Growing Hair on the extremal $BTZ$ black hole

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ABSTRACT

We show that the nonlinear $\sigma$–model in an asymptotically $AdS_3$ space-time admits a novel local symmetry. The field action is assumed to be quartic in the nonlinear $\sigma$–model fields and minimally coupled to gravity. The local symmetry transformation simultaneously twists the nonlinear $\sigma$–model fields and changes the space-time metric, and it can be used to map the extremal $BTZ$ black hole to infinitely many hairy black hole solutions.
Asymptotically AdS black holes are of current interest, in part due to their correspondence to thermal states in the conformal theory. Asymptotically AdS black holes have been shown to admit hair. Solutions with various types of hair have been constructed, and their properties have been investigated [1]-[8]. The search for solutions for AdS black holes with scalar hair can require introducing complicated expressions for the potential energy density. In the present work we are able to generate an infinite number of asymptotically AdS black hole solutions with scalar hair. This is possible due to the presence of a hidden local symmetry, rather than some potential energy term.

The hair examined in this article is associated with the nonlinear $\sigma$-model fields. Classical solutions to the nonlinear $\sigma$-model coupled to gravity were obtained previously, and they corresponded to topological solitons.[9] No hairy black solutions were found. The Lagrangian utilized in [9] was invariant under $SO(3)$ transformations in the target space, quadratic in the $\sigma$-model fields and minimally coupled to gravity in $2 + 1$ dimensions. In the present work we replace the $\sigma$-model action in [9] by one which is quartic in the nonlinear $\sigma$-model fields (and $SO(3)$ invariant). In flat space-time there are no static solutions to this dynamical system. On the other hand, one can find solutions in flat space-time when symmetry breaking terms are added.[10] No such symmetry breaking terms are needed for this purpose after coupling to gravity, as we show in this work. Using a general ansatz for a rotating field configuration in a stationary space-time, we find that the system of Einstein equations, field equations and AdS boundary conditions determine the fields only up to an arbitrary radial function. This indicates the existence of a local (radially dependent) transformation which leaves the combined gravity-$\sigma$-model action invariant. The symmetry transformation simultaneously twists the sigma model fields and changes the space-time metric. The transformation can be used to map the extremal BTZ black hole [11] to infinitely many hairy black hole solutions. Unlike the previously found hairy black hole solutions, here the energy-momentum tensor is traceless, and the Hawking temperature is zero (just as with the case of the extremal black hole). These results are unchanged under the action of the local symmetry transformation.

The outline for this article is the following: We first introduce the nonlinear $\sigma$-model with a quartic field action minimally coupled to gravity in $2 + 1$ dimensions and with a negative cosmological constant. We then write down an ansatz for the nonlinear $\sigma$-model fields in a rotating space-time. The fields and metric tensor are shown to be undetermined by the Einstein equations, field equations and AdS boundary conditions, and this is due to the presence of a local symmetry which simultaneously changes the metric tensor and twists the nonlinear $\sigma$-model fields. For different choices of an arbitrary radial function we show that we can obtain hairy black holes, which we argue are extremal hairy black holes. We find naked singularity solutions, as well. We conclude with some final remarks.

The nonlinear $\sigma$-model fields shall be denoted by $\Phi^a$, $a = 1, 2, 3$, where $\Phi^a \Phi^a = 1$. The fields are maps to unit vectors in a three-dimensional internal space i.e., the target space is $S^2$. The action is generally taken to be invariant under $SO(3)$ rotations of the fields in the internal space, as will be assumed here. Rather than applying the standard quadratic field action coupled to gravity, as was done in [9], we assume an action for $\Phi^a$ which is quartic in the fields and their derivatives. For the full action of the nonlinear $\sigma$-model coupled to gravity with a negative cosmological constant we take

$$S = \int d^3x \sqrt{-g} \left( \frac{1}{16\pi G} (R - 2\Lambda) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \lambda (\Phi^a \Phi^a - 1) \right) + S_{GHY},$$

(1)

where $G$ is the three-dimensional version of Newton’s constant (here in dimensionless units), $\Lambda < 0$ is the cosmological constant and $\lambda$ is a Lagrange multiplier. $F_{\mu\nu}^a$ may be interpreted as field strengths for the nonlinear $\sigma$-model, which can be formulated as a gauge theory.[12, 13] They can be expressed in
terms of derivatives of $\Phi^a$ according to

$$F_{\mu\nu}^a = \epsilon^{abc} \partial_\mu \Phi^b \partial_\nu \Phi^c,$$

where $\epsilon^{abc}$ is the Levi-Civita symbol. $\Phi^a$ and $F_{\mu\nu}^a$ transform as vectors under the action of the $SO(3)$ group, $\Phi^a \rightarrow \Phi'^a = R_{ab} \Phi^b$, $F_{\mu\nu}^a \rightarrow F'_{\mu\nu} = R_{ab} F_{b\mu \nu}^a$, $R \in SO(3)$, and hence $S$ is $SO(3)$ invariant. $S_{CHY}$ is the Gibbons–Hawking–York term [13] written on the boundary at spatial infinity $r \rightarrow \infty$. We assume that the space-time is asymptotically $AdS_3$. It may be necessary to add divergent counter terms to (1) in order for the action evaluated on the space of solutions to be finite. We shall not consider them here, as they do not affect the classical dynamics.

The Einstein equations and field equations resulting from variations of the action with respect to the metric tensor $g_{\mu\nu}$ and the fields $\Phi^a$ in (1) are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\partial_\mu J^{\mu a} \propto \Phi^a,$$

respectively, where the energy-momentum tensor $T_{\mu\nu}$ and currents $J^{\mu a}$ are

$$T_{\mu\nu} = F_{\mu\rho}^a F^{\rho a}_{\nu} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^a F^{\rho\sigma a}$$

$$J^{\mu a} = \sqrt{-g} \epsilon^{abc} F_{\mu\nu}^{bc} \partial_\nu \Phi^c.$$

The field equations [4] state that $J^{\mu a}$ is conserved in directions perpendicular to $\Phi^a$ in the target space. From [4] one easily obtains the conservation of the stress-energy tensor. The assumption that the space-time is asymptotically $AdS_3$ means that the invariant interval goes to

$$ds^2 \rightarrow \Lambda r^2 dt^2 - \frac{dr^2}{\Lambda r^2} + r^2 d\phi^2, \quad as \quad r \rightarrow \infty,$$

where $\Lambda$ is again the cosmological constant. Then from [5], the energy density at large distances becomes $T_{tt} \rightarrow \frac{1}{2} r^2 (\Lambda^2 F^2_{\phi\phi} - \Lambda F^2_{r\phi} + \frac{1}{r} F^2_{t\phi})$. It is positive definite since $\Lambda < 0$. The requirement that $T_{tt}$ vanishes in the large distance limit means that all the field strengths vanish in the limit. This in turn implies that the tangent vectors $\partial_\mu \Phi^a$ are parallel or zero when $r \rightarrow \infty$. From the former possibility it follows that the $\sigma$-model fields need not point along a constant direction in the target space at the $AdS_3$ boundary. This differs from the case of the $2 + 1$ dimensional $\sigma$-model with the usual quadratic Lagrangian density $\sim \partial_\mu \Phi^a \partial^\mu \Phi^a$. In that case energy finiteness necessarily compactifies any time-slice of the space-time to $S^2$, and field configurations belong to disjoint equivalence classes associated with $\Pi_2(S^2)$. While this restriction does not follow from energy finiteness in our case, here we shall only consider field configurations $\{\Phi^a(t, r, \phi)\}$ which do point along a constant direction in the internal space at the $AdS_3$ boundary and hence do belong to disjoint equivalence classes.

We restrict to stationary metrics, allowing for a nonvanishing angular momentum. Following [9], the most general such metric can be expressed in terms of three radially dependent functions $A$, $B$ and $\Omega$ according to

$$ds^2 = -A(r) dt^2 + \frac{B(r)}{A(r)} dr^2 + r^2 \left(d\phi + \Omega(r) dt\right)^2.$$
Since the space-time is asymptotically $AdS_3$ \(^1\)

\[ A \to -\Lambda r^2 \quad B \to 1 \quad \Omega \to 0, \quad \text{as } r \to \infty. \quad (9) \]

For $\Omega \neq 0$ the space-time has a nonvanishing angular momentum, i.e., it is rotating. In order to obtain nontrivial solutions to \((\ref{eq:3})\) and \((\ref{eq:4})\) it is necessary to assume that the nonlinear $\sigma$-model fields rotate as well in the target space. Following \([\ref{eq:13}]\) rotational invariance for the nonlinear $\sigma$-model fields means that $\partial_\phi \Phi^a + \epsilon^{ab} \Phi^b = 0$. The condition for rotating fields is $\partial_\phi \Phi^a = -\omega \partial_\phi \Phi^a$, where $\omega$ is the angular velocity. The solution can be written in terms of one radially dependent function $\chi$,

\[
\begin{align*}
\begin{pmatrix}
\Phi^1 \\
\Phi^2 \\
\Phi^3
\end{pmatrix} &= \begin{pmatrix}
\sin \chi(r) \cos (\phi - \omega t) \\
\sin \chi(r) \sin (\phi - \omega t) \\
\cos \chi(r)
\end{pmatrix},
\end{align*}
\quad (10)
\]

The tangent vector $\partial_\phi \Phi^a$ is orthogonal to $\partial_t \Phi^a$ and $\partial_\phi \Phi^a$, and so from the previous energy considerations it should vanish at the $AdS_3$ boundary. The field strengths \((\ref{eq:2})\) resulting from the ansatz are $F^a_{\mu\nu} = f_{\mu\nu} \Phi^a$, where $f_{\mu\nu}$ are independent of $t$, $f_{rt} = -\omega f_{\phi t} = \omega \partial_r \cos \chi$ and $f_{t\phi} = 0$, and in this sense the field configurations \((\ref{eq:10})\) are stationary. Using \((\ref{eq:13})\) and \((\ref{eq:9})\) the asymptotic behavior of the energy density for these configurations is $T_{tt} \to \frac{1}{2} \Lambda r^2 (\Lambda - \omega^2) (\partial_r \cos \chi)^2$ as $r \to \infty$. The requirement of finite energy means that $\cos \chi$ goes asymptotically to a constant. Then in fact all tangent vectors $\partial_\mu \Phi^a$ vanish at the $AdS_3$ boundary. We choose $\chi \to 0$ as $r \to \infty$. In order for $\Phi^a$ to be well-defined at the origin, $\chi(0)$ needs to be an integer multiple of $\pi$, and so field configurations \((\ref{eq:10})\) belong to $\Pi_2(S^2)$. Such configurations can yield an everywhere well-defined energy density $T_{tt}$ even in the presence of a space-time singularity.

Upon substituting \((\ref{eq:8})\) and \((\ref{eq:10})\) into the action (including the Gibbons–Hawking–York term) one gets

\[
S = \frac{\pi}{\kappa} \int dtdr \left\{ \partial_t A + \frac{r^3 (\partial_r \Omega)^2}{2} + \frac{2rB}{\ell^2} - \kappa \ell^2 r \left( \frac{A}{r^2} - (\omega + \Omega)^2 (\partial_r \chi)^2 \sin^2 \chi \right) \right\}, \quad (11)
\]

where $\kappa \ell^2 = 8\pi G$, and we set $\Lambda = -\frac{1}{\ell^2}$. It is convenient to introduce the dimensionless radial variable $x = r/\ell$. Then the action becomes

\[
S = \frac{\pi}{\kappa} \int \frac{dtdx}{\sqrt{B}} \left\{ A' + \frac{x^3 \tilde{\Omega}^2}{2} + 2xB - \kappa xH \chi'^2 \sin^2 \chi \right\}, \quad (12)
\]

where

\[
H = \frac{A}{x^2} - (\tilde{\omega} + \tilde{\Omega})^2, \quad (13)
\]

$\tilde{\Omega} = \ell \Omega$ and $\tilde{\omega} = \ell \omega$. The prime denotes a derivative with respect to $x$. The Einstein equations resulting from extremizing the action with respect to variations in $A$, $B$, $\tilde{\Omega}$ are

\[
\frac{x}{2} (\ln B)' = \kappa \chi'^2 \sin^2 \chi \quad (14)
\]

\[
A' = 2xB - \frac{x^3 \tilde{\Omega}^2}{2} + \kappa xH \chi'^2 \sin^2 \chi \quad (15)
\]

\[
\left( \frac{x^3 \tilde{\Omega}}{\sqrt{B}} \right)' = \frac{2\kappa x}{\sqrt{B}} (\tilde{\Omega} + \tilde{\omega}) \chi'^2 \sin^2 \chi \quad (16)
\]

\(^1\)More generally, $\tilde{\Omega}$ can tend to a nonzero constant. However, the constant vanishes if we transform to the co-rotating frame.
respectively, and the field equation resulting from variations with respect to $\chi$ is

$$\left(\frac{xH}{\sqrt{B}} \chi' \sin \chi\right)' = 0.$$  \hfill (17)

These equations can be obtained directly from (3) and (4). (17) is a consequence of the conservation of $J^{\mu a}$ in directions normal to $\Phi^a$. It means that $\frac{xH}{\sqrt{B}} \chi' \sin \chi$ equals a constant, $C$. From the asymptotic conditions (9), $(\cos \chi)' \to -\frac{C}{x}$ as $x \to \infty$. Since $\cos \chi$ cannot be logarithmically divergent in the limit, the constant $C$ must be zero. For nontrivial matter fields, this further means that $H$ must vanish for all $x$, and so

$$A = x^2(\tilde{\Omega} + \tilde{\omega})^2.$$  \hfill (18)

$A$ is then a positive function, and furthermore (17) cannot be used to determine the matter degree of freedom $\chi$. The angular velocity $\tilde{\omega}$ appearing in the ansatz for the nonlinear $\sigma$-model fields is not arbitrary. In order for (18) to be consistent with the $AdS_3$ limit at spatial infinity (9), we need that $\tilde{\omega}^2 = 1$. We choose $\tilde{\omega} = 1$. Using (18), the remaining equations (14-16) reduce to just two independent equations

$$\frac{x}{2}(\ln B)' = \kappa \chi'^2 \sin^2 \chi$$  \hfill (19)

$$\left(x^2(\tilde{\Omega} + 1)\right)' = 2x\sqrt{B}.$$  \hfill (20)

There are only three relations, (18)-(20) amongst the four functions $A$, $B$, $\tilde{\Omega}$ and $\chi$, and so solutions are parametrized by an arbitrary function. We can choose, for example, the arbitrary function to be the matter field $\chi$, and then use it to determine the space-time metric. Alternatively, the space-time metric can be expressed in terms of an arbitrary function, say $\tilde{\Omega}$, from which one can then determine the matter field.

The results can be compactly summarized after making the coordinate change to a frame which is co-rotating with the nonlinear $\sigma$-model fields: $(t, x, \phi) \to (t, y, \xi)$, where $y = x^2(\tilde{\Omega} + 1)$ and $\xi = \phi - t$. While $x$ is defined on the half-line, the range of $y$ depends on the function $\tilde{\Omega}(x)$. [From (20), the coordinate change is valid globally provided that $B$ does not vanish at some $x \neq 0$.] Using $\tilde{\omega} = 1$, the ansatz (10) for the $\sigma$-model fields depends only on $y$ and $\xi$, $\Phi^a = \Phi^a(y, \xi)$. In the $(t, y, \xi)$ coordinates, the metric tensor is given by

$$ds^2 = \frac{dy^2}{4y^2} + x(y)^2 d\xi^2 + 2y d\xi dt,$$  \hfill (21)

where here $x$ is regarded as a function of $y$. The $AdS_3$ vacuum has $x(y)^2 = y$ and $\chi(y) = \text{constant}$. More generally, $x(y)^2$ is related to $\chi(y)$ by

$$\frac{d^2}{dy^2} x(y)^2 + 2\kappa \left(\frac{d}{dy} \cos \chi(y)\right)^2 = 0,$$  \hfill (22)

which is the only condition that results from the Einstein equations. The only nonvanishing components of the field strength in these coordinates are $F^a_{\xi y} = (\frac{d}{dy} \cos \chi) \Phi^a$, and the only nonvanishing components of the current $J^{\mu a}$ are $J^{\xi a} = -2y \partial_\xi \Phi^a$. Since this current is identically conserved, no conditions on $\chi(y)$ result from the field equations (6).

The presence of infinitely many solutions shows the existence of a continuous symmetry of the action. This is evident from the result that the integrand of the action evaluated on the space of solutions is
independent of $x(y)^2$ or $\chi(y)$. In fact it is a constant\footnote{The action evaluated on the space of solutions can be made to vanish by adding a divergent term to \cite{1}.} The infinitesimal version of the symmetry transformation is

$$x(y)^2 \rightarrow x(y)^2 + \delta x(y)^2, \quad \chi(y) \rightarrow \chi(y) + \delta \chi(y), \quad (23)$$

where $\frac{d^2}{dy^2} \delta x(y)^2 = -4\kappa(\frac{d}{dy} \cos \chi(y)) \frac{d}{dy} (\delta \cos \chi(y))$. It simultaneously changes the $g_{\xi \xi}$ component of the metric, and twists the nonlinear $\sigma$-model fields, $\Phi(y, \xi) \rightarrow \Phi(y, \xi) + \delta \Phi(y, \xi)$, where

$$\delta \Phi(y, \xi) = R_3(\xi) T_2 R_3(\xi)^{-1} \Phi(y, \xi) \delta \chi(y), \quad (24)$$

$$R_3(\xi) = \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (25)$$

The system may admit an event horizon. $y = \text{constant}$ hypersurfaces are space-like or null. This is because the normal one-form $dy$ has norm equal to $g^{yy} = 4y^2 \geq 0$. So a $y = 0$ slice, if it exists, is everywhere null and thus an event horizon. $K = \partial_t$ [or equivalently, $K = \partial_t + \partial_\phi$ in $(t, x, \phi)$ coordinates] is a null Killing vector on the event horizon, meaning that it is also a Killing horizon. In fact, $K = \partial_t$ is null on the entire space-time manifold. Furthermore, $K^\mu \nabla_\mu K^\nu$ is identically zero implying that a Killing horizon has zero surface gravity. From this one concludes that an event horizon would have zero Hawking temperature. This result holds for all possible matter contributions consistent with (10), including no matter contribution. The result is thus invariant under the local symmetry transformation \cite{23}. For the case of no matter contribution, the system reduces to the extremal BTZ black hole (or the $AdS_3$ vacuum), which we show next.

The simplest postulate for the nonlinear $\sigma$-model fields is $\Phi^a = \delta^{a3}$, or $\chi = 0$, yielding a vanishing energy-momentum tensor. Then from \cite{22} and the $AdS_3$ boundary condition one gets that $x(y)^2 = y + M_\infty/2$, where $M_\infty$ is an integration constant. The resulting range for $y$ is: $-M_\infty/2 \leq y < \infty$. While $M_\infty = 0$ gives the $AdS_3$ vacuum, the $M_\infty \neq 0$ solution describes an extremal BTZ black hole. This is obvious when re-expressed in the $(t, r, \phi)$ coordinates. The functions $A$, $B$ and $\tilde{\Omega}$ for the solution are

$$A = x^2 - M_\infty + \frac{M_\infty^2}{4x^2}; \quad B = 1; \quad \tilde{\Omega} = \frac{-M_\infty}{2x^2}, \quad (26)$$

which agrees with the standard expression for a BTZ black hole \cite{11}. Because $M_\infty$ corresponds to both the mass and the angular momentum, the black hole is extremal, with a single horizon at $x = x_0 = \sqrt{M_\infty/2}$.

More generally, $\chi$ is an arbitrary function satisfying $\chi \rightarrow 0$ at spatial infinity. For the metric tensor one can set $x(y)^2 = y + M(y)/2$ and impose that $M(y) \rightarrow M_\infty$ as $y \rightarrow \infty$. When $M(y)$ is not a constant, the space-time is not locally $AdS_3$, meaning that the Riemann tensor $R_{\mu \nu \rho \sigma}$ does not reduce to $\Lambda (g_{\mu \nu} g_{\rho \sigma} - g_{\mu \rho} g_{\nu \sigma})$. In the $(t, y, \xi)$ coordinates, the only contribution to the energy-momentum tensor is $T_{\xi \xi} = \frac{y}{\kappa} \frac{d^2}{dy^2} M(y)$, using \cite{22}. It follows that the trace of $T$ is zero (more generally, $\text{tr} T^n = 0$, for any positive integer $n$).

As stated above, $\chi$ at $x = 0$ is an integer multiple of $\pi$ for our field configurations \cite{10}, the integer being the winding number. (From previous arguments, energy finiteness does not require that the winding number be conserved.) Below we examine a winding number one field configuration. It is easiest to express it in $(t, r, \phi)$ coordinates. Take

$$\chi(x) = 2 \tan^{-1} \frac{a}{x}, \quad (27)$$

where $\kappa$ is an arbitrary function satisfying $\kappa \rightarrow 0$ at spatial infinity. For the metric tensor one can set $x(y)^2 = y + M(y)/2$ and impose that $M(y) \rightarrow M_\infty$ as $y \rightarrow \infty$. When $M(y)$ is not a constant, the space-time is not locally $AdS_3$, meaning that the Riemann tensor $R_{\mu \nu \rho \sigma}$ does not reduce to $\Lambda (g_{\mu \nu} g_{\rho \sigma} - g_{\mu \rho} g_{\nu \sigma})$. In the $(t, y, \xi)$ coordinates, the only contribution to the energy-momentum tensor is $T_{\xi \xi} = \frac{y}{\kappa} \frac{d^2}{dy^2} M(y)$, using \cite{22}. It follows that the trace of $T$ is zero (more generally, $\text{tr} T^n = 0$, for any positive integer $n$).

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The action evaluated on the space of solutions can be made to vanish by adding a divergent term to \cite{1}.\footnote{The action evaluated on the space of solutions can be made to vanish by adding a divergent term to \cite{1}.}
where \( a \) is a scale factor. Then \([\Phi^a]^{-1}\) defines the stereographic projection of the unit sphere to the plane. From (18)-(20) it follows that

\[
B = e^{-2\rho}, \quad \rho = \frac{8}{3} \frac{\kappa a^4}{(x^2 + a^2)^3}
\]

\[
y = x^2(\hat{\Omega} + 1) = \frac{1}{3} (x^2 + a^2) E_4(\rho) + C , \tag{28}
\]

where here \( E_n \) denotes the exponential integral \( E_n(\rho) = \int_1^\infty dt \frac{t^n e^{-\rho t}}{t} \), and \( C \) is an integration constant. The result (28) follows from the identity \((n-1)E_n(\rho) = e^{-\rho} - \rho E_{n-1}(\rho)\). To obtain the asymptotic expansion for \( A \) and \( \hat{\Omega} \) one can use \( E_4(\rho) = 3 + \Gamma[-\frac{1}{3}]\rho^{1/3} + O(\rho)\). In comparing the asymptotic form of \( A \) and \( \hat{\Omega} \) with (26) one then gets

\[
-\frac{M_\infty}{2} = (8\kappa) \frac{a^2}{3} E_4(\frac{8\kappa}{3a^2}). \tag{29}
\]

\( B \) approaches a finite nonzero value as \( x \) tends to zero, while \( A \) and \( \hat{\Omega} \) diverge as \( \sim 1/x^2 \) for generic values of \( C \). This leads to a singularity in the metric tensor at \( x = 0 \). Nevertheless, the energy density is well behaved near the origin. From \( \chi \sim \pi - \frac{2\kappa}{a} \), one gets that the energy density \( T_{tt} = \frac{A}{\hat{B}} \chi^2 \sin^2 \chi \) has a finite limit as \( x \to 0 \).

\( A \) and \( \hat{\Omega} \) are nonsingular at the origin for the special case

i) \( C = C_0 \),

\[
C_0 = -\frac{a^2}{3} E_4(\frac{8\kappa}{3a^2}). \tag{30}
\]

For this case, \( A \) vanishes at the origin, \( A \to 0 \) and \( \hat{\Omega} + 1 \to \frac{1}{3} E_4(\frac{8\kappa}{3a^2}) \) as \( x \to 0 \). A vanishing \( A(x) \) for some value of \( x \) indicates the existence of a horizon. Since \( C = C_0 \) means that \( A(x) \) vanishes at the origin, this corresponds to the limiting case of a horizon at the origin. \( A \) as a function of \( x \) is plotted for this case in figure one [solid curve]. \( \kappa \) and the asymptotic value of the mass \( M_\infty \) were set to one in the figure. \( a \) and \( C_0 \) can then be determined from (29) and (30), respectively. The result is \( a \approx 0.7264 \) and \( C_0 \approx -0.0018 \).

Aside from this limiting case, there exist two other classes of solutions which follow from (27):

ii) For \( C < C_0 \), \( A \), and also \( \hat{\Omega} + 1 \), have a zero at some finite nonzero value \( x_0 \) of \( x \), indicating an event horizon at \( x = x_0 \). The configuration corresponds to a hairy black hole. Even though it has a nonvanishing source, it shares properties with the extremal black hole, e.g. it has only one horizon (despite the possibility of having nonvanishing angular momentum), \( g^{yy} \geq 0 \), and the Hawking temperature is zero. We can then say that this is an extremal hairy black hole. In figure one \( A \) as a function of \( x \) is plotted for \( C = -0.15 \) [dot-dashed curve]. Again we have set \( \kappa = M_\infty = 1 \), while \( a \) is determined from (29). The result is \( a \approx 0.5988 \). Also from (30), \( C_0 \approx 0 \), and so \( C < C_0 \). For these values of the parameters there is a horizon at \( x = x_0 \approx .7406 \). This can be compared to an extremal BTZ black hole of mass equal to one [dotted curve in figure one], which has a horizon with radius \( \approx .7071 \).

iii) No horizons occur for \( C > C_0 \). Such configurations have a naked singularity at the origin. \( A \) as a function of \( x \) is plotted for \( C = .3 \) in figure one [dashed curve]. After setting \( \kappa = M_\infty = 1 \), (29) and (30) give \( a \approx .9475 \) and \( C_0 \approx -0.038 < C \), respectively.

Similar results are obtained after replacing (27) with other ansätze, for example, ones with winding number greater than one.
Figure 1: $A$ versus $x$ for a fixed value of the asymptotic mass and the coupling, $M_\infty = \kappa = 1$, and various values of $C$ and $a$ consistent with [29]. Horizons occur at zeros of $A$. Case i) [solid curve] has $C = C_0 = -0.00018$ and $a \approx 0.7264$, and it is the limiting case of a horizon at the origin. Case ii) [dot-dashed curve] has $C = -0.15 < C_0 \approx 0$ and $a \approx 0.5998$, and exhibits a horizon at $x = x_0 \approx 0.7406$. Case iii) [dashed curve] has $C = 0.3 > C_0 \approx -0.038$ and $a \approx 0.9475$ with no horizon. The plots are compared to $A$ versus $x$ for the extremal BTZ black hole [dotted curve] with $M_\infty = 1$ and horizon at $\approx 0.7071$.

We conclude with the following remarks:

Whether or not the local symmetry we found plays a role in the corresponding conformal theory is an open question. The symmetry applies only for extremal black holes and extremal hairy black holes. No analogous local symmetry which mixes the nonlinear $\sigma$-model and the space-time metric was found in the case of non-extremal black holes. We also found no evidence of this symmetry when the quadratic term in the nonlinear $\sigma$-model is included in the action.[9]

It has been noted that many properties of extremal black holes are distinct from non-extremal black holes, and that the extremal limit of non-extremal black holes is singular. For example, their topologies differ. Also unlike in the non-extremal case, extremal black holes have zero Hawking temperature and zero entropy.[10]-[15] It was also found that extremal and nonextremal black holes have distinct greybody factors and quasinormal mode structure.[16]-[21] The extremal hairy black hole solutions we obtained here also have properties which are distinct from that of other hairy black holes. Extremal hairy black holes have a traceless energy momentum tensor and zero Hawking temperature. It would be of interest to know if there are other properties which are unique to extremal hairy black holes.

The energy momentum tensor (6) for our system has the same form as for electromagnetism. It is therefore natural to ask whether or not similar solutions exist in the 2 + 1 Einstein–Maxwell system. Given analogous expressions for the electromagnetic field strengths $f_{\mu\nu}$ in a stationary space-time, namely $f_{xy} = f(y), f_{zt} = f_{yt} = 0$, one can solve the Einstein equations for the metric tensor given in (21). Instead of (22), the Einstein equations now give $\frac{d^2}{dy^2} x(y)^2 + 2\kappa f(y)^2 = 0$. While $F_{xy}$ is undetermined for the nonlinear $\sigma$-model we have been investigating, $f_{xy}$, and hence $x(y)^2$, is fixed in the electromagnetic theory. From the sourceless Maxwell equations one gets

$$f_{xy} = \frac{q}{y}, \quad x(y)^2 = y + \frac{M_\infty}{2} + 2\kappa q^2 \ln y, \quad (31)$$
where $q$ and $M_{\infty}$ are constants, and we again assume that the space-time is asymptotically $AdS_3$. The general set of rotationally invariant solutions to Einstein–Maxwell equations in $2+1$ dimensions (which applies to the nonextremal case as well) was found in [22].

Of course, it is natural to ask whether or not this system can be generalized to higher dimensions, and specifically if a local symmetry between nonlinear model fields and the space-time exists in that setting. A natural candidate in four space-time dimensions would be the $SU(N)$ chiral model coupled to gravity. The Skyrme model coupled to gravity in $3+1$ space-time has been studied for some time and shown to admit solitons and hairy black holes, [23]-[41]. The quadratic term in the chiral fields was present in these articles. Our work suggests that novel results, such as the existence of a hidden local symmetry between chiral fields and the space-time metric may appear if the quadratic term is absent, and the only matter contribution to the action is the quartic (Skyrme) term.

**Acknowledgments**

We are very grateful to M. Kaminski, A. Pinzul and S. Sarker for valuable discussions.

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