Cost of Einstein-Podolsky-Rosen steering in the context of extremal boxes

Debarshi Das,1, † Shounak Datta,2, ‡ C. Jebaratnam,2, † and A. S. Majumdar2, §

1Centre for Astroparticle Physics and Space Science (CAPSS), Bose Institute, Block EN, Sector V, Salt Lake, Kolkata 700 091, India
2S. N. Bose National Centre for Basic Sciences, Salt Lake, Kolkata 700 098, India

Einstein-Podolsky-Rosen steering is a form of quantum nonlocality which is weaker than Bell nonlocality, but stronger than entanglement. Here we present a method to check Einstein-Podolsky-Rosen steering in the scenario where the steering party performs two black-box measurements and the trusted party performs projective qubit measurements corresponding to two arbitrary mutually unbiased bases. This method is based on decomposing the measurement correlations in terms of extremal boxes of the steering scenario. In this context, we propose a measure of steerability called steering cost. We show that our steering cost is a convex steering monotone. We illustrate our method to check steerability with two families of measurement correlations and find out their steering cost.

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I. INTRODUCTION

Quantum entanglement admits stronger than classical correlations which can lead to quantum nonlocality. Local quantum measurements on a composite system lead to nonlocality if the statistics of the measurement outcomes cannot be explained by a local hidden variable (LHV) model [1, 2]. Such a nonclassical feature of quantum correlations termed as Bell nonlocality can be used to certify the presence of entanglement in a device-independent way and it finds applications in device-independent quantum information processing [2].

Quantum steering is a form of quantum nonlocality which was first noticed by Schrodinger [3] in the context of the famous Einstein-Podolsky-Rosen (EPR) paradox [4]. EPR steering arises in the scenario where local quantum measurements on one part of a bipartite system are used to prepare different ensembles for the other part. This scenario demonstrates EPR steering if these ensembles cannot be explained by a local hidden state (LHS) model [5]. The demonstration of the EPR paradox was first proposed by Reid [6] based on the Heisenberg uncertainty relation. Using tighter uncertainty relations such as entropic ones, corresponding entropic steering criteria have been subsequently proposed [7], leading to the demonstration of steering for more categories of states [8]. Oppenheim and Wehner [9] introduced fine-grained uncertainty relations that provide a direct way of linking uncertainty with nonlocality. In Ref. [10], Pramanik et. al. have derived steering inequalities based on fine-grained uncertainty relations, an approach that has been later extended for continuous variables too [11].

It is well-known that EPR steering lies in between entanglement and Bell nonlocality: quantum states that demonstrate Bell nonlocality form a subset of EPR steerable states which, in turn, form a subset of entangled states [5, 12]. The operational definition of EPR steering is that it certifies the presence of entanglement in a one-sided device-independent way in which the measurement device at only one of the two sides is fully trusted [13]. Steering inequalities which are analogous to Bell inequalities have been derived to rule out LHS description for the steering scenarios [14, 15]. Recently, it has been demonstrated that violation of a steering inequality is necessary for one-sided device-independent quantum key distribution [16]. EPR steering admits an asymmetric formulation: there exist entangled states which are one-way steerable, i.e., demonstrate steerability from one observer to the other observer but not vice-versa [17, 18]. Various other steering criteria have also been proposed such as all versus nothing proof of EPR steering [17] and hierarchy of steering criteria based on moments [19].

Motivated by the question of how much a steering scenario demonstrates steerability, a measure of steering called steering weight was defined in Ref. [20]. Quantitative characterization of steering has started receiving attention recently [21, 22]. In Ref. [21], Gallego and Aolita (GA) have developed the resource theory of steering. GA have observed that in the steering theory, local operations assisted by one-way classical communications (1W-LOCCs) from the trusted side to the black-box side are allowed operations. With 1W-LOCCs as free operations of steering, GA have introduced a set of postulates that a bona fide quantifier of steering should fulfill. Those functions that satisfy these postulates are called convex steering monotones. GA have proved that the first proposed measure of steering, i.e., steering weight is a convex steering monotone.

In the case of the Bell scenario with a finite number of settings per party and a finite number of outcomes per setting, it is well-known that the set of correlations that have a LHV model forms a convex polytope [23–25]. The nontrivial facet inequalities of this polytope are called Bell inequalities. For a given Bell scenario, a correlation has a LHV model if (if and only if) it satisfies all the Bell inequalities. In Ref. [26], Cavalcanti, Foster, Fuwa and Wiseman (CFFW) have considered an analogous characterization of EPR steering. Steering can also be understood as a failure of a hybrid local hidden variable-local hidden state (LHV-LHS) model to produce the

† debarshidas@jcbose.ac.in
‡ shounak.datta@bose.res.in
§ jebaratnim@bose.res.in
archan@bose.res.in
correlations between the black-box side and the trusted side. In Ref. [26], CFFW have shown that any LHV-LHS model can be written as a convex mixture of the extremal points of the unsteerable set.

In this work, we present a method to check EPR steering in the context of extremal points of the following steering scenario: Alice performs two black-box measurements and Bob performs projective qubit measurements corresponding to any two mutually unbiased bases (MUBs). This method provides a simple way to check the existence of a LHV-LHS model for the measurement correlations arising from the above steering scenario. Based on this formulation, we propose a measure of steerability which we call steering cost. We show that our steering cost is a convex steering monotone. We illustrate our method to check steerability with two families of measurement correlations and we find out the steering cost of these two families. Steering cost is also compared with another measure of steering, called “steering weight” [20]. The advantage in experimental determination of steering cost over that of steering weight is also discussed.

The organization of the paper is as follows. In Sec. II, we review the polytope of nonsignaling boxes which we use to provide a criterion for EPR steering and discuss some basic notions in EPR steering. In Sec. III, we present our quantifier of steering and we apply our method to check steerability of two families of measurement correlations. Comparison of steering cost with steering weight is presented in Sec. IV. In Sec. V, we present our concluding remarks.

II. PRELIMINARIES

A. Bell nonlocality

Consider the Bell scenario where two spatially separated parties, Alice and Bob, share a bipartite black box. Let us denote the inputs on Alice’s and Bob’s sides by $x$ and $y$, respectively, and the outputs by $a$ and $b$. The given Bell scenario is characterized by the set of joint probabilities, $P(ab|xy):=\{p(ab|xy)\}_{a,b,x,y}$, which is called correlation or box (also denoted by $P$). A correlation $P$ is Bell nonlocal if it cannot be reproduced by a LHV model, i.e.,

$$p(ab|xy) = \sum_\lambda p(\lambda)p(a|x,\lambda)p(b|y,\lambda) \forall a, b, x, y,$$  \hspace{1cm} (1)

where $\lambda$ denotes shared randomness which occurs with probability $p(\lambda)$; each $p(a|x,\lambda)$ and $p(b|y,\lambda)$ are conditional probabilities.

In the case of two-binary-inputs and two-binary-outputs per side, the set of nonsignaling boxes forms an 8 dimensional convex polytope with 24 extremal boxes [25], the 8 Popescu-Rohrlich (PR) boxes [24]:

$$P_{PR}^{ab}(ab|xy) = \begin{cases} \frac{1}{2}, & a \oplus b = x \cdot y \oplus ax \oplus by \oplus y, \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

and 16 local-deterministic boxes:

$$P_{DL}^{ab}(ab|xy) = \begin{cases} 1, & a = ax \oplus \beta \\ b = \gamma y \oplus \epsilon, & \text{otherwise.} \end{cases} \quad (3)$$

Here, $\alpha, \beta, \gamma, \epsilon \in \{0,1\}$ and $\oplus$ denotes addition modulo 2. All the deterministic boxes as defined above can be written as the product of marginals corresponding to Alice and Bob, i.e., $P_{DL}^{ab}(ab|xy) = P_{DL}^{a}(a|x)p_{DL}^{b}(b|y)$, with the deterministic box on Alice’s side given by,

$$P_{DL}^{a}(a|x) = \begin{cases} 1, & a = ax \oplus \beta \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

and the deterministic box on Bob’s side given by,

$$P_{DL}^{b}(b|y) = \begin{cases} 1, & b = \gamma y \oplus \epsilon \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The 8 PR boxes are equivalent under “local reversible operations” (LRO). Similarly, the 16 local-deterministic boxes are equivalent under LRO. By using LRO Alice and Bob can convert any PR box into any other PR box, or any local-deterministic box into any other local-deterministic box. LRO is designed [25] as follows: Alice may relabel her inputs: $x \rightarrow x \oplus 1$, and she may relabel her outputs (conditionally on the input) : $a \rightarrow a \oplus ax \oplus \beta$; Bob can perform similar operations.

The set of boxes which have a LHV model forms a sub-polytope of the full nonsignaling polytope whose extremal boxes are the local-deterministic boxes. A box with two-binary-inputs-two-binary-outputs is local iff it satisfies a Bell–Clauser-Horne-Shimony-Holt (CHSH) inequality [27] and its permutations [23] which are given by,

$$B_{\alpha\beta\gamma\epsilon} := (−1)^\alpha \langle A_0 \langle 0 \rangle \rangle + (−1)^\beta \langle A_0 \langle 1 \rangle \rangle + (−1)^\epsilon \langle B_0 \langle 1 \rangle \rangle + (−1)^\gamma \langle B_0 \langle 1 \rangle \rangle \leq 2,$$  \hspace{1cm} (6)

where $\langle A_0 \rangle := \sum_{a=0}^1 (-1)^{a\alpha}p_{NL}(ab|xy)$. The above inequalities form the facet inequalitiies for the local polytope formed by the extremal points given in Eq. (3).

Nonlocal cost is a measure of nonlocality [28] which is based on the Elitzur-Popescu-Rohrlich decomposition [29]. In this approach, a given box $P(ab|xy)$ is decomposed into a nonlocal part and a local part, i.e.,

$$P(ab|xy) = P_{NL}P_{NL}(ab|xy) + (1 - P_{NL})P_L(ab|xy), \quad (7)$$

where $P_{NL}(ab|xy)$ (or, simply, $P_{NL}$) is a nonsignaling box and $P_L(ab|xy)$ (or, simply, $P_L$) is a local box: $0 \leq P_{NL} \leq 1$. The nonlocal cost of the box $P(ab|xy)$, denoted $C_{NL}(P)$, is obtained by minimizing the weight of the nonlocal part over all possible decompositions of the form (7), i.e.,

$$C_{NL}(P) := \min_{\text{decompositions}} P_{NL}.$$  \hspace{1cm} (8)
nonlocal cost, i.e., \( C_{NL}(P_{NL}) = 1 \) since it is an extremal nonlocal box. An extremal nonlocal box in a given Bell scenario cannot be decomposed as a convex mixture of the other boxes in that given Bell scenario and violates a Bell inequality maximally [25]. In the case of two-binary-inputs and two-binary-outputs per side, for the optimal decomposition, the nonlocal part \( P_{NL}(ab|xy) \) is one of the PR-boxes given in Eq. (2).

\[
\sum_{\lambda} p(\lambda)p(a|x, \lambda)p(b|y, \rho_{\lambda}) \quad \forall a, x, b, y,
\]

where \( p(b|y, \rho_{\lambda}) = \text{Tr}(\Pi_{b\lambda}\rho_{\lambda}) \), which arises from some local hidden state \( \rho_{\lambda} \). The above decomposition is called a LHV-LHS model. Let us denote the set of all correlations that belongs to the given steering scenario \( \mathcal{N}_{E} \). The set of correlations that have a LHV-LHS model denoted by \( \mathcal{L}_{E} \) forms a convex subset of \( \mathcal{N}_{E} \) [14], which we call unsteerable set. In particular, any LHV-LHS model can be decomposed in terms of the extremal points of \( \mathcal{L}_{E} \) [26]. That is we can simplify the decomposition (12) as follows:

\[
p(ab|xy) = \sum_{\chi, \zeta} p(\chi, \zeta) D(a|x, \chi)(\psi_\chi|\Pi_{b\chi}|\psi_\zeta) \quad \forall a, x, b, y,
\]

with \( D(a|x, \chi) = \delta_{a, f(x, \chi)} \). Here, \( \chi \) are the variables which determine all values of Alice’s observables \( A_k \) through the function \( f(x, \chi) \) and \( \zeta \) determines a pure state \( |\psi_\zeta \rangle \) for Bob.

### III. QUANTIFYING EPR STEERING

Analogous to nonlocal cost, we now define steering cost of a box \( P(ab|xy) \in \mathcal{N}_{E} \). First, the given box \( P(ab|xy) \) is decomposed in a convex mixture of a steerable part and an unsteerable part, i.e.,

\[
P(ab|xy) = p_S P_S(ab|xy) + (1 - p_S) P_{US}(ab|xy),
\]

where \( P_S(ab|xy) \) (or, simply, \( P_S \)) is a steerable box and \( P_{US}(ab|xy) \) (or, simply, \( P_{US} \)) is an unsteerable box; \( 0 \leq p_S \leq 1 \). Second, the weight of the steering part minimized over all possible decompositions of the form (14) gives the steering cost of the box \( P(ab|xy) \) denoted by \( C_{steer}(P) \), i.e.,

\[
C_{steer}(P) := \min_{P(ab|xy)} \text{decompositions} p_S.
\]

Here, \( 0 \leq C_{steer}(P) \leq 1 \) (since, \( 0 \leq p_S \leq 1 \)). It follows that, for the optimal decomposition, the steerable part \( P_S(ab|xy) \) has the maximal steering cost, i.e., \( C_{steer}(P_S) = 1 \) since it is an extremal steerable box. An extremal steerable box \( P_{EUS}^{US}(ab|xy) \in \mathcal{N}_{E} \) cannot be decomposed as a convex mixture of the other boxes in the set \( \mathcal{N}_{E} \) and violates a steering inequality in the given steering scenario maximally.

We will now demonstrate that the steering cost \( C_{steer}(P) \) is a proper quantifier of steering, i.e., it is a convex steering monotone [21]. For this purpose, we introduce the following notations. A box \( P(ab|xy) \) which is obtained by Bob performing projective measurements \( \Pi_B \) on an assemblage \( \sigma \) is denoted by \( P[\sigma] \). Here, \( P[\sigma] := P(ab|xy) = \text{Tr}_{b} \text{[Tr}_{a} \text{[Tr}_{a} \sigma_{a|x}\text{]}_{a,b|x}\text{]}_{a,b} \). Consider the situation in which deterministic one-way local operations and classical communications (1W-LOCCs) [21] occur from Bob to Alice before Bob performs measurements on the assemblage. Following Ref. [32], we define the deterministic 1W-LOCC as a completely positive trace preserving (CPTP) map \( M \) that take an assemblage \( \sigma \) into a final assemblage \( M(\sigma) \), where

\[
M(\sigma) = \sum_{\omega} M_{\omega}(\sigma) := \sum_{\omega} \mathcal{K}_{\omega} W_{\omega}(\sigma) \mathcal{K}_{\omega}^{\dagger},
\]

Consider the situation in which deterministic one-way local operations and classical communications (1W-LOCCs) [21] occur from Bob to Alice before Bob performs measurements on the assemblage.
with $W_\omega$ being a deterministic wiring map which transforms one assemblage $\sigma = \{\sigma_{ax}\}_{a,x}$ to another assemblage $\tilde{\sigma} = \{\tilde{\sigma}_{ax'}\}_{a,x'}$ having different setting $x'$ and outcome $a'$ at Alice’s side in the following way:

$$[W_\omega(\sigma)]_{x'} := \tilde{\sigma}_{ax'}$$

$$= \sum_{a,x} p(x'|x,\omega)p(a'|x',a,x,\omega)\sigma_{ax} \quad \forall a', x'. \quad (17)$$

Define

$$D_\omega(\sigma) := \frac{M_\omega(\sigma)}{Tr[M_\omega(\sigma)]},$$

which is the set of normalized conditional states arising from the action of a subchannel $M_\omega$, labeled by $\omega$, of the CPTP map $M$ on the assemblage $\sigma$ at Bob’s end. Here, $T(\omega) := Tr[M_\omega(\sigma)]$ is the probability of transmitting the assemblage $\sigma$ through the $\omega$ with subchannel of $M$; and $\sum_\omega T(\omega) \leq 1$. Let us denote $D_\omega(\sigma) := \{[D_\omega(\sigma)]_{ax'|x,x'}, \omega \}$, where the normalized state $[D_\omega(\sigma)]_{ax'|x,x'}$ denotes an element of $D_\omega(\sigma)$. Hence, we can define $P[D_\omega(\sigma)]$ which is a box arising from any valid assemblage (steerable or unsteerable) $\sigma \in \Sigma^S$ after the action of a map $M_\omega$ as follows:

$$P[D_\omega(\sigma)] = P(a'b|x'y)$$

$$:= \{p(a'|x')Tr[\Pi_{b'y}[D_\omega(\sigma)]_{ax'|x,x'}] \}_{a',x',b,y}. \quad (18)$$

where $p(a'|x')$ is the conditional probability of obtaining the outcome $a'$, when Alice performed the measurement $x'$, and is given by

$$p(a'|x') = \sum_{a,x} p(x'|x,\omega)p(a'|x',a,x,\omega)p(a|x).$$

This can be obtained from Eq. (17), expressing the elements of the assemblages $\sigma_{ax}$ and $\tilde{\sigma}_{ax'}$ at Bob’s side as $p(a|x)p_{ax}$ and $p(a'|x')p_{a'x'}$ respectively (where $p(a|x)$ and $p(a'|x')$ are conditional probabilities and $p_{ax}$ and $p_{a'x'}$ are normalized states at Bob’s side) and taking trace on both side of the equation.

With the above notations, we now proceed to show that $C_{steer}(P[\sigma])$ satisfies the following two properties:

1. $C_{steer}(P[\sigma])$ does not increase on average under deterministic 1-W-LOCCs, i.e.,

$$\sum_\omega T(\omega)C_{steer}(P[D_\omega(\sigma)]) \leq C_{steer}(P[\sigma]), \quad \forall \sigma \in \Sigma^S. \quad (19)$$

Proof. Let us consider the following decomposition of an arbitrary assemblage $\sigma := \{\sigma_{ax}\}_{a,x} \in \Sigma^S$:  

$$\sigma_{ax} = p_S\sigma_{ax}^S + (1-p_S)\sigma_{ax}^{US} \quad \forall a, x, \quad (20)$$

where $\sigma_{ax}^S$ is an element of an assemblage $\sigma^S$ having steerability and $\sigma_{ax}^{US}$ is an element of an unsteerable assemblage $\sigma^{US}$. Now, one can write,

$$Tr[\Pi_{b'y}\sigma_{ax}] = p_S \quad Tr[\Pi_{b'y}\sigma_{ax}^S] + (1-p_S) \quad Tr[\Pi_{b'y}\sigma_{ax}^{US}] \quad \forall a, x, b, y. \quad (21)$$

Hence, for the box $P[\sigma]$ arising from the assemblage $\sigma$, one can write the following decomposition:

$$P[\sigma] = p_S P_S[\sigma^S] + (1-p_S) P_{US}[\sigma^{US}]. \quad (22)$$

Here, $P_S[\sigma^S]$ is a steerable box, produced from the steerable assemblage $\sigma^S$ and $P_{US}[\sigma^{US}]$ is an unsteerable box, produced from the unsteerable assemblage $\sigma^{US}$. The steering cost of the box $P[\sigma]$, i.e., $C_{steer}(P[\sigma])$ is obtained by minimizing $p_S$ in Eq. (22) over all such possible decompositions. Let the decomposition (22) denote the optimal decomposition, i.e., $p_S = C_{steer}(P[\sigma])$.

Now consider the set of normalized states $D_\omega(\sigma)$, where $D_\omega$ has been applied on the assemblage $\sigma$ producing the box $P[\sigma]$ with the optimal decomposition given by Eq. (22) with $p_S = C_{steer}(P[\sigma])$. From Eq. (20), we have,

$$D_\omega(\sigma) = D_\omega(p_S\sigma^S + (1-p_S)\sigma^{US})$$

$$= \frac{M_\omega(p_S\sigma^S + (1-p_S)\sigma^{US})}{Tr[M_\omega(\sigma)]}, \quad (23)$$

where

$$M_\omega(p_S\sigma^S + (1-p_S)\sigma^{US})$$

$$= \mathcal{K}_\omega W_\omega(p_S\sigma^S + (1-p_S)\sigma^{US}) \mathcal{K}_\omega^\dagger. \quad (24)$$

Now, consider the assemblage,

$$\tilde{\sigma} = [\tilde{\sigma}_{ax'}]_{a',x'} = W_\omega(p_S\sigma^S + (1-p_S)\sigma^{US}). \quad (25)$$

From Eq. (17), it follows that each element in the above assemblage $\tilde{\sigma}$ has the following decomposition:

$$\tilde{\sigma}_{ax'} = \sum_{a,x} p(x'|x,\omega)p(a'|x',a,x,\omega)(p_S\sigma_{ax'}^S + (1-p_S)\sigma_{ax'}^{US})$$

$$= p_S \sum_{a,x} p(x'|x,\omega)p(a'|x',a,x,\omega)\sigma_{ax'}^S$$

$$+ (1-p_S) \sum_{a,x} p(x'|x,\omega)p(a'|x',a,x,\omega)\sigma_{ax'}^{US} \quad \forall a', x'. \quad (26)$$

which implies that

$$W_\omega(p_S\sigma^S + (1-p_S)\sigma^{US})$$

$$= p_S W_\omega(\sigma^S) + (1-p_S) W_\omega(\sigma^{US}). \quad (27)$$

Hence, from Eqs. (24) and (27), we obtain

$$M_\omega(p_S\sigma^S + (1-p_S)\sigma^{US}) = \mathcal{K}_\omega[p_S W_\omega(\sigma^S) + (1-p_S) W_\omega(\sigma^{US})] \mathcal{K}_\omega^\dagger$$

$$= p_S \mathcal{K}_\omega W_\omega(\sigma^S) \mathcal{K}_\omega^\dagger + (1-p_S) \mathcal{K}_\omega W_\omega(\sigma^{US}) \mathcal{K}_\omega^\dagger$$

$$= p_S M_\omega(\sigma^S) + (1-p_S)M_\omega(\sigma^{US}). \quad (28)$$


Now, from Eqs. (23) and (28), we obtain
\[
D_{a}(\sigma) = pS \frac{M_{\omega}(\sigma^S)}{Tr[\mathcal{M}_{\omega}(\sigma)]} + (1 - pS) \frac{M_{\omega}(\sigma^{US})}{Tr[\mathcal{M}_{\omega}(\sigma)]} \\
= pS \frac{D_{a}(\sigma^S)}{Tr[\mathcal{M}_{\omega}(\sigma^S)]} + \frac{1}{Tr[\mathcal{M}_{\omega}(\sigma)]} \sum_{x} \{ \frac{\lambda_x}{\lambda^0} \} \left[ \sum_{x} \frac{\lambda_x^0}{\lambda} \right] ^{1/2} \\
(1 - pS) \frac{D_{a}(\sigma^{US})}{Tr[\mathcal{M}_{\omega}(\sigma^{US})]} + \sum_{x} \frac{\lambda_x}{\lambda^0} \right] ^{1/2} \\
(1 - pS) \frac{D_{a}(\sigma^{US})}{Tr[\mathcal{M}_{\omega}(\sigma^{US})]} + \sum_{x} \frac{\lambda_x}{\lambda^0} \right] ^{1/2} \\
\forall a', x'.
\] (30)

From Eq. (30), one can write,
\[
p(a|x') \sum_{x} \frac{\lambda_x}{\lambda^0} \right] ^{1/2} \\
(1 - pS) \frac{D_{a}(\sigma^{US})}{Tr[\mathcal{M}_{\omega}(\sigma^{US})]} + \sum_{x} \frac{\lambda_x}{\lambda^0} \right] ^{1/2} \\
\forall a', x', b, y.
\] (31)

Hence, from Eq. (31), we get the following decomposition for the box \(P[D_{a}(\sigma)]\):
\[
P[D_{a}(\sigma)] = pS \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \frac{P[D_{a}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma^S)]} + (1 - pS) \frac{Tr[\mathcal{M}_{\omega}(\sigma^{US})]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \frac{P[D_{a}(\sigma^{US})]}{Tr[\mathcal{M}_{\omega}(\sigma^{US})]}.
\] (32)

Note that the assemblage \(\{p(a|x') D_{a}(\sigma^S)\}_{a,x'} \in \Sigma^{US}\) is not unsteerable [21]. This implies that the box \(P[D_{a}(\sigma^{US})]\) in the decomposition (32) is an unsteerable box. There are now two cases which have to be checked to verify Eq. (19). (i) Suppose the assemblage \(\{p(a|x') D_{a}(\sigma^S)\}_{a,x'} \in \Sigma^{US}\) is unsteerable. Then from Eq. (32) it is clear that the box \(P[D_{a}(\sigma)]\) is a convex mixture of two unsteerable boxes and, hence, unsteerable. Therefore, in this case, the following inequality trivially holds:
\[
\sum_{a} T(\omega) C_{steer}(P[D_{a}(\sigma)]) = 0 \leq C_{steer}(P[\sigma]) \quad \forall \sigma \in \Sigma^{S}. \] (33)

(ii) Suppose the assemblage \(\{p(a|x') D_{a}(\sigma^S)\}_{a,x'} \in \Sigma^{US}\) is steerable and the box \(P[D_{a}(\sigma^S)]\) in the decomposition (32) is a steerable box. Then, the decomposition (32) may not be the optimal decomposition (for which the weight of the steerable part being the minimum over all possible decompositions of the box \(P[D_{a}(\sigma)]\)). Hence, one has to minimize the weight of the steerable part \(P[D_{a}(\sigma^S)]\) in Eq. (32) over all possible decompositions of the box \(P[D_{a}(\sigma)]\) to obtain the steering cost \(C_{steer}(P[D_{a}(\sigma)])\) of the box. Therefore, we have
\[
C_{steer}(P[D_{a}(\sigma)]) = \min_{\mathcal{M}} \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \\
\leq pS \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \\
= C_{steer}(P[\sigma]) \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]}. \] (34)

The last equality holds as we have assumed that the decomposition (22) denotes the optimal decomposition of the box \(P[\sigma]\), i.e., \(pS = C_{steer}(P[\sigma])\). As for all \(\omega, T(\omega) = Tr[\mathcal{M}_{\omega}(\sigma^S)] \geq 0 \) and \(\sum_{\omega} T(\omega) \leq 1\), from Eq. (34) we get for deterministic 1W-LOCCs,
\[
\sum_{\omega} T(\omega) C_{steer}(P[D_{a}(\sigma)]) \leq \sum_{\omega} T(\omega) C_{steer}(P[\sigma]) \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \\
= \sum_{\omega} C_{steer}(P[\sigma]) \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \\
= C_{steer}(P[\sigma]) \sum_{\omega} \frac{Tr[\mathcal{M}_{\omega}(\sigma^S)]}{Tr[\mathcal{M}_{\omega}(\sigma)]} \leq C_{steer}(P[\sigma]), \quad \forall \sigma \in \Sigma^{S}. \] (35)

The last inequality holds, because \(\sigma^S\) is the set of unnormalized conditional states and \(\mathcal{M}\) is a deterministic map, i.e., \(\sum_{\omega} Tr[\mathcal{M}_{\omega}(\sigma^S)] \leq 1\). This completes the proof for the monotonicity of \(C_{steer}(P)\) on average, under all W-LOCCs for all assemblages.

2. For all convex decompositions of \(\sigma = \mu \sigma' + (1 - \mu) \sigma''\), in terms of the other two assemblages \(\sigma'\) and \(\sigma''\) with \(0 \leq \mu \leq 1\),
\[
C_{steer}(P[\sigma]) \leq \mu C_{steer}(P[\sigma']) + (1 - \mu) C_{steer}(P[\sigma'']) \quad \forall \sigma \in \Sigma^{S}. \] (37)

Proof. Note that an arbitrary assemblage \(\sigma := \{\sigma_{a|z}\}_{a,z} \in \Sigma^{S}\) satisfies the following relation for all possible convex decompositions as in Eq. (36):
\[
\sum_{a} T(\omega) C_{steer}(P[\sigma_{a|z}]) = \mu \sum_{a} T(\omega) C_{steer}(P[\sigma'_{a|z}]) + (1 - \mu) \sum_{a} T(\omega) C_{steer}(P[\sigma''_{a|z}]) \quad \forall a, x, y. \] (38)

which implies that the box \(P[\sigma]\) arising from the assemblage \(\sigma\) has the following decomposition:
\[
P[\sigma] = \mu P[\sigma'] + (1 - \mu) P[\sigma''], \] (39)

where the box \(P[\sigma']\) arises from the assemblage \(\sigma' := \{\sigma'_{a|z}\}_{a,z}\) and the box \(P[\sigma'']\) arises from the assemblage \(\sigma'' := \{\sigma''_{a|z}\}_{a,z}\).
We write the optimal decompositions (with weight of the steerable part being the minimum over all possible decompositions) for the two boxes in the above decomposition (39) as follows:

\[ P[\sigma'] := C_{\text{steer}}(P[\sigma'])P_{1}^1 + (1 - C_{\text{steer}}(P[\sigma']))P_{\text{US}}^1, \]

where \( P_{1}^1 \) and \( P_{\text{US}}^1 \) are steerable and unsteerable boxes, respectively, and \( 0 \leq C_{\text{steer}}(P[\sigma']) \leq 1 \) is the steering cost of the box \( P[\sigma'] \), and

\[ P[\sigma''] := C_{\text{steer}}(P[\sigma''])P_{2}^1 + (1 - C_{\text{steer}}(P[\sigma'']))P_{\text{US}}^1, \]

where \( P_{2}^1 \) and \( P_{\text{US}}^1 \) are steerable and unsteerable boxes, respectively, and \( 0 \leq C_{\text{steer}}(P[\sigma'']) \leq 1 \) is the steering cost of the box \( P[\sigma''] \). Decomposing the boxes in the decomposition (39) with the above two optimal decompositions, we obtain

\[
P[\sigma] = \mu \left[ C_{\text{steer}}(P[\sigma'])P_{1}^1 + (1 - C_{\text{steer}}(P[\sigma']))P_{\text{US}}^1 \right]
+ (1 - \mu) \left[ C_{\text{steer}}(P[\sigma''])P_{2}^1 + (1 - C_{\text{steer}}(P[\sigma'']))P_{\text{US}}^1 \right]
\]

\[
:= \nu \mathbb{P} + (1 - \nu) \mathbb{P}_{\text{US}},
\]

with

\[ \nu := \mu C_{\text{steer}}(P[\sigma']) + (1 - \mu) C_{\text{steer}}(P[\sigma'']), \]

which satisfies \( 0 \leq \nu \leq 1 \) and

\[ \mathbb{P} := \frac{1}{\nu} \left[ \mu C_{\text{steer}}(P[\sigma'])P_{1}^1 + (1 - \mu) C_{\text{steer}}(P[\sigma''])P_{2}^1 \right], \]

which may be a steerable or an unsteerable box, and

\[ \mathbb{P}_{\text{US}} := \frac{1}{1 - \nu} \left[ \mu(1 - C_{\text{steer}}(P[\sigma']))P_{1}^1 + (1 - \mu)(1 - C_{\text{steer}}(P[\sigma'']))P_{2}^1 \right], \]

which is an unsteerable box since any convex mixture of two unsteerable boxes is unsteerable.

Suppose the box \( \mathbb{P} \) (45) is unsteerable. Then from Eq. (43) it is clear that the box \( P[\sigma] \) is a convex mixture of two unsteerable boxes and, hence, unsteerable. Therefore, in this case the following inequality trivially holds for all possible convex decompositions as in Eq. (36) of an arbitrary assemblage \( \sigma \in \Sigma^3 \):

\[ C_{\text{steer}}(P[\sigma]) = 0 \leq \nu = \mu C_{\text{steer}}(P[\sigma']) + (1 - \mu) C_{\text{steer}}(P[\sigma'']). \]

Suppose the box \( \mathbb{P} \) (45) is steerable. Then the decomposition (43) is not the optimal one if the weights of both the boxes \( P_{1}^1 \) and \( P_{2}^1 \) are nonzero (since the box \( \mathbb{P} \) is not an extremal box in this case, because an extremal steerable box in the set \( \mathcal{N}_2 \) cannot be decomposed as a convex mixture of the other boxes in the set \( \mathcal{N}_2 \)). Even if the weight of the box \( P_{1}^1 \) or that of the box \( P_{2}^1 \) is zero, the decomposition (43) may not be the optimal one. Hence, to obtain the steering cost \( C_{\text{steer}}(P[\sigma]) \) of the box \( P[\sigma] \), one has to minimize the weight of the steerable part \( \nu \) over all such possible decompositions of the box \( P[\sigma] \). So we have,

\[ C_{\text{steer}}(P[\sigma]) \leq \nu. \]  

From Eq. (48) together with Eq. (44), we can conclude that for all possible convex decompositions as in Eq. (36) of an arbitrary assemblage \( \sigma \in \Sigma^3 \),

\[ C_{\text{steer}}(P[\sigma]) \leq \mu C_{\text{steer}}(P[\sigma']) + (1 - \mu) C_{\text{steer}}(P[\sigma']). \]

Since the steering cost \( C_{\text{steer}}(P) \) satisfies the above two properties, it is a convex steering monotone.

In what follows, we will characterize steerability of two families of correlations which are called white-noise BB84 family and colored-noise BB84 family in the context of the following steering scenario: Alice performs two black-box dichotomic measurements on her part of an unknown d × 2 quantum state shared with Bob which produce the assemblage \( \{ |\psi_i\rangle \}_{i=1}^2 \) on Bob’s side. On this assemblage, Bob performs projective qubit measurements \( \{ |f_i\rangle \}_{i=1}^2 \), corresponding to any two mutually unbiased bases (MUBs), i.e. \( B_0 \equiv \{ |f_i\rangle \}_{i=1}^2 \) and \( B_1 \equiv \{ |g_j\rangle \}_{j=1}^2 \) such that, \( \langle f_i | g_j \rangle = \frac{1}{\sqrt{d}} \delta_{ij} \) (there, \( |f_i\rangle \) and \( |g_j\rangle \) are two sets of orthonormal basis). In this scenario, the necessary and sufficient condition for quantum steering from Alice to Bob is given by [26],

\[
\sqrt{\langle (A_0 + A_1)B_0 \rangle^2 + \langle (A_0 + A_1)B_1 \rangle^2 }
+ \sqrt{\langle (A_0 - A_1)B_0 \rangle^2 + \langle (A_0 - A_1)B_1 \rangle^2 } \leq 2.
\]

This inequality is called the analogous CHSH inequality for quantum steering.

The white-noise BB84 and colored-noise BB84 families belong to the local polytope of the two-binary-inputs and two-binary-outputs Bell scenario. In order to find out the existence of a LHV-LHS model for the given local correlation, we will consider a classical simulation model by using shared classical randomness, i.e., a local hidden variable model of finite dimension [33]. Suppose a local box \( P_L(ab|xy) := \{ p_L(ab|xy) \}_{a,x,b,y} \) admits the following decomposition:

\[ p_L(ab|xy) = \sum_{d=0}^{d_s-1} p(a|x,\lambda)p(b|y,\lambda) \quad \forall a, x, b, y. \]

Then it defines a classical simulation model by using shared randomness of dimension \( d_s \).

In Ref. [33], the upper bound on the minimum dimension of shared randomness required to simulate a local n-partite correlation is derived (see Proposition 5 in Ref. [33]). For the bipartite Bell scenario with two-binary inputs and two-binary outputs, shared randomness of dimension \( d_s \leq 4 \) is sufficient to simulate any local box.

Our method to check the existence of a LHV-LHS model for the local correlations in terms of the extremal boxes of the given steering scenario goes as follows. We first decompose the given local correlation in the form (51) where \( p(a|x,\lambda) \) are
different deterministic distributions and \( p(b|y, \lambda) \) may be nondeterministic in order to minimize the dimension of shared randomness. Then, we try to check whether each Bob’s distribution in this decomposition has a quantum realization in the context of the given steering scenario.

### A. White noise BB84 family

Consider the family of correlations defined as

\[
P_{\text{BB84}}(ab|xy) = 1 + (1 - V)\rho_\text{BB84}(ab|xy) + \frac{1 - V}{4},
\]

where \( 0 < V \leq 1 \). For \( V = 1 \), the above family of correlations corresponds to the BB84 correlation \(^1\) up to LRO. For this reason, we refer to the family of correlations given in Eq. (52) as white noise BB84 family. The white noise BB84 family is local as it does not violate a Bell-CHSH inequality (6). The white noise BB84 family violates the analogous CHSH inequality for quantum steering given by Eq.(50) for \( V > 1/\sqrt{2} \). Hence, the white noise BB84 family cannot be decomposed as a convex mixture of the extremal points of the unsteerable set as in Eq. (13) iff \( V > 1/\sqrt{2} \) in the given steering scenario, i.e., where Alice performs two black-box dichotomic measurements and Bob performs projective qubit measurements corresponding to any two mutually unbiased bases (MUBs). In the following we will demonstrate our procedure to find out in which range the white noise BB84 family can be written as a convex mixture of the extremal points of the unsteerable set as in Eq. (13) in the given steering scenario.

In the context of nonsignaling polytope, the BB84 family can be decomposed as follows:

\[
P_{\text{BB84}} = V\left(\frac{P_0^0 + P_{11}^{10}}{2}\right) + (1 - V)P_N,
\]

where \( P_N \) is the maximally mixed box, i.e., \( P_N(ab|xy) = \frac{1}{4}, \forall a, b, x, y \). Let us rewrite the above decomposition as follows:

\[
P_{\text{BB84}} = V\left(\frac{P_0^0}{2} + \frac{1}{2}P_N\right) + V\left(\frac{P_{11}^{10}}{2} + \frac{1}{2}P_N\right) + (1 - 2V)P_N \tag{54}
\]

By writing the each box in the above decomposition in terms of the local deterministic boxes, we obtain the following decomposition which defines a classical simulation protocol by using shared randomness of dimension 4:

\[
P_{\text{BB84}}(ab|xy) = \frac{V}{8} \sum_{a\beta y} P_a^{\beta y}(ab|xy) + \frac{V}{8} \sum_{b\alpha x} P_b^{\alpha x}(ab|xy) + (1 - 2V)P_N \tag{55}
\]

where \( \tilde{a} = a \oplus 1; \tilde{y} = y \oplus 1; P_{\text{BB84}}(a|x,y) = \{p(a|x,\lambda)\}_{x,y} \) is the set of conditional probability distributions \( p(a|x,\lambda) \) for all possible \( a, x; \) and \( P_{\text{BB84}}(b|y) = \{p(b|y,\lambda)\}_{b,y} \) is the set of conditional probability distributions \( p(b|y,\lambda) \) for all possible \( b, y \). In the LHV model given in Eq. (57), one of the parties (here, Alice) uses deterministic strategies given by:

\[
P_1(a|x) = P_D^0, P_2(a|x) = P_D^{10}, \\
P_3(a|x) = P_D^{10}, P_4(a|x) = P_D^{11}, \tag{56}
\]

while the other (here, Bob) uses nondeterministic strategies given by:

\[
P_1(b|y) = VP_D^{10}, P_2(b|y) = VP_D^{11}, \\
P_3(b|y) = VP_D^{00}, P_4(b|y) = VP_D^{01}. \tag{57}
\]

Let us now try to find in which range the BB84 family has a decomposition in terms of the extremal points of the unsteerable set as in Eq. (13) from the decomposition given in Eq. (57). For this purpose, we try to check in which range each nondeterministic strategy on Bob’s side in Eq. (59) can arise from a pure qubit state in the given steering scenario, i.e., for the measurements \( B_0 \) and \( B_1 \) in two mutually unbiased bases (MUB). With this aim, we note that each of Bob’s nondeterministic strategies can be written in the form, \( P_{\text{BB84}}(b|y) = \langle \phi_j^B|\Pi_B\rangle\phi_j^A \) \((\Pi_B := \{|\psi_{\lambda}\rangle\})\) corresponds to the set projective measurements at Bob’s side in any two mutually unbiased bases in Hilbert space \( C^2 \): \( B_0 \equiv \{|\psi_{\lambda}\rangle\}_{x=1}^2 \) and \( B_1 \equiv \{|\bar{\psi}_{\lambda}\rangle\}_{x=1}^2 \) such that, \(|\psi_{\lambda}\rangle|^2 = \frac{1}{2} \forall \lambda, j \), where \(|\psi_{\lambda}\rangle_{i=1}^2 \) and \(|\bar{\psi}_{\lambda}\rangle_{j=1}^2 \) are two sets of orthonormal basis, with the following pure states:

\[
|\psi_1\rangle = \frac{1}{\sqrt{1+V^2}}|f_1\rangle + e^{i\phi_1} \frac{1-V}{\sqrt{2}}|f_2\rangle, \tag{58}
\]

where \( \cos \phi_1 = -\frac{V}{\sqrt{1+V^2}} \),

\[
|\psi_2\rangle = \frac{1}{\sqrt{1-V^2}}|f_1\rangle + e^{i\phi_2} \frac{1+V}{\sqrt{2}}|f_2\rangle, \tag{59}
\]

1 The BB84 correlation satisfies \( p(a = b|xy) = 1 \) for \( x = y \) and \( p(a = b|xy) = 1/2 \) for \( x \neq y \), here \( p(a = b|xy) = p(00|xy) + p(11|xy) \) \([34]\).
where \( \cos \phi_2 = \frac{V}{\sqrt{V+V-1}} \),

\[ |\psi_1\rangle = \sqrt{\frac{1+V}{2}}|f_1\rangle + e^{i\phi_3} \sqrt{\frac{1-V}{2}}|f_2\rangle, \tag{62} \]

where \( \cos \phi_3 = \frac{V}{\sqrt{V+V-1}} \), and

\[ |\psi_4\rangle = \sqrt{\frac{1-V}{2}}|f_1\rangle + e^{i\phi_3} \sqrt{\frac{1+V}{2}}|f_2\rangle, \tag{63} \]

where \( \cos \phi_4 = -\frac{V}{\sqrt{V+V-1}} \). For any \( |\psi_4\rangle \) given above, \(|\cos \phi_4| \leq 1 \) iff \( V \leq 1/\sqrt{2} \). Note that, in the given steering scenario, the above states are the only pure states which give rise to the nondeterministic probability distributions on Bob’s side in Eq. (57). Therefore, we can conclude that the decomposition (57) represents convex mixture of the extremal points of the unsteerable set as in Eq. (13) in the given steering scenario iff \( V \leq 1/\sqrt{2} \).

**Theorem 1.** The steering cost of the white noise BB84 family is given by \( C_{\text{steer}}(P_{BB84}) = \max(0, \frac{\sqrt{V-1}}{\sqrt{V-1}}) \) in the given steering scenario.

**Proof.** Note that for \( V \geq 1/\sqrt{2} \), the BB84 family can be decomposed as follows:

\[ P_{BB84} = \frac{\sqrt{V}-1}{\sqrt{V}-1} P_S^{\text{ext}} + \frac{\sqrt{V}(1-V)}{\sqrt{V}-1} P_{US}, \tag{64} \]

where

\[ P_S^{\text{ext}} = \frac{1 + (-1)^{x+y} \delta_{x,y}}{4} \]

is an extremal steerable box as it violates the steering inequality (50) maximally, and \( P_{US} \) is an unsteerable box which has a decomposition as in Eq. (57) with \( V = 1/\sqrt{2} \) in terms of the extremal boxes of the given steering scenario. We see that the weight \( \frac{\sqrt{V-1}}{\sqrt{V-1}} \) in the decomposition (64) is nonzero iff \( P_{BB84} \) detects steerability. Therefore, the decomposition given in Eq. (64) is the optimal decomposition for the BB84 family for \( V \geq 1/\sqrt{2} \), because it is a convex mixture of the extremal steerable box (in the given steering scenario) and the unsteerable box with the weight of the steerable part going to zero iff the box is unsteerable.

\( \square \)

We will now verify that, \( C_{\text{steer}}(P_{BB84}) = \max(0, \frac{\sqrt{V-1}}{\sqrt{V-1}}) \) is a convex roof measure. From Eq. (9), we know that the assemblage \( \sigma_{|\psi}\rangle \) arising from the state \( \rho_W \) given in Eq. (53) can be decomposed as follows:

\[ \sigma_{|\psi}\rangle = \text{Tr}_A(M_A \otimes \mathbb{I}_{\rho_W}) = V \text{Tr}_A(M_A \otimes \mathbb{I}_{\Psi^-})(\Psi^-) \]

\[ + (1-V) \text{Tr}_A(M_A \otimes \mathbb{I}_{\Psi^-}) = \mathbb{I} + (1-V)\sigma_{|\Psi^-}\rangle, \tag{66} \]

Here, \( \sigma_{|\psi^-}\rangle \) is the assemblage arising from the state \( |\Psi^-\rangle \), and \( \sigma_{|\Psi^-}\rangle \) is the assemblage arising from the state \( \frac{1}{V} \). We now see that for any \( V \in [0,1] \), the following relation is satisfied:

\[ C_{\text{steer}}(P[\sigma_{|\psi^-\rangle}]) \leq VC_{\text{steer}}(P[\sigma_{|\Psi^-\rangle}]) + (1-V)C_{\text{steer}}(P[\sigma_{|\Psi^-\rangle}]), \tag{67} \]

for the measurements that generate the BB84 family. Here, \( C_{\text{steer}}(P[\sigma_{|\psi^-\rangle}]) = C_{\text{steer}}(P_{BB84}) = \max(0, \frac{\sqrt{V-1}}{\sqrt{V-1}}) \). \( C_{\text{steer}}(P[\sigma_{|\Psi^-\rangle}]) = 1 \) (since \( P[\sigma_{|\psi^-\rangle} \) violates the steering inequality (50) maximally for the aforementioned measurement settings) and \( C_{\text{steer}}(P[\sigma_{|\Psi^-\rangle}]) = 0 \) since \( \mathbb{I} \) does not have steerability. In another way, we can conclude that, if \( P[\sigma] \) and \( P[\sigma'] \) are two boxes belonging to the given steering scenario and obeying the following relation:

\[ P[\sigma] = \eta P[\sigma'] + (1-\eta)P_{US}, \tag{68} \]

with \( 0 \leq \eta \leq 1 \) and \( P_{US} \) being an unsteerable box, then \( P[\sigma'] \) is more steerable than \( P[\sigma] \) or equally steerable to \( P[\sigma] \), analogous to the case of Bell non-locality as demonstrated in Ref. [36].

**B. Colored noise BB84 family**

Let us now consider the colored-noise BB84 family defined as

\[ P_{BB84}^{\text{col}}(ab|xy) = \frac{1 + (-1)^{x+y} \delta_{x,y}}{4} V + \frac{1-V}{2} \]

\[ \frac{1-V}{2} \]

where \( 0 < V \leq 1 \). Note that for \( V = 1 \), the above family of correlations corresponds to the BB84 correlation [34] upto LRO. The colored-noise BB84 family can be obtained from the colored-noise two-qubit maximally entangled state,

\[ P_{col} = V|\Psi^-\rangle\langle\Psi^-| + (1-V)\mathbb{I}_{col}, \tag{70} \]

where the color noise \( \mathbb{I}_{col} = ([01] \langle 01| + [10] \langle 10|)/2 \), for suitable projective measurements. The colored-noise BB84 family is local as it does not violate a Bell-CHSH inequality.

The colored-noise BB84 family violates the analogous CHSH inequality for quantum steering (50) for \( V > 0 \). Hence, the colored-noise BB84 family cannot be decomposed as a convex mixture of the extremal points of the unsteerable set as in Eq. (13) iff \( V > 0 \) in the given steering scenario. In the following, adopting our procedure, we will find out in which range the colored-noise BB84 family can be written as a convex mixture of the extremal points of the unsteerable set as in Eq. (13) in the given steering scenario.
In the context of nonsignaling polytope, the colored-noise BB84 family can be decomposed as follows:

\[
P_{BB84}^{col} = V \left(\frac{p^0 + p_110}{2} + (1 - V)p_{US}\right). \tag{71}
\]

Here, \[P_{US} := \frac{p^0 + p_110 + p_101 + p_{110}}{4} \]

which belongs to the unsteerable set of the steering scenario that we have considered. There are many possible decompositions for the box \(P_{US}\) in terms of local deterministic boxes. But all of them do not lead to convex mixtures of extremal boxes of the unsteerable set in the given steering scenario for any two projective measurements \(\Pi_b := \{\Pi_{b_i}\}_{i=1}^2\) in any two mutually unbiased bases (at Bob’s side) in Hilbert space \(\mathcal{C}^2\): \(B_0 \equiv \{|f_i\}_{i=1}^2\) and \(B_1 \equiv \{|g_j\}_{j=1}^2\) such that, \(|\langle f_i|g_j\rangle|^2 = \frac{1}{2}\) \(\forall i, j\) (Here \(|f_i\}_{i=1}^2\) and \(|g_j\}_{j=1}^2\) are two sets of orthonormal basis). To obtain a such a convex mixture, we consider the following decomposition for the box \(P_{US}\):

\[
P_{US} = \frac{1}{4} P^0_D + \frac{1}{4} P^1_D + \frac{1}{4} P^0_D + \frac{1}{4} P^1_D + \frac{1}{4} P^0_D + \frac{1}{4} P^1_D + \frac{1}{4} P^0_D + \frac{1}{4} P^1_D, \tag{72}
\]

which is a LHV-LHS model in terms of extremal boxes of the unsteerable set.

By decomposing the first box in the decomposition (71) in terms of local deterministic boxes and using the decomposition (72) for the second box in the decomposition (71), we obtain the following LHV model for the colored-noise BB84 box by using shared randomness of dimension 4:

\[
P_{BB84}^{col}(ab|xy) = \sum_{a=1}^{4} P_a(a|x) P_b(b|y), \tag{74}
\]

where one of the parties (here, Alice) uses deterministic strategies given by:

\[
P_1(a|x) = P^0_D, P_2(a|x) = P^1_D, \tag{75}
\]

while the other (here, Bob) uses nondeterministic strategies given by:

\[
P_1(b|y) = V P^0_D + (1 - V) P^1_D, \tag{76}
\]

\[
P_2(b|y) = V P^1_D + (1 - V) P^0_D, \tag{77}
\]

\[
P_3(b|y) = V P^0_D + (1 - V) P^1_D, \tag{78}
\]

\[
P_4(b|y) = V P^1_D + (1 - V) P^0_D. \tag{79}
\]

Let us now try to find in which range the colored-noise BB84 family has a decomposition in terms of the extremal points of the unsteerable set as in Eq. (13) from the decomposition given in Eq. (74). For this purpose, we try to check in which range each nondeterministic strategy on Bob’s side in Eq. (76) can arise from a pure qubit state in the given steering scenario, i.e., for the measurements \(B_0 \) and \(B_1\) in two mutually unbiased bases (MUB). With this aim, we note that each of Bob’s nondeterministic strategies can be written in the form, \(P_{j}(b|y) = \langle \psi^j_{y} | \Pi_{b} \psi^j_{y} \rangle \) \(\Pi_{b} := \{\Pi_{b_i}\}_{i=1}^2\) corresponds to the set projective qubit measurements at Bob’s side in any two mutually unbiased bases: \(B_0 \equiv \{|f_i\}_{i=1}^2\) and \(B_1 \equiv \{|g_j\}_{j=1}^2\) as defined earlier, with the following pure states:

\[
|\psi^j_1\rangle = \sqrt{\frac{1 - V}{2}} |g_1\rangle + e^{i \phi^j_1} \sqrt{\frac{1 + V}{2}} |g_2\rangle, \tag{77}
\]

where \(\cos \phi^j_1 = \frac{1}{\sqrt{V + 1}} \). For any \(|\psi^j_1\rangle\) given above, \(|\cos \phi^j_1| \leq 1\) iff \(V = 0\). Note that, in the given steering scenario, the above states are the only pure states which give rise to the nondeterministic probability distributions on Bob’s side in Eq. (74). Therefore, we can conclude that the decomposition (74) represents convex mixture of the extremal points of the unsteerable set as in Eq. (13) in the given steering scenario iff \(V = 0\).

**Theorem 2.** The steering cost of the colored-noise BB84 family is given by \(C_{steer}(P_{BB84}^{col}) = V\) in the given steering scenario.

**Proof.** Note that the colored-noise BB84 family can be decomposed as follows:

\[
P_{BB84}^{col} = V P_{US}^{ext} + (1 - V) P_{US}, \tag{81}
\]

where \(P_{US}^{ext}\) is the extremal steerable box given in Eq. (65) and \(P_{US}\) is the unsteerable box given in Eq. (75). The decomposition given in Eq. (81) is the optimal decomposition for the BB84 family, because it is a convex mixture of the extremal steerable box (in the given steering scenario) and the unsteerable box with the weight of steerable part goes to zero iff the box is unsteerable. □
We will now verify that, $C_{\text{steer}}(P_{\text{BB84}}^{\text{col}}) = V$ is a convex roof measure. Note that the assemblage $\sigma_{\text{col}}$ arising from state $\rho_{\text{col}}$ given in Eq. (70) can be decomposed as follows:

$$
\sigma_{\text{col}} = \text{Tr}(M_A \otimes I_{\rho_{\text{col}}}) = V \text{ Tr}(M_A \otimes I_{\Psi}) + (1 - V) \text{ Tr}(M_A \otimes I_{\rho_{\text{col}}}) = V \sigma_{\text{col}}^{\Psi} + (1 - V) \sigma_{\text{col}}.
$$

(82)

Here, $\sigma_{\text{col}}^{\Psi}$ is the assemblage arising from the state $|\Psi\rangle$, and $\sigma_{\text{col}}$ is the assemblage arising from the state $I_{\rho_{\text{col}}}$. We now see that for any $V \in [0, 1]$, the following relation is satisfied:

$$
C_{\text{steer}}(P[\sigma_{\text{col}}]) = V C_{\text{steer}}(P[\sigma_{\text{col}}^{\Psi}]) + (1 - V) C_{\text{steer}}(P[\sigma_{\text{col}}^{\text{col}}]),
$$

(83)

for the measurements that generate the colored-noise BB84 family. Here, $C_{\text{steer}}(P[\sigma_{\text{col}}]) = C_{\text{steer}}(P_{\text{BB84}}^{\text{col}}) = V$, $C_{\text{steer}}(P[\sigma_{\text{col}}^{\Psi}]) = 1$ since $P[\sigma_{\text{col}}^{\Psi}]$ is an extremal box (it violates the steering inequality (50) maximally for the aforementioned measurement settings) and $C_{\text{steer}}(P[\sigma_{\text{col}}^{\text{col}}]) = 0$ since $I_{\rho_{\text{col}}}$ does not have steerable.

IV. STEERING COST VERSUS STEERING WEIGHT

For any assemblage $\sigma = \{\sigma_{\text{al}}\}_{a,x}$ arising from a given steering scenario, steering weight [20] which we denote by $W_{\text{steer}}(\sigma)$ is defined as follows. Consider the following decomposition of the given assemblage $\sigma$:

$$
\sigma_{\text{al}} = p_S \sigma_{\text{al}}^{\Psi} + (1 - p_S) \sigma_{\text{al}}^{US} \quad \forall a, x,
$$

(84)

where $\sigma_{\text{al}}^{\Psi}$ is an element of an assemblage $\sigma^{\Psi}$ having steerability and $\sigma_{\text{al}}^{US}$ is an element of an unsteerable assemblage $\sigma^{US}$. The weight of the steerable part $p_S$ minimized over all possible decompositions of the given assemblage $\sigma$ gives the steering weight $W_{\text{steer}}(\sigma)$.

**Proposition 1.** Let us assume the following optimal decomposition of the given assemblage $\sigma = \{\sigma_{\text{al}}\}_{a,x}$ with the weight of the steerable part being minimized over all possible decompositions of the assemblage, i.e., the weight of the steerable part being equal to the steering weight $W_{\text{steer}}(\sigma)$ of the assemblage $\sigma$:

$$
\sigma_{\text{al}} = W_{\text{steer}}(\sigma) \sigma_{\text{al}}^{\Psi} + (1 - W_{\text{steer}}(\sigma)) \sigma_{\text{al}}^{US} \quad \forall a, x
$$

(85)

and Bob performs a set of projective measurements $\Pi_B := \{\Pi_{\text{al}}\}_{a,b}$ on $\sigma$; we have

$$
C_{\text{steer}}(P[\sigma]) \leq W_{\text{steer}}(\sigma),
$$

(86)

where $P[\sigma] = P(ab|x) = |\text{ Tr}[\Pi_{\text{al}} \sigma_{\text{al}}]|_{a,b,x}$; $\sigma_{\text{al}}^{\Psi}$ is an element of an assemblage $\sigma^{\Psi}$ having steerability and $\sigma_{\text{al}}^{US}$ is an element of an unsteerable assemblage $\sigma^{US}$.

**Proof.** Suppose Bob performs a set of projective measurements $\Pi_B := \{\Pi_{\text{al}}\}_{a,b}$ on $\sigma$ given by the decomposition (85). Then, one can write,

$$
\text{Tr}[\Pi_{\text{al}} \sigma_{\text{al}}] = W_{\text{steer}}(\sigma) \text{Tr}[\Pi_{\text{al}} \sigma_{\text{al}}^{\Psi}] + (1 - W_{\text{steer}}(\sigma)) \text{Tr}[\Pi_{\text{al}} \sigma_{\text{al}}^{US}], \quad \forall a, x, b, y.
$$

(87)

Hence, for the box $P[\sigma]$ arising from the assemblage $\sigma$, one can write the following decomposition:

$$
P[\sigma] = W_{\text{steer}}(\sigma) P_S [\sigma^{\Psi}] + (1 - W_{\text{steer}}(\sigma)) P_{US} [\sigma^{US}].
$$

(88)

Here, $P_S [\sigma^{\Psi}]$ is a steerable box, produced from the steerable assemblage $\sigma^{\Psi}$ and $P_{US} [\sigma^{US}]$ is an unsteerable box, produced from the unsteerable assemblage $\sigma^{US}$.

Now in the decomposition (88) the weight $W_{\text{steer}}(\sigma)$ of the steerable correlation may not be minimum weight of the steerable correlation over all possible decompositions of the correlation $P[\sigma]$. Since the steering cost of the correlation $P[\sigma]$ is obtained by minimizing the weight of the steerable correlation over all possible decompositions of the correlation $P[\sigma]$, the steering cost $C_{\text{steer}}(P[\sigma])$ of the correlation $P[\sigma]$ satisfies the relationship given by $C_{\text{steer}}(P[\sigma]) \leq W_{\text{steer}}(\sigma)$.

We will now present two examples demonstrating the above proposition. Suppose Alice and Bob share the two-qubit Werner state $\rho_W(V)$ given by Eq.(53) and Alice performs projective measurements $M_A := \{M_{\text{al}}\}_{a,x}$ in the two bases: $\{1/2 (I - \sigma_z), 1/2 (I + \sigma_z)\}$ and $\{1/2 (I + \sigma_y), 1/2 (I - \sigma_y)\}$. Then the assemblage prepared on Bob’s side which we denote by $\sigma_{\text{al}}^{W}$ is steerable iff $V > \frac{1}{\sqrt{3}} [31, 14]$. For $V \geq \frac{1}{\sqrt{3}}$, this assemblage can be decomposed in the following way:

$$
\sigma_{\text{al}}^{W} |_{a,b} = 
\begin{cases}
\frac{\sqrt{2} V - 1}{\sqrt{2} - 1} \sigma_{\text{al}}^{W} |_{a,b} & \text{if } V > \frac{1}{\sqrt{3}} \\
\frac{\sqrt{2} (1 - V)}{\sqrt{2} - 1} \sigma_{\text{al}}^{W} |_{a,b} & \text{if } V \leq \frac{1}{\sqrt{3}}
\end{cases}
$$

(89)

where $\sigma_{\text{al}}^{W} |_{a,b}$ represents the element of assemblage prepared on Bob’s side when Alice performs the aforementioned measurements on the singlet state $|\Psi\rangle = (|01\rangle - |10\rangle) / \sqrt{2}$, which is an element of steerable assemblage and $\sigma_{\text{al}}^{W} |_{a,b} |_{V=1/\sqrt{3}}$ represents the elements of assemblage prepared on Bob’s side when Alice performs the aforementioned measurements on the shared two-qubit Werner state $\rho_W (53)$ for $V = \frac{1}{\sqrt{3}}$, which is an element of unsteerable assemblage [31]. It can be checked that, for all $a, x$, each element of the steerable assemblage $\sigma_{\text{al}}^{W} |_{a,b}$ is a pure state after normalization and hence cannot be written as a convex combination of steerable and unsteerable assemblage. The coefficient of the element of the steerable assemblage in the decomposition (89), therefore, cannot be reduced further. Moreover, the weight of steerable part goes to zero iff the assemblage $\sigma_{\text{al}}^{W}$ is unsteerable. Hence, the decomposition (89) is the optimal decomposition of the assemblage $\sigma_{\text{al}}^{W}$.

This implies that the steering weight of the two-qubit Werner state $\rho_W$, when Alice performs the aforementioned two measurements, is given by $\max(0, \frac{\sqrt{2} V - 1}{\sqrt{2} - 1})$.
If Bob performs projective measurements $\Pi_B := \{\Pi_{b}\}_{b=0}^3$ in the two mutually unbiased bases: $\{\frac{1}{2}(\mathbb{1} + \sigma_z), \frac{1}{2}(\mathbb{1} - \sigma_z)\}$ and $\{\frac{1}{2}(\mathbb{1} + \sigma_y), \frac{1}{2}(\mathbb{1} - \sigma_y)\}$ on the above assemblage $\sigma[12]$, then the white noise BB84 family is produced. The steering cost of the white noise BB84 family is given by $\max(0, \frac{\sqrt{V}}{\sqrt{5} - 1})$. Hence, with these measurements performed by Alice and Bob, the steering cost of the state $\rho_w$ is equal to the steering weight of the state $\rho_w$.

Now, instead of performing the above measurements, if Bob performs projective measurements $\Pi_B := \{\Pi_{b}\}_{b=0}^3$ in the two mutually unbiased bases: $\{\frac{1}{2}(\mathbb{1} + \cos\frac{\pi}{2}\sigma_z + \sin\frac{\pi}{2}\sigma_y), \frac{1}{2}(\mathbb{1} - \cos\frac{\pi}{2}\sigma_z + \sin\frac{\pi}{2}\sigma_y)\}$ and $\{\frac{1}{2}(\mathbb{1} + \sigma_z), \frac{1}{2}(\mathbb{1} - \sigma_z)\}$ on the above assemblage $\sigma[12]$, then the produced correlation violates analogous CHSH inequality for quantum steering (50) for $V > \sqrt{2/5}(11 - \sqrt{5})$. Hence, for $0 < V \leq \sqrt{2/5}(11 - \sqrt{5})$, the steering cost of the produced correlation is 0. In the range $\frac{1}{\sqrt{2}} < V \leq \sqrt{2/5}(11 - \sqrt{5})$, with these measurements performed by Alice and Bob, the steering cost of the correlation is, therefore, less than the steering weight of the assemblage from which this correlation has been produced in the given steering scenario.

Experimentally, the determination of the steering weight for the steering scenario that we have considered requires complete tomographic knowledge of the qubit assemblage prepared on the trusted side [37]. On the other hand, the steering cost proposed by us is determined from the observed tomographic knowledge of the assemblage prepared. Thus, the determination of our steering cost is experimentally less demanding than the determination of the steering weight.

V. CONCLUSIONS

In this work, we have presented a method to check steerability for the scenario where Alice performs two black-box dichotomic measurements and Bob performs two arbitrary projective qubit measurements in mutually unbiased bases (MUBs). This method is based on the decompositions of the measurement correlations in the context of the extremal boxes of the steering scenario. Our method provides a simple way to check the existence of a LHV-LHS model for the measurement correlations. Based on this formulation to check steerability, we have proposed a quantifier of steering called steering cost. The determination of our steering cost is experimentally less demanding than the determination of the steering weight. We have demonstrated that our steering cost is a convex steering monotone. We have illustrated our method to check steerability with two families of measurement correlations and obtained their steering cost. In Ref. [34], security of the device-independent quantum key distribution protocol with the nonlocal correlations arising from the two-qubit Werner states was studied in the context of extremal nonsignaling boxes. Similarly, it would be interesting to study security of the one-sided device-independent quantum key distribution protocol with the measurement correlations that we have considered in the context of extremal boxes of the steering scenario.

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