Dynamics of a Strongly Damped Two-Level System: Beyond the DBGA *

Tabish Qureshi†

Institute of Mathematical Sciences,
C.I.T. Campus, Taramani, Madras-600113, INDIA

Dynamics of a dissipative two-level system is studied using quantum relaxation theory. This calculation for the first time goes beyond the commonly used dilute bounce gas approximation (DBGA), even for strong damping. The new results obtained here deviate from the DBGA results at low temperatures, however, the DBGA form is recovered at high temperatures. The results in the parameter regime $1/2 < \alpha < 1$, where the model has connection with the Kondo Hamiltonian, are of particular significance. In this regime, the spin shows a cross-over to a slower exponential relaxation at intermediate times, which is roughly half the relaxation rate at short times, as also observed in Quantum Monte-Carlo simulation of the model. The asymptotic behavior of the spin in the Kondo regime is in agreement with the exact conformal field theory results for the Kondo model. A connection of the dissipative dynamics of the two-level system with the quantum Zeno effect is also presented.

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I. INTRODUCTION

For the task of describing the interesting phenomenon of dissipation in quantum systems, two-level systems have served as a simple and tractable prototype. Over the years, several experimental situations have been found which can be accurately described in terms of damped two-level systems. Superconducting Quantum Interference Devices (SQUIDs) for example, are systems where quantum coherence can be studied at a macroscopic level and description in terms of a two-level system is reasonably good. In metallic glasses, where atoms are quenched in random positions, certain atoms can end up finding themselves in a situation where two close by positions are energetically equivalent. So it is easiest for them to quantum mechanically tunnel between the two positions. Such a system, at very low temperatures, behaves like a two-level system. The sea of conduction electrons however impedes this motion. What one ends up with is a dissipative two-level system. Similar situation exists for light interstitial particles in metals, which has become an interesting field of its own and goes by the name of “hydrogen in metals” \cite{2}. An example from the realm of quantum optics is an atom in a radiation field. If one is interested in the radiative properties of just one excited state of the atom, one can imagine it to be a two-level system coupled to a dissipative “bath” of photons. A very well studied problem in condensed matter physics, that of a static spin $1/2$ magnetic impurity in a metal (the so-called Kondo problem) \cite{3}, can also be thought of as a two-level system coupled to the conduction electrons. Later in this paper, we shall see some detailed application of the results to this problem. Recent interest in “quantum computers”, has led to the introduction of some interesting models to address the problem of quantum coherence \cite{4}. These too can be recast in the language of dissipative two-state systems. Lately, in the active field of high temperature superconductivity, dissipative two-level systems have been recalled in the context of $c$-axis transport in the normal state of some high temperature superconductors \cite{5}.

A dissipative two-level system can be generally described by the so-called spin-boson Hamiltonian \cite{6}. It consists of a pseudo-spin $\sigma$, depicting the two-level system and a set of independent harmonic oscillators. The coordinates of the harmonic oscillators are linearly coupled to the $z$-component of the pseudo-spin $\sigma_z$. The Hamiltonian can be expressed as

$$H = \frac{1}{2}\hbar\Delta \sigma_x + \sum_{j=0}^{\infty} \left( \frac{p_j^2}{2m} + \frac{1}{2}m_j\omega_j^2 (x_j - \frac{c_j}{m_j\omega_j}\sigma_z)^2 \right),$$

(1)

where $\Delta$ is like a tunneling matrix element if one has a particle in a double-well kind of a system in mind, $x_j$s and $p_j$s are the coordinates and momenta of the harmonic oscillators, and $c_j$s are their respective coupling constants to the two-level system, i.e., the pseudo-spin. Looking at the Hamiltonian in (1) one would notice that the (pseudo-)spin by virtue of being in a state with $\sigma_z = \pm 1$ or -1, shifts the centers of the harmonic oscillators to the left or right. This results in a dissipative drag on the spin. It turns out that in order to analyse the influence of the “heat-bath” consisting of harmonic oscillators on the spin $\sigma$, one need not demand the individual knowledge of $c_j$s, $m_j$s and $\omega_j$s. One only requires them in a particular combination as they appear in the so-called spectral density function :

$$J(\omega) = \sum_{j=0}^{\infty} \frac{c_j^2}{m_j\omega_j} \delta(\omega - \omega_j).$$

(2)

The spectral density function contains all the information needed to specify the dissipative dynamics of the spin $\sigma$.

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In order to describe the dissipative behavior and introduce irreversibility one assumes a continuous spectrum of frequencies in the bath. The most common form of the spectral density is the “Ohmic” form where \( J(\omega) \) is linear in \( \omega \) for small frequencies:

\[
J(\omega) = \alpha \omega e^{-\omega/\omega_c},
\]

where \( \alpha \) is a dimensionless coupling constant parametrizing the strength of the interaction of the spin with the heat-bath, and \( \omega_c \) is a large cutoff frequency. The Ohmic form of the bath describes a variety of systems for various values of \( \alpha \). For example, for \( 0 \leq \alpha \leq 1/2 \), the spin-boson Hamiltonian describes the low-temperature tunneling dynamics of a positively charged particle in a double-well potential, in the presence of conduction electrons. The harmonic oscillators in this case represent charge density excitations of the electron gas, which behave like bosons. For the case \( 1/2 < \alpha < 1 \), the spin-boson Hamiltonian mimics the Kondo model. The harmonic oscillators here play the role of spin density excitations of the electron gas, and the pseudo-spin symbolizes the magnetic impurity spin. For most purposes, the quantity of interest is the symmetrized spin correlation function

\[
C(t) = \frac{1}{2} < [\sigma_z(0)\sigma_z(t) + \sigma_z(t)\sigma_z(0)] >,
\]

where the angular brackets denote thermal average, and the Heisenberg time evolution of \( \sigma_z \) is dictated by the Hamiltonian (1).

The spin-boson model is a very old one, but for reasons mentioned earlier, it still attracts considerable research attention. Its dynamics is what has been of most interest to researchers. It was first attacked by Leggett and collaborators using the path-integral formalism of Feynman and Vernon. Using the path-integral method, it possible to exactly integrate over the bath degrees of freedom to obtain a reduced dynamics of the pseudo-spin, at least formally. But in order to obtain closed expressions for the reduced dynamics, one has to make certain approximations. The authors employed the so-called dilute bounce gas approximation (DBGA) to the functional integral expression. The results thus obtained turned out to be very good for a wide range of values of the parameter \( \alpha \) and temperature. The functional integral analysis of the spin-boson Hamiltonian based on the instanton technique was considered by many as quite elaborate and formidable. This led to some simpler derivations of the DBGA results using second order Born approximation and resolvent expansion technique. These calculations demonstrated the precise manner in which the DBGA is connected to conventional perturbative techniques.

The DBGA was widely applauded for the simplicity of the results obtained and the accuracy with which they described the dynamics. For the case \( 0 \leq \alpha < 1/2 \), the spin shows weakly damped coherent oscillations at low temperatures. The physical picture is the following. The spin evolves quantum mechanically and is decohered by the effect of the heat-bath over a time which is much longer than \( \Delta^{-1} \). So the spin has time to evolve quantum mechanically before its coherence is destroyed. As the temperature is raised, the excitations of the heat-bath increase in strength, and at a particular temperature, the time over which coherence is destroyed is much smaller than \( \Delta^{-1} \). Here the coherent oscillations of the spin disappear completely, and what is left is an incoherent relaxation. Incoherence sets in faster if \( \alpha \) is large. When \( \alpha \) becomes equal to \( 1/2 \), coherence is destroyed completely at all temperatures. For \( 1/2 < \alpha < 1 \), the spin mimics a Kondo impurity spin and shows relaxation without any coherent behavior. In the region \( 0 \leq \alpha < 1/2 \), the DBGA is very good at not very low temperatures and at not very long times. In the incoherent relaxation behaviour, the spin decays with a rate which follows a power-law with temperature. At very low temperatures DBGA breaks down unless the coupling \( \alpha \) is very weak. There have been some calculations which have gone beyond the DBGA for weak coupling. It has been demonstrated that for an unbiased two-level system, weak coupling, those calculations essentially yield the DBGA result. Thus it seems that the so-called “inter-bounce” interactions, at low temperature, vanish as \( \alpha^2 \), which means that for strong coupling, inter-bounce interactions have to be taken into account. This territory has been explored in the present investigation. In the Kondo regime too, the DBGA is quite good at not very long times, or small frequencies. In fact, DBGA yielded new analytical results for the dynamics of the Kondo spin, which reproduce many old perturbative results for the Kondo problem, in various limits, and generate very good fits to spectroscopic data in dilute magnetic alloys. Inspite of this success, DBGA fails badly at long times, and the question regarding the low temperature, long times dynamics of the Kondo spin remains open. In fact, behavior of \( C(t) \) in the regime \( T = 0, 1/2 < \alpha \leq 1 \) is considered a currently unresolved problem (see section IV E of [3]).

In the present study, I put forward a calculation based on the resolvent expansion technique. The results achieved, detail the dynamics of the spin at low, as well as high temperatures, for all relevant values of \( \alpha \). I shall present some new results in the difficult regime mentioned in the last para. The asymptotic behavior of the Kondo spin seen here agrees with the exact asymptotic results derived by Affleck and Ludwig for the Kondo problem using conformal field theory. This serves as a stringent test of the approximation employed here.

### II. “NON-PERTURBATIVE” EXPANSION

I begin by making a unitary transformation on the Hamiltonian which pulls back the centers of the shifted oscillators by the amount \( c_j \sigma_z \). The unitary operator for
the purpose is given by \( S \equiv e^{-2\sigma_x\xi} \) where \( \xi = \sum c_j^2 p_j \).
The operation \( SHS^{-1} \) on (1) leads to
\[
H' = \frac{1}{2} \hbar \Delta (\sigma_- e^{-\xi} + \sigma_+ e^{\xi}) + \sum_{j=0}^{\infty} \left( \frac{p_j^2}{2m} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right).
\]
(5)

Now I expand the exponential factors in the first term in (1) in a Taylor series and write them as
\[
e^{\pm \xi} = 1 + \sum_{n=1}^{\infty} \frac{(\pm \xi)^n}{n!}.
\]
(6)
Substituting (6) in (5) and regrouping terms I obtain
\[
H = \frac{1}{2} \hbar \Delta \sigma_z + \frac{1}{2} \hbar \Delta \left\{ \sigma_z \sum_{n=1}^{\infty} \frac{\xi^{2n}}{(2n)!} + i \sigma_y \sum_{n=1}^{\infty} \frac{\xi^{2n-1}}{(2n-1)!} \right\}
+ \sum_{j=0}^{\infty} \left( \frac{p_j^2}{2m} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right).
\]
Finally, I rotate my spin space basis about the y-axis, by \( \pi/2 \) to arrive at
\[
H'' = \frac{1}{2} \hbar \Delta \sigma_z + \frac{1}{2} \hbar \Delta \left\{ \sigma_z \sum_{n=1}^{\infty} \frac{\xi^{2n}}{(2n)!} + i \sigma_y \sum_{n=1}^{\infty} \frac{\xi^{2n-1}}{(2n-1)!} \right\}
+ \sum_{j=0}^{\infty} \left( \frac{p_j^2}{2m} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right).
\]
(8)
Notice that in the new basis while \( \sigma_z \) has become entangled with the bath operators, \( \sigma_z \) is just rotated to \( -\sigma_z \).
Consequently, for evaluating \( C(t) \) one just has to replace \( \sigma_z \) by \( -\sigma_z \), and \( H \) by \( H'' \). Now the cleverest way to do perturbation theory would be to locate terms depending either on the spin coordinates or only on the bath coordinates, and treat them exactly. Following this idea I split the Hamiltonian in (8) into a Hamiltonian for the spin \( H_S \), Hamiltonian for the bath \( H_B \), and the Hamiltonian for the spin bath interaction \( H_I \), given by
\[
H_S = \frac{1}{2} \hbar \Delta \sigma_z,
\]
\[
H_B = \sum_{j=0}^{\infty} \left( \frac{p_j^2}{2m} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right).
\]
\[
H_I = \frac{1}{2} \hbar \Delta \left\{ \sigma_z \sum_{n=1}^{\infty} \frac{\xi^{2n}}{(2n)!} + i \sigma_y \sum_{n=1}^{\infty} \frac{\xi^{2n-1}}{(2n-1)!} \right\}.
\]
(9)
It is now possible to treat \( H_I \) as a perturbation, which does not amount to a weak coupling between the spin and the bath. One will notice that even if \( H_I \) is treated to second order, the final expression will contain \( \alpha \) to arbitrary order due to the infinite series in \( \xi \). The more important point is that, in addition to \( \alpha \), the final expression will also contain \( \Delta \) to arbitrary order, due to the exact treatment of \( H_S \). This is very important from the point of view of the Kondo problem, because \( \Delta \) is proportional to \( J_\perp \), the transverse part of the anisotropic exchange coupling in the Kondo Hamiltonian. This modus operandi will keep both \( J_\perp \) and \( J_\parallel \) to all orders and hence is expected to produce a much better perturbation theory.

Instead of looking at the spin correlation function \( C(t) \) itself, it is convenient to handle its Laplace transform \( \tilde{C}(s) \):
\[
\tilde{C}(s) = \int_0^\infty e^{-st} C(t) dt,
\]
(10)
where \( s \) is the Laplace transform variable. In order to write out the Laplace transform of (3) explicitly, the time evolution of \( \sigma_z \) in the Heisenberg representation can be expressed as \( \exp(i\hbar H''t) \sigma_z \exp(-i\hbar H''t) \). Introducing the Liouvillian associated with \( H \) as \( L(...) = \frac{1}{\hbar} [H, ...] \), the Heisenberg evolution of \( \sigma_z \) can be expressed in a more compact notation \( e^{iLt} \sigma_z \). The “superoperator” \( e^{iLt} \) completely specifies the time evolution of the system and is referred to as the time evolution operator, denoted by \( \tilde{U}(t) \). Further, I assume that the initial density matrix of the total system is factorized as \( \rho_S \cdot \rho_B \). The Laplace transformed correlation function then assumes the form
\[
\tilde{C}(s) = Tr[\rho \left\{ \sigma_z (\frac{1}{s-iL} \sigma_x) + (\frac{1}{s-iL} \sigma_x) \sigma_z \right\}],
\]
(11)
where the \( (s-iL)^{-1} \) is the Laplace transform of the time evolution operator. As the \( \sigma_z \)s do not depend on the bath variables, one can perform a trace over the bath variables to obtain a “bath averaged” time evolution operator \( \langle \tilde{U}(s) \rangle_B \), rather its Laplace transform. The tracing of the bath variables can be formally performed by introducing a projection operator \( P \) which when acting on an operator \( A \) is defined by \( PA \equiv Tr_{\rho_B}(\rho_B A) \). All the information regarding the effect of the bath on the spin is contained in the averaged time evolution. The major task then is to evaluate \( \langle \tilde{U}(s) \rangle_B \) in a suitable approximation. The average time evolution operator can be shown to satisfy the following integro-differential equation
\[
\frac{d}{dt} \langle \tilde{U}(t) \rangle_B = iL_S \langle \tilde{U}(t) \rangle_B + \int_0^t \langle M(t-t) \rangle_{av} \langle \tilde{U}(t) \rangle_B dt,
\]
(12)
where \( \langle M(t-t) \rangle_{av} \) is called the memory function. The Laplace transform of the memory function looks like
\[
\tilde{M}(s)_{av} = P(iL_I)(1-P)
\]
\[
\frac{1}{s-iL_S-iL_B-(1-P)(iL_I)(1-P)}
\]
\[
(1-P)(iL_I)P,
\]
(13)
where $L_S$, $L_B$ and $L_I$ are the Liouvillians associated with $H_S$, $H_B$ and $H_I$, respectively. In terms of the memory function, the average density operator has deceptively compact form

$$<\hat{U}(s)>_B = \frac{1}{s-iL_S - [M(s)]_{av}}.$$  

(14)

The form is deceptive because $<\hat{U}(s)>_B$ is still exact and a complicated object to calculate. I shall restrict myself to treating $[M(s)]_{av}$ to second order in $L_I$, which is manageable. The corresponding expression for the memory function takes up the subsequent form:

$$[M(s)]_{av} \approx i(PL_I) + (PL_I)\frac{1}{s-iL_S}(PL_I)$$

$$+ P\{L_I\frac{1}{s-iL_S-iL_BL_I}\}.$$  

(15)

The strategy is to first perform the trace over the bath states in (15), which will give a $4 \times 4$ matrix in the spin states. This matrix can then be plugged in the denominator of (14). A $4 \times 4$ matrix inversion then yields the averaged time evolution operator $<\hat{U}(s)>_B$ which one requires. The matrix elements of the last term in (15) can be written in terms of correlation functions of certain bath operators, which can be calculated using the properties of a bath of independent harmonic oscillators alone. In the ensuing analysis the spin states will be denoted by Greek indices, and the number states of a set of harmonic oscillators by $| n >, | n' >$ etc. We assume a canonical form of the density matrix at a temperature $T$. The Laplace transformed spin correlation function can be written in terms of the matrix elements of $<\hat{U}(s)>_B$:

$$\tilde{C}(s) = \frac{1}{2}\{(++|<\hat{U}(s)>_B|++) + (++|<\hat{U}(s)>_B|-+) + (+-|<\hat{U}(s)>_B|++),$$

$$\{(-+|<\hat{U}(s)>_B|++) + (-+|<\hat{U}(s)>_B|-+)\},$$  

(16)

which can be rewritten in terms of a Laplace transform

$$\chi''(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} <[\sigma_z(t),\sigma_z(0)]> dt,$$  

(17)

which is still $\chi''(\omega)$ as $<\hat{U}(s)>_B$, which is symmetric function of $\omega$. Now the stage is set to compute the time evolution operator and put the corresponding matrix elements in (16) and (19).

### III. RESULTS

I will skip the details of the algebra involved, which was handled using Mathematica, and present the final form of the averaged time evolution operator. Only the portion of the $4 \times 4$ matrix for $<\hat{U}(s)>_B$, within the space $| + +, | - + >$ is displayed:

$$<\hat{U}(s)>_B = \frac{1}{Det} \left(\begin{array}{cc}
F(s) + F(\omega_+) & F(\omega_-) \\
F(\omega) & s - F(\omega) + F(\omega_-)
\end{array}\right)$$  

(20)

where

$$Det = [s + F(\omega_+)] [s + F(\omega_-)] + F(\omega) [z + \frac{1}{2} (F(\omega_+) + F(\omega_-))].$$  

(21)

In (20) and (21) $\omega_\pm = \omega \pm \Delta$, and $F(\omega)$ is the Laplace transform (with $s = i\omega$) of the following quantity,

$$F(t) = \Delta^2 \exp \left(-2\alpha \int_0^\infty \frac{1}{\omega} e^{-\omega/\omega_c} \text{coth}(\hbar \beta \omega)$$

$$\{1 - \cos(\omega t)\} i \sin(\omega t) d\omega + c.c.\right)$$  

(22)

The correlation function (16) now looks like

$$\tilde{C}(s) = \frac{1}{s + F(\omega) + [F(\omega_+) + F(\omega_-)]/2 - \frac{1}{\pi} [F(\omega_+ - F(\omega_-))]^{1/2}}.$$  

(23)

The integral in (22) can be done for $t, \hbar \beta \gg 1/\omega_c$, so that its Laplace transform assumes the following structure

$$F(\omega) = \frac{\Delta^2}{2\omega_c} \left(\frac{2\pi}{\hbar \beta \omega_c}\right)^{2\alpha - 1} \frac{\Gamma(1 - 2\alpha) \Gamma(\alpha + i\omega \hbar \beta / 2\pi)}{\Gamma(1 - \alpha + i\omega \hbar \beta / 2\pi)} \cos(\pi \alpha),$$  

(24)

where $\Gamma$ is Euler’s Gamma function. Similarly, $\chi''(\omega)$ becomes

$$\chi''(\omega) = \frac{1}{\pi} Re \left[\frac{\tanh(\hbar \beta \Delta/2) [F(\omega_+) - F(\omega_-)]/2}{\text{Re} \left[\frac{s + F(\omega_+)}{s + F(\omega_-)}\right] + F(\omega) [s + \frac{1}{2}(F(\omega_+) + F(\omega_-))]}.\right]$$  

(25)

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The correlation function expressed by (23) describes the complete dynamics of the spin in the spin-boson model. For $T=0$, one can do better than this by taking the limit $T \to 0$ in (22) itself. In this case we do not need to make any approximation, and the exact form of $F(\omega)$ is given by

$$F(\omega) = -i(\omega_c/2)(\Delta/\omega_c)^2(\omega/\omega_c)^{2\alpha-1}e^{\omega/\omega_c}\Gamma(1-2\alpha,\omega/\omega_c)$$

$$-i(\omega_c/2)(\Delta/\omega_c)^2(-\omega/\omega_c)^{2\alpha-1}$$

$$e^{-\omega/\omega_c}\Gamma(1-2\alpha,-\omega/\omega_c),$$

where $\Gamma(a,b)$ is Euler’s incomplete Gamma function. Interestingly, $\omega_c$ can be completely “scaled out” of (26), and hence the correlation function.

### IV. DISCUSSION

The equations (23) and (25) describe the dynamics of the spin at all temperatures and all values of the coupling strength. For the sake of comparison let us look at the corresponding DBGA result, which has the following form:

$$\hat{C}(s) = \frac{1}{s + 2F(\omega)}.$$  

(27)

The reader will notice that if the $\Delta$ dependence in the arguments of the Gamma functions in (23) is ignored, it reduces to the DBGA result (27). Let us discuss the different parameter regimes in somewhat greater detail. Figure 1 Re$\hat{C}(s)$ plotted against $\omega$ for $\omega_c/\Delta = 1000$, $\alpha = 0.1$ and $k_B T/\hbar \Delta = 0.5$. The solid line denotes the present calculation whereas the dashed line denotes the DBGA result.

#### A. The case $0 \leq \alpha < 1/2$

For this range of values of $\alpha$, the spin shows a coherent evolution at low temperatures. This coherent evolution is damped by the dissipative interaction. In this situation the spin-boson model can describe the coherent tunneling behavior of a particle in a double well potential, interacting with conduction electrons. Figures 1 shows real part of $\hat{C}(s)$ which is related to the structure factor for neutron scattering off the tunneling particle. The correlation function shows two ‘inelastic’ peaks which is a signature of coherent evolution. The position of the peaks is clearly shifted from the DBGA values.

As the temperature is raised the peaks broaden and shift towards the origin. At a characteristic temperature the function assumes the form of a single peak at $\omega = 0$. The coherent behavior is completely destroyed and the spin relaxes incoherently with an exponential decay rate. Figure 2 shows the crossover as a function of temperature. It is convenient to introduce a frequency $\Delta_r \equiv \Delta(2\pi/\hbar \omega_c)^\alpha \sqrt{\Gamma(1-2\alpha)}$ which is approximately the effective tunnel splitting within the DBGA. For $T \gg \Delta_r$ the correlation function $\hat{C}(s)$ can be approximated by

$$\hat{C}(s) \approx \frac{1}{s + F(0) + [F(\Delta) + F(-\Delta)]/2 + \left[F(\Delta) - F(-\Delta)\right]^2}/2.$$  

(28)

The expression in (28) depicts a spin relaxing with an exponential decay rate which goes like $T^{2\alpha-1}$. The tunneling particle tends to localize as temperature is increased.

#### B. Relation to quantum Zeno effect

One way to understand this decrease in relaxation rate is in terms of the so-called quantum Zeno effect. This effect was proposed by Mishra and Sudarshan \[17\] for a quantum system on which a series of measurements are made. The limit of ‘continuous’ measurement results in a freezing of the ‘free’ dynamics of the system completely. In the problem at hand, the heat-bath consisting of the harmonic oscillators is sensitive to the position of the particle, the $\sigma_z$-state of the spin in the problem at hand, as is obvious from the form of the Hamiltonian (1). In some sense the heat bath monitors the position of the particle in the double-well. By virtue of its possessing infinite degrees of freedom, it is capable of ‘collapsing’ the wave function of the tunneling particle to a particular well. With increasing temperature the measurement becomes more and more efficient. In addition, the interaction being present all the time amounts to a continuous measurement of the particle being in one of the two wells. Thus the coherent dynamics of the particle between the wells is destroyed and it tends to ‘freeze’ in one of the wells. Here one can easily work out the ‘decoherence time scale’ which has been a topic of some recent controversy \[18,19\], in terms of the parameters of the Hamiltonian. Because of the finite relaxation time of the particle, complete freezing will not take place, and the particle will occasionally tunnel from one well to the other. I think this new way of looking at the dynamics of the tunneling particle helps in understanding the non-classical behavior seen here.

Coming back to the issue of the dynamics of the spin-boson model, the value of $\alpha$ is crucial in deciding whether
the spin can have coherent dynamics or not. The coherent behavior is destroyed at $\alpha = 1/2$. At $\alpha = 1/2$, $C(t)$ decays with a single relaxation rate $\pi \Delta^2/2\omega_c$, which is what DBGA also yields.

\[ \Gamma \]

C. The case $1/2 \leq \alpha < 1$

Beyond $\alpha = 1/2$ we enter the interesting regime where the spin-boson model is related to the Kondo Hamiltonian. As mentioned earlier, the results (23)-(25) contain $\Delta$, and hence $J_1$, to arbitrary order. Hence, the calculation is expected to be better than that in [14]. In order to obtain exponential relaxation rates which are relevant for experimentalist, I first neglect the frequency dependence in the arguments of the Gamma functions in (23). The function $\hat{C}(s)$ is anyway concentrated around $\omega = 0$. This results in the following form of the correlation function

\[ \hat{C}(s) \approx \frac{1}{s + F(0) + [F(\Delta) + F(-\Delta)]/2 + [F(\Delta) - F(-\Delta)]^2/4}, \]

\[ (29) \]

The above expression can be decomposed into partial fractions as

\[ \hat{C}(s) = \frac{A}{s + \Gamma_1} + \frac{B}{s + \Gamma_2}, \]

\[ (30) \]

where

\[ \Gamma_{1,2} = \frac{1}{2} \left[ F(0) + F(\Delta) + F(-\Delta) \right] \]

\[ \pm \frac{1}{2} \sqrt{F^2(0) - [F(\Delta) - F(-\Delta)]^2}, \]

\[ A, B = \frac{1}{2} \pm \frac{F(0)/2}{\sqrt{F^2(0) - [F(\Delta) - F(-\Delta)]^2}} \]

\[ (31) \]

The spin shows two exponential decay rates $\Gamma_1$ and $\Gamma_2$. The Quantum Monte Carlo simulation of the spin-boson model by Egger and Weiss revealed that after an initial exponential relaxation, the spin shows a slower exponential relaxation [24]. This second slower decay rate was estimated to be roughly half the initial decay rate. The initial decay rate is seen to match the DBGA value. In order to make contact with the QMC prediction I consider the special case $F^2(0) \gg -[F(\Delta) - F(-\Delta)]^2$ which is satisfied for the parameters chosen by Egger and Weiss. For this case I arrive at

\[ \Gamma_1 \approx F(0) + [F(\Delta) + F(-\Delta)]/2 \]

\[ \Gamma_2 \approx [F(\Delta) + F(-\Delta)]/2 \]

\[ A \approx 1 - \frac{-[F(\Delta) - F(-\Delta)]^2/2}{F^2(0) - [F(\Delta) - F(-\Delta)]^2} \]

\[ B \approx \frac{\mp (\Delta) - F(-\Delta)]^2/2}{F^2(0) - [F(\Delta) - F(-\Delta)]^2} \]

\[ (32) \]

Clearly in (32), $\Gamma_2 \approx \Gamma_1 / 2$. Moreover the quantity $B = -[F(\Delta) - F(-\Delta)]^2$ being small, $\Gamma_1$ dominates for small times. The second decay being slower, shows up at long times.

In the Kondo regime, expression (23) for $\hat{C}(s)$ is valid for all temperatures of practical interest. Thus the dynamics is also described at very low temperature and small frequency, a regime inaccessible in [22]. To stress this point let us look at the absorptive part of the spin susceptibility $\chi''(\omega)$. An exact relation derived by Shiba [27] for the general Anderson model says that at $T = 0$, $\chi''(\omega)$ should vanish linearly in $\omega$:

\[ \lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = 2\pi \chi_0'. \]

\[ (33) \]

This implies that $C(t)$ will asymptotically behave as $\sim t^{-2}$ [8]. DBGA predicts an asymptotic behavior $\sim t^{-2(1-\alpha)}$, and hence a divergent spectral function $S(\omega)$. Very recently, conformal field theory has been employed to calculate the exact asymptotic spin correlation function for the Kondo Hamiltonian $\hat{C}(s)$. The exact results show that the spin correlation function asymptotically decays as $\sim t^{-2}$. Let us take a closer look at the $\chi''(\omega)$ calculated here. In order to make the complicated expression (25) more tractable, I consider the case $\omega < k_B T/\hbar \ll \Delta$. In this situation $F(\omega_{\pm})$ become temperature independent, and $F(\omega)$ becomes negligible in their comparison. It is convenient to introduce a parameter $\epsilon \equiv 1 - \alpha$ which is related to the $J_1$ in the Kondo problem. In reality, it may be treated as a small parameter. For small $\omega$ one may expand the various terms involved in a Taylor series:

\[ F(\omega_{\pm}) + F(\omega_{-}) \approx 2\gamma \Delta^{1-2\epsilon} \sin(\pi \epsilon) + 2i\gamma \omega \Delta^{-2\epsilon} \cos(\pi \epsilon) \]

\[ F(\omega_{\pm}) - F(\omega_{-}) \approx 2i\gamma \Delta^{1-2\epsilon} \cos(\pi \epsilon) + 2\gamma \omega \Delta^{-2\epsilon} \sin(\pi \epsilon) \]

\[ (34) \]

where

\[ \gamma = \frac{\Delta^2}{4\omega_c} \frac{\pi}{\Gamma(2\epsilon) \sin(\pi \epsilon)}. \]

\[ (35) \]

In this approximation $\chi''(\omega)$ assumes the following form

\[ \chi''(\omega) \approx \frac{1}{\pi} Re \]

\[ \frac{i\Delta + \omega \tan(\pi \epsilon)}{(\gamma \cos(\pi \epsilon)) + \Delta \tan(\pi \epsilon) + i\omega} \]

\[ \frac{-\pi \omega}{\Delta} \cos(\pi \epsilon) [i\Delta + \omega \tan(\pi \epsilon)]^2 \]

\[ (36) \]

From the above expression it is clear that, $\lim_{\omega \to 0} \chi''(\omega) \sim \omega$. This implies a finite static susceptibility, and an asymptotic correlation function going as $\sim t^{-2}$, which is a rigorous result from conformal field theory [14]. The major achievement of this work is that the low temperature and long time dynamics of the spin in the Kondo regime is solved and the results seem quite satisfactory.

In conclusion, I have presented a calculation of the dynamics of a strongly damped two-level system which
goes beyond the DBGA. In the coherent regime the results substantially deviate from the corresponding DBGA ones. In the Kondo regime, the short and intermediate time behavior agrees with the Quantum Monte Carlo simulation of Egger and Weiss. The spin crosses over to a slower exponential relaxation after an initial fast decay. The asymptotic behavior of the spin is in agreement with the exact conformal field theory results. An interesting relation to the quantum Zeno effect is presented. Finally, it should be mentioned that it is straightforward to carry out a similar calculation for a biased two-level system.

† Electronic address: tabish@imsc.ernet.in
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