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Note

Coloring graphs with crossings

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\textbf{A B S T R A C T}

We generalize the Five-Color Theorem by showing that it extends to graphs with two crossings. Furthermore, we show that if a graph has three crossings, but does not contain $K_6$ as a subgraph, then it is also 5-colorable. We also consider the question of whether the result can be extended to graphs with more crossings.

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1. Introduction

In this paper, $n$ will denote the number of vertices, and $m$ the number of edges, of a graph $G$. A coloring of $G$ is understood to be a proper coloring; that is, one in which adjacent vertices always receive distinct colors.

We will consider drawings of graphs in the plane $\mathbb{R}^2$ for which no three edges have a common crossing. A crossing of two edges $e$ and $f$ is trivial if $e$ and $f$ are adjacent or equal, and it is non-trivial otherwise. A drawing is good if it has no trivial crossings. The following is a well-known lemma.

\textbf{Lemma 1.1.} A drawing of a graph can be modified to eliminate all of its trivial crossings, with the number of non-trivial crossings remaining the same.

To avoid complicating the notation, we will use the same symbol for a graph and its drawing in the plane. We will refer to the regions of a drawing of a graph $G$ as the maximal open sets $U$ of $\mathbb{R}^2 - G$ such that for every two points $x, y \in U$, there exists a polygonal $xy$-curve in $U$.

\textbf{Definition 1.2.} The crossing number of a graph $G$, denoted by $\nu(G)$, is the minimum number of crossings in a drawing of $G$. An optimal drawing of $G$ is a drawing of $G$ with exactly $\nu(G)$ crossings.

\textbf{Definition 1.3.} Suppose $G'$ and $G$ are graphs. A function $\alpha$ with domain $V(G') \cup E(G')$ is an immersion of $G'$ into $G$ if the following hold:

(1) the restriction of $\alpha$ to $V(G')$ is an injection into $V(G)$;
(2) for an edge $e$ of $G'$ incident to $u$ and $v$, the image $\alpha(e)$ is a path in $G$ with ends $\alpha(u)$ and $\alpha(v)$; and
(3) for distinct edges $e$ and $f$ of $G'$, their images $\alpha(e)$ and $\alpha(f)$ are edge-disjoint.
The crossing number of the complete graph $K_n$ is easy to draw if $n$ is even. It is worth noting that if, for every edge $e$ of $G'$, the path $\alpha(e)$ consists of a single edge, then $G'$ is a subgraph of $G$. All immersions considered in the remainder of this paper will be essential.

**Proposition 1.4.** If $n \geq 3$, then $v(G) \geq m - 3n + 6$.

**Proof.** As a consequence of Euler’s formula, since $m \leq 3n - 6$ in a planar graph, every edge in excess of this bound introduces at least one additional crossing. □

**Corollary 1.5.** The crossing number of the complete graph $K_6$ is 3.

**Proof.** It is easy to draw $K_6$ with exactly three crossings, while Proposition 1.4 implies that $v(K_6) \geq 3$. □

2. **Immersions and crossings**

In this section we present several results that relate crossings of a drawing with immersions of a graph.

**Lemma 2.1.** Suppose $G$ is a good drawing with exactly $k$ crossings and there is an essential immersion of $G'$ onto $G$. Then $G'$ has a good drawing with exactly $k$ crossings.

**Proof.** Let $\alpha$ be an essential immersion of $G'$ onto $G$. Draw $G'$ by placing each vertex $v$ at $\alpha(v)$, drawing each edge $e$ so that it follows $\alpha(e)$, and then perturbing the edges slightly so that no edge contains a vertex and no three edges cross at the same point. Each crossing of edges $e$ and $f$ in $G'$ arises from the corresponding paths $\alpha(e)$ and $\alpha(f)$ either crossing or sharing a vertex. In the latter case, the crossing is trivial as the immersion $\alpha$ is essential. The conclusion now follows immediately from Lemma 1.1. □

Thus we have the following:

**Corollary 2.2.** If $G'$ is essentially immersed into $G$, then $v(G') \leq v(G)$.

We may also use essential immersions to extend the Five-Color Theorem.

**Lemma 2.3.** Let $G$ be a graph and let $v$ be a vertex in $G$ of degree at most 5 such that there is no $v$-immersion of $K_6$ into $G$. If $G - v$ is 5-colorable, then so is $G$.

**Proof.** Suppose that $G$ is not 5-colorable, and let $c$ be a 5-coloring of $G - v$. Then $c$ must assign all five colors to the neighbors of $v$ and hence $\deg(v) = 5$; since otherwise we can extend $c$ to $G$. Let the neighbors of $v$ be $v_1, v_2, v_3, v_4$ and $v_5$; and use the notation $c(v_i) = i$ for each $i \in \{1, 2, 3, 4, 5\}$.

For each pair of distinct $i$ and $j$ in $\{1, 2, 3, 4, 5\}$, let $G_{i,j}$ denote the subgraph of $G - v$ whose vertices are colored by $c$ with $i$ or $j$. If, for one such pair of $i$ and $j$, the graph $G_{i,j}$ has $v_i$ and $v_j$ in distinct components, then the colors $i$ and $j$ can be switched in one of the components so that two neighbors of $v$ are colored the same. In this case, the coloring $c$ can be extended to $v$ so that $G$ is 5-colorable; a contradiction.

Hence, for each pair of distinct $i$ and $j$, the graph $G - v$ has a path joining $v_i$ and $v_j$ whose vertices are alternately colored $i$ and $j$ by $c$, and thus $G$ contains a $v$-immersion of $K_6$; again, a contradiction. □
Corollary 2.4 (Generalized Five-Color Theorem). Every graph with crossing number at most two is 5-colorable.

Proof. Suppose not and consider a counterexample \( G \) on the minimum number of vertices. Proposition 1.4 implies that \( m \leq 3n - 4 \), and so \( G \) has a vertex \( v \) whose degree is at most 5. From Corollaries 1.5 and 2.2 we conclude that there is no \( v \)-immersion of \( K_6 \) into \( G \). The minimality of \( G \) implies that \( G - v \) is 5-colorable, and the conclusion follows from Lemma 2.3. □

Lemma 2.3 establishes that a graph \( G \) with \( v(G) \leq 3 \) is 5-colorable if there is no \( v \)-immersion of \( K_6 \) into \( G \). The next lemma addresses the case of graphs with \( v(G) \leq 3 \) for which there is a \( v \)-immersion of \( K_6 \) into \( G \) for some vertex \( v \) in \( G \). The following corollary of a result of Kleitman \([1]\) will be used in its proof.

Proposition 2.5. Every good drawing of \( K_5 \) has an odd number of crossings.

Lemma 2.6. If \( G \) is a drawing with exactly three crossings and \( \alpha \) is a \( v \)-immersion of \( K_6 \) into \( G \) for some vertex \( v \) in \( G \), then \( v \) is incident with exactly two crossed edges.

Proof. Without loss of generality, we may assume by Lemma 1.1 that all crossings of \( G \) are non-trivial. Let \( H \) be the subgraph of \( G \) that is the image of \( K_6 \) under \( \alpha \), and let \( u \) be the vertex in \( K_6 \) such that \( \alpha(u) = v \). From Corollary 1.5 and Lemmas 2.1 and 2.3, we conclude that \( H \) is a good drawing with three non-trivial crossings, and so all crossings of \( G \) occur in \( H \).

If \( v \) were incident with one or three crossed edges in \( H \), then \( H - v \) would be a good drawing with zero or two crossings with \( K_6 \) essentially immersed onto it. This, together with Lemma 2.1, would imply that there is a good drawing of \( K_5 \) with zero or two crossings, which would contradict Proposition 2.5.

Moreover, if \( v \) were incident with no crossed edges in \( H \), then \( H - v \) would be a drawing with a region \( R \) that is incident with all vertices in the set \( S = \{\alpha(w) : w \in V(K_6 - u)\} \). The boundary of \( R \) then induces a cyclic order on the set \( S \), and hence also on \( V(K_6 - u) \). If \( e \) and \( f \) are distinct non-adjacent edges of \( K_6 - u \) and each joins a pair of non-consecutive vertices, then \( \alpha(e) \) and \( \alpha(f) \) must cross. It follows that \( H \) would have at least five crossings; a contradiction. □

3. Colorings and crossings

Lemmas 2.3 and 2.6, respectively, characterize a graph \( G \) when it does and does not contain a \( v \)-immersion of \( K_6 \). With these, we now proceed to the main theorem. We will use \( \omega(G) \) to denote the clique number of \( G \), that is, the largest \( n \) for which \( K_n \) is a subgraph of \( G \).

Main Theorem 3.1. If \( v(G) \leq 3 \) and \( \omega(G) \leq 5 \), then \( G \) is 5-colorable.

Proof. Let \( \mathcal{G} \) denote the class of all graphs with crossing number at most three that are not 5-colorable, and let \( G \) be a member of \( \mathcal{G} \) with the minimum number of vertices. Suppose that \( \omega(G) \leq 5 \) and that \( G \) is drawn optimally in the plane.

If \( G \) contains a vertex \( v \) of degree less than 5, then \( G \) is not a minimal member of \( \mathcal{G} \), since a 5-coloring of \( G - v \) extends to a 5-coloring of \( G \). Hence, the minimum degree of \( G \) is 5. By Proposition 1.4, the graph \( G \) has at most \( 3n - 3 \) edges, and thus has at least six vertices of degree 5.

Let \( v \) be a vertex of degree 5. Lemma 2.3 implies that there is a \( v \)-immersion of \( K_6 \) into \( G \), and Corollary 2.2 implies that the image of \( \alpha \) in \( G \) contains three crossed edges. Then Lemma 2.6 implies that two crossed edges of \( G \) are incident with \( v \). Since \( G \) is not \( K_6 \), it contains a vertex \( w \) of degree 5 not adjacent to \( v \). However, Lemma 2.3 implies that there is also a \( w \)-immersion of \( K_6 \) into \( G \), and so \( w \) is also incident with two crossed edges. Since \( v \) and \( w \) are not adjacent, these two crossed edges are different from the crossed edges incident with \( v \), which implies that \( G \) contains four crossings; a contradiction. □

We also show that when Theorem 3.1 is applied to a 4-connected graph \( G \) other than \( K_6 \), then the assumption \( \omega(G) \leq 5 \) may be discarded. More precisely, we have:

Corollary 3.2. If \( G \) is 4-connected, \( v(G) \leq 3 \) and \( G \neq K_6 \), then \( G \) is 5-colorable.

Proof. Let \( G \) be an optimal drawing of a 4-connected graph with at most three crossings and not isomorphic to \( K_6 \). We show that \( \omega(G) \leq 5 \), from which the conclusion follows immediately from Theorem 3.1.

Suppose, to the contrary, that \( G \) has a complete subgraph \( K \) on six vertices. Let \( v \) be a vertex of \( G \) that is not in \( K \), and let \( K' \) be the plane drawing obtained from \( K \) by replacing each crossing with a new vertex. By Corollary 1.5, all three crossings of \( G \) are in \( K \), and so \( |V(K')| = 9 \) and \( |E(K')| = 21 \). Thus, since every plane graph in which \( m = 3n - 6 \) is a triangulation, \( K' \) must be a triangulation, and so every region of \( K \) contains at most three vertices in its boundary. But this is impossible, as \( G \), being 4-connected, has four paths from \( v \) to vertices of \( K \), with each pair of paths having only \( v \) in common. □

Lastly, note that \( C_3 \lor C_5 \), the graph in which every vertex of \( C_3 \) is adjacent to every vertex of \( C_5 \), contains no \( K_6 \) subgraph and is not 5-colorable.

Proposition 3.3. The crossing number of \( C_3 \lor C_5 \) is 6.
Fig. 2. $C_3 \lor C_5$ drawn with an optimal number of crossings.

**Proof.** Let $G$ be an optimal drawing of $K \lor L$, where $K$ and $L$ are cycles on, respectively, three and five vertices. Suppose that $G$ has fewer than six crossings. Note that $G \setminus (E(K) \cup E(L))$ is isomorphic to $K_{3,5}$, which has crossing number 4 [2]. This implies that the edges of $K \cup L$ are involved in at most one crossing, and thus $L$ has at most three regions, one of which contains $K$. Thus at least one region of $L$ avoids $K$ and has two non-adjacent vertices of $L$ in its boundary. These two vertices of $L$ can be joined by a new edge that crosses no edges of $G$ thereby creating a graph with 8 vertices, 24 edges, and 5 crossings; a contradiction to Proposition 1.4. Hence, $G$ has six crossings. Fig. 2 shows a drawing which achieves this bound, proving that $\nu(C_3 \lor C_5) = 6$. □

We do not currently know whether the main theorem extends to graphs with four or five crossings, and hence conclude with the following question:

**Question 3.4.** Does a graph $G$ have a 5-coloring if $\nu(G) \leq 5$ and $\omega(G) \leq 5$?

**References**

[1] D.J. Kleitman, A note on the parity of the number of crossings of a graph, Journal of Combinatorial Theory Series B 21 (1976) 88–89.
[2] D.J. Kleitman, The crossing number of $K_{5,n}$, Journal of Combinatorial Theory 9 (1970) 315–323.