 \lambda_2^2 \xi^2 } } \right) e^{\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t} + \left( \frac{1}{2} + \frac{1}{2 \sqrt{1 - 4 \lambda_2^2 \xi^2 } } \right) e^{-\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t}
\]

\[
\approx -\lambda_2^2 \xi^2 e^{\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t} + \left( \frac{1}{2} + \frac{1}{2 \sqrt{1 - 4 \lambda_2^2 \xi^2 } } \right) e^{-\frac{t}{2} \sqrt{1 - 4 \lambda_2^2 \xi^2 } t}.
\]

Therefore, using it in (3.6) for $[\xi] \in \mathcal{R}_1$, we get
\[ |\partial_t \hat{u}(t, \xi)_{k\ell}| \lesssim \lambda_2^2 e^{-\lambda_2^2 t} \left( |\tilde{u}_0(\xi)_{k\ell}| + |\tilde{u}_1(\xi)_{k\ell}| \right) + e^{-t} \left( |\tilde{u}_0(\xi)_{k\ell}| + |\tilde{u}_1(\xi)_{k\ell}| \right)
\]

\[ \lesssim (1 + t)^{-1} \left( |\tilde{u}_0(\xi)_{k\ell}| + |\tilde{u}_1(\xi)_{k\ell}| \right). \]

**Estimate on $\mathcal{R}_2$**: When $\frac{1}{10} \leq \lambda_2^2 \xi^2 \leq \frac{1}{10}$, by following the similar calculation, there exists a suitable positive constant $c_1$ independent of $[\xi]$ such that
\[ |\hat{u}(t, \xi)_{k}\ell| \lesssim e^{-ct} \left( |\tilde{u}_0(\xi)_{k\ell}| + |\tilde{u}_1(\xi)_{k\ell}| \right). \]

When $\lambda_2^2 \geq \frac{1}{10}$, it is easy to note that $|A_0(t, \xi)| \leq 1$ and $|A_1(t, \xi)| \leq t$. Therefore from (3.3), there exists a suitable $c_2 > 0$ independent of $[\xi]$ such that
\[ |\hat{u}(t, \xi)_{k}\ell| \leq e^{-\frac{t}{2} \hat{u}_0(\xi)_{k}\ell} + te^{-\frac{t}{2} \hat{u}_1(\xi)_{k}\ell}
\]

\[ \lesssim (1 + t)e^{-\frac{t}{2} \left( |\tilde{u}_0(\xi)_{k}\ell| + |\tilde{u}_1(\xi)_{k}\ell| \right)}
\]

\[ \lesssim e^{-c t} \left( |\tilde{u}_0(\xi)_{k}\ell| + |\tilde{u}_1(\xi)_{k}\ell| \right). \]

Thus from (3.10) and (3.11), we have
\[ |\hat{u}(t, \xi)_{k}\ell| \lesssim e^{-ct} \left( |\tilde{u}_0(\xi)_{k}\ell| + |\tilde{u}_1(\xi)_{k}\ell| \right). \]
Moreover, for \([\xi] \in \mathcal{R}_2\), it follows that

\[
\lambda_\xi^\alpha |\hat{u}(t, \xi)_{k\ell}| \lesssim e^{-ct} \left( \lambda_\xi^\alpha |\hat{u}_0(\xi)_{k\ell}| + |\hat{u}_1(\xi)_{k\ell}| \right),
\]

for a suitable positive constant \(c\).

On the other hand, for \([\xi] \in \mathcal{R}_2\), we get the estimate

\[
|\partial_t \hat{u}(t, \xi)_{k\ell}| \lesssim e^{-ct} \left( \lambda_\xi^\alpha |\hat{u}_0(\xi)_{k\ell}| + |\hat{u}_1(\xi)_{k\ell}| \right),
\]

where \(c > 0\) is a suitable constant.

**Estimate for \(\|u(t, \cdot)\|_{L^2(G)}\):** Using the Plancherel formula along with the equations (3.7) and (3.12), it follows that

\[
\|u(t, \cdot)\|_{L^2(G)}^2 = \sum_{[\xi] \in \mathcal{G}} d_\xi \sum_{k, \ell = 1} d_\xi |\hat{u}(t, \xi)_{k\ell}|^2
\]

\[
= \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} |\hat{u}(t, \xi)_{k\ell}|^2 + \sum_{[\xi] \in \mathcal{R}_2} d_\xi \sum_{k, \ell = 1} |\hat{u}(t, \xi)_{k\ell}|^2
\]

\[
\lesssim \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} \left( |\hat{u}_0(\xi)_{k\ell}|^2 + |\hat{u}_1(\xi)_{k\ell}|^2 \right)
\]

\[
+ \sum_{[\xi] \in \mathcal{R}_2} d_\xi \sum_{k, \ell = 1} e^{-2ct} \left( |\hat{u}_0(\xi)_{k\ell}|^2 + |\hat{u}_1(\xi)_{k\ell}|^2 \right)
\]

\[
\lesssim \sum_{[\xi] \in \mathcal{G}} d_\xi \sum_{k, \ell = 1} \left( |\hat{u}_0(\xi)_{k\ell}|^2 + |\hat{u}_1(\xi)_{k\ell}|^2 \right)
\]

\[
= \|u_0 \|^2_{L^2(G)} + \|u_1 \|^2_{L^2(G)}.
\]

**Estimate for \(\|(-\mathcal{L})^{\alpha/2}u(t, \cdot)\|_{L^2(G)}\):** Using the Plancherel formula, we get

\[
\left\|(-\mathcal{L})^{\alpha/2}u(t, \cdot)\right\|^2_{L^2(G)} = \sum_{[\xi] \in \mathcal{G}} d_\xi \left\|\sigma_{(-\mathcal{L})^{\alpha/2}}(\xi)\hat{u}(t, \xi)\right\|^2_{HS}
\]

\[
= \sum_{[\xi] \in \mathcal{G}} d_\xi \sum_{k, \ell = 1} \lambda_\xi^{2\alpha} |\hat{u}(t, \xi)_{k\ell}|^2
\]

\[
= \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} \lambda_\xi^{2\alpha} |\hat{u}(t, \xi)_{k\ell}|^2 + \sum_{[\xi] \in \mathcal{R}_2} d_\xi \sum_{k, \ell = 1} \lambda_\xi^{2\alpha} |\hat{u}(t, \xi)_{k\ell}|^2
\]

\[
\lesssim (1 + t)^{-1} \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} \left( |\hat{u}_0(\xi)_{k\ell}|^2 + |\hat{u}_1(\xi)_{k\ell}|^2 \right)
\]

\[
+ e^{-ct} \sum_{[\xi] \in \mathcal{R}_2} d_\xi \sum_{k, \ell = 1} \left( \lambda_\xi^{2\alpha} |\hat{u}_0(\xi)_{k\ell}|^2 + |\hat{u}_1(\xi)_{k\ell}|^2 \right)
\]

\[
\lesssim (1 + t)^{-1} \left( \|u_0 \|^2_{H^\alpha(G)} + \|u_1 \|^2_{L^2(G)} \right).
\]
Estimate for $\|\partial_t u(t, \cdot)\|_{L^2(G)}$: From (3.9) and (3.14), the Plancherel formula yields that

$$\|\partial_t u(t, \cdot)\|_{L^2(G)}^2 = \sum_{[\xi] \in \mathcal{G}} d_\xi \sum_{k, \ell = 1} d_\xi |\partial_t \widehat{u}(t, \xi)_{k\ell}|^2$$

$$= \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} |\partial_t \widehat{u}(t, \xi)_{k\ell}|^2 + \sum_{[\xi] \in \mathcal{R}_2} d_\xi \sum_{k, \ell = 1} |\partial_t \widehat{u}(t, \xi)_{k\ell}|^2$$

$$\lesssim (1 + t)^{-2} \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} \left( |\widehat{u}_0(\xi)_{k\ell}|^2 + |\widehat{u}_1(\xi)_{k\ell}|^2 \right)$$

$$+ e^{-2ct} \sum_{[\xi] \in \mathcal{R}_1} d_\xi \sum_{k, \ell = 1} \left( \lambda^2_{\xi} |\widehat{u}_0(\xi)_{k\ell}|^2 + |\widehat{u}_1(\xi)_{k\ell}|^2 \right)$$

$$\lesssim (1 + t)^{-2} (\|u_0\|_{H^2(G)}^2 + \|u_1\|_{L^2(G)}^2). \quad (3.17)$$

Now, we are in a position to prove Proposition 1.1.

Proof of Proposition 1.1. The proof of Theorem 1.1 follows from the estimates (3.15), (3.16), and (3.17) for $\|u(t, \cdot)\|_{L^2(G)}$, $\|(-\mathcal{L})^{\alpha/2}u(t, \cdot)\|_{L^2(G)}$, and $\|\partial_t u(t, \cdot)\|_{L^2(G)}$, respectively. \hfill \Box

3.2. Local in time existence. In this subsection we will prove Theorem 1.2, i.e., the local well-posedness of the Cauchy problem (1.1) in the energy evolution space $\mathcal{C}([0, T], H^2_v(G)) \cap C^1([0, T], L^2(G))$.

First, we recall some notations to present the proof of Theorem 1.2. Consider the space

$$X(T) := \mathcal{C}([0, T], H^2_v(G)) \cap C^1([0, T], L^2(G)), \quad (3.18)$$

equipped with the norm

$$\|u\|_{X(T)} := \sup_{t \in [0, T]} (\|u(t, \cdot)\|_{L^2(G)} + \|(-\mathcal{L})^{\alpha/2}u(t, \cdot)\|_{L^2(G)} + \|\partial_t u(t, \cdot)\|_{L^2(G)}).$$

Here we will briefly recall the notion of mild solutions in our framework to the Cauchy problem (1.1) and will analyze our approach to prove Theorem 1.2. Applying Duhamel’s principle, the solution to the nonlinear inhomogeneous problem

$$\begin{cases}
\partial_t^2 u + (-\mathcal{L})^\alpha u + \partial_t u = F(t, x), & x \in G, t > 0, \\
u(0, x) = u_0(x), & x \in G, \\
\partial_t u(0, x) = u_1(x), & x \in G,
\end{cases} \quad (3.19)$$

can be expressed as

$$u(t, x) = u_0(x) *_{(x)} E_0(t, x) + u_1(x) *_{(x)} E_1(t, x) + \int_0^t F(s, x) *_{(x)} E_1(t-s, x) \, ds,$$

where $*_{(x)}$ is the group convolution product on $G$ with respect to the $x$ variable. Here $E_0(t, x)$ and $E_1(t, x)$ represent the fundamental solutions to the homogeneous problem, i.e., (3.19) with $F = 0$ and the initial data $(u_0, u_1) = (\delta_0, 0)$ and $(u_0, u_1) = (0, \delta_0)$, respectively. For any left-invariant differential operator $L$ on the compact Lie group $G$, we apply the property that it commute with the group convolution, that is, $L \left( v *_{(x)} E_1(t, \cdot) \right) = v *_{(x)} L(E_1(t, \cdot))$ and the invariance by time translations for the wave operator $\partial_t^2 + (-\mathcal{L})^\alpha$, to get the previous representation formula.
We say that a function $u$ is a mild solution to (3.19) on $[0, T]$ if $u$ is a fixed point for the integral operator, $N : u \in X(T) \to Nu(t, x)$, given by
\[
Nu(t, x) := \varepsilon u_0(x) *_{(x)} E_0(t, x) + \varepsilon u_1(x) *_{(x)} E_1(t, x) + \int_0^t |u(s, x)|^p *_{(x)} E_1(t - s, x) \, ds
\]
in the energy evolution space $X(T) = C([0, T], H^0_\infty(G)) \cap C^1([0, T], L^2(G))$, equipped with the norm defined in (3.18).

In order to show a uniquely determined fixed point of $N$ for a sufficiently small $T = T(\varepsilon)$, we use the Banach fixed point theorem with respect to the norm on $X(T)$ as defined by (3.18). In fact, for the small enough initial data $\|(u_0, u_1)\|_{H^2_\infty(G) \times L^2(G)}$, we will establish the following two inequalities
\[
\|Nu\|_{X(T)} \leq C\|(u_0, u_1)\|_{H^2_\infty(G) \times L^2(G)} + C\|u\|_{X(T)}^p,
\]
and
\[
\|Nu - Nv\|_{X(T)} \leq C\|u - v\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|v\|_{X(T)}^{p-1}\right),
\]
for any $u, v \in X(T)$ and for some suitable constant $C > 0$ independent of $T$. Then the Banach fixed point theorem immediately gives a uniquely determined fixed point $u$ on $N$. This fixed point $u$ will be our mild solution to (3.19) on $[0, T]$.

In order to prove the local existence result, an essential tool is the following Gagliardo-Nirenberg type inequality proved in general Lie groups [35].

**Lemma 3.1.** [35] Let $G$ be a connected unimodular Lie group with topological dimension $n$. For any $1 < q_0 < \infty$, $0 < q, q_1 < \infty$ and $0 < \alpha < n$ such that $q_0 < \frac{n}{\alpha}$, the following Gagliardo-Nirenberg type inequality holds
\[
\|f\|_{L^q(G)} \lesssim \|f\|^{\theta(n, q, q_0, G)}_{H^{n, q_0}_\infty(G)}\|f\|^{1-\theta(n, q, q_1, G)}_{L^{q_1}(G)},
\]
for all $f \in H^{n, q_0}_\infty(G) \cap L^{q_1}(G)$, provided that
\[
\theta = \theta(n, \alpha, q, q_0, q_1) = \frac{1}{q} - \frac{1}{q_0} + \frac{\alpha}{n} \in [0, 1].
\]

We refer to [35, 27] for several immediate important remarks from Lemma 3.1. The next corollary is a version of Lemma 3.1, which is useful in our setting.

**Corollary 3.2.** Let $G$ be a connected unimodular Lie group with topological dimension $n \geq 2[\alpha] + 2$. For any $q \geq 2$ such that $q \leq \frac{2n}{n - 2\alpha}$, the following Gagliardo-Nirenberg type inequality holds
\[
\|f\|_{L^q(G)} \lesssim \|f\|^{\theta(n, q, \alpha, G)}_{H^{n, q}_\infty(G)}\|f\|^{1-\theta(n, q, \alpha, G)}_{L^2(G)},
\]
for all $f \in H^{n, q}_\infty(G)$, where $\theta(n, q, \alpha) = \frac{n}{\alpha} \left(\frac{1}{2} - \frac{1}{q}\right)$.

**Proof of Theorem 1.2.** The expression (3.20) can be written as $Nu = u^\sharp + I[u]$, where
\[
u^\sharp(t, x) = \varepsilon u_0(x) *_{(x)} E_0(t, x) + \varepsilon u_1(x) *_{(x)} E_1(t, x),
\]
and
\[
I[u](t, x) := \int_0^t |u(s, x)|^p *_{(x)} E_1(t - s, x) \, ds.
\]
Now, for the part $u^\sharp$, Theorem 1.1, immediately implies that
\[
\|u^\sharp\|_{X(T)} \lesssim \varepsilon\|(u_0, u_1)\|_{H^0_\infty(G) \times L^2(G)},
\]

On the other hand, for the part $I[u]$, using Minkowski's integral inequality, Young's convolution inequality, Theorem 1.1, and by time translation invariance property of the Cauchy problem (1.1), we get

$$
\|\partial_t^i (-\mathcal{L})^{\alpha/2} I[u]\|_{L^2(G)} = \left( \int_G \left| \partial_t^i (-\mathcal{L})^{\alpha/2} \int_0^t |u(s,x)|^p \ast_x E_1(t-s,x) ds \right|^2 dg \right)^{\frac{1}{2}}
$$

$$
= \left( \int_G \left| \int_0^t |u(s,x)|^p \ast_x \partial_t^i (-\mathcal{L})^{\alpha/2} E_1(t-s,\cdot) ds \right|^2 dg \right)^{\frac{1}{2}}
$$

$$
\lesssim \int_0^t \|u(s,\cdot)|^p \ast_x \partial_t^i (-\mathcal{L})^{\alpha/2} E_1(t-s,\cdot)\|_{L^2(G)} ds
$$

$$
\lesssim \int_0^t \|u(s,\cdot)|^p \|L^2(G)\| \|\partial_t^i (-\mathcal{L})^{\alpha/2} E_1(t-s,\cdot)\|_{L^2(G)} ds
$$

$$
\lesssim \int_0^t (1 + t - s)^{-j - \frac{\alpha}{2} + \frac{1}{p}} \|u(s,\cdot)|^p \|_{L^2^p(G)} ds
$$

$$
\lesssim \int_0^t (1 + t - s)^{-j - \frac{\alpha}{2} + \frac{1}{p}} \|u(s,\cdot)|^{p(0,2p,\alpha)} \|u(s,\cdot)|^{p(1-\theta(0,2p,\alpha))} \|_{L^2(G)} ds
$$

$$
\lesssim \int_0^t (1 + t - s)^{-j - \frac{\alpha}{2}} \|u\|_{X(s)}^p ds \lesssim t \|u\|_{X(t)}^p,
$$

(3.26)

for $i, j \in \{0, 1\}$, such that $0 \leq i + j \leq 1$. Again for $i, j \in \{0, 1\}$, such that $0 \leq i + j \leq 1$, a similar calculations as in (3.26) together with Hölder’s inequality and (3.24), we get

$$
\|\partial_t^i (-\mathcal{L})^{\alpha/2} (I[u] - I[v])\|_{L^2(G)}
$$

$$
\lesssim \int_0^t (1 + t - s)^{-j - \frac{\alpha}{2}} \|u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^2(G)} ds
$$

$$
\lesssim \int_0^t (1 + t - s)^{-j - \frac{\alpha}{2}} \|u(s,\cdot) - v(s,\cdot)\|_{L^2^p(G)} \left( \|u(s,\cdot)\|_{L^2^p(G)}^{p-1} + \|v(s,\cdot)\|_{L^2^p(G)}^{p-1} \right) ds
$$

$$
\lesssim t \|u - v\|_{X(t)} \left( \|u\|_{X(t)}^{p-1} - \|v\|_{X(t)}^{p-1} \right),
$$

(3.27)

Thus combining (3.25), (3.26), and (3.27), we have

$$
\|Nu\|_{X(T)} \leq D\varepsilon \|(u_0, u_1)\|_{H^2(G) \times L^2(G)} + DT \|u\|_{X(t)}^p
$$

(3.28)

and

$$
\|Nu - Nv\|_{X(T)} \leq DT \|u - v\|_{X(t)} \left( \|u\|_{X(t)}^{p-1} - \|v\|_{X(t)}^{p-1} \right),
$$

(3.29)

where $D$ is a constant independent of $t$. Choose $T$ (sufficiently small) in such a way that the map $N$ turns out to be a contraction in some neighborhood of 0 in the Banach space $X(T)$. Therefore, Banach’s fixed point theorem gives us the uniquely determined fixed point $u$ for the map $N$, which is our mild solution. This completes the proof.

From the above local existence result, we have the following remark.
Remark 3.3. We note that in the statement of Theorem 1.2, the restriction on the upper bound for the exponent $p$ which is $p \leq \frac{n}{n-2\alpha}$ is necessary in order to apply Gagliardo-Nirenberg type inequality (3.24) in (3.26). Also, the other restriction $n \geq 2[\alpha] + 2$ is made to fulfill the assumptions for the employment of such inequality.

3.3. Blow-up result. In this subsection, we prove Theorem 1.5 using a comparison argument for ordinary differential inequality of second order. Now we are ready to prove our main result of this section using an iteration argument.

Proof of Theorem 1.5. According to Definition 1.4, let $u$ be a local in-time energy solution to (1.1) with lifespan $T$. Let $t \in (0, T)$ be fixed. Suppose that $\phi \in C_0^\infty([0, T] \times G)$, is a cut-off function such that $\phi = 1$ on $[0, t] \times G$ in (1.6). Then

$$
\int_G \partial_t u(t, x) \, dx + \int_G u(t, x) \, dx - \varepsilon \int_G u_0(x) \, dx - \varepsilon \int_G u_1(x) \, dx = \int_0^t \int_G |u(s, x)|^p \, dx \, ds
$$

(3.30)

Let us introduce the time-dependent functional

$$
U_0(t) = \int_G u(t, x) \, dx.
$$

Then the equality (3.30) can be rewritten in the following way:

$$
U_0'(t) - U_0'(0) + U_0(t) - U_0(0) = \int_0^t \int_G |u(s, x)|^p \, dx \, ds.
$$

We also remark that, from the assumptions on the initial data, we obtain

$$
U_0(0) = \varepsilon \int_G u_0(x) \, dx \geq 0 \quad \text{and} \quad U_0'(0) = \varepsilon \int_G u_1(x) \, dx \geq 0.
$$

Using Jensen’s inequality, we have

$$
U_0'(t) - U_0'(0) + U_0(t) - U_0(0) \geq \int_0^t |U_0(s)|^p \, ds.
$$

(3.31)

Multiplying both sides of (3.31) by $e^t$ and then integrating over $[0, t]$, we obtain

$$
e^t U_0(t) \geq (U_0'(0) + U_0(0))(e^t - 1) + U_0(0) + \int_0^t e^\eta \int_0^\eta |U_0(s)|^p \, ds \, d\eta,
$$

i.e.,

$$
U_0(t) \geq U_0(0) + U_0'(0)(1 - e^{-t}) + \int_0^t e^{\eta-t} \int_0^\eta |U_0(s)|^p \, ds \, d\eta.
$$

Since $U_0(0)$ and $U_0'(0)$ are non-negative, the above expression implies that $U_0$ is a positive function. Moreover, we also can say that

$$
U_0(t) \geq U_0(0) + U_0'(0)(1 - e^{-t}) \geq C\varepsilon \quad \text{for} \ t \geq 0,
$$

where the multiplicative constant $C$ depends on $u_0, u_1$ and we also have the following iteration scheme

$$
U_0(t) \geq \int_0^t e^{\eta-t} \int_0^\eta |U_0(s)|^p \, ds \, d\eta.
$$

Now proceeding similarly as in Subsection 3.1 and 3.2 of [27] for the iteration argument, we conclude the proof of Theorem 1.5. □
4. A GLOBAL EXISTENCE RESULT

In this section, we study the global in-time existence of small data solutions for the nonlinear fractional dumped wave equation with mass and the power type nonlinearity. More precisely, for $0 < \alpha < 1$, we consider the Cauchy problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t^2 u + (-\mathcal{L})^\alpha u + b\partial_t u + m^2 u = |u|^p, & x \in G, \, t > 0, \\
u(0, x) = u_0(x), & x \in G, \\
\partial_t u(x, 0) = u_1(x), & x \in G,
\end{array} \right.
\end{aligned}
\]

where $p > 1$, $b, m^2$ are positive constants, $u_0(x)$ and $u_1(x)$ are two given functions on $G$.

4.1. Fourier multiplier expressions and $L^2(G) - L^2(G)$ estimates. In this subsection, we derive $L^2(G) - L^2(G)$ estimates for solutions of the homogeneous problem (4.1). We employ the group Fourier transform on the compact group $G$ with respect to the space variable $x$ together with the Plancherel identity in order to estimate $L^2$-norms of $u(t, \cdot), (-\mathcal{L})^{\frac{\sigma}{2}}u(t, \cdot)$, and $\partial_t u(t, \cdot)$.

Let $u$ be a solution to (4.1). Let \( \hat{u}(t, \xi) = (\hat{u}(t, \xi)_{kl})_{1 \leq k, l \leq d_\xi} \in \mathbb{C}^{d_\xi \times d_\xi}, [\xi] \in \hat{G} \) denote the Fourier transform of $u$ with respect to the $x$ variable. Applying the group Fourier transform with respect to $x$ on (4.1), we deduce that \( \hat{u}(t, \xi) \) is a solution to the following Cauchy problem for the system of ODE’s (with the size of the system that depends on the representation $\xi$)

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t^2 \hat{u}(t, \xi) + (-\mathcal{L}(\xi))^{\alpha} \hat{u}(t, \xi) + b\partial_t \hat{u}(t, \xi) + m^2 \hat{u}(t, \xi) = 0, & [\xi] \in \hat{G}, \, t > 0, \\
\hat{u}(0, \xi) = \hat{u}_0(\xi), & [\xi] \in \hat{G}, \\
\partial_t \hat{u}(0, \xi) = \hat{u}_1(\xi), & [\xi] \in \hat{G},
\end{array} \right.
\end{aligned}
\]

where $\mathcal{L}(\xi)$ is the symbol of the Laplace-Beltrami operator $\mathcal{L}$. Using the identity (2.2), the system (4.2) can be written in the form of $d_\xi^2$ independent ODE’s, namely,

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t^2 \hat{u}(t, \xi)_{kl} + b\partial_t \hat{u}(t, \xi)_{kl} + \lambda_{\xi}^{2\alpha} \hat{u}(t, \xi)_{kl} + m^2 \hat{u}(t, \xi)_{kl} = 0, & [\xi] \in \hat{G}, \, t > 0, \\
\hat{u}(0, \xi)_{kl} = \hat{u}_0(\xi)_{kl}, & [\xi] \in \hat{G}, \\
\partial_t \hat{u}(0, \xi)_{kl} = \hat{u}_1(\xi)_{kl}, & [\xi] \in \hat{G},
\end{array} \right.
\end{aligned}
\]

for all $k, l \in \{1, 2, \ldots, d_\xi \}$. Then, the characteristic equation of (4.3) is given by

\[
\lambda^2 + b\lambda + \lambda_{\xi}^{2\alpha} + m^2 = 0,
\]

and consequently the characteristic roots are $\lambda = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - \lambda_{\xi}^{2\alpha} - m^2}$. Thus the solution to the homogeneous problem (4.3) is given by

\[
\hat{u}(t, \xi)_{kl} = e^{-\frac{b}{2} t} A_0(t, \xi) \hat{u}_0(\xi)_{kl} + e^{-\frac{b}{2} t} A_1(t, \xi) \left( \hat{u}_1(\xi)_{kl} + \frac{b}{2} \hat{u}_0(\xi)_{kl} \right),
\]

where

\[
A_0(t, \xi) = \begin{cases} 
\cosh \left( \frac{\sqrt{\frac{b^2}{4} - \lambda_{\xi}^{2\alpha} - m^2} \cdot t}{2} \right), & \text{if } \lambda_{\xi}^{2\alpha} < \frac{b^2}{4} - m^2, \\
1, & \text{if } \lambda_{\xi}^{2\alpha} = \frac{b^2}{4} - m^2, \\
\cos \left( \frac{\sqrt{\lambda_{\xi}^{2\alpha} - \frac{b^2}{4} + m^2} \cdot t}{2} \right), & \text{if } \lambda_{\xi}^{2\alpha} > \frac{b^2}{4} - m^2,
\end{cases}
\]

\[
A_1(t, \xi) = \begin{cases} 
\sinh \left( \frac{\sqrt{\frac{b^2}{4} - \lambda_{\xi}^{2\alpha} - m^2} \cdot t}{2} \right), & \text{if } \lambda_{\xi}^{2\alpha} < \frac{b^2}{4} - m^2, \\
0, & \text{if } \lambda_{\xi}^{2\alpha} = \frac{b^2}{4} - m^2, \\
\sin \left( \frac{\sqrt{\lambda_{\xi}^{2\alpha} - \frac{b^2}{4} + m^2} \cdot t}{2} \right), & \text{if } \lambda_{\xi}^{2\alpha} > \frac{b^2}{4} - m^2,
\end{cases}
\]
and

$$A_1(t, \xi) = \begin{cases} 
\frac{2 \sinh \left( \sqrt{\frac{b^2}{4} - \lambda^2_\xi - m^2 - t} \right)}{\sqrt{\frac{b^2}{4} - \lambda^2_\xi - m^2}}, & \text{if } \lambda^2_\xi < \frac{b^2}{4} - m^2, \\
t, & \text{if } \lambda^2_\xi = \frac{b^2}{4} - m^2, \\
\frac{\sin \left( \sqrt{\lambda^2_\xi + \frac{b^2}{4} - m^2} \right)}{\sqrt{\lambda^2_\xi + \frac{b^2}{4} - m^2}}, & \text{if } \lambda^2_\xi > \frac{b^2}{4} - m^2.
\end{cases}$$

(4.6)

We notice that $A_0(t, \xi) = \partial_t A_1(t, \xi)$ for any $[\xi] \in \hat{G}$. Moreover, we have the following representation for the time derivative

$$\partial_t \hat{u}(t, \xi)_{kl} = e^{-\frac{bt}{2}} A_0(t, \xi) \hat{u}_1(\xi)_{kl} - e^{-\frac{bt}{2}} A_1(t, \xi) \left[ \frac{b}{2} \hat{u}_1(\xi)_{kl} + (\lambda^2_\xi + m^2) \hat{u}_0(\xi)_{kl} \right].$$

(4.7)

Next we will estimate the values of $|\hat{u}(t, \xi)_{kl}|, |\partial_t \hat{u}(t, \xi)_{kl}|$ and $\lambda_\xi |\hat{u}(t, \xi)_{kl}|$ by considering the relation between $b$ and $m^2$.

**When $b^2 < 4m^2$:** The only case is to consider that $\lambda^2_\xi > \frac{b^2}{4} - m^2$ by considering the fact that all eigenvalues $\{\lambda^2_\xi\}_{[\xi] \in \hat{G}}$ of $(-L)^a$ are nonnegative. Thus, by the similar calculus done in Subsection 3.1, we have

$$|\hat{u}(t, \xi)_{kl}| \lesssim e^{-\frac{bt}{2}} \left[ |\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}| \right],$$

(4.8)

$$\lambda^a_\xi |\hat{u}(t, \xi)_{kl}| \lesssim e^{-\frac{bt}{4}} \left[ (1 + \lambda^a_\xi) |\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}| \right],$$

(4.9)

and

$$|\partial_t \hat{u}(t, \xi)_{kl}| \lesssim e^{-\frac{bt}{2}} \left[ \lambda^a_\xi |\hat{u}_0(\xi)_{kl}| + m^2 |\hat{u}_0(\xi)_{kl}| \right]$$

$$\lesssim e^{-\frac{bt}{4}} \left[ (1 + \lambda^a_\xi) |\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}| \right],$$

(4.10)

for any $t \geq 0$. Thus, using the Plancherel formula along with the equations (4.8), (4.9) and (4.10), it follows that

$$\|\partial_t (-L)^{(i+a)/2} u(t, \cdot)\|^2_{L^2(G)} = \sum_{[\xi] \in \hat{G}} d_\xi \sum_{k, \ell = 1}^d \lambda^{2ai}_\xi \left| \partial_t \hat{u}(t, \xi)_{kl} \right|^2$$

$$\lesssim e^{-bt} \sum_{[\xi] \in \hat{G}} d_\xi \sum_{k, \ell = 1}^d \left( (1 + \lambda^2_\xi)^{i+j} |\hat{u}_0(\xi)_{kl}|^2 + |\hat{u}_1(\xi)_{kl}|^2 \right)$$

$$= e^{-bt} \left[ \|u_0\|^2_{H_{+}^{i+j}(G)} + \|u_1\|^2_{L^2(G)} \right],$$

(4.11)

for any $i, j \in \{0, 1\}$, such that $0 \leq i + j \leq 1$, with the convention that $H^0_G(G) = L^2(G)$.

**When $b^2 = 4m^2$:** In this case, we only have to consider the cases when $\lambda^2_\xi = 0$ and $\lambda^2_\xi > 0$. Then, from (4.4), (4.5), and (4.6), the solution can be written as

$$\hat{u}(t, \xi)_{kl} = \begin{cases} 
- \frac{bt}{2} \cos \left( \lambda^a_\xi t \right) \hat{u}_0(\xi)_{kl} + e^{-\frac{bt}{2} \sin \left( \lambda^a_\xi t \right)} \left( \hat{u}_1(\xi)_{kl} + \frac{b}{2} \hat{u}_0(\xi)_{kl} \right), & \text{if } \lambda^2_\xi > 0, \\
- \frac{bt}{2} \hat{u}_0(\xi)_{kl} + t e^{-\frac{bt}{2}} \left( \hat{u}_1(\xi)_{kl} + \frac{b}{2} \hat{u}_0(\xi)_{kl} \right), & \text{if } \lambda^2_\xi = 0.
\end{cases}$$

The second case $\lambda^2_\xi = 0$ needs to be included as 0 is the eigenvalue for the trivial representation $G$. Thus

$$|\hat{u}(t, \xi)_{kl}| \lesssim (1 + t) e^{-\frac{bt}{4}} \left[ |\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}| \right],$$

$$\lambda^a_\xi |\hat{u}(t, \xi)_{kl}| \lesssim e^{-\frac{bt}{4}} \left[ (1 + \lambda^a_\xi) |\hat{u}_0(\xi)_{kl}| + |\hat{u}_1(\xi)_{kl}| \right],$$

(4.12)
and
\[ |\partial_t \tilde{u}(t, \xi)_{\xi \ell}| \lesssim (1 + t) e^{-\frac{b t}{2}} \left[ (1 + \lambda_\xi^2) |\tilde{u}_0(\xi)_{\xi \ell}| + |\tilde{u}_1(\xi)_{\xi \ell}| \right], \]
for any \( t \geq 0 \). Thus using the Plancherel formula along with the above estimates, we get
\[
\| \partial_t^j (-L)^{\alpha/2} u(t, \cdot) \|_{L^2(G)}^2 = \sum_{|\xi| \in \mathbb{G}} d_\xi \sum_{k, \ell = 1}^d \lambda_\xi^{2\alpha i} |\partial_t^j \tilde{u}(t, \xi)_{\xi \ell}|^2
\]
\[
\lesssim (1 + t)^2 e^{-bt} \sum_{|\xi| \in \mathbb{G}} d_\xi \sum_{k, \ell = 1}^d \left( (1 + \lambda_\xi^{2\alpha})^{(i+j)} |\tilde{u}_0(\xi)_{k \ell}|^2 + |\tilde{u}_1(\xi)_{k \ell}|^2 \right)
\]
\[
= (1 + t)^2 e^{-bt} \left[ \|u_0\|_{H^{\alpha+i+1}(G)}^2 + \|u_1\|_{L^2(G)}^2 \right], \tag{4.12}
\]
for any \( i, j \in \{0, 1\} \), such that \( 0 \leq i + j \leq 1 \).

**When \( b^2 > 4m^2 \):** In this case, depending on the range of \( \lambda_\xi^2 \), the characteristic roots may be complex conjugate or real distinct, or they may coincide. But comparing all possible cases in (4.5) and (4.6) and keeping in mind that the regularity is provided from the case with complex conjugate characteristic roots, whereas the decay rate is given by the continuous irreducible unitary representations with \( \lambda_\xi^2 = 0 \), we obtain
\[
|\tilde{u}(t, \xi)_{\xi \ell}| \lesssim e^{\left( -\frac{b}{2} + \sqrt{\frac{4}{d_\xi^2} - m^2} \right) t} (|\tilde{u}_0(\xi)_{\xi \ell}| + |\tilde{u}_1(\xi)_{\xi \ell}|),
\]
\[
\lambda_\xi^{2\alpha} |\tilde{u}(t, \xi)_{\xi \ell}| \lesssim e^{\left( -\frac{b}{2} + \sqrt{\frac{4}{d_\xi^2} - m^2} \right) t} \left[ (1 + \lambda_\xi^{2\alpha}) |\tilde{u}_0(\xi)_{\xi \ell}| + |\tilde{u}_1(\xi)_{\xi \ell}| \right],
\]
and
\[
|\partial_t \tilde{u}(t, \xi)_{\xi \ell}| \lesssim e^{\left( -\frac{b}{2} + \sqrt{\frac{4}{d_\xi^2} - m^2} \right) t} \left[ (1 + \lambda_\xi^{2\alpha}) |\tilde{u}_0(\xi)_{\xi \ell}| + |\tilde{u}_1(\xi)_{\xi \ell}| \right],
\]
for any \( t \geq 0 \). Thus using the Plancherel formula along with the above estimates, we get
\[
\| \partial_t^j (-L)^{\alpha/2} u(t, \cdot) \|_{L^2(G)}^2 = \sum_{|\xi| \in \mathbb{G}} d_\xi \sum_{k, \ell = 1}^d \lambda_\xi^{2\alpha i} |\partial_t^j \tilde{u}(t, \xi)_{\xi \ell}|^2
\]
\[
\lesssim e^{\left( -\frac{b}{2} + \sqrt{\frac{4}{d_\xi^2} - m^2} \right) t} \sum_{|\xi| \in \mathbb{G}} d_\xi \sum_{k, \ell = 1}^d \left( (1 + \lambda_\xi^{2\alpha})^{(i+j)} |\tilde{u}_0(\xi)_{k \ell}|^2 + |\tilde{u}_1(\xi)_{k \ell}|^2 \right)
\]
\[
= e^{\left( -\frac{b}{2} + \sqrt{\frac{4}{d_\xi^2} - m^2} \right) t} \left[ \|u_0\|_{H^{\alpha+i+1}(G)}^2 + \|u_1\|_{L^2(G)}^2 \right], \tag{4.13}
\]
for any \( i, j \in \{0, 1\} \), such that \( 0 \leq i + j \leq 1 \).

Now, we are in a position to prove Proposition 1.7.

**Proof of Proposition 1.7.** The proof of Proposition 1.7 follows from the estimates (4.11), (4.12), and (4.13) for \( \|u(t, \cdot)\|_{L^2(G)} \), \( \|(-L)^{\alpha/2} u(t, \cdot)\|_{L^2(G)} \), and \( \|\partial_t u(t, \cdot)\|_{L^2(G)} \), respectively. \( \square \)

4.2. **Global in time existence.** This subsection is devoted to prove Theorem 1.8, i.e., the global existence of small data solutions for the fractional Cauchy problem (4.1) in the energy evolution space \( C ([0, T], H^2_0(G)) \cap C^1 ([0, T], L^2(G)) \).

First, we recall some notations to present the proof of Theorem 1.8. Consider the space \( X(T) := C ([0, T], H^2_0(G)) \cap C^1 ([0, T], L^2(G)), \)
equipped with the norm
\[
\|u\|_{X(T)} := \sup_{t \in [0,T]} (A_{b,m^2}(t))^{-1} \left( \|u(t,\cdot)\|_{L^2(G)} + \|(-\mathcal{L})^{\alpha/2}u(t,\cdot)\|_{L^2(G)} + \|\partial_t u(t,\cdot)\|_{L^2(G)} \right),
\]
where \(A_{b,m^2}(t)\) is given by
\[
A_{b,m^2}(t) = \begin{cases} 
    e^{-\frac{b^2}{4}t} & \text{if } b^2 < 4m^2, \\
    (t+1)e^{-\frac{b^2}{4}t} & \text{if } b^2 = 4m^2, \\
    e^{-\frac{b^2}{4}t + \sqrt{\frac{b^2}{4} - m^2} t} & \text{if } b^2 > 4m^2.
\end{cases}
\]

Here we briefly recall the notion of mild solutions in our framework to the Cauchy problem (4.1) and will analyze our approach to prove Theorem 1.8. Applying Duhamel’s principle, the solution to the nonlinear inhomogeneous problem
\[
\begin{aligned}
\partial_t^2 u + (-\mathcal{L})^\alpha u + b\partial_t u + m^2 u = F(t,x), & \quad x \in G, t > 0, \\
\partial_t u(0,x) = u_0(x), & \quad x \in G, \\
\partial_t u(0,x) = u_1(x), & \quad x \in G,
\end{aligned}
\]
(4.15)
can be expressed as
\[
u(t,x) = u_0(x) *_{(x)} E_0(t,x) + u_1(x) *_{(x)} E_1(t,x) + \int_0^t F(s,x) *_{(x)} E_1(t-s,x) \, ds,
\]
where \(*_{(x)}\) denotes the group convolution product on \(G\) with respect to the \(x\) variable. Here, \(E_0(t,x)\) and \(E_1(t,x)\) are the fundamental solutions to the homogeneous problem (4.15), i.e., when \(F = 0\) with initial data \((u_0, u_1) = (\delta_0, 0)\) and \((u_0, u_1) = (0, \delta_0)\), respectively.

For a function \(u\) on \([0,T]\) to be a mild solution to (4.15), we refer to subsection 3.2. Furthermore, if the estimates (3.21) and (3.22) hold uniformly with respect to \(T\) then the solution can be prolonged and defined for any \(t \in (0, \infty)\) which will be our global solution. Now we present the proof of Theorem 1.8.

**Proof of Theorem 1.8.** The expression (3.20) can be written as \(Nu = u^r + I[u]\), where
\[
u^r(t,x) = \varepsilon u_0(x) *_{(x)} E_0(t,x) + \varepsilon u_1(x) *_{(x)} E_1(t,x)
\]
and
\[
I[u](t,x) := \int_0^t |u(s,x)|^p *_{x} E_1(t-s,x) \, ds.
\]

Now, for the part \(u^r\), Theorem 1.7, immediately implies that
\[
\|u^r\|_{X(T)} \lesssim \|(u_0, u_1)\|_{H^2_0(G) \times L^2(G)}.
\]
(4.16)
On the other hand, for the part \(I[u]\), using Minkowski’s integral inequality, Young’s convolution inequality, Theorem 1.7, and by time translation invariance property of the Cauchy problem (4.1), we get
\[
\|\partial_t^j (-\mathcal{L})^{\alpha/2} I[u]\|_{L^2(G)} = \left( \int_G |\partial_t^j (-\mathcal{L})^{\alpha/2} \int_0^t |u(s,x)|^p *_{x} E_1(t-s,x) \, ds|^2 \, dg \right)^{\frac{1}{2}} \\
= \left( \int_G \int_0^t |u(s,x)|^p *_{x} \partial_t^j (-\mathcal{L})^{\alpha/2} E_1(t-s,x) \, ds|^2 \, dg \right)^{\frac{1}{2}}.
\]
determined fixed point

This shows that the map without any conditions on the Banach space $X$ have the uniform boundedness of the integral

Thus combining (4.16) for similar calculations as in (4.17) together with Hölder’s inequality and (3.24), we get

$$
\|\partial_t^1 (-\mathcal{L})^{\alpha/2} (I[u] - I[v])\|_{L^2(G)} \\
\lesssim \int_0^t A_{b,m^2}(t-s)\|u(s,\cdot)|^p - |v(s,\cdot)|^p\|_{L^2(G)} ds \\
\lesssim \int_0^t A_{b,m^2}(t-s)\|u(s,\cdot) - v(s,\cdot)\|_{L^2(G)} \left(\|u(s,\cdot)\|^{p-1}_{L^2(G)} + \|v(s,\cdot)\|^{p-1}_{L^2(G)}\right) ds \\
\lesssim \|u - v\|_{X(t)} \left(\|u\|^{p-1}_{X(t)} - \|v\|^{p-1}_{X(t)}\right) \int_0^t A_{b,m^2}(t-s)A_{b,m^2}(s)^p ds \\
\leq \|u\|^{p}_{X(t)} A_{b,m^2}(t). \tag{4.18}
$$

Thus combining (4.16), (4.17), and (4.18), we have

$$
\|Nu\|_{X(t)} \leq D\|(u_0, u_1)\|_{H^2(G) \times L^2(G)} + D\|u\|^{p}_{X(t)} \tag{4.19}
$$

and

$$
\|Nu - Nv\|_{X(T)} \leq D\|u - v\|_{X(T)} \left(\|u\|^{p-1}_{X(T)} - \|v\|^{p-1}_{X(T)}\right). \tag{4.20}
$$

This shows that the map $N$ turns out to be a contraction in some neighborhood of 0 in the Banach space $X(T)$. Therefore, Banach’s fixed point theorem gives us the uniquely determined fixed point $u$ on $[0, T]$ for the map $N$, which is our mild solution. Not that, thanks to the exponential decay rate $A_{b,m^2}(t)$ both in (4.17) and (4.18) we have the uniform boundedness of the integral

$$
A_{b,m^2}(t)\int_0^t A_{b,m^2}(t-s)A_{b,m^2}(s)^p ds,
$$

without any conditions on $p$. This completes the proof of Theorem 1.8.

We have the following remark regarding Theorem.
5. Final remarks

In [12], we already seen that for the fractional wave operator $\partial^2_t + (-\mathcal{L})^\alpha$ and for the damped wave operator $\partial^2_t + (-\mathcal{L})^\alpha + \partial_t$ defined in Section 3, under some suitable assumptions on the initial data, the local in-time solutions to these Cauchy problem blow up in finite time for any $p > 1$. In other words, we do not get any global in-time existence result in this case. However, in Section 4 of this paper, we have seen that the presence of a positive damping term and a positive mass term in the Cauchy problem completely reverses the scenario. In a similar manner, the fractional damped wave equation on the Heisenberg group will be considered in a forthcoming paper.

6. Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

References

[1] B. Ahmad, A. Alsaedi and M. Kirane, Nonexistence of global solutions of some nonlinear space-nonlocal evolution equations on the Heisenberg group, Electron. J. Differ. Equ., 2015(227), 1–10 (2015).
[2] J.-Ph. Anker and H.-W. Zhang, Wave equation on general noncompact symmetric spaces, (to appear in) Amer. J. Math, 2022. arXiv:2010.08467
[3] D. Applebaum, Lévy processes from probability to finance quantum groups, Notices Amer. Math. Soc., 51, 1336-1347 (2004).
[4] G. Autuori and P. Pucci, Elliptic problems involving the fractional Laplacian in $\mathbb{R}^n$, J. Differential Equations 255(8), 2340-2362 (2013).
[5] A. K. Bhardwaj, V. Kumar, and S. S. Mondal, Estimates for the nonlinear viscoelastic damped wave equation on compact Lie groups, arXiv:2207.06645 (2022).
[6] G. M. Bisci and V. D. Rădulescu, Applications of local linking to nonlocal Neumann problems, Commun. Contemp. Math. 17(1), 1450001 (2015).
[7] G. M. Bisci and V. D. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations, Calc. Var. Partial Differential Equations 54, 2985-3008 (2015).
[8] G. M. Bisci, V. D. Rădulescu, and R. Servadei, Variational methods for nonlocal fractional problems, Encyclopedia of Mathematics and its Applications, 162, Cambridge University Press, Cambridge (2016).
[9] N. Burq, P. Gerard, and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Am. J. Math. 126(3), 569-605 (2004)
[10] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32, 1245-1260 (2007).
[11] M. Caponi and P. Pucci, Existence theorems for entire solutions of stationary Kirchhoff fractional $p$-Laplacian equations, Ann. Mat. Pura Appl. 195, 2099-2129 (2016).
[12] A. Dasgupta, V. Kumar, and S. S. Mondal, Nonlinear fractional wave equation on compact Lie groups, arXiv:2207.04422 (2022).
[13] J. L. A. Dubbeldam, A. Milchev, V. G. Rostiashvili, and T. A. Vilgis, Polymer translocation through a nanopore: a showcase of anomalous diffusion, Phys. Rev. E 76, 010801 (2007).
[14] M. R. Ebert and M. Reissig, Methods for Partial Differential Equations, Birkhäuser, Basel (2018).
[15] V. Georgiev, H. Lindblad, and C. D. Sogge, Weighted Strichartz estimates and global existence for semi-linear wave equations, Am. J. Math. 119(6), 1291–1319 (1997).
[16] C. Garetto and M. Ruzhansky, Wave equation for sums of squares on compact Lie groups, J. Differential Equations 258(12), 4324-4347 (2015).
[17] R. Herrmann, Fractional Calculus: An Introduction for Physicists, World Scientific, Singapore (2011).
[18] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore (2000).
[19] R. Ikehata and K. Tanizawa, Global existence of solutions for semilinear damped wave equations in $\mathbb{R}^n$ with noncompactly supported initial data, Nonlinear Anal. 61(7), 1189-1208 (2005).
[20] L. Kapitanski, Minimal compact global attractor for a damped semilinear wave equation, Commun. Partial. Differ. Equ. 20(7/8), 1303–1323 (1995).
[21] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E, 66, 056108 (2002).
[22] Yu. Luchko and A. Punzi, Modeling anomalous heat transport in geothermal reservoirs via fractional diffusion equations, GEM Int. J. Geomath. 1, 257-276 (2011).
A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, *Publ. Res. Inst. Math. Sci.* 12(1), 169-189 (1976/77).

A. I. Nachman, The wave equation on the Heisenberg group, *Comm. Partial Differential Equations* 7(6), 675-714 (1982).

E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker’s Guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136, 521-573 (2012).

K. B. Oldham and J. Spanier, *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*, Academic Press, London (1974).

A. Palmieri, On the blow-up of solutions to semilinear damped wave equations with power nonlinearity in compact Lie groups, *J. Differential Equations* 281, 85-104 (2021).

A. Palmieri, Semilinear wave equation on compact Lie groups, *J. Pseudo-Differ. Oper. Appl.* 12, 43 (2021).

A. Palmieri, A global existence result for a semilinear wave equation with lower order terms on compact Lie groups, *J. Fourier Anal. Appl.* 28, Article number: 21 (2022).

I. Podlubny, *Fractional differential equations*, Academic press, New York (1999).

M. Ruzhansky and N. Tokmagambetov, Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups, *J. Differential Equations* 265(10), 5212-5236 (2018).

M. Ruzhansky and C. Taranto, Time-dependent wave equations on graded groups, *Acta Appl. Math.* 171, Article number: 21 (2021).

M. Ruzhansky and V. Turunen, *Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics*, Birkhäuser-Verlag, Basel (2010).

M. Ruzhansky and V. Turunen, Global quantization of pseudo-differential operators on compact Lie groups, *SU(2), 3-Sphere, and homogeneous spaces, Int. Math. Res. Not. IMRN* (11), 2439-2496 (2013).

M. Ruzhansky and N. Yessirkegenov, Hardy, Hardy-Sobolev, Hardy-Littlewood-Sobolev and Caffarelli-Kohn-Nirenberg inequalities on general Lie groups, arXiv:1810.08845 (2019).

M. Ruzhansky and N. Yessirkegenov, Very weak solutions to hypoelliptic wave equations, *J. Differential Equations* 268(5), 2063-2088 (2020).

S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives of the fractional-order and their some applications*, Minsk: Nauka i Tekhnika (1987).

R. Servadei and E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.* 389, 887-898 (2012).

Y. Sire, C. D. Sogge, and C. Wang, The Strauss conjecture on negatively curved backgrounds, *Discrete Contin. Dyn. Syst.*, 39:7081–7099, 2019.

C. Taranto, Wave equations on graded groups and hypoelliptic Gevrey spaces, Imperial College London Ph.D. thesis, 2018, arXiv:1804.03544 (2018).

G. Todorova and B. Yordanov, Critical exponent for a nonlinear wave equation with damping, *J. Differ. Equ.* 174(2), 464-489 (2001).

P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, *J. Appl. Mech.* 51(2), 294-298 (1984).

Q. S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, *C. R. Acad. Sci. Paris, Ser. I* 333(2), 109-114 (2001).

H.-W. Zhang, Wave and Klein-Gordon equations on certain locally symmetric spaces, *J. Geom. Anal.*, 30(4):4386–4406, 2020.

H.-W. Zhang, Wave equation on certain noncompact symmetric spaces, *Pure Appl. Anal.* 3:363-386, 2021.

APARAJITA DASGUPTA, ASSISTANT PROFESSOR
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
DELHI, 110016 INDIA.
Email address: adasgupta@maths.iitd.ac.in

VISHVESHWAR KUMAR, PH. D.
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
Ghent University
Krijgslaan 281, Building S8, B 9000 Ghent, Belgium.
Email address: vishveshmishra@gmail.com

SHYAM SWARUP MONDAL
DEPARTMENT OF MATHEMATICS
Indian Institute of Technology Delhi
Delhi, 110 016, India.
Email address: mondalshyam055@gmail.com