A REMARK ON BEAUVILLE’S SPLITTING PROPERTY

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ABSTRACT. Let $X$ be a hyperkähler variety. Beauville has conjectured that a certain subring of the Chow ring of $X$ should inject into cohomology. This note proposes a similar conjecture for the ring of algebraic cycles on $X$ modulo algebraic equivalence: a certain subring (containing divisors and codimension 2 cycles) should inject into cohomology. We present some evidence for this conjecture.

1. INTRODUCTION

For a smooth projective variety $X$ over $\mathbb{C}$, let $A^i(X) = CH^i(X)_{\mathbb{Q}}$ denote the Chow group of codimension $i$ algebraic cycles modulo rational equivalence with $\mathbb{Q}$–coefficients. Intersection product defines a ring structure on $A^*(X) = \bigoplus_i A^i(X)$. In the case of $K3$ surfaces, this ring structure has a remarkable property:

**Theorem 1.1** (Beauville–Voisin [6]). Let $S$ be a $K3$ surface. Let $D_i, D'_i \in A^1(S)$ be a finite number of divisors. Then

$$\sum_i D_i \cdot D'_i = 0 \text{ in } A^2(S) \iff \sum_i [D_i] \cup [D'_i] = 0 \text{ in } H^4(S, \mathbb{Q}).$$

In the wake of this result (combined with results concerning the Chow ring of abelian varieties [4]), Beauville has asked which varieties have behaviour similar to theorem 1.1. This is the problem of determining which varieties verify the “splitting property” of [5]. We briefly state this problem here as follows:

**Problem 1.2** (Beauville [5]). Find a class $C$ of varieties (containing $K3$ surfaces, abelian varieties and hyperkähler varieties), such that for any $X \in C$, the Chow ring of $X$ admits a multiplicative bigrading $A^*_{(\cdot)}(X)$, with

$$A^i(X) = \bigoplus_{j=0}^i A^i_{(j)}(X) \text{ for all } i.$$  

This bigrading should split the conjectural Bloch–Beilinson filtration, in particular

$$A^i_{\text{hom}}(X) = \bigoplus_{j \geq 1} A^i_{(j)}(X).$$

2010 Mathematics Subject Classification. Primary 14C15, 14C25, 14C30.

Key words and phrases. Algebraic cycles, Chow groups, Bloch–Beilinson filtration, hyperkähler varieties, multiplicative Chow–Künneth decomposition, splitting property.
This question is hard to answer in practice, since we do not have the Bloch–Beilinson filtration at our disposal. However, as noted by Beauville, the class $C$ has some nice properties that can be tested in practice. In particular, the conjecture that hyperkähler varieties are in $C$ leads to the so-called weak splitting property conjecture, which is the following falsifiable statement:

**Conjecture 1.3** (Beauville [5], Voisin [18]). Let $X$ be a hyperkähler variety, and let $D^*(X) \subset A^*(X)$ be the $\mathbb{Q}$-subalgebra generated by divisors and Chern classes. The cycle class map induces an injection

$$D^i(X) \hookrightarrow H^{2i}(X, \mathbb{Q})$$

for all $i$.

(cf. [18], [19], [20], [21], [8], [14], [23], [7] for extensions and partial results concerning conjecture 1.3)

An interesting novel approach to problem 1.2 (as well as a reinterpretation of theorem 1.1) is provided by the concept of multiplicative Chow–Künneth decomposition, giving rise to unconditional constructions of a bigraded ring structure on the Chow ring of certain varieties [15], [17], [16], [9] (The bigrading constructed in these works should be seen as a candidate for the (only ideally existing) bigrading evoked in problem 1.2; in particular, it is not known whether property (1) holds for these candidates.)

This note does not directly address problem 1.2 or conjecture 1.3. Instead, our aim is to propose a modified version of conjecture 1.3. The modification consists in considering the groups $B^*(X)$ of cycles with $\mathbb{Q}$-coefficients modulo algebraic equivalence. For any $X \in C$ (in particular, for a hyperkähler variety), the conjectural bigrading $A^*(X)$ is expected to be of motivic origin (i.e., induced by a Chow–Künneth decomposition). As such, one expects the bigrading to pass to algebraic equivalence and induce a bigrading $B^*(X)$. Now, it has been conjectured that (for any smooth projective variety) the deepest level $F^3A^*(X)$ of the conjectural Bloch–Beilinson filtration should be algebraically trivial [10], and so $B^3_{(2)}(X) = 0$. For a hyperkähler variety, one expects that also $B^3_{(1)}(X) = 0$ (this is clear when $X$ is of $K3[n]$ type; for general hyperkähler varieties, one can reason as in the proof of proposition 3.2 below), and so conjecturally

$$B^3(X) = B^3_{(0)}(X).$$

This leads to the following variant of conjecture 1.3:

**Conjecture 1.4.** Let $X$ be a hyperkähler variety. Let $E^*(X) \subset B^*(X)$ be the $\mathbb{Q}$-subalgebra generated by $B^1(X), B^2(X)$ and the Chern classes. The cycle class map induces injections

$$E^i(X) \hookrightarrow H^{2i}(X, \mathbb{Q}) \ \forall i.$$

Here is some evidence we have found for conjecture 1.4:

**Theorem** (=theorem 2.1). Let $X$ be either

(i) a Hilbert scheme $X = S^{[2]}$, where $S$ is a projective $K3$ surface, or

(ii) a Fano variety of lines $X = F(Y)$, where $Y \subset \mathbb{P}^5(\mathbb{C})$ is a very general cubic fourfold.

The cycle class map induces an injection

$$E^3(X) \hookrightarrow H^6(X, \mathbb{Q}).$$
Theorem. Let $X = K_m(A)$ be a generalized Kummer variety of dimension $2m$. The cycle class map induces an injection

$$E^{2m-1}(X) \hookrightarrow H^{4m-2}(X, \mathbb{Q}).$$

Our evidence is, alas, restricted to 1–cycles. The reason for this restriction is that in proving theorems 2.1 and 3.1, we rely on the bigrading of the Chow ring of $X$ constructed unconditionally in [15] resp. [9]. In both cases, it is not known whether the bigrading satisfies property (1) for all $i$ (this is only known for $i \geq \dim X - 1$).

Conventions. In this article, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All groups of cycles will be with rational coefficients: we will denote by $A^j(X)$ the Chow group of $j$–dimensional algebraic cycles on $X$ with $\mathbb{Q}$–coefficients; for $X$ smooth of dimension $n$ we will write $A^i(X) := A_{n-i}(X)$. Likewise, we will write $B^i(X) := B_{n-i}(X)$ for $X$ smooth.

The notations $A^i_{\text{hom}}(X)$, $A^i_{\text{alg}}(X)$ will be used to indicate the subgroups of homologically trivial, resp. algebraically trivial cycles. Likewise, we write $B^i_{\text{hom}}(X)$ for what is commonly known as the Griffiths group of $X$.

We will write $H^j(X)$ to indicate singular cohomology $H^j(X, \mathbb{Q})$.

2. Some Hyperkähler Fourfolds

Theorem 2.1. Let $X$ be either
(i) a Hilbert scheme $X = S^{[2]}$, where $S$ is a projective $K3$ surface, or
(ii) a Fano variety of lines $X = F(Y)$, where $Y \subset \mathbb{P}^5(\mathbb{C})$ is a very general cubic fourfold.

Let $E^i(X) \subset B^i(X)$ be the $\mathbb{Q}$–subalgebra generated by $B^1(X)$, $B^2(X)$ and the Chern classes. Then the cycle class map induces an injection

$$E^i(X) \hookrightarrow H^{2i}(X) \quad \text{for } i \geq 3.$$  

Proof. Since algebraic and homological equivalence coincide for 0–cycles, the $i = 4$ case is trivially true. The interesting part of the statement is thus only the injectivity of $E^3(X) \to H^6(X)$.

In both cases (i) and (ii), there exists a bigraded ring structure $A^*_*(X)$ induced by the Fourier transform constructed in [15]. In both cases, the bigrading is also described by the action of a Chow–Künneth decomposition, and therefore the ring $B^*(X)$ inherits a bigrading $B^*_*(X)$. The Chern classes of $X$ are in $A^1_*(X)$ (in case (i), this is [17] Theorem 2); in case (ii), this follows from the fact that the Chern classes are polynomials in the classes labelled $l \in A^1(X), c \in A^2(X)$ in [15] (coming from the tautological bundle on the Grassmannian), and it is known that $c \in A^2_0(X)$ [15] Theorem 21.9(iii)].

The theorem now follows from the following claim:

Claim 2.2. One has $B^2(X) = B^2_0(X)$. 

Indeed: the claim, combined with the above remarks, implies that $E^*(X) \subset B^*_0(X)$. But we know (lemma 2.3 below) that $B^*_0(X) \to H^6(X)$ is injective, and so theorem 2.1 is proven.

The claim follows from the fact, proven by Shen–Vial [15, Theorems 2.2 and 2.4], that there exists a correspondence $L \in A^2(X \times X)$ with the property that

$$A^2_{(2)}(X) = L_* A^4_{(2)}(X).$$

Indeed, any 0-cycle $a \in A^3_{(2)}(X)$ is (homologically trivial hence) algebraically trivial. As algebraic equivalence is an adequate equivalence relation, it follows that $L_*(a)$ is algebraically trivial and so

$$A^2_{(2)}(X) \subset A^2_{alg}(X).$$

This proves the claim:

$$B^2(X) = A^2(X)/A^2_{alg}(X) = (A^0_{(0)}(X) \oplus A^2_{(2)}(X))/A^2_{alg}(X) = A^0_{(0)}(X)/A^2_{alg}(X) = B^2_{(0)}(X).$$

It only remains to prove the following lemma:

**Lemma 2.3** (Shen–Vial [15]). Let $X$ be either

(i) a Hilbert scheme $X = S^{[2]}$, where $S$ is a projective K3 surface, or

(ii) a Fano variety of lines $X = F(Y)$, where $Y \subset \mathbb{P}^5(\mathbb{C})$ is any smooth cubic fourfold.

Then $A^3_{(0)}(X) \cap A^3_{hom}(X) = 0$.

**Proof.** This is contained in [15]. A quick way of proving the lemma is as follows: let $F$ be the Fourier transform of $A^3_{(0)}(X) = A^1(X)$ [15, Theorem 2]. Suppose $a \in A^3_{(0)}(X)$ is homologically trivial. Then also $F(a) \in A^1(X)$ is homologically trivial, hence $F(a) = 0$ in $A^1(X)$. But then, using [15, Theorem 2.4], we find that

$$\frac{25}{2} a = F \circ F(a) = 0 \quad \text{in } A^3(X).$$

□

□

In theorem 2.1(ii), we restricted to very general cubic fourfolds. The reason is that for the Fano variety $X$ of lines on any given smooth cubic fourfold, it is not yet known that the Fourier decomposition $A^*_v(X)$ is a bigraded ring structure (cf. [15, Remark 22.9]). If we abandon the hypothesis “very general”, we can obtain a weaker statement:

**Definition 2.4.** Let $Y \subset \mathbb{P}^5(\mathbb{C})$ be a smooth cubic fourfold, and let $X$ be the Fano variety of lines on $Y$. One defines

$$A^1(X)_{prim} := P_*(A^2(Y)_{prim}) \subset A^1(X),$$

where $P \in A^3(Y \times X)$ is the universal family of lines, and $A^2(Y)_{prim} := \{ c \in A^2(Y) \mid [c] \in H^4(Y)_{prim} \}$. We set $B^1(X)_{prim} = A^1(X)_{prim}$.
Proposition 2.5. Let \( Y \subseteq \mathbb{P}^5(\mathbb{C}) \) be any smooth cubic fourfold, and let \( X = F(Y) \) be the Fano variety of lines in \( Y \). Let \( b \in B^3(X) \) be a cycle of the form

\[
b = \sum_{k=1}^r a_k \cdot d_k \in B^3(X),
\]

where \( a_k \in B^2(X) \) and \( d_k \in B^1(X)_{\text{prim}} \). Then \( b \) is algebraically trivial if and only if \( b \) is homologically trivial.

Proof. Claim 2.2 still applies to \( X \), and so the \( a_k \) are in \( B^2_{(0)}(X) \). As such, they can be lifted to \( \bar{a}_k \in A^2_{(0)}(X) \). One knows that \( A^2_{(0)}(X) \cdot A^1(X)_{\text{prim}} \subset A^3_{(0)}(X) \) \([15, \text{Proposition 22.7}]\), and thus \( \bar{a}_k \cdot d_k \in A^3_{(0)}(X) \). It follows that \( b \in B^3_{(0)}(X) \), and one concludes using lemma 2.3. \( \square \)

3. Generalized Kummer varieties

Theorem 3.1. Let \( A \) be an abelian surface, and let \( X = K_m(A) \) be a generalized Kummer variety of dimension \( 2m \). Let \( E^*(X) \subset B^*(X) \) be the \( \mathbb{Q} \)-subalgebra generated by \( B^1(X) \), \( B^2(X) \) and the Chern classes. The cycle class map induces injections

\[
E^i(X) \hookrightarrow H^{2i}(X)
\]

for \( i \geq 2m - 1 \).

Proof. Thanks to \([9, \text{Theorem 7.9}]\), there exists a multiplicative Chow–Künneth decomposition for \( X \) and so the Chow ring has a bigrading \( A^*_*(X) \). Moreover, the Chern classes of \( X \) are in \( A^*_*(X) \) \([9, \text{Proposition 7.13}]\). Let \( B^*_*(X) \) denote the induced bigrading modulo algebraic equivalence.

Proposition 3.2. We have

\[
B^2(X) = \bigoplus_{j \leq 0} B^2_{(j)}(X).
\]

Proof. One knows that \( B^i_{(j)}(X) = 0 \) for all \( i > 0 \) \([13, \text{Corollary 18}]\). It remains to check that \( B^2_{(1)}(X) = 0 \). By definition, we have

\[
B^2_{(1)}(X) = (\Pi^X_1)_* B^2(X),
\]

where \( \{\Pi^X_j\} \) is the multiplicative Chow–Künneth decomposition furnished by \([9]\). Since \( B^2_{(1)}(X) \subset B^2_{\text{hom}}(X) \), and \( \Pi^X_3 \) is idempotent, we also have

\[
B^2_{(1)}(X) = (\Pi^X_3)_* B^2_{\text{hom}}(X). \tag{2}
\]

Next, we observe that (as \( X \) is hyperkähler) \( H^3(X, \mathcal{O}_X) = 0 \). Since the generalized Hodge conjecture is known to hold for self–products of abelian surfaces \([1, 7.2.2], [2, 8.1(2)]\), and generalized Kummer varieties are motivated by abelian surfaces in the sense of \([3]\), the generalized Hodge conjecture is true for generalized Kummer varieties (for the usual Hodge conjecture, this
was noted in [22, Theorem 3.3]). In particular, \( H^3(X) \) is supported on a divisor \( D \subset X \), and \( H^{2n-3}(X) \) is supported on a 2–dimensional subvariety \( S \subset X \). Using the Lefschetz \((1,1)\) theorem, one can find a cycle \( \gamma \in A^{2m}(X \times X) \) representing the Künneth component \( \pi_X^X \) and supported on \( S \times D \). For dimension reasons, we have

\[
\gamma_* B^2_{\text{hom}}(X) = 0 .
\]

(Indeed, the action of \( \gamma \) on \( B^2_{\text{hom}}(X) \) factors over \( B^2_{\text{hom}}(\tilde{S}) = 0 \), where \( \tilde{S} \rightarrow S \) denotes a desingularization.) Applying lemma 3.3 below, this implies that also

\[
(\Pi^X_S)_* B^2_{\text{hom}}(X) = 0 ,
\]

and we are done in view of (2).

Here, we have used the following lemma. (The lemma applies to our set–up, because generalized Kummer varieties have finite–dimensional motive [22], [9].)

**Lemma 3.3.** Let \( X \) be a smooth projective variety of dimension \( n \), and assume \( X \) has finite–dimensional motive. Let \( \Pi \) and \( \pi \in A^n(X \times X) \) be such that \( \Pi \) is idempotent and \( \Pi = \pi \) in \( H^{2n}(X \times X) \). Then

\[
\pi_* B^i_{\text{hom}}(X) = 0 \Rightarrow \Pi_* B^i_{\text{hom}}(X) = 0 .
\]

**Proof.** We have

\[
\Pi - \pi \in A^n_{\text{hom}}(X \times X) .
\]

From Kimura’s nilpotence theorem [12], it follows that there exists \( N \in \mathbb{N} \) such that

\[
(\Pi - \pi)^{\otimes N} = 0 \quad \text{in} \quad A^n(X \times X) .
\]

Developing this expression, we obtain

\[
\Pi = \Pi^{\otimes N} = P_1 + P_2 + \cdots + P_m \quad \text{in} \quad A^n(X \times X)_\mathbb{Q} ,
\]

where each \( P_j \) is a composition of correspondences containing at least one copy of \( \pi \). But then (by hypothesis) the right–hand side acts as zero on \( B^i_{\text{hom}}(X) \), and hence so does the left–hand side. \( \square \)

This ends the proof of proposition 3.2. \( \square \)

Proposition 3.2 combined with the fact that the Chern classes are in \( B^*_0(X) \), implies that there is an inclusion

\[
E^*(X) \subset \bigoplus_{j \leq 0} B^*_j(X) .
\]

Theorem 3.1 follows from this inclusion, combined with the following lemma:

**Lemma 3.4.** Let \( X \) be a generalized Kummer variety of dimension \( 2m \). Then

\[
B^j_{(j)-1}(X) = 0 \quad \text{for} \quad j < 0 , \quad \text{and} \quad B^j_{(0)-1}(X) \cap B^{2m-1}_{\text{hom}}(X) = 0 .
\]
To prove the lemma, we note that by definition,

\[ B^{2m-1}(X) = (\Pi^{X}_{4m-2-\cdot}) \ast B^{2m-1}(X), \]

where \(\{\Pi^X_k\}\) is a multiplicative Chow–Künneth decomposition [9]. Inspecting the construction in [9], one finds that \(\Pi^{X}_{4m-1} = 0\) and \(\Pi^{X}_{4m}\) is of the form \(X \times x\), where \(x \in X\). This proves the first statement.

As for the second statement of the lemma, we observe that there exists a cycle \(\gamma \in A^{2m}(X \times X)\) representing the Künneth component \(\pi^{X}_{4m-2}\) and supported on \(X \times S\), where \(S \subset X\) is a smooth surface (this is a general fact, for any variety \(X\) verifying the Lefschetz standard conjecture \(B(X)\), cf. [11, Theorem 7.7.4]). For dimension reasons, we have

\[ \gamma_* B^{2m-1}(X) = 0. \]

(Indeed, the action of \(\gamma\) on \(B^{2m-1}_{\text{hom}}(X)\) factors over \(B^{1}_{\text{hom}}(S) = 0\). By lemma [5,3], this implies that

\[ (\Pi^{X}_{4m-2})_* B^{2m-1}_{\text{hom}}(X) = 0. \]

On the other hand, \(\Pi^{X}_{4m-2}\) is a projector on \(B^{2m-1}_{(0)}(X)\), and so

\[ B^{2m-1}_{(0)}(X) \cap B^{2m-1}_{\text{hom}}(X) = (\Pi^{X}_{4m-2})_* B^{2m-1}_{\text{hom}}(X). \]

\[ \square \]

Acknowledgements. Thanks to all participants of the Strasbourg 2014/2015 “groupe de travail” based on the monograph [20] for a stimulating atmosphere. Thanks to Yoyo, Kai and Len for wonderful Christmas holidays. Thanks to the referee for highly pertinent remarks.

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