ARNOLD-TYPE INVARIANTS OF CURVES ON SURFACES

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Abstract. Recently V. Arnold introduced Strangeness and $J^\pm$ invariants of generic immersions of an oriented circle to $\mathbb{R}^2$. Here these invariants are generalized to the case of generic immersions of an oriented circle to an arbitrary surface $F$. We explicitly describe all the invariants satisfying axioms, which naturally generalize the axioms used by V. Arnold.

By a surface we mean any smooth two-dimensional manifold, possibly with boundary.

1. Introduction

Consider the space $\mathcal{F}$ of all curves (immersions of an oriented circle) on a surface $F$. We call a curve generic, if its only multiple points are double points of transversal self-intersection. Nongeneric curves form a discriminant hypersurface in $\mathcal{F}$. There are three main strata of the discriminant. They are formed by curves with a triple point, curves with a self-tangency point, at which the velocity vectors of the two branches are pointing to the same direction (direct self-tangency) and curves with a self-tangency point, at which the velocity vectors of the two branches are pointing to the opposite directions (inverse self-tangency). The union of these strata is dense in the discriminant. In [1] V. Arnold associated a sign to a generic crossing of each of these strata. He also introduced $\text{St}$, $J^+$ and $J^-$ invariants of generic curves on $\mathbb{R}^2$, which change by a constant under a positive crossing of the triple point, direct self-tangency and inverse self-tangency strata, respectively, and do not change under crossings of the other two strata. These invariants give a lower bound for the number of crossings of each part of the discriminant, which are necessary to transform one generic curve on $\mathbb{R}^2$ to another.

We construct generalizations of these invariants to the case when $F$ is any surface (not necessarily $\mathbb{R}^2$). The fact, that for most surfaces the fundamental group is nontrivial, allows us to subdivide each of the three strata of the discriminant into pieces. We show that this subdivision is natural from the point of view of the singularity theory. We take an integer valued function $\psi$ on the set of pieces obtained from one stratum, and try to construct an invariant which increases by $\psi(P)$ under a positive crossing of $P$ and does not change under crossings of the other two strata. In an obvious sense $\psi$ is a derivative of such an invariant and the invariant is an integral of $\psi$. We introduce a condition on $\psi$ which is necessary and sufficient for existence of such an invariant. Any integrable, in the sense above, function $\psi$ defines this kind of an invariant up to an additive constant.

If the surface $F$ is orientable, then the condition which corresponds to the generalizations of $J^+$ and $J^-$ is automatically satisfied and such an invariant exists for any function $\psi$. For the generalization of $\text{St}$ the condition is not trivial. We reduce
it to a simple condition on $\psi$ which is sufficient for existence of such an invariant. All these conditions are satisfied in the case of orientation reversing curves.

When this work was complete and the main results of it were published as preprints of Uppsala University and I received a preprint of A. Inshakov containing similar results, obtained by him independently.

2. Arnold’s Invariants

2.1. Basic facts and definitions. A curve is a smooth immersion of (an oriented circle) $S^1$ into a (smooth) surface $F$.

A generic curve has only ordinary double points of transversal self-intersection. All nongeneric curves form in the space of all curves a discriminant hypersurface, or for short, the discriminant.

A self-tangency point of (an oriented) curve is called a point of a direct self-tangency, if the velocity vectors at this point have the same direction; otherwise it is called a point of an inverse self-tangency.

A coorientation of a smooth hypersurface in a functional space is a local choice of one of the two parts, separated by this hypersurface, in a neighborhood of any of its points. This part is called positive.

The coorientation of the smooth part of a singular hypersurface is called consistent, if the following consistency condition holds in a neighborhood of any singular point of any stratum of codimension one on the hypersurface (of codimension two in the ambient functional space):

The intersection index of any generic small oriented closed loop with a hypersurface (defined as a difference between the numbers of positive and negative intersections) should vanish.

A hypersurface is called cooriented, if a consistent coorientation of its smooth part is chosen, and coorientable, if such a coorientation exists.

There are three parts of the discriminant hypersurface formed by the curves having triple points, having direct self-tangencies, and having inverse self-tangencies, respectively.

Lemma 2.1.1 (Arnold). Each of these three parts of the discriminant hypersurface is coorientable.

Consider a transversal crossing of the triple point stratum of the discriminant. A vanishing triangle is the triangle formed by the three branches of the curve, corresponding to a subcritical or to a supercritical value of the parameter near the triple point of the critical curve.

The sign of a vanishing triangle is defined by the following construction. The orientation of the immersed circle defines the cyclic order on the sides of the vanishing triangle. (It is the order of the visits of the triple point by the three branches.) Hence, the sides of the triangle acquire orientations induced by the ordering. But each side has also its own orientation, which may coincide, or not, with the orientations defined by the ordering.

For each vanishing triangle we define a quantity $q \in \{0, 1, 2, 3\}$ to be the number of sides of the vanishing triangle equally oriented by the ordering and their direction. The sign of the vanishing triangle is $(-1)^q$. 
Definition 2.1.2 (of the sign of a crossing of a stratum). A transversal crossing of the direct self-tangency or of the inverse self-tangency stratum of the discriminant is positive, if the number of double points increases (by two).

A transversal crossing of the triple point stratum of the discriminant is positive, if the new-born vanishing triangle is positive.

2.2. Invariants $\text{St}, J^+$ and $J^-$. The index of an immersion of an oriented circle into an oriented plane is the number of turns of the velocity vector. (The degree of the mapping sending a point of the circle to the direction of the derivative of the immersion at this point.) The Whitney Theorem [5] says that the connected components of the space of oriented planar curves are counted by the indices of the curves.

Consider one of these components, that is, the space of immersions of a fixed index.

Theorem 2.2.1 (Arnold [1]). There exists a unique (up to an additive constant) invariant of generic planar curves of a fixed index, whose value remains unchanged under crossings of the self-tangency strata of the discriminant, but increases by one under the positive crossing of the triple point stratum of the discriminant.

This invariant is denoted by $\text{St}$ (from Strangeness), when normalized by the following conditions:

$$\text{St}(K_0) = 0, \quad \text{St}(K_{i+1}) = i \quad (i = 0, 1, \ldots),$$

where $K_0$ is the figure eight curve and $K_{i+1}$ is the simplest curve with $i$ double points (see Figure 1). The curve $K_j$ has index $\pm j$, depending on the orientation.

Theorem 2.2.2 (Arnold [1]). There exists a unique (up to an additive constant) invariant of generic planar curves of a fixed index, whose value remains unchanged under a crossing of the inverse self-tangency or of the triple point strata of the discriminant, but increases by two under the positive crossing of the direct self-tangency stratum of the discriminant.

This invariant is denoted by $J^+$, when normalized by the following conditions:

$$J^+(K_0) = 0, \quad J^+(K_{i+1}) = -2i \quad (i = 0, 1, \ldots),$$

where $K_0$ and $K_{i+1}$ are the curves shown in Figure 1.

Theorem 2.2.3 (Arnold [1]). There exists a unique (up to an additive constant) invariant of generic planar curves of a fixed index, whose value remains unchanged under a crossing of the direct self-tangency or of the triple point strata of the discriminant, but decreases by two under the positive crossing of the inverse self-tangency stratum of the discriminant.
This invariant is denoted by $J^-$, when normalized by the following conditions:

$$J^-(K_0) = -1, \quad J^-(K_{i+1}) = -3i \quad (i = 0, 1, \ldots),$$

(3)

where $K_0$ and $K_{i+1}$ are the curves shown in Figure 1.

These normalizations of the three invariants were chosen to make them independent of the orientation of the parameterizing circle and additive under the connected summation of planar curves.

3. Strangeness-type Invariant of Curves on Surfaces.

3.1. Natural decomposition of the triple point stratum.

Definition 3.1.1. Let $F$ be a surface. We say that a curve $\xi \subset F$ with a triple point $q$ is a generic curve with a triple point, if its only nongeneric singularity is this triple point, at which every two branches are transverse to each other.

3.1.2. Let $F$ be a surface. Let $B_3$ be a bouquet of three oriented circles with a fixed cyclic order on the set of them, and let $b$ be the base point of $B_3$. Let $s : S^1 \to F$ be a generic curve with a triple point $q$.

Let $\alpha : S^1 \to B_3$ be a continuous mapping such that:

a) $\alpha(s^{-1}(q)) = b$.

b) $\alpha$ is injective on the complement of $s^{-1}(q)$.

c) The orientation induced by $\alpha$ on $B_3 \setminus b$ coincides with the orientation of the circles of $B_3$.

d) The cyclic order induced on the set of circles of $B_3$ by traversing $\alpha(S^1)$ according to the orientation of $S^1$ coincides with the fixed one.

The mapping $\phi : B_3 \to F$ such that $s = \phi \circ \alpha$ is called an associated with $s$ mapping of $B_3$.

Note, that the free homotopy class of the mapping of $B_3$ to $F$ realized by $\phi$ is well defined, modulo an automorphism of $B_3$ which preserves the orientation of the circles and the cyclic order on the set of them.

Definition 3.1.3 (of $T$-equivalence). Let $s_1$ and $s_2$ be two generic curves with a triple point (see 3.1.1). We say, that these curves are $T$-equivalent, if there exist associated with them mappings of $B_3$ which are free homotopic. The triple point stratum is naturally decomposed into parts corresponding to different $T$-equivalence classes.

We denote by $[s]$ the $T$-equivalence class corresponding to $s$, a generic curve with a triple point. We denote by $T$ the set of all the $T$-equivalence classes.

3.2. Axiomatic description of $\text{St}$. A natural way to introduce St type invariants of generic curves on a surface $F$ is to take a function $\psi : T \to \mathbb{Z}$ and to construct an invariant of generic curves from a fixed connected component $C$ of the space $F$ (of all the curves on $F$) such that:

1. It does not change under crossings of the self-tangency strata of the discriminant.

2. It increases by $\psi([s])$ under a positive crossing of the part of the triple point stratum, which corresponds to a $T$-equivalence class $[s]$.

If for a given function $\psi : T \to \mathbb{Z}$ there exists such an invariant of curves from $C$, then we say that there exists a $\text{St}$ invariant of curves in $C$, which is an integral of $\psi$. Such $\psi$ is said to be $\text{St}$-integrable in $C$. 
3.2.1. Obstructions for the integrability. Fix $\psi : T \to \mathbb{Z}$. Let $\xi \subset F$ be a generic curve and $\gamma \subset C$ be a generic loop starting at $\xi$. We denote by $I^\xi$ the set of moments, when $\gamma$ crosses the triple point stratum. We denote by $\{t^i_{\xi}\}_{i \in I^\xi}$ the $T$-equivalence classes corresponding to the parts of the stratum, where the crossings occur, and by $\{\sigma^i_{\xi}\}_{i \in I^\xi}$ the signs of the crossings. Put

$$\Delta_{St}(\gamma) = \sum_{i \in I^\xi} \sigma^i_{\xi}(t^i_{\xi})$$

We call $\Delta_{St}(\gamma)$ the change of $St$ along $\gamma$. If $\Delta_{St}(\gamma) = 0$, then $\psi$ is said to be integrable along $\gamma$. It is clear, that if a function $\psi$ is integrable in $C$, then it is integrable along any generic loop $\gamma \subset C$.

Below we describe the two loops $\gamma_1$ and $\gamma_2$ such that the integrability of $\psi$ along them implies integrability of $\psi$ in $C$. In a sense, the changes along these loops are the only obstructions for the integrability. (The loop $\gamma_2$ is going to be well defined (and needed) only in the case of $F$ being a Klein bottle and $C$ consisting of orientation reversing curves on it.)

3.2.2. Loop $\gamma_1$. Let $\xi \in C$ be a generic curve and $\gamma_1 \subset C$ be the loop starting at $\xi$, which is constructed below.

Deform $\xi$ along a generic path $t$ in $F$ to get two opposite kinks, as it shown in Figure 2. Make the first kink very small and slide it along the curve (in such a way that at each moment of time points of $\xi$ located outside of a small neighborhood of the kink do not move) till it comes back. (See Figure 3.) Finally deform $\xi$ to its original shape along $t^{-1}$.

Note, that if $\xi$ represents an orientation reversing loop on $F$, then the kink slides twice along $\xi$ before it returns to the original position.

![Figure 2](image_url)

3.2.3. Loop $\gamma_2$. Let $\xi$ be a generic orientation preserving curve on the Klein bottle $K$. Let $\gamma_2 \subset C$ be the loop starting at $\xi$ which is constructed below.

Consider $K$ as a quotient of a rectangle modulo the identification on its sides shown in Figure 4. Let $p$ be the orientation covering $T^2 \to K$. There is a loop $\alpha$ in the space of all autodiffeomorphisms of $T^2$, which is the sliding of $T^2$ along the unit vector field parallel to the lifting of the curve $c \subset K$ (see Figure 5). Since $\xi$ is an orientation preserving curve it can be lifted to a curve $\xi'$ on $T^2$. The loop $\gamma_2$ is the composition of $p$ and of the sliding of $\xi'$ induced by $\alpha$. (To make $\gamma_2$ well-defined for each $\xi$ we choose which one of the two possible liftings of $\xi$ to a curve on $T^2$ is $\xi'$.)
Theorem 3.2.4. Let $F$ be a surface (not necessarily compact or orientable), $T$ be the set of all the $T$-equivalence classes, $C$ be a connected component of $F$ and $\xi \in C$ be a generic curve. Let $\psi : T \to \mathbb{Z}$ be a function.

Then the following two statements I and II are equivalent.

I: There exists an invariant $\overline{S}_t$ of generic curves from $C$ which is an integral of $\psi$.

II: If $F \neq K$ (Klein bottle) then $\psi$ is integrable along the loop $\gamma_1 \subset C$ starting at $\xi$.

If $F = K$ and $C$ consists of orientation reversing curves on $K$, then $\psi$ is integrable along the loop $\gamma_1 \subset C$ starting at $\xi$. If $F = K$ and $C$ consists of orientation preserving curves on $K$, then $\psi$ is integrable along the loops $\gamma_1, \gamma_2 \subset C$ starting at $\xi$.

For the Proof of Theorem 3.2.4 see Section 6.1.

Remarks. 1. If for a given function $\psi : T \to \mathbb{Z}$ there exists an invariant $\overline{S}_t$ which is an integral of $\psi$, then it is unique up to an additive constant. (This statement follows from the proof of Theorem 3.2.4.)

Note, that if statement II holds for one generic $\xi \in C$, then statement I holds, which implies that II holds for all generic $\xi' \in C$. 
A straightforward modification of the proof of Theorem 3.2.4 shows that it holds for \( \psi \) taking values in any torsion free Abelian group.

The connected components of \( \mathcal{F} \) admit a rather simple description. One can show (cf. 6.3.3) that they are naturally identified with the connected components of the space of free loops in \( STF \) (the spherical tangent bundle of the surface) or, which is the same, with the conjugacy classes of \( \pi_1(STF) \).

### 3.2.5. Cases, when \( \psi \) is automatically integrable

Theorem 3.2.4 says, that in the cases of orientable \( F \), or of \( C \) consisting of orientation reversing curves on \( F \), integrability of \( \psi \) along \( \gamma_1 \) is sufficient for the existence of the \( \text{St} \) invariant.

Clearly, all the crossings of the triple point stratum, which occur along \( \gamma_1 \) (sliding of a kink along \( \xi \)) happen, when the kink passes through a double point of \( \xi \). (See Figure 3.)

If \( F \) is orientable, then the kink passes twice through each double point. A straightforward check shows that the signs of the corresponding triple point stratum crossings are opposite. The mappings of \( B_3 \) associated with these crossings are different by an orientation preserving automorphism of \( B_3 \), which does not preserve the cyclic order on the circles. For both crossings the restriction of an associated mapping to one of the circles of \( B_3 \) represents a contractible loop. Thus, if \( F \) is orientable then statement II (and hence statement I) of Theorem 3.2.4 is true for any function \( \psi \), provided that it takes the same value on any two \( T \)-equivalence classes, for which there exist mappings \( \phi_1, \phi_2 \) of \( B_3 \) representing them, such that:

a) The restriction of \( \phi_i \) \((i \in \{1, 2\})\) to one of the circles of \( B_3 \) represents a contractible loop on \( F \).

b) There exists \( \alpha \), an orientation preserving automorphism of \( B_3 \) (not preserving the cyclic order on the circles) such that \( \phi_1 = \phi_2 \circ \alpha \).

If \( \xi \) represents an orientation reversing loop on \( F \), then the kink has to slide twice along \( \xi \) before it comes to the original position. Thus, it passes four times through each double point of \( \xi \). One can show, that the corresponding crossings of the triple point stratum can be subdivided into two pairs, such that the \( T \)-equivalence classes corresponding to the crossings inside each pair are equal and the signs of the two crossings in each pair are opposite. Thus, if \( \xi \) represents an orientation reversing loop on \( F \), then statement II (and hence statement I) of Theorem 3.2.4 is true for any function \( \psi \), provided that it takes the same value on any two \( T \)-equivalence classes, for which there exist mappings \( \phi_1, \phi_2 \) of \( B_3 \) representing them, such that:

Remark . The following example shows that on nonorientable surfaces even a constant (nonzero) function \( \psi \) is not necessarily integrable.

Let \( F \) be a nonorientable surface. Let \( \xi \in C \) be an orientation preserving curve with a single double point \( x \) which separates \( \xi \) into two orientation reversing loops. Then \( \Delta_{\text{St}}(\gamma_1) \neq 0 \) for a constant nonzero function \( \psi : T \to \mathbb{Z} \). (The reason is that the signs of the two crossings of the triple point stratum corresponding to a kink passing through \( x \) have the same sign.)

### 3.2.6. Connection with the standard \( \text{St} \)-invariant

Since \( \mathbb{R}^2 \) is simply connected, there is just one \( T \)-equivalence class of singular curves on \( \mathbb{R}^2 \). Thus, the construction of \( \text{St} \) does not give anything new in the classical case of planar curves.

### 3.3. Singularity theory interpretation of \( \text{St} \) for orientable \( F \)
Definition 3.3.1 (of $\mathcal{T}$-equivalence). Let $S^1(3)$ be the configuration space of unordered triples of distinct points on $S^1$. Consider a space $S^1(3) \times \mathcal{F}$. Let $\mathcal{M}$ be the subspace of $S^1(3) \times \mathcal{F}$ consisting of $t \times f \in S^1(3) \times \mathcal{F}$, such that $f$ maps the three points from $t$ to one point on $F$. (This is a sort of singularity resolution for strata involving points of multiplicity greater than two.)

We say that $m_1, m_2 \in \mathcal{M}$ are $\mathcal{T}$-equivalent, if they belong to the same path connected component of $\mathcal{M}$. For $s$, a generic curve with a triple point (see 3.1.1), there is a unique $\mathcal{T}$-equivalence class associated to it. We denote this class by $[\tilde{s}]$.

Thus, the $\mathcal{T}$-equivalence relation induces a decomposition of the triple point stratum of the discriminant hypersurface.

Let $\mathcal{T}$ be the set of all the $\mathcal{T}$-equivalence classes. There is a natural mapping $\phi: \mathcal{T} \to \mathcal{T}$. It maps $\tilde{t} \in \mathcal{T}$ to such $\tilde{t} \in \mathcal{T}$, that there exists $s$, a generic curve with a triple point, for which $[s] = t$ and $[\tilde{s}] = \tilde{t}$.

Let $F$ be an orientable surface. Let $\mathcal{C}$ be a connected component of $\mathcal{F}$ and $\mathcal{T}_C \subset \mathcal{T}$ be the set of all the $\mathcal{T}$-equivalence classes, corresponding to generic curves (from $\mathcal{C}$) with a triple point.

Theorem 3.3.2. The mapping $\phi|_{\mathcal{T}_C}$ is injective.

For the Proof of Theorem 3.3.2 see Section 6.4.

3.3.3. Interpretation of $\mathcal{ST}$. Let $F$ be an orientable surface. Let $\mathcal{C}$ be a connected component of $\mathcal{F}$. Let $\mathcal{ST}$ be an invariant of generic curves from $\mathcal{C}$ such that:

a) It does not change under crossings of the self-tangency strata of the discriminant.

b) Under the positive crossing of a part of the triple point stratum of the discriminant it increases by a constant depending only on the $\mathcal{T}$-equivalence class corresponding to this part of the stratum.

Theorem 3.3.2 implies, that this $\mathcal{ST}$ invariant is an $\mathcal{ST}$ invariant for some choice of the function $\psi: \mathcal{T} \to \mathbb{Z}$.

4. $J^+$-type Invariant of Curves on Surfaces.

4.1. Natural decomposition of the direct self-tangency point stratum.

Definition 4.1.1. Let $F$ be a surface. We say that a curve $\xi \subset F$ with a direct self-tangency point $q$ is a generic curve with a direct self-tangency point, if its only nongeneric singularity is this point.

4.1.2. Let $F$ be a surface. Let $B_2$ be a bouquet of two oriented circles, and $b$ be its base point. Let $s: S^1 \to F$ be a generic curve with a direct self-tangency point $q$. It can be lifted to the mapping $\tilde{s}$ from the oriented circle to $STF$ (the spherical tangent bundle of $F$), which sends a point $p \in S^1$ to the point in $STF$ corresponding to the direction of the velocity vector of $s$ at $s(p)$. (Note, that $q$ lifts to a double point $\tilde{q}$ of $\tilde{s}$.)

Let $\alpha: S^1 \to B_2$ be a continuous mapping such that:

a) $\alpha(\tilde{s}^{-1}(\tilde{q})) = b$.

b) $\alpha$ is injective on the complement of $\tilde{s}^{-1}(\tilde{q})$.

c) The orientation induced by $\alpha$ on $B_2 \setminus b$ coincides with the orientation of the circles of $B_2$. 
The mapping $\phi : B_2 \to STF$ such that $\bar{s} = \phi \circ \alpha$ is called an associated with $s$ mapping of $B_2$.

Note, that the free homotopy class of a mapping of $B_2$ to $STF$ realized by $\phi$ is well defined, modulo the orientation preserving automorphism of $B_2$ which interchanges the circles.

**Definition 4.1.3** (of $T^+$-equivalence). Let $s_1$ and $s_2$ be two generic curves with a point of direct self-tangency (see 4.1.1). We say that these curves are $T^+$-equivalent if there exist associated with the two of them mappings of $B_2$, which are free homotopic. The direct self-tangency point stratum is naturally decomposed into parts corresponding to different $T^+$-equivalence classes.

We denote by $[s^+]$ the $T^+$-equivalence class corresponding to $s$, a generic curve with a point of direct self-tangency. We denote by $T^+$ the set of all the $T^+$-equivalence classes.

**4.2. Axiomatic description of $J^+$.** A natural way to introduce $J^+$ type invariant of generic curves on a surface $F$ is to take a function $\psi : T^+ \to \mathbb{Z}$ and to construct an invariant of generic curves from a fixed connected component $C$ of the space $F$ (of all the curves on $F$) such that:

1. It does not change under crossings of the triple point or of the inverse self-tangency strata of the discriminant.
2. It increases by $\psi([s^+])$ under a positive crossing of the part of the direct self-tangency point stratum, which corresponds to a $T^+$-equivalence class $[s^+]$.

If for a given function $\psi : T^+ \to \mathbb{Z}$ there exists such an invariant of curves from $C$, then we say that there exists a $J^+$ invariant of curves in $C$, which is an integral of this function. Such $\psi$ is said to be $\overline{T^+}$-integrable in $C$.

Similarly to 3.2.1 we introduce the notions of the change of $J^+$ along a loop and of the integrability of $\psi$ along a loop.

**Theorem 4.2.1.** Let $F$ be a surface (not necessarily compact or orientable), $T^+$ be the set of all the $T^+$-equivalence classes, $C$ be a connected component of $F$ and $\xi \in C$ be a generic curve. Let $\psi : T^+ \to \mathbb{Z}$ be a function.

Then the following two statements I and II are equivalent.

**I:** There exists an invariant $\overline{T^+}$ of generic curves from $C$, which is an integral of $\psi$.

**II:** If $F \neq K$ (Klein bottle) then $\psi$ is integrable along the loop $\gamma_1 \subset C$ starting at $\xi$.

If $F = K$ and $C$ consists of orientation reversing curves on $K$, then $\psi$ is integrable along the loop $\gamma_1 \subset C$ starting at $\xi$. If $F = K$ and $C$ consists of orientation preserving curves on $K$, then $\psi$ is integrable along the loops $\gamma_1, \gamma_2 \subset C$ starting at $\xi$.

For the Proof of Theorem 4.2.1 see Section 6.5.

**Remarks 2.** If for a given function $\psi : T^+ \to \mathbb{Z}$ there exists an invariant $\overline{T^+}$ which is an integral of $\psi$, then it is unique up to an additive constant. (This statement follows from the proof of Theorem 4.2.1)

Note, that if statement II holds for one generic $\xi \in C$, then statement I holds, which implies that II holds for all generic $\xi' \in C$.

A straightforward modification of the proof of Theorem 4.2.1 shows that it holds for $\psi$ taking values in any torsion free Abelian group.
The connected components of $\mathcal{F}$ admit a rather simple description. One can show (cf. 6.3.3) that they are naturally identified with the connected components of the space of free loops in $\text{STF}$ (the spherical tangent bundle of the surface) or, which is the same, with the conjugacy classes of $\pi_1(\text{STF})$.

4.2.2. Cases, when $\psi$ is automatically integrable. Theorem 4.2.1 says, that in the cases of orientable $F$, or of $C$ consisting of orientation reversing curves on $F$, integrability of $\psi$ along $\gamma_1$ is sufficient for the existence of the $\overline{J}^+$ invariant.

Clearly, all the crossings of the direct self-tangency stratum, which occur along $\gamma_1$ (sliding of a kink along $\xi$) happen, when the kink passes through a double point of $\xi$.

If $F$ is orientable, then the kink passes twice through each double point of $\xi$. A straightforward check shows, that the signs of the corresponding direct self-tangency stratum crossings are opposite, and the $T^+$-equivalence classes corresponding to them are equal. Thus, if $F$ is orientable, then statement II (and hence statement I) of Theorem 4.2.1 is true for any function $\psi : T^+ \to \mathbb{Z}$.

If $\xi$ is an orientation reversing curve on $F$, then the kink slides twice along $\xi$, before it comes to its original position. Thus, it passes four times through each double point of $\xi$. One can show, that the corresponding four crossings of the direct self-tangency stratum can be subdivided into two pairs, such that the $T^+$-equivalence classes corresponding to the crossings inside the same pair are equal and the signs of the two crossings in each pair are opposite. Thus, if $\xi$ represents an orientation reversing loop on $F$, then statement II (and hence statement I) of Theorem 4.2.1 is true for any function $\psi : T^+ \to \mathbb{Z}$. Another way of proving this is based on the fact, that for such $\xi$ the loop $\gamma_1 = 1 \in \pi_1(F, \xi)$, see Section 6.3.3.

Remark. Similarly to the case of $\overline{ST}$, even a constant (nonzero) function $\psi$ is not necessarily integrable in the case of orientation preserving $\xi$ on a nonorientable surface $F$.

4.2.3. Connection with the standard $J^+$-invariant. Since $\pi_1(\text{STR}^2) = \mathbb{Z}$, there are countably many $T^+$-equivalence classes of singular curves on $\mathbb{R}^2$, which can be obtained from a curve of the fixed index. (Note, that the index of a curve $\xi$ defines the connected component of the space of all curves on $\mathbb{R}^2$, which $\xi$ belongs to.) Thus, the construction of $\overline{J}^+$ gives rise to a splitting of the standard $J^+$ invariant of V. Arnold. This is the splitting introduced by V. Arnold [6] in the case of planar curves of index zero and generalized to the case of arbitrary planar curves by F. Aicardi [7].

4.3. Singularity theory interpretation of $\overline{J}^+$ for orientable $F$.

Definition 4.3.1 (of $\overline{T}^+$-equivalence). Let $S^1(2)$ be the configuration space of unordered pairs of distinct points on $S^1$. Consider a space $S^1(2) \times \mathcal{F}$. Let $\mathcal{M}^+$ be the subspace of $S^1(2) \times \mathcal{F}$ consisting of $t \times f \in S^1(2) \times \mathcal{F}$, such that $f$ maps the two points from $t$ to one point on $F$ and the velocity vectors of $f$ at these two points have the same direction. (This is a sort of singularity resolution for the strata, involving points of direct self-tangency.)

We say that $m_1^+$ and $m_2^+$ from $\mathcal{M}^+$ are $\overline{T}^+$-equivalent, if they belong to the same path connected component of $\mathcal{M}^+$. 
Clearly, for \(s\), a generic curve with a direct self-tangency point (see 4.1.1), there is a unique \(T^+\)-equivalence class associated with it. We denote this class by \([s^+]\). Thus, the \(T^+\)-equivalence relation induces a decomposition of the direct self-tangency point stratum of the discriminant hypersurface.

Let \(T^+\) be the set of all the \(T^+\)-equivalence classes. There is a natural mapping \(\phi : T^+ \to T^+\). It maps \(t^+ \in T^+\) to such \(t^+ \in T^+\), that there exists \(s\) (a generic curve with a direct self-tangency point), for which \([s^+] = t^+\) and \([s^+] = t^+\).

Let \(F\) be an orientable surface. Let \(C\) be a connected component of \(F\) and \(T^+_C \subset T^+\), be the set of all the \(T^+\)-equivalence classes corresponding to generic curves (from \(C\)) with a point of direct self-tangency.

**Theorem 4.3.2.** The mapping \(\phi |_{T^+_C}\) is injective.

For the Proof of Theorem 4.3.2 see Section 6.6.

4.3.3. Interpretation of \(J^+\). Let \(F\) be an orientable surface. Let \(C\) be a connected component of \(F\). Let \(J^+\) be an invariant of generic curves from \(C\), such that:

a) It does not change under crossings of the inverse self-tangency and of the triple point strata of the discriminant.

b) Under the positive crossing of a part of the direct self-tangency point stratum of the discriminant it increases by a constant, depending only on the \(T^+\) equivalence class corresponding to this part of the stratum.

Theorem 4.3.2 implies, that this \(J^+\) invariant is a \(J^+\) invariant for some choice of the function \(\psi : T^+ \to \mathbb{Z}\).

5. **\(J^-\)-Type Invariant of Curves on Surfaces.**

5.1. **Natural decomposition of the inverse self-tangency point stratum.**

**Definition 5.1.1.** Let \(F\) be a surface. We say that a curve \(\xi \subset F\) with an inverse self-tangency point \(q\) is a **generic curve with an inverse self-tangency point**, if its only nongeneric singularity is this point.

5.1.2. Let \(F\) be a surface. Let \(B_2\) be a bouquet of two oriented circles, and \(b\) be its base point. Let \(s : S^1 \to F\) be a generic curve with an inverse self-tangency point \(q\). It can be lifted to the mapping \(\tilde{s}\) from the oriented circle to \(PTF\) (the projectivized tangent bundle of \(F\)), which sends a point \(p \in S^1\) to the point in \(PTF\) corresponding to the tangent line containing the velocity vector of \(s\) at \(s(p)\). (Note, that \(q\) lifts to a double point \(\tilde{q}\) of \(\tilde{s}\).)

Let \(\alpha : S^1 \to B_2\) be a continuous mapping such that:

a) \(\alpha(\tilde{s}^{-1}(\tilde{q})) = b\).

b) \(\alpha\) is injective on the complement of \(\tilde{s}^{-1}(\tilde{q})\).

c) The orientation induced by \(\alpha\) on \(B_2 \setminus b\) coincides with the orientation of the circles of \(B_2\).

The mapping \(\phi : B_2 \to PTF\) such that \(\tilde{s} = \phi \circ \alpha\) is called an **associated** with \(s\) mapping of \(B_2\).

Note, that the free homotopy class of a mapping of \(B_2\) to \(PTF\) realized by \(\phi\) is well defined, modulo the orientation preserving automorphism of \(B_2\) which interchanges the circles.
Definition 5.1.3 (of $T^-$-equivalence). Let $s_1$ and $s_2$ be two generic curves with a point of an inverse self-tangency (see 5.1.1). We say, that these curves are $T^-$-equivalent, if there exist associated with the two of them mappings of $B_2$, which are free homotopic. The inverse self-tangency point stratum is naturally decomposed into parts corresponding to different $T^-$-equivalence classes.

We denote by $[s^-]$ the $T^-$-equivalence class corresponding to $s$, a generic curve with a point of an inverse self-tangency. We denote by $T^-$ the set of all the $T^-$-equivalence classes.

5.2. Axiomatic description of $J^-$. A natural way to introduce $J^-$ type invariant of generic curves on a surface $F$ is to take a function $\psi : T^- \to \mathbb{Z}$ and to construct an invariant of generic curves from a fixed connected component $C$ of the space $\mathcal{F}$ (of all the curves on $F$) such that:

1. It does not change under crossings of the triple point or of the direct self-tangency strata of the discriminant.
2. It increases by $\psi([s^-])$ under a positive crossing of the part of the inverse self-tangency point stratum, which corresponds to a $T^-$-equivalence class $[s^-]$.

If for a given function $\psi : T^- \to \mathbb{Z}$ there exists such an invariant of curves from $C$, then we say that there exists a $\mathcal{J}^-$ invariant of curves in $C$, which is an integral of this function. Such $\psi$ is said to be $\mathcal{J}^-$-integrable in $C$.

Similarly to 3.2.1 we introduce the notions of the change of $J^-$ along a loop and of the integrability of $\psi$ along a loop.

Theorem 5.2.1. Let $F$ be a surface (not necessarily compact or orientable), $T^-$ be the set of all the $T^-$-equivalence classes, $C$ be a connected component of $\mathcal{F}$ and $\xi \in C$ be a generic curve. Let $\psi : T^- \to \mathbb{Z}$ be a function.

Then the following two statements I and II are equivalent.

I: There exists an invariant $\mathcal{J}^-$ of generic curves from $C$, which is an integral of $\psi$.

II: If $F \neq K$ (Klein bottle) then $\psi$ is integrable along the loop $\gamma_1 \subset C$ starting at $\xi$.

If $F = K$ and $C$ consists of orientation reversing curves on $K$, then $\psi$ is integrable along the loop $\gamma_1 \subset C$ starting at $\xi$. If $F = K$ and $C$ consists of orientation preserving curves on $K$, then $\psi$ is integrable along the loops $\gamma_1, \gamma_2 \subset C$ starting at $\xi$.

The Proof of Theorem 5.2.1 is a straightforward generalization of the Proof of Theorem 4.2.1.

Remarks 3. If for a given function $\psi : T^- \to \mathbb{Z}$ there exists an invariant $\mathcal{J}^-$ which is an integral of $\psi$, then it is unique up to an additive constant. (This statement follows from the proof of Theorem 5.2.1.)

Note, that if statement II holds for one generic $\xi \in C$, then statement I holds, which implies that II holds for all generic $\xi' \in C$.

A straightforward modification of the proof of Theorem 5.2.1 shows that it holds for $\psi$ taking values in any torsion free Abelian group.

The connected components of $\mathcal{F}$ admit a rather simple description. One can show (cf. 5.3.3) that they are naturally identified with the connected components of the space of free loops in $STF$ (the spherical tangent bundle of the surface) or, which is the same, with the conjugacy classes of $\pi_1(STF)$. 
5.2.2. Cases, when $\psi$ is automatically integrable. Similarly to 4.2.2 one can show, that statement II of Theorem 5.2.1 is true for any function $\psi: T^\sim \rightarrow \mathbb{Z}$, provided that $F$ is orientable, or that $C$ consists of orientation reversing curves on $F$.

Remark. Similarly to the case of $\mathrm{St}$, even a constant (nonzero) function $\psi$ is not necessarily integrable in the case of orientation preserving $\xi$ on a nonorientable surface $F$.

5.2.3. Connection with the standard $J^-$-invariant. Since $\pi_1(P\mathbb{R}^2) = \mathbb{Z}$, there are countably many $T^-$-equivalence classes of singular curves, which can be obtained from a curve of the fixed index. (Note, that the index of a curve $\xi$ defines the connected component of the space of all curves on $\mathbb{R}^2$, which $\xi$ belongs to.) Thus, the construction of $\mathcal{J}^-$ gives rise to a splitting of the standard $J^-$ invariant of $V$. Arnold. This splitting is analogous to the splitting of $J^+$ introduced by V. Arnold in the case of planar curves of index zero and generalized to the case of arbitrary planar curves by F. Aicardi.

5.3. Singularity theory interpretation of $\mathcal{J}^-$ for orientable $F$.

Definition 5.3.1 (of $\mathcal{T}^-$-equivalence). Let $S^1(2)$ be the configuration space of unordered pairs of distinct points on $S^1$. Consider a space $S^1(2) \times \mathcal{F}$. Let $\mathcal{M}^-$ be the subspace of $S^1(2) \times \mathcal{F}$ consisting of $t \times f \in S^1(2) \times \mathcal{F}$, such that $f$ maps the two points from $t$ to one point on $F$ and the velocity vectors of $f$ at these two points have opposite directions. (This is a sort of singularity resolution for the strata, involving points of an inverse self-tangency.)

We say, that $m^-_1$ and $m^-_2$ from $\mathcal{M}^-$ are $\mathcal{T}^-$-equivalent, if they belong to the same path connected component of $\mathcal{M}^-$. Clearly, for $s$, a generic curve with a point of an inverse self-tangency (see 5.1.1), there is a unique $\mathcal{T}^-$-equivalence class associated to it. We denote it by $[s^-]$.

Thus, the $\mathcal{T}^-$-equivalence relation induces a decomposition of the inverse self-tangency point stratum of the discriminant hypersurface.

Let $\mathcal{T}^-$ be the set of all the $\mathcal{T}^-$-equivalence classes.

There is a natural mapping $\phi: \mathcal{T}^- \rightarrow \mathcal{T}^-$. It maps $t^- \in \mathcal{T}^-$ to such $t^- \in \mathcal{T}^-$, that there exists $s$ (a generic curve with an inverse self-tangency point), for which $[s^-] = t^-$ and $[\overline{s}] = t^-$. Let $F$ be an orientable surface. Let $\mathcal{C}$ be a connected component of $\mathcal{F}$, and $\mathcal{T}_C^-$ be the set of all the $\mathcal{T}^-$-equivalence classes corresponding to generic curves (from $\mathcal{C}$) with a point of an inverse self-tangency.

Theorem 5.3.2. The mapping $\phi|_{\mathcal{T}_C^-}$ is injective.

The Proof of Theorem 5.3.2 is a straightforward generalization of the proof of Theorem 4.3.3.

5.3.3. Interpretation of $\mathcal{J}^-$. Let $F$ be an orientable surface. Let $\mathcal{C}$ be a connected component of $\mathcal{F}$. Let $\mathcal{J}^-$ be an invariant of generic curves from $\mathcal{C}$, such that:

a) It does not change under crossings of the direct self-tangency and of the triple point strata of the discriminant.

b) Under the positive crossing of a part of the inverse self-tangency point stratum of the discriminant it increases by a constant, depending only on the $\mathcal{T}^-$ equivalence class corresponding to this part of the stratum.
Theorem 5.3.2 implies, that this $\tilde{J}^-$ invariant is a $\tilde{J}^-$ invariant for some choice of the function $\psi : \mathcal{T}^- \to \mathbb{Z}$.

6. Proofs

6.1. Proof of Theorem 3.2.4. Clearly, in order for $S_t$ to be well defined, the change of it along any generic closed loop in $\mathcal{C}$ should be zero. Thus, we have proved that statement I implies statement II.

To prove that statement II implies statement I we imitate the approach developed by V. Arnold [1] in the case of planar curves.

Fix any value of $S_t(\xi) \in \mathbb{Z}$. Let $\xi' \in \mathcal{C}$ be another generic curve. Take a generic path $p$ in $\mathcal{C}$, which connects $\xi$ with $\xi'$. When we go along this path we see a sequence of crossings of the self-tangency and of the triple point strata of the discriminant. Let $I$ be the set of moments when we crossed the triple point stratum. Let $\{\sigma_i\}_{i \in I}$ be the signs of the corresponding new born vanishing triangles and $\{[s_i]\}_{i \in I}$ be the $T$-equivalence classes represented by the corresponding generic curves with a triple point. Put $\Delta S_t(p) = \sum_{i \in I} \sigma_i \psi([s_i])$ and $S_t(\xi') = S_t(\xi) + \Delta S_t(p)$. To prove the Theorem it is sufficient to show, that $S_t(\xi')$ does not depend on the generic path $p$, we used to define it. The last statement follows from Lemma 6.1.1 and Lemma 6.1.2. Thus, we have proved Theorem 6.1 modulo these two lemmas.

Lemma 6.1.1 (Cf. V. Arnold [1]). Let $p$ be a generic path in $\mathcal{F}$, which connects $\xi$ to itself. Then $\Delta S_t(p)$ depends only on the class in $\pi_1(\mathcal{F}, \xi)$ represented by $p$.

Lemma 6.1.2. If statement II of Theorem 3.2.4 is true, then for every element of $\pi_1(\mathcal{F}, \xi)$ there exists a generic loop $q$ in $\mathcal{F}$, representing this element, such that $\Delta S_t(q) = 0$.

6.2. Proof of Lemma 6.1.1. It is sufficient to show that, if we go around any stratum of codimension two along a small generic loop $r$ (not necessarily starting at $\xi$), then $\Delta S_t(r) = 0$. The only strata of codimension two in the bifurcation diagram of which triple points are present are: a) two distinct triple points, b) triple point and distinct self-tangency point, c) triple point at which two branches are tangent (of order one) and d) quadruple point (at which every two branches are transverse). (All the codimension two singularities and bifurcation diagrams for them were described by V. Arnold [1].)

If $r$ is a small loop which goes around the stratum of two distinct triple points, then in $\Delta S_t(r)$ we have each of the two $T$-equivalence classes twice, once with the plus sign of the newborn vanishing triangle, once with the minus. Hence $\Delta S_t(r) = 0$.

If $r$ is a small loop which goes around the stratum of one triple and one self-tangency point, then the two $T$-equivalence classes participating in $\Delta S_t(r)$ are equal and the signs with which they participate are opposite. Hence $\Delta S_t(r) = 0$.

Let $r$ be a small loop which goes around the stratum of a triple point with two tangent branches. We can assume, that it corresponds to a loop on Figure 3 directed clockwise. (The colored triangles are the newborn vanishing triangles.) As we can see from Figure 3 there are just two terms in $\Delta S_t(r)$. It is clear, that the $T$-equivalence classes in them coincide. A direct check shows that the signs of the two terms are opposite. (Note, that if they are not always opposite, then Arnold’s St invariant is not well defined.)

Finally, let $r$ be a small loop, which goes around the stratum of a quadruple point (at which every two branches are transverse). We can assume, that it corresponds
to a loop in Figure 3 directed counter clockwise. There are eight terms in $\Delta_{ST}(r)$. We split them into pairs I, II, III, IV, as it is shown in Figure 3. One can see, that the $T$-equivalence classes of the two curves in each pair are the same. For each branch the sign of the colored triangle is equal to the sign of the triangle, which died under the triple point stratum crossing shown on the next (in the counterclockwise direction) branch. The sign of the dying vanishing triangle is minus the sign of the newborn vanishing triangle. Finally, one can see that the signs of the colored triangles inside each pair are opposite. Thus, all these eight terms cancel out.

This finishes the Proof of Lemma 6.1.1.

6.3. Proof of Lemma 6.1.2

6.3.1. In $\mathbb{Z}$ there are no elements of finite order. Thus, if $m \neq 0$, then $\Delta_{ST}(q) \neq 0 \iff m\Delta_{ST}(q) = \Delta_{ST}(q^m) \neq 0$. Hence, to prove Lemma 6.1.2 it is sufficient to show that $\Delta_{ST}(q^m) = 0$ for a certain power $m \neq 0$ of $q \in \pi_1(F, \xi)$.

Proposition 6.3.2. Let $F$ be a surface, $STF$ be its spherical tangent bundle and $p \in STF$ be a point. Let $f \in \pi_1(STF, p)$ be the class of an oriented (in some way) fiber of the $S^1$-fibration $pr : STF \to F$.

If $\alpha \in \pi_1(STF, p)$ is a loop projecting to an orientation preserving loop on $F$, then

$$\alpha f = f \alpha. \quad (5)$$

If $\alpha \in \pi_1(STF, p)$ is a loop projecting to an orientation reversing loop on $F$, then

$$\alpha f = f^{-1} \alpha. \quad (6)$$
The proof of this Proposition is straightforward.

6.3.3. Parametric $h$-principle. The parametric $h$-principle, see [8] page 16, implies that $F$ is weak homotopy equivalent to the space $\Omega STF$ of free loops in $STF$. The corresponding mapping $h : F \to \Omega STF$ sends a curve $\xi \in F$ to a loop $\tilde{\xi} \in \Omega STF$ by mapping a point $y \in S^1$ to a point in $STF$, to which points the velocity vector of $\xi$ at $\xi(y)$.

Fix a point $a$ on $S^1$ (which parameterizes the curves). Let $q$ be a loop in $F$ starting at $\xi$. At any moment of time $q(t)$ is a curve, which can be lifted to a loop in $STF$. Thus, $q$ gives rise to the mapping $q_h : S^1 \times S^1 \to STF$ (the lifting of $q$ by $h$). (In the product $S^1 \times S^1$ the first copy of $S^1$ corresponds to the parameterization of a curve and the second to the parameterization of the loop $q$.) The mapping $q_h$ restricted to $a \times S^1$ gives rise to the loop $t_a(q)$ in $STF$. (It is a trajectory of the lifting of $a$.) One can check, that the mapping $t_a : \pi_1(F, \xi) \to \pi_1(STF, \tilde{\xi}(a))$ is a homomorphism.

Note, that if $q \in \pi_1(F, \xi)$ is the sliding of a kink (see 3.2.2) along a curve $\xi$ representing an orientation preserving loop on $F$, then the velocity vector of $\xi$ at $\xi(a)$ is rotated by $2\pi$ under this sliding. Thus, $t_a(q) \in \pi_1(STF, \tilde{\xi}(a))$ is equal to $f$, the homotopy class of the fiber of the $S^1$-fibration $pr : STF \to F$.

One can check, that if $q \in \pi_1(F, \xi)$ is the sliding of a kink along a curve $\xi$ representing an orientation reversing loop on $F$, then $t_a(q) = 1 \in \pi_1(STF, \tilde{\xi}(a))$. (In this case the kink has to slide twice along $\xi$, before it returns to its original position and the total angle of rotation of the velocity vector of $\xi$ at $\xi(a)$ appears to be zero.)
Proposition 6.3.4 (Cf. V. Hansen [1]). The group $\pi_1(\Omega STF, \lambda)$ is isomorphic to $Z(\lambda)$, the centralizer of the element $\lambda \in \pi_1(STF, \lambda(a))$.

6.3.5. Proof of Proposition 6.3.4. Let $p : \Omega STF \to STF$ be the mapping, which sends $\omega \in \Omega STF$ to $\omega(a) \in STF$. (One can check, that this $p$ is a Serre fibration, with the fiber of it isomorphic to the space of loops based at the corresponding point.)

A Proposition proved by V.L. Hansen [1] says that: if $X$ is a topological space with $\pi_2(X) = 0$, then $\pi_1(\Omega X, \lambda) = Z(\lambda) < \pi_1(X, \lambda(a))$. (Here $\Omega X$ is the space of free loops in $X$ and $\lambda$ is an element of $\Omega X$.) One can check that $\pi_2(STF) = 0$ for any surface $F$. Thus, we get that $\pi_1(\Omega STF, \lambda)$ is isomorphic to $Z(\lambda) < \pi_1(STF, \lambda(a))$. From the proof of the Hansen Proposition it follows that the isomorphism is induced by $p_*$.

The following statement is an immediate consequence of Proposition 6.3.4 and the $h$-principle (see Section 6.3.3).

Corollary 6.3.6. Let $F$ be a surface and $\xi$ be a curve on $F$, then $\pi_1(F, \xi)$ is isomorphic to $Z(\xi)$, the centralizer of $\xi \in \pi_1(STF, \xi(a))$. The isomorphism is given by $t_a : \pi_1(F, \xi) \to Z(\xi)$, which sends $q \in \pi_1(F, \xi)$ to $t_a(q)$. (See Section 6.3.3.)

Proposition 6.3.7. Let $F \neq S^2, T^2$ (torus), $\mathbb{R}P^2, K$ (Klein bottle) be a surface (not necessarily compact or orientable) and $G'$ be a nontrivial commutative subgroup of $\pi_1(F)$. Then $G'$ is infinite cyclic and there exists a unique maximal infinite cyclic $G < \pi_1(F)$, such that $G' < G$.

6.3.8. Proof of Proposition 6.3.7. It is well known, that any closed $F$, other than $S^2, T^2, \mathbb{R}P^2, K$, admits a hyperbolic metric of a constant negative curvature. (It is induced from the universal covering of $F$ by the hyperbolic plane $H.$) The Theorem by A. Preissman (see [4]) says, that if $M$ is a compact Riemannian manifold with a negative curvature, then any nontrivial Abelian subgroup $G' < \pi_1(M)$ is isomorphic to $Z$. Thus, if $F \neq S^2, T^2, \mathbb{R}P^2, K$ is closed, then any nontrivial commutative $G' < \pi_1(F)$ is infinite cyclic.

The proof of the Preissman Theorem given in [4] is based on the fact, that if $\alpha, \beta \in \pi_1(M)$ are nontrivial commuting elements, then there exists a geodesic in $M$ (the universal covering of $M$) which is mapped to itself under the action of these elements considered as deck transformations on $M$. Moreover, these transformations restricted to the geodesic act as translations. This implies, that if $F \neq S^2, T^2, \mathbb{R}P^2, K$ is a closed surface, then there exists a unique maximal infinite cyclic $G < \pi_1(F)$, such that $G' < G$. This gives the proof of Proposition 6.3.7 for closed $F$.

If $F$ is not closed then the statement of the Proposition is also true, because in this case $F$ is homotopy equivalent to a bouquet of circles.

We first prove Lemma 6.1.2 for $F \neq S^2, \mathbb{R}P^2, T^2, K$ and then separately for the cases $F = S^2, \mathbb{R}P^2, T^2, K$.

6.3.9. Case $F \neq S^2, T^2, \mathbb{R}P^2, K$ Corollary 6.3.6 says, that $\pi_1(F, \xi) = Z(\xi) < \pi_1(STF, \xi(a))$. The corresponding isomorphism (see Section 6.3.3) maps $q \in \pi_1(F, \xi)$ to $t_a(q) \in \pi_1(STF, \xi(a))$. 
Thus, for any $q \in \pi_1(F, \xi)$ the elements $t_a(q)$ and $\xi$ commute in $\pi_1(STF, \xi(a))$. Hence, $\xi = \text{pr}_* (\vec{\xi})$ commutes with $\text{pr}_* (t_a(q))$ in $\pi_1(F, \xi(a))$. Proposition 6.3.7 implies that there exists an infinite cyclic subgroup of $\pi_1(F, \xi(a))$ generated by some $g \in \pi_1(F, \xi(a))$, which contains both of these loops. Then there exist $m, n \in \mathbb{Z}$ such that $\xi = g^m$ and $\text{pr}_* (t_a(q)) = g^n$.

Consider a curve $l$ (direct tangent to $\xi$ at $\xi(a)$) which represents $g$. We can lift it to an element $\vec{g} \in \pi_1(STF, \xi(a))$.

The kernel of $\text{pr}_*$ is generated by $f$, the class of an oriented fiber. Using (3) and (1) one can interchange $f$ with the other elements of $\pi_1(STF, \xi(a))$. We get that $\vec{\xi} = \vec{g}^nf^k$ and $t_a(q) = \vec{g}^nf^l$, for some $k, l \in \mathbb{Z}$. We prove Lemma 6.1.2 separately for cases $m \neq 0$ and $m = 0$ in Section 6.3.11 and Section 6.3.11 respectively. (These two cases correspond to $\xi \neq 1 \in \pi_1(F, \xi(a))$ and to $\xi = 1 \in \pi_1(F, \xi(a))$, respectively.)

6.3.10. Case $m \neq 0$. To prove Lemma 6.1.2 it is sufficient to show, that $\Delta_{\pi^T}(q^m) \neq 0$ (see 6.3.1).

One can show that $t_a(q^m) = \vec{\xi}^n f^j$ for some $j \in \mathbb{Z}$. For $g$ which is an orientation preserving loop this follows from the following calculation (which uses (3)):

$$t_a(q^m) = (t_a(q))^m = (\vec{g}^nf^j)^m = (\vec{g}^nf^j)^{lm-nk} = \vec{\xi}^n f^{lm-nk}.$$  (7)

For $g$, which is an orientation reversing loop on $F$, this follows from the similar calculation (which uses (1)). The fact that $t_a(q^m)$ should commute with $\vec{\xi}$ (since it is the $m$-th power of $t_a(q) \in Z(\vec{\xi})$) and the identity (3) imply, that $t_a(q^m) = \vec{\xi}^n$, provided that $\vec{\xi}$ represents an orientation reversing loop on $F$.

Let $\gamma_1$ be the sliding of a kink along $\xi$ (see 3.2.3). If $\xi$ represents an orientation preserving loop on $F$, then the velocity vector of $\xi$ at $\xi(a)$ is rotated by $2\pi$ under $\gamma_1$. Thus, $t_a(\gamma_1) = f$. Hence, the loop $\alpha \in \pi_1(F, \xi)$ for which $t_a(\alpha) = t_a(q^m)$ is: $n$ times sliding of $\xi$ along itself according to the orientation, composed with $\gamma_1$.

As it was said above, if $\vec{\xi}$ represents an orientation reversing loop, then $t_a(q^m) = \vec{\xi}^n$. Hence, the loop $\alpha \in \pi_1(F, \xi)$, for which $t_a(\alpha) = t_a(q^m)$ is: $n$ times sliding of $\xi$ along itself. (In 6.3.3 it was shown that $\gamma_1 = 1 \in \pi_1(F, \xi)$ for $\xi$ representing an orientation reversing loop on $F$.)

No triple points appear during the sliding of $\xi$ along itself. The inputs of the triple point stratum crossings which occur under $\gamma_1$ cancel out, by the assumption of the Lemma. Hence, $\Delta_{\pi^T}(q^m) = 0$.

Thus, we have proved (see 6.3.1) Lemma 6.1.2 for $F \neq S^2, \mathbb{R}P^2, T^2, K$ and $m \neq 0$.

6.3.11. Case $m = 0$. If $m = 0$, then $\xi$ represents $1 \in \pi_1(F, \xi(a))$. For any $q \in \pi_1(F, \xi)$ the projection of $t_a(q^2) \subset STF$ to $F$ is an orientation preserving loop on $F$. A straightforward check shows that for any $q \in \pi_1(F, \xi)$ the element $q^2$ can be obtained by a composition of $\gamma_1^\pm$ (see 3.2.2) and loops obtained by the following construction.

Push $\xi$ into a small disc by a generic regular homotopy $r$. Slide this small disc along some orientation preserving curve in $F$ and return $\xi$ to its original shape along $r^{-1}$.

Clearly, the inputs of $r$ and $r^{-1}$ into $\Delta_{\pi^T}$ cancel out and no triple point stratum crossings happen, when we slide a small disc along a path in $F$. Thus, loops
obtained by this construction do not give any input to $\Delta_{ST}$. By the assumption of the Lemma $\Delta_{ST}(\gamma_1) = 0$.

This implies that $\Delta_{ST}(q^2) = 0$ for any $q \in \pi_1(\mathcal{F}, \xi)$, and we have proved (see 6.3.1) Lemma 6.1.2 for $F \neq S^2, \mathbb{R}P^2, T^2, K$.

6.3.12. Case $F = S^2$. One checks that $\pi_1(STS^2) = \mathbb{Z}_2$. (Note that $STS^2 = \mathbb{R}P^3$.) Corollary 6.3.4 implies that $\pi_1(\mathcal{F}, \xi) = \mathbb{Z}_2$ for $F = S^2$. Thus, $\Delta_{ST}(q^2) = \Delta_{ST}(1) = 0$. (Here $1$ is a trivial loop in $\mathcal{F}$.) This finishes (see 6.3.1) the proof of Lemma 6.1.2 for $F = S^2$.

6.3.13. Case $F = T^2$. Using identity (5) we get, that $\pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Corollary 6.3.6 implies that $\pi_1(\mathcal{F}, \xi) = \pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. The generators of this group are:

1) The loop $\gamma_1$, which is the sliding of a kink along $\xi$ (see 3.2.2).

2) The loops $\gamma_3$ and $\gamma_4$, which are slidings of $\xi$ along the unit vector fields parallel to the meridian and longitude of $T^2$, respectively.

By the assumption of the Lemma $\Delta_{ST}(\gamma_1) = 0$. Since no discriminant crossings occur during $\gamma_3$ and $\gamma_4$ we get, that $\Delta_{ST}(\gamma_3) = \Delta_{ST}(\gamma_4) = 0$. This finishes the proof of Lemma 6.1.2 for $F = T^2$.

6.3.14. Case $F = \mathbb{R}P^2$. One checks that $\pi_1(ST\mathbb{R}P^2) = \mathbb{Z}_4$. Corollary 6.3.6 implies that $\pi_1(\mathcal{F}, \xi) = \mathbb{Z}_4$ for $F = \mathbb{R}P^2$. Thus, $\Delta_{ST}(q^4) = \Delta_{ST}(1) = 0$. (Here $1$ is a trivial loop in $\mathcal{F}$.) This finishes (see 6.3.1) the proof of Lemma 6.1.2 for $F = \mathbb{R}P^2$.

Remark. As it follows from 6.3.14 the condition $\Delta_{ST}(\gamma_1) = 0$ is automatically satisfied in the case of curves on $\mathbb{R}P^2$.

6.3.15. Case $F = K$. Corollary 6.3.6 says, that $\pi_1(\mathcal{F}, \xi)$ is isomorphic to $Z(\xi) < \pi_1(STK, \xi(a))$.

Consider $K$ as a quotient of a rectangle modulo the identification on its sides, which is shown in Figure 4. We can assume that $\xi(a)$ coincides with the image of a corner of the rectangle and that $\xi$ is direct tangent to the curve $c$ at $\xi(a)$. Let $g$ and $h$ be the curves such that: $\xi(a) = \tilde{g}(a) = \tilde{h}(a)$, $g = c \in \pi_1(K, \xi(a))$ and $h = d \in \pi_1(K, \xi(a))$. (Here $c$ and $d$ are the elements of $\pi_1(K)$ realized by the sides of the rectangle used to construct $K$, see Figure 3.) Let $f$ be the class of an oriented fiber of the fibration $pr : STK \to K$. One can show that:

$$\pi_1(STK, \xi(a)) = \{ \tilde{g}, \tilde{h}, f | \tilde{g}^\pm 1 = \tilde{g} \tilde{h}^\pm 1 = \tilde{g} \tilde{f} \tilde{g}^{-1} \tilde{h}, \tilde{h} f \tilde{h}^{-1} = f \tilde{h} f = f \}.$$  (8)

The second and the third relations in this presentation follow from (3) and (2). To get the first relation one notes that the identity $dc^{\pm 1}$ implies $\tilde{g}^\pm 1 = \tilde{g} \tilde{h} f \tilde{g}$ for some $k \in \mathbb{Z}$. But $\tilde{h}^2$ commutes with $\tilde{g}$, since they can be lifted to $STT^2$, the fundamental group of which is Abelian. Hence, $k = 0$.

Using relations (3) one can calculate $Z(\xi) = \pi_1(\mathcal{F}, \xi)$. (Note that these relations allow one to present any element of $\pi_1(STK, \xi(a))$ as $\tilde{g}^k \tilde{h}^m f^l$, for some $k, l, m \in \mathbb{Z}$.)

This group appears to be:

a) The whole group $\pi_1(STK, \xi(a))$, provided that $\xi = \tilde{h}^{2l}$ for some $l \in \mathbb{Z}$.

b) An isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(STK, \xi(a))$, provided that $\xi = \tilde{g}^k \tilde{h}^l f^m$ for some $k, l, m \in \mathbb{Z}$, such that $k \neq 0$ or $m \neq 0$. This subgroup is generated by $\{ \tilde{g}, \tilde{h}^2, f \}$.
c) An isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(STK, \tilde{\xi}(a))$, provided that $\tilde{\xi} = \tilde{g}^k \tilde{h}^{2l+1} f^m$ for some $k, l, m \in \mathbb{Z}$. This subgroup is generated by $\{\xi, \tilde{h}\}$.

A straightforward check (which uses (8)) shows that:

a) If $\xi$ represents an orientation preserving loop on $K$, then a certain degree of any loop $\gamma \in \pi_1(F, \xi)$ can be expressed as a product of $\gamma_1$ (see 3.2.2), $\gamma_2$ (see 3.2.3), $\gamma_3$, described below, and their inverses.

b) If $\xi$ represents an orientation reversing loop on $K$, then a certain degree of any loop $\gamma \in \pi_1(F, \xi)$ can be expressed as a product of $\gamma_3, \gamma_4$, described below, and their inverses.

Consider a loop $\beta$ in the space of all the autodiffeomorphisms of $K$, which is the sliding of $K$ along the unit vector field parallel to the curve $d$ on $K$. (Note that $K$ has to slide twice along itself under this loop before all points of $K$ come to the original position.) The loop $\gamma_3$ is the sliding of $\xi$ induced by $\beta$.

The loop $\gamma_4$ is the sliding of $\xi$ along itself.

No triple point stratum crossings occur under $\gamma_3$ and $\gamma_4$. By the assumption of the Lemma $\Delta_{St}(\gamma_1) = 0$ and $\Delta_{St}(\gamma_2) = 0$ (when $\gamma_2$ is well defined).

Thus, $\Delta_{St}(\gamma_i) = 0$ ($i \in \{1, 2, 3, 4\}$) and we have proved (see 6.3.1) Lemma 6.1.2 in the case of $F = K$.

Remark. One can check that for the curve on the Klein bottle shown in Figure 7 the equations $\Delta_{St}(\gamma_1) = 0$ and $\Delta_{St}(\gamma_2) = 0$ are independent. This means that both of the corresponding conditions are needed for the integrability of $\psi$.

![Figure 7](image_url)

This finishes the proof of Lemma 6.1.2 for all the cases. □

6.4. **Proof of Theorem 3.3.2.** Let $S$ be the space of all the smooth mappings (not necessarily immersions) from $S^1$ to $F$. Consider the subspace $\mathcal{N}$ of $S^1(3) \times S$ consisting of $t \times f$, such that $f$ maps the three points from $t$ to one point on $F$. Clearly $\mathcal{M}$ is a subspace of $\mathcal{N}$.

Let $s_1, s_2 \in \mathcal{C}$ be two generic curves with a triple point, such that $[s_1] = [s_2]$. Let $m_1, m_2 \in \mathcal{M}$ be the elements corresponding to $s_1$ and $s_2$. To prove the Theorem we need to show that $m_1$ and $m_2$ belong to the same path connected component of $\mathcal{M}$.

Since $[s_1] = [s_2]$, we see that $m_1$ can be transformed to $m_2$ (in $\mathcal{N}$) by the sequence of moves $S_1, S_2, S_3, S_4, S_5$ (shown in Figure 8) and their inverses (and a continuous change of the parameterization). Note, that $S^+_{1}$ is the only move in this list which happens not in $\mathcal{M}$. 

We can imitate the $S_1$-move staying in $\mathcal{M}$ by creating two opposite kinks (see Figure 2) and then making one of these kinks very small. (Similarly, we can imitate the $S_1^{-1}$-move staying in $\mathcal{M}$.)

![Diagram of kinks and S operations]

Figure 8.

We use $S_2, S_3, S_4, S_5$ and their inverses and the imitations of $S_1^\pm 1$ to deform $m_1$ inside $\mathcal{M}$ so that it looks exactly as $m_2$, except some number of small extra kinks located on the three loops of $m_1$ adjacent to the triple point.

As it is shown below, we can create two opposite extra kinks, the first on one of the three loops of $m_1$, the second on another.

The order, in which a small loop going around the triple point crosses the three branches of $m_1$ passing through the triple point, induces a cyclic order on the branches. We use $S_2$ to deform Figure 3 to Figure 4, then we use $S_5$ to deform it to Figure 5. Note, that under this procedure the branches I and II get interchanged in the cyclic order. Then in a similar way we interchange the branches in the pairs {I, III}; {I, II} and {I, III}. After this the local picture around the triple point is the same as before. One can check, that what happened with $m_1$ globally is equivalent to the addition of two opposite kinks, the first to one branch of $m_1$, the second to another.

It is clear, that using this procedure and the cancelation of two opposite kinks (see Figure 2) we can concentrate all the extra kinks on one of the three loops of
Slide these extra kinks along the loop, so that they are all concentrated on a small arc. Cancel out all the pairs of opposite extra kinks. Now all the small extra kinks are pointing to one side of the loop.

The $h$-principle (see page 16) implies that the space of all the curves on $F$ is weak homotopy equivalent to the space of all the free loops in $STF$ (the spherical tangent bundle of $F$). The corresponding mapping $h$ sends a curve $\xi \in F$ to a loop $\tilde{\xi} \subset STF$ by mapping a point $y \in S^1$ to a point in $STF$, to which points the velocity vector of $\xi$ at $\xi(y)$. Since by our assumption both $s_1$ and $s_2$ belong to the same connected component of $F$, we get that their liftings to loops in $STF$ are free homotopic.

Let $f$ be the homotopy class of the fiber of the $S^1$-fibration $pr : STF \to F$. An extra small kink corresponds under $h$ to a multiplication by $f \pm 1$, depending on the side of the loop the kink points to. Let $n$ be the number of small extra kinks which are present on $m_1$.

Fix a point $a$ on $S^1$. We can assume that after the process described above the curves $s_1$ and $s_2$ (corresponding to $m_1$ and $m_2$, respectively) are direct tangent at the image of $a$. Now we can consider $s_1$ and $s_2$ as elements of $\pi_1(F,s_1(a))$ and the liftings $\tilde{s}_1$ and $\tilde{s}_2$ (see 6.3.3) as elements of $\pi_1(STF,\tilde{s}_1(a))$. By the initial assumption $s_1$ and $s_2$ belong to the same connected component of $F$. The $h$-principle implies, that $\tilde{s}_1$ is free homotopic to $\tilde{s}_2$. Hence, we get that for some element $\alpha \in \pi_1(STF,\tilde{s}_1(a))$

$$\tilde{s}_1 = \alpha \tilde{s}_1 f^n \alpha^{-1}. \quad (9)$$

Consider the case of $F = S^2$. One checks that $\pi_1(STS^2) = \mathbb{Z}_2$ is commutative and $f$ has order two in $\pi_1(STS^2)$. (Note that $STS^2 = \mathbb{R}P^3$.) From (9) we get that $n$ (the number of extra kinks) is even. We take one of the kinks and evert it, by expanding it till it goes around $S^2$ and comes back as a kink pointing to the other side of the loop. Then we cancel it out with one of the other extra kinks. In order to deform $m_1$ to $m_2$ we perform this operation until there are no extra kinks left.

In the case of $F = T^2$ the group $\pi_1(STT^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ is commutative. From (9) we get that $f^n = 1$. But $f \in \pi_1(STT^2)$ has infinite order, thus $n = 0$ and there were no extra kinks that survived the process. This means, that we have constructed the desired path from $m_1$ to $m_2$.

For $F \neq S^2,T^2$ the element $f \in \pi_1(STF)$ has infinite order. Combining identities (9) and (9) (recall that $F$ was assumed to be orientable) we get that $\tilde{s}_1^{-1} \alpha^{-1} \tilde{s}_1 \alpha = f^n$. Thus, the projections of $\tilde{s}_1$ and $\alpha$ commute in $\pi_1(F,s_1(a))$. Proposition 6.3.7 implies that these projections can be expressed as powers of some $g \in \pi_1(F,s_1(a))$. Let $g_{s_1}$ be a curve representing this $g$, which is direct tangent to $s_1$.

![Figure 9](image-url)
at $s_1(a)$. The kernel of the homomorphism $pr_*$ is generated by $f$. Using identity (5) we can present $\alpha$ as $\vec{g}_i f^j \in \pi_1(STF,s_1(a))$ and $s_1$ as $\vec{g}_k f^l \in \pi_1(STF,s_1(a))$, for some $i,j,k,l \in \mathbb{Z}$. This means (see (5)) that $\alpha$ commutes with $s_1$ in $\pi_1(STF,s_1(a))$. From the identity (9) we get that $f^n = 1$. But $f$ has infinite order in $\pi_1(STF)$. Hence, $n = 0$ and there were no extra kinks, that survived the process. This means that we have constructed the desired path from $m_1$ to $m_2$.

6.5. **Proof of Theorem 4.2.1.** The proof of Theorem 4.2.1 is analogous to the proof of Theorem 3.2.4.

One can easily formulate and prove the corresponding versions of Lemma 6.1.1 and Lemma 6.1.2.

The strata you have to go around, in order to prove the analogue of Lemma 6.1.1, are: a) two self-tangency points, b) self-tangency point and distinct triple point, c) triple point at which exactly two branches are tangent (of order one), d) self-tangency point of order two. The bifurcation diagrams for the last two cases are shown in Figure 5 and Figure 10, respectively.

6.6. **Proof of Theorem 4.3.2.** The proof of this Theorem is analogous to the proof of Theorem 3.3.2.

Let $s_1, s_2 \in \mathcal{C}$ be two generic curves with a point of direct self-tangency, such that $[s_1^+] = [s_2^+]$. Let $m_1, m_2 \in \mathcal{M}^+$ be the elements corresponding to $s_1$ and $s_2$, respectively. Since $[s_1^+] = [s_2^+]$ we can choose the mappings of $B_2$ to $STF$ associated with $s_1$ and $s_2$ so that they are free homotopic. Hence, the projections of them to $F$ are also free homotopic. One can show, that the projections of the two circles of $B_2$ can be assumed to be direct tangent (at the base point) under
this homotopy. Clearly the only moves needed for this homotopy are $S_1, S_2, S_3$ (see Figure 8), $S_6, S_7$ (see Figure 11) and their inverses.

We use the imitation of $S_1$ (described in 6.4) $S_3, S_4, S_6, S_7$ and their inverses (and a continuous change of the parameterization) to deform $m_1$ in the space $\mathcal{M}^+$ to an element, which looks nearly as $m_2$, except a number of small extra kinks located on the two loops of $m_1$ adjacent to the point of a direct self-tangency.

![Diagram](image)

**Figure 11.**

For each of the two loops we slide all the extra kinks, so that they are located on a small arc of the loop. We cancel all the pairs of opposite kinks by reversing the process shown in Figure 8. Now the kinks on each loop are pointing to the same side of it.

We note, that the $T^+$-equivalence class corresponding to $m_1$ did not change under all these deformations. An extra kink located on a loop of $m_1$ corresponds (under the lifting of the loop to a loop in $STF$) to the multiplication by the class of an oriented fiber of $pr : STF \to F$. Similarly to 6.4 we get, that for $F \neq S^2$, the number of extra kinks on each of the two loops of $m_1$ is zero. This means that we have constructed the desired path connecting $m_1$ to $m_2$. For $F = S^2$ we use the process described in Section 6.4 to cancel out all the extra kinks on each of the two loops, and obtain a path connecting $m_1$ to $m_2$.

This finishes the Proof of Theorem 4.3.2.

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