Matrix Representation of Special Relativity

Wolfgang Köhler, Potsdam, Germany*

March 31, 2022

Abstract

I compare the matrix representation of the basic statements of Special Relativity with the conventional vector space representation.

It is shown, that the matrix form reproduces all equations in a very concise and elegant form, namely: Maxwell equations, Lorentz-force, energy-momentum tensor, Dirac-equation and Lagrangians. The matrix representation requires fewer assumptions, uses fewer parameters and may lead to new insights into physical reality.

A new result is a matrix form of Dirac’s equation. It can explain the non-existence of right-handed neutrinos and can be generalized to include a new variant of Yang-Mills gauge fields, which possibly express unified electro-weak interactions.

Contents

1 Introduction 1
2 Matrix Representation of Minkowski-Vectors 2
3 Transformations and Covariant Forms 3
4 Relativistic Electromagnetics 4
  4.1 Maxwell-Equations 4
  4.2 Lorentz-Force 4
  4.3 Energy-Momentum-Tensor of Electromagnetic Field 5
5 Relativistic Quantum Mechanics 6
  5.1 Dirac-Equation 6
  5.2 Matrix-Dirac-Equation 7
  5.3 Lagrangian of Coupled Dirac- and EM-Field 8
  5.4 Yang-Mills Gauge-Fields in Matrix Form 9
  5.5 2-Spinors and Minkowski-Matrices 10
6 Conclusions 10
7 Acknowledgments 11

1 Introduction

The possibility of representing Minkowski spacetime vectors with 2x2-matrices has been known since the 1920ies (e.g. [10], [8], pp. 61). It is a consequence of the fact, that the Lorentz-group is homomorphic to the group of unimodular binary matrices $SL(2,C)$.

This matrix representation is mostly used to show, how covariant equations for spinors can be derived. There is a general consensus, that both representations (matrix form and usual component form) are

*E-mail address: wolfk@gfz-potsdam.de
Homepage: http://icgem.gfz-potsdam.de/QM
actually equivalent methods to express the equations of Special Relativity, and the matrix form is used very rarely in publications.¹

One principal reason for this is, that conventional component formulae can be formally applied to an arbitrary number of dimensions, while the matrix form is only possible for the four-dimensional case.

In this article I show, that the matrix form gives an elegant means to derive all equations of special relativity but with significantly less prerequisites.

The most important prerequisite is the existence of a metric tensor with the signature \(+−−−\), that has to be postulated for the vector space (in principle, any metric signature would be conceivable). This metric is automatically determined for the matrix formalism.

To value this fact, one should note, that the metric tensor is - at least implicitly - contained in every relativistic equation.

Moreover, the homogeneous MAXWELL equations, which have to be introduced independently in SRT in component form, are a direct consequence of the inhomogeneous MAXWELL eqs. here.

The last and most important argument gives the reformulation of the DIRAC equation in matrix form. All arbitrary free parameters without physical content, which arise in the 4-spinor form, vanish here, because the remaining similarity transformations can be understood as gauge transformations.

Thus, the main aim of this paper is a change of perspective: Physical spacetime is primarily to represent by a matrix algebra and the component formulation is a derived one, which also has its disadvantages.

I will denote this perspective as ”matrix spacetime“ (MST) compared to ”vector spacetime“².

Please note that, if this point of view is adopted, this is not only a formal aspect, but it has far-reaching consequences for many other physical theories. E.g. obviously all theories with more than four spacetime dimensions are excluded. The matrix form also has applicability in General Relativity, but this goes beyond the scope of this paper.³

This new perspective also may lead to new theories, e.g. if possible generalizations of this form are considered. One might also look for an underlying spinor structure for the matrix algebra, which is e.g. the main thesis of the “twistor-theory” presented in [9] (Vol. II) but has not led to a satisfactory physical theory yet.

In conclusion I have to say, that many of the equations presented here, can also be found scattered in other publications.

New in any case, is the notation of DIRACS eq. as “matrix equation”. Also the corresponding LAGRANGIAN, I have not found in another publication. This new form perhaps allows new insights in particle physics, esp. unified electro-weak theory.

2 Matrix Representation of MINKOWSKI-Vectors

Let me start with the 4-dimensional vector space of real numbers \(V^4 = \{(x^0, x^1, x^2, x^3)\}\). This can be mapped one-to-one to the set of hermitean 2x2 matrices \(M = \{x\}\), when a basis of 4 linearly independent hermitean matrices \(\tau_\mu = (\tau_0, \ldots, \tau_3)\) is given by (as usual, over double upper and lower indices \(\mu = 0,\ldots,3\) is to sum):

\[
x = x^\mu \tau_\mu.
\] (1)

These hermitean matrices \(x\) build a well defined subset of the binary matrix algebra. In the following they are denoted as MINKOWSKI-matrices and represented by boldface letters (except the Greek letters \(\rho, \tau, \sigma\) and the partial operator \(\partial\)).

---

¹ One of the first fundamental papers on this topic is [11], where the idea of em. gauge symmetry was invented and some of the concepts and eqs. below can be found. However, he focuses there on gravitation and curved space-time. A newer perspective can be found in [2], where the authors try to give a fundamental overview from a mathematical point of view. They also focus on GR and various kinds of generalizations.

² Compare e.g. [4], where a similar concept with a four-dimensional algebra based on the CLIFFORD-matrices is presented. He uses a similar term ”spacetime algebra“ (STA).

³ Then the 4 basis matrices \(\{\tau_\mu\}\) introduced below, or equivalently the 16 coefficients \(a_\nu^\mu\), which play the role of tetrades, have to be used instead the metric tensor as field variables. More detailed discussions of this can be found again in [11] and [9].
Since this is a one-to-one map, it is clear that all relations written in one form can also be transcribed into the other, and in principle no form can be given preference.

However, the crucial difference is, that one has to put a postulated metric tensor on top of the vector space, to define a vector norm and get covariant equations there (this is the definition of a tensor space). As shown below, for the matrix representation the existence and form of this tensor is a consequence of the algebraic structure.

For binary matrices holds \[ x = x^\mu \tau^\mu \] and consequently the matrix determinant naturally defines a quadratic norm in \((x^\mu)\). This norm can now be identified with the norm of the vector space. This is only for \(2 \times 2\) matrices possible, and vector dimensions greater than four are excluded.

The metric tensor \( g = (g_{\mu\nu}) \) is then given by:

\[
|x| = x^\mu x^{\nu} \frac{1}{2} T(\tau^\mu \tau^\nu) = x^\mu x^{\nu} g_{\mu\nu}.
\]  

(2)

Obviously, symmetry follows \( g_{\mu\nu} = g_{\nu\mu} \) and all are real numbers, as required.

On the other hand, the four matrices \( \tau^\mu \) (like every hermitean matrix) can be expressed as linear combinations of the 3 Pauli-matrices \( \sigma_1, \sigma_2, \sigma_3 \) and a fourth matrix \( \sigma_0 \) follows, that all possible metric tensors are transformations of \( g^{(0)} \) and locally this metric can always be chosen. If the restriction of metric invariance \( (g = g^{(0)}) \) is made, then the \( \{a^\mu_\nu\} \) are identical to the Lorentz-group.

Consequently for simplification, the set of Pauli-matrices \( \sigma_\mu \) is used in the following as basis. In this case the components can be simply recovered from the matrix form \( x = x^\mu \sigma_\mu \) by

\[
x^\mu = \frac{1}{2} T(x \sigma_\mu) \quad \iff \quad x = x^\mu \sigma_\mu.
\]  

(5)

Explicitly it has the simple form \( x = \begin{pmatrix} t + z, & x - iy, \\ x + iy, & t - z \end{pmatrix} \).

Because the matrix algebra includes addition and subtraction operations, also trivially the symmetry under spacetime translations holds, i.e. it shows the complete Poincare group symmetry.

3 Transformations and Covariant Forms

A Lorentz-transformation is represented here by an unimodular \( 2 \times 2 \) matrix \( T \in SL(2, \mathbb{C}), |T| = 1 \) and a Minkowski-matrix transforms with \( \bar{T} \):

\[
x \to x' = T x T^\dagger.
\]  

(6)

4 The “bar” operation \( \tau \to \bar{\tau} \) stands for matrix adjunction and \( |\tau| \) for the determinant of the matrix \( \tau \), i.e. \( |\tau| \tau^{-1} = \bar{\tau} \) holds. \( T(\tau) \) here denotes the scalar trace of \( \tau \), and from \( \tau \bar{\tau} = \bar{\tau} \tau = |\tau| I \) follows \( |\tau| = \frac{1}{2} T(\bar{\tau}) \).

5 with the usual representation \( \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) one easily checks for all pairs \( \mu, \nu = 0, \ldots, 3 \) the orthogonality relation: \( \sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu = 2 g^{(0)}_{\mu\nu} \).

6 \( T^\dagger \) denoting the conjugate transpose (or hermite conjugate) of \( T \).
which obviously preserves the hermitecity and the MINKOWSKI-invariant $|x|$. It has of course 6 free real
(3 complex) parameters.

The general product $AB$ (obviously $AB$ is not covariant under proper LT) of any two MINKOWSKI-
matrices $A, B$ is then apparently a covariant matrix, because it transforms with:

$$AB \rightarrow (TA^T)(T^B^T) = T(AB)^T. \quad (7)$$

The general scalar-product is the invariant expression, which is evidently always real:

$$\frac{1}{2}T(AB) = A_\mu B^\mu. \quad (8)$$

Space rotations, as important special case, are the subgroup of matrices, obeying $T^\dag = T(\equiv T^{-1})$. They also preserve the trace, which represents the time component $x^0 = \frac{1}{2}T(x)\{\}$.

Another important transformation, which cannot be represented with any matrix $T$ of this group, is spatial inversion $\mathcal{P}$. It is obviously described by

$$x \rightarrow x_{sp} = x. \quad (9)$$

It is remarkable, that $\mathcal{P}$ is closely connected to the matrix multiplication order, since a general covariant equation of the form $AB = C$ transforms to $BA = C_{sp}$.

## 4 Relativistic Electromagnetics

### 4.1 Maxwell-Equations

At first I will shortly list Maxwell's equations in component notation. Contemporary textbooks usually start the derivation of relativistic electrodynamics with the 4-vector potential $A^\mu$, where $A^0 = V$ is the electric and $A = (A^1, A^2, A^3)$ the magnetic potential.

The antisymmetric field strength tensor $F_{\mu\nu}$ is then derived from $A_\mu$ with the ansatz $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}$.

It is composed from electric and magnetic field vectors $\vec{E}, \vec{B}$:

$$F_{01} = E_1, \ldots \quad \text{and} \quad F_{12} = B_3, \ldots.$$ 

Then the 4 inhomogeneous Maxwell-eqs. are (with $J_\mu$ as 4-vector of current, see e.g. [1], p. 42)

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = J_\mu.$$ 

The 4 homogeneous eqs. however, can be derived from the above potential ansatz for $F$

$$\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\sigma\nu}}{\partial x_\mu} + \frac{\partial F_{\mu\sigma}}{\partial x_\nu} = 0.$$ 

In matrix form the vector potential is obviously represented by the MINKOWSKI-matrix $A = A^\mu\sigma_\mu$, using the general mapping formula (5).

The field strength matrix $F$ is here derived from a general covariant product, similar to (1), with the partial derivation operator $\partial$ in the form $\partial A$. $\partial$ is according to (5) a hermitean (MINKOWSKI-) matrix with the explicit form

$$\partial \overset{\text{def}}{=} \sigma_\mu \partial^\mu = \sigma_\mu \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_0} + \sigma_1 \frac{\partial}{\partial x_1} + \cdots = \frac{\partial}{\partial t} + \nabla. \quad (10)$$

---

7 This group is homomorph to the restricted Lorentz-group and the homomorphism possesses the kernel $T \in \{I, -I\}$ (see e.g. [1], pp. 16).
8 Above product matrix [1], can be decomposed into two covariant expressions: a scalar commutator (which is this scalar-product) and a traceless anti-commutator $AB = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA)$.
9 Since $T$ is then a similarity transformation, $T^T T^{-1}$, it is also clear that both eigenvalues of $x$ are invariant.
10 since $\sigma_0 = \sigma_3$ and $\sigma_1 = -\sigma_1, \ldots$
We use the anticommutator of this form to get a traceless matrix $F$, $(F + \bar{F} = 0)$:

$$F \overset{\text{def}}{=} \frac{1}{2}(\partial A - \bar{A} \partial).$$

It is then easy to show, that $F$ (it has 3 complex = 6 real components) combines the field vectors, here both as traceless, hermitean matrices $E = E^k \sigma_k$, $(k = 1, \ldots, 3)$ and $B = B^k \sigma_k$:

$$F = E + iB.$$  \hfill (11)

Now Maxwell’s equations are represented by only \underline{one matrix equation}, which includes either homog. and inhomog. eqs.:

$$\partial F = J.$$  \hfill (12)

Proof: The l.h.s. of eq. (12) can be decomposed into an hermitean and anti-hermitean term (vanishing, since $J$ is hermitean), which are \underline{both} Maxwell eqs.

$$\partial F = (\partial / \partial t + \nabla)(E + iB) = \dot{E} + \nabla \cdot E + i\nabla \times B + \nabla \times E + i\dot{B} + i\nabla \cdot B \Rightarrow$$

$q.e.d.$

The \underline{Lorentz-covariance} of (12) is guaranteed, when the following transformation rule for $F$ is assumed:

$$F \rightarrow F' = \bar{T}^\dagger FT^\dagger.$$  \hfill (13)

For checking the mirror-invariance of (12) one must realize, that $E, B$ transform as proper- and pseudo-vectors, resp. under spatial inversion: $E_{sp} = \bar{E} = -E$ and $B_{sp} = -\bar{B} = +B$. Thus $F_{sp} = -F^\dagger = \bar{F}^\dagger$ holds and consequently (12) is mirror-invariant.$^{13}$

Discussion: The main difference between matrix- and component form is, that the homogeneous eqs. are not needed as independent assumptions. They are fulfilled in any case, regardless of the potential ansatz.

4.2 Lorentz-Force

In conventional component form the Lorentz-force is $K_\mu = F_{\mu\nu}J^\nu$. Here one has the matrix form, which obviously gives a hermitean force matrix $K$:

$$K = \frac{1}{2}(JF + F^\dagger J).$$  \hfill (14)

Of course, it is Lorentz-covariant and mirror-invariant.

4.3 Energy-Momentum-Tensor of Electromagnetic Field

Although it is not strictly necessary for the main thesis of this paper, I included this chapter, because it shows quite impressively the power of the matrix formalism.$^{14}$

Inserting the Maxwell eq. (12) into the Lorentz-force (14) immediately gives:

$$K = \frac{1}{2}((F^\dagger \partial)F + F^\dagger (\partial F)) = \frac{1}{2}F^\dagger \partial F = \frac{\partial}{\partial x_\mu} \frac{1}{2}(F^\dagger \sigma_\mu F) = \partial T_{\mu}^\mu.$$  \hfill (15)

---

$^{11}$ This matrix eq. consists of 4 complex, i.e. 8 real eqs. In chapter \& this is shortly sketched, how it can be derived from a LAGRANGIAN.

$^{12}$ Like necessary, for space rotations $T = T^\dagger$ then $E, B$ transform independently as 3-vectors, but for proper LT, they get mixed.

$^{13}$ From $\partial_{sp}F_{sp} = J_{sp} \rightarrow \partial \bar{F}^\dagger = \bar{J}$, and after bar-operation and herm. conj. one gets the original eq. again, q.e.d.

$^{14}$ Of course, a general tensor with 16 real components, or a symmetric tensor with 10, cannot be represented by a single 2x2-matrix, but only by a set of matrices.

$^{15}$ the parentheses in the first terms denote the differential-operands of $\partial$, while in the underlined term it operates both to the left and right.
This derivation, consisting only of two simple reorderings, is significantly more concise than the corresponding component form (see e.g. [1], p. 50). Obviously the four hermitean matrices $T_\mu$ (with 16 real components) here represent the energy-momentum tensor. To get the corresponding component form, one uses the general mapping formula (5), which here leads to the 16 real components:

$$T_\nu^\mu = \frac{1}{2} T^\sigma (T_\mu \sigma_\nu)$$

Then with the following explicit formula the symmetry of $T_{\mu\nu} = T_{\nu\mu}$ can be easily shown, with usual formulas for the trace:

$$T_{\mu\nu} = \frac{1}{2} T^\sigma (T_\mu \sigma_\nu) = \frac{1}{4} T^\sigma (F^\dagger \sigma_\mu F \sigma_\nu).$$

(16)

5 Relativistic Quantum Mechanics

In this section I will show, that relativistic quantum mechanics can be readily expressed with $2 \times 2$ matrices (which is well-known for a great part), but useless degrees of freedom are significantly suppressed.

This is an especially important case, since in our contemporary understanding, quantum mechanics and esp. Dirac’s eq. (with its various generalizations) is the fundament of the physical world. On the other hand, this theory is surely not yet finished, and it is to expect that new insights will evolve in the future, possibly within the framework of the matrix formalism.

Here closes the circle: the matrix formulation was first introduced for the description of quantum mechanical spin and can now hopefully lead to a better understanding of physics.

5.1 Dirac-Equation

In most modern textbooks Dirac’s eq. is presented in the conventional component notation, with the four Clifford matrices $\gamma_\mu$ (and $\partial^\mu \equiv \frac{\partial}{\partial x^\mu}$) for the 4-spinor wave function as column vector $\psi = (\psi_1, \ldots, \psi_4)^T$: (see [3], p. 50, [8] pp. 110)

$$i\gamma^\mu \partial_\mu \psi = m\psi.$$  

(17)

This is a mathematically very elegant form, but it is achieved at the price of loss of physical reality of $\psi$. It exposes a great amount of ambiguity, since it is obviously invariant under the so called similarity transformation [17] (see e.g. [3], p. 55):

$$\gamma_\mu \rightarrow U \gamma_\mu U^{-1} \quad \text{and} \quad \psi \rightarrow U \psi.$$  

Here $U$ is an arbitrary $4 \times 4$ matrix, containing 16 free complex parameters. This means, the formula (17) allows a linear transformation, leading to different representations, with 16 complex parameters without any change of the physical meaning. The components of $\psi$ thus cannot represent any physical entities directly. In my opinion, this is a great disadvantage of this formula.

For the derivation of the matrix form, I start with the Weyl-representation of the $\gamma_\mu$

$$\gamma_0 = \begin{pmatrix} 0, & -I_2 \\ -I_2, & 0 \end{pmatrix}, \quad \text{and} \quad \gamma_k = \begin{pmatrix} 0, & \sigma_k \\ -\sigma_k, & 0 \end{pmatrix}, \quad k = 1, 2, 3.$$  

(18)

This form has the important special feature, that here the 4-spinor can be decomposed into two 2-spinors $\Psi, \Phi$: $\psi = (\Psi, \Phi)^T$, which transform independently under Lorentz-transformations (see below), and (17) reads with them:

$$i\partial \Phi = -m \Psi \quad \text{and} \quad i\partial \Psi = -m \Phi.$$  

(19)

An additional, external electromagnetic vector potential field $A$ is as usual introduced by the substitution $\partial \rightarrow \partial - ieA$:

$$(i\partial + eA)\Phi = -m \Psi \quad \text{and} \quad (i\partial + eA)\Psi = -m \Phi.$$  

(20)
This bi-spinor form of Dirac’s eq. is well-known (although in most cases given in slightly different notation, see e.g. [8], p. 70) and sometimes referred to as ”zigzag” model of the electron (e.g. [9]). From (20) the Lorentz-transformation rules for the 2-spinors can be derived as:

\[ \Psi \rightarrow T^T \Psi \quad \text{and} \quad \Phi \rightarrow \bar{T}^T \Phi, \]

leading to obviously covariant eqs. (20). Under spatial inversions both eqs. and consequently the spinors are interchanging: \( \Psi \leftrightarrow \Phi \).

### 5.2 Matrix-Dirac-Equation

It is not yet commonly known, however, that both parts of (20) can be combined in one single matrix equation. This representation must be considered as the natural form of Dirac’s eq. in the MST context, and it opens up new possibilities for its generalization.

To develop this matrix eq., the second equation of (20) is converted in the following manner. With \( M \overset{\text{def}}{=} i\partial + eA \) it reads

\[ \bar{M} \Psi = -m \Phi. \]

Now one uses the general identity for every 2x2 matrix \( M \):

\[ \bar{M} = \rho M^T \rho, \quad \text{with} \quad \rho \overset{\text{def}}{=} \begin{pmatrix} 0,1 \\ -1,0 \end{pmatrix} \]

The auxiliary matrix \( \rho \) is sometimes denoted as ”spinor metric”, because it defines an invariant spinor determinant (see chapter 5.5). Inserting the above identity leads to

\[ M^T \rho \Psi = -m \rho \Phi. \]

Of this one takes the complex conjugate, using \( (M^T)^* = M^\dagger = -i\partial + eA \):

\[ (-i\partial + eA) \rho \Psi^* = -m \rho \Phi^*. \]

Here it is obviously useful to define a new ”tilde-operator” for 2-spinors: \( \tilde{\Psi} \overset{\text{def}}{=} \rho \Psi^* \) and the last eq. then writes \( (-i\partial + eA) \tilde{\Psi} = -m \rho \Phi \).

Then it is possible to combine this equation and the first of (20) as 2 columns into one 2x2 matrix equation:

\[ eA(\Phi, \tilde{\Psi}) + i\partial(\Phi, -\tilde{\Psi}) = -m(\Psi, \tilde{\Phi}). \]

Now one defines the “spinor-matrix” \( P \overset{\text{def}}{=} (\Phi, \tilde{\Psi}) \) (which is the replacement of the 4-spinor \( \psi \)) and notes \( \tilde{P}^\dagger = -(\Psi, \tilde{\Phi}) \), and with the constant matrix \( S \overset{\text{def}}{=} \begin{pmatrix} i,0 \\ 0,-i \end{pmatrix} \) finally gets:

\[ eAP + \partial PS = m\tilde{P}^\dagger. \]

Although this formula at a first glance looks somewhat uncommon, esp. the right-side factor \( S \) in the derivation term, it possesses all features and solutions of the original 4-spinor equation (17). The 2x2-matrix \( S \) together with the operator on the r.h.s here ”magically absorb” all 4 Clifford matrices \( \gamma_\mu \). It should be clear from the above, that the special form of \( S \) (\( S = i\sigma_3 \)), is the consequence of the choice of \( \gamma_\mu \). A more general form shall be discussed below.

To demonstrate the novel power of this matrix eq., one can derive an equivalent bilinear form by multiplying it from left with \( P^\dagger \), resulting in

\[ eP^\dagger AP + P^\dagger (\partial P)S = m|P|^*. \]

Note, that this is still a matrix eq., although the r.h.s. is scalar (\( \sim I \)) and the l.h.s. terms are Lorentz-invariants. And it is still equivalent to (22), provided \( P \) is not singular (\( |P| \neq 0 \)).

---

18 consider again that \( \partial \) and \( A \) transform like \( \partial \rightarrow T\partial T^\dagger \)
19 A geometric explanation is, that the bar-operation as mentioned already, means spatial inversion, which is equal to the combined operation of transposing (i.e. \( y \rightarrow -y \)) and a rotation around \( y \) of 180, which is performed by the transformation \( T = \rho = i\sigma_2 \).
20 this operator obeys \( \tilde{\Psi} = -\Psi \), since \( \rho^2 = -1 \)
21 multiplication from right produces another eq. with the same r.h.s.
This direct way is only possible by using matrix algebra. By utilizing this bilinear form, esp. many computations, e.g. regarding gauge invariance, LAGRANGIAN and conservation laws can be performed much simpler.

According to above definitions, $P$ transforms consistently with $P \rightarrow T^\dagger P$ under LORENTZ-transformations and (22) is obviously covariant. Since $T^\dagger$ operates only from the left on $P$, the two column 2-spinors of $P$ transform equally and independently.

The mirror-invariance is guaranteed with $P \rightarrow P^s = P^\dagger$ (since $S = S^\dagger$).

Here also a similarity transformation is possible by right-side multiplication of $P$ with a matrix $U$ obeying $U = U^\dagger$

$$P \rightarrow PU \quad \text{and} \quad S \rightarrow U^{-1}SU,$$

but this 2x2-matrix $U$ has only 2 free complex parameters (4 real), compared to 16 above (since one of the 4 real parameters is only a constant factor, there actually remain only 3 real free parameters). Essentially this transformation says, that $S$ (like $U$) can be any matrix obeying the condition $S = S^\dagger$, which describes a subalgebra of matrices, which is isomorphic to the algebra of quaternions.

An obvious possibility to explain the remaining ambiguity physically, is discussed in chapter 5.4.

The gauge invariance of (22) and the corresponding LAGRANGIAN (20) below is a bit different to the conventional form, because $P$ cannot be multiplied with a scalar complex phase factor $e^{i\lambda}$, because the mass-term would then transform with $e^{-i\lambda}$. This impossibility to apply scalar phase factors is probably the reason, that this quite simple and obvious formula has never been considered before. Also the usual covariant replacement of the derivation operator $\partial^\mu \rightarrow D^\mu = \partial^\mu - ieA^\mu$ cannot simply be transcribed to $D = \partial - iA$, but must be modified here.

However, one easily checks, that e.g. the gauge transformation

$$P \rightarrow Pe^{i\lambda S} \quad \text{and} \quad eA \rightarrow eA + \partial \lambda,$$

where $\lambda(x)$ is an arbitrary real spacetime function, is the correct form (25). The matrix $S$ can be thus seen as replacement of the imaginary unit $i$, since it also obeys $S^2 = -1$.

Stationary states, which represent bound states in atoms, are here similarly described by the ansatz $P = P_0(e^{-e\epsilon S})$ (with $\epsilon$ as energy), which results in (since $\frac{\partial}{\partial t}P = -ePS$)

$$(\epsilon + eA)P_0 + \nabla P_0S = mP_0^\dagger,$$

and it is easy to show, that it has the the same solutions as the original DIRAC-eq.

An important special case regards massless, uncharged fermions, i.e. neutrinos. It is known from experiments, that only left-handed neutrinos exist, right-handed ones have never been observed.

DIRACs original eq., eg. written in the form (10), leaves this fact unexplained, because both parts decouple with $m = 0$, and so they have independent solutions for $\Psi$ and $\Phi$, representing both types of neutrinos.

The non-existence of right-handed neutrinos is a direct consequence of the matrix eq. (22), however. With $m = 0, e = 0$ it simplifies to $\partial PS = 0$ (for this singular case, (22) is no longer equivalent to the original DIRAC-eq.). Here the factor $S$ can be eliminated (by rhs multiplication with $S^{-1}$) giving $\partial P = 0$. All their solutions have left-handed chirality, which is easy to show by transforming it into momentum space.

In close connection to this, also weak interactions in the V-A-theory are most simply expressed in this form (22). It follows from the fact, that for the used Weyl-representation the matrix $\gamma_5 \gamma_0 \gamma_1 \gamma_2 \gamma_3$ is a diagonal matrix: $\gamma_5 = (I_4 - I_4^s)$. And since weak interaction couples in the 4-spinor form with $I_4 \pm \gamma_5$, so always in one of the eq-pair (20) the respective term vanishes.

Further considerations, regarding electro-weak gauge theory are done in chapter 5.4.

---

22 left-side multiplication always describes a LORENTZ-transformation
23 Note that $e^{i\lambda S}$ commutes with $S$ and $\partial e^{i\lambda S} = (\partial \lambda)e^{i\lambda S}$. 
5.3 LAGRANGIAN of Coupled Dirac- and EM-Field

LAGRANGIANS play a very important role in modern field theory. They can readily be written in matrix form using the above entities. For the combined Dirac- and em-field it is the sum of four scalar terms:

\[ \mathcal{L} = \mathcal{T}(P^1(\partial P)S) + e\mathcal{T}(AP^P) - 2\Re|F| - 2m\Re|P|. \]  

This form demonstrates another advantage of the matrix representation. It can reveal subtle similarities between some terms (here e.g. the 3. and 4. term), which are hidden in the component form.

The validity of (25) can be proved by transforming it into component form, or better by deriving the field eqs., namely \( (12) \) and (22) from it. This complete derivation must be omitted here, only some basic steps should be stated.

In the first (differential) term, the partial operator should only operate to the right (as indicated by the parentheses). Furthermore one notices, that this term is not real (as normally required for a LAGRANGIAN and is the case for the other three terms). However, the actually relevant spacetime integral \textit{is real}:

\[ I = \int d^4x \mathcal{T}(P^1(\partial P)S) = \text{real}, \quad \text{i.e.} \quad \Im(I) = 0, \]

which is proved with the vanishing of the integral \( \int \mathcal{T}((P^1\partial P)S) = \int \partial_\mu \mathcal{T}(P^1\sigma_\mu PS) = 0 \) by Gauss’ law and partial integration.

The third term is the well-known LAGRANGIAN of the electromagnetic field \( \mathcal{L}_{em} = -2\Re|F| = E^2 - B^2 \), since \( F = \frac{1}{2}(\partial A - A \partial) = E + iB \).

Consequently, the variation of \( A \) in the second and third term leads to MAXWELLS eq. (12), if the 4-current of the DIRAC-field is defined as

\[ J_\mu \overset{\text{def}}{=} eP^\mu \overset{(27)}{=} (eP^\mu \hat{P}). \]

Variation of \( P^\mu \) (or independently \( P \)) in the terms 1, 2 and 4 leads to DIRACS-eq. (22).

5.4 Yang-Mills Gauge-Fields in Matrix Form

The remaining possibility of similarity transformations \( P \rightarrow PU \) \((U = U^\dagger)\) leads to an obvious generalization of the DIRAC eq. in matrix form (22) with four vector fields \( B^\mu, \mu = 0, \ldots, 3 \):

\[ \partial PS + B^\mu P\sigma_\mu = m\hat{P}. \]  

This also resolves the remaining ambiguities of \( P \). Here \( B^0 = A \) is apparently again the em. vector potential, which is invariant under this transformation. The other three fields \( B^1, B^2, B^3 \) mix however (under Lorentz-trafos they act as normal MMs, like \( A \)), since with \( U\sigma_\mu U^{-1} = \delta^a_\mu \sigma_m \), \((m, k = 1, 2, 3)\) follows \( B^k \rightarrow \delta_m^k B^m \). With the additional restriction \( |U| = 1 \) this is the \( SU(2) \) group and its \( SO(3) \) representation acts on the \( B^k \).

One can now formulate the interesting hypothesis, that by introducing a local non-abelian gauge field \( U(x) \) the unified electro-weak field may be represented, similar to Yang-Mills theory (see e.g. [2]).

It is striking, that this gauge field shows remarkable similarities to the symmetry \( SU(2) \times U(1) \) as proposed by Weinberg and Salam for the unified theory, although it is evidently not equivalent [24]. To get all 4 gauge fields, it is obviously necessary to use the infinitesimal generator of the complete quaternionic algebra \( \mathcal{B} \) for \( U \) instead of the subset \( SU(2) \), which is:

\[ U(x) = e^{i\lambda^0 \sigma_0 + i\lambda^k \sigma_k} \approx I + \lambda^0 \sigma_0 + i\lambda^k \sigma_k, \quad k = 1, 2, 3, \]

with 4 real spacetime functions \( \lambda^\mu(x) \), \( (|\lambda^\mu| \ll 1) \).

Then \( \lambda^\mu(x) \) represents the em. gauge field, coupled with \( A \equiv B^0 \) (if \( S = i\sigma_3 \), as before [25] explained).

\[ \text{See again chapter 5.1; the only required condition for } U \text{ is actually } \hat{U} = U^\dagger, \text{ which is fulfilled by this algebra.} \]

Remember, that \( \sigma_0 = I \), so one gets \( \hat{U} = U^\dagger = e^{i\lambda^0 - i\lambda^k \sigma_k} \).
The gauge fields \( \lambda^{1,2}(\mathbf{x}) \) couple with \( \pm B^{2,1} \), respectively. The gauge field \( \lambda^0(\mathbf{x}) \) obviously represents a boost (since \( |U| \neq 1 \)) and couples with \( B^3 \), very different to the standard theory.

In conclusion should be emphasized the remarkable fact, that the gauge symmetry here is an intrinsic feature of Dirac’s eq. in matrix form and its group structure is automatically determined. Moreover, gauge- and Lorentz-symmetry here turn out to be “two sides of one coin” in the general transformation formula for the spinor-matrix \( P \rightarrow TPU \).

Further discussions, regarding covariant field equations for the associated generalized em. field tensors \( F^\mu \), the complete Lagrangian and a possible Higgs mechanism for symmetry breaking, go beyond the scope of this article and shall be considered in a subsequent paper.

5.5 2-Spinors and Minkowski-Matrices

At the end, some general remarks about the relations of spinors and matrices should be added. As stated above, 2-spinors are represented by binary column matrices \( \Psi = (\alpha \beta) \), \( \Phi = (\gamma \delta) \), ..., which transform under LT as \( \Psi \rightarrow T\Psi \). Then for any pair of spinors \( P = (\Psi, \Phi) \) the determinant

\[
|P| = |\Psi, \Phi| = \Psi^T \rho \Phi = \alpha\delta - \beta\gamma
\]

is obviously a Lorentz-invariant, because \( TP = (T\Psi, T\Phi) \).

Also note the important fact, that spinor products, like e.g. the matrix

\[
\begin{align*}
H &= \Psi\Psi^\dagger = (\alpha \beta)(\alpha^*, \beta^*) \\
   &= (|\alpha|^2, \alpha \beta^*), (\beta^*, |\beta|^2)
\end{align*}
\]

is obviously a Minkowski-matrix (in this special example a null-matrix: \(|H| = 0\)). That says, that matrices can be constructed by spinors, but the opposite does not hold. Only null-matrices can be uniquely (up to a phase-factor) decomposed into spinors.

It is also a fascinating feature of forms like (29), that they have a positive definite time-component, which might help to explain the direction of time. From the realization, that 2-spinors are the algebraic basis of all, it will be possibly feasible to develop a complete theory of spacetime only with spinors.

The crucial problem is however, how to retain the spacetime translation symmetry in such constructs.

6 Conclusions

In this paper I have presented the most important concepts of Special Relativity in 2x2-matrix form, namely the entities and equations of electromagnetic interactions and the Dirac equation. Essentially this form uses another algebraic concept of spacetime, rather than the conventional vector space.

Although the equations are obviously equivalent to conventional component formulation, I have showed that the matrix form has several striking advantages, which suggest that this form should be considered as the primary description of the physical world.

The main advantages can be shortly summarized:

- The metric tensor needs not to be postulated and spacetime can have no more than four dimensions
- The Maxwell equations are represented by a single equation rather than two independent
- The Dirac spinor field in the novel Dirac eq. in matrix form has much less of degrees of freedom without any physical meaning and this form can explain the non-existence of right-handed neutrinos
- A new type of Yang-Mills gauge fields arises from the generalization of this matrix Diracs eq., which possibly can describe electro-weak interactions

26 In most textbooks a “dotted index” notation is used to describe conjugated spinors like \( \Psi^\dagger \), that goes back to the first publications on this topic. I do not adopt it here.
From a heuristic point of view, from a bunch of theories which describe the same phenomena with equal accuracy, the one with the least prerequisites should be given preference.

Another major intention of writing this paper was, to encourage other theoretical physicists, to find extensions of this concept for new theories. Also I hope to be able, to present a new concept for quantum mechanics on the basis of this algebra, which can replace the wave function by a discrete model. Some first ideas can be found in [6].

7 Acknowledgments

I want to thank Dr. Charles Francis and Dr. Peter Enders for their interest and helpful tips in preparing this paper.

References

[1] Albert Einstein, Grundzüge der Relativitätstheorie, Akademie Verlag, Berlin 1973.
[2] Dietmar Ebert, Eichtheorien (Gauge-theories), Akademie Verlag, Berlin 1989.
[3] R.P. Feynman, Quanten-Elektrodynamik, R. Oldenburg Verlag, München 1992.
[4] David Hestenes, Zitterbewegung Modeling, Foundations of Physics, 20, 365-387, (1993).
[5] E. Herlt, N. Salie, Spezielle Relativitätstheorie, Akademie Verlag, Berlin 1978.
[6] W. Köhler, New Discrete View to Quantum Mechanics, http://arxiv.org/pdf/quant-ph/0601080
[7] Karl Lanius, Physik der Elementarteilchen, Akademie Verlag, Berlin 1981.
[8] Landau, Lifschitz, Quanten-Elektrodynamik, Akademie Verlag, Berlin 1991.
[9] Roger Penrose, Wolfgang Rindler, Spinors and space-time, Vol. 1 + 2., Cambridge University Press, Cambridge 1984.
[10] B. L. van der Waerden, Spinoranalyse, Nachr. Akad. Wiss. Göttingen, 1929.
[11] Hermann Weyl, Elektron und Gravitation, Zeitschr. für Physik, 1929.