On a backward problem for multidimensional Ginzburg–Landau equation with random data

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Received 31 May 2017, revised 10 October 2017
Accepted for publication 21 November 2017
Published 15 December 2017

Abstract

In this paper, we consider a backward in time problem for the Ginzburg–Landau equation in a multidimensional domain associated with some random data. The problem is ill-posed in the sense of Hadamard. To regularize the unstable solution, we develop a new regularized method combined with statistical approach. We prove an upper bound on the rate of convergence of the mean integrated squared error in $L^2$ and $H^1$ norms.

Keywords: backward problem, regularization, ill-posed problem

1. Introduction

The Ginzburg–Landau equation has been applied in various areas in physics, including phase transitions in non-equilibrium systems, instabilities in hydrodynamic systems, chemical turbulence, and thermodynamics (see [2, 14]). The degenerate Ginzburg–Landau equation with variable diffusion coefficients in the following form

$$u_t - \nabla \cdot (a(x,t)\nabla u) = b(x,t)u - c(x,t)u^3 + G(x,t), \quad x \in \mathbb{R}^d, \quad t > 0,$$

(1.1)
(b and c are modeling parameters) has many applications in a macroscopic description of superconductivity [7]. In the theory of superconductivity equation (1.1) may arise as a simplified model, where the variable coefficient represents the equilibrium density of superconducting electrons in a material. In this case, the positive, smooth function \( a(x, t) \) represents the geometrical variation of a thin superconducting film of variable thickness. For more details on the physical meaning of the appearance of the variable diffusion coefficient, we refer to the references [6, 7, 11, 12, 13] and the section 4 of this paper.

In this paper, we consider the backward problem of finding the initial value \( u(x, 0) \) for a simplified version of (1.1)

\[
\begin{align*}
\dot{u} - \Lambda(t)\Delta u &= u - u^3 + G(x, t), & (x, t) \in \Omega \times (0, T), \\
u(x, t) &= 0, & x \in \partial\Omega, \\
u(x, T) &= H(x), & x \in \Omega,
\end{align*}
\]  

(1.2)

where \( \Lambda \in C([0, T]) \) and \( H \in L^2(\Omega) \). Here the domain \( \Omega = (0, \pi)^d \) is a subset of \( \mathbb{R}^d \) and \( x := (x_1, \ldots, x_d) \). The function \( G \in L^2(\Omega) \) is called the source function. The function \( H \) is given and is often called a final value data.

It is well-known that the backward in time problem (1.2) is severely ill-posed in the sense of Hadamard [2] and [10]. Hence solutions do not depend continuously on the data. In reality, the exact values of the source \( G \) and the final data \( H \) are not available. We only have observation data \( G^{obs} \) and \( H^{obs} \) that give construction of regularization method. In the majority of recent papers, the data \( G^{obs} \) and \( H^{obs} \) are given on the whole space domain, \( \mathbb{R}^d \) and they are used to construct an approximation for \( u(x, 0) \). In [2], the author considered continuous dependence of the solutions on the parameter \( c(x, t) \) of a Ginzburg–Landau equation (1.1) in the case \( a(x, t) = 1 \). Ames [2] introduced the backward in time problem for Ginzburg–Landau equation. But she did not give a regularization result. In 2015, Trong et al [27] have given a regularization method for (1.2) when \( \Lambda(t) = 1 \). Their idea is to approximate the function \( f(u) = u - u^3 \) with a globally Lipschitz function and using this function to find the regularized solution of the problem. To the best of our knowledge, the regularization results for Ginzburg–Landau are still limited.

In the present paper, we will consider the data from a different point of view in which the source \( G \) and the final data \( H \) will be measured with errors at a discrete set of points. These errors may be generated from controllable sources or uncontrollable sources. If the errors are generated from uncontrollable sources as wind, rain, humidity, etc, then the model is random. On a first thought, such small errors seem to be of no importance. However, the accumulation of the small errors in the data of an ill-posed problem can make the noise of the solution large and, hence, it cannot be ignored. This effect is considered in the theory of statistical inverse problems [1], section 2.1.5, p 48.

The backward problem for linear parabolic equations has a long history and is a special form of statistical inverse problems. From [4], we know that equation (1.1) with \( a = 1 \), \( b = c = 0 \) can be transformed by a linear operator with random noise

\[
H = Ku_0 + \text{‘noise’},
\]  

(1.3)

where \( u_0 := u(x, 0) \) is the initial data and \( K \) is a bounded linear operator that does not have a continuous inverse. A large part of the literature focuses on methods which require the explicit knowledge of a spectral decomposition of the operator \( K^*K \). The simplest of these methods is spectral cut-off (or truncated singular value decomposition (SVD) for compact operators): see, for example, [4, 5, 16, 19, 26]. This method is applied to solve many practical problems: for example, the Radon transform, Deconvolution problem, etc. Recently, the number...
of articles on statistical inverse problems and the backward problem with random data has increased rapidly. We list here some other well-known methods such as the Tikhonov method [8, 20–22], iterative regularization methods [9]. A well known fact is that, when the ‘noise’ in these models is modeled as a random quantity, the convergence of estimators \( \hat{u}(x,0) \) of \( u(x,0) \) should be studied by statistical methods. Methods applied to the deterministic cases cannot be applied directly to this case. The main idea in using the random noise is to find suitable estimators \( \hat{u}(x,0) \) and to consider the expected square error \( \mathbb{E}\left[\left\|\hat{u}(\cdot,0) - u(\cdot,0)\right\|^2\right] \) in a suitable space \( V \) which is also called the mean integrated square error (MISE).

In this paper, we describe the relationship between observed data and the sources \( G \) and \( H \) by means of nonparametric regression models. Indeed, if we measure the function \( H(x) \) and \( G(x,t) \) at non-random design points \( x_i \) with index \( i = (i_1, i_2, ..., i_d) \in \mathbb{N}^d, 1 \leq i_k \leq n_k \) for \( k = 1, d \)

\[
x_i = (x_{i_1}, ..., x_{i_d}) = \left(\frac{\pi(2i_1 - 1)}{2n_1}, \frac{\pi(2i_2 - 1)}{2n_2}, ..., \frac{\pi(2i_d - 1)}{2n_d}\right), \quad i_k = 1, n_k, \quad k = 1, d,
\]

then we obtain a set of values

\[
\tilde{D}_i = \tilde{D}_{i_1, i_2, ..., i_d} \approx H(x_{i_1}, ..., x_{i_d}), \quad \tilde{G}_i(t) = \tilde{G}_{i_1, i_2, ..., i_d}(t) \approx G(x_{i_1}, ..., x_{i_d}, t).
\]

The real measurements are always observed with errors. We consider the nonparametric regression model of data as follows

\[
\tilde{D}_i = \tilde{D}_{i_1, i_2, ..., i_d} := H(x_{i_1}, ..., x_{i_d}) + A_{i_1, i_2, ..., i_d} \Theta_{i_1, i_2, ..., i_d} = H(x_i) + A_i \Theta_i
\]

\[
\tilde{G}_i(t) = \tilde{G}_{i_1, i_2, ..., i_d}(t) := G(x_{i_1}, ..., x_{i_d}, t) + \vartheta \Psi_{i_1, i_2, ..., i_d}(t) = G(x_i, t) + \vartheta \Psi_i(t),
\]

for \( i_k = 1, n_k, \quad k = 1, d \). Here \( \Theta_i := \Theta_{i_1, i_2, ..., i_d} \sim \mathcal{N}(0, 1) \) and \( \Psi_i(t) := \Psi_{i_1, i_2, ..., i_d}(t) \) are Brownian motions. Here \( A_{i_1, i_2, ..., i_d} \) and \( \vartheta \) are positive constants which are bounded by a positive constant \( V_{\text{max}} \). We assume furthermore that they are mutually independent. For the similar random model as above, we refer to the works of Trong et al [26]. In [26], the authors deal with a discrete random model in 1-D case for Helmholtz equation. In [27], the authors considered a discrete random model in 2D case. In both papers, the spectral methods together with trigonometric least squares method in nonparametric regression has been applied.

In this paper, since the model in (1.2) is nonlinear, we cannot transform it into the operator form defined in equation (1.3). This makes the nonlinear problem (1.2) more challenging. Another difficulty arises when \( \Lambda \) is noisy by random observation of the following form

\[
\Lambda_{\epsilon}(t) = \Lambda(t) + \epsilon \xi(t),
\]

where \( \epsilon \) is a deterministic noise level and \( \xi(t) \) is a Brownian motion. If \( \Lambda(t) \) is a constant (independent of \( t \)) then we can apply well-known methods such as the spectral method (see section 3 [15]) for solving the problem (1.2). However, when \( \Lambda \) depends on \( t \) and is noisy as in equation (1.7), the problem is more challenging. Although spectral method is effective for linear random models, it is not possible to approximate the solution of problem (1.2) using the spectral method. Until now, to the best of our knowledge, there do not exist any results for approximating the solution of the problem (1.2) with the random model (1.6) and (1.7). We want to emphasize that our random model here is multidimensional case which is a generalization of results in [26, 27].

Our main goal in this paper is to provide a new regularized method that give a regularized solution that is called estimators for approximating \( u(x,t), 0 \leq t < T \). In this paper, we do
not investigate the existence of the solution of backward problem (1.2). Applying the results in [17], we can show the uniqueness of the solution of problem (1.2) (see more details in section 2 and section 3 of [17]).

The backward problem for Ginzburg–Landau equation with random data has not been studied before. Our main idea in this paper is to apply a modified quasi-reversibility method. However, our method is original and very different from the methods in [18]. First, we approximate $H$ and $G$ by the approximating functions $\hat{H}_s$ and $\hat{G}_s$, that are defined in theorem (2.1). Next, our task is to find the approximating operator for $\Lambda(t)\Delta$. We will not approximate directly the operator $\Lambda(t)\Delta$ as introduced in [18]. We introduce a new approach by giving an unbounded time independent operator $\mathbf{P}$ that is defined in lemma 3.1. Then, we approximate $\mathbf{P}$ by a bounded operator $\mathbf{P}_{\rho,\gamma}$ in order to establish an approximation for the regularized problem (3.10). Here $\beta_n$ satisfies $\lim_{n \to \infty} \beta_n = +\infty$, and we choose $\rho_n$ that depends on $\beta_n$ suitably to obtain the convergence rate. In contrast to the initial value problem, for the final value (inverse) problem, we need to assume that problem (1.2) has a unique solution $u$. In particular, the main purpose in our error estimates is to show that the norm of the difference between the regularized solution of the problem (3.10) and the solution of the problem (1.2) in $L^2(\Omega)$ and $H^1(\Omega)$ tends to zero when $|n| = \sqrt{n_1^2 + \cdots + n_d^2} \to +\infty$. To study the mean integrated square error $\mathbb{E} \left[ \| \tilde{u}(\cdot, 0) - u(\cdot, 0) \|_{L^2}^2 \right]$ we must investigate the errors $\mathbb{E}\| \hat{H}_s - H \|^2$ and $\mathbb{E}\| \hat{G}_s - G \|^2$. This makes the study of the random model more difficult than the deterministic one. Our analysis and techniques are new and different from the previously used methods.

2. Constructing a function from discrete random data

In this section, we recall a new tool that was developed in [15] for constructing a function in $L^2(\Omega)$ from the given discrete random data.

First, since the Laplace operator $-\Delta$ is a linear, densely defined self-adjoint and positive definite elliptic operator on the connected bounded domain $\Omega$ with Dirichlet boundary condition, the eigenvalues of $-\Delta$ are given by $\lambda_p = |p|^2 = p_1^2 + p_2^2 + \cdots + p_d^2$; their corresponding eigenfunctions are denoted respectively by

$$\psi_p(x) = \left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)^d \sin(p_1 x_1) \sin(p_2 x_2) \cdots \sin(p_d x_d). \quad (2.1)$$

The functions $\psi_p$ are normalized so that $\{\psi_p\}_{p \in \mathbb{N}^d}$ is an orthonormal basis of $L^2(\Omega)$.

We will use the following notation: $|p| = |(p_1, \ldots, p_d)| = \sqrt{p_1^2 + \cdots + p_d^2}$, $|n| = |(n_1, \ldots, n_d)| = \sqrt{n_1^2 + \cdots + n_d^2}$.

Definition 2.1. For $\gamma > 0$, we define

$$\mathcal{H}^\gamma(\Omega) := \left\{ h \in L^2(\Omega) : \sum_{p_1=1}^\infty \cdots \sum_{p_d=1}^\infty |p|^2\gamma \langle h, \psi_p \rangle^2 < \infty \right\}. \quad (2.2)$$

The norm on $\mathcal{H}^\gamma(\Omega)$ is defined by

$$\| h \|^2_{\mathcal{H}^\gamma(\Omega)} := \sum_{p_1=1}^\infty \cdots \sum_{p_d=1}^\infty |p|^2\gamma \langle h, \psi_p \rangle^2. \quad (2.3)$$
For any Banach space $X$, we denote by $L^p (0, T; X)$, the Banach space of measurable real functions $v : (0, T) \rightarrow X$ such that

$$
\|v\|_{L^p(0, T; X)} = \left( \int_0^T \|v(\cdot, t)\|_X^p \, dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty,
$$

$$
\|v\|_{L^\infty(0, T; X)} = \text{esssup}_{0 < t < T} \|v(\cdot, t)\|_X < \infty, \quad p = \infty.
$$

Let $\beta : \mathbb{N}^d \rightarrow \mathbb{R}$ be a function. The next result gives an error estimate between $H$ and $\hat{H}_{\beta_n}$, and error estimate between $\hat{G}_{\beta_n}$ and $G$.

**Theorem 2.1 (Theorem 2.1 in Kirane et al [15]).** Define the set $\mathcal{W}_{\beta_n}$ for any $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$

$$
\mathcal{W}_{\beta_n} = \mathcal{W}_{\beta_n} = \left\{ p = (p_1, \ldots, p_d) \in \mathbb{N}^d : |p|^2 = \sum_{k=1}^d p_k^2 \leq \beta_n = \beta(n_1, \ldots, n_d) \right\}
$$

(2.4)

where $\beta_n$ satisfies

$$
\lim_{|\mathbf{n}| \rightarrow +\infty} \beta_n = +\infty.
$$

For a given $\mathbf{n}$ and $\beta_n$ we define functions that are approximating $H$, $G$ as follows

$$
\hat{H}_{\beta_n} (x) = \sum_{\mathbf{p} \in \mathcal{W}_{\beta_n}} \left[ \frac{\pi^d}{\prod_{k=1}^d n_k} \prod_{i=1}^{n_1} \cdots \prod_{i_d}^{n_d} \hat{D}_{\mu, \ldots, \mu}(x_i) \right] \varphi_p (x).
$$

(2.5)

and

$$
\hat{G}_{\beta_n} (x, t) = \sum_{\mathbf{p} \in \mathcal{W}_{\beta_n}} \left[ \frac{\pi^d}{\prod_{k=1}^d n_k} \prod_{i=1}^{n_1} \cdots \prod_{i_d}^{n_d} \hat{G}_{\mu, \ldots, \mu}(t) \psi_p (x_i, \ldots, x_d) \right] \varphi_p (x).
$$

(2.6)

where $\hat{D}_{\mu, \ldots, \mu}$, $\hat{G}_{\mu, \ldots, \mu}(t)$ are defined in (1.5) and (1.6) respectively.

Let $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$ with $\mu_k > 1$ for any $k = 1, d$. Let us choose $\mu_0 \geq d \max(\mu_1, \ldots, \mu_d)$.

If $H \in \mathcal{H}^{\mu_0}(\Omega)$ and $G \in L^\infty(0, T; \mathcal{H}^{\mu_0}(\Omega))$ then the following estimates hold

$$
E \left\| \hat{H}_{\beta_n} - H \right\|_{L^2(\Omega)}^2 \leq \mathcal{C}(\mu_1, \ldots, \mu_d, H) \beta_n^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_0} + 4 \beta_n^{-\mu_0} \left\| H \right\|_{\mathcal{H}^{\mu_0}(\Omega)}^2,
$$

$$
E \left\| \hat{G}_{\beta_n}(..., t) - G(..., t) \right\|_{L^\infty(0,T,L^2(\Omega))}^2 \leq \mathcal{C}(\mu_1, \ldots, \mu_d, H) \beta_n^{d/2} \prod_{k=1}^d (n_k)^{-4\mu_0} + 4 \beta_n^{-\mu_0} \left\| G \right\|_{L^\infty(0,T; \mathcal{H}^{\mu_0}(\Omega))}^2,
$$

where

$$
\mathcal{C}(\mu_1, \ldots, \mu_d, H) = 8\pi^d V_{\max} \frac{2\pi^{d/2}}{d!^{(d/2)}} + \frac{16\pi^2 (\mu_1, \ldots, \mu_d)}{d!^{(d/2)}} \left\| H \right\|_{\mathcal{H}^{\mu_0}(\Omega)}^2.
$$
Corollary 2.1 (Corollary 2.1 in Kirane et al [15]). Let $H, G$ be as in theorem (2.1). Then the error term $\mathbb{E}\left|\hat{H}_{\beta_n} - H\right|_{L^2(\Omega)}^2 + \mathbb{E}\left|\hat{G}_{\beta_n} - G\right|_{L^\infty(0,T;L^2(\Omega))}^2$ is of order
\[
\max\left(\frac{\beta_n^{d/2}}{d}, \beta_n^{-\mu_d}\right).
\]

3. Backward problem for parabolic equations with random coefficients

In this section, we assume that $\Lambda_\gamma, \Lambda$ are continuous functions on $[0, T]$, hence there exist two positive numbers $A_0, A_1$ such that
\[
A_0 \leqslant \|\Lambda_\gamma\|_{C([0,T])} = \sup_{0 \leqslant t \leqslant T} |\Lambda_\gamma(t)| < A_1, \quad A_0 \leqslant \|\Lambda\|_{C([0,T])} = \sup_{0 \leqslant t \leqslant T} |\Lambda(t)| < A_1. \tag{3.1}
\]

We approximate the function $F(u) = u - u^3$ with $\mathcal{F}_Q$ defined by
\[
\mathcal{F}_Q(u(x,t)) = \begin{cases} 
Q - Q^3, & u(x,t) > Q, \\
- Q - Q^3, & u(x,t) < -Q, \\
u - u^3, & -Q \leqslant u(x,t) \leqslant Q. 
\end{cases}
\]

for all $Q > 0$. In the sequel, we use a parameter sequence $Q_n := Q(n_1, n_2, \ldots, n_d) \to +\infty$ as $|n| \to +\infty$ so that when $|n|$ is large enough, we have that $Q_n \geqslant \|u\|_{L^\infty(0,T;L^2(\Omega))}$. Moreover, we also have
\[
\mathcal{F}_{Q_n}(u) = F(u) = u - u^3, \quad \text{for } |n| \text{ large enough.} \tag{3.2}
\]

Using [28], we also obtain that $\mathcal{F}_{Q_n}$ is a globally Lipschitz source:
\[
\|\mathcal{F}_{Q_n}(v_1) - \mathcal{F}_{Q_n}(v_2)\|_{L^2(\Omega)} \leqslant (2 + 6Q_n^2) \|v_1 - v_2\|_{L^2(\Omega)}, \tag{3.3}
\]

for any $v_1, v_2 \in L^2(\Omega)$.

Lemma 3.1. Define the following space:
\[
\mathcal{Z}_{\gamma,B}(\Omega) := \left\{ f \in L^2(\Omega) : \sum_{\mathbf{p} \in \mathbb{N}^d} |\mathbf{p}|^{2+2\gamma} e^{2B|\mathbf{p}|^2} \left\langle f, \psi_\mathbf{p} \right\rangle_{L^2(\Omega)}^2 < +\infty \right\}, \tag{3.4}
\]

for any $\gamma \geqslant 0$ and $B \geqslant 0$. Let $\rho : \mathbb{N}^d \to \mathbb{R}$ be a function. Define also the operator $P = A_1 \Delta$ ($A_1$ is the upper bound in (3.1)) and $P_{\rho_n}$ as follows
\[
P_{\rho_n}(v) = A_1 \sum_{|\mathbf{p}| \leqslant \sqrt{\frac{2\gamma}{\rho_n}}} |\mathbf{p}|^2 \left\langle v, \psi_\mathbf{p} \right\rangle_{L^2(\Omega)} \psi_\mathbf{p}, \tag{3.5}
\]

for any function $v \in L^2(\Omega)$. Then for any $v \in L^2(\Omega)$
\[
\|P_{\rho_n}(v)\|_{L^2(\Omega)} \leqslant \rho_n \|v\|_{L^2(\Omega)}, \tag{3.6}
\]

and for $v \in \mathcal{Z}_{\gamma,B}(\Omega)$
\[
\|Pv - P_{\rho_n}v\|_{L^2(\Omega)} \leqslant A_1 \rho_n^{-\gamma} e^{-T \rho_n} \|v\|_{\mathcal{Z}_{\gamma,B}(\Omega)} \cdot \tag{3.7}
\]
\textbf{Proof.} First, for any \( v \in L^2(\Omega) \), we have
\[
\| P_{\rho_n}(v) \|_{L^2(\Omega)}^2 = A_1^2 \sum_{|p| \leq \sqrt{n}} |p|^4 \left\langle v, \psi_p \right\rangle_{L^2(\Omega)}^2 \\
\leq \rho_n^2 \sum_{|p| \leq \sqrt{n}} \left\langle v, \psi_p \right\rangle_{L^2(\Omega)}^2 = \rho_n^2 \| v \|_{L^2(\Omega)}^2 ,
\]
(3.8)
and
\[
\| P v - P_{\rho_n}(v) \|_{L^2(\Omega)}^2 = A_1^2 \sum_{|p| > \sqrt{n}} |p|^{-4} e^{-2T \lambda_1 |p|^2} |p|^{-4} e^{2T \lambda_1 |p|^2} \left\langle v, \psi_p \right\rangle_{L^2(\Omega)}^2 \\
\leq A_1^2 \rho_n^{-2} e^{-2T \lambda_1 \rho_n} \sum_{|p| > \sqrt{n}} |p|^{-4} e^{2T \lambda_1 |p|^2} \left\langle v, \psi_p \right\rangle_{L^2(\Omega)}^2 \\
= A_1^2 \rho_n^{-2} e^{-2T \lambda_1 \rho_n} \| v \|_{L^2(\Omega)}^2 .
\]
(3.9)

Applying a modified quasi-reversibility method as in section 4.1 in Kirane et al [15], we introduce the following regularized problem
\[
\begin{aligned}
\frac{\partial \hat{U}_{\rho_n, \beta_0}}{\partial t} - \Lambda_e(t) \Delta \hat{U}_{\rho_n, \beta_0} - P \hat{U}_{\rho_n, \beta_0} + P_{\rho_n, \beta_0} \hat{U}_{\rho_n, \beta_0} \\
= F_{\hat{Q}_n}(\hat{U}_{\rho_n, \beta_0}(x, t)) + \hat{G}_{\beta_0}(x, t), \quad 0 < t < T, \\
\hat{U}_{\rho_n, \beta_0}(x, t) = 0, \quad x \in \partial \Omega, \\
\hat{U}_{\rho_n, \beta_0}(x, T) = \hat{H}_{\beta_0}(x).
\end{aligned}
\]
(3.10)
Since the first equation of the system (3.10) contains the term \( \Lambda_e(t) \) which depends on \( \epsilon \), it is suitable to denote the solution of problem (3.10) by \( \hat{U}_{\rho_n, \beta_0} \) with three variables \( \rho_n, \beta_0, \epsilon \). Now, we give convergence rates between the regularized solution \( \hat{U}_{\rho_n, \beta_0} \) of problem (3.10) and the solution \( u \) of problem (1.2). Furthermore, we show that \( \hat{U}_{\rho_n, \beta_0} \) converges to \( u \) when \( |n| \to +\infty \) and \( \epsilon \to 0 \).

The next proposition gives existence and uniqueness of solutions of equation (3.10).

\textbf{Proposition 3.1.} The problem (3.10) has a unique solution \( \hat{U}_{\rho_n, \beta_0} \in C([0, T] ; L^2(\Omega)) \cap L^2(0, T ; H^1_0(\Omega)) \).

\textbf{Proof.} Let \( \hat{V}_{\rho_n, \beta_0}(x, t) = \hat{U}_{\rho_n, \beta_0}(x, T - t) \) and set \( \bar{\Lambda}_e(t) = A_1 - \Lambda_e(t) \). Then by (3.10), \( \hat{V}_{\rho_n, \beta_0}(x, t) \) satisfies the following problem
\[
\begin{aligned}
\frac{\partial \hat{V}_{\rho_n, \beta_0}}{\partial t} - \bar{\Lambda}_e(t) \Delta \hat{V}_{\rho_n, \beta_0} = \mathcal{G} \hat{V}_{\rho_n, \beta_0}(x, t) - \hat{G}_{\beta_0}(x, t), \quad 0 < t < T, \\
\hat{V}_{\rho_n, \beta_0}(x, t) = 0, \quad x \in \partial \Omega, \\
\hat{V}_{\rho_n, \beta_0}(x, 0) = H_{\beta_0}(x),
\end{aligned}
\]
(3.11)
where \( \mathcal{G} \) is given by
\[
\mathcal{G} w(x, t) = P_{\rho_n} w(x, t) - F_{\hat{Q}_n}(w(x, t)),
\]
for any \( w \in C([0, T]; L^2(\Omega)) \). For any \( w_1, w_2 \in C([0, T]; L^2(\Omega)) \), we have the following estimate
\[
\|G w_1(x, t) - G w_2(x, t)\|_{L^2(\Omega)} \leq \|F Q_n(w_1(x, t)) - F Q_n(w_2(x, t))\|_{L^2(\Omega)} + \|P \rho_n w_1(x, t) - P \rho_n w_2(x, t)\|_{L^2(\Omega)}.
\]
(3.12)

where we have used (3.3) and (3.6). So \( G \) is a Lipschitz function. Using the results of chapter 12, theorem 12.2, p 211 of [3], the proof of the proposition is completed. \( \square \)

We state our main theorem in this paper.

**Theorem 3.1.** Assume that problem (1.2) has unique solution \( u \in L^\infty(0, T; Z_{\gamma, T_n}(\Omega)) \).

(a) **Error estimate in \( L^2 \).** Let \( H, G \) be as in theorem 2.1. Let \( \beta_n, \rho_n \) be such that
\[
\lim_{|n| \to +\infty} \beta_n = \lim_{|n| \to +\infty} \rho_n = +\infty, \quad \lim_{|n| \to +\infty} \frac{e^{2\rho_n T} \beta_n^{d/2}}{\prod_{k=1}^d (n_k)^d \nu_k} = \lim_{|n| \to +\infty} e^{2\rho_n T} \beta_n^{-\mu_0} = 0,
\]
(3.13)

and
\[
\rho_n \leq \frac{1}{T} \log \left( \frac{1}{E(\epsilon)} \right)
\]
(3.14)

for any \( 0 < E(\epsilon) < 1 \) that satisfies
\[
\lim_{\epsilon \to 0} \frac{\epsilon}{E(\epsilon)} = 0.
\]
(3.15)

Let \( Q_n \) be such that
\[
\lim_{|n| \to +\infty} e^{6Q_n T} \rho_n^{-2\gamma} = \lim_{|n| \to +\infty} e^{6Q_n T} \frac{e^{2\rho_n T} \beta_n^{d/2}}{\prod_{k=1}^d (n_k)^d \nu_k} = \lim_{|n| \to +\infty} e^{6Q_n T} e^{2\rho_n T} \beta_n^{-\mu_0} = 0,
\]
(3.16)

and
\[
Q_n \leq \sqrt{\frac{1}{6T} \log \left( \frac{1}{E_0(\epsilon)} \right)}
\]
(3.17)

where \( 0 < E_0(\epsilon) < 1 \) satisfies
\[
\lim_{\epsilon \to 0} \frac{\epsilon}{E(\epsilon)E_0(\epsilon)} = 0.
\]
(3.18)

Then for \( |n| \) large enough, and \( \epsilon \) small enough, the error \( \| U^\epsilon_{\rho_n, \beta_n}(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \) is of order
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\[ e^{6Q_T} \max \left( \frac{\mu^2}{d} \prod_{k=1}^{d} (n_k)^{4\mu_2}, e^{-2\rho_n^{-2\gamma} \cdot 2^2(\gamma-1) \beta_n^{-\mu_0}} \right) + \frac{\epsilon}{E(\epsilon)E_0(\epsilon)}. \]  \hspace{1cm} (3.19)

(b) Error estimate in \( H^1(\Omega) \). Let \( G \) be as in theorem 2.1 and \( H \in H^{\mu_0+1}(\Omega) \). Let \( \beta_n, \rho_n \) be such that

\[ \lim_{|n| \to +\infty} \beta_n = \lim_{|n| \to +\infty} \rho_n = +\infty, \quad \lim_{|n| \to +\infty} \frac{e^{2\rho_n T \beta_n^{d+2}}}{\prod_{k=1}^{d} (n_k)^{4\mu_2}} = \lim_{|n| \to +\infty} e^{2\rho_n T \beta_n^{-\mu_0}} = 0, \]

and

\[ \rho_n \leq \frac{1}{T} \log \left( \frac{1}{E(\epsilon)} \right). \]

Assume that \( Q_n \) satisfies

\[ \lim_{|n| \to +\infty} \exp \left( \frac{48TQ_n^2}{A_1-A_0} \right) \rho_n^{-2\gamma} = \lim_{|n| \to +\infty} \exp \left( \frac{48TQ_n^2}{A_1-A_0} \right) \frac{\rho_n^{d+2}}{\prod_{k=1}^{d} (n_k)^{4\mu_2}} = 0 \]

and

\[ Q_n \leq \left( \frac{A_1-A_0}{48T} \right) \log \left( \frac{1}{E(\epsilon)} \right). \]

Then for \( |n| \) large enough, and \( \epsilon \) small enough, the error \( E \left\| \hat{U}_{\rho_n, \beta_n} - u \right\|_{H^1(\Omega)} \) is of order

\[ \exp \left( \frac{48TQ_n^2}{A_1-A_0} \right) \max \left( \frac{\mu^2}{d} \prod_{k=1}^{d} (n_k)^{4\mu_2}, e^{-2\rho_n^{-2\gamma} \cdot 2^2(\gamma-1) \beta_n^{-\mu_0}} \right) + \frac{\epsilon}{E(\epsilon)E_0(\epsilon)}. \]

(3.24)

Remark 3.1.

1. When \( t = 0 \), by (3.19) and (3.24), it follows from theorem 3.1 that the error

\[ E \left\| \hat{U}_{\rho_n, \beta_n}(., 0) - u(., 0) \right\|_{L^2(\Omega)} \] is of order

\[ e^{6Q_T} \max \left( \frac{\mu^2}{d} \prod_{k=1}^{d} (n_k)^{4\mu_2}, e^{-2\rho_n^{-2\gamma} \cdot 2^2(\gamma-1) \beta_n^{-\mu_0}} \right) + \frac{\epsilon}{E(\epsilon)E_0(\epsilon)}. \]

(3.25)

From (3.13), (3.18) and (3.16), we conclude that (3.25) tends to zero as \( |n| \to +\infty \) and \( \epsilon \to 0 \). Similarly, from (3.20) and (3.22), we conclude that the error (3.24) also tends to zero as \( |n| \to +\infty \) and \( \epsilon \to 0 \).

2. One choice of \( \beta_n \) is

\[ \beta_n = \left( \prod_{k=1}^{d} (n_k)^{\frac{1}{4\mu_0+\gamma}} \right)^{\frac{1}{d}}. \]

(3.26)
then $\rho_n$ is chosen such that
\begin{equation}
\rho_n = \frac{\alpha_0}{T(2\alpha_0 + d/2)} \log \left( \prod_{k=1}^{d} n_k \right). \tag{3.27}
\end{equation}
Since $\lim_{\epsilon \to 0} \frac{\epsilon}{\log(1/\epsilon)} = 0$, we can choose $\mathcal{E}(\epsilon) = \epsilon^{m_0}$ for any $0 < m_0 < 1$. Since $\rho_n \leq \frac{1}{T} \log \left( \frac{1}{\epsilon} \right)$, we have
\begin{equation}
\frac{\alpha_0}{T(2\alpha_0 + d/2)} \log \left( \prod_{k=1}^{d} n_k \right) \leq m_0 \log \left( \frac{1}{\epsilon} \right).
\end{equation}
We can choose $Q_n$ such that $e^{\beta_n \epsilon_0} \leq (\Pi(n))^{-\delta_0 - 1}$ for any $0 < \delta_0 < 1$. So, we have
\begin{equation}
Q_n = \sqrt{\frac{\rho_n - 1}{6T}} \log \left( \prod_{k=1}^{d} n_k \right). \tag{3.28}
\end{equation}
Here $\Pi(n)$ is defined by
\begin{equation}
\Pi(n) = \max \left( e^{2\beta_n^2/\delta_0} \prod_{k=1}^{d} (n_k)^{-2|\mu_k|} e^{2\beta_n^2/\delta_0} \left| \mathcal{M} \left( \sqrt{T} \beta_n \right) \right|^{-2} \right). \tag{3.29}
\end{equation}
We also make a choice of $\mathcal{E}(\epsilon) = \epsilon^{m_1}$ for any $0 < m_1 < 1 - m_0$. Since $Q_n \leq \sqrt{\frac{1}{T}} \log \left( \frac{1}{\epsilon} \right)$, we have
\begin{equation}
\log \left( \frac{1}{\prod(n)} \right) \leq \frac{m_1}{1 - \delta_0} \log \left( \frac{1}{\epsilon} \right).
\end{equation}
Hence with these choices of $\beta_n, \rho_n, Q_n$, we get the error estimates stated in part (a) of theorem 3.1.

**Proof of theorem 3.1. Part 1. Error estimate in $L^2$:**

Let $\Lambda_\epsilon(t) = A_1 - \Lambda_\epsilon(t)$. The main equation in (1.2) can be rewritten as follows
\begin{equation}
\frac{\partial u}{\partial t} - \Lambda_\epsilon(t) \Delta u = F(u(x, t)) + G(x, t) - \left( \Lambda_\epsilon(t) - \Lambda(t) \right) \Delta u.
\end{equation}
For $\nu_n > 0$, we put
\begin{equation}
Y^\epsilon_{\nu_n, \nu_n, \beta_n}(x, t) = e^{\nu_n(t-T)} \left[ \bar{U}^\epsilon_{\nu_n, \beta_n}(x, t) - u(x, t) \right].
\end{equation}
Then from the last two equations, we have
\begin{align}
\frac{\partial Y^\epsilon_{\nu_n, \nu_n, \beta_n}(x, t)}{\partial t} + \Lambda_\epsilon(t) \Delta Y^\epsilon_{\nu_n, \nu_n, \beta_n}(x, t) - \nu_n Y^\epsilon_{\nu_n, \nu_n, \beta_n}(x, t)
&= -e^{\nu_n(t-T)} P_{\nu_n} Y^\epsilon_{\nu_n, \nu_n, \beta_n}(x, t) + e^{\nu_n(t-T)} (P_{\nu_n} - P) u(x, t) \\
&+ e^{\nu_n(t-T)} \left( \Lambda_\epsilon(t) - \Lambda(t) \right) \Delta u(x, t) \\
&+ e^{\nu_n(t-T)} \left[ \bar{G}_{\nu_n}(\bar{U}^\epsilon_{\nu_n, \beta_n}(x, t)) - F(u(x, t)) \right] + e^{\nu_n(t-T)} \left[ \bar{G}_{\beta_n}(x, t) - G(x, t) \right]. \tag{3.30}
\end{align}
and

\[ Y_{p_{w,v},\beta}^e(\mathbf{x}, t)|_{\partial \Omega} = 0, \quad Y_{p_{w,v},\beta}^e(\mathbf{x}, T) = \hat{H}_\beta(\mathbf{x}) - H(\mathbf{x}). \]

By taking the inner product of the two sides of (3.30) with \( Y_{p_{w,v},\beta}^e \) and integrating over \((t, T)\) one deduces that

\[ \|Y_{p_{w,v},\beta}^e(\cdot, T)\|_{L^2(\Omega)}^2 = \|Y_{p_{w,v},\beta}^e(\cdot, t)\|_{L^2(\Omega)}^2 - 2\int_t^T \tilde{X}_e(\tau) \|Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{H^1(\Omega)}^2 d\tau - 2\nu_n \int_{\Omega} \|Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau = \mathcal{L}_1(t) + \mathcal{L}_2(t) + \mathcal{L}_3(t) + \mathcal{L}_4(t) + \mathcal{L}_5(t), \]

where

\[ \mathcal{L}_1(t) := -2\int_{\Omega} \int_t^T e^{i\eta(t-T)} P_{\mathbf{a}} Y_{p_{w,v},\beta}^e(\mathbf{x}, \tau) Y_{p_{w,v},\beta}^e(\mathbf{x}, \tau) d\mathbf{x} d\tau \]

(3.32)

\[ \mathcal{L}_2(t) := 2\int_{\Omega} \int_t^T e^{i\eta(t-T)} (P_{\mathbf{a}} - \mathbf{P}) u(\mathbf{x}, \tau) Y_{p_{w,v},\beta}^e(\mathbf{x}, \tau) d\mathbf{x} d\tau \]

(3.33)

\[ \mathcal{L}_3(t) := -2\int_{\Omega} \int_t^T e^{i\eta(t-T)} \left( \lambda_e(\tau) - \lambda_e(\tau) \right) \Delta u(\mathbf{x}, \tau) Y_{p_{w,v},\beta}^e(\mathbf{x}, \tau) d\mathbf{x} d\tau \]

(3.34)

\[ \mathcal{L}_4(t) := 2\int_{\Omega} \int_t^T e^{i\eta(t-T)} \left[ F_{\mathbf{a}}(\hat{\mathbf{G}}_{\beta_{p_{w,v},\beta}}(\mathbf{x}, \tau)) - F(u(\mathbf{x}, \tau)) \right] Y_{p_{w,v},\beta}^e(\mathbf{x}, \tau) d\mathbf{x} d\tau \]

(3.35)

\[ \mathcal{L}_5(t) := 2\int_{\Omega} \int_t^T e^{i\eta(t-T)} \left( \hat{G}_{\beta_e}(\mathbf{x}, \tau) - G(\mathbf{x}, \tau) \right) Y_{p_{w,v},\beta}^e(\mathbf{x}, \tau) d\mathbf{x} d\tau. \]

(3.36)

Applying the Cauchy–Schwarz, the expectation of \(|\mathcal{L}_1(t)|\) is bounded as follows

\[ E|\mathcal{L}_1(t)| \leq 2E\left( \int_{\Omega} \|P_{\mathbf{a}} Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{L^2(\Omega)} \|Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{L^2(\Omega)} d\tau \right) \]

\[ \leq 2\nu_n \int_{\Omega} E\|Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \]

(3.37)

where we have used (3.6) in the last inequality. For the term \( \mathcal{L}_2(t) \), it follows by (3.7) and Cauchy–Schwarz inequality that

\[ E|\mathcal{L}_2(t)| \leq E\left( \int_{\Omega} \int_t^T e^{i\eta(t-T)} A_1 \|u\|^2_{L^2([0,T,\mathbb{R}^2,\Omega])} d\tau \right) \]

\[ + E\left( \int_{\Omega} \|Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \right) \]

\[ \leq T A_1 \nu_n \|u\|^2_{L^2([0,T,\mathbb{R}^2,\Omega])} + \int_{\Omega} E\|Y_{p_{w,v},\beta}^e(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. \]

(3.38)

Next, we find an upper for \( \mathcal{L}_3(t) \). Using the inequality \( 2\langle a_1, a_2 \rangle_{L^2(\Omega)} \leq \|a_1\|_{L^2(\Omega)}^2 \) for any \( a_i \in L^2(\Omega) \), \( i = 1, 2 \), we infer that
Combining (3.30), (3.37), (3.41), we conclude that

\[ E[L_{\text{fin}}(t)] \leq \mathbb{E} \left( \int_{t}^{T} e^{\alpha (\cdot - T)} \left( \Lambda_\tau (\cdot) - \Lambda (\cdot) \right)^2 \| \Delta u \|^2_{L^2(\Omega)} \, d\tau \right) + \mathbb{E} \left( \int_{t}^{T} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \right) \]

\[ \leq \| u \|^2_{L^\infty ([0,T];H^1(\Omega))} \int_{t}^{T} \mathbb{E} \left( (\Lambda_\tau (\cdot) - \Lambda (\cdot))^2 \right) \, d\tau + \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \]

\[ \leq \epsilon^2 \left( \int_{0}^{T} \mathbb{E} \left( \| \mathbf{\tilde{z}} (\tau) \|^2 \right) \, d\tau \right) \| u \|^2_{L^2 ([0,T];H^1(\Omega))} + \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \]

\[ \leq \epsilon^2 T^2 \| u \|^2_{L^\infty ([0,T];H^1(\Omega))} + \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau. \]  

where we have used the fact that \( \Lambda_\tau (\cdot) - \Lambda (\cdot) = \epsilon \mathbf{\tilde{z}} (\tau) \), and \( \mathbb{E} \| \mathbf{\tilde{z}} (\tau) \|^2 = t \).

For \( L_{\text{fin}} \), thanks to (3.2) and (3.3), we deduce that

\[ E[L_{\text{fin}}(t)] \leq \mathbb{E} \left( \int_{t}^{T} e^{\alpha (\cdot - T)} F(u, (\mathbf{\tilde{G}}_{\beta_{\text{fin}}} (\cdot, \tau))) - F(u, (\cdot, \tau)) \right) \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \]

\[ \leq 2 \left( 1 + 3Q^2 \right) \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau. \]  

The term \( E[L_{\text{fin}}(t)] \) can be bounded by

\[ E[L_{\text{fin}}(t)] \leq \mathbb{E} \left( \int_{t}^{T} 2e^{\alpha (\cdot - T)} \| \mathbf{\tilde{G}}_{\beta_{\text{fin}}} (\cdot, \tau) - G (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau + \int_{t}^{T} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \right) \]

\[ \leq 2T \mathbb{E} \left( \| \mathbf{G}_{\beta_{\text{fin}}} - G \|^2_{L^\infty ([0,T];L^2(\Omega))} + \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \right). \]

Combining (3.30), (3.37), (3.41), we conclude that

\[ E\| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, T) \|^2_{L^2(\Omega)} = E\| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, T) \|^2_{L^2(\Omega)} \]

\[ - 2 \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \]

\[ \geq -2\rho_n \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau - T \rho_n^{-2} e^{-2T\rho_n} \| u \|^2_{L^\infty ([0,T];H^1(\Omega))} \]

\[ - 2T \mathbb{E} \| \mathbf{G}_{\beta_{\text{fin}}} - G \|^2_{L^\infty ([0,T];L^2(\Omega))} - \epsilon^2 (T - t) \| u \|^2_{L^\infty ([0,T];H^1(\Omega))} \]

\[ - (5 + 6Q^2) \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau. \]

Whereupon

\[ E\| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, t) \|^2_{L^2(\Omega)} + (2\rho_n - 2\rho_n - 5 - 6Q^2) \int_{t}^{T} \mathbb{E} \| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, \tau) \|^2_{L^2(\Omega)} \, d\tau \]

\[ \leq E\| \mathbf{Y}_{\rho,\nu,\beta_{\text{fin}}} (\cdot, T) \|^2_{L^2(\Omega)} + T \rho_n^{-2} e^{-2T\rho_n} \| u \|^2_{L^\infty ([0,T];H^1(\Omega))} \]

\[ + \epsilon^2 (T - t) \| u \|^2_{L^\infty ([0,T];H^1(\Omega))} + 2T \mathbb{E} \| \mathbf{G}_{\beta_{\text{fin}}} - G \|^2_{L^\infty ([0,T];H^1(\Omega))}. \]
Choosing \( \nu_n = \rho_n \) then
\[
e^{2j\omega(T-t)}E\|\tilde{U}_{\rho_n}(., t) - u(., t)\|_{L^2(\Omega)}^2
\leq (5 + 6Q_n^2) \int_0^T e^{2j\omega(T-\tau)}E\|\tilde{U}_{\rho_n}(., \tau) - u(., \tau)\|_{L^2(\Omega)}^2 d\tau
\]
\[+ E\|\tilde{H}_{h_n} - H\|_{L^2(\Omega)}^2 + TE\|\tilde{G}_{h_n} - G\|_{L^\infty(0,T,L^2(\Omega))}^2
\]
\[+ TA^2 \rho_n^2 e^{-2\gamma T \rho_n} \|u\|_{L^\infty(0,T,H^F(\Omega))}^2 + e^2(T - t) \|u\|_{L^2(0,T,H^F(\Omega))}^2.\]

Multiplying both sides by \( e^{2j\omega T} \) we get
\[
e^{2j\omega T}E\|\tilde{U}_{\rho_n}(., t) - u(., t)\|_{L^2(\Omega)}^2
\leq (5 + 6Q_n^2) \int_0^T e^{2j\omega T}E\|\tilde{U}_{\rho_n}(., \tau) - u(., \tau)\|_{L^2(\Omega)}^2 d\tau
\]
\[+ e^{2j\omega T}E\|\tilde{H}_{h_n} - H\|_{L^2(\Omega)}^2 + e^{2j\omega T}TE\|\tilde{G}_{h_n} - G\|_{L^\infty(0,T,L^2(\Omega))}^2
\]
\[+ e^2 e^{2j\omega T} \|u\|_{L^2(0,T,H^F(\Omega))}^2 + TA^2 \rho_n^2 e^{-2\gamma T \rho_n} \|u\|_{L^\infty(0,T,H^F(\Omega))}^2.\]

Hence, Gronwall’s lemma yields the desired estimate
\[
e^{2j\omega T}E\|\tilde{U}_{\rho_n}(., t) - u(., t)\|_{L^2(\Omega)}^2
\leq \left[ e^{2j\omega T}E\|\tilde{H}_{h_n} - H\|_{L^2(\Omega)}^2 + Te^{2j\omega T}E\|\tilde{G}_{h_n} - G\|_{L^\infty(0,T,L^2(\Omega))}^2
\]
\[+ TA^2 \rho_n^2 \|u\|_{L^\infty(0,T,H^F(\Omega))}^2 + e^2 e^{2j\omega T} \|u\|_{L^\infty(0,T,H^F(\Omega))}^2 \right] e^{(5+6Q_n^2)(T-t)}.
\]

Consequently
\[
E\|\tilde{U}_{\rho_n}(., t) - u(., t)\|_{L^2(\Omega)}^2
\leq e^{2j\omega(T-t)} \left[ E\|\tilde{H}_{h_n} - H\|_{L^2(\Omega)}^2 + TE\|\tilde{G}_{h_n} - G\|_{L^\infty(0,T,L^2(\Omega))}^2
\]
\[+ TA^2 \rho_n^2 e^{-2\gamma T \rho_n} \|u\|_{L^\infty(0,T,H^F(\Omega))}^2 + e^2 e^{2j\omega(T-t)} \|u\|_{L^\infty(0,T,H^F(\Omega))}^2 \right] e^{(5+6Q_n^2)(T-t)}.
\]

From corollary 2.1, we see that the error term \( E\|\tilde{H}_{h_n} - H\|_{L^2(\Omega)}^2 + TE\|\tilde{G}_{h_n} - G\|_{L^\infty(0,T,L^2(\Omega))}^2 \) is of order
\[
\max_{k=1} \left( \frac{j^{d/2}}{(n_k)^{d+\mu_0}} \right).
\]

This together with (3.44), implies (3.19).

**Part 2.** *Estimate in \( H^1(\Omega) \):
Recall that \( H^1(\Omega) \) is the space of the function \( f \) such that \( f \) and \( \nabla f \) belong to \( L^2(\Omega) \) with the norm defined by \( \|f\|_{H^1} = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2. \)
By taking the inner product of the two sides of (3.30) with \(-\Delta Y^e_{\rho,\nu,\beta_n}(\cdot, \tau)\), and integrating over \((t, T)\), one deduces that

\[
\|Y^e_{\rho,\nu,\beta_n}(\cdot, T)\|_{L^2(\Omega)}^2 - \|Y^e_{\rho,\nu,\beta_n}(\cdot, t)\|_{L^2(\Omega)}^2 = 2\int_t^T \Lambda_e(\tau)\|\Delta Y^e_{\rho,\nu,\beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau - 2\nu_n\int_t^T \|Y^e_{\rho,\nu,\beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau
\]

\[
= \mathcal{L}_7.n(t) + \mathcal{L}_8.n(t) + \mathcal{L}_9.n(t) + \mathcal{L}_{10.n}(t) + \mathcal{L}_{11.n}(t)
\]

(3.46)

where

\[
\mathcal{L}_7.n(t) := 2\int_t^T \int_\Omega e^{\nu_n(\tau - T)} P_{\rho,\nu} Y^e_{\rho,\nu,\beta_n}(x, \tau) \Delta Y^e_{\rho,\nu,\beta_n}(x, \tau) dx d\tau,
\]

(3.47)

\[
\mathcal{L}_8.n(t) := -2\int_t^T \int_\Omega e^{\nu_n(\tau - T)} (P_{\rho,\nu} - P) u(x, \tau) \Delta Y^e_{\rho,\nu,\beta_n}(x, \tau) dx d\tau,
\]

(3.48)

\[
\mathcal{L}_9.n(t) := 2\int_t^T \int_\Omega e^{\nu_n(\tau - T)} \left(\Lambda_e(\tau) - \Lambda(\tau)\right) \Delta u(x, \tau) \Delta Y^e_{\rho,\nu,\beta_n}(x, \tau) dx d\tau,
\]

(3.49)

\[
\mathcal{L}_{10.n}(t) := -2\int_t^T \int_\Omega e^{\nu_n(\tau - T)} \left[F_{\rho,\nu}(\hat{U}^e_{\rho,\nu,\beta_n}(x, \tau)) - F(u(x, \tau))\right] \Delta Y^e_{\rho,\nu,\beta_n}(x, \tau) dx d\tau,
\]

(3.50)

\[
\mathcal{L}_{11.n}(t) := -2\int_t^T \int_\Omega e^{\nu_n(\tau - T)} \left[G_{\beta_n}(x, \tau) - G(x, \tau)\right] \Delta Y^e_{\rho,\nu,\beta_n}(x, \tau) dx d\tau.
\]

(3.51)

Using the bound from lemma 3.1, we can easily deduce that

\[
E[\mathcal{L}_n(t)] \leq 2\nu_n\int_t^T E\|Y^e_{\rho,\nu,\beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau.
\]

(3.52)

Using Cauchy–Schwarz inequality and lemma 3.1, we estimate \(|\mathcal{L}_{8.n}(t)|\) as follows

\[
E[\mathcal{L}_{8.n}(t)] \leq E \left(\frac{16}{A_1 - A_0} \int_t^T e^{\nu_n(\tau - T)} A_1^2 \rho_n^{-2} e^{-2T\rho_n}\|u\|_{L^\infty(0,T;Z(\delta_1(\Omega)))}^2 d\tau\right) + E \left(\frac{A_1 - A_0}{4} \int_t^T \|\Delta Y^e_{\rho,\nu,\beta_n}(x, \tau)\|_{L^2(\Omega)}^2 d\tau\right)
\]

\[
\leq \frac{16A_1^2}{A_1 - A_0} \rho_n^{-2} e^{-2T\rho_n}\|u\|_{L^\infty(0,T;Z(\delta_1(\Omega)))}^2 + \frac{A_1 - A_0}{4} \int_t^T E[\|\Delta Y_{\rho,\nu,\beta_n}(\cdot, \tau)\|_{L^2(\Omega)}^2] d\tau,
\]

(3.53)
and similarly using the fact that $\Lambda_c(\tau) - \Lambda(\tau) = \epsilon \xi(t)$ we get

$$
E[|L_{10,n}(t)|] \leq E \left( \frac{A_1 - A_0}{4} \int_t^T e^{2\nu \epsilon (\tau - t)} \left( \Lambda_c(\tau) - \Lambda(\tau) \right)^2 \|\Delta u\|^2_{L^2(\Omega)} d\tau \right)
$$

$$
+ E \left( \frac{A_1 - A_0}{4} \int_t^T \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau \right)
$$

$$
\leq \frac{16}{A_1 - A_0} \|u\|^2_{L^\infty(0,T)H^2(\Omega)} \int_t^T E \left( \Lambda_c(\tau) - \Lambda(\tau) \right)^2 d\tau
$$

$$
+ \frac{A_1 - A_0}{4} \int_t^T E \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau
$$

$$
\leq \frac{16}{A_1 - A_0} e^{2T^2} \|u\|^2_{L^\infty(0,T)H^2(\Omega)} + \frac{A_1 - A_0}{4} \int_t^T E \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau.
$$

The term $E[|L_{10,n}(t)|]$ is estimated using the fact that for $|u|$ large enough $F_{\rho}(u(x, \tau)) = F(u(x, \tau))$

$$
E[|L_{10,n}(t)|] \leq \frac{16}{A_1 - A_0} \int_t^T e^{2\nu \epsilon (\tau - t)} \left\| F_{\rho}(\tilde{U}_{\rho_{\nu,\lambda}}(x, \tau)) - F(u(\cdot, \tau)) \right\|^2_{L^2(\Omega)} d\tau
$$

$$
+ \frac{A_1 - A_0}{4} \|\Delta Y_{\rho_{\nu,\lambda}}\|^2_{L^2(\Omega)}
$$

$$
\leq \frac{16}{A_1 - A_0} (1 + 3Q_0^2) \int_t^T E \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau + \frac{A_1 - A_0}{4} \int_t^T E \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau.
$$

The term $E[|L_{11,n}(t)|]$ can be bounded as follows

$$
E[|L_{11,n}(t)|] \leq E \left( \frac{16T}{A_1 - A_0} \int_t^T e^{2\nu \epsilon (\tau - t)} \left\| \tilde{G}_{\beta,\lambda}(\cdot, \tau) - G(\cdot, \tau) \right\|^2_{L^2(\Omega)} d\tau \right)
$$

$$
+ E \left( \frac{A_1 - A_0}{4} \int_t^T \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau \right)
$$

$$
\leq \frac{16T}{A_1 - A_0} E \left\| G_{\beta,\lambda} - G_{\Omega} \right\|^2_{L^\infty(0,T,L^2(\Omega))} + \frac{A_1 - A_0}{4} \int_t^T E \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau.
$$

Combining (3.46), (3.52)–(3.55) gives

$$
E[\|Y_{\rho_{\nu,\lambda}}(\cdot, T)\|^2_{H^1(\Omega)}] - E[\|Y_{\rho_{\nu,\lambda}}(\cdot, t)\|^2_{H^1(\Omega)}]
$$

$$
= - 2 \int_t^T \Lambda_c(\tau) E[\|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)}] d\tau - 2\nu \int_t^T \int E[\|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)}] d\tau
$$

$$
\geq - 2\nu \int_t^T \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau
$$

$$
- \frac{16}{A_1 - A_0} \int_t^T \|\tilde{G}_{\beta,\lambda} - G_{\Omega}\|^2_{L^\infty(0,T,L^2(\Omega))} d\tau - \frac{16}{A_1 - A_0} e^{2(T - t)} \|u\|^2_{L^\infty(0,T)H^2(\Omega)} d\tau
$$

$$
\geq - \frac{16}{A_1 - A_0} \int_t^T \|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)} d\tau - (A_1 - A_0) \int_t^T E[\|\Delta Y_{\rho_{\nu,\lambda}}(\cdot, \tau)\|^2_{L^2(\Omega)}] d\tau.
$$
Choosing $\nu_n = \rho_n$ we obtain

$$e^{2\rho_n(t-T)}E\|\hat{\tau}_{\rho_n}\beta_n(\cdot, t) - \mathbf{u}(\cdot, t)\|^2_{H^1(\Omega)}$$

$$\leq \frac{16}{A_1 - A_0} (1 + 3Q_n^2) \int_T^T e^{2\rho_n(t-T)}E\|\hat{\tau}_{\rho_n}\beta_n(\cdot, \tau) - \mathbf{u}(\cdot, \tau)\|^2_{L^2(\Omega)}d\tau$$

$$+ E\|\hat{\beta}_{\beta_n} - H\|^2_{H^1(\Omega)} + \frac{16T}{A_1 - A_0} E \|\hat{\mathbf{G}}_{\beta_n} - G\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$+ \frac{16TA_1}{A_1 - A_0} \rho_n^{-2\gamma} e^{-2T\rho_n} E\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{E}_{\rho_n})} + \frac{16T}{A_1 - A_0} e^2(T - t) E\|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))}.$$ 

(3.56)

Multiplying both sides of the last inequality by $e^{2T\rho_n}$, we obtain

$$e^{2T\rho_n}E\|\hat{\tau}_{\rho_n}\beta_n(\cdot, t) - \mathbf{u}(\cdot, t)\|^2_{H^1(\Omega)}$$

$$\leq \frac{16}{A_1 - A_0} (1 + 3Q_n^2) \int_T^T e^{2\rho_n(t-T)}E\|\hat{\tau}_{\rho_n}\beta_n(\cdot, \tau) - \mathbf{u}(\cdot, \tau)\|^2_{L^2(\Omega)}d\tau$$

$$+ e^{2T\rho_n}E\|\hat{\beta}_{\beta_n} - H\|^2_{H^1(\Omega)} + \frac{16e^{2T\rho_n}}{A_1 - A_0} E \|\hat{\mathbf{G}}_{\beta_n} - G\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$+ \frac{16TA_1}{A_1 - A_0} \rho_n^{-2\gamma} E\|\mathbf{u}\|^2_{L^\infty(0,T;\mathcal{E}_{\rho_n})} + \frac{16}{A_1 - A_0} e^2(T - t) E\|\mathbf{u}\|^2_{L^\infty(0,T;H^1(\Omega))}.$$ 

(3.57)

Now, we need to find an upper bound of $E\|\hat{\beta}_{\beta_n} - H\|^2_{H^1(\Omega)}$. By theorem 2.1 [15], we get

$$E\|\hat{\beta}_{\beta_n} - H\|^2_{H^1(\Omega)} = 4 \sum_{p \in \mathcal{W}_{\rho_n}} |p|^2 \left[ \sum_{i=1}^{n_1} \cdots \sum_{i_d}^{n_d} A_{i_1,i_2,...,i_d} \tau_{i_1,i_2,...,i_d}(x_{i_1},...,x_{i_d}) - \mathbf{\Gamma}_{n,p} \right]^2$$

$$+ 4 \sum_{p \in \mathcal{W}_{\rho_n}} |p|^2 |H_p|^2$$

$$\leq \frac{8\pi^{2d}}{\left( \prod_{k=1}^{d} n_k \right)^2} \sum_{p \in \mathcal{W}_{\rho_n}} |p|^2 \left[ \sum_{i=1}^{n_1} \cdots \sum_{i_d}^{n_d} A_{i_1,i_2,...,i_d} \tau_{i_1,i_2,...,i_d} \right]^2$$

$$+ 8 \sum_{p \in \mathcal{W}_{\rho_n}} |p|^2 |\mathbf{\Gamma}_{n,p}|^2 + 4 \sum_{p \in \mathcal{W}_{\rho_n}} |p|^2 |H_p|^2.$$ 

(3.58)

The expectation of $C_{111}$ is bounded as follows

$$EC_{111} \leq \frac{8\pi^{2d}}{\left( \prod_{k=1}^{d} n_k \right)^2} \sum_{p \in \mathcal{W}_{\rho_n}} \beta_n \prod_{k=1}^{d} n_k \sum_{i=1}^{n_1} \cdots \sum_{i_d}^{n_d} V_{\max}^2 V_{\max}^2 \beta_n \text{card} (\mathcal{W}_{\rho_n}).$$
It follows from theorem 2.1 [15] that
\[
EC_{111} \leq 8\pi^d V_2^\text{max} \frac{2\pi^{d/2}}{d!} \frac{\beta_n^{2d/2}}{\prod_{k=1}^d n_k}.
\]
(3.59)
for which we used (2.30) of [15]
\[
\text{card} (\mathcal{W}_\beta) \leq \frac{2\pi^{d/2}}{d!} \beta_n^{d/2}.
\]
(3.60)
From (2.37) of [15], we obtain
\[
C_{222} = 8 \sum_{p \in \mathcal{W}_\beta} |p|^2 |T_{n,p}|^2
\leq 8\beta_n C^2(\mu_1, \ldots, \mu_d) \left\| H \right\|_{H^{\mu}(\Omega)}^2 \prod_{k=1}^d (n_k)^{-4\mu_k} \text{card} (\mathcal{W}_\beta)
\leq \frac{16C^2(\mu_1, \ldots, \mu_d)\pi^{d/2}}{d!} \left\| H \right\|_{H^{\mu}(\Omega)}^2 \beta_n^{\frac{2d}{2}} \prod_{k=1}^d (n_k)^{-4\mu_k}.
\]
(3.61)
For $A_{333}$ on the right hand side of (3.58), noting that $|p|^2 \geq \beta_n$ if $p \not\in \mathcal{W}_\beta$, we have the following estimation
\[
C_{333} = 4 \sum_{p \in \mathcal{W}_\beta} |p|^{-2\mu_0} |p|^{2\mu_0+2} \left\| H_p \right\|^2 \leq 4\beta_n^{-\mu_0} \left\| H \right\|_{H^{\mu_0+1}(\Omega)}^2.
\]
(3.62)
Combining (3.58)–(3.62), we deduce that
\[
E \left\| H_{\beta_n} - H \right\|_{H^\mu(\Omega)}^2 \leq EA_{111} + A_{222} + A_{333}
\leq 8\pi^d V_2^\text{max} \frac{2\pi^{d/2}}{d!} \frac{\beta_n^{2d/2}}{\prod_{k=1}^d n_k}
+ \frac{16C^2(\mu_1, \ldots, \mu_d)\pi^{d/2}}{d!} \left\| H \right\|_{H^{\mu}(\Omega)}^2 \beta_n^{\frac{2d}{2}} \prod_{k=1}^d (n_k)^{-4\mu_k} + 4\beta_n^{-\mu_0} \left\| H \right\|_{H^{\mu_0+1}(\Omega)}^2,
\]
(3.63)
which implies that $E \left\| H_{\beta_n} - H \right\|_{H^\mu(\Omega)}^2$ is of order $\max \left( \frac{\beta_n^{2d}}{\prod_{k=1}^d (n_k)^{4\mu_k}}, \beta_n^{-\mu_0} \right)$. This together with corollary 2.1 yields that the term
\[
e^{2T\rho_n} E \left\| H_{\beta_n} - H \right\|_{H^\mu(\Omega)}^2 + \frac{16e^{2T\rho_n}}{A_1 - A_0} E \left\| G_{\beta_n} - G \right\|_{L^\infty(0,T;L^2(\Omega))}^2
+ \frac{16T\rho_n^2}{A_1 - A_0} \rho_n^{2\gamma} \left\| u \right\|_{L^2(0,T;L^2(\Omega))}^2 + \frac{16}{A_1 - A_0} e^{2T\rho_n} \left\| u \right\|_{L^\infty(0,T;H^\mu(\Omega))}^2
\]
(3.64)
is of order
\[ \max \left( \frac{\varepsilon^2 \| \hat{u} \|_{L^d}}{\prod_{k=1}^{d} (n_k)^{2\mu_k}}, e^{2\rho T} \right) + e^{2\rho T}. \] (3.64)

Combining (3.57) and (3.64) gives that the error \( E \left\| \hat{U}_{\mu,0} - \hat{u} \right\|_{H^1(\Omega)} \) is of order
\[ \exp \left( \frac{48TQ_n^2}{A_1 - A_0} \right) \max \left( \frac{\varepsilon^2 \| \hat{u} \|_{L^d}}{\prod_{k=1}^{d} (n_k)^{2\mu_k}}, e^{2\rho T(2^{-\beta} - 1)} \right) + \exp \left( \frac{48TQ_n^2}{A_1 - A_0} \right) e^{2\rho T}. \] (3.65)

This implies the desired result (3.24).

4. Further discussion

In this section, we discuss some specific models for which our methods apply.

4.1. Fractional Ginzburg–Landau equation

Our analysis and techniques in this section can be applied to consider a space fractional version of the Ginzburg–Landau type equation
\[
\begin{cases}
u_t + \Lambda(t)(-\Delta)^\beta u = B(x,t)u - C(x,t)u^3 + G(x,t), & (x,t) \in \Omega \times (0, T), \\
u(x,t) = 0, & x \in \partial \Omega, \\
u(x,T) = H(x), & x \in \Omega,
\end{cases}
\] (4.1)

where the fractional Laplacian \((-\Delta)^\beta\) is defined by the spectral theorem for the functions \( h \) as follows
\[ (-\Delta)^\beta h(x) = \sum_{p \in \mathbb{N}^d} \lambda_p^\beta (h, \psi_p) \psi_p(x) = \sum_{p \in \mathbb{N}^d} |p|^{2\beta} \langle h, \psi_p \rangle \psi_p(x). \]

See [23] for more about this operator. Here \( B \) and \( C \) are randomly perturbed observations
\[ \tilde{B}_{i_1, i_2, \ldots, i_d}(t) := B(x_{i_1}, \ldots, x_{i_d}, t) + \vartheta \Psi_{i_1, i_2, \ldots, i_d}(t), \quad i_k = 1, n_k, \quad k = 1, d \]
and
\[ \tilde{C}_{i_1, i_2, \ldots, i_d}(t) := C(x_{i_1}, \ldots, x_{i_d}, t) + \vartheta \Psi_{i_1, i_2, \ldots, i_d}(t), \quad i_k = 1, n_k, \quad k = 1, d. \]

First, as in theorem 2.1, we define the following functions
\[ \tilde{B}_{\mu} (x, t) = \sum_{p \in \mathcal{W}_{\mu}} \left[ \frac{\pi^d}{d} \prod_{k=1}^{d} n_k i_1 = 1 \ldots n_d \tilde{B}_{i_1, i_2, \ldots, i_d} \psi_p(x_{i_1}, \ldots, x_{i_d}) \right] \psi_p(x), \]
and
\[ \tilde{C}_{\mu} (x, t) = \sum_{p \in \mathcal{W}_{\mu}} \left[ \frac{\pi^d}{d} \prod_{k=1}^{d} n_k i_1 = 1 \ldots n_d \tilde{C}_{i_1, i_2, \ldots, i_d} \psi_p(x_{i_1}, \ldots, x_{i_d}) \right] \psi_p(x). \]
We approximate the function $F(v) = B(x, t)v - C(x, t)v^3$ by $\mathcal{F}_{Q_n}$ defined by

$$
\mathcal{F}_{Q_n}(v(x, t)) = \begin{cases}
B_{\beta_n}(x, t)Q_n - C_{\beta_n}(x, t)Q_n^3, & v(x, t) > Q_n, \\
B_{\beta_n}(x, t)v - C_{\beta_n}(x, t)v^3, & -Q_n \leq v(x, t) \leq Q_n, \\
-\hat{B}_{\beta_n}(x, t)Q_n + \hat{C}_{\beta_n}(x, t)Q_n^3, & v(x, t) < -Q_n.
\end{cases}
$$

where $Q_n$ is given as in theorem 3.1. Let $P = -A_1(-\Delta)^\beta$ and $P_{\rho_n}$ be defined by

$$
P_{\rho_n}(v) = \sum_{|p| \leq M_A \rho_n} |p|^{2\beta} \langle v, \phi_p \rangle_{L^2(\Omega)} \psi_p(x),
$$

for any function $v \in L^2(\Omega)$. We can study a regularized solution $\tilde{U}_{\rho_n, \beta_n}$ which satisfies

$$
\begin{align*}
\frac{\partial \tilde{U}_{\rho_n, \beta_n}}{\partial t} + A_1(t)(-\Delta)^\beta \tilde{U}_{\rho_n, \beta_n} - P\tilde{U}_{\rho_n, \beta_n} + P_{\rho_n} \tilde{U}_{\rho_n, \beta_n} &= \mathcal{F}_{Q_n}(\tilde{U}_{\rho_n, \beta_n}(x, t)) + \tilde{G}_{\rho_n}(x, t), 0 < t < T, \\
\tilde{U}_{\rho_n, \beta_n}(x, t) &= 0, x \in \partial \Omega, \\
\tilde{U}_{\rho_n, \beta_n}(x, T) &= \tilde{H}_{\rho_n}(x).
\end{align*}
$$

The convergence is not mentioned here. But it can be similarly worked out as in theorem 3.1.

4.2. Ginzburg–Landau equation with variable coefficient

Our analysis and techniques in this paper can also be applied to the Ginzburg–Landau type equation with variable coefficient

$$
\begin{align*}
\frac{\partial u}{\partial t} - \nabla (b(x) \nabla u) &= B(x, t)u - C(x, t)u^3 + G(x, t), (x, t) \in \Omega \times (0, T), \\
u(x, t) &= 0, x \in \partial \Omega, \\
u(x, T) &= H(x), x \in \Omega,
\end{align*}
$$

(4.2)

We assume that $b(x)$ is randomly perturbed by the following observations

$$
\hat{b}_1 = b(x_1, \ldots, x_d) + A_{i_1, i_2, \ldots, i_d} \gamma_{i_1, i_2, \ldots, i_d} = b(x) + A_1 \gamma_1
$$

First, as in theorem 2.1, we define the following functions that are approximating $b$ as follows

$$
\hat{b}_{\beta_n}(x) = \sum_{p \in V_{\beta_n}} \left( \frac{\pi}{d} \prod_{k=1}^{d} n_k \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} \hat{b}_p(x_{i_1}, \ldots, x_{i_d}) \right) \psi_p(x)
$$

(4.3)

Now, using a similar method as in section 3, we give a regularized problem as follows

$$
\begin{align*}
\frac{\partial \tilde{U}^r_{\rho_n, \beta_n}}{\partial t} - \nabla \left( \hat{b}_{\beta_n}(x) \nabla \tilde{U}^r_{\rho_n, \beta_n} \right) - P\tilde{U}^r_{\rho_n, \beta_n} + P_{\rho_n} \tilde{U}^r_{\rho_n, \beta_n} &= \mathcal{F}_{Q_n}(\tilde{U}^r_{\rho_n, \beta_n}(x, t)) + \tilde{G}_{\rho_n}(x, t), 0 < t < T, \\
\tilde{U}^r_{\rho_n, \beta_n}(x, t) &= 0, x \in \partial \Omega, \\
\tilde{U}^r_{\rho_n, \beta_n}(x, T) &= \tilde{H}_{\beta_n}(x).
\end{align*}
$$

(4.4)

The convergence rate of the error term can be worked out similarly as in theorem (3.1).
4.3. Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP) type equation

Our results and method in this paper can be applied for the Fisher–Kolmogorov–Petrovsky–Piskunov (Fisher–KPP) equation in the following form

$$u_t - \nabla \left( a(t) \nabla u \right) = \gamma(x) u^2 - \mu(x) u, \quad (x, t) \in \Omega \times (0, T),$$ (4.5)

with the following condition

$$\begin{cases} u(x, T) = g(x), & (x, t) \in \Omega \times (0, T), \\ u|_{\partial \Omega} = 0, & t \in (0, T), \end{cases}$$ (4.6)

As mentioned in Skellam [24, 25], the equation (4.5) has many applications in population dynamics and periodic environments. In these references, the quantity $u(x, t)$ generally stands for a population density, and the coefficients $a(t)$, $\gamma(x)$, $\mu(x)$ respectively, correspond to the diffusion coefficient, the intrinsic growth rate coefficient and a coefficient measuring the effects of competition on birth and death rates.

Acknowledgments

The authors would like to thank the editor and two reviewers for their very helpful comments which have led to the improvement of the manuscript. M Kirane is supported by the Ministry of Education and Science of the Russian Federation (Agreement 02.a03.21.0008).

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